MATRIX UNITS AND SCHUR ELEMENTS
FOR THE DEGENERATE CYCLOTOMIC HECKE ALGEBRAS

DEKE ZHAO

ABSTRACT. The paper uses the cellular basis of the (semi-simple) degenerate cyclotomic Hecke algebras to investigate these algebras exhaustively. As a consequence, we describe explicitly the “Young’s seminormal form” and an orthogonal bases for Specht modules and determine explicitly the closed formula for the natural bilinear form on Specht modules and Schur elements for the degenerate cyclotomic Hecke algebras.

1. INTRODUCTION

The cyclotomic Hecke algebras or Ariki-Koike algebras were introduced independently by Ariki and Koike [AK] who were interested in them because they are a natural generalization of the Iwahori-Hecke algebras of types $A$ and $B$, and by Brundan and Malle [BM] who conjectured these algebras should play a role in the modular representations of finite groups of Lie types. The cyclotomic Hecke algebras also appear in a different guise in the work of Cherednik [C] as a family of cyclotomic quotient of the (extended) affine Hecke algebras. Now the cyclotomic Hecke algebras play a fundamental role in many branches of mathematics and were investigated in various settings.

The cyclotomic Hecke algebras have a natural rational degeneration—degenerate cyclotomic Hecke algebras, which is a family of finite dimensional quotient algebras of the degenerate (extend) affine Hecke algebras, see [K, Chap. 7]. Now it turns out that the category of integral representations of degenerate affine Hecke algebras consists precisely of all inflations of all its cyclotomic quotients and the study of degenerate cyclotomic Hecke algebras has some independent interests, see for example [BK1, BK4]. The purpose of this paper is to investigate exhaustively the degenerate cyclotomic Hecke algebras using the cellular basis of these algebras, some investigations are outlined in Ariki, Mathas and Rui’s work [AMR, §6], which should lead to a better understanding of these algebras.

The paper is largely inspired by Mathas’s work [M04], which originates form Murphy’s classical work [Mu95, Mu92]. For semi-simple degenerate cyclotomic Hecke algebras, we explicitly construct the matrix unites of it (Theorem 3.20). To do this we give a detailed investigation on Specht modules, which enables us to determine explicitly the “Young’s seminormal form” and an orthogonal bases for Specht modules (Theorem 3.15 and Corollary 3.16). Furthermore, we obtain a closed formula on the natural bilinear forms on Specht modules (Theorem 3.18), which answer a question of Ariki, Mathas and Rui [AMR, §6.9]. As an application, we obtain a complete set of primitive idempotents of semi-simple degenerate cyclotomic Hecke algebras (Theorem 3.27), and describe ‘explicitly’ the Brundan-Kleshchev isomorphism ([BK2, Corollary 1.3]) between degenerate cyclotomic Hecke algebras and cyclotomic Hecke algebras in the semi-simple case.

It is well-known that the Schur elements play a powerful role in the representation theory of symmetric algebras, see for example [CR, Chap.9] and [GP, Chap.7]. In the case of the degenerate cyclotomic Hecke algebras, Brundan and Kleshchev’s work [BK1, Theorem A2] showed that these algebras are symmetric algebras for all parameters, which enable us to use the Schur elements to determine when Specht modules are projective irreducible and whether the algebra is semi-simple. In the paper we determine explicitly Schur elements for the degenerate cyclotomic Hecke algebras by computing the trace form on some nice idempotents of these algebras (Theorem 7.9). In the process of our computation, we introduce firstly the operation for the degenerate cyclotomic Hecke algebras (Definition 4.1), which is very different from that one introduced by Mathas [M04, §3] for cyclotomic Hecke algebras, and then by a straightforward computation based on two key Lemmas 6.3 and 7.8.

The lay-out of this paper as follows. In section 2 we recall the definition of the degenerate cyclotomic Hecke algebras and the non-degenerate trace form on these algebras. In the long but fundamental Section 3, use the cellular bases we give a thorough investigation of the (semi-simple) degenerate cyclotomic Hecke algebras, for example the “Young’s seminormal form” and a orthogonal bases for Specht modules and the primitive idempotents of these algebras. The dual Specht module, or equivalently,
a dual cellular basis of the degenerate cyclotomic Hecke algebras is considered in Section 4, which is applied in Section 5 to give some nice idempotents of these algebras. Section 6 gives a direct computation of the trace form on these nice idempotents without any assumption, which gives a different proof of the non-degeneration of the trace form. Finally, a closed formula for Schur elements for the degenerate cyclotomic Hecke algebras is given in Section 7. Throughout this paper, we assume that \( R \) is a commutative ring and that \( m \) and \( n \) are positive integers unless otherwise stated.

2. Degenerate cyclotomic Hecke algebras

In this section, we recall that the definitions of degenerate cyclotomic Hecke algebras and of degenerate affine Hecke algebras. The non-degenerate trace form on the degenerate cyclotomic Hecke algebras and some basic facts are reviewed briefly.

2.1. Let \( m, n \) be positive integers. Recall from [ST] or [Co] that the complex reflection group \( W_{m,n} \) of type \( G(m, 1, n) \) is the finite group generated by elements \( s_0, s_1, \ldots, s_{n-1} \) subject to the relations

\[
\begin{align*}
\langle s_0^m = 1, & \quad s_0 s_1 s_0 = s_1 s_0 s_1 \\
\langle s_i^2 = 1, & \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad i > 1 \\
& \quad s_i s_j = s_j s_i, \quad |i - j| > 1.
\end{align*}
\]

In particular, the subgroup \( \langle s_1, \ldots, s_{n-1} \rangle \) of \( W_{m,n} \) is isomorphic to the symmetric group \( S_n \) of order \( n \) with simple transposition \( s_i = (i, i+1) \) for \( i = 1, \ldots, n-1 \). It is well-known that \( W_{m,n} \cong (\mathbb{Z}/m\mathbb{Z})^n \times S_n \).

Clearly, \( W_{1,n} \) is the Weyl group of type \( A_n \) and \( W_{2,n} \) is the Weyl group of type \( B_n \).

2.2. Definition. Let \( R \) be a commutative ring and \( Q = \{q_1, \ldots, q_m\} \subset R \). The degenerate cyclotomic Hecke algebra is the unital associative \( R \)-algebra \( \mathcal{H} := \mathcal{H}_{m,n}(Q) \) generated by \( s_0, s_1, \ldots, s_{n-1} \) and subjected to relations

\[
\begin{align*}
(i) \quad & (s_0 - q_1) \ldots (s_0 - q_m) = 0, \\
(ii) \quad & s_0(s_1 s_0 + s_1) = (s_1 s_0 + s_1)s_0, \\
(iii) \quad & s_i^2 = 1, \quad 1 \leq i < n, \\
(iv) \quad & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad 1 \leq i < n - 1, \\
(v) \quad & s_i s_j = s_j s_i, \quad |i - j| > 1.
\end{align*}
\]

The elements \( x_1 := s_0 \) and \( x_{i+1} := s_i x_i s_i + s_i \) for \( i = 1, \ldots, n-1 \) of \( \mathcal{H} \) are called the Jucys-Murphy elements of \( \mathcal{H} \).

Clearly, \( \mathcal{H}_{1,n}(Q) \) is exactly the group algebra \( R S_n \) and \( x_1, \ldots, x_n \) are algebraically dependent, moreover, for all \( i \), the minimal polynomial of \( x_i \) can be determined explicitly, see Corollary 3.28(ii). The following facts will be used frequently.

2.3. Lemma. Suppose that \( 1 \leq i < n \) and \( 1 \leq j, k \leq n \). Then

\[
\begin{align*}
(i) \quad & s_j x_j - x_{j+1} s_j = -1 \quad \text{and} \quad s_{j-1} x_j - x_{j-1} s_{j-1} = 1, \\
(ii) \quad & s_i x_j = x_j s_i \quad \text{if} \quad i \neq j - 1, j, \\
(iii) \quad & x_j x_k = x_k x_j \quad \text{if} \quad 1 \leq j, k \leq n, \\
(iv) \quad & s_j(s_j x_{j+1}) = x_j s_{j+1} s_j \quad \text{and} \quad s_j(x_j + x_{j+1}) = (x_j + x_{j+1}) s_j, \\
(v) \quad & \text{if} \quad a \in R \quad \text{and} \quad i \neq j \quad \text{then} \quad s_i \quad \text{commutes with} \quad (x_1 - a)(x_2 - a) \cdots (x_j - a).
\end{align*}
\]

Proof. (i) follows directly by the definition of \( x_{j+1} \) and 2.2(iii).

(ii) If \( j = 1 \) then \( s_1 x_1 = s_1 x_0 = x_1 s_1 \) whenever \( i \neq 1 \) according to 2.2(v). Assume that for all \( j = l \geq 1 \), \( s_1 x_1 = x_1 s_1 \) if \( i \neq l - 1, l \). Suppose that \( j = l + 1 \). Then, by induction and 2.2(v), \( s_1 x_j = s_1(s_l + s_l x_l) = x_j s_l \) if \( i \neq l - 1, l, l + 1 \). On the other hand,

\[
\begin{align*}
s_{l-1} x_{l+1} - x_{l+1} s_{l-1} & = s_{l-1} s_l - s_l s_{l-1} + s_{l-1} s_l x_l s_l - s_l x_l s_{l-1} \\
& = s_{l-1} s_l x_{l-1} s_{l-1} s_l - s_l s_{l-1} x_{l-1} s_{l-1} s_l \quad \text{since} \quad (s_l x_l) s_{l-1} = s_{l-1} x_{l-1} s_l, \\
& = s_l s_{l-1} (s_l x_{l-1} - x_{l-1} s_l) s_{l-1} s_l,
\end{align*}
\]

so, by induction, \( s_{l-1} x_{l+1} = x_{l+1} s_{l-1} \) and (ii) is proved.

(iii) Without loss of generality, we may assume that \( 1 \leq j \leq k \leq n \). Then 2.2(ii) and 2.2(v) imply that \( x_1 x_k = x_k x_1 \) for \( k = 1, \ldots, n \). Assume that for all \( j = l \geq 1 \) and for all \( l \leq k \leq n \), \( x_k x_k = x_k x_k \).
Suppose that \( j = l + 1 \). Then, by (ii) and induction, \( x_j x_k = (s_l + s_l x_l s_l) x_k = x_k x_i \) for all \( k \geq i = l + 1 \). Hence \( x_j x_k = x_k x_j \) for all \( 1 \leq j, k \leq n \).

(iv) The first equality follows by \( 2.2(\text{iii}) \) and (iii), and the second one follows by (ii).

(v) Let \( X_i(a) := (x_1 - a) \cdots (x_j - a) \). Obviously \( s_i X_i(a) = X_i(a) s_i \) if \( i \neq j \). Assume that for all \( j = l + 1 \), \( s_i X_i(a) = X_i(a) s_i \) if \( i \neq j \). Then, by induction and (ii),

\[
\begin{align*}
\hat{X}_{l+1}(a) &= \hat{X}_l(a)(x_{l+1} - a) = X_{l+1}(a) s_i,
\end{align*}
\]

if \( i \neq l, l + 1 \). On the other hand, by induction and (iv),

\[
\begin{align*}
\hat{X}_{l+1}(a) &= X_{l+1}(a) s_i(x_l - a)(x_{l+1} - a) = X_{l+1}(a)(s_i x_l x_{l+1} - a s_l x_l x_{l+1} - a^2 s_l) = X_{l+1}(a) s_i.
\end{align*}
\]

Thus \( s_i X_{l+1}(a) = X_{l+1}(a) s_i \) for \( i \neq l + 1 \). The proof is completed. □

Recall from [D] or [L89] that the degenerate affine Hecke algebra \( H_n^{aff} \) of \( GL_n(\mathbb{C}) \) is the associated \( R \)-algebra which is equal as an \( R \)-module to the tensor product \( R[y_1, \ldots, y_n] \otimes_R RS_n \) of the polynomial algebra \( R[y_1, \ldots, y_n] \) and the group algebra \( RS_n \). Multiplication is defined so that \( R[y_1, \ldots, y_n] \) and \( RS_n \) are subalgebras, and in addition

\[
\begin{align*}
s_i y_{i+1} &= y_i s_i, \\
s_i y_j &= y_j s_i & \text{if } i \neq j - 1, j.
\end{align*}
\]

The following lemma shows that the degenerate cyclotomic Hecke algebras are the cyclotomic quotients of the degenerate affine Hecke algebra, which play a fundamental role in the study of the degenerate affine Hecke algebra, see for example [K, Chap. 7].

2.4. Lemma. Let \( J_Q \) denote the two-sided ideal of \( H_n^{aff} \) generated by \( (y_1 - q_1) \cdots (y_n - q_m) \). Then there is a surjective homomorphism of algebras \( \pi : H_n^{aff} \rightarrow \mathcal{H} \) such that \( y_i \mapsto x_i \) for each \( i \) and \( s_j \mapsto s_j \) for each \( j \). Then \( \pi \) is a surjective homomorphism and \( \text{Ker}(\pi) = J_Q \).

Proof. The Lemma follows by definitions and Lemma 2.3. □

For degenerate cyclotomic Hecke algebra \( \mathcal{H} \), we have the following important theorem.

2.5. Theorem ([K], Theorem 7.5.6). The degenerate cyclotomic Hecke algebra \( \mathcal{H} \) is a free \( R \)-module with basis \( \{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} w \mid 0 \leq i_1, \ldots, i_n < m, w \in S_n \} \).

Now we recall the non-degenerate trace form on \( \mathcal{H} \) constructed by Brundan and Kleshchev [BK1], which is our main investigation in this paper. To describe this one and for the computation in Section 6, we introduce some notation. Let \( R_m[x_1, \ldots, x_n] \) be the level \( m \) truncated polynomial algebra, that is, the quotient of the polynomial algebra \( R[y_1, \ldots, y_n] \) by the two-sided ideal generated by \( y_1^m, \ldots, y_n^m \). Also define a grading on the twist tensor algebra \( R_m[y_1, \ldots, y_n] \otimes_R RS_n \) by declaring that each \( y_i \) is of degree 1 and each \( w \in S_n \) is of degree 0.

2.6. Lemma ([BK1], Lemma A1). Let \( \hat{\tau} : R_m[y_1, \ldots, y_n] \otimes_R RS_n \rightarrow R \) be the \( R \)-linear map defined by

\[
\hat{\tau}(y_1^{i_1} \cdots y_n^{i_n} w) := \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = m - 1 \text{ and } w = 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( \hat{\tau} \) is a non-degenerate trace form on \( R_m[y_1, \ldots, y_n] \otimes_R RS_n \).

Now let \( l = (m - 1)n \) and define a filtration

\[
RS_n = F_0 \mathcal{H} \subseteq F_1 \mathcal{H} \subseteq \cdots \subseteq F_l \mathcal{H} = \mathcal{H}
\]

of \( \mathcal{H} \) by defining that

\[
F_r \mathcal{H} := \text{Span}_R \{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} w \mid i_1 + \cdots + i_n \leq r, w \in S_n \}.
\]

For any \( 0 \leq i \leq l \), let

\[
gr_i : \mathcal{H} \rightarrow F_i \mathcal{H} / F_{i-1} \mathcal{H}
\]

be the map sending an element of \( \mathcal{H} \) to its degree \( i \) graded component.
2.7. Lemma ([BK1], Lemma 3.5). There is a well defined isomorphism of graded algebras
\[ \psi_n : R_m[y_1, \ldots, y_n] \otimes RS_n \rightarrow \text{gr}\mathcal{H} \]
such that \( y_i \mapsto \text{gr}_1 x_i \) for each \( i \) and \( s_j \mapsto \text{gr}_0 s_j \) for each \( j \).

The following surprising theorem says that \( \mathcal{H} \) is a symmetric algebra for all parameters \( q_1, \ldots, q_m \) in \( R \), Theorem 6.4 gives a different proof on the non-degeneration of the trace form \( \tau \) on \( \mathcal{H} \).

2.8. Theorem ([BK1], Theorem A2). Let \( \tau : \mathcal{H} \rightarrow R \) be the \( R \)-linear map determined by
\[ \tau(x_1^{i_1} \cdots x_n^{i_n} w) := \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = m - 1 \text{ and } w = 1, \\ 0, & \text{otherwise.} \end{cases} \]
Then \( \tau = \hat{\tau} \circ \text{gr}_1 \), which is a non-degenerate trace form on \( \mathcal{H} \).

2.9. Note that \( \tau \) is essentially independent of the choice of basis of \( \mathcal{H} \). We will need the following easy verified facts.

(i) Suppose that \( h_1, h_2 \in \mathcal{H} \). Then \( \tau(h_1 h_2) = (h_2 h_1) \).
(ii) Suppose that \( w, v \in S_n \) and that \( 0 \leq i_1, \ldots, i_n < m \). Then
\[ \tau(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} wv) = \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = m - 1 \text{ and } w = v^{-1}, \\ 0, & \text{otherwise.} \end{cases} \]

2.10. Assumption. In this paper we will mainly be concerned with the semi-simple degenerate cyclotomic Hecke algebras; these were classified by Ariki-Mathas-Rui [AMR, Theorem 6.11] who showed that when \( R \) is a field \( \mathcal{H} \) is semisimple if and only if
\[ P_\mathcal{H}(Q) = n! \prod_{1 \leq i < j \leq n} \prod_{|d| < n} (d + q_i - q_j) \neq 0. \]
For most of what we do it will be enough to assume that \( R \) is a ring in which \( P_\mathcal{H}(Q) \) is invertible.

3. A orthogonal basis of the degenerate cyclotomic Hecke algebra

In this section, we review the cellular basis and Specht modules of the degenerate cyclotomic Hecke algebra \( \mathcal{H} \). Using the cellular basis, we first give the “Young’s seminorma form” for Specht modules (Theorem 3.15), which enables us to give a orthogonal bases of Specht modules (Corollary 3.16) and a orthogonal bases and primitive idempotents of \( H \) (Theorems 3.20 and 3.27). Furthermore, we obtain a closed formula for the natural bilinear form on Specht modules, which answer a question of Ariki, Mathas and Rui [AMR, §6.9], and give a differential proof of the Brundan-Kleshchev isomorphism between the cyclotomic Hecke algebras and the degenerate ones in [BK2, Corollary 1.3] when these algebras are semi-simple.

3.1. Definition (Graham and Lehrer [GL96]). Let \( A \) an \( R \)-algebra. Fix a partially ordered set \( \Lambda = (\Lambda, \geq) \) and for each \( \lambda \in \Lambda \) let \( T(\lambda) \) be a finite set. Finally, fix \( C_\lambda^\Lambda \in A \) for all \( \lambda \in \Lambda \) and \( s, t \in T(\Lambda) \). Then the triple \((\Lambda, T, C)\) is a cell datum for \( A \) if:

(i) \( \{C_\lambda^\Lambda| \lambda \in \Lambda \text{ and } s, t \in T(\Lambda)\} \) is an \( R \)-basis for \( A \);
(ii) the \( R \)-linear map \(* : A \rightarrow A\) determined by \( (C_\lambda^\Lambda)^* = C_{\lambda t}^\Lambda \) for all \( \lambda \in \Lambda, s, t \in T(\Lambda) \) is an anti-automorphism of \( A \);
(iii) for all \( \lambda \in \Lambda, s \in T(\Lambda) \) and \( a \in A \) there exist scalars \( r_{su}(a) \in R \) such that
\[ aC_\lambda^\Lambda = \sum_{w \in T(\Lambda)} r_{su}(a)C_{sw}^\Lambda \pmod{A^{>\lambda}}, \]
where \( A^{>\lambda} = \text{Span}_R \{C_\mu^\lambda| \mu \gg \lambda \text{ and } a, b \in T(\mu)\} \).

An algebra \( A \) is a cellular algebra if it has a cell datum and in this case we call \( \{C_\lambda^\Lambda| s, t \in T(\Lambda), \lambda \in \Lambda\} \) a cellular basis of \( A \).
3.2. Recall that an $m$-multipartition of $n$ is a ordered $m$-tuple $\lambda = (\lambda^1; \ldots; \lambda^m)$ of partitions $\lambda^i$ such that $n = \sum_{i=1}^m |\lambda^i|$. Denote by $\mathcal{P}(m, n)$ the set of all $m$-multipartitions of $n$, which is a poset under dominance $\triangleright$, where $\lambda \triangleright \mu$, if

$$
\sum_{k=1}^{i-1} |\lambda^k| + \sum_{l=1}^{j} \lambda^i_l \geq \sum_{k=1}^{i-1} |\mu^k| + \sum_{l=1}^{j} \mu^i_l,
$$

for all $1 \leq i \leq m$ and $j \geq 1$. We write $\lambda \triangleright \mu$ if $\lambda \triangleright \mu$ and $\lambda \neq \mu$.

Suppose that $\lambda$ is an $m$-multipartition of $n$ and let $m = \{1, \ldots, m\}$. The diagram of $\lambda$ is the set $[\lambda] := \{(i, j, c) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times m | 1 \leq j \leq \lambda^i_c\}$. The elements of $[\lambda]$ are the nodes of $\lambda$; more generally, a node is any element of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times m$. We may and will identify $[\lambda]$ with the $m$-tuple of diagrams of the partitions $\lambda^i$, for $1 \leq c \leq m$. Recall that a node $y \notin [\lambda]$ is an addable node for $\lambda$ if $[\lambda] \cup \{y\}$ is the diagram of an $m$-multipartition, and denote by $\mathcal{A}(\lambda)$ the set of all addable nodes for $\lambda$; similarly, $y \in [\lambda]$ is a removable node for $\lambda$ if $[\lambda] \setminus \{y\}$ is the diagram of an $m$-multipartition, and denote by $\mathcal{R}(\lambda)$ the set of all removable nodes for $\lambda$.

A $\lambda$-tableau is a bijection $t : [\lambda] \to \{1, 2, \ldots, n\}$, and if $t$ is a $\lambda$-tableau write $\text{Shape}(t) = \lambda$. As with diagrams, we may and will think of a tableau $t$ as an $m$-tuple of tableaux $t = (t^1; \ldots; t^m)$, where $t^i$ is a $\lambda^i$-tableau, for $1 \leq c \leq m$. The tableaux $t^i$ are called the components of $t$. A tableau is standard if in each component the entries increase along the rows and down the columns; let $\text{Std}(\lambda)$ be the set of standard $\lambda$-tableaux.

Given an $m$-multipartition $\lambda$ let $t^\lambda$ be the $\lambda$-tableau with the numbers $1, 2, \ldots, n$ ordered first along the rows of $t^\lambda$ and then the rows and down the columns. The symmetric group $S_n$ acts from the right on the set of $\lambda$-tableaux; let $S_\lambda = S_{\lambda^1} \times \cdots \times S_{\lambda^m}$ be the row stabilizer of $t^\lambda$. For any $\lambda$-tableau $t$ let $d(t)$ be the unique element of $S_n$ such that $t = t^\lambda d(t)$ and denote by $\ell(t)$ the length of $d(t)$.  

3.3. Example. Let $\lambda = (3 \cdot 2; 2 \cdot 1; 1)$ be a 3-multipartition of 9. Then $S_\lambda = S_{(1, 2, 3)} \times S_{(4, 5)} \times S_{(6, 7)} \times S_{(8)} \times S_{(9)}$, 

$\lambda = \begin{pmatrix} 3 & 2 & 1 \hline 4 & 5 \end{pmatrix}$, $t^\lambda = \begin{pmatrix} 1 & 2 & 3 & 6 & 7 & 9 \hline 4 & 5 \end{pmatrix}$, $t = \begin{pmatrix} 2 & 4 & 6 & 7 & 8 & 9 \hline 1 \end{pmatrix}$

$\mathcal{R}(\lambda) = \begin{pmatrix} & & + \hline & + \end{pmatrix}$, $\mathcal{A}(\lambda) = \begin{pmatrix} + & + \hline + & + & + \hline \end{pmatrix}$, $d(t) = (1, 2, 4, 3, 6, 7, 8, 9)$,

where $\square$ (resp., $\square^+$) means the node is removable (resp., addable) and $S_{(1, 2, 3)}$ is the symmetric group on letters 1, 2, 3 and so on.

Let $\prec$ be the Bruhat order on $S_n$. The following fact was first proved by Ehresmann and then rediscovered by Dipper and James [DJ86], see for example, [M99, Theorem 3.8].

3.4. Lemma (Ehresmann Theorem). Suppose that $s$ and $t$ are standard tableaux of the same shape. Then $d(s) < d(t)$ if and only if $s \triangleright t$.

3.5. Let $\ast$ be the $R$-linear anti-automorphism of $\mathcal{H}$ determined by $x^*_i = x_i$ for all $1 \leq i \leq n$, $s^*_j = s_j$ for all $1 \leq j \leq n - 1$. Then $w^\ast = w^{-1}$ for all $w \in S_n$ and $f^\ast = f$ for all $f \in R[x_1, \ldots, x_n] \subset \mathcal{H}$.

3.6. Definition. Suppose that $\lambda = (\lambda^1; \ldots; \lambda^m)$ is an $m$-multipartition of $n$ and define $a_i = \sum_{j=1}^{i-1} |\lambda^j|$ for $1 \leq i \leq m$ with $a_1 = 0$. Let $m_\lambda = x_\lambda u^+_\lambda$, where

$$x_\lambda := \sum_{w \in S_\lambda} w \quad \text{and} \quad u^+_\lambda := \prod_{i=2}^{m} \prod_{k=1}^{a_i} (x_k - q_i).$$

Finally, given standard $\lambda$-tableaux $s$ and $t$ let $m_{st} = d(s)^* m_\lambda d(t)$.

It follows from Lemma 2.3(v) that all of elements in $RS_\lambda$ commute with $u^+_\lambda$, in particular, $m_\lambda = x_\lambda u^+_\lambda = u^+_\lambda x_\lambda$. Observe that $m_\lambda = m_{\lambda^1 \lambda^2}$ and $m_{\lambda^2} = m_{\lambda}$ for standard $\lambda$-tableaux $s$ and $t$. Whenever we write $m_{st}$ in what follows $s$ and $t$ will be standard tableaux of the same shape (and similarly, for $f_{st}$ etc.).
3.7. **Theorem** ([AMR], THEOREM 6.3). The degenerate cyclotomic Hecke algebra $\mathcal{H}$ is free as an $R$-module with cellular basis $\{m_{\alpha}|s, t \in \text{Std}(\lambda)\}$ for $\lambda$ an $m$-multipartition of $n$.

If $t$ is any tableau and $k \geq 0$ is an integer, let $t \downarrow k$ be the subtableau of $t$ which contains the integers $1, \ldots, k$. Observe that $t$ is standard if and only if $\text{Shape}(t \downarrow k)$ is an $m$-multipartition for all $k = 1, \ldots, n$. We extend the dominance order $\succeq$ on the set of $m$-multipartitions to the set of standard tableaux by defining $s \succeq t$ if $\text{Shape}(s \downarrow k) \succeq \text{Shape}(t \downarrow k)$ for all $k = 1, \ldots, n$, and write $s \succeq t$ if $s \succeq t$ and $s \neq t$. Define the residue of $k$ in $t$ to be $\text{res}_t(k) = j - i + q$, if $k$ appears in node $(i, j, c) \in t$.

3.8. **Lemma** (cf. [M99], Lemma 3.34). Assume that $R$ is a field and that Assumption 2.10 holds. Suppose that $\lambda$ and $\mu$ are $m$-multipartition of $n$ and let $s \in \text{Std}(\lambda)$ and $t \in \text{Std}(\mu)$.

(i) $s = t$ (and $\lambda = \mu$) if and only if $\text{res}_s(k) = \text{res}_t(k)$ for $k = 1, \ldots, n$.

(ii) Suppose that $\lambda = \mu$ and there exists an $i$ such that $\text{res}_s(k) = \text{res}_t(k)$ for all $k \neq i, i + 1$. Then either $s = t$ or $s = t(i, i + 1)$.

**Proof.** Note that the nodes of $s$ and $t$ containing entry $n$ are removable nodes, which have distinct residues if they are distinct nodes. Then both parts of Lemma follow by using the induction argument on $n$. \qed

The Lemma says that when Assumption 2.10 holds the residues separate the standard tableaux; this enables us to give the orthogonal basis of degenerate cyclotomic Hecke algebras and Specht modules. Notice that both parts of the Lemma can be fail if the assumption does not hold.

If $\lambda$ is an $m$-multipartition then let $\mathcal{H}^{\lambda}$ be the free $R$-submodule of $\mathcal{H}$ with basis $\{m_{\alpha}|s, t \in \text{Std}(\mu)\}$ for $\mu \triangleright \lambda$. It follows from Theorem 3.7 and 3.1(iii) that $\mathcal{H}^{\lambda}$ is a two-sided ideal of $\mathcal{H}$.

3.9. **Definition.** The Specht module $S^\lambda$ is the left $\mathcal{H}$-module $\mathcal{H}m_\lambda/(\mathcal{H}m_\lambda \cap \mathcal{H}^{\lambda})$, which is a submodule of $\mathcal{H}/\mathcal{H}^{\lambda}$.

Theorem 3.7 implies that $S^\lambda$ is a free $R$-module with basis $\{m_t|t \in \text{Std}(\lambda)\}$, where $m_t = m_{\alpha} + \mathcal{H}^{\lambda}$. Further, by the general theory of cellular algebras, there is a natural associative bilinear form $\langle , \rangle$ on $S^\lambda$ which is determined by either

$$\langle m_a, m_b \rangle S^\lambda \equiv m_a m_b \text{ mod } \mathcal{H}^{\lambda}, \quad \text{or } \langle m_a, m_b \rangle S^\lambda \equiv m_a m_b \text{ mod } \mathcal{H}^{\lambda},$$

for $a, b, s, t \in \text{Std}(\lambda)$.

Now we begin to determine the “Young seminorm form” for Specht modules. Our start point is the following fact.

3.10. **Lemma** (cf. [M99], Lemma 3.29). Let $\lambda$ be an $m$-multipartition of $n$ and let $s$ and $t$ be standard $\lambda$-tableaux such that $s \triangleright t = s(i, i + 1)$ for some $i$ with $1 \leq i < n$. For $k = 1, \ldots, n$, suppose that there exist elements $r_s(k) \in R$ such that

$$x_k m_s = r_s(k) m_s + \sum_{a > s} r_a m_a \text{ for some } r_a \in R.$$

Then for $k = 1, \ldots, n$, there exist elements $r_b \in R$ such that

$$x_k m_t = r_t(k) m_t + \sum_{b > t} r_b m_b,$$

where $r_s(k) = r_t(k)$ if $k \neq i, i + 1$, $r_s(i) = r_s(i + 1)$ and $r_v(i + 1) = r_v(i)$.

**Proof.** First note that $m_t = s_i m_a$ because $s \triangleright t$ which implies that $\ell(s) < \ell(t)$ by Lemma 3.4. Therefore, by Lemma 2.3(ii), if $k \neq i, i + 1$ then

$$x_k m_t = x_k s_i m_a = s_i x_k m_a = r_s(k) m_t + \sum_{a > t} r_a m_a.$$

Next suppose that $k = i + 1$. Then using the first equality of Lemma 2.3(ii)

$$x_{i+1} m_t = x_{i+1} s_i m_a = (1 + s_i x_i) m_a = r_s(i) m_t + \sum_{b > t} r_b m_b.$$

The case $k = i$ is similar to the case $k = i + 1$ by using the second equality of Lemma 2.3(ii). Finally, equating the coefficients of these formulae, we complete the proof. \qed
3.11. **Theorem** ([AMR], Lemma 6.6). Suppose that \( \lambda \) is an \( m \)-multipartition of \( n \) and that \( s \) and \( t \) are standard \( \lambda \)-tableaux. Suppose that \( k \) is an integer with \( 1 \leq k \leq n \). Then there exist \( r_a \in R \) such that

\[
\begin{align*}
(i) & \quad x_km_\ast = \text{res}_s(k)m_\ast + \sum_{a > s} r_\ast m_a \mod H^{>\lambda}, \\
(ii) & \quad x_km_s = \text{res}_s(k)m_s + \sum_{a > s} r_a m_a \mod H^{>\lambda}.
\end{align*}
\]

or equivalently, there exist \( r_a \in R \) such that

\[
\begin{align*}
(i) & \quad x_km_\ast = \text{res}_s(k)m_\ast + \sum_{a > s} r_\ast m_a \mod H^{>\lambda}, \\
(ii) & \quad x_km_s = \text{res}_s(k)m_s + \sum_{a > s} r_a m_a \mod H^{>\lambda}.
\end{align*}
\]

**Proof.** Note that \( m_\ast d(t)^\ast = m_{a^\lambda} \) for any standard tableaux \( s \) and \( t \), we only need to show (ii). First consider the case where \( s = t^\lambda \), then \( m_s = m_\lambda + H^{>\lambda} \). Suppose that \( k \) appears in node \((i, j, c) \in s \) and that \( l \) is the smallest integer appears in component \( t^\lambda \). Then \( l \leq k \). Working modulo \( H^{>\lambda} \) and using Lemma 2.3 (ii), (iv), we obtain that

\[
x_k m_\lambda = x_k m_{k-1} + x_k m_{k-2} \cdots x_k m_{k-1}
\]

where \( h = s_{k-1} \cdots s_{l-1}(t_{l-1} \cdots t_{k-1}) m_s = s_{k-1} \cdots s_{l-1}(t_{l-1} \cdots t_{k-1})(u_s s_{k-1} \cdots s_{l-1} x_k \in H^{>\lambda}). \]

Therefore

\[
x_k m_\lambda = (s_{k-1} + t_{k-1} m_s + s_{k-2} \cdots s_{l-1} \cdots s_{k-2} m_s) m_\lambda \in H^{>\lambda}.
\]

So it is sufficient to show that, for \( k = 1, \ldots, n \),

\[
(\ast) \quad s_{k-1} m_s + s_{k-2} \cdots s_{l-1} m_s = (\text{res}_s(k) - c_m) m_s \equiv (j - i) m_s.
\]

When \( n = 1 \) there is nothing to prove so by induction we assume that (\ast) holds for all smaller integers of \( n \). We now proceed by induction on \( n \). The case \( k = 1 \) being trivial because \((x_1 - q_c) m_s = 0 \) and \( \text{res}_s(k) = q_c \). Suppose first that \( \lambda_{c+1} = \lambda_{c} \neq 0 \). Let \( \mu = (\lambda_1, \ldots, \lambda_k) \) and \( \mu = (\lambda_{c+1}^\mu, \ldots, \lambda_{c}^\mu) \). Then \( a = \mu < n \) and \( m_s = m_s + H^{>\lambda} \) for some element \( h \) of the subalgebra of \( R S \lambda \) generated by \( s_{k+1}, \ldots, s_n \). Furthermore, \( x_k m_\mu \in H^{>\lambda} \) and \( h, x_k \) commute by Lemma 2.3(ii). Hence working modulo \( H^{>\lambda} \) and arguing by induction on \( n \),

\[
x_k m_\lambda = x_k m_\mu h = \text{res}_s(k) m_\mu h = \text{res}_s(k) m_\lambda.
\]

Thus we may assume that \( h \) is in the last row of the \( c \)-component of \( t^\lambda \).

Next suppose that \( k = 1 \) is not in the first column of \( t^\lambda \) and that we have known the result for small \( k \). Then \( k > 1 \) and \( s_{k-1} \) is an element of \( S_{\lambda^c} \), so \( s_{k-1} m_\lambda = m_\lambda \) and therefore, by induction on \( k \),

\[
x_k m_\lambda = (s_{k-1} + s_{k-2} \cdots s_{k-1}) m_\lambda = (1 + \text{res}_s(k - 1)) m_\lambda.
\]

This reduces us to the case that \( k \) appears in the last row and the first column of \( t^\lambda \). Thus we can assume that \( k = a_{c+1} = \sum_{\ell=1}^c |\lambda^\ell| \) and that \( \lambda^\ell = (\lambda_1^\ell, \ldots, \lambda_{c+1}^\ell) \). Let \( p \) be the integer appears in the node \((i - 1, 1, c) \) and let \( u \) be the standard \( \lambda \)-tableau \( t^\lambda \). Then the second last row of \( u \) contains the integers \( p, p+2, \ldots, a_{c+1} \) and the last row of \( u \) contains the single integer \( p + 1 \).

So far we have shown that \( x_k m_\lambda = \text{res}_s(k) m_\lambda \) for \( a_c \leq k < a_{c+1} \). We also know that \( (x_{a_{c+1}} + \cdots + x_{a_{c+1}}) m_\lambda = r_m \) for some \( r \in R \) since \( x_{a_{c+1}} + \cdots + x_{a_{c+1}} \) belongs to the center of \( H \). Consequently, \( x_{a_{c+1}} m_\lambda = r_m \) for some \( r \in R \). Therefore, by Lemma 3.10

\[
x_k m_\lambda = \text{res}_a(k)m_\lambda + \sum_{a > u} r_\ast m_a \quad \text{and} \quad x_k m_\lambda = r_m m_\lambda + \sum_{a > b} r_a m_b.
\]

Now, by induction, we have

\[
x_{p+1} m_s = (1 + s_p x_p) s_p m_s = - (1 + s_p x_p) m_s + \sum_{a > s} r_a m_a
\]

\[
= (\text{res}_a(p) - 1)m_s + \sum_{a > s} r_a m_a
\]

\[
= \text{res}_a a_{c+1} m_s + \sum_{a > s} r_a m_a.
\]
Consequently, \( r_c = \text{res}_\lambda(a_{c+1}) \) as required. The general case follows by Lemma 3.10 and induction argument on \( \ell(s) \).

3.12. Remark. (i) If \( u \) is a standard tableau and \( \text{res}_u(k) \neq \text{res}_s(k) \) for some integer \( 1 \leq k \leq n \), then

\[
\frac{x_k - \text{res}_u(k)}{\text{res}_s(k) - \text{res}_u(k)} m_{st} = m_{st} + \sum_{a \succ s, b} r_{ab} m_{ab} \quad \text{for some } r_{ab} \in R.
\]

(ii) Apply the \( * \)-anti-automorphism, we get that

\[
m_{st} x_k = \text{res}_t(k) m_{st} + \sum_{a \succ t} r_a m_{st} \mod \mathcal{H}^{s \lambda} \quad \text{for some } r_a \in R.
\]

Let \( \mathcal{R}(k) \) be the complete set of possible residues \( \text{res}_t(k) \) as \( t \) runs over the set of all standard tableaux. Note that the residues in \( \mathcal{R}(k) \) are all distinct.

3.13. Definition ( [AMR, DEFINITION 6.7]). Suppose that \( \lambda \) is an \( m \)-multipartition of \( n \) and that \( s \) and \( t \) are standard \( \lambda \)-tableaux.

(i) Let \( F_t := \prod_{k=1}^{n} \prod_{\mathcal{R}(k) \ni c \neq \text{res}_s(k)} \frac{x_k - c}{\text{res}_s(k) - c} \).

(ii) Let \( f_{st} := F_s m_{st} F_t \).

Note that all of the factors in \( F_t \) commute so there are no need to specify an order of the terms in the product. The elements \( F_t \) are defined for any choice of field \( R \), regardless of whether or not Assumption 2.10 holds.

Extend the dominance order to the pairs of standard tableaux by defining \( (s, t) \succeq (a, b) \) if \( s \succeq a \) and \( t \succeq b \), and write \( (s, t) \succeq (a, b) \) if \( (s, t) \succeq (a, b) \) and \( (s, t) \neq (a, b) \).

3.14. Proposition. Assume that Assumption 2.10 holds and that \( \lambda \) is an \( m \)-multipartition of \( n \). Suppose that \( s \) and \( t \) are standard \( \lambda \)-tableaux.

(i) \( f_{st} = m_{st} + \sum_{(a, b) > (s, t)} r_{ab} m_{ab} \) for some \( r_{ab} \in R \);

(ii) If \( u \) is a standard \( \lambda \)-tableau, then \( F_u f_{st} = \delta_{su} f_{st} \) and \( f_{st} F_u = \delta_{tu} f_{st} \);

(iii) If \( k \) is an integer with \( 1 \leq k \leq n \), then \( x_k f_{st} = \text{res}_s(k) f_{st} \);

(iv) If \( a \) and \( b \) are standard \( \lambda \)-tableaux, then \( f_{st} f_{ab} = \delta_{at} r_f f_{ab} \) for some \( r_f \in R \).

Proof. (i) follows directly from Theorem 3.11(i), Remark 3.12, and Definition 3.13. Indeed, we have

\[
f_{st} = F_s m_{st} F_t
\]

\[
= \left( \prod_{k=1}^{n} \prod_{\mathcal{R}(k) \ni c \neq \text{res}_s(k)} \frac{x_k - c}{\text{res}_s(k) - c} \right) m_{st}
\]

\[
= \left( m_{st} + \sum_{a \succ s} r_a m_{st} \right) \prod_{k=1}^{n} \prod_{\mathcal{R}(k) \ni c \neq \text{res}_s(k)} \frac{x_k - c}{\text{res}_s(k) - c}
\]

\[
= m_{st} + \sum_{(a, b) > (s, t)} r_{ab} m_{ab}.
\]

(ii) and (iii) can be proved by using a variation argument of the proof of [M99, Proposition 3.35]. For the convenience of the reader especially for myself, we contain the proof. By (i) \( f_{st} = m_{st} + \sum_{(a, b) > (s, t)} r_{ab} m_{ab} \) for some \( r_{ab} \in R \). First suppose that \( u \) is a standard \( \lambda \)-tableau such that \( u \succeq s \), by Lemma 3.8(ii), there exists an integer \( k_1 \) such that \( \text{res}_u(k_1) \neq \text{res}_s(k_1) \). Then \( x_{k_1} - \text{res}_s(k_1) \) is a factor of \( F_s \), so \( F_u m_{st} = \sum_{b \succ a} r_b m_{st} \) for some \( r_b \in R \) by Theorem 3.11(i). Extended the dominance order \( \succeq \) to the total order \( > \) on the set of all standard \( \lambda \)-tableaux, and chose \( c_2 \) with respect to \( > \) with \( r_{c_2} \neq 0 \). Then \( c_2 > s \) and, as before, there exists an integer \( k_2 \) such that \( \text{res}_{c_2}(k_1) \neq \text{res}_s(k_2) \). Therefore \( F_u^2 m_{st} = \)}
Continuing in this way shows that \( F_u^N m_{st} = 0 \) whenever \( u \triangleright s \), where \( N = \sum_\lambda |\text{Std}(\lambda)| \).
So \( F_u^N f_{st} = \delta_{us} f_{st} \). Similarly, \( f_{st} F_u^N = \delta_{su} f_{st} \) by using Remark 3.12(ii).

Next let \( \tilde{f}_{st} = F_u^N f_{st} F_i \). Then
\[
\begin{align*}
x_k \tilde{f}_{st} &= x_k F_u^N f_{st} F_i = F_u^n(x_k f_{st}) \\
&= F_u^n(\text{res}_s(k) f_{st} + \sum_{a > s} r_a m_{st}) \\
&= \text{res}_s(k) \tilde{f}_{st}.
\end{align*}
\]
Hence, \( F_u \tilde{f}_{st} = \tilde{f}_{st} \). Furthermore, if \( u \neq s \) then, by Lemma 3.8(ii), there exists an integer \( k \) such that \( \text{res}_u(k) \neq \text{res}_s(k) \). So \( (x_k - \text{res}_u(k)) f_{st} = 0 \) and \( F_u \tilde{f}_{st} = 0 \) because \( x_k - \text{res}_u(k) \) is a factor of \( F_u \).

Finally, note that there exist \( r_a \in R \) such that \( m_{st} = \tilde{f}_{st} + r_a \tilde{f}_{st} \). So \( f_{st} = F_u m_{st} F_i = F_u (\tilde{f}_{st} + r_a \tilde{f}_{st}) F_i = \tilde{f}_{st} \). Thus (ii) and (iii) are proved.

(iv) By definition and (iii),
\[
f_{st} f_{ab} = F_u m_{st} F_i f_{ab} = \delta_{at} F_u m_{st} m_{ab} F_i = \delta_{at} F_u \left( r_t m_{ab} + \sum_{(s', b') > (a, b)} m_{s'b'} \right) F_i = \delta_{at} r_{ta} f_{ab}.
\]

Now the “Young’s seminormal form” for Specht modules can be given as follows.

3.15. **Proposition.** Suppose that \( \lambda \) is a multipartition of \( n \) and that \( s \) and \( u \) are standard \( \lambda \)-tableaux. Let \( t = s(i, i + 1) \) for some integer \( i \) with \( 1 \leq i < n \).

(i) If \( t \) is standard then
\[
s_i f_{su} = \begin{cases} 
\frac{1}{\text{res}_s(i) - \text{res}_u(i)} f_{su} + f_{tu}, & \text{if } s \triangleright t; \\
\frac{1}{\text{res}_s(i) - \text{res}_u(i)} f_{su} + \frac{(\text{res}_s(i) - \text{res}_u(i) - 1)(\text{res}_s(i) - \text{res}_u(i) + 1)}{(\text{res}_s(i) - \text{res}_u(i))^2} f_{tu}, & \text{if } t \triangleright s.
\end{cases}
\]

(ii) If \( t \) is not standard then
\[
s_i f_{su} = \begin{cases} 
-f_{su}, & \text{if } i \text{ and } i + 1 \text{ are in the same row of } s; \\
f_{su}, & \text{if } i \text{ and } i + 1 \text{ are in the same column of } s.
\end{cases}
\]

**Proof.** By Theorem 3.7 and Proposition 3.14(i), \( \{f_{su}\} \) is a basis of \( H \), so \( s_i f_{su} = \sum_{a,b} r_{ab} f_{ab} \) for some \( r_{ab} \in R \). By Proposition 3.14(ii), \( f_{ab} F_u = \delta_{bu} f_{au} \). Therefore, multiplying the equation for \( s_i f_{su} \) on the right by \( F_u \) shows that \( r_{ab} = 0 \) whenever \( a \neq u \); in particular, \( r_{ab} = 0 \) if \( \text{Shape}(b) \neq \lambda \). Hence, \( s_i f_{su} = \sum_a r_a f_{au} \), for some \( r_a \in R \), where \( a \) runs over the set of standard \( \lambda \)-tableaux.

Suppose that \( k \) is an integer such that \( k \neq i, i + 1 \). Then, by Lemma 2.3(ii) and Proposition 3.14(iii),
\[
x_k s_i f_{su} = s_i x_k f_{su} = \text{res}_s(k) s_i f_{su} = \text{res}_s(k) \sum_{a \in \text{Std}(\lambda)} r_a f_{au},
\]

On the other hand, by Proposition 3.14(iii), we have
\[
x_k s_i f_{su} = \sum_{a \in \text{Std}(\lambda)} r_a x_k f_{au} = \sum_{a \in \text{Std}(\lambda)} r_a \text{res}_a(k) f_{au}.
\]
Equating coefficients, \( r_a \text{res}_a(k) = r_a \text{res}_{s(k)}(k) \) for all \( k \neq i, i + 1, a \in \text{Std}(\lambda) \). Therefore, by Lemma 3.8(ii), \( r_a = 0 \) unless either \( a = s \) or \( a = t \) and \( t \) is standard.

Suppose that \( t \) is not standard . Then we have shown \( s_i f_{su} = r_s f_{su} \) for some \( r_s \in R \). By Proposition 3.14(i), \( f_{su} = m_{su} + \sum_{(a, b) > (s, u)} r_{ab} f_{ab} \) for some \( r_{ab} \in R \). Because \( t \) is not standard, either \( i \) and \( i + 1 \) are in the same row of \( t \) or they are in the column. In the first case, by [DJ86, Lemma 1.1(iv)], \( s_i \in S_n \cap S_\lambda \) and \( d(s) = s_i d(s) s_i \). Therefore \( s_i m_{su} = s_i d(s)^* m_\lambda d(u) = d(s)^* x_\lambda u_\lambda^* d(u) = m_{su} \).

In the second case, there is a unique standard tableau \( c \) such that \( t = s(i, i + 1) = cs_jw \) for some \( j \) and \( w \in S_n \) with \( \ell(t) = \ell(c) + 1 + \ell(w) \), and if \( b \) is any (standard) tableau with \( b \triangleright cs_j \) then
Corollary. Hence \( r \) which implies that (iv). □

Therefore
\[
s_i m_{su} = w^* m_{cs_i u} u = (s_i w^* s_j) m_{cs_j u} + h \quad \text{for some } h \in \mathcal{H} m_\mu, \mathcal{H} \text{ and } \mu \triangleright \lambda
\]

\[
= -s_i w^* d(c)^* m_{tu} u - s_i w^* s_j d(b)^* m_{tu} u + h
\]

\[
= -m_{su} + \sum_{b \triangleright s} r_b m_{bu}.
\]

So \( s_if_{su} = -f_{su} \) in the second case.

Now suppose that \( t = s(i, i + 1) \) is standard, we have shown that \( s_if_{su} = r_s f_{su} + f_{tu} \) for some \( r_s, r_t \in R \). First suppose that \( s \triangleright t \). Then \( s_i m_{su} = m_{tu} \) for any \( u \in \text{Std}(\lambda) \) since \( d(t) = d(s)s_i \). Therefore

\[
s_i f_{su} = s_i(m_{su} + \sum_{(a,b) \triangleright (s, u)} r_{ab} f_{ab}) \quad \text{for some } r_{ab} \in R
\]

\[
= m_{tu} + \sum_{(a,b) \triangleright (s, u)} r_{ab} s_i f_{ab}
\]

\[
= m_{tu} + \sum_{(a,b) \triangleright (1, u)} r_{ab} f_{ab}.
\]

Hence \( r_t = 1 \), that is, \( s_if_{su} = r_s f_{su} + f_{tu} \). Now, by Lemma 2.3(ii), we get

\[
x_{i+1}(s_if_{su}) = x_r x_{i+1} f_{su} + x_{i+1} f_{tu} = r_s f_{su} + r_t f_{tu} \text{ and } x_{i+1} f_{su} = s_i x_i f_{su} + f_{tu} = (r_s f_{su} + f_{tu} + r_s(i + 1) f_{tu}.
\]

Note that \( \text{res}_s(i) = \text{res}_s(i + 1) \) and \( \text{res}_s(i + 1) = \text{res}_s(i) \), we yield that \( r_s = \frac{1}{\text{res}_s(i) - \text{res}_s(i)} \).

Suppose that \( t \triangleright s \). Then \( t \triangleright t(i, i + 1) = s \) and \( s_{iu} f_{su} = \frac{1}{\text{res}_s(i) - \text{res}_s(i)} f_{su} \) by the same argument as above. Thus

\[
f_{su} = s_i^2 f_{su} = s_i(r_s f_{su} + r_t f_{tu}) = (r_s + r_t) f_{su} + r_t(r_s - \frac{1}{\text{res}_s(i) - \text{res}_s(i)}) f_{tu},
\]

which implies that \( r_s = \frac{1}{\text{res}_s(i) - \text{res}_s(i)} \) and \( r_t = \frac{(\text{res}_s(i) - \text{res}_s(i)) (\text{res}_s(i) - \text{res}_s(i) + 1)}{(\text{res}_s(i) - \text{res}_s(i))^2} \). □

Our next step is to construct an orthogonal basis of Specht modules with respect to the bilinear form \( \langle \rangle \). For each standard \( \lambda \)-tableau \( s \) let \( f_s = f_{st \lambda} + \mathcal{H} ^{\otimes \lambda} \). We have the following facts.

3.16. Corollary. Assume that Assumption 2.10 holds. Suppose that \( \lambda \) is an \( m \)-multipartition of \( n \).

(i) Suppose that \( t \) is a standard \( \lambda \)-tableau.

(a) There exist \( r_s \in R \) such that \( f_t = m_t + \sum_{s \triangleright t} r_s f_s \).

(b) Suppose that \( k \) is an integer with \( 1 \leq k \leq n \). Then \( x_k f_{t_1} = \text{res}_s(k) f_{t_1} \).

(c) Suppose that \( s \) is a standard \( \lambda \)-tableau. Then \( f_{s_1} f_{t} = \delta_{st} f_{t} \).

(ii) Suppose that \( s \) and \( t \) are standard \( \lambda \)-tableaux.

(a) If \( s = s(i, i + 1) \triangleright t \) then \( f_t = (s_i - \alpha) f_s \), where \( \alpha = \frac{1}{\text{res}_s(i) - \text{res}_s(i)} \).

(b) \( \langle f_s, f_t \rangle = \delta_{st} r_t \) for some \( r_t \in R \).

(c) \{ \langle f_s | s \in \text{Std}(\lambda) \rangle \} is an orthogonal basis of the Specht module \( S^\lambda \).

Proof. (i) follows directly by Proposition 3.14. Theorem 3.15(ii) implies (ii.a). Now, by Proposition 3.14(i) and (iv), (ii.b) is proved. (ii.c) is proved by using Theorem 3.7 and Proposition 3.14(i) and (iv). □
The inner products \( \langle f_s, f_t \rangle \), for \( s, t \in \text{Std}(\lambda) \) will be computed explicitly (as rational functions) in the following. To describe this we need some more notation. Given two nodes \( x = (i, j, k) \) and \( y = (a, b, c) \), write \( y < x \), if either \( c < k \), or \( c = k \) and \( b > j \).

Suppose that \( \lambda \) is an \( m \)-multipartition of \( n \) and that \( s \) be a standard \( \lambda \)-tableau. Then for each integer \( i \) with \( 1 \leq i \leq n \) there is a unique node \( x \in [\lambda] \) such that \( s(x) = i \). Let \( A_s(i) \) be the set of addable nodes for the \( m \)-multipartition \( \text{Shape}(s \downarrow i) \) which are strictly greater than \( x \) (with respect to \( \prec \)); similarly, let \( R_s(i) \) be the set of removable nodes which are strictly greater than \( x \) for the \( m \)-multipartition \( \text{Shape}(s \downarrow i - 1) \). If \( y = (i, j, s) \) is either an addable or a removable node, then we define its residue to be \( \text{res}(y) = j - i + q_s \). Finally, if \( \lambda \) is an \( m \)-multipartition let \( \lambda! = \prod_{s=1}^{m} \prod_{i \geq 1} \lambda_s! \).

3.17. Example. Let \( \lambda = (3\cdot 1; 1) \). Then \( A_\lambda(1) = \{(1, 1, 2)\} \), \( A_\lambda(2) = \{(2, 1, 1), (1, 1, 2)\} \), \( A_\lambda(3) = \{(2, 1, 1), (1, 1, 2)\} \), \( A_\lambda(4) = \{(1, 1, 2)\} \), \( A_\lambda(5) = \emptyset \). It follows directly that
\[
\frac{\prod_{i=1}^{5} \prod_{x \in A_\lambda(i)} (\text{res}_\lambda(i) - \text{res}(x))}{\prod_{y \in R_\lambda(i)} (\text{res}_\lambda(i) - \text{res}(y))} = 3!(-1 + q_1 - q_2)(q_1 - q_2)(1 + q_1 - q_2)(2 + q_1 - q_2),
\]
which equals exactly to \( \gamma_\lambda = \langle f_\lambda, f_\lambda \rangle \) determined by Theorem 3.18(ii.a).

Now we can give a closed formula of \( \gamma : = \langle f_t, f_\lambda \rangle \) for any \( m \)-multipartition \( \lambda \) of \( n \) and any standard \( \lambda \)-tableau \( t \). For a moment, we write \( i = (a, b, c) \in t \) if the integer \( i \) with \( 1 \leq i \leq n \) appears in the unique node \( (a, b, c) \) of \([\lambda]\) such that \( t(a, b, c) = i \).

3.18. Theorem. Assume that Assumption 2.10 holds. Suppose that \( \lambda \) is an \( m \)-multipartition of \( n \).

(i) Suppose that \( t \) is a standard \( \lambda \)-tableau. Then \( \gamma_t \) is uniquely determined by the two conditions
(a) \( \gamma_t = \lambda! \prod_{1 \leq s < t \leq m} (j - i + q_s - q_t) \); and
(b) if \( s = t(i, i + 1) \triangleright t \) then \( \gamma_s = \frac{\prod_{x \in A_\lambda(i)} (\text{res}_\lambda(i) - \text{res}(x))}{\prod_{y \in R_\lambda(i)} (\text{res}_\lambda(i) - \text{res}(y))} \).

(ii) Let \( s \) be a standard \( \lambda \)-tableau. Then \( \gamma_s = \prod_{i=1}^{n} \frac{\prod_{x \in A_\lambda(i)} (\text{res}_\lambda(i) - \text{res}(x))}{\prod_{y \in R_\lambda(i)} (\text{res}_\lambda(i) - \text{res}(y))} \).

Proof. (i.a) By definition,
\[
f_\lambda f_\lambda = (F_\lambda m_\lambda F_\lambda)(F_\lambda m_\lambda F_\lambda) = m_\lambda m_\lambda = \lambda! \prod_{t=2}^{m} \prod_{k=1}^{a_t} (x_k - q_t)m_\lambda
\]
\[
= \lambda! \prod_{t=2}^{m} \prod_{k=1}^{a_t} \prod_{s \in \lambda} (\text{res}_\lambda(k) - q_t)m_\lambda
\]
\[
= \lambda! \prod_{t=2}^{m} \prod_{s \in \lambda} \prod_{i \in \lambda} (j - i + q_s - q_t)m_\lambda
\]
\[
= \gamma_t m_\lambda.
\]
So (i.a) is proved.

Suppose that \( s = t(i, i + 1) \triangleright t \) and let \( \alpha = \frac{1}{\text{res}_\lambda(i) - \text{res}_\lambda(i)} \). Applying Corollary 3.16(ii), \( f_t = (s_t - \alpha)f_s \), moreover \( \langle s_t f_s, f_s \rangle = \alpha \gamma_s \) and \( \langle s_t f_s, s_t f_s \rangle = \gamma_s \) since \( \langle f_s, f_t \rangle = 0 \). Thus
\[
\gamma_t = \langle s_t f_s, s_t f_s \rangle - 2\alpha \langle s_t f_s, f_s \rangle + \alpha^2 \gamma_s = (1 - \alpha^2) \gamma_s.
\]
So (i.b) is proved.

(ii) We proceed by induction on \( \ell(s) \) for standard \( \lambda \)-tableaux \( s \). First, let \( s = t^\lambda \) and assume that the integer \( i = (a, b, c) \) with \( 1 \leq i \leq n \), that is \( a_c < i \leq a_{c+1} \). First consider the contribution that the addable and removable nodes in \([\lambda^\lambda] \) make to \( t^\lambda \). By definition, these nodes occur in pairs
\[(x, y) = ((a + 1, 1, c), (a, b - 1, c)).\] Therefore
\[
\prod_{x \in \mathcal{A}(i) \cap [\lambda]}(\text{res}_A(i) - x) = b \quad \text{and} \quad \prod_{y \in \mathcal{A}(i) \cap [\lambda]}(\text{res}_A(i) - y) = \lambda^c!.
\]

Next, consider the contribution that admissible and removable nodes in some \(t\)-component of \([\lambda]\) with \(t > c\) (there are no such node for \(t < c\)). In this case, we have \(\mathcal{A}(i) = \{(1, 1, c + 1), \ldots, (1, 1, m)\}\) and \(\mathcal{R}(i) = \{\emptyset\}\). Therefore
\[
\prod_{a \geq i} \prod_{y \in \mathcal{A}(i) \cap [\lambda]}(\text{res}_A(i) - y) = \prod_{i, j \in [\lambda]}(j - i + q_c - q_t).
\]

Hence
\[
\prod_{i = 1}^{m} \prod_{y \in \mathcal{A}(i) \cap [\lambda]}(\text{res}_A(i) - y) = \lambda^c! \prod_{1 \leq s < t \leq m}(j - i + q_s - q_t) = \gamma_{t^c}.
\]

Now assume that \(t^4 \triangleright \mathfrak{s}\) and there exists an integer \(k\) with \(1 \leq k < n\) such that \(t^4 = \mathfrak{s}(k, k + 1)\). Let \(\alpha = \frac{1}{\text{res}_A(k) - \text{res}_A(k)}\) and let
\[
L := \prod_{i = k, k + 1} \prod_{y \in \mathcal{A}(i) \cap [\lambda]}(\text{res}_A(i) - y) \quad \text{and} \quad R := \prod_{i = k, k + 1} \prod_{y \in \mathcal{A}(i) \cap [\lambda]}(\text{res}_A(i) - y).
\]

Since \(\mathcal{A}(i) = \mathcal{A}(i)\) and \(\mathcal{R}(i) = \mathcal{R}(i)\) for all \(i \neq k, k + 1\). Therefore, by (ii.b) and Assumption 2.10, we only need to show that \(L = (\alpha^2 - 1)R\).

Note that either both \(k\) and \(k + 1\) appear in the same component of \(\mathfrak{s}\), say the \(c\)-component of \(\mathfrak{s}\), or \(k\) and \(k + 1\) appear in the different components of \(\mathfrak{s}\). In the first case, we have that \(a_c < k, k + 1 \leq a_{c+1}\), moreover, \(k = (a + 1, 1, c) \in \mathfrak{s}\) and \(k + 1 = (a, \lambda^c_a, c) \in \mathfrak{s}\) for some \(a\) with \(\lambda^c > 1\) and \(a < \ell(\lambda^c)\). Now by definition,
\[
\mathcal{A}(k) = \{(1, 1, c + 1), \ldots, (1, 1, m)\}, \quad \mathcal{R}(k) = \{\emptyset\};
\]
\[
\mathcal{A}(k + 1) = \{(a + 1, 2, c), (a + 2, 1, c), (1, 1, c + 1), \ldots, (1, 1, m)\},
\]
\[
\mathcal{R}(k + 1) = \{(a + 1, 1, c), (a, \lambda^c - 1, c)\}.
\]

On the other hand, since \(k = (a, \lambda^c_a, c) \in t^k\) and \(k + 1 = (a + 1, 1, c) \in t^\lambda\),
\[
\mathcal{A}(k) = \{(a + 1, 1, c), (1, 1, c + 1), \ldots, (1, 1, m)\}, \quad \mathcal{R}(k) = \{(a, \lambda^c_a - 1, c)\};
\]
\[
\mathcal{A}(k + 1) = \{(1, 1, c + 1), \ldots, (1, 1, m)\}, \quad \mathcal{R}(k + 1) = \{\emptyset\}.
\]

Hence \(\alpha^2 - 1 = \frac{(\lambda^c_a)^2 - 1}{(\lambda^c_a)^2}\) and
\[
L = \frac{(\lambda^c_a)^2 - 1}{\lambda^c_a} \prod_{j = 1}^{m-c}(-a + q_c - q_{c+j})(\lambda^c_a - a + q_c - q_{c+j}),
\]
\[
R = \lambda^c_a \prod_{j = 1}^{m-c}(-a + q_c - q_{c+j})(\lambda^c_a - a + q_c - q_{c+j}).
\]

It follows directly that \(L = (\alpha^2 - 1)R\).

In the second case, clearly \(k = a_{c+1}\) for some \(c\) with \(1 \leq c < m\). Therefore \(k + 1 = (a, \lambda^c_a, c) \in \mathfrak{s}\) where \(a = \ell(\lambda^c)\) and \(k = (1, 1, d) \in \mathfrak{s}\) where \(d = \min\{m \geq d > c \mid \lambda^d \neq \emptyset\}\). Then \(\text{res}_\mathfrak{s}(k) = q_d\), \(\text{res}_\mathfrak{s}(k + 1) = \lambda^c_a - a + q_c = \text{res}_\mathfrak{y}(k)\), and
\[
\mathcal{A}(k) = \{(1, 1, d + 1), \ldots, (1, 1, m)\}, \quad \mathcal{R}(k) = \{\emptyset\};
\]
\[
\mathcal{A}(k + 1) = \{(a + 1, 1, c), (1, 1, c + 1), \ldots, (1, 1, d - 1), (1, 2, d), (2, 1, d), (1, 1, d + 1), \ldots, (1, 1, m)\},
\]
\[
\mathcal{R}(k + 1) = \{(a, \lambda^c_a - 1, c), (1, 1, d)\}.
\]

On the other hand, since \(k = (a, \lambda^c_a, c) \in t^k\) and \(k + 1 = (1, 1, d) \in t^\lambda\),
\[
\mathcal{A}(k) = \{(a + 1, 1, c), (1, 1, c + 1), \ldots, (1, 1, m)\}, \quad \mathcal{R}(k) = \{(a, \lambda^c_a - 1, c)\};
\]
\[
\mathcal{A}(k + 1) = \{(1, 1, d + 1), \ldots, (1, 1, m)\}, \quad \mathcal{R}(k + 1) = \{\emptyset\}.
\]
Therefore
\[
\alpha^2 - 1 = \frac{\lambda_a^c(\lambda_a^c - a + 1 + q_d - q_d)\lambda_a^c - a - 1 + q_c - q_d)}{(\lambda_a^c - a + q_c - q_d)^2},
\]
\[
L = \frac{\lambda_a^c(\lambda_a^c - a + 1 + q_d - q_d)\lambda_a^c - a - 1 + q_c - q_d)}{\lambda_a^c - a + q_c - q_d} \prod_{j=1}^{m-d} (q_d - q_{d+j})(\lambda_a^c - a + q_c - q_{d+j}),
\]
\[
R = \lambda_a^c \prod_{j=0}^{m-d} (\lambda_a^c - a + q_c - q_{d+j}) \prod_{j=1}^{m-d} (q_d - q_{d+j}).
\]
It follows directly that \( L = (\alpha^2 - 1)R \) and we complete the proof by induction argument. \( \square \)

3.19. Remark. The assertion (i) has given in [AMR, LEMMA 6.10] with a skeleton outline of proof. The assertion (ii) answers a question of Ariki-Mathas-Rui [AMR, §6.9].

Let \( (, ) \) be the inner product on \( \mathcal{H} \) given by \( (h_1, h_2) = \tau(h_1 h_2^\ast) \), for \( h_1, h_2 \in \mathcal{H} \). It follows from Theorem 2.8 and §3.5 that the trace form \( \tau \) satisfying that \( \tau(h) = \tau(h^\ast) \) for all \( h \in \mathcal{H} \). Therefore \( (, ) \) is a symmetric bilinear form on \( \mathcal{H} \). Furthermore, \( (, ) \) is associative in the sense that \( (ab, c) = (a, cb^\ast) \) for all \( a, b, c \in \mathcal{H} \).

3.20. Theorem ([AMR], PROPOSITION 6.8). Assume that Assumption 2.10 holds.

(i) If \( s, t, a \) and \( b \) are standard tableaux then \( f_{st}f_{ab} = \delta_{at}\gamma_tf_{sb} \).

(ii) \( \{f_{st} | s, t \in \text{Std}(\lambda) , \lambda \in \mathcal{P}(m, n) \} \) is an orthogonal basis of \( \mathcal{H} \) with respect to the trace form \( \tau \).

Proof. By proposition 3.14(iv), \( f_{st}f_{ab} = \delta_{at}r_{st}f_{sb} \) for some \( r_{st} \in R \). So we only need to show that \( r_{st} = \gamma_t \) for all standard tableaux \( s, b \). Note that \( f_t = f_{tt}^\lambda = \mathcal{H}^{\ast \lambda} \) and \( \gamma_t = (f_t, f_t) \). Therefore, the inner product \( (, ) \) on the Specht module \( S^\lambda \) gives \( \gamma_t m_t = (f_t, f_t)m_t \equiv f_{tt}f_{tt} \mod \mathcal{H}^{\ast \lambda} \). Hence \( f_{tt}f_{tt} = \gamma_t f_{tt}f_{tt} \) since \( f_{tt}f_{tt} = m_t \). Thus
\[
f_{st}f_{ab} = \delta_{at}F_d(\ast)\gamma_t F_t | F_t f_{tt}f_{tt} d(t) F_b
\]
\[
= \delta_{at}F_d(\ast)_{f_{tt}f_{tt}} d(t) F_b
\]
\[
= \delta_{at}\gamma_t F_d(\ast) m_t(d(b) F_b
\]
\[
= \delta_{at}\gamma_t f_{st} f_{ab}.
\]
(i) is proved.

By Theorem 3.7 and Proposition 3.14(i), \( \{f_{ta} | a, t \in \text{Std}(\lambda) , \lambda \in \mathcal{P}(m, n) \} \) is a basis of \( \mathcal{H} \). Now, by (i), \( (f_{st}, f_{ba}) = \tau(f_{st}f_{ab}) = \tau(f_{st}f_{ba}) = \delta_{at}\gamma_t(f_{ba}) \). On the other hand, \( \tau \) is a trace form, so we also have that \( \tau(f_{st}f_{ba}) = \tau(f_{ba}f_{st}) = \delta_{at}\gamma_t(f_{ba}) \). Thus \( f_{st}, f_{ab} = \delta_{at}\gamma_t(f_{ba}) = \delta_{at}\gamma_t(f_{ba}) = \delta_{at}\gamma_t(f_{tt}) \). Consequently, \( \{f_{st} | s, t \in \text{Std}(\lambda) \} \) is an orthogonal basis of \( \mathcal{H} \) with respect to the trace form \( \tau \) and \( \tau(f_t) \neq 0 \) for all standard tableaux \( t \). Indeed, if there is a standard tableau \( t \) such that \( \tau(f_t) = 0 \). Then \( \gamma_t f_{tt} = \gamma_t f_{tt} = 0 \) and \( \gamma_t \neq 0 \) implies that \( \gamma_t f_{tt} = 0 \) for all standard tableaux \( s \) since \( \gamma_s \neq 0 \) for all standard tableaux \( s \). It is impossible since \( \tau \) is non-degenerate. We complete the proof. \( \square \)

3.21. Remark. In [BK1, Theorem A2] it was shown that \( \mathcal{H} \) is a symmetric algebra with respect to the trace form \( \tau \) for all parameters \( q_1, \ldots, q_m \); however, this was proved indirectly without constructing a pair of dual bases. The Theorem gives a self-dual basis of the semisimple degenerate cyclotomic Hecke algebras. In general, no such basis is known in general.

Now we may identify the Specht module \( S^\lambda \) with a submodule of \( \mathcal{H} \) up to isomorphism.

3.22. Corollary. Suppose that \( s \) and \( t \) be standard \( \lambda \)-tableaux. Then \( S^\lambda \cong \mathcal{H} f_{st} = \sum_{a \in \text{Std}(\lambda)} R f_{at} \).

Proof. The Theorem implies that \( \{f_{at} | a \in \text{Std}(\lambda) \} \) is a basis of \( \mathcal{H} f_{at} \). On the other hand, by §3.9 and Proposition 3.14(i), \( \{f_a | a \in \text{Std}(\lambda) \} \) is a basis of the Specht module \( S^\lambda \). Now the \( R \)-linear map \( f_a \mapsto f_{at} \) gives the desired isomorphism. \( \square \)
Let $G(\lambda) = \det(\langle m_s, m_t \rangle)$, for $s, t \in \text{Std}(\lambda)$, be the Gram determinant of this form, which is well-defined up to a unit in $R$. As an application of the closed formula for $\gamma_s$, we obtain a closed formula for the Gram determinant, which is differential but equivalent to that one given in [AMR, COROLLARY 6.9].

3.23. Corollary. Suppose that $R$ is a field and that Assumption 2.10 holds. Let $\lambda$ be an $m$-multipartition of $n$. Then

$$G(\lambda) = \prod_{t \in \text{Std}(\lambda)} \gamma_t = \prod_{t \in \text{Std}(\lambda)} \prod_{i=1}^n \prod_{x \in A(i)} (\text{res}_t(i) - \text{res}(x)) \prod_{y \in A(i)} (\text{res}_t(i) - \text{res}(y)).$$

Proof. Fix $t \in \text{Std}(\lambda)$. Then Specht module $S^\lambda$ is isomorphic to the submodule of $H_{/H^\lambda}$ which is spanned by $\{m_s + H_{/H^\lambda} s | s \in \text{Std}(\lambda)\}$, where the isomorphism is given by $\theta : H_{/H^\lambda} \to S^\lambda ; m_s + H_{/H^\lambda} m_s$. On the other hand, by Corollary 3.16(ii.c) and (i.a), $\{f_s | s \in \text{Std}(\lambda)\}$ is a basis of $S^\lambda$ and the transition matrix between the two bases $\{m_s\}$ and $\{f_s\}$ of $S^\lambda$ is unitriangular. Consequently, $G(\lambda) = \det(\langle f_s, f_t \rangle)$, where $s, t \in \text{Std}(\lambda)$. However, it follows from Theorem 3.20(i) and Corollary 3.16(ii.b) that $\langle f_s, f_t \rangle = \delta_{st} \gamma_t$. Hence the result follows directly by Theorem 3.18(ii).

Set $\tilde{f}_s = \gamma_t^{-1} f_s$. Then $\tilde{f}_s \tilde{f}_{st} = \delta_{st} \tilde{f}_{st}$ and $\{\tilde{f}_{st}\}$ is a basis of $H_{/H}$. Hence, we have the following.

3.24. Corollary. Assume that Assumption 2.10 holds. Then $\{\tilde{f}_{st} | s, t \in \text{Std}(\lambda) \}$ for $\lambda \in \mathcal{P}(m, n)$ is a bases of matrix units in $H_{/H}$.

The last result yields an explicit isomorphism from $H_{/H}$ to the group ring of $W_{m,n}$ when $R_{/H}(Q)$ is invertible. Assume that $R$ contains a primitive $m$-th root of unity $\zeta$; then, $H_{/H} \cong RW_{m,n}$ when $q_s = \zeta^i$ for $s = 1, \ldots, m$. Write $\tilde{f}_s^{(t)}$ for the element of $RW_{m,n}$ corresponding to $\tilde{f}_s \in H_{/H}$ under the canonical isomorphism $H_{/H} \to RW_{m,n}$.

3.25. Corollary. Assume that $R$ contains a primitive $m$-th root of unity and that Assumption 2.10 holds. Then $H_{/H} \cong RW_{m,n}$ via the $R$-algebra homomorphism determined by $\tilde{f}_s \mapsto \tilde{f}_s^{(t)}$.

By parts (i) and (iii) of Theorem 3.27 below, $s_i = \sum_l \tilde{f}_s s_i$, for $0 \leq i < n$; so, in principle, we can determine the image of the generators of $H_{/H}$ under this isomorphism.

3.26. Remark. (i) Lusztig [L81] has shown that there exists a homomorphism $\Phi$ from the Hecke algebra $H(W)$ of any finite Weyl group $W$ to the group ring $RW$ and he shows that $\Phi$ induces an isomorphism when $H(W)$ is semisimple. Ding and Hu [DH] have given a generalization of Lusztig’s isomorphism theorem for cellular algebras, especially the cyclotomic Hecke algebras. Our map is not an analogue of Lusztig’s isomorphism.

(ii) Brundan and Kleshchev [BK2, Corollary 1.3] showed that when $R$ is a field of characteristic zero, there is an isomorphism $\Psi$ between the cyclotomic Hecke algebras $H_{m,n}(q, Q)$ with $q$ not a root of unity and the degenerate cyclotomic Hecke algebras $H_{m,n}(Q)$. On the other hand, Lusztig [L89] showed that there is a completion isomorphism $\Theta$ between the affine Hecke algebras $H_{m,n}(Q)$ and the degenerate affine Hecke $H_{m,n}(Q)$. Then the following diagram commutes

$$\begin{array}{ccc}
H_{m,n}(Q) & \cong & H_{m,n}(q, Q) \\
\downarrow & & \downarrow \\
H_{m,n}(Q) & \cong & RS_n.
\end{array}$$

where the other homomorphisms are the natural ones. Moreover, the isomorphism $\Psi$ can be given by the Mathas [M04, Corollary 2.14] and Corollary 3.25 when both $H_{m,n}(q, Q)$ and $H_{m,n}(Q)$ are semisimple over a field of characteristic zero.

3.27. Theorem. Suppose that $R$ is a field and that $p_{/H}(Q) \neq 0$. Let $\lambda$ be an $m$-multipartition of $n$.

(i) Let $t$ be a standard $\lambda$-tableau. Then $F_t = \frac{1}{\gamma_t} f_t$ and $F_t$ is a primitive idempotent with $S^\lambda \cong H_{/H} F_t$.
(ii) Let $F_\lambda = \sum_{t \in \text{Std}(\lambda)} F_t$. Then $F_\lambda$ is a primitive central idempotent.
Proof. Note that $F_i = \sum a_b r_{ab} f_{st}$ by Theorem 3.20(ii). Now by Proposition 3.14(ii) and Theorem 3.20(i) $f_{st} = f_{st} F_i = \sum a_b r_{ab} f_{st} f_{ab} = \sum_b \gamma_{tb} r_{ab} f_{st}$. Equating coefficients, $r_{tb} = 0$ if $b \neq t$ and $r_{ta} \gamma_1 = 1$. Since $F_i^t = F_i$, we also have $r_{st} = 0$ if $b \neq t$. Hence $F_i = \frac{1}{\gamma_1}$. By Theorem 3.20(i), $F_i = \frac{1}{\gamma_1}$ is an idempotent. Further, $F_i$ is primitive since $S^1$ is irreducible and $S^1 \cong H F_i$.

(ii) and (iii) now follows because $H = \bigoplus_{\lambda \in P(m,n)} \bigoplus_{t \in \mathrm{Std}^\lambda} H F_i$ is a decomposition of $H$ into a direct sum of simple modules $\{S^\lambda \cong H F_i \mid \lambda \in P(m,n), t \in \mathrm{Std}^\lambda\}$. □

3.28. Corollary. Let $t$ be a standard tableau and let $k$ be an integer with $1 \leq k \leq n$. Then

(i) $x_k F_i = \mathrm{res}_t(k) F_i$ and $x_k f_{st} = \mathrm{res}_s(k) f_{st}$.

(ii) $\prod_{c \in R(k)} (x_k - c) = 0$ and this is the minimum polynomial for $x_k$ acting on $H$.

(iii) $x_k = \sum_t \mathrm{res}_t(k) F_i$, where the sum is over the set of all standard tableaux (of arbitrary shape).

Proof. (i) follows directly by Theorem 3.27(i) and Proposition 3.14(iii).

(ii) First, $\prod_{c \in R(k)} (x_k - c) = \sum_i \prod_{c \in R(k)} (x_k - c) F_i = \sum_i \prod_{c \in R(k)} \mathrm{res}_t(k) - c = 0$. Now if we remove any factor $x_k - c$ from the product $\prod_{c \in R(k)} (x_k - c)$. Then there exists a standard tableau $t$ such that $\mathrm{res}_t(k) = c'$ and for any standard tableau $s \neq t$, $\mathrm{res}_s(k) \neq c'$. Hence $\prod_{s \neq t} \prod_{c \neq c'} (x_k - c) F_i = \prod_{R(k) = c \neq c'} (x_k - c) F_t + \prod_{R(k) = c = c'} (x_k - c) F_i \neq 0$, that is, $\prod_{c \in R(k)} (x_k - c)$ is the minimal polynomial for $x_k$ acting on $H$.

(iii) is follows directly by (i) and Theorem 3.27(iii). □

Now we have a remark on the center of $H$ which described explicitly by Brundan [B, Theorem 1].

3.29. Remark. The results in this section can be used to give a differential proof of the center of $H$, which is the set of symmetric polynomials in $x_1, \ldots, x_n$, when $H$ is semisimple.

4. Dual Specht modules

Let $\mathcal{D} = \mathbb{Z}[\hat{q}_1, \ldots, \hat{q}_m]$, where $\hat{q}_1, \ldots, \hat{q}_m$ are indeterminates over $\mathbb{Z}$, and let $H_{\mathcal{D}}$ be the degenerated cyclotomic Hecke algebra with parameters $\hat{q}_1, \ldots, \hat{q}_m$ over $\mathcal{D}$. Consider the ring $R$ as a $\mathcal{D}$-module by letting $\hat{q}_i$ act on $R$ as multiplication by $q_i$, for $1 \leq i \leq m$. Then $H \cong H_{\mathcal{D}} \otimes_{\mathcal{D}} R$, since $H$ is free as an $R$-module; we say that $H$ is a specialization of $H_{\mathcal{D}}$ and call the map which sends $h \in H_{\mathcal{D}}$ to $h \otimes 1 \in H$ the specialization homomorphism.

4.1. Definition. Let $\tilde{\cdot}: \mathcal{D} \to \mathcal{D}$ be the $\mathcal{D}$-linear map given by $\hat{q}_i \mapsto -\hat{q}_{m-i+1}$, $x_i \mapsto -x_i$ for $1 \leq i \leq m$, and $s_k = -s_k$ for $1 \leq k < n$.

Using the relations of $H_{\mathcal{D}}$ it is easy to verify that $\tilde{\cdot}$ now extends to a $\mathcal{D}$-linear ring involution $\tilde{\cdot}: H_{\mathcal{D}} \to H_{\mathcal{D}}$ of $H_{\mathcal{D}}$. Hereafter, we drop the distinction between $\hat{q}_i$ and $q_i$. Suppose that $h \in H$. Then there exists a (not necessarily unique) $h_{\mathcal{D}} \in H_{\mathcal{D}}$ such that $h = h_{\mathcal{D}} \otimes 1$ under specialization; we sometimes abuse notation and write $h = h_{\mathcal{D}} \otimes 1 \in H$. As the map $\tilde{\cdot}$ does not in general define a semilinear involution on $R$, this notation is not well-defined on elements of $H$; however, in the cases where we employ it there should be no ambiguity. For example, $\tilde{w} = (-1)^{|w|} w$ for all $w \in S_n$.

4.2. Remark. The operation $\tilde{\cdot}$ is very differential form the one defined by Mathas in [M04, §3] in the case of the cyclotomic Hecke algebras.

4.3. Definition. Suppose that $\lambda = (\lambda^1; \ldots; \lambda^m)$ is an $m$-multipartition of $n$. Let $y_{\lambda} = \sum_{w \in S_{\lambda}} (-1)^{|w|} w$ and define $\lambda = y_{\lambda} u_{\lambda}$ where

$$u_{\lambda} = (-1)^{n(\lambda)} \prod_{i=2}^{m} \prod_{k=1}^{a_i} (x_k - q_{m-i+1}) = (-1)^{n(\lambda)} \prod_{i=1}^{m-1} \prod_{k=1}^{a_{m-i+1}} (x_k - q_i),$$

where $a_i = |\lambda^1| + \cdots + |\lambda^{i-1}|$ for $i = 1, \ldots, m$ and $n(\lambda) = \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} (i - 1)|\lambda^i|$. 

(iii) $\{F_\lambda \mid \lambda \in P(m,n)\}$ is a complete set of primitive central idempotents; in particular,

$$1 = \sum_{\lambda \in P(m,n)} F_\lambda = \sum_{t \text{ standard}} F_t.$$
For standard tableaux $s, t \in \text{Std}(\lambda)$ set $n_{st} = \overline{d(s)} n_{\lambda} d(t)$.

4.4. Lemma. Keep notations as above. Then $y_{\lambda} = x_{\lambda}, u_{\lambda}^- = u_{\lambda}^+$, and $m_{st} \in \mathcal{H}_2$ is mapped to $n_{st}$ under specialization.

Proof. All statements follows directly by definitions and the computations. \hfill \Box

4.5. Theorem. The degenerate cyclotomic Hecke algebra $\mathcal{H}$ is free as an $R$-module with cellular basis \{ $n_{st} | s, t \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}(m, n)$ \}.

Proof. The assertion follows by Theorem 3.7 and Lemma 4.4. \hfill \Box

Let $\lambda$ be an $m$-multipartition of $n$. Then $\mathcal{H}_{>\lambda}$ is a two-sided ideal of $\mathcal{H}$ which is free as an $R$-module with basis \{ $n_{ab} | a, b \in \text{Std}(\mu) \text{ for } \mu > \lambda$ \}. For simplicity, we write $\mathcal{H}_{>\lambda}$ for $\mathcal{H}_{>\lambda}$.

4.6. Definition. Let $\tilde{S}^\lambda$ be the Specht module corresponding to $\lambda$ determined by the basis \{ $n_{st}$ \}, which is call the dual Specht module corresponding to $\lambda$.

It is clearly that $\tilde{S}^\lambda \cong \mathcal{H} n_{\lambda}/(\mathcal{H} n_{\lambda} \cap \mathcal{H}_{>\lambda})$ and $\tilde{S}^\lambda$ is free as an $R$-module with basis \{ $n_{st} | t \in \text{Std}(\lambda)$ \}, where $n_t = n_{st} + \mathcal{H}_{>\lambda} = n_{tt} + \mathcal{H}_{>\lambda}$ for all $t \in \text{Std}(\lambda)$.

In order to compare the two modules $S^\lambda$ and $\tilde{S}^\lambda$ we need to introduce some more notation. Given a partition $\sigma$ let $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots)$ be the partition which is conjugate to $\sigma$; thus, $\tilde{\sigma}_i$ is the number of nodes in column $i$ of the diagram of $\sigma$. If $\lambda = (\lambda^1; \ldots; \lambda^m)$ is an $m$-multipartition then the conjugate $m$-multipartition to $\lambda$ is the $m$-multipartition $\tilde{\lambda} = (\tilde{\lambda}^1, \ldots, \tilde{\lambda}^m)$ with $\tilde{\lambda}^i = \lambda^{m-i+1}$ for $1 \leq i \leq m$. Now suppose that $t = (t^1; \ldots; t^m)$ is a standard $\lambda$-tableau. Then the conjugate of $t$ is the standard $\lambda$-tableau $\tilde{t} = (\tilde{t}^1; \ldots; \tilde{t}^m)$ where $\tilde{t}^i$ is the tableau obtained by interchanging the rows and columns of $t^m-i$.

4.7. Example. Let $\lambda = (3 \cdot 1; 4 \cdot 2) \in \mathcal{P}(2, 10)$. Then $\tilde{\lambda} = (2 \cdot 2 \cdot 1 \cdot 1; 2 \cdot 1 \cdot 1) \in \mathcal{P}(2, 10)$, and

$$[\lambda] = \begin{pmatrix}
\begin{array}{c|c|c}
\hline
3 & 4 & 8 \\
\hline
2 & 5 & 7 \\
\hline
\hline
1 & 6 \\
\end{array}
\end{pmatrix}
\quad t^\lambda = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 \\
\end{pmatrix}
\quad \tilde{t}^\lambda = \begin{pmatrix}
5 & 9 \\
6 & 10 \\
7 & 2 \\
8 & 3 \\
\end{pmatrix}.
$$

$$[\tilde{\lambda}] = \begin{pmatrix}
\begin{array}{c|c|c}
\hline
5 & 9 & 1 \\
\hline
6 & 10 & 2 \\
\hline
7 & 3 & 4 \\
\hline
8 & 5 & 6 \\
\hline
\hline
1 & 3 & 5 \\
\end{array}
\end{pmatrix}
\quad \tilde{t}^\tilde{\lambda} = \begin{pmatrix}
7 & 9 & 10 \\
8 & 2 & 4 \\
5 & 6 \\
\end{pmatrix}.
$$

4.8. Lemma. Let $t$ be a standard $\lambda$-tableau. Then $\overline{\text{res}_k}(t) = -\overline{\text{res}_k}(t)$ in $\mathcal{Z}$, for $1 \leq k \leq n$.

Proof. The lemma follows immediately form the definitions. \hfill \Box

The point of Lemma 4.8 is that the expression $\text{res}_k(t)$ is always well-defined; whereas $\overline{\text{res}_k}(t)$ is ambiguous for certain rings $R$. As a first consequence we have the following fact.

4.9. Proposition. Let $s$ and $t$ be standard $\lambda$-tableaux and suppose that $k$ is an integer with $1 \leq k \leq n$. Then there exist $r_t \in R$ such that

$$x_k n_{st} = \text{res}_k(t)n_{st} + \sum_{\text{Std}(\lambda) \ni \beta \ni \sigma} r_{\beta} n_{\beta t} \mod \mathcal{H}_{>\lambda}.$$
Proof. First assume that \( R = \mathcal{Z} \). Then \( - \) is a \( \mathbb{Z} \)-linear ring involution on \( \mathcal{H}_Z \) and \( \overline{x}_k = -x_k \); therefore, by Theorem 3.11,
\[
x_k \overline{m}_{st} = - \overline{x}_k m_{st} = - \left( \text{res}_s(k) m_{st} + \sum_{\text{Std}(\lambda) \ni \delta > s} r_{\delta} m_{\delta t} \mod \mathcal{H}^{> \lambda} \right)
\]
\[
= - \text{res}_s(k) m_{st} + \sum_{\text{Std}(\lambda) \ni \delta > s} (-r_{\delta}) m_{\delta t} \mod \mathcal{H}^{> \lambda}
\]
\[
= \text{res}_s(k) n_{st} + \sum_{\text{Std}(\lambda) \ni \delta > s} (-r_{\delta}) n_{\delta t} \mod \mathcal{H}^{> \lambda}.
\]
The general case now follows by specialization since \( \mathcal{H} \cong \mathcal{H}_Z \otimes \mathcal{Z} R \).

Next consider the orthogonal basis \( \{ f_{st} \} \) of \( \mathcal{H} \) in the case where \( P_\mathcal{H}(Q) \) is invertible. Let \( \mathcal{Z} P \) be the localization of \( \mathcal{Z} \) at \( P_\mathcal{H}(Q) \) and let \( \mathcal{H}_P \) be the corresponding degenerate cyclotomic Hecke algebra. The involution \( - \) extends to \( \mathcal{H}_P \), and \( \mathcal{H} \) is a specialization of \( \mathcal{H}_P \), whenever \( P_\mathcal{H}(Q) \) is invertible in \( R \). (Note that \( q_1, \ldots, q_m \) are indeterminates in \( \mathcal{Z} \).

In general, \( f_{st} \notin \mathcal{H}_P \); however, if \( t \neq u \) then \( \text{res}_s(k) = \text{res}_u(k) \) is a factor of \( P_\mathcal{H}(Q) \) for all \( k \), so \( f_{st} \in \mathcal{H}_P \), and we can speak of the elements \( F_i \) and \( f_{st} \in \mathcal{H}_P \). More generally, whenever \( P_\mathcal{H}(Q) \) is invertible in \( R \) we have an element \( f_{st} \in \mathcal{H} \) via specialization because \( \mathcal{H} \cong \mathcal{H}_P \otimes \mathcal{Z} R \).

4.10. Proposition. Suppose that \( t \) is a standard tableau. Then \( \overline{F_i} = F_i \) in \( \mathcal{H}_P \).

Proof. Note that \( \overline{R(k)} = R(k) \) and \( \text{res}_s(k) \neq \text{res}_u(k) \) if and only if \( \text{res}_u(k) \neq \text{res}_s(k) \). Now applying the definitions together with Lemma 4.8 gives
\[
\overline{F_i} = \frac{\prod_{k=1}^n \prod_{c \in R(k)} \frac{x_k - c}{\text{res}_s(k) - c}}{\prod_{k=1}^n \prod_{c \in \text{res}_u(k)} \frac{x_k - \text{res}_u(k)}{\text{res}_s(k) - \text{res}_u(k)}}
\]
\[
= \frac{\prod_{k=1}^n \prod_{c \in \text{res}_u(k)} \frac{x_k - \text{res}_u(k)}{\text{res}_s(k) - \text{res}_u(k)}}{\prod_{k=1}^n \prod_{c \in \text{res}_s(k)} \frac{x_k - \text{res}_s(k)}{\text{res}_s(k) - \text{res}_u(k)}}
\]
\[
= \prod_{k=1}^n \prod_{c \in \text{res}_u(k)} \frac{x_k - \text{res}_u(k)}{\text{res}_s(k) - \text{res}_u(k)} \frac{x_k - \text{res}_s(k)}{\text{res}_s(k) - \text{res}_u(k)} = F_i.
\]

Now we let \( g_{st} = F_s f_{st} ; \) then in \( \mathcal{H}_P \), \( g_{st} = \overline{F_s f_{st}} \overline{F_i} = \overline{f_{st}} \). Applying \( - \) to \( \{ f_{st} \} \) and using Theorem 3.20(ii) (and a specialization argument) shows that \( \{ g_{st} \mid s, t \in \text{Std}(\lambda) \} \) for \( \lambda \in \mathcal{P}(m,n) \) is a basis of \( \mathcal{H} \). Consequently, as in Corollary 3.22, \( S^\lambda \cong \mathcal{H} g_{st} \) for any standard \( \lambda \)-tableaux \( s, t \in \text{Std}(\lambda) \).

4.11. Remark. By the Proposition and Theorem 3.27(i),
\[
g_{st} = R_{ft} = \gamma_t F_i = \overline{F_i} = \overline{\gamma_t} f_{st}.
\]

More generally, we can write \( g_{st} = \sum_{a,b} r_{ab} f_{ab} \) for some \( r_{ab} \in R \). By Propositions 3.14 and 4.10, \( F_s f_{st} = g_{st} \); so it follows that \( r_{ab} = 0 \) unless \( a = s \) and \( b = t \). Therefore, \( g_{st} = \alpha_{st} f_{st} \) for some \( \alpha_{st} \in R \).

Applying the \( * \)-involution shows that \( \alpha_{st} = \alpha_{ts} \). Finally, by looking at the product \( g_{st} g_{ts} \) we see that \( \alpha_{st} = \overline{\gamma_t} / \overline{\gamma_s} \).
Combining Proposition 4.10 with Corollary 3.22 and the corresponding result for the $g$-basis shows that $S^\lambda \cong \mathcal{H} f_{t^\lambda} = \mathcal{H} g_{t^\lambda} \cong \tilde{S}^\lambda$, for any $t \in \text{Std}(\lambda)$. Hence, we have the following.

4.12. Corollary. Assume that Assumption 2.10 holds. Then $\tilde{S}^\lambda \cong S^\lambda$.

4.13. Remark. When $R$ is field the assumption that Assumption 2.10 holds is equivalent to $\mathcal{H}$ being semisimple. This assumption is necessary because, in general, $S^\lambda$ and $\tilde{S}^\lambda$ are not isomorphic; rather, we can show that $S^\lambda$ is isomorphic to the dual of $\tilde{S}^\lambda$, the detail will be appear elsewhere. In the semisimple case both $S^\lambda$ and $\tilde{S}^\lambda$ are irreducible, and hence self-dual, since they carry a non-degenerate bilinear form. Accordingly, we call the module $\tilde{S}^\lambda$ the dual Specht module.

4.14. Corollary. Let $\lambda$ and $\mu$ be $m$-multipartitions of $n$. Suppose that $s$ and $t$ are standard $\lambda$-tableaux and that $a$ and $b$ are standard $\mu$-tableaux. Then $f_{st}g_{ab} = \delta_{st}f_{ab}$ for some $r_{ab} \in R$.

Proof. Applying the definitions and Remark 4.11,

$$ f_{st}g_{ab} = F_s m_{st} F_t F_a F_b = F_s m_{st} F_t F_a \overline{F}_b = \delta_{st} F_s m_{st} F_t \overline{F}_b = \delta_{st} r_{ab} f_{ab} $$

for some $r_{ab} \in R$. It completes the proof. \qed

The Specht modules $S^\lambda$ and the dual Specht modules $\tilde{S}^\lambda$ are both constructed as quotient modules using the cellular bases $\{m_{st}\}$ and $\{n_{st}\}$ respectively (see Corollary 3.22). Using the orthogonal basis $\{f_{st}\}$ and $\{g_{st}\}$ we have also constructed these modules as submodules of $\mathcal{H}$.

Recall that $t^\lambda$ is the $\lambda$-tableau which has the numbers $1, 2, \ldots, n$ entered in order first along the rows of $t^\lambda$, and then the rows of $t^\lambda$ and so on. Let $t_\lambda = t^\lambda$, that is, $t_\lambda$ is the $\lambda$-tableau and by applying the involution $\overline{}$ (in $\mathcal{H}_x^n$ and then specializing) we see that there exist $r_{\alpha\beta} \in R$ such that

$$ (\dagger) \quad n_\lambda = g_{\lambda t_\lambda} + \sum_{\alpha, \beta \succ \lambda} r_{\alpha\beta} g_{\alpha\beta} = \frac{\gamma_{\lambda t_\lambda}}{\gamma_{\lambda}} f_{t_\lambda t_\lambda} + \sum_{t_\lambda \succ \alpha, \beta} r_{\alpha\beta} g_{\alpha\beta} $$

where for the second equality we have used Remark 4.11 (note that $t^\lambda = t_\lambda$) and the observation that $\alpha, \beta \succ t^\lambda$ if and only if $t_\lambda \succ \bar{\alpha}, \bar{\beta}$. Now $m_\lambda \mathcal{H} n_\lambda$ is spanned by the elements $m_\lambda f_{st} n_\lambda$, where $s$ and $t$ range over all pairs of standard tableaux of the same shape. Now, (\dagger) and Corollary 4.14 imply that

$$ m_\lambda f_{st} n_\lambda = \left( f_{t^\lambda} + \sum_{a, b \succ t^\lambda} r_{ab} f_{ab} \right) f_{st} \left( \frac{\gamma_{\lambda t_\lambda}}{\gamma_{\lambda}} f_{t_\lambda t_\lambda} + \sum_{t_\lambda \succ \alpha, \beta} r_{\alpha\beta} g_{\alpha\beta} \right) $$

$$ = \frac{\gamma_{\lambda t_\lambda}}{\gamma_{\lambda}} f_{t^\lambda} f_{st} f_{t_\lambda t_\lambda} + \sum_{t_\lambda \succ \alpha, \beta} r_{ab} r_{\alpha\beta} f_{ab} f_{st} g_{\alpha\beta}, $$

$$ = \frac{\gamma_{\lambda t_\lambda}}{\gamma_{\lambda}} f_{t^\lambda} f_{st} f_{t_\lambda t_\lambda} + \sum_{a, b \succ t^\lambda} r_{ab} r_{\alpha\beta} f_{ab} f_{st} g_{\alpha\beta}, $$

$$ \begin{cases} \frac{\gamma_{\lambda t_\lambda}}{\gamma_{\lambda}} f_{t^\lambda} f_{t_\lambda t_\lambda}, & \text{if } s = t^\lambda \text{ and } t = t_\lambda, \\ 0, & \text{otherwise,} \end{cases} $$

With the third equality following the facts that $f_{ab} f_{st} g_{\alpha\beta} = \delta_{ab} \delta_{st} r f_{\alpha\beta}$ for some $r \in R$ and that $t_\lambda \succ \bar{\alpha} = t \succ t_\lambda$ is impossible. \qed

Let $w_\lambda = d(t_\lambda)$. Then $w_\lambda$ is the unique element of $S_n$ such that $t_\lambda = t^\lambda w_\lambda$.

4.16. Definition. Suppose that $\lambda$ is an $m$-multipartition of $n$. Let $z_\lambda = m_\lambda w_\lambda n_\lambda$.

The element $z_\lambda$ and the following result are crucial to our computation of the Schur elements.
4.17. Proposition. Assume that Assumption 2.10 holds. Then \( z_\lambda = \gamma_\lambda f_{t\lambda} \) and \( m_\lambda \mathcal{H} n_\lambda = Rz_\lambda \).

Proof. By definition and applying the equality (\dagger) in the proof of Proposition 4.15,

\[
\begin{align*}
z_\lambda &= m_\lambda w_\lambda n_\lambda = m_{t\lambda} n_\lambda \\
&= \left( f_{t\lambda} + \sum_{a,b > t\lambda} r_{ab} f_{ab} \right) \left( \frac{\gamma_\lambda f_{t\lambda}}{\gamma_\lambda} + \sum_{t\lambda > \alpha, \beta} r_{\alpha\beta} y_{\alpha\beta} \right) \\
&= \frac{\gamma_\lambda f_{t\lambda}}{\gamma_\lambda} f_{t\lambda} f_{t\lambda} \\
&= \frac{\gamma_\lambda f_{t\lambda}}{\gamma_\lambda},
\end{align*}
\]

where the last equality follows by Theorem 3.20(i). \(\square\)

Now, \( z_\lambda \) is an element of \( \mathcal{H} \), so \( \frac{\gamma_\lambda f_{t\lambda}}{\gamma_\lambda} \in \mathcal{H} \). By definition, \( \mathcal{H} z_\lambda \) is a quotient module of \( \mathcal{H} m_\lambda \) and a submodule of \( \mathcal{H} n_\lambda \).

4.18. Remark. Over an arbitrary ring \( R \), along the line of Du and Rui [DR, Remark 2.5], we can show that \( S^\lambda \cong \mathcal{H} z_\lambda^\lambda \) and \( \tilde{S}^\lambda \cong \mathcal{H} z_\lambda \) as \( \mathcal{H} \)-modules, the isomorphisms being given by the natural quotient maps \( \mathcal{H} m_\lambda \to \mathcal{H} z_\lambda^\lambda \) and \( \mathcal{H} n_\lambda \to \mathcal{H} z_\lambda \). Note that \( S^\lambda \cong \tilde{S}^\lambda \) when \( \mathcal{H} \) is semisimple by Corollary 3.22.

5. SOME NICE PRIMITIVE IDEMPOTENTS

In this section we give a simple formula for the primitive idempotents \( F_\lambda = \frac{1}{\gamma_\lambda} f_{t\lambda} \).

5.1. Proposition. Suppose that \( t \) is a standard \( \lambda \)-tableau. Then there exist invertible elements \( \Phi_t \) and \( \Psi_t \) in the group algebra \( RS_n \) such that

- (i) \( \Psi_t F_t = F_t \Phi_t \).
- (ii) \( \Phi_t = d(t) + \sum_{w < d(t)} r_{tw} w \), for some \( r_{tw} \in R \); and
- (iii) \( \gamma_t \Phi_t \Psi_t^* = \gamma_t \).

Proof. We prove all three statements by induction on \( t \) with respect to the dominance \( \triangleright \). When \( t = t^\lambda \) there is nothing to prove as we may take \( \Phi_{t^\lambda} = \Psi_{t^\lambda} = 1 \). Suppose then that \( t \neq t^\lambda \). Then there exists an integer \( i \), with \( 1 \leq i < n \), such that \( s = t(i, i+1) \triangleright t \), or equivalently \( t = s(i, i+1) \triangleright s \).

Let \( \alpha = \frac{1}{\text{res}(s) - \text{res}(t)} \) and \( \beta = \frac{-1}{\text{res}(s) - \text{res}(t)} = -\alpha \). Then \( \alpha + \beta = 0 \), \( 1 + \alpha \beta = \frac{2\alpha}{\gamma_s} \), and \( (s_i - \beta)f_t = (\gamma_t/\gamma_s)f_{st} \) according to Theorems 3.18(i,b) and 3.15. Similarly, by the right hand analogue of Theorem 3.15 (interchanging the roles of \( s \) and \( t \)), \( f_{ss}(s_i - \alpha) = f_{st} \). Therefore,

\[(\dagger) \quad (s_i - \beta) F_t = \frac{1}{\gamma_t} (s_i - \beta) f_t = \frac{1}{\gamma_s} f_{st} = \frac{1}{\gamma_s} f_{ss}(s_i - \alpha) = F_t (s_i - \alpha).\]

By induction, there exist invertible elements \( \Phi_s \) and \( \Psi_s \) in \( RS_n \) satisfying properties (i)–(iii).

Now define \( \Psi_t = \Psi_s(s_i - \beta) \) and \( \Phi_t = \Phi_s(s_i - \alpha) \). Then, by induction and the equation (\dagger),

\[
\Psi_t F_t = \Psi_s(s_i - \beta) F_t = \Psi_s F_s(s_i - \alpha) = F_t \Phi_s(s_i - \alpha) = F_t \Phi_t.
\]

Hence, (i) holds. Next, again by induction,

\[
\Psi_t = \Psi_s (s_i - \beta) = \left( d(s) + \sum_{w < d(s)} p_{sw} w \right) (s_i - \beta) = d(t) + \sum_{w < d(t)} p_{tw} w,
\]

by the standard properties of the Bruhat order since \( d(t) = d(s) s_i > d(s) \) according to Lemma 3.4. Furthermore, by induction, \( \Psi_t \) and \( \Phi_t \) are invertible because \( s_i - \alpha \) and \( s_i - \beta \) are invertible in \( RS_n \). (ii) is proved.

Finally, using induction once more (and a quick calculation for the second equality),

\[
\gamma_t \Phi_t \Psi_t^* = \gamma_t \Phi_s (s_i - \alpha) (s_i - \beta) \Psi_s^* = \gamma_t \Phi_s (1 + \alpha \beta - (\alpha + \beta) s_i) \Psi_s^* = \gamma_t,
\]

In this section we give a simple formula for the primitive idempotents \( F_\lambda = \frac{1}{\gamma_\lambda} f_{t\lambda} \).
This proves (iii) and so completes the proof. □

5.2. Remark. We are not claiming that the elements $\Phi_t$ and $\Psi_t$ are uniquely determined by the conditions of the Proposition; ostensibly, these elements depend upon the choice of reduced expression for $d(t)$. Indeed, if $s_{i_1} \ldots s_{i_k}$ is a reduced expression of $d(t)$, then we may choose that

$$\Psi_t = (s_{i_1} - \beta_{i_1}) \ldots (s_{i_k} - \beta_{i_k}) \quad \text{and} \quad \Phi_t = (s_{i_1} - \alpha_{i_1}) \ldots (s_{i_k} - \alpha_{i_k}),$$

where $t_j = t(s_{i_1} \ldots s_{i_j})$, $\alpha_{i_j} = \frac{1}{\text{res}_{s_{i_j}^{-1}(t_j)} \cdot \text{res}_{s_{i_j}^{-1}(t_j)}}$, and $\beta_{i_j} = -\alpha_{i_j}$.

5.3. Corollary. Suppose that $t$ is a standard $\lambda$-tableau. Then

(i) $F_{\lambda} = \frac{\gamma_{\lambda}}{\gamma_t} \Psi_t F_i \Psi_t^*$; and,
(ii) $F_t = \frac{\gamma_{\lambda}}{\gamma_t} F_i \Phi_t F_i$.

Proof. Using parts (iii) and (i) of the Proposition, shows that

$$F_{\lambda} = \left( \frac{\gamma_{\lambda}}{\gamma_t} \Psi_t \Phi_t \right)^* F_{\lambda} = \frac{\gamma_{\lambda}}{\gamma_t} \Psi_t F_i \Psi_t^* \quad \text{and} \quad F_t = F_i \left( \frac{\gamma_{\lambda}}{\gamma_t} \Psi_t \Phi_t \right) = \frac{\gamma_{\lambda}}{\gamma_t} \Phi_t F_i \Phi_t,$$

where we use the ‘conjugating’ version of Proposition 5.1 for the proof of the part (ii). □

The main reason why we are interested in $\Psi_t$ and $\Phi_t$ is the following.

5.4. Lemma. Suppose that $s$ and $t$ are standard $\lambda$-tableaux. Then

$$f_{st} = \Psi_s^* f_{t}\Psi_t.$$

Proof. By the definition of $f_{st}$ and Proposition 5.1(ii) we have

$$f_{st} = F_d(s)^* m_\lambda d(t) F_i = F_s \Phi_s m_\lambda \Phi_t F_i - \sum_{(v,w)<(d(s),d(t))} p_{vw} p_{tw} F_s v^* m_\lambda w F_t.$$

Now if $(v, w) < (d(s), d(t))$ then, by Lemma 3.4, $v^* m_\lambda w$ belongs to the span of the $m_{ab}$ where $(a, b) \triangleright (s, t)$. Therefore, by Proposition 3.14(i), $v^* m_\lambda w$ belongs to the span of the $f_{ab}$ where either $a$ and $b$ are standard $\lambda$-tableaux and $(a, b) \triangleright (s, t)$, or $\text{Shape}(a) = \text{Shape}(b) \triangleright \lambda$; consequently, $F_s v^* m_\lambda w F_t = 0$ by Proposition 3.14(iii). Hence, by Theorem 2.27(i) and Proposition 5.1(i),

$$f_{st} = F_s \Phi_s^* m_\lambda \Phi_t F_i = \Psi_s^* F_{st} m_\lambda \Phi_t F_i = \Psi_s^* f_{st} \Psi_t$$

as required. □

Recall that, see Proposition 4.15, $z_\lambda = m_\lambda w_\lambda n_\lambda = u_\lambda^{-1} x_\lambda w_\lambda y_\lambda u_\lambda^{-1} = \bar{z}_\lambda f_{t}\Phi_t$. The following fact is crucial to our computation of the Schur elements of the degenerate cyclotomic Hecke algebras.

5.5. Proposition. Suppose that $\lambda$ is an $m$-multipartition of $n$. Then

$$F_{\lambda} = \frac{1}{\gamma_t} f_{t}\Phi_t = \frac{1}{\gamma_s \gamma_t} z_\lambda w_\lambda$$

is a primitive idempotent of $\mathcal{H}$ with $\mathcal{H} F_{\lambda} \cong S^\lambda$.

Proof. By Corollary 5.3(i) and Proposition 5.1(iii), $F_{\lambda} = \frac{\gamma_{\lambda}}{\gamma_t} \Psi_t F_i \Psi_t^* = \gamma_\lambda F_i \Phi_t$. Now, by Proposition 4.17, $z_\lambda = \frac{\gamma_{\lambda}}{\gamma_t} f_{t}\Phi_t = \frac{\gamma_{\lambda} \gamma_t \gamma_s}{\gamma_t} F_s \Phi_t$. Therefore $F_{\lambda} = \frac{1}{\gamma_t \gamma_s} z_\lambda \Phi_t$, which implies that $z_{\lambda} \in F_{\lambda} \mathcal{H}$, moreover $F_{\lambda} z_{\lambda} = z_{\lambda}$ since $F_{\lambda}$ is an idempotent. As a consequence,

$$z_\lambda w_\lambda = F_{\lambda} z_\lambda w_\lambda = \frac{1}{\gamma_t \gamma_s} z_\lambda \Phi_t z_\lambda w_\lambda = \frac{1}{\gamma_t \gamma_s} m_\lambda w_\lambda n_\lambda \Phi_t m_\lambda w_\lambda n_\lambda w_\lambda.$$

Applying Proposition 5.1(ii), $\Phi_t = w_\lambda + \sum_{w < w_\lambda} r_{tw_\lambda} w$, for some $r_{tw} \in R$. On the other hand, note that the permutation $w_\lambda$ has the “trivial intersection property”, that is, $S_\lambda \cap w S_\lambda w^{-1} = \{ 1 \}$ if and only if $S_\lambda w S_\lambda = S_\lambda$, see for example [DJ86, §4.9]. As a consequence, $y_\lambda x_\lambda w_\lambda \neq 0$ if and only if $w \in S_\lambda w_\lambda S_\lambda$. Thus $y_\lambda x_\lambda w_\lambda$ is the unique element of minimal length in $S_\lambda w_\lambda S_\lambda$, and $z_\lambda \Phi_t z_\lambda = (x_\lambda^{-1} u_\lambda^r x_\lambda y_\lambda) \Phi_t (x_\lambda^{-1} u_\lambda^r x_\lambda y_\lambda) = x_\lambda u_\lambda^r x_\lambda y_\lambda x_\lambda^{-1} u_\lambda^r x_\lambda y_\lambda (x_\lambda^{-1} u_\lambda^r x_\lambda y_\lambda) = z_\lambda w_\lambda$. Therefore, $z_\lambda w_\lambda = \frac{1}{\gamma_t \gamma_s} (z_\lambda w_\lambda)^2$. Furthermore, $\frac{1}{\gamma_t \gamma_s} z_\lambda w_\lambda$ is an idempotent in $\mathcal{H} F_{\lambda}$. Hence, $F_{\lambda} = \frac{1}{\gamma_t \gamma_s} z_\lambda w_\lambda$ since $F_{\lambda}$ is a primitive idempotent. □
6. Computation of $\tau(\zeta_\lambda w_\lambda)$

In this section we determine $\tau(\zeta_\lambda w_\lambda)$, which is the key to the computation of the Schur elements of the degenerate cyclotomic Hecke algebras $\mathcal{H}$. We will see that for all parameters $q_1, \ldots, q_m$, $\tau(\zeta_\lambda w_\lambda)$ is a unit in $R$, which answer partly that for all parameters $q_1, \ldots, q_m$, $\mathcal{H}$ is a symmetric algebra. This fact has important consequences for the representation theory of degenerate Hecke algebras. For example, it can be used to show that $S^\lambda = \mathcal{H} m_\lambda$ is a self dual $\mathcal{H}$-module and that $S^\lambda$ is isomorphic to the dual of $S^\lambda$. The details will be appear elsewhere.

6.1. Fix an $m$-multipartition $\lambda = (\lambda^1; \ldots; \lambda^m)$ of $n$. Let $a_i = \sum_{j=1}^{i} |\lambda^j|$ and $b_i = \sum_{j=i+1}^{m} |\lambda^j|$ for $1 \leq i \leq m$. Set $n(\lambda) = \sum_{i=1}^{m} (i-1)|\lambda^j|$ and $\alpha(\lambda) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j \geq 1} (\lambda^j_i - 1)\lambda^j_i$. For each integer $k$ with $1 \leq k \leq n$, we let $c^\lambda(k)$ be $c$ (resp., $c^\lambda(k)$) if $k$ appears in the $c$-component of $t^\lambda$ (resp., $t^\lambda$). Then $c^\lambda(k) = \min\{1 \leq c \leq m \mid k \leq |\lambda| + 1 + \ldots + |\lambda^m|\}$. Clearly, $u^\lambda_\lambda$ is a polynomial (in variables $x_1, \ldots, x_n$) of degree $\alpha(\lambda)$ and $u^-_\lambda$ is a polynomial (in variables $x_1, \ldots, x_n$) of degree $n(\lambda)$. Furthermore, for each integer $k$ with $1 \leq k \leq n$, $u_\lambda^k$ is a polynomial in $x_k$ of degree $m - c^\lambda(k)$.

Recall that $u^\lambda_\lambda = \prod_{i=2}^{m} \prod_{k=1}^{n(\lambda)} (x_k - q_i)$ and $u^-_\lambda = (-1)^{n(\lambda)} \prod_{i=2}^{m} \prod_{k=1}^{n(\lambda)} (x_k - q_i)$ (Definitions 3.6 and 4.3). Clearly, $u^\lambda_\lambda$ is a polynomial in the polynomial $f(y_1, \ldots, y_n)$ defined by

$$\sigma \cdot f(y_1, \ldots, y_n) = f(y_{\sigma(1)}, \ldots, y_{\sigma(n)})$$

for all $\sigma \in S_n$ and for all $f(y_1, \ldots, y_n) \in R[y_1, \ldots, y_n]$. For a moment, we denote by $L(f)$ the leading term of a polynomial $f(y_1, \ldots, y_n) \in R[y_1, \ldots, y_n]$.

Before we state our key lemma, we consider an example.

6.2. Example. Let $\lambda = (2 \cdot 1; 1; 1)$ be a 3-multipartition of 8. Then $\lambda = (1; 2 \cdot 1; 1)$, $a_1 = 0 = b_3$, $a_2 = 4$, $a_3 = 7$, $b_1 = 4$, $b_2 = 1$, $n(\lambda) = 5$. Furthermore

$u^\lambda_\lambda = (x_1 - q_2)(x_2 - q_2)(x_3 - q_2)(x_4 - q_2)(x_5 - q_3)(x_6 - q_3)(x_7 - q_3)$,

$w_\lambda = \left(1, 2, 3, 4, 5, 6, 7, 8\right)(1, 2, 3, 4, 5, 6, 7, 8) = (1, 5, 2, 8)(3, 6, 4, 7) = w^-_\lambda$,

and $x^k \cdot u^\lambda_\lambda = (x_1 - q_2)(x_2 - q_2)(x_3 - q_2)(x_4 - q_2)(x_5 - q_3)(x_6 - q_3)(x_7 - q_3)$.

Thus $L(u^\lambda_\lambda(w^\lambda_\lambda \cdot u^-_\lambda)) = -x_1^2 x_2^2 \cdots x_8^2$, similarly, it follows by direct computation that $L((w^-_\lambda u^\lambda_\lambda)\cdot u^-_\lambda) = -x_1^2 x_2^2 \cdots x_8^2$.

The following fact is the key to our computation of $\tau(\zeta_\lambda w_\lambda)$.

6.3. Lemma. Let $\lambda$ be an $m$-multipartition of $n$ and let $l = (m-1)n$. Then the polynomials $u^\lambda_\lambda(w^\lambda_\lambda \cdot u^-_\lambda)$ and $(w^-_\lambda \cdot u^\lambda_\lambda)u^-_\lambda$ are of degree $(m-1)n$, moreover,

$$\text{gr}_l u^\lambda_\lambda(w^\lambda_\lambda \cdot u^-_\lambda) = \text{gr}_l (w^\lambda_\lambda \cdot u^\lambda_\lambda)u^-_\lambda = (-1)^{n(\lambda)}x_1^{m-1}x_2^{m-1} \cdots x_n^{m-1}.$$ 

Proof. It suffices to show that $\text{gr}_l (w^\lambda_\lambda \cdot u^\lambda_\lambda)u^-_\lambda = (-1)^{n(\lambda)}x_1^{m-1}x_2^{m-1} \cdots x_n^{m-1}$. Indeed, if this is done, then the first part follows immediately from that the degrees of $u^\lambda_\lambda(w^\lambda_\lambda \cdot u^-_\lambda)$ and $(w^-_\lambda \cdot u^\lambda_\lambda)u^-_\lambda$ are at most $(m-1)n$; on the other hand, note that for any $w \in S_n$, $\text{gr}_l (w^\lambda_\lambda \cdot u^\lambda_\lambda)u^-_\lambda = (-1)^{n(\lambda)}x_1^{m-1}x_2^{m-1} \cdots x_n^{m-1}$.

In particular, let $w = w^\lambda$, then $\text{gr}_l (w_\lambda \cdot (w^\lambda_\lambda \cdot u^\lambda_\lambda)u^-_\lambda) = \text{gr}_l (w_\lambda(w^\lambda_\lambda \cdot u^\lambda_\lambda)u^-_\lambda) = \text{gr}_l (w^\lambda_\lambda(w^\lambda_\lambda \cdot u^-_\lambda)) = (-1)^{n(\lambda)}x_1^{m-1}x_2^{m-1} \cdots x_n^{m-1}$ since $w^\lambda w_\lambda = 1$.

We proceed by induction on $n$. Then, by definitions, $L(u^\lambda_\lambda) = \prod_{i=1}^{n} x_i^{m-c^\lambda(i)}$ and $L(u^-_\lambda) = \prod_{i=1}^{n} x_i^{c^\lambda(i)-1}$. Now, if there are some $i$ with $1 \leq i \leq m$ such that $|\lambda^i| = 0$, for simplicity, we assume that $\lambda^1 = \emptyset$ and $\lambda^2 \neq \emptyset \neq \lambda^m$, then

$$c^\lambda(i) = \begin{cases} 2, & \text{if } 1 \leq i \leq |\lambda^2|; \\ m, & \text{if } n - |\lambda^m| < i \leq n; \end{cases} \quad \text{and} \quad c_\lambda(i) = \begin{cases} 2, & \text{if } n - |\lambda^2| < i \leq n; \\ m, & \text{if } 1 \leq i \leq |\lambda^m|. \end{cases}$$
Therefore
\[ L(u^+_\lambda) = \prod_{i=1}^{\lfloor \lambda \rfloor} x_i^{m-1} \prod_{i>|\lambda|} x_i^{-c_\lambda(i)} \quad \text{and} \quad L(u^-_\lambda) = (-1)^{n(\lambda)} \prod_{i=1}^{\lfloor \lambda \rfloor} x_i^{m-1} \prod_{i>|\lambda|} x_i^{-c_\lambda(i)} \prod_{i>n-|\lambda|} x_i. \]

On the other hand,
\[ w_\lambda = \begin{pmatrix} 1 & 2 & \cdots & |\lambda| & \cdots & |\lambda| & +1 & \cdots & n \\ i_1 & i_2 & \cdots & i_{|\lambda|} & \cdots & j_1 & \cdots & j_{|\lambda|} \end{pmatrix}, \]
where \( \{i_1 = n - |\lambda|^2 + 1, i_2, \ldots, i_{|\lambda|}\} = \{n - |\lambda|^2 + 1, \ldots, n\} \) and \( \{j_1 = 1, j_2, \ldots, j_{|\lambda|}\} = \{1, \ldots, |\lambda|^2\} \).

As a consequence,
\[ L((w_\lambda \cdot u^+_\lambda) u^-_\lambda) = (-1)^{n(\lambda)} \prod_{i=1}^{\lfloor \lambda \rfloor} x_i^{m-1} \prod_{i>|\lambda|} x_i^{-c_\lambda(i)} \prod_{i>n-|\lambda|} x_i. \]

Now, by induction, \( L((w_\lambda \cdot u^+_\lambda) u^-_\lambda) = (-1)^{n(\lambda)} x_1^{m-1} x_2^{m-1} \cdots x_n^{m-1}. \)

Now assume that \( \lambda = (\lambda^1; \ldots; \lambda^m) \) with \( |\lambda| > 0 \). Then
\[ L(u^+_\lambda) = \prod_{i=1}^{\lfloor \lambda \rfloor} x_i^{m-1} \prod_{i>|\lambda|} x_i^{-c_\lambda(i)} \quad \text{and} \quad L(u^-_\lambda) = (-1)^{n(\lambda)} \prod_{i=1}^{\lfloor \lambda \rfloor} x_i^{m-1} \prod_{i>|\lambda|} x_i^{-c_\lambda(i)}. \]

Using the same arguments, we yield that \( L((w_\lambda \cdot u^+_\lambda) u^-_\lambda) = (-1)^{n(\lambda)} x_1^{m-1} x_2^{m-1} \cdots x_n^{m-1} \).

Now we can obtain the main result of this section.

6.4. **Theorem.** Let \( \lambda \) be an \( m \)-multipartition of \( n \) and \( n(\lambda) = \sum_{i=1}^{m} (i-1) |\lambda|^i \). Then \( \tau(\zeta_\lambda w_\lambda) = (-1)^{n(\lambda)} \).

**Proof.** First, note that for any \( w \in S_n \) and any polynomial \( f(x_1, \ldots, x_n) \) (in \( x_1, \ldots, x_n \)), \( f \) and \( w \cdot f \) have the same degree, furthermore, the leading terms of \( w \cdot f \) are obtained from those of \( f \) via the \( w \)-action. Let \( l = (m-1)n \). Therefore
\[ \tau(\zeta_\lambda w_\lambda) = \tau(x_\lambda u^+_\lambda w_\lambda x_\lambda y_\lambda w_\lambda) \] (Definition 4.16)
\[ = \tau(u^+_\lambda w_\lambda y_\lambda w_\lambda x_\lambda) \] (2.9(i))
\[ = \tilde{\tau}(g_{r_1}(u^+_\lambda w_\lambda y_\lambda w_\lambda x_\lambda)) \] (Theorem 2.8)
\[ = \tilde{\tau}(g_{r_1}(u^+_\lambda(w_\lambda \cdot u^-_\lambda))) w_\lambda y_\lambda w_\lambda x_\lambda \] (Lemma 2.7)
\[ = (-1)^{n(\lambda)} \sum_{u \in S_\lambda} (-1)^{\ell(u)} \tau(x_1^{m-1} \cdots x_n^{m-1} w_\lambda uw_\lambda v) \] (Lemma 6.3)
\[ = (-1)^{n(\lambda)} \sum_{u \in S_\lambda} (-1)^{\ell(u)} \tau(x_1^{m-1} \cdots x_n^{m-1} w_\lambda uw_\lambda^{-1} v) \] (w_\lambda w_\lambda = 1)
\[ = (-1)^{n(\lambda)} \sum_{(u,v) \in S_\lambda \times S_\lambda} (-1)^{\ell(u)} \tau(x_1^{m-1} \cdots x_n^{m-1} w_\lambda uw_\lambda^{-1} v) \] (S_\lambda \cap w_\lambda S_\lambda w_\lambda^{-1} = \{1\})
\[ = (-1)^{n(\lambda)} \tau(x_1^{m-1} \cdots x_n^{m-1}). \] (S_\lambda \cap w_\lambda S_\lambda w_\lambda^{-1} = \{1\})

\[ \square \]

6.5. **Remark.** Note that we do not use the Assumption 2.10 in this section. The Theorem shows that for all parameters \( q_1, \ldots, q_m \), \( \tau \) is a non-degenerate trace form on the degenerate cyclotomic Hecke algebra \( H_{m,n}(Q) \), that is, \( H_{m,n}(Q) \) is a symmetric algebra for all parameters \( q_1, \ldots, q_m \), which gives a differential proof of the non-degeneration of the trace form \( \tau \).
7. The Schur elements

In this section we compute the Schur elements of the degenerate cyclotomic Hecke algebra $\mathcal{H}$. Assume that Assumption 2.10 holds. In this semisimple case $\{S^\lambda | \lambda \in \mathcal{P}(m,n)\}$ is a complete set of pairwise non-isomorphic irreducible $\mathcal{H}$-modules. Let $\chi^\lambda$ be the character of $S^\lambda$. Following Geck’s results on symmetrizing form (see [GP, Theorem 7.2.6]), we obtain the following definition for the Schur elements of $\mathcal{H}$ associated to the irreducible representations of $\mathcal{H}$.

7.1. Definition. Suppose that $R$ is a field and that $P(\mathcal{H})(Q) \neq 0$. The Schur elements of $\mathcal{H}$ are the elements $s_\lambda(Q) \in R$ such that

$$
\tau = \sum_{\lambda \in \mathcal{P}(m,n)} \frac{1}{s_\lambda(Q)} \chi^\lambda.
$$

The rational functions $\frac{1}{s_\lambda(Q)}$ are also called the weights of $\mathcal{H}$.

Recall that $F_\lambda$ is a primitive idempotent in $\mathcal{H}$ such that $S^\lambda \cong \mathcal{H}F_\lambda$ for each $m$-multipartition of $n$ and $\{F_\lambda | \lambda \in \mathcal{P}(m,n)\}$ is the set of primitive idempotents in $\mathcal{H}$ (see Proposition 5.5). By applying a well-known fact about symmetric algebras (see [CR, Proposition 9.17]), we can compute the Schur elements of $\mathcal{H}$ by following the Lemma.

7.2. Lemma. Assume that $R$ is a field and that $\mathcal{H}$ is semisimple. Let $\lambda$ be an $m$-multipartition of $n$. Then $s_\lambda(Q) = \frac{1}{\tau(F_\lambda)}$.

Before we begin our computation, we consider first an example.

7.3. Example. Fix $i$ with $1 \leq i \leq m$ and let $\eta_i = (\eta_i^1; \ldots; \eta_i^n) \in \mathcal{P}(m,n)$ with $\eta_i^j = \delta_{ij}n$. We will compute the Schur elements $s_{\eta_i}(Q)$. Let $x_\eta = \sum_{w \in S_n} w$ and $u_\eta = \prod_{i \neq j} \prod_{k=1}^n (x_k - q_i)$, and set $e_\eta = u_\eta x_\eta = x_\eta u_\eta$ (cf (3.7)). It follows from Lemma 2.3 that $u_\eta$ is central in $\mathcal{H}$. Further, the relations imply that $x_1 u_\eta = q_i u_\eta$ and $w x_\eta = x_\eta$ for $w \in S_n$; it follows that $x_k e_\eta = (k - 1 + q_i) e_\eta$ for $k = 1, \ldots, n$. Thus the module $\mathcal{H}m_{\eta_i}$ is one dimensional and, in particular, irreducible; in fact by Theorem 3.11(ii) and $\|S^\eta \| \cong \mathcal{H}e_\eta = \mathcal{H}e_{\eta_i}$. Moreover, by what we have said

$$e_{\eta_i}^2 = n! \prod_{j \neq i} \prod_{k=1}^n (k - 1 + q_i - q_j) \cdot e_{\eta_i}.
$$

So $e_{\eta_i}$ is a scalar multiple of the primitive idempotent which generates $S^\eta$. Hence, by the Lemma,

$$s_{\eta_i}(Q) = n! \prod_{j \neq i} \prod_{k=0}^{n-1} (k + q_i - q_j).
$$

Similar arguments give the Schur elements for the multipartition which is conjugate to $\eta_i$; alternatively, they are given by Corollary 7.5 and the calculation above.

7.4. Remark. There is an action of $S_m$ on the set of $m$-multipartitions of $n$ (by permuting components) and also on the rational functions in $q_1, \ldots, q_m$ (by permuting parameters). When $\mathcal{H}$ is semisimple the Specht modules are determined up to isomorphism by the action of $x_1, \ldots, x_n$; as the relation $\prod_{k=1}^n (x_1 - q_i) = 0$ is invariant under the $S_m$-action it follows that $s_{v^\lambda}(Q) = v \cdot s_\lambda(Q)$ for all $m$-multipartitions $\lambda$ and all $v \in S_m$; this is also clear from Theorem 3.27(i). In the case where $\lambda = \eta_i$, this symmetry is evident in the formulæ above.

By Lemma 7.2 and Theorem 3.27 the Schur elements are given by $s_\lambda(Q) = \tau(F_\lambda)^{-1}$. Proposition 4.10 implies that the Schur elements have the following “palindromy” property.

7.5. Corollary. Suppose that $\lambda$ is an $m$-multipartition of $n$. Then $s_{\lambda^\tau}(Q) = s_\lambda(Q)$.

The following fact gives the formula for the Schur elements of $s_\lambda(Q)$ of $\mathcal{H}$.

7.6. Proposition. Assume that Assumption 2.10 holds. Let $\lambda$ be an $m$-multipartition of $n$. Then $s_\lambda(Q) = (-1)^{\text{d}(\lambda)} \eta_{\lambda^\tau} Q^{\lambda^\tau}$.

Proof. The Proposition follows directly by Proposition 5.5, Lemma 7.2, and Theorem 6.4.

□
A closed formula for $\gamma_{k_\lambda}$ can be given by Theorem 3.18(ii.a). Now we determine $\gamma_{k_\lambda}$ which enables us to give an explicit formula for $s_\lambda(Q)$. Before to do this, we need some notation.

7.7. Recall that the $(i, j)$-th hook in the diagram $[x^\lambda]$ is the collection of nodes to the right of and below the node $(i, j, s)$, including the node $(i, j, s)$ itself. The $(i, j)$-th hook length $h_{ij}^{\lambda^c} = \lambda^c_i + \lambda^c_j - i - j + 1$ is the number of nodes in the $i$th hook and the leg length $h_{ij}^{\lambda^c} = \lambda^c_j - j + 1$, is the number of nodes in the “leg” of this hook. Observe that if $(a, b, c)$ and $(i, j, c)$ are two removable nodes in $[\lambda^{(c)}]$ with $a \leq i$ and $j \leq b$ then $h_{aj}^{\lambda^c} = b - a - j + i + 1$.

7.8. Lemma. Suppose that $\lambda$ is an $m$-multipartition of $n$. Then

$$\gamma_{k_\lambda} = \prod_{(i, j, s) \in [\lambda]} h_{ij}^{\lambda^c} \prod_{t=s+1}^m \left( \frac{(j-i+q_s-\lambda^c_i-q_t)}{(j-i+q_s-k+1+\lambda^c_k-q_t)} \right).$$

Proof. We proceed by induction on $n$. If $n = 0$, both sides are 1 and there is nothing to prove (by convention, empty products are 1). Suppose that $n > 0$. Let $\mu = \text{Shape}(t_\lambda \downarrow n - 1)$. Then $\mu$ is an $m$-multipartition of $n - 1$. Applying Theorem 3.18(1),

$$\frac{\gamma_{k_\lambda}}{\gamma_{k_\mu}} = \frac{\prod_{x \in \mathcal{A}_\lambda(n)} (\text{res}_{\lambda^c}(n) - \text{res}(x))}{\prod_{y \in \mathcal{A}_\mu(n)} (\text{res}_{\lambda^c}(n) - \text{res}(y))}.$$

Assume that the integer $n$ appears in node $(a, b, c)$ of $t_\lambda$. First consider the contribution that the addable and removable nodes in $[\lambda^c]$ make to $\gamma_{k_\lambda}$. By definitions, these nodes occur in pairs $(x, y)$ where $y \succ (a, b, c)$ is a removable node in row $t$ and $x \succ (a, b, c)$ is an addable node in row $t + 1$ for some $i \geq a$. If $x$ is in column $d$ of $[\lambda^c]$ and $y$ is in column $d'$ then $d \leq d' < b$ and

$$\frac{\text{res}_{\lambda^c}(n) - \text{res}_{\lambda^c}(x)}{\text{res}_{\lambda^c}(n) - \text{res}_{\lambda^c}(y)} = \frac{b-a+q_c-d+(i+1)-q_c}{b-a+q_c-d'+i-q_c} = \prod_{j=d}^{d'} \frac{b-a-j+(i+1)}{b-a-j+i} = \prod_{j=d}^{d'} \frac{j-i-1}{j-i} = \prod_{j=d}^{d'} \frac{h_{aj}^{\lambda^c}-1}{h_{aj}^{\lambda^c}} - 1.$$

Therefore,

$$\frac{\prod_{(i, j, c) \in \mathcal{A}_\lambda(n)} (\text{res}_{\lambda^c}(n) - \text{res}_{\lambda^c}(i, j, c))}{\prod_{(i, j, c) \in \mathcal{A}_\lambda(n)} (\text{res}_{\lambda^c}(n) - \text{res}_{\lambda^c}(i, j, c))} = \prod_{j=1}^{b-1} \frac{h_{aj}^{\lambda^c}}{h_{aj}^{\lambda^c}} - 1 = \prod_{j=1}^{b-1} \frac{h_{aj}^{\lambda^c}}{h_{aj}^{\lambda^c}} - 1.$$

Now $\gamma_{k_\mu}$ is known by induction and it contains as a factor the left hand term in the product below. Further, $\ell_{ij}^{\mu^c} = \ell_{ij}^{\lambda^c}$ and $h_{ij}^{\mu^c} = h_{ij}^{\lambda^c}$ if $(i, s) \neq (a, c)$, and $\ell_{aj}^{\mu^c} = \ell_{aj}^{\lambda^c}$ for $1 \leq j < b$, so

$$\left( \prod_{(i, j, s) \in [\mu]} h_{ij}^{\mu^c} \right) \left( \prod_{j=1}^{b-1} h_{aj}^{\lambda^c} \right) \left( \prod_{(i, j, s) \in [\lambda]} \frac{h_{ij}^{\lambda^c}}{h_{ij}^{\mu^c}} \right) \left( \prod_{(i, j, s) \in [\lambda]} \frac{h_{ij}^{\lambda^c}}{h_{ij}^{\mu^c}} \right) = \prod_{(i, j, s) \in [\lambda]} h_{ij}^{\lambda^c},$$

since $h_{ab}^{\lambda^c} = 1 = \ell_{ab}^{\lambda^c}$. This accounts for the left hand factor in the expression for $\gamma_{k_\lambda}$ given in the statement of the Lemma.

Finally, consider the nodes in $\mathcal{R}_\lambda(n)$ and $\mathcal{R}_\lambda(n)$ which are in component $t$ for some $t > c$ (there are no such nodes for $t < c$). Again, almost all of the addable and removable nodes in component $t$ occur in pairs placed in consecutive rows; however, this time there is also an additional addable node at the end of the first row of $\lambda^i$. As above, it is easier to insert extra factors which cancel out and so take a
product over all of the columns of $\lambda^{(t)}$. An argument similar to that above shows that the nodes in $\mathcal{A}_\lambda(n)$ and $\mathcal{B}_\lambda(n)$ which do not belong to component $c$ contribute the factor
\[
\prod_{t=c+1}^r \frac{(b - a + q_c - \lambda_1 - q_t) \prod_{k=1}^{\lambda_1^t} \frac{(b - a + q_c - k + 1 + \lambda_c^t - q_t)}{(b - a + q_c - k + \lambda_c^t - q_t)}}
\]
to $\gamma_\lambda$. Using induction to combine the formulae above proves the Lemma.

Now we obtain the explicit formulae for the Schur elements of the degenerate Hecke algebras $H$.

**7.9. Theorem.** Let $\lambda$ be an $m$-multipartition of $n$. Then
\[
s_\lambda(Q) = \prod_{(i,j,s) \in [\lambda]} h_{ij}^{s} \prod_{1 \leq s < t \leq m} X_{st}^{\lambda}
\]
where, for $1 \leq s < t \leq m$,
\[
X_{st}^{\lambda} = \prod_{(k,l) \in [\lambda]} (l - k + q_t - q_s) \prod_{(i,j) \in [\lambda]} \frac{(j - i + q_s - \lambda_1 - q_t) \prod_{r=1}^{\lambda_1^t} \frac{(j - i + q_s - r + 1 + \lambda_r^t - q_t)}{j - i + q_s - r - q_t}}{(j - i + q_s - r + \lambda_r^t - q_t)}
\]

**Proof.** By applying Theorem 3.18(ii.a), we obtain that
\[
\bar{\gamma}^{\lambda} = \tilde{\lambda}! \prod_{1 \leq s < t \leq m} \prod_{(i,j,s) \in [\lambda]} (j - i + q_t - q_s).
\]
Observe that $\tilde{\lambda}! = \prod_{(i,j,s) \in [\lambda]} \ell_{ij}^{s}$. Further $(i, j) \in [\lambda^s]$ if and only if $(j, i) \in [\lambda^{m-s+1}]$. Therefore applying the $\bar{\gamma}$ operation on the equality $(\bar{\gamma}^{\lambda})$, and swapping the roles $s$ and $t$ in the right-hand factor,
\[
\bar{\gamma}^{\lambda} = \tilde{\lambda}! \prod_{1 \leq s < t \leq m} \prod_{(i,j) \in [\lambda^s]} (j - i + q_t - q_s)
\]
\[
= \tilde{\lambda}! \prod_{1 \leq s < t \leq m} \prod_{(j,i) \in [\lambda^{m-s+1}]} (j - i + q_{m-l+1} - q_{m-s+1})
\]
\[
= (-1)^{n(\lambda)} \tilde{\lambda}! \prod_{1 \leq s < t \leq m} \prod_{(j,i) \in [\lambda^{m-s+1}]} (i - j + q_{m-s+1} - q_{m-l+1})
\]
\[
= (-1)^{n(\lambda)} \prod_{(i,j,s) \in [\lambda]} \ell_{ij}^{s} \prod_{1 \leq s < t \leq m} \prod_{(i,j) \in [\lambda^s]} (j - i + q_t - q_s).
\]
Now using Proposition 7.6 and Lemma 7.8,
\[
s_\lambda(Q) = \prod_{s=1}^m \prod_{(i,j) \in [\lambda^s]} h_{ij}^{s} \prod_{t=s+1}^m \prod_{(k,l) \in [\lambda^t]} (l - k + q_t - q_s)
\]
\[
\prod_{(i,j) \in [\lambda]} \frac{(j - i + q_s - q_t) \prod_{r=1}^{\lambda_1^t} \frac{(j - i + q_s - r + 1 + \lambda_r^t - q_t)}{j - i + q_s - r - q_t}}{(j - i + q_s - r + \lambda_r^t - q_t)}
\]
\[
= \prod_{s=1}^m \prod_{(i,j) \in [\lambda^s]} h_{ij}^{s} \prod_{t=s+1}^m X_{st}^{\lambda}.
\]

**7.10. Example.** It is straightforward to check that the Theorem gives the same rational functions for the Schur elements $s_\mu(Q)$ as were obtained in Example 7.3.

**7.11. Remark.** In a subsequent paper, we will give a symmetric and cancellation-free formula for the Schur elements for the degenerate cyclotomic Hecke algebras and some applications of our formula.
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*School of Applied Mathematics, Beijing Normal University at Zhuhai, Zhuhai, 519087, China
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
E-mail address: deke@amss.ac.cn*