CONTACT HAMILTONIAN DYNAMICS
AND PERTURBED CONTACT INSTANTONS
WITH LEGENDRIAN BOUNDARY CONDITION

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Abstract. This is the first of a series of papers in preparation in which we
study the Hamiltonian perturbed contact instantons with Legendrian bound-
ary condition and its applications. In this paper, we establish nonlinear el-
lipticity of this boundary value problem by proving the a priori elliptic coercive
estimates for the contact instantons with Legendrian boundary condition, and
prove an asymptotic exponential $C^\infty$-convergence result at a puncture under
the uniform $C^1$ bound. We prove that the asymptotic charge of contact in-
stantons at the punctures under the Legendrian boundary condition vanishes,
which eliminates the phenomenon of the appearance of spiraling cusp instan-
ton along a Reeb core. This removes the only remaining obstacle towards
the compactification and the Fredholm theory of the moduli space of contact
instantons in the open string case, which plagues the closed string case.

In sequels to the present paper, we study the $C^1$ estimates by defining a
proper notion of energy for the contact instantons, and develop a Fredholm
theory and construct a Gromov-type compactification of the moduli space of
contact instantons with Legendrian boundary condition and of finite energy,
and apply them to problems in contact topology and dynamics.

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1. **Introduction**

This is the first part of a series of papers in preparation in which we study the Hamiltonian perturbed contact instantons with Legendrian boundary condition. The purpose of our study of this problem is two-fold. The first one is to construct a Floer theoretic construction of Legendrian spectral invariants on the one-jet bundle given by Théret in [T], Bhupal [B] and Sandon [Sa1], which are constructed using the generation functions quadratic at infinity (g.f.q.i.). This is the Legendrian version of Viterbo’s g.f.q.i. spectral invariants constructed in [V] for the Lagrangian submanifolds in the cotangent bundle.

Recall that the Floer theoretic construction of Viterbo’s invariant is given by the present author in [Oh1, Oh2]. The starting point of this Floer theoretic construction was a remarkable observation of Weinstein [We] which reads that the classical action functional is a generating function of the time-one image $\phi^1_H(0_{J^1B})$ of the zero section $0_{J^1B}$ under the Hamiltonian flow of $H = H(t,x)$.

Therefore the natural first step towards Floer theoretic construction of Legendrian spectral invariants is to find a similar formulation of the contact version of Weinstein’s observation. More precisely, let

$$\lambda = dz - pdq$$

be the standard contact one-form on $J^1B$ and $R = \psi^1_H(0_{J^1B})$ is the Legendrian submanifold which is the time-one image of the contact flow $\psi^t_H$ associated to the time-dependent function $H = H(t,y)$ with $y = (x,z) \in J^1B$. We have found the contact counterpart of Weinstein’s observation whose detailed explanation will be given in a sequel [OY] to this paper.

Then towards the construction of a Legendrian counterpart of the construction in [Oh1 Oh2] it turns out that one needs to generalize the notion of contact instantons introduced by Wang and present author [OW1, OW3]. In a series of papers, [OW2, OW3] jointed with Wang and in [Oh4], the present author developed analysis of contact Cauchy-Riemann maps without taking symplectization. There is the phenomenon of the appearance of spiraling contact instantons along the Reeb core, even for finite $\pi$-energy instantons, which is caused by a puncture at which the asymptotic charge of a contact instanton is nonzero while the asymptotic period is zero. The first main result of the present paper is a vanishing result of this
charge for the instanton with Legendrian boundary condition, and the asymptotic convergence theorem stated below in Theorem 1.6.

Let $(\Sigma, j)$ be a compact Riemann surface with boundary and $\hat{\Sigma}$ a punctured Riemann surface with a finite number of punctures which may be either interior or boundary.

Then we extend the theory of contact instantons in two directions. One is to establish the Legendrian boundary condition for the contact instanton equation for a map $w : \hat{\Sigma} \to M$ with $\hat{\Sigma}$ from a punctured bordered Riemann surface as an elliptic boundary condition by establishing the a priori coercive elliptic estimates. For a $(k+1)$-tuple $(R_0, R_1, \ldots, R_k)$ of Legendrian submanifolds, we consider the boundary value problem

\[
\begin{cases}
\bar{\partial} w = 0, & d(w^*\lambda \circ j) = 0, \\
 w(\Sigma_i, z_{i+1}) \subset R_i.
\end{cases}
\]

(1.1)

The other is to introduce a Hamiltonian-perturbed contact instanton equation again as an elliptic boundary value problem. Consider a time-dependent function $H = H(t, y) : \mathbb{R} \times M \to \mathbb{R}$. Denote by $X_H$ the associated contact vector field. Then we have $\lambda(X_H) = -H$. For each given coorientation preserving contact diffeomorphism $\psi$ of $(M, \xi)$, we have $\psi^*\lambda = e^g\lambda$ for some function $g$ which we denote by $g = g_\psi$.

The following perturbed contact instanton equation is the contact counterpart of the celebrated Floer’s Hamiltonian-perturbed Cauchy-Riemann equation in symplectic geometry.

**Definition 1.1.** Let $(M, \lambda)$ be contact manifold equipped with a contact form, and consider the (time-dependent) contact triad $$(M, \lambda, J), \quad J = \{J_t\}_{t \in [0,1]}.$$ Let $H = H(t, x) : \mathbb{R} \times M \to \mathbb{R}$ be a time-dependent Hamiltonian. We say $u : \mathbb{R} \times [0,1] \to M$ is a $X_H$-perturbed Legendrian Floer trajectory if it satisfies

\[
\begin{cases}
(du - X_H \otimes dt)^{\pi(0,1)} = 0, & d(e^{g_H(u)}(u^*\lambda + H dt \circ j)) = 0 \\
u(\tau, 0) \in R_0, & u(\tau, 1) \in R_1
\end{cases}
\]

(1.2)

where the function $g_H(u) : \mathbb{R} \times [0,1] \to \mathbb{R}$ is defined by

\[
g_H(u)(t, x) := g_{(\psi_H^t(\psi_H^1)^{-1})(u(t, x))}.
\]

(1.3)

The presence of the second equation in (1.2) may look somewhat mysterious but it turns out to be a natural equation for the instanton connecting two contact Hamiltonian (Moore) trajectories: The contact Hamilton’s equation can be decomposed

\[
\dot{x} = X_H(t, x) \iff \begin{cases}
(\dot{x} - X_H(t, x))^\pi = 0, \\
g^*(\lambda + H dt) = 0, \\
g(0) \in R_0, & g(1) \in R_1
\end{cases}
\]

(1.4)

into the $\xi$-component and the Reeb component of the equation. See Section 3 for more discussion on this.

**Remark 1.2.** In physics literature ([W] for example), the term instanton is used to mean a connecting gradient trajectory of two critical points of a function (or a functional). Regarding a contact Hamiltonian trajectory as a ‘critical point’ of
the contact action functional, we would like to regard a contact instanton with two
punctures as such a connecting 'gradient trajectory'.

We can regard a contact Hamiltonian vector field $X_{-H}$ for a positive time-
dependent Hamiltonian $H$ as the time-dependent Reeb vector field of the time-
dependent contact form $H^{-1}\lambda$ and then (1.2) itself as the associated (unperturbed)
time-dependent contact instanton equation. In the present paper, we will focus on
the analysis of unperturbed equation (1.3) leaving necessary detailed discussion on
the Hamiltonian-perturbed equation (1.2) and its application to [Oh5]. We also
refer readers to [Oh5] for the explanation on what the choice of the conformal
exponent function $g_{H}$ in (1.3) is about.

As the first step towards the analytic study of the above boundary value problem
of the contact instanton, we prove the following elliptic $W^{2,2}$-estimates in the
general context of maps $w : \Sigma \to M$ for a Riemann surface of genus zero with a
finite number of boundary punctures.

Let $R_{0}, \ldots, R_{k}$ be a tuple of Legendrian submanifolds and $\Sigma = \Sigma \setminus \{z_{0}, \ldots, z_{k}\}$
with $z_{0}, \ldots, z_{k} \subset \partial \Sigma$. Let $w : \hat{\Sigma} \to M$ satisfy

$$\begin{cases}
\bar{\partial}^* w = 0 & d(w^* \lambda \circ j) = 0 \\
w(z) \in R_{i}, & \text{for } z \in \Sigma_{i+1}
\end{cases} \quad (1.5)$$

**Theorem 1.3.** Let $w : \mathbb{R} \times [0,1] \to M$ satisfy (1.5). Then for any relatively
compact domains $D_{1}$ and $D_{2}$ in $\hat{\Sigma}$ such that $\overline{D_{1}} \subset D_{2}$, we have

$$\|dw\|_{W^{1,2}(D_{1})} \leq C_{1}\|dw\|_{L^{2}(D_{2})}^{2} + C_{2}\|dw\|_{L^{4}(D_{2})}^{4} + C_{3}\|dw\|_{L^{\lambda}(\partial D_{2})}^{3}$$

where $C_{1}, C_{2}$ are some constants which depend only on $D_{1}, D_{2}$ and $(M, \lambda, J)$
and $C_{3}$ is a constant which also depends on $R_{i}$ as well.

Starting from Theorem 1.3 and using the embedding $W^{2,2} \hookrightarrow C^{0,\alpha}$ with $0 < \alpha < 1/2$,
we also establish the following higher local $C^{k,\alpha}$-estimates on punctured surfaces $\Sigma$
in terms of the $W^{2,2}$-norms with $\ell \leq k + 1$.

**Theorem 1.4.** Let $w$ satisfy (1.5). Then for any pair of domains $D_{1} \subset D_{2} \subset \hat{\Sigma}$
such that $\overline{D_{1}} \subset D_{2}$,

$$\|dw\|_{C^{k,\alpha}} \leq C\|dw\|_{W^{1,2}(D_{2})}$$

where $C > 0$ depends on $J, \lambda$ and $D_{1}, D_{2}$ but independent of $w$.

In particular, any weak solution of (1.5) in $W^{1,4}_{\text{loc}}$ automatically becomes a classical
solution.

Next we study the asymptotic convergence result of contact instantons of finite
$E^{\pi}$-energy with Legendrian pair $(R_{0}, R_{1})$ near the punctures of a Riemann surface
$\Sigma$.

Let $\hat{\Sigma}$ be a punctured Riemann surface with punctures $\{p_{i}^{+}\}_{i=1,\ldots,t} \cup \{p_{j}^{-}\}_{j=1,\ldots,t-}$
equipped with a metric $h$ with cylindrical ends outside a compact subset $K_{\Sigma}$. Let
$\hat{w} : \hat{\Sigma} \to M$ be any smooth map. As in [OW2], we define the total $\pi$-harmonic
energy $E^{\pi}(w)$ by

$$E^{\pi}(w) = E^{\pi}(w, \Sigma, h) = \frac{1}{2} \int_{\Sigma} |d^{\pi}w|^{2} \quad (1.6)$$

where the norm is taken in terms of the given metric $h$ on $\hat{\Sigma}$ and the triad metric
on $M$. 

We put the following hypotheses in our asymptotic study of the finite energy contact instanton maps \( w \) as in [OW2].

**Hypothesis 1.5.** Let \( h \) be the metric on \( \dot{\Sigma} \) given above. Assume \( w : \dot{\Sigma} \to M \) satisfies the contact instanton equations (1.5), and

1. \( E^\pi_{(\lambda, J, \dot{\Sigma}, h)}(w) < \infty \) (finite \( \pi \)-energy);
2. \( \|dw\|_{C^0(\dot{\Sigma})} < \infty \).
3. Image \( w \subset K \subset M \) for some compact set \( K \).

Let \( w \) satisfy Hypothesis 1.5. We can associate two natural asymptotic invariants at each puncture defined as

\[
T := \frac{1}{2} \int_{[0, \infty) \times [0, 1]} |d^\pi w|^2 + \int_{\{0\} \times [0, 1]} (w|_{\{0\}} \times [0, 1])^* \lambda \quad (1.7)
\]

\[
Q := \int_{\{0\} \times [0, 1]} ((w|_{\{0\}} \times [0, 1])^* \lambda \circ j) \quad (1.8)
\]

(Here we only look at positive punctures. The case of negative punctures is similar.) As in [OW2], we call \( T \) the *asymptotic contact action* and \( Q \) the *asymptotic contact charge* of the contact instanton \( w \) at the given puncture.

**Theorem 1.6** (Vanishing Charge and Subsequence Convergence). Let \( w : [0, \infty) \times [0, 1] \to M \) satisfy the contact instanton equations (1.5) and Hypothesis 1.5. Then for any sequence \( s_k \to \infty \), there exists a subsequence, still denoted by \( s_k \), and a massless instanton \( w_\infty(\tau, t) \) (i.e., \( E^\pi(w_\infty) = 0 \)) on the cylinder \( \mathbb{R} \times [0, 1] \) such that

\[
\lim_{k \to \infty} w(s_k + \tau, t) = w_\infty(\tau, t)
\]

in the \( C^l(K \times [0, 1], M) \) sense for any \( l \), where \( K \subset [0, \infty) \) is an arbitrary compact set. Furthermore, \( w_\infty \) has \( Q = 0 \) and the formula \( w_\infty(\tau, t) = \gamma(Tt) \) with asymptotic action \( T \), where \( \gamma \) is some Reeb chord joining \( R_0 \) and \( R_1 \) of period \( |T| \).

**Remark 1.7.** The vanishing \( Q = 0 \) of the asymptotic charge of contact instanton \( w \) with (nonempty) Legendrian boundaries is a big deviation from the closed string case studied in [OW2, OW3, Oh4]. This vanishing removes the only remaining obstruction, the so called the appearance of spiraling instantons along the Reeb core, to a Fredholm theory and to a compactification of the moduli space of contact instantons. This obstruction has been the main reason why the paper [Oh4] has not been released in public but being held in the author’s homepage until recently since 2013.

The coercive elliptic regularity estimates and the vanishing theorem of asymptotic charge in the present work will be the foundation of the analytic package of Gromov-Floer-Hofer compactification and the Fredholm theory of Hamiltonian-perturbed contact instantons with Legendrian boundary condition and its applications. In [Oh5] the author developed this analytic package for the study of the moduli space of contact instantons perturbed by a (parameterized) cut-off Hamiltonian of the type used in [Oh2]. Using the analytic package, we prove a conjecture of Shelukhin [She], which concerns existence of translated points of contactomorphisms [Shc], as a consequence of its Legendrianization, a general existence result of Reeb chords between the pair \( (R, \psi(R)) \) for any compact Legendrian submanifolds.
In a sequel [Oh6], we will develop the coercive estimates for the perturbed contact instantons extending the current estimates to the case of with non-zero Hamiltonians, and a relevant Fredholm theory elsewhere by adapting the one from [Oh4] to the current case of contact instantons with boundary. In other sequels, we will also apply these for the construction of the Fukaya-type category on contact manifolds whose objects are Legendrian submanifolds and morphisms and products which will be constructed by counting appropriate contact instantons satisfying some prescribed asymptotic conditions and Legendrian boundary conditions. It will be interesting to see if Chekanov-Eliashberg-type DGA can be constructed through such a study, which we will investigate elsewhere in a future.

In another sequel with Yu [OY], utilizing these background geometric preparation, we will give the Floer theoretic construction of Legendrian spectral invariants and provide some applications thereof.

Now is the organization of the paper in order. In Section 2, we provide a systematic exposition on the basic contact Hamiltonian geometry which will be needed for the a priori elliptic estimates. Majority of the results in this section are widely known to experts but are scattered around in literature with various different sign conventions. Partly to fix our sign conventions, we organize them in the way that will be useful for our tensorial calculations. In Section 3 we compute the first variation of the contact action functional and explain the relation between its critical point and the contact Hamiltonian trajectories. In Section 4 and 5, we establish the local $W^{2,2}$ and $C^{k,\alpha}$ estimates for $k \geq 1$ with $0 < \alpha < 1/2$ respectively for the contact instantons with Legendrian boundary conditions. In Section 6 we prove that the presence of Legendrian barrier forces the asymptotic charge to vanish and that for any contact instantons with derivative bound there exists a sequence $\tau_j \to \infty$ for which $u(\tau_j, t) \to \gamma(Tt)$ uniformly as $j \to \infty$ for some $T > 0$ in the strip-like coordinates $(\tau, t)$ near each puncture of the domain $\dot{\Sigma}$ of the map $u$. In Section 7, we prove that when the Legendrian boundary $(R_0, R_1)$ near a puncture are transversal in a suitable sense then the above convergence is exponentially fast. In 8, we apply the contact instanton equation to the one-jet bundle $J^1B$ and rewrite the equation in terms of the product coordinates $w = (u, f)$ of $J^1B = T^*B \times \mathbb{R}$ with respect to CR almost complex structure lifted from the cotangent bundle $T^*B$, which gives rise to an interesting elliptic system.

We thank Rui Wang for her previous collaboration on the contact instantons with the closed string case [OW1, OW2, OW3], and for her useful comments on the preliminary version of the present paper.

Convention:

- (Contact Hamiltonian) The contact Hamiltonian of a time-dependent contact vector field $X_t$ is given by
  \[ H := -\lambda(X_t). \]
  We denote by $X_H$ the contact vector field whose associated contact Hamiltonian is given by $H = H(t, x)$. This convention is consistent with that of [B, BCT, dMV] but is the negative to that of [Ar, Appendix 4] and [MS].

- The Hamiltonian vector field on symplectic manifold $(P, \omega)$ is defined by $X_H|\omega = dH$. We denote by
  \[ \psi_H : t \mapsto \psi^t_H \]
its Hamiltonian flow.

- Contact action functional denoted by \( A_H \) is given by

\[
A_H(\gamma) = \int \gamma^*(\lambda - H dt) = -\int \gamma^*\lambda - \int_0^1 H(t, \gamma(t)) dt.
\]

The choice of the negative sign in front of \( \int \gamma^*\lambda \) is to be consistent with the classical action functional on the cotangent bundle \( T^*B \) which is given by

\[
A_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(t, \gamma(t)) dt
\]

where \( \lambda = dz - \theta \) is the canonical contact form on \( J^1B \).

- We denote by \( R_\lambda \) the Reeb vector field associated to \( \lambda \). We denote by \( \phi^t_{R_\lambda} \) its flow.

2. Some Contact Hamiltonian Geometry and Dynamics

In this section, we state some useful results concerning the contact Hamiltonian dynamics. Partly to fix our sign conventions, we organize them in the way that will be useful for us to systematically perform geometric calculations that enter in our elliptic estimates for contact instantons in the spirit of \([OW2, OW3]\). This systematic treatment will be needed for the analytical study of Hamiltonian-perturbed contact instanton equations \([153]\). For this reason, we here include the contact Hamiltonian calculi more than what we need for the purpose of the present paper (the case with \( H = -1 \)), but develop them as the contact Hamiltonian calculi similarly as in the Hamiltonian calculus in the symplectic case. These will be needed in the sequel as in \([?]\), where we develop the coercive elliptic estimates for the perturbed equation \([153]\), and also applications thereof to contact Hamiltonian dynamics in \([OY]\) and in others.

We also refer readers to \([BCT, dMV]\) for some expositions on contact Hamiltonian calculus (especially \([dMV]\) with the same sign conventions as ours in both the symplectic and the contact contexts), which partially overlap with the exposition of the present section.

2.1. Conformal exponents of contact diffeomorphisms. We begin with the definition of contact diffeomorphisms.

**Definition 2.1.** Let \((M, \xi)\) be a contact manifold. A diffeomorphism \( \psi : M \to M \) is called a contact diffeomorphism if \( d\psi(\xi) \subset \xi \). Denote by \( \text{Cont}(M, \xi) \) (resp. \( \text{Cont}_0(M, \xi) \)) the set of contact diffeomorphisms (resp. the identity component thereof).

When \((M, \xi)\) is coorientable, we can choose a contact form \( \lambda \) with \( \xi = \ker \lambda \). In the present paper, we will always assume \((M, \xi)\) is cooriented and all contact forms represent the given coorientation.

**Remark 2.2.** When \((M, \xi)\) is not coorientable, a natural treatment of such a contact manifold is to regard it as a Jacobi manifold \((M, L, \{\cdot, \cdot\})\) where \( L \to M \) is the line bundle \( L = TM/\xi \) and \( \{\cdot, \cdot\} \) is a Lie bracket on the set of sections \( \Gamma(L) \) which is a first order differential operator on each entry. See \([K, LOTV]\) Definition
Furthermore for a different choice of contact form $\lambda$ and $g$ as $\xi$ in the same orientation class, we make $\lambda$, $\lambda'$ of $G$ consider the function $\phi$ can be written as $Z$ on the right if $\psi(\phi)$ for some smooth function $\psi$. Furthermore we have $\lambda' = e^{h(\lambda', \lambda)} \lambda$ for some function $h(\lambda, \lambda')$ depending on $\lambda$, $\lambda'$. Then a straightforward calculation leads to $g_{\phi'} = h_{\lambda', \lambda} \circ \psi + g_{\phi}$. This itself can be written as $\varphi_{\lambda'} - \varphi_{\lambda} = h_{\lambda, \lambda'}$ where we regard $h(\lambda, \lambda')$ as a zero-cochain in the group cohomology complex $C^*(G, C^\infty(M))$. In other words, the two cochains $\varphi_{\lambda'}$, $\varphi_{\lambda}$ are cohomologous to each other.

The following is an immediate corollary of Lemma 2.4 which plays an important role in the study of an invariance property of Legendrian spectral invariants of contact diffeomorphisms under the conjugation. (See [3], [Sa2], for example.)

**Corollary 2.5.** Let $\lambda$ be any contact form of $(M, \xi)$ and consider the conformal exponent map $\psi \mapsto g_{\psi}$ associated to $\lambda$. Then

$$g_{\phi \psi} = g_{\phi} \circ \psi \circ \phi^{-1} - g_{\phi} \circ \phi^{-1} + g_{\psi} \circ \phi^{-1}.$$  \hspace{1cm} (2.3)

Note that this formula is reduced to a simple one $g_{\phi \psi} = g_{\psi} \circ \phi^{-1}$ when $\phi$ is a strict contactomorphism.
2.2. Contact vector fields and their Hamiltonians.

**Definition 2.7.** A vector field $X$ on $(M, \xi)$ is called contact if $[X, \Gamma(\xi)] \subset \Gamma(\xi)$ where $\Gamma(\xi)$ is the space of sections of the vector bundle $\xi \to M$. We denote by $\mathfrak{X}(M, \xi)$ the set of contact vector fields.

When $\lambda$ is a contact form of $\xi$, it uniquely defines the Reeb vector field $R_\lambda$ by the defining condition

$$R_\lambda d\lambda = 0, \quad \lambda(R_\lambda) = 1 \quad (2.4)$$

and it admits a decomposition

$$TM = \xi \oplus \mathbb{R} \langle R_\lambda \rangle$$ $$T^*M = (\mathbb{R} \langle R_\lambda \rangle) ^\perp \oplus (\xi) ^\perp$$

where $(\cdot) ^\perp$ denotes the annihilator of $(\cdot)$. Then the condition $[X, \Gamma(\xi)] \subset \Gamma(\xi)$ is equivalent to the condition that there exists a smooth function $f : M \to \mathbb{R}$ such that

$$\mathcal{L}_X \lambda = f \lambda.$$ 

**Definition 2.8.** Let $\lambda$ be a contact form of $(M, \xi)$. The associated function $H$ defined by

$$H = -\lambda(X) \quad (2.5)$$

is called the $\lambda$-contact Hamiltonian of $X$. We also call $X$ the $\lambda$-contact Hamiltonian vector field associated to $H$.

We alert readers that under our sign convention, the $\lambda$-Hamiltonian $H$ of the Reeb vector field $R_\lambda$ as a contact vector field becomes the constant function $H = -1$.

**Remark 2.9.** Unlike the symplectic case, the vanishing locus of contact Hamiltonian $H$ has a very natural geometric meaning: The locus is precisely the set of points at which the contact vector field is tangent to the contact distribution $\xi$.

Conversely, any smooth function $H$ associates a unique contact vector field that satisfies the relationship spelled out in the above definition. We highlight the fact that unlike the symplectic case, this correspondence is one-one with no ambiguity of addition by constant. Existence and derivation of the formula of $X_H$ from $H$ is not as straightforward as that of Hamiltonian vector field in symplectic geometry. Since we will mostly fix the contact form $\lambda$ in the present paper, we will simply call $X_H$ a contact Hamiltonian vector field associated to $H$ instead of $\lambda$-contact Hamiltonian vector field.

The following lemma is a special case of [OW3, Lemma 2.1]. Since there is a sign difference between the current paper and [OW3] for the definitions of Hamiltonian vector fields and that of contact Hamiltonian, we provide the full proof of the lemma for the readers’ convenience. Also see [MS, Section 2] for some relevant discussions.

We denote by $R_\lambda$ the Reeb vector field of $\lambda$.

**Proposition 2.10.** Let $H$ be any smooth function on $M$ equipped with contact form $\lambda$. Then the contact Hamiltonian vector field $X_H$ is uniquely determined by $H$, and satisfies the equation

$$dH = X_H d\lambda + R_\lambda[H] \lambda. \quad (2.6)$$
Proof. Recall the contact form $\lambda$ provides a canonical splitting

$$TM = \xi \oplus \text{span}_\mathbb{R}\{R_\lambda\}$$

for the Reeb vector field $R_\lambda$. Therefore we can decompose

$$X = X^\pi + X^\perp$$

where $X^\pi$ (resp. $X^\perp$) is the component of $\xi$ (resp. of $\text{span}_\mathbb{R}\{R_\lambda\}$).

The relation $H = -\lambda(X)$ already uniquely specifies the $R_\lambda$-component $X^\perp$ by

$$X^\perp = -HR_\lambda$$

since $\lambda(X^\pi) = 0$. It remains to determine $\xi$-component $X^\pi$. Thanks to the nondegeneracy of $d\lambda$ on $\xi$, it is enough to determine $X^\pi|d\lambda$. For this, we first take the differential of $H = -\lambda(X)$ and apply Cartan’s formula to obtain

$$dH = X|d\lambda - \mathcal{L}_X\lambda.$$ 

Then we note $X|d\lambda = X^\pi|d\lambda$ and that $\mathcal{L}_X\lambda$ is uniquely determined from the contact property of $X$ 

$$\mathcal{L}_X\lambda = f\lambda$$

where $f$ is uniquely determined by $H$ as follows.

**Lemma 2.11.** Let $X$ be a contact vector field with $\mathcal{L}_X\lambda = f\lambda$, and let $H$ be the associated contact Hamiltonian. Then $f = -R_\lambda[H]$.

**Proof.** Using the defining condition (2.4) of the Reeb vector field $R_\lambda$ of $\lambda$, and the formula $\lambda(X) = -H$, we obtain

$$f = (\mathcal{L}_X\lambda)(R_\lambda) = -\lambda([X,R_\lambda]) = \mathcal{L}_{R_\lambda} (\lambda(X)) = -R_\lambda[H].$$

Then $X^\pi|d\lambda$ is given by

$$X^\pi|d\lambda = dH + \mathcal{L}_X\lambda = dH - R_\lambda[H]\lambda.$$ 

Finally it follows from (2.7) and (2.9), the right hand sides of which are uniquely determined by $H$, that $X$ satisfies (2.6) which finishes the proof.

In fact, the above proof proves an explicit formula for $X_H$

$$X_H = (\overline{d\lambda})^{-1}(dH)^{+R_\lambda} - HR_\lambda$$

with $(dH)^{+R_\lambda} = dH - R_\lambda[H]\lambda$ where $\overline{d\lambda} : \xi \to \xi^* \cong \{R_\lambda\}^\perp \subset T^*M$ is the natural isomorphism $Y \mapsto Y|d\lambda$ followed by the isomorphism $\xi^* \cong \{R_\lambda\}^\perp$ induced by the splitting $TM \cong \xi \oplus \text{span}\{R_\lambda\}$. Here $\{R_\lambda\}^\perp$ is the annihilator of $R_\lambda$. Note that the one form $dH - R_\lambda[H]\lambda$ is indeed contained in $\{R_\lambda\}^\perp$. In the canonical coordinates $(q, p, z)$ on $\mathbb{R}^{2n+1}$, this expression is reduced to the well-known coordinate formula for the contact Hamiltonian vector field below. (See [Ar, Appendix 4], [OW3, Lemma 2.1], [B, Lemma 4.1], [BCT, (3.6)] for example.)

**Example 2.12.** Following [LOTV, Section 6], we introduce a vector field

$$\frac{D}{\partial q_i} := \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial z}$$
which are tangent to $\xi$. Together with $\frac{\partial}{\partial p_i}$, the set $\{\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}\}$ provides a frame of $\xi$ on the chart of a canonical coordinates $(q_i, p_i, z)$. (In [LOTV], $\frac{\partial}{\partial q_i}$ is denoted by $D_i$ and there is a sign difference in the choice of canonical symplectic form on $T^*B$.)

Let $H : \mathbb{R}^{2n+1} \to \mathbb{R}$ be a smooth function on $\mathbb{R}^{2n+1}$. Then the contact Hamiltonian vector field $X_t = X_{H_t}$ is given by

$$X_t = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{D}{\partial q_i} - \frac{DH}{\partial q_i} \frac{\partial}{\partial p_i} \right) - H \frac{\partial}{\partial z}$$

(2.11)

**Remark 2.13.**

1. In the formula [OW3, Lemma 2.1] applied to one-form $dH$ has no negative sign in front of $\mathbb{R}^\lambda[H] \lambda$ of (2.10). This is because here and in [BCT] the contact Hamiltonian is defined by $H = -\lambda(X)$ while in [OW3], it is defined by $\lambda(X)$.

2. The formula (2.11) explicitly shows that the assignment $H \mapsto X_H$ involves a differential operator of mixed order, a combination of orders zero and one, unlike the symplectic case. This difference makes contact Hamiltonian geometry and dynamics exhibit quite different phenomena from the symplectic case. This will be seen even in the study of the associated Hamilton’s equation and the contact instanton equation.

### 2.3. Contact vector field as a Reeb vector field.

Let $(M, \xi)$ be a contact manifold. For each given contact form $\lambda$, any smooth function $H$ gives rise to a contact vector field, temporarily denoted as

$$X_H = X^\lambda_H$$

to emphasize the $\lambda$-dependence of the expression $X_H$. For another co-oriented contact form $\lambda' = f \lambda$ with $f > 0$, one might want to express $X^\lambda_H$ in terms of $X^\lambda_H$. For general function $H$, the relationship between them is rather complicated and does not seem to deserve writing it down. But for the or the $f\lambda$-Reeb vector field $R_{f\lambda}$, which corresponds to $H = -1$, we have the following.

**Proposition 2.14.** Let $f = f(x)$ be a positive function on $(M, \lambda)$. Consider its inverse $G = -1/f$ and consider a new contact form $f\lambda$. Then we have

$$R_{f\lambda} = X^\lambda_G \left( \frac{1}{f} R_\lambda + X^\pi_{-1/f} \right).$$

**Proof.** For the notational simplicity, we write $X_G = X^\lambda_G$ omitting the superscript $\lambda$ in the following discussion.

Let $X^G_G$ be the $\xi$-projection of $X_G$. Then we have

$$X_G = X^G_G - GR_\lambda \in \xi \oplus \mathbb{R}(R_\lambda).$$

We evaluate

$$\lambda(R_{f\lambda}) = \frac{1}{f}(f\lambda)(R_{f\lambda}) = 1/f = -G$$

which shows that $X_G$ and $R_{f\lambda}$ have the same Reeb component. We can uniquely express $R_{f\lambda}$ as

$$R_{f\lambda} = \frac{1}{f}(R_\lambda + \eta)$$
for some \( \eta \in \xi \). Then we compute

\[
\begin{align*}
-1 f R f \lambda \eta R f \lambda d \lambda &= R f \lambda (d(f \lambda) = R f \lambda)(df \wedge \lambda) \\
&= -R f \lambda (df \wedge \lambda) \\
&= -R f \lambda [f] \lambda + \lambda(R f \lambda) df \\
&= -\frac{1}{f}(R \lambda + \eta)[f] \lambda + \frac{1}{f} \lambda(R \lambda + \eta) df \\
&= -\frac{1}{f} R \lambda[f] \lambda - \frac{1}{f} \eta[f] \lambda + \frac{1}{f} df.
\end{align*}
\]

Take value of \( R \lambda \) for both sides, we derive that \( \eta \) satisfies \( \eta[f] = 0 \). Therefore we have

\[
R f \lambda \eta R f \lambda d \lambda = -\frac{1}{f} R \lambda[f] \lambda - \frac{1}{f} \eta[f] \lambda + \frac{1}{f} df.
\]

This proves \((X_{1/f})^\pi = (R f \lambda)^\pi\). Combining this with \( \lambda(X_{1/f}) = -R f \lambda \), we have finished the proof. \( \square \)

In this regard, for a given contact manifold \((M,\xi)\) equipped with a contact form \( \lambda \), Reeb vector fields for different contact form \( f \lambda \) correspond to the special \( \lambda \)-contact vector fields whose Hamiltonian \( G = -1/f \) is nowhere vanishing.

2.4. Contact isotopy, conformal exponent and contact Hamiltonian. Next let \( \psi_t \) be a contact isotopy of \((M,\xi = \ker \lambda)\) with \( \psi^*_t \lambda = e^{g_t} \lambda \) and let \( X_t \) be the time-dependent vector field generating the isotopy. Let \( H : [0,1] \times \mathbb{R}^{2n+1} \rightarrow \mathbb{R} \) be the associated time-dependent contact Hamiltonian \( H_t = -\lambda(X_t) \). One natural question to ask is explicitly what the relationship between the Hamiltonian \( H = H(t,x) \) and the conformal exponent \( g = g(t,x) \) of \( \Psi = \psi_t \) is.

Similarly as in the symplectic case [OM], we first name the parametric assignment \( \psi_t \rightarrow -\lambda(X_t) \) of \( \lambda \)-Hamiltonians the \( \lambda \)-developing map \( \text{Dev}_{\lambda} \). We will just call it the developing map when there is no need to highlight the \( \lambda \)-dependence of the contact Hamiltonian.

**Definition 2.15 (Developing map).** Let \( T \in \mathbb{R} \) be given. Denote by

\[
\mathcal{P}_0([0,T],\text{Cont}(M,\xi))
\]

the set of contact isotopies \( \Psi = \{\psi_t\}_{t \in [0,T]} \) with \( \psi_0 = \text{id} \). We define the \( \lambda \)-developing map

\[
\text{Dev}_{\lambda} : \mathcal{P}([0,T],\text{Cont}(M,\xi)) \rightarrow C^\infty([0,T] \times M)
\]

by the timewise assignment of Hamiltonians

\[
\text{Dev}_{\lambda}(\Psi) := -\lambda(X), \quad \lambda(X)(t,x) := \lambda(X_t(x)).
\]

We also state the following as a part of contact Hamiltonian calculus, whose proof is a straightforward calculation.

**Lemma 2.16.** Let \( \Psi = \{\psi_t\} \in \mathcal{P}([0,T],\text{Cont}(M,\xi)) \) be a contact isotopy satisfying \( \psi^*_t \lambda = e^{g_t} \lambda \) with \( g_\Psi := g(t,x) \) and generated by the vector field \( X_t \) with its contact Hamiltonian \( H(t,x) = H_t(x) \), i.e., with \( \text{Dev}_{\lambda}(\Psi) = H \).
(1) Then the (timewise) inverse isotopy \( \Psi^{-1} := \{ \psi_t^{-1} \} \) is generated by the (time-dependent) contact Hamiltonian

\[
\text{Dev}_\lambda(\Psi^{-1}) = -e^{-g_\psi \circ \Psi} \text{Dev}_\lambda(\Psi)
\]  

(2.12)

where the function \( g_\psi \circ \Psi \) is given by \((g_\psi \circ \Psi)(t, x) := g_\psi(t, \psi_t(x))\).

(2) If \( \Psi' = \{ \psi'_t \} \) is another contact isotopy with conformal exponent \( g_{\Psi'} = \{ g'_t \} \), then

*the timewise product* \( \Psi' \Psi \) is generated by the Hamiltonian

\[
\text{Dev}_\lambda(\Psi' \Psi) = \text{Dev}_\lambda(\Psi') - e^{-g_\psi \circ (\Psi')^{-1}} \text{Dev}_\lambda(\Psi) \circ (\Psi')^{-1}.
\]  

(2.13)

In particular, with the more standard notation as in [MS], we have

\[
\mathcal{T}' # H(t, x) := \text{Dev}_\lambda(\Psi^{-1} \Psi) = e^{-\delta_t(\psi_t')(H - H')(t, \psi'_t(x))}
\]  

(2.14)

when \( H = \text{Dev}_\lambda(\Psi) \) and \( H' = \text{Dev}_\lambda(\Psi') \) and \( g_{\Psi'}(t, x) = g'_t(x) \). We remark that these formulae are reduced to the standard formulae in the Hamiltonian dynamics in symplectic geometry if \( g_0 \equiv 1 \equiv g'_0 \), i.e., if \( \psi_t, \psi'_t \) are \( \lambda \)-strict contactomorphism.

Finally the following formula provides an explicit relationship between the contact Hamiltonians and the conformal exponents of a given contact isotopy.

**Proposition 2.17.** Let \( \Psi = \{ \psi_t \} \) be a contact isotopy of \( (M, \xi = \ker \lambda) \) with \( \psi_t^* \lambda = e^{g_t} \lambda \) with \( \text{Dev}_\lambda(\Psi) = H \). We write \( g_\psi(t, x) = g_t(x) \). Then

\[
\frac{\partial g_\psi}{\partial t}(t, x) = -R_\lambda[H](t, \psi_t(x)).
\]  

(2.15)

In particular, if \( \psi_0 = \text{id} \),

\[
g_\psi(t, x) = \int_0^t -R_\lambda[H](u, \psi_u(x)) \, du.
\]  

(2.16)

**Proof.** Clearly the latter equality at \( t = 0 \) holds since \( \psi_0 \) is the identity and so \( g_0 \equiv 0 \). We differentiate \( \psi_t^* \lambda = e^{g_t} \lambda \) to obtain

\[
\psi_t^*(L_{\mathcal{X}_t} \lambda) = e^{g_t} \frac{\partial g_\psi}{\partial t} \lambda.
\]

But by Corollary 2.11 we have

\[
\psi_t^*(L_{\mathcal{X}_t} \lambda)(t, x) = \psi_t^*(-R_\lambda[H] \lambda)
\]

\[
= -(R_\lambda[H] \circ \psi_t) \psi_t^* \lambda = -(R_\lambda[H] \circ \psi_t) e^{g_t} \lambda.
\]

Comparing the two sides, we have derived

\[
\frac{\partial g_\psi}{\partial t}(x) = -R_\lambda[H](t, \psi_t(x))
\]

which is (2.15). By integrating it from \( 0 \) to \( t \), we have derived (2.16). \( \square \)

**Remark 2.18.** Bhupal [B] derived the corresponding formula (2.16) for the contact structure \( \lambda = dz - pdq \) in \( \mathbb{R}^{2n+1} \) by a coordinate calculation. Since any contact structure admits canonical coordinates, his coordinate calculation also derives (2.16). Here we provide a simple coordinate-free proof by utilizing the basic properties of contact Hamiltonian vector fields spelled out in this section.
2.5. Miscellaneous contact Hamiltonian calculi. We also mention a few more interesting contact calculi, although they will not be used in the present paper. These calculi are scattered around in various literature with different contexts. (See [LOTV], [dMV], [MS] for example.)

As a special case of Jacobi manifold \((M, L, \{\cdot, \cdot\})\) [K], each contact manifold carries a natural Jacobi bracket

\[ \{\cdot, \cdot\} : \Gamma(L) \times \Gamma(L) \rightarrow \Gamma(L) \]

which is the natural Lie bracket on \(X(M, \xi) \sim T_{id} \text{Cont}(M, \xi)\). (See [K], [LOTV].)

When \((M, \xi)\) is cooriented, the line bundle \(L = \mathbb{R}_M\) is trivial and the associated Jacobi bracket is also called the Lagrange bracket in some literature. (See [AG].)

When a contact form \(\lambda\) is given the map \(X \mapsto -\lambda(X)\) induces a Lie algebra isomorphism given as follows. (Its existence is well-known, e.g., stated explicitly in [Ar], [LOTV], [dMV] in a slightly different form but equivalent up to sign convention.)

Partly due to different sign conventions literature-wise, we state them here to fix the sign convention of ours.

**Proposition 2.19** (Compare with Proposition 5.6 [LOTV], Proposition 9 [dMV]). Let \((M, \xi)\) be cooriented and \(\lambda\) be an associated contact form. Define a bilinear map

\[ \{H, G\} := -\lambda([X_H, X_G]) \quad (2.17) \]

by

\[ \{H, G\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M). \]

Then it satisfies Jacobi identity and the assignment \(X \mapsto -\lambda(X)\) defines a Lie algebra isomorphism \(X(M, \xi)\) to \(C^\infty(M)\).

We state the following special cases explicitly for a future purpose.

**Proposition 2.20.** Let \(H : M \rightarrow \mathbb{R}\) be a function and \(X_H\) the contact Hamiltonian vector field. Then

1. \(dH(X_H) = -HR_\lambda[H]\).
2. Let \(X\) be a contact vector field and let \(H\) be its Hamiltonian. Then \([R_\lambda, X_H] = X = R_\lambda[H]\). In other words, we have \(\{1, H\} = -R_\lambda[H]\).

**Proof.** For the identity (1), we just evaluate \(dH\) against the formula (2.11) of \(X_H\) in canonical coordinates for the time-independent \(H\).

For the identity (2), we first note that the Hamiltonian of \(R_\lambda\) as a contact vector field is the constant function \(-1\). We need to compute \(\lambda([R_\lambda, X_H])\). We have

\[ \lambda([R_\lambda, X_H]) = R_\lambda[\lambda(X_H)] - (L_{R_\lambda} \lambda)(X_H) = R_\lambda[-H] = -R_\lambda[H] \]

which finishes the proof of the first statement. For the second, we recall \(\lambda(R_\lambda) \equiv 1\) and hence \(R_\lambda = X_{-1}\). Therefore we have

\[ \{1, H\} = -\lambda([X_1, X_H]) = -\lambda([-R_\lambda, X_H]) = \lambda([R_\lambda, X_H]) = -R_\lambda[H] \]

where the last equality comes from the first statement. \(\Box\)

**Remark 2.21.**

1. The identity (1) above is interpreted as ‘energy dissipation’ in [dMV]. One interesting consequence of this identity is that the contact Hamiltonian vector field \(X_H\) is tangent to the zero-level set \(H^{-1}(0)\). In this sense, the zero-level set forms an equilibrium status of the flow of \(X_H\). The same also holds for the zero-level set of the function \(R_\lambda[H]\).
The non-vanishing of the bracket \( \{1, H\} = -R_\lambda[H] \) exhibits quite a contrast from the case of symplectic Poisson bracket, which also indicates that the Jacobi bracket does not satisfy the Leibniz rule.

The equality \( \psi^*X_H = X_{H_{\psi^*}} \) holds for any strict contactomorphism \( \psi \) not just for the Reeb flow. In other words, we have

\[
\text{Dev}_\lambda(\psi^{-1} \circ \Psi \circ \psi) = \text{Dev}_\lambda(\Psi) \circ \psi
\]

for any strict contactomorphism \( \psi \). (See [MS, Lemma 2.2].)

### 3. First Variation of the Contact Action Functional

Let \( X_t \) be a contact Hamiltonian vector field and its associated Hamiltonian \( H(t, x) = -\lambda(X_t) \).

We denote by \( \psi^t_H \) its flow and \( (\psi^t_H)^* \lambda = e^{g_t} \lambda \). Consider the free path space \( P := C^\infty([0, 1], M) = \{ \gamma : [0, 1] \to M \} \) and consider the action functional

\[
A_H(\gamma) = -\int \gamma^* \lambda - \int_0^1 H(t, \gamma(t)) \, dt
\]

A straightforward calculation derives the following first variation formula of the action functional \( A_H \) on a free path space \( P \).

**Proposition 3.1.** For any vector field \( \eta \) along \( \gamma \), we have

\[
\delta A_H(\gamma)(\eta) = -\int_0^1 (d\lambda(\dot{\gamma} - X_H(t, \gamma(t)), \eta) - R_\lambda[H](\gamma)(\lambda(\eta))) \, dt
\]

\[
-\langle \lambda(\gamma(1)), \eta(\gamma(1)) \rangle + \langle \lambda(\gamma(0)), \eta(\gamma(0)) \rangle - a(1) + a(0).
\]

**Proof.** Let \( \gamma_s \) with \(-\varepsilon < s < \varepsilon \) such that

\[
\gamma_0 = \gamma, \quad \frac{\partial \gamma_s}{\partial s} \bigg|_{s=0} = \eta
\]

and compute

\[
\frac{d}{ds} \bigg|_{s=0} A(\gamma_s) = -\frac{d}{ds} \bigg|_{s=0} \int_0^1 \gamma_s^* \lambda - \frac{d}{ds} \bigg|_{s=0} \int_0^1 H(t, \gamma_s(t)) \, dt.
\]

For the first term, we have

\[
\frac{d}{ds} \bigg|_{s=0} \gamma_s^* \lambda = d(\eta(\lambda)) + \eta(d\lambda).
\]

On the other hand,

\[
dH = X_H \lambda + R_\lambda[H] \lambda.
\]

Combining the above, we have finished the proof.

We recall that \( \lambda \) provides a natural decomposition

\[
TM = \xi \oplus \mathbb{R}\langle R_\lambda \rangle
\]

which induces a decomposition \( \eta(t) = \eta^\pi(t) + a(t)R_\lambda(\gamma(t)) \) with \( a(t) = \lambda(\eta(t)) \) and hence the splitting

\[
T_\gamma P = \Gamma(\gamma^* TM) = \Gamma(\gamma^* \xi) \oplus C^\infty([0, 1], \mathbb{R})\langle R_\lambda \rangle.
\]
This in turn provides a decomposition of the first variation
\[ \delta A_H(\eta) = \delta A_H(\eta^\pi) + \delta A_H(aR) \]
In terms of this decomposition, we can rewrite (3.2) into
\[ \delta A_H(\eta) = -\int_0^1 d\lambda((\dot{\gamma} - X_H(t, \gamma(t)))^\pi, \eta) dt - \int_0^1 R_\lambda[H](\gamma)a(t) dt \]
for \( \eta = \eta^\pi + aR \). More precisely, we have
\[ \delta A_H(\eta^\pi) = -\int_0^1 d\lambda((\dot{\gamma} - X_H(t, \gamma(t)))^\pi, \eta) dt - \langle \lambda(\gamma(1)), \eta^\pi(\gamma(1)) \rangle + \langle \lambda(\gamma(0)), \eta^\pi(\gamma(0)) \rangle \]
(3.4)
\[ \delta A_H(aR) = -\int_0^1 R_\lambda[H](\gamma)a(t) dt - a(1) + a(0) \]
(3.5)

Summarizing the above discussion and considering the (Moore) path space starting from a Legendrian submanifold \( R_0 \) with free end at \( T \)
\[ \mathcal{L}(M; R_0) = \bigcup_{T \in \mathbb{R}} \{ T \} \times \{ \gamma : [0, T] \to M \mid \gamma(0) \in R_0 \} \]
we have proved that the \( \pi \)-critical point, i.e., a path satisfying \( \delta A_H(\gamma) = 0 \) with fixed \( T \) satisfies
\[ (\dot{\gamma} - X_H(t, \gamma(t)))^\pi = 0, \quad \gamma(0) \in R_0. \]
(3.6)
Then taking the \( R_\lambda \) component (3.9) (with 1 replaced by \( T \)), we obtain
\[ \int_0^T R_\lambda[H](\gamma)a(t) dt + a(T) = 0. \]
(3.7)

**Proposition 3.2.** Consider the restriction of \( A_H \) on the level set curves \( \gamma \) satisfying \( A_H^{-1}(0) \cap \mathcal{L}(M; R_0). \)
(3.8)
If \( (\dot{\gamma} - X_H(t, \gamma))^{\pi} = 0, \quad \gamma(0) \in R_0 \) is a critical point thereof in the level set \( A_H^{-1}(0) \) \( \subset \mathcal{L}(M; R_0) \) of codimension 1.

**Proof.** Let \( \eta \) be a first variation of \( \gamma \) under the variational constraint (3.8). Then any such \( \eta = \eta^\pi + aR \) satisfies
\[ 0 = \delta A_H(\gamma)(\eta) \]
\[ = -\int_0^1 d\lambda((\dot{\gamma} - X_H(t, \gamma(t)))^\pi, \eta) dt - ((\lambda(\gamma(1)), \eta(1)) \]
\[ - \int_0^1 R_\lambda[H](\gamma)a(t) dt - a(1) \]
where we use the Legendrian boundary condition \( \gamma(0) \in R_0 \) applied to (3.3). In particular if \( (\dot{\gamma} - X_H(t, \gamma))^\pi = 0 \), then we obtain
\[ 0 = -(\lambda(\gamma(1)), \eta(1)) - \int_0^1 R_\lambda[H](\gamma)a(t) dt - a(1). \]
By substituting these into (3.2), we have obtained
\[ \delta A_h(\gamma) = 0 \]
inside the level set $A_H^{-1}(0)$. This finishes the proof. □

**Remark 3.3.**

1. In fact, we can decompose contact Hamilton’s equation $\dot{x} = X_H(t, x)$ into

$$
\dot{x} = X_H(t, x) \iff \begin{cases}
(\dot{x} - X_H(t, x))^\pi = 0 \\
\gamma^*(\lambda + H dt) = 0
\end{cases}
$$

with respect to the decomposition of the tangent bundle $TM = \xi \oplus \mathbb{R} \langle R_\lambda \rangle$. This explains somewhat mysterious appearance of the Reeb component $d(w^*\lambda \circ j) = 0$ for the contact instanton equation, the one whose asymptotic limiting equation is the Reeb component $\gamma^*(\lambda + H dt)$, contact Hamilton’s equation.

2. Since any contact Hamiltonian trajectory satisfies $\gamma^*(\lambda + H dt) = 0$, we have $A_H(\gamma) = 0$ for any contact Hamiltonian trajectory $\gamma$. Therefore Proposition 3.2 shows that any contact Hamiltonian trajectory is a critical point of the aforementioned constrained variational problem in Proposition 3.2.

3. For general contact manifolds, there does not seem to exist a 0-order constraint for $\gamma$ at $t = T$ that naturally provides a gradient structure for the contact instanton equation. In Section 8, we will show that there is such a boundary condition for the case of one-jet bundles.

We now examine the relationship between the critical points of the aforementioned constrained action functional and the contact Hamiltonian trajectories.

**Proposition 3.4.** Suppose a path $\gamma : [0, 1] \to M$ satisfies

$$
(\dot{\gamma} - X_H(t, \gamma(t)))^\pi = 0.
$$

Define the function $\rho : [0, 1] \to \mathbb{R}$ by $\rho(t) := -A_H(\gamma|_{[0,t]})$ and consider the Reeb-translated Hamiltonian

$$
\tilde{H}(t, x) := H(t, \phi_R^\rho(t)(x)).
$$

Then the Reeb-translated curve

$$
\tilde{\gamma}(t) := \phi_R^\rho(t)(\gamma(t))
$$

satisfies the Hamilton’s equation $\dot{x} = X_{\tilde{H}}(t, x)$ for $\tilde{H}$.

**Proof.** Since $\gamma$ satisfies (3.10), we can write $\dot{\gamma}(t) - X_H(t, \gamma(t)) = b(t)R_\lambda(\gamma(t))$, i.e.,

$$
\dot{\gamma}(t) = b(t)R_\lambda(\gamma(t)) + X_H(t, \gamma(t))
$$

for some function $b = b(t)$. In fact, we have

$$
b(t) = \lambda(\dot{\gamma}(t) - X_H(t, \gamma(t))) = \lambda(\dot{\gamma}(t)) + H(t, \gamma(t)).
$$

Now consider the flow of the time-dependent vector field $b(t)R_\lambda$ which is just a reparameterization of the Reeb flow

$$
t \mapsto \phi_R^\rho(t), \quad \rho(t) := \int_0^t b(u) \, du = -A_H(\gamma|_{[0,t]}).
$$

By definition, we have $b(t) = \rho'(t)$. We define

$$
\tilde{\gamma}(t) := \phi_R^\rho(t)(\gamma(t))
$$

inside the level set $A_H^{-1}(0)$. This finishes the proof. □
and compute its derivative
\[
\frac{d}{dt} \tilde{\gamma}(t) = -\rho'(t) R_\lambda(\tilde{\gamma}(t)) + d\psi_{R_\lambda}^{-\rho(t)} \tilde{\gamma}(t)
\]
\[
= -\rho'(t) R_\lambda(\tilde{\gamma}(t)) + d\psi_{R_\lambda}^{-\rho(t)} (\rho'(t) R_\lambda(\gamma(t)) + X_H(t, \gamma(t)))
\]
\[
= d\psi_{R_\lambda}^{-\rho(t)} (X_H(t, \gamma(t))) = (\psi_{R_\lambda}^{\rho(t)})^* X_H(t, \tilde{\gamma}(t))
\]
\[
= X_{H, \psi_{R_\lambda}^{\rho(t)}}(t, \tilde{\gamma}(t)).
\]
where we use the identity
\[
R_\lambda(\tilde{\gamma}(t)) = d\psi_{R_\lambda}^{\rho(t)} R_\lambda(\gamma(t))
\]
for the penultimate equality and (2.18) for the last equality, respectively. This finishes the proof. \qed

**Remark 3.5.** In particular, the asymptotic limit \( \gamma \) of any finite \( \pi \)-energy solution of (1.2) satisfies
\[
(\dot{\gamma} - X_H(t, \gamma(t)))^\pi = 0.
\]
It is shown in [Oh5] that the asymptotic limit indeed of the form
\[
t \mapsto \phi_H^t \circ (\phi_H^1)^{-1} \circ \phi_{R_\lambda}^t (\phi_H^1(p))
\]
forsome point \( p \in R_0 \).

4. **Contact instantons with Legendrian boundary condition**

In this section, consider the Hamiltonian-perturbed contact instanton equation (1.5) with the finite \( \pi \)-energy condition
\[
\int_{\Sigma} |(dw - X_H \otimes dt)^\pi|^2 < \infty
\]
on general contact manifolds \((M, \lambda)\) equipped with a \( \lambda \)-adapted CR-almost complex structure called a contact triad in [OW1, OW2, OW3].

To highlight the main points of the a priori estimates of perturbed contact instantons with Legendrian boundary condition, we will restrict to the case with \( H = -1 \) in the rest of the present paper postponing the discussion of the modifications needed to handle the Hamiltonian term to [OY] and other sequels. We start with a review of the contact triad connection introduced in [OW1].

**4.1. Review of the contact triad connection.** Assume \((M, \lambda, J)\) is a contact triad for the contact manifold \((M, \xi)\), and equip with it the contact triad metric \( g = g_\xi + \lambda \otimes \lambda \). In [OW1], the authors introduced the contact triad connection associated to every contact triad \((M, \lambda, J)\) with the contact triad metric and proved its existence and uniqueness.

**Theorem 4.1 (Contact Triad Connection [OW1]).** For every contact triad \((M, \lambda, J)\), there exists a unique affine connection \( \nabla \), called the contact triad connection, satisfying the following properties:

1. The connection \( \nabla \) is metric with respect to the contact triad metric, i.e., \( \nabla g = 0 \);
2. The torsion tensor \( T \) of \( \nabla \) satisfies \( T(R_\lambda, \cdot) = 0 \);
3. The covariant derivatives satisfy \( \nabla_{R_\lambda} R_\lambda = 0 \), and \( \nabla_Y R_\lambda \in \xi \) for any \( Y \in \xi \);
The projection \( \nabla^\pi := \pi|_\xi \nabla \) defines a Hermitian connection of the vector bundle \( \xi \to M \) with Hermitian structure \( (d\lambda|_\xi, J) \);

The \( \xi \)-projection of the torsion \( T \), denoted by \( T^\pi := \pi T \) satisfies the following property:

\[
T^\pi(Y, Y) = 0
\] (4.1)

for all \( Y \) tangent to \( \xi \);

For \( Y \in \xi \), we have the following

\[
\partial^\pi Y \rho := \frac{1}{2}(\nabla Y \rho - J\nabla J Y \rho) = 0.
\] (4.2)

From this theorem, we see that the contact triad connection \( \nabla \) canonically induces a Hermitian connection \( \nabla^\pi \) for the Hermitian vector bundle \( (\xi, J, g_{\xi}) \), and we call it the contact Hermitian connection. This connection will be used to study estimates for the \( \pi \)-energy in later sections.

Moreover, the following fundamental properties of the contact triad connection was proved in [OW1], which will be useful to perform tensorial calculations later.

**Corollary 4.2.** Let \( \nabla \) be the contact triad connection. Then

1. For any vector field \( Y \) on \( M \),
   \[
   \nabla Y \rho = \frac{1}{2}(\mathcal{L}_Y J J Y) Y;
   \] (4.2)

2. \( \lambda(T) = d\lambda \).

We refer readers to [OW1] for more discussion on the contact triad connection and its relation with other related canonical type connections.

### 4.2. \( \varepsilon \)-regularity and interior density estimate

Now we establish the elliptic a priori estimates for the boundary value problem (1.5) which extends the interior estimates proved in [OW2].

The following \( \varepsilon \)-regularity and interior density estimate was proved in [OW2, Corollary 5.2].

**Theorem 4.3.** There exist constants \( C, \varepsilon_0 \) and \( r_0 > 0 \), depending only on \( J \) and the Hermitian metric \( h \) on \( \Sigma \), such that for any \( C^1 \) contact instanton \( w : \Sigma \to M \) with

\[
E(r_0) := \frac{1}{2} \int_{D(r_0)} |dw|^2 \leq \varepsilon_0,
\]

and discs \( D(2r) \subset \text{Int } \Sigma \) with \( 0 < 2r \leq r_0 \), \( w \) satisfies

\[
\max_{\sigma \in (0, r)} \left( \sigma^2 \sup_{D(\sigma-\sigma)} e(w) \right) \leq CE(r)
\] (4.3)

for all \( 0 < r \leq r_0 \). In particular, letting \( \sigma = r/2 \), we obtain

\[
\sup_{D(r/2)} |dw|^2 \leq \frac{4CE(r)}{r^2}
\] (4.4)

for all \( r \leq r_0 \).

The proof of this theorem is a consequence of the following differential inequality proved therein

\[
\Delta e(w) \leq Ce(w)^2 + \|K\|_{L^\infty(\Sigma)} e(w),
\]
compact domains we have

\[C = 2\|\mathcal{L}_R J\|_{C^0(M)} + \|\nabla^\tau (\mathcal{L}_R J)\|_{C^0(M)} + \|\text{Ric}\|_{C^0(M)} + 1\]

which is a positive constant independent of \(w\). (See [OW2] Theorem 5.1.) Next we prove the local a priori estimates.

4.3. Local coercive \(W^{2,2}\) estimate. The following pointwise inequality is derived in [OW2].

**Lemma 4.4** (Equation (5.13), [OW2]). Let \(w\) be any contact instanton \(w : (\tilde{\Sigma}, j) \to (M; \lambda, J)\) i.e., any map satisfying the equation

\[\overline{\partial}^* w = 0, \quad d(w^* \lambda \circ j) = 0.\]  

Then we have

\[|\nabla (dw)|^2 \leq C_1 |dw|^4 - 4K|dw|^2 - 2|\Delta w|^2\]  

(4.5)

We recall \(|dw|^2 = |d^\tau w|^2 + |w^* \lambda|^2 = |\partial^\tau w|^2 + |w^* \lambda|^2\) for the contact instanton \(w\) which satisfies \(\overline{\partial}^* w = 0\).

We now establish the following a priori estimate

**Theorem 4.5.** Let \(w : \mathbb{R} \times [0, 1] \to M\) satisfy (4.5). Then for any relatively compact domains \(D_1\) and \(D_2\) in \(\tilde{\Sigma}\) such that \(\overline{\partial} \subset D_1 \subset D_2\), we have

\[\|dw\|_{W^{1,2}(D_i)} \leq C_1 \|dw\|_{L^2(D_2)}^4 + C_2 \|dw\|_{L^1(D_2)}^4 + C_3 \|dw\|_{L^1(\partial D_2)}^4\]

where \(C_1, C_2, C_3\) are some constants which depend only on \(D_1, D_2\) and \((M, \lambda, J)\) and \(C_3\) is a constant which also depends on \(R_\lambda\) with \(w(\partial D_2) \subset R_i\) as well.

**Proof.** The proof is similar to that of [OW2], Proposition 5.3 given in [OW2], Appendix C] and so we just indicate necessary modification needed to handle the Legendrian boundary condition in the estimates.

We have only to consider the case of a pair of semi-discs \(D_1, D_2 \subset \tilde{\Sigma}\) with \(\overline{D}_1 \subset D_2\) such that \(\partial D_2 \subset \partial D_1 \subset \partial \tilde{\Sigma}\). (The open disc cases of \(D_1 \subset D_2\) are already treated in [OW2], Appendix C.)

For the pair of given domains \(D_1\) and \(D_2\), we choose another domain \(D\) such that \(\overline{D}_1 \subset D \subset \overline{D}_2 \subset D_2\) and a smooth cut-off function \(\chi : D_2 \to \mathbb{R}\) such that \(\chi \geq 0\) and \(\chi \equiv 1\) on \(\overline{D}_1\), \(\chi \equiv 0\) on \(D_2 - D\). Multiplying (4.6) by \(\chi^2\) and integrating over \(D\), we get

\[
\int_{D_1} |\nabla (dw)|^2 \leq \int_D \chi^2 |\nabla (dw)|^2
\]

\[
\leq C_1 \int_D \chi^2 |dw|^4 - 4 \int_D K \chi^2 |dw|^2 - 2 \int_D \chi^2 \Delta e
\]

\[
\leq C_1 \int_{D_2} |dw|^4 + 4 \|K\|_{L^\infty(\Sigma)} \int_{D_2} |dw|^2 - 2 \int_D \chi^2 \Delta e
\]

where \(C_1\) is the same constant as the one appearing in (4.6).

We now deal with the last term \(\int_{D_2} \chi^2 \Delta e\). We rewrite

\[
\chi^2 \Delta e \, dA = (\chi^2 \Delta e) = \chi^2 \Delta e = -\chi^2 d \ast de
\]

\[
= -d(\chi^2 \ast de) + 2\chi d\chi \wedge (\ast de).
\]

By the same argument as in [OW2], p.677, we get the estimate

\[
\left| \int_D \chi d\chi \wedge (\ast de) \right| \leq \frac{1}{\epsilon} \int_D \chi^2 |\nabla (dw)|^2 \, dA + \epsilon \|\chi\|^2_{C^0(D)} \int_D |dw|^2 \, dA
\]
for any $\epsilon > 0$.

We now estimate the integral
\[
\int_D -d(\chi^2 \ast de) = \int_{\partial D} -\chi^2 \ast de
\]
by Stokes' formula. This is where the current estimate deviates from that of [OW2] p.677.

**Lemma 4.6.** $w$ satisfies the Neumann boundary condition, i.e., $\frac{\partial w}{\partial t} \perp TR_i$.

**Proof.** Since $R_i$ are Legendrian, we have
\[
\frac{\partial w}{\partial \tau} = (\frac{\partial w}{\partial \tau})^\pi.
\]
Since $\bar{\nabla}^\pi w = 0$, we have
\[
-J \left( \frac{\partial w}{\partial t} \right)^\pi = \frac{\partial w}{\partial \tau} \in TR_i.
\]
Therefore $\left( \frac{\partial w}{\partial t} \right)^\pi \in NR_i$. Since $R_\lambda \in NR_i$, this proves
\[
\frac{\partial w}{\partial t} = \left( \frac{\partial w}{\partial t} \right)^\pi + \lambda \left( \frac{\partial w}{\partial t} \right) R_\lambda
\]
is contained in $NR_i$. \qed

**Remark 4.7.** We recall that contact triad connection preserves the metric but may have nonzero torsion, i.e., is not the Levi-Civita connection of the triad metric. The definition of the second fundamental form of a submanifold $S \subset (M,g)$ for such a Riemannian connection is still a bilinear map $B : TS \times TS \to NS$ defined by the symmetric average
\[
B(X_1, X_2) = \frac{1}{2}((\nabla X_1 X_2)^\perp + (\nabla X_2 X_1)^\perp).
\]
(4.7)

Since
\[
\frac{\partial w}{\partial \tau} = \left( \frac{\partial w}{\partial \tau} \right)^\pi = -J \left( \frac{\partial w}{\partial t} \right)^\pi,
\]
we have
\[
e := \left| \frac{\partial w}{\partial \tau} \right|^2 + \left| \frac{\partial w}{\partial t} \right|^2 = 2 \left| \frac{\partial w}{\partial \tau} \right|^2 + \left| \lambda \left( \frac{\partial w}{\partial t} \right) \right|^2.
\]
We then compute
\[
* de|_{\partial D} = -\frac{de}{dt} = -4 \left\langle \nabla_t \frac{\partial w}{\partial \tau}, \frac{\partial w}{\partial \tau} \right\rangle - 2 \frac{\partial}{\partial t} \left( \lambda \left( \frac{\partial w}{\partial t} \right) \right) \cdot \lambda \left( \frac{\partial w}{\partial t} \right).
\]
(4.8)

Since $w$ satisfies Neumann boundary condition, the first term becomes
\[
-4 \left\langle B \left( \frac{\partial w}{\partial \tau}, \frac{\partial w}{\partial \tau} \right), \frac{\partial w}{\partial t} \right\rangle
\]
with the second fundamament form (4.7). For the second, we compute
\[
\frac{\partial}{\partial t} \left( \lambda \left( \frac{\partial w}{\partial t} \right) \right) = \nabla_t \lambda \left( \frac{\partial w}{\partial t} \right) + \lambda \left( \nabla_t \frac{\partial w}{\partial t} \right).
\]
We write
\[
\frac{\partial w}{\partial t} = \left( \frac{\partial w}{\partial t} \right)^\pi + \lambda \left( \frac{\partial w}{\partial t} \right) R_\lambda
\]
and then
\[ \nabla_t \frac{\partial w}{\partial t} = \nabla (\frac{\partial w}{\partial \tau})^\pi + \lambda \left( \frac{\partial w}{\partial \tau} \right) \nabla R \frac{\partial w}{\partial \tau}. \]

Using the torsion property and \( \nabla R X \in \xi \) for all \( X \) (see Theorem 4.1 (2) and (3) respectively), we compute
\[ \lambda \left( \nabla R \frac{\partial w}{\partial t} \right) = \lambda \left( \nabla \frac{\partial w}{\partial \tau} R \frac{\partial w}{\partial \tau} \right) = 0. \]

Therefore using these vanishing and the torsion properties of \( \nabla \) again, we derive
\[ \lambda \left( \nabla_t \frac{\partial w}{\partial t} \right) = \lambda \left( \nabla (\frac{\partial w}{\partial \tau})^\pi \right) = \lambda \left( \nabla J (\frac{\partial w}{\partial \tau})^\pi J \left( \frac{\partial w}{\partial \tau} \right) \right) = \lambda \left( J^2 \nabla (\frac{\partial w}{\partial \tau})^\pi \right) \]
on \( \partial D \). Here the penultimate equality follows from the following lemma

**Lemma 4.8.**
\[ \lambda \left( \nabla (\frac{\partial w}{\partial \tau})^\pi \right) = -\lambda \left( \nabla (\frac{\partial w}{\partial \tau})^\pi \right). \]

**Proof.** From the equation
\[ (\overline{\partial w})^\pi \left( \frac{\partial}{\partial \tau} \right) = 0 \]
we get \( \frac{\partial w}{\partial \tau} = J \frac{\partial w}{\partial \tau} \). Therefore
\[ \lambda \left( \nabla (\frac{\partial w}{\partial \tau})^\pi \right) = \lambda \left( J \left( \nabla (\frac{\partial w}{\partial \tau})^\pi \right) \right) = \lambda \left( J \nabla (\frac{\partial w}{\partial \tau})^\pi \right) = \lambda \left( J^2 \nabla (\frac{\partial w}{\partial \tau})^\pi \right) \]
where we repeatedly use \( \nabla Y J = 0 \) for any \( Y \in \xi \) and also use the torsion property (4.1) for the penultimate equality. This finishes the proof. \( \square \)

This proves
\[ \left| \lambda \left( \nabla_t \frac{\partial w}{\partial t} \right) \right| \leq \| \lambda \|_{C^0} \| B \|_{C^0} \left| \frac{\partial w}{\partial t} \right|^2. \]

On the other hand, we compute
\[ \left| \nabla_t \lambda \left( \frac{\partial w}{\partial t} \right) \right| \leq \| \lambda \|_{C^1} \left| \frac{\partial w}{\partial t} \right|^2. \]

By substituting these and (4.9) into (4.8), we obtain
Lemma 4.9.
\[
| \partial w | \leq 4 \left( \left\| B \left( \frac{\partial w}{\partial t}, \frac{\partial w}{\partial \tau} \right) \right\| \partial w \left( \frac{\partial w}{\partial \tau} \right) \right) + \left\| \lambda \right\|_{C^0} \left\| B \right\|_{C^0} \left\| \frac{\partial w}{\partial \tau} \right\| + \left\| \lambda \right\|_{C^1} \left\| \lambda \right\|_{C^0} \left\| \frac{\partial w}{\partial t} \right\|
\]
for \( C_3 := 4 \left\| B \right\|_{C^0} + \left\| \lambda \right\|_{C^0}^2 \left\| B \right\|_{C^0} + \left\| \lambda \right\|_{C^1} \left\| \lambda \right\|_{C^0} \) where \( B = B_i \) is the second fundamental form of \( R_i \).

Then we can sum all the estimates above and get
\[
\int_{D_3} | \nabla (\partial w) |^2 \leq \int_{D_3} \frac{2 \chi^2}{\epsilon} | \nabla (\partial w) |^2
\]
\[
+ \left( 4 \left\| K \right\|_{L^\infty (\Sigma)} + 2 \left\| d \chi \right\|_{C^0 (D)} \epsilon \right) \int_{D_2} | \partial w |^2
\]
\[
+ C_1 \int_{D_2} | \partial w |^4 + C_3 \int_{\partial D} | \partial w |^3
\]
We take \( \epsilon = 4 \). Then
\[
\int_{D_3} | \nabla (\partial w) |^2 \leq \int_{D_3} \chi^2 | \nabla (\partial w) |^2
\]
\[
\leq \left( 8 \left\| K \right\|_{L^\infty (\Sigma)} + 16 \left\| d \chi \right\|_{C^0 (D)}^2 \right) \int_{D_2} | \partial w |^2 + 2 C_1 \int_{D_2} | \partial w |^4 + C_3 \int_{\partial D} | \partial w |^3.
\]
By setting
\[
C_1(D_1, D_2) = 8 \left\| K \right\|_{L^\infty (\Sigma)} + 16 \left\| d \chi \right\|_{C^0 (D)}^2
\]
\[
C_3(D_1, D_3) = 4 \left\| B \right\|_{C^0} + \left\| \lambda \right\|_{C^0}^2 \left\| B \right\|_{C^0} + \left\| \lambda \right\|_{C^1} \left\| \lambda \right\|_{C^0}
\]
and \( C_2 = C_2(D_1, D_2) = 2 C_1 \) the constant given in [110], we have finished the proof. \( \square \)

5. \( C^{k, \alpha} \) Coercive Estimates for \( k \geq 1 \)

Once we have established \( W^{2,2} \) estimate, we could proceed with the \( W^{k+2,2} \) estimate \( k \geq 1 \) inductively as in [OW2 Section 5.2]. The effect of the Legendrian boundary condition on the higher derivative estimate is not quite straightforward, although it should be doable. Instead we take an easier path by expressing the following fundamental equation in the isothermal coordinates of \( (\Sigma, j) \).

**Theorem 5.1 (Theorem 4.2, [OW1]).** Let \( w \) satisfy \( \partial w = 0 \). Then
\[
d^{\pi^*} (\partial w) = -w^* \lambda \wedge j \wedge \left( \frac{1}{2} (L_{R, J}) \partial w \right).
\]

In an isothermal coordinates \( z = x + iy \), with the evaluation of \( (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \) into [51], the equation becomes
\[
\nabla_x^* \zeta + J \nabla_y^* \zeta + \frac{1}{2} \lambda \left( \frac{\partial w}{\partial y} \right) (L_{R, J}) \zeta - \frac{1}{2} \lambda \left( \frac{\partial w}{\partial x} \right) (L_{R, J}) J \zeta = 0.
\]
(See [OW1 Corollary 4.3].)

We note that by the Sobolev embedding, \( W^{2,2} \subset C^{0, \alpha} \) for \( 0 \leq \alpha < 1/2 \). Therefore we start from \( C^{0, \alpha} \) bound with \( 0 < \alpha < 1/2 \) and will inductively bootstrap.
it to $C^{k,\alpha}$ bounds for $k \geq 1$ by a Schauder-type estimates, instead of the $W^{k+2,2}$-estimates. The current approach also simplifies the higher derivative estimate given in [OW2, Section 5.2] which do the $W^{k+2,2}$-estimates instead.

We recall that $(\Sigma, j)$ is equipped with strip-like coordinates near a punctured neighborhood $U_i \setminus \{z_i\}$ for each $z_i$. We also equip a Kähler metric $h$ on $\Sigma$ that is strip-like, i.e., $h = dt^2 + d\tau^2$ thereon.

**Theorem 5.2.** Let $w$ be a contact instanton satisfying (1.5). Then for any pair of domains $D_1 \subset D_2 \subset \Sigma$ such that $D_1 \subset D_2$ and equipped with an isothermal coordinates $(x, y)$ such that

$$D_2 = \{(x, y) \mid |x|^2 + |y|^2 < \delta, y \geq 0\}$$

for some $\delta > 0$ and so $\partial D_2 \subset \{(x, y) \in D \mid y = 0\}$. Assume $D_1 \subset D_2$ is the semi-disc with radius $\delta/2$. We denote $\zeta = \pi \partial w/\partial x$, $\eta = \pi \partial w/\partial y$ as in [OW3, Subsection 11.5]. We note that since $w$ satisfies the Legendrian boundary condition, we have

$$\lambda \left( \frac{\partial w}{\partial x} \right) = 0 \tag{5.2}$$

on $\partial D_2$.

**Lemma 5.3 (Lemma 11.19 [OW3]).** Let $\zeta = \pi \partial w/\partial x$. Then

$$\ast(d(w^* \lambda)) = |\zeta|^2.$$

Combining Lemma 5.3 together with the equation $d(w^* \lambda \circ j) = 0$, we notice that $\alpha$ satisfies the equations

$$\begin{aligned}
\bar{\partial} \alpha &= \nu, \quad \nu = \frac{1}{2}|\zeta|^2 + \sqrt{-1} \cdot \frac{1}{2} \lambda \left( \frac{\partial w}{\partial x} \right) \\
\alpha(z) &\in \mathbb{R} \quad z \in \partial D_2 \tag{5.3}
\end{aligned}$$

thanks to (5.2), where $\bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right)$ is the standard Cauchy-Riemann operator for the standard complex structure $J_0 = \sqrt{-1}$.

Then we arrive at the following system of equations for the pair $(\zeta, \alpha)$

$$\begin{aligned}
\nabla R^x \zeta + J \nabla R^y \zeta + \frac{1}{2} \lambda \left( \frac{\partial w}{\partial y} \right) (\mathcal{L}_{R^x} J) \zeta - \frac{1}{2} \lambda \left( \frac{\partial w}{\partial x} \right) (\mathcal{L}_{R^y} J) J \zeta &= 0 \\
\zeta(z) &\in TR_i \quad z \in \partial D_2 \tag{5.4}
\end{aligned}$$

for some $i = 0, \ldots, k$, and

$$\begin{aligned}
\bar{\partial} \alpha &= \frac{1}{2}|\zeta|^2 \\
\alpha(z) &\in \mathbb{R} \quad z \in \partial D_2. \tag{5.5}
\end{aligned}$$

These two equations form a nonlinear elliptic system for $(\zeta, \alpha)$ which are coupled: $\alpha$ is fed into (5.4) through its coefficients and then $\zeta$ provides the input for the
equation (5.5) and then back and forth. Using this structure of coupling, we obtain the higher derivative estimates by alternating bootstrap arguments between $\zeta$ and $\alpha$ which is now in order.

It is obvious to see that (5.4) is a linear elliptic equation for $\zeta$ with $W^{1,2}$ coefficients by the above $W^{2,2}$-estimate for $w$ where we have

$$\frac{1}{2} \lambda \left( \frac{\partial w}{\partial y} \right) (\mathcal{L}_R, J) = \frac{1}{2} \left( \text{Re} \alpha + T \right) (\mathcal{L}_R, J)$$

$$\frac{1}{2} \lambda \left( \frac{\partial w}{\partial x} \right) (\mathcal{L}_R, J) = \frac{1}{2} \left( \text{Im} \alpha \right) (\mathcal{L}_R, J).$$

On the other hand, by the standard estimate for the Riemann-Hilbert problem with the real (or imaginary) boundary condition, it follows from (5.5) that $\alpha$ is in fact in $W^{2,2}$ and so is in $C^0,\delta$ with say $\delta = \frac{2}{5} < \frac{1}{2}$. By the standard Schauder estimate applied to (5.4), $\eta$ is indeed in $C^{1+\delta}$. By substituting $\eta$ back into (5.5), we get $\alpha$ in $C^{1+\delta}$. Repeating this alternating process between (5.4) and (5.5), we have established the $C^k$-estimate for all $k \geq 1$. This finishes the proof. \hfill \Box

6. Vanishing of asymptotic charge and subsequence convergence

In this section, we study the asymptotic behavior of contact instantons on the Riemann surface $(\dot{\Sigma}, j)$ associated with a metric $h$ with strip-like ends. To be precise, we assume there exists a compact set $K_\Sigma \subset \dot{\Sigma}$, such that $\dot{\Sigma} - \text{Int}(K_\Sigma)$ is a disjoint union of punctured semi-disks each of which is isometric to the half strip $[0, \infty) \times [0, 1]$ or $(-\infty, 0] \times [0, 1]$, where the choice of positive or negative strips depends on the choice of analytic coordinates at the punctures. We denote by $\{p^+_i\}_{i=1, \ldots, l^+}$ the positive punctures, and by $\{p^-_j\}_{j=1, \ldots, l^-}$ the negative punctures. Here $l = l^+ + l^-$. Denote by $\phi^+_i$ such strip-like coordinates. We first state our assumptions for the study of the behavior of boundary punctures. (The case of interior punctures is treated in [OW2, Section 6].)

Definition 6.1. Let $\dot{\Sigma}$ be a boundary-punctured Riemann surface of genus zero with punctures $\{p^+_i\}_{i=1, \ldots, l^+} \cup \{p^-_j\}_{j=1, \ldots, l^-}$ equipped with a metric $h$ with strip-like ends outside a compact subset $K_\Sigma$. Let $w: \dot{\Sigma} \to M$ be any smooth map with Legendrian boundary condition. We define the total $\pi$-harmonic energy $E^\pi(w)$ by

$$E^\pi(w) = E^\pi_{(\lambda, J; \Sigma, h)}(w) = \frac{1}{2} \int_{\Sigma} |d^\pi w|^2$$

where the norm is taken in terms of the given metric $h$ on $\dot{\Sigma}$ and the triad metric on $M$.

We put the following hypotheses in our asymptotic study of the finite energy contact instanton maps $w$ as in [OW2], except not requiring the charge vanishing condition $Q = 0$, which itself we will prove here under the hypothesis using the Legendrian boundary condition:

Hypothesis 6.2. Let $h$ be the metric on $\dot{\Sigma}$ given above. Assume $w: \dot{\Sigma} \to M$ satisfies the contact instanton equation (1.5) and

1. $E^\pi_{(\lambda, J; \Sigma, h)}(w) < \infty$ (finite $\pi$-energy);
2. $\|dw\|_{C^0(\dot{\Sigma})} < \infty$.
3. Image $w \subset K \subset M$ for some compact set $K$. 

Throughout this section, we work locally near one boundary puncture \( p \), i.e., on a punctured semi-disc \( D^3(p) \setminus \{ p \} \). By taking the associated conformal coordinates \( \phi^+ = (\tau, t) : D^3(p) \setminus \{ p \} \to [0, \infty) \times [0, 1] \) such that \( h = d\tau^2 + dt^2 \), we need only look at a map \( w \) defined on the half strip \([0, \infty) \times [0, 1] \to M \) without loss of generality.

The above finite \( \pi \)-energy and \( C^0 \) bound hypotheses imply

\[
\int_{[0, \infty) \times [0, 1]} |d^\pi w|^2 \, d\tau \, dt < \infty, \quad \|dw\|_{C^0([0, \infty) \times [0, 1])} < \infty \quad (6.2)
\]
in these coordinates.

Let \( w \) satisfy Hypothesis 6.2. We can associate two natural asymptotic invariants at each puncture defined as

\[
T := \frac{1}{2} \int_{[0, \infty) \times [0, 1]} |d^\pi w|^2 + \int_{\{0\} \times [0, 1]} (w|_{\{0\} \times [0, 1]})^* \lambda \quad (6.3)
\]

\[
Q := \int_{\{0\} \times [0, 1]} ((w|_{\{0\} \times [0, 1]})^* \lambda \circ j) \quad (6.4)
\]

(Here we only look at positive punctures. The case of negative punctures is similar.)

We will use the following equality which is derived in [OW1]. (See [OW1, Equation (3.1)].)

**Lemma 6.3.** Let \( h \) be a Kähler metric of \((\Sigma, j)\). Suppose \( w \) satisfies \( \partial^\pi w = 0 \). Then

\[
d(w^* \lambda) = \frac{1}{2} |d^\pi w|^2 dA \quad (6.5)
\]

where \( dA \) is the area form of \( h \).

**Remark 6.4.** In particular (6.3) holds for any contact instanton \( w \). By Stokes’ formula, we can express

\[
T = \frac{1}{2} \int_{[s, \infty) \times [0, 1]} |d^\pi w|^2 + \int_{\{s\} \times [0, 1]} (w|_{\{s\} \times [0, 1]})^* \lambda, \quad \text{for any } s \geq 0.
\]

Moreover, since \( w \) satisfies \( d(w^* \lambda \circ j) = 0 \) and the Legendrian boundary condition, it follows that the integral

\[
\int_{\{s\} \times [0, 1]} (w|_{\{s\} \times [0, 1]})^* \lambda \circ j, \quad \text{for any } s \geq 0
\]
does not depend on \( s \) whose common value is nothing but \( Q \).

We call \( T \) the *asymptotic contact action* and \( Q \) the *asymptotic contact charge* of the contact instanton \( w \) at the given puncture.

The proof of the following subsequence convergence result largely follows that of [OW2, Theorem 6.4]. Since we need to take care of the Legendrian boundary condition and also need to prove that the charge \( Q \) vanishes in the course of the proof, we duplicate the details of the proof therein in the first half of our proof and then explain in the second half how the charge vanishing occurs under the Legendrian boundary condition. One may say that the presence of Legendrian barrier prevents the instanton from spiraling.

**Theorem 6.5** (Subsequence Convergence). Let \( w : [0, \infty) \times [0, 1] \to M \) satisfy the contact instanton equations (1.5) and Hypothesis 6.2.
Then for any sequence \( s_k \to \infty \), there exists a subsequence, still denoted by \( s_k \), and a massless instanton \( w_\infty(\tau, t) \) (i.e., \( E^\tau(w_\infty) = 0 \)) on the cylinder \( \mathbb{R} \times [0, 1] \) that satisfies the following:

1. \( \overline{\mathcal{J}}^T w_\infty = 0 \) and 
   \[ \lim_{k \to \infty} w(s_k + \tau, t) = w_\infty(\tau, t) \]

   in the \( C^1(K \times [0, 1], M) \) sense for any \( l \), where \( K \subset [0, \infty) \) is an arbitrary compact set.

2. \( w_\infty \) has vanishing asymptotic charge \( Q = 0 \) and satisfies \( w_\infty(\tau, t) = \gamma(T t) \) for some Reeb chord \( \gamma \) is some Reeb chord joining \( R_0 \) and \( R_1 \) with period \( T \) at each puncture.

3. \( T > 0 \) either for single Legendrian i.e., with the case of one puncture or for a generic nonempty collection \( \{R_0, \ldots, R_k\} \) with \( k \geq 1 \) with repetition allowed so that \( R_i \cap R_j = \emptyset \) for \( R_i \neq R_j \).

**Proof.** For a given contact instanton \( w : [0, \infty) \times [0, 1] \to M \), we consider the family of maps \( w_s : [-s, \infty) \times [0, 1] \to M \) defined by \( w_s(\tau, t) = w(\tau + s, t) \) with Legendrian boundary condition

\[
w_s(\tau, 0) \in R, \quad w_s(\tau, 1) \in R'
\]

for \( R, R' \in \{R_0, \ldots, R_k\} \). For any compact set \( K \subset \mathbb{R} \), there exists some \( s_0 \) large enough such that \( K \subset [-s, \infty) \) for every \( s \geq s_0 \). For such \( s \geq s_0 \), we can also get an \([s_0, \infty)-family of maps by defining \( w_s^K := w_s|_{K \times [0, 1]} : K \times [0, 1] \to M \).

The asymptotic behavior of \( w \) at infinity can be understood by studying the limiting behavior of the sequence of maps \( \{w_s^K : K \times [0, 1] \to M\}_{s \in [s_0, \infty)} \), for each given compact set \( K \subset \mathbb{R} \).

First of all, it is easy to check that under Hypothesis \( 6.2 \) the family \( \{w_s^K : K \times [0, 1] \to M\}_{s \in [s_0, \infty)} \) satisfies the following

1. For every \( s \in [s_0, \infty) \),
   \[
   \left\{ \begin{array}{l}
   \overline{\mathcal{J}}^T w_s^K = 0, \\
   d((w_s^K)^* \alpha) = 0 \\
   w_s(\tau, 0) \in R, \\
   w_s(\tau, 1) \in R'.
   \end{array} \right.
   \]

2. \( \lim_{s \to \infty} \|d^T w_s^K\|_{L^2(K \times [0, 1])} = 0 \)

3. \( \|dw_s^K\|_{C^{0}(K \times [0, 1])} \leq \|dw\|_{C^{0}(\infty \times [0, 1])} < \infty. \)

From (1) and (3) together with the compactness of \( \text{image } w \subset M \) (which provides a uniform \( L^2(K \times [0, 1]) \) bound) and Theorem \( 5.2 \) we obtain

\[ \|w_s^K\|_{C^{1,\alpha}(K \times [0, 1])} \leq C_{K,1,\alpha} < \infty, \]

for some constant \( C_{K,1,\alpha} \) independent of \( s \). Then \( \{w^K_s : K \times [0, 1] \to M\}_{s \in [s_0, \infty)} \) is sequentially pre-compact. Therefore, for any sequence \( s_k \to \infty \), there exists a subsequence, still denoted by \( s_k \), and some limit \( w^K_\infty \in C^1(K \times [0, 1], M) \) (which may depend on the subsequence \( \{s_k\} \) ) such that

\[ w^K_{s_k} \to w^K_\infty, \quad \text{as } k \to \infty, \]

in the \( C^1(K \times [0, 1], M) \)-norm sense. Furthermore, combining this with (2) and \( (b.3) \), we get

\[ dw_s^K \to dw^K_\infty \quad \text{and} \quad dw^K_\infty = (w^K_\infty)^* \lambda \otimes R_\lambda, \]
and both \((w^K_\infty)^* \lambda\) and \((w^K_\infty)^* \lambda \circ j\) are harmonic one-forms. Closedness of \((w^K_\infty)^* \lambda\) follows from (6.5) and the convergence
\[
|d^\pi K_\infty| = \lim_{k \to \infty} |d^\pi K_s k| = 0.
\]
Closedness of \((w^K_\infty)^* \lambda \circ j\), which is equivalent to the coclosedness of \((w^K_\infty)^* \lambda\), follows from the hypothesis. The limit map \(w^K_\infty\) also satisfies Legendrian boundary conditions
\[
w^K_\infty(\tau, 0) \in R, \quad w^K_\infty(\tau, 1) \in R'
\]
since \(w^K_s \to w^K_\infty\) in \(C^1\)-topology and each \(w^K_s\) satisfies the same Legendrian boundary condition.

Note that these limiting maps \(w^K_\infty\) have a common extension \(w_\infty\) by a diagonal sequence argument where, one takes a sequence of compact sets \(K\) one including another and exhausting \(\mathbb{R}\). Then it follows that \(w_\infty\) is \(C^1\), satisfies
\[
\|dw_\infty\|_{C^0(\mathbb{R} \times [0, 1])} \leq \|dw\|_{C^0([0, \infty) \times [0, 1])} < \infty
\]
and \(d^\pi w_\infty = 0\). Therefore
\[
dw_\infty = w_\infty^* \lambda \otimes R_\lambda
\]
Then we derive from Theorem 5.2 that \(w_\infty\) is actually in \(C^\infty\). Also notice that both \(w_\infty^* \lambda\) and \(w_\infty^* \lambda \circ j\) are bounded harmonic one-forms on \(\mathbb{R} \times [0, 1]\).

**Lemma 6.6.** We have
\[
w_\infty^* \lambda = b dt, \quad w_\infty^* \lambda \circ j = b d\tau
\]
for some constant \(b\).

**Proof.** We have \(w_\infty^* \lambda = f \, dt + g \, d\tau\) for some bounded functions \(f, g\). Furthermore \(w_\infty^* \lambda\) is harmonic if and only if \((f, g)\) satisfies the Cauchy-Riemann equation
\[
\frac{\partial f}{\partial \tau} = \frac{\partial g}{\partial t}, \quad \frac{\partial f}{\partial t} = -\frac{\partial g}{\partial \tau}
\]
for the complex coordinate \(\tau + it\). This in particular implies \(\Delta f = 0\). On the other hand, the Legendrian boundary condition
\[
w_\infty(\tau, 0) \in R, \quad w(\tau, 1) \in R'
\]
implies
\[
\lambda \left( \frac{\partial w_\infty}{\partial \tau}(\tau, 0) \right) = 0 = \lambda \left( \frac{\partial w_\infty}{\partial \tau}(\tau, 1) \right).
\]
Therefore \(f\) is bounded and satisfies
\[
\begin{cases}
\Delta f = 0 & \text{on } R \times [0, 1] \\
f(\tau, 0) = f(\tau, 1) = 0 & \text{for all } \tau \in \mathbb{R}.
\end{cases}
\]
It follows from standard results on the harmonic functions on the strip \(\mathbb{R} \times [0, 1]\) satisfying the Dirichlet boundary condition that \(f = 0\). The equation (6.6) in turn implies \(\frac{\partial f}{\partial \tau} = 0 = \frac{\partial g}{\partial t}\) from which we derive \(g\) is a constant function. Therefore the forms \(w_\infty^* \lambda\) and \(w_\infty^* \lambda \circ j\) must be written in the form
\[
w_\infty^* \lambda = b \, dt, \quad w_\infty^* \lambda \circ j = b \, d\tau
\]
for some constants \(b\). This finishes the proof.
We now show that

\[ Q = 0, \quad T = b. \]

By taking an arbitrary point \( r \in K \), since \( w_\infty |_{\{r\} \times [0,1]} \) is the limit of some sequence \( w_{s_k} |_{\{r\} \times [0,1]} \) in the \( C^1 \) sense, we derive

\[
\begin{align*}
 b &= \int_{\{r\} \times [0,1]} (w_\infty |_{\{r\} \times [0,1]})^* \lambda = \int_{\{r\} \times [0,1]} \lim_{k \to \infty} (w_{s_k} |_{\{r\} \times [0,1]})^* \lambda \\
 &= \lim_{k \to \infty} \int_{\{r\} \times [0,1]} (w_{s_k} |_{\{r\} \times [0,1]})^* \lambda = \lim_{k \to \infty} \int_{\{r\} \times [0,1]} (w |_{\{r\} \times [0,1]})^* \lambda \\
 &= \lim_{k \to \infty} \left( T - \frac{1}{2} \int_{[r+s_k, \infty) \times [0,1]} |d^Tw|^2 \right) \\
 &= T - \lim_{k \to \infty} \frac{1}{2} \int_{[r+s_k, \infty) \times [0,1]} |d^Tw|^2 = T; \\
 0 &= \int_{\{r\} \times [0,1]} (w_\infty |_{\{r\} \times [0,1]})^* \lambda \circ j = \int_{\{r\} \times [0,1]} \lim_{k \to \infty} (w_{s_k} |_{\{r\} \times [0,1]})^* \lambda \circ j \\
 &= \lim_{k \to \infty} \int_{\{r\} \times [0,1]} (w_{s_k} |_{\{r\} \times [0,1]})^* \lambda \circ j \\
 &= \lim_{k \to \infty} \int_{\{r\} \times [0,1]} (w |_{\{r\} \times [0,1]})^* \lambda \circ j = Q.
\end{align*}
\]

Here in the derivation, we use Remark 6.4.

As in [Ab], [OW2], we conclude that the image of \( w_\infty \) is contained in a single leaf of the Reeb foliation by the connectedness of \( \mathbb{R} \times [0,1] \). Let \( \gamma : \mathbb{R} \to M \) be a parametrisation of the leaf so that \( \dot{\gamma} = R_\lambda(\gamma) \). Then we can write \( w_\infty(\tau, t) = \gamma(s(\tau, t)) \), where \( s : \mathbb{R} \times [0,1] \to \mathbb{R} \) and \( s = Tt + c_0 \) since \( ds = Tdt \), where \( c_0 \) is some constant.

From this we derive that, if \( T \neq 0 \), \( \gamma \) is a Reeb chord of period \( T \) joining \( R \) and \( R' \). Of course, if \( T \) also vanishes, \( w_\infty \) is a constant map and so corresponds to an intersection point \( R \cap R' \).

The latter case will not occur for a single Legendrian and for a generic \( k+1 \) tuple \( \{R_0, \ldots, R_k\} \) since \( R_i \)'s are pairwise disjoint for generic tuples by the dimensional reason, because \( R_\lambda \) has no zero and so cannot have a constant solution. This finishes the proof. \( \square \)

**Remark 6.7.** This charge vanishing of \( Q = 0 \) is a big advantage for the current open string case which is a big deviation from the closed string case studied in [OW2, OW3, Oh4]. This vanishing removes the only remaining difficulty, the so called the appearance of spiraling instantons along the Reeb core, towards the Fredholm theory and the compactification of the moduli space of contact instantons. One could say that the presence of the Legendrian obstacle blocks this spiraling phenomenon of the contact instantons. In sequels to the present paper as in [Oh8], we will develop a Fredholm theory and a compactification in the spirit of [Oh4] and apply this compactification for the construction of the Fukaya type category on contact manifolds whose objects are Legendrian submanifolds and morphisms and products will be constructed by counting appropriate instantons with prescribed asymptotic conditions and Legendrian boundary conditions.
From the previous theorem, we immediately get the following corollary as in [OW2, Section 8].

**Corollary 6.8.** Let \( w : \hat{\Sigma} \to M \) satisfy the contact instanton equation (1.5) and Hypothesis (6.2). Then on each strip-like end with strip-like coordinates \((\tau, t) \in [0, \infty) \times [0, 1]\) near a puncture

\[
\lim_{s \to \infty} \left| \pi \frac{\partial w}{\partial \tau}(s + \tau, t) \right| = 0, \quad \lim_{s \to \infty} \left| \pi \frac{\partial w}{\partial t}(s + \tau, t) \right| = 0
\]

\[
\lim_{s \to \infty} \lambda \left( \frac{\partial w}{\partial \tau}(s + \tau, t) \right) = 0, \quad \lim_{s \to \infty} \lambda \left( \frac{\partial w}{\partial t}(s + \tau, t) \right) = T
\]

and

\[
\lim_{s \to \infty} |\nabla^{\ell} dw(s + \tau, t)| = 0 \quad \text{for any} \quad l \geq 1.
\]

All the limits are uniform for \((\tau, t)\) in \(K \times [0, 1]\) with compact \(K \subset \mathbb{R}\).

### 7. Exponential \(C^\infty\) convergence

In this section, we improve the subsequence convergence to the exponential \(C^\infty\) convergence under the transversality hypothesis. Suppose that the tuple \(R_0, \ldots, R_k\) are transversal in the sense all pairwise Reeb chords are nondegenerate. In particular we assume that the tuples are pairwise disjoint.

We closely follow the arguments used in [OW3, Section 11] which treats the closed string case under the hypothesis of vanishing charge \(Q = 0\). Since in the current open string case, this vanishing is proved and so all the arguments used therein can be repeated verbatim after adapting them to the presence of the boundary condition in the arguments. So we will outline the main arguments of the proofs referring the detailed arguments to [OW3, Section 11] but indicate only the necessary changes needed.

#### 7.1. \(L^2\)-exponential decay of the Reeb component of \(dw\)

In this subsection, we prove the exponential decay of the Reeb component \(w^*\lambda\). We focus on a punctured neighborhood around a puncture \(z_i \in \partial \Sigma\) equipped with strip-like coordinates \((\tau, t) \in [0, \infty) \times [0, 1]\).

We again consider a complex-valued function

\[
\alpha(\tau, t) = \left( \lambda \left( \frac{\partial w}{\partial t} \right) - T \right) + \sqrt{-1} \left( \lambda \left( \frac{\partial w}{\partial \tau} \right) \right).
\]

By the Legendrian boundary condition, we know \(\alpha(\tau, i) \in \mathbb{R}\) for \(i = 0, 1\).

The following lemma was proved in the closed string case in [OW3]. For readers’ convenience, we provide some details by indicating how we adapt the argument with the presence of boundary condition.

**Lemma 7.1** (Compare with Lemma 11.20 [OW3]). Suppose the complex-valued functions \(\alpha\) and \(\nu\) defined on \([0, \infty) \times [0, 1]\) satisfy

\[
\begin{align*}
\overline{\partial} \alpha &= \nu, \\
\|\nu\|_{L^2([0,1])} + \|\nabla \nu\|_{L^2([0,1],\mathbb{R}^n)} &\leq C e^{-\delta \tau} \\
\lim_{\tau \to +\infty} \alpha(\tau, t) &= 0
\end{align*}
\]

then \(\|\alpha\|_{L^2(S^1)} \leq \overline{C} e^{-\delta \tau}\) for some constant \(\overline{C}\).
Proof. Notice that from previous section we have already established the $W^{1,2}$-exponential decay of $\nu = \frac{1}{2} |\zeta|^2$. Once this is established, the proof of this $L^2$-exponential decay result can be proved again by the standard three-interval method and so omitted. (See [OW3, Theorem 10.11] for a general abstract framework or the Appendix of the arXiv version of [OW2] for friendly details.) □

7.2. $C^0$ exponential convergence. Now the $C^0$-exponential convergence of $w(\tau, \cdot)$ to some Reeb chord as $\tau \to \infty$ can be proved from the $L^2$-exponential estimates presented in previous sections by the same argument as the proof of [OW3, Lemma 11.22] whose proof is omitted.

Proposition 7.2 (Compare with Lemma 11.22 [OW3]). Under Hypothesis 6.2, for any contact instanton $w$ satisfying the Legendrian boundary condition, there exist a unique Reeb orbit $z(\cdot) = \gamma(T \cdot) : [0, 1] \to M$ with period $T > 0$, such that

$$\|d(w(\tau, \cdot), z(\cdot))\|_{C^0([0,1])} \to 0,$$

as $\tau \to +\infty$, where $d$ denotes the distance on $M$ defined by the triad metric.

We just mention that the proof is based on the following lemma whose proof we refer readers to that of [OW3, Lemma 11.22].

Lemma 7.3. Let $t \in [0, 1]$ be given. Then for any given $\epsilon > 0$, there exists sufficiently large $\tau_1 > 0$ such that

$$d(w(\tau, t), w(\tau', t)) < \epsilon$$

for all $\tau, \tau' \geq \tau_1$.

Then the following $C^0$-exponential convergence is also proved.

Proposition 7.4 (Compare with Proposition 11.23 [OW3]). There exist some constants $C > 0$, $\delta > 0$ and $\tau_0$ large such that for any $\tau > \tau_0$,

$$\|d(w(\tau, \cdot), z(\cdot))\|_{C^0([0,1])} \leq C e^{-\delta \tau}$$

7.3. $C^\infty$-exponential decay of $dw - R_\lambda(w)\, dt$. So far, we have established the following:

- $W^{1,2}$-exponential decay of $w$,
- $C^0$-exponential convergence of $w(\tau, \cdot) \to z(\cdot)$ as $\tau \to \infty$ for some Reeb chord $z$ between two Legendrians $R, R'$.

Now we are ready to complete the proof of $C^\infty$-exponential convergence $w(\tau, \cdot) \to z$ by establishing the $C^\infty$-exponential decay of $dw - R_\lambda(w)\, dt$. The proof of the latter decay is now in order which will be carried out by the bootstrapping arguments applied to the system [125].

Combining the above three, we have obtained $L^2$-exponential estimates of the full derivative $dw$. By the bootstrapping argument using the local uniform a priori estimates on the cylindrical region, we obtain higher order $C^{k,\alpha}$-exponential decays of the term

$$\frac{\partial w}{\partial t} - TR_\lambda(z), \quad \frac{\partial w}{\partial \tau}$$

for all $k \geq 0$, where $w(\tau, \cdot)$ converges to $z$ as $\tau \to \infty$ in $C^0$ sense. This, combined with local elliptic $C^{k,\alpha}$-estimates given in Theorem [124], then completes proof of $C^\infty$-convergence of $w(\tau, \cdot) \to z$ as $\tau \to \infty$. 
8. The case with one-jet bundles and future works

In the case of one-jet bundles $J^1B$, it is shown in [OY] that the contact Hamilton's equation $\dot{x} = X_H(t,x)$ is the critical point equation of the reduced action functional

$$\tilde{A}_H(\gamma) = \int (\pi \circ \gamma)^*\theta - H(t,\gamma(t)) \, dt$$

defined on the path space

$$\Omega_0(T^*B, 0_{T^*B})$$

$$= \left\{ \gamma : [0,1] \to T^*B \mid \gamma(0) \in 0_{T^*B}, \gamma(1) \in Z, z(\gamma(t)) = - \int_{[0,t]} \gamma^* (\tilde{E}[H] - H) \right\}$$

Let $J$ be an $\omega_0$-compatible almost complex structure on $T^*B$ with $\omega_0 = -d\theta$. This induces a natural $\lambda$-adapted CR-almost complex structure on $J^1B$ in the sense of [OW2] by pulling it back to $\xi$ by the isomorphism $\xi \to T(T^*B)$ induced by the restriction to $\xi$ of the projection $d\pi : T(J^1B) \to T(T^*B)$. Since this class of CR-almost complex structures on $J^1B$ will play an important role for the construction of Legendrian spectral invariants in [OY], we assign a name to them.

**Definition 8.1** (Lifted CR-almost complex structures). We call a CR-almost complex structure on $J^1B$ a $T^*B$-lift if it is lifted to $\xi$ by $d\pi$ above from $\omega_0$-compatible almost complex structure $J$ on for $\omega_0 = -d\theta$ on $T^*B$. We denote the associated CR-almost complex structure by the same letter $J$ by an abuse of notation.

Let us first express such a CR-almost complex structure $J$ on $J^1B$ in terms of the coordinate $w = (u, f)$ where $u = \pi \circ w$ and $f = z \circ w$, we can express $dw = Du + df$ where $Du$ is $\xi$-valued one-form and $df$ is $\mathbb{R} (\frac{\partial}{\partial t})$-valued one form respectively. More specifically we have

$$Du = (dw)^2 : T\Sigma \to \xi$$

is the horizontal lifting of $du \in \Omega^1(u^*T(T^*B))$ to one in $\Omega^1(u^*\xi)$ which induced by the map

$$\frac{\partial}{\partial q_i} \to D \frac{\partial}{\partial q_i}, \quad \frac{\partial}{\partial p_i} \to \frac{\partial}{\partial p_i}.$$ 

Then we have the decomposition

$$Du = (Du)^{(1,0)} + (Du)^{(0,1)}$$

with the complex linear and the anti-complex linear part of $Du : (T\Sigma, j) \to (\xi, J)$. By definition, we have

$$\overline{D} w = (Du)^{(0,1)}, \quad \partial^w w = (Du)^{(1,0)} = D^Du.$$ 

Let $J$ be a $T^*B$-lifted CR almost complex structure on $J^1B$ and consider the case $\tilde{\Sigma} = \mathbb{R} \times [0,1]$. In terms of the splitting $w = (u, f)$, we can write \cite{15} into

$$\begin{cases}
\overline{D} u - (X_H^u \otimes dt)^{(0,1)} = 0, \\
d \left( e^{g_H(u)}(w^*\lambda + w^* H, dt) \circ j \right) = 0, \\
u(\tau, 0), \quad u(\tau, 1) \in 0_{T^*B}
\end{cases}$$

with $g_H(w)(z) = g(\psi^u)^{-1}(w(z))$ when the pair $(R_0, R_1) = (0,J^1B, 0,J^1B)$ is considered. We would like to emphasize that such a splitting $w = (u, f)$ does not exist on general contact manifolds which is a special feature of one-jet bundle.
For a $T^*B$-lifted CR-almost complex structure $J$, we can rewrite into the unperturbed equation in terms of $(v, g)$ similarly as in the cotangent bundle case \cite{Oh1, Oh2}:

$$
\begin{cases}
\partial v = 0, 
& d(g \circ j) = d(v^* \theta \circ j) \\
(v(\tau, 0), g(\tau, 0)) \in R, 
& v(\tau, 1) \in 0_{T^*B}.
\end{cases}
$$

(8.1)

Using this equation, we will develop a Floer theoretic construction of Legendrian spectral invariants and their applications in a sequel \cite{OY}.

**Appendix A. Generic transversality of contact Hamiltonian chords**

We start with the definition

**Definition A.1.** Let $X_t$ be a contact vector field and $H = -\lambda(X_t)$ be its contact Hamiltonian. We call a trajectory $\gamma$ thereof transversal to $\xi$ if $\lambda(\dot{\gamma}) \neq 0$.

**Remark A.2.**

1. Note that any Reeb vector field which corresponds to $H = 1$ is transversal to $\xi$ and the set of transversal trajectories for a given contact vector field are open.

2. However not every contact Hamiltonian trajectory is transversal to $\xi$. Note that for a general hypersurface, it carries a codimension one hypersurface along which the contact vector field is tangent to $\xi$ which is called the dividing set of the hypersurface.

Now let us given a pair of Legendrian submanifolds $(R_0, R_1)$. In the rest of the present section, we show that for generic choice of $H$ all Hamiltonian chords satisfying the Legendrian boundary condition for $(R_0, R_1)$ are transversal.

**Theorem A.3.** Let $R_0, R_1$ be a Legendrian pair. Then there exists a dense subset of time-dependent function $H = H(t, x) \in C^\infty(\mathbb{R} \times M)$ as follows:

- There is a discrete set of times $t_i \in \mathbb{R}$ satisfying $\psi_{H_i}^t \cap R_1 \neq \emptyset$.
- The associated Hamiltonian chords from $R_0$ to $R_1$ are nondegenerate and transversal to $\xi$.

We call any such $H$ a $(R_0, R_1)$-transversal Hamiltonian.

**Proof.** We write $H = C^\infty([0, 1] \times M)$. Define a vector bundle $\mathcal{X} \to \Omega = \Omega(M; R, R_1)$ by

$$
\mathcal{X} = \bigsqcup_{\gamma \in \Omega} \{ \gamma \} \times \Gamma(\gamma^* TM)
$$

and write $\mathcal{X}_\gamma := \Gamma(\gamma^* TM)$.

We first note that if $\dot{\gamma} = X_H(\gamma)$, then $\lambda(\dot{\gamma}(t)) = -H(t, \gamma(t))$ and hence $\dot{\gamma} \in \xi$ is equivalent to $H(t, \gamma(t)) \neq 0$. Motivated by this, we consider the map

$$
\Phi : \Omega_0(M; R_0) \times \mathcal{H} \times \mathbb{R}^2 \to \mathcal{X} \times \mathbb{R} \times M \times M
$$

defined by

$$
\Phi(\gamma, H, t_0, t_1) = (\dot{\gamma} - X_H(\gamma), H(t_0, \gamma(t_0)), \gamma(t_1), \gamma(0))
$$

(A.1)

**Lemma A.4.** The map $\Phi$ is transversal to the submanifold

$$
0_\mathcal{X} \times \{ 0 \} \times R_1 \times R_0.
$$
Proof. The scheme of the proof used to prove this kind of statement is standard. (See [Oh3, Section 3] for example.) We just mention the reason how the Hamiltonian chords can be made transversal. For this purpose, it is enough to show that there is no pair \((t_0, t_1)\) with \(0 \leq t_0 \leq t_1\) that satisfy
\[
\begin{cases}
\dot{\gamma}(t_0) - X_H(\gamma(t_0)) = 0, & H(t_0, \gamma(t_0)) = 0 \\
\gamma(0) \in R_0, & \gamma(t_1) \in R_1.
\end{cases}
\]
We have the linearization of \(\Phi\) at \((\gamma, H, t_0, t_1) \in \Phi^{-1}(0_X, 0, R_1, R_0)\)
\[D\Phi(\gamma, H, t_0, t_1)(\zeta, g, a_0, a_1) = (\nabla_t \zeta - DX_H(\gamma(t))(\zeta), g(t_0, \gamma(t_0)), \zeta(a_1), \zeta(0)).\]
From this explicit expression, it is easy to show that \(D\Phi(\gamma, H, t_0, t_1)\) is surjective after taking a suitable completion of relevant function spaces which is standard and so omit. (See [Oh3, Section 3] for such details in a similar context for symplectic Hamiltonian trajectories.)

Now we combine this generic transversality statement with a dimension counting argument as follows. Thanks to \(\dim R_0 + \dim R_1 = \dim M - 1\), the index of the projection map
\[\Pi : \Phi^{-1}(0_X \times \{0\} \times R_1) \to H\]
has index \(-1\) and hence \(\Pi^{-1}(H) = \emptyset\) for the regular values \(H\) of \(\Pi\), i.e., all chords are transversal to \(\xi\).

This finishes the proof of the theorem. □

References

[Ab] Abbas, C., Holomorphic open book decompositions, Duke Math. J. 158 (2011), 29–82.
[Ar] Arnold, V. I., Mathematical Methods of Classical Mechanics, Second edition, Translated from the Russian by K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics, 60. Springer–Verlag, New York, 1989. xvi+508 pp.
[AG] Arnold, V., Givental, A., Symplectic Geometry, Encyclopedia of Mathematical Sciences, vol IV, Springer, New York, 2001.
[B] Bhupal, M., A partial order on the group of contactomorphisms of \(\mathbb{R}^{2n+1}\) via generating functions, Turkish J. Math. 25 (2001), 125–235.
[BCT] Bravetti, A., Cruz, H., Tapias, D., Contact Hamiltonian mechanics, Ann. Physics 376 (2017), 17–39.
[K] Kirillov, A. A, Local Lie algebras, (Russian) Uspehi Mat. Nauk 31 (1976), no. 4(190), 57–76.
[LOTV] Le, Hong Van, Oh, Yong-Geun, Tortorella, Alfonso G., Vitagliano, Luca, Deformations of coisotropic submanifolds in Jacobi manifolds, J. Symplectic Geom. 16 (2018), no. 4, 1051–1116.
[MS] Müller, S., Spaeth, P., Topological contact dynamics I: symplectization and applications of the energy-capacity inequality, Adv. Geom. 15 (2015), no. 3, 349–380.
[Oh1] Oh, Y.-G., Symplectic topology as the geometry of action functional. I. Relative Floer theory on the cotangent bundle, J. Differential Geom. 46 (1997), no. 3, 499–577.
[Oh2] Oh, Y.-G., Gromov-Floer theory and disjunction energy of compact Lagrangian embeddings, Math. Res. Lett., 4 (1997), no. 6, 895–905.
[Oh3] Oh, Y.-G., Symplectic topology as the geometry of action functional. II. Pants product and cohomological invariants, Comm. Anal. Geom. 7 (1999), no. 1, 1–54.
[Oh4] Oh, Y.-G., Analysis of contact Cauchy-Riemann maps III: energy, bubbling and Fredholm theory, preprint, arXiv:2103.15376.
[Oh5] Oh, Y.-G., Geometry and analysis of contact instantons and entanglement of Legendrian links I, preprint, 2021.
[Oh6] Oh, Y.-G., Perturbed contact instantons with Legendrian boundary condition: a priori $C^{k,\alpha}$ estimates, asymptotic convergence and Fredholm theory, in preparation.
[OM] Oh, Y.-G.; Müller, S., The group of Hamiltonian homeomorphisms and $C^0$-symplectic topology, J. Symplectic Geom. 5 (2007), no. 2, 167–219.
[OW1] Oh, Y.-G., Wang, R., Canonical connection on contact manifolds, Real and Complex Submanifolds, Springer Proceedings in Mathematics & Statistics, vol. 106, 2014, 43–63, its full version in [arXiv:1212.4817].
[OW2] Oh, Y.-G., Wang, R., Analysis of contact Cauchy-Riemann maps I: a priori $C^k$ estimates and asymptotic convergence, Osaka J. Math. 55 (2018), no. 4, 647–679.
[OW3] Oh, Y.-G., Wang, R. Analysis of contact Cauchy-Riemann maps II: Canonical neighborhoods and exponential convergence for the Morse-Bott case, Nagoya Math. J. 231 (2018), 128–223.
[OY] Oh, Y.-G., Yu, S.-W., in preparation.
[Sa1] Sandon, S., Contact homology, capacity and non-squeezing in $\mathbb{R}^{2n} \times S^1$ via generating functions, Ann. Inst. Fourier 61 (2011), 145–185.
[Sa2] Sandon, S., On iterated translated points for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$, Internat. J. Math., 23 (2012), no 2, 1250042, 14 pp.
[She] Shelukhin, E., The Hofer norm of a contactomorphism, J. Symplectic Geom., 15 (2017), no 4, 1173–1208.
[T] Théret D., Thèse de doctorat, Université Denis Diderot (Paris 7), 1995.
[V] Viterbo, C., Symplectic topology as the geometry of generating functions, Math. Ann. 292 (1992), no. 4, 685–710.
[We] Weinstein, A., A graduate course in University of California, Berkeley, 1987.
[Wi] Witten, E., Supersymmetry and Morse theory, J. Differ. Geom. 17 (1982), 661–692.

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