The order of a homotopy invariant in the stable case

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Abstract. Let $X, Y$ be cell complexes, let $U$ be an Abelian group, and let $f: [X, Y] \to U$ be a homotopy invariant. By definition, the invariant $f$ has order at most $r$ if the characteristic function of the $r$th Cartesian power of the graph of a continuous map $a: X \to Y$ determines the value $f([a])$ $\mathbb{Z}$-linearly. It is proved that, in the stable case (that is, when $\dim X < 2n - 1$, and $Y$ is $(n - 1)$-connected for some natural number $n$), for a finite cell complex $X$ the order of the invariant $f$ is equal to its degree with respect to the Curtis filtration of the group $[X, Y]$.

Bibliography: 9 titles.

Keywords: invariants of finite order, stable homotopy, Curtis filtration.

§ 1. Introduction

The order of a homotopy invariant. Let $X$ and $Y$ be given spaces. Let $E_r$ be the group of all functions $(X \times Y)^r \to \mathbb{Z}$, $r \in \mathbb{N} = \{0, 1, \ldots \}$. Given a map $a \in C(X, Y)$, denote its graph by $\Gamma_a \subset X \times Y$, and the characteristic function of the set $\Gamma^r_a \subset (X \times Y)^r$ by $I_r(a) \in E_r$. Let $D_r \subset E_r$ be the subgroup generated by the functions $I_r(a), a \in C(X, Y)$.

Assume an Abelian group $U$ and a map $f: [X, Y] \to U$ are given. We define the order of the map $f$, $\text{ord } f = \inf \{r \in \mathbb{N} \cup \{\infty\} : \exists l: D_r \to U$ such that $f([a]) = l(I_r(a)), a \in C(X, Y)\}$. (If there is no such $r$, then we set $\text{ord } f = \inf \emptyset = \infty$.)

The main result. Let $X$ be a finite cell complex and $Y$ a cell complex such that $\dim X \leq m$, $Y$ is $(n - 1)$-connected, and $m < 2n - 1$ (the stable case). Then the set $[X, Y]$ is naturally an Abelian group. The Curtis filtration $B = (B_s)_{s=1}^{\infty}$ is defined in $[X, Y]$,

$[X, Y] = B_1 \supset B_2 \supset \cdots$

(see § 3). It is known (see [1], Theorem 1.4) that $B_s = 0$ for $s > 2^m - n$. The degree $\deg_B f \in \widehat{\mathbb{N}}$ of the map $f$ with respect to the filtration $B$ is defined (see below). If $f$ is a nonzero homomorphism, then $\deg_B f$ equals the largest $s$ for which $f|B_s \neq 0$.

Theorem 1.1. The identity $\text{ord } f = \deg_B f$ holds.

The inequality $\text{ord } f \leq \deg_B f$ is not hard to prove, the key results for this are the algebraic Lemma 5.1 and its corollary, Lemma 7.1. Sections 8–16 are required to prove the reverse inequality. It is known (see [2]) that if $X = S^m$ and $f$ is a homomorphism, then $\text{ord } f < \infty$.

AMS 2010 Mathematics Subject Classification. Primary 55Q05, 55P42.
The equivalence of the inequalities \( \text{ord} f \leq 1 \) and \( \deg_B f \leq 1 \) follows from the characterization of invariants of order one (see [3], Theorem II). For the case \( Y = S^1 \) this characterization is also illustrated by Theorem 8 from [4]. A similar argument can be used to show that the degree of the stable Hopf invariant \([S^{n+1}, S^n] \to \mathbb{Z}_2, n \geq 2\), is greater than one. The Brouwer degree \([S^n, S^n] \to \mathbb{Z}\) has order one. This can be proved by considering a topological Abelian group of homotopy type \( K(\mathbb{Z}, n) \) (see [5]) and applying the following lemma.

**Lemma 1.2.** Let \( M \) be a topological Abelian group, let \( c: Y \to M \) be a continuous map, and \( c_*: [X, Y] \to [X, M] \) the induced map. Then \( \text{ord} c_* \leq 1 \).

*Proof.* Let \( F \subset E_1 \) be the subgroup formed by those functions \( e: X \times Y \to \mathbb{Z} \) for which the set \( \{ y \in Y : e(x, y) \neq 0 \} \) is finite for any \( x \in X \). Obviously, \( D_1 \subset F \). Consider the homomorphism \( k: F \to M^X \) defined by

\[
k(e)(x) = \sum_{y \in Y} e(x, y)c(y), \quad x \in X, \quad e \in F.
\]

Given \( a \in C(X, Y) \), we have \( k(I_1(a)) = c \circ a \in C(X, M) \subset M^X \). Therefore, \( k(D_1) \subset C(X, M) \). Define the homomorphism \( l: D_1 \to [X, M] \) by \( l(e) = [k(e)] \). Then for \( a \in C(X, Y) \) we have \( l(I_1(a)) = [c \circ a] = c_*([a]) \).

The degree of maps between Abelian groups with respect to a filtration. Assume that Abelian groups \( T \) and \( U \) are given, along with a map \( f: T \to U \) and a filtration \( P = (P_s)_{s \geq 1} \) of the group \( T: T = P_1 \supset P_2 \supset \cdots \). Define the degree of the map \( f \) with respect to \( P \), \( \deg_P f \in \mathbb{N} \), as the infimum of those \( r \in \mathbb{N} \) for which any set of elements \( t_1, \ldots, t_k \in T, k \in \mathbb{N} \), with \( t_i \in P_{s_i} \) and \( s_1 + \cdots + s_k > r \), satisfies

\[
\sum_{e_1, \ldots, e_k = 0, 1} (-1)^{e_1 + \cdots + e_k} f(e_1 t_1 + \cdots + e_k t_k) = 0.
\]

**§ 2. Preliminary material**

**Simplicial theory.** We refer to [6] and [7] for information about (co)simplicial objects. A simplicial map between pointed simplicial sets is pointed if it preserves the basepoints. A simplicial Abelian group \( D \) is free if Abelian groups \( D_n \) are free for all \( n \in \mathbb{N} \). Given a simplicial set \( E \), we denote its \( m \)-skeleton by \( E_{(m)} \subset E \), for \( m \in \mathbb{N} \).

**Polyhedra.** A polyhedron \( L \) is a finite set of affine simplices in \( \mathbb{R}^\infty \) which satisfies the ‘axioms of a simplicial complex’ and whose vertices are linearly ordered, so that the order of vertices of a simplex induces the order of vertices for each of its faces. The body \( |L| \) of a polyhedron \( L \) is the union of its simplices. A polyhedral body is the body of a polyhedron.

**Morphisms of polyhedra.** A map \( f: K \to L \) between polyhedra is called a morphism if it maps vertices to vertices, the image of a simplex is the convex hull of the images of its vertices, and the nonstrict order of vertices is preserved under \( f \). A morphism \( f: K \to L \) induces a continuous map \( |f|: |K| \to |L| \).

**Generation.** A simplex \( y \in L \) generates the subpolyhedron \( \bar{y} \subset L \). A set \( T \subset L \) generates the subpolyhedron \( \bar{T} \subset L \).
Small sets. A set $T \subset L$ is small if there is a simplex $y \in L$ such that $\bar{y} \supset T$; the smallest simplex with this property is said to be spanned by $T$.

The distance $\rho_L$. Let $\rho_L(x, y) \in \mathbb{N}$ be the infimum of the lengths of chains of edges connecting $x, y \in L$. (The orientation of the edges is not taken into account; the length of a chain is the number of edges in it.) If $\rho_L(x, y) < a$ and $\rho_L(y, z) < b$ for $x, y, z \in L$, $a, b \in \mathbb{N}$, then $\rho_L(x, z) < a + b$.

Neighbourhoods $O_L$. Let

$$O_L(y, d) = \{z \in L: \rho_L(y, z) < d\}, \quad y \in L, \quad d \in \mathbb{N}.$$ 

Given a set $T \subset L$, we denote by $O_L(T, d)$ the union of sets $O_L(y, d)$, $y \in T$.

Sparseness $\varepsilon_L$. Set $\varepsilon_L(T) = \inf\{\rho_L(x, y): x, y \in T, x \neq y\} \in \mathbb{N}$ for $T \subset L$.

Subdivisions $\varepsilon_L$. Given a polyhedron $L$, we consider its barycentric subdivision $\delta L$ with the following order of its vertices: the greater the dimension of a simplex, the higher the order of its barycentre. Let $\varphi_L: \delta L \to L$ be the morphism sending the barycentre of a simplex to its highest order vertex, let $\delta' L$ be the barycentric subdivision of $L$ with the order reversed, and let $\varphi'_L: \delta' L \to L$ be the morphism sending the barycentre of a simplex to its lowest order vertex. Set $\Delta L = \delta' \delta L$, $\Phi_L = \varphi_L \circ \varphi'_L: \Delta L \to L$. The map $|\Phi_L|: |L| = |\Delta L| \to |L|$ is homotopic to the identity. The image of the star of any simplex of the polyhedron $\Delta L$ under the morphism $\Phi_L$ is small. Therefore, if $\rho_{\Delta L}(x, y) \leq 2d$ for $x, y \in \Delta L$ and $d \in \mathbb{N}$, then $\rho_L(\Phi_L(x), \Phi_L(y)) \leq d$.

The empty simplex. Given a polyhedron $L$, set $L^0 = L \cup \{\emptyset\}$. We assume that the empty simplex spans the empty subpolyhedron: $\emptyset = \emptyset$. For any $x, y \in L^0$ we have $x \cap y \in L^0$.

Completion. A polyhedron $L$ defines the simplicial set $\hat{L}$ (obtained by adding degenerate simplices). We have $L \subset \hat{L}_0 \cup \hat{L}_1 \cup \cdots$. The spaces $|L|$ and $|\hat{L}|$ are canonically homeomorphic. A morphism $f: K \to L$ of polyhedra induces a simplicial map $\hat{f}: \hat{K} \to \hat{L}$. The map $f \mapsto \hat{f}$ is bijective.

Sections. Given a polyhedron $L$ and a simplicial set $E$, denote by $E(L)$ the set of simplicial maps $v: \hat{L} \to E$ (called sections). A section $v \in E(L)$ induces a map $|v| \in C(|L|, |E|)$. For any subpolyhedron $K \subset L$ the restriction $v|_K \in E(K)$ is defined. For any morphism $f: K \to L$ of polyhedra we have the composition $v \circ f \in E(K)$. A simplicial map $t: D \to E$ (where $E$ is a simplicial set) induces a map $t_\#: D(L) \to E(L)$. Given a section $v \in G(L)$ (where $G$ is a simplicial group), set $\sigma(v) = \{y \in L: v|_y \neq 1\}$.

Quasisections. Given a set $T \subset L$ and a simplicial set $E$, define

$$E_T = \prod_{y \in T} E(\bar{y}).$$ 

Set $v|_T = (v|_{\bar{y}})_{y \in T} \in E_T$ for $v \in E(L)$. For an element $w \in E_L$ (called a quasisection) and a morphism $f: K \to L$ of polyhedra, define the composition $w \circ f \in E_K$.
by the following rule:\footnote{Given a map $t : X \to Y$ and $t(A) \subset B$, where $A \subset X$, $B \subset Y$, we denote by $t\big|_{A \to B}$ the map $A \to B$, $x \mapsto t(x)$.}
\[(w \circ f)_x = w_{f(x)} \circ (f|_{\overline{I \to J}}), \quad x \in K.\]

We have the map $f^\#: E_L \to E_K$, $f^\#(w) = w \circ f$. For a simplicial map $t : D \to E$ (where $D$ is a simplicial set) and a quasisection $v \in D_L$, the composition $t \circ v \in E_L$ is defined.

Free groups. For a set $E$ with the basepoint $\ast$, define the group $FE$ with generators $e, e \in E$, and one relation $e_\ast = 1$. The map $i : E \to FE$, $i(e) = e$, is referred to as canonical.

Lower central series and Abelianisation. Given a group $G$, denote its lower central series by $(\gamma_s G)_{s=1}^{\infty}$ (see \cite{1}, \S 1.3), and set $G^+ = G/\gamma_2 G$.

Free Abelian groups. A set $E$ defines the Abelian group $\langle E \rangle$ with basis $\langle e' \rangle_{e \in E}$. The map $j : E \to \langle E \rangle$, $j(e) = e'$, is called canonical. Let $\langle E \rangle_\triangle$ be the kernel of the homomorphism $\langle E \rangle \to \mathbb{Z}$, $e' \mapsto 1$. A map $t : D \to E$ (where $D$ is a set) induces a homomorphism $\langle t \rangle : \langle D \rangle \to \langle E \rangle$.

Assume that a polyhedron $L$, a simplicial set $E$, and an element $V \in \langle E(L) \rangle$ are given. Let $|V| \in \langle (C(|L|, E)) \rangle$ be the image of $V$ under the homomorphism induced by the map\footnote{Here $|?| : v \mapsto |v|$. In what follows we shall also use the notation $|?|_T : v \mapsto |v|_T$, etc.} $|?| : E(L) \to C(|L|, E)$. For a subpolyhedron $K \subset L$ the element $V|_K \in \langle E(K) \rangle$ can be defined similarly; for a set $T \subset L$ we have the element $V|_T \in \langle E_T \rangle$; for sets $X, Y$ and an element $A \in \langle C(X, Y) \rangle$ we have the element $[A] \in \langle [X, Y] \rangle$; and for a set $Z \subset X$ we have the element $A|_Z \in \langle C(Z, Y) \rangle$.

Given a simplicial group $G$ and an element $V \in \langle G(L) \rangle$,
\[V = \sum_{v \in G(L)} m_v v', \quad m_v \in \mathbb{Z},\]
we set
\[\Sigma(V) = \bigcup_{v \in G(L) : m_v \neq 0} \sigma(v).\]

Group rings. Given a group $G$, $\langle G \rangle$ is the group ring and $\langle G \rangle_\triangle$ is its (two-sided) ideal. For each $s \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ the ideal $\langle G \rangle^s_\triangle$ is generated additively by all elements of the form $\langle g_1' \rangle \cdots \langle g_s' \rangle$, $g_1, \ldots, g_s \in G$.

Simplicial application. Natural constructions can be applied fibrewise to simplicial objects. Given a pointed simplicial set $E$, we have the simplicial group $FE$ and the canonical simplicial map $i : E \to FE$. The map $i$ is a model for the canonical map from a pointed space to the loop space on its suspension (the Milnor model, see \cite{6}, \S V.6). Given a simplicial group $G$, we have the simplicial Abelian group $G^+$, the simplicial ring $\langle G \rangle$, the canonical simplicial map $j : G \to \langle G \rangle$ and the simplicial subgroups $\gamma_s G \subset G$ for $s \in \mathbb{N}_+$, and $\langle G \rangle_\triangle \subset \langle G \rangle$ for $s \in \mathbb{N}$.

Superposition maps. Let $L$ be a polyhedron, let $G$ be a simplicial group and let $j : G \to \langle G \rangle$ be the canonical map. The ring homomorphism $J : \langle G(L) \rangle \to \langle G \rangle(L)$, $J('v') = j \circ v$ is called the superposition map.
§ 3. The Curtis filtration in the stable case

Assume cell complexes $X$ and $Y$ are given such that $\dim X \leq m$, the space $Y$ is $(n-1)$-connected, and $m < 2n - 1$. There is a filtration $B = (B_s)_{s=1}^\infty$ in the Abelian group $[X, Y]$, $[X, Y] = B_1 \supset B_2 \supset \cdots$, called the Curtis filtration (it is a particular case of the filtration introduced in [1], §1.6). This filtration is constructed as follows. Take a simplicial set $E$ and consider the simplicial group $s$-connectivity $\gamma_s$.

Theorem, the canonical simplicial map $a$ basepoint in $E$ follows. Take a simplicial set $s$-case of the filtration introduced in [1], §1.6. This filtration is constructed as follows. Take a simplicial set $E$ and a homotopy equivalence $k: Y \to |E|$. We choose a basepoint in $E$ and consider the simplicial group $G = FE$. By the Freudenthal Theorem, the canonical simplicial map $i: E \to G$ is $(2n-1)$-connected. The map $h = |i| \circ k: Y \to |G|$ is also $(2n-1)$-connected. Let $j_s: \gamma_s G \to G$, $s \in \mathbb{N}_+$, be the simplicial inclusion homomorphisms. For each $s \in \mathbb{N}_+$ we have the following chain of groups and homomorphisms:

\[
[X, Y] \xrightarrow{h_s} [X, |G|] \xleftarrow{j_s \circ k} [X, |\gamma_s G|].
\]

Since $m < 2n - 1$, the map $h_s$ is an isomorphism. Set $B_s = h_s^{-1}(\text{im } |j_s|)$. (The filtration $B$ does not depend on the choice of $E$ and $k$.)

§ 4. A lemma on Lie rings

Let $U$ be the universal enveloping ring functor.

Lemma 4.1. Suppose that Lie rings $L$ and $M$ which are free as Abelian groups are given, together with an injective homomorphism $k: L \to M$. Then the homomorphism $Uk: UL \to UM$ is also injective.

This lemma follows easily from the Poincaré-Birkhoff-Witt Theorem (see [8], Part I, Ch. III, Theorem 4.3).

§ 5. A lemma on group rings

Assume groups $V$, $W$ and a homomorphism $t: V \to W$ are given. We have the ring homomorphism $(t): \langle V \rangle \to \langle W \rangle$. For $s \in \mathbb{N}$ let $I_s \subset \langle V \rangle$ be the subgroup generated by all elements of the form $(v_1 - 1) \cdots (v_k - 1)$ where $k \in \mathbb{N}$, $v_i \in t^{-1}(\gamma_{s_i} W)$, and $s_1 + \cdots + s_k \geq s$. It is easy to see that all the $I_s$ are ideals, $I_s \supset I_{s+1}$, and $I_r I_s \subset I_{r+s}$.

Lemma 5.1. Let $W$ be a product of finitely many free groups. Then

\[
(t)^{-1}(\langle W \rangle_A) = I_s
\]

for each $s \in \mathbb{N}$.

Proof. If $w \in \gamma_s W$, then $w - 1 \in \langle W \rangle_A$ (this is valid for any group $W$, see [9], Ch. III, Corollary 1.3). This implies that $(t)^{-1}(\langle W \rangle_A) \supset I_s$.

We introduce the graded ring $P$ with homogeneous components $P_s = I_s/I_{s+1}$ for $s \geq 0$, and the graded ring $Q$ with $Q_s = \langle W \rangle_A/\langle W \rangle_A^{s+1}$. Since $(t)(I_s) \subset \langle W \rangle_A$, the homomorphism $(t)$ induces a homomorphism of graded rings $l: P \to Q$. We shall show that $l$ is injective. Then by using induction on $s$ and the five lemma we shall show that the induced homomorphism $\langle V \rangle/I_s \to \langle W \rangle/\langle W \rangle_A^s$ is injective, and hence prove the required identity.
We define the graded Lie rings $L$ with $L_s = t^{-1}(\gamma_s W)/t^{-1}(\gamma_{s+1} W)$, and $M$ with $M_s = \gamma_s W/\gamma_{s+1} W$, $s \geq 1$; the product is induced by the group commutation (see [9], Ch. VIII, Theorem 6.2). The homomorphism $t$ induces a homomorphism $k: L \to M$ of graded Lie rings, which is clearly injective.

We have the commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{k} & M \\
\downarrow{f} & & \downarrow{g} \\
P & \xrightarrow{l} & Q
\end{array}
$$

where $f$ and $g$ are the representations with components

- $f_s: L_s \to P_s$, $f_s(v) = 'v' - 1$, $v \in t^{-1}(\gamma_s W)$,
- $g_s: M_s \to Q_s$, $g_s(w) = 'w' - 1$, $w \in \gamma_s W$.

By extending the representations $f$ and $g$ to homomorphisms of universal enveloping rings, we obtain the commutative diagram

$$
\begin{array}{ccc}
UL & \xrightarrow{Uk} & UM \\
\downarrow{\tilde{f}} & & \downarrow{\tilde{g}} \\
P & \xrightarrow{l} & Q
\end{array}
$$

It follows from [9], Ch. VIII, Theorem 6.2 that $\tilde{g}$ is an isomorphism. Then using Corollary I.4.6.2 from [8] implies that $M$ is free as an Abelian group. By Lemma 4.1, the homomorphism $Uk$ is injective. The ring $P$ is generated by the elements of the form $'v' - 1 \in P_s$, where $s \in \mathbb{N}_+$, $v \in t^{-1}(\gamma_s W)$. They are contained in the image of the representation $f$, and hence in the image of the homomorphism $\tilde{f}$, which is therefore surjective. Thus, $l$ is injective. (Furthermore, $\tilde{f}$ is an isomorphism.)

§ 6. Ideals in the group ring of a product of groups

Suppose we are given a finite collection of groups $(G_i)_{i \in I}$. Given $J \subset I$, set

$$G_J = \prod_{i \in J} G_i,$$

and let $p_J: G_I \to G_J$ be the projection homomorphism. We have the ring homomorphisms $\langle p_J \rangle: \langle G_I \rangle \to \langle G_J \rangle$.

**Lemma 6.1.** For each $s \in \mathbb{N}$ we have

$$\bigcap_{\# J < s} \ker\langle p_J \rangle \subset \langle G_I \rangle^s.$$

**Proof.** We have

$$\langle G_I \rangle = \bigotimes_{i \in I} \langle G_i \rangle.$$
Since \( \langle G_i \rangle = \langle G_i \rangle_\Delta \oplus \langle 1 \rangle \),

\[
\langle G_I \rangle = \bigoplus_{J \subset I} S(J), \quad S(J) = \bigotimes_{i \in I} T_i(J),
\]

where the subgroup \( T_i(J) \subset \langle G_i \rangle \) is \( \langle G_i \rangle_\Delta \) for \( i \in J \) and is \( \langle 1 \rangle \) otherwise. Clearly, \( \langle p_J \rangle|S(J') \) is a monomorphism for \( J' \subset J \) and is zero for \( J' \not\subset J \). This implies that

\[
\bigcap_{\#J < s} \ker \langle p_J \rangle = \bigoplus_{\#J \geq s} S(J).
\]

It remains to observe that \( S(J) \subset \langle G_I \rangle_{\#J} \).

§ 7. The functions \( \eta \) and \( \theta \)

Assume we are given a polyhedron \( L \) and a simplicial group \( G \). We have the homomorphism \( \tilde{\gamma}_L : G(L) \to G_L \). Given \( V \in \langle G(L) \rangle \), define

\[
\eta(V) = \sup \{ s \in \mathbb{N} : V\|_L \in \langle G_L \rangle^s \} \in \mathbb{N}.
\]

For each \( s \in \mathbb{N} \) we consider the subgroup \( I_s \subset \langle G(L) \rangle \) generated by all elements of the form \( ('v_1' - 1) \cdots ('v_k' - 1) \), where \( k \in \mathbb{N} \), \( v_l \in \langle \gamma_s, G \rangle(L) \subset G(L) \) and \( s_1 + \cdots + s_k > s \). (This is an ideal.)

**Lemma 7.1.** Assume that the group \( G_n \) is free for each \( n \in \mathbb{N} \). Then

\[
\{ V \in \langle G(L) \rangle : \eta(V) \geq s \} = I_s, \quad s \in \mathbb{N}.
\]

This follows from Lemma 5.1.

Given a simplicial set \( E \) and an element \( V \in \langle E(L) \rangle \), set

\[
\theta(V) = \inf \{ \#T : T \subset L, V\|_T \neq 0 \} \in \mathbb{N}.
\]

**Lemma 7.2.** If \( V \in \langle G(L) \rangle \) then \( \theta(V) \leq \eta(V) \).

This follows from Lemma 6.1.

§ 8. A product of affine functions

**Lemma 8.1.** Suppose that a group \( V \), a ring \( H \), and homomorphisms \( a_1, \ldots, a_r : V \to H \) (mapping into the additive group; \( r \in \mathbb{N} \)) are given. Consider the additive homomorphism \( Q : \langle V \rangle \to H \) given by

\[
Q('v') = \prod_{s=1}^{r} (1 + a_s(v)).
\]

Then \( Q|\langle V \rangle_{\Delta}^{r+1} = 0 \).

This follows from [9], Ch. V, Theorem 2.1.
§ 9. Strict and \( r \)-strict homomorphisms

Assume we are given groups \( V \) and \( W \). An additive homomorphism \( h: \langle V \rangle \to \langle W \rangle \) is called \textit{strict} if \( h(\langle V \rangle^s) \subset \langle W \rangle^s \) for all \( s \in \mathbb{N} \), and is called \textit{\( r \)-strict} if the previous inclusion holds for \( s \leq r, r \in \mathbb{N} \).

**Lemma 9.1.** Given a homomorphism \( t: V \to W \), the homomorphism 
\[
\langle t \rangle: \langle V \rangle \to \langle W \rangle
\]
is strict.

**Lemma 9.2.** Assume \( r \)-strict homomorphisms \( f, g: \langle V \rangle \to \langle W \rangle \), \( r \in \mathbb{N} \), are given. Then the homomorphism 
\[
h: \langle V \rangle \to \langle W \rangle, \quad h(\langle v \rangle) = f(\langle v \rangle)g(\langle v \rangle),
\]
is \( r \)-strict.

**Proof.** Take \( s \in \mathbb{N}_+ \), \( s \leq r \), and \( v_1, \ldots, v_s \in V \). Let \( x_t = \langle v_t \rangle - 1 \in \langle V \rangle^s \). We shall check that \( h(x_1 \cdots x_s) \in \langle W \rangle^s \). We have 
\[
(-1)^s h(x_1 \cdots x_s) = \sum_{e_1, \ldots, e_s = 0, 1} (-1)^{e_1 + \cdots + e_s} h(\langle v_1^{e_1} \cdots v_s^{e_s} \rangle)
\]
\[
= \sum_{e_1, \ldots, e_s = 0, 1} (-1)^{e_1 + \cdots + e_s} f(\langle v_1^{e_1} \cdots v_s^{e_s} \rangle) g(\langle v_1^{e_1} \cdots v_s^{e_s} \rangle)
\]
\[
= \sum_{e_1, \ldots, e_s = 0, 1} (-1)^{e_1 + \cdots + e_s} \left( \sum_{a_1, \ldots, a_s = 0, 1} e_1^{a_1} \cdots e_s^{a_s} f(\langle x_1^{a_1} \cdots x_s^{a_s} \rangle) \right)
\]
\[
\times \left( \sum_{b_1, \ldots, b_s = 0, 1} e_1^{b_1} \cdots e_s^{b_s} g(\langle x_1^{b_1} \cdots x_s^{b_s} \rangle) \right)
\]
\[
= \sum_{a_1, b_1, \ldots, a_s, b_s = 0, 1} \left( \sum_{e_1, \ldots, e_s = 0, 1} (-1)^{e_1 + \cdots + e_s} e_1^{a_1 + b_1} \cdots e_s^{a_s + b_s} \right) f(\langle x_1^{a_1} \cdots x_s^{a_s} \rangle)
\]
\[
\times g(\langle x_1^{b_1} \cdots x_s^{b_s} \rangle).
\]
Fix \( a_1, b_1, \ldots, a_s, b_s \). We need to check that the corresponding summand in the external sum belongs to \( \langle W \rangle^s \). Set \( a = a_1 + \cdots + a_s, b = b_1 + \cdots + b_s \). Since \( a, b \leq s \leq r \) and the homomorphisms \( f, g \) are \( r \)-strict, hence 
\[
f(\langle x_1^{a_1} \cdots x_s^{a_s} \rangle)g(\langle x_1^{b_1} \cdots x_s^{b_s} \rangle) \in \langle W \rangle^{a+b}.
\]
If \( a+b \geq s \) we are done. Otherwise there is \( t \) such that \( a_t = b_t = 0 \). Then the value \( e_1^{a_1+b_1} \cdots e_s^{a_s+b_s} \) does not depend on \( e_t \), and therefore the internal sum is zero.
§ 10. Group ring of a free group

Let $E$ be a pointed set, $G = FE$, and let $i: E \to G$ be the canonical map. For each $s \in \mathbb{N}$ we consider the pointed set $E^{\wedge s} = E \wedge \cdots \wedge E$ (where $E^{\wedge 0}$ is a 0-sphere) and the homomorphism $k_s: \langle E^{\wedge s} \rangle_{\Delta} \to \langle G \rangle_{\Delta}^s$ given by

$$k_s('({e_1, \ldots, e_s})' - '*') = \prod_{t=1}^{s} (e_t - 1),$$

where $* \in E^{\wedge s}$ is the basepoint. By [9], Ch. VIII, Theorem 6.2, the composite map $\langle E^{\wedge s} \rangle_{\Delta} \overset{k_s}{\longrightarrow} \langle G \rangle_{\Delta}^s \overset{\text{projection}}{\longrightarrow} \langle G \rangle_{\Delta}^s / \langle G \rangle_{\Delta}^{s+1}$ is an isomorphism. Therefore, $\langle G \rangle_{\Delta}^s = D^s \oplus \langle G \rangle_{\Delta}^{s+1}$, where $D^s \cong \langle E^{\wedge s} \rangle_{\Delta}$.

§ 11. Lifting a simplicial homomorphism

Lemma 11.1. Assume the diagram

$$\begin{array}{ccc}
Q & \xrightarrow{f} & P \\
\downarrow & & \uparrow \\
D & \xrightarrow{s} & P
\end{array}$$

of simplicial Abelian groups and homomorphisms is given. Assume further that the group $D$ is free and $m$-connected for some $m \in \mathbb{N}$, and the homomorphism $f$ is surjective. Then there exists a simplicial homomorphism $t: D \to Q$ such that $f \circ t|D_{(m)} = s|D_{(m)}$.

Proof. We set $\bigodot$ to be the Dold-Kan simplicial normalization functor (see [6], Ch. III, § 2). The complex $D^{\bigodot}$ is free. Hence, $D^{\bigodot} = C_0^{\bigodot} \oplus C_1^{\bigodot} \oplus \cdots$, where $C^n$ is a free complex with $C^n_i = 0$ for $i \neq n, n+1$ and with injective differential $\partial: C^n_{n+1} \to C^n_n$. The complex $D^{\bigodot}$ is $m$-connected. Therefore the differential $\partial: C^n_{n+1} \to C^n_n$ is an isomorphism for $n \leq m$. The morphism $f^{\bigodot}: Q^{\bigodot} \to P^{\bigodot}$ is surjective. Hence, for $n \leq m$ there is a morphism $g^n: C^n \to Q^{\bigodot}$ such that $f^{\bigodot} \circ g^n = s^{\bigodot} |C^n$. Define a morphism $h: \bigodot \to Q^{\bigodot}$ by setting $h|C^n$ to be equal to $g^n$ for $n \leq m$ and to zero for $n > m$. Clearly, $(f^{\bigodot} \circ h)_n = s^{\bigodot}_n$ for $n \leq m$. The Dold-Kan correspondence gives rise to a simplicial homomorphism $t: D \to Q$ with $t^{\bigodot} = h$. It has the required property.

§ 12. The function $\mu_L$

Let $L$ be a polyhedron. For $x \in L^{\circ}$ define

$$\mu_L(x) = 1 - \chi(\text{lk}_L x),$$

where $\chi$ is the Euler characteristic, lk denotes the link; we shall assume that $\text{lk}_L \varnothing = L$. 

Lemma 12.1. Given \( y, z \in L^o \), we have

\[
\sum_{x \in L^o : x \cap y = z} \mu_L(x) = \begin{cases} 
1 & \text{for } y = z, \\
0 & \text{for } y \neq z.
\end{cases}
\]

Proof. For any \( t \in L^o \) we have

\[
\sum_{x \in L^o : x \subset t, x \cap y = z} (-1)^{\dim x} = \begin{cases} 
(-1)^{\dim z} & \text{for } z \subset t \subset y, \\
0 & \text{otherwise}
\end{cases}
\]

(we assume that \( \dim \emptyset = -1 \)). For \( x \in L^o \) we have

\[
\chi(\text{lk}_L x) = \sum_{t \in L^o : x \subset t} (-1)^{\dim t - \dim x - 1}
\]

and therefore,

\[
\mu_L(x) = \sum_{t \in L^o : x \subset t} (-1)^{\dim t - \dim x}.
\]

We have

\[
\sum_{x \in L^o : x \cap y = z} \mu_L(x) = \sum_{x, t \in L^o : x \subset t, x \cap y = z} (-1)^{\dim t - \dim x} = \sum_{t \in L^o} (-1)^{\dim t} \sum_{x \in L^o : x \subset t, x \cap y = z} (-1)^{\dim x} = \sum_{t \in L^o : z \subset t \subset y} (-1)^{\dim t + \dim z} = \begin{cases} 
1 & \text{for } y = z, \\
0 & \text{for } y \neq z.
\end{cases}
\]

§ 13. The figure of a simplicial group

A model for the path fibration. We introduce the following cosimplicial simplicial pointed set\(^3\) \( B \): let \( B^n_m, m, n \in \mathbb{N} \), be the set of nonstrictly increasing partial maps \( b: [m] \rightarrow [n] \) (we have \( \text{dom } b \subset [m] \)) with the basepoint \( o^n_m \), \( \text{dom } o^n_m = \emptyset \). The structure maps are obvious (compare the definition of the cosimplicial simplicial set \( \Delta \) in [6], Ch. VIII, §1). Then \( B^n \) is a pointed simplicial set for each \( n \in \mathbb{N} \).

Assume we are given a simplicial group \( G \). We consider the simplicial group \( \tilde{G} \) (the figure of \( G \)) in which \( \tilde{G}_n, n \in \mathbb{N} \), is the group of pointed simplicial maps \( B^n \rightarrow G \), and the structure homomorphisms are induced by the cosimplicial structure.

Lemma 13.1. The space \( |\tilde{G}| \) is contractible.

Proof. Let \( I \) be the standard simplicial 1-simplex: \( I_n \) is the set of nonstrictly increasing maps \( s : [n] \rightarrow [1] \), for each \( n \in \mathbb{N} \). The set of maps \( I_n \times B^n_m \rightarrow B^n_m, (s, b) \mapsto b((s \circ b)^{-1}(1)), m, n \in \mathbb{N} \), induces a contracting homotopy \( I \times \tilde{G} \rightarrow \tilde{G} \).

Evaluation at \( i_n \in B^n_n, i_n = \text{id} : [n] \rightarrow [n] \), gives rise to a simplicial homomorphism \( p : \tilde{G} \rightarrow G \), which is called the projection.

\(^3\)That is, a cosimplicial object in the category of simplicial pointed sets (see [7], Ch. II, Definition 2.4 and [6], Ch. VII).
Lemma 13.2. Assume that $G_0 = 1$. Then the homomorphism $p$ is surjective.

Proof. Take an element $g \in G_n$, $n \in \mathbb{N}$. We need to find an element $\bar{g} \in \tilde{G}_n$ with $p_n(\bar{g}) = g$, that is, a pointed simplicial map $\bar{g} : B^n \to G$ with $\bar{g}_n(i_n) = g$.

Let $V \subset B^n$ be the simplicial subset generated by the elements $i_n$ and $l_n \in B^n$, dom$l_n = \{0\}$, $l_n(0) = 0$. This is a bouquet of the standard $n$-simplex and the 1-simplex. Consider the simplicial map $f : V \to G$ given by $f_n(i_n) = g$ and $f_1(l_n) = 1$. Since $V$ is contractible and $G$ is a Kan set (see [6], Ch. I, §3), the map $f$ extends to $B^n$, providing the required map $\bar{g}$.

Extension of sections. Let $L$ be a polyhedron. Take simplices $x, y \in L$ with dimension $r, s$ respectively. Let $i : [r] \to L$, $j : [s] \to L$ be increasing orderings of their vertices. We have a partial map $t = i^{-1} \circ j : [s] \to [r]$. Given a pointed simplicial map $\bar{g} : B^r \to G$, we introduce the pointed simplicial map $e_{xy}(\bar{g}) : B^s \to G$ for $b : [mi] \to [s]$, $m \in \mathbb{N}$, by setting $e_{xy}(\bar{g})_m(b) = \bar{g}_m(t \circ b)$ (the composition of partial maps is understood in the usual sense). This defines a homomorphism $e_{xy} : \tilde{G}_r \to \tilde{G}_s$.

Given $x \in L$ with dim $x = r$, we define the homomorphism $E_x : \tilde{G}(\bar{x}) \to \tilde{G}(L)$ by setting $E_x(v)_s(y) = e_{xy}(v_r(x))$ for $y \in L$, dim $y = s$. We extend this construction to $x \in L^\circ$ by setting $E_\emptyset(1) = 1$ (we have $\tilde{G}(\emptyset) = 1$).

Lemma 13.3. For $x \in L^\circ$, $v \in \tilde{G}(\bar{x})$, we have

(a) $E_x(v)|_{\bar{x}} = v$;

(b) $E_x(v)|_{\bar{y}} = E_{x \cap y}(v|_{\bar{x} \cap \bar{y}})|_{\bar{y}}$, $y \in L^\circ$;

(c) $\sigma(E_x(v)) \subset O_L(x, 1)$ for $x \neq \emptyset$.

The realization. Let $\tilde{J} : \langle \tilde{G}(L) \rangle \to \langle \tilde{G}(L) \rangle$, $\tilde{J}_x : \langle \tilde{G}(\bar{x}) \rangle \to \langle \tilde{G}(\bar{x}) \rangle$, $x \in L$, be the superposition maps. Clearly, $\tilde{J}_x$ is an isomorphism. Define an additive homomorphism $R : \langle \tilde{G}(L) \rangle \to \langle \tilde{G}(L) \rangle$, called the realization, by setting

$$R(w) = \sum_{x \in L} \mu_L(x)(E_x) \circ \tilde{J}_x^{-1}(w|_{\bar{x}}).$$

We have $R(\langle \tilde{G}(L) \rangle) \subset \langle \tilde{G}(L) \rangle$.

Lemma 13.4. For each $w \in \langle \tilde{G}(L) \rangle$, $\tilde{J}(R(w)) = w$.

Proof. Given $z \in L^\circ$, define the homomorphism $H_z : \langle \tilde{G}(\bar{z}) \rangle \to \langle \tilde{G}(L) \rangle$ by setting $H_z = \langle E_z \rangle \circ \tilde{J}_z^{-1}$, $z \neq \emptyset$, and $H_\emptyset = 0$. It follows from Lemma 13.3(b) that $H_z(u)|_{\bar{y}} = H_{x \cap y}(u|_{\bar{x} \cap \bar{y}})|_{\bar{y}}$ for any $x, y \in L^\circ$ and $u \in \langle \tilde{G}(L) \rangle$. For a given $y \in L$ we have

$$\tilde{J}(R(w))|_{\bar{y}} = \tilde{J}_y(R(w)|_{\bar{y}}) = \sum_{x \in L^\circ} \mu_L(x) \tilde{J}_y(H_x(w|_{\bar{x}})|_{\bar{y}})$$

$$= \sum_{x \in L^\circ} \mu_L(x) \tilde{J}_y(H_{x \cap y}(w|_{\bar{x} \cap \bar{y}})|_{\bar{y}})$$

$$= \sum_{z \in L^\circ} \left( \sum_{x \in L^\circ : x \cap y = z} \mu_L(x) \tilde{J}_y(H_z(w|_{\bar{y}})|_{\bar{y}}) \right)$$

$$= \tilde{J}_y(\langle E_y \rangle \tilde{J}_y^{-1}(w|_{\bar{y}}))$$

$$= \tilde{J}_y(\langle J_y \rangle(w|_{\bar{y}})) = w|_{\bar{y}}.$$
Lemma 13.5. For each \( w \in \langle \widetilde{G}(L) \rangle \), \( \Sigma(R(w)) \subset O_L(\sigma(w), 1) \).

This follows from Lemma 13.3(c).

Lemma 13.6. For each \( s \in \mathbb{N} \), \( R(\langle \widetilde{G}^s(\Delta)(L) \rangle) \subset \langle \widetilde{G}(L) \rangle^s_\Delta \).

Proof. For \( w \in \langle \widetilde{G}^s(\Delta)(L) \rangle \), \( x \in L \) we have
\[
\left. w \right|_{\widetilde{x}} \in \langle \widetilde{G}^s(\Delta)(\widetilde{x}) \rangle, \quad \widetilde{J}^{-1}_x(\left. w \right|_{\widetilde{x}}) \in \langle \widetilde{G}(\widetilde{x}) \rangle^s_\Delta
\]
and \( \langle E_x \rangle(\widetilde{J}^{-1}_x(\left. w \right|_{\widetilde{x}})) \in \langle \widetilde{G}(L) \rangle^s_\Delta \) by Lemma 9.1. Summing over \( x \in L \) we obtain \( R(w) \in \langle \widetilde{G}(L) \rangle^s_\Delta \).

§ 14. Partitions

Let \( L \) be a polyhedron and \( D \) a simplicial Abelian group. A collection of homomorphisms \( (h_z : D(\widetilde{z}) \to D(L))_{z \in L} \) is called a partition if for each \( w \in D(L) \) we have
\[
\sum_{z \in L} h_z(w|_{\widetilde{g}}) = w,
\]
and \( \sigma(h_z(w)) \subset O_L(z, 1) \) for all \( z \in L \).

Lemma 14.1. Assume that \( \dim L \leq m \) for some \( m \in \mathbb{N} \), and the group \( D \) is free and \( m \)-connected. Then there is a partition \( (h_z : D(\widetilde{z}) \to D(L))_{z \in L} \).

Proof. We use the Dold-Kan correspondence. There is a decomposition
\[
D = D^0 \oplus D^1 \oplus \cdots,
\]
where \( D^n \) is a simplicial Abelian group whose normalization \( C^n \) is concentrated in dimensions \( n, n + 1 \) and has injective differential \( \partial : C^n_{n+1} \to C^n_n \) (compare the proof of Lemma 11.1). It suffices to construct a partition \( (h^n_z : D^n(\widetilde{z}) \to D^n(L))_{z \in L} \) for each \( n \). If \( n \leq m \) then \( \partial : C^n_{n+1} \to C^n_n \) is an isomorphism since the group \( D \) is \( m \)-connected. It follows that sections on a polyhedron with values in \( D^n \) are the same as \( n \)-cochains on this polyhedron with coefficients in \( C^n_n \). For \( \dim z = n \) we define \( h^n_z \) to be the homomorphism extending a cochain by zero; in other cases we set \( h^n_z = 0 \). If \( n > m \) then \( D^n(L) = 0 \) (since \( \dim L \leq m \)) and we have a zero partition.

§ 15. Modifying a set of sections

Fix numbers \( b_1, \ldots, b_5, \ e \in \mathbb{N} \) each of which is sufficiently large in comparison with the ones preceding it: \( b_1 \geq 2, b_2 \geq b_1 + 2, b_3 \geq 2b_2, b_4 \geq 2b_1 + b_3, b_5 \geq 2b_2 + b_4, 2^{e-1} \geq 2b_5 + 1 \).

The morphism \( e : L \to K \). Assume we are given a polyhedron \( K \), \( \dim K \leq m, \ m \in \mathbb{N} \). We set \( L = \Delta^c K, e = \Phi_K \circ \cdots \circ \Phi_{\Delta^{c-1} K} : L \to K \). For each \( z \in L \) the set \( e(O_L(z, b_5)) \subset K \) is small (this follows from the properties of the operation \( \Delta \) and the inequality \( 2^{e-1} \geq 2b_5 + 1 \)).
The morphisms $e_z$. Take a simplex $z \in L$. Since $b_2 \leq b_5$, the set $e(O_L(z, b_2))$ is small. Let $x \in K$ be the simplex spanned by this set, and let $u \in K$ be the highest vertex of $x$. We shall construct a morphism $e_z: L \to K$ with the following properties:

1. $e_z(O_L(z, b_1)) = \{u\}$;
2. $e_z(O_L(z, b_2)) \subseteq \bar{x}$;
3. $e_z$ coincides with $e$ beyond $O_L(z, b_2)$.

Set $L_1 = \delta\Delta^{e-1}K$. We have $L = \delta' L_1$. Let $B_1 \subset L_1$ be the subpolyhedron spanned by the simplices whose centres (which are vertices of the polyhedron $L$) belong to the set $O_L(z, b_1 + 1)$. Set $B = \delta' B_1$. Then $B \subset L$ is a subpolyhedron. We have $O_L(z, b_1) \subset B$ and $O_L(B, 1) \subset O_L(z, b_2)$ since $b_2 \geq b_1 + 2$. The polyhedron $L$ does not contain edges pointing to or from $B$. Given a vertex $t \in L$, we define $e_z(t)$ to be equal to $u$ for $t \in B$ and to $e(t)$ for $t \notin B$. It is easy to see that the morphism $e_z$ is well-defined and has the required properties.

The morphisms $e_Z$. Take a set $Z \subset L$ with $e(Z) \geq b_3$. We define the morphism $e_Z: L \to K$ by the following conditions:

1. for $z \in Z$ the morphism $e_Z$ coincides with $e_z$ on the set $O_L(z, b_2)$;
2. $e_Z$ coincides with $e$ beyond $O_L(Z, b_2)$.

Since $b_3 \geq 2b_2$, this morphism is well defined.

The simplicial groups $G$ and $D$. Let $E$ be an $(n - 1)$-connected simplicial set with one vertex, $n \in \mathbb{N}$. Assume that $m \leq 2n - 1$. Let $G = FE$, $i: E \to G$ and $j: G \to \langle G \rangle$ be the canonical simplicial maps, and let $q: G \to G^+$ be the simplicial projection homomorphism. We shall use the decomposition $\langle G \rangle \cong \langle 1 \rangle \oplus G^+ \oplus D$ and simplicial homomorphisms determined by it. Let $d: \langle G \rangle \to \langle G \rangle$ be the simplicial homomorphism whose restriction to $\langle G \rangle_\Delta$ is the identity and which is zero on $\langle 1 \rangle$. We define the simplicial homomorphism $f: \langle G \rangle \to G^+$ by the condition $f \circ j = q$ and define the homomorphism $g: G^+ \to \langle G \rangle$ by the condition $g \circ q \circ i = d \circ j \circ i$. We have $f \circ g = id$. Let $D = \langle G \rangle_\Delta^* \subset \langle G \rangle$, and let $k: D \to \langle G \rangle$ be the inclusion. We define the simplicial homomorphism $l: \langle G \rangle \to D$ by the condition $k \circ l + g \circ f = d$. We have $l \circ k = id$. The homomorphisms constructed above are shown in the following diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{i} & G \\
\downarrow{j} & & \downarrow{q} \\
D & \xrightarrow{k} & \langle G \rangle \\
\downarrow{l} & & \downarrow{f} \\
& & G^+. \\
\end{array}
$$

The group $D$ is free. By Freudenthal’s Theorem, the map $i: E \to G$ is $(2n - 1)$-connected. Since $m \leq 2n - 1$, this map is also $m$-connected. Using the Dold-Thom Theorem we see that the simplicial homomorphism $\langle i \rangle: \langle E \rangle \to \langle G \rangle$ is $m$-connected. It is easy to see that $\langle \langle i \rangle, k \rangle: \langle E \rangle \oplus D \to \langle G \rangle$ is an isomorphism. Therefore, the group $D$ is $m$-connected.

For each $s \in \mathbb{N}$ we define the simplicial subgroup $D^{(s)} \subset D$ which is equal to $\langle G \rangle_\Delta^s$ for $s \geq 2$ and equal to $D$ for $s < 2$.

The decomposition of the group $D$. Assume we are given $r \in \mathbb{N}$, $r \geq 2$. Using the final formula in §10 we write out the decomposition $D = D^2 \oplus \cdots \oplus D^r$ in
which $\langle G \rangle^s_\Delta = D^s \oplus \cdots \oplus D^r$, $s = 2, \ldots, r$. (We have $D^s \cong \langle E^s \rangle_\Delta$ for $s < r$ and $D^r = \langle G \rangle^r_\Delta$.) Since the group $D$ is free and $m$-connected, the groups $D^s$ are also free and $m$-connected.

**The partition $h$.** By Lemma 14.1, for each $s = 2, \ldots, r$ there is a partition $(h_s^x : D^s(\bar{z}) \to D^s(L))_{z \in L}$. Taking their union we obtain a partition

$$(h_z : D(\bar{z}) \to D(L))_{z \in L}.$$ We have $h_z(D^s(\bar{z})) \subset D^s(L)$ for $s \in \mathbb{N}, s \leq r$.

**The simplicial homomorphism $X$.** Let $\tilde{G}$ be the figure of the group $G$, and let $p : \tilde{G} \to G$ be the projection. By Lemma 13.2, the homomorphism $p$ is surjective. Therefore, the simplicial homomorphism $\langle p \rangle : \langle G \rangle \to \langle G \rangle$ satisfies $\langle p \rangle(\langle \tilde{G} \rangle^s_\Delta) = \langle G \rangle^s_\Delta$, $s \in \mathbb{N}$. Applying Lemma 11.1 to each component $D^s$ of the decomposition of $D$, we obtain a simplicial homomorphism $X : D \to \langle \tilde{G} \rangle$ with the following properties:

1. the diagram

$$\begin{array}{ccc}
D(m) & \xrightarrow{X} & \langle \tilde{G} \rangle \\
\text{inclusion} & \downarrow & \langle G \rangle \\
\langle D(m) \rangle & \xrightarrow{\langle p \rangle} & \langle G \rangle
\end{array}$$

is commutative;

2. $X(D^s(\bar{z})) \subset \langle \tilde{G} \rangle^s_\Delta$ for $s \in \mathbb{N}, s \leq r$.

We have $\text{im} \, X \subset \langle \tilde{G} \rangle_\Delta$.

**The homomorphism $V$.** Let $J : \langle G(L) \rangle \to \langle G \rangle(L)$ be the restriction, and let

$$R : \langle \tilde{G} \rangle(L) \to \langle \tilde{G} \rangle(L)$$

be the realization. Consider the composition

$$V : D(L) \xrightarrow{X^\#} \langle \tilde{G} \rangle(L) \xrightarrow{R} \langle \tilde{G} \rangle(L) \xrightarrow{\langle p^\# \rangle} \langle G \rangle(L).$$

We have $\text{im} \, V \subset \langle G \rangle(L)_\Delta$.

**Lemma 15.1.** The diagram

$$\begin{array}{ccc}
\langle G(L) \rangle & \xrightarrow{V} & \langle G \rangle(L) \\
\downarrow & \nearrow & \downarrow \\
D(L) & \xrightarrow{k^\#} & \langle G \rangle(L)
\end{array}$$

is commutative.

**Proof.** Let $\tilde{J} : \langle \tilde{G} \rangle(L) \to \langle \tilde{G} \rangle(L)$ be the restriction. The diagram

$$\begin{array}{ccc}
\langle \tilde{G} \rangle(L) & \xrightarrow{R} & \langle \tilde{G} \rangle(L) \xrightarrow{\langle p^\# \rangle} \langle G \rangle(L) \\
\downarrow & \nearrow & \downarrow \\
D(L) & \xrightarrow{X^\#} & \langle \tilde{G} \rangle(L) \xrightarrow{\langle p^\# \rangle} \langle G \rangle(L)
\end{array}$$
is commutative (use Lemma 13.4 and the fact that im $X \subset \langle \tilde{G} \rangle_\Delta$). We have

$$J \circ V = \langle p \rangle_\# \circ X_\# = k_\#$$

by property (1) of the homeomorphism $X$.

**Lemma 15.2.** If $w \in D(L)$, then $\Sigma(V(w)) \subset O_L(\sigma(w), 1)$.

This follows from Lemma 13.5.

**Lemma 15.3.** The inclusion $V(D^{(s)}(L)) \subset \langle G(L) \rangle_\Delta^s$ holds for $s \in \mathbb{N}$, $s \leq r$.

This follows from property (2) of the homeomorphism $X$ and Lemmas 13.6 and 9.1.

The maps $P_z$, $P$. For $z \in L$ we define the map $P_z: G(K) \to \langle G(L) \rangle$ by

$$P_z(u) = (V \circ h_z)(l \circ j \circ u \circ e\big|_z).$$

We have $P_z(u) \in \langle G(L) \rangle_\Delta$ since im $V \subset \langle \tilde{G}(L) \rangle_\Delta$; furthermore, $\Sigma(P_z(u)) \subset O_L(z, b_1)$ (this follows from the definition of the partition, Lemma 15.2 and the inequality $b_1 \geq 2$).

Define the map $P: G(K) \to \langle G(L) \rangle$ by $P(u) = V(l \circ j \circ u \circ e)$. Then

$$\sum_{z \in L} P_z(u) = P(u).$$

The homomorphism $M$. Define the additive homomorphism $M: \langle G(K) \rangle \to \langle G(L) \rangle$ by

$$M(\langle u \rangle) = \sum_{Z \subset L: \varepsilon_L(Z) \geq b_3} (-1)^{\#Z}u \circ e_Z \prod_{z \in Z} P_z(u).$$

Here and below we assume that the order of factors in the product $\prod$ is determined by a certain fixed order on $L$ (however, it can be seen that the factors in all the formulae below commute).

**Lemma 15.4.** If $U \in \langle G(K) \rangle$, then $\theta(M(U)) \geq \min(\theta(U) + 1, \eta(U))$.

**Proof.** Suppose that $\theta(U) \geq s - 1$, $\eta(U) \geq s$ for some $s \in \mathbb{N}_+$; we need to show that $\theta(M(U)) \geq s$. We shall show that $M(U)\|_T = 0$ for any subset $T \subset L$ with $\#T < s$.

Consider the case $\varepsilon_L(T) \geq b_4$. Let $I = \{Z \subset L: \varepsilon_L(Z) \geq b_3\}$. We have

$$M(\langle u \rangle)\|_T = \sum_{Z \in I} (-1)^{\#Z}u \circ e_Z \|_T \prod_{z \in Z} P_z(u)\|_T, \quad u \in G(K).$$

The sets $O_L(y, b_1)$, $y \in T$ (which we refer to as balls) do not intersect; the distance $\rho_L$ between simplices from different balls is at least $b_3$ (since $b_4 \geq 2b_1 + b_3$). The distance between simplices in the same ball is less than $b_3$ since $b_3 \geq 2b_1$. Let $I_0$ be the collection of sets $Z \subset L$ which are contained in the union of the balls and contain at most one simplex in each ball. We shall show that the sum over $Z \in I$ can be replaced by the sum over $Z \in I_0$. We have $I_0 \subset I$. If $Z \in I \setminus I_0$, then there
is a simplex \( z \in Z \setminus O_L(T, b_1) \). Therefore, \( P_z(u)\|_T = 0 \) since \( P_z(u) \in \langle G(L) \rangle_{\Delta} \), and \( \Sigma(P_z(u)) \subset O_L(z, b_1) \), but \( O_L(z, b_1) \cap T = \emptyset \). It follows that the corresponding summand is zero.

Set

\[
I'_0 = \coprod_{S \subseteq T} W_S,
\]

where \( W_S \) is the set of maps \( w: S \to L \) such that \( w(y) \in O_L(y, b_1) \) for any \( y \in T \).

We have a bijection \( I'_0 \to I_0, (S, w) \mapsto w(S) \). Therefore,

\[
M('u')\|_T = \sum_{(S, w) \in I'_0} (-1)^{\#S} u \circ e_w(S)\|_T \prod_{y \in S} P_w(y)(u)\|_T.
\]

Let \( t_y: G(\bar{y}) \to G_T, y \in T, \) be the canonical monomorphism of a factor into the product. We shall show that for each \( (S, w) \in I'_0 \) we have the following identity:

\[
(u \circ e_w(S))\|_T = \prod_{y \in T \setminus S} t_y(u \circ e|_{\bar{y}}).
\]

If \( y \in S \) then \( y \in O_L(w(y), b_1) \) and the morphism \( e_w(S) \) maps the simplex \( y \) to a vertex of the polyhedron \( K \). Then \( u \circ e_w(S)|_{\bar{y}} = 1 \) since \( G_0 = 1 \). If \( y \in T \setminus S \), then \( y \notin O_L(w(S), b_2) \) (since \( b_4 > b_1 + b_2 \)) and \( e_w(S)|_{\bar{y}} = e|_{\bar{y}} \). This implies the required identity.

For any \( (S, w) \in I'_0 \) and \( y \in S \) we have \( P_w(y)(u)\|_T = \langle t_y(P_w(y)(u)|_{\bar{y}} \rangle. \) This follows from the fact that \( \Sigma(P_w(y)(u)) \subset O_L(w(y), b_1) \), and \( O_L(w(y), b_1) \cap T = \{ y \} \), since \( b_4 > 2b_1 \).

Therefore,

\[
M('u')\|_T = \sum_{(S, w) \in I'_0} (-1)^{\#S} \left( \prod_{y \in T \setminus S} t_y(u \circ e|_{\bar{y}}) \right) \left( \prod_{y \in S} \langle t_y(P_w(y)(u)|_{\bar{y}} \rangle \right)
\]

\[
= \sum_{S \subseteq T} (-1)^{\#S} \left( \prod_{y \in T \setminus S} t_y(u \circ e|_{\bar{y}}) \right) \left( \sum_{w \in W_S} \prod_{y \in S} \langle t_y(P_w(y)(u)|_{\bar{y}} \rangle \right)
\]

\[
= \sum_{S \subseteq T} (-1)^{\#S} \left( \prod_{y \in T \setminus S} \langle t_y(u \circ e|_{\bar{y}}) \rangle \right) \left( \prod_{y \in S} \sum_{z \in O_L(y, b_1)} \langle t_y(P_z(u)|_{\bar{y}}) \rangle \right)
\]

\[
= \prod_{y \in T} \langle t_y(u \circ e|_{\bar{y}}) \rangle - \sum_{z \in O_L(y, b_1)} P_z(u)|_{\bar{y}}.
\]

The range of summation of the last sum can be extended to \( z \in L \). Indeed, for \( z \in L \setminus O_L(y, b_1) \) we have \( P_z(u)|_{\bar{y}} = 0 \) since \( P_z(u) \in \langle G(L) \rangle_{\Delta} \), and \( \Sigma(P_z(u)) \subset O_L(z, b_1) \), but \( O_L(z, b_1) \cap \bar{y} = \emptyset \) for such \( z \). We obtain

\[
M('u')\|_T = \prod_{y \in T} \langle t_y(u \circ e|_{\bar{y}}) - P(u)|_{\bar{y}} \rangle.
\]
Let $J_y: \langle G(\tilde{y}) \rangle \to \langle G \rangle(\tilde{y})$, $y \in T$, be the superposition map. It is clearly an isomorphism. We have the following commutative diagram

\[
\begin{array}{ccc}
(G(L)) & \xrightarrow{\gamma} & \langle G(\tilde{y}) \rangle \\
\downarrow V & & \downarrow J_y \\
D(L) & \xrightarrow{k#} & \langle G \rangle(L) \xrightarrow{\gamma} \langle G \rangle(\tilde{y})
\end{array}
\]

(use Lemma 15.1). We have

\[
\begin{align*}
J_y(\langle u \circ e \rangle \big|_\tilde{y}) &= J_y(\langle u \circ e \rangle \big|_\tilde{y}) - V(l \circ j \circ u \circ e \big|_\tilde{y}) \\
&= j \circ u \circ e \big|_\tilde{y} - k \circ l \circ j \circ u \circ e \big|_\tilde{y} = 1 + g \circ f \circ j \circ u \circ e \big|_\tilde{y} = 1 + g \circ q \circ u \circ e \big|_\tilde{y}.
\end{align*}
\]

Define the homomorphism $a_y: G_K \to \langle G_T \rangle$ (mapping to the additive group) by $a_y(v) = (\langle t_y \circ J_y^{-1} \rangle ((g \circ q \circ v \circ e)_y))$. We have

\[
M(\langle u \rangle) = \prod_{y \in T} (1 + a_y(u \parallel K)).
\]

Since $\eta(U) > \#T$, Lemma 8.1 implies that $M(U) \parallel_T = 0$.

Now consider the case $\varepsilon_L(T) < b_4$. There are two different simplices $y_0, y_1 \in T$ with $\rho_L(y_0, y_1) < b_4$. For each $y \in T \setminus \{y_1\}$ we consider the simplex $x \in K$ spanned by the set $e(O_L(y, b_5))$. Let $S \subset K$ be the set of these simplices. We have $\#S < s - 1$. For each $y \in T$ there exists $y' \in T \setminus \{y_1\}$ such that $O_L(y, 2b_2) \subset O_L(y', b_5)$: we can take $y'$ to be equal to $y_0$ for $y = y_1$ and equal to $y$ for $y \neq y_1$ (we use the fact that $b_5 \geq 2b_2 + b_4$). Therefore, for each $y \in T$ there is $x \in S$ such that $e(O_L(y, 2b_2)) \subset \tilde{x}$. Set $e' = e \parallel_{O_L(T, b_1) \to \tilde{S}}$ (we use the fact that $b_1 \leq b_2$).

Take a subset $Z \subset L$, $\varepsilon_L(Z) \geq b_3$. We shall show that $e_Z(T) \subset \tilde{S}$. It suffices to check that $e_Z(y) \in \tilde{S}$ for $y \in T$. If $y \notin O_L(Z, b_2)$, then $e_Z(y) = e(y) \notin \tilde{S}$. Otherwise, $y \in O_L(z, b_2)$ for some $z \in Z$, and $e_Z(y) = e_z(y) \in \tilde{x}$, where $x \in K$ is the simplex spanned by $e(O_L(z, b_2))$. But then $e(O_L(z, b_2)) \subset e(O_L(y, 2b_2))$ implies that $e_Z(y) \in \tilde{S}$. We set $e \parallel_{T \to \tilde{S}}$.

We define the additive homomorphism $\widetilde{M}: \langle G(\tilde{S}) \rangle \to \langle G(\tilde{T}) \rangle$ by the formula

\[
\widetilde{M}(\langle \tilde{u} \rangle) = \sum_{Z \subset O_L(T, b_1): \varepsilon_L(Z) \geq b_3} (-1)^{\# Z} \tilde{u} \circ e_Z \prod_{z \in Z} (V \circ h_z)(l \circ j \circ \tilde{u} \circ e'_z) |_{\tilde{T}}.
\]

We shall show that the diagram

\[
\begin{array}{ccc}
\langle G(K) \rangle & \xrightarrow{\widetilde{M}} & \langle G(L) \rangle \\
\downarrow \gamma \big|_T & & \downarrow \gamma \big|_T \\
\langle G(\tilde{S}) \rangle & \xrightarrow{\widetilde{M}} & \langle G(\tilde{T}) \rangle
\end{array}
\]

is commutative. We have

\[
M(\langle u \rangle) = \sum_{Z \subset L: \varepsilon_L(Z) \geq b_3} (-1)^{\# Z} u \circ e_Z \prod_{z \in Z} P_\tau(u) |_{\tilde{T}}.
\]
The summands with $Z \not\subset O_L(T, b_1)$ are zero (indeed, $z \in Z \setminus O_L(T, b_1)$ implies that $P_z(u)|_{\overline{T}}=0$ since $P_z(u) \in \langle G(L) \rangle_\Delta$ and $\Sigma(P_z(u)) \subset O_L(z, b_1)$, but $O_L(z, b_1) \cap \overline{T} = \emptyset$). We obtain

$$M('u')|_{\overline{T}} = \sum_{Z \subset O_L(T, b_1): \varepsilon_L(Z) \supset b_3} (-1)^{\#Z} u \circ e_Z|_{\overline{T}} \prod_{z \in Z} (V \circ h_z)(l \circ j \circ u \circ e)|_{\overline{T}} = \overline{M('u')}_S.$$ 

The inequality $\theta(U) > \#S$ implies that $U|_S = 0$. Hence, $U|_S = 0$. We obtain $M(U)|_{\overline{T}} = \overline{M(U)}_S = 0$. Thus, $M(U)|_{\overline{T}} = 0$.

**Lemma 15.5.** If $U \in \langle G(K) \rangle$, then $\eta(M(U)) \geq \min(\eta(U), r)$.

**Proof.** Consider the additive homomorphism $N: \langle G_K \rangle \to \langle G_L \rangle$ given by

$$N('v') = \sum_{Z \subset L: \varepsilon_L(Z) \supset b_3} (-1)^{\#Z} v \circ e_Z \prod_{z \in Z} (V \circ h_z)(l \circ j \circ v \circ e)|_L.$$ 

The diagram

$$\begin{array}{ccc}
\langle G(K) \rangle & \xrightarrow{M} & \langle G(L) \rangle \\
\downarrow{?|_K} & & \downarrow{?|_L} \\
\langle G_K \rangle & \xrightarrow{N} & \langle G_L \rangle 
\end{array}$$

is commutative. It suffices to prove that the homomorphism $N$ is $r$-strict. For $z \in L$ we define the homomorphism $t_z: G_K \to G(\overline{z})$ by $t_z(v) = (v \circ e)_{\overline{z}}$, and define the additive homomorphism $B_z: \langle G(\overline{z}) \rangle \to \langle G(L) \rangle$ by $B_z('v') = (V \circ h_z)(l \circ j \circ v)$. We have the homomorphisms $e^\#: G_K \to G_L$ and $?|_L: G(L) \to G_L$. For $v \in G_K$,

$$N('v') = \sum_{Z \subset L: \varepsilon_L(Z) \supset b_3} (-1)^{\#Z} (e^\#_{\overline{z}}('v')) \prod_{z \in Z} (B_z \circ \{t_z\})('v')|_L.$$ 

By Lemmas 9.1 and 9.2, it suffices to show that the homomorphisms $B_z$ are $r$-strict. The homomorphism $B_z$ is given by the composition

$$\begin{array}{ccc}
\langle G(\overline{z}) \rangle & \xrightarrow{J_z} & \langle G(\overline{z}) \rangle \\
\downarrow{l^\#} & & \downarrow{h_z} \\
D(\overline{z}) & \xrightarrow{V} & \langle G(L) \rangle 
\end{array}$$

where $J_z$ is the restriction. For $s \in \mathbb{N}$ we have

$$J_z(\langle G(\overline{z}) \rangle^s) = \langle G(\overline{z}) \rangle^s, \quad l^\#(\langle G(\overline{z}) \rangle^s) = D^{(s)}(\overline{z})$$

(since the restriction of $l$ to $D$ is the identity), $h_z(D^{(s)}(\overline{z})) \subset D^{(s)}(L)$ for $s \leq r$ (by the property of the partition $h$), $V(D^{(s)}(L)) \subset \langle G(L) \rangle^s$ for $s \leq r$ by Lemma 15.3. Therefore, $B_z(\langle G(\overline{z}) \rangle^s) \subset \langle G(L) \rangle^s$ for $s \leq r$, as required.

Let $Q = |K| (= |L|)$.

**Lemma 15.6.** For $U \in \langle G(K) \rangle$ the identity $[|M(U)|] = [|U|]$ holds in the ring $\langle Q, |G| \rangle$. 
Lemma 17.1. For a morphism $e$ in $\langle t \rangle$, define the map $\Delta(t) \circ e : \langle t \rangle \circ e \circ \Delta(t) = \langle t \rangle$. The properties of $L$ are as follows:

1. The required pair $(L, V)$ is obtained by the $s$-fold application of the pair of operations $(\Delta^e, M)$ from §15 to the pair $(K, U)$. We take $r$ to be no smaller than $s$. The properties of $L$ and $V$ listed above follow from Lemmas 15.4–15.6.

§16. The main procedure

Let $K$ be a polyhedron with $\dim K \leq m, m \in \mathbb{N}$, let $E$ be an $(n-1)$-connected simplicial set with one vertex, $n \in \mathbb{N}$, and let $m \leq 2n-1$. Set $Q = |K|, G = FE$.

Lemma 16.1. Given an element $U \in \langle G(K) \rangle$ with $\eta(U) \geq s, s \in \mathbb{N}$, there is a polyhedron $L$ with body $Q$ and an element $V \in \langle G(L) \rangle$ with $\theta(V) \geq s$ and $[V] = [U]$ in $\langle [Q], [G] \rangle$.

Proof. The required pair $(L, V)$ is obtained by the $s$-fold application of the pair of operations $(\Delta^e, M)$ from §15 to the pair $(K, U)$. We take $r$ to be no smaller than $s$. The properties of $L$ and $V$ listed above follow from Lemmas 15.4–15.6.

§17. The function $\theta$: the topological version

Assume two spaces $X, Y$ are given. We set

$$\theta(A) = \inf \{ \#V : V \subset X, V \text{ is finite, and } A|_V \neq 0 \} \in \widehat{\mathbb{N}}$$

for $A \in \langle C(X, Y) \rangle$.

Assume we are also given two more spaces $X', Y'$ and continuous maps

$$g : X' \to X, \quad h : Y \to Y'.$$

Define the map $t : C(X, Y) \to C(X', Y')$ by $t(a) = h \circ a \circ g$. We have the homomorphism $\langle t \rangle : \langle C(X, Y) \rangle \to \langle C(X', Y') \rangle$.

Lemma 17.1. For $A \in \langle C(X, Y) \rangle$, the inequality $\theta(\langle t \rangle(A)) \geq \theta(A)$ holds.

Proof. Take a finite $V' \subset X', \#V' < \theta(A)$; we need to show that $\langle t \rangle(A)|_{V'} = 0$. Let $V = g(V') \subset X$. We have $\#V < \theta(A)$. Hence, $A|_V = 0$. Let $\tilde{g} = g|_{V \to V'}$. Consider the map $\tilde{t} : C(V, Y) \to C(V', Y')$ given by $\tilde{t}(a) = h \circ a \circ \tilde{g}$. The diagram

$$\begin{array}{ccc}
C(X, Y) & \xrightarrow{t} & C(X', Y') \\
\downarrow^{\gamma|_V} & & \downarrow^{\gamma|_{V'}} \\
C(V, Y) & \xrightarrow{\tilde{t}} & C(V', Y')
\end{array}$$

is commutative. We have $\langle t \rangle(A)|_{V'} = (\tilde{t})(A)|_{V'} = 0$.

Characterization of the order. Assume that an Abelian group $U$ and a map $f : [X, Y] \to U$ are given. Define the homomorphism $\overline{f} : \langle [X, Y] \rangle \to U$ by $\overline{f}(w) = f(w)$.
Lemma 17.2. The condition \( \text{ord} f \leq r \) for some \( r \in \mathbb{N} \) is equivalent to the identity \( \mathcal{I}([A]) = 0 \) for each \( A \in \langle C(X,Y) \rangle \) with \( \theta(A) > r \).

Proof. Suppose that \( X \neq \emptyset \) (otherwise the statement is trivial). Let \( E_r, I_r \) and \( D_r \) be the same as in the first paragraph of §1. Consider the homomorphism \( h: \langle C(X,Y) \rangle \to D_r \) given by \( h(a') = I_r(a) \). It is surjective. We shall show that for \( A \in \langle C(X,Y) \rangle \) the conditions \( h(A) = 0 \) and \( \theta(A) > r \) are equivalent.

Let \( \theta(A) > r \). Take \( z = (x_1, y_1, \ldots, x_r, y_r) \in (X \times Y)^r \); we need to show that \( h(A)(z) = 0 \). Let \( V = \{x_1, \ldots, x_r\} \subseteq X \). The inequality \#\( V < \theta(A) \) implies that \( A|_V = 0 \). Consider the homomorphism \( k: \langle C(V,Y) \rangle \to \mathbb{Z} \) given on the generators by the rule\(^4 \) \( k(\bar{a}') = [\bar{a}(x_s) = y_s, s = 1, \ldots, r] \). The diagram

\[
\begin{array}{ccc}
\langle C(X,Y) \rangle & \xrightarrow{h} & D_r \\
\downarrow & & \downarrow \\
\langle C(V,Y) \rangle & \xrightarrow{k} & \mathbb{Z}
\end{array}
\]

is commutative. We have \( h(A)(z) = k(A|_V) = 0 \).

Suppose that \( h(A) = 0 \). We shall show that \( \theta(A) > r \). Take \( V \subseteq X \) with \#\( V < r \); we need to check that \( A|_V = 0 \). We may assume that \( V \neq \emptyset \) or \( r = 0 \) (otherwise we add a point to \( V \), we can do this as \( X \neq \emptyset \)). Then \( V = \{x_1, \ldots, x_r\} \) for some \( x_1, \ldots, x_r \in X \). Let \( i: \langle C(V,Y) \rangle \to \mathbb{Z}^{C(V,Y)} \) be the standard monomorphism: \( i(\bar{a}') = [\bar{a} = \bar{b}], \bar{a}, \bar{b} \in C(V,Y) \). Consider the homomorphism \( q: D_r \to \mathbb{Z}^{C(V,Y)} \) given by \( q(e)(\bar{b}) = e(x_1, \bar{b}(x_1), \ldots, x_r, \bar{b}(x_r)) \) for \( \bar{b} \in C(V,Y), e \in D_r \). The diagram

\[
\begin{array}{ccc}
\langle C(X,Y) \rangle & \xrightarrow{h} & D_r \\
\downarrow & & \downarrow q \\
\langle C(V,Y) \rangle & \xrightarrow{i} & \mathbb{Z}^{C(V,Y)}
\end{array}
\]

is commutative. We have \( i(A|_V) = q(h(A)) = 0 \), which implies that \( A|_V = 0 \).

The equivalence is therefore verified. Consider the homomorphism

\( \widehat{f}: \langle C(X,Y) \rangle \to U \)

given by \( \widehat{f}(A) = \mathcal{I}([A]) \). The existence of a homomorphism \( l: D_r \to U \) with \( l \circ h = \widehat{f} \) is equivalent to the condition \( f|_{\ker h} = 0 \), that is, to the identity \( \mathcal{I}([A]) = 0 \) for each \( A \in \langle C(X,Y) \rangle \) with \( \theta(A) > r \).

This implies the required assertion since the order of the invariant \( f \) is equal to the infimum of those \( r \) for which there exists a homomorphism \( l \) as above.

\[\text{§ 18. The geometric realization and simplicial approximation}\]

Let \( K \) be a polyhedron, \( E \) a simplicial set, and \( Q = |K| \).

Lemma 18.1. The identity \( \theta(|U|) = \theta(U) \) holds for \( U \in \langle E(K) \rangle \).

\(^4\)Here and below \([\text{true}] = 1, [\text{false}] = 0\).
Lemma 18.2. Assume an element $B \in \langle C(Q,E)\rangle$ is given. Then there exists a polyhedron $L$ with body $Q$ and an element $V \in \langle E(L)\rangle$ with $\theta(V) \geq \theta(B)$ and $\|[V]\| = [B]$ in $\langle [Q,E]\rangle$.

Proof. We can find a finite set $I$, a map $k: I \to C(Q,E)$ and an element $g \in \langle I \rangle$ such that $\langle k \rangle(g) = B$. Let $b_i = k(i)$, $i \in I$. For $q \in Q$ we define the equivalence relation $R_q = \{(i,j): b_i(q) = b_j(q)\}$ on $I$. Given a finite subset $W \subset Q$, define

$$R_W = \bigcap_{q \in W} R_q.$$

The map $i \mapsto b_i|_W$ is constant on the equivalence classes of $R_W$. We have a commutative diagram

$$\begin{array}{ccc}
I & \xrightarrow{k} & C(Q,E) \\
p_W \downarrow & & \downarrow ?|_W \\
I/R_W & \xrightarrow{k_W} & C(W,E)
\end{array}$$

where $p_W$ is the projection. The map $k_W$ is injective. We have

$$\langle k_W \rangle(\langle p_W \rangle(g)) = \langle k \rangle(g)|_W = B|_W.$$ 

If $\#W < \theta(B)$ then $B|_W = 0$, hence, $\langle p_W \rangle(g) = 0$.

We have a continuous map $b = (b_i)_{i \in I}: Q \to |E|^I$. Let $h: |E^I| \to |E|^I$ be the canonical continuous bijection. Since the set $I$ is finite and the space $Q$ is Hausdorff and compact, the map $c = h^{-1} \circ b: Q \to |E|^I$ is continuous.

For each equivalence relation $R$ on $I$ there is the associated simplicial subset $D(R) \subset E^I$. Let $D(R) = \{(e_i)_{i \in I} \in E^I_q: (i,j) \in R \Rightarrow e_i = e_j\}$ (the diagonal). For $q \in Q$ we have $c(q) \in |D(R_q)| \subset |E|^I$. Define the simplicial subset $M \subset E^I$ by

$$M = \bigcup_{q \in Q} D(R_q).$$

We have $c(Q) \subset |M| \subset |E|^I$. Let $c' = c|_{Q \to |M|}$. By the simplicial approximation theorem, there is a polyhedron $L$ with body $Q$ and a section $u' \in M(L)$ such that the map $|u'|: Q \to |M|$ is homotopic to the map $c'$. Let $u \in E^I(L)$ be the composition of the section $u'$ and the inclusion $M \to E^I$. We have $u = (u_i)_{i \in I}$, $u_i \in E(L)$. The map $|u_i|: Q \to |E|$ is homotopic to the map $b_i$. Define the map $l: I \to E(L)$ by $l(i) = u_i$. Let $V = \langle l \rangle(g)$. We have $\|[V]\| = [B]$.

Given $y \in L$, $\dim y = s$, we have $u_s(y) \in M_s$, that is, there is a point $q = q_y \in Q$ such that $u_s(y) \in D(R_q)_s$. Hence, $u_i|_{q_y} = u_j|_{q_y}$ for $(i,j) \in R_q$, which implies that the map $i \mapsto u_i|_{q_y}$ is constant on the equivalence classes of $R_q$.

Take a subset $T \subset L$. Set $W = \{q_y: y \in T\}$. Then $\#W \leq \#T$. The map $i \mapsto u_i|_T$ is constant on the equivalence classes of $R_W$. There is a commutative diagram

$$\begin{array}{ccc}
I & \xrightarrow{l} & E(L) \\
p_W \downarrow & & \downarrow ?|_T \\
I/R_W & \xrightarrow{l_T} & E_T
\end{array}$$
We have $V\|_T = \langle l(g) \|_T = \langle l_T \rangle (dpW(g))$. If $\#T < \theta(B)$, then $\#W < \theta(B)$, $\langle pW \rangle(g) = 0$, and $V\|_T = 0$. Thus, $\theta(V) \geq \theta(B)$.

§ 19. Subgroups of $\langle [Q, |G|] \rangle$

Suppose that a polyhedral body $Q$, dim $Q \leq m$, $m \in \mathbb{N}$, and an $(n-1)$-connected simplicial set $E$ with one vertex, $n \in \mathbb{N}$, are given where $m \leq 2n - 1$. Let $G = FE$. We consider the subgroups $P, M_s, J_s \subset \langle C(Q, |G|) \rangle$, $s \in \mathbb{N}$, which are defined as follows:

$$P = \langle C(Q, |G(m)|) \rangle$$

(we assume that $C(Q, |G(m)|) \subset C(Q, |G|)$).

$M_s = \{B: \theta(B) \geq s\},$

$J_s$ is generated by the elements of the form $\langle 'b_1' - 1 \rangle \cdots \langle 'b_k' - 1 \rangle$, where $k \in \mathbb{N}$, $b_l \in C(Q, |\gamma_{s_l}G|) \subset C(Q, |G|)$ and $s_1 + \cdots + s_k \geq s$. (Note that $M_s$ and $J_s$ are ideals; we conjecture that $M_s \subset J_s$.) Given a subgroup $S \subset \langle C(Q, |G|) \rangle$, we denote by $[S] \subset \langle [Q, |G|] \rangle$ its image under the homomorphism $\langle \gamma \rangle: \langle C(Q, |G|) \rangle \rightarrow \langle [Q, |G|] \rangle$.

Lemma 19.1. For each $s \in \mathbb{N}$ we have $[M_s] = [P \cap M_s] = [J_s]$.

Proof. We first prove the inclusion $[M_s] \subset [J_s]$. Take an element $B \in M_s$. Then $\theta(B) \geq s$. By Lemma 18.2, there is a polyhedron $L$ with body $Q$ and an element $V \in \langle G(L) \rangle$ such that $\theta(V) \geq s$ and $[V] = [B]$. It suffices to show that $[V] \in J_s$. By Lemma 7.2, we have the inequality $\eta(V) \geq s$. Let $I_s \subset \langle G(L) \rangle$ (cf. § 7) be the subgroup generated by all elements of the form $\langle 'v_1' - 1 \rangle \cdots \langle 'v_k' - 1 \rangle$, where $k \in \mathbb{N}$, $v_l \in \langle \gamma_{s_l}G \rangle(L) \subset G(L)$, and $s_1 + \cdots + s_k \geq s$. It follows from Lemma 7.1 that $V \in I_s$. Obviously, $[V] \in J_s$.

Now we prove the inclusion $[P \cap M_s] \supset [J_s]$. Take an element $B \in \langle C(Q, |G|) \rangle$, $B = \langle 'b_1' - 1 \rangle \cdots \langle 'b_k' - 1 \rangle$, where $k \in \mathbb{N}$, $b_l \in C(Q, |\gamma_{s_l}G|) \subset C(Q, |G|)$, and $s_1 + \cdots + s_k \geq s$. These elements generate the subgroup $J_s$, so it is enough to show that $[B] \in [P \cap M_s]$.

Let $K$ be a polyhedron with body $Q$. Since the $\gamma_{s_l}G$ are Kan sets, there are sections $u_l \in (\gamma_{s_l}G)(K)$ with $[\{u_l\}] = [b_l]$ in $[Q, |G|]$. Let $U = \langle 'u_1' - 1 \rangle \cdots \langle 'u_k' - 1 \rangle \in \langle G(L) \rangle$. We have $[U] = [B]$ in $\langle [Q, |G|] \rangle$. Lemma 7.1 implies that $\eta(U) \geq s$. By Lemma 16.1, there is a polyhedron $L$ with body $Q$ and an element $V \in \langle G(L) \rangle$ such that $\theta(V) \geq s$ and $[V] = [U]$ in $\langle [Q, |G|] \rangle$. Obviously, $[V] \in P$. Lemma 18.1 implies that $\theta([V]) \geq s$. Thus, $[B] = [V]$ and $[V] \in P \cap M_s$.

§ 20. The passage from $[Q, |G|]$ to $[X, Y]$

Suppose we are given a finite cell complex $X$, dim $X \leq m$, $m \in \mathbb{N}$, and an $(n-1)$-connected cell complex $Y$, $n \in \mathbb{N}$, where $m < 2n - 1$. Consider the subgroups $L_s \subset \langle C(X, Y) \rangle$, $s \in \mathbb{N}$, given by $L_s = \{A: \theta(A) \geq s\}$. Let $B = (B_s)_{s=1}^\infty$ be the Curtis filtration in the group $[X, Y]$. For each $s \in \mathbb{N}$ we define the subgroup $H_s \subset \langle [X, Y] \rangle$ generated by all elements of the form $\langle 'w_1' - 1 \rangle \cdots \langle 'w_k' - 1 \rangle$, where $k \in \mathbb{N}$, $w_l \in B_{s_l}$ and $s_1 + \cdots + s_k \geq s$. (This is an ideal.) For each subgroup $R \subset \langle C(X, Y) \rangle$ we denote its image under the homomorphism $\langle \gamma \rangle: \langle C(X, Y) \rangle \rightarrow \langle [X, Y] \rangle$ by $[R] \subset \langle [X, Y] \rangle$. 
Lemma 20.1. For each $s \in \mathbb{N}$, $[L_s] = H_s$.

Proof. There exists a polyhedral body $Q$ with $\dim Q \leq m$ and a homotopy equivalence $g: Q \to X$. Let $g': X \to Q$ be the homotopy inverse map. There exists a simplicial set $E$ with one vertex and a homotopy equivalence $k: Y \to |E|$. Let $G = FE$ and let $i: E \to G$ be the canonical simplicial map. It is $(2n-1)$-connected by the Freudenthal Theorem. The map $h = [i] \circ k: Y \to |G|$ is also $(2n-1)$-connected. Since $m \leq 2n-1$, there is a map $h': |G(m)| \to Y$ such that the map $h \circ h'$ is homotopic to the inclusion $|G(m)| \to |G|$. Define the map $t: C(X,Y) \to C(Q,|G|)$ by $t(a) = h \circ a \circ g$. Since $m < 2n-1$, it induces an isomorphism $\tilde{t}: [X,Y] \to [Q,|G|]$. Define the map $t': C(Q,|G(m)|) \to C(X,Y)$ by $t'(b) = h' \circ b \circ g'$. For each $b \in C(Q,|G(m)|) \subset C(Q,|G|)$ we have $[t'(b)] = t^{-1}([b])$.

It is easy to see that

$$t(B_s) = \{ [b] \in [Q,|G|]: b \in C(Q,|\gamma_s G|) \subset C(Q,|G|) \}, \quad s \in \mathbb{N}. \tag{1}$$

Let $P, M_s, J_s \subset \langle C(Q,|G|) \rangle$ be as in the first paragraph of §19. We have the homomorphisms

$$\langle t \rangle: \langle C(X,Y) \rangle \to \langle C(Q,|G|) \rangle \quad \text{and} \quad \langle t' \rangle: P = \langle C(Q,|G(m)|) \rangle \to \langle C(X,Y) \rangle.$$

By Lemma 17.1, $\langle t \rangle(L_s) \subset M_s$ and $\langle t' \rangle(P \cap M_s) \subset L_s$. We have the ring homomorphism $\langle \tilde{t} \rangle: \langle [X,Y] \rangle \to \langle [Q,|G|] \rangle$. It follows from (1) that $\langle \tilde{t} \rangle(H_s) = [J_s]$. Using Lemma 19.1 we obtain that

$$\langle \tilde{t} \rangle([L_s]) = \langle \langle t \rangle(L_s) \rangle \subset [M_s] = [J_s] = \langle \tilde{t} \rangle(H_s).$$

Hence, $[L_s] \subset H_s$ and $[L_s] \supset [\langle t' \rangle(P \cap M_s)] = \langle \tilde{t} \rangle^{-1}([P \cap M_s]) = \langle \tilde{t} \rangle^{-1}([J_s]) = H_s$.

Proof of Theorem 1.1. We introduce the homomorphism $\overline{f}: \langle [X,Y] \rangle \to U$ where $\overline{f}(\overline{w'}) = f(w')$. By Lemma 17.2, the condition $\text{ord } f < s$ for $s \in \mathbb{N}$, is equivalent to the condition $\overline{f}([L_s]) = 0$. It is clear that the conditions $\deg_B f < s$ and $\overline{f}|_{H_s} = 0$ are equivalent. It remains to observe that $[L_s] = H_s$ by Lemma 20.1.

Bibliography

[1] E. B. Curtis, “Some relations between homotopy and homology”, Ann. of Math. (2) 82:3 (1965), 386–413.

[2] S. S. Podkorytov, “Mappings of the sphere to a simply connected space”, Geometry and topology. 9, Zap. Nauchn. Semin. S.-Peterburg. Otd. Mat. Inst. Steklov., vol. 329, S.-Peterburg. Otd. Mat. Inst. Steklov., St. Petersburg 2005, pp. 159–194; English transl. in J. Math. Sci. (N.Y.) 140:4 (2007), 589–610.

[3] S. S. Podkorytov, “An alternative proof of a weak form of Serre’s theorem”, Geometry and topology. 4, Zap. Nauchn. Semin. S.-Peterburg. Otd. Mat. Inst. Steklov., vol. 261, S.-Peterburg. Otd. Mat. Inst. Steklov., St. Petersburg 1999, pp. 210–221; English transl. in J. Math. Sci. (N.Y.) 110:4 (2002), 2875–2881.

[4] S. S. Podkorytov, “Order of a function on the Bruschinsky group”, Geometry and topology. 4, Zap. Nauchn. Semin. S.-Peterburg. Otd. Mat. Inst. Steklov., vol. 261, S.-Peterburg. Otd. Mat. Inst. Steklov., St. Petersburg 1999, pp. 222–228; English transl. in J. Math. Sci. (N.Y.) 110:4 (2002), 2882–2885.
[5] A. Dold and R. Thom, “Quasifaserungen und unendliche symmetrische Produkte”, 
Ann. of Math. (2) 67:2 (1958), 239–281.

[6] P. G. Goerss and J. F. Jardine, Simplicial homotopy theory, Progr. Math., vol. 174, 
Birkhäuser, Basel 1999.

[7] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, 
Springer-Verlag, Berlin–Heidelberg–New York 1967.

[8] J.-P. Serre, Lie algebras and Lie groups, Benjamin, New York–Amsterdam 1965.

[9] I. B. S. Passi, Group rings and their augmentation ideals, Lecture Notes in Math., 
vol. 715, Springer-Verlag, Berlin–Heidelberg–New York 1979.

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Received 25/FEB/10 and 11/JAN/11
Translated by T. PANOV