Abstract

Let $F$ be a nonarchimedean local field of characteristic zero and let $G = \text{SL}(N) = \text{SL}(N, F)$. This article is devoted to studying the influence of the elliptic representations of $\text{SL}(N)$ on the $K$-theory. We provide full arithmetic details. This study reveals an intricate geometric structure. One point of interest is that the $R$-group is realized as an isotropy group. Our results illustrate, in a special case, part (3) of the recent conjecture in [2].

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1 Introduction

Let $F$ be a nonarchimedean local field of characteristic zero and let $G = \text{SL}(N) = \text{SL}(N, F)$. This article is devoted to studying subspaces of the tempered dual of $\text{SL}(N)$ which have an especially intricate geometric structure, and to computing, with full arithmetic details, their $K$-theory. Our results illustrate, in a special case, part (3) of the recent conjecture in [2].

The subspaces of the tempered dual which are especially interesting for us contain elliptic representations. A tempered representation of $\text{SL}(N)$ is elliptic if its Harish-Chandra character is not identically zero on the elliptic set.

An element in the discrete series of $\text{SL}(N)$ is an isolated point in the tempered dual of $\text{SL}(N)$ and contributes one generator to $K_0$ of the reduced $C^*$-algebra of $\text{SL}(N)$.

Now $\text{SL}(N)$ admits elliptic representations which are not discrete series: we investigate, with full arithmetic details, the contribution of the elliptic...
representations of $\text{SL}(N)$ to the $K$-theory of the reduced $C^*$-algebra $A_N$ of $\text{SL}(N)$.

According to [9], $A_N$ is a $C^*$-direct sum of fixed $C^*$-algebras. Among these fixed algebras, we will focus on those whose duals contain elliptic representations. Let $n$ be a divisor of $N$ with $1 \leq n \leq N$ and suppose that the group $U_F$ of integer units admits a character of order $n$. Then the relevant fixed algebras are of the form $C(T^n/T, R)^{\mathbb{Z}/n\mathbb{Z}} \subset A_N$.

Here, $R$ is the $C^*$-algebra of compact operators on standard Hilbert space, $T^n/T$ is the quotient of the compact torus $T^n$ via the diagonal action of $T$. The compact group $T^n/T$ arises as the maximal compact subgroup of the standard maximal torus of the Langlands dual $\text{PGL}(n, \mathbb{C})$. We prove (Theorem 3.1) that this fixed $C^*$-algebra is strongly Morita equivalent to the crossed product $C(T^n/T) \rtimes \mathbb{Z}/n\mathbb{Z}$.

The reduced $C^*$-algebra $A_N$ is liminal, and its primitive ideal space is in canonical bijection with the tempered dual of $\text{SL}(N)$. Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual of $\text{SL}(N)$, see [5, 3.1.1, 4.4.1, 18.3.2].

Let $\mathfrak{X}_n$ denote the $C^*$-dual of $C(T^n/T, R)^{\mathbb{Z}/n\mathbb{Z}}$. Then $\mathfrak{X}_n$ is a non-Hausdorff space, and has a very special structure as topological space. When $n$ is a prime number $\ell$, then $\mathfrak{X}_\ell$ will contain multiple points. When $n$ is non-prime, $\mathfrak{X}_n$ will contain not only multiple points, but also multiple subspaces. This crossed product $C^*$-algebra is a noncommutative unital $C^*$-algebra which fits perfectly into the framework of noncommutative geometry. In the tempered dual of $\text{SL}(N)$, there are connected compact non-Hausdorff spaces, laced with multiple subspaces, and simply described by crossed product $C^*$-algebras.

The $K$-theory of the fixed $C^*$-algebra is then given by the $K$-theory of the crossed product $C^*$-algebra. To compute (modulo torsion) the $K$-theory of this noncommutative $C^*$-algebra, we apply the Chern character for discrete groups [3]. This leads to the cohomology of the extended quotient $(T^n/T)//(\mathbb{Z}/n\mathbb{Z})$. This in turn leads to a problem in classical algebraic topology, namely the determination of the cyclic invariants in the cohomology of the $n$-torus.

The ordinary quotient will be denoted by $\mathfrak{X}(n)$:

$$\mathfrak{X}(n) := (T^n/T)/(\mathbb{Z}/n\mathbb{Z})$$

This is a compact connected orbifold. Note that $\mathfrak{X}(1) = pt$. The orbifold $\mathfrak{X}(n, k, \omega)$ which appears in the following theorem is defined in section 4.
The notation is such that $X(n, n, 1)$ is the ordinary quotient $X(n)$ and each $X(n, 1, \omega)$ is a point. The highest common factor of $n$ and $k$ is denoted $(n, k)$.

**Theorem 1.1.** The extended quotient $(T^n/T)/(\mathbb{Z}/n\mathbb{Z})$ is a disjoint union of compact connected orbifolds:

$$(T^n/T)/(\mathbb{Z}/n\mathbb{Z}) = \bigsqcup X(n, k, \omega)$$

The disjoint union is over all $1 \leq k \leq n$ and all $n/(k,n)$th roots of unity $\omega$ in $\mathbb{C}$.

We apply the Chern character for discrete groups [3], and obtain

**Theorem 1.2.** The $K$-theory groups $K_0$ and $K_1$ are given by

$$K_0 (C(T^n/T), \mathbb{R})^{\mathbb{Z}/n\mathbb{Z}} \otimes \mathbb{C} \simeq \bigoplus H^{ev}(X(n, k, \omega); \mathbb{C})$$

$$K_1 (C(T^n/T), \mathbb{R})^{\mathbb{Z}/n\mathbb{Z}} \otimes \mathbb{C} \simeq \bigoplus H^{odd}(X(n, k, \omega); \mathbb{C})$$

The direct sums are over all $1 \leq k \leq n$ and all $n/(k,n)$th roots of unity $\omega$ in $\mathbb{C}$.

For the ordinary quotient $X(n)$ we have the following explicit formula (Theorems 6.1 and 6.3). Let $H^* := H^{ev} \oplus H^{odd}$ and let $\phi$ denote the Euler totient.

**Theorem 1.3.** Let $X(n)$ denote the ordinary quotient $(T^n/T)/(\mathbb{Z}/n\mathbb{Z})$. Then we have

$$\dim_\mathbb{C} H^* (X(n); \mathbb{C}) = \frac{1}{2n} \sum_{d|n, d \text{ odd}} \phi(d) 2^{n/d}.$$

Theorem 1.1 lends itself to an interpretation in terms of representation theory. When $n = \ell$ a prime number, the elliptic representations of SL($\ell$) are discussed in section 2. The extended quotient $(T^\ell/T)/(\mathbb{Z}/\ell\mathbb{Z})$ is the disjoint union of the ordinary quotient $X(\ell)$ and $\ell(\ell - 1)$ isolated points. We consider the canonical projection $\pi$ of the extended quotient onto the ordinary quotient:

$$\pi : (T^\ell/T)/(\mathbb{Z}/\ell\mathbb{Z}) \longrightarrow X(\ell)$$

The points $\tau_1, \ldots, \tau_\ell$ constructed in section 2, are precisely the $\mathbb{Z}/\ell\mathbb{Z}$ fixed points in $T^\ell/T$. These are $\ell$ points of reducibility, each of which admits $\ell$ elliptic constituents. Note also that, in the canonical projection $\pi$, the fibre $\pi^{-1}(\tau_j)$ of each point $\tau_j$ contains $\ell$ points. We may say that the extended
quotient encodes, or provides a model of, reducibility. This is a very special case of the recent conjecture in [2].

When \( n \) is non-prime, we have points of reducibility, each of which admits elliptic constituents. In addition to the points of reducibility, there is a subspace of reducibility. There are continua of \( L \)-packets. Theorem 1.2 describes the contribution, modulo torsion, of all these \( L \)-packets to \( K_0 \) and \( K_1 \).

Let the infinitesimal character of the elliptic representation \( \epsilon \) be the cuspidal pair \((M, \sigma)\), where \( \sigma \) is an irreducible cuspidal representation of \( M \) with unitary central character. Then \( \epsilon \) is a constituent of the induced representation \( i_G M(\sigma) \). Let \( s \) be the point in the Bernstein spectrum which contains the cuspidal pair \((M, \sigma)\). To conform to the notation in [2], we will write \( E^s := \mathbb{T}^n/\mathbb{T}, \ W^s = \mathbb{Z}/n\mathbb{Z} \). The standard projection will be denoted

\[ \pi^s : E^s/\!//W^s \to E^s/\!//W^s. \]

The space of tempered representations of \( G \) determined by \( s \) will be denoted Irr\(_{\text{temp}}\)(\( G \))^s, and the infinitesimal character will be denoted \( \text{inf.ch.} \).

**Theorem 1.4.** There is a continuous bijection

\[ \mu^s : E^s/\!//W^s \to \text{Irr}_{\text{temp}}(G)^s \]

such that

\[ \pi^s = (\text{inf.ch.}) \circ \mu^s. \]

This confirms, in a special case, part (3) of the conjecture in [2].

In section 2 of this article, we review elliptic representations of the special linear algebraic group \( \text{SL}(N, F) \) over a \( p \)-adic field \( F \). Section 3 concerns fixed \( C^* \)-algebras and crossed products. Section 4 computes the extended quotient \( (\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}) \). The formation of the \( R \)-groups is described in section 5. In section 6 we compute the cyclic invariants in the cohomology of the \( n \)-torus.

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## 2 The elliptic representations of \( \text{SL}(N) \)

Let \( F \) be a nonarchimedean local field of characteristic zero. Let \( G \) be a connected reductive linear group over \( F \). Let \( G = G(F) \) be the \( F \)-rational points of \( G \). We say that an element \( x \) of \( G \) is **elliptic** if its centralizer is
compact modulo the center of $G$. We let $G^e$ denote the set of regular elliptic elements of $G$.

Let $E_2(G)$ denote the set of equivalence classes of irreducible discrete series representations of $G$, and denote by $E_t(G)$ be the set of equivalence classes of irreducible tempered representations of $G$. If $\pi \in E_t(G)$, then we denote its character by $\Theta_\pi$. Since $\Theta_\pi$ can be viewed as a locally integrable function, we can consider its restriction to $G^e$, which we denote by $\Theta_\pi^e$. We say that $\pi$ is elliptic if $\Theta_\pi^e \neq 0$. The set of elliptic representations includes the discrete series.

Here is a classical example where elliptic representations occur [1]. We consider the group $\text{SL}(\ell, F)$ with $\ell$ a prime not equal to the residual characteristic of $F$. Let $K/F$ be a cyclic of order $\ell$ extension of $F$. The reciprocity law in local class field theory is an isomorphism

$$F^\times / N_{K/F} K^\times \cong \Gamma(K/F) = \mathbb{Z}/\ell \mathbb{Z}$$

where $\Gamma(K/F)$ is the Galois group of $K$ over $F$. Let now $\mu_\ell(\mathbb{C})$ be the group of $\ell$th roots of unity in $\mathbb{C}$. A choice of isomorphism $\mathbb{Z}/\ell \mathbb{Z} \cong \mu_\ell(\mathbb{C})$ then produces a character $\kappa$ of $F^\times$ of order $\ell$ as follows:

$$\kappa : F^\times \to F^\times / N_{K/F} K^\times \cong \mathbb{Z}/\ell \mathbb{Z} \cong \mu_\ell(\mathbb{C})$$

Let $B$ be the standard Borel subgroup of $\text{SL}(\ell, F)$, let $T$ be the standard maximal torus, and let $B = T \cdot N$ be its Levi decomposition. Let $\tau$ be the character of $T$ defined by

$$\tau := 1 \otimes \kappa \otimes \cdots \otimes \kappa^{\ell-1}$$

and let

$$\pi(\tau) := \text{Ind}_B^G(\tau \otimes 1)$$

be the unitarily induced representation of $\text{SL}(\ell)$.

Now $\pi(\tau)$ is a representation in the minimal unitary principal series of $\text{SL}(\ell)$. It has $\ell$ distinct irreducible elliptic components and the Galois group $\Gamma(K/F)$ acts simply transitively on the set of irreducible components. The set of irreducible components of $\pi(\tau)$ is an $L$-packet.

Let

$$\pi(\tau) = \pi_1 \oplus \cdots \oplus \pi_\ell$$

be the $\ell$ components of $\pi(\tau)$. The character $\Theta$ of $\pi(\tau)$, as character of a principal series representation, vanishes on the elliptic set. The character $\Theta_1$ of $\pi_1$ on the elliptic set is therefore cancelled out by the sum $\Theta_2 + \cdots + \Theta_\ell$ of the characters of the relatives $\pi_2, \ldots, \pi_\ell$ of $\pi_1$. 5
Let $\omega$ denote an $\ell$th root of unity in $\mathbb{C}$. All the $\ell$th roots are allowed, including $\omega = 1$. In the definition of $\tau$, we now replace $\kappa$ by $\kappa \otimes \omega^{v_{\text{val}}}$. This will create $\ell$ characters, which we will denote by $\tau_1, \ldots, \tau_\ell$, where $\tau_1 = \tau$. For each of these characters, the $R$-group is given as follows:

$$R(\tau_j) = \mathbb{Z}/\ell \mathbb{Z}$$

for all $1 \leq j \leq \ell$, and the induced representation $\pi(\tau_j)$ admits $\ell$ elliptic constituents.

If $P = MU$ is a standard parabolic subgroup of $G$ then $i_{GM}(\sigma)$ will denote the induced representation $\text{Ind}_{MU}^G(\sigma \otimes 1)$ (normalized induction). The $R$-group attached to $\sigma$ will be denoted $R(\sigma)$.

Let $P = MU$ be the standard parabolic subgroup of $G := \text{SL}(N,F)$ described as follows. Let $N = mn$, let $\tilde{M}$ be the Levi subgroup $\text{GL}(m)^n \subset \text{GL}(N,F)$ and let $M = M \cap \text{SL}(N,F)$.

We will use the framework, notation and main result in [6]. Let $\sigma \in \mathcal{E}_2(M)$ and let $\pi_\sigma \in \mathcal{E}_2(\tilde{M})$ with $\pi_\sigma|M \supset \sigma$. Let $W(M) := N_G(M)/M$ denote the Weyl group of $M$, so that $W(M)$ is the symmetric group on $n$ letters. Let

$$\mathcal{T}(\pi_\sigma) := \{ \eta \in \hat{F}^\times | \pi_\sigma \otimes \eta \simeq w \pi_\sigma \text{ for some } w \in W \}$$

$$X(\pi_\sigma) := \{ \eta \in \hat{F}^\times | \pi_\sigma \otimes \eta \simeq \pi_\sigma \}$$

By [6] Theorem 2.4], the $R$-group of $\sigma$ is given by

$$R(\sigma) \simeq \mathcal{T}(\pi_\sigma)/X(\pi_\sigma).$$

We follow [6] Theorem 3.4]. Let $\eta$ be a smooth character of $F^\times$ such that $\eta^n \in X(\pi_1)$ and $\eta^j \notin X(\pi_1)$ for $1 \leq j \leq n - 1$. Set

$$\pi_\sigma \simeq \pi_1 \otimes \eta \pi_1 \otimes \eta^2 \pi_1 \otimes \cdots \otimes \eta^{n-1} \pi_1, \quad \pi_\sigma|M \supset \sigma \quad (1)$$

with $\pi_1 \in \mathcal{E}_2(\text{GL}(m))$, $\eta \pi_1 := (\eta \circ \text{det}) \otimes \pi_1$. Then we have

$$\mathcal{T}(\pi_\sigma)/X(\pi_\sigma) = \langle \eta \rangle$$

and so $R(\sigma) \simeq \mathbb{Z}/n\mathbb{Z}$. The elliptic representations are the constituents of $i_{GM}(\sigma)$ with $\pi_\sigma$ as in equation (1).
3 Fixed algebras and crossed products

Let \( M \) denote the Levi subgroup which occurs in section 2. Denote by \( \Psi^1(M) \) the group of unramified unitary characters of \( M \). Now \( M \subset \text{SL}(N, F) \) comprises blocks \( x_1, \ldots, x_n \) with \( x_i \in \text{GL}(m, F) \) and \( \prod \det(x_i) = 1 \). Each unramified unitary character \( \psi \in \Psi^1(M) \) can be expressed as follows,

\[
\psi : \text{diag}(x_1, \ldots, x_n) \to \prod_{j=1}^n z_j^{\text{val}(\det x_j)}
\]

with \( z_1, z_2, \ldots, z_n \in T \), i.e. \( |z_i| = 1 \). Such unramified unitary characters \( \psi \) correspond to coordinates \( (z_1 : z_2 : \cdots : z_n) \) with each \( z_i \in T \). Since

\[
\prod_{i=1}^n (zz_i)^{\text{val}(\det x_i)} = \prod_{i=1}^n z_i^{\text{val}(\det x_i)}
\]

we have homogeneous coordinates. We have the isomorphism

\[
\Psi^1(M) \cong \{(z_1 : z_2 : \cdots : z_n) : |z_i| = 1, 1 \leq i \leq n\} = T^n / T.
\]

If \( M \) is the standard maximal torus \( T \) of \( \text{SL}(N) \) then \( \Psi^1(T) \) is the maximal compact torus in the dual torus

\[
T^\vee \subset G^\vee = \text{PGL}(N, \mathbb{C})
\]

where \( G^\vee \) is the Langlands dual group.

Let \( \sigma, \pi, \pi_1 \) be as in equation (1). Let \( g \) be the order of the group of unramified characters \( \chi \) of \( F^\times \) such that \( (\chi \circ \det) \otimes \pi_1 \simeq \pi \). Now let

\[
E := \{\psi \otimes \sigma : \psi \in \Psi^1(M)\}.
\]

The base point \( \sigma \in E \) determines a homeomorphism

\[
E \simeq T^n / T, \quad (z_1^{\text{val}\det} \otimes \cdots \otimes z_n^{\text{val}\det}) \otimes \sigma \mapsto (z_1^g : \cdots : z_n^g).
\]

From this point onwards, we will require that the restriction of \( \eta \) to the group \( \mathcal{U}_F \) of integer units is of order \( n \). Let \( W(M) \) denote the Weyl group of \( M \) and let \( W(M, E) \) be the subgroup of \( W(M) \) which leaves \( E \) globally invariant. Then we have \( W(M, E) = W(\sigma) = R(\sigma) = \mathbb{Z} / n \mathbb{Z} \).

Let \( \mathcal{R} = \mathcal{R}(H) \) denote the \( C^* \)-algebra of compact operators on the standard Hilbert space \( H \). Let \( a(w, \lambda) \) denote normalized intertwining operators. The fixed \( C^* \)-algebra \( C(E, \mathcal{R})^{W(M, E)} \) is given by

\[
\{f \in C(E, \mathcal{R}) | f(w\lambda) = a(w, \lambda^*) f(\lambda) a(w, \lambda^*)^{-1}, w \in W(M, E)\}.
\]

This fixed \( C^* \)-algebra is a \( C^* \)-direct summand of the reduced \( C^* \)-algebra \( \mathfrak{A}_N \) of \( \text{SL}(N) \), see [9].
Theorem 3.1. Let $G = \text{SL}(N,F)$, and $M$ be a Levi subgroup consisting of $n$ blocks of the same size $m$. Let $\sigma \in \mathcal{E}_2(M)$. Assume that the induced representation $i_{GM}(\sigma)$ has elliptic constituents, then the fixed $C^*$-algebra $C(E,\mathcal{R})^{W(M,E)}$ is strongly Morita equivalent to the crossed product $C^*$-algebra $C(E) \rtimes \mathbb{Z}/n\mathbb{Z}$.

Proof. For the commuting algebra of $i_{MG}(\sigma)$, we have \cite{12}:

$$\text{End}_G(i_{MG}(\sigma)) = \mathbb{C}[R(\sigma)].$$

Let $w_0$ be a generator of $R(\sigma)$, then the normalized intertwining operator $a(w_0, \sigma)$ is a unitary operator of order $n$. By the spectral theorem for unitary operators, we have

$$a(w_0, \sigma) = \sum_{j=0}^{n-1} \omega^j E_j$$

where $\omega = \exp(2\pi i/n)$ and $E_j$ are the projections onto the irreducible subspaces of the induced representation $i_{MG}(\sigma)$. The unitary representation

$$R(\sigma) \to U(H), \quad w \mapsto a(w, \sigma)$$

contains each character of $R(\sigma)$ countably many times. Therefore condition (***) in \cite{10} p. 301 is satisfied. The condition (**) in \cite{10} p. 300 is trivially satisfied since $W(\sigma) = R(\sigma)$.

We have $W(\sigma) = \mathbb{Z}/n\mathbb{Z}$. Then a subgroup $W(\rho)$ of order $d$ is given by $W(\rho) = k\mathbb{Z}$ mod $n$ with $dk = n$. In that case, we have

$$a(w_0, \sigma)|_{W(\rho)} = \sum_{j=0}^{n-1} \omega^{kj} E_j.$$ 

We compare the two unitary representations:

$$\phi_1 : W(\rho) \to U(H), \quad w \mapsto a(w, \sigma)|_{W(\rho)}$$
$$\phi_2 : W(\rho) \to U(H), \quad w \mapsto a(w, \rho).$$

Each representation contains every character of $W(\rho)$. They are quasi-equivalent as in \cite{10}. Choose an increasing sequence $(e_n)$ of finite-rank projections in $L(H)$ which converge strongly to $I$ and commute with each projection $E_j$. The compressions of $\phi_1, \phi_2$ to $e_n H$ remain quasi-equivalent. Condition (*) in \cite{10} p. 299 is satisfied.

All three conditions of \cite{10} Theorem 2.13 are satisfied. We therefore have a strong Morita equivalence

$$(C(E) \otimes \mathcal{R})^{W(M,E)} \simeq C(E) \rtimes R(\sigma) = C(E) \rtimes \mathbb{Z}/n\mathbb{Z}.$$
We will need a special case of the Chern character for discrete groups \([3]\).

**Theorem 3.2.** We have an isomorphism

\[
K_i(C(E) \rtimes \mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{C} \cong \bigoplus_{j \in \mathbb{N}} H^{2j+i}(E/(\mathbb{Z}/n\mathbb{Z}); \mathbb{C})
\]

with \(i = 0, 1\), where \(E/(\mathbb{Z}/n\mathbb{Z})\) denotes the extended quotient of \(E\) by \(\mathbb{Z}/n\mathbb{Z}\).

When \(N\) is a prime number \(\ell\), this result already appeared in \([8, 10]\).

### 4 The formation of the fixed sets

Extended quotients were introduced by Baum and Connes \([3]\) in the context of the Chern character for discrete groups. Extended quotients were used in \([7, 8]\) in the context of the reduced group \(C^*\)-algebras of \(GL(N)\) and \(SL(\ell)\) where \(\ell\) is prime. The results in this section extend results in \([8, 10]\).

**Definition 4.1.** Let \(X\) be a compact Hausdorff topological space. Let \(\Gamma\) be a finite abelian group acting on \(X\) by a (left) continuous action. Let

\[
\tilde{X} = \{(x, \gamma) \in X \times \Gamma : \gamma x = x\}
\]

with the group action on \(\tilde{X}\) given by

\[
g \cdot (x, \gamma) = (gx, \gamma)
\]

for \(g \in \Gamma\). Then the extended quotient is given by

\[
X/\Gamma := \tilde{X}/\Gamma = \bigsqcup_{\gamma \in \Gamma} X^\gamma/\Gamma
\]

where \(X^\gamma\) is the \(\gamma\)-fixed set.

The extended quotient will always contain the ordinary quotient. The standard projection \(\pi : X/\Gamma \to X/\Gamma\) is induced by the map \((x, \gamma) \mapsto x\). We note the following elementary fact, which will be useful later (in Lemma 5.2): let \(y = \Gamma x\) be a point in \(X/\Gamma\). Then the cardinality of the pre-image \(\pi^{-1} y\) is equal to the order of the isotropy group \(\Gamma_x\):

\[
|\pi^{-1} y| = |\Gamma_x|.
\]

We will write \(X = E = \mathbb{T}^n/\mathbb{T}\), where \(\mathbb{T}\) acts diagonally on \(\mathbb{T}^n\), i.e.

\[
t(t_1, t_2, \cdots, t_n) = (tt_1, tt_2, \ldots, tt_n), \quad t, t_i \in \mathbb{T}.
\]

We have the action of the finite group \(\Gamma = \mathbb{Z}/n\mathbb{Z}\) on \(\mathbb{T}^n/\mathbb{T}\) given by cyclic permutation. The two actions of \(\mathbb{T}\) and of \(\mathbb{Z}/n\mathbb{Z}\) on \(\mathbb{T}^n\) commute. We will write \((k, n)\) for the highest common factor of \(k\) and \(n\).
Theorem 4.2. The extended quotient $(\mathbb{T}^n/\mathbb{T})//\mathbb{Z}/n\mathbb{Z}$ is a disjoint union of compact connected orbifolds:

$$(\mathbb{T}^n/\mathbb{T})//\mathbb{Z}/n\mathbb{Z} \simeq \bigsqcup_{1 \leq k \leq n, \omega^{\phi(n)/k(n)} = 1} X(n, k, \omega)$$

Here, $\omega$ is a $n/(k, n)$th root of unity in $\mathbb{C}$.

Proof. Let $\gamma$ be the standard $n$-cycle defined by $\gamma(i) = i + 1 \mod n$. Then $\gamma^k$ is the product of $n/d$ cycles of order $d = n/(n, k)$. Let $\omega$ be a $d$th root of unity in $\mathbb{C}$. All $d$th roots of unity are allowed, including $\omega = 1$. The element $t(\omega) = t(\omega; z_1, \ldots, z_n) \in \mathbb{T}^n$ is defined by imposing the following relations:

$$z_{i+k} = \omega^{-1}z_i$$

all suffices $\mod n$. This condition allows $n/d$ of the complex numbers $z_1, \ldots, z_n$ to vary freely, subject only to the condition that each $z_j$ has modulus 1. The crucial point is that

$$\gamma^k \cdot t(\omega) = \omega t(\omega)$$

Then $\omega$ determines a $\gamma^k$-fixed set in $\mathbb{T}^n/\mathbb{T}$, namely the set $\mathcal{Y}(n, k, \omega)$ of all cosets $t(\omega) \cdot \mathbb{T}$. The set $\mathcal{Y}(n, k, \omega)$ is an $(n/d - 1)$-dimensional subspace of fixed points.

Note that $\mathcal{Y}(n, k, \omega)$, as a coset of the closed subgroup $\mathcal{Y}(n, k, 1)$ in the compact Lie group $E$, is homeomorphic (by translation in $E$) to $\mathcal{Y}(n, k, 1)$. The translation is by the element $t(\omega : 1, \ldots, 1)$. If $\omega_1, \omega_2$ are distinct $d$th roots of unity, then $\mathcal{Y}(n, k, \omega_1), \mathcal{Y}(n, k, \omega_2)$ are disjoint.

We define the quotient space

$$X(n, k, \omega) := \mathcal{Y}(n, k, \omega)/\mathbb{Z}/n\mathbb{Z}$$

and apply definition 4.1. \qed

When $k = n$, we must have $\omega = 1$. In that case, the orbifold is the ordinary quotient: $X(n, n, 1) = X(n)$.

Let $(n, k) = 1$. The number of such $k$ in $1 \leq k \leq n$ is $\phi(n)$. In this case, $\omega$ is an $n$th root of unity and $X(n, k, \omega)$ is a point. There are $n$ such roots of unity in $\mathbb{C}$. Therefore, the extended quotient $(\mathbb{T}^n/\mathbb{T})//\mathbb{Z}/n\mathbb{Z}$ always contains $\phi(n)n$ isolated points.

Theorem 1.1 is a consequence of Theorems 3.1, 3.2 and 4.2. If, in Theorem 1.1, we take $n$ to be a prime number $\ell$, then we recover the following result in [3], p. 30: the extended quotient $(\mathbb{T}^\ell/\mathbb{T})//\mathbb{Z}/\ell\mathbb{Z}$ is the disjoint union of the ordinary quotient $X(\ell)$ and $(\ell - 1)\ell$ points.
5 The formation of the $R$-groups

We continue with the notation of section 3. Let $\sigma, \pi, \pi_1, \eta$ be as in equation (1). The $n$-tuple $t := (z_1, \ldots, z_n) \in \mathbb{T}^n$ determines an element $[t] \in E$. We can interpret $[t]$ as the unramified character

$$\chi_t := (z_1^{\text{valdet}}, \ldots, z_n^{\text{valdet}})$$

Let $\Gamma = \mathbb{Z}/n\mathbb{Z}$, and let $\Gamma[t]$ denote the isotropy subgroup of $\Gamma$.

**Lemma 5.1.** The isotropy subgroup $\Gamma[t]$ is isomorphic to the $R$-group of $\chi_t \otimes \sigma$:

$$\Gamma[t] \cong R(\chi_t \otimes \sigma)$$

**Proof.** Let the order of $\Gamma[t]$ be $d$. Then $d$ is a divisor of $n$. Let $\gamma$ be a generator of $\Gamma[t]$. Then $\gamma$ is a product of $n/d$ disjoint $d$-cycles, as in section 4. We must have $t = t(\omega)$ with $\omega$ a $d$th root of unity in $\mathbb{C}$. Note that $\gamma \cdot t(\omega) = \omega t(\omega)$. Then we have

$$R(\chi_t \otimes \sigma) = \overline{L(\chi_t \otimes \pi_\sigma)} / X(\chi_t \otimes \pi_\sigma)
= \{ \alpha \in \overline{\mathbb{F}}^\times : \omega \pi_\sigma \simeq \pi_\sigma \otimes \alpha \text{ for some } \omega \text{ in } W \} / X(\chi_t \otimes \pi_\sigma)
= \langle \omega^{\text{valdet}} \otimes \eta^{n/d} \rangle
= \mathbb{Z}/d\mathbb{Z}
= \Gamma[t]$$

since, modulo $X(\chi_t \otimes \pi_\sigma)$, the character $\eta^{n/d}$ has order $d$. \qed

**Lemma 5.2.** In the standard projection $p : E//\Gamma \rightarrow E/\Gamma$, the cardinality of the fibre of $[t]$ is the order of the $R$-group of $\chi_t \otimes \sigma$.

**Proof.** This follows from Lemma 5.1. \qed

We will assume that $\sigma$ is a cuspidal representation of $M$ with unitary central character. Let $\mathfrak{s}$ be the point in the Bernstein spectrum of $\text{SL}(N)$ which contains the cuspidal pair $(M, \sigma)$. To conform to the notation in [2], we will write $E^\mathfrak{s} := \mathbb{T}^n/\mathbb{T}$, $W^\mathfrak{s} = \mathbb{Z}/n\mathbb{Z}$. The standard projection will be denoted

$$\pi^\mathfrak{s} : E^\mathfrak{s} / W^\mathfrak{s} \rightarrow E^\mathfrak{s} / W^\mathfrak{s}.$$

The space of tempered representations of $G$ determined by $\mathfrak{s}$ will be denoted $\text{Irr}_{\text{temp}}(G)^\mathfrak{s}$, and the infinitesimal character will be denoted $\text{inf.ch.}$
Theorem 5.3. We have a commutative diagram:

\[ E/W^s \xrightarrow{\mu^s} \text{Irr}^{\text{temp}}(G)^s \]

\[ \pi^s \downarrow \quad \downarrow \text{inf.ch.} \]

\[ E/W^s \longrightarrow E/W^s \]

in which the map \( \mu^s \) is a continuous bijection. This confirms, in a special case, part (3) of the conjecture in [2].

Proof. We have

\[ \mathbb{C}[R(\sigma)] \simeq \text{End}_G(i_{GM}(\sigma)) \]

This implies that the characters of the cyclic group \( R(\sigma) \) parametrize the irreducible constituents of \( i_{GM}(\sigma) \). This leads to a labelling of the irreducible constituents of \( i_{GM}(\sigma) \), which we will write as \( i_{GM}(\sigma : r) \) with \( 0 \leq r < n \).

The map \( \mu^s \) is defined as follows:

\[ \mu^s : (t, \gamma^{rd}) \mapsto i_{GM}(\chi_t \otimes \sigma : r) \]

We now apply Lemma 5.2.

Theorem 3.2 in [9] relates the natural topology on the Harish-Chandra parameter space to the Jacobson topology on the tempered dual of a reductive \( p \)-adic group. As a consequence, the map \( \mu^s \) is continuous.

\[ 6 \quad \text{Cyclic invariants} \]

We will consider the map

\[ \alpha : \mathbb{T}^n \rightarrow (\mathbb{T}^n/\mathbb{T}) \times \mathbb{T}, \quad (t_1, \ldots, t_n) \rightarrow ((t_1 : \ldots : t_n), t_1 t_2 \cdots t_n) \]

where \( (t_1 : \ldots : t_n) \) is the image of \( (t_1, \ldots, t_n) \) via the map \( \mathbb{T}^n \rightarrow \mathbb{T}^n/\mathbb{T} \). The map \( \alpha \) is a homomorphism of Lie groups. The kernel of this map is

\[ \mathcal{G}_n := \{ \omega I_n : \omega^n = 1 \} \]

We therefore have the isomorphism of compact connected Lie groups:

\[ \mathbb{T}^n/\mathcal{G}_n \cong (\mathbb{T}^n/\mathbb{T}) \times \mathbb{T} \quad (2) \]

This isomorphism is equivariant with respect to the \( \mathbb{Z}/n\mathbb{Z} \)- action, and we infer that

\[ (\mathbb{T}^n/\mathcal{G}_n)/(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}) \times \mathbb{T} \quad (3) \]

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Theorem 6.1. Let $H^\bullet(-; \mathbb{C})$ denote the total cohomology group. We have

$$\dim_{\mathbb{C}} H^\bullet(\mathcal{X}(n); \mathbb{C}) = \frac{1}{2} \cdot \dim_{\mathbb{C}} H^\bullet(\mathbb{T}^n; \mathbb{C})^{\mathbb{Z}/n\mathbb{Z}}.$$  

Proof. The cohomology of the orbit space is given by the fixed set of the cohomology of the original space [4, Corollary 2.3, p.38]. We have

$$H^j(\mathbb{T}^n/\mathcal{G}_n; \mathbb{C}) \cong H^j(\mathbb{T}^n; \mathbb{C})^{\mathcal{G}_n} \cong H^j(\mathbb{T}^n; \mathbb{C})$$  (4)

since the action of $\mathcal{G}_n$ on $\mathbb{T}^n$ is homotopic to the identity. We spell this out.

Let $z := (z_1, \ldots, z_n)$ and define $H(z, t) = \omega^t \cdot z = (\omega^t z_1, \ldots, \omega^t z_n)$. Then $H(z, 0) = z$, $H(z, 1) = \omega \cdot z$. Also, $H$ is equivariant with respect to the permutation action of $\mathbb{Z}/n\mathbb{Z}$. That is to say, if $\epsilon \in \mathbb{Z}/n\mathbb{Z}$ then $H(\epsilon \cdot z, t) = \epsilon \cdot H(z, t)$. This allows us to proceed as follows:

$$H^j(\mathbb{T}^n; \mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \cong H^j(\mathbb{T}^n/\mathcal{G}_n; \mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \cong H^j((\mathbb{T}^n/\mathbb{T}) \times \mathbb{T}; \mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \cong H^j((\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}) \times \mathbb{T}; \mathbb{C})$$  (5)

We apply the Kunneth theorem in cohomology (there is no torsion):

$$(H^j(\mathbb{T}^n; \mathbb{C}))^{\mathbb{Z}/n\mathbb{Z}} \cong H^j(\mathcal{X}(n); \mathbb{C}) \oplus H^{j-1}(\mathcal{X}(n); \mathbb{C}) \quad \text{with } 0 < j \leq n$$

$$(H^n(\mathbb{T}^n; \mathbb{C}))^{\mathbb{Z}/n\mathbb{Z}} \cong H^{n-1}(\mathcal{X}(n); \mathbb{C}), \quad H^0(\mathbb{T}^n; \mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \cong H^0(\mathcal{X}(n); \mathbb{C}) \cong \mathbb{C}$$

$$H^{ev}(\mathbb{T}^n; \mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} = H^\bullet(\mathcal{X}(n); \mathbb{C}), \quad H^{odd}(\mathbb{T}^n; \mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} = H^\bullet(\mathcal{X}(n); \mathbb{C})$$

We now have to find the cyclic invariants in $H^\bullet(\mathbb{T}^n; \mathbb{C})$. The cohomology ring $H^\bullet(\mathbb{T}^n, \mathbb{C})$ is the exterior algebra $\bigwedge V$ of a complex $n$-dimensional vector space $V$, as can be seen by considering differential forms $d\theta_1 \wedge \cdots \wedge d\theta_n$. The vector space $V$ admits a basis $\alpha_1 = d\theta_1, \ldots, \alpha_n = d\theta_n$. The action of $\mathbb{Z}/n\mathbb{Z}$ on $\bigwedge V$ is induced by permuting the elements $\alpha_1, \ldots, \alpha_n$, i.e. by the regular representation $\rho$ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. This representation of $\mathbb{Z}/n\mathbb{Z}$ on $\bigwedge \rho$ will be denoted $\bigwedge \rho$. The dimension of the space of cyclic invariants in $H^\bullet(\mathbb{T}^n, \mathbb{C})$ is equal to the multiplicity of the unit representation 1 in $\bigwedge \rho$. To determine this, we use the theory of group characters.
Lemma 6.2. The dimension of the the subspace of cyclic invariants is given by
\[
(\chi_{\Lambda,\rho}, 1) = \frac{1}{n}(\chi_{\Lambda,\rho}(0) + \chi_{\Lambda,\rho}(1) + \cdots + \chi_{\Lambda,\rho}(n-1)).
\]

Proof. This is a standard result in the theory of group characters [11]. \qed

Theorem 6.3. The dimension of the space of cyclic invariants in \( H^\bullet(\mathbb{T}^n, \mathbb{C}) \) is given by the formula
\[
g(n) := \frac{1}{n} \sum_{d|n, d \text{ odd}} \phi(d)2^{n/d}
\]

Proof. We note first that
\[
\chi_{\Lambda,\rho}(0) = \text{Trace } 1_{\Lambda V} = \dim \mathbb{C} \bigwedge V = 2^n.
\]

To evaluate the remaining terms, we need to recall the definition of the elementary symmetric functions \( e_j \):
\[
\prod_{j=1}^n (\lambda - \alpha_j) = \lambda^n - \lambda^{n-1}e_1 + \lambda^{n-2}e_2 - \cdots + (-1)^n e_n.
\]

When we need to mark the dependence on \( \alpha_1, \ldots, \alpha_n \) we will write \( e_j = e_j(\alpha_1, \ldots, \alpha_n) \). Set \( \alpha_j = \omega^{j-1}, \omega = \exp(2\pi i/n) \). Then we get
\[
\lambda^n - 1 = \prod_{j=1}^n (\lambda - \alpha_j) = \lambda^n - \lambda^{n-1}e_1 + \lambda^{n-2}e_2 - \cdots + (-1)^n e_n.
\]

Let \( d|n \), let \( \zeta \) be a primitive \( d \)th root of unity. Let \( \alpha_j = \zeta^{j-1} \). We have
\[
(\lambda^d - 1)^{n/d} = (\lambda^d - 1) \cdots (\lambda^d - 1) = \prod_{j=1}^n (\lambda - \alpha_j)
\]  
(6) Set \( \lambda = -1 \). If \( d \) is even, we obtain
\[
0 = 1 + e_1(1, \zeta, \zeta^2, \ldots) + e_2(1, \zeta, \zeta^2, \ldots) + \cdots + e_n(1, \zeta, \zeta^2, \ldots)
\]  
(7) If \( d \) is odd, we obtain
\[
2^{n/d} = 1 + e_1(1, \zeta, \zeta^2, \ldots) + e_2(1, \zeta, \zeta^2, \ldots) + \cdots + e_n(1, \zeta, \zeta^2, \ldots)
\]  
(8)

We observe that the regular representation \( \rho \) of the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) is a direct sum of the characters \( m \mapsto \omega^{rm} \) with \( 0 \leq r \leq n \). This direct
sum decomposition allows us to choose a basis \( v_1, \ldots, v_n \) in \( V \) such that the representation \( \Lambda \rho \) is diagonalized by the wedge products \( v_j \wedge \cdots \wedge v_j \). This in turn allows us to compute the character of \( \Lambda \rho \) in terms of the elementary symmetric functions \( e_1, \ldots, e_n \).

With \( \zeta = \omega^r \) as above, we have

\[
\chi_{\Lambda \rho}(r) = 1 + e_1(1, \zeta, \zeta^2, \ldots) + e_2(1, \zeta, \zeta^2, \ldots) + \cdots + e_n(1, \zeta, \zeta^2, \ldots)
\]

We now sum the values of the character \( \chi_{\Lambda \rho} \). Let \( d := n/(r, n) \). Then \( \zeta \) is a primitive \( d \)th root of unity. If \( d \) is even then \( \chi_{\Lambda \rho}(r) = 0 \). If \( d \) is odd, then \( \chi_{\Lambda \rho}(r) = 2^{n/d} \). There are \( \phi(d) \) such terms. So we have

\[
\chi_{\Lambda \rho}(0) + \chi_{\Lambda \rho}(1) + \cdots + \chi_{\Lambda \rho}(n - 1) = \sum_{d|n, d\text{ odd}} \phi(d)2^{n/d} \tag{9}
\]

We now apply Lemma 6.2.

The sequence \( n \mapsto g(n)/2, n = 1, 2, 3, 4, \ldots \), is

1, 1, 2, 2, 4, 6, 10, 16, 30, 52, 94, 172, 316, 586, 1096, 2048, 3856, 7286, \ldots

as in [www.research.att.com/~njas/sequences/A000016](http://www.research.att.com/~njas/sequences/A000016). Thanks to Kasper Andersen for alerting us to this web site.

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