RANDOMIZED SUBSPACE REGULARIZED NEWTON METHOD FOR THE UNCONSTRAINED NON-CONVEX OPTIMIZATION

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Abstract. While there already exist randomized subspace Newton methods that restrict the search direction to a random subspace for a convex function, we propose a randomized subspace regularized Newton method for a non-convex function and more generally we investigate thoroughly the local convergence rate of the randomized subspace Newton method. In our proposed algorithm using a modified Hessian of the function restricted to some random subspace, with high probability, the function value decreases even when the objective function is non-convex. In this paper, we show that our method has global convergence under appropriate assumptions and its convergence rate is the same as that of the full regularized Newton method. Furthermore, we can obtain a local linear convergence rate under some additional assumptions, and prove that this rate is the best we can hope, in general, when using random subspace. We furthermore prove that if the Hessian at the local optimum is rank deficient then superlinear convergence holds.

Key words. random projection, Newton method, non-convex optimization, local convergence rate

AMS subject classifications. 90C26, 90C30

1. Introduction. While first-order optimization methods such as stochastic gradient descent methods are well studied for large-scale machine learning optimization, second-order optimization methods have not received much attention due to the high cost of computing second-order information until recently. However, in order to overcome relatively slow convergence of first-order methods, there has been recent interest in second-order methods that aim to achieve faster convergence speed by utilizing subsampled Hessian information and stochastic Hessian estimate (see e.g., [3, 30, 31] and references therein).

In this paper, we develop a Newton-type iterative method with random projections for the following unconstrained optimization problem:

\[
\min_{x \in \mathbb{R}^n} f(x),
\]

where \(f : \mathbb{R}^n \to \mathbb{R}\) is a possibly non-convex twice differentiable function. In our method, at each iteration, we restrict the function \(f\) to a random subspace and compute the next iterate by choosing a descent direction on this random subspace.

There are some existing studies on developing second-order methods with random subspace techniques for convex optimization problems (1.1). Let us now review randomized subspace Newton (RSN) existing work [11], while gradient-based randomized subspace algorithms are reviewed in Section 2.1. RSN computes the descent direction
$d_k$ and the next iterate as
\[
d_{RSN}^k = -P_k^T (P_k \nabla^2 f(x_k) P_k^T)^{-1} P_k \nabla f(x_k),
\]
\[
x_{k+1} = x_k + \frac{1}{L} d_{RSN}^k,
\]
where $P_k \in \mathbb{R}^{s \times n}$ is a random matrix with $s < n$ and $\hat{L}$ is some fixed constant. RSN is expected to be highly computationally efficient with respect to the original Newton method, since it does not require computation of the full Hessian inverse. RSN is also shown to achieve a global linear convergence for strongly convex $f$. We first note that the second-order Taylor approximation around $x_k$ restricted in the affine subspace $\{x_k\} + \text{Range}(P_k^T)$ is expressed as
\[
f(x_k + P_k^T u) \simeq f(x_k) + \nabla f(x_k)^T P_k^T u + \frac{1}{2} u^T P_k \nabla^2 f(x_k) P_k^T u,
\]
and the direction $d_{RSN}^k$ is obtained as $d_{RSN}^k = P_k^T u_k^*$ where $u_k^*$ is the minimizer of the above subspace Taylor approximation, i.e.,
\[
u_k^* = \arg \min_{u \in \mathbb{R}^s} \left( f(x_k) + \nabla f(x_k)^T P_k^T u + \frac{1}{2} u^T P_k \nabla^2 f(x_k) P_k^T u \right).
\]
Hence, we can see that the next iterate of RSN is computed by using the Newton direction for the function
\[
(1.2) \quad f_{x_k} : u \mapsto f(x_k + P_k^T u).
\]

Other second-order subspace descent methods, as well cubically-regularized subspace Newton methods, [13], have been studied in the literature. More precisely, the method in [13] can be seen as a stochastic extension of the cubically-regularized Newton method [21] and also as a second-order enhancement of stochastic subspace descent [17]. In [16], a random subspace version of the BFGS method is proposed. The authors prove local linear convergence if the function is assumed to be self-concordant. Apart in recent Shao’s Ph.D thesis [24] and the associated papers [9, 8] which have been done parallelly to this paper, to the best of our knowledge, existing second-order subspace methods have iteration complexity analysis only for convex optimization problems.

The thesis [24] and the paper [8] propose a random subspace adaptive regularized cubic method for unconstrained non-convex optimization and show a global convergence property with sub-linear rate to a stationary point\(^1\). In this paper we propose a new subspace method based on the regularized Newton method and discuss the local convergence rate together with global iteration complexity.\(^2\) Notice indeed that, to the best of our knowledge, the local convergence of such methods never seen to have been thoroughly studied\(^3\); one would expect super-linear convergence for second order method and no papers discuss whether super-linear convergence holds or not for second order methods. Indeed any iterative algorithm can be easily adapted to a random subspace method as it suffices to apply it to the function restricted to the

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\(^1\)The author also proves that if the Hessian matrix has low rank and scaled Gaussian sketching matrices are used, then the Hessian at the stationary point is approximately positive semidefinite with high probability.

\(^2\)Just as the ordinary cubic method is superior to the Newton method in terms of iteration complexity, similar observation seems to hold between the subspace cubic method [24] and ours.

\(^3\)Some papers, as we will see later, investigate when local linear convergence holds.
subspace: \( u \mapsto f(x_k + P_k^T u) \). We therefore believe that it is important to investigate thoroughly whether the properties of such full-space algorithms are preserved or not when adapted to the random subspace setting.

The topic around subspace methods in optimization has received much attention recently. Besides random subspace methods, deterministic subspace methods are also investigated. In [20], a limited-memory common-directions method is proposed for smooth optimization problems. At each iteration, a subspace is constructed using first-order information from previous iterations, and a Newton method is used to obtain an approximate minimizer within this subspace. They prove sub-linear convergence in the non-convex case and linear convergence if the function satisfies the Polyak-Lojasiewicz (PL) inequality:

\[
\forall x \in \mathbb{R}^n, \quad \|\nabla f(x)\|^2 \geq c_0(f(x) - f(x^*)).
\]

If the objective function \( f \) is not convex, the Hessian is not always positive semi-definite and \( d_k^{RSN} \) is not guaranteed to be a descent direction so that we need to use a modified Hessian. Based on the regularized Newton method (RNM) for the unconstrained non-convex optimization [26, 27], we propose the randomized subspace regularized Newton method (RS-RNM):

\[
d_k = -P_k^T (P_k \nabla^2 f(x_k) P_k^T + \eta_k I_s)^{-1} P_k \nabla f(x_k),
\]

\[
x_{k+1} = x_k + t_k d_k,
\]

where \( \eta_k \) is defined to ensure that search direction \( d_k \) is a descent direction and the step size \( t_k \) is chosen so that it satisfies Armijo’s rule. As with RSN, this algorithm is expected to be computationally efficient since we use projections onto lower-dimensional spaces. In this paper, we first show that RS-RNM has global convergence under appropriate assumptions, more precisely, we have \( \|\nabla f(x_k)\| \leq \varepsilon \) after at most \( O(\varepsilon^{-2}) \) iterations with some probability, which is the same as the global convergence rate shown in [26] for the full regularized Newton method. We then prove that under additional assumptions, we can obtain a linear convergence rate locally. In particular, one contribution of the paper is to propose, to the best of our knowledge, the weakest conditions until now for local linear convergence. To do so we will extensively use the fact that the subspace is chosen at random. From these conditions, we can derive a random-projection version of the PL inequality (1.3), which will be satisfied when the function is restricted to a random subspace. One other contribution of this paper is to prove that, in general, linear convergence is the best rate we can hope for this method. Furthermore, we also prove that if the Hessian at the local optima is rank deficient, then one can achieve super-linear convergence using a subspace dimension \( s \) large enough.

The rest of this paper is organized as follows. After reviewing gradient-based randomized subspace algorithms and introducing properties of random projections in Section 2, we introduce our random subspace Newton method for non-convex functions in Section 3. In Section 4, we prove global convergence properties for our method. In Section 5, we investigate local linear convergence as well as local super-linear convergence. Finally, in Section 6, we show some numerical examples to illustrate the theoretical properties derived in the paper. In Section 7 we conclude the paper.

2. Preliminaries.

Notation: In this paper we call a matrix \( P \in \mathbb{R}^{s \times n} \) a random projection matrix or a random matrix when its entries \( P_{ij} \) are independently sampled from the normal
distribution \( N(0, 1/s) \). Let \( I_n \) be the identity matrix of size \( n \). We denote by \( g_k \) the gradient of the \( k \)-th iterate of the obtained sequence and by \( H_k \) its Hessian.

### 2.1. Related optimization algorithms using random subspace

As introduced in Section 1, random subspace techniques are used for second-order optimization methods with the aim of reducing the size of Hessian matrix. Here we refer to other types of subspace methods focusing on their convergence properties.

Cartis et al. [5] proposed a general framework to investigate a general random embedding framework for global optimization of a function \( f \). The framework projects the original problem onto a random subspace and solves the reduced subproblem in each iteration:

\[
\min_u f(x_k + P_k^T u) \text{ subject to } x_k + P_k^T u \in C.
\]

These subproblems need to be solved to some required accuracy by using a deterministic global optimization algorithm. This study is further expanded in [6] and [4], when \( f \) has low effective dimension.

There are also various subspace first-order methods based on coordinate descent methods (see e.g. [29]). In [7] a randomized coordinate descent algorithm is introduced assuming some subspace decomposition which is suited to the \( A \)-norm, where \( A \) is a given preconditioner. In [19], minimizing \( f(\tilde{A}x) + \frac{1}{2}\|x\|^2 \), where \( f \) is a strongly convex smooth function and \( \tilde{A} \) is a high-dimensional matrix, is considered and a new randomized optimization method that can be seen as a generalization of coordinate descent to random subspaces is proposed. The paper [12] deals with a convex optimization problem \( \min_x f(x) + g(x) \), where \( f \) is convex and differentiable and \( g \) is assumed to be convex, non-smooth and sparse inducing such as \( \|x\|_1 \). To solve the problem, they propose a randomized proximal algorithm leveraging structure identification: the variable space is sampled according to the structure of \( g \). The approach in [25] is to optimize a smooth convex function by choosing, at each iteration a random direction on the sphere. Recently, in some contexts such as iteration complexity analysis, the assumption of strong convexity has been replaced by a weaker one, the PL inequality (1.3). Indeed, [18] has introduced a new first-order random subspace and proved that if the non-convex function is differentiable with a Lipschitz first derivative and satisfies the PL inequality (1.3) then linear convergence rate is obtained in expectation. Notice that in all these papers a local linear convergence rate is only obtained when assuming that the objective function is, at least locally, strongly convex or satisfies the PL inequality.

From above, without (locally) strong convexity nor the PL inequality, it seems difficult to construct first-order algorithms having (local) linear convergence rates. Indeed, the probabilistic direct-search method [22] in reduced random spaces is applicable to both convex and non-convex problems but it obtains sub-linear convergence.

In this paper, we will prove that our algorithm achieves local linear convergence rates without locally strong convexity nor the PL inequality assumption on the full space. This is due to randomized Hessian information used in our algorithm. More precisely, our assumptions will allow us to prove that the function, restricted to a random subspace, satisfies a condition similar to the PL inequality.

### 2.2. Properties of random projection

In this section, we recall basic properties of random projection matrices. One of the most important features of a random projection defined by a random matrix is that it nearly preserves the norm of any given vector with arbitrary high probability. The following lemma is known as a variant of the Johnson-Lindenstrauss lemma [15].
Lemma 2.1 ([28, Lemma 5.3.2, Exercise 5.3.3]). Let $P \in \mathbb{R}^{s \times n}$ be a random matrix whose entries $P_{ij}$ are independently drawn from $N(0, 1/s)$. Then for any $x \in \mathbb{R}^n$ and $\varepsilon \in (0, 1)$, we have

$$\text{Prob} \left[ (1 - \varepsilon) \|x\|^2 \leq \|Px\|^2 \leq (1 + \varepsilon) \|x\|^2 \right] \geq 1 - 2 \exp(-C_0 \varepsilon^2 s),$$

where $C_0$ is an absolute constant.

The next lemma shows that when $P$ is a Gaussian matrix, we can obtain a bound on the norm of $PP^\top$.

Lemma 2.2. For a $s \times n$ random matrix $P$ whose entries are sampled from $N(0, 1/s)$, there exists a constant $\bar{C} > 0$ such that

$$\|PP^\top\| = \|P^\top P\| = \|P\|^2 \leq \bar{C} \frac{n}{s},$$

with probability at least $1 - 2e^{-s}$.

Proof. By [28, Theorem 4.6.1], there exists a constant $\bar{C}$ such that

$$\left\| \frac{s}{n} PP^\top - I_s \right\| \leq 2\bar{C} \sqrt{\frac{n}{s}},$$

holds with probability at least $1 - e^{-s}$. Therefore, we have

$$\|PP^\top\| \leq \left\| PP^\top - \frac{n}{s} I_s \right\| + \left\| \frac{n}{s} I_s \right\| \leq 2\bar{C} \sqrt{\frac{n}{s}} + \frac{n}{s} \leq 2\bar{C} \frac{n}{s} + \frac{n}{s} = (2\bar{C} + 1) \frac{n}{s}.$$ 

Setting $\bar{C} = 2\bar{C} + 1$ ends the proof.

All the results of this paper are stated in a probabilistic way. In the proofs we will constantly use the following fact:

(2.1) For any two events $E_1$ and $E_2$: \(\text{Prob}(E_1 \cap E_2) \geq 1 - ((1 - \text{Prob}(E_1)) + (1 - \text{Prob}(E_2)))\).

3. Randomized subspace regularized Newton method. In this section, we describe a randomized subspace regularized Newton method (RS-RNM) for the following unconstrained minimization problem,

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f$ is a twice continuously differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$. In what follows, we denote the gradient $\nabla f(x_k)$ and the Hessian $\nabla^2 f(x_k)$ as $g_k$ and $H_k$, respectively.

In [26], the authors develop one of the regularized Newton methods (RNM):

$$x_{k+1} = x_k - t_k(H_k + c_1 \Lambda_k I_n + c_2 \|g_k\|^{\gamma'} I_n)^{-1} g_k,$$

where $\Lambda_k = \max(0, -\lambda_{\min}(H_k))$, $c_1, c_2, \gamma'$ are some positive parameter values and $t_k$ is the step-size chosen by Armijo’s step size rule, and show that this algorithm achieves $\|g_k\| \leq \varepsilon$ after at most $O(\varepsilon^{-2})$ iterations and it has a super-linear rate of convergence in a neighborhood of a local optimal solution under appropriate conditions.

To increase the computational efficiency of this algorithm using random projections, based on the randomized subspace Newton method [11], we propose the randomized subspace regularized Newton method (RS-RNM) with Armijo’s rule, which
Since $\Omega$ is a bounded set and (4.1), there exists $t_k = \beta t_k, x_{k+1} = x_k + t_k d_k$ and $k \leftarrow k + 1$

until some stopping criteria is satisfied

return the last iterate $x_k$

The level set of $f(\Omega) := \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$ is bounded.

Assumption 4.1. The level set of $f$ at the initial point $x_0$ is bounded, i.e., $\Omega := \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$ is bounded.

By (3.4), we have that for any $k \in \mathbb{N}$, $f(x_{k+1}) \leq f(x_k)$, implying all $x_k \in \Omega$. Since $\Omega$ is a bounded set and $f$ is continuously differentiable, there exists $U_g > 0$ such that

\[ \|g_k\| \leq U_g, \forall k \geq 0. \]
Similarly, there exists $L > 0$ such that for all $x \in \Omega$,
\begin{equation}
\|\nabla^2 f(x)\| \leq L.
\end{equation}
In particular, for all $k > 0$,
\begin{equation}
\|H_k\| \leq L.
\end{equation}
Notice that by (4.2), $\nabla f$ is $L$-Lipschitz continuous. We also define $f^\ast = \inf_{x \in \Omega} f(x)$.

4.1. Global convergence. We first show that the norm of $d_k$ can be bounded from above.

**Lemma 4.2.** Suppose that $\|d_k\| \neq 0$. Then, $d_k$ defined by 3.3 satisfies
\begin{equation}
\|d_k\| \leq C \frac{n}{s} \frac{\|g_k\|^{1-\gamma}}{c_2},
\end{equation}
with probability at least $1 - 2e^{-s}$.

**Proof.** By Lemma 2.2 we have $\|P_k^T P_k\| \leq C \frac{n}{s}$, holds with probability at least $1 - 2e^{-s}$. Then, it follows from 3.3 that
\begin{align*}
\|d_k\| &= \|P_k^T M_k^{-1} P_k g_k\| \\
&= \|P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k g_k\| \\
&\leq \|P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k\| \|g_k\| \\
&\leq \|P_k^T\| \|P_k\| \|(P_k H_k P_k^T + \eta_k I_s)^{-1}\| \|g_k\| \\
&= \frac{\|P_k^T P_k\| \|g_k\|}{\lambda_{\min}(P_k H_k P_k^T + c_1 \Lambda_k I_s + c_2 \|g_k\|^\gamma I_s)} \\
&\leq C \frac{n}{s} \frac{\|g_k\|^{1-\gamma}}{c_2}.
\end{align*}

We next show that, when $\|g_k\|$ is at least $\varepsilon$ away from 0, $\|d_k\|$ is bounded above by some constant.

**Lemma 4.3.** Suppose that Assumption 4.1 holds. Suppose also that there exists $\varepsilon > 0$ such that $\|g_k\| \geq \varepsilon$. Then, with probability at least $1 - 2e^{-s}$, $d_k$ defined by 3.3 satisfies
\begin{equation}
\|d_k\| \leq r(\varepsilon),
\end{equation}
where
\[ r(\varepsilon) := \frac{\bar{C}_n}{c_2 s} \max \left( U_g^{1-\gamma}, \frac{1}{\varepsilon^{\gamma-1}} \right). \]

**Proof.** If $\gamma \leq 1$, it follows from Lemma 4.2 and 4.1 that
\[ \|d_k\| \leq \frac{\bar{C}_n}{s} \frac{U_g^{1-\gamma}}{c_2}. \]
Meanwhile, if $\gamma > 1$, it follows from Lemma 4.2 and $\|g_k\| \geq \varepsilon$ that
\[ \|d_k\| \leq \frac{\bar{C}_n}{s} \frac{1}{c_2 \varepsilon^{\gamma-1}}. \]
This completes the proof.
When \( \|g_k\| \geq \varepsilon \), we have from Lemma 4.3 that
\[
x_k + \tau d_k \in \Omega + B(0, r(\varepsilon)), \forall \tau \in [0, 1].
\]
By boundedness of \( \Omega + B(0, r(\varepsilon)) \) and by using the fact that \( f \) is twice continuously differentiable, we deduce that there exists \( U_H(\varepsilon) > 0 \) such that
\[
\|\nabla^2 f(x)\| \leq U_H(\varepsilon), \forall x \in \Omega + B(0, r(\varepsilon)).
\]

The following lemma indicates that a step size smaller than some constant satisfies Armijo’s rule when \( \|g_k\| \geq \varepsilon \).

**Lemma 4.4.** Suppose that Assumption 4.1 holds. Suppose also that there exists \( \varepsilon > 0 \) such that \( \|g_k\| \geq \varepsilon \). Then, with probability at least \( 1 - 2e^{-\gamma} \), a step size \( t_k' > 0 \) such that
\[
t_k' \leq \frac{2(1 - \alpha)c_2^2 \varepsilon^2 \gamma s}{((1 + c_1) \tilde{C} n U_H(\varepsilon) + c_2 U_T^2) U_H(\varepsilon) \tilde{C} n}
\]
satisfies Armijo’s rule, i.e.,
\[
f(x_k) - f(x_k + t_k' d_k) \geq -\alpha t_k' g_k^T d_k.
\]

**Proof.** From Taylor’s theorem, there exists \( \tau_k' \in (0, 1) \) such that
\[
f(x_k + t_k' d_k) = f(x_k) + \alpha t_k' g_k^T d_k + \frac{1}{2} t_k'^2 d_k^T \nabla^2 f(x_k + \tau_k' t_k' d_k) d_k.
\]
Then, we have
\[
\text{where the first inequality derives from the fact that}
\]
\[
g_k^T P_k^T M_k^{-1} P_k \nabla^2 f(x_k + \tau_k' t_k' d_k) P_k^T M_k^{-1} P_k g_k \leq \lambda_{\max}(M_k^{-1} P_k \nabla^2 f(x_k + \tau_k' t_k' d_k) P_k^T M_k^{-1}) \|P_k g_k\|^2
\]
\[
\leq \lambda_{\max}(\nabla^2 f(x_k + \tau_k' t_k' d_k)) \lambda_{\max}(M_k^{-1} P_k P_k^T M_k^{-1}) \|P_k g_k\|^2.
\]
By Lemma 2.2, we have that, with probability at least \( 1 - 2e^{-\gamma} \), \( \|P_k P_k^T\| \leq \frac{\tilde{C} n}{\varepsilon} \).
\[
f(x_k) - f(x_k + t_k' d_k) + \alpha t_k' g_k^T d_k
\]
\[
= (\alpha - 1)t_k' g_k^T d_k - \frac{1}{2} t_k'^2 \|d_k\|^2 \nabla^2 f(x_k + \tau_k' t_k' d_k) d_k
\]
\[
= (1 - \alpha)t_k' g_k^T P_k^T M_k^{-1} P_k g_k - \frac{1}{2} t_k'^2 g_k^T P_k^T M_k^{-1} P_k \nabla^2 f(x_k + \tau_k' t_k' d_k) P_k^T M_k^{-1} P_k g_k
\]
\[
\geq (1 - \alpha) t_k' \lambda_{\min}(M_k^{-1}) \|P_k g_k\|^2
\]
\[
- \frac{1}{2} t_k'^2 \lambda_{\max}(\nabla^2 f(x_k + \tau_k' t_k' d_k)) \lambda_{\max}(M_k^{-1} P_k P_k^T M_k^{-1}) \|P_k g_k\|^2
\]
\[
\geq (1 - \alpha) t_k' \lambda_{\min}(M_k^{-1}) \|P_k g_k\|^2 - \frac{1}{2} t_k'^2 U_H(\varepsilon) \lambda_{\max}(M_k^{-1} P_k P_k^T M_k^{-1}) \|P_k g_k\|^2,
\]
(by 3.3)
\[
\geq (1 - \alpha) t_k' \lambda_{\min}(M_k^{-1}) \|P_k g_k\|^2 - \frac{1}{2} t_k'^2 U_H(\varepsilon) \lambda_{\max}(M_k^{-1} P_k P_k^T M_k^{-1}) \|P_k g_k\|^2,
\]
(by 4.5)
In addition, we have \( \|H_k\| \leq U_H(\varepsilon) \) from 4.5, which gives us \( \|P_k H_k P_k^T\| \leq \frac{\tilde{C} n}{\varepsilon} U_H(\varepsilon) \).
For these reasons, we obtain evaluation of the values of \( \lambda_{\min}(M_k^{-1}) \) and \( \lambda_{\max}(M_k^{-1}P_kP_k^T M_k^{-1}) \):

\[
\lambda_{\min}(M_k^{-1}) = \frac{1}{\lambda_{\max}(M_k)} = \frac{1}{\lambda_{\max}(P_kH_kP_k^T + c_1\Lambda_k I_s + c_2 \|g_k\|^7 I_s)} \geq \frac{c_n}{s} U_H(\epsilon) + c_1 \frac{c_n}{s} U_H(\epsilon) + c_2 \|g_k\|^7,
\]

\[
\lambda_{\max}(M_k^{-1}P_kP_k^T M_k^{-1}) \leq \|P_kP_k^T\| \lambda_{\max}(M_k^{-1})^2 \leq \frac{\bar{C}_n}{s} \lambda_{\min}(P_kH_kP_k^T + c_1\Lambda_k I_s + c_2 \|g_k\|^7 I_s)^2 \leq \frac{\bar{C}_n}{s} \frac{1}{c_2^2 \|g_k\|^{2\gamma}},
\]

so that we have

\[
f(x_k) - f(x_k + t_k' d_k) + \alpha t_k' g_k^T d_k \geq \frac{(1 - \alpha)t_k'}{c_n \bar{C}_n} \|P_k g_k\|^2 - \frac{1}{2} t_k' \frac{\bar{C}_n}{s} U_H(\epsilon) \|P_k g_k\|^2 \geq \frac{(1 - \alpha)t_k'}{c_n \bar{C}_n} \|P_k g_k\|^2 - \frac{1}{2} t_k' \frac{\bar{C}_n}{s} U_H(\epsilon) \|P_k g_k\|^2 \quad \text{(by 4.1 and } \|g_k\| \geq \epsilon) \]

\[
= \frac{\bar{C}_n \gamma n}{2c_2^2 \epsilon^{2\gamma} s} t_k' \left( \frac{2(1 - \alpha)c_2^2 \epsilon^{2\gamma} s}{((1 + c_1) \frac{c_n}{s} U_H(\epsilon) + c_2 U_H(\epsilon)) \bar{C}_n} - t_k' \right) \|P_k g_k\|^2 \geq 0,
\]

which completes the proof.

As a consequence of this lemma, it turns out that the step size \( t_k \) used in RS-RNM can be bounded from below by some constant.

**Corollary 4.5.** Suppose that Assumption 4.1 holds. Suppose also that there exists \( \epsilon > 0 \) such that \( \|g_k\| \geq \epsilon \). Then, with probability at least, \( 1 - 2e^{-s} \) the step size \( t_k \) chosen in Line 6 of RS-RNM satisfies

\[
t_k \geq t_{\min}(\epsilon),
\]

where

\[
t_{\min}(\epsilon) = \min \left( 1, \frac{2(1 - \alpha)c_2^2 \epsilon^{2\gamma} s}{((1 + c_1) \frac{c_n}{s} U_H(\epsilon) + c_2 U_H(\epsilon)) \bar{C}_n} \right).
\]

**Proof.** Proof. If

\[
\frac{2(1 - \alpha)c_2^2 \epsilon^{2\gamma} s}{((1 + c_1) \frac{c_n}{s} U_H(\epsilon) + c_2 U_H(\epsilon)) \bar{C}_n} > 1,
\]

then...
we know that \( t_k = 1 \) satisfies Armijo’s rule 3.4 from 4.4. If not, there exists \( l_k \in \{0, 1, 2, \ldots\} \) such that

\[
\beta_{l_k+1} < \frac{2(1 - \alpha)c^2 \varepsilon^2 s}{(1 + c_1)\epsilon_2 U_H(\varepsilon) + c_2 U_0} \leq \beta_{l_k},
\]

and by Lemma 4.4, we have that the step size \( \beta_{l_k+1} \) satisfies Armijo’s rule 3.4. Then, from the definition of \( \beta_{l_k} \) in Line 6 of RS-RNM, we have

\[
t_k = \beta_{l_k} \geq \beta_{l_k+1} = \beta \cdot \beta_{l_k} \geq \frac{2(1 - \alpha)\beta c^2 \varepsilon^2 s}{(1 + c_1)\epsilon_2 U_H(\varepsilon) + c_2 U_0}.
\]

This completes the proof. \( \square \)

Using Corollary 4.5, we can show the global convergence of RS-RNM under Assumptions 4.1.

**Theorem 4.6.** Suppose that Assumption 4.1 holds. Let

\[
m = \left\lfloor \frac{f(x_0) - f^*}{p(\varepsilon)\varepsilon^2} \right\rfloor + 1, \quad \text{where} \quad p(\varepsilon) = \frac{\alpha \lambda_{\min}(\varepsilon)}{2C(1 + c_1)\epsilon_2 U_H(\varepsilon) + 2c_2 U_0}.
\]

Then, with probability at least \( 1 - 2m \left( \exp\left(\frac{C}{16} s\right) + \exp(-s)\right) \), there exists \( k \in \{0, 1, \ldots, m-1\} \) such that \( \|g_k\| < \varepsilon \).

**Proof.** We first notice that, by Lemma 2.1, applied with \( \varepsilon = 1/2 \), and Lemma 2.2, we have, using (2.1), that \( \|P_k g_k\|^2 \geq \frac{1}{2} \|g_k\|^2 \) and \( \|P_k P_k^T\| \leq C_{||\gamma||} \) holds for all \( k \in \{0, 1, \ldots, m-1\} \) with the given probability.

Suppose, for the sake of contradiction, that \( \|g_k\| \geq \varepsilon \) for all \( k \in \{0, 1, \ldots, m-1\} \). From Armijo’s rule 3.4, we can estimate how much the function value decreases in one iteration:

\[
f(x_k) - f(x_{k+1}) \geq -\alpha t^T g^T d_k
\]

\[
= \alpha t^T g^T P_k^T M_k^{-1} P_k g_k
\]

\[
\geq \alpha t_{\min}(M_k^{-1}) \|P_k g_k\|^2
\]

\[
\geq \frac{\alpha t_{\min}(\varepsilon)}{2(1 + c_1)\epsilon_2 U_H(\varepsilon) + 2c_2 \|g_k\|^2} \|g_k\|^2
\]

\[
\geq p(\varepsilon)\varepsilon^2.
\]

(by \( \|P_k g_k\|^2 \geq \frac{1}{2} \|g_k\|^2 \))

\[
\geq p(\varepsilon)\varepsilon^2
\]

Adding up all these inequalities from \( k = 0 \) to \( k = m-1 \), we get

\[
f(x_0) - f(x_m) \geq mp(\varepsilon)\varepsilon^2,
\]

and thus,

\[
f(x_0) - f^* \geq f(x_0) - f(x_m)
\]

\[
\geq mp(\varepsilon)\varepsilon^2
\]

\[
> \frac{f(x_0) - f^*}{p(\varepsilon)\varepsilon^2} \cdot p(\varepsilon)\varepsilon^2
\]

\[
= f(x_0) - f^*.
\]
which is a contradiction.

Because of the dependency of $p(\varepsilon)$ on $\varepsilon$, the above discussion can not lead to the iteration complexity analysis, as we need to quantify the exact dependency of the iteration complexity bound with respect to $\varepsilon$. This will be done, under a few additional assumptions, in the next subsection.

4.2. Global iteration complexity. We now estimate the global iteration complexity of the RS-RNM under Assumption 4.1 and the following assumption.

**Assumption 4.7.**
(i) $\gamma \leq 1/2$,
(ii) $\alpha \leq 1/2$,
(iii) There exists $L_H > 0$ such that
\[
\| \nabla^2 f(x) - \nabla^2 f(y) \| \leq L_H \| x - y \|, \quad \forall x, y \in \Omega + B(0, r_1),
\]
where $r_1 := \frac{\bar{C}_U^{1-\gamma_n}}{c_2 s}$.

The following inequality can also be derived from (iii):

\[
\|
\nabla^2 f(y)(y - x) - (\nabla f(x) - \nabla f(y))
\| \leq \frac{L_H}{2} \| x - y \|^2, \quad \forall x, y \in \Omega + B(0, r_1).
\]

From the definition of $r_1$ in (iii), Lemma 4.2 and 4.1, we have
\[
\| d_k \| \leq \frac{\bar{C}_n}{s} \| g_k \|^{1-\gamma} \leq \frac{\bar{C}_n}{s} \frac{U_g^{1-\gamma}}{c_2} = r_1.
\]

Note that unlike (4.4), the bound has no dependency on $\varepsilon$. For this reason, we have
\[
x_k + \tau d_k \in \Omega + B(0, r_1), \quad \forall \tau \in [0, 1].
\]
Moreover, since $\Omega + B(0, r_1)$ is bounded and $f$ is twice continuously differentiable, there exists $U_H > 0$ such that

\[
\|
\nabla^2 f(x)
\| \leq U_H, \quad \forall x \in \Omega + B(0, r_1).
\]

Similar to the result of Lemma 4.4, we can show that a step size smaller than some constant satisfies Armijo’s rule and therefore, $t_k$ can be bounded from below by some constant.

**Lemma 4.8.** Suppose that Assumptions 4.1 and 4.7 hold. Then, with probability at least $1 - 2e^{-s}$, a step size $t_k' > 0$ such that
\[
t_k' \leq \min \left( 1, \frac{c_2 s^2}{L_H U_g^{1-2\gamma n^2}} \right),
\]
satisfies Armijo’s rule, i.e.,
\[
f(x_k) - f(x_k + t_k' d_k) \geq -\alpha t_k' g_k^T d_k.
\]
Proof. Proof. As 4.6 is obtained in the proof of Lemma 4.4, there exists $\tau^*_k \in (0, 1)$ such that
\[
f(x_k) - f(x_k + t'_k d_k) + \alpha t'_k g_k^T d_k = (1 - \alpha) t'_k g_k^T P_k^T M_k^{-1} P_k g_k - \frac{1}{2} t'_k^2 g_k^T P_k^T M_k^{-1} P_k \nabla^2 f(x_k + \tau'_k t'_k d_k) P_k^T M_k^{-1} P_k g_k.
\]

Since we have $1 - \alpha \geq 1/2 \geq t'_k/2$ from Assumption 4.7 (ii), we obtain
\[
f(x_k) - f(x_k + t'_k d_k) + \alpha t'_k g_k^T d_k \\
\geq \frac{1}{2} t'_k^2 g_k^T P_k^T M_k^{-1} P_k g_k - \frac{1}{2} t'_k^2 g_k^T P_k^T M_k^{-1} P_k \nabla^2 f(x_k + \tau'_k t'_k d_k) P_k^T M_k^{-1} P_k g_k \\
= \frac{1}{2} t'_k^2 g_k^T P_k^T M_k^{-1} P_k (1 - M_k^{-1} P_k H_k P_k^T M_k^{-1}) P_k g_k \\
\geq \frac{1}{2} t'_k^2 \eta_k M_k^{-1} P_k g_k.
\]

We next evaluate the first and second terms respectively. Since we have
\[
M_k^{-1} - M_k^{-1} P_k H_k P_k^T M_k^{-1} = M_k^{-1} (M_k^{-1} (M_k - \eta_k I) M_k^{-1}) \\
= \eta_k (M_k^{-1})^2,
\]
the first term can be bounded as follows:
\[
\frac{1}{2} t'_k^2 g_k^T P_k^T (M_k^{-1} - M_k^{-1} P_k H_k P_k^T M_k^{-1}) P_k g_k = \frac{1}{2} t'_k^2 \eta_k M_k^{-1} P_k g_k \\
\geq \frac{1}{2} t'_k^2 c_2 \eta_k M_k^{-1} P_k g_k^2.
\]

Using Lemma 2.2 and Assumption 4.7 (iii), we also obtain, with probability at least $1 - 2e^{-x}$, the bound of the second term:
\[
\frac{1}{2} t'_k^2 g_k^T P_k (\nabla^2 f(x_k + \tau'_k t'_k d_k) - H_k) P_k^T M_k^{-1} P_k g_k \\
\leq \frac{1}{2} t'_k^2 \|\nabla^2 f(x_k + \tau'_k t'_k d_k) - H_k\| P_k P_k^T \| M_k^{-1} P_k g_k \|^2 \\
\leq \frac{c_n}{2s} \| \nabla^2 f(x_k + \tau'_k t'_k d_k) - H_k \| P_k P_k^T \| M_k^{-1} P_k g_k \|^2.
\]

Thus, we have
\[
f(x_k) - f(x_k + t'_k d_k) + \alpha t'_k g_k^T d_k \geq \frac{1}{2} t'_k^2 \left( c_2 \| g_k \|^\gamma - \frac{c_n}{s} \| \nabla^2 f(x_k + \tau'_k t'_k d_k) - H_k \| P_k P_k^T \| M_k^{-1} P_k g_k \|^2 \right) \\
= \frac{c_n}{2s} \| \nabla^2 f(x_k + \tau'_k t'_k d_k) - H_k \| P_k P_k^T \| M_k^{-1} P_k g_k \|^2.
\]

Moreover, from (4.1), (4.2) and Assumption 4.7 (i), we have
\[
\| g_k \|^{\gamma} \geq \frac{c_2 s}{c_n} \| g_k \|^{\gamma-2\gamma} \geq \frac{c_2 s}{CU^2_{\beta} n},
\]
so that we finally obtain
\[
f(x_k) - f(x_k + t'_k d_k) + \alpha t'_k g_k^T d_k \geq \frac{c_n}{2s} \| \nabla^2 f(x_k + \tau'_k t'_k d_k) - H_k \| P_k P_k^T \| M_k^{-1} P_k g_k \|^2 \geq 0.
\]
This completes the proof. \qed
Corollary 4.9. Suppose that Assumptions 4.1 and 4.7 hold. Then, with probability at least \(1 - 2e^{-s}\), the step size \(t_k\) chosen in Line 6 of RS-RNM satisfies
\[
t_k \geq t_{\min},
\]
where
\[
t_{\min} = \min \left(1, \frac{\beta c_2^2 s^2}{C^2 L_H U_\eta^{1-2s_2 n^2}}\right).
\]

Proof. We get the conclusion in the same way as in the proof of Corollary 4.5 using Lemma 4.8. \(\square\)

Remark 4.10. Since 4.13 is equivalent to \(\beta l_k \geq t_{\min}\), and moreover \(l_k \leq \log t_{\min}/\log \beta\), Corollary 4.9 tells us that the number of the backtracking steps is bounded above by some constant independent of \(k\).

Now, we can obtain the global iteration complexity of RS-RNM.

Theorem 4.11. Suppose that Assumptions 4.1 and 4.7 hold. Let
\[
m = \left\lceil \frac{f(x_0) - f^*}{p\varepsilon^2} \right\rceil + 1, \quad \text{where} \quad p = \frac{\alpha t_{\min}}{2C(1 + c_1)n^2 U_H + 2c_2 U_\eta^2}.
\]
Then, with probability at least
\[
1 - 2m \left( \exp(-\frac{C_0}{4} s) - \exp(-s) \right),
\]
we have
\[
\sqrt{\frac{f(x_0) - f^*}{mp}} \geq \min_{k=0,1,\ldots,m-1} \|g_k\|.
\]

Proof. Proof. Replacing \(U_H(\varepsilon)\) and \(t_{\min}(\varepsilon)\) with \(U_H\), in (4.10), and \(t_{\min}\) respectively in the argument in the proof of Theorem 4.6, we have
\[
f(x_k) - f(x_{k+1}) \geq p \|g_k\|^2 \quad (k = 0, 1, \ldots, m - 1),
\]
with the given probability. Therefore, we obtain
\[
f(x_0) - f^* \geq f(x_0) - f(x_m) \geq \sum_{k=0}^{m-1} (f(x_k) - f(x_{k+1})) \geq p \sum_{k=0}^{m-1} \|g_k\|^2 \geq mp \left( \min_{k=0,1,\ldots,m-1} \|g_k\| \right)^2,
\]
which is equivalent to the conclusion. \(\square\)

If we ignore the probability, Theorem 4.11 shows that we get \(\|g_k\| \leq \varepsilon\) after at most \(O(\varepsilon^{-2})\) iterations. This global complexity \(O(\varepsilon^{-2})\) is the same as that obtained in [26] for the regularized Newton method.
4.3. Discussion on the probability bound. Let us now discuss the probability bound, (4.14), that appears in Theorem 4.11. In the following proposition, we prove that, under some basics assumptions, this probability is in the order of $1 - o(1)$ by taking $s = O(\log^2(n))$.

**Proposition 4.12.** Assuming that $f, \nabla f$ and $\nabla^2 f$ are respectively $L_f, L_g$ and $L_H$-Lipschitz on $\Omega$ and that the value $\sup_{k}\|x_k - x^*\|_{\infty}$, where $x^*$ denotes an optimal solution, is independent of the dimension $n$. Then by choosing $s = O(\log^2(n))$, we have that the probability of success of Theorem 4.11 tends to 1 as $n$ tends to infinity.

Proof. Notice that since $f$ is Lipschitz, and that $\|x_0 - x^*\| \leq \|x_0 - x^*\|_{\infty}\sqrt{n}$, we have that

$$f(x_0) - f^* \leq L_f\|x_0 - x^*\|_{\infty}\sqrt{n}$$

for some constant. Hence assuming that $\|x_0 - x^*\|_{\infty}$ is independent of the ambient dimension $n$, we deduce that

$$f(x_0) - f^* \leq \hat{L}\sqrt{n},$$

for some constant $\hat{L}$. Similarly since

$$p = \frac{\alpha t_{\min}}{2\tilde{C}(1 + c_1)\frac{\|U_H\|}{2} + 2c_2U_g^2} \quad \& \quad t_{\min} = \min \left(1, \frac{\beta c_2s^2}{\bar{C}^2L_HU_1^{1-2\gamma}n^2}\right),$$

we have that

$$\frac{1}{p} \leq \max \left(1, \frac{\beta c_2s^2}{\bar{C}^2L_HU_1^{1-2\gamma}n^2}\right) \frac{2\tilde{C}(1 + c_1)\frac{\|U_H\|}{2} + 2c_2U_g^2}{\alpha}.$$

Since $\nabla f(x)$ is $L_g$-Lipschitz and $\bar{x}$ is a stationary point, we have that $\|\nabla f(x_k)\| \leq L_g\|x_k - \bar{x}\|_2 \leq L_g\sqrt{n}\|x_k - \bar{x}\|_{\infty}$.

Furthermore we also have that $\|\nabla^2 f(x_k)\| \leq \|\nabla^2 f(\bar{x})\| + L_H\sqrt{n}\|x_k - \bar{x}\|_{\infty}$. Hence $U_g$ in (4.1) and $U_H$ in (4.10) can be bounded by $O(\sqrt{n})$. We can therefore bound $m$ by $\tilde{C}\frac{n^2}{\alpha}$ for some constant $\tilde{C}$ large enough. Hence, by taking $s = \tilde{C}\log^2(n)$ for any constant $\tilde{C}$, we see that for such $s$, the term $2m \left(\exp(-\frac{\tilde{C}}{2}s) - \exp(-s)\right)$ in (4.14) tends to 0 as $n$ tends to infinity. 

Notice that, by a cubic regularization, the R-ARC algorithm in [24] achieves $O(\varepsilon^{-3/2})$ to obtain a first order stationary point.

5. Local convergence. In this section, we investigate local convergence properties of the sequence $\{x_k\}$ assuming that it converges to a strict local minimal $\bar{x}$. First we will show that the sequence converges locally linearly to the strict local minimizer; then we will prove that, when $f$ is strongly convex, we cannot aim at local super-linear convergence using random subspace. Finally, we will prove that when the Hessian at $\bar{x}$ is rank deficient then we can attain super-linear convergence for $s < n$ large enough.

**Assumption 5.1.** For all $x, y$

$$\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L_H\|x - y\|$$

holds in some neighborhood $B_H$ of $\bar{x}$.
5.1. Local linear convergence. In this subsection we will show the sequence \( \{f(x_k) - f(\bar{x})\} \) converges locally linearly, i.e. there exists \( \kappa \in (0,1) \) such that for \( k \) large enough,
\[
 f(x_{k+1}) - f(\bar{x}) \leq (1 - \kappa)(f(x_k) - f(\bar{x})).
\]

We will further prove that \( \kappa \) can be expressed as \( \kappa = O(\frac{s}{n\hat{\kappa}(\nabla^2 f(\bar{x}))}) \), where \( \hat{\kappa}(\nabla^2 f(\bar{x})) \) is the ratio of the largest eigenvalue value over the smallest non-zero eigenvalue of \( \nabla^2 f(\bar{x}) \). Notice that, to the best of our knowledge, until now, local linear convergence is always proved for subspace algorithms assuming that the function is locally strongly convex or satisfies some PL-inequality (1.3). In this section we prove that under a Hölderian error bound condition, and an additional mild assumptions on the rank of the Hessian at the local minimizer, we can prove local linear convergence. More precisely let us denote by \( r = \text{rank}(\nabla^2 f(\bar{x})) \), which measures the number of positive eigenvalues of \( \nabla^2 f(\bar{x}) \). We will first prove that, for any \( \sigma \in (0,1) \), there exists \( n_0 \) such that if \( n \geq n_0 \) and \( r \geq \sigma n \), then for any \( x \) in the a neighborhood of \( \bar{x} \), the function
\[
 \tilde{f}_x : u \mapsto f(x + P^\top u)
\]
is strongly convex with high probability in a neighborhood of 0. Let us fix \( \sigma \in (0,1) \). We recall here that \( P \in \mathbb{R}^{s \times n} \) is equal to \( \frac{1}{\sqrt{s}} \) times a random Gaussian matrix. In this subsection, we make the following additional assumptions:

**Assumption 5.2.** We have that \( s = o(n) \), that is, \( \lim_{n \to +\infty} \frac{s}{n} = 0 \).

**Assumption 5.3.**

(i) There exists \( \sigma \in (0,1) \) such that \( r = \text{rank}(\nabla^2 f(\bar{x})) \geq \sigma n \).

(ii) There exist \( \rho \in (0,3) \) and \( \tilde{C} \) such that in a neighborhood of \( \bar{x} \), \( f(x_k) - f(\bar{x}) \geq \tilde{C}\|x_k - \bar{x}\|^\rho \) holds.

From Assumption 5.3 (i), \( \nabla^2 f(\bar{x}) \) has \( r \) positive eigenvalues, i.e, \( \lambda_1(\bar{x}) \geq \cdots \lambda_r(\bar{x}) > 0 \). By continuity of the eigenvalues, there exists a neighborhood \( \tilde{B} \) of \( \bar{x} \) such that for any \( x \in \tilde{B} \), \( \lambda_r(x) \geq \frac{\lambda_r(\bar{x})}{2} \). Here, we assume, w.l.o.g. that \( \tilde{B} \subseteq B_H \), where \( B_H \) is defined in Assumption 5.1. Let us denote
\[
 \tilde{\lambda} := \frac{\lambda_r(\bar{x})}{2}.
\]

Assumption 5.3 (ii) is called a Hölderian growth condition or a Hölderian error bound condition [14]. The condition is weaker than local strong convexity in the sense that it holds with \( \rho = 2 \) if \( f \) is locally strongly convex.

**Proposition 5.4.** Let \( 0 < \varepsilon_0 < 1 \). Then under Assumptions 5.2 and 5.3 (i), there exist \( n_0 \in \mathbb{N} \) (which depends only on \( \varepsilon_0 \) and \( \sigma \)) and a neighborhood \( B^* \subseteq \tilde{B} \) such that if \( n \geq n_0 \), for any \( x \in B^* \),
\[
 \nabla^2 \tilde{f}_x(0) \succeq \frac{(1 - \varepsilon_0)^2 n}{2s} \sigma \tilde{\lambda} I_s
\]
holds with probability at least \( 1 - 6 \exp(-s) \).

**Proof.** Proof. Let \( x \in \tilde{B} \) be fixed. Because of \( \nabla^2 \tilde{f}_x(0) = P\nabla^2 f(x)P^\top \), we have \( u^\top \nabla^2 \tilde{f}_x(0) u = (P^\top u)^\top \nabla^2 f(x)(P^\top u) \) for any \( u \in \mathbb{R}^s \). Let \( \nabla^2 f(x) = U(x)D(x)U(x)^\top \) be the eigenvalue decomposition of \( \nabla^2 f(x) \). Since \( \nabla^2 f_x(0) = (PU(x))D(x)(PU(x))^\top \)
and $PU(x)$ has the same distribution as $P$, we can assume here w.l.o.g. that $PU(x) = P$. Here

$$D(x) = \begin{pmatrix} \lambda_1(x) & 0 & \cdots & 0 \\ 0 & \lambda_2(x) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(x) \end{pmatrix},$$

where $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ and $\lambda_r(x) \geq \bar{\lambda}$ (since $x \in \bar{B}$).

Let us decompose $P^\top$ such that

$$P^\top = \begin{pmatrix} P^1 \\ P^2 \end{pmatrix}$$

where $P^1 \in \mathbb{R}^{n_1 \times s}$ and $P^2 \in \mathbb{R}^{n_2 \times s}$, where $n_1$ and $n_2$ are chosen such that $n_1 = r$ and $n_2 = n - r$. Furthermore let $D_1(x)$ and $D_2(x)$ be respectively the $n_1 \times n_1$ and $n_2 \times n_2$ diagonal matrix such that $D(x) = \begin{pmatrix} D_1(x) & 0 \\ 0 & D_2(x) \end{pmatrix}$. We have

$$\begin{equation} (P^\top u)^\top D(x) (P^\top u) = (P^1 u)^\top D_1(x) (P^1 u) + (P^2 u)^\top D_2(x) (P^2 u). \end{equation}$$

By Assumption 5.3 (i), and by definition of $\bar{B}$, we have that $D_1(x) \succeq \lambda_r(x) I_{n_1} \succeq \bar{\lambda} I_{n_1} > 0$, and $D_2(x) \succeq \lambda_n(x) I_{n_2}$. Hence from (5.2), we have

$$\begin{equation} (P^\top u)^\top D(x) (P^\top u) \geq \bar{\lambda} \|P^1 u\|^2 + \lambda_n(x) \|P^2 u\|^2. \end{equation}$$

Let $\sigma_{\text{max}}(\cdot)$ and $\sigma_{\text{min}}(\cdot)$ denote respectively the largest and the smallest singular value of a matrix. Using [28, Theorem 4.6.1], there exists a constant $C$ such that with probability at least $1 - 6 \exp(-s)$:

$$\begin{align} \sqrt{n/s} - C &\leq \sigma_{\text{min}}(P^\top) \leq \sigma_{\text{max}}(P^\top) \leq \sqrt{n/s} + C, \\ \sqrt{n_1/s} - C &\leq \sigma_{\text{min}}(P^1) \leq \sigma_{\text{max}}(P^1) \leq \sqrt{n_1/s} + C, \\ \sqrt{n_2/s} - C &\leq \sigma_{\text{min}}(P^2) \leq \sigma_{\text{max}}(P^2) \leq \sqrt{n_2/s} + C. \end{align}$$

More precisely, since all the three matrices $P^\top, P^1$ and $P^2$ are Gaussian random matrices, we can apply [28, Theorem 4.6.1] and deduce that each of the three inequalities above holds with probability $1 - 2 \exp(-s)$. The probability that all the three equations hold is derived using (2.1). Hence, with probability at least $1 - 6e^{-s}$, for any $u \in \mathbb{R}^s$,

$$\|P^1 u\| \geq \sqrt{n/s} \left( \frac{\sqrt{n_1/s} - C}{\sqrt{n_1/s}} \right) \|u\|,$$

$$\|P^2 u\| \leq \sqrt{n/s} \left( \frac{\sqrt{n_2/s} + C}{\sqrt{n_2/s}} \right) \|u\|.$$

We have that $n_1 \geq \sigma n$ and $n_2 \leq (1 - \sigma)n$. Since $s = o(n)$ by Assumption 5.2, we also have $s = o(n_1)$. Hence

$$\lim_{n \to \infty} \sqrt{n/s} \sqrt{n_1/s} \geq \sqrt{\sigma} \quad \text{and} \quad \lim_{n \to \infty} \sqrt{n/s} \sqrt{n_2/s} \leq \sqrt{(1 - \sigma)}.$$
Let $0 < \varepsilon_0 < 1$, there exists $n_0$ such that if $n \geq n_0$, 
\[
\|P^1u\| \geq (1 - \varepsilon_0)\sqrt{\sigma(n/s)\|u\|}, \\
\|P^2u\| \leq \varepsilon_0\sqrt{(1 - \sigma)(n/s)\|u\|}.
\]

Hence, from (5.3), we have that 
\[
(P^T u)^\top D(x)(P^T u) \geq n/s ((1 - \varepsilon_0)^2 \sigma \lambda + \varepsilon_0^2 (1 - \sigma) \min(\lambda_n(x), 0)) \|u\|^2.
\]

We conclude the proposition by noticing that $\min(\lambda_n(x), 0)$ tends to 0, hence the claim holds by considering a neighborhood $B^* \subseteq \hat{B}$ of $\bar{x}$ small enough. \qed

We deduce the following PL inequality for $\hat{f}_x$ when $x \in B^*$.

**Proposition 5.5.** Assume that Assumptions 5.2 and 5.3 (i) hold, there exist $n_0 \in \mathbb{N}$ (which depends only on $\varepsilon_0$ and $\sigma$) and neighborhoods $\hat{B} \subset B^*$ and $B_0$ (a neighborhood of $0 \in \mathbb{R}^s$) such that if $n \geq n_0$, for any $x \in \hat{B}$,
\[
\nabla \hat{f}_x(0)^\top (P_k \nabla^2 f(x) P_k^\top)^{-1} \nabla \hat{f}_x(0) \geq f(x) - \min_{u \in B_0} f(x + P^T u)
\]
holds with probability at least $1 - 6 \exp(-s)$.

**Proof.** Proof. Let $\hat{B} \subset B^*$, and let $x \in \hat{B}$. By the Taylor expansion of $\hat{f}_x$ at 0 we have by Proposition 5.4 that for any $u \in \mathbb{R}^s$ such that $x + P^T u \in B^*$,
\[
f(x + P^T u) \geq f(x) + (P \nabla f(x))^\top u + \frac{1}{4} u^\top P_k \nabla^2 f(x) P_k^\top u.
\]

Let $B_0$ be a neighborhood of $0 \in \mathbb{R}^s$ such that $x + P^T u \in B^*$ for any $x \in \hat{B}$. Let $g(u) = (P \nabla f(x))^\top u + \frac{1}{4} u^\top P_k \nabla^2 f(x) P_k^\top u$. By the above inequality we have that
\[
\min_{u \in B_0} f(x + P^T u) \geq f(x) + \min_{u \in B_0} g(u).
\]

By Proposition 5.4 we know that for any $u \in \mathbb{R}^s$ such that $x + P^T u \in B^*$, $g$ is convex, and the minimum is attained at the point $u^*$ satisfying
\[
\nabla g(u^*) = P \nabla f(x) + \frac{1}{2} P_k \nabla^2 f(x) P_k^\top u^* = 0.
\]

Hence, since $\|\nabla f(x)\|$ tends to 0 as $x$ tends to $\bar{x}$, we can ensure, by taking $\hat{B}$ small enough, that $u^* \in B_0$. Hence
\[
\min_{u \in B_0} g(u) = -2(P_k \nabla f(x))^\top (P_k \nabla^2 f(x) P_k^\top)^{-1} P_k \nabla f(x) + \frac{1}{4} (P_k \nabla^2 f(x) P_k^\top)^{-1} P_k \nabla f(x)
\]
holds and (5.5) yields the desired inequality. \qed

Before proving local linear convergence, we prove the following technical proposition.

**Proposition 5.6.** Assume that Assumptions 5.1, 5.2, and 5.3 hold. There exists $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $k \geq k_0$, we have with probability $1 - 6 \exp(-s) + \exp(-\frac{\lambda_0}{2s})$:
\[
f(x_k) - \min_{u \in B_0} \hat{f}_x(u) \geq \frac{\lambda_0}{4 \lambda_{\max}(\nabla^2 f(x_k - f(\bar{x})),
\]
where $\lambda_0$ is the minimal non-zero eigenvalue of $\nabla^2 f(\bar{x})$. 

Proof. Using a Taylor expansion around $\bar{x}$, we have that for all $y \in \bar{B}$,

\begin{equation}
|f(y) - f(\bar{x}) - \frac{1}{2}(y - \bar{x})^T \bar{H}(y - \bar{x})| \leq L_H \|y - \bar{x}\|^3,
\end{equation}

where we define

\begin{equation}
\bar{H} := \nabla^2 f(\bar{x}).
\end{equation}

Also, for $u \in \mathbb{R}^d$ small enough, we have by setting $y = x_k + P_k^T u$ in (5.6), that for $k$ large enough such that $x_k + P_k^T u \in \bar{B}$,

\begin{align*}
&|f(x_k + P_k^T u) - f(\bar{x}) - \frac{1}{2}(x_k - \bar{x})^T \bar{H}(x_k - \bar{x}) - \frac{1}{2}u^T P_k \bar{H} P_k^T u - (P_k \bar{H}(x_k - \bar{x}))^T u| \\
&\leq L_H \|x_k - \bar{x} + P_k^T u\|^3
\end{align*}

holds.

Let $g(u) = \frac{1}{2}u^T P_k \bar{H} P_k^T u + (P_k \bar{H}(x_k - \bar{x}))^T u$. By a reasoning similar to that of Proposition 5.4, $g$ is strongly convex with probability $1 - 6e^{-s}$ and hence is minimized at

\begin{equation}
u^* = -(P_k \bar{H} P_k^T)^{-1} P_k \bar{H}(x_k - \bar{x}).
\end{equation}

Notice that as $k$ tends to infinity $\|u^*\|$ tends to 0, hence for $k$ large enough we have $x_k + P_k^T u^* \in \bar{B}$ and $u^* \in B_0$. Plugging (5.9) in (5.8) yields

\begin{align*}
f(x_k + P_k^T u^*) &\leq f(\bar{x}) + \frac{1}{2}(x_k - \bar{x})^T \bar{H}(x_k - \bar{x}) - \\
&\quad \frac{1}{2}(x_k - \bar{x})^T \bar{H} P_k (P_k \bar{H} P_k^T)^{-1} P_k \bar{H}(x_k - \bar{x}) + L_H \|x_k - \bar{x} + P_k^T u^*\|^3,
\end{align*}

from which we deduce

\begin{align*}
f(x_k) - f(x_k + P_k^T u^*) &\geq \\
f(x_k) - f(\bar{x}) - \frac{1}{2}(x_k - \bar{x})^T \bar{H}(x_k - \bar{x}) + \frac{1}{2}(x_k - \bar{x})^T \Pi(x_k - \bar{x}) - L_H \|x_k - \bar{x} + P_k^T u^*\|^3,
\end{align*}

where $\Pi = \bar{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \bar{H}$. Using (5.6), we further obtain

\begin{align}
f(x_k) - f(x_k + P_k^T u^*) &\geq \frac{1}{2}(x_k - \bar{x})^T \Pi(x_k - \bar{x}) - L_H (\|x_k - \bar{x} + P_k^T u^*\|^3 + \|x_k - \bar{x}\|^3).
\end{align}

We have $(x_k - \bar{x})^T \Pi(x_k - \bar{x}) = (\bar{H}^{1/2}(x_k - \bar{x}))^T \Pi(\bar{H}^{1/2}(x_k - \bar{x}))$, where $\Pi := \bar{H}^{1/2} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \bar{H}^{1/2}$ is an orthogonal projection matrix into $\text{Range}(\bar{H}^{1/2} P_k^T)$ parallel to $\ker P_k \bar{H}^{1/2}$. Hence

\begin{align*}
(x_k - \bar{x})^T \Pi(x_k - \bar{x}) = \|\Pi \bar{H}^{1/2}(x_k - \bar{x})\|^2.
\end{align*}
Since \( \|P_k \tilde{H}^{1/2}\||\|\Pi \tilde{H}^{1/2}(x_k - \bar{x})\|\|^2 \geq \|P_k \tilde{H}^{1/2} \Pi \tilde{H}^{1/2}(x_k - \bar{x})\|^2 \), we have

\[
(x_k - \bar{x})^\top \Pi(x_k - \bar{x}) \geq \frac{1}{\|P_k \tilde{H}^{1/2}\|^2} \|P_k \tilde{H}(x_k - \bar{x})\|^2
= \frac{1}{\|P_k \tilde{H}^{1/2}\|^2} \|P_k \tilde{H}(x_k - \bar{x})\|^2
\geq \frac{\lambda_0}{2\|P_k \tilde{H}^{1/2}\|^2} \|\tilde{H}^{1/2}(x_k - \bar{x})\|^2
= \frac{\lambda_0}{2\lambda_{\text{max}}(P_k \tilde{H} P_k)} \|\tilde{H}^{1/2}(x_k - \bar{x})\|^2
= \frac{\lambda_0}{2\lambda_{\text{max}}(P_k \tilde{H} P_k)} (x_k - \bar{x})^\top \tilde{H}(x_k - \bar{x}).
\]

(5.11)

where the second inequality holds, with probability at least \( 1 - 2\exp(-\frac{n}{2\|s\|^2}) \) (by Lemma 2.2, as \( \lambda_0 \) is the smallest non-zero eigenvalue of \( \tilde{H} \). The second equality holds as \( \sigma_{\text{max}}(P_k \tilde{H}^{1/2}) = \lambda_{\text{max}}(P_k \tilde{H} P_k) \). We have therefore proved that

\[
\tilde{H}^{1/2}(x_k - \bar{x})^\top \Pi(\tilde{H}^{1/2}(x_k - \bar{x})) \geq \frac{\lambda_0}{2\lambda_{\text{max}}(P_k \tilde{H} P_k)} (x_k - \bar{x})^\top \tilde{H}(x_k - \bar{x}).
\]

(5.12)

Hence, by (5.10), we have

\[
f(x_k) - f(x_k + P_k^\top u^*) \geq \frac{\lambda_0}{4\lambda_{\text{max}}(P_k \tilde{H} P_k)} (x_k - \bar{x})^\top \tilde{H}(x_k - \bar{x})
- L_H(\|x_k - \bar{x} + P_k^\top u^*\|^3 + \|x_k - \bar{x}\|^3).
\]

(5.13)

From (5.9), we have that \( \|x_k - \bar{x} + P_k^\top u^*\| = \|(I_n - P_k^\top (P_k \tilde{H} P_k^\top)^{-1} P_k \tilde{H})(x_k - \bar{x})\| \). Hence

\[
\|x_k - \bar{x} + P_k^\top u^*\| \leq \|I_n - P_k^\top (P_k \tilde{H} P_k^\top)^{-1} P_k \tilde{H}\|\|x_k - \bar{x}\|.
\]

(5.14)

Since \( P_k^\top (P_k \tilde{H} P_k^\top)^{-1} P_k \tilde{H} \) is projection matrix (along \( \text{Im}(P_k^\top) \) parallel to \( \text{Ker}(P_k \tilde{H}) \)), we have by [1] that

\[
\|I_n - P_k^\top (P_k \tilde{H} P_k^\top)^{-1} P_k \tilde{H}\| = \|P_k^\top (P_k \tilde{H} P_k^\top)^{-1} P_k \tilde{H}\|.
\]

(5.15)

Furthermore, by Proposition 5.4, we have that with probability at least \( 1 - 6\exp(-s) \),

\[
P_k \tilde{H} P_k^\top \geq \frac{(1 - \varepsilon_0)^2 n}{2s} \sigma \tilde{\lambda} I_s.
\]

Hence, we deduce from (5.15) that

\[
\|I_n - P_k^\top (P_k \tilde{H} P_k^\top)^{-1} P_k \tilde{H}\| \leq \frac{\|P_k^\top\|^2 \|\tilde{H}\|}{(1 - \varepsilon_0)^2 n \sigma \tilde{\lambda}}.
\]

(5.16)

Therefore, we deduce by (5.13), (5.14) and (5.16) for \( \beta_1 > 0 \) suitably chosen, we have

\[
f(x_k) - f(x_k + P_k^\top u^*) \geq \frac{\lambda_0}{4\lambda_{\text{max}}(P_k \tilde{H} P_k)} (x_k - \bar{x})^\top \tilde{H}(x_k - \bar{x}) - \beta_1 \|x_k - \bar{x}\|^3.
\]

(5.17)

By taking \( y = x_k \) in (5.6), we have that

\[
\frac{1}{2} (x_k - \bar{x})^\top \tilde{H}(x_k - \bar{x}) \geq f(x_k) - f(\bar{x}) - L_H \|x_k - \bar{x}\|^3.
\]
Hence, by 5.17

\[ f(x_k) - f(x_k + P_k^T u^*) \geq \frac{\lambda_0}{2\lambda_{\max}(P_k^T HP_k)} (f(x_k) - f(\bar{x})) - \left( \frac{\lambda_0}{2\lambda_{\max}(P_k^T HP_k)} L_H + \beta_1 \right) \|x_k - \bar{x}\|^3. \]

By Assumption 5.3 (ii),

\[ f(x_k) - f(x_k + P_k^T u^*) \geq \left( \frac{\lambda_0}{2\lambda_{\max}(P_k^T HP_k)} \right) - \left( \frac{\lambda_0}{2\lambda_{\max}(P_k^T HP_k)} L_H + \beta_1 \right) \frac{1}{C} \|x_k - \bar{x}\|^3 \]

where the first inequality holds as, by (5.9),

\[ \eta \left( \frac{5}{4} \right) \]

recall that

\[ d_k \]

the iteration parameter of Algorithm 3.1.

\[ \varepsilon \]

whole proof we only use Lemma 2.1 with

\[ B \]

ability bound in the statement of the theorem is obtained by using (2.1): in the

\[ C \]

hence by the first equation of (5.4)

\[ \|x_k - \bar{x}\| \]

tends to 0 as \( k \) tends to infinity and \( \rho < 3 \), we have that for \( k \) large enough

\[ f(x_k) - \min_{u \in B_0} f(x_k + P_k^T u) \geq f(x_k) - f(x_k + P_k^T u^*) \geq \frac{\lambda_0}{4\lambda_{\max}(P_k^T HP_k)} (f(x_k) - f(\bar{x})), \]

where the first inequality holds as, by (5.9), \( u^* \in B_0 \) for \( k \) large enough. The probability bound in the statement of the theorem is obtained by using (2.1): in the whole proof we only use Lemma 2.1 with \( \varepsilon = \frac{1}{2} \), which holds with probability at least \( 1 - 2 \exp(-\frac{C}{\varepsilon}) \), and the inequalities (5.4) which hold with probability at least \( 1 - 6 \exp(-s) \). We also factorize the expression, using that \( 1 - 2 \exp(-\frac{C}{\varepsilon}) > 1 - 6 \exp(-\frac{C}{\varepsilon}) \). We end the proof by noticing that \( \lambda_{\max}(P_k^T HP_k) \leq \lambda_{\max}(\bar{H}) \sigma_{\max}(P_k)^2 \), hence by the first equation of (5.4)

\[ \lambda_{\max}(P_k^T HP_k) \leq \lambda_{\max}(\bar{H}) \left( \sqrt{\frac{n}{s}} + C \right)^2. \]

\[ (5.18) \]

We are now ready to prove the main theorem of this section.

**Theorem 5.7.** Assume that Assumptions 5.1, 5.2, and 5.3 hold. There exist \( 0 < \kappa < 1, k_0 \in \mathbb{N}, \) and \( n_0 \in \mathbb{N} \) such that if \( n \geq n_0 \) and \( k \geq k_0 \), then

\[ f(x_{k+1}) - f(\bar{x}) \leq \left( 1 - \frac{1}{2} \alpha (1 - \alpha) \right) \frac{\lambda_0}{4\lambda_{\max}(\bar{H}) (\sqrt{\frac{n}{s}} + C)^2} (f(x_k) - f(\bar{x})) \]

holds with probability at least \( 1 - 6(\exp(-s) + \exp(-\frac{C}{\varepsilon} s)) \). Here \( \alpha \in (0, 1) \) is a parameter of Algorithm 3.1.

**Proof.** We recall that we use a backtracking line search to find at each iteration \( k \) a step-size \( t_k \) such that

\[ f(x_k + t_k d_k) \leq f(x_k) + \alpha t_k \nabla f(x_k)^T d_k, \]

with \( d_k = P_k^T u_k \) and the update rule \( t_k \leftarrow \beta t_k \) for \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). We recall that

\[ u_k = -((P_k^T H_k P_k^T + \eta_k I_s)^{-1} P_k g_k, \]

where we recall that \( \eta_k = c_1 \Lambda_k + c_2 \|g_k\|^2 \). By a Taylor expansion of \( f \) around \( x_k \), there exists \( x_k^* \in [x_k, x_{k+1}] \) such that

\[ f(x_k + t_k P_k^T u_k) = f(x_k) + t_k (P_k^T g_k)^T u_k + \frac{t_k^2}{2} u_k^T P_k \nabla^2 f(x_k^*) P_k u_k. \]
Notice that $\nabla^2 f$ is Lipschitz continuous (by Assumption 5.1). Furthermore, by Proposition 5.4, for $k$ large enough, $P_k H_k P_k$ is positive definite with probability at least $1 - 6 \exp(-s)$ as the sequence $\{x_k\}$ converges to $\bar{x}$. Hence, we deduce that there exists $n_0$ such that for $n \geq n_0$ and for $k$ large enough

$$u_k^T P_k \nabla^2 f(x_k^*) P_k^T u_k \leq u_k^T P_k H_k P_k^T u_k + \|P_k^T u_k\|^2 \|H_k - \nabla^2 f(x_k^*)\|$$

$$\leq u_k^T P_k H_k P_k^T u_k + L_H \|P_k^T u_k\|^2 \|x_k - x_{k+1}\| \leq 2 u_k^T P_k H_k P_k^T u_k$$

holds with probability at least $1 - 6 \exp(-s) + \exp(-\frac{C_0 s}{4})$. By (5.20), we deduce that for $k$ large enough and $n \geq n_0$:

$$f(x_k + t_k P_k^T u_k) \leq f(x_k) + t_k (P_k g_k)^\top u_k + 2 \frac{t_k^2}{2} u_k^T P_k H_k P_k u_k$$

$$\leq f(x_k) + t_k (P_k g_k)^\top u_k + t_k^2 u_k^T (P_k H_k P_k + \eta_k I_s) u_k,$$

where that second inequality holds as $\eta_k \geq 0$. Let

$$\mu_k^2 = -g_k^\top d_k = (P_k g_k)^\top (P_k H_k P_k^T + \eta_k I_s)^{-1} (P_k g_k).$$

Since $(P_k g_k)^\top u_k = g_k^\top (P_k^T u_k) = -\mu_k^2$, and by definition of $u_k$, (5.19), we can write

$$f(x_k + t_k P_k^T u_k) \leq f(x_k) - t_k \mu_k^2 + t_k^2 u_k^T (P_k H_k P_k + \eta_k I_s) u_k = f(x_k) - t_k \mu_k^2 + t_k^2 \mu_k^2.$$

Hence, we have

$$f(x_{k+1}) \leq f(x_k) - t_k (1 - t_k) \mu_k^2.$$

Thus the step-size $t_k = 1 - \alpha$ satisfies the exit condition, $f(x_k) - f(x_k + t_k d_k) \geq -\alpha t_k g_k^\top d_k$, in the backtracking line search as we have

$$(1 - t_k) = \alpha$$

for such $t_k$. Therefore, the backtracking line search stops with some $t_k \geq 1 - \alpha$. Thus, when the line search exits we have

$$f(x_{k+1}) \leq f(x_k) - \alpha (1 - \alpha) \mu_k^2.$$

Notice that since $\eta_k$ tends to 0, we have that

$$\mu_k^2 = (P_k g_k)^\top (P_k H_k P_k^T + \eta_k I_s)^{-1} (P_k g_k) \geq \frac{1}{2} (P_k g_k)^\top (P_k H_k P_k^T)^{-1} (P_k g_k).$$

Hence, by Proposition 5.5, we have that when $n$ and $k$ are large enough,

$$f(x_{k+1}) - f(\bar{x}) \leq f(x_k) - f(\bar{x}) - \frac{1}{2} \alpha (1 - \alpha) \left( f(x_k) - \min_{u \in B_0} f_{x_k}(u) \right)$$

holds with probability at least $1 - 6 \exp(-s) + \exp(-\frac{C_0 s}{4})$. By Proposition 5.6, we have that $f(x_k) - \min_{u \in B_0} f_{x_k}(u) \geq \lambda_0 \frac{\lambda_0}{4 \lambda_{max}(H)(\sqrt{\frac{C}{s}} + C)} (f(x_k) - f(\bar{x}))$ holds with probability at least $1 - 6 \exp(-s) + \exp(-\frac{C_0 s}{4})$. Hence

$$f(x_{k+1}) - f(\bar{x}) \leq \left( 1 - \frac{1}{2} \alpha (1 - \alpha) \frac{\lambda_0}{4 \lambda_{max}(H)(\sqrt{\frac{C}{s}} + C)} \right) (f(x_k) - f(\bar{x})),$$

which proves the theorem.  \qed
Remark 5.8. Notice that the rate we obtain corresponds to a high probability estimation of the local convergence rate derived, when \( f \) is assumed to be strongly convex, in the Stochastic Subspace Cubic Newton Method [13]. This can be seen in the proof of Proposition 5.6, where the rate \( \frac{\lambda_0}{4\lambda_{\max}(\hat{H})(\sqrt{\frac{s}{\delta}} + C)^2} \) corresponds to a lower bound of \( \lambda_{\min}(\hat{H}^{1/2} P_k^T (P_k \hat{H} P_k^T)^{-1} P_k \hat{H}^{1/2}) \), as seen in (5.12) and (5.18). More specifically, this corresponds to a high probability lower bound of the parameter \( \zeta = \lambda_{\min}[\mathbb{E}(\bar{H})] = \lambda_{\min}[\mathbb{E}(\hat{H}^{1/2} P_k^T (P_k \hat{H} P_k^T)^{-1} P_k \hat{H}^{1/2})] \) that appears in the local convergence rate in Theorem 6.2 of [13].

Let us define
\[
\kappa := \frac{1}{2} \alpha (1 - \alpha) \frac{\lambda_0}{4\lambda_{\max}(\hat{H})(\sqrt{\frac{s}{\delta}} + C)^2} < 1.
\]

We have the following direct corollary:

**Corollary 5.9.** Assume that Assumptions 5.1, 5.2, and 5.3 hold. There exist \( k_0 \in \mathbb{N} \) and \( n_0 \in \mathbb{N} \) such that if \( n \geq n_0 \) and \( k \geq k_0 \), then, for any \( m \in \mathbb{N} \),
\[
f(x_{k+m}) - f(\bar{x}) \leq (1 - \kappa)^m (f(x_k) - f(\bar{x}))
\]
holds with probability at least \( 1 - 6m(\exp(-s) + \exp(-\frac{\lambda_0}{4}s)) \).

**Proof.** This is a direct consequence of Theorem 5.7 where the success probability is obtained by union bound, using (2.1).

Notice that one can also derive an expectation version of Theorem 5.7 as follows.

**Corollary 5.10.** Assume that Assumptions 5.1, 5.2, and 5.3 hold. There exist \( k_0 \in \mathbb{N} \) and \( n_0 \in \mathbb{N} \) such that if \( n \geq n_0 \) and \( k \geq k_0 \), then
\[
\mathbb{E}[f(x_{k+1}) - f(\bar{x})] \leq (1 - p^2 \kappa) \mathbb{E}[f(x_k) - f(\bar{x})],
\]
where \( p := 1 - 6(\exp(-s) + \exp(-\frac{\lambda_0}{4}s)) \). Here the expectation is taken with respect to the random variables \( P_1, P_2, \ldots, P_k \).

**Proof.** By (5.24) we have that
\[
f(x_{k+1}) - f(\bar{x}) \leq f(x_k) - f(\bar{x}) - \frac{1}{2} \alpha (1 - \alpha) \left( f(x_k) - \min_{u \in B_0} \tilde{f}_{x_k}(u) \right)
\]
holds with probability \( p = 1 - 6(\exp(-s) + \exp(-\frac{\lambda_0}{4}s)) \). Let us denote by \( \mathcal{E} \) the event, with respect to \( P_k \), on which the above equation holds. Since \( f(x_{k+1}) - f(\bar{x}) \leq f(x_k) - f(\bar{x}) \) holds with probability one, we can write that
\[
f(x_{k+1}) - f(\bar{x}) \leq f(x_k) - f(\bar{x}) - \frac{1}{2} \alpha (1 - \alpha) \left( f(x_k) - \min_{u \in B_0} \tilde{f}_{x_k}(u) \right) 1_{\mathcal{E}},
\]
where \( 1_{\mathcal{E}} \) is the indicator function over \( \mathcal{E} \). Let us consider the following conditional expectation: \( \mathbb{E}[\cdot \mid P_1, \ldots, P_{k-1}] \). We have that
\[
\mathbb{E}[f(x_{k+1}) - f(\bar{x}) \mid P_1, \ldots, P_{k-1}] \leq f(x_k) - f(\bar{x}) - \frac{1}{2} \alpha (1 - \alpha) \mathbb{E} \left[ \left( f(x_k) - \min_{u \in B_0} \tilde{f}_{x_k}(u) \right) 1_{\mathcal{E}} \mid P_1, \ldots, P_{k-1} \right]
\]
holds as \( f(x_k) - f(\bar{x}) \) is measurable with respect to the sigma algebra generated by \( P_1, \ldots, P_{k-1} \). Let us define the event
\[
\mathcal{E}' = \left\{ f(x_k) - \min_{u \in B_0} \tilde{f}_{x_k}(u) \geq \frac{\lambda_0}{4\lambda_{\max}(\hat{H})(\sqrt{\frac{s}{\delta}} + C)^2} (f(x_k) - f(\bar{x})) \mid x_k \right\},
\]
on this sigma algebra, which holds by probability at least \( p = 1 - 6(\exp(-s) + \exp(-\frac{Cs}{4}))) \), by Proposition 5.6. By conditioning the right-hand-side of (5.26) with respect to this event, we obtain that when \( n \) and \( k \) are large enough

\[
\mathbb{E} [f(x_{k+1}) - f(\bar{x}) \mid P_1, \ldots, P_{k-1}] \leq f(x_k) - f(\bar{x}) - \frac{1}{2} \alpha (1-\alpha) \mathbb{E} \left[ \frac{\lambda_0}{4\lambda_{\max}(\bar{H}) (\sqrt{\frac{p}{s}} + C)^2} (f(x_k) - f(\bar{x}))^2 \right] p
\]

\[
\leq (f(x_k) - f(\bar{x})) \left( 1 - \frac{1}{2} \alpha (1-\alpha) \frac{\lambda_0}{4\lambda_{\max}(\bar{H}) (\sqrt{\frac{p}{s}} + C)^2} \right).
\]

Where the first inequality holds as in any case we have that \( f(x_k) - \min_{u \in B_0} \tilde{f}_x(u) \geq 0 \). By taking the expectation with respect to \( P_1, \ldots, P_{k-1} \) we deduce the corollary. \( \square \)

Let the consider the following assumption.

**Assumption 5.11.** There exists \( \rho > 0 \) such that for \( k \) large enough

\[
\|\nabla f(x_k)\| \geq \rho \|x_k - \bar{x}\|.
\]

Notice that Assumption 5.11 is actually stronger than Assumption 5.3(i).

**Lemma 5.12.** We have, under Assumptions 5.1 and 5.11, that for \( k \) large enough:

\[
\frac{\rho}{2\sqrt{\lambda_{\max}(\bar{H})}} \|x_k - \bar{x}\| \leq \|\sqrt{\bar{H}}(x_k - \bar{x})\|.
\]

**Proof.** Using a Taylor expansion of \( t \mapsto \nabla f(\bar{x} + t(x_k - \bar{x})) \) around 0, we have that

\[
\nabla f(x_k) = \nabla f(\bar{x}) + \int_0^1 \nabla^2 f(\bar{x} + t(x_k - \bar{x})) (x_k - \bar{x}) dt = \int_0^1 \nabla^2 f(\bar{x} + t(x_k - \bar{x})) (x_k - \bar{x}) dt.
\]

By Assumption 5.1, for any \( t \in [0,1] \) we have \( \|\nabla^2 f(\bar{x} + t(x_k - \bar{x})) - \bar{H}\| \leq tL_H \|x_k - \bar{x}\| \). Hence we deduce that

\[
\|\nabla f(x_k)\| \leq \|\bar{H}(x_k - \bar{x})\| + \|\nabla^2 f(\bar{x} + t(x_k - \bar{x})) - \bar{H}\| \|x_k - \bar{x}\| \leq \|\bar{H}(x_k - \bar{x})\| + L_H \|x_k - \bar{x}\|^2.
\]

Therefore, by (5.27), we deduce that

\[
\rho \|x_k - \bar{x}\| - L_H \|x_k - \bar{x}\|^2 \leq \|\nabla f(x_k)\| - L_H \|x_k - \bar{x}\|^2 \leq \|\bar{H}(x_k - \bar{x})\|. \tag{5.29}
\]

Since \( \|x_k - \bar{x}\| \) tends to 0, we deduce that for \( k \) large enough:

\[
\frac{\rho}{2} \|x_k - \bar{x}\| \leq \|\bar{H}(x_k - \bar{x})\| \leq \sqrt{\lambda_{\max}(\bar{H})} \|\sqrt{\bar{H}}(x_k - \bar{x})\|.
\]

Let us now define the semi-norm:

\[
\|x\|_{\bar{H}} := x^{\top} \bar{H} x. \tag{5.30}
\]

Notice that by Lemma 5.12, under Assumption 5.11, when \( k \) is large enough, \( \|\cdot\|_{\bar{H}} \) is a norm for \( x_k - \bar{x} \) as we have that \( \|x_k - \bar{x}\|_{\bar{H}} = 0 \) if and only if \( \|x_k - \bar{x}\| = 0 \).
Proposition 5.13. Assume that Assumptions 5.1, 5.3(i) and 5.11 hold. Then there exists \( n_0 \) such that if \( n \geq n_0 \) and for \( k \) large enough:

\[
\| x_{k+1} - \bar{x} \|_H \leq \left( 1 - \frac{\lambda_0}{4\lambda_{\text{max}}(H) (\sqrt{s} + C)^2} \right) \| x_k - \bar{x} \|_H
\]

holds with probability at least \( 1 - 6(\exp(-s) + \exp(-\frac{\alpha_n}{2}) \)).

Proof. Proof.

\[
\sqrt{H}(x_{k+1} - \bar{x}) = \sqrt{H}(x_{k+1} - x_k) + \sqrt{H}(x_k - \bar{x})
\]

\[
= -\sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k g_k + \sqrt{H}(x_k - \bar{x})
\]

\[
= -\sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k (x_k - \bar{x}) + \sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k (g_k - H_k(x_k - \bar{x})) + \sqrt{H}(x_k - \bar{x})
\]

(5.31)

\[
= -A + B + \sqrt{H}(x_k - \bar{x}),
\]

where \( A := \sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k (x_k - \bar{x}) \) and \( B := \sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k (g_k - H_k(x_k - \bar{x})) \).

First let us bound \( B \). In order to do so, we bound \( \|P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k\| \). Notice that by Assumption 5.3(i) and Proposition 5.4, \( P_k H_k P_k^T > 0 \) holds for \( n \geq n_0 \). Hence, since \( \eta_k \geq 0 \),

\[
\|P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k\| \leq \|P_k^T (P_k H_k P_k^T)^{-1} P_k\|.
\]

Hence we deduce that with probability at least \( 1 - 6\exp(-s) \),

(5.32)

\[
\|P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k\| \leq \frac{\|P_k^T\|^2}{2s \sigma^2 n \sigma 
\]

holds. Therefore, by Lemma 2.2, we have

(5.33)

\[
\|P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k\| \leq \frac{2C}{(1 - \epsilon_0)^2 \sigma \lambda}.
\]

By Taylor expansion at \( \bar{x} \) of \( \nabla f \), as in (5.28), and by subtracting \( H_k(x_k - \bar{x}) \) to both side, we obtain by Assumption 5.1 that

(5.34)

\[
\|g_k - H_k(x_k - \bar{x})\| \leq \int_0^1 \|\nabla^2 f(\bar{x} + t(x_k - \bar{x})) - \nabla^2 f(\bar{x})\||x_k - \bar{x}||dt = O(\|x_k - \bar{x}\|^2).
\]

Hence, by (5.33) and (5.34), there exists a constant \( \beta_1 > 0 \) such that

(5.35)

\[
B \leq \|\sqrt{H}||P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k\| (g_k - H_k(x_k - \bar{x}))\| \leq \beta_1 \|x_k - \bar{x}\|^2.
\]

Let us now bound \( A = \sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k (x_k - \bar{x}) \). Let us furthermore decompose \( A = A_1 + A_2 \) such that

(5.36)

\[
\sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k (x_k - \bar{x})
\]

\[
= \sqrt{H}P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k (x_k - \bar{x}) + \sqrt{H}P_k^T ((P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k
\]

\[
- (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k)(x_k - \bar{x})).
\]
Notice that by Assumption 5.1, we have that \(||(\bar{H} - H_k)||\) tends to 0. Therefore, we deduce from (5.33) and (5.35) that

\[ \| \sqrt{H} P_k^T ((P_k H_k P_k^T + \eta_k I_s)^{-1} P_k H_k - (P_k \bar{H} P_k^T + \eta_k I_s)^{-1} P_k \bar{H}) (x_k - \bar{x}) \| = o(||x_k - \bar{x}||) . \]

Therefore by (5.31), (5.35) and (5.36), we deduce that

\[ \sqrt{H} (x_k+1 - \bar{x}) = - A + B + \sqrt{H} (x_k - \bar{x}) = - A_1 + \sqrt{H} (x_k - \bar{x}) + o(||x_k - \bar{x}||) . \]

Hence, by evaluating the norm of \(A_2\) as \(o(||x_k - \bar{x}||)\), we deduce that with probability at least \(1 - 6(exp(-s) + exp(-\frac{\eta_0}{4}s))\)

\[ \| \sqrt{H} (x_{k+1} - \bar{x}) \| \leq \left\| (I_n - \sqrt{H} P_k^T (P_k \bar{H} P_k^T + \eta_k I_s)^{-1} P_k \sqrt{H}) \sqrt{H} (x_k - \bar{x}) \right\| + o(||x_k - \bar{x}||) . \]

We can write

\[ \left( I_n - \sqrt{H} P_k^T (P_k \bar{H} P_k^T + \eta_k I_s)^{-1} P_k \sqrt{H} \right) \sqrt{H} (x_k - \bar{x}) = \left( I_n - \sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \sqrt{H} \right) \sqrt{H} (x_k - \bar{x}) \]

\[ - \sqrt{H} P_k^T ((P_k \bar{H} P_k^T + \eta_k I_s)^{-1} - (P_k \bar{H} P_k^T)^{-1}) P_k \sqrt{H} (x_k - \bar{x}) . \]

Hence, using the same reasoning as before, we obtain that

\[ \| \sqrt{H} (x_{k+1} - \bar{x}) \| \leq \left\| (I_n - \sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \sqrt{H}) \sqrt{H} (x_k - \bar{x}) \right\| + o(||x_k - \bar{x}||) . \]

Notice that \(\sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \sqrt{H}\) is an orthogonal projection, hence

\[ \left\| (I_n - \sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \sqrt{H}) \sqrt{H} (x_k - \bar{x}) \right\|^2 = \| \sqrt{H} (x_k - \bar{x}) \|^2 - \| \sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \sqrt{H} (x_k - \bar{x}) \|^2 . \]

Then similarly to the proof of Proposition 5.6 and similarly to (5.11), we have that with probability at least \(1 - 6(exp(-s) + exp(-\frac{\eta_0}{4}s))\),

\[ \| \sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \bar{H} (x_k - \bar{x}) \|^2 = (x_k - \bar{x})^T \Pi (x_k - \bar{x}) , \]

and

\[ \| \sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \bar{H} (x_k - \bar{x}) \|^2 \geq \frac{\lambda_0}{2\lambda_{\max} (P_k \bar{H} P_k^T)} \| \sqrt{H} (x_k - \bar{x}) \|^2 , \]

where \(\lambda_0\) is the first non-zero eigenvalue of \(\bar{H}\). Therefore, we have that

\[ \left\| (I_n - \sqrt{H} P_k^T (P_k \bar{H} P_k^T)^{-1} P_k \sqrt{H}) \sqrt{H} (x_k - \bar{x}) \right\| \leq \sqrt{1 - \frac{\lambda_0}{2\lambda_{\max} (P_k \bar{H} P_k^T)}} \| \sqrt{H} (x_k - \bar{x}) \| . \]

Therefore, by (5.37), we have that

\[ \| \sqrt{H} (x_{k+1} - \bar{x}) \| \leq \sqrt{1 - \frac{\lambda_0}{2\lambda_{\max} (P_k \bar{H} P_k^T)}} \| \sqrt{H} (x_k - \bar{x}) \| + o(||x_k - \bar{x}||) . \]

By Lemma 5.12, we have \(o(||x_k - \bar{x}||) = o(||\sqrt{H} (x_{k+1} - \bar{x})||)\), hence we deduce that when \(k\) is large enough,

\[ \| \sqrt{H} (x_{k+1} - \bar{x}) \| \leq \sqrt{1 - \frac{\lambda_0}{4\lambda_{\max} (P_k \bar{H} P_k^T)}} \| \sqrt{H} (x_k - \bar{x}) \| . \]

We complete the proof using (5.18).
5.2. Impossibility of local super-linear convergence in general. In this section we will prove that when $f$ is strongly convex locally around the strict local minimizer $\bar{x}$, we cannot aim, with high probability, at local super-linear convergence using random subspace. More precisely, the goal of this section is to prove that there exists a constant $c > 0$ such that when $k$ is large enough, we have that with probability $1 - 2\exp(-\frac{c^2}{4}) - 2\exp(-s)$,

$$\|x_{k+1} - \bar{x}\| \geq c\|x_k - \bar{x}\|.$$ 

From that, we will easily deduce that there exists a constant $c'$ such that

$$f(x_{k+1}) - f(\bar{x}) \geq c'(f(x_k) - f(\bar{x}))$$

holds with high probability when $k$ is large enough. This will prove that the results obtained in the previous section are optimal when $f$ is locally strongly-convex. Indeed, by local strong-convexity of $f$ and Hessian Lipschitz continuity (i.e. Assumption 5.1), there exists $l_2 \geq l_1 > 0$ such that for $k$ large enough,

$$l_1\|x_k - \bar{x}\|^2 \leq f(x_k) - f(\bar{x}) \leq l_2\|x_k - \bar{x}\|^2.$$ 

This immediately proves the existence of the constant $c'$ described above. In this subsection we make the following additional assumption.

**Assumption 5.14.** We assume that

$$(C + 2)^2s < n,$$

where $C$ is the constant that appears in (5.4).

We recall here that for all $k$:

$$x_{k+1} = x_k - t_kP_k^\top((P_k\nabla^2 f(x_k)P_k^\top) + \eta_kI_s)^{-1}P_k\nabla f(x_k),$$

where $t_k$ is the step-size and $\eta_k > 0$ is a parameter that tends to 0 when $k$ tends to infinity.

Let us fix $k$. Using a Taylor expansion of $t \mapsto \nabla f(\bar{x} + t(x_{k+1} - \bar{x}))$ around 0, as in (5.28), we have that

$$\|\nabla f(x_{k+1})\| \leq \int_0^1 \|\nabla^2 f(\bar{x} + t(x_{k+1} - \bar{x}))\|\|x_{k+1} - \bar{x}\|dt \leq \int_0^1 2\lambda_{\text{max}}(\nabla^2 f(\bar{x}))\|x_{k+1} - \bar{x}\|dt,$$

where the second inequality holds for $k$ large enough under Assumption 5.1. Hence, for $k$ large enough and under Assumption 5.1:

$$\|x_{k+1} - \bar{x}\| \geq \frac{1}{2\lambda_{\text{max}}(\nabla^2 f(\bar{x}))}\|\nabla f(x_{k+1})\|,$$

where $\lambda_{\text{max}}(\cdot)$ denotes the largest eigenvalue. Using a Taylor expansion of $\nabla f$ around $x_k$, we have that

$$\nabla f(x_{k+1}) = \nabla f(x_k) + \int_0^1 \nabla^2 f(x_k + t(x_{k+1} - x_k))(x_{k+1} - x_k)dt.$$ 

Hence,

$$\nabla f(x_{k+1}) = \nabla f(x_k) + \nabla^2 f(x_k)(x_{k+1} - x_k) + \int_0^1 (\nabla^2 f(x_k + t(x_{k+1} - x_k)) - \nabla^2 f(x_k))(x_{k+1} - x_k).$$
We deduce therefore that
\[ \| \nabla f(x_{k+1}) \| \geq \| \nabla f(x_k) + \nabla^2 f(x_k)(x_{k+1} - x_k) \| - \int_0^1 \| \nabla^2 f(x_k + t(x_{k+1} - x_k)) - \nabla^2 f(x_k) \| (x_{k+1} - x_k) \|.
\]
By Assumption 5.1, the Hessian is \( L_H \)-Lipschitz in \( B_H \). Since \( x_k \) and \( x_k + t(x_{k+1} - x_k) \in B_H \) for \( k \) large enough, we have that
\[ \| (\nabla^2 f(x_k + t(x_{k+1} - x_k)) - \nabla^2 f(x_k))(x_{k+1} - x_k) \| \leq L_H \| x_{k+1} - x_k \|^2.
\]
Hence (5.39) leads to
\[ (5.40) \quad \| x_{k+1} - \bar x \| \geq \frac{1}{2\lambda_{\text{max}}(\nabla^2 f(\bar x))} \left( \| g_k + H_k(x_{k+1} - x_k) \| - L_H \| x_{k+1} - x_k \|^2 \right).
\]

**Proposition 5.15.** Assume that Assumptions 5.1 and 5.14 hold and that \( f \) is strongly convex locally around \( \bar x \). There exists a constant \( \beta > 0 \) such that if \( k \) is large enough, then with probability at least \( 1 - 2\exp(-\frac{\alpha}{4}s) - 2\exp(-s) \), we have
\[ \| g_k + H_k(x_{k+1} - x_k) \| \geq \beta \| x_{k+1} - x_k \|.
\]

**Proof.** We have
\[ \| g_k + H_k(x_{k+1} - x_k) \| = \| (I_n - t_k H_k P_k^T M_k^{-1} P_k) g_k \|,
\]
where \( M_k \) is defined in (3.2). If \( k \) is large enough, \( H_k \) is invertible by strong convexity of \( f \). Notice that \( \| (I_n - t_k H_k P_k^T M_k^{-1} P_k) g_k \| = \| H_k(H_k^{-1} - t_k P_k^T M_k^{-1} P_k) g_k \| \). Hence since for any invertible matrix \( A \) we have \( \| Ax \| \geq \| x \| \), we deduce that
\[ \| (I_n - t_k H_k P_k^T M_k^{-1} P_k) g_k \| \geq \frac{1}{\| H_k^{-1} \|} \| (H_k^{-1} - t_k P_k^T M_k^{-1} P_k) g_k \|.
\]
Furthermore, we have
\[ (5.41) \quad \| (H_k^{-1} - t_k P_k^T M_k^{-1} P_k) g_k \|^2 = \| H_k^{-1} g_k \|^2 + \| t_k P_k^T M_k^{-1} P_k g_k \|^2 - 2\langle H_k^{-1} g_k, t_k P_k^T M_k^{-1} P_k g_k \rangle.
\]
Let \( H_k^{-1} g_k = P_k^T z_1 + z_2 \) be the orthogonal decomposition of \( H_k^{-1} g_k \) on \( \text{Im}(P_k^T) \) parallel to \( \text{Ker}(P_k) \). Since \( P_k z_2 = 0 \), we have
\[ \langle H_k^{-1} g_k, t_k P_k^T M_k^{-1} P_k g_k \rangle = \langle P_k^T z_1, t_k P_k^T M_k^{-1} P_k g_k \rangle.
\]
Hence, by (5.41), we deduce that
\[ (5.42) \quad \| (H_k^{-1} - t_k P_k^T M_k^{-1} P_k) g_k \|^2 \geq \| H_k^{-1} g_k \|^2 + \| t_k P_k^T M_k^{-1} P_k g_k \|^2 - 2\| P_k^T z_1 \| \| t_k P_k^T M_k^{-1} P_k g_k \|.
\]
Since \( H_k^{-1} g_k = P_k^T z_1 + z_2 \) with \( P_k z_2 = 0 \), we have that \( P_k H_k^{-1} g_k = P_k P_k^T z_1 \). Hence
\[ \| P_k^T z_1 \| = \| P_k^T (P_k P_k^T)^{-1} P_k H_k^{-1} g_k \| \leq \| P_k^T (P_k P_k^T)^{-1} \| \| P_k H_k^{-1} g_k \|.
\]
By Lemma 2.1, we have that with probability at least \( 1 - 2\exp(-\frac{\alpha}{4}s) \) that \( \| P_k H_k^{-1} g_k \| \leq 2 \| H_k^{-1} g_k \| \). Furthermore, by writing the singular value decomposition, \( U \Sigma V^T \), of \( P_k^T \),
we have that \( \|P_k^\top (P_k P_k^\top)^{-1}\| = \|US^{-1}V^\top\| = \frac{1}{\sigma_{\min}(P_k)} \). Since \( \sigma_{\min}(P_k) \geq \sqrt{\frac{2}{5}} - C \) holds with probability at least \( 1 - 2e^{-s} \) (we only consider the first equation of (5.4)), we deduce that
\[
\|P_k^\top z_1\| \leq \frac{2}{\sqrt{\frac{2}{5}} - C} \|H_k^{-1}g_k\|.
\]

Hence, from (5.42) we have
\[
(5.43) \quad \|H_k^{-1} - t_k P_k^\top M_k^{-1} P_k g_k\|^2 \\
\geq \|H_k^{-1}g_k\|^2 + \|t_k P_k^\top M_k^{-1} P_k g_k\| - \frac{4}{\sqrt{\frac{2}{5}} - C} \|H_k^{-1}g_k\| \|t_k P_k^\top M_k^{-1} P_k g_k\| \\
\geq \left(1 - \frac{2}{\sqrt{\frac{2}{5}} - C}\right) \|H_k^{-1}g_k\|^2 + \left(1 - \frac{2}{\sqrt{\frac{2}{5}} - C}\right) \|t_k P_k^\top M_k^{-1} P_k g_k\|^2,
\]
where we used that \( 2ab \leq a^2 + b^2 \) in the last inequality, and that \( \left(1 - \frac{2}{\sqrt{\frac{2}{5}} - C}\right) > 0 \) holds by Assumption 5.14. Hence, from (5.43) we proved that
\[
\|(I_n - t_k H_k P_k^\top M_k^{-1} P_k)g_k\|^2 \geq \frac{1}{\|H_k^{-1}\|^2} \left(1 - \frac{2}{\sqrt{\frac{2}{5}} - C}\right) \|t_k P_k^\top M_k^{-1} P_k g_k\|^2 = \frac{1}{\|H_k^{-1}\|^2} \left(1 - \frac{2}{\sqrt{\frac{2}{5}} - C}\right) \|x_{k+1} - x_k\|^2.
\]

That is
\[
(5.44) \quad \|g_k + H_k(x_{k+1} - x_k)\| \geq \sqrt{1 - \frac{2}{\sqrt{\frac{2}{5}} - C}} \|H_k^{-1}\| \|x_{k+1} - x_k\|.
\]

Considering \( k \) large enough, as \( x_k \) tends to \( \bar{x} \), we can bound, using Assumption 5.1, \( \|H_k^{-1}\|^2 \geq \frac{1}{\|H_k\|^2} \), where we recall that \( H = \nabla^2 f(\bar{x}) \), which ends the proof.

**Theorem 5.16.** Assume that Assumptions 5.1 and 5.14 hold and that \( f \) is locally strongly convex around \( \bar{x} \). There exists a constant \( c > 0 \) such that for \( k \) large enough,
\[
\|x_{k+1} - \bar{x}\| \geq c\|x_k - \bar{x}\|
\]
holds with probability at least \( 1 - 2 \exp(-\frac{C_0}{4}s) - 2 \exp(-s) \).

**Proof.** Proof. From (5.40) and Proposition 5.15 we deduce that with probability at least \( 1 - 2 \exp(-\frac{C_0}{4}s) - 2 \exp(-s) \), when \( k \) is large enough
\[
\|x_{k+1} - \bar{x}\| \geq \frac{1}{2\lambda_{\max}(\nabla^2 f(\bar{x}))} (\beta - L_H \|x_{k+1} - x_k\|) \|x_{k+1} - x_k\|.
\]
Since \( \beta > 0 \), we have that for \( k \) large enough so as to yield \( L_H \|x_{k+1} - x_k\| \leq \beta/2 \),
\[
\|x_{k+1} - \bar{x}\| \geq \frac{1}{2\lambda_{\max}(\nabla^2 f(\bar{x}))} \beta \|x_{k+1} - x_k\| = \frac{\beta}{2} \|x_{k+1} - x_k\|.
\]
Hence
\[
\|x_{k+1} - \bar{x}\| \geq \frac{\beta}{4\lambda_{\max}(H)} \|x_{k+1} - x_k\|.
\]
Now we notice that
\begin{equation}
\|x_{k+1} - x_k\| = \|P_k^T M_k^{-1} P_k g_k\| \geq \sigma_{\text{min}}(P_k^T) \|M_k^{-1}\| \|P_k g_k\|.
\end{equation}
Using Lemma 2.1 and the bound (5.4) on \(\sigma_{\text{min}}(P_k^T)\), we have that
\begin{equation}
\sigma_{\text{min}}(P_k^T) \|M_k^{-1}\| \|P_k g_k\| \geq \left(\sqrt{n/s} - C\right) \|M_k^{-1}\| \frac{1}{2} \|g_k\|.
\end{equation}
Since \(x_k\) converges to \(\bar{x}\) and the Hessian is Lipschitz continuous, we have that \(H_k\) converges to \(\bar{H}\). Therefore, when \(k\) is large enough, we have \(\|M_k^{-1}\| \geq \frac{1}{2} \|(P_k \bar{H} P_k^T)^{-1}\| = \frac{1}{2} \|M^{-1}\|\), where \(M := P_k \bar{H} P_k^T\). Since
\begin{equation}
0 \prec \bar{M} \preceq \lambda_{\text{max}}(\bar{H}) P_k P_k^T,
\end{equation}
we deduce by Lemma 2.2
\begin{equation}
\|M_k^{-1}\| \geq \frac{1}{2C\lambda_{\text{max}}(\bar{H})}.
\end{equation}
Hence, by (5.45),(5.46) and (5.47) we have that there exists a constant \(\kappa_2 > 0\) such that
\begin{equation}
\|x_{k+1} - \bar{x}\| \geq \kappa_2 \|g_k\|.
\end{equation}
By (5.28) we have that
\begin{equation*}
g_k = \bar{H}(x_k - \bar{x}) + \int_0^1 (\nabla^2 f(\bar{x} + t(x_k - \bar{x})) - \bar{H})(x_k - \bar{x}) dt.
\end{equation*}
Hence, since \(f\) is assumed to be locally strongly convex, by Assumptions 5.1 we have that for \(k\) large enough:
\begin{equation}
\|g_k\| \geq \frac{\lambda_{\text{min}}(\bar{H})}{2} \|\bar{x} - x_k\|.
\end{equation}
Hence we have for \(k\) large enough that
\begin{equation}
\|x_{k+1} - \bar{x}\| \geq \frac{1}{2} \kappa_2 \lambda_{\text{min}}(\bar{H}) \|x_k - \bar{x}\|,
\end{equation}
which concludes the proof. \(\square\)

We have the following deterministic corollary:

**Corollary 5.17.** Assume that Assumptions 5.1 and 5.14 hold and that \(f\) is locally strongly convex around \(\bar{x}\). There exists a constant \(c > 0\) such that for \(k\) large enough,
\begin{equation*}
\mathbb{E}(\|x_{k+1} - \bar{x}\|) \geq \bar{c} \mathbb{E}(\|x_k - \bar{x}\|),
\end{equation*}
where \(\bar{c} = (1 - 2 \exp(-\frac{\delta}{4}s) - 2 \exp(-s))c\), and where the expectation is taken with respect to the random variables \(P_1, \ldots, P_k\).

**Proof.** Proof. The proof is very similar to the proof of Corollary 5.10. Let us consider the random variable \(\mathbb{E}(\|x_{k+1} - \bar{x}\| \mid P_1, \ldots, P_{k-1})\). Let \(\mathcal{E} = \{\|x_{k+1} - \bar{x}\| \geq \bar{c}\|x_k - \bar{x}\| \mid x_k\}\) be an event with respect to the random variable \(P_k\). Using the fact that \(\|x_{k+1} - \bar{x}\| \geq 0\), we obtain that
\begin{equation}
\mathbb{E}(\|x_{k+1} - \bar{x}\| \mid P_1, \ldots, P_{k-1}) = \mathbb{E}(\|x_{k+1} - \bar{x}\| \mid P_1, \ldots, P_{k-1}, \mathcal{E}) P(\mathcal{E}) + \mathbb{E}(\|x_{k+1} - \bar{x}\| \mid P_1, \ldots, P_{k-1}, \mathcal{E}^c) (1 - P(\mathcal{E}))
\end{equation}
Taking the expectation with respect to \(P_1, \ldots, P_{k-1}\) leads to the result. \(\square\)
5.3. The rank deficient case. Previously we proved that when $f$ is locally strongly convex, super-linear convergence cannot hold for RS-RNM. Here we prove that when the Hessian $\bar{H}$ at the local optimum $\bar{x}$ is rank deficient, then RS-RNM can achieve super-linear convergence. In this whole subsection, we assume that Assumptions 5.1 and 5.11 are satisfied. We also denote by $r (< n)$ the rank of $\bar{H}$. We first prove the following lemma which is very similar to Lemma 5.12.

**Lemma 5.18.** We have, under Assumptions 5.1 and 5.11, that for $k$ large enough:

$$\rho^2 \|x_k - \bar{x}\| \leq \|\bar{H}(x_k - \bar{x})\|.$$  

Furthermore,

$$\|g_k\| \leq 2\lambda_{\text{max}}(\bar{H})\|x_k - \bar{x}\|.$$  

**Proof.** As in the proof of Lemma 5.12, we have that

$$\rho\|x_k - \bar{x}\| - L_H \|x_k - \bar{x}\|^2 \leq \|\nabla f(x_k)\| - L_H \|x_k - \bar{x}\|^2 \leq \|\bar{H}(x_k - \bar{x})\|.$$  

Since $\|x_k - \bar{x}\|$ tends to 0, we deduce that for $k$ large enough:

$$\rho^2 \|x_k - \bar{x}\| \leq \|\bar{H}(x_k - \bar{x})\|.$$  

The other inequality is easy to deduce from (5.28), as in (5.38):

$$\|g_k\| \leq \|\bar{H}\|\|x_k - \bar{x}\| + L_H \|x_k - \bar{x}\|^2 \leq 2\lambda_{\text{max}}(\bar{H})\|x_k - \bar{x}\|,$$

when $k$ is large enough such that $L_H \|x_k - \bar{x}\| \leq \lambda_{\text{max}}(\bar{H})$ holds. \qed

The next lemma is the key to prove super-linear convergence. Notice that since $s \geq r$, we have that with probability one $\sigma_{\text{min}}(P_k^1) > 0$.

**Lemma 5.19.** Under Assumptions 5.1 and 5.11. If $s \geq r$, we have that for $k$ large enough, with probability at least $1 - 2\exp(-s)$:

$$\|P_k g_{k+1}\| \geq \rho \frac{\sigma_{\text{min}}(P_k^1)}{8\lambda_{\text{max}}(\bar{H})}\|g_{k+1}\|,$$

where $P_k^1 \in \mathbb{R}^{s \times r}$ is an $s \times r$ i.i.d. Gaussian matrix having the same distribution with $P_k$.

**Proof.** Proof. By (5.28) applied at $k + 1$, we have that

$$\nabla f(x_{k+1}) = \int_0^1 \nabla^2 f(\bar{x} + t(x_{k+1} - \bar{x}))(x_{k+1} - \bar{x})dt.$$  

Hence,

$$P_k g_{k+1} = P_k \bar{H}(x_{k+1} - \bar{x}) + \int_0^1 P_k(\nabla^2 f(\bar{x} + t(x_{k+1} - \bar{x})) - \bar{H})(x_{k+1} - \bar{x}),$$

which leads to

$$\|P_k g_{k+1}\| \geq \|P_k \bar{H}(x_{k+1} - \bar{x})\| - L_H \|P_k\|\|x_{k+1} - \bar{x}\|^2.$$  

Let $UDU^T = \bar{H}$ be the diagonal decomposition of $\bar{H}$. Since $\bar{x}$ is a strict local minimizer, by Assumption 5.11, for $k$ large enough, $U$ is an orthogonal matrix independent
of $P_k$, and hence, $\tilde{P}_k := P_k U$ is an i.i.d. random Gaussian matrix with the same distribution as $P_k$. Let $y_{k+1} = U^T (x_{k+1} - \tilde{x})$. We have that

$$
(5.49) \quad \tilde{H}(x_{k+1} - \tilde{x}) = Dy_{k+1} \quad \& \quad P_k \tilde{H}(x_{k+1} - \tilde{x}) = \tilde{P}_k Dy_{k+1}.
$$

Furthermore, since $D$ has rank $r < n$, we can write $Dy_{k+1} = \begin{pmatrix} z_{k+1}^T \\ 0 \end{pmatrix}$, where $z_{k+1} \in \mathbb{R}^r$. We have therefore that

$$
(5.50) \quad \|P_k \tilde{H}(x_{k+1} - \tilde{x})\| = \|P_k^1 z_{k+1}\|,
$$

where $P_k^1 \in \mathbb{R}^{s \times r}$ is a submatrix of $\tilde{P}_k$, i.e., $\tilde{P}_k = \begin{pmatrix} P_k^1 & P_k^2 \end{pmatrix}$. Notice that from the definition of $y_{k+1}$ and $z_{k+1}$, we have, by orthogonality of $U$, that

$$
\|z_{k+1}\| = \|Dy_{k+1}\| = \|\tilde{H}(x_{k+1} - \tilde{x})\| \geq \frac{\rho}{2} \|x_{k+1} - \tilde{x}\|,
$$

where the inequality follows from Lemma 5.18. Hence, from (5.48) and (5.50), we deduce that

$$
\|P_k g_{k+1}\| \geq \rho \frac{\sigma_{\min}(P_k^1)}{2} \|x_{k+1} - \tilde{x}\| - L_H \|P_k\| \|x_{k+1} - \tilde{x}\|^2.
$$

Using that $\|P_k\|$ is bounded, with probability at least $1 - 2 \exp(-s)$, by Lemma 2.2, we deduce, as in the proof of Lemma 5.18, that for $k$ large enough:

$$
\|P_k g_{k+1}\| \geq \rho \frac{\sigma_{\min}(P_k^1)}{4} \|x_{k+1} - \tilde{x}\| \geq \rho \frac{\sigma_{\min}(P_k^1)}{4} \|g_{k+1}\| \frac{\|H_{k+1}\|}{2 \lambda_{\max}(H)}.
$$

That is:

$$
\|P_k g_{k+1}\| \geq \rho \frac{\sigma_{\min}(P_k^1)}{8 \lambda_{\max}(H)} \|g_{k+1}\|.
$$

Similarly, we have the following lemma.

**Lemma 5.20.** Let $M \in \mathbb{R}^{n \times n}$ be any matrix. Under Assumptions 5.1 and 5.11, if $k$ is large enough and $s \geq r$, we have

$$
\frac{\sigma_{\min}(P_k^1)}{2} \|H_k M\| \leq \|P_k H_k M\|.
$$

**Proof.** The proof is very similar to the proof of Lemma 5.19. We have

$$
(5.51) \quad \|P_k H_k M\| \geq \|P_k \tilde{H} M\| - \|P_k(H_k - \tilde{H}) M\|.
$$

Let $UDU^T = \tilde{H}$ be the diagonal decomposition of $H$. Similarly to the proof of Lemma 5.19, $\tilde{P}_k := P_k U$ is an i.i.d. random Gaussian matrix with the same distribution as $P_k$. Let $N = U^T M$. We have that $P_k \tilde{H} M = \tilde{P}_k D N$. Furthermore, since $D$ has rank $r < n$, we can write $D N = \begin{pmatrix} \hat{N} \\ 0 \end{pmatrix}$, where $\hat{N} \in \mathbb{R}^{r \times n}$. We have therefore that

$$
(5.52) \quad \|P_k \tilde{H} M\| = \|P_k^1 \hat{N}\|,
$$

where $P_k^1 \in \mathbb{R}^{s \times r}$ is a submatrix of $\tilde{P}_k$, i.e., $\tilde{P}_k = \begin{pmatrix} P_k^1 & P_k^2 \end{pmatrix}$. Therefore

$$
(5.53) \quad \|P_k \tilde{H} M\| \geq \sigma_{\min}(P_k^1) \|\hat{N}\| = \sigma_{\min}(P_k^1) \|D N\| = \sigma_{\min}(P_k^1) \|\tilde{H} M\|,
$$

where
where the last equality holds by orthogonality of $U$. We deduce therefore, from (5.51) and (5.53) that

$$
\|P_k H_k M\| \geq \sigma_{\min}(P_k^T)\|H_k M\| - \sigma_{\min}(P_k^T)||H - H_k)||M| - \|P_k (H_k - \bar{H}) M||.
$$

Since $H_k$ tends to $\bar{H}$, we have the desired result for $k$ large enough.

The next lemma, similar to Lemma 5.2 of [26], is needed to control $\|\eta_k\| = \|c_1 \Lambda_k + c_2 \|g_k\|^T\|$, where $\Lambda_k = \max(0, -\lambda_{\min}(P_k H_k P_k^T))$.

**Lemma 5.21.** Under Assumption 5.1, for $k$ large enough, we have that with probability at least $1 - 2 \exp(-s)$,

$$
\Lambda_k \leq \frac{\tilde{C}_n}{s} L_H \|x_k - \bar{x}\|.
$$

**Proof.** The result is obvious when $\Lambda_k = 0$. Let us consider the case $\Lambda_k > 0$. Let $\lambda = (\lambda_k^{(1)}, \ldots, \lambda_k^{(s)})$ be a vector of eigenvalues of $P_k \bar{H} P_k^T$ and we write the eigenvalue decomposition of $P_k \bar{H} P_k^T$ as follows:

$$
P_k \bar{H} P_k^T = U_k^T diag(\lambda_k) U_k.
$$

Since $\lambda_{\min}(P_k H_k P_k^T) I_s - U_k P_k H_k P_k^T U_k^T$ is singular, therefore

$$
\lambda_{\min}(P_k H_k P_k^T) I_s - diag(\lambda_k)
$$

is not singular. We define

$$
A_k = (\lambda_{\min}(P_k H_k P_k^T) I_s - diag(\lambda_k))^{-1}(\lambda_{\min}(P_k H_k P_k^T) I_s - U_k P_k H_k P_k^T U_k^T),
$$

which is therefore singular. Notice furthermore that since $\lambda_{\min}(P_k H_k P_k^T) < 0$,

$$
(5.54) \quad \|\lambda_{\min}(P_k H_k P_k^T) I_s - diag(\lambda_k))^{-1}\| \leq \frac{1}{-\lambda_{\min}(P_k H_k P_k^T)} = \frac{1}{\Lambda_k},
$$

Hence we have

$$
1 \leq \|I_k - A_k\|
$$

$$
= \|I_s - (\lambda_{\min}(P_k H_k P_k^T) I_s - diag(\lambda_k))^{-1}(\lambda_{\min}(P_k H_k P_k^T) I_s - U_k P_k H_k P_k^T U_k^T)\|
$$

$$
= \|I_s - (\lambda_{\min}(P_k H_k P_k^T) I_s - diag(\lambda_k))^{-1}

(\lambda_{\min}(P_k H_k P_k^T) I_s - diag(\lambda_k)) - U_k P_k (H_k - \bar{H}) P_k^T U_k^T||
$$

$$
= \|\lambda_{\min}(P_k H_k P_k^T) I_s - diag(\lambda_k))^{-1}U_k P_k (H_k - \bar{H}) P_k^T U_k^T||
$$

$$
\leq \frac{1}{\Lambda_k} \|P_k P_k^T\|\|H_k - \bar{H}\|
$$

$$
\leq \frac{2}{\Lambda_k} \frac{\tilde{C}_n}{s} L_H \|x_k - \bar{x}\|,
$$

where the first inequality is a well known inequality for a singular matrix and is proved in [26, Lemma 5.1].

Let us recall that

$$
d_k = -P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1} P_k g_k,
$$
and

\[ M_k = P_k H P_k^\top + \eta_k I_s. \]

**Lemma 5.22.** Under Assumptions 5.1 and 5.11, if \( s \geq r \), we have that for \( k \) large enough, we have that with probability at least \( 1 - 2\exp(-s) \),

\[ \|d_k\| \leq \frac{4}{\sigma_{\min}(P_k^s)} \left( 2 + \frac{1}{c_1 - 1} \right) \sqrt{C_n/s} \|x_k - \bar{x}\|, \]

where \( P_k^s \in \mathbb{R}^{s \times r} \) is an \( s \times r \) i.i.d. Gaussian matrix having the same distribution with \( P_k \).

**Proof.** Notice first that by Taylor expansion of \( t \mapsto \nabla f(\bar{x} + t(x_k - \bar{x})) \) and by Assumption 5.1, we have that

\[ \|g_k - \nabla f(\bar{x}) - H_k(x_k - \bar{x})\| \leq \frac{L_H}{2} \|x_k - \bar{x}\|^2. \]

(5.55)

The definition of \( d_k \) leads to

\[ \|d_k\| = \|P_k^\top M_k^{-1} P_k g_k\| \]

\[ \nabla f(\bar{x}) = 0 \|P_k^\top M_k^{-1} P_k (g_k - \nabla f(\bar{x}) - H_k(x_k - \bar{x}) + H_k(x_k - \bar{x}))\| \]

\[ \leq \|P_k\|^2 \|M_k^{-1}\| \|g_k - \nabla f(\bar{x}) - H_k(x_k - \bar{x})\| + \|P_k^\top M_k^{-1} P_k H_k\| \|x_k - \bar{x}\| \]

(5.56)

\[ \leq \frac{L_H}{2} \|P_k\|^2 \|M_k^{-1}\| \|x_k - \bar{x}\|^2 + \|P_k^\top M_k^{-1} P_k H_k\| \|x_k - \bar{x}\|. \]

Let us first bound the first term in the right-hand side of (5.56). When \( k \) is large enough, with probability at least \( 1 - 2\exp(-s) \), we have by Lemma 2.2

\[ \frac{L_H}{2} \|P_k\|^2 \|M_k^{-1}\| \|x_k - \bar{x}\|^2 \leq \frac{L_H}{2} \frac{\tilde{C}_n}{s} \frac{1}{\lambda_{\min}(P_k H_k P_k^\top + c_1 \Lambda_k I_s + c_2 \|g_k\| I_s)} \]

\[ \leq \frac{L_H \tilde{C}_n}{2c_2 \|g_k\|^s} \]

(5.27)

\[ \leq \frac{L_H \tilde{C}_n}{2c_2 s \rho^s \|x_k - \bar{x}\|^s}. \]

Hence

(5.57)

\[ \frac{L_H}{2} \|P_k\|^2 \|M_k^{-1}\| \|x_k - \bar{x}\|^2 \leq \frac{L_H \tilde{C}_n}{2c_2 s \rho^s \|x_k - \bar{x}\|^2 - s}. \]

Next, we consider the second term \( \|P_k^\top M_k^{-1} P_k H_k\| \|x_k - \bar{x}\| \). Notice that

\[ \|P_k^\top M_k^{-1} P_k H_k\| = \|H_k P_k^\top M_k^{-1} P_k\| \leq \frac{2}{\sigma_{\min}(P_k)} \|P_k H_k P_k^\top M_k^{-1} P_k\| \|P_k\|, \]

where \( P_k \) is a random matrix with probability at least \( 1 - 2\exp(-s) \), we have that for \( k \) large enough, we have that with probability at least \( 1 - 2\exp(-s) \),
where the inequality follows from Lemma 5.20. We have

\[
\|P_k H_k P_k^T M_k^{-1}\| = \|P_k H_k P_k^T (P_k H_k P_k^T + \eta_k I_s)^{-1}\|
\]

\[
\leq \|(P_k H_k P_k^T + \eta_k I_s)^T (P_k H_k P_k^T + \eta_k I_s)^{-1}\| + \eta_k\|(P_k H_k P_k^T + \eta_k I_s)^{-1}\|
\]

\[
\leq 1 + \frac{\eta_k}{\lambda_{\min}(P_k H_k P_k^T + \eta_k I_s)}
\]

\[
\leq 1 + \frac{c_1 \Lambda_k + c_2 \|g_k\|^6}{(c_1 - 1)\Lambda_k + c_2 \|g_k\|^6}
\]

\[
\leq 2 + \frac{1}{c_1 - 1}.
\]

Therefore,

\[
\|P_k^T M_k^{-1} P_k H_k\|\|x_k - \bar{x}\| \leq \frac{2}{\sigma_{\min}(P_k^T)} \left(2 + \frac{1}{c_1 - 1}\right) \|P_k\|\|x_k - \bar{x}\| \leq \frac{2}{\sigma_{\min}(P_k^T)} \left(2 + \frac{1}{c_1 - 1}\right) \sqrt{\frac{C_n}{s}} \|x_k - \bar{x}\|,
\]

where the second inequality follows from Lemma 2.2. The results follows from (5.56) and (5.57) noticing that \(\|x_k - \bar{x}\|^2 - \delta\) tends to 0, as \(\delta < 1\), hence for \(k\) large enough

\[
\frac{L_H \tilde{C}_n}{2c_2 s \rho^3} \|x_k - \bar{x}\|^{2 - \delta} \leq \frac{2}{\sigma_{\min}(P_k^T)} \left(2 + \frac{1}{c_1 - 1}\right) \sqrt{\frac{C_n}{s}} \|x_k - \bar{x}\|.
\]

**Theorem 5.23.** Under Assumptions 5.1 and 5.11, for \(k\) large enough and for any \(s \geq r\), we have that with probability at least \(1 - 2\exp(-s)\)

\[
\|x_{k+1} - \bar{x}\| \leq \frac{c_2 \Gamma}{\sigma_{\min}(P_k^T)} \|x_k - \bar{x}\|^{1+\delta},
\]

where \(\Gamma\) is some constant depending on \(n\) and \(s\), and where \(P_k^T \in \mathbb{R}^{s \times r}\) is an \(s \times r\) i.i.d. Gaussian matrix having the same distribution with \(P_k\).

**Proof.** We have

\[
\|x_{k+1} - \bar{x}\| \overset{(5.27)}{\leq} \frac{1}{\rho} \|g_{k+1}\|
\]

\[
\leq \frac{8\lambda_{\max}(H)}{\rho^2 \sigma_{\min}(P_k)} \|P_k g_{k+1}\|
\]

\[
\leq \frac{8\lambda_{\max}(H)}{\rho^2 \sigma_{\min}(P_k)} \left(\|P_k(g_{k+1} - g_k - H_k(x_{k+1} - x_k))\| + \|P_k g_k + P_k H_k(x_{k+1} - x_k)\|\right).
\]

Where the first inequality holds by (5.27), and the second holds by Lemma 5.19.

By Lemma 5.22 and a similar equation similar to (5.55)(where \(x_k\) is replaced by \(x_{k+1}\) and \(\bar{x}\) is replaced by \(x_k\), we have that

\[
\|P_k(g_{k+1} - g_k - H_k(x_{k+1} - x_k))\| \leq L_H \|P_k\| \left(\frac{4}{\sigma_{\min}(P_k^T)} \left(2 + \frac{1}{c_1 - 1}\right) \sqrt{\frac{C_n}{s}}\right)^2 \|x_k - \bar{x}\|^2.
\]
Furthermore,
\[
\|P_k g_k + P_k H_k (x_{k+1} - x_k)\| = \|(I_s - P_k H_k P_k^\top (P_k H_k P_k^\top + \eta_k I_s)^{-1}) P_k g_k\|
\leq \|\eta_k (P_k H_k P_k^\top + \eta_k I_s)^{-1} P_k g_k\|
\leq \frac{\eta_k}{\sigma_{\min}(P_k)} \|P_k^\top (P_k H_k P_k^\top + \eta_k I_s)^{-1} P_k g_k\|
= \frac{\eta_k}{\sigma_{\min}(P_k)} \|d_k\|.
\]
Using that \(\|\eta_k\| \leq c_1 \|A_k\| + c_2 \|g_k\|\) and that \(\|g_k\| = O(\|x_k - \bar{x}\|)\) by Lemma 5.18, we deduce, by Lemmas 5.21 and 5.22, that there exists some constants \(\alpha, \beta, \beta' > 0\) such that with probability at least \(1 - 2 \exp(-s)\),
\[
\frac{\eta_k}{\sigma_{\min}(P_k)} \|d_k\| \leq \frac{1}{\sigma_{\min}^2(P_k)} \left( c_1 \alpha \|x_k - \bar{x}\|^2 + c_2 \beta \|x_k - \bar{x}\| \|g_k\|^\delta \right)
\leq \frac{1}{\sigma_{\min}^2(P_k)} \left( c_1 \alpha \|x_k - \bar{x}\|^2 + c_2 \beta' \|x_k - \bar{x}\|^{1+\delta} \right),
\]
where we have used in the second inequality that \(\|g_k\| \leq O(\|x_k - \bar{x}\|)\). Putting everything together, we obtain the desired result.

Notice that by using [23], we can furthermore bound \(\frac{1}{\sigma_{\min}(P_k)}\), with high probability, by \(O(\frac{1}{\sqrt{s} - \sqrt{r} - 1})\). Hence one can derive a version, in expectation of Theorem 5.16, where no probability bound appear, as we did for Corollary 5.10.

Finally notice that Assumption 5.11 can be easily satisfied by considering functions with low effective dimension, that is function of the form \(y \mapsto f(Ry)\), where \(y \in \mathbb{R}^r\) and where \(R \in \mathbb{R}^{n \times r}\) is an unknown matrix.

**6. Numerical illustration.** In this section, we illustrate numerically the randomized subspace regularized Newton method (RS-RNM). All results are obtained using Python scripts on a 12th Gen Intel(R) Core(TM) i9-12900HK 2.50 GHz with 64GB of RAM. As a benchmark, we compare it against the gradient descent method (GD) and the regularized Newton method (RNM) [26]. Here we do not aim to prove that our method is faster to the state-of-the-art methods but rather to illustrate the theoretical results that have been proved in the previous sections.

The methods are tested on a support vector regression problem formulated as

\[
(6.1) \quad f(w) = \frac{1}{m} \sum_{i=1}^{m} \ell(y_i - x_i^\top w) + \lambda \|w\|^2.
\]

Here, \((x_i, y_i) \in \mathbb{R}^n \times \{0, 1\} (i = 1, 2, \ldots, m)\) denote the training example and \(\ell\) is the loss function, \(\lambda\) is a constant of the regularizer and is fixed to 0.01 in the numerical experiments below. We note that 6.1 is a type of (generalized) linear model used in the numerical experiments of [11] and [13]. As the loss function \(\ell\), we use the following two functions known as robust loss functions: the Geman-McClure loss function \((\ell_1)\) and the Cauchy loss function \((\ell_2)\) [2] defined as

\[
\ell_1(t) = \frac{2t^2}{t^2 + 4},
\ell_2(t) = \log \left( \frac{1}{2} t^2 + 1 \right).
\]
Since both loss functions $\ell_1$ and $\ell_2$ are non-convex, the objective function 6.1 is non-convex.

The search directions at each iteration in GD and RNM are given by

$$d_{k}^{GD} = -\nabla f(w_k),$$
$$d_{k}^{RNM} = -(\nabla^2 f(w_k) + c_1' \Lambda_k' I_n + c_2' \|\nabla f(w_k)\|^{\gamma'} I_n)\nabla f(w_k),$$

$$(\Lambda_k' = \max(0, -\lambda_{\text{min}}(\nabla^2 f(w_k))))$$

and the step sizes are all determined by Armijo backtracking line search (3.4) with the same parameters $\alpha$ and $\beta$ for the sake of fairness. The parameters shown above and in Section 3 are fixed as follows:

$$c_1 = c_1' = 2, c_2 = c_2' = 1, \gamma = \gamma' = 0.5, \alpha = 0.3, \beta = 0.5, s \in \{100, 200, 400\}.$$

We test the methods on internet advertisements dataset from UCI repository\[10\] that is processed so that the number of instances is 600 ($= m$) and the number of data attributes is 1500 ($= n$), and the results until the stop condition $\|\nabla f(w_k)\| < 10^{-4}$ is satisfied are shown in 6.1. Our first observation is that RS-RNM converges faster than GD. GD does not require the calculation of Hessian or its inverse, making the time per iteration small. However, it usually needs a large number of iterations, resulting in slow convergence. Next, we look at the comparison between RNM and RS-RNM. From (a)(b) in 6.1, we see that RNM has the same or a larger decrease in the function value in one iteration than RS-RNM, and it takes fewer iterations to converge. This is possibly due to the fact that RNM determines the search direction in full-dimensional space. In particular, it should be mentioned that RNM converges rapidly from a certain point on, as it is shown that RNM has a super-linear rate of convergence near a local optimal solution. However, as shown in (c)(d) of 6.1, since RNM takes a long time to get close to the local solution along with the heavy calculation of the full regularized Hessian, RS-RNM results in faster convergence than RNM. We also confirm on figure (b) that for small dimensions $s = 100, 200$ a linear convergence rate seems to be achieved. However for $s = 400$ it seems that the method converges super-linearly.

7. Conclusions. Random projections have been applied to solve optimization problems in suitable lower-dimensional spaces in various existing works. In this paper, we proposed the randomized subspace regularized Newton method (RS-RNM) for a non-convex twice differentiable function in the expectation that a framework for the full-space version [26, 27] could be used; indeed, we could prove the stochastic variant of the same order of iteration complexity, i.e., the global complexity bound of the algorithm: the worst-case iteration number $m$ that achieves $\min_{k=0,\ldots,m-1} \|\nabla f(x_k)\| \leq \varepsilon$ is $O(\varepsilon^{-2})$ when the objective function has Lipschitz Hessian. On the other hand, although RS-RNM uses second-order information similar to the regularized Newton method having a super-linear convergence, we proved that it is not possible, in general, to achieve local super-linear convergence and that local linear convergence is the best rate we can hope for in general. We were however able to prove super-linear convergence in the particular case where the Hessian is rank deficient at a local minimizer. In this paper we choose to thoroughly investigate local convergence rate for the Newton-based method. One could possibly, in a future work, extend these results to a state-of-the-art second order iterative method and compare the resulting subspace method with other state-of-the-art algorithms.
Fig. 6.1. Results by GD, RNM and RS-RNM ($s = 100, 200, 400$) under the parameter setting: $c_1 = c_1' = 2, c_2 = c_2' = 1, \gamma = \gamma' = 0.5, \alpha = 0.3, \beta = 0.5$ for RNM and RS-RNM.

REFERENCES

[1] R. Andreev, A note on the norm of oblique projections, Applied Mathematics E-Notes, 14 (2014), pp. 43–44.
[2] J. T. Barron, A general and adaptive robust loss function, in Proc. IEEE Comput. Soc. Conf. Comput. Vis. Pattern Recognit., 2019, pp. 4331–4339.
[3] L. Bottou, F. E. Curtis, and J. Nocedal, Optimization methods for large-scale machine learning, SIAM REVIEW, 60 (2018), pp. 223–311.
[4] C. Cartis, E. Massart, and A. Otemissov, Bound-constrained global optimization of functions with low effective dimensionality using multiple random embeddings, Mathematical Programming, (2022), pp. 1–62.
[5] C. Cartis, E. Massart, and A. Otemissov, Global optimization using random embeddings, Mathematical Programming, 200 (2023), pp. 781—829.
[6] C. Cartis and A. Otemissov, A dimensionality reduction technique for unconstrained global optimization of functions with low effective dimensionality, Information and Inference: A Journal of the IMA, 11 (2021), pp. 167–201.
[7] L. Chen, X. Hu, and H. Wu, Randomized fast subspace descent methods, arXiv preprint arXiv:2006.06589, (2020).
[8] C. Coralia, F. Jaroslav, and S. Zhen, Randomised subspace methods for non-convex optimization, with applications to nonlinear least-squares, arXiv preprint arXiv:2211.09873v1, (2022).
[9] F. Coralia, Cartis andJarošlav and S. Zhen, A randomised subspace gauß-newton method for nonlinear least-squares, In Thirty-seventh International Conference on Machine Learning, 2020. In Workshop on Beyond First Order Methods in ML Systems, (2020).

[10] D. Dua and C. Graff, UCI machine learning repository, 2017, http://archive.ics.uci.edu/ml.

[11] R. Gower, D. Kovalev, F. Lieder, and P. Richtárik, RSN: Randomized Subspace Newton, Adv. Neural Inf. Process. Syst., 32 (2019), pp. 616–625.

[12] D. Grishchenko, F. Iutzeler, and J. Malick, Proximal gradient methods with adaptive subspace sampling, Mathematics of Operations Research, 46 (2021), pp. 1303–1323.

[13] F. Hanzeley, N. Doikov, P. Richtárik, and Y. Nesterov, Stochastic subspace cubic Newton method, in Proceedings of the 37th International Conference on Machine Learning, H. D. III and A. Singh, eds., vol. 119 of Proceedings of Machine Learning Research, PMLR, 13–18 Jul 2020, pp. 4027–4038.

[14] R. Jiang and X. Li, Hölderian error bounds and kurdyka-Lojasiewicz inequality for the trust region subproblem, Mathematics of Operations Research, 47 (2022), pp. 3025–3050, https://doi.org/10.1287/moor.2021.1243.

[15] W. Johnson and J. Lindenstrauss, Extensions of Lipschitz mappings into a Hilbert space, in Conference in Modern Analysis and Probability, G. Hedlund, ed., vol. 26 of Contemporary Mathematics, Providence, 1984, American Mathematical Society, pp. 189–206.

[16] D. Kovalev, R. M. Gower, P. Richtárik, and A. Rogozin, Fast linear convergence of randomized bfgs, arXiv preprint arXiv:2002.11337, (2020).

[17] D. Kozak, S. Becker, A. Doostan, and L. Tenorio, A stochastic subspace approach to gradient-free optimization in high dimensions, Computational Optimization and Applications, 79 (2021), pp. 339–368.

[18] D. Kozak, S. Becker, A. Doostan, and L. Tenorio, A stochastic subspace approach to gradient-free optimization in high dimensions, Computational Optimization and Applications, 79 (2021), pp. 339–368.

[19] J. Lacotte, M. Pilanci, and M. Pavone, High-Dimensional Optimization in Adaptive Random Subspaces, Curiat Associates Inc., Red Hook, NY, USA, 2019.

[20] C.-P. Lee, P.-W. Wang, and C.-J. Lin, Limited-memory common-directions method for large-scale optimization: convergence, parallelization, and distributed optimization, Mathematical Programming Computation, (2022), pp. 1–49.

[21] Y. Nesterov and B. Polyak, Cubic regularization of newton method and its global performance, Mathematical Programming, 108 (2006), pp. 177–205.

[22] L. Roberts and C. W. Royer, Direct search based on probabilistic descent in reduced spaces, arXiv preprint arXiv:2204.01275, (2022).

[23] M. Rudelson and R. Vershynin, Smallest singular value of a random rectangular matrix, Communications on Pure and Applied Mathematics, 62 (2009), pp. 1707–1739, https://doi.org/10.1002/cpa.20294.

[24] Z. Shao, On random embeddings and their application to optimisation, arXiv preprint arXiv:2206.03371, (2022).

[25] S. U. Stich, C. L. Müller, and B. Gärtner, Optimization of convex functions with random pursuit, SIAM Journal on Optimization, 23 (2013), pp. 1284–1309.

[26] K. Ueda and N. Yamasita, Convergence properties of the regularized newton method for the unconstrained nonconvex optimization, Appl. Math. Optim., 62 (2010), pp. 27–46.

[27] K. Ueda and N. Yamasita, A regularized newton method without line search for unconstrained optimization, Comput. Optim. Appl., 59 (2014), pp. 321–351.

[28] R. Vershynin, High-dimensional probability: An introduction with applications in data science, vol. 47, Cambridge university press, 2018.

[29] S. J. Wright, Coordinate descent algorithms, Math. Program., 151 (2015), pp. 3–34.

[30] P. Xu, F. Roosta, and M. W. Mahoney, Second-order optimization for non-convex machine learning: an empirical study, Proceedings of the 2020 SIAM International Conference on Data Mining (SDM), (2020), pp. 199–207.

[31] Z. Yao, Efficient second-order methods for non-convex optimization and machine learning, UC Berkeley Electronic Theses and Dissertations, (2021), https://escholarship.org/uc/item/0431q1ws.