Continuous quantum error correction as classical hybrid control

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Abstract. The standard formulation of quantum error correction (QEC) comprises repeated cycles of error estimation and corrective intervention in the free dynamics of a qubit register. QEC can thus be seen as a form of feedback control, and it is of interest to seek a deeper understanding of the connection between the associated theories. Here we present a focused case study within this broad program, connecting continuous QEC with elements of hybrid control theory. We show that canonical methods of the latter engineering discipline, such as recursive filtering and dynamic programming approaches to solving the optimal control problem, can be applied fruitfully in the design of separated controller structures for quantum memories based on coding and continuous syndrome measurement.

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1. Introduction

Prospects for quantum computation have motivated the development of an extensive theory of quantum error correction (QEC) [1], which thus far has been founded largely upon related aspects of classical information theory. One of the most prominent recent trends in QEC research has been the development of enlightening connections with quantum many-body physics, for example through Kitaev’s work on toric codes and non-Abelian anyons [2]. The associated influx of new ideas is expanding our understanding of how decoherence can be suppressed, and could lead directly to the discovery of especially powerful and/or efficient quantum memory architectures in certain amenable experimental implementations. The primary motivation for this paper is to advance a similar strategy of broadening our scope of thinking about QEC, now in the direction of systems engineering rather than theoretical physics. For reasons that will be discussed at the end of the paper, by this strategy we hope in particular to facilitate the development of efficient and robust quantum memory architectures in experimental implementations that can provide electromagnetic coupling of physical qubits via networks of optical [3] or microwave [4] waveguides and resonators.

We specifically revisit the framework of QEC via coding and continuous syndrome measurement proposed by Ahn and co-workers [5]–[10], and show how an associated problem of optimizing the correction protocol can be cast in the conceptual framework of hybrid control theory. The feedback loop is decomposed into the sequential action of an error state estimator and an optimal controller, where the latter component executes a simple threshold decision policy. While we believe that the design methods utilized are new in the context of quantum information science and quantum control, they are textbook material in classical control theory. We therefore hope that these and similar techniques can easily be adopted by the QEC and quantum control communities if they prove to be of interest.

2. Continuous QEC model

For efficiency of presentation we will focus in this paper mainly on a continuous-time relaxation of the bit-flip code as introduced by Ahn et al [5], although our methods can be adapted straightforwardly to arbitrary Stabilizer codes and general single-qubit errors (some numerical simulations for the five-qubit code will be presented). A logical qubit state $|\Psi\rangle = c_0|0\rangle + c_1|1\rangle$ is encoded in the joint state of three physical qubits via $|\Psi\rangle \mapsto c_0|000\rangle + c_1|111\rangle \equiv |\Psi_E\rangle$. The system is coupled to decay channels that can cause independent Markovian bit flips and also to probe channels that allow for continuous (finite-strength) syndrome measurement (see [11] for a proposed physical implementation of the continuous syndrome measurements). Following [5, 7], we assume that the syndrome measurements are constructed by coupling two probe fields to the syndrome generators [1] $M_1 = ZZI = \sigma_z \otimes \sigma_z \otimes 1$ and $M_2 = ZIZ = \sigma_z \otimes 1 \otimes \sigma_z$. The total system dynamics can be described by the quantum stochastic differential equation (e.g. [12])

$$dU_t = \left\{ \sqrt{\gamma} \left( X_1 dB_{1t}^\dagger + X_2 dB_{2t}^{2\dagger} + X_3 dB_{3t}^{3\dagger} - \text{h.c.} \right) + \sqrt{\kappa} \left( M_1 dB_{1t}^{1\dagger} + M_2 dB_{2t}^{2\dagger} - \text{h.c.} \right) \right\} U_t,$$

(1)
where $B_i^t$ are the bit-flip channels, $A_i^t$ are the probe channels and $X_1 = X I I = \sigma_x \otimes 1 \otimes 1$, $X_2 = I X I$ etc. The unitary evolution $U_i$ is a Schrödinger picture propagator for the entire system including the probe fields. From a control theoretic perspective, equation (1) represents the dynamics of the plant that we will be trying to control. From a QEC perspective we can simply think of this as a quantum physical model that realizes the idea of simultaneous bit-flip decoherence and continuous syndrome measurements.

For homodyne-type detection of the two probe channels we obtain two observation processes $Y_i^t = U_i^+ (A_i^+ + A_i^+) U_i$ that can be written (via quantum Itô rules) as

$$dY_i^t = 2\sqrt{\kappa} U_i^+ M_i U_i \, dt + dA_i^+ + dA_i^+. \tag{2}$$

These are the real-time signals from which recursive estimates of the error syndrome are derived in the QEC setting. Each is the sum of an informative part $2\sqrt{\kappa} U_i^+ M_i U_i \, dt$ and an information-less part $dA_i^+ + dA_i^+$, which can be thought of as a Gaussian white noise. The ratio of these two contributions reflects the strength (signal-to-noise ratio) of the continuous syndrome measurements, and should be finite in any physically realistic setting. Note that the smaller the signal-to-noise ratio the longer we must average such observations in order to determine the syndrome with confidence, and if the signal-to-noise ratio is too low for a given error rate it will not generally be possible to keep track of all the changes in the syndrome (and fidelity will thus be lost quite rapidly). This is the manifestation in the continuous QEC setting of the familiar fact that in discrete QEC the time interval between syndrome measurements must be small compared with the timescale on which errors occur. Here, unlike in the discrete setting, we see that the finite strength of realistic physical measurement interactions is naturally captured by finite parameters in a continuous-time dynamical model (see [11] for more details in an atomic physics setting).

Theoretically speaking we could also consider observation of the bit-flip channels $B_i^t$, for example by direct detection, which would give rise to additional (counting) observations $Z_i^t$ that are independent Poisson processes with rate $\gamma$ (see e.g. [13]); every ‘click’ in the observation signal $Z_i^t$ would then indicate the occurrence of an environmental bit flip of the $i$th register qubit. Of course the $Z_i^t$ are generally assumed to be unobservable in practical QEC scenarios—if we could actually measure them we would not need the quantum code and syndrome measurements—but below we will keep track of them in numerical simulations in order to correlate the behavior of the syndrome measurement signals $dY_i^t$ with the ‘actual’ timing of bit-flip errors.

Our QEC goal is to detect and to correct bit-flip errors by making use of the probe currents $Y_i^t$, $i = 1, 2$ only. Hence we must understand how best to extract information about error events from the noisy observed signals. We begin by calculating the least mean-square estimator for timing of bit-flip errors.

For every system observable $A$, $\text{Tr}[A \rho_t]$ is the function of the observation history that minimizes the estimation error $\langle (U_i^+ A U_i - \text{Tr}[A \rho_t])^2 \rangle$. If we observe only $Y_i^t$, then $\rho_t$ obeys the quantum filtering equation [13]

$$d\rho_t = \sum_{k=1}^{3} \gamma T[X_k] \rho_t \, dt + \sum_{i=1}^{2} \kappa T[M_i] \rho_t \, dt + \sum_{i=1}^{2} \sqrt{\kappa} \mathcal{H}[M_i] \rho_t (dY_i^t - 2\sqrt{\kappa} \, \text{Tr}[M_i \rho_t] \, dt), \tag{3}$$

where $T[X] \rho = X \rho X^\dagger - \rho$ and $\mathcal{H}[X] \rho = X \rho + \rho X^\dagger - \text{Tr}[(X + X^\dagger) \rho] \rho$. This is equivalent to the stochastic master equation used in [5]. If we were additionally to observe the bit flip channels $Z_i^t$, we could obtain an improved estimator $\tilde{\rho}_t$ that would obey a ‘jump-unraveled’
version of equation (3) with \( \gamma \mathcal{T}[X_k] \, dt \) replaced by \( \mathcal{T}[X_k] \, dZ_i' \). An important feature of these equations [13] is that the so-called innovations processes \( dW_i^j = dY_i^j - 2 \sqrt{\kappa} \text{Tr}[M_j \rho_i] \, dt \) are independent and have the law of a Wiener process.

We can use the fact that the innovations are Wiener processes, in the absence of ‘real’ (physically generated) measurement signals \( Y_i^j \), to generate the latter through Monte Carlo simulations. By driving equation (3) with random sample paths of a Wiener process, we can reconstruct observation processes \( Y_i^j \) that sample the space of measurement records with the correct probability measure. The evolution of the filter variables in each simulation fairly represents what would have happened if physically generated measurement records \( Y_i^j \) had actually been presented to the filter and it derived \( W_i^j \) from them. Similarly, we can reconstruct \( Y_i^j \) from the jump-unraveled version of equation (3) by driving it with Wiener processes \( \tilde{W}_i^j \) and independent Poisson processes \( Z_i \) with rate \( \gamma \). The innovations theorem guarantees that both simulations will generate sample paths \( Y_i^j \) with the same probability measure. We will invoke this property later.

3. Error-state estimator

In the usual setting of discrete quantum codes, an instantaneous measurement of \( M_1 \) and \( M_2 \) after a period of free evolution is used to determine the recovery operation (if any) that should be applied. Of course, the notion of perfect instantaneous measurement is a theoretically convenient abstraction but not physically realistic. If \( (M_1, M_2) = (+1, +1) \) no correction is necessary; outcomes \((-1, +1)\) mean that \( I X I \) should be applied; \((+1, -1) \Rightarrow I I X \) and \((-1, -1) \Rightarrow X I I \). These outcomes are called the error syndromes. In the continuous setting, we introduce the orthogonal projectors \( \Pi_0, \ldots, \Pi_3 \) onto the eigenspaces corresponding to each syndrome. The quantity \( p_{i}^m = \text{Tr}[\Pi_m \rho_i] \) is then the conditional probability (given the noisy probe observations up to time \( t \)) that, had we actually measured \( M_1 \) and \( M_2 \), we would have obtained the syndrome corresponding to \( \Pi_m \). Plugging \( p_{i}^m \) into equation (3) we obtain the syndrome filter

\[
\frac{dp_i}{dt} = \Lambda^T p_i \, dt + \sum_{j=1}^{2} (H_i - h_i^T p_i \, \mathbb{1}) p_i \, (dY_i^j - h_i^T p_i \, dt),
\]

where \( h_i^m/2\sqrt{\kappa} \) is the outcome of \( M_i \) corresponding to the syndrome \( \Pi_m \), \( H_i = \text{diag} \, h_i \), and \( \Lambda_{mn} = \gamma (1 - 4\delta_{mn}) \). The \( p_i^j \) form a closed set of equations, as the observations are uninformative on the logical state of the qubit and the coherences (if any) between the syndromes.

The syndrome filter, equation (4), is a familiar equation from classical probability theory; we will gain important insight by reducing our problem to a classical one (recall that the essence of the Stabilizer formalism lies in reducing the problem of error identification to one involving only a commutative set of observables, thereby making it formally classical in nature). Consider a system that can be in one of four states labeled 0, \ldots, 3. Suppose the system is in some known state at time \( t \). After an infinitesimal time increment \( dt \) the system can switch to one of the other three states, each of which occurs with probability \( \gamma \, dt \). This defines a Markov jump process on a graph, as depicted in figure 1(a). Unfortunately we cannot observe the state directly; instead, we are given two observation processes of the form \( dY_i^j = h_i^m \, dt + dV_i^j \) where \( V_i^{1,2} \) are two independent Wiener processes that corrupt our observations, and \( m_i \) is the state of the Markov jump process at time \( t \). As our observations are noisy we cannot know the system state with certainty at any time. However, we can calculate the \textit{conditional} probability \( p_{i}^{m} \) that it is in state...
Figure 1. Markov chains associated with the three-qubit code: (a) the syndrome chain, and (b) the extended correction chain with the corresponding syndromes labeled between brackets. All transitions are independent with rate $\gamma$.

$m$ at time $t$, based on our knowledge of the $dY_i^j$. This classical estimation problem is precisely solved by equation (4), known as the Wonham filter [14, 15].

To help interpret this result, consider the jump-unraveled version of equation (3). In the same way that we obtained equation (4), we can substitute $\tilde{p}_t = \text{Tr} [\Pi_m \tilde{\rho}_t]$, and get a closed form expression. Assuming the initial state lies inside one of the syndrome spaces (this is not an essential restriction, as the probability measure on the space of measurement records can be written for any initial state as the corresponding mixture of such measures given a fixed initial syndrome), it is readily verified that $[M_1, \tilde{\rho}_t] = [M_2, \tilde{\rho}_t] = 0$ for all $t$ and we obtain

$$d\tilde{p}_t = \sum_{k=1}^3 (\tilde{X}_k - \tilde{\pi}) \tilde{p}_t \, dZ_k^t,$$

where $\tilde{\pi}_0$ is one of the unit vectors $e_n^m = \delta_{nm}$. Here $\tilde{X}_1$ is a matrix such that $\tilde{X}_1 e_0 = e_1$, $\tilde{X}_1 e_1 = e_0$, $\tilde{X}_1 e_2 = e_3$ and $\tilde{X}_1 e_3 = e_2$, and $X_{2,3}$ are defined similarly as shown in figure 1(a). The solution of this equation is of the form $\tilde{p}_t = e_m$, where $m_t$ is the Markov jump process defined above. Since

$$d\tilde{W}_i^t = dY_i^t - h_i^T \tilde{p}_t \, dt = dY_i^t - h_i^m \, dt,$$

must be a Wiener process that is independent from all $Z_i^t$, the statistics of the probe observations obtained from the quantum system are precisely described by the classical model of the previous paragraph. The Markov process $m_t$ is simply the error syndrome obtained by observing the bit flips directly, and the Wonham filter above has a natural classical interpretation as the best estimate of $m_t$ given only the noisy probe observations.

4. Feedback strategies for error correction

We now turn to the problem of error correction. Suppose that we let the system evolve for some time while propagating the filter equation (4) with the observations. At some time $T$ we pose the question: what operation, if any, should we perform on the system to maximize the probability of restoring the initial logical state $|\Psi_t\rangle$? We will assume that we can pulse the system sufficiently strongly (as compared to $\kappa, \gamma$) to perform effectively instantaneous bit flips on any of the physical qubits (this is of course an approximation, but one that is much easier to justify than that of instantaneous measurements since here we require only very strong control fields and not very strong Hamiltonian couplings between register qubits and ancillae). The most obvious decision strategy simply mimics discrete error correction by considering the
probabilities generated by the syndrome filter at regular intervals $T$—given the most likely syndrome state $m_s = \arg \max_mp_t^m$, we do nothing if $m_s = 0$ and otherwise we perform a bit flip on physical qubit $m_s$.

But it is possible to do better. Recall that the discrete error correction strategy is based on an assumption that at most one bit flip occurs in the interval $[0, T]$. This assumption may not hold in practice. With our continuous syndrome measurement, we do actually have some basis for estimating the total number (and kind) of bit flips that have occurred. Unfortunately this information does not reside in the statistic $p_t$, which only gives the conditional probabilities of the syndromes at the current time. We are seeking a non-Markovian decision policy that knows something about the history of the bit flips.

The classical machinery introduced above allows us to solve this problem optimally. To do this we simply extend the Markov jump process $m_t$ as shown in figure 1(b). The states of the extended chain $\hat{m}_t$ are no longer the four syndromes but the eight error states that may obtain at any given time. Every syndrome corresponds to two error states, as is shown in figure 1(b). We still consider the same observation processes, so error states that belong to the same syndrome give rise to identical observations. Thus on the basis of the observations, the two Markov chains are indistinguishable. Nonetheless the extended chain gives rise to a different estimator that provides precisely the information we want. As by construction the Wonham filter provides the optimal estimate, we conclude that the optimal solution to our problem is given by the Wonham filter for the extended chain, i.e. the eight-dimensional (8D) equation

$$d\hat{p}_t = \hat{\Lambda}^T \hat{p}_t \, dt + \sum_{i=1}^{2} (\hat{H}_i - \hat{h}_i^T \hat{p}_t, \mathbb{1}) \hat{p}_t \, (dY_i^t - \hat{h}_i^T \hat{p}_t \, dt),$$

where $\hat{\Lambda}$, $\hat{h}_i$ are the intensity matrix and observation vector for the chain $\hat{m}_t$ (see [14, 15] for details). The optimal correction policy is now simple: at time $T$, we perform the correction that corresponds to the state $\hat{m}_s = \arg \max_{\hat{m}} \hat{p}_T^\hat{m}$. Hence if $\hat{m}_s = III$ we do nothing, if $\hat{m}_s = IXX$ we flip physical qubits 2 and 3, etc. This maximizes the probability of restoring $|\Psi_E\rangle$. Alternatively, note that since we are keeping track of the probabilities of every possible error state of the code and the fact that all of these are invertible (thanks to the magic properties of the quantum code), we can simply let errors accumulate for the entire logical qubit storage period and then correct only the final error state at the end. Of course if the desired storage period is long compared to the signal-to-noise ratio we have in our continuous syndrome measurement, we expect that fidelity should be lost (this is again not a unique defect of the continuous QEC approach—the same is true of discrete QEC if the total storage period is long compared to the square of the product of the error rate and measurement interval).

From figure 1(b) we can see how information is lost from the quantum memory in our continuous QEC scenario. At time $t = 0$, we begin in the no-error state $III$. A bit flip might occur which puts us, e.g. in the state $XII$, then $XIX$, etc. But as we are observing these changes in white noise there is always a chance that when two bit flips happen in rapid succession, we ascribe the corresponding observations to a fluctuation in the white noise background rather than to the occurrence of two bit flips. In essence, successive bit flips are resolvable only if they are separated by a sufficiently long interval that the filter can average away the white noise fluctuations (the signal-to-noise ratio $\sim 4\kappa$ determines this timescale). Occasionally, multiple bit flips occur too rapidly and information is lost (e.g. $III \rightarrow XII \rightarrow XIX$ may be mistaken for $III \rightarrow IXI$ since the two final states have identical syndromes). It is evident from figure 1(b) that this rate of information loss is independent of the error state. Hence as noted above there
is no essential benefit in applying corrective bit flips at intermediate times \( t < T \), as this cannot slow the loss of information. As the Wonham filter is by construction the optimal method for tracking the error state given the information in the continuous syndrome measurement signals \( dY_t \), we conclude that either of the correction strategies described above can be considered optimal if our only figure of merit is the fidelity at the end of the storage period, and (in case we follow a strategy of correcting whenever \( m_s \neq \text{III} \)) we allow ourselves an unlimited number of control pulses. (This assumes that we trust equation (1) completely; if there is some uncertainty in the model it is sometimes advantageous to consider different estimators [16].) In the following section, we will consider a more sophisticated optimal control scenario in which more subtle feedback strategies are required.

How can we quantify the information loss from the system? By construction \( \hat{p}^*_t = \max_m \hat{p}^*_tm^\ast \) is the probability of correct recovery at time \( t \). Unfortunately, as one can see in figure 2, the quantity \( \hat{p}^*_t \) fluctuates rather wildly in time. Thus it is not a good measure of the information content of the system, as it is very sensitive to the whims of the filter: the filter may respond to fluctuations in the measurement record by adjusting the state as if a bit flip had occurred, but then correct itself when it becomes evident that nothing happened. We actually want to find some quantity that gives a (sharp) upper bound on all future values of \( \hat{p}^*_t \). This would truly measure the information content of the system, as it bounds the probability of correct recovery that can be achieved.

We claim that the quantity \( \tilde{\gamma}_t = \max_m (\hat{p}^*_t m^\ast / (\hat{p}^*_tm^\ast )) \), which is a function of filter variables, provides a suitable measure of the information content at time \( t \). Here \( m \neq m^\ast \) is the error state that corresponds to the same syndrome as \( m \), so \( \hat{p}^*_t m^\ast + \hat{p}^*_tm^\ast = P^m_t \) is the probability of the syndrome corresponding to \( m \). Hence we can interpret \( \tilde{\gamma}_t \) as the conditional probability of the error state \( m \), given that the system is in the corresponding syndrome. Define \( I_t^m = \hat{p}^*_t m^\ast / (\hat{p}^*_tm^\ast ) \) so that \( \tilde{\gamma}_t = \max_m I_t^m \). Direct application of the Itô rules gives

\[
\frac{dI_t^m}{dr} = - \sum_{n \neq m} \tilde{\Lambda}_{nm} \frac{P_n}{P_t} (I_t^m - I_t^n).
\]

If we define \( m^*_t = \arg \max_m I_t^m \), then we obtain

\[
\frac{d\tilde{\gamma}_t}{dr} = - \sum_{n \neq m^*_t} \tilde{\Lambda}_{nm^*_t} \frac{P_n}{P_t} (\tilde{\gamma}_t - I_t^n) \leq 0.
\]

(To make the argument completely rigorous one must check that this equation is well defined, i.e. that \( d\tilde{\gamma}_t / dr \) exists. This can be done using the methods in [17].) Hence \( \tilde{\gamma}_t \) decreases monotonically, and moreover by construction we must have \( \hat{p}^*_t m^\ast \leq I_t^m \), so \( \hat{p}^*_t \leq \tilde{\gamma}_t \leq \tilde{\gamma}_t \) for all \( s < t \). Thus evidently \( \tilde{\gamma}_t \) bounds all future values of \( \hat{p}^*_t \). One would expect the bound to be tight for sufficiently high signal-to-noise (as then \( P_t^m T \) will be close to one), which is indeed the case as can be seen in figure 2.

The procedure we have outlined can be generalized to other stabilizer codes, such as the five-qubit code [1]. This code protects one logical qubit against single-qubit errors by encoding in five physical qubits and measuring four stabilizer generators. For ‘Pauli channel’ decoherence described by Lindblad terms \( \sum_{k=1}^5 \gamma (T[X_k] + T[Y_k] + T[Z_k]) \), the error state graph can be constructed by considering all possible assignments of an error state \( e \in \{ I, X, Y, Z \} \) to each qubit and by connecting every pair of states that are related by the action of a Pauli operator \( e \in \{ X, Y, Z \} \) on one qubit. One thus has a graph with \( 4^5 = 1024 \) nodes, with each node connected to \( 3 \times 5 = 15 \) other nodes (we have validated this construction by comparing simulations of the

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corresponding Wonham filter with an appropriate stochastic master equation). The total error rate is $\Gamma = 15\gamma$. Figure 2 shows a portion of a single Monte Carlo simulation of the Wonham filter for the five-qubit code; figure 3 shows averages of $3_t$ over tens of trajectories each for $\kappa/\Gamma \in \{10, 30, 100\}$.

5. Optimal controller

In order to further explore the connections between continuous QEC and control theory, we consider in this section a modified version of the above quantum memory scenario and show that the associated optimal control problem can be solved using standard methods of hybrid control. We assume that a logical state is loaded into the quantum memory at time $t = 0$. At some later time $\tau$, which we do not know in advance, we will need to deliver the contents of
the memory for immediate use and will not be allowed any delay for additional processing of the qubit register in response to the recall instruction. As is standard in queueuing theory we assume that $\tau$ is an exponentially distributed random variable. With these specifications it is clear that corrective feedback pulses will be required ‘on the fly’ (as opposed to deferring inversion of the error state to the end of the storage period), but we will assess a finite cost per correction pulse in order to constrain the number that are utilized. Such a penalty could be warranted for example in an atomic physics scenario if there is a finite probability per correction pulse of inducing transitions out of the qubit subspace (leakage error). The total cost functional is then a linear combination of the probability that the memory is not in a trivial error state at time $\tau$, and the expected number of corrections prior to time $\tau$. The optimal control problem we consider is to find an impulse control strategy that minimizes this cost.

Our technical approach will be to reduce the impulse control problem to a sequence of optimal stopping problems, as described for example in chapter 7 of [18]. The value function of the impulse control problem is the limit as $n \to \infty$ of the value functions of a sequence of control problems where the control strategy is constrained to consist of at most a fixed number $n$ of corrections. Using the dynamic programming principle this can be reduced to a sequence of optimal stopping problems. There are well-established methods to compute numerically the solution to such problems; we have used Markov chain approximations [19] for the example described below.

Exploiting the insights discussed in the previous sections, we formulate the optimal control problem directly in terms of the classical stochastic error state dynamics. Hence we define the syndrome process $(X_t)_{t \geq 0}$ as a finite-state Markov chain on the state space $\{0, 1, 2, 3\}$. The syndrome 0 is the ‘correct’ syndrome (trivial error state) and the others are the undesirable ones. The rates $\lambda_{ij}$ of jumping from state $i$ to state $j$ are assumed constant, $\lambda_{ij} = \gamma$ for all $i \neq j$. The control inputs act as follows (see figure 1(b)). Flipping the first qubit permutes $0 \leftrightarrow 1$ and $2 \leftrightarrow 3$; flipping the second qubit permutes $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$; and flipping the third qubit permutes $0 \leftrightarrow 3$ and $1 \leftrightarrow 2$. The 3D observation process is

$$
\begin{align*}
\frac{dY^1_t}{dt} &= 2\sqrt{h} \frac{d}{dt} h^1(X_t) dt + dW^1_t, \\
\frac{dY^2_t}{dt} &= 2\sqrt{h} \frac{d}{dt} h^2(X_t) dt + dW^2_t, \\
\frac{dY^3_t}{dt} &= 2\sqrt{h} \frac{d}{dt} h^3(X_t) dt + dW^3_t,
\end{align*}
$$

where $W^i$ are independent Wiener processes and the $h^i$ are functions defined as follows:

$$
\begin{align*}
&h^1([0, 1, 2, 3]) = \{1, -1, -1, 1\}, \\
&h^2([0, 1, 2, 3]) = \{1, -1, 1, -1\}, \\
&h^3([0, 1, 2, 3]) = \{1, 1, -1, -1\}.
\end{align*}
$$

Note that we are now assuming simultaneous measurement of all three parity functions rather than the minimal two (which was our choice above); it is in fact fine to set $h^3 = 0$ but the symmetry of the case with all three measurements will help to reduce somewhat the computations required for our optimal control solution. We denote the observation filtration as $\mathcal{F}_t = \sigma\{Y^1_s, Y^2_s, Y^3_s : s \leq t\}$.

We presume that the stored qubit will be recalled at a random time $\tau$, which is independent of $(X_t, Y_t)_{t \geq 0}$ and is exponentially distributed $\tau \sim \lambda e^{-\lambda t} dt$ with rate $\lambda > 0$. One term in our cost function will be the probability of being in an incorrect state at time $\tau$, which we note can...
be written
\[
P(X_t \neq 1) = \int_0^\infty \lambda e^{-\lambda t} P(X_t \neq 1) \, dt = \mathbb{E} \left[ \int_0^\infty \lambda e^{-\lambda t} P(X_t \neq 1 | \tilde{S}_t^Y) \, dt \right],
\]
where we have used independence of \(\tau\) and \((X_t)_{t \geq 0}\). Next, let \(\{\vartheta_n\}_{n \geq 1}\) be an impulse control policy (i.e. a sequence of stopping times at which we perform a correction such that \(0 < \vartheta_n < \vartheta_{n+1} \not\sim \infty\) a.s.) which is independent of \(\tau\) (as it must be, otherwise \((X_t, Y_t)_{t \geq 0}\) would not be independent of \(\tau\) when the control is active). Then
\[
\mathbb{E}(\#\{n : \vartheta_n \leq \tau\}) = \int_0^\infty \lambda e^{-\lambda t} \mathbb{E}(\#\{n : \vartheta_n \leq t\}) \, dt = \mathbb{E} \left[ \sum_{n=1}^{\infty} e^{-\lambda \vartheta_n} \right],
\]
where we have used independence and we have integrated by parts (to be completely precise, we must be careful only to allow admissible controls such that \(e^{-\lambda t} \#\{n : \vartheta_n \leq t\} \to 0\) as \(t \to \infty\) a.s.). Note that our control strategy must of course tell us not only the times \(\vartheta_n\) at which pulses much be applied but also which qubits should be flipped; we suppress this in our notation for the sake of readability. Likewise, it is implicit that \(X_t\) and therefore \(P(X_t \neq 1 | \tilde{S}_t^Y)\) depend on the control strategy.

We now have two competing goals: (i) we want to minimize the probability \(P(X_t \neq 1)\) that the memory is in the incorrect syndrome at the readout time; and (ii) we want to minimize the expected number of corrections prior to the readout time \(\mathbb{E}(\#\{n : \vartheta_n \leq \tau\})\). We therefore define a cost function that is a linear combination of these two quantities:
\[
J[\{\vartheta_n\}_{n \geq 1}] = \mathbb{E} \left[ \int_0^\infty \lambda e^{-\lambda t} P(X_t \neq 1 | \tilde{S}_t^Y) \, dt + \sum_{n=1}^{\infty} ce^{-\lambda \vartheta_n} \right],
\]
where the constant \(c > 0\) determines the relative importance we ascribe to our competing control goals. Note that \(J\) takes the standard form of a ‘discounted cost functional’. Our aim is to find an impulse control strategy \(\{\vartheta_n\}_{n \geq 1}\) that minimizes this cost, subject to the additional constraint that the correction time \(\vartheta_n\) is an \(\tilde{S}_t^Y\)-stopping time for every \(n\) (i.e. the optimal strategy is causally determined by the observations only). Note that as long as we can solve this optimal control problem for any value of \(c\), we can also solve the variational version of the problem: find the strategy that minimizes \(P(X_t \neq 1)\), subject to the constraint \(\mathbb{E}(\#\{n : \vartheta_n \leq \tau\}) \leq C\) (for some fixed \(C > 0\)). This is simply done by varying the constant \(c\) until the control that minimizes the cost \(J\) above satisfies \(\mathbb{E}(\#\{n : \vartheta_n \leq \tau\}) = C\) (thus \(c\) acts as a Lagrange multiplier).

As usual in measurement-feedback control theory, we separate the problem into a filtering and a control design step. The filter design follows methods analogous to those presented above in section 3 (we omit the details for brevity) and results in a set of conditional probabilities \(\pi_t^i = P(X_t = i | \tilde{S}_t^Y)\). Note that \(\pi_t^0 = 1 - \pi_t^1 - \pi_t^2 - \pi_t^3\). In terms of these, we may write
\[
J[\{\vartheta_n\}_{n \geq 1}] = \mathbb{E} \left[ \int_0^\infty \lambda e^{-\lambda t} (\pi_t^1 + \pi_t^2 + \pi_t^3) \, dt + \sum_{n=1}^{\infty} ce^{-\lambda \vartheta_n} \right].
\]
To distinguish between the controlled and the uncontrolled filter, we will denote by \( \tilde{\pi} \) a copy of the filter which has no controls applied to it. We now proceed to solve the optimal control problem via a sequence of optimal stopping problems. First, we define

\[
V^0(\pi) = \mathbb{E} \left[ \int_0^\infty e^{-\lambda t} (\tilde{\pi}_1^1 + \tilde{\pi}_1^2 + \tilde{\pi}_1^3) \, dt \mid \tilde{\pi}_0 = \pi \right].
\]  

(16)

This is the value function for the trivial control problem where the only admissible control is zero. We can next construct the value function \( V^1 \) for the optimal control problem in which we restrict ourselves to those control strategies that exercise at most one correction. To this end, define the *correction operator*

\[
\mathcal{M} h(\pi) = \min_{\tau} \{ h(\Gamma \pi) + c \},
\]

(17)

where \( \Gamma \) is one of the three permutations that correspond to the possible control actions. Then

\[
V^1(\pi) = \inf_{\tilde{\pi}} \mathbb{E} \left[ \int_0^\infty e^{-\lambda t} (\tilde{\pi}_1^1 + \tilde{\pi}_1^2 + \tilde{\pi}_1^3) \, dt + e^{-\lambda \tau} \mathcal{M} V^0(\tilde{\pi}) \mid \tilde{\pi}_0 = \pi \right],
\]

(18)

which is an optimal stopping problem. Proceeding recursively we can compute \( V^n \) in terms of \( V^{n-1} \) in the same way. It turns out that (see chapter 7 of [18]) the value functions converge monotonically,

\[
V^n(\pi) \geq V(\pi) = \inf_{\{\tilde{\pi}, \tilde{\pi}_{n}\geq 1\}} \mathbb{E} \left[ \int_0^\infty \lambda e^{-\lambda t} (\pi_1^1 + \pi_1^2 + \pi_1^3) \, dt + \sum_{n=1}^{\infty} ce^{-\lambda \delta_n} \mid \pi_0 = \pi \right].
\]

Likewise \( D^n \supset D^{n+1} \supset \cdots \supset D = \{ \pi : V(\pi) < \mathcal{M} V(n)(\pi) \} \), where \( D^n = \{ \pi : V^n(\pi) < \mathcal{M} V^{n-1}(\pi) \} \) and \( D \) is the *continuation region* of the optimal impulse control (see below). The numerical approach is now simple. We begin by evaluating \( V^0 \), which can be done analytically. We subsequently discretize the optimal stopping problem for \( V^1 \) and solve it numerically, via solution of the associated dynamic programming equation by Gauss–Seidel iteration (using an approximating Markov chain for the filter dynamics [19]). We then proceed recursively until \( V^n \) and \( V^{n-1} \) no longer differ substantially, in which case we use the continuation region \( D^n \) to define our approximately optimal strategy.

In figure 4 we show the results of a numerical example with the following parameter values: \( \gamma = \kappa = 1, \lambda = 0.1, c = 0.0035 \). We work on a grid with spacing 1/18 and choose a time step that is equal to the grid spacing; at this level of coarseness the required computations are easily performed on a desktop computer in several minutes. The axes of the 3D plot are the conditional probabilities \( \pi_1^1, \pi_1^2 \) and \( \pi_1^3 \); the expected permutation symmetry (due to the fact that we make all three parity measurements) is apparent. The green surface is the edge of the simplex (\( \pi_1^1 + \pi_1^2 + \pi_1^3 = 1 \)); the filter evolves ‘beneath’ this surface. The solid (red) volume is the continuation region for the optimal impulse control. The optimal policy is as follows: run the filter until it hits the boundary of the red region (it starts at \( t = 0 \) at the origin), then perform the correction that minimizes \( \pi_1^1 + \pi_1^2 + \pi_1^3 \) (the appropriate condition is easily read off from the plot—there are three correction regions/facets, one for each possible qubit-flip correction). The jaggedness of the boundary of the continuation region is an artifact of the grid size.

6. Proposed physical realizations

Although the results presented in this paper have been somewhat abstract in nature, we would like to close by noting that quantum memories based on quantum coding and continuous
Figure 4. Continuation region for the optimal impulse control strategy example (red); axes are $\pi^1$, $\pi^2$ and $\pi^3$. Green surface is the boundary of the simplex ($\pi_1 + \pi_2 + \pi_3 = 1$).
	syndrome measurement may be a natural and resource-efficient approach in implementations based on nanophotonics or circuit quantum electrodynamics (circuit QED). In [11] we have proposed a concrete physical setup—based on straightforward ideas of cavity quantum electrodynamics (cavity QED)—for realizing continuous parity syndrome measurements for the continuous quantum bit-flip code. We have derived analogous schemes for continuous phase-parity measurements as well, which should enable quantum memories for protecting against arbitrary single-qubit errors using Bacon–Shor-type codes [20].

The central requirement of our continuous syndrome measurement schemes is that each register qubit be strongly coupled to its own electromagnetic resonator; continuous parity measurements (for example) are then obtained by reflecting weak coherent-state optical probes sequentially from pairs of qubit cavities (see figure 5). The optical probes are detected using phase-quadrature homodyne receivers, which generates the signals $dY_t^i$ of equation (2). If the performance requirements for the quantum memory (in terms of tolerable recall latency) are such that it is sufficient simply to track the error state of the quantum register, assuming a single correction operation at the end of the memory storage period, then the homodyne
Figure 5. Schematic arrangement of three qubit cavities for an implementation of the continuous quantum bit-flip code [11].

photocurrents are processed using the eight-state Wonham filter, equation (7). Alternatively, if the requirements are such that a discounted cost functional applies (as discussed in section 5), the homodyne photocurrents can be processed using a syndrome filter along the lines of equation (4) to produce conditional probabilities \((\pi_1^t, \pi_2^t, \pi_3^t)\) with application of control impulses whenever the \(\pi_i^t\) reach a boundary of the continuation region (as in figure 4). If the qubits are stored in ground-state hyperfine coherences of an atom, for example, such corrections could be implemented by driving Raman transitions with control lasers focused directly onto the atoms.

Although the appearance of figure 5 suggests a physical implementation based on gas-phase atoms in Fabry–Perot optical cavities, which would be prohibitively cumbersome to realize on a large scale, our basic scheme should transfer straightforwardly to solid-state implementations based on, for example, nitrogen-vacancy center qubits in diamond coupled to nanophotonic resonators and waveguides [21]. In such systems, the most natural way to exploit modern fabrication techniques in support of the development of practical quantum information technologies lies in architectures in which each solid-state qubit is embedded in a photonic microcavity and the cavities are optically networked using waveguides (for example, planar photonic crystal cavities and waveguides [3]). Nanophotonic ‘circuits’ for routing optical probes in the configuration of figure 5 can easily be envisioned, potentially even connecting to on-chip light sources and photodetectors. Although the technical details of the circuit QED paradigm [4] are less familiar to us, it seems intuitive that super-conducting qubits coupled to high-Q stripline
resonators and waveguides could be fabricated in configurations that mimic figure 5 in the rf/microwave regime. Indeed, the possibility of solid-state implementations of continuous QEC has been discussed previously by Sarovar et al in [7].

Apart from the intrinsic interest of exploring ‘non-standard’ implementations of QEC, we believe that the approach advocated here and in [11] may have important practical advantages for future work in quantum information science. On the theoretical side, now that we have begun to understand how ‘preserving the fidelity of a stored qubit’ can in principle be understood as a control objective in completely classical terms (via the Wonham filter), it becomes reasonable to ask whether we can formulate the design of quantum memories as a feedback-control problem without first assuming the structure of a Stabilizer code and syndrome measurements. It could likewise be fruitful further to explore the fundamental tradeoffs among memory storage time, final fidelity, and the cost of control actions using the optimal impulse control framework introduced in section 5, or to investigate issues of robustness using what is now known about filter stability [22]. On the experimental side it seems clear that the type of arrangement depicted in figure 5 is physically much simpler than any literal instantiation of the usual quantum circuit diagram of repetitive discrete QEC, as the ancillary qubits (which would presumably need to be a clocked stream of single atoms or photons) are replaced by cw laser beams and the synchronous decoding gates (presumably controlled–controlled-not’s for the bit-flip code) are replaced by elementary Hamiltonian interactions that do not need to be modulated in any way (note that the strength of the continuous syndrome measurements can be increased or decreased in our scheme simply by adjusting the power of the optical probes). Setting aside the specific details of our proposed implementation, we hope that it is at least now possible to appreciate the viability and potential advantages of our general approach of trying to ‘push down’ the essential insights of standard QEC to work directly at a more physical level of modeling, analysis and design.

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