SOME RIGIDITY RESULTS FOR NONCOMPACT GRADIENT
STEADY RICCI SOLITONS AND RICCI-FLAT MANIFOLDS

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Abstract. Gradient steady Ricci solitons are natural generalizations of Ricci-flat manifolds. In this article, we prove a curvature gap theorem for gradient steady Ricci solitons with nonconstant potential functions; and a curvature gap theorem for Ricci-flat manifolds, removing the volume growth assumptions in known results. We also prove a rigidity result for asymptotically cylindrical Ricci-flat manifolds, we show that such a manifold contains a closed embedded orientable minimal hypersurface if and only if it is isometric to a Ricci-flat cylinder.

1. Introduction

A gradient Ricci soliton is a triple \((M, g, f)\), where \(g\) is a Riemannian metric on a smooth manifold \(M\), and \(f\) is a smooth function satisfying

\[\text{Ric}(g) + \text{Hess}(f) = \lambda g.\]

It is called shrinking, steady or expanding when \(\lambda\) is positive, 0 or negative respectively. As the name suggested, Ricci solitons came from the study of Ricci flow, as self-similar solutions and important singularity models. Meanwhile, they are natural generalizations of Einstein manifolds, i.e. Riemannian manifolds \((M, g)\) satisfying

\[\text{Ric}(g) = \lambda g,\]

for some constant \(\lambda\), this is the viewpoint that we take in this article. Unless otherwise stated, all manifolds and their boundaries here are assumed to be smooth and orientable.

We will study gradient steady Ricci solitons, including Ricci-flat manifolds as a special case. Indeed, any Ricci-flat manifold can be viewed as a gradient steady Ricci soliton with constant potential function. However, the potential function for a Ricci-flat steady Ricci soliton is not necessarily constant, for example, \((\mathbb{R}^n, g_e, x_1)\), where \(x_1\) is the first coordinate function. We call a gradient steady Ricci soliton nontrivial if its potential function is nonconstant. Since it is known that compact steady Ricci solitons are Ricci-flat, our focus here is on the noncompact case.

Gradient Ricci solitons and Einstein manifolds are very rigid because their Riemann curvature tensors satisfy elliptic systems. We will not survey the various rigidity results, instead, we only recall those closely related to ours. For Ricci-flat manifolds with maximal volume growth, a curvature gap theorem has been implied by the work of Bando, Kasue and Nakajima\(^{2}\), and independently proved by Shen.
in [23], the proof depends on a Euclidean type Sobolev inequality. Minerbe generalized this gap theorem to Ricci-flat manifolds with much weaker volume growth assumptions in [18], by establishing weighted Sobolev and Poincare inequalities. Similar results with no volume growth assumption have been obtained by Carron ([5]), in the more general setting of critical metrics, the proof only relies on local regularity estimates of the Riemann curvature tensor. Our first goal is to generalize these results to noncompact gradient steady Ricci solitons. We prove the following:

**Theorem 1.1.** There exist constants \( c(n) \) and \( \epsilon(n) \) depending only on \( n \), such that for any complete noncompact gradient steady Ricci soliton \((M, g, f)\) with dimension \( n \geq 3 \), and \( R + |\nabla f|^2 = \Lambda > 0 \) where \( R \) is the scalar curvature, if

\[
\int_M |Rm|^2 r^n V_f(r)^{-1} e^{c(n) \sqrt{\Lambda} r} dv_f < \epsilon(n) e^{-f(p)},
\]

where \( r(x) = \text{dist}(p, x) \) for some \( p \in M \), \( V_f(r) = \int_{B_p(r)} dv_f \), \( dv_f = e^{-f} dv \), then \((M, g)\) is flat, and the pullback of \( f \) to the universal cover is a linear function.

Recall the cigar soliton discovered by Hamilton ([9]):

\[
\left( \mathbb{R}^2, ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}, f = -\log(1 + x^2 + y^2) \right).
\]

Its product with any flat manifold is a non-flat gradient steady Ricci soliton with exponential curvature decay, therefore the \( L^2 \) integral of its sectional curvature, weighted as in Theorem 1.1 with probably smaller \( c(n) \), is finite.

In the following, the term Sobolev inequality will be used to refer to Sobolev inequalities possibly with an \( L^2 \) term on the right hand side, i.e. inequalities of the form:

\[
\left( \int \phi^n \right)^{\frac{2}{n-2}} \leq C \left( \int |
abla \phi|^2 + \int \phi^2 \right).
\]

And the term Euclidean type Sobolev inequality refers to inequalities of the form:

\[
\left( \int \phi^n \right)^{\frac{2}{n-2}} \leq C \int |
abla \phi|^2.
\]

We will prove Theorem 1.1 by a weighted Sobolev inequality of Euclidean type. Recall that Riemannian manifolds with nonnegative Ricci curvature has a uniform volume doubling constant, and the validity of Euclidean type Sobolev inequalities is equivalent to maximal volume growth. If volume growth is only super-quadratic, Minerbe showed there are weighted Euclidean type Sobolev inequalities ([18]), the proof of which also depends sensitively on the volume growth assumption and the uniform volume doubling constant. However, in the case of gradient steady Ricci soliton, the volume doubling constants are not uniform, they depend exponentially on the radius of geodesic balls. But on the other hand, Munteanu and Wang showed in [19] that nontrivial gradient steady Ricci solitons have positive \( f \)-spectrum. We can apply this result to improve a Sobolev inequality with \( L^2 \) term into Euclidean type, provided we use the weighted measure \( e^{-f} dv \). Therefore, we only need to prove a weighted Sobolev inequality, which can be obtained by gluing up local Sobolev inequalities via a well-chosen partition of unity.

Also, as shown in [18], (weighted) Poincare inequalities directly imply curvature gap theorems. So Munteanu and Wang’s result can be used to prove the following:
Theorem 1.2. There exists a constant $C(n)$, such that for any complete noncompact gradient steady Ricci soliton $(M, g, f)$ with dimension $n \geq 3$, and $R + |\nabla f|^2 = \Lambda > 0$, if for some $p \in M$ and $\alpha \geq 1$,

$$\int_{B_p(2R) \setminus B_p(R)} |Rm|^{2\alpha} dv_f = o(R^2) \text{ as } R \to \infty,$$

and

$$\sup_M |Rm| < \frac{\Lambda}{4\alpha C(n)},$$

then $(M, g)$ is flat, and the pullback of $f$ to the universal cover is a linear function.

We would like to remark that since $|\nabla f|$ is bounded, the proof in [5] with little modification shows that gradient steady Ricci solitons are $(\Gamma, k)$ regular in the definition of Carron (see section 4). Therefore, some results in [5] which only depend on the local regularity property still hold for gradient steady Ricci solitons, these results imply that the curvature cannot decay too fast locally.

Our method for nontrivial gradient steady Ricci solitons doesn’t apply to Ricci-flat manifolds since the later necessarily have zero bottom spectrum. However, we can apply local Dirichlet type Sobolev and Poincare inequalities to prove some rigidity results, which roughly tell that the curvature cannot decay too fast in integral sense relative to its $L^\infty$ or $L^{n/2}$ norm. Moreover, we can combine local Sobolev inequalities and the regularity property of Ricci-flat metrics to prove the following:

Theorem 1.3. There exists an $\epsilon$ depending only on $n$, such that for any complete noncompact Ricci-flat Riemannian manifold $(M, g)$ of dimension $n$, if there exists a point $p \in M$ such that

$$\int_M |Rm|^{\nabla^2} \rho_p^{-1}(r) < \epsilon,$$

where

$$\rho_p(r) = \frac{V(B_p(r))}{r^n},$$

then $Rm \equiv 0$.

This theorem has been proved in [18] with an additional assumption that the volume growth is super-quadratic, which is removed here. One major motivation to study Ricci-flat manifolds with small volume growth is the existence of a large class of asymptotically cylindrical Ricci-flat manifolds, which has been studied in [23, 26, 13, 11, 6]. These manifolds have exactly linear volume growth. Let’s first explain the definition:

Definition 1.4. A Riemannian manifold $(M, g)$ is called asymptotically cylindrical (ACyl), if for each end $E \subset M$, there is a closed Riemannian manifold $(N, g_N)$ and a diffeomorphism $\Phi$, such that $E = \Phi(N \times [0, \infty))$, and

$$|\nabla^l (\Phi^* g - (g_N + dt^2))| \leq O(e^{-\delta t})$$

for all integer $l \geq 0$, where $\delta$ is a positive number and the covariant derivatives are with respect to $g_N + dt^2$. We will call $t$ the cylindrical coordinate function, and $N \times \{t\}$ a cross section.
Examples of non-flat ACyl Ricci-flat manifolds asymptotic to flat metrics at infinity are given by [11]. Such manifolds have their sectional curvature integrable in $L^2$ norm weighted by $\rho^{-1}$ as in Theorem 1.3. Moreover, their products with scaled circles $\varepsilon S^1$ have arbitrarily small $L^2$ integral of curvature as $\varepsilon \to 0$, justifying the volume ratio term in Theorem 1.3.

For ACyl Ricci-flat manifolds, we can consider a different kind of rigidity phenomenon, which is related to the existence of closed embedded orientable minimal hypersurfaces. It is obvious that any cross section $N \times \{s\}$ in a cylinder $N \times \mathbb{R}$ is an embedded minimal hypersurface, which is actually totally geodesic. A natural question is, if we perturb the cylinder a little bit, is there still a minimal hypersurface near a cross section? It is easy to see the answer depends on the perturbation by looking at the following metrics

$$ds_1^2 = dt^2 + (1 + e^{-t})^2 d\theta^2,$$

and

$$ds_2^2 = dt^2 + (1 + \sin^2(t)e^{-t})^2 d\theta^2,$$

defined on $S^1 \times \mathbb{R}$. However, the Ricci-flat condition will put strong restrictions on the existence of closed embedded minimal hypersurfaces in an ACyl manifold.

**Theorem 1.5.** If $(M, g)$ is ACyl and Ricci-flat, then $(M, g)$ contains a closed embedded orientable minimal hypersurface if and only if $(M, g)$ is isometric to $N \times \mathbb{R}$, where $N$ is a closed Ricci-flat Riemannian manifold.

**Remark 1.6.** The Ricci-flat condition can be replaced by $Ric(g) \geq 0$, provided $g$ is real analytic.

Using the Fredholm theory for elliptic operators on noncompact manifolds developed by Lockhart and McOwen in [15, 16], Salur showed in [22] how to construct harmonic functions exponentially asymptotic to the cylindrical coordinate functions on an ACyl manifold with at least 2 ends. The Ricci-flat and ACyl condition will force such a function to be Hessian-flat, so it yields information about the reduction of the Holonomy, and hence of the manifold. We will borrow this technique, but we solve Dirichlet problems on an ACyl end $E$ instead of on the whole manifold, the solutions provide information about the geometry of $\partial E$ with the induced metric.

It follows directly from the maximum principle that closed minimal hypersurfaces in a cylinder $N \times \mathbb{R}$ are cross sections, where $N$ is closed. The method mentioned above provides an alternative proof when $N$ has nonnegative Ricci curvature, see Corollary 5.3. For the case where $N$ is complete, see for e.g. [20] and references therein.

The organization of this paper is as the following: we will prove weighted Sobolev inequalities on smooth metric measure spaces in section 2 and apply it to prove Theorem 1.1 in section 3, Ricci-flat manifolds will be discussed in section 4, and ACyl manifolds will be studied in section 5.

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2. **Weighted Sobolev inequalities**

In this section, we let \((M, g, d\lambda)\) be a weighted manifold with dimension \(n \geq 3\), where \(d\lambda = wd\nu\) for some positive smooth function \(w\) on \(M\). We can write the volume form in polar coordinates as

\[
dv(\exp_x(r, \theta)) = J(x, r, \theta) dr d\theta,
\]

and let \(J_\lambda(x, r, \theta) = wJ(x, r, \theta)\). We say that \((M, g, d\lambda)\) satisfies the **exponential Jacobian comparison property** with respect to \(p \in M\) if there is a constant \(c_0\), such

\[
J_\lambda(x, r_2, \theta) \leq e^{c_0 R} \left( \frac{r_2}{r_1} \right)^{n-1},
\]

for any \(R > 0\), \(x \in B_p(R)\) and \(0 < r_1 < r_2 \leq R\).

Condition \((1)\) immediately implies the **exponential volume comparison property** that

\[
\frac{V_\lambda(x, r_2)}{V_\lambda(x, r_1)} \leq e^{c_0 R} \left( \frac{r_2}{r_1} \right)^n,
\]

for any \(R > 0\), \(x \in B_p(R)\) and \(0 < r_1 < r_2 \leq R\), where

\[
V_\lambda(x, r) = \int_{B_x(r)} d\lambda.
\]

Typical examples satisfying \((1)\) are smooth metric measure spaces \((M, g, e^{-f} d\nu)\) with linear potential function and nonnegative \(\infty\)-Bakry-Emery Ricci tensor

\[
Ric(g) + \Hess(f) \geq 0,
\]

which include gradient steady Ricci solitons as special cases.

We can use \((1)\) and the method of Buser ([3]) to prove Neumann Poincare inequalities on geodesic balls:

**Lemma 2.1.** There exist \(C_1\) and \(C_2\) depending only on \(n\) and \(c_0\), such that for any \(R > 0\), \(x \in B_p(R)\), \(0 < r \leq R\) and \(\phi \in C^1(B_x(r))\), we have

\[
\int_{B_x(r)} |\phi - \phi_{B_x(r)}|^2 d\lambda \leq C_1 e^{C_2 R^2} \int_{B_x(r)} |\nabla \phi|^2 d\lambda,
\]

where \(\phi_{B_x(r)} = V_\lambda(x, r)^{-1} \int_{B_x(r)} \phi d\lambda\).

The proof is omitted here, one can refer to [19] for a detailed proof in the smooth metric measure space setting. Using \((2)\) and Lemma 2.1, we can apply the method of Maheux-Sallof-Coste ([17]) to prove Neumann Sobolev inequalities on balls:

**Lemma 2.2.** There exist \(C_1\) and \(C_2\) depending only on \(n\) and \(c_0\), such that for any \(R > 0\), \(x \in B_p(R)\), \(0 < r \leq R\) and \(\phi \in C^1(B_x(r))\), we have

\[
\left( \int_{B_x(r)} |\phi - \phi_{B_x(r)}|^2 d\lambda \right)^{\frac{n}{n-2}} \leq C_1 e^{C_2 R} \frac{r^2}{V_\lambda(x, r)^{2/n}} \int_{B_x(r)} |\nabla \phi|^2 d\lambda,
\]

where \(\phi_{B_x(r)} = V_\lambda(x, r)^{-1} \int_{B_x(r)} \phi d\lambda\).
We have to estimate the following quantity, take any

\[ \text{where we can take} \]

\[ A_0 = B; 2A_i \cap 2A_{i+1} \neq \emptyset; A_i \in \mathcal{F}_{1,s} \text{ for } i = 0, 1, \ldots, l(B) - 1; A_{l(B)} \in \mathcal{F}_{0,s}; \]

and \( A_i \cap \gamma_B \neq \emptyset \), where \( \gamma_B \) is a minimal geodesic joining \( p \) and the center of \( B \). For each \( B \in \mathcal{F}_s \), define

\[ \mathcal{F}_s(B) = \{ A_i | i = 0, 1, \ldots, l(B) \}; \]

and for each \( A \in \mathcal{F}_s \), define

\[ A(\mathcal{F}_s) = \{ B \in \mathcal{F}_s | A \in \mathcal{F}_s(B) \}. \]

It is estimated in [12] that

\[
\sharp \mathcal{F}_s(B) \leq C(n) D \log \left( \frac{s}{r(B)} \right),
\]

and

\[
\sum_{B \in A(\mathcal{F}_s), r \leq r(B) \leq 2r} V_\lambda(B) \leq C_1(n) D^{C_2(n)} \left( \frac{r}{r(A)} \right)^\epsilon V_\lambda(A),
\]

where we can take

\[ \epsilon = \log_2(1 + C(n) D^{-1}). \]

We have to estimate the following quantity, take any \( p > 1 \),

\[
\sum_{B \in A(\mathcal{F}_s)} \sharp \mathcal{F}_s(B)^{p-1} V_\lambda(B)
\]

\[
\leq \sum_{i=1}^\infty \left( \frac{1}{2} \right)^{10^2 r(A) \leq r(B) \leq (1/2)^{-1} 10^2 r(A)} \sum_{B \in A(\mathcal{F}_s), r \leq r(B) \leq 2r} \sharp \mathcal{F}_s(B)^{p-1} V_\lambda(B)
\]

\[
\leq \sum_{i=1}^\infty \left( C(n) D \log \left( \frac{s}{(1/2)^{10^2 r(A)}} \right) \right)^{p-1} C_1(n) D^{C_2(n)} \left( \frac{r}{r(A)} \right)^\epsilon V_\lambda(A)
\]

\[
\leq \sum_{i=1}^\infty C(n, p) D^{C(n, p)} \left( \log \frac{s}{r(A)} - \log 10^2 \left( \frac{1}{2} \right)^i \right)^{p-1} \left( 10^2 \left( \frac{1}{2} \right)^i \right)^\epsilon V_\lambda(A).
\]
Denote $l_i = 10^2 \left( \frac{1}{2} \right)^i$.

$$\left( \log \frac{s}{r(A)} - \log l_i \right)^{p-1} l_i^r \leq \left( \frac{p-1}{\epsilon} \right)^{p-1} \left( \frac{\epsilon}{p-1} \right)^{p-1} \left( \frac{\epsilon}{p-1} \right)^{p-1} l_i^r \left( \log \frac{s}{r(A)} \right)^{p-1} + 2^{p-1} \epsilon^{1-R_i^r/2}.$$

Since

$$\sum_{i=1}^{\infty} l_i^r = 10^2 \sum_{i=1}^{\infty} \left( \frac{1}{2} \right)^i \leq 10^2 (\ln 2) \epsilon^{-1},$$

we get

$$\sum_{B \in A(F_i)} \mathbb{F}_s(B)^{p-1} V_\lambda(B) \leq C(n, p) D^{C(n, p)} \max \{ \epsilon^{-1}, \epsilon^{-p} \} \left( \log \frac{s}{r(A)} \right)^{p-1} V_\lambda(A).$$

Note that $\ln(1 + x) > x^2$ when $0 < x < 1/4$, so we get from the above estimates:

$$\sum_{B \in A(F_i)} \mathbb{F}_s(B)^{p-1} V_\lambda(B) \leq C(n, p) D^{C(n, p)} \left( \log \frac{s}{r(A)} \right)^{p-1} V_\lambda(A),$$

which is Lemma 4.3 in [17]. The dependency of constants is then clear in the rest of the proof, see [17][1].

Note we only need to take $p = 2$ in this lemma.

With this lemma, we can prove weak Dirichlet Sobolev inequalities on connected components of annuli by using the method in [8].

**Lemma 2.3.** There exist constants $C_1(n, \delta)$ and $C_2(n, c_0)$, for any connected component $E$ of annulus $A(R_1, R_2) := B_p(R_2) \setminus B_p(R_1)$, $R_2 > R_1 > 0$, and any $\delta << 1$, we have

$$\left( \int_E |\phi|^2 d\lambda \right)^{n-2} \leq C_1 e^{C_2 R_2} \frac{R_2^2}{V_\lambda(B_p(R_2))^{2/n}} \int_{E_\delta} |\nabla \phi|^2 d\lambda,$$

for any $\phi \in C^1(E)$.

**Proof.** Let $E$ be a connected component of $A(R_1, R_2)$. Choose a $\delta$-lattice of $E$, i.e. a maximal set $(x_i)_{i \in I}$ in $E$ such that $d(x_i, x_j) \geq \delta(R_2 - R_1)/3$ when $i \neq j$. For convenience, let $r = \delta(R_2 - R_1)/6$, and let $B_i, B_i', B_i$ denote $B_{x_i}(r), B_{x_i}(2r), B_{x_i}(6r)$ respectively. Then $B_i \cap B_j = \emptyset$ whenever $i \neq j$, and $(B_i)_{i \in I}$ is a finite cover for $E$. By [2], we can estimate

$$\mathbb{I} \leq e^{2c_0 R_2} \left( \frac{2R_2}{r} \right)^n.$$

Let $N_I$ be the maximal intersection number, i.e. any $B_i$ can intersect nontrivially with at most $N_I$ balls in the family $(B_i)_{i \in I}$. Then

$$N_I \leq e^{c_0 R_2} 6^n.$$

Similarly we can define and estimate

$$\bar{N}_I \leq e^{c_0 R_2} 18^n.$$
We also need to compare the volume of different balls in this family, let
\[ C_V = \sup_{i,j \in I} V_\lambda(x_i, 2r), \]
then
\[ C_V \leq e^{2c_0 R_2} \left( \frac{R_2}{r} \right)^n. \]

For any \( \phi \in C^1(E), \)
\[
\int_E |\phi|^{\frac{2n}{n-2}} d\lambda \\
\leq \sum_i \int_{B_i} |\phi|^{\frac{2n}{n-2}} d\lambda \\
\leq 2^{\frac{n+2}{2}} \sum_i \int_{B_i} |\phi - \phi_{B_i}|^{\frac{2n}{n-2}} d\lambda + 2^{\frac{n+2}{2}} \sum_i \int_{B_i} \phi_{B_i}^{\frac{2n}{n-2}} d\lambda \\
= J_1 + J_2.
\]

By Lemma 2.2,
\[
J_1 \leq 2^{\frac{n+2}{2}} \sum_i \left( C_1 e^{C_2 R_2} \frac{(2r)^2}{V_\lambda(B_i)^{2/n}} \int_{B_i} |\nabla \phi|^2 d\lambda \right)^{\frac{n-2}{2}} \\
\leq 2^{\frac{n+2}{2}} N I \left( C_1 e^{(4c_0/n + C_2)} R_2 \frac{R_2^2}{V_\lambda(B_p(R_2))^{2/n}} \int_{E_\delta} |\nabla \phi|^2 d\lambda \right)^{\frac{n-2}{2}},
\]
where \( E_\delta \) is the \( \delta(R_2 - R_1) \)-neighbourhood of \( E \). To estimate \( J_2 \), we need a lemma from [18]:

**Lemma 2.4 (V. Minerbe).** Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) be a finite graph, \( \mathcal{V} \) is the set of vertices and \( \mathcal{E} \) is the set of edges. Fix \( p \geq 1 \), then for any function \( f \) on \( \mathcal{V} \),
\[
\sum_{i \in \mathcal{V}} |f(i)|^p \leq (2\mathcal{V})^p \sum_{(i,j) \in \mathcal{E}} |f(i) - f(j)|^p.
\]

In our setting, we define a graph by letting \( \mathcal{V} = I \) and \( (i,j) \in \mathcal{E} \) if and only if \( B_i \cap B_j \neq \emptyset \). Let’s denote \( \phi_i = \phi_{B_i} \).
\[
2^{-\frac{n+2}{2}} J_2 = \sum_i \phi_i^{\frac{2n}{n-2}} V_\lambda(B_i) \\
\leq \max_{i \in I} V_\lambda(B_i) \sum_i \phi_i^{\frac{2n}{n-2}} \\
\leq \max_{i \in I} V_\lambda(B_i) (\sharp I)^{\frac{2n}{n-2}} \sum_{(i,j) \in \mathcal{E}} |\phi_i - \phi_j|^{\frac{2n}{n-2}} \\
\leq C_V (\sharp I)^{\frac{2n}{n-2}} \sum_{(i,j) \in \mathcal{E}} |\phi_i - \phi_j|^{\frac{2n}{n-2}} \max (V_\lambda(B_i), V_\lambda(B_j)).
\]
where we used Lemma 2.4 in the second inequality.
\[
\sum_{E} |\phi_i - \phi_j| \frac{2^n}{\lambda} \max(V_\lambda(B_i), V_\lambda(B_j))
\]
\[
= \sum_{E} \frac{\max(V_\lambda(B_i), V_\lambda(B_j))}{V_\lambda(B_i) V_\lambda(B_j)} \left| \int_{B_i} \int_{B_j} (\phi(x) - \phi(y)) d\lambda(x) d\lambda(y) \right| \frac{2^n}{\lambda}
\]
\[
\leq \sum_{E} \frac{1}{V_\lambda(B_i)} \int_{B_i} \int_{B_i} |\phi(x) - \phi(y)| \frac{2^n}{\lambda} d\lambda(x) d\lambda(y)
\]
\[
\leq 2 \frac{2^{(n-2)}}{n-2} N_1 \frac{\lambda}{\lambda} \frac{2^n}{\lambda} e^{C_9 R_2^3 n} \sum_{I} \frac{C_1 e^{C_2 R_2}}{V_\lambda(B_i)^{2/n}} \int_{B_i} |\nabla \phi|^2 d\lambda \frac{n}{n-2}
\]
\[
\leq 2 \frac{2^{(n-2)}}{n-2} N_1 \frac{\lambda}{\lambda} \frac{2^n}{\lambda} e^{C_9 R_2^3 n} \left( \frac{C_1 e^{(4C_9/n+C_2)R_2}}{V_\lambda(B_p(R_2))^{2/n}} \int_{E_\delta} |\nabla \phi|^2 d\lambda \right) \frac{n}{n-2}.
\]

Using the above estimates, we can finish the proof of Lemma 2.3. \qed

Now we can glue up local sobolev inequalities to get a global weighted sobolev inequality.

**Theorem 2.5.** Let \((M, g, d\lambda)\) be a smooth weighted Riemannian manifold satisfying property [1], then there exist constants \(\alpha(n, c_0)\) and \(C_S(n)\), such that for any \(\phi \in C_0^1(M)\),
\[
\left( \int_M |\phi|^{\frac{2^n}{n-2}} \frac{V_\lambda(r)}{\lambda} \frac{2^n}{n} e^{-\alpha r} d\lambda \right)^{\frac{n-2}{n}} \leq C_S \int_M \phi^2 + |\nabla \phi|^2 d\lambda.
\]

**Proof.** First we decompose \(M\) into connected components of annuli. Choose
\[
R_0 = 0, R_i = 2^i, i = 1, 2, 3, ...
\]
let
\[
A_i = B_p(R_{i+1}) - B_p(R_i), i = 0, 1, 2, ...
\]
and denote the connected components of \(A_i\) as
\[
E_{i1}, E_{i2}, ..., E_{id_i}.
\]
If \(E_{ij}\) is not connected to \(A_{i+1}\), then it must be connected to \(E_{i-1,k}\) for some \(k\), in this case, we delete \(E_{ij}\) from the list and merge it to \(E_{i-1,k}\). Similarly, if \(E_{ij}\) is not connected to \(A_{i-1}\), we merge it to some connected component of \(A_{i+1}\).

Finally we let \(U_{ij} = E_{ij} \cup E_{i-1,k} \cup E_{i+1,l}\) where the union is for all \(E_{i-1,k}\) and \(E_{i+1,l}\) connected to \(E_{ij}\) in \(A(R_{i-1}, R_{i+2})\).
Since every \(E_{ij}\) contains a ball of radius \(R_{i-1}\), we can use the volume comparison property \(2\) to estimate
\[
l_i \leq e^{c_0 R_{i+3}} 16^n.
\]

We choose smooth nonnegative cut-off functions \(\psi_{ij} = 1\) on \(E_{ij}\) and \(\psi_{ij} = 0\) on \(M \setminus U_{ij}\), and define a partition of unity
\[
\eta_{ij} = \frac{\psi_{ij}}{\sum \psi_{ij}}.
\]

\(\eta_{ij}\) is well-defined since the sum is finite and positive at every point on \(M\). We can choose \(\psi_{ij}\) properly so that \(|\nabla \psi_{ij}| \leq 1\) everywhere. Then we can estimate the gradient of \(\eta_{ij}\)
\[
|\nabla \eta_{ij}| = \frac{|\nabla \psi_{ij} \sum \psi_{pq} \psi_{ij} - \psi_{ij} \sum \nabla \psi_{pq}|}{(\sum \psi_{ij})^2} \leq 1 + \#I(ij),
\]
where \(I(ij) = \{(pq)|U_{pq} \cap U_{ij} \neq \emptyset\}\). It’s clear to see
\[
N_i := \sup_j \#I(ij) \leq l_{i-2} + l_{i-1} + l_i + l_{i+1} + l_{i+2}.
\]

Let
\[
\rho(r) = \frac{V_{\lambda}(p, r) \frac{2\alpha}{n-2}}{r^{\frac{n-2}{2}}},
\]
and let \(\alpha\) be a number to be determined. For any \(\phi \in C_0^1(M)\),
\[
\int_M |\phi|^{\frac{2n}{n-2}} \rho(r)e^{-\alpha r} d\lambda = \int_M \left| \sum \eta_{ij} \phi \right|^{\frac{2n}{n-2}} \rho(r)e^{-\alpha r} d\lambda
\]
\[
\leq \sum_{i=0}^{\infty} N_i^{\frac{n+2}{n}} \int_{U_{ij}} |\eta_{ij} \phi|^{\frac{2n}{n-2}} \rho(r)e^{-\alpha r} d\lambda
\]
\[
\leq \sum_{i=0}^{\infty} N_i^{\frac{n+2}{n}} \rho(R_{i-1}) e^{\frac{2c_0}{n-2} R_{i+2} - \alpha R_{i-1}} \sum_{j=0}^{l_i} \int_{U_{ij}} |\eta_{ij} \phi|^{\frac{2n}{n-2}} d\lambda.
\]

By Lemma \(2.3\) and the gradient estimate for the partition of unity, we get
\[
\int_{U_{ij}} |\eta_{ij} \phi|^{\frac{2n}{n-2}} d\lambda \leq \left( C_1 e^{C_2 R_{i+2} \rho(R_{i+2}) - \frac{n-2}{n}} \int_{(A_i)_{ij}} |\nabla (\eta_{ij} \phi)|^2 d\lambda \right)^{\frac{n-2}{2n}},
\]
and
\[
\int_{U_{ij}} |\nabla (\eta_{ij} \phi)|^2 d\lambda \leq N_i \int_{(A_i)_{ij}} \phi^2 d\lambda + \int_{(A_i)_{ij}} \eta_{ij}^2 |\nabla \phi|^2 d\lambda.
\]

Hence using the estimates for \(N_i\) we derive
\[
\int_M |\phi|^{\frac{2n}{n-2}} \rho(r)e^{-\alpha r} d\lambda \leq \sum_{i=0}^{\infty} C(n, C_1) e^{(C(n,c_0,C_2) - \alpha) R_{i-1}} \left( \int_{(A_i)_{ij}} \phi^2 + |\nabla \phi|^2 d\lambda \right)^{\frac{n-2}{2n}}.
\]
Let $\alpha > C(n, c_0, C_2)$ and fix $\delta << 1$, we get
\[
\left( \int_M |\phi|^{\frac{2n}{n-2}} \rho(r)e^{-\alpha r} d\lambda \right)^{\frac{n-2}{n}} \leq 2C(n, C_1) \int_M \phi^2 + |\nabla \phi|^2 d\lambda.
\]

If in addition, there is another measure $d\mu$ on $M$, such that $(M, g)$ satisfies a weighted Poincaré inequality:
\[
\int_M \phi^2 d\lambda \leq C_P \int_M |\nabla \phi|^2 d\mu.
\]
Then there is a Euclidean type weighted Sobolev inequality on $M$:
\[
\left( \int_M |\phi|^{\frac{2n}{n-2}} \rho(r)e^{-\alpha r} d\lambda \right)^{\frac{n-2}{n}} \leq C_S(1 + C_P) \int_M |\nabla \phi|^2(1 + \frac{d\lambda}{d\mu}) d\mu.
\]

3. ON NONTRIVIAL GRADIENT STEADY RICCI SOLITONS

Let’s first recall some facts about gradient steady Ricci solitons. We say a Ricci soliton $(M, g, f)$ is nontrivial if the potential function $f$ is nonconstant. Let $R$ be the scalar curvature, it is well-known that on a gradient steady Ricci Soliton, $R + |\nabla f|^2$ is a constant [10], which is non-zero if and only if $f$ is nonconstant. We can normalize the metric so that
\[
R + |\nabla f|^2 = 1.
\]
This equation implies that $f$ has at most linear growth. By possibly adding a constant we can let
\[
|f(x)| \leq r_p(x)
\]
for some $p \in M$.

Denote $dv_f = e^{-f} dv$ where $dv$ is the Riemannian volume form. Since a gradient steady Ricci soliton has nonnegative $\infty$-Bakry-Emery Ricci tensor
\[
Ric + Hess(f) = 0,
\]
the result in [27] implies that $(M, g, dv_f)$ satisfies the exponential Jacobian comparison property [1] in section 2 with $c_0 = 12$.

Moreover, Munteanu and Wang proved in [19] that gradient steady Ricci solitons with $R + |\nabla f|^2 = \Lambda > 0$ have positive bottom $f$-spectrum $\frac{\Lambda}{4}$, i.e.
\[
\inf_{\phi \in C_0^1(M)} \frac{\int_M |\nabla \phi|^2 dv_f}{\int_M \phi^2 dv_f} = \frac{\Lambda}{4}.
\]
Therefore we can use results in section 2 to prove

**Theorem 3.1.** Let $(M, g, f)$ be a nontrivial gradient steady Ricci soliton with dimension $n \geq 3$, suppose it is normalized that $|f(x)| \leq r_p(x)$ for some $p \in M$, then for any $\phi \in C_0^1(M)$,
\[
\left( \int_M |\phi|^{\frac{2n}{n-2}} \left( \frac{V_f(p, r)}{r^n} \right)^{\frac{2}{n}} e^{-\alpha r} dv_f \right)^{\frac{n-2}{n}} \leq C_S \int_M |\nabla \phi|^2 dv_f,
\]
where $\alpha$ and $C_S$ are constants depending only on $n$, and $V_f(p, r) = \int_{B_p(r)} dv_f$. 

Now recall that the Riemann curvature tensor of a gradient steady Ricci soliton satisfies an elliptic partial differential system:

\[(4)\quad -\Delta f Rm = Q(Rm),\]

where \(\Delta f = \Delta - \langle \nabla f, \cdot \rangle\) is the self-adjoint Laplacian with respect to \(dv_f\), \(Q(Rm)\) is a quadratic term in \(Rm\).

*Proof of Theorem 1.1.* We can scale the metric to let \(\Lambda = 1\), and modify \(f\) by adding a constant so that \(f(p) = 0\). Equation (4) implies that

\[-\Delta f |Rm| \leq C(n)|Rm|^2.\]

For each \(R > 1\), choose a radial cut-off function \(\psi\) such that \(\psi(r) = 1\) when \(0 \leq r \leq R\), \(\psi(r) = 0\) when \(r > 2R\), and \(|\nabla \psi| \leq 1\). Let \(u = |Rm|^{\frac{n}{2}}\), then integrate by parts to get

\[
\int_M \psi^2 |\nabla u|^2 dv_f = \int_M -2\psi u \langle \nabla \psi, \nabla u \rangle - \psi^2 u \Delta f u dv_f
\]

\[
= \int_M -2\psi u \langle \nabla \psi, \nabla u \rangle - \frac{n}{4}|Rm|^\frac{n}{2} - 1 \psi^2 u \Delta f |Rm| - (1 - \frac{4}{n})\psi^2 |\nabla u|^2 dv_f
\]

\[
\leq C(n) \int_M \psi^2 u^2 |Rm| dv_f + (\frac{4}{n} - \frac{1}{2}) \int_M \psi^2 |\nabla u|^2 dv_f + 2 \int_M |\nabla \psi|^2 u^2 dv_f.
\]

The assumption \(n \geq 3\) implies \(\frac{4}{n} - \frac{1}{2} \leq \frac{5}{6}\), hence

\[
\frac{1}{6} \int_M \psi^2 |\nabla u|^2 dv_f \leq C(n) \int M \psi^2 u^2 |Rm| dv_f + 2 \int_M |\nabla \psi|^2 u^2 dv_f.
\]

Theorem 3.1 implies

\[
\left( \int_M (\psi u)^{\frac{2n}{n-2}} \left( \frac{V_f(r)}{r^n} \right)^{\frac{2}{n-2}} e^{-\alpha r} dv_f \right)^{\frac{n-2}{n}} \leq C_S \int_M |\nabla (\psi u)|^2 dv_f
\]

\[
= 2C_S \int_M \psi^2 |\nabla u|^2 + |\nabla \psi|^2 u^2 dv_f
\]

\[
\leq C_S C(n) \int M \psi^2 u^2 |Rm| dv_f + 2C_S \int M |\nabla \psi|^2 u^2 dv_f
\]

\[
\leq C_S C(n) \left( \int_M |Rm|^\frac{n}{2} \frac{r^n}{V_f(r)} e^{\alpha - 2n} dv_f \right)^{\frac{2}{n}} \left( \int_M (\psi u)^{\frac{2n}{n-2}} \left( \frac{V_f(r)}{r^n} \right)^{\frac{2}{n-2}} e^{-\alpha r} dv_f \right)^{\frac{n-2}{n}}
\]

\[
+ 2C_S \int M |\nabla \psi|^2 u^2 dv_f.
\]

Suppose

\[
\left( \int_M |Rm|^\frac{n}{2} \frac{r^n}{V_f(r)} e^{\alpha - 2n} dv_f \right)^{\frac{2}{n}} \leq \epsilon < \frac{1}{C(n)C_S},
\]
then
\[
\left(\int_M (\psi u)^{\frac{2n}{n-2}} \left(\frac{V_f(r)}{r^n}\right)^{\frac{2}{n-2}} e^{-\alpha r} dv_f\right)^{\frac{n-2}{2}} \leq \frac{2CS}{1-CS^C(n)e} \int_{A(R,2R)} u^2 dv_f.
\]
Observe that
\[
\frac{1}{V_f(1)} \int_{A(R,2R)} u^2 dv_f \leq \int_{A(R,2R)} |Rm|^\frac{n}{2} \frac{r^n}{V_f(r)} e^{(8+\frac{n-2}{2})\alpha r} dv_f.
\]
Therefore if we take \( c = 8 + \frac{n-2}{2} \alpha \), then
\[
\int_{A(R,2R)} u^2 dv_f \to 0, \text{ as } R \to \infty,
\]
thus \( u \equiv 0 \) and \((M,g)\) has to be flat. The universal cover of a flat manifold is \( \mathbb{R}^n \) with the Euclidean metric. Since \( \nabla \nabla f \equiv 0 \) and \( |\nabla f| \equiv 1 \), the pullback of \( f \) to the covering map will be a linear function. □

Proof of theorem 1.2. For any \( R > 0 \), let \( \psi \) be a cut-off function such that \( \psi = 1 \) on \( B_p(R) \), \( \psi = 0 \) on \( M \setminus B_p(2R) \) and \( |\nabla \psi| \leq \frac{2}{R} \). Let \( u = |Rm|^\alpha \). Apply the Poincare inequality by Munteanu-Wang to get
\[
\int_M \psi^2 u^2 dv_f \leq \frac{4}{\Lambda} \int_M |\nabla (\psi u)|^2 dv_f.
\]
Integration by parts yields
\[
\int_M \psi^2 u^2 dv_f \leq \frac{4}{\Lambda} \left( \int_M |\nabla \psi|^2 u^2 dv_f - \int_M \psi^2 u \Delta f u dv_f \right)
\]
\[
\leq \frac{4}{\Lambda} \left( \frac{4}{R^2} \int_{B_p(2R) \setminus B_p(R)} |Rm|^{2\alpha} dv_f - \int_M \psi^2 u \Delta f u dv_f \right).
\]
Since \( \alpha \geq 1 \), we can compute
\[
-\Delta f u \leq \alpha C(n) |Rm| u.
\]
Let \( R \to \infty \), we get
\[
\int_M u^2 dv_f \leq \frac{4}{\Lambda} \alpha C(n) \int_M |Rm|^2 dv_f.
\]
The condition \( \sup_M |Rm| < \frac{\Lambda}{4\alpha C(n)} \) will force \( u \) to be identically 0. □

4. On Ricci-flat manifolds

Before turning our attention to Ricci-flat manifolds, let’s first define \((M,g)\) to be a complete Riemannian manifold of dimension \( n \) with \( \text{Ric}(g) \geq 0 \). For simplicity of statements, let’s fix an arbitrary point \( p \in M \), denote the geodesic ball centered at \( p \) with radius \( r \) as \( B(r) \), and denote its volume as \( V(r) \). Integrations in this section are all with respect to the Riemannian volume form. It is well-known that \((M,g)\) has infinite volume, and the volume ratio \( \frac{V(r)}{r^n} \) is nonincreasing.

Poincare inequalities on geodesic balls have been proved by P. Li and R. Schoen ([14]). For any \( r > 0 \) and \( \phi \in C_0^\infty(B(r)) \),
\[
\int_{B(r)} \phi^2 \leq C_P r^2 \int_{B(r)} |\nabla \phi|^2,
\]
where \( C_P \) depends only on \( n \). By a well-known result of Saloff-Coste ([21]) and ([5]), there are Euclidean type Sobolev inequalities on geodesic balls: For any \( \phi \in C_0^\infty(B(r)) \),

\[
\left( \int_{B(r)} \phi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_S \frac{r^2}{V(r)^{\frac{n}{2}}} \int_{B(r)} |\nabla \phi|^2,
\]

where \( C_S \) depends only on \( n \).

Let \( u \) and \( h \) be nonnegative functions on a complete Riemannian manifold \( (M, g) \) with nonnegative Ricci curvature, suppose \( u \) satisfies almost everywhere a partial differential inequality

\[-\Delta u \leq C_0 hu.
\]

We can apply (6) to prove the following vanishing theorem for \( u \):

**Theorem 4.1.** Suppose \(|u|_{L^{\frac{2n}{n-2}}(M\setminus B(R))} < \infty\). Let

\[ R_0 = \inf \{ R | |u|_{L^{\frac{2n}{n-2}}(B(R))}^2 \geq 32C_S |u|_{L^{\frac{2n}{n-2}}(M\setminus B(R))}^2 \}. \]

If

\[ \int_M |h|^{\frac{n}{2}} < \frac{1}{C_0^\frac{n}{2}(\frac{1}{8} + 4C_S)\frac{n}{n-2}(2R_0)^n} \frac{V(2R_0)}{V(2R_0)^{\frac{n}{2}}}, \]

then \( u \equiv 0 \).

**Proof.** For any \( R > 0 \), choose a cut-off function \( \psi \leq 1 \) supported on \( B(2R) \) with \( \psi = 1 \) on \( B(R) \) and \( |\nabla \psi| \leq \frac{2}{R} \). Integration by parts yields

\[
\int_M |\nabla (\psi u)|^2 = \int_{B(2R)\setminus B(R)} |\nabla \psi|^2 u^2 - \int_M \psi^2 u \Delta u
\leq \int_{B(2R)\setminus B(R)} |\nabla \psi|^2 u^2 + C_0 \int_{B(2R)} hu^2
\lessapprox \frac{4V(2R) - V(R)}{R^2} \left( \int_{B(2R)\setminus B(R)} u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + C_0 \int_{B(2R)} hu^2.
\]

By letting \( R \to \infty \), we get

\[
\int_M |\nabla u|^2 \leq C_0 \int_M hu^2,
\]

when the righthand side is integrable.
Now let $\phi$ be a cut-off function supported on $B(2R_0)$, such that $\phi = 1$ on $B(R_0)$, $|\nabla \phi| \leq \frac{C}{R_0}$ \text{ Apply the Sobolev inequality (5) to get }

\[
\left( \int_{B(R_0)} u^{\frac{2}{n-2}} \right)^{\frac{n-2}{n}} \leq \left( \int_{B(2R_0)} (\phi u)^{\frac{2}{n-2}} \right)^{\frac{n-2}{n}} \\ \leq C_S \frac{4R_0^2}{V(2R_0)^{\frac{n}{2}}} \int_{B(2R_0)} |\nabla (\phi u)|^2 \\ \leq C_S \frac{4R_0^2}{V(2R_0)^{\frac{n}{2}}} \int_{B(2R_0)} 2|\nabla \phi|^2 u^2 + 2\phi^2 |\nabla u|^2 \\ \leq C_S \frac{8R_0^2}{V(2R_0)^{\frac{n}{2}}} \left( \frac{2}{R_0^2} \int_{B(2R_0)\setminus B(R_0)} u^2 + \int_{B(2R_0)} |\nabla u|^2 \right) \\ \leq C_S \frac{8R_0^2}{V(2R_0)^{\frac{n}{2}}} \left( \frac{2}{R_0^2} \int_{B(2R_0)\setminus B(R_0)} u^{\frac{2}{n-2}} \right)^{\frac{n-2}{n}} + \int_{B(2R_0)} |\nabla u|^2 \right)
\]

By the assumption, we get

\[
\left( \int_{B(R_0)} u^{\frac{2}{n-2}} \right)^{\frac{n-2}{n}} \leq \frac{16C_SR_0^2}{V(2R_0)^{\frac{n}{2}}} \int_{B(2R_0)} |\nabla u|^2.
\]

Hence

\[
\int_M |\nabla u|^2 \leq C_0 \left( \int_M |h|^2 \right)^{\frac{2}{n}} \left[ \left( \int_{B(R_0)} u^{\frac{2}{n-2}} \right)^{\frac{n-2}{n}} + \left( \int_{M\setminus B(R_0)} u^{\frac{2}{n-2}} \right)^{\frac{n-2}{n}} \right] \\ \leq C_0 \left( 1 + \frac{1}{32C_S} \right) \left( \int_M |h|^2 \right)^{\frac{2}{n}} \left( \int_{B(R_0)} u^{\frac{2}{n-2}} \right)^{\frac{n-2}{n}} \\ \leq C_0 \left( 1 + \frac{1}{32C_S} \right) \frac{16C_SR_0^2}{V(2R_0)^{\frac{n}{2}}} \left( \int_M |h|^2 \right)^{\frac{2}{n}} \int_{B(2R_0)} |\nabla u|^2.
\]

Under the assumption on $h$, we get $\nabla u \equiv 0$, hence $u \equiv 0$ given its integrability. □

Similarly we can apply (5) to prove:

**Theorem 4.2.** Suppose $u \in L^2$. Let

\[ R_0 = \inf \{ R \mid |u|^2_{L^2(B(R))} \geq 32C_F |u|^2_{L^2(M\setminus B(R))} \}. \]

If

\[ \sup_M |h| < C_0^{-1} \left( \frac{1}{2} + 16C_F \right)^{-1} R_0^{-2}, \]

then $u \equiv 0$.

From now on we will focus on Ricci-flat manifolds. The Riemann curvature tensor of a Ricci-flat manifold satisfies an elliptic system:

\[-\Delta Rm = Q(Rm),\]

where $Q(Rm)$ is a quadratic term of 0th order. Moreover, there is a refined Kato’s inequality (2)[4]:
where $n \geq 4$ is the dimension. This inequality leads to the following lemma:

**Lemma 4.3.** Let $(M, g)$ be a Ricci-flat Riemannian manifold of dimension $n \geq 4$, then for any $\alpha \geq \frac{n-3}{n-1}$,

$$\Delta |\!\!\!| Rm |\!\!\!| \alpha \leq C(\alpha) \cdot |\!\!\!| Rm |\!\!\!|^{\alpha+1},$$

where $C(\alpha)$ is a constant depending only on $n$.

**Proof.** A detailed proof can be found in [18]. □

With the help of Lemma 4.3, Theorem 4.1 and 4.2 directly yield the following rigidity results:

**Theorem 4.4.** Let $(M, g)$ be a complete Ricci-flat Riemannian manifold with $|\!\!\!| Rm |\!\!\!| \alpha \in L^2(M)$, for some $\alpha \geq \frac{n-3}{n-1}$. Let

$$R_0 = \inf_{p \in M} \{ R |\!\!\!| Rm |\!\!\!|^{2\alpha} L^{\frac{2\alpha}{\alpha}} (B_p(R)) \geq 32C_S |\!\!\!| Rm |\!\!\!|^{2\alpha} L^{\frac{2\alpha}{\alpha}} (M \setminus B_p(R)) \}.$$

If

$$\int_M |\!\!\!| Rm |\!\!\!|^{\frac{2\alpha}{\alpha}} < \frac{1}{\alpha C(n) (\frac{1}{\delta} + 4C_S)^{\frac{2\alpha}{\alpha}}} V(B_p(2R_0))^\alpha,$$

then $(M, g)$ is flat. Here $C(n)$ is the same constant in [7], $C_S$ is the constant in the local Sobolev inequality [7].

**Theorem 4.5.** Let $(M, g)$ be a complete Ricci-flat Riemannian manifold with $|\!\!\!| Rm |\!\!\!| \in L^{2\alpha}(M)$ for some $\alpha \geq \frac{n-3}{n-1}$. Let

$$R_0 = \inf_{p \in M} \{ R |\!\!\!| Rm |\!\!\!|^{2\alpha} L^{\frac{2\alpha}{\alpha}} (B_p(R)) \geq 32C_P |\!\!\!| Rm |\!\!\!|^{2\alpha} L^{\frac{2\alpha}{\alpha}} (M \setminus B_p(R)) \}.$$

If

$$\sup_{M} |\!\!\!| Rm |\!\!\!| < \alpha^{-1} C(n)^{-1} (\frac{1}{\delta} + 16C_P)^{-1} R_0^{-2},$$

then $(M, g)$ is flat. Here $C(n)$ is the same constant as in [7], $C_P$ is the constant in the local Poincaré inequality [7].

**Proof of Theorem 4.4 and 4.5.** By Lemma 4.3 we can directly apply Theorem 4.3 and 4.2 with $u = |\!\!\!| Rm |\!\!\!|^{\alpha}$, $h = |\!\!\!| Rm |\!\!\!|$ to prove Theorem 4.4 and 4.5. □

To prove Theorem 4.4 we need to recall the regularity estimates in [5], where Carron defined the following:

**Definition 4.6.** A Riemannian manifold $(M, g)$ is $(\Gamma, k)$ regular if for any $x \in M$, $r > 0$ and $\delta \in (0, 1)$ such that

$$\sup_{B_x(\delta r)} |\!\!\!| Rm |\!\!\!| \leq \frac{1}{r^2},$$

there are estimates for up to $k^{th}$ order covariant derivatives of the curvature:

$$\sup_{B_x(\delta r)} |\!\!\!| \nabla^j Rm |\!\!\!| \leq \frac{\Gamma}{\delta^{j+2}}, j = 1, 2, ..., k.$$
Lemma 4.7 (G. Carron). If $(M,g)$ is Ricci-flat, then $(M,g)$ is $(\Gamma,1)$ regular with \( \Gamma \) depending only on the dimension \( n \).

Proof. This lemma is proved in the more general setting of critical metrics in [5]. In the special case of Einstein metrics, it can be seen as an elliptic version of Shi’s estimate [24]. See [5] for details. \( \square \)

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. When \( n \leq 3 \), a Ricci-flat manifold is necessarily flat. When \( n \geq 4 \), observe that \( n^2 - 2 \frac{n-2}{n-1} > n^2 - 3 \frac{n-1}{n-1} \), so we can apply Theorem 4.4 with \( \alpha = \frac{n-2}{n-1} \), provided we can estimate both the smallness and the decay rate of \(|Rm|_{L^\alpha} \).

To find a region near \( p \) with large curvature, we use a point picking process. Let

\[ r = \sup \{ t > 0 \mid \sup_{B_t(p)} |Rm| \leq \frac{1}{Lt^2} \}, \]

for some \( L \geq 1 \) to be determined later. Then there exists \( x_0 \in B_r(p) \) such that

\[ |Rm|(x_0) = \frac{1}{Lt^2}. \]

Let \( r_0 = \frac{r}{2} \). For each integer \( i \geq 0 \), if \( \sup_{B_{r_i}(\frac{r}{2})} |Rm| \leq \frac{4}{Lr_i^2} \), then stop. Otherwise there exists \( x_{i+1} \in B_{r_i}(\frac{r}{2}) \) such that

\[ |Rm|(x_{i+1}) = \frac{4}{Lr_i^2}. \]

Let \( r_{i+1} = \frac{r}{2} \). If this process doesn’t stop in finite steps, then we can find a sequence \( r_i, i = 0, 1, 2, \ldots \) such that

\[ r_i = \frac{r}{2^{i+1}}, \]

\[ |Rm|(x_{i+1}) = \frac{1}{Lr_i^2} = \frac{4^{i+1}}{Lr^2}, \]

\[ d(x_0, x_{i+1}) \leq \sum_{j=1}^{i+1} r_j \leq \frac{r}{2}, \]

leading to a contradiction since \( \sup_{B_{r_{i}}(\frac{r}{2})} |Rm| < \infty \). Therefore the point picking process must stop at a finite step \( N \), denote \( \bar{x} = x_N \) and \( \bar{r} = r_{N+1} \), then

\[ |Rm|(\bar{x}) = \frac{1}{4L\bar{r}^2}, \]

\[ \sup_{B_{\bar{r}}(\bar{x})} \leq \frac{1}{L\bar{r}^2}. \]

By Lemma 4.7

\[ \sup_{B_{\bar{r}}(\bar{x})} |\nabla Rm| \leq \frac{\Gamma}{\bar{r}^3}. \]

Hence for any \( y \in B_{\bar{r}}(\bar{x}) \),

\[ |Rm|(y) \geq \frac{1}{4L\bar{r}^2} - \frac{\Gamma}{\bar{r}^3} \frac{\bar{r}}{8LT} = \frac{1}{8L\bar{r}^2}. \]
Denote $\tilde{r} = \frac{r}{8L}$. Since $B_2(\tilde{r}) \subset B_p(2r)$, we have
\[
\int_{B_p(2r)} |Rm|^\frac{p}{2} \geq \int_{B_2(\tilde{r})} |Rm|^\frac{p}{2} \geq \frac{1}{(8L)^n \Gamma_n} \rho_\tilde{x}(\tilde{r}) \geq \frac{1}{(\frac{r}{2^n})^n (8L)^n \Gamma_n} \rho_p(2r),
\]
the last inequality comes from the Bishop-Gromov volume comparison theorem. Since $\rho_p(r)$ is non-increasing, the above inequality implies
\[
\int_{B_p(2r)} |Rm|^\frac{p}{2} \geq \frac{1}{\epsilon (\frac{r}{2^n})^n (8L)^n \Gamma_n} \rho_p(2r) \int_M |Rm|^\frac{p}{2} \rho_p^{-1}
\]
\[
\geq \frac{1}{\epsilon (\frac{r}{2^n})^n (8L)^n \Gamma_n} \int_{M \setminus B_p(2r)} |Rm|^\frac{p}{2}.
\]
On the other hand,
\[
\int_{B_p(r)} |Rm|^\frac{p}{2} \leq \left( \frac{1}{L^2} \right)^{\frac{p}{2}} \frac{V(B_p(r))}{L^2 \rho_p(2r)} \leq \frac{2^n}{L^2 \rho_p(2r)},
\]
and
\[
\int_{M \setminus B_p(r)} |Rm|^\frac{p}{2} \leq \rho_p(r) \int_{M \setminus B_p(r)} |Rm|^\frac{p}{2} \rho_p^{-1} \leq \epsilon 2^n \rho_p(2r),
\]
hence
\[
\int_M |Rm|^\frac{p}{2} \leq \left( \frac{2^n}{L^2} + \epsilon 2^n \right) 2^n \rho_p(4r).
\]

If we first choose $L$ large enough, and then choose $\epsilon$ small enough, all depending only on $n$, then (8) and (9) together with the monotonicity of $\rho_p(r)$ imply that the conditions in Theorem 4.4 are satisfied, in particular $R_0 \leq 2r$. Therefore $Rm \equiv 0$.  \[\Box\]

5. ACyl Ricci-flat manifolds

We need to recall the Fredholm theory developed by Lockhart and McOwen ([15, 16]). On ACyl manifolds with $l$ ends, they defined weighted Sobolev spaces $W^p_{k,\alpha}$ by using norms of the form
\[
|f|^p_{k,\alpha} = \left( \sum_{i=0}^{k} \int |\nabla^i f|^p e^{\chi \cdot t} \right)^{\frac{1}{p}}
\]
where $1 < p < \infty$, $k \in \mathbb{N}$, $t$ is the cylindrical coordinate function on each ACyl end, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ with $\alpha_i > 0$, $i = 1, 2, \ldots, l$, and $\chi = (\chi_1, \chi_2, \ldots, \chi_l)$ with $\chi_i$ being a characteristic function on the $i$th end. The Laplacian extends to a bounded linear operator
\[
\Delta^p_{k,\alpha} : W^p_{k,\alpha} \to W^p_{k-2,\alpha}.
\]
It is proved in [15] that $\Delta^p_{k,\alpha}$ is Fredholm for $\alpha \in \mathbb{R}^l \setminus D$, where $D$ is a discrete set.

We begin with a lemma:

Lemma 5.1. Let $(E, g)$ be a connected ACyl manifold with closed smooth boundary $\Sigma$, then the Dirichlet problem
\[
\begin{cases}
\Delta f = 0 & \text{on } E; \\
f|\Sigma = 0.
\end{cases}
\]
has a solution of the form $f = t - C + O(e^{-\alpha t})$, for some $\alpha > 0$ and any given constant $C$. Note that $\Sigma$ need not to be connected.

Proof. Choose a smooth function $\phi = 0$ on $\Sigma$ and $\phi = 1$ when $t > \sup_{\Sigma} t + 1$. Let $f = \phi(t - C) + u$, we only need to solve

\[
\begin{cases}
\Delta u = -\Delta (\phi(t - C)) & \text{on } E; \\
u|_{\Sigma} = 0.
\end{cases}
\]

By the ACyl condition, $\Delta (\phi(t - C))$ has all derivatives decaying like $O(e^{-\delta t})$. Choose $\alpha < \delta$ noncritical, by Fredholm alternative we only need to show that the homogeneous problem

\[
\begin{cases}
\Delta u = 0 & \text{on } E; \\
u|_{\Sigma} = 0.
\end{cases}
\]

has only zero solution, which is true by the maximum principle. Therefore we can find a solution $u$ which has all derivatives decaying like $O(e^{-\alpha t})$. □

Let $\nu$ be the outward unit normal vector of $\Sigma$. Direct calculation shows

\[
\frac{\partial}{\partial \nu} |\nabla f|^2 = -2 |\frac{\partial f}{\partial \nu}|^2 H_{\Sigma}
\]

on $\Sigma$, where $H_{\Sigma}$ is the mean curvature of $\Sigma$. Our sign convention is that unit sphere in Euclidean space has positive mean curvature with respect to outer normal.

Lemma 5.2. Suppose that $(E, g)$ is an ACyl manifold with boundary $\Sigma$ and $\text{Ric}(g) \geq 0$, then

(1) There exists some point $p \in \Sigma$, such that $H_{\Sigma}(p) \leq 0$;
(2) If $\text{Ric} > 0$ at some point $q \in E$, then there exists $p \in \Sigma$, such that $H_{\Sigma}(p) < 0$;
(3) If $\Sigma$ is minimal, i.e. $H_{\Sigma} \equiv 0$, then $\Sigma$ is connected, and $(E, g)$ is isometric to a half cylinder $\Sigma \times [0, \infty)$. In particular, $E$ has only one end.

Proof. Let $f$ be the harmonic function found in Lemma 5.1. Let $E_T$ be the subset of $E$ bounded by $\Sigma$ and $N_T$, where $N_T$ is the image of the cross section $N \times \{T\}$. Note that $N_T$ may not be connected when $E$ has multiple ends. Integrate by parts to get the following Reilly’s formula:

\[
\int_{E_T} |\nabla \nabla f|^2 = \frac{1}{2} \int_{\Sigma} \frac{\partial}{\partial \nu} |\nabla f|^2 + \frac{1}{2} \int_{N_T} \frac{\partial}{\partial \nu} |\nabla f|^2 - \int_{E_T} \langle \Delta \nabla f, \nabla f \rangle.
\]

By the Ricci formula,

\[
\Delta \nabla f = \nabla \Delta f + \text{Ric}(\nabla f, \cdot).
\]

By (10) and the fact that $f$ is harmonic,

\[
\int_{E_T} |\nabla \nabla f|^2 = -\int_{\Sigma} \frac{\partial f}{\partial \nu}^2 H_{\Sigma} + \frac{1}{2} \int_{N_T} \frac{\partial}{\partial \nu} |\nabla f|^2 - \int_{E_T} \text{Ric}(\nabla f, \nabla f).
\]

Let $T \to \infty$, the boundary term on $N_T$ goes to zero by the exponential decay of the derivatives of $f$. Hence we have

\[
\int_{E_T} |\nabla \nabla f|^2 = -\int_{\Sigma} \frac{\partial f}{\partial \nu}^2 H_{\Sigma} - \int_{E_T} \text{Ric}(\nabla f, \nabla f).
\]

By the Hopf lemma, there exist some point $p \in \Sigma$, where $\frac{\partial f}{\partial \nu} < 0$, then (1) and (2) follow from the Ricci-nonnegative condition. To prove (3), let’s assume $H_{\Sigma} \equiv 0$, then

\[
\text{Ric}(\nabla f, \nabla f) \equiv 0,
\]
and
\[ \nabla \nabla f \equiv 0, \]
which implies that \( \nabla f \) is parallel. Also, the asymptotic behaviour \( f = t - C + O(e^{-\alpha t}) \) guarantees that \( \nabla f \) is nontrivial. In fact \( |\nabla f| \equiv 1 \) since it’s constant and approaches 1 at infinity.

Then \( (E, g) \) is isometric to a cylinder. Indeed, \( f \) is a Morse function with no critical point, hence \( E \) is diffeomorphic to \( \Sigma \times [0, \infty) \). Since \( E \) is connected, \( \Sigma \) also has to be connected. To see that the metric splits, we can choose local coordinates \( (x_i)_{i=1}^{n-1} \) on a level surface of \( f \), which is diffeomorphic to \( \Sigma \) by Morse theory, then \( (x_1, x_2, ..., x_{n-1}, f) \) gives local coordinates on \( E \). Since \( \frac{\partial}{\partial f} = \nabla f \) is parallel
\[ \frac{\partial}{\partial f} g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) = 2g(\nabla \frac{\partial}{\partial f}, \frac{\partial}{\partial f} + [\frac{\partial}{\partial f}, \frac{\partial}{\partial x_1}], \frac{\partial}{\partial x_2}) = 0, \]
where \([,] \) is the Lie bracket. Therefore \( g = g|\Sigma + df^2 \) in these coordinates. □

Very roughly speaking, Lemma 5.2 suggests that ACyl manifolds with nonnegative Ricci curvature should have some similarity with Hamilton’s cigar soliton near infinity, or split. This viewpoint will be discussed elsewhere. Now let’s first prove a direct corollary of Lemma 5.2, these simple facts also follow directly from the maximum principle.

**Corollary 5.3.** Let \( \Sigma \) be a closed embedded orientable hypersurface in a cylinder \( N \times \mathbb{R} \), where \( N \) is a closed Riemannian manifold with nonnegative Ricci curvature.

1. If \( \Sigma \) is minimal, then \( \Sigma \) is a cross section \( N \times \{s\} \) for some \( s \in \mathbb{R} \).
2. If \( \Sigma \) has constant mean curvature, and \( N \times \mathbb{R} \setminus \Sigma \) has two infinite components, then \( \Sigma \) is minimal.

**Proof of Corollary 5.3.** For (1), let \( \Sigma \) be a closed embedded orientable minimal hypersurface in \( N \times \mathbb{R} \), let \( E \) be an infinite component of \( N \times \mathbb{R} \setminus \Sigma \). By Lemma 5.2 \( E \) is isometric to \( \Sigma \times [0, \infty) \) with metric \( g|\Sigma + df^2 \), where \( f \) is the harmonic function found in Lemma 5.1. Let \( t \) be the coordinate function on the \( \mathbb{R} \) factor of \( N \times \mathbb{R} \). Since
\[ |\nabla f| = |\nabla t| = 1 \]
on \( E \), and \( f = t - C + O(e^{-\alpha t}) \), we have
\[ \langle \nabla f, \nabla t \rangle = 1 - \frac{1}{2} |\nabla (f - t)|^2 = 1 - O(e^{-\alpha t}). \]
But both \( \nabla f \) and \( \nabla t \) are parallel, so
\[ \langle \nabla f, \nabla t \rangle \equiv 1. \]
Therefore \( \nabla f \equiv \nabla t \) on \( E \), and \( \Sigma = f^{-1}(0) = t^{-1}(s) \) for some \( s \in \mathbb{R} \).
(2) follows directly from Lemma 5.2(1). □

Now we consider a Ricci-flat ACyl manifold \( (M, g) \). By the Cheeger-Gromoll splitting theorem, \( (M, g) \) either has only one end or is isometric to \( N \times \mathbb{R} \), for some closed Ricci-flat manifold \( N \). Since a cylinder obviously contains closed minimal hypersurfaces, we only need to show the other direction of Theorem 1.5

**Proof of Theorem 1.5.** Let \( \Sigma \) be a closed embedded orientable minimal hypersurface in \( (M, g) \), Let \( E \) be an infinite component of \( M \setminus \Sigma \). By Lemma 5.2 \( E \) is isometric to \( \Sigma \times [0, \infty) \). Moreover, there are local coordinates \( (x_1, x_2, ..., x_{n-1}, f) \) on \( E \), where \( (x_1, x_2, ..., x_{n-1}) \) are local coordinates on \( \Sigma \), and \( f \) is the harmonic
function obtained in Lemma 5.1. The metric on \( E \) can be written as \( g = g|_{\Sigma} + df^2 \). First we show that this product structure extends to an \( \varepsilon \)-neighbourhood of \( E \).

Let \( d_\Sigma(x) = \inf \{ \text{dist}(x, y) | y \in \Sigma \} \), \( x \in M \). Since \( \Sigma \) is compact, \( d_\Sigma(x) \) is a smooth function on an \( \varepsilon \)-neighbourhood of \( E \), which we denote as \( E_\varepsilon \). Clearly \( d_\Sigma(x) = f(x) \) for \( x \in E \), so we can extend the domain of \( f \) by defining

\[
 f(x) = \begin{cases} 
 -d_\Sigma(x), & x \in M \setminus E, \\
 f(x), & x \in E.
\end{cases}
\]

\( f \) is now a smooth function on \( E_\varepsilon \), with \( |\nabla f| \equiv 1 \). By Morse theory, every level set of \( f \) is diffeomorphic to \( f^{-1}(0) = \Sigma \), so \( E_\varepsilon \) is diffeomorphic to \( \Sigma \times (-\varepsilon, \infty) \).

Moreover, for any \( p \in \Sigma \), the integral curve of \( \nabla f \) through \( p \) is a minimizing geodesic in \( E_\varepsilon \) when \( \varepsilon \) is small enough. Therefore \( f \) is a coordinate function in the geodesic normal coordinates centered at \( p \). By the well-known result of DeTurck ([7]), \( g \) is real analytic in the geodesic normal coordinates since \( g \) is Einstein. Hence

\[
 \frac{\partial}{\partial f} g = 0
\]

in a neighbourhood of \( p \). Again by the compactness of \( \Sigma \), \( (E_\varepsilon, g) \) is isometric to \( \Sigma \times (-\varepsilon, \infty) \).

Let \( \tau \) be the supremum of all \( \varepsilon \) such that \( (E_\varepsilon, g) \) is isometric to \( \Sigma \times (-\varepsilon, \infty) \), we only need to show that \( \tau = \infty \).

If \( \tau < \infty \), by the continuity of \( g \) we see that \( \partial E_\tau \) is isometric to \( \Sigma \) with the induced metrics. Then previous arguments show that there is a small \( \varepsilon > 0 \) such that \( (E_{\tau+\varepsilon}, g) \) is isometric to \( \Sigma \times (-\tau - \varepsilon, \infty) \), which contradicts the definition of \( \tau \).

\[
\square
\]

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