Learning to be adversarially robust and differentially private

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ABSTRACT
We study the difficulties in learning that arise from robust and differentially private optimization. We first study convergence of gradient descent based adversarial training with differential privacy, taking a simple binary classification task on linearly separable data as an illustrative example. We compare the gap between adversarial and nominal risk in both private and non-private settings, showing that the data dimensionality dependent term introduced by private optimization compounds the difficulties of learning a robust model. After this, we discuss what parts of adversarial training and differential privacy hurt optimization, identifying that the size of adversarial perturbation and clipping norm in differential privacy both increase the curvature of the loss landscape, implying poorer generalization performance.

1 CONVERGENCE RATES OF THE LOGISTIC LOSS ON A LINEARLY SEPARABLE PROBLEM WITH GRADIENT DESCENT

We begin by analyzing logistic regression as a motivating example. Let \( \ell(\cdot, (\cdot, \cdot)) : W \times (X, Y) \rightarrow \mathbb{R} \) be the logistic loss \( \ell(\theta; x, y) = \log(1 + e^{-yx^T \theta}) \). We assume \( X \subseteq \mathbb{R}^d \) and \( \forall (x, y) \in (X, Y), \|x\| \leq 1 \) and \( y \in \{\pm 1\} \). We further assume the data is linearly separable with margin \( \gamma \), and set \( u = \arg\max_{x \in [-1, 1]} \min_{c \in [0, 2]} y^T x^T \theta \), the optimal hyperplane that classifies all \((x, y)\) correctly with margin at least \( \gamma \). We let \( t \in \mathbb{N} \) denote the training step, and \( \eta \in \mathbb{R} \) denote the learning rate. We will see that the difference in rate of convergence between non-robust and robust optimization grows with the dimensionality \( d \) of the input space \( X \).

Following a similar analysis in [6], we compute convergence rates for the loss under gradient descent with and without adversarial training and differential privacy. Without robustness we optimize the loss \( L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; x_i, y_i) \), while in adversarial training we minimize the adversarial loss \( L_a(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_a(\theta; x_i, y_i) \), where \( f_a(\theta) = \max_{\|\delta\| \leq \epsilon} \log(1 + e^{-(y + \delta)^T \theta}) = \log(1 + e^{-(y + \epsilon^T \theta)^T \delta}) \), where \( \epsilon \) is the size of adversarial perturbations. Throughout our analysis we use the \( \ell_2 \) norm on \( L_a \), however we also conduct experiments with the \( \ell_0 \) norm. Under differential privacy the gradient at each time step is perturbed by Gaussian noise with variance \( \sigma \) – the privacy guarantees can be computed based on the amount of noise, number of iterations and Lipschitz constant of the loss in a standard way [2].

Table 1 gives the convergence rates in each of our four possible optimization settings, with proofs given in appendix A. 1 We observe that optimizing the adversarial loss introduces a dependence on the adversarial budget \( \epsilon \) in the convergence bound, while making the optimizer differentially private introduces terms \( d\sigma^2 \) depending on the dimension and amount of noise. Combining adversarially robust and private optimization introduces the two modifications on the bound almost independently – the only cross-contribution is through a negative term. In general, two factors dominate the rate of convergence in robust and private optimization: (i) the dimensionality of data \( d \); and (ii) the size of the adversarial budget \( \epsilon \) with respect to the size of the margin \( \gamma \).

We discuss some interesting observations that arise from considering these different optimization settings. Hierarchies of empirical risk bounds. We inspect the hierarchy of empirical nominal and adversarial risk in non-private and private settings. Firstly, empirical nominal risk is always smaller than empirical adversarial risk in either setting. The non-private empirical adversarial risk can be smaller or larger than private empirical nominal risk. The position of overlap depends upon respective hyperparameters \( \epsilon \) and \( \sigma \), and data dependent parameters such as dimensionality. These effects are shown in fig. 1 for a simple binary classification task.

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1We stress that we do not claim our bounds are tight. For example, we should be able to improve the bound in the non-private non-robust setting by a factor of \( \log t \) by appealing to Theorem 3.3 in [3].
this gap tends to zero over training given a sufficient number of steps. We measure the gap between adversarial and nominal risk throughout training in either the private or non-private setting. Figure 2 shows that this gap tends to zero as $t \to \infty$, and that the gap increases in the private setting as $d$ increases.

**Empirical adversarial risk comparison.** It is worth noting that fig. 1 is not an apples-to-apples comparison because we are comparing standard loss ($L$) under standard training against robust loss ($L_a$) under adversarial training. We can compare with robust accuracy under standard training by noting that under gradient descent, we can upper bound on the robust loss by $L_a(\theta^t) \leq \frac{8-\eta}{\eta} \left( 1 + \log \frac{L^2}{T} \right) + \left( \frac{8-\eta}{\eta^2} \right) \log \left( \frac{L^2}{T} \right) + c(1 + \eta(t - 1))$ (a full derivation can be found in appendix A). We can then plot the robust loss under gradient descent and adversarial training + gradient descent in fig. 3, where the benefits of adversarial training are clear to see.

![Figure 3: Comparison of robust loss under gradient descent with and without adversarial training, using the same parameters as described in fig. 1.](image)

**Hierarchy of convergence on non-linear models.** We now evaluate how well our bounds on logistic regression match the empirically observed convergence rates of non-linear models. We train a three layer convolutional neural network on the MNIST dataset, and evaluate canonical measures as introduced previously. Firstly, fig. 4 shows the convergence hierarchy as previously introduced is preserved in practice; nominal training has the lowest loss while private and robust optimization incurs the largest loss over training. Nearly identical observations were made for the test loss.

The nominal model has a test set accuracy of 99.03%, the robust model a test set accuracy of 99.20%, the private model a test set accuracy of 96.10%, and the robust and private model a test accuracy of 94.43%. The private models are $(\epsilon, \delta)$-DP, with $\epsilon = 3.8$ and $\delta = 10^{-5}$, where individual gradients were clipped to norm one (cf. Abadi et al. [1]). The robust models were trained with $\ell_\infty$ perturbations of size 0.3 using 40 steps of PGD. All models are trained with standard SGD. Additionally, we train analogous models on CIFAR-10 using six layers with max-pooling and tanh activations, and observe similar descent curves. The private models are $(\epsilon, \delta)$-DP, with $\epsilon = 8$ and $\delta = 10^{-5}$, where individual gradients were clipped to norm 0.1. The robust models are trained to be robust to $\ell_\infty$ perturbations of size $8/255$ using 20 steps of PGD.

**Gap in practice.** We now evaluate the gap between empirical adversarial and nominal risk on this non-linear model. From fig. 2, we would expect that this gap shrinks as $t \to \infty$, with the gap being strictly larger in the private setting. This is precisely what we observe in fig. 5. Of course, we can no longer be confident the gap will tend to zero in this non-toy setting.

![Figure 5: Gap in loss between robust and non-robust optimization on MNIST.](image)

### Table 1: Convergence rates of empirical nominal and adversarial risk of logistic loss in non-private and private settings.

| Optimizer | Loss | Bound |
|-----------|------|-------|
| $\times$ | $L(\theta^t)$ | $\frac{8-\eta}{\eta^2} \left( \frac{\log L^2}{T} + 1 \right) + \left( \frac{8-\eta}{\eta^2} \right) \log \left( \frac{L^2}{T} \right)$ |
| $\times$ | $\mathbb{E}[L(\theta^t)]$ | $\frac{8-\eta}{\eta^2} \left( \frac{\log L^2}{T} + 1 + d\sigma^2 \right) + \left( \frac{8-\eta}{\eta^2} \right) \log \left( \frac{L^2}{T} \right) + \eta d\sigma^2$ |
| $\checkmark$ | $L_a(\theta^t)$ | $\frac{8-\eta}{\eta^2} \left( \frac{\log L^2}{T} + (1 + c)^2 \right) + \left( \frac{8-\eta}{\eta^2} \right) \log \left( \frac{L^2}{T} \right)$ |
| $\checkmark$ | $\mathbb{E}[L_a(\theta^t)]$ | $\frac{8-\eta}{\eta^2} \left( \frac{\log L^2}{T} + (1 + c)^2 + d\sigma^2 \right) + \left( \frac{8-\eta}{\eta^2} \right) \frac{(1 + c)^2 - 2\eta T}{2\eta^2} \log \left( \frac{L^2}{T} \right) + \eta d\sigma^2$ |

![Figure 4: Training loss of a three-layer CNN on MNIST.](image)
From fig. 6b, we note the relative improvement in robust accuracy when going from a non-robust model to a robust model is slightly higher in the non-private setting in comparison to the private setting. This implies that learning to be both robust and private is a more difficult learning task, and as a result robust accuracy suffers. We also note, from fig. 6c and fig. 6d, the relative gap in improvement in robust accuracy between private and non-private optimization on CIFAR-10 models is larger than on MNIST. This is to be expected as our previous analysis suggested learning to be robust and private is more difficult for datasets of larger dimensions.

2 THE ROLE OF CLIPPING, NOISE AND ADVERSARIAL BUDGET

In the previous section we have seen examples of the increased risk brought about by robust and private optimization, and as the clipping parameter, \( c \), decreases we achieve smoother solutions (comparatively) small test set accuracy. Similar to the purely adversarial training affect the learned model. Importantly, the trends serve that less smooth solutions are more likely to have smaller test set accuracy. Decreasing \( c \) finding a smooth model with high test accuracy decreases.

Adversarial training. We again consider binary logistic regression with loss \( f_\ell(\theta) = \max_{\|\delta\|_2 \leq c} \log(1 + e^{-y(x+\delta)T\theta}) = \log(1 + e^{-yxT\theta+c\|\theta\|_2}) \). We show in appendix B that:

\[
\nabla^2 f_\ell(\theta^*) = \frac{c}{2\|\theta^*\|^2} \left( I - \frac{\theta^*\theta^*^T}{\|\theta^*\|^2} \right),
\]

where \( \theta^* \) is the optimal solution, perfectly separating data from the two classes. Clearly \( \nabla^2 f_\ell(\theta^*) \) is a positive semi-definite matrix with eigenvalues at 0 and \( \frac{2}{\|\theta^*\|^2} \) - we can take the eigenvector \( v \) to be \( \theta^* \) and then \( (I - \frac{\theta^*\theta^*^T}{\|\theta^*\|^2})\theta^* = 0 \), and we can also take the eigenvector \( v \) to be orthogonal to \( \theta^* \) resulting in:

\[
\frac{c}{2\|\theta^*\|^2} \left( I - \frac{\theta^*\theta^*^T}{\|\theta^*\|^2} \right) v = \frac{c}{2\|\theta^*\|^2} v
\]

Thus, the curvature of our loss landscape is completely determined by the ratio of adversarial budget to the magnitude of the optimal parameters, and as \( c \) decreases we achieve smoother solutions.

Role of clipping. We define the clipping operation on a function, \( f \), as \( \text{clip}(f, k) = \min(1, \frac{k}{\|k\|})f \), for some \( k \in \mathbb{R}_+ \). Xie et al. [12] showed that that gradient clipping implies bounded weight parameters. We can assume the optimal parameter takes the form \( \theta_{clip}^* = h(k) \frac{\theta^*}{\|\theta^*\|} \), where \( h(k) \) is a monotonically increasing function of the gradient clipping parameter, \( k \). Note, we also assume we can find the optimum under the clipping norm \( k > 0 \), and so we can repeat the above analysis. This leaves the smoothness of the loss landscape determined by \( \frac{c}{2\|\theta^*\|^2} \). Similar to the purely adversarial setting, this has the intuitive property that decreasing the clipping norm decreases the smoothness of the loss landscape.

Role of noise. The landscape sharpness is governed by the ratio of adversarial noise to weight norm. If we perform noisy gradient descent: \( \theta^{t+1} = \theta^t - \eta(V + b) \), where \( b \sim N(0, \sigma^2I) \), then \( \theta^{t+1} \sim N(\theta^t - \eta \nabla, \eta^2\sigma^2I) \). We can use Jensen’s inequality to upper bound the expected norm of \( \theta^{t+1} \):

\[
E[\|\theta^{t+1}\|^2] \leq E[\|\theta^{t+1} - \theta\|^2] = \sqrt{\|\theta^t - \eta \nabla\|^2 + 2\sigma^2}
\]

This implies the expected norm is larger with an additional Gaussian noise term added to gradients.

Empirical validation. We now demonstrate that these effects are observed in practice using multi-class logistic regression on MNIST. We show that smaller adversarial budgets, \( c \), and larger clipping norms, \( k \), lead to smoother solutions with smaller generalization error.

Firstly, we train without differentially private noise; only clipping and adversarial training are activated. This means our models are not differentially private but allows us to show our analysis holds empirically (c.f. fig. 7). In total, we train 2500 multi-class logistic regression on MNIST, sweeping over different training attack budget (c) and clipping threshold (k) configurations. Each point in the fig. 7 represents a model trained with a specific (c, k) configuration.

After this, we will show that moving to differentially private models has virtually no difference empirically, implying our analysis is useful in these settings (c.f. fig. 8). Again, in total, we train 2500 multi-class logistic regression on MNIST, sweeping over different training attack budget (c) and differentially private \( \epsilon_{DP} \) configurations. Each point in the fig. 8 represents a model trained with a specific \( c, \epsilon_{DP} \) configuration.

In fig. 7a, we plot the maximum eigenvalue of the Hessian for different training attack budgets, \( c \), and maximum gradient norm, \( k \). As expected, the smallest eigenvalues belong to solutions with small \( c \) and large \( k \). For values of \( k \) close to zero, effective learning becomes increasingly difficult, resulting in non-smooth solutions regardless of the choice of attack budget, \( c \). Likewise, for attack budgets close to 0.1, the maximum Hessian eigenvalue remains high even if we increase the clipping threshold, \( k \), (i.e. the space of learnable solutions). In fig. 8a, we plot analogous results but for fully differential private models, and observe similar results; small attack budgets, \( c \), and less privacy (large \( \epsilon_{DP} \)) results in smaller eigenvalues.

In fig. 7b and fig. 8b, we fix \( c \) and \( \epsilon_{DP} \), respectively, and visualize the relationship between maximum Hessian eigenvalue and test accuracy as a function of the attack budget \( c \). We can immediately note that smaller eigenvalues (that imply smoother solutions) result in smaller generalization error, confirming that smooth solutions are preferable. Unsurprisingly, as \( c \) increases, the probability of finding a smooth model with high test accuracy decreases.

Similarly, in fig. 7c and fig. 8c, we fix the attack budget \( c \), and visualize the relationship between maximum Hessian eigenvalue and test accuracy as a function of either \( k \) or \( \epsilon_{DP} \). Again, we observe that less smooth solutions are more likely to have smaller test set accuracy. Decreasing \( k \) or \( \epsilon_{DP} \) (more privacy) increase the chances of finding a model with a large maximum eigenvalue and (comparatively) small test set accuracy.

We have a theoretical understanding of how clipping and adversarial training affect the learned model. Importantly, the trends
observed in fig. 7 are mirrored in fig. 8, implying that our analysis is useful in the fully differentially private and robust setting.

3 DISCUSSION

Recent work by Song et al. [10] has empirically shown a tension between privacy and robust learning. In particular, it is shown that six state-of-the-art defense methods designed to reduce the success of adversarial examples increase the risk of membership inference attacks due to overfitting on the training set [9]. The phenomenon of overfitting in adversarially robust optimization has also been observed by Rice et al. [7]. If Song et al. [10] showed that robust models are less private, Tursynbek et al. [11] recently identified the contrapositive relation, empirically showing that differentially private models come at the expense of robustness. Meanwhile, Ghazi et al. [5] shows that the sample complexity of learning both robust and private halfspaces is worse than learning with either property by itself.

Figure 6: Robustness comparison of MNIST and CIFAR-10. For fig. 6b and fig. 6d, given an attack budget, $c$, we plot the difference in accuracy between a robust model and a nominal model, in either a non-private or private setting. We refer to this as the improvement of the robust model over the nominal model.

Figure 7: Robust and clipped logistic regression. Each point in the figures represents a different model trained to be robust to perturbations of size $c$ with gradients clipped to be smaller than $k$.

Figure 8: Private and robust logistic regression. Each point in the figures represents a different model trained to be robust to perturbations of size $c$ and differentially private with $\epsilon_{DP}$. 
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A CONVERGENCE RATES OF LOGISTIC LOSS ON A LINEARLY SEPARABLE PROBLEM WITH GRADIENT DESCENT

Let \( \ell(\cdot, (\cdot, \cdot)) : \mathcal{W} \times (X, Y) \to \mathbb{R} \) be the logistic loss, we assume \( X \subseteq \mathbb{R}^d \) and \( Y(x, y) \in (X, Y), \|x\| \leq 1 \) and \( y \in \{ \pm 1 \} \).

Following a similar analysis in [6], we compute convergence rates for gradient descent with and without adversarial training and differential privacy. We assume the data is linearly separable with margin \( \gamma \), and set \( u = \arg \max_{\|\theta\|=1} \min_{i \in [n]} y_i x_i^T \theta \), the optimal hyperplane that classifies all \((x, y)\) correctly with margin at least \( \gamma \).

No privacy / No robustness. We find convergence rates under the logistic loss \( \ell(\theta; y, x) = \log(1 + e^{-yx^T \theta})^2 \). The logistic loss has the following first and second derivatives:

\[
\nabla \ell(\theta) = \frac{-yx e^{-yx^T \theta}}{1 + e^{-yx^T \theta}}
\]

\[
\nabla^2 \ell(\theta) = \frac{e^{-yx^T \theta} xx^T}{(1 + e^{-yx^T \theta})^2}
\]

Assuming \( \|x\| \leq 1 \), we have \( \|\nabla \ell(\theta)\| \leq 1 \) and \( \|\nabla^2 \ell(\theta)\| \leq \frac{1}{4} \). We let \( L(\theta) = \frac{1}{n} \sum_{i=1}^n \ell(\theta)_i \), and so \( \nabla L(\theta) = \frac{1}{n} \sum_{i=1}^n \frac{-yx e^{-yx^T \theta} x_i}{1 + e^{-yx^T \theta}} \). By Taylor expansion we have:

\[
L(\theta^{t+1}) = L(\theta^t - \eta_t \nabla L(\theta^t)) \leq L(\theta^t) - \eta_t \|\nabla L(\theta^t)\|^2 + \frac{\eta_t^2}{8} \|\nabla^2 L(\theta^t)\|^2
\]

where \( \eta_t \) is the learning rate at step \( t \). For simplicity, we assume gradient descent is performed with a constant learning rate \( \eta_t := \eta \) for every step \( t \). Our following analysis will depend on a sufficiently small learning rate, which we specify at the relevant places. Because \( L \) is smooth (bounded Hessian), we can apply standard gradient descent convergence analysis. For any \( \theta \in \mathbb{R}^d \):

\[
\|\theta^{t+1} - \theta\|^2 \leq \|\theta^t - \theta\|^2 - 2\eta \|\nabla L(\theta^t)\|^2 + \eta^2 \|\nabla^2 L(\theta^t)\|^2
\]

(8)

\[
\leq \|\theta^t - \theta\|^2 - 2\eta (L(\theta^t) - L(\theta)) + \eta^2 \|\nabla L(\theta^t)\|^2 \quad \text{(by convexity of } \ell)\]

(9)

\[
\leq \|\theta^t - \theta\|^2 - 2\eta (L(\theta^t) - L(\theta)) + \frac{\eta}{1 - \frac{8}{\eta} \|\nabla L(\theta^t)\|^2} (L(\theta^t) - L(\theta^{t+1})) \quad \text{(by Taylor expansion)}
\]

(10)

\[
= \|\theta^t - \theta\|^2 + \left( \frac{\eta}{1 - \frac{8}{\eta}} - 2\eta \right) L(\theta^t) + 2\eta L(\theta) - \frac{\eta}{1 - \frac{8}{\eta}} L(\theta^{t+1})
\]

(11)

If \( \eta < 4 \), the contribution of the \( L(\theta^t) \) term is negative and so:

\[
\|\theta^{t+1} - \theta\|^2 \leq \|\theta^t - \theta\|^2 + 2\eta L(\theta) - \frac{\eta}{1 - \frac{8}{\eta}} L(\theta^{t+1})
\]

(12)

If we set \( \theta^0 = 0 \), then \( \theta^1 = -\frac{\eta}{n} \sum_{i=1}^n \frac{-yx i}{2} \) and \( \|\theta^1\| \leq \frac{\eta}{2} \leq 1 \) as long as \( \eta < 2 \). Under the assumption that \( L(\theta^{t+1}) \leq L(\theta^t) \), summing the above between \( s = 1, \ldots, t \), we get:

\[
\|\theta^{t+1} - \theta\|^2 \leq \|\theta^1 - \theta\|^2 + t \left( \frac{\eta}{1 - \frac{8}{\eta}} L(\theta^1) - 2\eta L(\theta) \right)
\]

(13)

\[
\implies L(\theta^{t+1}) \leq \frac{1 - \frac{8}{\eta t} \|\theta^1\|^2 + \|\theta\|^2}{t} + 2(1 - \frac{\eta}{8}) L(\theta)
\]

(14)

Since we are free to choose \( \theta \), we set it to \( \theta := \frac{\log t}{t} u \), then:

\[
L(\theta) = \frac{1}{n} \sum_{i=1}^n \log(1 + e^{-yx^T u}) \leq \frac{1}{n} \sum_{i=1}^n \log(1 + e^{-\log t}) = \log(t + 1)
\]

(15)

Together with the fact that \( \|\theta\| = \frac{\log t}{t} \), we get the final upper bound:

\[\text{Ongoing work. Appeared in PPML, 2021}\]
\[ L(\theta^{t+1}) \leq \frac{1 - \frac{7}{8} \left( 1 + \left( \frac{\log t}{y} \right)^2 \right)}{t} + 2 \left( 1 - \frac{7}{8} \right) \log \left( \frac{t+1}{t} \right) \] 
\[ = \frac{8}{\delta t n} \left( 1 + \left( \frac{\log t}{y} \right)^2 \right) + \frac{8}{\delta t} \left( 1 + \left( \frac{\log t}{y} \right)^2 \right) \] 

(16)

(17)

**No privacy / robustness.** We next find convergence rates under gradient-based adversarial training. Let \( \ell_a(\theta) = \max_{||\delta|| \leq \epsilon} \log (1 + e^{-\theta^T (\delta + \theta)} = \log (1 + e^{-yx^T \theta + \epsilon} ||\theta||) \). We first find the first and second derivative of this loss.

If we let \( h(\theta) := e^{-yx^T \theta + \epsilon} ||\theta|| \), then \( \ell_a(\theta) = \log(1 + h(\theta)) \) and \( \nabla \ell_a(\theta) = \frac{h'(\theta)}{1+h(\theta)} \), where \( h'(\theta) = h(\theta) r(\theta) \) if we take \( r(\theta) := -yx + e \frac{\partial}{\partial ||\theta||} \).

Similarly:

\[ \nabla^2 \ell_a(\theta) = \frac{h(\theta) r(\theta) r(\theta)^T + h(\theta)^2 r'(\theta) + h(\theta) r'(\theta)}{(1 + h(\theta))^2} \] 

(18)

To bound the above we must bound \( r'(\theta) = c(\frac{l}{\|\theta\|} - \frac{\delta \theta^T \theta}{\|\theta\|^2}) \). We first note that the problem can be reduced to finding a lower bound to \( \|\theta^t\| \) for any \( t > 0 \) because \( \|r'(\theta)\| \leq \frac{2\epsilon}{\|\theta\|} \). If we let \( \theta^0 = 0 \), then \( \ell_a \) is no longer differentiable at zero, so we take the sub-gradient to be:

\[ e^{-yx^T \theta + \epsilon ||\theta||} \left( -yx + cr \right) \in \partial \ell_a(\theta^1) \]

where \( r \) is a vector with \( ||r|| \leq 1 \). For example, we can take \( r = 0 \), and then the sub-derivative becomes \( \frac{-yx}{2} \in \partial \ell_a(\theta^1) \), and is bounded above like \( \theta^1 = \frac{\eta}{2L} \sum_{i=1}^n y_i \), and \( \langle \theta^1, u \rangle \geq L \eta \eta \). We then note the inner product between \( -\nabla \ell_a(\theta) \) and \( u \) is strictly positive:

\[ \langle -\nabla \ell_a(\theta), u \rangle \geq \frac{e^{-yx^T \theta + \epsilon ||\theta||}}{1 + e^{-yx^T \theta + \epsilon ||\theta||}} (\gamma - c) > 0 \]

(19)

(20)

and so

\[ \langle \theta^2, u \rangle = \langle \theta^1 - \eta \nabla \ell_a(\theta^1), u \rangle = \langle \theta^1, u \rangle + \eta \langle -\nabla \ell_a(\theta^1), u \rangle \geq \langle \theta^1, u \rangle \geq \eta \gamma \]

(21)

By a similar argument for any \( t > 1 \), \( \langle \theta^t, u \rangle \geq \eta \gamma \). Because \( u \) is the global minimizer it follows that \( \|\theta^t\| \geq \eta \gamma \). We have a lower bound on \( \theta^t \) for any \( t \) and so \( \|r'(\theta)\| \leq \frac{2\epsilon}{\gamma} \).

The other term we upper bound is \( r(\theta)r(\theta)^T = xx^T - 2cyx^T \frac{\partial}{\partial ||\theta||} + \gamma^2 \frac{\partial^2}{\partial ||\theta||^2} \), and since we assume bounded data we can bound this like \( \|r(\theta)r(\theta)^T\| \leq (1 + c)^2 \). Piecing this altogether gives:

\[ \|\nabla^2 \ell_a(\theta)\| \leq \frac{\|h(\theta)r(\theta) r(\theta)^T\|}{(1 + h(\theta))^2} + \|h(\theta)(1 + h(\theta))r'(\theta)\| \]

\[ = \frac{h(\theta)r(\theta) r(\theta)^T}{(1 + h(\theta))^2} + \|h(\theta) r'(\theta)\| \]

\[ \leq \frac{h(\theta)}{(1 + h(\theta))^2} \|r(\theta)r(\theta)^T\| + \|h(\theta)\| \|r'(\theta)\| \]

\[ \leq (1 + c)^2 + \frac{2\gamma}{\eta} \]

(22)

(23)

(24)

(25)

Letting \( s := \frac{(1+c)^2}{4} + \frac{2\gamma}{\eta} \), now that we have a bound on the second derivatives we can again use the Taylor expansion of gradient descent:

\[ L_a(\theta^{t+1}) = L_a(\theta^t - \eta \nabla \ell_a(\theta^t)) \]

\[ \leq L_a(\theta^t) - \eta \|\nabla \ell_a(\theta^t)\|^2 + \frac{s \eta^2}{2} \|\nabla \ell_a(\theta^t)\|^2 \]

(26)

(27)

Then for any \( \theta \in \mathbb{R}^d \),
we do not clip gradients as they are already 1-Lipschitz and we assume here the clipping norm is larger than one. Let $\nabla L(\theta) = \nabla L(\theta) + b$, where $b \sim \mathcal{N}(0, \sigma^2 I_d)$. Throughout the following analysis we find convergence results under expectations taken over the noise added for differential privacy.

Our analysis is almost identical to the non-private case, and we sometimes write $\nabla$ or $\nabla$ where the context is clear. Firstly, the Taylor expansion follows:

$$L(\theta^{t+1}) = L(\theta^t - \eta \nabla) \leq L(\theta^t) - \eta \nabla^T \nabla + \frac{\eta^2}{2} \nabla^T \nabla \nabla$$

Note that $\|\nabla\|^2 \leq \frac{1}{4}$ and $\nabla = \nabla + b$. Although we take expectations over the random noise in following, we omit the notation $\mathbb{E}$ for conciseness:

$$L(\theta^{t+1}) \leq L(\theta^t) - \eta \|\nabla\|^2 + \frac{\eta^2}{8}(\|\nabla\|^2 + d\sigma^2)$$

Then for any $\theta \in \mathbb{R}^d$, and again taking expectations over the noise,

$$\|\theta^{t+1} - \theta\|^2 = \|\theta^t - \theta\|^2 - 2\eta(\nabla + b - \theta) + \eta^2 \|\nabla + b\|^2 \leq \|\theta^t - \theta\|^2 + 2\eta L(\theta) - \frac{8\eta}{8 - \eta} L(\theta^{t+1}) + \frac{8d\eta^2 \sigma^2}{8 - \eta}$$

Repeating the same process as in the non-private case we get:

$$L(\theta^{t+1}) \leq \frac{8 - \eta}{8t\eta} \left(1 + d\sigma^2 + \left(\frac{\log t}{\gamma}\right)^2\right) + \frac{8 - \eta}{4t} \log \left(\frac{t + 1}{\gamma}\right) + \eta d\sigma^2$$

**Privacy / No robustness.** We next find convergence rates under gradient descent with differential privacy. At each step we add noise; we do not clip gradients as they are already 1-Lipschitz and we assume here the clipping norm is larger than one. Let $\nabla L(\theta) = \nabla L(\theta) + b$, where $b \sim \mathcal{N}(0, \sigma^2 I_d)$. Throughout the following analysis we find convergence results under expectations taken over the noise added for differential privacy.

Our analysis is almost identical to the non-private case, and we sometimes write $\nabla$ or $\nabla$ where the context is clear. Firstly, the Taylor expansion follows:

$$\|\theta^{t+1} - \theta\|^2 \leq \|\theta^t - \theta\|^2 - 2\eta L_a(\theta^t) + \frac{\eta}{1 - \frac{\eta}{2}} (L_a(\theta^t) - L_a(\theta^{t+1}))$$

As long as $\eta$ satisfies $s\eta < 1$, the $L_a(\theta^t)$ contribution is negative and we get:

$$\|\theta^{t+1} - \theta\|^2 \leq \|\theta^t - \theta\|^2 + 2\eta L_a(\theta) - \frac{2\eta}{2 - s\eta} L_a(\theta^{t+1})$$

Isolating $\eta$ in the condition $s\eta < 1$ gives $\eta < \frac{t}{t(1+c)^2}$ which also implies that we require $c < \frac{1}{4}$ for our results to hold. Summing over the iterations as before gives:

$$\|\theta^{t+1} - \theta\|^2 \leq \|\theta^1 - \theta\|^2 - t\left(\frac{2\eta}{2 - s\eta} L_a(\theta^{t+1}) - 2\eta L_a(\theta)\right)$$

By choosing $\theta = \frac{\log t}{t-c} u$ we again get $L_a(\theta) = \log(1 + \frac{1}{t})$ and $\|\theta\| = \frac{\log t}{t-c}$, giving the final upper bound:

$$L_a(\theta^{t+1}) \leq \frac{2 - s\eta}{2t\eta} (\|\theta\|^2 + \|\theta\|^2 + 2t\eta L_a(\theta))$$

**Privacy / robustness.** We can repeat the analysis as in non-private and robust case, while taking expectations over the random noise to give the upper bound:
The following analysis only considers a single input, but can be extended to a batch of inputs through simple averaging. We again consider learning with differential privacy.

So far we have concentrated our analysis on logistic regression. We now give a simple risk analysis for general convex losses for robust binary logistic regression with loss noting that under gradient descent, if \( \| \theta \| \leq 1 \), then \( \| \theta' \| \leq 1 + \eta(t - 1) \), and together with:

\[
\ell_a(\theta) = \log(1 + e^{-yx^T \theta + c\|\theta\|}) + c \| \theta \| + c \| \theta \|, \\
= \ell(\theta) + c \| \theta \| \leq \ell(\theta) + c \| \theta \|, \\
= \ell(\theta) + c \| \theta \|,
\]

we get the following upper bound on the robust loss under gradient descent:

\[
L_a(\theta^t) \leq \frac{8 - \eta}{8t\eta} \left( 1 + \frac{\log t}{\gamma} \right) + \frac{(8 - \eta)(t + 1)}{4t^2} + c(1 + \eta(t - 1))
\]

### B Binary Logistic Regression with Clipping and Adversarial Training.

The following analysis only considers a single input, but can be extended to a batch of inputs through simple averaging. We again consider binary logistic regression with loss \( \ell_a(\theta) = \max_{\|\delta\| \leq e} \log(1 + e^{-yx + \langle \delta \rangle^T \theta}) = \log(1 + e^{-yx^T \theta + c\|\theta\|}) \). As stated previously the first and second derivatives are given by:

\[
\nabla \ell_a(\theta) = \frac{h(\theta) r(\theta)}{1 + h(\theta)}, \\
\nabla^2 \ell_a(\theta) = \frac{h(\theta) r(\theta) r(\theta)^T + h(\theta) r(\theta) r(\theta)^T + h(\theta) r(\theta)}{(1 + h(\theta))^2}
\]

where \( h(\theta) := e^{-yx^T \theta + c\|\theta\|} \), \( r(\theta) := -yx + \frac{\theta}{\| \theta \|} \), and \( r'(\theta) = c(\frac{1}{\| \theta \|} - \frac{\theta^T \theta}{\| \theta \|^2}) \). At the optimal \( \theta^* = \arg \min_\theta \ell_a(\theta) \), we have \( -yx + \frac{\theta^*}{\| \theta^* \|} = 0 \Rightarrow \theta^* = \frac{yx}{c} \). The \( h(\theta) r(\theta) r(\theta)^T \) term in the second derivative vanishes at the optimum and \( h(\theta^*) = 1 \) and we are left with:

\[
\nabla^2 \ell_a(\theta^*) = \frac{r'(\theta^*)}{2} = \frac{c}{2 \| \theta^* \|^2} \left( I - \frac{\theta^* \theta^T}{\| \theta^* \|^2} \right)
\]

Clearly \( \nabla^2 \ell_a(\theta^*) \) is a positive semi-definite matrix with eigenvalues at 0 and \( \frac{c}{2 \| \theta^* \|^2} - \) we can take the eigenvector \( \nu \) to be \( \theta^* \) and then \( (I - \frac{\theta^* \theta^T}{\| \theta^* \|^2}) \theta^* = 0 \), and we can also take the eigenvector \( \nu \) to be orthogonal to \( \theta^* \) resulting in:

\[
\frac{c}{2 \| \theta^* \|} \left( I - \frac{\theta^* \theta^T}{\| \theta^* \|^2} \right) v = \frac{c}{2 \| \theta^* \|} v
\]

Thus, the curvature of our loss landscape is completely determined by the ratio of adversarial noise to the magnitude of the optimal parameters, and as \( c \) decreases we achieve smoother solutions.

### C Excess Risk Analysis for Robust & Private Learning.

So far we have concentrated our analysis on logistic regression. We now give a simple risk analysis for general convex losses for robust learning with differential privacy.

Let \( \ell(\cdot, \cdot) : \mathcal{W} \times (\mathcal{X}, \mathcal{Y}) \rightarrow \mathbb{R} \) be a loss function with the following properties: \( L \)-Lipschitz, \( G \)-smooth, convex and linear in its parameters, \( \ell(w, (x, y)) = \ell(yw^T x), \forall w \in \mathcal{W}, \| w \| \leq R, X \subseteq \mathbb{R}^d, \forall (x, y) \in (\mathcal{X}, \mathcal{Y}), \| x \| \leq 1 \) and \( y \in \{ \pm 1 \} \).

\[
L_a(\theta^{t+1}) \leq \frac{2 - s\eta}{2t\eta} \left( 1 + c \right)^2 + d\sigma^2 + \left( \frac{\log t}{1 - c} \right)^2 + \left( 2 - s\eta \right) \log(1 + \frac{1}{t}) + \eta(d\sigma^2)
\]
Consider two batch updates that differ in their final element:

\[
P = \sum_{i=1}^{n-1} y_i \nabla \ell(y_i w_i^T x_i) x_i + y_n \nabla \ell(y_n w^T x_n) x_n + z
\]

(52)

\[
Q = \sum_{i=1}^{n-1} y_i \nabla \ell(y_i w_i^T x_i) x_i + y_n' \nabla \ell(y_n' w_i^T x_n') x_n' + z
\]

(53)

Setting \(A = \sum_{i=1}^{n-1} y_i \nabla \ell(y_i w_i^T x_i) x_i, B = y_n \nabla \ell(y_n w_i^T x_n) x_n, B' = y_n' \nabla \ell(y_n' w_i^T x_n') x_n'\), we have \(P = N(A + B, \sigma^2 I_d)\) and \(Q = N(A + B', \sigma^2 I_d)\). To achieve \((\epsilon, \delta) - DP\), we must bound \(\alpha_M(\lambda) = \max_{X, X'} \alpha_M(\lambda; X, X')\) as defined in [1], where \(\alpha_M(\lambda) = \lambda D_{\lambda+1}(P||Q)\) is as defined in [4]. This results in the following:

\[
\alpha_M(\lambda) = \lambda D_{\lambda+1}(P||Q) = \frac{\lambda(\lambda + 1)\|B - B'\|^2}{2\sigma^2}
\]

(54)

Under the assumption that \(\|x\| \leq 1, y \in \{\pm 1\}\), and \(\ell\) is L-Lipschitz, we have the following upper bound:

\[
\|B - B'\| \leq 2L \Rightarrow \alpha_M(\lambda) \leq \frac{2\lambda(\lambda + 1)L^2}{\sigma^2}
\]

(55)

Summing over all iterations gives:

\[
\alpha_M(\lambda) \leq \sum_{k=1}^{T} \alpha_M(\lambda) \leq \frac{2\lambda(\lambda + 1)L^2T}{\sigma^2} \leq \frac{c\lambda^2L^2T}{\sigma^2}
\]

(56)

for some constant \(c\). Then for some \(\epsilon'\), taking \(\sigma^2 = \frac{c\epsilon'^2L^2T\log(\frac{1}{\delta})}{\epsilon^2}\) will give \((\epsilon, \delta) - DP\) according to Theorem 2.2 in [1].

Turning to the robust setting, we wish to solve \(\min_w w \max_{\|w\| \leq \epsilon} \ell(yw^T(x + \epsilon))\), and in the convex setting the inner-maximization can be given in closed-form as:

\[
\max_{\|w\| \leq \epsilon} \ell(yw^T(x + \epsilon)) = \ell(yw^T x - r \|w\|)
\]

(57)

where \(\frac{1}{\lambda} + \frac{1}{\lambda} = 1\). If we set \(p = \infty\), then squared Euclidean distance between \(B\) and \(B'\) becomes

\[
\|B - B'\| \leq \|y_n \nabla t_n x_n - y_n' \nabla t_n' x_n' + r \text{sign}(w) \nabla t_n + r \text{sign}(w) \nabla t_n'\|
\]

(58)

\[
\leq 2\sqrt{d}(1 + \epsilon)L
\]

(59)

This similarly holds for \(p = 2\) by noting that that the derivative of \(\|w\|\) is of unit length and replacing \(\text{sign}(w)\) with this unit length value. Then \((\epsilon, \delta) - DP\) holds by taking \(\sigma^2 = \frac{c'(1+\epsilon)^2L^2T\log(\frac{1}{\delta})}{\epsilon^2}\).

First note that if \(\ell\) is L-Lipschitz, then \(\mathbb{E}[\|\hat{y}\|^2] = \|y\|^2 + \mathbb{E}[\|\hat{z}\|^2] \leq L^2 + d\sigma^2\). Assuming \(\ell\) is \(\lambda\)-strongly convex, then by Theorem 1 of [8] the bound on excess risk is given by the following:

\[
\mathbb{E}[\ell(w_T) - \ell(w^*)] \leq \frac{17(L^2 + d\sigma^2)(1 + \log(T))}{\lambda T}
\]

(60)

where for \(t > 0\) the per step learning rate satisfies \(t_T = \frac{1}{T}\). For private learning we can plug in \(\sigma^2 = \frac{c\epsilon^2L^2T\log(\frac{1}{\delta})}{\epsilon^2}\) and for robust and private learning we can plug in \(\sigma^2 = \frac{c'(1+\epsilon)^2L^2T\log(\frac{1}{\delta})}{\epsilon^2}\).