Robust comparisons of variation using ratios of interquantile ranges

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Abstract

There are two major shortcomings of the $F$-test for testing the equality of two variances. Firstly, underlying normality of the populations from which the data is drawn is assumed and a violation of this assumption can lead to unreliable inference. Secondly, the usual sample variance estimators are non-robust and may be heavily influenced by outliers. In our study, we propose to use confidence intervals of ratios of interquantile ranges to compare the variation between two populations. We introduce interval estimators for the ratio that have excellent coverage properties for a wide range of distributions. Robustness properties of the estimator are studied using the influence function.

Keywords: Bounded influence, Coverage probability, Partial influence function

1 Introduction

The most commonly use method to test the equality of two variances is the $F$-test, so called because the ratio of two independent sample variance estimators is $F$-distributed when the populations are normally distributed. An analogous confidence interval estimator for the ratio is also available and provides more insight than simply rejecting or not rejecting a null hypothesis of equality. If the normality assumption is violated, then the reliability of the test and interval estimator of the ratio needs to be questioned.

Shoemaker [15] introduced a test for comparing populations using differences in estimated interquantile ranges defined to be the difference between two symmetric quantiles. Shoemaker finds that the test is reliable for many underlying distributions, including heavily skewed distributions. Motivated by these findings, we propose new interval estimators for the ratio of two interquantile ranges. Such ratios have the advantage of being scale-free and therefore readily interpretable. Additionally, for many distributions that include, for example members of the location-scale family, the squared population ratio of the two interquantile ranges is exactly equal to the ratio of variances. We begin by providing a brief summary of the aforementioned tests of variance equality.
1.1 The F-test for testing equality of variances

Let $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$ denote simple random samples from the $N(\mu_x, \sigma^2_x)$ and $N(\mu_y, \sigma^2_y)$ respectively. Then the sample mean and sample variance estimators of $\mu_x$ and $\sigma^2_x$ are $\bar{X} = \frac{\sum_{i=1}^{n_1} X_i}{n_1}$ and $S^2_x = \frac{\sum_{i=1}^{n_1} (X_i - \bar{X})^2}{n_1 - 1}$ with similarly defined $\bar{Y}$ and $S^2_y$ as estimators of $\mu_y$ and $\sigma^2_y$.

1.2 Shoemaker’s test

Let $F_1$ denote the distribution function for random variable $X$. For a $p \in [0, 1]$, let the $p$th quantile be denoted $x_p = F^{-1}(p) = \inf\{x : F(x) \geq p\}$. For $p \in (0, 0.5)$, the interquantile range is defined to be $x_{1-p} - x_p$. We will denote this interquantile range as $\text{IQR}_p(X)$ and note that $\text{IQR}_{0.25}(X)$ is the commonly used interquantile range. For random variable $Y$ that is independent of $X$, similarly define $F_2, y_p$ and $\text{IQR}_p(Y)$. Let $\hat{x}_p, \hat{x}_{1-p}, \hat{y}_p, \hat{y}_{1-p}$ denote the quantile estimators arising from simple random samples from $F_1$ and $F_2$ of size $n_1$ and $n_2$ respectively. Then Shoemaker’s [15] test statistic is defined to be

$$T = \frac{(\hat{x}_{1-p} - \hat{x}_p) - (\hat{y}_{1-p} - \hat{y}_p)}{\sqrt{\omega^2_x/n + \omega^2_y/m}}$$

(1.1)

where $\omega^2_x$ and $\omega^2_y$ are asymptotic variances of the $\text{IQR}_p$ estimators derived using Hampel’s [4] influence function. When $\text{IQR}_p(X) = \text{IQR}_p(Y)$, $T$ in (1.1) is asymptotically $N(0, 1)$ distributed and can therefore be used to test equality of the $\text{IQR}_p$’s when the sample sizes are large. Shoemaker’s results indicate vastly superior Type I error for skewed distributions when compared to the $F$-test. Marozzi [6] introduced a permutation version of this test that improved power and a combined interquantile test which does not require a choice of $p$. Additional to the $F$-test, Marozzi highlights improved performance compared to the W50 test [1], M50 and L50 tests [8], the R test [7] and the modified Fligner-Killeen (FK) test [2].

1.3 Outline of this paper

In Section 2 we derive the partial influence functions for the ratio of variances and ratio of interquantile range estimators. We use these influence functions to derive asymptotic variances of the estimators which leads to the introduction of confidence intervals in Section 3 which are assessed via simulation. In Section 4 we consider applications to a real data set before providing a brief discussion in Section 5.

2 Ratios of interquantile ranges

2.1 Some preliminary definitions

Throughout we continue with the notations introduced in Sections 1.1 and 1.2 and we continue to restrict $p \in (0, 1)$. We define the squared ratio of interquantile ranges to be

$$R_p(X,Y) = \left[\frac{\text{IQR}_p(X)}{\text{IQR}_p(Y)}\right]^2.$$

(2.1)

We have decided to focus mainly on the squared ratio of IQRs since it is analogous to the ratio of variances. Further, the ratio of IQRs is non-negative, it is simple to obtain estimates, including
interval estimates, for the ratio of IQRs by a simple square-root back-transformation. For now we consider the population-based squared ratio of IQRs for location-scale family distributions.

**Lemma 2.1.** Let \( G(\mu, \eta) \) denote the distribution function of a location family, scale family or a location-scale family random variable with location parameter \( \mu \) and scale parameter \( \eta \). Then, if \( X \sim G(\mu_x, \eta_x) \) and \( Y \sim G(\mu_y, \eta_y) \),

\[
R_p(X, Y) = \frac{\text{Var}(X)}{\text{Var}(Y)}
\]

for any \( p \in (0, 1/2) \).

**Proof.** We will provide the proof for the location-scale family and the other proofs are similar. Let \( Z_x \sim G(0, 1) \) and \( Z_y \sim G(0, 1) \). Then we can write \( X = \eta_x Z_x + \mu_x \) and \( Y = \eta_y Z_y + \mu_y \). Since quantile functions are equivariant with respect to monotone transformations, the quantile functions for \( X \) and \( Y \) may each be written as \( \eta_x Q(p) + \mu_x \) and \( \eta_y Q(p) + \mu_y \) where \( Q(p) \) is the quantile function for \( G(0, 1) \). Consequently, \( R_p(X, Y) = \frac{\eta_x^2}{\eta_y^2} \). For \( Z \sim G(0, 1) \), we have \( \text{Var}(X) = \eta_x^2 \text{Var}(Z) \) and \( \text{Var}(Y) = \eta_y^2 \text{Var}(Z) \) so that \( R_p(X, Y) \) is equal to the ratio of variances.

From Lemma 2.1, the \( R_p(X, Y) \) is equal to the ratio of variances when the distributions are from the same location-scale family of distributions but differing with respect to the location and scale parameters. This means that an estimator of the \( R_p(X, Y) \) is a direct competitor for the ratio of variances for the same population comparison of variation.

As in Section 1.2, let \( \hat{x}_p \) and \( \hat{y}_p \) denote sample estimates of \( x_p \) and \( y_p \) arising for \( n_1 \) observations drawn from \( F_1 \) and \( n_2 \) from \( F_2 \). We define our estimator of \( R_p(X, Y) \) as

\[
r_p = R_p(\hat{X}, \hat{Y}) = \left( \frac{\hat{x}_{1-p} - \hat{x}_p}{\hat{y}_{1-p} - \hat{y}_p} \right)^2.
\]

(2.2)

### 2.2 Robustness properties

For background material on robustness concepts such as breakdown points, influence functions and partial influence functions, see Hampel et al. [4] or Staudte and Shether [17] and Pires and Branco [10].

Define the univariate contamination distribution to be

\[
F_\epsilon = (1 - \epsilon) F + \epsilon \Delta_{x_0}
\]

where \( \Delta_{x_0} \) is the probability density function with all of its weighting at the contaminant \( x_0 \). The influence function [IF, 3] for any statistical functional, \( T \), is then defined for each \( x \) as

\[
\text{IF}(x; T, F) \equiv \lim_{\epsilon \to 0} \frac{T(F_\epsilon) - T(F)}{\epsilon} = \left. \frac{\partial}{\partial \epsilon} T(F_\epsilon) \right|_{\epsilon=0}.
\]

In the context of ratios of interquantile ranges, we have two populations and therefore consider partial influence functions [PIF, 10]. For the case of two populations, we have two PIFs,
where contamination is introduced to each of the populations while the other population remains uncontaminated. The PIFs of the functional \( T \) at \((F_1, F_2)\) are

\[
PINF_1(x; T, F_1, F_2) = \lim_{\epsilon \to 0} \left[ \frac{T[(1-\epsilon)F_1 + \epsilon \Delta x_0, F_2] - T(F_1, F_2)}{\epsilon} \right],
\]

\[
PINF_2(x; T, F_1, F_2) = \lim_{\epsilon \to 0} \left[ \frac{T[F_1, (1-\epsilon)F_2 + \epsilon \Delta x_0] - T(F_1, F_2)}{\epsilon} \right].
\]

### 2.2.1 Partial Influence Functions for Ratio of Variances

Let \( T \) denote the function for the mean estimator such that \( T(F) = \int xdF_i = \mu \). Let \( V \) the function for the variance estimator where \( V(F) = \int [x - T(F)]^2 dF = \sigma^2 \). It is straightforward to show that \( T(F_i) = \int xd[(1-\epsilon)F + \epsilon \Delta x_0] = (1-\epsilon)\mu + \epsilon x_0 \) and, similarly, \( V(F_e) = \sigma^2 + \epsilon(1-\epsilon)(x_0 - \mu)^2 - \sigma^2 \). Consequently, \( IF(V, F, x) = (x - \mu)^2 - \sigma^2 \) is the influence function for the variance estimator with functional \( V \).

Let \( R \) denote the functional for the ratio of variances such that at \( F_1 \) and \( F_2 \), we have \( R(F_1, F_2) = V(F_1)/V(F_2) = \sigma_1^2/\sigma_2^2 = \rho \). Then the partial influence functions of \( R \) for contamination introduced to each of \( F_1 \) and \( F_2 \) are

\[
PINF_1(x_0; R, F_1, F_2) = \frac{\sigma_1^2}{\sigma_2^2} \left[ \frac{(x - \mu_1)^2}{\sigma_1^2} - 1 \right] = \rho[z_1^2 - 1]
\]

\[
PINF_2(x_0; R, F_1, F_2) = -\frac{\sigma_2^2}{\sigma_2^2} \left[ \frac{(x - \mu_2)^2}{\sigma_2^2} - 1 \right] = -\rho[z_2^2 - 1]
\]

where \( z_i = (x_0 - \mu_i)/\sigma_i \) \((i = 1, 2)\) denotes the standardized contaminant with respect to each of \( F_1 \) and \( F_2 \). Hence, as expected, outliers can be detrimental to estimation of ratio of variances since the PIFs are unbounded with respect to \( x_0 \).

### 2.2.2 Partial Influence Functions for squared IQR Ratio

Let \( f \) denote the probability density function of \( F \) and let \( Q_p \) denote the functional for the \( p \)th quantile so that \( Q_p(F) = x_p \). The influence function of the \( p \)th quantile at \( F \) is well known [e.g., p.59 17] to be

\[
IF(x_0; Q_p, F) = [p - I(x_p \geq x_0)]g(p)
\]

where \( Q'(p) = g(p) = 1/f(x_p) \) is known as the quantile density (Parzen, [9]) at \( p \). For \( X \sim F \), it can be shown that \( E_F[IF(x_0; Q_p, F)] = 0 \) and

\[
Var_F[IF(X; Q_p, F)] = E_F[IF^2(X; Q_p, F)] = p(1-p)g^2(p).
\]

When the statistical functional is applied to the empirical distribution denoted \( F_n \), then the asymptotic variance of \( Q_p(F_n) \) is \( nVar[Q_p(F_n)] = p(1-p)g^2(p) \). This variance will be revisited when we consider the asymptotic variance for the quantile estimators later.

Let \( R_p \) denote the function for the squared ratio of two interquantile ranges. Then, for a \( p \in (0,0.5) \),

\[
R_p(F_1, F_2) = \left[ \frac{Q_{1-p}(F_1) - Q_p(F_1)}{Q_{1-p}(F_2) - Q_p(F_2)} \right]^2 = \rho_p.
\]
Recall that we denote $Q_p(F_1) = x_p$ and $Q_p(F_2) = y_p$ to distinguish between the quantiles from the two populations.

**Theorem 2.1.** The partial influence functions of $R_p$ for contamination introduced to each of $F_1$ and $F_2$ are

\[
PIF_1(x_0; R_p, F_1, F_2) = \frac{2p}{x_{1-p} - x_p} \left[ IF(x_0; Q_{1-p}, F_1) - IF(x_0; Q_p, F_1) \right],
\]

\[
PIF_2(x_0; R_p, F_1, F_2) = -\frac{2p}{y_{1-p} - y_p} \left[ IF(x_0; Q_{1-p}, F_2) - IF(x_0; Q_p, F_2) \right]
\]

where $IF(x_0; Q_p, F)$ is defined in (2.4).

The proof of Theorem 2.1 is in Appendix A.

**2.2.3 Partial influence function comparisons**

In this section we compare the partial influence functions of the ratio of variances and the squared IQRs ratio.

Figure 1: PIF\(_1\) comparisons for (A) two exponential populations with rates 1 and 2 and (B) two log-normal populations both with $\mu = 0$ and $\sigma = 1$. For both examples we set $p = 0.2$.

Figure 1 depicts the partial influence functions for the two estimators for two exponential populations with different rates (plot A) and two log-normal populations with the same parameters (plot B). For both examples we set $p = 0.2$. Since the exponential distribution is a member of the scale family, both the ratio of variances and squared IQRs are estimators of the same quantity. Since the parameters for the log-normal example are set the same values for the two populations, the estimators for this example are both estimators of one. The partial influence function of the ratio of variance (shown in red) is quadratically unbounded in $x_0$ highlighting that outliers can be highly influential. However, the partial influence function of the squared IQR ratio (shown in black) is bounded in $x_0$ with discontinuities at $x_{0.2}$ and $x_{0.8}$. The partial influence functions for the second population are simply the negative of partial influence functions for the first so we do not show them here.
2.3 Asymptotic variance derivations and comparisons

Consider \( n_1 \) and \( n_2 \) independent and identically distributed (iid) random variables from \( F_1 \) and \( F_2 \) respectively. For \( F_{n_1} \) and \( F_{n_2} \) denoting the empirical distribution functions for these (iid samples), from [10]

\[
\sqrt{n_1 + n_2} \left[ R(F_{n_1}, F_{n_2}) - R(F_1, F_2) \right]
\]

is asymptotically normal with mean zero and asymptotic variance

\[
\text{ASV}(R; n_1, n_2) = \frac{1}{w_1} E_{F_1}[\text{PIF}_1(X; R, F_1, F_2)^2] + \frac{1}{w_2} E_{F_2}[\text{PIF}_2(X; R, F_1, F_2)^2]
\]  \hspace{1cm} (2.6)

where \( w_i = n_i/(n_1 + n_2) \) (\( i = 1, 2 \)) and where \( E_F(X; .) \) denotes expectation when \( X \sim F \).

Recall that \( \mu_i \) and \( \sigma_i \) denote the mean and standard deviation of \( F_i \) (\( i = 1, 2 \)) and that \( \rho = \sigma_2^2/\sigma_1^2 \). Then from (2.3) and (2.6), it is straightforward to show that the asymptotic variance for the ratio of variances estimator is

\[
\text{ASV}(R; n_1, n_2) = \rho^2 \left\{ \frac{1}{w_1} \left[ E_{F_1}(Z_1^4) - 1 \right] + \frac{1}{w_2} \left[ E_{F_2}(Z_2^4) - 1 \right] \right\} \]  \hspace{1cm} (2.7)

where \( Z_i = (X - \mu_i)/\sigma_i \) so that \( E_{F_i}(Z_i^4) \) is the scaled fourth central moment of \( F_i \) (\( i = 1, 2 \)). We now provide the asymptotic variance for the squared ratio of interquantile ranges estimator.

**Theorem 2.2.** The asymptotic variance of \( \sqrt{n_1 + n_2} R_p(F_{n_1}, F_{n_2}) \) is

\[
\text{ASV}(R_p; n_1, n_2) = 4p \rho_p^2 \left\{ \frac{g_1^2(p) + g_1^2(1-p) + p [g_1(p) - g_1(1-p)]^2}{w_1(x_{1-p} - x_p)^2} + \frac{g_2^2(p) + g_2^2(1-p) + p [g_2(p) - g_2(1-p)]^2}{w_2(y_{1-p} - y_p)^2} \right\}
\]

The proof of Theorem 2.2 is in Appendix B.

**Corollary 2.1.** Suppose that \( X \) and \( Y \) are both random variables from the same location-scale family such that the density of \( X \) may be written \( f(x; \mu_x, \eta_x) \) and the density of \( Y \) \( f(y; \mu_y, \eta_y) \) where \( \mu_x \), \( \mu_y \) and \( \eta_x \), \( \eta_y \) are the respective location and scale parameters. Let \( q_{1-p} \) and \( q_p \) denote the \((1-p)\)th and \( p\)th quantiles of the distribution with density \( f(.; 0, 1) \) and \( g_0(1-p) = 1/f(q_{1-p}; 0, 1) \) and \( g_0(p) = 1/f(q_p; 0, 1) \) the respective quantile densities. Then

\[
\text{ASV}(R_p; n_1, n_2) = 4p \eta_x^4 \eta_y^2 \left\{ \frac{g_0^2(p) + g_0^2(1-p) + p [g_0(p) - g_0(1-p)]^2}{w_1(1 - w_1)(q_{1-p} - q_p)^2} \right\}
\]

**Proof.** Since \( X \) and \( Y \) are from the same location-scale family, then \( x_{1-p} - x_p = \eta_x(q_{1-p} - q_p) \), \( y_{1-p} - y_p = \eta_y(q_{1-p} - q_p) \) and

\[
f(x; \mu_x, \eta_x) = \frac{1}{\eta_x} f \left( \frac{\eta_x x - \mu_x}{\eta_x} ; 0, 1 \right), \ f(y; \mu_y, \eta_y) = \frac{1}{\eta_y} f \left( \frac{\eta_y y - \mu_y}{\eta_y} ; 0, 1 \right).
\]

Using these results \( g_1(p) = g_0(p)/\eta_x \) and \( g_2(p) = g_0(p)/\eta_y \). The result follows after some simplification and noting that \( w_2 = 1 - w_1 \). \( \square \)
Corollary 2.1 is a useful result that may provide insights in the behavior of the ASV for location-scale families.

**Remark 2.1.** Since the ASV in Corollary 2.1 depends on the location and scale parameters only though the term $\eta^4_x/\eta^4_y$ which is a common factor to all terms, then the choice of $p$ that minimises the ASV is independent of the location and scale parameters.

As examples, we have selected the LN$(0, 1)$, EXP$(1)$ and Uniform$(2, 5)$ distributions to compare the asymptotic variances of the ratio of variances and squared ratio of interquantile range estimators.

As shown in the above Figure 2, the ASV of the squared IQR ratio (black curve) can vary greatly with $p$. Here the plots are over the domain $p \in [0.01, 0.45]$ and for choices of $p$ for the log-normal distribution we can see that the ASV is smaller than that for the ratio of variances (red line). For the exponential distribution, a choice of $p$ of around 0.15 will result in a smaller ASV for the squared IQR although if $p$ is either very small or moderately large then ASV for the ratio of variances is smaller. An interesting finding arises for the continuous uniform distribution. The ASV is minimised when $p$ is chosen to be as small as possible. This implies that, in practice, have should choose to select the range (max $-$ min) as the interquantile range to decrease estimator variability.

We now explore the choices of $p$ that result in the minimum ASV of the squared IQR ratio estimator for selected distributions; namely the log-normal, exponential, continuous uniform, Normal, Chi-square, Beta, Weibull, Gamma and Pareto type II (Lomax) distributions.
Table 1: Choice of $p$ related to minimise the ASV for several distributions.

| Distribution          | $p$   |
|-----------------------|-------|
| Exp($\lambda$)        | 0.128 |
| Unif($a,b$)            | 0     |
| Log Normal(0,1)        | 0.193 |
| $N(\mu,\sigma^2)$     | 0.069 |
| Chi-Square(1)          | 0.127 |
| Chi-Square(2)          | 0.128 |
| Chi-Square(25)         | 0.079 |
| Beta(0.1,0.1)          | 0     |
| Beta(0.5,0.5)          | 0     |
| Beta(1,1)              | 0     |
| Beta(10,10)            | 0.055 |
| Weibull(0.5)           | 0.181 |
| Weibull(1)             | 0.128 |
| Weibull(2)             | 0.069 |
| Weibull(10)            | 0.081 |
| Gamma(1)               | 0.128 |
| Gamma(2)               | 0.110 |
| Gamma(10)              | 0.081 |

Table 2: Choice of $p$ to minimise the ASV when the numerator and denominator distributions are both Pareto type II.

| Shape Parameter($\alpha$) | $p$   |
|---------------------------|-------|
| 1                         | 0.282 |
| 2                         | 0.224 |
| 3                         | 0.198 |
| 4                         | 0.183 |
| 5                         | 0.173 |
| 6                         | 0.167 |
| 7                         | 0.161 |

As shown in Table 1, the choice of $p$ related to the minimum ASV of the squared IQR ratio estimator vary for different distributions. The choice of $p$ that minimises the ASV for the exponential, uniform and normal distributions does not depend on the parameters of these distributions. This is true when the two distributions are members of the same location, scale or location-scale families (see Remark 2.1 which is a consequence of Corollary 2.1) We also considered various choices of shape parameter of the Pareto type II distribution (2) when the scale parameter is equal to one. An advantage of using the squared IQR ratio of this distribution is that it exists for all $\alpha$ while the ratio of variances only exists for $\alpha > 2$. For all cases in Tables 1 and 2 with the exception of small shape parameter for the Pareto type II, choosing a $p$ less than 0.25 results in a smaller ASV than if one were to use the ratio of interquartile ranges. This finding agrees with the observations of [16].

3 Inference

While we will be using asymptotic results to create approximate 95% confidence intervals, recent research has shown that excellent interval coverage for estimators of functions of quantiles can be obtained for sample sizes even as low as 30 to 50 [see 12, 14, 13]. These works use the Quantile Optimality Ratio [QOR 11] to choose optimal bandwidths for kernel density estimators that are needed to estimate the unknown quantile densities, denoted $g$, in the variance formulations. Throughout, we let $\hat{g}_i$ ($i = 1, 2$) denote the quantile density estimated using the QOR approach.
3.1 Approximate variance of the ratio of variances estimator

Let $\hat{\rho} = \frac{S_1^2}{S_2^2}$ denote the estimator of $\rho = \frac{\sigma_1^2}{\sigma_2^2}$ where $S_i^2 = \mathcal{V}(F_{n_i})$ ($i = 1, 2$) are the sample variance estimators. Let $\text{Var}(\hat{\rho})$ denote the finite sample variance of $\hat{\rho}$ based on the combined sample observations $n_1 + n_2$. Let $\{X_i\}_{i=1}^{n_1}$ denote the simple random sample for the first sample with sample mean $\bar{X} = \mathcal{T}(F_{n_1})$ and $\{Y_i\}_{i=1}^{n_2}$ denote the simple random sample for the second with $\bar{Y} = \mathcal{T}(F_{n_2})$.

From (2.6), we can obtain an approximate variance

$$\text{Var}(\hat{\rho}) \approx \frac{\hat{\rho}^2}{n_1 + n_2} \left[ \frac{1}{w_1} (\bar{Z}_1^4 - 1) + \frac{1}{w_2} (\bar{Z}_2^4 - 1) \right]$$

where

$$w_i = \frac{n_i}{n_1 + n_2}, \quad \bar{Z}_1^4 = \frac{1}{n_1} \sum_{i=1}^{n_1} \left( \frac{X_i - \bar{X}}{S_1} \right)^4, \quad \bar{Z}_2^4 = \frac{1}{n_1} \sum_{i=1}^{n_1} \left( \frac{Y_i - \bar{Y}}{S_2} \right)^4, \quad \hat{\rho} = \frac{S_1^2}{S_2^2}.$$

3.2 Approximate variance of the squared IQR ratio estimator

Let $\text{Var}(\hat{\rho}_p)$ denote the variance of the $\hat{\rho}_p$ estimator. From Theorem 2.2

$$\text{Var}(\hat{\rho}_p) \approx \frac{4 p \hat{\rho}_p^2}{n_1 + n_2} \left\{ \frac{\hat{g}_1^2(p) + \hat{g}_1^2(1-p) + p [\hat{g}_1(p) - \hat{g}_1(1-p)]^2}{w_1(\bar{x}_{1-p} - \bar{x}_p)^2} + \frac{\hat{g}_2^2(p) + \hat{g}_2^2(1-p) + p [\hat{g}_2(p) - \hat{g}_2(1-p)]^2}{w_2(\bar{y}_{1-p} - \bar{y}_p)^2} \right\},$$

where $\hat{g}_i(p)$ ($i = 1, 2$) is the estimated quantile density using the QOR method [11].

3.3 Asymptotic confidence intervals

As is often the case, in constructing out interval estimators for the ratios we use the log transformation and exponentiate the interval to return to the ratio scale. Given a strictly positive random variable $W > 0$, using the Delta method it can easily be shown that $\text{Var}[\ln(W)] \approx \text{Var}(W)/W^2$.

Hence, approximate $(1 - \alpha) \times 100\%$ confidence intervals for $\rho$ and $\rho_p$ are

$$\exp \left[ \ln(\hat{\rho}) \pm z_{1-\alpha/2} \frac{1}{\hat{\rho}} \sqrt{\text{Var}(\hat{\rho})} \right],$$

$$\exp \left[ \ln(\hat{\rho}_p) \pm z_{1-\alpha} \frac{1}{\hat{\rho}_p} \sqrt{\text{Var}(\hat{\rho}_p)} \right],$$

respectively where $z_{1-\alpha/2}$ is the $(1 - \alpha/2) \times 100\%$ percentile of the standard normal distribution.

We begin by comparing coverage probabilities for two intervals for the ratio of variances. The first is the standard interval based on the $F$ distribution which is analogous to the $F$-test and the second is the interval in (3.3). We have simulated data of varying sample sizes, $n_1$ and $n_2$, using 10,000 simulation trials. Table 3 provides the simulated coverage probabilities from two methods for log normal and exponential distributions. The $F$-test interval showed poor throughout which is expected to do the reliance on underlying normality of the populations. The interval in
(3.3) provides acceptable coverage probabilities for the LN(0,1) distribution only with very large sample sizes (e.g. in the thousands) and good coverage probabilities for the EXP(1) distribution with moderately large sample sizes. To investigate the robustness properties of the interval, we introduced an outlier by replacing one data point in our simulated data with a very large outlier. The effect of introducing the single outlier was that sample sizes need to be very large before adequate coverage is achieved. This suggests that the number of outliers needs to be very small relative to the sample size to produce reliable results.

We now consider coverage for confidence interval in (3.4) for the squared IQR ratio. We again conducted 10,000 simulations and varied sample sizes from 50 to 10,000. We considered four different values for $p$, 0.01, 0.05, 0.1 and 0.2. Table 4 shows the resulting coverage probabilities. The coverage for the squared IQR ratio interval is very good for our chosen skewed distributions with the exception of a combination of small sample sizes and small $p$.

### 4 An example: Doctor visits

We selected the doctor visits data set used in [5] to apply our findings to a real world problem. The doctor visits data is a subsample of 3066 individuals of the AHEAD cohort (born before 1924) for wave 6 (year 2002) from the Health and Retirement Study (HRS) which surveys more than 22,000 Americans over the age of 50 every 2 years. We grouped this data in to two groups by taking the gender as the grouping variable. The response variable that we were interested is the number of doctor visits. Table 5 provides summary statistics of the response variable for the two gender groups.

From Table 5, the summary statistics suggest that the doctor visits distributions are positively skewed which is common for count variables. There is also a large outlier in the female group with a number of doctor visits equal to 750. We removed the outlier from the data set and again calculated the descriptive statistics for female group as shown in the 3rd column of the above Table 5. The mean for the female group reduces after the removal of the outlier and the summary statistics still suggest positive skew.

Our objective was to compare the variation of the number of doctor visits between males and
females. We used the ratio of variance approach and the squared IQR ratio to compare the variation of the number of doctor visits between males and females with and without the outlier.

Table 6 provides the confidence interval bounds of the 95 percent confidence intervals using the two methods. It can be clearly see that there is a large difference between the ratio of variance confidence intervals depending on whether the outlier is included. On the other hand, the confidence interval for the squared IQR ratio is hardly influenced by the outlier. Additionally, in comparison, the interval for the ratio of variances is wide compared to interval for the squared IQR ratio with the exception of when $p = 0.01$ is chosen for the latter. This suggests that the IQR approach is a better choice to compare variation between the two groups for this example.

5 Summary and Discussion

We have considered two alternative approaches basically to the often used $F$-test to compare the variation between two populations. The intervals were derived, asymptotically, from the derivation of the partial influence function. The first considered the ratio of variances and the second the squared ratio of IQRs. In comparison to the ratio of variances interval, the interval for the squared ratio of IQRs provides very good coverage for the distributions that we considered for moderate to large sample sizes. The interval is also robust to outliers which may be encountered in skewed data sets and which can be detrimental to the ratio of variances approach. Future work will consider how to best choose $p$ or the creation of a combined interval that does not require $p$ to be chosen as was done recently by [6] for hypothesis tests of variation.

6 Acknowledgement

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A Proof of Theorem 2.1

A power series expansion for $Q_{1-p}(F_{\epsilon}) - Q_p(F_{\epsilon})$ can be written as

$$Q_{1-p}(F) - Q_p(F) + \epsilon [IF(x_0; Q_{1-p}, F) - IF(x_0; Q_p, F)] + O(\epsilon^2).$$

Note, setting $F_\epsilon = (1 - \epsilon)F + \epsilon \Delta x_0$ where $Q_p(F_1) = x_p$, we have that $[Q_{1-p}(F_{\epsilon}) - Q_p(F_{\epsilon})]^2$ can be written

$$(x_{1-p} - x_p)^2 + 2\epsilon(x_{1-p} - x_p) [IF(x_0; Q_{1-p}, F_1) - IF(x_0; Q_p, F_1)] + O(\epsilon^2). \tag{A.1}$$

For simplicity below write $PIF_1 = PIF_1(x_0; R_p, F_1, F_2)$ and $IF_1,p = IF(x_0; Q_p, F_1)$ and recall that $Q_p(F_2) = y_p$. Then the first partial influence function is

$$PIF_1 = \lim_{\epsilon \downarrow 0} \left\{ \frac{(x_{1-p} - x_p)^2 + 2\epsilon(x_{1-p} - x_p) [IF(x_0; Q_{1-p}, F_1) - IF_1,p] + O(\epsilon^2) - (x_{1-p} - x_p)^2}{\epsilon(y_p - y_{1-p})^2} \right\}$$

$$= \frac{2\rho_p}{(x_{1-p} - x_p)^2 [IF_1,1-p - IF_1,p]},$$

since $\rho_p = (x_{1-p} - x_p)^2/(y_{1-p} - y_p)^2$. 

11
Let $IQR_p(F) = Q_{1-p}(F) - Q_p(F)$ denote the function for the interquantile range at $p$ and for the second partial influence function set $F = (1 - \epsilon)F_2 + \epsilon\Delta_{x_0}$. Then the second partial influence function is

$$
\text{PIF}_2 = \lim_{\epsilon \to 0} \left\{ \frac{(x_{1-p} - x_p)^2 [IQR^2(F)]}{\epsilon} - \frac{(x_{1-p} - x_p)^2/(y_{1-p} - y_p)^2}{\epsilon} \right\}
$$

$$
= \lim_{\epsilon \to 0} \left\{ \frac{(x_{1-p} - x_p)^2(y_{1-p} - y_p)^2 - (x_{1-p} - x_p)^2 IQR^2(F_e)}{\epsilon(y_{1-p} - y_p)^2 IQR^2(F_e)} \right\}
$$

$$
= \lim_{\epsilon \to 0} \left\{ \frac{-2\epsilon(y_{1-p} - y_p) [IF_{1-p} - IF_{2,p}] + O(\epsilon^2)}{\epsilon(y_{1-p} - y_p)^2 IQR^2(F_e)} \right\}
$$

when using (A.1) but evaluated at $F_2$ and letting $IF_{2,p} = IF(x_0; Q_p, F_2)$. The proof concludes after canceling the $\epsilon$ terms and taking the limit.

**B Proof of Theorem 2.2**

First, note that for $g_1(p) = 1/f_1(x_p)$ where $f_1$ is the density function for distribution $F_1$,

$$
\text{IF}(x_0; Q_p, F_1)^2 = [y^2 + (1 - 2p)I(x_p \geq x_0)] g_1^2(p),
$$

$$
\text{IF}(x_0; Q_{1-p}, F_1)^2 = (1 - p) [1 - I(x_{1-p} \geq x_0)] g_1^2(1 - p)
$$

and, since $p < 1 - p$, $\text{IF}(x_0; Q_p, F_1)\text{IF}(x_0; Q_{1-p}, F_1)$ is equal to

$$
p [(1 - p) - I(x_{1-p} \geq x_0) + I(x_p \geq x_0)] g_1(p)g_1(1 - p).
$$

For simplicity let $IF_p(X) = IF(X; Q_p, F_1)$. Then, from above and Theorem 2.1 and noting that, for example, $E_{F_1}[I(x_p \geq X)] = p$,

$$
E_{F_1} [\text{PIF}^2] = \frac{4p^2}{(x_{1-p} - x_p)^2} \left\{ E_{F_1} [IF_{1-p}^2(X)] + E_{F_1} [IF_p^2(X)] \right\}
$$

$$
- 2E_{F_1} [IF_{1-p}(X)IF_p(X)]
$$

$$
= \frac{4pp^2}{(x_{1-p} - x_p)^2} \left\{ (1 - p)[g_1^2(p) + g_1^2(1 - p)] - 2pg_1(p)g_1(1 - p) \right\}
$$

$$
= \frac{4pp^2}{(x_{1-p} - x_p)^2} \left\{ g_1^2(p) + g_1^2(1 - p) + p [g_1(p) - g_1(1 - p)]^2 \right\}.
$$

$E_{F_2} [\text{PIF}^2]$ is the same as the above but where $x_{1-p} - x_p$ is replaced by $y_{1-p} - y_p$ and $g_2(p) = 1/f_2(y_p)$ replaces $g_1(p)$.

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Table 4: Coverage for 95 percent interval estimator of the squared IQR ratio in (3.4).

| Sample size (n₁, n₂) | p   | X ~ LN(0,1) | Y ~ LN(0,1) | X ~ EXP(1) | Y ~ EXP(1) | X ~ χ²₃ | Y ~ χ²₃ | X ~ PAR(1,7) | Y ~ PAR(1,3) |
|----------------------|-----|-------------|-------------|------------|------------|---------|---------|-------------|-------------|
| 50,50                | 0.01| 0.8249      | 0.9861      | 0.9958     | 0.8339     |         |         |             |             |
|                      | 0.05| 0.9751      | 0.9706      | 0.9693     | 0.9771     |         |         |             |             |
|                      | 0.1 | 0.9776      | 0.9668      | 0.9684     | 0.9767     |         |         |             |             |
|                      | 0.2 | 0.9784      | 0.9706      | 0.9705     | 0.9744     |         |         |             |             |
| 100,100              | 0.01| 0.6329      | 0.6479      | 0.7157     | 0.6241     |         |         |             |             |
|                      | 0.05| 0.9769      | 0.9701      | 0.9698     | 0.9757     |         |         |             |             |
|                      | 0.1 | 0.9752      | 0.9701      | 0.9654     | 0.9749     |         |         |             |             |
|                      | 0.2 | 0.9753      | 0.9624      | 0.9672     | 0.9695     |         |         |             |             |
| 200,200              | 0.01| 0.9575      | 0.9493      | 0.9428     | 0.9521     |         |         |             |             |
|                      | 0.05| 0.9780      | 0.9682      | 0.9652     | 0.9754     |         |         |             |             |
|                      | 0.1 | 0.9729      | 0.9663      | 0.9649     | 0.9679     |         |         |             |             |
|                      | 0.2 | 0.9728      | 0.9612      | 0.9627     | 0.9650     |         |         |             |             |
| 200,500              | 0.01| 0.9598      | 0.9515      | 0.9593     | 0.9735     |         |         |             |             |
|                      | 0.05| 0.9705      | 0.9646      | 0.9621     | 0.9719     |         |         |             |             |
|                      | 0.1 | 0.9686      | 0.9630      | 0.9617     | 0.9677     |         |         |             |             |
|                      | 0.2 | 0.9634      | 0.9604      | 0.9605     | 0.9625     |         |         |             |             |
| 500,500              | 0.01| 0.9808      | 0.9712      | 0.9689     | 0.9753     |         |         |             |             |
|                      | 0.05| 0.9724      | 0.9627      | 0.9621     | 0.9666     |         |         |             |             |
|                      | 0.1 | 0.9653      | 0.9621      | 0.9600     | 0.9638     |         |         |             |             |
|                      | 0.2 | 0.9639      | 0.9587      | 0.9597     | 0.9600     |         |         |             |             |
| 500,1000             | 0.01| 0.9785      | 0.9695      | 0.9708     | 0.9790     |         |         |             |             |
|                      | 0.05| 0.9664      | 0.9624      | 0.9623     | 0.9695     |         |         |             |             |
|                      | 0.1 | 0.9633      | 0.9563      | 0.9617     | 0.9641     |         |         |             |             |
|                      | 0.2 | 0.9616      | 0.9607      | 0.9540     | 0.9578     |         |         |             |             |
| 1000,1000            | 0.01| 0.9746      | 0.9689      | 0.9684     | 0.9768     |         |         |             |             |
|                      | 0.05| 0.9667      | 0.9627      | 0.9609     | 0.9684     |         |         |             |             |
|                      | 0.1 | 0.9630      | 0.9589      | 0.9584     | 0.9597     |         |         |             |             |
|                      | 0.2 | 0.9590      | 0.9551      | 0.9563     | 0.9608     |         |         |             |             |
| 1000,5000            | 0.01| 0.9656      | 0.9564      | 0.9587     | 0.9711     |         |         |             |             |
|                      | 0.05| 0.9627      | 0.9558      | 0.9611     | 0.9576     |         |         |             |             |
|                      | 0.1 | 0.9622      | 0.9562      | 0.9577     | 0.9581     |         |         |             |             |
|                      | 0.2 | 0.9554      | 0.9558      | 0.9546     | 0.9583     |         |         |             |             |
| 5000,5000            | 0.01| 0.9644      | 0.9643      | 0.9627     | 0.9646     |         |         |             |             |
|                      | 0.05| 0.9605      | 0.9535      | 0.9566     | 0.9524     |         |         |             |             |
|                      | 0.1 | 0.9561      | 0.9538      | 0.9538     | 0.9544     |         |         |             |             |
|                      | 0.2 | 0.9523      | 0.9515      | 0.9567     | 0.9553     |         |         |             |             |
| 10000,10000          | 0.01| 0.9633      | 0.9606      | 0.9605     | 0.9622     |         |         |             |             |
|                      | 0.05| 0.9577      | 0.9568      | 0.9530     | 0.9579     |         |         |             |             |
|                      | 0.1 | 0.9545      | 0.9532      | 0.9498     | 0.9555     |         |         |             |             |
|                      | 0.2 | 0.9536      | 0.9513      | 0.9534     | 0.9528     |         |         |             |             |
Table 5: Summary Statistics of number of doctor visits between Male and Female

| Summary Statistic | Male | Female (without outlier) |
|-------------------|------|--------------------------|
| Sample Size       | 987  | 2079                     |
| Minimum           | 0    | 0                        |
| 1st Quartile      | 4    | 4                        |
| Median            | 8    | 8                        |
| Mean              | 12.08| 12.8                     |
| 3rd Quartile      | 14   | 15                       |
| Maximum           | 300  | 750                      |

Table 6: 95 percent confidence interval lower bounds (LB) and upper bounds (UB) for the doctor visits data.

| Confidence Interval Method | x = male, y = female | With outlier | Without outlier |
|----------------------------|----------------------|--------------|-----------------|
| Ratio of Variance          | 0.2155               | 1.8582       | 0.5367          | 2.3671          |
| Squared IQR ratio, $p = 0.01$ | 0.2489               | 2.5223       | 0.2814          | 2.8186          |
| Squared IQR ratio, $p = 0.05$ | 0.4861               | 1.2268       | 0.5356          | 1.3759          |
| Squared IQR ratio, $p = 0.1$ | 0.8789               | 1.1378       | 0.8799          | 1.1365          |
| Squared IQR ratio, $p = 0.2$ | 0.4610               | 0.9454       | 0.4610          | 0.9454          |
| Squared IQR ratio, $p = 0.25$ | 0.5983               | 1.1416       | 0.5985          | 1.1412          |