Bounding Pinch Point Schemes of Projected Surfaces

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Abstract

Let \( X \) be a smooth surface and let \( \varphi : X \to \mathbb{P}^N \), with \( N \geq 4 \), be a finitely ramified map which is birational onto its image \( Y = \varphi(X) \), with \( Y \) non-degenerate in \( \mathbb{P}^N \). In this paper, we produce a lower bound for the length of the pinch scheme of a general linear projection of \( Y \) to \( \mathbb{P}^3 \). We then prove that the lower bound is realized if and only if \( Y \) is a rational normal scroll.

1 Introduction

The classical General Projection Theorem (see [2, Theorem 2.5], [13, Section 2], [12, Section 1], [14, Theorem 1]) states that there are at most three types of singular points (often called ordinary singularities) contained in the image of a general linear projection of a smooth surface to \( \mathbb{P}^3 \). These include: A curve of double points, finitely many triple points, and finitely many pinch points. It is natural to wonder what types of numerical limitations there are on these singularities.

In keeping with classical notation, let \( X \) be a smooth projective surface in \( \mathbb{P}^N \). (Later on, we will exchange this notation for a more general setup.) In this paper, we focus on the third type of singularity, pinch points. In particular, we will present a lower bound for the length of the pinch point scheme of a general projection of \( X \) to \( \mathbb{P}^3 \). The length of this scheme, much like the class of \( X \) (that is, the degree of its dual variety \( X^* \) inside \( \mathbb{P}^{N^*} \)), is a number which measures something about the geometry of \( X \) as it sits inside \( \mathbb{P}^N \). While there is clearly no upper bound on the pinch point scheme length, it can be bounded from below in terms of \( N \).

The pinch scheme length admits another interpretation: It is one of two coefficients defining the Gauss class \( [\gamma(X)] \in A_n^*(G(n,N)) \). To answer this question, take a \( \mathbb{Z} \)-basis for the Chow group \( A_n(G(n,N)) \) in terms of Schubert classes \( \sigma_a^* \), where \( a \) is an integer partition of \( n \), and where \( \sigma_a^* \) is the dual to the generator \( \sigma_a \in A^n(G(n,N)) \). Then we can write

\[
[\gamma(X)] = \sum \gamma_a \cdot \sigma_a^*,
\]

for coefficients \( \gamma_a \in \mathbb{Z} \). The integers \( \gamma_a \) roughly measure the “twistedness” of the tangent space of \( X \), and we choose to make them the central objects of study. Each coefficient is some natural expression involving the Chern classes of the bundle of principal parts for \( \mathcal{O}_X(1) \), and their study gives rise to a particular bipartite classification problem.
I. What is the minimum value that $\gamma_\alpha$ can take for each partition $\alpha$ of $n$?
II. Which varieties $X$ realize this minimum?

1.1 Historical Context

The case where $n = 1$ (so that $X$ is a degree $d$ curve of genus $g$) is rather straightforward. Since $A_1(G(1,N))$ is freely generated by $\sigma_1^*$, the class of a line in the Plücker embedding of the Grassman-
nian, there is only one coefficient to consider. To find $\gamma_1$, where $[\gamma(X)] = \gamma_1 \cdot \sigma_1^*$, we compute

$$[\gamma(X)] \cdot \sigma_1 = \gamma_1 \cdot \sigma_1^* \cdot \sigma_1 = \gamma_1.$$  

But $\sigma_1$ is the class of a line in $\mathbb{P}^N$ which meets a linear space of codimension 2. Geometrically, the
intersection product above amounts to projecting away from a general codimension 2 plane; this is a
general linear projection from $X$ to $\mathbb{P}^1$. We are asking: How many lines tangent to $X$ meet a general
codimension 2 plane? Equivalently, at how many point of $\gamma$ of minimal degree $[3]$ implies that
$\gamma$ is a rational normal scroll. Moreover, every non-degenerate curve in $\mathbb{P}^N$ satisfies $d \geq N$, so that
$\gamma_1 \geq 2N - 2$, and equality holds when $X$ is a curve of minimal degree. In other words, the absolute
minimal value for $\gamma_1$ in $\mathbb{P}^N$ (for any $N$) is realized by the rational normal curve of degree $N$.

The case where $n = 2$ (so that $X$ is a surface), is significantly more involved. The Chow group $A_2(G(2,N))$ is freely generated by $\sigma_{1,1}^*$ and $\sigma_2^*$, and so computing

$$[\gamma(X)] = \gamma_{1,1} \cdot \sigma_{1,1}^* + \gamma_2 \cdot \sigma_2^*$$

requires two distinct analyses. Tracing through the definition of $\gamma_2$, we see that it is none other than the
class of $X$, i.e. the degree of its dual variety, classically denoted $\mu_2$. In 1955, E. Marchionna proved
that if $d = \deg X$, then $\gamma_2 \geq d - 1$, with equality holding if and only if $X \subset \mathbb{P}^5$ is the Veronese
surface [10]. The next year, in 1956, D. Gallarati proved a similar result, namely that for all other
smooth projective surfaces, $\gamma_2 \geq d$ with equality holding if and only if $X$ is a scroll over a smooth
curve [5]. One year later, (in 1957 [6]), Gallarati was investigating the difference between degree and
class, and he proved that if $\gamma_2 > d$, then in fact $\gamma_2 \geq d + 3$. He also classifies the surfaces satisfying
$3 \leq \gamma_2 - d \leq 10$. A concise statement of this result can be found in [8].

Since the minimal degree for a non-degenerate surface in $\mathbb{P}^N$ is $N - 1$, the classification of varieties of
minimal degree [3] implies that $\gamma_2 \geq N - 2$ with equality holding if and only if $X$ is the Veronese
surface in $\mathbb{P}^5$, and $\gamma_2 = N - 1$ if and only if $X$ is a rational normal scroll.

1.2 The Main Result

In this paper, we investigate what remains of the case where $n = 2$. Specifically, we give a lower bound
for $\gamma_{1,1}$ and classify surfaces which meet that bound using techniques which are very different from
those used by Marchionna and Gallarati. Geometrically, $\gamma_{1,1}$ measures the number of pinch points
contained in the image of a general linear projection of the surface $X$ to $\mathbb{P}^3$. Indeed, we are asking
how many planes tangent to $X$ meet a generally chosen codimension 4 linear subspace of $\mathbb{P}^N$.

Our setup is as follows. For any finitely ramified map $f : X \to \mathbb{P}^3$, where $X$ is a smooth surface
and $f$ is birational onto its image $f(X)$, the ramification scheme or pinch point scheme of $f$ is
the subscheme of $X$ defined locally by the $2 \times 2$ minors of a matrix representing $df$, the induced
map on tangent spaces. We let \( \mathcal{P}(f) \) denote the length of the ramification scheme of \( f \). When \( X \) is a non-degenerate surface in \( \mathbb{P}^N \) with \( N \geq 4 \), and \( \pi : X \to \mathbb{P}^3 \) is a general linear projection, from [4, Proposition 12.6] we have a formula for \( \mathcal{P}(\pi) \):

\[
\mathcal{P}(\pi) = \deg \left( 6\xi^2 + 4\xi K_X + K_X^2 - c_2 \right),
\]

(1)

where \( \xi \) represents the class of a hyperplane section, and \( c_2 = c_2(T_X) \) is the second Chern class of the tangent bundle of \( X \), and \( \mathcal{P}(\pi) \) is the number of pinch points contained in the image surface \( \pi(X) \).

Moving forward, we will work in a more general setting. Let \( X \) be a smooth surface and let \( \varphi : X \to \mathbb{P}^N \) with \( N \geq 4 \) be a finitely ramified map which is birational onto its image \( Y = \varphi(X) \), with \( Y \) non-degenerate in \( \mathbb{P}^N \) (we will call such a map uncrumpled, see Definition 2.1). Then for a general projection \( \pi : Y \to \mathbb{P}^3 \), Equation (1) measures the length of the finite scheme \( \text{Ram}(\pi \circ \varphi) \). Within this context, we use the method of inner projection with a proof by induction on \( N \) for the main result:

**Theorem 1.1.** Let \( X \) be a smooth surface, and let \( \varphi : X \to \mathbb{P}^N \) be an uncrumpled map to \( Y = \varphi(X) \) with \( N \geq 4 \). Then for a general linear projection \( \pi : Y \to \mathbb{P}^3 \), we have

\[
\mathcal{P}(\pi \circ \varphi) \geq 2N - 6,
\]

with equality holding if and only if \( Y \) is a rational normal scroll.

Theorem 1.1 is proved as Corollary 3.10 in the text. The rest of the paper will proceed as follows. In Section 2, we will list out the specific definitions, hypotheses and conventions which we will use throughout the rest of the paper, along with some preliminary results which we will need later on. Section 3 will house the actual proof for Theorem 1.1 preceded by a series of constituent lemmas. In Section 4, we give a generalization of our result which classifies, for \( N \) large enough, the surfaces which have a near-minimal pinch point scheme length. Finally, in Section 5 we present some other possible generalizations and outstanding curiosities.

## 2 Definitions and Conventions

Throughout what follows, we work over an algebraically closed field \( k \) of characteristic 0. The word *variety* will be taken to mean a reduced finite type \( k \)-scheme, and the words *curve* and *surface* will refer to varieties of dimension one and two respectively. We adopt the pre-Grothendieck convention for projective spaces, i.e. \( \mathbb{P}^N_k \) will refer to the set of one dimensional subspaces of \( k^{N+1} \).

Let \( X \) be an irreducible smooth surface. For a map \( \varphi : X \to \mathbb{P}^N \), we write \( d\varphi : T_X \to \varphi^*T_{\mathbb{P}^N} \) for the derivative of \( \varphi \), and we denote by \( \text{Ram}(\varphi) \) the ramification scheme of \( \varphi \), i.e. the subscheme of \( X \) for which \( \text{rank}(d\varphi) \leq 1 \).

**Definition 2.1.** For a smooth surface \( X \), we say that a map \( \varphi : X \to \mathbb{P}^N \) is **uncrumpled** if

- \( \varphi(X) \subset \mathbb{P}^N \) is non-degenerate,
- \( X \) is birational onto its image \( \varphi(X) \), and
- \( \text{Ram}(\varphi) \) is finite.

Observe in particular that an uncrumpled map \( \varphi \) is necessarily finite. Since \( X \) is smooth, it follows that \( X \) is the normalization of \( \varphi(X) \).

**Setting 2.2** (The Inner Projection Setting). Let \( X \) be an irreducible smooth surface, and let \( \varphi : X \to \mathbb{P}^N \), \( N \geq 4 \), be an uncrumpled map whose image we denote \( Y = \varphi(X) \). Let \( x \in X \) be a generally chosen
point, and put \( y = \varphi(x) \) (so that \( y \) is a general point in \( Y \)). Let \( \beta_x : \tilde{X} \to X \) be the blow-up of \( X \) at \( x \), let \( E = \beta_x^{-1}(x) \) be the exceptional curve, and let \( \beta_y : \tilde{Y} \to Y \) the blow-up of \( Y \) at \( y \).

Next, let \( \pi_y : Y \to \mathbb{P}^{N-1} \) be the inner projection from \( y \), i.e. the projection map from \( y \in \mathbb{P}^N \) to \( \mathbb{P}^{N-1} \) restricted to \( \mathbb{P}^N \setminus \{y\} \). Let \( \varphi : \tilde{X} \to \mathbb{P}^{N-1} \) and \( \pi_Y : Y \to \mathbb{P}^{N-1} \) be the resolutions of \( \pi_X \) and \( \pi_Y \) to the corresponding blow-ups.

We use \( \zeta \) in the Chow ring \( A(X) \) and \( \tilde{\zeta} \in A(\tilde{X}) \) to denote the pullback of the hyperplane class along \( \varphi \) and \( \tilde{\varphi} \) respectively. Lastly, we define the pinch point scheme length of \( \varphi \) as

\[
\Psi(\varphi) = \deg \left( 6\zeta^2 + 4\zeta K_X + K_X^2 - c_2(T_X) \right).
\]

**Remark 2.3.** Note that since \( \varphi \) is uncrumpled, if \( \pi : Y \to \mathbb{P}^3 \) is a general linear projection then \( \Psi(\varphi) \) is precisely the length of \( \text{Ram}(\pi \circ \varphi) \). If \( \varphi \) is an embedding, then \( \Psi(\varphi) \) gives the number of pinch points in \( \pi(Y) \).

Setting 2.2 can be summarized in the following diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\
\downarrow{\beta_x} & & \downarrow{\beta_y} \\
X & \xrightarrow{\varphi} & Y \\
\downarrow{\varphi} & & \downarrow{\pi_y} \\
& \mathbb{P}^{N-1} & \\
\end{array}
\]

For the remainder of this section, we wish to collect several classical results which will arise in the proofs in the following sections.

**Theorem 2.4** (General Position Theorem \([7, 11]\)). Let \( C \subset \mathbb{P}^r, r \geq 2, \) be an irreducible non-degenerate, possibly singular, curve of degree \( d \). Then a general hyperplane meets \( C \) in \( d \) points any \( r \) of which are linearly independent.

**Theorem 2.5** (Surfaces with a two-dimensional family of plane curves \([11, 17, 15] \)). The only surfaces \( S \subset \mathbb{P}^N, N > 3, \) containing a 2-dimensional family of plane curves are either the Veronese surface in \( \mathbb{P}^5 \) or its (maybe singular) projection to \( \mathbb{P}^4 \), the rational normal scroll in \( \mathbb{P}^4 \) and the cones.

**Theorem 2.6** (Projective Duality \([18] \)). For any irreducible variety \( X \subset \mathbb{P}^N, (X^*)^* = X \).

**Definition 2.7.** A surface \( X \subset \mathbb{P}^N \) is **ruled by lines** if its Fano scheme is positive dimensional.

One consequence of Theorem 2.6 is that a surface whose dual variety has codimension 2 in \( \mathbb{P}^{N^*} \) is ruled by lines. The next lemma can be proved with standard techniques, and so we omit its proof.

**Lemma 2.8.** Maintain the context of Setting 2.2. If \( Y \) is ruled by lines, then \( X \) is a \( \mathbb{P}^1 \)-bundle over a smooth curve, and \( \varphi \) maps the rulings to lines.

**Theorem 2.9** (A Consequence of Fulton-Hansen Connectedness \([9, \text{Theorem 3.4.1}] \)). Let \( X \) be a complete irreducible variety of dimension \( n \), and let \( f : X \to \mathbb{P}^r \) be an unramified morphism. If \( 2n > r \), then \( f \) is a closed embedding.
3 Proof of Theorem 1.1

In this section, we maintain the conventions established in Section 2. As we proceed to the proof of Theorem 1.1, we shall require a series of lemmas, which we establish presently. Our first pair of lemmas serve as observations about the inner projection construction, where we address the corresponding properties of uncrumpled maps. Note that Lemma 3.1 is stated without proof.

Lemma 3.1. In the context of Setting 2.2 the image surface $\tilde{\pi}_y(\tilde{Y}) \subset \mathbb{P}^{N-1}$ is again non-degenerate.

Lemma 3.2. In the context of Setting 2.2 the map $\tilde{\pi}_y : \tilde{Y} \to \mathbb{P}^{N-1}$ is birational onto its image $\tilde{\pi}_y(\tilde{Y})$.

Proof. Since $\tilde{Y}$ is an irreducible projective variety and $\tilde{\pi}_y$ is regular, $\tilde{\pi}_y(\tilde{Y})$ is a closed subset of $\mathbb{P}^{N-1}$. If $\dim \tilde{\pi}_y(\tilde{Y}) = 1$, then $Y$ is a cone with vertex $y$. Since this is true for a general $y \in Y$, then joining any two points on $Y$ is a line, which implies that $\tilde{Y}$ is an impossible 2-plane. If $\dim \tilde{\pi}_y(\tilde{Y}) = 0$, then $Y$ would be entirely contained in a line, which is absurd. Thus $\tilde{\pi}_y(\tilde{Y})$ must be two-dimensional.

Suppose now that $\tilde{\pi}_y : \tilde{Y} \to \tilde{\pi}_y(\tilde{Y})$ has degree strictly greater than 1. By the construction of $\pi_y$, this means that if a line is generally chosen among those containing $y$ which are also secant to $Y$, it is in fact trisecant to $Y$. Moreover, since $y$ itself was chosen generally, it follows that a general secant line to $Y$ is trisecant.

Now, let $C$ be a general hyperplane section of $Y$. Since $N \geq 4$, $d = \deg Y \geq 3$, and $C$ is a non-degenerate irreducible curve in $\mathbb{P}^r$ with $r \geq 3$. By Theorem 2.4, a general hyperplane meets $C$ in $d$ points, any $r$ of which are linearly independent. This is a direct contradiction to the hypothesis that every secant line to $Y$ and hence $C$ is trisecant. Thus $\deg \tilde{\pi}_y = 1$. 

We wish to investigate the situation where the map $\tilde{\varphi}$ is ramified entirely along the exceptional curve $E$. To that end, we prove a significantly more general result:

Proposition 3.3. Let $X$ be any surface, and let $\varphi : X \to \mathbb{P}^N$ be birational onto its image $Y = \varphi(X)$. If the inner projection map $\tilde{\varphi} : X \to \mathbb{P}^{N-1}$ from a general point $y = \varphi(x) \in Y$ is ramified along the exceptional curve $E \subset X$, then $Y$ is a 2-plane.

Proof. The complete local ring $\mathcal{O}_{X,x}$ is isomorphic to $\mathbb{C}[s,t]$, the power series ring with analytic local coordinates $s$ and $t$, with maximal ideal $m_x = (s,t)$. In terms of $s$ and $t$, a power series $\varphi$ has an expression

$$\varphi = [S_0 : S_1 : \cdots : S_N],$$

where the $S_j$ are formal power series in $s$ and $t$. Since $\varphi$ is well-defined at $x$, we can without loss of generality assume $S_0$ is a unit in $\mathbb{C}[s,t]$, and put $\varphi = [1 : f_1 : \cdots : f_n]$, where $f_j = S_j / S_0$. We can also assume without loss of generality that each $f_j$ vanishes at $x$. Since $\varphi$ is unramified at $x$, two of the $f_j$, say $f_1$ and $f_2$, have linearly independent linear terms. By choosing a different pair of local coordinates, we can assume that $f_1 = s$ and $f_2 = t$. After applying a suitable automorphism of $\mathbb{P}^N$, we can express $\varphi$ as

$$\varphi = [1 : s : t : g_3(s,t) : \cdots : g_N(s,t)],$$

where $g_j \in \mathfrak{m}_x^2$ for $j \geq 3$ (meaning that all terms of each $g_j$ have degree at least 2).

The inner projection map $\pi_y \circ \varphi$ from $x = \varphi^{-1}(y)$ has a local expression

$$[s : t : g_3(s,t) : \cdots : g_N(s,t)] \in \mathbb{P}^{N-1},$$

which is undefined precisely at $x$. To resolve this indeterminacy, we blow-up the surface $X$ at $x$ to obtain from the relation $sV - tU$ an affine chart of the blow-up $\tilde{X}$, given by $s = tu$, with $u := U/V$ and $V \neq 0$. Then the regular map $\tilde{\varphi}$, the resolution of $\pi_y \circ \varphi$, has a local description

$$\tilde{\varphi}(\tilde{X}) = [1 : u : \frac{g_3(tu,t)}{t} : \cdots : \frac{g_N(tu,t)}{t}].$$
Note that since $g_j \in m_2^2$, all terms of the power series $\frac{g_j(u,t)}{t}$ contain a factor of $t$. Consequently, on the affine chart of the target $\mathbb{P}^{N-1}$ where the first homogeneous coordinate is nonzero, we can express the resolution $\tilde{\phi}$ as mapping $\tilde{X}$ locally near $E$ to 

$$(u, t \cdot h_3(t,u), \ldots, t \cdot h_N(t,u)) \in \mathbb{A}^{N-1} \subset \mathbb{P}^{N-1},$$

where the terms of $h_j(t,u)$ that are constant with respect to $t$ have degree at most 2 in $u$. Since $\tilde{\phi}$ is expressed as a map between affine charts, its derivative $d\tilde{\phi}$ evaluated at $t=0$ is given by the matrix

$$d\tilde{\phi}|_{t=0} = \begin{vmatrix} 1 & t \cdot \frac{\partial h_3}{\partial u}(t,u) & h_3(t,u) + t \cdot \frac{\partial h_3}{\partial t}(t,u) \\ \vdots & \vdots & \vdots \\ t \cdot \frac{\partial h_N}{\partial u}(t,u) & h_N(t,u) + t \cdot \frac{\partial h_N}{\partial t}(t,u) \end{vmatrix}_{t=0} = \begin{bmatrix} 1 & 0 \\ 0 & h_3(0,u) \end{bmatrix}.$$

At $t=0$, the map $d\tilde{\phi}$ fails to be full rank precisely when every term in each $h_j(0,u)$ vanishes. Each term in the original power series $g_j(s,t)$ whose degree was at least 3 will have a corresponding term that vanishes in $h_j(0,u)$. Thus, the hypothesis that $\tilde{\phi}$ is ramified along $E$ imposes the condition that the $g_j(s,t)$ used to express $\phi$ were members of $m_3^2 \subset m_2^2$. But then the second fundamental form at $y = \phi(x)$ is identically 0, and since $x$ was chosen generally, $Y$ is a 2-plane.

In the next proposition, we address the third property of uncrumpled maps, concerning whether $\tilde{\phi}$ is finitely ramified.

**Proposition 3.4.** In the context of Setting 2.2, either $\tilde{\phi} : \tilde{X} \to \mathbb{P}^{N-1}$ is finitely ramified, or $Y$ is ruled by lines.

**Proof.** We begin by asserting that $\text{Ram}(\tilde{\phi})$ is properly contained inside $\tilde{X}$. Indeed, if $\dim \text{Ram}(\tilde{\phi}) = 2$, then $\text{Ram}(\tilde{\phi}) = \tilde{X}$, since the ramification scheme of a map is a closed subscheme of the domain. Then $Y$ is a cone over a curve with vertex $\phi(x)$. But since $x$ is chosen generally, $Y$ is a degenerate 2-plane.

Suppose instead that $\tilde{\phi}$ is ramified along a curve $R \subset \tilde{X}$, not containing $E$ by Proposition 1.3. Then $\beta_x(R) \subset X$ is a curve. For any $p \in X \setminus \text{Ram}(\phi)$, we define $\Lambda_p \subset \mathbb{P}^N$ to be the projective 2-plane

$$\Lambda_p := (d\phi(T_p X)) \subset \mathbb{P}^N,$$

the linear span of the Zariski tangent plane at $\phi(p)$. We also define the incidence variety $\Sigma$ as the closure of the set

$$\{(p,q) \mid p,q \notin \text{Ram}(\phi), \text{ and } \phi(p) \notin \Lambda_q \setminus \{\phi(q)\}\} \subset X \times X.$$ 

Let $\pi_1, \pi_2 : \Sigma \to X$ be the projection maps to the first and second factor respectively. A generic fiber of $\pi_1$ is one-dimensional; this is a restatement of the hypothesis that $\tilde{\phi}$ is ramified along the curve $R$. By dimension counting, it follows that $\dim \Sigma = 3$.

But now consider the map $\pi_2$. The fiber $\pi_2^{-1}(q)$ over a point $q \in X$ is the set of points in $X$ whose image is contained in a line tangent to $Y$ at $\phi(q)$. Equivalently,

$$\pi_2^{-1}(q) = \phi^{-1}(\Lambda_q \cap Y).$$

Again by dimension counting, we see that $\dim (\pi_2(\Sigma)) \geq 1$. If $\pi_2(\Sigma)$ is a curve $C \subset X$, then the fiber over a general point $c \in C$ will be two-dimensional. Since $\text{Ram}(\phi)$ is finite, $\Lambda_c$ is a well-defined 2-plane. But if $\pi_2^{-1}(c)$ is two-dimensional, then $Y \subset \Lambda_c$, so $Y$ is again degenerate.

Suppose now that $\pi_2$ is one-dimensional (and hence surjective). Then a general fiber of $\pi_2$ is one-dimensional, meaning that $\Lambda_q \cap Y$, denoted $C_q$, is a plane curve. Since $\pi_1$ is surjective, the fibers of
If $q \in X$, then $\mathcal{F}$ is two-dimensional. By Theorem 2.5, if $Y$ is the Veronese surface in $\mathbb{P}^3$, $Y$ is a cubic scroll or $Y$ is a cone. In the first case, the resolved inner projection is an embedding and is hence unramified, and in both other cases $Y$ is ruled by lines.

Suppose instead that $\dim \mathcal{F} = 1$. Then for a general point $q \in X$, there are infinitely many points in $X$ whose fiber under $\pi_2$ is $C_q$. Let $r \in X$ be general among such points. Then

$$C_q \subset \Lambda_q \cap \Lambda_r.$$ 

If the planes $\Lambda_q$ and $\Lambda_r$ are distinct, then $C_q$ is their line of intersection, implying $Y$ is ruled by lines. Instead, let $\pi : \mathbb{P}^N \rightarrow \mathbb{P}^3$ be a general linear projection, let $Z = \pi(Y)$, and consider the dual variety $Z^* \subset \mathbb{P}^3$. If $\Lambda_q = \Lambda_r$, then there are infinitely many points on $Z$ which share a projective tangent plane, so $Z^*$ must be at most a curve. By Theorem 2.5, $Z$ is ruled by lines, and since $Z$ is a projection of $Y$, $Y$ is also ruled by lines.

Taken together, the assumption that $Y$ is not ruled by lines, along with Lemmas 3.1 and 3.2 and Proposition 3.4 imply the following key fact which we use in the proof of Theorem 1.1.

**Theorem 3.5.** If $\varphi : X \rightarrow \mathbb{P}^N$ with $N \geq 4$ is an uncrumpled map on a smooth surface $X$ with $\varphi(X)$ not ruled by lines, then the general inner projection $\overline{\varphi} : \overline{X} \rightarrow \mathbb{P}^{N-1}$ is uncrumpled.

**Lemma 3.6.** Maintain the context of Section 2.2. If $Y$ is not ruled by lines, then

$$\mathfrak{P}(\overline{\varphi}) = \mathfrak{P}(\varphi) - 4.$$

**Proof.** Since $Y$ is not ruled, $\overline{\varphi}$ is uncrumpled by Theorem 3.5. To keep the notation manageable, we write $E = [E]$, $\beta = \beta_x$, and we omit the degree map on zero cycles. Recall the following familiar properties from intersection theory on the blow-up:

- $(\beta^* \zeta)^2 = \zeta^2$
- $(\beta^* \zeta) \cdot E = 0$
- $\zeta = \beta^* \zeta - E$
- $(\beta^* K_X)^2 = K_X^2$
- $(\beta^* K_X) \cdot E = 0$
- $K_X = \beta^* K_X + E$
- $(\beta^* \zeta) \cdot (\beta^* K_X) = \zeta \cdot K_X$
- $E^2 = -1$
- $c_2(T_X) = c_2(T_X) + 1$.

Applying Equation (1) to $\overline{\varphi}$, we execute a straightforward computation:

$$\mathfrak{P}(\overline{\varphi}) = 6\zeta^2 + 4\zeta \cdot K_X + K_X^2 - c_2(T_X)$$

$$= 6(\beta^* \zeta - E)^2 + 4(\beta^* \zeta - E) \cdot (\beta^* K_X + E) + (\beta^* K_X + E)^2 - (c_2(T_X) + 1)$$

$$= 6 \left( (\beta^* \zeta)^2 - 2\beta^* \zeta \cdot E + E^2 \right) + 4 \left( \beta^* \zeta \cdot \beta^* K_X + \beta^* \zeta \cdot E - \beta^* K_X \cdot E - E^2 \right)$$

$$+ \left( (\beta^* K_X)^2 + 2\beta^* K_X \cdot E + E^2 \right) - c_2(T_X) - 1$$

$$= 6 \left( \zeta^2 - 2 \cdot 0 - 1 \right) + 4 \left( \zeta \cdot K_X + 0 - 0 + 1 \right) + \left( K_X^2 + 2 \cdot 0 - 1 \right) - c_2(T_X) - 1$$

$$= \left( 6\zeta^2 + 4\zeta \cdot K_X + K_X^2 - c_2(T_X) \right) - 4$$

$$= \mathfrak{P}(\varphi) - 4.$$ 

\[\square\]
Remark 3.7. Lemma [3.6] generalizes [16, Ex. 4] from the setting of a generic surface in $\mathbb{P}^r$ with $r > 4$ to any surface in $\mathbb{P}^N$ with $N \geq 4$ which arose as the image of an uncrumpled map. Moreover, the technique we use in this proof is an intersection theoretic computation, rather than the purely geometric argument used in the literature.

Lemma 3.8. Maintain the context of Setting [2.2] and suppose that $Y$ is ruled by lines. Then

$$\Psi(\varphi) \geq 2N - 6,$$

with equality holding precisely when $Y$ is a rational normal scroll.

Proof. By Lemma [2.8], $X$ is a projective bundle over a smooth base curve $B$. Let $\rho : X \to B$ be the bundle map, and let $\eta$ represent the class of a section of $\mathcal{O}_X(1)$. The Picard group $\text{Pic}(X)$ is isomorphic to $\rho^* \text{Pic}(B) \oplus \mathbb{Z} \cdot \langle \eta \rangle$. The Whitney sum formula applied to the relative Euler exact sequence on $X$ yields the canonical class $K_X = -2\eta - \rho^* K_B$. But the hyperplane class $\xi$ pulls back to a class in $\text{Pic}(X)$ which is of the form $\eta + aF$, for some $a \in \mathbb{Z}$ and $F \in \rho^* \text{Pic}(B)$. It follows that $2\xi + K_X \in \rho^* \text{Pic}(B) \oplus \{0\}$, and so $(2\xi + K_X)^2 = 0$.

Moreover, $\xi^2 = \text{deg} Y$, and in characteristic zero we have

$$c_2(T_X) = \chi(X) = \chi(\mathbb{P}^1) \cdot \chi(B) = 2(2 - 2g),$$

where $\chi$ is the topological Euler characteristic and $g$ is the genus of $B$. Applying Equation (1),

$$\Psi(\varphi) = \text{deg} \left( 6\xi^2 + 4\xi K_X + K_X^2 - c_2(T_X) \right) = \text{deg} \left( 2\xi + K_X \right)^2 + 2\xi^2 - c_2(T_X).$$

Substituting $(2\xi + K_X)^2 = 0$, $\xi^2 = \text{deg} Y$, and $c_2(T_X) = 4 - 4g$ into the expression above yields

$$\Psi(\varphi) = 2\text{deg} Y + 4g - 4. \quad (3)$$

Since $Y$ is non-degenerate in $\mathbb{P}^N$, $\text{deg} Y \geq N - 1$, so $\Psi(\varphi) \geq 2N + 4g - 6$. If $Y$ simultaneously minimizes its degree and the (non-negative) genus of $B$, $Y$ is a variety of minimal degree. Since $Y$ is ruled by lines, it must be a rational normal scroll of degree $N - 1$. In this case, since $Y$ is smooth, $X \cong Y$ and $\varphi$ is an embedding. \hfill \Box

Remark 3.9. We now prove Theorem [1.1]. The inequality in this theorem holds trivially for $N = 3$ as well as for $N \geq 4$, so we use $N = 3$ as the base case for induction. The last part of the theorem (classifying surfaces that meet the lower bound) fails in $\mathbb{P}^3$ however, since it would claim that the only smooth surface in $\mathbb{P}^3$ is the quadric surface, which is absurd.

Corollary 3.10. In the context of Setting [2.2],

$$\Psi(\varphi) \geq 2N - 6,$$

with equality holding if and only if $Y$ is a rational normal scroll.

Proof. We will proceed via induction on $N$, beginning with the observation that a finitely ramified map $\varphi : X \to \mathbb{P}^3$ has $\Psi(\varphi) \geq 0$. By Lemma [3.8] if $Y$ is ruled, then $\Psi(\varphi) \geq 2N - 6$. Assume from now on that $Y$ is not ruled, and suppose for induction that $\Psi(\varphi) \geq 2N - 8$. We know from Theorem [3.5] that $\varphi$ is uncrumpled, and by Lemma [3.6] since $\Psi(\varphi) = \Psi(\varphi) - 4$,

$$\Psi(\varphi) = \Psi(\varphi) + 4 \geq 2N - 8 + 4 > 2N - 6.$$

The strict inequality forces the classification result to follow from Lemma [3.8]. \hfill \Box
4 Generalization

In this section, we wish to strengthen the classification result of Theorem 1.1. The length of the pinch scheme is always even (this is a consequence of Nöther’s theorem and Equation (1)). A general projection of the Veronese surface to \( \mathbb{P}^3 \) admits 6 pinch points; this is the smallest non-minimal number for a surface in \( \mathbb{P}^3 \). It is natural to wonder, are there other surfaces which are close to minimal with respect to the length of their pinch point scheme?

Theorem 4.2 below gives a characterization of surfaces \( Y = \varphi(X) \), where \( \varphi : X \to \mathbb{P}^N \) is an uncrumpled map on a smooth surface \( X \) and where \( N \) is large enough, such that the difference between \( \Psi(\varphi) \) and \( 2N - 6 \) is fixed. Before we state and prove that result, Lemma 4.1 characterizes surfaces whose inner projection is ruled.

**Lemma 4.1.** In the context of Setting 2.2 if \( \varphi(\tilde{X}) \) is ruled by lines, then either \( Y \) is ruled by lines or \( Y \) is the Veronese surface in \( \mathbb{P}^3 \).

**Proof.** Since \( \varphi \) is uncrumpled by Theorem 3.5 and since \( \varphi(\tilde{X}) \) is ruled, \( \tilde{X} \) is a \( \mathbb{P}^1 \)-bundle over a smooth curve \( B \) with bundle map \( \rho : \tilde{X} \to B \). For a general point \( b \in B \), let \( R = \rho^{-1}(b) \subset \tilde{X} \) be a general ruling, so that \( \zeta \cdot R = 1 \). By the push-pull formula,

\[
\deg(\beta_X(R) \cdot \zeta) = \deg(\beta_X(\zeta) \cdot R) = E \cdot R + \zeta \cdot R = E \cdot R + 1. \tag{4}
\]

Suppose \( x \notin \beta_X(R) \), meaning that \( E \cdot R = 0 \). From Equation (4), it follows that \( (\varphi \circ \beta_X)(R) \) is a line. Since a general point of \( \tilde{X} \) is contained in a general ruling line, we conclude that \( X \) (and hence \( Y \)) is ruled by lines.

Suppose instead that \( x \in \beta_X(R) \), so that \( E \cdot R \geq 1 \). Then in fact, \( E \cdot R = 1 \), since otherwise there is a subscheme of \( E \) of length at least 2 that embeds under \( \varphi \), which implies that a general ruling line of \( \varphi(\tilde{X}) \) is contained in \( E \), a contradiction. Since \( R \) meets \( E \) transversely, Equation (4) implies that \( \beta_X(R) \) is a degree 2 smooth curve containing \( x \). Then \( (\varphi \circ \beta_X)(R) \) is a smooth conic containing \( y \) since \( \varphi \) is birational onto its image. Moreover, because a general pair of ruling lines of \( \tilde{X} \) is disjoint, no pair of the corresponding conics intersect at a point in \( Y \setminus \{y\} \). By varying \( x \in X \), we get a two-dimensional family of plane curves in \( Y \), so by Theorem 2.5 \( Y \) is the Veronese surface in \( \mathbb{P}^5 \).

**Theorem 4.2.** Maintain the context of Setting 2.2. For any positive integer \( i \), if \( N \geq 3 + i \geq 4 \), then \( \Psi(\varphi) = 2N - 6 + 2i \) if and only if

1. \( Y \) is ruled by lines, or
2. \( N = 5 \), \( i = 1 \), and \( Y \) is the Veronese surface, or
3. \( 4 \leq N \leq 9 \), \( i = N - 3 \), and \( Y \) is a del Pezzo surface of degree \( N \).

**Proof.** Suppose \( \Psi(\varphi) = 2N - 6 + 2i \). By Theorem 3.5 \( \varphi \) is again uncrumpled, and by Lemma 3.6

\[
\Psi(\tilde{\varphi}) = \Psi(\varphi) - 4 = 2N - 10 + 2i = 2(N - 1) - 6 + 2(i - 1). \tag{5}
\]

We now have two cases.

**Case I.** Suppose \( N \geq 4 + i \), with \( i \geq 0 \). If \( i = 0 \), Theorem 1.1 implies that \( Y \) is a rational normal scroll and is hence ruled by lines. Suppose for induction that for if \( \Psi(\varphi) = 2(N - 1) - 6 + 2(i - 1) \) and \( i \geq 1 \), then either \( Y \) is ruled or \( Y \) is the Veronese surface in \( \mathbb{P}^5 \). The inductive step follows immediately from Equation (5), with \( Y \) the Veronese surface precisely when \( N = 5 \) and \( i = 1 \).

**Case II.** Suppose \( N = 3 + i \), with \( i \geq 1 \). If \( i = 1 \), then \( N = 4 \), and therefore \( \Psi(\varphi) = 4 \). By Equation (5), \( \Psi(\tilde{\varphi}) = 0 \), so \( \varphi \) is an unramified map from a smooth surface to \( \mathbb{P}^3 \). By Theorem 2.9 \( \tilde{X} \) embeds...
into \( \mathbb{P}^3 \). Moreover, \( \varphi(\tilde{X}) \) contains a line whose self intersection is \(-1\), and so by adjunction, it must be the cubic del Pezzo surface.

Suppose for induction that if \( \mathcal{P}(\mathfrak{g}) = 2(N - 1) - 6 + 2(i - 1) \), then \( Y \) is a del Pezzo surface of degree \( N - 1 \). By Equation 4.3, \( \varphi(X) \) is precisely such a del Pezzo surface. By a characterization of del Pezzo surfaces, \( \varphi(X) \) is a smooth linearly normal surface in \( \mathbb{P}^{N - 1} \) of degree \( N - 1 \). But generic inner projection preserves linear normality and decreases degree by 1, so \( \deg Y = N \), and \( Y \) is linearly normal in \( \mathbb{P}^N \). Thus, \( Y \) is a del Pezzo surface of degree \( N \), with \( 4 \leq N \leq 9 \).

\[ \square \]

**Remark 4.3.** By tracing through the computation in Lemma 3.8, we can gain a little more clarity in the cases where \( Y \) is ruled. Indeed, one can easily deduce the bound \( g \leq \frac{1}{2} \) for the genus \( g \) of the base curve over which \( X \) is a projective bundle, limiting the complexity of the ruled surface \( Y \).

5 Future Work

Returning to the context of the larger problem, we have seen that for a smooth surface \( X \), its Gauss class can be written as
\[
[\gamma(X)] = \gamma_{1,1} \cdot \sigma^{*}_{1,1} + \gamma_{2} \cdot \sigma^{*}_{2},
\]
where \( A_2(G(2, N)) = \mathbb{Z}[\sigma^{*}_{1,1}, \sigma^{*}_{2}] \), the integer \( \gamma_{1,1} = \mathcal{P}(\pi) \) for a projection map \( \pi \) from a general codimension 4 linear space, and the integer \( \gamma_{2} \) is the degree of \( X^* \). Theorem 1.1 completes the case where \( X \) is a surface.

For concreteness, we can consider the next case where \( X \subset \mathbb{P}^N \) is a smooth irreducible projective variety of dimension three, with the intent to extend into higher dimensions. Then the setting of the problem changes to the Chow ring \( A_3(G(3, N)) = \mathbb{Z}[\sigma^{*}_{1,1,1}, \sigma^{*}_{2,1}, \sigma^{*}_{3}] \), and in this ring we have
\[
[\gamma(X)] = \gamma_{1,1,1} \cdot \sigma^{*}_{1,1,1} + \gamma_{2,1} \cdot \sigma^{*}_{2,1} + \gamma_{3} \cdot \sigma^{*}_{3}.
\]
This case is still largely unexplored. We know that \( \gamma_{3} \) still represents the degree of \( X^* \), and in general, \( \gamma_{n} \) always represents the degree of \( X^* \) for \( X^n \subset \mathbb{P}^N \) for \( 0 < n < N \). Similarly, \( \gamma_{1,1,1} \) represents the (finite) number of pinch points in the image of \( X \) under general linear projection to \( \mathbb{P}^3 \), and again the pattern holds for \( \gamma_{1^n} \) representing the pinch point number for a general linear projection. On the other hand, the coefficients in between these extremes, such as \( \gamma_{2,1} \), have a more complicated geometric meaning.

Extending the problem upwards into higher dimensions, we see that an entire family of classification problems emerges: For each of the generators \( \sigma^{*}_{a} \) of \( A_n(G(n, N)) \), where \( a \) is an integer partition of \( n \), we may ask the following questions:

I. What is the minimum value that \( \gamma_{a} \) can take?

II. Which varieties \( X \) realize this minimum?

A very different enumerative problem arises, in the case where \( X \) is a surface, from the classification given by Theorem 1.1. First, observe that the Hilbert scheme of finite sets of \( 2N - 6 \) points on \( X \), denoted \( \text{Hilb}^{2N-6}(X) \), has dimension \( 4N - 12 \). This is precisely the same as \( \dim G(N - 4, N) \). When \( X \) is a rational normal scroll \( S(a, b) \subset \mathbb{P}^N \) with \( N = a + b + 1 \), it admits \( 2N - 6 \) pinch points under general linear projection from a codimension 4 linear space. Then we get a map
\[
\Phi : G(N - 4, N) \longrightarrow \text{Hilb}^{2N-6}(X),
\]
given by $\Phi(\Lambda) = \text{Ram}(\pi_\Lambda)$. A natural question we might ask is: What is the degree of $\Phi$? In particular, how does the answer depend on the splitting type ($a$ and $b$) of $X$?

For $N = 4$, $\deg \Phi = 1$, two generic points on the cubic scroll $S(1,2)$ uniquely determine a point of projection which realizes them as pinch points. For $N = 5$ however, we see a disparity between the two quartic scrolls $S(2,2)$ and $S(1,3)$: The directrix of $S(1,3)$ is a line in $\mathbb{P}^5$ which meets every tangent plane to $S(1,3)$, but which is nevertheless an invalid source of projection. Investigating higher dimensional spaces reveals even more invalid linear spaces which satisfy the appropriate enumerative properties. The invalid sources of projection cause an excess intersection problem when trying to execute the enumerative computation. This excess gets worse as $N$ or the eccentricity of the scroll increases.

**Remark 5.1.** In the case where $X$ is a rational normal curve and $\Phi : G(N - 2, N) \to \text{Hilb}^{2N-2}$, it is well known that $\deg \Phi$ is given by the $N$-th Catalan number,

$$\deg \Phi = \frac{1}{N} \binom{2n-2}{n-1}.$$

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**References**

[1] Enrico Arbarello, Maurizio Cornalba, and Phillip A Griffiths. *Geometry of algebraic curves: volume II with a contribution by Joseph Daniel Harris*. Springer, 2011.

[2] Ciro Ciliberto and Flaminio Flamini. On the branch curve of a general projection of a surface to a plane. *Transactions of the American Mathematical Society*, 363(7):3457–3471, 2011.

[3] David Eisenbud and Joe Harris. On varieties of minimal degree. In *Proc. Sympos. Pure Math*, volume 46, pages 3–13, 1987.

[4] David Eisenbud and Joe Harris. *3264 and all that: A second course in algebraic geometry*. Cambridge University Press, 2016.

[5] Dionisio Gallarati. Una proprietà caratteristica delle rigate algebriche. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)*, 21:55–56, 1956.

[6] Dionisio Gallarati. Ancora sulla differenza tra la classe e l’ordine di una superficie algebrica. *Ricerche Mat.*, 6:111–124, 1957.

[7] Joe Harris. Galois groups of enumerative problems. *Duke Mathematical Journal*, 46(4):685–724, 1979.

[8] Antonio Lanteri. On the class of a projective algebraic surface. *Archiv der Mathematik*, 45(1):79–85, 1985.
[9] Robert K Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, volume 48. Springer, 2017.

[10] Ermanno Marchionna. Sopra una disuguaglianza fra i caratteri proiettivi di una superficie algebraica. *Bollettino dell’Unione Matematica Italiana*, 10:478–480, 1955.

[11] Emilia Mezzetti and Dario Portelli. A tour through some classical theorems on algebraic surfaces. *An. Științ. Univ. Ovidius Constanța Ser. Mat.*, 5(2):51–78, 1997.

[12] Ragni Piene. Some formulas for a surface in $\mathbb{P}^3$. In *Algebraic geometry*, pages 196–235. Springer, 1978.

[13] Ragni Piene. Singularities of some projective rational surfaces. In *Computational methods for algebraic spline surfaces*, pages 171–182. Springer, 2005.

[14] Joel Roberts. Generic projections of algebraic varieties. *American Journal of Mathematics*, 93(1):191–214, 1971.

[15] Corrado Segre. Le superficie degli iperspazi con una doppia infinità di curve piane o spaziali. *Atti Acc. Torino*, 56:143–517, 1921.

[16] J. G. Semple and L. Roth. *Introduction to algebraic geometry*, chapter IX, page 198. Clarendon Press Oxford, 1949.

[17] José Carlos Sierra and Andrea Luigi Tironi. Some remarks on surfaces in $\mathbb{P}^4$ containing a family of plane curves. *Journal of Pure and Applied Algebra*, 209(2):361–369, 2007.

[18] Evgueni A Tevelev. *Projective duality and homogeneous spaces*, volume 133. Springer Science & Business Media, 2006.

[19] F. L. Zak. *Tangents and secants of algebraic varieties*, volume 127 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by the author.