Every Borel automorphism without finite invariant measures admits a two-set generator

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Abstract

We show that if an automorphism of a standard Borel space does not admit finite invariant measures, then it has a two-set generator. This implies that if the entropies of invariant probability measures of a Borel system are all less than \( \log k \), then the system admits a \( k \)-set generator, and that a wide class of hyperbolic-like systems are classified completely at the Borel level by entropy and periodic points counts.

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1 Introduction

1.1 Background and statement of results

Borel dynamics is the study of the action of an automorphism \( T \), or a group of automorphisms, on a standard Borel space \( (X, \mathcal{B}) \). These objects appear throughout dynamical systems theory, lying at the intersection of ergodic theory and topological dynamics, in the first of which the system is additionally endowed with an invariant measure, and in the second with topology which makes \( T \) continuous. But it is perhaps more accurate to say that Borel dynamics lies somewhere between the two: since measurable maps are far more abundant than continuous ones the category is “looser” than the topological one, but, in the absence of a reference measure, maps are defined everywhere, rather than almost-everywhere, and so morphisms preserve substantially more of the structure than in ergodic theory. The systematic development of the theory as a branch of dynamics began with the work of Shelah and Weiss, and over the
past few decades a close parallel has been established between Borel dynamics and the ergodic theory of conservative transformations [23, 26, 27, 19]. Another notable direction in the theory is the study of the orbit relation of Borel actions, see e.g. [12] (but this will not concern us here). Recently, and more directly related to our work, Borel dynamics has come up in connection with the classification of hyperbolic-like dynamics following Buzzi’s work on entropy conjugacy [6, 10].

In this paper we resolve a longstanding problem on the existence and size of generators for Borel automorphisms (i.e. actions of \( \mathbb{Z} \)). Let us begin by describing the situation in ergodic theory, which is classical and intimately related to entropy theory. Given a probability measure \( \mu \) on \((X, \mathcal{B})\), a measurable partition \( \alpha = \{ A_i \} \) is called a generator (for \( \mu \)) if the family of iterates \( \{ T^j \alpha \}_{j \in \mathbb{Z}} \) generates the \( \sigma \)-algebra \( \mathcal{B} \) up to \( \mu \)-null sets. The size of the smallest generator is a reflection of the complexity of the system, and when \( \mu \) is \( T \)-invariant it is a classical theorem of Krieger that a \( k \)-set generator for \( \mu \) exists if (and, essentially, only if) \( h_\mu(T) < \log k \), where \( h_\mu(T) \) denotes the Kolmogorov-Sinai entropy.\(^1\) When \( h_\mu(T) = \infty \) no finite generator can exist, but, by a theorem of Rohlin, a countable one does.

When one moves away from invariant probability measures to more general ones the picture changes drastically. A measure \( \mu \) is conservative for \( T \) if \( T \) preserves the class of \( \mu \)-null sets and \( \mu(A) = 0 \) for every wandering set \( A \), where \( A \) is wandering if its iterates \( T^n A, n \in \mathbb{Z} \), are pairwise disjoint. Krengel [17] showed that every ergodic conservative measure \( \mu \) that is not equivalent to an invariant probability measure admits a two-set generator. This is another manifestation of the absence of a good entropy theory for conservative transformations.\(^2\)

We now return to the Borel setting. Here, a partition \( \alpha = \{ A_i \}_{i \in I} \) is called a (Borel) generator for \((X, \mathcal{B}, T)\) if the \( \sigma \)-algebra generated by \( \{ T^j \alpha \}_{j \in \mathbb{Z}} \) is equal to \( \mathcal{B} \). Such a partition clearly must be a generator for every conservative measure of \( T \), so the presence of invariant probability measures of high entropy poses an obstruction to the existence of finite Borel generators, but the theorems of Rohlin and Krengel make it plausible that countable Borel generators could exist. That they always do exist for free actions\(^3\) was established by Benjamin

\(^1\)In fact, \( h_\mu(T) \) can be expressed purely in terms of the size of generators: writing \( g_\mu(T) \) for the cardinality of the smallest \( \mu \)-generator of \( T \), we have

\[
h_\mu(T) = \lim_{n \to \infty} \frac{1}{n} \log g_\mu(T^n)
\]

\(^2\)Several notions of entropy have been suggested for conservative transformation, e.g. [10, 21, 13], but these generally lack many important properties present in the classical notion.

\(^3\)A free \( \mathbb{Z} \)-action is one without periodic points. In order for countable generators to exist, some assumption on periodic points is needed, since a countable generator cannot exist if there are more than countably many periodic points. If there are only countably many periodic points, then they pose no obstruction.
Weiss [27], who showed that every free Borel system \((X, T)\) admits a countable generator. Weiss proved his theorem modulo a the sigma-filter generated by wandering sets, but this qualification can be removed, see e.g. [23, Corollary 7.6]. We shall later use the existence of such a generator.

Weiss’s theorem left open the question of finite generators when there is no obstruction from the finite invariant measures. Specifically, Weiss asked in [27] whether, in the total absence of invariant probability measures, finite generators must exist (again, he allowed wandering sets to be neglected. The stronger version appears in [12, Problem 5.7 and 6.6(A)]. The question was partly answered recently by Tserunyan [24], who gave an affirmative answer when \(T\) is a continuous map of a locally compact separable metric space. Our main result of the present paper is an answer to the problem in general:

**Theorem 1.1.** Every Borel system without invariant probability measure\(^4\) admits a two-set generator.

More generally, given a non-trivial mixing shift of finite type \(Y \subseteq \Sigma^Z\), one can find a generator \(\alpha = \{A_i\}_{i \in \Sigma}\) such that the itineraries lie in \(Y\).

In [10, Theorem 1.5] we showed how to obtain a uniform Krieger generator theorem; more precisely, if \((X, T)\) is a free Borel system with \(h_\mu(T) < \log k\) for every \(T\)-invariant probability measure \(\mu\), then there is an invariant Borel subset \(X_0 \subseteq X\) supporting all finite \(T\)-invariant measures, and \((X_0, T|_{X_0})\) admits a \(k\)-set generator. Combining this with Theorem 1.1 to find a generator for \(X \setminus X_0\), and working a little to make the images disjoint, we get the following corollary (see Section 9):

**Corollary 1.2.** Suppose that \((X, T)\) is a free Borel system with \(h_\mu(T) < \log k\) for every \(T\)-invariant probability measure \(\mu\) (alternatively, for every such measure \(\mu\) with a single exception which is Bernoulli of entropy \(\log k\)). Then there exists a \(k\)-set Borel generator for \(T\).

We also note the following related dichotomy, which was conditionally derived in [24, Theorem 9.5] from Theorem 1.1. A Borel system admits a finite generator if and only if it admits no invariant probability measure of infinite entropy.

### 1.2 Application to hyperbolic-like dynamics

It is classical and well known that hyperbolic-like maps are “essentially” determined by their periodic points counts and entropy. The “top” of the system usually consists of a unique invariant probability measure of maximal entropy, which is ergodically isomorphic to a Bernoulli shift. Thus, by Ornstein theory

\(^4\)This assumption automatically ensures that \(T\) acts freely, since a finite orbit would carry an invariant probability measure.
[20], when the entropies of two such systems are equal, their entropy-maximizing measures are isomorphic. In many special cases, e.g. for mixing shifts of finite type, this isomorphism can be made continuous on a set of full measure, as in the finitary isomorphism theory of Keane and Smorodinsky [15], and even extended farther “down” to some of the “low-entropy” part of the phase, as is the almost-conjugacy theorem of Adler and Marcus [1].

More recently Buzzi introduced the notion of entropy conjugacy [6], whereby in the problem above one replaces continuity by measurability in the hope of extending the isomorphism results to a larger class of systems [4, 6]. One also hopes to extend the isomorphisms farther into the low-entropy part of the systems, ideally to all of the “free part” of the system, that is, to the complement of the periodic points. This possibility was raised in [10], where it was partly achieved for a large family of systems on sets supporting all non-atomic invariant probability measures (but not all conservative ones). See also [7]. Isomorphisms between the entire free parts of equal-entropy strongly positively recurrent Markov shifts were constructed recently by Boyle, Buzzi and Gómez [5], using the special presentations of such subshifts. Using the arguments from [10] together with Theorem 1.1 one can give a quite general result in this direction.

Corollary 1.3. Let \( h > 0 \). Then, up to Borel isomorphism, there is a unique homeomorphism \( T \) of a Polish space satisfying the following properties: (a) \( T \) acts freely, (b) every \( T \)-invariant probability measure has entropy \( \leq h \), and equality occurs for a unique measure which is Bernoulli, (c) \( T \) admits embedded mixing SFTs of topological entropy arbitrarily close to \( h \).

In particular, if two systems from the classes listed below have the same topological entropy, then they are isomorphic, as Borel systems, on the complements of their periodic points. The classes are: Mixing positively-recurrent countable-state shifts of finite type, mixing sofic shifts, Axiom A diffeomorphisms, intrinsically ergodic mixing shifts of quasi-finite type.

It remains an open problem whether, on the complement of the periodic points, the isomorphism can be made continuous in any non-trivial cases, e.g. between equal-entropy mixing shifts of finite type which are not topologically conjugate [10] Problem 1.9].

1.3 Remark about the role of wandering sets and conservative measures

Although it has no direct bearing on the proof, we digress to say a few words about the role of conservative measures and wandering sets. In ergodic theory one generally neglects nullsets. In Borel dynamics, the appropriate class

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5 The periodic points themselves can be dealt with separately to give an isomorphism if their numbers are compatible.
of dynamically negligible sets is the family $\mathcal{W} \subseteq \mathcal{B}$ of all countable unions of (measurable) wandering sets. It is easy to check that $\mathcal{W}$ is closed under taking measurable subsets and countable unions, i.e. it is a $\sigma$-ideal. Many results from ergodic theory, including Poincaré recurrence, Rohlin’s tower lemma, and hyperfiniteness of the orbit relation for an automorphism, can be proved in the Borel setting if we work modulo $\mathcal{W}$.

The $\sigma$-ideal $\mathcal{W}$ is closely related to conservative measures: by definition, every $A \in \mathcal{W}$ is a nullset for every $T$-conservative measure, and Shelah and Weiss [23, 26] proved the converse, showing that $\mathcal{W}$ consists of precisely those $A \in \mathcal{B}$ which are nullsets for every $T$-conservative measure. In particular, $T$ is dissipative (i.e. $X \in \mathcal{W}$) if and only if it admits no conservative measures. This implies that results which hold a.e. for conservative measures hold everywhere modulo $\mathcal{W}$, because the set of points where the property fails is null for every conservative measure, and hence the set of these points is in $\mathcal{W}$.

Now, Krengel’s generator theorem says that in a Borel system without invariant probability measures, we can find a two-set generator for every conservative measure. The discussion above hints that one should be able to find a finite Borel generator, at least modulo $\mathcal{W}$. Unfortunately, it is unclear how to glue these generators together. One might hope to partition the space into invariant Borel sets, each of which supports a unique conservative measure; then, at least, the partitions given by Krengel’s theorem would be of disjoint sets and we could take their union, leaving only the measurability question. Unfortunately, if the system admits conservative measures at all, then no such partition exists: see e.g. Weiss [26] (alternatively, this is a consequence of the Glimm-Effros theorem and standard results on topologizing Borel systems). This makes it highly unlikely that this “divide and conquer” strategy can work.

### 1.4 Structure of the proof of Theorem [1.1]

Our proof of Theorem [1.1] is made up of three separate generator theorems, each of which applies to points exhibiting a different form of “non-stationary” statistical behavior. By statistical behavior we mean the asymptotics of the number of visits to a set: For $A \in \mathcal{B}$ and $x \in X$ let

$$S_n(x, A) = \frac{1}{n} \sum_{i=0}^{n-1} 1_A(T^i x)$$

If the limit as $n \to \infty$ exists, we denote it by

$$s(x, A) = \lim_{n \to \infty} S_n(x, IA)$$

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6 This is in contrast to the theorem of Varadarajan [25], which gives a partition into invariant Borel sets each supporting a unique invariant probability measure.
We write \( s(x, A) \) and \( \bar{s}(x, A) \) for the upper and lower limits.

**Definition 1.4.** Let \( x \in X \) and \( A \in \mathcal{B} \), and let \( \alpha = \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{B} \) be a measurable partition of \( X \). We say that

- \( x \) is \( A \)-null if \( x \in \bigcup_{n=-\infty}^{\infty} T^n A \) and \( s(x, A) = 0 \).
- \( x \) is \( A \)-divergent if \( S_n(x, A) \) diverges.
- \( x \) is \( \alpha \)-deficient if \( s(x, A_i) \) exists and is positive for all \( i \) and \( \sum s(x, A_i) < 1 \).

The sets of points satisfying each of the above conditions are denoted \( \text{null}(A) \), \( \text{div}(A) \) and \( \text{def}(\alpha) \), respectively.

These behaviors are “non-stationary” in the following sense. If \( \mu \) is an ergodic probability measure for \( T \) and \( \mu(A) > 0 \), then by the ergodic theorem the frequencies \( s(x, A) \) exist \( \mu \)-a.s. and are equal to \( \mu(A) \), which is positive. Hence \( x \) is neither \( A \)-divergent nor \( A \)-deficient. Similarly, for any partition \( \alpha = \{A_i\} \), for \( \mu \text{-a.e.} \) \( x \) we have \( \sum s(x, A_i) = \sum \mu(A_i) = 1 \), so \( x \) is not \( \alpha \)-defective.

The sets \( \text{null}(A), \text{div}(A) \) and \( \text{def}(\alpha) \) are measurable and \( T \)-invariant, and the core of this paper is devoted to proving that the restriction of \( T \) to each of them has a finite generator. These constructions share some common infrastructure (see Section 4), but the underlying mechanism in each case is rather different. The construction for null points is quite simple, and related to the construction of generators for infinite invariant measures. We give the details in Section 5. The construction for divergent points is new, and of independent interest: it gives an effective and optimal (though in no sense efficient) source coding algorithm for sequences that do not have a limiting mean value. This may be seen as another manifestation of the necessity of stationary statistics for the existence of an entropy theory. The details appear in Section 6. The deficient case, given in Section 7, is the most involved of the three, though also in a sense the most classical. It partly relies on the other two cases, and it is the only one where entropy makes an appearance. In fact a crucial component will be a version of the Krieger generator theorem, given in Section 8 that uses only empirical statistics to find a finite partition generating the same \( \sigma \)-algebra as a given countable partition of finite empirical entropy.

Let us now explain how all this comes together to give Theorem 1.1. The starting point is Nadkarni’s beautiful characterization of Borel systems which do not admit finite invariant measures. Recall that a set \( D \in \mathcal{B} \) is called a sweeping out set if \( \bigcup_{i \in \mathbb{Z}} T^i D = X \).

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7We have chosen to work with forward averages because it has some mild simplifying effects, though also some odd side-effects. In some places we will need to consider two-sided averages as well and it would have been possible to use these exclusively.
Theorem 1.5 (Nadkarni, [19]). Let \((X, B, T)\) be a Borel system. Then \(T\) does not admit an invariant probability measure if and only if there exists a sweeping out set \(D \in B\), a measurable partition \(\{D_n\}_{n=1}^{\infty}\) of \(X\), and integers \(n_1, n_2, \ldots\), such that the sets \(T^{n_i}D_i\) are pairwise disjoint, and \(T^{n_i}D_i \subseteq X \setminus D\) for all \(i\).

Given sets \(D, D_1, D_2, \ldots \in B\) and integer \(n_1, n_2, \ldots \in \mathbb{Z}\) as in the theorem above, let \(\mathcal{A}\) denote the (countable) algebra generated by \(D, D_1, D_2, \ldots\). We claim that every \(x \in X\) is either null for some \(A \in \mathcal{A}\), divergent for some \(A \in \mathcal{A}\), or deficient for the partition \(\alpha = \{D_i\}_{i=1}^{\infty}\). Indeed suppose \(x\) is not null or divergent for any \(A \in \mathcal{A}\). By non-divergence, the frequencies \(s(x, A)\) exists for all \(A \in \mathcal{A}\), so the set-function \(\mu_x(A) = s(x, A)\) is a well-defined finitely additive measure on \(\mathcal{A}\) that is invariant in the sense that \(\mu_x(TA) = \mu_x(A)\). Also, \(x\) is not \(D\)-null, i.e. \(\mu_x(D) > 0\). Therefore the inclusion \(\bigcup_{i=1}^{\infty} T^{n_i}D_i \subseteq X \setminus D\) implies that for every \(N\),

\[
\sum_{i=1}^{N} \mu_x(D_i) = \mu_x\left(\bigcup_{i=1}^{N} T^{n_i}D_i\right) \leq 1 - \mu_x(D)
\]

Hence \(\sum_{i=1}^{\infty} \mu_x(D_i) \leq 1 - \mu_x(D) < 1\), showing that \(x\) is \(\alpha\)-deficient.

All this goes to show that if a Borel system \((X, B, T)\) does not admit invariant probability measures then we can cover the space by a set of the form \(\text{def}(\alpha)\) together with countably many sets of the form \(\text{div}(A)\) and \(\text{null}(A)\). By a standard disjointification argument the cover can be turned into a partition by sets of the same form, and we then can merge sets with common forms to obtain a partition \(X = \text{null}(A') \cup \text{div}(A'') \cup \text{def}(\alpha)\) (See Section 4.3). We will show that the restriction of \(T\) to each of these three invariant sets admits a \(k\)-set generator for some universal constant \(k\). Then, by taking the union of these generators, we obtain a \(3k\)-set generator for \(T\).

The final step of the proof is to reduce the size of the generator from \(3k\) to \(2\). This uses the observation that if \(T\) has no invariant probability measures then neither does the induced map \(T_C\) on any sweeping-out set \(C \in B\) with bounded return times. Applying the argument above gives a \(3k\)-set generator for \(T_C\). Then, by a version of the Abramov entropy formula for induced maps, and assuming (as one may) that the first return time to \(C\) takes values that are large enough relative to \(k\), one obtains a \(2\)-set generator for \(T\). A similar argument gives a generator whose itineraries lie in a given mixing shift of finite type. The details of this argument are given in Section 4.4.

1.5 Further remarks

It would be quite interesting if it were enough to consider the null or divergent cases alone. In other words, does a Borel system without invariant probability
measures always admit a set $A$ such that $X = \text{null}(A)$? Or a set $B$ such that $X = \text{div}(B)$? Besides simplifying the proof of Theorem 1.1 this would give new characterizations of such systems, and the existence of such a set $B$ would also give an elegant converse to the ergodic theorem. We do not know whether such sets exist, but we point out that for every non-singular measure $\mu$ in the system (which, by assumption, is not equivalent to to an invariant probability measure) there is a set $A$ such that $X = \text{null}(A)$ modulo $\mu$, and a set $B$ such that $X = \text{div}(B)$ modulo $\mu$, so by the Shelah-Weiss characterization of $W$ it is plausible that our question has a positive answer.

Finally, it is very natural to ask the question about generators in the context of more general group actions. The work of Tserunyan mentioned earlier [24] is restricted by topological assumptions, but it has the remarkable feature that it applies to actions of arbitrary countable groups. Our argument relies on statistical properties of orbits and entropy considerations, and we see no reason why in principle it should not extend to countable amenable groups, but anything beyond this will probably require substantial new ideas. We remark that Tserunyan’s proof works for actions of general countable groups, and shows that if a finite generator does not exist, then there is a finitely additive, finite invariant measure for the action [24 Theorem 4.1 and Corollary 4.4]; the topology is used to extend this to a countably additive measure. If these measures are not $\sigma$-additive, then, in a sense, they are deficient, and perhaps this could be ruled out using some coding procedure similar to ours to show that deficiency implies a finite generator. However, the coding would need to be done without access to the machinery of Følner sets, empirical frequencies etc., so in fact quite a different methods would be required.

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## 2 Notation and conventions

A standard Borel space is a measurable space arising from a complete separable metric space and its Borel $\sigma$-algebra. An automorphism of a measure space is a measurable injection with measurable inverse (for standard Borel spaces measurability of the inverse is automatic). A Borel system $(X, \mathcal{B}, T)$ consists of a standard Borel space $(X, \mathcal{B})$ and a Borel automorphism $T$ of it. Given a family of sets $\alpha \subseteq \mathcal{B}$ we write $\sigma(\alpha) \subseteq \mathcal{B}$ for the $\sigma$-algebra generated by $\alpha$, and $\sigma_T(\alpha) = \sigma(\bigcup_{n \in \mathbb{Z}} T^n \alpha)$ for the smallest $T$-invariant $\sigma$-algebra containing $\alpha$. Similarly for a measurable map $f$ defined on $X$ we write $\sigma(f)$ for the smallest $\sigma$-algebra with respect to which $f$ is measurable and $\sigma_T(f)$ for the smallest such
$T$-invariant $\sigma$-algebra. A factor map from a Borel system $(X, \mathcal{B}, T)$ to a Borel system $(Y, \mathcal{C}, S)$ is a map $\pi : X \to Y$ such that $\pi$ is equivariant: $S\pi = \pi T$. Note that the map need not be onto, and the image need not be measurable (which is why we emphasize factor maps rather than factors). Such a map gives rise to a $T$-invariant sub-$\sigma$-algebra by pulling back $\mathcal{C}$ through $\pi$.

For a finite or countable alphabet $\Sigma$ we write $\Sigma^n$ for the set of words of length $n$ over $\Sigma$, i.e. sequences $w = w_1 \ldots w_n$ with symbols from $\Sigma$. We write $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$. A word $a = \Sigma^n$ appears in $b \in \Sigma^*$ if there is an index $i$ such that $b_i b_{i+1} \ldots b_{i+n-1} = a$. We then say that $a$ appears in $b$ at $i$ or that there is an occurrence of $a$ in $b$ at $i$. We also say that $a$ is a subword of $b$.

For intervals we mean integer intervals, so $[u, v] = \{i \in \mathbb{Z} : u \leq i \leq v\}$ (and similarly for half-open intervals and intervals that are unbounded on one or two sides). Given $a \in \Sigma^*$ and an interval $[u, v]$ such that $a_i$ is defined for $i \in [u, v]$, the subword of $a$ on $[u, v]$ is $a|[u,v] = a_ua_{u+1} \ldots a_v$. We denote concatenation of words $a \in \Sigma^m$, $b \in \Sigma^n$ by $ab = a_1 \ldots a_mb_1 \ldots b_n$. We write $a^n$ for the $n$-fold self concatenation of a symbol or word $a$.

For a countable set $\Sigma$ we frequently work in the space $\Sigma^\mathbb{Z}$ of bi-infinite sequences over $\Sigma$ and less frequently in $\Sigma^\mathbb{N}$, the space of one-sided sequences. The notation and terminology used for finite sequences generalizes to infinite sequences where appropriate. By taking the discrete topology on $\Sigma$ and the product topology on the product spaces we find that $\Sigma^\mathbb{N}$ and $\Sigma^\mathbb{Z}$ are separable metrizable spaces, and compact when $\Sigma$ is finite. In particular they carry the Borel $\sigma$-algebra and together with it form standard Borel spaces. The shift maps $S : \Sigma^\mathbb{N} \to \Sigma^\mathbb{N}$ and $S : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z}$ is defined by

$$(Sx)_i = x_{i+1}$$

$S$ is onto and with respect to the product topology it is continuous, and hence measurable. It is a bijection of $\Sigma^\mathbb{Z}$. For simplicity we use the same letter $S$ to denote shifts on sequence spaces over different alphabets and different index sets ($\mathbb{N}$ or $\mathbb{Z}$).

We have already defined the frequency $s(x, A)$ of visits of the orbit of $x \in X$ to $A \subseteq X$, including upper and lower versions. We introduce similar notation in the symbolic setting and for subsets of $\mathbb{Z}$. For $x \in \Sigma^\mathbb{Z}$ and $a \in \Sigma^*$ let

$$S_N(x, a) = \frac{1}{N} \# \{0 \leq i < N : a \text{ appears in } x \text{ at } i\}$$

and define the upper and lower frequencies of $a$ in $x$ by

$$\overline{s}(x, a) = \limsup_{N \to \infty} S_N(x, a)$$

$$\underline{s}(x, a) = \liminf_{N \to \infty} S_N(x, a)$$
If the two agree their common value is denoted $s(x,a)$ and called the frequency of $a$ in $x$.

The upper and lower densities of a subset $I \subseteq \mathbb{Z}$ is defined in the same way: take
\[
S_N(I) = \frac{1}{N} |I \cap [0, N-1]|
\]
and
\[
\overline{s}(I) = \limsup_{N \to \infty} S_N(I) \\
\underline{s}(I) = \liminf_{N \to \infty} S_N(I)
\]
The common value, if it exists, is denotes $s(I)$ and called the density of $I$. Note that this is the same as the frequency of 1 in $1_I$.

We will also need to use uniform densities. The version we need is the two-sided one. For $I \subseteq \mathbb{Z}$, the upper and lower uniform densities of $I \subseteq \mathbb{Z}$ are
\[
\overline{s}^u(I) = \limsup_{N \to \infty} \left( \sup_{n \in \mathbb{Z}} \frac{1}{N} |I \cap [n, n+N-1]| \right) \\
\underline{s}^u(I) = \liminf_{N \to \infty} \left( \inf_{n \in \mathbb{Z}} \frac{1}{N} |I \cap [n, n+N-1]| \right)
\]
We write $s^u(I)$ for the common value if they coincide, and call it the uniform density of $I$. In a Borel system $(X,T)$ and $x \in X$, $A \subseteq X$, we write $s^u(x,A)$ for $s^u \left( \{i : T^i x \in A \} \right)$, and similarly $s^u^*(x,A)$.

Finally, we note that obvious fact that
\[
\underline{s}^u(I) \leq s(I) \leq \overline{s}(I) \leq \overline{s}^u(I)
\]
and that the set-functions $\overline{s}$ and $\overline{s}^u$ are sub-additive.

### 3 Preliminary constructions

In this section we establish some basic machinery for manipulating orbits. We first prove some technical results that reformulate our problem in symbolic terms, and establish a marker lemma. We then show how to manipulate subsets of an orbit in a stationary and measurable manner. One result will say that if $A$ is a subset of an orbit with density $\alpha$ and $\beta < \alpha$ then we can select a subset $B$ of $A$ whose density is approximately $\beta$. Another allows us to construct an injection between subsets $C,D$ of an orbit, assuming that the density of $C$ is less than that of $D$. These are rather elementary observations but will play an important role in our coding arguments, since they allow to “move data around” inside an orbit. We also prove some other auxiliary results of a technical nature.
3.1 Factor maps and generators

A factor map from a Borel system into $\Sigma^\mathbb{Z}$ for a finite set $\Sigma$ is called a symbolic factor map. Given a finite or countable partition $\alpha = \{A_i\}_{i \in \Sigma}$ of a Borel system $(X, B, T)$, write $\alpha(x) = i$ if $x \in A_i$, and define $\alpha_* : X \to \Sigma^\mathbb{Z}$ by $\alpha_*(x)_n = \alpha(T^n x)$. This is a measurable equivariant map, and defines a symbolic factor map if $\alpha$ is finite.

The problem of finding a finite generator is equivalent to finding an injective symbolic factor map. To see the equivalence, note that if $\alpha = \{A_1, \ldots, A_r\}$ is a finite generator then the itinerary map $\alpha_*$ is a symbolic factor map and injective. Conversely, if $\pi : X \to \Delta^\mathbb{Z}$ is an injective symbolic factor map, then the partition $\{[i]\}_{i \in \Delta}$ is a finite generator for $(\Delta^\mathbb{Z}, S)$, and equivariance of the factor map implies that $\alpha = \{\pi^{-1}[i]\}_{i \in \Delta}$ is a generator for $X$.

3.2 The space $2^\mathbb{Z}$

Let $2^\mathbb{Z}$ denote the set of all subsets of $\mathbb{Z}$. We identify each $I \subseteq \mathbb{Z}$ with its indicator sequence $1_I \in \{0, 1\}^\mathbb{Z}$, where

$$1_I(n) = \begin{cases} 1 & \text{if } n \in I \\ 0 & \text{otherwise} \end{cases}$$

In this way $2^\mathbb{Z}$ inherits both a structure and the shift map. We shall apply the shift directly to subsets of $\mathbb{Z}$ and note that it is given by

$$SI = I - 1 = \{i \in \mathbb{Z} : i + 1 \in I\}$$

Also, given a Borel system $(X, B, T)$, we can speak of measurable and equivariant $X \to 2^\mathbb{Z}$, specifically, $I : X \to 2^\mathbb{Z}$ is equivariant if $I(Tx) = SI(x)$.

3.3 Aperiodic sequences and a marker lemma

Let $\Sigma$ be a countable alphabet, and write

$$\Sigma_{AP}^\mathbb{Z} = \{x \in \Sigma^\mathbb{Z} : x \text{ is not periodic}\}$$

This is an invariant Borel set. In this section and those that follow we construct various factor maps whose domain involves $\Sigma_{AP}^\mathbb{Z}$. We note that, instead, one could take any aperiodic Borel system $(X, T)$. Indeed, by Weiss’s countable generator theorem [27] (strengthened so as not to exclude a $W$-set using [24] Corollary 7.6), one can embed $(X, T)$ in $(\Sigma_{AP}^\mathbb{Z}, S)$.

Lemma 3.1. For every $x \in \Sigma_{AP}^\mathbb{Z}$ and every $\varepsilon > 0$ there is a block $a \in \Sigma^*$ that occurs in $x$ and satisfies $s(x, a) < \varepsilon$. 

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Proof. For a finite or infinite word \( y \) let \( L_n(y) \) denote the set of words of length \( n \) appearing in \( y \) and \( N_n(y) = |L_n(y)| \) their number. It is well known that \( x \) is periodic if and only if \( \sup_n N_n(x) < \infty \), so by assumption there is an \( n \) such that \( N_n(x) > 1/\varepsilon \). If for this \( n \) we had \( \underline{x}(x, a) \geq \varepsilon \) for all \( a \in L_n(x) \) then we would arrive at a contradiction, since

\[
1 \geq \sum_{a \in L_n(x^+)} \underline{x}(x, a) \geq N_n(x^+) \cdot \varepsilon > 1
\]

Hence there is \( a \in L_n(x) \) such that \( \underline{x}(x, a) < \varepsilon \). \( \square \)

Note that a word \( a \in \Sigma^* \) as in the lemma can be chosen measurably from \( x \in \Sigma_{AP}^2 \) and in a manner that is constant over \( S \)-orbits, since one can simply choose the lexicographically least word satisfying the conclusion. Also note that the hypothesis of the lemma holds automatically if \( x \in \Sigma^2 \) contains infinitely many distinct symbols.

We say that \( I \subseteq \mathbb{Z} \) is \( N \)-separated if \( |j - i| \geq N \) for all distinct \( i, j \in I \), and that it is \( N \)-dense if every interval \( [i, i + N - 1] \) intersects \( I \) non-trivially. Equivalently, the gap between consecutive elements is no larger than \( N \). We say that \( z \in \{0, 1\}^\mathbb{Z} \) is \( N \)-separated or \( N \)-dense if \( z = 1_I \) for an \( N \)-separated or \( N \)-dense set \( I \), respectively. We say that \( z \) is an \( N \)-marker if it is \( N \)-separated and \( (N + 1) \)-dense. More concretely, this means that the distance between consecutive \( 1 \)'s is \( N \) or \( N + 1 \).

We require the following version of the Alpern-Rohlin lemma [2], which we state in symbolic terms.

**Lemma 3.2.** For every \( N \in \mathbb{N} \) there is a factor map \( \Sigma_{AP}^2 \to \{0, 1\}^\mathbb{Z} \) whose image is contained in the \( N \)-markers.

Proof. Fix \( N \) and \( x \in \Sigma_{AP}^2 \). Choose \( a \in \Sigma^* \) which occurs in \( x \) but \( \underline{s}(x, a) < 1/N^2 \). Let

\[
I = \{i \in \mathbb{Z} : a \text{ appears in } x \text{ at } i\}
\]

Then \( I \) is non-empty and \( \underline{s}(I) < 1/N^2 \). Therefore the set

\[
I' = \{i \in I : (i, i + N^2) \cap I = \emptyset\}
\]

is non-empty and \( N^2 \)-separated. If \( I' \) has a least element \( i_0 \) add to \( I' \) the numbers \( i_0 - kN^2 \) for \( k = 1, 2, 3, \ldots \), and if \( I' \) has a maximal element \( i_1 \) add to \( I' \) the numbers \( i_1 + kN^2 \), \( k = 1, 2, 3, \ldots \). The resulting set \( I'' \) is now unbounded above and below and still \( N^2 \)-separated. Finally, for each consecutive pair \( u < v \) in \( I'' \), let \( L = v - u \) so \( L \geq N^2 \). There is a (unique) representation \( L = mN + n(N + 1) \) with \( m, n \in \mathbb{N} \). Now add to \( I'' \) all the numbers of the form \( u + m'N + n'(N + 1) \) for \( 0 < m' \leq m \) and \( 0 < n' \leq n \). Doing this for every
consecutive pair $u, v \in I''$, we obtain a set $I'''$ which is measurably determined by $x$, is $N$-separated and $(N + 1)$-dense. Set $π(x) = 1_{I'''}. This is the desired map.

The proposition above might produce a periodic factor; the next one ensures that the image is aperiodic, i.e. it takes an aperiodic sequence on a countable alphabet, and “reduces” the number of symbols to two, preserving aperiodicity. This will be used when we construct symbolic factors to ensure that the factors are themselves aperiodic. The proposition may be viewed as a baby version of the generator theorem: it gives a symbolic factor map, which, while not injective, at least preserves the aperiodicity of points in the domain. It appears in a more general setting in [24, Theorem 8.7]

**Proposition 3.3.** For every $N \in \mathbb{N}$ there is a factor map $π : Σ_AP \to \{0, 1\}_AP$ whose image is contained in the aperiodic $N$-markers.

**Proof.** Fix $x \in Σ_AP$. We construct inductively a decreasing sequence of sets $I_n \subseteq \mathbb{Z}$ with $I_n$ periodic of period $p_n$, and $p_{n+1} \geq p_n! + p_n$. Start by applying the previous lemma to $x$ and $N_1 = N$ to obtain an $N_1$-marker and let $I_1 \subseteq \mathbb{Z}$ denote the sequence of indices where this marker is 1. Then $I_1$ is $N$-separated.

If it is aperiodic set $π(x) = 1_{I_1}$. Otherwise, denote its period by $p_1$ and note that $p_1 \geq N_1 = N$.

Assume that after $n$ steps we have constructed $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n$ and that $I_n$ is periodic with period $p_n$. Apply the previous lemma to $x$ and $N_{n+1} = p_n! + p_n$, to obtain an $N_{n+1}$-marker, and let $I_{n+1}'$ denote the positions of the 1s in it, so the gaps in $I_{n+1}'$ are of length at least $N_{n+1} \geq N$. Let $k = \min\{i \in \mathbb{N} : S^i I_{n+1}' \cap I_n \neq \emptyset\}$ and set

$$I_{n+1} = S^k I_{n+1}' \cap I_n$$

If $I_{n+1}$ is aperiodic, define $π(x) = 1_{I_{n+1}}$. Otherwise continue the induction.

Suppose we did not stop at a finite stage of the construction. First, we claim that $p_n \to \infty$. To see this, note the gaps in $I_{n+1}$ are of size at least $2p_n$, so, since it is periodic, its least period $p_{n+1}$ greater than $p_n$.

Observe that there is at most one $i \in \mathbb{Z}$ contained in infinitely many (equivalently all) of the $I_n$’s, because the gaps in $I_n$ tend to infinity. Define $π(x)$ by setting $π(x)_i = 1$ if $i$ is in infinitely many $I_n$ and for any other $i$ set

$$π(x)_i = \max\{n : i \in I_n\} \mod 2$$

$$= \#\{k : i \in I_k\} \mod 2$$

It is clear that $x \mapsto π(x)$ is measurable and equivariant.

We claim that $π(x)$ is aperiodic. Indeed, suppose it was periodic with least
period $q$. Choose $n$ such that $p_n > q$ and define $y \in \{0, 1\}^\mathbb{Z}$ by

$$y_i = \max\{m \leq n : i \in I_m\} \mod 2$$

Clearly $y$ is periodic with period at most $p' = \text{lcm}_{k \leq n} p_k \geq p_n$. Also, $\pi(x)$ and $y$ agree everywhere except, possibly, on $I_{n+1}$. But the gaps in $I_{n+1}$ are at least $p_n + p'$, and in these gaps $\pi(x)$ and $y$ agree, so there is a $j$ such that $\pi(x)$ and $y$ agree on $[j, j + p' + q]$. But then for $i \in [j, j + p - 1]$ we have $y_i = y_{i+q}$, and since $y$ is $p'$-periodic this means that $y$ is $q$-periodic, a contradiction.

We note that $I_n \setminus \bigcup_{k \geq n} I_k$ is infinite and unbounded above and below for each $n$, from which it follows easily that $\pi(x)$ contains infinitely many 1s in both directions.

The sequences $\pi(x)$ are $N$-separated, aperiodic and contains infinitely many 1 in each direction, but the gaps can still be large. To get $N$-markers, begin with $2N^2$ instead of $N$. Then replace each block $10^n1$ in $\pi(x)$ with a sequence of the form $1(0^{N-1}1)^{k_1}(0^N1)^{k_2}$, where $k_1, k_2 \geq 1$ and $k_2$ is chosen to be minimal. Since $m \geq 2N^2$ there exists such a choice of $k_1, k_2$. The original location of 1s is the location of the central 1s in the sequences $10^N10^{N-1}$, so the new sequence is aperiodic, and is clearly a measurable equivariant function of $x$, as desired.

3.4 Stationary selection

In this section we show that one can select a subset of given approximate density from a set of higher density in a shift-invariant manner. Denote

$$[0, 1]^2_\infty = \{(t_1, t_2) \in [0, 1]^2 : t_1 < t_2\}$$

**Lemma 3.4.** There is a measurable map $\Sigma_{AP}^2 \times 2^2 \times [0, 1]^2_\infty \to 2^2$ that assigns to each $y \in \Sigma_{AP}^2$, $I \subseteq \mathbb{Z}$ and $t_1 < t_2$ a subset $J \subseteq I$ in a manner that is equivariant in the sense that $(Sy, SI, t_1, t_2) \mapsto SJ$, and which satisfies $d(J) \geq t_1 d(I)$ and $d(I \setminus J) \geq (1 - t_2) d(I)$, and similarly for upper densities.

**Remark 3.5.** The parameter $y$ may seem superfluous, and it would certainly be less cumbersome if we could define the set $J$ using only $I$ and $t_1 < t_2$. But if $I$ is periodic then any equivariant choice of $J \subseteq I$ must be periodic then any set $J$ determined from it equivariantly must have the same period and so the density of these sets must be a multiple of $1/p$, where $p$ is the period of $I$. The role of the parameter $y$ is precisely to break any such periodicity. Also, note that $d(I \setminus J) \geq (1 - t_2) d(I)$ implies $d(J) \leq t_2 d(I)$, and similarly $d(I \setminus J) \leq (1 - t_1) d(I)$, or with upper and lower densities reversed. But we will not need these upper bounds.
Proof. We may assume that $s(I) > 0$, otherwise there is nothing to prove. Choose rational $t_1 < \beta_1 < \beta_2 < t_2$ and $N \in \mathbb{N}$ large enough that $\beta_1 s(I) + \frac{1}{N} < \beta_2 s(I)$. For each finite subset $\emptyset \neq U \subseteq \mathbb{Z}$ choose once and for all a subset $\hat{U} \subseteq U$ such that $|\hat{U}| = \lfloor \beta_2 |U| \rfloor$, so that

$$\beta_2 \frac{|U|}{N} - \frac{1}{N} \leq \frac{|\hat{U}|}{N} \leq \beta_2 \frac{|U|}{N}$$

Let $z = z(y) \in \{0,1\}^\mathbb{Z}$ be the $N$-marker derived from $y$ as in Lemma 3.3. Let $U = \{ \ldots < u_1 < u_0 < u_2 < \ldots \}$ denote the positions of 1’s in $z$, so $u_{n+1} - u_n \in \{N, N+1\}$, and let $U_n = [u_n, u_{n+1})$. For each $n$ let

$$I_n = I \cap U_n \quad J_n = \hat{I}_n$$

and set

$$J = \bigcup_{n \in \mathbb{Z}} J_n$$

Evidently $J \subseteq I$ and the definition is measurable and equivariant in the stated sense. It remains to estimate the density of $J$. Using the fact that the lengths of $U_n$ are uniformly bounded we see that the sequences

$$\frac{1}{n} |I \cap [0,n)| \quad \text{and} \quad \frac{1}{u_n} \sum_{i=1}^{n} |I_i|$$

have the same lim sup and lim inf as $n \to \infty$. Therefore

$$s(J) = \lim \inf_{n \to \infty} \frac{1}{u_n} \sum_{i=1}^{n} |J_i| \geq \lim \inf_{n \to \infty} \frac{1}{u_n} \sum_{i=1}^{n} (\beta_2 |I_i| - 1) = \beta_2 s(I) - \lim \sup_{n \to \infty} \frac{n}{u_n} \geq \beta_2 s(I) - \frac{1}{N} > \beta_1 s(I)$$

where we used the fact that $u_n \geq nN - O(1)$. The calculation for $I \setminus J$ is similar, using the fact that $(1 - \beta_2)|U| \leq |U \setminus \hat{U}| \leq (1 - \beta_1)|U|$.

We also will need a version for uniform frequencies:

**Lemma 3.6.** There is a measurable map $\Sigma_{AP}^Z \times 2^Z \times [0,1]_Z^2 \to 2^Z$ that assigns
to each \( y \in \Sigma^Z_{AP}, I \subseteq \mathbb{Z} \) and \( t_1 < t_2 \) a subset \( J \subseteq I \) so that the assignment is is equivariant in the sense that \((Sy, SI, t_1, t_2) \mapsto SJ\), and satisfies \( \mathfrak{s}^*(J) \geq t_1 \mathfrak{s}^*(I) \)
and \( \mathfrak{s}^*(I \setminus J) \geq (1 - t_2) \mathfrak{s}^*(I) \), and similarly for upper uniform densities.

The proof is almost exactly the previous one, using the fact that for large enough \( N \), for all \( n \) we have \( \frac{1}{N} < \frac{1}{N} |I_n| < \frac{1}{N} \mathfrak{s}^*(I) + \frac{1}{N} \), and similarly for lower uniform densities. We omit the details.

Finally, the following lemma encapsulates the recursive application of the lemmas above. We state the uniform case but the non-uniform one is identical with the requisite changes. Let \( Q \subseteq [0,1]^{\mathbb{N}} \) denote the set of sequences \((t_n)_{n=1}^{\infty}\) such that \( \sum t_n < 1 \).

**Lemma 3.7.** There is a measurable map \( \Sigma^Z_{AP} \times \mathbb{Z} \times Q \to (2^{\mathbb{Z}})^{\mathbb{N}} \) that assigns to every \( y \in \Sigma^Z_{AP}, I \subseteq \mathbb{Z} \), and \((t_n)_{n=1}^{\infty} \in Q \) a sequence of disjoint subsets \( J_1, J_2, \ldots \subseteq I \) satisfying \( \mathfrak{s}^*(J_n) \geq t_n \mathfrak{s}^*(I) \), and the assignment is equivariant in the sense that \((Sy, SI, t) \mapsto (SJ_n)_{n \in \mathbb{N}} \).

**Proof.** Fix \( y \in \Sigma^Z_{AP} \) and \( I \subseteq \mathbb{Z} \). First, suppose we are given a sequence \( 0 < r_n^- < r_n^+ < 1 \). Choose intervals \( J_n \) recursively applying the previous lemma at stage \( n \) to \((y, I \setminus \bigcup_{i<n} J_i, r_n^-, r_n^+)\). Writing \( I_n = I \setminus \bigcup_{i<n} J_i \) for the interval from which \( J_n \) was chosen, we have the relations

\[
\mathfrak{s}^*(J_n) \geq r_n^- \cdot \mathfrak{s}^*(I_n)
\]
\[
\mathfrak{s}^*(I_n) \geq (1 - r_{n-1}^+) \cdot \mathfrak{s}^*(I_{n-1})
\]

Therefore

\[
\mathfrak{s}^*(I_n) \geq \prod_{i<n} (1 - r_i^+) \cdot \mathfrak{s}^*(I)
\]
\[
\mathfrak{s}^*(J_n) \geq r_n^- \cdot \prod_{i<n} (1 - r_i^+) \mathfrak{s}^*(I)
\]

Now let \( t_n > 0 \) be given satisfying \( \sum t_n < 1 \). We claim that we can choose \( 0 < r_n^- < r_n^+ < 1 \) to satisfy

\[
\frac{\prod_{i<n} (1 - r_i^+)}{\prod_{i<n} (1 - r_i^+)} > t_n
\]
\[
\sum_{i>n} t_i > \sum_{i>n} t_i
\]

This is done by induction: For \( n = 1 \), the requirements simplify to \( t_1 < r_1^- < r_1^+ < 1 - \sum_{i>1} t_i \), and the existence of such \( r_1^+ \) follows from the inequality \( t_1 < 1 - \sum_{i>1} t_i \), which is our hypothesis. Next, assuming that the inequalities above hold for \( n - 1 \), write \( a = \prod_{i<n} (1 - r_i^+) \), so by assumption \( a > \sum_{i>n-1} t_i \).
We are looking for $r_n^\pm$ satisfying $\frac{1}{n} t_n < r_n^- < r_n^+ < 1 - \frac{1}{n} \sum_{i > n} t_i$, and they exist provided that $\frac{1}{n} t_n < 1 - \frac{1}{n} \sum_{i > n} t_i$, which, after rearranging, is just the inequality $\sum_{i > n} t_i < a$, which we know to hold.

In conclusion, we have shown how to find $r_n^\pm$ as above, and by the discussion at the start of the proof we obtain $\mathsf{I}^* (I_n) > t_n \mathsf{I}^* (I)$, as desired. \[
\]

### 3.5 Equivariant partial injections

Next, we show how to construct injections between subsets of $\mathbb{Z}$ in a measurable and equivariant manner. The space of all partially defined maps between countable sets $A$ and $B$ can be represented as $(B \cup \{\ast\})^A$, where $\ast$ is a symbol not already in $B$, and a sequence $(z_i)$ in this space represents the map $\{i \in A : z_i \neq \ast\} \rightarrow B$ given there by $i \mapsto z_i$. We write $\text{Inj}_x(A,B)$ for the space of all partially defined injections (the $\ast$ implying that the maps are partially defined), and note that with the structure above $\text{Inj}_x(A,B)$ is a Borel set.

It is useful to extend the “shift” action from sets to functions: for $I, J \subseteq \mathbb{Z}$ and $f : I \rightarrow J$ let $Sf : SI \rightarrow SJ$ be given by $Sf(i) = f(i + 1) - 1$. We say that a map $X \rightarrow \text{Inj}_x(\mathbb{Z},\mathbb{Z})$, $x \mapsto f_x$, is equivariant if $f_{tx} = Sf_x$, which is just another way of saying that $f_{tx}(i) = f_x(i + 1) - 1$.

**Lemma 3.8.** Let $y \in \Sigma_{AP}^\omega$ and $I, J \subseteq \mathbb{Z}$ sets such that $\mathfrak{r}(I) < \mathfrak{r}(J)$ (or $\mathfrak{s}(I) < \mathfrak{s}(J)$). Then there exists a measurable map $(y, I, J) \mapsto f_{(y, I, J)} \in \text{Inj}(I, J)$ that is equivariant in the sense that $(Sy, SI, SJ) \mapsto Sf_{(y, I, J)}$. Furthermore, for any $\mathfrak{r}(I) < s < \mathfrak{r}(J)$ (respectively $\mathfrak{s}(I) < s < \mathfrak{s}(J)$) we can ensure that $\mathfrak{r} (\text{image}(f_{(y, I, J)})) < s$ (respectively $\mathfrak{s}(\text{image}(f_{(y, I, J)})) < s$).

**Proof.** We prove the statement for upper densities, the lower density case being similar. Fixing $y, I, J$ as in the statement, we first show how to construct $f = f_{(y, I, J)}$ without control over the image density. We define $f$ by induction. At the $k$-th stage we say that $i \in I$ and $j \in J$ are free if $f$ is not yet defined on $i$ and $j$ is not yet in the image. Start with $f = \emptyset$. For each $k$, define $f(i) = i + k$ if $i \in I$ and $i + k \in J$ are free, otherwise leave $f$ undefined on $i$. We claim that $f$ is eventually defined on every $i \in I$. To see this note that by the assumption $\mathfrak{r}(J) > \mathfrak{r}(I)$ (or $\mathfrak{s}(J) > \mathfrak{s}(I)$), there exists a $k$ such that $|I \cap [i, i + k]| < |J \cap [i, i + k]|$; it is clear that this $i$ must have been assigned in one of the first $k$ steps.

It is clear that $f_y : I \rightarrow J$ is injective and that the construction is shift-invariant and measurable, as required.

For the second statement, given $s > \mathfrak{r}(I)$ let $t_1 = \mathfrak{r}(I)/\mathfrak{r}(J)$ and $t_2 = s/\mathfrak{r}(J)$, and apply Lemma 3.4 to $(y, J, t^-, t^+)$. We obtain a subset $J' \subseteq J$ depending measurable and equivariantly on the data and satisfying $\mathfrak{r}(I) = t_1 \mathfrak{r}(J) < \mathfrak{r}(J) < t_2 \mathfrak{r}(J) = s$. Now apply the first part of this lemma to $(y, I, J')$ to obtain $f_y \in \text{Inj}(I, J') \subseteq \text{Inj}(I, J)$. Since image $f_y \subseteq J'$ we have $\mathfrak{r}(\text{image}(f_y)) < s$. \[
\]
We require a variant of Lemma 3.8 that uses uniform densities and produces partial injections \( f \in \text{Inj}_u(Z, Z) \) with bounded displacement. Here \( f \in \text{Inj}_u(Z, Z) \) is said to have bounded displacement if there is a constant \( M = M(f) \) (the displacement) such that \( |n - f(n)| < M \) for all \( n \) in the domain. When \( f = f_z \) depends on a parameter \( z \) the statement that \( f_z \) has bounded displacement does not indicate that the constant \( M(f_z) \) is uniform in \( z \).

**Lemma 3.9.** Let \( y \in \Sigma_{AP}^Z \) and let \( I, J \subseteq \mathbb{Z} \) be sets such that \( s^*(I) < s^*(J) \). Then there exists a measurable map \((y, I, J) \mapsto f = f_{(y, I, J)} \in \text{Inj}(I, J)\) that is equivariant in the sense that \((Sy, SI, SJ) \mapsto Sf_{(y, I, J)}\), and such that \( f_{(y, I, J)} \) has bounded displacement and satisfies \( s^*(\text{image } f_{(y, I, J)}) = s^*(I) \).

The proof is identical to the previous, noting that, because of uniformity, \( k \) can be chosen from a fixed bounded set and hence \( f_{(y, I, J)} \) has bounded displacement. Then use the fact that the last conclusion of the lemma is a consequence of the earlier ones, because:

**Lemma 3.10.** Let \( I \subseteq \mathbb{Z} \). Suppose that \( f : I \to \mathbb{Z} \) is an injection with bounded displacement and let \( J = \text{image}(f) \). Then \( s^*(J) = s^*(I) \) and \( s^*(J) = s^*(I) \).

The proof is immediate and we omit it.

## 4 General strategy

In this section we set the stage for the proof of the main theorem, proving a variety of technical results. The main one is Proposition 4.3, which gives a sufficient condition for the existence of a finite generator that will underly the generator theorems in later sections. It also gives a new characterization of Borel systems without invariant probability measures (see the discussion after the proof).

### 4.1 Constructing generators using allocations

For the following discussion it is convenient to have a concrete representation of \( X \). To this end fix a measurable (but not equivariant!) bijection \( \eta : X \to \{0, 1\}^\mathbb{N} \), which can be done because all standard Borel spaces are isomorphic. We then have, for each \( x \in X \), a sequence \( \eta(x) \) of bits identifying it uniquely. We call \( \eta(x) \) the static name of \( x \) (static because its definition does not depend on \( T \) in any way).

Now, if one wants to produce an injective symbolic factor map \( X \to \{0, 1\}^\mathbb{Z} \), then one must somehow encode the binary sequence \( \eta(x) \) in \( \pi(x) \). Since the map is equivariant, this means that \( \eta(T^n x) \) is encoded in \( \pi(T^n x) \), which is just a shift of \( \pi(x) \), so in fact \( \pi(x) \) must encode all the sequences \( \eta(T^n x) \). Thus, what we
want to do is encode the binary array \( \hat{x} \in \{0, 1\}^{\mathbb{Z} \times N} \) given by \( \hat{x}_{i,j} = \eta(T^i x)_j \) into a linear binary sequence \( \pi(x) \in \Delta^\mathbb{Z} \), in a measurable and equivariant manner.

The most direct approach, which is the one we shall use, is to construct an injection \( F_x : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Z} \). Then we can define \( \pi : X \rightarrow \{0, 1\}^\mathbb{Z} \) by

\[
\pi(x)_{F_x(i,j)} = \hat{x}_{i,j} = \eta(T^i x)_j
\]

and fill in any unused bits with 0. Then every bit in \( \hat{x} \) has been written somewhere in \( \pi(x) \).

In order to make the map \( \pi \) above measurable and equivariant, we must require the same from \( F_x \). Endow the space of functions between countable sets \( A, B \) with the product structure on \( B^A \), which makes it into a standard Borel space. The subset consisting of injective maps \( A \rightarrow B \) is measurable, and we denote it \( \text{Inj}(A,B) \). We say that a map \( X \rightarrow \text{Inj}(\mathbb{Z} \times \mathbb{N}, \mathbb{Z}) \), \( x \mapsto F_x \), is equivariant if

\[
F_{T^x}(i,j) = F_x(i+1,j) - 1
\]

Given \( x \mapsto F_x \), for each \( n \in \mathbb{N} \) we can define functions \( F_{x,n} : \mathbb{Z} \rightarrow \mathbb{Z} \) by \( F_{x,n}(i) = F_x(i,n) \), and then equivariance in the sense above is the same as equivariance, in the sense of Section 3.5 of each of the maps \( X \mapsto \text{Inj}(\mathbb{Z}, \mathbb{Z}) \), \( x \mapsto F_{x,n} \).

**Definition 4.1.** A map \( X \rightarrow \text{Inj}(\mathbb{Z} \times \mathbb{N}, \mathbb{Z}) \) that is measurable and equivariant is called an allocation.

If \( F_x \) is an allocation, then the map \( \pi : X \rightarrow \{0, 1\}^\mathbb{Z} \) given by \( \boxed{1} \) is easily seen to be measurable, and a short calculation shows that it is also equivariant: To see this, let \( y = T^x \) and fix \( k \in \mathbb{Z} \) and \( n \in \mathbb{N} \), let \( i = F_y(k+1,n) \) and \( j = F_y(k,n) \), so \( \pi(x)_i = \eta(T^{k+1}x)_n \) and \( \pi(y)_j = \eta(T^kx)_n \), and note that

\[
j = F_y(k,n) = F_{T^x}(k,n) = F_x(k+1,n) - 1 = i - 1
\]

This means that \( \pi(y)_{i-1} = \pi(x)_i \) for \( i \) in the image of \( F_x \). Since clearly \( \text{image}(F_y) = \text{image}(F_x) - 1 \), we have \( \pi(y)_{i-1} = \pi(x)_i = 0 \) for \( i \in \mathbb{Z} \setminus \text{image} F_x \). Thus we have shown that \( \pi(T^x) = \pi(y) = S\pi(x) \).

This procedure for encoding \( \hat{x} \) in \( \pi(x) \) is not yet reversible, but by \( \boxed{1} \), if we know both \( \pi(x) \) and \( F_x \) then we can recover the sequence \( \eta(x) \) (and in fact \( \eta(T^j x) \) for all \( j \)), and therefore recover \( x \). Thus we have established the following proposition:

**Proposition 4.2.** Let \((X, B, T)\) be a Borel system and \( F : x \mapsto F_x \) an allocation. Then there is a symbolic factor map \( \pi : X \rightarrow \{0, 1\}^\mathbb{Z} \) (equivalently, a two-set partition \( \beta \)) such that \( \sigma(\pi) \vee \sigma(F) = B \) (respectively \( \sigma_T(\beta) \vee \sigma(F) = B \)).
4.2 Constructing generators from deficient ω-covers

We say that a collection $\alpha = \{A_i\}$ of sets is an ω-cover of $X$ if every point $x \in X$ belongs to infinitely many of the $A_i$. Our main technical tool for constructing generators is the following:

**Proposition 4.3.** Let $(X, B, T)$ be a Borel system. Let $\alpha = \{A_i\}^{\infty}_{i=1} \subseteq B$ be an ω-cover of $X$ and suppose that either

(a) $\sum_{i=1}^{\infty} \mathfrak{s}(x, A_i) < 1$ for all $x \in X$, or

(b) There is a partition $\mathbb{N} = \bigcup_{u=1}^{\infty} I_u$ such that for each $u \in \mathbb{N}$ the collection $\{A_i\}_{i \in I_u}$ is pairwise disjoint, and for any finite $J \subseteq \mathbb{N}$ we have

$$\sum_u \mathfrak{s}(\bigcup_{i \in I_u \cap J} A_i) < 1 - \sup_{i \in \mathbb{N}} \mathfrak{s}(A_i).$$

Then there exists a two-set partition $\beta$ such that $\sigma_T(\beta) \lor \sigma_T(\alpha) = B$. In particular, if there exists a finite partition $\gamma$ such that $\alpha \subseteq \sigma_T(\gamma)$, then $\beta \lor \gamma$ is a finite generator.

One can prove many variants using other conditions than (a) or (b), but these are the ones we will need.

**Proof.** We shall show how to construct an allocation from an ω-cover $\alpha = \{A_i\}$ satisfying one of the hypotheses of the proposition. The proposition then follows from Proposition 4.2.

We begin with case (a). Fix $x \in X$, and suppress it in the notation below except when needed, in which case it is indicated with a superscript. Let

$$J_i = J_i^x = \{n \in \mathbb{Z} : T^nx \in A_i\}$$

Note that $x \mapsto J_i$ is equivariant and that, since $\alpha$ is an ω-cover, every $n \in \mathbb{Z}$ belongs to infinitely many of the $J_i$.

We next want to define injections $f_i^x : J_i \to \mathbb{Z} \setminus \bigcup_{j<i} \text{image}(f_j^x)$ so that $x \mapsto f_i^x$ is measurable and equivariant. Assuming we have done this, given $n \in \mathbb{Z}$ and $j \in \mathbb{N}$ let $i(n, j)$ denote the $j$-th index $i$ such that $n \in J_i$ (which is well defined since $n$ belongs to infinitely many of the sets $J_i$), and define

$$F_x(n, j) = f_i^x(n, j)(n)$$

Since the images of the $f_i^x$’s are disjoint, $F_x \in \text{Inj}(\mathbb{Z} \times \mathbb{N}, \mathbb{Z})$. Clearly $x \mapsto F_x$ is measurable. To see that it is equivariant, note that $i^{T^nx}(n, j) = i^x(n+1, j)$, because $J_i = J_i - 1$, so using equivariance of $x \mapsto f_i^x$,

$$F_{T^nx}(n, j) = f_{i^{T^nx}(n, j)}(n) = f_{i^x(n+1, j)}(n) = f_{i^x(n+1, j)}(n+1) - 1 = F_x(n+1, j) - 1$$

So $x \mapsto F_x$ is an allocation.
It remains to construct the $f_i^x$. By Weiss’s countable generator theorem [27] (for the version which does not exclude a $W$-set see [12, Theorem 5.4] or [24, Corollary 7.6]), we may assume that $X \subseteq \Sigma_Z$ for a countable alphabet $\Sigma$, and $T$ is the shift map. Choose $s_i = s_i(x) \in (0, 1)$ measurably satisfying $\overline{s}(J_i) < s_i < 1 - \sum_{j<i} s_j$ and $\sum_{i=1}^{\infty} s_i < 1$, which can be done because of the hypothesis $\sum_{i=1}^{\infty} \overline{s}(J_i) < 1$. Now for $i = 1, 2, \ldots$ apply Lemma 5.8 inductively to $(x, J_i, \mathbb{Z} \setminus \bigcup_{j<i} \text{image}(f_j^x))$ and $s_i$. We can do this because by induction we have

$$\overline{s}(J_i) < 1 - \sum_{j<i} s_j < 1 - \sum_{j<i} \overline{s}(\text{image}(f_j^x)) \leq \mathbb{S}(\mathbb{Z} \setminus \bigcup_{j<i} \text{image}(f_j^x))$$

and the construction can be carried through.

The construction of $F_x$ under assumption (b) follows the same lines with some minor changes. There is no need to introduce the $s_i$, but rather proceed directly, using Lemma 3.9 to construct the maps, which will have bounded displacement, and Lemma 3.10 to control the density of the images. At stage $i$, note that there is some $N_i = N_i(x)$ such that

$$\bigcup_{j<i} \text{image } f_j = \bigcup_{u=1}^{N_i} \left( \bigcup_{j<i, j \in I_u} \text{image } f_j \right).$$

For each $u$, as $j$ ranges over $j \in I_u$, the domains $J_j$ of $f_j^x$ are disjoint, so we can define

$$f_{i,u} = f_{i,u}^x = \bigcup_{j<i, j \in I_u} f_j.$$

As the union of finitely many maps with bounded displacement, this map has the same property. Thus, noting that

$$\text{image } f_{i,u} = \bigcup_{j<i, j \in I_u} \text{image } f_j,$$

and using Lemma 3.10 by (3), we have

$$\overline{s}^*(\bigcup_{j<i} \text{image } f_j) \leq \sum_{u=1}^{N_i} \overline{s}^*(\text{image } f_{i,u}) = \sum_{u=1}^{N_i} \overline{s}^*(\bigcup_{j<i, j \in I_u} \text{dom } f_j) \leq \sum_{u=1}^{N_i} \overline{s}^*(\bigcup_{j \in I_u} J_j)$$

(note that the first inequality is valid since the sum is actually over finitely many $u$). But recalling $J_j = \{n : T^n x \in A_j\}$ and the definition of the sets $I_u$ in assumption (b) of the proposition, the last sum is less than $1 - \overline{s}^*(J_i)$, so we
\[ \mathfrak{N}(J_i) < 1 - \mathfrak{N}\left(\bigcup_{j<i} \text{image}(f^i_j)\right) \leq 2^i (\mathbb{Z} \setminus \bigcup_{j<i} \text{image}(f^i_j)) \]

Thus, Lemma 3.9 lets the construction proceed, and finishes the proof. \(\Box\)

In the following sections we show that, given a Borel system \((X, \mathcal{B}, T)\) without invariant probability measures, one can partition \(X\) into two measurable invariant sets (modulo \(\mathcal{W}\)) such that the first admits an \(\omega\)-cover satisfying condition (a) of the proposition above, and the second admits an \(\omega\)-cover satisfying condition (b). Clearly if a system admits an invariant measure then no such partition can exist. Thus, we have arrived at another characterizations of Borel systems without invariant probability measures. It would be nicer to eliminate the need to partition the space: perhaps there is always an \(\omega\)-cover (modulo \(\mathcal{W}\)) that satisfies (a) (or that satisfies (b)), but we have not been able to show this.

### 4.3 Generators for unions of \(\text{def}(\alpha)\), \(\text{null}(A_i)\)s and \(\text{div}(A_i)\)s

Our strategy, as explained in the introduction, is to divide \(X\) into sets of points that are null or divergent for countably many sets \(A_i\), or deficient for some partition \(\alpha\). We now indicate how to modify these sets so as to obtain a partition of \(X\) into finitely many sets of the same forms. The following is elementary:

**Lemma 4.4.** Let \(A, B \in \mathcal{B}\) and assume that \(B\) is \(T\)-invariant. Then \(\text{null}(A) \setminus B = \text{null}(A \setminus B)\) and \(\text{div}(A) \setminus B = \text{div}(A \setminus B)\).

As an immediate consequence, we have

**Lemma 4.5.** Let \(A_1, A_2, \ldots \in \mathcal{B}\) and set \(D_1 = A_1\) and \(D_n = A_n \setminus \bigcup_{i=-\infty}^{n-1} T^i D_{n-1}\). Then

\[ \bigcup_{n=1}^{\infty} \text{null}(A_i) = \text{null}\left(\bigcup_{n=1}^{\infty} D_n\right) \]

and similarly if we replace \(\text{null}(\cdot)\) by \(\text{div}(\cdot)\).

Thus, noting that \(\text{def}(\alpha)\) is invariant, we have

**Lemma 4.6.** Let \(\alpha \subseteq \mathcal{B}\) and \(A_i, B_i \in \mathcal{B}\) and suppose that

\[ X = \text{def}(\alpha) \cup \bigcup_{i=1}^{\infty} \text{null}(A_i) \cup \bigcup_{i=1}^{\infty} \text{div}(B_i) \]

Then there are sets \(A, B \in \mathcal{B}\) such that \(X = \text{def}(\alpha) \cup \text{null}(A) \cup \text{div}(B)\) and the union is disjoint.

**Proof.** By Lemma 4.4 we can replace each \(A_i\) by \(A_i \setminus \text{def}(\alpha)\) and the hypothesis remains. Use the previous lemma to find \(A \in \mathcal{B}\) such that \(\bigcup_{n=1}^{\infty} \text{null}(A_i) = \bigcup_{n=1}^{\infty} \text{null}(A) \cup \bigcup_{n=1}^{\infty} \text{null}(A_i) \cup \bigcup_{n=1}^{\infty} \text{null}(A_n) \setminus \bigcup_{n=1}^{\infty} T^i D_{n-1}\).
null(A). By the same reasoning we can replace \( B_i \) with \( B_i \setminus (\text{def}(\alpha) \cup \text{null}(A)) \) without affecting the hypothesis and find \( B \in \mathcal{B} \) with \( \bigcup_{i=1}^{\infty} \text{div}(B_i) = \text{div}(B) \). But note that \( \text{def}(\alpha), \text{null}(A) \) and \( \text{div}(B) \) are pairwise disjoint and their union is \( X \), as desired.

4.4 From finite to two-set generators

As explained in the introduction, most of the work in the proof of Theorem 1.1 goes towards proving the following theorem:

**Theorem 4.7.** There is a natural number \( K \) such that every Borel system without invariant probability measures admits a \( K \)-set generator.

This is good enough to get two-set generators, because

**Proposition 4.8.** Theorem 4.7 implies Theorem 1.1. Furthermore the generator may be chosen so that the itineraries lie in a given mixing non-trivial shift of finite type.

**Proof.** We first prove the existence of a two-set generator without requirements on the itineraries. The proof is basically a variant of Abramov’s formula for entropy of an induced map. Taking a set \( A \) with large but bounded return times, the induced map will not have invariant probability measures (because such a measure would lift to one on \( X \)), and so has a \( K \)-set generator, which can be converted to a 2-set generator of \( X \) by coding each symbol in the space between returns to \( A \).

Here is the detailed proof. Fix \((X, \mathcal{B}, T)\) without invariant probability measures. By hypothesis we can assume that \( X \subseteq \Sigma_{AP}^\infty \) for \( \Sigma = \{1, \ldots, K\} \), with \( T \) being the shift.

Let \( N = 4 + 2 \lfloor \log_2 K \rfloor \) and let \( \pi : X \to \{0, 1\}^\infty \) be an equivariant measurable map into \( N \)-markers, as provided by Lemma 3.3.

Let \( A = \{ x \in X : \pi(x)_0 = 1 \} \) and \( r_A(x) = \min\{ n > 0 : T^n x \in A \} \) the entrance time map. By the \( N \)-marker property, \( r_A(x) \leq N + 1 \) for every \( x \in X \), and in particular every forward orbit meets \( A \). Let \( T_A(x) = T^{r_A(x)} x \) denote the induced map on \( A \) and consider the induced system \((A, \mathcal{B}|_A, T_A)\). Then \((X, \mathcal{B}, T)\) is isomorphic to the suspension of \((A, \mathcal{B}|_A, T_A)\) with the bounded roof function \( r_A \).

If \((A, \mathcal{B}|_A, T_A)\) admitted a finite invariant measure then the measure could be lifted to the suspension, and the result would be a finite measure because the roof function is bounded, giving a finite invariant measure on \((X, \mathcal{B}, T)\). This is impossible, so by our hypothesis, \((A, \mathcal{B}|_A, T_A)\) admits a \( K \)-set generator \( \alpha \).

We next define a measurable equivariant map \( \tilde{\pi} : X \to \{0, 1\}^\infty \). Fix \( x \in X \) and \( i \) with \( \tilde{\pi}(x)_i = 1 \). Set \( \tilde{\pi}(x)_i = \tilde{\pi}(x)_{i+1} = \ldots = \tilde{\pi}(x)_{N/2} = 1 \) and \( \tilde{\pi}(x)_{1+N/2} = 0 \). Then in the next \( N/2 - 1 \) symbols of \( \tilde{\pi}(x) \) write a binary string...
identifying $\alpha(T^i x)$, using some fixed coding of the elements of $\alpha$ (we can do this because there are $K$ possible values for $\alpha(T^i x)$ and $N/2 - 1 > \log_2 K$ available symbols). After doing this for every $i$ with $\pi(x)_i = 1$, set any undefined symbols in $\tilde{\pi}(x)$ to 0. By the $N$-marker property the gap between 1s in $\pi(x)$ is at least $N$, so we have not tried to define any symbol more than once, and $\tilde{\pi}(x)$ is well defined. Evidently $x \mapsto \tilde{\pi}(x)$ is measurable and equivariant.

Now, the word $1^{N/2}0$ occurs only at indices $i$ with $\pi(x)_i = 1$, so $\pi(x)$ can be recovered from $\tilde{\pi}(x)$, hence given $\pi(x)$ we can find all the $i$ such that $T^i x \in A$. For such an $i$, we recover $\alpha(T^i x)$ by reading off the $N/2 - 1$ binary digits in $\tilde{\pi}(x)$ starting at $i + N/2 + 2$. Thus, $\tilde{\pi}(x)$ determines $\alpha(T^i x)$ for all $i$ such that $T^i x \in A$, and since $\alpha$ generates for $T_A$, this determines $T^i x$ for such $i$, and therefore determines $x$. We have shown that $x \mapsto \tilde{\pi}(x)$ is an injection, completing the proof of Theorem 1.1.

Now assume that $Y \subseteq \Lambda^Z$ is a non-trivial mixing shift of finite type (SFT). The modification of the previous proof is rather standard; for definitions and basic techniques related to SFTs can be found e.g. in [18]. We modify the construction above as follows. Using the mixing property of $Y$, choose words $a_0, a'_0$ and $a_1$ in $Y$ such that any concatenation of the words appears in $Y$, and every infinite concatenation has a unique parsing into these words. Also require that the length of $a'_0$ is greater by one than the length of $a_0$. Choose $N$ now to be large relative to the lengths of these words as well, and proceed as before, except that when building the image $\tilde{\pi}(x)$ we write copies of $a_0, a'_0$ instead of 0 and $a_1$ instead of 1; the choice between $a_0, a'_0$ is made in such a way that the length of the final concatenation is precisely the distance between occurrences of visits to $A$. The remaining details are left to the reader.

5 A generator theorem for null points

Recall that $x \in \text{null}(A)$ if $s(x, A) = 0$ and $x \in \bigcup_{n=-\infty}^{\infty} T^n A$. In this section we prove:

**Theorem 5.1.** Let $(X, B, T)$ be a Borel system and $A \in B$. Then $\text{null}(A)$ has a 4-set generator.

Heuristically, this result is a Borel version of the generator theorem for infinite invariant measures. Indeed if $\mu$ is such a measure and $A$ is a set with $0 < \mu(A) < \infty$, then by Hopf’s ratio ergodic theorem $x \in \text{null}(A)$ for $\mu$-a.e. $x$. In fact the theorem above recovers (most aspects of) Krengel’s generator theorem for such measures.

**Proof.** For $i, j \in \mathbb{Z}$ define

$$A_{i,j} = T^{-i} A$$

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(note that this does not actually depend on \( j \)). Then for each \( j \) the union 
\[ \bigcup_{i \in \mathbb{Z}} A_{i,j} \]
includes all \( x \in \text{null}(A) \) such that \( T^n x \in A \) for some \( n \in \mathbb{Z} \), so 
\[ \bigcup_{i \in \mathbb{N}} A_{i,j} = \text{null} A. \]
Clearly \( (A_{i,j})_{i,j \in \mathbb{N}} \) is an \( \omega \)-cover of \( \text{null}(A) \). But also \( A_{i,j} = T^{-i} A \) so 
\[ \mathcal{F}(x, A_{i,j}) \leq \mathcal{F}(x, A) = 0 \]
for every \( x \in \text{null}(A) \), hence 
\[ \sum_{i,j \in \mathbb{Z}} \mathcal{F}(x, A_{i,j}) = \sum_{i,j \in \mathbb{Z}} 0 = 0 < 1 \]
for all \( x \in \text{null}(A) \).

The hypotheses of Proposition 4.3 are satisfied for the system \((\text{null}(A), \mathcal{B}|_{\text{null}(A)}, T|_{\text{null}(A)})\), so there is a two-set partition \( \beta \) of \( \text{null}(A) \) such that 
\[ \sigma_T(\beta) \lor \sigma_T(\{A_{i,j}\}_{i,j \in \mathbb{N}}) = \mathcal{B}|_{\text{null}(A)}. \]
But setting \( \gamma = \{A, \text{null}(A) \setminus A\} \), clearly \( A_{i,j} \in \sigma_T(\gamma) \), so \( \beta \lor \gamma \) is a generating partition with four sets.

We remark that, up to removing an invariant set from the wandering ideal \( \mathcal{W} \), it is possible to define a partition of \( \text{null}(A) \) which, in a sense, is deficient. Specifically, let \( \tilde{A}_i \subseteq \text{null}(A) \) with \( T^j x \in A \) and \( T^j x \notin A \) for \( 0 \leq j < i \). Then 
\[ \text{null}(A) \setminus \bigcup \tilde{A}_i \]
consists of points which do not enter \( A \) in the future, but, by definition of \( \text{null}(A) \), enter it in the past, so \( \text{null}(A) \setminus \bigcup \tilde{A}_i \in \mathcal{W} \). One might hope to apply our coding of deficient partitions to \( \alpha = \{\tilde{A}_i\} \). Formally this is not possible, since in our definition of deficient partitions we required positive frequencies. With some adjustment this approach could be made to work. But, in any event, the construction for the deficient case is far more complex than the one above, and such a reduction would not be very enlightening.

The construction above applies to many examples of Borel systems without invariant measures. A popular construction of such a system, for example, is to begin with the dyadic odometer \( G \) and build the suspension \( X \) with respect to a functions that is continuous except at one point, and has infinite integral with respect to Haar measure on the base. In such constructions we have \( X = \text{null}(G) \), and the short proof above provides a generator. As noted in the introduction, we don’t know whether every Borel system without invariant probability measures is of the form \( \text{null}(A) \) for some measurable set \( A \).

### 6 A generator theorem for divergent points

Our purpose in this section is to construct a finite generator for the set \( \text{div}(A) \) of points which do not have well-defined visit frequencies to \( A \). The key to this is Bishop’s quantitative result on the decay of the frequency of repeated fluctuations of ergodic averages.
6.1 Bishop’s theorem

Birkhoff’s ergodic theorem states that, in a probability preserving system, the ergodic averages of an $L^1$ function converge a.e. It is well known that this convergence does not admit a universal rate, even if one fixes the system and varies only the function. Nevertheless, there is an effective version of Birkhoff’s theorem, due originally to E. Bishop and subsequently extended by various authors, stated in terms of the probability that there occur many fluctuations of the ergodic averages across a given gap. More precisely, for a map $T : X \to X$ and $f : X \to \mathbb{R}$, we say that $x \in X$ has $k$ upcrossings of a real interval $(a, b) \subseteq \mathbb{R}$ (w.r.t. $f$) if there is a sequence $0 \leq m_1 < n_1 < m_2 < n_2 < \ldots < m_k < n_k$ such that

$$S_{m_i}(x, f) < a < b < S_{n_i}(x, f) \quad \text{for } i = 1, \ldots, k$$

and $S_n(x, f) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$. If there is an infinite such sequence we say there are infinitely many upcrossings. Clearly when $X$ carries a measurable structure the set of points with $k$ upcrossings is measurable and we can choose $m_i(x), n_i(x)$ measurably, e.g. taking the lexicographically least sequence. Bishop’s theorem reads as follows.

**Theorem 6.1** (Bishop [3]). Let $(X, \mathcal{B}, \mu, T)$ be an ergodic probability-preserving system and $f \in L^1(\mu)$. Then for every $a < b$,

$$\mu(x \in X : x \text{ has } k \text{ upcrossings of } (a, b) \text{ (w.r.t. } f)) \leq \frac{\|f\|_1}{k(b - a)}$$

The point is that the rate of decay is universal, depending only on the magnitude of the gap and the norm of $f$ (this normalization or one like it is unavoidable in order for the rate to be invariant under scaling of $f$ and $a, b$).

What we need is not precisely the last theorem, but a finitistic variant that is used its proof. We give the statement for indicator functions. Given $T : X \to X$, a set $A \subseteq X$, write

$$U_{a,b,k,N} = \{ x \in X : x \text{ has } k \text{ upcrossings of } (a, b) \text{ w.r.t. } 1_A \text{ up to time } N \}$$

($k$ upcrossings up to time $N$ means that we can choose the times $m_1, n_1, \ldots, m_k, n_k$ in the definition with $n_k \leq N$).

**Theorem 6.2.** Let $T : X \to X$ be a map and $A \subseteq X$. For every $k$, every $a < b$, every $N$ and every $x \in X$,

$$\mathcal{F}(x, U_{a,b,k,N}) < \frac{2}{k(b - a)}$$

In fact this holds with an exponential decay rate [11]. We do not need this stronger result, all we will use is that the rate is universal (i.e., depends only
on $a, b$), but the proof is easier; we give a sketch below. Fix $y \in X$ and a large $L \gg N$, and consider the set of times $I \subseteq \{1, \ldots, L\}$ such that $T^y I \subseteq A$. Consider the function $f : I \to \{0, 1\}$ such that $f(i) = 1_A(T^y i)$. For each such $i \in I$ there are times $1 \leq m_1(i) < n_1(i) < m_2(i) < n_2(i) < \ldots < m_k(i) < n_k(i) \leq N$, witnessing the fact that $T^y i$ has $k$ upcrossings up to time $N$. This means that on each of the intervals $A_{i,j} = [i, i + m_j(i)]$ the average of $f$ is less than $a$, and on each of the intervals $B_{i,j} = [i, i + n_j(u))$, the average of $f$ is greater than $b$; and $A_{i,1} \subseteq B_{i,1} \subseteq A_{i,2} \subseteq \ldots \subseteq A_{i,k} \subseteq B_{i,k}$. Given this combinatorial structure, one now shows that one can obtain disjoint families of intervals $A_1, B_1, \ldots, A_k, B_k$, with the $A_i$ family consisting of intervals of the form $A_{i,j}$ and the $B_i$ families consisting of intervals $B_{i,j}$, such that

$$\cup A_1 \subseteq \cup B_1 \subseteq \cup A_2 \subseteq \ldots \subseteq \cup A_k \subseteq \cup B_k$$

and such that $\cup A_1$ is of size comparable to $I$. This is a variation on the Vitali covering lemma (observe that for each $1 \leq j \leq k$, the original intervals $\{A_{i,j}\}_{i \in I}$ may overlap quite a lot). Finally, we observe that the average of $f$ on each $A_j$ is less than $a$, while the average over $B_{j-1}$ is greater than $b$. Since $f$ is bounded between 0 and 1, this says that $|\cup A_j| \geq \frac{b}{a} |\cup B_{j-1}|$. Thus $|A_k| \geq \left(\frac{b}{a}\right)^{k-1}|B_1| \geq |I|$. Finally $A_k \subseteq [1, L + N] \subseteq [1, \frac{b}{a} L]$ (since $N \ll L$), and we conclude that $|I| \leq \left(\frac{b}{a}\right)^k L$, as desired.

Other versions can be found in [8, Section 2] and [14, Section 2], where one can also read off versions of the statement above.

### 6.2 Construction of the generator

**Theorem 6.3.** Let $(X, B, T)$ be a Borel system and $A \in B$. Then $\text{div}(A)$ has a 4-set generator.

**Proof.** We can assume (by restriction if necessary) that $X = \text{div}(A)$. For $x \in X$ write $\delta = \delta(x) = \overline{\delta}(x, A) - \underline{\delta}(x, A)$ and let $a = a(x) = \underline{\delta}(x, A) + \delta/3$ and $b = b(x) = \overline{\delta}(x, A) - \delta/3$, so $a(\cdot), b(\cdot)$ are measurable and shift invariant and $\underline{\delta}(x, A) < a < b < \overline{\delta}(x, A)$.

Let $k_p = \lfloor 2^{p+2}/(b - a) \rfloor$. For $x \in X$ let $m_i(x), n_i(x)$ be the lexicographically least upcrossing sequence of $x$ with respect to $(a, b)$, and let $A_{p,n}$ denote the set of points whose $k_p$-th upcrossing occurs at time $n$, i.e.

$$A_{p,n} = \{x \in X : n_{k_p}(x) = n\}$$

These sets are measurable, and we claim that $\alpha = \{A_{p,n}\}_{p,n \in \mathbb{N}}$ satisfies is an $\omega$-cover of $X$ satisfying the hypothesis of Proposition [4,3].

Indeed, for each $p$ and $x \in X$ the $k_p$-th upcrossing of $(a(x), b(x))$ occurs at some time or other, i.e. $X = \bigcup_{n=1}^{\infty} A_{p,n}$ for all $p$, which shows that $\alpha$ is an
ω-cover of X.

We now verify the hypothesis of Proposition 4.3 (b). For the index set \( \mathbb{N} \times \mathbb{N} \) of \( \{ A_{p,n} \} \) we choose the partition \( I_p = \{ p \} \times \mathbb{N}, \ p \in \mathbb{N} \). We first claim that for every \( N \),

\[
\overline{s}^\ast(x, \bigcup_{n=1}^N A_{p,n}) < \frac{1}{2^{p+1}} \tag{4}
\]

This is enough because for any finite \( J \in \mathbb{N} \times \mathbb{N} \) there is an \( N \) such that \( n \leq N \) for every \( (p, n) \in J \), and therefore for every \( (q, m) \in \mathbb{N} \times \mathbb{N} \),

\[
\sum_{p=1}^{\infty} \overline{s}^\ast(x, \bigcup_{(p,n) \in J \cap I_p} A_{p,n}) \leq \sum_{p=1}^{\infty} \overline{s}^\ast(x, \bigcup_{n=1}^N A_{p,n}) < \frac{1}{2^{p+1}} = \frac{1}{2} < 1 - \overline{s}^\ast(x, A_{q,m})
\]

where the last inequality is because, by (4) again, \( \overline{s}^\ast(x, A_{q,m}) < 1/2 \).

It remains to prove (4). Fix \( \alpha < \beta \) and consider the \( T \)-invariant set

\( X_{\alpha,\beta} = \{ x \in X : a(x) = \alpha, \ b(x) = \beta \} \)

Fix \( p, N \) and set \( k'_p = 2^{p+2}/(\beta - \alpha) \). Using the notation of Theorem 6.2 we have

\( X_{\alpha,\beta} \cap \bigcup_{n=1}^N A_{p,n} \subseteq U_{\alpha,\beta,k'_p,N} \) \( \tag{5} \)

Therefore for \( x \in X_{\alpha,\beta} \), by Theorem 6.2 and monotonicity of \( \overline{s}^\ast(x, \cdot) \) we have

\[
\overline{s}^\ast(x, \bigcup_{n=1}^N A_{p,n}) \leq \overline{s}^\ast(x, U_{\alpha,\beta,k'_p,N}) < \frac{2}{k'_p(\beta - \alpha)} \leq \frac{1}{2^{p+1}}
\]

This holds for \( x \in X_{\alpha,\beta} \), for all \( \alpha < \beta \). But every \( x \in X \) belongs to some \( X_{\alpha,\beta} \) for some \( \alpha < \beta \), and the last inequality gives (4).

To conclude the proof, apply Proposition 4.3 which gives a two-set partition \( \beta \) of \( X \) such that \( \sigma_T(\beta) \vee \sigma_T(\alpha) = \mathcal{B} \). Taking \( \gamma = \{ A, X \setminus A \} \), we note that \( \alpha \) is \( \sigma_T(\gamma) \)-measurable, so by the same proposition \( \beta \vee \gamma \) is a 4-set generator for \( (X, \mathcal{B}, T) \). \( \square \)
7 A generator theorem for deficient points, and putting it all together

We say that \( x \in \Sigma \) is regular if the frequency \( s(x, a) \) exists for every \( a \in \Sigma^* \), and otherwise call it divergent. Let \( \text{Reg}(\Sigma) \) and \( \text{Div}(\Sigma) \) denote the sets of regular and divergent points. If \( x \in \Sigma \) is such that \( s(x, a) \) exists and is positive for all \( a \in \Sigma^* \), write
\[
\rho(x) = \sum_{a \in \Sigma} s(x, a)
\]
and in this case say that \( x \) is deficient if \( \rho(x) < 1 \). We then define the defect to be \( 1 - \rho(x) \). Denote for the set of deficient points by \( \text{Def}(\Sigma) \). Finally, say that \( x \in \Sigma \) is null if \( s(x, a) = 0 \) for some \( a \in \Sigma^* \) and write \( \text{Null}(\Sigma) \) for the set of null points.

Given a Borel system \((X, \mathcal{B}, T)\) and a partition \( \alpha = \{A_i\}_{i \in \Sigma} \), associate to every \( x \in X \) its \( \alpha \)-itinerary, \( \alpha^*(x) = (\alpha(T^n x))_{n \in \mathbb{Z}} \). We say that \( x \in X \) is \( \alpha \)-regular, \( \alpha \)-divergent, \( \alpha \)-deficient or \( \alpha \)-null if \( \alpha^*(x) \) is regular, divergent, deficient or null, respectively (this characterization of \( \alpha \)-deficient points is consistent with the one in the introduction).

7.1 Increasing the defect

The goal of this section is to show that, given a defective partition (relative to some point), we can measurably produce another partition which, relative to the point, either has defect arbitrarily close to one, or is divergent. We formulate this in symbolic language.

**Proposition 7.1.** For every \( \delta > 0 \) there factor map,
\[
\pi_\delta : \text{Def}(\Sigma) \to \text{Def}(\Sigma) \cup \text{Div}(\Sigma),
\]
such that if \( x \in \text{Def}(\Sigma) \) and \( \pi_\delta(x) \) is regular, then \( \rho(\pi_\delta(x)) < \delta \).

**Proof.** The scheme of the proof is as follows. We describe a (measurable) construction which either produces a divergent point \( y \in \Sigma \), in which case we can set \( \pi_\delta(x) = y \), or else produces an integer \( p \) and disjoint subsets \( J^{(0)}, \ldots, J^{(p)} \subseteq \mathbb{Z} \) and partitions \( \{J_j^{(k)}\} \) of \( J^{(k)} \), such that

(i) \( s(\mathbb{Z} \setminus \bigcup_{k=0}^p J^{(k)}) < \delta/2 \),

(ii) \( \sum s(J_j^{(k)}) < \frac{\delta}{2} s(J^{(k)}) \) for each \( k = 0, \ldots, p \).

Then, identifying \( \Sigma \) with \( \mathbb{N} \times \mathbb{N} \), we can define
\[
\pi_\delta(x)_i = \begin{cases} (k, j) & i \in J_j^{(k)} \\ (p + 1, 0) & i \in \mathbb{Z} \setminus \bigcup_{k=0}^p J^{(k)} \end{cases}
\]
and we have

\[
\rho(\pi_\delta(x)) = \sum_{k=0}^{p} \sum_{j} s(J_j^{(k)}) + s(\mathbb{Z} \setminus \bigcup_{k=0}^{p} J^{(k)}) \leq \sum_{k=0}^{p} \frac{\delta}{2} s(J^{(k)}) + \frac{\delta}{2} \leq \delta
\]

so \(\pi_\delta(x)\) has defect at least \(1 - \delta\).

We turn to the construction. Without loss of generality we assume that \(\delta < 1/8\) and \(\Sigma = \mathbb{N}\). Let \(x \in \mathbb{N}^\mathbb{Z}\) be regular and deficient. Note that deficiency implies that \(x\) is aperiodic.

**Constructing \(J^{(0)}\) and \(\{J_j^{(0)}\}\):** Let \(n_0 = n_0(x)\) denote the least integer such that

\[
\sum_{j > n_0} s(x, j) < \delta^4(1 - \rho(x))
\]

(there exists such \(n_0\) since \(\sum s(x, j) < \infty\) and \(1 - \rho(x) > 0\) by assumption), and let

\[
J^{(0)} = \{i \in \mathbb{Z} : x_i > n_0\}
\]

and

\[
J_j^{(0)} = \{i \in \mathbb{Z} : x_i = j\}
\]

so that \(\{J_j^{(0)}\}_{j > n_0}\) partitions \(J^{(0)}\). Note that

\[
s(J^{(0)}) = 1 - \sum_{j=0}^{n_0} s(J_j) \geq 1 - \rho(x) > 0
\]

Thus, by choice of \(n_0\),

\[
\sum_{j > n_0} s(J_j^{(0)}) < \delta^4(1 - \rho(x)) < \delta^4 s(J^{(0)}) \tag{6}
\]

so (ii) is satisfied.

**Constructing \(J^{(k)}, \{J_j^{(k)}\}\) for \(k = 1, \ldots, p\):** Our strategy is now to copy a substantial subset of \(J^{(0)}\), and the partition induced on it from \(\{J_j^{(0)}\}\), into the complement of \(J^{(0)}\), and repeat this until most of the complement is exhausted. We would like to do this by mapping \(J^{(0)}\) to \(\mathbb{Z} \setminus J^{(0)}\) using Lemma 3.8 but in the process one loses control of the densities of the images of \(J_j^{(0)}\). But one can
control the frequencies if one works with points in \(J^{(0)}\) that are moved by at most some large \(M\). The details are worked out in the following lemma, which provides the basic step of the strategy:

**Lemma 7.2.** Let \(J \subseteq \mathbb{Z}\) and suppose that \(s(J)\) exists and satisfies

\[
\frac{1}{2} \delta s(J^{(0)}) < s(J) < \delta s(J^{(0)})
\]

Then there exists a set \(J' \subseteq J\) and a partition \(\{J_j\}\) of \(J'\), all determined measurably by \(x\) and \(J\), such that one of the following holds:

(a) \(s(J')\) does not exist,

(b) \(s(J') > \frac{\delta}{4}s(J^{(0)})\) and \(\sum s(x, J_j) < \delta s(J')\).

**Proof.** By assumption \(s(J) < \delta s(J^{(0)}) < s(J^{(0)})\), so we can apply Lemma 3.8 to \(x, J, J^{(0)}\), and obtain an injection \(f : J \rightarrow J^{(0)}\), determined measurably by \(x, J^{(0)}, J\), and hence by \(x, J\) (since \(J^{(0)}\) is itself determined measurably by \(x\)). For \(m = 0, 1, 2, \ldots\) set

\[
U_m = \{n \in J : |f(n) - n| = m\}
\]

If one of the densities \(s(U_m)\) doesn’t exist we define \(J' = U_m\) and we are in case (a).

Thus assume these densities exist. If \(\sum s(U_m) < \delta s(J)\), we define \(J' = J\) and \(J_j = U_j\), so \(\{J_j\}\) partitions \(J'\), and we are in case (b).

Thus, assume that \(\sum s(U_m) \geq \delta s(J)\). Choose \(M \in \mathbb{N}\) such that

\[
\sum_{m=0}^{M} s(U_m) > \frac{\delta}{2}s(J) \quad (7)
\]

Set

\[
J' = \bigcup_{m \leq M} U_m
\]

Note that by the hypothesis \(s(J) \geq \frac{1}{2} \delta s(J^{(0)})\) we have

\[
s(J') = \sum_{m \leq M} s(U_m) > \frac{\delta}{2}s(J) \geq \frac{\delta^2}{4}s(J^{(0)}) \quad (8)
\]

Next, for \(j > n_0\) define

\[
J_j = J' \cap (f^{-1}(J_j^{(0)}))
\]

(we leave it undefined for \(j \leq n_0\)). Clearly \(\{J_j\}\) is a partition of \(J'\), and by (8) we have the first part of (b). Furthermore, the map \(f|_{J'}\), and hence also
\((f|_{J'})^{-1} = f^{-1}|_{f(J')},\) displaces points by at most \(M\), so these maps preserve densities, and we have

\[
s(J_j) = s(J_j^{(0)} \cap f(J')) \leq s(J_j^{(0)})
\]

Therefore, using (6) and (8) and the standing assumption \(\delta < 1/8\),

\[
\sum_{j>n_0} s(J_j) \leq \sum_{j>n_0} s(J_j^{(0)}) < \delta s(J') < \frac{\delta}{2} s(J')
\]

which is the second part of (b).

Returning to the proof of the proposition, suppose that \(s(Z \setminus J^{(0)}) > \delta \geq 1/4 \geq \frac{\delta}{2} s(J^{(0)})\) (as explained earlier, if not, we are done). Applying Lemma 3.4 to \(x\) and \(I = Z \setminus J^{(0)}\) to obtain a set \(J \subseteq I\) with \(\frac{1}{2} \delta s(J^{(0)}) < s(I) < \delta s(J^{(0)})\).

To this we apply the previous lemma, either obtaining the set \(E\) from (a) in the lemma, in which case we define \(\pi_{\delta}(x) = 1\) \(E \in \text{Div}(N_Z)\), or else obtaining \(J^{(1)} \subseteq J \subseteq Z \setminus J^{(0)}\) and \(\{J^{(1)}_j\}\) satisfying (b) of the lemma, which gives property (ii) above, and furthermore, \(s(J^{(1)}) > \frac{\delta^2}{4} s(J^{(0)})\), which is a definite increment.

We can repeat this inductively: assuming that we have defined \(J^{(k)}\) and \(\{J^{(k)}_j\}\) for \(\ell < k\) and \(s(\bigcup_{\ell<k} J^{(\ell)}) \geq \frac{\delta}{2}\), we either define \(\pi_{\delta}(x) \in \text{Div}(N_Z)\) or obtain \(J^{(k)}_j\) and \(\{J^{(k)}_j\}\) as required by (ii). At each step the total mass of the \(J^{(k)}_j\)s increases by \(\delta^2 s(J^{(0)})/4\), so unless the process terminates early with \(\pi_{\delta}(x) \in \text{Div}(N_Z)\), after a finite number \(p\) of steps we cover a set of density \(1-\delta/2\), and are done.

We reformulate the proposition in the language of partitions.

**Corollary 7.3.** Let \(\alpha\) be a countable partition of a Borel system \((X, B, T)\). Then for every \(\delta > 0\) there is a partition \(\alpha'\) of \(X\) such that every \(x \in \text{Def}(\alpha)\) is either \(\alpha'\)-divergent or else \(\sum_{A \in \alpha'} s(x, A) < \delta\).

**Proof.** Compose the itinerary map \(\alpha_*\) with the factor map \(\pi_{\delta}\) from the previous proposition, and pull back the standard generating partition of \(\Sigma^\mathcal{Z}\) (consisting of length-1 cylinders). This is \(\alpha'\).

### 7.2 Deficient partitions of finite empirical entropy

For \(x \in \text{Reg}(\Sigma^\mathcal{Z})\) set

\[
\tilde{H}(x) = -(1 - \rho(x)) \log(1 - \rho(x)) - \sum_{a \in \Sigma} s(x, a) \log s(x, a)
\]

with the usual convention that \(0 \log 0 = 0\) and logarithms are in base 2. This is just the entropy of the infinite probability vector whose coordinates are \(s_a(x)\)
and $1 - \rho(x)$. This quantity in general may be infinite, but by merging finite sets of atoms one can always reduce the entropy as much as one wants.

**Lemma 7.4.** There exists a factor map $\pi : \text{Def}(\Sigma^\mathbb{Z}) \to \text{Def}(\mathbb{N}^\mathbb{Z})$ such that $H(\pi(x)) < 2$ for every $x \in \text{Def}(\Sigma^\mathbb{Z})$. Furthermore, $\pi$ maps regular points to regular points.

One could replace the upper bound $H(\pi(x)) < 2$ by $1 + \varepsilon$ for any $\varepsilon > 0$, but one cannot ask for $H(\pi(x)) \leq 1$ because this is impossible in the case that $\rho(x) = 1/2$. An alternative approach would be to use Proposition 7.1 to decrease the defect, but possibly produce an irregular point.

**Proof.** Fix an ordering of $\Sigma$. Fix $x \in \text{Def}(\Sigma^\mathbb{Z})$ and partition $\Sigma$ into finite sets $\Sigma_1, \Sigma_2, \ldots$ inductively: writing $s(x, \Sigma_n) = \sum_{a \in \Sigma_n} s(x, a)$, we choose $\Sigma_1$ to be the shortest initial segment such that $s(x, \Sigma_1) > \frac{9}{10} \sum_{a \in \Sigma} s(x, a)$, and assuming we have chosen $\Sigma_1, \ldots, \Sigma_{n-1}$ choose $\Sigma_n$ to be the largest initial segment of $\Sigma \setminus \bigcup_{i < n} \Sigma_i$ such that $s(x, \Sigma_n) > \frac{9}{10} \sum_{a \in \Sigma \setminus \left( \bigcup_{i < n} \Sigma_i \right)} s(x, a)$. Since $\Sigma_n, \Sigma_{n+1} \subseteq \Sigma \setminus \left( \bigcup_{i < n} \Sigma_i \right)$ and $\Sigma_n$ takes up at least $9/10$ of the set on the right, it is clear that

$$\sigma(x, \Sigma_{n+1}) < \frac{9}{10} s(x, \Sigma_n)$$

so

$$\sigma(x, \Sigma_n) < \frac{1}{10^{n-1}} \rho(x)$$

Evidently the choice of the $\Sigma_n$ is measurable. Now define $\pi(x)$ by

$$\pi(x)_i = n \quad \text{if } x_i \in \Sigma_n$$

Clearly (using finiteness of $\Sigma_n$),

$$s(\pi(x), n) = \sum_{a \in \Sigma_n} s(x, a)$$

so $\sum_{n \in \mathbb{N}} s(\pi(x), n) = \sum_{a \in \Sigma} s(x, a) = \rho(x) < 1$, and $\pi(x)$ is deficient. Also, by the above $s(\pi(x), n) < \rho(x)/10^{n-1}$, so, using $-t \log t \leq 1/2$ for $t \in (0, 1]$,

$$-\sum_{n \in \mathbb{N}} s(\pi(x), n) \log(s(\pi(x), n)) < -\sum_{n=1}^{\infty} \rho(x) 10^{-n+1} \log \rho(x) 10^{-n+1}$$

$$< -\rho(x) \log \rho(x) \sum_{n=0}^{\infty} \frac{10^{-n}}{10^n} + \sum_{n=1}^{\infty} \frac{n \log_2 10}{10^n}$$

$$< -\frac{1}{2} \cdot \frac{10}{9} + \frac{10}{81} \cdot \log_2 10$$

$$< 0.9656\ldots$$

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Since also $-(1 - \rho(x)) \log(1 - \rho(x)) \leq 1/2$, we obtain \( \tilde{H}(\pi(s)) < 2 \).

Finally, if \( x \) is regular then so is \( \pi(x) \), since each symbol in \( \pi(x) \) corresponds to the occurrences of a finite set of symbols in \( x \).

The reason we are interested in partitions with finite empirical entropy is the following:

**Theorem 7.5.** For every countable alphabet \( \Sigma \), the shift-invariant Borel set \( \{ x \in \text{Reg}(\Sigma^\mathbb{Z}) : \tilde{H}(x) < 2 \} \) admits a 4-set generator.

This is a consequence of the more general Krieger-type theorem that we state and prove given in Section 8.

We summarize the discussion above in the language of partitions.

**Corollary 7.6.** Let \( \alpha \) be a countable partition of a Borel system \( (X, \mathcal{B}, T) \). Let \( X' \) denote the set of points that are \( \alpha \)-regular and \( \alpha \)-deficient. Then there exist partitions \( \alpha', \beta \in \sigma_T(\alpha) \) of \( X' \) such that every \( x \in X' \) is \( \alpha' \)-regular and \( \alpha' \)-deficient, \( \beta \) has only four sets, and \( \alpha' \in \sigma_T(\beta) \).

**Proof.** Compose the itinerary map \( \alpha_* \) with the map from Lemma 7.4 so that the image of \( X' \) is contained in the set \( Y \subseteq \mathbb{N}^\mathbb{Z} \) of deficient, regular points \( y \) satisfying \( \tilde{H}(y) < 2 \). Let \( \alpha' \) be the pull-back to \( X' \) of the standard generating partition of \( \mathbb{N}^\mathbb{Z} \). Now apply the last theorem to find a four-set partition for \( Y \), and let \( \beta \) be its pull-back to \( X' \).

### 7.3 Constructing the generator

**Theorem 7.7.** Let \( (X, \mathcal{B}, T) \) be a Borel system and \( \alpha = \{ A_i \}_{i=1}^\infty \) a partition. Then \( \text{def}(\alpha) \) admits a 16-set generator.

**Proof.** Fix the partition \( \alpha \). In the course of the proof we encounter various sets with respect to which the statistical properties of a given point are a-priori, not known. Every time we encounter such a set \( A \) we implicitly separate out the points that are null or divergent for it, and continue to work in the complement of \( \text{null}(A) \cup \text{div}(A) \). At the end we will be left with an invariant measurable set \( Y \subseteq \text{def}(\alpha) \) and a sequences of sets \( A_1, A_2, \ldots \subseteq \text{def}(\alpha) \setminus Y \) such that

\[
\text{def}(\alpha) = Y \cup \bigcup_{i=1}^\infty \left( \text{null}(A_i) \cup \text{div}(A_i) \right)
\]

By Lemmas 4.5 and 4.6 we can find disjoint invariant measurable sets \( A', A'' \) such that

\[
\bigcup_{i=1}^\infty \left( \text{null}(A_i) \cup \text{div}(A_i) \right) = \text{null}(A') \cup \text{div}(A'')
\]
By Theorems 5.1 and 6.3 there are 4-set generators $\gamma'$ and $\gamma''$ of null($A'$) and div($A''$), respectively. Below we shall construct an 8-set generator $\gamma$ for $Y$. It then will follow that $\gamma \cup \gamma' \cup \gamma''$ is a 12-set generator for def($\alpha$), as desired.

We turn to the construction.

First, separate out the points that are not $\alpha$-regular, i.e. points that are not regular for some set in the countable algebra generated by the $T$-translates of $\alpha$ (note that $x \in \text{def}(\alpha)$ only ensures that $x$ is regular for every atom of $\alpha$). Denote the complement of these points by $X'$.

Using Corollary 7.6, we find a 4-set partition $\beta$ of $X'$ and a countable partition $\alpha' \subseteq \sigma_T(\beta)$ such that every $x \in X'$ is $\alpha'$-deficient.

Applying Corollary 7.3 to $\alpha'$ with $\delta_k = 2^{-(k+1)}$, we obtain partitions $\alpha_k' \subseteq \sigma_T(\alpha')$ of such that every $x \in X'$ is $\alpha_k'$-divergent or else it is $\alpha_k'$-regular and satisfies $\rho(\alpha_k') < 2^{-(k+1)}$. We separate out the points in the divergent case. Let $Y$ denote the $T$-invariant set that is left after doing this for all $k$.

Now, $\bigcup_{k=1}^{\infty} \alpha_k'$ is an $\omega$-cover of $Y$, satisfies the hypothesis of Proposition 4.3, and is measurable with respect to $\sigma_T(\beta)$. Since $\beta$ has four sets, by Proposition 4.3 there exists an 8-set generator $\gamma$ for $T|_Y$, as claimed.

8 A generator theorem for countable partitions of finite empirical entropy

In this section we present a version of the Krieger generator theorem for sequences over a countable alphabet which, in a certain sense, have finite entropy. The main novelty is that the statement is “measureless”, and uses empirical frequencies. For points that are generic for an ergodic shift-invariant probability measure this is a slight improvement over the usual Krieger generator theorem, since it gives some additional control of the exceptional set. More significantly, it applies also in other cases. One non-trivial case is when the point is generic for a non-ergodic measure of finite entropy, but the entropy of the ergodic components is unbounded. Another interesting case occurs when a point has well-defined frequencies for all words but is not generic for any measure, e.g. the empirical frequencies of symbols do not sum to 1. It is this last case that is relevant in the proof of Theorem 1.1.

The theorem below is stronger than necessary for the application to Theorem 1.1 since for that purpose it would have been enough to find a finite generator for the set of sequences $x$ satisfying $\tilde{H}(\alpha_\omega(x)) < 2$. But we have not found a significantly simpler argument for this case. It is worth noting that recently Seward [22] proved a theorem of this type for probability-preserving actions (of arbitrary groups) using an elegant argument that bears some similarities to ours in the way data is “moved around an orbit”. However, he uses $\sigma$-additivity of the measure in an apparently crucial way to bound the probability of those symbols.
that require more than $n$ bits to encode, and this fails in our setting, where the (implicit) measures are only finitely additive. This appears to prevent his argument from working in the Borel category.

8.1 Coding shift-invariant data

The following will be used to encode information about the orbit of a point $x \in \Sigma^\mathbb{Z}$, i.e. information that is shift invariant. For instance, in a measure-preservation system it could be used to encode the ergodic component to which $x$ belongs, or, in our setting, the empirical frequencies of words in $x$. A similar coding result for shift-invariant functions was obtained in a more general setting in [24, Section 9].

It is convenient to consider the space of partially defined infinite sequences over a finite alphabet $\Sigma$, that is, elements of $\Sigma^I$ for $I \subseteq \mathbb{Z}$. Given $x \in \Sigma^I$, we define the shift on it by $Sx \in \Sigma^{SI}$, $Sx(i) = x(i+1)$. The space of partially defined sequences carries the usual measurable structure.

**Lemma 8.1.** Let $\Sigma$ be a finite alphabet, $f : \Sigma^\mathbb{Z} \to \{0,1\}^\mathbb{N}$ a shift-invariant function. Then to each $x, y \in \Sigma^\mathbb{Z}$ and $I \subseteq \mathbb{Z}$ with $\mathbb{P}(I) > 0$ one can associate $z = z(x,y,I) \in \{0,1\}^I$ measurably and equivariantly (i.e. $(Sx, Sy, SI) \mapsto Sz$), and such that $(x,z)$ determines $f(y)$.

**Proof.** Fix $y \in \Sigma^\mathbb{Z}$ and $I \in \mathbb{Z}$ with $\mathbb{P}(I) > 0$. Let $\varepsilon_n = 3^{-n}$ so that $\sum_{n=1}^{\infty} \varepsilon_n < 1$. Apply Lemma 3.7 to $x, I$, and $(\varepsilon_n)_{n=1}^{\infty}$. We obtain disjoint sets $J_0, J_1, J_2, \ldots \subseteq I$ with $\mathbb{P}(J_n) \geq \varepsilon_n \mathbb{P}(I)$ and in particular $J_n \neq \emptyset$. Let $J = \bigcup_{n=1}^{\infty} J_n$ and define $z \in \{0,1\}^I$ by $z|_{J_n} \equiv f(y)_n$ and $z|_{I \setminus J} \equiv 0$. Since $z$ determines $I$, and $x$ and $I$ determine $J_1, J_2, \ldots$, and $z|_{J_n}$ determines $f(y)_n$ for all $n$, we see that $(x,z)$ determines $f(y)$. Measurability and equivariance are immediate. \qed

8.2 A finite coding lemma

We require some standard facts from the theory of types. Let $\Delta$ be a finite set and for $x \in \Delta^n$ let $P_x \in \mathcal{P}(\Delta)$ denote the empirical distribution of digits in $x$, i.e.

$$P_x(a) = \frac{1}{n} \# \{1 \leq i \leq n : x_i = a\}$$

This is sometimes called the type of $x$. The type class of $x$ is the set of all sequences with the same empirical distribution:

$$T_x^n = T_x^n(\Delta) = \{ y \in \Delta^n : P_x = P_y \}$$

The set of type classes of sequences of length $n$ is

$$\mathcal{P}_n = \mathcal{P}_n(\Delta) = \{ P_y : y \in \Delta^n \}$$
The following standard combinatorial facts can be found e.g. in [9, Theorems 11.1.1 and 11.1.3]:

**Proposition 8.2.** For every finite set $\Delta$ and $n \in \mathbb{N}$,

$$|P_n| \leq (n+1)^{|\Delta|}$$

For every $x \in \Delta^n$,

$$\frac{1}{(n+1)^{|\Delta|}} \cdot 2^{nH(P_x)} \leq |T_x^n| \leq 2^{nH(P_x)}$$

It follows that

**Corollary 8.3.** For every finite set $\Delta$, $n \in \mathbb{N}$ and $h > 0$,

$$\#\{x \in \Delta^n : H(P_x) < h\} \leq O(n^{|\Delta|}) \cdot 2^{nh}$$

For $x \in \Delta^n$ it is convenient to introduce a $\Delta$-valued random variable $\xi_x$ whose distribution is $P_x$; i.e.

$$\mathbb{P}(\xi_x = a) = P_x(a)$$

Now suppose that $\Delta = \Delta_1 \times \Delta_2$. Write $\xi_1, \xi_2$ for the coordinate projections. These become random variables once a probability measure is given on $\Delta$. For $x \in \Delta^n$ we identify $x$ with the pair of sequences $(x_1, x_2) \in \Delta_1^n \times \Delta_2^n$ obtained from the first and second coordinates of each symbol, respectively. Then $P_x \in \mathcal{P}(\Delta_1^n \times \Delta_2^n)$ and $\xi_x = (\xi_1, \xi_2)$ is a coupling of $\xi_1, \xi_2$, which we denote for ease of reading by $(\xi_1^x, \xi_2^x)$.

Given a pair of discrete random variables $X, Y$, we use the slightly non-standard notation

$$H(X|Y = y) = -\sum_x \mathbb{P}(X = x|Y = y) \log \mathbb{P}(X = x|Y = y)$$

so that $H(X|Y) = \sum_y \mathbb{P}(Y = y)H(X|Y = y)$. We also use subscripts to indicate the probability distribution when necessary, as in $H_P(\xi_1^a|\xi_2^a = a)$.

Finally, we endow $\mathcal{P}(\Delta)$ with the $\ell^1$ metric: for $P, Q \in \mathcal{P}(\Delta)$ let

$$\|P - Q\| = \sum_{a \in \Delta} |P(a) - Q(a)|$$

**Proposition 8.4.** Let $\Delta = \Delta_1 \times \Delta_2$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $n$ the following holds. Let $P \in \mathcal{P}(\Delta)$ and let $I_1, \ldots, I_m \subseteq [1, n]$ be disjoint intervals such that $J = [1, n] \setminus \bigcup I_i$ satisfies $|J| > \varepsilon n$. Let $y \in \Delta_1^n$ be a fixed sequence, and let $\Lambda = \Lambda(y, I_1, \ldots, I_m) \subseteq \Delta^n$ denote the set of sequences

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Proof. Using the continuity of the entropy function and the marginal probability function on the simplex of measures on \( \Delta \), we can choose \( \delta_0 > 0 \) so that if \( Q \in \mathcal{P}(\Delta) \) and \( |Q - P| < \delta_0 \), then \( Q(\xi^1 = a) \neq 0 \) if and only if \( P(\xi^1 = a) \neq 0 \), and 

\[
|H_P(\xi^1 = a) - H_Q(\xi^1 = a)| < \varepsilon/2
\]

for these \( a \). We also assume that \( \delta_0 \log |\Delta_2| \leq \varepsilon/2 \).

Set \( \delta = \varepsilon\delta_0/3 \).

Fix \( y \) and consider \( x = (y, z) \) as in the statement. Consider \( u = |x|_J \) and \( v = x|_{[0,n]\setminus J} \) as new sequences. Note that 

\[
P_v = \sum \alpha_i \cdot P_{x|I_i}, \quad \text{where } \alpha_i = |I_i|/\sum |I_i|
\]

so 

\[
\|P_v - P\| \leq \sum \alpha_i \cdot \|P_{x|I_i} - P\| < \sum \alpha_i \delta = \delta
\]

Therefore 

\[
\|P_x - P_v\| \leq \|P_x - P\| + \|P - P_v\| < 2\delta
\]

Similarly 

\[
P_x = \frac{|J|}{n} P_u + (1 - \frac{|J|}{n}) P_v,
\]

so

\[
P_u = \frac{n}{|J|} (P_x - (1 - \frac{|J|}{n}) P_v)
\]

\[
= \frac{n}{|J|} (P_x - (1 - \frac{|J|}{n}) P_x + (1 - \frac{|J|}{n})(P_x - P_v))
\]

\[
= P_x + (\frac{n}{|J|} - 1)(P_x - P_v)
\]

\[
= P + (P_x - P) + (\frac{n}{|J|} - 1)(P_x - P_v)
\]

Since \( |J| > \varepsilon n \),

\[
\|P_u - P\| < \delta + (\frac{1}{\varepsilon} - 1)2\delta < \frac{3\delta}{\varepsilon} < \delta_0
\]

Now for \( a \in \Delta \) let \( J_a = \{ j \in J : z = a \} \). By choice of \( \delta_0 \) and the fact that 

\[
\|P - P_u\| < \delta_0
\]

we have \( J_a \neq \emptyset \) if and only if \( P(\xi^1 = a) \neq 0 \), and for such \( a \),

\[
|H(\xi^2_a|\xi^1_a = a) - H_P(\xi^2|\xi^1 = a)| < \frac{\varepsilon}{2}
\]
Writing $u = (u^1, u^2) \in \Delta_1^J \times \Delta_2^J$, this means that
\[ u^2|_{J_u} \in \{ w \in \Delta_2^{J_u} : H(P_w) < H_P(\xi^2|\xi^1 = a) + \frac{\varepsilon}{2} \} \]
so by Corollary 3.3 the number of choices for $u^2|_{J_u}$ is $O(|J_u|^{\Delta_2^J}) = 2^{H_P(\xi^2|\xi^1 = a) + \varepsilon/2}$. Multiplying over all $a$ such that $J_u \neq \emptyset$, the number of possible values for $u^2$ is
\[
\prod_a \{ z|_{J_u} : (x, z) \in \Lambda \} = \prod_a O(|J_u|^{\Delta_2^J})2^{H_P(X_1|X_2 = a) + \varepsilon/2}
\]
where in second line we used the identity $P(\xi_u^1 = a) = |J_u|/|J|$, and in the last line we used the fact that $\|P_a - P\| < \delta_0$ implies that $|P(\xi_u^1 = a) - P(X_1 = a)| < \delta_0$ and $H_P(\xi^2|\xi^1 = a) \leq \log |\Delta_2^J|$. Since we chose $\delta_0$ to satisfy $\delta_0 \log |\Delta_2| < \varepsilon/2$, the proof of the first statement is complete. The second statement follows, since by the assumption $|J| > \varepsilon n$ we have $O(n^{\Delta_1^J}|\Delta_2^J| = 2^{O(\log n)} = 2^{o(|J|)}$. \hfill \square

We shall require a slightly stronger version of the proposition above that works with the empirical frequencies of $k$-tuples, rather than of individual symbols. For $x = x_1 \ldots x_n$ and $k \leq n$ define the $k$-th higher block code of $x$ to be the sequence $x^{(k)} = x_1^{(k)} \ldots x_n^{(k)}$ where
\[ x_i^{(k)} = x_i x_{i+1} \ldots x_{i+k-1} \]

**Proposition 8.5.** Let $\Delta = \Delta_1 \times \Delta_2$. For every $\varepsilon > 0$ and $k$ there exists a $\delta > 0$ such that for every $n$ the following holds. Let $P \in P(\Delta^n)$ and let $I_1, \ldots, I_m \subseteq [1, n - k + 1]$ be disjoint intervals of length at least $\ell$ such that $J = [1, n - k + 1] \setminus \bigcup I_i$ satisfies $|J| > \varepsilon n$. Let $y \in \Delta_1^k$ be a fixed sequence, and let $A = A(y, I_1, \ldots, I_m) \subseteq \Delta^n$ denote the set of sequences $x = (y, z) \in \Delta^n$ whose first component is the given sequence $y$, and such that $\|P_{x^{(k)}} - P\| < \delta$ and $\|P_{y^{(k)}} - P\| < \delta$ for every $1 \leq i \leq m$. Then
\[ |\{ z|_J : (y, z) \in A \}| < O(n^{\Delta_1^{|\Delta_2^J|}}) \cdot 2^{|J|H_P(\xi^2|\xi^1) + \varepsilon} \]
In particular if $n$ is large enough relative to $\varepsilon$, then we can ensure
\[ |\{ z|_J : (y, z) \in A \}| < 2^{|J|H_P(\xi^2|\xi^1) + \varepsilon} \]

**Proof.** The idea of the proof is very similar to the previous one, we mention only the new ingredients.

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As before, using uniform continuity of the functions involved on the simplex of measures on $\Delta$, choose $\delta_0 > 0$ so that if $Q \in \mathcal{P}(\Delta^k)$ and $|Q - P| < \delta_0$ then $Q(\xi^1 = a) \neq 0$ if and only if $P(\xi^1 = a) \neq 0$, and $|H_P(\xi^2 | \xi^1 = a) - H_Q(\xi^2 | \xi^1 = a)| < \varepsilon/4$ for these $a$. Assume further that $k\delta_0 \log |\Delta| \leq \varepsilon/2$.

Choose $\delta$ small enough that the hypothesis implies $\|P_{x^{(k)}}|_{J_r} - P\| < \delta_0$. This argument is identical to the one in the previous proof and is based on writing $P_{x^{(k)}}$ as a convex combination of $P_{x^{(k)}}|_{J_r}$ and the $P_{x^{(k)}}|_{J_r}$.

Split $J$ into congruence classes modulo $k$: For each $0 \leq r < k$, let $J_r = J \cap (k\mathbb{Z} + r)$. Observe that $x^{(k)}|_{J_r}$ determines $x|_{J_r + [0,k-1]}$, which does not yet determine $x|_J$, but almost: one easily checks that $J \setminus (J_r + [0,k-1])$ is contained in $m$ intervals of length $k$ that share an endpoint with one of the intervals $I_i$, and these have total length at most $mk$. Therefore the symbols in $x$ that are not determined by $x^{(k)}|_{J_r}$ constitute at most a $mk/n$-fraction of the symbols in $[1,n]$. Since the intervals $I_1, \ldots, I_m$ each have length at least $\ell$ and are contained in $[1,n]$, we have $m \leq n/\ell$, so $mk/n$ can be made arbitrarily small by making $\ell$ large. Thus we can assume that for each $j$ the number of possibilities for $x|_{J \triangle (J_r + [0,k-1])}$ is at most $2^{O(n/2)}$.

Using the relation $P_{x^{(k)}}|_{J_r} = \sum_{i=0}^{k-1} \frac{1}{k} P_{x^{(k)}}|_{J_i}$ and $\|P_{x^{(k)}}|_{J_r} - P\| < \delta_0$, it follows that there is some $i$ with $\|P_{x^{(k)}}|_{J_i} - P\| < \delta_0$. Arguing exactly as in the previous proof, it follows that for this $i$,

$$\# \text{ possibilities for } z|_{J_i + [0,k-1]} = \# \text{ possibilities for } z^{(k)}|_{J_i}$$
$$= O(n^{\|\Delta_1\|\|\Delta_2\|} 2^{J_i/\ell}) + O(n^{\|\Delta_1\|\|\Delta_2\|} 2^{J_i + [0,k-1]/\ell})$$
$$= O(n^{\|\Delta_1\|\|\Delta_2\|} 2^{J_i/\ell}) + O(n^{\|\Delta_1\|\|\Delta_2\|} 2^{J_i/\ell})$$

where in the last line we used the fact that $k|J_i| \leq |J| + O(mk/n)$ and, as explained earlier, by making $\ell$ we can ensure $mk/n < \varepsilon/4$. Putting it all together, recalling that the number of possibilities for $x|_{J \setminus (J_r + [0,k-1])}$ is $2^{O(n/2)} < 2^{2|J|/2}$, we have the desired bound.

8.3 A relative generator theorem

In this section we consider regular points $x \in (\Sigma_1 \times \Sigma_2)^\mathbb{Z}$ with $\Sigma_1, \Sigma_2$ finite, writing them as $x = (y,z) \in \Sigma_1^\mathbb{Z} \times \Sigma_2^\mathbb{Z}$. Regularity means in particular that $x$ determines a distribution on $\Sigma_1 \times \Sigma_2$ by $P_x(a,b) = s(x,(a,b))$, and also a function $P_x^* : (\Sigma_1 \times \Sigma_2)^* \to [0,1]$ given by $a \mapsto s(x,a)$ which extends to a shift-invariant $\sigma$-finite probability measure $\mu_x$ on $(\Sigma_1 \times \Sigma_2)^\mathbb{Z}$ (this extensibility relies crucially on the fact that the alphabet is finite). As in the last section, we write $\xi_x = (\xi_x^1, \xi_x^2)$ for the random variable with distribution $P_x$. Regularity of
x = (y, z) implies regularity of y and z, so we have \( \xi^1_x = \xi_y \) and \( \xi^2_x = \xi_z \). Write

\[
H(x) = H(\xi_x)
\]

Extending the notation of the previous section we denote by \( x^{(k)} \) the (infinite) sequence whose \( i \)-th symbol is \( x_i^{(k)} = x_i x_{i+1} \ldots x_{i+k-1} \). Regularity of \( x \) implies that also \( x^{(k)} \) is regular for all \( k \), hence \( H(\xi_{x^{(k)}}) \) is defined. Since \( \xi_{x^{(k+m)}} \) is a coupling of \( \xi_{x^{(k)}} \) and \( \xi_{x^{(m)}} \) we have \( H(\xi_{x^{(k+m)}}) \leq H(\xi_{x^{(k)}}) + H(\xi_{x^{(m)}}) \) and so the limit

\[
h(x) = \lim_{k \to \infty} \frac{1}{k} H(\xi_{x^{(k)}})
\]

exists by sub-additivity. Of course, this is just the Kolmogorov-Sinai entropy of \( \mu_x \). In the same manner we define \( h(z) \), and set

\[
h(x|z) = h(x) - h(z)
\]

\[
= \lim_{k \to \infty} \left( \frac{1}{k} H(\xi_{x^{(k)}}) - \frac{1}{k} H(\xi_{z^{(k)}}) \right)
\]

\[
= \lim_{k \to \infty} \frac{1}{k} (H(\xi_{x^{(k)}}|\xi_{z^{(k)}}))
\]

which is, again, the entropy of \( \mu_x \) relative to the factor determined by the second coordinate.

**Theorem 8.6.** Let \( 2 \leq Q \in \mathbb{N} \) and \( \Sigma_1, \Sigma_2 \) finite alphabets. To every \( x = (y, z) \in (\Sigma_1 \times \Sigma_2)^\mathbb{Z} \), such that \( z \) is aperiodic and \( h(x|z) < \log_2 Q \), and to every \( I \subseteq \mathbb{Z} \) such that \( s^*(I) > \frac{1}{\log_2 Q} h(x|z) \), one can associate \( w \in \{1, \ldots, Q\}^I \) such that the map \( (x, I) \mapsto w \) is measurable and equivariant, and \( (P_x^*, z, w) \) determines \( x \) (equivalently, \( y \)).

The statement probably remains true if we replace the uniform density \( s^*(I) \) with \( s(I) \), but the uniform assumption allows for a simpler proof that is good enough for our application.

Theorem 8.6 falls short of being a true relative generator theorem, since in order to recover \( x \) from \( z, w \) we must also know \( P_x^* \). In the probability-preserving category, knowing \( P_x^* \) is analogous to knowing the ergodic component of \( x \), and the corresponding theorem would be one that gives a partition that generates for every ergodic component of the measure without guaranteeing that different ergodic components have distinct images under the itinerary map. This shortcoming can be overcome by encoding \( P_x^* \) in \( w \) (the information carried by \( P_x^* \) is invariant under the shift, so it can be coded efficiently using Lemma 8.1). But this would lengthen an already long proof, and we prefer to postpone this step to the more general theorem for countable partitions.

For simplicity we show how to prove the theorem using a larger output alphabet: we introduce two additional symbols, [ and ], and produce \( w \in \{1, \ldots, Q\}^I \).
\{1,\ldots,Q,[]\}^I$ with the desired properties. We comment at the end how to make do without the extra symbols.

In the proof we will build up $w$ gradually, starting with all symbols “blank”. Formally one could introduce a new symbol with this name and set $w_i = \text{blank}$ for $i \in I$. As the construction progresses we will re-define more and more occurrences of the “blank” symbols to have values from $\{1,\ldots,Q,[]\}$.

We omit the routine verification that the constructions are equivariant and measurable.

Choosing parameters $\epsilon, \delta, k$

By hypothesis $s^*(I) > h(x|z)$, so setting

$$\epsilon = \frac{1}{10 \log Q |\Sigma_1|} \left( s^*(I) - \frac{1}{\log Q} h(x|z) \right)$$

we have $\epsilon > 0$.

Choose $\delta$ associated to $\epsilon$ as in Proposition 8.5. We can assume that $\delta < \epsilon$.

Since

$$h(x|z) = \lim_{k \to \infty} \frac{1}{k} H(\xi_{x^{(k)}}|\xi_{z^{(k)}})$$

we can choose $k$ such that

$$\frac{1}{k} H(\xi_{x^{(k)}}|\xi_{z^{(k)}}) < h(x|z) + \epsilon \log Q$$

Choosing $I', I''$

Relying on the definition of $\epsilon$ and choosing a suitable small $0 < \eta_1 < \eta_2 < 1$, apply Lemma 3.6 to $z, I, \eta_1, \eta_2$. We obtain disjoint subsets $I', I'' \subseteq I$ satisfying

$$s^*(I') > \frac{1}{\log Q} h(x|z) + 7\epsilon \left[ \log Q |\Sigma_1| \right] \quad (9)$$

$$s^*(I'') > 3\epsilon \left[ \log Q |\Sigma_1| \right] \quad (10)$$

We will use each of these sets to encode a different portion of the word $y$. The first, $I'$, will be used to encode “most” (a $(1 - 3\epsilon)$-fraction) of the symbols of $y$, namely, those that we succeed in covering by intervals with good empirical statistics in a sense to be defined below. The second set, $I''$, will encode the remaining (at most $3\epsilon$-fraction) symbols of $y$.

Intervals with good empirical statistics

Observe that

$$P_{x^{(k)}} = \lim_{\ell \to \infty} P_{x^{(k)}}|_{[\ell,\ell]}$$

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in the pointwise sense (as functions on $\Sigma^k_1 \times \Sigma^k_2$). Since empirical frequencies are invariant under the shift, for every $i \in \mathbb{Z}$ the same limit holds with $S^i x$ in place of $x$. It follows that for every $i$ there exists an $\ell_0(i) \in \mathbb{N}$ such that

$$\| P_{x^{(k)}}|_{[i, i + \ell]} - P_{x^{(k)}} \| < \frac{1}{2} \delta$$

for all $\ell \geq \ell_0(i)$.

**The good scales $L_n$, intervals $J_r(i)$, and sets of candidate points $U_r$**

Choose $L_0 \geq 2$ large enough that every interval $J$ of length at least $L_0$ satisfies

$$\frac{1}{|J|} |J \cap I'| > \frac{1}{\log Q} h(x|z) + 6\varepsilon \left[ \log Q \left| \Sigma_1 \right| \right]$$

as can be done by (9). Define $L_1, L_2, \ldots \in \mathbb{N}$ by the recursion

$$L_{r+1} = \left\lceil 4L_r^2 / \varepsilon^4 \right\rceil$$

These will serve as the lengths of the intervals we deal with from now on. We abbreviate

$$J_r(i) = [i, i + L_r - 1]$$

For a given length $L_r$ we are only interested in points $i \in \mathbb{Z}$ for which this length is long enough to ensure good empirical statistics: set

$$U_r = \{ i \in \mathbb{Z} : L_r \geq \ell_0(i) \}$$

Thus, for $i \in U_r$ we have $\| P_{x^{(k)}}|_{[i, i + \ell_s]} - P_{x^{(k)}} \| < \delta/2$ for all $s \geq r$. Note that $U_1 \subseteq U_2 \subseteq \ldots$ and $\bigcup_{r=1}^{\infty} U_r = \mathbb{Z}$.

**Choosing the good intervals: $V_r, J_r, E_r$**

Below we will define, for every $r = 1, 2, 3, \ldots$, subsets

$$V_r \subseteq U_r$$

of “good” points and the associated family of intervals

$$\mathcal{J}_r = \{ J_r(i) \}_{i \in V_r}$$

whose union we denote

$$E_r = \cup \mathcal{J}_r = \bigcup_{i \in V_r} J_r(i)$$

Similarly let $\mathcal{J}_{<r} = \{ J_s(i) : i \in U_s, s < r \}$ and $E_{<r} = \cup \mathcal{J}_{<r} = \bigcup_{s<r} E_s$. 

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The construction will satisfy the following properties (note that in (4) the even and odd stages are treated differently):

1. \( V_r \subseteq U_r \setminus E_{<r} \).
2. For each \( r \), the collection of intervals \( \mathcal{J}_r = \{ J_i \}_{i \in V_r} \) is pairwise disjoint.
3. For each \( i \in V_r \),
   \[
   \frac{1}{|J_r(i)|} |J_r(i) \setminus E_{<r}| > 3\varepsilon
   \]
4. For odd \( r \), if \( i \in U_r \setminus (E_{<r} \cup E_r \cup E_{r+1}) \), then either
   \[
   \frac{1}{|J_r(i)|} |J_r(i) \setminus E_{<r}| \leq 3\varepsilon
   \]
   or
   \[
   \frac{1}{|J_{r+1}(i)|} |J_{r+1}(i) \setminus E_{<r+1}| \leq 3\varepsilon
   \]

For the construction we induct over odd \( r = 1, 3, 5, \ldots \) and at step \( r \) define \( V_r \) and \( V_{r+1} \). Fix an odd \( r \) and assume we have defined \( V_s \) for \( s < r \). Set

\[
U_r' = U_r \setminus E_{<r}
\]

Recall that \( z \) is the second component of \( x \) and is assumed to be aperiodic. Apply Lemma 3.3 to \( z \) to obtain an \( (L_r + L_{r+1}) \)-marker and let \( W_r \) denote the set of 1s in the marker, so \( W_r \) is unbounded above and the gap between consecutive elements in it is at least \( 2L_r \).

Let \( i, i' \in W_r \) be consecutive elements of \( W_r \). We define \( V_r \cap [i, i' - L_r] \) inductively: assuming we have defined \( i_p \) for \( 1 \leq p < q \), define \( i_q \) to be the least element of \( (U'_r \cap [i, i' - L_r]) \setminus \bigcup_{p < q} J_r(i_p) \) that satisfies \( \frac{1}{|J_r(i_q)|} |J_r(i_q) \setminus E_{<r}| > 3\varepsilon \). Stop when no such element exists. Since we chose \( i_p \) from the set \( U'_r \), (1) is immediate. Also, the intervals \( J_r(i_p) \) chosen for a given \( i, i' \in W_r \) are pairwise disjoint by construction, and since we only choose elements of \([i, i' - L_r)\) we have \( J_r(i_p) \subseteq [i, i') \), so the intervals are disjoint from those constructed from other consecutive pairs \( j, j' \in W_r \). This verifies (2). Property (3) and the first alternative in property (4) are immediate from the construction.

Now, to define \( V_{r+1} \), for each consecutive \( i, i' \in W_r \) do exactly the same in the intervals \([i' - L_r, i')\), using \( r + 1 \) instead of \( r \). Since this interval has length \( L_r < L_{r+1} \) we see that \( V_{r+1} \) will contain at most one element, namely the least element \( i \in (U'_r \cap [i' - L_r, i')) \setminus E_{<r+1} \) such that \( \frac{1}{|J_{r+1}(i)|} |J_{r+1}(i) \setminus E_{<r+1}| \geq 3\varepsilon \), if such an element exists (there can be no more because the interval \([i' - L_r, i')\) is shorter than the length of the interval \( J_{r+1}(i) \), so after one iteration of the induction there are no candidates left). Again (1) is automatic, (3) is like before, and the second alternative of (4) is clear (using \( U_r \subseteq U_{r+1} \)). As for (2), note
that the gaps between consecutive elements of $W_r$ are at least $L_r + L_{r+1}$, so if $i \in W_r$ and $j \in [i - L_r, i)$, then $J_{r+1}(j) \cap [i' - L_r, i') = \emptyset$ for every $i' \in W_r \setminus \{i\}$. This easily implies (2).

Decomposing $E_{<r}$ into components

Define a component of $E_{<r}$ to be an interval $J$ that satisfies

$$J = \cup \{ J' \in J_{<r} : J' \cap J \neq \emptyset \}$$

and which is minimal in the sense that no proper subinterval of $J$ satisfies the same condition. Clearly the intersection of components is a component, so by the minimality property any two components are either equal or disjoint. We remark that $J$ is just the union of intervals in the intersection graph of $J_{<r}$, where the graph is defined by connecting two intervals if they intersect nontrivially.

Lemma 8.7. Every component $[a, b] \subseteq E_{<r}$ is of the form $[a, b] = J_{r_1}(i_1) \cup J_{r_2}(i_2) \cup \ldots \cup J_{r_m}(i_m)$, where $r > r_1 > r_2 > \ldots > r_m$ and $a = i_1 < i_2 < \ldots < i_m$. In particular $m < r$ and $|a, b| \leq \sum_{s < r_1} L_s < (1 + \varepsilon/2)L_{r_1-1}$.

Proof. If two intervals from $J_{<r}$ intersect, then by properties (1) and (2) they do not have the same length, and the left endpoint of the shorter one lies inside the longer one but not vice versa. Thus if neither of the intervals is contained in the other, the shorter must protrude beyond the right side of the longer one.

Now fix a component $J$ of $E_{<r}$. Let $J_{r_1}(i_1) \in J_{<r}$ be the interval (necessarily unique by disjointness of the $V_i$s) that has the same left endpoint as $J$. It must be contained in $J$ because $J$ is a component. If $J_{r_1}(i_1) = J$ we are done, otherwise let $J_{r_1}(i_2)$ be the longest interval in $J_{<r}$ that intersects $J_{r_1}(i_1)$ non-trivially, and note that by the previous paragraph it must be shorter ($r_2 < r_1$), and contained in $J$ because $J$ is a component. Continuing inductively we exhaust $J$. The two last conclusions follow immediately from the first and (12). □

Lemma 8.8. Every $J \in J_r$ intersects at most $1 + L_r/L_1$ components of $E_{<r}$.

Proof. Let $J = J_r(i) \in J_r$ and let $J'_1, \ldots, J'_m$ denote the components in $E_{<r}$ that intersect it non-trivially. They are disjoint, and each has length $\geq L_1$ (because it contains some interval from $J_{<r}$). Also, all except possibly the rightmost component are contained in $J$ (the leftmost one must be contained in $J$ by property (1)). Therefore $|J'_j \cap J| \geq L_1$ for at least $m - 1$ of the intervals. Taken together, this shows that $L_r \geq (m-1)L_1$, which is what was claimed. □

Lemma 8.9. If $[a, b] \subseteq E_{<r}$ is a component, then $\left\| P_{x^{(k)}}|_{[a,b]} - P_{x^{(k)}} \right\| < \delta$. 

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Proof. Write $[a, b]$ as a union as in Lemma 8.7. Let $[a, c] = J_r(i_1) \subseteq [a, b]$. We know that $\left\| P_{x(k)}|_{[a, c]} - P_{x(k)}\right\| < \delta/2$, and

\[ P_{x(k)}|_{[a, b]} = \frac{c-a}{b-a} P_{x(k)}|_{[a, c]} + \frac{b-c-1}{b-a} P_{x(k)}|_{[c+1, b]} \]

By Lemma 8.7 $|c-a| > (1-\delta/2)|b-a|$, and the conclusion follows. \(\square\)

Lemma 8.10. For every $J = J_r(i) \in \mathcal{J}_r$,

\[ \frac{1}{|J \setminus E_{<r}|} |(J \setminus E_{<r}) \cap I_0| > \frac{1}{\log Q} h(x|z) + 5\varepsilon \left\lceil \log Q \right\rceil |1| \]

Proof. Let $J'_1, \ldots, J'_m$ be an enumeration of the components of $E_{<r}$ that intersect $J = J_r(i)$ non-trivially, and let $J''_1, \ldots, J''_m$ denote those maximal intervals in $J \setminus \bigcup_{j=1}^m J'_j$ whose length is $< \frac{1}{2} \sqrt{L_1}$. Then $m' \leq m + 1$ and by the previous lemma $m < 1 + L_r/L_1$, so the total length of the $J''_j$'s is

\[ \sum_{j=1}^{m'} |J''_j| \leq \frac{\sqrt{L_1}}{2} \cdot m' < \frac{\sqrt{L_1}}{2} \frac{L_r}{L_1} + 2 < \frac{L_r}{L_1} \]

By property (3), $|J_r(i) \setminus E_{<r}| > 3\varepsilon L_r$, and $L_r/\sqrt{L_1} < \varepsilon^2 L_r$ by (12), so

\[ \begin{align*}
|J_r(i) \setminus (\bigcup_{j=1}^m J'_j \cup \bigcup_{j=1}^{m'} J''_j)| &= |J_r(i) \setminus E_{<r}| - |\bigcup_{j=1}^{m'} J''_j| \\
&\geq |J_r(i) \setminus E_{<r}| - \frac{L_r}{\sqrt{L_1}} \\
&> (1-\varepsilon)|J_r(i) \setminus E_{<r}| 
\end{align*} \]

Each maximal interval in $J_r(i) \setminus (\bigcup_{j=1}^m J'_j \cup \bigcup_{j=1}^{m'} J''_j)$ has length at least $\frac{1}{2} \sqrt{L_1} > L_0$. Thus by (11) and the above,

\[ |(J_r(i) \setminus E_{<r}) \cap I'| > \left( \frac{1}{\log Q} h(x|z) + 6\varepsilon \left\lceil \log Q \right\rceil |1| \right) (1-\varepsilon)|J_r(i) \setminus E_{<r}| \]

\[ > \left( \frac{1}{\log Q} h(x|z) + 5\varepsilon \left\lceil \log Q \right\rceil |1| \right) \cdot |J_r(i) \setminus E_{<r}| \]

as claimed. \(\square\)

Encoding $y|_J$ for $J \in \mathcal{J}_r$.

We now define $w|_{I' \cap E_{<r}}$ by induction on $r$. Upon entering stage $r$, we will have already defined all symbols in $I' \cap E_{<r}$, and define all remaining symbols in
Our objective is that the information recorded at step $r$, together with $y|_{E_{<r}}$, $z$ and $P^*_x$, will suffice to recover $E_r$ and $y|_{E_r}$, no matter what additional information is written later in $w$. If this is accomplished then $w|_{I'}$, $z$ and $P^*_x$ uniquely determine $y|_{E_{<\infty}}$.

The actual encoding of information is done as follows. For each component $J$ of $E_{<r}$, we write symbols in the portion of $I' \setminus E_{<r}$ that falls within $J$. To identify this set we mark its beginning and end with brackets, and denote the remainder by $J'$. We want to define the symbols in $J'$ so as to determine $y|_J$.

We do this by first enumerating all possibilities for $J$ and $y|_J$ that are consistent with the region $J'$, and, if the actual word $y|_J$ is $N$-th in this list, we record this index $N$ in $J'$. To be able to carry this out, the number of possible values of $N$ must be less than the $Q|_{J'}$. The estimates below show that this is indeed the case.

Let $r$ be given and fix $J \in J_r$. First, let $i_{\text{min}} = \min((J \setminus E_{<r}) \cap I')$ and $i_{\text{max}} = \max((J \setminus E_{<r}) \cap I')$ and set $w_{i_{\text{min}}} = [ \ldots]$ and $w_{i_{\text{max}}} = ] \ldots$.

Next, we estimate the number of choices for $J$. Using (11) and property (2) of $V_r$,

$$3\varepsilon|L_r| = 3\varepsilon|J| \leq i_{\text{max}} - i_{\text{min}} \leq L_r$$

It follows from (12) that $i_{\text{min}}, i_{\text{max}}$ determine $r$. In order to determine $J$, it thus suffices to specify its left endpoint, whose distance from $i_{\text{min}}$ is at most $L_r$. Thus given $i_{\text{min}}, i_{\text{max}}$ there are at most $L_r$ possibilities for $J$ (this is a slight over-estimate but we can afford to make it).

Next, we estimate the number of choices for $y|_{J \setminus E_{<r}}$. Let $J'_1, \ldots, J'_m$ denote the components of $E_{<r}$ that intersect $J$ non-trivially. Let

$$F = \cup \{J'_i : J'_i \subseteq J\}$$

The union consists of all but at most one of the intervals $J'_i$, the possible exception occurring at the right end of $J$. By Lemma 8.9 each $J'_i \subseteq J$ satisfies

$$\|P_{x^{(k)}}|J'_i - P_{x^{(k)}}\| < \delta$$

and by definition of $U_r$ and $J_r(i)$,

$$\|P_{x^{(k)}}|J - P_{x^{(k)}}\| < \delta$$
Also, by Lemma 8.10, the complement of $F$ in $J$ is at least an $\varepsilon$-fraction of $J$. Thus by Proposition 8.5,

$$
\# \text{ possibilities for } y | J \setminus E_{<r} \leq \# \text{ possibilities for } y | J \setminus F \\
\leq 2^{|J \setminus F|(h(x|z)+\varepsilon)}
$$

Since $J \setminus E_{<r}$ and $J \setminus F$ differ by at most two intervals from $J_{<r}$, whose combined length is $\leq 2L_{r-1} < \frac{1}{2} \varepsilon^2 L_r$, and since by (3) we have $|J \setminus E_{<r}| > 3\varepsilon |J|$, we have

$$ |J \setminus F| \leq |J \setminus E_{<r}| + \frac{1}{2} \varepsilon^2 L_r < (1 + \varepsilon) |J \setminus E_{<r}| $$

so, using the trivial bound $h(x|z) \leq \log |\Sigma_1|$, we have

$$ \# \text{ possibilities for } y | J \setminus E_{<r} < 2^{(|J \setminus E_{<r}| + \varepsilon \log |\Sigma_1|)} $$

Combining the estimates above, the number of possibilities for the pair $(J, y | J \setminus E_{<r})$ satisfies

$$ \# \text{ possibilities for } (J, y | J \setminus E_{<r}) < L_r \cdot 2^{(|J \setminus E_{<r}| + \varepsilon \log |\Sigma_1|)} $$

The number of symbols we have available to write in is $|(J \setminus E_{<r}) \cap I'| - 2$ (since the symbols $i_{\text{min}}, i_{\text{max}}$ were used for the brackets), and by Lemma 8.10 we know that

$$ |(J \setminus E_{<r}) \cap I'| \geq \left( \frac{1}{\log Q} h(x|z) + 5\varepsilon \log_Q |\Sigma_1| \right) \cdot |J \setminus E_{<r}| $$

so, since we are using the alphabet $\{1, \ldots, Q\}$,

$$ \# \text{ different sequences we can produce } \geq \frac{Q^{\frac{1}{\log Q} h(x|z) + 5\varepsilon \log_Q |\Sigma_1|} - 2}{2^{(|J \setminus E_{<r}| + \varepsilon \log_Q \log_Q |\Sigma_1|)}} $$

Comparing these two expressions and noting that $|J \setminus E_{<r}| > 3\varepsilon L_r$ and $\log L_r / 3\varepsilon L_r < \varepsilon$, we find that there are enough undefined symbols in $J \setminus E_{<r}$ to uniquely encode $J$ and $y | J \setminus E_{<r}$.

**Decoding $w|I$.**

For each $r$ and $J \in J_r$, the symbols $[,]$ were only to surround an interval $[j, j'] \subseteq J$ which was later completely filled in with other symbols from $1, \ldots, Q$. It follows that the pattern of brackets in $w|I$ forms a legal bracket expression, i.e. each bracket has a unique matching one. Furthermore, as we noted during the construction, $j' - j$ determines the stage $r$ at which they were written. Thus
[j, j'] \cap E_{<r}$ can be recognized as the union of interiors of bracketed intervals contained in $[j, j']$, and the data in the pattern $(j, j'] \setminus E_{<r}) \cap I'$ together with $z$ and $P^*_x$ determines the (unique) interval $J \in J_r$ such that $j = \min(J \setminus E_{<r}) \cap I'$ and $j' = \max(J \setminus E_{<r}) \cap I'$, and also determines $y|_{J \setminus E_{<r}}$. In this way $w|_{I'}$ determines $E_{<\infty}$ and $y|_{E_{<\infty}}$.

**Encoding $y|_{Z \setminus E_{<\infty}}$**

It remains to encode $y|_{Z \setminus E_{<\infty}}$ in $w|_{I''}$. If $E_{<\infty} = Z$ there is nothing to do and we set $w|_{I''} \equiv 0$. Otherwise let $i \in Z \setminus E_{<\infty}$. Then it belongs to $U_r$ for all large enough $r$ and hence, for all large enough $r$, we have $i \in U_r \setminus E_{<r+2}$. By (3) either

$$\frac{1}{J_r(i)} |J_r(i) \setminus E_{<r}| \leq 3\varepsilon \quad \text{or} \quad \frac{1}{|J_{r+1}(i)|} |J_{r+1}(i) \setminus E_{<r+1}| \leq 3\varepsilon$$

It follows that

$$g(Z \setminus E_{<\infty}) \leq 3\varepsilon$$

Since $g(I''') > 3\varepsilon \lceil \log_Q |\Sigma_1| \rceil$ (equation (10)), we can apply Lemma 3.7 to $z$ and $I''$ to obtain disjoint $I'' \subseteq I''', i = 1, \ldots, \lceil \log_Q |\Sigma_1| \rceil$, with $g(I''') > 3\varepsilon$, and apply Lemma 3.8 to $z$ and each $I''$ to obtain an injection $f_i : Z \setminus E_{<\infty} \rightarrow I''$. For each $i \in Z \setminus E_{<\infty}$ represent $y_i \in \Sigma_1$ as a string $a_1 \ldots a_{\lceil \log_Q |\Sigma_1| \rceil}$ and set $w_{f_i(i)} = a_j$.

**Decoding $w|_{I''}$**

Since $E_{<r}$ can be recovered from $w$, we can recover $Z \setminus E_{<r}$ and hence the sets $I''$ and the injection $f_i$. Then for $i \in Z \setminus E_{<\infty}$ we recover $y_i$ by reading off the sequence $w_{f_1(i)}, w_{f_2(i)}, \ldots$.

**Reducing the alphabet from size $Q + 2$ to $Q$**

To make do with $Q$ symbols of output instead of $Q+2$, Choose long enough words $a_i, a_j \in \{1, \ldots, Q\}^*$ so that the SFT in $\{1, \ldots, Q\}^2$ that omits them has entropy greater than $h(x|z)$, and the words cannot overlap themselves or each other. Then we use them in place of the symbols $[,]$ and choose all other sequences in the encoding so that they omit $a_i, a_j$, i.e., so that they are admissible for the SFT defined by omitting these two symbols. We can arrange this SFT to be mixing, and all encoding can be seen to occur in long blocks, which makes this possible. The fact that enough legal sequences exist for the encoding can be ensured, because by choosing $a_i, a_j$ long enough, we can ensure that the topological entropy of the SFT omitting them is still larger than the empirical entropy of $y$. We omit the standard details (for the application to the main theorem of this paper, the version with $Q + 2$ symbols suffices).
8.4 Constructing the generator

Fix a countable alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$. Let * be a symbol not in $\Sigma$, let $
abla_n = \{\sigma_1, \ldots, \sigma_n, *\}$, let $\pi_n : \Sigma \to \nabla_n$ denote the map that collapses the symbols $\sigma_{n+1}, \sigma_{n+2}, \ldots$ to *, and extend $\pi_n$ pointwise to sequences. Thus $\pi_n(x) \in \nabla_n^\infty$ is the sequence

$$\pi_n(x) = \begin{cases} x_i & \text{if } x_i \in \{\sigma_1, \ldots, \sigma_n\} \\ * & \text{otherwise} \end{cases}$$

It is clear that if $x \in \nabla_n^\infty$ is regular then so is $\pi_n(x)$ for every $n$, so using the notation of the previous section we can define

$$h_n(x) = h(\pi_n(x))$$

Since $\pi_n(x)$ is obtained from $\pi_{n+1}(x)$ by merging occurrences of $\sigma_{n+1}$ and * into the symbol *, it is easy to see that $h(\pi_{n+1}(x)) \geq h(\pi_n(x))$ (either directly, or using the fact that $\pi_n$ is a factor map from $(\nabla_{n+1}, \mu_{\pi_{n+1}(x)}, S)$ to $(\nabla_n, \mu_{\pi_n(x)}, S)$, and that Kolmogorov-Sinai entropy is non-increasing under factors). Thus we can set

$$h(x) = \lim_{n} h_n(x) = \sup_{n} h_n(x)$$

For the same reason that $h_n(x)$ is non-decreasing, this definition of $h(x)$ is independent of the ordering of $\Sigma$: If we choose a different ordering $\Sigma = \{\sigma'_1, \sigma'_2, \ldots\}$ and define corresponding $\Sigma'_n$ and $\pi'_n$ and $h'_n(x)$, then for every $n$ we have $\pi_n(x) = \pi'_n(\pi'_n(x))$ for large enough $n'$, so $h_n(x) \leq h'_n(x)$, hence if $h'(x) = \sup h'_n(x)$ then $h(x) \leq h'(x)$, and reversing the argument we see the two are the same. Thus $h(x)$ does not depend on the particular ordering we chose for $\Sigma$ (although $h(\pi_n(x))$ does).

Finally, observe that $h(x) \leq \overline{H}(x)$. Indeed, write $P_x$ for the probability vector on $\nabla \cup \{*\}$ that gives mass $s_a(x)$ to $a \in \Sigma$ and $1 - \sum_{a \in \Sigma} s_a(x)$ to *. Note that $P_{\pi_n(x)} = \pi_n P_x$, which implies

$$h_n(x) = \inf_k H(\xi_{\pi_n(x)}(\theta)) \leq H(\xi_{\pi_n(x)}(\theta)) = H(P_{\pi_n(x)}) \leq H(P_x) = \overline{H}(x)$$

Therefore $h(x) = \lim_{n \to \infty} h_n(x) \leq \overline{H}(x)$.

**Theorem 8.11.** Let $Q \in \mathbb{N}$ and let $Y_Q \subseteq \nabla_{AP}$ denote the set of aperiodic and regular sequences $y$ satisfying $h(y) < \log Q$. Then $Y_Q$ has a $Q$-set generator.

Our aim is to construct an injective factor map $\tau : Y_Q \to \{1, \ldots, Q\}^\infty$. As in the proof of Theorem 6.6, we begin with $w = \tau(y)$ “blank”, and define it inductively.

Fix $x \in Y_Q$. We begin with a preliminary step, which we call step 0, in which we choose a rather sparse subset $I_0 \subseteq \mathbb{Z}$ and record all of the frequencies
$(s_u(x))_{u \in \Sigma^*}$ on $w|_{I_0}$ (in particular $w|_{I_0}$ determines the entropies $h(\pi_n(x)))$. We also determine a sequence of disjoint sets $I_1, I_2, \ldots \subseteq \mathbb{Z}\setminus I_0$ with having uniform densities which we will specify later. We do this in a manner that $I_0, I_1, \ldots$ can be recovered from $w$ irrespective of what is written later.

After the preliminary step is complete, we apply the relative generator theorem (Theorem 8.6) inductively to define $w|_{I_n}$ in such a way that given $\pi_{n-1}(x)$ and $P_{\pi_{n-1}}(x)$, we can recover $\pi_n(x)$ (in fact a minor modification of this strategy is necessary, see below).

**Definition of $\rho_n$**

For $n = 1, 2, 3, \ldots$ choose numbers $\rho_n$ in the range

$$\frac{h_n(x) - h_{n-1}(x)}{\log Q} < \rho_n < 1$$

(here $h_0(x) = 0$) and another rational number $0 < \rho_0 < 1$, in such a way that

$$\sum_{n=0}^{\infty} \rho_n < 1.$$  

This is possible since by hypothesis, $\sum_{n=1}^{\infty} (h_n(x) - h_{n-1}(x)) = h(x) < \log Q$.

**Encoding an aperiodic sequence in $w$**

The first thing we do will be to encode an aperiodic sequence in $w$. This sequence will be used in both the encoding and decoding of $w$, so we must ensure that it can be recovered no matter what additional data is later written to $w$.

Choose $M$ large enough that $1/M < \rho_0/4$ and let $w' \in \{0, 1\}^\mathbb{Z}$ denote the sequence derived from $y$ using Proposition 3.3 and parameter $M^2$, so $w'$ is aperiodic and the gaps between consecutive 1s in $w'$ is at least $M^2$.

For every $i$ such that $w'_i = 1$, set $w_i = w_{i+1} = \ldots = w_{i+M-1} = 1$.

Next, let $i < j$ be the positions of a pair of consecutive occurrences of 1 in $w'$, let $k$ be the largest index such that $i + kM < j$, and set $w_{i+M} = w_{i+2M} = \ldots = w_{i+kM} = 2$.

The point now has the property that the blank symbols appear in blocks of length at most $M - 1$, and each such block is preceded by a 2 and is terminated by either a 2 or the word $1^{M_2}$. Thus, no matter what symbols are eventually written in the blank sites in $w$, no new occurrences of $1^{M_2}$ can be formed, and $w'$ can be recovered:

$$w' = \begin{cases} 
1 & \text{if } 1^{M_2} \text{ occurs in } w \text{ at } i \\
0 & \text{otherwise}
\end{cases}$$

We estimate the density of undefined symbols in $w$: Since the gap between 1’s in $w'$ are at least $M^2$ we have $\overline{s}(w', 1) \leq 1/M^2$, so $s'(w, 1) = M \cdot s'(w', 1) \leq \frac{1}{M^2}$.
1/M. Also, since the distance between 2’s in $w$ is at least $M$, we have $s^*(w, 2) \leq 1/M$. Therefore by choice of $M$,

$$\sum s^*(w, \text{blank}) \geq 1 - s^*(w, 1) - s^*(w, 2) \geq 1 - \frac{2}{M} > 1 - \frac{1}{2}\rho_0$$

### Encoding the empirical distribution

Let

$$U = \{ n \in \mathbb{N} : w_n \text{ is blank} \}$$

and

$$\rho = 1 - s_n(U)$$

so $\rho < \frac{1}{2}\rho_0$ and, as explained above, $\rho$ can be recovered from $w$. Apply Lemma 3.3 to $(w', U, \frac{1}{3}\rho, 1 - \frac{1}{2}\rho)$ to obtain a subset $I_0 \subseteq U$ with

$$s_n(I_0) > \frac{1}{3}\rho$$

$$> 0$$

$$s_n(U \setminus I_0) > (1 - \frac{1}{2}\rho)s_n(U)$$

$$> 1 - \rho_0$$

The sets $I_0, U \setminus I_0$ are recoverable from $w$.

Choose a measurable map $f : Y_Q \to \{0, 1\}^\mathbb{N}$ such that $f(x)$ encodes the sequence of frequencies $(s(x, a))_{a \in \Sigma^*}$. This is a shift-invariant function, so we can apply Lemma 8.1 with the function $f$ to $w', x, I_0$ to define $w|_{I_0}$ in such a way that $w'$ and $w|_{I_0}$ determine $f(x)$, and hence the frequencies $(s(x, a))_{a \in \Sigma}$. Thus no matter what information is later written in $w$, we can use $w$ to recover $w', I_0$, hence $w|_{I_0}$, hence $(s(x, a))_{a \in \Sigma}$. In particular $w$ determines $h_n(x)$ and $\rho_n$ for all $n \geq 1$.

### Choosing $I_1, I_2, \ldots$

Since $\sum \rho_n < 1 - \rho_0 < s(U \setminus I_0)$ we can apply Lemma 3.7 to $(w', U \setminus I_0, (\rho_n/(1 - \rho_0))_{n=1}^\infty)$ to obtain disjoint subsets $I_1, I_2, \ldots \subseteq U \setminus I_0$ such that

$$s_n(I_n) \geq \frac{\rho_n}{1 - \rho_0} s_n(U \setminus I_0) > \rho_n$$

Since $w', U, I_0, \rho_n$ are all recoverable from $w$ no matter what is written later, the sets $I_n$ are recoverable as well.
Defining $\tilde{\pi}_n$

Let

$$\tilde{\pi}_n(x) = (\pi_n(x), w') \in (\Sigma_n \times \{0, 1\})^\mathbb{Z}$$

(note that $w'$ depends measurably and equivariantly on $x$).

Coding $\pi_n(x)$

For each $n = 1, 2, 3, \ldots$, observe that

$$h(\tilde{\pi}_n(x)|\tilde{\pi}_{n-1}(x)) = h(\pi_n(x)|\pi_{n-1}(x), w') \leq h(\pi_n(x)|\pi_{n-1}(x)) = h_n(x) - h_{n-1}(x) < \rho_n$$

Furthermore, $\tilde{\pi}_{n-1}(x)$ is aperiodic, since $w'$ is (this is the reason we introduced the $w'$: the sequence $\pi_n(x)$ might be periodic). Apply the relative generator theorem (Theorem 8.6) to $(y, z) = (\tilde{\pi}_n(x), \tilde{\pi}_{n-1}(x))$ and $I_n$. We obtain a pattern $w|I_n$ from which, together with $\tilde{\pi}_{n-1}(x)$ and $\tilde{P}_{\tilde{\pi}_{n-1}(x)}$, we can recover $\tilde{\pi}_n(x)$ and in particular $\pi_n(x)$.

Summary

Let us review what has transpired: We encoded an aperiodic sequence $w'$ in $w$, with the property that it can be recovered later no matter how the rest of $w$ is defined. The density of $w'$ itself indicates the density of symbols needed for the encoding, so constructions based on this number can be reproduced knowing only $w'$. Using this we reserved a low-density set $I_0$ of blank sites, and encoded the empirical distribution of $x$ in it. We then reserved subsets $I_1, I_2, \ldots$ of the remaining blank symbols of sufficient density that in $I_n$ one we could record the sequence $\pi_n(x)$, the coding being unequivocal given $\pi_{n-1}(x)$ (which is encoded in $I_{n-1}$).

Decoding

We have already said almost everything about this. From $w$ we can recover $w'$, hence $\rho$, hence $I_0$, hence the frequencies $(s_a(x))_{a \in \Sigma^*}$. These determine $P_{\pi_{n}(x)}$, and also $\rho_n$ and hence $I_n$. Now inducing on $n = 1, 2, \ldots$ we recover $\pi_n(x)$ from $\tilde{\pi}_{n-1}(x)$, $P_{\tilde{\pi}_{n-1}(x)}$ and $w|I_n$. The sequences $\pi_n(x), n = 1, 2, \ldots$, determine $x$.  

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9 Proofs of Corollaries 1.2 and 1.3

Here we fill in some details about Corollaries 1.2 and 1.3. We begin with Corollary 1.2. In [10, Theorem 1.5] it was shown that if $Y$ is a mixing shift of finite type of topological entropy $h$ and $(X, T)$ is a free Borel system whose invariant measures are all of smaller entropy, or if this holds with one exception which is Bernoulli of entropy $h$, then there is a $T$-invariant Borel set $X_0 \subseteq X$ such that $X \setminus X_0$ supports no invariant probability measure, and a Borel embedding $\pi : X_0 \rightarrow Y$.

Let $Y_1, Y_2, \ldots \subseteq Y$ be a pairwise disjoint sequence of mixing shifts of finite type (constructing such a sequence is elementary and we omit the details). We define a sequence of pairwise disjoint $T$-invariant Borel subsets $X_1, X_2, \ldots \subseteq X$ supporting no $T$-invariant probability measures, and Borel embeddings $\pi_i : X_i \rightarrow Y_i$, such that $\pi_0|_{X \setminus \bigcup X_i \cup \bigcup \pi_i}$ is an embedding of $X$ into $Y$.

Set $X_1 = X \setminus X_0$ and apply Theorem 1.1 to obtain an embedding $X_1 \rightarrow Y_1$. Let $X_2 = \pi_0^{-1}(\text{Image}(\pi_0) \cap \text{Image}(\pi_1))$. Note that $X_2 \subseteq X_0$ and hence $X_2 \cap X_1 = \emptyset$. Also, $\pi_1^{-1} \pi_0$ is an isomorphism of $X_2$ to a subset of $X_1$ and hence $X_2$ supports no invariant probability measures. Thus, we can apply Theorem 1.1 to obtain an embedding $X_2 \rightarrow Y_2$.

Proceeding inductively, we define $X_{n+1} = \pi_0^{-1}(\text{Image}(\pi_0) \cap \text{Image}(\pi_n))$, noting that it is disjoint from the previous sets because the $Y_i$ are disjoint; and is isomorphic to $(X_n, T|_{X_n})$ and hence supports no $T$-invariant probability measures. Thus, by Theorem 1.1 we can find an embedding $\pi_{n+1} : X_{n+1} \rightarrow Y_{n+1}$.

It is now easy to see that $\pi_0|_{X \setminus \bigcup X_i \cup \bigcup \pi_i}$ is a Borel injection and of course $T$-equivariant, as required.

Turning now to Corollary 1.3, we proceed similarly. By [10, Theorem 1.5] (together with the Ornstein isomorphism theorem to deal with the measure of maximal entropy), if $(X, T), (Y, S)$ are Borel systems as in the statement of Corollary 1.3 and for the same $h$, then there is a $T$-invariant Borel subset $X_0 \subseteq X$ and Borel embedding $\pi_0 : X_0 \rightarrow Y$. This can be improved to an embedding $X \rightarrow Y$ by the same argument above, using the fact that there are mixing SFTs embedded in $Y$. By symmetry there are also embeddings $Y \rightarrow X$. One now applies a Cantor-Bernenstein argument, as in [10, Proof of Proposition 1.4], to obtain an isomorphism.

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