CHARACTERISTIC FUNCTION OF SYMMETRIC DAMEK-RICCI SPACE

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Abstract. A Damek-Ricci space is a typical locally harmonic manifold, which is a generalization of the rank one symmetric space of the non-compact type. In this paper, we determine explicitly the characteristic function of a Damek-Ricci space by calculating the determinant of a Jacobi tensor.

1. Introduction

A Riemannian manifold \((M, g)\) is locally harmonic at \(p \in M\) if every volume density \(\sqrt{\det(g_{ij})}\) is a function of the Riemannian distance from \(p\) on some neighborhood of \(p\) in \(M\). There are several equivalent definitions for locally harmonic manifolds ([2], p. 156). One of them is the following:

Theorem 1. A Riemannian manifold \(M = (M, g)\) is locally harmonic at \(p \in M\) if and only if the equality

\[
\triangle \Omega_p = f_p(\Omega_p) \quad \left(\Omega_p = \frac{1}{2} r_p^2\right)
\]

holds for a certain smooth function \(f_p\) on \([0, \varepsilon(p))\), where \(\varepsilon(p)\) is the injectivity radius at \(p \in M\) and \(r_p\) is the Riemannian distance from \(p\).

It is known that the function \(f_p\) in Theorem 1 does not depend on the choice of \(p \in M\) ([2], Proposition 6.16). The function \(f = f_p\) \((p \in M)\) is called the characteristic function of a harmonic manifold \(M = (M, g)\). The characteristic function plays an important role in the geometry of harmonic manifolds and there are many applications such as [5, 8, 9].

The characteristic function has been determined previously for all the rank one symmetric spaces except for the Cayley projective plane and the Cayley...
hyperbolic plane \([8, 9]\). Very recently, this function of the Cayley projective plane and the Cayley hyperbolic plane has just been determined explicitly \([6]\).

Damek-Ricci spaces are important noncompact harmonic manifolds. Non-symmetric Damek-Ricci spaces are counterexamples of the Lichnerowicz conjecture: "every locally harmonic manifold is a locally symmetric space." and there are many nonsymmetric Damek-Ricci spaces, in addition to symmetric Damek-Ricci spaces. Complex hyperbolic space \(\mathbb{C}H^n(-1)\), quaternion hyperbolic space \(\mathbb{Q}H^n(-1)\), and Cayley hyperbolic plane \(\mathbb{C}H^2(-1)\) are all symmetric Damek-Ricci spaces.

Damek-Ricci spaces are harmonic, so they have the characteristic function. In \([1]\), the characteristic function of a Damek-Ricci space was discovered but it is not the explicit form. In this paper, we can get the explicit form of the characteristic function of a Damek-Ricci space by using another method. In this article, we shall prove the following theorem by using the Jacobi tensor.

**Theorem 2.** Let \(S\) be a symmetric Damek-Ricci space. Then, the characteristic function as a harmonic manifold is given by

\[
\triangle \Omega = 1 + \sqrt{\frac{\Omega}{2}}((n + m) \coth(\sqrt{\frac{\Omega}{2}}) + m \tanh(\sqrt{\frac{\Omega}{2}})).
\]

We aimed our paper to be self-contained as much as possible. The authors thank to the referee for the kind suggestions.

2. Preliminaries

In this section, we prepare a brief review on the geometry of the Damek-Ricci space.

A Damek-Ricci space is a one-dimensional extension of a generalized Heisenberg group and a Lie group with the Lie algebra of Iwasawa type. It is a solvable Lie group with a left invariant metric, and is a Riemannian homogeneous Hadamard manifold which is harmonic (see \([1]\) for details). Concretely, the definition of a Damek-Ricci space is the following \([1, 4]\).

**Definition.** Let \(\mathfrak{s}\) be a Lie algebra with inner product \(\langle \cdot, \cdot \rangle\) satisfying

\[
\mathfrak{s} = \mathfrak{o} \oplus \mathfrak{h} \oplus \mathfrak{a}
\]

orthogonally where \(\mathfrak{a}\) is a one-dimensional subspace of \(\mathfrak{s}\), \(\mathfrak{o} \oplus \mathfrak{h} = [\mathfrak{s}, \mathfrak{s}]\), and the linear maps

\[
J : \mathfrak{h} \to \text{End}(\mathfrak{o}), \quad J_X := J(X),
\]

\[
J_X^2 = -Id_\mathfrak{o}, \quad \forall X \in \mathfrak{h}
\]

are given. The simply connected Lie group \(S\) with the Lie algebra \(\mathfrak{s}\) is called a Damek-Ricci space.

We will use the notations \(n := \dim \mathfrak{o}\), \(m := \dim \mathfrak{h}\), and the Jacobi operator \(R_v(w) := R(w, v)v\) for all tangent vectors \(v, w \in TS\) in \([1]\) for a Damek-Ricci
space $S$. For a symmetric Damek-Ricci space, the Jacobi operator has constant eigenvalues with multiplicities [1]:

**Theorem 3.** Let $S$ be a symmetric Damek-Ricci space. Let $V + Y + sA$ be a unit vector in $s$ where $V \in \mathfrak{a}, Y \in \mathfrak{h}, sA \in \mathfrak{a}$. The eigenvalues and multiplicities of $R_{V + Y + sA}$ are 0, 1, $-1/4, n; -1, m$.

3. Proof of Theorem 2

This chapter uses the same method as [6]. First, we denote by $\gamma = \gamma(t)$ the normal geodesic in $(S, g)$ through the identity $e = \gamma(0)$ with the initial direction $\gamma'(0) = y_0$. By Theorem 3, there is an orthonormal basis $\{y_0, y_1, \ldots, y_n, y_1, \ldots, y_m\}$ of $s$ such that

$$R_{y_0}(y_j) = -\frac{1}{4}y_j, \quad R_{y_0}(y_k) = -y_k$$

for $1 \leq j \leq n$ and $1 \leq k \leq m$ when identifying $s$ and $T_e S$. Then there is a parallel frame field $\{y_1(t), \ldots, y_n(t), y_1(t), \ldots, y_m(t)\}$ along $\gamma$ such that

$$y_j(0) = y_j(1 \leq j \leq n), \quad y_k(0) = y_k(1 \leq k \leq m).$$

Since $S$ is locally symmetric, by (3.1) and (3.2),

$$R_{\gamma'}(y_j(t)) = -\frac{1}{4}y_j(t), \quad R_{\gamma'}(y_k(t)) = -y_k(t)$$

for $1 \leq j \leq n$ and $1 \leq k \leq m$.

Now, let $Y_i(t)$ ($1 \leq i \leq n$) and $Y_l(t)$ ($1 \leq l \leq m$) be the Jacobi vector fields along $\gamma$ satisfying the following conditions

$$Y_i(0) = 0, \quad Y_l(0) = 0,$$

$$Y_i'(0) = (\nabla_{\gamma'} Y_i)(0) = y_i, \quad Y_l'(0) = (\nabla_{\gamma'} Y_l)(0) = y_l$$

for $1 \leq i \leq n$ and $1 \leq l \leq m$. We set as follows along $\gamma$:

$$Y_i(t) = \sum_{j=1}^{n} a_{ji}(t)y_j(t) + \sum_{k=1}^{m} a_{ki}(t)y_k(t),$$

(3.5)

$$Y_l(t) = \sum_{j=1}^{n} a_{jl}(t)y_j(t) + \sum_{k=1}^{m} a_{kl}(t)y_k(t)$$

for $1 \leq i \leq n$ and $1 \leq l \leq m$.

Since $Y_i(t)$ ($1 \leq i \leq n$) and $Y_l(t)$ ($1 \leq l \leq m$) are Jacobi vector fields along the geodesic, from (3.5), taking account of (3.3), we have the following system
of differential equations along $\gamma$:

$$a''_{ji} - \frac{1}{4}a_{ji} = 0,$$

$$a''_{ki} = a''_{ji} = 0,$$

$$a''_{kl} - a_{kl} = 0$$

for $1 \leq i, j \leq n$ and $1 \leq k, l \leq m$.

Solving (3.6) under the initial conditions (3.4), we have

$$a_{ji}(t) = 2\delta_{ji} \sinh\left(\frac{t}{2}\right),$$

$$a_{ki}(t) = a_{jl}(t) = 0,$$

$$a_{kl}(t) = \delta_{kl} \sinh t$$

for $1 \leq i, j \leq n$ and $1 \leq k, l \leq m$.

Now, we define $(n + m) \times (n + m)$-matrix $A(t)$ by

$$A(t) = \begin{pmatrix} a_{ij} & a_{ik} \\ a_{lj} & a_{lk} \end{pmatrix}$$

for $1 \leq i, j \leq n$ and $1 \leq k, l \leq m$. Let $\Theta_e(q)$ be the volume density function of the geodesic sphere centered at $e$ through $q$ for each point $q$ in a normal neighborhood centered at $e$. Then, it is well-known that the following equality holds along the geodesic $\gamma$ for small $t$.

$$\Theta_e(\gamma(t)) = \det A(t)$$

from (3.8) with (3.7), we have

$$\det A(t) = \left(2 \sinh \frac{1}{2}t\right)^n (\sinh t)^m.$$ 

Thus, from (3.9) and (3.10), we have

$$\ln \Theta_e(\gamma(t)) = n \ln 2 + n \ln \sinh \frac{1}{2}t + m \ln \sinh t.$$ 

Here, since a Damek-Ricci space $S$ is a harmonic manifold, the volume density function $\theta_e$ (and hence, the function $\Theta_e$) is a radial function on a normal neighborhood $U_e$ centered at the identity $e$. Thus, $\Theta_e$ is determined by its value along the geodesic $\gamma$. Thus, from (3.11), we easily see that the function $\Theta_e$ is given by

$$\ln \Theta_e(q) = n \ln 2 + n \ln \sinh \frac{1}{2}t + m \ln \sinh t,$$

where $q = \gamma(t) \in U_e - \{e\}$ (3, p. 269).

Now, let $\phi(t)$ be a smooth function of $t$ ($0 < t < \epsilon$, $\epsilon > 0$), and consider the function $f(q)$ on $U_e$ defined by $f(q) = \phi(t)$, $t = d(e, q)$, $q \in U_e$. Then, the following equality holds as in [7] with the sign difference:

$$\triangle f = \phi''(t) + \left(\frac{\Theta_e(\gamma(t))}{\Theta_e(\gamma(t))}\right)' \phi(t), \quad q \in \gamma(t),$$
where $\triangle$ denotes the Laplace-Beltrami operator of $(S, g)$. Here, from (3.12), we get

\begin{align}
\frac{(\Theta_e(\gamma(t)))'}{\Theta_e(\gamma(t))} &= (\ln \Theta_e(\gamma(t)))' \\
&= \frac{n}{2} \coth \frac{1}{2} t + m \coth t \\
&= \frac{n + m}{2} \coth \frac{1}{2} t + \frac{m}{2} \tanh \frac{1}{2} t.
\end{align}

We here consider the special case where $\phi(t) = \frac{1}{2} t^2$ $(t > 0)$. Then, from (3.13) and (3.14), by direct calculation, we see that

\( \triangle \Omega = 1 + \sqrt{\Omega} \left\{ (n+m) \coth \frac{\Omega}{2} + m \tanh \frac{\Omega}{2} \right\} \)

holds on $U_e - \{e\}$. This completes the proof of Theorem 2.

**Remark 1.** From Theorem 2, taking account of the discussion in [6], we may reconfirm the statement that symmetric Damek-Ricci spaces are isometric to the complex hyperbolic space, the quaternionic hyperbolic space or a Cayley hyperbolic plane with the canonical Riemannian metrics, respectively ([1], p. 79).

**Remark 2.** In the proof, $y_j(t)$ or $\bar{y}_k(t)$ are eigenvectors of the Jacobi operator at each point along the geodesic since they are eigenvectors at the initial point and the Damek-Ricci space $S$ is locally symmetric. But, if $S$ is nonsymmetric, usually, neither $R_{\gamma(t)}(y_j(t))$ nor $R_{\gamma(t)}(\bar{y}_k(t))$ are parallel, so we need to calculate $R_{\gamma(t)}(y_j(t))$ and $R_{\gamma(t)}(\bar{y}_k(t))$. Thus, the following question remains.

**Question.** How can we determine the characteristic function of a nonsymmetric Damek-Ricci space by the Jacobi tensor in the explicit form?

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