Linear Magnetic Response of Disordered Metallic Rings: Large Contribution from Forward Scattering Interactions

Long Phi Chau and Peter Kopietz

Institut für Theoretische Physik, Universität Frankfurt, Robert-Mayer-Strasse 8, 60054 Frankfurt, Germany

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We calculate the effect of electron-electron interactions involving vanishing momentum transfer (forward scattering) on the orbital linear magnetic response of disordered metal rings pierced by a magnetic flux $\phi$. Using the bulk value of the Landau parameter $F_0$ for copper, we find that in the experiment by Lévy et al. [Phys. Rev. Lett. 64, 2074 (1990)] the forward scattering contribution to the linear magnetic response is larger than the corresponding contribution from large momentum transfers considered by Ambegaokar and Eckern [Phys. Rev. Lett. 65, 381 (1990)]. However, outside the regime of validity of linear response and to first order in the effective screened interaction the persistent current is dominated by scattering processes involving large momentum transfers.

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I. INTRODUCTION

More than a decade ago the measurement by Lévy et al.\textsuperscript{1} of persistent currents in mesoscopic normal metal rings pierced by an Aharonov-Bohm flux $\phi$ has triggered a lot of theoretical activity.\textsuperscript{2,3} Yet, up until now a truly convincing and generally accepted theoretical explanation of the surprisingly large persistent currents observed in Ref.\textsuperscript{1} and in subsequent experiments\textsuperscript{4,5} has not been found. It has become clear, however, that this effect cannot be explained within a model of non-interacting electrons. Ambegaokar and Eckern (AE)\textsuperscript{6} were the first to examine the effect of electron-electron interactions on mesoscopic persistent currents: they realized that, to first order in the screened Coulomb-interaction, the dominant contribution to the disorder averaged persistent current can be obtained from the two diagrams shown in Fig. \ref{fig:1}, representing a special correction $\Pi_{\text{AE}}(\phi)$ to the disorder averaged thermodynamic potential which depends strongly on the Aharonov-Bohm flux $\phi$. Here the overline denotes averaging over the disorder. Given the grand canonical potential $\Omega(\phi)$, the corresponding persistent current $I(\phi)$ can be obtained from the thermodynamic relation

$$I(\phi) = -e \frac{\partial \Omega(\phi)}{\partial \phi}. \quad (1)$$

In a bulk metal at high densities the bare Coulomb-interaction $V_0(q) = 4\pi e^2/q^2$ is strongly screened. A simple way to take the screening into account diagrammatically is the random-phase approximation (RPA). Following this procedure, AE approximated the effective interaction (in the imaginary frequency formalism) as follows

$$\nabla_{\text{RPA}}(q, i\omega) = V_0(q) \frac{1}{1 + \Pi_0(q, i\omega)V_0(q)}. \quad (2)$$

For momentum transfers $|q|$ small compared with the inverse elastic mean free path $\ell^{-1}$, and for frequency transfers $|\omega|$ small compared with the inverse elastic lifetime $\tau^{-1}$ the disorder averaged polarization is given by

$$\Pi_0(q, i\omega) \approx 2\nu_0 \frac{D_0 q^2}{D_0 q^2 + |\omega|}, \quad (3)$$

where $D_0$ is the diffusion coefficient and $\nu_0$ is the average density of states at the Fermi energy (per spin) in the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Feynman diagrams representing the flux-dependent part of the grand canonical potential to first order in the screened interaction. (a) Hartree diagram; (b) Fock diagram. Solid arrows represent non-interacting disorder averaged Green functions and thick wavy lines represent the effective density-density interaction. The Cooperon (shaded symbol) is defined in Fig. \ref{fig:2}.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Diagrammatic definitions of the Cooperon (C) and the Diffuson (D). A dashed line represents the covariance of the impurity potential.}
\end{figure}
absence of interactions. Note that \( \nu_0 = (\Delta_0 V)^{-1} \), where \( V \) is the volume of the system and \( \Delta_0 \) is the average level spacing (per spin) at the Fermi energy. It turns out that both diagrams in Fig. 1 are dominated by momentum transfers of the order of the Fermi momentum \( k_F \), which for a metallic system is large compared with \( \ell^{-1} \). Eqs. (2) and (3) are therefore not suitable for a quantitatively accurate calculation of persistent currents. To make some progress analytically, AE estimated the contribution from the diagrams in Fig. 1 by replacing the effective interaction by a constant

\[
\mathbf{V}_{\text{RPA}}(k - k', i\omega) \rightarrow \langle \mathbf{V}_{\text{RPA}}(k_F - k_F', i\omega) \rangle \equiv \mathbf{V},
\]

where \( \langle \cdots \rangle \) denotes the Fermi surface average over \( k_F \) and \( k_F' \). For simplicity, it is assumed that the ring is quasi one-dimensional, with transverse thickness \( L_\perp \) in the range \( k_F^{-1} \ll L_\perp \ll \ell \ll L \), where \( L \) is the circumference of the ring. Then diffusive motion is only possible along the circumference. At temperature \( T = 0 \) the resulting average persistent current can be written as

\[
\mathbf{T}^{AE}(\phi) = \sum_{k=1}^{\infty} I_k^{AE} \sin(4\pi k\phi/\phi_0),
\]

where \( \phi_0 = hc/e \) is the flux quantum and the Fourier coefficients of the current are

\[
I_k^{AE} = \frac{c 16\lambda_c}{\phi_0 k^2} E_c e^{-k\sqrt{\gamma}[1 + k\sqrt{\gamma}]},
\]

where \( \lambda_{\text{BCS}} < 0 \) is the attractive dimensionless interaction in the BCS model, and the coherence energy \( E_{\text{co}} \) is the smaller energy of \( \hbar/\ell \) and the Debye energy \( \hbar\omega_D \). Eq. (3) should be compared with the corresponding result for the local interaction model used by AE, which implies according to Eqs. (5) and (6),

\[
\frac{\partial \mathbf{T}^{AE}}{\partial \phi} \bigg|_{\phi=0} = \frac{c}{\phi_0^2} 32\pi \lambda_c E_c \ln \left( \frac{E_c}{\Delta_0} \right),
\]

where we have used \( \Gamma = \Delta_0/\pi \) and retained only the leading logarithmic order. Note that the logarithm is due to the slow decay \( \propto k^{-1} \) of the Fourier coefficients \( 4\pi k I_k^{AE}/\phi_0 \) of \( \partial \mathbf{T}^{AE}/\partial \phi \), so that all coefficients with \( k \lesssim 1/\sqrt{\gamma} \) contribute to the linear response. For \( E_{\text{co}} \gg E_c \) the linear magnetic response in the BCS model is parametrically larger than the linear response in the local interaction model. Whether or not this remains true beyond the linear response has not been clarified. Note also that in the BCS model the linear magnetic response is diamagnetic because the effective interaction is attractive \( (\lambda_{\text{BCS}} < 0) \), whereas the linear response in the local interaction model is paramagnetic, corresponding to a repulsive effective interaction \( (\lambda_c > 0) \).

II. MAGNETIC RESPONSE DUE TO FORWARD SCATTERING

An interesting observation made by the authors of Ref. 14 is that an effective interaction different from the local interaction used by AE can lead to a much larger persistent current, at least for sufficiently small flux \( \phi \), where it is allowed to calculate the current from the linear response. Given the rather crude approximations in the microscopic derivation of the local interaction model, it seems worth while to explore the magnetic response for other types of effective interactions. A possibility which so far has not been thoroughly analyzed is an interaction which is dominated by small momentum transfers. Note that the assumption that only forward scattering processes (corresponding to vanishing momentum transfer) have to be taken into account for a consistent description of the low-energy and long-wavelength properties of normal metals lies at the heart of the Landau’s Fermi liquid theory. The Landau model is in a sense the opposite extreme of the local interaction model, because the effective interaction in the Landau model is proportional to a Kronecker-delta in momentum space,

\[
\mathbf{V}_{\text{eff}}(q, i\omega) \rightarrow \delta_{q,0} f_0,
\]

where the Landau parameter \( f_0 \) can be determined from experiments. In fact, the dimensionless Landau parameter \( F_0 \equiv 2\nu_0 f_0 \) can be written as \( F_0 = \frac{B}{\mu_0 m_B} - 1 \), where \( B \) is the bulk modulus, \( m_B \) is the effective mass, and \( \mu_0 \) and \( m_0 \) are the corresponding quantities in the absence of interactions. Inserting the known bulk values
where $\langle i\omega \rangle = -\frac{2\Delta}{\pi} \sum_q \frac{1}{\hbar D_0 q^2 + |\omega| + \Gamma}$. \hfill (14)

Essentially we have used the equation of continuity to replace the charge vertices in Fig. 3 by current vertices, which cannot be renormalized by singular diffusion corrections. The fact that a gauge transformation replacing charge vertices by current vertices can be used to avoid the explicit calculation of vertex corrections has also been employed in Ref. 12 to calculate the zero bias anomaly in the tunneling density of states of two-dimensional disordered electrons interacting with Coulomb forces.

The evaluation of the contribution of the three Fock diagrams in Fig. 3 to the persistent current is now straightforward. Note that for a thin ring with $L_\perp \ll \ell \ll L$ the $q$-summation is one-dimensional, with quantized wavevectors $2\pi(n + 2\phi/\phi_c)/\ell$, $n = 0, \pm 1, \pm 2, \ldots$. Then we obtain for the $k$-th Fourier component of the average current due to the Fock diagrams (b)–(d) in Fig. 3 for the Landau model,

$$I_{F, k} = \frac{F_{k, Fock}}{k^{-1} f_0 \nu}.$$ \hfill (15)

Due to the extra factor of inverse volume, this contribution is, for experimentally relevant parameters, negligible compared with corresponding result in the local interaction model given in Eq. (6).

The Hartree diagram in the Landau model is more interesting. The fact that the diagram with two Cooperons shown in Fig. 3(a) dominates the persistent current due to electron-electron interactions with momentum transfers $|q| \lesssim \ell^{-1}$ has already been pointed out in Ref. 14. A similar diagram with two Cooperons (but without interaction line) dominates the fluctuations of the number of energy levels in a fixed energy window centered at the Fermi energy. Using the approximate relation

$$I_N(\phi) = -\frac{e}{2\Delta} \frac{\partial^2 (\delta N)^2}{\partial \phi^2}$$ \hfill (16)

between the persistent current $I_N(\phi)$ at constant particle number and the fluctuation $(\delta N)^2$ of the particle number at constant chemical potential $\mu$, several authors have realized that without interactions the two-Cooperon diagram determines the average persistent current in a canonical ensemble. Note that the Hartree diagram in Fig. 3(a) does not contain any vertex corrections analogous to the diffusion corrections of the vertices in the Fock diagrams (b)–(d). This is due to the fact that the interaction line in the Hartree process does not transfer any energy. Hence, the two Green functions attached to the vertex of a Hartree interaction are either both retarded or both advanced, so that it is impossible to attach a singular Diffuson to the vertex.

For the Landau model the Hartree diagram in Fig. 3(a) yields at finite temperature $T$ the following correction

\begin{align*}
\end{align*}
to the disorder averaged grand canonical potential,

\[
\Omega_k^{\text{L,Hartree}}(\phi) = \frac{f_0}{2V} \sum_{\mathbf{q}} T^2 \sum_{\omega_n} \theta(-\omega_n\omega_n') (2\pi/\Delta_0)^2 \left[ \frac{\hbar}{2}\frac{2}{\pi} \frac{\omega_n - \omega_n'}{\omega_n' + \tilde{\omega}(\mathbf{q})} \right]^2 \times \sum_{\mathbf{k}} [G_0(\mathbf{k}, i\omega_n)]^2 \tilde{G}_0(-\mathbf{k} + \mathbf{q}, i\omega_n') \times \sum_{\mathbf{k}'} [\tilde{G}_0(\mathbf{k}', i\omega_n')]^2 \tilde{G}_0(-\mathbf{k}' + \mathbf{q}, i\omega_n').
\]  

(17)

Here \( \tau = \ell/v_F \) is the elastic lifetime, \( \tilde{\omega}_n = 2\pi(n + \frac{1}{2})T \) are fermionic Matsubara frequencies, and

\[
\tilde{G}_0(\mathbf{k}, i\omega_n) = \frac{1}{\omega_n - \hbar k^2/2m + \mu + i\hbar/2\tau \text{sign}\omega_n}.
\]  

(18)

is the disorder averaged non-interacting Matsubara Green function. Since the Cooperons (i.e. the second line in Eq. 17) are only singular for \( |\mathbf{q}| \lesssim \ell^{-1} \), because the \( \mathbf{k} \) and \( \mathbf{k}' \) sums are dominated by momenta of the order of the Fermi momentum, we may approximate \( \tilde{G}_0(-\mathbf{k} + \mathbf{q}, i\omega_n') \approx \tilde{G}_0(-\mathbf{k}, i\omega_n') \) and \( \tilde{G}_0(-\mathbf{k}' + \mathbf{q}, i\omega_n') \approx \tilde{G}_0(-\mathbf{k}', i\omega_n) \) in Eq. 17. The product of the last two lines of Eq. 17 gives then rise to a factor of \( [(2\pi/\Delta_0)(\tau/\hbar)^2]^2 \), so that we obtain

\[
\Omega_k^{\text{L,Hartree}}(\phi) = \frac{f_0}{2V} P(\phi),
\]  

(19)

with the dimensionless coefficient

\[
P(\phi) = \frac{4T}{\pi} \sum_{0 < \omega_m < \hbar/\tau} \sum_{\mathbf{q}} \frac{\omega_m}{[hD_0\mathbf{q}^2 + \omega_m + 1]^2},
\]  

(20)

where \( \omega_m = 2\pi m T \) are bosonic Matsubara frequencies. Assuming again a thin ring with \( L \ll \ell \ll L \), we find in the limit \( T \to 0 \) for the Fourier components of the persistent current,

\[
I_k^{\text{L,Hartree}} = \frac{16}{\pi} \frac{f_0}{\phi_0} \frac{\hbar}{2V} e^{-k\sqrt{\tau}} = \frac{c}{\phi_0} 8F_0 \frac{\Delta_0}{\pi} e^{-k\sqrt{\tau}}.
\]  

(21)

Comparing this expression with the corresponding result of the local interaction model, we see that in the Landau model the Fourier components \( I_k^{\text{L,Hartree}} \) are independent of \( k \) as long as \( k \ll 1/\sqrt{\tau} \). Therefore the linear magnetic response is determined by all Fourier components up to \( k \lesssim \sqrt{E_c/\Gamma} \),

\[
\frac{\partial \Omega_k^{\text{L,Hartree}}}{\partial \phi} \bigg|_{\phi=0} = \frac{4\pi}{\phi_0} \sum_{k=1}^{\infty} k I_k^{\text{L,Hartree}} \approx \frac{c}{\phi_0} 16\pi F_0 E_c,
\]  

(22)

where we have assumed that \( \gamma = \Gamma/E_c \ll 1 \), so that

\[
\sum_{k=1}^{\infty} ke^{-k\sqrt{\tau}} = \frac{e^{\sqrt{\tau}}}{[1 - e^{-\sqrt{\tau}}]^2} \approx \frac{1}{\gamma} \approx \frac{\pi E_c}{\Delta_0}.
\]  

(23)

Note that the small energy scale \( \Delta_0 \) of Eq. 21 has disappeared on the right-hand side of Eq. 22, and is replaced by the much larger Thouless energy \( E_c \). Due to the faster decay of the Fourier components of the Fock contribution in the Landau model, the linear response due to the Fock diagrams shown in Fig. 3 (b)–(d) is a factor of \( \sqrt{\Gamma/E_c} \) smaller than the corresponding Hartree contribution. Interestingly, the anomalously large linear magnetic response in the BCS model given in Eq. 5 is also dominated by the Hartree process. Thus, the importance of Hartree interactions for persistent currents is some extent independent of a specific model for the interaction. For the Cu-rings used in the experiment, we estimate \( F_0 \approx 1.7 \), \( \lambda_c \approx 0.06 \) and \( E_c/\Delta_0 \approx 25 \); with these values the linear magnetic response due to forward scattering is more than four times larger than the linear response in the local interaction model considered by AE. To take both contributions into account one should parameterize the total effective interaction as

\[
V_{\text{eff}}(\mathbf{r} - \mathbf{r}') = V(\mathbf{r} - \mathbf{r}') + \frac{f_0 - V}{\sqrt{V}}
\]  

(24)

which in momentum space amounts to

\[
V_{\text{eff}}(\mathbf{q}) = \begin{cases} 
V & \text{for } \mathbf{q} = 0 \\
\frac{f_0}{\sqrt{V}} & \text{for } \mathbf{q} \neq 0
\end{cases}.
\]  

(25)

Using the estimate for bulk Cu given above, \( F_0 = 2\nu_0 f_0 \approx 1.7 \) and \( \lambda_c = \nu_0 \sqrt{V} \approx 0.06 \), we find \( f_0/\sqrt{V} \approx 14 \), which supports our assumption that there is indeed a strong enhancement of the effective interaction in the forward scattering channel.

III. CONCLUSIONS

In summary, we have shown that a forward scattering excess interaction, involving a vanishing momentum transfer, yields the dominant contribution to the linear magnetic response of mesoscopic metal rings for experimentally relevant parameters. On the other hand, outside the linear response regime the persistent current is dominated by the term \( V \) involving large momentum transfers, at least if we use the bulk estimates for \( V \) and \( F_0 \) for Cu. However, for a mesoscopic disordered Cu-ring it is not obvious that the bulk estimates are reliable. Note also that in the bulk the normal Fermi liquid is stable as long as \( F_0 > -1 \), so that for \( 0 > F_0 > -1 \) the linear magnetic response in the normal state can be diamagnetic, in spite of the fact that the effective coupling \( \lambda_c \) in the Cooper channel is positive. Moreover, in the vicinity of an s-wave Pomeranchuk instability22,23, where \( F_0 < 0 \) and \( |1 + F_0| \ll 1 \), one should replace \( F_0 \) by \( F_0/(1 + F_0) \). In this case we predict a strongly enhanced diamagnetic linear response. In fact, the forward scattering channel might then dominate the persistent current even beyond the linear order in the flux \( \phi \). To
clarify this point, a better microscopic theory of the effective electron-electron interaction in mesoscopic disordered metals is necessary. In particular, a microscopic theory should properly treat the problem of screening in a finite system and incorporate the breakdown of Fermi liquid theory in quasi one-dimensional disordered metals at sufficiently low temperatures.\textsuperscript{24}

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