Graphostructural modeling of discrete-argument systems

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Abstract. Graphostructural modeling is an effective tool for the analysis and control of social and economics systems. Graphs, hypergraphs and metagraphs are used to simulate complex hierarchical structures. In this paper we consider metagraphs and their matrices associated with 2D and 3D systems and cellular automatons; and their relationship with the transition from basic to associated models of cellular automatons. The transition to an associated model is a key step in developing the theory of discrete argument systems. In this paper with the simple examples of discrete-argument systems we demonstrate that matrix characteristics of the models, associated with such systems, are closely related to the matrix characteristics of associated metagraphs.

1. Introduction
The study [1] presents the math theory foundations of the wide class of the discrete-argument systems. The argument of such systems formalizes real time and space, on which real distributed objects, processes and systems evolve. The theory of discrete-argument systems is based on the conversion to associated discrete-time model with variable structure; this transformation allows one to apply results of classical Control Theory to discrete-argument systems.

The aim of this paper is to show on a simple examples, such as 2D-systems and 3D-systems and cellular automatons, that matrix characteristics of such systems are closely related to matrix characteristics (first of all, to incidence matrix) of discrete-argument graph structures, such as graphs, hypergraphs [2, 3, 4, 5, 6] and metagraphs [7, 8, 9, 10, 11, 12]. Since this task has already been solved for the case of 2D-systems, and the solution for 3D-systems is same as 2D, we show the decision in a short form; for the case of cell machine we show the decision in details.

2. Graphostructural modeling of 2D-systems and 3D-systems
2D-systems are the simplest representatives of subclass of discrete-argument multidimensional systems; the decision of the stated problem for this class of systems is given in the paper [13].

For the sake of completeness we will summarize the results below.

The base model of autonomic 2D-system is described with equation
\[ x[p, q] = \Phi_1 x[p - 1, q] + \Phi_2 x[p, q - 1]. \]

Associated model of 2D-system is described with equation
\[ x[t] = \Phi [t, t - 1] x[t - 1]. \] (1)
Figure 1. An example of 2D-system digraph.

Its matrix characteristics are described in [1]:

\[ \Phi_{1,0} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \]

\[ \Phi_{2,1} = \begin{bmatrix} \Phi_1 & 0 \\ \Phi_2 & \Phi_1 \\ 0 & \Phi_2 \end{bmatrix}. \]

The digraph of 2D-system and its transition to associated time is given on figure 1. Incidence matrices for the metagraphs, associated with 2D-system, are given in [13]

| x[p, q, r] | me₁ | me₂ | me₃ |
|------------|-----|-----|-----|
| x[0, 0]    | -1  |     |     |
| x[1, 0]    | 1   | -1  |     |
| x[0, 1]    | 1   |     | -1  |
| x[2, 0]    | 1   |     |     |
| x[1, 1]    | 1   | 1   |     |
| x[0, 2]    | 1   |     | 1   |

Metagraphs, associated with 2D-system, is shown on figure 2.

There are an accordance between bottom 2 × 1 submatrix of the first matrix and matrix \( \Phi_{1,0} \), an accordance between bottom 3 × 2 submatrix of the second matrix and matrix \( \Phi_{2,1} \), and so on. This observation could be continued.

One more simple example of multidimensional systems would be autonomic 3D-system, which are described by the equation

\[ x[p, q, r] = \Phi_1 x[p - 1, q, r] + \Phi_2 x[p, q - 1, r] + \Phi_3 x[p, q, r - 1]. \]

Its associated model is described by the same equation
Figure 2. An example of 2D-system metagraph.

\[ \dot{x}[t] = \Phi[t, t-1] x[t-1], \]

and its matrix characteristics are given in [1]:

\[ \Phi[1,0] = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}, \]

\[ \Phi[2,1] = \begin{bmatrix} \Phi_1 & 0 & 0 \\ \Phi_2 & \Phi_1 & 0 \\ \Phi_3 & 0 & \Phi_4 \\ 0 & \Phi_2 & 0 \\ 0 & \Phi_3 & \Phi_2 \\ 0 & 0 & \Phi_3 \end{bmatrix}. \]

Incidence matrix of the metagraph associated with 3D-system is

\[
\begin{array}{c|cccc}
\text{node} & m_{e_1} & m_{e_2} & m_{e_3} & m_{e_4} \\
\hline
x[0,0,0] & 1 & -1 & -1 & -1 \\
\end{array}
\]
There are an accordance between $3 \times 1$ submatrix of the incidence matrix (elements two to four of the first column) and matrix $\Phi[1, 0]$, an accordance between bottom $6 \times 3$ submatrix of the incidence matrix (elements from fifth to tenth of the second, third and fourth column) and matrix $\Phi[2, 1]$, and so on. This observation could be continued.

3. Graphostructural modeling of linear cellular automata

Linear cellular automaton (LCA) is described by the following equation [1]

$$x[t, c_1, c_2] = \Phi_1 \cdot x[t - 1, c_1, c_2] + \Phi_2 \cdot x[t - 1, c_1 - 1, c_2] + \Phi_3 \cdot x[t - 1, c_1, c_2 - 1] + \Phi_4 \cdot x[t - 1, c_1 + 1, c_2] + \Phi_5 \cdot x[t - 1, c_1, c_2 + 1].$$

Here $t$ is the LCA evolving time, $c_1, c_2$ – cell position (coordinates) in two-dimensional cellular space, $x[t; c_1, c_2]$ – LCA state vector.

Let us suppose, that start LCA configuration in cellular space, composed of single cell with coordinates 0, 0, and start LCA state $[0; 0, 0]$ are specified. On the first step of the LCA evolving the first configuration – LCA template $(1) = \{[0, 0], [-1, 0], [0, -1], [1, 0], [0, 1]\}$ in cellular space – will be completed; on the second step – second configuration $(2)$, and so on. Unlike multidimensional systems, it is not required to define associated time – LCA evolving time is used instead of it.

Associated LCA model is described by the equation (1) too; its matrix characteristics are given in [8]. We restrict ourselves to configurations (1) and (2).

The cell numbers of the configuration $P(1)$ are 0, 1, 2, 3, 4. The cells of the configuration $P(1)$ are $[1; 0, 0], [1; -1, 0], [1; 0, -1], [1; 1, 0], [1; 0, 1]$. Matrix $\Phi[1, 0]$ of the transition from configuration $P(0)$ to configuration $P(1)$ is obvious.

The cell numbers of the configuration $P(2)$ are 0, 1, 2, 3, ..., 12. The cells of the configuration $P(2)$ are $[2; 0, 0], [2; -1, 0], [2; 0, -1], ..., [2; 2, 0], [2; 0, 2]$. Matrix $\Phi[2, 1]$ of the transition from configuration $P(1)$ to configuration $P(2)$ is

$$\Phi[2, 1] = \begin{bmatrix}
\Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \\
\Phi_4 & \Phi_1 & \Phi_4 & \Phi_5 & \Phi_2 \\
\Phi_5 & \Phi_1 & \Phi_4 & \Phi_5 & \Phi_2 \\
\Phi_4 & \Phi_4 & \Phi_4 & \Phi_5 & \Phi_2 \\
\Phi_5 & \Phi_2 & \Phi_5 & \Phi_2 & \Phi_3
\end{bmatrix}.$$
Incidence matrix $I(MG)$ of the metagraph which is associated with LCA

| $x[1,0,0]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $x[1,-1,0]$ | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    |
| $x[1,0,-1]$ | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| $x[2,0,0]$  | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |

The incidence matrix $I(MG)$ of the associated metagraph has some advantages in comparison with $I(DG)$: it is more compact – it consists of 5 columns (that is 20 fewer); there are an accordance between elements from fifth to tenth of its first column and matrix $\Phi [1,0]$; an accordance between its bottom $13 \times 5$ submatrix and matrix $\Phi [2,1]$. It is notable that top submatrices (with -1) in all metagraph incidence matrices are minus unitary arrays. All observations could be continued.

4. Conclusion

So, in this paper on the simple examples of discrete-argument systems it was shown that matrix characteristics of the models, associated with such systems, are closely related to the matrix
characteristics of associated metagraphs.

In conclusion it should be noted, that metavertice of the metagraph is a subset of the set of its
vertices, consists of one or more elements. In other words, metavertices are the hyperedges of the
hypergraph with the same set of vertices. Metaedges of the metagraph are defined as a disordered
pairs of metavertices, like in a graph. Metaedges (or metaarcs) of the directed metagraph, by
analogy with digraph, are defined as an ordered pairs of metavertices: one of them is defined as
a beginning of the edge, and another – as a end. Metaedges determine the relationships between
metavertices and the nature of these relationships.

Metagraphs is one of the links in the chain from classical graphs to iterated hypergraphs,
which used to simulate complex hierarchical social and economic structures. Thus grapho-
structural modeling is a relevant mathematical tool for the analysis and control of social and
economic systems.

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