Discrete, q-difference deformations of associative algebras and integrable systems

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Abstract

Discrete and q-difference deformations of the structure constants for a class of associative noncommutative algebras are studied. It is shown that these deformations are governed by a central system of discrete or q-difference equations which in particular cases represent discrete and q-difference versions of the oriented associativity equation. It is demonstrated also that the celebrated Hirota-Miwa bilinear equation for the AKP and BKP hierarchies describes discrete deformations of certain finite-dimensional algebras.

1 Introduction

One of the approaches within the deformation theory for associative algebras proposed by Gerstenhaber in his seminal papers [1,2] consists in the treatment of ”the set of structure constants of an algebra in a given basis as parameter space for the deformation theory”. A remarkable class of such deformations was discovered by Witten [3], Dijkgraaf-Veride-Verlinde [4] and beautifully formalized by Dubrovin [5,6] in terms of Frobenius manifolds and subsequently by Hertling and Manin [7,8] in terms of F-manifolds (see also [9,10]).

A different method to describe classes of deformations for associative algebras, namely, coisotropic and quantum deformations has been proposed recently in [11-13]. For the quantum deformations [13] this approach consists 1) in putting the correspondence between the table of multiplication for an associative algebra in the basis $P_0, P_1, ..., P_{N-1}$, i.e.

$$P_j P_k = C_{jk}^l P_l, \quad j, k = 0, 1, ..., N - 1$$

and the set of operators

$$f_{jk} = -p_j p_k + \left( x^0, x^1, ..., x^{N-1} \right) p_l, \quad j, k = 0, 1, ..., N - 1$$
where $x^0, x^1, ..., x^{N-1}$ stand for the deformation parameters and summation over repeated index (from 0 to N) is assumed, 2) the requirement that the operators $p_0, p_1, ..., p_{N-1}$ and $x^0, x^1, ..., x^{N-1}$ are elements of the Heisenberg algebra, i.e.

$$[p_j, p_k] = 0, \quad [x^j, x^k] = 0, \quad [p_j, x^k] = \hbar \delta^k_j, \quad j, k = 0, 1, ..., N - 1$$

where $\hbar$ is a constant and $\delta^k_j$ is the Kronecker symbol and 3) the requirement that the functions $C_{jk}^l(x)$ are such that the set of equations

$$f_{jk} \mid \Psi \rangle = 0, \quad j, k = 0, 1, ..., N - 1$$

has a nontrivial common solution (Dirac’s prescription) where $\mid \Psi \rangle$ are elements of a certain linear space.

The requirement (4) gives rise to the set of equations (quantum central system (QCS)) [13]

$$\hbar \frac{\partial C_{jk}^l}{\partial x^l} - \hbar \frac{\partial C_{kl}^m}{\partial x^j} + C_{jk}^m C_{lm}^n - C_{kl}^m C_{jm}^n = 0, \quad j, k, l, n = 0, 1, ..., N - 1$$

which governs quantum deformations of the structure constants $C_{jk}^l(x^0, ..., x^{N-1})$. The QCS (5) has a simple geometrical meaning of vanishing Riemann curvature tensor and contains the oriented associativity equation, WDVV equation, Boussinesq equation, Gelfand-Dikii and Kadomtsev-Petviashvili (KP) hierarchies as the particular cases [13].

In the present paper we will define and study discrete and q-difference deformations of associative algebras. The basic steps of the construction are quite similar to those for quantum deformations. First, we “identify” the elements $P_0, P_1, ..., P_{N-1}$ of a basis and deformation parameters $x^0, x^1, ..., x^{N-1}$ with the elements of the algebra of shifts or q-shifts. Then, the requirement of existence of nontrivial common solutions of equations (4) provides us with the central system (DCS) which governs discrete or q-difference deformations of the structure constants $C_{jk}^l(x)$. This DCS is the discrete or q-difference version of the QCS (5). As the particular cases it contains the discrete and q-difference versions of the oriented associativity equations and other integrable equations. The construction provides us with the discrete version of the curvature tensor connected in a simple way with the associator of the algebra.

We demonstrate also that the discrete deformations of an algebra for which the multiplication of only distinct elements are admitied are described, in particular, by the discrete Darboux system and by the famous bilinear Hirota-Miwa equations for the AKP and BKP hierarchies.

The paper is organized as follows. In section 2 discrete and q-difference deformations of associative algebras are defined and the corresponding DCSs are derived. Discrete associator, discrete version of the curvature tensor and their interrelation are discussed in section 3. Reduction of the DCS to the
discrete versions of the oriented associativity equation are studied in section 4. Discrete deformations governed by the Hirota-Miwa bilinear equations are considered in section 5.

2 Discrete and q-difference deformations

Thus, we consider a finite-dimensional associative noncommutative algebra $A$ with (or without) unite element $P_0$. We will restrict ourselfs to a class of algebras which possess a basis composed by pairwise commuting elements. Denoting elements of such a basis as $P_0, P_1, ..., P_{N-1}$, one has the corresponding multiplication table

$$P_j P_k = C^l_{jk} P_l, \quad j, k = 0, 1, ..., N - 1$$

The commutativity of for the basis implies that $C^l_{jk} = C^l_{kj}$.

In order to define deformations $C^l_{jk}(x^0, x^1, ..., x^{N-1})$ of the structure constants we first identify the elements of the basis $P_j$ and deformation parameters $x^j$ with the elements of the algebra defined by the commutation relations

$$[p_j, p_k] = 0, \quad [x^j, x^k] = 0, \quad [p_j, x^k] = \delta^k_j (\hat{I} + p_j), \quad j, k = 0, 1, ..., N - 1$$

where $\hat{I}$ denotes the identity operator. It is easy to notice that the algebra of shifts, i.e. $\Delta_j = T_j - 1, T_j(x^k) = x^k + \delta^k_j, j, k = 0, 1, ..., N - 1$ with the rule $\Delta_j(f \cdot g) = \Delta_j f \cdot g + T_j f \cdot \Delta_j g$ is a realization of this algebra: $p_j = \Delta_j$.

Next steps are to introduce the operators

$$f_{jk} \equiv -p_j p_k + C^l_{jk}(x) p_l, \quad j, k = 0, 1, ..., N - 1$$

and to require that the functions $C^l_{jk}(x)$ are such that these operators have common nontrivial kernel or, equivalently, that the equations

$$f_{jk} |\Psi\rangle = 0, \quad j, k = 0, 1, ..., N - 1$$

are compatible. Here $|\Psi\rangle$ are elements of a linear space where the action of the operators $p_j$ and $x^j$ is defined.

**Definition.** The structure constants $C^l_{jk}(x)$ of the associative algebra $A$ are said to define deformations generated by the algebra (7) if the operators $f_{jk}$ given by (8) have a common nontrivial kernel.

This definition can be converted into the set of equations for the structure constants. The basic tools are given by the following two identities. The first is

$$[p_j, \varphi(x)] = \Delta_j \varphi(x) \cdot (\hat{I} + p_j), \quad j = 1, 2, ..., N - 1$$

where $\varphi(x)$ is an arbitrary function, $\Delta_j \varphi(x^0, x^1, ..., x^{N-1}) = (T_j - 1) \varphi(x^0, x^1, ..., x^{N-1})$ and $T_j \varphi(x^0, x^1, ..., x^{N-1}) = \varphi(x^0, x^j + 1, ..., x^{N-1})$. The second is
\[(p_j p_k) p_l - p_l (p_k p_l) = -p_l f_{jk} + p_l f_{kl} - T_l C_{jk}^m \cdot f_{lm} + T_l C_{kl}^m \cdot f_{jm} +\]
\[+ (\Delta j C_{jk}^m - \Delta j C_{kl}^m + T_j C_{jk}^m \cdot C_{lm} - T_j C_{kl}^m \cdot C_{jm}) p_n, \quad j, k, l = 1, 2, \ldots, N - 1 (11)\]

The identity (11) implies that

\[\{ (p_j p_k) p_l - p_l (p_k p_l) \} | \Psi \rangle = A^n_{klj}(x) p_n | \Psi \rangle = \]
\[= (\Delta j C_{jk}^n - \Delta j C_{kl}^n + T_j C_{jk}^n \cdot C_{lm} - T_j C_{kl}^n \cdot C_{jm}) p_n | \Psi \rangle \quad (12)\]

where \( | \Psi \rangle \subset \text{linear subspace } H_\Gamma \text{ defined by equations (9) and } A^n_{klj} \text{ is the associator for the algebra } A \). Weak associativity, i.e. the requirement that l.h.s. of (12) vanishes, means that the r.h.s. of (16) should vanishes too. This is valid for all values of the deformation parameters if

\[A^n_{klj} = \Delta j C_{jk}^n - \Delta j C_{kl}^n + T_j C_{jk}^n \cdot C_{lm} - T_j C_{kl}^n \cdot C_{jm} = 0, \quad j, k, l, n = 0, 1, \ldots, N - 1 (13)\]

If the subspace \( H_\Gamma \) does not contain elements linear in \( p_j | \Psi \rangle \), then equation (13) represents also the necessary condition.

Thus, we have

**Proposition.** The structure constants \( C^l_{jk}(x) \) define deformations driven by the algebra (7) if they obey equation (13).

We will refer to the system (13) as the discrete central system (DCS). We note that though the DCS (13) seems directly connected with the algebra of shifts, it defines deformations generated by the abstract algebra (7).

In a similar manner one defines q-difference deformations of associative algebras. Considering the same class of associative noncommutative algebras and following the same scheme, one, instead of the algebra (7), takes the algebra of q-shifts, i.e.

\[ [p_j, p_k] = 0, \quad [x^j, x^k] = 0, \quad [p_j, x^k] = \delta^k_j (\hat{I} + qx^j p_j), \quad j, k = 0, 1, \ldots, N - 1 (14)\]

where \( q \) is an arbitrary number. A realization of this algebra is given by

\[ p_j = \Delta q_j = \frac{1}{q x^j} (T_{qj} - 1) \quad (15)\]

where

\[ T_{qj}(x^k) = x^k + q \delta^k_j x^k, \quad j, k = 0, 1, \ldots, N - 1 \]

and

\[ T_{qj} \varphi(x^0, \ldots, x^{N-1}) = \varphi(x^0, \ldots, (1 + q)x^j, \ldots, x^{N-1}). \]

Since
\[ \Delta_{qj}(f \cdot g) = \Delta_{qj}f \cdot g + T_{qj}f \cdot \Delta_{qj}g \]

the central system which governs the deformations driven by the algebra (14) is quite similar to the DCS (13). It is

\[ \Delta_{ql}C^n_{jk} - \Delta_{qj}C^n_{kl} + T_{ql}C^m_{jk} \cdot C^n_{lm} - T_{qj}C^m_{kl} \cdot C^n_{jm} = 0. \]  (16)

In spite of this similarity solutions of the central systems (13) and (16) are rather different. So, algebras (7) and (14) generate quite different deformations of the structure constants for the same algebra.

We will refer to the algebra of the type (7) and (14) which generates deformations of associative algebra within our scheme as the Deformation Driving Algebra (DDA) to avoid possible confusion with the other already existing abbreviations like DGA (Deformation Generating Algebra) and so on (see e.g. [14]).

Similar to the other cases one can present the DCS (13) in a compact matrix form. Using the standard matrices \( C_j \), \( A^l_{ij} \) defined by \((C_j)_l^k = C^l_{jk}\) and \((A^l_{ij})_k^m = A^l_{klj} \), one rewrites the DCS (13) and (16) as

\[ A^d_{lj} \oplus \Delta_l C_j - \Delta_j C_l + C_l T_l C - C_j T_j C_l = 0 \]  (17)

and

\[ A^q_{lj} \oplus \Delta_q C_j - \Delta_q C_l + C_l T_q C - C_j T_q C_l = 0 \]  (18)

or

\[ A^d_{lj} = (1 + C_{l})T_{l} (1 + C_{j}) - (1 + C_{j})T_{j} (1 + C_{l}) = 0 \]  (19)

and

\[ A^q_{lj} = (1 + C_{l})T_{q} (1 + C_{j}) - (1 + C_{j})T_{q} (1 + C_{l}) = 0. \]

At last, we would like to notice that the DCS (13) which is the consequence of the weak associativity condition

\[ \{ (p_j p_k) p_l - p_j (p_k p_l) \} \mid |\Psi\rangle = 0 \]  (20)

at the realization \( p_j = \Delta_j \) eventually coincides with the compatibility condition for the linear problems \( f_{jk} \mid |\Psi\rangle = 0 \), i.e.

\[ \{ \Delta_j \Delta_k - C^d_{jk}(x) \Delta_l \} \mid |\Psi\rangle = 0. \]
3 Associator, discrete curvature and linear problems.

In the continuous limit \( \Delta_j \rightarrow \varepsilon \hbar \frac{\partial}{\partial x_j}, C_j \rightarrow \varepsilon C_j, \varepsilon \rightarrow 0 \) all the above equations are reduced to those for quantum deformations [13]. In particular, the DCS (13) (or (17)) is converted into the QCS (5) which has the geometrical meaning of vanishing Riemann curvature tensor \( R_{ijkl} \) given by the l.h.s. of equation (5) with the structure constants \( C_{jk} \) identified with the Christoffel symbols. We recall that in the continuous case the matrix \( R_{jk}^{\text{class}} \) with the matrix elements

\[
(R_{jk}^{\text{class}})_n^l = R_{njk}
\]

is the commutator

\[ R_{jk}^{\text{class}} = [\nabla_j, \nabla_k], \quad j, k = 0, 1, ..., N - 1 \]  
(21)

where \( \nabla_j = \hbar \frac{\partial}{\partial x_j} + C_j \) and the equation \( R_{jk}^{\text{class}} = 0 \) is equivalent to the compatibility condition for the linear problems

\[
\nabla_j \Psi = \left( \hbar \frac{\partial}{\partial x_j} + C_j \right) \Psi = 0, \quad j = 0, 1, ..., N - 1. \]  
(22)

In the continuous case the corresponding version of the relation (12) implies that the associator \( A_{jk}^{\text{class}} \) coincides with the Riemann curvature matrix \( R_{jk}^{\text{class}} \).

The situation is quite different in the discrete case. The discrete associator \( A_{jk}^d \) is given by the formula (17) or (19). In order to introduce the discrete analog of the curvature tensor we observe that equations (19) are equivalent to the compatibility condition for the linear system

\[
L_j | \Psi \rangle = \left( T_j - (1 + C_j)^{-1} \right) | \Psi \rangle = \left( \Delta_j - (1 + C_j)^{-1} C_j \right) | \Psi \rangle = 0, \quad j = 0, 1, ..., N - 1. \]  
(23)

Indeed, in virtue of the relation

\[
[L_j, L_k] | \Psi \rangle = \left\{ T_k (1 + C_j)^{-1} \cdot (1 + C_k)^{-1} - T_j (1 + C_k)^{-1} \cdot (1 + C_j)^{-1} \right\} | \Psi \rangle, \quad j, k = 0, 1, ..., N - 1 \]  
(24)

equations (23) are compatible if

\[
T_k (1 + C_j)^{-1} \cdot (1 + C_k)^{-1} - T_j (1 + C_k)^{-1} \cdot (1 + C_j)^{-1} = 0, \quad j, k = 0, 1, ..., N - 1. \]  
(25)

These equations are obviously equivalent to equations (19) provided all matrices \( 1 + C_j \) are nondegenerate. Here and in the rest of this section there is no summation over repeated indices. The relation (24) in analogy with the continuous case suggests to treat the expression in the bracket of the r.h.s., i.e.
\[ R_{jk}^d = T_k(1 + C_j)^{-1} \cdot (1 + C_k)^{-1} - T_j(1 + C_k)^{-1} \cdot (1 + C_j)^{-1} = \Delta_k(1 + C_j)^{-1} \cdot (1 + C_k)^{-1} - \Delta_j(1 + C_j)^{-1} \cdot (1 + C_j)^{-1} + [(1 + C_j)^{-1}, (1 + C_k)^{-1}] \] (26)

The relation (24) provided us with the weak definition of the curvature "tensor"

\[ R_{jk}^d \mid \Psi \rangle = [L_j, L_k] \mid \Psi \rangle, \quad j, k = 0, 1, ..., N - 1. \] (27)

Using the explicit form of \( L_j \), one derives the following operator expression for the discrete curvature "tensor"

\[ R_{jk}^d = [L_j, L_k] + \Delta_j(1 + C_k)^{-1} \cdot L_j - \Delta_k(1 + C_j)^{-1} \cdot L_k, \quad j, k = 0, 1, ..., N - 1. \] (28)

Comparing the formulae (19) and (26), one also concludes that

\[ A_{jk}^d = (1 + C_j)T_j(1 + C_k) \cdot R_{jk}^d \cdot (1 + C_k)T_k(1 + C_j). \] (29)

In the continuous limit \( T_j = 1 + \varepsilon \frac{\partial}{\partial x^j}, C_j \rightarrow \varepsilon C_j, (1 + C_j)^{-1} \rightarrow 1 - \varepsilon C_j, L_j \rightarrow \varepsilon \left( \frac{\partial}{\partial x^j} + C_j \right) \) and

\[ R_{jk}^d \rightarrow \varepsilon^2 [\nabla_j, \nabla_k] + \varepsilon^3 \ldots = \varepsilon^2 A_{jk}^{\text{class}} + \varepsilon^3 \ldots, \] (30)

\[ A_{jk}^d \rightarrow \varepsilon^2 A_{jk}^{\text{class}} + \ldots, \] (31)

and

\[ A_{jk}^{\text{class}} = R_{jk}^{\text{class}}. \]

The above formulae also indicate that in the situation in which one ignores the relation with associative algebras, it is natural to define a discrete curvature "tensor" associated the discrete "connection" \( L_j = \Delta_j + B_j \) as follows

\[ R_{jk}^d = [L_j, L_k] - \Delta_j B_k \cdot L_j + \Delta_k B_j \cdot L_k = \Delta_j B_k \cdot (1 - B_k) - \Delta_k B_j \cdot (1 - B_j) + [B_j, B_k], \]

\[ j, k = 0, 1, ..., N - 1. \] (32)

The connection with the original matrices \( C_j \) is given by \( (1 + C_j)(1 - B_j) = 1 \).

Finally, we present the linear problems with the spectral parameter \( \lambda \) for the DCS (19). It is

\[ L_j(\lambda) \mid \Psi \rangle = (T_j - \lambda(1 + C_j)^{-1}) \mid \Psi \rangle = 0. \] (33)
4 Discrete oriented associativity equation

For general discrete or q-difference deformations, similar to the quantum deformations [13], the global associativity condition \([C_j, C_k] = 0\) is not preserved for all values of the deformation parameters. Deformations of associative algebras for which the associativity condition is globally valid (associative deformations) form an important class of all possible deformations [5-13]. Within the theories of Frobenius and F-manifolds [5-10] and also for the coisotropic and quantum deformations [11-13] such deformations are characterized by the existence of a set of functions \(\Phi^l, l = 0, 1, ..., N - 1\) such that

\[
C^l_{jk} = \frac{\partial^2 \Phi^l}{\partial x^j \partial x^k}, \quad j, k, l = 0, 1, ..., N - 1. \quad (34)
\]

These functions obey the oriented associativity equation [5,15]

\[
\frac{\partial^2 \Phi^n}{\partial x^j \partial x^m} \frac{\partial^2 \Phi^m}{\partial x^j \partial x^k} - \frac{\partial^2 \Phi^n}{\partial x^j \partial x^m} \frac{\partial^2 \Phi^m}{\partial x^l \partial x^k} = 0, \quad j, k, l, n = 0, 1, ..., N - 1. \quad (35)
\]

Here we will present discrete versions of this equation. So, we consider the isoassociative deformations for which

\[
[C_j(x), C_k(x)] = 0, \quad j, k = 0, 1, ..., N - 1. \quad (36)
\]

For such deformations the DCS (17) takes the form

\[
\Delta_l C_j - \Delta_j C_l + C_l \Delta_l C_j - C_j \Delta_j C_l = 0 \quad (37)
\]

or

\[
(1 + C_l) \Delta_l (1 + C_j) - (1 + C_j) \Delta_j (1 + C_l) = 0. \quad (38)
\]

There is no summation over repeated indices in the formulae (37), (38) and also in the formula (43). General solution of these equations is

\[
C_j = g^{-1} \Delta_j g \quad (39)
\]

where \(g(x)\) is a matrix-valued function. Since \(C^l_{jk} = C^l_{kj}\) one has

\[
\Delta_j g^n_k = \Delta_k g^n_j, \quad j, k, n = 0, 1, ..., N - 1 \quad (40)
\]

and hence

\[
g^n_k = g^n_0 + \alpha \Delta_k \Phi^n, \quad k, n = 0, 1, ..., N - 1 \quad (41)
\]

where \(g^n_0\) and \(\alpha\) are arbitrary constants and \(\Phi^n\) are functions. Substitution of (39) and (41) into (36) gives

\[
\Delta_l \Delta_j \Phi^n \cdot (g^{-1})^l_m \Delta_j \Delta_k \Phi^m - \Delta_l \Delta_l \Phi^n \cdot (g^{-1})^l_m \Delta_j \Delta_k \Phi^m = 0, \quad j, k, l, n = 0, 1, ..., N - 1. \quad (42)
\]
Since in the continuous limit $\Delta_j \to \varepsilon \frac{\partial}{\partial x}, g^n_{0k} = \delta^k_n, \alpha = 0, \varepsilon \to 0$ the system (42) is reduced to (35), it represents a discrete isoassociative version of the oriented associativity equation.

Different discrete version of equation (35) arises if one relaxes the condition (36) and requires that the following quasi-associativity condition

$$C_lT_lC_j = C_jT_jC_l, \quad j, l = 0, 1, ..., N - 1$$

(43)
is valid for all values of deformation parameters. In this case the DCS (17) is reduced to the system

$$\Delta_lC_j - \Delta_jC_l = 0, \quad j, l = 0, 1, ..., N - 1$$

which implies the existence of the matrix-valued function $\Phi$ such that

$$C_j = \Delta_j\Phi, \quad j = 0, 1, ..., N - 1.$$  

Since $C_{jk} = \Delta_j\Phi_k = C_{kj}$ one has

$$\Phi_l^k = \Delta_k\Phi_l^k, \quad l, k = 0, ..., N - 1$$

where $\Phi_l^k, l = 0, 1, ..., N - 1$ are functions. So

$$C_{jk}^l = \Delta_j\Delta_k\Phi_l^k.$$  

Finally, the quasi-associativity condition (43) takes the form

$$\Delta_j\Delta_kT_l^m\Phi^m - \Delta_l\Delta_m\Phi^m - \Delta_l\Delta_kT_j^m\Phi^m - \Delta_j\Delta_m\Phi^m = 0, \quad j, k, l, n = 0, 1, ..., N - 1.$$  

(44)

which is a discrete version of the oriented associativity equation (35). Any solution of the systems (42) and (44) defines discrete deformation of the structure constants $C_{jk}^l$.

In a similar manner one derives q-difference versions of the oriented associativity equation.

5 Hirota-Miwa bilinear equations, discrete Darboux system and discrete deformations

Here we will study discrete deformations of algebras for which products of only distinct elements of the basis $P_0, P_1, ..., P_{N-1}$ are defined and the table of multiplication is of the form

$$P_jP_k = C_{jk}^l P_l + C_{jk}^0 P_0, \quad j \neq k, j, k = 0, 1, ..., N - 1.$$  

(45)

Deformation driving algebra is given by the commutation relations

$$[p_j, p_k] = 0, \quad [x^j, x^k] = 0, \quad [p_j, x^k] = \delta^k_j p_j, \quad j, k = 0, 1, ..., N - 1.$$  

(46)
Algebra of shifts $p_j = T_j$ gives a realization of this abstract algebra. For any three distinct indices $j,k,l$ one has a closed subtable (45). Denoting these three indices as 1,2,3, we present a corresponding subtable of multiplication as

$$P_1P_2 = AP_1 + BP_2 + LP_0,$$

$$P_1P_3 = CP_1 + DP_3 + MP_0,$$

$$P_2P_3 = EP_2 + GP_3 + NP_0.$$  

Discrete central system for the structure constants $A,B,...,N$ is given by the system of equations

$$A_3 = \frac{C_2}{C}, \quad B_1 = \frac{E_1}{E}, \quad D_2 = \frac{G_1}{G},$$  

$$A_3 - E_1L + B_3N - G_1M = 0, \quad (A_3 - G_1)D - E_1A - N_1 = 0,$$

$$C_2 - E_1D + B_3G + L_3 = 0, \quad (C_2 - G_1)L + D_2N - G_1M = 0,$$

$$E_3 + G_1N + (D_2 - B_3)N = 0$$  

where we denote $A_j = T_jA, B_j = T_jB$ and so on. Equations (50) imply that there exist three functions $U,V,W$ such that

$$A = \frac{U_2}{U}, B = \frac{V_1}{V}, C = \frac{U_3}{U}, D = \frac{W_1}{W}, E = \frac{V_3}{V}, G = \frac{W_2}{W}.$$  

We will consider here three different reductions of this general system. The first reduction is associated with the constraints

$$A + B + L = 1,$$

$$C + D + M = 1,$$

$$E + G + N = 1.$$  

In the terms of the functions $H^1, H^2, H^3$ defined by

$$U = H^1_1, V = H^2_2, W = H^3_3$$  

the DCS (50-53) under the constraints (55-57) takes the form

$$H^j_{ik} - \frac{H^k_i}{H^k_i}H^j_{ik} - \frac{H^k_i}{H^k_i}H^j_{ik} + \frac{H^k_i}{H^k_i}H^j + \frac{H^k_i}{H^k_i}H^j - H^j = 0$$  

or equivalently

$$\Delta_l\Delta_kH^j - \frac{\Delta_lH^k_i}{H^k_i} \cdot \Delta_kH^j - \frac{\Delta_kH^l_i}{H^l_i} \cdot \Delta_lH^j = 0$$  

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where indices $j,k,l=1,2,3$ are all distinct. It is the well-known discrete Darboux system which was first derived in [16] and then found various applications in the discrete geometry (see e.g. [17,18]).

For the general $n$-dimensional case (45) with the constraints

$$C^k_{jk} + C^j_{jk} + C^0_{jk} = 1, \quad j \neq k, \quad (61)$$

one has

$$C^k_{jk} = \frac{H^k_{kj}}{H^k_j}, \quad C^j_{jk} = \frac{H^j_{kj}}{H^j_j}, \quad C^0_{jk} = 1 - \frac{H^k_{kj}}{H^k_j} - \frac{H^j_{kj}}{H^j_j} \quad (62)$$

and the DCS is given by equations (59) or (60).

So, the discrete Darboux system describes discrete deformations of the structure constants for the algebras of the above type. Interrelation between such algebras and geometrical constructions for the quadrilateral lattices will be discussed elsewhere.

Second reduction is given by the constraints

$$L = M = N = 0, \quad (63)$$

$$A + B = 0, \quad C + D = 0, \quad E + G = 0. \quad (64)$$

Under these constraints the DCS (50-53) becomes

$$\frac{A_3}{A} = \frac{C_2}{C} = \frac{E_1}{E}, \quad (65)$$

$$A_3C + E_1C - E_1A = 0. \quad (66)$$

Equations (65) imply that there exists a function $\tau$ such that

$$A = -\frac{\tau_1\tau_2}{\tau\tau_{12}}, \quad C = -\frac{\tau_1\tau_4}{\tau\tau_{13}}, \quad E = -\frac{\tau_2\tau_4}{\tau\tau_{23}} \quad (67)$$

where $\tau_j = T_j\tau$ etc. In terms of the function $\tau$ equation (66) takes the form

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} = 0. \quad (68)$$

It is the famous Hirota discrete bilinear equation for the Kadomtsev-Petviashvili (AKP) hierarchy [19]. Thus, the Hirota bilinear equation governs the discrete deformations of the structure constants from the table of multiplication (47-49) under the constraints (63-64).

It is worth to note that these deformations are isoassociative one. Indeed, the only associativity condition for the ”algebra” (47-49) with the constraints (63-64) is given by the relation

$$AC + EC - AE = 0. \quad (69)$$
The Hirota equation (68) is exactly the associativity condition (69) with the structure constants A,C,E given by the formulae (67).

The third reduction corresponds to the constraints

\[ L = M = N = 1, \]  

\[ A + B = 0, \quad C + D = 0, \quad E + G = 0. \]  

(70)  

(71)

In this case the DCS (50-53) becomes

\[ \frac{A_3}{A} = \frac{C_2}{C} = \frac{E_1}{E}, \]  

\[ A_3 C + E_1 C - E_1 A - 1 = 0. \]  

(72)  

(73)

Equations (72) again lead to the expressions (67) for A,C,E. Equation (73) takes the form

\[ \tau_1 \tau_{23} - \tau_2 \tau_{13} + \tau_3 \tau_{12} - \tau \tau_{123} = 0. \]  

(74)

This equation is nothing but the Hirota-Miwa bilinear discrete equation for the KP hierarchy of B type (BKP hierarchy) [20]. So, the Hirota-Miwa equation (74) together with the formulae (67) describe discrete deformations of the "algebra" (47-49) under the constraints (70-71). In contrast to the previous case these deformations are not isoassociative.

Solutions of the Hirota-Miwa equations (68) and (74) are given by the AKP and BKP \( \tau \)-functions [19,20]. Thus, any \( \tau \)-function of the AKP and BKP hierarchies defines discrete deformations of the structure constants for the corresponding algebras.

Finally, we note that the linear equations (9) for all three above reductions give rise to the linear problems for the systems (60), (68) and (74). For the last two cases they are

\[ \left( T_j T_k + \frac{\tau_j \tau_k}{\tau \tau_{jk}} (T_j - T_k) - \alpha \right) | \Psi \rangle = 0, \quad j \neq k, j, k = 0, 1, ..., N - 1. \]  

(75)

where \( \alpha = 0 \) for the AKP case and \( \alpha = 1 \) for the BKP case, that coincides with the well-known linear problems [21,20].

6 Conclusion

Discrete equations and corresponding deformations considered in the paper represent only a part of the vast variety. Choosing different algebras to deform and deformation driving algebras, one can get most of the known discrete equations within the described scheme. For instance, discrete deformations of the infinite-dimensional algebra in the Faa’ de Bruno basis for which

\[ C_{j,k}^l = \delta_{j,k} + H_{j+1}^{k-l} + H_{k-l}^1 [11-13] \]  

are described by the discrete KP hierarchy.
The results presented above reveal also a deep connection of the theory of discrete deformations for associative algebras with the algebraic geometry (Fay’s trisecant formulae, addition formulae for $\tau$-function) and discrete geometry (quadrilateral lattices and all that).

References

1. Gerstenhaber M., 1964 On the deformation of rings and algebras, Ann. Math., 79, 59-103.
2. Gerstenhaber M., 1966 On the deformation of rings and algebras. II, Ann. Math., 84, 1-19.
3. Witten E., 1990 On the structure of topological phase of two-dimensional gravity, Nucl. Phys., B 340, 281-332.
4. Dijkgraaf R., Verlinde H. and Verlinde E., 1991 Topological strings in $d < 1$, Nucl. Phys., B 352, 59-86.
5. Dubrovin B., 1992 Integrable systems in topological field theory, Nucl. Phys., B 379, 627-689.
6. Dubrovin B., 1996 Geometry of 2D topological field theories, Lecture Notes in Math., 1620, 120-348, Springer, Berlin.
7. Hertling C. and Manin Y.I., 1999 Weak Frobenius manifolds, Int. Math. Res. Notices, 6, 277-286.
8. Manin Y.I., 2005 F-manifolds with flat structure and Dubrovin’s duality, Adv. Math., 198, 5-26.
9. Manin Y.I., 1999 Frobenius manifolds, quantum cohomology and moduli spaces, AMS, Providence.
10. Hertling C., 2002 Frobenius manifolds and moduli spaces for singularities, Cambridge Univ. Press.
11. Konopelchenko B.G. and Magri F., 2007 Coisotropic deformations of associative algebras and dispersionless integrable hierarchies, Commun. Math. Phys., 274, 627-658.
12. Konopelchenko B.G. and Magri F., 2007 Dispersionless integrable equations as coisotropic deformations: extensions and reductions, Theor. Math. Phys., 151, 803-819.
13. Konopelchenko B.G. 2008 Quantum deformations of associative algebras and integrable systems, arXiv:0802.3022.
14. Merkulov S.A. 2004 Operads, deformation theory and F-manifolds, in: Frobenius manifolds, quantum cohomology and singularities, (Eds. C. Hertling and M. Marcoli), Aspects of Math., E36 213-251.
15. Losev A. and Manin Y.I. 2004 Extended modular operads, in: Frobenius manifolds, quantum cohomology and singularities, (Eds. C. Hertling and M. Marcoli), Aspects of Math., E36 181-211.
16. Bogdanov L.V. and Konopelchenko B.G. 1995 Lattice and q-difference Darboux-Zakharov-Manakov system via $\mathcal{P}$-dressing method, J. Phys. A:Math. Gen., 28, L173-178.
17. Doliwa A. and Santini P. 1997 Multidimensional quadrilateral lattices are integrable, Phys. Lett. A 233, 365-372.
18. Konopelchenko B.G. and Schief W.K. Three-dimensional integrable lattices in Euclidean spaces: conjugacy and orthogonality, Proc. Royal Soc. London, Ser. A., 454, 3075-3104.

19. Hirota R, 1981 Discrete analogue of a generalized Toda equation, J. Phys. Jpn., 50, 3785-3791.

20. Miwa T., 1982 On Hirota’s difference equations, Proc. Japan Acad., 58A, 8-11.

21. Date E., Jimbo M. and Miwa T., 1982, Method for generating discrete soliton equations I, J. Phys. Jpn., 51, 4116-4127.