The spectrum of Bogomol’nyi solitons
in gauged linear sigma models

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Abstract

Gauged linear sigma models with $\mathbb{C}^m$-valued scalar fields and gauge group $U(1)^d$, $d \leq m$, have soliton solutions of Bogomol’nyi type if a suitably chosen potential for the scalar fields is also included in the Lagrangian. Here such models are studied on $(2 + 1)$-dimensional Minkowski space. If the dynamics of the gauge fields is governed by a Maxwell term the appropriate potential is a sum of generalised Higgs potentials known as Fayet-Iliopoulos D-terms. Many interesting topological solitons of Bogomol’nyi type arise in models of this kind, including various types of vortices (e.g. Nielsen-Olesen, semilocal and superconducting vortices) as well as, in certain limits, textures (e.g. $\mathbb{C}P^m$ textures and gauged $\mathbb{C}P^{m-1}$ textures). This is explained and general results about the spectrum of topological defects both for broken and partially broken gauge symmetry are proven. When the dynamics of the gauge fields is governed by a Chern-Simons term instead of a Maxwell term a different scalar potential is required for the theory to be of Bogomol’nyi type. The general form of that potential is given and a particular example is discussed.

March 1996

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1. Introduction

In the study of topological solitons, field theories of Bogomol’nyi type occupy a special place. Mathematically such field theories are characterised by the fact that their soliton solutions may be obtained by solving first order differential equations (called the Bogomol’nyi equations) instead of the second order Euler-Lagrange equations. A further characteristic property is the existence of static multisoliton solutions made up of arbitrarily placed single solitons. Physically one may think of this property as reflecting the absence of static forces between well-separated single solitons. It follows that, in theories of Bogomol’nyi type, the space of static multisoliton solutions is larger and more interesting than in generic field theories with topological solitons. Defining the moduli spaces $M_N$ to be the spaces of static soliton solutions of soliton number $N$, one finds for example that in theories of Bogomol’nyi type the dimension of $M_N$ increases linearly with $N$. Starting with the proposal by Manton [1] in the context of magnetic monopoles, moduli spaces have become a powerful tool for investigating the physics of several interacting solitons in theories of Bogomol’nyi type. This program is particularly advanced in the case of vortices in the abelian Higgs model. Here moduli space techniques have been used successfully to study the interacting dynamics of several vortices [2] and, more recently, the statistical mechanics of vortices [3].

The purpose of this paper is to explore a certain class of (2+1)-dimensional field theories with topological solitons of Bogomol’nyi type which provide a natural generalisation of the abelian Higgs model. In two spatial dimensions, it is useful to distinguish two types of topological solitons: vortices and textures (depending on the context the latter are also often called lumps or baby-Skyrmions). Generally we may distinguish the two by the origin of their topological stability. If the stability is due to a non-trivial first homotopy group of the vacuum manifold we speak of vortices, if it is due to a non-trivial second homotopy group of the vacuum manifold we speak of textures (though care has to be taken in interpreting these statements, as first pointed out in ref. [4] and as we shall also see later). More specifically, vortices arise in field theories with charged scalar fields when a $U(1)$ gauge symmetry is spontaneously broken, whereas textures typically arise in non-linear sigma models. Typical and much studied examples of such textures are the lumps in $\text{CP}^n$ sigma models.

Remarkably, both vortices and textures can be studied in a unified manner in the
framework of gauged linear sigma models (GLSM’s) with a judiciously chosen Higgs potential. The special potentials which are required here are known to supersymmetry theorists as Fayet-Iliopoulos D-terms, and GLSM’s with these potentials have been studied intensively in recent years in the context of topological sigma models and string theory, see for example refs. [5][6]. It has also been emphasised in that context that, in suitable limits, GLSM’s with Fayet-Iliopoulos D-terms become non-linear sigma models whose target spaces are certain special manifolds called toric varieties (of which the projective spaces $\mathbb{CP}^n$ are particular examples). Thus there exists a unifying framework for studying textures and vortices of Bogomol’nyi type, with a host of beautiful mathematical results, whose implications for the study of topological defects in (2+1) dimensional field theories do not appear to have been fully exploited.

However, while there is significant overlap between the questions studied by string theorists and those which one might ask in the context of (2+1)-dimensional field theories, there are also important differences. While in the context of topological sigma models and string theory GLSM’s have typically been studied on compact Riemann surfaces, the field theorist would naturally study the models on flat Minkowski space. Thus, in the (2+1)-dimensional case static fields are defined on (non-compact) $\mathbb{R}^2$, and it is natural also to consider time-dependent fields. This leads to additional questions. On a non-compact domain the convergence of integrals needs to be checked carefully. This results in extra constraints which affect the spectrum of Bogomol’nyi solitons. The inclusion of the time coordinate allows one to study time-dependent fields and to consider gauge fields whose dynamics is governed by a Chern-Simons term. The latter is physically interesting because topological solitons in such theories display anyonic statistics.

This paper begins with two general sections in which GLSM’s are introduced and general results about the spectrum of Bogomol’nyi solitons are derived. Most of the algebraic manipulations in these sections are standard, but the parts of the analysis which deal with the non-compactness of $\mathbb{R}^2$ appear to be new. In sect. 4 we show how the non-linear $\mathbb{CP}^1$ sigma model can be understood in a very precise way as a limiting case of the semilocal vortex model in the context of GLSM’s, and we explain how this limiting procedure can be generalised. Our contribution in this section is mainly an expository one, linking work done in the study of topological defects with the mathematical framework of toric varieties. In sect. 5 we study in detail a family of models with two complex scalar fields and gauge group $U(1) \times U(1)$. We show that the topological solitons in this family include, for various
parameter values and in various limits, superconducting vortices and gauged CP$^1$ lumps. The latter are solitons in the gauged $O(3)$ sigma model introduced in [7]. The present paper developed out of an attempt better to understand the mathematical structure and physical interpretation of that model, and we will see that the framework of GLSM’s with Fayet-Iliopoulos D-terms provides a satisfactory understanding of both. In sect. 6 we write down a general Lagrangian for GLSM’s of Bogomol’nyi type with Chern-Simons terms for the gauge fields. Again we illustrate the general results in a particular model with two complex scalar fields. Finally, the purpose of sect. 7 is to draw together the rather diverse viewpoints which enter this paper and to highlight some open questions.

2. Gauged linear sigma models

Perhaps the most natural way to introduce GLSM’s with Fayet-Iliopoulos D terms without invoking supersymmetry is to recall the manipulations that lead to the establishment of the Bogomol’nyi equations for the abelian Higgs model. The basic fields in the abelian Higgs model are a complex scalar and a $U(1)$ gauge field, but as we shall see presently Bogomol’nyi equations can still be established, mutatis mutandis, if we consider instead a $\mathbb{C}^m$-valued scalar field $w$ and gauge any subgroup of the maximal torus of the unitary group $U(m)$ acting on $w$ in the fundamental representation. The maximal torus of $U(m)$ is $m$-dimensional and a choice of generators $t_a, a = 1, \ldots, m$, defines an isomorphism between it and $U(1)^m$. Concretely the generators are diagonal $m \times m$ matrices with integer entries, so we can write in components

$$t_{\alpha\beta}^a = Q_{\alpha\delta}^a \delta_{\beta\alpha}, \quad (\alpha, \beta = 1, \ldots, m, \text{ no sum over } \alpha). \quad (2.1)$$

Introducing a $U(1)$ gauge fields $A_\mu^a$ for each of the generators we define the covariant derivative

$$D_\mu w = \partial_\mu w + i \sum_{a=1}^m t_a A_\mu^a w, \quad (2.2)$$

and the curvatures

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a. \quad (2.3)$$

Here the space-time indices $\mu, \nu, \ldots$ run over $\{0, 1, 2\}$. In the following we shall exclusively work on (2+1)-dimensional Minkowski space with signature $(-, +, +)$, whose points we denote by $x^\mu$. Occasionally we also use latin indices $i, j, k, \ldots \in \{1, 2\}$ to label the spatial
components of $x^\mu$, or polar coordinates $(\rho, \theta)$ for $(x_1, x_2)$. Finally we should define a suitable “Higgs potential” for each generator $t_a$ of the gauge group $U(1)^m$. The appropriate potential turns out to be the Fayet-Iliopoulos D-term $(R_a - w^\dagger t_a w)^2$, where $R_a$ is a parameter of dimension mass. There are a number of ways to think about this term, but we only point out as an aside that in terms of the symplectic geometry of $\mathbb{C}^m$, with symplectic form $dw^\dagger \wedge dw$, $H_a = R_a - w^\dagger t_a w$ is a Hamiltonian for the $U(1)$ action on $\mathbb{C}^m$ generated by $t_a$. The interested reader is referred to the book [8] for more details on this point of view.

Thus we can write down the general Lagrangian density which is the main subject of this paper. In natural units $c = \hbar = 1$ it reads

$$L = -\frac{1}{2}(D_\mu w)^\dagger D^\mu w - \sum_{a=1}^m \frac{1}{4e_a^2}(F_{\mu\nu}^a)^2 - \sum_{a=1}^m \frac{e_a^2}{8}(R_a - w^\dagger t_a w)^2. \quad (2.4)$$

It depends on $m$ coupling constants $e_a$ of dimension $(\text{mass})^2$, and on the $m$ parameters $R_a$. The dependence of the theory on these parameters is one of our main interests. Note in particular that we can trivially eliminate any of the gauge fields $A_\mu^a$ by setting them and the corresponding coupling constants $e_a$ to zero.

We are mostly interested in static fields, for which the energy functional has the form

$$E = \frac{1}{2} \int d^2 x \frac{1}{2}(D_1 w)^\dagger D_1 w + (D_2 w)^\dagger D_2 w + \sum_{a=1}^m \frac{1}{4e_a^2}(F_{12}^a)^2 + \sum_{a=1}^m \frac{e_a^2}{4}(R_a - w^\dagger t_a w)^2. \quad (2.5)$$

To ensure that the energy of a configuration is finite we also impose the boundary conditions

$$\lim_{\rho \to \infty} (R_a - w^\dagger t_a w) = 0 \quad \quad (2.6)$$
$$\lim_{\rho \to \infty} D_i w = 0. \quad \quad (2.7)$$

Below we will state more precisely how quickly these limits should be attained. For now we proceed, assuming that the decay of the fields at infinity is fast enough to justify the following manipulations. Thus using the algebraic identity

$$E = \frac{1}{2} \int d^2 x |(D_1 \pm iD_2) w|^2 + \sum_{a=1}^m \frac{1}{4} e_a^2 |F_{12}^a| + \frac{e_a^2}{2}(R_a - w^\dagger t_a w)|^2$$
$$\pm \frac{1}{2} \int d^2 x i(D_1 w)^\dagger D_2 w - i(D_2 w)^\dagger D_1 w + \sum_{a=1}^m F_{12}^a (R_a - w^\dagger t_a w) \quad (2.8)$$

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and integrating by parts we deduce

\[ E = \frac{1}{2} \int d^2x |(D_1 \pm iD_2)w|^2 + \sum_{a=1}^{m} \left[ \frac{1}{e_a} F_{12}^a \pm \frac{e_a}{2} (R_a - w^\dagger t_a w) \right]^2 \]

\[ \mp \frac{1}{2} \sum_{a=1}^{m} R^a \int d^2x F_{12}^a, \]

(2.9)

where we have also used that

\[ w^\dagger (D_1 D_2 - D_2 D_1) w = i \sum_{a=1}^{m} w^\dagger t_a w F_{12}^a. \]

(2.10)

Finally defining suitably normalised magnetic fluxes

\[ \Phi_a = -\frac{1}{2\pi} \int d^2x F_{12}^a \]

(2.11)

and

\[ T = \sum_{a=1}^{m} R_a \Phi_a \]

(2.12)

we deduce the inequality

\[ E \geq \pi |T|. \]

(2.13)

More precisely we have

\[ E = \pi T \]

(2.14)

if and only if the Bogomol’nyi equations hold

\[ (D_1 + iD_2)w = 0 \]

(2.15a)

\[ F_{12}^a + \frac{e_a^2}{2} (R_a - w^\dagger t_a w) = 0, \]

(2.15b)

and

\[ E = -\pi T \]

(2.16)

if and only if the (anti)-Bogomol’nyi equations hold

\[ (D_1 - iD_2)w = 0 \]

(2.17a)

\[ F_{12}^a - \frac{e_a^2}{2} (R_a - w^\dagger t_a w) = 0. \]

(2.17b)
The quantity $T$ deserves two further comments. First note that, by converting the surface integral into a line integral round the circle $C$ at spatial infinity and using the boundary conditions (2.6) and (2.7), $T$ can be written as

$$T = \frac{1}{2\pi i} \oint_C w^\dagger dw.$$  

(2.18)

The second comment is a caveat. Since we are working on non-compact $\mathbb{R}^2$ the fluxes $\Phi_a$ are not necessarily integers. Thus neither the fluxes nor $T$ have, in general, a clear topological meaning. Nonetheless the magnetic fluxes are interesting quantities to consider because they are conserved if one rules out infinite energy configurations. This follows from Faraday’s law of induction

$$\frac{d\Phi_a}{dt} = -\frac{1}{2\pi} \oint_C E_i^a dx_i,$$

(2.19)

where $E_i^a = F_{0i}^a$ is the electric field of the $a$-th gauge field. The integral on the right hand side is only non-zero if the electric field falls off for large $\rho$ no faster than $1/\rho$, which is precisely the condition for the electric field to have infinite energy.

3. The Bogomol’nyi equations on $\mathbb{R}^2$

For the rest of this paper we focus on the Bogomol’nyi equations (2.15a) and (2.15b). We are interested in finite energy solutions of these equations, and we need to to specify more carefully the boundary condition which ensure that the energy and flux integrals written down in the previous section converge. For this purpose it is convenient to introduce the complex notation $z = x_1 + ix_2$, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $A_a = \frac{1}{2}(A_{1}^a - iA_2^a)$. Further we define gauge potentials and curvatures

$$a_\alpha = \sum_{a=1}^{m} Q_{a\alpha} A_a$$

(3.1)

$$f_{12}^\alpha = \sum_{a=1}^{m} Q_{a\alpha} F_{12}^a$$

and we introduce the set of parameters $r_\alpha$ through

$$R_a = \sum_{\alpha=1}^{m} Q_{a\alpha} r_\alpha$$

(3.2)
(note that the matrix $Q$ with entries $Q_{aa}$ is invertible by virtue of the generators $t_a$ being independent). The parameters $r_\alpha$ are convenient for discussing the spontaneous symmetry breaking in this model. The potential

$$\sum_{a=1}^{m} \frac{e_a^2}{8} (R_a - w^\dagger t_a w)^2$$

vanishes if and only if $w^\dagger t_a w = R_a$ for all $a$ or, equivalently if

$$|w_\alpha|^2 = r_\alpha, \quad \alpha = 1, ..., m.$$  

Thus, in order to have any chance of finding finite energy solutions we must require

$$r_\alpha \geq 0, \quad \alpha = 1, ..., m.$$  

Further we see that the $U(1)$ gauge group which rotates the phase of $w_\beta$ for some given $\beta \in \{1, ..., m\}$ is spontaneously broken if $r_\beta > 0$ but unbroken if $r_\beta = 0$.

In the new notation the Bogomol’nyi equations (2.15a) and (2.15b) become

$$\bar{\partial}_z + i\bar{a}_\alpha) w_\alpha = 0$$

$$f_{12}^{\alpha} + \frac{1}{2} \sum_{a,\beta=1}^{m} e_a^2 Q_{aa} Q_{a\beta} (r_\beta - |w_\beta|^2) = 0.$$  

These two first order equations imply one second order equation for $w_\alpha$ as follows. At points where $w_\alpha \neq 0$ the first of the above equations is equivalent to

$$\bar{a}_\alpha = i\partial_\bar{z} \ln w_\alpha,$$

which in turn implies

$$f_{12}^{\alpha} = \frac{1}{2} \Delta \ln |w_\alpha|^2.$$  

To extend this equation to the whole plane one uses the $\bar{\partial}$-Poincaré lemma in a standard fashion, see e.g. [10]. Supposing that $w_\alpha$ has zeros at $\{z_{\alpha,s}| s = 1, ..., n_\alpha\}$ and using that in two dimensions

$$\Delta \ln |z - z_s|^2 = 4\pi \delta(z - z_s)$$

one concludes

$$f_{12}^{\alpha} = \frac{1}{2} \Delta \ln |w_\alpha|^2 - 2\pi \sum_{s=1}^{n_\alpha} \delta(z - z_{\alpha,s}).$$
Combining this with (3.6b) we arrive at the promised second order equation for $w_\alpha$:

$$
\Delta \ln |w_\alpha|^2 + \sum_{a, \beta = 1}^{m} e_a^{2} Q_{a \alpha} Q_{a \beta} (r_\beta - |w_\beta|^2) = 4\pi \sum_{s=1}^{n_\alpha} \delta(z - z_{\alpha,s}).
$$

(3.11)

This equation should be thought of as a generalised vortex equation, in the sense that in the case $m = 1$ and $t_1 = 1$ it reduces to the equation

$$
\Delta \ln |w|^2 + e^2 (r - |w|^2) = 4\pi \sum_{s=1}^{N} \delta(z - z_s)
$$

(3.12)

(we have omitted the label $\alpha = 1$ and written $N = n_1$), which is the key to the study of Nielsen-Olesen vortices in the abelian Higgs model at critical coupling. As mentioned in the introduction many of the questions we will address in this paper are motivated by what is known about the abelian Higgs model; it may thus be useful briefly to summarise some salient results.

The first question that one needs to settle is that of existence and uniqueness of solutions. In the case of (3.12) this question is completely answered in [10]. It is shown there that no finite-energy solutions exist if $r \leq 0$, but that for $r > 0$, (3.12) has a unique finite energy solution for each integer $N \geq 0$ and given zeros $z_s$, $s = 1, \ldots, N$. It follows that for given $N \geq 0$ (and thus fixed energy) the Bogomol’nyi equations in the abelian Higgs model have an $N$ complex parameter family of solutions, the $N$ complex parameters determining the positions of the zeros of the $w$-field up to permutations. This parameter space is called the moduli space of $N$-vortices. The moduli space is in fact a differentiable manifold and has a natural Riemannian metric, inherited from the field theory kinetic energy functional. This metric is crucial in the so-called moduli space to vortex dynamics. In that scheme the interacting dynamics of $N$ vortices is approximated by geodesic motion on the moduli space of $N$-vortices. In [2] it is shown that the Bogomol’nyi property of the field theory implies that the metric on the moduli space is Kähler. The Kähler property together with symmetries is then used to compute the metric explicitly in the case $N = 2$; by studying the geodesics of this metric much can be learnt about the interaction of two vortices. The Kähler property is also crucial in the study of statistical mechanics of vortices on Riemann surfaces in ref. [3].

The example of vortices in the abelian Higgs model shows paradigmatically how the Bogomol’nyi property is the key to understanding difficult and interesting aspects of vortex...
physics, such as vortex interactions and statistical mechanics of vortices. The family of theories defined in this and the previous section also have soliton solutions of Bogomol’nyi type whose physical properties can be studied with similar methods. In this paper we will not attempt to do this in full generality but instead go through some of the steps outlined above for a few specific model. However, to end this general section we show how one can deduce some general information about the spectrum of Bogomol’nyi solitons directly from (3.11).

In particular we can now state the conditions for a solution of the Bogomol’nyi equations to have finite energy. The energy of solutions of the equations (3.6a) and (3.6b) can be written conveniently in terms of the fluxes

$$\varphi_\alpha = -\frac{1}{2\pi} \int d^2 x f^{\alpha}_{12}. \quad (3.13)$$

It is

$$E = \pi T = \pi \sum_{\alpha=1}^{m} r_\alpha \varphi_\alpha. \quad (3.14)$$

It thus follows from (3.6d) that all fluxes and the energy are well-defined if \((r_\alpha - |w_\alpha|^2)\) is integrable over \(\mathbb{R}^2\) for all \(\alpha\). If \(r_\alpha > 0\) for all \(\alpha\) the gauge symmetry is completely broken. Then all scalar and gauge fields approach their vacuum values exponentially

$$|w_\alpha|^2 \approx r_\alpha + C_\alpha e^{-m_\alpha \rho} \quad \text{for large } \rho, \quad (3.15)$$

with \(m_\alpha > 0\) and \(C_\alpha\) arbitrary constants, and there are no convergence problems. On the other hand if \(r_\beta = 0\) for some (at least one) \(\beta \in \{1, \ldots, m\}\) then the corresponding scalar fields \(w_\beta\) and (because of the coupling) possibly some other scalar fields approach their vacuum values slower than exponentially. The precise asymptotic behaviour depends on the matrix \((\sum_{a=1}^{m} e^2_a Q_{aa} Q_{a\beta})\) which appears in (3.11) and which couples the various components of \(w\); we will give a more detailed discussion of some special cases later in this paper. Here we note that it is consistent with (3.11) for \(|w_\alpha|\) to approach the vacuum according to a power law

$$|w_\alpha|^2 \approx r_\alpha + C_\alpha \rho^{-2\tilde{\eta}_\alpha} \quad \text{for large } \rho. \quad (3.16)$$

In that case we impose

$$\tilde{\eta}_\alpha > 1 \quad (3.17)$$
to ensure finiteness of the fluxes and the energy. Finally it is possible that the power-law decay is modified by logarithmic terms\footnote{1}. In particular we should also allow for the asymptotic form

$$|w_\alpha|^2 \approx r_\alpha + \frac{C_\alpha}{\rho^2 \ln^2 \rho} \quad \text{for large } \rho,$$

(3.18)

which (for suitable $e_\alpha$ and $Q_{\alpha\alpha}$) is consistent with (3.11) and the finite energy requirement.

We can obtain explicit formulae for the fluxes from (3.10) as follows. Using Stokes’s theorem we first obtain

$$\int d^2 x \Delta \ln |w_\alpha|^2 = 2\pi \lim_{\rho \to \infty} \rho \frac{\partial}{\partial \rho} \ln |w_\alpha|^2.$$

(3.19)

Then using $\ln(r_\alpha + \epsilon) \approx \ln r_\alpha + \epsilon/r_\alpha$ if $r_\alpha \neq 0$ and $\epsilon$ small, and integrating (3.10) we deduce

$$r_\alpha > 0 \Rightarrow \varphi_\alpha = n_\alpha \in \mathbb{Z}^{\geq 0}$$

(3.20)

regardless of how $|w_\alpha|^2$ approaches $r_\alpha$. If on the other hand $r_\alpha = 0$ we know that $|w_\alpha|^2$ tends to zero according to (3.16) or (3.18). Then the result of the integration can be summarised in

$$r_\alpha = 0 \Rightarrow \varphi_\alpha = n_\alpha + \eta_\alpha, \quad n_\alpha \in \mathbb{Z}^{\geq 0} \quad \text{and} \quad \eta_\alpha \in \mathbb{R}^{\geq 1}.$$

(3.21)

The real number $\eta_\alpha$ equals $\tilde{\eta}_\alpha$ if $|w_\alpha|^2$ decays according to the power law (3.16) and is 1 if $|w_\alpha|^2$ approaches zero according to (3.18).

The above formulae show that the flux $\varphi_\alpha$ counts the zeros of $w_\alpha$ with multiplicity. If $r_\alpha > 0$, all zeros are at finite $\rho$ and have integer multiplicity. If $r_\alpha = 0$ then $w_\alpha$ has a zero at infinity whose multiplicity is $\geq 1$ but not necessarily integer. (One shows similarly that for square-integrable solution of eqs. (2.17a) and (2.17b), $\varphi_\alpha \leq 0$ if $r_\alpha > 0$ and $\varphi_\alpha \leq -1$ if $r_\alpha = 0$.) A further useful condition can be deduced from (3.6d) if $r_\beta = 0$ for some $\beta \in \{1, \ldots, m\}$. It then follows that

$$\sum_{a=1}^{m} (Q^{-1})_{\beta a} \frac{F_{\alpha a}}{e_\alpha^2} \geq 0$$

(3.22)

so that

$$\sum_{a=1}^{m} (Q^{-1})_{\beta a} \frac{\Phi_{\alpha}}{e_\alpha^2} \leq 0.$$

(3.23)

Here the equality holds only if $w_\beta$ vanishes everywhere.

\footnote{1 This was pointed out to me by Trevor Samols}
4. Semilocal vortices, CP\(^1\) lumps and toric varieties

For specific calculations we will mostly concentrate on the GLSM with two scalar fields \(w_1\) and \(w_2\) in this paper. Our favourite choice of generators of \(U(1)\) subgroups of the maximal torus in this case is

\[
t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad t_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (4.1)

(we will mostly omit writing the identity matrix \(t_1\) in the following formulae). Then, to avoid crowded notation, we write \(A_\mu\) and \(B_\mu\) for the gauge fields \(A_1^\mu\) and \(A_2^\mu\), \(A\) and \(B\) for the corresponding complex fields \(A^1\) and \(A^2\), and \(F_{\mu\nu}\) and \(G_{\mu\nu}\) for the corresponding field strengths.

In this section we restrict attention to the case where one of the gauge fields is set to zero. Specifically consider the case \(e_2 = 0\) and \(B_\mu = 0\). The resulting model has been much studied in the recent literature and has vortex solutions known as semilocal vortices, see refs. [4], [11] and [12]. One interesting aspect of this model is that it contains stable vortices although the vacuum manifold, defined as the submanifold of \(\mathbb{C}^2\) where the potential \((R_1 - w^\dagger w)\) vanishes, is a three-sphere (provided \(R_1 > 0\)) and therefore has trivial first homotopy group. The reason why there are nonetheless stable vortex solution, first explained in [4], is that in addition to condition (2.6) we have the condition (2.7) and this forces \(w\) to lie both on the vacuum manifold and on a gauge orbit of the gauged \(U(1)\) at spatial infinity. Thus, in terms of spatial polar coordinates \((\rho, \theta)\) \(w\) has to be of the form \((w_1, w_2) = (c_1 e^{iN_1 \theta}, c_2 e^{iN_1 \theta})\) for large \(\rho\) with \(N_1\) some integer and \(c_1\) and \(c_2\) complex constants satisfying \(c_1^2 + c_2^2 = R_1\). The point is that the gauge group acts without fixed points on the vacuum manifold, so that gauge orbits are necessarily loops and never just a point. The integer \(N_1\) is the degree of the map \(w|_{\rho=\infty}\) from the circle at spatial infinity into one of these gauge orbits. Evaluating the formula (2.18) for the topological lower bound in this case we find \(T = R_1 N_1\), showing that vortex solutions of the Bogomol’nyi equations with \(N_1 \neq 0\) cannot decay into the vacuum.

It has also been observed in the literature [13] that there is a close connection between semilocal vortices and topological solitons in the \(\mathbb{CP}^1\) (or \(O(3)\)) sigma model. This connection is usually discussed in terms of energy scales, with the sigma model being thought of as a low-energy effective theory of the vortex model. However, from the present point of view it is more convenient to keep the energy under consideration fixed and vary the
parameter $e_1$ (recall that this has dimensions $(\text{mass})^{\frac{1}{2}}$). In fact it is easy to see that in limit $e_1 \to \infty$ the semilocal vortex model reduces to the $\mathbb{CP}^1$ model. To keep the energy finite we simultaneously impose the constraint

$$R_1 - w^\dagger w = 0.$$  \hfill (4.2)

Then, since the kinetic term of the gauge field $A_\mu$ disappears from the Lagrangian in the limit $e_1 \to \infty$ we can eliminate the gauge field altogether from the energy functional via the equation

$$\frac{\partial L}{\partial A_\mu} = \frac{1}{2i}((D_\mu w)^\dagger w - w^\dagger D_\mu w) = 0,$$

which implies

$$R_1 A_\mu = i w^\dagger \partial_\mu w.$$  \hfill (4.3)

It is a standard result, see e.g. [14], that the resulting model is equivalent to the $\mathbb{CP}^1$ sigma model. Geometrically, the condition (4.2) forces $w$ to lie on a three-sphere $S^3$, and the appearance of the covariant derivative with the gauge potential given by (4.4) means that the Lagrangian depends on $w$ only up to an overall phase. Thus, defining the equivalence relation $\sim$ via $(w_1, w_2) \sim e^{i\chi}(w_1, w_2)$, $\chi \in [0, 2\pi)$, the limit $e_1 \to \infty$ leads to the non-linear sigma model with target space $(S^3/\sim) \cong \mathbb{CP}^1 \cong S^2$. For later use, we note that this can be made explicit by introducing the $\mathbb{CP}^1$-valued field

$$u = \frac{w_2}{w_1}$$  \hfill (4.5)

or the $S^2$-valued field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, defined via

$$\phi_l = w^\dagger \tau_l w, \quad l = 1, 2, 3,$$

where $\tau_1, \tau_2$ and $\tau_3$ are the three Pauli matrices. The Lagrangian density can then be written in terms of $u$ or $\vec{\phi}$, see ref. [14].

Note also that the topological character of the topological defect changes as we take the limit $e_1 \to \infty$. For the semilocal vortex we have already interpreted the topological bound $T$ as ($R_1$ times) the degree of the $w$ viewed as a mapping from the circle at spatial infinity to a gauge orbit on the vacuum manifold. However, in the limit $e_1 \to \infty$, points on gauge orbits should be identified, so that now $w$ maps the entire circle at spatial infinity into on point on the target space $\mathbb{CP}^1$. This allows us to regard $u = w_2/w_1$ as a map
from compactified space $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ to $\mathbb{C}P^1$. The flux $\Phi_1 = -\frac{1}{2\pi} \int d^2xF_{12}$, with $F_{12}$ computed from (4.4), equals the degree of that map. Thus, in the general terminology of the introduction topological solitons in the $\mathbb{C}P^1$ model are textures. However, in this particular context they are more usually called lumps.

The moduli spaces of both semilocal vortices and $\mathbb{C}P^1$ lumps have been studied in some detail. In [11] it is shown that the moduli space of semilocal vortices with magnetic flux number $N$ is diffeomorphic to the space of polynomials of the form $P_N(z) = z^N + \sum_{s=0}^{N-1} a_s z^s$ and $Q_N(z) = \sum_{s=0}^{N-1} b_s z^s$. The coordinatisation of that space in terms of the complex coefficients $a_s$ and $b_s$, $s = 0, ..., N - 1$ shows that it can be identified with $\mathbb{C}^{2N}$.

The translation from a point in the moduli space to an actual field configuration, however, is best done in terms of the zeros of $P_N$ and $Q_N$. They are also the zeros of the scalar fields $w_1$ and $w_2$.

In the $\mathbb{C}P^1$ model, a lump of degree $N$ is a rational function on $\mathbb{R}^2$ of degree $N$ which tend to zero at infinity. Explicitly such a function can again be written in terms of the polynomials $P_N$ and $Q_N$

$$u(z) = \frac{Q_N(z)}{P_N(z)}.$$  

but now we have to require in addition that $P_N$ and $Q_N$ have no common zeros, or equivalently that the resultant of $P_N$ and $Q_N$ is non-vanishing. Writing $R_N$ for the set where the resultant does vanish we conclude that the moduli space of degree $N$ lumps is $\mathbb{C}^{2N} - R_N$. The interpretation of the moduli is now quite different from the vortex case. Writing for example a single lump configuration as $b/(z - a)$, the complex number $a$ is the lump’s position in $\mathbb{R}^2$ and the modulus and phase of the complex number $b$ are its size and orientation respectively. Note that $b$ has to be non-vanishing and that in the limit $|b| \to 0$ the lump becomes infinitely spiky. These features generalise to lumps of degree $N$.

The moduli encode information about internal as well as position degrees of freedom. The condition of the non-vanishing resultant removes precisely those points from the moduli space which correspond to infinitely spiky configurations.

As mentioned earlier, the Riemannian metric which the moduli spaces inherit from the field theory kinetic energy is crucial in the moduli space approximation to soliton dynamics. In ref. [15] this approximation is applied to lumps in the $\mathbb{C}P^1$ model and it turns out that the moduli space metric has two problematic features in this case (see also [16] for further discussion of this point). The metric is not finite because changes in the
coefficient $q_{N-1}$ require infinite kinetic energy. Furthermore it is not complete: there are geodesics which reach infinitely spiky configurations in finite time. By contrast the metric on the moduli space for semilocal vortices, studied in ref. [12] in the case $e_1 = 1$, is finite and complete. Thus the moduli space of semilocal vortices can be thought of in a very precise sense as a regularised version of the moduli space of $\mathbb{CP}^1$ lumps, to which it tends in the limit $e_1 \to \infty$.

Let us briefly consider the case where only the $U(1)$-factor generated by $t_2$ is gauged. Thus we set $e_1 = 0$ and $A_\mu = 0$. The Bogomol’nyi equations (2.15a) and (2.15b) are this case:

\begin{align}
  (\partial_\bar{z} + i\bar{B})w_1 &= 0 \\
  (\partial_\bar{z} - i\bar{B})w_2 &= 0 \\
  G_{12} + \frac{1}{2}(R_2 - |w_1|^2 + |w_2|^2) &= 0.
\end{align}

(4.8a) (4.8b)

The energy of a solution of these equations is $E = R_2\Phi_2$. Thus, if $R_2 < 0$ and $E \neq 0$ we necessarily have $\Phi_2 < 0$. However, it then follows from a standard vanishing theorem (a line bundle of negative degree cannot have a non-zero holomorphic section) that $w_1 = 0$. Similarly if $R_2 > 0$ we deduce that for any solution with non-vanishing energy $w_2 = 0$ In either case the vortex solutions of this model are just embedded Nielsen-Olesen vortices. In the limit $e_2 \to \infty$ we again obtain a non-linear sigma model, but this time the target space is $\{(w_1, w_2) \in \mathbb{C}^2 \mid R_2 = |w_1|^2 - |w_2|^2\} / \sim$, where $\sim$ is the equivalence relation $(w_1, w_2) \sim (e^{i\chi}w_1, e^{-i\chi}w_2)$. This space is known in mathematics as the weighted projective space $\mathbb{CP}^1_{(1,-1)}$ and is isomorphic to $\mathbb{C}$. Bogomol’nyi solitons in the corresponding non-linear sigma model are holomorphic maps from $S^2$ to $\mathbb{CP}^1_{(1,-1)}$. However, such maps are necessarily constant maps. Thus there are no non-trivial textures of Bogomol’nyi type in this model.

The phenomena encountered in this section form part of a very general story, much discussed in mathematics and string theory, see refs. [5] and [6]. Briefly, it goes as follows. Returning to the general notation of sect. 2, consider taking the limit

\begin{equation}
  e_a \to \infty, \quad a \in I
\end{equation}

(4.9) and simultaneously imposing

\begin{equation}
  (R_a - w^\dagger t_aw) = 0, \quad a \in I
\end{equation}

(4.10)
for some subset of indices $I \subset \{1, \ldots, m\}$ which we can without loss of generality take to be $I = \{1, \ldots, d\}$, $d < m$. Assume that the $R_a$ are such that the entire gauge symmetry is spontaneously broken. Also let us at first restrict our attention to the situation where the couplings and gauge fields labelled by the complementary indices are set to zero

$$e_a = 0, \quad A^a_\mu = 0 \quad \text{for} \quad a = d + 1, \ldots, m. \quad (4.11)$$

Defining the currents

$$j^a_\mu = \frac{1}{2i} \left( (D_\mu w)^\dagger t^a w - w^\dagger t^a D_\mu w \right), \quad (4.12)$$
the equations of motion for the gauge fields $A^a_\mu$, $a \in I$ are then

$$j^a_\mu = 0 \quad \text{for} \quad a \in I \quad (4.13)$$

which one can solve explicitly for $A^a_\mu$, $a \in I$:

$$\sum_{b=1}^d w^\dagger t_a t_b w A^b_\mu = iw^\dagger t_a \partial_\mu w. \quad (4.14)$$

The gauge fields $A^a_\mu$, $a \in I$, can thus be eliminated, but the Lagrangian (2.4) is still invariant under transformations $w(x) \rightarrow e^{i\chi^a(x)t_a} w(x)$, $a \in I$, $\chi_a \in [0, 2\pi)$. Then defining the equivalence relation

$$w \sim e^{i\chi^a t_a} w, \quad a \in I, \quad \chi_a \in [0, 2\pi) \quad (4.15)$$

the fields $w$ may be thought of as taking values in the non-linear space

$$Z = \{w \in \mathbb{C}^m | w^\dagger t_a w = R_a, a \in I\} / \sim. \quad (4.16)$$

The combined operation of imposing (4.10) and dividing by the action (4.15) of the torus $U(1)^d$ is called the symplectic quotient of $\mathbb{C}^m$ by $U(1)^d$ and is often written $\mathbb{C}^m // U(1)^d$. In fact what we are looking at here is a very special sort of symplectic quotient. With the original space being $\mathbb{C}^m$ and the group action being a $U(1)^d$-action the resulting quotient is a so-called toric variety of complex dimension $m - d$. The complex projective space and the weighted complex projective space which we encountered above are special examples of toric varieties. Toric varieties are naturally Kähler manifolds, and their topology (which depends on the values of the $R_a$) is well-studied. In particular it is
known that under some restriction on the \( Q^a_\alpha \) \(^{(2.1)}\) the quotient space \( Z \) is compact. This holds for example if for some \( a \in I \) all the \( Q^a_\alpha \) are positive. (For more general conditions see \(^{[4]}\).) Moreover, the second homology group \( H^2(Z, \mathbb{R}) \) is \( d \)-dimensional for generic values of \( R_a \) (note that as long as \( d < m \), the real dimension of \( Z \) is \( \geq 2 \)).

In analogy to our discussion of the \( \mathbb{C}P^1 \) sigma model static fields which obey the condition \(^{(2.7)}\) can here be regarded as maps from \( \mathbb{R}^2 \cup \{\infty\} \cong S^2 \) into \( Z \). Such maps can be classified topologically by their multi-degree, the integral of the pull-back of the \( d \) generators of \( H^2(Z, \mathbb{R}) \). This generalises the observation made in the simple case of the \( \mathbb{C}P^1 \) sigma model. For every gauge field which we eliminate by taking the corresponding coupling constant to infinity the topological meaning of the magnetic flux number changes. If before elimination this number counts the number of times the circle at spatial infinity is wrapped round a certain loop in the vacuum manifold then, after elimination, it counts the number of times compactified space get wrapped around a certain generator of the second homology of the (now non-linear) target space.

Note that the argument in the last paragraph depends crucially on the fact that all gauge fields are either eliminated through the limit \(^{(4.9)}\) or set to zero \(^{(4.11)}\). Only in this case does the boundary condition \(^{(2.7)}\) allow us to identify points at infinity to one point. However, it is also interesting to consider the mixed situation, where some gauge fields are eliminated and others remain as dynamical fields. This leads to gauged non-linear sigma models, where the geometric interpretation of the magnetic fluxes \( \varphi_\alpha \) is more subtle. We shall see this in a particular example in the next section.

5. Topological solitons in \( U(1) \times U(1) \) gauge theory

We now have all the ingredients necessary for the study of the case where the entire maximal torus of \( U(2) \) is gauged. There are eight independent parameters in this model: the real numbers \( e_1, e_2, R_1, R_2 \) and the integer entries in the generators \( t_a \):

\[
t_1 = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{12} \end{pmatrix}, \quad t_2 = \begin{pmatrix} Q_{21} & 0 \\ 0 & Q_{22} \end{pmatrix}.
\]

It is instructive to write out the Lagrangian density in components for this model

\[
\mathcal{L} = -\frac{1}{2} |D_\mu w_1|^2 - \frac{1}{2} |D_\mu w_2|^2 - \frac{1}{4e_1^2} F_{\mu\nu}^2 - \frac{1}{4e_2^2} G_{\mu\nu}^2 \\
- \frac{e_1^2}{8} (R_1 - Q_{11} |w_1|^2 - Q_{12} |w_2|^2)^2 - \frac{e_2^2}{8} (R_2 - Q_{21} |w_1|^2 - Q_{22} |w_2|^2)^2.
\]

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This Lagrangian density is of the type studied first by Witten in the seminal paper [17] and later by many others (see the book [18] and references therein) as a model for bosonic superconducting vortices. In that context the discussion is usually conducted in terms of the gauge fields \( a_\mu = Q_{11} A_\mu + Q_{21} B_\mu \) and \( b_\mu = Q_{12} A_\mu + Q_{22} B_\mu \) which couple only to the scalar fields \( w_1 \) and \( w_2 \) respectively. Writing \( U(1)_a \) and \( U(1)_b \) for the gauge groups which rotate the phases of \( w_1 \) and \( w_2 \) respectively the basic observation can be stated as follows.

By adjusting parameters suitably one can arrange for one of the two gauge groups, say \( U(1)_a \), to be spontaneously broken while \( U(1)_b \) remains unbroken. Thus \( U(1)_b \) may be interpreted as the gauge group of electromagnetism (there is a caveat which we explain below). Then there is a range of parameters for which the model has \( U(1)_a \)-vortex solutions in whose core the Higgs field \( w_2 \) of \( U(1)_b \) is non-zero, thereby breaking the electromagnetic gauge group there. Witten showed that under these circumstances the core of the vortex becomes superconducting.

The Lagrangian density (5.2) is more general than the Lagrangians usually studied in the context of superconducting vortices because it allows for interactions between the gauge fields \( a_\mu \) and \( b_\mu \) (through the curvature terms) but it also has many special properties because it of Bogomol’nyi type. One general feature of Bogomol’nyi solitons in gauge theories with scalar fields is the cancellation between the static forces mediated by the gauge fields and those mediated by the scalar fields. Such a cancellation is of course only possible if the scalar and gauge fields are either both massless or both massive. Thus we should expect on general grounds that in our model, too, the scalar field \( w_2 \) should acquire a long range component when \( U(1)_b \) is unbroken. In fact it is easy to check this explicitly. By the general condition stated after eq. (3.5), \( U(1)_b \) is unbroken if and only if \( r_2 = 0 \), which is equivalent to \( Q_{21} R_1 = Q_{11} R_2 \). The vacuum expectation value of \( |w_1|^2 \) is then \( R_1/Q_{11} = R_2/Q_{21} \) and, by collecting the terms quadratic in \( |w_2| \), the mass of \( w_2 \) is found to be zero. (A similar calculation shows that if we fix the parameters so that \( w_1 \) vanishes in the vacuum then \( w_1 \) becomes massless.) From the point of view of superconducting vortex physics this result is disappointing: one requires an exponentially localised \( w_2 \) condensate in order to interpret the unbroken gauge group \( U(1)_b \) as the gauge group of electromagnetism. If the condensate only decays according to some power law the \( U(1)_b \) gauge invariance is never properly restored outside the core of the vortex and the electromagnetic interpretation is not appropriate.

By adding a suitable perturbation to the Lagrangian (5.2) one could ensure an ex-
ponentially localised condensate, of course at the expense of destroying the Bogomol’nyi property. However, properties such as interactive dynamics and thermodynamics of such ‘almost Bogomol’nyi’ models can also be studied with relative ease by perturbation methods \[19\]. This may be interesting from the point of view of the phenomenology of superconducting vortices but we will not pursue it here. Instead we want to exhibit some detailed properties of the model \([3.2]\) by picking a particular set of generators \(t_a\).

Thus we return to the generators \([4.1]\), i.e. we set \(Q_{11} = Q_{12} = 1\) and \(Q_{21} = -Q_{22} = 1\) so that \(r_1 = (R_1 + R_2)/2\), \(r_2 = (R_1 - R_2)/2\). We also continue with the index saving terminology and write \(a_i = A_i + B_i\) and \(b_i = A_i - B_i\), \(a = (a_1 - i a_2)/2\) and \(b = (b_1 - i b_2)/2\) as well as \(f_{12} = F_{12} + G_{12}\) and \(g_{12} = F_{12} - G_{12}\). Then the Bogomol’nyi equations \((3.6)\) read

\[
(\partial \bar{z} + i a)w_1 = 0 \\
(\partial \bar{z} + i b)w_2 = 0 \\
f_{12} + \frac{1}{2}(e_1^2 + e_2^2)(r_1 - |w_1|^2) + \frac{1}{2}(e_1^2 - e_2^2)(r_2 - |w_2|^2) = 0 \\
g_{12} + \frac{1}{2}(e_1^2 - e_2^2)(r_1 - |w_1|^2) + \frac{1}{2}(e_1^2 + e_2^2)(r_2 - |w_2|^2) = 0,
\]

and the generalised vortex equations \((3.11)\) are

\[
\Delta \ln |w_1|^2 + (e_1^2 + e_2^2)(r_1 - |w_1|^2) + (e_1^2 - e_2^2)(r_2 - |w_2|^2) = 4\pi \sum_{s=1}^{n_1} \delta(z - z_{1,s}) \tag{5.4a}
\]

\[
\Delta \ln |w_2|^2 + (e_1^2 - e_2^2)(r_1 - |w_1|^2) + (e_1^2 + e_2^2)(r_2 - |w_2|^2) = 4\pi \sum_{t=1}^{n_2} \delta(z - z_{2,t}). \tag{5.4b}
\]

We are interested in the way the properties of solutions, such as energies and magnetic fluxes, depend on the coupling constants \(e_1\) and \(e_2\) and on the values of the parameters \(r_1\) and \(r_2\). The latter determine the symmetry pattern. If \(r_1 > 0\) and \(r_2 > 0\) the gauge symmetry is completely broken. Then the fluxes \(\varphi_1\) and \(\varphi_2\) equal the integers \(n_1\) and \(n_2\) respectively. Although we do not prove this rigorously here we expect that given these integers there is a \((n_1 + n_2)\) complex parameter family of solutions of \((5.4d)\) and \((5.4d)\). Each solution is fully characterised by the unordered set of parameters \(\{z_{1,1}, ..., z_{1,n_1}\}\) and \(\{z_{2,1}, ..., z_{2,n_2}\}\) which label the zeros of the scalar fields \(w_1\) and \(w_2\). Note that, contrary to the situation for a single vortex, the magnitudes of the magnetic fields are not necessarily maximal at these zeros.

If \(r_1 > 0\) and \(r_2 = 0\), \(U(1)_a\) is broken but \(U(1)_b\) remains unbroken (and vice-versa for \(r_1 = 0\) and \(r_2 > 0\)). Thus \(\varphi_1\) equals again the integer \(n_1\) which counts the zeros of \(w_1\), but
\( \varphi_2 \) may have a non-integral part. The interpretation, explained in sect. 3, is that \( w_2 \) has a zero at infinity whose order is at least one but may be non-integral. Thus only \( [\varphi_2 - 1] \) zeros (where \([\varphi_2]\) denotes the integral part of \( \varphi_2 \)) can be placed arbitrarily in \( \mathbb{R}^2 \cup \{\infty\} \).

For given \( \varphi_1 = n_1 \) and \( \varphi_2 \) we therefore expect there to be a \( n_1 + [\varphi_2] - 1 \) complex parameter family of solutions of (5.4a) and (5.4b).

Finally in the special case \( r_1 = r_2 = 0 \) the full gauge symmetry survives, but we shall see that the model has no finite energy solutions in this case. Before we discuss the spectrum of solitons in more detail we will now show that our model contains, in the limit \( e_1 \to \infty \), the gauged \( O(3) \) sigma model introduced in ref. [7] and analysed further in ref. [20].

### 5.1. The gauged \( O(3) \) sigma model revisited

In taking the limit \( e_1 \to \infty \) we can now follow the general recipe of sect. 4. Thus we impose simultaneously the constraint \( |w_1|^2 + |w_2|^2 = R_1 > 0 \). Then we solve for \( A_\mu \) according to (4.14):

\[
R_1 A_\mu = iw^\dagger \partial_\mu w - w^\dagger t_2 w B_\mu. \tag{5.5}
\]

Since the field \( (w_1, w_2) \) is now only defined up to an overall phase we can again discuss the model in terms of the ratio \( u = w_2/w_1 \) (4.5). In particular the energy functional for static fields takes the form

\[
E = \frac{1}{2} \int d^2 x (r_1 + r_2) \frac{|D_1 u|^2 + |D_2 u|^2}{(1 + |u|^2)^2} + \frac{1}{e_2^2} G_{12}^2 + e_2^2 \left( \frac{r_1 |u|^2 - r_2}{1 + |u|^2} \right)^2, \tag{5.6}
\]

where the covariant derivative on \( u \) is \( D_i u = (\partial_i - 2iB_i)u \). This functional is a generalised form of the energy functional of the gauged \( O(3) \) sigma model introduced in ref. [4] and formulated there in terms of the field \( \vec{\phi} \) (1.6). To recover the formulae in ref. [4] one should set \( r_1 = 1, r_2 = 0 \) and \( e_2 = 1 \), and identify \( -2B_i \) and \( -2G_{12} \) with what is called \( A_i \) and \( F_{12} \) there. Then the energy functional (5.6) is 4 times the energy functional studied in ref. [7] and the magnetic flux

\[
\Phi_2 = -\frac{1}{4\pi} \int d^2 x G_{12} \tag{5.7}
\]

is \( \frac{1}{4\pi} \times \) the magnetic flux defined in ref. [7], which is denoted \( \Phi \) there.

Again we have the inequality

\[
E \geq \pi (r_1 \varphi_1 + r_2 \varphi_2), \tag{5.8}
\]
and from eq. (5.3) one finds the following formulae for the fluxes

\[ \varphi_1 = \frac{1}{2\pi i} \int d^2 x \frac{\bar{D}_1 u D_2 u - \bar{D}_2 u D_1 u}{(1 + |u|^2)^2} - \frac{1}{2\pi} \int d^2 x \frac{2G_{12} |u|^2}{1 + |u|^2}, \]  

(5.9)

and

\[ \varphi_2 = \frac{1}{2\pi i} \int d^2 x \frac{\bar{D}_1 u D_2 u - \bar{D}_2 u D_1 u}{(1 + |u|^2)^2} + \frac{1}{2\pi} \int d^2 x \frac{2G_{12} |u|^2}{1 + |u|^2}. \]  

(5.10)

Such integral formulae are useful for explicit computations, but to understand the geometrical meaning of the magnetic fluxes \( \varphi_1 \) and \( \varphi_2 \) in this model it is best to recall how the magnetic fluxes are related to the numbers of zeros of the scalar fields in the vortex model.

Since \( r_1 \) and \( r_2 \) should not both be zero let us for definiteness assume that \( r_1 > 0 \). Then \( \varphi_1 \) is an integer (called \( n_1 \) above) which counts with multiplicity the number of zeros of \( w_1 \) or equivalently the number of poles of \( u \). The flux \( \varphi_2 \) counts the zeros of \( w_2 \) and hence of \( u \). If \( r_2 > 0 \), i.e. if the only remaining \( U(1) \) gauge symmetry is spontaneously broken, then \( |u|^2 = r_2/r_1 \) at spatial infinity and all zeros of \( u \) are at finite \( \rho \). Then \( \varphi_2 \) is also an integer (called \( n_2 \) above) which counts these zeros with multiplicity. If \( r_2 = 0 \), i.e. in the case of unbroken gauge symmetry, the condition \( \varphi_2 \in \mathbb{R}^{\geq 1} \) tells us that \( u \) must have a zero at infinity of order at least 1.

As an aside we note that integer \( \varphi_1 \) is called degree of \( u \) in ref. [7]. When \( \varphi_2 \) is not an integer this is somewhat misleading because the scalar field \( u \) then decays at spatial infinity according to some power law with a non-integral exponent. Such a field cannot be viewed as a continuous map from \( \mathbb{R}^2 \cup \{\infty\} \) to \( \mathbb{C}P^1 \) and therefore does not have a well-defined degree. As explained here, \( \varphi_1 \) should be thought of more generally as the number of poles of \( u \), counted with multiplicity.

The Bogomol’nyi equations in this model may be derived either by determining the condition for the equality in (5.8) to hold, or by subtracting (5.3a) from (5.3b) and (5.3c) from (5.3d). In either case the result is

\[ (\partial^2_x - 2i\bar{B})u = 0 \]  

(5.11a)

\[ G_{12} + e^2_2 \left( \frac{r_1 |u|^2 - r_2}{1 + |u|^2} \right) = 0. \]  

(5.11b)

These equations imply a second order elliptic equation for \( u \) which can again be derived in two ways. Either, as first shown in ref. [20], from (5.11a) and (5.11b) via the \( \bar{\partial} \)-Poincaré lemma or by subtracting eq. (5.4a) from eq. (5.4b). In either case the result is

\[ \Delta \ln |u|^2 - 4e^2_2 \left( \frac{r_1 |u|^2 - r_2}{1 + |u|^2} \right) = -4\pi \sum_{s=1}^{n_1} \delta(z - z_{1,s}) + 4\pi \sum_{t=1}^{n_2} \delta(z - z_{2,t}). \]  

(5.12)
This equation with $r_2 = 0$ is analysed carefully in ref. [20]. It is shown there that in that situation and for given $\varphi_1$ and $\varphi_2$ it has a $\varphi_1 + [\varphi_2] - 1$ complex parameter family of solutions, which agrees with our general counting argument given in the discussion of (5.4a) and (5.4b) above.

Physically one may think of a solution of the Bogomol’nyi equations (5.11a) and (5.11b) as a texture carrying magnetic flux $2\Phi_2 = \varphi_1 - \varphi_2$ which counts the difference between the number (with multiplicity) of poles and the number of zeros of $u$. In the case of spontaneously broken gauge symmetry the flux is quantised and can be positive, negative or zero. In the unbroken case, however, it is clear from (5.7) and (5.11d) with $r_2 = 0$ that the magnetic flux $\Phi_2$ is positive for non-trivial solutions. Thus we deduce that $\Phi_2$ takes values in a finite interval

$$0 < 2\Phi_2 = \varphi_1 - \varphi_2 \leq \varphi_1 - 1.$$  
(5.13)

This formula was derived independently by Samols (as referred to in ref. [3]) and by Yang in ref. [20]. In the next section we shall see that it is a consequence of the general inequality (3.23).

At the end of the introduction I mentioned that the wish better to understand the mathematical structure and physical significance of the gauged $O(3)$ sigma model was the starting point of this paper. In this section we have seen that mathematically this model is a limiting case of a certain gauged linear sigma model with Fayet-Iliopoulos potential terms. Since the relevant gauged linear sigma model has solutions which describe superconducting vortices of Bogomol’nyi type this observation also sheds light on the physics. When $r_2 = 0$ the gauged $O(3)$ sigma model has an unbroken $U(1)$ gauge group which we may identify with the gauge group of electromagnetism. In the core of a soliton solution the gauge symmetry is broken and in this sense we may think of the solitons in the gauged $O(3)$ sigma model as superconducting textures. As in our discussion of superconducting vortices in the previous section the Bogomol’nyi property of the model leads to a power-law localisation of the scalar condensate, which is phenomenologically disastrous. However, this problem can again be solved by moving away from the Bogomol’nyi limit. In fact a model of the required type is studied in [21]. The soliton solutions, called Skyrme-Maxwell solitons there, are exponentially localised. Asymptotically there is an unbroken $U(1)$ gauge group which is broken in the centre of a soliton. Such a topological soliton may thus properly be called a superconducting texture. These textures only involve one scalar field and one
gauge field; from a mathematical point of view they therefore appear to provide a more economical model for superconducting topological defects than superconducting vortices.

5.2. Symmetry breaking patterns and the soliton spectrum

The purpose of this subsection is to highlight some consequences of the simple energy formula

\[ E_{\text{BPS}} = \pi (r_1 \varphi_1 + r_2 \varphi_2) \]  (5.14)

for Bogomol’nyi solitons in the \( m = 2 \) GLSM. As we saw above, this formula is valid for all values of \( e_1 \) and \( e_2 \), including the non-linear sigma model limit \( e_1 \to \infty \) (although the interpretation of \( \varphi_1 \) and \( \varphi_2 \) changes in this limit). Again we organise the discussion according to the symmetry breaking pattern.

First consider the case of completely broken symmetry, i.e. \( r_1 > 0 \) and \( r_2 > 0 \). Then fluxes \( \varphi_1 \) and \( \varphi_2 \) take arbitrary non-negative integer values \( n_1 \) and \( n_2 \). The resulting spectrum depends in an interesting way on whether \( r_1/r_2 \) is irrational or rational. In the former case there is no degeneracy in the energy level for given \( \varphi_1 \) and \( \varphi_2 \) other than that coming from the arbitrary positions of the \( n_1 \) and \( n_2 \) zeros of \( w_1 \) and \( w_2 \) discussed earlier. When \( r_1/r_2 \) is rational, however, solutions with different flux quantum numbers may be degenerate in energy. More precisely, writing \( r_1/r_2 = p/q \), where \( p \) and \( q \) are two integers which are relatively prime, we deduce from formula (5.14) that the solution with \( (\varphi_1, \varphi_2) = (n_1, n_2) \) has the same energy as the solution with \( (\varphi_1, \varphi_2) = (n_1 + q, n_2 - p) \) (provided these integers are still positive). In the case \( e_1 = e_2 \), where the equations (5.4a) and (5.4b) decouple this degeneracy is easily understood: if the masses of a single \( w_1 \)-vortex and a single \( w_2 \)-vortex have a rational ratio, then we expect degeneracies in the energy levels of superpositions of \( w_1 \)- and \( w_2 \) vortices. However, for \( e_1 \neq e_2 \), and particularly in the limit \( e_1 \to \infty \), this degeneracy is more remarkable. In this limit we interpret \( \varphi_1 \) as the degree of \( u \) and \( 2\Phi_2 \) as the magnetic flux carried by the texture; in terms of these (5.14) reads

\[ E_{\text{BPS}} = \pi ((r_1 + r_2)\text{degree}[u] - 2r_2 \Phi_2) . \]  (5.15)

Thus the degeneracy for rational \( r_1/r_2 \) means physically that in the gauged \( O(3) \) sigma model changing the degree of a configuration is energetically equivalent to changing, by a suitable number of units, the magnetic flux it carries.
If \( r_1 > 0 \) and \( r_2 = 0 \) then \( U(1)_a \) is broken and \( U(1)_b \) remains unbroken; in that case \( \varphi_1 \) is still quantised as a non-negative integer, but \( \varphi_2 \) may now have a non-integer part. The energy (5.14) depends only on \( \varphi_1 \) and is degenerate with respect to changes in \( \varphi_2 \). Note, however, that the range of \( \varphi_2 \) is restricted by the finite-energy condition \( \varphi_2 \geq 1 \) and the further constraint coming from (3.23) for non-vanishing solutions. In this case it reads

\[
\varphi_2 < \frac{e_1^2 - e_2^2}{e_1^2 + e_2^2} \varphi_1.
\]  

Together these conditions force \( \varphi_2 \) to lie in a finite interval. In particular they may force \( \varphi_2 \) to vanish for certain values of \( e_1 \) and \( e_2 \). This holds in particular for \( e_1 = e_2 \) where this result, too, is easily understood. The equation (5.4b) for \( w_2 \) is then just the abelian Higgs vortex equation (3.12). As mentioned in sect. 3 this equation has no solution other than the trivial solution when \( r = 0 \). In the non-linear sigma model limit \( e_1 \to \infty \) the formula (5.16) yields the promised re-derivation of (5.13). It now reads \( \varphi_2 < \varphi_1 \), which we combine with the earlier condition \( \varphi_2 \in \mathbb{R} \geq 1 \) to

\[
1 \leq \varphi_2 < \varphi_1.
\]

This is equivalent to the inequality (5.13). Note in particular that this condition cannot be satisfied if \( \varphi_1 = 1 \) and that there is therefore no solution with precisely one pole in the gauged \( O(3) \) sigma model. This result was first derived for the spherically symmetric case in ref. [7] and proved in general in ref. [20].

The reverse situation, with \( r_1 = 0 \) and \( r_2 > 0 \) can be discussed in an analogous fashion, and the case \( r_1 = r_2 = 0 \) does not lead to any interesting solution: now the condition (3.23) implies, for non-trivial solutions, (5.16) as well as

\[
\varphi_1 < \frac{e_1^2 - e_2^2}{e_1^2 + e_2^2} \varphi_2
\]

which is incompatible with (5.16). Thus the trivial solutions \( w_1 = w_2 = 0 \) is the only finite energy solution of the equations (5.4a) and (5.4b) in this case.

6. Abelian Chern-Simons models of Bogomol’nyi type

Consider the following Lagrangian density for a theory where the dynamics of all the gauge fields are governed by Chern-Simons terms:

\[
\mathcal{L}_{CS} = -\frac{1}{2}(D_\mu w)^\dagger D^\mu w - \sum_{a=1}^{m} \frac{1}{2\kappa_a} A_\mu^a \partial_\nu A_\lambda^a \epsilon^{\mu\nu\lambda} - V_{CS}(\kappa_a, w, R_a),
\]  

(6.1)
with \( m \) dimensionless coupling constants \( \kappa_a \). In this section we want to determine the potentials \( V_{CS} \) such that the model has static soliton solutions of Bogomol’nyi type.

The Chern-Simons term is independent of the metric, so it does not contribute to the energy-momentum tensor. As a result the energy functional reads

\[
E_{CS} = \frac{1}{2} \int d^2 x (D_0 w)^\dagger D_0 w + (D_1 w)^\dagger D_1 w + (D_2 w)^\dagger D_2 w + V_{CS}. 
\] (6.2)

However, the energy does depend on the gauge fields through the covariant derivatives. In particular the time-component of the gauge fields is determined as a function of the scalar fields and the spatial components of the field strength by the Chern-Simons version of Gauss’s law:

\[
j^a_0 = \frac{1}{\kappa_a} F_{12}^a, \quad (6.3)
\]

where \( j^a_\mu \) is defined as in (4.12). This constraint means that, even for static fields we cannot set \( A_0^a = 0 \). Instead we find, for time-independent \( w_a \),

\[
\sum_{b=1}^m w^\dagger t_a t_b w A_0^b = \frac{1}{\kappa_a} F_{12}^a. \quad (6.4)
\]

Thus the energy functional \( E_{CS} \) can be expressed entirely in terms of spatial components of the gauge fields and the spatial part of the field strengths. To do this it is convenient to introduce the notation \( \mathbf{F} \) for the column vector with components \( F_{12}^a/\kappa_a \), \( \mathbf{V} \) for the column vector with components \( \kappa_a (R_a - w^\dagger t_a w) \) and \( \mathbf{W} \) for the \( m \times m \) matrix with components \( W_{ab} = w^\dagger t_a t_b w, \, a, b = 1, \ldots, m \). We claim that by setting

\[
V_{CS} = \frac{1}{8} \mathbf{V}^\dagger \mathbf{W} \mathbf{V} = \frac{1}{8} \sum_{a,b=1}^m \kappa_a (R_a - w^\dagger t_a w) w^\dagger t_a t_b w \kappa_b (R_b - w^\dagger t_b w) \quad (6.5)
\]

we obtain an energy functional of Bogomol’nyi type:

\[
E_{CS} = \frac{1}{2} \int d^2 x (D_1 w)^\dagger D_1 w + (D_2 w)^\dagger D_2 w + \mathbf{F}^\dagger \mathbf{W}^{-1} \mathbf{F} + \frac{1}{4} \mathbf{V}^\dagger \mathbf{W} \mathbf{V}
\]

\[
= \frac{1}{2} \int d^2 x \left| (D_1 \pm i D_2) w \right|^2 + \left( \mathbf{F} \pm \frac{1}{2} \mathbf{W} \right)^\dagger \mathbf{W}^{-1} \left( \mathbf{F} \pm \frac{1}{2} \mathbf{W} \right) \quad (6.6)
\]

\[
\pm \frac{1}{2} \sum_{a=1}^m R^a \int d^2 x F_{12}^a.
\]

Here the superscript \( t \) means transposition and we have again used the boundary condition (2.7) to integrate by parts and exploited the identity (2.10). As in sect. 2 we therefore deduce the inequality

\[
E_{CS} \geq \pi |T| \quad (6.7)
\]

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with equality if and only if one of the Bogomol’nyi equations holds

\[(D_1 \pm iD_2)w = 0\]  \hspace{1cm} (6.8a)

\[\frac{F_{12}^{a}}{\kappa_{a}} \pm \frac{1}{2} \sum_{b=1}^{m} w^{\dagger}t_{b}w \kappa_{b}(R_{b} - w^{\dagger}t_{b}w) = 0.\]  \hspace{1cm} (6.8b)

In the case \(m = 1\) the general prescription we have discussed leads to the abelian Chern-Simons-Higgs model with a sextic potential, which has been much discussed in the literature. One reason why this model has attracted attention is that it contains both topological and non-topological solitons \([22]\). The generalised models described here include, for \(m = 2\), the choice of generators \((4.1)\) and \(\kappa_2 = 0\), the semilocal Chern-Simons model discussed in ref. \([23]\), and have in general a variety of both topological and non-topological soliton solutions whose properties deserve to be studied further. However, here we now turn to another point of principal interest.

Like in the case of GLSM’s with the gauge field governed by a Maxwell term it is interesting to consider limits in which some of the gauge fields are eliminated, thus leading to a (possibly gauged) non-linear sigma model. Specifically we can eliminate the Chern-Simons terms for the gauge fields \(A_{a}^{a}\), where \(a\) runs over the subset of indices \(I = \{1, \ldots, d\}\), \(d < m\), by taking the limit

\[\kappa_{a} \rightarrow \infty, \hspace{1cm} a \in I\]  \hspace{1cm} (6.9)

and simultaneously imposing

\[(R_{a} - w^{\dagger}t_{a}w) = 0, \hspace{1cm} a \in I.\]  \hspace{1cm} (6.10)

Then the equations of motion for the gauge fields \(A_{a}^{a}\), \(a \in I\), are again \((4.13)\), allowing us to express these gauge fields in terms of the scalar fields and the complementary set of gauge fields:

\[\sum_{b=1}^{d} w^{\dagger}t_{b}w A_{a}^{b} = iw^{\dagger}t_{a}(\partial_{\mu} + i \sum_{b=d+1}^{m} t_{b}A_{a}^{b})w.\]  \hspace{1cm} (6.11)

However, there is a subtlety here: in the Chern-Simons case, unlike in the Maxwell case, simply taking the value of the potential in the limit \((6.9)\) and \((6.10)\) as the potential for the remaining degrees of freedom does not appear to lead to a model with interesting Bogomol’nyi solitons. The quickest way to see this is naively to apply the limit \((6.9)\) and \((6.10)\) to the Bogomol’nyi equation \((6.8b)\). The equations labelled by \(a \in I\) then
lead to purely algebraic constraints on the scalar fields which eliminate many solutions. Here we therefore follow a different path and define a new potential as follows. Let $\tilde{I} = \{d + 1, \ldots, m\}$ be the complementary index set to $I$ and let $\tilde{F}$ and $\tilde{V}$ be the column vectors with components $F_{12}/\kappa_\tilde{a}$ and $\kappa_\tilde{a}(R_{\tilde{a}} - w^t_{\tilde{a}}w)$ respectively, where $\tilde{a} \in \tilde{I}$. Further write $\tilde{W}^{-1}$ for the matrix with elements $(W^{-1})_{\tilde{a}\tilde{b}}, \tilde{a}, \tilde{b} \in \tilde{I}$. Then the kinetic term for the gauge field in the Chern-Simons energy functional in the limit (6.9) is

$$\tilde{F}^t \tilde{W}^{-1} \tilde{F}$$

(6.12)

and thus the appropriate potential is

$$V_{CS} = \frac{1}{8} \tilde{V}^t \left( \tilde{W}^{-1} \right)^{-1} \tilde{V}.$$  

(6.13)

With this choice the energy functional

$$E_{CS} = \frac{1}{2} \int d^2x (D_1w)^tD_1w + (D_2w)^tD_2w + \tilde{F}^t \tilde{W}^{-1} \tilde{F} + \frac{1}{4} \tilde{V}^t \left( \tilde{W}^{-1} \right)^{-1} \tilde{V}$$

(6.14)

satisfies

$$E_{CS} \geq \pi |T|$$

(6.15)

with equality if and only if one of the Bogomol’nyi equations holds

$$(D_1 \pm iD_2)w = 0$$

(6.16a)

$$\frac{F_{12}}{\kappa_{\tilde{a}}} \pm \frac{1}{2} \sum_{b=d+1}^{m} (W^{-1})^{-1}_{\tilde{a}b} V_b = 0, \quad \tilde{a} \in \tilde{I}.$$  

(6.16b)

In the first of these equations the gauge potentials $A^a_i, a \in I$ are again determined by (5.11).

The potential (6.13) is, in general, of a more complicated form than the sextic polynomial that is familiar in Chern-Simons theories of Bogomol’nyi type. Thus it may be useful to illustrate the general discussion by considering again the case $m = 2$, with entire maximal torus of $U(2)$ gauged. We will again use the index saving notation of sect. 5.

With $t_1$ and $t_2$ as in (4.1), the matrix $W$ is

$$W = \begin{pmatrix} w^t w & w^t t_2 w \\ w^t t_2 w & w^t w \end{pmatrix}$$

(6.17)
so that in the limit \( \kappa_1 \to \infty \) and with \( R_1 - w^\dagger w = 0 \), the matrix \( \widetilde{W}^{-1} \) is just a number:

\[
\widetilde{W}^{-1} = \frac{R_1}{R_1^2 - (w^\dagger t_2 w)^2}.
\]

(6.18)

Thus the Chern-Simons potential is

\[
V_{CS} = \frac{\kappa_2^2}{8R_1}(R_1^2 - (w^\dagger t_2 w)^2)(R_2 - w^\dagger t_2 w)^2
\]

(6.19)

and the Bogomol'nyi equations are

\[
(D_1 \pm iD_2)w = 0
\]

(6.20a)

\[
G_{12} \pm \frac{\kappa_2^2}{2R_1}(R_1^2 - (w^\dagger t_2 w)^2)(R_2 - w^\dagger t_2 w) = 0.
\]

(6.20b)

Formulating this model in terms of the field \( u \) (4.5) is again instructive. Using also again the parameters \( r_1 \) and \( r_2 \) defined in sect. 5, the potential (6.19) takes the form

\[
V_{CS} = 2\kappa_2^2(r_1 + r_2)|u|^2 \frac{(r_1|u|^2 - r_2)^2}{(1 + |u|^2)^4}.
\]

(6.21)

Choosing the upper sign in the Bogomol'nyi equations they now read

\[
(\partial \bar{z} - 2i\bar{B})u = 0
\]

(6.22a)

\[
G_{12} + 4\kappa_2^2(r_1 + r_2)|u|^2 \frac{(r_1|u|^2 - r_2)}{(1 + |u|^2)^3} = 0.
\]

(6.22b)

In fact these equations and their soliton solutions are studied with slightly different notation in ref. [24] and ref. [25] for the case of unbroken \( U(1) \) gauge symmetry \( r_2 = 0 \). In ref. [26] the case of broken gauge symmetry, \( r_2 > 0 \) is also considered. As often in Chern-Simons theories there are both topological and non-topological solitons in this model. All soliton solutions carry magnetic flux, and in the case of broken gauge symmetry the topological solitons may derive their topological stability from the (quantised) magnetic flux, from the number of poles of \( u \) (the “degree”) or from a combination of both. For more details we refer the reader to the papers [24], [25] and [26] and also to ref. [27], where a related model is discussed. However, it should be clear from this brief case study that the family of gauged linear sigma models with Chern-Simons terms defined in this section have an even richer spectrum of solitons than their Maxwell cousins.
7. Discussion and Outlook

In (2+1) dimensions, many interesting topological solitons of Bogomol’nyi type can be studied in a unified way in terms of the gauged linear sigma models discussed in this paper. Thinking of the GLSM’s with Maxwell term and Fayet-Iliopoulos potential terms as a generalisation of the abelian Higgs model and keeping in mind the phenomena encountered here, it would be interesting to address the following points in full generality.

Can one extend the analysis of ref. [10] and ref. [20] to establish rigorous general existence and uniqueness theorems for the generalised vortex equation (3.11)? If in the general model discussed in sect. 3 the gauge symmetry is completely broken (i.e. if all \( r_\alpha > 0 \)) then all the fluxes \( \varphi_\alpha \) are non-negative integers and the examples studied above suggest that for given fluxes there is a \( \sum_{\alpha=1}^{m} \varphi_\alpha \) complex parameter family of solutions. If the gauge symmetry is only partially broken, say \( r_\alpha > 0 \) for \( 1 \leq \alpha \leq d \) and \( r_\alpha = 0 \) for \( d < \alpha \leq m \), then I conjecture that there is generically a \( (\sum_{\alpha=1}^{d} \varphi_\alpha + \sum_{\alpha=d+1}^{m} (\varphi_\alpha - 1)) \) complex parameter family of solutions for given fluxes. However, in this case the fluxes also have to satisfy the constraints (3.23). Here we have only shown that these constraints are necessary conditions for finite-energy solutions. It would be interesting to know whether they (or possibly some stricter version of them) are also sufficient for the existence of finite energy solutions.

Having counted the solutions it is then interesting to look at the structure - differentiable and metric - of the moduli spaces of solutions. The examples studied here suggest that the moduli spaces of vortices on \( \mathbb{R}^2 \) are typically diffeomorphic to \( \mathbb{C}^D \) for some \( D \) (determined by the above counting arguments) and have smooth metrics. However, in the limit \( e_a \to \infty \) for some \( a \in \{1, ..., m\} \), where vortices turn into (possibly gauged) textures a number of interesting things happen. The interpretation of the moduli changes. Whereas vortex moduli simply characterise the (unordered) zeros of the scalar fields, the moduli for textures include internal as well as position parameters. Taking certain limits of the internal parameters corresponds to making the texture infinitely spiky, and these limiting values are therefore forbidden as moduli for textures. Thus the moduli spaces for textures are typically obtained from those of vortices by removing certain lower dimensional algebraic submanifolds. Metrically, the moduli spaces of textures are also worse behaved than those of vortices. The example of the \( \mathbb{CP}^1 \) lumps suggest that non-complete and non-finite metrics arise. In studying moduli spaces of textures it is therefore useful to bear in mind...
that they may be thought of as singular limits of smooth vortex moduli spaces. In that way, singularities may be understood and, if desired, circumvented.

Some of the questions raised so far are completely answered in the case where GLSM’s are defined on compact Riemann surfaces. There the counting of solutions can be done using standard theorems in algebraic geometry, and much can be deduced about the topological and (complex) differentiable structure of the moduli spaces from the key observation that they (like the target spaces of non-linear sigma models encountered in sect. 4, and for similar reasons) are toric varieties; see ref. [6] for a pedagogical explanation of this statement and for further references. In particular the intersection ring of the moduli spaces is described in the literature, see again [6]. This is of interest from the point of view of vortex physics because knowing the intersection ring of the moduli spaces is the crucial ingredient in the study of the statistical mechanics of vortices in ref. [3]. There the statistical mechanics of Nielsen-Olesen vortices is studied on (amongst other Riemann surfaces) the torus, which is equivalent to studying vortices on the plane with periodic boundary conditions. Thus it appears that much of the information needed similarly to study the statistical mechanics of the more general types of vortices described here can be found in the mathematical and string theory literature. However, one important ingredient in the analysis of ref. [3], a certain normalisation factor, is lacking. Computing it requires some knowledge of the Riemannian structure that the moduli space inherits from the kinetic energy of the field theory. Since that metric is only natural from a (2+1)-dimensional point of view it has not been considered in the string theory context.

It is also interesting to ask some of the above questions for the Chern-Simons version of the GLSM’s introduced in this paper. Already the first step, the general analysis of the soliton spectrum, is more complicated in this case. There are now both topological and non-topological solitons and since the scalar potential no longer has a universal polynomial structure the finite energy condition needs to be analysed for each model separately. It is also natural to contemplate further generalisations, such as mixed models, where some of the $U(1)$ gauge fields are governed by Maxwell terms and others by Chern-Simons terms. It should even be possible to construct general abelian GLSM’s of Bogomol’nyi type where both a Maxwell and a Chern-Simons term is included for each gauge field. As shown first for the abelian Higgs model in ref. [28], later extended in ref. [29] and recently demonstrated in the context of the gauged $O(3)$ sigma model in refs. [24] and [26], this requires the introduction of additional neutral scalar fields. In such models the gauge
fields can get their masses from the Chern-Simons term (“topological mass”) or through the Higgs mechanism. The analysis in ref. [29] suggests that generalising this construction for abelian GLSM’s will lead to models which can accommodate a truly bewildering array of massless, massive, topological and non-topological particles. Thus it seems that the framework of abelian GLSM’s provides an extremely versatile kit for the construction of Bogomol’nyi type field theories in (2+1) dimensions. The Maxwell version mainly studied here also shows that this kit comes equipped with powerful tools for exploring the physics of these field theories.

**Acknowledgements**

I thank Robbert Dijkgraaf and Jae-Suk Park for discussions about the connection between GLSM’s and toric varieties and acknowledge financial support through a Pioneer Fund of the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).
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