Methods for solving the problem of the biological population in the two-case

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Abstract. Considered the methods for solving the problem of the Kolmogorov-Fisher type diffusion reaction in the two-dimensional case. Obtained invariant properties of solutions and a two-sided estimate of the solution of the problem. Presented the results of numerical experiments for various values of the parameters entering the equation in the two-dimensional case.

1. Introduction
To date, studies of linear mathematical models of biological processes are convenient for analysis, since for their underlying linear partial differential equations, general methods for their solution have been developed. [5] In applied problems, real biological processes are nonlinear, and the use of nonlinear mathematical models is relevant for their adequate description [4-12]. The hyperbolic growth of mankind, occurring in a regime with peaking and exceeding by tens of thousands of times all comparable processes, becomes the dominant function in solving the differential growth equation. For estimates of the Earth's population in the foreseeable future, modeling results can be compared with the calculations of the International Institute for Applied Systems Analysis (NASA), the United Nations and other agencies [6].

A regime with peaking is a process in which at one point or some area of space, or in the whole space, the temperature becomes infinite in a finite time, called the time of exacerbation [7]. Interest in regimes with exacerbation arose in the mid-seventies of the last century in connection with the study of non-stationary processes occurring in high-temperature plasma [8].

In the world, the widespread use of mathematical models of processes, described by quasilinear parabolic equations, is explained by the fact that they are derived from the fundamental conservation laws. Therefore, a situation is possible where the process of the biological population and the physical process, which do not seem to have anything in common, are described by the same nonlinear diffusion equation, only with different numerical parameters. Studies show that nonlinearities change not only the quantitative characteristics of processes, but also a qualitative picture of their course. It is interesting, from the point of view of applications, to study classes of nonlinear differential equations in which the unknown function and the derivative of this function enter the power-law fashion. Then, thanks to the comparison theorems for solutions, this class can be extended. In this case, it is simpler to find a suitable solution of the differential inequality than any exact solution of the parabolic equation describing the nonlinear processes of the biological population.

In the last decade, in connection with the growing interest in the problems of structuration, the study of models of multicomponent competing biological systems in the class of systems of nonlinear equations of the reaction-diffusion type has received a new impulse.

In [1], for a model of one population, it was shown that the nonlinear dependence of the migration stream of the species on the local density of the population makes it possible to adequately describe the characteristic behaviour of population dynamics observed during the development of a population outbreak. In the Malthusian description of demographic processes, the outbreak develops at large times exponentially both in terms of local density and over the captured area of the outbreak for any nonnegative values of the exponents of migration flows. For all the considered variants of describing
demographic processes, the development of the flare at small times qualitatively coincides with the Malthusian kinetics at the same times. However, for a long time its development takes place in a different way. The key here is the relationship between the exponents in the terms describing migration and demographic processes.

A review of foreign scientific research shows that starting with Turing's work, mathematical models for the formation and propagation of nonlinear waves and processes of structural self-organization in physical, chemical, biological and social systems of reaction-diffusion type, where nonlinear terms describe kinetics and transport processes are represented by isotropic diffusion (Oxford University, University of Cambridge) [9]. However, in many systems, more complex mechanisms of the diffusion type-nonlinear, anisotropic and cross-diffusion (cross-diffusion) are no less important. In the majority of works, direct diffusion (self-diffusion), described by equations whose diffusion coefficients are constant (Ecole des Hautes Etudes en Sciences Sociales, Institut de Math´ematiques de Toulouse, Universit´e Paul Sabatier, Universit`a degli Studi di Padova Dipartimento di Matematica [10]. In this class of systems, the processes of formation of space-time structures are determined by the diffusion coefficients and specific forms of the kinetic function of the reaction processes. The case when the diffusion coefficients are not constants, but depend on dynamic variables, corresponds to nonlinear diffusion. Examples of nonlinear diffusion occur in mass transfer processes in porous media, as well as in population models (Consortium of the Americas for Interdisciplinary Science and Department of Physics and Astronomy of University of New Mexico) [11]. Mathematical models with the coefficient of diffusion, depending on the density of bacteria, describe the formation of complex spatial structures with the growth of bacterial colonies. The regimes with peaking in spatial localization in open dissipative systems are described by models with nonlinear diffusion (Institute of Applied Mathematics, Institute of Theoretical and Experimental Biophysics, Tomsk State University) [12].

2. Statement of the problem

We consider the two-dimensional reaction problem with diffusion of the Kolmogorov-Fisher type.

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1} \left[ u^\sigma \frac{\partial}{\partial x_1} u \right] + \frac{\partial}{\partial x_2} \left[ u^\sigma \frac{\partial}{\partial x_2} u \right] + k(t, x)u \left(1 - u^\rho \right)
\]

(1)

In the region \( D = \Omega \times (0, T), \Omega \subset R^N, \Omega = \{ -b_\alpha < x_\alpha < b_\alpha, \alpha = 1, 2 \} \) with initial and boundary conditions

\[
u(0, x) = u_0(x) \geq 0,
\]

(2)

\[ u|_G = \mu(x, t), \ t \in (0, T), G - the boundary \ \Omega. \]

To solve this problem (1), (2) we use the initial approximation

\[
u_0(t, x) = \psi(t) \left( a - \frac{\sigma}{4} \xi^2 \right)^{1/\delta}; \quad a = 1; \quad \xi = \frac{x}{\tau^{1/2}}; \quad \tau(t) = \int_0^{\psi(\eta)} \frac{d\eta}{\beta^{1/\beta}}.
\]

when \( k(t, x) = k(t); \quad k(t) = \frac{1}{(1 + t)^\alpha}, \quad \alpha > 1; \quad \psi(t) = \left( 1 + e^{-\int_0^t k(t) dt} \right)^{1/\beta}. \)

In this case, we obtain the following condition is for the localization of the solution of problem (1),

\[
u(\infty) < +\infty, \quad q\int_0^1 e^{\eta} d\eta < +\infty.
\]

Note that the method of constructing the initial approximation is based on the splitting of equation (1) [1].

This problem was studied in a number of works in the case \( m = 0 \) and \( k(t, x) = \text{const} \ [2, 3-5]. \)
In [6], invariant properties of solutions are obtained. And in [1] the properties of the finite perturbation velocity were obtained for the models of Malthus, Olli, and Verhulst. In [4], the localization properties of the flare are established.

3. Properties of invariance of the solution

We show that the function

$$u(t, x) = \psi(t)w(\tau(t), x),$$

(3)

where \(\psi(t)\) the solution of the equation without the diffuse part of equation (1)

$$\frac{d\psi}{dt} = k(t)\psi(1 - \psi^\beta),$$

(4)

again satisfies an equation of the form (1).

In fact, after setting (3) in (1), taking into account (3), (4), it is easy to calculate that for \(w(t, x)\) we have equation

$$w = w(x_1, x_2, \tau(t)), \quad |x| = \sqrt{(x_1)^2 + (x_2)^2},$$

where \(w(\tau(t), x)\) the new unknown function, and \(\tau(t)\) is the function to be determined.

\(\psi(t)\) we find from the equation:

$$\frac{d\psi}{dt} = k(t)\psi(1 - \psi^\beta),$$

(4)

Substituting (3) into (1) and choosing from \(\frac{d\tau}{dt} = \psi^\alpha\) we have:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial x_1} \left[ w^\rho \frac{\partial}{\partial x_1} w \right] + \frac{\partial}{\partial x_2} \left[ w^\rho \frac{\partial}{\partial x_2} w \right] + \psi_1(t)w(1 - w^\beta),$$

(5)

where \(\psi_1(t) = k(t)\psi^\beta-a\).

However, it is obvious that by (4) \(\lim_{t \to \infty} \psi(t) = 1\), if \(\int_0^t k(t)dt\) exists. Therefore, it is possible to assume that for sufficiently large \(t\), \(\psi_1(t) \sim k(t)\), i.e. we again obtain equation (1). By virtue of this, we call the function \(w(t, x)\) the solution of equation (4), and \(w(\tau(t), x)\) the solution of equation (5) an invariant of equation (1).

Estimation of the solution of the problem (1), (2).

To solve the problem (1), (2), we have

Theorem 1. Let \(0 \leq u_0(x) \leq 1, \quad x \in R, \quad |x|/\tau^{1/2}.\) Then for the solution of problem (1), (2) in \(Q = \{t, x); t > 0; x \in R^N\}\) there is a two-sided estimate

$$\psi(t)(T + \tau)^{-N/(2 + \sigma N)} \left( a - \frac{\sigma}{4} \right)^{1/2} \leq u(t, x) \leq \psi(t)(T + \tau)^{-N/(2 + \sigma N)} \left( a - \frac{\sigma}{4} \right)^{1/2} \psi(t),$$

where \(\psi(t)\) the function defined above, \(N=2\).

Proof. To prove the theorem, we first obtain an upper bound. For this purpose, (1) make the change
\[
\begin{align*}
u(t, x) &= e^{\int_{j_t}^x \mu(y)dy} \ w(\tau(t), x) . \tag{6}
\end{align*}
\]

Then for \( w(\tau(t), x) \) we have equation
\[
\begin{align*}
\frac{\partial w}{\partial \tau} &= \nabla \left( w^\beta \nabla w \right) - k(t)e^{-\beta \int_{\mu}^x \eta(y)dy} \ w^\beta . \tag{7}
\end{align*}
\]
The function \( w_*(\tau(t), x) = (T + \tau(t))^{-N/(2+\sigma N)} \left( a - \frac{\sigma}{4} \xi^2 \right)^{1/\sigma} \) is an upper solution of equation (7), since
\[
w_*(\tau(t), x) \text{ is a solution of the equation } \frac{\partial w}{\partial \tau} = \nabla \left( w^\beta \nabla w \right) \text{ and } -k(t)e^{-\beta \int_{\mu}^x \eta(y)dy} \ w^\beta \leq 0 \text{ in } Q \text{ for any constant } T > 0 . \text{ Therefore, by the comparison theorem for solutions of [1], we have an upper bound}
\]
\[
\begin{align*}
u(t, x) &\leq e^{\int_{j_t}^x \mu(y)dy} \ w_*(\tau(t), x) \tag{8}
\end{align*}
\]
in \( Q \) if \( w_*(0, x) \leq u_0(x), x \in R^N \).

In order to obtain an estimate from below, we apply the nonlinear splitting method [1]. According to this method, the lower solution is sought in the form
\[
u(t, x) = \psi(t)w_*(\tau(t), x) , \tag{9}
\]
where \( \psi(t) \) the function defined above by formula (4).

Then from (5) we have
\[
\begin{align*}
\frac{\partial w}{\partial \tau} &= \frac{\partial}{\partial x_1} \left[ w^\sigma \frac{\partial}{\partial x_1} w \right] + \frac{\partial}{\partial x_2} \left[ w^\sigma \frac{\partial}{\partial x_2} w \right] + k(t)\psi^{\beta-\sigma}w_*(1-w_*^\beta) . \tag{10}
\end{align*}
\]
For a function \( (T + \tau(t))^{-N/(2+\sigma N)} \left( a - \frac{\sigma}{4} \xi^2 \right)^{1/\sigma} \) in \( k(t)\psi^{\beta-\sigma}w_*(1-w_*^\beta) \geq 0 \), if the constant \( T \geq 1 \).

Then, applying the comparison theorem for solutions [1], by (10), we have
\[
u(t, x) \geq \psi(t)(T + \tau(t))^{-N/(2+\sigma N)} \left( a - \frac{\sigma}{4} \xi^2 \right)^{1/\sigma} , \tag{9}
\]
(9) proves the validity of Theorem 1.

It seems important to generalize this problem to different cases. In particular, it has not been studied in the case of a heterogeneous medium (the diffusion coefficient is a function of the spatial variable), the response coefficient depends on time or is more complex.

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x_1} \left[ x^m u^\sigma \frac{\partial}{\partial x_1} u \right] + \frac{\partial}{\partial x_2} \left[ x^m u^\sigma \frac{\partial}{\partial x_2} u \right] + k(t, x)u(1-u^\beta) , \tag{11}
\end{align*}
\]
in case \( k(t, x) := k(t) ; \ k(t) = \frac{1}{(1+t)^{\alpha}} , \ \alpha > 1 ; \ \psi(t) = \left( 1 + e^{-\beta \int_{\mu}^x \eta(y)dy} \right)^{1/\beta} .
\]
in the region \( D = \Omega \times (0, T) , \ \Omega \subset R^N , \ \Omega = \{-b_a < x_a < b_a , \ \alpha = 1, 2\} \) with initial and boundary conditions
\[
u(0, x) = u_0(x) \geq 0 , \tag{12}
\]
\[
u|_{\Gamma} = \mu(x, t) , \ t \in (0, T) , \ \Gamma \text{ – the boundary } \Omega .
\]
To solve this problem (11), (12) we use the initial approximation
\[ u_0(t,x) = \psi(t) \left( a - \frac{\sigma}{4} \xi^2 \right)^{1/\alpha}; \quad a = 1; \quad \xi = \frac{|x|}{\tau^{1/2}}; \quad \tau(t) = \int_0^t \left[ \psi(\eta) \right]^{\eta} d\eta, \]

when \( k(t,x) := k(t); \quad k(t) = \frac{1}{(1+t)^\alpha}, \quad \alpha > 1; \quad \psi(t) = \left( 1 + e^{-\beta \int_0^t k(\eta)d\eta} \right)^{-1/\beta}. \)

In this case, the following condition is obtained for the localization of the solution of problem (11), (12):
\[ \tau(\infty) < +\infty, \quad q\int e^{\xi} d\eta < +\infty. \]

We note that the method of constructing the initial approximation is based on the splitting of equation (11), first leading it to a radially symmetric form [1].

Below we study the effect of inhomogeneity at \( m = 2(N-1), \quad N \geq 2, \) where \( N \) is the dimensionality of the space in the following cases:
\[ \beta < \sigma + 1, \quad \beta = \sigma + 1 \text{ and } \beta > \sigma + 1. \]

A picture of the evolution of the process over time on a computer using MathCAD is simulated. Below are the results of numerical experiments for different values of the parameters entering into the equation in the two-dimensional case:
\[ t \in [0,1]; \quad x_1 \in [-6,6]; \quad x_2 \in [-6,6]; \quad n = 1.1; \quad n_{x_1} = 30; \quad n_{x_2} = 30; \quad n_t = 100. \]

In all the cases considered, the number of iterations on the average did not exceed three for the given accuracy \( \varepsilon. \)

The program created in the input language Matlab allows you to trace visually the evolution of the process for different values of parameters and data.
The given data of numerous experiments show that the use of computer technology in population problems for wide ranges of changes in input parameters makes it possible to obtain results with a high degree of accuracy and with a small expenditure of computer time.

4. Population models with cross-diffusion with double nonlinearity

Consider the following system of two partial differential equations for the two-dimensional case:

$$\frac{\partial u_i}{\partial t} = f(u_1, u_2) + D_{11} \frac{\partial^2 u_i}{\partial x_1^2} + D_{12} \frac{\partial^2 u_i}{\partial x_2^2} + h_{11} \frac{\partial}{\partial x_1} \left( Q_i(u_1, u_2) \frac{\partial u_i}{\partial x_1} \right) + h_{12} \frac{\partial}{\partial x_2} \left( Q_i(u_1, u_2) \frac{\partial u_i}{\partial x_2} \right),$$

$$\frac{\partial u_j}{\partial t} = g(u_1, u_2) + D_{21} \frac{\partial^2 u_i}{\partial x_1^2} + D_{22} \frac{\partial^2 u_i}{\partial x_2^2} + h_{21} \frac{\partial}{\partial x_1} \left( Q_i(u_1, u_2) \frac{\partial u_i}{\partial x_1} \right) + h_{22} \frac{\partial}{\partial x_2} \left( Q_i(u_1, u_2) \frac{\partial u_i}{\partial x_2} \right).$$

When \( h_{11} = h_{12} = h_{21} = h_{22} = 0 \) the mathematical model (1) is a reaction-diffusion type system with diffusion coefficients \( D_{11} \geq 0, D_{12} \geq 0, D_{21} \geq 0, D_{22} \geq 0 \) (at least one \( D_{ij} \neq 0 \)). In the case when at least one of the coefficients \( h_{ij} \neq 0 \) (the sign can be arbitrary), system (1) is cross-diffusion. The linear cross-diffusion corresponds \( Q_{ij}(u, v) \neq const \) to \( i, j = 1, 2 \) \( i = 1, 2 \); nonlinear cross-diffusion, \( Q_{ij}(u, v) \neq const \) at least for one \( i \) and \( j \).

Cross-diffusion means that the spatial displacement of one object, described by one of the variables, occurs due to the diffusion of another object, described by another variable. At the population level, the simplest example-parasite (the first object) that is inside the "master" (the second object) is moved by diffusion of the host. The term "self-diffusion" (diffusion, direct diffusion, ordinary diffusion) implies the movement of an individual due to the diffusion flux from the region of high concentration, especially to the region of their low concentration. The term "cross-diffusion" refers to the movement / flow of one species of individuals / substances due to the presence of a gradient of other individuals /
substances. The magnitude of the cross-diffusion coefficient can be positive, negative or zero. A positive cross-diffusion coefficient indicates that the movement of individuals occurs in the direction of a low concentration of other individuals occurs in the direction of a high concentration of other species of individuals / substances. In nature, systems with cross diffusion are quite common and play an important role, especially in biophysical and biomedical systems.

Equation (13) is a generalization of the simplest diffusion model for the logistic model of population growth [1-12] of the Malthus type \( (1) \), of the Furschulst type \( (1) \), and the Ollie type \( (1) \) for the case of double nonlinear diffusion. In the case where, it can also be considered as an equation for nonlinear filtration, thermal conductivity with simultaneous source and absorption, whose powers are equal, respectively \( u_1 \), \( u_2 \), \( u_3 \), \( u_4 \).

Consider the spatial analogue of the Volterra-Lotka competition system with a nonlinear power-law dependence of the diffusion coefficient on the population density. In the case of the simplest Volterra competitive interactions between populations, it is possible to build numerically, and in some cases analytically, spatially inhomogeneous solutions [12].

5. Localization of the wave solution of reaction-diffusion systems with a double nonlinearity

Consider the parabolic system of two quasilinear reaction-diffusion equations for the problem of a biological population of the Kolmogorov-Fisher type in the domain \( Q=\{(t,x): 0< t < \infty, \ x \in \mathbb{R}^2\} \)

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \frac{\partial}{\partial x_1}\left(D_{11}u_1^{m_1-1}\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(D_{12}u_2^{m_2-1}\frac{\partial u_2}{\partial x_2}\right) + l(t)\frac{\partial u_1}{\partial x_1} + l(t)\frac{\partial u_2}{\partial x_2} + k(t)u_1(1-u_2^\beta), \\
\frac{\partial u_2}{\partial t} &= \frac{\partial}{\partial x_1}\left(D_{21}u_1^{m_1-1}\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(D_{22}u_2^{m_2-1}\frac{\partial u_2}{\partial x_2}\right) + l(t)\frac{\partial u_1}{\partial x_1} + l(t)\frac{\partial u_2}{\partial x_2} + k(t)u_2(1-u_1^\beta),
\end{align*}
\]

\( u_1 \big|_{x=0}(x) = u_{10}(x), \quad u_2 \big|_{x=0}=u_{20}(x), \)

which describes the process of the biological population in a nonlinear two-component medium whose diffusion coefficient is equal to \( D_{ij}u_{ij}^{m_{ij}-1}\frac{\partial u_{ij}}{\partial x_{ij}} \) and convective transfer with velocity \( l(t) \), where \( m_1, m_2, p, \beta_1, \beta_2 \) are positive real numbers, \( u_1 = u_1(t, x_1, x_2) \geq 0, \quad u_2 = u_2(t, x_1, x_2) \geq 0 \) are the desired solutions.

The Cauchy problem and boundary-value problems for system (13) in one-dimensional and multidimensional cases have been investigated by many authors.

The aim of this paper is to investigate the qualitative properties of the solution of problem (14) on the basis of self-similar analysis and its numerical solutions using the methods of modern computer technologies, to study the ways of linearization to the convergence of the iterative process with further visualization. Estimates of solutions and the resulting free boundary are found, which makes it possible to choose suitable initial approximations for each value of the numerical parameters.

It is known that nonlinear equations have wave solutions in the form of diffusion waves. By wave we mean a self-similar solution of equation (17) of the form

\[
\begin{align*}
u(t, x) &= f(\xi), \quad \xi = ct \pm x \\
u(t, x) &= u(t, x_1, x_2), \quad x = \sqrt{(x_1)^2 + (x_2)^2},
\end{align*}
\]

where the constant \( c \) is the wave velocity.

We now turn to the construction of a self-similar system of equations for (2), a system of equations that is simpler for research.
We construct a self-similar system of equations by the method of nonlinear splitting. Substitution in (17)

\[ u_i(t, x_1, x_2) = e^{-\int \frac{k_i(\zeta) d\zeta}{t}} v_i(t, \eta_1, \eta_2), \quad \eta_1 = x_1 - \int_0^t l_i(\zeta) d\zeta, \]

\[ u_2(t, x_1, x_2) = e^{-\int \frac{k_i(\zeta) d\zeta}{t}} v_2(t, \eta_1, \eta_2), \quad \eta_2 = x_2 - \int_0^t l_2(\zeta) d\zeta, \]

leads to the form:

\[
\begin{align*}
&\left( \frac{\partial v_1}{\partial \tau} + \frac{\partial v_1}{\partial \eta_1} \right) + \left( \frac{\partial v_1}{\partial \eta_2} \right) - k_1(t)v_1^{1,1}v_1^{1,1} - k_2(t)v_2^{1,1}v_2^{1,1} = 0, \\
&\left( \frac{\partial v_2}{\partial \tau} + \frac{\partial v_2}{\partial \eta_1} \right) + \left( \frac{\partial v_2}{\partial \eta_2} \right) - k_1(t)v_1^{2,1}v_1^{2,1} - k_2(t)v_2^{2,1}v_2^{2,1} = 0,
\end{align*}
\]

(15)

If \( k_i(p - (m_i + 1)) = k_i(p - (m_i + 1)), \) then choosing \( \tau(t) = \frac{e^{(m_i-1)k_i(p-2)k_i}}{e^{(m_i-1)k_i(p-2)k_i}} = \frac{e^{(m_i-1)k_i(p-2)k_i}}{e^{(m_i-1)k_i(p-2)k_i}}, \) we obtain the following system of equations:

\[
\begin{align*}
&\left( \frac{\partial v_1}{\partial \tau} + \frac{\partial v_1}{\partial \eta_1} \right) + \left( \frac{\partial v_1}{\partial \eta_2} \right) - a_1(t)v_1^{1,1}v_1^{1,1} = 0, \\
&\left( \frac{\partial v_2}{\partial \tau} + \frac{\partial v_2}{\partial \eta_1} \right) + \left( \frac{\partial v_2}{\partial \eta_2} \right) - a_2(t)v_2^{1,1}v_2^{1,1} = 0,
\end{align*}
\]

(16)

where \( a_1 = k_i((p-2)k_i + (m_i-1)k_2), \) \( b_1 = \frac{(2-p)k_i + (m_i-1)k_2}{(m_i-1)k_i + (p-2)k_2}, \)

\( a_2 = k_i((m_i-1)k_i + (p-2)k_2), \) \( b_2 = \frac{(2-p)k_i + (m_i-1)k_2}{(m_i-1)k_i + (p-2)k_2}. \)

If \( b_1 = 0, \) and \( a_1(t) = \text{const}, i = 1, 2, \) then the system has the form:

\[
\begin{align*}
&\left( \frac{\partial v_1}{\partial \tau} + \frac{\partial v_1}{\partial \eta_1} \right) + \left( \frac{\partial v_1}{\partial \eta_2} \right) - a_1v_1^{1,1}v_1^{1,1} = 0, \\
&\left( \frac{\partial v_2}{\partial \tau} + \frac{\partial v_2}{\partial \eta_1} \right) + \left( \frac{\partial v_2}{\partial \eta_2} \right) - a_2v_2^{1,1}v_2^{1,1} = 0,
\end{align*}
\]

The Cauchy problem for system (16) in the case \( b_1 = b_2 = 0 \) learned in [1] and the existence of wave global solutions and blow-up solutions is proved.

Below, we describe one of the methods for obtaining a self-similar system for the system of equations (16). It consists in the following. We first find a solution of the system of ordinary differential equations

\[
\begin{align*}
\frac{d\tau_i}{d\tau} &= -a_i\tilde{v}_i^{b_i}, \\
\frac{d\tilde{v}_i}{d\tau} &= -a_i\tilde{v}_i^{b_i},
\end{align*}
\]

of the form
\[ F_1(t) = c_1(t + T_0)^{-\gamma}, \quad F_2(t) = c_2(t + T_0)^{-\gamma_2}, \quad T_0 > 0, \]

where \( c_1 = 1, \quad \gamma_1 = \frac{1}{\beta_1}, \quad c_2 = 1, \quad \gamma_2 = \frac{1}{\beta_2}. \)

And then the solution of the system (14) - (16) is sought in the form

\[
\begin{align*}
  v_1(t, \eta_1, \eta_2) &= v_1(t)w_1(t, \eta_1, \eta_2), \\
  v_2(t, \eta_1, \eta_2) &= v_2(t)w_2(t, \eta_1, \eta_2),
\end{align*}
\]

and \( \tau = \tau(t) \) is chosen so

\[
\tau_1(t) = \int_0^t \frac{1}{1-\gamma_1(p-2) + \gamma_2(m_1-1)} \left(T + \tau\right)^{-1-\gamma_1(p-2) + \gamma_2(m_1-1)} dt,
\]

if \( \gamma_1(p-2) + \gamma_2(m_1-1) = \gamma_2(p-2) + \gamma_1(m_1-1) \).

Then for \( w_i(\tau, x), \ i = 1, 2 \) we get the system of equations

\[
\begin{align*}
  &\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \eta_1} \left( D_{11}w_{11}^{-1} \frac{\partial}{\partial \eta_1} \right) + \frac{\partial}{\partial \eta_2} \left( D_{12}w_{12}^{-1} \frac{\partial}{\partial \eta_2} \right) + \psi_1(w_1w_1^2 - w_2), \\
  &\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \eta_1} \left( D_{21}w_{21}^{-1} \frac{\partial}{\partial \eta_1} \right) + \frac{\partial}{\partial \eta_2} \left( D_{22}w_{22}^{-1} \frac{\partial}{\partial \eta_2} \right) + \psi_2(w_2w_2^2 - w_1),
\end{align*}
\]

where

\[
\begin{align*}
  \psi_1 &= \left(1 - \gamma_1(p-2) + \gamma_2(m_1-1)\right)^{-1}, \quad \text{if } 1 - \gamma_1(p-2) + \gamma_2(m_1-1) > 0, \\
  \psi_2 &= \left(1 - \gamma_2(p-2) + \gamma_1(m_1-1)\right)^{-1}, \quad \text{if } 1 - \gamma_2(p-2) + \gamma_1(m_1-1) > 0.
\end{align*}
\]

The representation of system (14) in the form (17) allows us to assume that, when \( \tau \to \infty, \ \psi_i \to 0 \) and

\[
\begin{align*}
  &\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \eta_1} \left( D_{11}w_{11}^{-1} \frac{\partial}{\partial \eta_1} \right) + \frac{\partial}{\partial \eta_2} \left( D_{12}w_{12}^{-1} \frac{\partial}{\partial \eta_2} \right) \psi_1 w_1w_1^2 w_2, \\
  &\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \eta_1} \left( D_{21}w_{21}^{-1} \frac{\partial}{\partial \eta_1} \right) + \frac{\partial}{\partial \eta_2} \left( D_{22}w_{22}^{-1} \frac{\partial}{\partial \eta_2} \right) \psi_2 w_2w_2^2 w_1.
\end{align*}
\]

Therefore, the solution of system (13) with conditions (17) tends to the solution of system (20).

If \( 1 - \gamma_1(p-2) + \gamma_2(m_1-1) = 0 \), the wave solution of system (6) has the form

\[ w_i(\tau(t), \eta_1, \eta_2) = y_i(\xi), \quad \xi = c \tau \pm \eta, \quad \eta = \sqrt{\left(\eta_1\right)^2 + \left(\eta_2\right)^2} \quad i = 1, 2, \]

where \( c \) is the velocity of the wave, and taking into account that the equation for \( w_i(\tau, \eta_1, \eta_2) \) without the lowest terms always has a self-similar solution in the case \( 1 - \gamma_1(p-2) + \gamma_2(m_1-1) \neq 0 \) we obtain the system
After integrating (20), we obtain a system of nonlinear differential equations of the first order

\[
\begin{aligned}
\frac{dy_1}{d\xi} + \frac{y_1}{m_1} + \psi_1(y_1 - y_2, y_2^\beta) &= 0, \\
\frac{dy_2}{d\xi} + \frac{y_2}{m_2} + \psi_2(y_2 - y_1, y_1^\beta) &= 0.
\end{aligned}
\]

After integrating (20), we obtain a system of nonlinear differential equations of the first order

\[
\begin{aligned}
\frac{dy_1}{d\xi} + \frac{y_1}{m_1} + cy_1 &= 0, \\
\frac{dy_2}{d\xi} + \frac{y_2}{m_2} + cy_2 &= 0.
\end{aligned}
\]

The system (21) has an approximate solution of the form

\[
\bar{y}_1 = A(a - \xi)^\gamma_1, \quad \bar{y}_2 = B(a - \xi)^\gamma_2,
\]

where

\[
\gamma_1 = \frac{(p - 1)(p - (m_1 + 1))}{(p - 2)^2 - (m_1 - 1)(m_2 - 1)},
\gamma_2 = \frac{(p - 1)(p - (m_2 + 1))}{(p - 2)^2 - (m_1 - 1)(m_2 - 1)}.
\]

And the coefficients A and B are determined from the solution of the system of nonlinear algebraic equations

\[
\begin{aligned}
(p^{m_1 - 1} A^{m_1 - 1} B^{m_1 - 1} - c), \\
(p^{m_2 - 1} A^{m_2 - 1} B^{m_2 - 1} - c).
\end{aligned}
\]

Then, taking expression

\[
u_1(t,x_1,x_2) = e^{-\frac{t}{\xi}} v_1(\tau(t),\eta_1,\eta_2),
\nu_2(t,x_1,x_2) = e^{-\frac{t}{\xi}} v_2(\tau(t),\eta_1,\eta_2)
\]

we have

\[
u_1(t,x_1,x_2) = Ae^{-\frac{t}{\xi}} (c\tau(t) - \xi)^\gamma_1, \\
u_2(t,x_1,x_2) = Be^{-\frac{t}{\xi}} (c\tau(t) - \xi)^\gamma_2, \quad C > 0.
\]

Due to the fact that

\[
[b\tau(t) - \int_0^t l_1(\eta)d\eta - x_i] = 0,
\]

if

\[x_i \geq [b\tau(t) - \int_0^t l_1(\eta)d\eta - x_i] < 0, \quad \forall t > 0,
\]

then

\[u_1(t,x) \equiv 0, \quad u_2(t,x) \equiv 0, \quad x_i \geq [b\tau(t) - \int_0^t l_1(\eta)d\eta - x_i] < 0, \quad \forall t > 0, \quad i = 1, 2.
\]

Therefore, the condition for localization of solutions of system (17) is conditions

\[
\int_0^\tau l_1(y)dy < 0, \quad \tau(t) < \infty \text{ for } \forall t > 0, \quad i = 1, 2.
\]
Condition (22) is a condition for the appearance of a new effect—the localization of the wave solutions (14). If condition (22) is not satisfied, then the phenomenon of finite propagation velocity of the disturbance takes place, i.e.

\[ u_i(t, x) = 0 \quad \text{at} \quad |x| \geq b(t), \quad \tau(t) = \int_{0}^{t} e^{-(m_1 + p-3)\int_{0}^{b(t)} dy} d\zeta, \] 

and the front goes away arbitrarily far, with increasing time, since \( \tau(t) \rightarrow \infty \) for \( t \rightarrow \infty \).

Investigation of the qualitative properties of the system (14) made it possible to carry out a numerical experiment, depending on the values entering into the system of numerical parameters. For this purpose, the constructed asymptotic solutions were used as the initial approximation. In the numerical solution of the problem for the linearization of system (14), linearizations were used by the methods of Newton and Picard. To construct a self-similar system of equations for a biological population, the nonlinear splitting method [3].

6. Computational experiment

For a numerical solution of problem (17) we construct a uniform grid

\[ \omega_h = \{x_i = ih, \quad h > 0, \quad i = 0, 1, ..., n, \quad hn = l\}, \]

and a time grid

\[ \omega_{h_1} = \{t_j = jh_1, \quad h_1 > 0, \quad j = 0, 1, ..., n, \quad \tau m = T\}. \]

We replace problem (17) with an implicit difference scheme and obtain a difference problem with an error \( O(h^2 + h_1) \).

As is known, the main problem for the numerical solution in nonlinear problems is the appropriate choice of the initial approximation and the method for linearizing the system (17).

Consider functions:

\[ v_{i0}(t, x_1, x_2) = v_i(t) \cdot \left(a - \xi\right)^{k_1}, \]

\[ v_{20}(t, x_1, x_2) = v_2(t) \cdot \left(a - \xi\right)^{k_2}, \]

where \( v_i(t) = e^{\mu \xi} \bar{v}_i(t) \) and \( v_2(t) = e^{2\mu \xi} \bar{v}_2(t) \) the functions defined above.

Recording \( (a)_+ \) means \( (a)_+ = \max(0, a) \). These functions have the property of a finite rate of propagation of perturbations [1]. Therefore, in the numerical solution of problem (16) - (17) at \( \beta_1 > \sigma_1 \), as functions of the initial approximation \( v_{i0}(t, x_1, x_2), i = 1, 2 \).

The program created in the input language MathCad allows you to trace visually the evolution of the process for different values of parameters and data.

Numerical calculations show that in the case of arbitrary values \( \sigma_0, \beta_0 > 0 \), the qualitative properties of solutions do not change. Below are the results of numerical experiments for different values of the parameters (Table 1).
Table 1. Results of the computational experiment

| Parameter Values | Results of the computational experiment |
|------------------|------------------------------------------|
| $m_1 = 0.3$, $m_2 = 0.7$, $p = 2.1$ | ![Graph 1](image1.png) |
| $\beta_1 = 5$, $k_1 = 2$ | ![Graph 2](image2.png) |
| $\beta_2 = 7$, $k_2 = 3$ | |
| $\epsilon p_x = 10^{-3}$ | |
| $m_1 = 2.2$, $m_2 = 2.2$, $p = 2.5$ | |
| $\beta_1 = 1$, $k_1 = 7$ | |
| $\beta_2 = 1$, $k_2 = 2$ | |
| $\epsilon p_x = 10^{-3}$ | |

7. Conclusion

It can be expected that further theoretical and experimental studies of excitable cross-diffusion systems will make an important contribution to the study of self-organization phenomena in all nonlinear systems from micro- and astrophysical systems to social systems.

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