Complexity behaviors of volatility dynamics for stochastic Potts financial model

Jie Wang

Abstract To investigate the price fluctuation mechanism of stock markets, this research aims to develop a novel stochastic financial model based on Potts dynamics and compound Poisson process. The new model considers two aspects: information interaction among traders and the uncertain events outside the system. Then, three different volatility statistics (return series $r_t$, absolute return series $|r_t|$ and volatility duration average intensity $V_t$) are introduced to explore the volatility and complexity properties of the proposed model. The descriptive statistical methods, such as basic statistical properties and distribution analysis, are studied to validate the practicable of the proposed stochastic financial model. The permutation Lempel-Ziv complexity of moving average series is referred to different volatility sequences to evaluate the complexity of the simulative data from proposed model and the real data from stock market. Moreover, the complexity analysis of fractional sample entropy and multiscale fractional sample entropy is improved to illustrate the complexity of volatility behaviors in different scales. Compared with the real stock data, the empirical results demonstrate that the new model could reproduce the fluctuation and volatility behaviors of real stock markets to some extent.

Keywords Stochastic Potts dynamics · Volatility analysis · Nonlinear analysis · Complexity behaviors · Fractional calculus

1 Introduction

Recently, nonlinear fluctuation and volatility dynamics of financial markets have attracted a lot of attention, because of the nonlinear characteristics of time-varying, non-stationary, and chaotic. Financial market is regarded as a complex dynamic system; then, rehabilitating and investigating nonlinear complexity characteristics of financial data have been active topics for deep comprehension and management of stock markets [1–5]. In recent decades, theoretical and practical studies show that price variations reveal significant statistical properties, such as power-law behavior, fat-tailed distribution, multifractality and complexity dynamics, etc. [6–11]. Numerous financial price dynamics models have been effectively constructed in simulating the volatility complex behaviors of financial markets based on statistical physics systems and stochastic particle systems [12–22]. Lux [12] applied sequential Monte Carlo estimation to two selected agent-based models of speculative dynamics with somewhat different flavors. Boswijk et al. [13] proposed a behavioral asset pricing model with endogenous evolutionary switching of investors between different forecasting strategies according to their relative past performances. Mike and Farmer [16] built an
empirical agent-based model for order placement to study the endogenous dynamics of liquidity and price formation in the order book. Stauffer and Penna [17] considered the Cont and Bouchaud herding model based on percolation theory for marker fluctuations. The results showed power-law distributions for short times and exponential truncation for longer time intervals, if they were made at the percolation threshold in 2-7 dimensions. Wang and Wang [18,21] developed a financial price model based on 3D Potts dynamics to investigate the volatility behaviors of return interval time series. Wang et al. [19] improved a novel nonlinear random interacting price dynamics by the combination of lattice oriented percolation and Potts dynamics, which concerns with the instinctive random fluctuation and the fluctuation caused by the spread of the investors’ trading attitudes, respectively. Zhang and Wang [20], Bornholdt and Wagner [22] used the Ising model to construct a financial price model, where the essential trading framework can be recast into an Ising-type spin model. The practical consequences reveal that these simulative models could reflect the statistical characteristics of real financial systems.

Price fluctuation and volatility dynamics are important factors in the research of financial system. Security logarithmic return is usually selected to discuss the fluctuation statistical characteristics. A variety of statistics for the measurement of price volatility behaviors are explored, such as return interval [18,23], volatility duration [24], continuous fluctuation intensity [25] and volatility duration-based series derived from volatility difference components [26]. Wang et al. [23] studied the volatility return intervals above a certain threshold \( q \) for 31 stock intraday data sets and found the scaling function \( f(x) \) is consistent for all 31 intraday data sets in various time resolutions, and the function is well approximated by the stretched exponential \( f(x) \sim e^{-ax^\gamma} \), with \( \gamma = 0.38 \pm 0.05 \) and \( a = 3.9 \pm 0.5 \). Yang et al. [24] proposed an approach to build a link between volatility intensity and duration, and the empirical research of volatility duration series of returns for Shanghai Stock Exchange Composite Index and the simulative data derived from stochastic contact model are analyzed by comparison. Wang and Li et al. [25,26] defined different fluctuation intensity analyses based on the volatility duration method. In this work, a novel volatility component named volatility duration average intensity is developed on the basis of volatility duration and the average change rate series, which can provide a new method on the statistical characteristics of financial market fluctuations. In addition, the research on the complexity analysis of financial system has drawn wide attention, which helps investors to better understand the variation statistics properties during investment decision. Many information theory methods to estimate the complexity have been used to different dynamic systems, such as multiscale sample entropy [27,28], Lempel-Ziv complexity [29–31], weighted fractional permutation entropy and fractional sample entropy [32], etc. Particularly, LZC put forward by Lempel and Ziv [33] and its derivatives have found numerous applications in characterizing the randomness of finite data series by measuring the number of distinct sub-strings.

Inspired by the above research results, we establish a new stochastic financial dynamic model based on Potts system and random theory. \( Q \)-state Potts model is a well-known statistical mechanics which is an generalized form of Ising model, and has been the focus research for decades [34–37]. Due to continuously affected by uncertain events, financial markets occasionally appear abnormal volatility characteristics. Then, employing random Poisson theory to simulate stock market extreme volatilities is helpful in researching the asset pricing model and financial risk management [38]. In order to estimate the fluctuation and volatility mechanism of proposed financial model and real financial markets, three different statistics are investigated, including the logarithmic return series \( r_t \), absolute return volatility series \( |r_t| \) and volatility duration average intensity series \( V_t \). The specific volatility analysis methods mainly involve three aspects, the statistical characteristics analysis, the nonlinear analysis and the complexity behavior analysis. We first analyze the descriptive statistics, fat-tail phenomena and power-law behaviors of volatility sequences for simulative data and real stock data. Then, nonlinear characteristics are a reflection of inherent behaviors in the time history of one (or more) of system variables, which may assume external messages for indicating their behaviors. Chaos analysis, such as the phase space reconstruction, the Lyapunov exponent, Kolmogorov–Sinai entropy, the mutual information method, the largest Lyapunov exponent, and so on, are powerful nonlinear methods which enable the extraction of characteristic parameters. In the present paper, we employ several classical methods to comparatively study the nonlinear properties of financial volatility series for the real
data and simulative data. Furthermore, the complexity behaviors of $r_i$, $|r_i|$ and $V_t$ are followed up with permutation Lempel-Ziv complexity (PLZC) method, which is a fresh complexity measure utilized to inspect the relevant uncertainty and complexity behaviors for financial time series [26,31,39]. In addition, the fractional sample entropy (FSE) [32] and the advanced method multiscale fractional sample entropy (MFSE) are adopted as complexity estimations for simulative data and real data.

In summary, the significant contributions in this research enumerate at least three points. First, our investigation has established a novel financial price model based on Potts dynamics and the compound Poisson, which enriches the modelling theory of financial markets and helps to further clarify the nonlinear and complex financial price dynamics. Second, we firstly introduce the volatility statistic (volatility duration average intensity series, $V_t$) into the stock market to survey the financial volatility dynamics. This provides a new tool to explore the fluctuation and nonlinear properties of financial markets. Third, a multiscale complexity entropy method is generalized based on FSE analysis, called MFSE. This method is then employed to financial simulative data and real data, which constitutes a new idea to uncover additional complexity behaviors of financial dynamics.

## 2 Stochastic Potts financial model

### 2.1 Brief description of Potts model

Potts system is a famous statistical physics dynamics [34–37] proposed by Potts in the early 1950s. It is an promotion system of the Ising model, which considers more than two components in the mechanism. The Potts model is related to a number of outstanding problems in lattice statistics, and its critical behavior is more abundant and more generalized than that of the Ising system. Consider the $d$-dimensional integer lattice $\mathbb{Z}^d$ and denote by $\mathcal{B}$ the set of bonds of the lattice (pairs of nearest neighbors). In the $Q$-state Potts model, let $\Omega_{2^d} = \{1, 2, \ldots, Q\}^{2^d}$ denote the space of spin configurations on $\mathbb{Z}^d$, an element of $\Omega_{2^d}$ usually notated $\sigma = \{\sigma_i : i \in \mathbb{Z}^d\}$. The spin $\sigma_i$ takes on one-integer values from 1 to $Q$, and the $Q$ is a parameter of the model. For every $\sigma \in \Omega_{2^d}$, the Hamiltonian system of the $Q$-state Potts model ($J > 0$) is

\begin{equation}
H_{2^d,b}(\sigma) = -J \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} - b \sum_i \delta_{\sigma_i,1},
\end{equation}

where $\delta$ is the Kronecker symbol, $\delta_{\sigma_i,\sigma_j} = 1$ only when $\sigma_i = \sigma_j$, $< i, j >$ denotes pairs of nearest-neighbor spins on the lattice, $J > 0$ is the ferromagnetic interaction between the nearest-neighboring vertices of the dynamic systems and the applied magnetic field $b$ acts on the (arbitrarily chosen) state 1. Consider a system with states $\sigma$ and Hamiltonian $H_{2^d,b}(\sigma)$, the partition function is [35,40]

\begin{equation}
Z_{2^d,b}(\sigma) = \sum_{\sigma} \exp(-\beta H_{2^d,b}(\sigma))
\end{equation}

\begin{equation}
= \sum_{\sigma} \exp\left(K \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} + h \sum_i \delta_{\sigma_i,1}\right).
\end{equation}

where $K = \beta J$ and $h = \beta b$, $\beta = 1/(k_B T)$, $k_B$ is the Boltzmann constant and $T$ is the temperature. The finite Gibbs state $P_{\mathbb{Z}^d}^\beta$ at inverse temperature $\beta$ is a probability measure given by

\begin{equation}
P_{\mathbb{Z}^d}^\beta = [Z_{2^d}^\beta]^{-1} \exp\{-\beta H_{2^d,b}(\sigma)\}.
\end{equation}

The free energy $F$ can be calculated by

\begin{equation}
F = -\frac{1}{\beta} \ln Z_{2^d,b}.
\end{equation}

So, if $X$ is some observable property of the system, such as its total energy or magnetization, with value $X(\sigma)$ for state $\sigma$, then its observed average dynamic value is

\begin{equation}
<X> = [Z_{2^d}^\beta]^{-1} \sum_{\sigma} X(\sigma) \exp\{-\beta H_{2^d,b}(\sigma)\}.
\end{equation}

Here, the $Q$-state Potts model with no external magnetic field ($b = 0$ and $h = 0$) is studied. The model stands an order-disorder transformation when $d \geq 2$, and the critical value is $\beta_c = \ln(1 + \sqrt{Q})$ in $d = 2$. When $\beta > \beta_c$, the $Q$-fold permutation symmetry of Eq. (1) is broken, and one of the $Q$ different ground states is picked over.
2.2 Modelling Stochastic Potts financial dynamics

We here establish a new stochastic financial model by integrating the two-dimensional three-state Potts dynamics and compound Poisson process with random jumps on a $L \times L$ regular lattice. Figure 1 shows the geometry of the proposed model, where “S” denotes the trader holds the selling opinion, “B” represents the trader with buying opinion and “N” stands for trader with neutral attitude. The proposed model premeditates the fluctuation caused by the interaction intensity among investors and the inherent randomness of the system, respectively. In the first part, the intensity of interactivity between adjoining components is quite fundamental. It changes on the basis of their position in the lattice and, as conventional of Potts models, little changes in interaction regulations do not alter the group attributes. Although the optimal number of states existed in Potts model is thus not restricted to 3, we apply a three-state Potts model that simulates (i) traders with the selling decision, (ii) traders with the buying decision, and (iii) traders with no trading decision, which we categorize as type 1, type 2, and type 3, respectively. Consider that financial volatility behavior is highly influenced by the quantity of traders $\omega^{(1)}(t)$ (traders of type 1), $\omega^{(2)}(t)$ (traders of type 2), and $\omega^{(3)}(t)$ (traders of type 3). We assess a single stock and suppose that there are $L^2$ traders of this stock who are situated in a square-lattice $L \times L \subset \mathbb{Z}^2$, and each trader can deal a unit number of stock at each time $t \in \{1, 2, \ldots, T\}$. At this time, the fluctuation of stock price is extremely affected by the number of traders who take buying or selling locations. When the number of traders in selling locations is smaller than the number of traders in buying locations, the market participants believe that if the stock price is undervalued, then the stock price gradually increases. The similar is correct in the opposite situation. Let $\omega_{ij}$ be the trading location of a trader ($1 \leq i \leq L, 1 \leq j \leq L$) at time $t$, and $\omega(t) = (\omega_{11}(t), \ldots, \omega_{1L}(t), \ldots, \omega_{L1}(t), \ldots, \omega_{LL}(t))$ be the configuration of locations for $L^2$ traders. A space of all configurations of locations for $L^2$ traders from time 1 to $t$ is defined by $W = \{\omega : \omega = (\omega(1), \ldots, \omega(t))\}$. For a given configuration $\omega \in W$ and a trading day $t$, let

$$M^{(k)}(\omega(t)) = |\omega^{(k)}(t)| \mid (k = 1, 2, 3),$$

which represents the number of $\omega^{(1)}(t), \omega^{(2)}(t),$ and $\omega^{(3)}(t)$ at time $t$, respectively. Suppose that the price changes are in accordance with the difference between demand and supply $M^{(k)}(\omega(t)) (k = 1, 2, 3)$, which is controlled by the strength parameter $\beta$, where $\beta$ stands for the intensity of information dissemination. Then define a random variable $\xi_i$ with values 1, −1, 0 when an trader is buying, selling or neutral with probabilities $p_1$, $p_{-1}$ or $1 - (p_1 + p_{-1})$, respectively. Meanwhile, these traders send a bullish, bearish, or neutral message into the market. From the above depiction and [41, 42], the stock price at trading day $t$ is given as

$$\mathcal{P}^1(t) = \exp \left[ \alpha_1 \sum_{k=1}^{3} \frac{\mathcal{M}^{(k)}(t)}{L^2} \right] \mathcal{P}^1(t - 1)$$

(8)

$$\mathcal{M}^{(k)}(t) = M^{(k)}(\omega(t)) \times \gamma_k \times \xi_i^k,$$

(9)

where $\alpha_1(> 0)$ is the depth indicator of the market, $\xi_i^k = -1, 0, 1 (k = 1, 2, 3)$ denote the trading opinion, and $\gamma_k$ is the effective strength of trading attitudes in the stock market such that $\gamma_1 + \gamma_2 + \gamma_3 = 1$. Then we have

$$\mathcal{P}^1(t) = \mathcal{P}(0) \exp \left[ \alpha_1 \sum_{s=1}^{T} \sum_{k=1}^{3} \frac{\mathcal{M}^{(k)}(s)}{L^2} \right],$$

(10)

where $\mathcal{P}(0)$ is the stock price at time 0.

Next, we discuss the sharp fluctuations of the financial market from the inherent randomness of the system. Let $B(\xi_i)$ be the $s$-th price jump, which is an independent identical distributed (i.i.d.) series, and suppose that $\eta_t$ is a Poisson process with intensity $\lambda$. The two
stochastic processes $B(\zeta_t)$ and $\eta_t$ are independent of each other. Then, the random sharp fluctuations for the financial system are described as

$$P_{\text{jump}}(t) = P(0) \exp \left\{ \alpha_2 \sum_{s=1}^{\eta_t} B(\zeta_s) \right\},$$

(11)

where $\alpha_2 (\geq 0)$ is the sensitivity parameter of the market.

In the final stage of the modelling, we assume the market volatility formation mechanism is caused by two parts. The first part is the interaction behaviors among market traders explained by Potts dynamics. The second part is the randomness of the sharp fluctuations which is expressed by compound Poisson process with random jumps. Combining the above two price models, the stock price process on the $t$-th trading day is defined as

$$P(t) = P(0) \exp \left\{ \alpha_1 \sum_{s=1}^{t} \sum_{k=1}^{3} \mathcal{M}(k)(s) + \alpha_2 \sum_{s=1}^{\eta_t} B(\zeta_s) \right\} + \mathcal{N},$$

(12)

where $P(0)$ is the stock price at time 0. The corresponding formula of the stock logarithmic return is

$$r(t) = r_t = \ln P(t) - \ln P(t-1), \quad t \in \{1, 2, \ldots, T\}.$$

(13)

### 3 Data description and statistical analysis

#### 3.1 Data preparation and processing

To investigate the nonlinear and complexity behaviors of financial price fluctuations, we select daily closing data of four real global indexes from [http://www.wind.com.cn](http://www.wind.com.cn), including Shanghai Stock Exchange Composite Index (SSE), Hang Seng Index (HSI), Nikkei 225 (N225) and Dow Jones Industrial Average (DJIA) from January 3, 2011, to January 4, 2021. The simulative data derived from the proposed stochastic Potts financial model are determined jointly by several parameters: the intensity parameter $\beta$ for Potts system, the intensity parameter $\lambda$ in the compound Poisson jump process and the trading days $T$, etc. Among them, the intensity parameter $\beta$ represents the intensity of information disseminations strength among investors, and parameter $\lambda$ represents the strength of sudden jump fluctuations. The selection of parameter sets is based on a large number of simulative experiments, and the researcher’s prior knowledge and expertise to ensure the simulative data can reflect the main characteristics of real market price dynamics. After plenty of simulative experiments, the more appropriate parameter sets are set to $\{\beta = \beta_c, \lambda = 10\}, \{\beta = 3, \lambda = 5\}$ and $\{\beta = 3, \lambda = 10\}$, and the number of trading days $T$ is set to 2500. The software used in this article is MATLAB R2018a. Figure 2 displays the price series and the corresponding returns for the real data and the simulative data.

In order to study the statistical and complexity behaviors of volatility sequences, we analyze the logarithmic returns $r_t$, absolute returns sequences $|r_t|$ and a new return volatility component, volatility duration average intensity $V_t$ for the daily price changes. The daily logarithmic return series $r_t$ is the most commonly used statistic of fluctuations analysis in financial theory, which can directly express the price changes between two consecutive trading days. $|r_t|$ can be viewed as corresponding volatility sequences. $V_t$ is a new method to estimate the volatility of stock prices, which is an improved form of the volatility duration and the average variation intensity of volatility. In the following, we here introduce the volatility duration length series $I(t)$ at trading day $t$. If $|r_{t+1}| > |r_t|$, it indicates that there is a local upward movement at trading day $t$. If $|r_{t+1}| < |r_t|$, the volatility sequences shows sectional downward trend at day $t$. $I(t)$ is defined to record the local trend length of duration volatility intensity, which is described as

$$I(t) = \begin{cases} \max \{\tau : |r_{t+i}| > |r_t|, i = 1, 2, \ldots, \tau\} \\ \text{if } |r_{t+i}| - |r_t| > 0 \\ \max \{\tau : |r_{t+i}| < |r_t|, i = 1, 2, \ldots, \tau\} \\ \text{if } |r_{t+i}| - |r_t| < 0. \end{cases}$$

(14)

In the case of $|r_{t+1}| = |r_t|$, $I(t) = 0$. Then, let $\Delta|r_t(i)|$ denote the volatility difference module at time $i$ in the duration $I(t)$ at time $t$, which is described as

$$\Delta|r_t(i)| = |r_t(t + i)| - |r_t(i)|, \quad 1 \leq i \leq I(t).$$

(15)
Fig. 2  a Price time series of real indexes and the simulative data of the stochastic Potts financial model. b Plots of return series $r_t$ for real indexes and the simulative data of the stochastic Potts financial model.

Fig. 3 Illustrations of $I(t), \Delta|r_t(i)|$ in the volatility series

The average variation intensity of volatility $A_t$ in the volatility duration series $I(t)$ is defined as

$$A_t = \frac{1}{I(t)} \sum_{i}^{I(t)} \Delta|r_t(i)|, \quad t = 1, 2, \ldots, N - 1. \quad (16)$$

The volatility duration average intensity $V_t$, $t = 1, 2, \ldots, N - 1$ based on series $I(t)$ and $A_t$ is defined as

$$V_t = \text{sign}(|r_{t+1}| - |r_t|) \times (I(t) \times A_t)^p \quad (17)$$

$$= \text{sign}(|r_{t+1}| - |r_t|) \times \left( \sum_{i}^{I(t)} \Delta|r_t(i)| \right)^p \quad (18)$$

where parameter $p$ can adjust extreme values. We set $p = 1/2$ in this paper. Figure 4 shows an example of volatility duration $I(t)$ and volatility difference component $\Delta|r_t(i)|$ in $V_t$. The final $V_t$ sequences of real data for different stock markets and the simulative data for proposed model are demonstrated in Fig. 3.

3.2 Statistical properties of financial volatility series

The statistical properties of financial fluctuation and volatility series have always been the key basis of the theory and practical in asset portfolio, asset pricing, financial risk management, stock volatility forecasting, etc. [41,43,44]. Empirical results have indicated that the probability density of returns are not follow Gaussian distribution, but demonstrate the significant features of excess kurtosis and fat tail. Actually, the logarithmic returns obeys a power-law distribution, $P(|r_t| > x) \sim x^{-\nu}$, where $r_t$ is the logarithmic return of price series and the exponent of $\nu$ is approximately equal to 3 [45]. The descriptive statistical analysis of different return series and volatility duration average intensity sequences is illustrated in Table 1.

Through the statistical research, the values for return series are between (-0.15,0.15), the mean values are basically around 0, the values of kurtosis are bigger than the value of standard normal distribution 3 and the corresponding skewness values are inconsistent with 0 of the Gaussian distribution. For sequences $V_t$, the

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Fig. 4 a The volatility duration average intensity sequences $V_t$ of real data for different market indexes. b The volatility duration average intensity sequences $V_t$ of the simulative data with $\{\beta = \beta_c, \lambda = 5\}$, $\{\beta = \beta_c, \lambda = 10\}$, $\{\beta = 3, \lambda = 5\}$, $\{\beta = 3, \lambda = 10\}$ for the stochastic Potts financial model

Table 1 Descriptive statistics and Kolmogorov–Smirnov test for returns and $V_t$ series of simulation data and real data

| Returns | Descriptive statistics | Kolmogorov–Smirnov test |
|---------|------------------------|-------------------------|
|        | Min        | Max        | Mean   | Std       | Kurtosis  | Skewness | $\nu$ | H   | CV | K-S stat |
| SSE    | -0.0887    | 0.0560     | $9.0996 \times 10^{-5}$ | 0.0135 | 9.6991    | -0.9574 | 2.6809 | 1   | 0.0275 | 0.4788 |
| HSI    | -0.0602    | 0.0552     | $7.0010 \times 10^{-5}$ | 0.0118 | 5.6250    | -0.3366 | 3.3788 | 1   | 0.0273 | 0.4819 |
| N225   | -0.1059    | 0.0773     | $3.8861 \times 10^{-4}$ | 0.0133 | 8.7601    | -0.4524 | 3.0038 | 1   | 0.0272 | 0.4792 |
| DJIA   | -0.1384    | 0.1076     | $3.8158 \times 10^{-4}$ | 0.0111 | 26.2062   | -1.0124 | 2.4492 | 1   | 0.0270 | 0.4811 |
| $\beta = \beta_c, \lambda = 5$ | -0.0910 | 0.0792 | $-2.7427 \times 10^{-4}$ | 0.0137 | 8.3853 | -0.0516 | 3.0174 | 1 | 0.0271 | 0.4767 |
| $\beta = \beta_c, \lambda = 10$ | -0.0575 | 0.0594 | $1.2529 \times 10^{-4}$ | 0.0114 | 5.8287 | -0.3582 | 3.3935 | 1 | 0.0275 | 0.4827 |
| $\beta = 3, \lambda = 5$ | -0.0899 | 0.0894 | $7.0701 \times 10^{-5}$ | 0.0109 | 10.4073   | -0.3832 | 3.2587 | 1 | 0.0270 | 0.4840 |
| $\beta = 3, \lambda = 10$ | -0.0704 | 0.0658 | $5.4794 \times 10^{-4}$ | 0.0133 | 6.3236 | -0.1251 | 3.3717 | 1 | 0.0272 | 0.4786 |
| $V_t$  | Min        | Max        | Mean   | Std       | Kurtosis  | Skewness | $\nu$ | H   | CV | K-S stat |
| SSE    | -10.2538   | 4.1989     | -0.0298 | 0.5722    | 115.0238 | -7.4699 | 1.5338 | 1   | 0.0275 | 0.3022 |
| HSI    | -8.3644    | 3.3896     | -0.0267 | 0.5189    | 89.7396 | -6.4875 | 1.6376 | 1   | 0.0273 | 0.3023 |
| N225   | -15.3126   | 2.7794     | -0.0310 | 0.5902    | 231.3262 | -11.1066 | 1.6884 | 1   | 0.0272 | 0.2958 |
| DJIA   | -10.4831   | 2.8552     | -0.0323 | 0.5096    | 141.7198 | -8.6897 | 1.5085 | 1   | 0.0270 | 0.3216 |
| $\beta = \beta_c, \lambda = 5$ | -10.2509 | 2.8552 | -0.0255 | 0.4894    | 119.0151 | -6.9690 | 1.7480 | 1 | 0.0271 | 0.2970 |
| $\beta = \beta_c, \lambda = 10$ | -8.3276 | 2.9808 | -0.0278 | 0.4774    | 70.5508 | -5.3898 | 1.5257 | 1 | 0.0275 | 0.3070 |
| $\beta = 3, \lambda = 5$ | -6.8464 | 4.3942 | -0.0195 | 0.4364    | 61.8496 | -3.8948 | 1.6454 | 1 | 0.0270 | 0.3093 |
| $\beta = 3, \lambda = 10$ | -6.6106 | 3.9170 | -0.0233 | 0.4831    | 62.6157 | -4.7977 | 1.6531 | 1 | 0.0272 | 0.2987 |

Specific values are among (-15,5), the mean values are close to 0 but less than 0, the standard deviation values are extremely approaching, the value of kurtosis are bigger than 3, and skewness are nonzero values. It evidences that the simulative data and the real data have excess kurtosis and nonzero skewness for return series and volatility duration average intensity sequences. The distributions of volatility return series $r_t$ and volatility sequences $V_t$ are power-law and different from the Gaussian distribution. In addition, the exponent $\nu$ for
returns series all is approximately equal to 3, and for $V_t$ sequences the values of $\nu$ are close to 1.6. Figure 5a and b exhibits the probability distributions of the $r_t$ and $V_t$ sequences for the simulative data, the real data and the Gaussian distribution, where the simulative data are with parameters $\{\beta = \beta_c, \lambda = 5\}$, $\{\beta = \beta_c, \lambda = 10\}$, $\{\beta = 3, \lambda = 5\}$, $\{\beta = 3, \lambda = 10\}$, the real data are SSE, HSI, N225 and DJIA. Figure 5c and d depicts the power-law distributions and the cumulative distributions of $|r_t|$ and $|V_t|$ for real data and the simulative data. Moreover, single-sample Kolmogorov–Smirnov (K-S) test of the real market data and the simulative data is investigated [46]. In K-S test, the significance level in the paper is set to 5%. The logical value $H = 1$ is returned in the K-S test if it rejects the null hypothesis that the distribution of normalized $r_t$ (or $V_t$) series follows the distribution of Gaussian at the given significance level, while $H = 0$ if it cannot. The empirical results are illustrated in Table 1, where ‘CV’ represents the critical value of whether to accept the hypothesis and ‘K-S stat’ is the value of the test statistics. It is observed that all the $H$ values are 1 and ‘K-S stat’ values are greater than the corresponding ‘CV’ for all $r_t$ and $V_t$ series; thus, the hypothesis is refused that the logarithmic $r_t$ and $V_t$ series distributions of simulative data and real data follow the Gaussian distribution.

The empirical results of basic statistical analysis demonstrate that the volatility return series ($r_t$) and volatility sequences ($V_t$) of the simulative data have the similar statistical characters with the real financial data. The results of the exponent $\nu$, K-S test, kurtosis and skewness show that the distribution of volatility series
$r_t$ and $V_t$ is power-law and different from the Gaussian distribution. Then, the statistics properties of simulative financial volatility series $r_t$ and $V_t$ with parameters \{\beta \neq \beta_c, \lambda = 5\}, \{\beta = \beta_c, \lambda = 10\}, \{\beta = 3, \lambda = 5\}, \{\beta = 3, \lambda = 10\} are closer to the real financial data.

3.3 Nonlinear analysis of financial volatility series

In this part, nonlinear techniques are applied to analyze the chaotic behaviors of the simulative series and the real financial volatility series. Chaos analysis, such as the phase space reconstruction, the Lyapunov exponent, Kolmogorov–Sinai entropy, and so on, are powerful nonlinear methods which enable the extraction of nonlinear dynamics intrinsic in the dynamical system from which the data series arises. Phase space reconstruction is to represent approximately the variables of the original system using reconstructed component, which can completely mirror the dynamic characteristics of the system. Applications of the phase space reconstruction and the theory behind have been highly focused on by medical scientists, physicists, mathematicians and engineers [21,47,48]. Phase space is reconstructed in a multidimensional phase space, and a proxy for the full multivariate phase space that ensures a full knowledge of the behavior over a system can be constructed from the single time series. Values of \( m \) and \( \tau \) can be constructed from the single time series, \( \tau \) is the time delay and \( m \) is the embedding dimension of the vector \( X_t \). The embedding theorem [49,50] guarantees a full knowledge of the behavior over a system contained in the time series of any one measurement and a proxy for the full multivariate phase space that can be constructed from the single time series. \( \tau \) and \( m \) are two key parameters in the phase space reconstruction. Values of \( \tau \) and \( m \) are obtained by the average mutual information method \[51\] and false nearest neighbor method \[50\], respectively. Figure 6 shows three-dimensional phase portraits of the reconstructed attractor of simulative data (\{\beta \neq \beta_c, \lambda = 5\}) and real data (SSE) for volatility series \( r_t \) and \( V_t \). The graphs correspond to reconstruction in three dimensions \( (m = 3) \) with delay time \( \tau = 2 \).

Largest Lyapunov exponent (LLE) is a typically nonlinear parameter used to quantify the chaos of the system [52]. It indicates the average rate of divergence of two neighboring attractor trajectories. The existence of a positive exponent for almost all initial conditions in a bounded dynamic system is a widely used definition of deterministic chaos. Any system containing with at least one positive Lyapunov exponent is considered to be chaotic. As all the LLE values are positive and close to 0, it denotes that the dynamical system is weakly chaotic. The LLE usually is computed by the Wolf’s algorithm \[53\] and used to analysis experimental data as a simple and robust method. The \( i \)-th Lyapunov exponent is defined by

\[
\lambda_i = \lim_{t \to \infty} \frac{\ln |U(t)|}{t} \tag{19}
\]

where \( U(t) \) is the average deviation from the unperturbed trajectory at time \( t \). Then, the LLE can be calculated by using a least-squares fit to the average rate at which these pairs split, along the trajectory, which is defined by

\[
y(i) = \frac{1}{n \Delta t} \sum_{j=1}^{n} \ln d_j(i), \tag{20}\]

where \( n \) is the total number of non-zero \( d_j(i) \). \( \Delta t \) is the sampling period of the time series. \( d_j(i) \) is defined as follows,

\[
d_j(i) = |X_{j+i} - X_{j+i}|, i = 1, 2, \ldots, \min\{N_m - j, N_m - \hat{j}\}. \tag{21}\]

The nearest neighbor \( X_j \) is found by searching for the point that minimizes the distance to the particular reference point \( X_j \). The spatiotemporal chaotic system of the proposed system can be considered as \( K \) dimensions dynamics. The mean Lyapunov exponent is defined as follows \[54\] :

\[
mLE = \frac{\sum_{i=1}^{K} \lambda_i}{K}. \tag{22}\]

Kolmogorov–Sinai entropy (KSE) evaluates the uncertainty of the signal with respect to time \[54,55\] and is computed from the embedded time series signal. It is also defined as metric entropy which is zero for non-chaotic signals and is greater than zero for chaotic
Fig. 6  a b 3-D phase space of SSE and \( \{ \beta = \beta_c, \lambda = 5 \} \) for \( r_t \) series. c d 3-D phase space of SSE and \( \{ \beta = \beta_c, \lambda = 5 \} \) for \( V_t \) sequences.

signals. Without loss of generality, the Kolmogorov–Sinai entropy density is employed here to eliminate the effect of number of lattices, which is presented as follows [56]:

\[
d_{KSE} = \frac{\sum_{i=1}^{K} \lambda_i^+}{K}
\]  

(23)

where \( \lambda_i^+ \) are positive Lyapunov exponents of \( K \) lattices. \( d_{KSE} = 0 \) indicates no positive Lyapunov exponent, and hence, the spatiotemporal chaotic dynamics is in regular behavior. On the other hand, \( d_{KSE} > 0 \) indicates the system in chaos. The Kolmogorov–Sinai entropy universality is the percentage of lattices in chaos, which evaluates the space complexity in \( K \) dimensions of dynamics. Additionally, Kolmogorov–Sinai entropy universality \( u_{KSE} \) that can reveal the space complexity in \( K \) dimensions of dynamics is defined as follows [56]:

\[
u_{KSE} = \frac{K'}{K}
\]  

(24)

where \( K' \) is the number of positive Lyapunov exponent in spatiotemporal chaotic system of the proposed system.

Table 2 exhibits the results of LLE, mLE, \( d_{KSE} \) and \( u_{KSE} \) for the volatility series \( r_t, V_t \) of simulative and real data. From this table, all LLE exponents of volatility series are positive and finite, and the values of LLE are within the interval of [0.0681, 0.0924], which indi-
Table 2  The results of nonlinear analysis for the simulative and real volatility system

| Data sets        | Time delay | ED  | LLE  | mLE  | \(d_{KSE}\) | \(u_{KSE}\) |
|------------------|------------|-----|------|------|-------------|------------|
| Volatility series \(r_t\) | 2          | 3   | 0.0768 | 0.0051 | 0.0264 | 0.7249   |
| HSI              | 2          | 4   | 0.0793 | 0.0078 | 0.0189 | 0.7084   |
| N225             | 2          | 4   | 0.0782 | 0.0056 | 0.0195 | 0.6516   |
| DJIA             | 2          | 3   | 0.0869 | 0.0103 | 0.0297 | 0.6908   |
| \(\beta = \beta_c, \lambda = 5\) | 2          | 3   | 0.0857 | 0.0109 | 0.0253 | 0.6157   |
| \(\beta = \beta_c, \lambda = 10\) | 2         | 4   | 0.0736 | 0.0071 | 0.0201 | 0.7129   |
| \(\beta = 3, \lambda = 5\) | 2          | 4   | 0.0681 | 0.0045 | 0.0173 | 0.7351   |
| \(\beta = 3, \lambda = 10\) | 2          | 3   | 0.0765 | 0.0078 | 0.0206 | 0.6812   |
| Volatility series \(V_t\) | 2          | 3   | 0.0924 | 0.0074 | 0.0303 | 0.7313   |
| SSE              | 2          | 4   | 0.0822 | 0.0066 | 0.0215 | 0.7255   |
| HSI              | 2          | 3   | 0.0887 | 0.0071 | 0.0229 | 0.8047   |
| N225             | 2          | 3   | 0.0904 | 0.0089 | 0.0271 | 0.6991   |
| DJIA             | 2          | 3   | 0.0860 | 0.0097 | 0.0236 | 0.5939   |
| \(\beta = \beta_c, \lambda = 5\) | 2          | 4   | 0.0800 | 0.0069 | 0.0223 | 0.7516   |
| \(\beta = \beta_c, \lambda = 10\) | 2         | 4   | 0.0821 | 0.0084 | 0.0245 | 0.7469   |
| \(\beta = 3, \lambda = 5\) | 2          | 4   | 0.0835 | 0.0062 | 0.0257 | 0.6916   |

Notes: “ED” represents embedding dimension.

The empirical results show the existence of chaotic property of the real and simulative financial volatility dynamics and also demonstrate they possess similar chaotic property to some extent.

4 Complexity behaviors for Potts financial dynamics

4.1 Permutation Lempel-Ziv complexity algorithm

The Lempel-Ziv complexity (LZC) proposed by Lempel and Ziv [33] is a nonparametric estimation of complexity in a one-dimensional time series that is corresponding to the number of distinct substrings and the frequency of their recurrence. The LZC is applied to measure the randomness and complexity of a finite length sequence and is relevant with such theoretical characteristics as entropy and compression ratio. Due to its advantage of capacity of anti-interference to noise, nonparametric needing and easily computing, LZC is applied to measure complexity of dynamical systems in many fields [29–31,57]. In this research, we employ a novel method, permutation Lempel-Ziv complexity (PLZC) to study the volatility complexity of the real data and the simulative data from the proposed financial model. For calculating the PLZC, the original series must be transformed into a finite symbol string \(s(t)\), this measurement is called coarse-graining. Here we introduce the permutation pattern method to generate symbolic string \([31,57,58]\). For a time series \(x(t) (t = 1, 2, \ldots, N)\), given an embedding dimension \(m\) and a delay \(\kappa\), \(x(t)\) is embed to an \(m\)-dimensional space \(X(t) = [x(t), x(t + \kappa), \ldots, x(t + (m - 1)\kappa)] (t = 1, 2, \ldots, N - (m + 1)\kappa)\). Then, arrange the components of \(X(t)\) in an increasing order \(x^1(t + (h_1 - 1)\kappa) \leq x^2(t + (h_2 - 1)\kappa) \leq \cdots \leq x^m(t + (h_m - 1)\kappa)\). In order to ensure all data points are mapped into one of the possible permutations, when values are equal we order by the index of the data. For example, if \(x(t + (h_i - 1)\kappa) = x(t + (h_j - 1)\kappa)\) and \(h_i \leq h_j\), let
At the beginning, set the complexity counter \( c(n) = 1 \). Any vector \( X(t) \) has a permutation \( \pi_1 = [h_1, h_2, \ldots, h_m] \), which is one of permutations of \( m \) distinct symbol set \( \{1, 2, \ldots, m\} \). For each permutation \( \pi_l(l \in \{1, 2, \ldots, m!\}) \) of symbol set \( \{1, 2, \ldots, m\} \), it has only one corresponding character \( c_l(c_l \in \{c_1, c_2, \ldots, c_m!\}) \). So, the original series \( x(t) \) is transformed to a character sequence \( s(t) \) with no more than \( m! \) sorts of different characters in. Moreover, \( s(t) \) labels the local shape of \( x(t) \), since every character in \( s(t) \) mirrors a local permutation pattern of \( x(t) \).

For a time series \( x(t)(t \in 1, 2, \ldots, n) \), the PLZC of \( x(t) \) is estimated by the following steps [18, 29, 30]:

1. Let \( S \) and \( Q \) represent two subsequences of the original subsequence and \( S \) be a concatenation of \( S \) and \( Q \). Here \( SQ\pi \) denotes the sequence extracted from \( S \) in which the last character is deleted.

2. Let \( \pi(SQ\pi) \) represent the set comprising all the different subsequences of \( SQ\pi \).

3. Set dimension indicator \( m \) and time delay parameter \( \kappa \). Convert the original series \( x(t) \) to a character sequence \( s(t) \) whose length is \( n \) in accordance with the permutation pattern method.

4. At the beginning, set the complexity counter \( c(n) = 1 \), \( S = s(1) \), \( Q = s(2) \), \( SQ = \{s(1), s(2)\} \), and \( SQ\pi = s(1) \).

5. Generally, consider that \( S = \{s(1), \ldots, s(r)\} \), \( Q = s(r + 1), \) so \( SQ\pi = \{s(1), \ldots, s(r)\} \). If \( Q \in \pi(SQ\pi) \), then \( Q \) is a subsequence of \( SQ\pi \), not a new sequence.

6. \( S \) does not change and renew \( Q \) by adding \( s(r + 2) \) to \( Q \), i.e., \( Q = \{s(r + 1), s(r + 2)\} \), then judge whether \( Q \) belongs to \( \pi(SQ\pi) \) or not.

7. Repeat steps (5) and (6) until \( Q \) is no longer a part of \( \pi(SQ\pi) \) and \( Q = \{s(r + 1), \ldots, s(r + i)\} \) is no longer a subsequence of \( SQ\pi = \{s(1), \ldots, s(r + i - 1)\} \) but a new sequence. We thus add \( c(n) \) by one.

8. Subsequently, \( S \) and \( Q \) are aggregated and renewed, denoting as \( \{s(1), \ldots, s(r + i)\} \), and \( s(r + i + 1) \).

9. Repeat the above steps until \( Q \) contains the last character. \( c(n) \) is the complexity of symbol sequence \( s(t) \) which implies the number of distinct new patterns in the original time series.

The total number of subsequences present in \( s(t) \) has an upper bound, denoted as \( L(n) \)

\[
L(n) = c(n)[\log_m!c(n) + 1].
\]

Then, the PLZC output is the normalized \( c(n) \), defined as

\[
C(n) = \frac{c(n)[\log_m!c(n) + 1]}{n}.
\]

4.2 Empirical research by PLZC

The PLZC complexity behaviors of different volatility sequences for the real data and the simulative data are investigated in this part. The moving averages (MA) price is widely used to analyze the trend of stock markets. The function of moving average is to smooth out short-term fluctuations and highlight long-term trends or cycles. In general, many financial investors usually refer to MA prices and the daily close prices to make investment decisions. Then, we utilize the MA prices to study the fluctuations of the short-term and long-term financial series. The MA prices are defined as follows

\[
r^d_i = \frac{1}{d} \sum_{k=t}^{t+d-1} r_k, \quad t = 1, 2, \ldots, N - d + 1,
\]

where \( r(t \in 1, 2, \ldots, N) \) is the return series. In addition, the PLZC of the volatility sequences with different exponents is expressed by \( (r_t)^q \), \( |r_t|^q \), \( (V_t)^q \). For different values of \( q \) (\( q = 1, 2, \ldots, 7 \)), different series manifest different levels of volatility. We explore the PLZC values of \( r^d_t \), \( |r_t|^d \), and the corresponding sequences \( V^d_t \) for the real data and the simulative data. The parameters for calculating PLZC are set to \( m = 4 \) and \( \kappa = 1 \) respectively.

Figure 7 demonstrates the PLZC curves of volatility series \( r^d_t, |r_t|^d, \) and \( V^d_t \) for SSE and simulative data (\( \beta = \beta_c, \lambda = 5 \)). Figure 8 shows the PLZC analysis results of \( r^d_t, |r_t|^d, \) and \( V^d_t \) for all real data and simulative data. Increasing \( d(d > 5) \) decreases the PLZC values, indicating that volatility series becomes regular and periodic with long-term MA series, and that the generation rate of new volatility behaviors also decreases when \( d(d > 5) \) increases. But for short-term MA series (when \( d \leq 5 \)), the PLZC values of real data and simulative data are irregular fluctuated. In addition, for different parameters \( q \), the PLZC analysis of volatility series \( r_t \) and \( V^d_t \) for all real data and simulative data shows significantly different complexity results. With an increase in time window length \( d \), the complexity value of PLZC decreases much slower for
Fig. 7 a b c PLZC plots of $r_i^{d,q}, |r_i|^{d,q}$ and $V_i^{d,q}$ for SSE, respectively. (d)(e)(f) PLZC plots of $r_i^{d,q}, |r_i|^{d,q}$ and $V_i^{d,q}$ for simulative data with parameter set $\{\beta = \beta_c, \lambda = 5\}$, respectively.
Fig. 8  a b PLZC plots of $r_t^{d,q}$ for real data and simulative data, respectively.  c d PLZC plots of $|r_t|^{d,q}$ for real data and simulative data, respectively.  e f PLZC plots of $V_t^{d,q}$ for real data and simulative data, respectively
$q = 1$ than the complexity values for $q > 1$. Furthermore, the PLZC curves of the simulative data are very close to the curves of the real data, which means the simulative data have similar complexity behaviors with the real data for fluctuation and volatility dynamics.

Table 3 provides the PLZC values of $r_i$, $|r_i|$ and $V_i$ of the real data (SSE, HSI, N225 and DJIA) and four groups of simulative data, where ‘Simu. 1’, ‘Simu. 2’, ‘Simu. 3’ and ‘Simu. 4’, respectively, represent the simulative data with $\{\beta = \beta_1, \lambda = 5\}$, $\{\beta = \beta_2, \lambda = 10\}$, $\{\beta = 3, \lambda = 5\}$ and $\{\beta = 3, \lambda = 10\}$. The PLZC of $r_i$ for the real data and the simulative data is all around 0.6300, which means the simulative data exist similar randomness and complexity. The PLZC of $|r_i|$ for DJIA is 0.9426, other 7 data sets (SSE, HSI, N225, ‘Simu. 1’, ‘Simu. 2’, ‘Simu. 3’ and ‘Simu. 4’) falls in (0.6250,0.6470). The PLZC of $V_i$ for all the data sets is close to 0.5700. Table 4 reveals the PLZC numerical results of volatility sequences $r_i$, $|r_i|$ and $V_i$ for moving average. Similarly, ‘Simu. 1’, ‘Simu. 2’, ‘Simu. 3’, ‘Simu. 4’ in Table 3 stand for the simulative data with $\{\beta = \beta_1, \lambda = 5\}$, $\{\beta = \beta_2, \lambda = 10\}$, $\{\beta = 3, \lambda = 5\}$ and $\{\beta = 3, \lambda = 10\}$, respectively. The PLZC of $r_i$, $|r_i|$ and $V_i$ decreases with the increase of $d$; this indicates that the MA of volatility sequences becomes more regular and periodic with the increase of $d$, and it also exhibits that there are less new pattern changes happening in these series. When $d = 5$, the PLZC value of $|r_i|$ is greater than the values of $r_i$, $V_i$. It suggests that the randomness and complexity in absolute returns are relatively stronger, followed by the sequence $V_i$. The above results show that the series $r_i$, $|r_i|$ and $V_i$ with different exponents $q$ and $d$ of the real data and the simulative data have similar complexity behaviors for fluctuation and volatility dynamics.

### 4.3 Complexity entropy analysis

In this section, we perform FSE and MFSE methods to investigate the complexity of real stock series and simulative series. Moreover, we make a comparison between the results and discuss their differences and similarities between these different data sets.

#### 4.3.1 Fractional sample entropy

Fractional sample entropy (FSE) is a new generalized method based on Sample entropy, which combining traditional sample entropy with fractional calculus [59,60]. Sample entropy, as a measure of degree of uncertainty, has a wide application in assessing the system complexity of time series signals [27,61]. The fractional calculus (FC) has been proposed by Leibniz in mathematics and has emerged as an important tool in the area of dynamic systems with complex behavior [62–68]. It deals with the generalization of integrals and derivatives to a non-integer order [69,70]. The generalized concepts motivate further development and new research avenues emerge. Based on the theory of Sample entropy and FC, FSE can explore fractional order dynamics and evolutionary information in a non-linear system, then much more accurate information of the non-stationary series can be obtained [32,71,72]. In general, a small fractional sample entropy value represents large regularity of time series.

For a time series $x(t)$ and its $r$-dimensional time-delay, embedding can be expressed as

$$X^r_{j} = \{x(j), x(j + 1), \ldots, x(j + (r - 1)\tau)\}$$

(j = 1, 2, \ldots, N - r \tau).

(28)

where $N$ is the length of the time series, $r$ is the embedding dimension and $\tau$ is the time-delay parameter. Two vectors $X^r_i$ and $X^r_j$ are considered as close if their distance

$$d(X^r_i, X^r_j) = \max \left\{ |x(i + k\tau) - x(j + k\tau)| : 0 \leq k \leq r - 1 \right\}$$

(29)

is smaller than a specified tolerance level $\hat{\tau}$, where $i \neq j$ to eliminate self-matches. Let $N^r_i(\hat{\tau})$ represent the number of vectors $X^r_j$ that are close to the vector $X^r_i$. Then, the probability $C^r(\hat{\tau})$ that any vector $X^r_i$ is close to the vector $X^r_j$ within a tolerance level $\hat{\tau}$ is described as

$$C^r(\hat{\tau}) = \frac{1}{n - r} \sum_{i}^{n-r} \frac{N^r_i(\hat{\tau})}{(n - r - 1)}.$$

(30)

The sample entropy of time series $x(t)$ is given by

$$H_{\text{SampEn}}(x, r, \hat{\tau}) = -\ln \left[ \frac{C^{r+1}(\hat{\tau})}{C^r(\hat{\tau})} \right].$$

(31)
Then, the fractional sample entropy of time series \( x(t) \) is defined as

\[
H_{FSE}(x, r, \alpha, \hat{\tau}) = -\frac{\ln C^{r+1}(\hat{\tau}) - \ln C^r(\hat{\tau}) + \psi(1) - \psi(1 - \alpha)}{\Gamma(\alpha + 1)} \\
\times \left[ \frac{C^{r+1}(\hat{\tau})}{C^r(\hat{\tau})} \right]^{-\alpha}, \tag{32}
\]

where the fractional order \( \alpha \in \mathbb{R} \), \( r \), and \( \hat{\tau} \) are critical in calculating the value of sample entropy, \( \Gamma(\cdot) \) and \( \psi(\cdot) \) represent the gamma and digamma functions. We here set parameters \( r = 2 \) and \( \hat{\tau} = l \times SD(l = 0.15) \), where \( SD \) is the standard deviation of the original time series [32,72].

### 4.3.2 Multiscale fractional sample entropy

Multiscale fractional sample entropy (MFSE) analysis takes into account the multiple time scales while measuring complexity, and it is applied to the financial time series analysis for the real data and the simulative data in this research. The complexity of dynamic sequences is not limited to the single scale. Costa et al. [27,73] first introduced the idea of multiscale entropy (MSE) analysis, for measuring the complexity of finite length.
Step 1: For one-dimensional time series \( \{x_t\} \) (\( t \in 1, 2, \ldots, N \)), construct the coarse-grained time series \( y_{j}^{(\tau)} \), \( \tau \) is the scale factor [73, 74], according to the equation below

\[
y_{j}^{(\tau)} = \frac{1}{\tau} \sum_{i=(j-1)\tau+1}^{j} x_i, \quad 1 \leq j \leq \left\lfloor \frac{N}{\tau} \right\rfloor.
\]  

Step 2: The multiscale fractional sample entropy is computed for each coarse-grained time series by using the FSE analysis; then the complexity entropy of MFSE can be researched as the function of scale factor \( \tau \).

4.4 Empirical results by FSE and MFSE

The FSE and MFSE analyses are applied to measure the complexity behaviors of the financial stock data (SSE, HSI, N225 and DJIA) and the simulative data from the proposed Potts financial model with parameter \( \{\beta = \beta_c, \lambda = 5\}, \{\beta = \beta_c, \lambda = 10\}, \{\beta = 3, \lambda = 5\} \) and \( \{\beta = 3, \lambda = 10\} \). Figure 9 depicts the FSE plots versus parameter \( \alpha \) of volatility sequences \( r_t, |r_t| \) and \( V_t \) for real stock data and simulative data, where \( \alpha \) increases from -1 to 0.6. The corresponding co-

Fig. 9  a b c FSE curves versus \( \alpha \) of volatility sequences \( r_t, |r_t| \) and \( V_t \) for real stock data and simulative data

Fig. 9  a b c FSE curves versus \( \alpha \) of volatility sequences \( r_t, |r_t| \) and \( V_t \) for real stock data and simulative data
crete results with $\alpha = -0.8, -0.5, -0.2, 0, 0.2, 0.5$ are shown in Table 5. In Fig. 9, the FSE entropy values for these volatility sequences first ascend with the increase of $\alpha$, and then starts to descend after reaching the maximum value. The return series get the maximum value when $\alpha$ is around 0.4, the absolute return series takes the maximum value when $\alpha$ is close to 0.3, and the $V_t$ sequences achieves the maximum value when $\alpha$ is in the vicinity of 0.2. Moreover, the FSE measures of DJIA for the three volatility sequences exhibit the lowest degree of complexity. The complexity and randomness of simulative data with parameter $\{\beta = \beta_c, \lambda = 5\}$ are very similar to DJIA. For the return series and the absolute series, HSI has the highest complexity than other 7 data sets, and as $\alpha$ increases, the complexity intensity decreases relative slowly. The complexity of simulative data with parameter $\{\beta = \beta_c, \lambda = 10\}$, $\{\beta = 3, \lambda = 5\}$ and $\{\beta = 3, \lambda = 10\}$ is in the middle of the complexity of SSE and HSI. For the sequences $V_t$, the simulative data and the real data show almost uniform complexity. From Table 5, the FSE reveals more differences for volatility sequences when $\alpha > 0$, but when $\alpha < 0$ the complexity of the volatility sequences has approximate values. The FSE values for all the real data and the simulative data are in [0.12, 0.19] when $\alpha = -0.8$. When $\alpha = 0.5$, the FSE values of return series are in [0.8, 2.9], and for the sequences $V_t$, the FSE values is between [-0.4, -0.1].
Figure 10 displays the MFSE curves for volatility sequences of the real financial stock data (SSE, HSI, N225 and DJIA) and the simulative data from the proposed Potts financial model with parameter \( \{ \beta = \beta_c, \lambda = 5 \}, \{ \beta = \beta_c, \lambda = 10 \}, \{ \beta = 3, \lambda = 5 \} \) and \( \{ \beta = 3, \lambda = 10 \} \). It can be seen from Fig. 10 that the MFSE results of different volatility series vary as the scale factor \( \tau \) increases, indicating the multiscale structure of the complexity and the necessity of studying the financial time series on multiple scales. With the scale factor increases, the MFSE complexity fluctuates sharply. For return series \( r_t \) of simulative data and real data, the complexity fluctuations are more severe than the other two volatility sequences. The complexity behavior of \( |r_t| \) changes relatively slightly as the scale factor increases, but fluctuation is also beginning to be obvious after \( \tau > 20 \). The MFSE complexity of sequences \( V_t \) has the similar results with \( |r_t| \), both for the real data and the simulative data. From the empirical results of Figs. 9 and 10, for the return series, the absolute return series and the sequences \( V_t \), the FSE and MFSE complexity entropy of the simulative data are approximate to those of real financial data.

Table 6 provides the numerical results of the MFSE analysis, where \( \tau = 1, 5, 10 \). The table shows that the MFSE values of return series \( r_t \) are much larger than the values of \( |r_t| \) and \( V_t \) with the same scale factor \( \tau \). In the case of \( \tau = 1 \), the MFSE values of \( r_t \), \( |r_t| \) and \( V_t \) for the simulative data from the proposed financial model are very close to the real financial data. When \( \tau = 5 \), the MFSE numerical results of return series for the real data and the simulative data nearly fluctuate in [2.2162, 2.8590]. For the sequences \( V_t \), the MFSE values of simulative data are a little bigger than the values of \( r_t \) and \( V_t \) for the real financial data. When \( \tau = 10 \), the complexity results of \( |r_t| \) for real data SSE, DJIA and the simulative data \( \{ \beta = 3, \lambda = 5 \} \) are smaller than the other five data sets, which means that the MFSE analysis for the simulative data and the real data are strongly persistent. In summary, all the return series show higher complexity and randomness for both the simulative data and real data, the complexity results of \( V_t \) at \( \tau = 1 \) have no obvious difference for the simulative data and real data. The simulative data sets from the proposed stochastic Potts model have the similar MFSE complexity behaviors with the real stock data by different scale factors \( \tau \).

5 Conclusion

In this study, a new stochastic financial price model is developed inspired by Potts dynamics and random jump process. The new model considers the fluctuations formation mechanism by combining the information interaction among traders with the randomness coming from the financial system and the external uncertain facts. Compared to the existing model based on Potts system, the proposed model comprehensively integrates the influence of micro- and macro-environmental factors.

To explore the volatility dynamics of the simulative data from the proposed model and the real financial data, return series \( r_t \), absolute return series \( |r_t| \) and a novel statistics volatility duration average intensity sequences \( V_t \) are employed. First, through basic statistical analysis, the volatility sequences \( r_t \), \( |r_t| \) and \( V_t \) of all simulative data are different from the Gaussian distribution, displaying significant features of excess kurtosis and fat tail, which are conforming to the relevant features of the real financial volatility series. Besides, it can be obtained that the sequences \( r_t \) and \( V_t \) obey the power-law distribution \( P(|r_t| > x) \sim x^{-\nu} \), where \( \nu \approx 3 \) for the return series and \( \nu \approx 1.6 \) for sequences \( V_t \). Then, the classical nonlinear methods are applied to analyze the chaotic behaviors of volatility dynamics for real and simulative data. After the estimation of the phase space reconstruction, largest Lyapunov exponent, Kolmogorov–Sinai entropy density and universality, the simulative volatility series with different parameters \( \beta \) and \( \lambda \) show similar nonlinear and chaotic properties with the real financial volatility series. Furthermore, the complexity behaviors of different volatility series are studied by applying PLZC, FSE and MFSE analysis. The PLZC results show that the volatility sequences \( r_t \), \( |r_t| \) and \( V_t \) with different parameters \( q \) and \( d \) of the real data and the simulative data have comparable permutation complexity for volatility dynamics. Especially for \( r_t \) and \( V_t \), the PLZC values with \( q = 1 \) are much larger than other exponent cases. The results of FSE reveal large differences when \( \alpha > 0 \) for real and simulative volatility dynamics. But when \( \alpha < 0 \), the complexity of real and simulative volatility dynamics has approximative values. The MFSE results of different volatility series vary as the scale factor \( \tau \) increases for all the real data and simulative data, which implies the indispensability of studying the financial volatility series on multiple scales.
Fig. 10  **a** **b** MFSE plots of $r_f$ for real data and simulative data. **c** **d** MFSE plots of $|r_f|$ for real data and simulative data. **e** **f** MFSE plots of $V_f$ for real data and simulative data
The empirical conclusions of the above statistical attributes, nonlinear analysis and complexity behaviors prove that the proposed financial price model is feasible, and it can beneficially reproduce the volatility dynamics of real stock markets to some extent. Further studies can extend the present study by considering the financial price model establishment based on other complex forms of space lattice, which can clearly uncover the heterogenous characters for the intensity and the mechanism of information interaction.

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Data availability statement The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declaration

Conflict of interest The authors declare no conflict of interest.

References

1. Farmer, J.D., Joshi, S.: The price dynamics of common trading strategies. J. Econ. Behav. Organ. 49, 149–171 (2002)
2. Lux, T., Marchesi, M.: Scaling and criticality in a stochastic multiagent model of a financial market. Nature. 397, 498–500 (1999)
3. Bouchaud, J.P., Potters, M.: Theory of Financial risk and derivative pricing: from statistical physics to risk management. Cambridge University Press, Cambridge (2003)
4. Stanley, H.E., Gabaix, X., Gopikrishnan, P., Plerou, V.: Economic fluctuations and statistical physics: the puzzle of large fluctuations. Nonlinear Dynam. 44, 329–340 (2006)
5. Gontis, V., Havlina, S., Kononovicuis, A., et al.: Stochastic model of financial markets reproducing scaling and memory in volatility return intervals. Physica A. 462, 1091–1102 (2016)
6. Gabaix, X., Gopikrishnan, P., Plerou, V., Stanley, H.E.: A theory of power-law distributions in financial market fluctuations. Nature. 423, 267–270 (2003)
7. Calvet, L., Fisher, A.: Multifractal Volatility: Theory, Forecasting, and Pricing. Academic Press, New York, USA (2008)
8. Grech, D., Pamula, G.: Multifractality of nonlinear transformations with application in finances. Acta Phys. Pol. A. 123, 529–537 (2013)
9. Amaral, L.A.N., Buldyrev, S.V., Havlin, S., et al.: Power law scaling for a system of interacting units with complex internal structure. Phys. Rev. Lett. 80, 1385–1388 (1998)
10. Wang, J., Wang, J.: Cross-correlation complexity and synchronization of the financial time series on Potts dynamics. Physica A. (2020). https://doi.org/10.1016/j.physa.2019.123286
11. Li, J., Shang, P., Zhang, X.: Financial time series analysis based on fractional and multifractal permutation entropy. Communications in Nonlinear Science and Numerical Simulation. 78, 104880 (2019)
12. Lux, T.: Estimation of agent-based models using sequential Monte Carlo methods. J. Econ. Dyn. Control. 91, 391–408 (2018)
13. Boswijk, H.P., Hommes, C.H., Manzan, S.: Behavioral heterogeneity in stock prices. J. Econ. Dyn. Control. 31(6), 1938–1970 (2007)
14. Durrett, R.: Lecture Notes on Particle Systems and Percolation. Wadsworth & Brooks, Pacific Grove-California (1998)
15. Zivot, E., Wang, J.H.: Modeling Financial Time Series with SPLUS. Springer, New York (2006)
16. Mike, S., Farmer, J.D.: An empirical behavioral model of liquidity and volatility. J. Econ. Dyn. Control. 32, 200–234 (2008)
17. Stauffer, D., Penna, T.J.P.: Crossover in the Cont-Bouchaud percolation model for market fluctuations. Physica A. 256, 284–290 (1998)
18. Wang, J., Wang, J., Stanley, H.E.: Multiscale multifractal DCCA and complexity behaviors of return intervals for Potts price model. Physica A. 492, 889–902 (2018)
19. Wang, Y., Zheng, S., Zhang, W., et al.: Modeling and complexity of stochastic interacting Lévy type financial price dynamics. Physica A. 499, 498–511 (2018)
20. Zhang, B., Wang, G., Wang, Y., et al.: Multiscale statistical behaviors for Ising financial dynamics with continuum percolation jump. Physica A. 525, 1012–1025 (2019)

21. Wang, J., Wang, J.: Measuring the correlation complexity between return series by multiscale complex analysis on Potts dynamics. Nonlinear Dyn. 89, 2703–2721 (2017)

22. Bornholdt, S., Wagner, F.: Stability of money: phase transitions in an Ising economy. Physica A. 316, 453–468 (2002)

23. Wang, F., Yamasaki, K., Havlin, S., Stanley, H.E.: Scaling and memory of intraday volatility return intervals in stock markets. Phys. Rev. E. (2006). https://doi.org/10.1103/PhysRevE.73.026117

24. Yang, G., Wang, J., Deng, W.: Nonlinear analysis of volatility duration financial series model by stochastic interacting dynamic system. Nonlinear Dyn. 80(1), 701–713 (2015)

25. Wang, H., Wang, J., Wang, G.: Nonlinear continuous fluctuation intensity financial dynamics and complexity behavior. Chaos: An Interdisciplinary Journal of Nonlinear Science. 28(8), 083122 (2018)

26. Li, R., Wang, J.: Symbolic complexity of volatility duration and volatility difference component on voter financial dynamics. Digital Signal Process. 63, 56–71 (2017)

27. Costa, M., Goldberger, A.L., Peng, C.K.: Multiscale entropy analysis of biological signals. Phys. Rev. E. (2005). https://doi.org/10.1103/PhysRevE.71.021906

28. Wu, Y., Shang, P., Li, Y.: Multiscale sample entropy and crosssample entropy based on symbolic representation and similarity of stock markets. Commun. Nonlinear Sci. Numer. Simul. 56, 49–61 (2018)

29. Hong, H., Liang, M.: Fault severity assessment for rolling element bearings using the Lempel-Ziv complexity and continuous wavelet transform. J. Sound Vib. 320, 452–468 (2009)

30. Fernández, A., LópezIbor, M.I., Turrero, A., et al.: Lempel-Ziv complexity in schizophrenia: a MEG study. Clin. Neurophys. 122, 2227–2325 (2011)

31. Bai, Y., Liang, Z., Li, X.: A permutation Lempel-Ziv complexity measure for EEG analysis. Biomed. Signal Process. Control. 19, 102–114 (2015)

32. Xu, K., Wang, J.: Weighted fractional permutation entropy and fractional sample entropy for nonlinear Potts financial dynamics. Phys. Lett. A. 381(8), 767–779 (2017)

33. Lempel, A., Ziv, J.: On the complexity of finite sequences. Inform. Theory IEEE Trans. 22, 75–81 (1976)

34. Potts, R.: Some generalized order-disorder transformations. Math. Proc. Camb. Philos. Soc. 48(1), 106–109 (1952)

35. Wu, F.Y.: The potts model. Rev. Modern Phys. 54(1), 235 (1982)

36. Gliozzi, F.: Simulation of potts models with real $q$ and no critical slowing down. Phys. Rev. E. (2002). https://doi.org/10.1103/PhysRevE.66.016115

37. Deng, Y., Blote, H.W.J., Nienhuis, B.: Backbone exponents of the two-dimensional $q$-state Potts model: a Monte Carlo investigation. Phys. Rev. E. (2004). https://doi.org/10.1103/PhysRevE.69.026114

38. Cont, R., Tankov, P.: Financial modeling with jump processes. Chapman & Hall CRC, USA (2004)

39. Tosun, P.D., Abássolo, D., Stenson, G., et al.: Characterisation of the Effects of Sleep Deprivation on the Electroencephalogram Using Permutation Lempel-Ziv Complexity. a Non-Linear Analysis Tool. Entropy 19(12), 673 (2017)

40. Baxter, R.J.: Exactly solved models in statistical mechanics. Elsevier, Amsterdam (2016)

41. Tsay, R.S.: Analysis of Financial Time Series. Wiley, Hoboken, USA (2005)

42. Ross, S.M.: An Introduction to Mathematical Finance. Cambridge University Press, Cambridge (1999)

43. Mantegna, R.N., Stanley, H.E.: An Introduction to Econophysics: Correlations and Complexity in Finance. Cambridge University Press, Cambridge (1999)

44. Gao, Z., Jin, N.: Complex network from time series based on phase space reconstruction. Chaos: An Interdisciplinary Journal of Nonlinear Science 19(3), 033137 (2009)

45. Han, L., Romero, C.E., Yao, Z.: Wind power forecasting based on principle component phase space reconstruction. Renew. Energy 81, 737–744 (2015)

46. Weber, M.D., Leemis, L.M., Kincaid, R.K.: Minimum Kolmogorov- Smirnov test statistic parameter estimates. J. Statist. Comput. Simul. 76(3), 195–206 (2006)

47. Banerjee, A., Chatterjee, S., Chatterjee, S., et al.: Simple simulation of two-dimensional Ising model with real $q$-state using Monte Carlo method. Phys. Rev. E. 83, 026120 (2011)

48. Bai, Y., Liang, Z., Li, X.: A permutation Lempel-Ziv complexity measure for EEG analysis. Biomed. Signal Process. Control. 19, 102–114 (2015)

49. Cont, R., Tankov, P.: Financial modeling with jump processes. Chapman & Hall CRC, USA (2004)

50. Lopes, A.M., Machado, J.A.T.: Multidimensional scaling analysis of generalized mean discrete-time fractional order
Complexity behaviors of volatility dynamics for stochastic Potts financial model

61. Richman, J.S., Lake, D.E., Moorman, J.R.: Sample entropy. Methods Enzymol. 384, 172–184 (2004)
62. Baleanu, D., Diethelm, K., Scalas, E., et al.: Fractional calculus: models and numerical methods. World Scientific, Singapore (2012)
63. Tarasov, V.E.: Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles. Fields and Media. Springer, Dordrecht (2010)
64. Machado, J.A.T., Silva, M.F., Barbosa, R.S., et al.: Some applications of fractional calculus in engineering. Math. Prob Eng. 2010, (2010)
65. Momani, S., Abu Arqub, O., Maayah, B.: Piecewise optimal fractional reproducing kernel solution and convergence analysis for the Atangana-Baleanu model of the Lienard’s equation. Fractals 28(8), 2040007 (2020)
66. Momani, S., Arqub, O.A., Maayah, B.: The reproducing kernel algorithm for numerical solution of Van der Pol damping model in view of the Atangana-Baleanu fractional approach. Fractals 28(8), 2040010 (2020)
67. Arqub, O.A.: Computational algorithm for solving singular Fredholm time-fractional partial integrodifferential equations with error estimates. J. Appl. Math. Comput. 59(1), 227–243 (2019)
68. Arqub, O.A., Rashaidah, H.: The RKHS method for numerical treatment for integrodifferential algebraic systems of temporal two-point BVPs. Neural Comput. Appl. 30(8), 2595–2606 (2018)
69. Miller, K.S., Ross, B.: An introduction to the fractional calculus and fractional differential equations. Wiley, Hoboken (1993)
70. Podlubny, I.: Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier, Amsterdam (1998)
71. Ubriaco, M.R.: Entropies based on fractional calculus. Phys. Lett. A 373(30), 2516–2519 (2009)
72. Zunino, L., Pérez, D.G., Martín, M.T., Garavaglia, M., Plastino, A., Rosso, O.A.: Permutation entropy of fractional Brownian motion and fractional Gaussian noise. Phys. Lett. A. 372, 4768–4774 (2008)
73. Costa, M., Goldberger, A.L., Peng, C.K.: Multiscale entropy analysis of complex physiological time series. Phys. Rev. Lett. (2002). https://doi.org/10.1103/PhysRevLett.89.068102
74. Thuraisingham, R.A., Gottwald, G.A.: On multiscale entropy analysis for physiological data. Phys. A. 366, 323–332 (2006)

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