Star-shaped Centrosymmetric Curves Under Gage’s Area-preserving Flow

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Abstract
It is proved that Gage’s area-preserving flow can evolve a centrosymmetric star-shaped initial curve smoothly, make it convex in a finite time and deform it into a circle as time tends to infinity.

Keywords Curve shortening flow · Area-preserving flow · Star-shaped

Mathematics Subject Classification 35C44 · 35K05 · 53A04

1 Introduction
Let \( X : S^1 \rightarrow \mathbb{R}^2(\varphi \mapsto (x(\varphi), y(\varphi))) \) be a \( C^1 \), regular, closed and embedded curve in the Euclidean plane. Let \( L \) be the length of \( X \) and \( s \) be the arc length parameter. Denote by \( T(s) \) the unit tangent vector at a point \( X(s) \) and by \( N(s) = N_{in}(s) \) the inward unit normal vector such that every ordered pair \( (T(s), N(s)) \) determines a positive orientation of the plane. The curve \( X \) is called star-shaped if there exists a point \( O \) inside the region bounded by \( X \) such that for every \( s \)

\[
\det(X(s), T(s)) = \begin{vmatrix}
  x(s) & y(s) \\
  x'(s) & y'(s)
\end{vmatrix} > 0.
\]

And the point \( O \) is called a star center of the curve. If \( X \) is a \( C^2 \) curve then the (relative) curvature is defined as

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\( \kappa(s) := \left\langle \frac{dT}{ds}(s), N(s) \right\rangle. \)

In this paper, we investigate the evolution behaviour of a star-shaped centrosymmetric plane curve under Gage’s area-preserving flow (GAPF) [5]

\[
\begin{align*}
\frac{\partial X}{\partial t}(\varphi, t) &= (\kappa(\varphi, t) - \frac{2\pi}{L(t)})N(\varphi, t) \quad \text{in} \quad S^1 \times (0, \omega), \\
X(\varphi, 0) &= X_0(\varphi) \quad \text{on} \quad S^1,
\end{align*}
\tag{1.1}
\]

where \( X : S^1 \times [0, \omega) \to \mathbb{R}^2((\varphi, t) \mapsto (x, y)) \) is a family of smooth, closed and star-shaped curves with \( X_0 \) being star-shaped and centrosymmetric about the origin \( O \), \( \kappa = \kappa(\varphi, t) \) the curvature and \( L = L(t) \) the length of \( X(\cdot, t) \). In 1984, Gage [5] proved that the evolving curve \( X(\cdot, t) \) under this flow can be deformed into a circle if the initial curve \( X_0 \) is convex. As Gage pointed out that there is a kind of simple closed curves which may develop singularities in a finite time under the flow (1.1). This fact is verified by Mayer’s numerical experiment [19]. So, unlike Grayson’s theorem [10] for the curve shortening flow (CSF), there is no convergence result of GAPF for generic simple closed initial curves. A natural question is whether there is a class of non-convex initial curves which may become convex and converge to a circle under the flow (1.1). To guarantee such a convergence, referring to Gage’s example [5, 19], we should assume that the initial curve \( X_0 \) can not concave wildly. In this paper, we consider \( X_0 \) to be star-shaped and centrosymmetric and obtain the following main result.

**Theorem 1.1** Let \( X_0 : S^1 \to \mathbb{R}^2 \) be a smooth, embedded and star-shaped curve in the plane. If \( X_0 \) is centrosymmetric with respect to the origin \( O \), then Gage’s area-preserving flow (1.1) with this initial curve exists globally and makes the evolving curve convex in a finite time and deforms it into a circle centered at \( O \) as time tends to infinity.

A flow is called global if the evolving curve is smooth for all \( t \in [0, +\infty) \). For an initial star-shaped curve in the plane, it is still unknown whether Gage’s area-preserving flow exists globally or not (see Lemma 2.5 and Remark 3.4). If \( X_0 \) is only star-shaped, it seems that the flow (1.1) may not preserve the star-shapedness of the evolving curve. This causes essential difficulties to understand the asymptotic behavior of \( X(\cdot, t) \). The extra symmetric property of \( X_0 \) is inspired by Michael Gage [21] (also see the early work [6, 8] by Gage and Li).

**Remark 1.2** The result in Theorem 1.1 is of special interest. On one hand its conclusion can not be generalized to higher dimensional cases. As is well known that the dumbbell-shaped surface will split into two pieces with singularities under mean curvature type flows. On the other hand, the result in Theorem 1.1 does not always hold if the initial curve \( X_0 \) is an immersed centrosymmetric curve with \( \det(X(\varphi), T(\varphi)) > 0 \) everywhere. See Remark 3.4 for more details.

The proof of Theorem 1.1 is divided into two parts. In the first part, i.e. the global existence of the flow, the star-shapedness of the evolving curve plays an essential role.
(see Lemma 3.8 and Corollary 3.9). If $X_0$ is centrosymmetric with respect to the point $O$, the key idea in this part is to show that the evolving curve never touches the point $O$ via a comparison to the evolution behavior of the famous CSF (see Lemmas 3.1-3.7 for the details). In the second part, i.e. the convergence of the evolving curve, some ideas in Grayson’s papers [10, 11] are adopted (see Lemmas 4.2-4.3).

Once $X(\cdot, t)$ is proved to be star-shaped with respect to $O$, the polar angle $\theta$ can be used as a parameter of the evolving curve. In order to make $\theta$ independent of time, one can add a tangent component to the origin flow to get a new one:

\[
\begin{align*}
\frac{\partial X}{\partial t} (\varphi, t) &= \alpha(\varphi, t) T(\varphi, t) + (\kappa(\varphi, t) - \frac{2\pi}{\ell(t)}) N(\varphi, t) \quad \text{in } S^1 \times (0, \omega), \\
X(\varphi, 0) &= X_0(\varphi) \quad \text{on } S^1.
\end{align*}
\]

(1.2)

The tangent component $\alpha(\varphi, t) T(\varphi, t)$ will be determined in the next section. By Proposition 1.1 on page 6 of the monograph [2], this term does not influence the shape of the evolving curve.

GAPF [5] has also been considered by Wang et al. [25] if the initial curve is closed and locally convex. They have studied the convergence for global flows and some blow-up properties. For expanding flows of star-shaped curves, one may consult Tsai [23] and Yagisita [26]. In higher dimensional spaces, one can refer to Huisken’s volume-preserving flow of convex hypersurfaces [14] and its generalization by Kim and Kwon [16] to the case of star-shaped hypersurfaces with $\rho$-reflection property.

This paper is organized as follows. In Sect. 2, some basic properties of the flow (1.1) are obtained, including the short time existence and a property of $X(\cdot, t)$ which implies its star-shapeness. In Sect. 3, it is proved that the flow (1.1) exists on the time interval $[0, +\infty)$. And in Sect. 4, the proof of Theorem 1.1 is completed.

## 2 Preparation

Given a curve $X$ in the plane, its “support function” is defined by\(^1\)

\[p(s) = -(X(s), N(s)).\]

If we express the curve as $X(s) = (x(s), y(s))(s \in [0, L])$, then its unit tangent and normal vector fields can be written as

\[T(s) = (\dot{x}(s), \dot{y}(s)), \quad N(s) = (-\dot{y}(s), \dot{x}(s)),\]

where “$\cdot$” stands for derivation with respect to the arc length parameter $s$. Since

\[p(s) = x(s)\dot{y}(s) - \dot{x}(s)y(s) = \det(X(s), T(s)),\]

$X$ is star-shaped with respect to the origin $O$ if and only if $p(s) > 0$ for all $s$.

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\(^1\) $p$ is usually called the support function of $X$ in convex geometry, for example [12, 13, 20].
Using the polar coordinate system \((r, \theta)\) for the plane, a smooth and closed curve can be expressed as

\[
X(s) = r(\theta(s)) P(\theta(s)),
\]

where \(P(\theta) = (\cos \theta, \sin \theta)\). Following Tsai [23], if we set \(Q(\theta) = (\sin \theta, \cos \theta)\) then the unit tangent vector of \(X\) is given by

\[
T = \frac{dr}{ds} P + r \frac{d\theta}{ds} Q
\]

and furthermore

\[
\det \langle X(s), T(s) \rangle = r^2(s) \frac{d\theta}{ds}(s). \tag{2.1}
\]

If \(X\) is closed and star-shaped then one can choose an origin \(O\) so that \(r(s) > 0\) and \(\det \langle X(s), T(s) \rangle > 0\) for every \(s\). The equation (2.1) implies that one can use the polar angle \(\theta\) to parameterize a star-shaped plane curve.

Now let us deal with the flow (1.1) with initial \(X_0\) star-shaped with respect to \(O\). We shall first derive some evolution equations and determine the tangent component \(\alpha T\) to make the polar angle \(\theta\) independent of time \(t\). Then the flow (1.1) can be reduced to a Cauchy problem of a single equation for the radial function \(r = r(\theta, t)\). After that, some basic properties of the flow (1.1) will be explored in this section.

Let \(g := \left( \frac{\partial X}{\partial \phi}, \frac{\partial X}{\partial \phi} \right)\) be the metric of the evolving curve. Set \(\beta = \kappa - \frac{2\pi}{L}\). Under the flow (1.2), \(g\) evolves according to

\[
\frac{\partial g}{\partial t} = g \left( \frac{1}{g} \left( \frac{\partial}{\partial t} \frac{\partial X}{\partial \phi} \right), \frac{\partial X}{\partial \phi} \right) = g \left( \frac{\partial}{\partial s} (\alpha T + \beta N), T \right) = \left( \frac{\partial \alpha}{\partial s} - \beta \kappa \right) g.
\]

The interchange of the operators \(\partial/\partial s\) and \(\partial/\partial t\) is given by

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial s} = \frac{\partial}{\partial t} \left( \frac{1}{g} \frac{\partial}{\partial \phi} \right) = \frac{\partial}{\partial s} \frac{\partial}{\partial t} - \left( \frac{\partial \alpha}{\partial s} - \beta \kappa \right) \frac{\partial}{\partial s}.
\]

And we have the evolution equation of \(T\) and \(N\):

\[
\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} \frac{\partial X}{\partial s} = \frac{\partial}{\partial s} \frac{\partial X}{\partial t} - \left( \frac{\partial \alpha}{\partial s} - \beta \kappa \right) T = \left( \alpha \kappa + \frac{\partial \beta}{\partial s} \right) N,
\]

\[
\frac{\partial N}{\partial t} = \left( \frac{\partial N}{\partial t}, T \right) T + \left( \frac{\partial N}{\partial t}, N \right) N = - \left( \alpha \kappa + \frac{\partial \beta}{\partial s} \right) T.
\]

If there is a family of star-shaped curves evolving under the flow (1.2) then we can express the evolving curve as

\[
X(\theta, t) = r(\theta, t) P(\theta). \tag{2.2}
\]
Noticing that \( \frac{\partial X}{\partial \theta} = \frac{\partial r}{\partial \theta} P + r Q \), we obtain
\[
g = \left\| \frac{\partial X}{\partial \theta} \right\| = \left( r^2 + \left( \frac{\partial r}{\partial \theta} \right)^2 \right)^{1/2}, \quad T = \frac{\partial r}{\partial s} P + \frac{r}{g} Q, \quad N = -\frac{r}{g} P + \frac{\partial r}{\partial s} Q.
\]
(2.3)

Differentiating the right hand side of (2.2) and using (1.2) and (2.3), one gets
\[
\frac{\partial r}{\partial t} P + r \frac{\partial \theta}{\partial t} Q = \alpha T + \beta N = \left( \frac{\alpha}{g} - \frac{r \beta}{g} \right) P + \left( \frac{\alpha r}{g} + \beta \frac{\partial r}{\partial s} \right) Q.
\]
Comparing the coefficients of both sides can yield the following evolution equations:
\[
\frac{\partial r}{\partial t} = \frac{\alpha r}{g} - \frac{r \beta}{g} \frac{\partial r}{\partial s}, \quad \frac{\partial \theta}{\partial t} = \frac{\alpha}{g} + \frac{\beta}{r} \frac{\partial r}{\partial s}.
\]
(2.4)

From now on, we choose
\[
\alpha = -\frac{\beta}{r} \frac{\partial r}{\partial s}, \quad g = -\frac{\beta}{r} \frac{\partial r}{\partial \theta}
\]
so that \( \frac{\partial \theta}{\partial t} \equiv 0 \), i.e., the polar angle \( \theta \) is independent of the time \( t \). Since the curvature of the evolving curve is
\[
\kappa = \frac{1}{g^3} \left( -r \frac{\partial^2 r}{\partial \theta^2} + 2 \left( \frac{\partial r}{\partial \theta} \right)^2 + r^2 \right),
\]
one can immediately obtain the evolution equation of \( r \),
\[
\frac{\partial r}{\partial t} = \frac{1}{g^2} \frac{\partial^2 r}{\partial \theta^2} - \frac{2}{rg^2} \left( \frac{\partial r}{\partial \theta} \right)^2 - \frac{r}{g^2} + \frac{2 \pi g}{rL}.
\]
(2.5)

Now, if \( r = r(\theta, t) > 0 \) is defined on \([0, 2\pi] \times [0, \omega)\) and satisfies the equation (2.5), then a family of curves \( \{ X = r P | t \in [0, \omega) \} \) satisfies the flow (1.2). So we can reduce the flow (1.2) to the equation (2.5) with initial value \( r_0(\theta) > 0 \):

**Lemma 2.1** Suppose \( X_0 \) is star-shaped with respect to \( O \). The flow (1.2) in some time interval is equivalent to the quasi-linear parabolic equation (2.5) with a positive initial value \( r_0 \).

The length of the curve can be calculated according to
\[
L(t) = \int_0^{2\pi} g(\theta, t) d\theta = \int_0^{2\pi} \sqrt{r^2 + \left( \frac{\partial r}{\partial \theta} \right)^2} d\theta.
\]
Let us define an operator $F$ from the space $C^{2,\alpha}([0, 2\pi] \times [0, \omega))$ to $C^\beta([0, 2\pi] \times [0, \omega))$, for $0 < \beta < \alpha \leq 1$ according to

$$F(r) = \frac{\partial r}{\partial t} - \frac{1}{g^2} \frac{\partial^2 r}{\partial \theta^2} + \frac{2}{rg^2} \left( \frac{\partial r}{\partial \theta} \right)^2 + \frac{r}{g^2} - \frac{2\pi g}{rL}.$$  

Since the Fréchet derivative of $F$ at some point $r_0 > 0$ is

$$DF(r_0)f = \frac{\partial f}{\partial t} - \frac{1}{r_0^2 + \left( \frac{\partial r}{\partial \theta} \right)^2} \frac{\partial^2 f}{\partial \theta^2} + \text{lower linear terms of } f,$$

the equation (2.5) is uniformly parabolic near its initial value $r_0$. It follows from the implicit function theorem of Banach spaces that the Cauchy problem (2.5) has a unique solution in some small time interval (See Section 1.2 in [2]). Using Lemma 2.1 can give us the short time existence.

**Lemma 2.2** The flow (1.1) has a unique smooth solution in some time interval $[0, \omega)$, where $\omega > 0$.

Next, we derive some basic properties of the flow (1.1) by using the above evolution equations.

**Lemma 2.3** Under the flow (1.1), the area $A$ of the evolving curve is a constant, that is, $A(t) = A_0$; and the length $L$ satisfies $\sqrt{4\pi A_0} \leq L(t) \leq L_0$.

**Proof** Using the equations (1.18)–(1.19) in [2], one obtains

$$\frac{dA}{dt} = -\int_0^L (\kappa - \frac{2\pi}{L}) ds = 0,$$

$$\frac{dL}{dt} = -\int_0^L \kappa^2 ds + \frac{4\pi^2}{L} \leq 0.$$  

The last inequality follows from the Cauchy-Schwarz inequality. So the area enclosed by $X$ is invariant and the length $L$ is decreasing, which together with the classical isoperimetric inequality can give us the desired result. \(\square\)

Under the flow (1.2), the “support” function is

$$p = -\langle X, N \rangle = -\left( rP, -\frac{r}{g} P + \frac{1}{g} \frac{\partial r}{\partial \theta} Q \right) = \frac{r^2}{g},$$

so a closed curve is star-shaped if and only if $r > 0$ and $\left| \frac{\partial r}{\partial \theta} \right|$ is bounded everywhere. Since $r$ and $\left| \frac{\partial r}{\partial \theta} \right|^2$ satisfy parabolic equations, one can apply the comparison principle to bound these two functions. For a continuous function $f = f(\theta, t)$, set

$$f_{\min}(t) = \min\{ f(\theta, t) | \theta \in [0, 2\pi] \}, \quad f_{\max}(t) = \max\{ f(\theta, t) | \theta \in [0, 2\pi] \}.$$  

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Lemma 2.4  Given a star-shaped curve \( X_0 \), we choose a point \( O \) in the plane such that \( r_0(\theta) > 0 \) for all \( \theta \). If \( r(\theta, t) \geq c > 0 \) holds for every \( (\theta, t) \in [0, 2\pi] \times [0, \omega) \) under the flow (1.2) then

\[
 r(\theta, t) \leq \frac{L_0}{2},
\]

and

\[
 \left| \frac{\partial r}{\partial \theta}(\theta, t) \right| \leq C_1,
\]

where \( C_1 = \max \left\{ \max_{\theta} \left| \frac{\partial r}{\partial \theta}(\theta, 0) \right|, \frac{3L_0}{\pi} \right\} \) is a constant relying on the initial curve \( X_0 \).

**Proof**  Fix a \( t_0 \in [0, \omega) \). Let \( t \in [0, t_0) \), if \( r(\cdot, t) \) attains its maximum value \( r_{\text{max}}(t) \) at \((\theta_*, t)\), then

\[
 \frac{\partial^2 r}{\partial \theta^2}(\theta_*, t) \leq 0, \quad \frac{\partial r}{\partial \theta}(\theta_*, t) = 0, \quad g(\theta_*, t) = r(\theta_*, t).
\]

By the evolution equation of \( r \) (see (2.5)),

\[
 \frac{\partial r}{\partial t}(\theta_*, t) \leq \frac{2\pi}{L(t)} \leq \frac{\sqrt{\pi}}{A_0}.
\]

So the maximum principle implies that if \( t \in [0, t_0) \) then

\[
 r(\theta, t) \leq r_{\text{max}}(0) + \sqrt{\frac{\pi}{A_0}}t_0.
\]  

(2.8)

Differentiating the equation (2.5), one gets

\[
 \frac{\partial^2 r}{\partial t \partial \theta} = \frac{1}{g^2} \frac{\partial^3 r}{\partial \theta^3} - \frac{2}{g^4} \frac{\partial r}{\partial \theta} \left( \frac{\partial^2 r}{\partial \theta^2} \right)^2 - \frac{4}{r^2 g^2} \frac{\partial r}{\partial \theta} \frac{\partial^2 r}{\partial \theta^2} + \frac{2}{r^2 g^2} \left( \frac{\partial r}{\partial \theta} \right)^3
\]

\[
 + \frac{4}{g^4} \left( \frac{\partial r}{\partial \theta} \right)^3 + \frac{4}{r^4 g^4} \left( \frac{\partial r}{\partial \theta} \right)^3 \frac{\partial^2 r}{\partial \theta^2} - \frac{1}{g^2} \frac{\partial r}{\partial \theta} + \frac{2r^2}{g^4} \frac{\partial r}{\partial \theta}
\]

\[
 + \frac{2\pi}{Lg} \frac{\partial r}{\partial \theta} + \frac{2\pi}{rL} \frac{\partial r}{\partial \theta} \frac{\partial^2 r}{\partial \theta^2} - \frac{2\pi g}{r^2 L} \frac{\partial r}{\partial \theta}.
\]

Set \( w = (\frac{\partial r}{\partial \theta})^2 \). Then \( w \) evolves according to

\[
 \frac{1}{2} \frac{\partial w}{\partial t} = \frac{1}{2g^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{g^2} \left( \frac{\partial^2 r}{\partial \theta^2} \right)^2 - \frac{2w}{g^4} \left( \frac{\partial^2 r}{\partial \theta^2} \right)^2 - \frac{2}{r^2 g^2} \frac{\partial r}{\partial \theta} \frac{\partial w}{\partial \theta}
\]

\[
 + \frac{2w^2}{r^2 g^2} + \frac{4w^2}{g^4} + \frac{2w}{r^4 g^2} \frac{\partial r}{\partial \theta} \frac{\partial w}{\partial \theta} - \frac{w}{g^2} + \frac{2r^2}{g^4}.
\]
\[ + \frac{2\pi w}{Lg} + \frac{\pi}{rLg} \frac{\partial r}{\partial \theta} \frac{\partial w}{\partial \theta} - \frac{2\pi gw}{r^2L}. \]

If \( w \) attains its maximum value \( w_{\text{max}}(t) \) at \( (\theta_*, t) \) then

\[ \frac{\partial w}{\partial \theta}(\theta_*, t) = 0, \quad \frac{\partial^2 w}{\partial \theta^2}(\theta_*, t) \leq 0. \]

So at the point \( (\theta_*, t) \) one obtains

\[ \frac{1}{2} \frac{\partial w}{\partial t} \leq \left( \frac{2w^2}{r^2g^2} + \frac{4w^2}{g^4} \right) + \left( -\frac{w}{g^2} + \frac{2r^2w}{g^4} \right) + \left( \frac{2\pi w}{Lg} - \frac{2\pi gw}{r^2L} \right) \]

\[ = \frac{2w^2(3r^2 + w)}{r^2g^4} + \frac{w}{g^4}(r^2 - w) - \frac{2\pi w^2}{r^2Lg} \]

\[ = \frac{2w^2(3Lr^2 + wL)}{r^2Lg^4} - 2\pi w^2(3r^2 + w)g + \frac{w}{g^4}(r^2 - w) \]

\[ = \frac{2w^2(3r^2 - \pi g)}{r^2Lg^4} + \frac{2w^3(L - \pi g)}{r^2Lg^4} + \frac{w}{g^4}(r^2 - w). \]

If \( w(\theta_*, t) \geq \max \{(r_{\text{max}}(t))^2, \left( \frac{3L_0}{\pi} \right)^2 \} \) then

\[ 3L(t) - \pi g(\theta_*, t) \leq 3L_0 - \pi \sqrt{w(\theta_*, t)} \leq 0 \]

and

\[ r^2(\theta_*, t) - w(\theta_*, t) \leq (r_{\text{max}}(t))^2 - w(\theta_*, t) \leq 0. \]

And furthermore one gets \( \frac{\partial w}{\partial r}(\theta_*, t) \leq 0 \). By the maximum principle,

\[ w(\theta, t) \leq \max \left\{ \max_{\theta} w(\theta, 0), \quad (r_{\text{max}}(t))^2, \quad \left( \frac{3L_0}{\pi} \right)^2 \right\}, \]

which implies

\[ \left| \frac{\partial r}{\partial \theta} \right| \leq \max \left\{ \max_{\theta} \left| \frac{\partial r}{\partial \theta}(\theta, 0) \right|, \quad r_{\text{max}}(t), \quad \frac{3L_0}{\pi} \right\}. \tag{2.9} \]

Combining (2.8) and (2.9), the support function \( p = \frac{r^2}{g} \) is positive on time interval \([0, t_0]\). The evolving curve \( X(\cdot, t) \) is star-shaped with respect to \( O \) for \( t \in [0, t_0] \).

Since \( r \) is the distance of \( O \) to \( X(\theta, t) \), one obtains

\[ r(\theta, t) \leq \frac{L(t)}{2} \leq \frac{L_0}{2} \]

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which gives us (2.6) and enables us to revise the estimate (2.9) as
\[
\left| \frac{\partial r}{\partial \theta} \right| \leq \max_{\theta} \left\{ \frac{\partial r}{\partial \theta} (\theta, 0) \right\} \cdot \frac{L_0}{2}, \frac{3L_0}{\pi} \} = \max_{\theta} \left\{ \frac{\partial r}{\partial \theta} (\theta, 0) \right\} \cdot \frac{3L_0}{\pi} .
\]

(2.10)

So we have done.

It follows from Lemma 2.4 that the support function of the evolving curve is positive everywhere once a positive lower bound of \( r \) is given. So the flow (1.1) preserves star-shapedness of the evolving curve under the condition of Lemma 2.4.

**Lemma 2.5** If the flow (1.1) does not blow up in the time interval \([0, \omega)\) and \( r_{\min}(t) > 0 \) for \( t \in [0, \omega) \) then the point \( O \) is the star center of every evolving curve \( X(\cdot, t) \).

### 3 Global Existence

In this section, it is shown that Gage’s area-preserving flow (1.1) exists in the time interval \([0, +\infty)\) if the initial smooth curve \( X_0 \) is centrosymmetric and star-shaped with respect to the origin \( O \).

#### 3.1 Star-Shaped Curves Under the CSF

The curve shortening flow with a smooth, closed and embedded initial \( X_0 \) is defined by
\[
\begin{align*}
\begin{cases}
\frac{\partial Y}{\partial t}(\varphi, t) = \kappa(\varphi, t)\tilde{N}(\varphi, t) \quad &\text{in } S^1 \times (0, \omega), \\
Y(\varphi, 0) = X_0(\varphi) \quad &\text{on } S^1,
\end{cases}
\end{align*}
\]

(3.1)

where \( \kappa(\varphi, t) \) is the relative curvature with respect to the Frenet frame \( \{ \tilde{T}, \tilde{N} \} \). Grayson has proved [10] that the evolving curve \( Y(\cdot, t) \) is smooth, preserves its embeddedness and becomes convex on the time interval \([0, \frac{A_0}{2\pi})\), where \( A_0 \) is the area bounded by \( X_0 \). Gage–Hamilton Theorem [3, 4, 7] says that a convex curve \( X_0 \) evolving under the CSF (3.1) becomes asymptotically circular as \( t \to \frac{A_0}{2\pi} \).

If \( X_0 \) is star-shaped with respect to the origin \( O \), it follows from the continuity of the evolving curve that there exists \( t_0 > 0 \) such that \( Y(\cdot, t) \) is star-shaped with respect to \( O \) for \( t \in [0, t_0) \). So one can add a proper tangent component to the flow (3.1)
\[
\begin{align*}
\begin{cases}
\frac{\partial Y}{\partial t}(\varphi, t) = \alpha \tilde{T} + \kappa \tilde{N} \quad &\text{in } S^1 \times (0, \omega), \\
Y(\varphi, 0) = X_0(\varphi) \quad &\text{on } S^1
\end{cases}
\end{align*}
\]

(3.2)

to make the polar angle \( \theta \) of \( Y(\cdot, t) \) independent of time. Now let us parameterize \( Y(\cdot, t) \) by \( \theta \) and set \( Y(\theta, t) = \rho(\theta, t) P(\theta) \) where \( P(\theta) = (\cos \theta, \sin \theta) \). As is well known that the solution to the flow (3.1) differs from that of the flow (3.2) by a...
Lemma 3.1 Suppose the initial smooth curve $X_0$ is star-shaped and centrosymmetric with respect to $O$. Under the CSF (3.2), if the evolving curve $Y(\cdot, t)$ is star-shaped with respect to $O$ then it is centrosymmetric with respect to $O$.

Proof For $t_s \in [0, \frac{4\pi}{27})$, the above mentioned Gage-Hamilton-Grayson Theorem tells us that there exist constants $M_i(t_s) > 0$ such that

$$\left| \frac{\partial^i \rho}{\partial \theta^i} (\theta, t) \right| \leq M_i(t_s), \quad (\theta, t) \in [0, 2\pi] \times [0, t_*], \quad i = 1, 2, \ldots. \quad (3.4)$$

Define $\tilde{\rho}(\theta, t) = \rho(\theta + \pi, t)$ and $\varphi_1(\theta, t) = \tilde{\rho}(\theta, t) - \rho(\theta, t)$. Since $X_0$ is centrosymmetric, $\varphi_1(\theta, 0) \equiv 0$. By the equation (3.3), $\varphi_1(\theta, t)$ evolves according to

$$\frac{\partial \varphi_1}{\partial t} = \frac{1}{(g_\rho)\rho(\theta, t)^2} \frac{\partial^2 \varphi_1}{\partial \theta^2} + \frac{1}{(g_\rho)^2 \rho(\theta, t)^2} \frac{\partial^2 \varphi_1}{\partial \theta^2} - \frac{1}{\rho(\rho, t)^2} \frac{\partial^2 \varphi_1}{\partial \theta^2} + \frac{2}{\rho(\rho, t)^2} \left[ \left( \frac{\partial \rho}{\partial \theta} \right)^2 - \left( \frac{\partial \rho}{\partial \theta} \right)^2 \right]
+ \frac{2}{\rho(\rho, t)^2} \left( \frac{\partial \tilde{\rho}}{\partial \theta} \right)^2 - \frac{2}{\tilde{\rho}(\rho, t)^2} \left( \frac{\partial \tilde{\rho}}{\partial \theta} \right)^2 + \rho - \rho + \frac{\tilde{\rho}}{\rho(\rho, t)^2} - \frac{\tilde{\rho}}{\rho(\rho, t)^2} \frac{\partial \varphi_1}{\partial \theta}
+ \frac{2}{\rho(\rho, t)^2} \left( \frac{\partial \tilde{\rho}}{\partial \theta} \right)^2 \left( \frac{\partial \tilde{\rho}}{\partial \theta} + \frac{\partial \tilde{\rho}}{\partial \theta} \right) \frac{\partial \varphi_1}{\partial \theta} - \frac{2}{\tilde{\rho}(\rho, t)^2} \left( \frac{\partial \tilde{\rho}}{\partial \theta} + \frac{\partial \tilde{\rho}}{\partial \theta} \right) \frac{\partial \varphi_1}{\partial \theta}
- \frac{\partial \tilde{\rho}}{\rho(\rho, t)^2} \frac{\partial \tilde{\rho}}{\partial \theta} \varphi_1 + 2 \left( \frac{\partial \tilde{\rho}}{\partial \theta} \right)^2 \varphi_1 + \frac{2}{\rho(\rho, t)^2} \left( \frac{\partial \tilde{\rho}}{\partial \theta} \right)^2 \varphi_1
- \frac{\varphi_1}{\rho(\rho, t)^2} + \frac{\tilde{\rho}}{\rho(\rho, t)^2} \frac{\partial \varphi_1}{\partial \theta}. \quad (3.5)$$

The equation (3.5) is linear with smooth coefficients (see (3.4)) and zero initial value. By the uniqueness of the solution to linear parabolic equations, one has

$$\varphi_1(\theta, t) \equiv 0. \quad (3.6)$$

So the evolving curve $Y(\cdot, t)$ is centrosymmetric with respect to $O$ for $t \in [0, t_*]$. The proof is completed by the arbitrary choice of $t_s$. \hfill \Box

Lemma 3.2 Let the initial smooth curve $X_0$ be star-shaped and centrosymmetric with respect to $O$, then under the CSF (3.2), the evolving curve $Y(\cdot, t)$ is star-shaped.
Proof Suppose that there is a time \( t^* \in [0, \frac{A_0}{2\pi}) \) such that \( Y(\cdot, t) \) is star-shaped with respect to \( O \) for \( t \in [0, t^*) \) but \( Y(\cdot, t^*) \) is not star-shaped with respect to \( O \). One can claim that

\[
\rho(\theta, t) > 0, \quad (\theta, t) \in [0, 2\pi] \times [0, t^*), \tag{3.7}
\]

and

\[
\rho_{\text{min}}(t^*) := \min\{\rho(\theta, t^*)|\theta \in [0, 2\pi]\} = 0. \tag{3.8}
\]

In fact, \( Y(\cdot, t) \) is star-shaped with respect to \( O \) for \( t \in [0, t^*) \) so (3.7) holds. If (3.8) does not hold then, for some \( \delta > 0 \), one obtains

\[
\rho_{\text{min}}(t^*) \geq \delta. \tag{3.9}
\]

By the evolution equation (3.3), the quantity \( v = \frac{1}{2} \left( \frac{\partial \rho}{\partial \theta} \right)^2 \) evolves according to

\[
\frac{\partial v}{\partial t} = \frac{1}{(g_\rho)^2} \frac{\partial^2 v}{\partial \theta^2} - \frac{1}{(g_\rho)^4} \left( \frac{\partial^2 \rho}{\partial \theta^2} \right)^2 - \frac{4v}{(g_\rho)^2} \left( \frac{\partial^2 \rho}{\partial \theta^2} \right)^2 - \frac{4}{\rho(g_\rho)^2} \frac{\partial \rho}{\partial \theta} \frac{\partial v}{\partial \theta} + \frac{8v^2}{\rho^2} + \frac{16v^2}{(g_\rho)^4} + \frac{8v}{\rho(g_\rho)^4} \frac{\partial \rho}{\partial \theta} \frac{\partial v}{\partial \theta} - \frac{2v}{(g_\rho)^2} + \frac{4\rho^2 v}{(g_\rho)^4}.
\]

At the point \((\theta^*, t)\) where \( v(\theta, t) \) attains \( v_{\text{max}}(t) \) one gets

\[
\frac{\partial^2 v}{\partial \theta^2}(\theta^*, t) \leq 0, \quad \frac{\partial v}{\partial \theta}(\theta^*, t) = 0,
\]

and thus

\[
\frac{\partial v}{\partial t}(\theta^*, t) \leq \frac{8v^2}{\rho^2(g_\rho)^2}(\theta^*, t) + \frac{16v^2}{(g_\rho)^4}(\theta^*, t) - \frac{2v}{(g_\rho)^2}(\theta^*, t) + \frac{4\rho^2 v}{(g_\rho)^4} (\theta^*, t) \leq \frac{4v}{\rho^2}(\theta^*, t) + 4 + 0 + \frac{1}{2} \leq \frac{4v}{\delta^2} + \frac{9}{2},
\]

where \( t \in [0, t^*) \). It follows from the maximum principle that

\[
v_{\text{max}}(t) \leq \left( v_{\text{max}}(0) + \frac{9}{8} \delta^2 \right) e^{\frac{4}{\delta^2} t^*},
\]

i.e.,

\[
\left| \frac{\partial \rho}{\partial \theta} \right|_{\text{max}}(t) \leq \sqrt{\left( 2v_{\text{max}}(0) + \frac{9}{4} \delta^2 \right)} e^{\frac{4}{\delta^2} t^*}. \tag{3.10}
\]
By (3.9) and (3.10), the support function of $Y(\cdot, t)$ (i.e., $p_Y(\theta, t_\ast) = \frac{\rho^2(\theta, t_\ast)}{\hat{g}_p(\theta, t_\ast)}$) has a positive lower bound. So $Y(\cdot, t)$ is star-shaped with respect to $O$ for $t \in [0, t_\ast + \varepsilon)$. This is a contradiction to the assumption that $Y(\cdot, t_\ast)$ is not star-shaped with respect to $O$. Therefore, the claim (3.8) holds. There exists a $\theta_\ast \in [0, 2\pi]$ so that

$$\rho(\theta_\ast, t_\ast) = 0. \quad (3.11)$$

By the choice of $t_\ast \in (0, \frac{A_0}{2\pi})$, the evolving curve $Y(\cdot, t_\ast)$ does not blow up. By Lemma 3.1, $Y(\cdot, t)$ is centrosymmetric with respect to $O$. So

$$\rho(\theta_\ast + \pi, t_\ast) = 0. \quad (3.12)$$

The equations (3.11) and (3.12) contradict to Huisken’s monotonic formula [15] or the fact that the evolving curve keeps embedded before it shrinks to a point (see Corollary 3.2.4 of [7]). Therefore, the equation (3.8) can not hold. For every $t \in \left[0, \frac{A_0}{2\pi}\right)$, the evolving curve $Y(\cdot, t)$ is star-shaped with respect to $O$. \hfill \Box

As a direct corollary of Lemma 3.1 and Lemma 3.2, we have

**Corollary 3.3** Suppose the initial smooth curve $X_0$ is star-shaped and centrosymmetric with respect to $O$. Under the CSF (3.2), the evolving curve $Y(\cdot, t)$ shrinks to the point $O$ as $t \to \frac{A_0}{2\pi}$.

In 2010, Mantegazza in his note [18] showed that a star-shaped (with respect to $O$) initial curve remains so under the CSF till the point $O$ is contained in the open region bounded by the evolving curve.

**Remark 3.4** There exist some smooth, closed, star-shaped but not embedded curves which do not always preserve the star-shapedness of evolving curves under the CSF. Figure 1 presents such a curve $X_0$ which is of positive curvature everywhere and is star-shaped with respect to $O$, but the two small loops shrink so that they do not intersect each other after some seconds under the CSF. This makes the evolving curve no longer star shaped. This example also indicates that the embeddedness of the initial curve in Theorem 1.1 can not be omitted.

### 3.2 Star-Shapedness of the Evolving Curve Under the Flow (1.1)

It is proved in this subsection that Gage’s area-preserving flow (1.1) preserves the star-shapedness of the evolving curve if the initial smooth curve $X_0$ is both centrosymmetric and star-shaped with respect to the origin $O$.

**Lemma 3.5** Suppose the initial smooth curve $X_0$ is star-shaped and centrosymmetric with respect to $O$ and Gage’s area-preserving flow (1.1) exists on the time interval $[0, \omega)$. If the evolving curve $X(\cdot, t)$ under (1.1) is star-shaped with respect to $O$ for all $t \in [0, \omega)$ then it is centrosymmetric with respect to $O$.\hfill \copyright Springer
Proof For every \( t_\ast \in [0, \omega) \), the radial function \( r(\theta, t) \) evolves according to the equation (2.5) under Gage’s area-preserving flow (1.1). By assumption, \( X(\cdot, 0) \) is star-shaped with respect to \( O \). So there exists a constant \( c = c(t_\ast) > 0 \) such that for all \( (\theta, t) \in [0, 2\pi] \times [0, t_\ast) \),

\[
r(\theta, t) \geq c(t_\ast).
\]  
(3.13)

Lemma 2.4 tells us that (2.6) and (2.7) hold for \( t \in [0, t_\ast] \). By the classical theory of parabolic equations [17], there exist constants \( C_i(t_\ast) > 0 \) so that

\[
\left| \frac{\partial^i r}{\partial \theta^i}(\theta, t) \right| \leq C_i(t_\ast),
\]  
(3.14)

where \( (\theta, t) \in [0, 2\pi] \times [0, t_\ast] \) and \( i = 1, 2, \ldots \).

Set \( \tilde{r}(\theta, t) := r(\theta + \pi, t) \). By the evolution equation (2.5), the function \( \varphi_2(\theta, t) \equiv \tilde{r}(\theta, t) - r(\theta, t) \) satisfies a linear, uniformly parabolic equation which has smooth coefficients (similar to the equation (3.5)) and a non-local term \( L = L(t) \):

\[
\frac{\partial \varphi_2}{\partial t} = \frac{1}{(g_r)^2} \frac{\partial^2 \varphi_2}{\partial \theta^2} - \frac{1}{g^2(g_r)^2} \frac{\partial^2 r}{\partial \theta^2} \left( \frac{\partial r}{\partial \theta} + \frac{\partial \tilde{r}}{\partial \theta} \right) \frac{\partial \varphi_2}{\partial \theta} - \frac{2}{rg^2} \left( \frac{\partial r}{\partial \theta} + \frac{\partial \tilde{r}}{\partial \theta} \right) \frac{\partial \varphi_2}{\partial \theta} \\
+ \frac{2}{rg^2(g_r)^2} \left( \frac{\partial \tilde{r}}{\partial \theta} \right)^2 \left( \frac{\partial r}{\partial \theta} + \frac{\partial \tilde{r}}{\partial \theta} \right) \frac{\partial \varphi_2}{\partial \theta} + \frac{\tilde{r}}{g^2(g_r)^2} \left( \frac{\partial r}{\partial \theta} + \frac{\partial \tilde{r}}{\partial \theta} \right) \frac{\partial \varphi_2}{\partial \theta} \\
- \frac{\tilde{r} + r}{g^2(g_r)^2} \frac{\partial^2 \varphi_2}{\partial \theta^2} + \frac{2(\tilde{r} + r)}{rg^2(g_r)^2} \left( \frac{\partial \tilde{r}}{\partial \theta} \right)^2 \varphi_2 + \frac{2}{r \tilde{r}(g_r)^2} \left( \frac{\partial \tilde{r}}{\partial \theta} \right)^2 \varphi_2 \\
- \frac{\varphi_2}{(r_r)^2} + \frac{\tilde{r}(\tilde{r} + r)}{g^2(g_r)^2} \varphi_2 + \frac{2\pi(\tilde{r} + r)}{r \tilde{r}(g_r)^2} \varphi_2
\]
Suppose Gage’s area-preserving flow (1.1) with initial

\[ \frac{2\pi}{\tilde{r} L (g + \tilde{g}^2)} \left( \frac{\partial \tilde{r}}{\partial \theta} + \frac{\partial r}{\partial \theta} \right) \frac{\partial \varphi_2}{\partial \theta} - \frac{2\pi g}{\tilde{r} r L} \varphi^2, \]

where \( \tilde{g}^2 := \sqrt{\tilde{r}^2 + \left( \frac{\partial \tilde{r}}{\partial \theta} \right)^2} \). Since \( \varphi_2(\theta, 0) \equiv 0 \), one can obtain

\[ \varphi_2(\theta, t) \equiv 0, \]

that is to say, the evolving curve \( X(\cdot, t) \) is symmetric with respect to \( O \).

\[ \square \]

**Corollary 3.6** Suppose the initial smooth curve \( X_0 \) is star-shaped and centrosymmetric with respect to \( O \). If the evolving curve \( X(\cdot, t) \) under Gage’s area-preserving flow (1.1) is star-shaped then \( O \) is one of its star centers.

**Proof** Suppose Gage’s area-preserving flow (1.1) with initial \( X_0 \) exists on a time interval \([0, \omega)\). By continuity, there exists a small \( t_0 \in [0, \omega) \) such that \( X(\cdot, t) \) is star-shaped with respect to \( O \) if \( t < t_0 \). Lemma 3.5 implies that \( X(\cdot, t) \) is symmetric with respect to \( O \) if \( t < t_0 \).

Suppose there exists a \( t_* \in (0, \omega) \) such that (i) \( X(\cdot, t_*) \) is star-shaped with some point but not star-shaped with respect to \( O \) and (ii) \( X(\cdot, t) \) is star-shaped with respect to \( O \) for all \( t \in [0, t_*) \). By Lemma 3.5 and the continuity of \( X(\cdot, t) \), \( X(\cdot, t_*) \) is centrosymmetric with respect to \( O \). So one can conclude that

\[ r_{\min}(t_*) > 0. \]

Otherwise, \( X(\cdot, t_*) \) is not star-shaped with respect to any points, since it is centrosymmetric with respect to \( O \).

Let \( P \) be a star center of \( X(\cdot, t_*) \), then the symmetry of the curve implies that \(-P\) is also a star center (see [22]). Since the set of all star centers of \( X(\cdot, t_*) \) is convex, \( O \) is one of its star centers, which leads to a contradiction.

\[ \square \]

Next we show that Gage’s area-preserving flow (1.1) preserves star-shapedness of the evolving curve.

**Lemma 3.7** If the initial smooth curve \( X_0 \) is star-shaped and centrosymmetric with respect to \( O \) and Gage’s area-preserving flow (1.1) exists on a time interval \([0, \omega)\), then the evolving curve \( X(\cdot, t) \) is star-shaped with respect to \( O \) for all \( t \in [0, \omega) \).

**Proof** Suppose we have a \( t_* \in (0, \omega) \) and \( X(\cdot, t) \) is star-shaped for all \( t \in [0, t_*) \) but \( X(\cdot, t_*) \) is not star-shaped. It follows from Corollary 3.6 that \( X(\cdot, t) \) is star-shaped with respect to \( O \) for all \( t \in [0, t_*) \). By Lemma 3.5, \( X(\cdot, t) \) is centrosymmetric with respect to \( O \) on the same time interval \([0, t_*) \). Set \( \varepsilon_0 = \min \left\{ \frac{\varepsilon}{2}, \frac{A_0}{2\pi} \right\} \). Then \( X(\cdot, t_* - \varepsilon_0) \) is a centrosymmetric and star-shaped curve with respect to \( O \).

Let \( X(\cdot, t_* - \varepsilon_0) \) evolve according to the CSF, then we obtain a family of smooth curves \( Y(\cdot, t) \) for \( t \in \left[ t_* - \varepsilon_0, t_* - \varepsilon_0 + \frac{A_0}{2\pi} \right] \). Set \( Y(\theta, t) = \rho(\theta, t) P(\theta) \), where \( \theta \) is the polar angle independent of \( t \). By Lemma 3.1, Lemma 3.2 and Corollary 3.3,
$Y(\cdot, t)$ is star-shaped and centrosymmetric with respect to $O$ and it shrinks to $O$ as $t$ tends to the time $\left(t_0 - \varepsilon_0 + \frac{A_0}{2\pi}\right)$. Therefore, there exists a $\delta = \delta(t_0) > 0$ such that

$$\rho(\theta, t) \geq \delta(t_0)$$

(3.15)

for all $(\theta, t) \in [0, 2\pi] \times [t_0 - \varepsilon_0, t_0 + \frac{3A_0}{8\pi}]$.

Under Gage’s area-preserving flow, the radial function $r(\theta, t)$ of $X(\theta, t)$ satisfies the equation (2.5) for $t \in [t_0 - \varepsilon_0, t_0]$. Set $\varphi_3(\theta, t) = r(\theta, t) - \rho(\theta, t)$, where $(\theta, t) \in [0, 2\pi] \times [t_0 - \varepsilon_0, t_0]$. By the equations (2.5) and (3.3), $\varphi_3$ evolves according to

$$
\frac{\partial \varphi_3}{\partial t} = \frac{1}{g^2} \frac{\partial^2 \varphi_3}{\partial \theta^2} - \frac{1}{(g\rho)^2} \frac{\partial^2 \rho}{\partial \theta^2} \left( \frac{\partial \rho}{\partial \theta} + \frac{\partial r}{\partial \theta} \right) \frac{\partial \varphi_3}{\partial \theta} - \frac{2}{\rho(g\rho)^2} \left( \frac{\partial \rho}{\partial \theta} + \frac{\partial r}{\partial \theta} \right) \frac{\partial^2 \varphi_3}{\partial \theta^2} \\
+ \frac{2}{(g\rho)^2} \frac{\partial \rho}{\partial \theta} \left( \frac{\partial \rho}{\partial \theta} + \frac{\partial r}{\partial \theta} \right) \frac{\partial \varphi_3}{\partial \theta} + \frac{r}{(g\rho)^2} \frac{\partial \rho}{\partial \theta} \left( \frac{\partial \rho}{\partial \theta} + \frac{\partial r}{\partial \theta} \right) \frac{\partial \varphi_3}{\partial \theta} \\
- \frac{r + \rho}{(g\rho)^2} \frac{\partial^2 \varphi_3}{\partial \theta^2} + \frac{2(r + \rho)}{\rho(g\rho)^2} \left( \frac{\partial \rho}{\partial \theta} \right)^2 \varphi_3 + \frac{2}{\rho r} \frac{\partial \rho}{\partial \theta} \left( \frac{\partial \rho}{\partial \theta} \right)^2 \varphi_3 \\
- \frac{\varphi_3}{g^2} + \frac{2(r + \rho)}{\rho g^2} \varphi_3 + \frac{2\pi g}{rL}.
$$

(3.16)

Noticing that $\frac{2\pi g}{rL} > 0$ and (3.16) is a linear parabolic equation with smooth and bounded coefficients, the maximum principle implies that there exists a constant $\tilde{c} > 0$ such that for $t \in [t_0 - \varepsilon_0, t_0]$,

$$\varphi_3(\theta, t) \geq \min \{\varphi_3(\theta, t_0 - \varepsilon_0) | \theta \in [0, 2\pi]\} e^{-\tilde{c}t}.$$ 

Since $\varphi_3(\theta, t_0 - \varepsilon_0) \equiv 0$, we have $\varphi_3(\theta, t) \geq 0$. Thus

$$r(\theta, t_0) \geq \rho(\theta, t_0) \geq \delta(t_0) > 0.$$ 

(3.17)

By Lemma 2.4, Lemma 3.5 and Corollary 3.6, $X(\cdot, t_0)$ is star-shaped, which contradicts with the assumption. \qed

### 3.3 Extending Gage’s Area-Preserving Flow

Now let us extend Gage’s area-preserving flow (1.1). Once the curvature of the evolving curve is bounded on the time interval $[0, T_0]$ for any $T_0 > 0$, one can prove the smoothness of $X(\cdot, t)$ and the flow (1.1) can be extended globally.

Suppose the initial smooth curve $X_0$ is star-shaped and centrosymmetric with respect to $O$ and Gage’s area-preserving flow (1.1) with initial $X_0$ exists on the time interval $[0, T_0)$, where $T_0 > 0$ is a finite number. By Lemma 3.5, Corollary 3.6 and Lemma 3.7, the evolving curve $X(\cdot, t)$ is star-shaped and centrosymmetric with respect to $O$. Set $\varepsilon_0 = \min\{\frac{T_0}{2}, \frac{A_0}{4\pi}\}$. $X(\cdot, t)$ is star-shaped with respect to $O$ for $t \in [0, T_0)$, so there exists a $\delta_1 = \delta_1(T_0) > 0$ such that $r(\theta, t) \geq \delta_1$ holds for
t ∈ [0, T_\ast - \varepsilon_0]. By the proof of Lemma 3.7, there exists a \( \delta_2 = \delta_2(T_\ast) > 0 \) such that \( r(\theta, t) \geq \delta_2 \) for \( t \in [T_\ast - \varepsilon_0, T_\ast) \). Set \( \delta = \delta(T_\ast) = \min\{\delta_1, \delta_2\} > 0 \). Then for all \((\theta, t) \in [0, 2\pi] \times [0, T_\ast)\), one can obtain
\[
   r(\theta, t) \geq \delta > 0,
\]
which together with Lemma 2.4 and Lemma 2.5 tells us that the evolving curve \( X(\cdot, t) \) is star-shaped with respect to \( O \) if \( t \in [0, T_\ast) \). So there exists an \( h = h(T_\ast) > 0 \) such that the “support function” with respect to \( O \) satisfies that
\[
   p(\theta, t) \geq 2h(T_\ast) > 0,
\]
for all \((\theta, t) \in [0, 2\pi] \times [0, T_\ast)\).

Under the flow (1.1), the curvature evolves according to
\[
   \frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3 - \frac{2\pi}{L} \kappa^2,
\]
where \( s \) stands for the arc length parameter. Following Tso \([24]\), we define
\[
   \varphi_4(s, t) = \frac{\kappa(s, t)}{p(s, t) - h(T_\ast)}, \quad t \in [0, T_\ast),
\]
then
\[
   \frac{\partial \varphi_4}{\partial s} = \frac{1}{p - h} \frac{\partial \kappa}{\partial s} - \frac{\kappa}{(p - h)^2} \frac{\partial p}{\partial s},
\]
\[
   \frac{\partial^2 \varphi_4}{\partial s^2} = \frac{1}{p - h} \frac{\partial^2 \kappa}{\partial s^2} - \frac{2}{(p - h)^2} \frac{\partial \kappa \partial p}{\partial s} - \frac{\kappa}{(p - h)^2} \frac{\partial^2 p}{\partial s^2} + \frac{2\kappa}{(p - h)^3} \left( \frac{\partial p}{\partial s} \right)^2.
\]
Since the support function evolves according to
\[
   \frac{\partial p}{\partial t} = -\frac{\partial}{\partial t} \langle X, N \rangle = -\left( \left( \kappa - \frac{2\pi}{L} \right) N, N \right) + \left( X, \frac{\partial \kappa}{\partial s} T \right) = \frac{2\pi}{L} - \kappa + \langle X, T \rangle \frac{\partial \kappa}{\partial s},
\]
the evolution equation of \( \varphi_4 \) is given by
\[
   \frac{\partial \varphi_4}{\partial t} = \frac{\partial^2 \varphi_4}{\partial s^2} + \frac{2}{p - h} \frac{\partial p \partial \varphi_4}{\partial s} - \frac{\kappa}{p - h} \langle X, T \rangle \frac{\partial \varphi_4}{\partial s} + \frac{\kappa}{(p - h)^2} \frac{\partial^2 p}{\partial s^2} \\
   + (p - h)^2 (\varphi_4)^3 - \frac{2\pi}{L} (p - h)(\varphi_4)^2 + (\varphi_4)^2 - \frac{2\pi}{L} \frac{\varphi_4}{p - h} \left( X, T \right) \frac{\partial p}{\partial s}.
\]
Using
\[
\frac{\partial p}{\partial s} = \kappa(X, T), \quad \frac{\partial^2 p}{\partial s^2} = \frac{\partial \kappa}{\partial s} (X, T) + \kappa - \kappa^3 p,
\]
we can compute that
\[
\frac{\kappa}{(p - h)^2} \frac{\partial^2 p}{\partial s^2} - \frac{\kappa^2}{(p - h)^3} (X, T) \frac{\partial p}{\partial s} = \varphi(X, T) \frac{\partial \varphi}{\partial s} + (\varphi)^2 - p(p - h)(\varphi)^3.
\]
(3.23)

Substituting (3.23) into the evolution equation of \( \varphi \) can give us
\[
\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial s^2} + \frac{2}{p - h} \frac{\partial p}{\partial s} \frac{\partial \varphi}{\partial s} - h(p - h)(\varphi)^3 - \frac{2\pi}{L} (p - h)(\varphi)^2
\]
\[+ 2(\varphi)^2 - \frac{2\pi}{L} \frac{\varphi}{p - h}.
\]
(3.24)

As is well known that the arc length parameter \( s \) depends on the time \( t \), one cannot apply the maximum principle directly. In Section 2, the parameter \( \varphi \) (independent of \( t \)) is used to parametrize \( X(\cdot, t) \). Now, let \( g = \frac{\partial s}{\partial \varphi} \) and suppose that \( \varphi(\cdot, t) \) attains \( (\varphi)_{\text{max}}(t) \) at the point \( (\varphi_*, t) \). Denote \( s_* = s(\varphi_*) \). Then we have
\[
\frac{\partial \varphi}{\partial s}(s_*, t) = \left( \frac{\partial \varphi}{\partial \varphi} \frac{1}{g} \right)(s(\varphi_*), t) = 0
\]
and
\[
\frac{\partial^2 \varphi}{\partial s^2}(s_*, t) = \left( \frac{\partial^2 \varphi}{\partial \varphi^2} \frac{1}{g^2} \right)(s(\varphi_*), t) - \left( \frac{\partial \varphi}{\partial \varphi} \frac{\partial g}{\partial \varphi} \frac{1}{g^2} \right)(s(\varphi_*), t) \leq 0.
\]

By the equation (3.24),
\[
\frac{\partial \varphi}{\partial t}(s_*, t) \leq -h(p - h)(\varphi(s_*, t))^3 - (p - h) \frac{2\pi}{L} (\varphi(s_*, t))^2
\]
\[+ 2(\varphi(s_*, t))^2 - \frac{2\pi}{L} \frac{\varphi(s_*, t)}{p - h}
\]
\[< -h(p - h)(\varphi(s_*, t))^3 + 2(\varphi(s_*, t))^2
\]
\[< -h^2(\varphi(s_*, t))^3 + 2(\varphi(s_*, t))^2.
\]

Once \( \varphi(s_*, t) \) is greater than \( \frac{2}{h^2} \), we have \( \frac{\partial \varphi}{\partial t}(s_*, t) < 0 \). By the maximum principle, we get
\[
\varphi(s, t) \leq \max \left\{ (\varphi)_{\text{max}}(0), \frac{2}{h^2(T_*)} \right\}
\]
(3.25)
for \((s, t) \in [0, L] \times [0, T_\ast + \frac{A_0}{8\pi}]\). Since the support function \(p(s, t) \leq \frac{L(t)}{2} \leq \frac{L(0)}{2}\), we have

\[
\kappa(s, t) \leq \left(\frac{L(0)}{2} - h(T_\ast)\right) \cdot \max \left\{ (\varphi_4)_{\min}(0), \quad \frac{2}{h^2(T_\ast)} \right\}. \tag{3.26}
\]

Similarly, if \(\varphi_4\) attains \((\varphi_4)_{\min}(t)\) at a point \((s_\ast, t)\) then \(\frac{\partial \varphi_4}{\partial s}(s_\ast, t) = 0\), \(\frac{\partial^2 \varphi_4}{\partial s^2}(s_\ast, t) \geq 0\). Once \(\varphi_4(s_\ast, t) \leq -\frac{2\pi}{\sqrt{4\pi A_0 h(T_\ast)}}\), we have

\[
\frac{\partial \varphi_4}{\partial t}(s_\ast, t) \geq -h(p - h)(\varphi_4(s_\ast, t))^3 - (p - h)\frac{2\pi}{L(t)}(\varphi_4(s_\ast, t))^2 \\
+ 2(\varphi_4(s_\ast, t))^2 \frac{2\pi}{L(t)} - \frac{\varphi_4(s_\ast, t)}{p - h} \\
\geq - \left( h\varphi_4(s_\ast, t) + \frac{2\pi}{L(t)} \right) (p - h)(\varphi_4(s_\ast, t))^2 > 0.
\]

So, by the maximum principle,

\[
\varphi_4(s, t) \geq \min \left\{ (\varphi_4)_{\min}(0), \quad -\frac{2\pi}{\sqrt{4\pi A_0 h(T_\ast)}} \right\}
\]

for \(t \in [0, L] \times [0, T_\ast + \frac{A_0}{8\pi}]\). Therefore we can get

\[
\kappa(s, t) \geq \left(\frac{L(0)}{2} - h(T_\ast)\right) \cdot \min \left\{ (\varphi_4)_{\min}(0), \quad -\frac{2\pi}{\sqrt{4\pi A_0 h(T_\ast)}} \right\}. \tag{3.27}
\]

Therefore, one can conclude that:

**Lemma 3.8** Let the initial smooth curve \(X_0\) be star-shaped and centrosymmetric with respect to \(O\). If Gage’s area-preserving flow (1.1) exists on a time interval \([0, T_\ast]\) for some \(T_\ast > 0\), then the curvature of \(X(\cdot, t)\) is bounded on time interval \([0, T_\ast]\) uniformly.

As a consequence of Lemma 3.8, one can extend Gage’s area-preserving flow (1.1) globally.

**Corollary 3.9** Gage’s area-preserving flow (1.1) exists on the time interval \([0, +\infty)\) if the initial smooth curve \(X_0\) is star-shaped and centrosymmetric with respect to \(O\).

**Proof** Suppose Gage’s area-preserving flow (1.1) with initial \(X_0\) exists on the maximal time interval \([0, T_\ast]\), where \(T_\ast\) is a finite positive number. By the proof of Lemma 3.8, the smooth evolving curve \(X(\cdot, t)\) is star-shaped and centrosymmetric for \(t \in [0, T_\ast]\). And there is a constant \(\delta(T_\ast)\) such that \(r(\cdot, t) \geq \delta(T_\ast) > 0\) for all \(t \in [0, T_\ast]\) (see the equation (3.18)).

By Lemma 2.4, both \(r\) and \(\frac{\partial r}{\partial \theta}\) are bounded uniformly on the time interval \([0, T_\ast]\). Applying Lemma (3.8), \(\frac{\partial^2 r}{\partial \theta^2}\) is also bounded uniformly on the same time interval.
All the higher derivatives \( \frac{\partial^i r}{\partial \theta^i} \) \((i \geq 3)\) satisfy linear parabolic equations, they also have uniform bounds. Therefore, integrating the flow equation gives a smooth, centrosymmetric and star-shaped curve

\[
X(\theta, T_*) = X_0(\theta) + \int_0^{T_*} \frac{\partial X}{\partial t}(\theta, \tau) d\tau.
\]

Using Lemma 2.2, one can extend the flow to a larger time interval \([0, T_* + \varepsilon]\). This conflicts to the assumption that \(T_*\) is maximal.

\[\Box\]

4 **Convergence and a Convexity Theorem**

In this section, it is shown that Gage’s area preserving flow can deform every smooth, closed and embedded curve into a convex one if the flow exists globally.

Since we have proved that the flow (1.1) with initial curve \(X_0\) being smooth, centrosymmetric and star-shaped does not blow up at any finite time in Section 3, the main result in this paper can be obtained by the following theorem immediately:

**Theorem 4.1** Given a smooth, embedded and closed initial curve \(X_0\), if the flow (1.1) does not blow up in time interval \([0, +\infty)\), then the evolving curve converges to a finite circle as time goes to infinity.

Before the proof of this theorem, two lemmas will be established. From now on, we use the subindex to stand for partial derivatives, such as \(\kappa_t = \frac{\partial \kappa}{\partial t}, \kappa_s = \frac{\partial \kappa}{\partial s}, \kappa_{ss} = \frac{\partial^2 \kappa}{\partial s^2}, \ldots\).

The length of the evolving curve is decreasing with respect to time and bounded by \(\sqrt{4\pi A_0}\) from below. Following the parlance from [10], the time derivative of the length must approach zero at an \(\varepsilon\)-dense set of sufficiently large time. Now one can use this fact in the following special case. Denote by \(I_n\) the time interval \([n - \frac{1}{10}, n + \frac{1}{10}]\), where \(n\) is a positive integer. For a positive number \(\varepsilon\), all of the intervals which satisfy that \(\inf I_n \int_0^L \kappa^2 ds - \frac{4\pi^2}{L(t_n)} > \varepsilon\) consists of a finite set. So there exists an \(N > 0\) such that whenever \(n > N\) we have \(t_n \in I_n\) satisfying

\[
\int_0^L \kappa(s, t_n)^2 ds - \frac{4\pi^2}{L(t_n)} \leq \varepsilon. \tag{4.1}
\]

**Lemma 4.2** The \(L^2\) norm of the difference \((\kappa - \frac{2\pi}{L})\) converges to zero as \(t \to \infty\).

**Proof** Integrating the evolution equation of \(L\) can give us that

\[
L(t) - L(0) = -\int_0^t \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 ds dt.
\]

Thus letting \(t\) tend to infinity, we obtain

\[
\int_0^\infty \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 ds dt < L(0). \tag{4.2}
\]
We consider the time derivative of the integral $\int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds$:

\[
\frac{d}{dt} \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds = -2 \int_0^L (\kappa_0)^2 \, ds + \int_0^L \kappa^2 \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds + \frac{2\pi}{L} \int_0^L \kappa \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds.
\]

(4.3)

One only needs to deal with the case that the evolving curve is not convex. In this case, \( \inf_s \kappa^2 = 0 \). Otherwise, the convex curve \( X(\cdot, t) \) will converge to a finite circle (see [3]) and the proof is completed. Suppose \( \kappa(s_1, t) = 0 \) and \( |\kappa|^2(s_2, t) = \sup_s \kappa^2(s, t) \).

Observing

\[
\kappa(s_2, t) = \kappa(s_2, t) - \kappa(s_1, t) \leq \int_{s_1}^{s_2} |\kappa_s| \, ds \leq \sqrt{L} \sqrt{\int_0^L (\kappa_0)^2 \, ds},
\]

one obtains \( \sup_s \kappa^2 \leq L \int_0^L (\kappa_0)^2 \, ds \), i.e.,

\[
-\int_0^L (\kappa_0)^2 \, ds \leq -\frac{1}{L} \sup_s \kappa^2.
\]

By the Cauchy-Schwarz inequality, one may estimate

\[
\left| \int_0^L \kappa \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds \right| \leq \sqrt{\int_0^L \kappa^2 \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds} \cdot \sqrt{\int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds} \leq \sqrt{\int_0^L \sup_s \kappa^4 \, ds} \cdot \sqrt{\int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds} = \sup_s \kappa^2 \cdot \sqrt{L(t)} \cdot \sqrt{\int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds}.
\]

Taking the above two estimates into the equation (4.3) can yield

\[
\frac{d}{dt} \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds \leq -\frac{2}{L(t)} \sup_s \kappa^2 + \sup_s \kappa^2 \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds \leq \sum_s \kappa^2 \left( \frac{2}{L(t)} + \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds \right) + \frac{2\pi}{\sqrt{L(t)}} \sqrt{\int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 \, ds}.
\]
where \( L_\infty = \sqrt{4\pi A} \). Choose \( \varepsilon > 0 \) in the equation (4.1) small enough such that
\[
-\frac{2}{L_0} + \frac{3}{2} \varepsilon + \frac{2\pi}{\sqrt{L_\infty}} \sqrt{\frac{3\varepsilon}{2}} \leq 0.
\]

We now claim that \( \int_0^L (\kappa(s, t) - \frac{2\pi}{L(t)})^2 ds < \frac{3\varepsilon}{2} \) for all large \( t \).

Firstly, by (4.1), we have the following at \( t = t_n \):
\[
\int_0^L \left( \kappa(s, t_n) - \frac{2\pi}{L(t_n)} \right)^2 ds \leq \varepsilon.
\]

Secondly, for \( t > t_n \), suppose there exists a \( t_* \in (t_n, t_n + \frac{11}{10}] \) such that
\[
\int_0^L \left( \kappa(\cdot, t_*) - \frac{2\pi}{L(t_*)} \right)^2 ds = \frac{3\varepsilon}{2}
\]
and
\[
\int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 ds < \frac{3\varepsilon}{2}
\]
for all \( t \in [t_n, t_*] \). In the time interval \((t_n, t_*]\), we have
\[
\frac{d}{dt} \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 ds \leq \sup_s \kappa^2 \left( -\frac{2}{L_0} + \frac{3}{2} \varepsilon + \frac{2\pi}{\sqrt{L_\infty}} \sqrt{\frac{3\varepsilon}{2}} \right) \leq 0.
\]

Thus
\[
\int_0^L \left( \kappa(\cdot, t_*) - \frac{2\pi}{L(t_*)} \right)^2 ds \leq \int_0^L \left( \kappa(\cdot, t_n) - \frac{2\pi}{L(t_n)} \right)^2 ds < \frac{3\varepsilon}{2},
\]
which contradicts with the definition of \( t_* \).

Now for any \( \varepsilon > 0 \), there exists an \( N > 0 \) such that \( t > N \) implies that
\[
\int_0^L (\kappa - \frac{2\pi}{L})^2 ds < \frac{3\varepsilon}{2}.
\]
That is to say, one has the limit
\[
\lim_{t \to \infty} \int_0^L \left( \kappa - \frac{2\pi}{L} \right)^2 ds = 0.
\]

So we have done. \( \square \)

Next, the \( L^2 \)-norm of \((\kappa - \frac{2\pi}{L})_s\) will be proved converging to 0 as time goes to \( +\infty \).
Lemma 4.3  Under the assumption of Theorem 4.1, we have the limit

$$\lim_{t \to \infty} \int_{0}^{L} (\kappa_s)^2 ds = 0.$$  

Proof  Denote $u = \kappa - \frac{2 \pi}{L}$. By the evolution of $\kappa$ and $L(t)$, that

$$u_t = u_{ss} + \left( u + \frac{2 \pi}{L} \right)^2 u - \frac{2 \pi}{L^2} \int_{0}^{L} u^2 ds$$

and

$$u_{st} = u_{sss} + 3 \left( u + \frac{2 \pi}{L} \right) u u_s + \left( u + \frac{2 \pi}{L} \right)^2 u_s.$$  

Using these equations, one can compute

$$\frac{d}{dt} \int_{0}^{L} (u_s)^2 ds = -2 \int_{0}^{L} (u_{ss})^2 ds + 7 \int_{0}^{L} u^2 (u_s)^2 ds + \frac{18 \pi}{L} \int_{0}^{L} u (u_s)^2 ds + \frac{8 \pi^2}{L^2} \int_{0}^{L} (u_s)^2 ds.$$  

(4.4)

If $\int_{0}^{L} (u_s)^2 ds > C \int_{0}^{L} u^2 ds$ holds for some positive $C$ then it follows from

$$\int_{0}^{L} (u_s)^2 ds = - \int_{0}^{L} u u_{ss} ds \leq \sqrt{\frac{1}{C} \int_{0}^{L} (u_s)^2 ds} \int_{0}^{L} (u_{ss})^2 ds$$

that

$$\int_{0}^{L} (u_s)^2 ds \leq \frac{1}{C} \int_{0}^{L} (u_{ss})^2 ds.$$  

(4.5)

We now estimate the terms in the right hand side of the equation (4.4).

$$\int_{0}^{L} u^2 (u_s)^2 ds \leq \int_{0}^{L} u^2 ds \cdot \sup_s (u_s)^2 \leq \int_{0}^{L} u^2 ds \left( \int_{0}^{L} |u_{ss}| ds \right)^2$$

$$\leq \int_{0}^{L} u^2 ds L \int_{0}^{L} (u_{ss})^2 ds.$$  

(4.6)

Using (4.5), (4.6) can yield

$$\int_{0}^{L} u (u_s)^2 ds \leq \frac{1}{2} \int_{0}^{L} u^2 (u_s)^2 ds + \frac{1}{2} \int_{0}^{L} (u_s)^2 ds$$
\[ \leq \frac{L}{2} \int_0^L u^2 ds \int_0^L (u_{ss})^2 ds + \frac{1}{2C} \int_0^L (u_{ss})^2 ds. \quad (4.7) \]

Taking the estimates (4.5)–(4.7) into the equation (4.4), one obtains

\[
\frac{d}{dt} \int_0^L (u_s)^2 ds = \left( -2 + (7L + 9\pi) \right) \int_0^L u^2 ds + \frac{9\pi}{CL} + \frac{8\pi^2}{CL^2} \int_0^L (u_{ss})^2 ds.
\]

Lemma 4.2 tells us that there exists a positive \( t_0 \) such that

\[ (7L_0 + 9\pi) \int_0^L u^2 ds \leq \frac{1}{2} \]

for all \( t > t_0 \). If \( C \geq \max\{ \frac{36\pi}{L_\infty}, \frac{32\pi^2}{L^2_\infty} \} \) then \( \frac{9\pi}{CL_\infty} + \frac{8\pi^2}{CL^2_\infty} \leq 1/2 \) and furthermore

\[
\frac{d}{dt} \int_0^L (u_s)^2 ds \leq - \int_0^L (u_{ss})^2 ds
\]

if \( t > t_0 \). For a \( C^1 \) and \( 2\pi \)-periodic function \( f \) defined on \( \mathbb{R} \) such that \( \int_0^{2\pi} f dx = 0 \), Wirtinger's inequality tells us

\[
\int_0^{2\pi} (f')^2 dx \geq \int_0^{2\pi} f^2 dx.
\]

Set \( s = \frac{L}{2\pi} \phi \). Then

\[
u_s = u\phi \frac{2\pi}{L} \quad \text{and} \quad u_{ss} = u_{\phi\phi} \left( \frac{2\pi}{L} \right)^2.
\]

By Wirtinger's inequality, we get

\[
\int_0^L (u_{ss})^2 ds = \left( \frac{2\pi}{L} \right)^3 \int_0^{2\pi} (u_{\phi\phi})^2 d\phi \geq \left( \frac{2\pi}{L} \right)^3 \int_0^{2\pi} (u_\phi)^2 d\phi
\]

\[
= \left( \frac{2\pi}{L} \right)^2 \int_0^L (u_s)^2 ds.
\]

If \( C \geq \max\{ \frac{36\pi}{L_\infty}, \frac{32\pi^2}{L^2_\infty} \} \), then

\[
\frac{d}{dt} \int_0^L (u_s)^2 ds \leq - \left( \frac{2\pi}{L_0} \right)^2 \int_0^L (u_s)^2 ds, \quad t > t_0.
\]
Thus, for $t > t_0$,

$$\int_0^L (u_s)^2 ds \leq \int_0^L (u_s(t_0))^2 ds \cdot \exp \left[-\left(\frac{2\pi}{L_0}\right)^2 t\right]. \quad (4.8)$$

For large time, the quantity $\int_0^L (u_s)^2 ds$ either decays exponentially or is bounded,

$$\int_0^L (u_s)^2 ds \leq \left(1 + \max \left\{\frac{36\pi}{L_\infty}, \frac{32\pi^2}{L_{2,\infty}^2}\right\}\right) \int_0^L u^2 ds.$$

In either event, it decreases to zero. □

Now we go back to the proof of Theorem 4.1. It follows from Sobolev’s inequality that $|\kappa - \frac{2\pi}{L}|$ tends to 0 as $t \to \infty$. Thus there is a time $T_0 > 0$ such that $\kappa > 0$ for all $t > T_0$. The evolving curve becomes a convex one. By the result in Gage’s original paper [5] (see also the note [1]), we know that the curvature of the evolving curve converges to $\sqrt{\frac{\pi}{A_0}}$ in the $C^\infty$ metric. In the section 4 of the paper [9] by the first author and Zhang, the evolving curve of GAPF is proved to converge to a fixed limiting circle not escaping to infinity or oscillating indefinitely as $t \to +\infty$. So the proof of Theorem 4.1 is completed.

Under the conditions of Theorem 1.1, the initial curve $X_0$ is smooth, star-shaped and centrosymmetric with respect to $O$. By Lemma 3.5, Lemma 3.7 and Corollary 3.9, the evolving curve $X(\cdot, t)$ is both centrosymmetric and star-shaped with respect to $O$ for all $t > 0$. Combining with Theorem 4.1, we conclude that the evolving curve under Gage’s area-preserving flow converges as $t \to +\infty$ to a limiting circle with radius $\sqrt{\frac{A_0}{\pi}}$ and center $O$. So the proof of Theorem 1.1 is completed.

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Data availability Data openly available in a public repository.

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