Polynomial scaling of QAOA for ground-state preparation: taming first-order phase transitions

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We show that the quantum approximate optimization algorithm (QAOA) can construct with polynomially scaling resources the ground state of the fully-connected p-spin Ising ferromagnet, a problem that notoriously poses severe difficulties to a Quantum Annealing (QA) approach, due to the exponentially small gaps encountered at first-order phase transition for \( p \geq 3 \). For a generic target state, we find that an appropriate QAOA parameter initialization is necessary to achieve a good performance of the algorithm when the number of variational parameters \( 2P \) is much smaller that the system size \( N \), because of the large number of sub-optimal local minima. We find that when \( P > P^*_N \propto N \), the structure of the parameter space simplifies, as all minima become degenerate. This allows to achieve the ground state with perfect fidelity with a number of parameters scaling extensively with \( N \), and with resources scaling polynomially with \( N \).

Introduction — Efficient optimization and ground state preparation are two of the most prominent issues in the growing field of quantum technology. Optimization is a long-standing problem in physics and in computer science, and lies at the roots of the efforts to show a possible “quantum supremacy” over classical algorithms. A robust state preparation strategy, in turn, would be a crucial tool for quantum technologies, and would also allow to “solve”, using quantum hardware, many long-standing problems in condensed matter theory or quantum chemistry. The two are intimately connected, as many optimization tasks can be reformulated in terms of finding the classical ground state of an appropriate spin-glass Hamiltonian.

A traditional tool in this field has been Quantum Annealing (QA) — alias Adiabatic Quantum Computation (AQC), which relies on the adiabatic theorem to find the ground state of a target Hamiltonian, starting from a trivial initial state. Although QA appeared to be more efficient than its classical counterpart for certain problems, it is limited by the smallest gap encountered during the evolution, which vanishes, in the thermodynamic limit, when the system crosses a phase transition. In this context, the fully-connected p-spin Ising ferromagnet is a simple but useful benchmark for optimization, because QA fails due to the exponentially small gap at the first-order phase transition encountered for \( p \geq 3 \). The introduction of non-stoquastic terms has been advocated to overcome the slowness induced by such an exponentially small gap.

Recent alternative ground state preparation approaches rely on hybrid quantum-classical variational techniques to tackle such problems, avoiding the limitations imposed by a QA adiabatic evolution. In this Letter we will focus on one such scheme, the Quantum Approximate Optimization Algorithm (QAOA).

The core idea of QAOA is to write a trial wavefunction as a product of many unitary operators, each depending on a classical variational parameter, applied to a simple to construct state, usually a product state with spins aligned in the \( x \)-direction. A quantum hardware performs the discrete quantum dynamics and measures the expectation value of the target Hamiltonian, which is then minimized by an external classical algorithm, as a real function in a high dimensional parameter space.

Although QAOA is a universal computational scheme, its performance strongly depends on the details of the target Hamiltonian. QAOA seems to perform rather well on Max-Cut problems and on short-range spin systems. The Grover search problem has also been studied within QAOA, showing that it leads to the optimal square root speed-up with respect to classical algorithms. For generic long-ranged Hamiltonians, however, many open questions remain. The questions concern, in particular, the efficiency of the algorithm when a large number of unitaries are employed, or the ability to overcome large energy barriers in presence of first-order phase transitions, or the presence of “smooth” sets of optimal parameters.

Addressing these issues, an essential step towards experimental implementations of QAOA in realistic problems, will be the goal of our Letter. We will show that QAOA can construct with polynomially scaling resources the ground state of the fully-connected p-spin Ising ferromagnet for all \( p \geq 2 \), hence including the case where a first-order phase transition occurs. For a generic target state, we find that an appropriate QAOA parameter initialization is necessary to achieve a good performance of the algorithm when the number of variational parameters \( 2P \) is much smaller that the system size \( N \), because of the large number of sub-optimal local minima. Finally, we show that when \( P > P^*_N \propto N \), the structure of the parameter space simplifies, and all minima become degenerate. This allows to achieve the ground state with perfect fidelity with a number of parameters scaling extensively with \( N \), and with resources scaling polynomially with \( N \).

Model and QAOA algorithm — As a benchmark for QAOA on long-range models we focus on the ferromag-
The QAOA algorithm is a variational method to find the ground state of a target Hamiltonian $\hat{H}_{\text{target}}$. Starting from an initial spin state polarized along the $x$ direction $|\rho\rangle = (|\uparrow\rangle + |\downarrow\rangle)^N$, QAOA writes the following variational Ansatz

$$|\psi_P(\gamma, \beta)\rangle \equiv e^{-i\beta_0 \hat{H}_x}e^{-i\gamma_0 \hat{H}_x}\cdots e^{-i\beta_p \hat{H}_x}e^{-i\gamma_p \hat{H}_x}|\uparrow\rangle$$

in terms of $2^P$ variational parameters $\gamma = (\gamma_1, \ldots, \gamma_P)$ and $\beta = (\beta_1, \ldots, \beta_P)$, where $\hat{H}_x$ and $\hat{H}_z$ are non-commuting Hamiltonians depending on the problem we wish to solve. Here we take $\hat{H}_x = -\sum_j \hat{\sigma}_j^x$, the standard transverse field term, and an interaction term $\hat{H}_z$

$$\hat{H}_z = \left(\sum_j \hat{\sigma}_j^z\right)_P,$$

chosen for convenience to have a super-extensive form with an integer spectrum. These choices allow us to restrict the parameter space for $\gamma_m$ and $\beta_m$ to the interval $[0, \pi]$. In each QAOA run the variational energy cost function

$$E_P(\gamma, \beta) = \langle \psi_P(\gamma, \beta)|\hat{H}_{\text{target}}|\psi_P(\gamma, \beta)\rangle,$$

is minimized, until convergence to a local minimum $(\gamma^*, \beta^*)$ is obtained. The quality of the variational solution is gauged by computing the residual energy difference

$$\epsilon^{\text{res}}(\gamma^*, \beta^*) = \frac{E_P(\gamma^*, \beta^*) - E_{\text{min}}}{E_{\text{max}} - E_{\text{min}}},$$

where $E_{\text{min}}$ and $E_{\text{max}}$ are the lowest and largest eigenvalues, respectively, of the target Hamiltonian.

The connection with a QA approach is interesting. In QA one would be interpolating Hamiltonians $\hat{H}(s) = s\hat{H}_{\text{target}} + (1-s)\hat{H}_z$, with $s(t)$ driven from $s(0) = 0$ to $s(\tau) = 1$ in a sufficiently large annealing time $\tau$. A lowest-order Trotter decomposition of the corresponding step-discretized evolution operator — with $s_m = 1 - p$ constant for a time-interval $\Delta t_m = 1 - p$ — would then result in a state of the form of Eq. (2) with:

$$\gamma_m = \frac{s_m \Delta t_m}{\hbar} \frac{1}{N^p - 1},$$
$$\beta_m = \frac{\Delta t_m}{\hbar} \frac{1 - s_m (1 - h)}{}$$

where the total evolution time would be given by:

$$\tau = \sum_{m=1}^P \frac{\Delta t_m}{\hbar} = \sum_{m=1}^P \left(\beta_m + (1 - h)\gamma_m N^p - 1\right).$$

While an optimization of the parameters $s_m$ and $\Delta t_m$ is in principle possible, the standard linear schedule $s(t) = t/\tau$ would result in a digitized QA scheme were $s_m = m/P$ and $\Delta t_m = \tau/P$. With these choices, a convenient starting point for the QAOA optimization algorithm would be to take $\gamma_m^0 = \Delta m / \tau$ and $\beta_m^0 = \Delta / \tau (1 - s_m (1 - h))$ with a possible addition of a small noise term. Alternatively, we might choose a completely random initial point with $\gamma_m^0, \beta_m^0 \in [0, \pi]$. These two alternative choices will be henceforth referred to as l-init and r-init.

**Results** — Ref. [35] has shown that the target ground state of the $p = 2$ fully connected Ising ferromagnet with $h = 0$, the so-called Lipkin-Meshov-Glick model, can be perfectly constructed, with unit fidelity, with the smallest QAOA circuit, $P = 1$. Ref. [10] has recently shown that a whole class of spin-glass models can be constructed where QAOA shows such a property. Here we show — see detailed proof in the Supplementary material (SM) — that the general $p$-spin model in Eq. (1) belongs, for $h = 0$, to the class of $P = 1$ QAOA-solvable problems, for $N$ odd. The core idea of the proof starts from observing that for $P = 1$ the target state fidelity reads:

$$F(\gamma, \beta) = \left| \langle \psi_{\text{targ}} | e^{-i\beta_0 \hat{H}_x}e^{-i\gamma_0 \hat{H}_x} | \uparrow \rangle \right|^2 = \left| \frac{1}{\sqrt{2^N}} \sum_{l} e^{-i\beta_0 \gamma_0 \hat{H}_x} | | \psi_{\text{targ}} | e^{-i\beta_0 \gamma_0 \hat{H}_x} \rangle \right|^2,$$

where $|\psi_{\text{targ}}\rangle$ is the $h = 0$ target ground state, and the sum in the second line runs over the $2^N$ basis states $|l\rangle$ of the computational basis, with $\hat{H}_z |l\rangle = E_l |l\rangle$. Eq. (5) shows that $F$ is the scalar product of two $2^N$-dimensional unit vectors of components $v(\gamma_l) = e^{i\gamma_0 E_l} / \sqrt{2^N}$ and $u(\beta_l) = e^{-i\beta_0 E_l} |l\rangle$, i.e., $F = |v \cdot u|^2$. To ensure $F = 1$, the Cauchy-Schwarz inequality requires $v(\gamma)$ and $u(\beta)$ to be parallel up to an overall phase factor. As discussed in the SM, this requires $\beta = \frac{\pi}{2}$. A unit fidelity further imposes $\gamma$ so that all terms appearing in the sum in Eq. (8) are pure phase factors, which have to be identical for all $l$, modulo $2\pi$. In the SM we perform the calculation explicitly, showing that $\gamma = \frac{\pi}{2}$ guarantees $F = 1$ for $p$ odd, while for $p$ even the precise value of $\gamma$ depends on $p$.

This result is already noteworthy, as it suggests that one can construct the exact $h = 0$ classical ground state with an algorithm whose equivalent computational time, see Eq. (7), scales as $N^{p-1}$. On the contrary, for any finite $N$, a QA algorithm would need to cope with a minimum spectral gap at the transition point $\Delta_c \approx N^{1/3}$, which scales as $\Delta_c \sim N^{-1/3}$ if $p = 2$ and $\Delta_c \sim e^{-\alpha_p N}$ if $p \geq 3$; with a linear schedule annealing, this would imply a total annealing time $\tau \propto \Delta_c^{-2}$, hence $\tau \sim N^{2/3}$ for $p = 2$ and $\tau \sim e^{2\alpha_p N}$.
for $p > 2$. Therefore, QAOA shows an exponential speed-up with respect to a linear-schedule QA for $p > 2$.

Such a remarkable property is however lost as soon as one targets a ground state with $h \neq 0$, where QAOA is no longer able to find the exact ground state with a shallow quantum circuit with $P = 1$ or 2. We find that the energy landscape $E_P(\gamma, \beta)$ is extremely rugged for $P > 2$, making local optimizations — specifically, we use the Broyden-Fletcher-Goldfard-Shanno (BFGS) algorithm — highly dependent on the initial set of parameters $(\gamma^0, \beta^0)$. We observe a very different behavior if the minimization is initialized with parameters $\gamma_m^0$ and $\beta_m^0$ chosen randomly in $[0, \pi]$ (r-init), or rather with an initial guess based on a linear schedule, $\gamma_m^0 = \frac{\Delta m}{N} \frac{1}{h}$ and $\beta_m^0 = \frac{4}{h} \left(1 - \frac{P}{Z} (1 - h)\right)$ (l-init). The results for the random initialization are summarized in Fig. 1 where we show the normalized residual energy, Eq. (6), versus the number of QAOA steps $P$ for $h = \frac{\sqrt{5} - 1}{2}$, whose target state lies in the ferromagnetic phase for any value of $p$. Data for different system sizes $N$ collapse perfectly after rescaling $P \to (P - 2)/N$ (see inset of Fig. 1) and drop below machine precision at $P = P^*_N = 2 + \frac{N}{2}$. Correspondingly, the variance of the residual energy distribution, which is rather large for $P < P^*_N$ as witnessed by the error bars, drops to 0 at $P^*_N$, implying that all local minima become degenerate. This behavior holds for any value of the transverse field $h$, if the QAOA minimization is initialized with random parameters. In general, we find that the residual energy follows:

$$\epsilon_P^{\text{res}} = \begin{cases} 
(1 - \frac{P}{P^*_N})^b & \text{if } P < P^*_N \\
0 & \text{if } P \geq P^*_N 
\end{cases}, \quad (9)$$

with $b \approx 3$. Remarkably, this scaling holds also for $p > 2$, with similar values of $b$, with the only difference that $P^*_N = N + 1$ for $p$ odd, because of the lack of the $Z_2$ symmetry. This in turn implies that for finite $N$ one can attain a perfect control of the state with a circuit depth $P = P^*_N \propto N$, physically corresponding to a total evolution time that scales as a power-law with $N$. Once again, this is at variance with a standard linear-schedule QA, where the total evolution time has to scale exponentially with $N$ when the transition is first order, i.e., for $p > 2$.

A linear initialization of QAOA parameters, with a small noise (see caption of Fig. 2 for details), improves drastically the QAOA performance. This is illustrated in Fig. 2 where the results of the two competing schemes, random (r-init) versus linear (l-init) initialization, are shown for a system with $N = 64$ for both $p = 2$ (main plot) and $p = 3$ (inset), and three fixed values of $P = 5, 15, 25$. Notice how the linear initialization is able to “detect” the quantum paramagnetic phase, for $h > h_c$, as being “easy”, with the QAOA minima found having vanishingly small residual energy, almost to machine precision, even if $P < P^*_N$. This occurs not only in the second-order transition case with $p = 2$, but also in the more “difficult” first-order case with $p = 3$. At variance with that, a random initialization performs on average quite independently of the target transverse field $h$, and knows nothing about the location of the critical field.

The linear initialization displays better efficiency, compared to the random one, also when the target state be-
A smooth change of the control parameters is required, or at least useful, for experimental implementations of QAOA algorithms. Finding local minima $(\gamma^*, \beta^*)$ which can be seen as the discretization of some continuous function, proves however to be a difficult task for this model. In contrast with Refs. 28,31 an iterative procedure that initializes $(\gamma^0, \beta^0)$ from an interpolation of a smooth set obtained for a smaller parameter space does not seem to work in a straightforward way. Our failed attempts do not exclude that smart smooth choices for $(\gamma^0, \beta^0)$ can be constructed: they only signal that finding them is a non-trivial task, due to the extreme roughness of the energy landscape. The linear initialization we have adopted is able to find reasonably smooth $(\gamma^*, \beta^*)$ only for small values of $P$. As the dimensionality of the parameter space increases, and so does the roughness and the number of local minima, the optimal parameters obtained starting from a linear initialization scheme appear to be increasingly irregular (data not shown).

**Discussion and conclusions** — We analyzed the performance of QAOA on the fully connected p-spin model, showing that it is able to find exactly the ferromagnetic ground state with polynomial resources, even when the system encounters a first order phase transition. In particular, the algorithm prepares the ground state of $\tilde{H}$ with only $P = 1$ (if $N$ is odd) or $P = 2$ (if $N$ is even) steps, with a corresponding evolution time that scales as $N^{P-1}$, while QA would require an exponentially long annealing time. This exact minimum however exists only for zero transverse field, $h = 0$. Interestingly, the exact minimum, which clearly survives for $P \geq 2$, is very hard to find with gradient-based optimization schemes due to the extreme roughness of the energy landscape, especially for $p > 2$. The “hardness” of the problem for $p > 2$ is thus reflected in the difficulty in finding the correct absolute minimum, rather than in the resources (i.e. the computational time) needed.

The performance of the optimization itself strongly depends on the initialization of the variational parameters $(\gamma^0, \beta^0)$. For a random initialization, the residual energy drops below machine precision as $(P_N^* - P)^b$, with $b \sim 3$ and $P_N^*$ growing linearly with $N$. This behavior is independent from the target transverse field $h$ and from $p$, with the only difference that $P_N^* = N/2 + 2$ for $p$ even and $P_N^* = N + 1$ for $p$ odd. With a linear initialization, the algorithm performs much better, and is able to detect the presence of a phase transition, although the improvement deteriorates rapidly as $P$ increases, because of the growing number of “bad” local minima.

At variance with Refs. 28,31 we are unable to construct minima in the energy landscape associated with smooth parameters $(\gamma^*, \beta^*)$. In particular individual set of optimized parameters do not approach any regular function neither if they are initialized with a continuous function — as the linear annealing schedule we have adopted — nor if we try to interpolate solutions obtained with smaller values of $P$, as done in Ref. 34. Preliminary results 33 with reinforcement learning methods applied to the QAOA evolution suggest however that smooth choices of $(\gamma^*, \beta^*)$ do indeed exists, but they are hard to find with local optimizations. Whether global minima are related to smooth values of $(\gamma^*, \beta^*)$ remains an open and interesting question.

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Appendix A: Exact ground state with $P = 1$

In this Supplementary Material we will show that one can get the exact target ground state of the p-spin model with a single QAOA step, $P = 1$, starting from the fully x-polarized state $|+\rangle = \frac{1}{\sqrt{2^N}} \bigotimes_{j=1}^{N} (|\uparrow\rangle_j + |\downarrow\rangle_j)$, provided the number of sites $N$ is odd. This holds true for all possible values of $p$, and generalizes the result of Ref. 38 to $p > 2$.

1. $P = 1$: requirements on $\beta$

For $P = 1$ the QAOA state has only two parameters, which we will denote by $\gamma$ and $\beta$, without an index. Let $|\psi_{targ}\rangle$ denote the (target) ground state of the model, and define the fidelity:

\[ F(\gamma, \beta) = \left| \langle \psi_{targ} | \psi_{P=1}(\gamma, \beta) \rangle \right|^2 = \left| \langle \psi_{targ} | e^{-i\beta \hat{H}_x} e^{-i\gamma \hat{H}_x} | + \rangle \right|^2 = \left| \frac{1}{\sqrt{2^N}} \sum_l e^{-i\gamma E_l} \langle \psi_{targ} | e^{-i\beta \hat{H}_x} | l \rangle \right|^2, \quad (A1) \]

where we have expanded the initial state $|+\rangle = \frac{1}{\sqrt{2^N}} \sum_l |l\rangle$ as an equal superposition of all possible $2^N$ classical $z$-basis configurations $|l\rangle$, and we used that $\hat{H}_x |l\rangle = E_l |l\rangle$, where $E_l$ is the energy of the configuration $|l\rangle$. Let us now define the following two $2^N$ dimensional complex vectors:

\[ (v(\gamma))_l = \frac{1}{\sqrt{2^N}} e^{i\gamma E_l} \quad \text{and} \quad (u(\beta))_l = \langle \psi_{targ} | e^{-i\beta \hat{H}_x} | l \rangle. \quad (A2) \]

Simple algebra shows that they have unit norm, $\|v(\gamma)\| = 1$ and $\|u(\beta)\| = 1$, and that the fidelity can be expressed as a scalar product of them: $F(\gamma, \beta) = |v(\gamma) \cdot u(\beta)|^2$. Hence, by the Cauchy-Schwarz inequality:

\[ 1 = F(\gamma, \beta) = |v(\gamma) \cdot u(\beta)|^2 \iff \exists \theta \in \mathbb{R} \quad \text{such that} \quad u(\beta) = e^{i\theta} v(\gamma), \quad (A3) \]

i.e., the two vectors coincide, up to an overall phase factor. Since $|(v(\gamma))_l|^2 = \frac{1}{2^N}$, this in turn implies that we must have

\[ \left| \langle \psi_{targ} | e^{-i\beta \hat{H}_x} | l \rangle \right|^2 = |(u(\beta))_l|^2 = \frac{1}{2^N} \quad \forall l. \quad (A4) \]

So far, our arguments have been rather general. We now specialize our discussion to the case where $|\psi_{targ}\rangle$ is the ground state of the classical p-spin ferromagnet.

For $p$ odd, we have $|\psi_{targ}\rangle = |\uparrow \cdots \uparrow\rangle$, and a simple calculation shows that:

\[ \langle \psi_{targ} | e^{-i\beta \hat{H}_x} | l \rangle = \prod_{j=1}^{N} \langle \uparrow | \cos \beta \hat{1}_j + i \sin \beta \hat{\sigma}_j^x | l \rangle = (\cos \beta)^{N_\uparrow^l} (i \sin \beta)^{N_\downarrow^l}, \quad (A5) \]

where $N_\uparrow^l$ and $N_\downarrow^l$ denote the number of $\uparrow$ and $\downarrow$ spins in the configuration $|l\rangle$. Hence the requirement given by Eq. (A4) is satisfied only if

\[ \cos^2 \beta = \sin^2 \beta = \frac{1}{2} \quad \Rightarrow \beta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \quad (A6) \]

Similar arguments have been used, see Ref. 40 for the more general case in which $|\psi_{targ}\rangle$ is the classical ground state of a generic spin-glass Hamiltonian.

For $p$ even the calculation is slightly more involved, since the target state is now a non-classical superposition of the two degenerate ferromagnetic states

\[ |\psi_{targ}\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow \cdots \uparrow\rangle + |\downarrow \cdots \downarrow\rangle \right). \quad (A7) \]

Hence:

\[ \left| \langle \psi_{targ} | e^{-i\beta \hat{H}_x} | l \rangle \right|^2 = \frac{1}{2} \left( (\cos \beta)^{N_\uparrow^l} (i \sin \beta)^{N_\downarrow^l} + (\cos \beta)^{N_\downarrow^l} (i \sin \beta)^{N_\uparrow^l} \right)^2. \quad (A8) \]

Once again, one easily verifies that $\beta = \frac{\pi}{4}$ satisfies the requirement (A4), provided $N$ is odd, so that $N_\uparrow^l$ and $N_\downarrow^l$ have opposite parity and therefore $|i^{N_\uparrow^l} + i^{N_\downarrow^l}|^2 = 2$. 

From now on we will therefore restrict our choice of $\beta$ to $\beta = \frac{\pi}{4}$, a necessary condition for unit fidelity, and study the conditions that $\gamma$ has to verify. Essentially, the value of $\gamma$ will have to be chosen in such a way that the various phase factors interfere constructively in a way that is independent of $l$. To this goal, we notice that the energy $E_l$ of the configuration $|l\rangle$ can be expressed as:

$$E_l = - \langle l | \left( \sum_j \hat{\sigma}_j^z \right)^p | l \rangle = -(N_l^\uparrow - N_l^\downarrow)^p = -M_l^p,$$

where $M_l = N_l^\uparrow - N_l^\downarrow$ is the total magnetization of the configuration.

2. $P = 1$ and $p$ odd: requirements on $\gamma$

Here we prove that for odd values of $p$ and $N$, the $p$-spin QAOA circuit of depth $P = 1$ and parameters $(\gamma = \frac{\pi}{4}, \beta = \frac{\pi}{4})$ is sufficient to prepare the ferromagnetic target state $|\psi_{\text{targ}}\rangle = |\uparrow \cdots \uparrow\rangle$. Substituting $\beta = \frac{\pi}{4}$ and $i = e^{i\pi/2}$ in Eq. (A5), and using $E_l = -M_l^p$ in Eq. (A1) we get:

$$F(\gamma, \frac{\pi}{4}) = \frac{1}{2N} \sum_l e^{i\gamma M_l^p} e^{i\pi N_l^\uparrow \sqrt{2}},$$

Taking $\gamma = \frac{\pi}{4}$ gives

$$F(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2N} \sum_l \exp \left( \left( \frac{2\pi}{8} [\left( M_l^p + 2N_l^\downarrow \right) \mod 8] \right)^2 \right) = 1,$$

where we have used the fact that for $N$ odd, $M_l = N - 2N_l^\downarrow$ is also odd and the following property of arithmetic congruences holds:

$$M_l^{p-1} = 1 \mod 8 \implies M_l^p = M_l \mod 8 \quad \text{if } p \text{ is odd}.$$ (A12)

Eq. (A11) proves our initial claim, that the QAOA protocol $(\gamma = \frac{\pi}{4}, \beta = \frac{\pi}{4})$ prepares the target ground state of $\hat{H}_z$ for the $p$-spin ferromagnet with unit fidelity, provided $N$ and $p$ are both odd.

3. $P = 1$ and $p$ even: requirements on $\gamma$

For even values of $p$, the system is $\mathbb{Z}_2$ symmetric. The $p$-spin QAOA circuit preserves such symmetry. Therefore, the targeted ground state of $\hat{H}_z$ is $|\psi_{\text{targ}}\rangle$ in Eq. (A7). As we did in the previous section, we compute the fidelity between the output $|\psi_{P=1}(\gamma, \beta = \frac{\pi}{4})\rangle$ of the QAOA circuit and the (non-classical) target state $|\psi_{\text{targ}}\rangle$ in Eq. (A7):

$$F(\gamma, \beta = \frac{\pi}{4}) = \frac{1}{2N} \sum_l e^{i\gamma M_l^p} \left( \frac{e^{i\frac{\pi}{4} N_l^\uparrow} + e^{i\frac{\pi}{4} N_l^\downarrow}}{\sqrt{2}} \right)^2.$$

We observe that for $N = N_l^\uparrow + N_l^\downarrow$ odd, $N_l^\uparrow$ and $N_l^\downarrow$ must have opposite parity, and the term inside parenthesis is a pure phase factor, which can be expressed as:

$$\left( \frac{e^{i\frac{\pi}{4} N_l^\uparrow} + e^{i\frac{\pi}{4} N_l^\downarrow}}{\sqrt{2}} \right) = e^{i\frac{\pi}{4} N_l e^{-i\pi f(M_l)}},$$ (A14)
where

\[ f(M) = \begin{cases} 
0 & \text{for } M \mod 8 = \pm 1 \\
1 & \text{for } M \mod 8 = \pm 3
\end{cases}. \tag{A15} \]

Hence, omitting the irrelevant \( l \)-independent common factor \( e^{i\pi N} \), we can rewrite the fidelity as:

\[ \mathcal{F}(\gamma, \frac{\pi}{4}) = \left| \frac{1}{2^N} \sum_l e^{i(\gamma M_l^p - \pi f(M_l))} \right|^2. \tag{A16} \]

The arithmetics to prove that the various phase factors can be made \( l \)-independent for a judicious choice of \( \gamma \) is now, for even \( p \), slightly more involved. By experimenting with this expression for \( p \leq 10 \), we have come out with the following unconventional parameterization of an even value of \( p \): for every even \( p \), two natural numbers \( n \) and \( k \) can be found such that:

\[ p = 2^k + n2^k. \tag{A17} \]

Correspondingly, given the value of \( k \) in Eq. (A17), we will set the value of \( \gamma \) to:

\[ \gamma_k = \frac{2\pi}{2^k + 4}. \tag{A18} \]

The crucial arithmetic identity which we will use — see Sec. A3 a for a proof — is the following:

\[ m^{2^{k+1} + n2^k} \mod 2^{k+4} = f(m) 2^{k+3} + 1 \quad \forall \, m \in \mathbb{Z} \text{ with } m \text{ odd}, \tag{A19} \]

where \( f(m) \) is the function given in Eq. (A15).

With these definitions, it is immediate to verify that:

\[ \mathcal{F}(\gamma_k, \frac{\pi}{4}) = \left| \frac{1}{2^N} \sum_l e^{-i\pi f(M_l)} \exp \left( i \frac{2\pi}{2^k + 4} (M_l)^{2^{k+1} + n2^k} \right) \right|^2 = \left| \frac{1}{2^N} \sum_l e^{-i\pi f(M_l)} \exp \left( i \frac{2\pi}{2^k + 4} \left[(M_l)^{2^{k+1} + n2^k} \mod 2^{k+4}\right]ight) \right|^2 = \left| \frac{1}{2^N} \sum_l e^{-i\pi f(M_l)} \exp \left( i \frac{2\pi}{2^k + 4} \left[f(M_l) 2^{k+3} + 1\right]\right) \right|^2 = \left| \frac{1}{2^N} \sum_l e^{-i\pi f(M_l)} e^{i\pi f(M_l) + i\frac{2\pi}{2^k + 4}} \right|^2 = 1. \tag{A20} \]

\[ a. \] Proof of identity in Eq. (A19) For completeness, we also present a proof of the arithmetic identity Eq. (A19). To prove Eq. (A19), it is sufficient to show that

\[ \forall k \in \mathbb{N}, m \in \mathbb{Z}, m \text{ odd} : \quad (m^{2^{k+1}} - 1) \mod 2^{k+4} = f(m) 2^{k+3}. \tag{A21} \]

We prove Eq. (A21) by induction over \( k \):

(i) We show that Eq. (A21) holds for \( k = 0 \).

For \( k = 0 \), a direct computation, for odd \( m \), gives:

\[ (m^{2^1} - 1) \mod 2^{0+4} = (m^2 - 1) \mod 16 = (m - 1)(m + 1) \mod 16 = f(m) 2^3. \tag{A22} \]

(ii) We show that if Eq. (A21) holds for a given \( k \in \mathbb{N} \) and for all odd \( m \in \mathbb{N} \), then it holds also for \( k + 1 \).
Using Eq. (A21), we write
\[
m^{2^{k+1}} = a_m 2^{k+4} + f(m) 2^{k+3} + 1,
\]
with \(a_m \in \mathbb{Z}\). Then, we have
\[
(m^{2^{(k+1)+1}} - 1) = (m^{2^{k+1}} - 1)(m^{2^{k+1}} + 1) = (a_m 2^{k+4} + f(m) 2^{k+3})(a_m 2^{k+4} + f(m) 2^{k+3} + 2) = (a_m 2^{k+5} + f(m) 2^{k+4})(a_m 2^{k+3} + f(m) 2^{k+2} + 1). \tag{A24}
\]
From this, we derive
\[
(m^{2^{(k+1)+1}} - 1) \mod 2^{(k+1)+4} = f(m) 2^{k+4}(a_m 2^{k+3} + f(m) 2^{k+2} + 1) \mod 2^{k+5} = f(m) 2^{k+4}, \tag{A25}
\]
where we have used that \(f(m) = 0, 1\) for all odd \(m \in \mathbb{Z}\). This indeed implies that for all \(k \in \mathbb{N}\):
\[
(m^{2^{k+1}} - 1) \mod 2^{k+4} = f(m) 2^{k+3} \implies (m^{2^{(k+1)+1}} - 1) \mod 2^{(k+1)+4} = f(m) 2^{(k+1)+3}. \tag{A27}
\]
This concludes the proof by induction of Eq. (A21).

Incidentally, as an immediate consequence of Eq. (A21) we get that, for any \(n \in \mathbb{N}\):
\[
m^{2^{k+1}} \mod 2^{k+4} = f(m) 2^{k+3} + 1 \tag{A28}
\]
\[
m^{2^{k+2}} \mod 2^{k+4} = 1 \tag{A29}
\]
\[
m^{2^{k+1}+n^{2^k}} \mod 2^{k+4} = f(m) 2^{k+3} + 1. \tag{A30}
\]
Notice that Eq. (A29) also follows from the properties of the multiplicative group of integers modulo \(2^k\) discussed in Refs. [44,45] (e.g. \((\mathbb{Z}/2^{k+4}\mathbb{Z})^\times \cong C_2 \times C_{2^{k+2}}\)).

**Appendix B: Symmetries of the parameter space for general \(P, N\) and \(p\)**

We discuss here the symmetries in the parameter space of the function
\[
E_P(\gamma, \beta) = \langle \psi_P(\gamma, \beta)|\hat{H}_{\text{target}}|\psi_P(\gamma, \beta)\rangle. \tag{B1}
\]
A first trivial operation that leaves the energy unaltered is the inversion \((\gamma, \beta) \to -(\gamma, \beta)\), which corresponds to the complex conjugate of Eq. (B1). Indeed it is immediate to see that
\[
|\psi_P(-\gamma, -\beta)\rangle = \prod_{m=1}^P e^{i\beta_m \hat{R}_z} e^{i\gamma_m \hat{R}_x} |\psi_P(\gamma, \beta)\rangle^*, \tag{B2}
\]
given that \(|\psi_P\rangle = |+\rangle\) is a real wavefunction in the basis of \(\hat{S}_z\).

The symmetries on the \(\beta\) parameters are shared by all QAOA wavefunctions where quantum fluctuations are induced by a magnetic field transverse to the computational basis. We can write a single evolution operator \(e^{-i\beta_m \hat{R}_z}\) as a set of rotation on each individual spin
\[
e^{-i\beta_m \hat{R}_z} = e^{i\beta_m \sum_{j=1}^{N} \hat{\sigma}_z} = \bigotimes_{j=1}^{N} (\cos \beta_m + \hat{\sigma}_z \sin \beta_m). \tag{B3}
\]
A shift \(\beta_m \to \beta_m + \pi\) changes the sign of each term in the product, leading to
\[
e^{-i(\beta_m + \pi) \hat{R}_z} = \bigotimes_{j=1}^{N} (-\cos \beta_m - \hat{\sigma}_z \sin \beta_m) = (-1)^N \bigotimes_{j=1}^{N} (\cos \beta_m + \hat{\sigma}_z \sin \beta_m), \tag{B4}
\]
\[
\forall \, N, \, p \quad E(-\gamma, -\beta) = E(\gamma, \beta)
\]

\[
p \text{ odd} \quad E(\gamma, \beta + \pi) = E(\gamma, \beta)
\]

\[
p \text{ even} \quad E(\gamma, \beta + \frac{\pi}{2}) = E(\gamma, \beta)
\]

\[
N \text{ odd} \quad E(\gamma + \pi, \beta) = E(\gamma, \beta)
\]

\[
N \text{ even} \quad E(\gamma + \frac{\pi}{2p - 1}, \beta) = E(\gamma, \beta)
\]

Table I: Symmetry operations for the QAOA process of the p-spin model. It is understood that any component of \(\gamma\) or \(\beta\) can be modified.

which is a trivial global phase that does not change the energy in Eq. (B1). Moreover, if \(p\) is even, the target Hamiltonian is \(\mathbb{Z}_2\) symmetric. Recall that \(\hat{H}_z = -\hat{S}_z\) (twice the total spin), which implies that:

\[
e^{i\pi \hat{S}_z} \hat{H}_{\text{target}} e^{-i\pi \hat{S}_z} = \hat{H}_{\text{target}},
\]

because \(e^{-i\pi \hat{S}_z}\) is a \(\pi\)-rotation around the \(x\)-direction which gives a global spin flip \(\hat{\sigma}_j^z \rightarrow -\hat{\sigma}_j^z\), leading to \(E_p(\gamma, \beta + \frac{\pi}{2}) = E_p(\gamma, \beta)\).

The symmetry for \(\gamma\) is subtler and is model-specific. Notice first that \(\hat{S}_z = \sum_j \hat{\sigma}_j^z\) has integer eigenvalues, even or odd depending on \(N\), and so does \(\hat{H}_z = -\hat{S}_z\). Following the same notation introduced previously, we write a single QAOA evolution operator as

\[
e^{-i\gamma_m \hat{H}_z} = \sum_l e^{i\gamma_m M_l^p \langle l \rangle} \langle l |.
\]

If \(N\) is odd the eigenvalues \(M_l^p\) of \(\hat{S}_z^p\) are also odd, and the periodicity of \(\gamma_m\) is \(\pi\), because

\[
(\gamma_m + \pi) M_l^p = \gamma_m M_l^p + \pi \mod 2\pi.
\]

Hence the shift \(\gamma_m \rightarrow \gamma_m + \pi\) introduces a global phase \(e^{-i(\gamma_m + \pi) \hat{H}_z} = -e^{-i\gamma_m \hat{H}_z}\), which is irrelevant in the expectation value of the energy. If \(N\) is even, the eigenvalues \(M_l^p\) of \(\hat{S}_z^p\) are multiples of \(2^p\), hence

\[
(\gamma_m + \frac{\pi}{2^p - 1}) M_l^p = \gamma_m M_l^p \mod 2\pi,
\]

which means that \(e^{i(\gamma_m + \frac{\pi}{2^p - 1}) \hat{S}_z^p} = e^{i\gamma_m \hat{S}_z^p}\). In Table I we summarize the symmetries we have discussed.
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