Stabilizing inverse problems by internal data

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Abstract
Several newly developing hybrid imaging methods (e.g., those combining electrical impedance or optical imaging with acoustics) enable one to obtain some auxiliary interior information (usually some combination of the electrical conductivity and the current) about the parameters of the tissues. This information, in turn, happens to stabilize the exponentially unstable and thus low-resolution optical and electrical impedance tomography. Various known instances of this effect have been studied individually. We show that there is a simple general technique (covering all known cases) that shows what kinds of interior data stabilize the reconstruction, and why. Namely, we show when the linearized problem becomes an elliptic pseudo-differential one, and thus stable. Stability here is meant as the problem being Fredholm, so the local uniqueness is not shown and probably does not hold in such generality.

1. Introduction

The fast developing, in recent years, hybrid imaging methods refer to a range of techniques in medical imaging in which different modalities are used in concert to benefit from the strengths of each while mitigating their individual weaknesses [4, 7, 9, 30, 31, 34, 50, 51]. For example, ultrasound tomography provides high resolution, while not necessarily providing high contrast. On the other hand, electrical impedance tomography (EIT) and optical tomography (OT) can provide high contrast but are typically plagued by high instability and thus poor resolution [15, 18, 22]. Acousto-electric tomography (AET), also called ultrasound modulated EIT (UMEIT) [4, 5, 26, 30, 32, 33, 57], uses focused (physically or synthetically) ultrasound waves to perturb the conductivity inside the object of interest, and these perturbations can be measured using the EIT techniques. A similar approach is used in ultrasound modulated OT (UMOT), which combines OT with acoustics [3, 13, 40, 41, 51]. The MREIT and CDI/CDII techniques use various combinations of EIT and MRI [37–39, 42, 53]. What is common for these (physically rather different) techniques is that after some manipulations (see [4, 6–8, 11–14, 26, 33, 37–39, 42, 53] for details), the values through the interior of the object of...
a function (often of the form $\sigma(x)|\nabla u(x)|^p$) can be obtained. Here, $\sigma$ is the conductivity to be determined, and $u$ is the corresponding electric potential. The problem then becomes to determine $\sigma$ from these interior data.

Surveys of the recent results on hybrid methods can be found, e.g., in [4, 7, 30, 31]. It has been observed that a common feature of these methods is that they provide significantly better stability and resolution than more conventional EIT and OT techniques. The opinion has been expressed by several experts that some meta-statement should exist that claims that ‘appropriate’ interior information stabilizes the exponentially unstable problems, such as EIT or OT. Our aim is to provide a version (in fact, several versions of various generality) of such a statement. Doing this, we can address the stability of different hybrid methods with internal data in a unified way.

Our considerations are local, i.e. in a neighborhood of a known smooth background. We thus use linearization. If one could prove that the Fréchet derivative of the corresponding nonlinear mapping exists, has zero kernel and is a (semi-)Fredholm operator in some scale of (say, Sobolev) Banach spaces, by standard functional analysis arguments this would have implied local uniqueness and stability of the nonlinear problem. In this text, we provide very general statements on the (semi-)Fredholm property of such Fréchet derivatives (see, e.g., [27–29, 55] for basics on Fredholm and semi-Fredholm operators). This is done by reducing them to elliptic pseudo-differential operators. This achieves necessary stability estimates modulo the possible finite-dimensional kernel. The authors doubt that the uniqueness claim (i.e. absence of non-trivial kernel) can be made in such wide generality in which we obtain ellipticity and thus the Fredholm property. Thus, the absence of kernel should be dealt with on a case-by-case basis (as it has been done before). However, the existence of the Fredholm property shows that the interior data do have a stabilizing effect on the initially exponentially unstable problem.

Let us describe the structure of this paper. In section 2, we describe the framework for later sections, and we state basic definitions and lemmas. In section 3, we investigate inverse conductivity problems with different types of additional interior data. We first consider the data of the form $\sigma(x)|\nabla u(x)|^p$, which are known to arise in a variety of hybrid problems [4, 6–8, 11–14, 26, 30, 33, 37–39, 42, 53]. For $p \in (0, 1)$, we show that ellipticity, and thus the stability, arise with a single set of interior data. For $p > 1$, two such sets are needed. Finally, we look at a rather general type of interior data of the form $F(\sigma(x), u(x), \nabla u(x))$ and obtain sufficient conditions under which one gets the problem stabilized. In section 4, we treat a problem coming from the so-called quantitative photo-acoustic tomography (QPAT) [10–12, 23, 25, 30, 33], where the equation becomes of diffusion type (with an absorption term) rather than just a divergence-type equation, as in the previous section. The main results of the paper are contained in theorems 3.2, 3.6, 3.8, and 4.1. Sections 5 and 6 contain the proofs of some technical lemmas used and final remarks and conclusions correspondingly.

2. Preliminaries

Let $\Omega$ be a bounded open region in $\mathbb{R}^n$ with smooth boundary, and let $\Omega'$ be an open region compactly contained within $\Omega$. We will also need an intermediate domain $\Omega''$:

$$\Omega' \subset \Omega'' \subset \Omega.$$
We will frequently need to use two cutoff functions: $\chi_1 \in C^\infty_0(\Omega)$ that is equal to 1 in a neighborhood of $\overline{\Omega}$ and $\chi_2 \in C^\infty_0(\Omega^\prime)$ that is equal to 1 in a neighborhood of $\partial \Omega$.

In section 3, we will address various inverse conductivity problems, in which the goal is to determine the unknown log conductivity $\sigma$ in the elliptic problem

$$L_0 u = -\nabla \cdot (e^\sigma \nabla u) = 0$$

in $\Omega$ from some boundary data. For instance, in the EIT, the data might be the whole Dirichlet-to-Neumann map on $\partial \Omega$ [18, 19, 22, 46–49].

We thus will have to work frequently with the Dirichlet boundary value problem

$$\left\{ \begin{array}{l} -\nabla \cdot (e^\sigma \nabla u) = 0, \\
|u|_{\partial \Omega} = f. \end{array} \right.$$

(2)

It will be important for us that the solution of (2) depends on log conductivity $\sigma$. To emphasize this dependence, we will sometimes write $u(\sigma)$ for the solution to (2).

Analogously to [21], we define the affine space of admissible log conductivities as

$$L_0^\infty(\Omega) = \{ \sigma \in L^\infty(\Omega) \mid \sigma|_{\partial \Omega} = 0 \}. \quad (3)$$

Functions in $L_0^\infty(\Omega)$ can be considered as defined on $\mathbb{R}^n$ by extending them by zero. We assume that $\sigma \in L_0^\infty(\Omega)$ and $f \in H^{1/2}(\partial \Omega)$ in (2), so $u \in H^1(\Omega)$ (see e.g. [18]).

In section 4, we will study the following more general problem:

$$\left\{ \begin{array}{l} L_{\sigma, \gamma} u := -\nabla \cdot (e^\sigma \nabla u) + e^\gamma u = 0 \\
|u|_{\partial \Omega} = f. \end{array} \right.$$

(4)

The coefficients $\sigma$ and $\gamma$ are the log-diffusion and log-attenuation coefficients, respectively.

**Lemma 2.1.** The map from $(\sigma, \gamma)$ to $u$, defined by (4), is Fréchet differentiable as a mapping from $L^\infty(\Omega) \times L^\infty(\Omega)$ to $H^1(\Omega)$ at any point $(\sigma_0, \gamma_0) \in C^\infty(\Omega) \times C^\infty(\Omega)$.

This fact is well known, but we supply in section 5 its proof. Infinite differentiability of the coefficients $(\sigma_0, \gamma_0)$ is in fact an overkill assumption here, but we will use (and thus prove) the lemma only in the smooth case.

In many hybrid imaging methods (see, e.g., [4, 5, 7, 26, 33, 37–39, 42, 53]), internal information of the form

$$F(\sigma(x), u(x), \nabla u(x)),$$

(5)

where $u$ is the solution of (2) corresponding to some specific boundary data $f$, can be derived from the measured data. Thus, the next goal is to recover the log-conductivity function $\sigma(x)$ from knowledge of $F(\sigma(x), u(x), \nabla u(x))$ for all $x \in \Omega$ (which explains the name ‘interior data’). Since, for the fixed boundary Dirichlet data $f$, the functions $u$ and $\nabla u(x)$ are determined by $\sigma$, we can consider $F$ as a nonlinear operator acting on $\sigma: F: \sigma \mapsto F(\sigma)$.

The pseudo-differential technique that we use makes the statements simple and their proofs rather transparent. We make use of basic facts about pseudo-differential operators on $\mathbb{R}^n$ (see, e.g., [44] or [45, chapter 7]). We will use the standard symbol classes $S^m(\mathbb{R}^n)$ comprising smooth functions $a(x, \xi)$ that satisfy for any multi-indices $\alpha, \beta$ and for sufficiently large $|\xi|$ the estimates

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{m, \beta} (1 + |\xi|)^{m-|\beta|}$$

(6)

It will be more convenient for us to work with log conductivity, which we denote $\sigma$, rather than with the true conductivity $e^\sigma$. 
with some constants $C_{a \beta}$. (Such symbol classes are often denoted $S_{1,0}^{m}(\mathbb{R}^n)$, but we will omit the subscripts as we will not be considering more general symbol classes.) The corresponding classes of pseudo-differential operators, which we denote by $\text{OPS}^m(\mathbb{R}^n)$, are given by
\[
 a(x, D)u = \frac{1}{(2\pi)^n} \int \int a(x, \xi) e^{i(x-y)\cdot \xi} u(y) \, dy \, d\xi , \tag{7}
\]
where $a(x, \xi) \in S^m(\mathbb{R}^n)$ (with the standard regularization of this expression, see, e.g., [45, 44]).

If a symbol $a(x, \xi) \in S^m(\mathbb{R}^n)$ satisfies
\[
 a(x, \xi) = a_m(x, \xi) + r(x, \xi), \tag{8}
\]
where $r(x, \xi) \in S^{m-1}(\mathbb{R}^n)$, we will call $a_m(x, \xi)$ the principal symbol of $a(x, \xi) \in S^m(\mathbb{R}^n)$. The principal symbol is determined modulo $S^{m-1}(\mathbb{R}^n)$. If, for some $R > 0$, the principal symbol $a_m(x, \xi)$ satisfies the estimate
\[
 |a_m(x, \xi)| \geq C|\xi|^m \text{ for } |\xi| \geq R, \tag{9}
\]
then the symbol $a(x, \xi)$ is called elliptic, and the corresponding operator $a(x, D)$ is called elliptic as well.

We will also need some facts about (square) matrix pseudo-differential operators. Let $A_{i,j}(x, \xi)$ for $i, j = 1, \ldots, p$ be a matrix of classical symbols of pseudo-differential operators. Suppose that there exist two $p$-tuples $(s_1, \ldots, s_p)$ and $(t_1, \ldots, t_p)$ of real numbers, such that $A_{i,j} \in S^{s_i-t_j}(\mathbb{R}^n)$. Let also $A_{i,j}^0 \in S^{s_i-t_j}$ be their principal symbols. The system $A = \{A_{i,j}\}$ is said to be elliptic in the Douglis–Nirenberg (DN) sense$^5$ (see [1] or [45, chapter 7], [52, chapter 9]) if the determinant $\det(A_{i,j}(x, \xi))$ does not vanish for $|\xi| > R$, for a suitable $R$.

3. Stability in inverse conductivity problems with internal data

Here, we address functionals $F$ of the more specific form
\[
 F(\sigma) = e^{\sigma} |\nabla u(\sigma)|^p. \tag{10}
\]
Several hybrid imaging methods provide this kind of internal data (see, e.g., [4, 7, 6, 8, 11–14, 26, 33, 37–39, 42, 53]). For example, it arises with $p = 2$ in AET (sometimes also called UMEIT, or impedigraphy).

When $0 < p \leq 2$ and $\sigma \in L^\infty_\ad(\Omega)$, one concludes that $F(\sigma)(x)$ belongs to $L^1(\Omega)$. Let us consider $F$ as a nonlinear mapping from $L^\infty_\ad(\Omega)$ to $L^1(\Omega)$. As such, it is Fréchet differentiable at any smooth log conductivity $\sigma_0$, as long as the corresponding solution $u(\sigma_0)$ has a gradient that is bounded below by a positive constant. This is a direct consequence of the following lemma.

**Lemma 3.1.** Let $F(y, z, w)$ be a function of three variables that is smooth when $y, z \in \mathbb{R}$, and $w > 0$. Assume that $F$ satisfies the bound
\[
 |F(y, z, w)| \leq C(y)(z^2 + w^2), \tag{11}
\]
where $C(y)$ depends continuously on $y$. Then, the mapping $\sigma \mapsto F(\sigma, u, |\nabla u|)$ is Fréchet differentiable at the smooth background log conductivity $\sigma_0$, as a mapping from $L^\infty_\ad(\Omega)$ to $L^1(\Omega)$.

$^5$ Sometimes it is also called Agmon–Douglis–Nirenberg ellipticity. An equivalent, although differently formulated, notion was also introduced by L. Volevich (see, e.g., [52, chapter 9]).
The proof is provided in section 5.

Since the Fréchet derivative exists, it can be found by a formal linearization. Consider a small perturbation of $\sigma_0$:
\[
\sigma = \sigma_0 + \epsilon \rho, \\
u = u_0 + \epsilon v + o(\epsilon),
\]
where $u_0 = u(\sigma_0)$. A simple substitution, as in [21], shows that $v \in H^1_0(\Omega)$ solves the boundary value problem
\[
\begin{aligned}
L_0 v &= \nabla \cdot (\rho e^{\sigma_0} \nabla u_0) \\
v |_{\Omega} &= 0.
\end{aligned}
\]
(13)

One notes that the dependence of $v$ on $\rho$ is linear. We will indicate it as $v(\rho)$.

The Fréchet derivative $dF$ of $F$, which we will denote by $dF$, is a linear bounded operator from $L^\infty_0(\Omega)$ to $L^1(\Omega)$. Applying the chain rule and (31) to $e^\sigma |\nabla u(\sigma)|^p$, one finds $dF$ as
\[
dF(\rho) = \rho e^{\sigma} |\nabla u_0|^p + \rho e^{\sigma} \frac{\nabla u_0 \cdot \nabla v(\rho)}{|\nabla u_0|^2}. \\
\]
(14)

We introduce a cutoff version of $dF$, which extends to a pseudo-differential operator on $\mathbb{R}^n$. Let $\chi_1$ be a smooth cutoff function supported in $\Omega$, which is identically equal to 1 on a neighborhood of $\overline{\Omega}$. We define an operator $A$ mapping $L^\infty(\Omega)$ to $L^1(\mathbb{R}^n)$ by
\[
A(\rho) = \chi_1 \, dF(\chi_1 \rho).
\]
(15)

Because of the presence of the cutoff by $\chi_1$ before applying $dF$, the operator $A$ has a natural extension to $L^\infty(\mathbb{R}^n)$.

In order to show that $A$ is a pseudo-differential operator, we analyze equation (31). The expression $\nabla \cdot (\chi_1 \rho e^{\sigma_0} \nabla u_0)$, as a differential operator acting on $\rho$, has the principal symbol $\imath e^{\sigma_0} \chi_1 \xi \cdot \nabla u_0$. This operator, when acting on functions that vanish outside of $\Omega'$, does not depend on the choice of $\chi_1$.

The principal symbol of the elliptic differential operator $L_0$, is $e^{\sigma} |\xi|^2$. Hence, $L_0$ has a pseudo-differential parametrix $P \in OPS^{-2}(\mathbb{R}^n)$ with the principal symbol $(\imath e^{\sigma} |\xi|^2)^{-1}$. This means that $L_0 P = I + S$, where $I$ is the identity operator on $\mathbb{R}^n$ and $S$ is a smoothing operator on $\mathbb{R}^n$, and analogously for $PL_0$. Let us define the following function:
\[
u := P(\nabla \cdot (\chi_1 \rho e^{\sigma_0} \nabla u_0)).
\]

Then, we have
\[
L_0 (v - w) = \nabla \cdot (\rho e^{\sigma_0} \nabla u_0) - L_0 P(\nabla \cdot (\chi_1 \rho e^{\sigma_0} \nabla u_0)) \\
= \nabla \cdot (\rho e^{\sigma_0} \nabla u_0) - (I + S)(\nabla \cdot (\chi_1 \rho e^{\sigma_0} \nabla u_0)) \\
= \nabla \cdot ((1 - \chi_1) \rho e^{\sigma_0} \nabla u_0) + S(\nabla \cdot (\chi_1 \rho e^{\sigma_0} \nabla u_0)).
\]
(16)

The expression in (16) is a smooth function, so $L_0 (v - w) \in C^\infty(\Omega)$. By elliptic regularity, $v \equiv w \mod C^\infty(\Omega)$. Because of this equivalence, the mapping $\rho \mapsto v$ is a pseudo-differential operator modulo infinitely smoothing operators on $\Omega$. All other operations in equation (14) are simply multiplication operators, so we see that after multiplying by $\chi_1$, $A$ is a pseudodifferential operator on $\mathbb{R}^n$.

Let $A_0(x, \xi)$ denote the principal symbol of $A$ so that $A(x, \xi) = A_0(x, \xi) + R_{-1}(x, \xi)$, where $R_{-1}$ is a symbol of order $-1$. The symbol $A_0$ is easily derived. The principal symbol of a composition of operators is the product of the individual principal symbols. Applying this to the composition of the operators
\[
\rho \mapsto \nabla \cdot (\chi_1 \rho e^{\sigma_0} \nabla u_0) \\
u \mapsto P(\rho)
\]
shows that the mapping $\rho \mapsto v(\rho)$ has a principal symbol given by

$$
(ie^{\theta_0}\chi_{1,\xi} \cdot \nabla u_0)(e^{\theta_0}|\xi|^2)^{-1} = \frac{-ie^{\theta_0}\chi_{1,\xi} \cdot \nabla u_0}{|\xi|^2}.
$$

From equation (14) we then find that $A_0$ is given by

$$
A_0(x, \xi) = \chi_1^{\xi} e^{\theta_0}|\nabla u_0|^p + p \chi_1^{\xi} e^{\theta_0} \frac{(i\xi \cdot \nabla u_0)^2}{|\nabla u_0|^2 + p|\xi|^2}.
$$

Let $\theta$ denote the angle between $\xi$ and $\nabla u_0$. Then, the principal symbol $A_0$ becomes

$$
A_0(x, \xi) = \chi_1^{\xi} e^{\theta_0}|\nabla u_0|^p (1 - p \cos^2 \theta)
$$

on $\Omega'$. Since we are interested in determining under what conditions $A$ is elliptic, we are therefore motivated to consider the case when $p < 1$ separately from $p \geq 1$.

### 3.1. The $p < 1$ case

#### Theorem 3.2. If $p < 1$, then

(i) $A(x, D)$, as defined above, is a pseudo-differential operator of order zero, which is elliptic in a neighborhood of $\overline{\Omega'}$.

(ii) $dF$, as an operator acting in $L^2(\Omega')$, is Fredholm.

(iii) Let $K$ be the kernel of $dF$ as an operator on $L^2(\Omega')$, and let $R$ be its range ($K$ is finite dimensional and $R$ is of a finite co-dimension). Then, $dF$, considered as an operator from $L^2(\Omega')/K$ onto $R$, is a topological isomorphism, i.e. there exists a constant $C$, such that for $\rho \in L^2(\Omega')$,

$$
\frac{1}{C} ||dF(\rho)||_{L^2(\Omega')} \leq ||\rho||_{L^2(\Omega')/K} \leq C ||dF(\rho)||_{L^2(\Omega')}.
$$

#### Remark 3.3.

(i) When we consider $dF$ as an operator from $L^2(\Omega')$ to itself, we are really considering the operator $T \circ dF$, where $T : L^2(\mathbb{R}^n) \to L^2(\Omega')$ is the restriction operator. The same goes for $A$.

(ii) Sometimes we need to consider $T$ as acting into $L^2(\mathbb{R}^n)$, in which case the action of $T$ is simply multiplication by the characteristic function of $\Omega'$. The operator $T$ also maps the corresponding Sobolev spaces:

$$
T : H^s(\mathbb{R}^n) \to H^s(\Omega').
$$

#### Proof. Since $A_0(x, \xi) = \chi_1^{\xi} e^{\theta_0}|\nabla u_0|^p (1 - p \cos^2 \theta)$ and $p < 1$, $A$ is elliptic on a neighborhood of $\overline{\Omega'}$. This proves the first claim of the theorem.

Let $\chi_2$ be a smooth cutoff function supported in $\Omega'$ that is equal to 1 on a neighborhood of $\Omega'$. We define a symbol $Q(x, \xi) \in S^0(\mathbb{R}^n)$ by setting

$$
Q(x, \xi) = \frac{\chi_2(x)}{A_0(x, \xi)}.
$$

Then, letting $I$ be the identity operator, $Q(x, D)A(x, D) - \chi_2(x)I$ is a pseudodifferential operator of order $-1$ on $\mathbb{R}^n$.

We claim that $TQ$ is a left inverse for $TA$ modulo compact operators on $L^2(\Omega')$. Indeed,

$$
TQTA - I = (TQA - I) + TQ(T - I)A.
$$

(22)
Since the pseudo-differential operator \( Q(x, D)A(x, D) - \chi_2(x)I \) is of order \(-1\) on \( \mathbb{R}^n \), the function \((TQA - I)f\) belongs to \( H^1(\Omega')\) for any \( f \in L^2(\Omega')\). In addition, the function \((T - I)f\) is equal to zero on \( \Omega' \), so by the microlocal property of pseudodifferential operators, \( TQ(T - I)f\) is a smooth function on \( \Omega' \). Thus, the right-hand side of equation (22) is a bounded operator from \( L^2(\Omega') \) into \( H^1(\Omega') \). The imbedding operator of \( H^1(\Omega') \) into \( L^2(\Omega') \) is compact, proving the claim.

We also observe that
\[
A(x, D)Q(x, D) = Q(x, D)A(x, D)
\] modulo operators of order \(-1\), so \( TQA - I \) maps \( L^2(\Omega') \rightarrow H^1(\Omega') \). Therefore, \( TQA \) is also right-invertible modulo compact operators. This means that \( TQA \) is a Fredholm operator on \( L^2(\Omega') \). Then, \( T \circ dF \) is a Fredholm operator on \( L^2(\Omega') \) as well, because \( dF = A \) as operators on \( L^2(\Omega') \). This proves the second claim of the theorem.

The third claim of the theorem is an immediate consequence of the second one. \( \square \)

3.2. The \( p \geq 1 \) case

If \( p \geq 1 \), the formula (18) shows that there are directions \( \xi \) at each point \( x \) in which the principal symbol \( A_0(x, \xi) = \chi_2^2(\xi)|\nabla u_0|^p(1 - p \cos^2 \theta) \) vanishes. In order to make the problem elliptic, we need to assume availability of two measurements. Namely, for two different boundary conditions \( f_1, f_2 \) in (2), let \( u^{(i)} \) for \( i = 1, 2 \) be the corresponding solutions:
\[
\begin{align*}
L_0 u^{(i)} &= 0 \\
u^{(i)}|_{\partial \Omega} &= f_i.
\end{align*}
\]
(24)
Assume that we are given knowledge, for all \( x \in \Omega \), of the following sets of internal data:
\[
\begin{align*}
F_{11} &= e^{\alpha_n} |\nabla u^{(1)}|^p \\
F_{22} &= e^{\alpha_n} |\nabla u^{(2)}|^p \\
F_{12} &= e^{\alpha_n} |\nabla u^{(1)} \cdot \nabla u^{(2)}|^p/2.
\end{align*}
\]
(25)
Such functionals have been extracted from the measured data in hybrid imaging methods; see, e.g., [8, 21, 33].

Our (vector) internal measurement function will be now
\[
F(\sigma) := \begin{pmatrix} F_{11}(\sigma) \\ F_{22}(\sigma) \\ F_{12}(\sigma) \end{pmatrix}
\]
(26)
We again consider small perturbations of a smooth background log conductivity \( \sigma_0 \) as in (12), and let \( u_0^{(i)} \) be the corresponding solutions of (24) with \( \sigma_0 \) as the log conductivity:
\[
\begin{align*}
L_0 u_0^{(i)} &= 0 \\
u_0^{(i)}|_{\partial \Omega} &= f_i.
\end{align*}
\]
(27)
We need to assume in addition that the gradients of \( u_0^{(1)} \) and \( u_0^{(2)} \) are nowhere parallel:
\[
u_0^{(1)} \parallel u_0^{(2)}.
\]
(28)
This is known to be possible \([2]\) in 2D under an appropriate choice of \( f_1 \) and \( f_2 \). However, as shown in \([35]\), in 3D, it is not always possible to choose boundary conditions such that (28) is satisfied. Condition (28) means in particular that the gradients are non-vanishing.

**Lemma 3.4.**

(i) As in the previous subsection, functionals \( F_{11} \) and \( F_{22} \) are Fréchet differentiable with respect to \( \sigma \) as mappings from \( L^\infty_{\text{ad}} \) to \( L^1 \).
(ii) When $p < 2$, the mapping $F_{12}$ is Fréchet differentiable if $u_0^{(1)}$ and $u_0^{(2)}$ satisfy
\[
\nabla u_0^{(1)} \cdot \nabla u_0^{(2)} \geq \alpha
\]
for some $\alpha > 0$. If this condition fails at some points, it is still true that the functional $\phi F_{12}$ is Fréchet differentiable if $\phi$ is a smooth cutoff function and (29) is satisfied on its support.

(iii) When $p = 2$, mapping $F_{12}$ is Fréchet differentiable without condition (29).

**Remark 3.5.**

- We will handle the case $1 < p < 2$ and note that the case $p = 2$ follows by the same argument, with the part concerning the smoothness of $F_{12}$ omitted.
- For the case $p = 1$, the functional $F_{12}$ is simply not needed, though the following arguments still apply. Thus, the full range $1 \leq p \leq 2$ will be covered.

The proof of this lemma can be found in section 5.

Given a vector $\xi$, let $\theta_1$ be the angle between $\xi$ and $\nabla u_0^{(1)}$, $\theta_2$ be the angle between $\xi$ and $\nabla u_0^{(2)}$, and $\theta$ be the angle between $\nabla u_0^{(1)}$ and $\nabla u_0^{(2)}$. As in the case when $p < 1$, we define cutoff versions of $dF_{12}$ by setting $A_{12} = \chi_{\Omega} dF_{12} \chi_{\Omega}$. The principal symbols of $A_{11}$ and $A_{22}$ near $\partial\Omega$, calculated in the same manner as before, are
\[
A_{11}(x, \xi) = \chi_{\Omega}^2 e^{\theta_0} |\nabla u_0^{(1)}|^p (1 - p \cos^2 \theta_1),
\]
\[
A_{22}(x, \xi) = \chi_{\Omega}^2 e^{\theta_0} |\nabla u_0^{(2)}|^p (1 - p \cos^2 \theta_2).
\]

The principal symbol of $A_{12}$ at points where $\nabla u_0^{(1)}$ and $\nabla u_0^{(2)}$ are not perpendicular is also easily derived from the formula
\[
dF_{12} = e^{\theta_0} \rho |\nabla u_0^{(1)} \cdot \nabla u_0^{(2)}|^p/2 + e^{\theta_0} \rho/2 |\nabla u_0^{(1)} \cdot \nabla u_0^{(2)}|^{p/2 - 1}
\times (\nabla u_0^{(1)} \cdot \nabla u_0^{(2)}(\rho) + \nabla u_0^{(2)} \cdot \nabla u_0^{(1)}(\rho))
\]
with
\[
\begin{align*}
L_{\rho} v^{(i)} &= \nabla \cdot (\rho e^{\theta_0} \nabla u_0) \\
v^{(i)}|_{\partial\Omega} &= 0,
\end{align*}
\]
which readily follows from formal linearization. Hence,
\[
A_{12}(x, \xi) = \chi_{\Omega}^2 e^{\theta_0} |\nabla u_0^{(1)}|^{p/2} |\nabla u_0^{(2)}|^{p/2} |\cos^{p/2} \theta| (1 - p \cos \theta_1 \cos \theta_2 |\cos \theta|).
\]

If, near each point $(x, \xi) \in \partial\Omega \times \mathbb{R}^n \setminus 0$, at least one of these symbols is non-vanishing, then the (vector) operator $(A_{11} A_{22} A_{12})$ is over-determined elliptic.

We note that $A_{12}(x, \xi)$ vanishes when $\cos \theta_1 = \pm 1/\sqrt{n}$. Near points $(x, \xi)$, where both $A_{11}(x, \xi)$ and $A_{22}(x, \xi)$ vanish, the symbol $A_{12}(x, \xi)$ will have to save the situation. Near such a point, i.e., where $|\cos \theta_1| = |\cos \theta_2| = 1/\sqrt{n}$, a simple trigonometric estimate shows that the expression $|\cos \theta|$ is separated from zero, and thus the symbol $A_{12}(x, \xi)$ is smooth. The non-vanishing of $A_{12}(x, \xi)$ then boils down to $\cos \theta \neq 1$ at the point $x$, which is guaranteed by (28).

Let us define three symbols homogeneous of order 0 on $\Omega \times \mathbb{R}^n$, $\psi_{11}(x, \xi)$, $\psi_{22}(x, \xi)$, and $\psi_{12}(x, \xi)$, such that
\[
\Psi(x, \xi) = \psi_{11}(x, \xi) A_{11}(x, \xi) + \psi_{22}(x, \xi) A_{22}(x, \xi) + \psi_{12}(x, \xi) A_{12}(x, \xi)
\]
is a non-vanishing symbol of order 0 near $\partial\Omega \times \mathbb{R}^n \setminus 0$, and therefore bounded away from zero by the compactness of the cosphere bundle of $\partial\Omega$. The above arguments imply that $\psi_{12}$
can be taken to be zero near points where condition (29) is not satisfied. (For \( p = 1 \), we do not need the functional \( F_{12} \), so we set \( \psi_{12} \equiv 0 \) in this case.) The operators \( \psi_{ij}(x, D) \) with symbols \( \psi_{ij}(x, \xi) \) are bounded on \( L^2(\Omega) \). This means that the operator norm of \( \Psi(x, D) \) can be controlled by the sum of the operator norms of \( A_{ij}(x, D) \). We thus have the following theorem.

**Theorem 3.6.**

(i) For \( 1 \leq p \leq 2 \), the operator

\[
\frac{dF}{L^2(\Omega')} \Rightarrow [L^2(\Omega')]
\]

is semi-Fredholm with a possible finite-dimensional kernel \( K \).

(ii) Letting \( K \) be the finite-dimensional kernel of \( dF \), the estimate

\[
C^{-1} ||\rho||_{L^2(\Omega')} \leq ||dF(\rho)||_{L^2(\Omega')} \leq C ||\rho||_{L^2(\Omega')/K}
\]

holds for \( \rho \in L^2(\Omega') \) and some constant \( C \).

**Remark 3.7.** As we have already mentioned, such functionals arise naturally and have been studied previously. When \( p = 2 \), a similar local stability estimate was proved in [21, 33] in the space \( C^{1,\alpha}(\Omega') \), and in [8], a global estimate was established in \( W^{1,\infty}(\Omega) \). A stability estimate for a single functional of the form \( e^\sigma |\nabla u|^2 \) was also established in [6] on a part of \( \Omega \). In the case \( p = 1 \), inversion procedures and reconstructions for a single functional were obtained in [37–39].

**Proof.** As in the proof of theorem 3.2, let

\[
Q(x, \xi) = \frac{\chi_2(x)}{\Psi(x, \xi)}.
\]

Then, the operator \( Q(x, D) \) lies in \( OPS^0(\mathbb{R}^n) \). Arguments identical to the ones in the proof of theorem 3.2 show that \( T\Psi(\Psi_{11}, \Psi_{12}, \Psi_{12}) \) is left regularizer\(^6\) for the operator

\[
\begin{pmatrix}
F_{11} \\
F_{22} \\
F_{12}
\end{pmatrix} : L^2(\Omega') \mapsto (L^2(\Omega'))^3.
\]

Thus, the operator in (36) is semi-Fredholm with a finite-dimensional kernel (and infinite-dimensional co-kernel), which implies all the statements of the theorem.

\footnote{Operator \( B \) is a left regularizer to operator \( A \), if \( BA - I \) is a compact operator [55, 29].}

### 3.3. More general interior data

We next consider a single, rather general functional and formulate a sufficient condition for the corresponding linearized problem being elliptic (and thus Fredholm).

Let \( F(y, z, w) \) be a function of three variables satisfying the conditions of lemma 3.1, i.e. \( F \) is smooth when \( y, z \in \mathbb{R} \), and \( w > 0 \), and satisfies the bound

\[
|F(y, z, w)| \leq C(y)(z^2 + w^2),
\]

where \( C(y) \) depends continuously on \( y \).

The Fréchet derivative can be derived by a formal calculation as before and is given by

\[
dF(\rho) = \frac{\delta F}{\delta y} \rho + \frac{\delta F}{\delta z} v + \frac{\delta F}{\delta w} \nabla u_0 \cdot \nabla v
\]

\footnote{Operator \( B \) is a left regularizer to operator \( A \), if \( BA - I \) is a compact operator [55, 29].}
As calculated before, the principal symbol of the operator mapping \( \rho \) to \( v \) is \( i\xi \cdot \nabla u_0|\xi|^{-2} \), which is of order \(-1\). The middle term on the right-hand side in expression (37) is therefore of lower order than the other two terms on the right-hand side and does not influence the ellipticity of the overall principal symbol. Also, as before, we set \( A = \chi_1 dF \). From the other two terms in (37), we find that the principal symbol of \( A \) is

\[
A_0(x, \xi) = \frac{\partial F}{\partial y} - \frac{\partial F(\xi \cdot \nabla u_0)^2}{\partial w} \frac{|\nabla u_0||\xi|^2}{|\nabla u_0|} = \frac{\partial F}{\partial y} - \frac{\partial F}{\partial w} |\nabla u_0| \cos^2 \theta
\]  

(38)

near \( \Omega' \) (we omit the cutoff function \( \chi_1 \) because it is identically equal to 1 there). This leads to a sufficient condition for the ellipticity of \( A_0 \).

**Theorem 3.8.** If

\[
\left| \frac{\partial F(\sigma_0, u_0, |\nabla u_0|)}{\partial y} \right| > \left| \frac{\partial F(\sigma_0, u_0, |\nabla u_0|)}{\partial w} \right| \frac{|\nabla u_0|}{L^\infty(\Omega)} > \delta > 0 \quad (39)
\]

pointwise in a neighborhood of \( \Omega' \), then

(i) \( A(x, \xi) \) is elliptic of order 0 on a neighborhood of \( \Omega' \);

(ii) \( dF \) as a Fredholm operator in \( L^2(\Omega') \);

(iii) letting \( K \) be the finite-dimensional kernel of \( dF \), the estimate

\[
\frac{1}{C} \left| \frac{dF(\rho)}{L^2(\Omega')} \right| \leq \left| \frac{\rho}{L^2(\Omega')} \right| K \leq C \left| \frac{dF(\rho)}{L^2(\Omega')} \right|
\]  

(40)

holds for \( \rho \in L^2(\Omega') \).

**Remark 3.9.**

(i) The functional \( F(\sigma) = \sigma |\nabla u|^p \) satisfies assumption (39) for \( p < 1 \), so theorem 3.8 generalizes theorem 3.2.

(ii) If \( \sigma_0 \equiv c \) for some constant \( c \), we can have \( |\nabla u_0| \equiv 1 \) by selecting an appropriate boundary condition, e.g., \( f = x_1 \). Then, (39) just means \( \left| \frac{\partial F}{\partial y} \right| > \left| \frac{\partial F}{\partial w} \right| \) pointwise on \( \Omega \).

**Proof.** We may assume that \( \chi_1 \equiv 1 \) on the region where condition (39) is satisfied. If (39) holds, then there exists \( \delta > 0 \), such that

\[
\left| \frac{\partial F}{\partial y} - \frac{\partial F}{\partial w} \frac{|\nabla u_0|}{L^2(\Omega)} \right| > \delta > 0 \quad (41)
\]

on \( \Omega' \) by the continuity of the terms involved. This implies that \( A_0(x, \xi) > \delta \), and so \( A \) is elliptic on the same neighborhood of \( \Omega' \). By setting

\[
Q(x, \xi) = \frac{\chi_1(x)}{A_0(x, \xi)} \quad (42)
\]

and composing with the restriction operator \( T \), we obtain an operator \( T \circ Q(x, D) \), which is a left and right inverse for \( TA \) on \( L^2(\Omega') \) (and hence for \( T \circ dF \) as well) modulo compact operators. Estimate (40) then follows. \( \square \)
4. Stability in QPAT

The standard model for diffusive regime photon propagation in biological tissues is

\[
\begin{align*}
L_{\sigma,\gamma} u := - \nabla \cdot (e^\sigma \nabla u) + e^\gamma u &= 0 \\
|u|_{\Omega} &= f
\end{align*}
\]  
(43)

(see, e.g., [51]). The coefficients \(\sigma\) and \(\gamma\) are the log-diffusion and log-attenuation coefficients, respectively. We will assume in this section that \(\sigma \in L^\infty(\Omega)\) and \(\gamma \in H^1(\Omega) \cap L^\infty(\Omega)\).

The PAT procedure performed first provides one with the values inside \(\Omega\) of the function

\[F(x) = \Gamma(x) e^{\gamma(x)} u(x) \]

(44)

Here, \(\Gamma(x)\) is the so-called Gr"uneisen coefficient\(^7\) describing the transfer of electromagnetic into acoustic energy, which we assume here to be identically equal to 1.

This function is the initial data for the QPAT, which strives to reconstruct the coefficients \(\sigma\) and \(\gamma\) from the data \((44)\).

We will denote by \(F_j(x)\), \(j = 1, 2, \ldots, J\), the internal data \((44)\) that correspond to solutions of \((43)\) with different boundary conditions \(f_j\).

For a pair of such measurements \((F_1, F_2)\), according to lemma 2.1, the mapping \((\sigma, \gamma) \mapsto F := (F_1, F_2)\) is Fréchet differentiable at smooth background coefficients \((\sigma_0, \gamma_0)\).

The derivative can be computed formally as before:

\[
\begin{align*}
\sigma &= \sigma_0 + \epsilon \rho, \\
\gamma &= \gamma_0 + \epsilon \nu, \\
\tilde{u}^{(j)} &= u^{(j)}_0 + \epsilon v^{(j)} + o(\epsilon),
\end{align*}
\]  
(45)

where \(\nu, \rho \in L^\infty(\Omega)\). Substitution into \((43)\) shows that \(v^{(j)} \in H^1(\Omega)\) solves the boundary value problem

\[
\begin{align*}
-\nabla \cdot (e^{\sigma_0} \nabla v^{(j)}) + e^{\gamma_0} v^{(j)} &= \nabla \cdot (\rho e^{\sigma_0} \nabla u^{(j)}_0) - \nu e^{\gamma_0} u^{(j)}_0 \\
\vbar{v^{(j)}} &= 0.
\end{align*}
\]  
(46)

We thus find that the differential of the mapping \(F_j\) is

\[dF_j(\rho, \nu) = \nu u^{(j)}_0 - L^{-1}_{\sigma_0,\gamma_0} (\nu u^{(j)}_0) + L^{-1}_{\sigma_0,\gamma_0} (\nabla \cdot (\rho e^{\sigma_0} \nabla u^{(j)}_0)).\]  
(47)

A calculation similar to the one in the previous section shows that the operator \(L_{\sigma_0,\gamma_0}\) has a parametrix with the principal symbol

\[
\frac{1}{e^{\sigma_0} |\xi|^2 + e^{\gamma_0}},
\]  
(48)

which is equivalent to \((e^{\sigma_0} |\xi|^2)^{-1}\) modulo lower order terms. Hence, according to \((47)\), the matrix of the principal symbols of the operator \((\rho, \nu) \mapsto \chi_1 dF \chi_1\) is given by

\[
A(x, \xi) := \chi_1 \begin{bmatrix}
\frac{i \xi \cdot \nabla u^{(1)}_0}{|\xi|^2} & u^{(1)}_0 \\
\frac{i \xi \cdot \nabla u^{(2)}_0}{|\xi|^2} & u^{(2)}_0
\end{bmatrix} \chi_1,
\]  
(49)

modulo lower order terms.

\(^7\) The Gr"uneisen coefficient is in principle also not known, so one might want to include it as an unknown in the reconstruction procedure, e.g., [12]. We are not doing this here.
We consider here the DN parameters
\[ s = (s_1, s_2) = (-1, -1), \quad t = (t_1, t_2) = (0, 1) \]
and attempt to check the DN ellipticity of \( dF \).

The determinant \( \det A(x, \xi) \) is non-vanishing near \( \Omega' \), if
\[ \xi \cdot (\nabla u_0^{(1)} - \nabla u_0^{(2)}) \neq 0. \]
Thus, ellipticity fails at the vectors \( \xi \) orthogonal to the field \( (\nabla u_0^{(1)} - \nabla u_0^{(2)}) \).

The natural idea is to have more measurements that would provide a basis of vector fields and thus preserve ellipticity. This leads to the following result.

**Theorem 4.1.** Suppose that one has access to \( 2n \) measurements
\[ (F_{1,1}, F_{1,2}), \ldots, (F_{n,1}, F_{n,2}), \]
such that the vector fields
\[ V_k := (k, 1) \nabla u_0^{(1)} - (k, 2) \nabla u_0^{(2)} \]
for \( k = 1, \ldots, n \) form a basis at each point \( x \) in \( \Omega \). We define the operator \( F \) as follows:
\[ F := (F_{1,1}, F_{1,2}, \ldots, F_{n,1}, F_{n,2}). \]

Then, the operator
\[ dF : L^2(\Omega') \bigoplus H^1_0(\Omega') \Rightarrow \{ H^1(\Omega') \}^{2n} \]
is semi-Fredholm with a finite-dimensional kernel.

Letting \( K \) be the (finite-dimensional) kernel of \( dF \), the estimate
\[ \frac{1}{C} \left\| \begin{pmatrix} \rho \\ v \end{pmatrix} \right\|_{H/K} \leq \left\| dF \begin{pmatrix} \rho \\ v \end{pmatrix} \right\|_{H^1(\Omega')}^{2n} \leq C \left\| \begin{pmatrix} \rho \\ v \end{pmatrix} \right\|_{H/K} \]
holds for some constant \( C > 1 \).

Here, we used the shorthand notation
\[ H := L^2(\Omega') \bigoplus H^1_0(\Omega'). \]

**Remark 4.2.** The assumptions we made on the vector fields \( V_k \), as the reader could see, arose naturally. It is interesting to note that they are the same that were also arising in the study of QPAT in [14], in which the complex geometric optics solutions technique was used. The authors of [14] derive a global estimate that is somewhat similar to (51):
\[ ||\delta \sigma||_{C^l(\Omega)} + ||\delta \gamma||_{C^l(\Omega)} \leq C \sum_{k=1}^{2} \sum_{j=1}^{n} \left| F_{k,j}(\sigma_1, \gamma_1) - F_{k,j}(\sigma_2, \gamma_2) \right|_{C^{l+1}(\Omega)} \]
for \( l \geq 2 \), where \( \delta \sigma = \sigma_1 - \sigma_2, \delta \gamma = \gamma_1 - \gamma_2 \).

Under an additional convexity assumption on \( \partial \Omega \), the authors of [14] also derive such an estimate with only two vector fields, when \( l \geq 3 \).

The reader might also note that in (51), in comparison with (52), different orders of smoothness are used, which allows us to get the two-sided estimate.
Proof. The $2n \times 2$ matrix operator $\chi_2 dF \chi_2$ has the principal symbol

$$A(x, \xi) := \begin{pmatrix}
\frac{i\xi \cdot \nabla u_0^{(1,1)}}{|\xi|^2} & u_0^{(1,1)} \\
\frac{i\xi \cdot \nabla u_0^{(1,2)}}{|\xi|^2} & u_0^{(1,2)} \\
\cdots & \cdots \\
\frac{i\xi \cdot \nabla u_0^{(n,1)}}{|\xi|^2} & u_0^{(n,1)} \\
\frac{i\xi \cdot \nabla u_0^{(n,2)}}{|\xi|^2} & u_0^{(n,2)}
\end{pmatrix} \chi_2,$$

Here, the principal symbol is understood in the DN sense with parameters $s = (-1, -1, \ldots, -1, -1), t = (0, 1),$ and $\chi_2$, as before, is a smooth cutoff function that is equal to 1 in a neighborhood of $\tilde{\Omega}$ and vanishes outside $\Omega^\prime$.

Under the assumptions of the theorem, at every $(x, \xi) \in \tilde{\Omega} \times (\mathbb{R}^n \setminus 0)$, the symbol $A(x, \xi)$ is injective (since at least one of the square $2 \times 2$ blocks

$$A_k := \begin{pmatrix}
\frac{i\xi \cdot \nabla u_0^{(k,1)}}{|\xi|^2} & u_0^{(k,1)} \\
\frac{i\xi \cdot \nabla u_0^{(k,2)}}{|\xi|^2} & u_0^{(k,2)}
\end{pmatrix}$$

for $k = 1, \ldots, n$ is invertible). Thus, the operator is over-determined elliptic in the DN sense.

Thus, there exists a left parametrix with the principal $2 \times 2n$ symbol $B(x, \xi)$ with DN parameters $s = (0, -1), t = (1, 1, \ldots, 1, 1)$ (this is a well-known construction, which we indicate at the end of section 5). In other words, for the corresponding pseudo-differential operators $dF$ and $B$, one has $B \chi_2 dF \chi_2 = I + K$ in a neighborhood of $\tilde{\Omega}$, where $K$ is a smoothing operator.

Let us agree to extend both $\rho \in L^2(\Omega^\prime)$ and $v \in H^1_0(\Omega^\prime)$ as equal to zero outside $\tilde{\Omega}$, without changing notation for these extended functions. Then, according to the DN parameters, $\chi_2 dF \chi_2$ belongs to $(H^1_0(\Omega^\prime))^{2n}$, and the corresponding mapping from $L^2(\Omega^\prime) \bigoplus H^1_0(\Omega^\prime)$ to $(H^1_0(\Omega^\prime))^{2n}$ is continuous. This proves the right-hand side inequality in (51).

Similarly, $B$ acts continuously from $(H^1_{comp}(\mathbb{R}^n))^{2n}$ to $L^2_{comp}(\mathbb{R}^n) \bigoplus H^1_{comp}(\mathbb{R}^n)$.

Then, for the restriction of $(\rho, v)$ to $\Omega^\prime$, we can write

$$(\rho, v) = \chi_2 (I + K) (\rho, v) - \chi_2 K (\rho, v)$$

$$= \chi_2 B \chi_2 dF \chi_2 (\rho, v) - \chi_2 K (\rho, v),$$

or

$$(I + \chi_2 K) (\rho, v) = \chi_2 B dF (\rho, v).$$

This gives us the estimate from above of the following kind:

$$\| (I + \chi_2 K) (\rho, v) \|_{L^2(\Omega^\prime) \bigoplus H^1_0(\Omega^\prime)} \leq C \| dF (\rho, v) \|_{H^1(\Omega^\prime)^{2n}}.$$ 

Since the operator $\chi_2 K$ is compact in $H = L^2(\Omega^\prime) \bigoplus H^1_0(\Omega^\prime)$, the operator $I + \chi_2 K$ is Fredholm in $H$.

This implies the remaining statements of the theorem.
5. Proofs of some lemmas

After proving lemma 2.1, we then proceed to give a proof of lemma 3.1. Finally, we prove lemma 3.4, making use of lemma 3.1.

5.1. Proof of lemma 2.1

Proof. First of all, we reduce (4) to a problem with homogeneous boundary conditions. Let $E$ be the operator of harmonic extension from $\partial \Omega$ to $\Omega$. Then, replacing the solution $u$ with $v_{\sigma, y} := u - Ef$, we reduce (4) to

\[
\begin{aligned}
L_{\sigma, y} v_{\sigma, y} &= f_{\sigma, y}, \\
v_{\sigma, y}|_{\partial \Omega} &= 0,
\end{aligned}
\]

where

\[f_{\sigma, y} := \nabla \cdot (e^n \nabla E f) - e^n E f \in H^{-1}(\Omega).\]

Now the map $(\sigma, y) \mapsto (e^n, e^y)$ factors as the composition of the following maps:

$(\sigma, y) \mapsto (e^n, e^y), f_{\sigma, y} \mapsto L_{\sigma, y} \mapsto \frac{1}{\sigma} L_{\sigma, y}^{-1} f_{\sigma, y} + Ef.$

The first map in (55) is clearly Fréchet differentiable as a mapping from $L^\infty(\Omega) \times L^\infty(\Omega)$ to $L^\infty(\Omega) \times L^\infty(\Omega) \times H^{-1}(\Omega)$. Indeed, $f_{\sigma, y}$ depends linearly and continuously on $(\sigma, y)$.

The second map $(e^n, e^y) \mapsto L_{\sigma, y}$ is linear and continuous from $L^\infty(\Omega) \times L^\infty(\Omega)$ to $L(H_0^1(\Omega), H^{-1}(\Omega))$.

The operator $L_{\sigma, y, y_0} \in L(H_0^1(\Omega), H^{-1}(\Omega))$ is invertible (see the simplest case of this statement in [45, chapter 5, proposition 1.1] and general results in [16, 36]); $L_{\sigma, y}$ is thus invertible in a neighborhood of $(y_0, \sigma_0)$ in $L^\infty(\Omega) \times L^\infty(\Omega)$.

The mapping

\[L(H_0^1(\Omega), H^{-1}(\Omega)) \to L(H^{-1}(\Omega), H_0^1(\Omega))\]

of taking inverse operator is known (e.g., [55]) to be analytic on the domain of invertible operators. This implies differentiability of the last two mappings in (55) and thus proves the lemma.

5.2. Proof of lemma 3.1

Proof. The function $F(\sigma(x), u(x), |\nabla u(x)|)$ lies in $L^1(\Omega)$. Indeed, we have $|F(\sigma, u, |\nabla u|)| \leq C(||\sigma||_{L^\infty(\Omega)})(u^n + |\nabla u|^2)$, and $u$ and $|\nabla u|$ are both square-integrable functions.

As a result of lemma 2.1, the map

\[\sigma \mapsto (\sigma, u, \nabla u)\]

is Fréchet differentiable from $L^\infty(\Omega) \to L^\infty(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^n$. (The middle space could be taken to be $H^1(\Omega)$ as before, but we will not need this here.) We claim that the map

\[L^\infty(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^n \to L^1(\Omega)\]

$(f, g, h) \mapsto F(f, g, |h|)$

is Fréchet differentiable at $(f_0, g_0, h_0) \in L^\infty(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^n$ where $f_0$, $g_0$ and $h_0$ are bounded and smooth with $|h_0(x)| \geq m$ for $x \in \Omega$ and some positive constant $m$. The boundedness of $f_0$, $g_0$ and $h_0$ implies that for any multi-index $\alpha$, the function

\[D^\alpha F(y, z, w)|_{y=f_0(x), z=g_0(x), w=h_0(x)}\]

is a bounded function on $\Omega$. 

\[\square\]
Let \((f, g, h)\) be a triplet of functions in \(L^\infty_0(\Omega) \times L^2(\Omega) \times (L^2(\Omega))^n\). Consider the function 
\[ E = E(x) \] defined on \(\Omega\) by
\[
E = F(f_0 + f, g_0 + g, h_0 + h) - F(f_0, g_0, h_0) - \nabla F(f_0, g_0, h_0) \cdot \left( f, g, \frac{h_0 \cdot h}{|h_0|} \right)
\]
\[
= F(f, g, h) - F(f_0, g_0, h_0) - \frac{\partial F}{\partial y}(f_0, g_0, h_0) f
\]
\[
- \frac{\partial F}{\partial z}(f_0, g_0, h_0) g - \frac{\partial F}{\partial w}(f_0, g_0, h_0) \frac{h_0 \cdot h}{|h_0|}.
\]
The function \(E\) lies in \(L^1(\Omega)\), since each individual term does. We estimate the \(L^1\)-norm of \(E\) as follows: let
\[
U = \{ x \in \Omega \mid \max(|f(x)|, |g(x)|, |h(x)|) \geq m \}.
\]
On \(\Omega \setminus U\), we can apply Taylor’s theorem to \(F\) to find that
\[
|E(x)| \leq C(f(x)^2 + g(x)^2 + |h(x)|^2),
\]
where \(C\) depends on an upper bound for the second-order partial derivatives in \((58)\). As a result,
\[
\int_{\Omega \setminus U} |E(x)| \, dx \leq C \int_{\Omega \setminus U} (f(x)^2 + g(x)^2 + |h(x)|^2) \, dx
\]
\[
\leq C(\text{Vol}(\Omega)) ||f||_{L^\infty(\Omega)}^2 + ||g||_{L^2(\Omega)}^2 + ||h||_{L^2(\Omega)^n}^2.
\]
On \(U\), we have
\[
\int_U |E| \leq C \int_U (|f_0 + f| + |g_0 + g| + |h_0 + h|^2
\]
\[
+ |F(f_0, g_0, h_0)| + |\nabla F(f_0, g_0, h_0)|(|f| + |g| + |h|)).
\]
The constant in front depends only on \(F\) and \(||f_0||_{L^\infty(\Omega)}\), as we need to consider only \(||f||_{L^\infty(\Omega)} \leq m\), say, which makes the constant \(C(y)\) in \((11)\) bounded. Using the fact that \(f_0, g_0\) and \(h_0\) are bounded functions, we have
\[
\int_U |E| \leq C \int_U (1 + |f| + |g| + |g|^2 + |h| + |h|)^2
\]
\[
\leq C \left( 1 + ||f||_{L^\infty(\Omega)} + \text{Vol}(U) + \left( ||g||_{L^2(\Omega)} + ||h||_{L^2(\Omega)^n} \right) \text{Vol}(U)^{1/2}
\]
\[
+ ||g||_{L^2(\Omega)}^2 + ||h||_{L^2(\Omega)^n}^2 \right).
\]
Owing to the inequality
\[
\text{Vol}(U) \leq \frac{2}{m^2} \left( ||g||_{L^2(\Omega)}^2 + ||h||_{L^2(\Omega)^n}^2 \right)
\]
for \(||f||_{L^\infty(\Omega)} < m\), we have
\[
\int_U |E(x)| \, dx \leq C \left( ||f||_{L^\infty(\Omega)} + ||g||_{L^2(\Omega)}^2 + ||h||_{L^2(\Omega)^n}^2 \right).
\]
This proves the Fréchet differentiability of the map \((f, g, h) \mapsto F(f, g, h)\) at \((f_0, g_0, h_0)\). Hence, by the chain rule for Fréchet derivatives, \(F(\sigma, u, |\nabla u|)\) is Fréchet differentiable as a function of \(\sigma\) at \(\sigma_0\).
5.3. Proof of lemma 3.4

Proof. The Fréchet differentiability of $F_{11}$ and $F_{22}$ follows from lemma 3.1. Only Fréchet differentiability of $F_{12}$ needs to be proven.

As a result of lemma 2.1, the map

$$\sigma \mapsto \left( \frac{\nabla u^{(1)}}{\nabla u^{(2)}} \right)$$

(65)
is Fréchet differentiable from $L^\infty_{\text{ad}}(\Omega)$ to $L^2(\Omega)^{2n}$ at $\sigma_0$. We claim that the map

$$L^2(\Omega)^{2n} \rightarrow L^1(\Omega)$$

$$\left( v_1, v_2 \right) \mapsto \phi_1^2 - |v_1 \cdot v_2|^{p/2}$$

(66)
is Fréchet differentiable at a pair of smooth vector fields $v_1$ and $v_2$ that satisfy

$$|v_1|, |v_2| \leq M, |v_1 \cdot v_2| \geq \alpha > 0$$

(67)
on the support of $\phi$ for some $M > 1$. To show this, let $w_1, w_2 \in L^2(\Omega)^n$ and define a function $E \in L^1(\Omega)$ by

$$E = \phi_1^2 - |v_1 \cdot v_2|^{p/2}$$

(68)

In order to estimate the $L^1(\Omega)$-norm of $E$, define a set $U$ by

$$U = \left\{ x \in \Omega \mid \max_{i=1,2} |w_i(x)| \geq \frac{\alpha}{4M} \right\}.$$

(69)

On $\Omega \setminus U$, both $w_1$ and $w_2$ are bounded above by $\alpha/4M$, so we have

$$|(v_1 + w_1) \cdot (v_2 + w_2)| \geq \alpha - |v_1 \cdot w_2| - |v_2 \cdot w_1| - |w_1 \cdot w_2|$$

$$\geq \alpha - |v_1| \frac{\alpha}{4M} - |v_2| \frac{\alpha}{4M} - \left( \frac{\alpha}{4M} \right)^2$$

$$\geq \frac{\alpha}{4}.$$  

(70)

It therefore follows from (68) and Taylor’s formula applied to the function of $2n$ variables $|x \cdot y|^{p/2}$ (which is smooth and has bounded derivatives when the arguments satisfy $\alpha/4 \leq x, y \leq M$) that

$$\int_{\Omega \setminus U} |E(x)| \, dx \leq C \left( \|w_1\|^2_{L^2(\Omega)^n} + \|w_2\|^2_{L^2(\Omega)^n} \right),$$

(71)

where $C$ depends on $m, M, p, n$ and $\alpha$.

On $U$, we use the triangle inequality

$$|E| \leq |v_1 \cdot v_2|^{p/2} \left| 1 + \frac{v_1 \cdot w_2 + v_2 \cdot w_1 + w_1 \cdot w_2}{v_1 \cdot v_2} \right|^{p/2} - 1$$

$$+ \frac{p}{2} |v_1 \cdot v_2|^{p/2-1} |v_2 \cdot w_1 + v_1 \cdot w_2|.$$  

(72)

We make use of the inequality

$$||1 + z|^{p/2} - 1| \leq C_p |z|$$  

(73)
to estimate the term in the first line of (72). (For $|z| \leq 1/2$, this inequality follows from the fact that the function $z \mapsto |1 + z|^{p/2}$ has bounded derivative for $|z| \leq 1/2$, while for $|z| > 1/2$, it follows easily from the fact that $p/2 \leq 1$.) After using (73), we obtain

$$
\int_U |E(x)| \, dx \leq \int_U C_p |v_1 \cdot v_2|^{p/2-1} |v_1 \cdot v_2 + v_2 \cdot w_1 + w_1 \cdot w_2| + \frac{p}{2} |v_1 \cdot v_2|^{p/2-1} |v_2 \cdot w_1 + v_1 \cdot w_2|.
$$

(74)

By the Cauchy–Schwarz inequality and the bounds on $v_1$, $v_2$ and $v_1 \cdot v_2$,

$$
\int_U |E(x)| \, dx \leq C \int_U |w_1| + |w_2| + |w_1 \cdot w_2|
\leq C \left( ||w_1||_{L^2(U)}^2 \text{Vol}(U)^{1/2} + ||w_2||_{L^2(U)}^2 \text{Vol}(U)^{1/2} + ||w_1||_{L^2(U)^p} ||w_2||_{L^2(U)^p} \right).
$$

The volume of $U$ can be estimated by

$$
\text{Vol}(U) \leq \text{Vol} \left( \left\{ |w_1| \geq \frac{\alpha}{4M} \right\} \right) + \text{Vol} \left( \left\{ |w_2| \geq \frac{\alpha}{4M} \right\} \right)
\leq 2 \left( \frac{4M}{\alpha} \right)^2 \left( ||w_1||_{L^2(\Omega)^p}^2 + ||w_2||_{L^2(\Omega)^p}^2 \right).
$$

(75)

Hence,

$$
\int_U |E(x)| \, dx \leq C \left( ||w_1||_{L^2(\Omega)^p}^2 + ||w_2||_{L^2(\Omega)^p}^2 \right).
$$

(77)

This proves that the map (66) is Fréchet differentiable. Since condition (67) holds for the vector fields $\nabla u_0^{(1)}$ and $\nabla u_0^{(2)}$ on the support of $\phi$, the Fréchet differentiability of $F_{12}$ at $s_0$ follows from the chain rule.

5.4. Left parametrix for over-determined DN elliptic operator

We provide here a sketch of the classical construction of the left parametrix $B$ used in the proof of theorem 4.1.

According to the assumptions of theorem 4.1, there exists a finite open covering $\{ U_i \}$ of the compact subset $\overline{\Omega'} \times S^{n-1} \subset \Omega' \times (\mathbb{R}^n \setminus \{0\})$, such that for each $i$, there exists a number $k_j$, such that $A_j(x, \xi)$ is invertible for $(x, \xi) \in U_j$. Here, $S^{n-1}$ is the unit sphere in the $\xi$-space $\mathbb{R}^n$.

Consider a smooth partition of unity $\psi_j(x, \xi)$ on $\overline{\Omega'} \times S^{n-1}$ subordinated to the covering. We can always assume that it is positively homogeneous of order zero with respect to $\xi$ outside a neighborhood of the origin and smooth on $\Omega' \times \mathbb{R}^n$. Let us also denote by $P_j: \mathbb{C}^{2n} \rightarrow \mathbb{C}^2$ the operator, such that $P_j(a_1, a_2, \ldots, a_{2n-1}, a_{2n}) = (a_{2j-1}, a_{2j})$. One sees that $A_j = P_j A$. Then, one can check that the symbol $B(x, \xi) = \sum_j \psi_j(x, \xi) A_j(x, \xi)^{-1} P_j$ is the one we require.

6. Final remarks

(i) As we have mentioned, some of the models discussed in this paper have been studied previously. Reconstructions from the functionals (25) with $p = 2$ have been performed, e.g., in [8, 21, 33], with similar stability estimates being obtained in [8, 33], the approach in [8] being global. Global reconstruction from a single functional was considered in [6] by solving a Cauchy problem inward from parts of $\partial \Omega$ for a nonlinear hyperbolic equation. In this case, stability results were obtained in two dimensions, and in parts of $\Omega$ in higher dimensions.
(ii) The $p = 1$ case has been studied in [37–39]. In [38], an iterative reconstruction procedure was provided, whose effectiveness was demonstrated in numerical experiments. Our analysis in terms of pseudo-differential operators implies that although the inversion of the corresponding linearized operator from a single functional is not an elliptic problem, ellipticity only fails at points $(x, \xi)$, where $\nabla u_0 \parallel \xi$. Since most singularities of $\rho$ would likely not satisfy that condition, we should expect accurate reconstruction of sharp features in almost all cases, as the numerical experiments demonstrate.

(iii) While we have shown the infinitesimal Fredholm property of the problems with internal data, infinitesimal uniqueness has not been shown and we suspect that it does not hold under our very general conditions. However, uniqueness should hold generically, which we plan to address elsewhere.

(iv) Besides the absence of an infinitesimal uniqueness result, there is another obstacle for obtaining the local uniqueness and stability result for the nonlinear problem. Namely, Fréchet differentiability is proven in worse function spaces than the Fredholm property. We also plan to address this discrepancy in a future work.

(v) We suggest that parametrices constructed in this paper could be used for approximate reconstructions and for pre-conditioning iterative methods.

(vi) In the cases when our ellipticity analysis asks for multiple measurements, this does not mean that reconstruction with a smaller number of measurements is impossible. In contrast, such reconstructions have been achieved by solving hyperbolic and degenerate elliptic problems in [7, 32, 33]. However, such approaches naturally lead to correspondingly error propagation and some blurring effects for the parts of the wavefront sets where ellipticity is lost [33, 30].

(vii) The types of the internal data functionals $F$ considered in this paper do not cover all the needs of hybrid methods. For instance, some non-local functionals of $\sigma$ arise in UOT [3]. We plan to address these in a subsequent work.

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