Solving the EPR paradox with pseudo-classical paths

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We discuss a novel interpretation of Quantum Mechanics, which can resolve the outstanding conflict between the principles of locality and realism and offers new insight on the so-called weak values of physical observables. The discussion is presented in the context of Bohm’s system of two photons in their singlet polarization state in which the Einstein-Podolski-Rosen paradox is commonly addressed. It is shown that quantum states can be understood as statistical mixtures of non-interfering pseudo-classical paths, in a way that overcomes the implicit assumptions of Bell’s theorem and reproduces all expected values and correlations. Every path corresponds to a post-selection of the quantum system in one of the eigenstates of a complete set of commuting observables, so that any other physical observable takes along that path its corresponding weak value. This interpretation follows from the observation that in the Heisenberg picture of a closed quantum system in state $|\Psi>\rangle$ every physical observable $O(t) = e^{+iHt}Oe^{-iHt}$ can be represented by an operator $P_{a(t)}$ within a commutative algebra, such that $O(t)|\Psi>\rangle = P_{a(t)}|\Psi>\rangle$.

1. Quantum Mechanics is widely believed to be the ultimate theoretical framework within which all fundamental laws of Nature are to be formulated. Its postulates have been extensively and accurately tested in a very broad class of physical systems, including optics, atomic and molecular physics, condensed matter physics and high-energy particle physics. Indeed, one of the greatest challenges in physics at present is to formulate Einstein’s general relativity theory of gravitation within this framework.

Nevertheless, there remain crucial questions about the interpretation of Quantum Mechanics that have not been properly understood yet. In particular, it has been known since long ago that the current interpretation of the quantum formalism cannot accommodate together two fundamental principles of modern physics usually taken for granted, namely, the
principle of locality and the principle of physical realism. The principle of locality states that physical events cannot affect or be affected by other events in space-like separated regions. Besides the principle of realism claims that all measurable physical observables correspond to intrinsic properties of the physical world, which a final theory should be able to completely account for. The clash between these two principles was first noticed by Einstein, Podolsky and Rosen [1] and it has been known since then as the EPR paradox. The paradox is commonly formulated as follows [2]:

Consider a massive particle that decays into two distinguishable photons, which travel in opposite directions along the Z axis. The two photons are assumed to be emitted in their singlet polarization state:

$$|\Psi> = \frac{1}{\sqrt{2}} (|\uparrow \downarrow> - |\downarrow \uparrow>),$$

where \{|\uparrow>, |\downarrow>\} is an orthonormal linear basis in the single particle polarization Hilbert space. If the photons travel freely, their polarization state remains entangled once they have travelled far away from each other.

In this state the polarizations of the photons are perfectly anti-correlated when they are measured along parallel directions: if we would perform a measurement of the polarization of photon A, we would know with certainty also the polarization of photon B along that direction. Hence, accepting the principle of locality implies that we can gain certainty on the polarization of photon B along any direction without perturbing it in any sense. Furthermore, accepting the principle of physical realism implies then that the polarization properties of photon B were set at emission. Obviously, the same could be said about the polarization of photon A. Nevertheless, according to the current interpretation of Quantum Mechanics nothing can be said with certainty about the polarization of each of the photons from their wavefunction at emission (1). This observation lead Einstein, Podolsky and Rosen [1] to claim that the description of the physical system provided by the wavefunction is not complete.

The description of the quantum system would be complete if we could interpret the wavefunction as a statistical mixture of classical paths defined in some more fundamental hidden variables phase space, so that a measurement of the polarization of any of the two photons would imply nothing but a mere update of our knowledge about the state of the system. Unfortunately, Bell’s theorem [3] rules out the possibility to build such a statistical classical
interpretation of the wavefunction based on the currently accepted notions of physical realism and locality. Indeed, the theorem proves that in any model of classical hidden variables based on these premises there exist certain constraints on the statistical correlations between physical observables (Bell’s inequalities), which are nonetheless not necessarily fulfilled by their quantum mechanical counterparts. Additional inequalities of this kind (CHSH inequalities) discovered later on by Clauser, Horne, Shimony and Holt and Clauser and Horne actually allowed to experimentally verify the predictions of quantum mechanics against these statistical models of classical local hidden variables.

Formally, Bell’s theorem can be stated as follows. Let us assume that the two photons system could be described as a statistical mixture of classical paths with well defined probabilities \( \rho(\lambda) \), where the label \( \lambda \in \mathcal{S} \) sets the final conditions of the paths in the hidden variables phase space \( \mathcal{S} \). Along each path every physical observable \( O \) has a well defined value \( o(\lambda) \), which is assumed to belong to its spectrum of eigenvalues. In particular, the spin polarization of photon A along any direction \( \vec{a} \) in the unit sphere - denoted as \( \sigma^{(A)}(\vec{a}, \lambda) \) - is assumed to get values either +1 or −1 on each of the paths. Similarly, the spin polarization of photon B along any other direction \( \vec{b} \) - denoted as \( \sigma^{(B)}(\vec{b}, \lambda) \). The stated constraint that requires the photons polarizations to be perfectly anti-correlated when measured along the same direction demands \( \sigma^{(B)}(\vec{a}, \lambda) = -\sigma^{(A)}(\vec{a}, \lambda) \). Therefore, the expected correlation between the spin polarizations of the two photons when measured along any two arbitrary directions is given by:

\[
E(\vec{a}, \vec{b}) \equiv \int d\lambda \, \rho(\lambda) \, \sigma^{(A)}(\vec{a}, \lambda) \, \sigma^{(B)}(\vec{b}, \lambda) = - \int d\lambda \, \rho(\lambda) \, \sigma^{(A)}(\vec{a}, \lambda) \, \sigma^{(A)}(\vec{b}, \lambda). \tag{2}
\]

Bell’s theorem states that for any triplet of unit vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) the following inequality holds:

\[
|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c})| \leq 1 + E(\vec{b}, \vec{c}). \tag{3}
\]

The importance of Bell’s statement lies on the fact that the correlations between the spin polarizations of the two photons predicted by Quantum Mechanics \( E(\vec{a}, \vec{b}) = \langle \Psi | \sigma^{(A)}_{\vec{a}} \cdot \sigma^{(B)}_{\vec{b}} | \Psi \rangle = -\vec{a} \cdot \vec{b} \) are not constrained by this inequality. For example, for \( \vec{b} = \vec{c}_x = (1, 0, 0) \), \( \vec{c} = \vec{c}_y = (0, 1, 0) \) and \( \vec{a} = \frac{1}{\sqrt{2}}(1, -1, 0) \), we find that, \( |E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c})| = |\vec{a} \cdot (\vec{c} - \vec{b})| = \sqrt{2} \).
while $1 + E(\vec{b}, \vec{c}) = 1$. Hence, the statistical models of classical hidden variables considered by Bell cannot completely reproduce the predictions of Quantum Mechanics.

In the past, most of the efforts to solve the issue of the completeness of the quantum mechanical description of the physical world focused on exploring the consequences of giving up in some way the principle of locality. Nevertheless, it has been shown that neither a general class of appealing non-local theories of classical hidden variables \cite{6} can reproduce all quantum correlations \cite{7}. The proof of this statement is based on a generalized version of the Clauser-Horne-Shimony-Holt inequality \cite{4} that holds within all these non-local hidden variables models, but it is not necessarily fulfilled by quantum mechanics. These results have led some leading physicists \cite{7} to suggest that it might be necessary to abandon certain intuitive aspects of the currently accepted notion of physical realism.

In this paper we demonstrate that the difficulties to integrate together within the framework of Quantum Mechanics both the principle of locality and the principle of physical realism are a consequence of an untested implicit assumption about the latter. These difficulties are removed once this assumption is lifted. Namely, most of the attempts made to date to interpret the quantum wavefunction as a statistical mixture of non-interfering random paths have implicitly assumed that the values of all physical observables along each of these paths must be equal to one of the eigenvalues of the given observable. Indeed, the proof of Bell’s inequality \cite{3} crucially relies on this assumption, as $\sigma^{(A)}(\vec{b}, \lambda) \cdot \sigma^{(A)}(\vec{c}, \lambda) = 1$ and $|\sigma^{(A)}(\vec{b}, \lambda) \cdot \sigma^{(A)}(\vec{c}, \lambda)| = 1$. Hence,

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c})| = \left| \int d\lambda \rho(\lambda) \left[ \sigma^{(A)}(\vec{a}, \lambda) \sigma^{(A)}(\vec{b}, \lambda) - \sigma^{(A)}(\vec{a}, \lambda) \sigma^{(A)}(\vec{c}, \lambda) \right] \right| = \left| \int d\lambda \rho(\lambda) \left[ \sigma^{(A)}(\vec{a}, \lambda) \sigma^{(A)}(\vec{b}, \lambda) - \sigma^{(A)}(\vec{a}, \lambda) \sigma^{(A)}(\vec{b}, \lambda) \sigma^{(A)}(\vec{b}, \lambda) \sigma^{(A)}(\vec{c}, \lambda) \right] \right| = \left| \int d\lambda \rho(\lambda) \left[ \sigma^{(A)}(\vec{a}, \lambda) \sigma^{(A)}(\vec{b}, \lambda) \sigma^{(A)}(\vec{b}, \lambda) \sigma^{(A)}(\vec{c}, \lambda) \right] \right| \leq \int d\lambda \rho(\lambda) \left( 1 - \sigma^{(A)}(\vec{b}, \lambda) \sigma^{(A)}(\vec{c}, \lambda) \right) = 1 + E(\vec{b}, \vec{c}).$$

This implicit assumption may not be justified. It stems from the Von Neumann paradigm of strong (projective) measurements, whose only possible outcomes are any of the eigenvalues of the measured observable. Notwithstanding, Bell’s theorem requires to estimate simultaneously the polarization of photon A along three distinct directions, $\vec{a}$, $\vec{b}$ and $\vec{c}$, while we can
actually perform simultaneous strong measurements of its polarization along at most two directions, say $\vec{a}$ and $\vec{b}$: one component is obtained by directly measuring on this photon and the second component is obtained by measuring on the second photon and exploiting the perfect anti-correlation between the two. The only experimental access that we can have to the photon polarization along a third direction $\vec{c}$ is through weak measurements, whose output can have absolute values larger and smaller than one and may even be complex [8]. Such weak values of the polarization of single photons have been experimentally measured and confronted with theoretical predictions in different setups [9, 10].

By giving up this implicit assumption we generalize the notion of physical realism. It is then fairly easy to give to the quantum wavefunction (1) a statistical interpretation that reproduces the average values and correlations of all physical observables. We proceed as follows: Let denote by $|\zeta_{\pm,\pm}\rangle$ the quantum eigenstates for two simultaneously strongly measured observables $\sigma_{a}^{(A)}$, $\sigma_{b}^{(B)}$, where the labels $\pm, \pm$ indicate their corresponding eigenvalues. The singlet Bell state (1) can be described as a statistical mixture of four pseudo-classical paths, which we will label as $\lambda = ++, +-, -+, --$, such that:

- Each one of the four paths happens with probability

$$\rho(\pm, \pm) = |< \zeta_{\pm,\pm} | \Psi >= |^2, \hspace{1cm} (4)$$

so that, $\sum_{\lambda=\pm,\pm} \rho(\lambda) = 1$.

- Along each of these paths every physical observable $O(t)$ in the Heisenberg picture is given its corresponding weak value [11–15]:

$$O_w(t)(\pm, \pm) = \frac{< \zeta_{\pm,\pm} | O(t) | \Psi >=}{< \zeta_{\pm,\pm} | \Psi >}. \hspace{1cm} (5)$$

It is straightforward to show that this statistical model reproduces the quantum average value of any physical observable:

$$\sum_{\pm,\pm} \rho(\pm, \pm) \cdot O_w(t)(\pm, \pm) = \sum_{\pm,\pm} |< \zeta_{\pm,\pm} | \Psi >= |^2 \frac{< \zeta_{\pm,\pm} | O(t) | \Psi >=}{< \zeta_{\pm,\pm} | \Psi >} = \sum_{\pm,\pm} < \Psi | \zeta_{\pm,\pm} > < \zeta_{\pm,\pm} | O(t) | \Psi >= < \Psi | O(t) | \Psi >.$$
and also the quantum correlation between any two physical observables \( O_1(t_1) \) and \( O_2(t_2) \):

\[
\sum_{\pm,\pm} \rho(\pm, \pm) (O_1) w(t_1)^\ast(\pm, \pm) (O_2) w(t_2)(\pm, \pm) = \\
\sum_{\pm,\pm} |< \zeta_{\pm,\pm} | \Psi > |^2 \left( \frac{< \zeta_{\pm,\pm} | O_1(t_1) | \Psi >}{< \zeta_{\pm,\pm} | \Psi >} \right)^\ast \left( \frac{< \zeta_{\pm,\pm} | O_2(t_2) | \Psi >}{< \zeta_{\pm,\pm} | \Psi >} \right) = \\
= \sum_{\pm,\pm} < \Psi | O_1(t_1) | \zeta_{\pm,\pm} > < \zeta_{\pm,\pm} | O_2(t_2) | \Psi > = < \Psi | O_1(t_1) \cdot O_2(t_2) | \Psi > .
\]

These pseudo-classical paths might still be coarse descriptions of an even finer hidden reality, as we will argue later on in section 3. Nonetheless, our pseudo-classical paths are essentially different from the coarse paths envisioned in the consistent histories interpretation of Quantum Mechanics \([16–22]\) in the sense that along each pseudo-classical path every quantum observable \( O(t) \) has a well defined value \((5)\).

This statistical interpretation of the quantum wavefunction \( |\Psi > \) can be more easily understood in the Heisenberg picture, in which the dynamics of the system is described through the time dependent operators that describe physical observables. We shall show in section 2 that every such observable \( O(t) = e^{+iHt} O e^{-iHt} \) can be represented by an operator \( P_{O(t)} \) within a commutative algebra, such that \([23, 24]\)

\[
O(t) |\Psi > = P_{O(t)} |\Psi > . \tag{6}
\]

Thus, our statistical interpretation is explicitly local, in the sense that the values on these paths of observable properties of one of the photons, say photon \( A \), do not change when photon \( B \) interacts with the external world. This can be easily proven by noticing that the hermitic operators \( O_A(t) \) that describe the physical properties of photon \( A \) do commute with the hamiltonian \( H_B \) that describe the interaction of photon \( B \) and, therefore, \( O_A(t) = e^{+iH_B^t} O_A e^{-iH_B^t} = O_A \).

Hence, once we choose an orthonormal basis of eigenstates of this algebra we can associate to every observable well defined values. In fact, the eigenvalues of the operator \( P_{O(t)} \) are equal to the weak values \([5]\) of its corresponding observable \( O(t) \) when the system is post-selected to each of these eigenstates.

The commutative algebra that represents the quantum system in this picture is not unique. Indeed, it is closely related to the complete set of commuting observables chosen to \textit{strongly} measure the quantum system. Different choices of this set lead to equivalent
commutative algebras to represent the system according to (6). It is important to stress, nonetheless, that these algebras, in spite of being commutative, are not classical agebras, because for any pair of observables, $\mathcal{O}_1(t)$ and $\mathcal{O}_2(t)$, the commutative operator representing their product, either $P_{\{\mathcal{O}_1(t),\mathcal{O}_2(t)\}}/2$ or $P_{i[\mathcal{O}_1(t),\mathcal{O}_2(t)]}/2$, is not necessarily equal to the product of the operators representing each one of them, $\{P_{\mathcal{O}_1(t)},P_{\mathcal{O}_2(t)}\}/2$ or $i[P_{\mathcal{O}_1(t)},P_{\mathcal{O}_2(t)}]/2$. In section 6 we shall discuss the conditions under which these equalities are recovered. These conditions can thus be understood as the onset of classicality, without any reference to external observers or environment.

The paper is organized as follows. In sections 2,3 and 4 we build the formalism of pseudo-classical paths for Bohm’s system of two entangled photons in their singlet polarization state. In section 5 we discuss within this formalism the EPR paradox and the problem of the collapse of the wavefunction due to measurement and in section 6 we briefly explore the onset of classicality. In section 7 we summarize our conclusions.

2. We consider a system of two distinguishable photons travelling in opposite directions and whose polarizations are described by the singlet Bell state $\Pi$. The axis along which the photons travel is labelled without any loss of generality as $Z$ axis.

We start our programme by choosing a complete set of commuting observables on the Hilbert space $\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$ of two photons polarization states. Such observables can be simultaneously measured through strong (projective) measurements and their common eigenstates set up an orthonormal basis in the Hilbert space. Let us pick, for example, the polarization components of each of the two photons along two arbitrary directions in the plane $XY$ orthogonal to the direction they are moving along:

$$\{\sigma_1^{(A)}, \sigma_\phi^{(B)} \equiv \cos(\phi) \sigma_1^{(B)} + \sin(\phi) \sigma_2^{(B)}\},$$

(7)

with $\phi \in (0, \pi) \cup (\pi, 2\pi)$ and where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices acting on single particle polarization states defined in the orthogonal basis $\{|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ and the upper index $(A, B)$ indicates on which of the two photons the observable is defined. Without any loss of generality we have labelled the direction along which the polarization of photon A is strongly defined as $X$ axis.
We now obtain the orthonormal basis of common eigenstates to the chosen complete set of commuting operators (7):

\[ |+;+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)_A \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle + e^{i\phi} |\downarrow\rangle)_B \]  
\[ |+;->\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)_A \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle - e^{i\phi} |\downarrow\rangle)_B \]  
\[ |-;+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle)_A \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle + e^{i\phi} |\downarrow\rangle)_B \]  
\[ |-;-\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle)_A \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle - e^{i\phi} |\downarrow\rangle)_B \]

Each one of the two observables in the complete set that we have chosen has eigenvalues ±1. The notation chosen for their common eigenvectors indicates their eigenvalues for each one of the two observables: the first sign refers to operator \( \sigma_1^{(A)} \) and the second to operator \( \sigma_\phi^{(B)} \).

The singlet Bell state (1) is now written in this basis:

\[ |\Psi\rangle = q_{++}|+;+\rangle + q_{+-}|+;-\rangle + q_{-+}|-;+\rangle + q_{--}|-;-\rangle, \]  

where

\[ q_{++} = <+;+|\Psi\rangle = -\frac{1}{2\sqrt{2}} (1 - e^{-i\phi}), \quad q_{+-} = <+;-|\Psi\rangle = -\frac{1}{2\sqrt{2}} (1 + e^{-i\phi}), \]
\[ q_{-+} = <-;+|\Psi\rangle = +\frac{1}{2\sqrt{2}} (1 + e^{-i\phi}), \quad q_{--} = <-;-|\Psi\rangle = +\frac{1}{2\sqrt{2}} (1 - e^{-i\phi}), \]

such that,

\[ |q_{++}|^2 = |q_{--}|^2 = \frac{1}{4} (1 - \cos (\phi)), \quad |q_{+-}|^2 = |q_{-+}|^2 = \frac{1}{4} (1 + \cos (\phi)). \]  

The next and crucial step is to show how the wavefunction (12) can be understood as a statistical mixture of four non-interfering paths, denoted as ++, +−, −+, −−, each one occurring with well defined probability \(|q_{\pm\pm}|^2\). In order to do it we will lay down a well defined set of rules that will allow us to assign to each physical observable a time-dependent c-value on each one of these paths. We will refer to it as the pseudo-classical value of the physical observable along the path.
To physical observables that commute with $\sigma^{(A)}_1$, $\sigma^{(B)}_\phi$ we obviously assign their eigenvalues for each one of the four eigenvectors. For example,

\begin{align}
\sigma^{(A)}_1 (\pm, \pm) &= +1, & \sigma^{(A)}_1 (-, \pm) &= -1, \\
\sigma^{(B)}_\phi (\pm, +) &= +1, & \sigma^{(B)}_\phi (-, -) &= -1,
\end{align}

and also,

\begin{align}
(1)_{cl} (\pm, \pm) &= +1, \\
\sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi (\pm, \pm) &= \sigma^{(A)}_1 (\pm, \pm) \cdot \sigma^{(B)}_\phi (\pm, \pm).
\end{align}

We now need to define the way to assign pseudo-classical values on paths to observables that do not commute with $\sigma^{(A)}_1$, $\sigma^{(B)}_\phi$. To do it we notice that

\begin{align}
1|\Psi > &= \frac{1}{\sqrt{2}} (|\uparrow \downarrow> - |\downarrow \uparrow>), \\
\sigma^{(A)}_1 |\Psi > &= \frac{1}{\sqrt{2}} (|\downarrow \uparrow> - |\uparrow \downarrow>), \\
\sigma^{(B)}_\phi |\Psi > &= \frac{1}{\sqrt{2}} (e^{-i\phi} |\uparrow \uparrow> - e^{+i\phi} |\downarrow \downarrow>), \\
\sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi |\Psi > &= \frac{1}{\sqrt{2}} (e^{-i\phi} |\downarrow \uparrow> - e^{+i\phi} |\uparrow \downarrow>).
\end{align}

are four linearly independent vectors, whenever $\phi \in (0, \pi) \cup (\pi, 2\pi)$. Therefore, for any given observable $O$ there exists one and only one linear combination $\mathcal{P}_o\left(1, \sigma^{(A)}_1, \sigma^{(B)}_\phi, \sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi\right)$ of the four commuting operators $1$, $\sigma^{(A)}_1$, $\sigma^{(B)}_\phi$ and $\sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi$, such that,

\begin{equation}
O|\Psi > = \mathcal{P}_o\left(1, \sigma^{(A)}_1, \sigma^{(B)}_\phi, \sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi\right)|\Psi >.
\end{equation}

Hence, we assign to each observable $O$ on each path the corresponding eigenvalue of the operator $\mathcal{P}_o\left(1, \sigma^{(A)}_1, \sigma^{(B)}_\phi, \sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi\right)$. That is,

\begin{equation}
(O)_{cl} (\pm, \pm) = \mathcal{P}_o\left(+1, \sigma^{(A)}_1 (\pm, \pm), \sigma^{(B)}_\phi (\pm, \pm), \sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi (\pm, \pm)\right).
\end{equation}

Since the linear operators $\mathcal{P}_o\left(1, \sigma^{(A)}_1, \sigma^{(B)}_\phi, \sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi\right)$ are not necessarily hermitic, their eigenvalues are, in general, complex numbers and, hence, also the $c$-values associated to these physical observables along the paths.
The c-values assigned to physical observables according to these rules are actually their weak values for the corresponding post-selected states \([11–14]\),

\[
(O)_{cl}(\pm, \pm) \equiv \frac{<\pm, \pm|P_o|\Psi>}{<\pm, \pm|\Psi>} = \frac{<\pm, \pm|O|\Psi>}{<\pm, \pm|\Psi>},
\]

(20)

Weak values of physical observables were first introduced as average values of post-selected weak measurements. Henceforth, weak measurements could be interpreted as protocols to experimentally access these pseudo-classical non-interfering paths.

In the particular case that we are considering here, in which both photons travel freely after being emitted, the hermitic operators describing physical observables in the Heisenberg picture do not evolve in time and, therefore, neither their c-values on paths do. More generally, operators describing physical observables evolve in time as

\[
O(t) \equiv e^{+i H t} O e^{-i H t},
\]

where \(H\) is the hamiltonian of the system and time \(t = 0\) is set at the instant of post-selection, so that, their corresponding c-values on paths do also evolve,

\[
(O)_{cl}(t)(\pm, \pm) = P_o(t) \left(1, (\sigma^{(A)}_1)_{cl}(\pm, \pm), (\sigma^{(B)}_\phi)_{cl}(\pm, \pm), (\sigma^{(A)}_1 \cdot \sigma^{(B)}_\phi)_{cl}(\pm, \pm) \right).
\]

(21)

From the equation

\[
\frac{dP_o(t)}{dt}|\Psi> = \frac{dO(t)}{dt}|\Psi> = i[H, O(t)]|\Psi> = iP[H, O(t)]|\Psi>
\]

(22)

we can obtain generic differential equations of motion for the pseudoclassical values on paths of any observable,

\[
\frac{d(O(t))_{cl}}{dt} = i ([H, O(t)])_{cl}.
\]

(23)

We have thus formally set a statistical interpretation of the quantum wavefunction as a mixture of non-interfering paths that obey pseudo-classical differential equations of motion. We showed at the end of the previous section that this statistical interpretation reproduces the quantum average values of all physical observables,

\[
<\Psi|O|\Psi> = \sum_{\pm, \pm} |q_{\pm, \pm}|^2 (O)_{cl}(\pm, \pm),
\]

(24)

as well as all two-points quantum correlations,

\[
<\Psi|O_1 \cdot O_2|\Psi> = \sum_{\pm, \pm} |q_{\pm, \pm}|^2 (O_1)_{cl}^*(\pm, \pm) (O_2)_{cl}(\pm, \pm).
\]

(25)
3. Actually, the freedom to build upon any complete set of commuting observables \( \{ \sigma_i^{(A)}, \sigma_i^{(B)} \} \) provides us with a continuous family of equivalent statistical representations of the same quantum state. This freedom can be understood as a global symmetry of the pseudo-classical equations of motion \([23]\) that describe the quantum state. Under such symmetry transformations the probabilities of different paths to occur get modified, as well as the pseudo-classical c-values of physical observables along them.

Indeed, let \( \{ |\zeta_i>\}_{i \in I} \) and \( \{ |\xi_j>\}_{j \in J} \) be families of common orthonormal eigenstates for two complete sets of commuting observables and

\[
\left\{ (O_\zeta(t))_{cl}^{(i)} = \frac{< \zeta_i | O(t) | \Psi >}{< \zeta_i | \Psi >} \right\}_{i \in I}, \quad \left\{ (O_\zeta(t))_{cl}^{(j)} = \frac{< \zeta_j | O(t) | \Psi >}{< \zeta_j | \Psi >} \right\}_{j \in J}
\]

be the pseudo-classical values of the observable \( O(t) \) along the paths associated to each of them. In these expressions is implicit the assumption that in both representations all paths have non zero probabilities to occur, \( p_i = |< \zeta_i | \Psi >|^2 \neq 0 \) for all \( i \in I \) and \( p_j = |< \xi_j | \Psi >|^2 \neq 0 \) for all \( j \in J \).

It is straightforward to obtain then the following relationship:

\[
(O_\zeta(t))_{cl}^{(i \in I)} = \frac{< \zeta_i | O(t) | \Psi >}{< \zeta_i | \Psi >} = \sum_{j \in J} \frac{< \zeta_i | \xi_j > < \xi_j | O(t) | \Psi >}{< \zeta_i | \Psi >} = \\
= \sum_{j \in J} \frac{< \zeta_i | \xi_j > < \xi_j | \Psi >}{< \zeta_i | \Psi >} (O_\xi(t))_{cl}^{(j)}.
\]

Hence, when we change the set of commuting observables used to represent the system the pseudo-classical values of a physical observable along the different paths transform linearly as the components of a vector in \( \mathbb{C}^n \), where \( n = 4 \) is the dimension of the Hilbert space of the system. The matrix of this transformation has complex coefficients:

\[
\tilde{p}_{j/i} = \frac{< \zeta_i | \xi_j > < \xi_j | \Psi >}{< \zeta_i | \Psi >} = \frac{< \xi_j | \Psi > < \zeta_i | \Psi >}{< \zeta_i | \Psi >} \in \mathbb{C},
\]

such that:

\[
(O_\zeta(t))_{cl}^{(i \in I)} = \sum_{j \in J} \tilde{p}_{j/i} \cdot (O_\xi(t))_{cl}^{(j)}.
\]
This relationship can be rewritten as

$$0 = \sum_{j \in J} \tilde{p}_{j/i} \cdot \left( (\mathcal{O}_\xi(t))^{(j)}_{cl} - (\mathcal{O}_\zeta(t))^{(i \in I)}_{cl} \right), \quad (29)$$

by noticing that

$$\sum_{j \in J} \tilde{p}_{j/i} = \sum_{j \in J} \frac{<\Psi|\xi_j><\xi_j|\Psi>}{<\Psi|\xi_j><\xi_j|\Psi>} = \frac{<\Psi|\xi_j><\xi_j|\Psi>}{<\Psi|\xi_j><\xi_j|\Psi>} = 1. \quad (30)$$

That is, for all physical observables $\mathcal{O}(t)$ the vector $\vec{\Omega}_{\mathcal{O}(t)}^{(j \in I)} \equiv \left( (\mathcal{O}_\xi(t))^{(j)}_{cl} - (\mathcal{O}_\zeta(t))^{(i \in I)}_{cl} \right)_{j \in J} \in \mathbb{C}^n$ is contained within the hyperplane of dimension $n - 1$ orthogonal to the vector $\vec{p}_{j/i} \in \mathbb{C}^n$. Moreover, the vectors $\left\{ \vec{\Omega}_{\sigma_1}^{(i \in I)}, \vec{\Omega}_{\sigma_2}^{(i \in I)}, \vec{\Omega}_{\sigma_3}^{(i \in I)} \right\}$ associated to the linear generators of a commutative algebra (18) of all the operators with common eigenvectors (excluding from them the vector $\vec{\Omega}_{I}^{(i \in I)} = 0$ associated to the identity operator) form a basis in this hyperplane [30].

The complex coefficients $\tilde{p}_{j/i}$ in (28) can also be understood as conditional pseudo-probabilities in terms of an extended version of Bayes law, as in addition to (30) they also fulfill the constraint

$$\sum_{i \in I} p_i \cdot \tilde{p}_{j/i} = \sum_{i \in I} \frac{<\Psi|\xi_j><\xi_j|\Psi>}{<\Psi|\xi_j><\xi_j|\Psi>} = \frac{<\Psi|\xi_j><\xi_j|\Psi>}{<\Psi|\xi_j><\xi_j|\Psi>} = p_{j \in J}. \quad (31)$$

The extension of the standard probability theory to the case in which pseudo-probabilities can take values outside the real interval $[0, 1]$ has been largely discussed in the literature since the very early stages of the development of Quantum Mechanics [25]. In particular, the complex pseudo-probabilities $\tilde{p}_{j/i}$ defined in (27) were first introduced in a different context in [26]. A formal axiomatic formulation of extended pseudo-probability models was first laid down in [27, 28] and more recently in [29]. In general, it is widely recognized that pseudo-probabilities outside the interval $[0, 1]$ can be formally associated to events that are not physical, as long as probabilities of all physical events lay within this interval. This is indeed the case of the conditional complex pseudo-probabilities $\tilde{p}_{j/i}$, whose naive interpretation would be the probability of obtaining the set $j \in J$ of eigenvalues of a given complete family of strongly measured commuting observables given that the set $i \in I$ of eigenvalues of a different complete family of commuting observables, not commuting with the former, had been previously strongly measured.
The argument that follows tries to offer a more intuitive understanding of these complex conditional pseudo-probabilities. This argument suggests that pseudo-probabilities are required when trying to statistically describe coarsely tested physical systems and, therefore, it implies that pseudo-classical paths may indeed be thick representations of an even finer hidden reality. In order to make our argument clearer we will present it with the help of a simple example.

We consider a probability space consisting of $n \times n$ possible single events $\{(i, j)\}_{i,j=1,...,n}$. For the sake of simplicity we assume that all these single events occur with equal probability, $p_{i,j} = 1/n^2$. The average value of any real random variable $z_{i,j} \in \mathbb{R}$ defined on this space is given, as usual, by $Z = \frac{1}{n^2} \cdot \sum_{i,j=1,...,n} z_{i,j}$. We can define also partial averages along rows $Z_{\text{row}}^{(i=1,...,n)} = \frac{1}{n} \cdot \sum_{j=1,...,n} z_{i,j}$ or columns $Z_{\text{col}}^{(j=1,...,n)} = \frac{1}{n} \cdot \sum_{i=1,...,n} z_{i,j}$.

Let us now assume that we have no access to the values of the random variables on individual cells, but only to their partial average values along complete rows or columns:

Figure 1. Schematic representation of the finely described probabilistic toy model.

Figure 2. Experimentally accessible coarse subsets within the probabilistic toy model.
If we now try to express the average values along rows as a linear mean of the average values along columns,

\[ Z_{\text{row}}^i = \sum_{j=1}^{n} \tilde{p}_{j/i} \cdot Z_{\text{col}}^j, \tag{32} \]

we cannot any longer require the real linear coefficients \( \tilde{p}_{j/i} \) to lay within the unit interval \([0, 1]\) if they are required to fulfill also

\[ \sum_j \tilde{p}_{j/i} = 1, \quad \sum_i \tilde{p}_{j/i} = 1, \tag{33} \]

because \( Z_{\text{col}}^j \) is not necessarily equal to \( z_{i,j} \). Nonetheless, if we do not demand from these linear coefficients to belong to the unit interval, the linear equation (32) always admits a solution \( (\tilde{p}_{j/i})_{i,j=1,\ldots,n} \in \mathcal{M}(\mathbb{R}, n) \) that fulfills the constrains (33), except for the singular cases in which either \( (Z_{\text{row}}^i)_{i=1,\ldots,n} \propto 1 \in \mathbb{R}^n \) or \( (Z_{\text{col}}^j)_{j=1,\ldots,n} \propto 1 \in \mathbb{R}^n \) and \( (Z_{\text{row}}^i)_{i=1,\ldots,n} \neq (Z_{\text{col}}^j)_{j=1,\ldots,n} \).

Actually, constrains (33) define a \((n - 1)^2\)-dim affine linear manifold within the linear space \( \mathcal{M}(\mathbb{R}, n) \) of all \( n \times n \) square matrices with real coefficients. To any matrix \( (\tilde{p}_{j/i})_{i,j=1,\ldots,n} \) within this linear manifold we can associate the linear subspace of all random real variables \( z_{i,j} \) in the finely described probability model (Fig. 1) whose average values along rows and columns are related by the corresponding linear relationship (32). Hence, we can interpret the matrices \( (\tilde{p}_{j/i})_{i,j=1,\ldots,n} \) within this affine linear manifold as representing different "states" of the coarsely described classical statistical system (Fig. 2).

4. In order to demonstrate how the formalism of pseudo-classical paths works, we will explicitly show in this section how to assign pseudo-classical values on paths to observables defined on the Hilbert space of two photons polarization states.

For example, in order to assign values to the observable \( \sigma_1^{(B)} \) we rely on the linear relationship

\[ \sigma_1^{(B)}|\Psi > = -\sigma_1^{(A)}|\Psi > \]
such that,
\[
(\sigma^{(B)}_1)_{cl}(\pm, \pm) = - (\sigma^{(A)}_1)_{cl}(\pm, \pm).
\]
The pseudo-classical values in the right hand side of this expression where defined in [14,15].

Likewise, in order to assign values to \(\sigma^{(B)}_2\) we notice that:
\[
\sigma^{(B)}_2 |\Psi> = \sin^{-1}(\phi) \left[ \sigma^{(B)}_\phi - \cos(\phi) \sigma^{(B)}_1 \right] |\Psi> = \sin^{-1}(\phi) \left[ \sigma^{(B)}_\phi + \cos(\phi) \sigma^{(A)}_1 \right] |\Psi>,
\]
which implies
\[
\left(\sigma^{(B)}_2\right)_{cl}(\pm, \pm) = \sin^{-1}(\phi) \left[ \left(\sigma^{(B)}_\phi\right)_{cl}(\pm, \pm) + \cos(\phi) \left(\sigma^{(A)}_1\right)_{cl}(\pm, \pm) \right].
\]
In order to assign values to the observable \(\sigma^{(B)}_3\) we notice first that \(\sigma^{(B)}_3 = -i\sigma^{(B)}_1 \sigma^{(B)}_2\) and, therefore,
\[
\sigma^{(B)}_3 |\Psi> = -i\sigma^{(B)}_1 \sigma^{(B)}_2 |\Psi> = i\sigma^{(B)}_2 \sigma^{(B)}_1 |\Psi> = -i\sigma^{(B)}_2 \sigma^{(A)}_1 |\Psi> = -i\sigma^{(A)}_1 \sigma^{(B)}_2 |\Psi> =
\]
\[
= -i \sin^{-1}(\phi) \left[ \sigma^{(A)}_1 \sigma^{(B)}_\phi + \cos(\phi) 1 \right] |\Psi>,
\]
which leads us to the following assignments:
\[
\left(\sigma^{(B)}_3\right)_{cl}(\pm, \pm) = -i \sin^{-1}(\phi) \left[ \left(\sigma^{(A)}_1\right)_{cl}(\pm, \pm) \cdot \left(\sigma^{(B)}_\phi\right)_{cl}(\pm, \pm) + \cos(\phi) \right].
\]
Any other observable defined on the Hilbert state of photon B can be written as a linear combination of \(\sigma^{(B)}_1, \sigma^{(B)}_2, \sigma^{(B)}_3\), and the identity operator \(1\) and, hence, its values on paths can be obtained from theirs.

In order to assign values to observables defined on the Hilbert space of photon A we proceed similarly. We have already given values to \(\sigma^{(A)}_1\). We can now exploit the perfect anti-correlation between the polarizations of photon A and photon B to assign values to \(\sigma^{(A)}_2\) and \(\sigma^{(A)}_3\). For example,
\[
\sigma^{(A)}_2 |\Psi> = -\sigma^{(B)}_2 |\Psi>,
\]
which implies
\[
\left(\sigma^{(A)}_2\right)_{cl}(\pm, \pm) = - \left(\sigma^{(B)}_2\right)_{cl}(\pm, \pm) = - \sin^{-1}(\phi) \left[ \left(\sigma^{(B)}_\phi\right)_{cl}(\pm, \pm) + \cos(\phi) \left(\sigma^{(A)}_1\right)_{cl}(\pm, \pm) \right].
\]
Any other observable defined on the Hilbert state of photon A can be written as a linear combination of \( \sigma_1^{(A)}, \sigma_2^{(A)}, \sigma_3^{(A)} \), and the identity operator \( 1 \) and, hence, its values on paths can be obtained from theirs.

Finally, we can obtain the values on paths for any observable \( \mathcal{O} \) defined on the Hilbert space of two photons polarization states by noticing that it can be written as \( \mathcal{O} = \alpha_{00} 1 + \sum_{i=1,2,3} \alpha_{i0} \sigma_i^{(A)} + \sum_{j=1,2,3} \alpha_{0j} \sigma_j^{(B)} + \sum_{i,j=1,2,3} \alpha_{ij} \sigma_i^{(A)} \cdot \sigma_j^{(B)} \) with \( \alpha_{00}, \alpha_{i0}, \alpha_{0j}, \alpha_{ij} \in \mathbb{R} \) and then using the rules set above. For example,

\[
\sigma_1^{(A)} \cdot \sigma_2^{(B)} |\Psi\rangle = \sin^{-1}(\phi) \left[ \sigma_1^{(A)} \sigma_0^{(B)} + \cos(\phi) 1 \right] |\Psi\rangle.
\] (34)

In the limit \( \phi \to 0 \), which corresponds to choosing almost parallel directions to define the polarization of the two photons, only \( |q_+|^2 \) and \( |q_-|^2 \) are different from zero and, therefore, only paths \((+,−)\) and \((−,+\)) has non-zero (and equal) probabilities to happen. The pseudo-classical values of the photons polarizations in this limit can be easily obtained by noticing that:

\[
\lim_{\phi \to 0} \frac{1 - \cos(\phi)}{\sin(\phi)} = 0.
\]

In particular, their polarizations along an arbitrary direction in the \( XY \) plane \( \sigma_x^{(A,B)} = \cos(\chi) \sigma_1^{(A,B)} + \sin(\chi) \sigma_2^{(A,B)} \) are given by:

\[
\begin{align*}
\left( \sigma_x^{(B)} \right)_{cl} (+,−) &= − \left( \sigma_x^{(A)} \right)_{cl} (+,−) = − \cos(\chi), \\
\left( \sigma_x^{(B)} \right)_{cl} (−,+\) &= − \left( \sigma_x^{(A)} \right)_{cl} (−,+) = + \cos(\chi),
\end{align*}
\]

which transform as classical vectors components. Besides the polarization along the \( Z \) direction is zero on both paths,

\[
\left( \sigma_3^{(B)} \right)_{cl} (+,−) = − \left( \sigma_3^{(A)} \right)_{cl} (+,−) = 0.
\]
Let us now show how the statistical description of the quantum wavefunction that we have introduced can solve the EPR paradox. We consider a measuring device that at time \( t = 0 \) interacts very briefly with one of the photons of the singlet state \( |\uparrow \downarrow\rangle \) and performs a *strong* measurement of its polarization along the X-direction. For the sake of simplicity we assume that the measuring device is an additional photon whose polarization is initially set in the state \( |\downarrow\rangle \). Hence, the composite system is described by the wave function:

\[
|\tilde{\Psi}\rangle = |\downarrow\rangle \otimes |\Psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle \otimes |\downarrow\rangle^{(B)} - |\downarrow\rangle \otimes |\uparrow\rangle^{(B)} \right),
\]

and its dynamics by the time dependent unitary operator:

\[
U(t) = e^{-iHt} = \begin{cases} 
1, & t < 0 \\
-\sigma_1^{(s)} \otimes \frac{1}{2} (1 + \sigma_1)^{(A)} \otimes 1^{(B)} - \sigma_3^{(s)} \otimes \frac{1}{2} (1 - \sigma_1)^{(A)} \otimes 1^{(B)}, & t \geq 0.
\end{cases}
\]

This transformation correlates the polarization of photon A along the X-direction with the polarization of the measuring device along the Z-direction:

\[
U(t)|\tilde{\Psi}\rangle = \begin{cases} 
|\tilde{\Psi}\rangle, & t < t_0 \\
\frac{1}{\sqrt{2}} |\uparrow\rangle \otimes |x+\rangle^{(A)} \otimes |\uparrow\rangle^{(B)} + \frac{1}{\sqrt{2}} |\downarrow\rangle \otimes |x-\rangle^{(A)} \otimes |x+\rangle^{(B)}, & t \geq t_0
\end{cases}
\]

where \( |x\pm\rangle \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle) \) are the eigenstates of the polarization operator \( \sigma_1 \).

Operators defined on the Hilbert space of photon B commute with the unitary transformation \( U(t) \) and, therefore, they do not get modified by the interaction of photon A with the measuring device (as it should be expected from locality).

\[
\sigma_{1,2,3}^{(B)}(t) = U(t)^\dagger \cdot \sigma_{1,2,3}^{(B)} \cdot U(t) = \sigma_{1,2,3}^{(B)}, \quad t \in \mathbb{R}.
\]

The operator that describe the polarization of photon A along the X-direction also commutes with the unitary transformation \( U(t) \) and, therefore, neither it does get modified by its interaction with the measuring device:

\[
\sigma_{1}^{(A)}(t) = U(t)^\dagger \cdot \sigma_{1}^{(A)} \cdot U(t) = \sigma_{1}^{(A)}, \quad t \in \mathbb{R}.
\]
Nevertheless, the interaction with the measuring device does modify the polarization of photon A along any other direction:

\[
\sigma_2^{(A)}(t) = U(t)^\dagger \cdot \sigma_2^{(A)} \cdot U(t) = \begin{cases} 
\sigma_2^{(A)}, & t < 0 \\
\sigma_2^{(A)} \otimes \sigma_3^{(A)} \otimes 1^{(B)}, & t \geq 0.
\end{cases}
\] (39)

\[
\sigma_3^{(A)}(t) = U(t)^\dagger \cdot \sigma_3^{(A)} \cdot U(t) = \begin{cases} 
\sigma_3^{(A)}, & t < 0 \\
-\sigma_2^{(A)} \otimes \sigma_2^{(A)} \otimes 1^{(B)}, & t \geq 0.
\end{cases}
\] (40)

Similarly, the polarization components of the measuring device gets also transformed by its interaction with photon A:

\[
\sigma_1^{(*)}(t) = U(t)^\dagger \cdot \sigma_1^{(*)} \cdot U(t) = \begin{cases} 
\sigma_1^{(*)}, & t < 0 \\
\sigma_1^{(*)} \otimes \sigma_1^{(A)} \otimes 1^{(B)}, & t \geq 0.
\end{cases}
\] (41)

\[
\sigma_2^{(*)}(t) = U(t)^\dagger \cdot \sigma_2^{(*)} \cdot U(t) = \begin{cases} 
\sigma_2^{(*)}, & t < 0 \\
-\sigma_2^{(*)}, & t \geq 0.
\end{cases}
\] (42)

\[
\sigma_3^{(*)}(t) = U(t)^\dagger \cdot \sigma_3^{(*)} \cdot U(t) = \begin{cases} 
\sigma_3^{(*)}, & t < 0 \\
-\sigma_3^{(*)} \otimes \sigma_1^{(A)} \otimes 1^{(B)}, & t \geq 0.
\end{cases}
\] (43)

In order to build the pseudo-classical paths that describe this composite system we choose an enlarged complete set of commuting observables \{\sigma_\rho^{(*)}, \sigma_1^{(A)}, \sigma_\phi^{(B)}\} and expand the wavefunction (35) in the orthonormal basis of their common eigenstates:

\[
|\pm^{(*)}, \pm^{(A)}, \pm^{(B)} \rangle \equiv |\pm^{(*)} \otimes \pm^{(A)}, \pm^{(B)} \rangle,
\] (44)

where \(|\pm^{(*)}\rangle\) are the polarization eigenstates of the measuring device under the operator \(\sigma_\rho^{(*)}\) and \(|\pm^{(A)}, \pm^{(B)}\rangle\) are the polarization eigenstates (8, 9, 10, 11) of the measured two photons system under the operators \(\{\sigma_1^{(A)}, \sigma_\phi^{(B)}\}\). Each of these eight paths occur with probability:

\[
p(\pm^{(*)}, \pm^{(A)}, \pm^{(B)}) = |<\pm^{(*)}, \pm^{(A)}, \pm^{(B)}|\bar{\Psi}>|^2 = |<\pm|\downarrow^{(*)}|\Psi>|^2 \cdot |<\pm, \pm|\Psi>|^2 = p^{(*)}(\pm) \cdot p(\pm, \pm).
\]
We then proceed to assign to any other physical observable $O(t)$ a polynomial operator

$$P_{o(t)} \left( 1, \sigma_{\rho}^{(s)}, \sigma_{\rho}^{(A)}, \sigma_{\rho}^{(B)}, \sigma_{\rho}^{(s)} \otimes \sigma_{\rho}^{(A)}, \sigma_{\rho}^{(s)} \otimes \sigma_{\rho}^{(B)}, \sigma_{\rho}^{(s)} \otimes \sigma_{\rho}^{(A)} \otimes \sigma_{\rho}^{(B)} \right),$$

according to the rule:

$$O(t)|\tilde{\Psi} > = P_{o(t)}|\tilde{\Psi} > .$$

The perfect correlation between the polarization of photon A along the X-direction and the polarization of the device along the Z-direction as a result of the measurement $^{36}$ is encoded in the relationship

$$\sigma_{3}^{(s)}(t)|\tilde{\Psi} > = -\sigma_{3}^{(s)} \otimes \sigma_{1}^{(A)} \otimes 1^{(B)}|\tilde{\Psi} > = \sigma_{1}^{(A)}|\tilde{\Psi} > = \sigma_{1}^{(A)}(t)|\tilde{\Psi} >, \quad t \geq 0,$$

which implies that along all paths

$$\left( \sigma_{3}^{(s)} \right)_{cl}(t) = \left( \sigma_{1}^{(A)} \right)_{cl}(t) = - \left( \sigma_{1}^{(B)} \right)_{cl}(t), \quad t \geq 0.$$

The measurement thus allows us to gain knowledge about the polarization of the photons A and B along the X-direction without actually disturbing it. Furthermore, we have seen $^{37}$ that the measurement does neither disturb the polarization of photon B along the two other directions $Y, Z$. Therefore, our statistical model should properly describe the state of photon B at emission once we have updated the probabilities of the pseudo-classical paths with the information provided by the measurement.

Let say, for example, that the measurement returns that after the interaction $\left( \sigma_{3}^{(s)} \right)_{cl}(t \geq 0) = +1$. Hence, $\left( \sigma_{1}^{(A)} \right)_{cl}(t \in \mathbb{R}) = - \left( \sigma_{1}^{(B)} \right)_{cl}(t \in \mathbb{R}) = +1$. The probabilities of all paths need to be updated in accordance with this information, discarding those four paths $(\pm^{(s)}, -(A), \pm^{(B)})$ along which $\left( \sigma_{3}^{(s)} \right)_{cl}(t \geq 0) = \left( \sigma_{1}^{(A)} \right)_{cl} = -1$ and re-evaluating according to Bayes law the probabilities to happen of the other four paths:

$$p(\pm^{(s)}, -(A), \pm^{(B)}) \rightarrow 0,$$

$$p(\pm^{(s)}, +(A), \pm^{(B)}) \rightarrow \frac{p(\pm^{(s)}, +(A), \pm^{(B)})}{P},$$
with $P = p(+(\ast), +(A), +(B)) + p(+(\ast), +(A), -(B)) + p(+(\ast), +(A), -(B)) + p(+(\ast), +(A), -(B)).$

As we are interested in the polarization of photon B along the latter four paths we integrate out the degrees of freedom of the measurement device. We are then left with two coarser paths $\pm$ occurring with probabilities,

\[
p_{(B)}(+) \equiv p(+(\ast), +(A), +(B)) + p(+(\ast), +(A), +(B)) = \frac{1}{2} (1 - \cos (\phi)),
\]

\[
p_{(B)}(-) \equiv p(+(\ast), +(A), -(B)) + p(+(\ast), +(A), -(B)) = \frac{1}{2} (1 + \cos (\phi)).
\]

Along these two paths the polarization of photon B is given by

\[
\left( \sigma_{1}^{(B)} \right)_{cl} (\pm) = -1, \quad (49)
\]

\[
\left( \sigma_{2}^{(B)} \right)_{cl} (\pm) = \sin^{-1} (\phi) [\pm 1 + \cos (\phi)], \quad (50)
\]

\[
\left( \sigma_{3}^{(B)} \right)_{cl} (\pm) = -i \sin^{-1} (\phi) [\pm 1 + \cos (\phi)]. \quad (51)
\]

Any other observable defined on the Hilbert state of photon B can be written as a linear combination of these three observables and their values on paths can then be obtained from theirs. It is straightforward to check that this family of pseudo-classical paths (49, 50, 51) actually describes the quantum state $|x- >^{(B)}$.

Similarly, if the measurement would have returned $\left( \sigma_{3}^{(\ast)} \right)_{cl} (t \geq 0) = -1$, we would obtain (after updating according to Bayes law the probabilities of all pseudo-classical paths) that photon B was emitted in the quantum state $|x+ >^{(B)}$. Hence, the notion of collapse of the wavefunction as a result of a strong measurement can be easily understood in the formalism of pseudo-classical paths that we have developed as an update of our knowledge of the system.

6. In the preceding sections we have shown how to build a statistical interpretation of the singlet polarization state of two entangled photons \cite{11} in terms of a few non-interfering pseudo-classical paths with well-defined probabilities. Every physical observable of the quantum theory is given along each one of these paths a well-defined time-dependent c-value, in
a way that reproduces the expectation values and two-points correlations of all observables
in the given quantum state.

Furthermore, the argument presented in section 3 suggests that these pseudo-classical
paths may indeed be coarse descriptions of an even finer hidden reality. Hence, it seems
natural to define the covariance of any pair of physical observables $O_1(t_1), O_2(t_2)$ along each
one of the paths:

$$
cov_{\pm, \pm} [O_1(t_1), O_2(t_2)] \equiv (O_1(t_1) \cdot O_2(t_2))_{cl} (\pm, \pm) - (O_1(t_1))_{cl}^* (\pm, \pm) \cdot (O_2(t_2))_{cl} (\pm, \pm).
$$

A path $\pi = \pm, \pm$ will appear to be a detailed thin path \cite{23, 24} if for any pair of physical observables

$$
|\text{cov}_\pi [O_1(t_1), O_2(t_2)]| < \Delta,
$$

the modulus of their covariance is smaller than some very small positive real number $\Delta$. In
particular, along such paths:

$$
\{O_1(t_1), O_2(t_2)\}_{cl} = (O_1(t_1) \cdot O_2(t_2) + O_2(t_2) \cdot O_1(t_1))_{cl} \simeq 2 \Re \left[ (O_1(t_1)_{cl}^* \cdot (O_2(t_2))_{cl} \right],
$$

$$
(-i [O_1(t_1), O_2(t_2)]_{cl} = -i (O_1(t_1) \cdot O_2(t_2) - O_2(t_2) \cdot O_1(t_1))_{cl} \simeq 2 \Im \left[ (O_1(t_1))_{cl}^* \cdot (O_2(t_2))_{cl} \right]
$$

Therefore, the equations of motion \cite{23} that describe the time-evolution of pseudo-classical values of physical observables along these paths can be approximated as follows:

$$
\frac{d(O(t))_{cl}}{dt} = i ([H, O(t)])_{cl} \simeq -2 \Im \left[ (\mathcal{H})_{cl}^* \cdot (O(t))_{cl} \right] = i ((\mathcal{H})_{cl}^* \cdot (O(t))_{cl} - (O(t))_{cl}^* \cdot (\mathcal{H})_{cl}).
$$
7. With the insight gained in previous sections we can now explicitly build a realistic and local (pseudo)classical statistical model that reproduces all quantum mechanical features of Bohm’s two photons system.

We start by choosing the frame of reference in which the observer of one of the photons, say \( A \), is going to describe the system. This frame is fixed by the direction along which this observer strongly measures the polarization of his photon. As we did above we will label this direction as \( X \)-axis and the direction along which the photon travels as \( Z \)-axis.

Let us now consider a physical system whose state is described statistically by a uniform distribution over the unit circle,

\[
\rho(\omega) = \frac{1}{2\pi}, \quad \omega \in [-\pi, \pi).
\] (56)

This phase space is divided into two halves \([-\pi, 0)\) and \([0, \pi)\), such that a strong measurement of the polarization \( \sigma^{(A)}_1 \) of photon \( A \) along \( X \)-axis would return a positive output if the hidden state of the system happens to be in the positive half of the phase space \([0, \pi)\) and a negative output if it happens to be in the second half \([-\pi, 0)\), while a strong measurement of the polarization of photon \( B \) along the same axis \( \sigma^{(B)}_1 \) would return just opposite values. That is,

\[
S^{(A)}_1(\omega) = -S^{(B)}_1(\omega) = \begin{cases} 
+1, & \omega \in [0, \pi), \\
-1, & \omega \in [-\pi, 0).
\end{cases}
\] (57)

Obviously, each of the two possible outputs happen with probability \(1/2\) and the average output is zero.

A strong measurement of the polarization of photon \( B \) along any other direction in the \( XY \)-plane, \( \sigma^{(B)}_\phi = \cos \phi \sigma^{(B)}_1 + \sin \phi \sigma^{(B)}_2 \), \( \phi \in [0, \pi) \), divides the unit circle into two new halves shifted with respect to the partition (57) by an angle

\[
\chi = \frac{\pi}{2} (1 - \cos \phi).
\] (58)

That is,

\[
S^{(B)}_\phi(\omega) = \begin{cases} 
+1, & \omega \in [\chi - \pi, \chi), \\
-1, & \omega \in [-\pi, \chi - \pi) \cup [\chi, \pi)
\end{cases}
\] (59)
Again, each of the two possible outputs happen with probability 1/2 and their average is also zero. Furthermore, the model reproduces the crossed probabilities \([13]\):

\[
p \left( S_1^{(A)} = +1, S_\phi^{(B)} = +1 \right) = p \left( S_1^{(A)} = -1, S_\phi^{(B)} = -1 \right) = \frac{\chi}{2\pi} = \frac{1}{4} (1 - \cos \phi) \\
p \left( S_1^{(A)} = +1, S_\phi^{(B)} = -1 \right) = p \left( S_1^{(A)} = -1, S_\phi^{(B)} = +1 \right) = \frac{\pi - \chi}{2\pi} = \frac{1}{4} (1 + \cos \phi).
\]

In this model strong measurements are certain, but very coarse, tests of the hidden state of the system. In order to build a realistic description of the physical system we need to assign actual values to the polarization properties to each one of the possible hidden states of the system. We do it by requiring that our (pseudo)classical statistical model will reproduce the weak values of the physical observables, as follows:

\[
s^{(A)}_1(\omega) = -s^{(B)}_1(\omega) = \begin{cases} 
  +1, & \omega \in [0, \pi), \\
  -1, & \omega \in [-\pi, 0). 
\end{cases} \quad (60)
\]

\[
s^{(A)}_2(\omega) = -s^{(B)}_2(\omega) = \begin{cases} 
  -\frac{\pi/2 - \omega}{\sqrt{\omega(\pi - \omega)}}, & \omega \in [0, \pi), \\
  -\frac{\pi/2 + \omega}{\sqrt{-\omega(\pi + \omega)}}, & \omega \in [-\pi, 0). 
\end{cases} \quad (61)
\]

where \(s^{(A,B)}_I(\omega)\) denotes the polarization properties of photons A or B along X-axis (I=1) and Y-axis (I=2) in the hidden space of the system denoted by coordinate \(\omega\).

Within this model weak values of physical observables can be naturally understood as their average values over the subset of the hidden phase space imposed by the post-selection conditions. For example, it can be easily checked that:

\[
\frac{1}{\chi} \int_0^\chi d\omega \ s^{(A,B)}_I(\omega) = \frac{\langle S^{(A)}_1 = +1; S^{(B)}_\phi = +1 | \sigma_I^{(A,B)} | \Psi \rangle}{\langle S^{(A)}_1 = +1; S^{(B)}_\phi = +1 | \Psi \rangle} \quad (62)
\]

\[
\frac{1}{\pi - \chi} \int_{\chi}^\pi d\omega \ s^{(A,B)}_I(\omega) = \frac{\langle S^{(A)}_1 = +1; S^{(B)}_\phi = -1 | \sigma_I^{(A,B)} | \Psi \rangle}{\langle S^{(A)}_1 = +1; S^{(B)}_\phi = -1 | \Psi \rangle} \quad (63)
\]

The model teaches us that the polarization components of the photons in their hidden states are not scalars, but transform with the frame of reference in which they are described. In each given frame of reference only their polarization components along a preferred axis get binary values, either +1 or −1, while all other components can get very different values.
and in order to know how they will appear in a different frame it is necessary to know the transformation laws.

8. We have developed a statistical interpretation of the Heisenberg picture of Quantum Mechanics, which allows to integrate the principle of locality and an extended notion of physical realism and, thus, it solves the Einstein-Podolsky-Rosen paradox about the completeness of the theoretical framework.

In this interpretation the quantum wavefunction $|\Psi>\rangle$ is described as a statistical mixture of non-interfering paths, which seem to be experimentally accessible through weak measurements. Each one of these paths represents a post-selection of the system in one of the eigenstates $|\Psi_{out}^{(i)}>\rangle$ of a complete set of commuting observables. Hence,

- The probability of each of the paths to occur is $p_i = |<\Psi_{out}^{(i)}|\Psi>\rangle|^2$

- Any other physical observable $\mathcal{O}(t)$ takes along the path its corresponding weak value:

$$\langle \mathcal{O}_d^{(i)}(t) \rangle = \frac{<\Psi_{out}^{(i)}|\mathcal{O}(t)|\Psi>\rangle}{<\Psi_{out}^{(i)}|\Psi>\rangle}. \quad (64)$$

It is straightforward to test (6) that this statistical description reproduces the average values and two-points correlations of all physical observables in the quantum theory. Indeed, this statistical description succeeds to overcome the constraints set by Bell’s theorem and reproduce all quantum mechanical correlations, because the values (64) of physical observables along these pseudo-classical paths are not constrained either to belong to their spectrum of eigenvalues nor to fulfill standard classical algebraic relationships and they may even be complex.

Moreover, the freedom to choose any complete set of post-selection conditions provides us with a continuous family of equivalent statistical representations of the same quantum state. The non-standard pseudo-probabilities needed to relate the values of physical observables on paths obtained for different choices of these conditions suggest that these pseudo-classical
paths are actually coarse descriptions of a finer hidden reality.

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[30] The proof is straightforward: the linear relationship
\[ \lambda_1 \Omega_{\sigma_1^{(A)}}^{(i \in I)} + \lambda_2 \Omega_{\sigma_2^{(B)}}^{(i \in I)} + \lambda_3 \Omega_{\sigma_1^{(A)} \cdot \sigma_2^{(B)}}^{(i \in I)} = 0 \]
means that
\[ \lambda_1 \cdot \langle \xi_j | \sigma_1^{(A)} \rangle |\Psi\rangle + \lambda_2 \cdot \langle \xi_j | \sigma_2^{(B)} \rangle |\Psi\rangle + \lambda_3 \cdot \langle \xi_j | \sigma_1^{(A)} \cdot \sigma_2^{(B)} \rangle |\Psi\rangle = \omega \cdot \langle \xi_j |\Psi\rangle \]
for all vectors \( \{ |\xi_j\rangle \}_{j \in J} \), with \( \omega = \lambda_1 \frac{\langle \xi_j | \sigma_1^{(A)} \rangle |\Psi\rangle}{\langle \xi_j |\Psi\rangle} + \lambda_2 \frac{\langle \xi_j | \sigma_2^{(B)} \rangle |\Psi\rangle}{\langle \xi_j |\Psi\rangle} + \lambda_3 \frac{\langle \xi_j | \sigma_1^{(A)} \cdot \sigma_2^{(B)} \rangle |\Psi\rangle}{\langle \xi_j |\Psi\rangle} \in \mathbb{C} \).
That is, \( \lambda_1 \cdot |\sigma_1^{(A)}\rangle |\Psi\rangle + \lambda_2 \cdot |\sigma_2^{(B)}\rangle |\Psi\rangle + \lambda_3 \cdot |\sigma_1^{(A)} \cdot \sigma_2^{(B)}\rangle |\Psi\rangle = \omega \cdot |\Psi\rangle \), which implies \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) because the vectors \( |\sigma_1^{(A)}\rangle |\Psi\rangle, |\sigma_2^{(B)}\rangle |\Psi\rangle, |\sigma_1^{(A)} \cdot \sigma_2^{(B)}\rangle |\Psi\rangle \) and \( |\Psi\rangle \) are linearly independent, see [18].