I. INTRODUCTION

Quantum physics manifests many properties different from classical physics, these properties are called quantum nonclassicality. There are diverse aspects and notions of quantum nonclassicality, such as noncommutativity of two operators, entanglement, coherence, uncertainty, non-reality, contextuality, and non-locality. These nonclassical properties remarkably deepen the understanding of quantum physics and provided fruitful applications in quantum technology.

Suppose $A = \{|a_j\rangle\}_{j=1}^d$, $B = \{|b_k\rangle\}_{k=1}^d$ are two orthonormal bases of a $d$-dimensional complex Hilbert space $H$. To avoid the freedom $|a_j\rangle \rightarrow e^{i\theta_j}|a_j\rangle$ with $\theta_j \in \mathbb{R}$ real number and $i = \sqrt{-1}$, we denote $\mathcal{A} = \{|a_j\rangle\langle a_j|\}_{j=1}^d$ and $\mathcal{B} = \{|b_k\rangle\langle b_k|\}_{k=1}^d$, that is, $\mathcal{A}$ and $\mathcal{B}$ are all rank-1 projective measurements. We adopt the notion “incompatibility” as in Ref. [1] that, when $\mathcal{A}$ and $\mathcal{B}$ commute we say $A$ and $B$ are compatible, otherwise we say $A$ and $B$ are incompatible. $\mathcal{A}$ and $\mathcal{B}$ commute means that $|a_j\rangle\langle a_j|$ and $|b_k\rangle\langle b_k|$ commute for any $j, k \in [1, d]$ with $[1, d]$ represents the set of consecutive integers $\{j\}_{j=1}^d$. Thus $A$ and $B$ are compatible iff (if and only if) $\mathcal{A} = \mathcal{B}$.

We provide a setting of consecutive measurements to understand this notion of incompatibility. Two persons, Alice and Bob, Alice has the orthonormal basis $A = \{|a_j\rangle\}_{j=1}^d$ (or say, the measurement $\mathcal{A} = \{|a_j\rangle\langle a_j|\}_{j=1}^d$) and Bob has the orthonormal basis $B = \{|b_k\rangle\}_{k=1}^d$. These mean that, for example, Alice has an apparatus which can measure the energy of the quantum system and $A = \{|a_j\rangle\}_{j=1}^d$ is the eigenvectors of energy, Bob has an apparatus which can measure the angular momentum of the quantum system and $B = \{|b_k\rangle\}_{k=1}^d$ is the eigenvectors of angular momentum. We assume that the eigenvalues of energy and angular momentum are all non-degenerate. Now Alice uses $A$ to measure the system which is in a pure state $|\psi\rangle$, suppose the resulting state is $|a_s\rangle$ with the corresponding probability $|\langle a_s|\psi\rangle|^2$. Next Bob uses $B$ to measure the state $|a_s\rangle$ and yields the state $|b_k\rangle$ with probability $|\langle a_s|b_k\rangle|^2$. We see that, if $\mathcal{A} = \mathcal{B}$, Bob can infer with certainty that Alice’s previous state is $|a_s\rangle = |b_k\rangle$. However, if $\mathcal{A} \neq \mathcal{B}$, when Bob finds his state is $|b_k\rangle$, he can not surely infer what the Alice’s previous state is. This is an interpretation of why we can regard $\mathcal{A} = \mathcal{B}$ as “compatible”. Evidently, $\mathcal{A} = \mathcal{B}$ is equivalent to that two operators of energy and angular momentum commute.

Note that the concept “incompatible” in the literature usually refers to the meaning that two measurements are not jointly measurable, such as in Refs. [2–8]. As a special case, when two measurements are two rank-1 projective measurements $(\mathcal{A}, \mathcal{B})$ above, we can check that $(\mathcal{A}, \mathcal{B})$ are jointly measurable iff $\mathcal{A} = \mathcal{B}$. In this work, we consider the incompatibility of two rank-1 projective measurements $(\mathcal{A}, \mathcal{B})$.

In Ref. [1], De Bièvre introduced the notion of complete incompatibility. Two orthonormal bases $A = \{|a_j\rangle\}_{j=1}^d$, $B = \{|b_k\rangle\}_{k=1}^d$ are completely incompatible, if for any nonempty subsets $\mathcal{A} \neq S_A \supseteq A$, $\mathcal{B} \neq S_B \supseteq B$, $|S_A| + |S_B| \leq d$, it holds that span$\{S_A\} \cap$ span$\{S_B\} = \emptyset$. Where $|S_A|$ stands for the number of elements in $S_A$, span$\{S_A\}$ is the subspace spanned by $S_A$ over the complex field $\mathbb{C}$. Although the definition of complete incompatibility is purely algebraic, it possesses the physical interpretation in terms of selective projective measurements [9–11]. It is shown that complete incompatibility closely links with the minimal support uncertainty [1], and also, it is useful to characterize the Kirkwood-Dirac nonclassicality [1].

In this work, we introduce the notion of $s$-order incompatibility with $s \in [2, d+1]$. Under this definition, complete incompatibility is just $(d+1)$-order incompatibility. This paper is organized as follows. In section II, we give the definition of $s$-order incompatibility, and establish a link between it and the minimal support uncertainty. In section III, we characterize $s$-order incompatibility via the transition matrix of the two orthonormal bases. In section IV, we propose a framework for quantification of incompatibility. Section V is a brief summary.
II. s-ORDER INCOMPATIBILITY AND MINIMAL SUPPORT UNCERTAINTY

In this section, we give the definition of s-order incompatibility, and establish a relation between it and the minimal support uncertainty.

Definition 1. s-order incompatibility. Suppose the integer s satisfies s ∈ [2, d + 1], A = \{a_j\}_{j=1}^d and B = \{b_k\}_{k=1}^d are two orthonormal bases of d-dimensional complex Hilbert space H. We say A and B are s-order incompatible if the following (1.1) and (1.2) hold.

(1.1). For any \(\emptyset \neq S_A \subseteq A\) and \(\emptyset \neq S_B \subseteq B\), if \(|S_A| + |S_B| < s\), then \(\text{span}\{S_A\} \cap \text{span}\{S_B\} = \{0\}\).

(1.2). There exist \(\emptyset \neq S_A \subseteq A\) and \(\emptyset \neq S_B \subseteq B\), such that \(|S_A| + |S_B| = s\) and \(\text{span}\{S_A\} \cap \text{span}\{S_B\} \neq \{0\}\).

We use \(\chi_{AB}\) to denote the incompatibility order of A and B. When \(\chi_{AB} = d + 1\), the \((d + 1)\)-order incompatibility just coincides with the complete incompatibility introduced in Ref. [1].

There is a physical interpretation for s-order incompatibility in terms of consecutive projective measurements. Let \(\Pi_A\) be a projective measurement on \(\text{span}\{S_A\}\), and \(\Pi_B\) be a projective measurement on \(\text{span}\{S_B\}\). \(\Pi_A\) and \(\Pi_B\) are not necessarily rank-1. Suppose A and B are s-order incompatible. If (1.1) holds then \(\Pi_A \Pi_B = 0\) for any \(\Pi_A\) and \(\Pi_B\); if (1.2) holds then there exist \(\Pi_A\) and \(\Pi_B\) such that \(\Pi_A \Pi_B \neq 0\).

We establish a link between s-order incompatibility and the minimal support uncertainty. For a pure state \(|\psi\rangle\), we express it in the orthonormal bases \(A = \{a_j\}_{j=1}^d\) and \(B = \{b_k\}_{k=1}^d\) as \(|\psi\rangle = \sum_{j=1}^d \langle a_j | (a_j \langle \psi | b_k \rangle) \rangle\). We use \(n_A(|\psi\rangle)\) to denote the number of nonzero elements in \(\{a_j \langle \psi | b_k \rangle\}_{k=1}^d\), use \(n_B(|\psi\rangle)\) to denote the number of nonzero elements in \(\{b_k \langle \psi | a_j \rangle\}_{j=1}^d\), and let

\[
\begin{align*}
    n_{AB}(|\psi\rangle) &= n_A(|\psi\rangle) + n_B(|\psi\rangle), \\
    n_{min}^{AB} &= \min_{|\psi\rangle \neq 0} n_{AB}(|\psi\rangle). 
\end{align*}
\]

We call \(n_{AB}(|\psi\rangle)\) the support uncertainty of \(|\psi\rangle\) with respect to A and B, and call \(n_{min}^{AB}\) the minimal support uncertainty of \(|\psi\rangle\) with respect to A and B. The support uncertainty \(n_{AB}(|\psi\rangle)\) has many applications in different situations [12–16]. Obviously, \(n_{min}^{AB} \in [2, d + 1]\). It is shown that \(\chi_{AB} = d + 1\) if \(n_{min}^{AB} = d + 1\) [1]. We now prove a more general result in Theorem 2.

Theorem 2. Suppose \(A = \{a_j\}_{j=1}^d\) and \(B = \{b_k\}_{k=1}^d\) are two orthonormal bases of d-dimensional complex Hilbert space H. The incompatibility order \(\chi_{AB}\) and minimal support uncertainty \(n_{min}^{AB}\) are defined in Definition 1 and Eq. (2), then it holds that

\[
\chi_{AB} = n_{min}^{AB}. 
\]

Proof. By the definition of \(n_{min}^{AB}\), if \(n_{min}^{AB} = s\), then there exists pure state \(|\psi\rangle\) such that \(n_{AB}(|\psi\rangle) = n_A(|\psi\rangle) + n_B(|\psi\rangle) = s\) and there does not exist pure state \(|\phi\rangle\) such that \(n_{AB}(|\phi\rangle) = n_A(|\phi\rangle) + n_B(|\phi\rangle) < s\). For such \(|\psi\rangle\), there exist \(\emptyset \neq S_A \subseteq A\) and \(\emptyset \neq S_B \subseteq B\), such that \(|S_A| = n_A(|\psi\rangle)\), \(|S_B| = n_B(|\psi\rangle)\), and \(|\psi\rangle \in \text{span}\{S_A\} \cap \text{span}\{S_B\}\). The nonexistence of such \(|\phi\rangle\) implies that there does not exist \(\emptyset \neq S_A \subseteq A\) and \(\emptyset \neq S_B \subseteq B\), such that \(|S_A| = n_A(|\phi\rangle)\), \(|S_B| = n_B(|\phi\rangle)\), and \(|\phi\rangle \in \text{span}\{S_A\} \cap \text{span}\{S_B\}\). These two conditions just coincide with (1.1) and (1.2) in Definition 1. Then the claim follows.

Again, when \(s = d + 1\), Theorem 1 returns to the corresponding result in Ref. [1].

III. s-ORDER INCOMPATIBILITY AND THE TRANSITION MATRIX

In this section, we introduce a quantity \(\tau_{AB}\) which can be directly calculated. We also establish a link between \(\chi_{AB}(n_{min}^{AB})\) and \(\tau_{AB}\), then \(\chi_{AB}\) can be determined via \(\tau_{AB}\).

For two orthonormal bases \(A = \{a_j\}_{j=1}^d\) and \(B = \{b_k\}_{k=1}^d\), the transition matrix \(U^{AB} = U^{AB}_{jk}\) is defined as \(U^{AB}_{jk} := \langle a_j | b_k \rangle\), that is

\[
U^{AB} = \begin{pmatrix}
\langle a_1 | b_1 \rangle & \langle a_1 | b_2 \rangle & \ldots & \langle a_1 | b_d \rangle \\
\langle a_2 | b_1 \rangle & \langle a_2 | b_2 \rangle & \ldots & \langle a_2 | b_d \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle a_d | b_1 \rangle & \langle a_d | b_2 \rangle & \ldots & \langle a_d | b_d \rangle 
\end{pmatrix}.
\]

We want to characterize s-order incompatibility via the transition matrix \(U^{AB}\). To do this, we introduce the definition of t-order rank deficiency of \(U^{AB}\).

Definition 3. t-order rank deficiency of \(U^{AB}\). For the transition matrix \(U^{AB}\) expressed in Eq. (4) and the integer \(t \in [0, d - 1]\), we define the t-order rank deficiency of \(U^{AB}\), denoted by \(R_t(U^{AB})\), as follows.

\[
R_t(U^{AB}) := \max \left\{ m - \text{rank} \left( \begin{array}{c}
k_1, k_2, \ldots, k_m; \\
j_1, j_2, \ldots, j_m; \\
k_1, k_2, \ldots, k_{m+t}
\end{array} \right) \right\}. 
\]

\[
R_{t,r}(U^{AB}) := \max \left\{ m - \text{rank} \left( \begin{array}{c}
k_1, k_2, \ldots, k_{m+t}; \\
j_1, j_2, \ldots, j_m; \\
k_1, k_2, \ldots, k_{m+t}
\end{array} \right) \right\}. 
\]

\[
R_t(U^{AB}) := \max \{ R_{t,r}(U^{AB}), R_{t,c}(U^{AB}) \}. 
\]

Where \(\begin{array}{c}
k_1, k_2, \ldots, k_{m+t} \\
j_1, j_2, \ldots, j_m \\
k_1, k_2, \ldots, k_{m+t}
\end{array}\) denotes the submatrix obtained by the \(j_1, j_2, \ldots, j_m\) rows and \(k_1, k_2, \ldots, k_m, k_{m+1}\) columns of \(U^{AB}\), for example \(\begin{pmatrix} 1, 3, 4 \end{pmatrix} = \begin{pmatrix} a_1 b_2 & a_1 b_3 & a_1 b_4 \\
a_2 b_2 & a_2 b_3 & a_2 b_4 \\
a_3 b_2 & a_3 b_3 & a_3 b_4 \end{pmatrix} \).

Clearly, the definitions of \(R_{t,r}(U^{AB})\), \(R_{t,c}(U^{AB})\), and \(R_t(U^{AB})\) above can be similarly defined for general matrices, not only the unitary matrices. Note that a similar definition of rank-deficient submatrices was proposed in Ref. [17].
Proposition 4. Suppose integers \( \{t, t_1, t_2\} \subseteq [0, d - 1] \), \( R_t(U^{AB}) \) is defined in Eqs. (5,6,7), then the following (4.1)-(4.4) hold.

(4.1). \( R_t(U^{AB}) \geq 0 \).

(4.2). If \( t_1 < t_2 \) then \( R_{t_1}(U^{AB}) \geq R_{t_2}(U^{AB}) \).

(4.3). \( R_{d-1}(U^{AB}) = 0 \).

(4.4). If \( R_0(U^{AB}) = 0 \) then \( R_t(U^{AB}) = 0 \) for any \( t \in [0, d - 1] \).

Proof. Recall that the matrix rank is defined as the rank of row vectors and which also equals the rank of column vectors, then \( R_t(U^{AB}) \geq 0 \) evidently holds since \( m \geq \text{rank}(j_1,j_2,...,j_m) \).

For \( t_2 \), according to Definition 3, there exist \( 1 \leq m \leq d - t_2 \) and \((j_1,j_2,...,j_m)\) such that \( R_{t_2}(U^{AB}) = m - \text{rank}(j_1,j_2,...,j_m) \), or there exist \( 1 \leq n \leq d - t_2 \) and \((j_1,j_2,...,j_n)\) such that \( R_{t_2}(U^{AB}) = n - \text{rank}(j_1,j_2,...,j_n) \). We consider the former case, the latter can be discussed similarly. For the former case, we see that

\[
R_{t_2}(U^{AB}) = m - \text{rank}(j_1,j_2,...,j_m) \\
\leq (m + t_2 - t_1) - \text{rank}(l_1,l_2,...,l_m,l_{m+1},...,l_{m+t_2-t_1}) \\
\leq R_{t_1}(U^{AB}),
\]

where \( 0 \leq l_1 < l_2 < ... < l_m < l_{m+1} < ... < l_{m+t_2-t_1} \leq d \) and \((j_1,j_2,...,j_m) \subseteq \{l_1,l_2,...,l_m,l_{m+1},...,l_{m+t_2-t_1}\} \).

The first inequality says the fact that adding \((t_2-t_1)\) rows can at most increase \((t_2-t_1)\) for the rank. The second inequality is from the definition of \( R_{t_1}(U^{AB}) \). This proves (4.2).

When \( t = d - 1 \), from Definition 3, \( m \) can only take \( m = 1 \). Since \( U^{AB} \) is unitary, then every row vector and every column vector of \( U^{AB} \) are all nonzero. Hence, \( R_{d-1}(U^{AB}) = 0 \). This proves (4.3).

\[
\begin{pmatrix}
\langle a_1 | b_1 \rangle & ... & \langle a_1 | b_{S_A} \rangle & \langle a_1 | b_{S_A+1} \rangle & ... & \langle a_1 | b_d \rangle \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
\langle a_{|S_A|} | b_1 \rangle & ... & \langle a_{|S_A|} | b_{S_A} \rangle & \langle a_{|S_A|} | b_{S_A+1} \rangle & ... & \langle a_{|S_A|} | b_{S_A+1} \rangle \\
\langle a_{|S_A|+1} | b_1 \rangle & ... & \langle a_{|S_A|+1} | b_{S_B} \rangle & \langle a_{|S_A|+1} | b_{S_B+1} \rangle & ... & \langle a_{|S_A|+1} | b_{S_B+1} \rangle \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
\langle a_d | b_1 \rangle & ... & \langle a_d | b_{S_B} \rangle & \langle a_d | b_{S_B+1} \rangle & ... & \langle a_d | b_d \rangle
\end{pmatrix}.
\]

(10)

Expanding \( |\psi\rangle \) in \( S_A \) and \( S_B \) we get that

\[
|\psi\rangle = \sum_{j=1}^{S_A} x_j |a_j\rangle = \sum_{k=1}^{S_B} y_k |b_k\rangle,
\]

where \( \{x_j\}_{j=1}^{S_A} \) are complex numbers not all vanishing, \( \{y_k\}_{k=1}^{S_B} \) are complex numbers not all vanishing. Consequently,

\[
\langle \psi | b_k \rangle = 0 \text{ for all } |S_B| + 1 \leq k \leq d,
\]

\[
\langle a_j | \psi \rangle = 0 \text{ for all } |S_A| + 1 \leq j \leq d.
\]
With Eq. (11), we have
\[
\sum_{j=1}^{|S_A|} x_j^* |a_j| b_k = 0 \quad \text{for all } |S_B| + 1 \leq k \leq d, \quad (14)
\]
and
\[
\sum_{k=1}^{n} y_k |a_j| b_k = 0 \quad \text{for all } |S_A| + 1 \leq j \leq d, \quad (15)
\]
where \(x_j^*\) is the complex conjugate of \(x_j\). Eq. (14) says that the \(|S_A| \times (d - |S_B|)\) submatrix \(\left( |S_B|, \ldots, |S_B| \right)\) has linearly dependent row vectors, Eq. (15) says that the \((d - |S_A|) \times |S_B|\) submatrix \(\left( |S_A| + 1, |S_B| + \ldots, d \right)\) has linearly dependent column vectors. Since \(2 \leq \chi_{AB} \leq d\), then \(|S_A| + |S_B| \leq d, |S_A| \leq d - |S_B|, \text{ and } |S_B| \leq d - |S_A|\). These further imply that \(R_{d - |S_A| - |S_B|}(U^{AB}) > 0\) and
\[
\tau_{AB} \geq d - \chi_{AB}. \quad (16)
\]

Conversely, suppose the index of rank deficiency is \(\tau_{AB}\), then there exist \(1 \leq m \leq d - \tau_{AB}\) and \((k_1, k_2, \ldots, k_{\tau_{AB}})\) such that \(m > \text{rank}(k_1, k_2, \ldots, k_{\tau_{AB}})\), or there exist \(1 \leq n \leq d - \tau_{AB}\) and \((j_1, j_2, \ldots, j_{\tau_{AB}})\) such that \(n > \text{rank}(j_1, j_2, \ldots, j_{\tau_{AB}})\). We consider the former case, the latter can be discussed similarly. For the former case of general loss of generality, we assume \(\left( |S_A| + 1, |S_B| + \ldots, d \right)\) is complex numbers and not all vanishing such that
\[
\sum_{j=1}^{m} z_j |a_j| b_k = 0 \quad \text{for all } 1 \leq k \leq m + \tau_{AB}. \quad (17)
\]
Let \(|\varphi| = \sum_{j=1}^{m} z_j^* |a_j|\), then \(|\varphi| \neq 0\) and
\[
\langle \varphi | b_k \rangle = 0 \quad \text{for all } 1 \leq k \leq m + \tau_{AB}. \quad (18)
\]
It follows that \(n_A(\varphi) \leq m, n_B(\varphi) \leq d - m - \tau_{AB}\), and
\[
\chi_{AB} = n_{AB}^{\min} \leq n_A(\varphi) + n_B(\varphi) \leq d - \tau_{AB}. \quad (19)
\]
Combing Eqs. (16,19) we then finished this proof. ■

By definition, \(\tau_{AB}\) can be directly calculated, then Theorem 6 and Theorem 2 provide a way to determine \(\chi_{AB}\) and \(n_{AB}^{\min}\) via \(\tau_{AB}\).

IV. QUANTIFICATION OF INCOMPATIBILITY

The incompatibility order \(\chi_{AB}\), or the minimal support uncertainty \(n_{AB}^{\min}\), provides only a classification for incompatibility of two orthonormal bases \(A\) and \(B\), not a quantification. For \(\chi_{AB} = n_{AB}^{\min} = 2\), it must hold that there is at least one element in the intersection of \(\overline{A} = \{ |a_j\rangle |a_j\rangle \}_{j=1}^{d} \text{ and } \overline{B} = \{ |b_k\rangle |b_k\rangle \}_{k=1}^{d}\). We assume that \(\{ |a_1\rangle |a_1\rangle \} \in \overline{A} \cap \overline{B}\). We write \(A = \{ |a_1\rangle \} \cup A'\) with \(A' = \{ |a_j\rangle \}_{j=2}^{d}, B = \{ |b_1\rangle \} \cup B'\) with \(B' = \{ |b_k\rangle \}_{k=2}^{d}\), then \(U^{AB} = (|a_1\rangle |b_1\rangle) + U^{A'B'}\) with \(\{ |a_1\rangle |b_1\rangle \} \) the \(1 \times 1\) matrix of element \(|a_1\rangle |b_1\rangle\). We see that \(\chi_{AB} = 2\) only grasps the fact \(\{ |a_1\rangle |a_1\rangle \} = \{ |b_1\rangle |b_1\rangle \} \in \overline{A} \cap \overline{B}\), without any further information about \(A' = \{ |a_j\rangle \}_{j=2}^{d}\) and \(B' = \{ |b_k\rangle \}_{k=2}^{d}\). To more precisely characterize incompatibility, we ask the question that how to quantitatively incompatibility, that is, how to find a real valued functional which can serve as an incompatibility measure. Two orthonormal bases \(A\) and \(B\) are compatible if \(\overline{A} = \overline{B}\), then the incompatibility measure should describe how far between \(\overline{A}\) and \(\overline{B}\). We propose three necessary conditions as follows which any incompatibility measure \(C(A,B)\) should satisfy.

(i). \(C(A,B) \geq 0\) and \(C(A,B) = 0\) if \(\overline{A} = \overline{B}\).

(ii). \(C(A, B) = C(A_1, B_1) + C(A_2, B_2)\) if \(A = A_1 \cup A_2, B = B_1 \cup B_2, |A_1| = |B_1|\) and \(U^{AB} = U^{A_1B_1} \oplus U^{A_2B_2}\).

(iii). For given dimension \(d, C(A, B)\) reaches the maximum if \(|\{ |a_j\rangle |b_k\rangle \} = 1/\sqrt{d}\}_{j,k=1}^{d}\).

Condition (i) is a natural requirement for \(C(A,B)\) being a faithful incompatibility measure. Condition (ii) is an additivity for direct sum of
\[
U^{AB} = U^{A_1B_1} \oplus U^{A_2B_2},
\]
where \(U^{AB} = U^{A_1B_1} \oplus U^{A_2B_2}\) means that \(\text{span} \{ |a_1\rangle \} \cap \text{span} \{ |b_1\rangle \} = \{0\}\) or \(\text{span} \{ |A_1\rangle \} = \text{span} \{ |B_1\rangle \} \text{ and } \text{span} \{ |A_2\rangle \} = \text{span} \{ |B_2\rangle \}.\) For this case, it is reasonable to think that incompatibility exists only between \(A_1\) and \(B_1\), between \(A_2\) and \(B_2\), but does not exist between \(A_1\) and \(B_2\), between \(A_2\) and \(B_1\). In condition (iii), two orthonormal bases \(A\) and \(B\) satisfying \(|\{ |a_j\rangle |b_k\rangle \} = 1/\sqrt{d}\}_{j,k=1}^{d}\) are called mutually unbiased bases (MUBs). See reviews on MUBs in Refs. [18, 19] and some applications of MUBs recently reported such as in Refs. [20–22]. We view two MUBs as maximally incompatible bases.

Below we propose some incompatibility measures satisfying (i)-(iii).

Definition 7. For \(0 < \alpha \neq 2\), we define the incompatibility measure based on \(l_\alpha\) norm as
\[
C_\alpha(A, B) = \frac{1}{2 - \alpha} \left( \sum_{k=1}^{d} \sum_{j=1}^{d} \{|a_j\rangle |b_k\rangle|^\alpha \right)^{1/\alpha} - d. \quad (21)
\]
In particular, when \(\alpha = 1\), we have the incompatibility measure based on \(l_1\) norm as
\[
C_1(A, B) = \sum_{j,k=1}^{d} \{|a_j\rangle |b_k\rangle| - d. \quad (22)
\]
We show that \(C_\alpha(A, B)\) satisfies (i)-(iii). For complex vector \(\overline{z} = (x_1, x_2, \ldots, x_d)\), the \(l_\alpha\) norm of \(\overline{z}\) is \(||\overline{z}||_\alpha = (\sum_{j=1}^{d} |x_j|^\alpha)^{1/\alpha}\). We know that when \(0 < \alpha < \beta\), it holds that \(||\overline{z}||_\alpha \geq ||\overline{z}||_\beta\) and \(||\overline{z}||_\alpha = ||\overline{z}||_\beta\) if there exists at most one nonzero element in \(\{x_j\}_{j=1}^{d}\). Since \(U^{AB}\) is unitary, then \(\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^2 = 1\), \(\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^\alpha \geq 1\) for \(0 < \alpha < 2\) and \(\sum_{j=1}^{d} |\langle a_j | b_k \rangle|^\alpha \leq 1\) for \(\alpha > 2\), and...
also \( \sum_{j=1}^{d} |\langle a_j | b_k \rangle|^2 = 1 \) iff there is only one nonzero element in \( \{ \langle a_j | b_k \rangle \}_{j,k=1}^{d} \), this nonzero element must have modulus 1. Consequently, \( C_{\alpha}(A, B) = 0 \) implies that every column of \( U_{AB} \) has just one nonzero element. \( U_{AB} \) thus has totally \( d \) nonzero elements, and every row also must have just one nonzero element since \( U_{AB} \) is of full rank. As a result, \( \overline{A} = \overline{B} \). We then showed \( C_{\alpha}(A, B) \) satisfies (i). \( C_{\alpha}(A, B) \) satisfying (ii) is obvious by the definition of \( C_{\alpha}(A, B) \), \( C_{\alpha}(A, B) \) satisfying (iii) can be proved by the method of Lagrange multipliers with the constraints \( \{ \sum_{j=1}^{d} |\langle a_j | b_k \rangle|^2 = 1 \}_{k=1}^{d} \).

**Definition 8.** We define the incompatibility measure based on noncommutability as

\[
C_{NC}(A, B) = \sum_{j,k=1}^{d} |\langle a_j | b_k \rangle| \sqrt{1 - |\langle a_j | b_k \rangle|^2}.
\]  

(23)

In Ref. 23, the authors defined a quantity \( \Upsilon_{\rho}(E, F) \) based on the noncommutativity of two measurements \( (E, F) \). In the case of two rank-1 projective measurements \( (\overline{A}, \overline{B}) \), this quantity \( \Upsilon_{\rho}(E, F) \) yields \( C_{NC}(A, B) \) in Eq. (23). We see that \( C_{NC}(A, B) \) evidently satisfies (i) and (ii), \( C_{NC}(A, B) \) satisfying (iii) is a result of \( \Upsilon_{\rho}(E, F) \) being maximized by MUBs.

Finally, we study two examples to illustrate the calculation of incompatibility order and incompatibility measure.

**Example 1.** For qubit system, \( d = 2 \),

\[
U_{AB} = \begin{pmatrix} \langle a_1 | b_1 \rangle & \langle a_2 | b_1 \rangle \\ \langle a_1 | b_2 \rangle & \langle a_2 | b_2 \rangle \end{pmatrix}.
\]

The incompatibility measure based on \( l_1 \) norm is

\[
C_1(A, B) = 2(\sin \theta + \cos \theta - 1),
\]

(24)

the incompatibility measure based on noncommutability is

\[
C_{NC}(A, B) = 2 \sin 2\theta,
\]

(25)

with \( \sin \theta = |\langle a_1 | b_1 \rangle| = |\langle a_2 | b_2 \rangle|, \cos \theta = |\langle a_1 | b_2 \rangle| = |\langle a_2 | b_1 \rangle|, \theta \in [0, \frac{\pi}{2}] \). When \( \theta = 0 \) or \( \frac{\pi}{2}, \) \( C_1(A, B) = 0, \overline{A} = \overline{B} \). When \( \theta = \frac{\pi}{4}, C_1(A, B) \) reaches the maximum.

\( U_{AB} \) is unitary, then, is of full rank. When \( \theta = 0 \) or \( \frac{\pi}{2}, \) \( \sin \theta = 0 \) or \( \cos \theta = 0, \) we have \( \tau_{AB} = 0, \chi_{AB} = 2, \overline{A} = \overline{B} \). When \( 0 \neq \theta \neq \frac{\pi}{2}, \) we have \( \tau_{AB} = -1, \chi_{AB} = 3 = d + 1, \) \( A \) and \( B \) are completely incompatible.

**Example 2.** Discrete Fourier transform (DFT) transition matrix \( F := U_{AB} \). For DFT,

\[
F_{jk} := U_{jk}^{AB} = |\langle a_j | b_k \rangle| = \frac{1}{\sqrt{d}} e^{\frac{i\pi}{d} jk},
\]

(26)

with \( i = \sqrt{-1} \). From Eqs. (22, 23), we get that \( C_1(A, B) = d(\sqrt{d} - 1) \) and \( C_{NC}(A, B) = d\sqrt{d} - 1 \). Since \( |\langle a_j | b_k \rangle| = \frac{1}{\sqrt{d}} \) for any \( j, k \), then \( A \) and \( B \) are maximally incompatible.

It is shown that \( A, B \) are completely incompatible \( (\chi_{AB} = d + 1) \) iff \( d \) is a prime number [1, 14]. We now consider the general case that \( d \) is not necessarily a prime number. Suppose

\[
d = d_1 d_2
\]

(27)

with \( d_1 |d, d_2 |d \) and \( 1 < d_1 \leq d _2 < d \), here \( d_1 \) is a factor of \( d \). We rewrite the index sets \( \{ j \}_{j=1}^{d} \) and \( \{ k \}_{j=1}^{d} \) as

\[
j = j_0 + j_1 d_2, j_0 \in [1, d_2], j_1 \in [0, d_1 - 1],
\]

\[
k = k_0 + k_1 d_1, k_0 \in [1, d_1], k_1 \in [0, d_2 - 1],
\]

then

\[
F_{jk} = \frac{1}{\sqrt{d}} e^{\frac{i\pi}{d} jk_0} e^{\frac{i\pi}{d} j_1 k_1} e^{\frac{i\pi}{d} k_1 j_2},
\]

(28)

where we have used the fact \( e^{\frac{i\pi}{d} j_1 k_1} e^{\frac{i\pi}{d} k_1 j_2} = 1 \). As pointed out in Ref. [17], Eq. (28) implies that

\[
\text{rank} \begin{pmatrix} \{ j_0 + j_1 d_2 \}_{j_1 \in [0, d_1 - 1]}, \{ k_0 + k_1 d_1 \}_{k_1 \in [0, d_2 - 1]} \end{pmatrix} = 1
\]

(29)

since

\[
\begin{pmatrix} \{ j_0 + j_1 d_2 \}_{j_1 \in [0, d_1 - 1]}, \{ k_0 + k_1 d_1 \}_{k_1 \in [0, d_2 - 1]} \end{pmatrix} = \frac{1}{\sqrt{d}} e^{\frac{i\pi}{d} jk_0} F_{\eta_0} F_{\eta_0},
\]

where we have denoted the row vector

\[
F_{\eta_0} = (1, e^{\frac{i\pi}{d} k_0 d_2}, e^{\frac{i\pi}{d} k_0 d_2}, \ldots, e^{(d_1 - 1)i\pi k_0 d_2}).
\]

The submatrix \( \begin{pmatrix} \{ j_0 + j_1 d_2 \}_{j_1 \in [0, d_1 - 1]}, \{ k_0 + k_1 d_1 \}_{k_1 \in [0, d_2 - 1]} \end{pmatrix} \) can be viewed as the column union of the submatrices \( \begin{pmatrix} \{ j_0 + j_1 d_2 \}_{j_1 \in [0, d_1 - 1]}, \{ k_0 + k_1 d_1 \}_{k_1 \in [0, d_2 - 1]} \end{pmatrix} \). From Eq. (29), we get that the column rank (and then the rank)

\[
\text{rank} \begin{pmatrix} \{ j_0 + j_1 d_2 \}_{j_1 \in [0, d_1 - 1]}, \{ k_0 + k_1 d_1 \}_{k_1 \in [0, d_1 - 1]} \end{pmatrix} \leq d_1 - 1.
\]

(30)

Since \( \begin{pmatrix} \{ j_0 + j_1 d_2 \}_{j_1 \in [0, d_1 - 1]}, \{ k_0 + k_1 d_1 \}_{k_1 \in [0, d_1 - 1]} \end{pmatrix} \) has \( d_1 \) rows, hence \( \begin{pmatrix} \{ j_0 + j_1 d_2 \}_{j_1 \in [0, d_1 - 1]}, \{ k_0 + k_1 d_1 \}_{k_1 \in [0, d_1 - 1]} \end{pmatrix} \) is rank deficient for rows. By the definition of \( \tau_{AB} \), it follows that \( \tau_{AB} \geq (d_1 - 1) d_2 - d_1 \), that is

\[
\tau_{AB} \geq d - (d_1 + d_2).
\]

With Theorem 6, we further get

\[
\chi_{AB} \leq d_1 + d_2.
\]

Minimizing \( d_1 + d_2 \) over all \( d_1 \) under Eq. (27) will yield

\[
\tau_{AB} \geq d - \left( \sqrt{d} + \frac{d}{\sqrt{d}} \right).
\]

(31)
\begin{equation}
\chi_{AB} \leq d' + \frac{d}{d'}, \tag{32}
\end{equation}
\begin{equation}
d' = \max\{d_1 | 1 < d_1 \leq \sqrt{d}, d_1 | d\}. \tag{33}
\end{equation}

We conjecture that
\begin{equation}
\tau_{AB} = d - (d' + \frac{d}{d'}), \tag{34}
\end{equation}
\begin{equation}
\chi_{AB} = d' + \frac{d}{d'}. \tag{35}
\end{equation}

V. SUMMARY

For two orthonormal bases $A, B$ of a quantum system, we generalized the notion of complete incompatibility to different order of incompatibility, which resulted in a classification for incompatibility. We established a link between $s$-order incompatibility and minimal support uncertainty. We characterized $s$-order incompatibility via the rank deficiency of the transition matrix $U^{AB}$. All these concepts and consequences included complete incompatibility as special case. We also proposed a framework to quantify the amount of incompatibility. Finally, we studied the example of discrete Fourier transform and raised a conjecture for its incompatibility order.

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