From potential modularity to modularity for integral Galois representations and rigid Calabi-Yau threefolds

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December 20, 2021

Abstract

In a previous article, we have proved a result asserting the existence of a compatible family of Galois representations containing a given crystalline irreducible odd two-dimensional representation. We apply this result, combined with the potential modularity results of Taylor, to prove modularity for any irreducible crystalline \( \ell \)-adic odd 2-dimensional Galois representation (with finite ramification set) unramified at 3 verifying an “ordinarity at 3” easy to check condition, with Hodge-Tate weights \( \{0, w\} \) such that \( 2w < \ell \) (and \( \ell > 3 \)) and such that the traces \( a_p \) of the images of Frobenii verify \( \mathbb{Q}(\{a_p\}) = \mathbb{Q} \). This result applies in particular to any motivic compatible family of odd two-dimensional Galois representations of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) if the motive has rational coefficients, good reduction at 3, and the “ordinarity at 3” condition is satisfied. As a corollary, this proves that all rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \) having good reduction at 3 and satisfying \( 3 \nmid a_3 \) are modular.

1 The result and its proof

The main tools in our proof are the existence of families proved in [D], together with the potential modularity results proved in [T1], [T2]. We will

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also need a result from [W] which controls ordinariness for Hilbert modular forms.

We will call field of coefficients of a Galois representation the field generated by the traces of the images of Frobenii elements.

**Theorem 1.1** Let $\ell > 3$ be a prime. Let $\sigma_\ell$ be a two dimensional odd irreducible $\ell$-adic Galois representation (of the absolute Galois group of $\mathbb{Q}$, continuous) ramified only at $\ell$ and at a finite set of primes $S$ not containing $3$, with field of coefficients $\mathbb{Q}$. Assume that $\sigma_\ell$ is crystalline at $\ell$, with Hodge-Tate weights $\{0, w\}$ ($w$ odd). Assume also that $\ell > 2w$. Then, if $a_3 := \text{trace}(\sigma_\ell(Frob 3))$ is not divisible by $3$, the representation $\sigma_\ell$ is modular.

**Corollary 1.2** If an odd, two-dimensional, compatible family of Galois representations attached to a motive defined over $\mathbb{Q}$, having rational coefficients and good reduction at $3$, verifies $3 \nmid a_3 := \text{trace}(\sigma_\ell(Frob 3))$, $\ell \neq 3$, then the family (thus, the motive) is modular. In particular, any rigid Calabi-Yau threefold defined over $\mathbb{Q}$ with good reduction at $3$ and $3 \nmid a_3$ is modular.

Remark: This modularity criterion for rigid Calabi-Yau threefolds is different than those obtained in [DM]. In particular, the criteria in [DM] required good reduction at $5$ or $7$.

Proof of corollary: Just apply the theorem to the $\ell$-adic representation in the family for a sufficiently large prime $\ell$ where the motive has good reduction.

Proof of theorem:
First, recall that from the existence of a family result in [D], we can insert $\sigma_\ell$ in a (strongly) compatible family $\{\sigma_q\}$, which has rational coefficients and is unramified at $3$ for any $q \neq 3$. Also, from the results in [T2], we know that this family is potentially modular, i.e., that when restricted to some totally real number field $F$ all representations in the family agree with those attached to some Hilbert modular form $h$ over $F$.

Now, we will translate the “easy to check condition” on the trace of $\sigma_\ell$ at Frob 3 into ordinariness of the 3-adic representation in the compatible family. We apply a result of Wiles [W], which tells us that we can read in the corresponding eigenvalue of a Hilbert modular form that the attached Galois
representation is ordinary. Our family of representations becomes modular when restricted to the Galois group of a totally real number field $F$ which can be assumed (using solvable base change) to be such that $3$ is totally split in $F/\mathbb{Q}$ (cf. [D]), so that ordinarity at $3$ of the restriction to the Galois group of $F$ is equivalent to ordinarity of the full $3$-adic representation. The condition $3 \nmid a_3$ implies (cf. [W]) that when we restrict $\sigma_3$ to the Galois group of $F$ we get a modular Galois representation which is ordinary at (every prime of $F$ dividing) $3$, therefore we conclude that $\sigma_3$ is ordinary.

The proof finishes using Skinner-Wiles results: since the residual mod $3$ representation has coefficients in $\mathbb{F}_3$, it is known that it is either modular or reducible (by results of Langlands and Tunnell). Knowing that $\sigma_3$ is ordinary, an application of [SW1] and [SW2] gives the modularity of $\sigma_3$, thus of $\sigma_\ell$ (because they are compatible).

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