A Polynomial-Time Algorithm for Solving the Minimal Observability Problem in Conjunctive Boolean Networks

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Abstract

Many complex systems in biology, physics, and engineering include a large number of state-variables, and measuring the full state of the system is often impossible. Typically, a set of sensors is used to measure part of the state-variables. A system is called observable if these measurements allow to reconstruct the entire state of the system. When the system is not observable, an important and practical problem is how to add a minimal number of sensors so that the system becomes observable. This minimal observability problem is practically useful and theoretically interesting, as it pinpoints the most informative nodes in the system.

We consider the minimal observability problem for an important special class of Boolean networks, called conjunctive Boolean networks (CBNs). Using a graph-theoretic approach, we provide a necessary and sufficient condition for observability of a CBN with \( n \) state-variables, and an efficient \( O(n^2) \)-time algorithm for solving the minimal observability problem. We demonstrate the usefulness of these results by studying the properties of a class of random CBNs.

Index Terms

Logical systems, observability, Boolean networks, computational complexity, systems biology, social networks, random graphs.

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I. INTRODUCTION

Real world systems often include a large number of state-variables (SVs). Measuring all these SVs to obtain the full state of the system is often impossible. For example, the function of a multipolar neuron may depend upon signals received from thousands of other interconnected neurons (see, e.g., [1]).

When the system has outputs, that is, functions of SVs that are directly measured using suitable sensors, an important and practical question is whether it is possible to determine the value of all the SVs by observing a sequence of the outputs. In the language of systems and control theory, a system that satisfies this property is called observable.

Establishing observability is the first step in the design of an observer, that is, a device that can reconstruct the entire state of the system based on a sequence of the outputs. A typical example is estimating the weather in a certain region based on a small set of measurements from local stations in this region [2]. Observers also play a crucial role in the implementation of full-state feedback controllers (see, e.g., [3]).

When a given system is not observable, it is sometimes possible to make it observable by placing additional sensors that measure more (functions of the) SVs. Of course, this may be costly in terms of time and money, so a natural question is: find the minimal number of measurements to add so that the resulting system is observable. This minimal observability problem is also interesting theoretically, as its solution means identifying the (functions of) SVs that provide the maximal information on the entire state of the system [4].

Minimal observability problems are recently attracting considerable interest. Examples include monitoring complex services by minimal logging [5], the optimal placement of
phasor measurement units in power systems (see, e.g., [6]), and the minimal sparse observability problem addressed in [7].

Here we solve the minimal observability problem for an important special class of Boolean networks (BNs). BNs are discrete-time dynamical systems with Boolean SVs. BNs have found many applications as models of dynamical systems. They have been used to capture the existence and directions of links in complex systems (see, e.g., [8]), to model social networks (see, e.g. [9], [10]), and the spread of epidemics [11]. In particular, BNs play an important role in the modeling of biological processes and networks, where the feasible set of states is assumed to be finite (see, e.g. [12], [13], [14]). A classical example is a gene regulation network, where each gene may be either ON or OFF (i.e., expressed or not) [15]. In this case, the state of each gene may be modeled by a Boolean SV, and interactions between the genes (through the proteins that they encode) determine the Boolean update function for each SV.

Let $S := \{0, 1\}$. For two integers $i, j$ let $[i, j] := \{i, i+1, \ldots, j\}$. A BN with $n$ SVs and $m$ outputs is a discrete-time dynamical system in the form:

$$
X_i(k+1) = f_i(X_1(k), \ldots, X_n(k)), \quad \forall i \in [1, n],
$$

$$
Y_j(k) = h_j(X_1(k), \ldots, X_n(k)), \quad \forall j \in [1, m].
$$

(1)

Every $X_i$ and $Y_j$ takes values in $S$, and $f_i, h_j$ are Boolean functions for all $i, j$, i.e., $f_i, h_j : S^n \to S$. If there exists an output $Y_j(k) = X_i(k)$ then we say that the SV $X_i$ is directly observable or directly measurable.

Denote the state of the system at time $k$ by $X(k) := \begin{bmatrix} X_1(k) & \ldots & X_n(k) \end{bmatrix}'$ and the output by $Y(k) := \begin{bmatrix} Y_1(k) & \ldots & Y_m(k) \end{bmatrix}'$. We say that (1) is observable on $[0, N]$, if any two different initial conditions $X(0)$ and $\tilde{X}(0)$ yield different output sequences $\{Y(0), \ldots, Y(N)\}$
and $\{\tilde{Y}(0), \ldots, \tilde{Y}(N)\}$. This means that given $\{Y(0), \ldots, Y(N)\}$ it is always possible to uniquely determine the initial condition $X(0)$ of the system. A BN is called *observable* if it is observable for some value $N \geq 0$.

If the output sequence $\{Y(0), \ldots, Y(N)\}$ is identical for two different initial states $X(0)$ and $\tilde{X}(0)$ then it is not possible to differentiate between them based on the given output sequence. In this case, we say that these two initial conditions are *indistinguishable* on the time interval $[0, N]$. Clearly, this implies that the BN is not observable on this time interval. Boolean control networks (BCNs) are BNs with inputs. There are several ways to extend the notion of observability to BCNs (see, e.g. [16]). Here, we first consider networks without inputs and then generalize the results to the case with inputs.

The observability of BNs has been analyzed using algebraic and graph-theoretic approaches (see, e.g., [17], [18]). It was proven that testing observability of BNs is NP-hard in the number of SVs in the system (see [18]). This means that, unless P=NP, it is computationally intractable to determine whether a large BN is observable. For a general survey on the computational complexity of various problems in systems and control theory, see [19].

Of course, the hardness results on determining observability in general BNs do not preclude the possibility that observability analysis is tractable for certain special classes of BNs. An important class of BNs are those with update functions that include nested canalyzing functions (NCFs) only [20]. A Boolean function is called canalyzing if there exists a specific value, called the canalyzing value, such that an input with this value uniquely determines the function’s output, regardless of the other variables. For example, 0 is a canalyzing value for the function AND, as $\text{AND}(0, X_1, \ldots, X_k) = 0$ for all $X_i \in \{0, 1\}$. BNs with NCFs are frequently used in modeling genetic networks [21], [22].
In this paper, we consider the subclass consisting of NCFs which are constructed exclusively by AND operators (i.e., by conjunctive functions). As models for gene regulation networks, conjunctive functions encode *synergistic* regulation of genes by transcription factors [24], and it seems that this mechanism indeed exists in certain regulatory networks [25], [26], [27].

A BN is called a *conjunctive Boolean network* (CBN) if every update function includes AND operations only, i.e., the dynamics is:

\[
X_i(k+1) = \prod_{j=1}^{n}(X_j(k))^\epsilon_{ji}, \quad \forall i \in [1, n],
\]

where \(\epsilon_{ji} \in \{0, 1\}\) for all \(i, j\). The special (but not very interesting) case \(X_i(k+1) = X_i(k)\) is called a constant updating function.

**Remark 1.** Note that a BN is called a *disjunctive Boolean network* (DBN) if every update function includes only OR operators. By applying De Morgan Law’s, it is possible to reduce DBNs to CBNs, and therefore all the results in this paper hold for DBNs as well.

A useful representation of a CBN is given by a *dependency graph* (also known as the wiring diagram). This is a directed graph (digraph) in which every vertex corresponds to an SV of the CBN, and a directed edge \((i \to j)\) exists if \(X_i(k)\) is one of the arguments in the update function of \(X_j(k+1)\). Thus, the dependency graph encodes the variable dependencies in the update functions.

There is a one-to-one correspondence between a CBN and its dependency graph, which enables a graph-theoretic analysis of CBNs. This has been used to analyze various properties of CBNs including: characterization of the periodic orbits [24], [28], robustness
of these orbits to single bit perturbations \([29]\), and controllability of CBNs \([30], [31]\).

However, observability and, in particular, the minimal observability problem in CBNs has not been studied before. We consider the following problem.

**Problem 1.** Given a CBN with \(n\) SVs determine a minimal set of indices \(I \subseteq [1, n]\), such that making each \(X_i(k), i \in I\), directly measurable yields an observable CBN.

Note that an efficient (i.e., polynomial time) solution to Problem 1 must entail an efficient algorithm for testing observability of a CBN.

**Example 1.** Consider Problem 1 for the CBN:

\[
X_1(k+1) = X_2(k),
\]

\[
X_2(k+1) = X_1(k)X_2(k).
\]

Suppose that we make \(X_1(k)\) directly measurable, that is, add an output \(Y_1(k) = X_1(k)\). Then the resulting one-output CBN is observable. Indeed, given \(\{Y_1(0), Y_1(1)\}\), the initial condition of the CBN is \(X_1(0) = Y_1(0), X_2(0) = Y_1(1)\). Since observability requires at least one output, it is clear that this is a minimal solution to Problem 1. □

To make things more concrete consider the following application. A graph describes a network of interacting agents with directed edges describing the neighboring relations. Every agent has two possible opinions on some matter. At time \(k\) the opinion of agent \(i\) is described by the state-variable \(x_i(k) \in \{0, 1\}\). Every agent is “conservative” in the sense that it tends to hold the opinion zero, unless all its neighbors hold the opinion one at time \(k\) and then he updates his opinion to \(x_i(k+1) = 1\). Initially, there are no observation nodes, but it is possible to recruit agents so that they provide reports on their opinion at any time \(k\). However, the recruitment of an agent is costly in terms of
money, time, etc. Then a natural question is: what is the minimal number of agents that must be recruited in order to be able to infer, using a time sequence of their reports, the entire state of the network? This is exactly Problem 1.

As another application, based on Remark 1, consider a model of epidemics that includes a set of agents that can be either susceptible or infected. The directed dependency network describes contacts between agents that can lead to infection. The infection is so contagious that a susceptible becomes infected if even a single neighbor is infected. In this context Problem 1 again has a natural interpretation.

The contributions of this paper are:

1) a necessary and sufficient condition for the observability of a CBN;
2) a procedure for designing an observer for an observable CBN; and
3) an $O(n^2)$-time algorithm for solving Problem 1.

The remainder of this paper is organized as follows. Section II reviews some standard definitions and notations from graph theory that will be used later on. Section III describes our main theoretical results. As already noted by Kauffman [15], there are good reasons to model various biological processes using networks of randomly connected binary devices. In Section IV we use our algorithm to solve Problem 1 for a class of random CBNs. These are described by random dependency graphs with equiprobable edges. Surprisingly, perhaps, we show that to make these CBNs observable, one must observe at least 69% of the nodes. Section V depicts two extensions of our results. Section VI concludes and presents directions for further research. A detailed description of the main algorithm introduced in the paper is given in the Appendix.
II. Preliminaries

Let $G = (V, E)$ be a digraph, with $V$ the set of vertices, and $E$ the set of directed edges (arcs). Let $e_{i \rightarrow j}$ (or $(v_i \rightarrow v_j)$) denote the arc from $v_i$ to $v_j$. When such an arc exists, we say that $v_i$ is an in-neighbor of $v_j$, and $v_j$ as an out-neighbor of $v_i$. The set of in-neighbors [out-neighbors] of $v_i$ is denoted by $N_{in}(v_i)$ [$N_{out}(v_i)$]. The in-degree [out-degree] of $v_i$ is $|N_{in}(v_i)|$ [$|N_{out}(v_i)|$]. A source [sink] is a node with in-degree [out-degree] zero.

Let $v_i$ and $v_j$ be two vertices in $V$. A walk from $v_i$ to $v_j$, denoted $w_{ij}$, is a sequence: $v_{i_0}v_{i_1} \ldots v_{i_q}$, with $v_{i_0} = v_i$, $v_{i_q} = v_j$, and $e_{i_k \rightarrow i_{k+1}} \in E$ for all $k \in [0, q - 1]$. A simple path is a walk with pairwise distinct vertices. We say that $v_i$ is reachable from $v_j$ if there exists a simple path from $v_j$ to $v_i$. A closed walk is a walk that starts and terminates at the same vertex. A closed walk is called a cycle if all the vertices in the walk are distinct, except for the start-vertex and the end-vertex.

Given a CBN in the form (2), the associated dependency graph is a digraph $G = (V, E)$ with $n$ vertices (corresponding to the SVs of the system), such that $e_{i \rightarrow j} \in E$ if and only if (iff) $\epsilon_{ij} = 1$. A node in the dependency graph that represents a [non] directly observable SV is called a [non] directly observable node.

III. Main Results

From hereon, we consider CBNs with $n$ SVs and $m \geq 0$ outputs:

$$X_i(k + 1) = f_i(X_1(k), \ldots, X_n(k)), \quad \forall i \in [1, n],$$

$$Y_j(k) = X_j(k), \quad \forall j \in [1, m],$$

(3)
where the $f_i$s are AND operators, and every output $Y_i$ is the value of an SV (without loss of generality, we assume that the $m$ outputs correspond to the first $m$ SVs). Thus, nodes $X_1, \ldots, X_m, [X_{m+1}, \ldots, X_n]$ in the dependency graph are [non] directly observable.

We begin by deriving two simple necessary conditions for observability of (3).

**Definition 1.** We say that a CBN has Property $O_1$ if for every non-directly observable node $X_i$ there exists some other node $X_j$, such that $\mathcal{N}_i(X_j) = \{X_i\}$.

**Fact 1.** If a CBN is observable then it has Property $O_1$.

**Proof of Fact 1.** Consider a CBN that does not satisfy Property $O_1$. Then it admits a non-directly observable node $X_i$ in its dependency graph, that is not the only element in the in-neighbors’ set of some other node. This implies one of the following two cases. Case 1: The node $X_i$ is a sink. Then clearly the CBN is not observable, as there is no way to determine $X_i(0)$.

Case 2: There exists some other node $X_j$ such that $\mathcal{N}_i(X_j)$ contains $X_i$ and at least one other node. Consider two initial conditions: one with all SVs equal to zero, and the second with all SVs equal to zero, except for $X_i(0)$ that is one. Then for both these conditions the value of every directly observable node will be zero for all time $k$, so these two states are indistinguishable. ■
Example 2. Consider the CBN:

\[
\begin{align*}
X_1(k+1) &= X_2(k)X_3(k), \\
X_2(k+1) &= X_1(k), \\
X_3(k+1) &= X_2(k), \\
Y_1(k) &= X_1(k).
\end{align*}
\] (4)

The dependency graph of this CBN does not satisfy Property \(O_1\) (see Fig. 1). Indeed, \(X_3\), which is a non-directly observable node, is not the only element in the in-neighbors set of some other node. It is clear that the two initial conditions \(X(0) = [0 \ 0 \ 0]'\) and \(X(0) = [0 \ 0 \ 1]'\) are indistinguishable, as for both conditions the output is \(Y_1(k) = 0\) for all \(k \geq 0\).

Definition 2. We say that a CBN has Property \(O_2\) if every cycle \(C\) in its dependency graph that is composed solely of non-directly observable nodes satisfies the following property. \(C\) includes a node \(X_i\) which is the only element in the in-neighbors set of some other node \(X_j\), i.e. \(N_{in}(X_j) = \{X_i\}\), and \(X_j\) is not part of the cycle \(C\).

Fact 2. If a CBN is observable then it satisfies Property \(O_2\).

Proof of Fact 2. Consider a CBN that does not satisfy Property \(O_2\). Then its dependency...
graph admits a cycle $C$, composed solely of non-directly observable nodes, and none of these nodes is the only element in the in-neighbors set of a node that is not part of the cycle $C$. Consider two initial conditions. One with all SVs equal to zero. The second with all SVs equal to zero, except for one SV that belongs to $C$, that is equal to one. Then these two initial conditions are indistinguishable.

**Example 3.** Consider the CBN:

\begin{align*}
X_1(k+1) &= X_2(k)X_4(k), \\
X_2(k+1) &= X_3(k), \\
X_3(k+1) &= X_2(k), \\
X_4(k+1) &= X_6(k), \\
X_5(k+1) &= X_4(k), \\
X_6(k+1) &= X_5(k), \\
Y_1(k) &= X_1(k).
\end{align*}

This CBN has Property $O_1$, but the cycle formed of $X_4, X_5, X_6$ implies that it does not satisfy Property $O_2$ (see Fig. 2). Here the two initial conditions $[0 \ 0 \ 0 \ 0 \ 0 \ 0]'$, and $[0 \ 0 \ 0 \ 1 \ 1 \ 1]'$ yield the same output sequence, namely, $Y_1(k) = 0$ for all $k \geq 0$, so this CBN is not observable.

A. Necessary and Sufficient Condition for Observability

Facts 1 and 2 provide two necessary conditions for observability of a CBN. The next result shows that the combination of these conditions provides a necessary and sufficient condition for observability.
Theorem 1. A CBN is observable iff it satisfies Properties $O_1$ and $O_2$.

To prove this, we introduce another definition and several auxiliary results. An observed path in the dependency graph is a non-empty ordered set of nodes such that: (1) the last element in the set is a directly observable node; and (2) if the set contains $p > 1$ elements, then for any $i < p$ the $i$-th element is a non-directly observable node, and is the only element in the in-neighbors set of node $i + 1$. Roughly speaking, an observed path corresponds to a shift register whose last cell is directly observable. Observed paths with non-overlapping nodes are called disjoint observed paths.

Proposition 1. Consider a CBN with a dependency graph $G$ that satisfies Properties $O_1$ and $O_2$. Then $G$ can be decomposed into disjoint observed paths, such that every vertex in the graph belongs to a single observed path (i.e., the union of the disjoint observed paths is a vertex cover of $G$).

Proof of Prop. 1. We give a constructive proof. Algorithm 1 below accepts such a graph $G$ and terminates after each vertex in the graph belongs to exactly one observed path.
Algorithm 1 Decompose the nodes of $G$ into disjoint observed paths

**Input:** Dependency graph $G$ of a CBN in the form (3) that satisfies Properties $O_1$ and $O_2$.

**Output:** A decomposition of $G$ into $m$ disjoint observed paths.

1: for $i = 1$ to $m$ do * every iteration builds a new path ending with $X_i$ *
2:     o-node $\leftarrow$ $X_i$ ; o-path $\leftarrow$ {$X_i$}
3:     if $|N_{in}(\text{o-node})| = 1$ then
4:         Let $v$ be such that {$v$} = $N_{in}(\text{o-node})$
5:         if $v$ does not belong to a previous path and is not directly observable then
6:             insert $v$ to o-path just before o-node
7:         o-node $\leftarrow$ $v$; goto 3
8:     else print o-path
9: end for

We now prove the correctness of this algorithm. To simplify the notation, let us say that $X_p$ points to $X_q$ if $p \neq q$ and $N_{in}(X_q) = X_p$, and denote this by $X_p \rightarrow X_q$. The special arrow indicates that the dependency graph includes an edge from $X_p$ to $X_q$ and that there are no other edges pointing to $X_q$.

If all the nodes are directly observable (i.e. if $m = n$) the algorithm will assign every node to a different observed path and this is correct. Thus, we may assume that $m < n$.

Pick a non directly observable node $X_j$. Then $m < j \leq n$. Our first goal is to prove the following result.

**Claim 1.** The algorithm outputs an observed path that contains $X_j$.

By Property $O_1$, there exists $k \neq j$ such that $X_j \rightarrow X_k$. We consider two cases.
Case 1. If $k \leq m$ then $X_k$ is directly observable and the algorithm will add $X_j$ to an observed path as it “traces back” from $X_k$ unless $X_j$ has already been included in some other observed path found by the algorithm. Thus, in this case Claim $\Box$ holds.

Case 2. Suppose that $k > m$, i.e. $X_k$ is non directly observable. By Property $O_1$, there exists $h \neq k$ such that $X_k \leftrightarrow X_h$, so $X_j \leftrightarrow X_k \leftrightarrow X_h$. If $h \leq m$ then we conclude as in Case 1 that the algorithm outputs an observed path that contains $X_j$. Thus, we only need to consider the case where as we proceed from $X_j$ using Property $O_1$ we never “find” a directly observable node. Then there exists a set of non directly observable nodes $X_{k_1}, \ldots, X_{k_\ell}$, with $k_1 = j$, such that

$$X_{k_1} \leftrightarrow X_{k_2} \leftrightarrow \cdots \leftrightarrow X_{k_\ell} \leftrightarrow X_{k_1}.$$  

This means that $X_j$ is part of a cycle $C$ of non directly observable nodes. By Property $O_2$, $C$ includes a node $X_{k_i}$ such that $X_{k_i} \leftrightarrow X_{s_1}$, where $X_{s_1}$ is not part of the cycle $C$. If $X_{s_1}$ is directly observable then we conclude that the algorithm will output an observed path that includes $X_j$. If $X_{s_1}$ is not directly observable then by Property $O_1$, there exists $s_2 \neq s_1$ such that $X_{s_1} \leftrightarrow X_{s_2}$. Furthermore, since every node in $C$ has in degree one, $X_{s_2} \not\in C$. Proceeding this way, we conclude that there exist $s_1, \ldots, s_p$ such that

$$X_{k_i} \leftrightarrow X_{s_1} \leftrightarrow X_{s_2} \leftrightarrow \cdots \leftrightarrow X_{s_p},$$

with $X_{s_p}$ a directly observable node. This means that the algorithm will output $X_j$ in an observed path as it traces back from $X_{s_p}$, unless it already included $X_j$ in another observed path. This completes the proof of Claim $\Box$

Summarizing, we showed that every non directly observable node $X_j$ is contained in an observed path produced by the algorithm. The fact that every $X_j$ will be in a single
observed path, and that the observed paths will be distinct is clear from the description of the algorithm. The algorithm’s correctness completes the proof of Prop. 1.

We can now prove Thm. 1.

Proof of Thm. 1. Consider the following set of statements.

(a) The CBN is observable;
(b) The dependency graph has Properties $O_1$ and $O_2$;
(c) There exists a decomposition of the dependency graph into a set of $m \geq 1$ disjoint observed paths $O^1, \ldots, O^m$, such that every vertex in the graph belongs to a single observed path.

We already know that (a) $\rightarrow$ (b). The correctness of Algorithm 1 implies that (b) $\rightarrow$ (c). If (c) holds then the values of the output of $O^i$ at times $0, \ldots, N_i - 1$ are the initial values of the SVs in $O^i$, organized in reverse order. Therefore it is possible to determine the initial condition of every SV in the CBN using the output sequence on $[0, \max_{i=1,\ldots,m}\{N_i\} - 1]$. Thus, the CBN is observable, so (c) $\rightarrow$ (a). We conclude that statements (a), (b), and (c) are all equivalent and this proves Thm. 1.

The proof of Thm. 1 implies the following.

Corollary 1. A CBN is observable iff its dependency graph can be decomposed into a set of disjoint observed paths.

The proof of Thm. 1 also provides a way to design an observer for an observable CBN. The procedure is as follows:

(a) Construct the dependency graph $G$;
(b) Apply Algorithm 1 to decompose the nodes of $G$ into a set of disjoint observed paths;
(c) Observe an output sequence of length equal to the longest observed path;
(d) Map the values observed at each output to the values of the SVs composing the observed paths, in reverse order, to obtain the initial state of the entire CBN.

Of course, once the initial condition is recovered, the known dynamics of the CBN allows to determine the state of the CBN at any time step.

**Example 4.** Consider the CBN:

\[
\begin{align*}
X_1(k+1) &= X_3(k), \\
X_2(k+1) &= X_5(k), \\
X_3(k+1) &= X_4(k), \\
X_4(k+1) &= X_2(k)X_3(k), \\
X_5(k+1) &= X_1(k)X_5(k), \\
Y_1(k) &= X_1(k), \\
Y_2(k) &= X_2(k).
\end{align*}
\]

The dependency graph of this CBN satisfies Properties $O_1, O_2$ (see Fig. 3), so Thm. 1 implies that it is observable, and decomposable to a set of disjoint observed paths. Applying Algorithm 1 to this CBN yields $O^1 = (X_4, X_3, X_1)$, $O^2 = (X_5, X_2)$, where $X_4 \mapsto X_3 \mapsto X_1$, $X_5 \mapsto X_2$.  

**B. Minimal Observability Problem**

We now use the conditions in Thm. 1 to efficiently solve Problem 1. We consider a CBN in the form (3), and the problem is to find a minimal number of additional SVs to measure so that the CBN becomes observable. Of course, if (3) is already observable then the solution to this problem is zero.
Algorithm 2 below solves this problem. For the sake of clarity, we provide here a high-level description of the algorithm. A more detailed description of the algorithm is given in the Appendix.
Algorithm 2 Solving the minimal observability problem: a high-level description

**Input:** A CBN (3) with \( n \) SVs and \( m \geq 0 \) outputs.

**Output:** A minimal set of SVs so that making these SVs directly observable yields an observable CBN.

1. generate the dependency graph \( G = (V, E) \)
2. create a list \( L_1 \) of all SVs that are not directly observable and are not the only element in the in-neighbors’ set of another node
3. create a list \( L_2 \) of all SVs that are not directly observable and are the only element in the in-neighbors set of another node
4. create a list \( L_C \) of cycles composed solely out of nodes in \( L_2 \)
5. for each cycle \( C \in L_C \), check if one of its elements appears as the only element in the in-neighbors set of another node that is not part of \( C \). If so, remove \( C \) from \( L_C \)
6. copy \( L_1 \) into a list \( I \); pick one element from each cycle \( C \in L_C \), and add these elements to \( I \)
7. return the list \( I \)

**Example 5.** Consider the CBN in Example 2. Applying Algorithm 2 to this CBN yields \( L_1 = \{X_3\} \), \( L_2 = \{X_2\} \), and \( L_C = \emptyset \). The algorithm thus returns \( L_1 = \{X_3\} \). Making this a directly observable node yields the CBN with dynamics (4) and outputs \( Y_i(k) = X_i(k), i = 1, 3 \). This CBN is indeed observable, and since the algorithm added a single output is is clear that this is a minimal solution.

Now consider the CBN in Example 3. Applying Algorithm 2 to this CBN yields \( L_1 = \emptyset \), \( L_2 = \{X_2, X_3, X_4, X_5, X_6\} \), and \( L_C = \{\{X_2, X_3\}, \{X_4, X_5, X_6\}\} \). Thus, Step 6 in Algorithm 2 yields, say, the output \( \{X_2, X_4\} \). Making these two nodes directly observable
yields the CBN with the dynamics in (5) and outputs \( Y_i(k) = X_i(k) \), \( i = 1, 2, 4 \). It is straightforward to verify that this CBN is indeed observable, and also that this addition of two outputs is a solution of the minimal observability problem.

\[ \square \]

**Theorem 2.** Algorithm 2 provides a solution to Problem [1]

**Proof of Thm.** [2] It is clear that the algorithm always terminates. Note that in step 2 of the algorithm all the SVs that do not satisfy Property \( O_1 \) are placed in the list \( L_1 \), and in steps 4 and 5 all the cycles that do not satisfy Property \( O_2 \) are placed in \( L_C \), and only those cycles. Step 6 initializes \( I \) as \( L_1 \) and then picks a representative of each cycle in \( L_C \) and then adds it to \( I \). Therefore, after making each of the SVs \( X_i(k) \), \( i \in I \), directly observable the modified CBN satisfies the conditions in Thm. [1] and hence is observable for some \( N \geq 0 \).

To prove that \( I \) is minimal, note that since \( L_C \) includes only nodes from \( L_2 \), it is clear that every cycle in \( L_C \) does not include nodes in \( L_1 \). Making a node from \( L_1 \) directly observable does not change the fact that every node in \( L_C \) does not satisfy Property \( O_2 \). Therefore, a minimal solution must be as composed by the algorithm. \[ \square \]

**Complexity Analysis of Algorithm 2:** Generating the dependency graph \( G \) requires going through \( n \) updating functions, and each function has at most \( n \) arguments, so the complexity of this step is \( O(n^2) \). The resulting graph satisfies \( |V| = n \), and \( |E| \leq n^2 \).

The complexity of each of the other steps in the algorithm is at most linear in \( |V|, |E| \), i.e., it is \( O(n^2) \). Summarizing, the complexity of the algorithm is linear in the length of the description of the CBN, and the latter is \( O(n^2) \).

Since the algorithm arbitrarily selects one element from each cycle in \( L_C \), it provides a specific solution to the minimal observability problem. It is straightforward to modify this
so that the algorithm will return the information needed to build all possible solutions. Note that if the algorithm returns an output list that is empty then the CBN is observable, so it can also be used to determine if a given CBN is observable or not.

IV. Minimal observability in random CBNs

Recall that we can represent a CBN via its dependency graph. In this section, we consider the case where the dependency graph is generated as a directed Erdős-Rényi graph [32], i.e., we fix the number of vertices \( n \) and a probability \( p \in [0, 1] \), and each possible directed edge in the graph is included with probability \( p \), independently of any other edge. We then study the minimal observability problem for such random CBNs via both simulations and analysis.

Simulations: We generated random dependency graphs with \( n = 1000 \) vertices for a set of \( p \) values. For every graph we ran the algorithm described here to obtain the solution \( k \) to the minimal observability problem, and calculated \( 100k/n \), i.e. the percentage of nodes that must be added as observed nodes in order to make the CBN observable. For each value of \( p \) we averaged the minimal number of outputs required over 100 independent trials to obtain the average value \( s := \langle 100k/n \rangle \). The middle curve in Fig. 4 depicts \( s \) as a function of \( p \).

It may be seen that \( s \) decreases sharply around \( p = 1/1000 (= 1/n) \), and achieves a minimum value \( s^* = 69.3\% \), suggesting that the optimal value for the probability is \( p^* = 1/n \). Similar results were found when simulating for other values of \( n \) in the range \([10^2, 10^4]\). In other words, even in the best possible random CBN, on average about 70% of the nodes must be added as outputs in order to obtain observability.

The polynomial complexity of Algorithm 2 makes it possible to solve the minimal observability problem even for large values of \( n \). For example, for a graph with \( n = 1000 \)
the typical running time of the algorithm, implemented in MATLAB using a standard PC (Intel core i5 processor, 4GB RAM memory) is about 0.03 seconds. For $n = 10^4$ the running time is about 2.8 seconds.

We now show how the analytical results described in Section III and, in particular, the notion of observed path allow to analyze the random CBNs simulated here.

Analysis: For a dependency graph $G = (V, E)$, with $|V| = n$, let $k$ denote the minimal number of nodes that must be made directly measurable in order to make the CBN observable, i.e. the size of the solution to Problem I. We begin with deriving a lower and upper bound on $k$.

The key point in the analysis is the set of vertices that have in-degree one and no self-loops. Denote this set by $W$. For the random graph described above the probability of a node to belong to $W$ is:

$$q(p) := (n - 1)p(1 - p)^{n-1},$$
so \( E(|W|) = q(p)n \).

The results in Section III imply that every node in \( V \setminus W \) can only be the first node in an observed path. Since the number of observed paths is equal to the number of outputs, \( k \geq n - |W| \). The exact number of outputs needed to achieve observability is determined by the topology of the graph. One optimal case is when all the vertices of \( W \) form one observed path starting with one of the nodes in \( V \setminus W \) yields the lower bound: \( k \geq n - |W| \). We conclude that

\[
(100/n)E(k) \geq 100(n - E(|W|))/n = \bar{s},
\]

with \( \bar{s}(p) := 100(1 - q(p)) \). The multiplication by \( 100/n \) is used to obtain the results in terms of percent.

To derive an upper bound on \( k \), note that every node \( v \in W \) which is located at the beginning of an observed path implies a needed output (in addition to those necessary for the nodes of \( V \setminus W \)). By the definition of \( W \), its elements have in-degree one. Therefore, a node \( v \in W \) might be located at the beginning of an observed path only when there is a cycle formed exclusively of nodes in \( W \). Let \( C(W) \) denote the number of cycles composed solely out of vertices belonging to \( W \). Then we conclude that \( k \leq n - |W| + C(W) \). Since the nodes of \( W \) do not have self-loops, the smallest possible cycle includes two vertices. Hence, \( C(W) \leq |W|/2 \) and this yields \( k \leq n - |W|/2 \). Thus,

\[
(100/n)E(k) \leq 100(n - \frac{E(|W|)}{2})/n = \bar{s},
\]

where \( \bar{s}(p) := 100(1 - \frac{q(p)}{2}) \).

\(^1\) One case where this bound is (almost) tight is when \( V = W \), with all the nodes forming a cycle, as in this case, \( k = 1 \).
Fig. 4 depicts $s(p)$ and $\bar{s}(p)$ as a function of $p$. It may be seen that these functions indeed provide a lower and upper bound for the value $s$ obtained in the simulations.

We now turn to determine the optimal value $p^*$, i.e. the edge probability that yields, on average, the smallest solution to the minimal observability problem. Clearly, the probability that maximizes $q(p)$ minimizes the bounds for $s$. It is straightforward to verify that $q(p)$ admits a unique maximum at

$$p^* := n^{-1},$$

and this agrees well with the simulation results.

The simulation results show that for the optimal value $p = p^*$ the corresponding minimal value is $s^* \approx 69.3\%$, that is, even in the optimal case about $70\%$ of the nodes must be observed in order to make the CBN observable. To explain this value, note that

$$q(p^*) = \left(1 - \frac{1}{n}\right)^n \approx e^{-1}.$$

Thus, for the optimal topology the percentage of outputs needed on average is:

$$\underline{s}(p) = 100(1 - e^{-1}) \approx 63.2\%,$$

(9)

Of course, this is a lower bound on the number of needed observation nodes, as there is no reason for the optimal topology to appear frequently in the random simulations. This analysis agrees well with the simulation results.

The graph of $s(p)$ in Fig. 4 shows a sharp rise near $p^* = 1/n$. To explain this, we
compute the second derivative of $q(p)$ at $p = p^*$, namely,

$$\frac{d^2 q(p)}{dp^2}|_{p=p^*} = -(n - 1)^2 (1 - p)^{n-3}(2 - np)|_{p=p^*}$$

$$= -(n - 1)^2 (1 - n^{-1})^{n-3}.$$

For $n \gg 1$ this yields

$$\left| \frac{d^2 q(p)}{dp^2} \right|_{p=p^*} = (n - 1)^2 (1 - n^{-1})^{n-3}$$

$$= (n - 1)^2 (1 - n^{-1})^{-3}(1 - n^{-1})^n$$

$$\approx \frac{n^3}{n - 1} e^{-1}.$$

This large value of the second derivative implies a rapid change in the slope of the curve near $p = p^*$. Roughly speaking, this means that outside a small interval of probability values around $1/n$ a very large number of outputs is needed to achieve observability. Again, this agrees well with the simulation results.

V. Extensions

We describe two simple extensions of the results above.

A. Observability of CBNs With Inputs

Consider CBNs with inputs, that is, conjunctive Boolean control networks (CBCNs). As noted in the introduction, there are several possible definitions for observability of BCNs with outputs (see, e.g., [16], [17]). For example, one definition of observability requires that for any two different initial conditions $a$ and $b$ there exists a control sequence (that may depend on $a$ and $b$) guaranteeing that the output sequences will be different.
A different possibility is to require that there exists a specific control sequence yielding an output sequence that distinguishes between any two different initial conditions.

For the case of CBCNs, it is clear that the “most informative” control sequence is when all inputs are one for all time. Indeed, any zero input may only obscure the value of an SV. Thus, we use the following definition.

**Definition 3.** A CBCN is said to be observable on $[0, N]$ if the CBN obtained by setting all inputs to one for all time $k \in [0, N - 1]$ is observable on $[0, N]$.

This means that we can first set all inputs $U_i(k)$ to one, simplify the network by using the fact that $\text{AND}(1, X_1, \ldots, X_k) = \text{AND}(X_1, \ldots, X_k)$, and then analyze observability and solve the minimal observability problem for the resulting CBN using the approach described in the previous section.

**B. Observability of CBNs With More General Output Functions**

Consider a CBN with outputs that are more general than in (3), namely,

$$Y_i(k) = g_i(X_1(k), \ldots, X_n(k)), \quad \forall i \in [1, m],$$

with every $g_i$ an AND operator.

Consider an augmented BN with $n + m$ SVs and $m$ outputs:

$$\begin{align*}
\bar{X}_i(k + 1) &= f_i(\bar{X}_1(k), \ldots, \bar{X}_n(k)), \quad \forall i \in [1, n], \\
\bar{X}_{n+j}(k + 1) &= g_j(\bar{X}_1(k), \ldots, \bar{X}_n(k)), \quad \forall j \in [1, m], \\
\bar{Y}_p(k) &= \bar{X}_{n+p}(k), \quad \forall p \in [1, m].
\end{align*}$$

(10)
This is a CBN in the form (3), and \( \bar{Y}_j(k+1) = \bar{X}_{n+j}(k+1) = g_j(\bar{X}_1(k), \ldots, \bar{X}_n(k)) = Y_j(k) \) for all \( j \) and \( k \). Thus, (11) is observable iff the CBN with outputs (10) is observable. In other words, any CBN with outputs in the form (10), where the \( g_i \)s are AND operators, can be reduced to the form (3), and then all of the above results on observability analysis and minimal observability can be applied.

VI. DISCUSSION

Observability is a fundamental property of dynamical systems, and it plays a crucial role in the design of observers, and full-state feedback controllers. When a system is not observable an important question is to determine a minimal set of measurements so that the system becomes observable. In the context of biological systems, this amounts to determining the minimal number of sensors to add so that the measurements will allow to determine the initial state of the biological system. This is important when the system includes a large number of SVs and adding sensors is costly in terms of time, money, etc.

We considered the minimal observability problem for CBNs. Using the dependency graph, we derived a necessary and sufficient condition for observability of CBNs, and an \( O(n^2) \)-time algorithm for solving the minimal observability problem for a CBN with \( n \) SVs. This also includes an explicit procedure which describes the construction of an observer for observable CBNs.

For LTI systems, it is well-known that controllability analysis and observability analysis are dual problems. For nonlinear dynamical systems, such as BNs, this is not true anymore. Indeed, it was recently shown that for CBNs the minimal controllability problem is NP-hard [30] (see also [31] for some related considerations), implying that there does not exist an algorithm solving it in polynomial time, unless \( \text{P}=\text{NP} \).
Although the necessary and sufficient conditions for controllability and observability of CBNs are quite analogous (see the definition of a controlled path in \cite{30}), a key difference is that adding a control input to a CBN in order to “improve” the controllability changes the dynamics of the CBN (and so changes its dependency graph). On the other-hand, adding an output in order to “improve” the observability does not change the dynamics. This is why the minimal observability problem is computationally more feasible than the minimal controllability problem.

The results here suggest several directions for further research. Recall that in undirected Erdős-Rényi graphs the size of the largest connected component undergoes a phase transition when the edge probability $p$ crosses the value $1/n$ \cite[Ch. 4]{32}. Our results show that $p^* = 1/n$ is the “best” value when considering the minimal observability problem for CBNs described by directed Erdős-Rényi graphs. It may be of interest to investigate if $p^*$ is also the “best” value when considering the minimal controllability problem for CBNs, and if other, more general, BNs demonstrate some special properties for this value of edge probability. Another natural direction for future research is the extension of the theoretical results described here to more general classes of BNs.

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Algorithm 2 Solving the minimal observability problem

**Input:** A CBN (3) with \( n \) SVs and \( m \geq 0 \) outputs.

**Output:** A minimal set of SVs so that making these SVs directly observable yields an observable CBN.

1. generate the dependency graph \( G = (V, E) \)
2. initialize lists \( L_1 \) and \( L_2 \) - each with \( n \) bits set to zero; initialize a matrix \( L_{pairs} \) of \( n \times 2 \) entries all set to zero
3. for \( i = 1 \) to \( n \) do * build \( L_2 \) and \( L_{pairs} \ *
   4. if \( |N_{in}(X_i)| = 1 \) and \( X_j \in N_{in}(X_i) \) is not directly observable and \( i \neq j \) then
   5. \( L_2(j) \leftarrow 1; \ L_{pairs}(i, 1) \leftarrow 1; \ L_{pairs}(i, 2) \leftarrow j \)
6. for \( i = m + 1 \) to \( n \) do
   * scan over non-directly observable nodes to build \( L_1 \ *
   7. if \( L_2(i) = 0 \) then \( L_1(i) \leftarrow 1 \)
8. copy the list \( L_2 \) into a list \( L_3 \)
9. for \( i = 1 \) to \( n \) do * build \( L_3 \ *
10. if \( L_3(i) = 0 \) then \( k \leftarrow i \)
   * \( X_k \) is directly observable or since it is in \( L_1 \) will become directly observable *
11. if \( L_{pairs}(k, 1) = 1 \) then
12. \( p \leftarrow L_{pairs}(k, 2) \)
13. \( L_{pairs}(k, 1) \leftarrow 0; \ L_{pairs}(k, 2) \leftarrow 0 \)
14. \( k \leftarrow p; \ L_3(k) \leftarrow 0 \)
   * trace back to \( X_p \) and remove it from \( L_3 \ *
15. goto \( 11 \)
16. generate a digraph \( \tilde{G} \) by removing from \( G \) all the vertices that are not in \( L_3 \) and all the incident edges
17. generate a list \( L_C \) of the cycles of \( \tilde{G} \)
18. copy \( L_1 \) into a list \( \mathcal{I} \); pick one element from each cycle in \( L_C \), and add to \( \mathcal{I} \)
19. return the list \( \mathcal{I} \)
The algorithm uses several data structures. $L_1$ is a list of $n$ bits. For any node $X_i$ in the dependency graph $L_1(i) = 1$ if $X_i$ is not directly observable and is not the only element in the in-neighbors’ set of some other node. Otherwise, $L_1(i) = 0$. $L_2$ is a list of $n$ bits with $L_2(i) = 1$ if $X_i$ is not directly observable and is the only element in the in-neighbors’ set of some other node. Otherwise, $L_2(i) = 0$. $L_{pairs}$ is a matrix of dimension $n \times 2$ such that $L_{pairs}(i, 1) = 1$ if node $X_i$ has in-degree one, i.e. $N_{in}(X_i) = \{X_j\}$ for some $j$, and in this case $L_{pairs}(i, 2) = j$. The list $L_3$ includes vertices that are part of a cycle, such that non of the elements composing the cycle appears as the only element in the in-neighbors’ set of another node that is not part of the cycle. This is created from $L_2$ using the auxiliary list $L_{pairs}$.

Steps 3-5 initialize $L_2$ and $L_{pairs}$. For convenience, denote the list of directly observable nodes by $L_{DON}$ (from the definition of the CBN (5) this is simply the first $m$ nodes). Using $L_2$ and $L_{DON}$, steps 6-7 form $L_1$ as $L_1 \leftarrow V \setminus \{L_{DON} \cup L_2\}$. Note that the sets $L_1$, $L_2$, and $L_{DON}$ are disjoint, with their union equal to the set of vertices of $G$. Steps 9-15 generate $L_3$ by a gradual reduction of $L_2$ using the matrix $L_{pairs}$, and a depth-first-search-like mechanism. From $L_3$ it is immediate to obtain the list of cycles $L_C$. Indeed, if $L_3$ is not empty at the end of the reduction process, then it is easy to verify that it contains exactly the list of vertices composing the desired cycles, but not yet divided to sets according to the different cycles. Steps 17-18 perform the division to the different sets. This can be implemented using a strongly connected components algorithm (which is linear in $|V|, |E|$, see, e.g., [33]), since every connected component in the digraph $\tilde{G}$ is a cycle.

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