On Generalized Minimum Distance Decoding
Thresholds for the AWGN Channel

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Abstract—We consider the Additive White Gaussian Noise channel with Binary Phase Shift Keying modulation. Our aim is to enable an algebraic hard decision Bounded Minimum Distance decoder for a binary block code to exploit soft information obtained from the demodulator. This idea goes back to Forney [1], [2] and is based on treating received symbols with low reliability as erasures. This erasing at the decoder is done using a threshold, each received symbol with reliability falling below the threshold is erased. Depending on the target overall complexity of the decoder this pseudo-soft decision decoding can be extended from one threshold \( T \) to \( z > 1 \) thresholds \( T_1 < \cdots < T_z \) for erasing received symbols with lowest reliability. The resulting technique is widely known as Generalized Minimum Distance decoding. In this paper we provide a means for explicit determination of the optimal threshold locations in terms of minimal decoding error probability. We do this for the one and the general \( z > 1 \) thresholds case, starting with a geometric interpretation of the optimal threshold location problem and using an approach from [3].

I. INTRODUCTION

The concept of concatenated codes was introduced by Forney in 1966 [1]. Concatenated codes consist of an inner and an outer code, a decoder for the concatenated code includes their associated decoders. Encoding is done such that the information block to be transmitted is first encoded using the outer code and then the symbols of the resulting outer codeword are encoded using the inner code. At the receiver side first the decoder for the inner code calculates estimates for the outer codeword symbols. Then, the decoder for the outer code tries to reconstruct the transmitted codeword utilizing the estimates from the inner decoder as inputs. In his original work, Forney proposed Generalized Minimum Distance (GMD) decoding, which extends simple single-trial decoding of concatenated codes to multiple decoding trials. More precisely, Forney specified GMD decoding for an integer \( z > (d - 1)/2 \) of decoding trials, where \( d \) is the minimum Hamming distance of the outer code. For smaller values of \( z \), Weber and Abdel-Ghaffar later introduced the term reduced GMD decoding [4]. GMD decoding relies on an outer error/erasure decoder and works as follows. In each decoding trial, an increasing set of most unreliable symbols obtained from the inner decoder are erased. The resulting intermediate word is fed into the outer error/erasure decoder, which calculates an outer codeword estimate. Potentially, each decoding trial results in a different outer codeword estimate so some means of selecting the “best” estimate needs to be provided.

Let the number of performed decoding trials be \( z \). We do not distinguish between reduced GMD decoding and full GMD decoding and allow \( z \) to be any non-zero natural number independent of the code parameters. In practice, erasing of the most unreliable symbols is accomplished using a set of real-valued thresholds \( \{T_1, \ldots, T_z\} \) with \( T_1 < \cdots < T_z \). If the reliability value of a symbol falls below threshold \( T_i \) in the \( i \)-th decoding trial, then this symbol is marked as erasure in this trial. The threshold version of GMD decoding was presented by Blokh and Zyablov [6].

In this paper we consider a special case of a code concatenation, i.e the case where the inner “code” is Binary Phase Shift Keying (BPSK) modulation and the outer code is a linear binary code with an error/erasure Bounded Minimum Distance (BMD) decoder. Such decoders are well-known for certain important code classes, e.g. for Bose–Chaudhuri–Hocquenghem (BCH) codes [5].

Our work is organized as follows. In Section II we give basic definitions and notations that are used in the remainder of the paper. Section III considers the most simple case of (reduced) GMD decoding, i.e. error/erasure BMD decoding with one single threshold. Its optimal location is derived using a geometric approach. Note that we use “optimal” as an abbreviation for “optimal in terms of minimal decoding error probability”. In Section IV we consider the general case of \( z > 1 \) thresholds before we finally wrap up the paper with conclusions and further research perspectives in Section V.

II. DEFINITIONS AND NOTATIONS

Assume an Additive White Gaussian Noise (AWGN) channel with BPSK modulation, let the transmitted symbols be \( x \in \{-1, +1\} \), i.e. the modulator performs for every transmitted binary value \( c \in \{0, 1\} \) the operation \( x = (-1)^c \) and the transmit signal power is fixed to \( E_s = 1 \). Hence, the standard deviation of the AWGN channel is \( \sigma = \sqrt{N_0/2} \). We define the probability that for given \( \sigma \) a transmitted symbol \( x \) results in a received symbol \( y \) within the real interval \([a, b]\) as

\[
p_a(a, b) := \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\chi - x)^2}{2\sigma^2}\right) d\chi.
\]
For simplicity we define the negative logarithmic probability
\[ l_\sigma(a, b) := -\ln(p_\sigma(a, b)). \]

As outer code we assume a linear binary \((n, k, d)\) code \(C\) with code length \(n\), dimension \(k\) and minimum Hamming distance \(d\). An error/erasure BMD decoder for \(C\) can decode error patterns with \(\tau\) erasures and \(\epsilon\) errors as long as
\[ 2\epsilon + \tau < d. \quad (1) \]

A codeword \(c = (c_0, \ldots, c_{n-1}) \in C\) is mapped to a vector \(x = (x_0, \ldots, x_{n-1}) \in \{-1, +1\}^n\) by the modulation function described above. At the receiver side, the vector \(y = (y_0, \ldots, y_{n-1}) \in \mathbb{R}^n\) is received. For each received symbol holds \(y_j = x_j + \xi\), where \(\xi\) is the realization of a Gaussian noise process with mean \(x_j\) and standard deviation \(\sigma\).

III. THE SINGLE THRESHOLD CASE

We start our considerations with the case of one single transmission of the all-zero codeword.

Let \(\phi_T\) be such that \(\phi_T(x) = y\) if and only if \(x\) is mapped to \(y\) by \(\phi_T\). The obvious extension of \(\phi_T\) to vectors is
\[ \phi_T(y) := (\phi_T(y_0), \ldots, \phi_T(y_{n-1})). \]

Note that since \(C\) is a linear code and the threshold location is symmetric, we can restrict our considerations in the following w.l.o.g. to the case \(\forall j = 0, \ldots, n-1 : x_j = +1\), i.e. transmission of the all-zero codeword.

Consider the probability \(P_{\sigma}\) that the decoder produces an error, i.e. the probability that it either returns no codeword or a wrong codeword. We use the abbreviated notation \(p_x := p_x(-T, T)\) and \(p_e := p_x(-\infty, -T)\) for the erasure and error probability, respectively. Similarly, we define the negative logarithms \(l_x := -\ln(p_x)\) and \(l_e := -\ln(p_e)\).

\[ P_{\sigma} = \sum_{\tau=0}^{n} \sum_{t_\tau=t_\tau}^{n-\tau} \left( \begin{array}{c} n \\ \tau, \epsilon, n-\tau-\epsilon \end{array} \right) \cdot p_x^{T_\tau} p_e^{n-T_\tau} (1-p_e)^{n-r-\epsilon}, \quad (2) \]

where \(t_\tau := \left\lfloor \frac{d-T}{\Delta} \right\rfloor\). For good channel conditions, i.e. small values of \(\sigma\), we obtain the approximation
\[ P_{\sigma} \approx \max_{0 \leq \tau \leq d} \left\{ \left( \begin{array}{c} n \\ \tau, l_\tau, n-\tau-\epsilon \end{array} \right) p_x^{T_\tau} p_e^{l_\tau} \right\}. \]

Note that the last term in \(2\) can be neglected since it is close to one. Transforming this approximation into negative logarithmic form we obtain
\[ -\ln(P_{\sigma}) \approx \min_{0 \leq \tau \leq d} \left\{ \tau l_x + t_\tau l_e - \ln(2) \left( n H(\tau/n) + (n-\tau) H \left( \frac{t_\tau}{n-\tau} \right) \right) \right\}, \]

where \(H(\cdot)\) denotes the binary entropy function. Since it only assumes values between 0 and 1 and \(l_x\) and \(l_e\) tend to infinity for small \(\sigma\), we can further approximate
\[ -\ln(P_{\sigma}) \approx \min_{0 \leq \tau \leq d} \left\{ \tau l_x + t_\tau l_e \right\}. \quad (3) \]

Now we return to the non-abbreviated notation and define the goal function
\[ g_\sigma(\tau, T) := \tau l_x(-T, T) + \frac{d-\tau}{2} l_e(-\infty, -T). \quad (4) \]

We omit the ceiling operation from \(t_\tau\) to obtain a function which is linear in \(\tau\). By means of \(3\) we observe that the minimum of the goal function over \(\tau\) approximates the negative logarithmic decoding error probability as long as the channel standard deviation \(\sigma\) is small. The behavior of the goal function for several thresholds is depicted in Figure I. The number of erasures \(\tau\) is spread on the abscissa and each straight line represents one threshold \(0 \leq T \leq 1 = E_x\), the minimum of each straight line represents the approximated negative logarithmic error probability for this specific threshold. The decoder’s aim is to select the threshold \(T\) such that the minimum is maximised since this yields the minimal decoding error probability.

The following theorem provides a necessary and sufficient criterion for the optimal high-SNR threshold \(T_{\sigma}\).

**Theorem 1** For good channel conditions, i.e. small channel standard deviation \(\sigma\), \(T_{\sigma}\) is the optimal threshold if and only if the following equation is fulfilled.

\[ \sqrt{\frac{1}{P_{\sigma}(-\infty, -T_{\sigma})}} = \frac{1}{g_{\sigma}(0, T_{\sigma})}. \quad (5) \]

**Proof:** Since the goal function is linear in \(\tau\), it assumes its minimum at one of the two extremal points \(g_{\sigma}(0, T)\) and \(g_{\sigma}(d, T)\) which means that \(3\) reduces to
\[ -\ln(P_{\sigma}) \approx \min \{g_{\sigma}(0, T), g_{\sigma}(d, T)\}. \]

Let \(T_{\sigma}\) be such that \(g_{\sigma}(0, T_{\sigma}) = g_{\sigma}(d, T_{\sigma})\). Inserting the definition of the goal function shows that this is equivalent to
\[ p_{\sigma}(-\infty, -T_{\sigma})^{\frac{1}{2}} = p_{\sigma}(-T_{\sigma}, T_{\sigma}) \]

Assume that threshold \(T' \neq T_{\sigma}\) is optimal. This gives
\[ p_{\sigma}(-T', T') = p_{\sigma}(-T_{\sigma}, T_{\sigma}) + \Delta \quad \text{and} \quad p_{\sigma}(-\infty, -T') = p_{\sigma}(-\infty, -T_{\sigma}) - \Delta, \]

where \(\Delta > 0\) if \(T' > T_{\sigma}\) and \(\Delta < 0\) if \(T' < T_{\sigma}\). Since both
\[ p_{\sigma}(-\infty, -T_{\sigma}) + p_{\sigma}(-T_{\sigma}, T_{\sigma}) + p_{\sigma}(T_{\sigma}, \infty) = 1 \]

and
\[ p_{\sigma}(-\infty, -T') + p_{\sigma}(-T', T') + p_{\sigma}(T', \infty) = 1 \]
must be fulfilled. If we transform (5) back to the non-
logarithmic domain we obtain
\[ P_\sigma \approx \max \left\{ p_\sigma(-\infty, -T) \hat{\tau}, p_\sigma(-T, T) d \right\}. \quad (7) \]

By using threshold \( T' \), we increase one of the two expressions
in (7) and thereby also the maximum of both expressions. But
this means that the decoding error probability is increased and
thus \( T' \neq T_\sigma \) cannot be the optimal threshold. Hence, \( T_\sigma \) is
optimal and the statement of the theorem is proved.

Theorem 1 allows for the following geometric interpretation.
The optimal threshold \( T_\sigma \) is the specific threshold for which
the goal function is a perfectly horizontal line in Figure 1.

Obtaining an analytic solution for equation (5) is non–trivial
since it essentially means solving
\[ \left( \text{Erf} \left( \frac{T - 1}{\sqrt{2} \sigma} \right) + \text{Erf} \left( \frac{T + 1}{\sqrt{2} \sigma} \right) \right)^2 = 2 \text{Erfc} \left( \frac{T + 1}{\sqrt{2} \sigma} \right), \]
where
\[ \text{Erf}(\alpha) := \frac{2}{\sqrt{\pi}} \int_0^\alpha e^{-x^2} dx \]
is the error function and \( \text{Erfc}(\alpha) := 1 - \text{Erf}(\alpha) \) is its
complementary counterpart. However, using the well–known
approximation
\[ \text{Erfc}(\alpha) \approx \sqrt{\frac{2}{\pi \alpha}} e^{-\frac{\alpha^2}{2}} \]
from [1] which is good for \( \alpha > 1 \) we can at least for good
channel conditions (i.e. small standard deviation \( \sigma \)) obtain the
analytic solution
\[ T_\sigma := 3 + 3\sigma^2 - \sqrt{9\sigma^4 + \left( 18 - \ln \left( \frac{2\pi}{\sigma^2} \right) \right) \sigma^2 + 8}, \quad (8) \]

which approximates the optimal high–SNR threshold location
for given \( \sigma \). Figure 2 compares the numerical and the anal-
lytical optimal high–SNR threshold locations with the general
optimal threshold. Note that the analytic approximation is only
valid for high SNR values. This imposes no problem since the
numerically calculated threshold given by Theorem 1 is also
valid in the high SNR regime.

We can utilize the analytic optimal threshold location to
show the gain of single–threshold error/erasure BMD decoding
over errors–only decoding for good channel conditions. If
the optimal threshold is used, (7) allows to approximate the
decoding error probability by
\[ P_\sigma \approx p_\sigma(-\infty, -T_\sigma) \hat{\tau}. \]

It is further well–known that the error probability of errors–
only BMD decoding can be approximated by
\[ P_{\text{BMD}} \approx p_\sigma(-\infty, 0) \hat{\tau}. \]

Now we let \( \sigma \to \infty \). From (8) we get \( T_\sigma = 3 - 2\sqrt{2} \). We can
then solve
\[ p_{\sigma_1}(-\infty, -3 + 2\sqrt{2}) \hat{\tau} = p_{\sigma_2}(-\infty, 0) \hat{\tau} \iff \]
\[ \text{Erfc} \left( \frac{2 \sqrt{2} - 1}{\sigma_1} \right) = \text{Erfc} \left( \frac{1}{\sqrt{2}\sigma_2} \right) \iff \]
\[ \sigma_1 = 2\sqrt{2} \left( \sqrt{2} - 1 \right) \sigma_2 \]
to see that the gain is 20 log_{10} \( 2\sqrt{2} \left( \sqrt{2} - 1 \right) \) \approx 1.4 dB.
This is in line with results obtained in the original works by
Forney [1], [2].

IV. THE GENERAL \( z \) THRESHOLDS CASE

We advance to the general case, where \( z > 1 \) thresholds
are used to determine which of the received symbols are
considered as unreliable and thus are erased. The situation
is depicted in Figure 3. We consider a set of \( z \) thresholds
The result of this approach can obviously be a list of codewords. In our simplified setting, where the inner code is BPSK modulation, the selection of the best guess from this result list is straightforward – it can be realized by applying the modulation operation to the binary symbols of all result list entries and choosing the one with the smallest Euclidean distance to the received vector \( y \). In the \( z > 1 \) thresholds case we denote the event that none of the list entries is within the received vector \( y \), that fall into the specific intervals.

\[
\begin{align*}
T := \{T_1, \ldots, T_z\} & \text{ fulfilling } 0 \leq T_1 < \cdots < T_z \leq 1 = E_s
\end{align*}
\]

and \( z \) trials of error/erasure decoding for the received vector \( y \) are performed. The first one with decoder input \( \phi_{T_1}(y) \), the second one with decoder input \( \phi_{T_2}(y) \) and so on, where the quantization–and–erasing function is

\[
\phi_T := \begin{cases} 
\mathbb{R} & \mapsto \{0, 1\} \cup \mathbb{X} \\
y_j & \mapsto \begin{cases} 1 & \text{if } y_j < -T_1 \\
0 & \text{if } y_j > T_1 \\
x & \text{if } -T_1 \leq y_j \leq T_1 
\end{cases}
\end{cases}
\]

where again \( i = 1, \ldots, z - 1 \). Some intervals and their corresponding abbreviated probability and number of symbols are depicted in Figure 3. With the previous definitions, the decoding error probability can be stated explicitly by

\[
P_\sigma = \sum_C \left( \left( t_1, t_c, t_r, L_1, \ldots, L_z, T_1, \ldots, T_z \right) \right) P(t_1) P(t_c) P(t_r) \prod_{i=1}^{z-1} \frac{L_i}{T_i} \frac{L_i}{T_i}
\]

where the sum is over all non-negative integers satisfying the two conditions

\[
C := \left[ \begin{array}{c} t_1 + t_c + t_r + \sum_{i=1}^{z-1} (L_i - \bar{T}_i) = n \\
\forall i = 1, \ldots, z : 2(t_1 + \sum_{\nu=1}^{i-1} L_\nu) + t_c + \sum_{\nu=1}^{i-1} (L_\nu + \bar{T}_\nu) \geq d \end{array} \right]
\]

The first condition in \( C \) is obvious, it simply states that the total number of received symbols must equal the code length \( n \). The second condition represents a decoding error for error/erasure BMD decoding of all input vectors \( \phi_{T_1}, i = 1, \ldots, z \). In this case, the number of errors for threshold \( T_i \in T \) is \( c_{T_i} = t_1 + \sum_{\nu=1}^{i-1} L_\nu \) and the number of erasures is \( \tau_{T_i} = t_c + \sum_{\nu=1}^{i-1} (L_\nu + \bar{T}_\nu) \) as can be easily seen by means of Figure 3. The second condition then follows from (9).

We can obtain an approximation of \( P_\sigma \) if we assume that the second condition is fulfilled with equality for all thresholds \( T_i \in T \). For \( i = 1, \ldots, z - 1 \) we can then substitute

\[
2(t_1 + \sum_{\nu=1}^{i-1} L_\nu) + t_c + \sum_{\nu=1}^{i-1} (L_\nu + \bar{T}_\nu) = d
\]

from

\[
2(t_1 + \sum_{\nu=1}^{z-1} L_\nu) + t_c + \sum_{\nu=1}^{z-1} (L_\nu + \bar{T}_\nu) = d
\]

and see that it holds

\[
\forall i = 1, \ldots, z - 1 : L_i = \bar{T}_i.
\]

Obeying this equality we obtain the new condition

\[
C^* := \left[ 2t_1 + t_c + 2 \sum_{i=1}^{z-1} L_i = d \right]
\]

for the sum in (9).

For good channel conditions, i.e. small values of the channel standard deviation \( \sigma \), the decoding error probability can be approximated by

\[
P_\sigma \approx \max_{C^*} \left\{ p(t_1) p(t_c) \prod_{i=1}^{z-1} \left( \frac{L_i}{T_i} \right) \right\}
\]

where \( i = 1, \ldots, z - 1 \). We also define the numbers of symbols

\[
\begin{align*}
p_i & := p_\sigma(-\infty, -T_2) \quad \text{and} \quad l_i := -\ln(p_i), \\
p_c & := p_\sigma(-T_1, T_1) \quad \text{and} \quad l_c := -\ln(p_c), \\
p_r & := p_\sigma(T_2, \infty) \quad \text{and} \quad l_r := -\ln(p_r), \\
p_1 & := p_\sigma(-T_{i+1}, -T_i) \quad \text{and} \quad l_i := -\ln(p_i), \\
\bar{p}_i & := p_\sigma(T_i, T_{i+1}) \quad \text{and} \quad \bar{l}_i := -\ln(\bar{p}_i),
\end{align*}
\]

In support of a dense notation we define the following abbreviated probabilities and their negative logarithmic counterparts.
The term $p^{r}_{i}$ in (9) can be neglected since for small $\sigma$ it is close to one. Furthermore, (10) is used to group the coefficients of the product under a single exponent. By transforming (11) into negative logarithmic form we obtain

$$- \ln(P_\sigma) \approx \min_{C^*} \left\{ t_{i}l_{i} + t_{c}l_{c} + \sum_{i=1}^{z-1} L_i (L_i + T_i) \right\},$$  \hspace{1cm} (12)

which contains the goal function

$$g_\sigma(t_i, t_c, L_1, \ldots, L_z-1, T_1, \ldots, T_z) := t_{i}l_{i} + t_{c}l_{c} + \sum_{i=1}^{z-1} L_i (L_i + T_i).$$  \hspace{1cm} (13)

The following theorem, whose proof exploits the linearity of the goal function in $t_i, t_c, L_1, \ldots, L_z$, provides a necessary and sufficient criterion for the optimal set of thresholds $T_\sigma$.

**Theorem 2** For good channel conditions, i.e. small channel standard deviation $\sigma$, $T_\sigma := \{ T_1, \sigma, \ldots, T_z, \sigma \}$ is the optimal set of thresholds if and only if the following system of equations is fulfilled.

$$\sqrt{p_\sigma(-\infty, -T_z, \sigma)} = p_\sigma(-T_{1, \sigma}, T_{1, \sigma}),$$

$$p_\sigma(-T_{1, \sigma}, T_{1, \sigma}) = \sqrt{p_\sigma(-T_{2, \sigma}, -T_{1, \sigma})p_\sigma(T_{1, \sigma}, T_{2, \sigma})}$$

and

$$\forall i = 1, \ldots, z - 2 :$$

$$p_\sigma(-T_{i+1, \sigma}, -T_{i, \sigma})p_\sigma(T_{i, \sigma}, T_{i+1, \sigma}) = p_\sigma(-T_{i+2, \sigma}, -T_{i+1, \sigma})p_\sigma(T_{i+1, \sigma}, T_{i+2, \sigma}).$$

**Proof:** Due to the linearity of the goal function in $t_i, t_c, L_1, \ldots, L_z$, it assumes its minimum at one of the extremal points given by condition $C^*$, i.e. (12) reduces to

$$- \ln(P_\sigma) \approx \min \left\{ g_\sigma \left( \frac{d}{2}, 0, 0, \ldots, 0, T_1, \ldots, T_z \right), \right.$$

$$g_\sigma(0, d, 0, \ldots, 0, T_1, \ldots, T_z),$$

$$g_\sigma(0, 0, d, \ldots, 0, T_1, \ldots, T_z),$$

$$\ldots$$

$$g_\sigma(0, 0, 0, \ldots, d, T_1, \ldots, T_z) \right\}. \hspace{1cm} (14)$$

Let $T_\sigma$ be the set of thresholds such that the value of the goal function is equal at all extremal points. Returning to the non-logarithmic representation, (14) becomes

$$P_\sigma \approx \max \left\{ p^{\frac{d}{2}}, p^{\frac{d}{2}}c, p^{\frac{d}{2}}cT_1^2, \ldots, p^{\frac{d}{2}}cT_z^2 \right\}. \hspace{1cm} (15)$$

Let $T'$ be a set of thresholds where at least one threshold is different than in $T_\sigma$. Assume that $T'$ is optimal. The only possible way for $T'$ to decrease $P_\sigma$ would be to decrease all terms in (15) simultaneously. This is impossible since the probabilities necessarily sum up to one, hence $T_\sigma$ is the optimal set of thresholds and the statement is proved.

V. CONCLUSIONS AND OUTLOOK

In this paper we considered a special case of (reduced) GMD decoding, i.e. transmission over an AWGN channel, BPSK modulation and error/erasure BMD decoding of a binary code. Starting from the single threshold case where only one decoding trial is performed, we generalized our considerations to the the $z > 1$ thresholds case. For both cases, we derived thresholds for erasing unreliable received symbols, that are optimal in terms of the achievable minimal decoding error probability. To simplify usage of our results in practical applications we gave the approximated analytic threshold location for the single threshold case.

We showed that a gain of 1.4 dB over errors-only BMD decoding can be achieved with single-trial error/erasure decoding. We did not address the error probability of GMD decoding with $z > 1$ thresholds in this paper. However, Forney showed that for good channel conditions the gain over error-residual decoding is approximately 3 dB if $z > (d - 1)/2$, i.e. in case of full GMD decoding.

Our work on the subject is continued with the goal to generalize the considerations from this paper to concatenated codes where the inner code is a binary block code and the outer code is a (potentially interleaved) Reed-Solomon code.

REFERENCES

[1] G. D. Forney, *Concatenated Codes*. Cambridge, MA, USA: M.I.T. Press, 1966.

[2] G. D. Forney, “Generalized minimum distance decoding,” *IEEE Trans. Inform. Theory*, vol. IT-12, pp. 125–131, April 1966.

[3] V. V. Zyablov, “Analysis of correcting properties of iterated and concatenated codes,” in *Transmission of Digital Information by Channels with Memory*, pp. 76–85, Moscow: Nauka, 1970. In Russian.

[4] J. H. Weber and K. A. S. Abdel-Ghaffar, “Reduced GMD decoding,” *IEEE Trans. Inform. Theory*, vol. IT-49, pp. 1013–1027, April 2003.

[5] R. E. Blahut, *Algebraic Codes for Data Transmission*. Cambridge: Cambridge University Press, first ed., 2003. ISBN 0-521-55374-1.

[6] E. L. Blokh and V. V. Zyablov, *Linear Concatenated Codes*. Nauka, 1982. In Russian.