SUPERCONVERGENCE OF $C^0-Q^k$ FINITE ELEMENT METHOD FOR ELLIPTIC EQUATIONS WITH APPROXIMATED COEFFICIENTS

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Abstract. We prove that the superconvergence of $C^0-Q^k$ finite element method at the Gauss Lobatto quadrature points still holds if variable coefficients in an elliptic problem are replaced by their piecewise $Q^k$ Lagrange interpolant at the Gauss Lobatto points in each rectangular cell. In particular, a fourth order finite difference type scheme can be constructed using $C^0-Q^2$ finite element method with $Q^2$ approximated coefficients.

Key words. Superconvergence, fourth order finite difference, variable coefficient Poisson equation, Gauss Lobatto points, approximated coefficients

AMS subject classifications. 65N30, 65N15, 65N06

1. Introduction.

1.1. Superconvergence of $C^0-Q^k$ finite element method. Consider solving a variable coefficient Poisson equation

\begin{equation}
- \nabla \cdot (a \nabla u) = f, \ a(x, y) > 0
\end{equation}

with homogeneous Dirichlet boundary conditions on a rectangular domain $\Omega$. Assume that the coefficient $a(x, y)$ and the solution $u(x, y)$ are sufficiently smooth. Let the Sobolev space be denoted by $W^{k,p}(\Omega)$ with the norm

$$
\|u\|_{k,p,\Omega} = \left( \sum_{i+j \leq k} \int_{\Omega} \left| \partial_x^i \partial_y^j u(x, y) \right|^p dxdy \right)^{1/p}.
$$

For $p = 2$, let $H^k(\Omega) = W^{k,2}(\Omega)$ and $\| \cdot \|_{k,\Omega} = \| \cdot \|_{k,2,\Omega}$. The subindex $\Omega$ will be omitted when there is no confusion, e.g., $\|u\|_0$ denotes the $L^2(\Omega)$-norm and $\|u\|_1$ denotes the $H^1(\Omega)$-norm. The variational form is to find $u \in H^1_0(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \}$ satisfying

\begin{equation}
A(u, v) = (f, v), \ \forall v \in H^1_0(\Omega),
\end{equation}

where $A(u, v) = \int_{\Omega} a \nabla u \cdot \nabla v dxdy$, $(f, v) = \int_{\Omega} f v dxdy$. Consider a rectangular mesh with mesh size $h$. Let $V^h_0 \subseteq H^1_0(\Omega)$ be the continuous finite element space consisting of piecewise $Q^k$ polynomials (i.e., tensor product of piecewise polynomials of degree $k$), then the $C^0-Q^k$ finite element solution of (1.2) is defined as $u_h \in V^h_0$ satisfying

\begin{equation}
A(u_h, v_h) = (f, v_h), \ \forall v_h \in V^h_0.
\end{equation}

Standard error estimates are $\|u - u_h\|_1 \leq Ch^k \|u\|_{k+1}$ and $\|u - u_h\|_0 \leq Ch^{k+1} \|u\|_{k+1}$ [8]. At some points the finite element solution or its derivatives have higher order accuracy, which is called superconvergence. Douglas and Dupont first proved that

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the $C^0$ Galerkin approximation of one-dimensional boundary problems using piecewise polynomial functions has $O(h^{2k})$ convergence at the knots \cite{11, 12},

\begin{equation}
\max_{(x,y) \in T_h} |(u_h - u)(x)| \leq Ch^{2k} \|u\|_{k+1},
\end{equation}

where $k$ is the polynomial degree and $T_h$ denote the set of cell ends in an one dimensional mesh. In \cite{12}, (1.4) was proven to be the best possible convergence rate. For $k \geq 2$, $O(h^{k+1})$ for the derivatives at Gauss quadrature points and $O(h^{k+2})$ for functions values at Gauss-Lobatto quadrature points were proven in \cite{15, 4, 2}.

For two dimensional cases, in \cite{13} Douglas et al. showed the superconvergence at the knots for the constant coefficient case $a \equiv 1$,

\begin{equation}
\max_{(x,y) \in T_h} |(u - u_h)(x,y)| \leq Ch^{k+2}, \quad k \geq 2,
\end{equation}

where $T_h$ is the set of vertices of all rectangular cells in a two dimensional rectangular mesh. Namely, the convergence rate at the knots is one order higher than the rate globally. For the constant coefficient case, Chen and Hu \cite{7} further proved for $k \geq 2$,

\begin{equation}
\left( h^2 \sum_{(x,y) \in T_h} |u(x,y) - u_h(x,y)|^p \right)^{1/p} \leq Ch^{2k} \|u\|_{2k,p,\Omega}, \quad 2 \leq p < \infty,
\end{equation}

\begin{equation}
\max_{(x,y) \in T_h} |u(x,y) - u_h(x,y)| \leq Ch^{2k} \ln h \|u\|_{2k,\infty,\Omega}.
\end{equation}

For the multi-dimensional variable coefficient case, when discussing the superconvergence of derivatives, it can be reduced to the Laplacian case. Superconvergence of tensor product elements for the Laplacian case can be established by extending one-dimensional results \cite{13, 19}. The superconvergence of function values in rectangular elements for the variable coefficient case were studied thoroughly in \cite{6} by Chen with M-type projection polynomials and in \cite{16} by Lin and Yan with the point-line-plane interpolation polynomials. In particular, let $Z_0$ denote the set of tensor product of $(k+1)$-point Gauss-Lobatto quadrature points for all rectangular cells, then the following superconvergence of function values for $Q^k$ elements was shown in \cite{6}:

\begin{equation}
\left( h^2 \sum_{(x,y) \in Z_0} |u(x,y) - u_h(x,y)|^2 \right)^{1/2} \leq Ch^{k+2} \|u\|_{k+2}, \quad k \geq 2,
\end{equation}

\begin{equation}
\max_{(x,y) \in Z_0} |u(x,y) - u_h(x,y)| \leq Ch^{k+2} \ln h \|u\|_{k+2,\infty,\Omega}, \quad k \geq 2.
\end{equation}

1.2. Approximated coefficients by interpolation. For implementing finite element method (1.3), either some quadrature is used or the coefficient $a(x,y)$ is approximated by polynomials for computing $\int_{\Omega} au_h \psi_h \, dx dy$. In practice, the most convenient choice of quadrature for $Q^k$ element is the tensor product of $(k+1)$-point Gauss Lobatto quadrature, since the standard error estimates still hold \cite{10, 8} and the numerical solution can be uniquely represented by its point values at these quadrature points. Such a quadrature scheme can be denoted as finding $u_h \in V_0^k$ satisfying

\begin{equation}
A_h(u_h, v_h) = \langle f, v_h \rangle_h, \quad \forall v_h \in V_0^k,
\end{equation}

where $A_h(\psi_h, \psi_h)$ and $\langle f, \psi_h \rangle_h$ denote using tensor product of $(k+1)$-point Gauss Lobatto quadrature for integrals $A(u_h, v_h)$ and $\langle f, v_h \rangle$ respectively.
We are interested in whether superconvergence for function values can be established for $Q^k$ element when the computation of integrals is simplified. Superconvergence of function values in (1.8) can be observed in numerical tests. For onedimensional problems, it was proven in [12] that (1.4) still holds if $(k+1)$-point Gauss-Lobatto quadrature is used for $P^2$ element. Superconvergence of the gradient for using quadrature in bilinear forms was studied in [15]. For multidimensional problems, even though it is possible to show (1.6) holds for (1.3) with accurate enough quadrature, it is difficult to extend the superconvergence proof to (1.8) using only $(k+1)$-point Gauss Lobatto quadrature in the bilinear form.

In this paper, we will show that (1.6) still holds if the coefficient $a(x, y)$ is approximated by a $Q^k$ interpolation polynomial in each cell. For instance, consider $Q^2$ element in two dimensions, tensor product of 3-point Lobatto quadrature form nine uniform points on each cell. By point values of $a(x, y)$ at these nine points, we can obtain a $Q^2$ Lagrange interpolation polynomial on each cell. Let $a_I(x, y)$ and $f_I(x, y)$ denote the piecewise $Q^2$ interpolation of $a(x, y)$ and $f(x, y)$ respectively. Assume the mesh is fine enough so that $a_I(x, y) \geq C > 0$. Consider the following scheme using the approximated coefficients $a_I(x, y)$: find $\tilde{u}_h \in V_0^h$ satisfying

\begin{equation}
I(\tilde{u}_h, v_h) = \langle f, v_h \rangle_h, \quad \forall v_h \in V_0^h,
\end{equation}

where

\begin{equation}
I(\tilde{u}, v) = \iint_{\Omega} a_I \nabla \tilde{u} \cdot \nabla v dx dy.
\end{equation}

One can also simplify the computation of the right hand side by using $f_I(x, y)$, so we also consider the scheme to find $\tilde{u}_h$ satisfying

\begin{equation}
I(\tilde{u}_h, v_h) = \langle f_I, v_h \rangle_h, \quad \forall v_h \in V_0^h.
\end{equation}

In this paper we will show that (1.6) still holds for both schemes (1.9) and (1.10).

1.3. A fourth order accurate finite difference scheme. Similar to the scheme (1.8), only grid point values of $a(x, y)$ and $f(x, y)$ are needed in the scheme (1.9) or (1.10) thus they can be implemented as finite difference type schemes. Consider a uniform $n_x \times n_y$ grid for a rectangle $\Omega$ with grid points $(x_i, y_j)$ and grid spacing $h$, where $n_x$ and $n_y$ are both odd numbers as shown in Figure 1(a). Then there is a straightforward $(n_x - 1)/2 \times (n_y - 1)/2$ $Q^2$ elements mesh $\Omega_h$ so that Gauss-Lobatto points for all cells in $\Omega_h$ are exactly the finite difference grid points. By using the scheme (1.9) or (1.10) on the finite element mesh $\Omega_h$ shown in Figure 1(b), we obtain a fourth order finite difference scheme in the sense that $\tilde{u}_h$ is fourth order accurate in the discrete 2-norm at all grid points.

1.4. Contributions and organization of the paper. The objective of this paper is to study the superconvergence of $C^0.Q^k$ finite element method when the coefficient is replaced by its $Q^k$ interpolant. The schemes (1.9) and (1.10) correspond to the equation

\begin{equation}
- \nabla \cdot (a_I(x, y) \nabla \tilde{u}(x, y)) = f(x, y).
\end{equation}

At first glance, one might expect $(k+1)$-th order accuracy for a numerical method applying to (1.11) due to the interpolation error $a(x, y) - a_I(x, y) = O(h^{k+1})$. But as we will show in Section 4.1, the difference between exact solutions $u$ and $\tilde{u}$ to the two elliptic equations (1.1) and (1.11) is $O(h^{k+2})$ in $L^2(\Omega)$-norm under suitable
assumptions. Via the superconvergence of finite element method, we can show (1.9) and (1.10) are \((k + 2)\)-th order accurate finite difference type schemes. In particular, for \(Q^2\) element, this provides a novel approach to construct a fourth order accurate finite difference method solving (1.1) since only \(Q^2\) polynomials are needed, even for the coefficient \(a(x, y)\). Moreover, the stiffness matrix of (1.9) and (1.10) is positive semi-definite, which is desired in applications.

The scheme (1.9) or (1.10) is also an efficient implementation of \(C^0\)-\(Q^k\) finite element method since only \(Q^k\) coefficient are needed to retain the \((k + 2)\)-th order accuracy of function values at the Lobatto points. The discussion in this paper cannot explain the superconvergence in the more efficient implementation, scheme (1.8). Numerical tests suggest that the approximated coefficient scheme (1.10) is more accurate and robust compared to the quadrature scheme (1.8) in some cases.

The paper is organized as follows. In Section 2, we introduce the notations and review standard interpolation and quadrature estimates. In Section 3, we review the tools to establish superconvergence of function values in \(C^0\)-\(Q^k\) finite element method (1.3) with a complete proof. In Section 4, we prove the main result on the superconvergence of (1.9) and (1.10) in two dimensions with extensions to a general elliptic equation. All discussion in this paper can be easily extended to the three dimensional case. Numerical results are given in Section 5. Section 6 consists of concluding remarks.

2. Notations and preliminaries.

2.1. Notations. In addition to the notations mentioned in the introduction, the following notations will be used in the rest of the paper:

- \(n\) denotes the dimension of the problem. Even though we discuss everything explicitly for \(n = 2\), all key discussions can be easily extended to \(n = 3\). The main purpose of keeping \(n\) is for readers to see independence/cancellation of the dimension \(n\) in the proof of some important estimates.
- We only consider a rectangular domain \(\Omega\) with its boundary \(\partial\Omega\).
- \(\Omega_h\) denotes a rectangular mesh with mesh size \(h\). Only for convenience, we assume \(\Omega_h\) is an uniform mesh and \(e = [x_e - h, x_e + h] \times [y_e - h, y_e + h]\) denotes any cell in \(\Omega_h\) with cell center \((x_e, y_e)\). The assumption of an uniform mesh is not essential to the proof.
- \(Q^k(e) = \left\{ p(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{k} p_{ij} x^i y^j, (x, y) \in e \right\}\) is the set of tensor product of polynomials of degree \(k\) on a cell \(e\).
The following are commonly used tools and facts:

- \( V^h = \{ p(x, y) \in C^0(\Omega_h) : p|_e \in Q^k(e), \; \forall e \in \Omega_h \} \) denotes the continuous piecewise \( Q^k \) finite element space on \( \Omega_h \).
- \( V_0^h = \{ v_h \in V^h : v_h = 0 \text{ on } \partial \Omega \} \).
- The norm and seminorms for \( W^{k,p}(\Omega) \) and \( 1 \leq p < +\infty \), with standard modification for \( p = +\infty \):

\[
\| u \|_{k,p,\Omega} = \left( \sum_{i+j \leq k} \int_{\Omega} |\partial_x^i \partial_y^j u(x,y)|^p dx dy \right)^{1/p},
\]
\[
|u|_{k,p,\Omega} = \left( \sum_{i+j = k} \int_{\Omega} |\partial_x^i \partial_y^j u(x,y)|^p dx dy \right)^{1/p},
\]
\[
[u]_{k,p,\Omega} = \left( \int_{\Omega} |\partial_x^k u(x,y)|^p dx dy + \int_{\Omega} |\partial_y^k u(x,y)|^p dx dy \right)^{1/p}.
\]

Notice that \( [u]_{k+1,p,\Omega} = 0 \) if \( u \) is a \( Q^k \) polynomial.

- \( \| u \|_{k,\Omega}, |u|_{k,\Omega} \) and \( [u]_{k,\Omega} \) denote norm and seminorms for \( H^k(\Omega) = W^{k,2}(\Omega) \).
- When there is no confusion, \( \Omega \) may be dropped in the norm and seminorms.
- For any \( v_h \in V_h \), \( 1 \leq p < +\infty \) and \( k \geq 1 \),

\[
\| v_h \|_{k,p,\Omega} := \left[ \sum_e \| v_h \|_{k,p,e}^p \right]^{1/p}, \quad |v_h|_{k,p,\Omega} := \left[ \sum_e |v_h|_{k,p,e}^p \right]^{1/p}.
\]

- Let \( Z_{0,e} \) denote the set of \( (k+1) \times (k+1) \) Gauss-Lobatto points on a cell \( e \).
- \( Z_0 = \bigcup_e Z_{0,e} \) denotes all Gauss-Lobatto points in the mesh \( \Omega_h \).
- Let \( \| u \|_{2,Z_0} \) and \( \| u \|_{\infty,Z_0} \) denote the discrete 2-norm and the maximum norm over \( Z_0 \) respectively:

\[
\| u \|_{2,Z_0} = \left[ \frac{h^2}{(x,y) \in Z_0} |u(x,y)|^2 \right]^{1/2}, \quad \| u \|_{\infty,Z_0} = \max_{(x,y) \in Z_0} |u(x,y)|.
\]

- For a smooth function \( a(x,y) \), let \( a_I(x,y) \) denote its piecewise \( Q^k \) Lagrange interpolant at \( Z_{0,e} \) on each cell \( e \), i.e., \( a_I \in V^h \) satisfies:

\[
a(x,y) = a_I(x,y), \quad \forall (x,y) \in Z_0.
\]

- \( P^k(t) \) denotes the polynomial of degree \( k \) of variable \( t \).
- \( (f,v) \) denotes the inner product in \( L^2(\Omega) \):

\[
(f,v) = \int_{\Omega} f v dx dy.
\]

- \( (f,v)_h \) denotes the approximation to \( (f,v) \) by using \( (k+1) \times (k+1) \)-point Gauss Lobatto quadrature for integration over each cell \( e \).

The following are commonly used tools and facts:

- \( \hat{K} = [-1,1] \times [-1,1] \) denotes a reference cell.
- For \( v(x,y) \) defined on \( e \), consider \( \hat{v}(s,t) = v(sh + x_e, th + y_e) \) defined on \( \hat{K} \).
For $n$-dimensional problems, 
\[ h^{k-n/p}|v|_{k,p,e} = |\hat{v}|_{k,p,K}, \quad h^{k-n/p}|v|_{k,p,e} = [\hat{v}]_{k,p,K}, \quad 1 \leq p \leq \infty. \]

- Sobolev’s embedding in two and three dimensions: $H^2(\hat{K}) \hookrightarrow C^0(\hat{K})$.
- The embedding implies 
\[
\|\hat{f}\|_{0,\infty,\hat{K}} \leq C\|\hat{f}\|_{2,\hat{K}}, \forall \hat{f} \in H^k(\hat{K}), k \geq 2, \\
\|\hat{f}\|_{1,\infty,\hat{K}} \leq C\|\hat{f}\|_{1,2,\hat{K}}, \forall \hat{f} \in H^{k+1}(\hat{K}), k \geq 2.
\]
- Cauchy Schwarz inequalities:
\[
\sum_e \|u\|_{k,e} \|v\|_{k,e} \leq \left( \sum_e \|u\|_{k,e}^2 \right)^{1/2} \left( \sum_e \|v\|_{k,e}^2 \right)^{1/2}, \quad \|u\|_{k,1,e} = \mathcal{O}(h^{k+1})\|u\|_{2,1,e}.
\]
- Poincaré inequality: let $\bar{u}$ be the average of $u \in H^1(\Omega)$ on $\Omega$, then 
\[ |u - \bar{u}|_{0,p,\Omega} \leq C|\nabla u|_{0,p,\Omega}, \quad p \geq 1. \]
If $\bar{u}$ is the average of $u \in H^1(e)$ on a cell $e$, we have 
\[ |u - \bar{u}|_{0,p,e} \leq Ch|\nabla u|_{0,p,e}, \quad p \geq 1. \]
- For $k \geq 2$, the $(k + 1) \times (k + 1)$ Gauss-Lobatto quadrature is exact for integration of polynomials of degree $2k - 1 \geq k + 1$ on $\hat{K}$.
- Any polynomial in $Q^k(\hat{K})$ can be uniquely represented by its point values at $(k + 1) \times (k + 1)$ Gauss Lobatto points on $\hat{K}$, and it is straightforward to verify that the discrete $2$-norm $\|p\|_{2,\hat{K}}$ and $L^2(\Omega)$-norm $\|p\|_{0,\Omega}$ are equivalent for a piecewise $Q^k$ polynomial $p \in V^k$.
- Define the projection operator $\Pi_1 : \hat{u} \in L^1(\hat{K}) \to \hat{\Pi}_1 \hat{u} \in Q^1(\hat{K})$ by
\[
(2.1) \quad \iint_{\hat{K}} (\hat{\Pi}_1 \hat{u}) w \, dx \, dy = \iint_{\hat{K}} \hat{u} w \, dx \, dy, \forall w \in Q^1(\hat{K}).
\]
Notice that $\hat{\Pi}_1$ is a continuous linear mapping from $L^2(\hat{K})$ to $H^1(\hat{K})$ (or $H^2(\hat{K})$) since all degree of freedoms of $\hat{\Pi}_1 \hat{u}$ can be represented as a linear combination of $\int_{\hat{K}} \hat{u}(s,t)p(s,t) \, ds \, dt$ for $p(s,t) = 1, s, t, st$ and by Cauchy Schwarz inequality $|\iint_{\hat{K}} \hat{u}(s,t)p(s,t) \, ds \, dt| \leq \|\hat{u}\|_{0,2,\hat{K}} \|p\|_{0,2,\hat{K}} \leq C\|\hat{u}\|_{0,2,\hat{K}}$.

2.2. The Bramble-Hilbert Lemma. By the abstract Bramble-Hilbert Lemma in [3], with the result $\|v\|_{m,p,\Omega} \leq C(\|v\|_{0,p,\Omega} + [v]_{m,p,\Omega})$ for any $v \in W^{m,p}(\Omega)$ [18, 1], the Bramble-Hilbert Lemma for $Q^k$ polynomials can be stated as (see Exercise 3.1.1 and Theorem 4.1.3 in [9]):

**Theorem 2.1.** If a continuous linear mapping $\Pi : H^{k+1}(\hat{K}) \to H^{k+1}(\hat{K})$ satisfies $\Pi v = v$ for any $v \in Q^k(\hat{K})$, then 
\[
(2.2) \quad \|u - \Pi u\|_{k+1,\hat{K}} \leq C\|u\|_{k+1,\hat{K}}, \quad \forall u \in H^{k+1}(\hat{K}).
\]
Thus if $l(\cdot)$ is a continuous linear form on the space $H^{k+1}(\hat{K})$ satisfying $l(v) = 0, \forall v \in Q^k(\hat{K})$, then 
\[ |l(u)| \leq C\|l\|_{k+1,\hat{K}} \|u\|_{k+1,\hat{K}}, \quad \forall u \in H^{k+1}(\hat{K}), \]
where $\|l\|_{k+1,\hat{K}}$ is the norm in the dual space of $H^{k+1}(\hat{K})$. 

2.3. Interpolation and quadrature errors. For $Q^k$ element ($k \geq 2$), consider $(k+1) \times (k+1)$ Gauss-Lobatto quadrature, which is exact for integration of $Q^{2k-1}$ polynomials.

It is straightforward to establish the interpolation error:

**Theorem 2.2.** For a smooth function $a$, $|a - a_I|_{0, \infty, \Omega} = \mathcal{O}(h^{k+1})|a|_{k+1, \infty, \Omega}$.

Let $s_j, t_j$ and $w_j$ ($j = 1, \ldots, k+1$) be the Gauss-Lobatto quadrature points and weight for the interval $[-1, 1]$. Notice $\hat{f}$ coincides with its $Q^k$ interpolant $\hat{f}_I$ at the quadrature points and the quadrature is exact for integration of $\hat{f}_I$, the quadrature can be expressed on $K$ as

$$\sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \hat{f}(s_i, t_j)w_iw_j = \int_K \hat{f}_I(x, y)dx, dy,$$

thus the quadrature error is related to interpolation error:

$$\int_K \hat{f}(x, y)dx, dy - \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \hat{f}(s_i, t_j)w_iw_j = \int_K \hat{f}(x, y)dx, dy - \int_K \hat{f}_I(x, y)dx, dy.$$

We have the following estimates on the quadrature error:

**Theorem 2.3.** For $n = 2$ and a sufficiently smooth function $a(x, y)$, if $k \geq 2$ and $m$ is an integer satisfying $k \leq m \leq 2k$, we have

$$\int_e a(x, y)dx, dy - \int_e a_I(x, y)dx, dy = O(h^{m+2})|a|_{m, e} = O(h^{m+n})|a|_{m, \infty, e}.$$

**Proof.** Let $E(a)$ denote the quadrature error for function $a(x, y)$ on $e$. Let $\hat{E}(\hat{a})$ denote the quadrature error for the function $\hat{a}(s, t) = a(sh + x_e, th + y_e)$ on the reference cell $\hat{K}$. Then for any $\hat{f} \in H^m(\hat{K})$ ($m \geq k \geq 2$), since quadrature are represented by point values, with the Sobolev’s embedding we have

$$|\hat{E}(\hat{f})| \leq C|\hat{f}|_{0, \infty, \hat{K}} \leq C\|\hat{f}\|_{m, 2, \hat{K}}.$$

Thus $\hat{E}(\cdot)$ is a continuous linear form on $H^m(\hat{K})$ and $\hat{E}(\hat{f}) = 0$ if $\hat{f} \in Q^{m-1}(\hat{K})$. The Bramble-Hilbert lemma implies

$$|E(a)| = h^n|\hat{E}(\hat{a})| \leq Ch^n[\hat{a}]_{m, 2, \hat{K}} = O(h^{m+2})|a|_{m, 2, e} = O(h^{m+n})|a|_{m, \infty, e}.$$

**Theorem 2.4.** If $k \geq 2$, $(f, v_h) - \langle f, v_h \rangle_h = O(h^{k+2})\|f\|_{k+2}\|v_h\|_2$, $\forall v_h \in V^h$.

**Proof.** This result is a special case of Theorem 5 in [10]. For completeness, we include a proof. Let $\hat{E}(\cdot)$ denote the quadrature error term on the reference cell $\hat{K}$. Consider the projection (2.1). Let $\Pi_1$ denote the same projection on $e$. Since $\Pi_1$ leaves $Q^m(\hat{K})$ invariant, by the Bramble-Hilbert lemma on $\Pi_1$, we get $||\hat{v}_h - \Pi_1\hat{v}_h||_{1, \hat{K}} \leq ||\hat{v}_h - \Pi_1\hat{v}_h||_{1, \hat{K}} \leq C[\hat{v}_h]_{1, \hat{K}}$ thus $[\Pi_1\hat{v}_h]_{1, \hat{K}} \leq [\hat{v}_h]_{1, \hat{K}} + [\hat{v}_h - \Pi_1\hat{v}_h]_{1, \hat{K}} \leq C[\hat{v}_h]_{1, \hat{K}}$. By setting $w = \Pi_1v_h$ in (2.1), we get $[\Pi_1\hat{v}_h]_{0, \hat{K}} \leq [\hat{v}_h]_{0, \hat{K}}$. For $k \geq 2$, repeat the proof of Theorem 2.3, we can get

$$|\hat{E}(\hat{f}\Pi_1\hat{v}_h)| \leq C[\hat{f}\Pi_1\hat{v}_h]_{k+2, \hat{K}} \leq C([\hat{f}]_{k+2, \hat{K}}[\Pi_1\hat{v}_h]_{0, \infty, \hat{K}} + [\hat{f}]_{k+1, \hat{K}}[\Pi_1\hat{v}_h]_{1, \infty, \hat{K}}).$$
where the fact $[\Pi_1 \hat{v}_h]_{l,\infty,\hat{K}} = 0$ for $l \geq 2$ is used. The equivalence of norms over $Q^1(\hat{K})$ implies
\[
|\hat{E}(f \Pi_1 \hat{v}_h)| \leq C([\hat{f}]_{k+2,K} [\Pi_1 \hat{v}_h]_{0,K} + [\hat{f}]_{k+1,K} [\Pi_1 \hat{v}_h]_{1,K})
\leq C([\hat{f}]_{k+2,K} [\hat{v}_h]_{0,K} + [\hat{f}]_{k+1,K} [\hat{v}_h]_{1,K}).
\]

Next consider the linear form $\hat{f} \in H^k(\hat{K}) \rightarrow \hat{E}(\hat{v}_h - \Pi_1 \hat{v}_h)$. Due to the embedding $H^k(\hat{K}) \hookrightarrow C^0(\hat{K})$, it is continuous with operator norm $\leq C\|v_h - \Pi_1 \hat{v}_h\|_{0,K}$ since
\[
|\hat{E}(\hat{v}_h - \Pi_1 \hat{v}_h)| \leq C\|\hat{v}_h - \Pi_1 \hat{v}_h\|_{0,K} \leq C\|\hat{f}\|_{k,K} \|\hat{v}_h - \Pi_1 \hat{v}_h\|_{0,K}
\leq C\|\hat{f}\|_{k,K} \|\hat{v}_h - \Pi_1 \hat{v}_h\|_{0,K}.
\]

For any $\hat{f} \in Q^{k-1}(\hat{K})$, $\hat{E}(\hat{f} \hat{v}_h) = 0$. By the Bramble-Hilbert lemma, we get
\[
|\hat{E}(\hat{f}(\hat{v}_h - \Pi_1 \hat{v}_h))| \leq C[f]_{k,K} \|\hat{v}_h - \Pi_1 \hat{v}_h\|_{0,K} \leq C[f]_{k,K} [\hat{v}_h]_{2,K}.
\]

So on a cell $e$, we get
\[
E(f v_h) = h^n \hat{E}(\hat{f} \hat{v}_h) = C h^{k+2} (f)_{k+2,e} v_h|_{0,e} + (f)_{k+1,e} v_h|_{1,e} + (f)_{k,e} v_h|_{2,e}).
\]

Summing over $e$ and use Cauchy Schwarz inequality, we get the desired result. \[ QED \]

**Theorem 2.5.** For $k \geq 2$, $(f, v_h) - (f_l, v_h) = O(h^{k+2}) \|f\|_{k+2} \|v_h\|_2$, $\forall v_h \in V_h$.

**Proof.** Repeat the proof of Theorem 2.4 for the function $f - f_l$ on a cell $e$, with the fact $[f_l]_{k+1,p,e} = [f_l]_{k+2,p,e} = 0$, we get
\[
E((f - f_l) v_h) = C h^{k+2} (f)_{k+2,e} v_h|_{0,e} + (f)_{k+1,e} v_h|_{1,e} + (f - f_l)_{k,e} v_h|_{2,e}).
\]

By (2.2) on the Lagrange interpolation operator and the fact $[f-f_l]_{k,e} \leq \|f-f_l\|_{k+1,e}$, we get $[f-f_l]_{k,e} \leq C h [f]_{k+1,e}$. Notice that $\langle f - f_l, v_h \rangle_h = 0$, we get
\[
(f, v_h) - (f_l, v_h) = (f - f_l, v_h) - (f - f_l, v_h) = O(h^{k+2}) \|f\|_{k+2} \|v_h\|_2, \forall v_h \in V_h.
\]

**3. The M-type Projection.** To establish the superconvergence of $C^0,Q^k$ finite element method for multi-dimensional variable coefficient equations, it is necessary to use a special polynomial projection of the exact solution, which has two equivalent definitions. One is the M-type projection used in [5, 6]. The other one is the point-line-plane interpolation used in [17, 16].

For the sake of completeness, we review the relevant results regarding M-type projection, which is a more convenient tool. Most results in this section were considered and established for more general rectangular elements in [6]. For simplicity, we use some simplified proof and arguments for $Q^k$ element in this section. We only discuss the two dimensional case and the extension to three dimensions is straightforward.

**3.1. One dimensional case.** The $L^2$-orthogonal Legendre polynomials on the reference interval $\hat{K} = [-1, 1]$ are given as
\[
l_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k : l_0(t) = 1, l_1(t) = t, l_2(t) = \frac{1}{2} (3t^2 - 1), \ldots
\]
Define their antiderivatives as M-type polynomials:

\[ M_{k+1}(t) = \frac{1}{2^{k+1}k!} \frac{d^{k-1}}{dt^{k-1}}(t^2-1)^k : M_0(t) = 1, M_1(t) = t, M_2(t) = \frac{1}{2}(t^2-1), M_3(t) = \frac{1}{2}(t^3-t), \ldots \]

which satisfy the following properties:

- \( M_k(\pm 1) = 0, \forall k \geq 2. \)
- If \( j - i \neq 0, \pm 2, \) then \( M_j(t) \perp M_i(t), \) i.e., \( \int_{-1}^{1} M_j(t)M_i(t)dt = 0. \)
- Roots of \( M_k(t) \) are the \( k \)-point Gauss-Lobatto quadrature points for \([-1, 1]\).

Since Legendre polynomials form a complete orthogonal basis for \( L^2([-1, 1]) \), for any \( f(t) \in H^1([-1, 1]) \), its derivative \( f'(t) \) can be expressed as Fourier-Legendre series

\[ f'(t) = \sum_{j=0}^{\infty} b_{j+1}l_j(t), \quad b_{j+1} = (j + \frac{1}{2}) \int_{-1}^{1} f'(t)l_j(t)dt. \]

Define the M-type projection

\[ f_k(t) = \sum_{j=0}^{k} b_jM_j(t), \]

where \( b_0 = \frac{f(1)+f(-1)}{2} \) is determined by \( b_1 = \frac{f(1)-f(-1)}{2} \) to make \( f_k(\pm 1) = f(\pm 1). \)

Since the Fourier-Legendre series converges in \( L^2 \), by Cauchy Schwarz inequality,

\[ \lim_{k \to \infty} |f_k(t) - f(t)| = \lim_{k \to \infty} \int_{-1}^{1} |f'_k(x) - f'(x)| dx \leq \lim_{k \to \infty} \sqrt{2}||f'_k(t) - f'(t)||_{L^2([-1, 1])} = 0. \]

We get the M-type expansion of \( f(t) : f(t) = \lim_{k \to \infty} f_k(t) = \sum_{j=0}^{\infty} b_jM_j(t). \) The remainder \( R_k(t) \) of M-type projection is

\[ R[f]_k(t) = f(t) - f_k(t) = \sum_{j=k+1}^{\infty} b_jM_j(t). \]

The following properties are straightforward to verify:

- \( f_k(\pm 1) = f(\pm 1) \) thus \( R_k(\pm 1) = 0 \) for \( k \geq 1. \)
- \( R[f]_k(t) \perp v(t) \) for any \( v(t) \in P^{k-2}(t) \) on \([-1, 1]\), i.e., \( \int_{-1}^{1} R[f]_k vdt = 0. \)
- \( R[f]^{\prime}_k(t) \perp v(t) \) for any \( v(t) \in P^{k-1}(t) \) on \([-1, 1]\).
- For \( j \geq 2, b_j = (j - \frac{1}{2})[f(t)l_{j-1}(t)]_{-1}^{1} - \int_{-1}^{1} f(t)l'_j(t-1)(t)dt. \)
- For \( j \leq k, |b_j| \leq C_k\|f\|_{0, \infty, K}. \)
- \( \|R[f]_k(t)\|_{0, \infty, K} \leq C_k\|f\|_{0, \infty, K}. \)

3.2. Two dimensional case. Consider a function \( \hat{f}(s, t) \in H^2(\hat{K}) \) on the reference cell \( \hat{K} = [-1, 1] \times [-1, 1], \) it has the expansion

\[ \hat{f}(s, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \hat{b}_{i, j}M_i(s)M_j(t), \]
where
\[
\hat{b}_{0,0} = \frac{1}{4} [\hat{f}(1, -1) + \hat{f}(-1, 1) + \hat{f}(1, 1) + \hat{f}(1, -1)],
\]
\[
\hat{b}_{0,j}, \hat{b}_{1,j} = \frac{2i - 1}{4} \int_{-1}^{1} [\hat{f}_j(t) \pm \hat{f}_j(t)] dt, \quad j \geq 1,
\]
\[
\hat{b}_{i,0}, \hat{b}_{i,1} = \frac{2i - 1}{4} \int_{-1}^{1} [\hat{f}_i(s, 1) \pm \hat{f}_i(s, -1)] ds, \quad i \geq 1,
\]
\[
\hat{b}_{i,j} = \frac{(2i - 1)(2j - 1)}{4} \int_{\mathcal{K}} \hat{f}_{st}(s, t) l_{i-1}(s) l_{j-1}(t) ds dt, \quad i, j \geq 1.
\]

Define the \(Q^k\) M-type projection of \(\hat{f}\) on \(\hat{K}\) and its remainder as
\[
\hat{f}_{k,k}(s, t) = \sum_{i=0}^{k} \sum_{j=0}^{k} \hat{b}_{i,j} M_i(s) M_j(t), \quad R[\hat{f}]_{k,k}(s, t) = \hat{f}(s, t) - \hat{f}_{k,k}(s, t).
\]

For \(f(x, y)\) on \(e = [x_e - h, x_e + h] \times [y_e - h, y_e + h]\), let \(\hat{f}(s, t) = f(sh + x_e, th + y_e)\) then the \(Q^k\) M-type projection of \(f\) on \(e\) and its remainder are defined as
\[
f_{k,k}(x, y) = \hat{f}_{k,k}(\frac{x - x_e}{h}, \frac{y - y_e}{h}), \quad R[f]_{k,k}(x, y) = f(x, y) - f_{k,k}(x, y).
\]

**Theorem 3.1.** The \(Q^k\) M-type projection is equivalent to the \(Q^k\) point-line-plane projection \(\Pi\) defined as follows:
1. \(\Pi u = \hat{u}\) at four corners of \(\hat{K} = [-1, 1] \times [-1, 1]\).
2. \(\Pi u - \hat{u}\) is orthogonal to polynomials of degree \(k - 2\) on each edge of \(\hat{K}\).
3. \(\Pi u - \hat{u}\) is orthogonal to any \(v \in Q^{k-2}(\hat{K})\) on \(\hat{K}\).

**Proof.** We only need to show that M-type projection \(\hat{f}_{k,k}(s, t)\) satisfies the same three properties. By \(M_j(\pm 1) = 0\) for \(j \geq 2\), we can derive that \(\hat{f}_{k,k} = \hat{f}\) at \((\pm 1, \pm 1)\).

For instance, \(\hat{f}_{k,k}(1, 1) = \hat{b}_{0,0} + \hat{b}_{1,0} + \hat{b}_{0,1} + \hat{b}_{1,1} = \hat{f}(1, 1)\).

The second property is implied by \(M_j(\pm 1) = 0\) for \(j \geq 2\) and \(M_j(t) \perp P^{k-2}(t)\) for \(j \geq k+1\). For instance, at \(s = 1\), \(\hat{f}_{k,k}(1, t) - \hat{f}(1, t) = \sum_{j=k+1}^{\infty} (\hat{b}_{0,j} + \hat{b}_{1,j}) M_j(t) \perp P^{k-2}(t)\) on \([-1, 1]\).

The third property is implied by \(M_j(t) \perp P^{k-2}(t)\) for \(j \geq k + 1\).

**Lemma 3.1.** Assume \(\hat{f} \in H^{k+1}(\hat{K})\) with \(k \geq 2\), then
1. \(|\hat{b}_{i,j}| \leq C_k \|\hat{f}\|_{0,\infty, \hat{K}}\), \quad \forall i, j \leq k.
2. \(|\hat{b}_{i,j}| \leq C_k \|\hat{f}\|_{1+i+j, 2, \hat{K}}\), \quad \forall i, j \geq 1, i + j \leq k + 1.
3. \(|\hat{b}_{i,k+1}| \leq C_k \|\hat{f}\|_{1+i, 2, \hat{K}}\), \quad 0 \leq i \leq k + 1.
4. If \(\hat{f} \in H^{k+2}(\hat{K})\), then \(|\hat{b}_{i,k+1}| \leq C_k \|\hat{f}\|_{k+2, 2, \hat{K}}\), \quad 1 \leq i \leq k + 1.

**Proof.** First of all, similar to the one-dimensional case, through integration by parts, \(\hat{b}_{i,j}\) can be represented as integrals of \(\hat{f}\) thus \(\hat{b}_{i,j} \leq C_k \|\hat{f}\|_{0,\infty, \hat{K}}\) for \(i, j \leq k\).

By the fact that the antiderivatives (and higher order ones) of Legendre polynomials vanish at \(\pm 1\), after integration by parts for both variables, we have
\[
|\hat{b}_{i,j}| \leq C_k \int_{\mathcal{K}} |\partial_{x}^i \partial_{y}^j \hat{f}| ds dt \leq C_k \|\hat{f}\|_{i+j, 2, \hat{K}}\), \quad \forall i, j \geq 1, i + j \leq k + 1.
\]
For the third estimate, by integration by parts only for the variable $t$, we get

$$\forall i \geq 1, |\hat{b}_{i,k+1}| \leq C_k \int_K |\partial_s \partial^k f| ds dt \leq C_k |\hat{f}|_{k+1,2,K}. $$

For $\hat{b}_{0,k+1}$, from the first estimate, we have $|\hat{b}_{0,k+1}| \leq C_k \|\hat{f}\|_{0,\infty,K} \leq C_k \|\hat{f}\|_{k+1,2,K}$

thus $\hat{b}_{0,k+1}$ can be regarded as a continuous linear form on $H^{k+1}(\hat{K})$ and it vanishes if $\hat{f} \in Q^k(\hat{K})$. So by the Bramble-Hilbert Lemma, $|\hat{b}_{0,k+1}| \leq C_k [\hat{f}]_{k+1,2,K}$.

Finally, by integration by parts only for the variable $t$, we get

$$|\hat{b}_{i,k+1}| \leq C_k \int_K |\partial_s \partial^k f| ds dt \leq C_k |\hat{f}|_{k+2,2,K}, \quad 1 \leq i \leq k + 1.$$  

**Lemma 3.2.** For $k \geq 2$, we have

1. $|\hat{R}[\hat{f}]|_{k,k,0,\infty,K} \leq C_k |\hat{f}|_{k+1,2,K}$, $|\hat{R}[\hat{f}]|_{k,k,0,2,\hat{K}} \leq C_k |\hat{f}|_{k+1,2,K}$.
2. $|\partial_s \hat{R}[\hat{f}]|_{k,k,0,\infty,K} \leq C_k |\hat{f}|_{k+1,2,K}$, $|\partial_s \hat{R}[\hat{f}]|_{k,k,0,2,\hat{K}} \leq C_k |\hat{f}|_{k+1,2,K}$.
3. $\int_K |\partial_s \hat{R}[\hat{f}]|_{k,k} ds dt = 0$.

**Proof.** Lemma 3.1 implies $|\hat{f}|_{k,k,0,\infty,K} \leq C_k \|\hat{f}\|_{0,\infty,K}$ and $|\partial_s \hat{f}|_{k,k,0,\infty,K} \leq C_k \|\hat{f}\|_{0,\infty,K}$.

Thus

$$\forall (s,t) \in \hat{K}, |\hat{R}[\hat{f}]|_{k,k} ds dt \leq |\hat{f}|_{k,k} ds dt + |\hat{f}|_{k,k} ds dt \leq C_k \|\hat{f}\|_{0,\infty,K} \leq C_k \|\hat{f}\|_{k+1,2,K}.$$ 

Notice that here $C_k$ does not depend on $(s,t)$. So $R[\hat{f}]|_{k,k}(s,t)$ is a continuous linear form on $H^{k+1}(\hat{K})$ and its operator norm is bounded by a constant independent of $(s,t)$. Since it vanishes for any $\hat{f} \in Q^k(\hat{K})$, by the Bramble-Hilbert Lemma, we get $|\hat{R}[\hat{f}]|_{k,k} \leq C_k [\hat{f}]_{k+1,2,K}$ where $C_k$ does not depend on $(s,t)$. So the $L^\infty$ estimate holds and it implies the $L^2$ estimate.

The second estimate can be established similarly since we have

$$|\partial_s \hat{R}[\hat{f}]|_{k,k} ds dt \leq |\partial_s \hat{f}|_{k,k} ds dt + |\partial_s \hat{f}|_{k,k} ds dt \leq C_k \|\hat{f}\|_{1,\infty,K} \leq C_k \|\hat{f}\|_{k+1,2,K}.$$ 

The third equation is implied by the fact that $M_j(t) \perp 1$ for $j \geq 3$ and $M'_j(t) \perp 1$ for $j \geq 2$. Another way to prove the third equation is to use integration by parts.

$$\int_K |\partial_s \hat{R}[\hat{f}]|_{k+1,2,K} ds dt = \int_{-1}^{1} \left( \hat{R}[\hat{f}]_{k+1,2,K+1}(1,t) - \hat{R}[\hat{f}]_{k+1,2,K+1}(-1,t) \right) dt,$$

which is zero the second property in Theorem 3.1.

For the discussion in the next few subsections, it is useful to consider the lower order part of the remainder of $\hat{R}[\hat{f}]|_{k,k}$:

**Lemma 3.3.** For $\hat{f} \in H^{k+2}(\hat{K})$ with $k \geq 2$, define $\hat{R}[\hat{f}]|_{k+1,k+1} - \hat{R}[\hat{f}]|_{k,k} = \hat{R}_1 + \hat{R}_2$ with

$$\hat{R}_1 = \sum_{i=0}^{k} \hat{b}_{i,k+1} M_i(s) M_{k+1}(t),$$

$$\hat{R}_2 = \sum_{i=0}^{k+1} \hat{b}_{k+1,i} M_{k+1}(s) M_i(t) = M_{k+1}(s) \hat{b}_{k+1}(t), \quad \hat{b}_{k+1}(t) = \sum_{j=0}^{k+1} \hat{b}_{k+1,j} M_j(t).$$

They have the following properties:
1. \( \iint_{\hat{K}} \partial_{s} \hat{R}_t \, dsdt = 0. \)
2. \( |\partial_{s} \hat{R}_t|_{0,0,\hat{K}} \leq C_k |\hat{f}|_{k+2,2,\hat{K}}; \quad |\partial_{s} \hat{R}_t|_{0,2,\hat{K}} \leq C_k |\hat{f}|_{k+2,2,\hat{K}}. \)
3. \( |\hat{b}_{k+1}(t)| \leq C_k |\hat{f}|_{k+2,\hat{K}}, \quad |\hat{b}'_{k+1}(t)| \leq C_k |\hat{f}|_{k+2,\hat{K}}, \quad \forall t \in [-1,1]. \)

**Proof.** The first equation is due to the fact that \( M_{k+1}(t) \perp 1 \) since \( k \geq 2. \)

Notice that \( M'_0(s) = 0, \) by Lemma 3.1, we have

\[
|\partial_{s} \hat{R}_1(s,t)| = \left| \sum_{i=1}^{k} \hat{b}_{i,k+1} M'_i(s) M_{k+1}(t) \right| \leq C_k |\hat{f}|_{k+2,\hat{K}}.
\]

So we get the \( L^\infty \) estimate for \( |\partial_{s} \hat{R}_1(s,t)| \) thus the \( L^2 \) estimate.

Similar to the estimates in Lemma 3.1, we can show \( |\hat{b}_{k+1,j}| \leq C_k |\hat{f}|_{k+1,\hat{K}} \) for \( j \leq k+1, \) thus \( |\hat{b}_{k+1}(t)| \leq C_k |\hat{f}|_{k+1,\hat{K}}. \) Since \( \hat{b}'_{k+1}(t) = \sum_{j=1}^{k+1} \hat{b}_{k+1,j} M'_j(t) \), by the last estimate in Lemma 3.1, we get \( |\hat{b}'_{k+1}(t)| \leq C_k |\hat{f}|_{k+2,\hat{K}}. \)

**3.3. The \( C^0-Q^k \) projection.** Now consider a function \( u(x,y) \in H^{k+2}(\Omega), \) let \( u_p(x,y) \) denote its piecewise \( Q^k \) M-type projection on each element \( e \) in the mesh \( \Omega_h. \) The first two properties in Theorem 3.1 imply that \( u_p(x,y) \) on each edge is uniquely determined by \( u(x,y) \) along that edge. Thus \( u_p(x,y) \) is continuous on \( \Omega_h. \)

The approximation error \( u - u_p \) is one order higher at all Gauss-Lobatto points \( Z_0. \)

**Theorem 3.2.**

\[
\|u - u_p\|_{2, Z_0} = O(h^{k+2})\|u\|_{k+2}, \quad \forall u \in H^{k+2}(\Omega).
\]

\[
\|u - u_p\|_{\infty, Z_0} = O(h^{k+2})\|u\|_{k+2,\infty}, \quad \forall u \in W^{k+2,\infty}(\Omega).
\]

**Proof.** Consider any \( e \) with cell center \( (x_e, y_e), \) define \( \hat{u}(s,t) = u(x_e + sh, y_e + th). \)

Since the \( (k+1) \) Gauss-Lobatto points are roots of \( M_{k+1}(t), \) \( \hat{R}_{k+1,k+1}[\hat{u}] - \hat{R}_{k,k}[\hat{u}] \) vanishes at \( (k+1) \times (k+1) \) Gauss-Lobatto points on \( \hat{K}. \) By Lemma 3.2, we have \( |\hat{R}_{k+1,k+1}[\hat{u}](s,t)| \leq C[\hat{u}]_{k+2,\hat{K}}. \)

Mapping back to the cell \( e, \) at the \( (k+1) \times (k+1) \) Gauss-Lobatto points on \( e, \)

\[
\|u - u_p\|_{2, Z_0} \leq C \left[ h^k \sum_{e} h^{2k+4-n[u]^2_{k+2,e}} \right]^{1/2} = O(h^{k+2})[u]_{k+2,\Omega}.
\]

If further assuming \( u \in W^{k+2,\infty}(\Omega), \) then at the \( (k+1) \times (k+1) \) Gauss-Lobatto points on \( e, \)

\[
\|u - u_p\|_{\infty, Z_0} \leq C h^{k+2} \frac{\|u\|_{k+2,\infty}}{\|u\|_{k+2,\infty,\Omega}}, \quad \text{which implies the second estimate.}
\]

**3.4. Superconvergence of bilinear forms.** For convenience, in this subsection, we drop the subscript \( h \) in a test function \( v_h \in V^h. \) When there is no confusion, we may also drop \( dx \) or \( ds \) in a double integral.

**Lemma 3.4.** Assume \( a(x,y) \in W^{2,\infty}(\Omega). \) For \( k \geq 2, \)

\[
\iint_{\Omega} a(u - u_p)xv_x \, dx \, dy = O(h^{k+2})\|u\|_{k+2}\|v\|_2, \quad \forall v \in V^h.
\]
Proof. For each cell \( e \), we consider \( \iint_e a(u - u_p)xv_x \, dx 
olabel{0} dy \). Let \( R[u]_{k,k} = u - u_p \) denote the M-type projection remainder on \( e \). Then \( R[u]_{k,k} \) can be split into lower order part \( R[u]_{k,k} = R[u]_{k+1,k+1} \) and high order part \( R[u]_{k+1,k+1} \).

\[
\iint_e a(u - u_p)xv_x \, dx = \iint_e a(R[u]_{k+1,k+1})xv_x + \iint_e a(R[u]_{k,k} - R[u]_{k+1,k+1})xv_x. 
\]

We first consider the high order part. Mapping everything to the reference cell \( \hat{K} \) and let \( \hat{\alpha}v_s \) denote the average of \( \alpha v_s \) on \( \hat{K} \). By the last property in Lemma 3.2, we get

\[
\begin{align*}
\hat{h}^{2-n} \left| \iint_\hat{K} a(R[u]_{k+1,k+1})xv_x \, dx \right| &= \left| \iint_\hat{K} \partial_s (\hat{R}[u]_{k+1,k+1})\hat{\alpha}v_s ds \right| \\
&= \left| \iint_\hat{K} \partial_s (\hat{R}[u]_{k+1,k+1})\hat{\alpha}v_s ds \right| \leq \left| \partial_s (\hat{R}[u]_{k+1,k+1}) \right|_{0,2,K} | \hat{\alpha}v_s - \hat{\alpha}v_s |_{0,2,K}.
\end{align*}
\]

By Poincaré inequality and Cauchy-Schwarz inequality, we have

\[
| \hat{\alpha}v_s - \hat{\alpha}v_s |_{0,2,K} \leq C | \nabla (\hat{\alpha}v_s) |_{0,2,K} \leq C | \hat{\alpha} |_{1,\infty,K} | \hat{v} |_{1,2,K} + C | \nabla \hat{\alpha} |_{0,\infty,K} | \hat{v} |_{2,2,K}.
\]

Mapping back to the cell \( e \), by Lemma 3.2, the higher order part is bounded by \( C\hat{h}^{k+2}[u]_{k+2,e} (| \hat{\alpha} |_{1,\infty,e} | v |_{1,2,e} + | \nabla \hat{\alpha} |_{0,\infty,e} | v |_{2,2,e} ) \) thus

\[
\sum_e \iint_e a(R[u]_{k+1,k+1})xv_x \, dx = O(\hat{h}^{k+2}) \| a \|_{1,\infty,\Omega} \sum_e \| u \|_{k+2,e} \| v \|_{2,e} \\
= O(\hat{h}^{k+2}) \| a \|_{1,\infty,\Omega} \| u \|_{k+2,\Omega} \| v \|_{2,\Omega}.
\]

Now we only need to discuss the lower order part of the remainder. Let \( R[u]_{k,k} - R[u]_{k+1,k+1} = R_1 + R_2 \) which is defined similarly as in (3.1). For \( R_1 \), by the first two results in Lemma 3.3, we have

\[
\begin{align*}
\iint_\hat{K} (\partial_s \hat{R}_1)\hat{\alpha}v_s &= \iint_\hat{K} (\partial_s \hat{R}_1)(\hat{\alpha}v_s - \hat{\alpha}v_s) \leq | \partial_s \hat{R}_1 |_{0,2,K} | \hat{\alpha}v_s - \hat{\alpha}v_s |_{0,2,K} \\
&\leq C | \hat{u} |_{k+2,\hat{K}} | \hat{\alpha}v_s - \hat{\alpha}v_s |_{0,2,K}.
\end{align*}
\]

By similar discussions above, we get

\[
\sum_e \iint_e (R_1)xv_x \, dx = O(\hat{h}^{k+2}) \| a \|_{1,\infty,\Omega} \| u \|_{k+2,\Omega} \| v \|_{2,\Omega}.
\]

For \( R_2 \), let \( N(s) \) be the antiderivative of \( M_{k+1}(s) \) then \( N(\pm 1) = 0 \). Let \( \tilde{\alpha} \) be the average of \( \hat{\alpha} \) on \( K \) then \( | \hat{\alpha} - \tilde{\alpha} |_{0,\infty,K} \leq C | \hat{\alpha} |_{1,\infty,K} \). Since \( M_{k+1}(s) \perp P^{k-2}(s) \), we have

\[
\iint_K \hat{b}_{k+1}(t)M_{k+1}(s)\hat{v}_s = 0. \quad \text{After integration by parts, by Lemma 3.3 we have}
\]

\[
\begin{align*}
\iint_K (\partial_s \hat{R}_2)\hat{\alpha}v_s &= -\iint_K \hat{b}_{k+1}(t)M_{k+1}(s)(\hat{\alpha}s_v + \hat{\alpha}v_s) \\
&= \iint_K \hat{b}_{k+1}(t)N(s)(\hat{\alpha}v_s + \hat{\alpha}v_s) - \iint_K \hat{b}_{k+1}(t)M_{k+1}(s)(\hat{\alpha} - \hat{\alpha})v_s \\
&\leq C | \hat{u} |_{k+1,\hat{K}} | \hat{\alpha} |_{2,\infty,\hat{K}} | \hat{v} |_{1,2,\hat{K}} + | \hat{\alpha} |_{1,\infty,\hat{K}} | \hat{v} |_{2,2,\hat{K}}.
\end{align*}
\]

Thus we can get

\[
\sum_e \iint_e (\partial_s \hat{R}_2)\hat{\alpha}v_x \, dx = O(\hat{h}^{k+2}) \| a \|_{2,\infty,\Omega} \| u \|_{k+1,\Omega} \| v \|_{2,\Omega}.
\]

So we have \( \iint_{\Omega} a(u - u_p)xv_x \, dx = O(\hat{h}^{k+2}) \| a \|_{2,\infty,\Omega} \| u \|_{k+2,\Omega} \| v \|_{2}, \quad \forall v \in V^h. \)
Lemma 3.5. Assume \( c(x, y) \in W^{1, \infty}(\Omega) \). For \( k \geq 2 \),
\[
\iint_{\Omega} c(u - u_p)v \, dx \, dy = O(h^{k+2})\|u\|_{k+1, \infty}\|v\|_1, \quad \forall v \in V^h.
\]

Proof. Let \( \bar{\hat{c}}v \) be the average of \( \hat{c}v \) on \( \hat{K} \). Then
\[
\left| \iint_{\hat{K}} \hat{R} \hat{[\hat{u}]_{k,k}} \hat{\hat{c}} \right| = \left| \iint_{\hat{K}} \hat{R} \hat{[\hat{u}]_{k,k}} (\hat{\hat{c}}v - \bar{\hat{c}}v) \right| \leq |\hat{R} \hat{[\hat{u}]_{k,k}}|_{0,2,\hat{K}} |\hat{\hat{c}} - \bar{\hat{c}}v|_{0,2,\hat{K}}
\leq C[u]_{k+1,2,\hat{K}} |\hat{\hat{c}}v|_{1,2,\hat{K}} \leq C[u]_{k+1,2,\hat{K}} (|\hat{\hat{c}}|_{0,\infty,\hat{K}} |\hat{v}|_{1,2,\hat{K}} + |\hat{\hat{c}}|_{1,\infty,\hat{K}} |\hat{v}|_{1,2,\hat{K}}).
\]
So we have
\[
\iint_{\Omega} cR[u]_{k,k}v \, dx \, dy = h^n \iint_{\hat{K}} (R[\hat{u}]_{k,k}) \bar{\hat{c}}vdsdt = O(h^{k+2})\|c\|_{1,\infty,\Omega}\|u\|_{k+1,\infty}\|v\|_{1,\infty},
\]
which implies the estimate.

Lemma 3.6. Assume \( b(x, y) \in W^{2, \infty}(\Omega) \). For \( k \geq 2 \),
\[
\iint_{\Omega} b(u - u_p)_x v \, dx \, dy = O(h^{k+2})\|u\|_{k+2, \infty}\|v\|_2, \quad \forall v \in V^h.
\]

Proof. Let \( \bar{\hat{b}}v \) be the average of \( \hat{b}v \) on \( \hat{K} \), then
\[
\left| \iint_{\hat{K}} \partial_s(\hat{R} \hat{[\hat{u}]_{k+1,k+1}}) \bar{\hat{b}}v \right| = \left| \iint_{\hat{K}} \partial_s(\hat{R} \hat{[\hat{u}]_{k+1,k+1}})(\bar{\hat{b}}v - \bar{\hat{b}}v) \right| 
\leq |\partial_s(\hat{R} \hat{[\hat{u}]_{k+1,k+1}})|_{0,2,\hat{K}} |\bar{\hat{b}}v - \bar{\hat{b}}v|_{0,2,\hat{K}} \leq C[u]_{k+2,2,\hat{K}} (|\hat{\hat{b}}|_{1,\infty,\hat{K}} |\hat{v}|_{1,2,\hat{K}} + |\hat{\hat{b}}|_{0,\infty,\hat{K}} |\hat{v}|_{1,2,\hat{K}}).
\]

Let \( N(s) \) be the antiderivative of \( M_{k+1}(s) \). After integration by parts, we have
\[
\iint_{\hat{K}} (\partial_s \hat{R}_2) \bar{\hat{b}}v = - \iint_{\hat{K}} b_{k+1}(t) M_{k+1}(s)(\bar{\hat{b}}s \hat{v} + \bar{\hat{b}}s) 
= \iint_{\hat{K}} b_{k+1}(t) N(s)(\bar{\hat{b}}s \hat{v} + \bar{\hat{b}}s) 
\leq C[u]_{k+1,2,\hat{K}} (|\hat{\hat{b}}|_{2,\infty,\hat{K}} |\hat{v}|_{0,2,\hat{K}} + |\hat{\hat{b}}|_{1,\infty,\hat{K}} |\hat{v}|_{1,2,\hat{K}} + |\hat{\hat{b}}|_{0,\infty,\hat{K}} |\hat{v}|_{2,2,\hat{K}}).
\]

After combining all the estimates, we have
\[
\iint_{\Omega} b(u - u_p)_x v \, dx \, dy = h^{n-1} \iint_{\hat{K}} \hat{b}(\hat{R}[\hat{u}]_{k,k}) \hat{s} \hat{v} = O(h^{k+2})\|b\|_{2,\infty,\Omega}\|u\|_{k+2,\infty}\|v\|_{2,\infty}.
\]

Lemma 3.7. Assume \( a(x, y) \in W^{2, \infty}(\Omega) \). For \( k \geq 2 \),
\[
\iint_{\Omega} a(u - u_p)_{x} v \, dx \, dy = O(h^{k+2-\frac{1}{2}})\|u\|_{k+2,2}\|v\|_2, \quad \forall v \in V^h,
\]
\[
\iint_{\Omega} a(u - u_p)_{x} v \, dx \, dy = O(h^{k+2})\|u\|_{k+2,2}\|v\|_2, \quad \forall v \in V^0_0.
\]
Proof. Similar to the proof of Lemma 3.4, we have
\[
\left| \int \int (a(R[u]_{k+1,k+1})x^y dxdy \right| = h^{n-2} \left| \int \int K(\hat{T}[u]_{k+1,k+1})\hat{a}\hat{v}_t dsdt \right|
\]
\[
= h^{n-2} \left| \int \int K(\hat{T}[u]_{k+1,k+1})(\hat{a}\hat{v}_t - \hat{a}\hat{v}_t) dsdt \right| \leq h^{n-2} |\partial_t(\hat{T}[u]_{k+1,k+1})|_{0,2,K} |\hat{a}\hat{v}_t|_{0,2,K}
\]
\[
\leq C h^{k+2} \|a\|_{1,\infty,\Omega} \|u\|_{k+2,\varepsilon} \|v\|_{2,\varepsilon},
\]
and
\[
\int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t = \int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t - \hat{a}\hat{v}_t \leq |\partial_t \hat{T}|_{0,2,K} |\hat{a}\hat{v}_t|_{0,2,K}.
\]
Following the proof of Lemma 3.4, we get
\[
\sum_{e} \int \int \alpha(R_1)x^y dxdy = O(h^{k+2}) \|a\|_{1,\infty,\Omega} \|u\|_{k+2,\Omega} \|v\|_{2,\Omega}.
\]
Let \(N(s)\) be the antiderivative of \(M_{k+1}(s)\). After integration by parts, we have
\[
\int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t = -\int \int K(\hat{T}[u]_{k+1,k+1})\hat{a}\hat{v}_t + \hat{a}\hat{v}_t)
\]
\[
= \int \int K(\hat{T}[u]_{k+1,k+1})\hat{a}\hat{v}_t + \hat{a}\hat{v}_t)
\]
\[
\int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t = \int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t + \hat{a}\hat{v}_t)
\]
\[
\int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t = \int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t + \hat{a}\hat{v}_t)
\]
\[
\int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t = \int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t + \hat{a}\hat{v}_t)
\]
By Lemma 3.3, we have the estimate for the two double integral terms
\[
\left| \int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t + \hat{a}\hat{v}_t) \right| \leq C |\hat{a}|_{k+1,2,K} |\hat{a}|_{2,\infty,\hat{K}} |\hat{v}|_{1,2,\hat{K}} + |\hat{a}|_{1,\infty,\hat{K}} |\hat{v}|_{2,2,\hat{K}}
\]
\[
\int \int K(\partial_t \hat{T})\hat{a}\hat{v}_t + \hat{a}\hat{v}_t) \leq C |\hat{a}|_{k+2,2,K} |\hat{a}|_{0,\infty,\hat{K}} |\hat{v}|_{1,2,\hat{K}} + |\hat{a}|_{1,\infty,\hat{K}} |\hat{v}|_{2,2,\hat{K}}
\]
which gives the estimate \(Ch^{k+2} \|a\|_{2,\infty,\Omega} \|u\|_{k+2,\varepsilon} \|v\|_{k+2,\varepsilon}\) after mapping back to \(e\).

So we only need to discuss the line integral term now. After mapping back to \(e\), we have
\[
\int_{t=-1}^{1} \hat{b}_{k+1}(t) M_{k+1}(s) \hat{a} \hat{v}_s ds \bigg|_{t=-1}^{t=1} = h \int_{x=-h}^{x=+h} \hat{b}_{k+1}(y) M_{k+1}(\frac{x-x_e}{h}) \hat{a} \hat{v}_x dx |_{y=y_e+h}^{y=y_e-h}.
\]
Notice that we have
\[ b_{k+1}(y_e + h) = \hat{b}_{k+1}(1) = \sum_{j=0}^{k+1} \hat{b}_{k+1,j} M_j(1) = \hat{b}_{k+1,0} + \hat{b}_{k+1,1} \]
and similarly we get \( b_{k+1}(y_e - h) = \hat{b}_{k+1}(-1) = (k + \frac{1}{2}) \int_{x_e - h}^{x_e + h} \partial_x u(x, y_e - h) l_k(\frac{x-x_e}{h}) dx \).
Thus the term \( b_{k+1}(y) M_{k+1}(\frac{x-x_e}{h}) \partial u_{xx} \) is continuous across the top/bottom edge of cells. Therefore, if summing over all elements \( e \), the line integral on the inner edges are cancelled out. Let \( L_1 \) and \( L_3 \) denote the top and bottom boundary of \( \Omega \). Then the line integral after summing over \( e \) consists of two line integrals along \( L_1 \) and \( L_3 \). We only need to discuss one of them.

Let \( l_1 \) and \( l_3 \) denote the top and bottom edge of \( e \). First, after integration by parts \( k \) times, we get
\[ \hat{b}_{k+1}(1) = (k + \frac{1}{2}) \int_{-1}^{1} \partial_x \hat{u}(s, 1) l_k(s) ds = (-1)^k (k + \frac{1}{2}) \int_{-1}^{1} \frac{\partial^{k+1}}{\partial s^{k+1}} \hat{u}(s, 1) \frac{1}{2^{2k}} (s^2 - 1)^k ds, \]
thus by Cauchy Schwarz inequality we get
\[ |\hat{b}_{k+1}(1)| \leq C_k \int_{-1}^{1} \left[ \frac{\partial^{k+1}}{\partial s^{k+1}} \hat{u}(s, 1) \right] ds \leq C_k h^{k+\frac{3}{2}} |u|_{k+1,2,l_1}. \]
Second, since \( v_{xx} \) is a polynomial of degree \( 2k \) w.r.t. \( y \) variable, by using \((k+2)\)-point Gauss Lobatto quadrature for integration w.r.t. \( y \)-variable in \( \iint_{e} v_{xx}^2 dxdy \), we get
\[ \int_{x_e-h}^{x_e+h} v_{xx}^2(x, y_e + h) dx \leq C h^{-1} \iint_{e} v_{xx}^2(x, y) dxdy. \]
So by Cauchy Schwarz inequality, we have
\[ \int_{x_e-h}^{x_e+h} v_{xx}^2(x, y_e + h) dx \leq \sqrt{2h} \int_{x_e-h}^{x_e+h} v_{xx}^2(x, y_e + h) dx \leq C |v|_{2,2,e}. \]
Thus the line integral along \( L_1 \) can be estimated by considering each \( e \) adjacent to \( L_1 \) in the reference cell:
\[ \sum_{e \cap L_1 \neq \emptyset} \left| \int_{-1}^{1} \hat{b}_{k+1}(1) M_{k+1}(s) \hat{u}(s, 1) \hat{v}_{ss}(s, 1) ds \right| \]
\[ \leq \sum_{e \cap L_1 \neq \emptyset} C |\hat{\alpha}|_{0, \infty, K} |\hat{b}_{k+1}(1)| \int_{-1}^{1} |\hat{v}_{ss}(s, 1)| ds \]
\[ = \mathcal{O}(h^{k+\frac{3}{2}}) \sum_{e \cap L_1 \neq \emptyset} |u|_{k+1,2,l_1} \int_{x_e-h}^{x_e+h} |v_{xx}(x, y_e + h)| dx \]
\[ = \mathcal{O}(h^{k+\frac{3}{2}}) \sum_{e \cap L_1 \neq \emptyset} |u|_{k+1,2,l_1} |v|_{2,2,e} \]
\[ = \mathcal{O}(h^{k+\frac{3}{2}}) \|u\|_{k+1,L_1} \|v\|_{2,2,\Omega} = \mathcal{O}(h^{k+\frac{3}{2}}) \|u\|_{k+2,\Omega} \|v\|_{2,\Omega}, \]
where the trace inequality \( \|u\|_{k+1,\partial\Omega} \leq C \|u\|_{k+2,\Omega} \) is used.

Combine all the estimates above, we get (3.2). Since the \( \frac{1}{2} \) order loss is only due to the line integral along \( L_1 \) and \( L_3 \), on which \( v_{xx} = 0 \) if \( v \in V_0^h \), we get (3.3). \( \square \)
4. The main result.

4.1. Estimates of bilinear forms with approximated coefficients. Even though standard interpolation error is \( a - a_I = O(h^{k+1}) \), as shown in the following discussion, the error in the bilinear forms is related to \( \int_{\Omega} (a - a_I) \, dx \, dy \) on each cell \( e \), which is the quadrature error thus the order is higher. We have the following estimate on the bilinear forms with approximated coefficients:

**Lemma 4.1.** Assume \( a(x, y) \in W^{k+2, \infty}(\Omega) \) and \( u(x, y) \in H^2(\Omega) \), then \( \forall v \in V^h \) or \( \forall v \in H^2(\Omega) \),

\[
\int_{\Omega} a u_x v_x \, dx \, dy - \int_{\Omega} a_I u_x v_x \, dx \, dy = O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|u\|_2 \|v\|_2,
\]

\[
\int_{\Omega} a u_x v_y \, dx \, dy - \int_{\Omega} a_I u_x v_y \, dx \, dy = O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|u\|_2 \|v\|_2,
\]

\[
\int_{\Omega} a u_x v_x \, dx \, dy - \int_{\Omega} a_I u x v_y \, dx \, dy = O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|u\|_2 \|v\|_1,
\]

\[
\int_{\Omega} a u v \, dx \, dy - \int_{\Omega} a_I u v \, dx \, dy = O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|u\|_1 \|v\|_1.
\]

**Proof.** For every cell \( e \) in the mesh \( \Omega_h \), let \( \overline{u_xv_x} \) be the cell average of \( u_xv_x \). By Theorem 2.2 and Theorem 2.3, we have

\[
\int_e (a_I - a) u_x v_x = \int_e (a_I - a) \overline{u_xv_x} + \int_e (a_I - a)(u_x v_x - \overline{u_xv_x})
\]

\[
= \frac{1}{4h^2} \int_e (a_I - a) \int_e u_x v_x + \int_e (a_I - a)(u_x v_x - \overline{u_xv_x})
\]

\[
= O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|u\|_{1,e} \|v\|_{1,e} + O(h^{k+1})\|a\|_{k+1, \infty, \Omega} \int_e |u_x v_x - \overline{u_xv_x}|.
\]

By Poincaré inequality and Cauchy-Schwarz inequality, we have

\[
\int_e |u_x v_x - \overline{u_xv_x}| = O(h)\|

\frac{\nabla(u_x v_x)}{1+e} = O(h)\|u\|_{2,e} \|v\|_{2,e}
\]

thus \( \int_{\Omega} (a_I - a) u_x v_x = O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|u\|_{2,e} \|v\|_{2,e} \). Summing over all elements \( e \), we have \( \int_{\Omega} (a_I - a) u_x v_x = O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|u\|_2 \|v\|_2 \). Similarly we can establish the other three estimates. \( \square \)

Lemma 4.1 implies that the difference in the solutions to (1.11) and (1.1) is \( O(h^{k+2}) \) in the \( L^2(\Omega) \)-norm:

**Theorem 4.1.** Assume \( a(x, y) \in W^{k+2, \infty}(\Omega) \) and \( a_I(x, y) \geq C > 0 \). Let \( u, \tilde{u} \in H^1_0(\Omega) \) be the solutions to

\[
A(u, v) := \int_{\Omega} a \nabla u \cdot \nabla v \, dx \, dy = (f, v), \quad \forall v \in H^1_0(\Omega)
\]

and

\[
A_I(\tilde{u}, v) := \int_{\Omega} a_I \nabla \tilde{u} \cdot \nabla v \, dx \, dy = (f, v), \quad \forall v \in H^1_0(\Omega)
\]

respectively, where \( f \in L^2(\Omega) \). Then \( \|u - \tilde{u}\|_0 = O(h^{k+2})\|a\|_{k+2, \infty, \Omega} \|f\|_0 \).
\textbf{Proof.} By Lemma 4.1, for any } v \in H^2(\Omega) \text{ we have } \\
A_I(u - \tilde{u}, v) = A_I(u, v) - A_I(\tilde{u}, v) = [A_I(u, v) - A(u, v)] + [A(u, v) - A_I(\tilde{u}, v)] \\
= A_I(u, v) - A(u, v) = O(h^{k+2})\|a\|_{k+2,\infty,\Omega}\|u\|_2\|v\|_2. \\
Let } w \in H^1_0(\Omega) \text{ be the solution to the dual problem } \\
A_I(w, v) = (u - \tilde{u}, v) \quad \forall v \in H^1_0(\Omega). \\
Since } a_I \geq C > 0 \text{ and } |a_I(x, y)| \leq C|a(x, y)|, \text{ the coercivity and boundedness of the bilinear form } A_I \text{ hold } [8]. \text{ Moreover, } a_I \text{ is Lipschitz continuous because } a(x, y) \in W^{k+2,\infty}(\Omega). \text{ Thus the solution } w \text{ exists and the elliptic regularity } \|w\|_2 \leq C\|u - \tilde{u}\|_0 \text{ holds on a convex domain, e.g., a rectangular domain } \Omega, \text{ see } [14]. \text{ Thus, } \\
\|u - \tilde{u}\|_0^2 = (u - \tilde{u}, u - \tilde{u}) = A_I(u - \tilde{u}, w) = O(h^{k+2})\|a\|_{k+2,\infty,\Omega}\|u\|_2\|w\|_2. \\
With elliptic regularity } \|w\|_2 \leq C\|u - \tilde{u}\|_0 \text{ and } \|u\|_2 \leq C\|f\|_0, \text{ we get } \\
\|u - \tilde{u}\|_0 = O(h^{k+2})\|a\|_{k+2,\infty,\Omega}\|f\|_0. \\
\hfill \Box \\
\textbf{4.2. The variable coefficient Poisson equation.} \text{ Let } u(x, y) \in H^1_0(\Omega) \text{ be the exact solution to } \\
A(u, v) := \iint_{\Omega} a \nabla u \cdot \nabla v \, dx \, dy = (f, v), \quad \forall v \in H^1_0(\Omega). \\
Let } \tilde{u}_h \in V^h(\Omega) \text{ be the solution to } \\
A_I(\tilde{u}_h, v_h) := \iint_{\Omega} a \nabla \tilde{u}_h \cdot \nabla v_h \, dx \, dy = (f, v_h)_h, \quad \forall v_h \in V^h_0(\Omega). \\
\textbf{Theorem 4.2.} For } k \geq 2, \text{ let } u_p \text{ be the piecewise } Q^h M\text{-type projection of } u(x, y) \text{ on each cell } e \text{ in } \Omega, \text{ Assume } a \in W^{k+2,\infty}(\Omega) \text{ and } u, f \in H^{k+2}(\Omega), \text{ then } \\
A_I(\tilde{u}_h - u_p, v_h) = O(h^{k+2})\|a\|_{k+2,\infty}\|u\|_{k+2} + \|f\|_{k+2}\|v_h\|_2, \quad \forall v_h \in V^h. \\
\textbf{Proof.} \text{ For any } v_h \in V^h, \text{ we have } \\
A_I(\tilde{u}_h, v_h) - A_I(u_p, v_h) \\
=(f, v_h) - A_I(u_p, v_h) + \langle f, v_h \rangle_h - (f, v_h) \\
= A(u, v) - A_I(u_p, v_h) + \langle f, v_h \rangle_h - (f, v_h) \\
= [A(u, v) - A_I(u, v)] + [A_I(u - u_p, v_h) - A(u - u_p, v_h)] + A(u - u_p, v_h) + \langle f, v_h \rangle_h - (f, v_h). \\
\text{Lemma 4.1 implies } A(u, v_h) - A_I(u, v_h) = O(h^{k+2})\|a\|_{k+2,\infty}\|u\|_2\|v_h\|_2. \text{ Theorem 2.4 gives } \langle f, v_h \rangle_h - (f, v_h) = O(h^{k+2})\|f\|_{k+2}\|v_h\|_2. \text{ By Lemma 3.4, } A(u - u_p, v_h) = O(h^{k+2})\|a\|_{2,\infty}\|u\|_{k+2}\|v_h\|_2. \\
\text{For the second term } A_I(u - u_p, v_h) - A(u - u_p, v_h) = \iint_{\Omega} (a - a_I) \nabla (u - u_p) \nabla v_h, \text{ by Theorem 2.2 and Lemma 3.2, we have } \\
\left| \iint_{\Omega} (a - a_I)(u - u_p) \partial_x v_h \right| \leq |a - a_I|_{\infty,\Omega} \sum_e \int_e |(u - u_p) \partial_x v_h| \\
\leq |a - a_I|_{\infty,\Omega} \sum_e \int_e |u - u_p|_{0,2,e} |v_h|_{1,2,e} \\
= O(h^{2k+1})\|a\|_{k+1,\infty,\Omega} \sum_e \|u\|_{k+1,e} \|v_h\|_1. \quad \Box
Theorem 4.3. Assume \( a(x, y) \in W^{k+2, \infty}(\Omega) \) is positive and \( u(x, y), f(x, y) \in H^{k+2}(\Omega) \). Assume the mesh is fine enough so that the piecewise \( Q^k \) interpolant satisfies \( a_I(x, y) \geq C > 0 \). Then \( \hat{u}_h \) is a \((k+2)\)-th order accurate approximation to \( u \) in the discrete 2-norm over all the \((k+1) \times (k+1)\) Gauss-Lobatto points:

\[
\| \hat{u}_h - u \|_{2, Z_0} = \mathcal{O}(h^{k+2})(\| a \|_{k+2, \infty} \| u \|_{k+2} + \| f \|_{k+2}).
\]

Proof. Let \( \theta_h = \hat{u}_h - u_p \). By the definition of \( u_p \) and Theorem 3.1, it is straightforward to show \( \theta_h = 0 \) on \( \partial \Omega \). By the Aubin-Nitsche duality method, let \( w \in H^1_0(\Omega) \) be the solution to the dual problem

\[
A_I(v, w) = (\theta_h, v) \quad \forall v \in H^1_0(\Omega).
\]

By the same discussion as in the proof of Theorem 4.1, the solution \( w \) exists and the regularity \( \| w \|_2 \leq C \| \theta_h \|_0 \) holds.

Let \( w_h \) be the finite element projection of \( w \), i.e., \( w_h \in V^h_0 \) satisfies

\[
A_I(v_h, w_h) = (\theta_h, v_h) \quad \forall v_h \in V^h_0.
\]

Since \( w_h \in V^h_0 \), by Theorem 4.2, we have

\[
(1.1) \quad \| \theta_h \|_0^2 = (\theta_h, \theta_h) = A_I(\theta_h, w_h) = \mathcal{O}(h^k)(\| a \|_{k+2, \infty} \| u \|_{k+2} + \| f \|_{k+2}) \| w_h \|_2.
\]

Let \( w_I = \Pi Iw \) be the piecewise \( Q^1 \) projection of \( w \) on \( \Omega_h \) as defined in (2.1). By the Bramble-Hilbert Lemma, we get \( \| w - w_I \|_{2, e} \leq C \| w \|_{2, e} \leq C \| w \|_2 \) thus

\[
\| w - w_I \|_2 \leq C \| w \|_2.
\]

By the inverse estimate on the piecewise polynomial \( w_h - w_I \), we have

\[
(4.2) \quad \| w_h \|_2 \leq \| w_h - w_I \|_2 + \| w_I - w \|_2 \leq C h^{-1} \| w_h - w_I \|_1 + C \| w \|_2.
\]

With coercivity, Galerkin orthogonality and Cauchy Schwarz inequality, we get

\[
C \| w_h - w_I \|_1^2 \leq A_I(w_h - w_I, w_h - w_I) = A_I(w_h - w_I, w_I - w_I) \leq C \| w - w_I \|_1 \| w_h - w_I \|_1,
\]

which implies

\[
(4.3) \quad \| w_h - w_I \|_1 \leq C \| w - w_I \|_1 \leq C h \| w \|_2.
\]

With (4.2), (4.3) and the elliptic regularity \( \| w \|_2 \leq C \| \theta_h \|_0 \), we get

\[
(4.4) \quad \| w_h \|_2 \leq C \| w \|_2 \leq C \| \theta_h \|_0.
\]

By (4.1) and (4.4) we have

\[
\| \theta_h \|_0^2 \leq \mathcal{O}(h^{k+2})(\| a \|_{k+2, \infty} \| u \|_{k+2} + \| f \|_{k+2}) \| \theta_h \|_0,
\]

i.e.,

\[
\| \hat{u}_h - u_p \|_0 = \| \theta_h \|_0 = \mathcal{O}(h^{k+2})(\| a \|_{k+2, \infty} \| u \|_{k+2} + \| f \|_{k+2}).
\]

Finally, by the equivalency between the discrete 2-norm on \( Z_0 \) and the \( L^2(\Omega) \) norm in the space \( V^h \), with Theorem 3.2, we obtain

\[
\| \hat{u}_h - u \|_{2, Z_0} = \mathcal{O}(h^{k+2})(\| a \|_{k+2, \infty} \| u \|_{k+2} + \| f \|_{k+2}).
\]
Remark 1. It is straightforward to extend Theorem 4.3 to homogeneous Neumann boundary conditions or mixed homogeneous Dirichlet and Neumann boundary conditions since all the estimates such as Theorem 4.2 hold not only for \( v \in V_h^0 \) but also for any \( v \in V_h \).

Remark 2. With Theorem 2.5, all the results hold for the scheme (1.10).

Remark 3. It is straightforward to verify that all results hold in three dimensions. Notice that the in three dimensions the discrete 2-norm is

\[
\|u\|_{2, Z_0} = \left[ \frac{h^3}{2} \sum_{x \in Z_0} |u(x)|^2 \right]^{\frac{1}{2}}.
\]

Remark 4. For discussing superconvergence of the scheme (1.8), we have to consider the dual problem of the bilinear form \( A \) instead and the exact Galerkin orthogonality in (1.8) no longer holds. In order for the proof above holds, we need to show the Galerkin orthogonality in (1.8) holds up to \( O(h^{k+2}) \|v_h\|_2 \) for a test function \( v_h \in V_h \), which is very difficult to establish. This is the main difficulty in attempting to extend the proof of Theorem 4.3 to the Gauss Lobatto quadrature scheme (1.8).

4.3. General elliptic problems. In this section, we discuss extensions to more general elliptic problems. Consider an elliptic variational problem of finding \( u \in H^1_0(\Omega) \) to satisfy

\[
A(u, v) := \iint_{\Omega} (\nabla v^T a \nabla u + b \nabla u v + c u v) \, dx dy = (f, v), \quad \forall v \in H^1_0(\Omega),
\]

where \( a(x, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) is positive definite and \( b = [b_1 \ b_2] \). Assume the coefficients \( a, b \) and \( c \) are smooth, and \( A(u, v) \) satisfies coercivity \( A(v, v) \geq C\|v\|_1 \) and boundedness \( |A(u, v)| \leq C\|u\|_1\|v\|_1 \) for any \( u, v \in H^1_0(\Omega) \).

By the estimates in Section 3.4, we first have the following estimate on the \( Q^k \) M-type projection \( u_p \):

Lemma 4.2. Assume \( a_{ij}(x, y), b_i(x, y) \in W^{2, \infty}(\Omega) \) and \( b_i(x, y) \in W^{2, \infty}(\Omega) \), then

\[
A(u - u_p, v_h) = \begin{cases} O(h^{k+2})\|u\|_{k+2}\|v_h\|_2, & \forall v_h \in V^h_0, \\ O(h^{k+1.5})\|u\|_{k+2}\|v_h\|_2, & \forall v_h \in V^h. \end{cases}
\]

If \( a_{12} = a_{21} \equiv 0 \), then

\[
A(u - u_p, v_h) = O(h^{k+2})\|u\|_{k+2}\|v_h\|_2, \quad \forall v_h \in V^h.
\]

Let \( a_I, b_I \) and \( c_I \) denote the corresponding piecewise \( Q^k \) Lagrange interpolation at Gauss-Lobatto points. We are interested in the solution \( \tilde{u}_h \in V^h_0 \) to

\[
A_I(\tilde{u}_h, v_h) := \iint_{\Omega} (\nabla v_h^T a_I \nabla \tilde{u}_h + b_I \nabla \tilde{u}_h v_h + c_I \tilde{u}_h v_h) \, dx dy = (f, v_h)_h, \quad \forall v_h \in V^h_0.
\]

We need to assume that \( A_I \) still satisfies coercivity \( A_I(v, v) \geq C\|v\|_1 \) and boundedness \( |A_I(u, v)| \leq C\|u\|_1\|v\|_1 \) for any \( u, v \in H^1_0(\Omega) \), so that the solution \( u \in H^1_0(\Omega) \) of the following problem exists and is unique:

\[
A_I(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega).
\]
We also need the elliptic regularity to hold for the dual problem:

\[ A_I(v, w) = (f, v), \quad \forall v \in H^1_0(\Omega). \]

For instance, if \( b \equiv 0 \), it suffices to require that eigenvalues of \( a_I + c_I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) has a uniform positive lower bound on \( \Omega \), which is achievable on fine enough meshes if \( a + c \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) are positive definite. This implies the coercivity of \( A_I \). The boundedness of \( A_I \) follows from the smoothness of coefficients. Since \( a_I \) and \( c_I \) are Lipschitz continuous, the elliptic regularity for \( A_I \) holds on a convex domain [14].

By Lemma 4.1 and Lemma 4.2, it is straightforward to extend Theorem 4.2 to the general elliptic case:

**Theorem 4.4.** For \( k \geq 2 \), assume \( a_{ij}, b_i, c \in W^{k+2,\infty}(\Omega) \) and \( u, f \in H^{k+2}(\Omega) \), then

\[ A_I(\tilde{u}_h - u_p, v_h) = \begin{cases} O(h^{k+2})(\|u\|_{k+2} + \|f\|_{k+2})\|v_h\|_2, & \forall v_h \in V^h_0, \\ O(h^{k+1.5})(\|u\|_{k+2} + \|f\|_{k+2})\|v_h\|_2, & \forall v_h \in V^h. \end{cases} \]

And if \( a_{12} = a_{21} = 0 \), then

\[ A_I(\tilde{u}_h - u_p, v_h) = O(h^{k+2})(\|u\|_{k+2} + \|f\|_{k+2})\|v_h\|_2, \quad \forall v_h \in V^h. \]

With suitable assumptions, it is straightforward to extend the proof of Theorem 4.3 to the general case:

**Theorem 4.5.** For \( k \geq 2 \), assume \( a_{ij}, b_i, c \in W^{k+2,\infty}(\Omega) \) and \( u, f \in H^{k+2}(\Omega) \). Assume the approximated bilinear form \( A_I \) satisfies coercivity and boundedness and the elliptic regularity still holds for the dual problem of \( A_I \). Then \( \tilde{u}_h \) is a \((k+2)\)-th order accurate approximation to \( u \) in the discrete 2-norm over all the \((k+1) \times (k+1)\) Gauss-Lobatto points:

\[ \|\tilde{u}_h - u\|_{2, Z_0} = O(h^{k+2})(\|u\|_{k+2} + \|f\|_{k+2}). \]

**Remark 5.** With Neumann type boundary conditions, due to Lemma 3.7, we can only prove \((k+1.5)\)-th order accuracy

\[ \|\tilde{u}_h - u\|_{2, Z_0} = O(h^{k+1.5})(\|u\|_{k+2} + \|f\|_{k+2}), \]

unless there are no mixed second order derivatives in the elliptic equation, i.e., \( a_{12} = a_{21} \equiv 0 \).

5. **Numerical results.** In this section we show some numerical tests of \( C^0-Q^2 \) finite element method on an uniform rectangular mesh and verify the order of accuracy at \( Z_0 \), i.e., all Gauss-Lobatto points. The following four schemes will be considered:
1. Full \( Q^2 \) finite element scheme (1.3) where integrals in the bilinear form are approximated by \( 5 \times 5 \) Gauss quadrature rule, which is exact for \( Q^5 \) polynomials thus exact for \( A(u_h, v_h) \) if the variable coefficient is a \( Q^5 \) polynomial.
2. The Gauss-Lobatto quadrature scheme (1.8): all integrals are approximated by \( 3 \times 3 \) Gauss-Lobatto quadrature.
3. The schemes (1.9) and (1.10).

The last three schemes are finite difference type since only grid point values of the coefficients are needed. In (1.9) and (1.10), \( A_I(u_h, v_h) \) can be exactly computed by
4 × 4 Gauss quadrature rule since coefficients are $Q^2$ polynomials. An alternative finite difference type implementation of (1.9) and (1.10) is to precompute integrals of Lagrange basis functions and their derivatives to form a sparse tensor, then multiply the tensor to the vector consisting of point values of the coefficient to form the stiffness matrix.

5.1. Accuracy. We consider the following example with either purely Dirichlet or purely Neumann boundary conditions:

\[ \nabla \cdot (a \nabla u) = f \quad \text{on } [0, 1] \times [0, 2] \]

where $a(x, y) = 1 + 0.1x^3y^5 + \cos(x^3y^2 + 1)$ and $u(x, y) = 0.1(\sin(\pi x) + x^3)(\sin(\pi y) + y^3) + \cos(x^4 + y^3)$. The nonhomogeneous boundary condition should be computed in a way consistent with the computation of integrals in the bilinear form. The errors at $Z_0$ are shown in Table 1 and Table 2. We can see that the four schemes are all fourth order in the discrete 2-norm on $Z_0$. Even though we did not discuss the max norm error on $Z_0$ in this paper, we should expect a $|\ln h|$ factor in the order of $l^\infty$ error over $Z_0$ due to (1.7), which was proven upon the discrete Green’s function.

### Table 1

*The errors of $C^0$-$Q^2$ for a Poisson equation with Dirichlet boundary conditions at Lobatto points.*

| Mesh   | $l^2$ error | $l^\infty$ error | order | $l^\infty$ error | order |
|--------|-------------|------------------|-------|------------------|-------|
| 10 × 10| 2.22E-1     | -                | -     | 3.96E-1          | -     |
| 20 × 20| 4.83E-2     | 2.20             | 1.51E-1| 3.96E-1          | -     |
| 40 × 40| 2.54E-3     | 4.25             | 1.10E-2| 3.71             | -     |
| 80 × 80| 1.49E-4     | 4.09             | 7.52E-4| 3.95             | -     |
| 160 × 160| 9.22E-6   | 4.01             | 5.14E-5| 3.87             | -     |

| Mesh   | $l^2$ error | $l^\infty$ error | order | $l^\infty$ error | order |
|--------|-------------|------------------|-------|------------------|-------|
| 10 × 10| 2.24E-1     | -                | -     | 4.30E-1          | -     |
| 20 × 20| 4.43E-2     | 2.34             | 1.37E-1| 1.65             | -     |
| 40 × 40| 2.27E-3     | 4.29             | 8.61E-3| 4.00             | -     |
| 80 × 80| 1.32E-4     | 4.11             | 4.87E-4| 4.14             | -     |
| 160 × 160| 8.13E-6   | 4.02             | 3.09E-5| 3.97             | -     |

| Mesh   | $l^2$ error | $l^\infty$ error | order | $l^\infty$ error | order |
|--------|-------------|------------------|-------|------------------|-------|
| 10 × 10| 2.78E-1     | -                | -     | 6.31E-1          | -     |
| 20 × 20| 2.76E-2     | 3.33             | 8.69E-2| 2.86             | -     |
| 40 × 40| 1.28E-3     | 4.43             | 3.77E-3| 4.53             | -     |
| 80 × 80| 8.96E-5     | 3.83             | 3.36E-4| 3.49             | -     |
| 160 × 160| 5.79E-6   | 3.95             | 2.41E-5| 3.80             | -     |

| Mesh   | $l^2$ error | $l^\infty$ error | order | $l^\infty$ error | order |
|--------|-------------|------------------|-------|------------------|-------|
| 10 × 10| 1.48E-2     | -                | -     | 3.79E-2          | -     |
| 20 × 20| 1.05E-2     | 0.50             | 3.76E-2| 0.01             | -     |
| 40 × 40| 7.32E-4     | 3.84             | 4.04E-3| 3.22             | -     |
| 80 × 80| 4.54E-5     | 4.01             | 2.83E-4| 3.83             | -     |
| 160 × 160| 2.85E-6   | 3.99             | 1.75E-5| 4.02             | -     |
Table 2
The errors of $C^0$-$Q^2$ for a Poisson equation with Neumann boundary conditions at Lobatto points.

| Mesh    | $l^2$ error  | order | $l^\infty$ error | order |
|---------|--------------|-------|------------------|-------|
| 10 $\times$ 10 | 3.44E0 | -     | 5.39E0 | -     |
| 20 $\times$ 20 | 1.83E-1 | 4.23  | 3.51E-1 | 3.93  |
| 40 $\times$ 40 | 1.38E-2 | 3.73  | 3.43E-2 | 3.36  |
| 80 $\times$ 80 | 8.37E-4 | 4.04  | 2.21E-3 | 3.96  |
| 160 $\times$ 160 | 5.13E-5 | 4.03  | 1.41E-4 | 3.96  |

FEM using Gauss Lobatto Quadrature (1.8)

| Mesh    | $l^2$ error  | order | $l^\infty$ error | order |
|---------|--------------|-------|------------------|-------|
| 10 $\times$ 10 | 3.43E0 | -     | 4.95E0 | -     |
| 20 $\times$ 20 | 1.81E-1 | 4.25  | 3.11E-1 | 3.99  |
| 40 $\times$ 40 | 1.37E-2 | 3.72  | 2.81E-2 | 3.47  |
| 80 $\times$ 80 | 8.33E-4 | 4.04  | 1.76E-3 | 4.00  |
| 160 $\times$ 160 | 5.11E-5 | 4.03  | 1.12E-4 | 3.97  |

FEM with Approximated Coefficients (1.10)

| Mesh    | $l^2$ error  | order | $l^\infty$ error | order |
|---------|--------------|-------|------------------|-------|
| 10 $\times$ 10 | 3.64E0 | -     | 5.06E0 | -     |
| 20 $\times$ 20 | 1.60E-1 | 4.51  | 2.54E-2 | 4.32  |
| 40 $\times$ 40 | 1.26E-2 | 3.67  | 2.39E-2 | 3.41  |
| 80 $\times$ 80 | 7.67E-4 | 4.03  | 1.67E-3 | 3.84  |
| 160 $\times$ 160 | 4.71E-5 | 4.03  | 1.09E-4 | 3.94  |

Full FEM Scheme

| Mesh    | $l^2$ error  | order | $l^\infty$ error | order |
|---------|--------------|-------|------------------|-------|
| 10 $\times$ 10 | 8.45E-2 | -     | 2.13E-1 | -     |
| 20 $\times$ 20 | 1.56E-2 | 2.43  | 5.66E-2 | 1.91  |
| 40 $\times$ 40 | 9.12E-4 | 4.10  | 5.14E-3 | 3.46  |
| 80 $\times$ 80 | 5.47E-5 | 4.06  | 3.24E-4 | 3.99  |
| 160 $\times$ 160 | 3.37E-6 | 4.02  | 2.22E-5 | 3.87  |

Next we consider an elliptic equation with purely Neumann boundary conditions:

$$
\nabla \cdot (a \nabla u) + cu = f \quad \text{on} \ [0,1] \times [0,2]
$$

where $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $a_{11} = 10 + 30y^5 + x \cos y + y$, $a_{12} = a_{21} = 2 + 0.5(\sin(\pi x) + x^3)(\sin(\pi y) + y^3) + \cos(x^3 + y^3)$, $a_{22} = 10 + x^5$, $c = 1 + x^4 y^3$ and $u(x, y) = 0.1(\sin(\pi x) + x^3)(\sin(\pi y) + y^3) + \cos(x^4 + y^3)$. The errors at $Z_0$ are listed in Table 3. Even though only $O(h^{3.5})$ can be proven due to the mixed second order derivatives and the Neumann boundary conditions as discussed in Remark 5, we still observe fourth order accuracy for (1.9) and (1.10) in this example.

5.2. Robustness. In Table 1 and Table 2, the errors of (1.9), (1.10) and the Gauss Lobatto quadrature scheme (1.8) are close to one another. We observe that the scheme (1.10) tends to be more accurate than (1.9) and (1.8) when the coefficient $a(x, y)$ is closer to zero in the Poisson equation. See Table 4 for errors of solving $\nabla \cdot (a \nabla u) = f \quad \text{on} \ [0,1] \times [0,2]$ with Dirichlet boundary conditions, $a(x, y) = 1 + \varepsilon x^3 y^5 + \cos(x^3 y^2 + 1)$ and $u(x, y) = 0.1(\sin(\pi x) + x^3)(\sin(\pi y) + y^3) + \cos(x^4 + y^3)$.
Table 3
An elliptic equation with mixed second order derivatives and Neumann boundary conditions.

| Mesh | $l^2$ error | order | $l^\infty$ error | order |
|------|-------------|-------|------------------|-------|
| $10 \times 10$ | 9.46E0 | - | 9.00E0 | - |
| $20 \times 20$ | 1.49E0 | 2.67 | 1.56E0 | 2.53 |
| $40 \times 40$ | 8.07E-2 | 4.21 | 1.15E-1 | 3.76 |
| $80 \times 80$ | 4.79E-3 | 4.08 | 7.30E-3 | 3.98 |
| $160 \times 160$ | 2.93E-4 | 4.03 | 4.77E-4 | 3.94 |

FEM using Gauss Lobatto Quadrature (1.8)

| Mesh | $l^2$ error | order | $l^\infty$ error | order |
|------|-------------|-------|------------------|-------|
| $10 \times 10$ | 9.36E0 | - | 8.24E0 | - |
| $20 \times 20$ | 1.51E0 | 2.63 | 1.12E0 | 2.88 |
| $40 \times 40$ | 8.18E-2 | 4.21 | 8.35E-2 | 3.74 |
| $80 \times 80$ | 4.88E-3 | 4.07 | 8.54E-3 | 3.29 |
| $160 \times 160$ | 3.05E-4 | 4.00 | 1.09E-3 | 2.97 |

FEM with Approximated Coefficients (1.10)

| Mesh | $l^2$ error | order | $l^\infty$ error | order |
|------|-------------|-------|------------------|-------|
| $10 \times 10$ | 9.37E0 | - | 8.32E0 | - |
| $20 \times 20$ | 1.51E0 | 2.63 | 1.12E0 | 2.89 |
| $40 \times 40$ | 8.17E-2 | 4.21 | 7.36E-2 | 3.93 |
| $80 \times 80$ | 4.84E-3 | 4.08 | 5.00E-3 | 3.88 |
| $160 \times 160$ | 2.96E-4 | 4.03 | 3.38E-4 | 3.89 |

Full FEM Scheme

| Mesh | $l^2$ error | order | $l^\infty$ error | order |
|------|-------------|-------|------------------|-------|
| $10 \times 10$ | 1.46E-1 | - | 4.31E-1 | - |
| $20 \times 20$ | 1.64E-2 | 3.16 | 6.55E-2 | 2.71 |
| $40 \times 40$ | 7.08E-4 | 4.53 | 3.42E-3 | 4.26 |
| $80 \times 80$ | 4.44E-5 | 4.06 | 4.84E-4 | 2.82 |
| $160 \times 160$ | 2.95E-6 | 3.85 | 7.96E-5 | 2.60 |

where $\varepsilon = 0.001$. Here the smallest value of $a(x, y)$ is around $\varepsilon = 0.001$. We remark that the difference among three schemes is much smaller for larger $\varepsilon$ such as $\varepsilon = 0.1$ as in Table 1.

6. Concluding remarks. We have shown that the classical superconvergence of functions values at Gauss Lobatto points in $C^0$-$Q^k$ finite element method for an elliptic problem still holds if replacing the coefficients by their piecewise $Q^k$ Lagrange interpolant at the Gauss Lobatto points. Such a superconvergence result can be used for constructing a fourth order accurate finite difference type scheme by using $Q^2$ approximated variable coefficients. Numerical tests suggest that this is an efficient and robust implementation of $C^0$-$Q^2$ finite element method without affecting the superconvergence of function values.

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REFERENCES
Table 4

| Mesh   | $l^2$ error | $l^\infty$ error | order | $l^\infty$ error | order |
|--------|-------------|-------------------|-------|-------------------|-------|
| 10 × 10| 2.78E-1     | 4.52E-1           | -     | -                 | -     |
| 20 × 20| 6.22E-2     | 2.16              | 1.12  | 2.08E-1           |       |
| 40 × 40| 1.09E-2     | 2.51              | 1.30  | 8.44E-2           |       |
| 80 × 80| 1.31E-3     | 3.05              | 2.22  | 1.81E-2           |       |
| 160 × 160| 1.08E-4    | 3.60              | 3.38  | 1.75E-3           |       |
| 160 × 160| 7.24E-6    | 3.90              | 3.53  | 1.52E-4           |       |

FEM using Gauss Lobatto Quadrature (1.8)

| Mesh   | $l^2$ error | $l^\infty$ error | order | $l^\infty$ error | order |
|--------|-------------|-------------------|-------|-------------------|-------|
| 10 × 10| 2.81E-1     | 4.59E-1           | -     | -                 | -     |
| 20 × 20| 4.69E-2     | 1.37E-1           | 1.74  | 1.87              |       |
| 40 × 40| 5.06E-3     | 3.21              | 1.87  | 3.75E-2           |       |
| 80 × 80| 7.04E-4     | 2.85              | 2.25  | 7.86E-3           |       |
| 160 × 160| 6.74E-5    | 3.39              | 2.70  | 1.21E-3           |       |
| 320 × 320| 4.94E-6    | 3.77              | 3.37  | 1.17E-4           |       |

FEM with Approximated Coefficients (1.10)

| Mesh   | $l^2$ error | $l^\infty$ error | order | $l^\infty$ error | order |
|--------|-------------|-------------------|-------|-------------------|-------|
| 10 × 10| 2.68E-1     | 5.48E-1           | -     | -                 | -     |
| 20 × 20| 2.91E-1     | 1.59E-1           | 1.78  | 4.02E-2           |       |
| 40 × 40| 3.21E-3     | 3.05              | 1.98  | 4.02E-2           |       |
| 80 × 80| 2.86E-4     | 3.62              | 3.48  | 3.60E-3           |       |
| 160 × 160| 1.86E-5    | 3.94              | 3.96  | 2.31E-4           |       |
| 320 × 320| 1.17E-6    | 4.00              | 3.91  | 1.53E-5           |       |

[1] S. Agmon, *Lectures on elliptic boundary value problems*, vol. 369, American Mathematical Soc., 2010.
[2] M. Bakker, *A note on $C^0$ Galerkin methods for two-point boundary problems*, Numerische Mathematik, 38 (1982), pp. 447–453.
[3] F. Brezzi and L. Marini, *On the numerical solution of plate bending problems by hybrid methods*, Revue française d’automatique, informatique, recherche opérationnelle. Analyse numérique, 9 (1975), pp. 5–50.
[4] C. Chen, *Superconvergent points of Galerkin’s method for two point boundary value problems*, Numerical Mathematics A Journal of Chinese Universities, 1 (1979), pp. 73–79.
[5] C. Chen, *Superconvergence of finite element solutions and their derivatives*, Numerical Mathematics A Journal of Chinese Universities, 3 (1981), pp. 118–125.
[6] C. Chen, *Structure theory of superconvergence of finite elements (In Chinese)*, Hunan Science and Technology Press, Changsha, 2001.
[7] C. Chen and S. Hu, *The highest order superconvergence for bi-$k$ degree rectangular elements at nodes: a proof of $2k$-conjecture*, Mathematics of Computation, 82 (2013), pp. 1337–1355.
[8] P. G. Ciarlet, *Basic error estimates for elliptic problems*, Handbook of Numerical Analysis, 2 (1991), pp. 17–351.
[9] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Society for Industrial and Applied Mathematics, 2002.
[10] P. G. Ciarlet and P.-A. Raviart, *The combined effect of curved boundaries and numerical integration in isoparametric finite element methods*, in The mathematical foundations of the finite element method with applications to partial differential equations, Elsevier, 1972, pp. 409–474.
[11] J. Douglas, *Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary problems*, Topics in numerical analysis, (1973), pp. 89–92.
[12] J. Douglas and T. Dupont, *Galerkin approximations for the two point boundary problem*
using continuous, piecewise polynomial spaces, Numerische Mathematik, 22 (1974), pp. 99–109.

[13] J. Douglas Jr, T. Dupont, and M. F. Wheeler, An $L^\infty$ estimate and a superconvergence result for a Galerkin method for elliptic equations based on tensor products of piecewise polynomials, (1974).

[14] P. Grisvard, Elliptic problems in nonsmooth domains, vol. 69, SIAM, 2011.

[15] P. Lesaint and M. Zlamal, Superconvergence of the gradient of finite element solutions, RAIRO. Analyse numérique, 13 (1979), pp. 139–166.

[16] Q. Lin and N. Yan, Construction and Analysis for Efficient Finite Element Method (In Chinese), Hebei University Press, 1996.

[17] Q. Lin, N. Yan, and A. Zhou, A rectangle test for interpolated finite elements, in Proc. Sys. Sci. and Sys. Eng. (Hong Kong), Great Wall Culture Publ. Co, 1991, pp. 217–229.

[18] K. Smith, Inequalities for formally positive integro-differential forms, Bulletin of the American Mathematical Society, 67 (1961), pp. 368–370.

[19] L. Wahlbin, Superconvergence in Galerkin finite element methods, Springer, 2006.