Exact solutions of the classical Boussinesq system

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ABSTRACT
In this paper, we study exact solutions of the classical Boussinesq (CB) system, which describes propagations of shallow water waves. By using the bilinear form, with exponential expansions, we obtain solitary wave solutions of the CB system. Based on asymptotic analysis method, we study the elastic and elastic-inelastic-coupled interactions of the obtained solitary wave solutions. With extended three-wave method, we obtain the periodic solitary solution of the CB system. And with polynomial expansions, we get the rational solutions of the CB system. These interesting exact solutions may be useful in the study of some phenomena appeared in shallow water waves.

1. Introduction
The study of the exact solutions has been an interesting issue in both experimental and theoretical research, which can explain the nonlinear phenomena of fluid dynamics, elastica dynamics, plasma, etc. In recent years, different types of exact solutions of nonlinear partial differential equations have been obtained, such as soliton (Cai, Bai, & Luo, 2017; Ekici, Mirzazadeh, & Eslami, 2016; Korkmaz, Akbar, & Alam, 2014; Liu, Ma, Chen, & Khalique, 2016a; Lü, Ma, Zhou, & Khalique, 2016b; Shi, Zhao, & Ma, 2015; Zhang & Ma, 2015), periodic solitary solution and rational solution (Dai, Lin, Fu, & Zeng, 2010; Fan & Hon, 2008; He & Abdou, 2007; Lax, 1976) and rational solutions (Lü, Ma, Zhou, & Khalique, 2016a; Liu, Ma, Zhou, & Khalique, 2016b; Shi, Zhao, & Ma, 2015; Zhang & Ma, 2015).

To seek the exact solutions of the nonlinear partial differential equations, many methods have been proposed, for instance the inverse scattering transformation (IST) (Ablowitz & Segur, 2000), Bäcklund transformation (Rogers & Schief, 2002), Painlevé analysis method (Chowdhury, 1999), Darboux transformation (DT) (Gu, Hu, & Zhou, 2005), Hirota direct method (Hirota, 2004), the tanh-function method (Hirota, 2004), the generalized Kudryashov method (Khan, Akbar, & Alam, 2015a, 2015b), the Exp-function method (He & Abdou, 2007; Khan & Akbar, 2014), the modified simple equation method (Akbar & Ali, 2011; Akbar, Ali, & Mohyud-Din, 2013; Alam & Akbar, 2013), the $G'/G$-expansion and extend $G'/G$-expansion method (Akbar, Ali, & Mohyud-Din, 2013; Alam & Akbar, 2013, 2015; Alam, Hafez, Belgacem, & Akbar, 2015b; Zayed & Shorog, 2010), the Exp-function method (He & Abdou, 2007; Khan & Akbar, 2014), the generalized Kudryashov method (Khan & Akbar, 2016), the exp$(-\Phi(\eta))$-expansion and exp$($Φ(η))$-method (Alam, Hafez, Akbar, & Roshid, 2015a; Roshid & Rahman, 2014), the extended three-wave method (Dai et al., 2010; Li, Dai, & Liu, 2011; Singh & Gupta, 2016; Wang, Dai, & Liang, 2010).

In this paper, based on the bilinear form, we consider exact solutions including solitary wave solution, periodic solitary solution and rational solution of the classical Boussinesq (CB) system (see Wu & Zhang, 1996)

$$
\begin{align*}
&u_t + [(1 + u)v]_x + \frac{1}{4}v_{xxx} = 0, \\
&v_t + v v_x + u_x = 0,
\end{align*}
$$

where, $u$ is the elevation of the water wave and $v$ is the surface velocity of water along $x$-direction. This system was introduced in Wu and Zhang (1996), which is derived from Euler equation. This system can be used to study the run-up of ocean waves such as tsunami waves on dykes and dams. A good understanding of exact solutions is helpful in harbour and coastal design. Therefore, finding more type of solutions is very important in fluid dynamics.

In Li, Ma, and Zhang (2000); Li and Zhang (2001, 2003); Zhang, Chang, and Li (2009); Zhang and Li (2003); Zhang, Zhao, and Chen (2015), by using Darboux transformation, the authors obtained the bidirectional solitons on water and the
elastic–fusion–coupled interaction of the CB system (1). In Roshid and Rahman (2014); Zayed and Shorog (2010); Zhang et al. (2002), by using of Tanh function, extended \((G'/G)-\) expansion and \(\exp(-\Phi(\eta))\)-expansion, the authors obtained the traveling wave solutions including hyperbolic solution, trigonometric solution, periodic solution of the CB system (1).

Through the following proper transformation
\[
u = -[\ln(f/g)]_x ,
\]
the CB system (1) can be transformed into the bilinear form
\[
(D_t - \frac{1}{2}D_x^2)f \cdot g = 0 , \quad (D_x D_t - \frac{1}{2}D_x^3)f \cdot g = 0.
\]

In Section 2, based on the bilinear form in (3) and assuming \(f\) and \(g\) have exponential expansions, we can obtain solitary wave solutions of the CB system (1), including the bidirectional soliton and the elastic-inelastic-coupled solitary wave solution. By using asymptotic analysis method (Chakravarty & Kodama, 1987; Wang, Tian, Li, Wang, & Jiang, 2014; Zhang & Chen, 2016), we analyze the interactions of the obtained solutions. In Section 3, by using extended three–wave method, we obtain single soliton solution, periodic solitary wave solution, and solitary wave solution with fission of the CB system (1). In Section 4, taking \(f\) and \(g\) as polynomial expansions (Lü et al., 2016a, 2016b; Shi et al., 2015; Zhang & Ma, 2015), we get rational solutions of the CB system (1). In Section 5, we give our conclusions.

2. Solitary wave solutions of the CB system

In this section, based on parameter expansions and taking \(f\) and \(g\) as exponential forms, solitary wave solutions of the CB system (1) are obtained according to the bilinear form (3). Then by using asymptotic analysis method, we study propagations and interactions of the solitary wave solutions of the CB system (1).

In order to get the bilinear derivative Equation (3), we assume that \(f\) and \(g\) have the following asymmetric form expansions of the bookkeeping parameter \(\varepsilon\)
\[
f = 1 + \varepsilon f_1(x, t) + \varepsilon^2 f_2(x, t) + \varepsilon^3 f_3(x, t) + \cdots,
\]
\[
g = \varepsilon g_1(x, t) + \varepsilon^2 g_2(x, t) + \varepsilon^3 g_3(x, t) + \cdots.
\]

Substituting (4) and (5) into the bilinear form (3), we have
\[
f = 1 + \sum_{i=1}^{N} h_i e^{\xi_i}, \quad g = \sum_{i=1}^{N} e^{\xi_i} + \sum_{1 \leq i < j \leq N} l_{ij} e^{\xi_i + \xi_j},
\]
where,
\[
\xi_i = k_i x - \frac{k_i^2}{2} t + \alpha_i, \quad h_i = \frac{k_i^2}{k_j^2}, \quad l_{ij} = \frac{k_i^2 (k_i - k_j)^2}{k_j^2 k_j^2}.
\]

\(k_i\) and \(\alpha_i\) \((i = 1, 2, \cdots, N)\) are arbitrary constants. Substituting (6) into the transformation (2), we can obtain the multi-solitary wave solutions of the CB system (1). In the following discussions, we give detailed analysis for the cases of \(N = 1, 2, 3\).

For \(N = 1\), based on (6), we obtain
\[
f = 1 + e^{\xi_i}, \quad g = e^{\xi_i}, \quad \xi_i = k_i x - \frac{k_i^2}{2} t + \alpha_i.
\]

Substituting \(f\) and \(g\) into (2), we get the single soliton solution of (1)
\[
u = \frac{k_i^2}{8} \text{sech}^2 \frac{\xi_i}{2} - 1, \quad \nu = \frac{k_i}{2} - \frac{k_i}{2} \tanh \frac{\xi_i}{2}.
\]

where, \(u\) is a bell-soliton solution and \(v\) is a kink-soliton solution with velocity \(k_i/2\) (see Figure 1).

For \(N = 2\), we have
\[
f = 1 + e^{\xi_1} + e^{\xi_2} + \frac{k_1^2}{k_2^2} e^{i_2}, \quad
\]
\[
g = e^{\xi_1} + e^{\xi_2} + \frac{(k_1 - k_2)^2}{k_2^2} e^{\xi_1 + \xi_2},
\]
\[
\xi_i = k_i x - \frac{k_i^2}{2} t + \alpha_i, \quad i = 1, 2.
\]

Substituting \(f\) and \(g\) into (2), we get the solution \((u, v)\) of the CB system (1). We consider the collisions

![Figure 1. Plots of the single soliton solution (9) of the CB system (1) with \(k_1 = 6, \alpha_1 = 3\).](image-url)
of the solution $u$ through the asymptotic analysis method. The analysis for $v$ is similar and is omitted for brevity. Without loss of generality, we let $0 < k_1 < k_2$ and we have the following asymptotic expressions.

1. Before the interactions ($t \to -\infty$):

   (a) $\xi_1 \sim 0, \xi_2 \sim +\infty, \quad u_{01} = \frac{k_1^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_2}{2} \right) + \Delta_{01}} - 1$, (11)

   (b) $\xi_2 \sim 0, \xi_1 \sim -\infty, \quad u_{02} = \frac{k_2^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_1}{2} \right) + \Delta_{02}} - 1$, (12)

   (c) $\xi_2 \sim 0, \xi_1 \sim -\infty, \quad u_{12} = \frac{(k_2 - k_1)^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_2 - \xi_1}{2} \right) - 1}$, (13)

2. After the interactions ($t \to +\infty$):

   (a) $\xi_1 \sim 0, \xi_2 \sim -\infty, \quad u_{01}^+ = \frac{k_1^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_1}{2} \right) - 1}$, (14)

   (b) $\xi_2 \sim 0, \xi_1 \sim +\infty, \quad u_{02}^+ = \frac{k_2^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_2}{2} \right) + \Delta_{02}^+} - 1$, (15)

   (c) $\xi_2 \sim 0, \xi_1 \sim +\infty, \quad u_{12}^+ = \frac{(k_2 - k_1)^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_2 - \xi_1}{2} + \Delta_{12}^+ \right) - 1}$, (16)

where, $\Delta_{01} = \ln((k_2 - k_1)/k_2), \quad \Delta_{02} = \ln(k_1/k_2), \quad \Delta_{02}^+ = \ln((k_2 - k_1)/k_2), \quad \Delta_{12}^+ = \ln(k_1/k_2).

Thus, the solution $u$ in (2) has the following asymptotic expression

$$u \to \begin{cases} u_{01}^+ + u_{02}^+ + u_{12}^+, & t \to -\infty, \\ u_{01}^+ + u_{02}^+ + u_{12}^+, & t \to +\infty. \end{cases} \quad (17)$$

From (11)–(17), we find that after the collision, the speeds and the amplitudes of the three-bell-type solitons of $u$ remain unchanged except the phase shifts and then the collision is elastic (see Figure 2). Therefore, the solution $u$ is a solitary wave solution with elastic interaction.

For $N = 3$, based on (6), we obtain

$$\begin{align*}
 f &= 1 + e^{\xi_1} + \frac{k_1^2}{k_2^2} e^{\xi_1} + \frac{k_1^2}{k_3^2} e^{\xi_3}, \\
 g &= e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + \left( \frac{(k_1 - k_2)^2}{k_2^2} e^{\xi_1} + \frac{(k_2 - k_3)^2}{k_3^2} e^{\xi_3} \right),
\end{align*} \quad (18)$$

where, $\xi_2 = k_3 x - tk_3^2/2 + \alpha_i (i = 1, 2, 3)$. Substituting $f$ and $g$ into (2), we get the solution $(u, v)$ of the CB system (1). Without loss of generality, we let $0 < k_1 < k_2 < k_3$ and then we have the following asymptotic expression

$$u \to \begin{cases} \tilde{u}_{02} + \tilde{u}_{03} + \tilde{u}_{13}, & t \to -\infty, \\ \tilde{u}_{01} + \tilde{u}_{02} + \tilde{u}_{12} + \tilde{u}_{13} + \tilde{u}_{23}, & t \to +\infty, \end{cases} \quad (19)$$

where,

$$\begin{align*}
 \tilde{u}_{02} &= \frac{k_2^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_2}{2} \right) + \Delta_{02}^+} - 1, \\
 \tilde{u}_{03} &= \frac{k_3^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_3}{2} \right) + \Delta_{03}^+} - 1, \\
 \tilde{u}_{13} &= \frac{(k_3 - k_1)^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_3 - \xi_1}{2} + \Delta_{13}^+ \right) - 1}, \\
 \tilde{u}_{01}^+ &= \frac{k_1^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_1}{2} \right) - 1}, \\
 \tilde{u}_{02}^+ &= \frac{k_2^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_2}{2} \right) + \Delta_{02}^+ - 1}, \\
 \tilde{u}_{12}^+ &= \frac{(k_2 - k_1)^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_2 - \xi_1}{2} + \Delta_{12}^+ \right) - 1}, \\
 \tilde{u}_{13}^+ &= \frac{(k_3 - k_1)^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_3 - \xi_1}{2} + \Delta_{13}^+ \right) - 1}, \\
 \tilde{u}_{23}^+ &= \frac{(k_3 - k_2)^2}{8} \frac{1}{\mathrm{sech}^2 \left( \frac{\xi_3 - \xi_2}{2} + \Delta_{23}^+ \right) - 1},
\end{align*} \quad (20)$$

and $\Delta_{02} = \ln((k_3 - k_2)/k_2), \quad \Delta_{03} = \ln(k_1/k_3), \quad \Delta_{12} = \ln((k_2 - k_1)/k_2), \quad \Delta_{13} = \ln(k_1/k_3), \quad \Delta_{23} = \ln(k_2/k_3).$

From the asymptotic expression in (19), we find that there are three waves before collision and there are five waves after collision. Taking $k_1 = 1, k_2 = 5/2, k_3 = 7/2, \alpha_1 = \alpha_3 = 0$, for different

![Figure 2](image_url)  
**Figure 2.** Plots for the three-soliton solution (17) of the CB system (1) with $k_1 = 3, k_2 = 5, k_3 = 7/2$. \alpha_1 = \alpha_3 = 0$. For different
values of \( \alpha_2 \), we will find different interaction phenomena. If we let \( \alpha_2 = -20 \), we find that \( \tilde{u}_{13} \) splits into \( \tilde{u}_{12} \) and \( \tilde{u}_{23} \); \( \tilde{u}_{03} \) splits into \( \tilde{u}_{01} \) and \( \tilde{u}_{13} \); \( \tilde{u}_{02} \) is transformed into \( \tilde{u}_{02} \) which is invariant except the phase shift. If \( \alpha_2 = 0 \), before and after collision, \( \tilde{u}_{02} \), \( \tilde{u}_{02} \), and \( \tilde{u}_{13}, \tilde{u}_{13} \) are the same in velocities and amplitudes except the phase shifts, and then the interaction is an elastic collision. After interaction, \( \tilde{u}_{03} \) splits into three solitary waves \( \tilde{u}_{01}, \tilde{u}_{12} \) and \( \tilde{u}_{23} \), and the interaction is an inelastic collision. If \( \alpha_2 = 20 \), \( \tilde{u}_{02} \) splits into \( \tilde{u}_{01} \) and \( \tilde{u}_{13} \); \( \tilde{u}_{03} \) splits into \( \tilde{u}_{02} \) and \( \tilde{u}_{23} \); \( \tilde{u}_{13} \) is transformed into \( \tilde{u}_{13} \) which is invariant except phase shift. Although there are different interaction behaviours depending on the values of \( \alpha_2 \), there are always three solitary waves before interaction and five after interactions. Therefore, the solution \( u \) is a solitary wave solution with elastic–inelastic–coupled interaction (see Figure 3).

### 3. Periodic solitary wave solutions of the CB System

In this section, based on the bilinear form (3), by using the extended three–wave method, we find the single soliton solution, periodic solitary solution and solitary wave solution of the CB system (1).

We formulate solutions of the bilinear form (3) as

\[
\begin{align*}
    f &= a_1 e^{\eta_1} + a_2 \cos \eta_2 + a_3 \cos \eta_3 + e^{-\eta_1}, \\
    g &= b_1 e^{\eta_1} + b_2 \cos \eta_2 + b_3 \cos \eta_3 + b_4 e^{-\eta_1},
\end{align*}
\]

where, \( \eta_i = k_i x + w_i t, i = 1, 2, 3 \). Substituting (23) into the bilinear form (3) and equating the coefficients of \( e^{\eta_1} \cos \eta_2, e^{-\eta_1} \cos \eta_3, \cdots \) to zeros, we obtain a system of algebraic equations with the unknowns of \( a_i, b_i, w_i (i = 1, 2, 3) \) and \( b_j (j = 1, 2, 3, 4) \) which can be solved by using Mathematica. And we have the following different sets of solutions.

Case 1. \( a_2 = b_1 = b_2 = b_3 = 0, w_1 = k_1^2, k_3 = k_1, w_3 = k_1^2 \).

In this case, from (23), we have

\[
\begin{align*}
    f &= a_1 e^{\eta_1} + a_3 \cosh(k_1 x + k_1^2 t) + e^{-(k_1 x + k_1^2 t)}, \\
    g &= b_4 e^{-(k_1 x + k_1^2 t)}.
\end{align*}
\]

Through the transformation (2), we get the solution of the CB system (1)

\[
\begin{align*}
    \left\{ \begin{array}{l}
        u = -1 + \frac{(2 + a_1)(2a_1 + a_2)k_1^2}{2((a_1 - 1)\sinh(k_1 x + k_1^2 t) + (1 + a_1 + a_2)\cosh(k_1 x + k_1^2 t))^2}, \\
        v = \frac{(2a_1 + a_2)k_1^2}{a_1 + e^{2k_1 x + k_1^2 t} + a_2 e^{2k_1 x + k_1^2 t} \cosh(k_1 x + k_1^2 t)}.
    \end{array} \right.
\]

If we let \( a_1 = 15, a_3 = 1, b_4 = 2 \) and \( k_1 = -2 \), \( u \) is a bell–soliton and \( v \) is a kink–soliton, whose velocity is different from that of the solution (9).

Case 2. \( a_1 = a_2 = a_3 = b_1 = b_2 = 0, w_2 = -k_1 k_2, w_1 = (k_2^2 - k_1^2)/2 \).

In this case, from (23), we have

\[
\begin{align*}
    f &= e^{-\eta_1}, \\
    g &= b_3 \cos \eta_2 + b_4 e^{-\eta_1}.
\end{align*}
\]

And from the transformation (2), we get the solution of the CB system (1)

\[
\begin{align*}
    \left\{ \begin{array}{l}
        u = -1 + \frac{b_3 e^{\eta_1} (b_3 k_2 e^{\eta_1} + 2 b_1 k_1 k_2 \sin \eta_2 + b_4 (k_2^2 - k_1^2) \cos \eta_2)}{(b_3 e^{\eta_1} \cos \eta_2 + b_4)^2}, \\
        v = \frac{b_3 e^{\eta_1} (k_1 \cos \eta_2 - k_3 \sin \eta_2)}{b_3 e^{\eta_1} \cos \eta_2 + b_4}.
    \end{array} \right.
\]

where, \( \eta_1 = k_1 x + (k_2^2 - k_1^2) t/2, \eta_2 = k_2 x - k_1 k_2 t \). This is a periodic solitary wave solution including exponential and trigonometric functions. The contour plots of the solution (27) are shown in Figure 4.

Case 3. \( a_1 = a_2 = a_3 = b_1 = b_2 = 0, w_1 = -(k_1^2 + k_2^2)/2, w_3 = -k_1 k_3 \).

Substituting these parameters into (23), we have

\[
\begin{align*}
    f &= e^{-\eta_1}, \\
    g &= b_3 \cosh \eta_3 + b_4 e^{-\eta_1}.
\end{align*}
\]
Then by the transformation (2), we get the solution of the CB system (1)

\[
\begin{align*}
  u &= -1 + \frac{b_4 e^{\alpha_1} (b_2 k_1 e^{\gamma} + 2 b_4 k_1 \sin h k_4 + b_4 (k_1^2 + k_4^2) \cosh k_4)}{2 (b_4 e^{\alpha_1} \cosh k_4 + b_4)^2}, \\
  v &= \frac{b_4 e^{\alpha_1} (k_1 \sin h k_4 + k_4 \sin h k_4)}{b_4 e^{\alpha_1} \cosh k_4 + b_4},
\end{align*}
\]

(29)

where, \( \eta_1 = k_1 x - (k_1^2 + k_4^2) t / 2 \) and \( \eta_2 = k_4 x - k_1 k_4 t \). When we take \( k_1 = 1 \), \( k_3 = 4 \), \( b_3 = 1 \), \( b_4 = 5 \), this is a solitary wave solution with fission (see Figure 5).

4. Rational solutions of the CB system

In this section, based on polynomial solutions of the bilinear form (3), we consider rational solutions of the CB system (1).

For simplicity, we formulate a polynomial solution of the standard bilinear form (3) as

\[
f = \sum_{i=0}^{n} f_i(t) x^i, \quad g = 1.
\]

Substituting (30) into the bilinear form (3), we get

\[
\begin{align*}
  \sum_{i=0}^{n} f_i'(t) x^i - \frac{\alpha_1 - 2}{2} \sum_{i=0}^{n} (i + 2)(i + 1) f_{i+1}(t) x^i &= 0, \\
  \sum_{i=0}^{n} (i + 1) f_{i+1}'(t) x^i - \frac{\alpha_1 - 2}{2} \sum_{i=0}^{n} (i + 3)(i + 2)(i + 1) f_{i+3}(t) x^i &= 0,
\end{align*}
\]

(31)

from which we can deduce the general formula of \( f \) as

\[
f = \sum_{k=0}^{n} x^{\alpha_k - k} \left( \sum_{i=0}^{n} \frac{\alpha_i^k b_i^k (n - 2i - \gamma)!}{(\alpha - i)! (2^{\alpha_1 - (n - k)!})^i} t^{i-j} \right),
\]

(32)

where, \( \alpha = [k/2], \beta = \text{mod}(k + 1, 2), \gamma = \text{mod}(k, 2), a, \) and \( b_i (i \geq 0) \) are arbitrary constants. From the transformation (2), we find the rational solutions of the CB system (1).

For \( n = 1 \), (32) reads

\[
f = a_1 x + b_1.
\]

(33)

Then the rational solution of the CB system (1) is

\[
u = -1 - \frac{a_1^2}{2(2a_1 x + b_1)^2} v = -\frac{a_1}{a_1 x + b_1}.
\]

(34)

For \( n = 2 \), (32) reads

\[
f = a_1 x^2 + b_1 x + a_1 t + a_2.
\]

(35)

The rational solution of the CB system is

\[
\begin{align*}
  u &= -1 - \frac{(2a_1 x + b_1)^2}{2(a_1 x^2 + b_1 x + a_1 t + a_2)^2} + \frac{a_1}{a_1 x^2 + b_1 x + a_1 t + a_2}, \\
  v &= -\frac{a_1}{a_1 x^2 + b_1 x + a_1 t + a_2}.
\end{align*}
\]

(36)

For \( n = 3 \), (32) is

\[
f = a_1 x^3 + b_1 x^2 + (3a_1 t + a_2)x + b_1 t + b_2.
\]

(37)

And the rational solution of the CB system is

\[
\begin{align*}
  u &= -1 - \frac{(3a_1 x^3 + 2b_1 x + 3a_1 t + a_2)^2}{2(a_1 x^3 + b_1 x^2 + (3a_1 t + a_2)x + b_1 t + b_2)^2} + \frac{a_1}{3a_1 x + b_1} + \frac{a_1 x^2 + b_1 x^2 + (3a_1 t + a_2)x + b_1 t + b_2}{3a_1 x^2 + 2b_1 x + 3a_1 t + a_2}, \\
  v &= -\frac{a_1}{a_1 x^2 + b_1 x^2 + (3a_1 t + a_2)x + b_1 t + b_2}.
\end{align*}
\]

(38)

The contour plots of \( u \) in (34), (36) and (38) are shown in Figure 6.

5. Conclusions

In this paper, based on the bilinear form, via the exponential expansion, the expanded three-wave method and polynomial expansion, explicit exact solutions of the classical Boussinesq system are derived, which include solitary wave solution, periodic solitary solution and rational solution. By using of the asymptotic analysis method, the interactions of the obtained solitary wave solutions are discussed in detail. In addition, plots of the obtained solutions are revealed with the help of Mathematica. These
interesting exact solutions may be useful in the study of some phenomena appeared in shallow water waves.

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References

Ablowitz, M. J., & Segur, H. (2000). Solitons and the inverse scattering transform. SIAM Studies in Applied Mathematics (Vol. 4). Philadelphia: SIAM.

Akbar, M. A., & Ali, N. H. M. (2011). The modified alternative \( (G'/G) \)-expansion method for finding the exact solutions of nonlinear PDEs in mathematical physics. International Journal of Physical Sciences, 6, 7910–7920.

Akbar, M. A., Ali, N. H. M., & Mohyud-Din, S.T. (2013). Further exact traveling wave solutions to the \((2 + 1)\)-dimensional Boussinesq and Kadomtsev-Petviashvili equation. Journal of Computational Analysis and Applications, 15, 557–571.

Akter, J., & Akbar, M. A. (2015). Exact solutions to the Benney-Luke equation and the Phi-4 equations by using modified simple equation method. Results in Physics, 5, 125–130.

Alam, M. N., & Akbar, M. A. (2013). Exact traveling wave solutions of the KP-BBM equation by using the new approach of generalized \((G'/G)\)-expansion method. Springerplus, 2, 617–623.

Alam, M. N., & Akbar, M. A. (2015). Some new exact traveling wave solutions to the simplified MCH equation and the \((1 + 1)\)-dimensional combined KdV-mKdV equations. Journal of the Association of Arab Universities for Basic and Applied Sciences, 17, 6–13.

Alam, M. N., Hafez, M. G., Akbar, M. A., & Roshid, H.O. (2015a). Exact solutions to the \((2 + 1)\)-dimensional Boussinesq equation via exp(\(\Phi/\eta\))–expansion method. Journal of Scientific Research, 7, 1–10.

Alam, M. N., Hafez, M. G., Belgacem, F. B. M., & Akbar, M. A. (2015b). Applications of the novel \((G'/G)\) expansion method to find new exact traveling wave solutions of the nonlinear coupled Higgs field equation. Nonlinear Studies, 22, 613–633.

Cai, Y. J., Bai, C. L., & Luo, Q. L. (2017). Exact soliton solutions for the \((2 + 1)\)-dimensional nonlinear Schrödinger equations in birefringent optical-Fiber communication. Communications in Theoretical Physics, 67, 273–279.

Chakravarty, S., & Kodama, Y. J. (2008). Classification of the line-soliton solutions of KP11. Journal of Physics A Mathematical and Theoretical, 49, 3140–3144.

Chowdhury, A. R. (2009). Painlevé analysis and its applications (Vol. 105). Boca Raton: Chapman and Hall/CRC.

Dai, Z. D., Lin, S. Q., Fu, H. M., & Zeng, X. P. (2010). Exact three-wave solutions for the KP equation. Applied Mathematics and Computation, 216, 1599–1604.

Ekici, M., Mirzazadeh, M., & Eslami, M. (2016). Solitons and other solutions to Boussinesq equation with power law nonlinearity and dual dispersion. Nonlinear Dynamics, 84, 669–676.

Fan, E. G., & Hon, Y. C. (2008). Quasiperiodic waves and asymptotic behavior for Bogoyavlenskii breaking soliton equation in \((2 + 1)\) dimensions. Physical Review E, 78, 036607.

Gu, C. H., Hu, H. S., & Zhou, Z. X. (2005). Darboux transformations in integrable systems. Dordrecht: Springer.

He, J. H., & Abdou. M. A. (2007). New periodic solutions for nonlinear evolution equations using Exp-function method. Chaos Solitons Fractals, 34, 1421–1429.

Hirota, R. (2004). The direct method in soliton theory. Cambridge: Cambridge University Press.

Islam, M. S., Khan, K., Akbar, M. A., & Mastroberardino, A. (2014). A note on improved F-expansion method combined with Riccati equation applied to nonlinear evolution equations. Royal Society Open Science, 1, 140038.

Khan, K., & Akbar, M. A. (2013). Exact and solitary wave solutions for the Tzitzeica-Dodd-Bullough and the modified KdV-Zakharov-Kuznetsov equations using the modified \(G'/G\)-expansion method. Ain Shams Engineering Journal, 4, 903–909.

Khan, K., & Akbar, M. A. (2014). Traveling wave solutions of the \((2 + 1)\)-dimensional Zoomeron equation and the Burgers equations via the MSE method and the Exp-function method. Ain Shams Engineering Journal, 5, 247–256.

Khan, K., & Akbar, M. A. (2016). Solving unsteady Korteweg-de Vries equation and its two alternatives. Mathematical Methods in the Applied Sciences, 39, 2752–2760.
Khan, K., Akbar, M. A., & Alam, M. N. (2013). Traveling wave solutions of the nonlinear Drinfeld-Sokolov-Wilson equation and modified Benjamin-Bona-Mahony equations. Journal of the Egyptian Mathematical Society, 21, 233–240.

Korkmaz, A. (2017a). Exact solutions of space-time fractional EW and modified EW equations. Chaos Solitons Fractals, 96, 132–138.

Korkmaz, A. (2017b). Exact solutions to $(3 + 1)$-dimensional Jimbo-Miwa, Zakharov-Kuznetsov and modified Zakharov-Kuznetsov equations. Communications in Theoretical Physics, 67, 479–482.

Lambert, F., Musette, M., & Kesteloot, E. (1987). Soliton resonances for the good Boussinesq equation. Inverse Problems, 3, 275–288.

Lax, P. D. (1976). Periodic solutions of the KdV equation. SIAM Review, 18, 351–375.

Li, Y. S., & Zhang, J. E. (2000). Darboux transformations of classical Boussinesq system and its new solutions. Physics Letters A, 275, 60–66.

Liu, Y. S., & Zhang, J. E. (2001). Darboux transformations of classical Boussinesq system and its multi-soliton solutions. Physics Letters A, 284, 253–258.

Li, Y. S., & Zhang, J. E. (2003). Bidirectional soliton solutions of the classical Boussinesq system and AKNS system. Chaos Solitons Fractals, 16, 271–277.

Li, Z. T., Dai, Z. D., & Liu, J. (2011). Exact three-wave solutions for the $(3 + 1)$-dimensional Jimbo-Miwa equation. Computers and Mathematics with Applications, 61, 2062–2066.

Ling, L. M., Feng, B. F., & Zhu, Z. N. (2016). Multi-soliton, multi-breather and higher order rogue wave solutions to the complex short pulse equation. Physica D: Nonlinear Phenomena, 327, 13–29.

Lü, X., Ma, W. X., Chen, S. T., & Khalique, C. M. (2016a). A note on rational solutions to a Hirota-Satsuma-like equation. Applied Mathematics Letters, 58, 13–18.

Lü, X., Ma, W. X., Zhou, Y., & Khalique, C. M. (2016b). Rational solutions to an extended Kadomtsev-Petviashvilli-like equation with symbolic computation. Computers and Mathematics with Applications, 71, 1560–1567.

Parke, E. J., & Duffy, B. R. (1996). An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations. Computer Physics Communications, 98, 288–300.

Rogers, C., & Schief, W. K. (2002). Bäcklund and Darboux transformations geometry and modern application in soliton theory. Cambridge: Cambridge University Press.

Roshid, H. O., & Rahman, M. A. (2014). The $\exp(-\Phi(\eta))$-expansion method with application in the $(1 + 1)$-dimensional classical Boussinesq equations. Results in Physics, 4, 150–155.

Shi, C. G., Zhao, B. Z., & Ma, W. X. (2015). Exact rational solutions to a Boussinesq-like equation in $(1 + 1)$-dimensions. Applied Mathematics Letters, 48, 170–176.

Singh, M., & Gupta, R. K. (2016). Bäcklund transformations, Lax system, conservation laws and multisolsiton solutions for Jimbo-Miwa equation with Bell-polynomials. Communications in Nonlinear Science and Numerical Simulation, 37, 362–373.

Vijayajayanthi, M., Kanna, T., Lakshmanan, M., & Murali, K. (2016). Explicit construction of single input-single output logic gates from three soliton solution of Manakov system. Communications in Nonlinear Science and Numerical Simulation, 36, 391–401.

Wang, C. J., Dai, Z. D., & Liang, L. (2010). Exact three-wave solution for higher dimensional KdV-type equation. Applied Mathematics and Computation, 216, 501–505.

Wang, D., & Zhang H. Q. (2005). Further improved F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equation. Chaos Solitons Fractals, 25, 601–610.

Wang, Y. F., Tian, B., Li, M., Wang, P., & Jiang, Y. (2014). Soliton dynamics of a discrete integrable Ablowitz-Ladik equation for some electrical and optical systems. Applied Mathematics Letters, 35, 46–51.

Wazwaz, A. M. (2017). Multiple soliton solutions and other exact solutions for a two-mode KdV equation. Mathematical Methods in the Applied Sciences, 40, 2277–2283.

Wu, T. Y., & Zhang, J. E. (1996). On modeling a nonlinear long wave. In L. P. Cook, V. Roythrud & M. Tulin (Eds.), Mathematics is for solving problems (pp. 233–241). Philadelphia: SIAM.

Zayed, E. M. E., & Shorog, A. J. (2010). Applications of an extended $(G'/G)$-expansion method to find exact solutions of nonlinear PDEs in mathematical physics. Mathematical Problems in Engineering, 2010, 768573.

Zhang, C. C., & Chen, A. H. (2016). Bilinear form and new multi-soliton solutions of the classical Boussinesq-Burgers system. Applied Mathematics Letters, 58, 133–139.

Zhang, J. E., & Li, Y. S. (2003). Bidirectional solitons on water. Physical Review E, 67, 016306.

Zhang, S. Q., Xu, G. Q., & Li, Z. B. (2002). General explicit solutions of a classical Boussinesq system. Chinese Physics B, 11, 993–995.

Zhang, Y., Chang, H., & Li, N. (2009). Explicit N-fold Darboux transformation for the classical Boussinesq system and multi-soliton solutions. Physics Letters A, 373, 454–457.

Zhang, Y., & Ma, W. X. (2015). Rational solutions to a KdV-like equation. Applied Mathematics and Computation, 256, 252–256.

Zhang, W. G., Zhao, Y. N., & Chen, A. H. (2015). The elastic-fusion-coupled interaction for the Boussinesq equation and new soliton solutions for the KP equation. Applied Mathematics and Computation, 259, 251–257.