GROWTH RATE OF DEHN TWIST LATTICE POINTS IN TEICHMÜLLER SPACE

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Abstract. Athreya, Bufetov, Eskin and Mirzakhani [2] have shown the number of mapping class group lattice points intersecting a closed ball of radius \( R \) in Teichmüller space is asymptotic to \( e^{hR} \), where \( h \) is the dimension of the Teichmüller space. In contrast we show the number of Dehn twist lattice points intersecting a closed ball of radius \( R \) is coarsely asymptotic to \( e^{h^2R} \). Moreover, we show the number multi-twist lattice points intersecting a closed ball of radius \( R \) grows coarsely at least at the rate of \( R \cdot e^{h^2R} \).

1. Introduction

1.1. Motivation. Let \( M \) be a compact, negatively curved Riemannian manifold and denote \( \tilde{M} \) its universal cover. Then its fundamental group \( \pi_1(M) \) acts on \( \tilde{M} \) by isometries. Given any \( x \in \tilde{M}, R > 0 \), let \( B_R(x) \) denote the ball of radius \( R \) in \( \tilde{M} \) centered at \( x \). It’s a classical result from G.A. Margulis that

\[
\text{Theorem 1.1 (Margulis \cite{16}). There is a function } c: M \times M \to \mathbb{R}^+ \text{ so that for every } x, y \in \tilde{M},
\]

\[
|\pi_1(M) \cdot y \cap B_R(x)| \sim c(p(x), p(y))e^{hR}
\]

where \( h \) equals to the dimension of the manifold, which is the topological entropy of the geodesic flow on the unit tangent bundle of \( M \).

Here and throughout, the notation \( f(R) \sim g(R) \) means \( \lim_{R \to \infty} \frac{f(R)}{g(R)} = 1 \).

Inspired by this result, Athreya, Bufetov, Eskin and Mirzakhani studied lattice point asymptotics in Teichmüller space. Let \( S_g, g \geq 2 \) denote the closed surface of genus \( g \), and let \( \text{Mod}_g, \mathcal{T}_g \) denote the corresponding mapping class group and Teichmüller space respectively, then \( \text{Mod}_g \) acts on \( \mathcal{T}_g \) by isometries. They showed

\[
\text{Theorem 1.2 (Athreya, Bufetov, Eskin and Mirzakhani \cite{2}). For any } \mathcal{X}, \mathcal{Y} \in \mathcal{T}_g, \text{ we have}
\]

\[
|\text{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \sim \Lambda(\mathcal{X})\Lambda(\mathcal{Y})e^{hR}
\]

where \( h = 6g - 6 \) is the dimension of \( \mathcal{T}_g \) and \( \Lambda: \mathcal{T}_g \to \mathbb{R}^+ \) is a bounded function called Hubbard-Masur function.

We note that the function \( \Lambda \) was later shown to be a constant by Mirzakhani, see \cite{6}.

The Nielsen-Thurston Classification \cite{22} says every element in \( \text{Mod}_g \) is one of the three types: periodic, reducible, or pseudo-Anosov. Letting \( PA \subset \text{Mod}_g \) denote the subset of pseudo-Anosov elements, Maher showed that, in terms of lattice points counting, pseudo-Anosov elements are generic in the following sense.
Theorem 1.3 (Maher [15]). For any \( \mathcal{X}, \mathcal{Y} \in T_g \), we have
\[
\frac{|PA \cdot \mathcal{Y} \cap B_R(\mathcal{X})|}{|\text{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})|} \sim 1.
\]

Note the above Theorems 1.2 and 1.3 also hold for punctured surfaces \( S_{g,n} \) satisfying \( 3g + n \geq 5 \).

Thus, it is natural to consider the asymptotic growth rate of reducible and periodic elements. A typical reducible element can be decomposed as a product of Dehn twists about disjoint simple closed curves and a partial pseudo-Anosov element on subsurfaces [4]. Dehn twists are also in a sense the most fundamental of Dehn twists about disjoint simple closed curves and a partial pseudo-Anosov periodic elements. A typical reducible element can be decomposed as a product

1.2. Statement of Main Results. Throughout this paper we let \( S_{g,n} \) denote a closed surface of genus \( g \) with \( n \) punctures such that \( 3g - 3 + n > 0 \), and we let \( \text{Mod}_{g,n}, T_{g,n} \) and \( \mathcal{M}_{g,n} \) denote the corresponding mapping class group, Teichmüller space and moduli space respectively. We use \( h = 6g + 2n - 6 \) to denote the dimension of \( T_{g,n} \). For any \( \epsilon > 0 \), we denote \( T_{g,n}^{\epsilon} \) the \( \epsilon \)-thick part of \( T_{g,n} \). By saying \( \alpha \) is a simple closed curve on \( S_{g,n} \), we mean it’s a non-trivial isotopy class of essential simple closed curves on \( S_{g,n} \).

A multicurve \( \alpha \) is a formal sum \( \alpha = \sum_{i=1}^{k} a_i \alpha_i \) where \( 1 \leq k \leq \frac{2g - 2}{3} \), \( a_i \in \mathbb{Z} \setminus \{0\} \), and \( \alpha_i \) are pairwise disjoint simple closed curves on \( S_{g,n} \). By this definition, simple closed curves are also multicurves. Let \( \mathcal{ML} (\mathbb{Z}) \) denote the set of multicurves on \( S_{g,n} \) and let \( S \subset \mathcal{ML} (\mathbb{Z}) \) denote the set of all simple closed curves. A multicurve \( \alpha = \sum_{i=1}^{k} a_i \alpha_i \) is said to be positive if all coefficients are positive and is said to be negative if all coefficients are negative. Otherwise, we say \( \alpha \) is of mixed sign. Two multicurves are of the same topological type if up to isotopy, there is an orientation-preserving homeomorphism taking one multicurve to another. For any \( \gamma \in \mathcal{ML} (\mathbb{Z}) \), we denote the multicurves of topological type \( \gamma \) by \( \mathcal{ML}(\gamma) \). In particular if \( \gamma \) is a simple closed curve, we denote \( \mathcal{ML}(\gamma) \) as \( S(\gamma) \). Since there are only finitely many topological types of simple closed curves on \( S_{g,n} \), \( S \) is a finite union of sets of the form \( S(\gamma) \). Meanwhile, there are infinitely many topological types of multicurves, as can be seen by looking at the coefficients.

For reasons we will see, let’s denote \( \mathcal{ML}^*(\mathbb{Z}) \) the set of multicurves \( \alpha = \sum_{i=1}^{k} a_i \alpha_i \) satisfying one of the following conditions:

1. \( \alpha \) is a weighted or unweighted simple closed curve, i.e., \( k = 1 \).
2. \( \alpha \) is positive or negative, i.e., all coefficients have the same sign.
3. \( \alpha \) is of mixed sign where each \( |a_i| \geq 2 \).

We write \( \mathcal{ML}^*(\gamma) \) instead of \( \mathcal{ML}(\gamma) \) when \( \gamma \in \mathcal{ML}^*(\mathbb{Z}) \).

For any simple closed curve \( \alpha \) we let \( T_\alpha \) denote the Dehn twist around \( \alpha \). In general, for any multicurve \( \alpha = \sum_{i=1}^{k} a_i \alpha_i \), we define \( T_\alpha = \prod_{i=1}^{k} T_{\alpha_i}^{a_i} \) and we call this a multi-twist. By this definition, Dehn twists are also multi-twists, and let’s call them as twists in general. In our Theorem 1.4 we will consider the following subsets of \( \text{Mod}_{g,n} \) consisting of twists:

1. \( D(\mathcal{ML}^*(\gamma)) = \{ T_\alpha \mid \alpha \in \mathcal{ML}^*(\gamma) \} = \{ fT_\gamma f^{-1} \mid f \in \text{Mod}_{g,n} \} \), the set of twists about curves of topological type \( \gamma \) or, equivalently, the conjugacy class of \( T_\gamma \).
(2) \(D(S) = \{T_\alpha \mid \alpha \in S\}\), the set of all Dehn twists without powers. \(D(S)\) is a finite union of sets of the form \(D(S(\gamma))\).

(3) \(M(S) = \{T^k_\alpha \mid \alpha \in S, k \in \mathbb{Z}\}\), the set of all Dehn twists with any powers. Similarly, \(M(S)\) is a finite union of \(M(S(\gamma))\). Each \(M(S(\gamma))\) can be realized as an infinite union of conjugacy classes of \(T^k_\gamma, k \in \mathbb{Z}\).

We now introduce some notations. Let \(A > 0\).

(1) We say \(f(x) \overset{+}{\sim} A \cdot g(x)\) if \(g(x) - A \leq f(x) \leq g(x) + A\) for any \(x\).

(2) We say \(f(x) \overset{\times}{\sim} A \cdot g(x)\) if \(A \cdot g(x) \leq f(x) \leq A \cdot g(x)\) for any \(x\).

(3) We say \(f(R) \overset{A}{\sim} g(R)\) if for any \(\lambda > 1\), there exists a \(M(\lambda)\) such that \(\frac{1}{\lambda^A} \cdot f(R) \leq g(R)\) for any \(R \geq M(\lambda)\).

(4) We say \(f(R) \overset{A}{\sim} g(R)\) if \(f(R) \overset{A}{\sim} g(R)\) and \(g(R) \overset{A}{\sim} f(R)\).

Moreover, we say \(f, g\) are coarsely asymptotic if \(f(R) \overset{A}{\sim} g(R)\) for some coefficient \(A\). Notice the notation \(f(R) \sim g(R)\) is the same as \(f(R) \overset{A}{\sim} g(R)\), i.e. \(f, g\) are asymptotic when they are coarsely asymptotic with coefficient 1.

Recall \(T^\epsilon_{g,n}\) denotes the \(\epsilon\)-thick part of \(T_{g,n}\) and \(h = 6g - 6 + 2n\) denotes the dimension of \(T_{g,n}\). For any \(\mathcal{X}, \mathcal{Y} \in T^\epsilon_{g,n}\), we define \(F(\mathcal{X}, \mathcal{Y}) = e^{\frac{2}{3}d_T(\mathcal{X}, \mathcal{Y})}\). For any multicurve \(\alpha = \sum_{i=1}^k a_i \gamma_i\), we denote the sum of absolute coefficients as \(c_\alpha = \sum_{i=1}^k |a_i|\) and we define \(F_\alpha(\mathcal{X}, \mathcal{Y}) = (c_\alpha)^{\frac{2}{3}} e^{\frac{2}{3}d_T(\mathcal{X}, \mathcal{Y})}\). Our main theorem below gives coarse asymptotics for \(D(\mathcal{M}L^\epsilon(\gamma))\).

**Theorem 1.4.** Given any \(S_{g,n}\) and given any \(\epsilon > 0\), there exists a \(J > 0\) such that for any multicurve \(\gamma \in \mathcal{M}L^\epsilon(\mathbb{Z})\) and for any \(\mathcal{X}, \mathcal{Y} \in T^\epsilon_{g,n}\), we have

\[
|D(\mathcal{M}L^\epsilon(\gamma)) \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \overset{A}{\sim} J F_\alpha(\mathcal{X}, \mathcal{Y}) \cdot n_\mathcal{X}(\gamma) \cdot e^{\frac{2}{3}R}
\]

where \(n_\mathcal{X}(\gamma)\) is the corresponding Mirzakhani constant, see section 2.7.

**Corollary 1.5.** Given \(S_{g,n}\) and given any \(\epsilon > 0\), for any \(\mathcal{X}, \mathcal{Y} \in T^\epsilon_{g,n}\), we have

\[
|D(S) \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \overset{A}{\sim} J F_\alpha(\mathcal{X}, \mathcal{Y}) \cdot n_\mathcal{X}(S) \cdot e^{\frac{2}{3}R}, \text{ if } h > 0,
\]

\[
|M(S) \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \overset{A}{\sim} J F_\alpha(\mathcal{X}, \mathcal{Y}) \cdot n_\mathcal{X}(S) \cdot e^{\frac{2}{3}R}, \text{ if } \frac{h}{2} > 1
\]

where \(n_\mathcal{X}(S)\) is the corresponding Mirzakhani constant.

We remark that when \(\frac{2}{3} = 1\), \(M(S) = \mathcal{M}L^\epsilon(\mathbb{Z}) = \mathcal{M}L(\mathbb{Z})\) is one dimensional. The coarse asymptotic for \(|M(S) \cdot \mathcal{Y} \cap B_R(\mathcal{X})|\) when \(\frac{2}{3} = 1\) is separated out as a special case and treated in Corollary 1.9.

The above results says for example, the number of Dehn twist lattice points intersecting a closed ball of radius \(R\) in the Teichmüller space is coarsely asymptotic to \(e^{\frac{2}{3}R}\). Note that any \(\mathcal{X} \in T_{g,n}\) lies in \(T^\epsilon_{g,n}\) for some \(\epsilon\), thus another way to phrase the theorem is by picking \(\mathcal{X}, \mathcal{Y} \in T_{g,n}\) first and then by picking any \(\epsilon > 0\) such that \(\mathcal{X}, \mathcal{Y} \in T^\epsilon_{g,n}\). The constant \(J\) and the above results follow. Recall that in the \(\epsilon\)-thick part of Teichmüller space, there is a uniformly bounded difference, depending on \(\epsilon\), between the Thurston metric and Teichmüller metric [14]. Thus the above results also hold for the Thurston metric after a slight variation.

Our argument hinges on studying how the length of any simple closed geodesic \(\tau\) on a hyperbolic structure \(\mathcal{X}\) changes after applying a twist \(T_\alpha\). To this end,
in Theorem 3.2 we obtain an explicit bound on the length of $\ell_{T,\alpha}(\tau)$ in terms of $\ell_X(\tau)$, $\ell_X(\alpha)$ and the intersection patterns between $\tau$ and $\alpha$, up to additive error. We then use this Theorem 3.2 together with results of Choi, Rafi [5] and Lenzhen, Rafi, Tao [14], to realize a precise relationship between $\ell_X(\alpha)$ and $d_T(X, T\alpha X)$. This relation is stated in the following theorem.

**Theorem 4.2** Fix some $S_{g,n}$ and given any $\epsilon > 0$, there exists a constant $H > 0$ such that given any $X \in T_{g,n}^\epsilon$, we have

$$d_T(X, T\alpha X) + H \leq \log \left( \sum_{i=1}^k |a_i| \ell_X(\alpha_i) \right)$$

for any $\alpha = \sum_{i=1}^k a_i \alpha_i \in ML^*(Z)$. Moreover, we have constructed a sequence of multicurves in $ML(Z) \setminus ML^*(Z)$ for which Theorem 4.2 does not hold, see Remark 4.3. There exists a $H' > 0$ depends on $S_{g,n}$ and $\epsilon$, so that for these multicurves the distances behave like

$$d_T(X, T\alpha X) + H' \leq \log \left( \sum_{i=1}^k |a_i| \ell_X(\alpha_i) \right)$$

for any $X \in T_{g,n}^\epsilon$. This leads to the following question.

**Question 1.6** For $\alpha \in ML(Z) \setminus ML^*(Z)$, how does the length of any simple closed geodesic $\tau$ on a hyperbolic structure $X$ changes after applying a twist $T_\alpha$? How far does a point move in Teichmüller space after applying the corresponding twist $T_\alpha$?

Another natural question prompted by Theorem 1.4 and Theorem 4.2 is

**Question 1.7** Let $D(ML(Z))$ denote the set of all twists. What are the coarse asymptotics for $D(ML(Z))$?

Recall that Mirzakhani’s Theorem 2.8 says

$$|\{\alpha \in ML(\gamma) \mid \ell_X(\alpha) \leq e^{2R}\}| \sim n_X(\gamma) \cdot e^{\frac{R}{2}}$$

which is at the same coarse asymptotic rate of $|D \cdot \gamma \cap B_R(X)|$ for the three cases as in Theorem 1.4 and Corollary 1.5. Moreover, Mirzakhani [17] also proves that for any $X \in M_{g,n}$, there exists a constant $n_X$ such that

$$|\{\alpha \in ML(Z) \mid \ell_X(\alpha) \leq e^{2R}\}| \sim n_X \cdot e^{\frac{R}{2}}.$$  

We may wonder whether $|D(ML(Z)) \cdot \gamma \cap B_R(X)|$ is coarsely asymptotic to $n_X \cdot e^{\frac{R}{2}}$ as well? This turns out to be false. Namely, we show there is a subset $ML(\gamma) \subset ML(Z)$ such that $|D(ML(\gamma)) \cdot \gamma \cap B_R(X)|$ is at least coarsely asymptotic to $R \cdot e^{\frac{R}{2}}$, forcing a lower bound for the coarse asymptotic rate of $|D(ML(Z)) \cdot \gamma \cap B_R(X)|$. At the end of section 3, we discuss the difficulty using Theorem 4.2 to obtain an upper bound estimate for the coarse asymptotic rate of $|D(ML(Z)) \cdot \gamma \cap B_R(X)|$.

Let $\gamma = \sum_{i=1}^k \gamma_i$ denote a multicurve with all coefficients equal to one and of maximal dimension $k = \frac{n}{2}$. We say $\gamma = \sum_{i=1}^k a_i \gamma_i \in [\gamma]$ if $\gamma$ and $\gamma_i$ are the same when without coefficients. Let’s denote

$$ML([\gamma]) = \bigcup_{\gamma \in [\gamma]} ML(\gamma).$$

Notice $ML(\gamma)$ consists of infinity many conjugacy classes of multicurves.
Theorem 1.8. Given any $S_{g,n}$ such that $h > 0$, $\epsilon > 0$, and $\gamma = \sum_{i=1}^{k} \gamma_i$ a multicurve with all coefficients equal to one and of maximal dimension $k = \frac{h}{2}$. There exists a number $f(\gamma)$ such that, for any $X, Y \in T_{g,n}^\epsilon$,

$$|D(\mathcal{ML}(Z)) \cdot Y \cap B_R(X)| \geq |D(\mathcal{ML}(\gamma)) \cdot Y \cap B_R(X)| \geq f(\gamma) \cdot R \cdot e^{\frac{h}{2} R}.$$ 

In particular, we can consider the case $\frac{h}{2} = 3g - 3 + n = 1$, where $S_{g,n}$ is either $S_{1,1}$ or $S_{0,4}$, and $\text{Mod}_{g,n}, T_{g,n}$ are $\text{SL}_2(\mathbb{Z}), \mathbb{H}^2$ respectively. In this case, $\mathcal{ML}(Z)$ is one dimensional and we have $\mathcal{ML}(Z) = \mathcal{ML}(\gamma)$ for any simple closed curve $\gamma$. In correspondence $D(\mathcal{ML}(Z))$ is the set of all parabolic elements of $\text{SL}_2(\mathbb{Z})$. There are many results about the asymptotic growth of lattice points in $\mathbb{H}^2$, see [10], [19] for example. The corollary below can also be interpreted as a coarse asymptotic for the number of parabolic lattice points of $\text{SL}_2(\mathbb{Z})$ intersecting a closed ball of radius $R$ in $\mathbb{H}^2$.

Corollary 1.9. Given $S_{g,n}$ equal to $S_{1,1}$ or $S_{0,4}$ and given any $\epsilon > 0$. For any $X, Y \in T_{g,n}^\epsilon$, we have

$$|D(\mathcal{ML}(Z)) \cdot Y \cap B_R(X)| \sim f(\gamma) n_X(S) \cdot R \cdot e^R.$$

The upper bound in this Corollary follows from an alternation of the proof of Corollary 1.5, see section 5, and the lower bound follows from previous Theorem 1.8, see section 6.

We conclude this introduction with two more questions for further study.

Question 1.10. For a set of twists $D$ with known coarse asymptotics, we may next ask for more precise asymptotics, i.e., what is the best coarse asymptotic coefficient $J$ we can achieve?

Question 1.11. In Theorem 1.4 we are essentially looking at the growth behavior of the orbit of a conjugacy class of twists. What about the asymptotics growth behavior of other conjugacy classes in $\text{Mod}_{g,n}$?

We explore Questions 1.11 for pseudo-Anosov conjugacy classes in the following paper [9].

The organization of the paper is as follows. In section 2 we briefly review some background and previous results. In section 3 we study how twists affect the lengths of simple closed curves and prove Theorem 3.2. In section 4 we prove Theorem 4.2 estimating how far a twist translates a point in Teichmüller space. In section 5 we prove our main results Theorem 1.4 and Corollary 1.5 using Mirzakhani’s result of counting simple closed geodesics [17]. In section 6 we prove Theorem 1.8 and Corollary 1.9 quickly follows. We remark that Theorem 3.2 and Theorem 4.2 are key ingredients in our argument and may be of independent interest.

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2. Background

We refer the reader to [7] for more background materials.

2.1. Mapping Class Group and Dehn twists. Let $\text{Homeo}_{g,n}^+$ denote the group of all the orientation-preserving homeomorphisms of $S_{g,n}$ preserving the set of punctures, and let $\text{Homeo}_{g,n}^0$ denote the connected component of the identity. The mapping class group of $S_{g,n}$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms:

$$\text{Mod}_{g,n} = \text{Homeo}_{g,n}^+/\text{Homeo}_{g,n}^0 = \text{Homeo}_{g,n}^+/\text{isotopy}$$

Let $A = S^1 \times [0, 1]$ be an oriented annulus, the twist map $T: A \to A$ is defined to be $(\theta, t) \mapsto (\theta + 2\pi t, t)$, so $T$ is a homeomorphism of $A$ relative to its boundary. Let $a$ be a representative of a simple closed curve $\alpha$ on $S_{g,n}$ and let $N$ be a regular neighborhood of $a$. Pick some orientation-preserving homeomorphism $\phi: A \to N$, the Dehn twist about $a$ is defined by

$$T_a(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in S_{g,n} \setminus N \end{cases}.$$ 

The isotopy class of $T_a$ does not depend on choice of $a$ in $\alpha$. Thus $T_\alpha$ is a well-defined mapping class. Now given any multicurve $\alpha = \sum_{i=1}^k a_i \alpha_i$, the composition $T_\alpha = \prod_{i=1}^k T_{a_i}$ is called a multi-twist.

Given two simple closed curves $\alpha, \beta$, the intersection number $i(\alpha, \beta)$ is defined to be $i(\alpha, \beta) = \min |a \cap b|$ where $a, b$ are in the isotopy classes $\alpha, \beta$ respectively and $|a \cap b|$ denotes how many times $a$ and $b$ intersect. The following proposition of Ivanov shows how twists effect intersection numbers.

**Proposition 2.1** (Intersection Formula [12]). Let $\alpha = \sum_{i=1}^k a_i \alpha_i$ be a multicurve on $S_{g,n}$, and $T_\alpha = \prod_{i=1}^k T_{a_i}$ the corresponding twist. Given $\beta, \gamma$ arbitrary simple closed curves on $S_{g,n}$. If $\alpha$ is positive or negative, we have

$$i(T_\alpha(\beta), \gamma) - \sum_{i=1}^n |k_i| i(\alpha_i, \beta) i(\alpha_i, \gamma) \leq i(\beta, \gamma).$$

If $\alpha$ is of mixed sign, we have

$$\sum_{i=1}^n (|k_i| - 2) i(\alpha_i, \beta) i(\alpha_i, \gamma) - i(\beta, \gamma) \leq i(T_\alpha(\beta), \gamma) \leq \sum_{i=1}^n |k_i| i(\alpha_i, \beta) i(\alpha_i, \gamma) + i(\beta, \gamma).$$

2.2. Teichmüller space and moduli space. A hyperbolic structure $\mathcal{X}$ on $S_{g,n}$ is a pair $(X, \phi)$ where $\phi: S_{g,n} \to X$ is a homeomorphism and $X$ is a hyperbolic surface. We say two hyperbolic structures $\mathcal{X} = (X, \phi), \mathcal{Y} = (Y, \psi)$ are isotopic if there is an isometry $I: X \to Y$ isotopic to $\psi \circ \phi^{-1}$. The Teichmüller space $T_{g,n}$ is the set of hyperbolic structures on $S_{g,n}$ modulo isotopy. We let $\mathcal{X} = (X, \phi), \mathcal{Y} = (Y, \psi)$ denote elements in $T_{g,n}$.
Given any $X, Y \in \mathcal{T}_{g,n}$, the Teichmüller distance between them is defined to be

$$d_T(X, Y) = \frac{1}{2} \inf_{f \sim \phi \circ \psi^{-1}} \log(K_f)$$

where the infimum is over all quasi-conformal homeomorphisms $f$ isotopic to $\phi \circ \psi^{-1}$ and $K_f$ is the quasi-conformal dilatation of $f$. Equipped with the Teichmüller metric, the Teichmüller space is a complete, unique geodesic metric space.

The mapping class group acts isometrically on $\mathcal{T}_{g,n}$ by changing the marking $(f, (X, \phi)) \mapsto (X, \phi \circ f^{-1})$. This action is properly discontinuous but not cocompact. The quotient $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\text{Mod}_{g,n}$ is called the moduli space, and it is a non-compact orbifold parameterizing hyperbolic surfaces homeomorphic to $S_{g,n}$.

Given any $X = (X, \phi) \in \mathcal{T}_{g,n}$ and given any isotopy class $\gamma$ of nontrivial simple closed curves on $X$, there exists a unique geodesic in this free homotopy class. We define the length function on $X$ by setting $\ell_X(\gamma)$ equal to the length of this unique geodesic. We also let $\ell_X(\alpha)$ denote $\ell_X(\phi(\alpha))$ for any simple closed curve $\alpha$ on $S_{g,n}$. For any multicurve $\alpha = \sum_{i=1}^{k} a_i \alpha_i$, we define $\ell_X(\alpha) = \sum_{i=1}^{k} |a_i| \ell_X(\alpha_i)$ to be its length.

A pair of pants is a closed surface of zero genus with three boundary components or punctures. A pants decomposition $\Gamma$ of the surface $S_{g,n}$ is a collection of pairwise disjoint non-trivial simple closed curves $\gamma_1, \cdots, \gamma_{3g-3+n}$ on $S_{g,n}$, together they decompose the surface $S_{g,n}$ into $2g + n - 2$ pairs of pants. Using pants decomposition and by introducing Fenchel-Nielsen coordinates, Fricke [8] showed that the dimension of $\mathcal{T}_{g,n}$ is $6g + 2n - 6$.

A theorem of Bers [3] says there exists a constant depending only on $S_{g,n}$ such that for every $X \in \mathcal{T}_{g,n}$, there is a pants decomposition $\Gamma_X$ of $X$ in which each simple closed curve has length bounded above by this Bers’ constant.

Given any $\epsilon > 0$, the $\epsilon$-thick part of Teichmüller space is defined to be

$$\mathcal{T}^\epsilon_{g,n} = \{X \in \mathcal{T}_{g,n} \mid \ell_X(\alpha) \geq \epsilon \text{ for any simple closed curve } \alpha \text{ on } S_{g,n} \}$$

and consequently the $\epsilon$-thick part of moduli space is $\mathcal{M}^\epsilon_{g,n} = \mathcal{T}^\epsilon_{g,n}/\text{Mod}_{g,n}$. The Mumford compactness criterion [18] says $\mathcal{M}^\epsilon_{g,n}$ is compact for any $\epsilon > 0$.

### 2.3. Short Marking

For any $X \in \mathcal{T}_{g,n}$, a short marking [5] $\mu_X$ is a collection of simple closed curves $\{\eta_i\}_{i=1}^{3g-3+n} \cup \{\delta_i\}_{i=1}^{3g-3+n}$ on $S_{g,n}$ picked in the following way:

First, choose a pant decomposition $\{\eta_i\}_{i=1}^{3g-3+n}$ by taking a curve $\eta_1$ on $S_{g,n}$ that is a shortest curve with respect to $X$, and then a next shortest disjoint curve from the first, and so on until we complete a pants decomposition. Next, for each $\eta_i$, pick a shortest curve $\delta_i$ that intersects $\eta_i$ and is disjoint from all other pants curves. For each $i$, we say $\eta_i, \delta_i$ is a pair. The collection of curves obtained in this way has the property that any two curves have intersection number bounded by 2. Note there could be a finite number of possible short markings corresponding to each $X \in \mathcal{T}_{g,n}$, we fix one such short marking and call it the short marking $\mu_X$. Moreover, given any $\epsilon > 0$, by Bers’ Theorem and trigonometry, there exists $N > 0$ depending on $\epsilon$ and $S_{g,n}$ such that for any $X \in \mathcal{T}^\epsilon_{g,n}$, all curves in the short marking $\mu_X$ have length bounded above by $N$ and bounded below by $\epsilon$.

We recall a result from Choi and Rafi [5] stating that for any $\epsilon > 0$, the Teichmüller distance in the $\epsilon$-thick part can be approximated by the maximum ratio of change of lengths of the short marking.
**Theorem 2.2** (Distance Formula [5]). For any $\epsilon > 0$, there exists $c > 0$ depending on $S_{g,n}$ and $\epsilon$ such that for any $X, Y \in T_{g,n}^\epsilon$,

$$d_T(X, Y) \lesssim \log \max_{\gamma \in \mu_X} \frac{\ell_Y(\gamma)}{\ell_X(\gamma)}.$$

We also recall that Lenzhen, Rafi, Tao [14] showed that for any simple closed curve on $S_{g,n}$, its length with respect to $X$ can be estimated via its intersection pattern with the short marking $\mu_X$.

**Proposition 2.3** (Length Formula [14]). There exists $C \geq 1$ depending on $S_{g,n}$ such that for any simple closed curve $\beta$ on $S_{g,n}$ and for any $X \in T_{g,n}$, we have

$$\ell_X(\beta) \lesssim \sum_{\gamma \in \mu_X} i(\beta, \gamma) \ell_X(\bar{\gamma})$$

where $\bar{\gamma}$ denotes the curve in the short marking paired with $\gamma$.

For a fixed $\epsilon$, any curve $\gamma$ in $\mu_X, X \in T_{g,n}$, satisfies $\epsilon \leq \ell_X(\gamma) \leq N$. We can therefore rewrite the above theorem and proposition for $T_{g,n}^\epsilon$.

**Lemma 2.4.** For any $S_{g,n}$ and $\epsilon > 0$, there exists $C$ depends on $S_{g,n}$ and $c, N$ depends on $S_{g,n}$ and $\epsilon$ such that

$$\log \left( \frac{1}{Ne^c} \max_{\gamma \in \mu_X} \ell_Y(\gamma) \right) \leq d_T(X, Y) \leq \log \left( \frac{e^c}{\epsilon} \max_{\gamma \in \mu_X} \ell_Y(\gamma) \right)$$

$$\frac{\epsilon}{C} \sum_{\gamma \in \mu_X} i(\beta, \gamma) \leq \ell_X(\beta) \leq CN \sum_{\gamma \in \mu_X} i(\beta, \gamma)$$

![Figure 1. A short marking $\mu_X$ and a simple closed curve $\beta$ on a hyperbolic surface $X$ homeomorphic to $S_2$.](image)

2.4. Properties of $\mathbb{H}^2$. We first recall the following useful lemma in hyperbolic geometry.

**Lemma 2.5** (Collar Lemma [7]). For any simple closed geodesic $\gamma$ of length $\ell$ on a hyperbolic surface, it is contained in an embedded cylinder of diameter of order $\ell^{-1}$, and the diameter is

$$W(\gamma) = \sinh^{-1} \left( \frac{1}{\sinh(\frac{1}{2}\ell)} \right)$$
For any two closed sets \( A, B \subset \mathbb{H}^2 \) we let \( d(A, B) \) denote the minimal distance between them. For any geodesic \( \eta \) in \( \mathbb{H}^2 \), we let \( \pi_\eta \) denote the closest point projection map, namely
\[
\pi_\eta(x) = \{ y \in \eta \mid d(x, y) = d(x, \eta) \}.
\]
For any two points \( x, y \in \mathbb{H}^2 \), we let \( [x, y] \) denote the unique geodesic connecting them. Given two points \( x, y \in \mathbb{H}^2 \) separated by a bi-infinite geodesic \( \eta \) and far away from \( \eta \), we let \( x_\eta \in \mathbb{H}^2 \) denote the first point that the geodesic \( [x, y] \) enters the \( L \)-neighborhood of \( \eta \) coming from the \( x \) side. Similarly, we can define \( y_\eta \in \mathbb{H}^2 \).

If \( x \) is in the \( L \)-neighborhood of \( \eta \) to begin with, we just let \( x_\eta = x \), and similarly for \( y \).

Being a hyperbolic space, geodesics are strongly contracting in \( \mathbb{H}^2 \), see [1] for example. That is, there exists a constant \( L \) such that for any geodesic \( \eta \) and for any geodesic \( \alpha \) that never enters the \( L \)-neighborhood of \( \eta \), the diameter of \( \pi_\eta(\alpha) \) is bounded by \( L \). As a consequence, we have

**Corollary 2.6.** There exists a constant \( L \) such that for any bi-infinite geodesic \( \eta \) in \( \mathbb{H}^2 \) and for any two points \( x, y \) separated by \( \eta \), we have
\[
d(x, \pi_\eta(x)) \leq 2L, d(y, \pi_\eta(y)) \leq 2L.
\]
This is because we have
\[
d(x, \pi_\eta(x)) \leq d(x, x_\eta) + d(x_\eta, \pi_\eta(x)) \leq 2L.
\]
Similarly for \( y \).

Another important property of the projection map in \( \mathbb{H}^2 \) is that it’s 1-Lipschitz. Viewing in the upper half plane model and up to isometry, we may assume \( \eta \) is the vertical line \( x = 0 \). For each point \((0, r) \in \eta \), the points projecting to \((0, r)\) are exactly the Euclidean semicircles of radius \( r \) centered at \((0, 0)\). Given two Euclidean semicircle centered at \((0, 0)\), the minimal distance between them are realized by the points intersecting the vertical line \( x = 0 \). This means

**Lemma 2.7.** \( \pi_\eta \) is 1-Lipschitz for any bi-infinite geodesic \( \eta \) in \( \mathbb{H}^2 \).

### 2.5. Lifts of twists

Given an oriented bi-infinite geodesic \( \beta \) in \( \mathbb{H}^2 \) and a number \( l_\beta \in \mathbb{R} \), we can decompose \( \mathbb{H}^2 \) into two open pieces, one to the left of \( \beta \) and one to the right of \( \beta \), and then regule the two pieces along \( \beta \) after translation according to \( l_\beta \). When \( l_\beta \) is positive, we regule the pieces along \( \beta \) after translating distance \( |l_\beta| \) to the left. When \( l_\beta \) is negative, we regule the pieces along \( \beta \) after translating distance \( |l_\beta| \) to the right. This process is called shearing along \( \beta \) according to \( l_\beta \), see [13] for more detail. We are mainly interested in what happens to geodesics after shearing. Let \( \tau \) be a bi-infinite geodesic in \( \mathbb{H}^2 \) transverse to \( \beta \) and let \( \tau' \) be the image of \( \tau \) after shearing along \( \beta \) according to \( l_\beta \), then \( \tau' \) is a concatenation of two geodesic rays with a sub-segment of \( \beta \) of length \( l_\beta \) connecting these two rays’ starting points, see Figure 2 for an illustration.

Given \( X = (X, \phi) \in T_{g,n} \) and let \( p : \mathbb{H}^2 \to X \) be the universal cover. For any multicurve \( \alpha = \sum_{i=1}^k a_i \alpha_i \), we let \( A = \{ \alpha_i \}_{i=1}^k \) and let \( \tilde{A} \) denote the set of lifts of curves in \( A \). For each curve \( \tilde{\alpha} \in \tilde{A} \), we let \( \alpha_{\pi(\tilde{\alpha})} \) denote the curve such that \( \tilde{\alpha} \) is a lift of \( \alpha_{\pi(\tilde{\alpha})} \). Note the complements of \( \cup_{\tilde{\alpha} \in \tilde{A}} \tilde{\alpha} \) are infinitely many open regions. Fixing one of these regions, we can shear along all these bi-infinite geodesics in \( \tilde{A} \) according to \( \alpha_{\pi(\tilde{\alpha})} \), and this is called shearing according to \( \alpha \).

Now, given any simple closed geodesic \( \beta \) on \( X \), we let \( \tau \) be a lift of \( \beta \) with a base point \( q_0 \in \tau \). Fixing the region containing \( q_0 \), we can shear according to \( \alpha \). Let
Figure 2. After shearing along $\beta$ according to $l_\beta$, $\tau$ becomes $\tau'$.

$\tau'$ denote the image of $\tau$ after shearing, then the projection of $\tau'$ is isotopic to the simple closed geodesic $T_\alpha(\beta)$. Let $q'_L, q'_R \in \partial \mathbb{H}^2$ denote the endpoints of $\tau'$. The two end points $q'_L, q'_R \in \partial \mathbb{H}^2$ define a unique bi-infinite geodesic $\sigma$ in $\mathbb{H}^2$ and $\sigma$ is in the same isotopy class of $\tau'$, see Figure 3. This means $\sigma$ is a lift of the simple closed geodesic $p(\sigma) = T_\alpha(\beta)$. Similarly, one can obtain the simple closed geodesic $T_{\alpha^{-1}}(\beta)$ by shearing in the opposite direction.

Figure 3. After shearing according to $\alpha$ (blue curves are in $\tilde{A}$), the geodesic $\tau$ becomes $\tau'$, and the geodesic $\sigma$ is uniquely defined by the endpoints of $\tau'$

2.6. Bass-Serre Tree. We briefly explain how to construct a Bass-Serre tree dual to an infinite collection of bi-infinite geodesics in $\mathbb{H}^2$ that arise from a covering map. In particular, one may imagine how to construct a Bass-Serre dual to the Figure 3. See [21] for more detail about Bass-Serre trees in general.

Let $p: \mathbb{H}^2 \to S_{g,n}$ be a universal cover. Given $\mathcal{A} = \{\alpha_i\}_{i=1}^n$ a collection of disjoint simple closed curves on $S_{g,n}$, we let $\tilde{\mathcal{A}}$ denote the set of all liftings of curves in $\mathcal{A}$ to $\mathbb{H}^2$, and we let $\cup \tilde{\mathcal{A}}$ denote the union of all elements in $\tilde{\mathcal{A}}$. Define $Z_\mathcal{A}$ to be the tree dual to $\tilde{\mathcal{A}}$ in $\mathbb{H}^2$. That is, $Z_\mathcal{A} = (V_\mathcal{A}, E_\mathcal{A})$ is a graph such that each vertex in $V_\mathcal{A}$ corresponds to a connected component in $\mathbb{H}^2 \setminus \cup \tilde{\mathcal{A}}$ and each edge is dual to an element in $\tilde{\mathcal{A}}$. We label each edge by the element in $\tilde{\mathcal{A}}$ that it is dual to.
Denote the connected component corresponding to a vertex \( v \) as \( C(v) \). Given two vertices \( v, w \in V_A \), \((v, w) \in E_A\) if and only if \( C(v), C(w) \) represent bordered connected components. Denote \( d_Z \) the metric on the tree \( Z_A \) where the length of each edge has length 1. \( (Z_A, d_Z) \) is a unique geodesic metric space.

By the Collar Lemma 2.5, there exists a \( r = \min\{W(\alpha_i)\}_{i=1}^n \) sufficiently small such that for any curve \( \alpha \in A \), \( N_r(\alpha) \) is an open annulus. We can define a \( \pi_1 \)-equivariant, continuous and surjective map \( \phi_A: \mathbb{H}^2 \rightarrow Z_A \) such that each \( N_r(\tilde{\alpha}) \) maps to an edge and each connected component in \( \mathbb{H}^2 \setminus \bigcup_{\alpha \in A} N_r(\tilde{\alpha}) \) gets mapped to a vertex.

Now, given any simple closed curve \( \tau \) on \( S_{g,n} \) and let \( \tilde{\tau} \) be a lift of \( \tau \) in \( \mathbb{H}^2 \). If \( \tau \) does not intersect any curve in \( A \), then \( \phi_A(\tilde{\tau}) \) is a vertex. Otherwise, denote

\[
i(\tau, A) = \sum_{i=1}^n i(\tau, \alpha_i)
\]

the intersection number of \( \tau \) with curves in \( A \), \( \phi_A(\tilde{\tau}) \) is a bi-infinite geodesic in \((Z_A, d_Z)\). The hyperbolic isometry of \( \mathbb{H}^2 \) along \( \tilde{\tau} \), with translation distance equals to the length of \( \tau \), is equivariant with respect to \( \phi_A \) and gives rise to an isometry \( \rho_{\tilde{\tau}} \) of \((Z_A, d_Z)\) with translation length \( i(A, \tau) \) and translation axis \( \phi_A(\tilde{\tau}) \). This means for any vertex \( s \) on the axis \( \phi_A(\tilde{\tau}) \), we have \( d_Z(s, \rho_{\tilde{\tau}}(s)) = i(\tau, A) \).

### 2.7. Counting Simple Closed Geodesics

Given \( \gamma \) a simple closed curve or multicurve on any \( X \in \mathcal{M}_{g,n} \), we denote

\[
s_X(L, \gamma) = |\{ \alpha \in \text{Mod}_{g,n} : \gamma \mid \ell_X(\alpha) \leq L \}|
\]

the number of simple closed geodesics on \( X \) of topological type \( \gamma \) and of hyperbolic length at most \( L \). The following formula is due to Mirzakhani.

**Theorem 2.8 (Counting Formula [17]).** Fix some \( S_{g,n} \), given \( \gamma \) a simple closed curve or a multicurve on any \( X \in \mathcal{M}_{g,n} \), we have

\[
s_X(L, \gamma) \sim n_X(\gamma) \cdot L^{6g+2n-6}
\]

where \( n_X(\gamma) \) depends on the hyperbolic structure \( X \) and the topological type of \( \gamma \).

Later in the paper, we will count the sum of several topological types of multicurves. Thus we phrase the above Theorem 2.8 in the following equivalent way.

**Remark 2.9.** For any \( \gamma, X \) and \( \lambda > 1 \), there exist constants \( n_X(\gamma) \) and \( r_X(\gamma, \lambda) \) such that

\[
\frac{1}{\lambda} \cdot n_X(\gamma) \cdot L^{6g+2n-6} \leq s_X(L, \gamma) \leq \lambda \cdot n_X(\gamma) \cdot L^{6g+2n-6}
\]

for any \( L \geq r_X(\gamma, \lambda) \).

It’s also necessary for us to know how \( n_X(\gamma) \) and \( r_X(\gamma, \lambda) \) behave with respect to scaling the curve \( \gamma \) for later purposes.

**Corollary 2.10.** For any \( \gamma, X, \lambda > 1 \) and \( c \in \mathbb{N} \), we have

\[
r_X(c \cdot \gamma, \lambda) = c \cdot r_X(\gamma, \lambda)
\]

\[
n_X(c \cdot \gamma) = \frac{n_X(\gamma)}{c^{6g+2n-6}}
\]
We can apply the signed intersection formula (2) to approximate
and that
\[
\frac{1}{\lambda} \cdot \frac{n_X(\gamma)}{c^{g+2n-6}} \cdot L^{6g+2n-6} \leq s_X \left( \frac{L}{c} \cdot \gamma \right) \leq \frac{n_X(\gamma)}{c^{g+2n-6}} \cdot L^{6g+2n-6}
\]
for any \( \frac{L}{c} \geq r_X(\gamma, \lambda) \). This gives us the desired result. \( \square \)

Since there are only finitely many topological types of simple closed curves, we denote \( n_X(S) \) the finite sum of \( n_X(\gamma) \) where \( \gamma \) ranges over all topological types of simple closed curves on \( S_{g,n} \). We will use the notation \( s_X(L, S) \) to denote the number of all simple closed geodesics that have length bounded by \( L \), and we will denote \( r_X(S, \lambda) = \max_{\gamma \in S} r_X(\gamma, \lambda) \) for any \( \lambda > 1 \).

### 3. The Effect of Twisting on Hyperbolic Length

In this section, we study how the length of simple closed geodesics on a hyperbolic surface change after applying a twist. In the next section, we use the results below to estimate how far a point in Teichmüller space moves after applying a twist.

As our first result, we may obtain the following estimate from the length formula (4) and intersection formula (1).

**Proposition 3.1.** Fix some \( \epsilon > 0 \). Given a multicurve \( \alpha = \sum_{i=1}^{k} a_i \alpha_i \) and a simple closed curve \( \tau \) on a hyperbolic surface \( X \in \tau_{g,n} \), there exists a constant \( A \) depends only on \( S_{g,n} \) and \( \epsilon \) such that
\[
A \left( \sum_{i=1}^{k} |a_i| \langle i(\alpha_i, \tau), \ell_X(\alpha_i) + \ell_X(\tau) \rangle \right) \geq \ell_{\alpha, X}(\tau) \geq \frac{1}{A} \left( \sum_{i=1}^{k} (|a_i| - 2) \langle i(\alpha_i, \tau), \ell_X(\alpha_i) - \ell_X(\tau) \rangle \right).
\]
Furthermore, if \( \alpha \) is positive or negative, the lower bound can be sharpened to
\[
\ell_{\alpha, X}(\tau) \geq \frac{1}{A} \left( \sum_{i=1}^{k} |a_i| \langle i(\alpha_i, \tau), \ell_X(\alpha_i) - \ell_X(\tau) \rangle \right).
\]

**Proof.** By the length formula (4), we know
\[
CN \sum_{\gamma \in \mu_X} i(T_{\alpha}^{-1}(\tau), \gamma) \geq \ell_{\alpha, X}(\tau) \geq \frac{\epsilon}{C} \sum_{\gamma \in \mu_X} i(T_{\alpha}^{-1}(\tau), \gamma).
\]
We can apply the signed intersection formula (2) to approximate \( i(T_{\alpha}^{-1}(\tau), \gamma) \). This allows us to expand the above inequality into the following:
\[
CN \sum_{\gamma \in \mu_X} \left( \sum_{i=1}^{k} \langle a_i | i(\alpha_i, \gamma), i(\alpha_i, \tau) \rangle \right) \geq \ell_{\alpha, X}(\tau) \geq \frac{\epsilon}{C} \sum_{\gamma \in \mu_X} \left( \sum_{i=1}^{k} (|a_i| - 2) \langle i(\alpha_i, \gamma), i(\alpha_i, \tau) - i(\tau, \gamma) \rangle \right).
\]
By switching the order of summations \( \sum_{\gamma \in \mu_X} \) and \( \sum_{i=1}^{k} \) and by applying the length formula (4) again, we obtain the result in the proposition.
If $\alpha$ is positive or negative, we can use the intersection formula \((1)\), and going through the same proof give us the sharpened lower bound. \(\square\)

Note the above proposition provides a good estimate for the length of multicurves up to a multiplicative error. This error arises from our repeated use of length formula \((\ref{formula})\). Below we propose a more generalized result that leads to removing this multiplicative error. Let \([\cdot]\) denote the 0 threshold function.

**Theorem 3.2.** Given a multicurve $\alpha = \sum_{i=1}^{k} a_i \alpha_i$ and a simple closed curve $\tau$ on any hyperbolic structure $X$, we have

\[
\ell_X(\tau) + \sum_{i=1}^{k} i(\tau, \alpha_i)|a_i|\ell_X(\alpha_i) \\
\geq \ell_{T_\alpha,X}(\tau) \\
\geq \sum_{i=1}^{k} i(\tau, \alpha_i) \cdot \left( |a_i| - 2 \right) \cdot \ell_X(\alpha_i) - 2\ell_X(\tau) - L \middle|_0
\]

where $L$ is a constant that depends on $\mathbb{H}^2$.

**Proof.** Fix the hyperbolic structure we may assume curves are geodesics. Given a multicurve $\alpha = \sum_{i=1}^{k} a_i \alpha_i$, we denote $A = \{\alpha_i\}_{i=1}^{k}$ and denote $\tilde{A}$ the set of all liftings of curves in $A$ to $\mathbb{H}^2$. Let $\mathcal{Z}_A$ denote the corresponding Bass-Serre tree, see section 2.6

For each $\beta \in \tilde{A}$, we denote $\psi_\beta \in \pi_1(S_{g,n})$ the corresponding hyperbolic isometry in $\mathbb{H}^2$. If $\beta, \gamma \in \tilde{A}$ are lifts of the same $\alpha \in A$, then $\psi_\beta, \psi_\gamma$ are conjugate to each other and have the same translation distance equal to $\ell_X(\alpha)$. This also means there exists an isometry $\psi$ in $\mathbb{H}^2$ that sends $\gamma$ to $\beta$. We can choose this isometry up to composing with any power of $\psi_\beta$ or pre-composing with any power of $\psi_\gamma$.

In particular, suppose there are geodesic segments $\beta' \subset \beta, \gamma' \subset \gamma$ such that their length are less than $\ell_X(\alpha)$, we can choose the isometry $\psi$ in a way such that $\beta'$ and $\psi(\gamma')$ both lie on $\beta$ and intersect the same fundamental domain of the action of $\psi_\beta$.

Given a simple closed curve $\tau$, we denote

\[
i(\tau, \alpha) = m = \sum_{i=1}^{k} i(\tau, \alpha_i) = \sum_{i=1}^{k} m_i.
\]

In the case of $i(\tau, \alpha) = 0$, $T_\alpha$ has no effect on $\tau$ and the theorem holds true. We may assume $i(\tau, \alpha) \geq 1$. Let $\tilde{\tau}$ be a lifting of $\tau$ and say it has end points $q_L, q_R \in \partial \mathbb{H}^2$. $\tilde{\tau}$ is therefore a bi-infinite geodesic in $\mathbb{H}^2$ transverse to $\tilde{A}$, and $\phi_A(\tilde{\tau})$ is a bi-infinite geodesic in $\mathcal{Z}_A$, say the edges are labeled by \[\mathcal{B} = \{\cdots, \beta_{-2}, \beta_{-1}, \beta_0, \beta_1, \beta_2, \cdots\} \].

For each $\beta_i$, let’s denote $\pi_{\beta_i}(\beta_{i-1}), \pi_{\beta_i}(\beta_{i+1})$ as $\beta_{i,L}, \beta_{i,R}$ respectively. Define a index function $s: \mathbb{N} \rightarrow \{1, \cdots, k\}$ so that $\alpha_{s(i)} \in A$ is the simple closed curve such that $\beta_i$ is a lift of $\alpha_{s(i)}$. For each $i$, we claim $d(\beta_{i,L}, \beta_{i,R}) \leq 2\ell_X(\tau)$. In the case of $i(\tau, A) = 1$, we pick $\kappa \subset \tilde{\tau}$ to be the geodesic segment between the points $\tilde{\tau} \cap \beta_{i-1}$ and $\tilde{\tau} \cap \beta_{i+1}$. Then $\kappa$ is a concatenation of two consecutive path liftings of $\tau$ and $\ell_X(\kappa) = 2\ell_X(\tau)$. For $i(\tau, A) \geq 2$, we pick $\kappa$ to be the path lifting of $\tau$ starting from $\beta_{i-1} \cap \tilde{\tau}$. In any case, $\kappa \subset \tilde{\tau}$ goes through $\beta_{i-1}, \beta_i, \beta_{i+1}$ and has length bounded by $2\ell_X(\tau)$. By Lemma 2.7 we know the projection maps $\pi_{\beta_i}$ are 1-Lipschitz, thus the
distance between projections of the two endpoints of $\kappa$ on $\beta_i$ is smaller than the length of $\kappa$, which is less than $2\ell_X(\tau)$. Since the projections of the two endpoints lie in $\beta_{i,L}, \beta_{i,R}$ respectively, we have $d(\beta_{i,L}, \beta_{i,R}) \leq 2\ell_X(\tau)$.

Fix some point $q_0 \in \hat{\tau}$ and let $\hat{\tau}'$ be $\hat{\tau}$ after shearing according to $-\alpha$ fixing the component of $q_0$, see section 2.4. The projection of $\hat{\tau}'$ to the surface $\mathcal{X}$ has length equal to $\ell_X(\tau) + \sum_{i=1}^{k} i(\tau, \alpha_i) |a_i| \ell_X(\alpha_i)$. Denote the end points of $\hat{\tau}'$ as $q'_{L}, q'_{R} \in \partial \mathbb{H}^2$. Let $\sigma$ be the geodesic with end points $q'_{L}, q'_{R} \in \partial \mathbb{H}^2$, then $\sigma$ is a lift of the geodesic $T^{-1}_{\alpha}(\tau)$ and its image $\phi_\mathcal{A}(\sigma)$ is a geodesic in $\mathcal{Z}_\mathcal{A}$. Since the projection of $\hat{\tau}'$ is in the isotopy class $T^{-1}_{\alpha}(\tau)$, the upper bound in (7) follows.

Once $\hat{\tau}'$ leaves a connected component of $\mathbb{H}^2 \setminus \cup \mathcal{A}$, it never comes back. This means $\phi_\mathcal{A}(\hat{\tau}')$ does not backtrack in $\mathcal{Z}_\mathcal{A}$ so $\phi_\mathcal{A}(\hat{\tau}')$ is a geodesic path in $\mathcal{Z}_\mathcal{A}$. Since $\sigma$ shares the same endpoints with $\hat{\tau}'$ and since $\mathcal{Z}_\mathcal{A}$ is a unique geodesic space, we have $\phi_\mathcal{A}(\hat{\tau}') = \phi_\mathcal{A}(\sigma)$. Denote the edge labels of $\phi_\mathcal{A}(\sigma)$ as

$$\mathcal{F} = \{ \cdots, \eta_{-2}, \eta_{-1}, \eta_0, \eta_1, \eta_2, \cdots \}.$$

For each $\eta_i$, let’s denote $\pi_{\eta_i}(\eta_{i-1}), \pi_{\eta_i}(\eta_{i+1})$ as $\eta_{i,L}, \eta_{i,R}$ respectively. Since each $\beta_i$ and $\eta_i$ are lifts of the same curve, we can use the same index function $s$ denoting $\alpha_{s(i)} \in \mathcal{A}$ the simple closed curve such that $\eta_i$ is a lift of $\alpha_{s(i)}$. Since for each $i$, the triples $\tau, \beta_i, \beta_{i+1}$ and $\eta_i, \eta_{i,L}, \eta_{i,R}$ realize the same intersection pattern on the surface, $\eta_{i,L}, \eta_{i,R}$ are translations of $\beta_{i,L}, \beta_{i,R}$ respectively and have the same diameters respectively.

The relative location of $\eta_{i,L}, \eta_{i,R}$ is the same as the relative location of $\beta_{i,L}$ and $\psi_{\beta_{i,L}}(\beta_{i,R})$, see Figure 4 for an illustration. Recall for any point $x$ on any $\beta_i$ and for any $t \in \mathbb{Z}$, we have $d(x, \psi^t_{\beta_{i,L}}(x)) = |t| \ell_X(\alpha_{s(i)})$. Since both $\text{diam}(\beta_{i,L}), \text{diam}(\beta_{i,R})$ are bounded by $\ell_X(\alpha_{s(i)})$, and since $d(\beta_{i,L}, \beta_{i,R}) \leq 2\ell_X(\tau)$, we have $\text{diam}(\beta_{i,L} \cup \beta_{i,R}) \leq 2\ell_X(\alpha_{s(i)}) + 2\ell_X(\tau)$. It follows that

$$d(\eta_{i,L}, \eta_{i,R}) \geq |a_{s(i)}| \ell_X(\alpha_{s(i)}) - \text{diam}(\beta_{i,L} \cup \beta_{i,R})$$

$$\geq (|a_{s(i)}| - 2) \ell_X(\alpha_{s(i)}) - 2\ell_X(\tau).$$

Denote

$$D_i = (|a_{s(i)}| - 2) \ell_X(\alpha_{s(i)}) - 2\ell_X(\tau)$$

so that $d(\eta_{i,L}, \eta_{i,R}) \geq D_i$ for any $i$.  

Figure 4. Before and after shearing
Denote the sequence of points \( \{w_i\}_{i \in \mathbb{N}} \) on \( \sigma \) such that each \( w_i \) is the first point on \( \sigma \) entering the \( L \)-neighborhood of \( \eta_i \) from left. Since \( \pi_{\eta_i}(q^*_L) \in \eta_i \), by Corollary 2.4, for each \( i \) we have
\[
d(\pi_{\eta_i}(w_i), \eta_i,L) \leq d(\pi_{\eta_i}(w_i), \pi_{\eta_i}(q^*_L)) \leq 2L.
\]
Moreover, all these points are equivalent under translation of \( \sigma \), i.e., for any \( i \) we have
\[
d(w_i, w_{i+m}) = \ell_x(T^{-1}_\alpha(\tau)) = \ell_{T\alpha x}(\tau).
\]
By Lemma 2.7, we know projection \( \pi_{\eta_i} \) is 1-Lipschitz for any \( i \). Since \( \pi_{\eta_i}(\eta_j) \subset \eta_i \), for any \( i < j \) we have
\[
d(w_i, w_j) \geq d(\eta_{i,L}, \eta_j) - 4L \geq [D_i - 4L]_0.
\]
See Figure 4 for an illustration.

We use \( a < b \) to denote that \( \sigma \) goes through the point \( a \) first and then the point \( b \) from left to right. We claim for any \( i \) such that \( D_i > 4L \) and for any \( j > i \), we have \( w_i < w_j \). Indeed, suppose \( w_j \leq w_i \), then by definition the geodesic segment from \( w_j \) to \( w_i \) completely lies outside the \( L \)-neighborhood of \( \eta_i \), and this means
\[
d(\pi_{\eta_i}(w_j), \pi_{\eta_i}(w_i)) \leq L
\]
because geodesics are strongly contracting in \( \mathbb{H}^2 \), see section 2.4. Meanwhile, we know \( d(\eta_{i,R}, \eta_i(w_j)) \leq d(\pi_{\eta_i}(\eta_{j,R}), \pi_{\eta_i}(w_j)) \leq d(\eta_j, w_j) = L \)
because \( \pi_{\eta_i}(\eta_j) \subset \eta_i \) and because the projection map \( \pi_{\eta_i} \) is 1-Lipschitz. Combining with the previous fact \( d(\pi_{\eta_i}(w_i), \eta_{i,L}) \leq 2L \), we conclude
\[
D_i \leq d(\eta_{i,L}, \eta_{i,R})
\]
\[
\leq d(\pi_{\eta_i}(w_i), \eta_{i,L}) + d(\pi_{\eta_i}(w_i), \pi_{\eta_i}(w_j)) + d(\eta_{i,R}, \pi_{\eta_i}(w_j))
\]
\[
\leq 2L + L + L = 4L.
\]
And this contradicts \( D_i > 4L \). Therefore, we have a pattern of ordering
\[
\cdots < w_0 < w_1 < \cdots < w_{m-1} < w_m < \cdots
\]
on \( \sigma \) provided that each \( D_i > 4L \). In this case we have
\[
\ell_{T\alpha x}(\tau) = d(w_0, w_m) \geq \sum_{i=1}^m d(w_{i-1}, w_i) \geq \sum_{i=1}^m [D_i - 4L]_0.
\]
If for some \( i \), \( D_i \leq 4L \), we can delete the point \( w_i \) from our sequence and we only need to measure \( d(w_{i-1}, w_{i+1}) \) instead. The same result (9) holds. Replacing 4L by \( L \) gives us the lower bound in (7). \( \square \)

While the above Theorem 3.2 no longer has multiplicative error, we are not yet able to provide an effective lower bound for multicurves with mixed sign and with each coefficient having absolute value \( \leq 2 \). The Proposition 3.3 below takes one more step and will lead to an effective lower bound for “long” multicurves with mixed sign and with each coefficient having absolute value \( \geq 2 \). Before that, we make the following two remarks that would help us establish Proposition 3.3.

Remark 3.3. We notice the following in the proof of Theorem 3.2.
Recall that we denote \( i(\tau, \alpha) = m = \sum_{i=1}^k i(\tau, \alpha_i) = \sum_{i=1}^k m_i \). Let’s fix \( m \)-many consecutive lifts in \( B \) and denote it as \( B_m \subset B \). Take any \( \alpha_i \) in the multicurve \( \alpha \) and without loss of generality, say \( \beta_{i(1)}, \ldots, \beta_{i(m_i)} \) are all the lifts of this \( \alpha_i \) in \( B_m \). As discussed in the proof of Theorem 3.2 for all \( 1 \leq j \leq m_i \), there exist isometries \( \phi_j \) sending \( \beta_{i(j)} \) to \( \beta_{i(1)} \) such that all \( \beta_{i(1),R}, \phi_j(\beta_{i(j),R}) \) lie on \( \beta_{i(1)} \). For
any distinct pair $\beta_{i(j_1)}, \beta_{i(j_2)}$ where $1 \leq j_1, j_2 \leq m_i$, the orbits $\langle \psi_{\beta_{i(j_1)}} \cdot \phi_j, (\beta_{i(j_1)}, R) \rangle$ and $\langle \psi_{\beta_{i(j_1)}} \cdot \phi_j, (\beta_{i(j_2)}, R) \rangle$ are either the same or completely disjoint. That is, for any distinct pair $\beta_{i(j_1)}, R, \beta_{i(j_2)}, R$, either $\phi_j(\beta_{i(j_1)}, R) = \phi_j(\beta_{i(j_2)}, R)$ or they are disjoint.

Thus except the repetitive ones, we can further assume all $\beta_{i(1), R}, \psi_j(\beta_{i(j)}, R)$ are disjoint, lie on $\beta_{i(1)}$, and lie in a same fundamental domain of the action of $\psi_{\beta_{i(j)}}$. Denote the intersection of this fundamental domain with $\beta_{i(1)}$ as $\beta_{i(1)}^R$, and it follows $\beta_{i(1)}^R \subset \beta_{i(1)}$ is a path lifting of $\alpha_i$ and $\text{diam}(\beta_{i(1)}^R) = \ell_X(\alpha_i)$. This means we have

$$\text{diam} \left( \beta_{i(1), R} \cup \phi_2(\beta_{i(2), R}) \cup \cdots \cup \phi_{m_i}(\beta_{i(m_i), R}) \right) \leq \ell_X(\alpha_i).$$

After removing the repetitive ones, the disjoint union of all these right neighbor projections $\phi_j(\beta_j, R)$ can be arranged into $\beta_{i(1)}^R$, a geodesic segment of diameter $\ell_X(\alpha_i)$. One can do the same thing to all the left neighbors, and the union of all these right neighbor projections $\phi_j(\beta_j, L)$ can be arranged into $\beta_{i(1)}^L$, a geodesic segment of diameter $\ell_X(\alpha_i)$.

**Remark 3.4.** Continuing on Remark 3.3, we recall there are $m_i$ many intersections points between $\alpha_i$ and $\tau$, and let’s denote the set of these points on the surface as $X_i = \{x_1, \ldots, x_{m_i}\}$. On one hand, we can lift $X_i$ to $Y_i = \{y_1, \ldots, y_{m_i}\}$, where each $y_j = \hat{\tau} \cap \beta_{i(j)}, \beta_{i(j)} \in B_m$. On the other hand, we can lift $X_i$ to $Z_i = \{z_1, \ldots, z_{m_i}\}$, where each $z_j = \phi_j(y_j)$, so all points in $Z_i$ lie in a geodesic segment of diameter $\ell_X(\alpha_i)$, namely, $\beta_{i(1)}^L$.

For any of these intersection points $z_j$, we denote the corresponding lift of $\tau$ as $\hat{\tau}_j$. The left neighbor of $z_j$ is defined to be the previous $\hat{\alpha} \in A$ that $\hat{\tau}_j$ intersects, and the right neighbor of $z_j$ is the next $\hat{\alpha} \in A$ that $\hat{\tau}_j$ intersects. Now, we notice the union of all these right neighbors projections of $Z_i$ are exactly the union of right neighbor projections we considered in Remark 3.3 and lie in $\beta_{i(1)}^R$. Similarly for the left neighbors.

**Proposition 3.5.** Given a multicurve $\alpha = \sum_{i=1}^k a_i \alpha_i$ and a simple closed curve $\tau$ on any hyperbolic structure $X$. Let $K \in (0, 1)$ be a constant, we have

\begin{equation}
\ell_{T_n X}(\tau) \geq \sum_{i=1}^k \min \{L_1^i, L_2^i \}
\end{equation}

where

$$L_1^i = i(\tau, \alpha_i) \cdot \left( |a_i| - 2 + K \cdot \ell_X(\alpha_i) - 2 \ell_X(\tau) - L \right)_0,$$

$$L_2^i = \left[ i(\tau, \alpha_i) - \frac{K \ell_X(\alpha_i) + 4 \ell_X(\tau)}{W(\tau)} \right]_0 \left( |a_i| - 1 - K \cdot \ell_X(\alpha_i) - 2 \ell_X(\tau) - L \right)_0.$$

**Proof.** We will use the similar notations and ideas from Theorem 3.2. Recall we denote $i(\tau, \alpha) = m = \sum_{i=1}^k i(\tau, \alpha_i) = \sum_{i=1}^k m_i$. And recall $\phi_{\alpha}(\hat{\tau})$ is a bi-infinite geodesic in $Z_A$, and its edges are labeled by

$$B = \{\ldots, \beta_{-2}, \beta_{-1}, \beta_0, \beta_1, \beta_2, \cdots \}.$$

Define the index function $s: \mathbb{N} \rightarrow \{1, \ldots, k\}$ so that $\alpha_{s(i)} \in A$ is the simple closed curve such that $\beta_{\ell}$ is a lift of $\alpha_{s(i)}$. Let $K \in (0, 1)$, and we consider two different scenarios.
If \( \text{diam}(\beta_{t,L} \cup \beta_{t,R}) \leq (2 - K)\ell_X(\alpha_{s(t)}) + 2\ell_X(\tau) \) for all \( t \), following the argument from Theorem 3.2 we can set
\[
D_t = (|s(t)| - 2 + K)\ell_X(\alpha_{s(t)}) - 2\ell_X(\tau),
\]
and we have \( d(\eta_{t,L}, \eta_{t,R}) \geq D_t \) for all \( t \). Following the same equation (9) and the same argument gives us the lower bound
\[
\ell_{T_t,X}(\tau) \geq \sum_{i=1}^{k} i(\tau, \alpha_i) \left( |\alpha_i| - 2 + K \cdot \ell_X(\alpha_i) - 2\ell_X(\tau) - L \right)_0.
\]

In the second scenario, we have \( \text{diam}(\beta_{t,L} \cup \beta_{t,R}) \geq (2 - K)\ell_X(\alpha_{s(t)}) + 2\ell_X(\tau) \) for some \( t \). Let’s \( i = s(t) \) for simplicity. For this \( t \), since both \( \text{diam}(\beta_{t,L}) \) and \( \text{diam}(\beta_{t,R}) \) are bounded by \( \ell_X(\alpha_i) \) respectively, we have both
\[
\text{diam}(\beta_{t,L}), \text{diam}(\beta_{t,R}) \geq (1 - K)\ell_X(\alpha_i).
\]

Since the \( \beta_{t,L} \) is exhausting an interval length of at least \( (1 - K)\ell_X(\alpha_i) \), as we discussed above in Remark 3.3, the diameter of the union of all other right neighbor projections is bounded by \( \ell_X(\alpha_i) - (1 - K)\ell_X(\alpha_i) \), that is, \( K\ell_X(\alpha_{s(t)}) \). Similarly, the diameter of the union of all left neighbor projections except \( \beta_{t,L} \), is bounded by \( K\ell_X(\alpha_i) \).

Denote \( i = s(t) \) and let \( \beta_{i(1)}, \ldots, \beta_{i(m_i)} \) denote distinct lifts of \( \alpha_i \) in \( B_m \) with \( \beta_i(1) = \beta_i \), see Remark 3.3. Define \( X_i, Y_i, Z_i \) just as in Remark 3.3. We say a \( z_j \) is in vain if its left neighbor is \( \beta_{i-1} \) and its right neighbor is \( \beta_{i+1} \), and we say \( z_j \) is effective otherwise. Notice any points in \( Z_i \) is within distance \( 2\ell_X(\tau) \) of its left neighbor projection and its right neighbor projection, see the proof of Theorem 3.2.

If \( \beta_{t,L} \cap \beta_{t,R} \) is empty, all points in vain lie in a geodesic segment of length \( 4\ell_X(\tau) \). If \( \beta_{t,L} \cap \beta_{t,R} \) is nontrivial, since
\[
(2 - K)\ell_X(\alpha_i) + 2\ell_X(\tau) \leq \text{diam}(\beta_{t,L} \cup \beta_{t,R}) \leq 2\ell_X(\alpha_i) + 2\ell_X(\tau),
\]
we have \( \text{diam}(\beta_{t,L} \cap \beta_{t,R}) \leq K\ell_X(\alpha_i) \), and all points in vain is within \( 2\ell_X(\tau) \)-neighborhood of \( \beta_{t,L} \cap \beta_{t,R} \). In any case, all points in vain can be arranged in a geodesic segment that has length bounded by \( \text{diam}(\beta_{t,L} \cap \beta_{t,R}) + 4\ell_X(\tau) \). By Collar Lemma 2.3 there are at most \( \frac{K\ell_X(\alpha_i) + 4\ell_X(\tau)}{W(\tau)} \) many intersections points in vain.

Thus, realized by \( \alpha_i \) and \( \beta_i \), there are at least \( T_i(K) \) many effective intersection points, where
\[
T_i(K) = \left\lfloor \frac{i(\alpha_i, \tau) - K\ell_X(\alpha_i) + 4\ell_X(\tau)}{W(\tau)} \right\rfloor_0.
\]

For each effective intersection point \( z_j \), it’s of exactly one of the following cases.

1. Its left neighbor is \( \beta_{i-1} \) where its projection is \( \beta_{t,L} \), and its right neighbor projection is being squeezed into an interval of length bounded by \( K\ell_X(\alpha_i) \).
2. Its left neighbor projection is being squeezed into an interval of length bounded by \( K\ell_X(\alpha_i) \), and its right neighbor is \( \beta_{i+1} \) where its projection is \( \beta_{t,L} \).
3. Both of its left and right neighbor projection are being squeezed into an interval of length bounded by \( K\ell_X(\alpha_i) \) respectively.

In any case, the diameter of the union of its left and right projection is bounded by
\[
(1 + K)\ell_X(\alpha_i) + 2\ell_X(\tau) = \max \{ \ell_X(\alpha_i) + K\ell_X(\alpha_i) + 2\ell_X(\tau), 2K\ell_X(\alpha_i) + 2\ell_X(\tau) \},
\]
In between the two dotted lines is the geodesic segment \( \overline{\beta_t} \). \( \beta_{t-1} \) and \( \beta_t \) differ by \( \psi_{\beta_t} \), thus their projections \( \beta_{t,L}, \beta_{t,R}' \) differ by \( \psi_{\beta_t} \). \( \tilde{\tau}_0 \) is in vain and thus is not “counted” in \( T_i(K) \). Lifts like \( \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4 \) are effective and hence will realize a translation distance no less than \( D_i(K) \) after twisting. \( \tilde{\tau}_1 \) is of case (1), \( \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4 \) is of case (2).

which is like the upper bound for \( \text{diam}(\beta_{i,L} \cup \beta_{i,R}) \) in the proof of Theorem 3.2. Apply the same argument from Theorem 3.2 about “\( D_i \)”, for any effective intersection point \( z_j \), and for its corresponding \( y_j \) and \( \beta_i(j) \), it realizes a distance no less than

\[
D_i(K) = \lfloor (|a_i| - 1 - K) \cdot \ell_X(\alpha_i) - 2\ell_X(\tau) - L \rfloor_0
\]

after twisting.

For example, let’s consider the situation in Figure 5, \( z_0 \) is the only one in vain and is not “included” in \( T_i(K) \). \( z_1, z_2 \) realize translation distances no less than

\[
\lfloor (|a_i| - 1) \cdot \ell_X(\alpha_i) - 2\ell_X(\tau) - L \rfloor_0
\]

after twisting, because the union of projections of their left and right neighbors are bounded by \( \ell_X(\alpha_i) \). \( z_3, z_4 \) realize translation distances no less than

\[
\lfloor (|a_i| - 1 - K) \cdot \ell_X(\alpha_i) - 2\ell_X(\tau) - L \rfloor_0
\]

after twisting since the union of projections of their left and right neighbors are bounded by \((1 + K)\ell_X(\alpha_i)\).

Finally, following the same procedure from Theorem 3.2 and only counting the sum of minimum distances realized by effective intersection points, we have

\[
\ell_{T_i(\tau)}(\tau) \geq \sum_{i=1}^{k} T_i(K) \cdot D_i(K).
\]

This gives us the desired result.

Notice that the bounds in Proposition 3.1, Theorem 3.2, Proposition 3.5 involve both the lengths \( \ell_X(\alpha_i) \) and the intersection numbers \( i(\tau, \alpha_i) \). Note also that the lower bounds we obtain are all vacuous in the case where the multicurve
\( \alpha \) is of mixed sign with all coefficients having absolute value 1. The following example shows that there cannot exist a general lower bound, on the order of \( \sum_{i=1}^{k} i(\tau, \alpha_i) \ell_X(\alpha_i) \) as in above results, that is effective in this case.

\[ \begin{align*}
\alpha \quad & \beta \\
\tau \quad & \eta
\end{align*} \]

Figure 6.

**Example 3.6.** Consider simple closed curves \( \alpha, \beta, \eta, \tau \) on a hyperbolic surface \( X \) as in the Figure 6 above. We can define two sequences of simple closed curves 
\[ \alpha_i = T^i_\eta(\alpha), \beta_j = T^j_\eta(\beta), \]
and note that the lengths \( \ell_X(\alpha_n), \ell_X(\beta_n) \) and intersection numbers \( i(\tau, \alpha_n), i(\tau, \beta_n) \) all tend to infinity with \( n \). Denote the multicurve \( \gamma_n = \alpha_n - \beta_n \) for each \( n \), then \( \{\gamma_n\}_{n \in \mathbb{N}} \) is a sequence of multicurves with mixed sign and with each coefficient having absolute value equal to 1. Hence neither Proposition 3.1, Theorem 3.2, nor Proposition 3.5 provides an effective lower bound on \( \ell_X(\tau) \). One can use train tracks \[20\] to study the images \( T_{\gamma_n}(\tau) \) is, and then use length formula (4) to verify that 
\[ \ell_{T_{\gamma_n}X}(\tau) \geq \ell_X(\gamma_n) = \ell_X(\alpha_n) + \ell_X(\beta_n) \]
up to an uniform multiplicative error for all \( n \in \mathbb{N} \). In particular, we notice the intersection numbers \( i(\tau, \alpha_n), i(\tau, \beta_n) \), which go to infinity as \( n \) goes to infinity, do not play any roles in the length of \( \ell_X(\gamma_n) \). Thus there does not exist a constant \( \lambda \) such that 
\[ \ell_{T_{\gamma_n}X}(\tau) \geq \lambda i(\tau, \alpha_n) \ell_X(\alpha_n) + \lambda i(\tau, \beta_n) \ell_X(\beta_n). \]

4. **Coarse distance**

In this section we will adopt the results from the previous section to establish a coarse distance formula, estimating how far a point in Teichmüller space moves after applying a twist.

The lemma below provides a lower bound of the length of \( \ell_{T_{\kappa}X}(\kappa) \) where \( \alpha \) is a multicurve of mixed sign where each coefficient has absolute value \( \geq 2 \), and \( \kappa \) is a particular curve in the short marking \( \mu_X \).

**Lemma 4.1.** Fix some \( \epsilon > 0 \), there exists \( E, Q > 0 \) depends on \( \epsilon \) so the following holds true. Given any \( X \in T_{g,n}^* \), let \( \alpha = \sum_{i=1}^{k} a_i \alpha_i \) be a multicurve where each coefficient has absolute value \( \geq 2 \), and satisfying \( \ell_X(\alpha) \geq E \). Let \( j \) denote an index
such that $|a_j|\ell_X^2(\alpha_j) = \max_{1 \leq i \leq k} |a_i|\ell_X^2(\alpha_i)$. Let $\kappa \in \mu_X$ be a marking curve such that $i(\alpha_j, \kappa) = \max_{\eta \in \mu_X} i(\alpha_j, \eta)$, we have

\begin{equation}
\ell_{T_n, X}(\kappa) \geq \frac{1}{Q} \cdot i(\alpha_j, \kappa)|a_j|\ell_X(\alpha_j).
\end{equation}

Proof. Since $k \leq \frac{1}{2}$, $\ell_X(\alpha) \geq E$ and since any simple closed curve has length $\geq \epsilon$, we have

$|a_j|\ell_X(\alpha_j) \geq \sqrt{|a_j|^2\ell_X^2(\alpha_j)} \geq \sqrt{|a_j|\ell_X^2(\alpha_j)} \geq \sqrt{\frac{2\epsilon E}{h}}$.

First, we consider the case $|a_j| \geq 3$. Since $|a_j| - 2 \geq \frac{1}{3}|a_j|$ for any $|a_j| \geq 3$, by assuming $E \geq \frac{18h(2N+L)^2}{\epsilon}$, we have

$|(a_j| - 2)\ell_X(\alpha_j) \geq \frac{1}{3}|a_j|\ell_X(\alpha_j) \geq \frac{1}{3} \sqrt{\frac{\epsilon E}{k}} \geq 2(2N + L)$,

$|(a_j| - 2)\ell_X(\alpha_j) - 2N - L \geq \frac{1}{6}|a_j|\ell_X(\alpha_j)$.

Thus for this particular $j$ and $\kappa$, since $\ell_X(\kappa) \leq N$ we have

$i(\alpha_j, \kappa)[|a_j| - 2)\ell_X(\alpha_j) - 2\ell_X(\kappa) - L] \geq \frac{1}{6} i(\alpha_j, \kappa)|a_j|\ell_X(\alpha_j)$.

Apply Theorem 3.2, we conclude

$\ell_{T_n, X}(\kappa) \geq \frac{1}{6} i(\alpha_j, \kappa)|a_j|\ell_X(\alpha_j)$.

Now, we consider the case $|a_j| = 2$. Recall by fixing $\epsilon$, any short marking curve corresponding to any $\mathcal{X} \in \mathcal{T}_{g,n}$ has length bounded on top by $N$. By Collar Lemma 2.5, there exists a $W$ depends on $\epsilon$ such that collar width of any short marking curve corresponding to any $\mathcal{X} \in \mathcal{T}_{g,n}$ has length bounded below by $W$. This means we have

$\frac{K\ell_X(\alpha_j) + 4\ell_X(\kappa)}{W(\kappa)} \leq \frac{hKCN_i(\alpha_j, \kappa) + 4N}{W}$.

Since we have $\ell_X(\alpha_j) \geq \frac{1}{2} \sqrt{\frac{2\epsilon E}{h}}$, by length formula (4),

$i(\alpha_j, \kappa) \geq \frac{\ell(\alpha_j)}{hCN} \geq \frac{1}{hCN} \sqrt{\frac{\epsilon E}{2h}}$.

Let $E \geq \frac{32hN^2}{\epsilon K^2}$ so that

$hKCNi(\alpha_j, \kappa) \geq 4N$,

and take the constant $K = \min\{\frac{W}{hCN}, \frac{1}{2}\}$, we have

\begin{equation}
\frac{hKCNi(\alpha_j, \kappa) + 4N}{W} \leq \frac{2hCNK}{W} \cdot i(\alpha_j, \kappa) \leq \frac{1}{2} i(\alpha_j, \kappa).
\end{equation}

Moreover, by further assuming $E \geq \frac{12h(2N+L)^2}{2\epsilon}$, we have $\ell_X(\alpha_j) \geq 6(2N + L)$. Since $|a_j| = 2$, $K \leq \frac{1}{2}$, we have

\begin{equation}
(|a_j| - 1 - K) \cdot \ell_X(\alpha_i) - 2N - L \geq \frac{1}{2} \ell_X(\alpha_j) - 2N - L \geq \frac{1}{3} \ell_X(\alpha_j).
\end{equation}
Now we can apply Proposition 3.5. In the $\ell_1$ case, $|a_j| = 2$ and the lower bound is
\[
\ell_{T_n X}(\kappa) \geq i(\alpha_j, \kappa) (K\ell_X(\alpha_j) - 2N - L).
\]
By further assuming
\[
E \geq \max \left\{ \frac{12^2 h (2N + L)^2}{2\epsilon}, \frac{h}{2\epsilon} \left( \frac{16hCN(2N + L)}{W} \right)^2 \right\},
\]
we have
\[
K\ell_X(\alpha_j) - 2N - L \geq \frac{1}{2} K\ell_X(\alpha_j),
\]
\[
\ell_{T_n X}(\kappa) \geq \min \left\{ \frac{W}{8hCN}, \frac{1}{4} \right\} \cdot i(\alpha_j, \kappa) \ell_X(\alpha_j).
\]
In the $\ell_2$ case, our previous formulas \[12\], \[13\] guarantee
\[
\ell_{T_n X}(\tau) \geq \left[ i(\kappa, \alpha_j) - \frac{K\ell_X(\alpha_j) + 2\ell_X(\kappa)}{W(\kappa)} \right]_0 \cdot \left[ (|a_j| - 1 - K) \cdot \ell_X(\alpha_j) - 2\ell_X(\kappa) - L \right]_0
\]
\[
\geq \frac{1}{2} i(\kappa, \alpha_j) \cdot \frac{1}{3} \ell_X(\alpha_j) \geq \frac{1}{6} \cdot i(\kappa, \alpha_j) \ell_X(\alpha_j).
\]
Let
\[
E = \max \left\{ \frac{12^2 h (2N + L)^2}{\epsilon}, \frac{32hN^2}{\epsilon K^2}, \frac{h}{2\epsilon} \left( \frac{16hCN(2N + L)}{W} \right)^2 \right\},
\]
\[
Q = \min \left\{ \frac{1}{6}, \frac{W}{8hCN} \right\}.
\]
The result follows. \[\square\]

Now we are ready to prove the main result of this section.

**Theorem 4.2** (Coarse Distance Formula). Fix some $S_{g,n}$ and given any $\epsilon > 0$, there exists a constant $H > 0$ such that given any $X \in T_{g,n}^\epsilon$, we have
\[
d_T(X, T_0 X) \approx^H \log \left( \sum_{i=1}^{k} |a_i| \ell_X^2(\alpha_i) \right)
\]
for any $\alpha = \sum_{i=1}^{k} a_i \alpha_i \in M\mathcal{L}^*(\mathbb{Z})$.  

Proof. By the distance formula (3) and our formula (7), we have

\[ d_T(\mathcal{X}, T_\alpha \mathcal{X}) \leq \log \left( \frac{e^c}{\epsilon} \max_{\tau \in \mu_X} \ell_{T_\alpha \mathcal{X}}(\tau) \right) \]

\[ \leq \log \left( \frac{N e^c}{\epsilon} + \frac{e^c}{\epsilon} \cdot \max_{\tau \in \mu_X} \sum_{i=1}^{k} i(\tau, \alpha_i)|a_i|\ell_{\mathcal{X}}(\alpha_i) \right) \]

\[ \leq \log \left( \frac{N e^c}{\epsilon} + \frac{e^c}{\epsilon} \cdot \sum_{\tau \in \mu_X} \sum_{i=1}^{k} i(\tau, \alpha_i)|a_i|\ell_{\mathcal{X}}(\alpha_i) \right) \]

\[ = \log \left( \frac{N e^c}{\epsilon} + \frac{e^c}{\epsilon} \cdot \sum_{i=1}^{k} \left( \sum_{\tau \in \mu_X} i(\tau, \alpha_i) \right) |a_i|\ell_{\mathcal{X}}(\alpha_i) \right) \]

\[ \leq \log \left( \frac{N e^c}{\epsilon} + \frac{e^c C}{\epsilon^2} \cdot \sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}^2(\alpha_i) \right) \]

where last inequality holds by applying the length formula (4). Since we always have \(|a_i|\ell_{\mathcal{X}}^2(\alpha_i) \geq \epsilon^2\), by using equality \(\log(a + b) = \log(1 + \frac{a}{b}) + \log(b)\) we have

\[ d_T(\mathcal{X}, T_\alpha \mathcal{X}) \leq \log \left( 1 + \frac{N e^c}{\epsilon} \right) + \log \left( \frac{e^c C}{\epsilon^2} \cdot \sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}^2(\alpha_i) \right) \]

\[ \leq \log \left( 1 + \frac{N}{\epsilon^2 c} \right) + \log \left( \frac{e^c C}{\epsilon^2} \right) + \log \left( \sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}^2(\alpha_i) \right). \]

This gives us the upper bound in (13) after setting appropriate \(H\).

Now we work toward the lower bound in (14). Let’s assume that \(\ell_{\mathcal{X}}(\alpha) \geq E\), \(E \) from Lemma 4.1. First, we consider the case that \(\alpha\) is positive or negative, i.e., all coefficients have the same sign. In this case, by applying Proposition 3.1 similar to the argument obtaining the upper bound, we have

\[ d_T(\mathcal{X}, T_\alpha \mathcal{X}) \geq \log \left( \frac{1}{N e^c} \max_{\tau \in \mu_X} \ell_{T_\alpha \mathcal{X}}(\tau) \right) \]

\[ \geq \log \left( \frac{1}{AN e^c} \right) + \log \left( \max_{\tau \in \mu_X} \sum_{i=1}^{k} |a_i| i(\alpha_i, \tau)\ell_{\mathcal{X}}(\alpha_i) - N \right) \]

\[ \geq \log \left( \sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}^2(\alpha_i) \right) - \log \left( 2hACN^2\epsilon \right) \]

Now, we consider the case where \(\alpha\) is of mixed sign where each coefficient has absolute value \(\geq 2\). Let \(j\) be the index of \(\alpha\) such that \(|a_j|\ell_{\mathcal{X}}^2(\alpha_j) = \max_{1 \leq i \leq k} |a_i|\ell_{\mathcal{X}}^2(\alpha_i)\) and let \(\kappa \in \mu_X\) be the curve realizes \(\max_{\eta \in \mu_X} i(\alpha_j, \eta)\), then by previous Lemma 4.1 we have

\[ \max_{\eta \in \mu_X} \ell_{T_\alpha \mathcal{X}}(\eta) \geq \ell_{T_\alpha \mathcal{X}}(\kappa) \geq \frac{1}{Q} \cdot i(\alpha_j, \kappa)|a_j|\ell_{\mathcal{X}}(\alpha_j). \]
For simplicity of notation, we use $\tau X$, $M$ using similar idea in Theorem 4.2, we would have


equation (15) for some $H$ of multicurves $T$

Since twists never stabilize any point in $T$, it follows that

Apply the distance formula (3), we have

Apply the length formula (4), we have

Remark 4.3 The result follows. □

We say $f$ above holds true for any $H$.

And this gives us the lower bound in (14) after setting appropriate $H$.

Finally, we set

$$H = \max\{\log\left(1 + \frac{N}{hC}\right) + \log\left(\frac{eC}{e^2}\right),$$

$$\log\left(2hACN^2\epsilon^c\right), \log\left(h^2QCN^2\epsilon^c\right), 2\log(E)\}.$$  

The result follows.

Remark 4.3. Consider Example 3.6 if $\tau$ is chosen to be a short marking curve, using similar idea in Theorem 1.2 we would have

$$d_T(\mathcal{X}, T_n\mathcal{X}) \overset{H}{\simeq} \log(\ell_X(\alpha_n) + \ell_X(\beta_n))$$  

for some $H$. This implies our coarse distance formula does not hold for this sequence of multicurves $\{\gamma_n\} \subset \mathcal{ML}(\mathbb{Z}) \setminus \mathcal{ML}^*(\mathbb{Z})$.

5. Proof of Theorem 1.3 and Corollary 1.5

Assume the conditions in Theorem 1.3, let $D$ be one of the three sets $D(S)$, $M(S)$, $D(M\mathcal{L}^*(\gamma))$. Since any mapping class is an isometry for $T_{g,n}$, given any $\mathcal{X}, \mathcal{Y} \in T_{g,n}$, we notice

$$|D \cdot \mathcal{X} \cap B_R - d_T(\mathcal{X}, \mathcal{Y})(\mathcal{X})| \leq |D \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \leq |D \cdot \mathcal{X} \cap B_{R+d_T(\mathcal{X}, \mathcal{Y})}(\mathcal{X})|.$$

Also, recall that

$$D \cdot \mathcal{X} \cap B_R(\mathcal{X}) = \{g \cdot \mathcal{X} \in T_{g,n} \mid g \in D, d_T(g \cdot \mathcal{X}, \mathcal{X}) \leq R\}.$$  

Since twists never stabilize any point in $T_{g,n}$, we have

$$|D \cdot \mathcal{X} \cap B_R(\mathcal{X})| = |\{g \in D \mid d_T(g \cdot \mathcal{X}, \mathcal{X}) \leq R\}|.$$

For simplicity of notation, we use $t = \frac{1}{A}$ to denote half the dimension of $T_{g,n}$. Recall we say $f(R) \sim^A g(R)$ if for any $\lambda > 1$, there exists a $M(\lambda)$ such that $\frac{1}{\lambda A} \leq \frac{g(R)}{f(R)} \leq \lambda A$.  

23
for any $R \geq M(\lambda)$. We are now ready to prove Theorem 1.4 and Corollary 1.5. For each case and for any $\lambda > 1$, we will compute the corresponding $M(\lambda)$.

**Proof of Theorem 1.4.** Let $\gamma = \sum_{i=1}^{k} a_{i}\gamma_{i}$ be a multicurve, then $c_{\alpha} = c_{\gamma}$ for any $\alpha \in \mathcal{ML}^{*}(\gamma)$. We consider the corresponding set of twists around curves of topological type $\gamma$

$$D(\mathcal{ML}^{*}(\gamma)) = \{ T_{\alpha} \mid \alpha \in \mathcal{ML}^{*}(\gamma) \}.$$

Define

$$\mathcal{S}_{R}^{+} = \left\{ \alpha = \sum_{i=1}^{k} a_{i}\alpha_{i} \in \mathcal{ML}^{*}(\gamma) \mid \sum_{i=1}^{k} |a_{i}|\ell_{\gamma}^{2}(\alpha_{i}) \leq e^{R\pm H} \right\},$$

$$\mathcal{S}_{R}^{++} = \left\{ \alpha = \sum_{i=1}^{k} a_{i}\alpha_{i} \in \mathcal{ML}^{*}(\gamma) \mid \ell_{\gamma}(\alpha) \leq \sqrt{c_{\alpha}} \cdot e^{(R+H)/2} \right\},$$

$$\mathcal{S}_{R}^{-} = \left\{ \alpha = \sum_{i=1}^{k} a_{i}\alpha_{i} \in \mathcal{ML}^{*}(\gamma) \mid \ell_{\gamma}(\alpha) \leq e^{(R-H)/2} \right\}.$$

By the coarse distance formula (14), we have

$$|\mathcal{S}_{R}^{-}| \leq |D(\mathcal{ML}^{*}(\gamma)) \cdot \mathcal{X} \cap B_{R}(\mathcal{X})| \leq |\mathcal{S}_{R}^{+}|.$$

Since

$$\sum_{i=1}^{k} |a_{i}|\ell_{\gamma}^{2}(\alpha_{i}) \leq \sum_{i=1}^{k} |a_{i}|^{2}\ell_{\gamma}^{2}(\alpha_{i}) \leq (\ell_{\gamma}(\alpha))^{2},$$

we have $\mathcal{S}_{R}^{-} \subseteq \mathcal{S}_{R}^{+}$. Moreover, by Schwartz inequality, we have

$$\ell_{\gamma}(\alpha)^{2} = \left( \sum_{i=1}^{k} |a_{i}|\ell_{\gamma}(\alpha_{i}) \right)^{2} \leq \left( \sum_{i=1}^{k} |a_{i}| \right) \cdot \left( \sum_{i=1}^{k} |a_{i}|\ell_{\gamma}^{2}(\alpha_{i}) = c_{\alpha} \cdot \sum_{i=1}^{k} |a_{i}|\ell_{\gamma}^{2}(\alpha_{i}) \right)$$

so $\mathcal{S}_{R}^{+} \subseteq \mathcal{S}_{R}^{++}$. Together this means

$$|\mathcal{S}_{R}^{-}| \leq |D(\mathcal{ML}^{*}(\gamma)) \cdot \mathcal{X} \cap B_{R}(\mathcal{X})| \leq |\mathcal{S}_{R}^{++}|.$$

Mirzakhani’s counting formula (6) tells us for any $\lambda > 1$, we have

$$|\mathcal{S}_{R}^{-}| = s_{\mathcal{X}}(e^{(R-H)/2}, \gamma) \geq \frac{1}{\lambda} \cdot n_{\mathcal{X}}(\gamma) \cdot e^{(R-H)};$$

$$|\mathcal{S}_{R}^{++}| = s_{\mathcal{X}}(\sqrt{c_{\alpha}} \cdot e^{(R+H)/2}, \gamma) \leq \lambda \cdot n_{\mathcal{X}}(\gamma) \cdot c_{\gamma} \cdot e^{(R+H)}$$

whenever $R \geq M(\lambda) = 2\log(r_{\mathcal{X}}(\gamma, \lambda)) + H$. This means

$$\frac{1}{\lambda} \cdot n_{\mathcal{X}}(\gamma) \cdot e^{(R-H)} \leq |D(\mathcal{ML}^{*}(\gamma)) \cdot \mathcal{X} \cap B_{R}(\mathcal{X})| \leq \lambda \cdot c_{\gamma} \cdot n_{\mathcal{X}}(\gamma) \cdot e^{(R+H)}$$

whenever $R \geq M(\lambda)$. By equation (15), we have

$$\frac{1}{\lambda} \cdot e^{(d_{\mathcal{X}}(\gamma, \mathcal{Y})+H)} \leq \frac{|D(\mathcal{ML}^{*}(\gamma)) \cdot \mathcal{Y} \cap B_{R}(\mathcal{X})|}{n_{\mathcal{X}}(\gamma)e^{HR}} \leq \lambda \cdot c_{\gamma} \cdot e^{(d_{\mathcal{X}}(\gamma, \mathcal{Y})+H)}$$

whenever $R \geq M(\lambda)$. Recall that we denote $F_{\gamma}(\mathcal{X}, \mathcal{Y}) = (c_{\gamma})^{t} e^{td_{\mathcal{X}}(\gamma, \mathcal{Y})}$. By setting $J = e^{tH}$, we are done with the case $D = D(\mathcal{ML}^{*}(\gamma))$ and Theorem 1.4 follows. \(\square\)
Proof of Corollary 1.5. We observe in the above proof of Theorem 1.4, when \( \gamma \) is a simple closed curve, \( c_\gamma = 1 \) and for \( R \geq 2 \log (r_X (\gamma, \lambda)) + H \) we have

\[
\frac{1}{\lambda \cdot e^{t (d_T (X, Y) + H)}} \leq \frac{|D(S(\gamma)) \cdot Y \cap B_R(X)|}{n_X(\gamma) e^{t R}} \leq \lambda \cdot e^{t (d_T (X, Y) + H)}.
\]

Summing up all the topological types of simple closed curves, we have

\[
\frac{1}{\lambda \cdot e^{t (d_T (X, Y) + H)}} \leq \frac{|D(S) \cdot Y \cap B_R(X)|}{n_X(S) e^{t R}} \leq \lambda \cdot e^{t (d_T (X, Y) + H)}
\]

whenever \( R \geq M(\lambda) = 2 \log (r_X (S, \lambda)) + H \). Recall we denote \( F(X, Y) = e^{t d_T (X, Y)} \) and \( J = e^{t H} \). Thus we are done with the case \( D = D(S) \) and the first result of Corollary 1.5 follows.

Now, we consider the set of Dehn twists with powers \( M(S) = \{ T^n_\alpha \mid \alpha \in S, n \in \mathbb{Z} \setminus \{0\} \} \).

Define

\[
M_R^+ = \{ T^n_\alpha \in M(S) \mid |n| \leq e^{R+H} \},
\]

\[
S_{R,n}^+ = \left\{ \alpha \in S \mid \ell_X(\alpha) \leq e^{(R+H)/2} \sqrt{|n|} \right\}
\]

so that

\[
|M_R^+| = \sum_{n \in \mathbb{Z} \setminus \{0\}} |S_{R,n}^+| = 2 \cdot \sum_{n \in \mathbb{N}} |S_{R,n}^+|.
\]

Thus we only need to consider \( n \in \mathbb{N} \). Since we are in \( T_{g,n} \), \( S_{R,n}^+ \) is empty when \( n \geq e^{\frac{R+H}{e^2}} \), we have

\[
|M_R^+| = 2 \cdot \sum_{n=1}^{e^{R+H}} |S_{R,n}^+|.
\]

Fix some \( \lambda > 1 \) and let’s assume \( r_X(S, \lambda) \geq \epsilon \). Let’s also assume that \( R \geq 2 \log (r_X (S, \lambda)) \) so that \( e^{R+H} \geq r_X^2(S, \lambda) \). For simplicity let’s denote

\[
a = \frac{e^{R+H}}{e^2}, b = \frac{e^{R+H}}{r_X^2(S, \lambda)}.
\]

Then the above assumptions say \( a \geq b \geq 1 \). By Corollary 2.10 and the Mirzakhani’s counting formula (6), we have

\[
S_{R,n}^+ = s_X\left(\frac{e^{R+H}/2}{\sqrt{n}}, S\right) \leq \lambda \cdot n_X(S) \cdot e^{t (R+H)} \cdot \frac{1}{n^t}
\]

provided that \( n \leq \min\{a, b\} = b \). Notice now we have

\[
|M_R^+| \leq 2 \cdot \sum_{n=1}^{a} |S_{R,n}^+| \leq 2 \cdot \sum_{n=1}^{b} |S_{R,n}^+| + 2 \cdot \sum_{n=b}^{a} |S_{R,n}^+|
\]
whenever $R \geq c_1 = 2 \log(r_X(S, \lambda))$.

When $t > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^t}$ converges and is bounded by 2. By assuming $R$ is even larger $(R \geq d_\lambda = \log \left( \frac{2X(r_X(S, \lambda), S)}{2e^2 n_X(S)} \right))$, bigger exponential wins and we have

$$2\lambda \cdot n_X(S) \cdot e^{t(R+H)} \geq \frac{8X(r_X(S, \lambda), S)}{e^2} \cdot e^{R+H}$$

Thus we have

$$\left| M^+_{R} \right| \leq 8\lambda \cdot n_X(S) \cdot e^{t(R+H)}$$

whenever $R \geq \max\{c_\lambda, d_\lambda\}$. By the coarse distance formula (14) we have

$$\left| S^+_R \right| \leq \left| M(S) \cdot X \cap B_R(X) \right| \leq \left| M^+_{R} \right|,$$

$\left| S^+_R \right|$ is from the proof of Theorem 1.4. Similar to the previous cases, we have

$$\frac{1}{\lambda e^{t(d_T(X, Y)+H)}} \leq \frac{\left| M(S) \cdot Y \cap B_R(Y) \right|}{n_X(S) e^{tR}} \leq \lambda \cdot 8e^{t(d_T(X, Y)+H)}$$

whenever $R \geq M(\lambda) = \max\{r_X(S, \lambda), c_\lambda, d_\lambda\}$. The second result of Corollary 1.5 follows.

**Proof of upper bound of Corollary 1.5** We now consider $t = \frac{b}{2} = 1$. Since in this case $\mathcal{M}(\mathbb{Z})$ is 1 dimensional, we have $\mathcal{M}(\mathbb{Z}) = M(S)$.

Notice from the above proof of $M(S)$, when $t = 1$, $\sum_{n=1}^{b} \frac{1}{n} \leq \log(b+1)$ where

$$\log(b+1) = \log\left( \frac{e^{R+H}}{r_X(S, \lambda)} \right) + 1 \leq R$$

by assuming $r_X(S, \lambda)$ sufficiently large. Moreover, we have

$$\lambda \cdot n_X(S) \cdot R \cdot e^{R+H} \geq \frac{8X(r_X(S, \lambda), S)}{e^2} \cdot e^{R+H}$$

when $R$ is large $(R \geq l_\lambda = \frac{8X(r_X(S, \lambda), S)}{\lambda e^2 n_X(S)})$. Thus we have

$$\left| M^+_{R} \right| \leq 4\lambda \cdot n_X(S) \cdot R \cdot e^{R+H}$$

whenever $R \geq M(\lambda) = \max\{c_\lambda, l_\lambda\}$. Similar to previous arguments, this shows

$$\left| M(S \cdot Y \cap B_R(X)) \right| \leq 4^b J_F(X, Y) \cdot n_X(S) \cdot R \cdot e^{\frac{b}{2}R}$$

when $\frac{b}{2} = 1$. 

\[ \square \]
6. Proof of theorem 1.8 and corollary 1.9

Let \( \gamma \) be a multicurve satisfying the conditions in Theorem 1.8. Given \( s \in \mathbb{N} \), we denote
\[
[\gamma, s] = \{ \gamma \in [\gamma] \mid c_\gamma = s \},
\]
\[
\mathcal{ML}(\gamma, s) = \bigsqcup_{\gamma \in [\gamma, s]} \mathcal{ML}(\gamma),
\]
and we denote \( \#[\gamma, s] \) the number of \( \gamma \in [\gamma], c_\gamma = s \). Indeed, this number equals, up to \( \text{Mod}_{g,n} \), the number of topological types of curves composing the set \( \mathcal{ML}(\gamma, s) \).

For any \( l < s \in \mathbb{N} \), we denote
\[
[\gamma, s, l] = \{ \gamma = \sum_{i=1}^{k} a_i \gamma_i \in [\gamma, s] \mid |a_i| \geq l \text{ for any } i \},
\]
and \( \mathcal{ML}(\gamma, s, l), \#[\gamma, s, l] \) are similarly defined as above.

**Lemma 6.1.** Let \( \gamma = \sum_{i=1}^{k} \gamma_i \) be a multicurve with all coefficients equal to one and of maximal dimension \( k = \frac{h}{2} \). For \( s \geq h - 2 \) we have
\[
\#[\gamma, s] \geq \frac{s^{k-1}}{2^{k-1}(k-1)!}.
\]
(17)

In particular, there exists a \( t \) such that for any \( s \geq h - 2 \), we have
\[
\#[\gamma, s, s] \geq \frac{1}{2} \#[\gamma, s] \geq \frac{s^{k-1}}{2^{k-1}(k-1)!}.
\]
(18)

**Proof.** The number \( \#[\gamma, s] \) equals to the number of ordered \( k \)-tuples \( (x_1, \cdots, x_k) \in \mathbb{N}^k \) such that \( \sum_{i=1}^{k} x_i = s \). It’s a standard combinatorics fact this number is \( \binom{s-1}{k-1} \), which is greater than \( \frac{s^{k-1}}{2^{k-1}(k-1)!} \) whenever \( s \geq 2(k-1) = h - 2 \).

For any \( x = (x_1, \cdots, x_k) \in \mathbb{R}_+^k \) we denote \( \delta_x \) the corresponding dirac measure and denote \( \|x\| = \sum_{i=1}^{k} x_i \). We define the following sets
\[
C = \{ x \in \mathbb{R}_+^k \mid \|x\| = 1 \},
\]
\[
C' = \{ x \in C \mid x_i \geq \frac{1}{l} \text{ for any } i \},
\]
\[
C_s = \{ x \in \mathbb{N}^k \mid \|x\| = s \},
\]
\[
C_s^t = \{ x \in C_s \mid x_i \geq \frac{s}{l} \text{ for any } i \}
\]
where \( t, s \in \mathbb{N} \). Define the measures \( \delta_s = \sum_{x \in C_s} \delta_x \) and \( \delta_s^t = \sum_{x \in C_s^t} \delta_x \). Denote the standard probability measure on \( C \) as \( \mu \), a classic measure theory result says the following ratio converges, and we have
\[
\lim_{s \to \infty} \frac{\#[\gamma, s, s]}{\#[\gamma, s]} = \lim_{s \to \infty} \frac{\delta_s^t(C)}{\delta_s(C)} = \frac{\mu(C^t)}{\mu(C)}.
\]

Thus by picking a \( t \) large enough the second equation (18) above holds true. \[ \square \]
Corollary 6.2. Let $\gamma = \sum_{i=1}^{k} \gamma_i$ be a multicurve with all coefficients equal to one and of maximal dimension $k = \frac{h}{2}$. For any $\gamma \in \left[\gamma, \frac{s}{t}, s\right]$, we have

$$\left(\ell_X(\gamma)\right)^2 \geq \frac{s}{t} \cdot \sum_{i=1}^{k} |a_i| \ell_X^2(\gamma_i),$$

(19)

$$\frac{s}{t} \cdot \ell_X(\gamma) \leq \ell_X(\gamma) \leq s \cdot \ell_X(\gamma).$$

This means for any $\gamma \in \left[\gamma, \frac{s}{t}, s\right]$ and any $\lambda > 1$, we have

$$s_X(L, \gamma) \geq s_X(L, s \cdot \gamma) \geq \frac{1}{\lambda} \cdot \frac{L^h}{s^h} n_X(\gamma),$$

(20)

for $L \geq s \cdot r_X(\gamma, \lambda)$.

Proof. The first two equations follow from the definition of $\left[\gamma, \frac{s}{t}, s\right]$. The third equation follows from Corollary 2.10. □

Now we are ready to prove the Theorem 1.8.

Proof of Theorem 1.8. Define

$$S_R = D \left(\mathcal{ML}(\gamma)\right) \cap X \cap B_R(X),$$

$$S_R^- = \left\{ \alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}(\gamma) \mid \sum_{i=1}^{k} |a_i| \ell_X^2(\alpha_i) \leq e^{R-H} \right\},$$

$$S_R^-(s) = \left\{ \alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}(\gamma, s) \mid \sum_{i=1}^{k} |a_i| \ell_X^2(\alpha_i) \leq e^{R-H} \right\},$$

$$S_R^-(s, \frac{t}{s}) = \left\{ \alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}(\gamma, \frac{s}{t}) \mid \ell_X(\alpha) \leq \sqrt{\frac{2}{t}} e^{R-H} \frac{n^h}{s^h} \right\}.$$

Notice for any $\alpha \in S_R^-(s, \frac{t}{s})$, by previous Corollary 6.2 we have

$$\frac{s}{t} \cdot \sum_{i=1}^{k} |a_i| \ell_X^2(\alpha_i) \leq (\ell_X(\alpha))^2 \leq \frac{s}{t} e^{R-H}$$

so that $S_R^-(s, \frac{t}{s}) \subset S_R^-(s)$. Fix some $\lambda > 1$, by Theorem 2.10 and formulas (18), (20), we have

$$\left| S_R^-(s, \frac{t}{s}) \right| \geq \sum_{\gamma \in \left[\gamma, \frac{s}{t}, s\right]} \frac{s}{t} \cdot \ell_X(\gamma) \cdot e^{R-H} \cdot \frac{n_X(\gamma)}{s^h} \cdot \frac{1}{\lambda} \cdot \frac{2^k (k-1)!}{s^k t^k} \cdot \frac{1}{s}.$$
provided that
\[\sqrt{\frac{s}{t}} \cdot e^{\frac{R-H}{t}} \geq s \cdot r_X(\gamma, \lambda) \text{ and } s \geq h - 2.\]
That is,
\[h - 2 \leq s \leq \frac{e^{R-H}}{t \cdot r_X(\gamma, \lambda)}\]
Thus, we have
\[|S_R| \geq |S_R^-| = \sum_{s \in \mathbb{N}} |S_R^-(s)| \]
\[\geq \sum_{s=h-2}^{e^{R-H}} \left| S_R^- (s, \frac{s}{t}) \right| \]
\[\geq \frac{1}{\lambda} \cdot \frac{n_X(\gamma) \cdot e^{k(R-H)}}{2^k(k-1)!t^k} \cdot \sum_{s=h-2}^{e^{R-H}} \frac{1}{s}\]
By assuming $R$ is large ($R \geq M(\lambda) = 2 \left( H + \log(tr_X(\gamma, \lambda)) + \log(h - 2) \right)$), the summation $\sum_{s=h-2}^{e^{R-H}} \frac{1}{s} \geq \frac{\ln^2}{2}$. We now have
\[|S_R| \geq \frac{1}{\lambda} \cdot \frac{n_X(\gamma)}{2^k(k-1)!t^k} \cdot R \cdot e^{\frac{4k}{2}}\]
Similar to the proof of Theorem 1.4, we have
\[|D(\mathcal{ML}(\gamma)) \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \geq |S_R - d_f(\mathcal{X}, \mathcal{Y})| \geq \frac{1}{\lambda JF(\mathcal{X}, \mathcal{Y})} \cdot f(\gamma) \cdot R \cdot e^{\frac{4k}{2}}\]
whenever $R \geq M(\lambda)$, and
\[f(\gamma) = \frac{n_X(\gamma)}{2^k(k-1)!t^k}\]
This concludes the proof of Theorem 1.6.8. \(\square\)

Finally, we discuss about how Corollary 1.9 follows from previous results.

Proof of Corollary 1.9. When $\frac{1}{k} = 1$, $\mathcal{ML}(\mathcal{Z})$ is one dimensional. Take any simple closed curve $\gamma$, then its maximal dimension and $\mathcal{ML}(\mathcal{Z}) = \mathcal{ML}(\gamma)$. As a special case of Theorem 1.6.8 we have $f(\gamma) = n_X(\gamma) = n_X(S)$ and
\[n_X(S) \cdot R \cdot e^{k \cdot JF(\mathcal{X}, \mathcal{Y})} \leq |D(\mathcal{ML}(\mathcal{Z})) \cdot \mathcal{Y} \cap B_R(\mathcal{X})|\]
This gives us the lower bound. The upper bound follows from an alternation of proof of Corollary 1.5.5 see section 5. This concludes the result. \(\square\)

Remark 6.3. We briefly discuss about the difficulty using Theorem 1.2 to obtain an upper bound estimate for the coarse asymptotic rate of $|D(\mathcal{ML}(\mathcal{Z})) \cdot \mathcal{Y} \cap B_R(\mathcal{X})|$. When we are dealing with a conjugacy class of multicurves, say the conjugacy class of $\gamma = \sum_{i=1}^{k} a_i \gamma_i$, we have a “bounding relation” (16) between $\ell^2_X(\gamma)$ and
\[ \sum_{i=1}^{k} |a_i| \ell^2_2(\gamma_i) \] depends only on the coefficients of \( \gamma \), and later we use this relation to estimate the number of corresponding lattice points inside a ball of radius \( R \). In the case of \( \mathcal{ML}(\mathbb{Z}) \), \( \gamma \) being a maximal dimensional multicurve with all coefficients equal to one, we consider a subset of \( \mathcal{ML}(\mathbb{Z}) \) with “balanced weights” so that a uniform “bounding relation” holds. This idea in fact works for multicurves with “balanced weights”. Namely, for any \( m \geq 0 \), we can define the following subset of multicurves

\[ \mathcal{ML}(\mathbb{Z}, m) = \{ \alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}(\mathbb{Z}) \mid |a_i| \geq \frac{c_{\alpha}}{m} \text{ for each } i \} \]

By using similar ideas one can show

\[ |D(\mathcal{ML}(\mathbb{Z}, m)) \cdot \mathcal{V} \cap B_R(\mathcal{X})| \leq s^{2h} \cdot n_{\mathcal{X}}(S) \cdot R \cdot e^{ \frac{h}{2} R} \]

However, for a sequence of multicurves \( \{ \gamma_j \}_{j \in \mathbb{N}} \) such that \( \gamma_j \) is outside \( \mathcal{ML}(\mathbb{Z}, j) \), the possible “bounding relations” get more and more coarse, and does not yield a uniform upper bound as above for \( \mathcal{ML}(\mathbb{Z}) \).

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