A GALOIS APPROACH TO KAPLANSKY RADICAL × HILBERT’S THEOREM 90

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ABSTRACT. This paper aims to prove a version of the Hilbert’s Theorem 90 for a field with non-trivial Kaplansky radical and the Galois group of its maximal 2-extension as a finitely generated elementary type pro-2 group.

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1. INTRODUCTION

The Kaplansky radical of a field $F$ (of characteristic not 2) [17] appears in the algebraic theory of quadratic forms [24] as the set $R(F) = \{a \in \hat{F} \mid D_F(1,-a) = \hat{F}\}$, where $D_F(1,-a)$ consists of all elements represented by the binary quadratic form $X^2 - aY^2$.

One can verify the basic fact that $R(F)$ always contains the squares group $\hat{F}^2 = \{\alpha^2 \mid \alpha \in \hat{F}\}$ by noting that any element of $\hat{F}$ is the difference between two squares.

In order to classify certain Witt rings, Cordes [5] noticed that it was necessary to build a field $F$ with strict inclusion $\hat{F}^2 \subsetneq R(F)$. Lately, Berman [3] and Kula [22, 23] provide many examples of such fields, which we call here as a field having a non-trivial radical.
Cordes also found that many results concerning quadratic forms and related subjects are still valid when replacing the squares group by $R(F)$. For instance, there is a version of 2-henselian valuations for fields with non-trivial radicals [9], useful in Galois Theory (see Example [23]). The next problem follows the same idea.

**Conjecture 1.** (Kijima and Nichi [18]) Let $F$ be a field of characteristic not 2 such that $\hat{F}/\hat{F}^2$ is finite and $K = F(\sqrt{a})$, $a \in \hat{F}\setminus\hat{F}^2$, be a quadratic extension of $F$ which norm map is $N : \hat{K} \to \hat{F}$. Then $N^{-1}(R(F)) = \hat{F}R(K)$.

Recalling that the inclusions $R(F) \subseteq R(K)$ and $N(\hat{F}R(K)) \subseteq R(F)$ are already automatic [2, Proposition 4.3], the problem also can be posed in terms of the exactnesses of the complex

$$\hat{F}/R(F) \xrightarrow{i} \hat{K}/R(K) \xrightarrow{N} \hat{F}/R(F)$$

where $i$ and $N$ are induced by the inclusion and the norm map, respectively. The classical Hilbert’s Theorem 90, applied to extension $K/F$, implies the exactness if the radicals are the squares group. For this reason, one can see Conjecture 1 as a version of this theorem for fields with non-trivial radicals.

Cordes and Ramsey [7, Theorems 3.10, 4.14] proved it for a field $F$ having only one, up to isomorphism, non-split quaternion algebra over $F$.

Kijima and Nichi also considered a version of Conjecture 1 without the finiteness hypothesis and studied it for quasi-Pythagorean fields [19], which have Kaplansky radical as the set of all non-zero finite sums of two squares.

Becher and Leep [2, Theorem 4.8] presented a quadratic extension $K/F$ such that the inclusion $\hat{F}R(K) \subseteq N^{-1}(R(F))$ is strict and $\hat{F}/\hat{F}^2$ is not finite.

In this paper, we consider Conjecture 1 in the context of Galois Theory, and we prove it for a large class of fields, conjecturally all fields having a finite number of square classes, as follows.

Denote by $F(2)$ the maximal 2-extension of $F$ inside a fixed separable closure of $F$. It is the composite of all finite Galois extensions of $F$, which degree is a power of 2. Its Galois group, denoted by $G_F(2)$, is a pro-2 group described as the inverse limit of the Galois groups of all finite Galois 2-extensions of $F$. The only case that it is finite is $\mathbb{Z}/2\mathbb{Z}$, and it occurs for an Euclidean field (e.g. $\mathbb{R}$).

Inspired in the original version for Witt rings [25], the long-standing elementary type conjecture for Galois groups [16] considers elementary operations in the category of pro-2 groups and basic pro-2 groups in order to describe the structure of $G_F(2)$.

The operations are the free pro-2 product and the semi-direct product having the action induced by the cyclotomic character [16, Theorems 2.2, 2.3]. The basic groups are free pro-2 groups and Demushkin Galois groups. Consider the family of finitely generated pro-2 groups resulting from a finite number of iterations of elementary operations between basic groups. A member of this family is realizable as a Galois group $G_K(2)$, for some field
K [11], and we call it an Elementary Type Galois group (ETG group), see Definition 10.

The elementary type conjecture claims that if $G_F(2)$ is topologically finitely generated, then $G_F(2)$ is an ETG group. The precise way to announce it is in terms of cyclotomic pairs, as explained in Section 4.

We are now finally ready to state our main result (Theorem 28):

**Theorem.** Conjecture 1 is true for a field $F$ such that $G_F(2)$ is an ETG group.

Since the number of topological generators of $G_F(2)$ equals the dimension of $\hat{F}/\hat{F}^2$ as a vector space over $\mathbb{F}_2$, our result fulfills the finiteness hypotheses in Conjecture 1.

In [9], we studied $G_F(2)$ assuming that $F$ has a $R(F)$-compatible valuation, as detailed in Example 23. Such a field is called pre-2-henselian. We prove Conjecture 1 for this case [8, Theorem 5.8] (in Portuguese). In this work, the arguments are very similar, but we avoid all the restrictions imposed by valuations. Theorem 28 generalizes the previously mentioned results for pre-2-henselian fields, quasi-Pythagorean fields and the examples obtained by Kula (Example 3) and Berman (Examples 3 and 4), provided the finiteness hypothesis.

The following section reviews examples concerning the Kaplansky radical and its properties. The basic Galois groups are briefly studied in Section 3. The radical behavior under the elementary operations between the basic groups appears in Section 5. Conjecture 1 is proved for ETG groups in Section 7 after preparatory results.

In this paper, we consider pro-2 groups. All subgroups are assumed to be closed subgroups, and homomorphisms are continuous. Fields are always be assumed to have characteristic not 2. Given two subgroups $S_1, S_2$ of $\hat{F}$ with $S_2 \leq S_1$, we denote by $(S_1 : S_2)$ the order of the quotient group $S_1/S_2$.

### 2. Preliminaries

Details and proofs omitted in this Section can be found in [24, Chapter XII, §6].

**Definition 1.** Let $F$ be a field of characteristic not 2 and $D_F(1,-a)$ be the image of the norm map $N : F(\sqrt{a})^\times \to \hat{F}$. The Kaplansky radical of $F$ is the set of all $a \in \hat{F}$ such that $D_F(1,-a) = \hat{F}$.

Now let $(F;a,b)$ be the quaternion algebra generated by $i, j$ such that $i^2 = a$, $j^2 = b$, $ij = -ji$ and $2\text{Br}(F)$ be the set of all classes of finitely generated central simple algebras in the Brauer group of $F$ having order dividing 2. By the basic equivalences

\begin{equation}
[F; a, b] = 0 \in 2\text{Br}(F) \iff a \in D_F(1,-b) \iff b \in D_F(1,-a),
\end{equation}

...
it follows that \( R(F) \) also is the radical of the symmetric bi-multiplicative pairing
\[
\hat{F}/\hat{F}^2 \times \hat{F}/\hat{F}^2 \to 2\text{Br}(F), \quad ([a],[b]) \mapsto [(F;a,b)].
\]

Therefore, putting all together, we have
\[
(3) \quad R(F) = \bigcap_{x \in \hat{F}} D_F\langle 1, x \rangle.
\]

In order to study the position of the radical, let us consider the chain of subgroups of \( \hat{F} \)
\[
(4) \quad \hat{F}^2 \subseteq R(F) \subseteq D_F\langle 1, 1 \rangle \subseteq \sum \hat{F}^2 \subseteq \hat{F},
\]
where \( \sum \hat{F}^2 \) is the set of all non-zero finite square sums.

Most fields are in the lower bound for the radical; that is, the radical is trivial. The pairing \((2)\) is non-degenerated in this case. It follows some well-known examples.

**Example 2.** Recall that \( F \) is a formally real field if it has at least one total order, or equivalently, \(-1 \notin \sum \hat{F}^2\). **Euclidean fields** are the formally real fields with \((\hat{F} : \hat{F}^2) = 2\) and therefore have trivial radicals. These are special cases of **Pythagorean fields**, for which \( \sum \hat{F}^2 = \hat{F}^2 \), by definition. A formally real Pythagorean field has trivial radical. Back to the case \((\hat{F} : \hat{F}^2) = 2\), we have \( R(F) = \hat{F} \) for finite fields and \( k((T)) \), the formal Laurent series field, where the field \( k \) is quadratically closed.

Other examples of fields with trivial radical are Demushkin fields, see Section 3 and 2-henselian fields (Example 19), according to Proposition 20.

Fields with non-trivial Kaplansky radical first appeared in the classification of Witt rings. In [5], Cordes mentioned that examples of those fields were necessary to complete the list of possible Witt rings of fields having eight squares classes.

By using quadratic form schemes, Kula [22] showed that given positive integers \( n \) and \( m \), there is a field \( F \) such that \((\hat{F} : R(F)) = 2^n\) and \((R(F) : \hat{F}^2) = 2^m\). In the same year, Berman [3] found more examples studying non-real extensions of Pythagorean fields. The answer to the Cordes’s question appears in [23].

Because Kula and Berman’s examples are particular cases of our results, see Theorem 25, we will explore it in more detail next, starting with Berman’s work.

**Example 3.** [3, Theorem 2.3] Let \( F \) be a formally real Pythagorean field. Suppose that the set of orders \( X(F) \) has \( 2^n \) elements, \( n \geq 2 \), and \((\hat{F} : \hat{F}^2) = 2^n+1\). The field \( K = F(\sqrt{-1}) \) has \( 2^n \) square classes and \((K : R(K)) = 4\).

**Example 4.** [3, Theorems 3.9 and 3.12] Consider two Pythagorean fields \( F_1 \) and \( F_2 \). The first one is a **super Pythagorean field**: it has \( 2^n \) squares...
classes and $2^{n-1}$ orders, $n \geq 3$. The second one satisfies the Strong Approximation Property (SAP); it has $2^m$ square classes and $m$ orders, $m \geq 2$. The latter is called a SAP Pythagorean field. Then there is a field $F$ such that $\hat{F}/\hat{F}^2 \cong \hat{F}_1/\hat{F}_1^2 \times \hat{F}_2/\hat{F}_2^2$ and for $K = F(\sqrt{-1})$, one has $K/R(K) \cong \hat{F}_1/\{\pm \hat{F}_1\}$ and $R(K)/K^2 \cong \hat{F}_2/\hat{F}_2^2$.

The main example of Kula constructions has a different approach, as follows.

**Example 5.** [22] Let $L$ be a field with a finite number of square classes and $F$ a field such that $G_F(2) \cong G_L(2) *_2 G_H(2)$, where $R(L) = \hat{L}$ and $H$ is the field of the iterated Laurent series $L((X_1)) \ldots ((X_n))$. Then $(\hat{F} : R(F)) = 2^n(\hat{L} : \hat{L}^2)$.

Examples 3, 4, and 5 provide several non-formally real fields with non-trivial Kaplansky radicals. It follows the main example of the real side.

**Example 6.** A field $F$ is called quasi-Pythagorean if $R(F) = D_F(1, 1)$. It actually implies $D_F(1, 1) = \sum \hat{F}^2$ [24, Corollary 6.5 (2), p. 452]. Unless that $R(F) = \hat{F}$, $F$ is formally real. Indeed, if $-1 \in R(F) = \sum \hat{F}^2$ one has $D_F(1, 1) = \hat{F}$, by definition. For a formally real quasi-Pythagorean field $F$, Ware [32, Corollary 1] proved that $G_F(2)$ is the free pro-$2$ product of a free pro-$2$ group and a pro-$2$ group generated by involutions, provided conditions that hold for $(\hat{F} : F^2)$ finite. As examples, one has pseudo-real closed fields and formally real generalized Hilbert fields [17].

Finally, from valuation theory, we have pre-2-henselian fields [9] as examples of fields with non-trivial Kaplansky radical, to be detailed in Section 4, Example [23].

### 3. Basic groups and the Kaplansky radical

As previously mentioned, free pro-$2$ groups and Demushkin groups are the basic groups necessary to understand the structure of $G_F(2)$, at least if it is of elementary type. Let us briefly describe the Kaplansky radical for this cases.

Let $p$ be a prime number and $\mathcal{F}$ be the free pro-$p$ group over the set $X$, defined as an inverse limit based in the ordinary free group over $X$ [20, p. 41] or equivalently, by its universal property [27, Definition 3.5.14]. The rank of $\mathcal{F}$ is the cardinality of $X$ and denoted by $rk(\mathcal{F})$.

For instance, the free pro-$p$ group of rank 1 is isomorphic to $\mathbb{Z}_p$, the additive group of $p$-adic integers, and it occurs as the Galois group $G_F(p)$ for $F = \mathbb{Q}((T))$ or if $F$ is a finite field [16, p. 392].

At the right end of the chain (4) we have

\begin{equation}
R(F) = \hat{F} \iff 2\text{Br}(F) = \{0\} \iff G_F(2) \text{ is a free pro-2 group.}
\end{equation}

The first equivalence follows from (1) and (3). Indeed, one only has to observe that $R(F) = \hat{F}$ occurs if and only if the quaternion algebras $(F; a, b)$
splits, for all $a, b \in \hat{F}$. By the Merkurjev’s Theorem, this is equivalent to $2\text{Br}(F) = \{0\}$. The second equivalence follows from [29, Corollary 3.8, p. 262 and Corollary 3.2, p. 255].

Now, let us turn to Demushkin groups. Denote by $H^i(G, \mathbb{F}_p)$ the $i$-th Galois cohomology group [27] of the pro-$p$ group $G$ with coefficients in $\mathbb{F}_p$.

**Definition 7.** A pro-$p$ group $G$ is a **Demushkin group** if the following three conditions are verified:

(a) The dimension of $H^1(G, \mathbb{F}_p)$ as a $\mathbb{F}_p$-vector space is finite.

(b) The cup product induces a non-degenerate bilinear form $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p)$.

(c) $H^2(G, \mathbb{F}_p) \cong \mathbb{Z}/p\mathbb{Z}$.

Labute and Serre completely described the structure of relations and generators of a Demushkin group [27, Theorem 3.9.11 and 3.9.19].

The only finite Demushkin group is $\mathbb{Z}/2\mathbb{Z}$ [27, Proposition 3.9.10].

We focus on $p = 2$ and a Demushkin group $G$ as a Galois group $G_F(2)$ for some field $F$. In this case, we say that $F$ is a **Demushkin field**.

Denoting $G = G_F(2)$, let us recall the canonical isomorphisms

(6) $H^1(G; \mathbb{F}_2) \rightarrow \hat{F}/\hat{F}^2$, $(a) \mapsto a\hat{F}^2$

(7) $H^2(G; \mathbb{F}_2) \rightarrow 2\text{Br}(F)$, $(a) \cup (b) \mapsto [(F; a, b)]$

Therefore, if $F$ is a Demushkin field, conditions (a), (b) in Definition 7 says that $G_F(2)$ is finitely generated and $F$ has trivial radical, respectively - see [2]. By (c), $G_F(2)$ has only one relation [27].

**Example 8.** For an odd prime number $p$, the $p$-adic field $\mathbb{Q}_p$ is a Demushkin field with $G_{\mathbb{Q}_p}(2) \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2$. The only relation is described in [16, Table 5.2]. A 2-adic local field is far more interesting. It is a finite extension $L$ of $\mathbb{Q}_2$, and its Galois group is a Demushkin group on $[L : \mathbb{Q}_2] + 2$ generators. For instance,

$$G_{\mathbb{Q}_2}(2) \cong \langle \alpha, \beta, \gamma | \alpha^2\beta^4[\beta, \gamma] \rangle,$$

where $[\beta, \gamma]$ is the commutator $\beta^{-1}\gamma^{-1}\beta\gamma$.

### 4. The Elementary Type Conjecture for Galois Groups

In this section, we define an ETG group as a finitely generated pro-$2$ Galois group built iterating only two elementary operations, the free product and the semi-direct product, between the basic pro-$2$ groups listed below. The elementary type conjecture for Galois groups claims that if $(\hat{F} : \hat{F}^2) < \infty$, $G_F(2)$ is an ETG group.

Let $\mu_\infty$ be the group of all $2^n$’s roots of unity in $F(2)^\times$, for all $n$. In order to properly describe pro-$2$ Galois groups, one has to consider the action of $G_F(2)$ over $\mu_\infty$, which leads to the definition of cyclotomic pairs [16], also called orientated pro-$2$ groups.
An orientated pro-2 group is a couple \((G, \theta)\) of a pro-2 group \(G\) and a continuous homomorphism 
\[
\theta : G \to \mathbb{Z}_2 \oplus (\mathbb{Z}/2\mathbb{Z}).
\]
One says that the pair \((G, \theta)\) is realizable for a field \(F\) of \(\text{char}(F) \neq 2\) if \(G = G_F(2)\) and \(\theta = \theta_F : G_F(2) \to \mathbb{Z}_2 \oplus (\mathbb{Z}/2\mathbb{Z})\) is the cyclotomic character of \(F\) defined as the composition of the canonical restriction map \(G_F(2) \to \text{Aut}(\mu_{\infty})\) with the isomorphism \(\text{Aut}(\mu_{\infty}) \cong \mathbb{Z}_2 \oplus (\mathbb{Z}/2\mathbb{Z})\), under which the first factor is generated by \(\tau\) defined by \(\tau(\xi) = \xi^s\) and the second is generated by the involution \(\sigma\) such that \(\sigma(\xi) = \xi^{-1}\).

Let \(\text{Im}(\theta)\) be the image of \(\theta\). It follows the list of elementary orientated pro-2 groups necessary to describe an ETG group.

### Definition 9. (List of elementary orientated pro-2 group)

- \((a)\) \((\mathbb{Z}/2\mathbb{Z}, \theta_{\mathbb{Z}/2\mathbb{Z}})\), where \(\text{Im}(\theta_{\mathbb{Z}/2\mathbb{Z}})\) is the cyclic subgroup generated by \(\sigma\).
- \((b)\) \((\mathbb{Z}_2, \theta)\), where the possible \(\text{Im}(\theta)\) are determined in [16, Definition 4.10].
- \((c)\) \((G_L(2), \theta_L)\), where \(L\) is a 2-adic local field (Example 8). See [16, Lemma 4.4 and Remark 5.5] for a description of the possible \(\text{Im}(\theta_L)\).

We now briefly introduce the operations between these pairs, starting with the free product.

### (A) Free products

Let \(G_1\) and \(G_2\) be pro-2 groups. The free pro-2 product \(G_1 \ast_2 G_2\) is defined as an inverse limit of finite quotients of the usual free product [27, Definition 4.1.1].

Now let \((G_1, \theta_1)\) and \((G_2, \theta_2)\) be oriented pro-2 groups. The free product is the pair \((G, \theta)\), where \(G = G_1 \ast_2 G_2\) and \(\theta : G \to \mathbb{Z}_2 \oplus (\mathbb{Z}/2\mathbb{Z})\) is induced by \(\theta_1, \theta_2\) via the universal property of the pro-2 product.

If \((G_1, \theta_1)\) and \((G_2, \theta_2)\) are realizable, also is the free product [11].

### (B) Semi-direct products

Given the pair \((\overline{G}, \theta)\) and a positive integer number \(n \geq 1\), we define the semi-direct product \((\mathbb{Z}_2^n \rtimes_{\hat{\theta}} \overline{G}, \hat{\theta})\), where
- \(\mathbb{Z}_2^n \rtimes_{\hat{\theta}} \overline{G}\) is the semi-direct product of pro-2 groups [27] with action \(\sigma a \sigma^{-1} = a^{\hat{\theta}(\sigma)}\), for every \(\sigma \in \overline{G}\) and \(a \in \mathbb{Z}_2^n\).
- \(\hat{\theta} = \theta \circ \pi\), where \(\pi : \mathbb{Z}_2^n \rtimes_{\hat{\theta}} \overline{G} \to \overline{G}\) is the canonical projection.

### Definition 10. The class \(\mathcal{C}\) of elementary type pro-2 groups is the smallest class of orientated pro-2 groups containing the pairs \((a), (b), (c)\) of Definition 9 and closed by operations (A) and (B), that is

- \((a)\) if the orientated pro-2 group \((\overline{G}, \theta)\) is in \(\mathcal{C}\), also the semi-direct product \((\mathbb{Z}_2^n \rtimes_{\hat{\theta}} \overline{G}, \hat{\theta})\) is in \(\mathcal{C}\).
(b) If \((G_1, \theta_1), (G_2, \theta_2)\) is in \(\mathcal{C}\), also the free product \((G_1 \ast_2 G_2, \theta)\) is in \(\mathcal{C}\).

We call an element of the class \(\mathcal{C}\) an ETG group. A more general version of the following conjecture appears in [26].

**Conjecture 11.** (Elementary Type Conjecture) The class of oriented pro-2 groups \((G_F(2), \theta_F)\), where \(F\) is a field such that \((\bar{\mathbb{F}}_F : \bar{\mathbb{F}}_F^2) < \infty\), is the same class \(\mathcal{C}\) of Definition 10.

Well-known examples of fields such that \(G_F(2)\) is an ETG group are Pythagorean fields, local fields, and finite extensions of \(\mathbb{Q}\). See [16], Tables 5.1, 5.2, 5.3, for a list of all possible ETG groups on three or fewer generators.

5. **Basic operations and the Kaplansky radical**

In this section, we describe the behavior of the Kaplansky radical under the basic operations (A) and (B) of Section 4, starting with free products. Since we focus on the radical, we can work only in the category of pro-2 groups.

5.1. **Free pro-2 products.** Let \(p\) be a prime number. Remember that \(H^1(G, \mathbb{F}_p)\) is the \(i\)-th Galois cohomology group of the pro-\(p\) group \(G\) with coefficients in \(\mathbb{F}_p\). By a well known result of Neukirch [28, Satze 4.2, 4.3], \(G\) is the free pro-\(p\) product \(G \cong G_1 \ast_p \ldots \ast_p G_n\) iff the restriction map \(\text{res}_j : H^j(G, \mathbb{F}_p) \rightarrow H^j(G_1, \mathbb{F}_p) \oplus \ldots \oplus H^j(G_n, \mathbb{F}_p)\) is an isomorphism for \(j = 1\) and a monomorphism for \(j = 2\).

In the following theorem, this criterion translates for field theory and our case \(p = 2\) by choosing \(G = G_F(2)\) and \(G_i = G_{F_i}(2)\), for field extensions \(F_i | F\) in \(F(2)\), \(i = 1, \ldots, n\).

**Theorem 12.** Let \(F_1, \ldots, F_n \subset F(2)\) be field extensions of \(F\). Then the Galois group \(G_F(2)\) decomposes as the free pro-2 product \(G_F(2) \cong G_{F_1}(2) \ast_2 \ldots \ast_2 G_{F_n}(2)\) if and only if it hold the following conditions

(i) the inclusions \(F \hookrightarrow F_i, i = 1, \ldots, n\), induces an isomorphism
\[
\hat{F}/\hat{F}^2 \rightarrow \hat{F}_1/(\hat{F}_1)^2 \times \ldots \times \hat{F}_n/(\hat{F}_n)^2
\]

(ii) let \(A\) be a finite dimensional central simple \(F\)-algebra of order dividing 2 in the Brauer group of \(F\), that is, the class \([A]\) is in the 2-torsion group \(2\text{Br}(F)\). The scalar extensions \(A \mapsto A \otimes_F F_i, i = 1, \ldots, n\), induces a monomorphism
\[
2\text{Br}(F) \rightarrow 2\text{Br}(F_1) \times \ldots \times 2\text{Br}(F_n)
\]

**Proof.** It follows from the previously mentioned Neukirch decomposition criterion and the canonical isomorphisms (6), (7), since

(i) \(\iff\) \(\text{res}_1\) is an isomorphism and (ii) \(\iff\) \(\text{res}_2\) is a monomorphism. 

\(\square\)
By Theorem 13, we can describe the radical of $F$ in terms of $R(F_i)$, $i = 1, \ldots, n$.

**Theorem 13.** If $F_1, \ldots, F_n \subset F(2)$ are field extensions of $F$ such that $G_F(2) \cong G_{F_1}(2) \ast_2 \cdots \ast_2 G_{F_n}(2)$, then

(a) $\hat{F} \cap \bigcap_{i=1}^n D_{F_i}(1, x) = D_F(1, x)$, for all $x \in \hat{F}$.

(b) $R(F) = \hat{F} \cap \bigcap_{i=1}^n R(F_i)$.

(c) If each $F_i$, $i = 1, \ldots, n$, has trivial radical (respec. $R(F_i) = \hat{F}_i$, $i = 1, \ldots, n$) then $F$ has trivial radical (respec. $R(F) = F$).

**Proof.** (a) Let $x \in \hat{F}$ and $-y \in \hat{F} \cap D_{F_i}(1, x)$, for $i = 1, \ldots, n$. Then the class $[(F; x, y)]$ is in the kernel of the monomorphism in (ii) of Theorem 12 since $[(F_i; x, y)] = 0$, for $i = 1, \ldots, n$. Thus, $[(F; x, y)] = 0$, that is, $-y \not\in D_F(1, x)$.

(b) It follows from (a) and 13.

(c) It is an immediate consequence of (b). \hfill \Box

**Corollary 14.** In Theorem 13 suppose, in addition, that $G_{F_1}(2)$ is a free pro-2 group and $F_2, \ldots, F_n$ have trivial radical. Then

(a) $R(F) = \hat{F} \cap (\hat{F}_2)^2 \cap \cdots \cap (\hat{F}_n)^2$.

(b) The inclusion $F \hookrightarrow F_1$ induces a monomorphism $R(F) / \hat{F}^2 \hookrightarrow \hat{F}_1 / (\hat{F}_1)^2$.

(c) $(R(F) : \hat{F}^2) = (\hat{F}_1 : (\hat{F}_1)^2)$. Therefore $rk(G_{F_1}(2)) = dim_{\mathbb{F}_2} R(F) / \hat{F}^2$.

**Proof.** (a) It follows from Theorem 13 and $R(F_1) = \hat{F}_1$.

(b) Let $H$ be the fixed field of $G_{F_1}(2) \ast_2 \cdots \ast_2 G_{F_n}(2)$ in $F(2)$. Then $G_F(2) \cong G_{F_1}(2) \ast_2 G_H(2)$. By Theorem 13 (c) and Theorem 12 (i), $H$ has trivial radical and $R(F) = \hat{F} \cap H^2$. Now $H / H^2 = F / (\hat{F} : R(F))$. Thus $H / H^2 = F / (\hat{F} : R(F))$. By Theorem 12 (i) we have $(\hat{F}_1 : (\hat{F}_1)^2)(\hat{F} : \hat{F}^2) = (\hat{F} : \hat{F}^2)$, which also is $(\hat{F} : R(F))(\hat{F} : \hat{F}^2)$. Putting all together, $(R(F) : \hat{F}^2) = (\hat{F}_1 : (\hat{F}_1)^2)$. \hfill \Box

5.2. **Semi-direct products and valuations.** While free pro-2 products in the Galois group $G_F(2)$ can produce a non-trivial $R(F)$, semi-direct products can do the opposite. Indeed, the latter operation indicates the existence of valuations on $F$, which are strongly related to rigid elements. Finally, in this section, we will see that rigid elements and a non-trivial Kaplansky radical usually do not exist together.

Remember that $D_F(1, a) = \{x^2 + ay^2 \neq 0, x, y \in F\}$ is the value group of the quadratic form $X^2 + aY^2$ over the field $F$.

**Definition 15.** An element $a \in \hat{F} \pm \hat{F}^2$ is called rigid if $D_F(1, a) = \hat{F}^2 \cup a\hat{F}^2$ and birigid if $a$ and $-a$ are rigid. 
The existence of enough rigid elements in $F$ allows us to build a valuation on $F$ [31]. Similarly, more general versions were developed in [11,13].

Now, let us see the behavior of Kaplansky radical under the presence of at least one rigid element.

**Proposition 16.** Suppose that there is a rigid element $a \in \hat{F}$. Then $(R(F) : \hat{F}^2) \leq 2$ and it occurs one, and only one, of the following alternatives:

(a) $\hat{F}^2 = R(F) \neq \hat{F}$, that is, $F$ has trivial radical and $G_F(2)$ is not free.

(b) $F$ is a formally real quasi-Pythagorean field and $R(F) = D_F(1,1) = \hat{F}^2 \cup a\hat{F}^2$. Moreover, $G_F(2) \cong \mathbb{Z}_2 \ast_2 \mathcal{H}$, where $\mathcal{H}$ is generated by involutions.

(c) $R(F) = \hat{F} = \hat{F}^2 \cup a\hat{F}^2$. In this case, $-1 \in \hat{F}^2$ and $G_F(2) \cong \mathbb{Z}_2$.

**Proof.** We have $R(F) \subseteq D_F(1,a) = \hat{F}^2 \cup a\hat{F}^2$, since $a$ is rigid. Thus, $(R(F) : \hat{F}^2) \leq 2$. If $a \notin R(F)$, then it occurs the alternative (a). Let us assume that $a \in R(F)$. If $R(F) = \hat{F}$, then $-1 \in \hat{F}^2$, because $a \notin -\hat{F}^2$. Then we have option (c). Now suppose $R(F) \neq \hat{F}$. By [24] Proposition 6.3 (1), p.451, $a \in R(F)$ implies $D_F(1,a) = D_F(1,1)$. Therefore $R(F) = D_F(1,1) = \hat{F}^2 \cup a\hat{F}^2$. Finally, the description of $G_F(2)$ in (b) follows from Example [6]. \hfill \Box

**Corollary 17.** Let $F$ be a filed such that $(\hat{F} : \hat{F}^2) \geq 4$ and $R(F) \neq \hat{F}$. If there is a birigid element $a \in \hat{F}$, the radical of $F$ is trivial.

**Proof.** It follows immediately from Definition [15] and the inclusion $R(F) \subseteq D_F(1,a) \cap D_F(1,-a)$. Note that Proposition [16] implies $-1 \notin \hat{F}^2$. \hfill \Box

Now, let us study briefly the compatibility conditions between the Kaplansky radical and valuations. We refer [15] for more details on field valuations.

We denote by $(F, v)$ a **valued field**, which means a field $F$ having a valuation $v : F \rightarrow \Gamma_v \cup \{\infty\}$, where $\Gamma_v$ is the value group and the associated valuation ring $A_v = \{x \in F : v(x) \geq 0\}$ has maximal ideal $m_v = \{x \in F : v(x) > 0\}$. The residue class field is $k_v = A_v/m_v$.

**Proposition 18.** [9] Prop. 2.4] Let $(F, v)$ be a valued field such that $R(F) \neq \hat{F}$ and $\text{char}(k_v) \neq 2$.

(a) If $R(F) \nsubseteq (1 + m_v)\hat{F}^2$, then $(\Gamma_v : 2\Gamma_v) = 2$. Moreover, if $(\Gamma_v : 2\Gamma_v) = 2$, $k_v$ is quadratically closed.

(b) If $(1 + m_v)\hat{F}^2 \nsubseteq R(F)$, then $\Gamma_v = 2\Gamma_v$.

**Example 19.** (2-henselian fields) Remember that the valued field $(F, v)$ is called 2-henselian field if $v$ has a unique extension to $F(2)$, or equivalently, it holds the Hensel’s Lemma for polynomials of degree 2. If $\text{char}(k_v) \neq 2$, it is also equivalent to the compatibility condition $1 + m_v \subseteq \hat{F}^2$ [15, Corollary 4.2.4].

**Proposition 20.** Let $(F, v)$ be a 2-henselian valued field. Suppose that $\text{char}(k_v) \neq 2$, $\Gamma_v \neq 2\Gamma_v$, and $R(F) \neq \hat{F}$. Then, $F$ has a trivial radical.
Proof. Assuming that \((F, v)\) is 2-henselian, note that Proposition 18 (a) implies \(R(F) = \hat{F}^2\), provided one of the following alternatives:

(a) \((\Gamma_v : 2\Gamma_v) > 2\).
(b) \((\Gamma_v : 2\Gamma_v) = 2 \text{ and } k_v \neq k_v(2)\).

It remains to consider the possibility \((\Gamma_v : 2\Gamma_v) = 2 \text{ and } k_v = k_v(2) = \hat{k}_v^2\).

In this case, \(\hat{F} = A_v^\times \hat{F}^2 \cup xA_v^\times \hat{F}^2\), for some \(x \in \hat{F}\). From \(\hat{k}_v = \hat{k}_v^2\) and \(1 + m_v \subset \hat{F}^2\) we have \(A_v^\times \subset \hat{F}^2\). Then \((\hat{F} : \hat{F}^2) = 2 \text{ and } R(F)\) is trivial, since we are assuming \(R(F) \neq \hat{F}\). \(\square\)

Proposition 20 also could be deduced from Corollary 17 since \(v(a) \in \Gamma_v \backslash 2\Gamma_v\) implies that \(a\) is birigid.

On the other hand, one can build 2-henselian valuations from the existence of enough rigid elements [31, [15, Theorem 2.2.7]. This approach was successfully applied in Galois Theory to identify 2-henselian valuations in \(G_F(2)\) [14], as well more generally in [13]. It will be helpful in the following theorem.

**Theorem 21.** Let \(F\) be a field of characteristic not 2 such that \(G_F(2) = \mathbb{Z}_2^m \rtimes G_K(2)\), for some integer number \(m \geq 1\) and \(K \neq K(2)\). If \(F\) is not a formally real Pythagorean field with \(G_F(2) \cong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}\), then \(F\) has a 2-henselian valuation \(v\) such that \(\Gamma_v \neq 2\Gamma_v\) and \(\text{char}(k_v) \neq 2\).

**Proof.** Since \(\mathbb{Z}_2^m\) is a normal abelian subgroup of \(G_F(2)\), it follows from [14, Theorem 4.4]. \(\square\)

**Corollary 22.** If \(F\) is a field as in Theorem 21, then \(R(F) = \hat{F}^2\).

**Proof.** It is enough to apply Proposition 20. Observe that the Pythagorean field excluded in Theorem 21 also has a trivial radical (Example 2). \(\square\)

To finish this section, let us see that from a 2-henselian field, we can build a field with a non-trivial radical.

**Example 23.** [pre-2-henselian-fields] A valued field \((F, v)\) with \(\text{char}(k_v) \neq 2\) is called a pre-2-henselian field when it holds the compatibility condition \(1 + m_v \subset R(F)\). By Proposition 18 (b), if \(R(F) \neq \hat{F}\) and \(\Gamma_v \neq 2\Gamma_v\), then \(R(F) = (1 + m_v)\hat{F}^2\). Therefore, it has a trivial radical if and only if it is 2-henselian. By [3] Theorem 4.2 \(G_F(2) \cong G_L(2) *_2 (\mathbb{Z}_2^m \rtimes G_{k_v}(2))\), for \(m = \dim_{\mathbb{F}_2} \Gamma_v / 2\Gamma_v\) and \(L\) a extension of \(F\) in \(F(2)\) such that \(G_F(2)\) is a free pro-2 group and \(\hat{L} / \hat{L}^2 \cong R(F) / \hat{F}^2\). Conversely, a field admitting such decomposition for \(G_F(2)\) is a pre-2-henselian field.

**6. The Kaplansky radical and ETG groups**

If the Galois group \(G_F(2)\) is an ETG group (Definition 10) the Kaplansky radical of \(F\) is associated with a free pro-2 factor of \(G_F(2)\) in a strong way, as we will describe in the following theorem, which is an essential ingredient to our main results.
Theorem 24. Let \( F \) be a field of characteristic not 2, \( \hat{F}^2 \neq R(F) \neq \hat{F} \), and suppose that \( G_F(2) \) is an ETG group. There are two field extensions \( L \) and \( H \) of \( F \) in \( F(2) \) such that \( G_2(F) \) admits a decomposition as the pro-2 free product \( G_F(2) \cong G_L(2) \ast_2 G_H(2) \) satisfying:

1. \( G_L(2) \) is a free pro-2 group of rank equal to \( \dim_{F_2} R(F) / \hat{F}^2 \).
2. The field \( H \) has trivial Kaplansky radical.

Proof. Let us consider the last operation made in the elementary construction of \( G_F(2) \). If it is a semi-direct product, then \( G_F(2) \cong \mathbb{Z}_2^m \rtimes G_1 \), for some pro-2 group \( G_1 \) and an integer \( m \geq 1 \). Taking the fixed filed of \( G_1 \) in \( F(2) \), Corollary 22 says that \( R(F) = \hat{F}^2 \). Therefore, we can assume \( G_F(2) \cong G_1 \ast_2 G_2 \), for two ETG groups \( G_1 \) and \( G_2 \). Taking the corresponding fixed fields, we have \( G_F(2) \cong G_K(2) \ast_2 G_N(2) \), for \( K, N \) field extensions of \( F \) in \( F(2) \), also having Galois groups as ETG groups. Let us proceed by induction over \( n = \dim_{F_2} \hat{F} / \hat{F}^2 \), the minimal number of (topological) generators of \( G_F(2) \), also denoted by \( d(G_F(2)) \). The hypotheses on \( R(F) \) exclude the case \( n = 1 \). There are six possible groups \( G_F(2) \) for the case \( n = 2 \), listed in [16] Table 5.2, p. 393. Only the case \( \mathbb{Z}_2 \ast_2 (\mathbb{Z}/2\mathbb{Z}) \) corresponds to a field \( F \) such that \( \hat{F}^2 \neq R(F) \neq \hat{F} \). Then, it is enough to take fixed fields and apply Theorem 13 For the general case, note that \( K \) and \( N \) cannot both have trivial radicals because it would imply \( R(F) = \hat{F}^2 \), again by Theorem 13 For the same reason, \( K \) or \( N \) must have the Galois group not free. Now, by Theorem 12 (1), \( d(G_F(2)) = d(G_K(2)) + d(G_N(2)) \). Then, we can apply the induction hypothesis over only two cases:

(a) \( R(K) = \hat{K} \) and \( R(N) = \hat{N}^2 \).
(b) \( \hat{K}^2 \neq R(K) \neq \hat{K} \) and \( \hat{N}^2 \neq R(N) \neq \hat{N} \).

For the first case, we take \( L = K \) and \( H = N \). For (b), induction hypotheses implies that \( G_K(2) \cong G_L(2) \ast_2 G_H(2) \) and \( G_N(2) \cong G_L(2) \ast_2 G_H(2) \), with \( H_1, H_2 \) having trivial radical and \( G_L(2) \) a pro-2 free group, \( i = 1, 2 \). Finally, choosing \( L \) the fixed field of \( G_L(2) \ast_2 G_L(2) \), \( H \) the fixed filed of \( G_H(2) \ast_2 G_H(2) \), and applying Theorem 13 we conclude the proof. \( \square \)

We can use Theorem 24 to revisit the examples in Section 2.

Theorem 25. If \( F \) is a pre-2 henselian field, or a field described in Examples 3, 4 or 5 and \( G_F(2) \) is finitely generated, then \( G_F(2) \) admits a decomposition as described in Theorem 24.

Proof. (a) (pre-2 henselian fields) As Example 23 it is a valued field \( (F, v) \) and \( G_F(2) \) admits such decomposition choosing \( H \) as a 2-henselization of \( (F, v) \) [3, Theorem 4.2].

(b) (quasi-Pythagorean fields - Example [32] Corollary 1] proved that \( G_F(2) \) is the free pro-2 product of a free pro-2 group and a pro-2 group \( H \) (topologically) generated by involutions, provided that \( (\hat{F} : \hat{F}^2) \) is finite. It is clear that the fixed field of \( H \) in \( F(2) \) has a trivial radical.
(c) Example 3 (Berman) It is a field \( K = F(\sqrt{-1}) \) with \( (\hat{K} : R(K)) = 4 \) and \( F \) a Pythagorean field. For every \( x \in \hat{K} \setminus R(K) \) we have \( (D_K(1,x) : R(K)) = 2 \). It means that such \( x \) is \( R(K) \)-rigid \cite[(27, Lemma 2.1)]{3} and by \cite[Theorem 2.2 (1)]{3} \( K \) is a pre-2 henselian field, since it is not formally real.

(d) Example 5 (Kula) It is immediate from Example 19 since the field \( L((X_1)) \ldots ((X_n)) \) is 2-henselian.

(e) Example 14 (Berman) Again we have \( K = F(\sqrt{-1}) \), but now \( \hat{F}/\hat{F}^2 \cong \hat{F}_1/(\hat{F}_1)^2 \times \hat{F}_2/(\hat{F}_2)^2 \), where \( F_1 \) is super Pythagorean and \( F_2 \) is SAP Pythagorean. It was show in \cite[Theorem 4.8]{9} that \( G(F) \) is isomorphic to \( G_{F_1(\sqrt{-1})}(2) \ast_2 G_{F_2(\sqrt{-1})}(2) \ast_2 \mathbb{Z}_2 \). Since \( F_2 \) is SAP, we have that \( G_{F_1(\sqrt{-1})}(2) \) is a free pro-2 group \cite[Remark 2]{3}. Thus, we choose the field \( L \) in Theorem 24 as the fixed field of \( \hat{G}_{F_2(\sqrt{-1})}(2) \ast_2 \mathbb{Z}_2 \). On the other side, by \cite{12}, \( H = F_1(\sqrt{-1}) \) has trivial radical.

\[ \square \]

7. The Hilbert’s Theorem 90

In order to prove Conjecture 1 for an ETG group, we need the Kurosh Subgroup Theorem for a pro-2 group \( G \) and its open subgroups (Theorem 26). Remember that a subgroup \( H \) is open when it is closed and \( G/H \) is finite. A proof of a more general case of Theorem 26 appears in \cite[Theorem 4.2.1, p.208]{27} or \cite{4}.

Theorem 26 will allow us to find \( G_2(F(\sqrt{a})) \) and \( R(F(\sqrt{a})) \), \( a \in \hat{F}/\hat{F}^2 \), from the decomposition of \( G_F(2) \). The two separated cases \( a \in F^x \setminus R(F) \) and \( a \in R(F) \) produce different descriptions, as Theorem 27 establish.

**Theorem 26** (Kurosh). Let \( G \) be a pro-2 group and \( H \) an open subgroup of \( G \). Suppose that \( G \cong G_1 \ast G_2 \), where \( G_1,G_2 \) are closed subgroups of \( G \). Let

\[
G = \bigcup_{i=1}^{n} G_1 a_i H \text{ and } G = \bigcup_{j=1}^{m} G_2 b_j H, \quad a_i, b_j \in G,
\]

be decompositions in double cosets of \( G \) w.r.t. \( H,G_1 \) and \( H,G_2 \), respectively. Then

\[
H = \left( \ast_{i=1}^{n} H \cap a_i G_1 a_i^{-1} \right) \ast_2 \left( \ast_{j=1}^{m} H \cap b_j G_2 b_j^{-1} \right) \ast \mathcal{F}
\]

where \( \mathcal{F} \) is a free pro-2 group of rank \( 1 + (G:H) - (m+n) \).

The existence of the fields \( F_0 \) and \( F_1 \) in the next theorem follows from Theorem 24.

**Theorem 27.** Let \( F \) be a field of characteristic not 2, with \( \hat{F}^2 \neq R(F) \neq \hat{F} \) and \( G_F(2) \) an ETG group. Choose extensions \( F_0,F_1 \subseteq F(2) \) such that \( G_F(2) \cong G_{F_0}(2) \ast G_{F_1}(2) \), with \( G_{F_0}(2) \) a free pro-2 group and \( F_1 \) having trivial radical. Let \( K = F(\sqrt{a}), a \in \hat{F}/\hat{F}^2 \). There is an extension \( L_0 \subseteq F(2) \) of \( F \), with \( G_{L_0}(2) \) a free pro-2 group, such that
Proof. It follows from Corollary 14 (a) that $R(F) = \hat{F}^2 \cap \hat{F}$. 

(1) Assume that $a \in \hat{F}\backslash R(F)$ and let us define $K_0 = F_0(\sqrt{a})$ and $K_1 = F_1(\sqrt{a})$.

First case: $a \notin \hat{F}^2_0$. Then $K \not\subseteq F_0$ and $G_{F_0}(2) \not\subseteq G_K(2)$. As $a \notin R(F) = \hat{F}^2_1 \cap \hat{F}$, it follows that $K_1 \neq F_1, G_{K_1}(2) \not\subseteq G_{F_1}(2)$. Since $(G_F(2) : G_K(2)) = 2$, it follows that $G_{F}(2) = G_{F_0}(2)G_{K}(2) = G_{F_1}(2)G_{K}(2)$ are two trivial decompositions of $G$ in double cosets w.r.t. the pairs $G_{K}(2), G_{F_0}(2)$ and $G_{K}(2), G_{F_1}(2)$, respectively. From Kurosh Subgroup Theorem (Theorem 26) we have

$$G_K(2) = \left(G_{F_0}(2) \cap G_K(2) \right) *_2 \left(G_{F_1}(2) \cap G_K(2) \right) *_2 F \cong G_{K_0}(2) *_2 G_{K_1}(2) *_2 \mathbb{Z}_2,$$

with $F$ a free pro-$2$ group of rank $1$, that is, $F \cong \mathbb{Z}_2$. Now, it is enough to chose $L_0 \subseteq F(2)$ as the fixed field of the free pro-$2$ group $G_{K_0}(2) *_2 \mathbb{Z}_2$.

Second case: $a \in \hat{F}^2_0$. Then $G_{K_0}(2) = G_{F_0}(2), G_F(2) = G_{F_1}(2)G_{K}(2)$ and we have the following decompositions in double cosets

$$G_{F}(2) = G_K(2) \cup \tau G_K(2) = G_{F_0}(2)G_K(2) \cup G_{F_0}(2)\tau G_K(2),$$

where $\tau$ is an automorphism such that $\tau|_K$ is the only non-trivial $K$-automorphism fixing $F$. Again by Theorem 26

$$G_K(2) = \left(G_{F_0}(2) \cap G_K(2) \right) *_2 \left(\tau G_{F_0}(2) \cap G_K(2) \right) *_2 \left(G_{F_1}(2) \cap G_K(2) \right).$$

The result now follows remembering that $G_{F_1}(2) \cap G_K(2) = G_{K_1}(2)$ and choosing $L_0$ as the fixed field of the free pro-$2$ group $(G_{F_0}(2) \cap G_K(2)) *_2 (\tau G_{F_0}(2) \cap G_K(2))$.

(2) Now let us assume $a \in R(F) = \hat{F}^2_1 \cap \hat{F}$. The case $a \in \hat{F}^2_0$ no longer occurs. Indeed, the decomposition $G_{F}(2) = G_{F_0}(2) *_2 G_{F_1}(2)$ implies the injectivity of $\text{res}_1 : \hat{F} / F^2 \to \hat{F}_0 / F^2_0 \times \hat{F}_1 / F^2_1$, according to Theorem 12. Then, $a \in \hat{F}^2_0 \cap \hat{F}$ would imply $a \in \hat{F}^2$. Therefore, $G_{F_0}(2)$ is not contained in
\(G_K(2)\) and we have \(G_F(2) = G_{F_1}(2)G_K(2)\), since \((G_F(2) : G_K(2)) = 2\).

By the other side, \(G_{F_1}(2) \subset G_K(2)\), because \(a \in \hat{F}_1^2\). Then \(G_F(2) = G_K(2) \cup \sigma G_K(2) = G_{F_1}(2)G_K(2) \cup G_{F_1}(2)\sigma G_K(2)\).

Again by Kurosh Subgroup Theorem,
\[
G_K(2) = \left( G_{F_0}(2) \cap G_K(2) \right) *_2 \left( G_{F_1}(2) \cap G_K(2) \right) *_2 \left( \sigma G_{F_1}(2) \sigma^{-1} \cap G_K(2) \right) = G_{F_0}(\sqrt{a})(2) *_2 G_{F_1}(2) *_2 G_{F_1}(2).
\]

In this case, \(L_0 = F_0(\sqrt{a})\).

Finally, we can prove Conjecture 1 for a field \(F\) such that \(G_F(2)\) is an ETG group. Once again, it applies to all examples in Theorem 25.

**Theorem 28.** Let \(F\) a field of characteristic not 2 and \(K = F(\sqrt{a})\), \(a \in F^\times \backslash \hat{F}^2\), be a quadratic extension of \(F\) with norm map \(N : \hat{K} \to \hat{F}\). If \(G_F(2)\) is an ETG group, then \(N^{-1}(R(F)) = \hat{F}R(K)\).

**Proof** The inclusion \(\hat{F}R(K) \subseteq N^{-1}(R(F))\) is already automatic and does not depends on \(G_F(2)\). Indeed, if \(r \in R(K)\), then \(r \in D_{K_1}(1,-x)\), for all \(x \in K^\times\). By the Knebusch Norm Principle \([24, \text{Theorem 5.1, p.206}]\), \(N(r) \in D_F(1,-x)\), for all \(x \in F^\times\). Therefore, \(N(r) \in R(F)\). For the other inclusion let us choose, according to Theorem 24, \(F_0, F_1 \subseteq F(2)\) such that \(G_F(2) \cong G_{F_0}(2) * G_{F_1}(2)\), with \(G_{F_0}(2)\) a free pro-2 group and \(F_1\) having trivial radical. It follows from Corollary 14 (a) that \(R(F) = \hat{F}_1^2 \cap \hat{F}\).

**First case:** Assume \(a \in \hat{F} \backslash R(F)\). Defining \(K_1 = F_1(\sqrt{a})\), Theorem 27 (1) says that \(G_K(2) \cong G_{L_0}(2) * G_{K_1}(2)\) and \(R(K) = \hat{K}_1^2 \cap \hat{K}\). Take \(x \in \hat{K}\) such that \(N(x) \in R(F) = \hat{F}_1^2 \cap \hat{F}\) and denote \(N_1 : \hat{K}_1 \to \hat{F}_1\) the correspondent norm application. By Hilbert’s Theorem 90, \(x \in \hat{F}_1\hat{K}_1^2\). Remember that \(\hat{F}_1 = \hat{F}\hat{F}_1^2\), according to Theorem 12. Putting all together, \(x \in \hat{F}K_1^2 \cap \hat{K} = \hat{F}R(K)\).

**Second case:** Now assume \(a \in R(F) = \hat{F}_1^2 \cap \hat{F} = (\hat{F}_1^2)^2 \cap \hat{F}\). Again we choose \(x \in \hat{K}\) such that \(N(x) \in R(F)\). Now we have \(K \subset F_1\) and \(\hat{F}_1 = \hat{F}\hat{F}_1^2\). Then \(x = ax^2\), for some \(a \in \hat{F}\) and \(t \in \hat{F}_1\). It follows that \(N(x) = \alpha N(t^2) = t^2 \sigma(t)^2\). Then \(t^2 \in \hat{F}_1^2\), because also are there \(N(x)\) and \(\sigma(t)^2\). Therefore \(x = ax^2 \in \hat{F}(\hat{F}_1^2 \cap (\hat{F}_1^2)^2 \cap \hat{K}) = \hat{F}R(K)\). \(\square\)
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REFERENCES

[1] Arason, J. K., Elman, R., Jacob, B. (1987). Rigid elements, valuations, and realization of Witt rings. J. Algebra. 110: 449-467.
[2] Becher, K. J., Leep, D. (2014). The Kaplansky radical of a quadratic field extension. J. Pure Appl. Algebra. 218(9): 1577-1582.
[3] Berman, L. (1979). Pythagorean fields and the Kaplansky radical. J. Algebra. 61: 497-507.
[4] Binz, E., Neukirch, J., Wenzel, G. (1971). A subgroup theorem for free products of profinite groups. J. Algebra. 19: 104-109.
[5] Cordes, C. M. (1973). The Witt ring and the equivalence of fields with respect to quadratic forms. J. Algebra. 26: 400-421.
[6] Cordes, C. M. (1975). Kaplansky’s radical and quadratic forms over non-real fields. Acta Arith. XXVII: 253-261.
[7] Cordes, C. M., Ramsey Jr, T. R. (1979). Quadratic forms over quadratic extensions of fields with two quaternion algebras. Canad. J. Math. 31: 1047-1058.
[8] Dario, R. P. (2010). Propriedades aritméticas de corpos com um anel de valorização compatível com o radical de Kaplansky, Thesis. UNICAMP, Campinas, BR.
[9] Dario, R. P., Engler, A. J. (2010). 2-Galois groups and the Kaplansky radical. Algebra Discrete Math. 10:9-50.
[10] Dario, R. P., Engler, A. J. (2015). Relative Brauer group and pro-p Galois group of pre-p-henselian fields. J. Algebra Appl. 14(1550087): 1-10.
[11] Efrat, I., Haran, D. (1994) On Galois groups over pythagorean and semi-real fields. Israel J. Math. 85(1-3): 57-78.
[12] Elman, R., Lam, T. Y. (1972). Quadratic forms over formally real fields and pythagorean fields. Amer. Journal of Math. 94(4): 1155-1194.
[13] Engler, A. J., Engler, Koenigsmann, J. (1996). Abelian subgroups of pro-2-Galois groups, Trans. Amer. Math. Soc. 350: 2473-2485.
[14] Engler, A. J., Nogueira, J. B. (1994). Maximal abelian normal subgroups of Galois pro-2-groups. J. Algebra. 166: 481-505.
[15] Engler, A. J., Prestel, A. (2005). Valued Fields, Berlin, DE: Springer.
[16] Jacob, B., Ware, R. (1989). A recursive description of the maximal pro-2 Galois group via Witt rings. Math. Z. 200: 379-396.
[17] Kaplansky, I. (1969). Fröhlich’s local quadratic forms (1969). J. Reine Angew. Mat. 239: 74-77.
[18] Kijima, D., Nishi, M. (1981). Kaplansky’s radical and Hilbert Theorem 90. Hiroshima Math. J. 11: 443-456.
[19] Kijima, D., Nishi, M. (1983). Kaplansky’s radical and Hilbert Theorem 90 II. Hiroshima Math. J. 13: 29-37.
[20] Koch, H. (2002). Galois theory of p-extensions, Berlin, DE: Springer.
[21] Koenigsmann, J., Strommen, K. (2019). Recovering valuations on Demushkin fields. Available at https://arxiv.org/abs/1506.05956v2
[22] Kula, M. (1979). Fields with prescribed quadratic form schemes. Math. Z. 167: 201-212.
[23] Kula, M. (1981) Fields with non-trivial Kaplansky’s radical and finite square class number. *Acta Arith.* XXXVIII: 411-418.
[24] Lam, T. Y. (2004). *Introduction to Quadratic Forms Over Fields*. Providence, RI: AMS.
[25] Marshall, M. (1980). *Abstract Witt Rings*. Queen’s Pap. Pure Appl. Math. n. 57. Queen’s University, Kingston, ON.
[26] Mináč, J., Pasini, F. W., Quadrelli, C., Tăn, N. D. (2018). Koszul algebras and quadratic duals in Galois cohomology. Available at http://arxiv.org/abs/1808.01695
[27] Neukirch, J., Schmidt, A., Wingberg, K., *Cohomology of Number Fields*, Berlin, DE: Springer, 2000.
[28] Neukirch, J. (1971). Freie Produkte Pro-endlicher gruppen und ihre kohomologie. *Arch. Math.*, 22: 337-357.
[29] Ribes, L. (1970). *Introduction of Profinite Groups and Galois Cohomology*. Queen’s Papers in Pure and Applied Math. n. 24. Queen’s University, Kingston, ON.
[30] Ribes, L., Zalesskii, P. (2000). *Profinite groups*, Berlin, DE: Springer, 2000.
[31] Ware, R. (1981). Valuations rings and rigid elements in fields. *Cannad. J. Math.* 33: 1338-1355.
[32] Ware, R. (1990). Witt rings and almost free pro-2-groups, *J. Algebra* 132: 377-383.

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