A robust multivariate linear non-parametric maximum likelihood model for ties

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Statistical analysis in applied research, across almost every field (e.g., biomedical, economics, computer science, and psychological) makes use of samples upon which the explicit error distribution of the dependent variable is unknown or, at best, difficult to linearly model. Yet, these assumptions are extremely common. Unknown distributions are of course biased when incorrectly specified, compromising the generalisability of our interpretations – the linearly unbiased Euclidean distance is very difficult to correctly identify upon finite samples and therefore results in an estimator which is neither unbiased nor maximally informative when incorrectly applied. The alternative common solution to the problem however, the use of non-parametric statistics, has its own fundamental flaws. In particular, these flaws revolve around the problem of order-statistics and the estimation in the presence of ties, which often removes the introduction of multiple independent variables and the estimation of interactions. We introduce a competitor to the Euclidean norm, the Kemeny norm, which we prove to be a valid Banach space, and construct a multivariate linear expansion of the Kendall-Theil-Sen estimator, which performs without compromising the parameter space extensibility, and establish its linear maximum likelihood properties. Empirical demonstrations upon both simulated and empirical data shall be used to demonstrate these properties, such that the new estimator is nearly equivalent in power for the glm upon Gaussian data, but grossly superior for a vast array of analytic scenarios, including finite ordinal sum-score analysis, thereby aiding in the resolution of replication in the Applied Sciences.

Introduction

Achieving the general construction of a non-parametric linear regression framework, wherein the distribution, linearity, and closed form expressions of the estimating equations between errors and covariates may be easily presented, has been a long desired result in applied statistics. The first major development was that of Kendall (1938) $\tau_a$ and the corresponding univariate Kendall-Theil-Sen estimator, which developed a locally consistent Gauss-Markov estimator insensitive to outliers, subject only to the requirement of order-ability to ensure these properties. This allows the estimator to compete well against least squares even for normally distributed data, while also allowing for linear single slopes to be applied to discrete ordinal data. The estimator, however, does not provide all of the necessary properties required in experimental statistical designs, in particular for use in the Applied Social Sciences. Specifically, we are referring to the higher-order factorial and polynomial design matrices, which are unable to effectively estimated with the introduction of ties (or collisions and surjective mappings), which preclude finite sample (strong) identification and convergence. To resolve this, we introduce a Banach norm metric topological vector space, which possesses the same structure of the Kendall $\tau$-metric, but which does not possess the selection bias upon the sample space, as it is naturally robust to the occurrence of ties, unlike Kendall’s $\tau_b$. Further, this same topology allows for the estimation of finite sample interactions, which are mathematically identical to ties, and therefore a substantial unresolved problem in applied research. We introduce the mathematical properties of a complete metric upon a linear sub-space, and compare performance in numerous scenarios to that of the traditional Gaussian linear regression model, which demonstrate support of the superiority of the Kemeny norm, in particular as an unbiased second-order consistent (i.e., replicable) estimator. In addition, we derive estimating equations similar to that of OLS regression in terms of the variance-covariance matrices for both parameter estimates and standard errors, and has been shown effective at addressing missing at random data with an EM-algorithm.
In applied analysis, researchers are often presented with a measurement of interest $y_m$, the dependent variable, along with a covariate set $X^m_n$ which we use to estimate and explore a stable relationship between the expected changes in the target relative to the differences observed or controlled upon our sample. In this manuscript, unless otherwise stated, we assume each column vector in $\{y, X\}$ is of length $m$, for which $y$ is a scalar while $X$ is a rectangular matrix of order $m \times n$, with the restriction that $m \gg n$, with uniform sampling selection independent and identical upon the population wrt the row-space. The goal of a regression framework is to remove the linear dependencies between all $n$ choose 2 features in the design matrix $X$ of $n$ features, and then to project the optimally weighted linear combination of these unique pieces of information onto $Y$. This unbiased Gauss-Markov optimality is such that we may interpret and approximate how the average fixed unit change in the similarity of $X \rightarrow y$ may be expected to correspond to an estimated observed change in $y$. Linear systems with complete metrics are extremely beneficial for such applications, both in terms of their parameter flexibility in establishing complicated yet estimable linear relations, but also in their ability to establish, upon relatively small samples, learned patterns which strongly generalise outside of the sample at hand.

This beneficence comes at a cost, however: the conditional normality and linearity of errors be correctly established, in order to maintain the orthonormal separability of the bias of incomplete sampling upon the sample from the bias in the parameters. Thus, we provide a robust multivariate mathematical framework, in the style of the general linear model, which may be applied to almost any partially orderable probability distribution function definable upon a common population; thus any distribution which is independently sampled, but which does not require linearity to be established across the column space of $X$. We will also demonstrate how the Kendall $\tau$ and similar non-parametric work (e.g., the Wilcoxon rank-sums test and the Friedman test) may be resolved to produce an unbiased linear estimator which is efficient and easily capable of addressing non-parametric multivariate families within a linear sub-space.

The Kemeny (1959) metric defines a complete mathematical framework whose methods are shown to be a maximum likelihood estimator, with both probabilistic and closed form solutions, for almost any sortable distribution. It is further demonstrated to be only mildly less efficient when in the presence of a truly normal distribution, and substantially more Gauss-Markov optimal when addressing non-normal data. We introduce a non-parametric linear regression system whose solutions and standard errors are demonstrably and theoretically a maximum likelihood estimator (MLE) which is robust to non-normality and more informative even under applications to estimation scenarios such as summative-scores and even applications of the polychoric correlations.

**Contribution and organisation of the paper**

Supposing the Kemeny metric $\rho_K$ to be a convex functional for a topological vector space $(X, \rho_K)$, we prove and provide empirical demonstrations that the relation between $(\hat{\alpha}_n, \epsilon)$ is both uniquely determined and linear under a relatively weak set of conditions, as well as being an estimator of minimum variance. We define the necessary characteristics of the parametric error family which satisfies this functional linear basis, and demonstrate how it enables minimum uncertainty with respect to $\hat{\alpha}_n$ as compared to other unbiased estimators. We conclude with several simulations and an applied data analysis, all validated under jackknife resampling, to demonstrate that the performance conditions expected under maximum likelihood are validated as a primal-dual characterisation for our introduced methodology, without the introduction of a non-identity link function, as a consequence of the affine relationship upon $(X, \rho_K)$ for a much wider array of the exponential family of distributions.

**Motivation and literature review**

We posit that the maximum likelihood properties of the Euclidean $\ell_2$ norm are non-robust in terms of their consistency and breakdown in finite samples. While asymptotic consistency in expectation is provably true, the ability for a finite sample to possess a sub-additive representation of the population, from said subset, is much less forthcoming, especially when the conditional distribution (i.e., the error distribution, $\epsilon$) is non-normally distributed. We argue that this empirical failing is largely a function of the over-generalisation of the normal distribution of errors as a continuous random field which is orthonormal to the covariate space, which directly implies that the finite sample selection and parameter biases are inconsistent with our asserted inductive interpretations in how to to understand a population.

A brief introduction to the foundational basis of this error may be found with the James and Stein (1961) lemma, decomposing bias into orthonormal components upon any arbitrary additive norm space, wherein $\gamma^2_2$ represents the total estimation bias and the Bayes error $\epsilon^2_0$, denoting the total irreducible error. $\gamma^2_T$ may be further expanded to denote bias with respect to the sampling upon the population $\gamma^2_m$, bias wrt the parameter estimation (e.g., scenarios for which the model is not correctly identified, as well as traditional Tikhonov ridge regression or restricted maximum likelihood estimation), and the interaction of these two pieces $\gamma^2_m \cdot \gamma^2_T$. With a complete metric topological vector space (TVS), under the limit wrt $m$ for uniform sampling, it is expected by definition that $\gamma^2_m$ is strongly convergent to 0, and therefore that the bias $\gamma^2_T$ is solely a function of the proportional representation of the population within the sample. This bias, if reflective of uniform sampling, should tend to 0 as well revealing the structure if the common population from which $\gamma^2_m$ arose.
Early non-parametric work arguably foundered upon the problem of model identification in the presence of ties, which naturally arise in both the sample and parameter spaces, such that the James and Stein (or bias-variance tradeoff) inequality for a Banach norm-space has both: (1) non-zero $\gamma_n$, or bias with respect to the sampling resulting from ties being excluded, and (2) non-ignorable bias with respect to the parameters $\gamma_n^*$ if the ties are averaged over. For finite samplings on a normal distribution of errors though, it naturally follows that $\gamma_n^* \to 0$ strongly, as previously shown, and therefore the interaction also cancels, leaving the sole term $\gamma_n^*$ converging to 0 in the population under the restriction that $\lim_{m \to \infty} \gamma_n^* = 0$. The Kemeny (1959) metric was constructed to explicitly resolve the problem of sub-additivity in the presence of ties, and from this metric space, the Kemeny distance function and a probability density function can be shown to be exponentially related, and in fact may be isometrically embedded. This will allow us to characterise the Kemeny distance within the Gaussian probability family, which may be shown to asymptotically converge to the same point. Unsurprisingly, the Euclidean metric is more informative for Gaussian data; however the linearity of the Kemeny metric as well as its minimal loss of power in favouring its selection, presents a convenient means of constructing a linear regression model space, without the assumption of the normality of errors and without losing the ability to estimate more complex terms as is otherwise typically observed.

When we know the distribution in which we are interested in modelling (i.e., $y$) the introduction of a such a complete metric, measuring the distance between our predictions (the estimands $\hat{y}$) and the true values $y$, is the definition of a maximum likelihood estimator, solely characterised by the minimisation of $\epsilon^2$ for which $\gamma_m^2$ tends to zero as the sample becomes the population, and $\epsilon^2 \to \epsilon_0^2$ when the regression model is correctly specified. The ability to leverage the inner-product space defined by the Euclidean norm enables a minimisation procedure of approximation for which $y_n$ strongly tends towards 0, and therefore so does $\gamma_n^* \cdot \gamma_n^* = 0$. Consequently assuming that we have representative sampling upon the population, the function we learn upon our data is very stable, independent of the specific objects recorded in our data: of course, $\gamma_n^* > 0$ as long as $|m| < \infty^*$. Therefore, these approximations are imperfect, but these imperfections do not compromise the estimation of the sample relations, merely the inductive capacity to link the understandings in our sample to the larger population with an unknown ‘truth’.

As stated, these techniques are valid for Euclidean spaces; however, knowing the appropriate transformations to establish additivity to allow the decomposition of the $\epsilon^2$ is much more difficult. If $\gamma_n > 0^2$, our ability to learn relations which are approximately correct is suddenly affected by every other uncertainty in the sample, under a model which asserts these terms are correctly fixed to zero. This increases the distance between our ability to fit a sample’s data, and our ability to understand a population, with the uniqueness of the likelihood function weakened and the sharpness of the convexity diminished as well. When well-posed, all bias in the population is 0, and therefore the correct modelling structure is solvable to produce a unique model solution. However, for any bias which is non-zero, the distance between the sample error $\epsilon^2$ and the true Bayes error $\epsilon_0^2$ grows as the uniqueness of our induced relationships (i.e., ‘the existence of a unique ‘truth’) is only established under the axiomatic veracity of said conjecture, the generalisability of all our interpretations is unknowingly compromised.

If we take Box (1976) in earnest, then $\epsilon^2 \neq \epsilon_0^2$, equivalent to stating that bias $\gamma_2^2 > 0$ is distributed randomly over all free-parameters in the model space, to a degree dependent upon which specific elements consist of $\gamma_m^2$, and that the pieces of the bias all interact together to deform the holomorphic mapping onto our parameter space upon our covariates. Of course, the incorrect application of a non-sub-additive metric introduces a positive third term in equation 2, in which the function learned is inseparably by a singular regularity criteria of error minimisation wrt the unique sample. Our ability to replicate interpretability across independent samplings without merely relying upon a weakly consistent cheat, is arguably a reason behind the replication crisis in the Social Sciences (Wald, 1949; White, 1982), since ordinal scales are certainly not continuous, let alone normally distributed even upon a population. Therefore, the likelihood tests and partial Wald tests may be presumed to not be strongly consistent under the conditions which they are commonly published if the normality of errors is false as well (Wald, 1949; Le Cam, 1953). Weak consistency (under which $\gamma_m^2 \to 0$ only when $m \approx \infty^*$) is an undesirable solution, since it requires the researchers to accurately characterise the function we are approximating only when the population is exhaustively sampled, which defeats the purpose of inductive argumentation in favour of description, and is therefore meaningless unless the population may be accurately collected. It should also be noted that the utilisation of meta-analysis does not pose an adequate solution, since the bias in the multiple levels amongst a collection of studies is typically not resolved, nor is it clearly addressed that the estimates themselves are biased. However, this presumption
remains the current default for non-normality in the use of both Kendall’s \( \tau \) and Spearman’s \( \rho \).

Consider a data sampling process which produces an \((m \times 1)\) vector \( \mathbf{y} = (y_1, y_2, \ldots, y_m)^T \) of observable real numbers, \( \mathbf{y} \in \mathbb{R}^m \). Said data is immutably capable of describing, with probability 1, the data in itself, the sample. However there exists no descriptive capacity to address anything beyond itself: no inductive inferences concerning the characteristics of either DSP or data generating process (DGP) are possible (Solomonoff, 1964). Functional data analysis is a process by which we may approximate upon an unknown function space, and a framework allowing us the ability to predict, within our sample. In Statistics, we are often presented with such an unknown data generating function drawn upon a finite sample, which we must approximate in an attempt to understand the population. The identification of a specific error structure (a parametric probability distribution family; pdf) which, conditionally, allows us to linearly separate a structure of interest (a model space amongst the universe of all possible identifiable parameters, \( \alpha_0 \subseteq \Omega \)) from the complete uncertainty of the system. Traditional solutions of maximum likelihood (ML) and ordinary least squares (OLS) have linearised the \( l_2 \)-metric space (see equation 4), for certain specific conditions

\[
y = \alpha_0 + \alpha \mathbf{X} + \epsilon.
\]

We define the data as independently and identically sampled upon a random variable from an unknown joint probability distribution, whose characteristics will be further expanded upon. These \( m \) independently distributed outcomes upon this endogenous process \( \mathbf{y} \), we wish to calculate estimates and conduct inference about unknown specifications of the relations between \( \mathbf{X} \) and \( \mathbf{y} \). The estimators of focus are upon a single level (constant within the population) vector \((a_0, a_\alpha) \in \mathbb{R}^{n+1}, \alpha \subseteq \Omega, \) wherein \( \Omega \) denotes the space of all identifiable parameters, and \( a_0 \) denotes the intercept. If we view the Euclidean distance as a characterisation of the Pearson correlation, then it immediately follows that a simple regression is another form of said correlation (see equation 4). Consider then an empirical scenario, wherein \( \mathbf{X} \sim \mathcal{N}(\mu, \sigma) \) and \( \mathbf{y} \sim \mathcal{N}(\mu, \sigma_\epsilon) \) upon \( m \) units. Within such a static (fixed) empirical system, the normal MLE is clearly applicable, and a provable minimum variance estimator. However, consider instead the same system of \( n \) random variables transformed by a copula, wherein the scores upon \( \mathbf{X} \) are such that minimising the Euclidean distance no longer satisfies the properties of the minimum variance maximum likelihood estimator which converges to an expected error of 0 for the population (by the smoothing theorem or the law of total expectation).

The specific transformations are completely arbitrary, however we assume that they continue to maintain the properties of a complete probabilistic metric space, as per Sklar’s theorem (Schweizer & Sklar, 2005; Menger, 1942). This ensures, by using a data generating function such as the the generalised partial credit model to link the original scores \( \mathbf{X} \rightarrow \mathbf{X}' \), that due to the probabilistic mapping, there is no guarantee of satisfying the triangle inequality upon the Euclidean topology. This is because distance between any adjacent ordinal values are no longer linear with three distinct points of origin (i.e., \( \rho(x, 0) - \rho(y, 0) \neq \rho(x', 0) - \rho(y', 0) \)), and must be necessarily transformed to ensure orthonormal additivity remains (i.e., unbiased estimation). Since neither addition or subtraction are orthonormal for non-Gaussian error spaces, the measurement error is also no longer capable of being averaged out. This consequently ensures that the error expectation of 0, by Slutsky’s theorem, is no longer satisfied, and the bias is a stochastic component of the sample, due to the necessary non-uniform weighting to ensure valid finite sample characterisation. It should be noted that this is entirely consistent with the asymptotic characterisation of the sum score as a valid description under the limits with respect to both \( m \) and \( n \). Therefore, the estimators \( \hat{\alpha} \) and \( \hat{\sigma}^2 \) are not independent, as required, a contradiction whose resolution is fundamental for construction of the valid classical t- and F-tests with respect to both first order approximations (coefficient bias) but more importantly, second order (standard error) bias. The estimator \( \alpha \) represents the coefficients of vector decomposition of \( \mathbf{y} = \mathbf{X} \hat{\alpha} = \mathbf{P} \mathbf{y} = \mathbf{X} \alpha + \mathbf{P} \epsilon \), from which follows that \( \alpha \) is a function of \( \mathbf{P} \epsilon \). Simultaneously, the estimator \( \hat{\sigma}^2 \) is a norm of vector \( \mathbf{M} \epsilon \) divided by \( n \), and thus also a function of \( \mathbf{M} \epsilon \). Now, random variables \( (\mathbf{P} \epsilon, \mathbf{M} \epsilon) \) are jointly normal as a linear transformation of \( \epsilon \), and also orthonormal because \( \mathbf{P} \mathbf{M} = 0 \), which means that \( \mathbf{P} \epsilon \) and \( \mathbf{M} \epsilon \) are not independent, and therefore that estimators \( \hat{\alpha} \) and \( \hat{\sigma}^2 \) are also not independent (Hoeffding, 1948). However, given established biases for finite samples, the minimum variance replicability of the Gaussian likelihood function is not a valid presumption, entirely consistent with current research findings. As the error cannot be linearly separated from the regularity parameters, it follows that the interaction from equation 1 as presented in equation 4 is non-zero, from which follows the introduction of non-zero bias wrt \( \gamma_{\epsilon}^2 \).

To address non-normal data, Nelder and Wedderburn (1972) introduced a linking function between the coefficients, \( \alpha \), and the error, \( \epsilon \), which linearised the function to allow the additive decomposition of \( \epsilon \) from \( \mathbf{y} \) as a function of \( \alpha \mathbf{X} \). This still maintains the parametric nature of the approximation distribution, such that we may correctly establish sub-additivity upon the parameter space in terms of our objective goal min \( \epsilon^2 \) which solely defines our learning process (Vapnik, 2013) by an appropriately selected monotonic transformation. Non-parametric functional families based upon data ranking (so-called order statistics; (Thurstone, 1927; Lipovetsky, 2007)) which are invariant to the specific distribution have been a popular alternative resource, seeking to define relations between relative data orderability, rather than the original data scores. However, the definition of a
proper complete metric upon the orderability of data scores has been unaddressed, and the parameter flexibility of a contractive mapping in the $\ell_2$ space has been widely preferable, conditional upon the correct selection of the empirical distribution wrt $y$.

Typical order-statistic methods, such as the Kendall $\tau_b$ and to a lesser extent, the Spearman footrule (Kendall, 1938; Spearman, 1906), rely upon a topological sub-space constructed upon a symmetric group $S_m$, for which each individual sample realisation is unique. This error structure was assumed to originate upon an explicitly (and without error) observed continuous random variable, such that $P(x_i = x_j) = 0$, almost surely, thereby precluding the existence of two subjects with different covariates possessing the same rank (ties). This characterisation excludes many common empirical measurements in both continuous and discrete empirical spaces, and results in a biased functional approximation due to the Heckman selection process which asserts non-uniform probability of representation within the sampling from the population, as a function of the characteristics of each $X_i$, demonstrating the non-ignorability of the $\gamma_n \cdot \gamma_m$ and $\gamma_m$ terms. This follows, as many multivariate and univariate probability distributions are consequently incapable of being uniquely embedded upon $S_m$ for finite (and generalisable) learning as a maximum likelihood problem, due to the lack of identifiability with respect to ties. These approaches further preclude higher-order polynomial terms and interactions, a substantive necessity in observational and experimental research. Moreover, the existence of ties is a substantially larger problem within the multivariate space, as they become substantially more frequent. Consider, for instance, the Rubin (1971) causal model, whose counterfactual approach is wholly constructed upon the existence of multivariate surjective mappings onto common points of non-identical multivariate $X$. The problem of ties (or collisions) in rank and order-statistic methods have been extensively detailed (Diaconis, 1988; Lehmann, 2009; Hollander, Wolfe, & Chicken, 2013; Harlow, 2013). However these resolutions rely upon the asymptotic weak order convergence wrt $m$, rather than properly defining a complete compact metric space. As interaction terms in the coefficient space are ties, almost all rank based techniques avoid estimating them, leading to their disuse in common experimental frameworks, which we address here.

**An unbiased linear metric estimator upon the expanded permutation space**

Complete metrics are an extensively studied field in theoretical statistics and topology (Schechter, 1997), although applied practice often limits use to the $\ell_2$, or Euclidean, metric space. Non-linear transformations in the form of the Generalised Linear Model (McCullagh & Nelder, 1989) induce linear additive separation between the model and error structures, while satisfying the primal-dual characterisation of error minimisation, from which follows the highly desirable generalisability of the unique learned patterns upon the sample. We prove that the Kemeny (1959) metric is such a linear metric for any sortable cumulative distribution function which is permutation non-invariant, thereby implying as a necessary characteristic, the ability to sort distributional probability as a direct linear function of the relative ordering in the sample, which grows linearly to become the population, and is therefore an MLE with minimum variance.

A realisation upon either $(X_i, y_i)$ which map to a non unique collision under either Spearman’s footrule and Kendall’s $\tau_b$ distances and respective correlational measures, fails to satisfy the properties of a complete metric (Fagin, Kumar, & Sivakumar, 2003), due to the uncertainty of the surjective mapping. For such a common empirical scenario, it follows that for finite sample ties, both distances are invalid maximum likelihood estimators. Said distances are finitely biased, due to the correlation which now exists between the error and the specific data realisations (Hoeffding, 1948), and thus the relative sparseness of the sample space restriction enables only weak convergence under the weak law of large numbers. Therefore, the development of a metric topological distance and corresponding quotient space for all reals upon $i \in \{1, \ldots, m\}$, $\forall 0 < \infty^m$ across the column rank spaces remains an unavoidably necessity; however such a measure has been largely neglected (Kemeny, 1959; Diaconis, 1988; Fagin et al., 2003). We present the utility of this Kemeny norm in Figure 1, wherein the relative size of the population permutation space is expanded for five observations, from a population of 24 unique permutations upon the sample space, to 256 (Good, 1975).

The Kemeny norm is constructed upon a score matrix, which denotes pairwise discretisation across all pairs of observed subjects as presented in equation 6 for comparison upon subjects $i$ and $i'$ in the space $\binom{m}{2}$. This score matrix describes a relative orderability to all other empirical observations, with the simple image of greater than (a), equal to (0), or less than (-a), upon which the fixed constant $a = 1$ is typically utilised. This is nearly identical to the familiar Kendall $\tau$, with the adjustment of a valid image for tied elements, which were merely assumed to occur with probability almost surely 0, for continuous non-normal data. However, empirical measure spaces such as ordinal survey responses, which contain fixed ordered sets of possible choices in response to prompts, are substantially more likely to incur such ties, thereby raising the loss of efficient MLE properties to an immediate point of concern; traditional approaches such as the polychoric correlation (Pearson & Pearson, 1922; Olsson, 1979; Savalei, 2011) fail to address this issue, as empty cells upon the cross-tabulation of responses (the inverse need to the original assumption of almost no ties) produce unstable approximations of the correlation matrix. Unsurprisingly,
A monotonically increasing CDF

A monotonically non-decreasing CDF

Figure 1

Comparison of the two rank metrics upon the permutation space $S_{m=4}$, in which ties are explicitly avoided, and then permitted, in order to demonstrate the empirical population space for unbiased estimators. It is seen that the latter, advocated, Kemeny metric is visually more dense, corresponding to faster convergence to the ECDF in the population under the strong law of large numbers.

The summation across the columns of the score matrix (equation 6) results in a complete metric, the Kemeny distance ($\rho_K$), as found in equation 5, (and re-expressed as a bijective linear cross-product in Emond and Mason (2002))

$$\rho_K(A, B) = \frac{1}{2} \sum_i \sum_j \text{sign}(\kappa_{ij}^0 - \kappa_{ij}')$$

$$\kappa_{ij}^0 = \begin{cases} 1 & \text{if } f(x_i) > f(x_j) \\ -1 & \text{if } f(x_i) < f(x_j) \\ 0 & \text{if } f(x_i) = f(x_j) \end{cases}$$

for which $\rho : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^1$, where $\rho$ is the space of possible monotonic metric functions which, smoothly approximate the Heaviside function in aggregation across $m$, producing the cumulative distribution function. In Table 1, we demonstrate a repeated random simulation of the bivariate correlation of a bivariate Poisson distribution, with a population correlation of 0 with 100 subjects. It is seen that consistent with our hypothesis, the Kemeny correlation does possess the smallest standard deviation under replication, with a minimum 25% greater concentration, and a maximum ratio nearly 250 times smaller. This serves to demonstrate our contention that numerous alternative metrics are less efficient in comparison to our proposed estimator, in both a univariate and multivariate space.

The Kemeny metric may be shown to be a continuous space from Schechter (1997), as it is a complete metric, and further to be strongly convergent wrt $m \rightarrow \infty$, presenting a sufficient basis equivalent to conventional Banach (1934)-norm spaces (Cauchy-Schwartz convergence is an explicit consequence upon any complete metric space). When combined with an observed empirically observable space measured with the Kemeny metric, the space is closed, complete, and continuous, and therefore compact for finite $m \in \mathbb{Z}^+$. We begin the proof here, with the assumption that any convex continuous metric space must be shown to be connected, as defined upon our population space, which here is the permutation space $H_m$, $X \in \mathbb{R}^m \equiv H_m$. We treat the graph of $m^2 - m$ distinct bands to reflect the unique distances of the permutations from an arbitrary origin $\rho(u, \pi) \in H_m$, for any orderable sequence $u_i \in \mathbb{R}^1$ for $i = \{1, \ldots, m^2 - m\}$, which may be compared to an arbitrary point of origin, $\pi$, upon the norm-space of $H_m$. However, it is recommended the $\pi = 1, 2, \ldots, m = \mathcal{I}_m$, the identity permutation, due to its uniqueness for any permutation group upon the sample of $m$ individual units. The inverse identity permutation, $\mathcal{I}_m^{-1}$ may also be utilised, under the same reasoning.

We call this graph of all non-isolated nodes $G$, with $m(m - 1)$ unique distances which contains $k$ elements, for which $k$ may be computed using a recursive summation of Stirling numbers (Good, 1975). $k$ may be less than or equal to $m$, denoting the presence of unique real valued measurements under equality, and a non-zero probability of ties occurring holds, as $k \rightarrow \infty$; the restriction that $k > 0$ holds in that if no bins exist to place at least one measure within, then the event is both observed and not observed in the sample space, a logical impossibility (Cox, 1972; Owen, 2001). A Cayley graph presents a specific sub-graph of the permutation space without replication, $S_m \subseteq H_m$, in which all distances are multiplied by a scalar of 2 for $a = 1$, obtaining a bijection between the Kendall and Kemeny metrics. This corresponds to the demonstrated point-wise equality upon the empirical cumulative distribution function (ECDF) for each specific distance in Figure 1. An adjacency swap of distance 1 (transposition of two rankings) under the Kendall distance now asserts a distance of 2 under the Kemeny distance, as the tied position is occupied, and then moved past indicating two distinct locations from $\rho_K(u, \mathcal{I}_m), \rho_K(v, \mathcal{I}_m) \rightarrow \{u \rightarrow v\} \rightarrow \{v, u\}$ upon said graph. $u - v$ denotes an equivalence (incomplete, or partial, ordering) for the two specific elements on a single random variable $X^i$. The elements in the compact support upon $\rho_K$ are connected by the underlying commonality of conditional exchangeability, adjusting for the generating function, thereby defining a residual, conditional upon the sufficient statistics as follows from the linearity of the metric. Said linearity enables a connected function upon an exhaustive permutation space $H_m$, as defined with the Kemeny metric, to converge to a normal distribution as the number of bins within which subjects’ measurements may be placed grows to equality with
the number of subjects asymptotically, and therefore for any continuous random variable. Compactness of the Kemeny distance may be easily shown on the space of \( n = \{1, 2\} \) and by induction to all finite sample sets. For \( n = 2 \) points in the set \( H_m \) on the graph \( G \) are realised \((u, v) \in V(G)\), we wish to identify a path from \( u \) to \( v \) termed the \((uv)\)-path. If \( u = v \), or \( u \neq v \), on the edges of \( G, E(G) \), then the distance between the points must be either 0 or \( a \), denoting either an isomorphism or adjacency. It is also trivially obvious that for \( n = 1 \) that the object ties with itself, and that the distance is therefore uniquely 0 from a population space of 1 element. Assuming that \( u \neq v \) and \( uv \ni E(G) \), the transitive property of the segments of the graph which does not contain both \((u, v)\) holds. Such a representation may be considered as a neighbourhood of each endpoint

\[
A = \{a \in V(G) | v_a \in E(G)\} \quad (7)
\]

\[
B = \{b \in V(G) | v_b \in E(G)\} \quad (8)
\]

and as long as these neighbourhoods possess a non-empty set intersection, a common element \( w \) must connect \( u \leftrightarrow w \) and \( w \leftrightarrow v \). \( u \leftrightarrow w \leftrightarrow v \) allowing construction of a continuous path from \( u \leftrightarrow v \) for any beginning and end point. The inclusion-exclusion principle operating upon the neighbourhood of each node guarantees a common neighbour must exist for all points (see (9)) upon all such \( H_m \) graphs, for \( n \geq 1 \), upon each random variable \( x_j \in X \), as the graph is connected.

\[
|A \cap B| = |A| + |B| - |A \cup B| \quad (9)
\]

\[
deg(u) + \deg(v) - |A \cup B| \quad (10)
\]

\[
\geq \frac{n-1}{2} + \frac{n-1}{2} - (n-2) \quad (11)
\]

\[= 1; |A \cap B| \neq \emptyset \quad (12)
\]

The metric is therefore uniquely defined (up to proportional constant \( a \)) to be a consistent, bounded distance defining a population topological vector space \((\Omega, \rho_K)\) which Kemeny (1959) treated as a functional mapping of the domain \( X \) onto a univariate real number. Said functional follows from the nesting of the image of the score matrix upon \( X \), which may be expanded to be considered as the design matrix without complication, within the summation in equation 5, resulting in the measurement of the metric distance between any two points. The diameter of \( G \) for all nodes \((u, v) \in V(G)\) are found within the realised finite countable interval \( 0 \leq \rho_K(u, v) \leq m(m-1) \), which is always known and fixed in the sample. As the distance is closed by the existence of the upper bound for all finite realisation of \( m \), an image of the support is both a closed and a bounded set. Therefore, given any two points \((u, v) \in X \) on \( H_m \), the pairwise distance is continuous and homogeneous for all finite sub-samples of the universe of populations \( u \in V(G) \) as the sample grows asymptotically under a uniform and independent sampling of all observable permutations in the population by Slutsky’s theorem, assuming a single population is sampled. We have thus shown that the Kemeny metric is a linear convex function upon the compact permutation support of \( H_m \), and is therefore continuous. A simple proof by contradiction may be used to establish distribution over additivity, and it will then be shown that by homogeneity, the evaluation of \( \rho = \rho_K \) commutes with multiplication by constant vector \( \alpha_n \), representing the coefficient parameters. Assume as a function of \( \alpha_n \) (for fixed intercept \( \alpha_0 \)) that \( \rho_K(x \alpha_n + y \alpha_n, \alpha_m) + \alpha_0 = \alpha_0 \rho_K(x, \alpha_m) + \alpha_0 \rho_K(y, \alpha_m) + 2 \alpha_0 \), which reduces to \( \alpha_0 = 2 \alpha_0 \), from this immediately follows that the solution is a linear equation \( w \forall \alpha_n \) for \( \alpha_0 = 0 \) and is therefore an unbiased and unique estimator, as otherwise a contradiction must follow. The use of the non-zero constant \( \alpha_0 \) is defined wrt to \( \alpha_m \), and therefore represents the normalised scores for which the central location is 0 for all variables under analysis: the introduction of the non-zero intercept term as an additive constant merely serves to translate the expectation of the errors in prediction as a Cauchy-Schwartz convergent function series under the limit as \( m \rightarrow 0\), demonstrating with probability 1 that the linear estimator is unbiased upon the Kemeny metric.

We next proceed to prove that the Kemeny metric is unbiased with minimum variance. This also demonstrates the conclusion that the Kendall rank distance is biased for finite samples as a direct result of the restriction to \( S_m \) for conventional data collection. However, this may also be seen by noting that all elements of \( x \in X \) cover \( \rho_K \) and therefore \( X : \bigcup_{m=1}^{\infty} x^m \leftrightarrow \text{Domain}(G) \), from which follows \( x_m \cap \overline{x_m} = \emptyset \), and therefore demonstrating both the completeness and compactness of the metric. Any norm space which establishes a Banach space for which the three properties of a metric must

| Statistic | Mean | St. Dev. | Skew | Min | 25% | 75% | Max |
|-----------|------|----------|------|-----|-----|-----|-----|
| \( \rho_K \) | 0.0111 | 0.0004 | -0.0121 | 0.0102 | 0.0109 | 0.0113 | 0.0120 |
| Pearson’s \( r \) | -0.0107 | 0.0007 | -0.0158 | -0.0158 | -0.0111 | -0.0104 | -0.0094 |
| Spearman \( \rho \) | -0.0108 | 0.0007 | -0.0133 | -0.0133 | -0.0111 | -0.0104 | -0.0094 |
| Kendall \( \tau \) | -0.0186 | 0.0005 | -0.0100 | -0.0097 | -0.0088 | -0.0083 | -0.0075 |
| Glass’ \( r \) | 0.0012 | 0.0012 | 0.0992 | -0.0015 | 0.0012 | 0.0012 | 0.0039 |
also be homogeneous, which provisions several useful properties, including the power-metric property. Kemeny (1959) proved the first three properties for a complete metric for \( \rho_K \), however we must also prove \( a\rho(P,Q) = \rho(a \cdot P, Q) \forall a \neq 0 \).

The linearity of a parameter space upon the Kemeny-space must also, unsurprisingly, be established in order to assert the valid estimation of a linear regression \( \text{wrt} \) \( \alpha, \varepsilon \), the regression parameters and its error, as necessary to establish both addition and multiplication as valid functions. Assume an intercept only regression model, \( x_{j=1} \equiv 1 \), for which both the properties of \( \rho(\alpha_0 x_i, I_m) = a_0 \rho(x_i, I_m), \forall x \in \mathbb{R}, a \in \mathbb{R} \), and \( \rho(x + y, I_m) = \rho(x, I_m) + \rho(y, I_m), \forall x \in \mathbb{R}, y \in \mathbb{R} \). By \( a \in \mathbb{R} \), the homogeneity property of the Kemeny metric follows such that \( \rho_K(aX, I_m) \equiv a \cdot \rho_K(X, I_m) \), by linear scaling of the penalty term, which forces the monoid scalar \( a \) to always be a finite non-zero real, but is otherwise unbounded, without affecting the relative ordering of all numbers. Exhaustive enumeration establishes that for \( H_2 \), the permutation group with repetition possesses cardinality \( 4 \rho_K(H_2, I_2) \in \{1, 0, 2, 1\} \), which by substitution of \( a \equiv 1 \), produces the set \( \{a, 0, 2a, a\} \). Simple induction by \( m + 1 \), where \( m \) is a finite number, demonstrates that any axiomatic conditions which hold upon \( S_2 \) must also hold upon \( S_m \), i.e., \( S_1 + S_2 + \cdots + S_m + S_{m+1} \). The validity of this induction is proven by seeking the equivalence from \( S_2 \) that all elements in the set \( \{S_{m+1} = S_m + S_{m+1}\} \). The cardinality of these two groups was proven in Good (1975), so it is immediately known that the groups are correctly sized, and always begin at \( 0 \), for the ascending sequence \( m \) (since by the property of indiscernibility, any group of size \( 1 \) must be equivalent to itself, and hence \( \rho(a, I_m) = 0 \)). Further, we know that the upper bound of the metric space is given by the expression \( a \cdot (m^2 - m) \), using the established multiplicity. From these, we see that, under the assumption that there is an element \( k \in S_m \) for \( m \geq 1 \), upon which the distance from \( I_m \) may be calculated according to equation 6 and equation 5, with \( I_m \) as the origin. As already established, \( \rho_K(x, I_m), \forall x \in S_m \) is correctly calculated upon the entire group, from both the left \( I_m \) and the right \( I_m \). As there is no finite number on the real for which \( S_m \) is not capable of calculating the Kemeny distance, due to the connectedness of \( G \), it is therefore immediately seen that \( \lim_{m \to \infty} S_1(x = aI) \subset S_2(ax) \subset \cdots \subset S_m(ax) \equiv aS_1(x = I_1) \subset S_2(x) \subset \cdots \subset S_m(x), \) by the distributivity of the linear multiplication. Thus, the Kemeny metric is shown to be a linear Banach space, and allows utilisation of the power metric property.

From these properties follows a means for consistent estimation of linear parameters (by the Cauchy-Schwarz convexity of all complete metrics), such as interaction with respect to almost any homogeneous error distribution whose cumulative distribution \( F \) is monotonically non-decreasing. For a function \( F \) to be monotonically non-decreasing is a complementary extension of the typical assumption of a monotonically increasing cumulative distribution function (as in (Mann & Whitney, 1947; Cox, 1972)). Under a monotonically increasing function, the probability of ordered indices or statistics, of the dependent variable are uniquely sortable, such that each of \( m \) observations possesses an relative ordering upon the sample with respect to its largest (or smallest) realised value. Assume \( F(x) \) explicitly characterises the space under the cdf with the point-wise inquality \( F_n(t) < F_m(t) \), wherein \( t \) satisfies the properties of the order-statistics which are exchangeable with realisations upon the raw observations \( x \) as a consequence of the unique probability metric, justified by Sklar’s theorem. A bijective relation therefore exists between the probability and empirically observed measure spaces for each individual \( x_i \in \{1, 2, \ldots, m\} \). Under the Kemeny metric, the inequality \( F \) is replaced with the relation \( F_K(t) \leq F_m(t) \), which induces a transformation of the finite sample correlation estimating equation, which will be later shown to also be a minimum variance maximum likelihood estimator. This expression is provided in equation 13, which, due to the Banach-norm properties of the Kemeny metric are linearly strongly consistent, unbiased, and invariant to monotonic transformations. The Kemeny correlation is later expanded to demonstrate a variance-covariance matrix, for which it is shown to enable the estimation of a multivariate linear non-parametric regression, for a parameter space which includes the introduction of polynomial terms and interactions, an immense improvement over current non-parametric estimators in terms of \( \gamma_n \):

\[
r_K = 1 - \frac{2 \cdot \rho_K(x_j, x_P)}{m(m - 1)},
\]

U-statistic estimator properties

Let \( P \) be an arbitrary family of probability distributions which are homogeneous upon the space \( (X, \rho_K) \), restricted only such that each distribution \( P_j \) is a vector of length \( m \) composed upon the family of weakly-orderable distributions. Said data is permutation non-invariant (i.e., it is orderable), but includes weak-orderings such that all pairwise elemental comparisons may be determined to be greater than, lesser than, or equal to an arbitrary point of origin, with Hermitian semi-positive definite distances. As the Kemeny distance is continuous and convex metric, the sole restriction to functional analysis lies upon the existence of a common and independent generating random probability function for the errors. In extension to multivariate distribution, it will not be expected that the parametric families be identical, but merely
that \( P_j \in \mathcal{P} \ \forall j = \{1, \ldots, n\} \). Let \( \lambda(P_j) \) be a real-valued function defined for \( P_j \in \mathcal{P} \), which is estimable for the observable data space \( \mathcal{X} \), a rectangular matrix of order \( m \times n \), upon which \( \mathcal{X} \subseteq \mathcal{X} \) is identically and independently randomly distributed. Further allow \( \lambda(P_j) \) be an estimable parameter for some integer \( m \) for which exists an unbiased estimator with property \( \lambda(P) \) which by the linearity of the space defined by \( \rho_K \) produces a symmetric function over all permutations upon the row-space of the data, which may be countably infinite as per the Axiom of Choice. The nature of the Kemeny metric as Borel measurable follows from the finite countable nature of \( H_n \sqcup m \) as established by Good (1975). The central location which minimises the distribution of distances upon the permutation space \( H_n, \rho_K(\pi_j, \pi_0) \forall \pi \in H_n \) is satisfied by the midpoint distance of the unique extrema and its inverse, expressed as a point of distance \( \frac{m(m-1)}{2} \) in the population, about which the distribution of distances is linearly symmetric. The expectation of the metric for said unique point of origin is equivalent to the exhaustive symmetric sampling amongst all permutation points upon \( \rho_K \) defining the order statistics, and therefore that the point of symmetry (the first moment or the expectation) is finite for all finite sample, \( m < \infty \). Therefore, \( U_j \) upon \( \rho_K \) produces a linear functional of the expectation, identical to the one-sample Wilcoxon rank-sum statistic upon \( S_m = \Omega \), thus establishing the expectation of the linear operator as the median. The variance for a finite mean is definable by the power metric property of the ultrametric (or any Hilbert space) as a quadratic Taylor expansion about the expectation. The variance of said \( U \)-statistic is expressed wherein \( \Xi_{jj} = \text{Var}(h(x_1, \ldots, x_k)) \), defined for all finite realisations upon \( x_m \in \mathbb{R}^n \). We express the variance of univariate variable \( j = \xi_j^2 = \xi^2(x_j) \) and as a multivariate set of variables as the diagonal of the \( n \times n \) matrix \( \Xi \). Consider two subsets of the population \( \mathcal{D} \) for which there are exactly \( k \) common subjects between two subsets. The distinct choices for the construction of both subsets are \( \binom{n}{k} \binom{n}{m-k} \); as the estimate \( \hat{h} \) is symmetric and independent of the construction of each subset (by the unbiasedness of said estimand upon the metric), it follows that the point of inflection in the probability distribution of the distance function \( \rho_K \) allows for the construction of a minimum variance estimator of order \( n^{-1} \) and therefore that the estimator amongst the exponential family \( P \in \mathcal{P} \) is \( n^{1/2} \)-consistent for a singular random variable. A closed multivariate solution for \( \alpha_n \) based upon \( \Xi \), the Kemeny metric variance-covariance upon the union of the design matrix \( X \) with \( n \) parameters and the dependent variable \( y \), with row and column \( n + 1 \) denoting the covariance of the dependent variable and the feature space to which \( y \) is linearly endogenous wrt to the residual \( \epsilon \). From this matrix the covariate parameters \( \alpha_n \), and their respective standard errors may be estimated. The \( n \) coefficients \( \alpha_j \) for \( j = \{1, 2, \ldots, n\} \) may be simultaneously estimated upon the sub-matrix \( \Xi_{1:n, 1:n} \), expressing the variance-covariance matrix of the feature space, along with the residual error variance \( \xi^2_\epsilon \) as the complement of the linear covariance between the dependent variable and regression function, subtracted from the total variance along with the intercept \( a_0 \) and the variance of the parameters \( \sigma^2_\alpha \): \[
abla \begin{align*}
\alpha_0 &= \nu(y) - \nu(X)\alpha_n \\
\alpha_n &= \Xi_{1:n, 1:n}^{-1}\Xi_{n+1, 1:n} \\
\xi^2_\epsilon &= \frac{1}{m-n-1}(\xi_{jj} - 2 \cdot \xi(\alpha_0 + X\alpha_n, y))_{1,2} \\
\sigma^2_\alpha &= (\xi^2_\epsilon \cdot \text{diag}(X^TX)^{-1})_{1:n, 1:n}.
\end{align*}
\]

From these established properties for linearly unbiased estimators, we conclude a valid realisation of the Gauss-Markov theorem, establishing the minimum variance properties of the unique linear model space parameters solved for under \( \rho_K \) by the continuity of the metric for a vector space which is of full column order rank, from which is justified the application of the Gram-Schmidt solution, for \( m \gg n \). This further allows us to define not only the correlation as a linear rescaling of the compact Kemeny distance about the expectation, but also to scale the correlation by the roots of the likelihood function, thereby defining both the variances and covariances by the inner-product scaling of the compact sufficient statistic. The cross-product \( X^TX \) requires no additional computational adjustment, due to the linear additive and multiplicative equivalence of the parameter space upon the Kemeny metric, and therefore is validly defined as such upon this topology as well.

**Probability distribution of** \( F \) **upon** \( y = f(x) \)**

As the Kemeny metric is henceforth definable in a unique linear metric space with established \( U \)-statistic properties with support \( f : aX \rightarrow 0 \leq \mathbb{R}^1 \leq m^2 - m \), a specific probability distribution for the population must be definable as well. As was previously shown, the first and second moments of the Kemeny metric are linearly expressible in closed form by the median \( \nu \) and dispersion about the median \( \xi^2 \), in lieu of the conventional mean and variance. Since the median naturally converges to the mean upon a population, the finite robustness of the median as the linear convex expectation under independent sampling is greatly beneficial, but asymptotically equivalent, while possessing a breakdown point of \( 50\% \), consistent with expectations regarding a closed form expression for the median.

We propose that the pdf, given its linear nature, be defined as a Gaussian function, which is strongly consistent as a linear function upon the Kemeny metric, and which may be
defined for the population of univariate reals \( x \in \mathbb{R}^1; |x| = m \)

\[
F(x_j) = \int_{-\infty}^{\infty} \xi_j \cdot \exp \left( -\frac{(x_j - \nu)^2}{2\xi^2} \right) dx = 1 \quad (18)
\]

\[
v = \frac{1}{m} \sum_{i=1}^{m} (\rho_K(x_i, \nu_m)) \quad (19)
\]

\[
\xi^2 = \frac{1}{m-1} \sum_{i=1}^{m} (\rho_K(x_i, \nu_m) - \frac{m^2 - m}{2})^2 \quad (20)
\]

for all finite orderable distributions of reals of length \( m \). This implies that a multinomial distribution is explicitly not well-posed, but any monotonic partial ordering would be. From this perspective, we may construct a robust Fisher z-score distribution wherein the expected value \( v \) is 0 and the scale \( \xi = 1 \) for any permutation invariant cumulative distribution in the population, as well as allowing for studentising to address finite sample Wald and partial Wald tests. More importantly, we may also thereby compare the information coverage and accuracy of the Euclidean and Kemeny metrics, to allow for comparative evaluations of the loss of power when estimated upon a continuous and homogeneous normal error distribution, which holds for the Kemeny metric under the general definition of a metric as a continuous field. To ensure that this is valid, we prove in Appendix C that under the additivity of the rank error distribution, the Kolmogorov axioms are satisfied under our distribution in equation 18, and therefore is valid.

**Probabilistic MLE**

It is assumed, for any MLE, that any real parameter on the interior of the parameter space, \( \hat{\alpha} = (\alpha_n, \alpha_0) \in \Omega \), possesses a cumulative distribution function, for which also exists a probability distribution function \( f(x_i; \hat{\alpha}) \) for random variable \( x_i \) associated with the \( l \)th empirical realisation within a study. We further assume that \( F(x_i, \hat{\alpha}) \) is either discrete for all \( \hat{\alpha} \) or absolutely continuous for all \( \hat{\alpha} \).

Under a linear pdf, an estimator \( \hat{\alpha} \) is regularly and linearly obtained to satisfy the likelihood score, or estimating equation which equals 0 and is invariant to logarithmic transformation, possessing a unique and singular optimum. Said optimum is defined by selection of the minimal sufficient statistics as equivalent to the already provided closed form expressions of \( \nu \) and \( \xi^2 \), for which the error term then becomes the object of minimisation wrt the joint set \( \alpha = (\alpha_n, \alpha_0) \). Said property is established by demonstration that the value of sufficient statistic \( zVT(z) \) may be consistently estimated, once the loss function \( L(\hat{\alpha}|X) \) is known. The likelihood function for \( \hat{\alpha} \) which arises under the assumption of independent and full rank data which is linearly and independently distributed in multivariate field of dimension \( n \) controlling for the sample estimates \( \hat{\nu}|X \) and \( \hat{\xi}|X \). To determine for each \( m \) the most likely estimate (wrt \( \alpha \)) or corresponding \( m \) predictions (wrt \( \alpha \)), we must demonstrate convergence with probability 1 to the local optima in the interior of the parameter space which individually characterise \( \hat{\alpha} \), and which is essentially unique (Perlmutter, 1983) under certain well-established conditions. We will demonstrate that our characterisation upon the space \( (X, y, \alpha; \rho_K) \) satisfies under the Kemeny metric for a matrix of sufficient order these requirements, such that even when the likelihood itself may be unbounded or the Bayes error may not coincide with zero, our proposal will remain a consistent estimator of \( \alpha_0 \). The Fisher expected information matrix about the interior parameter set \( \hat{\alpha} \) is defined such that \( \mathcal{I}(\hat{\alpha}) = E_{\hat{\alpha}}[S(\hat{\alpha}X)S^T(\hat{\alpha})X] \) for which \( S(\hat{\alpha}; x_i) = \partial \log L(\hat{\alpha})/\partial \hat{\alpha} \) is the gradient score statistic of the log-likelihood, and \( x_i = (x_1, \cdots, x_m) \) is the \( j \)th column vector of \( X \) from which the Fisher information is estimated, and from which follows the standard errors (SE),

\[
\text{SE}(\hat{\alpha}_r) \approx (\mathcal{I}^{-1}(\hat{\alpha}))_{rr}^{1/2} \quad (r = 1, \cdots, n + 1) \in \text{diag} \, X^T X \quad (21)
\]

as the cross-product of the score statistics for an \( n + 1 \) dimensional model space. This characterisation readily coincides with the results of the note published by Redner (1981), under which for a compact complete metric space \( \rho(X, \cdot) \), all points \( \alpha \) on the interior of the parameter space \( \Omega \) are well-posed.

The uniqueness clearly follows for any finite selection upon of \( \hat{\alpha} \), wherein for a fixed finite sample space \( X \) exists a function \( L(\alpha; x) = h(x) \cdot f(x; \alpha) \) defined upon the three-sequence of partial derivatives of equation 18 wrt \( \alpha_n \), utilising the probability field as indicated in equation 18, which is already established to be a linear function, with both derivatives of the same sign, and thus positive definite. This thereby satisfies the selection of \( \hat{\alpha}_r \), a unique point which maximises the likelihood and minimises of the Kemeny metric, invariant to logarithmic transformations said loss function, and the maximisation of the log of this same equation produce. It therefore immediately follows that \( \log[L(\hat{\alpha}; x, \rho_K)] = l(\hat{\alpha}; x, \rho_K) \), characterises a uniquely solvable score equation of the log of the unbiased linear estimate in terms of the pdf found in the derivative of equation 18 at point \( x \).

\[
l(\nu, \xi|x) = -\frac{m}{2} \left( \log 2\pi + \log \xi^2 \right) - \frac{1}{2\xi^2} \sum_{i=1}^{m} (x_i - \nu) \quad (22)
\]

The sufficiency of the two provided statistics may be established for both \( S_m \) and the expanded space \( H_m \), wherein \( m \) continues to denotes the unique set of permutations realisable upon vector \( x \), which may then be established for all \( m < \infty \) by recurrence. Begin upon \( H_1 = x = \{1\} \), for which using equations 19 and 20, we produce the estimates \( \nu_j = 1 \) and \( \xi_j = 0 \), respectively. This is immediately observably valid, as there is only one possible permutation upon
the finite and enumeratively exhaustible sample of one observation, and its constancy indicates a variance of 0 upon the population permutation space. By the unit cardinality of this space \( |H_1| \) then, the population probability is complete, integrable to 1, and for \( \sum_i (f(x) = 1) \cdot x \equiv x = \nu \). The variance expression \( \xi(x) \) is identically established, as \( f(x) = 1 \cdot (x_1 - \nu) = 0 \), thereby realising in expectation over the population, both necessary and sufficient statistics to satisfy our Gaussian characterisation of the probability random error. Application of the score function provides

\[
\hat{v} = \frac{1}{\xi^2} \sum_{i=1}^{m} (x_i - \nu) = \frac{m}{\xi^2} (\hat{v} - \nu),
\]

from which it may be immediately seen that there exists only one optima, at 0, with \( \hat{v} = \nu \). For the score function taken wrt \( \xi^2 \) for known \( \nu \) is obtained

\[
l(\nu, \xi^2 | x) = -\frac{m}{2\xi^2} + \frac{1}{2(\xi^2)^2} \sum_{i=1}^{m} (x_i - \nu)^2 = -\frac{m}{2(\xi^2)^2} (\xi^2 - \frac{1}{m} \sum_{i=1}^{m} (x_i - \nu)^2),
\]

which given the identity \( \nu(x) = E(x) \), follows

\[
\xi^2(x) = \frac{1}{m} \sum_{i=1}^{m} (x_i - \hat{v})^2.
\]

Said estimators are biased upon finite samples, in that neither \( \nu \), nor \( \xi^2 \) are typically known, and therefore a reduction in the degrees of freedom are necessary to correct for finite samples. The variance estimator may be corrected upon finite samples by substitution of \( m - 2 \) for \( m \), as with \( m - 1 \) for \( m \) for the median. Therefore, both necessary and sufficient statistics are producible from these two estimating score equations, for which the minimal sufficient conditions are thereby satisfied. The variance of the parameters, and the construction of the Fisher information matrix, is thereby produced by taking the derivatives for each equation wrt the target parameters, as already well established.

As a linear function space, the asymptotic sampling distribution of the MLE \( \hat{\alpha}(x) \) is expected to be (multivariate) normally distributed, with expectation \( \alpha \) and variance \( I(\alpha)^{-1} \) wherein the established properties of the linear metric space ensures that a quadratic approximation of \( I(\alpha) \) is sufficient as the sum of orthonormal variances (as previously utilised to demonstrate the bias of the \( \ell^2 \)-norm MLE) as calculated using equation 20. For a linear regression upon the Kemeny space, is a linear equation

\[
y_i = \alpha_0 + \alpha_n x_i + \epsilon_i,
\]

where \( \epsilon_i \) is distributed as an independent normal distribution with median 0 and and unknown error variance, as previously established for elements \( i = 1, \cdots, m \) in the sample. Therefore, the joint density for \( \epsilon_i \) upon \( (X, y; \rho_X) \) is as follows, as specified according to the likelihood equation

\[
\frac{1}{\sqrt{2\pi\epsilon^2}} \exp\left(-\frac{\epsilon_i^2}{2\epsilon^2}\right) = \frac{1}{\sqrt{2\pi\epsilon^2}} \exp\left(-\frac{\epsilon_i^2}{2\epsilon^2}\right).
\]

By substituting \( \epsilon_i = y_i - (\alpha_0 + \alpha_n x_i) \), the likelihood function is therefore

\[
L(\alpha_0, \alpha_n, \xi^2 | y, x) = \frac{1}{\sqrt{2\pi\epsilon^2}} \exp\left(-\frac{m}{2\epsilon^2} \sum_{i=1}^{m} (y_i - (\alpha_0 + \alpha_n x_i))^2 \right),
\]

from which follows the score function

\[
l(\alpha_0, \alpha_n, \xi^2 | y, x) = -\frac{m}{2} (\log(2\pi) + \log(\epsilon^2)) - \frac{1}{2\epsilon^2} \sum_{i=1}^{m} (y_i - (\alpha_0 + \alpha_n x_i))^2.
\]

Consequently, it is seen that optimising the likelihood function for the parameter space \( \alpha \) is equivalent to minimising the residual sum of squares as previously defined, with \( n + 1 \) parameters, thereby providing \( m - (n - 1) \) residual degrees of freedom for any well-posed regression scenario with an error distribution which is monotonically non-decreasing. It should be noted that in a regression problem for finite samples, both the expectation and the expected dispersion (i.e., the median and the variances and covariances) are unknown. Therefore, it is recommended that Students’ \( t \) statistics be employed, which is reasonable since the sharp convexity previously established for finite samples ensures that the sample mean and variances are both orthonormal and strongly consistent, thereby precluding the typical necessity for asymptotic weak convergence, as typically utilised for the Wilcoxon rank-sum statistic, the Kendall’s \( \tau_p \), and the Theil-Kendall-Sen non-parametric estimators when one or more ties occurs. The Hessian matrix is as follows, established separately for the parameters \( \hat{\alpha} \) and the residual sum of squares, for which a second derivative must each be taken upon the likelihood equations previously provided. These result in the estimators

\[
H = \frac{\partial l}{\partial \hat{\alpha} \partial \hat{\alpha}^T} = \left[ \begin{array}{c} \frac{\sum_i x_i y_i}{\xi^2} \end{array} \right] = \frac{X^T X}{\xi^2} - \frac{\sum_{i=1}^{m} \xi^2}{\xi^2}.
\]

The expectation of \( H(\hat{\alpha}) \) follows, under which the Gauss-Markov assumptions cancel out the covariances between \( \xi^2 \) and \( X \) on the off-diagonals. The expectation of the model variance (which is a fixed constant) is equivalent to the closed form expression already provided, therefore establishing asymptotic maximum likelihood estimator candidacy.
The second term, concerning the error variance may be reduced

\[
E(H) = \frac{\partial l}{\partial \hat{\alpha} \hat{\alpha}^T} = \begin{bmatrix}
\frac{X X^T}{2\epsilon^2} & 0 \\
0 & -\frac{m}{2\epsilon^2}
\end{bmatrix}
\]

(24)

and from which follows the typical relation estimator as previously established for linear estimators upon this topological manifold. Thus the Information matrix can be seen to be equivalently established for linear estimators upon this topological manifold. The Cramer-Rao lower bound is further shown to be equivalent to the closed form expressions already provided from which follows the typical relation estimator as previously established for linear estimators upon this topological manifold. Thus the Information matrix can be seen to be equivalently established for linear estimators upon this topological manifold.

Geometric perspective upon a non-parametric interaction

The idea of main effects does not necessarily guarantee that a collision upon the covariate space will occur, however the multiplication of two features, especially in the common problem of finite observation spaces (demographics or trial conditions) necessitates that ties will occur. We demonstrate that the Kemeny metric maintains its maximum likelihood properties in a linear framework while allowing for interactions to be estimated. This is a compelling improvement over all other current employed non-parametric techniques, as it allows for the estimation of linearly first and second order consistent interactions without a need to conduct sub-study stratification. A brief simulation study was conducted to demonstrate the superiority of the method proposed in this manuscript in comparison to established alternatives and traditional OLS regression, and the results are provided in Table 2. Also of note is the finding reported in Table 2, that the standard deviation of the estimated parameters is nearly equivalent for all parameters with the closed form expression, as expected for a linear estimator.

A cursory inspection of Figure 2 reveals the distribution of all coefficients under the Kemeny metric to be both normally distributed and less dispersed (i.e., more informative) compared to the alternative formulations, as expected of a minimum variance estimator. The predominant component of the calculation of the asymptotic standard errors, the sum of squared errors, is geometrically identical to the Kemeny distance between the regression predictions and the target. It also immediately follows from this geometric equivalence that the Kemeny metric is, both empirically and theoretically, a more powerful estimator for any cumulative distribution function upon homogeneous but non-Gaussian data samples. This is of course not true for instances in which a proper $L_2$ contraction may be imposed, however this typically induces a non-linearity, whereas our approach is linear. Computing the distance between the projection and the target, divided by the residual degrees of freedoms, provides the mean squared error, as would be expected for any linear functional basis. The product of the MSE by the individual parameter cross-product, produces the standard errors as a simple linear function. Further, all matrix multiplications which result in a $n \times n$ product are supplanted by the covariance matrix of all coe

| Statistic | Mean | St. Dev. | Min | 25% | 75% | Max | $\hat{\alpha}$ |
|-----------|------|---------|-----|-----|-----|-----|---------------|
| (Intercept) | 0.231 | 0.082 | -0.189 | 0.193 | 0.267 | 0.506 | 0.075 |
| socio | 0.126 | 0.091 | -0.036 | 0.0001 | 0.191 | 0.456 | 0.064 |
| sex | -0.327 | 0.128 | -0.838 | -0.419 | -0.251 | 0.056 | 0.103 |
| age | 0.163 | 0.184 | -0.476 | 0.000 | 0.294 | 0.906 | 0.165 |
| Sex:Age | 0.208 | 0.044 | 0.113 | 0.192 | 0.225 | 0.286 | 0.040 |
| socio | 0.097 | 0.037 | -0.040 | 0.074 | 0.122 | 0.253 | 0.0323 |
| sex | -0.131 | 0.044 | -0.277 | -0.161 | -0.100 | 0.017 | 0.0410 |
| age | 0.041 | 0.042 | -0.101 | 0.012 | 0.069 | 0.218 | 0.0375 |
| Sex:Age | 0.001 | 0.043 | -0.132 | -0.029 | 0.031 | 0.136 | 0.0389 |
| (Intercept) | 0.266 | 0.094 | -0.120 | 0.205 | 0.331 | 0.521 | 0.097 |
| socio | 0.144 | 0.073 | -0.091 | 0.095 | 0.192 | 0.480 | 0.075 |
| sex | -0.337 | 0.118 | -0.718 | -0.416 | -0.257 | 0.052 | 0.120 |
| age | 0.425 | 0.331 | -0.826 | 0.202 | 0.654 | 1.605 | 0.344 |
| Sex:Age | -0.321 | 0.409 | -1.937 | -0.595 | -0.043 | 1.288 | 0.417 |

Table 2

Estimation of interaction parameters over 2,500 jackknife resamplings under Kloke (2009), Kemeny (1957), and OLS metrics with mean bootstrapped linear standard errors reported as $\hat{\alpha}$, indicating expected higher efficiency of the parametric standard errors.

We also must demonstrate that the standard errors are (second-order) minimised with respect to the fixed data input into the model. To address this, we utilise the Anscombe (1973) dataset with all appropriate bivariate pairings, which are bootstrapped with replacement to produce 15,500 datasets of 550 subject measurements, for each pairing. We would expect under valid characterisations, that the standard deviations of the parametric formulas to be smaller than the corresponding bootstrapped estimates, unless the parametric assumptions were violated, and to otherwise approach a ratio of 1 between each of the two metric spaces. The substantively finding reflects the approximately constant scaling difference in the four bivariate data sets between the bootstrapped and formula estimated standard errors. Com-


Mardia distributional examination of regularity parameters with interaction

pared to the OLS formulation of the first and second order statistics (coefficients and standard errors, respectively, of the intercept and regression slope) which possesses a ratio approaching 1 (min88% for (x₁, y₁); the bivariate Gaussian distribution) between the bootstrapped standard deviations of the empirical coefficient distribution and mean standard errors as expected for the constant (but heavily biased) correlation of \( \hat{r} = 0.8164 \), our proposed methodology changes between the bivariate empirical distributions, although both ratios are correctable by a simple linear rescaling coefficient. However, as stated, this reported values in Table 3 greater magnitude (by a constant scaling) is expected, as the sum-of-squared-errors are in fact non-constant and does not affect the comparison of the relative variability. In Table 3 the standard errors show that the only approximately unit scaling between the two estimates are found for the sole case in which the Kemeny metric is invalidly applied, i.e., the quadratic function approximated by a single slope upon \((x_2, y_2)\). For all other cases, the standard errors are consistently smaller (approximately 92%), as would be expected for the scenario in which the parametric error family was correctly derived. More interestingly though is the comparison of \((x_1, y_1; \rho_E, \rho_K)\), which is the only instance of an unbiased estimator upon the Anscombe dataset. Here, in the first results for Table 3, the estimated standard errors are approximately twice the size of the empirically estimated standard deviation, whereas for \((x_4, y_4)\) the OLS-\(\rho_E\) ratio is again approximately 1, however the ratio upon \(\rho_K\) is nearly 5, but under replication, the constant scaling is found to approximate \(\sqrt{m-n}\), the residual rescaling of the degrees of freedom of the variance. Once this is corrected for, the linear relative efficiency we hypothesised is validated with a ratio of approximately 92% upon two or more coefficients in a linear system of \(m\) equations. This necessary correction however, for \((x_2, y_2)\) does not hold, as it over-corrects under the non-monotonic function approximation necessary to linearise the quadratic function despite holding for the other three linear-parameter relations, thus demonstrating the necessity to still maintain the parametric assumptions of the Kemeny error field.

Decomposition of Sums-of-Squares

A final connection for any theory of a general linear model requires the demonstration of decomposition by sums of squares. As such a decomposition upon a centred data set \(X_{mea}\) in the form of a square \(X^\top X\) or \(XX^\top\) matrix provides
the same fundamental information necessary to produce, in
terms of the design matrix, the total sum of squares and cross
products. As previously stated, the Sums of Squared Errors
(SSE) and the Mean-Squared Error (MSE) terms are already
defined as the distance and expected distance between the
predictions and the target. The intercept only model in turn
defines the origin of the coefficient space as the
mean, about which the regression terms minimise the dis-
crepancy in ordination of the prediction set
and a multinomial distribution. The dependent
measure in the data set is a total count of warp breaks per
tension (a three level factor; L, M, and H) and wool type
for an ANOVA based decomposition of Breakage counts per
sampling indicates significant non-zero effects which are concor-
dant with the conclusions of the Wald tests under the
Kemeny metric space. Comparisons to DISCO analysis (Rizzo
& Székely, 2010) provide one-way ANOVA results, how-
ever the tests are shown to be less powerful, for Minkowski
larger for the Frobenius norm.

Table 4, providing demonstration of bootstrapped consistency
and relative stability of all estimates, while providing both
an omnibus and local Wald-testing for each effect under the
assumption of a weakly-orderable measurement data space.

More interesting though is the recognition that, while the tension variable is an ordered magnitude (Low,Medium,High) which must be categorically coded under the Frobenius norm, the Kemeny norm allows for the ordinal nature of the data to assessed as part of the data. In order to compare this, we assessed the recoded the Warpbreaks design matrix to assess the tension variable and all interactions as single variables, reducing the necessary model degrees of freedom to freedom from 5 – the results are presented in Table 6. The interesting result, presented here with the 25,500 bootstrapped replications of the 54 subjects with replacement, was that the conclusions are in fact similar between the OLS and Kemeny approaches: both identified two of the same elements as significant, however the comparative range and standard deviations for the coefficients of determination and the MSEs are substantially larger for the Frobenius norm.

Provided in Table 5 are the empirical standard deviation of
the distribution of the Mean Squares, which show that non-
normality presents more widely dispersed ANOVA terms
than the proposed Kemeny metric. Interestingly in all cases of the application of a Wald test, the cross-validated resam-
pling indicates significant non-zero effects which are concor-
dant with the conclusions of the Wald tests under the
Kemeny metric space. Comparisons to DISCO analysis (Rizzo
& Székely, 2010) provide one-way ANOVA results, how-
ever the tests are shown to be less powerful, for Minkowski
adjustment $p = 1.5$, demonstrating that the Kemeny metric
provides a more robust and consistent alternative to the alter-
Table 4

4,500 times cross-validated estimates of parameter distributions and comparisons between standard deviation of estimates and median standard error estimate for the Main effects and Interaction upon $\rho_K$

| Statistic          | Mean | St. Dev. | Min  | 25%  | 75%  | Max  | $\sigma(\alpha)$ |
|--------------------|------|----------|------|------|------|------|------------------|
| (Intercept)        | 0.055| 0.017    | -0.014| 0.043| 0.067| 0.111| 0.144           |
| tensionM           | -0.192| 0.140    | -0.732| -0.285| -0.100| 0.350| 0.090           |
| tensionH           | -0.308| 0.135    | -0.826| -0.398| -0.223| 0.275| 0.090           |
| woolB              | -0.176| 0.119    | -0.609| -0.258| -0.096| 0.234| 0.088           |
| tensionM:woolB     | 0.162| 0.145    | -0.071| 0.431 | 0.081| 0.729| 0.144           |
| tensionH:woolB     | 0.003| 0.137    | -0.550| -0.121| 0.114| 0.826| 0.143           |

Table 5

ANOVA Wald test decomposition for $\rho_K$ over 2,500 cross-validations in comparison to two univariate DISCO analyses with $p = 1.5$ and traditional $\rho_E$ ANOVA

| Method Term     | $\text{df}$ | $MS$ | Min($MS$) | 25% $MS$ | Max($MS$) | $\sigma(MS)$ |
|-----------------|-------------|------|-----------|----------|-----------|-------------|
| Kemeny tension  | 2           | 872.13| 789.25    | 856.75   | 888.25    | 942.25      |
| wool            | 1           | 3,477.26| 2,699     | 3,295.8  | 3,582     | 4,063       |
| Error           |             | 48    | 2,098     | 18.67    | 25.24     | 36.31       |
| DISCO tension   | 2           | 127.06| 32.40     | 173.46   | 318.94    | 812.00      |
| wool            | 1           | 134.25| 5.69      | 56.86    | 187.18    | 656.73      |
| Error           |             | 48    | 2,644     | 16.76    | 25.24     | 36.31       |
| ANOVA tension   | 2           | 823.40| 119.05    | 1077.43  | 2083.32   | 5233.44     |
| wool            | 1           | 768.45| 0.02      | 299.67   | 1079.64   | 4883.08     |
| Error           |             | 48    | 243.59    | 15.65    | 24.78     | 29.71       |

Table 6

Empirical comparison of the standard errors upon the ordinal characterisation of the Tension feature

| $\alpha$ | $\sigma(\alpha)$ | $\sigma(\tau)$ | $t_{\alpha}(p_{\alpha})$ | $p(t_{\alpha})$ |
|----------|------------------|----------------|--------------------------|----------------|
| (Intercept)| 28.1528| 1.4780| 1.4865| 19.0484| 0.0000 |
| woolB    | -2.8857| 1.4780| 1.4786| -1.9525| 0.0282 |
| tensionM| -5.0127| 1.8049| 2.0120| -2.7773| 0.0038 |
| tensionH| -3.2424| 1.0444| 0.9212| -3.1046| 0.0016 |
| woolB:tensionM| 5.2573| 1.8049| 2.0098| 2.9129| 0.0027 |
| woolB:tensionH| 0.0000| 1.0444| 0.9242| -0.0000| 0.5000 |

Empirical Demonstrations

We conclude with several empirical data demonstrations, to validate the mathematical fact that the results in the presence of ties of several estimators are inconsistent and under-powered, even in the presence of jackknife resampling. In the presence of ties, conventional estimators do not possess the properties of an MLE, which is empirically explored by the application of jackknife resampling and statistical comparisons of the results, with the null hypothesis of comparable performance expected to produce no observable differences in the first order results of the coefficients. The second contention, that the application of a non-metric topology results in inconsistent (and therefore unrealisable representations of a population), is explored as well, with the null hypothesis of performance less than or equal to that of conventional methods. Thus, it would be found that the proposed methods fail to demonstrate any improvement over current methods for scenarios such as normal distributions. These conjectures, if validated, therefore demonstrate that even non-parametric tests do not provide adequate unbiased understanding outside the population, when we introduce tied subjects, a point of presumed necessity for empirical studies in any field. All simulations in this section are performed using R v.4.0.3 (R Core Team, 2020) and custom written software which is available from the authors.

We next will compare the Tukey-Siegel, Kendall-$\tau_b$, and Wilcoxon rank-sum performance to the Pearson correlation and the proposed estimator, with respect to the point estimate differences and empirical variance (power) of each estimator. We rely upon the natural relationship between a distance measure and a similarity measure as mutual reflections upon the arbitrary, but appropriate, point of origin, and thus estimations of similar and comparable bivariate relationships. This is a valid assertion, as otherwise the data must be multinomial distributed, and result in a substantially divergent perspective as to the nature of the data, for which all data analytic methods are invalid without a priori data processing in the form of binomial coding, and therefore would be expected to perform poorly by definition.

Minimum variability of rank-tests with ties

Table 7

Empirical distribution of relations estimated upon both jackknife and 7,500 times repeated resampling (CV).

| Statistic          | Mean | St. Dev. | Min  | 25%  | 75%  | Max  |
|--------------------|------|----------|------|------|------|------|-------|
| $\rho(x_1, x_2)$  | -0.033| 0.043    | -0.103| -0.061| 0.000| 0.051|
| $r(x_1, x_2)$     | -0.095| 0.074    | -0.180| -0.154| -0.054| 0.057|
| $\rho(x_1, x_2)$  | -0.111| 0.082    | -0.234| -0.170| -0.021| 0.022|
| $\tau(x_1, x_2)$  | -0.096| 0.070    | -0.202| -0.146| -0.018| 0.019|
| $Glass' r$        | 0.133| 0.099    | -0.028| 0.025| 0.198| 0.275|
| $r(x_1, x_2)$     | 0.042| 0.079    | -0.229| -0.013| 0.094| 0.341|
| $r(x_1, x_2)$     | -0.088| 0.137    | -0.552| -0.182| 0.001| 0.494|
| $p(x_1, x_2)$     | -0.109| 0.156    | -0.630| -0.214| -0.004| 0.425|
| $\rho(x_1, x_2)$  | -0.095| 0.136    | -0.557| -0.186| -0.004| 0.369|
| Glass' $r$        | -0.134| 0.187    | -0.594| 0.009| 0.263| 0.773|

A second brief demonstration with a 14 observation bivariate data set is presented; these data may be equivalently
viewed as an unpaired Wald t-test, an OLS regression, a point-biserial correlation, a rank-sum test of differences between groups, and any other functional linear basis analytic techniques – asymptotically, these are all consistent under Wilk’s theorem. In effect, it proposes the identification of a test in the shift between two groups’ measurements, for a one unit change in the group membership. Here, we are solely interested in the distributional properties of the various estimators under the conceptual meaning of the unbiased maximum likelihood estimator. We expect that a correctly identified estimator with appropriately chosen error distribution will possess a global representation of the bivariate relationship, with the point of minimum error in turn being the most informative (smaller standard error) estimate as well. Under these two definitions of the maximum likelihood estimator, the point is made that solving the minimum error of approximation must best approximate the population value. We demonstrate that this is not true for any estimator which is not bivariate normal even with standard ‘non-parametric’ testing procedures, due to the non-metric nature of the rank interpretation with ties. Further, not only does the Kemeny correlation coefficient most accurately approximate the population true value, but also appears to do so with the expected smallest distribution of scores, thereby demonstrating that this estimator, for any homogeneous score distribution stochastically dominates not only the Pearson based estimators (for scores which are approximately normally distributed) upon non-Gaussian spaces but also the bivariate non-parametric tests as well. This provides support for our conjecture that, in the presence of ties, even ‘tie-resolving’ estimators for both Spearman $\rho$ and Kendall $\tau_b$, are biased estimators of the population relationship as tested within a Null Hypothesis framework.

**Real world VHA dataset**

A data set from a Veteran Affair Office of Research and Development study approved by the VA Central Institutional Review Board (CIRB) is utilised to demonstrate both finite and bootstrap comparative performance upon an endogenous ordinal summative score in evaluating randomised treatment outcomes for $m = 594$ Major Depressive Disorder (MDD) complete case patients. All participants provided written informed consent and privacy authorization. This data was previously reported in Zisook et al. (2019) and the non-normality of the data was addressed with a Cox survival model of time to remission, where remission was defined as a quantised decrease of at least a fixed reduction in depression score. Here we instead provide a method of quantitative assessment of the reduction in overall depressive impairment over time for the three treatment strata, which is compared to the an equivalent model estimated under OLS. This allows for explicit shifts in the quantity of depressive impairment to be assessed as a continuous measure without explicit parametric assumptions beyond the existence of a suitable cumulative distribution function. However, the non-normality of the response variable is unresolved by both definition (a finite set of responses demonstrates only weakly consistent normality in the population) and in overall ECDF. As previously reviewed, traditional alternative techniques explicitly preclude the estimation of substantive parameters of explicit interest, such as interactions, resulting in choices in model selection favouring the general linear model, in spite of its inappropriateness. We demonstrate here how these conclusions are inconsistent with the data learning process, resulting in both false positives and false negatives, for both the complete cases sample and a bootstrapped resampling with replacement of 15,500 cases of 594 subjects for each sample.

The participants here were VHA patients aged 18 years or older who were both diagnosed and treated as an MDD patient. Diagnostic eligibility was supplemented with the 9-Item Patient Health Questionnaire, an ordered scale with support [0, 27]; worsening mental health was reflected in higher scores. However, our focus is upon the European Quality of Life (EuroQOL) scale, an ordinal measurement set of 16 items with 5 graded thresholds for each item. The scale distinctly non-normal, with a mean of 67.32, standard deviation of 18.67, median of 70, range of 99, skew of -0.76, kurtosis of 0.41, and a median absolute difference of 14.83.

Controlling for ancillary baseline covariates, it was desired to assess the change in score at week twelve upon the EuroQOL, with explicit focus being upon the interaction of social support (a categorical 4 level variable) and treatment approach arm. The hypothesis was held that greater availability of social support would present differential changes in the treatment efficacy, and further that these findings would not be found under the traditional use of an OLS regression. The results of the model are presented in Table 8, comparing the two techniques. It should be noted that if the normality assumption was met, all statistics would be expected to replicate with greater stability for the OLS model, and would present with smaller standard errors than those found under the methods proposed in this paper. However, as we seen from the disjoint findings between the cross-validation resampling performed for both models, while the OLS approach does imply a higher coefficient of determination and therefore better fit, the resampling demonstrates that this value is biased, resulting in a lower estimated MSE compared to the true MSE, and therefore also draws concerns wrt the partial Wald tests, which may be invalidly over-powered as a result.

As was hypothesised, the order-statistic based technique demonstrated greater power on the non-normally distributed data, and further, these findings replicated with greater success than the cross-validation performed upon under OLS regression. It was found that the non-parametric linear basis was a better substantive average of all observations, satisfy-
Table 8

VHA Data example, conducted upon 594 subjects; \((F_{18,575} = 5.566, R^2 = 17.3\%(\sigma_R^2 = 2.8\%); \rho_F)\) and \((F_{18,575} = 1.811, R^2 = .1%(\sigma_R^2 = .1\%); \rho_K)\), for 15,500 cross-validated resamplings.

|               | OLS Coefficients | OLS Standard Errors |
|---------------|------------------|---------------------|
|               | mean  | sd    | min   | max    | range  | mean  | sd    | min   | max    | range  |
| (Intercept)   | 55.0681 | 4.7306 | 35.9797 | 72.6146 | 36.6167 | 8.704  | 0.2188 | 3.8504 | 5.5423 | 1.6919 |
| AGE           | 0.0039 | 0.0729 | -0.2661 | 0.3009 | 0.567  | 0.068  | 0.0029 | 0.0583 | 0.082  | 0.0238 |
| TrtcodeB      | -1.1438 | 2.5716 | -10.9435 | 8.6931 | 19.6366 | 2.5985 | 0.1298 | 2.1816 | 3.1298 | 0.9481 |
| TrtcodeC      | -0.6499 | 2.4634 | -10.8852 | 9.4516 | 20.3368 | 2.6167 | 0.132  | 2.1693 | 3.1337 | 0.9644 |
| marital_status1 | -2.5109 | 2.9317 | -13.8083 | 9.161  | 22.9693 | 2.9556 | 0.1561 | 2.3653 | 3.8533 | 1.488  |
| marital_status2 | -0.9593 | 3.781  | -18.3705 | 13.344  | 31.7145 | 3.966  | 0.3353 | 3.0119 | 6.4659 | 3.454   |
| marital_status3 | -2.4362 | 6.3579 | -36.5888 | 22.6965 | 59.2853 | 5.796  | 1.3334 | 3.9924 | 17.8907 | 13.8983 |
| race1         | 0.6867 | 1.7443 | -6.6847 | 7.0143  | 13.699  | 2.7369 | 0.0779 | 1.4769 | 2.1382 | 0.6613  |
| race2         | 1.3162 | 2.6922 | -12.2875 | 12.2162 | 24.5037 | 2.6354 | 0.1898 | 2.0273 | 3.6687 | 1.6414  |
| education     | 0.9011 | 0.5914 | -1.399  | 3.1646  | 4.5636  | 1.6038 | 0.0242 | 0.5203 | 0.72   | 0.1997  |
| EUROHLTH_BASE | 0.2965 | 0.0376 | 0.1389  | 0.4393  | 0.3005  | 0.0336 | 0.0014 | 0.0287 | 0.04    | 0.0114  |
| F20totaIACE   | -0.0211 | 0.277  | -1.2014 | 0.9195  | 2.1208  | 0.2815 | 0.0121 | 0.2373 | 0.3318 | 0.0948  |
| CIRSscore     | 0.4233 | 0.1739 | -1.0677 | 1.992  | 2.0681  | 2.1561 | 0.0066 | 0.1835 | 0.3278 | 0.0537  |
| TrtcodeB:marital_status1 | -3.3911 | 3.8553 | -10.5904 | 17.7299 | 28.3203 | 3.869  | 0.1603 | 3.278  | 4.7566 | 1.4975  |
| TrtcodeB:marital_status2 | -3.3911 | 3.8553 | -10.5904 | 17.7299 | 28.3203 | 3.869  | 0.1603 | 3.278  | 4.7566 | 1.4975  |
| TrtcodeC:marital_status1 | 0.0388 | 4.0232 | -15.8494 | 15.2499 | 31.0992 | 3.8563 | 0.156  | 3.2354 | 4.5409 | 1.3075  |
| TrtcodeC:marital_status2 | 1.7688 | 4.9905 | -23.1863 | 24.3952 | 47.7814 | 6.2726 | 0.398 | 4.5075 | 7.4708 | 3.3632  |
| TrtcodeC:marital_status3 | 2.5762 | 4.9334 | -15.2903 | 24.2221 | 39.5124 | 5.539  | 0.3419 | 4.3294 | 7.5138 | 3.1844  |
| CIRSscore     | 0.0091 | 0.5914 | -1.399  | 3.1646  | 4.5636  | 1.6038 | 0.0242 | 0.5203 | 0.72   | 0.1997  |
| EUROHLTH_BASE | 0.2965 | 0.0376 | 0.1389  | 0.4393  | 0.3005  | 0.0336 | 0.0014 | 0.0287 | 0.04    | 0.0114  |
| F20totaIACE   | -0.0211 | 0.277  | -1.2014 | 0.9195  | 2.1208  | 0.2815 | 0.0121 | 0.2373 | 0.3318 | 0.0948  |
| CIRSscore     | 0.4233 | 0.1739 | -1.0677 | 1.992  | 2.0681  | 2.1561 | 0.0066 | 0.1835 | 0.3278 | 0.0537  |

...
We suggest the focus of attention be with respect to the effect TrtcodeB:marital_status2, as this term is explicitly significant and indicative of a contradictory finding than under OLS assumptions. This demonstrates evidence in favour of the interpretation that married individuals were less depressed under treatment B, and further that marriage in general provided a strongly significant positive effect upon the overall assessment of depression at treatment termination. In particular is the difference in MSE between the two models; while OLS does in fact establish a higher predictive accuracy within sample, the replicability or stability of this point estimate under bootstrapping is nearly four times greater, whereas while under resampling the median coefficient of determination for the two approaches was nearly halved for OLS, it remained nearly constant for the Kemeny methodology proposed herein. As well, the standard errors are found to perform exactly as expected, being in most cases smaller and producing Wald test statistics which are consistently found to be significant, unlike with the OLS approach. The consistency of these results are empirically demonstrated with the comparison of the standard deviations of the comparisons across bootstrapped data sets, which are, again, smaller for the Kemeny metric.

Discussion

This paper introduces both a closed form and maximum likelihood solution to an endogenous unbiased linear learning problem upon an ordinal, and more generally non-parametric, metric topological manifold, with several demonstrations of performance with arbitrary data sets. Further, this metric space was shown to be linear and unbiased, possessing MLE properties in the presence and absence of ties, thereby encompassing a much wider breadth of empirical applications in terms of the estimable parameters (Mukherjee, 2016; Chatterjee & Mukherjee, 2019). This allows for the calculation of indeterminate points (e.g., polynomial terms and interactions) for incomplete rankings and sum-of-square decompositions in a similar, but more comprehensive manner to the Theil-Kendall-Sen estimator in the model regularity parameter space $\alpha \in \Omega$, for which an intuitive and computationally simple deconstruction may be applied for finite samples. Pragmatically, this Kemeny distance estimator has been shown to satisfy the requirements of a linear maximum likelihood estimator in particular for ordinal data both in the absence and presence of all non-nominal data, for which we have derived and empirically shown that the local and global model parameters may be assessed under the conventionally Wald test framework. Further demonstrated in this text is the information superiority and reduced parametric assumptions necessary to validate the inductive generalisation beyond the sample to a homogeneous and potentially multivariate population. Explicit comparisons were made for nested models along with the computational stability in the estimation of a linear and well-posed parameter space for joint coefficients of determination, as well as the ability to examine the marginal distributions of the conditionally independent parameter space. Over all of these assertions, we follow through to have demonstrated the relative power gains which are expected to hold under the strong law of large numbers, under much weaker parametric conditions than those typically associated with maximum likelihood principles.

Importantly, this embodies an important empirical demonstration of the conjecture of the replication crisis, in that it is seen that even significant empirical findings upon improper error distributions fail to allow for replication to validly be a strongly consistent estimator. That is, the variability of replication efforts is greater than the standard errors typically estimated and reported. Since the assumption that a finite set of ordinal measurements, where the number of responses is greater than 2, was shown to produce biased estimates under the Gaussian estimator it questions the widespread validity of the assumption of Gaussian assumptions upon the Frobenius norm, as is commonly taught at both the undergraduate and graduate levels of the Social Sciences. Such a consideration is extremely important in the context of the near universal prevalence and reliance upon ordinal measurements in the Social Sciences, and more generally the failure to correctly establish distributions of normal errors upon finite samples under the $\ell_2$-space. This mathematical framework demonstrates that the false assumption of the sufficiency of the Euclidean norm upon data spaces can result in scenarios which are markedly similar to that of the replication crisis of statistical tests, wherein significant biased point estimates may be found, for which the variability is drastically underestimated. This follows from the simple fact that in the loss of MLE properties, the strong convexity and consistency of our estimators upon a finite sample are lost, and therefore may only be abstracted to a description of the population when exhaustive sampling has been enacted. Since this is infeasible, the choice of maximum likelihood procedures which are appropriate to our data is recommended. Similar arguments have been extended to contexts such as latent variable modelling and Psychometrics, a popular methodological topic employed. It has been demonstrated in another manuscript under submission that the techniques upon which this process is constructed allow us to remove the multivariate normality assumption, while maintaining first and second order consistency, thereby presenting an acceptable maximum likelihood estimator for ordinal data, which avoids said issues described with the polychoric correlation, and addresses the common failure of the goodness-of-fit statistics upon non-normal data. Similar work under submission has been employed to similarly address both missing data under the expectation-maximisation algorithm for finite samples upon the $(X, \rho_K)$-space, construct a non-parametric formulation of
Box’s M-statistic, and to address mixture models under non-parametric feature spaces.

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## Appendix A

**Kullback-Leibler divergence**

The Kullback-Leibler divergence is defined for any correctly posed probability distribution, from which follows a monotonically non-increasing primal problem (error minimisation) convergence sequence which is unbiased and converges to the target when the probability distribution is linear and orthonormal. For any measurable space on the reals which is sortable, then, it follows that the Kemeny metric is an unbiased estimator, which is linear but less informative than the logarithmic linearisation undergone by the use of the canonical linking function upon the $l_2$ metric. For conditions in which the family of distributions are not identified, the likelihood function is almost surely maximised in the neighbourhood of $\Omega$ as defined upon the sample as long as the quotient space is defined (Redner, 1981) and is asymptotically normal as long for each $\alpha$ with sufficiently small radius in the concave neighbourhood about the optima, $f(\cdot, \alpha, r)$ is measurable and the following inequalities are therefore true

$$\int_x \log f^*(x, \alpha, r)d\alpha^* < \infty$$

and

$$\int_x \log h^*(x, s)d\theta_0 < \infty,$$

assuming $\alpha^*$ denotes the true parameter set. If true, then as $\delta(\theta_0, \tilde{\theta}_i) \to \infty^+$, $f(x, \theta_0) \to 0$ for any identified set which has a non-zero Lebesgue measure orthonormal to $\theta_0$, and is thus satisfied for any sub-additive metric topology. Further assuming that

$$\int |\log f(x, \theta_0)|d\theta_0 < \infty,$$

and therefore finite, then if $\theta_i \to \theta$ and in turn $f(x, \theta) \to f(x, \tilde{\theta}_i)$ for a non-deterministic functional space, the with probability one the likelihood function converges to the true function, which for any linear function is best approximated by its expectation which minimises the error for all $M$. Upon any mixture family, which is linear $\delta(X, \Omega)$, then it follows that both extensions to Wald’s theorems (Redner, 1981, p. 226) are satisfied upon the Kemeny metric, and therefore $f(X, \tilde{\theta}_i) \to f(X, \theta_0)$ with probability 1. From Redner
Theorem 5, it follows that the Kemeny metric, as a compact ultrametric, is strongly consistent for any orderable and independently sampled distribution, including a multivariate normal distribution. Any distribution for which the $\ell_2$ metric is complete and compact is also linearly strongly consistent for the Kemeny metric, as it is invariant to any monotonic transformation (non-identity canonical link), and thus the properties generalise without any additional restrictions.

However, the selection of an appropriate non-linear transformation is subjective and subject to misjudgement due to idiosyncrasies such as cultural modelling norms, non-uniform sampling between the data and the population, and incomplete data observations. Analysing ordinal data as an endogenous measure space, as opposed to continuous data, requires fewer structural assumptions in the endogenous response to introduce maximum likelihood estimation (Friedman, 1937; Wilcoxon, 1945). However, this must be paired with the introduction of more assumptions upon the sampling procedure. This follows under the conventional construction of both Spearman’s $\rho$ and Kendall’s $\tau$, in which ties are explicitly excluded from the space with probability 1, from which directly follows by contradiction a tie in any sample upon the population with probability of 0. By the pigeon-hole principle though, ties almost surely are observed for any ordinal measure space since the number of empirical paired orderings is always less than the number of possible observations in the sample, as is demonstrated by the construction of a polychoric correlation matrix (Olsson, 1979; Pearson & Pearson, 1922).

Unfortunately, the polychoric correlation introduces an assumption concerning the latent variable linearity upon the $\ell_2$ metric, and requires greater sample sizes with respect to $g$ responses upon an ordinal scale (i.e., $g^2$ cells). Each bivariate relation pairing must possess sufficient cell size to just identify each latent variable, requiring at least two parameters; multidimensional latent spaces in turn require more parameters to be estimated, which quickly inflates the minimum sufficient sample size. As a continuous distribution upon a compact metric is discrete, Gibbs’ inequality may be directly leveraged to define for two any orderable distributions $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_m\}$ for which $P, Q \in [0, 1]$ with equality only holding when $P = Q$ for the metric space $\delta$. Since by our previous assumptions, the sample is defined for a finite support in the field $[0, m(m-1)]$ for which all occur with probability of at least $1/(1 + m^2 - m)$ in the population, thereby allowing for the Kullback-Leibler divergence to be constructed

$$D_{\text{KL}}(P||Q) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i} \geq 0.$$  

If one were to expand the definition of $P$ to be a function of parameters $\alpha \in \Omega$ in the parameter universe, then a Bregman divergence naturally follows $F : \Omega \rightarrow \mathbb{R}$, wherein the observed space is strictly non-negative $D_{\text{KL}}(P(Q)) \geq 0$ under the convexity of $F$ which is linear wrt the endogenous empirical process $P$ for which the expectation is the optimal sufficient statistic (Frigyik, Srivastava, & Gupta, 2008).

A further proof of uniqueness for any $F$-norm, which includes any Banach norm-space, may be constructed from the axiom of the excluded middle dependent choice, given a proper metric upon any random partially orderable field. Thus, the property of dependent choice holds upon it, unlike the sub-graph formed by the traditional rank methods of the $S_m$-space (Diaconis, 1988). As a normed space (which includes Banach spaces, $F$-spaces, and $G$-spaces) $X$ defined with the topology $(X, \rho_X)$ is provably both linear and continuous with finite dimensionality $n$ upon the stochastic space $(\epsilon, a_n)$. It is therefore a valid dream space, which is explicitly uniquely defined with an inner-product space (Lane, 1971). Let $X$ be complete and $\rho_X$ be compact as previously established. It follows then that if $Y$ is any topological vector space and $f : X \rightarrow Y$ is any linear operator, then $f$ is continuous. It also follows that should $\Omega$ be an open convex subset of $X$, $Y$ is a locally full space, and $f : \Omega \rightarrow Y$ is a convex operator, then $f$ is continuous. By Schechter (1997, p. 751), any two complete $F$-norms on a vector space are topologically equivalent; this is proven as the identity mapping $X \rightarrow X$ is a linear operator. Thus proves the uniqueness of the maximum likelihood estimator upon the Kemeny metric, which is also a proper MLE for any partially orderable distribution (Schechter, 1997), as also defined for the linear function space we have established. Further, since given the Gaussian nature of the probability space is always monotonically non-decreasing, the Hessian of the likelihood function is always non-negative, and therefore it follows that the likelihood function is a linear local maxima for any identically and independently distributed error function which is partially orderable.

Appendix B

Positive semi-definiteness of the Kemeny correlation matrix

For $m$ observations upon $n$ measures, the matrix $\Xi$ may be constructed, as a square matrix of order $m \times m$ or $n \times n$, summarising pairwise similarity over all subjects or variables, respectively. Assume that a square real matrix $A$ is positive semi-definite (p.s.d.) when, for any $m \times 1$ vector, $x^\top A x \geq 0$. Then if both $A$ and $B$ are p.s.d. upon the space $(X, \rho_X)$ then by the established additive and multiplicative properties of the Kemeny metric, so is $A + B$. We construct the following bounded equivalence $x^\top (A + B) x = x^\top A x + x^\top B x \geq 0$; from this follows the general statement that the sum of any p.s.d. matrices is itself positive semi-definite.

For a sample of vectors $x_i = (x_{i1}, \ldots, x_{im})^\top$, with $i = 1, \ldots, m$, the sample median $\hat{v}$ is estimated as per equation 19 and the sample correlation matrix $\Xi$ as given for each...
bivariate pair in equation 13 (with the covariance scaling following by the use of equation 20). For the non-zero vector \( \mathbf{z} \in \mathbb{R}^n \), it follows that each vector is non-constant, and therefore has positive variance from which we use equation 20:

\[
\xi^2 = \mathbf{z}^T \mathbf{P}(\mathbf{x}_n, \mathbf{x}_n) \mathbf{z} \preceq \mathbf{z}^T \mathbb{E} \mathbf{z} > 0.
\]

Allow \( z_i \) to be defined \( z_i = (x_i - \mathbb{E}(x)) \), for \( i = 1, \ldots, m \). Any non-zero \( x \in \mathbb{R}^n \) is therefore equal to zero if and only if \( \mathbf{P}(\mathbf{x}_i, \mathbb{I}_m) = 0 \), for each \( i = 1, \ldots, n \). Upon the set \( \{z_1, \ldots, z_n\} \) spanning \( \mathbb{R}^n \), there exist real numbers \( \beta_1, \ldots, \beta_m \) such that \( z = \beta_1 x_1 + \cdots + \beta_n x_m \), and we also possess \( \mathbf{z}^T x = \alpha_1 z_1^T x + \cdots + \alpha_n z_n^T x = 0 \), which induces a contradiction. It therefore follows that if the span of any random sampling distribution upon \( z_i \) spans \( \mathbb{R}^n \), then \( \mathbb{E} \) is positive definite. Positive semi-definiteness then may be established from \( \mathbb{E} \mathbf{z} \mathbf{z}' \geq 0 \) for any vector \( \mathbf{z} \). As \( \mathbb{E} \mathbf{z} \) is \( 1/m \) times the distance of \( \mathbf{P}(\mathbf{z}_1, \mathbb{I}_m) - \mathbf{P}(\mathbf{z}_1, \mathbb{I}_m) \), the squared length of vector \( \mathbf{z} \) is \( m \), and as \( m > 0 \in \mathbb{Z} \) and a sum of squares is strictly non-negative, \( \mathbb{E} \mathbf{z} \mathbf{z}' \geq 0 \), and thus when \( \mathbf{z} \) spans the continuous field, \( \mathbb{E} \mathbf{z} \) is definite, and therefore any \( \mathbb{E} \) may be inverted for the marginalised sampling distribution wrt either \( m \) or \( n \).

**Appendix C**

**Proof of the probability measure for the Kemeny metric**

For a measure space \((\Omega, \mathcal{F}, P)\), where \( P(x \in X) \) denotes a measure \( X \) upon which \( x \) occurs with probability \( P(x) \), for which \( P(\Omega) = 1 \), allowing us to define said measure space as a probability space, with sample space \( \Omega \), event space \( \mathcal{F} \), and probability measure \( P \). The first Kolmogorov axiom has already been proven, wherein each observed event occurs with positive probability, for each element in the event space, as seen in equation 9. Concurrently, the finite and compact nature of the Kemeny measure space ensures that \( P(x) \) is also finite, with bounded support \( P(-\infty) = 0 \leq P(x) \leq P(\infty) = 1 \). The second axiom, that that the probability that at least one of the elementary events in the entire sample space will occur is 1, and therefore that \( \Omega \) is complete, follows from finite and cumulative nature of the Stirling numbers wrt the Kemeny metric, from which we may define that for the space \(|\mathcal{H}_1|=1\), which is the exhaustive symmetric group, with repetitions, for one event, said observation must be observed for any non-negative sample set with probability 1. Since all samples are cumulatively composed from this set, it follows that all samples greater than size 1 must contain at least the symmetric group permutation for one unit, the identity permutation \( \mathbb{I}_1 \) as a partial ranking, and which therefore must occur with probability 1. If not, then the Identity permutation is not a valid origin for the sample space, and therefore the population is non-uniquely identified (as neither the identity permutation or its inverse are validly measurable events). Finally, \( \sigma \)-additivity must be validly present to ensure that a given event must occur with unique probability, and is therefore a function. In equation 9, it is shown that the graph of the metric space is connected, and thus therefore that the Kemeny metric space is a function, for which each occurrence is uniquely mapped onto a singular distance with respect to the arbitrary origin \( \pi_0 \) for the entire space \( \mathbb{X}_m \). As such, any element in the sample space must be measured upon the probability field, which for equation 18 contains all reals. Second, since the cumulative distribution is complete and must integrate to 1 over all disjoint events measured upon the Kemeny metric space, all disjoint subsets must also sum to 1, as shown in equation 9, which, together with the finite countability of the symmetric group \( \mathcal{H}_m \), ensures that the probability measure is always observed with probability 1, thereby satisfying the equality \( \mu(\bigcup_{m=1}^{\infty} A_m) = \sum_{m=1}^{\infty} \mu(A_m) \), to ensure \( \sigma \)-additivity.