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Abstract. It is known ([10], [27]) that there is a unique K3 surface $X$ which corresponds to a genus 2 curve $C$ such that $X$ has a Shioda-Inose structure with quotient birational to the Kummer surface of the Jacobian of $C$. In this paper we give an explicit realization of $X$ as an elliptic surface over $\mathbb{P}^1$ with specified singular fibers of type $II^*$ and $III^*$. We describe how the Weierstrass coefficients are related to the Igusa-Clebsch invariants of $C$.

1. Introduction

In this paper, we study some K3 surfaces of high rank. Precisely, we consider K3 surfaces which have a Shioda-Inose structure: that is, an involution which preserves a global 2-form, such that the quotient is a Kummer surface. Kummer surfaces are a special class of K3 surfaces which are quotients of abelian surfaces. The Kummer surface thus carries algebro-geometric information about the abelian surface. It has Néron-Severi rank at least 17, and so therefore do the K3 surfaces with Shioda-Inose structure.

These surfaces were first studied by Shioda and Inose [16], who give a description of singular K3 surfaces, i.e., those with rank 20, the maximum possible for a K3 surface over a field of characteristic zero. They prove that there is a natural one-to-one correspondence between the set of singular K3 surfaces up to isomorphism and the set of equivalence classes of positive definite even integral binary quadratic forms. The result follows that of Shioda and Mitani [19] who show that the set of singular abelian surfaces (that is, those having Picard number 4) is also in one-to-one correspondence with the equivalence classes of positive definite even integral binary quadratic forms. The construction of Shioda and Inose produces a singular K3 surface by taking a double cover of a Kummer surface associated to a singular abelian surface and with a specific type of elliptic fibration. The resulting K3 surface has an involution such that the quotient is the original Kummer surface. It also turns out that the lattice of transcendental cycles on the K3 surface (i.e. the orthogonal complement of the Néron-Severi group in the second singular cohomology group) is isomorphic to the lattice of transcendental cycles on the abelian surface.

Nikulin studied groups of automorphisms of K3 surfaces in [24], and Morrison studied Shioda-Inose structures more extensively in [20], giving necessary and sufficient
conditions for a K3 surface to have a Shioda-Inose structure, in terms of the Néron-Severi group of the K3 surface.

The Shioda-Inose setup takes a K3 surface and produces the data of a Kummer surface, or of an abelian surface. One may reverse the question, and ask: given an abelian surface $A$, can we produce a K3 surface with Shioda-Inose structure such that its quotient under the Nikulin involution is birational to $\text{Km}(A)$?

In particular, when $A$ is the Jacobian of a curve $C$ of genus 2, Galluzzi and Lombardo [10] show that there is a unique K3 surface (up to isomorphism) with Shioda-Inose structure which corresponds to $A$ under this construction. Dolgachev, in the appendix to the paper, proves an isomorphism between the moduli space of elliptic K3 surfaces with bad fibers of type $E_8$ and $E_7$, and the moduli space of principally polarized abelian surfaces. However, the isomorphism is not explicit, and uses a result of Naruki [27] to show that the quotient surface constructed is a Kummer surface.

In this paper we describe explicitly the relation between $C$ and the K3 surface $X$ with $E_8$ and $E_7$ fibers. There exists an alternative elliptic fibration on $X$, with a 2-torsion section. The Nikulin involution on $X$ is translation by the 2-torsion section. The quotient elliptic surface under the 2-isogeny is the Kummer surface of a unique principally polarized abelian surface (generically, the Jacobian of a curve of genus 2). We relate the Igusa-Clebsch invariants of the genus 2 curve to the moduli of the original K3 elliptic surface, thus giving an explicit description of the map on moduli spaces. Thus, we also partially answer a question of Kuwata and Shioda [18], who ask for explicit elliptic fibrations on a generic Kummer surface.

This work generalizes a construction of Elkies [8], who found a family of K3 elliptic surface with two $E_8$ fibers such that the quotient is birational to the Kummer surface of a product of two elliptic curves $E_1$ and $E_2$, and is related to it by a Galois-invariant map of degree 9. Earlier, Inose [15] had found the isogenous K3 surface to the Kummer surface of $E_1 \times E_2$, but with maps that were defined over an algebraically closed ground field. Recent work of Shioda [29] realizes elliptic K3 surfaces with two $E_8$ fibers as part of a “Kummer sandwich” with the Kummer surface of $E_1 \times E_2$ over an algebraically closed ground field. Clingher and Doran have also studied the K3 surfaces corresponding to $E_1 \times E_2$ in [6], as well as to $J(C)$ more recently.

Such explicit formulas have some Diophantine applications. Elkies has used his techniques from [8] to construct elliptic curves of high Mordell-Weil rank over $\mathbb{Q}(t)$ and $\mathbb{Q}$. Similarly, he used a K3 surface of Néron-Severi rank 19 corresponding to a rational point on a Shimura curve to find an elliptic curve of Mordell-Weil rank at least 28 over $\mathbb{Q}$. An exposition of these techniques may be found in [9]. As an application of the results of this paper, we can find explicit parametrization of some Hilbert modular surfaces. We hope to address these techniques and results in a future publication.

The outline of the paper is as follows: in Section 2, we recall facts about integral lattices which appear in this context as sublattices of Néron-Severi groups or cohomology groups of K3 surfaces. In Section 3, we cover the necessary background on K3 surfaces, whereas section 4 recalls the theory of genus 2 curves. In Section 5, we state and prove the main theorem. In section 6, we describe a connection with the moduli space of 6 points in $\mathbb{P}^1$. In section 7, we describe how the family of elliptic surfaces with two $E_8$ fibers may be obtained as a degeneration of the family of surfaces with
$E_8$ and $E_7$ fibers discussed here, and obtain the compatibility between our formulas and Shioda’s.

2. Preliminaries on lattices

Definition 1. A *lattice* will denote a finitely generated free abelian group $\Lambda$ equipped with a symmetric bilinear form $B : \Lambda \times \Lambda \to \mathbb{R}$. An *integral lattice* is a lattice whose form takes values in $\mathbb{Z}$. The *signature* of the lattice is the real signature of the form $B$, written $(r_+, r_-, r_0)$ where $r_+$, $r_-$ and $r_0$ are the number of positive, negative and zero eigenvalues of $B$, counted with multiplicity. Usually $r_0$ is omitted if it is zero, i.e., the form $B$ has zero kernel, in which case we say that $\Lambda$ is non-degenerate. We say $\Lambda$ is even if $u^2 = B(u, u) \in 2\mathbb{Z}$ for all $u \in \Lambda$. The dual lattice to $\Lambda$ is denoted $\Lambda^*$. The *discriminant* of a non-degenerate lattice is $|\det(B)| = |\Lambda^*/\Lambda|$. The lattice is said to be unimodular if its discriminant is 1. Note that in the literature, the discriminant is frequently defined to be $\det(B) = (-1)^{r_-} |\Lambda^*/\Lambda|$, in which case a lattice is unimodular iff its discriminant is $\pm 1$.

For a lattice $\Lambda$ and a real number $\alpha$, we denote by $\Lambda(\alpha)$ the lattice which has the same underlying group but with the bilinear form scaled by $\alpha$. The lattice of rank one with a generator of norm $\alpha$ will be denoted $\langle \alpha \rangle$.

A root of a positive definite lattice, we will mean an element $x$ such that $x^2 = 2$, whereas for a negative-definite or indefinite lattice, we will mean an element $x$ such that $x^2 = -2$.

A root lattice is a lattice that is generated (as an abelian group) by its roots. First, let us introduce some familiar root lattices, through their Dynkin diagrams. The subscript in the name of the lattice is the dimension of the lattice, which is also the number of (blank) nodes in the Dynkin diagram. Adding the extra (filled in) node gives the extended Dynkin diagram, which will arise later in connection with elliptic surfaces. The labels on the nodes are the coefficients of a vector in the kernel of the Cartan matrix of the extended Dynkin diagram.

![Dynkin Diagram](image)

Figure 1. $A_n (n \geq 1)$, signature $(n, 0)$, discriminant $n + 1$.

$A_n$ is the positive definite lattice with $n$ generators $v_1, \ldots, v_n$ with $v_i^2 = 2$ and $v_i \cdot v_j = -1$ if the vertices $i$ and $j$ are connected by an edge, and 0 otherwise. It may be realized as the set of integral points on the hyperplane $\{x \in \mathbb{R}^{n+1} | \sum x_i = 0 \}$.

$D_n$ can be realized as $\{x \in \mathbb{Z}^n | \sum_{i=1}^{n} x_i \equiv 0 \pmod{2} \}$. It has $2n(n-1)$ roots.

One realization of $E_8$ is as the span of $D_8$ and the all-halves vector $(1/2, \ldots, 1/2)$. It has 240 roots.
Taking the orthogonal complement of any root in $E_8$ gives us $E_7$. It has 126 roots. Taking the orthogonal complement of $e_1$ and $e_2$ in $E_8$, where $e_1, e_2$ are roots such that $e_1 \cdot e_2 = -1$, gives us $E_6$. It has 72 roots.

We let the Nikulin lattice $N$ be the lattice generated by $v_1, \ldots, v_8$ and $\frac{1}{2}(v_1 + \ldots + v_8)$, with $v_i^2 = -2$ and $v_i \cdot v_j = 0$ for $i \neq j$. It is isomorphic to $D_8^*(-2)$. It has signature $(0,8)$ and discriminant $2^6$. One checks easily that $N$ has 16 roots, namely $\pm v_i$. In particular, $N$ is not a root lattice.

Finally, let $U$ be the hyperbolic plane, i.e. the indefinite rank 2 lattice whose matrix is

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

3. K3 surfaces

3.1. Background. In this section we recall the definition and some basic properties of K3 surfaces. For more details we refer the reader to [1], [2] and [21]. Let $X$ be a smooth projective algebraic surface over a field $k$. 
Definition 2. We say that $X$ is a K3 surface if $H^1(X, \mathcal{O}_X) = 0$ and the canonical bundle of $X$ is trivial, i.e. $K_X \cong \mathcal{O}_X$.

In the sequel, we will assume that $k$ is the field $\mathbb{C}$ of complex numbers or some subfield of $\mathbb{C}$.

For a K3 surface, one can prove that the middle cohomology $H_X := H^2(X, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$, isomorphic to $E_8(-1)^2 \oplus U^3$.

The first Chern class map $H^1(X, \mathcal{O}_{X}^*) \to H^2(X, \mathbb{Z})$ is injective. Linear equivalence, algebraic equivalence, and numerical equivalence all coincide for an algebraic K3 surface.

Definition 3. The image of the classes of algebraic divisors $H^1(X, \mathcal{O}_{X}^*)$ in $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$ is a sublattice of $H^2(X, \mathbb{Z})$, which we call the Néron-Severi group of $X$, and denote by $\text{NS}(X)$ or $S_X$ or $\text{Pic}(X)$. The orthogonal complement of $\text{NS}(X)$ in $H^2(X, \mathbb{Z})$ is called the transcendental lattice $T_X$ of $X$.

We have the Hodge decomposition

$$H^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

with the vector spaces $H^{0,2}$ and $H^{2,0}$ being one-dimensional.

Definition 4. A $(-2)$-curve on a surface is an irreducible rational curve with self-intersection $-2$.

On a K3 surface, it is an easy exercise using the genus formula to see that curves of self-intersection $-2$ are exactly the smooth rational curves.

3.2. Kummer surfaces. Here we recall the construction of the Kummer surface associated to an abelian surface, which is an example of a K3 surface. More details may be found in [3],[11],[12] and [26].

Let $A$ be an abelian surface, and let $\iota$ be the involution which is multiplication by $-1$. Note that $\iota$ fixes any regular algebraic 2-form on $A$ (the space of 2-forms is one-dimensional since $h^0(X, K) = h^2(X, \mathcal{O}) = 1$). Now if we blow up the surface $A$ at its sixteen 2-torsion points, we get a surface $\tilde{A}$ on which $\iota$ extends to an involution $\tilde{\iota}$. The quotient surface $Y = \tilde{A}/\{1, \tilde{\iota}\}$ is in fact a K3 surface, and the regular 2-form is induced from the one on $A$. The surface $Y = \text{Km}(A)$ is called the Kummer surface of $A$. It has sixteen disjoint rational curves $F_1, \ldots, F_{16}$ which are the images of the sixteen exceptional divisors on $\tilde{A}$.

The Néron-Severi lattice of a Kummer surface has 16 linearly independent divisor classes coming from the sixteen rational curves above. These generate a negative definite lattice, and there is also a class of a polarization on $\text{Km}(A)$, since it is projective. Therefore its signature is $(1, r)$ for some $r \geq 16$. In fact, the Néron-Severi lattice always contains a particular lattice of signature $(0, 16)$ and discriminant $2^6$, called the Kummer lattice $K$. We describe its structure.

The set $I := A[2] = (\mathbb{Z}/2)^4 \cong \mathbb{F}_2^4$ of 16 elements has a natural structure of a vector space of dimension 4 over $\mathbb{F}_2$. Choose a labeling $I = \{1, 2, \ldots, 16\}$ (for instance, by writing $i - 1 = \sum_{j=0}^{3} b_j 2^j$ for $1 \leq i \leq 16$) and let $f_1, \ldots, f_{16}$ be the classes of the rational curves corresponding to the blowups at the 2-torsion points. Let $Q$ be the
set of 32 elements consisting of 30 affine hyperplanes (considered as subsets of $I$) as well as the empty set and all of $I$. This set has the structure of a vector space over $\mathbb{F}_2$, the addition operation being symmetric difference of sets. The set $Q$ is in fact the Reed-Muller code $\mathcal{R}(1, 4)$: the characteristic functions of the sets in $Q$, viewed as functions from $\mathbb{F}_2^4$ to $\mathbb{F}_2$, are exactly the polynomials in 4 variables of degree at most 1.

For every $M \in Q$, we have an element $f_M = \frac{1}{2} \sum_{i \in M} f_i$ of $\sum Q f_i \in \text{NS}(\text{Km}(A)) \otimes \mathbb{Q}$. These vectors actually lie in $\text{NS}(\text{Km}(A))$. The lattice generated by the $f_i$, $(i = 1, \ldots, 16)$ has discriminant $2^{16}/(2^5)^2 = 2^6$ and it is called the Kummer lattice.

In section 4, we will say more about the Kummer surface associated to a principally polarized abelian surface which arises as the Jacobian of a genus 2 curve.

3.3. Shioda-Inose structures. We now describe the data of a Shioda-Inose structure on a K3 surface.

**Definition 5.** An involution $\iota$ on a K3 surface $X$ is called a Nikulin involution if $\iota^*(\omega) = \omega$ for every $\omega \in H^{2,0}(X)$.

In fact, a Nikulin involution fixes $T_X$ pointwise. Every Nikulin involution has 8 isolated fixed points. As in the construction of the Kummer surface, we may blow up these points to get $\tilde{X}$, which has eight exceptional curves, and an involution $\tilde{\iota}$. The quotient $\tilde{X}/\{1, \tilde{\iota}\}$ of $X$ by a Nikulin involution is a K3 surface $Y$.

The images of the exceptional curves are $(-2)$-curves on the quotient K3 surface $Y$. The Néron-Severi lattice of $Y$ contains the Nikulin lattice $N$: it is generated by vectors $c_1, \ldots, c_8$ (the classes of the eight disjoint $-2$-curves) and $\frac{1}{2} \sum c_i$, with the form induced by $c_i \cdot c_j = -2\delta_{ij}$.

**Definition 6.** We say that $X$ admits a Shioda-Inose structure if there is a Nikulin involution $\iota$ on $X$ with rational quotient map $\pi: X \to Y$ such that $Y$ is a Kummer surface and $\pi_* \text{ induces a Hodge isometry } T_X(2) \cong T_Y$.

If $X$ has a Shioda-Inose structure, let $A$ be the abelian surface whose Kummer surface is $Y$. Then we have a diagram

$$
\begin{array}{ccc}
X & \searrow & A \\
\downarrow & & \downarrow \\
Y & \swarrow & \\
\end{array}
$$

of rational maps of degree 2, and Hodge isometries $T_X(2) \cong T_Y \cong T_A(2)$, thus inducing a Hodge isometry $T_X \cong T_A$.

The following theorem of Morrison characterizes K3 surfaces with Shioda-Inose structures in terms of the Néron-Severi lattice.

**Theorem 7** (Morrison [20]). Let $X$ be an algebraic K3 surface. The following are equivalent:

1. $X$ admits a Shioda-Inose structure.
2. There exists an abelian surface $A$ and a Hodge isometry $T_X \cong T_A$.
3. There is a primitive embedding $T_X \hookrightarrow U^3$.
4. There is an embedding $E_8(-1)^2 \hookrightarrow \text{NS}(X)$.
3.4. Elliptic K3 surfaces. We recall here a few facts about elliptic surfaces. References for these are [28] and [30].

**Definition 8.** An elliptic surface is a smooth projective algebraic surface $X$ with a proper morphism $\pi : X \to C$ to a smooth projective algebraic curve $C$, such that

1. There exists a section $\sigma : C \to X$.
2. The generic fiber $E$ is an elliptic curve.
3. $\pi$ is relatively minimal.

Concretely, we will be considering the case $C = \mathbb{P}^1$, since a K3 surface cannot have a non-constant map to a curve of positive genus. We will choose a Weierstrass equation for the generic fiber, which is an elliptic curve over the function field $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$, namely

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

where $a_i$ are rational functions of $t$. In fact, by multiplying $x$ and $y$ by suitable rational functions, we can make $a_i(t)$ polynomials in $t$. Furthermore, we may translate $x$ by a rational function, and $y$ by a $\mathbb{Q}(t)$-linear combination of $1$ and $x$. This can be done in such a way that the degree of the discriminant is minimal. We can read out some properties of the surface directly from the Weierstrass equation. For instance, if $p_a(X)$ is the arithmetic genus of $X$, then $p_a(X) + 1 = \chi(O_X)$ is the minimal $n$ such that $\deg a_i \leq ni$ for $i = 1, 2, 3, 4, 6$. In particular, for a K3 surface $X$, we need to have degree $a_i \leq 2i$ (the case $n = 1$ corresponds to a rational elliptic surface).

All but finitely many of the fibers of the elliptic surface are nonsingular and hence elliptic curves. Tate’s algorithm [31] allows us to compute the description of the singular fibers, according to the Kodaira-Néron classification. We note that the reducible fibers are unions of nonsingular rational curves with multiplicities. The dual graph of these components is an extended Dynkin diagram of type $A, D$ or $E$. The identity component is represented by the filled in node in the extended Dynkin diagram, whereas the multiplicities of the different components of the fiber are given by the labels on the nodes.

The Néron-Severi lattice of $X$ is generated by the classes of all the sections of $\pi$ (i.e. the Mordell-Weil group of $X$) considered as curves on the surface $X$, together with the class $F$ of a fiber, and all the non-identity components of the reducible fibers. Let $R = \{ v \in C(\mathbb{C}) \mid F_v$ is reducible $\}$, and for each $v \in R$, let $F_v = \pi^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v-1} \mu_{v,i} \Theta_{v,i}$, where $\Theta_{v,0}$ is the component which intersects the identity, and the other $\Theta_{v,i}$ are the non-identity components. Let $O$ be the zero section. The intersection pairing satisfies:

1. for any section $P$, $P^2 = O^2 = -\chi$,
2. $P \cdot F = O \cdot F = 1$,
3. $F^2 = 0$,
4. $O \cdot \Theta_{v,i} = 0$ for $i \geq 1$,
5. $\Theta_{v,i} \cdot \Theta_{w,j} = 0$ for $v \neq w$.

The intersection pairing for $\Theta_{v,i}$ and $\Theta_{v,j}$ is $-2$ if $i = j$, and $0, 1, 2$ and if $i \neq j$ according to the number of edges connecting the corresponding nodes in the extended
Dynkin diagram (2 occurs only for the types I\(_2\) and III, which have the extended Dynkin diagram of A\(_1\)). For a general section \(P\), the intersection pairing with each \(\Theta_{v,i}\) can be computed locally. In particular, for each \(v\) exactly one of the intersection numbers is 1, for some \(i\) such that \(\mu_{v,i} = 1\), and the others vanish. The rank of the Néron-Severi group is given by the formula

\[
\rho = r + 2 + \sum_v (m_v - 1).
\]

The discriminant of the sublattice \(T\) generated by the non-identity components of all the fibers is \(\prod_v m_v^{(1)}\), where \(m_v^{(1)}\) is the number of multiplicity one components of \(F_v\).

4. Genus 2 curves

4.1. Background: Moduli and invariants. Here we describe the basic geometry and moduli of curves of genus 2. For more background we refer the reader to [4], [5], [13], [22]. Let \(C\) be such a curve defined over a field \(k\) of characteristic zero. Then the canonical bundle \(K_C\) of \(C\) has degree 2 and \(h^0(C, K_C) = 2\). That is, the corresponding complete linear system is a \(g^1_2\) (and it is the unique \(g^1_2\)). We therefore have a map

\[
x : C \to \mathbb{P}^1
\]

which is ramified at 6 points by the Riemann-Hurwitz formula, and the function field of \(C\) is a quadratic extension of \(k(x)\). Therefore, we may write the equation of \(C\) as

\[
y^2 = f(x) = \sum_{i=0}^{6} f_i x^i.
\]

The roots of the sextic are the six ramification points of the map \(C \to \mathbb{P}^1\). Their preimages on \(C\) are the six Weierstrass points.

Now, the isomorphism class of \(C\) over \(\bar{k}\), the algebraic closure of \(k\), is determined by the isomorphism class of the sextic \(f(x)\), where two sextics are equivalent if there is a transformation in \(\text{PGL}_2(\bar{k})\) which takes the set of roots (considered inside \(\mathbb{P}^1\)) to the roots of the other.

Clebsch was the first to determine the invariants of binary sextics. He defined invariants of \(I_2, I_4, I_6, I_{10}\) of weights 2, 4, 6, 10 respectively. That is, \(I_d\) has degree \(d\) in the coefficients of \(f\), and if \(f\) transforms to \(g\) under the action of \(GL_2(\bar{k})\), then there is an element \(r \in \bar{k}\) such that \(I_d(g) = r^d I_d(f)\). Clebsch and Bolza showed that these invariants determined the sextic up to \(\bar{k}\)-equivalence. Therefore, the point \((I_2(f) : I_4(f) : I_6(f) : I_{10}(f))\) in weighted projective space determines the isomorphism class of \(C\). In fact, \(C\) and \(C'\) are isomorphic over \(k\) iff there is an \(r \in k^*\) such that \(I_d(f') = r^d I_d(f)\). Igusa generalized Clebsch’s theory to hold in all characteristics by choosing a different algebraic equation for the curve \(C\) (through an embedding as a quartic in \(\mathbb{P}^2\) with one node) and defining invariants \(J_2, J_4, J_6, J_8\) and \(J_{10}\). He thus obtained a moduli space of genus two curves defined over \(\text{Spec} \mathbb{Z}\). The invariants \(I_2, \ldots, I_{10}\) are called the Igusa-Clebsch invariants whereas \(J_2, \ldots, J_{10}\) are called the Igusa invariants of \(C\). We will use the former, since we are working over a field of characteristic zero.
Remark 9. If the Igusa-Clebsch invariants of a curve $C$ lie in a field $k$, it does not necessarily mean that $C$ can be defined over $k$: there is usually an obstruction in $Br_2(k)$. But $C$ can always be defined over a quadratic extension of $k$.

4.2. Kummer surface. Let $C$ be a curve of genus 2, which we can write as

$$y^2 = f(x) = \sum_{i=0}^{6} f_i x^i$$

Let $\theta_i, i = 1, \ldots, 6$ be the roots of the of the sextic, so that

$$f(x) = f_0 \prod_{i=1}^{6} (x - \theta_i).$$

We shall concern ourselves with the embedding of the singular Kummer surface as a quartic in $\mathbb{P}^3$, which comes from the complete linear system $2\Theta$, twice the theta divisor which defines the principal polarization. For a treatment of the quartic surface and the formulas we use, we refer the reader to [4],[7],[12] and [17]. The quartic is given by the equation

$$K(z_1, z_2, z_3, z_4) = K_2z_1^2 + K_1z_4 + K_0 = 0.$$  

where

$$K_2 = z_2^2 - 4z_1z_3,$$  

$$K_1 = -4z_1^3f_0 - 2z_1^2z_2f_1 - 4z_1^2z_3f_2 - 2z_1z_2z_3f_3 - 4z_1z_3^2f_4 - 2z_2z_3^2f_5 - 4z_3^2f_6,$$  

$$K_0 = -4z_1^3f_0f_2 + z_1^2f_1^2 + 4z_1^2z_2f_0f_3 - 2z_1^2z_3f_1f_3 - 4z_1z_2^2f_0f_4 + 4z_1z_2z_3f_0f_5 - 4z_1z_3^2f_0f_6 + 2z_1^2z_3^2f_1f_5 - 4z_1^2z_2f_2f_4 + z_1^2z_3^2f_3^2 - 4z_1z_3^2f_0f_5 + 8z_1z_2z_3f_0f_6 - 4z_1z_2^2z_3f_1f_5 + 4z_1z_2z_3^2f_1f_6 - 4z_1z_2z_3^2f_2f_5 - 2z_1z_3^2f_3f_5 - 4z_2^2f_0f_6 - 4z_2^2z_3f_1f_6 - 4z_2^2z_3^2f_2f_6 - 4z_2z_3^2f_3f_6 - 4z_3^2f_4f_6 + z_3^2f_5^2.$$  

The 16 singular points define ordinary double points on the quartic, which are called nodes. These are given explicitly by the coordinates

$$p_0 = (0 : 0 : 0 : 1)$$  

$$p_{ij} = (1: \theta_i + \theta_j: \theta_i\theta_j: \beta_0(i, j))$$  

for $1 \leq i < j \leq 6$.

Here $\beta_0(i, j)$ is defined as follows. Let

$$f(x) = (x - \theta_i)(x - \theta_j)h(x),$$  

with $h(x) = \sum_{n=0}^{4} h_n x^n$.

Then

$$\beta_0(i, j) = -h_0 - h_2(\theta_i\theta_j) - h_4(\theta_i\theta_j)^2.$$
The singular point \( p_0 \) comes from the 0 point of the Jacobian, whereas the \( p_{ij} \) comes from the 2-torsion point which is the difference of divisors \([\theta_i, 0] - [\theta_j, 0]\) corresponding to two distinct Weierstrass points on \( C \). The sixteen singular points are called **nodes**.

There are also sixteen hyperplanes in \( \mathbb{P}^3 \) which are tangent to the Kummer quartic. These are called **tropes**. Each trope intersects the quartic in a conic with multiplicity 2, and contains 6 nodes. Conversely, each node is contained in exactly 6 tropes. This beautiful configuration is called the \((16,6)\) Kummer configuration.

The explicit formulas for the tropes are as follows. Six of the tropes are given by

\[
\theta_i^2 z_1 - \theta_i z_2 + z_3 = 0.
\]

We call this trope \( T_i \). It contains the nodes \( p_0 \) and \( p_{ij} \). The remaining ten tropes are labeled \( T_{ijk} \) and correspond to partitions of \( \{1, 2, 3, 4, 5, 6\} \) into two sets of three, say \( \{i, j, k\} \) and its complement \( \{l, m, n\} \). Set

\[
G(X) = (x - \theta_i)(x - \theta_j)(x - \theta_k) = \sum_{r=0}^{3} g_r x^r,
\]

\[
H(X) = (x - \theta_l)(x - \theta_m)(x - \theta_n) = \sum_{r=0}^{3} h_r x^r.
\]

Then the equation of \( T_{ijk} \) is

\[
f_6(g_2 h_0 + g_0 h_2)z_1 + f_6(g_0 + h_0)z_2 + f_6(g_1 + h_1)z_3 + z_4 = 0.
\]

The Néron-Severi lattice of the nonsingular Kummer surface contains classes of rational curves \( E_0 \) and \( E_{ij} \) coming from the nodes, and \( C_i \) and \( C_{ijk} \) coming from the tropes. We will denote the lattice generated by these as \( \Lambda_{(16,6)} \). It has signature \((1,16)\) and discriminant \(2^6\) and is the Néron-Severi lattice of the Kummer surface of a generic principally polarized abelian surface.

Let \( L \) be the class of a hyperplane section. Furthermore, since \( T_{ijk} \) only depends on the partition \( \{1, 2, 3, 4, 5, 6\} = \{i, j, k\} \cup \{l, m, n\} \), we may assume \( i = 1 \) and set \( C_{jk} = C_{1jk} \) for \( 1 \leq j < k \leq 6 \). Also set \( C_{ij} = C_j \) for \( 1 < j \leq 6 \) and \( C_0 = C_1 \) to agree with the notation in [27]. We have the following intersection numbers and relations in the Néron-Severi lattice.

\[
\begin{align*}
L^2 &= 4, \\
E_0^2 &= -2, \\
E_{ij}^2 &= -2, \\
E_0 \cdot E_{ij} &= 0, \\
E_{ij} \cdot E_{kl} &= 0 \quad \text{for} \quad \{i, j\} \neq \{k, l\}, \\
C_0 &= \left( L - E_0 - \sum_k E_{1k} \right)/2, \\
C_{1j} &= \left( L - E_0 - \sum_{k \neq j} E_{jk} \right)/2, \\
C_{jk} &= \left( L - E_{ij} - E_{jk} - E_{ik} - E_{lm} - E_{mn} - E_{ln} \right)/2,
\end{align*}
\]

where \( \{l, m, n\} \) is the complementary set to \( \{1, j, k\} \).
Projection to a hyperplane from $p_0$ defines a 2 to 1 map of the Kummer to $\mathbb{P}^2$, and thus identifies the Kummer surface as a double cover of $\mathbb{P}^2$, ramified along the union of six lines, which are the projections of the conics $C_i$ (or the tropes $T_i$). The exchange of sheets gives an involution of the lattice, which acts by

$$
E_0 \mapsto 2L - 3E_0, \\
E_{ij} \mapsto E_{ij}, \\
L \mapsto 3L - 4E_0.
$$

We can explicitly write down the projection to $\mathbb{P}^2$ as $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$. The involution which is the exchange of sheets is $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, -x_4)$. Let $q_0, q_{ij}$ be the projections of the $p_0, p_{ij}$.

5. Family of elliptic K3 surfaces associated to genus 2 curves

5.1. Elliptic surfaces with $E_8$ and $E_7$ fibers. Let $X$ be an elliptic K3 surface with bad fibers of type $E_8$ and $E_7$ at $\infty$ and 0 respectively. A generic such K3 surface has a Néron-Severi lattice $\text{NS}(X) \cong U \oplus E_8(-1) \oplus E_7(-1)$ by Shioda’s explicit description of the Néron-Severi lattice of an elliptic surface. This lattice has rank 17, signature $(1,16)$ and discriminant 2. The transcendental lattice $T_X$ has rank 5, signature $(2,3)$ and discriminant 2. We deduce that $T_X \cong U^2 \oplus (-2)$.

The transcendental lattice of a generic principally polarized abelian surface, that is, the Jacobian $J(C)$ for $C$ a generic curve of genus 2, also satisfies the same property, since the Néron-Severi of $J(C)$ is generated by the theta divisor, which has self-intersection 2 by the genus formula on the abelian surface

$$2 = 2g - 2 = C \cdot (C + K) = C^2.$$

Therefore the orthogonal complement in $H^2(J(C), \mathbb{Z}) \cong U^3$ is exactly $U^2 \oplus (-2)$. We expect that the elliptic K3 surface $X$ has a Shioda-Inose structure such that the quotient by the Nikulin involution gives the Kummer surface of a principally polarized abelian surface $\text{Km}(J(C))$. Galluzzi, Lombardo and Dolgachev [10] prove that, in fact, $X$ corresponds to a unique $C$ up to isomorphism. However, an explicit identification of the quotient as a Kummer surface was not known. Below, we give an explicit construction of the correspondence.

We begin with the K3 surface $X$ given by the equation

$$y^2 = x^3 + t^3(at + a')x + t^5(b't^2 + bt + b').$$

It is easily checked that the surface $X$ has an $I_1^*$ or $E_8$ fiber at $t = \infty$ and a $I_1^*$ or $E_7$ fiber at $t = 0$. Now we describe another elliptic fibration on $X$.

Applying the transformation $(x, y, t) = (x't'^2/b'^2, y't'^2/b'^3, t'/b')$, we get

$$y^2 = t'^3 + (x'^3 + ax' + b)t'^2 + b''(a'x' + b')t'$$

and again replacing $(x', y', t')$ by $(x, y, t)$ for convenience gives finally

$$y^2 = t^3 + (x^3 + ax + b)t^2 + b''(a'x + b')t$$

which is an elliptic surface over the $x$-line with an $I_1^*$ or $D_{14}$ fiber at $x = \infty$, an $I_2$ or $A_1$ fiber at $x = -b'/a'$ and a 2-torsion section $(y, t) = (0, 0)$. The translation by the
2-torsion section is a Nikulin involution. We write down the isogenous elliptic surface $Y$ as

$$y^2 = t^3 - 2(x^3 + ax + b)t^2 + ((x^3 + ax + b)^2 - 4b''(a'x + b'))t.$$ 

This is an elliptic surface over the $x$-line with an $I_5^*$ or $D_9$ fiber at $x = \infty$ and $I_2$ or $A_1$ fibers at the roots of the sextic $(x^3 + ax + b)^2 - 4b''(a'x + b')$, and with a 2-torsion section $(t, y) = (0, 0)$. The Néron-Severi lattice of a generic such surface has signature $(1, 16)$ and discriminant $4 \cdot 2^6/2^2 = 2^6$. In fact, we will identify it with the Néron-Severi lattice of a generic Kummer surface (which we call the $(16, 6)$ lattice) in a later section. This will lead to the identification of the Kummer surface of $J(C)$ as an elliptic K3 surface with a bad fiber of type $I_5^*$ at $\infty$, with $I_2$ fibers at the roots of a sextic derived from $C$, and with a 2-torsion section.

**Remark 10.** If $E$ is an elliptic curve with a 2-torsion point, written in the Weierstrass form

$$y^2 = x^3 + ax^2 + bx$$

with a 2-torsion point $P = (0, 0)$, then the 2-isogenous curve $E' = E/\{O, P\}$ is given by

$$Y^2 = X^3 - 2aX^2 + (a^2 - 4b)X.$$ 

The isogeny $\phi : E \to E'$ is given by

$$(x, y) \to \left(\frac{y^2}{x^2}, \frac{y(b - x^2)}{x^2}\right).$$

The dual isogeny $\hat{\phi}$ is given by

$$(X, Y) \to \left(\frac{Y^2}{4X^2}, \frac{Y(a^2 - 4b - X^2)}{8X^2}\right).$$

### 5.2. Main theorem.

In this section, we state the main theorem, which gives the parameters $(a, a', b, b', b'')$ of the K3 elliptic surface with $E_8$ and $E_7$ fibers (so far conjecturally) associated to a genus 2 curve $C$ in terms of the Igusa-Clebsch invariants of $C$. The proof will be given in the following sections.

**Theorem 11.** Let $C$ be a curve of genus two, and $Y = \text{Km}(J(C))$ the Kummer surface of its Jacobian. Let $I_2, I_4, I_6, I_{10}$ be the Igusa-Clebsch invariants of $Y$. Then there is an elliptic fibration on $Y$ for which the Weierstrass equation may be written

$$y^2 = x^3 - 2\left(t^3 - \frac{I_4}{12} t + \frac{I_2 I_4 - 3I_6}{108}\right)x^2 + \left(\left(t - \frac{I_4}{12} t + \frac{I_2 I_4 - 3I_6}{108}\right)^2 + I_{10}\left(t - \frac{I_2}{24}\right)\right)x.$$ 

There is an elliptic K3 surface $X$ given by

$$y^2 = x^3 - t^3\left(\frac{I_4}{12} t + 1\right)x + t^5\left(\frac{I_{10}}{4} t^2 + \frac{I_2 I_4 - 3I_6}{108} t + \frac{I_2}{24}\right)$$

with fibers of type $E_8$ and $E_7$ at $t = \infty$ and $t = 0$ respectively, and a Nikulin involution on $X$, such that the quotient K3 surface is $Y$. 
Remark 12. Note that the correspondence of the K3 surface $X$ with the genus 2 curve $C$ is Galois invariant, i.e. $X$ is defined over the field of definition of $C$. The proof of the theorem will involve making some non-Galois invariant choices (for the level 2 structure of the Jacobian of $C$) but we will show that $X$ as well as the Shioda-Inose structure on $X$ is independent of these choices.

Remark 13. The Nikulin involution on $X$ may be written as follows:

$$(x, y, t) \mapsto \left( \frac{16x(-x + I_2t^2/24)^2}{I_2^{10}t^8}, \frac{-64y(-x + I_2t^2/24)^3}{I_3^{12}t^{12}}, \frac{4(-x + I_2t^2/24)}{I_{10}t^3} \right).$$

5.3. Néron-Severi lattices. In this and the next section, we give the details of how to put an elliptic fibration on the Kummer surface of a Jacobian of a curve of genus 2, with a 2-torsion section, and $I_2$ fibers. We first identify the Néron-Severi lattices involved (namely $\Lambda_{[16,6]}$ and $(D_9 \oplus A_1^6 \oplus U^+)$. We use the work of Naruki [27], which gives an embedding of the lattice $N \oplus E_8(-1)$ inside $\Lambda_{[16,6]}$.

First, we start with the Néron-Severi lattice of the K3 surface $X$ which has $E_8$ and $E_7$ fibers. The roots of the NS($X$) which correspond to the smooth rational curves on $X$ are drawn below (we use the notation from [10]). These are in fact all the roots corresponding to smooth rational curves on $X$, by a result of Nikulin [25].

There is an elliptic fibration on $X$ which has $R_8 + 2R_7 + 3R_6 + 4R_5 + 5R_4 + 6R_3 + 4R_2 + 2R_1 + 3R_0$ as an $II^*$ or $E_8$ fiber, $N_7 + 2N_6 + 3N_5 + 4N_4 + 3N_3 + 2N_2 + N_1 + 2N_0$ as a $III^*$ or $E_7$ fiber, and $S$ as the zero section. This is the fibration over $\mathbb{P}^1$. The fibration over $\mathbb{P}^1_2$ has the $I_{10}^*$ or $D_{14}$ fiber given by $R_0 + R_2 + 2(R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + S + N_1 + N_2 + N_3 + N_4) + N_0 + N_5$, an $I_2$ or $A_1$ fiber $A + N_7$, a 2-torsion section (say $R_1$) and a zero section $N_6$.

The Nikulin involution $\sigma$ is translation by the 2-torsion section. It reflects the above picture about its vertical axis of symmetry. There are two obvious copies of $E_8(-1)$ switched by $\sigma$, namely the sublattices of NS($X$) generated by the roots $\{S, N_1, N_2, N_3, N_4, N_5, N_6\}$ and $\{R_7, R_6, R_5, R_4, R_3, R_0, R_2, R_1\}$. Next, we write down some roots on NS($Y$), where $Y$ is the quotient K3 surface of $X$ by the involution. As we have described, $Y$ has six $I_2$ or $A_1$ fibers $Q_{13} + Q_{14}, \ldots, Q_{23} + Q_{24}$, a $I_5^*$ or $D_9$.
fiber, namely $Q_1 + Q_2 + 2(Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8) + Q_9 + Q_{10}$, with zero section $O = Q_{11}$ and a 2-torsion section $T = Q_{12}$.

It is easily checked that the rational components of the $E_8$ fiber described above map as following: $N_6 \mapsto O$ (recall that $N_6$ is the zero section of the $D_{14}$ fibration, on which the quotient map is an isogeny of elliptic surfaces), $N_5 \mapsto Q_9, N_4 \mapsto Q_8, N_0 \mapsto Q_{10}, N_3 \mapsto Q_7, N_2 \mapsto Q_6, N_1 \mapsto Q_5, S \mapsto Q_4$. Hence, we see a natural copy of $E_8$ within the Néron-Severi lattice of $Y$. On the other hand, we can also see eight roots orthogonal to all the generators of $E_8$ as well as to each other, namely $Q_{14}, Q_{16}, Q_{18}, Q_{20}, Q_{22}, Q_{24}, Q_1$ and $Q_2$.

Now, we use the calculation of [27] which gives an explicit embedding of $N \oplus E_8(-1)$ inside the Néron-Severi lattice of a Kummer surface of a generic principally polarized abelian surface, or $\Lambda_{(16,6)}$. We extend this embedding to get an identification of $\Lambda_{(16,6)}$ with $\text{NS}(Y)$, i.e. the lattice generated by the roots in the diagram above.

The identification is as follows:

Here $\alpha(C_{23}) = C_{23} + L - 2E_0$. 
The class of the fiber is
\[ F = C_{23} + \alpha(C_{23}) + 2(E_{23} + C_{12} + E_{26} + C_{16} + E_{16} + C_0) + E_{15} + E_{14} \]
\[ = 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45}) \]
and \( e_1, \ldots, e_6, f_1, \ldots, f_6 \) are given by
\[
\begin{align*}
e_1 &= (L - E_0) - (E_{12} + E_{46}) \\
e_2 &= 2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{46} + E_{56}) \\
e_3 &= 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{36} + E_{45} + E_{46} + E_{56}) \\
e_4 &= 4(L - E_0) - 2(E_{12} + E_{13} + E_{46}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{56}) \\
e_5 &= 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{34} + E_{36} + E_{45}) \\
e_6 &= E_{45} \\
f_i &= F - e_i \text{ for all } i \\
f_1 &= (L - E_0) - (E_{12} + E_{56}) \\
f_2 &= 2(L - E_0) - (E_{12} + E_{13} + E_{25} + E_{46} + E_{56}) \\
f_3 &= 3(L - E_0) - 2E_{12} - (E_{13} + E_{25} + E_{36} + E_{45} + E_{46} + E_{56}) \\
f_4 &= 4(L - E_0) - 2(E_{12} + E_{13} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{46}) \\
f_5 &= 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{34} + E_{36} + E_{45}) \\
f_6 &= E_{34}.
\]

Notice that under the simple transposition \((45)\) of indices we have the permutation of fibers \( \tau = (14)(23)(56) \) and in fact \( e_i \mapsto f_{\tau(i)}, f_i \mapsto e_{\tau(i)} \).

5.4. **Completion of proof.** Next, we describe how to use all this information from the Néron-Severi group to construct \( x, y \) and \( t \) in the Weierstrass equation for \( Y = \text{Km}(J(C)) \):
\[ y^2 = x^3 + a(t)x^2 + b(t)x. \]
Consider the class of the fiber \( F \in \text{NS}(\text{Km}(J(C))) \), given by
\[ F = 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45}). \]
We can write down the parameter on the base by computing explicitly the sections of \( H^0(Y, \mathcal{O}_Y(F)) \). This linear system consists of (the pullback of) quintics passing through the points \( q_0 \) and \( q_{ij} \) which pass through \( q_{24}, q_{25}, q_{36}, q_{45} \), having a double point at \( q_{13}, q_{46}, q_{56} \) and a triple point at \( q_{12} \). This linear system is two-dimensional, and taking the ratio of two linearly independent sections gives us the parameter \( t \) on the base, \( \mathbb{P}^1 \), for the elliptic fibration. Now, \( t \) is only determined up to the action of \( \text{PGL}_2 \), but the first restriction we make is to put the \( I_5^* \) fiber at \( t = \infty \), which fixes \( t \) up to affine linear transformations. Any elliptic K3 surface with a 2-torsion section can be written in the form
\[ y^2 = x^3 - 2q(t)x^2 + p(t)x \]
with \( p(t) \) of degree at most 8 and \( q(t) \) of degree at most 4. The 2-torsion section is \((x, y) = (0, 0)\). The discriminant of this elliptic surface is a multiple of \( p^2(q^2 - p) \). In fact, we see that \( p \) must have degree exactly 6, because there are six \( I_2 \) fibers that the zero and 2-torsion sections meet in different components. The positions \( t_1, \ldots, t_6 \) of the \( I_2 \) fibers are the roots of the polynomial \( p(t) = p_0 \prod_{i=1}^{6} (t - t_i) \). Now \( t \) is determined.
up to transformations of the form $t \mapsto at + b$. To have exactly a $I^*_5$ fiber at $\infty$, we must have $p(t) = q(t)^2 + r(t)$ where $q(t)$ is a monic cubic polynomial and $r(t)$ is a linear polynomial in $t$. We can further fix $t$ up to scalings $t \mapsto at$ by translating $t$ so that the quadratic term of $q(t)$ vanishes. We notice that the top coefficient $p_0$ of $p(t)$ is a square, and so by scaling $t, x, y$ appropriately, we may assume $p_0 = 1$, i.e. that $p(t)$ and $q(t)$ are monic.

Now we describe how to obtain $x$. It is a Weil function, so that the horizontal component of its divisor equals $2T - 2O$, and the vertical component is uniquely determined by that fact that $(x)$ is linearly (and hence numerically) equivalent to zero. So we deduce that the divisor of $x$ is $2T - 2O + Q_{10} - Q_9 + Q_{14} + Q_{16} + Q_{18} + Q_{20} + Q_{22} + Q_{24} - 3F_0$, where

$$F_0 = Q_1 + Q_2 + 2(Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8) + Q_9 + Q_{10}$$

is the $D_9$ fiber.

To convert this to formulas, we compute the functions which cut out $Q_{16}, \ldots, Q_{24}$, $F_0$, $T$, and $O$. There is a quintic $s_1$ which cuts out $O = C_{14}$. Now, notice that the $D_9$ fiber contains $C_{12}, C_{16}$ and $C_0$. Therefore $s_1$ is divisible by $T_2, T_6$ and $T_1$. We write

$$s_1 = q_1T_1T_2T_6$$

with a quadratic $q_1$. Next, we know that $T_3$ cuts out $C_{14} = O$ and $T_5$ cuts out $C_{15} = T$. To find, for instance, the function which cuts out $e_2$, we find the quadratic (unique up to constants) which passes through $q_{12}, q_{13}, q_{24}, q_{46}, q_{56}$. Call this function $e_2$, by abuse of notation. Similarly, we find $e_1, \ldots, e_5$. We also note that the factor of $T_5$ in the numerator of $x$, which gives a zero along $T$, also gives a zero along $e_6 = E_{35}$ owing to the fact that $T = C_{15}$ intersects $E_{35}$ (recall that we are working with the singular Kummer surface, on which the image of the curve $E_{35}$ is just a single point).

Putting everything together, we can write $x$ up to scaling as a quotient of two homogeneous polynomials of degree 16 as follows:

$$x = \frac{e_1e_2e_3e_4e_5T_5}{s_1^3T_4} = \frac{e_1e_2e_3e_4e_5T_5}{(T_1T_2T_6q_1)^3T_4}$$

Finally, we have to scale $x$ and $t$ so that $x^3 + a(t)x^2 + b(t)x$ becomes a square of a function $y$ on the Kummer.

We note that in the equation of the Kummer

$$K_2z_4^2 + K_1z_4 + K_0 = 0$$

we can complete the square for $z_4$ to obtain

$$(K_2z_4 + K_1/2)^2 = K_1^2/4 - K_0K_2 = 4T_1T_2T_3T_4T_5T_6$$

We let $y$ be a constant multiple of

$$\frac{e_1e_2e_3e_4e_5(K_2z_4 + K_1/2)}{T_1^3T_2^3T_3^4T_4^2q_1^2}$$

a quotient of two homogeneous polynomials of degree 18, and verify that this makes the Weierstrass equation hold. The computation is carried out in a Maxima program, which is available from the arXiv.org e-print archive. This paper is available as math.AG/0701669. To access the auxiliary file, download the source file for the paper.
That will produce not only the \LaTeX{} files for the paper but also the computer algebra code. The code takes about half an hour to run on a 2.1 GHz computer.

We noted earlier that the permutation (45) on the \(A_1\) fibers by \(\tau = (14)(23)(56)\), and takes \(e_i\) to \(f_{\tau(i)}\). That is, it switches the components intersecting the identity and 2-torsion sections as well. In addition, it switches the zero section \(C_{14}\) and the 2-torsion section \(C_{15}\), and on the \(D_9\) fibers it switches the two near leaves \(E_{15}\) and \(E_{14}\), namely, again the components intersecting \(T\) and \(O\). On the other hand, consider the action on \(\text{NS}(Y)\) induced by the translation by \(T\). Under this map, \(T\) and \(O\) get swapped, the 2-torsion and identity components of the \(D_9\) and \(A_1\) fibers all get switched, and the far leaves of the \(D_9\) fiber also get switched (this can be seen, for instance, from the fact that the simple components of the special \(D_9\) fiber form a group compatible with the group law on the generic elliptic curve). The locations of the \(A_1\) fibers themselves are fixed. Therefore the effect of the permutation 45 is the same as translation by 2-torsion composed with a pure involution \((14)(23)(56)\) of the \(A_1\) fibers and a switch of the far leaves of the \(D_9\) fiber. Since the far leaves of the \(D_9\) fiber are switched by the Galois involution that multiplies the square root of \(b'' = I_{10}/4\) by \(-1\), this tells us that we have the correct quadratic twist, since \(I_{10}\) is within a square factor of the discriminant of the sextic. That is, making a different choice of 2-level structure while describing the fibration on the Kummer surface \(Y\) would have given us the same answer for \(X\).

6. An application to the moduli of six points in \(\mathbb{P}^1\)

The construction above gave us a correspondence of sextics

\[
f(x) = \sum f_ix^i = f_6 \prod (x - x_i)
\]

and

\[
g(x) = \left(\frac{x^3}{12} - \frac{I_4}{12}x + \frac{I_2I_4 - 3I_6}{108}\right)^2 + \frac{I_1}{I_{10}}\left(x - \frac{I_2}{24}\right).
\]

Therefore, over an algebraically closed field, we get a birational map from the moduli space of 6 points in \(\mathbb{P}^1\) (i.e. the quotient of \((\mathbb{P}^1)^6\) under the action of \(\text{PGL}_2\) and \(S_6\)) with the space of roots up to scaling of

\[
(x^3 + ax + b)^2 + (a'x + b')
\]

as \(a, b, a', b'\) vary (we suppressed \(b''\) since it just scales \(a'\) and \(b')\). This latter space is cut out inside \(\mathbb{P}^5 = \{(X_1 : X_2 : X_3 : X_4 : X_5 : X_6)\}\) by the hyperplane \(\sigma_1(X) = X_1 + \ldots + X_6 = 0\) and the quartic hypersurface \(\sigma_2(X)^2 = 4\sigma_4(X)\), where \(\sigma_2\) and \(\sigma_4\) are the second and fourth elementary symmetric functions of the \(X_i\). Thus, we get a model as a singular quartic threefold in \(\mathbb{P}^4\), which is known in the literature as the Igusa quartic. Here the Igusa quartic arises naturally in the context of Shioda-Inose structures on K3 surfaces.

There is no simple one-one correspondence between the roots of \(f(x)\) and \(g(x)\), since the two actions of \(S_6\) acting by the permutation representation on the six roots of \(f(x)\) on the six roots of \(g(x)\) are related by an outer automorphism. To see this, we recall from the last section that the permutation (45) on the roots of \(f(x)\) (the Weierstrass points) acts on the roots of \(g(x)\) (which are the locations of the \(A_1\) fibers)
by the permutation (14)(23)(56). By symmetry, all the transpositions of $S_6$ act by a
product of three transpositions on the roots of $g(x)$. Thus we get a homomorphism
$S_6(f) \to S_6(g)$ which is an outer automorphism.

The curve $W$ defined by the equation $g(x) = 0$ is, as explained in [10], the genus 2
component of the fixed locus of the involution $\tau$ which is associated to the K3
surface $X$ by virtue of the fact that $\text{NS}(X)$ has a 2-elementary discriminant group
[25]. Explicitly, the involution $\tau$ is $(x, y, t) \mapsto (x, -y, t)$ on the model of $X$ with $D_{14}$
fiber, an $A_1$ fiber and a 2-torsion section:

$$y^2 = t^3 + \left(x^3 - \frac{I_4}{12}x + \frac{I_2I_4 - 3I_6}{108}\right)t^2 - \frac{I_{10}}{4}\left(x - \frac{I_2}{24}\right)t.$$  

Consider the map $\phi$ from the moduli space $M_2$ of genus 2 curves to itself which
takes $C = V(f)$ to $W = V(g)$. That is, we consider $g$ not only up to scaling of
roots, but up to the action of all elements of $\text{PGL}_2$ (on the roots) which fix the form
$g = p^2 + q$, where $p$ is a monic cubic polynomial with zero constant term, and $q$ is a
linear polynomial. Then, as noted in [10], $\phi$ is a rational self-map of $M_2$ of degree 16.

7. Compatibility with formulas for $\text{Km}(E_1 \times E_2)$

Here we check that the formulas we obtain degenerate to those of Shioda [29] for the
elliptic K3 surface corresponding to the Kummer surface of a product of two elliptic
curves, over an algebraically closed base field. For the rest of this section, let us fix
this field $k$.

It is shown in [29] that given two elliptic curves $E_1$ and $E_2$ with $j$-invariants $j_1$
and $j_2$, there is an elliptic K3 surface with two $E_8$ fibers which is 2-isogenous to
$\text{Km}(E_1 \times E_2)$. Its equation is given by

$$y^2 = x^3 - 3\alpha x + \left(t + \frac{1}{t} - 2\beta\right)$$

with $\alpha = \sqrt[3]{j_1j_2}$ and $\beta = \sqrt{(1 - j_1)(1 - j_2)}$. The choice of the square root and the
cube root here are arbitrary (since changing $x$ by a cube root of unity multiplies $\alpha$ by
it, whereas changing $y$ by a square root of $-1$ and $x$ and $t$ by $-1$ changes the sign of
$\beta$). Notice that Shioda’s $j$ is related to the $J$-invariant more commonly employed in
the literature by $j = J/1728$.

Now, the Igusa-Clebsch invariants $I_1, I_4, I_6, I_{10}$ (or $A, B, C, D$ in Igusa’s [14]
notation) define a weighted projective space. The $k$-points of this space with $I_{10} \neq 0$
correspond to curves of genus 2, therefore also their Jacobians with canonical po-
larizations. Blowing up around the point $I_4 = I_6 = I_{10} = 0$ gives a variety which
parametrizes Jacobians of genus 2 curves as well as their degenerations, products of
two elliptic curves. Igusa defines the coordinates

$$x_1 = \frac{2^{1/3}B}{A^2}, \quad x_2 = \frac{2^{6/3}(3C - AB)}{A^3}, \quad x_3 = \frac{2^{1/3}D}{A^5}$$

$$y_1 = \frac{x_1^3}{x_3}, \quad y_2 = \frac{x_2^3}{x_3}, \quad y_3 = \frac{x_1^3 x_2}{x_3}.$$
The point corresponding to a product of two elliptic curves with \( J \)-invariants \( J_1 \) and \( J_2 \) has \( x_1 = x_2 = x_3 = y_3 = 0 \) and \( y_1 = J_1 J_2, y_2 = (J_1 - 1728)(J_2 - 1728) \).

Using these formulas Shioda’s elliptic surface may be written as

\[
y^2 = x^3 - \frac{3}{12^2} \sqrt[3]{y_1} x + \left( t + \frac{1}{t} - \frac{2}{12^2} \sqrt{y_2} \right).
\]

Now let us transform the K3 surface

\[
y^2 = x^3 - t^3 \left( \frac{I_4}{12} t + 1 \right) x + t^5 \left( \frac{I_{10} t^2}{4} + \frac{I_2 I_4 - 3 I_6}{108} t + \frac{I_2}{24} \right)
\]

by first setting \( y = y t^6, x = x t^4 \). This gives the equation

\[
y^2 = x^3 - \left( \frac{I_4}{12} + \frac{1}{t} \right) x + \left( \frac{I_{10} t}{4} + \frac{I_2 I_4 - 3 I_6}{108} + \frac{I_2}{24 t} \right).
\]

Next, if we scale \( x, y, t \) as \( y = y/\lambda^3, x = x/\lambda^2, t = \mu t \), we get

\[
y^2 = x^3 - \lambda^4 \left( \frac{I_4}{12} + \frac{1}{\mu t} \right) x + \lambda^6 \left( \frac{I_{10} \mu t}{4} + \frac{I_2 I_4 - 3 I_6}{108} + \frac{I_2}{24 \mu t} \right).
\]

Now we let

\[
\mu = \left( \frac{I_2}{6 I_{10}} \right)^{1/2}, \quad \lambda = \left( \frac{96}{I_2 I_{10}} \right)^{1/12}.
\]

On simplification, the equation of the K3 surface becomes

\[
y^2 = x^3 - \left( \frac{3}{12^2} \sqrt[3]{\frac{2113 B^3}{AD}} + \frac{1}{t} \left( \frac{2^2 D}{3^2 A^3} \right)^{1/6} \right) x + \left( t + \frac{1}{t} + \frac{2}{12^3} \sqrt{\frac{2113 (AB - C^2)}{AD}} \right)
\]

or

\[
y^2 = x^3 - \left( \frac{3}{12^2} \sqrt[3]{y_1} + \frac{1}{t} \left( \frac{2 x_3}{3^7} \right)^{1/6} \right) x + \left( t + \frac{1}{t} + \frac{2}{12^3} \sqrt{y_2} \right).
\]

Now, noticing that \( x_3 = 0 \) for a product of two elliptic curves, we get Shioda’s formula, but with the sign of \( \beta \) changed. However, as we remarked earlier, this sign can be twisted away. Therefore, our formulas indeed degenerate to those of Shioda.

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