Hodge-elliptic genera and how they govern K3 theories

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Abstract

The (complex) Hodge-elliptic genus and its conformal field theoretic counterpart were recently introduced by Kachru and Tripathy, refining the traditional complex elliptic genus. We construct a different, so-called chiral Hodge-elliptic genus, which is expected to agree with the generic conformal field theoretic Hodge-elliptic genus, in contrast to the complex Hodge-elliptic genus as originally defined.

For K3 surfaces $X$, the chiral Hodge-elliptic genus is shown to be independent of all moduli. Moreover, employing Kapustin’s results on infinite volume limits it is shown that it agrees with the generic conformal field theoretic Hodge-elliptic genus of K3 theories, while the complex Hodge-elliptic genus does not. This new invariant governs part of the field content of K3 theories, supporting the idea that all their spaces of states have a common subspace which underlies the generic conformal field theoretic Hodge-elliptic genus, and thereby the complex elliptic genus. Mathematically, this space is modelled by the sheaf cohomology of the chiral de Rham complex of $X$. It decomposes into irreducible representations of the $N = 4$ superconformal algebra such that the multiplicity spaces of all massive representations have precisely the dimensions required in order to furnish the representation of the Mathieu group $M_{24}$ that is predicted by Mathieu Moonshine. This is interpreted as evidence in favour of the ideas of symmetry surfing, which have been proposed by Taormina and the author, along with the claim that the sheaf cohomology of the chiral de Rham complex is a natural home for Mathieu Moonshine.

These investigations also imply that the generic chiral algebra of K3 theories is precisely the $N = 4$ superconformal algebra at central charge $c = 6$, if the usual predictions on infinite volume limits from string theory hold true.
1 Introduction and summary

In conformal field theory (CFT), mathematical structures that have a counterpart in geometry play a key role. The success stories of orbifolding techniques [Hae90, Thu97, DHVW85, DHVW86] and mirror symmetry [LVW89, GP90, CdGP91, CLS90, Kon95] are examples of this. As a common feature, there are striking correspondences that allow to recover geometric invariants, on the one hand, in terms of quantum numbers in conformal field theory, on the other. Though rooted in string theory, the study of the relevant invariants and their imprint in conformal field theory is a fruitful mathematical enterprise, independently of string theory.

A concrete example of such a correspondence identifies the Euler characteristic of a compact Calabi-Yau manifold $X$ with the Witten index of an associated conformal field theory, obtained as non-linear sigma model on $X$ [Wit82]. Indeed, the predictive power of string theory motivates this correspondence [Wit82, LVW89], since a large volume limit of the sigma model is expected to recover the cohomology of the target manifold $X$. More generally, the complex elliptic genus [Hir88, Wit88, Kri90] of $X$, which can be defined as the holomorphic Euler characteristic of a certain virtual bundle $E_{q,-y} \rightarrow X$, is expected to be recovered from the part of the partition function of the associated superconformal field theory which a topological half-twist [EY90, Wit92] projects to $X$ [AKMW87, EOTY89, DY93, Wit94].

Recently, Kachru and Tripathy have defined a very interesting refined version of the complex elliptic genus of a compact Calabi-Yau manifold $X$, which they call the Hodge-elliptic genus [KT17]. The key idea is to introduce an additional parameter which keeps track of the grading on the cohomology of the virtual bundle $E_{q,-y} \rightarrow X$ that underlies the complex elliptic genus. This grading, in turn, has as its counterpart the natural grading of the space of states of an associated superconformal field theory by the right-moving $U(1)$ charge. Indeed, the Hodge-elliptic genus has a natural counterpart in conformal field theory, which is also introduced in [KT17], and which we call the conformal field theoretic Hodge-elliptic genus. It is important to keep these two versions of the Hodge-elliptic genus apart, as indeed they disagree, most of the time. To make the difference clearer, we will add the adjective complex to Kachru and Tripathy’s Hodge elliptic genus.

Both these new quantities are very promising, since the additional grading eliminates the typical cancellations that make it so difficult to reconstruct data from the complex elliptic genus and its conformal field theoretic counterpart. But preventing such cancellations, in general, causes a dependence on the moduli. Indeed, the complex Hodge-elliptic genus of a general compact Calabi-Yau manifold $X$ in dimensions greater than two is expected to depend on
the complex structure of $X$, while it is independent of the Kähler structure. We thus obtain meaningful geometric invariants by insisting on a fixed complex structure for any of our Calabi-Yau manifolds. For complex tori and for K3 surfaces, however, the complex Hodge-elliptic genus is independent of the complex structure \cite{KT17}.

The conformal field theoretic Hodge-elliptic genus, on the other hand, always severely depends on all the moduli. By definition, it is a power series in three formal variables with integral exponents. Its coefficients, up to signs, are just the dimensions of eigenspaces of certain natural linear operators on the space of states of the conformal field theory. The moduli dependence is thus reflected in a jumping behaviour of the coefficients. In particular, each such coefficient \textit{generically} attains its minimal value, on the moduli space, while it may jump to a higher value at non-generic points. To obtain a quantity that stands any chance of relating to some invariant geometric counterpart, instead of the conformal field theoretic Hodge-elliptic genus, one should therefore consider a \textit{generic conformal field theoretic Hodge-elliptic genus}. We shall define such a quantity, below, as is certainly in the spirit of \cite{KT17}. In string theory language, our definition amounts to an \textit{infinite volume limit} of the conformal field theoretic Hodge elliptic genus. However, we claim that the generic conformal field theoretic Hodge-elliptic genus of a CFT with geometric interpretation on some Calabi-Yau manifold $X$ also differs from the complex Hodge-elliptic genus of $X$, in general.

The reason for this discrepancy lies in the very definition of the complex Hodge-elliptic genus for a compact Calabi-Yau manifold $X$ by means of the cohomology of the virtual bundle $E_{q,-y} \to X$. Indeed, there is no reason to expect the cohomology of this bundle to describe conformal field theory data, not even in an infinite volume limit. According to Kapustin’s seminal insights \cite{Kap05}, such an infinite volume limit of a topological half-twist of a sigma model on $X$ should yield the sheaf cohomology of the \textit{(holomorphic) chiral de Rham complex} $\Omega_X^{\text{ch}}$ on $X$, instead. The sheaf of sections of the virtual bundle $(-y)^{\bullet} E_{q,y}$ is isomorphic to the \textit{graded object} of the chiral de Rham complex for a natural filtration of $\Omega_X^{\text{ch}}$. Therefore, the graded Euler characteristics of the two agree, but the separate degrees $H^i(X, E_{q,-y})$ of the cohomology, which enter crucially in the definition of the complex Hodge-elliptic genus, need not be isomorphic to those of $\Omega_X^{\text{ch}}$.

It is therefore natural to replace the complex Hodge-elliptic genus by a new invariant which is directly obtained from the sheaf cohomology of the \textit{(holomorphic) chiral de Rham complex} of $X$, by introducing an additional grading. That this is possible is shown in this note, and we call the resulting quantity the \textit{chiral Hodge-elliptic genus}. By construction, it does not depend on the Kähler structure of $X$, but it may depend on the complex structure. In general, we expect the chiral Hodge-elliptic genus of any compact Calabi-Yau manifold $X$ to agree with the generic conformal field theoretic Hodge-elliptic genus of CFTs with geometric interpretation by $X$ at some fixed complex structure. This allows to predict the behaviour of the chiral Hodge-elliptic genus under mirror symmetry.

The new quantities introduced so far turn out to be particularly useful in the context of K3 surfaces and K3 theories. Note that in the K3 setting, the notions of moduli spaces of CFTs and geometric interpretations are well understood \cite{Sei88, Cec91, AM94, NW01}, even if these concepts may seem a bit vague for other Calabi-Yau manifolds $X$.

Similarly to the complex Hodge-elliptic genus of \cite{KT17}, the chiral Hodge-elliptic genus of a K3 surface $X$ is independent of the complex structure, as is shown in this note by an explicit calculation. Under the assumption that the generic chiral algebra of all K3 theories is the $N = 4$ superconformal algebra at central charge $c = 6$, in addition, we show that as expected, the generic conformal field theoretic Hodge-elliptic genus of K3 theories agrees with the chiral Hodge-elliptic genus of K3 surfaces, while it disagrees with the complex Hodge-elliptic genus of \cite{KT17}. Note that a failure of the above assumption on the generic chiral algebra of K3 theories would be highly interesting in itself. Developing the representation theory of the relevant enhancement of the $N = 4$ superconformal algebra would amount to a major advance towards the construction of K3 theories beyond the families of examples that are known, so far. However, the investigation of Hodge-elliptic genera on K3 produces further pieces of evidence in favour of the expectation that the generic chiral algebra of K3 theories is \textit{not} extended beyond
the $N = 4$ superconformal algebra. Indeed, in this note we show that this already follows from the claims made in Kapustin’s work \cite{Kap05} on infinite volume limits.

Let us therefore assume, temporarily, that the generic chiral algebra of K3 theories is not enhanced beyond the $N = 4$ superconformal algebra at central charge $c = 6$. That the chiral Hodge-elliptic genus of K3 surfaces is an invariant which agrees with the generic conformal field theoretic Hodge-elliptic genus of K3 theories then means that certain quantities of our K3 theories are protected by this invariant. Concretely, one may work under the assumption that all K3 theories share a common space $\hat{\mathbb{H}}^R$ of protected states, all of which are Ramond ground states with respect to the right-moving superconformal algebra. At this level, $\hat{\mathbb{H}}^R$ is just a subrepresentation of the Ramond sector of our theory under the action of the $N = 4$ superconformal algebra at central charge $c = 6$, extended by the zero mode of the $U(1)$ current of the right-moving superconformal algebra. It is left for future work to equip it with further structure and to decide whether or not this smoothly varies with respect to the moduli. The sheaf cohomology of the (holomorphic) chiral de Rham complex of a K3 surface $X$ serves as a model for the subspace of protected states which is related to $\hat{\mathbb{H}}^R$ by spectral flow. Moreover, by explicitly determining the generic conformal field theoretic Hodge-elliptic genus, we can show that as a representation of the $N = 4$ superconformal algebra, $\hat{\mathbb{H}}^R$ splits into a direct sum of irreducible representations whose multiplicity spaces have precisely the right dimensions to accommodate Mathieu Moonshine according to \cite{EOT11, Che10, GHV10b, GHV10a, EH11, Gan16}. In particular, the multiplicity spaces of massive representations are never virtual, as is required by the results of \cite{Gan16}. Using the sheaf cohomology of the chiral de Rham complex of $X$ as a mathematically more established model, this means that the latter is the natural home for Mathieu Moonshine, in accord with \cite[\S 4.2]{Wen15}. In fact, this implication holds true independently of the above assumption on the generic chiral algebra of K3 theories. As such, our final conclusion agrees with results of Bailin Song in \cite{Son17}.

These findings support the idea of symmetry surfing, as proposed by Taormina and the author starting in \cite{TW13, LW15a}. In particular, in \cite{TW15a}, we show how to construct the leading order massive representation of Mathieu Moonshine for a maximal subgroup of $M_{24}$, implementing a twist. Our construction is based on a space of states that is common to all K3 theories obtained by a standard $Z_2$-orbifolding from a theory with target a complex torus, which we now may recover as subspace of the protected space $\hat{\mathbb{H}}^R$, above. Further evidence in favour of this idea is provided in \cite{GKPT17}.

Altogether, the chiral Hodge-elliptic genus of K3 turns out to be a surprisingly useful new invariant. It truly refines the complex elliptic genus to a three-parameter function in $(\tau, z, \nu) \in \mathfrak{f} \times \mathbb{C}^2$ which is still elliptic in $z$ with respect to $\Lambda_\tau = Z\tau + Z$, and which is Mock modular in $\tau$. The behaviour in the new parameter is polynomial in $u = \exp(2\pi i u)$ and $u^{-1}$. On the other hand, the meaning of the complex Hodge-elliptic genus of \cite{KTT17} for conformal field theory is not so obvious. Our findings may well bear some implications on the black hole counting formulas proposed in \cite{KTT17}, which were built on the assumption that the complex Hodge elliptic genus agrees with the generic conformal field theoretic Hodge-elliptic genus, which however contradicts Kapustin’s results on large volume limits, as we show. Nevertheless, we expect the complex Hodge-elliptic genus of \cite{KTT17} to be just as useful in geometry. We therefore also provide an implicit formula for the complex Hodge-elliptic genus of K3 surfaces, below.

In more detail, the structure of this note is as follows:

\footnote{The preprint \cite{Son17} reached me during the final stages of writing this note. In fact, I am grateful to Bailin Song for his comments on an earlier version of this work, as they allowed me to vastly expand the interpretation of my results.}

\footnote{Occasionally, this idea is attributed to the seminal paper \cite[p. 4]{EOT11}, where indeed, the authors ask “Is it possible that these automorphism groups at isolated points in the moduli space of K3 surface are enhanced to $M_{24}$ over the whole of moduli space when we consider the elliptic genus?”. Similarly, in \cite[p. 2]{GHV12}, one finds the statement “… the elliptic genus is independent of the specific point in the moduli space of K3 that is considered, and thus the symmetries of the elliptic genus are in some sense the union of all symmetries that are present at different points in moduli space.” These statements, however, do not anticipate any concrete constructions, let alone the symmetry surfing proposed in \cite{TW13, TW15b, TW15a}.}
Sect. 2 introduces the various inhabitants of our zoo of Hodge-elliptic genera. We begin by recalling the definitions of the conformal field theoretic elliptic and Hodge-elliptic genera in Sect. 2.1, and of their supposed geometric counterparts, namely the complex elliptic and Hodge-elliptic genera for compact Calabi-Yau manifolds in Sect. 2.2. In Sect. 2.3, we introduce the generic conformal field theoretic Hodge-elliptic genus, and we explain why it stands a chance to have a geometric counterpart. We then introduce the chiral Hodge-elliptic genus, after recalling some properties of the (holomorphic) chiral de Rham complex, which are needed for the definition of this final member of the Hodge-elliptic species. From this, we justify our expectation that the chiral Hodge-elliptic genus should play a more important role for conformal field theory than the complex Hodge-elliptic genus. Indeed, we argue that string theory predicts that the generic conformal field theoretic Hodge-elliptic genus agrees with the chiral Hodge-elliptic genus, and not with the complex Hodge-elliptic genus, in general.

The following Sect. 3 serves as a warmup for more serious calculations of Hodge-elliptic genera: for complex tori, we calculate each of the elliptic and Hodge-elliptic genera that were introduced in Sect. 2. This is a straightforward yet rewarding exercise, as it turns out that the generic conformal field theoretic Hodge-elliptic genus, the chiral Hodge-elliptic genus, as well as the complex Hodge-elliptic genus all agree in this case. This may explain why on first sight, one wouldn’t distinguish between the complex Hodge-elliptic genus and its chiral relative.

Sect. 4 is primarily devoted to the Hodge-elliptic genera for K3 surfaces and K3 theories and contains the main results of this work. As a first step, in Sect. 4.1, we calculate the conformal field theoretic Hodge-elliptic genus for standard $\mathbb{Z}_2$-orbifolds of non-linear sigma models with target a complex torus. This, again, is an easy exercise, which we find worthwhile in view of a comparison to the results of [KT17]. Sect. 4.2 addresses the generic conformal field theoretic Hodge-elliptic genus of K3 theories. We derive a closed formula, which yields the latter Hodge-elliptic genus if and only if the generic chiral algebra of K3 theories is precisely the $N = 4$ superconformal algebra at central charge $c = 6$. Under the assumption that this latter condition holds, in Sect. 4.3, we prove that in the K3 case, the complex Hodge-elliptic genus differs from the generic conformal field theoretic Hodge-elliptic genus. We also provide an implicit formula for the complex Hodge-elliptic genus of K3 surfaces.

Sect. 4.4 addresses the chiral Hodge-elliptic genus of K3 surfaces. We argue that the results up to this point yield a string theory proof of an explicit formula for the chiral Hodge-elliptic genus of K3 surfaces, relying on the assumption that the generic chiral algebra of K3 theories is as stated above, and that an infinite volume limit of the topological half-twist of K3 theories yields the (sheaf) cohomology of the (holomorphic) chiral de Rham complex, as claimed by Kapustin [Kap05]. We also provide a direct proof of this formula, which crucially uses Bailin Song’s result on the global holomorphic sections of the chiral de Rham complex [Son16, Thm. 1.2], thus supporting the belief that the string theory assumptions stated above hold true. Reversing the argument, one obtains a string theory proof of Bailin Song’s beautiful result that the global holomorphic sections of the chiral de Rham complex on a K3 surface precisely yield an $N = 4$ superconformal vertex operator algebra at central charge $c = 6$. Moreover, under the assumption that this vertex operator algebra yields the generic chiral algebra of K3 theories, it follows that the generic conformal field theoretic Hodge-elliptic genus of K3 theories agrees with the chiral Hodge-elliptic genus of K3 surfaces. That the assumption holds, in turn, is found to follow from Kapustin’s claims on infinite volume limits [Kap05].

We conclude in a final Sect. 4.5 explaining the consequences of our calculations for Mathieu Moonshine: we interpret the results of this note as strongly supporting the expectation [Wen15, §4.2] that the chiral de Rham complex might bear the key to understanding Mathieu Moonshine by means of symmetry surfing, as in [TW13, TW15b, TW15a, GKPT17]. It serves as a mathematical model of a common subspace of the space of states of all K3 theories, which is protected by the chiral Hodge-elliptic genus and which naturally carries the action of an $N = 4$ superconformal algebra at central charge $c = 6$, and of the finite symplectic automorphism groups of all K3 surfaces.

An Appendix lists the relevant formulae for Jacobi theta functions and characters of the irreducible representations of the (small) $N = 4$ superconformal algebra at central charge $c = 6$.
2 The setup: complex and Hodge-elliptic genera

In this section, we recall the setup and definitions of complex and Hodge-elliptic genera in the conformal field theoretic as well as the geometric context, and we extend these notions by a few further ideas. We begin by introducing some notations, and by stating the general assumptions that are made throughout this note. See, for example, [Wen15] for a recent review of the relevant notions, adapted to the applications that we have in mind, here.

Some of the definitions and statements given in this note hold in greater generality than claimed. However, to keep the exposition more accessible, throughout this note, we restrict our attention to a certain type of two-dimensional Euclidean unitary superconformal field theories (SCFTs):

**Assumption 2.1.** In this note, by a superconformal field theory (SCFT), a two-dimensional Euclidean unitary superconformal field theory with $N = (2, 2)$ worldsheet supersymmetry is meant, throughout. For the generators of the two commuting copies of $N = 2$ super-Virasoro algebras acting on the space of states $\mathbb{H}$ of such a theory, we use the standard notations and normalizations (see, for example, [LVW89] §2).

Furthermore, we assume that the central charges of the theory obey $c = \tau$, $c = 3D$, $D \in \mathbb{N}$, and that space-time supersymmetry holds. For the space of states $\mathbb{H}$, we assume $\mathbb{H} = \mathbb{H}^{\text{NS}} \oplus \mathbb{H}^{\text{R}}$, with $\mathbb{H}^{\text{NS}}$ denoting the Neveu-Schwarz sector and $\mathbb{H}^{\text{R}}$ the Ramond sector, referring to left- and right-moving boundary conditions simultaneously, as we require that the NS-R and R-NS sectors are trivial. We finally assume that all eigenvalues of $J_0$ and $\overline{J}_0$ on $\mathbb{H}^{\text{R}}$ belong to $\frac{1}{2} + \mathbb{Z}$.

We set $q := \exp(2\pi i \tau)$ for $\tau \in \mathbb{C}$, $\Im(\tau) > 0$, and $y := \exp(2\pi iz)$ for $z \in \mathbb{C}$, and we denote the standard partition function along with its $\overline{R}$-sector by

$$Z(\tau, z) = \text{tr}_{\mathbb{H}} \left( \frac{1}{2} \left(1 + (-1)^{J_0 - \overline{J}_0}\right) y^{J_0} q^{\overline{J}_0} \overline{y}^{\overline{J}_0} \overline{q}^{J_0} \overline{\tau}^2 \right),$$

$$Z_{\overline{R}}(\tau, z) = \text{tr}_{\mathbb{H}^{\overline{R}}} \left( (-1)^{J_0 - \overline{J}_0} y^{J_0} q^{\overline{J}_0} \overline{y}^{\overline{J}_0} \overline{q}^{J_0} \overline{\tau}^2 \right).$$

The assumption of space-time supersymmetry ensures that the linear operator $J_0 - \overline{J}_0$ on $\mathbb{H}$ possesses only integral eigenvalues and that the Ramond and the Neveu-Schwarz sector are related by spectral flow. By the additional assumptions on the spectra of $J_0$ and $\overline{J}_0$ on the Ramond sector, this implies that the eigenvalues of these two operators are integral on the Neveu-Schwarz sector. In a string theory interpretation, the properties listed in Assumption 2.1 are expected to be necessary to allow an interpretation of the theory as the internal CFT of a non-linear sigma model with a compact $D$-dimensional Calabi-Yau target space. It is worth noting, however, that these assumptions are stated here, and turn out to be useful, independently of such a string theory interpretation.

We are now ready to define and discuss the various versions of conformal field theoretic and geometric elliptic genera that are the topic of this note:

2.1 Conformal field theoretic elliptic genera

In this section, we recall the definitions of the conformal field theoretic elliptic genus and of its refinement to the conformal field theoretic Hodge-elliptic genus, and we briefly discuss some of their properties, in particular explaining the necessity to require Assumption 2.1.

**Definition 2.2.** Consider an $N = (2, 2)$ superconformal field theory at central charges $c, \tau$ and with space of states $\mathbb{H} = \mathbb{H}^{\text{NS}} \oplus \mathbb{H}^{\text{R}}$, which obeys Assumption 2.1. Then

$$e^{\text{CFT}}(\mathbb{H}; \tau, z) := \text{tr}_{\mathbb{H}^{\overline{R}}} \left( (-1)^{J_0 - \overline{J}_0} y^{J_0} q^{\overline{J}_0} \overline{y}^{\overline{J}_0} \overline{q}^{J_0} \overline{\tau}^2 \right)$$

is the conformal field theoretic elliptic genus of the theory. Now let

$$\mathbb{H}^{\overline{R}} := \{ \phi \in \mathbb{H}^{\overline{R}} \mid \overline{\tau}_0 \phi = \frac{\tau}{24} \phi \}.$$
denote the subspace of the Ramond sector given by those states which are Ramond ground states with respect to the right-moving superconformal algebra, and set \( u := \exp(2\pi i \nu) \) for \( \nu \in \mathbb{C} \). Then, following [KT17],

\[
\mathcal{E}^{\text{CFT}}_{\text{Hodge}}(\mathbb{H}; \tau, z, \nu) := \text{tr}_{\mathbb{H}} \left( (-1)^{J_0 - J_0} y^{J_0} u^{J_0} q^{L_0 - L_0} \right)
\]

is the conformal field theoretic Hodge-elliptic genus of the theory.

Note that \( \mathbb{H}^R \) is isomorphic to the space of states of the topological half-twist [EY90] [Wit92] of our SCFT, by construction: with \( Q := \mathcal{G}_0^+ \), we have \( \mathbb{H}^R = \ker Q \cap \ker Q^+ \), the space of “harmonic representatives” of the BRST cohomology \( \ker Q / \im Q \).

By the standard arguments for cancellations due to supersymmetry,

\[
\mathcal{E}^{\text{CFT}}(\mathbb{H}; \tau, z) = \text{tr}_{\mathbb{H}} \left( (-1)^{J_0} y^{J_0} q^{L_0 - L_0} \right),
\]

showing that indeed, the conformal field theoretic Hodge-elliptic genus \( \mathcal{E}^{\text{CFT}}_{\text{Hodge}}(\mathbb{H}; \tau, z, \nu) \) is a refinement of the conformal field theoretic elliptic genus \( \mathcal{E}^{\text{CFT}}(\mathbb{H}; \tau, z) \):

\[
\mathcal{E}^{\text{CFT}}_{\text{Hodge}}(\mathbb{H}; \tau, z, \nu = 0) = \mathcal{E}^{\text{CFT}}(\mathbb{H}; \tau, z).
\]  (2.1)

Let us remark that the very Assumption 2.1 on the type of superconformal field theories that enter the Def. 2.2 are necessary, in order to ensure that the conformal field theoretic Hodge-elliptic genus possesses a power series expansion (with integral exponents only) in \( q, y^{\frac{1}{2}}, \) and \( u^{\frac{1}{2}} \). For \( q \), this claim is a consequence of well-known properties of the spectral flow (see, for example, [Sen86] [Sen87] or [Gre97] [§3.4]), along the lines of an argument already presented in [TW17] [§3.2]: indeed, our assumptions ensure that every common eigenstate \( \phi \in \mathbb{H}^R \) of \( J_0, \bar{J}_0 \) and \( L_0 \) is the image, under spectral flow, of some state in the Neveu-Schwarz sector with conformal weights \((h, \bar{h})\) and \( U(1) \) charges \((Q, \bar{Q})\) with \( h \geq \frac{|Q|}{2}, \ 2\bar{h} = \pm Q, \ h - \bar{h} \in \frac{1}{2} \mathbb{Z} \) and \( Q, \bar{Q} \in \mathbb{Z} \). Moreover, by our assumption of space-time supersymmetry, \( Q - \bar{Q} \in 2\mathbb{Z} \) if and only if \( h - \bar{h} \in \mathbb{Z} \), or equivalently, if and only if the state is bosonic. The holomorphic conformal weight of \( \phi \) thus is

\[
h + \frac{Q}{2} + \frac{\nu}{24} = h - \bar{h} + \frac{Q - \bar{Q}}{2} + \frac{\nu}{24} \in \frac{c}{24} + \mathbb{N},
\]

as claimed. Note that the additional assumption on the eigenvalues of \( J_0 \) and \( \bar{J}_0 \) in Def. 2.2 is sometimes only tacitly made, in the literature. As pointed out, for example, in [Gre97] [§3.4] and [Wen00] [§3.1.1], it is equivalent to the requirement that the theory is invariant under the purely holomorphic and anti-holomorphic two-fold spectral flows. By [EOTY89], this condition should hold for all SCFTs that arise as non-linear sigma models with a compact Calabi-Yau target space.

The very fact that \( \mathcal{E}^{\text{CFT}}_{\text{Hodge}}(\mathbb{H}; \tau, z, \nu) \) has a formal power series expansion in \( q, y^{\frac{1}{2}}, \) and \( u^{\frac{1}{2}} \), where only integral exponents occur, seems to be the main advantage of this new quantity over the partition function of the underlying SCFT, as we shall see below. Indeed, the advantage of introducing \( \mathcal{E}^{\text{CFT}}_{\text{Hodge}}(\mathbb{H}; \tau, z, \nu) \) is not at all immediate, because just like the partition function, the conformal field theoretic Hodge-elliptic genus severely depends on the moduli of the SCFT chosen at the outset. In contrast, the traditional conformal field theoretic elliptic genus \( \mathcal{E}^{\text{CFT}}(\mathbb{H}; \tau, z) \) is invariant under deformations of the theory within the space of SCFTs of the type specialized to, above (see [AKMW87] [EOTY89] [DY93] [Wit94] for the original results and e.g. [Wen00] [§3.1] for a summary, including proofs).

If a SCFT that obeys Assumption 2.1 arises as a non-linear sigma model with some compact Calabi-Yau target space \( X \), then one expects \( \mathcal{E}^{\text{CFT}}(\mathbb{H}; \tau, z) \) to agree with the complex elliptic genus of \( X \), a topological invariant which generalizes the Hirzebruch \( \chi_y \) genus to a complex elliptic genus [Hit88] [Wit88] [Kr90] and whose definition we recall below, see Def. 2.4. This motivates the following definition, which is advocated, for example, in [NW01] [Wen15]:
Definition 2.3. A superconformal field theory is called a K3 theory, if the following conditions hold: the SCFT is an $N = (2,2)$ superconformal field theory at central charges $c = 6$, $\tau = 6$ with space-time supersymmetry, all the eigenvalues of the operators $J_0$ and $\overline{J}_0$ on the space of states $H$ are integral, and the conformal field theoretic elliptic genus of the theory agrees with the complex elliptic genus of a K3 surface,

$$\mathcal{E}^{\text{CFT}}(\mathbb{H}; \tau, z) = 8 \left( \left( \frac{\vartheta_2(\tau, z)}{\vartheta_2(\tau, 0)} \right)^2 + \left( \frac{\vartheta_3(\tau, z)}{\vartheta_3(\tau, 0)} \right)^2 + \left( \frac{\vartheta_4(\tau, z)}{\vartheta_4(\tau, 0)} \right)^2 \right), \quad (2.2)$$

where here and in the following, we use the standard Jacobi theta functions $\vartheta_k(\tau, z)$, $k \in \{1, \ldots, 4\}$, c.f. Appendix A.

Possibly, every K3 theory allows a non-linear sigma model interpretation with target given by some K3 surface, but a proof is far out of reach. For these SCFTs, it is not hard to see and well-known to the experts\footnote{See \cite{Wen15, §3} for a recent review, adapted to our applications.} that the assumptions on the representation content of such a K3 theory guarantee extended $N = (4,4)$ supersymmetry\footnote{More precisely, the relevant left- and right-moving superconformal algebras both yield a small $N = 4$ superconformal algebra according to \cite{ABD+76}, which for simplicity, in this note, we call the $N = 4$ superconformal algebra.}, resonating with the fact that every K3 surface is hyperkähler. Under an assumption on the integrability of certain deformations, which can be justified in string theory and which is demonstrated to all orders of perturbation theory in \cite{Dix88}, and based on the previous results \cite{Sei88, Cec91}, the moduli space of K3 theories has been determined in \cite{AM94, NW01}. The results of \cite{AM94} allow to give geometric interpretations by K3 surfaces to each theory that is accounted for in this moduli space. Vice versa, this provides a map that assigns a unique K3 theory to every choice of geometric moduli, in terms of a hyperkähler structure, volume and B-field on a K3 surface. To make this map explicit is a wide open problem, to date.

The expectation that the conformal field theoretic elliptic genus for a non-linear sigma model on some Calabi-Yau manifold $X$ should agree with the complex elliptic genus of $X$ is rooted in the expected large volume behaviour of such sigma models. Following \cite{Wit82, LVW89}, the space of Ramond ground states in such a large volume limit should recover the Dolbeault cohomology of $X$. The work of Kapustin \cite{Kap05} shows how this expectation extends to the additional states that are accounted for by the complex elliptic genus, as we shall recall in Sect. 2.3. Since the conformal field theoretic elliptic genus remains invariant under deformations within the moduli space of theories restricted to in Def. 2.2, the agreement must hold away from large volume limits, as well.

2.2 Geometric elliptic genera

While the conformal field theoretic Hodge-elliptic genus does not define an invariant under deformations of SCFTs, in any large-volume limit, according to \cite{KT17}, one should obtain a geometric version of the Hodge-elliptic genus. In this section, we therefore recall the definitions of the relevant geometric genera, according to the proposal of \cite{KT17}.

In the following, let $X$ denote a compact $D$-dimensional complex manifold, and $E \rightarrow X$ a holomorphic vector bundle on $X$. Recall that the holomorphic Euler characteristic of $E$ is given by

$$\chi(E) = \sum_{j=0}^{D} (-1)^j \dim H^j(X, E).$$

Following \cite{KT17}, for $u \in \mathbb{C}^*$, we may also introduce

$$\chi^u(E) := \sum_{j=0}^{D} (-u)^j \dim H^j(X, E),$$
which might be dubbed the Hodge-holomorphic Euler characteristic. For any formal variable \( x \), we use the shorthand notations

\[
\Lambda_x E := \bigoplus_{p=0}^{\infty} x^p \Lambda^p E, \quad S_x E := \bigoplus_{p=0}^{\infty} x^p S^p E,
\]

where \( \Lambda^p E \), \( S^p E \) denote the \( p \)th exterior and symmetric powers of \( E \), respectively. We are now ready to define the complex elliptic genus and the complex Hodge-elliptic genus:

**Definition 2.4.** Let \( X \) denote a compact complex \( D \)-dimensional manifold, and \( T := T^{1,0} X \) its holomorphic tangent bundle. With \( q \), \( y \) as in our Assumption 2.1 let

\[
\mathcal{E}_{q,-y} := y^{-\frac{D}{2}} \bigoplus_{n=1}^{\infty} \left( \Lambda_{-yq^{n-1}} T^* \otimes \Lambda_{-y^{-1}q^{n}} T \otimes S_q T^* \otimes S_q T \right),
\]

viewed as a formal power series with variables \( y^{\pm \frac{1}{2}}, q \), whose coefficients are holomorphic vector bundles on \( X \),

\[
\mathcal{E}_{q,-y} = y^{-\frac{D}{2}} \bigoplus_{\ell=0}^{D} \bigoplus_{m=-D}^{D} \mathcal{T}_{\ell,m} (-y)^m q^\ell.
\]

Then, following [Hir88, Wic88, Kri90], the complex elliptic genus of \( X \) is

\[
\mathcal{E}(X; \tau, z) := \chi(\mathcal{E}_{q,-y}) = y^{-\frac{D}{2}} \sum_{\ell=0}^{D} \sum_{m=-D}^{D} \chi(\mathcal{T}_{\ell,m})(-y)^m q^\ell.
\]

With \( u := \exp(2\pi i \nu) \) for \( \nu \in \mathbb{C} \), following [KT17], the complex Hodge-elliptic genus is\(^5\)

\[
\mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu) := u^{-\frac{D}{2}} \chi^u(\mathcal{E}_{q,-y}) = (uy)^{-\frac{D}{2}} \sum_{\ell=0}^{D} \sum_{m=-D}^{D} \chi^u(\mathcal{T}_{\ell,m})(-y)^m q^\ell.
\]

From the very definition, it is clear that \( \mathcal{E}_{\text{Hodge}} \) is a natural and interesting refinement of the complex elliptic genus:

\[
\mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu = 0) = \mathcal{E}(X; \tau, z).
\]

Note that the holomorphic vector bundles \( \mathcal{T}_{\ell,m} \rightarrow X \) in the above definition, by construction, are sums of tensor products of exterior and symmetric powers of the holomorphic tangent bundle \( T \) and its dual \( T^* \). Nevertheless, one should expect \( \mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu) \) to depend on the choice of the complex structure on \( X \), since the dimensions of the cohomology spaces \( H^j(X, \mathcal{T}_{\ell,m}) \) may jump, as the complex structure of \( X \) varies (see, e.g., [Huy95, BPH92, AMP12] for some examples of this phenomenon\(^4\)). For a general Calabi-Yau manifold \( X \), we therefore always assume that a fixed complex structure has been chosen. However, if \( X \) is a K3 surface, then the Bochner principle suffices to prove that \( \mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu) \) is independent of this choice, as is shown in [KT17]. Moreover, for complex tori \( X = T^D \), the holomorphic tangent bundle is trivial, yielding \( \mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu) \) independent of the choice of the complex structure, as well, as we shall confirm in Sect. 3.

In [KT17], it is claimed that one should expect that \( \mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu) \) agrees with the conformal field theoretic Hodge-elliptic genus of sigma models on \( X \) in a large volume limit. As we shall see in Prop. 4.3 below, this claim does not hold when \( X \) is a K3 surface, unless K3 theories generically have an extended chiral algebra beyond the \( N = 4 \) superconformal algebra at central charge \( c = 6 \). In fact, in Sect. 4.3 we show that such a generically extended chiral algebra would contradict the results of Kapustin [Kap05] on large volume limits of topologically half twisted sigma models. As we shall explain in Sect. 2.3, we do not expect the claim to hold in general, except for few examples, like complex tori.

\(^5\)up to the prefactor \( u^{-\frac{D}{2}} \) which presumably is omitted in [KT17] §3 only due to a typo

\(^6\)We thank Emanuel Scheidegger for pointing these references out to us.
2.3 Generic genera?

In this section, for a compact Calabi-Yau manifold $X$ that obeys certain additional assumptions, we introduce the notions of generic conformal field theoretic Hodge-elliptic genus and of chiral Hodge-elliptic genus. In the string theory literature, the prior would probably rather be called an infinite volume limit of the conformal field theoretic Hodge-elliptic genus of sigma models on $X$, and we expect it to agree with the chiral Hodge-elliptic genus of $X$, in general.

Assume that for some compact $D$-dimensional Calabi-Yau manifold $X$, we have a notion of a moduli space of SCFTs that obey our Assumption 2.1 and that allow a geometric interpretation by $X$. This is the case, for example, if $X$ is a complex torus or a K3 surface, using [CENT85, Nar80, AM94, NW01] and Def. 2.2 where the geometric moduli are naturally expressed in terms of a hyperkähler structure, the volume and a B-field on $X$. In general, we lack a clean mathematical definition of the appropriate moduli spaces. The more mathematically inclined reader is therefore invited to restrict their attention to tori and K3 surfaces, at least in the context of the generic conformal field theoretic Hodge-elliptic genus introduced in Def. 2.5 below.

However, it is largely believed that for general Calabi-Yau manifolds $X$, there is a notion of a moduli space of SCFTs arising as non-linear sigma models on $X$. If $X$ is not hyperkähler, then the moduli space, at least locally, is expected to decompose into a product of complex structure and (complexified) Kähler parameter spaces of $X$ [Dr88, DG88]. Mathematically, these two factors exhibit quite distinct behaviours. Since both the complex and the chiral Hodge elliptic genus of $X$ are expected to depend on the complex structure of $X$, in general, we then assume that $X$ is equipped with a fixed choice of complex structure, requiring the (complexified) Kähler moduli to be generic when referring to generic quantities. If $X$ is hyperkähler, then the situation is different, since similarly to the K3 case, the relevant moduli spaces are not even locally expected to decompose into a product of complex structure and (complexified) Kähler moduli spaces. For non-linear sigma models on hyperkähler manifolds $X$ we therefore consider all moduli, when we refer to generic quantities, so such $X$ is not assumed to be equipped with a fixed complex structure. This distinction is motivated by the treatment of infinite volume limits in the string theory literature, particularly in [KT17], and by the fact that for K3 surfaces, both the complex and the chiral Hodge-elliptic genus are independent of the complex structure, as we shall see below. In fact, the conformal field theoretic Hodge-elliptic genus at generic (complexified) Kähler moduli is also independent of all complex structure moduli for SCFTs that arise as non-linear sigma models with some complex torus as target (see Prop. 3.1). The same behaviour is predicted for K3 theories by string theory, as we shall see in the proof of Prop. 1.7. It is left for future work to decide whether for higher-dimensional hyperkähler manifolds $X$, a refinement of our notion of generic conformal field theoretic Hodge-elliptic genus is more adequate.

We add the assumption that spectral data depend on the moduli at least continuously, which is known to hold true for complex tori and orbifolds thereof. Since in any SCFT that obeys our Assumption 2.1 by Def. 2.2 $\mathbb{H}^R$ is the kernel of the linear operator $L_0 - \frac{D}{2}\text{id}$ on $\mathbb{H}^R$, the coefficients of the conformal field theoretic Hodge-elliptic genus $C^{CFT}_\text{Hodge}(H; \tau, z, \nu)$ are the dimensions of common eigenspaces of the linear operators $L_0 - \frac{D}{2}\text{id}$, $\vartheta L_0 - \frac{D\vartheta}{2}\text{id}$, $J_0$ and $\vartheta J_0$ on $\mathbb{H}^R$. By our assumptions, as was explained in the discussion of Def. 2.2 all the eigenvalues of these linear operators on $\mathbb{H}^R$ are restricted to values in $\frac{1}{2}\mathbb{Z}$. This means that the generic dimensions of these eigenspaces in $\mathbb{H}^R$ on our moduli space yield the maximal lower bounds of these dimensions\footnote{This behaviour is well known from dimension theory in commutative algebra and is sometimes called upper semicontinuity, e.g. in [KT17].}. Let us introduce a generating function for these generic dimensions:

**Definition 2.5.** Let $X$ denote a compact Calabi-Yau manifold of dimension $D$. Consider the moduli space of SCFTs that obey Assumption 2.1 and that allow a geometric interpretation by $X$. Here, we assume that $X$ is equipped with a fixed complex structure, unless $X$ is hyperkähler. In the hyperkähler case, we only fix the diffeomorphism type of $X$. 
For \( h \in \mathbb{R} + \mathbb{N} \) and \( Q, \overline{Q} \in \mathbb{R} + \mathbb{Z} \), and with notations as in Def. 2.2 let

\[
N_{h, Q, \overline{Q}} := \inf \left\{ \dim \left\{ \phi \in \mathbb{H}^R \mid L_0 \phi = h \phi, \ J_0 \phi = Q \phi, \ \mathcal{T}_y \phi = \overline{Q} \phi \right\} \right\},
\]

where the infimum is taken over all SCFTs within the moduli space. Then the generic conformal field theoretic Hodge-elliptic genus of \( X \) is given by

\[
\mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) := \sum_{h, Q, \overline{Q}} (-1)^Q N_{h, Q, \overline{Q}} \cdot y^Q u \overline{Q} q^h \frac{\phi}{R},
\]

where the sum runs over \( h \in \mathbb{R} + \mathbb{N} \) and \( Q, \overline{Q} \in \mathbb{R} + \mathbb{Z} \).

By the above definition of the generic conformal field theoretic Hodge-elliptic genus, for a Calabi-Yau manifold \( X \) and for every SCFT in the moduli space of theories with geometric interpretation by \( X \) and with space of states \( \mathbb{H} \) as in Def. 2.2 there is a space \( \mathbb{H}^R \) which injects into \( \mathbb{H}, \mathbb{H}^R \hookrightarrow \mathbb{H}^R \), with the following two properties: first,

\[
\mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) = \text{tr}_{\mathbb{H}} \left( (-1)^{J_0} - \mathcal{T}_0 y^{J_0} \mathcal{T}_0 q^{L_0} \frac{\phi}{R} \right),
\]

where \( \mathbb{H}^R \) is a representation of the left-moving \( N = 2 \) superconformal algebra, extended by \( \overline{J}_0 \). Second, if the superconformal field theories in our moduli space share a common extended chiral algebra \( \mathcal{A} \), then \( \mathbb{H}^R \) is a representation of \( \mathcal{A} \), extended by \( \overline{J}_0 \).

At fixed values of \( h \in \mathbb{R} + \mathbb{N} \), \( Q, \overline{Q} \in \mathbb{R} + \mathbb{Z} \), by the defining properties of our SCFTs, there are only finitely many values of \( \overline{Q} \in \mathbb{R} + \mathbb{Z} \) such that \( N_{h, Q, \overline{Q}} \neq 0 \). Hence generically, the common eigenspaces of \( L_0, J_0, \mathcal{T}_0 \) in \( \mathbb{H}^R \) with eigenvalues \( h, Q, \overline{Q} \) at fixed \( h, Q \), for all \( \overline{Q} \) have dimension \( N_{h, Q, \overline{Q}} \). Since the conformal-field theoretic elliptic genus is invariant on the moduli space, we therefore have

\[
\mathcal{E}^\text{CFT}(\mathbb{H}; \tau, z) = \text{tr}_{\mathbb{H}} \left( (-1)^{J_0} - \mathcal{T}_0 y^{J_0} q^{L_0} \frac{\phi}{R} \right) \overset{(2.3)}{=} \mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu = 0).
\]

As mentioned above, in string theory, \( \mathcal{E}^0_{\text{Hodge}} \) would probably be called an infinite volume limit of the conformal field theoretic Hodge-elliptic genus, which for general Calabi-Yau manifolds refers to a point in an appropriate boundary of the moduli space where the dependence on all (complexified) Kähler parameters is lost. For our discussion, it is more appropriate to focus on generic values of these parameters instead of a limit where they are infinite. For K3 surfaces, by our Definition 2.2 we consider all moduli of K3 theories at generic values. Then, the notion of infinite volume limit seems a little out of place, and we prefer the notion of generic space of states.

It is important to note that the very restrictions on the spectra of \( J_0 \) and \( \overline{J}_0 \) on \( \mathbb{H}^R \) in our Assumption 2.4 are crucial in order to expect any meaningful quantity to arise from Def. 2.3. Indeed, generic dimensions of common eigenspaces of \( \overline{L}_0, L_0, J_0 \) and \( \overline{J}_0 \), for generic choices of eigenvalues, are zero. In our view, this is why Kachru’s and Tripathy’s conformal field theoretic Hodge-elliptic genus \( \mathcal{E}^\text{CFT}_{\text{Hodge}}(\mathbb{H}; \tau, z, \nu) \) turns out to be so useful, in contrast to the partition function.

As was mentioned at the end of Sect. 2.2 in Prop. 4.4 under one additional assumption which is commonly believed to hold true, we will disprove the expectation of [KT17] that for K3 surfaces \( X \), the generic conformal field theoretic Hodge-elliptic genus agrees with the complex Hodge-elliptic genus \( \mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu) \) of Def. 2.3. Indeed, we see no reason for the two quantities to agree, in general. Let us now explain why this is so, and offer an alternative proposal.

In his beautiful work Kapustin has proposed the following relationship between the non-linear sigma model on a given compact Calabi-Yau manifold \( X \) and the (holomorphic) chiral de Rham complex \( \Omega^\bullet_X \) of \( X \) that was introduced by Malikov, Schectman, Vaintrob

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Footnote: see, for example, [KT17] §3, page 253]
Kapustin argues that an infinite volume limit of the model obtained from the sigma-model by a topological half-twist [EY90] can be identified with the sheaf cohomology of $\Omega^X_\mathfrak{ch}$. Let us recall and explain this in more detail. The chiral de Rham complex is a sheaf of vertex operator algebras which can be defined on any compact complex manifold $X$. It possesses a bigrading by globally defined operators $L_0^{\top}$ and $J_0$ and a compatible natural filtration. Both these operators descend to the (sheaf) cohomology $H^*(X, \Omega^X_\mathfrak{ch})$ of the chiral de Rham complex, inducing a $\mathbb{Z}_2$-grading by $(-1)^{h_0 + j}$ on $H^*(X, \Omega^X_\mathfrak{ch})$. According to [MSV99, Bor01, BL00, GM04, LL07, BHS08], the associated graded object for $\Omega^X_\mathfrak{ch}$ is isomorphic to the sheaf of sections of the virtual bundle $(-y)^{-\frac{D}{2}} \mathcal{E}_{q,y}$ on $X$ that was used in the Definition 2.4 of the complex elliptic genus of $X$. This implies that the latter, given by the holomorphic Euler characteristic of $\mathcal{E}_{q,-y}$, agrees with the graded Euler characteristic of the chiral de Rham complex,

$$\mathcal{E}(X; \tau, z) = y^{-\frac{D}{2}} \sum_{j=0}^D (-1)^j \text{tr}_{H^j(X, \Omega^X_\mathfrak{ch})} \left( (-1)^{h_0} y^{h_0} q^{L_0^{\top}} \right). \quad (2.5)$$

Note that $H^*(X, \Omega^X_\mathfrak{ch})$ carries the action of a topological $\mathfrak{N} = 2$ superconformal algebra at rank $3D$, according to [Bor01] Prop. 3.7 and Def. 4.1, which is extended to an $\mathfrak{N} = 4$ superconformal algebra if $X$ is hyperkähler [BHS08, Hel09, see also Son17 §2].

Kapustin’s interpretation of the cohomology of the chiral de Rham complex $\Omega^X_\mathfrak{ch}$ as infinite volume limit of the topologically half-twisted sigma model on $X$ explains why one might expect the complex elliptic genus of $X$ to agree with the conformal field theoretic elliptic genus of the sigma model. Note that $H^*(X, \Omega^X_\mathfrak{ch})$ is thus interpreted as infinite volume limit of the Neveu-Schwarz sector of the sigma model, after the topological half-twist. The latter is indicated in (2.5) by the lack of anti-holomorphic contributions. The use of $L_0^{\top} = L_0 - \frac{1}{2} J_0$, with $L_0$ the untwisted Virasoro zero-mode, accounts for the fact that $H^*(X, \Omega^X_\mathfrak{ch})$ naturally carries the action of a topologically twisted $\mathfrak{N} = 2$ superconformal algebra. To obtain the elliptic genus, one needs to perform a spectral flow from the Neveu-Schwarz to the Ramond sector. This is reflected in (2.5) by the fact that the trace of $(-1)^{h_0} y^{h_0} q^{L_0^{\top}} = (-1)^{h_0} q^{\frac{D}{2}} (y q^{-\frac{1}{2}})^{h_0} q^{L_0 - \frac{D}{2}}$ is taken, instead of $(-1)^{h_0} y^{h_0} q^{L_0^{\top}}$.

This reasoning also implies that $H^j(X, \mathcal{E}_{q,-y})$ cannot be expected to be isomorphic to (a graded object of) $H^j(X, \Omega^X_\mathfrak{ch})$. In fact, by the above, $H^j(X, \mathcal{E}_{q,-y})$ arises on the first sheet of the spectral sequence that is obtained from our filtered complex, while $H^j(X, \Omega^X_\mathfrak{ch})$ requires the limit of that spectral sequence. Therefore, we do not expect the respective Hodge-elliptic genera to agree, not even in an infinite volume limit. As an alternative, we propose to define a Hodge-elliptic genus using the (holomorphic) chiral de Rham complex. To fully appreciate such a definition, a few more details about this sheaf are helpful. First note that in [Kap05], a Dolbeault resolution of the chiral de Rham complex is introduced, which essentially extends $\Omega^X_{\mathfrak{ch}}$ by additional anti-holomorphic fields. All these additional fields are assumed to be constant. As Grimm explains in his thesis [Gri16 §5.1], this restricts one to the realm of real analytic differential forms, where one lacks a partition of unity. The relevant sheaves, therefore, are not fine. Also inspired by the works of Lian and Linshaw [LL07], Grimm instead constructs a resolution that sits in between the two that are suggested in [Kap05, LL07, respectively. He proves that the resulting sheaves yield an acyclic resolution, thus allowing him access to explicit calculations of chiral de Rham cohomology, in some examples. The important point, for our purposes, is the existence of a well-defined operator $\overline{J}_0$ on the sections of the chiral de Rham complex, which descends to cohomology, as follows from Grimm’s construction, with $H^j(X, \Omega^X_{\mathfrak{ch}})$ arising as the kernel of $\overline{J}_0 - j$ in $H^*(X, \Omega^X_{\mathfrak{ch}})$ by [Gri16 Thm. 5.1.7]. As was mentioned before, this is already implicit in the constructions of [Bor01, BL00], where the $\mathbb{Z}_2$-grading by what we now recognize as $(-1)^{h_0 - \overline{J}_0}$ is introduced. This motivates the following.

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Footnotes:

9: that is, topologically twisted, according to [EY90]
10: $(-1)^{h_0 - \overline{J}_0} = (-1)^{h_0 + \overline{J}_0}$ since by Assumption 2.1 all eigenvalues of $\overline{J}_0$ in the Neveu-Schwarz sector are integral.
Definition 2.6. Let $X$ denote a compact Calabi-Yau manifold of dimension $D$, and $\Omega^\text{ch}_X$ its (holomorphic) chiral de Rham complex. The chiral Hodge-elliptic genus of $X$ is defined by

$$\mathcal{E}^\text{ch}_{\text{Hodge}}(X; \tau, z, \nu) := (yu)^{-D} \operatorname{tr}_{H^* (X, \Omega^\text{ch}_X)} \left( (-1)^{j_0} y^{j_0} q^{j_0 \text{top}} \right)$$

$$= (yu)^{-D} \sum_{j=0}^D (-u)^j \operatorname{tr}_{H^* (X, \Omega^\text{ch}_X)} \left( (-1)^{j_0} y^{j_0} q^{j_0 \text{top}} \right).$$

Let us discuss some of the properties of this new Hodge-elliptic genus. The chiral Hodge-elliptic genus is a natural refinement of the complex elliptic genus, as is immediate from the above Def. 2.6 along with (2.5):

$$\mathcal{E}^\text{ch}_{\text{Hodge}}(X; \tau, z, \nu = 0) = \mathcal{E}(X; \tau, z). \quad (2.6)$$

Following Kapustin, $H^* (X, \Omega^\text{ch}_X)$ is interpreted as infinite volume limit of the Neveu-Schwarz sector of a topologically half-twisted sigma model on $X$, which as advertised above should be a space that can be injectively mapped into the space of states of any sigma model on $X$. In this sense, $H^* (X, \Omega^\text{ch}_X)$ is a common subspace of all such theories. Spectral flow maps this space to $\mathbb{H}^\text{R} \rightarrow \mathbb{H}^\text{R}$ in (2.3), where the restriction to $\ker (\hat{\mathcal{L}}_0 - \hat{\mathcal{T}} \cdot \text{id})$ with $\hat{\mathcal{T}} = 3D$ in the Definition 2.2 of $\hat{\mathcal{H}}^\text{R}$ implements the topological half-twist, as mentioned above. This explains why we expect the chiral Hodge-elliptic genus of Def. 2.7, rather than the complex Hodge-elliptic genus of [KT17], to agree with the generic conformal field theoretic Hodge-elliptic genus of $X$.

Conjecture 2.7. Consider a compact Calabi-Yau manifold $X$. The generic conformal field theoretic Hodge-elliptic genus of SCFTs with geometric interpretation by $X$ (Def. 2.6) agrees with the chiral Hodge-elliptic genus of $X$ (Def. 2.6),

$$\mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) = \mathcal{E}^\text{ch}_{\text{Hodge}}(X; \tau, z, \nu).$$

In Sect. 3 we will see that it is straightforward to prove this conjecture if $X$ is a complex torus, see Prop. 3.1. Moreover, Prop. 4.6 shows that for K3 surfaces $X$, this conjecture holds under the natural assumption that the generic chiral algebra of all K3 theories is precisely the $N = 4$ superconformal vertex operator algebra at central charge $c = 6$. On the other hand, Prop. 4.7 states that under the same assumption, the complex Hodge-elliptic genus of K3 differs from the generic conformal field theoretic Hodge-elliptic genus of K3 theories. Finally, Prop. 4.7 shows that our assumption on the generic chiral algebra of K3 theories holds true if the (sheaf) cohomology of the (holomorphic) chiral de Rham complex of a K3 surface $X$ can indeed be identified with the infinite volume limit of topologically half-twisted sigma models on $X$ in the sense explained above, and as argued by Kapustin [Kap05].

There is no reason to expect the chiral Hodge-elliptic genus to be independent of the choice of complex structure on $X$. However, in Props. 4.1 and 4.6 we will see that $\mathcal{E}^\text{ch}_{\text{Hodge}}(X; \tau, z, \nu)$ is independent of that choice if $X$ is a complex torus or a K3 surface. In this respect, the chiral Hodge-elliptic genus behaves analogously to the complex Hodge-elliptic genus of [KT17].

Finally recall that for superconformal field theories which obey our Assumption 2.1 on the level of the $N = (2, 2)$ superconformal algebra with generating fields $(T, J, G^+, G^-; \mathcal{T}, \mathcal{J}, \mathcal{G}, \mathcal{G}^\ast, \mathcal{G}^\ast)$, mirror symmetry acts as an outer automorphism induced by

$$(T, J, G^+, G^-; \mathcal{T}, \mathcal{J}, \mathcal{G}^\ast, \mathcal{G}^\ast) \mapsto (T, -J, G^-, G^+; \mathcal{T}, \mathcal{J}, \mathcal{G}, \mathcal{G}^\ast)$$

or

$$(T, J, G^+, G^-; \mathcal{T}, \mathcal{J}, \mathcal{G}^\ast, \mathcal{G}^\ast) \mapsto (T, J, G^+, G^-; \mathcal{T}, -\mathcal{J}, \mathcal{G}, \mathcal{G}^\ast),$$

where the choice between the two should solely amounts to a choice of normalization [LVWS89]. Assumption 2.1 includes the requirement that in the Ramond sector, all eigenvalues of $J_0$, $J_0$ belong to $\frac{D}{2} + \mathbb{Z}$. Therefore, if Conjecture 2.7 holds, then it in particular implies


**Conjecture 2.8.** Let \((X, \tilde{X})\) denote a mirror pair of compact Calabi-Yau manifolds of complex dimension \(D\). Then the corresponding chiral Hodge-elliptic genera obey

\[
\mathcal{E}_{\text{Hodge}}^{\text{ch}}(\tilde{X}; \tau, z, \nu) = (-1)^D \mathcal{E}_{\text{Hodge}}^{\text{ch}}(X; \tau, -z, \nu) = (-1)^D \mathcal{E}_{\text{Hodge}}^{\text{ch}}(X; \tau, z, -\nu).
\]

This is an immediate generalization of the corresponding behaviour of the elliptic genera for mirror pairs of Calabi-Yau manifolds, which in turn is well-established, see e.g. [BL00 §1(*)].

### 3 Warmup: Hodge-elliptic genera for tori

As a warm up, in this section we discuss the complex elliptic genus and the Hodge-elliptic genera for complex tori \(T_D\) of any dimension \(D\), all of which can be calculated explicitly with little effort. Although for the experts, this is an easy exercise, we find it useful to include its solution in this note, since for the Hodge-elliptic genera introduced in the previous section it may also explain some of the misconceptions in the literature.

First, since the holomorphic tangent bundle of \(T_D\) is trivial, by classical index theory, the complex elliptic genus of \(T_D\) is

\[
\mathcal{E}(T_D; \tau, z) = 0.
\]

The complex Hodge-elliptic genus \(\mathcal{E}_{\text{Hodge}}(T_D; \tau, z)\) can also be explicitly calculated (c.f. [KT17 §3], whose normalization of \(\vartheta_1(\tau, z)\) differs by a prefactor of \(-i\) from ours),

\[
\mathcal{E}_{\text{Hodge}}(T_D; \tau, z) = \left( \frac{-i\vartheta_1(\tau, z)}{\eta(\tau)^3} \cdot (u^{-\frac{1}{2}} - u^\frac{1}{2}) \right)^D.
\]

Furthermore, the sheaf cohomology of the (holomorphic) chiral de Rham complex for \(T_D\) consists of the classical Dolbeault cohomology of the torus, tensorized by polynomials in \(D\) copies of the modes \((b_m, a_m, \Phi_m, \Psi_m)_{m > 0}\) of a bc-\(\beta\gamma\) system [Gri16 Thm. 5.2.1]. On the classical Dolbeault cohomology \(H^{j,k}(T_D, \mathbb{C}) \hookrightarrow H^{j,k}(T_D, \Omega^D_{T_D})\), one has \(L_{\text{top}} \equiv 0\), \(J_0 \equiv j \cdot \text{id}\), \(\bar{J}_0 \equiv k \cdot \text{id}\), and thus

\[
\mathcal{E}_{\text{Hodge}}^{\text{ch}}(T_D; \tau, z, \nu) \quad \text{Def} = \sum_{j,k=0}^{D} (-1)^{j-k} \binom{D}{j} \binom{D}{k} y^j u^k \cdot \prod_{m=1}^{\infty} \frac{(1 - y q^m)(1 - y^{-1} q^m)}{(1 - q^m)^2}.
\]

App\(\mathbb{C}\) \(\left( \frac{-i\vartheta_1(\tau, z)}{\eta(\tau)^3} \cdot (u^{-\frac{1}{2}} - u^\frac{1}{2}) \right)^D\)

\[\equiv \mathcal{E}_{\text{Hodge}}(T_D; \tau, z, \nu).\]

Note that this result is compatible with Conjecture 2.8, since the mirror of a complex \(D\)-torus \(T_D\) is a complex \(D\)-torus.

To compare to the results obtained from the conformal field theoretic Hodge-elliptic genus, recall that any non-linear sigma model with target \(T_D\) and space of states \(\mathcal{H}\) has a partition function whose \(R\)-sector (c.f. Assumption 2.1) takes the following form:

\[
Z_{\text{R}}^{T_D}(\tau, z) = \mathcal{Z}_\Gamma(\tau) \left( \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right)^{2D}, \quad \text{where} \quad \mathcal{Z}_\Gamma(\tau) = \sum_{\gamma = (\gamma_L, \gamma_R) \in \Gamma} q^{\frac{1}{2} \sum_{\gamma \in \Gamma_L} \gamma \cdot \gamma_L} q^{\frac{1}{2} \sum_{\gamma \in \Gamma_R} \gamma \cdot \gamma_R} |\eta(\tau)|^{-4D},
\]

depends on the moduli of the theory via the charge lattice \(\Gamma \subset \mathbb{R}^{D,D}\) (see [Wen17 §2.1] for a recent review adapted to our purposes). Here and in the following, \(\mathbb{R}^{D,D} = \mathbb{R}^D \oplus \mathbb{R}^D\) is equipped with the scalar product

\[
\forall (Q, \overline{Q}), (Q', \overline{Q'}) \in \mathbb{R}^{D,D}; \quad (Q, \overline{Q}) \cdot (Q', \overline{Q'}) := Q \cdot Q' - \overline{Q} \cdot \overline{Q'},
\]

14
where $Q \cdot Q' \in \mathbb{R}$ denotes the standard scalar product of $Q$, $Q' \in \mathbb{R}^D$. According to Def. 2.2 we thus have

$$E_{\text{Hodge}}^\ast (\mathbb{H}; \tau, z, \nu) = \sum_{\gamma \in \mathbb{L} \cdot \gamma_L \in \Gamma} q^{\gamma_L \cdot \gamma_L} \eta(\gamma)^{2D} \cdot \left(\frac{-i\theta_1(\tau, z)}{\eta(\tau)^3} \cdot (u^{-\frac{1}{2}} - u^{\frac{1}{2}})\right)^D,$$

hence

$$E_{\text{Hodge}}^\ast (\mathbb{H}; \tau, z, \nu) = E_{\text{Hodge}}^\ast (\mathbb{H}; \tau, z, \nu = 0) \oplus E(T^D; \tau, z),$$

as expected by what was said in Sect. 4 and as in fact is well-known. Moreover, for any given value of $h \in \mathbb{R}$ with $h > 0$, and for almost all possible charge lattices $\Gamma$, one finds

$$\{ \gamma = (\gamma_L, \gamma_R) \in \Gamma \mid \frac{1}{2} \gamma_L \cdot \gamma_L = h, \gamma_R = 0 \} = \emptyset.$$

In fact, there is a dense subset of the moduli space of non-linear sigma models with target $T^D$ where the charge lattices $\Gamma$ obey

$$\{ \gamma = (\gamma_L, \gamma_R) \in \Gamma \mid \gamma_R = 0 \} = \{0\}. \quad (3.7)$$

The subset of the moduli space where the charge lattice $\Gamma$ disobeys condition (3.7) is the union of those subsets where $\Gamma$ contains some vector $(\gamma_L, \gamma_R)$ with $\gamma_R = 0$ and $\gamma_L \cdot \gamma_L = 2N, N \in \mathbb{N}$. This is a countable union of nowhere dense sets. Hence the theories in the dense subset of the moduli space where (3.7) holds are the generic ones\footnote{To see that this is true already when in any geometric interpretation, one only varies the volume parameter, using [NW01] (1.11) as standard convention for charge vectors, note that for $T^D \cong \mathbb{R}^{2D}/\Lambda$, every $\gamma_R$ has the form $\gamma_R = \frac{1}{\sqrt{2}}(\mu - B\lambda - \lambda)$ with $\lambda \in \Lambda, \mu \in \Lambda^\ast = \{ y \in \mathbb{R}^{2D} \mid y \cdot \lambda \in \mathbb{Z} \ \forall \lambda \in \Lambda \}$ and a skew-symmetric $B \in \text{End}(\mathbb{R}^{2D})$. Varying the volume parameter amounts to scaling $\Lambda$ by some $t \in \mathbb{R}$, where $(t\Lambda)^\ast = t^{-1}\Lambda^\ast$.}. For the generic conformal field theoretic Hodge-elliptic genus of Def. 2.5, (3.5) thus immediately implies

$$E_{\text{Hodge}}^0 (T^D; \tau, z, \nu) = q^h \left(\frac{-i\theta_1(\tau, z)}{\eta(\tau)^3} \right)^D. \quad (3.8)$$

This may be used to confirm the ideas presented in Sect. 2.3 in the case of complex tori. First,

$$E_{\text{Hodge}}^0 (T^D; \tau, z, \nu) \cong E_{\text{Hodge}}^0 (T^D; \tau, z, \nu),$$

that is, Conjecture 2.7 holds for complex tori. Now let us determine a candidate for the generic space $\mathbb{H}^R$ that was described in Sect. 2.3. By construction, SCFTs with target $T^D$ share a common chiral algebra that can be described as $N = 2D$ superextension $\mathcal{A}$ of a $\mathfrak{u}(1)^{2D}$ current algebra. In general, any irreducible representation $\mathbb{H}_h$ of $\mathcal{A}$ in the Ramond sector has lowest weight $h + \frac{1}{8}$ with $h \geq 0$, and one finds

$$\text{Tr}_{\mathbb{H}_h} \left( (-1)^{y_0} y^{y_0} q^{L_0 - \frac{3}{8}} \right) = q^h \left(\frac{-i\theta_1(\tau, z)}{\eta(\tau)^3} \right)^D = q^h \left( (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^D + O(q) \right). \quad (3.9)$$

In particular, if $\phi \in \mathbb{H}_h$ with $L_0 \phi = \frac{1}{8} \phi$, then $\phi \in \mathbb{H}_0$, and analogously for right-movers. Moreover, for a non-linear sigma model with target $T^D$ and charge lattice $\Gamma \subset \mathbb{R}^{2D}$, the Ramond sector has the form

$$\mathbb{H}^R = \bigoplus_{\gamma = (\gamma_L, \gamma_R) \in \Gamma} \mathbb{H}_{\frac{1}{2} \gamma_L \cdot \gamma_L} \otimes \mathbb{H}_{\frac{1}{2} \gamma_R \cdot \gamma_R}.$$

With $\mathcal{N} := \{ \phi \in \mathbb{H}_0 \mid L_0 \phi = \frac{1}{8} \phi \}$, viewed as a representation of the Lie algebra of type $\mathfrak{u}(1)$ generated by $L_0$, according to Def. 2.2 and the discussion in Sect. 2.3 we thus have

$$\mathbb{H}^R = \bigoplus_{\gamma = (\gamma_L, \gamma_R) \in \Gamma} \mathbb{H}_{\frac{1}{2} \gamma_L \cdot \gamma_L} \otimes \mathcal{N}.$$
Therefore, by what was said above about the generic behaviour of the charge lattices \( \Gamma \) on the moduli space,
\[
\hat{H}^R := H^0 \otimes \tilde{H} \subset \tilde{H}^R
\]
is a representation of the common chiral algebra \( \mathcal{A} \), extended by \( J_0 \), as required in Sect. 2.3.

The space \( \hat{H}^R \) also obeys the other requirement (2.3),
\[
\text{tr}_{\hat{H}^R} \left( (-1)^{\Lambda_0} y^{\Lambda_0} u^{\Lambda_0} q^{L_0 - \frac{D}{8}} \right) \equiv \left( \frac{-i \vartheta_1(\tau, z)}{\eta(\tau)^3} \right) \left( u^{-\frac{1}{2}} - u^{\frac{1}{2}} \right)^D \equiv \mathcal{E}_{\text{Hodge}}^0(T^D; \tau, z, \nu).
\]

We summarize the above results in

**Proposition 3.1.** The conformal field theoretic Hodge-elliptic genus of a non-linear sigma model with target a complex torus \( T^D \) depends severely on the moduli of the theory, as is readily seen from (3.5). The generic conformal field theoretic Hodge-elliptic genus in this case agrees with the chiral Hodge-elliptic genus, confirming Conjecture 2.7 for complex tori. In particular, there is a dense subset of the moduli space of non-linear sigma models with target \( T^D \) where
\[
\mathcal{E}_{\text{CFT}}^{\text{Hodge}}(\mathbb{H}; \tau, z, \nu) = \mathcal{E}_{\text{ch}}^{\text{Hodge}}(T^D; \tau, z, \nu).
\]

In fact, the above equation holds generically, within the moduli space of SCFTs that arise as non-linear sigma models with target \( T^D \).

Finally, for complex tori, the chiral Hodge-elliptic genus agrees with the complex Hodge-elliptic genus of \([KT17]\), and thus the latter agrees with the generic conformal field theoretic Hodge-elliptic-genus in this case.

Though the final statement of the above proposition is not claimed explicitly in \([KT17]\), it is surely known to the authors, as their discussions in \([KT18]\) indicate. It is probably the origin of the false expectation that the complex Hodge-elliptic genus should always agree with the generic conformal field theoretic Hodge-elliptic genus. Note also that the subset of the moduli space of SCFTs with target \( T^D \) where the charge lattice \( \Gamma \) does not obey (3.7) contains the set of rational toroidal SCFTs and thus is dense. There, the conformal field theoretic Hodge-elliptic genus differs from the complex Hodge-elliptic genus of \( T^D \). This is the basis of the discussions in \([KT18]\).

### 4 Hodge-elliptic genera for K3

This section provides results on the Hodge-elliptic genera introduced in Sect. 2, in the case of K3 theories and K3 surfaces. We establish a formula for each of these; for the generic conformal field theoretic Hodge-elliptic genus, the result is obtained under the assumption that generically, the chiral algebra of a K3 theory is precisely the \( N = 4 \) superconformal algebra at central charge \( c = 6 \). Apart from the fact that in string theory, this assumption is commonly believed to hold true, Sect. 4.4 provides further evidence in favour of this assumption, see Prop. 4.7. In the final section 4.5, we discuss the consequences of our findings for Mathieu Moonshine.

#### 4.1 The conformal field theoretic Hodge-elliptic genus of Kummer K3 theories

The *Kummer construction* is a classical construction of certain K3 surfaces as \( \mathbb{Z}_2 \)-orbifolds of complex two-tori (see e.g. \([Wen17]\) §1.3 for a recent summary adjusted to our purposes). Its conformal field theoretic counterpart yields examples of K3 theories that are obtained by the standard \( \mathbb{Z}_2 \)-orbifold construction from non-linear sigma models with target a complex two-torus \([EOTY89]\). In the current section, we investigate the conformal field theoretic Hodge-elliptic genera in this particular setting. For the experts, this again is an easy exercise; nevertheless, as we shall see, there seem to be some misconceptions about this in the literature.
Consider a SCFT with space of states $\mathbb{H}_{\text{orbifold}}$, obtained as a standard $\mathbb{Z}_2$-orbifold SCFT from a non-linear sigma model with target a complex torus $T^D$ of dimension $D$ with charge lattice $\Gamma$ as in [3.4]. Using standard conformal field theory techniques (reviewed, for example, in [Wen17] §2.2, in a form adjusted to our purposes), for the $R$-sector of the partition function (see Assumption 2.1), one obtains

\[
Z_{\text{orbifold}}^R(\tau, z) = \frac{1}{2} \left( Z_V(\tau) \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|^{2D} + \frac{2\vartheta_2(\tau, z)}{\vartheta_2(\tau, 0)} \right)^{2D} + \frac{2\vartheta_3(\tau, z)}{\vartheta_3(\tau, 0)} \right)^{2D} + \frac{2\vartheta_4(\tau, z)}{\vartheta_4(\tau, 0)} \right)^{2D} \right).
\]

This yields

\[
E_{\text{Hodge}}(\mathbb{H}_{\text{orbifold}}; \tau, z, \nu) = \frac{1}{2} \left( \sum_{\gamma=\gammaL \cdot \gammaR \in \Gamma \text{ with } \gammaR = 0} \frac{q^{\frac{1}{2} \gammaL \cdot \gammaL}}{\eta(\tau)^{2D}} \cdot \left( \frac{-i\vartheta_1(\tau, z)}{\eta(\tau)} \cdot (u^{-\frac{1}{2}} - u^{\frac{1}{2}}) \right)^D + \left( \frac{2\vartheta_2(\tau, z)}{\vartheta_2(\tau, 0)} \cdot (u^{-\frac{1}{2}} + u^{\frac{1}{2}}) \right)^D + \left( \frac{4\vartheta_3(\tau, z)}{\vartheta_3(\tau, 0)} \right)^D + \left( \frac{4\vartheta_4(\tau, z)}{\vartheta_4(\tau, 0)} \right)^D \right).
\]

For $D = 2$, inserting $\nu = 0$ correctly yields the elliptic genus $E(X; \tau, z)$ of a K3 surface $X$ as in [2.2]. Analogously to the derivation of Prop. 3.1 we arrive at

**Proposition 4.1.** There is a dense subset of the moduli space of $\mathbb{Z}_2$-orbifolds of non-linear sigma models with target $T^D$ where the charge lattices $\Gamma$ obey [3.4]. The corresponding SCFTs have conformal field theoretic Hodge-elliptic genus

\[
E_{\text{Hodge}}(\mathbb{H}_{\text{orbifold}}; \tau, z, \nu) = \frac{1}{2} \left( \left( \frac{-i\vartheta_1(\tau, z)}{\eta(\tau)^{3}} \cdot (u^{-\frac{1}{2}} - u^{\frac{1}{2}}) \right)^D + \left( \frac{2\vartheta_2(\tau, z)}{\vartheta_2(\tau, 0)} \cdot (u^{-\frac{1}{2}} + u^{\frac{1}{2}}) \right)^D + \left( \frac{4\vartheta_3(\tau, z)}{\vartheta_3(\tau, 0)} \right)^D + \left( \frac{4\vartheta_4(\tau, z)}{\vartheta_4(\tau, 0)} \right)^D \right).
\]

In fact, the above equation holds generically for $\mathbb{Z}_2$-orbifolds of non-linear sigma models with target $T^D$.

For $D = 2$, the formula proved in Prop. 4.1 agrees with the one stated in [KT17] §3, p. 253], where however it is claimed that this formula yields the conformal field theoretic Hodge-elliptic genus for the $\mathbb{Z}_2$-orbifold conformal field theory obtained from the toroidal theory with target the standard torus $\mathbb{R}^4/\mathbb{Z}_2$. This claim of [KT17] §3, p. 253] is a little misleading, since it can only hold true if a non-rational B-field is chosen, as to ensure condition [3.7], i.e. if the $\mathbb{Z}_2$-orbifold conformal field theory under inspection is generic. The choice of the standard torus is irrelevant.

### 4.2 The generic conformal field theoretic Hodge-elliptic genus of K3 theories

The current section establishes a formula for the generic conformal field theoretic Hodge-elliptic genus of K3 theories, under the following

**Assumption 4.2.** The chiral algebra of a generic K3 theory, according to Def. 2.3 agrees with the $N = 4$ superconformal algebra at central charge $c = 6$.\[12\]

\[\text{Note that odd } D \text{ yields } \mathbb{Z}_2\text{-orbifolds that do not allow a geometric interpretation on a compact Calabi-Yau target. Indeed, resolving the singularities of } T^D/\mathbb{Z}_2 \text{ does not yield a Calabi-Yau manifold, since the holomorphic volume form on } T^D \text{ is not preserved by the } \mathbb{Z}_2\text{-action. The resulting conformal field theories are well-defined nevertheless, with the formulae for the partition function and its } R\text{-sector as stated.}\]
This assumption is widely believed to hold true, in string theory, though probably the only solid piece of evidence in favour of this assumption is the lack of a better candidate for such a generic chiral algebra. Below, and in Sect. 4.4, we will present further evidence in favour of Assumption 1.2. see Prop. 4.1. The assumption allows us to determine the representation $\mathbb{H}^R$ used in (2.3), resulting in an explicit formula for the generic conformal field theoretic Hodge-elliptic genus of K3 theories. To explain this, in the following, a priori only assume that $\mathbb{H}$ is the space of states of a fixed K3 theory as in Def. 2.3 and let $X$ denote a K3 surface.

As mentioned in the discussion of Def. 2.3, all K3 theories enjoy $N = (4, 4)$ supersymmetry. Hence the subspace $\mathbb{H}^R$ of $\mathbb{H}^R$ is a representation of the left-moving $N = 4$ superconformal algebra, extended by $J_0$. Moreover, by our assumptions on SCFTs in general and K3 theories in particular, $\mathbb{H}^R$ decomposes into a direct sum of tensor products of irreducible representations of the left- and the right-moving $N = 4$ superconformal algebra. Such representations have been classified, and their characters have been determined in [ETS7, ET88a, ET88b, ET88c, Tao90].

There are three types of irreducible unitary representations of the $N = 4$ superconformal algebra at central charge $c = 6$, called the VACUUM REPRESENTATION, the MASSLESS MATTER REPRESENTATION, and finally the MASSIVE MATTER REPRESENTATIONS. The latter form a one-parameter family indexed by $h \in \mathbb{R}_{>0}$. For our purposes, we may focus on the Ramond sector. Alluding to the properties of the corresponding representations in the Neveu-Schwarz sector $\mathbb{H}^{NS}$, which are related to the representations in $\mathbb{H}^R$ by spectral flow, we denote the respective irreducible unitary representations by $\mathcal{H}_0$, $\mathcal{H}_{mm}$, $\mathcal{H}_h$ ($h \in \mathbb{R}_{>0}$). Indeed, the ground state of the vacuum representation in the NS-sector is the vacuum of the theory.

Using $\frac{2\pi}{h} = \frac{1}{2}$, the respective characters of the irreducible unitary representations of the $N = 4$ superconformal algebra at central charge $c = 6$ are denoted by

$$\chi_a(\tau, z) := \text{Tr}_{\mathcal{H}_a} \left( (-1)^J y^J q^{L_0 - \frac{c}{24}} \right), \quad a \in \mathbb{R}_{\geq 0} \cup \{mm\}.$$  

For the reader’s convenience, we state the explicit formulae of [ET88a] for these functions in Appendix A. But at this point, we only need the following properties:

$$\chi_0(\tau, z = 0) = -2, \quad \chi_{mm}(\tau, z = 0) = 1,$$

$$\forall h > 0: \quad \chi_h(\tau, z) = q^h \tilde{\chi}(\tau, z) \quad \text{with} \quad \tilde{\chi}(\tau, z) = \chi_0(\tau, z) + 2\chi_{mm}(\tau, z), \quad (4.1)$$

hence

$$\chi_h(\tau, z = 0) = \tilde{\chi}(\tau, z = 0) = 0.$$

The constant $\chi_a(\tau, z = 0)$ is called the WITTEN INDEX [Wit82, Wit87, Wit88] of the respective representation.

The transformation properties of the above characters under the action of the modular group in general are not modular, in contrast to the situation at lower supersymmetry, where an infinite class of characters of irreducible unitary representations does enjoy modularity. Instead, the massless $N = 4$ characters exhibit a so-called MOCK MODULAR behaviour, as was already observed – necessarily using different terminology, then – in [ETS7, ET88a, ET88b, ET88c, Tao90].

We may now make an ansatz for a decomposition of $\mathbb{H}^R$ into irreducible representations of the two commuting $N = 4$ superconformal algebras,

$$\mathbb{H}^R = \bigoplus_{a, \overline{a} \in \mathbb{R}_{\geq 0} \cup \{mm\}} m_{a, \overline{a}} \mathcal{H}_a \otimes \overline{\mathcal{H}_{\overline{a}}},$$

with appropriate non-negative integers $m_{a, \overline{a}}$. Then the $R$-sector of the partition function of our theory (see Assumption 2.1) reads

$$Z_R(\tau, z) = \sum_{a, \overline{a} \in \mathbb{R}_{\geq 0} \cup \{mm\}} m_{a, \overline{a}} \cdot \chi_a(\tau, z) \cdot \overline{\chi_{\overline{a}}(\tau, z)}.$$ 

\[ A large part of the following argument already occurs in [Wen13, §4.1] and thus is based on [Wen00] as well as the ideas of [EOTY89]. \]
which together with Def. 2.2 yields the conformal field theoretic elliptic genus of our CFT as

\[
\mathcal{E}^{CFT}(\mathbb{H}; \tau, z) = \sum_{a, \pi \in \mathbb{R}_{\geq 0} \cup \{mm\}} m_{a, \pi} \cdot \chi_a(\tau, z) \cdot \chi_{\pi}(\tau, z = 0). \tag{4.2}
\]

Inserting (4.1) as well as a number of known properties of K3 theories (see [Wen15] §4.1 for details), this ansatz simplifies to

\[
\mathbb{H}^R = \mathcal{H}_0 \otimes \tilde{\mathcal{H}}_0 \oplus 20 \mathcal{H}_{mm} \otimes \tilde{\mathcal{H}}_{mm} \oplus \bigoplus_{n, h \in \mathbb{R}_{\geq 0}} k_{h, n} \mathcal{H}_h \otimes \tilde{\mathcal{H}}_n
\]
\[
\oplus \bigoplus_{n=1}^{\infty} [f_n \mathcal{H}_n \otimes \tilde{\mathcal{H}}_0 + f_n \mathcal{H}_0 \otimes \tilde{\mathcal{H}}_n]
\]
\[
\oplus \bigoplus_{n=1}^{\infty} [g_n \mathcal{H}_n \otimes \tilde{\mathcal{H}}_{mm} + g_n \mathcal{H}_{mm} \otimes \tilde{\mathcal{H}}_n]. \tag{4.3}
\]

The coefficients \(k_{h, n}, f_n, \tilde{f}_n, g_n, \tilde{g}_n\) are all non-negative integers, and their precise values depend on the specific K3 theory under inspection. By (4.2), with (4.1) and the refined ansatz (4.3), we obtain

\[
\mathcal{E}^{CFT}(\mathbb{H}; \tau, z) = -2\chi_0(\tau, z) + 20\chi_{mm}(\tau, z) + \sum_{n=1}^{\infty} [-2f_n + gn] \chi_n(\tau, z)
\]
\[
= -2\chi_0(\tau, z) + 20\chi_{mm}(\tau, z) + e(\tau) \tilde{\chi}(\tau, z)
\]
\[
\text{with } e(\tau) = \sum_{n=1}^{\infty} a_n q^n := \sum_{n=1}^{\infty} [gn - 2f_n] q^n. \tag{4.4}
\]

While the multiplicities \(g_n, f_n\) vary within the moduli space of K3 theories, the coefficients \(a_n := gn - 2f_n\) of \(e(\tau)\) are invariant. Since closed formulas for \(\mathcal{E}^{CFT}(\mathbb{H}; \tau, z) = \mathcal{E}(X; \tau, z)\), \(\chi_0(\tau, z)\), \(\chi_{mm}(\tau, z)\) and \(\tilde{\chi}(\tau, z)\) are known (see Def. 2.3 and Appendix A), one may solve the above equation (4.4) for \(e(\tau)\) if need be.

We will now make use of our Assumption 4.2 for the first time, to argue that \(a_n \geq 0\) for all \(n \in \mathbb{N}\). This follows, since the spectral flow maps the irreducible representation \(\mathcal{H}_0\) to the representation of the \(N = 4\) superconformal algebra whose ground state is the vacuum. Hence the coefficients \(f_n\) in (4.3) determine those contributions to the space of states which are holomorphic but do not belong to the vacuum representation under the \(N = 4\) superconformal algebra. For any fixed value of \(n \in \mathbb{N}\) with \(n > 0\), Assumption 4.2 implies that generically, no such additional contributions occur. In other words, our assumption implies that generically \(f_n = 0\) and thus that the \(n^{th}\) coefficient \(a_n\) of \(e(\tau)\) agrees with \(g_n \geq 0\). Since on the moduli space of K3 theories, these coefficients \(a_n\) are invariant, \(a_n \geq 0\) follows.

That the coefficients \(a_n\) are non-negative was already conjectured in [Oog89] and independently in [Wen00 Conj. 7.2.2] and was later proved in [EH09 EOT11]. This gives evidence in favour of Assumption 4.2 to hold true.

In light of equation (2.2), we see that by definition, and independently of Assumption 4.2, the decomposition (4.3) induces an isomorphism of representations of the left-moving \(N = 4\) superconformal algebra, extended by \(\tilde{f}_n\),

\[
\mathbb{H}^R = \mathcal{H}_0 \otimes \tilde{\mathcal{H}}_0 \oplus 20 \mathcal{H}_{mm} \otimes \tilde{\mathcal{H}}_{mm} \oplus \bigoplus_{n=1}^{\infty} f_n \mathcal{H}_n \otimes \tilde{\mathcal{H}}_0 \oplus \bigoplus_{n=1}^{\infty} g_n \mathcal{H}_n \otimes \tilde{\mathcal{H}}_{mm}. \tag{4.3}
\]

Note that in [EOTY89], the elliptic genus \(\mathcal{E}^{CFT}(\mathbb{H}; \tau, z)\) was already decomposed in the spirit of (4.3).

This is independent of whether or not, within the moduli space, one should expect that there is a dense subset of K3 theories that possess some additional holomorphic states beyond the vacuum representation of the \(N = 4\) superconformal algebra.
Employing Assumption 4.2, the generic values of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, and $\tilde{e}$, we arrive at

\[ \tilde{H}_\text{max} := H_0 \otimes \tilde{H}_0 \oplus 20 H_{\text{mm}} \otimes \tilde{H}_{\text{mm}} \oplus \bigoplus_{n=1}^{\infty} a_n H_n \otimes \tilde{H}_{\text{mm}}, \quad (4.5) \]

we have

\[ \tilde{H}^R = \tilde{H}_{\text{max}}^R \oplus \bigoplus_{n=1}^{\infty} f_n \mathcal{H}_n \otimes \left( \tilde{H}_0 \oplus 2 \tilde{H}_{\text{mm}} \right), \quad (4.6) \]

with model dependent multiplicities $f_n \geq 0$, such that independently of Assumption 4.2.

$$ \tilde{H}^R_{\text{max}} \supseteq \tilde{H}^R. \quad (4.7) $$

Employing Assumption 4.2, the generic values of $f_n$, $g_n$ across the entire moduli space of K3 theories are $f_n = 0$ and $g_n = a_n$, implying $\tilde{H}^R = \tilde{H}_{\text{max}}^R$. In fact, we find

$$ \text{Assumption 4.2} \iff \tilde{H}^R = \tilde{H}_{\text{max}}^R. \quad (4.8) $$

Now recall that by construction,

\[ \mathcal{E}^\text{CFT}(\mathbb{H}; \tau, z) := \sum_{n=0}^{\infty} \mathcal{E}^\text{CFT}(\mathbb{H}; \tau, z), \quad (4.1) \]

\[ \mathcal{E}^\text{CFT}(\mathbb{H}; \tau, z) = \text{tr}_{\tilde{H}^R_{\text{max}}} \left( (-1)^{J_0} y^{J_0} q^{L_0 - \frac{j}{2}} \right), \quad (4.2) \]

\[ \mathcal{E}^\text{Hodge}(\mathbb{H}; \tau, z) \overset{\text{Def.}}{=} \sum_{n=0}^{\infty} \mathcal{E}^\text{Hodge}(\mathbb{H}; \tau, z), \quad (4.3) \]

\[ \mathcal{E}^\text{Hodge}(\mathbb{H}; \tau, z) = \text{tr}_{\tilde{H}^R_{\text{max}}} \left( (-1)^{J_0} y^{J_0} q^{L_0 - \frac{j}{2}} \right), \quad (4.4) \]

\[ \mathcal{E}^\text{Hodge}(X; \tau, z) := \text{Tr}_{\tilde{H}^R_{\text{min}}} \left( (-1)^{J_0} y^{J_0} q^{L_0 - \frac{j}{2}} \right), \quad (4.5) \]

such that

$$ \text{Assumption 4.2} \iff \mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) = \mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu). \quad (4.6) $$

Here, $\tilde{H}^R$, $\tilde{H}^R_{\text{min}}$, and $\tilde{H}^R_{\text{max}}$ are solely viewed as representations of the left-moving $N = 4$ superconformal algebra, extended by $J_0$. A failure of (4.8) would imply that K3 theories generically possess a chiral algebra which is extended beyond the $N = 4$ superconformal algebra at central charge $c = 6$. In fact, by the discussion of Sect. 2.3 all common eigenspaces of $J_0$, $\mathfrak{J}_0$ and $L_0$ in $\tilde{H}^R$ attain the minimal possible dimensions across the moduli space. A failure of (4.8) would thus mean that all K3 theories possess a chiral algebra which is a proper extension of the $N = 4$ superconformal algebra at central charge $c = 6$, which would in fact be quite an exciting result. We emphasize once again that we do not assume that there might not be a dense subset of K3 theories within the moduli space that do possess holomorphic states beyond the vacuum representation of the $N = (4, 4)$ superconformal algebra. Indeed, we see no reason to make such a strong assumption, the analogue of which fails for toroidal superconformal field theories, as we have seen in Sect. 4.3.

If one wishes to make a connection to Mathieu Moonshine, one may replace the multiplicities $a_n$ by representations of $M_{24}$ according to [Gan16], see also [Wen15 §4]. We will come back to this comment in Sect. 4.3.

It is now straightforward to calculate a closed formula for $\mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu)$: by (4.5) and using (4.1),

\[ \mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) = \chi_0(\tau, z) \cdot (-u - u^{-1}) + 20 \chi_{\text{mm}}(\tau, z) + \sum_{n=1}^{\infty} a_n \chi_n(\tau, z). \]

Inserting (4.4), we arrive at

\[ \mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) = (2 - u - u^{-1}) \cdot \chi_0(\tau, z) + \mathcal{E}(X; \tau, z), \quad (4.9) \]
where closed formulas for $\chi_0(\tau, z)$ and $\mathcal{E}(X; \tau, z)$ are given in Appendix A and (2.2), respectively. Note that $\chi_0(\tau, z)$ has Mock modular transformation properties, as mentioned above. For an arbitrary K3 theory, (4.6) yields

$$E^\text{CFT}_\text{Hodge}(H; \tau, z, \nu) = E^{00}_\text{Hodge}(X; \tau, z) + (2 - u - u^{-1}) \sum_{n=1}^{\infty} f_n \chi_n(\tau, z),$$

with model dependent multiplicities $f_n \geq 0$. In fact, the above derivation proves

**Proposition 4.3.** Let $X$ denote a K3 surface, and consider a K3 theory according to Def. 2.3 with space of states $\mathbb{H}$, whose Ramond sector $\mathbb{H}^R$ decomposes according to (1.3). Then the conformal field theoretic Hodge-elliptic genus of this theory is given by

$$E^\text{CFT}_\text{Hodge}(\mathbb{H}; \tau, z, \nu) = (2 - u - u^{-1}) \cdot \left( \chi_0(\tau, z) + \sum_{n=1}^{\infty} f_n \chi_n(\tau, z) \right) + \mathcal{E}(X; \tau, z).$$

Furthermore, precisely one of the following holds: either, the generic conformal field theoretic Hodge-elliptic genus of all K3 theories obeys

$$E^0_{\text{Hodge}}(X; \tau, z, \nu) = (2 - u - u^{-1}) \cdot \chi_0(\tau, z) + \mathcal{E}(X; \tau, z),$$

or all K3 theories possess a chiral algebra which is a proper extension of the $N = 4$ superconformal algebra at central charge $c = 6$.

Recall that all standard $\mathbb{Z}_2$-orbifold conformal field theories obtained from non-linear sigma models with target a complex two-torus possess a chiral algebra which is a strict enhancement of the $N = 4$ superconformal algebra at central charge $c = 6$. This is immediately reflected in the fact that the formula for the generic conformal field theoretic Hodge-elliptic genus in Prop. 4.3 differs from the formula stated in Prop. 4.1. On first sight, this observation may seem surprising, because the latter formula is obtained from the Hodge-elliptic genus of complex two-tori (3.2) by standard orbifold techniques. Hence the discrepancy between the formula in Prop. 4.3 and the orbifold one implies that orbifolding techniques do not apply to the calculation of the generic conformal field theoretic Hodge-elliptic genus. Indeed, even though the partition function as well as the elliptic genus for K3 theories may be obtained by this procedure from the respective quantities for two-tori, this idea cannot work for the Hodge-elliptic genus. The explanation lies in the action of the modular group: while orbifolding techniques both for the partition function and the elliptic genus heavily use the fact that both of them exhibit modular transformation properties, there is no reason to expect such modular behaviour for the generic conformal field theoretic Hodge-elliptic genus. For K3 theories, Prop. 4.6 unveils a Mock modular behaviour, instead, albeit elliptic in $z$ with respect to $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$.

### 4.3 The complex Hodge-elliptic genus of K3 surfaces

How the complex Hodge-elliptic genus of [KT17] can be of help in the investigation of K3 theories or sigma models is not clear to us. But it certainly yields a new, highly non-trivial and interesting invariant for K3 surfaces, which is investigated more closely in this section. Though we will leave it to future work to achieve this goal, one important aim is to state a closed formula for the complex Hodge-elliptic genus of K3 surfaces.

In the following, let $X$ denote a K3 surface. Recall that the virtual bundle $\mathcal{E}_{q, -y}$ on $X$, which crucially enters the Definition 2.4 of the complex Hodge-elliptic genus, according to [Wen15 §4, Conjecture 1] allows a decomposition which induces the decomposition (4.3) of the complex elliptic genus into characters of irreducible representations of the $N = 4$ superconformal algebra: with notations as in Def. 2.4

$$\mathcal{E}_{q, -y} = -\mathcal{O}_X \cdot \chi_0(\tau, z) - T \cdot \chi_{\text{mm}}(\tau, z) + \sum_{n=1}^{\infty} p_n(T) \cdot \chi_n(\tau, z),$$

(4.10)
where $p_n(T)$ is a virtual bundle of the form

$$p_n(T) = \sum_{k=0}^{N_n} \alpha_k T^{\otimes k}, \quad \alpha_k \in \mathbb{Z}, \quad \text{with} \quad a_n = \chi(p_n(T)), \quad \text{where} \quad c(\tau) = \sum_{n=1}^{\infty} a_n q^n \quad (4.11)$$

as before, in \cite{14}. This can be viewed as a generalization of a local index theorem \cite{PA, GL, ABP73, Ge83}. With Thomas Creutzig \cite{CW15}, we have proved the claims \cite{110, 111}, using ideas that have been developed in \cite{CH14}. In fact, our proof reveals a refinement of the above conjecture: up to a global sign, all the virtual bundles $p_n(T)$ turn out to be direct sums of symmetric tensor powers of the holomorphic tangent bundle $T$, thus yielding each $-p_n(T)$ as an honest holomorphic vector bundle rather than a virtual bundle. For the complex Hodge-elliptic genus of $K3$ we thus obtain

$$E_{\text{Hodge}}(X; \tau, z, \nu)$$

\begin{align*}
\text{Def.} & \quad u^{-1} \chi^u(E_{q, -y}) \\
\text{Eq.110} & \quad - u^{-1} \chi^u(O_X) \cdot \chi_0(\tau, z) - u^{-1} \chi^u(T) \cdot \chi_{mm}(\tau, z) + \sum_{n=1}^{\infty} u^{-1} \chi^u(p_n(T)) \cdot \chi_n(\tau, z) \\
& \quad = -(u^{-1} + u) \cdot \chi_0(\tau, z) + 20 \chi_{mm}(\tau, z) + \sum_{n=1}^{\infty} u^{-1} \chi^u(p_n(T)) \cdot \chi_n(\tau, z). \quad (4.12)
\end{align*}

Equipped with this information, we are now ready to prove

**Proposition 4.4.** Let $X$ denote a $K3$ surface, and suppose that Assumption 4.2 holds. Then the complex Hodge-elliptic genus of $X$ differs from the generic conformal field theoretic Hodge-elliptic genus of $K3$ theories.

**Proof.** We prove the claim by contradiction. So let us assume that $E^h_{\text{Hodge}}(X; \tau, z, \nu) = E_{\text{Hodge}}(X; \tau, z, \nu)$. Since Assumption 4.2 is supposed to hold, Prop. 4.3 yields a closed formula for $E^h_{\text{Hodge}}(X; \tau, z, \nu)$. Comparison with the formula (4.12) for the Hodge-elliptic genus of $X$ implies that for all $n \in \mathbb{N}$ with $n > 0$, we have $\chi(p_n(T)) = a_n = u^{-1} \chi^u(p_n(T))$, in other words, that the holomorphic vector bundle $E_n := -p_n(T)$ obeys $H^j(X, E_n) = \{0\}$ if $j \neq 1$. However, as one checks by a direct calculation, $E_1 = S^2(T)$, $E_2 = 2S^3(T) \oplus O_X$, and hence $H^0(X, E_2) \neq \{0\}$. \hfill $\square$

In principle, (4.12) yields a formula for the complex Hodge elliptic genus of $K3$ surfaces. However, so far, no closed formula for the vector bundles $E_n$, $n \in \mathbb{N}$, is known. To arrive at a less implicit presentation, note that the derivation of (4.12) solely uses the observation that the virtual bundle $E_{q, -y}$ possesses global holomorphic sections which on each fiber of this bundle yield the structure of an $N = 4$ superconformal algebra at central charge $c = 6$. But the crucial step in the above proof of Prop. 4.3 rests on the fact that the virtual bundle $E_{q, -y}$ possesses additional global holomorphic sections. In fact, \cite{CH14} Thm. 3.1 and Remark 3.6 imply that the global holomorphic sections of this bundle induce the structure of a module on each fiber of $E_{q, -y}$ for a super vertex operator algebra $V^{SU(2)}$ which extends the $N = 4$ superconformal algebra by 8 fields of conformal dimensions $2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{3}{2}, 3, 3, 3$, respectively. The explicit form of these fields is also stated in \cite{CH14} Remark 3.6). This additional structure of the virtual bundle $E_{q, -y}$ may be used to arrive at a more promising formula for the complex Hodge-elliptic genus. So far, it remains implicit, as we shall explain next.

**Remark 4.5.** According to \cite{CH14} Thm. 3.12, similarly to the situation for the $N = 4$ superconformal algebra at central charge $c = 6$, the super vertex operator algebra $V^{SU(2)}$ possesses three types of irreducible representations with characters $\text{ch}_m$, $m \in \mathbb{N}$, where $m \in \{0, 1\}$ gives two massless characters and $m > 1$ gives an infinite family of massive ones. These characters are related to those of the $N = 4$ superconformal algebra via

$$\begin{align*}
\text{ch}_0(\tau, z) &= \chi_0(\tau, z) + H_0(\tau) \cdot \tilde{\chi}(\tau, z), \\
\text{ch}_1(\tau, z) &= \chi_{mm}(\tau, z) + H_1(\tau) \cdot \tilde{\chi}(\tau, z), \\
\forall m > 1: \quad \text{ch}_m(\tau, z) &= H_m(\tau) \cdot \tilde{\chi}(\tau, z),
\end{align*}$$

which is consistent with the fact that $H^0(X, E_2) \neq \{0\}$. In this sense, the virtual bundle $E_{q, -y}$ is a virtual bundle for the complex Hodge-elliptic genus of $K3$ surfaces. This also suggests a more promising formula for the complex Hodge-elliptic genus, which we shall explain next.
where

\[ H_m(\tau) = h_m(\tau) - 2h_{m+1}(\tau) + 2h_{m+3}(\tau) - h_{m+4}(\tau) + \delta_{m,0}, \]

\[ h_m(\tau) = \frac{1}{\eta(\tau)^2} \sum_{k,r,s \in \mathbb{Z}^+} (-1)^{r+s+1} q^{rk+sk-m+1-kr+s} (\text{sgn}(r)+\text{sgn}(k-m+1)s)^2 - \frac{m-1}{4}, \]

given here already proves this formula if Assumption 4.2 and Conjecture 2.7 hold true. That formula for the chiral Hodge-elliptic genus, stated in Prop. 4.6, below. In fact, the approach in this section, we prove an explicit formula for the chiral Hodge-elliptic genus of K3 surfaces.

4.4 The chiral Hodge-elliptic genus of K3

choice of complex structure, as it should. It would be interesting to know its modular properties. Note that our formula for the complex Hodge-elliptic genus of K3 surfaces is independent of the de Rham complex \( \Omega^{\text{ch}} \) = 4 superconformal vertex operator algebra at central charge \( c = 6 \), which was established again by [CH14] Thm. 3.12. In particular, the Witten genera of these characters agree with those of the respective characters of the \( N = 4 \) superconformal algebra.

Applying the same arguments that prove the decomposition (4.12), one obtains a refined decomposition of the bundle \( E_{q,-y} \) of the form

\[ E_{q,-y} = -\mathcal{O}_X \cdot \text{ch}_0(\tau, z) - T \cdot \text{ch}_1(\tau, z) - \sum_{m=2}^{\infty} P_m(T) \cdot \text{ch}_m(\tau, z), \]

where \( P_m(T) \) is a virtual bundle of the form \( P_m(T) = \sum_{k=0}^{N_m} \beta_k T^k \), \( \beta_k \in \mathbb{Z} \). Moreover, [CH14] Prop. 3.10 implies that \( P_m(T) \) has no non-trivial global holomorphic sections. We therefore find the following formula for the Hodge-elliptic genus of any K3 surface \( X \):

\[ \mathcal{E}_{\text{Hodge}}(X; \tau, z, \nu) = -(u + u^{-1}) \text{ch}_0(\tau, z) + 20 \text{ch}_1(\tau, z) + \sum_{m=2}^{\infty} b_m \text{ch}_m(\tau, z), \]

where the coefficients \( b_m = \dim H^1(X, P_m(T)) \) are obtained by comparison to (4.4), which implies

\[ \sum_{m=2}^{\infty} b_m H_m(\tau) = e(\tau) + 2H_0(\tau) - 20H_1(\tau). \]

Note that our formula for the complex Hodge-elliptic genus of K3 surfaces is independent of the choice of complex structure, as it should. It would be interesting to know its modular properties.

4.4 The chiral Hodge-elliptic genus of K3

In this section, we prove an explicit formula for the chiral Hodge-elliptic genus of K3 surfaces.

First note that Prop. 4.3 together with Conjecture 2.7 immediately implies a conjectural formula for the chiral Hodge-elliptic genus, stated in Prop. 4.10 below. In fact, the approach given here already proves this formula if Assumption 4.2 and Conjecture 2.7 hold true. That this latter conjecture is satisfied follows from the identification of the infinite volume limit of the topologically half-twisted sigma model on a K3 surface \( X \) with the sheaf cohomology of the chiral de Rham complex \( \Omega_X^{\text{ch}} \) by Kapustin [Kap05]. Since Assumption 2.7 is expected to hold true, as well, the above already gives a derivation of this formula from string theory ingredients. In addition, we give a direct mathematical proof of this proposition, below.

**Proposition 4.6.** The chiral Hodge-elliptic genus of a K3 surface \( X \) (Def. 2.6) is given by

\[ \mathcal{E}_{\text{Hodge}}^\text{ch}(X; \tau, z, \nu) = (2 - u - u^{-1}) \cdot \chi_0(\tau, z) + \mathcal{E}(X; \tau, z), \]

where closed formulas for \( \chi_0(\tau, z) \) and \( \mathcal{E}(X; \tau, z) \) can be found in Appendix A and (2.2), respectively. In particular, if Assumption 2.7 holds, then Conjecture 2.7 is true, i.e. for K3 theories we have

\[ \mathcal{E}_{\text{Hodge}}^\text{ch}(X; \tau, z, \nu) = \mathcal{E}_{\text{Hodge}}(H; \tau, z, \nu). \]

**Proof.** The second statement follows from the first one by Prop. 4.3 while the first statement is almost immediate from Bailin Song’s result [Son16 Thm. 1.2] that the chiral de Rham complex for a K3 surface does not have global holomorphic sections other than those furnishing the \( N = 4 \) superconformal vertex operator algebra at central charge \( c = 6 \), which were established
Note that this even holds if the moduli space of K3 theories possesses more than one component, 
Indeed, from [Son16] Thm. 1.2 and using the notations introduced in Sect. 4.2 above, it follows that as a representation of the \( N = 4 \) superconformal algebra, \( H^0(X, \Omega_X^{ch}) \cong \mathcal{H}_0 \). Moreover, Poincaré duality holds for the chiral de Rham complex [MS99], hence \( H^2(X, \Omega_X^{ch}) \cong \mathcal{H}_0 \), as well. Extending the \( N = 4 \) superconformal algebra by \( J_0 \), more precisely we have \( H^0(X, \Omega_X^{ch}) \oplus H^2(X, \Omega_X^{ch}) \cong \mathcal{H}_0 \otimes \mathcal{H}_0 \). Thus by Def. 2.6
\[
\mathcal{E}^\text{ch}_{\text{Hodge}}(X; \tau, z, \nu) = -(u + u^{-1}) \chi_0(\tau, z) - y^{-1} \text{tr}_{H^1(X, \Omega_X^{ch})} \left( (-1)^{J_0} y^{J_0} q^{J_0} \right).
\]
Insertion of \( \nu = 0 \) according to (4.6) yields the complex elliptic genus \( \mathcal{E}(X; \tau, z) \), and since the only unknown contributions to the right hand side of the above formula are independent of \( u \), one can solve for those unknown contributions at \( \nu = 0 \), confirming the claim.

Reversing the above arguments, one may view the string theory derivation of Prop. 4.6 given previously as an alternative, not entirely mathematical derivation of the beautiful result [Son16] Thm. 1.2 that the global holomorphic sections of the chiral de Rham complex for K3 surfaces yield precisely the \( N = 4 \) superconformal vertex operator algebra at central charge \( c = 6 \).

Moreover, we have

**Proposition 4.7.** If the infinite volume limit of a topologically half-twisted sigma model on a K3 surface \( X \) yields the sheaf cohomology of the chiral de Rham complex of \( X \), as is argued in [Kap05], then the generic chiral algebra of K3 theories is the \( N = 4 \) superconformal vertex operator algebra at central charge \( c = 6 \).

**Proof.** Our assumption on the infinite volume limit of a topologically half-twisted sigma model on a K3 surface \( X \) implies that \( H^*(X, \Omega_X^{ch}) \) is the generic space of states of K3 theories at generic volume and some fixed choice of a hyperkähler structure and B-field on \( X \). On the other hand, \( \widehat{H}^R \) is the generic space of states at generic values of all moduli of K3 theories. Hence we have
\[
\widehat{\mathbb{H}}^R_{\text{min}} \subset \widehat{H}^R \subset H^*(X, \Omega_X^{ch}).
\]
Note that this even holds if the moduli space of K3 theories possesses more than one component, since by construction, the number \( N_{h,Q} \) in Def. 2.5 is obtained as infimum over the entire moduli space, and \( \widehat{\mathbb{H}}^R_{\text{min}} \) is constructed as to obey (4.7). By (4.9) and Prop. 4.4 the full characters of \( \widehat{\mathbb{H}}^R_{\text{min}} \) and \( H^*(X, \Omega_X^{ch}) \) agree, hence \( \widehat{\mathbb{H}}^R_{\text{min}} = H^*(X, \Omega_X^{ch}) \) and thus \( \widehat{\mathbb{H}}^R_{\text{min}} = \widehat{H}^R \).

We conclude that \( \mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) = \mathcal{E}^0_{\text{Hodge}}(X; \tau, z, \nu) \) in (4.9) and thus, by Prop. 4.3, that Assumption 4.2 holds, as claimed.

The above results explain why, in contrast to the chiral de Rham complex, the virtual bundle \( E_{q,-y} \) can only be of limited use for the investigation of the fine structure of K3 theories. Indeed, by what was said in Sect. 4.3, \( E_{q,-y} \) has global holomorphic sections that do not belong to the \( N = 4 \) superconformal vertex operator algebra at central charge \( c = 6 \). This is a profound difference to the sheaf cohomology of the chiral de Rham complex, which in turn seems to beautifully model a generic field content of all K3 theories.

Note furthermore that by Prop. 4.6 the chiral Hodge-elliptic genus of K3 surfaces is independent of the complex structure, i.e. it yields a new topological invariant in this case. Kapustin’s work only assumes the transition to an infinite volume limit. We have thus shown the surprising fact that in the case of K3 theories, such an infinite volume limit, viewed solely as a representation of the \( N = 4 \) superconformal algebra extended by \( J_0 \), yields a generic space of states for all K3 theories. Moreover, the result of Prop. 4.6 is compatible with Conjecture 2.8 as one immediately checks, since the mirror of a K3 surface is a K3 surface.
4.5 A geometric Mathieu Moonshine Module

We close this note by commenting on the consequences of our findings for Mathieu Moonshine[16]. To this end, consider a K3 surface $X$ with fixed complex structure. Then any finite symplectic automorphism group $G$ of $X$ has a natural induced action on the cohomology $H^*(X, \mathbb{E}_q, y)$ of the virtual bundle $\mathbb{E}_q, y$, but also on the cohomology of the chiral de Rham complex, according to [GM04 (2.1.3)]. By [Muk88 Thm. 0.3], $G$ is a subgroup of the one-point stabilizer $M_{23}$ of the Mathieu group $M_{24}$, Mathieu Moonshine, on the other hand, which was discovered by Eguchi, Ooguri and Tachikawa in [EOT11] and proved by Gannon in [Gan16], predicts that the decomposition (4.3) into characters of irreducible representations of the $N = 4$ superconformal algebra is governed by an action of $M_{24}$. Yet what $M_{24}$ should act on, has been a mystery, so far.

Up to now, $H^*(X, \mathbb{E}_q, y)$ and $H^*(X, \Omega_X)$ seemed equally auspicious to this effect, since both carry natural actions of an $N = 4$ superconformal algebra at central charge $c = 6$, and of the finite symplectic automorphism group $G$. By [CH14 Thm. 4.3] and [Son17 Thm. 3.3], the resulting twisted elliptic genera agree with those of Mathieu Moonshine. However, Prop. 4.6 now shows that $H^*(X, \Omega_X)$ is by far more promising. Indeed, it implies that as a module of the $N = 4$ superconformal algebra, none of the contributions coming from massive representations contains a virtual representation, in agreement with the results of [Gan16]. As was explained in Sect. 4.3 this is different for $H^*(X, \mathbb{E}_q, y)$. Since the focus of [CH14] is entirely on the fine structure of $\mathbb{E}_q, y$, this may clarify why their attempts of explaining Mathieu Moonshine failed. We also conclude that our conjecture [Wen15 Conjecture 1] is a red herring when it comes to a geometric realization of the representation of $M_{24}$ that is relevant for Mathieu Moonshine. However, since this conjecture is correct [CW15], we hope that it may prove useful in the study of the complex Hodge-elliptic genus of [KT17], instead. On the other hand, the above findings support the expectations that we stated in [Wen15 §4.2], namely that the (holomorphic) chiral de Rham complex might bear the key to understanding Mathieu Moonshine.

To arrive at a satisfactory explanation for Mathieu Moonshine, one must now find a natural way to equip $H^*(X, \Omega_X)$ with all the structures predicted by Mathieu Moonshine. As was pointed out above, for any choice of complex structure on our K3 surface, the corresponding finite symplectic automorphism groups act naturally on $H^*(X, \Omega_X)$ in a fashion that is compatible with Mathieu Moonshine. One must now find a way to combine the actions of all such finite symplectic automorphism groups to the action of $M_{24}$. To do so, in [TW13, TW15b, TW15a] Taormina and the author have proposed a technique called symmetry surfing. Focusing on standard $\mathbb{Z}_2$-orbifold conformal field theories obtained from non-linear sigma models with target a complex two-torus, in these works we have been able to show that symmetry surfing allows to combine all finite symplectic automorphism groups of Kummer surfaces to the maximal subgroup $\mathbb{Z}_2^4 \rtimes \mathcal{A}_8$ of $M_{24}$.

Furthermore, in [TW15a] Taormina and the author show that the action of this group on the leading order massive representation may be realized on a subspace of the space of states that is common to all standard $\mathbb{Z}_2$-orbifold conformal field theories obtained from non-linear sigma models with target a complex two-torus. The resulting representation is equivalent to the restriction of the corresponding Mathieu Moonshine action of $M_{24}$ to this subgroup. The symmetry groups from distinct points in moduli space must be combined with a twist. Further evidence in favour of symmetry surfing, including the twist, is provided in [GKP17]. The focus on a subspace of the space of states which is common to all K3 theories that have been accessible to these methods, so far, is in full accord with the expectation that the cohomology of the chiral de Rham complex might play a key role in the explanation of Mathieu Moonshine. The behaviour of the chiral Hodge-elliptic genus found in this note further supports this idea. Indeed, we view the chiral Hodge-elliptic genus as a refinement of the traditional complex elliptic genus. Its agreement with the generic conformal field theoretic Hodge-elliptic genus, addressed in Props. 4.4 and 4.7, supports the idea that the sheaf cohomology $H^*(X, \Omega_X)$ of the chiral de Rham complex might play a key role in the explanation of Mathieu Moonshine.

[16] Some aspects of this discussion can be found analogously in the preprint [Son17] by Bailin Song, which reached me during the final stages of writing this note, as mentioned already in the Introduction.

25
de Rham complex should be viewed as a model for a subspace of the space of states that is generically present in K3 theories, along the lines presented in Sect. 4.2. Whether or not this subspace varies smoothly with respect to the moduli remains unanswered, for the time being. As is emphasized in TW13, TW15b, symmetry surfing crucially requires to restrict attention to the geometric symmetry groups, i.e. to groups $G$ that arise as finite symplectic automorphism groups of K3 surfaces\(^1\). We have already pointed out in Wen15, §4.2 that this might have its explanation in a required compatibility with an infinite volume limit of topologically half-twisted K3 theories, where we expect to find the sheaf cohomology of the chiral de Rham complex by Kapustin’s claims [Kap05].

It remains an open problem, however, to extend the action of the maximal subgroup $\mathbb{Z}_2^2 \rtimes A_8$ to an action of the entire group $M_{24}$. Then, an interpretation must be found for the action of those elements of $M_{24}$ which cannot act as finite symplectic automorphisms on any K3 surface. Furthermore, the behaviour of the multiplicity spaces of the massless representations of the $N = 4$ superconformal algebra remains obscure. Not least, why of all groups the Mathieu group $M_{24}$ plays such a prominent role for K3, remains unknown.

Finally, Mathieu Moonshine predicts the structure of a vertex operator algebra on the representation space that underlies $H^*\left(X, \Omega_X^2\right)$. Indeed, according to [Bor01, Prop. 3.7 and Def. 4.1], the cohomology of the chiral de Rham complex bears the structure of a super vertex operator algebra. For the $\mathbb{Z}_2$-orbifold conformal field theories obtained from non-linear sigma models with target a complex two-torus, the results of BL03, FS07, TW15b, GKP17 give strong evidence in favour of compatibility with the combined symmetry group $\mathbb{Z}_2^2 \rtimes A_8$ of $M_{24}$, if one respects the twist. For any K3 surface $X$, by introducing a novel filtration on $H^*\left(X, \Omega_X^2\right)$, Song proves in Son17, Thm. 3.2 that the associated graded object is a unitary representation of the $N = 4$ superconformal vertex operator algebra at central charge $c = 6$.

Altogether, Mathieu Moonshine seems to gradually unveil its mysteries.

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A Some $N = 4$ characters

The characters of the irreducible representations of the $N = 4$ superconformal algebra have been determined explicitly in ET88a. Here, we restrict ourselves to stating the characters in the twisted Ramond sector $\tilde{R}$ at central charge $c = 6$.

For our purposes, the most convenient formulas use the standard Jacobi theta functions,

\(^{17}\)To lift the action of $G$ to a K3 theory with geometric interpretation on the respective K3 surface, one may allow a non-trivial B-field iff the latter can be represented by some $G$-invariant $B \in H^2(K3, \mathbb{R})$.  

26
where we take the following normalizations:

$$\vartheta_1(\tau, z) := i \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} n^{-\frac{1}{2}} y^n$$

$$\vartheta_2(\tau, z) := \sum_{n=-\infty}^{\infty} q^{n^2} n^{-\frac{1}{2}} y^n$$

$$\vartheta_3(\tau, z) := \sum_{n=-\infty}^{\infty} q^{n^2} n^{-\frac{1}{2}} y^n$$

Moreover, we need the Mordell function $h_3(\tau)$ [Mor33 p. 347],

$$h_3(\tau) := \frac{1}{\eta(\tau) \vartheta_3(\tau, 0)} \sum_{m \in \mathbb{Z}} \frac{q^{m^2} n^{-\frac{1}{2}}}{1 + q^{m^2} n^{-\frac{1}{2}}} = \frac{2}{\eta(\tau) \vartheta_3(\tau, 0)} \sum_{m \in \mathbb{N} \setminus \{0\}} \frac{q^{m^2} n^{-\frac{1}{2}}}{1 + q^{m^2} n^{-\frac{1}{2}}}.$$

Then, including the more commonly used notations for those characters,

$$\chi_0(\tau, z) = \text{ch}_0(\ell = 0; \tau, z)$$

$$\chi_{mm}(\tau, z) = \text{ch}_0(\ell = \frac{1}{2}; \tau, z)$$

$$\tilde{\chi}(\tau, z) = \text{ch}_0(\ell = \frac{1}{2}; \tau, z)$$

From these formulas, one reads the leading order contributions in $q$ of each character:

$$\chi_0(\tau, z) = -y - y^{-1} + O(q), \quad \chi_{mm}(\tau, z) = 1 + O(q), \quad \tilde{\chi}(\tau, z) = 2 - y - y^{-1} + O(q). \quad \text{(A.1)}$$

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