A multiscale modeling of cell mobility: from kinetic to hydrodynamics

J. Nieto & L. Urrutia*

Abstract

This paper concerns a model for tumor cell migration through the surrounding extracellular matrix by considering, up to previous model, mass balance phenomena involving the chemical interactions produced on the cell surface. The well-posedness of this model is proven. An asymptotic analysis via a suitable hydrodynamic limit completes the description of the macroscopic behaviour.

Keywords: Cell mobility, kinetic theory, multiscale models, multicellular systems, hyperbolic limits, chemotaxis.

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1 Introduction

There is a huge literature describing mathematical models for cell migration through the extracellular matrix (ECM), specially tumor cells, since they usually try to reach a blood vessel to obtain nutrients or simply invade other parts of the body in a metastatic process. There are a lot of biological and mechanisms involved in these cell movement such as signaling, diffusion, chemotaxis, haptotaxis, reorientation due to the surrounding tissue fibers, cell–cell interactions, etc, and also some mechanical considerations as balance laws, mechanical forces, pressure, etc, (see for example [3, 5, 6, 7, 8, 9, 10]).

In general, there is some analogy with the models for mechanical particles, where biological considerations are included in several ways. For example, reorientations of the particles due to biological interactions can be modeled by a Boltzmann–type equations where the usual collision kernel plays the role of reorientation kernel. Of course, macroscopic descriptions (Navier–Stokes or Keller–Segel models) are very common, and the connections between kinetic and hydrodynamical models by way of limiting procedures has been largely treated (see for example [1 2 3 5 7]).

Following the analogy with the mechanical models, it is remarkable the framework of Kinetic Theory of Active Particles KTAP introduced by Bellomo

*University of Granada. Departamento de Matemática Aplicada, 18071 Granada, Spain. e-mail: jjmnieto@ugr.es, lurutia@ugr.es
and coll. (see for example \[1\] and references therein) where active particles join the double role of mechanical entities and that of living being. This theory allows to construct models for cell movement that take into account the heterogeneity of the cells, the biological interactions, birth/death phenomena, and also different scales of description. In this frame, a recent paper by Kelkel and Surulescu [6], present a multiscale model describing the evolution of tumor cell population density where the movement of the cells is mainly due to receptor dynamics on the cell surface. It joins several processes such as the binding of the cell surface to the ECM fibers, the chemotaxis due to a substance originated from the degradation of tissue fibers and the action mass law of the receptor on the cell surface.

In this work, we start from the multiscale model presented in [6], and include some mechanical and biological considerations to improve it. In order to be self consistent, we briefly describe in the next subsection the elements involved in the cell motion. In Section 2, we prove the local existence and uniqueness of solution for the obtained model and, in Section 3, we perform the hyperbolic limit of the stated model. In particular, we will obtain a closed relations between the averaged chemical substances of the cells involved in their movement and the respective concentration in the ECM.

1.1 The multiscale model

There are two interesting chemical substances in the environment: an oriented proteic fiber, and a degenerated chemical compound, coming from degeneration of the said fibers. Both compounds have their own dynamics and modify the cells movement. We call $Q(t, x, \theta)$ to the density of oriented proteic fibers at time $t$ and position $x$, oriented in direction $\theta \in \mathbb{S}^{n-1}$. The density of proteic fibers at time $t$ and position $x$ is denoted by $\bar{Q}(t, x)$:

$$\bar{Q}(t, x) := \int_{\mathbb{S}^{n-1}} Q(t, x, \theta) d\theta.$$ 

At last, we define $L(t, x)$ to be the concentration of the other chemical compound. From now on, we will use the same notation for the compounds and for their densities and concentrations, being now the $\bar{Q}$ and $L$ compounds, respectively.

The final model we propose consists on a system of a kinetic model for the cell population coming from the KTAP and two macroscopic reaction and reaction–diffusion equations for both chemical compounds. With this objective, the cells population will be treated as a system of active particles, meanwhile we will use macroscopic models for the chemicals. Here, we will improve the model presented in [6], by including a reaction term which take into account the balance mass of the compounds due to the chemical reactions produced in the cell surface.

We first describe the cell population by way of a standard distribution function $f(t, x, v, y)$ depending on time $t$, space $x$, velocity $v$ and activity $y$ (which
will be described below), verifying the following equation deduced in [6],
\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_y \cdot (G(y, \bar{Q}, L)f) = \mathcal{H}(f, Q) + \mathcal{L}(f) + \mathcal{C}(f, L),
\]

where the right–hand side models the cell mobility by way of velocity changes and the \(y\)–divergence term is related with the cell membrane reactions. Concretely,

- the term \(\mathcal{H}\), modeling haptotaxis, is
  \[
  \mathcal{H}(f, Q)(t, x, v, y) := \int_V \int_{S^{n-1}} p_h(t, x, v', y)p(v; v', \theta)f(t, x, v', y)Q(t, x, \theta)d\theta dv' - p_h(t, x, v, y)f(t, x, v, y)\bar{Q}(t, x);
  \]

- the turning operator \(\mathcal{L}\) models random changes in velocity,
  \[
  \mathcal{L}(f)(t, x, v, y) := \int_V p_l(t, x, v', y)\alpha_1(y)T(v, v')f(t, x, v', y)dv' - p_l(t, x, v, y)\alpha_1(y)f(t, x, v, y);
  \]

- and the chemotactic term, \(\mathcal{C}\), reads
  \[
  \mathcal{C}(f, L)(t, x, v, y) := \int_V p_c(t, x, v', y)\alpha_2(y)K[\nabla L](v, v')f(t, x, v', y)dv' - p_c(t, x, v, y)\alpha_2(y)f(t, x, v, y).
  \]

Here \(p_h, p_l\) and \(p_c\) are the interaction frequencies, \(\psi, T\) and \(K\) are the interaction kernels, and \(\alpha_i\) are nonnegative weight functions verifying \(\alpha_1 + \alpha_2 = 1\). For the three loss terms we have used the normalization properties [13–17].

In order to define the activity \(y\) and the cell membrane reactions term, we need to recover the mass action laws produced at the cell membrane involving the two chemicals in the ECM and the receptors on the cell,
\[
\bar{Q} + R \xrightarrow{k_1} \bar{Q}R, \quad L + R \xrightarrow{k_2} LR.
\]

where \(R\) stands for the free enzyme on the cell surface and \(\bar{Q}R\) and \(LR\) represents the respective complexes once the enzyme binds the ECM chemical. Then, \(y\) is defined as the two–component vector of microscopic concentrations of the two cell–membrane compounds \(\bar{Q}R\) and \(LR\), respectively. It is defined in the set
\[
Y = \{(y_1, y_2) \in (0, R_0) \times (0, R_0) : y_1 + y_2 < R_0\},
\]
where \(R_0 > 0\) represents the maximum concentration of receptors on the cell surface. The function \(G\) is given by the expression
\[
G(y, q, l) := \left(\frac{k_1(R_0 - y_1 - y_2)q - k_{-1}y_1}{k_2(R_0 - y_1 - y_2)l - k_{-2}y_2}\right).
\]
whose rows represents the respective equations associated with (2).

Now, we introduce the macroscopic equations for both free chemicals $Q$ and $L$ in the ECM:

$$\frac{\partial Q}{\partial t} = -\kappa \left( \int_V \int_Y \left( 1 - \frac{\theta \cdot v}{|v|} \right) f dv dy \right) Q$$

$$- k_1 Q \int_V \int_Y (R_0 - y_1) f dv dy + \frac{k_{-1}}{S_{n-1}} \int_V \int_Y y_1 f dv dy,$$

and

$$\frac{\partial L}{\partial t} = \kappa \int_{S_{n-1}} \left( \int_V \int_Y \left( 1 - \frac{\theta \cdot v}{|v|} \right) f dv dy \right) Q d\theta - r_L L + D_L \Delta_x L$$

$$- k_2 L \int_V \int_Y (R_0 - y_1 - y_2) f dv dy + \frac{k_{-2}}{S_{n-1}} \int_V \int_Y y_2 f dv dy.$$

We first identify here the models deduced in [6, 7], where the first term represents, in both equations, the production of chemical $L$ by degradation of the fiber $Q$ after interaction with a cell. Also a decay and diffusion of chemical $L$ can be observed. Finally, the two last reaction terms are introduced in this paper, with the aim of adding the mass balance due to the cell membrane interactions, which completes the previous model.

Now, we are interested in the well–posedness of the whole system (1), (3), and (4) in an adequate space.

### 2 Existence and uniqueness of solution to the model

First, we show where the variables are defined: $x \in \mathbb{R}^n$ for some $n \geq 1$, $y \in Y$ and $v \in V = [s_1, s_2] \times S^{n-1}$ with $0 \leq s_1 < s_2 < \infty$. Now, we recall some useful estimations, easy to prove, which can be found in [6].

**Lemma 1 (Properties of integral operators)** Let $T_0 > 0$ and $0 \leq t \leq T_0$. Then, the following properties holds.

1. Let $p_h(t) \in L^\infty(\mathbb{R}^n \times V \times Y)$ and $\psi(v; v', \theta)$ be non-negative functions verifying:

$$\int_V \psi(v; v', \theta) dv = 1, \quad \int_V \psi(v; v', \theta) dv' \leq M. \quad (5)$$

Then the integral operator $\mathcal{H}$ is a continuous bilinear mapping from $L^p(\mathbb{R}^n \times V \times Y) \times L^\infty(\mathbb{R}^n \times S^{n-1})$ to $L^p(\mathbb{R}^n \times V \times Y)$ ($p = 1, \infty$), verifying,

$$\|\mathcal{H}(f(t), Q(t))\|_p \leq C\|p_h(t)\|_\infty \|\tilde{Q}(t)\|_\infty \|f(t)\|_p.$$ 

Furthermore, if $Q(t) \in L^1(\mathbb{R}^n \times S^{n-1})$,

$$\|\mathcal{H}(f(t), Q(t))\|_1 \leq C\|p_h(t)\|_\infty \|\tilde{Q}(t)\|_1 \|f(t)\|_\infty.$$
2. Assume \( p(t) \in L^\infty(\mathbb{R}^n \times V \times Y) \) with \( p(t) \geq 0, \alpha_1 \in L^\infty(Y) \) and \( T(v,v') \) be given functions verifying:
\[
\int_V T(v,v')dv = 1; \quad |T(\cdot,v)| \leq C|v|.
\]
(6)

Then, the integral operator \( \mathcal{L} \) is a continuous mapping from \( L^p(\mathbb{R}^n \times V \times Y) \) to \( L^p(\mathbb{R}^n \times V \times Y) \) \( (p = 1, \infty) \), with the inequality
\[
\|\mathcal{L}(f(t))\|_p \leq 2\|p(t)\|_\infty\|f(t)\|_p.
\]

3. Let \( p_0(t) \in L^\infty(\mathbb{R}^n \times V \times Y) \), \( \alpha_2 \in L^\infty(Y) \) and \( K[F](v,v') \) be given functions verifying:
\[
\int_V K(v,v')dv = 1, \quad |K(\cdot,v)| \leq C|v|, \quad |K[F] - K[G]| \leq C|F - G|.
\]
(7)

Then, the integral operator \( \mathcal{C} \) is a continuous mapping from \( L^p(\mathbb{R}^n \times V \times Y) \times L^\infty(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n \times V \times Y) \) \( (p = 1, \infty) \), verifying
\[
\|\mathcal{C}(f(t), L(t))\|_p \leq 2M\|p_0(t)\|_\infty\|f(t)\|_p.
\]
To start with the proof of existence and uniqueness of solution of the system (1)-(3)-(4), we first add suitable initial conditions to complete the Cauchy problem, \( f(t = 0) = f_0, Q(t = 0) = Q_0, \) and \( L(t = 0) = L_0 \). We will develop an analogous technique to those given in [6].

The first step is uncoupling the equations, by substitution on each equation of the functions for a given non-negative function:

\[
\begin{align*}
 f^* & \in L^\infty(0,T_0; L^1(\mathbb{R}^n \times V \times Y) \cap L^\infty(\mathbb{R}^n \times V \times Y)), \\
 Q^* & \in L^\infty(0,T_0; L^1(\mathbb{R}^n \times S^{n-1}) \cap L^\infty(\mathbb{R}^n \times S^{n-1})), \\
 L^* & \in L^\infty(0,T_0; W^{1,1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)).
\end{align*}
\]
(8)

The uncoupled system reads:

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_y \cdot (G(y,Q^*, L^*) f) = \mathcal{H}(f, Q^*) + \mathcal{L}(f) + \mathcal{C}(f, L^*) + g,
\]
(9)

\[
\begin{align*}
 \frac{\partial Q}{\partial t} &= -\kappa \left( \int_V \int_Y \left( 1 - \left| \theta \cdot \frac{v}{|v|} \right| \right) f^* dv dy \right) Q \\
 & \quad - k_1 Q \int_V \int_Y (R_0 - y_1 - y_2) f^* dv dy + \frac{k_{-1}}{|S^{n-1}|} \int_V \int_Y y_1 f^* dv dy + h, \\
 \frac{\partial L}{\partial t} &= \kappa \int_{S^{n-1}} \left( \int_V \int_Y \left( 1 - \theta \cdot \frac{v}{|v|} \right) f^* dv dy \right) Q^* d\theta - r_L L + D_L \Delta_x L \\
 & \quad - k_2 L \int_V \int_Y (R_0 - y_1 - y_2) f^* dv dy + k_{-2} \int_V \int_Y y_2 f^* dv dy,
\end{align*}
\]
(10)

where \( g(t, x, v, y) \) and \( h(t, x, \theta) \) are two additional functions.

We start with the following result for equation (9).
Moreover, if

$$f \in L^1(\mathbb{R}^n \times V) \cap L^\infty(\mathbb{R}^n \times V \times Y)$$

be a non-negative function, take $g \in L^1(0, t_0; L^1 \cap L^\infty)(\mathbb{R}^n \times V \times Y)$ and consider $Q^*$ and $L^*$ verifying $[\mathcal{X}]$. We assume that

$$f \in L^\infty(\mathbb{R}^n \times V; L^1(Y)) \cap L^\infty(\mathbb{R}^n \times V; W^{1,\infty}(Y)) \cap L^1(\mathbb{R}^n \times V; W^{1,1}(Y)).$$

We also suppose that we are in the hypothesis of Lemma $[\mathcal{Y}]$ that $p_c, p_h, \nabla_y p_h$ and $\nabla_y p_c$ are functions of $L^\infty(0, t_0; L^\infty(\mathbb{R}^n \times V \times Y))$, and that functions $\nabla_y \alpha_1, \nabla_y \alpha_2$ are bounded. Then, there exist an unique weak solution $f$ to the equation $[\mathcal{Z}]$ with initial condition $f_0$. Also, $f$ verifies $(p = 1, \infty)$

$$\|f(t)\|_p \leq \left(\|f_0\|_p + \int_0^{T_0} \|g(\tau)\|_p d\tau\right) (1 + Cte^{Ct})$$  \hspace{1cm} (12)

where $C$ is a positive constant, linear dependant of $\|Q^*\|_\infty$ and $\|L^*\|_\infty$. Furthermore, if $g \equiv 0$,

$$\|\nabla_y f(t)\|_\infty \leq (\|\nabla_y f(0)\|_\infty + C\|f_0\|_\infty (T_0 + CT_0^2 e^{CT_0})) e^{CT_0}.$$

Now we prove the next result for the equation to $Q$.

**Theorem 2** Let $Q_0, h(t) \in L^1(\mathbb{R}^n \times \mathbb{S}^{n-1}) \cap L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$, $Q_0$ be non negative and $f^*$ as in $[\mathcal{A}]$. Then there exists an unique function $Q(t) \in L^1(\mathbb{R}^n \times \mathbb{S}^{n-1}) \cap L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ solution of $(10)$ with initial condition $Q(t = 0) = Q_0$. Furthermore, $\forall t \in [0, T_0]$ this solution verifies $(p = 1, \infty)$:

$$\|Q(t)\|_p \leq \|Q_0\|_p + \int_0^{T_0} \|h(\tau)\|_p d\tau + C \int_0^{T_0} \|\rho^*(\tau)\|_p d\tau$$  \hspace{1cm} (13)

where

$$\rho^*(t, x) := \int_V \int_Y f^*(t, x, v, y) dv dy.$$

Moreover, if $h = 0$, then $Q(t) \geq 0$.

**Proof.** Existence and uniqueness of solution is straightforward, because it is a linear differential equation. Actually, the solution can be written as

$$Q(t) = e^{\int_0^t J(\tau) d\tau} Q_0 + \int_0^t e^{\int_\tau^t J(\sigma) d\sigma} \left(\frac{k_{-1}}{|\mathbb{S}^{n-1}|} \int_V \int_Y y f^*(s) dv dy + h(s)\right) ds,$$

with a function $J$ given by

$$J(t, x, y) := -\kappa \int_V \int_Y \left(1 - \left|\frac{\theta}{|\theta|}\right| \right) f^*(v, y) dv dy - k_1 \int_V \int_Y (R_0 - y_1 - y_2) f^*(v, y) dv dy.$$

Here, by takings absolute values, we deduce that

$$|Q(t)| \leq e^{\int_0^t J(\tau) d\tau} |Q_0| + \int_0^t e^{\int_\tau^t J(\sigma) d\sigma} \left(\frac{k_{-1}}{|\mathbb{S}^{n-1}|} \int_V \int_Y y f^*(s) dv dy + |h(s)|\right) ds.$$

Then, using that $J$ is non positive, we deduce $[\mathcal{Z}]$ with $C = k_{-1} R_0 / |\mathbb{S}^{n-1}|$.

Finally, the equation of $L$ is a linear perturbation of the heat equation. So, classic results lead to:

$$\int_V \int_Y f(t) dv dy = \int_V \int_Y f \left|\frac{\partial}{\partial t}\right|.$$
solution to the complete model (1)–(3)–(4). We introduce the following notation:

Furthermore, for all $t \in [0, T)$, we have,

$$
\|L(t)\|_p \leq C (\|f^*(t)\|_\infty \|Q^*(t)\|_p + \|\rho^*(t)\|_p),
$$

(14)

$$
\|\nabla L(t)\|_1 \leq C (\|f^*(t)\|_\infty \|Q^*(t)\|_1 + \|\rho^*(t)\|_1).
$$

With these results in mind, we can prove the existence and uniqueness of solution to the complete model (1)–(3)–(4). We introduce the following notation:

$$
\mathbb{X}_f := L^\infty(0, T_0; L^1(\mathbb{R}^n \times V \times Y) \cap L^\infty(\mathbb{R}^n \times V \times Y)),
$$

$$
\mathbb{X}_Q := L^\infty(0, T_0; L^1(\mathbb{R}^n \times S^{n-1}) \cap L^\infty(\mathbb{R}^n \times S^{n-1})),
$$

$$
\mathbb{X}_L := L^\infty(0, T_0; W^{1,1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)),
$$

$$
\mathbb{X} := \mathbb{X}_f \times \mathbb{X}_Q \times \mathbb{X}_L.
$$

endowed with its respective natural norms. We can now give the main result of this section.

**Theorem 4** Let $Q_0 \in L^1(\mathbb{R}^n \times S^{n-1}) \cap L^\infty(\mathbb{R}^n \times S^{n-1})$ a non–negative function and $f_0$ be in the conditions of theorem (3). Then, given the initial conditions $f(0) = f_0, Q(t = 0) = Q_0, L(t = 0) = 0$, there exists $T_0 > 0$ such that the system (2)–(4) have an unique weak solution $(f, Q, L) \in \mathbb{X}$ defined in $[0, T_0]$.

**Proof.** First of all, we construct the sequence $(f_j, Q_j, L_j)$, defined in $[0, T_0]$ as the corresponding solution of (2)–(4)–(11) with $g, h = 0$, initial data $(f_0, Q_0, 0)$, and $(f^*, Q^*, L^*) = (f^{j-1}, Q^{j-1}, L^{j-1})$ for any $j \geq 1$, starting with $f^0 = Q^0 = L^0 = 0$. Using Theorems 1, 2 and 3, it is straightforward to prove that the sequence is well–defined in $\mathbb{X}$. Actually, for a constant $R > 2 \|(f_0, Q_0, 0)\|_\mathbb{X}$, we can choose a small enough $T_0$ such that the sequence in the closed ball $B(R)$ of radius $R$ of the space $\mathbb{X}$.

Our objective is to prove that this sequence converges to a solution of (1)–(3)–(4) with initial data $(f_0, Q_0, 0)$. To do that, we study the difference between two consecutive elements of the sequence. First, we note that $(f^{j+1} - f^j)$ verifies equation (4) with initial condition 0, $(Q^*, L^*) = (Q^j, L^j)$, and

$$
g := \mathcal{H}(f^j, Q^j - Q^{j-1}) + C(f^j, L^j) - C(f^j, L^{j-1})
$$

$$
+ \nabla_y \cdot \left( (G(y, Q^{j-1}, L^{j-1}) - G(y, Q^j, L^j)) f^j \right).
$$

Using Theorem 1, Lemma 4 and the trivial inequality

$$
|G(y, q, l) - G(y, \tilde{q}, \tilde{l})| + |\nabla_y \cdot (G(y, q, l) - G(y, \tilde{q}, \tilde{l}))| \leq C(|q - \tilde{q}| + |l - \tilde{l}|),
$$
we can easily deduce
\[
\|f^{j+1} - f^j\|_{L^\infty(0,T_0;L^1(\mathbb{R}^n \times Y))} \leq C(R) \left( \|Q^{j+1} - Q^j\|_{L^\infty(0,T_0;L^1(\mathbb{R}^n \times \mathbb{S}^{n-1}))} + \|L^j - L^{j-1}\|_{L^\infty(0,T_0;W^{1,1}(\mathbb{R}^n))} \right),
\]
with \(C\) increasingly dependent on \(T_0\).

Analogously, for \((Q^{j+1} - Q^j)\), it can be proven that
\[
\|Q^{j+1} - Q^j\|_{L^\infty(0,T_0;L^1(\mathbb{R}^n \times \mathbb{S}^{n-1}))} \leq C(R) \left( \|f^{j+1} - f^j\|_{L^\infty(0,T_0;L^1(\mathbb{R}^n \times Y))} \right),
\]
by noticing that \((Q^{j+1} - Q^j)\) solves \([11]\) with \(f^* = (f^j - f^{j-1})\), with initial data 0, and
\[
h := -Q^j \int_V \int_Y \left( 1 - \left| \frac{v}{|v|} \right| \right) + k_1(R_0 - y_1 - y_2) (f^j - f^{j-1})dvdy
- (Q^{j+1} - Q^j) \int_V \int_Y \left( 1 - \left| \frac{v}{|v|} \right| \right) + k_1(R_0 - y_1 - y_2) f^{j-1}dvdy.
\]

Finally, for \((L^{j+1} - L^j)\), similar argument allows us to obtain
\[
\|L^{j+1} - L^j\|_{L^\infty(0,T_0;W^{1,1}(\mathbb{R}^n))} \leq C(R) \left( \|f^{j+1} - f^j\|_{L^\infty(0,T_0;L^1(\mathbb{R}^n \times Y))} + \|Q^j - Q^{j-1}\|_{L^\infty(0,T_0;L^1(\mathbb{R}^n \times \mathbb{S}^{n-1}))} \right).
\]

Analogous estimates can be obtained in \(L^\infty\), using the corresponding inequalities. We finally obtain
\[
\|(f^{j+1} - f^j, Q^{j+1} - Q^j, L^{j+1} - L^j)\|_X \leq C \|(f^{j+1} - f^j, Q^j - Q^{j-1}, L^j - L^{j-1})\|_X.
\]

In order to complete the proof, we can choose \(T_0\) small enough in such a way that \(C < 1\), and then, conclude the convergence of the sequence to a solution of the system.

The uniqueness comes from the same computation, noticing that the argument is implicitly a Banach fixed point theorem, which gives us the uniqueness of the limit, then that of the solution. \(\blacksquare\)

3 High-Field limit in the model

We want to study a macroscopic description of the previous model, by way of a suitable hyperbolic hydrodynamical limit. As seen before, haptotaxis and chemotaxis are the key to describe the evolution of the system, so we want them to retain its influence in the macroscopic description. To keep these properties in the limit is a powerful motivation to study the hyperbolic limit in the previous system. Some references about scale limits are, for example \([11, 2, 3]\), for parabolic and hyperbolic limits in a generic system with single/multiple populations modeled by the KTAP, or \([4]\) to understand the connections between parabolic and hyperbolic scales in general kinetic theory.
3.1 Hyperbolic scaling

In this section we perform the typical fluid description of a kinetic model by way of an macroscopic limit of hyperbolic type. We note that equations (3) and (4) are already macroscopic, so the scale should not change it. On the following, the interaction frequencies \( p_h, p_l, \) and \( p_c \) are considered to be constant (in other case, the scaling does not make sense). First of all, we define the dimensionless variables.

\[
\begin{align*}
  t &= \hat{t} \tau, \quad x := \hat{x} R, \quad v := \hat{v} s_2, \quad y := \hat{y} R_0, \\
  f(t, x, v, y) &= \hat{f}(\hat{t}, \hat{x}, \hat{v}, \hat{y}), \quad Q(t, x, \theta) := \hat{Q}(\hat{t}, \hat{x}, \hat{\theta}), \quad L(t, x) := \hat{L}(\hat{t}, \hat{x}), \\
  p_k(t, x, v, y) &= \hat{p}_k, (k = h, l, c), \quad G(t, Q, L) := \hat{G}(\hat{y}, \hat{Q}, \hat{L}), \\
  \alpha_j(y) &= \hat{\alpha}_j(\hat{y}), (j = 1, 2), \quad T(v, v') := \frac{1}{\hat{s}_2} \hat{T}(\hat{v}, \hat{v'}), \\
  \psi(v; v', \theta) &= \frac{1}{\hat{R}_0 \hat{s}_2} \hat{\psi}(\hat{v}; \hat{v'}, \hat{\theta}), \quad K(v, v') := \frac{1}{\hat{s}_2} \hat{K}(\hat{v}, \hat{v'}),
\end{align*}
\]

where the “hat” variables are dimension–less, and \( \tau, R, \hat{f}, \hat{p}_h, \) and \( \hat{G} \) are typical quantities of their respective variables. The new variables are defined in the sets

\[
\hat{V} := \frac{1}{s_2} V, \quad \hat{Y} := \frac{1}{\hat{R}_0} Y.
\]

Our system becomes:

\[
\begin{align*}
  \frac{\partial \hat{f}}{\partial t} + \frac{s_2 \tau}{R} \hat{v} \cdot \nabla_{\hat{x}} \hat{f} + \frac{\tau \hat{G}}{\hat{R}_0} \nabla_{\hat{y}} \cdot (\hat{G} \hat{f}) = \hat{p}_h \tau \hat{\mathcal{H}}(\hat{f}, \hat{Q}) + \hat{p}_l \tau \hat{\mathcal{L}}(\hat{f}) + \hat{p}_c \tau \hat{\mathcal{C}}(\hat{f}, \hat{L}), \\
  \frac{\partial \hat{Q}}{\partial t} &= - \tau R^2 \hat{f} \hat{k} \int_{\hat{V}} \int_{\hat{Y}} \left( 1 - \frac{\theta \cdot \hat{v}}{|\hat{v}|} \right) \hat{f} \hat{d} \hat{v} \hat{d} \hat{y} \hat{Q} \\
  &\quad - \tau R^3 \hat{s} \hat{k} \hat{Q} \int_{\hat{V}} \int_{\hat{Y}} (\hat{y}_1 - \hat{y}_2) \hat{f} \hat{d} \hat{v} \hat{d} \hat{y} + \tau R^2 \hat{s} \hat{f} \hat{k}_{-1} \int_{\hat{V}} \int_{\hat{Y}} \hat{y}_1 \hat{f} \hat{d} \hat{v} \hat{d} \hat{y}, \\
  \frac{\partial \hat{L}}{\partial t} &= \tau R^2 \hat{f} \hat{k} \int_{\hat{V}} \int_{\hat{Y}} (1 - \frac{\theta \cdot \hat{v}}{|\hat{v}|}) \hat{f} \hat{d} \hat{v} \hat{d} \hat{y} \hat{Q} \hat{d} \theta - \tau \hat{\tau}_L \hat{L} + \frac{\tau}{R^2} D_L \Delta_{\hat{x}} \hat{L} \\
  &\quad - \tau R^3 \hat{s} \hat{f} \hat{k}_{2} \hat{L} \int_{\hat{V}} \int_{\hat{Y}} (\hat{y}_1 - \hat{y}_2) \hat{f} \hat{d} \hat{v} \hat{d} \hat{y} + \tau R^2 \hat{s} \hat{f} \hat{k}_{-2} \int_{\hat{V}} \int_{\hat{Y}} \hat{y}_2 \hat{f} \hat{d} \hat{v} \hat{d} \hat{y}.
\end{align*}
\]

We impose first the normalization restrictions \( \frac{s_2 \tau}{R} = 1 \) and \( \frac{\tau}{R^3} D_L = 1 \). The hyperbolic scaling corresponds to the choice

\[
\tau \hat{p}_1 = \frac{1}{\hat{\varepsilon}},
\]

i. e., the turning time \( \frac{1}{\hat{\varepsilon}} \) is very small compared to the typical time \( \tau \). After substituting in the equation, there are three other phenomena (cell membrane reactions, haptotaxis and chemotaxis) to consider. We rescale the corresponding
terms, assuming also that their frequencies are small compared with the turning frequency \( \bar{p}_t \). More precisely, we choose the following relations:

\[
\frac{\bar{G}}{R_0} = \varepsilon^a \bar{p}_t, \quad \bar{p}_n = \varepsilon^b \bar{p}_t, \quad \bar{p}_c = \varepsilon^d \bar{p}_t
\]

where \( 0 < a, b, d \geq 1 \).

To scale the other two equations, we remember that they are actually macroscopic, and then, it will preserve their form. So we only define the scaled associated non–dimensional constants involved. Define then,

\[
\hat{\kappa} := \tau R_0^2 \int f \kappa, \quad \hat{r}_L := \tau r_L,
\]

\[
\hat{k}_i := \tau R_0^2 \int f k_i, \quad \hat{k}_{i-1} := \tau R_0^2 \int f k_{i-1}, \quad (i = 1, 2).
\]

Skipping the “hat” for the non–dimensional variables, our system becomes

\[
\varepsilon \left( \frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon \right) + \varepsilon^a \nabla_y \cdot (G(y, Q_\varepsilon, L_\varepsilon) f_\varepsilon) = \varepsilon^b \mathcal{H}(f_\varepsilon, Q_\varepsilon)
\]

\[
+ \mathcal{L}(f_\varepsilon) + \varepsilon^d \mathcal{C}(f_\varepsilon, L_\varepsilon)
\]

(15)

for the cell population, where

\[
\mathcal{H}(f_\varepsilon, Q_\varepsilon)(t, x, y, v) = \int_Y \int_Y \psi(v, v', \theta) f_\varepsilon(t, x, v', y) Q_\varepsilon(t, x, \theta) d\theta dv'
\]

\[
- f_\varepsilon(t, x, v, y) Q_\varepsilon(t, x);
\]

(16)

\[
\mathcal{L}(f_\varepsilon)(t, x, y, v, y') = \int_Y \alpha_1(y) T(v, v') f_\varepsilon(t, x, v', y) dv'
\]

\[
- \alpha_1(y) f_\varepsilon(t, x, v, y);
\]

(17)

\[
\mathcal{C}(f_\varepsilon, L_\varepsilon)(t, x, y, v, y') = \int_Y \alpha_2(y) K[\nabla L_\varepsilon](v, v') f_\varepsilon(t, x, v', y) dv'
\]

\[
- \alpha_2(y) f_\varepsilon(t, x, v, y).
\]

(18)

and for the chemicals,

\[
\frac{\partial Q_\varepsilon}{\partial t} = -\kappa \left( \int_Y \int_Y \left( 1 - \left| \frac{\theta - v}{|v|} \right| \right) f_\varepsilon dv dy \right) Q_\varepsilon
\]

\[
- k_1 Q_\varepsilon \int_Y \int_Y (1 - y_1 - y_2) f_\varepsilon dv dy + \frac{k_{-1}}{s^{n-1}} \int_Y \int_Y y_1 f_\varepsilon dv dy
\]

(19)

\[
\frac{\partial L_\varepsilon}{\partial t} = \kappa \left( \int_Y \int_Y \left| \frac{\theta - v}{|v|} \right| f_\varepsilon dv dy \right) Q_\varepsilon d\theta - r_L L_\varepsilon + \Delta_x L_\varepsilon
\]

\[
- k_2 L_\varepsilon \int_Y \int_Y (1 - y_1 - y_2) f_\varepsilon dv dy + k_{-2} \int_Y \int_Y y_2 f_\varepsilon dv dy,
\]

(20)

where

\[ V = [s, 1] \times S^{n-1}, \quad Y = \{(y_1, y_2) \in (0, 1) \times (0, 1) : y_1 + y_2 < 1\}, \]

and \( s := s_1/s_2 \).
3.2 Deducing the limiting equations

We first mention some standard assumptions on the turning operator $L$:

- **Solvability conditions.** $L$ satisfies
  \[ \int_V L(f)dv = 0, \quad \text{and} \quad \int_V vL(f)dv = 0. \]

- **Kernel of $L$.** For all $\rho \geq 0, U \in \mathbb{R}^n$, there exists a unique function $M_{\rho,U} \in L^1(V \times Y, (1 + |v|)dv + dy)$ such that
  \[ L(M_{\rho,U}) = 0, \quad \int_V \int_Y M dvdy = \rho, \quad \int_V \int_Y vM dvdy = \rho U. \] (21)

Let’s formally deduce the limit equations. Being interested in a macroscopic limit, we study the equations verified by the moments of $f_\varepsilon$:

\[ \rho_\varepsilon := \int_V \int_Y f_\varepsilon dvdy, \quad \rho_\varepsilon U_\varepsilon := \int_V \int_Y v f_\varepsilon dvdy, \quad \rho_\varepsilon W_\varepsilon := \int_V \int_Y y f_\varepsilon dvdy. \]

Then, taking $\varepsilon = 0$ in (15) we obtain $L(f_0) = 0$ and therefore we deduce that the limiting function has to be the form $f_0 = M_{\rho_0,U_0}$. Let us now to check for the macroscopic equations verified by these moments. As usual, integrating equation (15) in $v$ and $y$, we get the mass conservation:

\[ \frac{\partial \rho_\varepsilon}{\partial t} + \nabla_x \cdot (\rho_\varepsilon U_\varepsilon) = 0. \] (22)

Then, multiplying (15) by $v$ and again integrating, we obtain:

\[ \frac{\partial (\rho_\varepsilon U_\varepsilon)}{\partial t} + \nabla_x \cdot (\mathbb{P}_\varepsilon + \rho_\varepsilon U_\varepsilon \otimes U_\varepsilon) = \varepsilon \int_Y \int_Y vH(f_\varepsilon, Q_\varepsilon) dvdy + \varepsilon \int_Y \int_Y vC(f_\varepsilon, L_\varepsilon) dvdy, \] (23)

where $\mathbb{P}_\varepsilon$ is the pressure tensor

\[ \mathbb{P}_\varepsilon(t,x) := \int_V \int_Y (v - U_\varepsilon) \otimes (v - U_\varepsilon) f_\varepsilon dvdy. \]

Finally, multiplying equation (15) by $y$ and integrating, we obtain:

\[ \varepsilon \partial_t \rho_\varepsilon W_\varepsilon + \varepsilon \nabla_x \cdot \int_V \int_Y y \otimes v f dvdy + \varepsilon^a (A_\varepsilon W_\varepsilon - b_\varepsilon) \rho_\varepsilon = 0, \] (24)

where, the matrix $A_\varepsilon$ and the vector $b_\varepsilon$ are respectively given by,

\[ A_\varepsilon := \begin{pmatrix} k_1 Q_\varepsilon + k_{-1} & k_1 Q_\varepsilon \\ k_2 L_\varepsilon & k_2 L_\varepsilon + k_{-2} \end{pmatrix}, \quad \text{and} \quad b_\varepsilon = \begin{pmatrix} k_1 Q_\varepsilon \\ k_2 L_\varepsilon \end{pmatrix}. \]
Now, we assume that solutions are a small perturbation of the limit, \( f_\varepsilon = M_{\rho_0,u_0} + \varepsilon f_1 \), and then, a Hilbert expansion of \( f_\varepsilon \) around \( M_{\rho_0,u_0} \). Inserting it in \([22],[23]\) and \([24]\), we obtain:

\[
\begin{align*}
\frac{\partial \rho_0}{\partial t} + \nabla_x \cdot (\rho_0 & U_0) = 0, \\
\frac{\partial (\rho_0 U_0)}{\partial t} + \nabla_x \cdot (P_0 + \rho_0 U_0 \otimes U_0) = \varepsilon^{b-1} \int_Y \int_V v\mathcal{H}(M_{\rho_0,u_0},Q_\varepsilon) \, dvdy \\
&+ \varepsilon^{d-1} \int_Y \int_V v\mathcal{C}(M_{\rho_0,u_0},L_\varepsilon) \, dvdy + O(\varepsilon^b) + O(\varepsilon^d), \\
(A_{\varepsilon} W_\varepsilon - b_\varepsilon) \rho_0 &= O(\varepsilon^{1-a}).
\end{align*}
\]

Formally, calling \( Q_0, L_0, A_0 \) and \( b_0 \) to the respective limits of \( Q_\varepsilon, L_\varepsilon, A_\varepsilon \) and \( b_\varepsilon \) and taking the limit \( \varepsilon \to 0 \), we obtain:

\[
\begin{align*}
\frac{\partial \rho_0}{\partial t} + \nabla_x \cdot (\rho_0 U_0) &= 0, \\
\frac{\partial (\rho_0 U_0)}{\partial t} + \nabla_x \cdot (P_0 + \rho_0 U_0 \otimes U_0) &= \delta_{b,1} \int_Y \int_V v\mathcal{H}(M_{\rho_0,u_0},Q_0) \, dvdy \\
&+ \delta_{d,1} \int_Y \int_V v\mathcal{C}(M_{\rho_0,u_0},L_0) \, dvdy, \\
(A_0 W_0 - b_0) \rho_0 &= 0,
\end{align*}
\]

where different regimes can be observed (\( \delta_{i,j} \) stands for the Kronecker delta), depending on the choice of the parameters \( b \) and \( d \). Actually, if \( b, d > 1 \), it is a pure hyperbolic system, if \( b > 1 \) and \( d = 1 \), we get a system with an additional chemotactic term meanwhile for \( b = 1 \) and \( d > 1 \) the additional term concerns the haptotaxis. Finally, for \( b = d = 1 \), we obtain the whole hyperbolic system including both phenomena. In all the cases, the third equation stands for a linear system that can be solved, obtaining the limiting distributions of compounds, i.e. the \( y \)–activity–moment of \( M_{\rho_0,u_0} \), expressed as follows:

\[
\rho_0 W_0 = \int_Y \int_V y M_{\rho_0,u_0} \, dy \, dv = \frac{\rho_0}{k_1 k_{-2} Q_0 + k_{-1} k_2 L_0 + k_{-2} k_{-1} k_2 L_0} \left( k_{1} k_{-2} Q_0 \right). \tag{25}
\]

On the other hand, limiting equations for \( \{Q_\varepsilon\} \) and \( \{L_\varepsilon\} \) can be deduced. For example, in the \( \{Q_\varepsilon\} \) equation we introduce the expansion for \( f_\varepsilon \), and formally taking the limit \( \varepsilon \to 0 \), we obtain:

\[
\begin{align*}
\frac{\partial Q_0}{\partial t} &= -\kappa \left( \int_Y \int_V \left( 1 - |v| \right) |v| M_{\rho_0,u_0} \, dvdy \right) Q_0 \\
&- k_1 Q_0 \int_Y \int_V (1 - y_1 - y_2) M_{\rho_0,u_0} \, dvdy + \frac{k_{-1}}{|S_{n-1}|} \int_Y \int_V y_1 M_{\rho_0,u_0} \, dvdy.
\end{align*}
\]
By using now the expression for the compounds (25), we obtain:

\[
\frac{\partial Q_0}{\partial t} = -\kappa \left( \int_V \int_Y \left( 1 - \frac{\theta \cdot v}{|v|} \right) M_{\rho_0, U_0} dv dy \right) Q_0 \\
+ \frac{k_1 k_{-1} k_{-2} \rho}{k_1 k_{-2} Q_0 + k_{-1} k_2 L_0 + k_{-1} k_{-2}} \left( \frac{Q_0}{|S_{n-1}|} - Q_0 \right).
\]

Note here that the last term cancels when taking integral with respect to \(\theta\). The same argument holds for \(L_0\), obtaining

\[
\frac{\partial L_0}{\partial t} = \kappa \int_{S_{n-1}} \left( \int_V \int_Y \left( 1 - \frac{\theta \cdot v}{|v|} \right) M_{\rho_0, U_0} dv dy \right) Q_0 d\theta - r_L L_0 + \Delta_x L_0 \\
- k_2 L_0 \int_V \int_Y (1 - y_1 - y_2) M_{\rho_0, U_0} dv dy + k_2 \int_V \int_Y y_2 M_{\rho_0, U_0} dv dy,
\]

where using again (25), it becomes:

\[
\frac{\partial L_0}{\partial t} = \kappa \int_{S_{n-1}} \left( \int_V \int_Y \left( 1 - \frac{\theta \cdot v}{|v|} \right) M_{\rho_0, U_0} dv dy \right) Q_0 d\theta - r_L L_0 + \Delta_x L_0.
\]

The main result of this section is the following.

**Theorem 5** Let \(\{f_\varepsilon, Q_\varepsilon, L_\varepsilon\}\) be the solution of (13)–(14)–(20) verifying hypothesis of Lemma 7 and

\[
\{\|f_\varepsilon\|_{L^\infty(0,T; L^1 \cap L^\infty)(\mathbb{R}^n \times V \times Y))} + \|Q_0, \varepsilon\|_{L^\infty(\mathbb{R}^n \times S_{n-1})} + \|L_0, \varepsilon\|_{L^\infty(\mathbb{R}^n)} \}< C < \infty.
\]

Assume also that the sucession \(\{f_\varepsilon, Q_\varepsilon, L_\varepsilon\}\) converges a.e. Then, the a.e. limit of \(f_\varepsilon\) is the function \(M_{\rho, U}\) given by the properties of \(L\), where \(\rho, U, W\) are the respective \(L^1\)-strong limits of \(\rho_\varepsilon, U_\varepsilon\) and \(W_\varepsilon\). Also, the sequences \(\{Q_\varepsilon\}\) and \(\{L_\varepsilon\}\) converge \(L^\infty\)-weakly* to some functions \(Q, L\).

Moreover, they solve the next equations, depending of the parameters:

\[
\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho U) = 0,
\]

\[
\frac{\partial (\rho U)}{\partial t} + \nabla_x \cdot (P_0 + \rho U \otimes U) = \delta_{b-1} \int_V \int_Y \nabla H(M_{\rho, U}, Q) dv dy \\
+ \delta_{d-1} \int_V \int_Y \nabla C(M_{\rho, U}, L) dv dy,
\]

\[
\rho W = \frac{k_1 k_{-2} Q}{k_1 k_{-2} Q + k_{-1} k_2 L + k_{-1} k_{-2}},
\]

and for \(Q\) and \(L\) we have,

\[
\frac{\partial Q}{\partial t} = -\kappa \left( \int_V \int_Y \left( 1 - \frac{\theta \cdot v}{|v|} \right) M_{\rho, U} dv dy \right) Q \\
+ \frac{k_1 k_{-1} k_{-2} \rho}{k_1 k_{-2} Q + k_{-1} k_2 L + k_{-1} k_{-2}} \left( \frac{Q}{|S_{n-1}|} - Q \right).
\]
\[
\frac{\partial L}{\partial t} = \kappa \int_{\mathbb{R}^{n-1}} \left( \int_V \int_Y \left( 1 - \frac{\theta \cdot v}{|v|} \right) M_{\rho,U} dv dy \right) Q d\theta - r_L L + \Delta_x L.
\]

**Proof.** We first observe that variables \(v\) and \(y\) are defined on bounded sets and then, we deduce that the sequences of moments \(\{\rho_\varepsilon\}\), \(\{\rho_\varepsilon U_\varepsilon\}\), and \(\{\rho_\varepsilon W_\varepsilon\}\) are also uniformly bounded in \(L^\infty(0,T;L^\infty(\mathbb{R}^n))\). On the other hand, going back to the inequalities (13) and (14) for \(Q_\varepsilon\) and \(L_\varepsilon\) with \(h = 0\), it follows that the sequences \(\{Q_\varepsilon\}\) and \(\{L_\varepsilon\}\) are uniformly bounded. So, we can pass to the limit, up to a subsequence, in the weak\(^*\) topology of \(L^\infty\) in all of them.

Now, the integral operators \(\{H(f_\varepsilon,Q_\varepsilon)\}\), \(\{L(f_\varepsilon)\}\), and \(\{C(f_\varepsilon,L_\varepsilon)\}\), can be also uniformly bounded in \(L^\infty(0,T;L^\infty(\mathbb{R}^n \times V \times Y))\), and then, obtain their convergence, but we have to identify these limits, which involves quadratic terms.

The weak\(^*\) limit of the sequence \(\{f_\varepsilon\}\) has to be its pointwise limit, called \(f_0\). Then, via Dunford-Pettis theorem, this convergence holds also weakly in \(L^1([0,T] \times \mathbb{R}^n \times V \times Y)\) locally, so then, strongly. Then, also \(L(f_\varepsilon)\) converges strongly to \(L(f_0)\) in the same space.

The showed convergences are enough to take limits, at least in a distributional sense, in the linear terms involved into the equations. Then, we have to take care of the non linear terms. Actually we only have to observe that the strong convergence in \(L^1_{loc}([0,T] \times \mathbb{R}^n \times V \times Y)\) of \(f_\varepsilon\) combined with the weak\(^*\)–\(L^\infty\) convergence of \(Q_\varepsilon\) and \(L_\varepsilon\) produces the following convergence

\[
H(f_\varepsilon,Q_\varepsilon) \to H(f_0,Q), \quad \text{and} \quad C(f_\varepsilon,L_\varepsilon) \to C(f_0,L),
\]

in a distributional sense. Analogously, due to the fact that \(v\) and \(y\) are defined in sets of finite measure, so this convergence holds also for its moments

\[
\int_V \int_Y v H(f_\varepsilon,Q_\varepsilon) dv dy, \quad \int_V \int_Y v C(f_\varepsilon,L_\varepsilon) dv dy.
\]

We can now take limit in the distributional sense in equation (15), obtaining \(\mathcal{L}(f_0) = 0\). Using the properties of \(\mathcal{L}\), we deduce that \(f_0 = M_{\rho,U}\) for some functions \(\rho\) and \(U\) given by (21). Then, again, variables \(v\) and \(y\) are defined in sets of finite measure, so the previous argument can be rewritten for the sequences of the moments, deducing that

\[
\rho_\varepsilon \to \rho, \quad \rho_\varepsilon U_\varepsilon \to \rho U, \quad \mathbb{P}_\varepsilon \to \mathbb{P}_0, \quad \text{and} \quad \rho_\varepsilon W_\varepsilon \to \rho W.
\]

Finally, we can pass to the limit in the macroscopic equations (22), (23), (24), (19) and (20) to deduce the announced result.

### 3.3 A particular case

There are two “hidden” problems in the previous development: the system of macroscopic equations is not close. Actually, the pressure tensor \(\mathbb{P}_0\) and the integral operators appearing in the macroscopic equation for \(\rho U\), involve some integrals with respect to the microscopic state, that have not been expressed as a functions of the macroscopic variables. Solving this problem requires an explicit
expression of the function $M_{\rho,U}$. Here we present a particular case, as done in \cite{2}, choosing a particular turning operator $L$ for which all the computations can be done.

Consider that $\alpha_1$ is a constant function and a kernel in the form

$$T(v,v') := \lambda + \beta v \cdot v',$$

with positive constants $\lambda$ and $\beta$, verifying

$$\lambda = \beta \frac{1 - s^{n+2}}{(1 - s^n)(n + 2)}. \quad (26)$$

Then, the operator $L$ given in (17) can be written as follows:

$$L(f) := \lambda \int_Y f(t,x,v',y)dv' + \beta \int_Y v'f(t,x,v',y)dv' - \lambda |V|f(t,x,v,y). \quad (27)$$

Note that, following the definition of $L$, the free parameter $\lambda$ has to verify $\lambda |V| = \alpha_1$. Let us compile the required properties of this operator.

**Lemma 2** Let $L$ be given by (27), with $\lambda$ and $\beta$ verifying (26). Then, $L$ has the following properties:

- $L$ verifies the hypothesis of Lemma 1.
- It verifies the hyperbolic solvability conditions: $\int_Y L(f)dv = \int vL(f)dv = 0$.
- Given $\rho \geq 0$ and $U \in \mathbb{R}^n$, the associated function $M_{\rho,U}$ in the kernel of $L$ verifying (27) is given by

$$M_{\rho,U} := \frac{\rho}{|V||Y|} \left(1 + \frac{\beta}{\lambda} v \cdot U\right). \quad (28)$$

**Proof.** Checking that $L$ verifies the hypothesis of Lemma 1 is immediate.

The solvability conditions can be easily deduced integrating the previous expression and using

$$|V| = \frac{|S^{n-1}|}{n}(1 - s^n), \quad \int_Y vdv = 0, \quad \int_Y v_i v_k dv = \frac{\delta_{ik}}{n(n+2)}(1 - s^{n+2}).$$

We check the second one. Multiplying by $v$ and integrating:

$$\int_Y \int_Y vL(f)dvdy = \lambda \int_Y \int_Y v_f(t,x,v',y)dv'dvdy + \beta \int_Y \int_Y (v \otimes v)v'f(t,x,v',y)dv'dvdy - |V|\lambda \int_Y \int_Y v_f(t,x,v,y)dvdy.$$
The first term is 0, via Fubini and \( \int_V vdv = 0 \). Analogously,
\[
\beta \int_Y \int_V (v \otimes v') f(t, x, v', y)dv'dy = \frac{|S^{n-1}|}{n(n+2)}(1 - s^{n+2})\beta \int_Y \int_V vf(t, x, v, y)dvdy,
\]
so then
\[
\int_Y \int_V \nabla(\mathbf{f})dvdy = \frac{|S^{n-1}|}{n(n+2)}(1 - s^{n+2})\beta - |V|\lambda \int_Y \int_V vf(t, x, v, y)dvdy,
\]
which is 0 once considered the value of \(|V|\) and the relation (26).

Finally, it is straightforward to verify that the function \( M_{\rho, U} \) given by (28) verifies (21) and then, generates the kernel of \( L \).

Let us now compute the pressure tensor \( P_0 \) and the integral operators (16) and (18) for this choice of turning operator. First, we calculate the pressure tensor.
\[
\int_Y \int_V v \otimes v M_{\rho, U}dvdy = \frac{\rho}{|V||Y|} \int_Y \int_V v \otimes vdvdy
\]
\[
+ \frac{\rho}{|V||Y|} \frac{\beta}{\lambda} \int_Y \int_V (v \otimes v) \cdot U dvdy
\]
\[
= \frac{\rho}{1 - s^{n+2}} \frac{2}{(n + 2)} I + 0,
\]
where \( I \) is the identity matrix. Then, we conclude,
\[
P_0 = 2 \frac{1 - s^{n+2}}{(n + 2)(1 - s^n)} \rho I - \rho U \otimes U.
\]

Now, we can compute the \( v \)-moments of the integral operators. Define the following macroscopic quantities, which will appear in the development:
\[
\Psi^1(\theta) := \int_Y \int_V v\psi(v; v', \theta)dv'dv',
\]
\[
\Psi^2(\theta) := \int_Y \int_V v \otimes v'\psi(v; v', \theta)dv'dv',
\]
\[
\mathcal{K}^1[L] := \int_Y \int_V vK[\nabla L](v, v')dv'dv',
\]
\[
\mathcal{K}^2[L] := \int_Y \int_V v \otimes v' K[\nabla L](v, v')dv'dv',
\]
and construct the following macroscopic integral operators:
\[
H(\rho, U, Q) := \frac{\rho}{|V|} \left( \int \left( \Psi^1(\theta) + \frac{\beta}{\lambda} \Psi^2(\theta) \cdot U \right) Q(\theta)d\theta - \bar{Q}|V|U \right),
\]
\[
C(\rho, U, L) := \frac{\rho \alpha_2}{|V|} \left( \mathcal{K}^1[L] + \left( \frac{\beta}{\lambda} \mathcal{K}^2[L] - |V|I \right) \cdot U \right),
\]
16
using the expression (28) in (16) and (18), calculation leads to:

\[
\int_V \int_Y vH(M_{\rho,U}, Q)dvdy = H(\rho, U, Q),
\]

\[
\int_V \int_Y vC(M_{\rho,U}, L)dvdy = C(\rho, U, L).
\]

Finally, the main Theorem 5 can be here rewritten in this particular case as follows.

**Theorem 6** Under the hypothesis of Theorem 5 and with the turning operator given by (27), the limiting equations for \( \rho \) and \( \rho U \) are:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0,
\]

\[
\frac{\partial (\rho U)}{\partial t} + 2\frac{1 - s^{n+2}}{(n+2)(1 - s^n)} \nabla \cdot \rho = \delta_{b-1} H(\rho, U, Q) + \delta_{d-1} C(\rho, U, L),
\]

\[
\rho W = \frac{\rho}{k_1 k_{-2} Q + k_{-1} k_2 L + k_{-1} k_{-2}} \left( k_1 k_{-2} Q \right).
\]

Also, the limiting equations for \( Q \) and \( L \) can be rewritten as follows:

\[
\frac{\partial Q}{\partial t} = -\kappa \frac{\rho}{|V|} g(\theta) Q + \frac{k_1 k_{-1} k_{-2} \rho}{k_1 k_{-2} Q + k_{-1} k_2 L + k_{-1} k_{-2}} \left( -Q + \frac{Q}{|S_{n-1}|} \right),
\]

\[
\frac{\partial L}{\partial t} = \kappa \frac{\rho}{|V|} \int_{S_{n-1}} g(\theta) Q(\theta) d\theta - r_L L + \Delta L,
\]

where \( g(\theta) = \int_V (1 - |\theta \cdot v|)dv. \)

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