BLOCK TOEPLITZ MATRICES: SOME BASIC RESULTS

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ABSTRACT. Given $A, B, C$ and $D$ block Toeplitz matrices, we will prove the necessary and sufficient condition for $AB - CD = 0$, and $AB - CD$ to be a block Toeplitz matrix. In addition, with respect to change of basis, the characterization of normal block Toeplitz matrices with entries from the algebra of diagonal matrices is also obtained.

1. Introduction

A scalar Toeplitz matrix is an $n \times n$ matrix with the following structure:

\[
A = \begin{pmatrix}
    a_0 & a_1 & a_2 & \ldots & a_{n-1} \\
    a_{-1} & a_0 & a_1 & \ldots & a_{n-2} \\
    a_{-2} & a_{-1} & a_0 & \ldots & a_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{1-n} & a_{2-n} & a_{3-n} & \ldots & a_0
\end{pmatrix}.
\]

The entries depend upon the difference $i - j$ and hence they are constant down all the diagonals. These matrices are ubiquitous and play an important role in different fields of mathematics, as well as applied areas as signal processing or time series analysis. The subject is many decades old; among monograph dedicated to the subject are \[12, 16\] and \[10\].

The related area of block Toeplitz matrices is less studied, one of the reason being the new difficulties that appear with respect to the scalar case. Besides its theoretical interests, the subject is also important to multivariate control theory. As references for block Toeplitz matrices on can use \[1, 11, 13, 14, 15\]. In \[2\] the authors have proved the variety of algebraic results about Toeplitz matrices. They have formulated those algebraic results in terms of products of Toeplitz matrices. Given Toeplitz matrices $A, B, C, \text{ and } D$, their main result determines

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if the matrix $AB - CD$ is Toeplitz. The necessary and sufficient condition is a rank two matrix equation involving tensor products of the vectors defining $A, B, C,$ and $D$. They have also proved the necessary and sufficient condition for $AB - CD = 0$. In addition to that they have also completely characterized the normal Toeplitz matrices. The characterization of normal Toeplitz matrices has been discussed in [3, 17, 4, 5, 6, 7, 18].

The purpose of the present paper is to generalize some of the results of [2] concerning the product of block Toeplitz matrices by way of introducing the special structure of the displacement matrix of a Block Toeplitz matrix.

The paper is organized as follows: Notations and some basic facts about displacement matrices are presented in section 2. Then as we will be interested in block Toeplitz matrices, some basic properties concerning the product of these matrices are derived in section 3. The last section is devoted to the study of normal Block Toeplitz matrices with entries from the algebra of diagonal matrices.

2. Preliminaries

Let $\mathbb{C}$ denote the set of complex numbers. We designate by $\mathcal{M}_n$ the algebra of $n \times n$ matrices and by $\mathcal{D}_d$ the algebra of $d \times d$ diagonal matrices with entries from $\mathbb{C}$. Throughout this paper, we prefer to label the indices from 0 to $n - 1$; so $A \in \mathcal{M}_n$ is written $A = (a_{i,j})_{i,j=0}^{n-1}$ with $a_{i,j} \in \mathbb{C}$. Then $\mathcal{T}_n \subset \mathcal{M}_n$ is the space of Toeplitz matrices $A = (a_{i,j})_{i,j=0}^{n-1}$. We will mostly be interested in block matrices, that is, matrices whose elements are not necessarily scalars, but elements in $\mathcal{M}_d$. Thus a block Toeplitz matrix is actually an $nd \times nd$ matrix, but which has been decomposed in $n^2$ blocks of dimension $d$, and these blocks are constant parallel to the main diagonal. we will use the following notations:

- $\mathcal{M}_n \otimes \mathcal{M}_d$ is the collection of $n \times n$ block matrices whose entries all belong to $\mathcal{M}_d$;
- $\mathcal{T}_n \otimes \mathcal{M}_d$ is the collection of $n \times n$ block Toeplitz matrices whose entries all belong to $\mathcal{M}_d$;
- $\mathcal{D}_n \otimes \mathcal{M}_d$ is the collection of $n \times n$ diagonal block Toeplitz matrices whose entries all belong to $\mathcal{M}_d$.
- $\mathcal{C}_1 \otimes \mathcal{M}_d$ is the collection of all $n \times 1$ block matrices whose entries all belong to $\mathcal{M}_d$;
- $\mathcal{R}_1 \otimes \mathcal{M}_d$ is the collection of all $1 \times n$ block matrices whose entries all belong to $\mathcal{M}_d$. 


Obviously $\mathcal{D}_n \otimes \mathcal{M}_d \subset \mathcal{T}_n \otimes \mathcal{M}_d \subset \mathcal{M}_n \otimes \mathcal{M}_d$. For block diagonal matrices we will use the notation

$$
\text{diag}(A_1 \ A_2 \ \cdots \ \ A_n) = \begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_n
\end{pmatrix}
$$

In many cases it will suffice to consider Toeplitz matrices with zero diagonals. In other cases $\tilde{A} + \tilde{A}_0$ will describe the most general Toeplitz matrix, where $\tilde{A}$ is a Toeplitz matrix with 0 on the main diagonal and $\tilde{A}_0$ is the diagonal Toeplitz matrix.

If $a = \begin{pmatrix} 0 \\ a_{-1} \\ \vdots \\ a_{1-n} \end{pmatrix}$, then we define $\hat{a} = \begin{pmatrix} 0 \\ a_{1-n} \\ \vdots \\ a_{-1} \end{pmatrix}$ and if $b = \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$, then

$$
\bar{b} = \begin{pmatrix} 0 \\ a_{n-1} \\ \vdots \\ a_1 \end{pmatrix}
$$

Let $I \in \mathcal{M}_d$ be the identity matrix and $S \in \mathcal{M}_n \otimes \mathcal{M}_d$ consisting of zero matrices, except for identity matrices along the subdiagonal, i.e.,

$$
S = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
I & 0 & 0 & \ldots & 0 & 0 \\
0 & I & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & I & 0
\end{pmatrix}.
$$

For any $M \in \mathcal{M}_n \otimes \mathcal{M}_d$ the displacement matrix is defined as

$$
\Delta(M) = M - SMS^*.
$$

We will use this matrix to determine whether the difference of the matrix products is Toeplitz. See [9] and [19] for other types of displacement matrices. We denote the matrix \( \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{R}_1 \otimes \mathcal{M}_d \) by $\mathcal{P}_+$, then its adjoint is the vector $\mathcal{P}_+^* \in \mathcal{C}_1 \otimes \mathcal{M}_d$.

**Lemma 2.1.** If $M \in \mathcal{M}_n \otimes \mathcal{M}_d$ then $M = \sum_{k=0}^{n-1} S^k(\Delta(M)) S^k$. 
Proof.
\[ \sum_{k=0}^{n-1} S^k(\Delta(M))S^{k*} = \sum_{k=0}^{n-1} S^k(M - SMS^*)S^{k*} = \sum_{k=0}^{n-1} (S^kMS^{k*} - S^{k+1}MS^{k+1*}) = M - S^nMS^n* = M \]

Thus to determine \( M = 0 \), it is sufficient to study the simpler equation \( \Delta(M) = 0 \).

3. Block Toeplitz Product

In this section we will generalize some important results of [2]. The following lemma describes the necessary and a sufficient condition for a block matrix \( A \) to be a Toeplitz matrix.

**Lemma 3.1.** \( A \in \mathcal{M}_n \otimes \mathcal{M}_d \) is Toeplitz if and only if there exist \( X, X' \in \mathbb{C}_1 \otimes \mathcal{M}_d \) such that \( \Delta(A) = XP_+ + P_+X'^* \).

**Proof.** Suppose that \( A = (A_{i,j})_{i,j=0}^{n-1} \in \mathcal{T}_n \otimes \mathcal{M}_d \). Since the displacement matrix for \( A \) is defined as \( \Delta(A) = A - SAS^* \). Then simple computation yields that

\[
A = \begin{pmatrix}
A_0 & A_1 & A_2 & \ldots & A_{n-1} \\
A_{-1} & 0 & 0 & \ldots & 0 \\
A_{-2} & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{-(n-1)} & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

If we take \( X = \begin{pmatrix} A_0 \\ A_{-1} \\ \vdots \\ A_{1-n} \end{pmatrix} \) and \( X' = \begin{pmatrix} 0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix} \) then one can easily check that

\[ \Delta(A) = XP_+ + P_+X'^* . \]

For the converse, let \( A = (A_{i,j})_{i,j=0}^{n-1} \in \mathcal{M}_n \otimes \mathcal{M}_d \). Suppose then that

\[
X = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{pmatrix} \text{ and } X' = \begin{pmatrix} X'_0 \\ X'_1 \\ \vdots \\ X'_{n-1} \end{pmatrix}
\]

be matrices in \( \mathcal{C}_1 \otimes \mathcal{M}_d \), since we
have

$$\triangle(A) = XP_+ + P_+^*X''$$

$$\implies$$

$$A = SAS^* + XP_+ + P_+^*X''$$

$$\implies$$

$$A = \begin{pmatrix}
X_0 + X_0 & X_1 & X_2 & \cdots & X_{n-1} \\
X_1 & A_{0,0} & A_{0,1} & \cdots & A_{0,n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_{n-1} & A_{n-2,0} & A_{n-1,1} & \cdots & A_{n-2,n-2}
\end{pmatrix}$$

Comparing corresponding entries yields $A_{i_1,j_1} = A_{i_2,j_2}$, whenever $i_1 - j_1 = i_2 - j_2$, where $0 \leq i_1, i_2, j_1, j_2 \leq n - 1$, i.e., $A$ is a block Toeplitz matrix.

$$\square$$

In the rest of this section, if $A = (A_{i,j})_{i,j=0}^{n-1}$ is any block Toeplitz matrix then for simplification we write $A = \tilde{A} + \check{A}_0$, where $\check{A} \in \mathcal{T}_n \otimes \mathcal{M}_d$.

**Lemma 3.2.** Let $\tilde{A}_- = \begin{pmatrix} 0 & A_{-1} & \vdots & A_{-(n-1)} \end{pmatrix}$, $\check{B}_+ = \begin{pmatrix} 0 & B_1 & \vdots & B_{n-1} \end{pmatrix}^T$, $\check{A}_+ = \begin{pmatrix} 0 & A_{n-1} & \vdots & A_1 \end{pmatrix}$.

and $\check{B}_- = \begin{pmatrix} 0 & B_{1-n} & \vdots & B_{-1} \end{pmatrix}^T$. If $A = \tilde{A} + \check{A}_0$ and $B = \check{B} + \check{B}_0$ then there exist $Y, Y' \in \mathcal{C}_1 \otimes \mathcal{M}_d$ such that $\triangle(AB) = YP_+ + P_+^*Y' + \tilde{A}_-\check{B}_+ + \tilde{A}_+\check{B}_- - \check{A}_-\check{B}_+$.

**Proof.** We have

$$\triangle(AB) = \triangle[(\tilde{A} + \check{A}_0)(\check{B} + \check{B}_0)]$$

$$= \triangle(\tilde{A}\check{B} + \tilde{A}\check{B}_0 + \check{A}_0\check{B} + \check{A}_0\check{B}_0) + \triangle(\check{A}_-\check{B}) + \triangle(\check{A}_+\check{B}) + \triangle(\check{A}_0\check{B}_0)$$

Since $\check{B}_0 \in \mathcal{D}_n \otimes \mathcal{M}_d$ then $\tilde{A}\check{B}_0 \in \mathcal{T}_n \otimes \mathcal{M}_d$, therefore by Lemma 3.1 there exist $U, U' \in \mathcal{C}_1 \otimes \mathcal{M}_d$ such that $\triangle(\tilde{A}\check{B}_0) = UP_+ + P_+^*U''$. Similarly $\triangle(\check{A}_0\check{B}) = VP_+ + P_+^*V''$ and $\triangle(\check{A}_0\check{B}_0) = WP_+ + P_+^*W''$, with $V, V', W, W' \in \mathcal{C}_1 \otimes \mathcal{M}_d$.

$$\triangle(AB) = \triangle(\tilde{A}\check{B}) + (U + V + W)P_+ + (U'' + V'' + W'')P_+^*$$
Now we have entry at the position \((i, j)\) of \(\tilde{A}\tilde{B}\) is

\[
(\tilde{A}\tilde{B})_{i,j} = \sum_{k=0}^{n-1} A_{k-i}B_{j-k}, \quad 0 \leq i, j \leq n - 1,
\]

Using the formula (3.2) and definition of \(S\), one obtains

\[
(\tilde{A}\tilde{B} - S\tilde{A}\tilde{B}S^*)_{i,j} = \begin{cases} 
A_{-i}B_j - A_{n-i}B_{j-n} & \text{for } 1 \leq i, j \leq n - 1, \\
(\tilde{A}\tilde{B})_{0,j} & \text{for } 1 \leq j \leq n - 1, \\
(\tilde{A}\tilde{B})_{i,0} & \text{for } 0 \leq i \leq n - 1.
\end{cases}
\]

Therefore

\[
\Delta (\tilde{A}\tilde{B}) = XP_0 + P_0^*X'^* + \tilde{A}\tilde{B}_+ - \tilde{A}_+\tilde{B}_-.
\]

Where \(X = \begin{pmatrix} (\tilde{A}\tilde{B})_{0,0} \\ (\tilde{A}\tilde{B})_{1,0} \\ \vdots \\ (\tilde{A}\tilde{B})_{n-1,0} \end{pmatrix}\), \(X' = \begin{pmatrix} 0 \\ (\tilde{A}\tilde{B})_{0,0} \\ \vdots \\ (\tilde{A}\tilde{B})_{0,n-1} \end{pmatrix}\). Combining (3.1) and (3.4) yields

\[
\Delta(AB) = (U + V + W + X)P_+ + (U'^* + V'^* + W'^* + X'^*)P_+^* + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_-
\]

\[
= YP_+ + P_+^*Y'^* + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_-.
\]

Where \(Y = U + V + W + X_0\) and \(Y'^* = U'^* + V'^* + W'^* + X_0'^*\). \(\square\)

The next result is the most important result of this paper.

**Theorem 3.3.** Let \(\tilde{A}_-, \tilde{B}_- \in C_1 \otimes M_d\) and \(\tilde{A}_+, \tilde{B}_+ \in R_1 \otimes M_d\) with 0 in the zeroth component. If \(A = \tilde{A} + \tilde{A}_0, B = \tilde{B} + \tilde{B}_0, C = \tilde{C} + \tilde{C}_0\) and \(D = \tilde{D} + \tilde{D}_0\) then: \(AB - CD \in T_n \otimes M_d\) if and only if \(\tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_- = \tilde{C}_-\tilde{D}_+ - \tilde{C}_+\tilde{D}_-\).

**Proof.** (i) By Lemma 3.2

\[
\Delta(AB - CD) = \Delta(AB) - \Delta(CD)
\]

\[
= (Y - Z)P_+ + P_+^*(Y'^* - Z'^*) + \tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_- - \tilde{C}_-\tilde{D}_+ + \tilde{C}_+\tilde{D}_-
\]

Note that the last four terms on the right side of the above equation involve vectors with 0 in the zeroth component. By Lemma 3.1, \(AB - CD\) is Toeplitz if and only if \(\tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_- = \tilde{C}_-\tilde{D}_+ - \tilde{C}_+\tilde{D}_-\).

\(\square\)

**Corollary 3.4.** If \(A, B \in T_n \otimes M_d\), then \(AB \in T_n \otimes M_d\) if and only if \(\tilde{A}_-\tilde{B}_+ - \tilde{A}_+\tilde{B}_- = 0\).
Proof. The proof follows immediately from Theorem 3.3 by taking $C = D = 0$. □

Theorem 3.5. Let $Y, Y', Z$ and $Z'$ be vectors in $C_1 \otimes \mathcal{M}_d$ with 0 on the zero component of $X'$ and $Z'$. If $AB - CD \in T_n \otimes \mathcal{M}_d$, then $AB = CD$ if and only if $Y = Z$ and $Z' = Z'$.

Proof. If $AB - CD \in T_n \otimes \mathcal{M}_d$, then by Lemma 3.1 there exist $Y, Y', Z$ and $Z'$ in $C_1 \otimes \mathcal{M}_d$ such that

$$\triangle(AB) - \triangle(CD) = (Y - Z)P_+ + P_+^*(Y'^* - Z'^*).$$

Now $AB = CD$ if and only if $\triangle(AB - CD) = 0$. The latter equation holds if and only if the vectors which form the product with $P_+$ are 0. That is, $AB = CD$ if and only if $Y = Z$ and $Y' = Z'$. □

4. Normal block Toeplitz matrices

In this section we will obtain the characterization for normal block Toeplitz matrices with entries from $D_d$.

The next result from [2] characterized normal Toeplitz matrices among all Toeplitz matrices.

Theorem 4.1. Let $A = (a_{i-j})_{i,j=0}^{n-1} \in T_n$ then $A$ is normal if and only if either $a = \lambda \hat{b}$ for some $|\lambda| = 1$ or $a = \lambda \check{b}$ for some $|\lambda| = 1$.

Before starting the main work of this section we first need to quote the result from [13].

Lemma 4.2. Suppose $A = (A_{i-j})_{i,j=0}^{n-1} \in T_n \otimes \mathcal{D}_d$. Then there is a change of basis that brings $A$ into the following form

$$A' = \text{diag}\left( A'_1, A'_2, \cdots, A'_d \right),$$

where for every $k = 1, 2, \cdots, d$, $A'_k \in T_d$.

The following result is the main result of this section.

Theorem 4.3. If $A \in T_n \otimes \mathcal{D}_d$ then $A$ is normal if and only if there exist scalars $\lambda_1, \lambda_2, \cdots, \lambda_d$ with $|\lambda_k| = 1$, such that for any $k = 1, 2, \cdots, d$, either $a_k = \lambda_k \hat{b}$ or $a_k = \lambda_k \check{b}$.

Proof. If $A \in T_n \otimes \mathcal{D}_d$, then by Lemma 4.2 $A$ has the form

$$A' = \text{diag}\left( A'_1, A'_2, \cdots, A'_d \right),$$

where for every $k = 1, 2, \cdots, d$, $A'_k = (a_{r-s,k})_{r,s=0}^{d}$. Now $AA^* = A^*A$ if and only if $A'_k A'^*_k = A'^*_k A'_k$. Then by Theorem 4.1 each $A_k$ is normal if and only if there exist scalars $\lambda_1, \lambda_2, \cdots, \lambda_d$ with $|\lambda_k| = 1$, such that
either \( a_k = \lambda_k \hat{b} \) or \( a_k = \lambda_k b_k \), where \( a_k = \begin{pmatrix} a_{-1,k} \\
 a_{-2,k} \\
 \vdots \\
 a_{-d,k} \end{pmatrix} \), and \( b_k = \begin{pmatrix} a_{1,k} \\
 a_{2,k} \\
 \vdots \\
 a_{d,k} \end{pmatrix} \),

\( k = 1, 2, \ldots, d \).

\[ \square \]

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