Impact of water toxicity and acidity on dynamics of prey-predator aquatic populations: a mathematical model

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Abstract. Escalation in pollution and contamination of the aquatic bodies is one of the alarming issues in recent times. Discharge of agricultural and industrial effluents into the water bodies is deteriorating the quality of water. Further, acid rain and pollutants washed off from land directly into water bodies lead to increase in acidity of water. These phenomena cause a decline in dissolved oxygen level of water, thus, threatening the survival of aquatic organisms. A non-linear mathematical model has been proposed to investigate the effect of toxicity and acidity on a prey-predator system wherein the predator is assumed to be completely dependent on prey for food. Also, it is assumed that rise in water acidity and toxicity develops impairments and infection in the lower level of food chain i.e. prey, which has indirect detrimental consequences for the growth and survival of the higher level of food chain i.e. predator due to consumption of prey by predator in the feeding process. Stability analysis of the model has been carried out and stability conditions have been derived taking into account all parameters of the proposed model. Numerical simulations are performed using MATLAB to support the analytical results obtained.

1. Introduction
One of the most alerting environmental concerns of recent times is the contamination of water bodies caused by various anthropogenic activities such as uncontrolled discharge of household wastes, agricultural and industrial effluents into the water bodies. Due to intractable release of pollutants directly into water bodies, the toxicity and acidity level of water is rising rapidly. The increasing acidity of water can also be attributed to the carbon components released by pollutants present in water bodies as well as the inflating level of carbon dioxide in the atmosphere [1]. The acidifying water bodies are also leading to exacerbation of coral reef cover [2]-[3]. The loss in coral cover is harmful to the macro invertebrate species in the coral reefs [4]. The aquatic organisms exposed directly or indirectly to toxic and acidic environment face several stresses which alter and threaten their growth and survival in water [5] - [14]. The increasing chemicals and metal wastes also have prominent negative effects on the aquatic species [15]-[17]. The human population, which is dependent on the aquatic ecosystem for food, minerals, fertilizers etc. is also facing grave consequences owing to the undermining effects of increasing water toxicity and acidity. The top predators in an aquatic food chain become exposed to the toxicants by feeding on the prey affected by the toxicants [18]. Increased water acidity and
metal contamination can also cause disturbances in the trophic structure of aquatic species, loss in biodiversity and decrease in density of fish stock in the water bodies [19] - [20]. Several mathematical and experimental studies have also proved that the fish population have high sensitivity to toxicant and pollutant [21] and exhibit high mortality rates when exposed to them [22]. The presence of toxicants and pollutants in water not only has negative consequences for the aquatic species but also for their resource i.e. oxygen as they decrease the dissolved oxygen level in water [23] - [24]. The increased acidification and eutrophication lead to a rise in toxic algal bloom in water. This further reduces oxygen level in water due to consumption of dissolved oxygen in the decomposition process of these algal blooms [25] - [26]. Drop in the concentration of dissolved oxygen in water also causes disruption in the pelagic food web [27].

Various biological phenomena have been studied by using mathematical modelling as an efficient tool [28]-[31]. Khare et al. conducted a mathematical study to evaluate the negative effects of decreasing dissolved oxygen on interacting planktonic population [32]. A few mathematical studies have also studied the effect of toxicant on prey-predator aquatic population [33] - [36]. Several mathematical studies have also highlighted the harmful role of increasing acidity on aquatic communities [37] - [38]. These studies show that the aquatic populations and their interactions are highly sensitive to the toxicant and acid level in water. However, the focus of the mathematical studies carried out till now is on the individual role of pollution, toxicant or acidity in an aquatic ecosystem. But, the effects of these factors have not been taken into account together in the available studies. In order to study the effects of toxicant and acid taken together on prey-predator aquatic populations, in this paper, a non-linear mathematical model is proposed. In this model, it is presumed that the water toxicity is increasing from various processes such as eutrophication, release of industrial pollutants, household wastes etc. directly into the water bodies. The acidity of water is rising via two processes, one is direct discharge of chemicals and acids in water bodies and the second being the release of carbon from the pollutants present in the water bodies. The carbon reacts with the dissolved oxygen present in water and forms carbonic acid, thus, acidifying the water bodies. The rise in water acidity and toxicity cause a decline in the concentration of resource i.e. dissolved oxygen in water. This further has negative influences on the prey and predator populations residing in water.

2. Mathematical Model

In the proposed mathematical model, the population dynamics of a prey-predator in an aquatic environment is studied under the effect of increasing water toxicity and acidity. It is assumed that the incoming pollutants cause a rise in toxicity level of water. The carbon components present in the pollutants reacts with the dissolved oxygen and form carbonic acid. Also, the industrial and agricultural pollution lead to direct input of acid components into the water bodies. Thus, the water acidity increases by two phenomena, one is the direct discharge of acidic components and the second one being through the released pollutants in water. These phenomena lead to decrease in dissolved oxygen level in water, thus, threatening the survival of the aquatic species. It is further assumed that the predator population is solely dependent on prey population for its growth and survival through feeding on prey.

In view of the above, let A denote the acid concentration in water and T represent the concentration of toxicant in water. Let $D_0$ represent the concentration of dissolved oxygen in water. N denotes the density of aquatic prey population like small fishes and let P represent the density of predator population directly dependent on prey for food, like whales, seabirds etc. With the above mentioned notations, a mathematical model consisting a set of non-linear differential equations is formulated given as:

$$\frac{dA}{dt} = p_0 - \beta A + kAT,$$  \hspace{1cm} (1)
A positive initial values

Proof. From equation (2) we obtain,

\[
\frac{dT}{dt} = q_0 - a_0 T - kAT, \tag{2}
\]

\[
\frac{dD_0}{dt} = r - n_{11} D_0 - n_{12} D_0 T - \gamma N D_0, \tag{3}
\]

\[
\frac{dN}{dt} = h N + \gamma N D_0 - \alpha_{11} N P \frac{g + N}{g + N} - \delta_1 N^2, \tag{4}
\]

\[
\frac{dP}{dt} = \alpha_{12} N P \frac{g + N}{g + N} - a_1 P, \tag{5}
\]

where the initial conditions for variables are given as:

\[
A(0) > 0, T(0) > 0, D_0(0) > 0, N(0) > 0, P(0) > 0.
\]

The system parameters are given as follows:

\( p_0 \) represents the rate of input of acid components in water via industrial effluents, acid rain etc. Rate of washing out of acid out of water bodies through natural processes is given by \( \beta \). The bilinear interaction represented by the term \( kAT \) depicts the reaction of carbon present in the toxicants and pollutants with the dissolved oxygen in water leading to formation of carbonic acid. \( k \) represents the rate of increase of water acidity on account of formation of carbonic acid. The rate of input of toxicants and pollutants into the water bodies is represented by \( g_0 \). \( a_0 \) gives the depletion rate of the pollutants and toxicants from the water bodies due to natural washing out, decomposition and intake by aquatic populations etc. \( r \) represents the input rate of dissolved oxygen in water and \( n_{11} \) gives its natural depletion rate in water. The increase in acidity and toxicity of water gives rise to inflated algal bloom growth. These algal blooms use the dissolved oxygen in their decomposition process, thus depleting its level in water. The rate at which the dissolved oxygen decreases due the algal decomposition is represented by \( n_{12} \). The rate of uptake of dissolved oxygen by the prey population in water is given by \( \gamma \). \( h \) represents the natural growth rate of the prey population. \( \alpha_{11} \) represents the consumption rate of prey population by predator population. \( g \) is the extent to which the prey population is protected by the environment. \( \delta_1 \) represents the intraspecific competition between the prey population. \( \alpha_{12} \) represents the assimilation rate of predator and \( a_1 \) gives the natural mortality rate of the predator population. All the parameters \( p_0, \beta, k, q_0, a_0, r, n_{11}, n_{12}, g, \alpha_{11}, \delta_1, g, \alpha_{12} \) and \( a_1 \) are assumed to be positive constants.

The mathematical analysis of the model given by equations (1)-(5) shall be carried out in the subsequent sections.

3. Boundedness and Dynamical Behaviour of Model

In this section, we shall establish the boundedness of solutions of the mathematical model given by equations (1)-(5). The following lemma shows that the solutions are bounded in \( R^+_5 \).

**Lemma 1.** All solutions of the system given by equations (1)-(5) with positive initial conditions will lie in the region \( V_r \) where:

\[ V_r = \{(A, T, D_0, N, P) \in R^+_5 : 0 \leq A + T \leq M_{11u}, 0 \leq N + D_0 + \frac{\alpha_{11}}{\alpha_{12}} P \leq M_{12u}, T_1 \leq T \leq T_u, 0 \leq D_0 \leq D_{0u}, 0 \leq N \leq N_u \} \] for all \( t \to \infty \) for positive initial values \( A(0), T(0), D_0(0), N(0), P(0) \) where \( M_{11u} = \frac{p_0 + r \mu}{w_2} ; w_2 = \min(\beta, a_0) \); \( M_{12u} = \frac{r + N_u (h + \mu)}{\mu} ; \mu \); \( \alpha_{11} \alpha_{12} > \mu \); \( T_u = \frac{q_0}{a_0} ; T_1 = \frac{q_0}{a_0 + kM_{11u}} ; D_{0u} = \frac{r}{n_{11}} \) and \( N_u = \frac{h + \gamma D_{0u}}{\delta_1} \).

**Proof.** From equation (2) we obtain,

\[
\frac{dT(t)}{dt} \leq q_0 - a_0 T.
\]
Then, following the usual comparison theorem we obtain:

$$\limsup_{t \to \infty} (T,t) \leq \frac{q_0}{a_0},$$

and hence,

$$T(t) \leq \frac{q_0}{a_0} = T_u(t).$$

Similarly from equation (3) following the usual comparison theorem, we get,

$$\limsup_{t \to \infty} (D_0,t) \leq \frac{r}{n_1},$$

and,

$$D_0(t) \leq \frac{r}{n_1} = D_{0u}(t),$$

and from equation (4), following the usual comparison theorem, we obtain,

$$\limsup_{t \to \infty} (N,t) \leq \frac{h + \gamma D_0 u}{\delta_1},$$

hence,

$$N(t) \leq \frac{h + \gamma D_0 u}{\delta_1} = N_u(t).$$

Let us consider a function $M_{11}(t)$ given as ,

$$M_{11}(t) = A(t) + T(t).$$

Taking $w_2 = \min(\beta, a_0)$ and from equations (1) and (2) we obtain,

$$\frac{dM_{11}(t)}{dt} \leq p_0 + q_0 - w_2 M_{11}(t).$$

Then, by the usual comparison theorem we have:

$$\limsup_{t \to \infty} (M_{11},t) \leq \frac{p_0 + q_0}{w_2}.$$

Hence,

$$A(t) + T(t) \leq \frac{p_0 + q_0}{w_2} = M_{11u}(t).$$

Again, from equation(2) we get,

$$\frac{dT(t)}{dt} \geq q_0 - a_0 T - k M_{11u} T,$$

and by usual comparison theorem we get,

$$\liminf_{t \to \infty} (T,t) \geq \frac{q_0}{a_0 + k M_{11u}}.$$

Therefore,

$$T(t) \geq \frac{q_0}{a_0 + k M_{11u}} = T_l(t).$$
Now, consider a function $M_{12}(t)$ given as,

$$M_{12}(t) = N(t) + D_0(t) + \frac{\alpha_{11}P(t)}{\alpha_{12}}.$$ 

From equations (3), (4) and (5) we get,

$$\frac{dM_{12}(t)}{dt} \leq r - n_{11}D_0 + hN_u - \frac{\alpha_{11}\alpha_1}{\alpha_{12}}P(t),$$

For a positive constant $\mu$ and assuming $n_{11} > \mu$; $\frac{\alpha_{11}a_1}{\alpha_{12}} > \mu$ we get,

$$\frac{dM_{12}}{dt} + \mu M_{12} \leq r + (h + \mu)N_u.$$

Then by the usual comparison theorem we have:

$$\limsup_{t \to \infty} (M_{12}, t) \leq \frac{r + (h + \mu)N_u}{\mu} = M_{12,N}(t).$$

Hence we get,

$$N(t) + D_0(t) + \frac{\alpha_{11}P(t)}{\alpha_{12}} \leq \frac{r + (h + \mu)N_u}{\mu}.$$

**Positivity of solutions**: Since the positivity of solutions of the model given by equations (1)-(5) implies persistence, hence it is important to prove the solutions of the model studying the dynamical behaviour of aquatic prey-predator population under the effect of increasing acidity and toxicants, exhibit positivity for all times. The positivity of solutions shall be shown by the following lemma.

**Lemma 2.** The solutions of the model given by equations (1)-(5), $(A(t), T(t), D_0(t), N(t), P(t))$, with initial conditions, $A(0) > 0, T(0) > 0, D_0(0) > 0, N(0) > 0, P(0) > 0$, remain positive for all times $t > 0$.

**Proof.** From equation (1) we get,

$$\frac{dA}{dt} \geq -\beta A,$$

$$A \geq g_1e^{-\beta t},$$

where $g_1$ is an integration constant.

Hence, $A > 0$ as $t \to \infty$.

Similarly, from equation (2), we get

$$\frac{dT}{dt} \geq -(a_0 + kM_{11u})T,$$

$$T \geq g_2e^{-(a_0+kM_{11u})t},$$

where $g_2$ is an integration constant.

Hence, $T > 0$ as $t \to \infty$.

From equation (3), we get

$$\frac{dD_0}{dt} \geq -(n_{11} + n_{12}M_{11u} + \gamma M_{12u})D_0,$$
\[ D_0 \geq g_3 e^{-(n_{11} + n_{12}M_{11u} + \gamma M_{12u})t}, \]

where \( g_3 \) is an integration constant.

Hence, \( D_0 > 0 \) as \( t \to \infty \).

Similarly, from equation (4), we obtain
\[
\frac{dN}{dt} \geq -\left(\alpha_{11}M_{12u} + \delta_1 N_u\right)N,
\]

\[ N \geq g_4 e^{-(\alpha_{11}M_{12u} + \delta_1 N_u)t}, \]

where \( g_4 \) is an integration constant.

Hence, \( N > 0 \) as \( t \to \infty \).

From equation (5), we obtain
\[
\frac{dP}{dt} \geq -a_1 P,
\]

\[ P \geq g_5 e^{-a_1 t}, \]

where \( g_5 \) is an integration constant.

Hence, \( P > 0 \) as \( t \to \infty \).

This leads to completion of the proof of the lemma.

3.1. Possible equilibrium points and existence conditions

In this section, for the model defined by set of equations (1)-(5), we find the possible equilibrium points. The model has the following four equilibrium points:

1. Prey and predator population vanishing equilibrium point \( E(\hat{A}, \hat{T}, \hat{D}_0, 0, 0) \) where \( \hat{N} = 0 \) and \( \hat{P} = 0 \) i.e. when the toxicant and acid level in the water increases to such a high amount such that the prey and predator populations tends towards extinction.

\[ \hat{A} = \frac{p_0}{\beta - k\hat{T}}. \]

(6)

\[ \hat{A} > 0 \text{ if } \beta - k\hat{T} > 0. \]

(7)

\[ \hat{D}_0 = \frac{r}{n_{11} + n_{12}\hat{T}}. \]

(8)

Since \( \hat{T} > 0 \), \( \hat{D}_0 > 0 \).

\( \hat{T} \) is given as the positive root of the following quadratic equation,
\[
a_0k\hat{T}^2 - \hat{T}(q_0k + a_0\beta + kp_0) + q_0\beta = 0.
\]

2. Predator vanishing equilibrium point \( E(\tilde{A}, \tilde{T}, \tilde{D}_0, \tilde{N}, 0) \) i.e. \( \tilde{P} = 0 \).

\[ \tilde{A} = \frac{p_0}{\beta - k\tilde{T}}. \]

(9)

\[ \tilde{A} > 0 \text{ if } \beta - k\tilde{T} > 0. \]

(10)

\[ \tilde{N} = \frac{1}{\delta_1} \left( \frac{h}{1 + \tilde{A}} + \gamma \tilde{D}_0 \right). \]

(11)

\( \tilde{T} \) is given as the positive root of the following quadratic equation,
\[
a_0k\tilde{T}^2 - \tilde{T}(q_0k + a_0\beta + kp_0) + q_0\beta = 0.
\]
\( \tilde{D}_0 \) is given as the positive root of the following quadratic equation,
\[
\tilde{D}_0^2 + \frac{\tilde{D}_0}{\gamma^2(1 + A)}((n_{11} + n_{12}\tilde{D}_0)(1 + \tilde{A})\delta_1 + \gamma h) - \frac{r\delta_1}{\gamma^2} = 0.
\]

3. Interior equilibrium point \( E^*(A^*, T^*, D_0^*, N^*, P^*) \): The values of \( A^*, T^*, D_0^*, N^*, P^* \) are given as:

From equation (1) we have,
\[
A^* = \frac{p_0}{\beta - kT^*}.
\]
\( A^* > 0 \) if \( \beta - kT^* > 0 \).  
(12)

\[
N^* = \frac{a_{11}g}{\alpha_{12} - a_1}.
\]
\( N^* > 0 \) if \( \alpha_{12} - a_1 > 0 \).  
(14)

\[
D_0^* = \frac{r}{n_{11} + n_{12}T^* + \gamma N^*}.
\]
(16)

Since \( N^* > 0 \) and \( T^* > 0 \), hence \( D_0^* > 0 \).

\( T^* \) is given as the positive root of the following quadratic equation,
\[
a_0kT^* - T^*(q_0k + a_0\beta + kp_0) + q_0\beta = 0.
\]
(17)

\[
P^* = \frac{g + N^*}{\alpha_{11}} \left( \frac{h}{1 + A^*} + \gamma D_0^* - \delta_1 N^* \right).
\]
(18)

\( P^* > 0 \) if \( \frac{h}{1 + A^*} + \gamma D_0^* - \delta_1 N^* > 0 \).  
(19)

4. Acid vanishing equilibrium point \( \tilde{E}(0, \tilde{T}, \tilde{D}_0, \tilde{N}, \tilde{P}) \) i.e. \( \tilde{A} = 0 \).

The values of variables \( \tilde{T}, \tilde{D}_0, \tilde{N}, \tilde{P} \) are given as:

\[
\tilde{T} = \frac{q_0}{a_0}.
\]
(20)

From equation (4) we get,
\[
\tilde{N} = \frac{a_{11}g}{\alpha_{12} - a_1}.
\]
(21)

\( \tilde{N} > 0 \) only if,
\[ \alpha_{12} - a_1 > 0. \]
(22)

From equation (3) we get,
\[
\tilde{D}_0 = \frac{r}{n_{11} + n_{12}\tilde{T} + \gamma \tilde{N}}.
\]
(23)

Since \( \tilde{N} > 0 \) and \( \tilde{T} > 0 \), hence \( \tilde{D}_0 > 0 \).

From equation (4), \( \tilde{P} \) is given as,
\[
\tilde{P} = \frac{g + \tilde{N}}{\alpha_{11}} \left( h + \gamma \tilde{D}_0 - \delta_1 \tilde{N} \right).
\]
(24)

\( \tilde{P} > 0 \) only if,
\[ h + \gamma \tilde{D}_0 - \delta_1 \tilde{N} > 0. \]
(25)
Remark: The toxicant (T) enters the water bodies through various multiple sources such as agricultural, household and industrial wastes discharge. Hence, the amount of toxicant entering in the system in the system can not be nullified completely and consequently T cannot be taken as zero.

The dynamical behaviour of the model given by equations (1)-(5) in terms of local and global stability for these possible equilibrium points shall be studied in the next section.

### 3.2. Local Stability

(i) **For prey and predator population vanishing equilibrium point** $\hat{E}(\hat{A}, \hat{T}, \hat{D}_0, 0, 0)$:

The characteristic equation corresponding to variational matrix about prey and predator population vanishing equilibrium point $\hat{E}$ is given as:

$$(-a_1 - \lambda)(-n_{11} - n_{12}\hat{T} - \lambda)((-\beta + k\hat{T} - \lambda)(-a_0 - k\hat{A} - \lambda) + k^2\hat{A}\hat{T})$$

$$\left(\frac{h}{1 + \hat{A}} + \gamma\hat{D}_0 - \lambda\right) = 0. \quad (26)$$

The eigen values corresponding to the above characteristic equation are given as:

$$\lambda_1 = -a_1, \quad \lambda_2 = -n_{11} - n_{12}\hat{T}, \quad \lambda_3 = \frac{h}{1 + \hat{A}} + \gamma\hat{D}_0, \quad (27)$$

$\lambda_4$ and $\lambda_5$ are obtained by solving the following quadratic equation:

$$(-\beta + k\hat{T} - \lambda)(-a_0 - k\hat{A} - \lambda) + k^2\hat{A}\hat{T} = 0. \quad (28)$$

As the eigen value $\lambda_3 = \frac{h}{1 + \hat{A}} + \gamma\hat{D}_0$ is positive, hence the equilibrium point $\hat{E}(\hat{A}, \hat{T}, \hat{D}_0, 0, 0)$ is unstable.

(ii) **For predator population vanishing equilibrium point** $\tilde{E}(\tilde{A}, \tilde{T}, \tilde{D}_0, \tilde{N}, 0)$:

The characteristic equation associated with the variational matrix about predator population vanishing equilibrium point $\tilde{E}$ is given by:

$$(\tilde{Z}_5 - \lambda)(\lambda^2 + (\tilde{Z}_2 - \tilde{Z}_1)\lambda - \tilde{Z}_1\tilde{Z}_2 + k^2\tilde{A}\tilde{T})(\lambda^2 + (\tilde{Z}_3 - \tilde{Z}_4)\lambda - \tilde{Z}_3\tilde{Z}_4 + \gamma^2\tilde{D}_0\tilde{N}) = 0 \quad (29)$$

where $\tilde{Z}_1 = -\beta + k\tilde{T}; \quad \tilde{Z}_2 = a_0 + k\tilde{A}; \quad \tilde{Z}_3 = n_{11} + n_{12}\tilde{T} + \gamma\tilde{N};$

$\tilde{Z}_4 = \frac{h}{1 + \tilde{A}} + \gamma\tilde{D}_0 - 2\delta_1\tilde{N}; \quad \tilde{Z}_5 = \frac{a_{12}\tilde{N}}{g + \tilde{N}} - a_1; \tilde{Z}_6 = \frac{-h\tilde{N}}{(1 + \tilde{A})^2}; \tilde{Z}_7 = \frac{-a_{11}\tilde{N}}{g + \tilde{N}}.$

The eigen values corresponding to the above characteristic equation are given as:

$$\lambda_1 = \tilde{Z}_5, \quad (30)$$

$\lambda_2$ and $\lambda_3$ are obtained by solving the following quadratic equation:

$$\lambda^2 + (\tilde{Z}_2 - \tilde{Z}_1)\lambda - \tilde{Z}_1\tilde{Z}_2 + k^2\tilde{A}\tilde{T} = 0. \quad (31)$$

$\lambda_4$ and $\lambda_5$ are obtained by solving the following quadratic equation:

$$\lambda^2 + (\tilde{Z}_3 - \tilde{Z}_4)\lambda - \tilde{Z}_3\tilde{Z}_4 + \gamma^2\tilde{D}_0\tilde{N} = 0. \quad (32)$$

Using Routh-Hurwitz criteria, the boundary equilibrium state $\tilde{E}$ will be asymptotically stable subject to satisfying the following conditions,
For acid vanishing equilibrium point

\[ \tilde{Z}_2 > \tilde{Z}_1, \]
\[ k^2 A T > \tilde{Z}_1 \tilde{Z}_2, \]
\[ \tilde{Z}_3 > \tilde{Z}_4, \]
\[ \gamma^2 D_0 \tilde{N} > \tilde{Z}_3 \tilde{Z}_4, \]
\[ a_1 > \frac{\alpha_{12} \tilde{N}}{g + \tilde{N}}, \]
\[ k \tilde{T} > \beta_1, \]
and
\[ \frac{h}{1 + A^*} + \gamma D_0 > 2 \delta_1 \tilde{N}. \]

(iii) **For interior equilibrium point** \( E^*(A^*, T^*, D_0^*, N^*, P^*) \) : The characteristic equation associated with the variational matrix about interior equilibrium point \( E^* \) is given by:

\[ \left[ \lambda^2 + (Z_2^* - Z_1^*) \lambda - Z_1^* Z_2^* + k^2 A^* T^* \right] \left[ \lambda^3 - \lambda^2 (-Z_3^* + Z_4^*) - \lambda (Z_1^* Z_4^* + Z_2^* Z_3^*) - \gamma^2 D_0^* N^* \right] = 0 \]  

where
\[ Z_1^* = -\beta + k T^*; \quad Z_2^* = a_0 + k A^*; \quad Z_3^* = n_{11} + n_{12} T^* + \gamma N^*; \quad Z_4^* = \frac{h}{(1 + A^* T^*)} + \gamma D_0^* - 2 \delta_1 N^* - \frac{\alpha_{11} g P^*}{(g + N^*)^2}; \quad Z_5^* = -\frac{h N^*}{(1 + A^* T^*)}; \quad Z_6^* = -\frac{\alpha_{11} N^*}{g + N^*}; \quad Z_7^* = \frac{\alpha_{12} g P^*}{(g + N^*)^2}. \]

Using Routh Hurwitz criteria, the interior equilibrium state \( E^* \) will be asymptotically stable subject to satisfying the following conditions,

\[ Z_3^* > Z_1^*, \]  
\[ k^2 A^* T^* > Z_1^* Z_2^*, \]  
\[ \gamma^2 D_0^* N^* > Z_3^* Z_4^* + Z_2^* Z_3^*, \]  
\[ (-Z_3^* + Z_4^*)(Z_1^* Z_4^* + Z_2^* Z_3^* - \gamma^2 D_0^* N^*) + Z_3^* Z_4^* Z_5^* > 0, \]  
\[ k T^* > \beta_1 \]
and
\[ \frac{h}{1 + A^*} + \gamma D_0^* > 2 \delta_1 N^* + \frac{\alpha_{11} g P^*}{(g + N^*)^2}. \]

(iv) **For acid vanishing equilibrium point** \( \tilde{E}(0, \tilde{T}, \tilde{D}_0, \tilde{N}, \tilde{P}) \) : The characteristic equation corresponding to the variational matrix about acid vanishing equilibrium point \( \tilde{E} \) is given by:

\[ (\tilde{Z}_1 - \lambda)(-\tilde{Z}_2 - \lambda)[\lambda^3 - \lambda^2 (-\tilde{Z}_3 + \tilde{Z}_4) - \lambda (\tilde{Z}_3 \tilde{Z}_4) + \tilde{Z}_7 (\tilde{Z}_5 - \gamma^2 \tilde{D}_0 \tilde{N}) - \tilde{Z}_3 \tilde{Z}_7 \tilde{Z}_8] = 0 \]

where
\[ \tilde{Z}_1 = -\beta + k \tilde{T}; \quad \tilde{Z}_2 = a_0; \quad \tilde{Z}_3 = n_{11} + n_{12} \tilde{T} + \gamma \tilde{N}; \quad \tilde{Z}_4 = h + \gamma \tilde{D}_0 - 2 \delta_1 \tilde{N} - \frac{\alpha_{11} g \tilde{P}}{(g + \tilde{N})^2}; \]
\[ \tilde{Z}_6 = -h \tilde{N}; \quad \tilde{Z}_7 = \frac{\alpha_{11} \tilde{N}}{g + \tilde{N}}; \quad \tilde{Z}_8 = \frac{\alpha_{12} g \tilde{P}}{(g + \tilde{N})^2}. \]

The eigen values corresponding to the above characteristic equation are given as:

\[ \lambda_1 = \tilde{Z}_1, \quad \lambda_2 = -a_0, \]

\[ (49) \]
Using Routh’s criteria, the eigenvalues \( \lambda_3, \lambda_4 \) and \( \lambda_5 \) will have non-positive real parts subject to satisfying the following conditions,

\[
\dot{Z}_3 > \dot{Z}_4, \tag{50}
\]

\[
\gamma^2 \bar{D}_0 \bar{N} > \dot{Z}_3 \dot{Z}_4 + \dot{Z}_7 \dot{Z}_8, \tag{51}
\]

\[
(-\dot{Z}_3 + \dot{Z}_4)(\dot{Z}_3 \dot{Z}_4 + \dot{Z}_7 \dot{Z}_8 - \gamma^2 \bar{D}_0 \bar{N}) + \dot{Z}_3 \dot{Z}_7 \dot{Z}_8 > 0, \tag{52}
\]

\[
k \bar{T} < \beta_1, \tag{53}
\]

and

\[
h + \gamma \bar{D}_0 > 2\delta_1 \bar{N} + \frac{\alpha_{11}g \bar{P}}{(g + \bar{N})^2}. \tag{54}
\]

### 3.3. Global Stability

In this section, the global stability of the mathematical model given by equations (1)-(5) shall be established. The following theorems shall establish the global stability:

**Theorem 3.** The box given by \( V_r \) is a positive invariant and compact set in the space \( (A, T, D_0, N, P) \).

**Proof.** Let us consider the system of equations given by (1)-(5). Consider a box \( V_r \) in the phase space \( ATD_0NP \) with one vertex at the origin and the other vertex at a point \( \bar{\omega} = (\bar{A}, \bar{T}, \bar{D}_0, \bar{N}, \bar{P}) \). The point \( \bar{\omega} \) is considered outside the box \( V_r \) with \( \bar{A} > A_u, \bar{T} > T_u, \bar{D}_0 > D_{0u}, \bar{N} > N_u, \bar{P} > P_u \).

Now we shall compute the angle of the flow with each face of the box \( V_r \) not lying in the coordinate planes. Let \( \nu_1, \nu_2, \nu_3, \nu_4 \) and \( \nu_5 \) be the outward unit normal vectors to the planes \( \chi_1 : A = \bar{A}, \chi_2 : T = \bar{T}, \chi_3 : D_0 = \bar{D}_0, \chi_4 : N = \bar{N}, \chi_5 : P = \bar{P} \) in reference to the box \( V_r \).

Then from equation (3) we get,

\[
\nu_3 \frac{d\bar{\omega}}{dt}|_{\chi_3} \leq r - n_{11} \bar{D}_0 - n_{12} T_1 \bar{D}_0,
\]

since

\[
\bar{D}_0 > \frac{r}{n_{11}},
\]

\[
\nu_3 \frac{d\bar{\omega}}{dt}|_{\chi_3} \leq -n_{12} T_1 \bar{D}_0,
\]

hence

\[
\nu_3 \frac{d\bar{\omega}}{dt}|_{\chi_3} \leq 0.
\]

Similarly, we can prove

\[
\nu_1 \frac{d\bar{\omega}}{dt}|_{\chi_1} \leq 0, \nu_2 \frac{d\bar{\omega}}{dt}|_{\chi_2} \leq 0, \nu_4 \frac{d\bar{\omega}}{dt}|_{\chi_4} \leq 0, \nu_5 \frac{d\bar{\omega}}{dt}|_{\chi_5} \leq 0.
\]

From the above theorem it is clear that the trajectories of the system of equations given by (1)-(5) do not not cross \( V_r \) once they enter inside the box \( V_r \). The interior equilibrium \( E^* \) is also observed to lie inside the \( V_r \).

In the next theorem, we will prove that the only global attractor inside \( V_r \) is \( E^* \).
Theorem 4. For the interior equilibrium point $E^*$ to be globally asymptotically stable, the following inequalities should hold for $E^*$.

\[(\beta - kT)(a_0 + kA) > (kT^* - kA^*)^2,\]
\[(a_0 + kA)(n_{11} + \gamma N + n_{12}T) > (n_{12}D_0^*)^2,\]
\[2(n_{11} + \gamma N + n_{12}T)\left(\frac{-\alpha_{11}P^*}{(g + N)(g + N^*)} + \delta_1\right) > 3(\gamma D_0^* - \gamma)^2,\]
\[4\left(\frac{-\alpha_{11}P^*}{(g + N)(g + N^*)} + \delta_1\right)\left(a_1 - \frac{\alpha_{12}N}{g + N}\right) > 3\left(\frac{\alpha_{11}}{g + N} - \frac{\alpha_{12}gP^*}{(g + N)(g + N^*)}\right)^2,\]
\[2(\beta - kT)\left(\frac{-\alpha_{11}P^*}{(g + N)(g + N^*)} + \delta_1\right) > 3\left(\frac{h}{(1 + A)(1 + A^*)}\right)^2.\]

Proof. For establishing the global stability of the equilibrium state $E^*$, we shall assume the following positive definite function:

\[Z = \frac{1}{2}(A - A^*)^2 + \frac{1}{2}(T - T^*)^2 + \frac{1}{2}(D_0 - D_0^*)^2 + \left(N - N^* - N^*\ln\frac{N}{N^*}\right) + \frac{1}{2}(P - P^*)^2.\] 

(55)

Differentiating the above equation w.r.t. ‘t’ we get,

\[
\frac{dZ}{dt} = -a_{11}(A - A^*)^2 + a_{22}(T - T^*)^2 + a_{33}(D_0 - D_0^*)^2 + a_{44}(N - N^*)^2 + a_{55}(P - P^*)^2
\]
\[+ a_{12}(A - A^*)(T - T^*) + a_{23}(T - T^*)(D_0 - D_0^*) + a_{34}(N - N^*)(D_0 - D_0^*)
\]
\[+ a_{45}(P - P^*)(N - N^*) + a_{14}(A - A^*)(N - N^*)],
\]

where

\[a_{11} = \beta - kT, \quad a_{22} = a_0 + kA, \quad a_{12} = kT^* - kA^*, \quad a_{33} = n_{11} + \gamma N + n_{12}T, \quad a_{44} = \delta_1 - \frac{\alpha_{11}P^*}{(g + N)(g + N^*)}, \quad a_{55} = a_1 - \frac{\alpha_{12}N}{g + N}, \quad a_{23} = n_{12}D_0^*, \quad a_{34} = \gamma D_0^* - \gamma, \quad a_{45} = \frac{\alpha_{11}}{g + N} - \frac{\alpha_{12}gP^*}{(g + N)(g + N^*)}, \quad a_{14} = \frac{h}{(1 + A)(1 + A^*)}.
\]

Sufficient conditions for \(\frac{dZ}{dt}\) to be negative definite obtained by Sylvester’s criteria are:

\[a_{11}a_{22} > a_{12}^2, \quad a_{22}a_{33} > a_{23}^2, \quad 2a_{33}a_{44} > 3a_{44}^2, \quad 4a_{44}a_{55} > 3a_{55}^2, \quad 2a_{11}a_{44} > 3a_{14}^2, \quad (66)
\]
i.e.

\[(\beta - kT)(a_0 + kA) > (kT^* - kA^*)^2,\]
\[(a_0 + kA)(n_{11} + \gamma N + n_{12}T) > (n_{12}D_0^*)^2,\]
\[2(n_{11} + \gamma N + n_{12}T)\left(\frac{-\alpha_{11}P^*}{(g + N)(g + N^*)} + \delta_1\right) > 3(\gamma D_0^* - \gamma)^2,\]
\[4\left(\frac{-\alpha_{11}P^*}{(g + N)(g + N^*)} + \delta_1\right)\left(a_1 - \frac{\alpha_{12}N}{g + N}\right) > 3\left(\frac{\alpha_{11}}{g + N} - \frac{\alpha_{12}gP^*}{(g + N)(g + N^*)}\right)^2,\]
\[2(\beta - kT)\left(\frac{-\alpha_{11}P^*}{(g + N)(g + N^*)} + \delta_1\right) > 3\left(\frac{h}{(1 + A)(1 + A^*)}\right)^2.\] 

\(\blacksquare\)
4. Numerical Simulation and Sensitivity Analysis
To substantiate the analytical results obtained for the system of equations given by (1)-(5), numerical simulations have been supplemented in this section. The set of parametric values considered are given below:

\[ p_0 = 0.43\mu gL^{-1}day^{-1}, \beta = 0.2day^{-1}, q_0 = 1.0\mu gL^{-1}day^{-1}, a_0 = 0.1day^{-1}, r = 70mgL^{-1}day^{-1}, \]
\[ k = 0.01 \mu g^{-1}day^{-1}, n_{11} = 1.0day^{-1}, n_{12} = 1.0\mu g^{-1}day^{-1}, \gamma = 0.995Lmg^{-1}day^{-1}, \]
\[ h = 1.0 \mu gL^{-1}day^{-1}, g = 1.0mgL^{-1}, \alpha_{11} = 1.00025day^{-1}, \delta_1 = 1.1Lmg^{-1}day^{-1}, \]
\[ \alpha_{12} = 1.001day^{-1}, a_1 = 0.825day^{-1}. \] (62)

For the above mentioned parametric values, the equilibrium values corresponding to interior equilibrium \( E^* \) are given as:

\[ A^* = 3.4258\mu gL^{-1}, T^* = 7.4483\mu gL^{-1}, D_0^* = 5.3383mgL^{-1}, N^* = 4.6881mgL^{-1}, \]
\[ P^* = 2.1687mgL^{-1}. \] (63)

For these values of the model parameters, the conditions of feasibility, boundedness and stability corresponding to interior equilibrium point \( E^* \) are satisfied. Also, for the mentioned parametric values, interior equilibrium \( E^* \) is found to exhibit asymptotic stability as shown in fig.1.

| Variable | 1st initial value | 2nd initial value | 3rd initial value | 4th initial value | 5th initial value |
|----------|------------------|------------------|------------------|------------------|------------------|
| \( A \)  | 1                | 1                | 2                | 1                | 2                |
| \( T \)  | 1                | 1.5              | 1.5              | 1.5              | 5                |
| \( D_0 \)| 1                | 1                | 4                | 1                | 10               |
| \( N \)  | 1                | 10               | 4                | 10               | 1                |
| \( P \)  | 1                | 2.5              | 3.5              | 1.5              | 2                |

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| Variable | 1st initial value | 2nd initial value | 3rd initial value | 4th initial value | 5th initial value |
|----------|------------------|------------------|------------------|------------------|------------------|
| \( A \)  | 1                | 1                | 2                | 1                | 2                |
| \( T \)  | 1                | 1.5              | 1.5              | 1.5              | 5                |
| \( D_0 \)| 1                | 1                | 4                | 1                | 10               |
| \( N \)  | 1                | 10               | 4                | 10               | 1                |
| \( P \)  | 1                | 2.5              | 3.5              | 1.5              | 2                |

Figure 1. Trajectories plotted with respect to time showing the stability of interior equilibrium point \( E^* \).
Table 2. Sensitivity Indices\(^{(\gamma)}\) of \(A^*, T^*, D_0^*, N^*, P^*\) at \(E^*\) to parameters \(Z_p\).

| Parameters\((Z_p)\) | \(\gamma_{A^*}^{Z_p}\) | \(\gamma_{T^*}^{Z_p}\) | \(\gamma_{D_0^*}^{Z_p}\) | \(\gamma_{N^*}^{Z_p}\) | \(\gamma_{P^*}^{Z_p}\) |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| \(p_0\)            | 0.869               | -0.222              | 0.126               | 0                   | 1.354               |
| \(\beta\)          | -1.384              | 0.353               | -0.201              | 0                   | -2.157              |
| \(k\)              | 0.384               | -0.353              | 0.201               | 0                   | 2.6155              |
| \(q_0\)            | 0.515               | 0.868               | -0.494              | 0                   | -7.1022             |
| \(a_0\)            | -0.384              | -0.647              | 0.367               | 0                   | 5.293               |
| \(r\)              | 0                   | 0                   | 0.999               | 0                   | 13.927              |
| \(n_{11}\)         | 0                   | 0                   | -0.0763             | 0                   | -1.0617             |
| \(n_{12}\)         | 0                   | 0                   | -0.568              | 0                   | -7.9081             |
| \(\gamma\)         | 0                   | 0                   | -0.356              | 0                   | -4.953              |
| \(h\)              | 0                   | 0                   | 0                   | 0                   | 0.592               |
| \(\alpha_{11}\)    | 0                   | 0                   | 0                   | 0                   | -0.998              |
| \(g\)              | 0                   | 0                   | -0.355              | 0.999               | -17.477             |
| \(\delta_1\)       | 0                   | 0                   | 0                   | 0                   | -13.522             |
| \(\alpha_{12}\)    | 0                   | 0                   | 2.027               | -5.701              | 100.622             |
| \(a_1\)            | 0                   | 0                   | -2.026              | 5.698               | -100.578            |

The numerical simulations also show that the interior equilibrium point \(E^*\) is globally stable. These results are shown by the phase plane graphs given by fig.2, which show that with increasing time, the trajectories starting from varied initial conditions as given by table 1, finally converge to the equilibrium value at \(E^*\).

Further, for the model given by equations (1)-(5) sensitivity analysis has been carried out. Sensitivity analysis results provide a more mathematically sound study of the effect of the model parameters given by equation(62) on the resource i.e. dissolved oxygen \((D_0)\), prey population \((N)\) and predator population \((P)\). The sensitivity indices have been calculated to examine the relative variation in the variables \(A^*, T^*, D_0^*, N^*\) and \(P^*\) with respect to change in the values of parameters \(p_0, \beta, q_0, a_0, k, r, n_{11}, n_{12}, \gamma, h, \alpha_{11}, g, \delta_1, \alpha_{12}\) and \(a_1\), so that the parameters having more profound effect on the model variables can be identified and accordingly necessary control strategies can be developed.

The normalized forward sensitivity index of a variable \(Z\) with dependent on parameter “\(v\)” is given by the expression given below [39]:

\[
\gamma_{Z}^{v} = \frac{\partial Z}{\partial v} \cdot \frac{v}{Z}
\]
The sensitivity indices of each variable at equilibrium point $E^*$ with respect to model parameters given by equation (62) are given in table 2.

5. Conclusion
The mathematical model given by equations (1)-(5) studies the impact of rising toxicants and acid components in water on the resource i.e. dissolved oxygen ($D_0$), prey population (N) and predator population (P) in an aquatic ecosystem. From the stability analysis it is observed that the acid vanishing point $\tilde{E}$ is linearly asymptotically stable only when the interior equilibrium point $E^*$ and boundary equilibrium point $\tilde{E}$ are unstable and vice-versa. The same is supported by equations (38,46, 53). Further, it is observed that the interior equilibrium point $E^*$ is asymptotically stable as shown in fig.1 and the stability conditions given by equations (12)-(19), (41)-(47), (57-61) are satisfied. Also, when the interior equilibrium point is stable, the boundary equilibrium point $\tilde{E}$ shall be unstable as supported by equations (46,53). Further, it may be noted that the resource i.e. dissolved oxygen concentration decreases with increasing water toxicity.
as shown by fig.3 and supported by equation 16. Also, the prey population (N) decreases with rising toxicity and acidity level in water as shown by figs. 4 and 5. Consequently, the predator population which is dependent on prey population for its food also exhibits a decline in its density with rising toxicity and acidity as supported by equation 18. Also, it is shown by fig. 6 that the dissolved oxygen concentration ($D_0$) shall decline on account of rise in value of toxicant input rate ($q_0$). Further, it may be noted that as the value of toxicant input rate ($q_0$) increases, it leads to decrease in density of prey population as shown in fig.7. Similarly, as illustrated by fig.8, with increase in the toxicant input level ($q_0$) in water, the predator population tends towards extinction. Moreover, from the sensitivity analysis, it is further observed that dissolved oxygen ($D_0$) and predator population (P) are sensitive and negatively dependent on input rate of toxicant ($q_0$) in water. Also, the prey and predator populations are highly sensitive to assimilation rate of predator ($\alpha_{12}$). For values of $\alpha_{12}$ equal to or greater than 1.23 but less than 1.28, stable limit cycles are observed. Figs. 9 and 10 show the stable limit cycles observed for prey and predator populations for value of $\alpha_{12}$ equal to 1.25. It may also be noted that
on increasing value of $\alpha_{12}$ to 1.28, the stable co-existing behaviour is changed to co-existing oscillatory behaviour. The oscillatory behaviour observed for prey and predator populations is...
shown in figs.11-14.
Furthermore, as the amount of agricultural and organic pollutants rise in water, the algal bloom in water increases and more oxygen is consumed in its decomposition process. From our study, it is inferred that in the absence of any toxicant or acid input in water, the dissolved oxygen level is 12.3567 mgL$^{-1}$ which lies in the required optimal range [25]. However, under these conditions, when the toxicant input ($q_0$) is increased from 1.0 to 8.878 and the oxygen utilisation rate in the decomposition process ($n_{12}$) is increased to 1.495, the concentration of dissolved oxygen falls to 0.5182 as shown in fig.15, thus creating a hypoxic condition which threatens the survival of aquatic species. The results of our study are validated by the results of the study carried out by Chakraborty et al. [25]. They found that with rising agricultural pollutants, the algal bloom growth increases which leads to consumption of dissolved oxygen in its decomposition process. Due to this, the concentration of dissolved oxygen falls to 0.5 mgL$^{-1}$ which is comparable to the results of our study. Moreover, the case study carried out by San-Diego-Mc Glone et al. also support the results of our study. They showed that under the stress of eutrophication, the dissolved oxygen level in coastal waters of Bolino,Philippines fell to 2.0 mgL$^{-1}$ which caused fish kills in year 2002 [40]. However, since we are considering the effects of pollutants, eutrophication and acidification together , the level of dissolved oxygen obtained in our study is 0.5182mgL$^{-1}$ which is considerably low than 2.0mgL$^{-1}$.

It is further established that at the value of toxicant input rate ($q_0$)=1.3679, the predator population tends towards zero a shown in fig.16. It is also observed that at the parameter values $q_0=1.3679$, if the value of r i.e. input rate of dissolved oxygen is increased from r=70.00 to 74.555, the predator population again starts rising from zero level as shown in fig.16. Thus, for the predator populations to exist , either the toxicant input rate $q_0$ should be maintained less than 1.3679 or the dissolved oxygen input rate(r) should always be maintained above threshold value of r= 74.555. Thus, it is concluded from our study that to sustain the prey-predator aquatic populations while maintaining a healthy level of dissolved oxygen in water, the release of toxic and acid components in water needs to be mitigated and control strategies needs to be developed for the same in future.

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