The Maximal MAM, 

a Reasonable Implementation of 

the Maximal Strategy

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1 Introduction

This note is about a reasonable abstract machine, called Maximal MAM, implementing the maximal strategy of the \(\lambda\)-calculus, that is, the strategy that always produces a longest evaluation sequence. The abstract machine is a minor variation over the Useful MAM of [Acc16], that is a reasonable implementation of the leftmost-outermost strategy. Here reasonable is a technical term: an abstract machine \(M\) implementing a strategy \(\to\) is reasonable when its overhead on a given term \(t\) is polynomial with respect to the size of \(t\) and the number of \(\to\)-steps necessary to evaluate \(t\).

The result of this note is discussed and put in context in the author’s (In)Efficiency and Reasonable Cost Models [Acc17]. Essentially, the maximal strategy is a peculiar case-study, for the following reasons:

• Non-trivial reasonable implementation: it is the most inefficient strategy ever, and yet implementing it with a reasonable overhead requires the whole heavy machinery, namely useful sharing, needed for the leftmost-outermost strategy, the other only strong strategy known to be reasonable.

• Unreasonable cost model: it is reasonably implementable—that usually is the tricky part in the study of cost models—but its number of steps does not provide a reasonable cost model. Indeed the strategy is so inefficient that it cannot even simulate Turing machines, that usually is the easy part in the study of cost models. To be precise, there are no proofs of such impossibility, but the known encodings of Turing machines do not work when the evaluation strategy in the \(\lambda\)-calculus is the maximal one.

We first define the strategy, and then show how to implement it via the Max MAM. The technical details follow closely those in [Acc16] for the Useful MAM. In turn, the Useful MAM is a refinement of the Strong MAM, an unreasonable abstract machine for the leftmost-outermost strategy studied in [ABM15], that in turn is a simplification of Cregut’s machine [Cre90, Cre07, GNM13]. The literature about abstract machines for strong evaluation is extremely limited, we essentially already cited all existing papers on the subject. A few further papers [GL02, AC15, AG17] deal with abstract machines for weak evaluation.
with open terms, that is an intermediate framework between the one of weak evaluation with closed terms, to which the almost totality of the literature is devoted, and the almost inexistent one of strong evaluation.

2 \(\lambda\)-Calculus and Maximal Evaluation

The syntax of the ordinary \(\lambda\)-calculus is given by the following grammar for terms:

\[
\text{\(\lambda\)-Terms} \quad t, u, s := x \mid \lambda x . t \mid tu.
\]

We use \(t\{x\leftarrow u\}\) for the usual (meta-level) notion of substitution. An abstraction \(\lambda x . t\) binds \(x\) in \(t\), and we silently work modulo \(\alpha\)-equivalence of bound variables, e.g. \((\lambda y . (xy))\{x\leftarrow y\} = \lambda z . (yz)\). We use \(fv(t)\) for the set of free variables of \(t\).

\(\beta\)-reduction. We define \(\beta\)-reduction \(\rightarrow\) as follows:

\[
\text{Contexts}
\]
\[
C :: := \langle \cdot \rangle \mid \lambda x . C \mid Ct \mid tC
\]

\text{Rule at top level}

\[
(\lambda x . t)u \rightarrow_\beta t\{x\leftarrow u\}
\]

\text{Contextual closure}

\[
C(t) \rightarrow_\beta C(u) \quad \text{if } t \rightarrow_\beta u
\]

A context \(C\) is \textit{applicative} if \(C = D(\langle \cdot \rangle u)\) for some \(D\) and \(u\). A term \(t\) is a \textit{normal form}, or simply \textit{normal}, if there is no \(u\) such that \(t \rightarrow_\beta u\), and it is \textit{neutral} if it is normal and it is not of the form \(\lambda x . u\) (i.e. it is not an abstraction). The \textit{position} of a \(\beta\)-redex \(C(t) \rightarrow_\beta C(u)\) is the context \(C\) in which it takes place.

To ease the language, we will identify a redex with its position. A derivation \(d : t \rightarrow^k u\) is a finite, possibly empty, sequence of reduction steps. We write \(|t|\) for the size of \(t\) and \(|d|\) for the length of \(d\).

Maximal Evaluation. The maximal strategy is the variation over leftmost-outermost (LO) evaluation in which, when the LO redex \((\lambda x . t)u\) is erasing (that is, when \(x \notin fv(t)\)), the strategy first evaluates (maximally) \(u\) to \(u'\), and then fires the erasing redex \((\lambda x . t)u' \rightarrow_\beta t\), to avoid the erasure of the \(\beta\)-redexes of \(u\) before being reduced. Of course, if \(u\) diverges than the maximal strategy diverges, while the LO strategy would not (or, at least, not because of \(u\)). The LO strategy is inefficient but it has the key property of being \textit{normalizing}, i.e. it reaches a normal form whenever it exists. The maximal strategy, dually, is \textit{perpetual}, that is, it diverges whenever possible. See van Raamsdonk at al.’s \cite{RSSX99} for more about perpetual and maximal strategies.

We define the maximal strategy by first defining the notion of \textit{maximal context}, that is a context in which a maximal redex can appear (and not a context that cannot be extended).

\textbf{Definition 2.1 (Max Context).} \textit{Maximal (or max) contexts are defined by induction as follows:}
\[
\begin{align*}
\langle \cdot \rangle & \text{ is Max (ax)} \\
C \text{ is Max} & \quad Ct \text{ is Max} \quad \frac{C \neq \lambda x.D}{\text{(@l)}} \\
C \text{ is Max} & \quad \lambda x. C \text{ is Max} \quad \frac{x \notin \mathsf{fv}(t)}{(\lambda x. t)C \text{ is Max}} \quad \text{(gc)} \\
C \text{ is Max} & \quad \text{t is neutral} \quad \frac{tC \text{ is Max}}{\text{(@r)}}
\end{align*}
\]

We define the maximal $\beta$-reduction strategy $\Rightarrow_{\text{Max}\beta}$ as follows:

**Rule at top level**

\[
(\lambda x.t)u \Rightarrow_{\text{Max}\beta} t \{x \leftarrow u\} \quad \text{if } x \notin \mathsf{fv}(t) \text{ or } u \text{ is normal}
\]

**Contextual closure**

\[
C(t) \Rightarrow_{\text{Max}\beta} C(u) \quad \text{if } t \Rightarrow_{\text{Max}\beta} u
\]

As expected,

**Lemma 2.2** (Basic Properties of the Maximal Strategy, Proof at page 13). Let $t$ be a $\lambda$-term that is not normal. Then

1. **Completeness**: there exists $u$ such that $t \Rightarrow_{\text{Max}\beta} u$.
2. **Determinism**: moreover, such a $u$ is unique.

Let us point out that we talk about the maximal strategy despite the fact that there are many maximal strategies, that can be obtained from our definition by adding some more freedom in the choice of the redex. A typical example is given by $xtu$: our strategy first reduces $t$ and then (if $t$ terminates) it evaluates $u$, while one can interleave the evaluations of $t$ and $u$ and still be maximal (because $t$ cannot act on $u$ in $xtu$). We refer it as the maximal strategy, because it is the leftmost maximal strategy, i.e. it mostly behaves as the leftmost strategy, it only changes its behavior on erasing redexes (and the leftmost strategy is a sort of canonical strategy).

## 3 Preliminaries on Abstract Machines

We study two abstract machines, the Maximal Milner Abstract Machine (Max MAM) (Fig. 4) and an auxiliary machine called the Checking AM (Fig. 2). The Max MAM is a reasonable implementation of the maximal strategy resting on labeled environments to implement useful sharing, and on the Checking AM to produce these labels.

The Max MAM is meant to implement the maximal strategy via a decoding function $\cdot$ mapping machine states to $\lambda$-terms. Machine states $s$ are given by a code $t$, that is a $\lambda$-term $t$ *not considered up to $\alpha$-equivalence* (which is why it is over-lined), and some data-structures like stacks, frames, and environments. The data-structures are used to implement the search for the next maximal redex and a form of micro-steps substitution, and they decode to evaluation contexts for $\Rightarrow_{\text{Max}\beta}$. Every state $s$ decodes to a term $\overline{t}$, having the shape $Cs\langle \overline{t}\rangle$, where $\overline{t}$ is the code currently under evaluation and $Cs$ is the evaluation context given by the data-structures.

The Checking AM uses the same states and data-structures of the Max MAM.
The Data-Structures. First of all, our machines are executed on well-named terms, that are those $\alpha$-representants where all variables (both bound and free) have distinct names. Then, the data-structures used by the machines are defined in Fig. 1, namely:

1. Stack $\pi$: it contains the arguments of the current code;
2. Frame $F$: a second stack, that together with $\pi$ is used to walk through the term and search for the next redex to reduce. The items $\phi$ of a frame are of three kinds:
   1. Variables: a variable $x$ is pushed on the frame $F$ whenever the machines starts evaluating under an abstraction $\lambda x$.
   2. Head argument contexts: $t \triangleright \pi$ is pushed on $F$ every time evaluation enters in the right subterm $\overline{\tau}$ of an application $\overline{\tau}$. The entry saves the left part $\overline{\tau}$ of the application and the current stack $\pi$, to restore them when the evaluation of the right subterm $\overline{\tau}$ is over.
   3. Erasing Contexts: $\lambda x. t \triangleright \pi$ is pushed on $F$ every time evaluation finds an erasing redex ($\lambda x. \overline{t} \overline{\pi}$) (that is, a redex for which $x \notin \text{fv}(\overline{t})$). In this case evaluation enters in the argument $\overline{t}$ to normalize it (or, possibly, to diverge) before erasing it.
3. Global Environment $E$: it is used to implement micro-step substitution (i.e. on a variable occurrence at the time), storing the arguments of $\beta$-redexes that have been encountered so far. Most of the literature on abstract machines uses local environments and closures. Having just one global environment $E$ (used only in a minority of works [FS09, SGM02, DZ13, ABM14, AC15, ABM15, Acc16, AG17], and discussed at length in [AB17]) removes the need for closures and simplifies the machine. On the other hand, it forces to use explicit $\alpha$-renamings (the operation $t \triangleright \alpha \in E$ and $\triangleright \alpha \text{abs}$ in Fig. 4), but this does not affect the overall complexity, as it speeds up other operations, see [AB17]. The entries of $E$ are of the form $[x = t]_l$, i.e. they carry a label $l$ used to implement usefulness, to be explained later on in this section. We write $E(x) = [x = t]_l$ when $E$ contains $[x = t]_l$ and $E(x) = \perp$ when in $E$ there are no entries of the form $[x = t]_l$.

The Decoding. Every state $s$ decodes to a term $\overline{s}$ (see Fig. 3) of shape $C_s(\overline{t}|_E)$, where

1. $\overline{t}|_E$ is a $\lambda$-term, roughly obtained by applying to the code the substitution induced by the global environment $E$. More precisely, the operation $\overline{t}|_E$ is called unfolding and it is properly defined at the end of this section.
2. $C_s$ is a context, that will be shown to be a max context, obtained by decoding the stack $\pi$ and the dump $D$ and applying the unfolding. Note that, to improve readability, $\pi$ is decoded in postfix notation for plugging.

The Transitions. According to the distillation approach of [ABM14] (related to Danvy and Nielsen’s refocusing [DN04]) we distinguish different kinds of transitions, whose names reflect a proof-theoretical view, as transitions can be seen as cut-elimination steps [ABS09, ABM14]:

1. Multiplicatives $\rightsquigarrow_\rho$: they correspond to the firing of a $\beta$-redex ($\lambda x. t u$, except that if the argument $u$ is not a variable and the redex is not erasing (that happens when $x \notin \text{fv}(t)$) then $\overline{\pi}$ is not substituted but added to the environment;
2. Exponentials $\rightsquigarrow_\sigma$: they perform a clashing-avoiding substitution from the
Frames $F ::= ε | F :: φ$

Stacks $π ::= ε | T :: π$

Frame Items $φ ::= x | T φ | λx.T π$

Phases $φ ::= ▼ | ▲$

Labels $l ::= \text{abs} | (\text{red}, n ∈ N) | \text{neu}$

Environments $E ::= ε | [x ← t] :: E$

Figure 1: Grammars.

environment on the single variable occurrence represented by the current code. They implement micro-step substitution.

3. **Commutatives $⇝_c$:** they locate and expose the next redex according to the maximal strategy, by rearranging the data-structures.

Both exponential and commutative transitions are invisible on the $λ$-calculus.

Garbage collection of environment entries that are no longer necessary is here simply ignored, or, more precisely, it is encapsulated at the meta-level, in the decoding function.

**Labels for Useful Sharing** A label $l$ for a code in the environment can be of three kinds. Roughly, they are:

1. **Neutral**, or $l = \text{neu}$: it marks a neutral term, that is always useless as it is $β$-normal and its substitution cannot create a redex, because it is not an abstraction;

2. **Abstraction**, or $l = \text{abs}$: it marks an abstraction, that is a term that is at times useful to substitute. If the variable that it is meant to replace is applied, indeed, the substitution of the abstraction creates a $β$-redex. But if it is not applied, it is useless.

3. **Redex**, or $l = \text{red}$: it marks a term that contains a $β$-redex. It is always useful to substitute these terms.

Actually, the explanation we just gave is oversimplified, but it provides a first intuition about labels. In fact in an environment $[x ← t] :: E$ it is not really $T$ that has the property mentioned by its label, rather the term $t ↛ E$ obtained by unfolding the rest of the environment on $T$. The idea is that $[x ← t]^{red}$ states that it is useful to substitute $T$ to later on obtain a redex inside it (by potential further substitutions on its variables coming from $E$). The precise meaning of the labels will be given by Definition 4.2 and the properties they encode will be made explicit by Lemma 5.1 in the Appendix (page 31).

A further subtlety is that the label red for redexes is refined as a pair $(\text{red}, n)$, where $n$ is the number of substitutions in $E$ that are needed to obtain the maximal redex in $T ↛ E$. Our machines never inspect these numbers, they are only used for the complexity analysis of Sect. 6.

**Grafting and Unfoldings** The unfolding of the environment $E$ on a code $T$ is defined as the recursive capture-allowing substitution (called grafting) of the entries of $E$ on $T$. For lack of space, the precise definition has been moved to the Appendix (page 13).

4 **The Checking Abstract Machine**

The Checking Abstract Machine (Checking AM) is defined in Fig. 2 and it is a variation over the very similar auxiliary checking machine for the Useful MAM in [Acc16]. The difference between the two is in the two new transitions $→_c T$.
and \( \rightarrow_{o_5} \) (that is also why in Fig. 2 they are misplaced with respect to the progressing numbering), plus the side condition if \( x \in \text{fv}(\overline{t}) \) in \( \rightarrow_{o_1} \).

The Checking AM starts executions on states of the form \((ε, 7, ε, E, ▼)\), with the aim of checking the usefulness of \( 7 \) with respect to the environment \( E \), i.e. it walks through \( 7 \) and whenever it encounters a variable \( x \) it looks up its usefulness in \( E \).

The Checking AM has seven commutative transitions, noted \( \rightarrow_{c_i} \) with \( i = 1, \ldots, 7 \), used to walk through the term, and six output transitions, noted \( \rightarrow_{o_j} \) with \( j = 1, \ldots, 6 \), that produce the value of the test for usefulness, to be later used by the Max MAM. The exploration is done in two alternating phases, evaluation \( ▼ \) and backtracking \( ▲ \). Evaluation explores the current code (morally towards the head, except when it encounters an erasing redex, that is when it first explores the right subterm) storing in the stack and in the frame the parts of the code that it leaves behind. Backtracking comes back to an argument that was stored in the frame, when the current head has already been checked. Note that the Checking AM never modifies the environment, it only looks it up, nor it erases or duplicates any piece of code, it only walks through the data structures.

Let us explain the transitions. First the commutative ones:

- \( \rightarrow_{c_1} \): the code is an application \( \overline{π} \) and the machine starts exploring the left subterm \( 7 \), storing \( π \) on top of the stack \( π \).
• $\rightarrow v_{c_2}$: the code $\lambda x.\overline{t}$ and the first argument $\overline{\pi}$ on the stack form an erasing redex (by hypothesis $x \notin \text{fv}(\overline{t})$). The machine has found a $\beta$-redex but it cannot output yet, because it does not know what number $n$ to associate to the label $l = (\text{red}, n)$. Indeed, $(\lambda x.\overline{t})\overline{\pi}$ is the next $\beta$-redex to reduce only if $\overline{\pi}$ is normal, otherwise the next one will be in $\overline{\pi}$ and obtaining it may require many substitution steps. Therefore, the machine stores $\lambda x.\overline{t}$ and the current stack $\pi$ on the frame $F$ and starts checking $\overline{\pi}$ (with an empty stack).

• $\rightarrow v_{c_3}$: the code is an abstraction $\lambda x.\overline{t}$ and the machine goes under the abstraction, storing $x$ on top of the frame $F$.

• $\rightarrow v_{c_4}$: it is the inverse of $\rightarrow v_{c_2}$, it puts back on the code an abstraction that was previously stored in the frame.

• $\rightarrow s_{c_1}$: backtracking from the evaluation of an argument $\overline{\pi}$, it restores the application $\overline{t}\overline{\pi}$ and the stack $\pi$ that were previously stored in the frame.

• $\rightarrow s_{c_2}$: backtracking from the evaluation of the left subterm $\overline{t}$ of an application $\overline{t}\overline{\pi}$, the machine starts evaluating the right subterm (by switching to the evaluation phase $\nabla$) with an empty stack $\epsilon$, storing on the frame the pair $\overline{t}\theta\pi$ of the left subterm and the previous stack $\pi$.

Then the output transitions:

• $\rightarrow o_1$: the machine finds a non-erasing $\beta$-redex, namely $(\lambda x.\overline{t})\overline{\pi}$ (by hypothesis $x \notin \text{fv}(\overline{t})$), and thus outputs a label saying that it requires only one substitution step (namely substituting the term the machine was executed on) to eventually find a $\beta$-redex.

• $\rightarrow o_2$: the machine finds a variable $x$ whose associated entry $[x\leftarrow\overline{t}]^{(\text{red}, n)}$ in the environment is labeled with $(\text{red}, n)$, and so outputs a label saying that it takes $n + 1$ substitution steps to eventually find a $\beta$-redex ($n$ plus 1 for the term the machine was executed on).

• $\rightarrow o_3$: the machine finds a variable $x$ whose associated entry $[x\leftarrow\overline{t}]^{\text{abs}}$ in the environment is labeled with $\text{abs}$, so $\overline{t}$ is an abstraction, and the stack is non-empty. Since substituting the abstraction will create a $\beta$-redex, the machine outputs a label saying that it takes two substitution steps to obtain a $\beta$-redex, one for the term the machine was executed on and one for the abstraction $\overline{t}$.

• $\rightarrow o_6$: the machine went through the whole code $\overline{\pi}$ and found no redexes. Then $\overline{\pi}$ is normal and together with the erasing abstraction $\lambda x.\overline{t}$ in the
frame it forms an erasing redex ready to fire, and so the output is a label saying that it requires only one substitution step (namely substituting the term the machine was executed on) to eventually find a $\beta$-redex.

- $\mathbin{\rightarrow_{o4}}$: the machine went through the whole term, that is an application, and found no redex, nor any redex that can be obtained by substituting from the environment. Thus that term is neutral and so the machine outputs the corresponding label.

- $\mathbin{\rightarrow_{o5}}$: as for the previous transition, except that the term is an abstraction, and so the output is the $\text{abs}$ label.

The fact that commutative transitions only walk through the code, without changing anything, is formalized by the following lemma, that is crucial for the proof of correctness of the Checking AM (forthcoming Theorem 4.3).

**Lemma 4.1** (Commutative Transparency, Proof at P. [14]). Let $s = (F, \overline{t}, \pi, E, \varphi) \mathbin{\leadsto_{c1,2,3,4,5,6,7}} (F', \overline{t}', \pi', E, \varphi') = s'$. Then

1. Decoding Without Unfolding: $E'\langle\overline{t}\rangle_{\overline{\pi}} = E'\langle\overline{t}'\rangle_{\overline{\pi}'}$, and

2. Decoding With Unfolding: $s = s'$.

For the analysis of the properties of the Checking AM we need a notion of well-labeled environment, i.e. of environment where the labels are consistent with their intended meaning. It is a technical notion also providing enough information to perform the complexity analysis, later on. Moreover, it includes two structural properties of environments: 1) in $[x\leftarrow\overline{t}]_l$ the code $\overline{t}$ cannot be a variable, and 2) there cannot be two entries associated to the same variables.

**Definition 4.2** (Well-Labeled Environments). Well-labeled environments $E$ are defined by

1. Empty: $\epsilon$ is well-labeled;

2. Inductive: $[x\leftarrow\overline{t}]_l :: E'$ is well-labeled if $E'$ is well-labeled, $x$ is fresh with respect to $\overline{t}$ and $E'$, and

   (a) Abstractions: if $l = \text{abs}$ then $\overline{t}$ and $\overline{t}_l E'$ are normal abstractions;

   (b) Neutral Terms: if $l = \text{neu}$ then $\overline{t}$ is an application and $\overline{t}_l E'$ is neutral.

   (c) Redexes: if $l = (\text{red}, n)$ then $\overline{t}$ is not a variable, $\overline{t}_l E'$ contains a $\beta$-redex. Moreover, there is a Max-context $C$ such that $\overline{t} = C(\overline{w})$ and

   - if $n = 1$ then $\overline{w}$ is a $\vdash_{\text{max}\beta}$-redex,

   - if $n > 1$ then $\overline{w} = x$ and $E' = E'' :: [y\leftarrow\overline{w}]' :: E'''$ with

     - if $n > 2$ then $l = (\text{red}, n - 1)$

     - if $n = 2$ then $l = (\text{red}, 1)$ or ($l = \text{abs}$ and $C$ is applicative).

The study of the Checking AM requires some technical invariants proved in the appendix (page 55). The next theorem provides the main properties of the Checking AM, i.e. that when executed on $\overline{t}$ and $E$ it provides a label $l$ to extend $E$ with a consistent entry for $\overline{t}$ (i.e. such that $[x\leftarrow\overline{t}]_l :: E$ is well-labeled), and that such an execution takes time linear in the size of $\overline{t}$. 8
Let us explain an important point about the complexity of the machine transition. Variables are meant to be implemented as memory locations and variable occurrences as pointers to those location. Therefore, the global environment $E$ is a store and can be accessed randomly, that is, with no need to go through it sequentially. With this hypothesis on the representation of terms, see [AB17] for an actual implementation, all the transitions of the Checking AM but $\rightarrow_{v_c}$ and $\rightarrow_{o_1}$ can be implemented in constant time. The check $x \in / \notin \text{fv}(\overline{t})$ in $\rightarrow_{v_c}$ and $\rightarrow_{o_1}$ at first sight requires time proportional to the size of the initial term, but in fact it can be implemented in $O(1)$ if one assumes a stronger hypothesis on the representation of terms: a variable is a data-type with a memory location plus pointers to its occurrences. Then all transitions can be implemented in $O(1)$. Anyway, this hypothesis is not essential for the implementation to be reasonable. If dropped, the subterm property (Lemma 6.9 in the Appendix) guarantees that the size of terms does not explode and that the check $x \in / \notin \text{fv}(\overline{t})$ can be done in reasonable time. The hypothesis, however, simplifies considerably the analysis.

**Theorem 4.3** (Checking AM Properties, Proof at Page 18). Let $\overline{t}$ be a code and $E$ a global environment.

1. Determinism and Progress: the Checking AM is deterministic and there always is a transition that applies;
2. Termination and Complexity: the execution of the Checking AM on $\overline{t}$ and $E$ always terminates, taking $O(|\overline{t}|)$ steps, moreover
3. Correctness: if $E$ is well-labeled, $x$ is fresh with respect to $E$ and $\overline{t}$, and $l$ is the output then $[x \leftarrow l]_E :: E$ is well-labeled.

## 5 The Maximal Milner Abstract Machine

The Maximal Milner Abstract Machine, or Max MAM, is defined in Fig. 4 and it is a small variation over the Useful MAM of [Acc16] (as for the Checking AM, the difference is in the two new transitions, $\Rightarrow_{m_2}$ and $\Rightarrow_{m_3}$, plus the side condition if $x \in \text{fv}(\overline{t})$ in $\Rightarrow_{m_3}$). It is very similar to the Checking AM, in particular it has exactly the same commutative transitions, and the same organization in evaluating and backtracking phases. The difference with respect to the Checking AM is that the output transitions are replaced by micro-step computational rules that reduce $\beta$-redexes and implement useful substitutions. Let us explain them:

- **Multiplicative Transition $\Rightarrow_{m_2}$**: when the argument of the $\beta$-redex $(\lambda x. \overline{t}) y$ is a variable $y$ then it is immediately substituted in $\overline{t}$. This happens because 1) such substitution are not costly (by the subterm invariant, Lemma 6.9 in the Appendix, their cost is bound by the size of $\overline{t}$ that is bound by the size of the initial term); 2) because in this way the environment stays compact; 3) because in this way the labels for useful sharing are slightly simpler.
- **Multiplicative Transition $\Rightarrow_{m_3}$**: since the argument $\overline{t}$ is not a variable and the redex is not erasing ($x \in \text{fv}(\overline{t})$ by hypothesis) then the redex is fired by adding the entry $[x \leftarrow \overline{t}]$ to the environment, with $l$ obtained by running the Checking AM on $\overline{t}$ and $E$. 


$\pi$ is any code $\alpha$-equivalent to $\pi$ such that it is well-named and its bound names are fresh with respect to those in the other machine components.

**Figure 4:** The Maximum (Useful) Milner Abstract Machine (Max MAM).

| Frame | Code | Stack | Env | Ph | Frame | Code | Stack | Env | Ph |
|-------|------|-------|-----|----|-------|------|-------|-----|----|
| $F$   | $\pi$ | $E$   | $\updownarrow \sim c_1$ | $F$ | $\uparrow \pi$ | $E$ | $\updownarrow$ | $\uparrow$ | $\updownarrow$ |
| $F$   | $\lambda x. \pi$ | $y :: \pi$ | $E$ | $\updownarrow \sim a_1$ | $F$ | $\{x \leftarrow y\}$ | $\pi$ | $E$ | $\updownarrow$ |
| $F$   | $\lambda x. \pi$ | $\pi :: \pi$ | $E$ | $\updownarrow \sim a_3$ | $F$ | $\pi$ | $\{x \leftarrow \pi\}$ | $E$ | $\updownarrow$ |

If $x \in f v(\pi)$ and $\pi$ is not a variable and $i$ is the output of the Checking AM on $\pi$ and $E$.

| $F$   | $\lambda x. \pi$ | $\pi :: \pi$ | $E$ | $\updownarrow \sim c_2$ | $F$ | $\{x \leftarrow \pi\}$ | $\pi$ | $\pi$ | $\updownarrow$ |
| $F$   | $\lambda x. \pi$ | $\pi :: \pi$ | $E$ | $\updownarrow \sim a_{\beta \text{red}}$ | $F$ | $\pi$ | $\pi$ | $\pi$ | $\updownarrow$ |

If $E(x) = [x \leftarrow \pi](\text{red}, n)$.

| $F$   | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow \sim a_{\text{abs}}$ | $F$ | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow$ |

If $E(x) = [x \leftarrow \pi](\text{abs})$ and $\pi = \epsilon$.

| $F$   | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow \sim a_{\text{abs}}$ | $F$ | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow$ |

| $F$   | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow \sim a_{\text{abs}}$ | $F$ | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow$ |

| $F$   | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow \sim a_{\text{abs}}$ | $F$ | $\pi$ | $\pi :: \pi$ | $E$ | $\updownarrow$ |

**Multiplicative Transition** $\sim a_{\alpha}$: an invariant of the machine is that when it backtracks (phase $\uparrow$) the code is normal (see Lemma 6.6.2 in the Appendix, page 20). Moreover, every abstraction $\lambda x. \pi$ in an erasing redex entry $\lambda x. \pi$ in the frame is such that $x \notin f v(\pi)$ (Lemma 6.4.1). Then in $\sim a_{\alpha}$ the code $\pi$ is normal and together with $\lambda x. \pi$ it forms an erasing $\Rightarrow_{\alpha \text{red}}$ redex. Correctly, the machine throws away $\pi$ and switches to evaluating $\pi$ (note the change of phase from $\uparrow$ to $\downarrow$).

**Exponential Transition** $\sim e_{\alpha, \beta}$: the environment entry associated to $x$ is labeled with $(\text{red}, n)$; thus it is useful to substitute $\pi$. $\pi$ is the idea that at most $n$ additional substitution steps (shuffled with commutative steps) a $\beta$-redex will be obtained. To avoid variable clashes the substitution $\alpha$-renames $\pi$.

**Exponential Transition** $\sim e_{\alpha, \beta}$: the environment associates an abstraction to $x$ and the stack is non empty, so it is useful to substitute the abstraction (again, $\alpha$-replacing to avoid variable clashes). Note that if the stack is empty the machine rather backtracks using $\sim c_2$.

The Max MAM starts executions on initial states of the form $(\epsilon, \tilde{\pi}, \epsilon, \epsilon, \downarrow)$, where $\tilde{\pi}$ is such that any two variables (bond or free) have distinct names, and any other component is empty. A state $s$ is reachable if there are an initial state $s'$ and a Max MAM execution $\rho: s' \sim^* s$, and it is final if no transitions apply.

The theorem of correctness and completeness of the machine with respect to $\Rightarrow_{\alpha \text{red}}$ follows. It rests on a technical development that is in the appendix, starting at page 20. The bisimulation is weak because transitions other than $\sim a_{\alpha}$ are invisible on the $\lambda$-calculus. For a machine execution $\rho$ we denote with $|\rho|$ (resp. $|\rho|_x$) the number of transitions (resp. $x$-transitions for $x \in \{m, e, c, \ldots\}$) in $\rho$. 

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Theorem 5.1 (Weak Bisimulation, Proof at Page 25). Let $s$ be an initial Max MAM state of code $\overline{t}$.

1. Simulation: for every execution $\rho : s \rightarrow^* s'$ there exists a derivation $d : s \rightarrow_{\text{Max}\beta}^* s'$ such that $|d| = |\rho|_{\lambda}$.
2. Reverse Simulation: for every derivation $d : t \rightarrow_{\text{Max}\beta}^* u$ there is an execution $\rho : s \rightarrow^* s'$ such that $\overline{s'} = u$ and $|d| = |\rho|_{\lambda}$.

6 Quantitative Analysis

The complexity analysis of the Max MAM is omitted because it follows exactly, by changing only minimal details, the one for the Useful MAM in [Acc16]. All the details can be found in the Appendix, starting from page 25.

Let us anyway provide the schema of the analysis. First of all one proves a subterm invariant, proving that most codes in the state of the Max MAM are subcodes of the initial code. Then the proof of the polynomial bound of the overhead is in three steps.

1. Exponential vs Multiplicative Transitions: we bound the number $|\rho|_e$ of exponential transitions of an execution $\rho$ using the number $|\rho|_m$ of multiplicative transitions of $\rho$, that by Theorem 5.1 corresponds to the number of maximal $\beta$-steps on the $\lambda$-calculus. The bound is quadratic.
2. Commutative vs Exponential Transitions: we bound the number $|\rho|_c$ of commutative transitions of $\rho$ by using the number of exponential transitions and the size of the initial term. The bound is linear in both quantities.
3. Global bound: we multiply the number of each kind of transition for the cost of that kind (everything is constant time but for exponential transitions, that are linear in the size of the initial term), and then sum over the kind of transitions.

Concretely, one obtains the following theorem.

Theorem 6.1 (Max MAM Overhead Bound, Proof at Page 34). Let $d : t \rightarrow_{\text{Max}\beta}^* u$ be a maximal derivation and $\rho$ be the Max MAM execution simulating $d$ given by Theorem 5.1.2. Then:

1. Length: $|\rho| = O((1 + |d|^2) \cdot |t|)$.
2. Cost: $\rho$ is implementable on RAM in $O((1 + |d|^2) \cdot |t|)$ steps.

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Proof Appendix

6.1 Proof of the properties of the Maximal Strategy (L2.2, p. [3])

Proof. By induction on $t$. Cases:

- **Variable**, i.e. $t = x$. Then $t$ is normal, absurd.

- **Abstraction**, i.e. $t = \lambda x. r$. Then by i.h. there exists $s$ such that $r \rightarrow_{\text{Max}} s$. By rule $(\lambda)$, $t = \lambda x. r \rightarrow_{\text{Max}} \lambda x. s$, i.e. just take $u := \lambda x. s$. **Determinism**: it follows from the i.h.

- **Application**, i.e. $t = rs$. Two cases:
  
  - $r$ is an abstraction, i.e. $r = \lambda x. p$. Two sub-cases:
    
    * $x \in r$ or $s$ is normal. Then $t = (\lambda x. p)s \rightarrow_{\text{Max}} p[x \leftarrow s]$. **Determinism**: the rules for Max contexts do not allow to evaluate in $r$, because $\lambda z. r$ is applied, nor to evaluate in $s$ (if it is not normal) because then $x \in r$ and so rule (gc) cannot be applied.
    
    * $x \notin r$ and $s$ is not normal. By i.h. there exists a unique $q$ such that $s \rightarrow_{\text{Max}} q$, that is, there is a perpetual context $C$ such that $s = C(s') \rightarrow_{\text{Max}} C(q')$ with $s' \rightarrow_{\text{Max}} q'$. By rule (gc), $(\lambda x. p) C$ is a Max-context and $t = (\lambda x. p) C(s') \rightarrow_{\text{Max}} (\lambda x. p) C(q')$. Clearly, there cannot be any $\rightarrow_{\text{Max}}$ redex in $r$, so determinism holds.

  - $r$ is not an abstraction. Two sub-cases:
    
    * $r$ is not normal. Then by i.h. there exists unique $p$ such that $r \rightarrow_{\text{Max}} p$, $t = rs \rightarrow_{\text{Max}} ps$. Since $r$ reduces, it is not neutral and so there cannot be $\rightarrow_{\text{Max}}$ redexes in $s$ (because rule (gc) does not apply).
    
    * $r$ is normal and thus neutral. Then $s$ is not normal (otherwise $t$ is normal, absurd). By i.h. there exists unique $p$ such that $s \rightarrow_{\text{Max}} p$. By rule $(\@r)$, $t = rs \rightarrow_{\text{Max}} rp$, and since $r$ is neutral this is the unique $\rightarrow_{\text{Max}}$ redex of $t$.

\[\square\]

6.2 Definition of Grafting and Unfolding, and their Properties

The unfolding of the environment $E$ on a code $\overline{t}$ is defined as the recursive capture-allowing substitution (called grafting) of the entries of $E$ on $\overline{t}$. 

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Definition 6.2 (Grafting and Environment Unfolding). The operation of grafting \( \overline{t}[x \leftarrow \overline{π}] \) is defined by

\[
\begin{align*}
\langle ϖs \rangle [x \leftarrow \overline{π}] & := \overline{t} \langle x \leftarrow \overline{π} \rangle \langle x \leftarrow \overline{π} \rangle \\
x \overline{π} & := \overline{π} \\
(λy. \overline{π}) [x \leftarrow \overline{π}] & := λy. \overline{π} \langle x \leftarrow \overline{π} \rangle
\end{align*}
\]

Given an environment \( E \) we define the unfolding of \( E \) on a code \( \overline{t} \) as follows:

\[
\overline{t} \downarrow_E := \overline{t} \quad \overline{t} \downarrow_{[x \leftarrow \overline{π}]} := \overline{t} \downarrow_{[x \leftarrow \overline{π}]} \downarrow_E
\]

or equivalently as:

\[
\begin{align*}
\langle \overline{w} \rangle \downarrow_E & := \overline{w} \downarrow_E \overline{E} \\
(λx. \overline{π}) \downarrow_E & := λx. \overline{π} \downarrow_E \\
x \downarrow_{[x \leftarrow \overline{π}]} :&= x \downarrow_{[y \leftarrow \overline{π}]} :\overline{E} := x \downarrow_{[y \leftarrow \overline{π}]} \downarrow_E
\end{align*}
\]

For instance, \( (\lambda x. y) \downarrow_{[y \leftarrow xx]} = \lambda x. (xx) \). The unfolding is extended to contexts as expected (i.e. recursively propagating the unfolding and setting \( \epsilon \downarrow_E = E \)).

Let us explain the need for grafting. In \cite{ABM15}, the Strong MAM is decoded to the LSC, that is a calculus with explicit substitutions, i.e. a calculus able to represent the environment of the Strong MAM. Matching the representation of the environment on the Strong MAM and on the LSC does not need grafting but it is, however, a quite technical affair. Useful sharing adds many further complications in establishing such a matching, because useful evaluation computes a shared representation of the normal form and forces some of the explicit substitutions to stay under abstractions. The difficulty is such, in fact, that we found much easier to decode directly to the \( λ \)-calculus rather than to the LSC. Such an alternative solution, however, has to push the substitution induced by the environment through abstractions, which is why we use grafting.

The following easy properties will be used to prove the correctness of the machine (in Lemma 6.7).

Lemma 6.3 (Properties of Grafting and Unfolding).

1. If the bound names of \( t \) do not appear free in \( u \) then \( t[x \leftarrow u] = t[x \leftarrow u] \).
2. If moreover they do not appear free in \( E \) then \( t \downarrow_E \{ x \leftarrow u \} = t \downarrow_E \{ x \leftarrow u \} \).

6.3 Proof of the Commutative Transparency Lemma (L 4.4, p. 8)

1. Transitions:
   - Case \( (F, \overline{π}, π, E, \langle \rangle) \sim_\epsilon_c (F, \overline{π}, π, E, \langle \rangle) \). We have \( D(\overline{π}(\overline{π})) = D(\overline{π}(\overline{π})) \).
   - Case \( (F, \lambda x. \overline{π}, π, E, \langle \rangle) \sim_\epsilon_c (F : \lambda x. \overline{π}(\overline{π}, π, E, \langle \rangle)). \) We have \( D(\lambda x. \overline{π}(\overline{π})) = D(\overline{π}(\lambda x. \overline{π}(\overline{π}, π, E, \langle \rangle))) = (F : \lambda x. \overline{π}(\overline{π}, π, E, \langle \rangle)) \).
   - Case \( (F, \lambda x. \overline{π}, E, \langle \rangle) \sim_\epsilon_c (F : x, \overline{π}, E, \langle \rangle). \) We have \( F(\lambda x. \overline{π}) = F : x, \overline{π} \).
   - Case \( (F, x, π, E, \langle \rangle) \sim_\epsilon_c (F, x, π, E, \langle \rangle). \) Nothing to prove.
   - Case \( (F : x, \overline{π}, E, \langle \rangle) \sim_\epsilon_c (F, \lambda x. \overline{π}, E, \langle \rangle). \) Exactly as \( \sim_\epsilon_c \).
• Case \((F :: T \pi, \epsilon, E, \textbf{A}) \leadsto_{\text{c}} (F, T \pi, \pi, E, \textbf{A})\). We have \((F :: T \pi) = D(T \pi)\).

• Case \((F, T, \pi :: \pi, E, \textbf{A}) \leadsto_{\text{c}} (F :: T \pi, \pi, E, \textbf{A})\). We have \(D(T \pi) = (F :: T \pi)\).

2. We have \(s = F(T \pi) \downarrow_{E} = F(T) \downarrow_{E} = s'\).

### 6.4 The Checking AM Invariants Lemma (L.6.5, p. 15)

Before the invariants we need a lemma to be used in the proof of the decoding invariant.

**Lemma 6.4.** Let \(C\) be a context.

1. **Right Extension**: \(C(\cdot)\) is Max iff \(C(\pi \langle \cdot \rangle)\) is Max;
2. **Left Neutral Extension**: \(C(\cdot)\) is Max and \(\pi\) is neutral iff \(C(\pi \langle \cdot \rangle)\) is Max;
3. **Abstraction Extension**: \(C(\cdot)\) is Max and not applicative iff \(C(\lambda x. \langle \cdot \rangle)\) is Max.
4. **Erasing Redexes Extension**: \(C(\cdot)\) is Max and \(x \notin \text{fv}(t)\) iff \(C(\lambda x. t \langle \cdot \rangle)\) is Max.
5. **Unfolding Removal**: if \(C \downarrow_{E}\) is Max then \(C\) is Max.

**Proof.** By induction on the predicate \(C\) is Max (Definition 2.1, page 2).

Now, some terminology:

- A state \(s\) is **initial** if it is of the form \((\epsilon, T, \epsilon, E, \textbf{A})\) with \(E\) well-labeled, \(T\) well-named and such that the variables abstracted in \(T\) do not occur in \(E\).
- A state \(s\) is **reachable** if there are an initial state \(s'\) and a Checking AM execution \(\rho : s' \Rightarrow^{*} s\).

Finally:

**Lemma 6.5** (Checking AM Invariants, Proof at P. 15). Let \(F \mid \pi \mid \pi' \mid E \mid \varphi\) be a Checking AM state reachable from an initial state of code \(T_0\). Then

1. **Normal Form**:
   (a) Backtracking Code: if \(\varphi = \textbf{A}\), then \(\pi \downarrow_{E}\) is normal, and if \(\pi\) is non-empty, then \(\pi \downarrow_{E}\) is neutral;
   (b) Frame: if \(F = F' :: T \pi' :: F''\), then \(\pi \downarrow_{E}\) is neutral.

2. **Subterm**:
   (a) Evaluating Code: if \(\varphi = \textbf{N}\), then \(\pi\) is a subterm of \(T_0\);
   (b) Stack: any code in the stack \(\pi\) is a subterm of \(T_0\);
   (c) Frame:
      i. Head Contexts: if \(F = F' :: T \pi' :: F''\), then any code in \(\pi'\) is a subterm of \(T_0\);
ii. Erasing Redexes: if $F = F' :: \lambda x.\tau :: F''$, then $\lambda x.\tau$ and any code in $\pi'$ are subterms of $T_0$.

3. Erasing Redexes: if $F = F' :: \lambda x.\tau :: F''$ then $x$ does not occur in $E$, $x \notin fv(\tau)$, and $x \notin fv(T_1 \upharpoonright E)$.

4. Decoding: $C_s$ is a Max context.

Proof.

1. Normal Form: the invariant trivially holds for an initial state $\epsilon \mid T \mid \epsilon \mid E \mid \nabla$. For a non-empty evaluation sequence we list the cases for the last transition. We omit $\Rightarrow_{\varphi \leq 1}$ because it follows immediately from the i.h.

- **Case** ($F, x, \pi, E, \nabla$) $\Rightarrow_{\varphi \leq 1} (F :: x, \pi, E, \nabla)$.
  (a) **Backtracking Code**: trivial since $\varphi \neq \nabla$.
  (b) **Frame**: it follows by the i.h., since the transition only extends $F$ with an item that is not a head context.

- **Case** ($F, x, \pi, E, \nabla$) $\Rightarrow_{\varphi \leq 1} (F :: \lambda x.\pi, E, \nabla)$ with $x \notin fv(T)$.
  (a) **Backtracking Code**: trivial since $\varphi \neq \nabla$.
  (b) **Frame**: it follows by the i.h., since the transition only extends $F$ with an item that is not a head context.

- **Case** ($F, x, \pi, E, \nabla$) $\Rightarrow_{\varphi \leq 1} (F, E(x) = \perp$ or $E(x) = [x = \tau]^{neu}$ or $E(x) = [x = \tau]^{abs}$ and $\pi = \epsilon$).
  (a) **Backtracking Code**: three cases depending on $E$:
    i. $E(x) = \perp$: then $x \bot E = x$, that is both a normal and a neutral term.
    ii. $E(x) = [x = \tau]^{neu}$: more precisely $E = E_1 :: [x = \tau]^{neu} :: E_2$.
       Then we have $x \bot E = x \bot E_1 :: [x = \tau]^{neu} :: E_2 = \tau \bot E_2$ that is a neutral term because $E$ is well-labeled. Note that $E_1$ cannot bound $x$ because of the freshness requirements in the definition of well-labeled environment.
    iii. $E(x) = [x = \tau]^{abs}$ and $\pi = \epsilon$: similarly to the previous case we obtain that $x \bot E$ is a normal abstraction.
  (b) **Frame**: it follows from the i.h., as $F$ is unchanged.

- **Case** ($F :: x, \pi, E, \Delta$) $\Rightarrow_{\varphi \leq 1} (F, \lambda x.\pi, E, \Delta)$.
  (a) **Backtracking Code**: by i.h. we know that $T_1 \upharpoonright E$ is a normal term.
     Then $(\lambda x.\pi) \downharpoonright E = \lambda x.\pi \upharpoonright E$ is a normal term. The stack is empty, so we conclude.
  (b) **Frame**: it follows from the i.h., as $F$ is unchanged.

- **Case** ($F :: T \upharpoonright \pi, \pi, E, \Delta$) $\Rightarrow_{\varphi \leq 1} (F, \lambda \pi, E, \Delta)$.
  (a) **Backtracking Code**: by i.h. we have that $\pi \upharpoonright E$ is a normal term while by Point [11] of the i.h. $T_1 \upharpoonright E$ is a neutral term. Therefore $(\pi \upharpoonright E) \downharpoonright E = \lambda \pi \upharpoonright E$ is a neutral term.
  (b) **Frame**: it follows from the i.h., as $F$ is unchanged.

- **Case** ($F :: T \upharpoonright \pi, \pi, E, \Delta$) $\Rightarrow_{\varphi \leq 1} (F, \lambda \pi, E, \Delta)$.
(a) Backtracking Code: trivial since $\varphi \neq \Delta$.

(b) Frame: $T|_E$ is a neutral term by Point (1a) of the i.h., the rest follows from the i.h.

2. Subterm: This is a special case of the more refined subterm invariant of the Max MAM (Lemma 6.9.1, page 26).

3. Erasing Redexes: the invariant trivially holds for an initial state $\epsilon | T | E | \gamma$. For a non-empty evaluation sequence, for all transitions $\rho \vdash_{e_{c\gamma}}$ the statement follows immediately from the i.h. because they all leave untouched the erasing redex items in the frame. So consider $(F, \lambda x. \pi. \pi :: \pi, E, \gamma) \vdash_{e_{c\gamma}} (F :: \lambda x. \pi. \pi. \pi, E, \gamma)$. By the hypothesis on the transition, we have $x \notin \text{fv}(\pi)$. By the subterm invariant (Lemma 6.4.14), $\lambda x. \pi$ is a subterm of the initial term and so, by the hypotheses on initial states, $x$ does not occur in $E$. Last, since $x$ does not occur in $E$ nor in $\gamma$, clearly it does not occur in $T|_E$.

4. Decoding: the invariant trivially holds for an initial state $\epsilon | T | E | \gamma$. For a non-empty evaluation sequence we list the cases for the last transition. To simplify the reasoning in the following case analysis we let implicit that the unfolding spreads on all the subterms, i.e. that $D(\langle \gamma \rangle |_E) = F_{\gamma E}(\langle \gamma \rangle |_E)$. Cases:

- Case $s' = (F, \bar{T}, \pi, E, \gamma) \vdash_{e_{c\gamma}} (F, \bar{T}, \pi :: \pi, E, \gamma) = s$. By Lemma 6.4.11 $C_s = D(\langle \gamma \rangle |_E) | = \text{Max}$ iff $C_{s'} = D(\langle \gamma \rangle |_E)$ is Max, and $C_{s'}$ is Max by i.h.

- Case $(F, \lambda x. \pi. \pi :: \pi, E, \gamma) \vdash_{e_{c\gamma}} (F :: \lambda x. \pi. \pi. \pi, E, \gamma)$ with $x \notin \text{fv}(\bar{T})$.

  By i.h., $C_{s'} = F_{\gamma E}(\langle \gamma \rangle |_E) = F_{\gamma E}(\langle \gamma \rangle | pi E :: \pi | pi E)$. Max. By the erasing redexes invariant (Lemma 6.4.11), we have $x \notin \text{fv}(\pi E)$. Then by Lemma 6.4.11 $F_{\gamma E}(\langle \lambda x. \pi \rangle E :: \pi | pi E) = F_{\gamma E}(\langle \lambda x. \pi \rangle E :: \pi | pi E) = C_{s'}$. Max.

- Case $s' = (F, \lambda x. \bar{T}, \epsilon, E, \gamma) \vdash_{e_{c\gamma}} (F :: \lambda x. \bar{T}, \epsilon, E, \gamma) = s$. We have $C_s = D(\langle \gamma \rangle |_E)$ and by i.h. we know that $D(\langle \gamma \rangle |_E) = C_{s'}$. Note that by definition the decoding of frames cannot be applicative. Then $C_s$ is Max by Lemma 6.4.13.

- Case $s' = (F, x, \pi, E, \gamma) \vdash_{e_{c\gamma}} (F, x, \pi, E, \Delta) = s$ with $E(x) = \perp$ or $E(x) = [x \in \pi]_{\text{new}}$ or $(E(x) = [x \in \pi]_{\text{abs}}$ and $\pi = \epsilon$). We have $C_s = D(\langle \gamma \rangle |_E) | = C_{s'}$, and so the statement follows immediately from the i.h.

- Case $s' = (F :: x, \bar{T}, \epsilon, E, \Delta) \vdash_{e_{c\gamma}} (F, \lambda x. \bar{T}, \epsilon, E, \Delta) = s$.

  We have $C_s = D(\langle \gamma \rangle |_E)$ and by i.h. we know that $D(\langle \lambda x. \gamma \rangle |_E) = C_{s'}$ is Max. Then $C_s$ is Max by Lemma 6.4.13.

- Case $s' = (F :: \bar{T}, \pi, \epsilon, E, \Delta) \vdash_{e_{c\gamma}} (F, \bar{T}, \pi, \epsilon, E, \Delta) = s$. By i.h. $C_{s'} = D(\langle \gamma \rangle |_E) | = \text{Max}$, and so by Lemma 6.4.12 $D(\langle \gamma \rangle |_E) = C_s$ is Max.

- Case $s' = (F, \bar{T}, \pi :: \pi, E, \Delta) \vdash_{e_{c\gamma}} (F :: \bar{T}, \pi, \epsilon, E, \Delta) = s$. By i.h. $C_{s'} = D(\langle \gamma \rangle |_E) = F_{\gamma E}(\langle \gamma \rangle | pi E)$ is Max, and by Lemma 6.4.13...
applied to \( s' \) we obtain that \( T_{\downarrow E} \) is neutral (because the stack is non-empty). So by Lemma 6.4.2, \( F_{\downarrow E} (T_{\downarrow E} (t) \pi_{\downarrow E}) = F(T_{\downarrow E} (\pi_{\downarrow E}))_{\downarrow E} = C_s \) is Max.

\[ \square \]

6.5 Proof of the Checking AM Properties Theorem (Thm. 4.3, p. 9)

Proof.

1. An inspection of the transition rules shows that there always is one and exactly one transition of the Checking AM that applies: for each phase (\( \downarrow \) and \( \uparrow \)) consider each case of the code (application, abstraction, variable) and the various combinations stack/frame.

2. By the previous point, the executions of the Checking AM are sequences of commutative steps that either diverge or are followed by an output transition. We now introduce a measure and prove that the sequence of commutative steps is always finite. In particular, it is bounded by the size of the initial term.

Consider the following notion of size for stacks, frames, and states:

\[
|\varepsilon| := 0 \\
|T :: \pi| := |T| + |\pi| \\
|F :: x| := |F| \\
|F :: T \pi| := |\pi| + |F| \\
|F :: \lambda x. T \pi| := |\pi| + |F| + |T| \\
|(F, T, \pi, E, \downarrow)| := |F| + |\pi| + |T| \\
|(F, T, \pi, E, \uparrow)| := |F| + |\pi|
\]

The proof of the bound is in 3 steps:

(a) Evaluation Commutative Steps Are Bound by the Size of the Initial Term.

By direct inspection of the rules of the machine it can be checked that:

- Evaluation Commutative Rules Decrease the Size: if \( s \twoheadleftarrow_a s' \) with \( a \in \{\downarrow c_1, \downarrow c_2, \downarrow c_3, \downarrow c_7\} \) then \(|s'| < |s|\);
- Backtracking Transitions do not Change the Size: if \( s \twoheadleftarrow_a s' \) with \( a \in \{\uparrow c_4, \uparrow c_5, \uparrow c_6\} \) then \(|s'| = |s|\).

Then a straightforward induction on the length of an execution \( \rho \) shows that

\[ |s'| \leq |s| - |\rho|_{\downarrow} \]

i.e. that \(|\rho|_{\downarrow} \leq |s| - |s'| \leq |s| = |(\varepsilon, T, \epsilon, E, \downarrow)| = |T|\).

(b) Backtracking Commutative Steps Are Bound by the Evaluation Ones.

We have to estimate \(|\rho|_{\uparrow} = |\rho|_{\uparrow c_4} + |\rho|_{\uparrow c_5} + |\rho|_{\uparrow c_6}\). Note that

i. \(|\rho|_{\uparrow c_4} \leq |\rho|_{\downarrow c_2}\), as \( \downarrow c_4 \) pops variables from \( F \), pushed only by \( \twoheadleftarrow_{\downarrow c_4}\);

ii. \(|\rho|_{\uparrow c_5} \leq |\rho|_{\downarrow c_6}\), as \( \downarrow c_5 \) pops pairs \( T \pi \) from \( F \), pushed only by \( \twoheadleftarrow_{\downarrow c_5}\);
iii. \(|\rho|_{\Delta E} \leq |\rho|_{\Delta E_0}\), as \(\rightarrow_{\Delta c_0}\) ends backtracking phases, started only by \(\rightarrow_{\Delta c_2}\).

Then \(|\rho|_{\Delta E} \leq |\rho|_{\Delta E_2} + 2|\rho|_{\Delta E_0} \leq 2|\rho|_{\Delta E}.

(c) Bounding All Commutative Steps by the Size of the Initial Term. We have \(|\rho|_{c} = |\rho|_{\varepsilon} + |\rho|_{\Delta E} \leq |\rho|_{\varepsilon} + 2|\rho|_{\Delta E} \leq \rho \cdot t_{1} |l|.

3. By the previous points executions of the Checking AM are sequences of commutative transitions followed by an output transition, *i.e.* they have the form \(s \rightarrow_{\Delta} s' \rightarrow_{\rho} l\) where \(l\) is the output label of the machine. The initial state by hypothesis is \(s := (e, t, E, \downarrow)\). Since commutative transitions do not change the decoding (Lemma 1.11), we have \(s' = s = t_{1} E\). To prove that \([x \varepsilon \bar{T}^\rho] : E\) is well-labeled—that requires to prove properties of \(\bar{T}_{1} E\)—we will then look at \(s'\) and at the various possible output transitions. Five cases:

- **s' = \((F, \lambda y. \bar{T}, \bar{\pi} : \pi, E, \nabla) \rightarrow_{o_1} (\text{red}, 1)\) with \(y \in fv(\bar{T})\). Then \(\bar{T}_{1} E = s' = F(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = F(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E\) that has a \(\beta\)-redex.

Now, we show that \(\bar{T}\) decomposes as a \(\rightarrow_{\max \beta}\)-redex in a Max context. By Lemma 6.1.5 \(E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = C_{\varepsilon}\) is a Max context. By Lemma 6.3.15 \(E(\lambda y. \bar{T}, \bar{\pi} : \pi)\) is also Max. Then \(E(\lambda y. \bar{T}, \bar{\pi} : \pi)\) is Max by Lemma 6.4.5. Moreover, \((\lambda y. \bar{T}, \bar{\pi})\) is a \(\rightarrow_{\max \beta}\)-redex because \(y \in fv(\bar{T})\) by hypothesis. Finally, \(\bar{T} = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\) by Lemma 4.1.1.

The Checking AM is executed only on terms that are not variables, and so \([x \varepsilon \bar{T}](\text{red}, 1) : E\) is well-labeled.

- **s' = \((F, y, \pi, \pi, E, \nabla) \rightarrow_{o_2} (\text{red}, n + 1)\) with \(E(y) = [y \varepsilon \bar{T}]^{(\text{red}, n)}\). Then \(\bar{T}_{1} E = s' = E(\lambda y. \bar{T})\downarrow E = E(\lambda y. \bar{T})\downarrow E = E(\lambda y. \bar{T})\downarrow E = E(\lambda y. \bar{T})\downarrow E\).

Since \(E\) is well-labeled and \(E(y) = [y \varepsilon \bar{T}]^{(\text{red}, n)}\) we have that \(y_{1} E\) contains a \(\beta\)-redex, and so does \(\bar{T}_{1} E\).

Now, we show that \(\bar{T}\) decomposes as a variable in a Max context. By Lemma 6.3.15 \(E(\bar{T}, \bar{\pi} : \pi)\downarrow E = C_{\varepsilon}\) is a Max context. By Lemma 6.4.5 \(E(\bar{T}, \bar{\pi} : \pi)\) is also Max. Finally, \(\bar{T} = E(\bar{T}, \bar{\pi} : \pi)\) by Lemma 4.1.1.

Therefore \([x \varepsilon \bar{T}](\text{red}, n + 1) : E\) is well-labeled.

- **s' = \((F, y, \pi, \pi, E, \nabla) \rightarrow_{o_3} (\text{red}, 2)\) with \(E(y) = [y \varepsilon \bar{T}]^{abs}\). Then \(\bar{T}_{1} E = s' = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E\).

Since \(E\) is well-labeled and \(E(y) = [y \varepsilon \bar{T}]^{abs}\) we have that \(y_{1} E\) is an abstraction and so \(\bar{T}_{1} E\) contains the \(\beta\)-redex \(y_{1} E\bar{T}_{1} E\).

Now, we show that \(\bar{T}\) decomposes as a variable in an applicative Max context. By Lemma 6.3.15 \(E(\lambda y. \bar{T}, \bar{\pi} : \pi)\downarrow E = C_{\varepsilon}\) is a Max context. By Lemma 6.4.5 \(E(\lambda y. \bar{T}, \bar{\pi} : \pi)\) is also Max. Additionally, it is applicative. Finally, \(\bar{T} = E(\lambda y. \bar{T}, \bar{\pi} : \pi)\) by Lemma 4.1.1.

Therefore \([x \varepsilon \bar{T}](\text{red}, 2) : E\) is well-labeled.
• $s' = (ε, \overline{y}, ε, E, ▲) \rightarrow_{α_new}. Then \overline{t} = \overline{y} by the commutative transparency lemma (Lemma 4.1.1), and so \overline{t} is an application. By the normal form invariant (Lemma 6.5.1a), $\overline{t}_{|E} = (\overline{y})_{|E}$ is normal. Moreover, it is an application, because $(\overline{y})_{|E} = \overline{y}_{|E} \overline{y}_{|E}$, and so it is neutral. Then $[x \rightarrow \overline{t}]_{new} :: E$ is well-labeled.

• $s' = (ε, λy.\overline{u}, ε, E, ▲) \rightarrow_{α_obs}. Then \overline{t} = λy.\overline{u} by the commutative transparency lemma (Lemma 4.1.1), and so \overline{t} is an abstraction. By the normal form invariant (Lemma 6.5.1a), $\overline{t}_{|E} = (λy.\overline{u})_{|E}$ is normal. Moreover, it is an abstraction, because $(λy.\overline{u})_{|E} = λy.\overline{u}_{|E}$. Then $[x \rightarrow \overline{t}]_{abs} :: E$ is well-labeled.

• $s' = (F :: λy.\overline{r} \overline{π}, \overline{u}, ε, E, ▲) \rightarrow_{α_0} (red, 1). Then$ $\overline{t}_{|E} = s'_{|E} = E(\langle (λy.\overline{r})_{π} \rangle_{π})_{E} = E(\langle (λy.\overline{r})_{π} \rangle_{π})_E = E(\langle (λy.\overline{r})_{π} \rangle_{π})_E$ that has a $β$-redex. Now, we show that $\overline{t}$ decomposes as a $\rightarrow_\text{Max}\beta$-redex in a Max context. By Lemma 6.5.4, $E(\langle (λy.\overline{r})_{π} \rangle_{π})_E = Cs$ is a Max context. By Lemma 6.4.5, $E(\langle (λy.\overline{r})_{π} \rangle_{π})_E$ is also Max. Then $E(\langle (λy.\overline{r})_{π} \rangle_{π})_E$ is Max by Lemma 6.4.1. Moreover, by the normal form invariant (Lemma 6.5.1a), $π$ is normal and by the erasing redexes invariant (Lemma 6.5.3) $y$ does not occur in $π$. Then $(λy.\overline{r})π$ is a $\rightarrow_\text{Max}\beta$-redex. Finally, $\overline{t} = E(\langle (λy.\overline{r})_{π} \rangle_{π})_E$ by Lemma 4.1.1. The Checking AM is executed only on terms that are not variables, and so $[x \rightarrow \overline{t}]_{(red, 1)} :: E$ is well-labeled.

6.6 Useful MAM Qualitative Study (L.6.6, p. 20)

Four invariants are required. The normal form and decoding invariants are exactly those of the Checking AM (and the proof for the commutative transitions is the same). The environment labels invariant follows from the correctness of the Checking AM (Theorem 4.3.2). The name invariant is used in the proof of Lemma 6.7.

Lemma 6.6 (Max MAM Qualitative Invariants, Proof at Page 20). Let $s = F' | μ | π | E | φ$ be a state reachable from an initial term $\overline{t}_0$. Then:

1. Environment Labels: $E$ is well-labeled.

2. Normal Form:

(a) Backtracking Code: if $φ = ▲$, then $π_{|E}$ is normal, and if $π$ is non-empty, then $π_{|E}$ is neutral;

(b) Frame: if $F = F' :: π' :: F''$, then $π_{|E}$ is neutral.

3. Name:

(a) Substitutions: if $E = E'' :: [x \rightarrow \overline{t}] :: E''$ then $x$ is fresh wrt $\overline{t}$ and $E''$;

(b) Abstractions and Evaluation: if $φ = ▼$ and $λx.\overline{r}$ is a subterm of $π$, $π$, or $π'$ (if $F = F' :: π' :: F''$) or of a erasing redex item in $F$ then $x$ may occur only in $\overline{t}$;
(c) Abstractions and Backtracking: if \( \varphi = \triangledown \) and \( \lambda x. T \) is a subterm of \( \pi \) or \( \pi' \) (if \( F = F' :: T \triangledown \pi' :: F'' \)) or of a erasing redex item in \( F \) then \( x \) may occur only in \( T \).

4. Erasing Redexes: if \( F = F' :: \lambda x. T \triangledown \pi' :: F'' \), then \( x \notin T \) and \( x \notin \{ \pi', \epsilon \} \).

5. Decoding: \( C_\alpha \) is a Max context.

Proof.

1. Environment Labels: the only transition that extends the environment is \( \sim_{\alpha_2} \), and it preserves well-labeledness by Theorem [135].

2. Normal Form: for the commutative transitions the proof is exactly as for the Checking AM, see Lemma [451] whose proof is at page [15]. For the multiplicative and exponential transitions the invariant holds trivially: the backtracking code part because these transition cannot happen during backtracking (note that \( \sim_{\alpha_3} \) may happen during backtracking but it ends in a evaluating state, for which then nothing has to be proven), and the frame part because they do not touch the head context items in the frame.

3. Name: the invariant trivially holds for an initial state \( \epsilon | \triangledown | \epsilon | \epsilon | \triangledown \). For a non-empty evaluation sequence we list the cases for the last transition.

- Principal Cases:
  - Case \( s' = (F, \lambda x. T, \pi, E, \triangledown) \sim_{\alpha_1} (F, T[y/x], \pi, E, \triangledown) = s \). (a) Substitution: it follows from the i.h.
  - Case \( s' = (F, \lambda x. T, \pi, E, \triangledown) \sim_{\alpha_2} (F, \{ x \mapsto y \}, \pi, E, \triangledown) = s \) with \( \pi \not\in \text{fv}(T) \).
    (a) Substitution: it follows from the i.h. for abstractions in the evaluation phase (Point [22]).
    (b) Abstractions and Evaluation: it follows from the i.h.
  - Case \( s' = (F, x, \pi, E, \triangledown) \sim_{\alpha_3} (F, T[x/y], \pi, E, \triangledown) = s \) with \( E(x) = [x \mapsto \{ x \mapsto y \}]_{\text{red,n}} \).
    (a) Substitution: it follows from the i.h.
    (b) Abstractions and Evaluation: it follows by the i.h. and the fact that in \( T' \) the abstracted variables are renamed (wrt \( T \)) with fresh names.
  - Case \( s' = (F, \lambda x. T, \pi, E, \triangledown) \sim_{\alpha_4} (F, T[x/y], \pi, E, \triangledown) = s \) with \( E(x) = [x \mapsto \{ x \mapsto y \}]_{\text{abs}} \).
    (a) Substitution: it follows from the i.h.
    (b) Abstractions and Evaluation: it follows by the i.h. and the fact that in \( T' \) the abstracted variables are renamed (wrt \( T \)) with fresh names.
  - Case \( (F :: \lambda x. T, \pi, E, \triangledown) \sim_{\alpha_5} (F, T, \pi, E, \triangledown) \).
    (a) Substitution: it follows from the i.h.
    (b) Abstractions and Evaluation: it follows by the i.h.

- Commutative Cases:
4. Erasing: the invariant trivially holds for an initial state $\epsilon \vdash \top \vdash \epsilon \vdash \top$. For a non-empty evaluation sequence, for all transitions but $\sim_{\epsilon \top}$, the statement follows immediately from the i.h. because they all leave untouched the erasing redex items in the frame (but for $\sim_{\epsilon \top}$ that, however, simply removes one such entry and so trivially preserves the invariant). So consider $(F, \lambda x \bar{tt} \bar{tt} :: \pi, \bar{tt} \pi, \epsilon, E, \top) \sim_{\epsilon \top} (F :: \lambda x (\overline{\bar{tt}} \bar{tt} \bar{tt}) \pi, \bar{tt} \pi, \epsilon, E, \top)$. (a) Substitution: it follows from the i.h.

(b) Abstractions and Evaluation: it follows from the i.h.

5. Decoding: the invariant trivially holds for an initial state $\epsilon \vdash \top \vdash \epsilon \vdash \top$. For a non-empty evaluation sequence we list the cases for the last transition. To simplify the reasoning in the following case analysis we let implicit that the unfolding spreads on all the subterms, i.e. that $D(F(\overline{\bar{tt}})) \downarrow_E = F(\downarrow_E \langle \overline{\bar{tt}} \rangle \pi) \downarrow_E$.

- Principal Cases:

  - Case $s' = (F, \overline{\lambda x \bar{tt} :: \pi, \bar{tt} \pi, \epsilon, E, \top}) \sim_{\epsilon \top} (F, \bar{tt} \bar{tt} :: \pi, \bar{tt} \pi, \epsilon, E, \top) = s$.
    
    By Lemma 6.4.1, $C_s = D(F(\langle \overline{\bar{tt}} \rangle) \pi) \downarrow_E$ is Max if $C_{s'} = D(F(\langle \overline{\bar{tt}} \rangle y \pi) \downarrow_E$ is Max, and $C_{s'}$ is Max by i.h.
\begin{itemize}
\item **Case** \(s' = (F, \lambda x. \overline{T}, \overline{\pi} :: \pi, E, \downarrow) \sim_{\beta_2} (F, \overline{T}, \pi, [x \to \overline{\pi}] :: E, \downarrow) = s\) with \(x \notin \text{fv}(\overline{T})\).

We have to prove that \(C' = D(\langle \cdot \rangle)\downarrow_{[x \to \overline{\pi}] :: E} \) is Max. By the name invariant for abstractions (Lemma 6.6.3a) we have that \(x\) does not occur in \(E\) nor \(\pi\), and so \(D(\langle \cdot \rangle)\downarrow_{[x \to \overline{\pi}] :: E} = D(\langle \cdot \rangle)\downarrow_{E}\).

Now, by Lemma 6.3.1, \(D(\langle \cdot \rangle)\downarrow_{E}\) is Max if \(D(\langle \cdot \rangle)\downarrow_E\) is Max, that holds by \(i.h.\), because this is exactly \(C'\).

\item **Case** \(s' = (F, x, \pi, E, \downarrow) \sim_{\text{abs, red}} (F, T, \pi, E, \downarrow) = s\) with \(E(x) = [x \to \overline{\pi}]^{\text{red}, n}\). We have \(C_s = D(\langle \cdot \rangle)\downarrow_{E} = C_{s'}\), and so the statement follows immediately from the \(i.h.\).

\item **Case** \(s' = (F, x, \pi :: \pi, E, \downarrow) \sim_{\text{abs, red}} (F, T^x, \pi :: \pi, E, \downarrow) = s\) with \(E(x) = [x \to \overline{\pi}]^{\text{abs}}\). We have \(C_s = D(\langle \cdot \rangle)\downarrow_{E} = C_{s'}\), and so the statement follows immediately from the \(i.h.\).

\item **Case** \(s' = (F : \lambda x. \overline{T} \cdot \pi, \pi, E, \overline{\pi}) \sim_{\pi} (F, \overline{T}, \pi, \overline{\pi}, E, \downarrow) = s\).

We have \(C_s = D(\langle \cdot \rangle)\downarrow_{E} = F_{\overline{\pi}}E(\overline{T}_E)(\pi)\downarrow_{E}\). By the erasing redexes invariant (Lemma 6.6.4), \(x \notin \overline{T}_E\). Then, by Lemma 6.3.1, \(C_s\) is Max if and only if \(F_{\overline{\pi}}E(\langle \lambda y. \overline{T}_E \rangle \langle \cdot \rangle)\pi\downarrow_{E} = F_{\overline{\pi}}E(\langle \lambda y. \overline{T}_E \rangle \pi)\downarrow_{E} = C_{s'}\), that holds by \(i.h.\).

\end{itemize}

\[\square\]

We can now show how every single transition projects on the \(\lambda\)-calculus, in particular that multiplicative transitions project to Max \(\beta\)-steps.

**Lemma 6.7** (One-Step Weak Simulation). Let \(s\) be a reachable state.

1. **Commutative:** if \(s \sim_{e_1, 2, 3, 4, 5, 6, 7} s'\) then \(\overline{s} = \overline{s'}\).
2. **Exponential:** if \(s \sim_{\text{abs, red}} s'\) then \(\overline{s} = \overline{s'}\).
3. **Multiplicative:** if \(s \sim_{m_1, 2, 3} s'\) then \(\overline{s} \rightarrow_{\text{max } \beta} \overline{s'}\).

**Proof.**

1. **Commutative:** the proof is exactly as the one for the Checking AM (Lemma 4.12), that can be found at page 16.

2. **Exponential:**

   - **Case** \(s = (F, x, \pi, E, \downarrow) \sim_{\text{red}} (F, T^x, \pi, E, \downarrow) = s'\) with \(E(x) = [x \to \overline{T}]^{\text{red}, n}\). Then \(E = E' :: [x \to \overline{T}]^{\text{red}, n} :: E''\) for some environments \(E', E''\). Remember that terms are considered up to \(\alpha\)-equivalence.

     \[\overline{s} = C_{s'}(x)_{\overline{E}} = C_{s'}(\overline{T}_{\overline{E}'}) = C_{s'}(\overline{T}_{\overline{E}}) = \overline{s'}\]

     In the chain of equalities we can replace \(\overline{T}_{\overline{E}'}\) with \(\overline{T}_{\overline{E}}\) because by well-labeledness the variables bound by \(E'\) are fresh with respect to \(\overline{T}\).

   - **Case** \(s = (F, x, \overline{\pi} :: \pi, E, \downarrow) \sim_{\text{abs}} (F, T^x, \overline{\pi} :: \pi, E, \downarrow) = s'\) with \(E(x) = [x \to \overline{T}]^{\text{abs}}\). The proof that \(\overline{s} = \overline{s'}\) is exactly as in the previous case.
3. Multiplicative:

- **Case** \( s = (F, \lambda x. T, y :: \pi, E, \triangleright) \leadsto_{\pi_1} (F, \tilde{T}(x \leftarrow y), \pi, E, \triangleright) = s' \). Note that \( C_{s'} = \tilde{D}(\pi) \| E \) is Max by the decoding invariant (Lemma 6.6.5). Note also that by the name invariant (Lemma 6.6.3) \( x \) can only occur in \( \tilde{T} \). Then:

\[
(F, \lambda x. T, y :: \pi, E, \triangleright) = \quad F(\lambda x. T)y :: \pi \| E \\
= \quad F(\langle \lambda x. T \rangle y) \| E \\
= \quad C_s(\langle \lambda x. T \rangle y) \| E \\
\rightarrow_{\text{Max}} \quad C_s(\tilde{T}(x \leftarrow y) \| E) \\
= \quad L[6.3.2] \quad L[6.3.2] \\
\]

- **Case** \( s = (F, \lambda x. T, \pi :: \pi, E, \triangleright) \leadsto_{\pi_2} (F, \tilde{T}, \pi, [x \leftarrow \pi] \| E, \triangleright) = s' \) with \( \pi \) not a variable. By the name invariant (Lemma 6.6.3) \( x \) can only occur in \( \tilde{T} \) and so \( C_{s'} = \tilde{D}(\langle \pi \rangle) \| x \leftarrow \pi \| E = \tilde{D}(\langle \pi \rangle) \| E = \tilde{D}(\langle \pi \rangle) \| E \). Moreover, such a context is Max by the decoding invariant (Lemma 6.6.5). Then:

\[
(F, \lambda x. T, \pi :: \pi, E, \triangleright) = \quad F(\lambda x. T)\pi :: \pi \| E \\
= \quad F(\langle \lambda x. T \rangle \pi) \| E \\
= \quad F_\| E(\langle \lambda x. T \rangle \pi) \| E \| E \\
\rightarrow_{\text{Max}} \quad F_\| E(\tilde{T}(x \leftarrow \pi) \| E) \| E \\
= \quad L[6.6.2] \quad L[6.3.1] \\
= \quad L[6.6.2] \\
\]

- **Case** \( s = (F :: \lambda x. T \| \pi, \pi, \pi, E, \triangle) \leadsto_{\pi_3} (F, \tilde{T}, \pi, E, \triangleright) = s' \). Note that \( C_{s'} = \tilde{D}(\langle \pi \rangle) \| E = F_\| E(\langle \pi \rangle) \| E \) is Max by the decoding invariant (Lemma 6.6.5). Then:

\[
(F :: \lambda x. T \| \pi, \pi, \pi, E, \triangle) = \quad F(\langle \lambda x. T \rangle \pi) \| E \\
= \quad F_\| E(\langle \lambda x. T \rangle \pi) \| E \| E \\
\rightarrow_{\text{Max}} \quad F_\| E(\tilde{T}(\pi) \| E) \| E \\
= \quad \tilde{D}(\langle \pi \rangle) \| E \\
\]

We also need to show that the Max MAM computes \( \beta \)-normal forms.

**Lemma 6.8** (Progress). Let \( s \) be a reachable final state. Then \( s \) is \( \beta \)-normal.

**Proof.** A simple inspection of the machine transitions shows that final states have the form \( (\epsilon, \tilde{T}, \epsilon, E, \triangle) \). Then by the normal form invariant (Lemma 6.6.2a) \( s = \tilde{T} \| E \) is \( \beta \)-normal.

\[\square\]
6.7 Proof of the Weak Bisimulation Theorem (Thm 5.1, p. 10)

Proof.

1. By induction on the length |ρ| of ρ, using the one-step weak simulation lemma (Lemma 6.7). If ρ is empty then the empty derivation satisfies the statement. If ρ is given by σ : s ⊸ s'' followed by s'' ⊸ s' then by i.h. there exists e : u ⊸ u max s'' s'' s.t. |e| = |σ| s. Cases of s'' ⊸ s':

   (a) Commutative or Exponential. Then s'' = s' by Lemma 6.71 and Lemma 6.72 and the statement holds taking d := e because |d| = |e| = i.h. |σ| s = |ρ| s.

   (b) Multiplicative. Then s'' = s' by Lemma 6.73 and defining d as e followed by a step we obtain |d| = |e| + 1 = i.h. |σ| s + 1 = |ρ| s.

2. We use nf ec(s) to denote the normal form of s with respect to exponential and commutative transitions, that exists and is unique because ⊸ c ⊂ ⊸ e terminates (termination is given by forthcoming Theorem 6.12 and Theorem 6.13 that are postponed because they actually give precise complexity bounds, not just termination) and the machine is deterministic (as it can be seen by an easy inspection of the transitions). The proof is by induction on the length of d. If d is empty then the empty execution satisfies the statement.

   If d is given by e : t ⊸ r followed by r ⊸ s'' u then by i.h. there is an execution σ : s ⊸ s'' s.t. r = s'' and |σ| s = |e|. Note that since exponential and commutative transitions are mapped on equalities, σ can be extended as σ' : s ⊸ s'' ⊸ ec 1,2,3,4,5,6 nf ec(s'') with nf ec(s'') = r and |σ' s| = |e|. By the progress property (Lemma 6.8) nf ec(s'') cannot be a final state, otherwise r = nf ec(s'') could not reduce. Then nf ec(s'') ⊸ s' (the transition is necessarily multiplicative because nf ec(s'') is normal with respect to the other transitions).

   By the one-step weak simulation lemma (Lemma 6.73) nf ec(s'') = r ⊸ max s' and by determinism of ⊸ max (Lemma 2.2.12) s' = u. Then the execution ρ defined as σ' followed by nf ec(s'') ⊸ s' satisfy the statement, as |ρ| s = |σ' s| + 1 = |σ| s + 1 = |e| + 1 = |d|.

6.8 Quantitative Analysis

The complexity analyses of this section rely on two additional invariants of the Max MAM, the subterm and the environment size invariants.

The subterm invariant bounds the size of the duplicated subterms and it is crucial. For us, τ is a subterm of t if it does so up to variable names, both free and bound. More precisely: define τ as t in which all variables (including those appearing in binders) are replaced by a fixed symbol *. Then, we will consider u to be a subterm of t whenever u ◦ is a subterm of t ◦ in the usual sense. The key property ensured by this definition is that the size |τ| of τ is bounded by |τ|.
Lemma 6.9 (Max MAM Quantitative Invariants). Let \( s = F | \pi | \pi | E | \varphi \) be a state reachable by the execution \( \rho \) from the initial code \( t_0 \).

1. Subterm:
   
   (a) Evaluating Code: if \( \varphi = ▼ \), then \( \pi \) is a subterm of \( t_0 \);
   
   (b) Stack: any code in the stack \( \pi \) is a subterm of \( t_0 \);
   
   (c) Frame:
      
      i. Head Contexts: if \( F = F' :: \varphi \triangleright \pi' :: F'' \), then any code in \( \pi' \) is a subterm of \( t_0 \);
   
      ii. Erasing Redexes: if \( F = F' :: λx. \varphi \triangleright \pi' :: F'' \), then \( λx. \varphi \) and any code in \( \pi' \) are subterms of \( t_0 \);
   
   (d) Global Environment: if \( E = E' :: [x = \varphi] :: E'' \), then \( \varphi \) is a subterm of \( t_0 \);

2. Environment Size: the length of the global environment \( E \) is bound by \( |\rho|_n \).

Proof. 1. Subterm: the invariant trivially holds for the initial state \( \epsilon | t_0 | \epsilon | \epsilon | ▼ \). In the inductive case we look at the last transition:

   • Principal Cases:
     
     − Case \( s' = (F, λx. \overline{\rho}, y :: \pi, E, ▼) \xrightarrow{\text{red}} (F, \overline{\rho}y, \pi, E, ▼) = s \):
       
       (a) Evaluating Code: note that according to our definition of subterm \( \overline{\rho}y \) is a subterm of \( λx. \overline{\rho} \).
       
       (b) Stack: note that any piece code in \( \pi \) is also in \( \overline{\pi} :: \pi \).
       
       (c) Frame: it follows from the i.h., since \( F \) is not modified.
       
       (d) Environment: it follows from the i.h., since \( E \) is not modified.
     
     − Case \( s' = (F, λx. \overline{\rho}, \overline{\pi} :: \pi, E, ▼) \xrightarrow{\text{abs}} (F, \overline{T}, \pi, [x = \overline{\pi}] :: E, ▼) = s \) with \( \overline{\pi} \) not a variable.
       
       (a) Evaluating Code: note that \( \overline{T} \) is a subterm of \( λx. \overline{T} \).
       
       (b) Stack: note that any piece code in \( \pi \) is also in \( \overline{\pi} :: \pi \).
       
       (c) Frame: it follows from the i.h., since \( F \) is not modified.
       
       (d) Environment: the new environment is of the form \( [x = \overline{\pi}] :: E \). Pieces of code in \( E \) are subterms of \( t_0 \) by i.h. Moreover \( \overline{\pi} \) is the top of the stack \( \overline{\pi} :: \pi \) so it is also a subterm of \( t_0 \).
     
     − Case \( s' = (F, x, \pi, E, ▼) \xrightarrow{\text{abs}} (F, T', \pi, E, ▼) = s \) with \( E(x) = [x = \overline{\pi}] \xrightarrow{\text{abs}} (F, T', \pi, E, ▼) = s \).
       
       (a) Evaluating Code: note that \( T \) is bound by \( E \). By i.h., it is a subterm of \( t_0 \). So \( T' \) is also a subterm of \( t_0 \).
       
       (b) Stack: it follows from the i.h., since the stack \( \pi \) is unchanged.
       
       (c) Frame: it follows from the i.h., since the frame \( F \) is unchanged.
       
       (d) Environment: it follows from the i.h., since the environment \( E \) is unchanged.
(a) **Evaluating Code**: note that $\overline{t}$ is bound by $E$. By i.h., it is a subterm of $\overline{t}_0$. So $\overline{T}'$ is also a subterm of $\overline{t}_0$.

(b) **Stack**: it follows from the i.h., since the stack $\nu :: \pi$ is unchanged.

(c) **Frame**: it follows from the i.h., since the frame $F$ is unchanged.

(d) **Environment**: it follows from the i.h., since the environment $E$ is unchanged.

- **Case** $s = (F :: \lambda x.\overline{t} :: \pi, \nu, E, \Delta) \leadsto_{c_3} (F, \overline{t}, \pi, E, \nabla) = s'$.
  
  (a) **Evaluating Code**: by the frame part of the i.h. $\lambda x.\overline{t}$ is a subterm of $\overline{t}_0$, and so is $\overline{t}$. So $\overline{T}'$ is also a subterm of $\overline{t}_0$.

  (b) **Stack**: by the frame part of the i.h. the codes in $\pi$ are subterms of $\overline{t}_0$.

  (c) **Frame**: it follows from the i.h.

  (d) **Environment**: it follows from the i.h., since the environment $E$ is unchanged.

- **Commutative Cases**:
  
  - **Case** $s' = (F, \overline{t}, \pi, E, \nabla) \leadsto_{c_1} (F, \overline{T}, \pi :: \pi, E, \nabla) = s$.
    
    (a) **Evaluating Code**: by i.h., $\overline{T}$ is a subterm of $\overline{t}_0$, so $\overline{t}$ is also a subterm of $\overline{t}_0$.

    (b) **Stack**: by i.h., $\overline{T}$ is a subterm of $\overline{t}_0$, so $\overline{T}$ is also a subterm of $\overline{t}_0$. Moreover, any piece of code in $\pi$ is a subterm of $\overline{t}_0$ by i.h.

    (c) **Frame**: it follows from the i.h., since the frame $F$ is unchanged.

    (d) **Environment**: it follows from the i.h., since the environment $E$ is unchanged.

  - **Case** $s' = (F, \lambda x.\overline{t}, \pi :: \pi, E, \nabla) \leadsto_{c_2} (F :: \lambda x.\overline{t} :: \pi, \nu, E, \nabla) = s'$.
    
    (a) **Evaluating Code**: by i.h., $\lambda x.\overline{t}$ is a subterm of $\overline{t}_0$, so $\overline{t}$ is also a subterm of $\overline{t}_0$.

    (b) **Stack**: nothing to prove.

    (c) **Frame**: for all entries in $F$ it follows from the i.h., for the new entry it follows by the evaluating code and stack part of the i.h.

    (d) **Environment**: it follows from the i.h., since the environment $E$ is unchanged.

  - **Case** $s' = (F, \lambda x.\overline{t}, \nu, E, \nabla) \leadsto_{c_2} (F :: \nu, \pi', E, \nabla) = s$.
    
    (a) **Evaluating Code**: note that $\overline{t}$ is a subterm of $\lambda x.\overline{t}$ which is in turn a subterm of $\overline{t}_0$ by i.h.

    (b) **Stack**: trivial since the stack $\pi$ is empty.

    (c) **Frame**: any pair of the form $\nu :: \pi'$ in the frame $x :: F$ is also already present in $F$, so it follows by the i.h.

    (d) **Environment**: it follows from the i.h., since the environment $E$ is unchanged.
Case $s' = (F, x, \pi, E, \text{\textbf{\textbackslash}}) \leadsto_{\text{\textbf{\textbackslash}}} (F, x, \pi, E, \text{\textbf{\textbackslash}}) = s$ with $E(x) = \downarrow$ or $E(x) = [x\leftarrow \text{\textbf{\textbackslash}}]_{\text{new}}$ or $E(x) = [x\leftarrow \text{\textbf{\textbackslash}}]_{\text{abs}}$ and $\pi = \epsilon$.

(a) Evaluating Code: trivial since $\varphi \neq \text{\textbf{\textbackslash}}$.

(b) Stack: it follows from the i.h., since the stack $\pi$ is unchanged.

(c) Frame: it follows from the i.h., since the frame $F$ is unchanged.

(d) Environment: it follows from the i.h., since the environment $E$ is unchanged.

Case $s' = (F :: x, \tilde{t}, \epsilon, E, \text{\textbf{\textbackslash}}) \leadsto_{\text{\textbf{\textbackslash}}} (F, \lambda x. \tilde{t}, \epsilon, E, \text{\textbf{\textbackslash}}) = s$.

(a) Evaluating Code: trivial since $\varphi \neq \text{\textbf{\textbackslash}}$.

(b) Stack: trivial since the stack is empty.

(c) Frame: any pair of the form $\pi \circ \pi$ or $\lambda x. \pi \circ \pi'$ in the frame $F$ is also in the frame $x :: F$, so it follows from the i.h.

(d) Environment: any substitution of the form $[y\leftarrow \text{\textbf{\textbackslash}}]$ in the environment $\text{\textbf{\textbackslash}}x :: E$ is also in the environment $E$, so $\tilde{\pi}$ is a subterm of $\tilde{t}_0$ by i.h.

Case $s' = (F :: \tilde{t} \circ \pi, \tilde{\pi}, \epsilon, E, \text{\textbf{\textbackslash}}) \leadsto_{\text{\textbf{\textbackslash}}} (F, \tilde{t} \circ \pi, \epsilon, E, \text{\textbf{\textbackslash}}) = s$.

(a) Evaluating Code: trivial since $\varphi \neq \text{\textbf{\textbackslash}}$.

(b) Stack: the stack $\pi$ occurs at the left-hand side in the frame $\tilde{t} \circ \pi :: F$, so by i.h. we know that any piece of code in $\pi$ is a subterm of $\tilde{t}_0$.

(c) Frame: any pair $\tilde{t} \circ \pi$ or $\lambda x. \tilde{t} \circ \pi'$ in the frame $F$ is also in the frame $\tilde{t} \circ \pi :: F$, so it follows from the i.h.

(d) Environment: it follows from the i.h., since the environment $E$ is unchanged.

Case $s' = (F :: \tilde{t}, \tilde{\pi} :: \pi, \epsilon, E, \text{\textbf{\textbackslash}}) \leadsto_{\text{\textbf{\textbackslash}}} (F :: \tilde{t} \circ \pi, \pi, E, \text{\textbf{\textbackslash}}) = s$.

(a) Evaluating Code: note that $\tilde{\pi}$ is an element of the stack at the left-hand side of the transition, so by i.h. $\tilde{\pi}$ is a subterm of $\tilde{t}_0$.

(b) Stack: trivial since the stack is empty.

(c) Frame: any pair $\tilde{t} \circ \pi$ or $\lambda x. \tilde{t} \circ \pi'$ is also in the frame $F$ except for $\tilde{t} \circ \pi$. Consider a piece of code $\pi$ in the stack $\pi$. It is trivially also a piece of code in the stack $\text{\textbf{\textbackslash}} :: \pi$, so by the second point of the i.h. we have that $\pi$ is a subterm of $\tilde{t}_0$.

(d) Environment: it follows from the i.h., since the environment $E$ is unchanged.

2. Environment Size: simply note that the only transition that extends the environment is $\leadsto_{\text{\textbf{\textbackslash}}}$. 

The proof of the polynomial bound of the overhead is in three steps. First, we bound the number $|\rho|_e$ of exponential transitions of an execution $\rho$ using the number $|\rho|_m$ of multiplicative transitions of $\rho$, that by Theorem 5.1 corresponds to the number of Max $\beta$-steps on the $\lambda$-calculus. Second, we bound the number $|\rho|_c$ of commutative transitions of $\rho$ by using the number of exponential transitions and the size of the initial term. Third, we put everything together.
Multiplicative vs Exponential Analysis. This step requires two auxiliary lemmas. The first one essentially states that commutative transitions eat normal and neutral terms, as well as Max contexts.

**Lemma 6.10.** Let \( s = F \parallel t \parallel \pi \parallel E \parallel \triangledown \) be a state and \( E \) be well-labeled. Then

1. If \( \overline{t}_{\downarrow E} \) is a normal term and \( \pi = \epsilon \) then \( s \sim_{G}^{*} F \parallel t \parallel \pi \parallel E \parallel \triangledown \).
2. If \( \overline{t}_{\downarrow E} \) is a neutral term then \( s \sim_{G}^{*} F \parallel t \parallel \pi \parallel E \parallel \triangledown \).
3. If \( \overline{t} = C\langle \overline{\pi} \rangle \) with \( C\parallel_{E} \) a Max context then there exist \( F' \) and \( \pi' \) such that \( s \sim_{G}^{*} F' \parallel \overline{\pi} \parallel \pi' \parallel E \parallel \triangledown \). Moreover, if \( C \) is applicative then \( \pi' \) is non-empty.

The first two points rest on the following inductive mutually inductive definition of normal and neutral terms:

| x is neutral | t is neutral | u is normal | tu is neutral |
|-------------|-------------|-------------|---------------|
| t is neutral | t is normal | \( \lambda x.t \) is normal |

And the following two immediate properties:

1. If \( \overline{t}_{\downarrow E} \) is normal then \( \overline{t} \) is normal;
2. If \( \overline{t}_{\downarrow E} \) is neutral then \( \overline{t} \) is neutral.

Now we can proceed with the proof.

**Proof.**

1. If \( \overline{t}_{\downarrow E} \) is normal then \( \overline{t} \) is normal. By induction on the derivation of \( \overline{t} \) is normal. Cases:

   - **Neutral**, i.e. \( \overline{t} \) is neutral. If \( \overline{t}_{\downarrow E} \) is neutral then it follows by Point 2. Otherwise \( \overline{t} = x \) and \( \overline{t}_{\downarrow E} \) is an abstraction, i.e. \( E = E' :: [x\leftarrow\overline{\pi}]^{obs} :: E'' \). Then:

     \[
     F \mid x \mid \epsilon \mid E \mid \triangledown \sim_{G}^{*} F \mid x \mid \epsilon \mid E \mid \triangledown
     \]

   - **Abstraction**, i.e. \( \overline{t} = \lambda x.\overline{\pi} \) with \( \overline{\pi} \) normal. Since \( \overline{t}_{\downarrow E} = \lambda x.\overline{\pi}_{\downarrow E} \) is normal then \( \overline{\pi}_{\downarrow E} \) is normal and we can use the i.h. on it. Then

     \[
     (i.h.) \quad F \mid \lambda x.\overline{\pi} \mid \epsilon \mid E \mid \triangledown \sim_{G}^{*} F \mid \overline{\pi} \mid \epsilon \mid E \mid \triangledown \sim_{G}^{*} F \mid \lambda x.\overline{\pi} \mid \epsilon \mid E \mid \triangledown
     \]

2. If \( \overline{t}_{\downarrow E} \) is neutral then \( \overline{t} \) is neutral. By induction on the derivation of \( \overline{t} \) is neutral. Cases:

   - **Variable**, i.e. \( \overline{t} = x \). If \( \overline{t}_{\downarrow E} \) is neutral then either \( E(x) = \perp \) or \( E = E' :: [x\leftarrow\overline{\pi}]^{new} :: E'' \). In both cases:

     \[
     F \mid x \mid \pi \mid E \mid \triangledown \sim_{G}^{*} F \mid x \mid \pi \mid E \mid \triangledown
     \]
• Application, i.e. $\mathcal{T} = \overline{\tau}$ with $\overline{\pi}$ neutral and $\overline{\tau}$ normal. Since $\overline{\tau} \mathcal{E}$ is neutral, $\overline{\tau} \mathcal{E}$ is neutral and $\overline{\pi} \mathcal{E}$ is normal, so that we can use Point 1 and the i.h. on them. Then

$$F \mid \overline{\tau} \mid \pi \mid E \mid \blacktriangleleft \Rightarrow \forall_{\mathcal{E}_1} \ F \mid \overline{\pi} \mid \overline{\tau} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(i.h.)

$$\Rightarrow \forall_{\mathcal{E}_2} \ F \mid \overline{\pi} \mid \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

Point 1

$$\Rightarrow \forall_{\mathcal{E}_3} \ F \mid \overline{\pi} \mid \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

Now, if $D$ is empty then $C$ is applicative, the stack is non-empty, and the statement is proved. Otherwise, it follows from the i.h.:

$$F \mid D \overline{\pi} \mid \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(i.h.)

• Abstraction (rule (\lambda)), i.e. $C = \lambda x.D$ with $D$ Max. As in the previous case, it is immediately seen that we can apply the i.h. to $D$.

$$s = F \mid \lambda x.D \overline{\pi} \mid \pi \mid E \mid \blacktriangleleft \Rightarrow \forall_{\mathcal{E}_1} \ F \mid x \mid D \overline{\pi} \mid \pi \mid E \mid \blacktriangleleft$$

The moreover part follows from the i.h.

• Application Right (rule (@r)), i.e. $C = \overline{\tau}D$ with $D$ Max and $\overline{\tau}$ neutral. Since $\overline{\tau} \mathcal{E} = \overline{\tau} \mathcal{E} D \mathcal{E}$, we have that $\overline{\tau} \mathcal{E}$ is neutral (and so we can apply point 2) and $D \mathcal{E}$ is a Max context and so we can use the i.h. on it.

$$s = F \mid \overline{\tau}D \overline{\pi} \mid \pi \mid E \mid \blacktriangleleft \Rightarrow \forall_{\mathcal{E}_1} \ F \mid \overline{\tau} \mid \overline{\tau} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(Point 2)

$$\Rightarrow \forall_{\mathcal{E}_2} \ F \mid \overline{\tau} \mid \overline{\tau} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(i.h.)

$$\Rightarrow \forall_{\mathcal{E}_3} \ F \mid \overline{\tau} \mid \overline{\tau} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

The moreover part follows from the i.h.

• Erasing Redex (rule (gc)), i.e. $C = (\lambda x.\overline{\tau})D$ with $D$ Max and $x \notin \overline{\tau} \mathcal{E}$. Since $\overline{\tau} \mathcal{E} = (\lambda x.\overline{\tau}) \mathcal{E} D \mathcal{E}$ we have that $D \mathcal{E}$ is a Max context and so we can use the i.h. on it.

$$s = F \mid (\lambda x.\overline{\tau})D \overline{\pi} \mid \pi \mid E \mid \blacktriangleleft \Rightarrow \forall_{\mathcal{E}_1} \ F \mid \lambda x.\overline{\tau} \mid D \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

$$\Rightarrow \forall_{\mathcal{E}_2} \ F \mid \lambda x.\overline{\tau} \mid D \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(i.h.)

$$\Rightarrow \forall_{\mathcal{E}_3} \ F \mid \lambda x.\overline{\tau} \mid D \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(i.h.)

3. If $C \mathcal{E}$ is a Max context then $C$ is a Max context by Lemma 0.3.10. By induction on $C$ being Max (Definition 2.1 page 2). If $C$ is empty then it is immediate. The other cases:

• Application Left (rule (\@l)), i.e. $C = D \overline{\tau}$ with $D$ Max and $D \neq \lambda x.E$.

Since $C \mathcal{E} = D \mathcal{E} \overline{\tau} \mathcal{E}$, we have that $D \mathcal{E}$ is a Max context and we can apply the i.h. to it. Then:

$$s = F \mid D \overline{\pi} \mid \pi \mid E \mid \blacktriangleleft \Rightarrow \forall_{\mathcal{E}_1} \ F \mid D \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(i.h.)

Now, if $D$ is empty then $C$ is applicative, the stack is non-empty, and the statement is proved. Otherwise, it follows from the i.h.:

$$F \mid D \overline{\pi} \mid \overline{\pi} \mathcal{E} \mid \pi \mid E \mid \blacktriangleleft$$

(i.h.)
The moreover part follows from the i.h.

The second lemma uses Lemma 6.10 and the environment labels invariant (Lemma 6.6.1) to show that the exponential transitions of the Max MAM are indeed useful, as they head towards a multiplicative transition, that is towards β-redexes.

**Lemma 6.11** (Useful Exponentials Lead to Multiplicatives). Let \( s \) be a reachable state such that \( s \leadsto_{\text{red},n} s' \).

1. If \( n = 1 \) then \( s' \leadsto_{\text{red}}^{e} s'' \);
2. If \( n = 2 \) then \( s' \leadsto_{\text{red},2}^{e} s'' \) or \( s' \leadsto_{\text{red},1}^{e} s'' \);
3. If \( n > 1 \) then \( s' \leadsto_{\text{red}}^{e} s'' \).

**Proof.** We have \((F, x, \pi, E, ▼) \leadsto_{\text{red}}^{e} (F, ▼', \pi, E, ▼)\) with \( E(x) = [x\leftarrow ▼(\text{red},n)] \). By the labeled environment invariant (Lemma 6.6.1), \( E \) is well-labeled. Then \( ▼' = C(▼) \) with \( C \) Max context. By Lemma 6.10 we obtain

\[
\begin{align*}
s' &= F \mid C(▼) \mid \pi \mid ▼ \leadsto_{\text{red}}^{e} F' \mid ▼' \mid ▼' \mid ▼ \mid ▼
\end{align*}
\]

Three cases:

1. \( n = 1 \) then by well-labeledness ▼ is a \( \leadsto_{\text{Max},\beta} \)-redex \( (\lambda x. ▼) ▼' \), that is, either \( x \notin \mathcal{F}(▼) \) or \( ▼ \) is normal. First, we have

\[
\begin{align*}
F' &\mid (\lambda x. ▼) ▼' \mid ▼' \mid ▼ \leadsto_{\text{red}}^{e} F' \mid ▼' \mid ▼' \mid ▼ \leadsto_{\text{red}}^{e} s''
\end{align*}
\]

Then there are 3 possible cases:

(a) ▼ is a variable. Then \( \leadsto_{\text{red}_1} \) applies.

(b) \( x \in \mathcal{F}(▼) \). Then \( \leadsto_{\text{red}_2} \) applies.

(c) \( x \notin \mathcal{F}(▼) \) and ▼ is normal and not a variable. Then:

\[
\begin{align*}
F' &\mid (\lambda x. ▼) ▼' \mid ▼' \mid ▼ \leadsto_{\text{red}}^{e} F' \mid ▼' \mid ▼' \mid ▼ \leadsto_{\text{red}}^{e} s''
\end{align*}
\]

(6.10.1)

\[
\begin{align*}
F &\mid ▼' \mid ▼' \mid ▼ \leadsto_{\text{red}}^{e} F' \mid ▼' \mid ▼' \mid ▼ \leadsto_{\text{red}}^{e} ▼''
\end{align*}
\]

(6.10.3)

2. \( n = 2, E' = E'' : [y\leftarrow ▼]' : E''' \) with \( l = \text{abs}, \) and \( C \) is applicative) By well-labeledness ▼ is a variable \( y \) and ▼ = \( \lambda z. ▼ \). Then:

\[
\begin{align*}
s &\leadsto_{\text{red},2} F \mid C(y) \mid ▼ \mid ▼ \leadsto_{\text{red}}^{e} F' \mid ▼' \mid ▼' \mid ▼ \leadsto_{\text{red}}^{e} ▼''
\end{align*}
\]

(6.10.3)

Always by Lemma 6.10.3 the stack ▼ is non-empty, say ▼ = ▼' : ▼''. Then we continue with:

\[
\begin{align*}
\leadsto_{\text{abs}} F' \mid ▼' \mid ▼'' \mid ▼ \leadsto_{\text{red}}^{e} ▼''
\end{align*}
\]
Now we repeat the reasoning for the case $n = 1$ completing the execution with $\sim^*_{\rho} \sim^*_{\pi}$.

3. $n > 1$ and $E' = E'' :: [y \cdot \pi] :: E'''$ with $l = (\text{red}, n - 1)$) By well-labeledness $\pi$ is a variable $y$. Then:

$$
\begin{align*}
  s &\sim_{\pi(\text{red}, n)} F | C(y) | \pi | E | \triangleright \sim^*_{\rho} F' | y | \pi' | E | \triangleright \sim_{\pi(\text{red}, n-1)} (L.10.3)
\end{align*}
$$

Finally, using the environment size invariant (Lemma 6.11.2) we obtain the local boundedness property, that is used to infer a quadratic bound via a standard reasoning (already employed in [AL16]).

**Theorem 6.12 (Exponentials vs Multiplicatives).** Let $s$ be an initial Max MAM state and $\rho: s \sim^* s'$.

1. **Local Boundedness:** if $\sigma: s' \sim^* s''$ and $|\sigma|_a = 0$ then $|\sigma|_a \leq |\rho|_a$.

2. **Exponentials are Quadratic in the Multiplicatives:** $|\rho|_a \in O(|\rho|_a^2)$.

**Remark 6.13.** In the following proof we use the fact that in from the definition of well-labeled environment (Definition 6.3, page 5) it immediately follows that if $E = E' :: [x \cdot \pi] :: E''$ is well-labeled then the length of $E''$, and thus of $E$, is at least $n$.

**Proof.**

1. We prove that $|\sigma|_e \leq |E|$. The statement follows from the environment size invariant (Lemma 6.11.2), for which $|E| \leq |\pi|_a$.

   If $|\sigma|_e = 0$ it is immediate. Then assume $|\sigma|_e > 0$, so that there is a first exponential transition in $\sigma$, i.e. $\sigma$ has a prefix $s' \sim^* \sim^*_{\pi} s''$ followed by an execution $\tau: s'' \sim^* s'''$ such that $|\tau|_a = 0$. Cases of the first exponential transition $\sim^*_{\pi}$:

   - Case $\sim^*_{\pi(\text{red}, n)}$: the next transition is necessarily multiplicative, and so $\tau$ is empty. Then $|\sigma|_e = 1$. Since the environment is non-empty (otherwise $\sim^*_{\pi(\text{red}, n)}$ could not apply), $|\sigma|_e \leq |E|$ holds.

   - Case $\sim^*_{\pi(e(\text{red}, n))}$: we prove by induction on $n$ that $|\sigma|_e \leq n$, that gives what we want because $n \leq |E|$ by Remark 6.13.

   Cases:

   - $n = 1$ Then $\tau$ has the form $s'' \sim^* \pi s'''$ by Lemma 6.11.3 and so $|\sigma|_e = 1$.

   - $n = 2$ Then $\tau$ is a prefix of $\sim^*_{\pi} \sim^*_{\pi(\text{red}, n)} \sim^*_{\pi}$ or $\sim^*_{\pi} \sim^*_{\pi(e(\text{red}, 1)$ by Lemma 6.11.2. In both cases $|\sigma|_e \leq 2$.

   - $n > 2$ Then by Lemma 6.11.3 $\tau$ is either shorter or equal to $\sim^*_{\pi} \sim^*_{\pi(e(\text{red}, n-1)}$, and so $|\sigma|_e \leq 2$, or it is longer than $\sim^*_{\pi} \sim^*_{\pi(e(\text{red}, n-1)}$, i.e. it writes as $\sim^*_{\pi}$ followed by an execution $\tau'$ starting with $\sim^*_{\pi(e(\text{red}, n-1)}$. By i.h. $|\tau'| \leq n - 1$ and so $|\sigma| \leq n$.  

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2. This is a standard reasoning: since by local boundedness (the previous point) \( m \)-free sequences have a number of \( e \)-transitions that are bound by the number of preceding \( m \)-transitions, the sum of all \( e \)-transitions is bound by the square of \( m \)-transitions. It is analogous to the proof of Theorem 7.2.3 in \([AL16]\).

\[\square\]

**Commutative vs Exponential Analysis** The second step is to bound the number of commutative transitions. Since the commutative part of the Max MAM is essentially the same as the commutative part of the Strong MAM of \([ABM15]\), the proof of such bound is essentially the same as in \([ABM15]\). It relies on the subterm invariant (Lemma 6.9.1).

**Theorem 6.14** (Commutatives vs Exponentials). Let \( \rho : s \leadsto^* s' \) be a Max MAM execution from an initial state of code \( t \). Then:

1. **Commutative Evaluation Steps are Bilinear**: \( |\rho|_c \leq (1 + |\rho|_e) \cdot |t| \).
2. **Commutative Evaluation Bounds Backtracking**: \( |\rho|_{\uparrow c} \leq 2 \cdot |\rho|_c \).
3. **Commutative Transitions are Bilinear**: \( |\rho|_c \leq 3 \cdot (1 + |\rho|_e) \cdot |t| \).

**Proof.**

1. We use the following notion of size for stacks/frames/states:

\[
\begin{align*}
|\varepsilon| & := 0 \\
|\Gamma : \pi| & := |\Gamma| + |\pi| \\
|F : x| & := |F| \\
|F : \Gamma \odot \pi| & := |\pi| + |F| \\
|F : \lambda x.\, \Gamma \bullet \pi| & := |\pi| + |\Gamma| + |F| \\
|(F, \Gamma, \pi, E, \odot)| & := |F| + |\pi| + |\Gamma| \\
|(F, \Gamma, \pi, E, \bullet)| & := |F| + |\pi|
\end{align*}
\]

By direct inspection of the rules of the machine it can be checked that:

- **Exponentials Increase the Size**: if \( s \leadsto_a s' \) is an exponential transition, then \( |s'| \leq |s| + |t| \) where \( |t| \) is the size of the initial term; this is a consequence of the fact that exponential steps retrieve a piece of code from the environment, which is a subterm of the initial term by Lemma 6.9.1.

- **Commutative Evaluation Transitions Decrease the Size**: if \( s \leadsto_a s' \) with \( a \in \{\odot c_1, \odot c_2, \odot c_3, \odot c_7\} \) then \( |s'| < |s| \) (for \( \odot c_3 \) because the transition switches to backtracking, and thus the size of the code is no longer taken into account);

- **Multiplicative Transitions and Backtracking Transitions Decrease or do not Change the Size**: if \( s \leadsto_a s' \) with \( a \in \{m_1, m_2, m_3, \uparrow c_4, \uparrow c_5, \uparrow c_6\} \) then \( |s'| \leq |s| \).

Then a straightforward induction on \( |\rho| \) shows that

\[|s'| \leq |s| + |\rho|_e \cdot |t| - |\rho|_c \cdot |t| - |s'|.\]

\text{i.e. that} \( |\rho|_c \leq |s| + |\rho|_e \cdot |t| - |s'|.\)
Now note that $|\cdot|$ is always non-negative and that since $s$ is initial we have $|s| = |t|$. We can then conclude with

$$|\rho|_c \leq |s| + |\rho|_a \cdot |t| - |s'| \leq |s| + |\rho|_a \cdot |t| = |t| + |\rho|_a \cdot |t| = (1 + |\rho|_a) \cdot |t|$$

2. We have to estimate $|\rho|_{\mathcal{A}_C} = |\rho|_{\mathcal{A}_C^t} + |\rho|_{\mathcal{A}_C^c} + |\rho|_{\mathcal{A}_C^e}$. Note that

(a) $|\rho|_{\mathcal{A}_C^t} \leq |\rho|_{\mathcal{V}_C^t}$, as $\leadsto_{\mathcal{A}_C^t}$ pops variables from $F$, pushed only by $\leadsto_{\mathcal{V}_C^t}$;

(b) $|\rho|_{\mathcal{A}_C^c} \leq |\rho|_{\mathcal{A}_C^c}$, as $\leadsto_{\mathcal{A}_C^c}$ pops pairs $\mathcal{T} \circ \pi$ from $F$, pushed only by $\leadsto_{\mathcal{A}_C^c}$;

(c) $|\rho|_{\mathcal{A}_C^e} \leq |\rho|_{\mathcal{V}_C^e}$, as $\leadsto_{\mathcal{A}_C^e}$ ends backtracking phases, started only by $\leadsto_{\mathcal{V}_C^e}$.

Then $|\rho|_{\mathcal{A}_C} \leq |\rho|_{\mathcal{V}_C^t} + 2|\rho|_{\mathcal{V}_C^c} \leq 2|\rho|_{\mathcal{V}_C}.

3. We have $|\rho|_c = |\rho|_{\mathcal{V}_C^c} + |\rho|_{\mathcal{A}_C} \leq |\rho|_{\mathcal{V}_C} + 2|\rho|_{\mathcal{V}_C} = 3|\rho|_{\mathcal{V}_C} = O(1 + |\rho|_a) \cdot |t|.

\[\Box\]

**The Main Theorem** Putting together the matching between Max $\beta$-steps and multiplicative transitions (Theorem 5.1), the quadratic bound on the exponentials via the multiplicatives (Theorem 6.12.2) and the bilinear bound on the commutatives (Theorem 6.11.3) we obtain that the number of the Max MAM transitions to implement a Max $\beta$-derivation $d$ is at most quadratic in the length of $d$ and linear in the size of $t$. Moreover, the subterm invariant (Lemma 5.9.11) and the analysis of the Checking AM (Theorem 13.2) allow to bound the cost of implementing the execution on RAM.

**Theorem** (Max MAM Overhead Bound). Let $d : t \rightarrow_{\mathcal{MAM}}^* \alpha$ be a maximal derivation and $\rho$ the Max MAM execution simulating $d$ given by Theorem 7.1.8. Then:

1. **Length**: $|\rho| = O((1 + |d|^2) \cdot |t|)$.

2. **Cost**: $\rho$ is implementable on RAM in $O((1 + |d|^2) \cdot |t|)$ steps.

**Proof**

1. By definition, the length of the execution $\rho$ simulating $d$ is given by $|\rho| = |\rho|_a + |\rho|_a + |\rho|_c$. Now, by Theorem 6.12.2 we have $|\rho|_a = O(|\rho|_a^2)$ and by Theorem 6.11.4 we have $|\rho|_c = O((1 + |\rho|_a) \cdot |t|) = O((1 + |\rho|_a^2) \cdot |t|)$. Therefore, $|\rho| = O((1 + |\rho|_a) \cdot |t|) = O((1 + |\rho|_a^2) \cdot |t|)$. By Theorem 5.13 $|\rho|_a = |d|$, and so $|\rho| = O((1 + |d|^2) \cdot |t|)$.

2. The cost of implementing $\rho$ is the sum of the costs of implementing the multiplicative, exponential, and commutative transitions. Remember that the idea is that variables are implemented as references, so that environment can be accessed in constant time (i.e., they do not need to be accessed sequentially). Moreover, we assuming the strong hypothesis on the representation of terms explained in the paragraph before Theorem 4.3 (page 8), so that the tests $x \in \mathcal{V}(\alpha)$ and $x \notin \mathcal{V}(\alpha)$ in $\leadsto_{\mathcal{MAM}}$ and $\leadsto_{\mathcal{V}_C}$ can be done in constant time.
(a) **Commutative**: every commutative transition takes constant time. At the previous point we bounded their number with $O((1 + |d|^2) \cdot |t|)$, which is then also the cost of all the commutative transitions together.

(b) **Multiplicative**:

- A $\rightsquigarrow_{n_1}$ transition costs $O(|t|)$ because it requires to rename the current code, whose size is bound by the size of the initial term by the subterm invariant (Lemma 6.9.1a).
- A $\rightsquigarrow_{n_2}$ transition costs $O(|t|)$ because checking $x \in \text{fv}(t)$ can be done in constant time and executing the Checking AM on $\overline{\pi}$ takes $O(|\overline{\pi}|)$ commutative steps (Theorem 4.3.2), commutative steps take constant time, and the size of $\overline{\pi}$ is bound by $|t|$ by the subterm invariant (Lemma 6.9.1b).
- A $\rightsquigarrow_{e_2}$ transition takes constant time.

Therefore, all together the multiplicative transitions cost $O(|d| \cdot |t|)$.

(c) **Exponential**: At the previous point we bounded their number with $|\rho| = O(|d|^2)$. Each exponential step copies a term from the environment, that by the subterm invariant (Lemma 6.9.1d) costs at most $O(|t|)$, and so their full cost is $O((1 + |d|) \cdot |t|^2)$ (note that this is exactly the cost of the commutative transitions, but it is obtained in a different way).

Then implementing $\rho$ on RAM takes $O((1 + |d|) \cdot |t|^2)$ steps.

**Remark 6.15.** Our bound is quadratic in the number of the Max $\beta$-steps but we believe that it is not tight. In fact, our transition $\rightsquigarrow_{n_1}$ is a standard optimisation, used for instance in Wand’s [Wan07] (section 2), Friedman et al.’s [FGSW07] (section 4), and Sestoft’s [Ses97] (section 4), and motivated as an optimization about space. In Sands, Gustavsson, and Moran’s [SGM02], however, it is shown that it lowers the overhead for time from quadratic to linear (with respect to the number of $\beta$-steps) for call-by-name evaluation in a weak setting. Unfortunately, the simple proof used in [SGM02] does not scale up to our setting, nor we have an alternative proof that the overhead is linear. We conjecture, however, that it does.