CODIMENSION ONE INTERSECTIONS BETWEEN COMPONENTS OF A MODULI STACK OF TWO-DIMENSIONAL
GALOIS REPRESENTATIONS

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ABSTRACT. The Emerton-Gee stack for GL$_2$ is a stack of $(\varphi, \Gamma)$-modules that can be viewed as a moduli stack of mod $p$ representations of a $p$-adic Galois group. A closely related stack $Z$ was constructed in [CEGS1] as a closed substack of a stack of étale-$\varphi$ modules, with many of the same geometric and moduli-theoretic properties as the Emerton-Gee stack. We compute criteria for intersections of irreducible components of $Z$ in codimension 1 and relate them to extensions of Serre weights.

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1. INTRODUCTION

Let $p$ be an odd prime and let $K/\mathbb{Q}_p$ be a finite extension, with ring of integers $O_K$, residue field $k$ and absolute Galois group $G_K$. In [EG], Emerton and Gee constructed and studied the stack of rank $d$ étale $(\varphi, \Gamma)$-modules, denoted $\mathcal{X}_d$. Over Artinian coefficients, there exists an equivalence of categories between rank $d$ étale $(\varphi, \Gamma)$-modules and $d$-dimensional $G_K$-representations that allows one to view $\mathcal{X}_d$ as a moduli stack of Galois representations. The Emerton-Gee stack $\mathcal{X}_d$ is expected to play a central role in the $p$-adic Langlands program, occupying the position played by the moduli stack of $L$-parameters in the work of Fargues-Scholze on the classical Langlands correspondence.

Specializing to $d = 2$, the reduced part of $\mathcal{X}_2$, denoted $\mathcal{X}_{2,\text{red}}$, is an algebraic stack defined over a finite field $F$. The irreducible components of $\mathcal{X}_{2,\text{red}}$ are labelled by Serre weights, which are the irreducible mod $p$ representations of GL$_2(k)$. By [CEGS3, Cor. 7.2], the labelling is in such a manner that if $\mathcal{X}_{2,\text{red},\sigma}$ is the component labelled by $\sigma$, then its finite type points are precisely those representations that have $\sigma$ as a Serre weight, that is, they have crystalline lifts of Hodge-Tate weights specified in a particular way by $\sigma$ (see Section 1.4 for details).
A closely related stack $Z$, also a reduced algebraic stack over $F$, was constructed in [CEGS1] as a closed substack of a stack of étale $\varphi$-modules. The finite type points of $Z$ are naturally in bijection with those $G_K$-representations that have at least one non-Steinberg Serre weight (defined in Section 1.4). The irreducible components of $Z$ are labelled by the non-Steinberg Serre weights, and as with $X_{Z,\text{red}}$, the labelling is in such a manner that if $Z_\sigma$ is the component labelled by $\sigma$, then its finite type points are precisely those $G_K$-representations that have $\sigma$ as a Serre weight. Both $Z$ and $X_{Z,\text{red}}$ have pure dimension $[K : \mathbb{Q}_p]$, and conjecturally, $Z$ is equivalent to the reduced union of the non-Steinberg components of $X_{Z,\text{red}}$.

The main objective of this article is to compute criteria for pairs of Serre weights $\sigma$ and $\tau$ so that $Z_\sigma \cap Z_\tau$ is a substack of codimension 1. Our strategy rests on finding families of representations that have both $\sigma$ and $\tau$ as Serre weights, and therefore give points of $Z_\sigma \cap Z_\tau$. The sizes of these families can then be used to determine the dimension of $Z_\sigma \cap Z_\tau$. As employed in [CEGS2], a source of families of representations is provided by extensions of fixed $G_K$ characters together with extensions of their unramified twists. Every irreducible component of $Z$ can be obtained as the closure of such a family. Vector spaces of extensions of fixed $G_K$ characters are typically $[K : \mathbb{Q}_p]$-dimensional. Allowing various unramified twists of the fixed characters adds 2 to the dimension, while 1 dimension is taken away because a $\mathbb{G}_m$ orbit of an extension class gives the same representation and yet another dimension is taken away because of a $\mathbb{G}_m$ worth of endomorphisms of each extension. Thus a codimension 1 intersection of $Z_\sigma$ and $Z_\tau$ may be expected to correspond to the existence of a codimension 1 subfamily of extensions of fixed $G_K$ characters (as well as their unramified twists) with both $\sigma$ and $\tau$ as their Serre weights. This line of investigation gives us the required criteria (stated in Theorem 7.1). For pairs of non-isomorphic and weakly regular (a very mild genericity hypothesis, see definition in Section 1.4) Serre weights, the criteria are summarized below. The precise criteria for intersections involving components labelled by Serre weights that are not weakly regular are significantly less succinct and omitted from the statement below.

**Theorem 1.1.** If $\sigma$ and $\tau$ are a pair of non-isomorphic, weakly regular Serre weights, then

$$\dim Z_\sigma \cap Z_\tau = [K : \mathbb{Q}_p] - 1 \iff \text{Ext}^1_{F[GL_2(O_K)]}(\alpha, \beta) \neq 0.$$  

If either of $\sigma$ and $\tau$ is not weakly regular, then $Z_\sigma$ and $Z_\tau$ intersect in codimension 1 if and only if the pair $(\sigma, \tau)$ satisfies one of 23 criteria (omitted, c.f. Theorem 7.1).

This result can be motivated in terms of the conjectural categorical $p$-adic Langlands correspondence. Specifically, it has been conjectured ([EGH, Conj. 6.1.6]) that there exists a fully faithful functor $\mathcal{U}$ from a derived category of smooth representations of $GL_2(K)$ to a derived category of coherent sheaves on $X_2$ that witnesses the $p$-adic local Langlands. The functor $\mathcal{U}$ is expected to satisfy properties related to duality and support that imply the following:

- For $\sigma$ a non-Steinberg Serre weight, the support of $\mathcal{U}(\text{c-Ind}_{GL_2(K)}^{GL_2(O_K)})\sigma$ is the irreducible component of $X_{Z,\text{red}}$ (and presumably of $Z$) labelled by $\sigma$.
- For $\sigma, \tau$ Serre weights and $V \in \text{Ext}^1_{F[GL_2(O_K)]}(\sigma, \tau)$, $\mathcal{U} \circ \text{c-Ind}_{GL_2(K)}^{GL_2(O_K)}(\tau \rightarrow V \rightarrow \sigma)$ is a short exact sequence.
Therefore,
\[ \mathcal{U} \circ c\text{-Ind}_{GL_2(O_K)}^{GL_2(K)}(V)\mid_{Z_\sigma \cap Z_\tau} \cong \mathcal{U} \circ c\text{-Ind}_{GL_2(O_K)}^{GL_2(K)}(\sigma \oplus \tau)\mid_{Z_\sigma \cap Z_\tau}. \]

Since \( \mathcal{U} \) is fully faithful, if the intersection of \( Z_\sigma \) and \( Z_\tau \) is empty, then \( c\text{-Ind}_{GL_2(O_K)}^{GL_2(K)}(V) \) must be isomorphic to \( c\text{-Ind}_{GL_2(O_K)}^{GL_2(K)}(\sigma \oplus \tau) \). Thus, we obtain the following diagram of \( GL_2(O_K) \) representations where the right downward arrow splits:

\[
\begin{array}{ccc}
\sigma & \longrightarrow & c\text{-Ind}_{GL_2(O_K)}^{GL_2(K)}(\sigma) \\
\downarrow & & \downarrow \\
V & \longrightarrow & c\text{-Ind}_{GL_2(O_K)}^{GL_2(O_K)}(V)
\end{array}
\]

The horizontal arrows split as maps of \( GL_2(O_K) \) representations by Mackey’s decomposition theorem. The left vertical arrow must then split as well, and \( V \) must be isomorphic to \( \sigma \oplus \tau \). This shows that if the conjectured functor \( \mathcal{U} \) exists, then an empty intersection of \( Z_\sigma \) with \( Z_\tau \) implies that there are no non-trivial extensions of \( \tau \) by \( \sigma \) as \( GL_2(O_K) \) modules. Our theorem is a finer variant of this expectation.

In the course of our computations, we also find a cohomological criterion for the number of components of dimension \([K : \mathbb{Q}_p] - 1\) when \( Z_\sigma \cap Z_\tau \) is codimension 1, along with some naturally occurring triples of Serre weights. The theorem below summarizes the results for pairs of weakly regular Serre weights \( \sigma \) and \( \tau \).

**Theorem 1.2.** Let \( \sigma \) and \( \tau \) be two weakly regular Serre weights such that \( Z_\sigma \cap Z_\tau \) is of codimension 1. Then the following are true:

(i) When \( K \) is unramified over \( \mathbb{Q}_p \), the number of components of dimension \([K : \mathbb{Q}_p] - 1\) in \( Z_\sigma \cap Z_\tau \) is 1. When \( K \) is ramified over \( \mathbb{Q}_p \), this number is 2 if the \( GL_2(k) \)-extensions of \( \tau \) by \( \sigma \) are non-trivial, and 1 otherwise.

(ii) When \( K \) is unramified over \( \mathbb{Q}_p \), a component of dimension \([K : \mathbb{Q}_p] - 1\) in \( Z_\sigma \cap Z_\tau \) does not lie in an intersection of three irreducible components of \( Z \). In the ramified case, for sufficiently generic Serre weights (c.f. Theorem 7.3), each component of dimension \([K : \mathbb{Q}_p] - 1\) in \( Z_\sigma \cap Z_\tau \) lies in an intersection of three irreducible components of \( Z \).

Note that the criterion that appears in Theorem 1.1 has to do with \( GL_2(O_K) \)-extensions, while the criterion that appears in Theorem 1.2 has to do with \( GL_2(k) \)-extensions.

1.3. Outline of the paper. In Section 2, we compute explicit criteria for the existence of non-trivial extensions of Serre weights as \( GL_2(O_K) \) representations. In Section 3, we relate the dimensions of families of \( G_K \)-representations with both \( \sigma \) and \( \tau \) as Serre weights to the dimension of \( Z_\sigma \cap Z_\tau \). We also relate the number of sufficiently large families to the number of components of maximal dimension inside \( Z_\sigma \cap Z_\tau \). Section 4 recalls explicit criteria for computations of Serre weights of representations. Along with the results of Section 3, these criteria are used to restructure the problem as that of finding \( \sigma \) and \( \tau \) that satisfy a precise computable relationship. Sections 5 and 6 compute the solution to the problem laid out in Section 4. Finally, Section 7 collates all the findings.
1.4. Notation. Let $p$ be a fixed prime and let $K$ be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$, residue field $k$ and uniformizer $\pi$. Eventually, $p$ will be an odd prime, to be consistent with the construction in [CEGS1]. However, we will allow $p = 2$ for many of the intermediate steps.

We let $f := f(K/\mathbb{Q}_p)$ and $e := e(K/\mathbb{Q}_p)$. Let $G_K$ be the absolute Galois group of $K$, and $I_K$ the inertia group. $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$, with a fixed algebraic closure $\overline{\mathbb{F}}$. $\mathbb{F}$ is taken to be sufficiently large so that all embeddings of $k$ into $\overline{\mathbb{F}}$ are contained in $\mathbb{F}$.

Let $T := [0, f - 1]$. Fix an embedding $\sigma_{f-1} : k \to \overline{\mathbb{F}}$. Let $\sigma_{f-1-i} := \sigma_{f-1}^i$ for $i \in T$. Let $\omega_i$ be the $G_K$ character given by $\omega_i(g) = \sigma_i(\varphi(g, \sqrt[p^i - 1]{\pi}))$.

We let $V_{f,s}$ denote the irreducible $GL_2(k)$ representation

$$\bigotimes_{i=0}^{f-1} (\det^{\omega_i} \otimes \text{Sym}^{s_i} k^2) \otimes_{k,\sigma_i} \overline{\mathbb{F}}$$

where each $s_i \in [0, p-1]$. All irreducible $GL_2(k)$ representations with coefficients in $\mathbb{F}$ are of this form and are called Serre weights. We can uniquely identify each Serre weight by $\mathbb{F}$ and $\mathbb{F}$ if we demand that $t_i \in [0, p-1] \forall i$ and at least one of the $t_i$’s is not $p-1$. Following [Gee], we say $V_{f,s}$ is weakly regular, if each $s_i \in [0, p-2]$. We say that $V_{f,s}$ is Steinberg if each $s_i$ equals $p-1$.

Normalize Hodge-Tate weights in such a way that all Hodge-Tate weights of the cyclotomic character are equal to $-1$. Following [CEGS1, Defn. A.1], we say that a representation $\overline{\mathcal{F}} : G_K \to GL_2(\mathbb{F}_p)$ has Serre weight $V_{f,s}$ if $\mathbb{F}$ has a crystalline lift $r : G_K \to GL_2(\mathbb{Q}_p)$ that satisfies the following condition: For each embedding $\sigma_i : k \to \mathbb{F}$, there is an embedding $\tilde{\sigma}_i : K \to \mathbb{Q}_p$ lifting $\sigma_i$ such that the $\tilde{\sigma}_i$ labeled Hodge-Tate weights of $r$ are $\{-s_i - t_i, 1 - t_i\}$, and the remaining $(e-1)f$ pairs of Hodge-Tate weights of $r$ are all $\{0, 1\}$. In this situation, we say $V_{f,s} \in W(\overline{\mathcal{F}})$.

When $p > 2$, $\mathcal{Z}$ is the stack of two dimensional $G_K$-representations constructed in [CEGS1] and is defined over $\mathbb{F}$. It is of pure dimension $ef$. The irreducible components of $\mathcal{Z}$ are indexed by the non-Steinberg Serre weights. For a non-Steinberg Serre weight $V_{f,s}$ we denote the corresponding irreducible component by $\mathcal{Z}_{V_{f,s}}$. If $\mathbb{F}'$ is a finite field extension of $\mathbb{F}$, then $\mathcal{Z}_{V_{f,s}}(\mathbb{F}')$ is the groupoid of representations $\mathcal{P} : G_K \to GL_2(\mathbb{F}')$ with $V_{f,s} \in W(\mathcal{P})$.

We will consider the $s_i$’s and $t_i$’s associated to the Serre weight $V_{f,s}$ to have indices in $\mathbb{Z}/f\mathbb{Z}$ via the identification of the set $T$ with a set of representatives of $\mathbb{Z}/f\mathbb{Z}$. We will similarly consider the indexing set of the embeddings $\sigma_i$’s to be $\mathbb{Z}/f\mathbb{Z}$.

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2. Extensions of Serre weights

Denote by $\Gamma$ the group $GL_2(k)$, by $K$ the group $GL_2(\mathcal{O}_K)$ and by $K_\pi$ the group $1 + \pi^n M_2(\mathcal{O}_K)$ for $n \in \mathbb{Z}_{>0}$. Our objective in this section is to compute when extensions of Serre weights are non-trivial, for use later.
We will sometimes write \( V_{\ell, \sigma} \) as \( \bigotimes_{j=0}^{f-1} (\det^j \otimes \text{Sym}^j \mathbb{F}^2)^{F_{r,f-1-j}} \), where \( \Gamma \) acts on \( \text{Sym}^j \mathbb{F}^2 \) via the natural embedding \( \Gamma \to \text{GL}_2(\mathbb{F}) \) induced by \( \sigma_{f-1} \). The exponentiation by \( F_{r,f-1-j} \) denotes precomposition of the action of \( \Gamma \) by the \( (f-1-j) \)-th power of the (arithmetic) Frobenius map.

**Proposition 2.1.** The conditions for non-triviality of \( \text{Ext}_\Gamma(V_{\ell, \sigma}, V_{\ell', \sigma'}) \) are given as follows:

(i) If \( p > 2, f > 1 \), then \( \text{Ext}_\Gamma(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \neq 0 \) if and only if one of the following two conditions are satisfied:

(a) \( \exists j \in \mathbb{Z}/f\mathbb{Z} \) such that \( s_i' = s_i \) for \( i \neq j-1, j \); \( s_{j-1}' = s_{j-1} - 1 \);

\[
\text{mod } p - 1.
\]
(b) \( \exists j \in \mathbb{Z}/f\mathbb{Z} \) such that \( s_i' = s_i \) for \( i \neq j-1, j \); \( s_{j-1}' = s_{j-1} + 1 \);

\[
\text{mod } p - 1.
\]

(ii) If \( p > 2, f = 1 \), then \( \text{Ext}_\Gamma(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \neq 0 \) if and only if one of the following two conditions are satisfied:

(a) \( s_0 < p - 2; s_0' = p - s_0 - 3 \); and \( t_0' \equiv t_0 + s_0 + 1 \) \( \text{mod } p - 1 \).

(b) \( s_0' \equiv 0, p - 1; s_0' = p - s_0 - 1 \); and \( t_0' \equiv t_0 + s_0 \) \( \text{mod } p - 1 \).

(iii) If \( p = 2, f > 1 \), then \( \text{Ext}_\Gamma(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \neq 0 \) if and only if the central characters for \( V_{\ell, \sigma} \) and \( \overline{V}_{\ell', \sigma'} \) are the same, as well as, \( \exists j \in \mathbb{Z}/f\mathbb{Z} \) such that \( s_i' = s_i \) for \( i \neq j-1, j \); \( s_{j-1}' = s_{j-1} + 1 \); and \( s_{j}' = p - s_{j} - 2 \).

(iv) If \( p = 2, f = 1 \), then \( \text{Ext}_\Gamma(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \neq 0 \) if and only if \( s_0' = s_0 = 0 \); and \( t_0' = t_0 \).

Moreover, \( \text{Ext}_\Gamma(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \) always has dimension \( \leq 1 \).

**Proof.** In order to compute \( \text{Ext}_\Gamma(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \), we need to compute the second socle layer of the injective hull of \( V_{\ell, \sigma} \). Note that if an \( \mathbb{F} \)-vector space \( V \) with \( \Gamma \) action is injective as an \( \text{SL}_2(k) \) module, then it is also injective as a \( \Gamma \) module. This is because any \( \text{SL}_2(k) \) module map \( \phi \) can be lifted to a \( \Gamma \) module map by replacing it with \( \frac{1}{[\Gamma : \text{SL}_2(k)]} \sum_{g \in G/H} g(\phi) \). Therefore, we need to find a \( \Gamma \) module containing \( V_{\ell', \sigma'} \), so that it is the injective hull of \( V_{\ell', \sigma'} \) as an \( \text{SL}_2(k) \) module and compute its second socle layer.

Beyond this point, the steps are precisely as in [AJL], while carefully tracking through the twists by powers of the determinant. The final result (stated in the proposition) is then obtained in the same manner as [AJL, Cor. 4.5]. An alternative reference for the case \( p > 2 \) is [BP, Cor. 4.6].

For \( p = 2 \), note that if \( V \) is a non-trivial \( \Gamma \) extension of \( V_{\ell, \sigma} \) by \( V_{\ell', \sigma'} \), then the central characters of \( V_{\ell, \sigma} \) and \( V_{\ell', \sigma'} \) are the same (this holds true for any \( p \)). This is because \( \mathbb{F}[\mathcal{Z}(\Gamma)] \) is semisimple, where \( \mathcal{Z}(\Gamma) \) is the center of \( \Gamma \). Therefore, by twisting by a square root of the central character (possible since \( p = 2 \)), \( V \) can be assumed to be a non-trivial \( \mathbb{F}[\text{PGL}_2(k)] = \mathbb{F}[\text{SL}_2(k)] \) extension. And therefore, \( \text{Ext}_\Gamma(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \neq 0 \) implies that \( \text{Ext}_{\mathbb{F}[\text{SL}_2(k)]}(V_{\ell, \sigma}, \overline{V}_{\ell', \sigma'}) \neq 0 \). On the other hand, every non-trivial \( \mathbb{F}[\text{SL}_2(k)] \) extension is a non-trivial \( \mathbb{F}[\text{PGL}_2(k)] \) extension, and
therefore, a non-trivial \( \Gamma \) extension. It follows that \( \text{Ext}_\Gamma(V_{\ell}, V_{\ell'}) \neq 0 \iff \text{Ext}_{[\text{SL}_2(k)]}(V_{\ell}, V_{\ell'}) \neq 0 \). The conditions for the latter are described in the paragraph preceding Corollary 4.5 in [AJL].

Remark 2.2. \( \text{Ext}_{[\text{SL}_2(k)]}(V_{\ell}, V_{\ell'}) = 0 \) implies \( \text{Ext}_\Gamma(V_{\ell}, V_{\ell'}) = 0 \). This is because any \( \overline{\text{F}}[\text{SL}_2(k)] \) splitting \( \phi \) can be upgraded to a \( \Gamma \) splitting by replacing it with \( \sum_{g \in \Gamma/\text{SL}_2(k)} g(\phi) \).

In order to compute \( \text{Ext}_\mathcal{K}(V_{\ell}, V_{\ell'}) \), we will use the Grothendieck spectral sequence. Let \( \sigma \) be a \( \overline{\text{F}}[\Gamma] \) representation, seen via inflation as a \( \overline{\text{F}}[\mathcal{K}] \) representation. The Grothendieck spectral sequence gives us the following left exact sequence:

\[
(2.2.1) \quad 0 \to \text{Ext}_\mathcal{K}(V_{\ell}, \sigma) \to \text{Ext}_\mathcal{K}(V_{\ell}, \sigma) \to \text{Hom}_\mathcal{K}(V_{\ell}, H^1(\mathcal{K}, \sigma))
\]

By [BP, Prop. 5.1], we have the following description of \( H^1(\mathcal{K}, \sigma) \).

Proposition 2.3. \( (i) \)

\[
H^1(\mathcal{K}, \sigma) \cong \bigoplus_{i=0}^{f-1} \sigma \otimes (V_2 \otimes \det^{-1})^{F_{r}^{-1}-i} \bigoplus_{i=1}^{d} \sigma
\]

where \( V_2 \) is the subspace spanned by \( (\ell)^{2} \hat{x}^{2} \) in \( \text{Sym}^2 \overline{\text{F}}^2 \) where \( \Gamma \) acts via the embedding \( \Gamma \to \text{GL}_2(\overline{\text{F}}) \) induced by \( \sigma_{f-1} \);

\( d = \text{dim}_{\overline{\text{F}}} \text{Hom}(1 + \pi \mathcal{O}_{K}, \overline{\text{F}}) \) for \( p \neq 2 \) and \( d = \text{dim}_{\overline{\text{F}}} \text{Hom}(1 + \pi \mathcal{O}_{K}, \overline{\text{F}}) - f \) for \( p = 2 \).

\( (ii) \) Under the above isomorphism, an element of \( \sigma \otimes (V_2 \otimes \det^{-1})^{F_{r}^{-1}-i} \) can be seen explicitly as a map (cocycle) \( \mathcal{K}_1 \to \sigma \) via the following correspondence:

\[
\begin{align*}
\alpha \otimes \hat{x}^2 &\in \sigma \otimes (V_2 \otimes \det^{-1})^{F_{r}^{-1}-i} \quad \kappa \beta : \mathcal{K}_1 \to \sigma \\
\alpha \otimes 2 \hat{x} \hat{y} &\in \sigma \otimes (V_2 \otimes \det^{-1})^{F_{r}^{-1}-i} \quad \epsilon : \mathcal{K}_1 \to \sigma \\
\alpha \otimes \hat{y}^2 &\in \sigma \otimes (V_2 \otimes \det^{-1})^{F_{r}^{-1}-i} \quad \kappa : \mathcal{K}_1 \to \sigma
\end{align*}
\]

where

\[
\begin{align*}
\kappa &:= \begin{pmatrix} 1 + \pi a & \pi b \\ \pi c & 1 + \pi d \end{pmatrix} \in \mathcal{K}_1 & \mapsto & \sigma(c) \in \overline{\text{F}} \\
\epsilon &:= \begin{pmatrix} 1 + \pi a & \pi b \\ \pi c & 1 + \pi d \end{pmatrix} \in \mathcal{K}_1 \mapsto & \sigma(a - d) \in \overline{\text{F}} \\
\kappa &:= \begin{pmatrix} 1 + \pi a & \pi b \\ \pi c & 1 + \pi d \end{pmatrix} \in \mathcal{K}_1 \mapsto & \sigma(a) \in \overline{\text{F}}
\end{align*}
\]

\( (iii) \) The summand \( \bigoplus_{i=1}^{d} \sigma \subset H^1(\mathcal{K}_1, \sigma) \) corresponds to maps \( \mathcal{K}_1 \to \sigma \) that factor through the determinant and are not given by any of the cocycles appearing in \( \bigoplus_{i=0}^{f-1} \sigma \otimes (V_2 \otimes \det^{-1})^{F_{r}^{-1}-i} \).

Corollary 2.4. \( \text{Ext}_\mathcal{K}(V_{\ell}, V_{\ell'}) \neq 0 \) if and only if \( \text{Ext}_\Gamma(V_{\ell}, V_{\ell'}) \neq 0 \).

Proof. Using (2.2.1), \( \text{Ext}_\mathcal{K}(V_{\ell}, V_{\ell'}) \neq 0 \) implies that either \( \text{Ext}_\Gamma(V_{\ell}, V_{\ell'}) \neq 0 \), or \( \text{Hom}_\Gamma(V_{\ell}, H^1(\mathcal{K}, \sigma)) \neq 0 \).
Either way, the central character is the same for $V_{\vec{t},\vec{s}}$ and $V_{\vec{t}',\vec{s}',\vec{t}}$. This is automatically true if $\text{Ext}_p(\tilde{V}_{\vec{t},\vec{s}}, V_{\vec{t}',\vec{s},\vec{t}}) \neq 0$ because the group algebra of the center of $\Gamma$ is semisimple. If $\text{Hom}(\tilde{V}_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t}',\vec{s},\vec{t}})) \neq 0$, we use the description in Proposition 2.3 and the fact that $V_2 \otimes \det^{-1}$ has trivial central character. Therefore, assuming the central characters of $\tilde{V}_j$ and $V_{\vec{t}',\vec{s}',\vec{t}}$ are the same, we only need to find criteria for inclusion of $\tilde{V}_j$ in $H^1(\mathcal{K}_1, V_{\vec{t}',\vec{s},\vec{t}})$ as $\text{SL}_2(k)$ representations.

Proof. The proof for $p > 2$ is covered by Proposition 5.4 and Corollary 5.5 in [BP]). For $p = 2$, we first make the following observation. If $\text{Hom}(\tilde{V}_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t}',\vec{s},\vec{t}})) \neq 0$, we can twist both sides by the square root of the central character and obtain an inclusion of $\tilde{V}_{\vec{t},\vec{s}}$ into $H^1(\mathcal{K}_1, V_{\vec{t}',\vec{s},\vec{t}})$ as $\text{PGL}_2(k) = \text{SL}_2(k)$ representations. On the other hand, suppose $\tilde{V}_{\vec{t},\vec{s}} \hookrightarrow H^1(\mathcal{K}_1, V_{\vec{t}',\vec{s},\vec{t}})$ as $\text{SL}_2(k)$ representations and the central characters of $\tilde{V}_{\vec{t},\vec{s}}$ and $V_{\vec{t}',\vec{s},\vec{t}}$ are the same. Then this inclusion is easily seen to be an inclusion as $\Gamma$-representations.

Therefore, assuming the central characters of $V_{\vec{t},\vec{s}}$ and $V_{\vec{t}',\vec{s},\vec{t}}$ are the same, we only need to find criteria for inclusion of $V_{\vec{t},\vec{s}}$ in $H^1(\mathcal{K}_1, V_{\vec{t}',\vec{s},\vec{t}})$ as $\text{SL}_2(k)$ representations.
To emphasize disregarding the determinant twists, we will denote by \( L(\vec{r}) \) or by \( L(\sum p^{f-1-j} r_j) \) the irreducible \( \text{SL}_2(k) \) representation \( \otimes_{j=0}^{f-1} (\text{Sym}^r \mathbb{F}_2^2)^{Fr^{f-1-j}} \) where \( r_j \in [0, p-1] \) for each \( j \) and \( \text{SL}_2(k) \) acts on \( \text{Sym}^r \mathbb{F}_2^2 \) via \( \sigma_{f-1} : \text{SL}_2(k) \to \text{SL}_2(\mathbb{F}) \).

By Proposition 2.3, \( H^1(\mathcal{K}_1, \sigma) \cong \bigoplus_{i=0}^{f-1} (L(\vec{s}_i) \otimes V_2^{Fr^{f-2-i}}) \bigoplus_{i=1}^d L(\vec{s}) \) as \( \text{SL}_2(k) \) representations. As \( L(\vec{s}) \not\cong L(\vec{s}_i) \), we need to understand when \( L(\vec{s}) \) embeds into \( L(\vec{s}_i) \otimes V_2^{Fr^{f-2-i}} \cong L(\vec{s}) \otimes L(1)^{Fr^{f-1-i}} \) for a given \( i \).

1. \( s_i = 0 \). Then \( L(\vec{s}) \otimes L(1)^{Fr^{f-1-i}} \) is irreducible and isomorphic to \( L(\vec{s}) \), where \( s_i = 1 \) and \( s_j = s_j' \) for \( j \neq i \).
2. \( s_i = 1 \). Then

\[
L(\vec{s}_i) \otimes L(1)^{Fr^{f-1-i}} \cong L(s_0')^{Fr^{f-1-0}} \otimes \cdots \otimes L(s_{i-1}')^{Fr^{f-i}} \otimes (L(1) \otimes L(1))^{Fr^{f-1-i}} \\
\otimes L(s_{i+1}')^{Fr^{f-2-i}} \otimes \cdots \otimes L(s_{f-1}')^{Fr^{0}}
\]

\[
\cong L(s_0')^{Fr^{f-1-0}} \otimes \cdots \otimes L(s_{i-1}')^{Fr^{f-i}} \otimes Q_1(0)^{Fr^{f-1-i}} \\
\otimes L(s_{i+1}')^{Fr^{f-2-i}} \otimes \cdots \otimes L(s_{f-1}')^{Fr^{0}}
\]

where \( Q_1(0) \) is a self-dual representation of Loewy length 3, with composition factors \( L(0), L(2), L(0) \) by [AJL, Lem. 3.1]. In fact, [AJL, Lem. 3.1] says that \( Q_1(0) \) is a direct summand of \( (\text{Sym}^1 \mathbb{F}_2^2 \otimes \text{Sym}^1 \mathbb{F}_2^2)^{Fr^{f-1-i}} \), but by comparing dimensions, they are equal. As \( \text{SL}_2(\mathbb{F}) \) representations:

\[
\text{soc} \left( L(s_0')^{Fr^{f-1-0}} \otimes \cdots \otimes L(s_{i-1}')^{Fr^{f-i}} \otimes Q_1(0)^{Fr^{f-1-i}} \right) \\
\cong L(s_{i+1}')^{Fr^{f-2-i}} \otimes \cdots \otimes L(s_{f-1}')^{Fr^{0}}
\]

The isomorphism in the last step is by [AJL, Lem. 3.4]. Using [AJL, Cor. 4.2] and the assumption that \( L(\vec{s}) \) is not Steinberg, we conclude that \( L(\vec{s}) \) embeds into \( L(\vec{s}_i) \otimes L(1)^{Fr^{f-1-i}} \) if and only if \( L(\vec{s}) \cong L(s_0')^{Fr^{f-1-0}} \otimes \cdots \otimes L(s_{i-1}')^{Fr^{f-i}} \otimes L(0)^{Fr^{f-1-i}} \otimes L(s_{i+1}')^{Fr^{f-2-i}} \otimes \cdots \otimes L(s_{f-1}')^{Fr^{0}} \).

\[ \square \]

Remark 2.7. The explicit calculation of \( H^1(\mathcal{K}_1, V_{\vec{r}, \vec{s}}) \) in [BP, Prop. 5.1] shows that the inflation homomorphism \( H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{r}, \vec{s}}) \to H^1(\mathcal{K}_1, V_{\vec{r}, \vec{s}}) \) is an isomorphism, since each \( \mathcal{K}_1 \) cocycle is in fact a \( \mathcal{K}_1/\mathcal{K}_2 \) cocycle. By the construction of the
Grothendieck spectral sequence, this means exactly that

$$\text{socr} \left( \frac{\text{inj}_K(V_{\ell,x} \sigma)_{K_2}/V_{\ell,x} \sigma}{\text{inj}_K(V_{\ell,x} \sigma)_{K_1}/V_{\ell,x} \sigma} \right) = \text{socr} \left( \frac{\text{inj}_K(V_{\ell,x} \sigma)/V_{\ell,x} \sigma}{\text{inj}_K(V_{\ell,x} \sigma)_{K_1}/V_{\ell,x} \sigma} \right)$$

Lemma 2.8. Let $V_{\ell,x}$ and $V_{\ell,x} \sigma$ be a pair of Serre weights. Then the natural map $\text{Ext}^1_{K/K_2}(V_{\ell,x}, V_{\ell,x} \sigma) \to \text{Ext}^1_{K/K_2}(V_{\ell,x}, V_{\ell,x} \sigma)$ is an isomorphism.

Proof. For a group $G$ with an $F$-representation $\sigma$, let $\text{inj}_G(\sigma)$ denote the injective hull of $\sigma$ as a smooth $\overline{F}[G]$ module. Then, for each Serre weight $\sigma$, $\text{inj}_K(\sigma)^{K_n}$ is an injective $K/K_n$ module. By injectivity of $\text{inj}_{K/K_n}(\sigma)$, there exists a map $\text{inj}_K(\sigma)^{K_n} \to \text{inj}_{K/K_n}(\sigma)$. The kernel of this map must be trivial, by the hull property of $\text{inj}_K(\sigma)$. By the hull property of $\text{inj}_{K/K_n}(\sigma)$, it is forced to be an isomorphism. We will henceforth use $\text{inj}_K(\sigma)^{K_n}$ as the injective hull of $\sigma$ as a $K/K_n$ representation.

The explicit calculation of $H^1(K_1, V_{\ell,x} \sigma)$ in [BP, Prop. 5.1] shows that the inflation homomorphism $H^1(K_1/K_2, V_{\ell,x} \sigma) \to H^1(K_1, V_{\ell,x} \sigma)$ is an isomorphism, since each $K_1$ cocycle is in fact a $K_1/K_2$ cocycle. By the construction of the Grothendieck spectral sequence, this means exactly that

$$(2.8.1) \quad \text{socr} \left( \frac{\text{inj}_K(V_{\ell,x} \sigma)_{K_2}/V_{\ell,x} \sigma}{\text{inj}_K(V_{\ell,x} \sigma)_{K_1}/V_{\ell,x} \sigma} \right) = \text{socr} \left( \frac{\text{inj}_K(V_{\ell,x} \sigma)/V_{\ell,x} \sigma}{\text{inj}_K(V_{\ell,x} \sigma)_{K_1}/V_{\ell,x} \sigma} \right)$$

Now, suppose $V_{\ell,x}$ lives inside the $\Gamma$ socle of $\frac{\text{inj}_K(V_{\ell,x} \sigma)_{K_2}/V_{\ell,x} \sigma}{\text{inj}_K(V_{\ell,x} \sigma)_{K_1}/V_{\ell,x} \sigma}$ with multiplicity $n$. Equivalently, $\text{Hom}_\Gamma(V_{\ell,x}, H^1(K_1/K_2, V_{\ell,x} \sigma))$ is $n$-dimensional. Let $N$ denote the preimage in $\text{inj}_K(V_{\ell,x} \sigma)_{K_2}/V_{\ell,x} \sigma$ of $V_{\ell,x}^n \subset \text{socr} \left( \frac{\text{inj}_K(V_{\ell,x} \sigma)_{K_2}/V_{\ell,x} \sigma}{\text{inj}_K(V_{\ell,x} \sigma)_{K_1}/V_{\ell,x} \sigma} \right)$.

Since $K/K_2$ is a finite group, $\text{Ext}_{K/K_2}(V_{\ell,x}, V_{\ell,x} \sigma) \cong H^1(K/K_2, V_{\ell,x} \sigma \otimes V_{\ell,x} \sigma)$. The Grothendieck spectral sequence gives us the following left exact sequence:

$$(2.8.2) \quad 0 \to H^1(\Gamma, V_{\ell,x} \sigma \otimes V_{\ell,x} \sigma) \xrightarrow{\text{inf}} H^1(K/K_2, V_{\ell,x} \sigma \otimes V_{\ell,x} \sigma) \xrightarrow{\text{res}} H^1(K_1/K_2, V_{\ell,x} \sigma \otimes V_{\ell,x} \sigma)^\Gamma$$

Proposition 2.9. Suppose $e > 1$. Then the res map in (2.8.2) is a split surjection.

Proof. $e > 1$ implies that $p \in \pi^2 O_K$. Let $O_K^{ur}$ be the ring of integers for the maximal unramified subextension inside $E$ over $\mathbb{Q}_p$. Therefore $k \cong O_K^{ur}/p \to O_K/\pi^2$. This gives a splitting of the natural surjection $\text{GL}_2(O_K/\pi^2) \to \text{GL}_2(k) \cong \text{GL}_2(O_K^{ur}/p)$. We obtain the following split exact sequence:

$$1 \longrightarrow K_1/K_2 \longrightarrow K/K_2 \longrightarrow \Gamma \longrightarrow 1$$
Therefore, $K/K_2 \cong K_1/K_2 \rtimes \Gamma$. For $b \in \Gamma$ and $a \in K_1/K_2$, denote $bab^{-1}$ by $a^b$.

Suppose $\sigma$ is a $\Gamma$ representation (seen via inflation as a $K/K_2$ representation) and $\psi$ is a cocycle representing a nonzero element of $H^1(K_1/K_2, \sigma)^\Gamma$. As $K_1/K_2$ action is trivial on $\sigma$, $H^1(K_1/K_2, \sigma)^\Gamma = Z^1(K_1/K_2, \sigma)^\Gamma$. $\Gamma$-invariance means precisely that for $b \in \Gamma$ and $a \in K_1/K_2$, $b^{-1}\psi(a^b) = \psi(a)$.

We define a function $\delta$ on $K_1/K_2 \rtimes \Gamma$ by setting $\delta((a, b))$ equal to $\psi(a)$. I claim that $\delta$ is a cocycle, i.e., $\delta((a, b)(a', b')) = \delta((a, b)) + (a, b) \cdot \delta((a', b'))$. Evaluation of the left hand side gives us:

\[
L.H.S. = \delta((aa'^b, bb'))
= \psi(aa'^b)
= \psi(a) + \psi(a'^b)
\]

Evaluation of the right hand side gives us:

\[
R.H.S. = \psi(a) + (a, 1)(1, b) \cdot \psi(a')
= \psi(a) + (1, b) \cdot \psi(a')
= \psi(a) + (1, b) \cdot ((1, b^{-1}) \cdot \psi(a'^b)) \quad \text{ (as $\psi$ is $\Gamma$-invariant)}
= \psi(a) + \psi(a'^b)
= L.H.S.
\]

This establishes that $\delta$ is a cocycle and therefore, res map in (2.8.2) is a split surjection.

\[\square\]

**Corollary 2.10.** If $e > 1$,

\[\text{Ext}_K(V_{\tilde{t},\tilde{s}}, V_{\tilde{t}',\tilde{s}'} \sim \text{Ext}_\Gamma(V_{\tilde{t},\tilde{s}}, V_{\tilde{t}',\tilde{s}'}) \oplus \text{Hom}_\Gamma(V_{\tilde{t},\tilde{s}}, H^1(K_1/K_2, V_{\tilde{t}',\tilde{s}'}) )}.\]

**Proof.** This is immediate from Proposition 2.9 and the fact that $H^1(K_1/K_2, \text{Sym}^{r'}_{\tilde{t}'} \otimes V_{\tilde{t}',\tilde{s}'})^\Gamma \cong \text{Hom}_\Gamma(V_{\tilde{t},\tilde{s}}, H^1(K_1/K_2, V_{\tilde{t}',\tilde{s}'}) )$ by the explicit description in Proposition 2.3.

\[\square\]

**Lemma 2.11.** Let $p > 2$, $r \leq p - 3$. Then the following are true:

(i) $\text{Sym}^{\Gamma +2}_{\tilde{t}'}$ embeds into $\text{Sym}^{\Gamma}_{\tilde{t}'} \otimes \text{Sym}^{\Gamma}_{\tilde{t}'}$ as a direct summand of multiplicity 1.

(ii) Let the obvious basis of $\text{Sym}^{\Gamma +2}_{\tilde{t}'}$ be given by $\{w^k z^{r+2-k}\}_{k \in [0,r+2]}$. Further, let a basis of $\text{Sym}^{\Gamma}_{\tilde{t}'} \otimes \text{Sym}^{\Gamma}_{\tilde{t}'}$ be given by $\{x^j y^{2-j} \otimes x^k y^{r-k}\}_{(j,k) \in [0,2] \times [0, r]}$ if $r > 0$, and by $\{x^j y^{2-j} \otimes 1\}_{j \in [0,2]}$ if $r = 0$. The embedding is given.
(uniquely up to scalar multiplication) as follows:

\[
\begin{align*}
  w^k z^{r+2-k} & \mapsto x^2 \otimes \frac{k(k-1)}{(r+2)(r+1)} x^{k-2} y^{r+2-k} \\
  & + 2\bar{x}\bar{y} \otimes \frac{k(r+2-k)}{(r+2)(r+1)} x^{k-1} y^{r+1-k} \\
  & + \bar{y}^2 \otimes \frac{(r+2-k)(r+1-k)}{(r+2)(r+1)} x^{k} y^{r-k} \quad \text{for } k \in [2, r]
\end{align*}
\]

\[
\begin{align*}
  w z^{r+1} & \mapsto 2\bar{x}\bar{y} \otimes \frac{1}{r+2} y^{r} + \bar{y}^2 \otimes \frac{r}{r+2} x y^{r-1} \quad \text{if } r > 0
\end{align*}
\]

\[
\begin{align*}
  w^r z^2 & \mapsto \bar{x}^2 \otimes \frac{r}{r+2} x^{r-1} y + 2\bar{x}\bar{y} \otimes \frac{1}{r+2} x^{r} \quad \text{if } r > 0
\end{align*}
\]

\[
\begin{align*}
  w & \mapsto \bar{x}\bar{y} \otimes 1 \\
  z & \mapsto \bar{y}^2 \otimes y^r \\
  z^{r+2} & \mapsto \bar{y}^2 \otimes y^r
\end{align*}
\]

**Proof.** The first statement is from [BP, Prop. 5.4]. The second statement can be verified by direct computation. \qed

**Lemma 2.12.** Let \( V_{t,s} \), a Serre weight. Denote by \( \{ \otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j} \}_{(k_j)} \) the obvious basis of \( V_{t,s} \). Use the same notation to denote a basis of \( V_{-t-s, \sigma} \). Then \( V_{t,s}^\vee \cong V_{-t-s, \sigma} \) under the following map:

\[
V_{t,s}^\vee \rightarrow V_{-t-s, \sigma}
\]

\[
\otimes_{j} (w^{k_j} z^{s_j-k_j})^\vee \rightarrow \otimes_{j} \left( \frac{s_j}{k_j} \right) w^{s_j-k_j} (-z)^{k_j}
\]

**Proof.** By direct computation. \qed

**Lemma 2.13.** Let \( p > 2 \). Consider a pair of non-isomorphic, non-Steinberg Serre weights \( V_{t,s} \) and \( V_{t',s'} \) satisfying the condition in Proposition 2.6, that is, \( s_i = s'_i + 2 \), \( s_j = s'_j \) for \( j \neq i \) and \( \sum_{j \in T} p^{f-1-1-j} t_j \equiv -p^{f-1-1} + \sum_{j \in T} p^{f-1-1-j} t'_j \mod p^f - 1 \).

Denote by \( \otimes_{j=0}^{f-1} (w^{k_j} z^{s_j-k_j})^\vee \) the dual of \( \otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j} \), where \( \{ \otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j} \}_{(k_j)} \) gives a basis of \( V_{t,s} \). Let the basis of \( V_{t',s'} \) be given by \( \{ \otimes_{j=0}^{f-1} \bar{y}^{k'_j} z^{s'_j-k'_j} \}_{(k'_j)} \).

Then the \( \Gamma \)-invariant cocycles of \( H^1(K_1/K_2, V_{t,s}^\vee \otimes V_{t',s'}^\vee) \cong H^1(K_1/K_2, V_{-t-s, \sigma} \otimes V_{t',s'}^\vee) \) are a 1-dimensional subspace spanned by

\[
\kappa^1_t A + \epsilon_i B + \kappa^u_t C
\]

where \( \kappa^1_t \), \( \epsilon_i \) and \( \kappa^u_t \) are homomorphisms \( K_1/K_2 \rightarrow \mathbb{F} \) defined in Proposition 2.3 and \( A, B \) and \( C \) are elements of \( V_{t,s}^\vee \otimes V_{t',s'}^\vee \) defined below.

For \( s'_i > 0 \),
\[ A = - \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i} \left( \frac{s_j}{k_j} \right) w^{s_j-k_j}(-z)^{k_j} \otimes (\otimes_{j \neq i} x^{s_j} y^{s_j-k_j}) \right) \otimes (z)^{s_i} \otimes 1 \right) \]

\[
\left( \sum_{k_i=2}^{s_i'} \left( \frac{s_i}{k_i} w^{s_i-k_i}(-z)^{k_i} \otimes \frac{k_i(k_i-1)}{(s_i'+2)(s_i'+1)} x^{k_i-2} y^{s_i'+2-k_i} \right) \right. \\
+ \left( s_i w(-z)^{s_i-1} \otimes \frac{s_i'}{s_i'+2} y^{s_i'-1} \right) \\
+ \left. \left( (-z)^{s_i} \otimes x^{s_i'} \right) \right)
\]

\[ B = \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i} \left( \frac{s_j}{k_j} \right) w^{s_j-k_j}(-z)^{k_j} \otimes (\otimes_{j \neq i} x^{s_j} y^{s_j-k_j}) \right) \otimes \left( \sum_{k_i=2}^{s_i'} \left( \frac{s_i}{k_i} w^{s_i-k_i}(-z)^{k_i} \otimes \frac{k_i(s_i'+2-k_i)}{(s_i'+2)(s_i'+1)} x^{k_i-1} y^{s_i'+1-k_i} \right) \right. \\
+ \left( s_i w^{s_i-1}(-z) \otimes \frac{1}{s_i'+2} y^{s_i'} \right) \\
+ \left. \left( (-z)^{s_i} \otimes \frac{s_i'}{s_i'+2} y^{s_i'} \right) \right)
\]

\[ C = \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i} \left( \frac{s_j}{k_j} \right) w^{s_j-k_j}(-z)^{k_j} \otimes (\otimes_{j \neq i} x^{s_j} y^{s_j-k_j}) \right) \otimes \left( \sum_{k_i=2}^{s_i'} \left( \frac{s_i}{k_i} w^{s_i-k_i}(-z)^{k_i} \otimes \frac{(s_i'+2-k_i)(s_i'+1-k_i)}{(s_i'+2)(s_i'+1)} x^{k_i} y^{s_i'-k_i} \right) \right. \\
+ \left( s_i w^{s_i-1}(-z) \otimes \frac{s_i'}{s_i'+2} x y^{s_i'-1} \right) \\
+ \left. \left( w^{s_i} \otimes y^{s_i'} \right) \right)
\]

For \( s_i' = 0 \),

\[ A = - \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i} \left( \frac{s_j}{k_j} \right) w^{s_j-k_j}(-z)^{k_j} \otimes (\otimes_{j \neq i} x^{s_j} y^{s_j-k_j}) \right) \otimes (z)^{s_i} \otimes 1 \right) \]

\[ B = \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i} \left( \frac{s_j}{k_j} \right) w^{s_j-k_j}(-z)^{k_j} \otimes (\otimes_{j \neq i} x^{s_j} y^{s_j-k_j}) \right) \otimes (-z) \otimes 1 \right) \]

\[ C = \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i} \left( \frac{s_j}{k_j} \right) w^{s_j-k_j}(-z)^{k_j} \otimes (\otimes_{j \neq i} x^{s_j} y^{s_j-k_j}) \right) \otimes (w^{s_i} \otimes 1) \right) \]

\[ (\otimes_{j \neq i} x^{s_j} y^{s_j-k_j}) \]
Proof. The first step is to compute \( \left( H^1(\mathcal{K}_1/\mathcal{K}_2, V^\vee_{t,\varphi} \otimes V^\vee_{t,\varphi}) \right)^\Gamma \). By Proposition 2.3, this group is isomorphic to \( \text{Hom}_{\Gamma}(V^\vee_{t,\varphi} \bigoplus_{i=0}^{f-r-1} V^\vee_{t,\varphi} \otimes (V_2 \otimes \det^{-1})^{f-r-1}, \bigoplus_{i=1}^{d} V^\vee_{t,\varphi}) \). Using Lemma 2.11, \( V^\vee_{t,\varphi} \) has a unique (upto scalars) embedding into \( \bigoplus_{i=0}^{f-r-1} V^\vee_{t,\varphi} \otimes (V_2 \otimes \det^{-1})^{f-r-1} \bigoplus_{i=1}^{d} V^\vee_{t,\varphi} \). This embedding may be written as an element of \( V^\vee_{t,\varphi} \otimes V^\vee_{t,\varphi} \otimes (V_2 \otimes \det^{-1})^{f-r-1} \subset H^1(\mathcal{K}_1/\mathcal{K}_2, V^\vee_{t,\varphi} \otimes V^\vee_{t,\varphi}) \). Employing Proposition 2.3 to further write this element as an explicit map \( \mathcal{K}_1/\mathcal{K}_2 \rightarrow V^\vee_{t,\varphi} \otimes V^\vee_{t,\varphi} \), we obtain the following values of \( A, B \) and \( C \):

For \( s'_i > 0 \),

\[
A = - \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i}(w^{k_j} z^{s_j - k_j})^\vee) \otimes (\otimes_{j \neq i} x^{k_j} y^{s_j - k_j}) \right) \otimes \\
\left( \sum_{k_i = 2}^{s'_i} \left( (w^{k_i} z^{s_i - k_i})^\vee \otimes \frac{k_i(k_i - 1)}{(s'_i + 2)(s'_i + 1)} x^{k_i - 2} y^{s'_i + 2 - k_i} \right) \\
+ \left( (w^{s_i - 1} z^w) \otimes \frac{s'_i}{s'_i + 2} x^{s_i - 1} y^w \right) \\
+ \left( (w^{s_i} z^w) \otimes \frac{1}{s'_i + 2} x^{s'_i} \right) \right)
\]

\[
B = \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i}(w^{k_j} z^{s_j - k_j})^\vee) \otimes (\otimes_{j \neq i} x^{k_j} y^{s_j - k_j}) \right) \otimes \\
\left( \sum_{k_i = 2}^{s'_i} \left( (w^{k_i} z^{s_i - k_i})^\vee \otimes \frac{k_i(s'_i + 2 - k_i)}{(s'_i + 2)(s'_i + 1)} x^{k_i - 1} y^{s'_i + 1 - k_i} \right) \\
+ \left( (w^{s_i - 1} z^w) \otimes \frac{1}{s'_i + 2} y^{s'_i} \right) \\
+ \left( (w^{s_i} z^w) \otimes \frac{1}{s'_i + 2} x^{s'_i} \right) \right)
\]

\[
C = \left( \sum_{(k_j) \neq i} (\otimes_{j \neq i}(w^{k_j} z^{s_j - k_j})^\vee) \otimes (\otimes_{j \neq i} x^{k_j} y^{s_j - k_j}) \right) \otimes \\
\left( \sum_{k_i = 2}^{s'_i} \left( (w^{k_i} z^{s_i - k_i})^\vee \otimes \frac{(s'_i + 2 - k_i)(s'_i + 1 - k_i)}{(s'_i + 2)(s'_i + 1)} x^{k_i} y^{s'_i - k_i} \right) \\
+ \left( w z^{s_i - 1} \otimes \frac{s'_i}{s'_i + 2} x y^{s'_i - 1} \right) \\
+ \left( (z^w)^\vee \otimes y^{s'_i} \right) \right)
\]

For \( s'_i = 0 \),
\[ A = \left( \sum_{(k_j)_{j \neq i}} (\otimes_{j \neq i} (w^{k_j} z^{s_{ji} - k_j})) \otimes (\otimes_{j \neq i} x^{k_j} y^{s_{ji}' - k_j})) \right) \otimes ((w^{s_i}) \otimes 1) \]

\[ B = \left( \sum_{(k_j)_{j \neq i}} (\otimes_{j \neq i} (w^{k_j} z^{s_{ji} - k_j})) \otimes (\otimes_{j \neq i} x^{k_j} y^{s_{ji}' - k_j})) \right) \otimes \left( (wz)^{\otimes} \otimes \frac{1}{2} \right) \]

\[ C = \left( \sum_{(k_j)_{j \neq i}} (\otimes_{j \neq i} (w^{k_j} z^{s_{ji} - k_j})) \otimes (\otimes_{j \neq i} x^{k_j} y^{s_{ji}' - k_j})) \right) \otimes ((z^{s_i}) \otimes 1) \]

Using Lemma 2.12, we can rewrite elements in \( V_{\tilde{e}, \tilde{s}} \) as elements of \( V_{-\tilde{e}, -\tilde{s}, \tilde{z}} \), giving us the desired answer.

Our next order of business is to check if a \( \Gamma \)-invariant cocycle in \( H^1(K_1/K_2, V_{\tilde{e}, \tilde{s}}^\vee \otimes V_{\tilde{e}, \tilde{s}}^\vee) \cong H^1(K_1/K_2, V_{-\tilde{e}, -\tilde{s}, \tilde{z}} \otimes V_{-\tilde{e}, -\tilde{s}, \tilde{z}}) \) is in the image of the res map in (2.8.2). Therefore, we will try and extend such a cocycle to \( K \). However, instead of extending it to all of \( K \), we will first focus our attention on extending it to the subgroup of \( K \) generated by the upper unipotent and diagonal matrices.

**Proposition 2.14.** Let \( e = 1 \). Consider a pair of non-isomorphic, weakly regular Serre weights \( V_{\tilde{e}, \tilde{s}} \) and \( V_{\tilde{e}, \tilde{s}} \) satisfying the conditions in Proposition 2.6. In particular,

\[ \text{Hom}_\Gamma(V_{\tilde{e}, \tilde{s}}, H^1(K_1, V_{\tilde{e}, \tilde{s}})) \neq 0 \]

Then res is the zero map and in (2.8.2) is an isomorphism, implying that \( \text{Ext}_\Gamma(V_{\tilde{e}, \tilde{s}}, V_{\tilde{e}, \tilde{s}}) \neq 0 \) iff \( \text{Ext}_\Gamma(V_{\tilde{e}, \tilde{s}}, V_{\tilde{e}, \tilde{s}}) \neq 0 \).

**Proof.** Satisfying the conditions in Proposition 2.6 along with \( s_j, s_j' < p - 1 \) for each \( j \in [0, f - 1] \) (by weak regularity) force \( p > 2 \). Therefore, \( s_i = s_i' + 2 \), \( s_j = s_j' \) for \( j \neq i \) and \( \sum_{j \in T} p^{f - 1 - i} j \equiv -p^{f - 1 - i} + \sum_{j \in T'} p^{f - 1 - j} j' \mod p^{f - 1} \). We will assume without loss of generality that \( \tilde{t} = 0 \).

Our proof will show that there is no way to extend a \( \Gamma \)-invariant cocycle \( \psi \in H^1(K_1/K_2, V_{-\tilde{e}, -\tilde{s}, \tilde{z}} \otimes V_{\tilde{e}, \tilde{s}}) \) simultaneously to upper unipotent and diagonal matrices with 1 in the bottom right entry. We denote these two groups by \( U \) and \( D \) respectively. We let \( U(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \) and \( D(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \).

First, we give a basis of \( V_{-\tilde{e}, -\tilde{s}, \tilde{z}} \otimes V_{\tilde{e}, \tilde{s}} \). We note that:

\[ V_{-\tilde{e}, -\tilde{s}, \tilde{z}} \otimes V_{\tilde{e}, \tilde{s}} \cong \]

\[ \otimes (\det -s_0 \otimes \text{Sym}^2 \otimes \text{F}_{r-1} \otimes \cdots \otimes (\det -s_{i-1} \otimes \text{Sym}^{s_i-1} \otimes \text{F}_{r-1} \otimes \cdots) \otimes (\det -s_{i+1} \otimes \text{Sym}^{s_i+1} \otimes \text{F}_{r-2} \otimes \cdots) \otimes (\det -s_{i-1} \otimes \text{Sym}^{s_i-1} \otimes \text{F}_{r-2} \otimes \cdots) \otimes (\det 0 \otimes \text{Sym}^{s_i} \otimes \text{F}_{r-2} \otimes \cdots) \otimes (\det 0 \otimes \text{Sym}^{s_i-1} \otimes \text{F}_{r-2} \otimes \cdots) \otimes (\det 0 \otimes \text{Sym}^{s_i} \otimes \text{F}_{r-2} \otimes \cdots) \otimes (\det 0 \otimes \text{Sym}^{s_i-1} \otimes \text{F}_{r-2} \otimes \cdots)) \]
This can be viewed as a tensor of $2f$ terms, each term being a tensor of a determinant power and a symmetric power. The first $f$ terms correspond to those coming from $V_{\tilde{t} - x, z}$ and for each such term, a basis is given by homogeneous degree $s_j$ monomials in variables $w$ and $z$. Here, $w$ corresponds to the first standard basis element of $F^2$, while $z$ corresponds to the second standard basis element. The last $f$ terms correspond to those coming from $V_{\tilde{t}, \zeta}$ and for each such term, a basis is given by homogeneous degree $s_j'$ monomials in variables $x$ and $y$. As before, $x$ corresponds to the first standard basis element of $F^2$, while $y$ corresponds to the second standard basis element.

Denote by $W$ the $F$ subspace of $V_{\tilde{t} - x, z} \otimes V_{\tilde{t}, \zeta}$ spanned by \{(\bigotimes_{j=0}^{n-1} \omega^{k_j} w_{s_j - k_j}) \otimes y^{s_0} \otimes y^{s_1} \otimes \cdots \otimes y^{s_{j-1}}\}_{(k_j)_j}$. Evidently, $W$ is a quotient as a $(U, D) \subset K$ representation. We now define a partial order on the indexing set of the basis of $W$. Let $(k_j)_j$ and $(k'_j)_j$ be two indices, where $(k_j)_j$ corresponds to the basis element \((\bigotimes_{j=0}^{n-1} \omega^{k_j} w_{s_j - k_j}) \otimes y^{s_0} \otimes y^{s_1} \otimes \cdots \otimes y^{s_{j-1}}\) while $(k'_j)_j$ corresponds to the basis element \((\bigotimes_{j=0}^{n-1} \omega^{k'_j} z_{s_j - k'_j}) \otimes y^{s_0} \otimes y^{s_1} \otimes \cdots \otimes y^{s_{j-1}}\). If $k'_j \geq k_j$ for all $j$, then we say that $(k_j)_j$ is a descendant of $(k'_j)_j$. More precisely, if \(\sum_{j}(k'_j - k_j) = n \geq 0\), we say that $(k_j)_j$ is a $n$-descendant of $(m_j)_j$. Alternatively, we say $(k'_j)_j$ is an $n$-ascendant of $(k_j)_j$, or $(k_j)_j$ is a $-n$-ascendant of $(k'_j)_j$, or $(k'_j)_j$ is a $-n$-descendant of $(k_j)_j$.

Now, take $\kappa_i A + \epsilon_i B + \kappa_i^n C$ to be the cocycle defined in Lemma 2.13. Denote by $\psi$ the restriction of this cocycle to $(U, D) \cap K$. Then $\psi = \epsilon_i B + \kappa_i^n C$. Suppose it has an extension to $(U, D)$. On composing the extension with the quotient map $V_{\tilde{t} - x, z} \otimes V_{\tilde{t}, \zeta} \rightarrow W$, we obtain a cocycle valued in $W$, which we denote by $q$. Denote by $q_{(k_j)_j}$ the coordinates of $q$ corresponding to the basis vector \((\bigotimes_{j=0}^{n-1} \omega^{k_j} w_{s_j - k_j}) \otimes y^{s_0} \otimes y^{s_1} \otimes \cdots \otimes y^{s_{j-1}}\) \(\in W\) [Cautionary note about the notation: here the exponent of $w$ is $k_j$, whereas in Lemma 2.13, $s_j - k_j$ is the exponent of $w$].

From the definition of $\psi$, $q_{(k_j)_j}|_{U \cap K} \neq 0$ if and only if $k_j = s_j$ for all $j$. Further, $q_{(k_j)_j}|_{U \cap K} = 0$ if and only if $k_i - s_i - 1 = s'_i + 1$ and $k_j = s_j - s'_j$ for all $j \neq i$. Each \((\bigotimes_{j=0}^{n-1} \omega^{k_j} w_{s_j - k_j}) \otimes y^{s_0} \otimes y^{s_1} \otimes \cdots \otimes y^{s_{j-1}}\) \(\in W\) is an eigenvector of $D(t)$ with eigenvalue $t^{\lambda_{(k_j)_j}}$, where $\lambda_{(k_j)_j}$ is the unique number in $[0, p^f - 1]$ that is equivalent to \(\sum_{j \neq i} p^{f-1-i}(k_j - s'_j) + p^{f-1-i}(k_i - s'_i - 1) \mod p^f - 1\). We make some observations about $\lambda_{(k_j)_j}$:

1. $\lambda_{(k_j)_j} = 0$ if and only if $k_i = s'_i + 1$ and $k_j = s'_j$ for all $j \neq i$ if and only if $q_{(k_j)_j}|_{U \cap K} \neq 0$.
2. $\lambda_{(k_j)_j}$ are evidently pairwise distinct.
3. Suppose $(k_j)_j \neq (s_j)_j$. Then $\lambda_{(k_j)_j} \neq p^{f-1-i}$ for any $l \in [0, f - 1]$.

To see this, suppose on the contrary that \(\sum_{j \neq i} p^{f-1-i}(k_j - s'_j) + p^{f-1-i}(k_i - s'_i - 1) = p^{f-1-i} \mod p^f - 1\). Equivalently, \(\sum_{j \neq i} p^{f-1-i}k_j = \sum_{j \neq i} p^{f-1-i}s'_j + p^{f-1-i}(s'_i + 1) + p^{f-1-i}l\).

If $l = i$, then the right hand side is \(\sum_{j \neq i} p^{f-1-i}s_j\). As each $s_j$ is less than or equal to $p - 1$, and at least one $s_j$ is strictly less than $p - 1$ (by assumption), $k_j$ is forced to equal $s_j$ for each $j$, a contradiction.
If \( l \neq i \), then \( s'_i + 1 \leq p - 1 \) since \( s'_i < p - 1 \) by assumption. Further, \( s'_i + 1 < p - 1 \), because \( s_i = s'_i + 2 \leq p - 1 \). Therefore, both the right and left hand sides have all coefficients of \( p^{l-1-j} \) less than or equal to \( p - 1 \), and at least one coefficient strictly less than \( p - 1 \), forcing right and left hand side coefficients to be the same. But this is a contradiction, since the coefficient of \( p^{l-1-i} \) clearly differs as for each \( j, k_j \leq s_j \).

For each \((k_j)_j\), since \( D \) acts diagonally on \( W \), \( q_{(k_j)_j}|_D \) is a cocycle \( D \to \mathbb{F}(\lambda_{(k_j)_j}) \), where \( \mathbb{F}(\lambda_{(k_j)_j}) \) is a one-dimensional \( \mathbb{F} \)-vector space with action of \( D(t) \) given by multiplication with \( t^{\lambda_{(k_j)_j}} \). Note that \( D \cong \mathcal{O}_K^* \cong k^* \times (1 + \pi \mathcal{O}_K) \). Therefore, when \( q_{(k_j)_j}|_{D \cap \mathcal{O}_K} = 0 \), \( q_{(k_j)_j}|_D \) can be seen as a cocycle \( k^* \to \mathbb{F}(\lambda_{(k_j)_j}) \). For non-zero \( \lambda_{(k_j)_j} \), \( \sum_{\xi \in k^*} \xi^{\lambda_{(k_j)_j}} = 0 \), because if \( \tilde{\xi} \) is the generator of the cyclic group \( k^* \), \( \tilde{\xi}^{\lambda_{(k_j)_j}} = \sum_{\xi \in k^*} (\tilde{\xi}^{\xi})^{\lambda_{(k_j)_j}} = \sum_{\xi \in k^*} \xi^{\lambda_{(k_j)_j}} \). It follows that \( H^1(k^*, \mathbb{F}(\lambda_{(k_j)_j})) = \frac{\mathbb{F}}{(\sum_{\xi \in k^*} \xi^{\lambda_{(k_j)_j}} - 1)\mathbb{F}} = 0 \). Therefore, when \( q_{(k_j)_j}|_{D \cap \mathcal{O}_K} = 0 \), there exists \( a_{(k_j)_j} \in \mathbb{F} \) such that \( q_{(k_j)_j}(D(\xi)) = \xi^{\lambda_{(k_j)_j}} a_{(k_j)_j} \). When \( q_{(k_j)_j}|_{D \cap \mathcal{O}_K} \neq 0 \), let \( q_{(k_j)_j} = 0 \).

Adjust the cocycle \( q \) by the coboundary given by the vector whose coordinate corresponding to \( (\otimes_{j=0}^{s-1} y^{k_j} z^{s-1-k_j}) \otimes y^{k_j} \otimes \cdots \otimes y^{k_t-1} \) is \(-a_{(k_j)_j} \). Therefore, we may assume that when \( q_{(k_j)_j}|_{D \cap \mathcal{O}_K} = 0 \), \( q_{(k_j)_j}|_D = 0 \). When \( q_{(k_j)_j}|_{D \cap \mathcal{O}_K} \neq 0 \), since \( \lambda_{(k_j)_j} = 0 \), \( q_{(k_j)_j}|_D \) is a group homomorphism \( k^* \times (1 + \pi \mathcal{O}_K) \cong D \to \mathbb{F} \). Since order of \( k^* \) is prime to \( p \), \( q_{(k_j)_j}(D(k^*)) = 0 \). Therefore, regardless of \((k_j)_j\), we have \( q_{(k_j)_j}(D(k^*)) = 0 \).

Our next order of business is to understand each \( q_{(k_j)_j}|_U \). Note that \( U \cong \mathcal{O}_K \).

Except when \( k_j = s_j \) for all \( j \), \( q_{(k_j)_j}|_{U(\pi \mathcal{O}_K)} = 0 \) (as remarked earlier) and therefore, \( q_{(k_j)_j}|_U \) can be seen as a map on \( \mathcal{O}_K/\pi \). Since \( D(\xi) U(\alpha) = U(\xi) D(\xi) \) for \( \xi \in k^* \) and \( \alpha \in \mathcal{O}_K/\pi \), we have the following for all \((k_j)_j \neq (s_j)_j\):

\begin{equation}
(2.14.1) \quad \xi^{\lambda_{(k_j)_j}} q_{(k_j)_j}(U(\alpha)) = q_{(k_j)_j}(D(\xi)U(\alpha)) = q_{(k_j)_j}(U(\xi) D(\xi)) = q_{(k_j)_j}(U(\xi \alpha))
\end{equation}

Therefore, replacing \( \alpha \) with \( 1 \) and \( \xi \) with \( \alpha \) (this covers all the cases since \( q_{(k_j)_j}(U(0)) \) is already 0 because \( q_{(k_j)_j}|_{U \cap \mathcal{O}_K} = 0 \), we obtain for all \((k_j)_j \neq (s_j)_j\):

\begin{equation}
(2.14.2) \quad q_{(k_j)_j}(U(\alpha)) = \alpha^{\lambda_{(k_j)_j}} q_{(k_j)_j}(U(1))
\end{equation}

Now, we do an inductive argument to show that \( q_{(k_j)_j}|_U = 0 \) for all \((k_j)_j \neq (s_j)_j\).

Suppose \( q_{(k_j)_j}|_U = 0 \) for each \( m \)-ascendant \((k_j)_j\) of \((0)_j\), where \(-1 \leq m < \sum s_j - 1 \). The base case with \( m = -1 \) is automatic, because \((0)_j\) has no descendants. We will show that \( q_{(k_j)_j}|_U = 0 \) for each \( m + 1 \)-ascendant \((k_j)_j\) of \((0)_j\).

Fix an \( m + 1 \)-ascendant \((k_j)_j\) of \((0)_j\). Take \((k'_j)_j\) to be a 1-ascendant of \((k_j)_j\) (therefore, an \( m + 2 \)-ascendant of \((0)_j\)).
Since \( q \) is a cocycle, we have for each \( 0 \neq \alpha \in \mathcal{O}_K/\pi \):

\[
q(k'_j, (U(\alpha))) + q(k'_j, (U(1)) + \sum_{\{l_j\}_{j} \in 1\text{-descendants of } (k'_j)_j} \left( \prod_j \begin{pmatrix} s_j - l_j \\ s_j - k'_j \end{pmatrix} \sum_{\alpha} \prod_j \alpha^{p^{f-1-j}(k'_j - l_j)} q(l_j)_j(U(1)) \right) = q(k'_j, (U(\alpha + 1))
\]

\[
= q(k'_j, (U(1 + \alpha))
\]

\[
= q(k'_j, (U(1)) + q(k'_j, (U(\alpha)) + \sum_{\{l_j\}_{j} \in 1\text{-descendants of } (k'_j)_j} \left( \prod_j \begin{pmatrix} s_j - l_j \\ s_j - k'_j \end{pmatrix} q(l_j)_j(U(\alpha)) \right)
\]

Therefore,

\[
\sum_{\{l_j\}_{j} \in 1\text{-descendants of } (k'_j)_j} \left( \prod_j \begin{pmatrix} s_j - l_j \\ s_j - k'_j \end{pmatrix} \sum_{\alpha} \prod_j \alpha^{p^{f-1-j}(k'_j - l_j)} q(l_j)_j(U(1)) \right) = \sum_{\{l_j\}_{j} \in 1\text{-descendants of } (k'_j)_j} \left( \prod_j \begin{pmatrix} s_j - l_j \\ s_j - k'_j \end{pmatrix} q(l_j)_j(U(\alpha)) \right)
\]

\[
= \sum_{\{l_j\}_{j} \in 1\text{-descendants of } (k'_j)_j} \left( \prod_j \begin{pmatrix} s_j - l_j \\ s_j - k'_j \end{pmatrix} \alpha^{\lambda(l_j)} q(l_j)_j(U(1)) \right) \quad \text{(by (2.14.2))}
\]

Therefore, each \( \alpha \in k^* \) satisfies the following polynomial in \( x \):

\[
\sum_{\{l_j\}_{j} \in 1\text{-descendants of } (k'_j)_j} \left( \prod_j \begin{pmatrix} s_j - l_j \\ s_j - k'_j \end{pmatrix} q(l_j)_j(U(1)) x^{\lambda(l_j)} \right) - \sum_{\{l_j\}_{j} \in 1\text{-descendants of } (k'_j)_j} \left( \prod_j \begin{pmatrix} s_j - l_j \\ s_j - k'_j \end{pmatrix} q(l_j)_j(U(1)) x^{\sum_j p^{f-1-j}(k'_j - l_j)} \right)
\]

If non-zero, this polynomial is of degree less than \( p^f - 1 \), with at least \(|k^*|\) distinct roots, a contradiction. Note that \( p^{f-1-j}(k'_j - l_j) = p^{f-1-m(l_j)} \) for some \( m(l_j) \in [0, f - 1] \).
Since $\chi_{1} \neq \chi_{2}$ does not equal any of the $\sum_{j} p^{j-1} \delta \cdot (k_{j} - l_{j})$ terms, the coefficient of $x^{\chi_{1}(k_{j})}$ is $\left( \prod_{j} \left( s_{j} - k_{j} \right) \right) q(k_{j}) (U(1))$ and it must equal 0. This implies that $q(k_{j}) | U = 0$ by (2.14.2).

Finally, we come to the last leg of the proof. Because $q(k_{j}) | U = 0$ for each $(k_{j})_{j} \neq (s_{j})_{j}$, $q(s_{j}) (U(\alpha + \beta)) = q(s_{j}) (U(\alpha)) + q(s_{j}) (U(\beta))$. Therefore $q(s_{j}) (U(p)) = pq(s_{j}) (U(1)) = 0$. However, $q(s_{j}) | U = K$ has dimension $\kappa_{K}^{|U| \cap \mathbb{K}}$ (from the definition of $C$ in Lemma 2.13). As $p$ is the uniformizer of $O_{K}$, $\kappa_{K}^{|U| (U(p))} \neq 0$, giving a contradiction.

\[ \square \]

3. Stack dimensions and extensions of $G_{K}$ characters

For this section, we take $p > 2$, so that the moduli stack of $G_{K}$-representations can be defined as in [CEGS1]. We record a fact from [DDR] and [Ste] that we will use repeatedly in this section. Suppose $\chi_{1}$ and $\chi_{2}$ are distinct $G_{K}$ characters such that the subspace of $\text{Ext}^{1}_{G_{K}[\chi_{2}, \chi_{1}]}$ corresponding to representations with non-Steinberg Serre weights both $\sigma$ and $\tau$ has dimension $d$. Suppose $\chi_{1}'$ and $\chi_{2}'$ are unramified twists of $\chi_{1}$ and $\chi_{2}$ respectively. If $\chi_{1}' \neq \chi_{2}'$, then the subspace of $\text{Ext}^{1}_{G_{K}[\chi_{2}', \chi_{1}']} \chi_{1}'$ that corresponds to representations with Serre weights $\sigma$ and $\tau$ also has dimension $d$. If on the other hand, $\chi_{1}' = \chi_{2}'$, then a $(d+1)$-dimensional subspace of $\text{Ext}^{1}_{G_{K}[\chi_{2}', \chi_{1}]}$ corresponds to representations with Serre weights $\sigma$ and $\tau$.

We now make a few definitions, before stating our main propositions relating dimensions of closed substacks of $\mathbb{Z}$ to vector space dimensions of extensions of $G_{K}$ characters.

**Definition 3.1.** Let $\chi_{1}$ and $\chi_{2}$ be a pair of $G_{K}$ characters. We say that a set $\mathcal{F}_{\chi_{1}, \chi_{2}}$ of $G_{K}$-representations with $\mathbb{F}$-coefficients is a family of representations if each representation in $\mathcal{F}_{\chi_{1}, \chi_{2}}$ is an extension of an unramified twist of $\chi_{2}$ by an unramified twist of $\chi_{1}$.

**Definition 3.2.** Consider $\mathbb{G}_{m} \times \mathbb{G}_{m}$ as parametrizing the unramified twists of $\chi_{1}$ and $\chi_{2}$ via the value of the unramified characters on $\text{Frob}_{K}$, as in [CEGS2, Sec. 3.3]. We say that the family $\mathcal{F}_{\chi_{1}, \chi_{2}}$ is of dimension $\leq d$ (resp. of dimension $d$) if there exists a dense open subset $W$ of $\mathbb{G}_{m} \times \mathbb{G}_{m}$ such that the following condition is satisfied: if $\chi_{1}'$ and $\chi_{2}'$ are unramified twists of $\chi_{1}$ and $\chi_{2}$ (respectively) corresponding to an $\mathbb{F}$-point of $W$, then the extensions in $\text{Ext}^{1}_{G_{K}[\chi_{2}', \chi_{1}]}$ giving elements of $\mathcal{F}_{\chi_{1}, \chi_{2}}$ form a subspace of dimension $\leq d$ (resp. of dimension $d$).

**Definition 3.3.** We say that two families $\mathcal{F}_{\chi_{1}, \chi_{2}}$ and $\mathcal{F}_{\chi_{1}', \chi_{2}'}$ are separated if $\chi_{1}'$ and $\chi_{2}'$ are both unramified twists of $\chi_{1}$ and $\chi_{2}$ respectively.

We recall some essential constructions from [CEGS2, Sec. 3.3, 3.4], as well as set some notation for use in the rest of this section. See [CEGS2] for missing details. Let $\mathcal{I}$ be a finite set indexing all possible pairs $(\mathfrak{m}_{i}, \mathfrak{n}_{i}) \in \mathbb{I}$ of Breuil-Kisin modules over $\mathbb{F}$ up to unramified twists. Let $T$ be the functor from Breuil-Kisin modules to $G_{K}$-representations ([CEGS1, Defn. 2.3.2, Para. 2.3.4]). For $i \in \mathcal{I}$, if $T(\mathfrak{m}_{i})$ and $T(\mathfrak{n}_{i})$ are different representations upon restriction to the inertia subgroup of $G_{K}$, set $A_{\mathcal{I}} = \mathbb{G}_{m} \times \mathbb{G}_{m}$. Else, interpreting $\mathbb{G}_{m} \times \mathbb{G}_{m}$ as encoding unramified twists of $T(\mathfrak{m}_{i})$ and $T(\mathfrak{n}_{i})$, remove the integral codimension 1 closed subscheme
Let \( \xi \) be a family \( F \) of representations that are not extensions of a character by itself. In order to obtain all reducible representations in the literal image, we consider the map \( \xi \) for each \( i \in I \), upon restriction to the inertia group of \( K \), \( T(\mathfrak{M}) = T(\mathfrak{N}) \). For each \( i \in I \), let \( C_i := \text{Spec} B_{i}^{\text{twist}} \times_{G_m} \mathcal{A}_i \) and denote by \( \gamma_i \) the restriction of \( \xi \) to \( C_i \).

Fix non-Steinberg Serre weights \( \sigma \) and \( \tau \). Let \( \mathcal{E} := \mathcal{Z}_\sigma \cap \mathcal{Z}_\tau \), \( Y_i := (\xi)^{-1} \mathcal{E} \cap \text{Spec} B_{i}^{\text{dist}} \) and \( X_i := \gamma_i^{-1} \mathcal{E} \). Let \( \pi_i \) be the structure map \( Y_i \to \text{Spec} A_{i}^{\text{dist}} \) and let \( X_i \) be the structure map \( X_i \to A_{i}^{\text{dist}} \).

**Proposition 3.4.** Let \( d \geq 0 \). Suppose all families of representations contained in \( \mathcal{E}(\mathcal{F}) \) are of dimension \( \leq d \), and moreover, \( \mathcal{E}(\mathcal{F}) \) contains at least one family of dimension \( d \). Then the following are true:

(i) \( \mathcal{E} \) has dimension \( d \).

(ii) If \( d > 0 \), the number of \( d \)-dimensional components in \( \mathcal{E} \) equals the number of \( d \)-dimensional pairwise separated families contained in \( \mathcal{E} \).

The proof of this proposition will use the following lemmas.

**Lemma 3.5.** Let \( \chi_1 \) and \( \chi_2 \) be fixed \( G_K \) characters. Suppose \( \mathcal{E}(\mathcal{F}) \) contains a family \( \mathcal{F}_{\chi_1, \chi_2} \) of representations of dimension \( d \). Suppose moreover that there is no other family of extensions of \( \chi_2 \) by \( \chi_1 \) contained in \( \mathcal{E}(\mathcal{F}) \) with dimension \( > d \). Let \( j \in I \) be such that \( T(\mathfrak{M}) \) is an unramified twist of \( \chi_2 \), while \( T(\mathfrak{N}) \) is an unramified twist of \( \chi_1 \). Then the following are true:

(i) The dimension of the scheme-theoretic image of \( Y_j \) is \( \leq d \).

(ii) If the dimension of the scheme-theoretic image of \( Y_j \) is \( d \), then there exists a dense open subset \( W_j \) of \( A_j^{\text{dist}} \) such that each closed point \( q \) of this dense open satisfies the following property: suppose the twists of \( \chi_1 \) and \( \chi_2 \) obtained after applying the unramified twists encoded by \( q \) are \( \chi_1' \) and \( \chi_2' \) respectively. Then each \( \mathcal{F} \)-extension of \( \chi_2' \) by \( \chi_1' \) contained in \( \mathcal{E}(\mathcal{F}) \) is in the literal image of \( Y_j \).

(iii) The number of \( d \)-dimensional components in the scheme-theoretic image of \( Y_j \) is at most 1.

**Proof.**

(i) Let \( q \) be a closed point of \( A_j^{\text{dist}} \), and after fixing an embedding \( \kappa(q) \to \overline{\mathcal{F}} \), let \( \overline{q} \) be the corresponding \( \overline{\mathcal{F}} \)-point of \( A_j^{\text{dist}} \). We denote the map on global sections by \( q^\# \) or \( \overline{q}_i^\# \), depending on whether we are viewing \( q_i \) as a \( \kappa(q_i) \)-point or as an \( \overline{\mathcal{F}} \)-point. By the construction of \( A_j^{\text{dist}} \), representations coming from \( Y_j(\overline{\mathcal{F}}) \) are never an extension of a character by itself. Therefore, the hypotheses in the statement of the Lemma force the image of \( \pi_j^{-1}(\overline{q})/(\overline{\mathcal{F}}) \)
to form a vector space necessarily of dimension $\leq d$. We have:

$$\dim_{\mathbb{F}} \pi^{-1}_j(\overline{\mathbb{F}}) = \dim_{\mathbb{F}} (\xi^{\text{dist}}(\pi^{-1}_j(\overline{\mathbb{F}}))) + \dim_{\mathbb{F}} \ker-\text{Ext}_{\mathcal{K}(\mathbb{F})}((\mathcal{M}^j_{\mathbb{F},\pi}(x)), \mathcal{M}^j_{\mathbb{F},\pi}(y))$$

$$\leq d + \dim_{\mathbb{F}} \ker-\text{Ext}_{\mathcal{K}(\mathbb{F})}((\mathcal{M}^j_{\mathbb{F},\pi}(x)), \mathcal{M}^j_{\mathbb{F},\pi}(y)).$$

Since the $\mathbb{F}$-points of $\pi^{-1}_j(\overline{\mathbb{F}})$ form a vector space, the reduced induced closed subscheme of $\pi^{-1}_j(q)$ must be be cut out by homogeneous linear equations in $(\text{Spec} \mathcal{B}_j)^{\text{dist}}$ and thus be irreducible of dimension equal to the $\mathbb{F}$-vector space dimension of $\pi^{-1}_j(\overline{\mathbb{F}})$.

Let $S$ be an irreducible component of $Y_j$. Denote by $\overline{\xi^{\text{dist}}(S)}$ the scheme-theoretic image of $S$. By [Sta, Tag 0DS4], there exists a dense set $U \subset S$ such that for any $p \in U(\mathbb{F})$, the dimension of the scheme-theoretic image of $S$ under $\xi^{\text{dist}}$ is given by:

$$\dim \overline{\xi^{\text{dist}}(S)} = \dim S - \dim_p(S_{\xi^{\text{dist}}(p)}) = \dim_p S - \dim_p(S_{\xi^{\text{dist}}(p)})$$

where,

$$\dim_p S \leq \dim \pi^{-1}_j(\pi_j(p)) + \dim (\pi_j(S))$$

Restrict $U$ further if necessary so that it is disjoint from other irreducible components of $Y_j$. Then for $p$ a closed point in $U$, since $\pi^{-1}_j(\pi_j(p))$ is irreducible, it is contained entirely in some irreducible component of $Y_j$. By the conditions on $U$, $\pi^{-1}_j(\pi_j(p)) \subset S$. Therefore, $\dim_p(S_{\xi^{\text{dist}}(p)}) = \dim_p(Y_j)_{\overline{\xi^{\text{dist}}(p)}}$. Since $\xi^{\text{dist}}|_{Y_j}$ factors through the quotient $Y_j/(\mathcal{G}_m \times \mathcal{G}_m)$, we obtain:

$$\dim_p(S_{\xi^{\text{dist}}(p)}) = \dim_p(Y_j)_{\overline{\xi^{\text{dist}}(p)}} = \dim \ker-\text{Ext}_{\mathcal{K}(\mathbb{F})}((\mathcal{M}^j_{\mathbb{F},\pi}(x)), \mathcal{M}^j_{\mathbb{F},\pi}(y)) + 2$$

Therefore the dimension of scheme-theoretic image of $S$ is $\leq d - (2 - \dim (\pi_j(S))) \leq d$.

(ii) If this dimension is $d$, then $\overline{\pi_j(S)}$ must be 2-dimensional, and for each $p \in U(\mathbb{F})$, $\xi^{\text{dist}}(\pi^{-1}_j(\pi_j(p)))(\mathbb{F})$ gives a $d$-dimensional vector space. Since $\pi_j(U)$ is constructible and $\pi_j(U) = \pi_j(S) = A^j_{\text{dist}}$, it contains a dense open subset $W_j$ of $A^j_{\text{dist}}$. Let $q$ be a closed point of $W_j$. Let $\chi'_1$ and $\chi'^2_2$ be unramified twists of $\chi_1$ and $\chi_2$ respectively corresponding to the unramified twists of $\mathcal{W}$ and $\mathcal{W}$ specified by $q$. Viewing $q$ as an $\mathbb{F}$-point $\mathfrak{F}$ by fixing an embedding $\kappa(q) \to \mathfrak{F}$, $\xi^{\text{dist}}(\pi^{-1}_j(\mathfrak{F}))$ is a $d$-dimensional vector space space of extensions of $\chi'_2$ by $\chi'^1_1$, which is maximal for containment in $\mathcal{E}(\mathfrak{F})$. Therefore the image of $Y_j|_q$ under $\xi^{\text{dist}}$ contains all extensions of $\chi'^2_2$ by $\chi'^1_1$ that are contained in $\mathcal{E}(\mathfrak{F})$.

(iii) For $i \in \{1, 2\}$, suppose $S^i$ is an irreducible component with a scheme-theoretic image of dimension $d$. Let $U^i$ be the dense open subset of $S^i$ obtained by taking the complement of all other irreducible components of $Y_j$. Therefore, $\pi_j(U^i) = \pi_j(S^i) = A^i_{\text{dist}}$. Since $\pi_j(U^i)$ is constructible, it contains a dense open $W^i$ of $A^i_{\text{dist}}$. Let $W = W^1 \cap W^2$. If $q$ is a closed point of $W$, $\pi^{-1}_j(q)$ is irreducible and contained entirely in at least one
Lemma 3.5. For any field-valued point \( q \) of \( A' \), the fiber at \( q \) is irreducible. We have:

We now deal with the situation where we have extensions of a character by itself. We show that in this case, the image of \( X_j \) will have dimension strictly less than \( d \). Morally, the idea is that an extension of a character by itself has an extra set of endomorphisms given by upper unipotent matrices, which reduce the dimension by 1.

**Lemma 3.6.** Fix \( \chi \), a \( G_K \) character. Suppose \( \mathcal{E}(F) \) contains a family \( \mathcal{F}_{\chi} \) of representations of dimension \( d \). Suppose moreover that there is no other family of extensions of \( \chi \) by itself contained in \( \mathcal{E}(F) \) with dimension \( > d \). Let \( j \in I' \) be such that \( T(F(\mathfrak{M})) \) and \( T(F(\mathfrak{M}'(\mathfrak{V})) \) are unramified twists of \( \chi \). Then the dimension of the \( d \)-dimensional scheme-theoretic image of \( X_j \) is \( < d \).

**Proof.** For any field-valued point \( q \) of \( A' \), let \( \tilde{\psi}_q \) denote the composition of the fiber at \( q \) with the natural localization map to obtain \( \mathrm{étale}-\varphi \) modules from Breuil-Kisin modules. We have:

We denote by \( \ker-V_i.q \) the kernel of \( \tilde{\psi}_q \). We will often use \( \tilde{\psi}_q \) to also denote the base change of \( \tilde{\psi}_q \) by any \( \kappa(q) \)-algebra \( A \). This usage will be clear from the context.

Now let \( q \) be a closed point of \( A' \), and after fixing an embedding \( \kappa(q) \rightarrow F \), let \( \overline{q} \) be the corresponding \( \overline{F} \)-point of \( A' \). We denote the corresponding map on global sections by \( \overline{q}^\# \) or \( \overline{q}^\#_i \), depending on whether we are viewing \( q_i \) as a \( \kappa(q)_i \)-point or as an \( \overline{F} \)-point. The hypotheses in the statement of the Lemma force the image of \( \gamma_j^{-1}(q)(\overline{F}) \) to form a vector space necessarily of dimension \( \leq d + 1 \). As in the proof of Lemma 3.5, we have:

As before, \( \gamma_j^{-1}(q) \) is irreducible of dimension equal to the \( \overline{F} \)-vector space dimension of \( \gamma_j^{-1}(q)(\overline{F}) \).

Let \( S \) be an irreducible component of \( X_j \). Denote by \( \overline{\alpha_j(S)} \) the scheme-theoretic image of \( S \). By [Sta, Tag 0DS4], there exists a dense set \( U \subset S \) such that for any \( p \in U(\overline{F}) \), the dimension of the scheme-theoretic image of \( S \) under \( \alpha_j \) is given by:

where,

Restrict \( U \) further if necessary so that it is disjoint from other irreducible components of \( X_j \). Let \( p \) be a closed point in \( U \) and let \( q = \gamma_j(p) \). Since \( \gamma_j^{-1}(\gamma_j(p)) \) is irreducible, it is contained entirely in some irreducible component of \( X_j \). By the conditions on \( U \), \( \gamma_j^{-1}(q) \subset S \). Therefore, \( \dim_p S_{\alpha_j(p)} = \dim_p (X_j)_{\alpha_j(p)} = \dim_p (C_j)_{\alpha_j(p)} \).
For each idempotent \(e_i\) of \(\mathcal{F}_F\), fix a basis of \(\mathcal{M}_i^j\) so that \(\mathcal{M}^j\) can be presented in the form described in \([CEGS2, \text{Lem. 4.1.1}]\). This naturally gives a fixed basis for each \(\mathcal{M}_i\), where \(\mathcal{M}_i^j\) is the unramified twist of \(\mathcal{M}^j\) given by \(\lambda\). After twisting if necessary, \(\mathcal{M}_i^j\) is isomorphic to \(\mathcal{M}_i^{(3.6.3)}\). Fix such an isomorphism.

Let \(\mathcal{B}\) be an extension of \(\mathcal{M}_i^{(3.6.3)}\) by \(\mathcal{M}_i^{(3.6.1)}\). It is a direct sum of \(\mathcal{M}_i^{(3.6.1),\#(x)}\) and \(\mathcal{M}_i^{(3.6.3),\#(y)}\) as \(\mathcal{A}_i[\text{Gal}(K'/K)]\) modules. Fix such a decomposition. Using the fixed isomorphism between \(\mathcal{M}_i^{(3.6.3),\#(x)}\) and \(\mathcal{M}_i^{(3.6.1),\#(y)}\), \(\mathcal{B}\) can be written as an extension of \(\mathcal{M}_i^{(3.6.1),\#(x)}\) by itself, and in particular as a direct sum of two copies of \(\mathcal{M}_i^{(3.6.1),\#(x)}\) as \(\mathcal{A}_i[\text{Gal}(K'/K)]\) modules. Thus, the Frobenius action can be described by the data of \(\psi\) in terms of upper triangular matrices \(F_i\), where each \(F_i\) describes the Frobenius map \(\varphi^*(\mathcal{B}[\frac{1}{u}]_{l-1}) \to (\mathcal{B}[\frac{1}{u}]_{l})\) with respect to the fixed ordered basis obtained from the two copies of \(\mathcal{M}_i^{(3.6.1),\#(x)}\).

Let \(G = \mathbb{G}_{m,q} \times_{\mathbb{G}_{q}} \mathbb{G}_{m,q} \times_{\mathbb{G}_{q}} \mathbb{G}_{q}\). Given \(\lambda, \mu, a \in G(A)\), let

\[
\lambda_j(\mu, \lambda, a) := \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

If \((\lambda, \mu, a)\) are clear, we will write \(\lambda_j(\mu, \lambda, a)\) as simply \(\lambda_j\). The set of \(\lambda_j\)'s can be viewed as giving base change data modifying \(F_i\) to \(F_i\psi(\lambda_{l-1})^{-1} = F_i F_{l-1}^{-1}\).

By construction, \((C_j)_{\#} = \text{Spec} A_j^\prime[\mathcal{V}_{i,j}]\). The following map is well-defined:

\[
\begin{align*}
C_j\times_{\mathbb{G}_{q}} G & \to C_j\times_{\mathbb{G}_{q}} G \\
(v, \lambda, \mu, a) & \mapsto (v, \lambda \mu^{-1} v, \{\lambda_j(\mu, \lambda, a)\}_l)
\end{align*}
\]

where, \(\{\lambda_j(\mu, \lambda, a)\}_l\) are to be interpreted as giving the base change matrices that modify the Frobenius matrices coming from the data of \(\psi(v)\) (in the manner described above), to give the data of Frobenius matrices coming from the data of \(\psi(\lambda \mu^{-1} v)\).

Therefore, there exists a map \([C_j\times_{\mathbb{G}_{q}} G] \to \mathbb{Z}\). Thus

\[
(3.6.3) \quad \dim F\mathcal{S}_{\alpha_j(p)} = \dim F(\mathcal{C}_j)_{\alpha_j(p)} \\
\geq \dim \mathcal{S}_{\alpha_j(p)} \ker V_{i,j} + \dim G \\
= \dim \mathcal{S}_{\alpha_j(p)} \ker V_{i,j} + 3
\]

Using (3.6.1) to (3.6.3), \(\dim \alpha_j(S) \leq d - 1\). \(\square\)

**Lemma 3.7.** Let \(\chi_1\) and \(\chi_2\) be fixed \(G_K\) characters. Suppose \(E(\mathbb{F})\) contains a family \(\mathcal{F}_{\chi_1, \chi_2}\) of representations of dimension \(d\). Suppose moreover that there is no other family of extensions of \(\chi_2\) by \(\chi_1\) contained in \(E(\mathbb{F})\) with dimension \(> d\). Let \(D\) be the subset of \(I\) such that \(i \in D\) if and only if \(T(\mathcal{M}^i)\) is an unramified twist of \(\chi_2\), while \(T(\mathcal{M}^i)\) is an unramified twist of \(\chi_1\). Then the following can be said about the scheme-theoretic image of \(\prod_{i \in D} Y_i\):

(i) Its dimension is \(d\).

(ii) It contains exactly one \(d\)-dimensional irreducible component.

**Proof.**

(i) Let \(i \in D\). By the construction of \(A_i\), representations coming from \(Y_i(\mathbb{F})\) are never an extension of a character by itself. Therefore, for each unramified twist of \(\chi_1\) and \(\chi_2\) coming from twisting \(\mathcal{M}^i\) and \(\mathcal{N}^i\) by
unramified characters corresponding to closed points of $A^\text{dist}_j$, the space of extensions contained in $E(\overline{F})$ is precisely $d$-dimensional.

As $\pi_i$ is of finite type over an integral scheme, there exists a dense open $W_i$ of $\text{Spec} A^\text{dist}_i$ such that $\mathcal{O}_{Y_i}|_{\pi_i^{-1}W_i}$ is locally free over $\mathcal{O}_{W_i}$. Let $W := \cap W_i$.

Replace $\chi_1$ and $\chi_2$ by a specific choice of their unramified twists chosen in the following way. For each $i \in D$, if $q_i$ is the closed point of $\mathbb{G}_m \times \mathbb{G}_m$ parameterizing the corresponding choice of unramified twists of $T(\mathcal{M})$ and $T(\mathcal{N})$, then $q_i$ is a closed point in $W \subset A^\text{dist}_i \subset \mathbb{G}_m \times \mathbb{G}_m$. Since $|D|$ is finite, such a choice of unramified twists is possible and corresponds to choosing any closed point in a dense open subscheme of $\mathbb{G}_m \times \mathbb{G}_m$. Denote by $V$ the $d$-dimensional subspace of $\text{Ext}_G^1(\chi_2, \chi_1)$ that lies in $E(\overline{F})$.

Note that $\kappa(q_i) = \kappa(q_j)$ for any $i, j \in D$. We fix an embedding of this residue field into $\overline{F}$ to view each $q_i$ as an $\overline{F}$-point $\overline{q}_i$. We denote the map on global sections by $q_i^\#$ or $q_i^\#$, depending on whether we are viewing $q_i$ as a $\kappa(q_i)$-point or as an $\overline{F}$-point. The closed points of $(Y_j)_\pi$ form an $\overline{F}$-vector space mapping onto an $\overline{F}$-vector space $V_i$ inside $E(\overline{F})$. As $V \subset \cup V_i \subset V$, there exists such that $V = V_j$. Therefore,

$$
\dim (Y_j)_\pi(\overline{F}) = \dim V_j + \dim \ker \text{Ext}_{\overline{F}}(\mathcal{M}_j^{\pi_j}(x), \mathcal{N}_j^{\pi_j}(y)) \\
= \dim V_j + \dim \ker \text{Ext}_{\kappa(q_j)}(\mathcal{M}_j^{\kappa(q_j), q_j^\#}(x), \mathcal{N}_j^{\kappa(q_j), q_j^\#}(y))
$$

Since $(Y_j)_\pi(\overline{F})$ gives a vector space, $(Y_j)_\pi$ (and hence $(Y_j)_q$) are irreducible of dimension $d + \dim \ker \text{Ext}_{\kappa(q_j)}(\mathcal{M}_j^{\kappa(q_j), q_j^\#}(x), \mathcal{N}_j^{\kappa(q_j), q_j^\#}(y))$.

By flatness over $W$, $\dim Y_j|_W = \dim (Y_j)_q + 2$. Therefore, there exists an irreducible component $S$ of $Y_j|_W$ with dimension $\dim (Y_j)_q + 2$. Denote by $\xi^\text{dist}(S)$ the scheme-theoretic image of $S$. As in the proof of Lemma 3.5, there exists a dense open subset $U$ of $S$, such that for all $p \in U$,

$$
\dim \xi^\text{dist}(S) = \dim S - \dim \ker \text{Ext}_{\kappa(p)}(\mathcal{M}_j^{\kappa(p), p}, \mathcal{N}_j^{\kappa(p), p}) + 2
$$

Therefore, the dimension of $\xi^\text{dist}(S)$, and of $Y_j$, is precisely $d$ (it cannot exceed $d$ by Lemma 3.5).

(ii) Suppose there exists $j' \in D$, distinct from $j$, such that the scheme-theoretic image of $Y_{j'}$ is also $d$-dimensional. Let $W_j$ and $W_{j'}$ be the dense open subsets of $A^\text{dist}_j$ and $A^\text{dist}_{j'}$ respectively described in Lemma 3.5(ii). We may assume (after removing some codimension 1 subschemes if necessary) that the $\overline{F}$-points in the literal image of $Y_j|_{\pi_j^{-1}W_j}$ and of $Y_{j'}|_{\pi_j^{-1}W_{j'}}$ are the same. Replacing $A^\text{dist}_j$ with $W_j$ and $A^\text{dist}_{j'}$ with $W_{j'}$, and applying the arguments from part (i), we see that the scheme-theoretic image of $Y_j|_{\pi_j^{-1}W_j}$ is $d$-dimensional and the same is true for $Y_{j'}|_{\pi_j^{-1}W_{j'}}$. Since the literal images contain the same $\overline{F}$-points, the scheme-theoretic images must be the same and contain the unique (by Lemma 3.5 (iii)) $d$-dimensional component in the scheme-theoretic images of $Y_j$ and of $Y_{j'}$. Therefore the $d$-dimensional
Proof of Proposition 3.4. By construction, each reducible representation is in the literal image of either one of the $Y_i$’s or one of the $X_i$’s. Moreover, irreducible representations contribute to zero-dimensional substacks of $Z$ by the description in the proof of [CEGS2, Prop. 5.2.12]. The first statement follows from Lemmas 3.5 and 3.6, which also show that each $d$-dimensional family contains precisely one $d$-dimensional component in its closure.

Now let $d > 0$. To see that the $d$-dimensional components contained in the closures of pairwise separated $d$-dimensional families are pairwise distinct, assume otherwise and let $Y$ be a top dimensional component in $E$ contained in the closure of two separated $d$-dimensional families $F_{X_1, X_2}$ and $F_{X'_1, X'_2}$. Then $Y$ is the closure of a constructible set whose finite type points are contained in $F_{X_1, X_2}$, and another constructible subset whose finite type points are contained in $F_{X'_1, X'_2}$. Therefore, there exists a dense open inside $Y$ whose finite type points are contained in both $F_{X_1, X_2}$ and $F_{X'_1, X'_2}$, a contradiction.

We now state a proposition on the dimension of the closure of irreducible representations. Irreducible representations are the finite type points in the image of the finitely many $Y(\mathfrak{M}, \mathfrak{N})$’s constructed in [CEGS2, Lem. 5.2.7].

Proposition 3.8. The dimension of the scheme-theoretic image of each $Y(\mathfrak{M}, \mathfrak{N})$ is at most 0. Given an irreducible representation $\pi$ in $E(\mathfrak{M})$, let $Y(\mathfrak{M}_1, \mathfrak{N}_1)$ be all the suitable pairs such that the literal image of $Y(\mathfrak{M}_1, \mathfrak{N}_1)(\mathfrak{F})$ contains an unramified twist of $\pi$. Then the number of 0-dimensional irreducible components in the scheme-theoretic image of $\bigcup Y(\mathfrak{M}_i, \mathfrak{N}_i)$ is precisely one.

Proof. The unramified twists of each $(\mathfrak{M}_i, \mathfrak{N}_i)$ are parameterized by $\mathbb{G}_m$ instead of $\mathbb{G}_m \times \mathbb{G}_m$. The dimension of automorphisms of a representation drops by 1 as well because we are looking at irreducible representations. Apart from this, the proof is similar to the reducible case and omitted.

4. Computations of Serre weights

4.1. Linear algebraic formulation for Serre weights of $G_K$-representations.

In the subsequent text, we will write our $f$-tuples with decreasing indices. We recall some relevant results from [Ste] and [DDR] below. Let $\pi$ be a $G_K$ representation of the form $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \in \text{Ext}^1_{G_K}(\mathfrak{F}(\chi_2), \mathfrak{F}(\chi_1))$.

$V_{t,s}^\vee$ is a Serre weight of $\pi^\vee$ if and only if $V_{t,s}$ is a Serre weight of $\pi$ in the sense of [Ste] and [DDR]. Thus $V_{t,s}^\vee$ is a Serre weight of $\pi^\vee$ iff the following conditions are met:

1. There exists a subset $J$ of $T$, and for each $i \in T$ there exists $x_i \in [0, e - 1]$ such that:

\begin{equation}
\chi_1|_{I_K} = \prod_{i \in T} \omega_i^{x_i} \prod_{i \in J} \omega_i^{s_i+1+x_i} \prod_{i \in J^c} \omega_i^{x_i}.
\end{equation}
and
\begin{equation}
\chi_2|_{I_K} = \prod_{i \in T} \omega_i^{\ell_i} \prod_{i \in J} \omega_i^{e-1-x_i} \prod_{i \in F} \omega_i^{s_i+e-x_i}.
\end{equation}

(2) $\pi \in L_{V_{e,s}}(\overline{F}(\chi_1), \overline{F}(\chi_2)) \subset \text{Ext}^1_{\overline{G}_K}(\overline{F}(\chi_2), \overline{F}(\chi_1))$, where $L_{V_{e,s}}(\overline{F}(\chi_1), \overline{F}(\chi_2))$ (or just $L_{V_{e,s}}$ if $\chi_1$ and $\chi_2$ are understood) is a particular distinguished subspace.

Assuming (4.1.1) and (4.1.2), we now note the recipe for obtaining $L_{V_{e,s}}$ as given in [Ste], with slight differences in notation.

We first write $\chi_2|_{I_K} = \prod_{i \in T} \omega_i^{\ell_i} \prod_{i \in J} \omega_i^{m_i}$ for the unique $m_i \in \{0, p - 1\}$ with not all $m_i$ equal to $p - 1$. Let $\mathcal{S}$ be the set of $f$-tuples of non-negative integers $(a_f, 0_f, ..., a_0)$ satisfying $\chi_2|_{I_K} = \prod_{i \in T} \omega_i^{\ell_i} \prod_{i \in J} \omega_i^{m_i}$ and $a_i \in \{0, e - 1\} \cup |s_i + 1, s_i + e| _r$ for all $i$. Evidently, $\mathcal{S}$ is non-empty.

For $i \neq f - 1$, let $v_i$ be the $f$-tuple $(0, ..., 0_p, -1, 0, ..., 0)$ with $-1$ in $i$ position, $p$ in $i + 1$ position and 0 everywhere else. Let $v_{f-1}$ be $(-1, 0, ..., 0_p)$. Then there exists a subset $A \subset T$ such that
\begin{equation}
(m_{f-1}, ..., m_0) + \sum_{i \in A} v_i \in \mathcal{S}
\end{equation}

**Definition 4.2.** Define $A_{\text{min}}$ to be the minimal $A$ satisfying (4.1.3), in the sense that it is contained in any other subset of $T$ satisfying (4.1.3).

**Definition 4.3.** Given $(m_{f-1}, ..., m_0)$ and $A_{\text{min}}$ as above.
\begin{equation}
y_{f-1}, ..., y_0 := (m_{f-1}, ..., m_0) + \sum_{i \in A_{\text{min}}} v_i \in \mathcal{S}
\end{equation}

\begin{equation}
z_i := s_i + e - y_i \text{ for all } i
\end{equation}

The indices of $y_i$'s and $z_i$'s will be interpreted to be elements of $\mathbb{Z}/f\mathbb{Z}$.

**Remark 4.4.** $\chi_1 = \prod_{i \in T} \omega_i^{\ell_i} \prod_{i \in J} \omega_i^{m_i}$, $\chi_2 = \prod_{i \in T} \omega_i^{\ell_i} \prod_{i \in J} \omega_i^{m_i}$ and $\chi_2^{-1} \chi_1 = \prod_{i\in T} \omega_i^{s_i+e-y_i}$.

**Definition 4.5.** If $y_i \geq s_i + 1$, let $I_i := [0, z_i - 1]$, and if $y_i < s_i + 1$, let $I_i := \{y_i\} \cup |s_i + 1, z_i - 1|$. Here the interval $[0, z_i - 1]$ is interpreted as the empty set if $z_i - 1 < 0$. We follow similar convention for $|s_i + 1, z_i - 1|$ when $z_i - 1 < s_i + 1$.

**Remark 4.6.** When $e = 1$, $I_i = \{0\}$ if $y_i = 0$ and $I_i = \emptyset$ if $y_i = s_i + 1$.

**Remark 4.7.** If $y_i \geq s_i + 1$, then $|I_i| \leq e - 1$ with equality if and only if $y_i = s_i + 1$. If $y_i < s_i + 1$, then since $z_i \leq s_i + e$, $|I_i| \leq e$ with equality if and only if $z_i = s_i + e$ or equivalently, $y_i = 0$.

Suppose $\chi_2^{-1} \chi_1 = \prod_{i \in T} \omega_i^{a_i}$ for $a_i \in [1, p]$ and not all $a_i = p$. We will extend the indices of the $a_i$ to all of $\mathbb{Z}$ by setting $a_j = a_j' \text{ if } j \equiv j' \mod f$. We call the tuple $(a_{j-1}, ..., a_0)$ the tame signature of $\chi_2^{-1} \chi_1$. $\text{Gal}(k/\mathbb{F}_p) = \langle \text{Frob} \rangle \cong \mathbb{Z}/f\mathbb{Z}$ acts on such tuples $(a_{j-1}, ..., a_0)$ via
\begin{equation}
\text{Frob} \cdot (a_{j-1}, ..., a_0) = (a_0, a_{j-1}, ..., a_1)
\end{equation}

Let $f'$ be the cardinality of the orbit of $(a_{j-1}, ..., a_0)$ under the action of $\text{Gal}(k/\mathbb{F}_p)$, and let $f'' := f/f'$. 


Definition 4.8. Let \( n_i \in \{0, p^f - 1\} \) be such that \( \chi_2^{-1} \chi_1 |_{I_K} = \omega_i^{n_i} |_{I_K} \).

Note that \( n_i = n_j \) if \( i \equiv j \mod f' \).

Definition 4.9. For \( i \in T \), let
\[
(4.9.1) \quad \lambda_i := \sum_{j=0}^{f-1} (z_{i+j+1} - y_{i+j+1}) p^{f-1-j}
\]
\[
(4.9.2) \quad \xi_i := (p^f - 1) z_i + \lambda_i
\]

Definition 4.10. Let \( J_{\nu}^{AH}(\chi_1, \chi_2) \) denote the subset of all \( \alpha = (m, \kappa) \in \mathbb{Z} \times \{0, ..., f'' - 1\} \) satisfying:

(i) \( \exists i \in T \) and \( u \in I_i \), such that if \( \nu \) is the \( p \)-adic valuation of \( \xi_i - u(p^f - 1) \), then
\[
(4.10.1) \quad m = \frac{\xi_i - u(p^f - 1)}{p^\nu}
\]

(ii) Let \( i_m \in \{0, ..., f' - 1\} \) be such that \( m \equiv n_{i_m} \mod p^f - 1 \). It exists because by the above, \( p^\nu m \equiv n_i \mod p^f - 1 \), and so, \( m = n_{i-\nu} \). We require that \( \kappa \) satisfies
\[
(4.10.2) \quad i_m + \kappa f' \equiv i - \nu \mod f
\]

By [Ste, Prop. 3.13], for each \( i \in T \) and \( u \in I_i \), there exists a unique \( \alpha \) satisfying the conditions above. By [Ste, Thm. 3.16], each \( \alpha \) in \( J_{\nu}^{AH}(\chi_1, \chi_2) \) gives a unique basis element of \( L_{V_{\nu},t}^\times(\chi_1, \chi_2) \), denoted as \( c_{i,\alpha} \). \( L_{V_{\nu},t}^\times(\chi_1, \chi_2) \) is the span of \( c_{i,\alpha} \)'s together with additional, distinguished basis elements \( c_{un} \) if \( \chi_2^{-1} \chi_1 \) is trivial and \( c_{tr} \) if \( \chi_2^{-1} \chi_1 \) is cyclotomic. \( \prod_{i \in T} \omega^{-t_i} \otimes \chi_2 \) is unramified and \( s_i = p - 1 \) for all \( i \). A consequence of these results is that
\[
(4.10.3) \quad \dim_{\mathbb{F}} L_{V_{\nu},t}^\times(\chi_1, \chi_2) = \sum_{i \in T} |I_i| + \delta
\]

where \( \delta \) depends on the situation and could be 0 or 1 for \( p > 2 \), and 0, 1 or 2 for \( p = 2 \). It is always 0 if \( \chi_2^{-1} \chi_1 \) is neither trivial nor cyclotomic.

Consider two Serre weights \( V_{\nu,\tilde{r}} \) and \( V_{\nu,\tilde{r'}} \). Suppose there exist subsets \( J \) and \( J' \) of \( T \), and for each \( i \in T \), there exist \( x_i, x'_i \in [0, e - 1] \) such that:
\[
(4.10.4) \quad \chi_1 |_{I_K} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{s_i + 1 + x_i} \prod_{i \in J'} \omega_i^{x'_i} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{s_i + 1 + x_i} \prod_{i \in J'} \omega_i^{x'_i}
\]
and
\[
(4.10.5) \quad \chi_2 |_{I_K} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{e_1 - x_i} \prod_{i \in J'} \omega_i^{s_i + e - x_i} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{e_1 - x_i} \prod_{i \in J'} \omega_i^{s_i + e - x_i}
\]

Then a basis for the intersection of \( L_{V_{\nu,\tilde{r}},t}^\times(\chi_1, \chi_2) \) with \( L_{V_{\nu,\tilde{r'}},t}^\times(\chi_1, \chi_2) \) is given by \( c_{\alpha} \) for \( \alpha \in J_{\nu}^{AH}(\chi_1, \chi_2) \cap J_{\nu}^{AH}(\chi_1, \chi_2) \), together with \( c_{un} \) and/or \( c_{tr} \) if \( \chi_2^{-1} \chi_1 \) is trivial and/or cyclotomic with some additional conditions.
When \( e = 1 \), there is another algorithm to specify a basis of \( L_{\nu}^{\text{t,s}}(\chi_1, \chi_2) \), given in [DDR]. We recall some essentials of this algorithm because it will be convenient/shorter to use it for some of the calculations in the unramified case.

**Definition 4.11.** Let \( J_{\text{max}} := \{ i \in \mathbb{Z}/f\mathbb{Z} | y_i = 0 \} \), where \( y_i \) are as defined in Definition 4.3.

**Definition 4.12.** Let \( (a_{j-1}, ..., a_0) \) be the same signature of \( \chi_2^{-1} \chi_1 \). The function \( \delta : \mathbb{Z} \to \mathbb{Z} \) is defined in the following way: For \( j \in \mathbb{Z} \), \( \delta(j) = j \) unless \( (a_{i-1}, a_{i-2}, ..., a_j) = (p, p - 1, ..., p - 1) \) for some \( j < i \), in which case \( \delta(j) = i \). When \( j = i - 1 \), the condition \( (a_{i-1}, a_{i-2}, ..., a_j) = (p, p - 1, ..., p - 1) \) is interpreted as \( a_j = p \).

\( \delta \) induces a function \( \mathbb{Z}/f\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z} \), also denoted by \( \delta \).

Let \( J \) be a subset of \( \mathbb{Z}/f\mathbb{Z} \). If \( \delta(J) \subset J \), \( \mu(J) := J \). Else choose some \( [i_1] \in \delta(J) \setminus J \) and let \( j_1 \) be the largest integer such that \( j_1 < i_1, [j_1] \in J \) and \( \delta(j_1) = i_1 \). If \( J = \{ [j_1], ..., [j_1] \} \) with \( j_1 > j_2 > ... > j_r > j_{r - 1} > f \), define \( i_\kappa \) for \( \kappa \in [2, r] \) inductively as follows:

\[
i_\kappa = \begin{cases} 
\delta(j_\kappa), & \text{if } i_{\kappa - 1} > \delta(j_\kappa) \\
\delta(j_\kappa), & \text{otherwise}
\end{cases}
\]

Then \( \mu(J) := \{ [i_1], ..., [i_r] \} \).

When \( e = 1 \), \( L_{\nu}^{\text{t,s}}(\chi_1, \chi_2) \) has a basis given by certain elements of \( \text{Ext}^1_{\mathbb{O}_K} (\overline{\mathbb{F}}(\chi_2), \overline{\mathbb{F}}(\chi_1)) \) indexed by \( \tau \in \mu(J_{\text{max}}) \) along with \( c_{\text{un}} \) and/or \( c_{\tau} \) if \( \chi_2^{-1} \chi_1 \) is trivial and/or cyclotomic with some additional conditions.

We now state the criterion for determining the Serre weights associated to an irreducible \( G_K \) representation, when \( K = \mathbb{Q}_p \). It can be stated for arbitrary \( K \), but this paper will only need the case \( K = \mathbb{Q}_p \).

Let \( \overline{\tau} \) be an irreducible \( \mathbb{Q}_p \)-representation. Let \( \eta_1 \) and \( \eta_2 \) be the two level 2 fundamental characters of \( \mathbb{I}_{\mathbb{Q}_p} \). \( V_{\nu}^{\text{t,s}} \) is a Serre weight of \( \overline{\tau} \) if \( \overline{\tau} = \eta_1^{t+s} \eta_2 \eta_1^{r+s} \).

**4.13. Translation from Linear Algebra to Geometry of Stacks.** Let \( V_{\nu, t, s} \) and \( V_{\nu', t', s'} \) be non-isomorphic, non-Steinberg Serre weights. Let \( -\tilde{t} - \tilde{s} = \tilde{d} \) and \( -\tilde{t}' - \tilde{s}' = \tilde{d}' \), so that \( V_{\nu, t, s} = V_{d, s}^{\nu} \) and \( V_{\nu', t', s'} = V_{d', s'}^{\nu'} \).

The closure of irreducible \( \overline{\tau} \)-representations is 0-dimensional. Thus, unless \( K = \mathbb{Q}_p \), we only need to consider closures of families of reducible representations in order to detect codimension 1 intersections between irreducible components. On the other hand, if \( K = \mathbb{Q}_p \), then by Proposition 3.8, we additionally need to consider when \( Z_{V_{\nu, t, s}} \cap Z_{V_{\nu', t', s'}} \) contains irreducible finite type points. This last point is dealt with easily.

**Lemma 4.14.** When \( K = \mathbb{Q}_p \), \( Z_{V_{\nu, t, s}} \cap Z_{V_{\nu', t', s'}} \) contains irreducible finite type points if and only if \( s' = p - 1 - s \) and \( t' \equiv t + s \pmod{p - 1} \).

**Proof.** By the algorithm for computing Serre weights, we need to determine when

\[
\eta_1^{t+s} \eta_2 \eta_1^{r+s} = \eta_1^{t'+s} \eta_2 \eta_1^{r'+s'}.
\]

Since \( V_{\nu, t, s} \) and \( V_{\nu', t', s'} \) are non-isomorphic and non-Steinberg, the relationship between \( (t, s) \) and \( (t', s') \) follows immediately. \( \square \)
Remark 4.15. The criterion in the statement of Lemma 4.14 is the same as that in Proposition 2.1(ii)(b).

Using Proposition 3.4, we can state a sufficient (and necessary when \( K \neq \mathbb{Q}_p \)) condition for \( \mathcal{Z}_{V_{\mu, \nu}} \cap \mathcal{Z}_{V_{\rho, \eta}} \) to be codimension 1: there exist \( G_K \) characters \( \chi_1 \) and \( \chi_2 \) so that after replacing \( \chi_1 \) and \( \chi_2 \) by generic unramified twists, the subspace \( \{ \mathbf{F} | V_{\mu, \nu}, V_{\rho, \eta} \in W(\mathbf{F}) \} \subset \text{Ext}^1_{G_K}(\mathbf{F}(\chi_1^{-1}), \mathbf{F}(\chi_2^{-1})) \) has dimension \( ef - 1 \). By generic unramified twists we mean that if we let \( G_m \times G_m \) parametrize the unramified twists of \( \chi_1 \) and \( \chi_2 \) via the value of the unramified characters on \( \text{Frob}_K \), then the statement is true for the points of a dense open subset of \( G_m \times G_m \). Equivalently, \( L_{V_{\mu, \nu}}(\chi_1, \chi_2) \cap L_{V_{\rho, \eta}}(\chi_1, \chi_2) \subset \text{Ext}^1_{G_K}(\mathbf{F}(\chi_2), \mathbf{F}(\chi_1)) \) is spanned by \( ef - 1 \) basis elements excluding \( c_{un} \) and \( c_{tr} \).

Therefore, we must find \( G_K \) characters \( \chi_1 \) and \( \chi_2 \) such that there exist subsets \( J \) and \( J' \) of \( T \), and for each \( i \in T \), there exist \( x_i, x'_i \in [0, e - 1] \) such that (4.10.4) and (4.10.5) are satisfied. We next require that \( |J_{V_{\mu, \nu}}^H(\chi_1, \chi_2)| \cap J_{V_{\rho, \eta}}^H(\chi_1, \chi_2)| = ef - 1 \). This can happen in one of two ways.

Definition 4.16. We say that a pair of Serre weights \( V_{\mu, \nu} \) and \( V_{\rho, \eta} \) have a type I intersection witnessed by \( (\chi_1, \chi_2) \) if \( |J_{V_{\mu, \nu}}(\chi_1, \chi_2)| = ef \) while \( |J_{V_{\rho, \eta}}(\chi_1, \chi_2)| = ef - 1 \). The ordering of the pair of Serre weights is not important for this definition.

Definition 4.17. We say that a pair of Serre weights \( V_{\mu, \nu} \) and \( V_{\rho, \eta} \) have a type II intersection witnessed by \( (\chi_1, \chi_2) \) if \( J_{V_{\mu, \nu}}^H(\chi_1, \chi_2) = J_{V_{\rho, \eta}}^H(\chi_1, \chi_2) \) of cardinality \( ef - 1 \).

We will say that the number of separated families in a type I (resp. type II) intersection for the Serre weights \( V_{\mu, \nu} \) and \( V_{\rho, \eta} \) is \( n \) if there exist exactly \( n \) pairs of \( G_K \) characters that witness the type I (resp. type II) intersection, such that each pair is distinct from all others upon restriction to \( I_K \). By Proposition 3.4, if \( K \neq \mathbb{Q}_p \), the number of irreducible components of \( \mathcal{Z}_{V_{\mu, \nu}} \cap \mathcal{Z}_{V_{\rho, \eta}} \) of dimension \( [K : \mathbb{Q}_p] - 1 \) equals the number of separated families in either a type I or a type II intersection for the Serre weights \( V_{\mu, \nu} \) and \( V_{\rho, \eta} \).

For the remainder of this article, we may assume that \( \vec{s}, \vec{s}' \) do not have all components equal to \( p - 1 \) since we are excluding Steinberg components from analysis. Finally, since we are interested in intersections of different irreducible components, we may assume that \( V_{\mu, \nu} \neq V_{\rho, \eta} \).

Lemma 4.18. \( |J_{V_{\mu, \nu}}^H(\chi_1, \chi_2)| = ef \) if and only if \( \chi_1 = \prod_{i \in T} \omega_i^{s_i+e} \prod_{i \in T} \omega_i^{d_i} \) and \( \chi_2 | I_K = \prod_{i \in T} \omega_i^{d_i} \).

Proof. By Remarks 4.4 and 4.7, \( |J_{V_{\mu, \nu}}(\chi_1, \chi_2)| = ef \) implies that \( \chi_1 | I_K = \prod_{i \in T} \omega_i^{s_i+e} = \prod_{i \in T} \omega_i^{s_i+e} \) and \( \chi_2 | I_K = \prod_{i \in T} \omega_i^{s_i+e} = 1 \). On the other hand, starting with these \( \chi_1 \) and \( \chi_2 \), we can compute \( A_{\min} \) as in Definition 4.2. In this case, we observe that \( (m_{i-1},...,m_0) = (0,...,0) \) as \( \chi_2 | I_K = 1 \). If \( J = \emptyset \) satisfies the criterion for \( A_{\min} \), and we obtain that \( y_i = 0 \) and \( z_i = s_i + e \) for all \( i \). By Remark 4.7, we get that \( |J_{V_{\mu, \nu}}^H(\chi_1, \chi_2)| = ef \), as desired. \( \square \)

Definition 4.19. Let \( \chi_1, \chi_2 \) be two characters and \( V_{\mu, \nu} \) be a Serre weight satisfying the conditions in Lemma 4.18. Then, we say that \( V_{\mu, \nu} \) is the highest weight...
associated to the pair \((\chi_1, \chi_2)\). It is uniquely determined since not all \(s_i\) can be \(p - 1\). Moreover, knowing the highest weight determines the pair \((\chi_1, \chi_2)\) after restriction to \(I_K\).

Let \(V^{\vee}_{\ell, x} = V_{d', \ell}^\vee\). Then for any \(\tau^\vee \in \text{Ext}^1_{\text{ik}}(\overline{\mathbb{F}}(\chi_2), \overline{\mathbb{F}}(\chi_1))\), we say that \(Z_{V_{\ell, x}}\) is the highest weight component containing \(\tau\).

**Remark 4.20.** The number of separated families in a type I intersection can be at most 2, because one of the two Serre weights has to be the highest weight.

**Lemma 4.21.** \(|J^{AH}_{\nu, \nu}(\chi_1, \chi_2)| = ef - 1\) if and only if one of the following conditions is satisfied:

(i) There exists an \(i \in T\) such that \(\chi_1 = \omega_i^{s_i - 1} \prod_{j \neq i} \omega_j^{s_j + e} \prod_{j \in T} \omega_j^{d_j}\) and \(\chi_2 = \omega_i^{s_i + 1} \prod_{j \in T} \omega_j^{d_j}\), and moreover, \(s_i \leq p - 2\), and if \(f = 1\) then \(s_i < p - 2\).

(ii) \(e = 1\), \(f > 1\) and there exists an \(i \in T\) such that \(\chi_1 = \omega_i^{s_i - 1} \prod_{j \neq i} \omega_j^{s_j + e} \prod_{j \in T} \omega_j^{d_j}\), \(\chi_2 = \omega_i^{s_i + 1} \prod_{j \in T} \omega_j^{d_j}\), \(s_i = p - 1\) and \(s_i - 1 > 0\).

(iii) \(e > 1\) and there exists an \(i \in T\) such that \(\chi_1 = \omega_i^{s_i + e - 1} \prod_{j \neq i} \omega_j^{s_j + e} \prod_{j \in T} \omega_j^{d_j}\), \(\chi_2 = \omega_i \prod_{j \in T} \omega_j^{d_j}\), and \(s_i \neq 0\).

In the first two situations above, \(y_i = s_i + 1\) and \(y_j = 0\) for all \(j \neq i\) (recall Definition 4.9). On the other hand, if \(y_i = s_i + 1\) and \(y_j = 0\) for all \(j \neq i\), then one of the two above must be satisfied.

The third situation is equivalent to \(y_i = 1\), \(y_j = 0\) for all \(j \neq i\) along with \(s_i \neq 0\).

**Proof.** By Remark 4.7, if \(|J^{AH}_{\nu, \nu}(\chi_1, \chi_2)| = ef - 1\) then one of the following two conditions must be satisfied:

(i) There exists \(i \in T\) such that \(y_i = s_i + 1\) and for \(j \neq i\), \(y_j = 0\). This implies that \(\chi_1 = \omega_i^{s_i - 1} \prod_{j \neq i} \omega_j^{s_j + e} \prod_{j \in T} \omega_j^{d_j}\) and \(\chi_2 = \omega_i^{s_i + 1} \prod_{j \in T} \omega_j^{d_j}\). On the other hand, starting with such \(\chi_1\) and \(\chi_2\), twisting them by \(\prod_{j \in T} \omega_j^{-d_j}\), and applying the recipe to compute \(A_{\text{min}}\) (Definition 4.10), \(y_j\) and \(z_j\) (Definition 4.3), we branch into two scenarios:

(a) If \(s_i \leq p - 2\) for \(f > 1\) and \(p - 2 < f = 1\), then \(A_{\text{min}} = 0\), \(y_i = s_i + 1\) and \(y_j = 0\) for \(j \neq i\), giving \(|J^{AH}_{\nu, \nu}(\chi_1, \chi_2)| = ef - 1\).

(b) If \(s_i = p - 1\), then \(\chi_2 \otimes \prod_{j \in T} \omega_j^{d_j} = \prod_{j \in T} \omega_j^{m_j}\), where \(m_j = 1\) and \(m_j = 0\) if \(j \neq i - 1\). Note that this automatically implies that \(f > 1\), since we are assuming our Serre weights are non-Steinberg. We can obtain the desired values of \(y_j\)’s if and only if \((m_{f-1}, \ldots, m_0)\) is not already in \(\mathcal{S}\) of Definition 4.2. In other words, if and only if \(e = 1\) and \(s_{i-1} \neq 0\).

(ii) There exists \(i \in T\) such that \(y_i = 1\), \(s_i \neq 0\) (this is to enforce distinction from the condition above) and \(y_j = 0\) when \(j \neq i\). Note that this automatically implies that \(e > 1\), and that \(\chi_1 = \omega_i^{s_i + e - 1} \prod_{j \neq i} \omega_j^{s_j + e} \prod_{j \in T} \omega_j^{d_j}\) and \(\chi_2 = \omega_i \prod_{j \in T} \omega_j^{d_j}\). On the other hand, starting with such \(\chi_1\) and \(\chi_2\), twisting them by \(\prod_{j \in T} \omega_j^{-d_j}\) and applying the recipe to compute \(A_{\text{min}}\), \(y_j\) and \(z_j\), we obtain that \(A_{\text{min}} = 0\), \(y_i = 1\) and \(y_j = 0\) for \(i \neq j\), and we get the correct cardinality of \(J^{AH}_{\nu, \nu}(\chi_1, \chi_2)\).

□
Lemma 4.21(i) and Lemma 4.21(ii),

\[ \chi_2^{-1} \chi_1 = \omega_i^{-2} \prod_{j \neq i} \omega_j^{s_j + e} \]

In the case Lemma 4.21(iii),

\[ \chi_2^{-1} \chi_1 = \omega_i^{-2} \prod_{j \neq i} \omega_j^{s_j + e} \]

Remark 4.22. In the cases Lemma 4.21(i) and Lemma 4.21(ii),

\[ \chi_2^{-1} \chi_1 = \omega_i^{-2} \prod_{j \neq i} \omega_j^{s_j + e} \]

Remark 4.23. When \( e = 1 \), the cases Lemma 4.21(i) and Lemma 4.21(ii) are together equivalent to \( J_{\text{max}} = \mathbb{Z}/p\mathbb{Z} \setminus \{i\} \).

Before launching into computations of Type I and II intersections, we introduce some more notation. When comparing \( f \)-tuples \( \vec{s} \) and \( \vec{s}' \), we will often only state the values of \( s_i \) and \( s_i' \) that have specific constraints or are potentially different from each other. If the values of \( s_i \) or \( s_i' \) are not specified, then we assume that \( s_i = s_i' \). If no range is specified for \( s_i \), we mean that beyond any relations that it must satisfy with respect to \( s_i' \), the value of \( s_i \) can be anything in \([0, p - 1] \). Further, if we say \((..., s_i, ... \) \( = (..., \in [a, b], ... \) \), we mean that \( s_i \) can take any value \( \in [a, b] \). Similar notational assumptions apply with the roles of \( s_i \) and \( s_i' \) interchanged.

Finally, we say that a tuple \((b_{f-1}, b_{f-2}, ..., b_0)\) is equivalent to \((b'_{f-1}, b'_{f-2}, ..., b'_0)\) if \( \sum_{j=0}^{f-1} b_j p^{j-1-j} \equiv \sum_{j=0}^{f-1} b'_j p^{j-1-j} \) mod \( p^f - 1 \).

We will retain the symbols \( y_i \), \( z_i \), \( \mathcal{I}_i \), \( \lambda_i \) and \( \xi_i \) as defined in Definitions 4.3, 4.5 and 4.9 for \( V_{\vec{d}, \vec{s}} \), and will replace them respectively with \( y'_i \), \( z'_i \), \( \mathcal{I}'_i \), \( \lambda'_i \) and \( \xi'_i \) for \( V_{\vec{d'}, \vec{s'}} \), and with \( y''_i \), \( z''_i \), \( \mathcal{I}''_i \), \( \lambda''_i \) and \( \xi''_i \) for \( V_{\vec{d''}, \vec{s''}} \).

With this notation, we record another lemma which will be useful eventually in switching from Serre weights to their duals.

Lemma 4.24. Suppose \( V_{\vec{d}, \vec{s}} \) and \( V_{\vec{d'}, \vec{s'}} \) are a pair of Serre weights. Assume there exist characters \( \chi_1 \) and \( \chi_2 \) such that (4.1.1) and (4.1.2) are satisfied for both \( V_{\vec{d}, \vec{s}} \) and \( V_{\vec{d'}, \vec{s'}} \). That is, \( \chi_1 = \prod_{i \in T} \omega_i^{z_i} \prod_{i \in T} \omega_i^{\xi_i} = \prod_{i \in T} \omega_i^{z'_i} \prod_{i \in T} \omega_i^{\xi'_i} \) and \( \chi_2 = \prod_{i \in T} \omega_i^{y_i} \prod_{i \in T} \omega_i^{\lambda_i} = \prod_{i \in T} \omega_i^{y'_i} \prod_{i \in T} \omega_i^{\lambda'_i} \). Denote the duals of \( V_{\vec{d}, \vec{s}} \) and \( V_{\vec{d'}, \vec{s'}} \) by \( V_{\vec{t}, \vec{s}} \) and \( V_{\vec{t'}, \vec{s'}} \) respectively. Then, mod \( p^f - 1 \),

\[ \sum_{j \in T} p^{f-1-j}d_j' \equiv \sum_{j \in T} p^{f-1-j}d_j + c \implies \sum_{j \in T} p^{f-1-j}t_j' \equiv \sum_{j \in T} p^{f-1-j}t_j + c \]

Proof. Considering all equivalences mod \( p^f - 1 \),

\[ \sum_{j \in T} p^{f-1-j}d_j \equiv \sum_{j \in T} p^{f-1-j}(-t_j - s_j), \text{ and} \]

\[ \sum_{j \in T} p^{f-1-j}d'_j \equiv \sum_{j \in T} p^{f-1-j}(-t'_j - s'_j). \]
The equation for $\chi_1$ gives:
\[
\sum_{j \in T} p^{f-1-j}(-t_j - s_j) + \sum_{j \in T} p^{f-1-j}z_j \equiv \sum_{j \in T} p^{f-1-j}(-t'_j - s'_j) + \sum_{j \in T} p^{f-1-j}z'_j
\]
\[
\iff - \sum_{j \in T} p^{f-1-j}t_j - \sum_{j \in T} p^{f-1-j}s_j - \sum_{j \in T} p^{f-1-j}e + \sum_{j \in T} p^{f-1-j}z_j \equiv
\]
\[
- \sum_{j \in T} p^{f-1-j}t'_j - \sum_{j \in T} p^{f-1-j}s'_j - \sum_{j \in T} p^{f-1-j}e + \sum_{j \in T} p^{f-1-j}z'_j
\]
\[
\iff \sum_{j \in T} p^{f-1-j}t_j + \sum_{j \in T} p^{f-1-j}(s_j + e - z_j) \equiv \sum_{j \in T} p^{f-1-j}t'_j + \sum_{j \in T} p^{f-1-j}(s'_j + e - z'_j)
\]
\[
(4.24.1)
\]

By the equation for $\chi_2$, we know that $\sum_{j \in T} p^{f-1-j}d_j + \sum_{j \in T} p^{f-1-j}y_j \equiv \sum_{j \in T} p^{f-1-j}d'_j + \sum_{j \in T} p^{f-1-j}y'_j$. Comparing this with (4.24.1) settles the proof. □

5. Type I intersections

In this section, we will compute criteria for existence of a pair of characters $(\chi_1, \chi_2)$ witnessing a type I intersection for Serre weights $V_{\varphi,\varphi}$ and $V_{\varphi,\varphi}'$, with $|J_{\varphi,\varphi}^{\text{AH}}(\chi_1, \chi_2)| = ef$ and $|J_{\varphi,\varphi}'(\chi_1, \chi_2)| = ef - 1$ can happen via one of three ways as enumerated in Lemma 4.21. In all three situations, we may assume without loss of generality that $i$ in the statements of Lemma 4.21(i), Lemma 4.21(ii) and Lemma 4.21(iii) is $f - 1$. We will also count the number of families contributing to a type I intersection when $V_{\varphi}$ and $V_{\varphi}'$ are both weakly regular. (In the general case, the information can still be gleaned directly from the computations that follow, but we omit the explicit description for the sake of clarity).

5.1. Type I intersections when $f = 1$. We will omit subscripts of components of $f$-tuples in this section as $f = 1$.

5.1.1. Case 1. $|J_{\varphi,\varphi}'(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(i).

Suppose $p = 2$. The non-Steinberg condition requires that $s = s' = 0$. Plugging in $s$ and $s'$ in the expressions for $\chi_2$ (using Lemmas 4.18 and 4.21), we get $d \equiv d' + 1 \pmod{p - 1}$. This gives $d = d'$ and shows that $V_{\varphi}$ and $V_{\varphi}'$ are isomorphic, a contradiction. Therefore, we may assume $p > 2$.

Comparing the two ways of writing $\chi_2^t \chi_1$, we obtain:
\[
s + e \equiv e - 2 - s' \pmod{p - 1} \iff
\]
\[
s' \equiv -2 - s \equiv p - 3 - s \pmod{p - 1}
\]
This gives one of the following two situations:

\( (1) \ s \leq p - 3 \implies s' = p - s - 3. \)
\( (2) \ s = p - 2 \implies s' = p - 2. \)

In both situations, comparing the two ways of writing \( \chi_2 \), we obtain that \( d' + s' + 1 \equiv d \mod p - 1 \). In other words \( d' \equiv d + p - s' - 2 \equiv d + s + 1 \mod p - 1. \)

The second situation therefore implies that \( \text{Ext}_{\mathbb{Z}_p}^1(V_{d',s'}, V_{d,s}) \neq 0 \), which is a contradiction. The first situation is equivalent to the conditions in Proposition 2.1(ii)(a) implying that \( \text{Ext}_{\mathbb{Z}_p}^1([V_{d',s'}, V_{d,s}]) \neq 0 \). Since it is symmetric in \( s \) and \( s' \), whenever \( \text{Ext}_{\mathbb{Z}_p}^1([V_{d',s'}, V_{d,s}]) \neq 0 \), there exist two separated families witnessing a type I intersection for \( V_{d,s} \), and \( V_{d',s'} \).

5.1.2. Case 2. \( \varepsilon(f) = e_2 \). Implicit in this case is \( e > 1 \) and \( p > 2 \), the latter since \( s' \) is not allowed to be 0.

Comparing the two ways of writing \( \chi_2^{-1} \chi_1 \), we obtain:

\[
\begin{align*}
\quad & s + e \equiv e - 2 + s' \mod p - 1 \iff \quad s' \equiv s + 2 \mod p - 1
\end{align*}
\]

This gives one of the following two situations:

\( (1) \ s < p - 3 \implies s' = s + 2. \)
\( (2) \ s = p - 3 \implies s' = 0. \)
\( (3) \ s = p - 2 \implies s' = 1. \)

In both situations, comparing the two ways of writing \( \chi_2 \), we obtain that \( d' \equiv -1 + d \equiv p - 2 + d \mod p - 1. \) By comparing with Proposition 2.1, we notice that the first situation implies \( \text{Hom}_{\mathbb{Z}_p}([V_{d,s}, H^1(GK, V_{d',s'})]) \neq 0 \) (Proposition 2.6), the second implies \( \text{Ext}_{\mathbb{Z}_p}([V_{d,s}, V_{d',s'}]) \neq 0 \) via Proposition 2.1(ii)(a) and the third implies \( \text{Ext}_{\mathbb{Z}_p}([V_{d,s}, V_{d',s'}]) \neq 0 \) via Proposition 2.1(ii)(b). Notice that the relationship between \( s \) and \( s' \) is asymmetric in all three situations, unless \( p = 3 \) in which case the second and third situations are symmetric.

The above calculations may be summarized in the following proposition:

**Proposition 5.2.** Let \( f = 1 \). A Type I intersection occurs with non-isomorphic, non-Steinberg Serre weights \( V_{d,s} \) and \( V_{d',s'} \) if and only if one of the following holds true:

(i) \( \text{Ext}_{\mathbb{Z}_p}([V_{d,s}, V_{d',s'}]) \neq 0 \) \text{ via Proposition 2.1(ii)(a).} In this case, 2 families witness the type I intersection.

(ii) \( e > 1 \) and \( \text{Hom}_{\mathbb{Z}_p}([V_{d,s}, H^1(GK, V_{d',s'})]) \neq 0 \). In this case, 1 family witnesses the type I intersection.

(iii) \( e > 1 \) and \( s = p - 2, s' = 1, d' \equiv -1 + d \mod p - 1 \). In this case, the number of families witnessing the type I intersection is 1 unless \( p = 3 \), in which case the number is 2.

Note that the non-isomorphic, non-Steinberg condition automatically forces \( p > 2 \). Further, the last statement implies \( \text{Ext}_{\mathbb{Z}_p}([V_{d,s}, V_{d',s'}]) \neq 0 \) \text{ via Proposition 2.1(ii)(b).}

5.3. Type I intersections when \( f > 1 \).
5.3.1. Case 1. : $|J^{AH}_{x',s} (\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(i) or via Lemma 4.21(ii).

Comparing the two ways of writing $\chi_2^{-1}\chi_1$, we obtain the following equivalences mod $p^f - 1$ upto translating all indices by a fixed element of $\mathbb{Z}/f\mathbb{Z}$:

\[
\sum_{j \in r} p^{f-1-j}(s_j + e) = e - 2 - s'_{f-1} + \sum_{j=0}^{f-2} p^{f-1-j}(s'_j + e) \iff \\
\sum_{j \in r} p^{f-1-j}s_j = -2 - s'_{f-1} + \sum_{j=0}^{f-2} p^{f-1-j}s'_j \\
= p - s'_{f-1} - 2 + p(s'_{f-2} - 1) + \sum_{j=0}^{f-3} p^{f-1-j}s'_j
\]

Therefore, for a fixed $\vec{s}$, $\vec{s}'$ is forced to be unique since each $s'_i \in [0, p - 1]$, and the non-Steinberg condition requires that not all $s'_i$ can be $p - 1$. Similarly, for a fixed $s'_i$, $\vec{s}$ is forced to be unique.

We have a number of possible cases:

(i) Suppose $s'_{f-1} \leq p - 2$, $s'_{f-2} = s'_{f-3} = \ldots = s'_{f-i} = 0$ and $s'_{f-1-i} \geq 1$, where $i \geq 1$.

\[
(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, \ldots, s'_{f-i}, s'_{f-1-i}, s'_{f-2-i}, \ldots, s'_{0}) \equiv \\
(p - s'_{f-1} - 2, -1, 0, \ldots, 0, s'_{f-1-i}, s'_{f-2-i}, \ldots, s'_{0}) \equiv \\
(p - s'_{f-1} - 2, p - 1, p - 1, \ldots, p - 1, s'_{f-1-i} - 1, s'_{f-2-i}, \ldots, s'_{0})
\]

Therefore, $s_{f-1} = p - s'_{f-1} - 2$, $s_{f-2} = s_{f-3} = \ldots = s_{f-i} = p - 1$, $s_{f-1-i} = s'_{f-1-i} - 1 \leq p - 2$ and $s_j = s'_j$ for all the remaining $j$’s.

(ii) Suppose $s'_{f-1} \leq p - 3$, $s'_{f-2} = s'_{f-3} = \ldots = s'_{0} = 0$.

\[
(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, \ldots, s'_{0}) \equiv (p - s'_{f-1} - 2, -1, 0, \ldots, 0) \\
\equiv (p - s'_{f-1} - 3, p - 1, p - 1, \ldots, p - 1)
\]

We get $s_{f-1} = p - s'_{f-1} - 3 \leq p - 3$, $s_{f-2} = s_{f-3} = \ldots = s_0 = p - 1$.

(iii) Suppose $s'_{f-1} = p - 2$, $s'_{f-2} = s'_{f-3} = \ldots = s'_{0} = 0$.

\[
(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, \ldots, s'_{0}) \equiv (0, -1, 0, \ldots, 0) \\
\equiv (p - 1, p - 2, p - 1, \ldots, p - 1)
\]

Hence, $s_{f-2} = p - 2$ and all the other $s_j$’s equal $p - 1$.

The remaining cases require $|J^{AH}_{x',s} (\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(ii), and implicitly, $e = 1$.

(iv) Suppose $s'_{f-1} = p - 1$, $s'_{f-2} > 1$.

\[
(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, \ldots, s'_{0}) \equiv (-1, s'_{f-2} - 1, s'_{f-3}, \ldots, s'_{0}) \\
\equiv (p - 1, s'_{f-2} - 2, s'_{f-3}, \ldots, s'_{0})
\]

Therefore, $s_{f-1} = p - 1$, $s_{f-2} = s'_{f-2} - 2$ and $s_j = s'_j$ for the remaining $j$’s.
(v) Suppose \( f > 2, s'_{f-1} = p - 1, s'_{f-2} = 1, s'_{f-3} = s'_{f-4} = ... = s'_{f-i} = 0 \) and 
\( s'_{f-1-i} \geq 1 \) for some \( i > 2 \).

\[
(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, ..., s'_{f-i}, s'_{f-1-i}, s'_{f-2-i}, ..., s'_{0}) \equiv (-1, 0, 0, ..., 0, s'_{f-1-i}, s'_{f-2-i}, ..., s'_{0})
\]

\[
(p - 1, p - 1, p - 1, ..., p - 1, s'_{f-1-i} - 1, s'_{f-2-i}, ..., s'_{0})
\]

Therefore, \( s_{f-1} = s_{f-2} = ... = s_{f-i} = p - 1, s_{f-1-i} = s'_{f-1-i} - 1 \) and \( s_j = s'_j \) for all the other \( j \)'s.

(vi) Suppose \( f = 2, s'_{f-1} = p - 1, s'_{f-2} = 1 \).

\[
(p - s'_{f-1} - 2, s'_{f-2} - 1) \equiv (-1, 0) \equiv (p - 2, p - 1)
\]

Therefore, \( s_{f-1} = p - 2 \) and \( s_{f-2} = p - 1 \).

(vii) Suppose \( f > 2, s'_{f-1} = p - 1, s'_{f-2} = 1 \) and \( s'_{f-3} = ... = s'_{0} = 0 \).

\[
(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, ..., s'_{0}) \equiv (-1, 0, 0, ..., 0) \equiv (p - 2, p - 1, ..., p - 1)
\]

Therefore, \( s_{f-1} = p - 2 \) and \( s_j = p - 1 \) for all the other \( j \)'s.

The results of the computations are summarized in the proposition below.

**Proposition 5.4.** Let \( f > 1 \). Consider pairs \((\vec{s}, \vec{s'})\) satisfying:

- \( \prod_{j=0}^{f-1} \omega_j s_j = \omega_{f-1}^{-s'_{f-1}} \prod_{j=0}^{f-2} \omega_j s'_j \), where \( s_j, s'_j \in [0, p - 1] \);
- Not all \( s_j \), as well as not all \( s'_j \), are \( p - 1 \).
- \( y'_{f-1} = s'_{f-1} + 1 \) and \( y'_j = 0 \) for \( j \neq f - 1 \) (\( y'_j \) are as defined in Definition 4.3).

Below is an enumeration of all such pairs.

(i) \((s_{f-1}, s_{f-2}, ..., s_{f-i}, s_{f-1-i}) = (\in [0, p-2], p-1, ..., p-1, \in [0, p-2]), where \)
\( i \in [1, f - 1] \);
\((s'_{f-1}, s'_{f-2}, ..., s'_{f-i}, s'_{f-1-i}) = (p - s_{f-1} - 2, 0, ..., 0, s_{f-1-i} + 1) \).

(ii) \((s_{f-1}, s_{f-2}, ..., s_{0}) = (\in [0, p-3], p-1, ..., p-1) ;
\((s'_{f-1}, s'_{f-2}, ..., s'_{0}) = (p - 3 - s_{f-1}, 0, ..., 0).
This only makes sense if \( p \geq 3 \).

(iii) \((s_{f-1}, s_{f-2}, s_{f-3}, ..., s_{0}) = (p - 1, p - 2, p - 1, ..., p - 1) ;
\((s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_{0}) = (p - 2, 0, 0, ..., 0).
When \( e = 1 \), we additionally have:

(iv) \((s_{f-1}, s_{f-2}) = (p - 1, \in [0, p-3]) ;
\((s'_{f-1}, s'_{f-2}) = (p - 1, s_{f-2} + 2).
This only makes sense if \( p \geq 3 \).

(v) \( f > 2, \)
\((s_{f-1}, s_{f-2}, s_{f-3}, ..., s_{f-i}, s_{f-1-i}) = (p - 1, p - 1, p - 1, ..., p - 1, \in [0, p-2]), where \)
\( i \in [2, f - 1] ;
\((s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_{f-i}, s'_{f-1-i}) = (p - 1, 1, 0, ..., 0, s_{f-1-i} + 1) \).
(vi) \( f = 2 \),
\[
\begin{align*}
(s_{f-1}, s_{f-2}) &= (p - 2, p - 1); \\
(s'_{f-1}, s'_{f-2}) &= (p - 1, 1).
\end{align*}
\]

(vii) \( f > 2 \),
\[
\begin{align*}
(s_{f-1}, s_{f-2}, s_{f-3}, \ldots, s_0) &= (p - 2, p - 1, p - 1, \ldots, p - 1); \\
(s'_{f-1}, s'_{f-2}, s'_{f-3}, \ldots, s'_0) &= (p - 1, 1, 0, \ldots, 0).
\end{align*}
\]

Comparing the two ways of writing \( \chi_2 \), we obtain:
\[
\sum_{j \in T} p^{f-1-j} d'_j + (s'_{f-1} + 1) \equiv \sum_{j \in T} p^{f-1-j} d_j \pmod{p^f - 1} \iff
\]
\[
(5.4.1) \quad \sum_{j \in T} p^{f-1-j} d'_j \equiv -1 - s'_{f-1} + \sum_{j \in T} p^{f-1-j} d_j \pmod{p^f - 1}
\]

5.4.1. Case 2. : \( |J_{AH}^{\chi_1, \chi_2}(\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(iii). Implicit in this case is that \( e > 1 \).

Comparing the two ways of writing \( \chi_2^{-1} \chi_1 \), we obtain the following equivalences mod \( p^f - 1 \):
\[
\sum_{j \in T} p^{f-1-j} (s_j + e) \equiv e - 2 + s'_{f-1} + \sum_{j \neq f-1} p^{f-1-j} (s'_j + e) \iff
\]
\[
\sum_{j \in T} p^{f-1-j} (s_j + 1) \equiv s'_{f-1} - 1 + \sum_{j \neq f-1} p^{f-1-j} (s'_j + 1) \iff
\]
\[
(5.4.2) \quad \sum_{j \in T} p^{f-1-j} s_j \equiv s'_{f-1} - 2 + \sum_{j \neq f-1} p^{f-1-j} s'_j
\]

Proposition 5.5. Let \( f > 1 \) and \( e > 1 \).
Consider pairs \((\vec{s}, \vec{s}')\) satisfying:
- \( \prod_{j=0}^{f-1} \omega_j^{s_j} = \omega_{f-1}^{s'_{f-1}} \prod_{j=0}^{f-2} \omega_j^{s'_j} \), where \( s_j, s'_j \in [0, p - 1] \);
- Not all \( s_j \), as well as not all \( s'_j \), are \( p - 1 \). Also, \( s'_{f-1} \neq 0 \)
- \( y'_{f-1} = 1 \) and \( y'_j = 0 \) for \( j \neq f - 1 \) (\( y'_j \) are as defined in Definition 4.3).
- After reindexing if necessary, \( \vec{s} \) and \( \vec{s}' \) satisfy one of the below:
  (i) \( s_{f-1} \leq p - 3; \)
  \( s'_{f-1} = s_{f-1} + 2. \)
  This only makes sense if \( p \geq 3. \)
  (ii) \( (s_{f-1}, s_{f-2}, \ldots, s_{f-i}) = (p - 1, p - 1, \ldots, p - 1, \in [0, p - 2]), \) where \( i \geq 1; \)
  \( (s'_{f-1}, s'_{f-2}, \ldots, s'_{f-i}, s'_{f-1-i}) = (1, 0, \ldots, 0, s_{f-i-1} + 1). \)
  (iii) \( (s_{f-1}, s_{f-2}, \ldots, s_0) = (p - 2, p - 1, \ldots, p - 1); \)
  \( (s'_{f-1}, s'_{f-2}, \ldots, s'_0) = (1, 0, \ldots, 0). \)

Comparing the two ways of writing \( \chi_2 \), we obtain:
\[(5.5.1) \quad \sum_{j \in T} p^{f-1-i}d_j' \equiv -1 + \sum_{j \in T} p^{f-1-i}d_j \mod p^f - 1\]

It is evident that when each \(s_i\) and each \(s_i'\) is \(< p - 1\), the relationship between \(\vec{s}\) and \(\vec{s}'\) described in Propositions 5.4 and 5.5 is asymmetric.

The calculations for type I intersections are summarized below.

**Proposition 5.6.** Let \(f > 1\). \(V_{d,\vec{x}}\) and \(V_{d',\vec{x}'}\) be a pair of non-isomorphic, non-Steinberg Serre weights. Then there exist \(G_K\) characters \(\chi_1\) and \(\chi_2\) such that 
\[|J^{AH}_{V_{d,\vec{x}}}(\chi_1, \chi_2)| = ef, \quad |J^{AH}_{V_{d',\vec{x}'}}(\chi_1, \chi_2)| = ef - 1\] if and only if one of the following is satisfied:

(i) Upto translating the indices by any fixed number, \(\vec{s}\) and \(\vec{s}'\) satisfy one of the conditions in Proposition 5.4 while \(\vec{d}\) and \(\vec{d}'\) satisfy (5.4.1).

(ii) Upto translating the indices by any fixed number, \(\vec{s}\) and \(\vec{s}'\) satisfy one of the conditions in Proposition 5.5 while \(\vec{d}\) and \(\vec{d}'\) satisfy (5.5.1).

**Corollary 5.7.** Suppose \(f > 1\) and \(V_{d,\vec{x}}\) and \(V_{d',\vec{x}'}\) are two non-isomorphic weakly regular Serre weights. Then there exists a type I intersection for the pair if and only if one of the following holds (upto translating the indices by any fixed number and/or interchanging \(V_{d,\vec{x}}\) and \(V_{d',\vec{x}'}\) if necessary):

(i) \(\vec{s}\) and \(\vec{s}'\) satisfy Proposition 5.4(i) with \(i = 1\); while \(\vec{d}\) and \(\vec{d}'\) satisfy (5.4.1).

(ii) \(\vec{s}\) and \(\vec{s}'\) satisfy Proposition 5.5(i); while \(\vec{d}\) and \(\vec{d}'\) satisfy (5.5.1).

In other words, if and only if one of the following is true:

(i) \(\text{Ext}_{\text{GL}_2(k)}(V_{d,\vec{s}}, V_{d',\vec{s}'}) \neq 0\), or

(ii) \(e > 1\) and \(\text{Hom}_{\text{GL}_2(k)}(V_{d,\vec{s}}, H^1(G_K, V_{d',\vec{s}'})) \neq 0\)

Equivalently, if and only if \(\text{Ext}_{\text{GL}_2(k)}(V_{d,\vec{s}}, V_{d',\vec{s}'}) \neq 0\).

Moreover, exactly 1 family witnesses the type I intersection. When \(f = 2\), \(s_1 = \frac{p^2 - 1}{2}\), \(s_0 = \frac{p^2 - 3}{2}\), \(s_1' = \frac{p^2 - 3}{2}\), and \(s_0' = \frac{p^2 - 1}{2}\), interchanging \(\vec{s}\) and \(\vec{s}'\) also satisfies Proposition 5.4(i) after shifting the indices by 1. However, in this case the computations of \(\vec{d}\) and \(\vec{d}'\) show that the situation is not symmetric, and we still have just 1 family witnessing the intersection.

**Proof.** By Propositions 2.1, 2.6 and 2.14 and corollary 2.10. \(\square\)

6. Type II Intersections

In this section, we will compute criteria for existence of a pair of characters \((\chi_1, \chi_2)\) witnessing a type II intersection for (non-isomorphic and non-Steinberg) Serre weights \(V_{d,\vec{x}}\) and \(V_{d',\vec{x}'}\). Thus, we will determine if \(\chi_1\) and \(\chi_2\) exist such that 
\[J^{AH}_{V_{d,\vec{x}}}(\chi_1, \chi_2) = J^{AH}_{V_{d',\vec{x}'}}(\chi_1, \chi_2)\] of cardinality \(ef\). We will denote the highest weight associated to the pair by \(V_{d,\vec{x}}\). A family witnessing a type II intersection necessarily also witnesses two type I intersections, one for the Serre weights \(V_{d,\vec{x}}\) and \(V_{d',\vec{x}'}\), and the other for \(V_{d,\vec{x}}\) and \(V_{d',\vec{x}'}\), thus it gives a triple intersection of codimension 1. On the other hand, every such triple intersection must involve a type II intersection.

6.1. Type II intersections when \(f = 1\). We will omit subscripts of components of \(f\)-tuples in this section as \(f = 1\).
6.1.1. Case 1. \( |J_{V}^{AH}(\chi_1, \chi_2)| = |J_{V^{\prime}}^{AH}(\chi_1, \chi_2)| = ef - 1 \), both via Lemma 4.21(i) or both via Lemma 4.21(iii). It is immediate that this forces \( V_{d, s}^{\prime} \) and \( V_{d^{\prime}, s^{\prime}}^{\prime} \) to be isomorphic, a contradiction.

6.1.2. Case 2. \( |J_{V}^{AH}(\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(i) (Lemma 4.21(ii) is not possible because non-Steinberg); \( |J_{V^{\prime}}^{AH}(\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(iii). Lemma 4.21(iii) assumes that \( e > 1 \).

Comparing ways of writing \( \chi_2^{-1}\chi_1 \) using Remark 4.22, we have

\[
e - 2 - s' \equiv e - 2 + s'' \equiv s + e \mod p - 1
\]
(6.1.1)

\[
\iff s'' \equiv p - 1 - s',
\]
\[
s' \equiv p - 3 - s,
\]
\[
s'' \equiv s + 2
\]

Comparing ways of writing \( \chi_2 \) using Lemma 4.21, we obtain:

\[
s' + 1 + d' \equiv 1 + d'' \equiv d \mod p - 1
\]
(6.1.2)

\[
\iff d'' \equiv d' + s',
\]
\[
d' \equiv d + s + 1
\]
\[
d'' \equiv -1 + d
\]

By stipulation in Lemma 4.21(i), \( s' \neq p - 2 \). Therefore, \( s' \leq p - 3 \), and since \( (d', s') \neq (d'', s'') \), \( s < p - 3 \), the equivalences in (6.1.1) are equalities and \( p > 3 \).

Notice the nature of type I intersection for \( V_{d, s}^{\prime} \) and \( V_{d^{\prime}, s^{\prime}}^{\prime} \). It corresponds to \( \text{Ext}^{1}(K_{V_{d, s}^{\prime}}; V_{d^{\prime}, s^{\prime}}^{\prime}) \neq 0 \) via Proposition 2.1(ii)(a). On the other hand, the type I intersection for \( V_{d, s}^{\prime} \) and \( V_{d^{\prime}, s^{\prime}}^{\prime} \) corresponds to \( \text{Hom}^{0}(K_{V_{d, s}^{\prime}}; V_{d^{\prime}, s^{\prime}}^{\prime}) \neq 0 \).

Imposing the above conditions, we now calculate \( y', y'', z', z'', \mathcal{I}', \mathcal{I}'' \), \( \xi' \) and \( \xi'' \), and compare \( J_{V}^{AH}(\chi_1, \chi_2) \) with \( J_{V^{\prime}}^{AH}(\chi_1, \chi_2) \).

By Lemma 4.21, \( y' = s' + 1 \) and \( z' = e - 1 \), while \( y'' = 1 \) and \( z'' = s'' + e - 1 \). Therefore

\[
\mathcal{I}' = [0, e - 2]
\]
\[
\mathcal{I}'' = \{1\} \cup [s'' + 1, s'' + e - 2]
\]
\[
\xi' = (p - 1)(e - 1) + (e - 2 - s')
\]
\[
\xi'' = (p - 1)(s'' + e - 1) + (e - 2 + s'')
\]

As \( u' \) varies in \( \mathcal{I}' \), \( \xi' - u'(p - 1) = (p - 1)v' + (e - 2 - s') \) with \( u' \) taking up values in \( [1, e - 1] \). Similarly, as \( u'' \) varies in \( \mathcal{I}'' \),

\[
\xi'' - u''(p - 1) = (p - 1)v'' + (e - 2 + s'') \quad \text{where } v'' \in [1, e - 2] \cup \{s'' + e - 2\}
\]
\[
= (p - 1)v'' + e - 2 - s' + (p - 1) \quad \text{where } v'' \in [1, e - 2] \cup \{p - 3 - s' + e\}
\]
\[
= (p - 1)v'' + e - 2 + s', \quad \text{where } v'' \in [2, e - 1] \cup \{p - 2 - s' + e\}
\]

By Definition 4.10, \( J_{V}^{AH}(\chi_1, \chi_2) = J_{V^{\prime}}^{AH}(\chi_1, \chi_2) \) if and only if for all \( v' \in [1, e - 1] \), there exists a \( v'' \in [2, e - 1] \cup \{p - 2 - s' + e\} \) such that:
(6.1.3) \[
\frac{(p-1)v' + (e-2-s')}{p^v'} = \frac{(p-1)v'' + (e-2-s')}{p^{v''}}
\]

where \(v'\) is the \(p\)-adic valuation of the numerator on L.H.S, while \(v''\) is that of the numerator on R.H.S.

The only thing to check then is that (6.1.3) holds for \(v' = 1\) and \(v'' = p - 2 - s' + e\). Plugging in,

(6.1.4) \[
L.H.S. = \frac{p - 3 - s' + e}{p^v}
\]

(6.1.5) \[
R.H.S. = \frac{(p-1)(p-2-s'+e) + e-2-s'}{p^{v''+1}} = \frac{p(p-3-s'+e)}{p^{v''+1}} = L.H.S.
\]

Therefore, conditions (6.1.1) and (6.1.2) guarantee a type II intersection, and are equivalent to the conditions in Proposition 2.1(ii)(b). In this case, the relationship between the pairs \((d', s')\) and \((d'', s'')\) is symmetric except when \(s' = 1\) and \(s'' = p - 2\). Therefore the calculations show the existence of 2 separated families (because the highest weights are distinct) witnessing the type II intersection except when \(s' = 1\) and \(s'' = p - 2\). In the special case \(s' = 1\) and \(s'' = p - 2\), there is just 1 family.

Summarizing these findings, we have the proposition below.

**Proposition 6.2.** Let \(f = 1\). If \(V_{\vec{d},s'}\) and \(V_{\vec{d}'',s''}\) are a pair of non-isomorphic, non-Steinberg Serre weights, then a type II intersection occurs for the pair if and only if \(e > 1\), \(p > 3\) and \(\text{Ext}_{\text{GL}_2(k)}(V_{\vec{d},s'}, V_{\vec{d}'',s''}) \neq 0\) via Proposition 2.1(ii)(b). In addition, the following statements are true:

- If \((\chi_1, \chi_2)\) witness the type II intersection, then one of the two corresponding type I intersections witnessed by \((\chi_1, \chi_2)\) arises via Proposition 5.2(i). The other arises via Proposition 5.2(ii).
- Each type II intersection is witnessed by 2 separate families except when \(s' = 1\) and \(s'' = p - 2\), in which case just one family witnesses it.

### 6.3. Type II intersections when \(f > 1, e = 1\)

We will use the algorithm in [DDR] for this section. Our objective is to find the conditions on \(V_{\vec{d},s'}\) and \(V_{\vec{d}'',s''}\) so that \(\mu(J''_{max}) = \mu(J'_{max})\) of cardinality \(f - 1\), where \(J''_{max}\) is the subset of \(\mathbb{Z}/f\mathbb{Z}\) satisfying the conditions in Definition 4.11 for \(V_{\vec{d},s'}\) while \(J'_{max}\) is the corresponding subset for \(V_{\vec{d}'',s''}\).

We will find these intersections in two steps. First, we will find \(V_{\vec{d},s'}\) and \(V_{\vec{d}'',s''}\), such that \(J''_{max} = \mathbb{Z}/f\mathbb{Z} - \{[f-1]\}\), \(J'_{max} = \mathbb{Z}/f\mathbb{Z} - \{[f-1-i]\}\) for some \(i \in [0, f-1]\), and \(\omega_{j_f-1-1} \prod_{j \in T \cup (f-1)} \omega_{j_f} = \omega_{j_f'}\) for all \(j \in T \cup (f-1)\). The assumption that \(J''_{max} = \mathbb{Z}/f\mathbb{Z} - \{[f-1]\}\) does not cause any loss of generality. In the second step, we will compute \(\mu(J''_{max})\) and \(\mu(J'_{max})\), and identify the situations in which they are the same.

For the first step, we will use the results of Proposition 5.4. Specifically, if a \(V_{\vec{d},s'}\) exists with \(J'_{max} = \mathbb{Z}/f\mathbb{Z} - \{[f-1]\}\), then there exists a non-Steinberg \(V_{\vec{d},s''}\) so that the pair \((\vec{s}, \vec{s'})\) satisfies one of the conditions enumerated in Proposition 5.4. This is
Lemma 4.21 has constraints for Proposition 5.4(i) are our candidates for can be cycled with Proposition 5.4(i)

Proposition 5.4(v)
in the following

Proposition 5.4(iv)

Proposition 5.4

Proposition 5.4(v)
and Proposition 5.4(i)

\[ (2) \]

Clearly, outline. Each \( f \) must equal \( f \) until \( f \) is used.

\( f_1 \) is used.

If this shows up in another list item after reindexing by adding \( i \), then we see that \( \bar{x} \) can show up as the \( \bar{s} \) in Proposition 5.4(v).

That is, Proposition 5.4(i) can be cycled with Proposition 5.4(v). The corresponding two \( \bar{s} \) (in the notation of Proposition 5.4) that show up in Proposition 5.4(i) and Proposition 5.4(v) are our candidates for \( \bar{s} \) and \( \bar{s} \) respectively (in the notation of this proposition). The reindexing tells us that \( J'_{\text{max}} \) ought to be \( \mathbb{Z}/f\mathbb{Z} \setminus \{(f-1)\} \) and \( J''_{\text{max}} \) ought to be \( \mathbb{Z}/f\mathbb{Z} \setminus \{(f-1-i)\} \).

For the second step, we note that \( \delta(f-2-i) = f-1-i \). Similarly, \( \delta(f-3-i) = f-2-i \) and so on until \( \delta(f-i-k) = f-i-k+1 \). Therefore \( f-i-k \not\in \mu(J''_{\text{max}}) \), which implies that \( \mu(J''_{\text{max}}) = \mathbb{Z}/f\mathbb{Z} \setminus \{(f-i-k)\} \). Similarly, \( \delta(f-2) = f-1 \) and if \( m > 1 \), we observe that \( \delta \) causes an increase in index right until \( f-m \), so that \( \delta(f-m) = f+1-m \). This forces \( f-1-m \in \mu(J'_{\text{max}}) \) and eventually, \( f-i-k \in \mu(J'_{\text{max}}) \). If \( m = 1 \), \( f-2-m \) is in \( \mu(J'_{\text{max}}) \) again forcing \( f-i-k \in \mu(J'_{\text{max}}) \). Hence \( \mu(J'_{\text{max}}) \neq \mu(J''_{\text{max}}) \).

We repeat this process by finding all possible cyclings and computing \( \mu(J'_{\text{max}}) \) and \( \mu(J''_{\text{max}}) \). Instead of showing details for all computations, we will give an outline. Each \( \bar{s} \) showing up in the list items of Proposition 5.4 has constraints for the components positioned in some specific way relative to the indices \( f-1 \) and \( f-1-m \) for some \( m \) (In the notation of Proposition 5.4, the symbol \( i \) is used instead of \( m \). Here we are using \( i \) differently, to indicate the translation of indices). If this \( \bar{s} \) shows up in another list item after reindexing by adding \( i \) mod \( f \), the constraints for the reindexed second list item will have a description relative to indices \( f-1 \) and \( f-1-n \) for some \( n \). After undoing the reindexing, we may expect to see constraints on \( \bar{s} \) components positioned in a specific away around the indices given by \( f-1 \) and \( f-1-m \) (as posed by the specifications of the first list item), and \( f-1-i \) and \( f-1-i-n \) (as posed by the specifications of the second list item). We will use this notation in the outline below.

(1) Proposition 5.4(i) can be cycled with Proposition 5.4(iv) in the following possible ways:

(a) \( i \in [m+1,f-2] \). Then \( \mu(J''_{\text{max}}) \) excludes \( f-1-i \). If \( m > 1 \), \( \mu(J'_{\text{max}}) \) excludes \( f-m \). If \( m = 1 \), \( \mu(J'_{\text{max}}) \) excludes \( f-1 \). Therefore, \( \mu(J'_{\text{max}}) \neq \mu(J''_{\text{max}}) \).
(b) \( i = m - 1 \geq 1 \). Again, \( \mu(J''_{\text{max}}) \) excludes \( f - 1 - i = f - m \). The same holds true for \( \mu(J'_{\text{max}}) \) and we have \( \mu(J''_{\text{max}}) = \mu(J'_{\text{max}}) \).

(2) Proposition 5.4(i) can be cycled with Proposition 5.4(v) in the following possible ways:

(a) \( i \in [m + 1, f - 3] \) and \( f - 1 - i - n \neq f - 1 \) mod \( f \). The calculations in the example above show that \( \mu(J'_{\text{max}}) \neq \mu(J''_{\text{max}}) \).

(b) \( i \in [m + 1, f - 3] \) and \( f - 1 - i - n \equiv f - 1 \) mod \( f \). Here, \( \mu(J''_{\text{max}}) \) excludes 0, whereas \( \mu(J'_{\text{max}}) \) includes it, making them unequal.

(c) \( i \in [1, m - 2] \) and \( f - 1 - i - n = f - 1 - m \). Here \( \mu(J''_{\text{max}}) \) and \( \mu(J'_{\text{max}}) \) are both of cardinality \( f - 1 \) and exclude \( f - m \). Therefore, they are equal.

(3) Proposition 5.4(ii) can be cycled with Proposition 5.4(iv) with \( i = f - 1 \). \( \mu(J''_{\text{max}}) \) excludes \( f - 1 - i = 0 \). The same is true for \( \mu(J'_{\text{max}}) \), which is thus equal to \( \mu(J''_{\text{max}}) \).

(4) Proposition 5.4(ii) can be cycled with Proposition 5.4(v) with any \( i \in [1, f - 2] \). In this case, both \( \mu(J'_{\text{max}}) \) and \( \mu(J''_{\text{max}}) \) exclude 0. They are therefore equal.

(5) Proposition 5.4(iii) can be cycled with Proposition 5.4(v) with \( i \in [0, f - 1] \setminus \{1\} \). Both \( \mu(J'_{\text{max}}) \) and \( \mu(J''_{\text{max}}) \) exclude \( f - 1 \), and are equal.

(6) Proposition 5.4(iii) can be cycled with Proposition 5.4(vi) with \( i = 1 \). Both \( \mu(J'_{\text{max}}) \) and \( \mu(J''_{\text{max}}) \) exclude \( f - 1 \), and are equal.

(7) Proposition 5.4(iii) can be cycled with Proposition 5.4(vii) with \( i = 1 \). Both \( \mu(J'_{\text{max}}) \) and \( \mu(J''_{\text{max}}) \) exclude \( f - 1 \), and are equal.

(8) Proposition 5.4(iv) can be cycled with Proposition 5.4(v):

(a) \( i > 1, f - 1 - i - n = f - 2 \). Both \( \mu(J'_{\text{max}}) \) and \( \mu(J''_{\text{max}}) \) exclude \( f - 1 \) and are equal.

(b) \( i > 1, f - 1 - i - n \neq f - 2 \). \( \mu(J''_{\text{max}}) \) excludes \( f - 1 \), while \( \mu(J'_{\text{max}}) \) excludes \( f - i - n \). Therefore, \( \mu(J'_{\text{max}}) \neq \mu(J''_{\text{max}}) \).

(9) Proposition 5.4(v) can be cycled with Proposition 5.4(vii) with \( i = m \). Both \( \mu(J'_{\text{max}}) \) and \( \mu(J''_{\text{max}}) \) exclude \( f - i \) and are equal.

**Proposition 6.4.** Let \( f > 1, e = 1 \). There exists a pair of \( G_K \) characters \( (\chi_1, \chi_2) \) of highest weight \( V_{\mathfrak{g}, \mathfrak{s}}, V_{\mathfrak{g}, \mathfrak{s}'} \) witnessing a type II intersection for \( V_{\mathfrak{g}, \mathfrak{s}''} \) and \( V_{\mathfrak{g}, \mathfrak{s}'''} \). If and only if after translating the indices by adding some fixed integer, there exists an \( i \in T \) such that the following are true:

(i) \( s_{f-1}^i + 1 + \sum_{j=0}^{f-1} p^{f-1-j} d_j' \equiv p^i (s_{f-1-i}^i + 1) + \sum_{j=0}^{f-1} p^{f-1-j} d_j'' \equiv \sum_{j=0}^{f-1} p^{f-1-j} d_j \) mod \( p^f - 1 \).

(ii) The vectors \( \mathbf{s'}, \mathbf{s''} \) and \( \mathbf{s'} \) satisfy one of the following conditions:

(a) \( (s_{f-1}', s_{f-2}', \ldots, s_{f-1-i}', s_{f-2-i}') = (n, 0, \ldots, 0, \in [1, p - 2]) \) for some \( i \in [1, f - 2] \);

\[ (s_{f-1}', s_{f-2}', \ldots, s_{f-1-i}', s_{f-2-i}') = (p - s_{f-1}' - 2, p - 1, \ldots, p - 1, s_{f-2-i}' + 1); \]

\[ (s_{f-1}', s_{f-2}', \ldots, s_{f-1-i}', s_{f-2-i}') = (p - s_{f-1}' - 2, p - 1, \ldots, p - 1, s_{f-2-i}' - 1). \]
(b) \( f > 2 \) and \((s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{f-1-i}^{i'}, s_{f-2-i}^{i'}, \ldots, s_{f-m}^{i'}, s_{f-1-m}^{i'}) = (\in [0, p - 2], 0, \ldots, 0, 0, \ldots, 0, \in [1, p - 1]) \) for some \( m \in [3, f - 1] \):

\[
(s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{f-1-i}^{i'}, s_{f-2-i}^{i'}, \ldots, s_{f-m}^{i'}, s_{f-1-m}^{i'}) = (p - s_{f-1}^{i'}, \ldots, p - s_{f-1-i}^{i'}, s_{f-1-i}^{i'} - 1, p - 1, 1, 0, \ldots, 0, s_{f-1-m}^{i'}) \text{ where } i \in [1, m - 2];
\]

\[
(s_{f-1}, s_{f-2}, \ldots, s_{f-1-i}, s_{f-2-i}, \ldots, s_{f-m}, s_{f-1-m}) = (p - s_{f-1}^{i'}, \ldots, p - 1, p - 1, \ldots, p - 1, s_{f-1-m}^{i'} - 1).\]

(c) \( i = f - 1 \) and \((s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{1}^{i'}, s_{0}^{i'}) = (\in [0, p - 3], 0, \ldots, 0, 0)\):

\[
(s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{1}^{i'}, s_{0}^{i'}) = (p - 1 - s_{f-1}^{i'}, p - 1, \ldots, p - 1, p - 1);\]

\[
(s_{f-1}, s_{f-2}, \ldots, s_{1}, s_{0}) = (p - 3 - s_{f-1}^{i'}, p - 1, \ldots, p - 1, p - 1).\]

(d) \( f > 2 \) and \((s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{f-1-i}^{i'}, s_{f-2-i}^{i'}, s_{f-3-i}, \ldots, s_{0}^{i'}) = (\in [0, p - 3], 0, \ldots, 0, 0, \ldots, 0)\):

\[
(s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{f-1-i}^{i'}, s_{f-2-i}^{i'}, s_{f-3-i}, \ldots, s_{0}^{i'}) = (p - 2 - s_{f-1}^{i'}, p - 1, \ldots, p - 1, 1, 0, \ldots, 0) \text{ where } i \in [1, f - 2];
\]

\[
(s_{f-1}, s_{f-2}, \ldots, s_{f-1-i}, s_{f-2-i}, \ldots, s_{f-3-i}, \ldots, s_{0}) = (p - s_{f-1}^{i'}, p - 1, \ldots, p - 1, p - 1, p - 1, \ldots, p - 1);\]

(e) \( f > 2 \) and \((s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{f-1-i}^{i'}, s_{f-2-i}^{i'}, s_{f-3-i}, \ldots, s_{0}^{i'}) = (p - 2, 0, 0, 0, 0, 0, \ldots, 0)\):

\[
(s_{f-1}^{i'}, s_{f-2}^{i'}, \ldots, s_{f-1-i}^{i'}, s_{f-2-i}^{i'}, s_{f-3-i}, \ldots, s_{0}^{i'}) = (0, p - 1, \ldots, p - 1, p - 1, 1, 0, \ldots, 0) \text{ where } i \in [2, f - 1];
\]

\[
(s_{f-1}, s_{f-2}, s_{f-3}, \ldots, s_{0}) = (p - 1, p - 2, p - 1, \ldots, p - 1).\]

(f) \( f = 2 \) and \((s_{f-1}^{i'}, s_{f-2}^{i'}) = (p - 2, 0)\):

\[
(s_{f-1}^{i'}, s_{f-2}^{i'}) = (1, p - 1) \text{ where } i = 1;
\]

\[
(s_{f-1}, s_{f-2}) = (p - 1, p - 2).
\]

(g) \( f > 2 \) and \((s_{f-1}^{i'}, s_{f-2}^{i'}, s_{f-3}, s_{f-4}, \ldots, s_{0}^{i'}) = (p - 2, 0, 0, 0, \ldots, 0)\):

\[
(s_{f-1}^{i'}, s_{f-2}^{i'}, s_{f-3}, \ldots, s_{0}^{i'}) = (0, p - 1, 1, 0, \ldots, 0) \text{ where } i = 1;
\]

\[
(s_{f-1}, s_{f-2}, s_{f-3}, s_{f-4}, \ldots, s_{0}) = (p - 1, p - 2, p - 1, \ldots, p - 1).
\]
(h) if $f > 2$ and $(s'_{f-1}, s'_{f-2}, s'_{f-3}, \ldots, s'_{f-1-i}, s'_{f-2-i})$

\[ \equiv (p-1, 1, 0, \ldots, 0) \in [1, p-2), \]

where $i > 1$;

\[ (s''_{f-1}, s''_{f-2}, s''_{f-3}, \ldots, s''_{f-1-i}, s''_{f-2-i}) \equiv (p-1, p-1, p-1, \ldots, p-1, s''_{f-2-i} + 1); \]

\[ (s_{f-1}, s_{f-2}, s_{f-3}, \ldots, s_{f-1-i}, s_{f-2-i}) \equiv (p-1, p-1, p-1, \ldots, p-1, s_{f-2-i} - 1). \]

(i) if $f > 2$, $i = f - 1$ and $(s'_{f-1}, s'_{f-2}, s'_{f-3}, \ldots, s'_{f-1}, s'_0) = (p-1, 1, 0, \ldots, 0, p-1);

\[ (s''_{f-1}, s''_{f-2}, s''_{f-3}, \ldots, s''_{f-1}, s''_0) = (1, 0, 0, \ldots, 0, p-1); \]

\[ (s_{f-1}, s_{f-2}, s_{f-3}, \ldots, s_1, s_0) = (p-1, p-1, p-1, \ldots, p-1, p-2). \]

Proof. The conditions on $\tilde{s'}$, $\tilde{s''}$ and $\tilde{s}$ are a consequence of the preceding discussion along with explicit descriptions coming from the list in Proposition 5.4. The condition on $\tilde{d}$, $\tilde{d''}$ and $\tilde{d}$ follow from comparing descriptions of $\chi_2$ using Lemmas 4.18 and 4.21.

Remark 6.5. In each triple of $\tilde{s'}$, $\tilde{s''}$ and $\tilde{s}$ featuring in the list in Proposition 6.4, at least two of the vectors have some component equal to $p - 1$.

6.6. Type II intersections when $f > 1$, $i > 1$. We will compute the scenarios in which type II intersections occur for the pair $V_{\tilde{d}, \tilde{s}}$ and $V_{\tilde{d'}, \tilde{s'}}$. In the case of $V_{\tilde{d}, \tilde{s}}$, we will assume without loss of generality that $i = f - 1$ in the statements of Lemma 4.21 and that $\tilde{d} = 0$.

In the following calculations, we will use some extra notation and strategies for comparing $J_{\tilde{d}, \tilde{s}}^{AH} (\chi_1, \chi_2)$ and $J_{\tilde{d'}, \tilde{s'}}^{AH} (\chi_1, \chi_2)$ that we now explain. Given the Serre weights $V_{\tilde{d}, \tilde{s}}$ and $V_{\tilde{d'}, \tilde{s'}}$, and suitable $G_V$ vectors $\chi_1$ and $\chi_2$, we may compute $y_j^e, y_j^{e''}, z_j^e, z_j^{e''}, \lambda_j^e, \xi_j^e, \lambda_j^{e''}, \xi_j^{e''}$ using Definitions 4.3, 4.5 and 4.9.

Definition 6.7. Fix $j \in T$. $V_j^T \subset \mathbb{Z}$ is defined to satisfy:

\[ \{ \xi_j^e - u(p^f - 1)|u \in I_j^e \} = \{(p^f - 1)v + \lambda_j^e|v \in V_j^e \} \]

$V_j^{e''} \subset \mathbb{Z}$ is defined to satisfy:

\[ \{ \xi_j^{e''} - u(p^f - 1)|u \in I_j^{e''} \} = \{(p^f - 1)v + \lambda_j^{e''}|v \in V_j^{e''} \} \]

The above definition of $V_j^{e''}$ makes sense because $\lambda_j^{e''} \equiv \lambda_j^e \mod p^{f-1}$, since exponentiating $\omega_j$ with either gives the same character $\chi_2^{-1} \chi_1$.

Definition 6.8. Define $P', P'' \subset T \times \mathbb{Z}$ as follows:

\[ P' := \{(j, v) \in T \times \mathbb{Z}|v \in V_j^e \} \]

\[ P'' := \{(j, v) \in T \times \mathbb{Z}|v \in V_j^{e''} \} \]

We define two functions $\beta$ and $\alpha$ next.
Definition 6.9.

\[ \beta : T \times \mathbb{Z} \to \mathbb{Z} \]
\[ (j, v) \mapsto (p^j - 1)v + \lambda'_j \]
and,

\[ \alpha : T \times \mathbb{Z} \to \mathbb{Z} \times \{0, 1, \ldots, f'' - 1\} \]
\[ (j, v) \mapsto (m, \kappa) \]

where \( m = \frac{\beta(j, v)}{p^{\text{val}_p(\beta(j, v))}} \), and \( \kappa \) satisfies (4.10.2).

Remark 6.10. By Definitions 4.10 and 6.9, \( J_{V_{\varphi, \sigma}}^{AH} (\chi_1, \chi_2) = \{ \alpha(j, v) | (j, v) \in P' \} \) and \( J_{V_{\varphi, \sigma}}^{AH} (\chi_1, \chi_2) = \{ \alpha(j, v) | (j, v) \in P'' \} \).

Remark 6.11. By the comments following Definition 4.10, \( \alpha|_{P'} \) and \( \alpha|_{P''} \) are injective functions.

Remark 6.12. An examination of Definition 4.10 shows that if \( v \in V''_j \) for some \( j \in T \), then finding a pair \((\hat{j}, \hat{v})\) in \( P'' \) such that \( \alpha(j, v) = \alpha(\hat{j}, \hat{v}) \) is equivalent to finding \( \hat{j} \) and \( \hat{v} \in V''_j \) satisfying the following two conditions:

- \( (p^j - 1)v - \lambda'_j \) and \( (p^j - 1)\hat{v} - \lambda'_j \) differ by a factor of a \( p \)-power;
- the difference of \( p \)-adic valuations offsets the difference between \( j \) and \( \hat{j} \) in the formula for computing \( \kappa \) in (4.10.2).

Remark 6.13. Let \( \alpha(j, v) = \alpha(\hat{j}, \hat{v}) \). Then \( j = \hat{j} \iff \text{val}_p((p^j - 1)v - \lambda'_j) \equiv \text{val}_p((p^{\hat{j}} - 1)\hat{v} - \lambda'_j) \mod f \).

Remark 6.14. If \( j \neq \hat{j} \) and \( \text{val}_p((p^j - 1)v - \lambda'_j) = \text{val}_p((p^\hat{j} - 1)\hat{v} - \lambda'_j) = 0 \), then \( \alpha(j, v) \neq \alpha(\hat{j}, \hat{v}) \).

Definition 6.15. We will say that a pair \((j, v) \in P'\) matches \((\hat{j}, \hat{v}) \in P''\) if \( \alpha(j, v) = \alpha(\hat{j}, \hat{v}) \).

Remark 6.16. For the purposes of our calculations, we will classify the ways a pair \((j, v) \in P'\) can match a pair \((\hat{j}, \hat{v}) \in P''\) in the following manner:

1. \((j, v) = (\hat{j}, \hat{v})\).
2. \( j \equiv j + 1 \mod f \) and \( \hat{v} = pv' + z'_{j+1} - y'_{j+1} \); or \( j \equiv j + 1 \) and \( v = p\hat{v} + z'_{j} - y'_{j} \).

In these cases, \( |\text{val}_p((p^j - 1)v - \lambda'_j) - \text{val}_p((p^\hat{j} - 1)\hat{v} - \lambda'_j)| = 1 \).

3. Matches not classified by either of the above.

As we will see, the first two types will be easy to spot, whereas the third will need some verification.

We will use the notation and ideas above repeatedly in the calculations below. Because of the repetitiveness of the arguments, we will show the calculations in detail only for a few scenarios, and will only report the findings from the calculations for the rest.

6.16.1. Case 1. : \(|J_{V_{\varphi, \sigma}}^{AH} (\chi_1, \chi_2)| = |J_{V_{\varphi, \sigma}}^{AH} (\chi_1, \chi_2)| = ef - 1\), both via Lemma 4.21(i).

Case 1a. : \( i = f - 1 \). Comparing ways of writing \( \chi_2 \chi_1 \) and \( \chi_2 \) in terms of \( \tilde{s}' \), \( \tilde{s}'' \) and \( \tilde{d}'' \) using Remark 4.22, we obtain that \( V_{\varphi, \sigma} = V_{\varphi, \sigma'} \), a contradiction.
Case 1b: \( i < f - 1 \).
Comparing ways of writing \( \chi_{x^{-1}} \chi_1 \) in terms of \( \vec{s}' \) and \( \vec{s}'' \), we have:
\[
e - 2 - s'_{j-1} + \sum_{j \in T \cap \{f-1\}} p^{f-1-j}(s'_j + e) \equiv
p^{f-1-i}(e - 2 - s''_i) + \sum_{j \in T \setminus \{i\}} p^{f-1-j}(s''_j + e)
\]
\(\iff\)
\[
- 2 - s'_{j-1} + \sum_{j \in T \setminus \{f-1\}} p^{f-1-j}s'_j \equiv p^{f-1-i}(-2 - s''_i) + \sum_{j \in T \setminus \{i\}} p^{f-1-j}s''_j
\]
\(\iff\)
\[
p - 2 - s'_{j-1} + p(s'_{j-2} - 1) + \sum_{j \in T \setminus \{f-1, f-2\}} p^{f-1-j}s'_j \equiv
p^{f-1-i}(p - 2 - s''_i) + p^{f-1-i}(s''_{i-1} - 1) \sum_{j \in T \setminus \{i, i-1\}} p^{f-1-j}s''_j V'_{d', \vec{s}}
\]
\(\iff\)
\[
(p - 2 - s'_{j-1}, s'_{j-2} - 1, s'_{j-3}, \ldots, s'_0) \equiv (s''_{j-1}, \ldots, s''_{i+1}, p - 2 - s'', s''_{i-1} - 1, s''_{i-2}, \ldots, s''_0)
\]

Lemma 6.17. The above condition is satisfied if and only if (upto interchanging \( \vec{s}' \) with \( \vec{s}'' \)), one of the following pairs describe \( \vec{s}' \) and \( \vec{s}'' \):

(i) \((s'_{j-1}, s'_{j-2}, \ldots, s'_{k+1}, s'_k) = (\in [0, p - 2], 0, \ldots, 0, \in [1, p - 1])\) for some \( k \in \nobreak [i + 1, f - 2] \).
\((s'_i, s'_{i-1}, \ldots, s'_{i+1}, s'_j) = (\in [0, p - 2], p - 1, \ldots, p - 1, \in [0, p - 2])\) for some \( l \in [0, i - 1] \);
\((s''_{j-1}, s''_{j-2}, \ldots, s''_{k+1}, s''_k) = (p - 2 - s'_{j-1}, p - 1, \ldots, p - 1, s''_k - 1)\);
\((s''_i, s''_{i-1}, \ldots, s''_{i+1}, s''_j) = (p - 2 - s''_i, 0, \ldots, 0, s''_j + 1)\).

(ii) \((s'_{j-1}, s'_{j-2}, \ldots, s'_{k+1}, s'_k) = (\in [0, p - 2], 0, \ldots, 0, \in [1, p - 1])\) for some \( k \in \nobreak [i + 1, f - 2] \);
\((s'_i, s'_{i-1}, \ldots, s'_0) = (\in [0, p - 2], p - 1, \ldots, p - 1)\);
\((s''_{j-1}, s''_{j-2}, \ldots, s''_{k+1}, s''_k) = (p - 1 - s'_{j-1}, p - 1, \ldots, p - 1, s''_k - 1)\);
\((s''_i, s''_{i-1}, \ldots, s''_0) = (p - 2 - s''_i, 0, \ldots, 0)\).

(iii) \((s'_{j-1}, s'_{j-2}, \ldots, s'_{i+1}, s'_i, s'_{i-1}, \ldots, s'_0) = (\in [0, p - 2], 0, \ldots, 0, \in [1, p - 1], p - 1, \ldots, p - 1)\);
\((s''_{j-1}, s''_{j-2}, \ldots, s''_{i+1}, s''_i, s''_{i-1}, \ldots, s''_0) = (p - 1 - s'_{j-1}, p - 1, \ldots, p - 1, p - s'_i - 1, 0, \ldots, 0)\).

Proof. Easy verification upon recalling that \( s'_{j-1}, s''_i \leq p - 2 \) by Lemma 4.21(i). \(\square\)

Imposing the above conditions, we now calculate \( y'_f, y''_f, z'_f, z''_f, T'_f, T''_f, V'_f \) and \( V''_f \), and compare \( J_{y''_{d', \vec{s}'}}(\chi_1, \chi_2) \) with \( J_{y''_{d', \vec{s}'}}(\chi_1, \chi_2) \) using Remark 6.10.

For Lemma 6.17(i), we have:

\[
y'_f = \begin{cases} s'_j + 1 & \text{if } j = f - 1 \\ 0 & \text{if } j \in T \setminus \{f - 1\} \end{cases}
\]
Recall that \( J_{\mathcal{H}, 2}^{AH}(\chi_1, \chi_2) = \{ \alpha(j, v) | (j, v) \in P' \} \) and \( J_{\mathcal{H}, 2}^{AH}(\chi_1, \chi_2) = \{ \alpha(j, v) | (j, v) \in P'' \} \). In order to compare the two, there is no work to be done for \((j, v) \in P' \cap P''\). So, we must now examine the image of \( \alpha \) when restricted to the set \( P' - P'' \) and compare it to the image of \( \alpha \) when restricted to the set \( P'' - P' \).

To begin, consider \( \{ (j, 1) | j \in [k, f - 2] \} \subset P' - P'' \). One can immediately verify using Remark 6.12 that \( \alpha(j, 1) = \alpha(j + 1, p + z'_j - y'_j) \), where \( (j, 1) \in P' - P'' \) and
(j + 1, p + z_j’’ - y_j’’) \in P'' - P'. Similarly, for j \in [l + 1, i], \alpha(j, z_j’’ - y_j’’) = \alpha(j - 1, 0).

Here (j, z_j’ - y_j’’) \in P' - P'' (since z_j’ - y_j’’ = s_j’ + e) and (j - 1, 0) \in P''. These matches are of the type described in Remark 6.16(ii). After taking into account all matches of the types described in Remark 6.16(i) and Remark 6.16(ii), the only possibly unmatched pairs are (l, e - 1) \in P' - P'' and (k, e) \in P'' - P'. Now, \text{val}_p(\beta(l, e - 1)) = 0 as s_j’ \neq p - 1. As s_k’ \neq 1, \text{val}_p(\beta(k, e)) = 0. As l \neq k, \alpha(l, e - 1) \neq \alpha(k, e) by Remark 6.14.

Therefore, J_{\bar{F}_{\bar{F}}, \bar{F}, \bar{F}}^{1H}(\chi_1, \chi_2) \neq J_{\bar{F}_{\bar{F}}, \bar{F}, \bar{F}}^{2H}(\chi_1, \chi_2).

Calculations for Lemma 6.17(ii) are as follows:

\begin{align*}
y_j’ &= \begin{cases} s_j’ + 1 & \text{if } j = f - 1 \\ 0 & \text{if } j \in T \setminus \{f - 1\} \end{cases} \\
y_j’’ &= \begin{cases} p - 1 - s_j’ & \text{if } j = i \\ 0 & \text{if } j \in T \setminus \{i\} \end{cases} \\
z_j’ &= \begin{cases} e - 1 & \text{if } j = f - 1 \\ e + s_j’ & \text{if } j \in T \setminus \{f - 1\} \end{cases} \\
z_j’’ &= \begin{cases} p + e - 1 - s_j’ = p + (z_j’ - y_j’’) + 1 & \text{if } j = f - 1 \\ p + e - 1 = p - 1 + (z_j’ - y_j’’) & \text{if } j \in [k + 1, f - 2] \\ e - 1 + s_j’ = -1 + (z_j’ - y_j’’) & \text{if } j = k \\ e + s_j’ = (z_j’ - y_j’’) & \text{if } j \in [i + 1, k - 1] \\ e - 1 = -p + (z_j’ - y_j’’) + y_j’’ & \text{if } j = i \\ e = -(p - 1) + (z_j’ - y_j’’) & \text{if } j \in [0, i - 1] \end{cases} \\
\mathcal{I}_j’ &= \begin{cases} [0, e - 2] & \text{if } j = f - 1 \\ \{0\} \cup [s_j’ + 1, s_j’ + e - 1] & \text{if } j \in T \setminus \{f - 1\} \end{cases} \\
\mathcal{I}_j’’ &= \begin{cases} \{0\} \cup [p - s_j’, z_j’’ - 1] & \text{if } j = f - 1 \\ \{0\} \cup [p, z_j’’ - 1] & \text{if } j \in [k + 1, f - 2] \\ \{0\} \cup [s_j’’ + 1, z_j’’ - 1] & \text{if } j \in [i + 1, k - 1] \\ \{0, e - 2\} & \text{if } j = i \\ \{0\} \cup [1, z_j’’ - 1] & \text{if } j \in [0, i - 1] \end{cases} \\
V_j’ &= \begin{cases} [1, e - 1] & \text{if } j = f - 1 \\ [1, e - 1] \cup \{s_j’ + e\} & \text{if } j \in T \setminus \{f - 1\} \end{cases} \end{align*}
To verify \( J_{V_{d', \nu}, \beta}^{AH}(\chi_1, \chi_2) = J_{V_{d''}, \nu}(\chi_1, \chi_2) \), we apply the same strategy as we previously did. As before, for each \((j, v) \in P\) except \((f - 1, e - 1)\), we can get \((j, v)\) to match some \((j, \tilde{v})\) with \((j, \tilde{v}) \in P''\) via Remark 6.16(i) or Remark 6.16(ii). \((k, e)\) is the only pair in \( P'' \) not matched to anything in \( P - \{(f - 1, e - 1)\} \) via these two matching strategies. By Remark 6.14, \((f - 1, e - 1)\) cannot match \((k, e)\) because \( \val_{\beta}(\beta(f - 1, e - 1)) = \val_{\beta}(\beta(k, e)) \), since \( s'_f - 1 \neq p - 1 \) and \( s'_k \neq 0 \). Therefore, we do not get a type II intersection in the desired manner.

The calculations for Lemma 6.17(iii) are similar and left to the reader. The results from the calculations are also similar, and show that \( J_{V_{d', \nu}, \beta}^{AH}(\chi_1, \chi_2) \neq J_{V_{d''}, \nu}(\chi_1, \chi_2) \).

The findings are summarized below.

**Proposition 6.18.** Let \( e > 1, f > 1 \). Suppose \( V_{d', \nu} \) and \( V_{d''} \) are a pair of non-isomorphic, non-Steinberg Serre weights. There do not exist any \( \Gamma_K \) characters \( \chi_1 \) and \( \chi_2 \) such that \( |J_{V_{d', \nu}, \beta}^{AH}(\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(i), \( |J_{V_{d''}, \nu}(\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(i) and \( J_{V_{d', \nu}, \beta}^{AH}(\chi_1, \chi_2) = J_{V_{d''}, \nu}(\chi_1, \chi_2) \).

6.18.1. Case 2. : \(|J_{V_{d', \nu}, \beta}^{AH}(\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(iii); \(|J_{V_{d''}, \nu}(\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(iii).

**Case 2a.** : \( i = f - 1 \). Comparing ways of writing \( \chi_2^{-1} \chi_1 \) and \( \chi_2 \) in terms of \( s' \), \( s'' \) and \( s' \) using Remark 4.22, we obtain that \( V_{d', s'} = V_{d'', s''} \), a contradiction.

**Case 2b.** : \( i < f - 1 \).

Comparing ways of writing \( \chi_2^{-1} \chi_1 \) in terms of \( s' \), \( s'' \) and \( s' \), we have:

\[
e - 2 + s'_f - 1 + \sum_{j \in T \setminus \{f - 1\}} p^{f - 1 - j}(s'_j + e) = p^{f - 1 - i}(e - 2 + s''_i) + \sum_{j \in T \setminus \{i\}} p^{f - 1 - j}(s''_j + e) \equiv \sum_{j \in T} p^{f - 1 - j}(s_j + e)
\]

(6.18.1)

\[
\Leftrightarrow (-2 + s'_f - 1, s''_f - 2, ..., s''_0) \equiv (s''_{f - 1}, ..., s''_{i + 1}, 2 + s''_{i + 1}, s''_{i + 1}, ..., s''_0) \equiv (s_{f - 1}, s_{f - 2}, ..., s_0)
\]
Comparing ways of writing $\chi_2$, we have:
\[
p^{f-1-i} + \sum_{j \in T} p^{f-1-j} d_j'' = 1 \equiv \sum_{j \in T} p^{f-1-j} d_j \pmod{p^f-1}
\]
(6.18.2) \iff \sum_{j \in T} p^{f-1-j} d_j'' = 1 - p^{f-1-i}, \quad \sum_{j \in T} p^{f-1-j} d_j = 1 \pmod{p^f-1}

Lemma 6.19. The condition in (6.18.1) is satisfied for some $\vec{s}'$, $\vec{s}''$ and $\vec{s}$ if and only if one of the following pairs describe $\vec{s}'$ and $\vec{s}''$:

(i) $s'_{f-1} \in [2, p - 1], s'_i \in [0, p - 3]$;

\[ s''_{f-1} = s'_{f-1} - 2, \quad s''_i = s'_i + 2. \]

(ii) $s'_{f-1}, s'_{f-2}, \ldots, s'_{k+1}, s'_k = (\in [0, 1), 0, \ldots, 0, \in [1, p - 1])$ for some $k \in [i + 1, f - 2], s'_i \in [0, p - 3]$;

\[ s''_{f-1}, s''_{f-2}, \ldots, s''_{k+1}, s''_k = (p - 2 + s'_{f-1}, p - 1, \ldots, p - 1, s'_k - 1), \quad s''_i = s'_i + 2. \]

(iii) $s'_{f-1}, s'_{f-2}, \ldots, s'_{k+1}, s'_k = (1, 0, \ldots, 0, \in [1, p - 2]);$

\[ s''_{f-1}, s''_{f-2}, \ldots, s''_{k+1}, s''_k = (p - 1, p - 1, \ldots, p - 1, s'_i + 1). \]

(iv) $s'_{f-1} \in [2, p - 1], (s'_i, s'_{i-1}, \ldots, s'_{i+1}, s'_i) = (p - 1, p - 1, \ldots, p - 1, \in [0, p - 2])$

for some $l \in [0, i - 1]$;

\[ s''_{f-1} = s'_{f-1} - 2, (s''_i, s''_{i-1}, \ldots, s''_{i+1}, s''_i) = (1, 0, \ldots, 0, s'_i + 1). \]

(v) $s'_{f-1} \in [1, p - 1], (s'_i, s'_{i-1}, \ldots, s'_0) = (p - 1, p - 1, \ldots, p - 1);$

\[ s''_{f-1} = s'_{f-1} - 1, (s''_i, s''_{i-1}, \ldots, s''_0) = (1, 0, \ldots, 0); \]

\[ s'_{f-1} \in [1, p - 1], (s'_i, s'_{i-1}, \ldots, s'_0) = (p - 1, p - 1, \ldots, p - 1). \]

(vi) $s'_{f-1}, s'_{f-2}, \ldots, s'_{k+1}, s'_k = (1, 0, \ldots, 0, \in [1, p - 1])$ for some $k \in [i + 1, f - 2],$

\[ s'_i, s'_{i-1}, \ldots, s'_{i+1}, s'_i = (p - 1, p - 1, \ldots, p - 1, \in [0, p - 2])$ for some $l \in [0, i - 1];$

\[ s''_{f-1}, s''_{f-2}, \ldots, s''_{k+1}, s''_k = (p - 2 + s'_{f-1}, p - 1, \ldots, p - 1, s'_k - 1), \]

\[ s''_i, s''_{i-1}, \ldots, s''_{i+1}, s''_i) = (1, 0, \ldots, 0, s'_i + 1). \]

Proof. Easy verification upon recalling that $s'_{f-1}, s''_i \geq 1$ by Lemma 4.21(iii). \qed

Imposing the above conditions, we now calculate $y'_j$, $y''_j$, $z'_j$, $z''_j$, $I'_j$, $I''_j$, $V'_j$, and $V''_j$, and compare $J^{AH}_{\vec{d}, \vec{r}'}(\chi_1, \chi_2)$ with $J^{AH}_{\vec{d}, \vec{r}''}(\chi_1, \chi_2)$ using Remark 6.10.

For Lemma 6.19(i), we have:

(6.19.1) \quad y'_j = \begin{cases} 1 & \text{if } j = f - 1 \\ 0 & \text{if } j \in T \setminus \{f - 1\} \end{cases}

(6.19.2) \quad y''_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \in T \setminus \{i\} \end{cases}

(6.19.3) \quad z'_j = \begin{cases} e - 1 + s'_j & \text{if } j = f - 1 \\ e + s'_j & \text{if } j \in T \setminus \{f - 1\} \end{cases}
\[ z''_j = \begin{cases} e - 2 + s'_j = z'_j - y'_j & \text{if } j = f - 1 \\ e + 1 + s'_j = (z'_j - y'_j) + y''_j & \text{if } j = i \\ e + s'_j = z'_j - y'_j & \text{if } j \in T \setminus \{f - 1, i\} \end{cases} \]

\[ \mathcal{I}_j'' = \begin{cases} \{1\} \cup [s'_j + 1, z'_j - 1] & \text{if } j = f - 1 \\ \{0\} \cup [s'_j + 1, z'_j - 1] & \text{if } j \in T \setminus \{f - 1\} \end{cases} \]

\[ V''_j = \begin{cases} [1, e - 1] \cup \{z'_j - y'_j\} & \text{if } j = f - 1 \\ [1, e - 1] \cup \{z'_j - y'_j\} & \text{if } j \in T \setminus \{f - 1\} \end{cases} \]

The only pairs in \( P \) and \( P'' \) that are unmatched after applying matching strategy are \((i, e - 1) \in P\) and \((f - 1, e - 1) \in P''\). As \( s'_i \neq p - 1 \) and \( s'_{f - 1} \neq 1 \), \( \text{val}_\mathbb{p}(\beta(i, e - 1)) = 0 = \text{val}_\mathbb{p}(\beta(f - 1, e - 1)) \). By Remark 6.14, \((i, e - 1)\) cannot possibly match \((f - 1, e - 1)\), and therefore, \( J^{AH}_{\mathfrak{d}, \mathfrak{e}}(x_1, x_2) \neq J^{AH}_{\mathfrak{d}, \mathfrak{e}}(x_1, x_2) \).

For Lemma 6.19(ii), we have:

\[ y''_j = \begin{cases} 1 & \text{if } j = f - 1 \\ 0 & \text{if } j \in T \setminus \{f - 1\} \end{cases} \]

\[ z''_j = \begin{cases} e - 1 + s'_j & \text{if } j = f - 1 \\ e + s'_j & \text{if } j \in T \setminus \{f - 1\} \end{cases} \]

\[ \mathcal{I}_j = \begin{cases} \{1\} \cup [s'_j + 1, z'_j - 1] & \text{if } j = f - 1 \\ \{0\} \cup [s'_j + 1, z'_j - 1] & \text{if } j \in T \setminus \{f - 1\} \end{cases} \]
The only pairs in $P$ and $P''$ that are unmatched after applying matching strategies in Remark 6.16(i) and Remark 6.16(ii) are those in \{(f-1, z'_{f-1} - y'_{f-1}), (i, e-1)\} $\subseteq P$ and \{(f-1, e-1), (k, e)\} $\subseteq P''$. As $\text{val}_p(\beta(i, e-1)) = 0 = \text{val}_p(\beta(k, e))$ and $i \neq k$, $J^{AH}_{\nu_{\theta'}^p \varphi}(\chi_1, \chi_2) = J^{AH}_{\nu_{\phi}^p \varphi}(\chi_1, \chi_2)$ if and only if $(i, e-1)$ matches $(f-1, e-1)$, while $(f-1, z'_{f-1} - y'_{f-1})$ matches $(k, e)$. Suppose this is true and $(m, \kappa) = \alpha(f-1, z'_{f-1} - y'_{f-1}) = \alpha(k, e)$. Plugging this data into the formula for $\kappa$ in (4.10.2), we get:

\[
\text{(6.19.17)} \quad \text{val}_p(\beta(f-1, z'_{f-1} - y'_{f-1})) \equiv f-1 - k \mod f
\]

Therefore, $p^{f-1-k} | \beta(f-1, z'_{f-1} - y'_{f-1}) = p^2(z'_{f-2} - y'_{f-2}) + p^2(z'_{f-3} - y'_{f-3}) + \ldots + p^f(z'_{f-1} - y'_{f-1})$ and we have:

\[
m \leq \frac{p(z'_{f-2} - y'_{f-2}) + p^2(z'_{f-3} - y'_{f-3}) + \ldots + p^f(z'_{f-1} - y'_{f-1})}{p^{f-1-k}}
\]

\[
= (z'_k - y'_k) + p(z'_{k-1} - y'_{k-1}) + \ldots + p^{k+1}(z'_{f-1} - y'_{f-1}) + \frac{(z'_{f-2} - y'_{f-2})}{p^{f-2-k}} + \ldots + \frac{(z'_{k+1} - y'_{k+1})}{p}
\]

\[
< (p^f - 1)e + (z'_k - y'_k) + p(z'_{k-1} - y'_{k-1}) + \ldots + p^{f-1}(z'_{k+1} - y'_{k+1})
\]

\[
= \alpha(k, e) = m
\]

Contradiction. Therefore, $J^{AH}_{\nu_{\theta'}^p \varphi}(\chi_1, \chi_2) \neq J^{AH}_{\nu_{\phi}^p \varphi}(\chi_1, \chi_2)$.

For Lemma 6.19(iii), we have:

\[
\text{(6.19.18)} \quad y'_j = \begin{cases} 
1 & \text{if } j = f - 1 \\
0 & \text{if } j \in T \setminus \{f - 1\}
\end{cases}
\]

\[
\text{(6.19.19)} \quad y''_j = \begin{cases} 
1 & \text{if } j = i \\
0 & \text{if } j \in T \setminus \{i\}
\end{cases}
\]
Lemma 6.19

Remark 6.16(ii).

They proceed similar to the calculations above, and the findings can be summarized as follows:

Lemma 6.19(v).

\[ z_j' = \begin{cases} 
-1 + s_j' & \text{if } j = f - 1 \\
-1 + s_j' & \text{if } j \in T \setminus \{f - 1\} 
\end{cases} \]

Lemma 4.21(iii).

Proposition 6.20.

Lemma 6.19(vi).

Steinberg Serre weights.

\[ (6.19.21) \]

Remark 6.16(i).

We omit demonstrating the calculations for every pair in \( P' \) upon applying matching strategies in Remark 6.16(i) and Remark 6.16(ii). Therefore, \( J_{V_{e_1, \bar{v}_1}}^{AH} (\chi_1, \chi_2) = J_{V_{e_1, \bar{v}_1}}^{AH} (\chi_1, \chi_2)! \)

We omit demonstrating the calculations for Lemma 6.19(iv), Lemma 6.19(v) and Lemma 6.19(vi).

They proceed similar to the calculations above, and the findings for all pairs described in Lemma 6.19 can be summarized as follows:

\[ (6.19.20) \]

\[ (6.19.22) \]

\[ (6.19.23) \]

\[ (6.19.24) \]

\[ (6.19.25) \]

Every pair in \( P \) matches with some pair in \( P'' \) upon applying matching strategies in Remark 6.16(i) and Remark 6.16(ii). Therefore, \( J_{V_{e_1, \bar{v}_1}}^{AH} (\chi_1, \chi_2) = J_{V_{e_1, \bar{v}_1}}^{AH} (\chi_1, \chi_2)! \)

We omit demonstrating the calculations for Lemma 6.19(iv), Lemma 6.19(v) and Lemma 6.19(vi).

They proceed similar to the calculations above, and the findings for all pairs described in Lemma 6.19 can be summarized as follows:

Proposition 6.20. Suppose \( V_{d_1, \bar{v}_1} \) and \( V_{d_2, \bar{v}_2} \) are a pair of non-isomorphic, non-Steinberg Serre weights.

Then there exist \( G_K \) characters \( \chi_1 \) and \( \chi_2 \) such that \( |J_{V_{d_1, \bar{v}_1}}^{AH} (\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(iii), \( |J_{V_{d_2, \bar{v}_2}}^{AH} (\chi_1, \chi_2)| = ef - 1 \) via Lemma 4.21(iii) and \( J_{V_{d_1, \bar{v}_1}}^{AH} (\chi_1, \chi_2) = J_{V_{d_2, \bar{v}_2}}^{AH} (\chi_1, \chi_2) \) if and only if

- After reindexing if necessary, \( \bar{v}_1 \) and \( \bar{v}_2 \) satisfy either of the below for some \( i < f - 1 \):
  - \( (s_{f-i-1}^{(1)}, s_{f-i-2}^{(1)}, ..., s_{i+1}^{(1)}, s_i^{(1)}) = (1, 0, ..., 0, \in [0, p - 2]) \);
  - \( (s_{f-i-1}^{(2)}, s_{f-i-2}^{(2)}, ..., s_{i+1}^{(2)}, s_i^{(2)}) = (p-1, p-1, ..., p-1, s_i+1) \) (Lemma 6.19(iii)).
  - \( s_{f-i-1}^{(3)} \in [1, p-1] \), \( (s_i^{(3)}, s_{i+1}^{(3)}; s_0^{(3)}) = (p-1, p-1, ..., p-1) \);
  - \( s_{f-i-1}^{(4)} = s_{f-i-2}^{(4)}, (s_i^{(4)}, s_{i+1}^{(4)}, s_0^{(4)}) = (1, 0, ..., 0) \) (Lemma 6.19(v)).

- With \( i \) as above, \( \sum_{j \in T} d_j^{(3)} \equiv 1 - p^{f-1-i} + \sum_{j \in T} d_j^{(4)}, \) and \( \sum_{j \in T} d_j \equiv 1 + \sum_{j \in T} d_j^{(5)} \mod p^{f-1}. \)
Proposition 6.20, (s_{f-1}, s_{f-2}, ..., s_0) \equiv (s_{f-1}' - 2, s_{f-2}', ..., s_0'). We leave the precise specification of the highest weight to the reader.

6.21.1. Case 3. : \(|J_{\mathcal{V}^*}^{AH}(\chi_1, \chi_2)| = ef - 1\) via Lemma 4.21(i); \(|J_{\mathcal{V}^*}^{AH}(\chi_1, \chi_2)| = ef - 1\) via Lemma 4.21(iii).

Case 3a. : \(i = f - 1\).

Comparing ways of writing \(\chi_2^{-1}\chi_1\) in terms of \(s', \tilde{s}', \text{ and } \tilde{s}'\), we have:

\[
e - 2 - s'_{f-1} + \sum_{j \in T \setminus \{f-1\}} p^{f-1-j}(s'_j + e) = (e - 2 + s''_{f-1}) + \sum_{j \in T \setminus \{f-1\}} p^{f-1-j}(s''_j + e) = \sum_{j \in T} p^{f-1-j}(s_j + e)
\]

(6.21.1)

\[
\iff (-2 - s'_{f-1}, s'_{f-2}, ..., s_0') \equiv (-2 + s''_{f-1}, s''_{f-2}, ..., s_0'') \equiv (s_{f-1}, s_{f-2}, ..., s_0).
\]

Comparing ways of writing \(\chi_2\), we have:

\[
1 + \sum_{j \in T} p^{f-1-j}d''_j \equiv s'_{f-1} + 1 \equiv \sum_{j \in T} p^{f-1-j}d_j \mod p^f - 1
\]

(6.21.2)

\[
\iff \sum_{j \in T} p^{f-1-j}d''_j \equiv s'_{f-1}, \sum_{j \in T} p^{f-1-j}d_j \equiv s'_{f-1} + 1 \mod p^f - 1
\]

Lemma 6.22. The condition in (6.21.1) is satisfied for some \(s', \tilde{s}'\) and \(s''\) if and only if one of the following pairs describe \(s', \tilde{s}'\) and \(s'\):

(i) \((s'_{f-1}, s'_{f-2}, ..., s'_{k+1}, s_k') = (\in [1, p-2], 0, ..., 0, \in [1, p-1])\) where \(k \in [0, f - 2]\);

\((s''_{f-1}, s''_{f-2}, ..., s''_{k+1}, s''_k) = (p - s''_{f-1}, p - 1, ..., p - 1, s''_k - 1)\).

(ii) \((s''_{f-1}, s''_{f-2}, ..., s''_0) = (\in [1, p-2], 0, ..., 0);\)

\((s'_{f-1}, s'_{f-2}, ..., s_0) = (p - s'_{f-1} - 1, p - 1, ..., p - 1)\).

Proof. Easy verification upon recalling that \(s'_{f-1} \leq p - 2\) by Lemma 4.21(i) and \(s''_0 \geq 1\) by Lemma 4.21(iii).

We omit the calculations for Lemma 6.22(ii) which show that \(J_{\mathcal{V}^*}^{AH}(\chi_1, \chi_2) \neq J_{\mathcal{V}^*}^{AH}(\chi_1, \chi_2)\). Briefly, \((f - 1, e - 1) \in P\) and \((k, e) \in P''\) are the pairs in \(P\) and \(P''\) that don’t match using matching strategies Remark 6.16(i) and Remark 6.16(ii). Both \(\beta(f - 1, e - 1)\) and \(\beta(k, e)\) turn out to have \(p\)-adic valuation \(0\) and therefore, by Remark 6.14, \(\alpha(f - 1, e - 1) \neq \alpha(k, e)\).

For Lemma 6.22(ii), all pairs in \(P\) end up matching with some pair in \(P''\) via Remark 6.16(i) or Remark 6.16(ii) (details omitted). Therefore, in this situation, \(J_{\mathcal{V}^*}^{AH}(\chi_1, \chi_2) = J_{\mathcal{V}^*}^{AH}(\chi_1, \chi_2)\).

Case 3b. : \(i < f - 1\).
Comparing ways of writing $\chi_2^{-1}\chi_1$ in terms of $\vec{s}'$, $\vec{s}''$ and $\vec{s}$, we have:
\begin{align}
e - 2 - s'_{f-1} + \sum_{j \in T \setminus \{f-1\}} p^{f-1-j}(s'_j + e) \\
\equiv p^{f-1-i}(e - 2 + s''_i) + \sum_{j \in T \setminus \{i\}} p^{f-1-j}(s''_j + e) \equiv \sum_{j \in T} p^{f-1-j}(s_j + e)
\end{align}
(6.22.1)
\[\iff (-2 - s'_{f-1}, s'_{f-2}, \ldots, s'_0) \equiv (s''_{f-1}, \ldots, s''_{i+1}, 2 + s''_i, s''_{i-1}, \ldots, s''_0) \equiv (s_{f-1}, s_{f-2}, \ldots, s_0)\]

Comparing ways of writing $\chi_2$, we have:
\[p^{f-1-i} + \sum_{j \in T} p^{f-1-j}d''_j \equiv s'_{f-1} + 1 \equiv \sum_{j \in T} p^{f-1-j}d_j \mod p^f - 1\]
(6.22.2)
\[\iff \sum_{j \in T} p^{f-1-j}d''_j \equiv s'_{f-1} + 1 - p^{f-1-i}, \sum_{j \in T} p^{f-1-j}d_j \equiv s'_{f-1} + 1 \mod p^f - 1\]

Lemma 6.23. The condition in (6.22.1) is satisfied for some $\vec{s}'$, $\vec{s}''$ and $\vec{s}$ if and only if one of the following pairs describe $\vec{s}'$ and $\vec{s}''$:

(i) $(s'_{f-1}, s'_{f-2}, \ldots, s'_{k+1}, s'_k) = (\in [0, p-3], 0, \ldots, 0, \in [1, p-1])$ for some $k \in [i + 1, f - 2]$;

(ii) $(s'_{f-1}, s'_{f-2}, s'_{i+1}, s'_i) = (\in [0, p-3], 0, \ldots, 0, \in [1, p-1])$;

(iii) $(s'_{f-1}, s'_{f-2}, s'_{k+1}, s'_k) = (\in [0, p-3], 0, \ldots, 0, \in [1, p-1])$ for some $k \in [i + 1, f - 2]$;

(iv) $(s'_{f-1}, s'_{f-2}, s'_{k+1}, s'_k) = (\in [0, p-3], 0, \ldots, 0, \in [1, p-1])$ for some $k \in [i + 1, f - 2]$;

Proof. Easy verification upon recalling that $s'_{f-1} \leq p - 2$ by Lemma 4.21(i) and $s''_i \geq 1$ by Lemma 4.21(iii). □

For each of the pairs in the statement of Lemma 6.23, we omit the details of the calculations comparing $J^{AH}_{d'' \chi_1 \chi_2}(\chi_1, \chi_2)$ and $J^{AH}_{d'' \chi_1 \chi_2}(\chi_1, \chi_2)$. For pairs in Lemma 6.23 (i), (iii) and (iv), $J^{AH}_{d'' \chi_1 \chi_2}(\chi_1, \chi_2) \neq J^{AH}_{d'' \chi_1 \chi_2}(\chi_1, \chi_2)$. For the pair in Lemma 6.23(ii),
\[J^{AH}_{d'' \chi_1 \chi_2}(\chi_1, \chi_2) = J^{AH}_{d'' \chi_1 \chi_2}(\chi_1, \chi_2).\]

Proposition 6.24. Suppose $V_{d'', \vec{s}'}$ and $V_{d'' \vec{s}', \vec{s}}$ are a pair of non-isomorphic, non-Steinberg Serre weights.
Then there exist $G_K$ characters $\chi_1$ and $\chi_2$ such that $|J^AH_{\hat{\omega},i}(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(i), $|J^AH_{\hat{\omega},i}(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(ii) and $J^AH_{\hat{\omega},i}(\chi_1, \chi_2) = J^AH_{\hat{\omega},i}(\chi_1, \chi_2)$ if and only if (after reindexing if necessary) $\bar{s}', \bar{s}''$, $\bar{d}'$ and $\bar{d}''$ and $\bar{d}$ satisfy either of the conditions below (we describe $\bar{s}$ only up to equivalence for the sake of clarity):

\begin{itemize}
  \item (i) $f' = f - 2$ such that:
    \begin{itemize}
      \item $\sum_{j \in T} p^{f'-1-j} \bar{d}'_j \equiv s'_{f-1} + \sum_{j \in T} p^{f'-1-j} \bar{d}'_j \mod p^{f'} - 1$ and $\sum_{j \in T} p^{f'-1-j} \bar{d}'_j \equiv s'_{f-1} + \sum_{j \in T} p^{f'-1-j} \bar{d}'_j \mod p^{f'} - 1$ (Lemma 6.22(ii)).
      \item $\sum_{j \in T} p^{f'-1-j} \bar{d}'_j \equiv s'_{f-1} + \sum_{j \in T} p^{f'-1-j} \bar{d}'_j \mod p^{f'} - 1$ (Lemma 6.23(ii)).
    \end{itemize}

\end{itemize}

\begin{remark}
For $p > 2$ and $i = f - 2$, the second condition in the proposition above is identical to that required for $\text{Ext}_{\overline{\mathcal{G}L}_2(k)}(V_{\hat{d}, \bar{s}}, V_{\hat{d}', \bar{s}'})$ to be non-zero via Proposition 2.1(i)(b). The relationship between $(\bar{d}', \bar{s}')$ and $(\bar{d}'', \bar{s}'')$ is asymmetric showing that only 1 family sees the type II intersection. For such a family witnessing a type II intersection, the two associated type I intersections correspond to $\text{Ext}_{\overline{\mathcal{G}L}_2(k)}(V_{\hat{d}, \bar{s}}, V_{\hat{d}', \bar{s}'}) \neq 0$ via Proposition 2.1(i)(b) and $\text{Hom}_{\overline{\mathcal{G}L}_2(k)}(V_{\hat{d}, \bar{s}}, H^1(G_K, V_{\hat{d}', \bar{s}'})) \neq 0$. In particular, when $e > 1$, $f > 1$, $\text{Ext}_{\overline{\mathcal{G}L}_2(k)}(V_{\hat{d}, \bar{s}}, V_{\hat{d}', \bar{s}'}) \neq 0$ guarantees the existence of both a type I intersection and a type II intersection, while $\text{Hom}_{\overline{\mathcal{G}L}_2(k)}(V_{\hat{d}, \bar{s}}, H^1(G_K, V_{\hat{d}', \bar{s}'}) \neq 0$ only guarantees a type I intersection.
\end{remark}

7. Conclusion

\begin{theorem}
Let $p > 2$ be a fixed prime. Let $K$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field $k$. Set $e = e(K/\mathbb{Q}_p), f = f(K/\mathbb{Q}_p)$. Let $Z$ be the stack of $p$ $G_K$-representations constructed in [CEGS1], defined over a finite field $\mathbb{F}$. Let $V_{\hat{t}, \bar{s}}$ and $V_{\hat{t}', \bar{s}'}$ be a pair of non-isomorphic, non-Steinberg Serre weights for $\mathbb{G}L_2(k)$. Consider the irreducible component $\mathcal{Z}_{V_{\hat{t}, \bar{s}}}$ (resp. $\mathcal{Z}_{V_{\hat{t}', \bar{s}'}}$) of $Z$ with the property that $\mathfrak{r} \in Z(\mathbb{F})$ is a point of $\mathcal{Z}_{V_{\hat{t}, \bar{s}}}$ (resp. $\mathcal{Z}_{V_{\hat{t}', \bar{s}'}}$) if and only if $V_{\hat{t}, \bar{s}}$ (resp. $V_{\hat{t}', \bar{s}'}$) is a Serre weight of $\mathfrak{r}$.

Then $\mathcal{Z}_{V_{\hat{t}, \bar{s}}}$ and $\mathcal{Z}_{V_{\hat{t}', \bar{s}'}}$ intersect in codimension 1 if and only if one of the following list of criteria holds. Next to each criterion we indicate in parenthesis the type of intersection.

\begin{enumerate}
  \item $V_{\hat{t}, \bar{s}}$ and $V_{\hat{t}', \bar{s}'}$ are both weakly regular and $\text{Ext}_{\overline{\mathcal{G}L}_2(\mathcal{O}_K)}(V_{\hat{t}, \bar{s}}, V_{\hat{t}', \bar{s}'}) \neq 0$. (Type I if $e = 1$, Type I or II or both if $e > 1$).
\end{enumerate}

\end{theorem}
(ii) $V_{\tilde{c},z}$ and $V_{\tilde{d},\tilde{z}}$ are not both weakly regular, $f > 1$ and one of the following is true after possibly interchanging $V_{\tilde{c},z}$ and $V_{\tilde{d},\tilde{z}}$ and possibly changing the indices of $\{s_j\}_j$, $\{s'_j\}_j$, $\{t_j\}_j$ and $\{t'_j\}_j$ by adding a fixed integer. We also indicate when $\text{Ext}^1_{\mathfrak{L}_{\text{GL}_2}(k)}(V_{\tilde{c},z}, V_{\tilde{d},\tilde{z}})$ is non-vanishing, or when $\text{Ext}^1_{\mathfrak{L}_{\text{GL}_2}(O_K)}(V_{\tilde{c},z}, V_{\tilde{d},\tilde{z}})$ is non-vanishing, or when $\text{Hom}_{\mathfrak{L}_{\text{GL}_2}(k)}(V_{\tilde{c},z}, H^1(K_1, V_{\tilde{d},\tilde{z}}))$ is non-vanishing but it is not known whether it contributes to $\text{Ext}^1_{\mathfrak{L}_{\text{GL}_2}(O_K)}(V_{\tilde{c},z}, V_{\tilde{d},\tilde{z}})$ or not.

(a) $(s_{f-1}, s_{f-2},\ldots, s_f, s_{f-1-i}) = (\in [0, p-2], p-1,\ldots, p-1, \in [0, p-2])$, where $i \in [1, f-1]$;

$(s'_{f-1}, s'_{f-2},\ldots, s'_{f-1}, s'_{f-1-i}) = (p - s_{f-1} - 2, 0,\ldots, 0, s_{f-1-i} + 1);

\sum_{j \in T} p^{f-1-j}t'_j \equiv 1 - s'_{f-1} - \sum_{j \in T} p^{f-1-j}t_j \mod p^f - 1 \text{ (Type I.)}

When $i = 1$, this implies $\text{Ext}^1_{\mathfrak{L}_{\text{GL}_2}(k)}(V_{\tilde{c},z}, V_{\tilde{d},\tilde{z}}) \neq 0$.

(b) $(s_{f-1}, s_{f-2},\ldots, s_0) = (\in [0, p-3], p-1,\ldots, p-1)$;

$(s'_{f-1}, s'_{f-2},\ldots, s_0) = (p - 3 - s_{f-1}, 0,\ldots, 0);

\sum_{j \in T} p^{f-1-j}t'_j \equiv 1 - s'_{f-1} - \sum_{j \in T} p^{f-1-j}t_j \mod p^f - 1 \text{ (Type I.)}

(c) $(s_{f-1}, s_{f-2}, s_{f-3},\ldots, s_0) = (p-1, p-2, p-1,\ldots, p-1)$;

$(s'_{f-1}, s'_{f-2}, s'_{f-3},\ldots, s_0) = (p-2, 0, 0,\ldots, 0);

\sum_{j \in T} p^{f-1-j}t'_j \equiv 1 - s'_{f-1} - \sum_{j \in T} p^{f-1-j}t_j \mod p^f - 1 \text{ (Type I.)}

When $f = 2$, this implies $\text{Ext}^1_{\mathfrak{L}_{\text{GL}_2}(k)}(V_{\tilde{c},z}, V_{\tilde{d},\tilde{z}}) \neq 0$.

When $e = 1$, $f > 1$, we additionally have:

(d) $(s_{f-1}, s_{f-2}) = (p-1, \in [0, p-3])$;

$(s'_{f-1}, s'_{f-2}) = (p-1, s_{f-2} + 2);

\sum_{j \in T} p^{f-1-j}t'_j \equiv -p + \sum_{j \in T} p^{f-1-j}t_j \mod p^f - 1 \text{ (Type I.)}

This implies $\text{Ext}^1_{\mathfrak{L}_{\text{GL}_2}(k)}(V_{\tilde{c},z}, V_{\tilde{d},\tilde{z}}) = 0$, $\text{Hom}_{\mathfrak{L}_{\text{GL}_2}(k)}(V_{\tilde{c},z}, H^1(K_1, V_{\tilde{d},\tilde{z}})) \neq 0$.

(e) $f > 2$,

$(s_{f-1}, s_{f-2}, s_{f-3},\ldots, s_{f-i}, s'_{f-i}) = (p-1, p-1, p-1,\ldots, p-1, \in [0, p-2])$, where $i \in [2, f-1]$;

$(s'_{f-1}, s'_{f-2}, s'_{f-3},\ldots, s'_{f-i}, s'_{f-i}) = (p-1, 1, 0,\ldots, 0, s_{f-i} + 1);

\sum_{j \in T} p^{f-1-j}t'_j \equiv 1 - s'_{f-1} - \sum_{j \in T} p^{f-1-j}t_j \mod p^f - 1 \text{ (Type I.)}

(f) $f = 2$,

$(s_{f-1}, s_{f-2}) = (p-2, p-1)$;

$(s'_{f-1}, s'_{f-2}) = (p-1, 1);

\sum_{j \in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} - \sum_{j \in T} p^{f-1-j}t_j \mod p^f - 1 \text{ (Type I.)}

(g) $f > 2$,

$(s_{f-1}, s_{f-2}, s_{f-3},\ldots, s_0) = (p-2, p-1, p-1,\ldots, p-1)$;

$(s'_{f-1}, s'_{f-2}, s'_{f-3},\ldots, s_0) = (p-1, 1, 0,\ldots, 0);

\sum_{j \in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} - \sum_{j \in T} p^{f-1-j}t_j \mod p^f - 1 \text{ (Type I.)}

(h) $(s_{f-1}, s_{f-2},\ldots, s_{f-i}, s_{f-2-i}) = (\in [0, p-2], 0,\ldots, 0, \in [1, p-2])$ for some $i \in [1, f-2]$;

$(s'_{f-1}, s'_{f-2},\ldots, s'_{f-i}, s'_{f-2-i}) = (p - s_{f-1} - 2, p-1,\ldots, p-1, s_{f-2-i} + 1);

s_{f-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^{i+1} + \sum_{j=0}^{f-1} t'_j \mod p^f - 1 \text{ (Type II.)}$.
(i) \( f > 2 \),
\[
(s_{j-1}, s_{j-2}, \ldots, s_{j-1-i}, s_{j-2-i}, s_{j-3-i}, \ldots, s_{f-m}, s_{f-1-m}) = (\in [0, p - 2], \ldots, 0, 0, 0, \ldots, 0, \in [1, p - 1]) \text{ for some } m \in [3, f - 1];
\]
\[
(s_{j-1}', s_{j-2}', \ldots, s_{j-1-i}', s_{j-2-i}', s_{j-3-i}', \ldots, s_{f-m}', s_{f-1-m}') = (p - s_{j-1}, p - 1, \ldots, p - 1, 1, 0, \ldots, 0, s_{j-1-m}') \text{ where } i \in [1, m - 2];
\]
\[
s_{j-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^i(s_{j-1-i} + 1) + \sum_{j=0}^{f-1} t_j' \mod p^f - 1 \text{ (Type II).}
\]

(j) \( f > 2 \),
\[
(s_{j-1}, s_{j-2}, \ldots, s_1, s_0) = (\in [0, p - 3], 0, \ldots, 0, 0);
\]
\[
(s_{j-1}', s_{j-2}', \ldots, s_1', s_0') = (p - 1 - s_{j-1}, p - 1, \ldots, p - 1, p - 1);
\]
\[
s_{j-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^{f-1}(s_0' + 1) + \sum_{j=0}^{f-1} t_j' \mod p^f - 1 \text{ (Type II).}
\]

(k) \( f = 2 \),
\[
(s_{j-1}, s_{j-2}) = (p - 2, 0);
\]
\[
(s_{j-1}', s_{j-2}') = (1, p - 1);
\]
\[
s_{j-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^2 + \sum_{j=0}^{f-1} t_j' \mod p^f - 1 \text{ (Type II).}
\]

(n) \( f > 2 \),
\[
(s_{j-1}, s_{j-2}, s_{j-3}, s_{j-4}, \ldots, s_0) = (p - 2, 0, 0, \ldots, 0);
\]
\[
(s_{j-1}', s_{j-2}', s_{j-3}', \ldots, s_0') = (0, p - 1, 1, 0, \ldots, 0);
\]
\[
s_{j-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p(s_{j-2} + 1) + \sum_{j=0}^{f-1} t_j' \mod p^f - 1 \text{ (Type II).}
\]

(o) \( f > 2 \),
\[
(s_{j-1}, s_{j-2}, s_{j-3}, \ldots, s_{j-1-i}, s_{j-2-i}) = (p - 1, 1, 0, \ldots, 0, \in [1, p - 2]) \text{ where } i > 1;
\]
\[
(s_{j-1}', s_{j-2}', s_{j-3}', \ldots, s_{j-1-i}', s_{j-2-i}') = (p - 1 - p, 1 - p, \ldots, p - 1, s_{j-2-i} + 1);
\]
\[
s_{j-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^i(s_{j-1-i} + 1) + \sum_{j=0}^{f-1} t_j' \mod p^f - 1 \text{ (Type II).}
\]

(p) \( f > 2 \)
\[
(s_{j-1}, s_{j-2}, s_{j-3}, \ldots, s_1, s_0) = (p - 1, 1, 0, \ldots, 0, p - 1);
\]
\[
(s_{j-1}', s_{j-2}', s_{j-3}', \ldots, s_1', s_0') = (1, 0, 0, \ldots, 0, p - 1);
\]
\[
s_{j-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^{f-1}(s_0' + 1) + \sum_{j=0}^{f-1} t_j' \mod p^f - 1 \text{ (Type II).}
\]

When \( e > 1 \), we additionally have:

(q) \( s_{f-1} \leq p - 3 \),
\[
s_{f-1}' = s_{f-1} + 2;
\]
\[
\sum_{j \in T} p^{f-1-j} t_j' \equiv -1 + \sum_{j \in T} p^{f-1-j} t_j \mod p^f - 1 \text{ (Type I).}
\]

This implies \( \text{Ext}_{\mathbb{F}_p}^1(G_{L_2(k)})(V_{\ell, s}, V_{\ell, s'}) = 0 \), but \( \text{Ext}_{\mathbb{F}_p}^1(G_{L_2(k)})(V_{\ell, s}, V_{\ell, s'}) \neq 0 \).
(r) \( s_{f+1} = (p-1, p-1, \ldots, p-1, \in [0, p-2]) \), where
\[ \sum_{j \in T} p^{f-1-j} t_j' \equiv 1 - \sum_{j \in T} p^{f-1-j} t_j \mod p^f - 1 \] (Type I).
(s) \( (s_{f+1}, s_{f+2}, \ldots, s_0) = (p-2, p-1, \ldots, p-1) \);
\[ \sum_{j \in T} p^{f-1-j} t_j' \equiv 1 - \sum_{j \in T} p^{f-1-j} t_j \mod p^f - 1 \] (Type I).
(t) \( (s_{f+1}, s_{f+2}, \ldots, s_{i+1}, s_i) = (1, 0, \ldots, 0, \in [0, p-2]) \), where \( i < f - 1 \);
\[ \sum_{j \in T} t_j' \equiv 1 - p^{f-1-1} + \sum_{j \in T} t_j \mod p^f - 1 \] (Type II).
(u) \( s_{f+1} = s_{f+1} - 1, (s_{f+1}, s_{f+1}, \ldots, s_0) = (1, 0, \ldots, 0) \);
\[ \sum_{j \in T} t_j' \equiv 1 - p^{f-1-i} + \sum_{j \in T} t_j \mod p^f - 1 \] (Type II).
(v) \( (s_{f+1}, s_{f+2}, \ldots, s_0) = (1, 0, \ldots, 0, 0) \);
\[ \sum_{j \in T} p^{f-1-j} t_j' \equiv s_{f+1} - 1 + \sum_{j \in T} p^{f-1-j} t_j \mod p^f - 1 \] (Type II).
(w) \( (s_{f+1}, s_{f+2}, \ldots, s_{i+1}, s_i) = (1, 0, \ldots, 0, \in [0, p-2]) \) for some
\[ \sum_{j \in T} t_j' \equiv 1 - p^{f-1-i} + \sum_{j \in T} t_j \mod p^f - 1 \] (Type II).

**Proof.** By Section 4.13, we need to find the criteria for when there exist two \( G_K \)
characters \( \chi_1 \) and \( \chi_2 \) such that \( \text{LV}_{\bar{\rho}, \bar{\epsilon}', \bar{\epsilon}}(\chi_1, \chi_2) \cap \text{LV}_{\bar{\rho}, \bar{\epsilon}', \bar{\epsilon}}(\chi_1, \chi_2) \subset \text{Ext}^1_{G_K}(\chi_2, \chi_1) \) has dimension \( ef - 1 \) and the same is true for most unramified twists of \( \chi_1 \) and \( \chi_2 \). The criteria are covered in Propositions 5.2, 5.6, 6.2, 6.4, 6.18, 6.20 and 6.24. In each of these, we have constraints on \( \bar{s} \) and \( \bar{s}' \) that do not depend on \( \bar{t} \) and \( \bar{t}' \). We similarly have constraints on \( \sum_{j=0}^{f-1} p^{f-1-j} (t_j - s_j') - \sum_{j=0}^{f-1} p^{f-1-j} (t_j - s_j) \). Collectively, the two sets of constraints define the criteria completely. By Lemma 4.24, in this situation, \( \sum_{j=0}^{f-1} p^{f-1-j} (t_j - s_j') - \sum_{j=0}^{f-1} p^{f-1-j} (t_j - s_j) \equiv \sum_{j=0}^{f-1} p^{f-1-j} (t_j - s_j) \mod p^f - 1 \).

Therefore, there exist two \( G_K \) characters \( \chi_1 \) and \( \chi_2 \) such that \( \text{LV}_{\bar{\rho}, \bar{\epsilon}', \bar{\epsilon}}(\chi_1, \chi_2) \cap \text{LV}_{\bar{\rho}, \bar{\epsilon}', \bar{\epsilon}}(\chi_1, \chi_2) \) has dimension \( ef - 1 \) for most unramified twists of \( \chi_1 \) and \( \chi_2 \) if and only if there exist two \( G_K \) characters \( \chi'_1 \) and \( \chi'_2 \) such that \( \text{LV}_{\bar{\rho}, \bar{\epsilon}}(\chi'_1, \chi'_2) \cap \text{LV}_{\bar{\rho}, \bar{\epsilon}}(\chi'_1, \chi'_2) \subset \text{Ext}^1_{G_K}(\chi'_2, \chi'_1) \) has dimension \( ef - 1 \) for most unramified twists of \( \chi'_1 \) and \( \chi'_2 \). In other words, if and only if \( V_{\bar{\rho}, \bar{\epsilon}}(\chi'_1, \chi'_2) \) satisfies the criteria in one of Propositions 5.2, 5.6, 6.2, 6.4, 6.18, 6.20 and 6.24.

The above criteria are necessary and sufficient when \( K \neq \mathbb{Q}_p \). When \( K = \mathbb{Q}_p \), \( Z_{\bar{\rho}, \bar{\epsilon}}(\chi_1, \chi_2) \) is codimension 1 if and only if either the above criteria hold or the intersection contains an irreducible representation (by Proposition 3.8). The criterion for existence of irreducible representations in \( Z_{\bar{\rho}, \bar{\epsilon}}(\chi_1, \chi_2) \) is given in Lemma 4.14.
Putting all the criteria together gives the list in the statement of the Theorem, along with Proposition 2.14 and Corollary 2.10 on computations of extensions of Serre weights as \( \text{GL}_2(\mathcal{O}_K) \)-modules.

Remark 7.2. In fact, our criteria show that when \( K/\mathbb{Q}_p \) is ramified and \( \sigma \) and \( \tau \) are non-isomorphic, non-Steinberg Serre weights, then

\[
\text{Ext}^1_{\mathcal{F}_{\text{GL}_2(\mathcal{O}_K)}}(\sigma, \tau) \neq 0 \implies \dim Z_{\sigma} \cap Z_{\tau} = [K : \mathbb{Q}_p] - 1.
\]

When \( K/\mathbb{Q}_p \) is unramified, we cannot yet make such a statement because we don’t have a complete description of \( \text{Ext}^1_{\mathcal{F}_{\text{GL}_2(\mathcal{O}_K)}}(\sigma, \tau) \) when the Serre weights are not weakly regular (see Proposition 2.14).

\[ \square \]

Theorem 7.3. In the setup of Theorem 7.1, assume that \( V_{\vec{t}, \vec{s}} \) and \( V_{\vec{t}', \vec{s}'} \) are both weakly regular and that \( Z_{V_{\vec{t}, \vec{s}}} \) and \( Z_{V_{\vec{t}', \vec{s}'}} \) intersect in codimension 1. Let \( n \) be the number of \( [K : \mathbb{Q}_p] - 1 \) dimensional irreducible components in the intersection. Then the following are true:

(i) If \( e = 1 \), then \( n = 1 \).

None of these components of dimension \( [K : \mathbb{Q}_p] - 1 \) are contained in triple intersections of irreducible components of \( Z \).

(ii) If \( e > 1 \) and \( f = 1 \), then

\[
n = \begin{cases} 
2 & \text{if } \text{Ext}^1_{\mathcal{F}_{\text{GL}_2(k)}}(V_{\vec{t}, \vec{s}}, V_{\vec{t}', \vec{s}'}) \neq 0 \\
1 & \text{if } \text{Ext}^1_{\mathcal{F}_{\text{GL}_2(k)}}(V_{\vec{t}, \vec{s}}, V_{\vec{t}', \vec{s}'}) = 0
\end{cases}
\]

If \( s, s' < p - 3 \), then each component of dimension \( [K : \mathbb{Q}_p] - 1 \) is contained in a triple intersection.

(iii) If \( e > 1 \) and \( f > 1 \), then

\[
n = \begin{cases} 
2 & \text{if } \text{Ext}^1_{\mathcal{F}_{\text{GL}_2(k)}}(V_{\vec{t}, \vec{s}}, V_{\vec{t}', \vec{s}'}) \neq 0 \\
1 & \text{if } \text{Ext}^1_{\mathcal{F}_{\text{GL}_2(k)}}(V_{\vec{t}, \vec{s}}, V_{\vec{t}', \vec{s}'}) = 0
\end{cases}
\]

Each of these components of dimension \( [K : \mathbb{Q}_p] - 1 \) is contained in a triple intersection.

Proof. When \( e = 1 \), the statements are a consequence of collating criteria for type I and type II intersections in Corollary 5.7 and Proposition 6.4. When \( f = 1 \), the relevant results are in Propositions 5.2 and 6.2. When \( e > 1, f > 1 \), they are in Corollary 5.7 and Remark 6.25.

\[ \square \]

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