Anomalous dimensions and ghost decoupling in a perturbative approach to the generalized chiral Schwinger model (*)

A. Bassetto

*Dipartimento di Fisica “G. Galilei”, Via Marzolo 8 – 35131 Padova, Italy
and INFN, Sezione di Padova, Italy

L. Griguolo

*SISSA, Via Beirut 2 - 34100 Trieste, Italy and INFN, Sezione di Trieste, Italy

Abstract

A generalized chiral Schwinger model is studied by means of perturbative techniques. Explicit expressions are obtained, both for bosonic and fermionic propagators, and compared to the ones derived by means of functional techniques. In particular a consistent recipe is proposed to describe the ambiguity occurring in the regularization of the fermionic determinant. The role of the gauge fixing term, which is needed to develop perturbation theory and the behaviour of the spectrum as a function of the parameters are clarified together with ultraviolet and infrared properties of the model.

(*) This work is carried out in the framework of the European Community Research Programme “Gauge theories, applied supersymmetry and quantum gravity” with a financial contribution under contract SC1-CT92-D789.

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I. INTRODUCTION

It is well known that two–dimensional gauge theories admit a consistent interpretation even in presence of local anomalies: the unitarity is recovered exploiting non– standard regularization procedure [1] or describing, by means of an appropriate Wess–Zumino action, the new degrees of freedom introduced by the anomaly [2]. It is clear that an analogous higher–dimensional result might be crucial for an alternative, and perhaps deeper, understanding of the standard model and of the superstring theory.

Unfortunately at present we have no evidence for a satisfactory four–dimensional gauge theory with local anomalies: it fails perturbatively to be either renormalizable or unitary [3] while, beyond perturbation theory, there is no real control on it and its physical interpretation seems indeed very obscure [4].

These problems do not exist in $d = 1 + 1$, where the non–perturbative region can be explored by means of powerful techniques like bosonization [5], conformal field theory [6], form factor approach [7], $1/N$ expansion [8]; exact solutions for some classes of models are also available.

In particular we can study the relation between perturbative and non–perturbative solution of a simple two–dimensional abelian gauge model, namely the so–called generalized chiral Schwinger model [9]. This theory is not completely dull, presenting two quite different regions in its parameter space: in the first one the fermionic states are infraparticles described by a Thirring model while a massive and a massless state appear in the bosonic sector; in the second case fermions are confined. The interaction gives rise to an ultraviolet renormalization costant for the fermion field, encoding the information of ultraviolet scaling.

Previous partial investigations were concerned with the bosonic sector of the chiral Schwinger model ($r^2 = 1$ in our parameter space), which is less interesting from the spectrum point of view (no infrared dressing of the fermions [10]).

In this paper we firstly present the resummation of the perturbative expansion for the boson propagator, starting from the Feynman diagrams: in order to develop the Feynman
rules we have to introduce a gauge fixing.

In the non–perturbative context, where gauge invariance is naturally broken by the anomaly, this amounts to studying different theories for different gauge fixings. The limit of vanishing gauge fixing will be performed after resummation. A lot of interesting features will be hidden in this limit.

The same propagator will also be obtained by path–integral techniques (See Appendix). In both procedures we have developed a systematic method to control the ambiguity related to regularization, clarifying the way in which the Jackiw–Ramarajan parameter $\Gamma$ is produced. Then, studying the bosonic spectrum, we follow the decoupling of ghost particles from the theory in the limit of vanishing gauge fixing, to recover the known result $\Omega$.

The fermionic correlation functions are also examined, leading to the correct Thirring behaviour in the non–perturbative limit; nevertheless we find very different ultraviolet scalings before and after the gauge–fixing removal, related to the appearance of an ultraviolet renormalization constant.

Decoupling of heavy states is indeed not trivial when anomalies are present $\Pi$.

II. BOSON PROPAGATOR AND THE REGULARIZATION AMBIGUITY

We want to study the quantum theory in $d = 1 + 1$ related to the classical Langrangian density $\Pi$:

\[
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^{\mu}[i\partial_{\mu} + e\left(\frac{1 + r\gamma_5}{2}\right)A_{\mu}]\psi,
\]

$F_{\mu\nu}$ is the usual field tensor, $A_\mu$ the vector potential and $\psi$ a massless Dirac spinor. The quantity $r$ is a real parameter interpolating between the vector ($r = 0$) and the chiral ($r^2 = 1$) Schwinger model. Our notations are

\[
g_{00} = -g_{11} = 1, \quad \epsilon^{01} = -\epsilon_{01} = 1,
\]

\[
\gamma^0 = \sigma_1, \quad \gamma^1 = -i\sigma_2,
\]

\[
\gamma_5 = \sigma_3, \quad \tilde{\partial}_\mu = \epsilon_{\mu\nu}\partial^\nu,
\]

\[\text{(2)}\]
\( \sigma_i \) being the usual Pauli matrices.

The Green function generating functional is

\[
W[J; \bar{\eta}, \eta] = N \int D\!A \, \mathcal{D}\!\bar{\psi} \, \mathcal{D}\!\psi \, \exp i \int d^2 x (\mathcal{L} + \mathcal{L}_s),
\]

where \( N \) is a normalization constant and

\[
\mathcal{L}_s = J_\mu A^\mu + \bar{\eta} \psi + \bar{\psi} \eta,
\]

\( J_\mu, \eta \) and \( \bar{\eta} \) being vector and spinor sources respectively.

In order to start a perturbative expansion, one has to break the classical gauge invariance of eq.(1) adding a gauge-fixing term: we use a generalized Lorentz gauge:

\[
\mathcal{L}_{gf} = \frac{1}{2\alpha} (\partial_\mu A^\mu)^2,
\]

\( \alpha \in \mathbb{R} \).

In a standard gauge theory physical observables do not depend on the particular form of the gauge-fixing term. But the Ward identities of this theory are modified by the presence of an anomaly in the conservation law of the dynamical current

\[
J^\mu_r(x) = e \bar{\psi} \left( \frac{1 - r\gamma_5}{2} \right) \gamma^\mu \psi;
\]

at quantum level gauge invariance is broken and different values of \( \alpha \) do correspond to different theories. We will be eventually interested in the limit \( \alpha \to \infty \) (no gauge fixing).

The Feynman propagators associated with \( \mathcal{L} + \mathcal{L}_{gf} \) are given (in the momentum space) by:

\[
G^0_{\mu\nu}(k) = -\frac{i}{k^2 + i\varepsilon} \left[ g_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2} \right],
\]

\[
S^0_F(k) = \frac{i \gamma^\mu k^\mu}{k^2 + i\varepsilon},
\]

and the vertex is

\[
T_\mu = ie \left( \frac{1 - r\gamma_5}{2} \right) \gamma^\mu.
\]
Let us look at the perturbative expansion for the boson propagator: it is well known \[12\] that, in these kind of theories, the only non–vanishing one particle–irreducible graph, giving contribution to the two–point Green functions of \(A_\mu\), is:

\[
\Pi_{\mu\nu}(p) = -\int \frac{d^2 k}{(2\pi)^2} Tr \left[ T_\mu S^0_F(k) T_\nu S^0_F(p-k) \right].
\]  

(10)

The full propagator should be obtained by summing the geometrical series:

\[
G_{\mu\nu}(p) = G^0_{\mu\nu}(p) + G^0_{\mu\rho}(p)\Pi^{\rho\lambda}(p)G^0_{\lambda\nu}(p) + \\
+ G^0_{\mu\rho}(p)\Pi^{\rho\lambda}(p)G^0_{\lambda\gamma}(p)\Pi^{\gamma\delta}(p)G^0_{\delta\nu}(p) + \ldots
\]  

(11)

Actually there is an ambiguity in the calculation of \(\Pi_{\mu\nu}\) arising from the need of regularizing the logarithmically divergent integral in eq.(10): \(\Pi_{\mu\nu}\) does not obey the classical Ward identity, no matter the regularization we choose, so there is no privileged choice in fixing the local terms in eq.(11). Nevertheless dimensional regularization \[13\] provides a well defined and systematic way to compute divergent diagrams in absence of \(\gamma_5\) couplings. When \(\gamma_5\) occurs, Breitenlohner and Mason (B–M) have developed in \[14\] a consistent formalism to define \(\gamma_5\) as well as the totally antisymmetric tensor within dimensional regularization; chiral anomalies appear very naturally in this framework.

In order to reproduce the ambiguity which is intrinsic in the regularization, we generalize the B–M formalism, showing that there is a one–parameter family of consistent definitions of \(\gamma_5\) and \(\epsilon_{\mu\nu}\) in \(d = 2n\), reproducing the usual one at \(d = 2\). The parameter describing the regularization is the origin of the Jackiw–Ramarajan phenomenon; other schemes leading to analogous results are presented in \[15\], \[16\] but they are not obtained as generalizations of the B–M formalism.

We start from the usual properties in \(d = 2n\)

\[
g_{\mu\nu}g^{\nu}_\lambda = g_{\mu\lambda}, \quad g_{\mu\nu} = g_{\nu\mu}, \\
g_{\mu\nu}k^{\nu} = k_\mu, \quad g^{\mu} = 2n, \\
g_{\mu\nu}\gamma^{\nu} = \gamma_\mu, \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}1.
\]  

(12)
As in B–M we write

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{g}_{\mu\nu} \]  

(13)

with \( \hat{g}_{\mu\nu} \) carrying indices beyond the “physical” dimension \( d = 2 \) and we get:

\[ g_{\mu}^{\nu} \hat{g}_{\nu\lambda} = \hat{g}_{\nu}^{\mu} \hat{g}_{\nu\lambda} = \hat{g}_{\mu\lambda}, \]

\[ \hat{g}_{\mu\nu} = \hat{g}_{\nu\mu}, \]

\[ \hat{g}_{\mu\nu} k^{\nu} = \hat{k}_{\mu}, \]

\[ \bar{g}_{\mu\nu} \hat{\gamma}^{\nu} = \hat{\gamma}_{\mu}, \]

\[ \hat{g}_{\mu\nu} \hat{\gamma}^{\nu} = \hat{\gamma}_{\mu}, \]  

(14)

\( \hat{\gamma}_{\mu} \) running on the extra dimensions. Now we just modify the B–M definition of \( \epsilon_{\mu\nu} \), so as to obtain

\[ \epsilon_{\mu_1\nu_2} \epsilon_{\nu_1\nu_2} = -\sum_{\pi \in S_2} (-1)^{\pi} \Pi_{i=1}^{2} (g_{\mu_i\nu_{\pi(i)}} - b\bar{g}_{\mu_i\nu_{\pi(i)}}), \]  

(15)

\( S_2 \) being the permutation group of two objects \( (S_2 = Z_2) \) and \( b \) a real parameter; the B–M definition corresponds to \( b = 1 \). It is easy to prove that:

\[ g_{\mu\nu} \hat{\gamma}^{\nu} = \hat{g}_{\mu\nu} \hat{\gamma}^{\nu} = \hat{\gamma}_{\mu} \]  

(16)

\[ \{ \gamma_{\mu}, \hat{\gamma}_{\nu} \} = \{ \hat{\gamma}_{\mu}, \gamma_{\nu} \} = 2\hat{g}_{\mu\nu} \mathbb{I} \]  

(17)

\[ \epsilon_{\mu_1\nu_2} = -\epsilon_{\mu_2\nu_1} \]  

(18)

\[ \hat{g}_{\mu}^{\mu} = 2n - 2 \]  

(19)

We define \( \gamma_5 \) as:

\[ \gamma_5 = \frac{1}{2\beta} \epsilon_{\mu\nu} \hat{\gamma}^{\mu} \hat{\gamma}^{\nu}, \]

\[ \beta^2 = 2n^2(1 - b)^2 + n(1 - 5b)(b - 1) + (3b^2 - 2b). \]  

(20)
The normalization is chosen so as to get \( \gamma_5^2 = \mathbb{1} \).

This definition coincides with the B–M one \( (b = 1) \) and the limit \( n = 1 \) \( (d = 2) \) is smooth. Then we define a dual algebra by:

\[
\tilde{\gamma}_\mu = \frac{1}{2\beta} \epsilon_{\mu\nu} \gamma^\nu. \tag{21}
\]

It follows that

\[
\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_1 \hat{g}_{\mu\nu} + 2\delta_2 g_{\mu\nu}, \tag{22}
\]

\[
\{\tilde{\gamma}_\mu, \gamma_\nu\} = \frac{1}{\beta} \epsilon_{\mu\nu}. \tag{23}
\]

with

\[
\delta_1 = \frac{b}{4\beta^2} [2n - b(2n - 2) + b - 2],
\]

\[
\delta_2 = -\frac{1}{4\beta^2} [2n - b(2n - 2) - 1] \tag{24}
\]

and

\[
\gamma_5 = \gamma_\mu \tilde{\gamma}_\mu. \tag{25}
\]

From eq.(25) and the algebras in eqs.(17), (23), we are able to find the relevant anticommutator \( \{\gamma_5, \gamma_\mu\} \):

\[
\{\gamma_5, \gamma_\mu\} = 2\gamma_5 \gamma_\mu - 4\tilde{\gamma}_\mu. \tag{26}
\]

One can easily check that the B–M result is recovered for \( b = 1 \). For \( b \neq 1 \) we notice that the anticommutator \( \{\gamma_5, \gamma_\mu\} \) has a term involving also \( \tilde{\gamma}_\mu \) and vice versa, at variance with the case \( b = 1 \). Using eq.(25) and the algebra in eqs.(17), (22), (23), all the traces can be computed.

The parameter \( b \) actually describes a one–parameter family of consistent dimensional regularizations, which differ by the definition of \( \gamma_5 \) and \( \epsilon_{\mu\nu} \) and reduce to the ordinary one in physical dimensions.
The relevant Feynman integral is

$$\frac{e^2}{16\pi^2} \int d^2k \frac{i}{(p-k)^2 + i\varepsilon k^2 + i\varepsilon} (p-k)^\lambda k^\rho \cdot Tr[\gamma_\mu(1 + r\gamma_5)\gamma_\lambda\gamma_\nu(1 + r\gamma_5)\gamma_\rho].$$

(27)

One easily gets:

$$\frac{e^2}{16\pi^2} \int d^2k \frac{i}{(p-k)^2 + i\varepsilon k^2 + i\varepsilon} (p-k)^\lambda k^\rho (\mu^2)^{1-n} =$$

$$\frac{ie^{2\pi n}}{16\pi^2} \Gamma^2(n) \Gamma(2-n) \left[ \frac{g^{\rho\lambda}}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2} \right] (-p^2/\mu^2)^{n-1}$$

(28)

\(\mu\) being a subtraction mass introduced by dimensional regularization. The trace part gives:

$$Tr[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\lambda] \left[ g^{\rho\lambda} \frac{1}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2} \right] =$$

$$-4(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) + O(n-1),$$

(29)

$$Tr[\gamma_\mu\gamma_5\gamma_\lambda\gamma_\nu\gamma_\rho + \gamma_\mu\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho] \left[ g^{\rho\lambda} \frac{1}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2} \right] =$$

$$-4[\frac{p_\nu p_\mu}{p^2} + \frac{p_\mu p_\nu}{p^2}] + O(n-1),$$

(30)

that do not involve the “ambiguity” parameter, while

$$Tr[\gamma_\mu\gamma_5\gamma_5\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho] \left[ g^{\rho\lambda} \frac{1}{2(n-1)} + \frac{p^\rho p^\lambda}{p^2} \right]$$

(31)

consists of an “unambiguous” piece

$$Tr[\gamma_\mu\gamma_5\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho] \frac{p^\rho p^\lambda}{p^2} = -2g_{\mu\nu} + 4\frac{p_\mu p_\nu}{p^2} + O(n-1)$$

(32)

and a \(b\)-dependent one

$$Tr[\gamma_\nu\gamma_5\gamma_5\gamma_\lambda\gamma_\nu\gamma_5\gamma_\rho] g^{\rho\lambda} \frac{1}{2(n-1)} =$$

$$4 (n-1)[1 + 2(1 - b^2)] \frac{1}{2(n-1)} g_{\mu\nu} + O(n-1) =$$

$$2 [1 + 2(1 - b^2)] g_{\mu\nu} + O(n-1).$$

(33)

Collecting all the terms with the appropriate coefficients and taking the limit \(n = 1\), we obtain
\[ \Pi_{\mu\nu}(p) = \frac{ie^2}{4\pi} [g_{\mu\nu}(1 - r^2(1 - b^2)) - (1 + r^2) \frac{p_\mu p_\nu}{p^2} + \frac{1}{p^2}(p_\mu \tilde{p}_\nu + p_\nu \tilde{p}_\mu)]. \]  

(34)

The relation between the J–R parameter and \( b \) is:

\[ a = r^2(b^2 - 1) \]  

(35)

We notice that a natural bonus of this procedure is to get \( a = 0 \) for \( b = 1 \) (B–M scheme) and for \( r = 0 \) (gauge invariant theory). In the computation we have disregarded terms with \( \tilde{\gamma}_\mu \) and \( \tilde{p}_\mu \) on the external legs: we do not lose any information because the (geometrical) sum of the vacuum polarization does not involve overlapping divergences.

The resummation is now straightforward and is reported in appendix A. We define

\[ m^2_\pm = e^2 \mu^2_\pm, \]  

(36)

with

\[ \mu^2_\pm = \frac{1}{8\pi} \left[ \alpha(a - r^2) + (1 + a) \pm \sqrt{\left[ 1 + a - \alpha(a - r^2)^2 \right] - 4\alpha r^2} \right]. \]  

(37)

The final result is:

\[ G_{\mu\nu}(k) = i \frac{1}{p^2 - m^2_+} \frac{1}{p^2 - m^2_-} \left[ (-p^2 g_{\mu\nu} + (1 - \alpha)p_\mu p_\nu) + \frac{e^2}{4\pi} \right] \left[ (\alpha(a - r^2)g_{\mu\nu} + \alpha(1 + r^2) \frac{p_\mu p_\nu}{p^2} - \alpha r \frac{1}{p^2}(k_\mu \tilde{p}_\nu + p_\nu \tilde{p}_\mu) \right]. \]  

(38)

first in the regions \( \left| \frac{m^2_\pm}{p^2} \right| < 1 \) which correspond to the following convergence disc in the complex plane of the coupling constant \( e^2 < \frac{|p^2|}{\mu^2} \), and then everywhere by analytic continuation.

We can recover eq.(38) without resumming the perturbative series, by exactly computing the generating functional for the bosonic Green function. This approach is more efficient, especially in the fermionic case, where the perturbative expansion is much more involved.

We feel however instructive to obtain the result by summing Feynman graphs for the bosonic propagator, deferring the functional integration to the Appendix B.
III. THE BOSONIC SPECTRUM

This section is devoted to the study of the bosonic spectrum of the theory and of its behaviour in the non-perturbative limit ($|\alpha| \to \infty$). We can compare the present result with the exact non-perturbative solution obtained in [9]: a non trivial decoupling takes place in the Hilbert space of the model to recover the spectrum. We can even understand the emerging of a consistent theory from one which violates unitarity.

We briefly recall the bosonic content of the non-perturbative solution: two different regions on the parameters space $(r, a)$ admit “physical” interpretation (no tachyons)

$$a > r^2,$$  \hspace{1cm} (39)

$$r^2 > 1, \quad 0 < a < r^2 - 1,$$

$$r^2 < 1, \quad r^2 - 1 < a < 0.$$  \hspace{1cm} (40)

In the region described by eq.(39) a boson of mass

$$m^2 = \frac{e^2}{4\pi} \frac{a(a + 1 - r^2)}{a - r^2}$$  \hspace{1cm} (41)

exists together with a massless excitation.

In the other region only the massive excitation is physical, the massless one being a probability ghost that however can be consistently expunged from the Hilbert space by means of a subsidiary condition.

Now the easiest way of reading the physical content in the bosonic sector of the theories with gauge fixing is to study the singularities of the propagator eq.(38): $G_{\mu\nu}$ exhibits three different poles respectively at $k^2 = m_+^2$, $k^2 = m_-^2$ and $k^2 = 0$. First of all we have to impose the condition

$$m_+^2 \geq 0$$  \hspace{1cm} (42)

which is necessary to have a particle interpretation for these poles (no tachyons): obviously inequality (42) selects a particular subregion of the whole parameter space $(\alpha, a, r)$. 
It leads to two different sets of inequalities:

\[ \alpha > 0, \]
\[ a(a + 1 - r^2) > 0, \]
\[ 1 + a + \alpha(a - r^2) > 0, \]
\[ [(1 + a) + \alpha(a - r^2)] - 4a\alpha(1 + a - r^2) > 0 \] (43)

and

\[ \alpha < 0, \]
\[ a(a + 1 - r^2) < 0, \]
\[ 1 + a + \alpha(a - r^2) < 0, \]
\[ [(1 + a) + \alpha(a - r^2)] - 4a\alpha(1 + a - r^2) > 0, \] (44)

the last inequality being forced from the reality condition of \( m_{\pm}^2 \); we do not consider the limiting situation of vanishing or equal masses.

It is not too difficult to solve inequalities (43) and (44) and the allowed regions of the parameters turn out to be:

**\( r^2 < 1: \)**

\[ \alpha > \frac{1}{r^2}; \quad a > \frac{1}{\alpha - 1}(1 + \sqrt{\alpha r^2})^2; \]
\[ 1 < \alpha < \frac{1}{r^2}; \quad 0 < a < \frac{1}{\alpha - 1}(1 - \sqrt{\alpha r^2})^2; \]
\[ 1 < \alpha < \frac{1}{r^2}; \quad a > \frac{1}{\alpha - 1}(1 + \sqrt{\alpha r^2})^2; \]
\[ r^2 < \alpha < 1; \quad a > 0; \]
\[ 0 < \alpha < r^2; \quad \frac{1}{\alpha - 1}(1 - \sqrt{\alpha r^2})^2 < a < r^2 - 1; \]
\[ 0 < \alpha < r^2; \quad a > 0; \]
\[ \alpha < 0; \quad r^2 - 1 < a < 0. \] (45)

**\( r^2 > 1: \)**
\[ \alpha > r^2 \ ; \quad a > \frac{1}{\alpha-1} (1 + \sqrt{\alpha} r^2)^2; \]
\[ 1 < \alpha < r^2 \ ; \quad r^2 - 1 < a < \frac{1}{\alpha-1} (1 - \sqrt{\alpha} r^2)^2; \]
\[ 1 < \alpha < r^2 \ ; \quad a > \frac{1}{\alpha-1} (1 + \sqrt{\alpha} r^2)^2; \]
\[ \frac{1}{r^2} < \alpha < 1 \ ; \quad a > r^2 - 1; \]
\[ 0 < \alpha < \frac{1}{r^2} \ ; \quad \frac{1}{\alpha-1} (1 - \sqrt{\alpha} r^2)^2 < a < 0; \]
\[ 0 < \alpha < \frac{1}{r^2} \ ; \quad a > r^2 - 1; \]
\[ \alpha < 0 \ ; \quad 0 < a < r^2 - 1. \] (46)

For any choice of \( r \) and \( \alpha \) a particular range of \( a \) is free from tachyons.

The next step is to study the unitarity on these poles by taking the residues of \( G_{\mu\nu}(k) \) at \( k^2 = m_+^2 \) and \( k^2 = 0 \) and forcing their positivity: we do not give the general result of this analysis, being the final parameter space rather involved. Because we are interested in the large \( |\alpha| \) behaviour, we give the details of the unitarity restrictions in the limit \( |\alpha| \to \infty \).

However one can easily verify that for any region in the parameter space, it never happens that all the three excitations are “physical”.

We notice that different regions are selected according to the sign of \( \alpha \).

For \( \alpha \to +\infty \) eqs. (45) and (46) implies:
\[ a > r^2 + O(\frac{1}{\sqrt{\alpha}}) \] (47)
while for \( \alpha \to -\infty \) we get exactly:
\[ r^2 - 1 < a < 0 \quad (r^2 < 1) \]
\[ 0 < a < r^2 - 1 \quad (r^2 > 1). \] (48)

The masses become, considering the appropriate range in the two limits:
\[ m_+^2 = \frac{e^2}{4\pi} (a - r^2)\alpha + \frac{e^2}{4\pi} \frac{r^2}{a - r^2} + O(\frac{1}{\alpha}), \] (49)
\[ m^2_+ = m^2 + O(\frac{1}{\alpha}). \] (50)

It is evident that \( m^2_+ \) goes to infinity with \(|\alpha|\), while \( m^2_- \) approaches the generalized J–R mass eq.\((41)\): the regions \((47)\) and \((48)\) coincide with \((A11), (38)\) respectively.

By taking in \( G_{\mu\nu}(k) \) the residue at \( k^2 = m^2_+ \), one gets

\[
-\text{Res} \ G_{\mu\nu}(k) \bigg|_{k^2 = m^2_+} = T^+_{\mu\nu}(k)
\]

\[
T^+_{\mu\nu}(k) = \frac{1}{e^2/4\pi \sqrt{[(1 + a) - \alpha(a - r^2)]^2 - 4\alpha r^2}} \cdot \left[ g_{\mu\nu}(-m^2_+ + \frac{e^2}{4\pi} \alpha(a - r^2)) + (1 - \alpha) k_{\mu}k_{\nu} + \alpha \frac{e^2}{4\pi} (1 + r^2) \frac{k_{\mu}k_{\nu}}{m^2_+} \right].
\] (51)

The determinant of \( T^+ \) vanishes, so that one eigenvalue is always zero: this corresponds to the decoupling of the would–be related excitation. The trace of \( T^+ \) gives the other eigenvalue:

\[
\text{Tr}(T^+) = \frac{1}{e^2/4\pi \sqrt{[(1 + a) - \alpha(a - r^2)]^2 - 4\alpha r^2}} \cdot \left[ (k^2_0 + k^2_1)(1 - \alpha) + \alpha \frac{e^2}{4\pi} \frac{(1 + r^2)}{m^2_+} \right] - 4\alpha \frac{e^2}{4\pi} \frac{k_0 k_1}{m^2_+}.
\] (52)

that, for \( \alpha \to \pm\infty \) becomes

\[
\text{Tr}(T^+) = -\alpha \frac{(k^2_0 + k^2_1)}{m^2_+} - 4 \frac{r}{a - r^2} \frac{k_0 k_1}{m^2_+} + O(\frac{1}{\alpha})
\] (53)

In the first region \((\alpha \to \infty)\) \( \text{Tr}(T^+) \) is negative and therefore the excitation of mass \( m^2_+ \) is a probability ghost, while, when \( \alpha \to -\infty \), it has “physical” meaning. We notice that the residue does not approach a finite value as \( m^2_- \) goes to infinity: it does not look like the naive decoupling one could expect.

The analysis for \( m^2_- \) is similar: we define \( T^-_{\mu\nu} \) as in eq.\((52)\) and \( \text{det}(T^-) \) turns and to be zero. For large \(|\alpha|\):

\[ \text{Tr}(T^-) = -\alpha \frac{(k^2_0 + k^2_1)}{m^2_-} - 4 \frac{r}{a - r^2} \frac{k_0 k_1}{m^2_-} + O(\frac{1}{\alpha}) \] (54)
\[ Tr(T^-) = \frac{1}{e^2/4\pi a - r^2} \left[ (k_0^2 + k_1^2) \left( 1 - \frac{e^2}{4\pi} \frac{(1 + r^2)}{m^2} \right) - 4r \frac{e^2 k_0 k_1}{4\pi m^2} \right] + O\left( \frac{1}{\alpha} \right). \]  \hspace{1cm} (54)

One can easily prove that in both limits

\[ Tr(T^-) > 0. \]  \hspace{1cm} (55)

The pole at \( m_-^2 \) is a “physical” particle and can be identified with the massive boson of eq.(41).

We are left with the massless pole at \( k^2 = 0 \): the definition of \( T^0_{\mu\nu} \) implies again a vanishing determinant and

\[ Tr[T^0] = \frac{\alpha e^2/4\pi}{m_+^2 m_-^2} \left[ (1 + r^2)(k_0^2 + k_1^2) - 4r k_0 k_1 \right]_{k^2=0} = \frac{1}{4} \frac{1}{(a - r^2)} (1 \pm r)^2 k_0^2. \]  \hspace{1cm} (56)

The massless pole appears to be “physical” in the first range \( \alpha \to +\infty, a > r^2 \) and a probability ghost in the second one: this is exactly the massless particle of [9].

It is quite unexpected that the residue at the massless pole does not depend an \( \alpha \): the massless sector is totally equivalent to the one in the non perturbative case. We shall find a similar behaviour in the fermionic sector.

In conclusion the non–perturbative bosonic spectrum of the generalized chiral Schwinger model is recovered, starting from the perturbation theory, in a subtle way. The first window in the parameter space \( (a > r^2) \) corresponds to the situation \( \alpha \to +\infty \). We obtain the boson of mass \( m_+^2 \) from \( m_-^2 \) and the massless excitation, together with a ghost of infinite mass and “infinite” residue.

The opposite regime \( \alpha \to -\infty \) leads to the window in eq.(40) where the massless boson is a ghost and the infinite massive state has a positive residue. If we perform the limits \( \alpha \to \pm \infty \), while keeping \( k_\mu \) fixed, the propagator eq.(A11) in the two cases, coincides with the non–perturbative one obtained in [9]: the infinite–mass boson seems to disappear from the theory if we look at the bosonic Green function.
But we have seen that its residue grows with \(|\alpha|\) and we do not expect a complete decoupling for more general Green functions (the fermionic ones for example), as we will see in the next section.

We end by recalling that in the first region unitarity is obtained by disregarding an infinite–massive ghost (decoupling in the bosonic Hilbert space), while in the second window no dynamical mechanism of this type is present and we have to expunge the ghost excitation by means of a subsidiary condition.

IV. THE FERMIONIC SPECTRUM

One of the most interesting feature of the generalized chiral Schwinger model is the appearance of a dynamically generated massless Thirring model, describing the fermionic sector of the spectrum in the first range of the parameters. One can prove [9] that fermionic correlation functions behave in the infrared limit as the ones of a massless Thirring model, in the spin–\(\frac{1}{2}\) representation, with coupling constant

\[ g^2 = \frac{1 - r^2}{a}; \quad (57) \]

The fermionic operator solving the quantum equation of motion was explicitly constructed in the form

\[ \psi(x) = \exp[F(m^2, x^2)]\psi_T(x) \quad (58) \]

with \(F(m^2, x^2)\) describing short range bosonic interaction and \(\psi_T(x)\) being the solution of the relevant Thirring theory [17].

The ultraviolet limit exhibits a different behaviour, due to the contribution of the massive boson state: a non–trivial scale dimension was found, related to an ultraviolet renormalization costant (different for left and right fermions)

\[ Z_{L(R)} = (\frac{\Lambda^2}{m^2})^{-\frac{1}{2} \frac{(1\pm r^2)}{a^2 - r^2}}, \quad (59) \]
while the c–theorem \[18\] trivially gives the flow between the two conformally invariant situations (labelled by their central charge \(c\))

\[
\Delta c = 1. \tag{60}
\]

For \(\alpha \neq \infty\) our solution reproduces only partially this scenario: as we will see the limit \(\alpha \to \infty\) drastically changes the small distance behaviour of the theory.

In order to study the fermions of the Lagrangian eq.(1) we compute the two point function

\[
S_F(x - y) = \langle T(\psi(x)\bar{\psi}(y)) \rangle. \tag{61}
\]

We recall that local gauge invariance is broken, hence we can extract meaningful information from the propagator. This Green function can be computed exactly summing “by hands” the perturbative expansion or using its definition in term of \(\zeta\)–function, namely by an explicit path–integral calculation. Obviously both methods give the same result: the functional integration is very simple, due to the possibility of decoupling the fermions from the gauge field with a clever change of variables, while the resummation of the Feynman graphs is rather involved but possible, following the arguments of \[19\]. The result is:

\[
S_F(x) = S_L(x) + S_R(x),
\]

\[
S_L(x) = Z_{\alpha}^L S_0^L(x) \exp \left\{ -i \frac{e^2(1 - r)^2}{16\pi(m_+^2 - m_-^2)} \cdot \left[\right. \right.
\]

\[
\left. \left. \left[ (4m_-^2(1 - \alpha) + e^2(1 + r)^2\alpha) \frac{1}{m_-^2} \Delta_F(x; m_-^2) - (4m_+^2(1 - \alpha) + e^2(1 + r)^2\alpha) \frac{1}{m_+^2} \Delta_F(x; m_+^2) \right] \right. \right.
\]

\[
\left. \left. \exp \left\{ \frac{ie^2(1 - r^2)^2\alpha}{16\pi(m_+^2 - m_-^2)} \left[ \frac{D_F(x; m_-^2)}{m_-^2} - \frac{D_F(x; m_+^2)}{m_+^2} \right] \right\} \right\}, \tag{62}
\]

where

\[
(i\gamma^\mu\partial_\mu)(\frac{1 - \gamma_5}{2})S_0^0(x) = (\frac{1 - \gamma_5}{2})\delta^2(x),
\]

\[
\Delta_F(x; m^2) = \frac{i}{2\pi} K_0(m\sqrt{-x^2 + i\varepsilon}),
\]

\[
D_F(x; m^2) = -\frac{i}{4\pi} \log(-m^2x^2 + i\varepsilon),
\]
and $S_R$ is obtained by changing $r \rightarrow -r$ and $S_L^0$ with $S_R^0$.

We notice that the perturbative summation for the fermionic Green function entails the exchange of bosons with propagator given by eq.(38), which is itself the sum of a geometrical series in the coupling constant $e^2$. The fermionic Green function requires a convolution of bosonic propagators in the momentum space; so doing one needs a continuation beyond the natural analyticity region $e^2 < \frac{|k^2|}{\mu^2}$. As a consequence we do not expect analyticity of the fermionic propagator at $e^2 = 0$ and indeed eq. (52) exhibits a branch point at $e^2 = 0$.

If instead one would compute the fermionic propagator directly starting from the massless quanta appearing in the free Lagrangian, one would immediately be confronted with IR singularities of the perturbative contributions.

We notice that no divergences arise unless $\alpha$ becomes infinite; only a finite renormalization of the wave function, described by $Z_{\alpha}^{L(R)}$, is present

$$Z_{\alpha}^{L(R)} = \left(\frac{m_+^2}{m_-^2}\right)^{\gamma_{L(R)}},$$

$$\gamma_{L(R)} = \frac{(1 \mp r)^2}{4} \frac{(1-\alpha)}{\sqrt{[(1+a) - \alpha(a-r^2)]^2 - 4\alpha r^2}}. \quad (63)$$

In order to identify the asymptotic states of the theory we study the large space–like limit of $(x - y)^2$ in eq.(52): the massive propagators do not contribute and we expect $\alpha$–independence in the scaling law, the massless sector being unaware of the presence of the gauge fixing

$$\lim_{x^2 \rightarrow \infty} S_{L,R}(x) = Z_{\alpha}^{L,R} \left(\frac{m_+^2}{m_-^2}\right)^{\rho_1} (x^2)^{-\frac{1}{4}\frac{(1-r^2)^2}{a(a+1-r^2)}} S_{L,R}^0(x), \quad (64)$$

where

$$\rho_1 = \frac{e^4 \ (1-r^2)^2 \ \alpha}{64\pi^2 \ m_+^2 - m_-^2 \ m_+^2},$$

$$\rho_2 = \frac{e^4 \ (1-r^2)^2 \ \alpha}{64\pi^2 \ m_+^2 - m_-^2 \ m_+^2}.$$
If we rescale the fermion fields

\[ \psi_R \rightarrow (Z_R^\alpha)^{-1} \psi_R, \]
\[ \psi_L \rightarrow (Z_L^\alpha)^{-1} \psi_L, \]

and we define

\[ \frac{(m_+^2)^{\rho_1}}{(m_+^2)^{\rho_2}} = [\mu^2(\alpha)]^{-\frac{1}{4} \frac{(1-r^2)^2}{a(a+1-r^2)}}, \]  

we can easily check the dimensional balance in eq.(65); the Thirring–like behaviour at large distances is recovered:

\[ \lim_{x^2 \rightarrow \infty} S(x) = (-\mu^2 x^2)^{-\frac{1}{4} \frac{(1-r^2)^2}{a(a+1-r^2)}} S^0(x). \]  

The fermionic asymptotic states are the ones found in [9]: they are constructed with Wick exponentials of the massless field: no dependence from \( \alpha \) can occur.

Let us turn our attention to the opposite regime of the theory, namely the limit \( x^2 \rightarrow 0 \). One finds from eq.(62)

\[ \lim_{x^2 \rightarrow 0} S_{L,R}(x) = S^0_{L,R}(x). \]  

Fermions are asymptotically free at variance with the result [4], where non trivial scaling was found even in the ultraviolet regime. It is very easy to check that also the boson propagator eq.(38) reduces to the free one eq.(7) in this situation: we conclude that at small distances the theory looks like the one of two free Weyl fermions (with a different normalization forced by our renormalization condition Eqs.(65)), carrying central charge \( c = 1 \), and of a free abelian gauge field carrying vanishing total central charge. We remark that unless \( \alpha \rightarrow \infty \) we are working with a non–unitary theory and therefore c–theorems does not hold: no central charge flow exists, the central charge being 1 in the ultraviolet regime as well as in the infrared theory (massless Thirring model).

We also observe that the limit \( e^2 \rightarrow 0 \) is possible in the correlation functions eq.(62) as well as in eq.(38) and it leads to the “free” propagators.
The high-energy regime of the present solution is very different from the non-perturbative one: the recovering of the unitarity is crucially linked to a change of the ultraviolet behaviour.

Let us take the limit \( \alpha \to +\infty \) in eq.(62)

\[
\Delta_F (x; m_+^2) \to 0,
\gamma_{L,R} \to -\frac{1}{4} \frac{(1 \mp r)^2}{(a - r^2)},
\]

(68)

\[
\lim_{\alpha \to +\infty} S_{L,R}(x) = S^0_{L,R}(x) \left[ \frac{e^2}{4\pi} \frac{(a - r^2)}{m^2} \alpha \right]^{-\frac{1}{4} \frac{(1 + \mp r)^2}{a - r^2}} \exp \left[ -i\pi \frac{(1 \mp r)^2(a - r^2 \mp r)^2}{a(a + 1 - r^2)(a - r^2)} \Delta_F(x, m^2) \right] \exp \left[ -\frac{1}{4} \frac{(1 - r^2)^2}{a(a + 1 - r^2)} \log(-m^2x^2 + i\epsilon) \right].
\]

(69)

We get the propagator of \([9]\) confirming that the non-perturbative solution is recovered. The large \( x^2 \) behaviour of eq.(69) is the same of eq.(62): after renormalization of eq.(65), that now is of an infinite type, we get

\[
\mu^2 = m^2.
\]

(70)

The opposite limit is \( (\alpha \to \infty, x^2 \to 0) \) gives:

\[
S_{L,R}(x) \to (-m^2x^2 + i\epsilon)^{-\frac{1}{4} \frac{(1 + \mp r)^2}{a - r^2}} S^0_{L,R}(x),
\]

(71)

\[
G_{\mu\nu}(x) \to \frac{4\pi}{e^2} \frac{1}{(a - r^2)} \partial_\mu \partial_\nu D_F(x).
\]

(72)

Eqs.(71) and (72) show two important features: asymptotic freedom is definitely lost as well as the analyticity of eq.(38) in \( e^2 \) (we notice the appearance of \( 1/e^2 \) terms). We can say that, sending \( \alpha \) to infinity, we shrink to zero the convergence radius of the power series in \( e^2 \). One can check that, in this case, the variation of the central charge \( \Delta c \) from the ultraviolet to the infrared situation is equal to one: with unitarity \( c \)-theorem is recovered.

We can trace the mechanism of the restored unitarity in this way: the original field \( A_\mu \) has no physical degrees of freedom, only a longitudinal zero norm state made by a physical
and a ghost particle (we can check it in Feynman gauge, for example). This is the original ultraviolet theory (together with free fermions): the interaction gives to the $A_\mu$ components $\alpha$ dependent different masses. As $\alpha \to +\infty$ the ghost decouples, leaving the physical particle of mass $m^2$; the long range interaction of Coulomb–type creates the infrared dressing for the fermions, leading to a Thirring model.

The drastic change of the dynamical content reflects itself in the doubling of the U–V central charge and in the divergent character of the renormalization costant:

$$Z_{\alpha}^{L,R} = \left[ \frac{e^2}{4\pi} \frac{(a - r^2)}{m^2} \alpha \right]^{-\frac{1}{4}} \frac{(1 \mp r^2)}{4} \left( a - r^2 \right)^{73}$$

with the identification

$$\frac{e^2}{4\pi} (a - r^2) \alpha = \Lambda$$

in eq.(74). Actually in the limit $\alpha \to +\infty$ renormalization constant is zero showing that there is no overlap between the naive perturbative asymptotic states and the effective solution of the theory.

Looking at expression eq.(74) we can give to $\alpha$ a different interpretation: we can look at it not as a free parameter in the perturbative approach but as a cut–off on the non perturbative theory. One can easily check that

$$\lim_{\alpha \to +\infty} (\frac{\partial}{\partial \alpha}) \log Z_{\alpha}^{L,R} = -\frac{1}{4} \frac{(1 \mp r^2)}{4 (a - r^2)}$$

is the ultraviolet scaling, obtained in the usual form of an anomalous dimension. The regularizing character of $\alpha$ becomes transparent if we look at the perturbative expansion of the propagator eq.(62).

Following the suggestions of [19] we could sum graphs of the type

$$\Sigma(p) = S^0_F(p) \int \frac{d^2k}{(2\pi)^2} Tr \left[ T_\mu G^{\mu\nu}(p - k) T^\nu S^0_F(k) \right] \cdot S^0_F(p),$$

where $G_{\mu\nu}$ is the propagator eq.(88). $G_{\mu\nu}$ can be written as
\[ G_{\mu\nu}(k) = \frac{1}{m_+^2 - m_-^2} [G^+_{\mu\nu}(m_+^2; k) - G^-_{\mu\nu}(m_-^2; k)] . \] (77)

The contributions of \( G^+_{\mu\nu} \) and \( G^-_{\mu\nu} \) are separately ultraviolet divergent in eq.\( (77) \), but the divergence actually cancels in their sum: \( \alpha \to \infty \) corresponds only to the contribution of \( G^-_{\mu\nu}(m^2; k) \). The boson \( m^2_+ (\alpha) \) behaves in this scenario as a kind of Pauli–Villars regulator.

V. CONCLUSIONS

The perturbative solution of the generalized chiral Schwinger model has been discussed, showing how the spectrum and the ultraviolet behaviour of Green’s functions depend not only on the couplings (\( e \) and \( r \)) and the regularization ambiguity (\( a \)) but also on a gauge–fixing parameter \( \alpha \) which is necessary in order to define the free vector propagator. However, since gauge symmetry turns out to be broken in the final solutions owing to the presence of the local anomaly, the introduction of a gauge fixing term actually amounts to considering inequivalent theories. The model discussed in [9] corresponds to the limits \( \alpha \to \pm \infty \) (no gauge fixing).

Accordingly particular regions of the parameter space have been considered in order to recover the non perturbative solutions in those limits. The decoupling of a massive ghost state and the change of the ultraviolet properties have been discussed when \( \alpha \to +\infty \): we have observed the transition from an asymptotically free theory to a theory that exhibits non–trivial scaling behaviour at small distances. This is related to the non–analyticity in \( e^2 \) of our result (after the limit \( \alpha \to +\infty \)) and with the doubling of the ultraviolet central charge.

The appearance of a divergent renormalization constant is intimately linked to a drastical change of the number of degrees of freedom. In the infrared regime the massless Thirring model is recovered, independently of the values of \( \alpha \).
APPENDIX A:

We present the resummation of the perturbative series for the vector propagator.

At zero order on $\frac{\alpha^2}{4\pi}$ we have only the “free propagator” eq.(7).

At the first order on $\frac{\alpha^2}{4\pi}$ we define the quantity $A_{\mu\nu}$ as

$$\Pi_{\mu\nu}(k) = A_{\mu\nu} + i(m_+^2 + m_-^2) \frac{1}{k^2} (G^{-1}_0)^{\mu\nu}$$

(A1)

where

$$A_{\mu\nu} = -ie^2 \frac{4\pi}{\alpha(a-r^2)(g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}) + (1+a) \frac{k_{\mu}k_{\mu}}{k^2} - r \frac{1}{k^2}(\tilde{k}_{\mu}k_{\nu} + \tilde{k}_{\nu}k_{\mu})},$$

(A2)

$$m_+^2 = e^2 \mu_+^2,$$

(A3)

being

$$\mu_\pm^2 = \frac{1}{8\pi} \left[ \alpha(a-r^2) + (1+a) \pm \right.
\pm \sqrt{[1 + \alpha(a-r^2)]^2 - 4ar^2},$$

(A4)

We introduce the quantities

$$B_{\pm}^{\mu\nu} = \frac{m_\pm^2}{k^2} G_0^{\mu\nu},$$

$$\hat{A} = G_0^{\mu\nu},$$

(A5)

It is easy to prove the identity

$$(\hat{A})^2_{*} = \left(-\frac{m_+^2 + m_-^2}{k^2}\right) \hat{A} - \frac{m_+^2 m_-^2}{k^4} G_0,$$

(A6)

where we have defined the * product of matrices $\hat{A}$ and $B_{\pm}$ as:

$$(\hat{A}B)_{*} = \hat{A} * B = \hat{A} G_0^{-1} B.$$
\[(A \hat{\uparrow})^2 = -\hat{A} \ast B_+ - \hat{A} \ast B_- - B_+ \ast B_- \quad \text{(A8)}\]

With the help of eq.(A8) we can write the n-th order of the perturbative expansion as:

\[(\hat{A} + B_+ + B_-)^n = \sum_{m=0}^{n} (B_+)^m \ast (B_-)^{n-m} + \]
\[+ \sum_{m=0}^{n-1} (B_+)^m \ast (B_-)^{n-1-m} \ast \hat{A} \quad \text{(A9)}\]

and the complete propagator as

\[\sum_{n=0}^{\infty} (\hat{A} + B_+ + B_-)^n = \sum_{n=0}^{\infty} \sum_{l=0}^{n} (B_+)^l \ast (B_-)^{n-l} + \]
\[+ \hat{A} \ast \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} (B_+)^l \ast (B_-)^{n-l-1}. \quad \text{(A10)}\]

Taking Lorentz indices into account we get:

\[G_{\mu\nu}(k) = G^0_{\mu\nu} \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \left(\frac{m_+^2}{k^2}\right)^l \left(\frac{m_-^2}{k^2}\right)^{n-l} + \]
\[+ \frac{e^2}{4\pi} k^4 \left[ \alpha(a - r^2)g_{\mu\nu} + \alpha(1 + r^2) \frac{k_{\mu}k_{\nu}}{k^2} - \right. \]
\[- \alpha r \frac{1}{k^2} (\tilde{k}_{\mu}k_{\nu} + \tilde{k}_{\nu}k_{\mu}) \right] \cdot \]
\[\sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \left(\frac{m_+^2}{k^2}\right)^l \left(\frac{m_-^2}{k^2}\right)^{n-l-1}. \quad \text{(A11)}\]

The series are of geometrical type, and the result is just eq.(38).

**APPENDIX B:**

In this appendix we show how to compute the boson propagator eq.(38), using a functional method. This kind of calculation is rather standard in the topologically trivial case, the only subtle point being the way to implement the Jackiw–Ramarajan ambiguity. The principal aim of the following discussion is to develop a systematic formalism to describe the regularization freedom in the functional approach. Putting to zero the fermionic sources in eq.(38) we get:
\[ Z[J_\mu] = \int \mathcal{D}A_\mu \exp \left[ i \int d^2 x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^{\mu})^2 + J_\mu A^\mu \right) \right] \det[D(A; r)], \quad (B1) \]

\[ D(A; r) = \gamma^\mu [i\partial_\mu + e(\frac{1-r\gamma^5}{2})A_\mu]. \quad (B2) \]

The functional determinant is obtained by integrating the fermionic degrees of freedom: \( \zeta \)-function technique \[20\] provides a well defined method to treat determinants of elliptic operators in any dimension. Really the operator \( D(A; r) \) is of hyperbolic type; so one has to make the computation in the euclidean space, where the principal symbol \[21\] is elliptic, and then to continue back the solution to Minkowski. In two dimensions the calculation can be performed exactly, even in the non–abelian case \[23\]: the usual procedure selects a precise value of the parameter \( a \) (as in the B–M scheme): therefore we propose a generalization of the \( \zeta \)-function regularization, depending on a real parameter.

Actually we study a slightly more general problem, considering the operator

\[ D(A_L, A_R) = \gamma^\mu [i\partial_\mu + P_L A_{R\mu} + P_R A_{L\mu}], \]

\[ P_{R,L} = (\frac{1 \pm \gamma^5}{2}), \quad (B3) \]

with \( A_{L\mu} \) and \( A_{R\mu} \) independent fields. The identification

\[ e(\frac{1 + r}{2})A_\mu = A_{L\mu} \]
\[ e(\frac{1 - r}{2})A_\mu = A_{R\mu} \]

leads immediately to the result eq.(B2).

For connections belonging to a trivial \( U(1) \) principal–bundle over the compactified euclidean space we can define

\[ \hat{\det}[D(A_L, A_R)] = \frac{\det[D(A_L, A_R)\hat{D}(A_L, A_R)]}{\det[\hat{D}(A_L, A_R)]}. \quad (B4) \]

With \( \det \) we mean the standard determinant constructed by \( \zeta \)-function and
\[ \hat{D}(A_L, A_R) = \gamma^\mu \left[ i\partial_\mu + P_R(aA_{R\mu} + cA_{L\mu}) + P_L(dA_{R\mu} + bA_{L\mu}) \right] \]  

(B5)

\(a, b, c, d\), being real parameters. Were the naive factorization property holding

\[ \det[AB] = \det[A] \det[B], \]  

(B6)

no dependence on \(a, b, c, d\) would appear in eq. (B4); in this sense the definition is allowed.

We expect that only local terms on the external fields \(A_{\mu L}\) and \(A_{\mu R}\) could depend on our parameters, according to the general claim that different regularizations cannot modify the non–local part of the effective action.

Using the properties of the complex powers of an elliptic operator \([21]\), the relevant Seeley-de Witt coefficient \([22]\) and the Hodge decomposition for \(A_{L,R\mu}\), we get (after analytical continuation to Minkowski space):

\[
\frac{\det[D(A_L, A_R)]}{\det[i\gamma^\mu \partial_\mu]} = \exp - \frac{i}{8\pi} \int d^2 x \mathcal{L}_{\text{eff.}}(A_R, A_L),
\]  

(B7)

\[\mathcal{L}_{\text{eff.}}(A_R, A_L) = A_{L\mu}^\nu g_{\mu\nu}(1 + a_L) - 2 \frac{\partial_\mu \partial_\nu}{\Box} + \]

\[+ \left( \frac{\partial_\mu \partial_\nu + \partial_\nu \partial_\mu}{\Box} \right) A_{L\nu}^\mu + A_{R\mu}^\nu g_{\mu\nu}(1 + a_R) - \]

\[- 2 \frac{\partial_\mu \partial_\nu}{\Box} - \left( \frac{\partial_\mu \partial_\nu + \partial_\nu \partial_\mu}{\Box} \right) A_{R\nu}^\mu + \]

\[+ 2A_{L\mu}^\nu g_{\mu\nu}(1 + b_1) - \epsilon_{\mu\nu}(1 + b_2)]A_{R\nu}^\mu, \]  

(B8)

with

\[a_L = 2b(1 - c),\]

\[a_R = 2a(1 - d),\]

\[b_2 = ab - (1 - c)(1 - d),\]

\[b_1 = -ab - (1 - c)(1 - d).\]

Actually we have only three independent parameters because eq. (B9) implies
\[b_2^2 - b_1^2 = 2a_L a_R.\]  

Eq. (B4) leads to the desired result:

\[
\frac{\det[D(A; r)]}{\det[i\gamma^\mu \partial_\mu]} = \exp \frac{-ie^2}{8\pi} \int d^2 x A^\mu [g_\mu\nu (1 + a) - (1 + r^2) \frac{\partial_\mu \partial_\nu}{\Box} + r \left( \frac{\partial_\mu \tilde{A}_\nu + \partial_\nu \tilde{A}_\mu}{\Box} \right)] A^\nu,
\]

\[a = \frac{(1 + r)^2}{2} a_R + \frac{(1 - r)^2}{2} a_L + b_1 \frac{(1 - r^2)}{2} \]  

The functional \(Z[J_\mu]\) turns out to be (normalized to \(\det[i\gamma^\mu \partial_\mu]\)):

\[
Z[J_\mu] = \int D A_\mu \exp \int d^2 x \left[ -\frac{1}{2} A_{\mu\nu} K_{\mu\nu} A^\nu + J_\mu A^\mu \right],
\]

with

\[
K_{\mu\nu} = -g_{\mu\nu} \Box + (1 - \alpha) \partial_\mu \partial_\nu - \frac{e^2}{4\pi} [g_{\mu\nu} (1 + a) - (1 + r^2) \frac{\partial_\mu \partial_\nu}{\Box} + r \left( \frac{\partial_\mu \tilde{A}_\nu + \partial_\nu \tilde{A}_\mu}{\Box} \right)].
\]

The propagator is nothing but the inverse of this operator

\[G_{\mu\nu}(x, y) = i K^{-1}_{\mu\nu}(x, y).\]

The inversion of \(K_{\mu\nu}\) is performed by means of the Fourier transform and is very simple although tedious:

\[
G_{\mu\nu}(x, y) = \frac{1}{m_+^2 - m_-^2} \left[ -\frac{e^2}{4\pi \alpha} (a - r^2) g_{\mu\nu} - g_{\mu\nu} \Box + \frac{1}{m_+^2} (1 - \alpha) m_+^2 + \alpha \frac{e^2}{4\pi} (1 + r^2) \partial_\mu \partial_\nu - \frac{e^2 \alpha r}{4\pi m_+^2} (\partial_\mu \tilde{A}_\nu + \partial_\nu \tilde{A}_\mu) \right] \Delta_F(x, y; m_+^2) + \frac{1}{m_-^2 - m_+^2} \frac{e^2 \alpha}{4\pi m_+^2} \left[ r (\partial_\mu \tilde{A}_\nu + \partial_\nu \tilde{A}_\mu) - (1 + r^2) \partial_\mu \partial_\nu \right] D_F(x, y) + m_+^2 \leftrightarrow m_-^2.
\]

The Fourier transform of \(G_{\mu\nu}\) coincides with eq. (38).
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