Matching preclusion for vertex-transitive networks*

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Abstract

In interconnection networks, matching preclusion is a measure of robustness when there is a link failure. Let $G$ be a graph of even order. The matching preclusion number $mp(G)$ is defined as the minimum number of edges whose deletion results in a subgraph without perfect matchings. Many interconnection networks are super matched, that is, their optimal matching preclusion sets are precisely those induced by a single vertex. In this paper, we obtain general results of vertex-transitive graphs including many known networks. A $k$-regular connected vertex-transitive graph has matching preclusion number $k$ and is super matched except for six classes of graphs. From this many previous results can be directly obtained and matching preclusion for some other networks, such as folded $k$-cubes, Hamming graphs and halved $k$-cubes, are derived.

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1 Introduction

A network (or graph) is a collection of points or nodes, called vertices, and a collection of links, called edges, each connecting two nodes. The number of vertices of a graph $G$ is its order, written $|G|$; its number of edges is denoted by $||G||$. We use $V(G)$ and $E(G)$ denote the vertex-set and edge-set of $G$ respectively. Throughout this article, all graphs are assumed to be connected and of even order. The matching preclusion, viewed as a measure of the robustness of graphs, of many networks has been investigated. By summarizing these results, we can see that almost all the networks considered are vertex-transitive and surprisingly, their matching preclusion almost act in the same way. A natural question arises: What does the matching preclusion of vertex-transitive graphs act? More precisely, can we obtain a unified property on the matching preclusion of vertex-transitive graphs?

A perfect matching in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. For $S \subseteq E(G)$, if $G - S$ has no perfect matchings, where $G - S$ denotes the subgraph of $G$ by deleting $S$ from it, then we call $S$ a matching preclusion set. The matching preclusion number

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of a graph $G$, denoted by $mp(G)$, is the minimum cardinality among all matching preclusion sets. Correspondingly, the matching preclusion set attaining the matching preclusion number is called an optimal matching preclusion set (or in short, optimal solution). The concept of matching preclusion was introduced by Brigham et al. for “measuring the robustness of a communications network graph which is a model for the distributed algorithm that require each node of it to be matched with a neighboring partner node” [1].

Until now, the matching preclusion numbers of lots of networks (graphs) have been computed, such as Petersen graph, hypercube, complete graphs and complete bipartite graphs [1], Cayley graphs generalized by transpositions and $(n, k)$-star graphs [2], augmented cubes [3], $(n, k)$-buddle-sort graphs [6], tori and related Cartesian products [8], burnt pancake graphs [10], balanced hypercubes [12], restricted HL-graphs and recursive circulant $G(2^m, 4)$ [15], and $k$-ary $n$-cubes [19]. Their optimal solutions have been also classified.

By deleting the edges incident with a given vertex in a graph, the resulting subgraph has no perfect matchings. Hence the matching preclusion number is bounded by the minimum degree.

**Theorem 1.1** ([2]). Let $G$ be a graph of even order. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of $G$.

In a network, a vertex with a special matching vertex after edge failure any time implies that tasks running on a fault vertex can be shifted onto its matching vertex. Thus under this fault assumption, larger $mp(G)$ signifies higher fault tolerance. Fortunately, matching preclusion numbers of many regular interconnection networks of degree $k$ attained the maximum value, the minimum degree $k$. Moreover, the optimal solutions are precisely those induced by a single vertex except the ones with small order. Formally, we call the optimal solution incident with a single vertex a trivial optimal solution (non-trivial optimal solution otherwise) and the graphs with all optimal solutions trivial super matched. Generally, in the event of a random link failure, it is very unlikely that all of the links incident to a single vertex fail simultaneously. From this point of view, that a graph is super matched implies that it has higher fault tolerance.

Recalling that the networks whose matching preclusion have been considered, we can see that many of them are vertex-transitive graphs. A graph $H$ is called vertex-transitive if for any two vertices $x, y$ in $V(H)$, there exists an automorphism $\varphi$ of $H$ such that $\varphi(x) = y$. From the known results, we can see that almost all of them are super matched. Fortunately, we obtain that almost all vertex-transitive graphs have such properties, too. Precisely, we get the following result, in which, $Z_{4n}(1, 4n, 2n)$ stands for the Cayley graph on $Z_{4n}$, the additive group modulo $4n$, with the generating set $S = \{1, 4n - 1, 2n\}$. $Z_{4n+2}(2, 4n, 2n+1)$ and $Z_{4n+2}(1, 4n + 1, 2n, 2n + 2)$ are defined similarly.

**Theorem 1.2.** A $k$-regular connected vertex-transitive graph $G$ of even order is super matched if and only if it doesn’t contain cliques of size $k$ when $k$ is odd and $k \leq |G| - 2$ or it is not isomorphic to a cycle of length at least six or $Z_{4n}(1, 4n - 1, 2n)$ or $Z_{4n+2}(2, 4n, 2n+1)$ or $Z_{4n+2}(1, 4n + 1, 2n, 2n + 2)$ or the Petersen graph.

This article is organized as follows. In Section 2, we will analyse some structural properties of
vertex-transitive graphs. In Section 3, we present the proof of Theorem 1.2. In Section 4, we make a conclusion and several applications to obtain the matching preclusion of some networks.

2 Preliminaries

In this section, we shall present several results that will be used later. An edge set $S \subseteq E(G)$ is called an edge-cut if there exists a set $X \subseteq V(G)$ such that $S$ is the set of edges between $X$ and $V(G) \setminus X$. The edge-connectivity $\lambda(G)$ of $G$ is the minimum cardinality over all edge-cuts of $G$. Mader proved the following result.

**Lemma 2.1** ([13]). If $G$ is a $k$-regular connected vertex-transitive graph, then $\lambda(G) = k$.

The following lemma makes a step further by characterizing the minimum edge-cuts of vertex-transitive graphs, where a clique of a graph $G$ is a subset of its vertices such that every two vertices in the subset are connected by an edge.

**Theorem 2.2** ([11], Lemma 5.5.26). Let $G$ be a $k$-regular connected vertex-transitive graph. Then $\lambda(G) = k$ and either  
(i) every minimum edge-cut of $G$ is the star of a vertex, or  
(ii) $G$ arises from a (not necessarily simple) vertex- and edge-transitive $k$-regular graph $G_0$ by a $k$-clique (cliques of size $k$) insertion at each vertex of $G_0$. Moreover, every minimum edge-cut of $G$ is the star of a vertex of $G$ or a minimum edge-cut of $G_0$.

The following corollary that will be used in Section 3 follows immediately. An edge-cut is called trivial if it isolates a vertex and non-trivial otherwise.

**Corollary 2.3.** For a $k$-regular connected vertex-transitive graph $G$, every $k$-edge-cut (an edge-cut of size $k$) is either trivial or the deletion of it results in two components, and each component’s vertices are partitioned into several $k$-cliques.

For a $k$-regular graph $G$, if every minimum edge-cut of it is trivial, then we say it is super-edge-connected (or simply super-\(\lambda\)). For vertex-transitive graphs, J. Meng has presented a characterization with respect to the cliques.

**Theorem 2.4** ([14]). Let $G$ be a $k$-regular connected vertex-transitive graph which is neither a complete graph nor a cycle. Then $G$ is super-\(\lambda\) if and only if it does not contain $k$-cliques.

Theorem 2.4 is used to characterize the structure of 3-regular connected non-bipartite vertex-transitive graphs with respect to the length of the minimum odd cycles. As we will see, minimum odd cycles play a crucial role in the following proof in this section. Here we make a convention that is suitable throughout this paper. For a cycle drawn on the plane without crossings, let $a, b \in V(C)$, denote $P_{ab}$ by the path of $C$ from $a$ to $b$ along a clockwise direction. A cycle $C$ is called a minimum odd cycle if $|C|$ is odd and it is the minimum among lengths of all odd cycles. For a minimum odd cycle, we usually say it is minimum. The following two results (Lemmas 2.5 and 2.6) will be used to prove Lemma 2.7, which play an important role in the proof of Theorem 1.2 in the next section.
Lemma 2.5. Let $G$ be a non-bipartite connected vertex-transitive graph of even order and $C$ be a minimum odd cycle in $G$ with $|C| \geq \frac{|G|}{2}$. If $G$ is not isomorphic to $K_4$ or $K_6$, then any vertex in $V(G) \setminus V(C)$ is adjacent to at most two vertices in $V(C)$. If in addition, a vertex $v \in V(G) \setminus V(C)$ is adjacent to two vertices $u$ and $w$ in $V(C)$, then either $P_{uw}$ or $P_{wu}$ in $C$ is of length 2 and further $G$ has a quadrangle containing $v$, $u$, and $w$.

Proof. We first draw $C$ on the plane without crossings. Suppose by the contrary that there exists a vertex $v \in V(G) \setminus V(C)$ with at least three neighbors in $V(C)$. Let $a$, $b$ and $c$ be three neighbors of $v$ in $V(C)$ and they are placed in $C$ in a successive order along the clockwise direction. Since $||P_{ab}|| + ||P_{bc}|| + ||P_{ca}|| = ||C||$ is odd, at least one of $||P_{ab}||$, $||P_{bc}||$ and $||P_{ca}||$ is odd, we may assume that $||P_{ab}||$. Hence $||P_{bc}|| + ||P_{ca}||$ is even. If $||P_{bc}|| + ||P_{ca}|| = 2$, then $vac$ is an odd cycle of length three, so is $C$ since $C$ is minimum. Further, $|G| \leq 2||C|| = 6$, we can easily check that $G$ is isomorphic to $K_4$ or $K_6$, which contradicts the hypothesis. If $||P_{bc}|| + ||P_{ca}|| \geq 4$, then $E(P_{ab}) \cup \{va, vb\}$ induces an odd cycle of smaller length than $C$, which contradicts that $C$ is minimum.

By the above arguments, any vertex in $V(G) \setminus V(C)$ is adjacent to at most two vertices in $V(C)$. If $v \in V(G) \setminus V(C)$ is adjacent to exactly two vertices $u$ and $w$ in $V(C)$, then by $||P_{uw}|| + ||P_{wu}|| = ||C||$ is odd, either $||P_{uw}||$ or $||P_{wu}||$ is even. We may assume that $||P_{uw}||$ is even. Further, the edges of $P_{wu}$ and $\{vu, vw\}$ form an odd cycle of length at least $||C||$. Hence $||P_{uw}|| = 2$ and $E(P_{uw}) \cup \{vu, vw\}$ induces a quadrangle containing $v$, $u$, and $w$. \hfill \Box

For a graph $G$, let the girth (odd girth) of $G$, denoted by $g(G)$ ($g_o(G)$), be the length of a shortest cycle (odd cycle) in $G$. We say two quadrangles in $G$ adjacent if they share vertices or edges.

Lemma 2.6. Let $G$ be a 3-regular connected non-bipartite vertex-transitive graph of girth 4. If $G$ has adjacent quadrangles, then it is isomorphic to $Z_{4n}(1,4n-1,2n)$ or $Z_{4n+2}(2,4n,2n+1)$ with $n \geq 2$. Otherwise $g_o(G) \leq \frac{|G|}{2}$.

Proof. By Theorem 2.1, $G$ is 3-edge-connected. Further, since $G$ is of girth 4, that is, $G$ does not contains 3-cliques (triangles), every 3-edge-cut is trivial by Theorem 2.4.

If $G$ has adjacent quadrangles $q_1$ and $q_2$, then by the 3-regularity of $G$, $q_1$ and $q_2$ should share at least one edge. If $q_1$ and $q_2$ share exactly three edges, then we obtain multiple edges, contradicting that $G$ is simple; If $q_1$ and $q_2$ share exactly two edges, let $H$ be the subgraph of $G$ induced by the edges of $q_1$ and $q_2$, then $H$ has exactly three vertices of degree 2 and the others of degree 3. Since every 3-edge-cut of $G$ is trivial, we obtain that the three degree-2 vertices in $H$ are adjacent to a common neighbor, that is, $G$ is $K_{3,3}$, which contradicts that $G$ is non-bipartite. By the above arguments, $q_1$ and $q_2$ should share exactly one edge, that is, $G$ contains $K_2 \times P_3$ as a subgraph, where ‘$\times$’ means the Cartesian product of graphs and $P_3$ denotes a path $P_m$ with $m = 3$ vertices. Let $K_2 \times P_m$ with $m \geq 3$ be a subgraph of $G$ with $m$ maximum (see Figure 1 (left)). For the four vertices $x_1, y_1, x_m$ and $y_m$, if any two of them are adjacent to each other, then the remaining two of them are adjacent by the 3-connectivity of $G$, this implies that $G$ is isomorphic to $Z_{2m}(1,2m-1,m)$ when $m$ is even or $Z_{2m}(2,2m-2,m)$ when $m$ is odd with $m \geq 3$ (see Figure 1 (the middle and right ones)). If no two of them are adjacent to each other, then we may suppose that $x_m$ is adjacent to $x_{m+1}$ and $y_m$
is adjacent to \(y_{m+1}\). By the vertex-transitivity of \(G\), we know that there are at least two adjacent quadrangles containing \(y_m\). Then by a simple check, we can obtain that \(x_{m+1}\) and \(y_{m+1}\) should be adjacent, then we find a subgraph \(K_2 \times P_{m+1}\) in \(G\), which contradicts our selection.

If \(G\) does not contain adjacent quadrangles, then every vertex lies in exactly one quadrangle. Suppose to the contrary that \(g_o(G) \geq \frac{|G|}{2} + 1\). Let \(C\) be a minimum odd cycle with length \(g_o(G)\). We shall obtain a contradiction by proving that there are at least \(|C|\) vertices in \(V(G) \setminus V(C)\). Since \(G\) is not \(K_4\) or \(K_6\) (by the hypothesis that \(G\) is of girth 4) and \(C\) is minimum, by Lemma 2.5, any vertex in \(V(G) \setminus V(C)\) is adjacent to at most two vertices in \(V(C)\).

Since every vertex lies in a quadrangle and \(C\) is minimum, each quadrangle containing vertices in \(V(C)\) can only contain one or two edges in \(C\). Denote \(G'\) by the subgraph of \(G\) induced by the edges in \(C\) and the quadrangles having a non-empty intersection with \(C\) (see Figure 2 (left)). For any vertex \(a \in V(C)\), if it is of degree 2 in \(G'\), then we claim that all its neighbors in \(G\) cannot lie entirely in \(V(G')\). Suppose not. Since \(C\) is minimum, \(C\) has no chords, we obtain the two structures shown in Figure 2 (the middle and right ones). For each case, by Lemma 2.5 \(v, a\) and \(b\) lie in a quadrangle. Then we obtain two adjacent quadrangles, a contradiction.

In \(G'\), for all the degree-2 vertices in \(V(C)\), we collect the edges incident to them in \(E(G) \setminus E(G')\), denoted by \(F\) (the bold edges in Figure 3 (left)). We claim that \(F\) is a matching. If not, then we obtain a vertex \(v\) adjacent to two vertices \(a\) and \(b\) in \(V(C)\). By Lemma 2.5 \(v, a\) and \(b\) lie in a quadrangle. Then we see that both \(a\) and \(b\) lie in two quadrangles, a contradiction.

Denote \(G''\) by the subgraph induced by \(E(G') \cup F\) in \(G\), and denote the set of degree-1 vertices in \(G''\) be \(A\). It is obvious that \(|A| = |F|\). We are going to show that there are at least \(|A|\) vertices in
$V(G) \setminus V(G')$. If so, then by a simple computation (for the vertices in $V(C)$ lying in the quadrangles sharing exactly one edge with $E(C)$, there is the same number of vertices in the quadrangles in $V(G) \setminus V(C)$; for the vertices in $V(C)$ lying in the quadrangles sharing exactly two edges with $E(C)$, by counting the vertices in quadrangles not in $V(C)$, the vertices contained in $F$ not in $V(C)$ and the vertices in $V(G) \setminus V(G')$, we obtain the same conclusion too), there are at least $|C|$ vertices in $V(G) \setminus V(C)$, which contradicts that $C$ is of length at least $\frac{|G|}{2} + 1$.

![Figure 3.](image)

We are left to prove that there are at least $|A|$ vertices in $V(G) \setminus V(G')$. For any vertex $v \in A$, let the quadrangle containing it be $Q_v$. If $Q_v$ contains three vertices in $V(G) \setminus V(G')$, then any vertex $u$ in $A$ other than $v$ cannot be adjacent to any vertex in this quadrangle (otherwise, $Q_u$ and $Q_v$ are adjacent, a contradiction), and we count three for $V(G) \setminus V(G')$; If the quadrangle containing it contains exactly two vertices $g$ and $h$ in $V(G) \setminus V(G')$ (see the middle one in Figure 3), similarly, any vertex in $A$ other than $e$ and $f$ can not be adjacent to $g$ and $h$ (if there does exist such a vertex $w$, then by considering the quadrangle containing $w$, we obtain two adjacent quadrangles, a contradiction), then we count two for these two vertices $e$ and $f$ in $A$. Moreover, since $C$ is a minimum odd cycle, by a similar argument as above, we can deduce that $|P_{cd}| = 3$ or $|P_{de}| = 3$; If the quadrangle containing it contains at most one vertex in $V(G) \setminus V(G')$ (see the right one in Figure 3), then we can deduce that $|P_{ba}| = |P_{ba}| = 3$. By substituting the edges in $P_{ba}$ in $C$ with the bold edges, we obtain an odd cycle of length smaller than $C$, which contradicts that $C$ is minimum. Therefore, there are at least $|A|$ vertices in $V(G) \setminus V(G')$. This completes the proof. 

Now everything is ready to prove the following key lemma which characterizes the vertex-transitive graphs with provided structure.

**Lemma 2.7.** Let $G$ be a $k$-regular connected non-bipartite vertex-transitive graph. Let $S \subseteq V(G)$ with $|S| = |S| + 2$, where $S = V(G) \setminus S$, and $S$ is an independent set of $G$. Then $G$ is isomorphic to $Z_{4n+2}(1,4n+1,2n,2n+2)$ or $Z_{4n}(1,4n-1,2n)$ or $Z_{4n+2}(2,4n,2n+1)$ or the Petersen graph.

**Proof.** Obviously, $S \neq \emptyset$. Let $C$ be a minimum odd cycle and $l = |C|$.

**Claim 1.** $l = g_o(G) \geq \frac{|G|}{2}$.

Suppose that there are $n_l$ minimum odd cycles in $G$ and each vertex is contained in $n_l$ minimum odd cycles. Since $G[S]$ is empty, each minimum odd cycle contains at most $\frac{1}{2}$ vertices in $S$ and at
least \( \frac{l+1}{2} \) vertices in \( S \). By counting the number (repeated by re-number calculation) of minimum odd cycles passing through the vertices in \( \overline{S} \), we have

\[
v_l \times |\overline{S}| \leq \frac{l-1}{2} n_l.
\]

Similarly for \( S \), we have

\[
v_l \times |S| \geq \frac{l+1}{2} n_l.
\]

Combining the above two inequalities with \( |S| = |\overline{S}| + 2 \), we have \( v_l \geq n_l \). Consequently, by counting the number of times (repeated by re-number calculation) that the minimum odd cycles passing through all the vertices in \( G \), we obtain that \( n_l \times l = v_l \times |G| \geq \frac{n_l}{2} \times |G| \), which implies \( l \geq \frac{|G|}{2} \). So Claim 1 holds.

Now we show that only vertex-transitive graphs with small \( k \) satisfy the conditions in the lemma.

**Claim 2.** \( 3 \leq k \leq 4 \).

We first show that \( k \geq 3 \). If not, then \( G \) is an odd cycle, which contradicts that \( G \) is of even order \( (|G| = 2|\overline{S}| + 2) \).

Next we will show \( k \leq 4 \). Since \( C \) is minimum, \( C \) has no chords. So there are \( l(k-2) \) edges between \( V(C) \) and \( \overline{V(C)} = V(G) \setminus V(C) \). If \( G \) is isomorphic to \( K_4 \), then \( k = 3 \leq 4 \). \( G \) is not isomorphic to \( K_6 \). Otherwise for any \( S \subseteq V(G) \) with \( |S| = |\overline{S}| + 2 \), we have \( |\overline{S}| = 2 \) and hence \( G[\overline{S}] \) is not empty, which contradicts the hypothesis. Hence we only need to consider that \( G \) is not isomorphic to \( K_4 \) or \( K_6 \). By Lemma 2.3, any vertex in \( \overline{V(C)} \) can only be adjacent to at most two vertices in \( V(C) \). So there are at least \( \frac{l(k-2)}{2} \) vertices in \( \overline{V(C)} \).

By Claim 1, \( |C| = l \geq \frac{|G|}{2} \). On one hand, \( |\overline{V(C)}| \geq \frac{l(k-2)}{2} \geq \frac{|G| \times (k-2)}{4} \). On the other hand, \( |\overline{V(C)}| = |G| - l \leq \frac{|G|}{2} \). Combining these two inequalities, that is, \( \frac{|G| \times (k-2)}{4} \leq |\overline{V(C)}| \leq \frac{|G|}{2} \), we have \( k \leq 4 \).

In the following, we divide the remaining proof into two cases: \( k = 4 \) and \( k = 3 \).

**Case 1.** \( k = 4 \).

**Figure 4.** \( Z_{4n+2}(1, 4n+1, 2n, 2n+2) \) and the labeling of it.

For any minimum odd cycle \( C \), by the above proof, \( \frac{|G|}{2} \leq \frac{|G| \times (k-2)}{4} \leq |\overline{V(C)}| \leq \frac{|G|}{2} \). Thus all
equalities hold, that is, \( l = \frac{|G|}{2} \) and every vertex in \( V(C) \) is adjacent to exactly two vertices in \( V(C) \). So every vertex in \( V(C) \) is of degree 2 in \( G[V(C)] \), the subgraph of \( G \) induced by \( V(C) \), and hence \( G[V(C)] \) is a union of disjoint cycles. By \( |V(C)| = |C| = l \) and \( C \) is minimum, \( G[V(C)] \) is indeed a minimum odd cycle. Therefore, we conclude that the deletion of any minimum odd cycle results in another minimum odd cycle.

Let \( C_1 \) be a minimum odd cycle and \( C_2 \) be the minimum odd cycle of \( G \) by deleting \( V(C_1) \) from it, we draw \( C_1 \) and \( C_2 \) on the plane as in Figure 3. Suppose that \( a \in V(C_2) \) is adjacent to \( b \) and \( c \) in \( V(C_1) \). Then by Lemma 2.5 either \( P_{cb} \) or \( P_{bc} \) in \( C_1 \) is of length 2, we may assume that \( P_{bc} \). The edges of \( P_{cb} \) in \( C_1 \), \( ca \) and \( ab \) form a minimum odd cycle. By deleting it, we obtain another minimum odd cycle. That is, \( d \) should be adjacent to \( e \) and \( f \), where \( e \) and \( f \) are neighbors of \( a \) in \( C_2 \). By continuing this process repeatedly, we obtain \( G \) is isomorphic to the graph shown in Figure 4 (left), by labeling it as shown in Figure 4 (right), we can see \( G \) is isomorphic to \( Z_{4n+2}(1, 4n + 1, 2n, 2n + 2) \) for some integer \( n \).

**Case 2.** \( k = 3 \).

For \( g(G) = 3 \) or 5, Claim 1 implies that \( |G| \leq 2l \leq 10 \). Read and Wilson [17] have enumerated all connected cubic vertex-transitive graphs on 34 and fewer vertices, from which, we deduce that all connected non-bipartite cubic vertex-transitive graphs of at most 10 vertices are either \( Z_{4n}(1, 4n − 1, 2n) \) or \( Z_{4n+2}(2, 4n, 2n + 1) \) with \( n = 1 \) and 2, or the Petersen graph. We are done.

For \( g(G) = 4 \), \( G \) has adjacent quadrangles. Otherwise, \( |G| = 4m \) for some integer \( m \). But Claim 1 and Lemma 2.6 imply \( l = \frac{|G|}{2} \) and \( |G| = 4t + 2 \) for some integer \( t \), a contradiction. By Lemma 2.6 G is isomorphic to a graph in \( Z_{4n}(1, 4n − 1, 2n) \) or \( Z_{4n+2}(2, 4n, 2n + 1) \) for some integer \( n \).

**Claim 3.** \( g(G) \leq 5 \). Suppose to the contrary that \( g(G) \geq 6 \).

There is no vertex in \( V(C) \) adjacent to two vertices in \( V(C) \). If not, then by Lemma 2.5, we obtain a quadrangle, a contradiction. Thus the edges sending out from \( V(C) \) form a matching. That is, there are at least \( l \) vertices in \( V(C) \). By Claim 1, \( l \geq \frac{|G|}{2} \), we obtain \( l = \frac{|G|}{2} \). Now, we focus on the graph \( G' = G[V(C)] \). Since \( |G'| \) is odd and each vertex in \( G' \) is of degree 2, \( G' \) is a union of disjoint cycles. Further, because \( l \) is the length of minimum odd cycles, \( G' \) is a cycle of length \( l \).

By the above arguments, we conclude that

(i) \( V(G) \) can be decomposed into two parts \( V_1 \) and \( V_2 \) such that both \( G[V_1] \) and \( G[V_2] \) are minimum odd cycles.

(ii) The deletion of any minimum odd cycle from \( G \) results in a minimum cycle, too.

Since \( S \) is an independent set of \( G \), by a simple computation, we have that \( G[S] \) contains three edges, and every odd cycle contains either one or three edges in \( G[S] \). Hence the following holds.

(iii) Every minimum odd cycle contains exactly one edge in \( G[S] \). (Otherwise, suppose there is a minimum odd cycle containing three edges in \( G[S] \). By (ii), the deletion of it results in a minimum odd cycle which contains no edges in \( G[S] \), a contradiction.)

Denote the three edges in \( G[S] \) by \( e_1, e_2 \) and \( e_3 \). Then by (i) and (iii), we may assume that \( C_1 \) and \( C_2 \) are two vertex-disjoint minimum odd cycles containing \( e_1 \) and \( e_2 \) respectively. Also by (iii), \( e_3 \) does not lie in \( C_1 \) or \( C_2 \). For the simplicity of description, we color the vertices in \( S \) white and
the vertices in $S$ black. So two white vertices are adjacent if and only if they are the end-vertices of some $e_i$, for $i = 1, 2$ or 3. There are three cases to consider.

**Figure 5.** The illustration of proof of Case 2.

**Subcase 2.1.** $e_1$, $e_2$ and $e_3$ are independent.

We draw $C_1$ and $C_2$ on the plane as shown in Figure 5 (left). Let $c$ be an end-vertex of $e_1$, $b$ the neighbor of $c$ in $V(C_2)$ and $a$ the end-vertex of $e_3$ in $V(C_2)$. We may assume that $e_2$ lies on $P_{ab}$ in $C_2$. Otherwise we may redraw $C_2$ by interchanging $P_{ab}$ and $P_{ba}$. The union of the edges of $P_{ab}$ in $C_2$, $e_3$, the edges of $P_{dc}$ in $C_1$ and $cb$ form a cycle, denoted by $C_3$. Since $C_3$ contains three edges in $G[S]$, it is an odd cycle of length at least $l + 2$. Simultaneously, the union of the edges of $P_{ba}$ in $C_2$, $e_3$, the edges of $P_{cd}$ in $C_1$ and $cb$ form a cycle, denoted by $C_4$. Since $C_4$ contains exactly one edge in $G[S]$, it is an odd cycle of length at least $l$. Clearly, we have $|C_1| + |C_2| + 4 = |C_3| + |C_4|$. That is, $|C_3| + |C_4| = 2l + 4$. Hence $l + 2 \leq |C_5| \leq l + 4$ by $|C_4| \geq l$.

Now we show that $C_3$ has no chords. Suppose by the contrary that $C_3$ has a chord. Then by using the chord and $E(C_3)$, we get two cycles, one is of odd length denoted by $C_5$ and one is of even length. By $g \geq 6$, the one with even length is of length at least 6 and $C_5$ is of length at least $l$. Since the sum of the lengths of these two cycles is at most $l + 6$, we obtain that $C_5$ is a minimum odd cycle and the other even cycle is of length 6. By deleting $V(C_5)$ from $G$, we can see the resulting graph has at least one vertex in $\{a, b, c, d\}$ of degree 3 and further cannot be a cycle, which contradicts (ii).

Since $C_3$ has no chords and also $C_2$ has no chords, the end-vertices of $e_2$ are adjacent to vertices in $V(C_1) \setminus V(C_3)$, denoted by $e$ and $f$. Then the bold edges in Figure 5 (left) form a new cycle $C_6$. Since $C_6$ contains only one edge in $G[S]$, it is an odd cycle. Recall that two white vertices are adjacent if and only if they are the end-vertices of some $e_i$, $P_{ef}$ in $C_1$ are of length at least five. $||P_{ef}|| + ||P_{fe}|| = l$, so for $C_1$, by substituting the edges in $P_{ef}$ with the bold edges $eg$, $e_2$ and $hf$, we obtain that $C_6$ is an odd cycle of length smaller than $l$, a contradiction.

**Subcase 2.2.** $e_1$, $e_2$ and $e_3$ form a path of length three (see Figure 5 (middle)).

Similarly to Subcase 2.1, we may assume that $e_1$, $e_2$ and $e_3$ are placed exactly like in Figure 5 (middle). Suppose that $c$ is adjacent to a vertex $a \in V(C_2)$. We consider the cycle (the bold edges in Figure 5 (middle)) consisting of the edges of $P_{da}$ in $C_2$, $ac$, $e_1$ and $e_3$. This cycle, denoted by $C_7$, contains exactly three edges in $G[S]$, so it is an odd cycle and of length at least $l + 2$ by (iii). Hence $P_{da}$ in $C_2$ is of length at least $l - 1$ and further $da$ is an edge. Then the four edges $da, ac, e_1, e_3$ form
a quadrangle, which contradicts that \( g(G) \geq 6 \).

**Subcase 2.3.** \( e_1, e_2 \) and \( e_3 \) form a union of an independent edge and a path of length two (see Figure 5 (right)).

Similarly to Subcase 2.1, we may assume that \( e_1, e_2 \) and \( e_3 \) are placed exactly like in Figure 5 (right). Suppose that the neighbor of \( g \) in \( V(C_2) \) be \( b \) and the neighbor of \( a \) in \( V(C_2) \) be \( c \). Let \( C_8 \) be the cycle formed by the edges of \( P_{db} \) in \( C_2 \), \( bg \), the edges of \( P_{hg} \) in \( C_1 \) and \( hd \). By the similar argument as for \( C_3 \) in Subcase 2.1, we know that \( C_8 \) has no chords. Hence \( c \) lies in \( P_{bd} \) in \( C_2 \). Let \( C_9 \) (the bold edges in Figure 5 (right)) be the cycle formed by the edges of \( P_{bc} \) in \( C_2 \), \( ca \), \( e_1 \) and \( gb \). Since \( P_{cb} \) in \( C_2 \) is of length at least three, by a similar argument as above, we get that \( C_9 \) is a minimum odd cycle. By (ii), the deletion of \( C_9 \) results in a minimum odd cycle having no vertices of degree 3. Hence \( ha \) is an edge, but \( h \) and \( a \) receive the same white color, a contradiction.

\[\square\]

### 3 Matching Preclusion

In this section, we shall prove Theorem 1.2. We first present the Plesník’s Theorem which is used to estimate the lower bound of matching preclusion number.

**Theorem 3.1** ([11]). *If \( G \) is a \( k \)-regular \((k-1)\)-edge-connected graph of even order, then \( G - F \) has a perfect matching for every \( F \subseteq E(G) \) with \( |F| \leq k - 1 \).*

For the classification of optimal solutions, Hall’s Theorem (for the bipartite case) and Tutte’s Theorem (for the non-bipartite case) are used.

**Theorem 3.2** (Hall’s Theorem [16]). *Let \( G \) be a bipartite graph with bipartition \( W \) and \( B \). Then \( G \) has a perfect matching if and only if \( |W| = |B| \) and for any \( U \subseteq W \), \( |N(U)| \geq |U| \) holds.*

**Theorem 3.3** (Tutte’s Theorem [18]). *A graph \( G \) has a perfect matching if and only if \( c_o(G - U) \leq |U| \) for any \( U \subseteq V(G) \), where \( c_o(G - U) \) is the number of odd components of \( G - U \).*

Now we are ready to prove Theorem 1.2. Note that for the classification of optimal solutions of the bipartite vertex-transitive graphs, the authors [4] presented a sufficient condition for regular bipartite graphs to be super matched with respect to the concept “super edge-connected”, the method here is similar.

**Proof of Theorem 1.2** By Lemma 2.1 \( G \) is \( k \)-edge-connected. Then by Theorem 3.1 \( mp(G) \geq k \). Combining this with Theorem 1.1 we obtain \( mp(G) = k \). We are left to classify the optimal solutions.

**Necessity.** We prove by contradiction that if \( G \) is isomorphic to one of the six classes, then it has a non-trivial optimal solution.

(a) \( G \) contains a clique \( S \) of size \( k \) when \( k \) is odd and \( k \leq |G| - 2 \). Since \( k \leq |G| - 2 \), \( \overline{S} \) consists of at least two vertices. The edges between \( S \) and \( \overline{S} \) is a non-trivial optimal solution.

(b) \( G \) is isomorphic to a cycle of length at least six. Suppose that \( G = v_1v_2 \ldots v_nv_1 \) with \( n \geq 6 \). Then the edge set \( \{v_1v_2, v_4v_5\} \) forms a non-trivial optimal solution.
(c) $G$ is isomorphic to $Z_{4n}(1, 4n-1, 2n)$. Pick $F = \{12, 1(2n+1), (2n+1)2n\}$ and $S = \{2i+1|1 \leq i \leq n-1\} \cup \{2i|n+1 \leq i \leq 2n\}$. Then $G-F-S$ consists of $2n+1$ isolated vertices. By $|S| = 2n-1$ and Tutte’s Theorem, $G-F$ has no perfect matchings. Hence $F$ is a non-trivial optimal solution.

(d) $G$ is isomorphic to $Z_{4n+2}(2, 4n, 2n+1)$. Let $F = \{(4n+1)1, 1(2n+2), (2n+2)(2n+4)\}$ and $S = \{4i-1|1 \leq i \leq n\} \cup \{2n+2+4i|1 \leq i \leq n\}$. Similarly to (c), one can check that $F$ is a non-trivial optimal solution.

(e) $G$ is isomorphic to $Z_{4n+2}(1, 4n+1, 2n, 2n+2)$. Make $F = \{12, 2(2n+2), (2n+2)(2n+3), (2n+3)1\}$ and $S = \{2i+1|1 \leq i \leq n\} \cup \{2i|n+2 \leq i \leq 2n+1\}$. Similarly to (c), one can check that $F$ is a non-trivial optimal solution.

(f) $G$ is isomorphic to the Petersen graph. Let $F$ be the set of three bold edges and $S$ the set of bold vertices as shown in Figure 6. Similarly to (c), one can show $F$ is a non-trivial optimal solution.

![Figure 6. The Petersen graph is not super matched.](image_url)

Sufficiency. Let $F$ be an optimal solution. Then $|F| = mp(G) = k$ and $G-F$ has no perfect matchings. We shall show that $F$ isolates a singleton. There are two cases to consider.

**Case 1.** $G$ is bipartite.

We are to show that $F$ is an edge-cut. Assume that $W$ and $B$ are the bipartition of $G$. By Hall’s theorem, there exists $S \subseteq W$ such that $|N_{G-F}(S)| \leq |S|-1$. On the other hand, since $F$ is a matching preclusion set with the smallest cardinality, for each edge $e \in F$, $G-F+e$ has perfect matchings and also by Hall’s Theorem, $|N_{G-F+e}(S)| \geq |S|$ holds. Note that by adding one edge $e$ to $G-F$, the neighborhood of $S$ increases at most one vertex. Hence $|N_{G-F+e}(S)| \leq |N_{G-F}(S)| + 1$.

Combining the above three inequations, we obtain that $|S| = |N_{G-F}(S)| + 1$. Denote $S' = N_{G-F}(S)$. The edges sending out from $S$ are divided into two parts: One goes into $F$ and one goes into $S'$. Thus $S$ sends exactly $k|S| - |F| = k|S| - k$ edges to $S'$. Since $|S'| = |S| - 1$, there are no edges connecting $S'$ to $W - S$. This implies that $F$ is an edge-cut.

If $k = 1$, then $G$ is isomorphic to $K_2$ and $G$ is super matched; If $G$ is a cycle of length four, then $G$ is super matched; If $k \geq 2$, then since $G$ is bipartite, $G$ is triangle-free, that is, $G$ is not isomorphic to a complete graph. By hypothesis, $G$ is not isomorphic to a cycle of length at least six. Therefore, by Theorem 2.4 $F$ is a trivial edge-cut, that is, $F$ isolates a singleton.

**Case 2.** $G$ is non-bipartite.

By Tutte’s Theorem, there exists $S \subseteq V(G-F)$ such that $c_o(G-F-S) \geq |S| + 2$. Now we count the number $N$ of edges between $S$ and $\overline{S}$ in $G$. Since every component of $G-S$ sends out at least
Theorem 4.1. A folded $k$-cube graph is super matched if and only if $k \geq 4$.
Proof. Clearly, $FQ_k$ is $k$-regular. $FQ_3$ is $K_4$ and $FQ_4$ is the complete bipartite graph $K_{4,4}$. By the vertex-transitivity of the $k$-hypercube graph, one can easily check that $FQ_k$ is vertex-transitive. If $k = 3$, $FQ_3 = K_4$ is isomorphic to $Z_4(1,3,2)$, then by Theorem 1.2, $FQ_3$ is not super matched. If $k = 4$, $FQ_4$ is a bipartite graph other than a cycle, then by Corollary 3.3, it is super matched. If $k \geq 5$, then $FQ_k$ is of minimum degree at least five and does not contain a $k$-clique, and by Corollary 3.5, it is super matched.

The Hamming graph $H(d,q)$ can be viewed as the Cartesian product of $d$ complete graphs $K_q$.

**Theorem 4.2.** A Hamming graph $H(d,q)$ with even order is super matched if and only if $(d,q) \notin \{(1,4),(2,2)\}$.

The halved $k$-cube graph or half $k$-cube graph ($k \geq 3$) is the graph of the demihypercube, formed by connecting pairs of vertices at distance exactly two from each other in the $k$-hypercube graph. This connectivity pattern produces two isomorphic graphs, disconnected from each other, each of which is the halved $k$-cube graph.

**Theorem 4.3.** A halved $k$-cube graph is super matched if and only if $k \geq 4$.

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