Quantifying the mesoscopic quantum coherence of approximate NOON states and spin-squeezed two-mode Bose-Einstein condensates

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We examine how to distinguish and quantify the mesoscopic quantum coherence of approximate two-mode NOON states and spin-squeezed two-mode Bose-Einstein condensates (BEC). We identify two criteria that verify a nonzero quantum coherence between states with quantum number different by \( n \). These criteria negate certain mixtures of quantum states, thereby signifying a generalized \( n \)-scopic Schrödinger cat-type paradox. The first criterion is the correlation \( \langle \hat{a}^{\dagger n} \hat{b}^r \rangle \neq 0 \) (here \( \hat{a} \) and \( \hat{b} \) are the boson operators for each mode). The correlation manifests as interference fringes in \( n \)-particle detection probabilities and is also measurable via quadrature phase amplitude and spin-squeezing measurements. Measurement of \( \langle \hat{a}^{\dagger n} \hat{b}^r \rangle \) enables a quantification of the overall \( n \)-th order quantum coherence, thus providing an avenue for high efficiency verification of high-fidelity photonic NOON states. The second criterion is based on a quantification of the measurable spin-squeezing parameter \( \xi_n \). We apply the criteria to theoretical models of NOON states in lossy interferometers and double-well trapped BECs. By analyzing existing BEC experiments, we demonstrate generalized atomic “kitten” states and atomic quantum coherence with \( n \gtrsim 10 \) atoms.

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I. INTRODUCTION

In 1935 Schrödinger considered the preparation of a macroscopic system in a quantum superposition of two macroscopically distinguishable states [1]. Such systems are called “Schrödinger cat states” after Schrödinger’s example of a cat in a superposition of dead and alive states. The preparation of such states in the laboratory is difficult due to the existence of external couplings, which cause the superposition state to decohere to a classical mixture [2,3]. While for the mixture the “cat” is probabilistically “dead” or “alive,” the paradox is that for the superposition the cat is apparently neither dead or alive. Developments in quantum optics and the cooling of atoms and mechanical oscillators have made the generation of mesoscopic cat states feasible [4,5]. This is interesting for atomic systems where a superposition of states with a different mass location may be created. Ghirardi, Rimini, and Weber [6], Diosi [7], and Penrose [8] have proposed that for such systems decoherence mechanisms would prevent the formation of the Schrödinger cat superposition states. To carry out tests, firm proposals are required for the creation and detection of cat states.

A major consideration for cat-state experiments is that the generation of the cat state is not likely to be ideal. This is especially true for larger \( N \). One of the most well-studied cat states is the NOON state [9–14]

\[
|\psi_{\text{NOON}}\rangle = \frac{1}{\sqrt{2}} \left( |N\rangle_a |0\rangle_b + e^{i\phi} |0\rangle_a |N\rangle_b \right),
\]

where \( N \) photons or particles are in a superposition of being in the spatial mode \( a \) or the spatial mode \( b \). Ideally, a cat state requires \( N \rightarrow \infty \) but “\( N \)-scopic kitten state” realizations focus on finite \( N > 1 \). Here \( |n\rangle_a (|m\rangle_b) \) are the eigenstates of particle number \( \hat{n}_a = \hat{a}^{\dagger} \hat{a} \) (\( \hat{n}_b = \hat{b}^{\dagger} \hat{b} \)), respectively, and \( \hat{a}, \hat{a}^{\dagger} \) (\( \hat{b}, \hat{b}^{\dagger} \)) are the boson operators for mode \( a \) (\( b \)). The NOON states have been generated in optics for \( N = 5 \) (and just recently up to 9) [13], and with atoms for \( N = 2 \) [15]. At low \( N \), however, photon detection efficiencies are usually very low and results are often obtained by postselection processes.

Generating for higher \( N \) is challenging. Proposals exist to exploit the nonlinear interactions formed from Bose-Einstein condensates (BEC) trapped in the spatially separated wells of an optical lattice [16–24]. Under some conditions, theory shows that the atoms can tunnel between the wells, resulting in the formation of a NOON superposition. However, it is known that for realistic parameters, the states generated are in fact of the type

\[
|\psi\rangle = \sum_{m=0}^{N} d_m |N-m\rangle_a |m\rangle_b,
\]

where there exist nonzero probabilities for numbers other than zero or \( N \) [19,21–23,25]. (The \( d_m \) are probability amplitudes.) Oscillation between two BEC states with significantly different mode numbers has been experimentally observed [26], presumably resulting in the formation of a superposition of type (2) at intermediate times.

A key question (raised by Leggett and Garg [27]) is how to rigorously signify the Schrödinger cat–like property of the state in such a nonideal scenario. In this paper we propose quantifiable “catness” signatures that can be applied to nonideal NOON-type states generated in photonic and cold atom experiments. The signatures that we examine exclude all classical mixtures of sufficiently separated quantum states, so that it is possible to exclude all classical interpretations where the cat is dead or alive (see the Conclusion for a qualification). For the cat state that can ultimately be detected in one of two macroscopically distinguishable states \( \rho_D \) and \( \rho_A \), a rigorous signature must negate all mixtures of the form

\[
\rho_{\text{mix}} = P_D \rho_D + P_A \rho_A,
\]

where \( P_D \) and \( P_A \) are probabilities and \( P_D + P_A = 1 \). In our treatment, the \( \rho_D (\rho_A) \) are density operators for quantum states otherwise unspecified except that they give macroscopically distinct outcomes (alive or dead) for a measurement of quantum number difference \( \hat{n}_a - \hat{n}_b \).
The first step is to generalize this signature for the nonideal case (2), where there are nonzero probabilities for obtaining outcomes in an intermediate (“sleepy”) domain over which the cat cannot be identified as either dead or alive. In Secs. II and III, we follow Refs. [27–29] and consider states \( \rho_{DS} \) and \( \rho_{SA} \) that give outcomes in the combined “dead and/or sleepy” and “sleepy and/or alive” regions, respectively. The states have overlapping outcomes (for the number difference) indistinguishable over the range of the intermediate (“sleepy”) region. Denoting the width of this intermediate region by \( 2n \), we explain how the negation of all mixtures of the type

\[
\rho_{\text{mix}} = P_+\rho_{DS} + P_+\rho_{SA} \tag{4}
\]

(where \( P_- \) and \( P_+ \) are probabilities and \( P_- + P_+ = 1 \)) will imply a generalized \( n \)-scopic cat-type paradox, in the sense that the system cannot be explained by any mixture of quantum superpositions of states different by up to \( n \) quanta. It is proved that the observation of the nonzero \( n \)-scopic quantum coherence term

\[
\langle 0|n|\rho|0\rangle|n\rangle \neq 0 \tag{5}
\]

(\( \rho \) is the density operator) will negate all mixtures of type (4), thus signifying a generalized \( n \)-scopic “kitten” quantum superposition. We identify two criteria that verify the \( n \)-scopic quantum coherence (5). In a separate paper, we examine a third criterion based on uncertainty relations and Einstein-Podolsky-Rosen steering [30].

Previous studies have proposed signatures (criteria or measures) for mesoscopic cat states (see for instance [3,4,22,25,28–38]). These include proposals based on interference fringes, entanglement measures, uncertainty relations, negative Wigner functions, and state fidelity. Not all of these signatures however provide a direct negation of all the mixtures (3), (4). Further, most of these studies do not address the nonideal case where there may be a range of outcomes not binnable as either dead or alive. Exceptions include the work of Refs. [22,27–29,31–34,39] which (like the work of this paper) are based on the observations of a nonzero quantum coherence.

The first criterion that we consider is a nonzero \( n \)th-order correlation,

\[
\langle \hat{a}^n\hat{b}^n \rangle \neq 0. \tag{6}
\]

This criterion is necessary and sufficient for the \( n \)th-order quantum coherence (5). While normally evidenced for NOON states by fringe patterns formed from \( n \)-fold photon count coincidences, we show in Sec. VII how this moment can also be measured for small \( N \) using highly efficient homodyne detection and Schwinger-spin moments. In Sec. IV, we show that the value of the coherence \( \langle \hat{a}^n\hat{b}^n \rangle \) when suitably normalized translates to an effective fidelity measure of the \( n \)-scopic “catness” property of the state (which is not given directly by the state fidelity). We provide (in Sec. V) a theoretical model for the NOON state with losses, thus examining the degradation of the fidelity measure in that case. We also show how the fidelity measure can be applied to quantify mesoscopic quantum coherence for the case of number states \( |N\rangle \) incident on a beam splitter (the linear beam splitter model). The quantum coherence (5) is optimally robust with respect to losses when \( n \ll N \), but this can be achieved for high \( n \).

The criterion (6) was proposed by Haigh et al. to signify NOON-type superposition states created from nonlinear interactions in two-well BECs [22]. In Sec. VI, we evaluate \( \langle \hat{a}^n\hat{b}^n \rangle \) in dynamical regimes suitable for the formation of approximate NOON states, using a two-mode Josephson model (the nonlinear beam splitter). In Sec. VII, we analyze the measurement strategy using multiparticle interferometry. For the case of \( n = 2,3 \), the moment (6) is readily measured in terms of Schwinger spin observables. In fact by analyzing spin-squeezing data reported from the atomic BEC experiment of Esteve et al. [40], we infer (in Sec. VIII) the existence of two-atom (\( n = 2 \)) generalized (sometimes called “embedded” [21]) kitten states.

The second criterion that we consider for an \( n \)-scopic quantum coherence (5) is based on spin squeezing [41,42]. The amount of squeezing observed for a given number of atoms \( N \) is quantified by a squeeze parameter \( \xi_N \) [40,43–45]. In Sec. III B, we apply the methods of Ref. [29] and prove that a given measured amount of squeezing places a lower bound of \( \sqrt{\frac{N}{\xi_N}} \) on the value of \( n \) for which the quantum coherence \( \langle 0|n|\rho|0\rangle|n\rangle \) is nonzero:

\[
n > \sqrt{\frac{N}{\xi_N}}. \tag{7}
\]

This criterion requires \( \langle \hat{a}^n\hat{b}^n \rangle \neq 0 \) and is not therefore useful to identify ideal NOON states. However, the squeezing signature (7) is very effective in confirming a high degree of mesoscopic quantum coherence for states (2) where adjacent \( a_n \) and \( a_{n+1} \) are nonzero. This occurs in systems with high losses or linear couplings. In Sec. III B we apply this signature to published experimental data, and confirm a mesoscopic coherence (with \( n \sim 10 \) atoms) in two-mode BEC systems. We note this is consistent with the recent work of Refs. [31,34], which proposes quantifiers (measures) of mesoscopic quantum coherence based on Fisher information and reports significant values of atomic coherence for BEC systems.

## II. \( n \)-SCOPIC QUANTUM COHERENCE

We begin by considering the outcomes of the observable \( 2\hat{J}_z = (\hat{a}^\dagger\hat{a} − \hat{b}^\dagger\hat{b}) \). For the ideal NOON state (1) these are \( −N \) and \( N \). In the limit of large \( N \), we identify the two outcomes as “dead (\( D \))” and “alive (\( A \))” in order to make a simplistic analogy with the Schrödinger cat example. How does one signify the superposition nature of the NOON state? The density matrix \( \rho \) for the superposition \( |\psi_{\text{NOON}}\rangle \) has nonzero off-diagonal coherence terms \( \langle 0|N|\rho|0\rangle|N\rangle \neq 0 \) that distinguish it from the classical mixture (\( P_D \) and \( P_A \) are probabilities, \( P_D + P_A = 1 \))

\[
\rho_{\text{mix}} = P_D|0\rangle\langle N|0\rangle + P_A|N\rangle\langle 0|0\rangle\langle N|. \tag{8}
\]

Thus the detection of the nonzero coherence \( \langle 0|N|\rho|0\rangle|N\rangle \) serves to signify an \( N \)-scopic cat state in this case.

The ultimate objective of a “Schrödinger cat” experiment is to negate classical realism at a macroscopic level. The accepted definition of macroscopic realism is that a system must be in a classical mixture of two macroscopically distinguishable states [27]. Similarly, we take as the definition of \( N \)-scopic realism that a system must be in a classical mixture of two
states that give predictions different by \( N \) quanta. In this paper, the meaning of classical mixture is in the quantum sense only: that the density operator \( \rho \) for the system is equivalent to a classical mixture of the two (quantum) states, as in (8).

More generally, the states generated in the experiments give outcomes for \( \hat{J}_z \) and \( \hat{J}_x \) different to zero and \( N \), as a result of even a small amount of loss or noise in the system (real predictions are illustrated in Fig. 1). The question becomes how to confirm by experiment that the system is indeed in a superposition of two mesoscopically distinguishable quantum states, as opposed to any alternative classical description where there would be no mesoscopic cat paradox. This question was examined in Refs. [28,29] for continuous outcomes realized from quadrature phase amplitude measurements and we apply the approach given there. The following is a result found in that paper as applied to this case.

**Result 1.** An \( n \)-scopic quantum coherence and generalized \( n \)-scopic cat paradox. Consider the following mixture sketched in Fig. 1(b):

\[
\rho_{\text{mix}} = P_- \rho_{DS} + P_+ \rho_{SA},
\]

where \( P_- \) and \( P_+ \) are probabilities and \( P_- + P_+ = 1 \). Here, \( \rho_{DS} \) is a quantum state for which (for some fixed \( j_c \)) the two-mode number state expansion may only include eigenstates with outcome \( 2J_z < j_c + n \); and \( \rho_{SA} \) is a quantum state for which the expansion only includes eigenstates with \( 2J_z > j_c - n \). The outcome \( 2J_z \geq j_c + n \) is interpreted as alive and the outcome \( 2J_z \leq j_c - n \) is interpreted as dead. The intermediate overlapping regime is sleepy. Usually, we consider \( j_c = 0 \) as depicted in Fig. 1(b). The negation of all mixtures of the type (9) will imply a generalized \( n \)-scopic “cat-type” paradox, in the sense that the system cannot be viewed as either “dead and/or sleepy”, or “alive and/or sleepy”—and cannot therefore be explained by any mixture of superpositions of states different by up to \( n \) quanta. This implies that superpositions of states with larger differences are required to explain the system. If (9) can be negated, then there is an \( n \)-scopic generalized quantum coherence (cat-type paradox). The negation of (9) implies that for some \( n', m' \):

\[
\rho_{n' m'}(n' m'|n m') \neq 0 \tag{10}
\]

(in fact \( j_c = n' - m' \)). The converse is also true. Conditions that negate (9) equivalently demonstrate (10), and we refer to these conditions as signatures of \( n \)-scopic quantum coherence, or of an \( n \)-scopic generalized cat paradox.

**Proof.** The justification is that if (9) fails, then the system cannot be thought of as being in one quantum state \( \rho_1 \) or the other \( \rho_2 \). We have not constrained the \( \rho_1 \) or \( \rho_2 \), except to specify they cannot include both dead and alive states (each one is orthogonal to either the dead or alive state). Hence there is a negation of the premise that the system must always be either dead or alive. In that sense we have an analogy with the Schrödinger cat paradox at the level where the dead and alive states are different by \( n \) quanta in a given mode. That the coherence is nonzero follows on expanding the density matrix in the number state basis.

We note that the nonzero \( n \)-scopic quantum coherence has a physical significance, in that it is possible (in principle) to filter out the intermediate “sleepy states” using measurements of \( |J_z| > n/2 \) to create a conditional cat state where the separation in \( J_z \) of the dead and alive states is of order \( n \). This method of preparation has been carried out experimentally [10]. However, where the \( n \)-scopic coherences are small, the heralding probability for the cat state also becomes small, making the states increasingly difficult to generate. This motivates Sec. V which examines how to quantify the quantum coherence through experimental signatures. First, we identify two criteria for the condition Eq. (10).

## III. TWO CRITERIA FOR \( n \)-SCOPIC QUANTUM COHERENCE

### A. Correlation test

It is well known that higher-order correlations can detect NOON states. We clarify with the following result.

**Result 2.** The \( n \)-th-order correlation test. Restricting to two-mode quantum descriptions for \( \rho \), the observation of

\[
\langle \hat{a}^{\dagger n} \hat{b}^{\dagger n} \rangle \neq 0 \tag{11}
\]

is a signature of the \( n \)-scopic quantum coherence (10).

The Result can be proved straightforwardly by expanding the operator \( \hat{a}^{\dagger n} \hat{b}^{\dagger n} \) in terms of the Fock basis elements \( |m_n\rangle |n_0\rangle |m_b\rangle |n_a\rangle \) or equivalently by expanding an arbitrary density matrix \( \rho \) written in the two-mode Fock basis and noting that the condition (11) is equivalent to (10). We will find it useful to note the following: if the moment \( \langle \hat{a}^{\dagger n} \hat{b}^{\dagger n} \rangle \) is nonzero then there is a nonzero probability that the system is in the following generalized \( n \)-scopic superposition state:

\[
|\psi_n\rangle = a_{n'm'}^{(n)} |n' m' + n\rangle + b_{n'm'}^{(n)} |n' m' + n\rangle |m'\rangle + d |\psi_0\rangle \tag{12}
\]

The \( a_{n'm'}^{(n)}, b_{n'm'}^{(n)}, d \) are probability amplitudes satisfying \( a_{n'm'}^{(n)}b_{n'm'}^{(n)} \neq 0 \), the \( d \) being unspecified. \( |\psi_0\rangle \) is an unspecified quantum state orthogonal to the states \( |n' m' + n\rangle |m'\rangle \) and \( |n' + n\rangle |m'\rangle \). The meaning of “nonzero probability that the system is in” in this context is that the density operator for the quantum system is necessariley of the form \( \rho = \sum_k P_k |\psi_k \rangle \langle \psi_k | \) where at least one of the states \( |\psi_k \rangle \) with nonzero \( P_k \) is an \( n \)-scopic superposition state \( |\psi_n\rangle \).

Appendix A gives a detailed explanation of this last result.
B. Spin-squeezing test and application to experiment

Significant nth-order quantum coherence can also in some cases be detected by observation of spin squeezing. We define the standard Schwinger operators:

\[
J_X = (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger)/2,
\]

\[
J_Y = (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger)/(2i),
\]

\[
J_Z = (\hat{a}^\dagger \hat{a} - b^\dagger b)/2,
\]

\[
\hat{N} = \hat{a}^\dagger \hat{a} + b^\dagger b.
\]

We consider a system described by a superposition of two-mode number states as in (2). Thus we specify a generalized superposition as

\[
|\psi\rangle = \sum_{i,j} c_{ij} |n_i\rangle |m_j\rangle = \sum_k d_k |\psi_k\rangle,
\]

where the last line relabels (for convenience) all states of the ij array by an index k. In (2) we have a superposition \(|\psi\rangle = \sum_{n,m} d_n |n\rangle |N - n\rangle\) where N (the total number of particles) is fixed. This case for large N and where \(d_n \neq 0\) for some \(n \neq 0\), N has been described as a superposition of dead, alive, and sleepy cats. Considering the general case (14), we can define for each term \(|\psi_k\rangle\) such that \(d_k \neq 0\) the spin number difference \(j_k = (n_i - m_j)/2\). The aim is to put a lower bound on the spread of possible \(j_k\) values (depicted in Fig. 1). We define the spread as

\[
\delta = \max |j_k - j_k'|,
\]

such that for \(j_k\) and \(j_k'\), the coefficients \(d_k\), \(d_k'\) not 0. For the ideal NOON state, \(\delta = N\). Here max denotes the maximum of the set.

We can show that a certain amount of squeezing in \(J_Y\) determines a lower bound in the spread of eigenstates of \(J_Z\). The method is similar to that given in Ref. [29] which studied quadrature phase amplitude squeezing. The spin Heisenberg uncertainty relation is

\[
(\Delta \hat{J}_Z)(\Delta \hat{J}_Z) \geq |(\hat{J}_X)|/2.
\]

Spin squeezing is obtained when [41,42]

\[
(\Delta \hat{J}_Z)^2 < |(\hat{J}_X)|/2.
\]

It is clear that in that case a low variance \((\Delta \hat{J}_Z)^2\) will always imply a high variance in \(\hat{J}_Z\). For many spin-squeezing experiments, \(\langle \hat{J}_X \rangle \sim \langle N \rangle/2\), which means the Bloch vector lies on the surface or near the surface of the Bloch sphere, so that the system is close to a pure state. Squeezing is then obtained when \((\Delta \hat{J}_Z)^2 < \langle N \rangle/4\).

For pure states, the high variance in \(\hat{J}_Z\) is associated with a minimum spread \(\delta\) of the superposition of eigenstates of \(\hat{J}_Z\). The spread is linked to the quantum coherence value \(n\). In the Appendix B, following the methods of Ref. [29], this connection is generalized for mixed states. We prove the following result.

Result 3. Spin-squeezing test for nth-order quantum coherence. An experimentally measured amount of spin squeezing in \(J_Y\) is defined in terms of a “squeezing parameter”

\[
\xi_N = \frac{(\Delta \hat{J}_Y)}{\sqrt{\langle (\hat{J}_X) \rangle/2}} \rightarrow \frac{(\Delta \hat{J}_Y)}{(N)^{1/2}/2},
\]

where \(\xi_N < 1\) implies spin squeezing and \(\xi_N = 0\) is the optimal possible squeezing (achievable as \(N \rightarrow \infty\)). Here we have taken the case where \(\langle \hat{J}_X \rangle \sim \langle N \rangle/2\). We can conclude that there exists a nonzero coherence \(\langle 0\rangle|n\rangle|\rho|0\rangle|n\rangle \neq 0\) for a value \(n\) where

\[
n > \sqrt{N}/\xi_N.
\]

Proof. The proof is given in the Appendix B.

The particular test given by Result 3 requires \(\langle \hat{J}_X \rangle \neq 0\). This would imply nonzero single atom coherence terms given as \((\hat{a}^\dagger \hat{b}) \neq 0\). We note that the final result (19) indicates that the coherence size is of order \(\sqrt{N}\). Spin squeezing with a considerable number \(N\) of atoms has been observed in several atomic experiments and excellent agreement has been obtained for \(N \sim 100\) with a two-mode model [40,43,44]. Typically, the number of atoms is \(N \sim 100\) or more, indicating values of quantum coherence of order \(n > 10\) atoms.

IV. MEASURABLE QUANTIFICATION OF THE MESOSCOPIC QUANTUM COHERENCE

A. Catness fidelity and quantum coherence

The observation of \((\hat{a}^\dagger \hat{b}) \neq 0\) certifies the existence of the (nonzero) n-sopic quantum coherence, but does not specify the magnitude of the quantum coherence (QC), originating from terms like

\[
C_n^{(n',m')} = 2|b_n|^m |\langle n'\rangle|\rho|n + n'\rangle |\langle m'\rangle|b_n|^m.
\]

taken from Eq. (10). In fact, we can easily identify states (such as \(|\alpha\rangle|\beta\rangle\) for which the n-sopic quantum coherence vanishes as \(n \rightarrow \infty\) (for any \(m', n'\)), but for which the moment \((\hat{a}^\dagger \hat{b})^{n'}\) increases. To put it another way, the observation \((\hat{a}^\dagger \hat{b})^{n'}\) does not tell us the probability \(P_{\rho}\) that the system will be found in an associated n-sopic superposition Eq. (12), nor the values of the probability amplitudes \(a_n^{(m')} b_n^{(m')}\).

We explain in this section that the measured correlation \((\hat{a}^\dagger \hat{b})^{n'}\) when suitably normalized places a lower bound on the sum of the magnitudes of the nth-order quantum coherences, defined as

\[
C_n = \sum_{n',m'} C_n^{(n',m')}.
\]

Here \(N\) is a normalization factor that ensures the maximum value of \(C_n = 1\) for the optimal case. The normalized correlation thus gives measurable information about \(C_n\) which is an effective “catness fidelity.”

The catness fidelity contrasts with the standard state-fidelity measure \(F\) (defined as the overlap between an experimental state \(\rho_{exp}\) and the desired superposition state [46]). The standard measure is not directly sufficient to quantify a cat state since it may be possible for mixtures that are not cat-type superpositions to give a high absolute \(F\) as \(N \rightarrow \infty\) [3].
B. General result for pure and mixed two-mode states

Defining a suitable catness fidelity is straightforward for pure states. Any two-mode state \( |\psi\rangle \) can be expanded in the number state basis and can thus be written in terms of a superposition of the states (12) but with \( a_{n,m}^{(a)}, b_{n,m}^{(a)} \) arbitrary. The state fidelity \( F \) of \( |\psi\rangle \) with respect to the symmetric \( n \)-scopic superposition

\[
|\psi_{\text{sup}}\rangle = (|m\rangle|m + n\rangle + e^{i\phi}|n\rangle|m + n\rangle)/\sqrt{2}
\]

is

\[
F = \frac{||\psi_{\text{sup}}\rangle||}{||\psi\rangle||^2} = \frac{1}{2} \left( |a_{n,m}^{(n)}|^2 + |b_{n,m}^{(n)}|^2 + 2|a_{n,m}^{(n)} b_{n,m}^{(n)}| \right),
\tag{22}
\]

where the phase \( \phi \) is chosen to maximize \( F \). We see that the magnitude of the quantum coherence of the pure-state density operator with respect to the states \( |n\rangle|m + n\rangle \) and \( |n\rangle|m + n\rangle \) is directly related to the fidelity:

\[
C_n^{(m',m)} = 2(|m' + n\rangle|n\rangle|m + n\rangle)|m'| \]

(23)

We note that \( F = 1 \) if and only if \( a_{n,m}^{(n)} b_{n,m}^{(n)} = 1 \), which implies \( C_n^{(m',m)} = 1 \). Similarly, \( C_n^{(m',m)} = 1 \) implies \( F = 1 \). An arbitrary two-mode pure state is a superposition of states over different \( n',m' \) and we may define as the total “\( n \)-scopic catness fidelity” the sum of the magnitudes of the \( n \)-th order coherences, i.e.,

\[
C_n = N \sum_{n',m'} C_n^{(n',m')} = 2N \sum_{n',m'} |a_{n,m}^{(n)} b_{n,m}^{(n)}|,
\tag{24}
\]

where \( N \) is a normalization factor to ensure the maximum value of \( C_n \) is \( 1 \). A pure two-mode state with fixed \( N \) as given in the Introduction can be written

\[
|\psi\rangle = \sum_{m=0}^{N} d_m |N - m\rangle_a |m\rangle_b
\]

\[
= \sum_{m' < N/2} d_{m'+n} |m'\rangle|m + n\rangle + d_{m'} |n\rangle|m + m'\rangle.
\tag{25}
\]

(The simplification in the last line is written for \( N \) odd.) The \( a_{n,m}^{(n)} \) and \( b_{n,m}^{(n)} \) can then be given in terms of \( d_{m+n} \) and \( d_{m'} \). For a pure state, we see that \( C_n \) can be inferred from the probabilities for the mode number. However, this is not useful for the practical case of mixtures.

With this motivation, we note that the catness-fidelity \( c_n \) can be expressed in terms of the measurable higher-order moments:

\[
\langle \hat{a}^m \hat{b}^n \rangle = \sum_{m',n'} a_{m',n'}^{(n)} b_{m',n'}^{(n)} \sqrt{(m + n)! \frac{1}{m} \frac{1}{n} \sqrt{n + 1}}.
\tag{26}
\]

For a general mixed two-mode state, using that \( \langle \hat{a}^m \hat{b}^n \rangle = \text{Tr}(\rho a^m b^n) = \sum_{m',n'} \langle n\rangle_{m'} |m\rangle \rho |n\rangle a^m b^n |m\rangle_{n'} \), we find

\[
\langle \hat{a}^m \hat{b}^n \rangle = \sum_{m',n'} \sqrt{\frac{(m + n)!}{n!} \frac{(m + n)!}{m!}} \times (n')|m'| + n\rangle|n' + n\rangle|m'\rangle.
\tag{27}
\]

This allows us to deduce the following result for mixed states.

Result 4. Measurable lower bound estimate to the \( n \)-scopic “catness fidelity,” defined as the sum of the magnitudes of \( n \)-th order quantum coherences.

The measurable quantity

\[
c_n = \frac{2}{N} \langle |\hat{a}^N \hat{b}^N| \rangle
\tag{28}
\]

gives a lower bound to the true catness fidelity \( C_n \). Here \( S = \sum_{m',m} \left\{ \sqrt{\frac{(m + 1)!}{m!}} \sqrt{\frac{n + 1}{n!}} \right\} \) over values of \( n', m' \) satisfying that the probability \( P_{m',n'} \) for detecting \( m' \) and \( n' \) particles in the respective modes \( b \) and \( a \) is nonzero, and also that the probability \( P_{m' + n',n'} \) for detecting \( m' + n \) and \( n' \) particles in the respective modes \( b \) and \( a \) is nonzero.

Proof. The proof follows from (27) using the definition (21).

Realistically, it is difficult in an experiment to truly verify that the probability for obtaining a certain mode number is zero. In light of this, we deduce in Appendix C a correction term to Result 4, assuming the experimentalist is at least able to verify that the “nonrelevant” probabilities \( P_{m',n'+n} \) are sufficiently small, and that there is a practical upper bound to the mode numbers (defined by an energy or atom number bound).

C. Ideal NOON case

In the ideal NOON case, an experimentalist would observe \( N \) particles in mode \( a \) or \( N \) particles in mode \( b \). Consider an experiment where indeed only such probabilities are nonzero. This is not unrealistic for photonic experiments with small \( N \) that use postselection. The experimentalist could deduce that the most general form of the density operator in this case is

\[
\rho = P_N \rho_N + P_{ah} \rho_{ah},
\tag{29}
\]

where \( \rho_{ah} \) is the density operator of a NOON superposition (12) with \( n' = m' = d = 0 \), and \( \rho_{ah} \) is an alternative density operator describing classical mixtures of number states (namely \( |N\rangle|0\rangle \) and \( |0\rangle|N\rangle \)). Here, \( P_N + P_{ah} = 1 \) and \( P_N, P_{ah} \) are probabilities.

We see from (12) that the quantity \( C_N \) defined as

\[
C_N = 2 |a_{00}^{(N)} b_{00}^{(N)}| P_N
\tag{30}
\]

gives an effective fidelity measure of the state \( \rho \) relative to the NOON cat state. We call the quantity \( C_N \) the catness fidelity, and note that \( 0 \leq C_N \leq 1 \). Clearly, the value of \( C_N = 1 \) is optimal and can only occur if the system \( \rho \) is the pure symmetric NOON state (1) for which \( |a_{00}^{(N)}| = |b_{00}^{(N)}| = \frac{1}{\sqrt{2}} \).

For the ideal NOON state, the prediction is \( \langle \hat{a}^m \hat{b}^n \rangle = \delta_{N/2} |N|!/2 \) and \( S = N! \) so that the catness fidelity is indeed 1:

\[
c_N = C_N = \frac{2}{N} \langle |\hat{a}^N \hat{b}^N| \rangle = 1.
\tag{31}
\]

The value of \( C_N \) reduces for asymmetric NOON states or for mixed states where \( P_N < 1 \).
Fidelity (24) (defined as the sum of the magnitudes of all the n-th-order quantum coherence) for the output state of the linear beam splitter with N particles incident in one arm, $C_N$ and $c_n$ vs $n$, for (a) $N = 5$, 10 and (b) $N = 50$, 100. $C_n \geq c_n$ as expected.

V. EXAMPLES OF QUANTIFICATION

A. Attenuated NOON states

Photonic NOON states have been reported experimentally for up to $N = 9$. For a rigorous detection of a catlike state, it is necessary to account for losses that may arise as a result of processes including detection inefficiencies. To model loss, we use a simple beam splitter approach [3]. We calculate the moments of final detected fields $\hat{b}^{\dagger n} \hat{b}^m$ given by $a_{\text{det}} = \sqrt{\eta} a + \sqrt{1 - \eta} a_{\text{env}}$, $b_{\text{det}} = \sqrt{\eta} b + \sqrt{1 - \eta} b_{\text{env}}$, where $a$, $b$ are the boson operators for the incoming field modes, prepared in a NOON state, and $a_{\text{env}}$, $b_{\text{env}}$ are boson operators for vacuum modes associated with the environment. Here $\eta$ is the probability that an incoming photon or particle is detected. We find

$$\langle \hat{b}^{\dagger n} \hat{b}^m \rangle = \eta^n \langle \hat{a}^{\dagger n} \hat{a}^m \rangle = \eta^n \delta_{nN} N^n! / 2.$$  (32)

The system is a mixture of type $\rho = P_N \rho_N + P_{\text{env}} \rho_{\text{env}}$ defined in (29). The catness-fidelity signature $C_N$ of Eq. (29) is measurable as $c_N$ ($S = N^n!$) defined by (28) and is plotted in Fig. 2. Comparing with the distributions of Fig. 1 which are generated for the attenuated NOON state, we see that only the extremes where $n = N$ have a nonzero coherence. As loss increases, the Nth quantum coherence remains in principle rigorously certifiable since it is predicted that $\langle \hat{a}^{\dagger n} \hat{a}^m \rangle \neq 0$ for all values $\eta$. However, the fidelity $C_N$ is greatly reduced with decreasing $\eta$, particularly for larger $N$ (Fig. 2).

B. States formed from number states incident on a linear beam splitter

Next we consider a two-mode number state $|N\rangle \langle 0|$ incident at the two single-mode input ports of a beam splitter, so that $N$ quanta are incident at one port only. The output state is the $N$-scopic superposition (2) with binomial coefficients:

$$|\text{out}\rangle = \sum_{m=0}^{N} d_m |m\rangle |N-m\rangle_a |a\rangle |a\rangle |b\rangle,$$  (33)

where $d_m = \sqrt{N! / \sqrt{2^N m!(N-m)!}}$. Different from the NOON states, nonzero quantum coherences $\langle \hat{a}^{\dagger n} \hat{a}^m \rangle \neq 0$ exist for all $m \leq N$.

Calculation shows that the pure state $n$-scopic catness fidelity (24) (defined as the sum of the magnitudes of all the...
normalization $N_{n,N}$ are as above. Figure 4 plots the values of the catness fidelity $c_n$ versus efficiency $\eta$. We note that the first-order coherence $n = 1$ is much more robust with respect to loss, as compared to the higher-order coherences. It is interesting that, for a fixed $n$, the robustness with respect to loss improves quite dramatically if one increases the value of $N$. At high $N$, the highest-order coherences are almost immeasurable, e.g., for $N = 100$, the quantum coherence becomes measurable at $n < 20$. We note also that the cutoff for a measurable $n$ increases with increasing $N$, making generation of $n$-scopic cat states in this generalized sense quite feasible.

VI. MESOSCOPIC QUANTUM COHERENCE IN DYNAMICAL TWO-WELL BOSE-EINSTEIN CONDENSATES

A. Hamiltonian and model

A mesoscopic NOON state can in principle be created from the nonlinear interaction modeled by the two-mode Josephson (LMG) Hamiltonian [47,48]:

$$H = \kappa \hat{a} \hat{b} + \kappa \hat{b} \hat{a} + \frac{g}{2} [\hat{a}^2 \hat{a}^2] + \frac{g}{2} [\hat{b}^2 \hat{b}^2]$$

(37)

($\hbar = 1$). This Hamiltonian is well described in the literature and models a Bose-Einstein condensate (BEC) constrained to two potential wells of an optical lattice [16,17,19,21–23,40,44,49]. The occupation of each well is modeled as a single mode (boson operators $\hat{a}$, $\hat{a}$ and $\hat{b}$, respectively). The nonlinearity is quantified by $g$ and the tunneling between wells by $\kappa$. We consider a system prepared with a definite number $N$ of atoms in one mode (well) (denoted by $\hat{a}$). Since the number of particles is conserved, the state at any later time is of the form (2). The Hamiltonian can be represented in matrix form and the time dependence of the $d_m$ solved as explained in Refs. [16,21,22,24,49].

B. Two-state oscillation and creation of NOON states

Solutions give the probability $P(m) = |d_m|^2$ of measuring $m$ particles in the well $A$ at a given time. For some parameters, the population oscillates between wells and there is an almost complete transfer to the well $B$ at some tunneling time $T_N$. For larger nonlinearity $g$, the system can approximate a dynamical two-state system, showing oscillations between the two distinguishable states $|N\rangle|0\rangle$ and $|0\rangle|N\rangle$ over long time scales (Fig. 5). At intermediate times ($\sim T_N/2$) before the complete tunneling from one state to the other, approximate NOON states can be formed. Figure 6 depicts the probabilities $P(m)$ at the intermediate times $T_N/6$ and $T_N/3$ that violate a Leggett-Garg inequality [27,50]. It is known however that, even for moderate $N$, the predicted tunneling times $T_N$ are typically much longer than practical decoherence times [19,21,51,52]. For instance, Carr et al. report impossibly long times for the typical parameters of Rb atoms [21].

C. Creation of $n$-scopic quantum superpositions

It is possible however to generate states with a significant mesoscopic coherence by preparing the system in an initial state $|n_L\rangle|N-n_L\rangle$, where $n_L \neq 0,N$. As pointed out by Gordon and Savage [19] and Carr et al. [21], the Hamiltonian (37) predicts (in some parameter regimes) an approximate two-state oscillation between the two states $|n_L\rangle|N-n_L\rangle$ and $|N-n_L\rangle|n_L\rangle$. At approximately half the time for oscillation from one state to the other, an $n$-scopic superposition state of the type given by (12) where $m',n' \neq 0$ is formed, i.e.,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n_L\rangle|N-n_L\rangle + |N-n_L\rangle|n_L\rangle).$$

(38)

Here, $n = N - 2n_L$. Such $n$-scopic superposition states have been called "embedded" cat states [21]. These embedded cat states are identical to those superpositions (12) discussed in the previous section. Calculations reveal that, for some parameters, the period of oscillation reduces to practical values [19,21]. Two-state oscillation of the BEC has been experimentally observed [26]. We present in Fig. 6 predictions for this type of oscillation with $N = 20$ and $n_L = 2$, where...
FIG. 6. Mesoscopic two-state oscillation and generation of NOON-type states. (a) $N = 100$, $g = 2$, and $n_L = 10$. (b) $N = 20$, $g = 4$, and $n_L = 4$. Time $t$ is in units $\kappa$.

The solutions indicate states (38) with a separation of $n = 16$ atoms.

The question becomes how to certify the quantum coherence of the embedded cat states (38) that may be generated in the experiment where such oscillation is observed. The value of the catness-fidelity signature $c_n$ is calculated and given in Fig. 7, for the parameters of Fig. 6(b). (See also Fig. 8.) The $c_n$ for moderate $n$ may be measurable using higher-order interference in multiatom detection, as described in Sec. VII.

VII. MEASUREMENT OF MESOSCOPIC QUANTUM COHERENCE VIA $\langle \hat{a}^\dagger \hat{b} \rangle$

Finally, we address how one may measure the correlation $\langle \hat{a}^\dagger \hat{b} \rangle$. The measurement of $J_X$ is a photon or atom number difference, achievable with counting detectors or imaging. Schwinger spin operators $J_X = (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a})/2, J_Y = (\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a})/2i$ are measured similarly as a number difference, after rotating to a different mode pair using polarizers [53], or

Rabi rotations with $\pi/2$ pulses [40,43,44], or beam splitters and phase shifts.

A. Interferometric detection

For instance, we consider the measurable output number difference $I_D$ after transforming the incoming modes $a, b$ to new modes $c, d$ via a 50:50 beam splitter and phase shift $\phi$:

$$I_D = \hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d} = \hat{a}^\dagger \hat{b}^\dagger e^{i\phi} + \hat{a} \hat{b} e^{-i\phi} = 2 J_X \cos \phi - 2 J_Y \sin \phi.$$  (39)

Here the transformed boson operators for the new modes are $\hat{c} = (\hat{a} + \hat{b} \exp^{i\phi})/\sqrt{2}, \hat{d} = (\hat{a} - \hat{b} \exp^{i\phi})/\sqrt{2}$. Selecting $\phi = 0$ or $\phi = -\pi/2$ measures $J_X$ or $J_Y$. For $N = 1$, $\langle \hat{a} \hat{b} \rangle = \langle J_X + i J_Y \rangle$. The first-order moment $\langle \hat{a} \hat{b} \rangle$ is thus measurable via the fringe visibility in $I_D$ as one varies $\phi$, i.e., $\langle \hat{a} \hat{b} \rangle$ is measurable via first-order interference. Similar transformations using atom interferometry give the same results as

FIG. 7. Signifying the creation of NOON-type states under the Hamiltonian (37). The $n$th-order quantum coherence measure $c_n$ vs time $t$ in units $\kappa$. (a) $N = 5, g = 10, and n_L = 0$. The NOON state $N = 5$ is signified by $c_5 = 1, c_i \sim 0 (i \neq 5)$ at $t = T_N/4$. (b) $N = 20$, $g = 4$, and $n_L = 4$ as for Fig. 6(b). The large quantum coherence $c_n$ for $n = 12$ signifies the superposition (38) at $t = T_N/4$.

FIG. 8. Plot of $P(m)$ and the $n$th-order quantum coherence $c_n$ for the state of Fig. 6(b) at $t = T_N/4$ [as in Fig. 7(b)].
explained in Ref. [54]. When a NOON state is incident on the interferometer, the nonzero value of \(\langle \hat{a}^{nN}\rangle\) can be deduced by observation of higher-order interference fringes that are signifyed by an \(e^{i\Phi}\) oscillation. This is the usual method for detecting NOON states [11–13,22]. The method can also be used to detect and quantify the nth-order quantum coherence \(c_n\). We consider that we have a fixed total number \(N\) of particles so the input state is of the form (2). The probability of detecting \(N\) quanta at the output denoted by mode \(c\) is \(\langle \hat{c}^{nN}\rangle/N!\). The probability of obtaining \(M\) particles at the port \(c\) is a calculable function of the correlation functions \(\langle \hat{c}^{nM}\rangle\), where \(n \geq M\). Suppose we measure \(\langle \hat{c}^{nM}\rangle\) for a given fixed \(n\). Expanding we find

\[
\langle \hat{c}^{nM}\rangle = \frac{1}{2^n} \langle \hat{a} \hat{b}^\dagger e^{-i\Phi} (\hat{a} + \hat{b} e^{i\Phi})^n \rangle
\]

\[
= \frac{1}{2^n} \sum_{m=0}^{n} \sum_{l=0}^{n} \sum_{\ell=0}^{n} (m,n,m,\ell) \times (\langle \hat{a} \hat{b}^\dagger \rangle)^m (\langle \hat{a} \hat{b}\rangle)^{n-m} (\langle \hat{b}^\dagger \hat{a}\rangle)^{n-\ell} e^{i\Phi(\ell-m)}. \tag{40}
\]

The terms that oscillate as \(e^{i\Phi}\) are proportional to the nth-order moment \(\langle \hat{a}^{nM}\rangle\). Hence if we measure \(\langle \hat{c}^{nM}\rangle\), the fringe visibility associated with this oscillation allows determination of the magnitude of \(\langle \hat{a}^{nM}\rangle\). Where \(\langle \hat{a}^{nM}\rangle\) is the only nonzero moment (as for the ideal NOON states with \(n = N\)), only the oscillation \(e^{i\Phi}\) will contribute and the higher-order interference enable a clear signature and quantification of the nth-order quantum coherence \(\langle \hat{a}^{nM}\rangle\).

For nonideal NOON states the interference method becomes less precise. However, the rapidly oscillating terms can only arise from moments that indicate a higher order of quantum coherence. This is evident by the last line of (40). The moments are of form \(\langle \hat{a} \hat{b}^\dagger \rangle^m, \langle \hat{a} \hat{b}\rangle^m \hat{a} \hat{b}^\dagger, \langle \hat{b}^\dagger \hat{a}\rangle^m \hat{a} \hat{b}^\dagger\rangle\) so the oscillation frequency where \(l - m = n\) requires \(l = n\) and \(m = 0\) and therefore has a nonzero amplitude only if \(\langle \hat{a} \hat{b}\rangle \neq 0\), which is a signature for a quantum coherence of order \(n\). Similarly, the oscillation frequency \(l - m = n - 1\) requires a nonzero quadratic coherence of order \(n - 1\). While the \(\langle \hat{c}^{nM}\rangle\) can be evaluated from the probabilities for particle counts, in practice for large numbers \(N\), resolution of atom or photon number is difficult. Here one can measure the probability that \(n\) is in a binned region e.g., the region \(n > M\). This probability is given by

\[
P(n \geq M) = \sum_{n=M}^{\infty} \zeta^{\langle \hat{c}^{nM}\rangle}/M!,
\]

where \(\zeta\) are constant calibrants. Here, measurement of a nonzero amplitude for oscillations \(e^{i\Phi}\) with frequency \(M\) or greater is evidence of quantum coherence of order \(M\). The high-frequency oscillation can only arise from the high-order quantum coherence terms. In Fig. 9, we plot \(P(n > M)\) and the Fourier analysis for the two-mode example given in Fig. 6(b).

B. Spin-squeezing observables and quadrature phase amplitudes

An alternative method for evaluating the higher order moments is given in Ref. [54]. We note that

\[
\langle \hat{a} \hat{b}^\dagger \rangle = \langle \hat{J}_X^3 \rangle - 3\langle \hat{J}_Y^3 \rangle + i\langle \hat{J}_X \hat{J}_Y \rangle,
\]

\[
\langle \hat{a} \hat{b} \rangle = \langle \hat{J}_X^3 \rangle + 3\langle \hat{J}_Y^3 \rangle - i\langle \hat{J}_X \hat{J}_Y \rangle,
\]

\[
\langle \hat{a}^\dagger \hat{b}^\dagger \rangle = \langle \hat{X}_A \hat{X}_B \rangle + \langle \hat{P}_A \hat{P}_B \rangle + i\langle \hat{X}_A \hat{P}_B - \hat{X}_B \hat{P}_A \rangle. \tag{41}
\]

FIG. 9. (a) Probability of measuring more or equal to \(M\) photons, for \(N = 20, g = 4,\) and \(n_L = 4\) at \(T/\pi = \sqrt{2}/2\) [same as in Fig. 7(b)] after a rotation. (b) The Fourier transform of the curves from (a), plotted against angular frequency (which is equivalent to the number of oscillations in the range of \(2\pi\), showing a significant peak at \(\omega = 12\) [the expected separation of the state, \(|4\rangle|16\rangle + |16\rangle|4\rangle)/\sqrt{2}\) for all \(M\).

where \(\{A, B\} = AB + BA\). The real part of \(\langle \hat{a} \hat{b} \hat{b}^\dagger \rangle\) can be evaluated by measurement of \(\langle \hat{J}_X^3 \rangle, \langle \hat{J}_Y^3 \rangle, \langle \hat{J}_X \hat{J}_Y \rangle\). We show that the moment is nonzero if we can show that \(\langle \hat{J}_X^3 \rangle \neq \langle \hat{J}_Y^3 \rangle\). If necessary, the imaginary part can be determined by measurement of suitably rotated spin observables defined by

\[
\hat{J}_0 = \hat{J}_X \cos \theta - \hat{J}_Y \sin \theta.
\]

For \(N = 3\) manipulation gives (see Appendix E for details)

\[
\langle \hat{a} \hat{b} \hat{b}^\dagger \rangle = 2\langle \hat{J}_X^3 \rangle - \sqrt{2}(\langle \hat{J}_Y^3 \rangle + \langle \hat{J}_X^3 \rangle) - 2i\langle \hat{J}_X \hat{J}_Y \rangle + i\sqrt{2}(\langle \hat{J}_X \hat{J}_Y \rangle + \langle \hat{J}_X^3 \rangle), \tag{42}
\]

where \(\langle \hat{J}_Y^3 \rangle\) is measurable by standard interferometry or atom interferometry techniques.

We note that similar expansions can be made expressing the \(a\) and \(b\) operators in terms of quadrature phase amplitudes \(\hat{X}\) and \(\hat{P}\). For optical NOON states, this may be a useful way to accurately measure the moments \(\langle \hat{a}^m \hat{b}^m \rangle\) since quadrature phase amplitudes can be measured with high efficiency. Specifically, we define the amplitudes \(\hat{X}\) and \(\hat{P}\) by \(\hat{a} = \hat{X}_A + i\hat{P}_A\) and \(\hat{b} = \hat{X}_B + i\hat{P}_B\). Hence

\[
\langle \hat{a}^\dagger \hat{b} \rangle = \langle \hat{X}_A \hat{X}_B \rangle + \langle \hat{P}_A \hat{P}_B \rangle + i\langle \hat{X}_A \hat{P}_B - \hat{X}_B \hat{P}_A \rangle, \tag{43}
\]
which is readily measurable. Continuing
\begin{equation}
\langle \hat{a}^{12} \hat{b}^{20} \rangle = \langle (\hat{X}^A - \hat{P}^A)(\hat{X}^B - \hat{P}^B) \rangle + \langle (\hat{X}^A, \hat{P}^A)(\hat{X}^B, \hat{P}^B) \rangle
- i\langle [\hat{X}^A, \hat{P}^B](\hat{X}^B - \hat{P}^B) \rangle + i\langle [\hat{X}^A, \hat{P}^A](\hat{X}^B, \hat{P}^B) \rangle.
\end{equation}

The anticommutator is measurable by rotation of the quadratures. We define the measurable rotated quadrature phase amplitudes as \(\hat{X}_\theta = \hat{X} \cos(\theta) + \hat{P} \sin(\theta)\) and \(\hat{P}_\theta = -\hat{X} \sin(\theta) + \hat{P} \cos(\theta)\). Hence \(\hat{X}_{\pi/4} = \frac{1}{\sqrt{2}}(\hat{X} + \hat{P})\) and \(\hat{P}_{\pi/4} = \frac{1}{\sqrt{2}}(-\hat{X} + \hat{P})\) and we note that \(\langle \hat{X}_{\pi/4}^2 \rangle = \langle \hat{X}^2 + \hat{P}^2 + \hat{X} \hat{P} + \hat{P} \hat{X} \rangle/2\). Thus we can deduce \(\hat{X}, \hat{P}\) by measuring the moments \(\langle \hat{X}^2 \rangle, \langle \hat{P}^2 \rangle, \) and \(\langle \hat{X}_{\pi/4}^2 \rangle\).

C. Experimental certification of atomic quantum coherence \(n \sim 2\) by inferring the correlation \(\langle \hat{a}^{12} \hat{b}^{20} \rangle\) from spin squeezing

Esteve et al. experimentally realize the system modeled by the two-mode Hamiltonian \([40]\). The ground-state solutions have been solved and studied in Ref. [23]. Esteve et al. report data obtained on cooling their two-well system, including measurements for the spin moments \(J^2\) associated with ultracold atomic mode populations of two wells of the optical lattice \([40]\). Their observations analyze the variances of the Heisenberg uncertainty principle
\begin{equation}
\Delta J^x, \Delta J^y \geq \frac{1}{2} \frac{\langle J^z \rangle}{N/4}.
\end{equation}

They report spin squeezing in \(J^z\), with enhanced noise in \(J^x, J^y\). They also report \(\langle J^z \rangle \sim 0\) and \(\langle J^x, J^y \rangle \sim 0\). Hence we can conclude
\begin{equation}
\langle J^2 \rangle \geq N/4 \geq \langle J^2 \rangle.
\end{equation}

Thus we deduce
\begin{equation}
\langle J_{\pi/4}^2 \rangle \neq 0,
\end{equation}

which implies \(\langle J_{\pi/4}^x, J_{\pi/4}^y \rangle \neq 0\) where \(J_{\pi/4}^x, J_{\pi/4}^y\) are Schwinger operators defined for the rotated modes \(\hat{e} = (\hat{a} + \hat{b})/\sqrt{2}\) and \(\hat{d} = e^{-i\phi/4} (\hat{a} - \hat{b})\). Hence we conclude
\begin{equation}
\langle [\hat{e}^{12}, \hat{d}^{20}] \rangle \neq 0,
\end{equation}

which (using the Results of Sec. III) gives evidence in their BEC system of a two-atom coherence, i.e., a generalized \(n\)-scop ic superposition with \(n = 2\) of type
\begin{equation}
|\psi_2\rangle = c_{20}|2\rangle|\psi_0\rangle + c_{11}|1\rangle|1\rangle + c_{02}|0\rangle|2\rangle + c_{00}|0\rangle|0\rangle + \psi_0, \quad (49)
\end{equation}

where the coefficients satisfy \(c_{00} \neq 0\) and \(c_{20} \neq 0\) and \(\psi_0\) is orthogonal to each of \(|2\rangle|\psi_0\rangle, |1\rangle|1\rangle, \) and \(|0\rangle|2\rangle\). We note that this is consistent with the predictions of [23]. This paper predicts nonzero moments \(\langle \hat{e}^{12}, \hat{d}^{20} \rangle \neq 0\) for the populations of modes \(c.d\) in atomic systems with \(\kappa < 0\). The observation of \(\langle \hat{a}^{12} \hat{b}^{20} \rangle \neq 0\) would be evidence of a superposition of states constrained to the modes of the wells. Such a state is written:
\begin{equation}
|\psi_2\rangle = c_{20}|2\rangle|\psi_0\rangle + c_{11}|1\rangle|1\rangle + c_{02}|0\rangle|2\rangle + c_{00}|0\rangle|0\rangle + \psi_0, \quad (50)
\end{equation}

where \(c_{02} \neq 0\) and \(c_{20} \neq 0\). This is predicted for atomic BEC with \(\kappa > 0\) \([23]\). Three-atom superpositions (for which \(\langle \hat{a}^{12} \hat{b}^{20} \rangle \neq 0\) and higher) are also predicted (up to \(N\)) and should be evident via higher-order fringe patterns, or else directly via the \(J_\theta\) measurements as above.

D. Entanglement

The observation of the \(n\)-th-order quantum coherence \(\langle \hat{a}^{12} \hat{b}^{20} \rangle \neq 0\) is not in itself sufficient to imply entanglement. For instance, \(|\psi_0\rangle\) in the expression \((49)\) might include contributions from terms such as \(|2\rangle|2\rangle\) and \(|0\rangle|0\rangle\). This means that a separable form for \(\langle |\psi\rangle\rangle\), e.g.,
\begin{equation}
|\psi\rangle = \frac{1}{2}(|2\rangle|0\rangle|0\rangle|2\rangle + |0\rangle|0\rangle|0\rangle|2\rangle + |0\rangle|0\rangle|2\rangle|2\rangle ),
\end{equation}

may be possible. The separable state contrasts with the “dead here—alive there” entangled superposition state whose ideal form is precisely the NOON state, e.g., for \(N = 2\),
\begin{equation}
|\psi_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle|0\rangle|0\rangle|2\rangle + |0\rangle|0\rangle|0\rangle|2\rangle ) .
\end{equation}

In this paper, we are only concerned with how to certify an \(n\)-scop ic quantum superposition, without regard to entanglement. However, the entangled case is of special interest, especially where the two modes are spatially separated. For the ideal NOON case, we therefore point out that one can make simple measurements to confirm the entanglement. If one measures the individual mode numbers \(n_a\) and \(n_b\), the results zero or \(N\) are obtained for each mode. The observations would be correlated, so that there is only a nonzero probability to obtain \(|N\rangle|0\rangle\) or \(|0\rangle|N\rangle\). This eliminates the possibility of nonzero contributions from terms in \(\psi_0\) and it remains only to confirm the nonzero quantum coherence in order to confirm the entanglement. The observation of \(\langle \hat{a}^{12} \hat{b}^{20} \rangle \neq 0\) then becomes sufficient to certify the entanglement of the NOON state. While simple in principle, this procedure is not so useful in practice. For example, the attenuated NOON state of Sec. V would predict a nonzero probability for obtaining \(|0\rangle|0\rangle\) and a more careful analysis is necessary to deduce entanglement.

VIII. CONCLUSION

We have examined how to rigorously confirm and quantify the mesoscopic quantum coherence of nonideal NOON states. In this paper, we link the observation of quantum coherence to the negation of certain types of mixtures, given as \((3)\) and \((4)\). However, it is stressed we are restricting to mixtures where the dead and alive states are quantum states that can therefore be represented by density operators \([\rho_A\rho_B\) in Eq. \((3)\). This contrasts with other possible signatures of a cat state where the dead and alive states might also be hidden variable states, as in Ref. [27].

In this paper, we have focused on two criteria for the \(n\)-th-order quantum coherence, defined as a quantum coherence between number states different by \(n\) quanta. The first criterion is a nonzero \(n\)-th-order moment \(\langle \hat{a}^{12} \hat{b}^{20} \rangle \neq 0\) and the second is a quantifiable amount of Schwinger spin squeezing. We have shown how the first criterion can be a quantifier of the overall \(n\)-th-order coherence. The second criterion can be a robust and effective signature for large \(n\), and can verify high orders of coherence in existing atomic experiments, but does not signify all cases of \(n\)-scopic quantum coherence. In Secs. V and VI, we have illustrated the use of the criteria with the examples of attenuated NOON states, number states \(|N\rangle\) that pass through beam spitters, and approximate NOON states formed from...
We note that any pure state cannot be a superposition of two states distinct by general two-mode quantum state for this two-mode system that and the superposition state \((12)\) in detail, consider the most two-mode number (Fock) state basis:

\[
|\psi_{\text{em}}\rangle = \sum_{n,m} c_{nm} |n_1\rangle |m_2\rangle
\]

\[
= c_{00} |0\rangle |0\rangle + c_{01} |0\rangle |1\rangle + c_{10} |1\rangle |0\rangle
\]

\[
+ c_{11} |1\rangle |1\rangle + c_{12} |1\rangle |2\rangle + c_{21} |2\rangle |1\rangle
\]

\[
+ c_{02} |0\rangle |2\rangle + c_{20} |2\rangle |0\rangle + \cdots .
\]

We see that if \((\hat{a}^{\dagger n} \hat{b}^m) \neq 0\), then the state is necessarily of the form \((12)\), which involves a superposition of two states distinct by \(n\) quanta. We note that when \((\hat{a}^{\dagger n} \hat{b}^m) \neq 0\), the density operator \(\rho\) for the system cannot be written in an alternative form except to provide a nonzero coherence \((10)\) between states \(|n\rangle |m\rangle + |n+1\rangle |m\rangle\). We conclude that the diagonal elements \(\rho_{nm} = \rho_{m+n} = \rho_{m+n+1} = \rho_{m+n+2} = \cdots\) and \(\rho_{m+n} = \rho_{m+n+1} = \rho_{m+n+2} = \cdots\) are also nonzero. Thus there is a nonzero probability \(P_D\) that the system is found in state \(|n\rangle |m\rangle + |n+1\rangle |m\rangle\) (that we call dead) and also a nonzero probability \(P_A\) that the system is found in state \(|n+1\rangle |m\rangle\) (that we call alive). Yet, the superposition state \((12)\) cannot be given as a classical mixture \((9)\) which has a zero coherence between the states \(|n\rangle |m\rangle + |n+1\rangle |m\rangle\) and \(|n+1\rangle |m\rangle\) whose \(2J\) values are different by \(n\).

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**APPENDIX A: RESULT 2**

To explain the connection between the condition \((\hat{a}^{\dagger n} \hat{b}^m) \neq 0\) and the superposition state \((12)\) in detail, consider the most general two-mode quantum state for this two-mode system that cannot be a superposition of two states distinct by \(n\) quanta. We note that any pure state \(|\psi_{\text{em}}\rangle\) can be expanded in the two-mode number (Fock) state basis:

\[
|\psi_{\text{em}}\rangle = \sum_{n,m} c_{nm} |n_1\rangle |m_2\rangle
\]

\[
= c_{00} |0\rangle |0\rangle + c_{01} |0\rangle |1\rangle + c_{10} |1\rangle |0\rangle
\]

\[
+ c_{11} |1\rangle |1\rangle + c_{12} |1\rangle |2\rangle + c_{21} |2\rangle |1\rangle
\]

\[
+ c_{02} |0\rangle |2\rangle + c_{20} |2\rangle |0\rangle + \cdots .
\]

We see that if \((\hat{a}^{\dagger n} \hat{b}^m) \neq 0\), then the state is necessarily of the form \((12)\), which involves a superposition of two states distinct by \(n\) quanta. We note that when \((\hat{a}^{\dagger n} \hat{b}^m) \neq 0\), the density operator \(\rho\) for the system cannot be written in an alternative form except to provide a nonzero coherence \((10)\) between states \(|n\rangle |m\rangle + |n+1\rangle |m\rangle\) and \(|n+1\rangle |m\rangle + |n\rangle |m+1\rangle\). We conclude that the diagonal elements \(\rho_{nm} = \rho_{m+n} = \rho_{m+n+1} = \rho_{m+n+2} = \cdots\) and \(\rho_{m+n} = \rho_{m+n+1} = \rho_{m+n+2} = \cdots\) are also nonzero. Thus there is a nonzero probability \(P_D\) that the system is found in state \(|n\rangle |m\rangle + |n+1\rangle |m\rangle\) (that we call dead) and also a nonzero probability \(P_A\) that the system is found in state \(|n+1\rangle |m\rangle\) (that we call alive). Yet, the superposition state \((12)\) cannot be given as a classical mixture \((9)\) which has a zero coherence between the states \(|n\rangle |m\rangle + |n+1\rangle |m\rangle\) and \(|n+1\rangle |m\rangle\) whose \(2J\) values are different by \(n\).

**APPENDIX B: PROOF OF RESULT 3 FOR SPIN-SQUEEZE TEST**

We follow from the main text and generalize to consider a two-mode description of the state as given for a mixed state by a density operator \(\rho\). We expand in terms of pure states \(|\psi_R\rangle\) so that \(\rho = \sum_R P_R |\psi_R\rangle \langle \psi_R|\) for some probability \(P_R\). Each pure state \(|\psi_R\rangle\) can be expressed as a superposition of number eigenstates given by \((14)\). We know that the variance \((\Delta J_y)^2\) of any mixture satisfies \((\Delta J_y)^2 \geq \sum R P_R (\Delta J_y^R)^2\). Thus

\[
(\Delta J_y)^2 \geq \sum_R P_R (\Delta J_y^R)^2 = \sum_R P_R |\langle J_y | R \rangle| \langle J_y | R \rangle = \frac{1}{\delta_0^2} |\langle J_y | R \rangle|^2.
\]

For all the possible mixtures denoted by a choice of set \(|\psi_R\rangle\) (where \(P_R \neq 0\)) we can determine the spread \(\delta_R\) for each state \(|\psi_R\rangle\) and then select the maximum of the set \(\delta_R\) and call it \(\delta_0\). We select the mixture set consistent with the density operator that has the minimum possible value of \(\delta_0\); that is, we determine that the density operator cannot be expanded in a set \(|\psi_R\rangle\) with a smaller \(\delta_0\). Then for the pure states of this set \(|\psi_R\rangle\), the maximum variance in \(J_y^R\) is \((\Delta J_y^R)^2 \leq \delta_0^2/4\). Then we see that the uncertainty relation \((16)\) implies a minimum value for the variance in \(J_y^R\):

\[
(\Delta J_y^R)^2 \geq \frac{1}{\delta_0^2} |\langle J_y | R \rangle|^2.
\]

Simplification gives

\[
(\Delta J_y)^2 \geq \frac{1}{\delta_0^2} |\langle J_y | R \rangle|^2.
\]

Taking the case of the spin-squeezing experiments where measurements give \(\langle J_y \rangle \sim |\langle N | 2 \rangle|/2\), we see that

\[
(\Delta J_y)^2 \geq \frac{1}{\delta_0^2} |\langle J_y | R \rangle|^2 \geq \frac{1}{\delta_0^2} |\langle J_y | R \rangle|^2 \geq \frac{\langle N | 2 \rangle^2}{4\delta_0^2}.
\]

Thus there is a lower bound on the best amount of squeezing determined by the maximum spread (extent) \(\delta_0\) of the superposition. We can now prove Result 3 as follows. The measured amount of squeezing places a lower bound on the extent \(\delta_0\) of the broadest superposition: thus if the measured squeezing is \(\xi_N\), then from \((B1)\) the underlying state has a minimum breadth \(\delta_0\) of superposition (in the eigenstates of \(J_y^R\)) given by \(\delta_0 > \frac{\langle N | 2 \rangle}{\sqrt{\xi_N}}\). The width \(\delta_0\) of the superposition gives the extent or size of the coherence, i.e., the value of \(n\) in expression \((19)\).

**APPENDIX C: CATNESS-FIDELITY QUANTIFIER FOR MIXED STATES**

Discussion in terms of superposition states. We give the proof of Result \((4)\) in terms of the superposition states. The experiment may confirm a range of values \(J_z\) for \(J_z\) for which \((\hat{a}^{\dagger n} \hat{b}^{+ n}) \neq 0\). Take one such value: \(2J_z = N_0\). Then we know there is a nonzero probability \(P_{N_0}\) that the system be

\[
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\]
in a superposition of form
\[ |\psi_N\rangle = a^{(n,m)}_{N_0} |m + N_0\rangle + b^{(n,m)}_{N_0} |n + N_0\rangle + c|\psi\rangle, \]
(C1)
where \( a^{(n,m)}_{N_0} \) and \( b^{(n,m)}_{N_0} \) are nonzero. Based on the measured moments, we can write the density operator in the general form
\[ \rho = \sum_{n,m} p^{(n,m)}_{\text{mix}} |n,m\rangle \langle n,m| + \rho_{\text{mix}} + \rho_n, \]
(C2)
where \( \rho_{\text{mix}} = |\psi_N\rangle \langle \psi_N| \) is a mixture of states \( |n\rangle \) \( |m + N_0\rangle \) and \( |n + N_0\rangle \). We note that the first term will contribute a nonzero value of \((|\psi_N\rangle \langle \psi_N|)\). The first term can also include superpositions of the different \(|\psi_N\rangle\) with different \(n,m\) but evaluation of the moment \(|\psi_N\rangle \langle \psi_N|\) will be the same if the system were in a mixture of those states (due to the orthogonality). The relevant \(Re\) values of \(n,m\) such that probabilities are nonzero can be determined from the measurements of mode number and we assume the sums only include those nonzero contributions. We note that the first term is written as a mixture of the NOON-type states. In some cases, such a mixture can be equivalent to (and therefore rewritten as) a classical mixture \(\rho_{\text{mix}}\), but the nonzero moment \(\langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle\) cannot arise in this case. The value of \(\langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle\) is zero for any \(\rho_{\text{mix}}\), and the prediction for \(\langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle\) given by \(\rho\) is
\[ \langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle = \sum_{n,m} p^{(n,m)}_{\text{mix}} P_{\text{mix}}(n,m) \]
\[ \times \sum_{n,m} \frac{|(m + N_0)!|^2}{m!} \frac{|(n + N_0)!|^2}{n!} \]
\[ \leq S \sum_{n,m} |a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} P_{\text{mix}}(n,m)|. \]
(C3)
We have used the prediction for \(\langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle\) for the state \(|\psi_N\rangle\) and the definitions of \(S\) as in the main text. The measurement of the moment \(\langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle\) thus allows the determination of a lower bound on an effective fidelity for the Schrödinger cat NOON state.

**Correction term.** Now we consider that the experimentalist can only confirm that the total probability of the “nonrelevant” \(\text{NRe}\) outcomes is less than or equal to \(\epsilon\). The contribution of the nonrelevant terms to the \(C_{N_0}\) (the sum of the \(N\)-th-order coherences) is bounded by the probabilities. For any density matrix, the off-diagonal elements are bounded by the diagonal elements that give the probabilities: Always \(a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \leq \frac{1}{2}\) and assuming \(\sum_{\text{NRe}} p^{(n,m)}_{\text{mix}} \leq \epsilon\), we find
\[ \sum_{\text{NRe}} |a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} P_{\text{mix}}(n,m)| \leq \epsilon/2. \]
Using that \(\frac{|(n + N_0)!!|^2}{n!} \leq (n + N_0)\), this implies
\[ \langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle \leq \sum_{\text{NRe}} |a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} P_{\text{mix}}(n,m)| \]
\[ \times \frac{|(m + N_0)!|^2}{m!} \frac{|(n + N_0)!|^2}{n!} \]
\[ \leq S \sum_{n,m} |a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} P_{\text{mix}}(n,m)|. \]
(C4)
Thus we know that
\[ C_{N_0} \geq \sum_{\text{Re}} |a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} P_{\text{mix}}(n,m)| \]
\[ \geq \left[ \langle a^{(n,m)}_{N_0} b^{(n,m)}_{N_0} \rangle - \epsilon \right] / S. \]
(C5)
where \(N_{ap}\) is the upper bound for the mode numbers, given that the system cannot have infinite mode or particle (atom) number. For the cases of interest to us on this paper, the total mode number is the atom number \(N\), which is fixed.

**APPENDIX D: EVALUATION OF NORMALIZATION**

We consider the state
\[ |\text{out}\rangle = \sum_{m=0}^{N} d_m |m\rangle \langle N - m|_b. \]
(D1)
We quantify the \(n\)-th order quantum coherence by the parameter \(C_n\) (that we have also called the catness fidelity)
\[ C_n = N_{n,N} \sum_{m=0}^{N-n} \sum_{d=0}^{N-n} |d_m d_{m+n}|. \]
(D2)
where \(N_{n,N}\) is a normalization constant to ensure the maximum value of \(C_n\) is 1. The normalization \(N_{n,N}\) is determined by the bounds on the coherences of the density matrix for a pure state. For example, where \(n = N\), the maximum \(|d_0 d_N^*|\) is obtained for \(d_0 = d_N = \frac{1}{\sqrt{2}}\) with all other amplitudes zero. Hence \(|d_0 d_N^*| \leq 1/2\) and \(N_{N,N} = 2\). Similarly, for \(n = N - 1\) and \(N > 3\) (so that the \(d\) terms in \(d_0 d_{0,n-1}^* + d_1 d_{1,n-1}^*\) are all different), we find \(|d_0 d_{0,n-1}^* + d_1 d_{1,n-1}^*| \leq 1/2\) where in this case the maximum \(N_{n,N}\) of \(|d_m d_{m+n}^*|\) is found taking \(d_0 = d_{n-1} = d_1 = d_N = \frac{1}{\sqrt{2}}\). The maximum value for more general \(n\) and \(N\) can be found numerically.

(1) We start by analyzing \(n = N\). Then \(C_n = N_{n,N} |d_0 d_N^*|\). There is only one term in the sum and therefore only two amplitudes contributing to the sum. The number of terms is independent of \(N\). We can show that the maximum value of the sum of the coherences (namely \(\sum_{m=0}^{N-n} |d_m d_{m+n}^*|\)) is given when \(d_0 = d_N = \frac{1}{\sqrt{2}}\), and all other amplitudes zeros. Hence \(C_n \leq N_{n,N} |d_0 d_N^*| = N_{n,N} \frac{1}{2}\) and the optimal normalization is \(N_{n,N} = 2\).

(2) Next we consider \(n > N/2\). Here \(N_{n,N} |d_0 d_N^*| = d_0 d_1 + d_1 d_2 + \cdots + d_{N-n} d_N\) and since \(n > N - n\) the terms in the summation involve different \(d\)’s, which can therefore be chosen independently apart from normalization requirements. Taking the \(2(N - n + 1)\) contributing amplitudes as equal, and all other as zero, \(N_{n,N} |d_0 d_N^*| = 2(N - n + 1) = \frac{1}{2}\), which we verify is the maximum value.
(3) For the remaining values, we determine the bounds numerically. We analyze all these cases and fit an expression for the maximum value of $\sum_{m=0}^{\frac{N}{n}} |d_{m}^n a_{n}|$. On numerically analyzing the cases $n < N/2$, we find that to a good approximation $\sum_{m=0}^{N-n} |d_{m}^n a_{n}| \leq \cos \left( \frac{\pi}{2N/n} \right)$, where $N/n$ denotes the integer part of $N/n$ and hence $N_{n,N} = 1/\cos \left( \frac{\pi}{2N/n} \right)$. We numerically verified this bound for all $N$ up to 500.

APPENDIX E: EVALUATION OF $\langle (\hat{a}^\dagger \hat{b})^3 \rangle$

For $N = 3$, we would like to measure the expectation value of the following observable:

$$\langle (\hat{a}^\dagger \hat{b})^3 \rangle = (J_x + i J_y)^3$$

$$= J_x^3 - i J_y^3 + i (J_x J_y J_x + J_x J_y^2 + J_y J_x^2)$$

$$- (J_x^2 J_y + J_x J_y^3 + J_y J_x^3).$$

In the expansion, we have dropped the “hats” and used lower case $x$ and $y$ in the subscripts of the $\hat{J}_x$ and $\hat{J}_y$ defined in (13) to simplify notation. The first and second terms can be measured in experiments. However, we need to express $J_x J_y J_x$, $J_x^2 J_y$, and $J_x J_y^3$ in terms of some other measurements that can be carried out in experiments. To this end, we define rotated Schwinger operators as follows:

$$J_0 = J_x \cos \theta + J_y \sin \theta,$$

$$J_{\theta + \frac{\pi}{2}} = G_\theta$$

$$= J_x \cos \left( \theta + \frac{\pi}{2} \right) + J_y \sin \left( \theta + \frac{\pi}{2} \right)$$

$$= -J_x \sin \theta + J_y \cos \theta.$$

For $\theta = \frac{\pi}{4}$, these rotated operators correspond to

$$J_0^3 = \frac{1}{\sqrt{2}} \left[ \left( J_x^3 J_y + J_x J_y^2 + J_y J_x^2 \right) + \left( J_x J_y J_x + J_x J_y^2 + J_y J_x^2 \right) \right]$$

$$+ \left( J_x J_y J_x + J_y J_x^2 + J_y J_x^3 \right) - J_x^3 + J_y^3.$$
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