UNIVERSAL RECURSIVE FORMULAE FOR $Q$-CURVATURES

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ABSTRACT. We formulate and discuss two conjectures concerning recursive formulae for Branson’s $Q$-curvatures. The proposed formulae describe all $Q$-curvatures on manifolds of all even dimensions in terms of respective lower order $Q$-curvatures and lower order GJMS-operators. They are universal in the dimension of the underlying space. The recursive formulae are generated by an algorithm which rests on the theory of residue families of [27]. We attempt to resolve the algorithm by formulating a conjectural description of the coefficients in the recursive formulae in terms of interpolation polynomials associated to compositions of natural numbers. We prove that the conjectures cover $Q_4$ and $Q_6$ for general metrics, and $Q_8$ for conformally flat metrics. The result for $Q_8$ is proved here for the first time. Moreover, we display explicit (conjectural) formulae for $Q$-curvatures of order up to 16, and test high order cases for round spheres and Einstein metrics.

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1. Introduction

For any Riemannian manifold \((M, h)\) of even dimension \(n\), there is a finite sequence \(P_{2N}(h)\) \((1 \leq N \leq \frac{n}{2})\) of natural differential operators on functions on \(M\) with leading part \(\Delta^N_h\) which transform as
\[
e^{(\frac{1}{2}+N)\varphi} \circ P_{2N}(h) \circ e^{-(\frac{1}{2}-N)\varphi} = P_{2N}(h)
\]
under conformal changes \(h = e^{2\varphi}h\) of the metric. These operators were derived in [22] from the powers of the Laplacian of the Fefferman-Graham ambient metric (see [14] and [13]). For \(2N > n\), the construction in [22] is obstructed by the Fefferman-Graham tensor. More sharply, in that range it is impossible to construct a conformally covariant operator (for all metrics) by adding lower order terms to \(\Delta^N\) ([19], [16]). On the other hand, if such operators exist, they are not uniquely determined by conformal covariance. In the following, \(P_{2N}\) will denote the operators constructed in [22], and they will be referred to as the GJMS-operators.

\(P_2\) and \(P_4\) are the well-known Yamabe and Paneitz operator which are given by
\[
P_2 = \Delta - \left(\frac{n}{2} - 1\right)J,
\]
\[
P_4 = \Delta^2 + \delta((n-2)J-4P)\#d + \left(\frac{n}{2} - 2\right)\left(\frac{n}{2}J^2 - 2|P|^2 - \Delta J\right),
\]
respectively. Here
\[
P = \frac{1}{n-2} \left(\text{Ric} - \frac{\tau}{2(n-1)}h\right)
\]
denotes the Schouten tensor of \(h\), \(\tau\) denotes the scalar curvature, and \(J = \frac{\tau}{2(n-1)}\) is the trace of \(P\). \# denotes the natural action of symmetric bilinear forms on 1-forms. Explicit expressions for the higher order operators \(P_{2N}\) for \(N \geq 3\) are considerably more complicated.

The GJMS-operators \(P_{2N}\) give rise to a finite sequence \(Q_{2N}\) \((1 \leq N \leq \frac{n}{2})\) of Riemannian curvature invariants according to
\[
P_{2N}(h)(1) = (-1)^N \left(\frac{n}{2} - N\right)Q_{2N}(h)
\]
(see [5]). \(Q_{2N}\) is a curvature invariant of order \(2N\), i.e., it involves \(2N\) derivatives of the metric. In the following, the quantities \(Q_{2N}(h)\) will be called the \(Q\)-curvatures of \(h\).

In particular, we find
\[
Q_2 = J \quad \text{and} \quad Q_4 = \frac{n}{2}J^2 - 2|P|^2 - \Delta J.
\]

Explicit formulae for \(Q_{2N}\) for \(N \geq 3\) are considerably more complicated.

The critical GJMS-operator \(P_n\) and the critical \(Q\)-curvature \(Q_n\) play a special role. In that case, (1.1) does not define \(Q_n\), however. Instead, \(Q_n\) arises by
continuation in dimension from the subcritical $Q$-curvatures $Q_{2N}$ ($2N < n$). The pair $(P_n, Q_n)$ satisfies the fundamental identity
\begin{equation}
    e^{n\phi} Q_n(\hat{h}) = Q_n(h) + (-1)^{\frac{n}{2}} P_n(h)(\phi).
\end{equation}

It shows that the transformation of $Q_n$ under conformal changes of $h$ is governed by the linear differential operator $P_n$. This is one of the remarkable properties of Branson’s $Q$-curvature $Q_n$. \eqref{1.3} implies that, for closed $M$, the total $Q$-curvature
\begin{equation}
    \int_M Q_n \text{vol}
\end{equation}
is a global conformal invariant.

Despite the simple formulae \eqref{1.2}, it remains notoriously difficult to find good expressions for $Q$-curvatures of higher order. Explicit formulae for $Q_6$ and $Q_8$ in arbitrary dimension were given in \cite{17}. For conformally flat metrics and general dimensions, $Q_6$ already appeared in \cite{5}.

It is natural to expect that the complexity of the quantities $Q_{2N}$ increases exponentially with the order. This is one of the aspects in which its behaviour resembles that of the heat coefficients of self-adjoint elliptic differential operators. The relations between both quantities are much more substantial, though. The problem to understand the structure of heat coefficients of conformally covariant operators was actually one of the origins of the notion of $Q$-curvature \cite{4}. Explicit formulae for heat coefficients are known only for sufficiently small orders. There is an extensive literature devoted to such formulae (see \cite{30} for a recent review).

The lack of information concerning the structure of high order $Q$-curvatures presently seems to obstruct the understanding of its nature and its proper role in geometric analysis (see \cite{28} for a review in dimension 4).

In the present work we propose a uniform description of all $Q$-curvatures with the following main features.

1. Any $Q$-curvature is the sum of two parts of different nature.
2. The main part is a linear combination of respective lower order GJMS-operators acting on lower order $Q$-curvatures with coefficients which do not depend on the dimension of the underlying space.
3. The second part is defined in terms of the constant term of a power of the Yamabe-operator of an associated Poincaré-Einstein metric.

These properties motivate to refer to the proposed formulae as universal and recursive.

In more detail, Conjecture 3.1 asserts that on manifolds of even dimension $n$,\begin{equation}
    Q_{2N} = \sum_I a_I^{(N)} P_{2I}(Q_{2N-2|I|}) + (-1)^{N-1} \frac{(2N-2)!}{(2N-3)!} \bar{P}_{2N-1}(\bar{Q}_{2})
\end{equation}
for all non-negative integers $N$ so that $2N \leq n$. The rational coefficients $a_I^{(N)}$ are generated by an algorithm which will be defined in Section 3. The sum in \eqref{1.5} runs over all compositions $I$ of integers in $[1, N - 1]$ as sums of natural numbers. Moreover, we use the following notation. For a composition $I = (I_1, \ldots, I_m)$ of
size $|I| = \sum_i I_i$, we set

$$P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_m}.$$ 

In (1.3) for the metric $h$, the operator $\bar{P}_2$ denotes the Yamabe operator of the conformal compactification $dr^2 + h_r$ of the Poincaré-Einstein metric of $h$ (the relevant constructions are reviewed in Section 2). Similarly, $\bar{Q}_2$ is $Q_2$ for the metric $dr^2 + h_r$, and $i^*$ restricts functions to $r = 0$.

Alternatively, the quantity $i^* \bar{P}_2^{N-1}(Q_2)$ can be written in the form

$$-\frac{n-1}{2} i^* \bar{P}_2^N(1).$$

However, we prefer to use the form (1.5) which hides the dimension $n$ of the underlying space.

The existence of recursive formulae for general $Q_{2N}$ has been an open problem since the invention of $Q$-curvature. (1.5) proposes some answer.

One might also ask for recursive formulae for $Q_{2N}$ which rest only on lower order GJMS-operators and lower order $Q$-curvatures of the given metric. In view of the contribution $i^* \bar{P}_2^{N-1}(\bar{Q}_2)$, the formula (1.5) is not of this form. However, already for $N = 2$ such formulae are unlikely to exist since $Q_4$ depends on the full Ricci tensor whereas $P_2$ and $Q_2$ only depend on scalar curvature.

The presentations (1.5) imply that the structure of the constant term of any GJMS-operator is influenced by all lower order GJMS-operators. This illustrates the enormous complexity of the GJMS operators. The recursive structure for $Q$-curvature seems to be a phenomenon which is not known to have analogs for related quantities as, for instance, the heat coefficients (see (1.17)).

Next, we make explicit (1.5) for $Q_4$, $Q_6$ and $Q_8$. In these cases, the asserted formulae are theorems and we briefly indicate their proofs. We start with a version for $Q_2$. It just says that

(1.6) $Q_2 = i^* \bar{Q}_2$

in all dimensions (see (3.3)). Next, the universal recursive formula for $Q_4$ states that

(1.7) $Q_4 = P_2(Q_2) - 2i^* \bar{P}_2(\bar{Q}_2).$

This formula is valid in all dimensions $n \geq 4$, i.e., (1.7) is universal. In fact, it reads

$$Q_4 = \left(\Delta - \frac{n-2}{2} J\right) (J) - 2i^* \left( (\partial / \partial r)^2 + \Delta_{h_r} - \frac{n-1}{2} \bar{Q}_2 \right) (\bar{Q}_2)$$

(see Section 2 for the notation). Using $i^* \bar{Q}_2 = Q_2 = J$ (see (1.6)) and

$$i^* (\partial / \partial r)^2 (\bar{Q}_2) = |P|^2,$$

the sum simplifies to

$$\frac{n}{2} J^2 - 2|P|^2 - \Delta J.$$

This shows the equivalence of (1.7) and the traditional formula (1.2) for $Q_4$. The presentation (1.7) is distinguished by the fact that it is uniform in all dimensions. A disadvantage of (1.7) is that the fundamental transformation law (1.3) in the
critical dimension $n = 4$ is less obvious from this formula. In this aspects, (1.7) resembles the holographic formula (1.20).

Next, we have the recursive formula

$$Q_6 = \frac{2}{3} P_2(Q_4) + \left[ -\frac{5}{3} P_2^2 + \frac{2}{3} P_4 \right] (Q_2) + \frac{8}{3} i^* \tilde{P}_2^2 (\bar{Q}_2)$$

for $Q_6$ in all dimensions $n \geq 6$. A detailed proof of (1.8) can be found in [27]. It is a special case of the algorithm of Section 3.

For $n = 6$, the holographic formula (1.19) of [23] presents $Q_6$ in the form

$$Q_6 = 16 \text{tr}(P^3) - 24 J |P|^2 + 8 J^3 + 8 (B, P) + \text{divergence terms},$$

where $B$ denotes a version of the Bach tensor. The recursive formula (1.8) covers the contribution $(B, P)$ in (1.9) by the term

$$\frac{8}{3} \left( \frac{\partial}{\partial r} \right)^4 |\bar{Q}_2|.$$ 

This illustrates the role of the term which involves $P_2$ and $\bar{Q}_2$. An extension of this observation to the general case will be discussed in Section 3.

We also note that (1.8) is equivalent to a formula of Gover and Peterson [17]. For a proof of this fact we refer to [27].

We continue with the description of the recursive formula for $Q_8$. In the critical dimension $n = 8$, the algorithm of Section 3 yields

$$Q_8 = \frac{3}{5} P_2(Q_6) + \left[ -4P_2^2 + \frac{17}{5} P_4 \right] (Q_4)$$

$$+ \left[ -\frac{22}{5} P_3^2 + \frac{8}{5} P_2 P_4 + \frac{28}{5} P_4 P_2 - \frac{9}{5} P_6 \right] (Q_2) - \frac{16}{5} i^* \tilde{P}_2^3 (\bar{Q}_2)$$

for locally conformally flat metrics (Proposition 3.1). Using a second algorithm, we prove that (1.10) holds true in all dimensions $n \geq 8$ (Proposition 3.2). It remains open, whether (1.10) extends to general metrics. The relation between (1.10) and the Gover-Peterson formula [17] for $Q_8$ is not yet understood.

For $N \geq 5$, Conjecture 3.1 enters largely unexplored territory. We outline the algorithm which generates the presentations (1.5). First, we generate such a presentation for the critical $Q$-curvature $Q_n$. For this, we apply an algorithm which rests on the relation of the critical $Q$-curvature $Q_n$ to the quantity

$$\hat{D}_n^{res}(0)(1)$$

and the recursive structure of all residue families $D_{2N}^{res}(\lambda)$ for $2N \leq n$. We refer to Section 2 for the definition of the relevant concepts. The details of the algorithm are explained in Section 3. An important argument which enters into the algorithm is the principle of universality. It plays the following role. The algorithm for $Q_n$ uses the assumption that the analogously generated presentations of all lower order $Q$-curvatures $Q_{2N}$, $N = 1, \ldots, \frac{n}{2} - 1$ hold true on manifolds of dimension $n$. In particular, the derivation of (1.5) in dimension $n = 6$, uses the facts that (1.6) and (1.7) hold true in dimension $n = 6$. Similarly, the derivation of (1.10) in dimension $n = 8$ applies the facts that the formulae (1.6), (1.7) and
(1.8) hold true in \( n = 8 \). Under the assumption of universality, the algorithm generates a formula for \( Q_n \). Since universality is open, the identification of the resulting formula with \( Q_n \) is only conjectural. Conjecture 3.1 asserts that the resulting formula for \( Q_n \) again is universal, i.e., holds true in all dimensions \( > n \).

In order to apply the factorization identities of residue families we restrict to conformally flat metrics. In low order cases, this restriction can be removed. It hopefully is superfluous in general.

With these motivations, it becomes important to describe the structure of the right-hand sides of (1.5) generated by the above algorithm. Although the algorithm only involves linear algebra, the complexity of calculations quickly increases with \( N \). In particular, we were unable to find closed formulae for the coefficients \( a^{(N)}_I \).

Instead, we describe an attempt to resolve the algorithm by relating it to another much simpler algorithm which deals with polynomials instead of operators. More precisely, we introduce an algorithm for the generation of a system of polynomials. It associates a canonical polynomial \( r_I \) to any composition \( I \). The degree of the polynomial \( r_I \) is \( 2|I| - 1 \). Conjecture 4.1 relates, for any \( I \), the restriction of \( r_I \) to \( \mathbb{N} \) to the function \( N \mapsto a^{(N)}_I \). The formulation of this conjecture results from an analysis of computer assisted calculations of the coefficients \( a^{(N)}_I \). In particular, such calculations indicate that the functions \( N \mapsto a^{(N)}_I \) can be described by interpolation polynomials. A deeper analysis of the numerical data leads to a description of these polynomials in terms of other interpolation problems.

We describe the content of Conjecture 4.1 for the coefficients of

\[
P_{2k}(Q_2 N - 2k), \quad N \geq k + 1
\]

and

\[
P_{2j}P_{2k}(Q_2 N - 2j - 2k), \quad N \geq j + k + 1.
\]

For \( k \geq 1 \), let \( r_{(k)} \) be the unique polynomial of degree \( 2k - 1 \) which is characterized by its \( 2k \) values

\[
r_{(k)}(-i) = 0, \quad i = 1, \ldots, k - 1,
\]

and

\[
r_{(k)} \left( \frac{1}{2} - i \right) = (-2)^{-(k-1)} \frac{\binom{1}{2}^{k-1}}{(k-1)!}, \quad i = 0, 1, \ldots, k.
\]

The second set of conditions can be replaced by the simpler requirement that \( r_{(k)} \) is constant on the set

\[
S(k) = \left\{ \frac{1}{2} - k, \ldots, -\frac{1}{2} \right\},
\]

together with the condition that

\[
r_{(k)}(0) = (-1)^{k-1}\frac{(2k-3)!!}{k!}.
\]
Now Conjecture 4.1 says that

\[ a^{(N)}_{(k)} = \prod_{i=1}^{k} \left( \frac{N-i}{2N-2i-1} \right) r_{(k)}(N-k), \quad N \geq k + 1. \]

For a composition \( I = (j, k) \) with two entries, we define a unique polynomial \( r_{(j,k)} \) of degree \( 2j + 2k - 1 \) by the \( j + k + 1 \) conditions

\[ r_{(j,k)}(-i) = 0, \quad i = 1, \ldots, j + k, \quad i \neq k, \]

the \( j + k + 1 \) conditions

\[ r_{(j,k)}(\cdot) + r_{(j)} \left( \frac{1}{2} \right) r_{(k)}(\cdot) = r_{(j,k)} \left( \frac{1}{2} \right) + r_{(j)} \left( \frac{1}{2} \right) r_{(k)} \left( \frac{1}{2} \right) \]

on the set

\[ S(j + k) = \left\{ \frac{1}{2}, \frac{1}{2} - 1, \ldots, \frac{1}{2} - (j + k) \right\}, \]

and the relation

\[ r_{(j,k)}(0) = -r_{(j)}(k)r_{(k)}(0). \]

(1.13) can be replaced by the simpler condition that the left hand side is constant on the set \( S(j + k) \). The value of that constant is determined by the additional relation (1.14) for the constant term of \( r_{(j,k)} \). Now Conjecture 4.1 says that

\[ a^{(N)}_{(j,k)} = \prod_{i=1}^{j+k} \left( \frac{N-i}{2N-2i-1} \right) r_{(j,k)}(N-(j+k)), \quad N \geq j + k + 1. \]

For general compositions \( I \), there are analogous interpolation polynomials \( r_{I} \). However, the interpolation data are more complicated. Indeed, those for \( r_{I} \) are recursively determined by those of polynomials \( r_{J} \) which are associated to sub-compositions \( J \) of \( I \). The corresponding recursive relations are non-linear (see (4.8), (4.7)). By iteration, they can be used to generate \( r_{I} \) from the polynomials \( r_{(k)} \), where \( k \) runs through the entries of \( I \). For the details we refer to Section 4.

We finish the present section with a number of comments. Branson introduced the quantity \( Q_{n} \) in order to systematize the study of extremal properties of functional determinants of the Yamabe operator \( P^{2} \) (and other conformally covariant differential operators). The central idea is to decompose the conformal anomalies of the determinants as sums of a universal part (given by Q-curvature), locally conformally invariant parts (which vanish in the conformally flat case) and divergence parts with local conformal primitives ([4], [5], [6], [7], [8]). The concept rests on the observation that the heat coefficients of conformally covariant differential operators display similar conformal variational formulae as the Q-curvatures \( Q_{2j} \). We briefly describe that analogy in the case of the Yamabe operator \( D = -P^{2} \). Assume that \( D \) is positive. The coefficients \( a_{j} \) in the asymptotics

\[ \text{tr}(e^{-tD}) \sim \sum_{j \geq 0} t^{-\frac{n+1}{2}} \int_{M} a_{j} \text{vol}, \quad t \to 0 \]
of the trace of its heat kernel are Riemannian curvature invariants which satisfy the conformal variational formulae

\[(1.16) \quad \left( \int_M a_j \ vol \right) \cdot [\varphi] = (n-j) \int_M \varphi a_j \ vol, \quad \varphi \in C^\infty(M).\]

Here the notation \(\cdot\) is used to indicate the infinitesimal conformal variation

\[\mathcal{F}^*(h)[\varphi] = (d/dt)|_0 \mathcal{F}(e^{2t\varphi} h)\]

of the functional \(\mathcal{F}\). In particular, the integral

\[(1.17) \quad \int_M a_n \ vol\]

is a global conformal invariant. The conformal variational formula

\[-(\log \det(D)) \cdot [\varphi] = 2 \int_M \varphi a_n \ vol\]

shows the significance of \(a_n\) as a conformal anomaly of the determinant. For the details we refer to [9], [10].

The conformal invariance of \((1.17)\) has strong implications. In fact, when combined with the Deser-Schwimmer classification of conformal anomalies (proved by Alexakis in the fundamental work [1]), it implies that \(a_n\) is a linear combination of the Pfaffian, a local conformal invariant and a divergence. The existence of such a decomposition also follows for the global conformal invariant \((1.4)\). The conformal invariance of \((1.4)\) is a consequence of

\[\left( \int_M Q_{2j} \ vol \right) \cdot [\varphi] = (n-2j) \int_M \varphi Q_{2j} \ vol.\]

The problem to find explicit versions of these decompositions is more difficult.

A third series of related scalar curvature quantities, which in recent years naturally appeared in connection with ideas around the AdS/CFT-correspondence, are the holographic coefficients \(v_{2j}\). These quantities describe the asymptotics of the volume form of Poincaré-Einstein metrics (Section 2). Here [11]

\[\left( \int_M v_{2j} \ vol \right) \cdot [\varphi] = (n-2j) \int_M \varphi v_{2j} \ vol,\]

and the integral

\[(1.18) \quad \int_M v_n \ vol\]

is a global conformal invariant [20]. \(v_n\) is the conformal anomaly of the renormalized volume of conformally compact Einstein metrics [20]. The problem to understand the parallel between renormalized volumes and functional determinants is at the center of the AdS/CFT-duality [12], [23].
Graham and Zworski [24] discovered that the global conformal invariants \((1.18)\) and \((1.4)\) are proportional. Moreover, the formula \((23)\), \((27)\)

\[
\frac{c_n}{2} Q_n = n v_n + \sum_{j=1}^{\frac{n}{2}-1} (n-2j) T^*_{2j}(0)(v_{n-2j})
\]

(with \(c_n = (-1)^{\frac{n}{2}} \left[ 2^n \left( \frac{n}{2}! \right) \left( \frac{n}{2} - 1 \right)! \right]^{-1} \)) for the critical \(Q\)-curvature completely expresses \(Q_n\) in terms of holographic data, \(v_{2j}\) and \(T_{2j}(0)\), of the given metric. For the definition of the differential operators \(T_{2j}(0)\) we refer to Section 2.

In dimension \(n = 4\), \((1.19)\) states that

\[
Q_4 = 16v_4 + 2\Delta v_2.
\]

Using \(v_4 = \frac{1}{8}(J^2 - |P|^2)\) and \(v_2 = -\frac{1}{2}J\), this is equivalent to \((1.2)\).

\((1.19)\) implies that in the conformally flat case the Pfaffian appears naturally in \(Q_n\) (as predicted by the Deser-Schwimmer classification). Although in that case all holographic coefficients \(v_{2j}\) are known, \(Q_n\) is still very complex. The complexity is hidden in the differential operators \(T_{2j}(0)\) which define the divergence terms. \((1.5)\) would shed new light on these divergence terms by replacing the coefficients \(v_{2j}\) by \(Q_{2j}\), and \(T^*_{2j}(0)\) by sums of compositions of GJMS-operators.

Finally, we note that the coefficients \(v_{2j}\) for \(2j \neq n\) give rise to interesting variational problems [11]. In the conformally flat case, \(v_{2j}\) is proportional to \(\text{tr}(\wedge^j P)\), and the functionals \(\int_M \text{tr}(\wedge^j P)\text{vol}\) were first studied by Viaclovski in [31]. The variational nature of the functionals \(\int_M \text{tr}(\wedge^j P)\) has been clarified by Branson and Gover in [3]. For a deeper study of the quantities \(v_{2j}\) see [21].

The paper is organized as follows. In Section 2, we describe the theoretical background from [27]. In Section 3 we formulate the universal recursive formula in full generality. We combine the detailed description of the algorithm with a clear accentuation of the conjectural input. For locally conformally flat metrics, we prove the universality of \((1.10)\) and the recursive formula for the critical \(Q_{10}\). We describe a part of the structure of the recursive formulae in terms of a generating function \(G\). Finally, we discuss a piece of evidence which comes from the theory of extended obstruction tensors [21]. In Section 4 we formulate a conjectural description of the functions \(N \mapsto a^{(N)}_I\) in terms of interpolation polynomials \(r_I\) which are generated by recursive relations (Conjecture 4.1). All formulated structural properties are obtained by extrapolation from numerical data (Section 6). The general picture is described in Section 4.1. Section 4.2 serves as an illustration. In particular, we reproduce all coefficients in the universal recursive formulae for \(Q_{2N}\) \((N \leq 5)\) in terms of the values of the polynomials \(r_I\). In Section 5 we emphasize some of the open problems raised by the approach. In the Appendix, we display explicit versions of the universal recursive formulae for \(Q_{10}, Q_{12}, Q_{14}\) and \(Q_{16}\), test the universality of these expressions by evaluation on round spheres of any even dimension, and list a part of the numerical data from which the conjectures have been distilled.

The present paper combines theoretical results of [27] with computer experiments using Mathematica with the NCAlgebra package. The computer allowed
to enter the almost unexplored world of $Q$-curvatures of order exceeding 8. The transformations of a large number of algorithms into effective programs is the work of the first named author.

2. The Recursive Structure of Residue Families

The algorithm which generates the proposed recursive formulae for all $Q$-curvatures rests on two central facts. One of these is the identity

$$Q_n(h) = -(-1)^{\frac{n}{2}}(d/d\lambda)|_0(D^\text{res}_n(h;\lambda)(1))$$

(2.1) which detects the critical $Q$-curvature $Q_n(h)$ in the linear part of the critical residue family $D^\text{res}_n(h;\lambda)$. The second fact is the recursive structure of residue families. We start by recalling the construction of residue families $D^\text{res}_{2N}(h;\lambda)$ and reviewing their basic properties [27]. The algorithm will be described in Section 3.

For $2N \leq n$, the families $D^\text{res}_{2N}(h;\lambda)$, $\lambda \in \mathbb{C}$ are natural one-parameter family of local operators

$$C^\infty([0, \varepsilon) \times M) \rightarrow C^\infty(M).$$

They are completely determined by the metric $h$. Their construction rests on the Poincaré-Einstein metrics with conformal infinity $[h]$ ([14], [13]).

A Poincaré-Einstein metric $g$ associated to $(M, h)$ is a metric on $(0, \varepsilon) \times M$ (for sufficiently small $\varepsilon$) of the form

$$g = r^{-2}(dr^2 + h_r),$$

where $h_r$ is a one-parameter family of metrics on $M$ so that $h_0 = h$ and

$$\text{Ric}(g) + ng = O(r^{n-2}).$$

The Taylor series of $h_r$ is even in $r$ up to order $n$. More precisely,

$$h_r = h(0) + r^2 h(2) + \cdots + r^n (h(n) + \log r h(n)) + \cdots.$$ 

(2.2)

In (2.3), the coefficients $h(2), \ldots, h(n-2)$ and $\text{tr}(h(n))$ are determined by $h(0) = h_0 = h$. These data are given by polynomial formulae in terms of $h$, its inverse, and the covariant derivatives of the curvature tensor. In particular, $h(2) = -P$. Let

$$v(r, \cdot) = \frac{\text{vol}(h_r)}{\text{vol}(h)} = v_0 + r^2 v_2 + \cdots + r^n v_n + \cdots, \ v_0 = 1.$$ 

Here $\text{vol}$ refers to the volume forms of the respective metrics on $M$. The coefficients $v_{2j} \in C^\infty(M)$ ($j = 0, \ldots, \frac{n}{2}$) are given by local formulae in terms of $h$, its inverse, and the covariant derivatives of the curvature tensor. $v_n$ is the holographic anomaly of the asymptotic volume of the Poincaré-Einstein metric $g$ [20].

**Definition 2.1 (Residue families).** For $2N \leq n$, let

$$D^\text{res}_{2N}(h;\lambda) : C^\infty([0, \varepsilon) \times M^n) \rightarrow C^\infty(M^n)$$

be defined by

$$D^\text{res}_{2N}(h;\lambda) = 2^{2N}N! \left[ (-\frac{n}{2} - \lambda + 2N - 1) \cdots (-\frac{n}{2} - \lambda + N) \right] \delta_{2N}(h;\lambda+n-2N)$$
with
\[\delta_{2N}(h; \lambda) = \sum_{j=0}^{N} \frac{1}{(2N-2j)!} \left[ T_{2j}(h; \lambda)v_0 + \cdots + T_{0}^*(h; \lambda)v_{2j} \right] i^* (\partial/\partial r)^{2N-2j}.\]

Here \(i^*\) restricts functions to \(r = 0\), and the holographic coefficients \(v_{2j}\) act as multiplication operators.

The rational families \(T_{2j}(h; \lambda)\) of differential operators on \(M\) arise by solving the asymptotic eigenfunction problem for the Poincaré-Einstein metric. In other words, \(T_{2j}(h; \lambda)\) is given by
\[T_{2j}(h; \lambda)f = b_{2j}(h; \lambda),\]
where
\[u \sim \sum_{j \geq 0} r^{\lambda+2j}b_{2j}(h; \lambda), \ r \to 0\]
describes the asymptotics of an eigenfunction \(u\) so that
\[-\Delta_g u = \lambda(n-\lambda) u\]
and \(b_0 = f\). In particular, the operators \(T_{2j}(h; 0)\) describe the asymptotics of solutions of the Dirichlet problem at infinity. Note that the asymptotics of an eigenfunction \(u\) for \(\Re(\lambda) = \frac{n}{2}\) contains a second sum with leading exponent \(n - \lambda\). This sum is suppressed in (2.4).

The renormalized families
\[P_{2j}(h; \lambda) = 2^{2j} j! \left( \frac{n}{2} - \lambda - 1 \right) \cdots \left( \frac{n}{2} - \lambda - j \right) T_{2j}(h; \lambda)\]
are polynomial in \(\lambda\). They satisfy \(P_{2j}(\lambda) = \Delta^j + \text{LOT}\) for all \(\lambda\) and
\[P_{2j}(h; \frac{n}{2}-j) = P_{2j}(h)\]

Formal adjoints of \(T_{2j}(h; \lambda)\) are taken with respect to the scalar product defined by \(h\).

The family \(D_{2N}^{\text{res}}(h; \lambda)\) is conformally covariant in the following sense. The Poincaré-Einstein metrics of \(h\) and \(\hat{h} = e^{2\varphi} h\) are related by
\[\kappa^* \left( r^{-2}(dr^2 + h_r) \right) = r^{-2}(dr^2 + \hat{h}_r),\]
where \(\kappa\) is a diffeomorphism which fixes the boundary \(r = 0\). Then we have
\[(2.5) \quad D_{2N}^{\text{res}}(\hat{h}; \lambda) = e^{(\lambda-2N)\varphi} \circ D_{2N}^{\text{res}}(h; \lambda) \circ \kappa^* \circ \left( \frac{\kappa^*(r)}{r} \right)^{\lambda}.\]

For the proof of (2.5) one interprets the family as a residue of a certain meromorphic family of distributions [27].

Now assume that \(h\) is conformally flat. Then for
\[\lambda \in \left\{ -\frac{n}{2} + N, \ldots, -\frac{n}{2} + 2N - 1 \right\} \cup \left\{ -\frac{n-1}{2} \right\},\]
the family \( D_{2N}^{res}(h; \lambda) \) factorizes into the product of a lower order residue family and a GJMS-operator:

\[
(2.6) \quad D_{2N}^{res} \left( h; -\frac{n}{2} + 2N - j \right) = P_{2j}(h) D_{2N-2j}^{res} \left( h; -\frac{n}{2} + 2N - j \right)
\]

for \( j = 1, \ldots, N \) and

\[
(2.7) \quad D_{2N}^{res} \left( h; -\frac{n-1}{2} \right) = D_{2N-2}^{res} \left( h; -\frac{n+3}{2} \right) P_{2}(dr^2 + h_r).
\]

The additional factorization identities which involve higher order GJMS-operators for \( dr^2 + h_r \) (see [27]) will not be important in the present paper. The factorization identities should be regarded as curved versions of multiplicity one theorems in representation theory.

For \( j = N \), (2.6) states that

\[
D_{2N}^{res} \left( h; -\frac{n}{2} + N \right) = P_{2N}(h)i^*.
\]

In particular, the critical residue family \( D_{n}^{res}(h; \lambda) \) specializes to the critical GJMS-operators at \( \lambda = 0 \):

\[
D_{n}^{res}(h; 0) = P_{n}(h)i^*.
\]

The factorization identities in (2.6) and the identity (2.7) are of different nature. The identities in (2.6) actually hold true without additional assumptions on \( h \). In [27] it is shown that this can be derived as a consequence of the identification of \( P_{2N} \) as the residue of the scattering operator [24]. (2.7) is more difficult and presently only known for general order under the assumption that \( h \) is conformally flat. In that case, the identity follows from the conformal covariance (2.5) of the family, together with a corresponding factorization in the flat case.

3. The universal recursive formulae

In the present section, we formulate conjectural recursive presentations of all \( Q \)-curvatures and describe their status.

**Conjecture 3.1 (Universal recursive formulae).** Let \( n \) be even and assume that \( 2N \leq n \). Then the \( Q \)-curvature \( Q_{2N} \) on Riemannian manifolds of dimension \( n \) can be written in the form

\[
(3.1) \quad Q_{2N} = \sum_{1 \leq |I| \leq N-1} a_I^{(N)} P_{2|I|}(Q_{2N-2|I|}) + (-1)^{N-1} \left( \frac{2N-2}{2N-3} \right)! \cdot i^* P_{N-1}(Q_2)
\]

with certain rational coefficients \( a_I^{(N)} \) which do not depend on \( n \). The sum in (3.1) runs over all compositions \( I = (I_1, \ldots, I_m) \) of integers in \([1, N-1]\) as sums of natural numbers. For \( I = (I_1, \ldots, I_m) \) of length \( m \) and size \(|I| = I_1 + \cdots + I_m\), the operator \( P_{2|I|} \) is defined as the composition \( P_{2I_1} \cdots P_{2I_m} \) of GJMS-operators. The coefficients \( a_I^{(N)} \) have the sign \((-1)^{|I|+m-1}\).
We emphasize that the sum in (3.1) runs over compositions \(I\) instead of partitions. This reflects the fact that the GJMS-operators do not commute. Since there are \(2^{N-1}\) compositions of size \(N\), the sum in (3.1) contains
\[
2^0 + 2^1 + \cdots + 2^{N-2} = 2^{N-1} - 1
\]
terms. The operator \(\bar{P}_2(h)\) denotes the Yamabe operator of the conformal compactification \(dr^2 + h_r\) of the Poincaré-Einstein metric of \(h\) (Section 2). \(Q_2\) is \(Q_2\) for the metric \(dr^2 + h_r\). In more explicit terms,
\[
(3.2) \quad \bar{Q}_2(h) = J(dr^2 + h_r) = -\frac{1}{2r} \text{tr}(h_r^{-1} \dot{h}_r)
\]
and \(\bar{P}_2(h) = \Delta_{dr^2+h_r} - (\frac{n}{2} - 1)\bar{Q}_2(h)\) with
\[
\Delta_{dr^2+h_r} = \partial^2 / \partial r^2 + \frac{1}{2} \text{tr}(h_r^{-1} \dot{h}_r) \partial / \partial r + \Delta h_r.
\]
Note that \(h_{(2)} = -P\) implies
\[
(3.3) \quad i^*\bar{Q}_2 = Q_2.
\]

We continue with the description of the algorithm which generates the presentations (3.1).

First of all, all formulae arise from the corresponding formulae for critical \(Q\)-curvatures by applying the principle of universality. The conjectural status of the formulae (3.1) is partly due to the unproven applicability of this principle.

As a preparation for the definition of the algorithm, we observe some consequences of the factorization identities for residue families. The family \(D_{2N}^{\text{res}}(h; \lambda)\) is polynomial of degree \(N\). The \(N + 1\) identities (2.6) and (2.7) imply that \(D_{2N}^{\text{res}}(h; \lambda)\) can be written as a linear combination of the right-hand sides of these identities. The lower order residue families which appear in this presentation, in turn, satisfy corresponding systems of factorization identities. These allow to write any of these families as a linear combination of the corresponding right-hand sides of the factorization relations they satisfy. The continuation of that process leads to a formula for \(D_{2N}^{\text{res}}(h; \lambda)\) as a linear combination of compositions of the GJMS-operators
\[
P_{2N}(h), \ldots, P_2(h)
\]
and the Yamabe operator \(\bar{P}_2(h) = P_2(dr^2 + h_r)\). The second reason for the conjectural status of (3.1) is that the full system of factorization identities is not yet available for general metrics (see the comments at the end of Section 2).

We apply the above method to the critical residue family \(D_n^{\text{res}}(h; \lambda)\) and combine the resulting formula with (2.1). This yields a formula for \(Q_n(h)\) as a linear combination of compositions of the GJMS-operators
\[
P_{n-2}(h), \ldots, P_2(h)
\]
and the Yamabe operator \(\bar{P}_2(h) = P_2(dr^2 + h_r)\) (acting on \(u = 1\)). That formula contains compositions of GJMS-operators with powers of \(\bar{P}_2(h)\) up to \(\frac{n}{2}\).

In the next step, we replace all quantities
\[
i^*\bar{P}_2^k(h)(1) \quad \text{for} \quad k = 1, \ldots, n/2 - 1
\]
by subcritical GJMS-operators and subcritical $Q$-curvatures $Q_{2k}$. For that purpose, we apply similar formulae for the subcritical $Q$-curvatures. Here the principle of universality becomes crucial. In fact, by assuming the universality of the respective formulae for $Q_2, \ldots, Q_{n-2}$, we regard these as formulae for $i^*F_2^k(1)$ ($1 \leq k \leq \frac{n}{2} - 1$), and plug them into the formula for $Q_n$. This finishes the algorithm.

The description shows that, for conformally flat metrics, the conjectural status of the presentations is only due to the principle of universality.

For the convenience of the reader, we illustrate the algorithm in two special cases.

We start with a proof of (1.7) in dimension $n = 4$. We consider the critical family $D^{res}_4(h; \lambda)$. We write this family in the form $A\lambda^2 + B\lambda + C$, and determine the operator coefficients by using the factorization identities

$$D^{res}_4(h; 0) = P_4(h)i^*,$$

$$D^{res}_4(h; 1) = P_2(h)D^{res}_2(h; 1),$$

$$D^{res}_4(h; -\frac{3}{2}) = D^{res}_2(h; -\frac{7}{2})P_2(dr^2 + h_r).$$

The first identity implies $C = P_4(h)i^*$. The remaining two relations yield

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{15} \begin{pmatrix} 4 & 6 \\ -4 & 9 \end{pmatrix} \begin{pmatrix} (D^{res}_2(h; -\frac{7}{2})P_2(dr^2 + h_r)) \\ P_2(h)D^{res}_2(h; 1) \end{pmatrix}.$$ 

Now by the factorization identities for $D^{res}_2(h; \lambda)$,

$$D^{res}_2(h; -\frac{7}{2}) = 5i^*P_2(dr^2 + h_r) - 4P_2(h)i^*,$$

$$D^{res}_2(h; 1) = -4i^*P_2(dr^2 + h_r) + 5P_2(h)i^*.$$ 

Thus, we find

$$(3.4) \quad A = 2P_2^2i^* - \frac{8}{3}P_2i^*P_2 + \frac{4}{3}i^*\bar{P}_2^2 \quad \text{and} \quad B = 3P_2^2i^* - \frac{4}{3}P_2i^*P_2 - \frac{4}{3}i^*\bar{P}_2^2.$$ 

Now the formula for $B$ in (3.4), together with (2.1), implies

$$Q_4 = -B(1) = -3P_2^2(1) + \frac{4}{3}P_2(i^*\bar{P}_2(1)) + \frac{4}{3}i^*\bar{P}_2^2(1)$$

$$= 3P_2(Q_2) - 2P_2(i^*\bar{Q}_2) - 2i^*P_2(\bar{Q}_2).$$

The last equality is a consequence of $P_2(1) = -Q_2$ and $\bar{P}_2(1) = -\frac{3}{2}\bar{Q}_2$ (see (1.1)). But using $i^*\bar{Q}_2 = Q_2$ (see (3.3)), we find

$$Q_4 = P_2(Q_2) - 2i^*P_2(\bar{Q}_2).$$
This is (1.7). Although, the above derivation is only valid in dimension \( n = 4 \),
the final formula for \( Q_4 \) is valid in all dimensions (see the discussion on page 4).
We also note that we simplified the contribution
\[ P_2(i^* \bar{P}_2(1)) \]
by using \( i^* \bar{Q}_2 = Q_2 \) in dimension \( n = 4 \) (see (3.3)). Since the latter identity can
be regarded as a version of the universal formula for \( Q_2 \), that argument is the
simplest special case of the application of universality of subcritical \( Q \)-curvatures
in the algorithm.

Similarly, the algorithm yields the recursive formula (1.8) for the critical \( Q \)-
curvature \( Q_6 \) for conformally flat metrics \( h \). The derivation makes use of the
relations \( i^* \bar{Q}_2 = Q_2 \) and (1.7) in dimension \( n = 6 \). Again, (1.8) holds true for
all metrics and in all dimensions \( n \geq 6 \). For detailed proofs of these results we
refer to [27]. A calculation using (1.8) shows that \( J^3 \) contributes to \( Q_6 \) with the
coefficient \( (\frac{n}{2} - 1)(\frac{n}{2} + 1) \).

Starting with \( Q_8 \), the theory is less complete. The following detailed descrip-
tion of this case will also point to the open problems. In this case, we use the
universality of \( i^* \bar{Q}_2 = Q_2 \), (1.7) and (1.8) to deduce the formula (1.10) for
\( Q_8 \) in dimension \( n = 8 \) for conformally flat \( h \). The starting point is the identity
\[ (3.5) \quad - \hat{D}_8^{\text{res}}(h, 0)(1) = Q_8(h). \]
The critical family \( D_8^{\text{res}}(h; \lambda) \) satisfies the factorization identities
\[
\begin{align*}
D_8^{\text{res}}(h; 0) &= P_8(h)i^*; \\
D_8^{\text{res}}(h; 1) &= P_6(h)D_2^{\text{res}}(h; 1), \\
D_8^{\text{res}}(h; 2) &= P_4(h)D_4^{\text{res}}(h; 2), \\
D_8^{\text{res}}(h; 3) &= P_2(h)D_6^{\text{res}}(h; 3),
\end{align*}
\]
and
\[ (3.6) \quad D_8^{\text{res}}\left(h; -\frac{7}{2}\right) = D_6^{\text{res}}\left(h; -\frac{11}{2}\right) \bar{P}_2(h). \]
In view of \( P_8(h)(1) = 0 \), it follows that \( Q_8(h) \) can be written as a linear combi-
nation of the four terms
\[ P_6(h)D_2^{\text{res}}(h; 1)(1), \quad P_4(h)D_4^{\text{res}}(h; 2)(1), \quad P_2(h)D_6^{\text{res}}(h; 3)(1) \]
and \( D_6^{\text{res}}(h; -\frac{11}{2}) \bar{P}_2(h)(1) \). The families \( D_2^{\text{res}}(h; \lambda) \) \((j = 1, 2, 3)\), in turn, can be
written as linear combinations of compositions of respective lower order GJMS-
operators and residue families. In order to obtain these presentations, we use
the corresponding systems of factorization identities which are satisfied by these
families. The continuation of the process leads to a presentation of \( Q_8(h) \) as
a linear combination of compositions of GJMS-operators with powers of \( \bar{P}_2(h) \)
(acting on 1). More precisely, the contributions which involve a non-trivial power
of \( \bar{P}_2 \) are of the form
\[ *(i^* \bar{P}_2^k(h)(1)) \quad \text{for } k = 1, \ldots, 4. \]
Now we apply the universality of $i^*\bar{Q}_2 = Q_2$, (1.7) and (1.8). In particular, in dimension $n = 8$ we regard these formulae as expressions for

$$i^*\bar{P}_2(h)(1), \quad i^*\bar{P}^2_2(h)(1) \quad \text{and} \quad i^*\bar{P}^3_2(h)(1)$$

by using $\bar{P}_2(h)(1) = -\frac{7}{2}\bar{Q}_2(h)$. These calculations prove

**Proposition 3.1.** On locally conformally flat Riemannian manifolds of dimension 8, $Q_8$ is given by (1.10).

It remains open whether, in dimension $n = 8$, the same formula yields $Q_8$ for general metrics. In the above proof, the restriction to conformally flat metrics is only due to the unproven validity of the factorization identity (3.5) for general metrics. We expect that the restriction can be removed.

However, more can be said in the locally conformally flat case. In this case, Proposition 3.2 yields the universality of (1.10). Before we prove this result, we describe a consequence.

The validity of (1.10) in dimension $n = 10$ (for locally conformally flat metrics) is the only new ingredient which is required for a proof that (for such a metric) $Q_{10}$ in dimension $n = 10$ coincides with the formula generated by the algorithm. In fact, in that proof, (1.10) is used as a formula for $i^*\bar{P}_2^4(1)$. The universality of (1.7) and (1.8) has been used already in the above constructions. In the present argument, these formulae are used in dimension $n = 10$ as formulae for the respective quantities $i^*\bar{P}_2^2(1)$ and $i^*\bar{P}_2^3(1)$. The resulting formula for $Q_{10}$ is displayed in Section 6.1.

The argument assumes conformal flatness since some of the factorization identities for $D_{8}^{\text{res}}(\lambda)$ and $D_{10}^{\text{res}}(\lambda)$ which enter into the algorithm are only known for such metrics. The problematic identities are those which contain the factor $P_2$ (see (2.7) and the comments at the end of Section 2).

Proving universality of (1.10) through comparison with the formula for $Q_8$ displayed in [17] seems to be a challenging task even for conformally flat metrics. Concerning a comparison of both formula for $Q_8$ we only note that a calculation using (1.10) shows that $J^4$ contributes to $Q_8$ with the coefficient $\left(\frac{3}{2} - 2\right)\frac{n}{2}(\frac{n}{2} + 2)$. This observation fits with [17].

Next, we describe a more conceptual approach towards universality. It rests on the systematic elaboration of the relations between the quantities

$$Q_{2N} \quad \text{and} \quad D_{2N}^{\text{res}}\left(-\frac{n}{2} + N\right)(1).$$

We first describe the method by proving the universality of the recursive formula

(3.7) $$Q_4 = P_2(Q_2) - 2i^*\bar{P}_2(\bar{Q}_2)$$

(for general metrics). For even $n \geq 8$, the polynomial

$$Q_4^{\text{res}}(\lambda) = -D_4^{\text{res}}(\lambda)(1)$$
can be characterized in two different ways. On the one hand, for all even \( n \geq 4 \), this quadratic polynomial satisfies the system

\[
Q_4^{\text{res}} \left( -\frac{n}{2} + 2 \right) = -P_4(1) = -\left( \frac{n}{2} - 2 \right) Q_4 \\
Q_4^{\text{res}} \left( -\frac{n}{2} + 3 \right) = -P_2 D_2^{\text{res}} \left( -\frac{n}{2} + 3 \right)
\]

and the relation

\[
Q_4^{\text{res}} \left( -\frac{n}{2} + 1 \right) = -D_2^{\text{res}} \left( -\frac{n}{2} + 1 \right) P_2(1).
\]

On the other hand, for even \( n \geq 8 \), the polynomial \( Q_4^{\text{res}}(\lambda) \) is characterized by \( \text{(3.8)} \) and \( \text{(3.9)} \). For \( n = 4 \) and \( n = 6 \), the condition \( \text{(3.10)} \) is contained in the conditions of \( \text{(3.8)} \). In particular, in the critical case, these conditions do not suffice to determine the polynomial.

For even \( n \geq 4 \), \( \text{(3.8)} \) and \( \text{(3.9)} \) imply that

\[
\dot{Q}_4^{\text{res}} \left( -\frac{n}{2} + 2 \right) = \frac{1}{3} \frac{n-4}{2} Q_4 + \frac{5n-14}{6} P_2(Q_2) - \frac{2(n-1)}{3} i^* P_2(Q_2).
\]

It leads to \( \text{(3.11)} \), when combined with \( \dot{Q}_4^{\text{res}}(0) = Q_4 \). This method has been used above. On the other hand, for even \( n \geq 8 \), \( \text{(3.8)} \) and \( \text{(3.10)} \) imply

\[
\dot{Q}_4^{\text{res}} \left( -\frac{n}{2} + 2 \right) = Q_4 + \left( \frac{n}{2} - 2 \right) (Q_4 + P_2(Q_2)).
\]

Subtracting \( \text{(3.11)} \) and \( \text{(3.12)} \) gives

\[
0 = \frac{n-1}{3} \left( Q_4 - P_2(Q_2) - 2i^* P_2(Q_2) \right).
\]

This proves the universality of \( \text{(3.7)} \). The cases \( n = 4, 6 \) are covered by analytic continuation in \( n \). The argument reverses an argument in [27], where \( \text{(3.12)} \) was derived from \( \text{(3.7)} \).

A similar argument can be applied for \( Q_6^{\text{res}} \). One formula for the polynomial \( Q_6^{\text{res}}(\lambda) = D_6^{\text{res}}(\lambda)(1) \) of degree 3 follows from the four factorization identities \( \text{(2.6)} \) and \( \text{(2.7)} \) (for \( N = 3 \)). The calculation extends the algorithm described above. It uses the universality of \( \text{(3.7)} \). On the other hand, for even \( n \geq 12 \), Lagrange’s interpolation formula yields a second formula for \( Q_6^{\text{res}}(\lambda) \) by using \( \text{(2.6)} \) (for \( N = 3 \)) and

\[
Q_6^{\text{res}}(0) = 0.
\]

For \( n = 6, 8, 10 \), the latter condition is contained in the system \( \text{(2.6)} \) (for \( N = 3 \)).

The comparison of both resulting formulae for \( \dot{Q}_6^{\text{res}} \left( -\frac{n}{2} + 3 \right) \) yields

\[
0 = \frac{n-1}{5} \left( Q_6 - \frac{2}{3} P_2(Q_4) - \frac{2}{3} P_4(Q_2) + \frac{5}{3} P_2^2(Q_2) - \frac{8}{3} i^* P_2^2(Q_2) \right).
\]
This proves the universality of (1.8). The cases \( n = 6, 8, 10 \) are covered by analytic continuation in \( n \). For the details (of the reversed argument) see [27], Theorems 6.11.7 – 6.11.8.

Similarly, we compare two formulae for

\[
\dot{Q}_{\text{res}}^{8}(\lambda) = -\frac{n}{2} + 4,
\]

where \( Q_{\text{res}}^{8}(\lambda) = -D_{\text{res}}^{8}(\lambda)(1) \). Under the assumption \( Q_{\text{res}}^{8}(0) = 0 \), we find

\[
0 = \frac{n-1}{7} \left[ Q_{8} - \frac{3}{5} P_{2}(Q_{6}) + 4P_{2}^{2}(Q_{4}) - \frac{17}{5} P_{4}(Q_{4}) + \frac{22}{5} P_{2}^{3}(Q_{2}) \right. \\
- \frac{8}{5} P_{2}P_{4}(Q_{2}) - \frac{28}{5} P_{4}P_{2}(Q_{2}) + \frac{9}{5} P_{6}(Q_{2}) + \frac{16}{5} i^{2} P_{2}^{3}(\tilde{Q}_{2})].
\]

We suppress the details of the calculations. The vanishing of the quantity in brackets is equivalent to (1.10).

The quantity \( Q_{\text{res}}^{8}(h; 0) \in C^\infty(M) \) is a scalar conformal invariant. In fact, the conformal transformation law (2.5) implies

\[
e^{2N\varphi} D_{2N}^{\text{res}}(\hat{h}; 0)(1) = D_{2N}^{\text{res}}(h; 0)(1), \quad \hat{h} = e^{2\varphi} h,
\]
i.e.,

\[
(3.13) \quad e^{2N\varphi} Q_{2N}^{\text{res}}(\hat{h}; 0) = Q_{2N}^{\text{res}}(h; 0)
\]
for \( Q_{2N}^{\text{res}}(h; \lambda) = -(-1)^{N} D_{2N}^{\text{res}}(h; \lambda) \). In particular,

\[
(3.14) \quad e^{\delta\varphi} Q_{8}^{\text{res}}(\hat{h}; 0) = Q_{8}^{\text{res}}(h; 0).
\]

By [13], Section 9 there are no such non-trivial invariants on locally conformally flat manifolds of dimension \( > 8 \). In other words, for locally conformally flat metrics \( h \), the condition \( Q_{8}^{\text{res}}(h; 0) = 0 \) is satisfied in dimension \( > 8 \). Thus, we have proved

**Proposition 3.2.** On locally conformally flat manifolds \( (M, h) \) of dimension \( n > 8 \), the recursive formula (1.10) for \( Q_{8}(h) \) holds true.

An alternative method is the following. We recursively determine \( Q_{8}^{\text{res}}(h; \lambda) \) by factorization identities at

\[
\lambda \in \left\{ -\frac{n}{2} + 4, -\frac{n}{2} + 5, -\frac{n}{2} + 6, -\frac{n}{2} + 7 \right\} \cup \left\{ -\frac{n-1}{2} \right\}
\]
(as described by the algorithm) and evaluate the result at \( \lambda = 0 \). For even \( n \geq 16 \), the condition \( Q_{8}^{\text{res}}(h; 0) = 0 \) is equivalent to the universal recursive formula. Again, the cases of even \( n \) such that \( 8 \leq n \leq 14 \) are covered by continuation.

As described above, Proposition 3.2 has the following consequence.

**Corollary 3.1.** On locally conformally flat Riemannian manifolds of dimension 10, the critical \( Q \)-curvature \( Q_{10} \) is given by the formula displayed in Section 6.1.
We continue with a number of supplementary comments on Conjecture 3.1. Alternatively, (3.1) can be viewed as a formula for the function

\[(3.15) \quad i^* \bar{P}_2^{N-1}(\bar{Q}_2) \in C^\infty(M)\]

which is associated to a Poincaré-Einstein metric on the space \((0, \varepsilon) \times M\). From that point of view, (3.1) states that the restriction of the function \(\bar{P}_2^{N-1}(\bar{Q}_2)\) \((N \geq 2)\) to \(M\) can be expressed in terms of boundary data:

\[(3.16) \quad (-1)^N \frac{(2N-2)!!}{(2N-3)!!} \circ i^* \bar{P}_2^{N-1}(\bar{Q}_2) = \sum_{j=0}^{N-1} \mathcal{P}_{2j}^{(N)}(Q_{2N-2j}),\]

where

\[(3.17) \quad \mathcal{P}_{2j}^{(N)} = \sum_{|I|=j} a_I^{(N)} P_{2I}.\]

Here we use the convention that \(\mathcal{P}_0^{(N)} = -1\). The identity \(i^* \bar{Q}_2 = Q_2\) should be regarded as the special case \(N = 1\) of these relations. The differential operators \(\mathcal{P}_{2j}^{(N)}\) are of the form

\[(3.18) \quad \alpha_j^{(N)} \Delta^j + \text{LOT}\]

with

\[(3.19) \quad \alpha_j^{(N)} = \sum_{|I|=j} a_I^{(N)}.\]

For the flat metric, the lower order terms in (3.18) vanish. In Table 3.1, we display the coefficients \(\alpha_j^{(N)}\) for \(N \leq 10\). An inspection suggests that

\[(3.20) \quad \alpha_j^{(N)} = \beta_j^{(N)},\]

where

\[(3.21) \quad \beta_j^{(N)} \overset{\text{def}}{=} (-1)^{j-1} \binom{N-1}{j} \frac{(2j-1)!!(2N-2j-3)!!}{(2N-3)!!}.\]

The relations (3.20) would imply the symmetry relations

\[(3.22) \quad \alpha_j^{(N)} = (-1)^{N-1} \alpha_{N-1-j}^{(N)}.\]

These are clearly visible in Table 3.1. The numbers \(\beta_j^{(N)}\) have a simple generating function. Let

\[(3.23) \quad G(z, w) = (1 - z)^{-\frac{1}{2}} (1 - w)^{-\frac{1}{2}}.\]

Then

\[(3.24) \quad G(z, w) = \sum_{0 \leq j \leq N-1} \beta_j^{(N)} \frac{(2N-3)!!}{(2N-2)!!} \frac{(-1)^{j-1} z^j w^{N-1-j}}{j!(N-1-j)!}.\]

In fact, (3.21) is equivalent to

\[(3.21) \quad \beta_j^{(N)} = (-1)^{j-1} \frac{(2N-2)!!}{j!(N-1-j)!} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_{N-1-j}}{(2N-3)!!},\]
where \((a)_n = a(a+1) \ldots (a+n-1)\). But using
\[
(1-z)^{-\frac{1}{2}} = \sum_{n \geq 0} \left(\frac{1}{2}\right)_n \frac{z^n}{n!}, \quad |z| < 1,
\]
we find that the coefficient of \(z^j w^{N-1-j}\) in \(G(z,w)\) is
\[
\frac{(\frac{1}{2})_j (\frac{1}{2})_{N-1-j}}{j!(N-1-j)!}.
\]
This proves (3.24). It follows that the conjectural relations (3.20) can be summarized in form of the identity
\[
G(z,w) = \sum_{0 \leq j \leq N-1} \alpha_j^{(N)} (2N-3)!! (2N-2)!! (-1)^{j-1} z^j w^{N-1-j}
\]
of generating functions. We do not attempt to prove this identity, but note only that it is compatible with (3.16) and the well-known fact that
\[
Q_{2N} = (-1)^{N-1} \Delta^{N-1}(J),
\]
up to terms with fewer derivatives (see [5]). Indeed, the assertion that \(\Delta^{N-1}(J)\) contributes on both sides of (3.16) with the same weight is equivalent to the relation
\[
\sum_{j=0}^{N-1} (-1)^{j-1} \alpha_j^{(N)} = \frac{(2N-2)!!}{(2N-3)!!}.
\]
But this identity follows from the restriction of (3.25) to \(z = w\) by comparing the coefficients of \(z^{N-1}\).

In the conformally flat case, the Taylor series of \(h_r\) terminates at the third term. More precisely,
\[
h_r = \left(1 - \frac{r^2}{2} P\right)^2
\]
([13], [27], [29]). Now (3.2) implies
\[
\bar{Q}_2 = \text{tr} \left( \left(1 - \frac{r^2}{2} P\right)^{-1} \right) = \sum_{k \geq 0} \left(\frac{r^2}{2}\right)^k \text{tr}(P^{k+1}) = Q_2 + \frac{r^2}{2} |P|^2 + \ldots,
\]
and it is not hard, although it becomes tedious for large \(N\), to determine the contribution \(i^* P_2^{N-1}(\bar{Q}_2)\) to \(Q_{2N}\). We shall apply this observation in Section 6.1.

We finish the present section with a brief discussion of a test of Conjecture 3.1 for general metrics. It deals with the contributions of the powers of the Yamabe operator \(P_2\) to \(Q_6\) in Section 1. Here we compare the contributions of
\[
(P, \Omega^{(N-2)})
\]
to \(Q_{2N}\) and
\[
(-1)^{N-1} \frac{(2N-2)!!}{(2N-3)!!} i^* P_2^{N-1}(\bar{Q}_2).
\]
The tensor $\Omega^{(N-2)}$ is one of Graham’s extended obstruction tensors [21]. In particular,

$$\Omega^{(1)} = \frac{B}{4-n}.$$  

On the right-hand side of (3.1), the contribution (3.28) only comes from the term $i^* P_2^{N-1}(\bar{Q}_2)$. On the other hand, its contribution to $Q_{2N}$ can be captured by its relation to $v_{2N}$:

$$(3.30) \quad Q_{2N} = \cdots + (-1)^N 2^{N-1} N! (N-1)! v_{2N}.$$  

For $2N = n$, the holographic formula (1.19) is such a relation. The suppressed lower order terms in (3.30) are not influenced by $\Omega^{(N-2)}$. In [27], such extensions of (1.19) were proposed and discussed in detail for subcritical $Q_2$, $Q_4$ and $Q_6$. For $Q_8$ in dimension $n \geq 8$ we expect that

$$(3.31) \quad \frac{1}{244!} Q_8 = 8 v_8 + 6 T_2^* \left(\frac{n}{2} - 4\right) (v_6) + 4 T_4^* \left(\frac{n}{2} - 4\right) (v_4) + T_6^* \left(\frac{n}{2} - 4\right) (v_2).$$

We combine (3.30) with the fact that (3.28) enters into $v_{2N}$ with the weight $\left(\frac{-1}{2} \right)^{N-1}$. Hence (3.28) contributes to $Q_{2N}$ through

$$(3.32) \quad - 2^{N} (N-1)! (P, \Omega^{(N-2)}).$$

Now in order to determine its contribution to (3.29), it suffices to trace its role in $i^* (\partial^2 / \partial r^2)^{N-1}(\bar{Q}_2)$, where $\bar{Q}_2$ is given by (3.2). Graham [21] proved that the expansion

$$h_r = h - P r^2 + h_{(4)} r^4 + \cdots + h_{(2N-2)} r^{2N-2} + h_{(2N)} r^{2N} + \cdots$$

has the structure

$$(3.33) \quad \frac{1}{2} h_{(2k)} = \frac{-1}{2^k k!} \left( \Omega^{(k-1)} + (k-1)(P \Omega^{(k-2)} + \Omega^{(k-2)} P) + \cdots \right).$$

Thus, it suffices to consider the contributions of

$$2(P, h_{(2N-2)}), \quad (2N-2)(P, h_{(2N-2)}) \quad \text{and} \quad 2N \text{ tr}(h_{(2N)})$$

to the Taylor-coefficients of $r^{2N-1}$ in $i^* h_{r}^{-1} h_r$. Using (3.33) we find the contribution

$$4 \frac{-1}{2^{N-1} (N-1)!} (P, \Omega^{(N-2)}).$$

It follows that

$$i^* (\partial^2 / \partial r^2)^{N-1}(\bar{Q}_2) = (-1)^N 2^{(2N-2)!} \frac{(2N-2)!}{2^{N-1} (N-1)!} (P, \Omega^{(N-2)}) + \cdots,$$

i.e., (3.29) yields the contribution

$$-2^N (N-1)! (P, \Omega^{(N-2)}).$$

It coincides with (3.32).
4. The structure of the coefficients $a^{(N)}_I$

The right-hand sides of (3.1) are generated by the algorithm described in Section 3. In the present section, we formulate a conjectural description of the coefficients $a^{(N)}_I$ in terms of polynomials $r_I$ which are canonically associated to compositions $I$. These polynomials are generated by a much simpler algorithm.

4.1. The polynomials $r_I$ and their role. The polynomials $r_I$ are defined recursively as interpolation polynomials on the sets

$$S(k) = \left\{ \frac{1}{2} - k, \ldots, -\frac{1}{2}, \frac{1}{2} \right\}, \quad k \in \mathbb{N}_0$$

of half-integers, and on certain sets of negative integers.

First of all, we define the polynomials $r^{(k)}$, $k \in \mathbb{N}$. These play the role of building blocks of the general case. Let $r^{(k)}$ be defined as the unique polynomial of degree $2k - 1$ with (simple) zeros in the integers in the interval $[-(k-1), -1]$ so that $r^{(k)}$ is constant on $S(k)$, and has constant term

$$r^{(k)}(0) = (-1)^{k-1} \frac{(2k-3)!!}{k!}.$$ 

Equivalently, $r^{(k)}$ can be defined as the interpolation polynomial which is characterized by its $2k$ values

$$r^{(k)}(-i) = 0 \quad \text{for all } i = 1, \ldots, k - 1,$$

$$r^{(k)}\left(\frac{1}{2} - i\right) = (-2)^{-(k-1)} \frac{\left(\frac{1}{2}\right)_{k-1}}{(k-1)!} \quad \text{for all } i = 0, 1, \ldots, k.$$

The equivalence of both characterizations follows from Lagrange’s formula.

In order to define $r_I$ for a general composition $I$, we introduce some more notation. For any $I$, we define the rational number

$$R_I = \sum_{I = (J_1, \ldots, J_M)} r_{J_1} \left(\frac{1}{2}\right) \cdots r_{J_M} \left(\frac{1}{2}\right),$$

where the sum runs over all compositions $J_1, \ldots, J_M$ which form a subdivision of $I$, i.e., the sequence of natural numbers which is obtained by writing the entries of $J_1$ followed by the entries of $J_2$ etc., coincides with the sequence which defines $I$. In particular, $R_{(k)}$, $R_{(j,k)}$ and $R_{(i,j,k)}$ are given by the values of the respective sums

$$r^{(k)}, \quad r^{(j)}r^{(k)} \quad \text{and} \quad r^{(i,j,k)} + r^{(i,j)}r^{(k)} + r^{(i)}r^{(j,k)} + r^{(i)}r^{(j)}r^{(k)}$$

at $x = \frac{1}{2}$. Next, using the polynomials $r_I$, we define

$$C_{(I_1, \ldots, I_m)}(x) = r_{(I_1, \ldots, I_m)}(x) + R_{(I_1)} \cdot r_{(I_2, \ldots, I_m)}(x) + \cdots + R_{(I_1, \ldots, I_{m-1})} \cdot r_{(I_m)}(x).$$

$C_I$ differs from $r_I$ by a lower degree polynomial.

Now let $r_I$ be a polynomial of degree $2|I| - 1$ so that

$$r_I(-i) = 0 \quad \text{for all } i = 1, \ldots, |I|, \ i \neq I_{\text{last}}$$
and

\[(4.7) \quad C_I(x) \text{ is constant on } S(\lvert I \rvert).\]

Here \(I_{\text{last}}\) denotes the last entry in the composition \(I = (I_{\text{first}}, \ldots, I_{\text{last}})\). The condition \((4.7)\) constitutes the first system of multiplicative recursive formulae for the values of the polynomials \(r_I\).

Now \((4.6)\) and \((4.7)\) determine \((\lvert I \rvert − 1) + (\lvert I \rvert + 1) = 2\lvert I \rvert\) values of \(r_I\). Since the value of \(C_I\) on \(S(\lvert I \rvert)\) was not chosen, one additional condition is required to characterize \(r_I\). For that purpose, we use the second system of multiplicative recursive formulae

\[(4.8) \quad r_{(j,k)}(0) + r_{j}(k) \cdot r_{(k)}(0) = 0\]

for the constant terms. The relations \((4.8)\) are required to hold true for all \(k \geq 1\) and all compositions \(J\). They describe how all values of the polynomials \(r_I\) on the natural numbers finally influence the constant terms of polynomials which are associated to compositions of larger sizes. The values \(r_{(k)}(0)\) are given by the explicit formula \((4.2)\).

It follows from the above definition that \(r_I\) is determined by the (values of the) polynomials \(r_J\) for all sub-compositions \(J\) of \(I\). By iteration, it follows that \(r_I\) is determined by the polynomials \(r_{(k)}\) for all \(k\) which appear as entries of \(I\).

Now we are ready to formulate the conjectural relation between the coefficients \(a_I^{(N)}\) and the values of \(r_I\) on \(\mathbb{N}\).

**Conjecture 4.1.** For all compositions \(I\) and all integers \(N \geq \lvert I \rvert + 1\),

\[(4.9) \quad a_I^{(N)} = \prod_{i=1}^{\lvert I \rvert} \left(\frac{N-i}{2N-2i-1}\right) r_I(N-\lvert I \rvert).\]

Conjecture \((4.1)\) is supported by the observation that all coefficients in the presentations \((3.1)\) of the \(Q\)-curvatures \(Q_{2N}\) with \(N \leq 14\) are correctly reproduced by \((4.9)\). We recall that for \(Q_{28}\) the sum in \((3.1)\) already contains \(2^{13} - 1\) terms.

In particular, we obtain uniform descriptions of all coefficients in the universal recursive formulae for \(Q_6, Q_8\) and \(Q_{10}\) in terms of the polynomials \(r_I\) for all compositions \(I\) with \(\lvert I \rvert \leq 4\). In Section 4.2 we shall discuss these examples in more detail.

Note that \((4.9)\) implies

\[\alpha_j^{(N)} = \sum_{\lvert I \rvert = j} a_I^{(N)} = \prod_{i=1}^{j} \left(\frac{N-i}{2N-2i-1}\right) \sum_{\lvert I \rvert = j} r_I(N-\lvert I \rvert).\]

Thus, under Conjecture \((4.1)\) the identity

\[\alpha_j^{(N)} = (-1)^{j-1} \binom{N-1}{j} \frac{(2j-1)!!(2N-2j-3)!!}{(2N-3)!!} \]

\[= (-1)^{j-1} \binom{2j-1}{j} \frac{(N-1) \ldots (N-j)}{j!} \frac{1}{(2N-3) \ldots (2N-2j-1)}\]
(see (3.20)) is equivalent to

\[ \sum_{|I|=j} r_I(x) = (-1)^{j-1}(2j-1)!! \cdot \frac{1}{j!}. \]

**Example 4.1.** The polynomial \( r_{(j,k)} \) is characterized by its zeros in\n\[ \{-(j+k), \ldots, -1\} \setminus \{-k\}, \]
the constancy of\n\[ C_{(j,k)}(x) = r_{(j,k)}(x) + r_{(j)} \left( \frac{1}{2} \right) r_{(k)}(x) \]
on \( S(j+k) \), and the relation\n\[ r_{(j,k)}(0) = -r_{(j)}(k) r_{(k)}(0). \]
Note that \( C_{(j,k)} \) is constant on \( S(j+k) \) iff\n\[ s_{(j,k)}(x) = -r_{(j)} \left( \frac{1}{2} \right) s_{(k)}(x) \]
on \( S(j+k) \), where\n\[ s_I(x) = r_I(x) - r_{\left( \frac{1}{2} \right)} s_{I_1}(x) \]
In particular, \( s_{(k,1)} = 0 \) on \( S(k+1) \).

In terms of \( s_I \), the condition (4.7) is equivalent to the condition that the polynomial\n
\[ s_{(I_1, \ldots, I_m)}(x) + R_{(I_1)} \cdot s_{(I_2, \ldots, I_m)}(x) + \cdots + R_{(I_1, \ldots, I_{m-1})} \cdot s_{(I_m)}(x) \]
vanishes on \( S(|I|) \). For instance, for compositions with three entries, (4.13) states that\n\[ s_{(i,j,k)}(x) = -r_{(i)} \left( \frac{1}{2} \right) s_{(j,k)}(x) - \left[ r_{(i,j)} + r_{(i)} r_{(j)} \right] \left( \frac{1}{2} \right) s_{(k)}(x) \]
on \( S(i+j+k) \). This generalizes (4.11).

(4.11) implies that \( s_{(j,k)} \) vanishes on \( S(k) \) and\n\[ s_{(j,k)} \left( -\frac{1}{2} - k \right) = -r_{(j)} \left( \frac{1}{2} \right) s_{(k)} \left( -\frac{1}{2} - k \right). \]
The latter relation is a special case of\n
\[ s_{(J,k)} \left( -\frac{1}{2} - k \right) = -r_J \left( \frac{1}{2} \right) \cdot s_{(k)} \left( -\frac{1}{2} - k \right) \]
which holds true for all compositions \( J \) and all \( k \geq 2 \). (4.15) is a formula for the value of \( r_{(J,k)} \) at the largest half-integer in the set \( \frac{1}{2} - N_0 \) for which this value differs from \( r_{(j,k)}(\frac{1}{2}) \). It is a consequence of (4.13).

Finally, we note that the values of \( r_I \) at \( x = -I_{\text{last}} \) satisfy the third system of multiplicative recursive relations \n\[ r_{(J,k,j)}(-j) = -r_J(k) \cdot r_{(k,j)}(-j) \]
for all \( j, k \geq 1 \) and all compositions \( J \). We summarize both relations (4.8) and (4.16) in
\[
(4.17) \quad r_{(j,k,j)}(-j) = -r_{j}(k) \cdot r_{(k,j)}(-j)
\]
for all \( j \geq 0, k \geq 1 \) and all \( J \). Here we use the convention \( r_{(I,0)} = r_I \). With the additional convention \( r_{(0)} = -1 \), (4.17) makes sense also for \( J = (0) \).

4.2. Examples. In the present section, we explicate and confirm Conjecture 4.1 in a number of important special cases.

4.2.1. The polynomials \( r_I \) for \( |I| \leq 4 \). We determine the polynomials \( r_I \) which are responsible for the coefficients in the universal recursive formulae for \( Q_{2N} \), \( N \leq 5 \). These are the polynomials \( r_I \) for all compositions \( I \) of size \( |I| \leq 4 \).

Example 4.2. We consider the polynomials \( r_I \) for compositions \( I \) of size \( |I| \leq 2 \). First of all, \( r_{(1)} = 1 \). The polynomials \( r_{(1,1)} \) and \( r_{(2)} \) for compositions \( I \) of size \( |I| = 2 \) are listed in Table 6.1. They are characterized as follows by their properties. Both polynomials are of degree \( 2|I| - 1 = 3 \) and satisfy the respective relations
\[
\begin{align*}
  r_{(1,1)} \left( \frac{1}{2} \right) &= r_{(1,1)} \left( -\frac{1}{2} \right) = r_{(1,1)} \left( -\frac{3}{2} \right) = -\frac{5}{4}, \\
  r_{(1,1)}(-2) &= 0,
\end{align*}
\]
and
\[
\begin{align*}
  r_{(2)} \left( \frac{1}{2} \right) &= r_{(2)} \left( -\frac{1}{2} \right) = r_{(2)} \left( -\frac{3}{2} \right) = -\frac{1}{4}, \\
  r_{(2)}(-1) &= 0
\end{align*}
\]
(see Table 6.3). The values \( -\frac{5}{4} \) and \( -\frac{1}{4} \) are given by
\[
-\frac{5}{4} = (-2)^{-1} \left( \frac{1}{2} + 2 \right)_1 \quad \text{and} \quad -\frac{1}{4} = (-2)^{-1} \left( \frac{1}{2} \right)_1,
\]
respectively (see (6.10)). Alternatively, the value of \( r_{(1,1)} \) on the set \( S(2) \) is determined by the recursive relation
\[
r_{(1,1)}(0) = -r_{(1,1)}(1) \cdot r_{(1)}(0) = -1
\]
(see (4.8)) for its constant term. Similarly, the value of \( r_{(2)} \) on the set \( S(2) \) can be determined by the relation \( r_{(2)}(0) = -\frac{1}{2} \) (see (4.2)).

Example 4.3. The polynomials
\[
r_{(1,1,1)}, r_{(1,2)}, r_{(2,1)}, r_{(3)}
\]
for compositions \( I \) of size \( |I| = 3 \) are listed in Table 6.2. These four polynomials of degree 5 are determined as follows by their properties. First of all, \( r_{(3)} \) and \( r_{(2,1)} \) are characterized by their respective zeros in \( x = -1, -2 \) and \( x = -2, -3 \), and their respective values
\[
\frac{3}{32} = (-2)^{-2} \left( \frac{1}{2} \right)_2 \quad \text{and} \quad \frac{7}{16} = (-2)^{-2} \left( \frac{1}{2} \right)_1 \left( \frac{3}{2} \right)_1.
\]
on the set \( S(3) \) (see (5.10) and Table 6.6). Alternatively, \( r(3) \) is constant on \( S(3) \), and the value of the constant is determined by its constant term \( r(3)(0) = \frac{1}{2} \) (see (4.2)). The values of \( r(2,1) \) on \( S(3) \) are determined by the constancy of

\[
C_{(2,1)} = r(2,1) + r(2) \left( \frac{1}{2} \right) r(1) = r(2,1) - \frac{1}{4}
\]
on this set, and the relation

\[
r(2,1)(0) = -r(2)(1) \cdot r(1)(0) = -1
\]
(see (4.8)). Similarly, \( r(1,2) \) is characterized by its zeros in \( x = -1, -3 \), the constancy of

\[
C_{(1,2)} = r(1,2) + r(1) \left( \frac{1}{2} \right) r(2) = r(1,2) + r(2),
\]
on the set \( S(3) \), and the relation

\[
r(1,2)(0) = -r(1)(2) \cdot r(2)(0) = -r(2)(0)
\]
(see (4.8)). These are special cases of Example 4.4. Finally, \( r(1,1,1) \) has zeros in \( x = -2, -3 \), \( C_{(1,1,1)} \) is constant on \( S(3) \), i.e., \( r(1,1,1) + r(1,1) \) is constant on \( S(3) \), and

\[
r(1,1,1)(0) = -r(1,1)(1) \cdot r(1)(0) = -r(1,1)(1)
\]
(see (4.8)).

**Example 4.4.** The polynomials \( r_I \) for compositions \( I \) of size \( |I| = 4 \) are listed in Table 6.3. We characterize these eight degree 7 polynomials in terms of their properties. Their values on \( S(4) \) are displayed in Table 6.7. First of all, the interpolation polynomial \( r(4) \) is defined as in (1.3). A special case of Example 4.1 yields a characterization of \( r(3,1) \). In particular, \( s(3,1) = 0 \) on \( S(4) \). Note also that \( r(4) \) and \( r(3,1) \) coincide with the averages \( \sigma_{(4,4)} \) and \( \sigma_{(3,1)} \) (see (4.4)). These polynomials can be characterized as in Section 6.3 by their zeros and their values on the set \( S(4) \). The polynomials \( r(2,2) \) and \( r(1,3) \) are also covered by Example 4.1. The central facts are that \( C_{(2,2)} \) and \( C_{(1,3)} \) are constant on \( S(4) \). We recall that this is equivalent to

\[
s_{(2,2)} = -R(2) \cdot s(2) \quad \text{and} \quad s_{(1,3)} = -R(1) \cdot s(3)
\]
on \( S(4) \). Next, the polynomials \( r_{(2,1,1)} \) and \( r_{(1,2,1)} \) both have zeros in \( \{-2, -3, -4\} \). Moreover, the functions

\[
C_{(2,1,1)} = r_{(2,1,1)} + R(2) \cdot r_{(1,1,1)} + R(2,1) \cdot r(1)
\]
and

\[
C_{(1,2,1)} = r_{(1,2,1)} + R(1) \cdot r_{(2,1,1)} + R(1,2) \cdot r(1)
\]
are constant on the set \( S(4) \) (see (4.5)), and we have the recursive relations

\[
r_{(2,1,1)}(0) = -r_{(2,1)}(1) \cdot r_{(1)}(0) \quad \text{and} \quad r_{(1,2,1)}(0) = -r_{(1,2)}(1) \cdot r_{(1)}(0)
\]
for the constant terms (see (4.8)). Note that \( C_{(2,1,1)} \) and \( C_{(1,2,1)} \) are constant on \( S(4) \) iff

\[
s_{(2,1,1)} = -R(2) \cdot s_{(1,1)} \quad \text{and} \quad s_{(1,2,1)} = -R(1) \cdot s_{(2,1)}
\]
on $S(4)$, respectively (see (4.13)). Similar arguments apply to $r_{(1,1,1,1)}$ and $r_{(1,1,2)}$. These polynomials vanish on the respective sets
\[ \{-2, -3, -4\} \quad \text{and} \quad \{-1, -3, -4\}, \]
the functions
\[ C_{(1,1,2)} = r_{(1,1,2)} + R_{(1)} \cdot r_{(1,2)} + R_{(1,2)} \cdot r_{(2)} \]
and
\[ C_{(1,1,1,1)} = r_{(1,1,1,1)} + R_{(1)} \cdot r_{(1,1,1)} + R_{(1,1,1)} \cdot r_{(1)} \]
are constant on $S(4)$, and
\[ r_{(1,1,1,1)}(0) = -r_{(1,1,1)}(1) \cdot r_{(1)}(0) \quad \text{and} \quad r_{(1,1,2)}(0) = -r_{(1,1)}(2) \cdot r_{(2)}(0) \]
(see (4.8)). Note that $C_{(1,1,2)}$ and $C_{(1,1,1,1)}$ are constant on $S(4)$ iff
\[ s_{(1,1,2)} = -R_{(1)} \cdot s_{(1,2)} \quad \text{and} \quad s_{(1,1,1,1)} = -R_{(1)} \cdot s_{(1,1,1)} - R_{(1,1,1)} \cdot s_{(1,1)}, \]
respectively (see (4.13)). The listed properties of $s_I$ and $r_I$ can be easily verified using Tables 6.5 – 6.7 and Tables 6.9 – 6.11. Here we use $R_{(1)} = -\frac{1}{4}$ and $R_{(1,1,1)} = \frac{1}{32}$.

These results can be used to confirm Conjecture 4.1 for the coefficients in the universal formulae for $Q_{2N}$ for $N \leq 5$. For the calculations of the values of the polynomials $r_I$ we apply the formulae in Table 6.1 – Table 6.3.

**Example 4.5.** By (4.9), the three coefficients in the formula (1.8) for $Q_6$ are given by
\[ a_{(1)}^{(3)} = \frac{2}{3} \cdot r_{(1)}(2) = \frac{2}{3}, \]
\[ a_{(1,1)}^{(3)} = \frac{2}{3} \cdot r_{(1,1)}(1) = \frac{2}{3} \cdot \left( -\frac{5}{2} \right) = -\frac{5}{3}, \]
\[ a_{(2)}^{(3)} = \frac{2}{3} \cdot r_{(2)}(1) = \frac{2}{3}. \]

**Example 4.6.** By (4.9), the seven coefficients in the formula (1.10) for $Q_8$ are given by the following formulae. First of all,
\[ a_{(1)}^{(4)} = \frac{3}{5} \cdot r_{(1)}(3) = \frac{3}{5}. \]

Next,
\[ a_{(1,1)}^{(4)} = \frac{3 \cdot 2}{5 \cdot 3} \cdot r_{(1,1)}(2) = -4, \]
\[ a_{(2)}^{(4)} = \frac{3 \cdot 2}{5 \cdot 3} \cdot r_{(2)}(1) = \frac{2}{5} \cdot \frac{17}{2} = \frac{17}{5} \]
and
\[ a_{(2,1)}^{(4)} = \frac{3 \cdot 2}{5 \cdot 3} \cdot r_{(2,1)}(1) = \frac{2}{5} \cdot \frac{14}{2} = \frac{28}{5}, \]
\[ a_{(2,2)}^{(4)} = \frac{3 \cdot 2}{5 \cdot 3} \cdot r_{(2,2)}(2) = \frac{2}{5} \cdot \frac{17}{2} = \frac{17}{5}. \]
Finally,
\[ a_{(1,1,1)}^{(4)} = \frac{3 \cdot 2}{5 \cdot 3} \cdot r_{(1,1,1)}(1) = -\frac{22}{5} \quad \text{and} \quad a_{(1,2)}^{(4)} = \frac{3 \cdot 2}{5 \cdot 3} \cdot r_{(1,2)}(1) = \frac{8}{5}. \]

**Example 4.7.** The fifteen coefficients in the universal recursive formula for \( Q_{10} \) (see Section 6.1) are determined by the values of the polynomials \( r_I \) with \(|I| \leq 4\) at certain integers. In particular,
\[ a_{(1,3)}^{(5)} = 4! \cdot \frac{105}{2} \cdot r_{(1,3)}(1) = -\frac{69}{35}, \]
\[ a_{(2,1)}^{(5)} = 4! \cdot \frac{105}{2} \cdot r_{(2,1)}(2) = \frac{176}{5}, \]
and
\[ a_{(2,2)}^{(5)} = \frac{12}{35} \cdot r_{(2)}(3) = \frac{312}{35}. \]

Similar straightforward calculations reproduce the remaining twelve coefficients.

4.2.2. *Some closed formulae.* For some compositions, Conjecture 4.1 allows to derive closed formulae for the coefficients in the universal recursive formulae. Here we discuss such formulae for the coefficients of the extreme contributions \( P_{2}(Q_{2N-2}) \) and \( P_{2N-2}(Q_{2}) \).

**Lemma 4.1.** Under Conjecture 4.1,
\[ a_{(1)}^{(N)} = \alpha_{(1)}^{(N)} = \frac{N-1}{2N-3} \quad \text{and} \quad a_{(N-1)}^{(N)} = (-1)^{N-1} \frac{N-1}{2N-3} (2N-5) \]
for \( N \geq 2 \).

**Proof.** The first formula follows from \( r_{(1)} = 1 \). The second claim is a consequence of the Lagrange representation of \( r_{(N-1)} \). By (4.9),
\[ (4.18) \quad a_{(N-1)}^{(N)} = \prod_{i=1}^{N-1} \left( \frac{N-i}{2N-2i-1} \right) r_{(N-1)}(1), \]
where the polynomial \( r_{(N-1)} \) is characterized by (4.3). Now by Lagrange’s formula,
\[ r_{(k)}(x) = (-2)^{-(k-1)} \frac{\frac{1}{2}(k-1)}{(k-1)!} \sum_{i=0}^{k} \prod_{j=0, j \neq i}^{k} \left( \frac{x + j - \frac{1}{2}}{j - i} \right)^{k-1} \prod_{j=1}^{k} \left( \frac{x + j}{j + 1 - i} \right). \]
Hence
\[ r_{(k)}(1) = (-2)^{-(k-1)} \frac{\frac{1}{2}(k-1)}{(k-1)!} \sum_{i=0}^{k} \prod_{j=0, j \neq i}^{k} \left( \frac{j + \frac{1}{2}}{j - i} \right)^{k-1} \prod_{j=1}^{k} \left( \frac{j + 1}{j + \frac{1}{2} - i} \right). \]
A calculation shows that the latter formula is equivalent to
\[ r_{(k)}(1) = (-1)^{k+2} 2^{-(2k-3)} \frac{(2k-3)!!}{(k-1)!} \sum_{i=0}^{k} (2i-1) \left( \frac{2k+1}{2i+1} \right). \]
It follows that

\[ r_{(N-1)}(1) = (-1)^{N-1}2^{-(2N-3)} \frac{(2N-5)!!}{(N-2)!} \sum_{i=0}^{N-1} (2i-1) \frac{(2N-1)}{2i+1}. \]

Hence by (4.18),

\[ a^{(N)}_{(N-1)} = (-1)^{N-1}2^{-(2N-3)} \frac{N-1}{2N-3} \sum_{i=0}^{N-1} (2i-1) \frac{(2N-1)}{2i+1}, \]

i.e., the assertion is equivalent to

\[ \sum_{i=0}^{N-1} (2i-1) \frac{(2N-1)}{2i+1} = (2N-5)2^{2N-3}. \]

The latter identity follows by subtracting

\[ 2 \sum_{i=0}^{N-1} \frac{(2N-1)}{2i+1} = 2^{2N-1} \]

from half of the difference of

\[ \sum_{i=0}^{2N-1} i \frac{(2N-1)}{i} = (2N-1)2^{2N-2} \quad \text{and} \quad \sum_{i=0}^{2N-1} (-1)^i i \frac{(2N-1)}{i} = 0. \]

The proof is complete. \( \square \)

4.2.3. On the multiplicative relations for the constant terms. The first system of multiplicative recursive relations concerns the values of the polynomials \( r_I \) on the set \( S(|I|) \) of half-integers. Their role was already exemplified in Examples 4.2–4.4. The second and the third system of multiplicative recursive relations concern the values of the polynomials \( r_I \) at \( x = 0 \) and \( x = -I_{\text{last}} \). The constant terms satisfy the relations

(4.19) \[ r_{(J,k)}(0) = -r_{J}(k) \cdot r_{(k)}(0). \]

Example 4.8. We use (4.19) to determine the constant values \( r_I(0) \) of the polynomials \( r_I \) for all compositions \( I \) with \( |I| = 5 \). These values are listed in Table 6.12. From this table it is evident that the values \( -r_{(J,1)}(0) \) with \( |J| = 4 \) coincide with the values which are listed in Table 6.11 for \( x = 1 \). Similarly, the values \( r_{(J,2)}(0) \) with \( |J| = 3 \) easily follow from the values in Table 6.10 for \( x = 2 \) using \( r_{(2)}(0) = -\frac{1}{2} \) and (4.19). Finally, the values \( r_{(J,3)}(0) \) with \( |J| = 2 \) follow from the values in Table 6.9 for \( x = 3 \) using \( r_{(3)}(0) = \frac{1}{2} \) and (4.19).

5. Further comments

The treatment of \( Q \)-curvatures in the present paper suggests a number of further studies. Some of these are summarized in the following.

Of course, the main open problems are Conjecture 3.1 and Conjecture 4.1.

The proposed universal recursive formulae for \( Q \)-curvatures involve respective lower order \( Q \)-curvatures and lower order GJMS-operators. These formulae can
be made more explicit by combining them with formulae for GJMS-operators. For the discussion of recursive formulae for these operators (as well as alternative recursive formulae for $Q$-curvatures) we refer to [26].

All recursive formulae for $Q$-curvatures involve a term which is defined through a power of the Yamabe operator $\bar{P}_2$. Its structure remains to be studied.

In Section 3, the universality of the recursive formulae for $Q_4$, $Q_6$ and $Q_8$ was proved (for locally conformally flat metrics) by comparing two formulae for the respective quantities $\bar{Q}_{2N}^k(-\frac{3}{2} + N)$, $N = 2, 3, 4$. This method deserves a further development. In fact, it should yield a full proof of the universality. Along this way, computer assisted calculations confirm the universality (in the locally conformally flat category) for not too large $N$.

Through Conjecture 4.1 the coefficients in the recursive formulae for $Q$-curvatures are linked to interpolation polynomials $r_I$ which are characterized by their values on integers and half-integers in $[-|I|, 1]$. A conceptual explanation of that description is missing.

The polynomials $r_I$ should be explored systematically. In particular, the identity (4.10) and the properties of the averages $\sigma_{(k,j)}$ formulated in Section 6.3 remain to be proved.

The coefficients $\alpha_j^{(N)}$ are expected to have a nice generating function $G$ (see (3.25)). Can one phrase the structure of the polynomials $r_I$ in terms of generating functions, too? In particular, it seems to be natural to study the generating function

$$ Q(x; y) = \sum_{0 \leq |I| \leq N-1} \frac{(2N-3)!!}{(2N-2)!!} a_I^{(N)} x^I y^{N-1-|I|}, \quad x = (x_1, x_2, \ldots). $$

This function refines $G$. In fact, for $x = \text{diag}(x) = (x, x, \ldots)$, (3.19) and (3.25) imply

$$ Q(\text{diag}(x); y) = -G(-x, y). $$

Under Conjecture 4.1 $Q(\cdot; \cdot)$ can be expressed in terms of the polynomials $r_I$. A calculation shows that

$$ Q(x; y) = \sum_I \frac{1}{2^{|I|}} \left( \sum_{N \geq 0} \left( \frac{1}{2} \right)_N r_I(N+1) \frac{y^N}{N!} \right) x^I. $$

6. Appendix

In the present section, we describe part of the numerical data which led to the formulation of Conjecture 3.1 and Conjecture 4.1. We start with explicit versions of the universal recursive formulae for $Q_{2N}$ with $N = 5, \ldots, 8$. Then we describe a test of the universality of the recursive formulae for round spheres. We display the polynomials $r_I$ and their values on integers and half-integers for compositions $I$ with $|I| \leq 5$. Finally, we formulate some remarkable properties of the averages of the polynomials $r_I$ over certain sets of compositions.
6.1. Explicit formulae for $Q_{2N}$ for $N \leq 8$. Explicit versions of the universal recursive formulae for $Q_{2N}$ for $N = 2, 3, 4$ were given in Section 1. Here we add the corresponding universal recursive formulae for $Q_{10}$, $Q_{12}$, $Q_{14}$ and $Q_{16}$. These formulae are generated by the algorithm of Section 3, i.e., the displayed formulae for higher order $Q$-curvatures $Q_{2N}$ are to be understood in the sense of Conjecture 3.1, stating that the generated expressions coincide with $Q$-curvature.

In dimension $n = 10$, the algorithm yields the following formula for $Q_{10}$ with 16 terms.

$$
\begin{align*}
\frac{4}{9} P_2(Q_8) &- \frac{66}{35} P_2^2(Q_6) - \frac{184}{35} P_2^3(Q_4) - \frac{2012}{35} P_4^4(Q_2) + \frac{312}{35} P_4(Q_6) - \frac{908}{35} P_4^2(Q_2) - \\
&- \frac{156}{35} P_6(Q_4) + \frac{9}{5} P_8(Q_2) + \frac{76}{5} P_2 P_4(Q_4) - \frac{69}{35} P_2 P_6(Q_2) + \frac{176}{5} P_2^2 P_4(Q_2) + \frac{176}{5} P_4 P_2(Q_4) + \\
&+ \frac{356}{35} P_2 P_2^2(Q_2) - \frac{554}{35} P_6 P_2(Q_2) + \frac{556}{35} P_4 P_2 P_2(Q_2) + \frac{128}{35} i^* P_4^2(Q_2).
\end{align*}
$$

The derivation of this formula assumes, in particular, that $Q_{10}$ for $Q_8$ holds true in dimension $n = 10$. By Proposition 3.2, this assumption is satisfied for conformally flat metrics. Hence the above formula is proved for such metrics (Corollary 3.1). Conjecture 3.1 states that the formula is universally true for $n \geq 10$.

Next, the algorithm yields the following conjectural formula for $Q_{12}$ in dimension $n = 12$ with 32 terms.

$$
\begin{align*}
\frac{5}{9} P_2(Q_{10}) &- \frac{1180}{63} P_2^2(Q_8) - \frac{442}{63} P_2^3(Q_6) - \frac{38312}{63} P_2^4(Q_4) - \frac{8360}{9} P_2^5(Q_2) + \frac{1150}{63} P_4(Q_8) - \\
&- \frac{1853}{63} P_4^2(Q_4) - \frac{356}{63} P_6(Q_4) + \frac{13960}{63} P_6(Q_4) - \frac{38}{9} P_{10}(Q_2) + \frac{208}{9} P_2 P_4(Q_6) - \frac{1576}{9} P_2^2 P_4(Q_2) - \\
&- \frac{388}{63} P_2 P_6(Q_2) + \frac{356}{63} P_4 P_2(Q_4) + \frac{3560}{63} P_4 P_2^2(Q_2) + \frac{39016}{63} i^* P_4^2(Q_2) - \\
&- \frac{1116}{63} P_8(Q_2) - \frac{1600}{63} P_2 P_8(Q_2) + \frac{6272}{63} P_8(Q_2) - \frac{2420}{63} P_8(Q_2) + \frac{1632}{63} P_2 P_8(Q_2) + \\
&+ \frac{22160}{63} P_4 P_2 P_2(Q_2) - \frac{1297}{21} P_2 P_2 P_2(Q_2) + \frac{25530}{63} P^2_4 P_2 P_2(Q_2) - \frac{22432}{63} P_4 P_2 P_2(Q_2) - \frac{256}{63} i^* P_4^2(Q_2).
\end{align*}
$$

This formula for $Q_{12}$ was derived under the assumptions that the above formulae for $Q_8$ and $Q_{10}$ hold true in dimension $n = 12$. Computer calculations confirm this assumption for locally conformally flat metrics (see the comment in Section 5). Conjecture 3.1 states that the above formula is universally true for $n \geq 12$.

The following formulae for $Q_{14}$ and $Q_{16}$ contain 64 and 128 terms, respectively. Their generation assumes that the above formulae for $Q_8, \ldots, Q_{12}$ hold true in the respective dimensions 14 and 16. Conjecture 3.1 asserts that these are universal.

$Q_{14}$ is given by

$$
\begin{align*}
\frac{6}{33} P_2(Q_{12}) &- \frac{1058}{33} P_2^2(Q_{10}) - \frac{14140}{37} P_2^3(Q_8) - \frac{256362}{37} P_2^4(Q_6) - \frac{444680}{37} P_2^5(Q_4) - \\
&- \frac{141658}{37} P_4(Q_{10}) - \frac{327606}{37} P_4^2(Q_8) - \frac{763066}{37} P_4^3(Q_6) - \frac{11260}{37} P_6(Q_4) - \\
&- \frac{141658}{21} P_2 P_6(Q_2) - \frac{2836}{21} P_2 P_6(Q_2) - \frac{576}{21} P_2 P_6(Q_2) - \frac{336}{21} P_2 P_6(Q_2) - \frac{13600}{21} P_2 P_6(Q_2) - \\
&- \frac{99842}{37} P_2^2 P_4(Q_4) - \frac{21594}{37} P_2^2 P_4(Q_4) - \frac{17000}{37} P_2 P_4(Q_4) - \frac{953}{37} P_2 P_6(Q_2) - \frac{113600}{21} P_2 P_6(Q_2) - \\
&- \frac{16578}{21} P_2^2 P_4(Q_4) - \frac{260526}{21} P_2 P_4(Q_4) + \frac{696}{77} P_2 P_4(Q_4) + \frac{54656}{77} P_2 P_4(Q_4) - \frac{121980}{77} P_2 P_4(Q_4) - \\
&- \frac{260526}{231} P_2^2 P_4(Q_4) + \frac{696}{33} P_2^2 P_4(Q_4) + \frac{696}{33} P_2 P_4(Q_4) - \frac{77}{231} P_2 P_4(Q_4).
\end{align*}
$$
\[Q_{16}\text{ is given by}\]
\[
\begin{align*}
P_7(Q_{14}) & = -\frac{7560}{143} P_2^5(Q_0) - \frac{7560}{143} P_2^3(Q_0) - \frac{6400}{299} P_2^3(Q_0) - \frac{1831120}{143} P_4^3(Q_0) - \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0) - \\
& \quad + \frac{429}{77} P_2^5(Q_0) + \frac{429}{77} P_2^3(Q_0) + \frac{429}{77} P_2^3(Q_0) - \frac{2395680}{143} P_4(Q_0) + \frac{3730786}{143} P_4(Q_0) - \\
& \quad - \frac{429}{77} P_2^5(Q_0) - \frac{429}{77} P_2^3(Q_0) - \frac{429}{77} P_2^3(Q_0) + \frac{2395680}{143} P_4(Q_0) - \frac{3730786}{143} P_4(Q_0)
\end{align*}
\]
\[
\begin{align*}
\frac{90070640}{143} P_1P_2P_2P_2(Q_2) + \frac{5624584}{143} P_1P_2P_3P_2(Q_2) - \frac{30578880}{143} P_1P_2P_4P_2(Q_2) + \\
\frac{46081964}{429} P_1P_3P_4P_2(Q_2) - \frac{60646472}{429} P_2P_4P_4P_2(Q_2) - \frac{2648}{429} i^*P_1^2(Q_2).
\end{align*}
\]

6.2. Tests on round spheres. Here we describe some details of a test which confirms the universality of the displayed formulae for \( Q_6, Q_8 \) on the spheres \( S^n \) of arbitrary even dimension \( n \) with the round metric \( h_0 \). Similar tests yield the correct values for \( Q_2N \) for all \( N \leq 10 \). Basically the same calculations cover the case of Einstein metrics. This test also illustrates the role of the terms \( i^*\bar{P}_2^k(\bar{Q}_2) \).

On \((S^n, h_0)\), the GJMS-operators are given by the product formulae

\[
P_{2N} = \prod_{j=\frac{n}{2}}^{\frac{n}{2} + N-1} (\Delta - j(n-1-j))
\]

(6.1)

(5, 2, 13). (6.1) implies

\[
P_{2N}(1) = (-1)^N \prod_{j=\frac{n}{2}}^{\frac{n}{2} + N-1} j(n-1-j).
\]

(6.2)

Using (1.1), i.e.,

\[
P_{2N}(1) = (-1)^N(m-N)Q_{2N}, \quad m = \frac{n}{2},
\]

we find

\[
Q_{2N} = m \prod_{j=1}^{N-1} (m^2 - j^2).
\]

(6.3)

These formulae suffice to determine the first \( 2^{N-1} - 1 \) terms in (3.1). In order to determine the contributions

\[
i^*\bar{P}_{2N-1}(\bar{Q}_2),
\]

we note that \( P = \frac{1}{2}h_0 \), i.e.,

\[
h_r = \left( 1 - \frac{i^2}{2} P \right)^2 = (1 - cr^2)^2h_0
\]

with \( c = \frac{1}{4} \) (by (3.26)). Hence

\[
\bar{P}_2 = \frac{\partial^2}{\partial r^2} - m(1-cr^2)^{-1} \frac{\partial}{\partial r} - m(m-2c)(1-cr^2)^{-1}
\]

on functions which are constant on \( M \). Moreover, we have

\[
\bar{Q}_2 = m(1-cr^2)^{-1}
\]

by (3.27).

Now straightforward calculations yield the results

\[
i^*\bar{P}_2(Q_2) = -2^{-1}m(m-1)(2m+1),
\]

\[
i^*\bar{P}_2^2(\bar{Q}_2) = 2^{-2}m(m-1)(4m^3 - 5m - 6)
\]
On the other hand, a calculation using \(6.2\) and \(6.3\) yields
\[
\frac{1}{3}(-5m^5 + 8m^4 - 5m^3 - 2m^2)
\]
for the sum of the first three terms in the universal formula (1.8) for \(Q_6\). Together with the contribution of \(i^*P_2^2(Q_2)\), we obtain
\[
m^5 - 5m^3 + 4m = (m^2 - 1)(m^2 - 4)
\]
which coincides with \(Q_6\) by \(6.3\).

Another calculation using \(6.2\) and \(6.3\) yields
\[
\frac{1}{5}(-11m^7 + 24m^6 - 34m^5 + 18m^4 + 133m^3 - 130m^2)
\]
for the first seven terms in the universal formula for \(Q_8\). Together with the contribution
\[
i^*P_3^3(Q_2) = -2^{-3}m(m - 1)(8m^5 - 4m^4 - 22m^3 - 31m^2 + 25m + 90)
\]
we find
\[
m^7 - 14m^5 + 49m^3 - 36m = (m^2 - 1)(m^2 - 4)(m^2 - 9)
\]
which coincides with \(Q_8\) by \(6.3\).

By [15], the product formula \((6.1)\) generalizes in the form
\[
P_{2N}(h) = \prod_{j=2}^{n+N-1} \left( \Delta - \frac{\tau(h)}{n(n-1)} j(n-1-j) \right)
\]
to Einstein metrics. In particular,
\[
Q_{2N} = \lambda^N m \prod_{j=1}^{N-1} (m^2 - j^2), \quad \lambda = \frac{\tau}{n(n-1)}.
\]
Moreover, \(P = \frac{1}{2} \lambda h\) and
\[
h_r = \left(1 - \frac{r^2}{2} P\right)^2 = (1-c\lambda r^2 h)^2, \quad c = \frac{1}{4}.
\]
Hence (on functions which are constant on \(M\),
\[
\bar{P}_2(h) = \frac{\partial^2}{\partial r^2} - m\lambda r(1-c\lambda r^2)^{-1} \frac{\partial}{\partial r} - m(m-2c)(1-c\lambda r^2)^{-1}
\]
and
\[
\bar{Q}_2(h) = m\lambda(1-c\lambda r^2)^{-1}.
\]
Therefore, for Einstein \(h\) with \(\tau = n(n-1)\), the same calculations as on round spheres, prove (3.1). For general scalar curvature, the result follows by rescaling. The assertion is trivial for \(\tau = 0\).
6.3. The averages $\sigma_{(k,j)}$. We consider averages of the polynomials $r_I$ over certain sets of compositions $I$ of the same size $|I|$. We speculate that these averages can be described in terms of standard interpolation polynomials.

**Definition 6.1 (Standard interpolation polynomials).** For given integers $M, N$ such that $N - 1 \geq M \geq 0$, let $I_{(M,N)}(x)$ be the interpolation polynomial of degree $2N - 1$ which satisfies

$$I_{(M,N)} \left( \frac{1}{2} - i \right) = 1, \quad i = 0, \ldots, N$$

and

$$I_{(M,N)} (-M - i) = 0, \quad i = 1, \ldots, N - 1.$$

We use the polynomials $r_I$ for compositions $I$ of size $|I| = j$ to define the $j$ averages $\sigma_{(k,j)}$, $1 \leq k \leq j$.

**Definition 6.2 (Averages).** For $j \geq 1$ and $1 \leq k \leq j$, let

$$\sigma_{(k,j)}(x) = \sum_{k+|J|=j} r_{(k,J)}(x).$$

In particular,

$$\sigma_{(1,j)} = \sum_{|J|=j-1} r_{(1,J)},$$

and

$$\sigma_{(j-2,j)} = r_{(j-2,1,1)} + r_{(j-2,2)},$$

$$\sigma_{(j-1,j)} = r_{(j-1,1)},$$

$$\sigma_{(j,j)} = r_{(j)}.$$

Now we expect that the averages $\sigma_{(k,j)}$ are related to the interpolation polynomials $I_{(M,N)}$ through the formula

$$\sigma_{(k,j)}(x) = (-2)^{-(j-1)} \left[ \left( \frac{1}{2} \right)_{k-1} \left( \frac{1}{2} + j \right)_{j-k} \right] \frac{1}{(k-1)!(j-k)!} I_{(j-k,j)}(x).$$

In other words, (6.5) states the equalities

$$\sigma_{(k,j)} \left( \frac{1}{2} \right) = \sigma_{(k,j)} \left( \frac{1}{2} - 1 \right) = \cdots = \sigma_{(k,j)} \left( \frac{1}{2} - j \right),$$

claims that this value coincides with

$$(-2)^{-(j-1)} \left[ \left( \frac{1}{2} \right)_{k-1} \left( \frac{1}{2} + j \right)_{j-k} \right] \frac{1}{(k-1)!(j-k)!},$$

and asserts that

$$\sigma_{(k,j)} \left( -(j - k) - i \right) = 0 \quad \text{for} \quad i = 1, \ldots, j - 1.$$

The $j - 1$ zeros in (6.8) are quite remarkable. In fact, (6.8) states that $\sigma_{(k,k)} = r_{(k)}$ has zeros in $x = -1, \ldots, -(k - 1)$. These are obvious by the definition of
But for \( k < j \), the zeros of \( \sigma_{(k,j)} \) in (6.8) are not obvious from the zeros of the individual terms \( r_I \) defining the sum.

Note that the obvious relation

\[
\sum_{|I|=j} r_I(x) = \sum_{k=1}^j \sigma_{(k,j)}(x)
\]

implies

\[
(6.9) \quad \sum_{k=1}^j \sigma_{(k,j)} \left( \frac{1}{2} \right) = (-1)^{j-1} \frac{(2j-1)!!}{j!} 2^{j-1}
\]

using the conjectural relation (6.10). On the other hand, (6.9) would be consequence of the explicit formula

\[
(6.10) \quad \sigma_{(k,j)} \left( \frac{1}{2} \right) = (-2)^{-(j-1)} \left[ \frac{\left( \frac{1}{2} \right)_{k-1} \left( \frac{1}{2} + j \right)_{j-k}}{(k-1)! (j-k)!} \right].
\]

In fact, comparing the coefficients of \( x^{j-1} \) on both sides of the identity

\[
(1 - x)^{-\frac{1}{2}} (1 - x)^{-(\frac{1}{2} + j)} = (1 - x)^{-(1+j)},
\]

we find

\[
\sum_{k=1}^j \left( \frac{1}{2} \right)_{k-1} \left( \frac{1}{2} + j \right)_{j-k} \frac{(k-1)! (j-k)!}{(k-1)! (j-k)!} = \frac{(j+1)!_{j-1}}{(j-1)!} = \frac{(2j-1)!}{(j-1)! j!} \frac{(2j-1)!!}{j!} 2^{j-1}.
\]

This yields the assertion (6.9).

6.4. The polynomials \( r_I \) for compositions of small size. In Table 6.1 – Table 6.4, we list the polynomials \( r_I \) for all compositions \( I \) with \( 2 \leq |I| \leq 5 \). In each case, we factorize off the zeros in the negative integers. We recall that \( r_{(1)} = 1 \).

6.5. Some values of \( r_I \). In Table 6.5 – Table 6.8, we list the values of \( r_I \) for \( 2 \leq |I| \leq 5 \) on the respective sets \( S(|I|) \) of half-integers. We write all values as perturbations by \( s_I \) of the respective values at \( x = \frac{1}{2} \). From that presentation it is immediate that the averages \( \sigma_{(k,j)} \) are constant on the respective sets of half-integers, and one can easily read off the values of \( s_I \). In Table 6.5 – Table 6.7, we also display some values of \( r_I \) on half-integers \( \notin S(|I|) \). These influence the values of corresponding polynomials for compositions of larger size through the multiplicative recursive relations. In particular,

\[
s_{(2,1)} + s_{(1,2,1)} = 0 \quad \text{and} \quad s_{(3)} + s_{(1,3)} = 0
\]

at \( x = -\frac{7}{2} \), and

\[
s_{(3,1)} + s_{(1,3,1)} = 0 \quad \text{and} \quad s_{(4)} + s_{(1,4)} = 0
\]

at \( x = -\frac{9}{2} \). These are special cases of \( s_{(1,k,1)} + s_{(k,1)} = 0 \) (see (4.14)) and \( s_{(1,k)} + s_{(k)} = 0 \) (see (4.13)).
Table 6.9 – Table 6.12 display the values of $r_I$ for $2 \leq |I| \leq 5$ on the respective sets of integers in $[-|I|, 2]$. One can easily confirm that the values $r_I(0)$ in Table 6.11 are determined by the values of $r_I(1)$ in Table 6.10 and $r_I(2)$ in Table 6.9 according to the relation $r_{(j,k)}(0) + r_{(j,k)}(0) = 0$ (see (4.8)). Similarly, the values $r_I(0)$ in Table 6.12 are determined by the values of $r_I(1)$ in Table 6.11, $r_I(2)$ in Table 6.10 and $r_I(3)$ in Table 6.9 (see Section 4.2.3).
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| $j$ | 9     | 8     | 7     | 6     | 5     | 4     | 3     | 2     | 1     | 0     |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $N = 1$ |       |       |       |       |       |       |       |       | 1     |       |
| $N = 2$ |       |       |       |       |       |       |       |       | 1     |       |
| $N = 3$ |       |       |       |       |       |       |       |       | -1    | $\frac{2}{3}$ |
| $N = 4$ |       |       |       |       |       |       |       |       | 1     | $\frac{5}{7}$ |
| $N = 5$ |       |       |       |       |       |       |       |       | -1    | $\frac{18}{35}$ |
| $N = 6$ |       |       |       |       |       |       |       |       | 1     | $\frac{10}{21}$ |
| $N = 7$ |       |       |       |       |       |       |       |       | -1    | $\frac{10}{21}$ |
| $N = 8$ |       |       |       |       |       |       |       |       | 1     | $\frac{5}{6}$ |
| $N = 9$ |       |       |       |       |       |       |       |       | -1    | $\frac{8}{13}$ |
| $N = 10$ | 1     |       |       |       |       |       |       |       | $\frac{36}{85}$ | $\frac{36}{85}$ |

**Table 3.1.** The coefficients $\alpha^{(N)}_j$ for $N \leq 10$

| $I$       | $r_I$ |
|-----------|-------|
| $(1, 1)$  | $-\frac{1}{6}(2 + x)(3 - 2x + 4x^2)$ |
| $(2)$     | $\frac{1}{6}(1 + x)(-3 + 2x + 4x^2)$ |

**Table 6.1.** $r_I$ for compositions $I$ with $|I| = 2$

| $I$       | $r_I$ |
|-----------|-------|
| $(1, 1, 1)$ | $-\frac{1}{60}(2 + x)(3 + x)(-25 + 2x + 30x^2 + 48x^3)$ |
| $(1, 2)$  | $\frac{1}{30}(1 + x)(3 + x)(5 - 12x + 6x^2 + 16x^3)$ |
| $(2, 1)$  | $\frac{1}{30}(2 + x)(3 + x)(-5 - 6x + 30x^2 + 16x^3)$ |
| $(3)$     | $-\frac{1}{60}(1 + x)(2 + x)(-15 + 2x + 42x^2 + 16x^3)$ |

**Table 6.2.** $r_I$ for compositions $I$ with $|I| = 3$
### Table 6.3. $r_I$ for compositions $I$ with $|I| = 4$

| $I$ | $r_I$ |
|-----|-------|
| $(1, 1, 1, 1)$ | $-\frac{1}{2520}(2 + x)(3 + x)(4 + x)(5 + x)(-1509 - 2140x + 4960x^2 + 8480x^3 + 4024x^4 + 640x^5)$ |
| $(1, 1, 1, 2)$ | $\frac{1}{360}(1 + x)(3 + x)(4 + x)(5 + x)\left(-10143 - 15270x + 34228x^2 + 58952x^3 + 28112x^4 + 4480x^5\right)$ |
| $(1, 1, 2, 1)$ | $\frac{1}{360}(2 + x)(3 + x)(4 + x)(5 + x)\left(-2079 - 3000x + 7000x^2 + 11680x^3 + 5600x^4 + 896x^5\right)$ |
| $(1, 1, 3)$ | $-\frac{1}{6720}(1 + x)(2 + x)(4 + x)(5 + x)\left(-2079 - 2808x + 6744x^2 + 11296x^3 + 5544x^4 + 896x^5\right)$ |
| $(1, 2, 1, 1)$ | $\frac{1}{360}(2 + x)(3 + x)(4 + x)(5 + x)\left(-8127 - 13110x + 26180x^2 + 50840x^3 + 26992x^4 + 4480x^5\right)$ |
| $(1, 2, 2)$ | $\frac{1}{5040}(1 + x)(3 + x)(4 + x)(5 + x)\left(-1197 - 1800x + 3712x^2 + 7208x^3 + 3848x^4 + 640x^5\right)$ |
| $(1, 3, 1)$ | $-\frac{1}{6720}(2 + x)(3 + x)(4 + x)(5 + x)\left(-4347 - 9672x + 9800x^2 + 38960x^3 + 25480x^4 + 4480x^5\right)$ |
| $(1, 4)$ | $\frac{1}{40320}(1 + x)(2 + x)(3 + x)(5 + x)(945 - 1776x - 3680x^2 + 4840x^3 + 4760x^4 + 896x^5)$ |
| $(2, 1, 1, 1)$ | $\frac{1}{360}(2 + x)(3 + x)(4 + x)(5 + x)\left(-14805 - 20172x + 50960x^2 + 79280x^3 + 34048x^4 + 4480x^5\right)$ |
| $(2, 1, 2)$ | $-\frac{1}{11088}(1 + x)(3 + x)(4 + x)(5 + x)\left(-14553 - 20010x + 49828x^2 + 78752x^3 + 33992x^4 + 4480x^5\right)$ |
| $(2, 2, 1)$ | $-\frac{1}{11088}(2 + x)(3 + x)(4 + x)(5 + x)\left(-2043 - 2808x + 6920x^2 + 11120x^3 + 4840x^4 + 640x^5\right)$ |
| $(2, 3)$ | $\frac{1}{2760}(1 + x)(2 + x)(4 + x)(5 + x)\left(-2457 - 3960x + 8520x^2 + 15040x^3 + 6720x^4 + 896x^5\right)$ |
| $(3, 1, 1)$ | $-\frac{1}{90720}(2 + x)(3 + x)(4 + x)(5 + x)\left(-18711 - 24000x + 65240x^2 + 95120x^3 + 37576x^4 + 4480x^5\right)$ |
| $(3, 2)$ | $\frac{1}{10080}(1 + x)(3 + x)(4 + x)(5 + x)\left(-3591 - 4866x + 12716x^2 + 18904x^3 + 7504x^4 + 896x^5\right)$ |
| $(4, 1)$ | $\frac{1}{25920}(2 + x)(3 + x)(4 + x)(5 + x)\left(-4725 - 5736x + 17080x^2 + 22480x^3 + 8120x^4 + 896x^5\right)$ |
| $(5)$ | $-\frac{1}{25920}(1 + x)(2 + x)(3 + x)(4 + x)(-945 - 888x + 3320x^2 + 3760x^3 + 1240x^4 + 128x^5)$ |
\[
I \quad \begin{array}{c|c|c|c|c|c}
& \frac{-7}{2} & \frac{-5}{2} & \frac{-3}{2} & \frac{-1}{2} & \frac{1}{2} \\
(1, 1) & \frac{-5}{4} + 16 & \frac{-5}{4} + 4 & \frac{-5}{4} & \frac{-5}{4} & \frac{-5}{4} \\
(2) & \frac{-1}{4} - 16 & \frac{-1}{4} - 4 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \\
\end{array}
\]

Table 6.5. Values of \( r_I (|I| = 2) \) on \( \frac{1}{2} - N_0 \)

\[
I \quad \begin{array}{c|c|c|c|c|c}
& \frac{-7}{2} & \frac{-5}{2} & \frac{-3}{2} & \frac{-1}{2} & \frac{1}{2} \\
(1, 1, 1) & \frac{49}{32} + 20 & \frac{49}{32} - 4 & \frac{49}{32} & \frac{49}{32} & \frac{49}{32} \\
(1, 2) & \frac{7}{16} - 24 & \frac{7}{16} + 4 & \frac{7}{16} & \frac{7}{16} & \frac{7}{16} \\
(2, 1) & \frac{7}{16} - 8 & \frac{7}{16} & \frac{7}{16} & \frac{7}{16} & \frac{7}{16} \\
(3) & \frac{3}{32} + 12 & \frac{3}{32} & \frac{3}{32} & \frac{3}{32} & \frac{3}{32} \\
\end{array}
\]

Table 6.6. Values of \( r_I (|I| = 3) \) on \( \frac{1}{2} - N_0 \)

\[
I \quad \begin{array}{c|c|c|c|c|c}
& \frac{-9}{2} & \frac{-7}{2} & \frac{-5}{2} & \frac{-3}{2} & \frac{-1}{2} & \frac{1}{2} \\
(1, 1, 1) & \frac{-123}{64} + 314 & \frac{-123}{64} - 16 & \frac{-123}{64} + 5 & \frac{-123}{64} & \frac{-123}{64} & \frac{-123}{64} \\
(1, 1, 2) & \frac{-15}{32} - 290 & \frac{-15}{32} + 20 & \frac{-15}{32} - 5 & \frac{-15}{32} & \frac{-15}{32} & \frac{-15}{32} \\
(1, 2, 1) & \frac{-3}{4} - 136 & \frac{-3}{4} + 8 & \frac{-3}{4} & \frac{-3}{4} & \frac{-3}{4} & \frac{-3}{4} \\
(1, 3) & \frac{-27}{128} + 118 & \frac{-27}{128} - 12 & \frac{-27}{128} & \frac{-27}{128} & \frac{-27}{128} & \frac{-27}{128} \\
(2, 1, 1) & \frac{-15}{32} - 110 & \frac{-15}{32} + 4 & \frac{-15}{32} + 1 & \frac{-15}{32} & \frac{-15}{32} & \frac{-15}{32} \\
(2, 2) & \frac{-39}{128} + 116 & \frac{-39}{128} - 4 & \frac{-39}{128} - 1 & \frac{-39}{128} & \frac{-39}{128} & \frac{-39}{128} \\
(3, 1) & \frac{-27}{128} + 18 & \frac{-27}{128} & \frac{-27}{128} & \frac{-27}{128} & \frac{-27}{128} & \frac{-27}{128} \\
(4) & \frac{-5}{128} - 30 & \frac{-5}{128} & \frac{-5}{128} & \frac{-5}{128} & \frac{-5}{128} & \frac{-5}{128} \\
\end{array}
\]

Table 6.7. Values of \( r_I (|I| = 4) \) on \( \frac{1}{2} - N_0 \)
Table 6.8. Values of \( r_I \) (\( |I| = 5 \)) on \( \frac{1}{2} - \mathbb{N}_0 \)

| \( I \) | \( -\frac{2}{3} \) | \(-\frac{1}{3} \) | \( 0 \) | \( \frac{1}{3} \) | \( \frac{2}{3} \) |
|-------|-----------|-----------|-----|-----|-----|
| \( 1, 1, 1, 1, 1 \) | \( 1155 - \frac{1025}{4} \) | \( 1155 + \frac{41}{2} \) | \( 1155 - \frac{49}{8} \) | \( 1155 - \frac{1}{8} \) | \( 1155 - \frac{11}{8} \) |
| \( 1, 1, 1, 2 \) | \( \frac{99}{128} + \frac{94}{4} \) | \( \frac{99}{128} - \frac{51}{2} \) | \( \frac{99}{128} + \frac{49}{8} \) | \( \frac{99}{128} + \frac{9}{8} \) | \( \frac{99}{128} + \frac{99}{8} \) |
| \( 1, 1, 2, 1 \) | \( \frac{55}{2048} + 110 \) | \( \frac{55}{2048} - 10 \) | \( \frac{55}{2048} \) | \( \frac{55}{2048} \) | \( \frac{55}{2048} \) |
| \( 1, 1, 3 \) | \( \frac{165}{1024} - 95 \) | \( \frac{165}{1024} + 15 \) | \( \frac{165}{1024} \) | \( \frac{165}{1024} \) | \( \frac{165}{1024} \) |
| \( 1, 2, 1, 1 \) | \( \frac{55}{2048} + \frac{205}{2} \) | \( \frac{55}{2048} - 7 \) | \( \frac{55}{2048} - 7 \) | \( \frac{55}{2048} - 7 \) | \( \frac{55}{2048} - 7 \) |
| \( 1, 2, 2 \) | \( \frac{231}{2048} - \frac{217}{2} \) | \( \frac{231}{2048} + 7 \) | \( \frac{231}{2048} + 7 \) | \( \frac{231}{2048} + 7 \) | \( \frac{231}{2048} + 7 \) |
| \( 1, 3, 1 \) | \( \frac{957}{2048} - 18 \) | \( \frac{957}{2048} \) | \( \frac{957}{2048} \) | \( \frac{957}{2048} \) | \( \frac{957}{2048} \) |
| \( 1, 4 \) | \( \frac{55}{2048} + 30 \) | \( \frac{55}{2048} \) | \( \frac{55}{2048} \) | \( \frac{55}{2048} \) | \( \frac{55}{2048} \) |

Table 6.9. The values of \( r_I \) (\( |I| = 2 \)) on \( \{-2, \ldots, 3\} \)

| \( I \) | \(-2\) | \(-\frac{3}{2}\) | \(-1\) | \( \frac{5}{2} \) | \(-10\) | \(-\frac{55}{2}\) |
|-------|------|--------|-----|-----|-----|-----|
| \( 1, 1 \) | 0 | \(-\frac{3}{2}\) | \(-1\) | \( \frac{5}{2} \) | \(-10\) | \(-\frac{55}{2}\) |
| \( 2 \) | \(-\frac{3}{2}\) | 0 | \(-\frac{1}{2}\) | 1 | \( \frac{17}{2}\) | 26 |
Table 6.10. The values of $r_I (|I| = 3)$ on $\{-3, \ldots, 2\}$

| $I$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 |
|-----|------|------|------|---|---|---|
| $(1, 1, 1)$ | 0 | 0 | $\frac{5}{2}$ | $\frac{5}{2}$ | $-11$ | $-161$ |
| $(1, 2)$ | $\frac{5}{2}$ | 0 | $\frac{1}{2}$ | 4 | $\frac{133}{2}$ | |
| $(2, 1)$ | 0 | 0 | 1 | $-1$ | 14 | 154 |
| $(3)$ | $\frac{5}{2}$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{9}{2}$ | $-57$ |

Table 6.11. The values of $r_I (|I| = 4)$ on $\{-4, \ldots, 1\}$

| $I$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 |
|-----|------|------|------|------|---|---|
| $(1, 1, 1, 1)$ | 0 | 0 | 0 | $-\frac{15}{4}$ | 11 | $-\frac{503}{2}$ |
| $(1, 1, 2)$ | 0 | 0 | $-\frac{5}{2}$ | 0 | $-5$ | 110 |
| $(1, 2, 1)$ | 0 | 0 | 0 | $-1$ | $-4$ | 86 |
| $(1, 3)$ | 0 | $-\frac{35}{8}$ | 0 | 0 | $-\frac{1}{2}$ | $-\frac{69}{8}$ |
| $(2, 1, 1)$ | 0 | 0 | 0 | $\frac{3}{2}$ | $-14$ | 235 |
| $(2, 2)$ | 0 | 0 | $-\frac{15}{8}$ | 0 | $\frac{17}{4}$ | $-\frac{227}{2}$ |
| $(3, 1)$ | 0 | 0 | 0 | $-\frac{9}{8}$ | $\frac{9}{2}$ | $-\frac{297}{4}$ |
| $(4)$ | $-\frac{35}{8}$ | 0 | 0 | 0 | $-\frac{5}{8}$ | $\frac{25}{2}$ |
| $I$          | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
|-------------|------|------|------|------|------|-----|
| $(1, 1, 1, 1)$ | 0    | 0    | 0    | 0    | $-\frac{33}{2}$ | $\frac{503}{2}$ |
| $(1, 1, 1, 2)$ | 0    | 0    | $\frac{25}{4}$ | 0    | $-\frac{494}{2}$ |        |
| $(1, 1, 2, 1)$ | 0    | 0    | 0    | 0    | 10   | $-110$ |
| $(1, 1, 3)$   | 0    | 0    | $\frac{35}{8}$ | 0    | $\frac{55}{4}$   |        |
| $(1, 2, 1, 1)$ | 0    | 0    | 0    | 0    | 6    | $-86$  |
| $(1, 2, 2)$   | 0    | 0    | $\frac{15}{8}$ | 0    | $\frac{133}{4}$  |        |
| $(1, 3, 1)$   | 0    | 0    | 0    | 0    | $\frac{9}{8}$    | $\frac{69}{8}$ |
| $(1, 4)$     | 0    | $\frac{63}{8}$ | 0    | 0    | 0    | $\frac{45}{8}$  |
| $(2, 1, 1, 1)$ | 0    | 0    | 0    | 0    | 21   | $-235$ |
| $(2, 1, 2)$   | 0    | 0    | $-\frac{5}{4}$ | 0    | 77   |        |
| $(2, 2, 1)$   | 0    | 0    | 0    | 0    | $-\frac{17}{2}$ | $\frac{227}{4}$ |
| $(2, 3)$     | 0    | 0    | $\frac{7}{2}$ | 0    | 0    | $-13$  |
| $(3, 1, 1)$   | 0    | 0    | 0    | 0    | $-\frac{27}{4}$ | $\frac{297}{4}$ |
| $(3, 2)$     | 0    | 0    | 0    | $\frac{9}{4}$ | 0    | $-\frac{57}{2}$ |
| $(4, 1)$     | 0    | 0    | 0    | 0    | $\frac{3}{2}$ | $-\frac{25}{2}$ |
| $(5)$        | $\frac{63}{8}$ | 0    | 0    | 0    | 0    | $\frac{7}{8}$   |

Table 6.12. The values of $r_I$ ($|I| = 5$) on $\{-5, \ldots, 0\}$