Research Article

Full and Reduced-Order Unknown Input Observer Design for Linear Time-Delay Systems with Multiple Delays

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In this work, we consider the design problem of a full-order and also reduced-order unknown input observers for a particular class of time-delay systems. Asymptotic stability and existence conditions for the designed observers are established. The quadruple-tank process is used as a benchmark to prove the efficiency of the proposed algorithms.

1. Introduction

Time delays emerge in many industrial applications such as hydraulic systems, automotive applications, and telecommunication networks [1–7]. Commonly, delays describe propagation phenomena, energy transfer, or data transmission. Furthermore, it is recognized that delays are the key sources of instability and poor dynamics [8]. Since the sixteen’s century, the research theme of time-delay systems has concerned lots of interest and has been considered as one of the essential research areas in control theory for which many significant research works have been devoted. The topical survey [9] and the recent books [10–12] as well as their related references prove the richness and maturity of this field. Time delays can appear in either the state variables, control inputs, or measurement outputs and the negligence of time delays in the controller design can lead to undesirable dynamics such as oscillations, bifurcations, chaos dynamics, and instability [8]. In this trend, stabilization of linear systems with delays has obtained a huge interest (see, for example, recent works [13–15] and related references). As reported in [16], for stabilization of continuous linear systems subject to time delays, most research works only focus in systems with input delays and solve the problem in the absence of state delays. To the best of the authors’ knowledge, few research works have considered the stabilization problem with multiple delays in states and inputs where the problem is generally solved by assuming fully available state variables [1] or by resorting to output-feedback controllers [17]. The design problem of observer-based stabilization for systems with multiple delays in states and inputs is rarely solved [16, 18]. Until now, this issue of theoretical and practical importance has been remained a challenging problem for systems subject to unknown inputs, uncertainties, and input saturation.

In control theory, modeling and controlling benchmark systems is another fundamental issue. Such systems are considered as challenging control problems, yet, in spite of their uncomplicated arrangement. In this outline, several systems are commonly used in control theory for testing novel control algorithms. We can cite in this framework the three-tank process [19], mass-spring-damper system [20], bouncing ball [21], TORA system [22], hard-disk drive system [23], magnetic levitation system [24], cart-inverted pendulum [25], Furuta pendulum [26], reaction wheel pendulum [27], beam-and-ball [28], two-link flexible manipulator [29], and so on. In the same direction, the well-known quadruple-tank benchmark [30–36] has attracted many attention since it can show elementary notions in estimation and control theory, particularly performance limitation due to the
nonminimum-phase zeros and their output directions for multivariable systems. This system is composed of four coupled tanks, two pumps, and two valves [35]. Many research papers have been devoted to the problem of observer and controller design of this benchmark system without consideration of time delays [37–43] and to the best of our information, no researches have been investigated for unknown input observer design of time-delay quadruple-tank systems.

Motivated by the authors’ early works for linear systems without delays [44], in this paper, we expand the problem of full-order unknown input observer (FOUIO) and reduced-order unknown input observer (ROUIO) design for linear systems with multiple time delays in states and inputs affected by unknown inputs. In this framework, it is important to note that a FOUIO is designed in [45–47] for linear systems with a single delay in the states and two delays in the inputs whereas another FOUIO is designed in [48] for linear systems with a single delay in the states and a single delay in the inputs. For the last works, only the state information is used to design observers in open loops and to the best of our knowledge, there are no related works for which both the state vector and the input vectors are used to design the observer. Furthermore, in our best knowledge, there is not in the vast literature ROUI design approach that attempts to solve such a problem.

The design procedures of both reduced-order and full-order observers are considered here. The key contributions of this work can be recapitulated as follows:

(i) The designed observers’ approaches do not require approximation of the infinite dimensional model of the delayed system in a lumped one as described in our previous works [45, 46].

(ii) Different from [46] where only the state vector is used to construct observers in open loop, in this paper, we construct the novel observers by using both state and input vectors. The structure of the proposed observers allows us to introduce more observer gains that can accomplish better control performance.

(iii) Different from the observer designed in [47] where the delayed system introduces a single delay in the states and a single delay in the inputs, in this paper, multiple time delays in the inputs are considered.

The rest of this paper is organized as follows: the problem statement is considered in Section 2. The full-order observer is designed in Section 3 whereas the reduced-order observer is developed in Section 4. Section 5 is devoted to the application of the proposed approaches to the quadruple-tank benchmark. Finally, conclusions are provided in Section 6.

2. Problem Statement

Consider a class of time-delay systems described by

\[
\dot{x}(t) = A_0x(t) + A_1x(t-\tau_0) + \sum_{i=0}^{k} (B_1u(t-\tau_{i+1})) + Dd(t),
\]

\[y(t) = Cx(t),\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}^m\) is the input vector, \(d(t) \in \mathbb{R}^p\) is the unknown input vector, and \(y(t) \in \mathbb{R}^q\) is the output vector; \(A_0 \in \mathbb{R}^{nxn}, A_1 \in \mathbb{R}^{nxn}, B_1 \in \mathbb{R}^{nxm}, C \in \mathbb{R}^{nxn}, D \in \mathbb{R}^{nxq}\), and \(\tau_i, 0 \leq i \leq k\) are known and constant time delays.

The objective of this paper is to design two different approaches of unknown input observers (UIOs) for system (1):

(1) An FOUIO is designed by

\[
\dot{\tilde{x}}(t) = C\dot{\tilde{x}}(t) + Ly(t) + Jy(t-\tau_0) + \sum_{i=0}^{k} G_i u(t-\tau_{i+1}),
\]

(2) An ROUIO is also designed in this paper for the estimation of the state vector \(x(t)\) by obtaining the estimation of the vector: \(\dot{\tilde{x}}(t) = Fx(t), F \in \mathbb{R}^{mxn}\), where \(\tilde{x}(t) \in \mathbb{R}^n\) is the vector enclosing a part of the system’s states or a combination of them. The dynamics of the ROUIO is

\[
\dot{\tilde{x}}(t) = H\tilde{x}(t) + Ey(t),
\]

where \(\tilde{x}(t) \in \mathbb{R}^n\) and \(\tilde{x}(t) \in \mathbb{R}^n\) are the observer state vector and estimated state vector, respectively. \(N, L, J, F, G_i, i=0,...,k\) are matrices of appropriate dimensions to be calculated such that \(\tilde{x}(t)\) converges asymptotically to \(x(t)\) under the following assumptions:

(A1) \(\text{rank}(D) = q\); (A2) \(\text{rank}(C) = p\); (A3) the pair \((A_0, C)\) is observable (detectable).

Remark 1. For the design procedure of ROUIO, a particular form for the matrix \(C\) can be chosen as \(C = [C_1 \ 0_{pq}]\), where \(C_1 \in \mathbb{R}^{mp}\) is a full rank (nonsingular) matrix. This is not restrictive as long as the matrix \(C\) is full row rank (conditions A2m A5), and there will always be an orthogonal transformation which leads to the equation \(y(t) = [C_1 \ 0] x(t)\).

Either for the FOUIO (2) or the ROUIO (3), system (1) is controlled using the state-feedback controller as

\[u = Kx(t),\]

where \(K \in \mathbb{R}^{mxn}\) is designed using one of the appropriate approaches for system (1), for example, the approach proposed in [1].
3. Design of the Full-Order Observer

**Theorem 2.** Consider system (1) with assumptions (A1-A3) presented above. The FOUIO (2) is convergent, i.e., \( e(t) \to 0 \) as \( t \to \infty \) with an arbitrary convergence rate for any \( x(0), d(t), \) and \( u(t) \), if there exists a positive-definite matrix \( R \in \mathbb{R}^{n \times n} \) verifying the matrix inequality

\[
N^T R + RN < 0
\]

and if the matrix \( P = (I + EC) \in \mathbb{R}^{n \times n} \) fulfills the following conditions:

\[
NP + LC - PA_0 = 0,
\]
\[
JC - PA_1 = 0
\]
\[
G_i - PB_i = 0 \quad \forall i = 0 \cdots k,
\]
\[
PD = 0.
\]

**Proof.** Define the observer reconstruction error vector as the difference between the estimated state \( \hat{x}(t) \) described by (2) and the state vector related to system (1) as

\[
e(t) = \hat{x}(t) - x(t) = \hat{\zeta}(t) - x(t) - Ey(t)
\]

or

\[
\dot{e}(t) = \dot{\zeta}(t) - (I + EC) \dot{x}(t),
\]

Using (1) and (2), the dynamics (2) can be written in the form:

\[
\dot{\zeta}(t) = (PA_0 - K_0 C) \zeta(t) + Ly(t) + Jy(t - \tau_0)
\]

\[+ \sum_{i=0}^{k} G_i u(t - \tau_{i+1}),
\]

\[
\hat{x}(t) = \zeta(t) - Ey(t).
\]

If the pair \((PA_0, C)\) is not observable, the calculation of matrix \(K_0\) is made such that the observer is asymptotically stable if and only if \((PA_0, C)\) is detectable [2]. By extending the approach from [2], we can conclude that the necessary and sufficient conditions to design a stable observer are given by the following theorem.

**Theorem 3** (see [2]). For system (1), the full-order observer (13) exists if and only if (1) \( \text{rank}(CD) = q \) (2) \( \text{rank} \left[ s I - PA_0 \right] = n, (\forall)s \in L, \text{Re}(s) \geq 0 \).

**Proof.** The proof of the theorem is an extension of the approach from [2]. Because system (13) is in the form of a standard observer equation, then the matrix \(K_0\) can be calculated such that the observer (2) is asymptotically stable if and only if the pair \((PA_0, C)\) is observable. In [2], it is proved that the above condition (2) is equivalent with the pair \((PA_0, C)\)– observable or at least detectable (first existence condition of the full-order observer). The second constraint is the condition (1) from Theorem 3; bearing in mind the assumptions A1 and A2, as well as the dimensions of the matrices \(C\) and \(D\), this condition is always fulfilled. Therefore, up to this point in the design of the unknown input observer, the only existence condition of FOUIO is related to the observability or at least detectability of the pair \((PA_0, C)\).

To calculate \(N, L, J, P, G_i\), and \(G_0\), we partition matrices \(A_0, A_1, C, N, P, L,\) and \(J\) as follows:

\[
A_0 = \begin{bmatrix} A_{0(11)} & A_{0(12)} \\ A_{0(21)} & A_{0(22)} \end{bmatrix},
\]
\[
A_1 = \begin{bmatrix} A_{1(11)} & A_{1(12)} \\ A_{1(21)} & A_{1(22)} \end{bmatrix},
\]
\[
C^T = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix},
\]
\[
N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},
\]
\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},
\]

the first term of (6) becomes

\[
N = PA_0 - K_0 C.
\]
\[ L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \]

\[ J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}, \]

where

\[ A_0 \in \mathbb{R}^{p \times n}, \]
\[ A_{0(11)} \in \mathbb{R}^{(n-p) \times (n-p)}, \]
\[ A_{0(12)} \in \mathbb{R}^{(n-p) \times p}, \]
\[ A_{0(21)} \in \mathbb{R}^{p \times (n-p)}, \]
\[ A_{0(22)} \in \mathbb{R}^{p \times p}, \]
\[ A_1 \in \mathbb{R}^{p \times n}, \]
\[ A_{1(11)} \in \mathbb{R}^{(n-p) \times (n-p)}, \]
\[ A_{1(12)} \in \mathbb{R}^{(n-p) \times p}, \]
\[ A_{1(21)} \in \mathbb{R}^{p \times (n-p)}, \]
\[ A_{1(22)} \in \mathbb{R}^{p \times p}, \]
\[ N \in \mathbb{R}^{n \times n}, \]
\[ N_{11} \in \mathbb{R}^{(n-p) \times (n-p)}, \]
\[ N_{12} \in \mathbb{R}^{(n-p) \times p}, \]
\[ N_{21} \in \mathbb{R}^{p \times (n-p)}, \]
\[ N_{22} \in \mathbb{R}^{p \times p}, \]
\[ P \in \mathbb{R}^{n \times n}, \]
\[ P_{11} \in \mathbb{R}^{(n-p) \times (n-p)}, \]
\[ P_{12} \in \mathbb{R}^{(n-p) \times p}, \]
\[ P_{21} \in \mathbb{R}^{p \times (n-p)}, \]
\[ P_{22} \in \mathbb{R}^{p \times p}, \]
\[ L \in \mathbb{R}^{p \times p}, \]
\[ I_1 \in \mathbb{R}^{(n-p) \times (n-p)}, \]
\[ I_2 \in \mathbb{R}^{p \times p}, \]
\[ J \in \mathbb{R}^{p \times p}, \]
\[ J_1 \in \mathbb{R}^{(n-p) \times p}, \]
\[ J_2 \in \mathbb{R}^{p \times p}, \]
\[ C \in \mathbb{R}^{p \times n}, \]

The observer's design is concentrated into the following theorem.

**Theorem 4.** If the rank conditions in Theorem 3 are satisfied, the UIO (2) can be designed for system (1) as follows:

\[ C_1 \in \mathbb{R}^{p \times (n-p)}, \]
\[ C_2 \in \mathbb{R}^{p \times p}. \]

(15)

The observer's design is concentrated into the following theorem.

\[ C_1 \in \mathbb{R}^{p \times (n-p)}, \]
\[ C_2 \in \mathbb{R}^{p \times p}. \]

(15)

The observer's design is concentrated into the following theorem.

\[ C_1 \in \mathbb{R}^{p \times (n-p)}, \]
\[ C_2 \in \mathbb{R}^{p \times p}. \]

(15)

The observer's design is concentrated into the following theorem.

\[ C_1 \in \mathbb{R}^{p \times (n-p)}, \]
\[ C_2 \in \mathbb{R}^{p \times p}. \]

(15)

The observer's design is concentrated into the following theorem.

\[ C_1 \in \mathbb{R}^{p \times (n-p)}, \]
\[ C_2 \in \mathbb{R}^{p \times p}. \]

(15)

The observer's design is concentrated into the following theorem.

\[ C_1 \in \mathbb{R}^{p \times (n-p)}, \]
\[ C_2 \in \mathbb{R}^{p \times p}. \]

(15)
\[ N_{21} P_{12} + N_{22} P_{22} + L_2 C_2 - P_{21} A_{0(12)} - P_{22} A_{0(22)} = 0_{p \times p} \]  

(19)

By using (17), system (19) is divided into the following systems:

\[ N_{12} P_{21} + L_1 C_1 = F_{11}, \]
\[ N_{12} P_{22} + L_1 C_2 = F_{12}, \]
\[ N_{22} P_{21} + L_2 C_2 = F_{22}, \]
\[ N_{22} P_{22} + L_2 C_1 = F_{21}. \]  

(20)

(21)

Working in the hypothesis: the matrix \((C_1 - C_2 P_{22}^{-1} P_{21}) \in \mathcal{R}^{p \times (n-p)}\) is full column rank. By solving systems (20) and (21), we obtain

\[ L_1 = (F_{11} - F_{12} P_{22}^{-1} P_{21})(C_1 - C_2 P_{22}^{-1} P_{21})^+, \]
\[ L_2 = (F_{21} - F_{22} P_{22}^{-1} P_{21})(C_1 - C_2 P_{22}^{-1} P_{21})^+, \]
\[ N_{12} = (F_{12} - (F_{11} - F_{12} P_{22}^{-1} P_{21})(C_1 - C_2 P_{22}^{-1} P_{21})^+ C_2) P_{22}^{-1}, \]
\[ N_{22} = (F_{22} - (F_{21} - F_{22} P_{22}^{-1} P_{21})(C_1 - C_2 P_{22}^{-1} P_{21})^+ C_2) P_{22}^{-1}. \]  

(22)

Now, solving the second equation of (6), we get

\[ J = PA_1 C^+. \]  

(23)

The unknown matrices of the FOUIO (2), i.e., \(N, L, J, P,\) and \(G,\) have been successfully determined with respect to the arbitrarily chosen matrices \(N_{11}\) and \(N_{21}\). According to Theorem 2, the obtained matrix \(N\) should satisfy two constraints: (1) there exists a symmetric and positive-definite matrix \(R \in \mathcal{R}^{n \times n}\) which verifies the LMI (5); (2) the pair \((PA_n, C)\) is observable or at least detectable; the matrix \(N\) has been obtained properly; otherwise, Steps 4–6 are repeated (in a “while” loop) until these conditions are fulfilled. Bearing in mind that there is no risk for infinite “while” loop, there can be concluded that the FOUIO’s design algorithm has no existence conditions, being characterized by lack of a priori restrictions on the class of systems to be considered.

Step 7. The observer described by (2) is completely designed and the time history of the system’s estimated states can be obtained.

4. Design of the Reduced-Order Observer

**Theorem 6.** Consider the LTI multivariable system (1) with multiple delays under the five assumptions (A4–A8) presented above; the ROUIO (3) is asymptotically stable, if and only if there exist symmetric and positive-definite matrix \(R \in \mathcal{R}^{n \times n}\) verifying the LMI:

\[ N^T R + R N < 0, \]  

(24)

with \(\bar{N} = Z^{-1} \bar{N}\), and if there exists a vector \(\bar{\zeta}(t) = Fx(t)\) such that the next conditions are satisfied:

\[ \bar{N} F + T \bar{C} - Z F A_0 = 0, \]
\[ \bar{J} C - Z F A_1 = 0, \]
\[ \bar{C}_i - Z F B = 0 \quad \forall i = 0 \cdots k, \]  

(25)

\[ Z F D = 0, \]
\[ \bar{E} C + H F = I_n; \]

Proof. Define the observer error vector defined as the difference between the estimated vector \(\bar{\zeta}(t)\) described by (3) and the state vector related to system (1) as

\[ \bar{e}(t) = \bar{\zeta}(t) - \bar{\zeta}(t) \]  

(26)
Using (1), (3), and (4), the dynamics of the estimation error becomes
\[ Z\dot{\epsilon} = \underbrace{\mathbf{N} \mathbf{r}}_{(27)} + \left[ \frac{\mathbf{N} \mathbf{F} - \mathbf{Z} \mathbf{F} \mathbf{A}_0 + \mathbf{G} \mathbf{C}}{2} \right] \mathbf{x}(t) \\
+ \left[ \mathbf{J} \mathbf{C} - \mathbf{Z} \mathbf{F} \mathbf{A}_1 \right] \mathbf{x}(t - \tau_0) \\
+ \sum_{i=0}^{k} \left( \left[ \mathbf{C}_i - \mathbf{Z} \mathbf{F} \mathbf{B}_i \right] \mathbf{u}(t - \tau_{i+1}) \right) - \mathbf{Z} \mathbf{F} \mathbf{D} \mathbf{d}(t). \]

where
\[ \mathbf{N} = \mathbf{Z}^{-1}. \]

Let the state estimation error be defined as \( \epsilon(t) = \bar{\mathbf{x}}(t) - \mathbf{x}(t). \) From (1) and (3), one obtains \( \epsilon(t) = (\mathbf{H} - \mathbf{I}_n + \mathbf{F} \mathbf{C})\mathbf{x}(t). \) If \( \mathbf{HF} = \mathbf{I}_n \mathbf{F} \mathbf{C} = 0, \epsilon(t) = 0 \) for any \( \mathbf{x}(t) \) results.

Furthermore, if condition (24) is satisfied then the observer's error dynamics (28) is asymptotically stable. Now, the proof of the Theorem 6 is complete.

To compute the matrices \( \mathbf{F}, \mathbf{Z}, \mathbf{N}, \mathbf{F}, \mathbf{J}, \mathbf{H}, \mathbf{C}, \mathbf{D}, \) and \( \mathbf{E} \) as follows:

\[
\mathbf{A}_0 = \begin{bmatrix} A_{0(11)} & A_{0(12)} \\ A_{0(21)} & A_{0(22)} \end{bmatrix}, \\
\mathbf{A}_1 = \begin{bmatrix} A_{1(11)} & A_{1(12)} \\ A_{1(21)} & A_{1(22)} \end{bmatrix}, \\
\mathbf{C} = \begin{bmatrix} C_1 & 0_{p 	imes q} \end{bmatrix}, \\
\mathbf{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \\
\mathbf{N} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \\
\mathbf{H} = \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}, \\
\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \\
\mathbf{J} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}, \\
\mathbf{F} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \\
\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \]

where

\[ A_{0(11)}, A_{0(12)}, A_{0(21)}, A_{0(22)}, A_{1(11)}, A_{1(12)}, A_{1(21)}, A_{1(22)}, C_1, D_1, D_2, N_{11}, N_{12}, N_{21}, N_{22}, H_{11}, L_1, L_2, J_1, J_2, F_{11}, F_{12}, F_{21}, F_{22}, Z_{11}, Z_{12}, Z_{21}, Z_{22}, \]

are matrices of appropriate dimensions.
existsthematrix if and only if the following conditions are satisfied: assumptions A4-A8; the ROUIO (3) is asymptotically stable.

Theorem 7. Consider the LTI multivariable (1) under the assumptions A4-A8; the ROUIO (3) is asymptotically stable if and only if the following conditions are satisfied: (1) there exists the matrix $\mathbf{R} > 0$ verifying the LMI (24); (2) the matrices $\overline{\mathbf{G}}$, $\overline{\mathbf{N}}$, $\overline{\mathbf{J}}$, $\overline{\mathbf{E}}$, and $\mathbf{H}$ are given by

$$
\overline{\mathbf{G}}_i = [0_{s \times q} \quad \overline{Z}_1 \quad \overline{Z}_2] \quad \forall i = 1 \ldots k,
$$

$$
\overline{\mathbf{N}} = \begin{bmatrix} Z_1 \left( A_{0[10]} - A_{0[13]} \overline{C}_1 \overline{C}_2 \right) & \overline{N}_2 \end{bmatrix},
$$

$$
\overline{\mathbf{L}} = Z_1 A_{0[21]} \overline{C}_1^+,
$$

$$
\overline{\mathbf{J}} = (\text{ZFA}_1) \overline{C}_1^+,
$$

$$
\overline{\mathbf{E}} = \begin{bmatrix} \overline{C}_1^+ \\
\mathbf{I}_p - \overline{C}_1 \left( C_1 \overline{C}_1 \right)^+ C_1 \\
\end{bmatrix},
$$

$$
\overline{\mathbf{H}} = \begin{bmatrix} -\overline{C}_1 \overline{C}_2 & 0_{q \times (s-p)} \\
\mathbf{I}_p - \overline{C}_2 + \overline{C}_1 \left( C_1 \overline{C}_1 \right)^+ C_1 \overline{C}_2 & 0_{p \times (s-p)} \\
\end{bmatrix},
$$

where the matrices $\overline{Z}_2 \in \mathbb{R}^{s \times (s-p)}$ and $\overline{N}_2 \in \mathbb{R}^{s \times (s-p)}$ are arbitrarily chosen, $\overline{C}_1 \in \mathbb{R}^{p \times q}$ and $\overline{C}_2 \in \mathbb{R}^{p \times p}$ are submatrices of the matrix $C$, and $\overline{D}_2 = I_{s \times q} \overline{D}_2$.

Proof. Using (29) and (30), the fourth equation of (25) becomes

$$
Z_1 \overline{D}_2 = 0_{s \times q},
$$

To compute the matrix $Z_1$ from this equation, one can choose

$$
Z_1 = I_{s \times p} + T_1 \overline{C}_1,
$$

with $T_1 \in \mathbb{R}^{s \times p}$ an unknown matrix to be computed. With the notation $\overline{D}_2 = I_{s \times p} \overline{D}_2$, one gets $T_1 = -\overline{D}_2 \overline{C}_1 \overline{D}_2^+$, where $(\overline{C}_1 \overline{D}_2)^+$ is the generalized pseudoinverse of $(\overline{C}_1 \overline{D}_2)$, given by $(\overline{C}_1 \overline{D}_2)^+ = (\overline{C}_1 \overline{D}_2)^T (\overline{C}_1 \overline{D}_2) -1 (\overline{C}_1 \overline{D}_2)^T$. According to the assumption A5 and to the fact that the matrix $C_1$ is full rank, we can conclude that rank$(C_1 \overline{D}_2) = q$, and the generalized pseudoinverse of $(\overline{C}_1 \overline{D}_2)$ can be always obtained.

Equation (33) can be then written as

$$
Z_1 = I_{s \times p} - \overline{D}_2 \overline{C}_1 \overline{D}_2^+ \overline{C}_1,
$$

and the matrix $\overline{Z}_2 \in \mathbb{R}^{s \times (s-p)}$ is chosen arbitrarily.

Now, using the expression of the matrix $Z$, we obtain the matrices $\overline{\mathbf{G}}$, in the third equation of (25). We arbitrarily choose the matrices $\overline{N}_{12}$ and $\overline{N}_{21}$: as a consequence, the matrix $\overline{N}_2$ is arbitrarily chosen. Using this as well as the form of matrix $C$ given in (29) and (30), i.e., $C = [\overline{C}_1 \quad 0_{p \times q}] = [\overline{C}_1 \quad \overline{C}_2]$, the first equation (25) can be written in a form of matrix system as follows:

$$
-Z_{11} A_{0[21]} + \overline{\mathbf{T}}_1 \overline{C}_1 = 0_{q \times q},
$$

$$
\overline{N}_{11} - Z_{11} A_{0[22]} + \overline{\mathbf{T}}_1 \overline{C}_2 = 0_{q \times p},
$$

$$
-Z_{21} A_{0[21]} + \overline{\mathbf{T}}_2 \overline{C}_1 = 0_{(s-q) \times q},
$$

$$
\overline{N}_{21} - Z_{21} A_{0[22]} + \overline{\mathbf{T}}_2 \overline{C}_2 = 0_{(s-q) \times p},
$$

by solving system (35), we obtain

$$
\overline{N}_{11} = Z_{11} \left( A_{0[22]} - A_{0[21]} \overline{C}_1 \overline{C}_2 \right),
$$

$$
\overline{T}_1 = Z_{11} A_{0[21]} \overline{C}_1^+,
$$

$$
\overline{N}_{21} = Z_{21} \left( A_{0[22]} - A_{0[21]} \overline{C}_1 \overline{C}_2 \right),
$$

$$
\overline{T}_2 = Z_{21} A_{0[21]} \overline{C}_1^+;
$$

bearing in mind the fact that $q < p$, it results that $\overline{C}_i$ is a submatrix of the nonsingular matrix $C_1$; from this, one deduces that rank$(\overline{C}_1) = q$, i.e., $\overline{C}_1$ is a full column rank matrix.

Now, solving the second equation of (25), we get

$$
\overline{\mathbf{J}} = (\text{ZFA}_1) \overline{C}_1^+.
$$
Using (29) and (30), the sixth equation (25) can be transformed into a matrix system as
\[
\begin{align*}
\mathbf{E}_1 \mathbf{C}_1 &= I_q, \\
\mathbf{E}_1 \mathbf{C}_2 + H_{11} &= 0_{q \times p}, \\
\mathbf{E}_2 \mathbf{C}_1 &= 0_{p \times q}, \\
\mathbf{E}_2 \mathbf{C}_2 + H_{21} &= I_p;
\end{align*}
\]
from the first and second equations of (38), we obtain \(\mathbf{E}_1 = \mathbf{C}_1^*\) and \(H_{11} = -\mathbf{E}_1 \mathbf{C}_2 = -\mathbf{C}_1^* \mathbf{C}_2\). On the other hand, the matrix \(\mathbf{E}_2\) is calculated from the third equation of (35) as follows:
\[
\mathbf{E}_2 = I_p + T_2 \mathbf{C}_4, \tag{39}
\]
where \(T_2 \in \mathbb{R}^{p \times p}\) is an unknown matrix to be computed. Because \(\mathbf{C}_4 \in \mathbb{R}^{p \times q}\) is a nonsingular matrix and \(\text{rank}(\mathbf{C}_4) = q\), one gets \(\text{rank}(\mathbf{C}_1 \mathbf{C}_4) = q\) i.e., \((\mathbf{C}_1 \mathbf{C}_4)\) is a full column rank matrix. Taking this into account, one successively gets
\[
T_2 = -\mathbf{C}_1 (\mathbf{C}_1 \mathbf{C}_4)^*\quad \text{and} \quad \mathbf{E}_2 = I_p - \mathbf{C}_1 (\mathbf{C}_1 \mathbf{C}_4)^* \mathbf{C}_1.
\]
From the fourth equation of system (38), we obtain \(H_{12} = I_p - \mathbf{C}_2 + \mathbf{C}_1 (\mathbf{C}_1 \mathbf{C}_4)^* \mathbf{C}_4 \mathbf{C}_2\); by concatenation of the matrices \(\mathbf{E}_1, \mathbf{E}_2, H_{11}, H_{12}, H_{21} = 0_{q \times (s-p)}\) and \(H_{22} = 0_{p \times (s-p)}\), the last two equations (31) result. Now, the proof of the Theorem 7 is complete.

**Algorithm 8.** Design of the ROUIO.

**Step 1.** Given system (1), verify the five related assumptions A4-A8.

**Step 2.** The matrix \(\mathbf{F}\) of from (29) is chosen; the matrices \(A_0, A_1, C, D, N, H, \mathcal{E}, \mathcal{F}, \mathcal{E}, \) and \(Z\) are partitioned with (29) and (30). The matrices \(Z_{12}\) and \(Z_{22}\) and thus the matrix \(Z_2\) are arbitrarily chosen; the matrices \(Z_1, \mathcal{C}_i, \forall i = 1 \cdots k\), are obtained via (31).

**Step 3.** The matrices \(N_{12}, N_{22}\) and thus the matrix \(N_2\) are arbitrarily chosen; by solving system (35) with respect to the matrices \(N_{11}, N_{21}, T_1, T_2\) or, directly, by means of (36), these four matrices are obtained; the matrices \(N, J\) and \(L\) are then calculated with (31). Matrices \(\mathcal{E}\) and \(H\) are computed by solving system (38).

**Step 4.** Using the above calculated matrix \(N\), one checks if there exists a symmetric and positive-definite matrix \(R \in \mathbb{R}^{m \times n}\) verifying the LMI (24); if so, the matrix \(N\) has been properly obtained; otherwise, Steps 2–4 are repeated (in a “while” loop) until this condition is fulfilled.

**Step 5.** The observer described by (3) is completely designed and the time history of the system’s estimated states can be obtained.

### 5. Numerical Simulation Results

#### 5.1. The Quadruple-Tank Process

The quadruple-tank process is shown in Figure 1. It is composed of four connected tanks and two pumps that split water into two tanks [30, 31, 49]. The inlet flow of each tank is measured by an electromagnetic flow-meter and regulated by a pneumatic valve whereas the level of each tank \(h_i\) (\(i = 1 \ldots 4\)) is measured by means of a pressure sensor. The process inputs are the input voltages of the pumps \(\psi_1\) and \(\psi_2\) and the output variables are the tank levels \(y_1\) and \(y_2\).

#### 5.2. The Quadruple-Tank Process Model

For practical considerations, we assume that the operating regime of the process is well known that the quadruple-tank process is
affected by delays and that the transport delays are perfectly symmetric. In such a case, the model of the process can be written as case study of model (1) and can be described by [30]

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + B_0 u(t - \tau_2) + B_1 u(t - \tau_3) + D d(t),
\]

\[
y(t) = C x(t),
\]

with \( x = [h_1 \ h_2 \ h_3 \ h_4]^{\top} \) being the state vector, \( u = [\theta_1 \ \theta_2]^{\top} \) being the input vector, and \( y = [y_1 \ y_2]^{\top} \) being the output vector, respectively, and with

\[
A_0 = \begin{bmatrix}
\frac{a_1}{A_1} \sqrt{\frac{g}{2h_{20}}} & 0 & 0 & 0 \\
0 & \frac{a_2}{A_2} \sqrt{\frac{g}{2h_{20}}} & 0 & 0 \\
0 & 0 & \frac{a_3}{A_3} \sqrt{\frac{g}{2h_{20}}} & 0 \\
0 & 0 & 0 & \frac{a_4}{A_4} \sqrt{\frac{g}{2h_{20}}} \\
\end{bmatrix},
\]

\[
B_0 = \begin{bmatrix}
y_1 k_2 & 0 \\
0 & y_2 k_2 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix},
\]

\[
A_1 = \begin{bmatrix}
0 & 0 & \frac{a_1}{A_1} \sqrt{\frac{g}{2h_{20}}} & 0 \\
0 & 0 & 0 & \frac{a_2}{A_2} \sqrt{\frac{g}{2h_{20}}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
(1 - y_1) k_1 \\
(1 - y_2) k_2 \\
0 & 0 \\
\end{bmatrix}.
\]

Using the following numerical values

\[
D_1 = D_2 = D_3 = D_4 = 9.2 \text{ cm},
\]

\[
A_1 = A_2 = A_3 = A_4 = \pi \left( \frac{d_1}{2} \right)^2,
\]

\[
h_{\text{max}} = 50 \text{ cm},
\]

\[
h_{00} = \frac{h_{\text{max}}}{2} = 25 \text{ cm},
\]

\[
g = 981 \text{ cm} \cdot \text{s}^{-2},
\]

\[
d_1 = 0.2 \text{ cm},
\]

\[
a_1 = a_2 = \pi \left( \frac{d_1}{2} \right),
\]

\[
d_3 = 0.9,
\]

\[
a_3 = a_4 = \pi \left( \frac{d_3}{2} \right)^2,
\]

\[
k_1 = 7.39,
\]

\[
k_2 = 6.92,
\]

the matrices from (1) are given by

\[
A_0 = \begin{bmatrix}
-0.0021 & 0 & 0 & 0 \\
0 & -0.0021 & 0 & 0 \\
0 & 0 & -0.0424 & 0 \\
0 & 0 & 0 & -0.0424 \\
\end{bmatrix},
\]

\[
B_0 = \begin{bmatrix}
0.1113 \times y_1 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\]

\[
A_1 = \begin{bmatrix}
0 & 0 & 0.0424 & 0 \\
0 & 0 & 0 & 0.0424 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & (1 - y_1) \times 0.1042 \\
(1 - y_2) \times 0.1042 & 0 \\
\end{bmatrix}.
\]

5.3. Software Implementation of the Full-Order Observer. For simulation results, the following numerical values are chosen: \( x(0) = [-4 4 6 -5], d(t) = 0.3 \sin(2 \pi t) \text{ ft}; D_T = \begin{bmatrix}
-10 & 20 & -12 & 10 \\
-3 \times 10^3 & 2 \times 10^3 & -1 \times 10^3 & 5 \times 10^3 \\
\end{bmatrix}, y_1 = 0.333, y_2 = 0.307, \alpha = 0.442, \) and \( \tau_1 = 5 \text{ s}, \tau_2 = 3 \text{ s}, \tau_3 = 4 \text{ s}. \) The control gain matrix is computed using the approach proposed in [1]

\[
K = \begin{bmatrix}
-0.1603 & -0.1765 & -0.0795 & -0.2073 \\
-0.1977 & -0.1579 & -0.2288 & -0.0772 \\
\end{bmatrix},
\]

For the FOUIO software implementation, one obtains

\[
E = \begin{bmatrix}
-1 & 0 \\
0 & -1 \\
1.264 & -0.032 \\
-1.54 & 0.02 \\
\end{bmatrix}.
\]
\[
P = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1.26 & -0.032 & 1 & 0 \\
-1.54 & -0.020 & 0 & 1
\end{bmatrix},
\]

\[
\bar{G}_0 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0.046 & -0.01 \\
-0.057 & 0.06
\end{bmatrix},
\]

\[
\bar{G}_1 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0.072 \\
0.074 & 0
\end{bmatrix},
\]

\[
N = 10^{13} \begin{bmatrix}
0 & 0 & 2.945 \\
0 & 0 & 0.46 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
L = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
-0.05 & -0.013 \\
-0.06 & 0.08
\end{bmatrix},
\]

\[
J = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
-0.043 & 0.095 \\
-0.025 & 0.017
\end{bmatrix},
\]

\[
Z = 10^{-14} \begin{bmatrix}
0.2220 & -0.0439 & 0.5255 \\
0.1377 & 0.0111 & 0.8992 \\
0 & 0 & 0.9452
\end{bmatrix}
\]

5.4. Software Implementation of the Reduced-Order Observer.

During the first step of the Algorithm 8, we find the matrices and vectors from (3). In this paper, the order is chosen as equal \(n=4\), \(m=2\), \(p=2\), \(q=2\), and \(s=3\) and the constant time delays are chosen as \(\tau_1 = 5s\), \(\tau_2 = 2s\), and \(\tau_3 = 4s\). We obtain

\[
E = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

5.4. Software Implementation of the Reduced-Order Observer.

During the first step of the Algorithm 8, we find the matrices and vectors from (3). In this paper, the order is chosen as equal \(n=4\), \(m=2\), \(p=2\), \(q=2\), and \(s=3\) and the constant time delays are chosen as \(\tau_1 = 5s\), \(\tau_2 = 2s\), and \(\tau_3 = 4s\). We obtain

\[
E = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

6. Conclusion

This study considers the design of two approaches of UIOs for systems with multiple time-delays: a full-order delay-dependent UIO and a reduced-order delay-dependent UIO. The quadruple-tank process was used as a benchmark to prove the effectiveness of the new observer algorithms for the case study of multivariable nonminimum phase systems with multiple delays.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.
Figure 2: FOUIO and ROUIO of the quadruple-tank process.

References

[1] F. El-Haoussi, E. H. Tissir, F. Tadeo, and A. Hamed, “Delay-dependent stabilisation of systems with time-delayed state and control: application to a quadruple-tank process,” *International Journal of Systems Science*, vol. 42, no. 1, pp. 41–49, 2011.

[2] M. Darouach, M. Zasadzinski, and S. J. Xu, “Full-order observers for linear systems with unknown inputs,” *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 39, no. 3, pp. 606–609, 1994.

[3] F. Zhang, S. Zong, X. Wang, and X. Ren, “An Expert PI Controller with Dead Time Compensation of Monitor AGC in Hot Strip Mill,” *Mathematical Problems in Engineering*, vol. 2016, Article ID 3041538, 8 pages, 2016.

[4] H. Li, X. Jing, and H. R. Karimi, “Output-feedback-based $H_{\infty}$ control for vehicle suspension systems with control delay,” *IEEE Transactions on Industrial Electronics*, vol. 61, no. 1, pp. 436–446, 2014.

[5] R. Sanz, P. Garcia, Q.-C. Zhong, and P. Albertos, “Predictor-Based Control of a Class of Time-Delay Systems and Its Application to Quadrotors,” *IEEE Transactions on Industrial Electronics*, vol. 64, no. 1, pp. 459–469, 2017.

[6] S. Bououden, M. Chadli, L. Zhang, and T. Yang, “Constrained model predictive control for time-varying delay systems: Application to an active car suspension,” *International Journal of Control, Automation, and Systems*, vol. 14, no. 1, pp. 51–58, 2016.

[7] W. Cao, Z. Liu, Y. Chang, and A. Szumanowski, “Direct Yaw-Moment Control of All-Wheel-Independent-Drive Electric Vehicles with Network-Induced Delays through Parameter-Dependent Fuzzy SMC Approach,” *Mathematical Problems in Engineering*, vol. 2017, Article ID 5170492, 15 pages, 2017.

[8] O. Boubaker, V. E. Balas, A. Benzaouia, M. Chaabane, M. S. Mahmoud, and Q. Zhu, “Time-Delay Systems: Modeling, Analysis, Estimation, Control, and Synchronization,” *Mathematical Problems in Engineering*, vol. 2017, Article ID 1398904, 3 pages, 2017.
[9] M. S. Mahmoud, “Recent Progress in Stability and Stabilization of Systems with Time-Delays,” *Mathematical Problems in Engineering*, vol. 2017, Article ID 7354654, 25 pages, 2017.

[10] E. Fridman, *Introduction to Time-delay Systems: Analysis and Control*, Springer, Berlin, Germany, 2014.

[11] B. Zhou, *Truncated predictor feedback for time-delay systems*, Springer, Berlin, Germany, 2014.

[12] E. Witrant, E. Fridman, O. Sename, and L. Dugard, *Recent Results on Time-Delay Systems*, vol. 5, Springer International Publishing, 2016.

[13] X.-M. Zhang, Q.-L. Han, A. Seuret, and F. Gouaisbaut, “An improved reciprocally convex inequality and an augmented Lyapunov-Krasovskii functional for stability of linear systems with time-varying delay,” *Automatica*, vol. 84, pp. 221–226, 2017.

[14] W. Qian, M. Yuan, L. Wang, X. Bu, and J. Yang, “Stabilization of systems with interval time-varying delay based on delay decomposing approach,” *ISA Transactions*, vol. 70, 6 pages, 2017.

[15] X. Cai, N. Bekiaris-Liberis, and M. Krstic, “Input-to-state stability and inverse optimality of linear time-varying-delay predictor feedbacks,” *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 63, no. 1, pp. 233–249, 2018.

[16] Q. Liu and B. Zhou, “Extended observer based feedback control of linear systems with both state and input delays,” *Journal of The Franklin Institute*, vol. 354, no. 18, pp. 8232–8255, 2017.

[17] W. Belhaj and O. Boubaker, “MIMO PI Controllers for LTI Systems with Multiple Time Delays Based on ILMs and Sensitivity Functions,” *Mathematical Problems in Engineering*, vol. 2017, Article ID 1241545, 25 pages, 2017.

[18] B. Zhou, Q. Liu, and F. Mazenc, “Stabilization of linear systems with both input and state delays by observer-predictors,” *Automatica*, vol. 83, pp. 368–377, 2017.

[19] R. Dormido, H. Vargas, N. Duro et al., “Development of a web-based control laboratory for automation technicians: the three-tank system,” *IEEE Transactions on Education*, vol. 51, no. 1, pp. 35–44, 2008.

[20] S. Di Cairano, A. Bemporad, I. V. Kolmanovsky, and D. Hrovat, “Model predictive control of magnetically actuated mass spring dampers for automotive applications,” *International Journal of Control*, vol. 80, no. 11, pp. 1701–1716, 2007.

[21] A. C. Luo and R. P. Han, “The dynamics of a bouncing ball with a sinusoidally vibrating table revisited,” *Nonlinear Dynamics*, vol. 10, no. 1, pp. 1–18, 1996.

[22] M. Jankovic, D. Fontaine, and P. V. Kokotovic, “TORA example: cascade- and passivity-based control designs,” *IEEE Transactions on Control Systems Technology*, vol. 4, no. 3, pp. 292–297, 1996.

[23] B. M. Chen, T. H. Lee, and V. Venkataaraman, *Hard disk drive servo systems*, Springer, Berlin, Germany, 2002.

[24] N. F. Al-Muthairi and M. Zribi, “Sliding mode control of a magnetic levitation system,” *Mathematical Problems in Engineering*, vol. 2004, no. 2, pp. 93–107, 2004.

[25] O. Boubaker and R. Iriarte, *The Inverted Pendulum in Control Theory and Robotics: From theory to new innovations*, Institution of Engineering and Technology, Stevenage, UK, 2017.

[26] I. Fantoni and R. Lozano, “Stabilization of the Furuta pendulum around its homoclinic orbit,” *International Journal of Control*, vol. 75, no. 6, pp. 390–398, 2002.

[27] D. J. Block, K. J. Åström, and M. W. Spong, “The Reaction Wheel Pendulum,” *Synthesis Lectures on Controls and Mechatronics*, vol. 1, no. 1, pp. 1–105, 2007.
[44] M. Lungu and R. Lungu, “Full-order observer design for linear systems with unknown inputs,” International Journal of Control, vol. 85, no. 10, pp. 1602–1615, 2012.

[45] S. Ben Warrad and O. Boubaker, “Robust observer for multiple time-delay systems with unknown inputs,” in Proceedings of the 16th International Conference on Sciences and Techniques of Automatic Control and Computer Engineering, (STA ’15), pp. 859–864, Tunisia, 2015.

[46] S. B. Warrad and O. Boubaker, “Full order unknown inputs observer for multiple time-delay systems,” International Journal On Smart Sensing and Intelligent Systems, vol. 9, no. 4, pp. 1750–1775, 2016.

[47] S. Ben Warrad and O. Boubaker, “Design of unknown input observers for linear systems with state and input delays,” in Proceedings of the 15th IEEE International Multi-Conference on Systems, Signals and Devices, Hammamet, Tunisia, 2018.

[48] S. Ben Warrad, O. Boubaker, M. Lungu, and Q. Zhu, “On unknown inputs observer design for linear systems with delays in states and inputs,” in New Trends in Observer-based Control: An Introduction to Design Approaches and Engineering Applications, Elsevier Academic Press, 1st edition, 2019.

[49] W. Belhaj and O. Boubaker, “Multivariable PID Control via LMIs: Performances Assessment,” International Journal on Smart Sensing & Intelligent Systems, vol. 8, no. 4, pp. 1896–1916, 2015.
