Fermionic coherent state path integral for ultrashort laser pulses and transformation to a field theory of coset matrices
(Classical field theory for the self-energy matrices from various Hubbard-Stratonovich transformations)

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12th April 2010

Abstract
A coherent state path integral of anti-commuting fields is considered for a two-band, semiconductor-related solid which is driven by an ultrashort, classical laser field. We describe the generation of exciton quasi-particles from the driving laser field as anomalous pairings of the fundamental, fermionic fields. This gives rise to Hubbard-Stratonovich transformations from the quartic, fermionic interaction to various Gaussian terms of self-energy matrices; the latter self-energy matrices are solely coupled to bilinear terms of anomalous-doubled, anti-commuting fields which are subsequently removed by integration and which create the determinant with the one-particle operator and the prevailing self-energy. We accomplish path integrals of even-valued self-energy matrices with Euclidean integration measure where three cases of increasing complexity are classified (scalar self-energy variable, density-related self-energy matrix and also a self-energy including anomalous doubled terms). According to the driving, anomalous-doubled Hamiltonian part, we also specify the case of a SSB with 'hinge' fields which factorizes the total self-energy matrix by a coset decomposition into density-related, block diagonal self-energy matrices of a background functional and into coset matrices with off-diagonal block generators for the anomalous pairings of fermions. In particular we investigate the transformation from the coset fields of a curved coset space, as the independent field degrees of freedom, to locally 'flat' fields with Euclidean integration measure. This allows to reduce the final path integral to solely 'Nambu'-doubled fields after a saddle point approximation for the density-related self-energy matrices and also allows to derive classical field equations for exciton quasi-particles from various kinds of gradient expansions of the determinant. Despite classical evolution equations for 'Nambu'-doubled terms, one thus incorporates a semi-classical notion with quantum properties, due to the transformation and reduction to self-energy matrices which represent the irreducible parts of a diagrammatic propagation. The derived equations with various kinds of classical self-energy matrices also allow to examine self-induced transparency effects and the area theorem for ultrashort coherent transients of matter in combination with holography and selective Fourier optics.

Keywords: 'Nambu' doubling of fields, Hubbard-Stratonovich transformations to self-energies, coset integration measure, coset decomposition for pair condensates, coherent state path integral, self-induced transparency, ultrashort coherent transients

1st PACS: 03.65.Db, 71.10.-w, 02.20.Qs;
2nd PACS: 42.50.Md, 42.65.Tg, 78.47.jb.

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1 Introduction

1.1 Hubbard-Stratonovich transformation of quartic fields to self-energies

Phenomena of many body physics can be mapped to different kinds of theories which can start from very direct approaches as in computational physics [1]-[4] or which can be based on various approximations. Apart from the YBE-related, integrable systems [5], there is a general lack of constant integrals of motion so that simple solutions of functions do usually not appear in many particle systems. In consequence, one has to rely on approximations, as by choosing a dependence of the self-energy on the irreducible diagrams or Green function parts or as by taking various manners of correlated field operator terms. The chosen approximation of the self-energy or correlated operators has a satisfactory, correct form if the prevailing physical phenomenon is considered within its computation; therefore, different manners of calculations can occur, each approximation being regarded as a particular theory aside from the original, beginning formulation with a fundamental Hamiltonian, Green function and Dyson equation or the original definition of the correlations with the 'full' operators [6]-[15]. We prefer those "approximating theories" as the more appealing kind in many particle physics where one only reduces to a few terms, but still regards the essentials of the corresponding phenomenon.
1.1 Hubbard-Stratonovich transformation of quartic fields to self-energies

In the present paper we investigate a coherent state path integral of fermions or anti-commuting fields, by way of example for a two-band, semiconductor-related solid which is driven by an external, classical electric field [6,16,17]. We apply various kinds of Gaussian transformations from the quartic interaction of Fermi fields to various self-energy matrices. The latter operation or Hubbard-Stratonovich transformation (‘HST’) maps the quartic Fermi-fields of the original coherent state path integral to a bilinear term of anti-commuting fields with linear coupling to a self-energy matrix and a separate Gaussian factor of the corresponding self-energy. In section 2 we define the total Hamiltonian from second quantized, fermionic operators in normal ordering whose time development is determined by a coherent state path integral [16,17]. Since the driving interaction with the classical electric field creates electron-hole pairs, one has to perform an anomalous doubling of anti-commuting fields for the presence of exciton quasi-particles within the original coherent state path integral. In section 3 subsequent HSTs lead to quadratic self-energy terms and to the linear coupling of the anomalous-doubled, bilinear Fermi fields with the self-energy. Three different kinds of HSTs are regarded within sections 3.1 to 3.3 which we abbreviate symbolically in advance:

**Section 3.1**: Auxiliary variables $\sigma(\chi^{(\eta)}_{j}, \bar{x})$ as scalar, real-valued self-energy

\[
\sum_{\mu,s} \chi_{\mu,s}^{*}(\chi^{(\eta)}_{j}, \bar{x}) q_{\mu} \chi_{\mu,s}(\chi^{(\eta)}_{j}, \bar{x}) \overset{\text{HST} \text{ section 3.1}}{\Longrightarrow} \sigma(\chi^{(\eta)}_{j}, \bar{x}) \quad \text{(cf. (3.1a, 3.5a))}
\]

**Section 3.2**: Hermitian self-energy matrix $\hat{\Sigma}_{\nu,s',\mu,s}(\chi^{(\eta)}_{j}, \bar{x}_2; \chi^{(\eta)}_{j}, \bar{x}_1)$ of densities

\[
\chi_{\nu,s'}(\chi^{(\eta)}_{j}, \bar{x}_2) \otimes \chi_{\mu,s}^{*}(\chi^{(\eta)}_{1}, \bar{x}_1) \overset{\text{HST} \text{ section 3.2}}{\Longrightarrow} \hat{\Sigma}_{\nu,s',\mu,s}(\chi^{(\eta)}_{j}, \bar{x}_2; \chi^{(\eta)}_{j}, \bar{x}_1) \quad \text{(cf. (3.9a, 3.13a))}
\]

**Section 3.3**: Self-energy matrix $\hat{\Sigma}_{\nu,s',\mu,s}(\chi^{(\eta)}_{j}, \bar{x}_2; \chi^{(\eta)}_{j}, \bar{x}_1)$ for densities and anomalous pairs

\[
\left( \begin{array}{c}
\chi_{\nu,s'}(\chi^{(\eta)}_{j}, \bar{x}_2) \\
\chi_{\mu,s}^{*}(\chi^{(\eta)}_{j}, \bar{x}_2)
\end{array} \right) \overset{b=1,2}{\BotHST} \left( \begin{array}{c}
\chi_{\mu,s}(\chi^{(\eta)}_{j}, \bar{x}_1) \\
\chi_{\nu,s'}(\chi^{(\eta)}_{j}, \bar{x}_1)
\end{array} \right) \overset{\text{HST} \text{ section 3.3}}{\Longrightarrow} \hat{\Sigma}_{\nu,s',\mu,s}(\chi^{(\eta)}_{j}, \bar{x}_2; \chi^{(\eta)}_{j}, \bar{x}_1) \quad \text{(cf. (3.16, 3.22a))}
\]

In section 3.1 we only transform to scalar self-energy variables with a contour time and spatial dependence, but without regard of any spin or electron-hole band labels. In section 3.2 we extend the HST to a self-energy matrix which replaces dyadic products of Fermi fields for pure density-related terms of field operators. Section 3.3 contains self-energy matrices, which follow from *dyadic products of anomalous-doubled*, anti-commuting fields and which thus consist of density-related, block diagonal matrix parts $\hat{\Sigma}_{\nu,s',\mu,s}(\chi^{(\eta)}_{j}, \bar{x}_2; \chi^{(\eta)}_{j}, \bar{x}_1)$ and also off-diagonal matrix parts $\hat{\Sigma}_{\nu,s',\mu,s}(\chi^{(\eta)}_{j}, \bar{x}_2; \chi^{(\eta)}_{j}, \bar{x}_1)$; the latter substitute anomalous-doubled pairings of Fermi operators [16-19]. After the various HSTs in sections 3.1 to 3.3 and the removal of bilinear Fermi fields by integration, we obtain path integrals with the corresponding self-energy variables and matrices as the independent field degrees of freedom whose first order and second order variation straightforwardly result in saddle point equations and corresponding solutions for fluctuation terms. We emphasize the precise time step order of the time evolution for a normal ordered Hamiltonian throughout the presented article and follow along the lines of the article [20] for a proper time step separation of field operators within coherent state path integrals. Section 3.4 has a more profound kind of HST where one transforms separately density-related and anomalous-doubled terms to corresponding self-energy matrices in combination with a spontaneous symmetry breaking (‘SSB’) and with ‘hinge’ fields [16]; this leads to a factorization of the total self-energy matrix into density-related, block diagonal self-energy matrices and into anomalous-doubled, off-diagonal blocks. The latter are taken into account by coset matrices within a coset decomposition $SO(\mathfrak{n}, \mathfrak{m})/U(\mathfrak{n}) \otimes U(\mathfrak{m})$ of the total self-energy matrix which itself has the detailed structure of a so(\mathfrak{n}, \mathfrak{m}) generator within the orthogonal
group SO(\(N, \mathcal{M}\)) (dimension \(\mathcal{M} = (\mu = 'e', 'h') \times (s = \uparrow, \downarrow) \times (\eta_\mu = \pm) \times (N = t_N/\Delta t) \times (N_x = (L/\Delta x)^d) = 2 \times 2 \times 2 \times N \times N_x\); a factor of two for the electron-hole, spin and contour time metric degrees of freedom, respectively, times the number of time-like and discrete, spatial points defines the relevant dimension \(\mathcal{M}\) within SO(\(N, \mathcal{M}\))/U(\(N\)) \(\otimes\) U(\(N\)). The coset decomposition combined with the factorization of the total self-energy allows for a projection onto the anomalous-doubled field degrees of freedom with the coset matrices whose path integral consists of the density-related self-energy as a background functional or as a solution from a saddle point equation. The coset decomposition involves various transformations, as the appropriate, invariant integration measure and as the transformation to Euclidean integration variables for first, second or higher order variations with classical coset matrices \([22, 23]\). These involved appearing, but straightforward transformations are described in sections 4 to 5 and are important for the appropriate consideration of the invariant integration measure, following from the square root of the coset metric tensor \([16, 21]\).

1.2 The importance of the appropriate, invariant integration measure

In section 4 we attain a coherent state path integral whose remaining field degrees of freedom, adapted by coset matrices for exciton-related parts, possess a nontrivial, non-Euclidean integration measure determined by the coset metric tensor of SO(\(N, \mathcal{M}\))/U(\(N\)) \(\otimes\) U(\(N\)). In analogy we illustrate this problem for a multidimensional integral \(Z[\vec{x}] = \int d[\vec{x}] \sqrt{\det(g(\vec{x}))} \exp \{ i \Lambda[\vec{x}] \} \); (1.4a)

\[ d[\vec{x}] : \text{Euclidean integration measure} \quad d[\vec{x}] = dx^1 \wedge \ldots \wedge dx^n ; \] (1.4b)

\[ (ds)^2 := dx^i \hat{g}_{ij}(\vec{x}) dx^j ; \] (1.5a)

\[ \hat{g}(\vec{x}) := \hat{g}_{ij}(\vec{x}) = \text{metric tensor with covariant indices} ; \] (1.5b)

\[ \hat{g}^{-1}(\vec{x}) := \hat{g}^{ij}(\vec{x}) = \text{inverted metric tensor with contravariant indices} . \] (1.5c)

The transformation to Euclidean variables \(dy^j = dy_j (1.6-1.8)\) is related to the inverse square root of the metric tensor \(\hat{g}^{-1/2}(\vec{x})\) as the appropriate Jacobi matrix where the symmetric property of the metric tensor allows a decomposition into orthogonal matrices \(\hat{O}_{ij}(\vec{x})\) with \(\det(\hat{O}(\vec{x})) = 1\) and real eigenvalues \(\hat{\lambda}^k(\vec{x})\) \((i,j,k,l = 1, \ldots, N)\)

\[ (ds)^2 = dx^i \hat{g}_{ij}(\vec{x}) dx^j = dx^i \hat{O}_{ik}(\vec{x}) \hat{\lambda}^k(\vec{x}) \hat{O}_{kj}(\vec{x}) dx^j = \] (1.6)

\[ = dx^i (\hat{O}^T(\vec{x}) \hat{\lambda}^{1/2}(\vec{x}))^k_i (\hat{\lambda}^{1/2}(\vec{x}) \cdot \hat{O}(\vec{x}))^k_j dx^j = dy^k dy_k = dy^k dy^k ; \]

\[ dy^j = dy_j = (\hat{\lambda}^{1/2}(\vec{x}) \cdot \hat{O}(\vec{x}))^j_i dx^i ; \] (1.7)

\[ \hat{O}_{ij}(\vec{x}) dx^i = (\hat{\lambda}^{-1/2}(\vec{x}) \cdot dy)^j_i ; \quad \Rightarrow \quad y^j = y^j(\vec{x}) \iff x^i = x^i(\vec{y}) . \] (1.8)

One obtains from \((1.3)\) an arbitrariness of further orthogonal transformations, e.g. to \(dy^j (1.9a)\), which are given by locally rotated Euclidean differentials, maintaining the local volume elements \((1.9b)\). These
1.2 The importance of the appropriate, invariant integration measure

transformations correspond to gauge transformations and to the gauge invariance of the coset metric integration measure, determined in section 4 for the anomalous-doubled field degrees of freedom

\[ dy'^i_j = dy'^i_j = (\dot{g}^{1/2}(\vec{x}))^i_j \] \[ dx^i_j = \left( \hat{\partial}^T(\vec{x}) \cdot \hat{\lambda}^{1/2}(\vec{x}) \cdot \hat{O}(\vec{x}) \right)^i_j \] \[ dx^i = \hat{\partial}^T_{jk}(\vec{x}) \left( \hat{\lambda}^{1/2}(\vec{x}) \cdot \hat{O}(\vec{x}) \right)_{ki} dx^i \] (1.9a)

\[ = \hat{\partial}^T_{jk}(\vec{x}) dy^k = \hat{\partial}^T_{jk}(\vec{x}) dy^k \; ; \; (dy'^i_j = dy'^i_j \; ; \; \text{locally rotated Euclidean differentials}); \]

\[ \implies d[\vec{y}'] = d[\vec{y}] \; ; \; (\text{invariance of volume elements under 'gauge' transformations}) . \] (1.9b)

The transformation to Euclidean differentials \( d\vec{y} \) (or to equivalently rotated versions \( d\vec{y}' \)) involves the additional Jacobi matrix \( \vec{J}' = (\partial x^i / \partial \vec{x}'^k) = (\hat{\partial}^T(\vec{x}) \cdot \hat{\lambda}^{-1/2}(\vec{x})')_{ki} \) whose determinant leads in combination with the original square root of the metric tensor to Euclidean integration variables. The simplified, Euclidean integration measure is accompanied by a transformation of field variables within the action \( A[\vec{x}] \rightarrow A'[\vec{y}] = A[\vec{x}(\vec{y})] \) which alters the dependence of the field degrees of freedom within the classical action of the exponential

\[ \hat{J}^i_k = \frac{\partial x^i}{\partial \vec{y}^k} = \left( \hat{\partial}^T(\vec{x}) \cdot \hat{\lambda}^{-1/2}(\vec{x}) \right)^i_k ; \] (1.10)

\[ \det(\hat{J}^i_k) = \det \left( \left( \hat{\partial}^T(\vec{x}) \cdot \hat{\lambda}^{-1/2}(\vec{x}) \right)^i_k \right) = \det(\hat{g}^{-1/2}(\vec{x})) = \left( \det(\hat{g}(\vec{x})) \right)^{-1/2} ; \] (1.11)

\[ Z[\vec{x}(\vec{y})] = \int d[\vec{y}] \det(\hat{g}^{-1/2}(\vec{x})) \sqrt{\det(\hat{g}(\vec{x}))} \exp \{ i A[\vec{x}(\vec{y})] \} = Z'[\vec{y}] = \int d[\vec{y}] \exp \{ i A'[\vec{y}] \} ; \] (1.12)

\[ A'[\vec{y}] = A[\vec{x}(\vec{y})] ; \quad Z'[\vec{y}] = Z[\vec{x}(\vec{y})] ; \quad d[\vec{y}] = dy^1 \wedge \ldots \wedge dy^N . \] (1.13)

In the following we compare the two cases 'I' and 'II' of variations for classical equations and their quadratic fluctuation terms 'without' (case 'I') and under inclusion of the transformation to Euclidean integration variables (case 'II'). Case 'I' disregards the nontrivial integration measure \((\det(\hat{g}(\vec{x})))^{1/2} \) in the first and second order variations so that one can simply expand the action \( A[\vec{x}] \). We thus attain from the first order variation the classical equations \( \partial A[\vec{x}] / \partial x^i \equiv 0 \) (‘in a transferred sense’) whose solution (or ’solutions’) determine the quadratic fluctuation term \[(1.14a)\] for the prevailing ’field’ configuration \( \vec{x} = \vec{x}_0 \)

\[ \text{Case 'I':} \quad Z[\vec{x}] = \int d[\vec{x}] \sqrt{\det(\hat{g}(\vec{x}))} \exp \{ i A[\vec{x}] \} ; \] \[ \text{neglect} \quad \sqrt{\det(\hat{g}(\vec{x}))} \quad \text{and expand} \quad A[\vec{x}] \quad \text{for} \quad \vec{x} = \vec{x}_0 \quad \text{following from:} \]

\[ \frac{\partial A[\vec{x}]}{\partial x^i} \equiv 0 ; \quad (i = 1, \ldots, N) \implies \vec{x} = \vec{x}_0 ; \quad (\text{extremum}); \] (1.14b)

\[ A[\vec{x}] = A[\vec{x}_0] + (x^i - x_0^i) \frac{\partial A[\vec{x}]}{\partial x^i} \bigg|_{\vec{x} = \vec{x}_0} + \frac{1}{2!} (x^k - x_0^k) \frac{\partial^2 A[\vec{x}]}{\partial x^l \partial x^k} \bigg|_{\vec{x} = \vec{x}_0} (x^l - x_0^l) + \ldots ; \] (1.14c)

\[ \frac{\partial^2 A[\vec{x}]}{\partial x^k \partial x^l} \bigg|_{\vec{x} = \vec{x}_0} \implies \text{quadratic fluctuation term} . \] (1.14d)

In case 'II' with the Euclidean integration variables we can similarly expand the action \( A'[\vec{y}] \) around a classical solution \( \vec{y}_0 \) up to second order, but have to take into account the chain rule compared to case 'I'. The first order variation \[(1.15b)\] contains the multiplicative Jacobi matrix, compared to \[(1.14b)\] of case 'I', so that the classical solution \( \vec{y}_0 \) directly follows from the transformation \( \vec{y}_0 = \vec{y}(\vec{x}_0) \) according to a necessary ’rank N’-property of the Jacobian for proper transformations (except the Jacobi determinant vanishes for particular
the classical field equations, according to the transformation of the field solution to Euclidean field degrees of freedom for the remaining points which causes separate ranges of integration and transformation intervals. Nevertheless, this implies a different, second order fluctuation term which is modified by a derivative and two multiplications of Jacobi matrices compared to case 'I'.

\[ \text{Case 'II':} \quad Z[\bar{x}(\tilde{y})] = Z'[\tilde{y}] = \int d[\tilde{y}] \exp\{\imath A'[\tilde{y}]\} = \int d[\tilde{y}] \exp\{\imath A[\bar{x}(\tilde{y})]\}; \quad (1.15a) \]

\[ \frac{\partial A'[\tilde{y}]}{\partial y^k} \bigg|_{\tilde{y}=\tilde{y}_0} = \frac{\partial x^i}{\partial y^k} \frac{\partial A[x]}{\partial x^i} \equiv 0 \iff \frac{\partial A[x]}{\partial x^i} \equiv 0; \quad (i=1,\ldots,N) \quad \Rightarrow \bar{x} = \bar{x}_0 = \bar{x}(\tilde{y}_0), \quad (1.15b) \]

because \( \hat{J}_k^i = (\frac{\partial x^i}{\partial y^k})_{\tilde{y}=\tilde{y}_0} \) has to be of rank \( N \) for a valid transformation to Euclidean coordinates.

\[ \begin{align*}
A'[\tilde{y}] &= A'[\bar{x}(\tilde{y}_0)] + (y^i - y_0^i) \frac{\partial A'[\tilde{y}]}{\partial y^i} \bigg|_{\tilde{y}=\tilde{y}_0} + \frac{1}{2!} (y^k - y_0^k) \frac{\partial^2 A'[\tilde{y}]}{\partial y^k \partial y^l} \bigg|_{\tilde{y}=\tilde{y}_0} (y^i - y_0^i) + \ldots \\
&= A[\bar{x}(\tilde{y}_0)] + (y^i - y_0^i) \frac{\partial A[\bar{x}]}{\partial x^i} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} + \frac{1}{2!} (y^k - y_0^k) \left( \frac{\partial A[\bar{x}]}{\partial x^i} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} \frac{\partial x^i}{\partial y^l} \bigg|_{\tilde{y}=\tilde{y}_0} \frac{\partial A[\bar{x}]}{\partial x^l} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} \frac{\partial x^l}{\partial y^k} \bigg|_{\tilde{y}=\tilde{y}_0} \right) (y^i - y_0^i) + \ldots;
\end{align*} \]

fluctuation term \( \Rightarrow \)

\[ \frac{\partial^2 A[x]}{\partial x^i \partial x^j} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} \frac{\partial A[x]}{\partial x^i} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} + \frac{\partial x^i}{\partial y^l} \bigg|_{\tilde{y}=\tilde{y}_0} \frac{\partial A[x]}{\partial x^l} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} \frac{\partial x^l}{\partial y^k} \bigg|_{\tilde{y}=\tilde{y}_0} = (1.15d) \]

In summary we can state that the classical solution, following from the first order variation, changes under the transformation to Euclidean integration variables because this induces a transformation of coordinates \( \tilde{y}_0 = \tilde{y}(\bar{x}_0) \) with the square root of the metric tensor. The second (and also higher n-th) order fluctuations are altered by multiplications and by derivatives of the inverse square root of the metric tensor due to the chain rule for the transformation \( \bar{x} = \bar{x}(\tilde{y}) \) aside from 'gauge' invariant transformations as in \( (1.9a, 1.9b) \).

2nd order fluctuations:

\[ \text{Case 'I':} \quad \frac{\partial^2 A[\bar{x}]}{\partial x^i \partial x^j} \bigg|_{x=\bar{x}_0}; \quad (1.16a) \]

\[ \text{Case 'II':} \quad \frac{\partial^2 (\tilde{g}^{-1/2}(\tilde{y}))^i_k}{\partial y^l} \bigg|_{\tilde{y}=\tilde{y}_0} \frac{\partial A[\bar{x}]}{\partial x^i} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} + (\tilde{g}^{-1/2}(\tilde{y}_0))^i_k \frac{\partial A[\bar{x}]}{\partial x^i \partial x^j} \bigg|_{x_0=\bar{x}(\tilde{y}_0)} (\tilde{g}^{-1/2}(\tilde{y}_0))^j_l. \quad (1.16b) \]

In analogy we perform in section transformations to Euclidean field degrees of freedom for the remaining anomalous-doubled self-energy parts within the coset matrices. In a similar manner one does attain changes for the classical field equations, according to the transformation of the field solution \( \tilde{y}_0 = \tilde{y}(\bar{x}_0) \) and to the 'gauge' symmetry \( (1.9a, 1.9b) \) in a 'transferred sense', but has to consider the inverse square root of the corresponding coset metric tensor in order to conclude for the appropriate second and higher order fluctuation terms. A further possibility is given by a modified classical action, according to

\[ \begin{align*}
Z[\bar{x}] &= \int d[\bar{x}] \exp\{\imath A[\bar{x}]\}, \quad (1.17a) \\
A[\bar{x}] &= -\frac{1}{2} \text{tr}[\ln(\tilde{g}(\bar{x}))] + A[\bar{x}], \quad (1.17b)
\end{align*} \]
1.3 A glimpse in advance for the case with locally Euclidean coset fields

In sections 3.1 to 3.3 we specify HSTs which range from the transformation with scalar, real-valued self-energy variables to total self-energy matrices with anomalous terms in additional, off-diagonal blocks. However, we extend the latter case with anomalous-doubled parts and perform a factorization of the self-energy matrix into density-related self-energy matrices of a background functional and into coset matrices having generators of off-diagonal, 'Nambu' parts. This factorization is taken by a coset decomposition for exciton-related quasi-density-related self-energy matrices of a background functional and into coset matrices having generators of off-diagonal, 'Nambu' parts. Although the various steps of section 4 and 5 are straightforward, we briefly list the resulting path integral of 'Euclidean' coset matrices for an overview in advance with regard to the familiar kind of a nonlinear sigma model on a coset space \( SO(N,\mathbb{R}) / U(\mathbb{R}) \otimes U(\mathbb{R}) \).  

\[
\mathcal{Z}[\hat{a}] \approx \int d[\hat{a}^{(k)}_p(\mathcal{J}^{(\eta_p)}_{j_p,\mu_p}; \mathcal{J}^{(\eta_j)}_{j,\bar{x}})]\ 3\left[\hat{T}^{-1}(\hat{a}); \hat{T}(\hat{a}); \hat{H}\right] \times \exp\left\{\frac{1}{2} N_{x} \sum_{j=0}^{2N+1} \sum_{\mu=e,L} \sum_{\bar{x}} \times \right.
\times \text{Tr}_{a,b} \left[ \ln \left( 1 - \left( \hat{\partial}_{(ln - \hat{\partial})} a \right)^{\mu a'} + \hat{P} \hat{\partial} \left[\hat{T}^{-1}(\hat{a}), \hat{T}(\hat{a})\right] \left(\hat{\partial}[\hat{J}_{j1}, \hat{J}_{j\bar{x}}] \right)^{-1} \hat{P}^{-1} \right) \right]_{\mu,\bar{x},\mu,\bar{x}}^{b=q,\hat{a}, \mathcal{J}^{(\eta_j)}, \mathcal{J}^{(\eta_j)}}, \bar{x} \right\}.
\]  

The independent field degrees of freedom are given by the matrices \( \hat{a}(\gamma', \gamma) \) whose derivative or variational increments '\( \Delta \)' are given in terms of quaternion-valued matrix elements \( \Delta \hat{a}^{(k)}_{s,s_1} (\mathcal{J}^{(\eta_j)}_{j,s_1}, \mathcal{J}^{(\eta_j)}_{j,s}) \) for off-diagonal elements and, aside from an additional phase factor (see second line in \( (1.19a) \)), by the field increments \( \Delta \hat{a}^{(2)}_{ss} (\mathcal{J}^{(\eta_j)}_{j,s_1}, \mathcal{J}^{(\eta_j)}_{j,s}) = \Delta \left( |\hat{a}|_{s=\pm 1} (\mathcal{J}^{(\eta_j)}_{j,s_1}) \right) \exp\left\{i |\hat{a}|_{s=\pm 1} (\mathcal{J}^{(\eta_j)}_{j,s_1}) \right\} \) for the particular elements along the main quaternion-diagonal with element \( (\hat{\tau}_2)_{\mu,s_1} \). Apart from the contour time \( \mathcal{J}^{(\eta_j)}_{j,s_1} \) \((j = 0, \ldots, 2N + 1)\) and space vector \( \bar{x} \) dependence, we include the semiconductor-related, two-band index \( \mu = \{e', l'\} \) and spin \( s = \uparrow, \downarrow \), \( S = 2s = +1, -1 \) degrees of freedom for the independent, locally Euclidean, 'Nambu' fields \( \Delta \hat{a}(\gamma', \gamma) \); the already listed abbreviations \( \gamma', \gamma' \ldots \) denote the total set of external and internal labels as the space-contour-time with additional band and spin indices, respectively.

\[
\Delta \hat{a}(\gamma', \gamma) = -\Delta \hat{a}(\gamma', \gamma) = \sum_{k=0}^{3} (\hat{\tau}_k)_{\mu,s_1} \Delta \hat{a}^{(k)}_{s,s_1} (\mathcal{J}^{(\eta_j)}_{j,s_1}, \mathcal{J}^{(\eta_j)}_{j,s_1}) + (\hat{\tau}_2)_{\mu,s_1} \exp\left\{ \int_{0}^{(\gamma')} \text{d} \mathcal{J}^{(\eta_j)}_{j,s_1} \left( \frac{2 |\hat{a}|_{s_1}}{\sinh (2 |\hat{a}|_{s_1})} - 1 \right) \frac{\partial \hat{a}_{s,s_1} (\mathcal{J}^{(\eta_j)}_{j,s}, \mathcal{J}^{(\eta_j)}_{j_1,s_1})}{\partial \mathcal{J}^{(\eta_j)}_{j,s_1}} \right\} \Delta \hat{a}^{(2)}_{s,s_1} (\mathcal{J}^{(\eta_j)}_{j,s_1}, \mathcal{J}^{(\eta_j)}_{j,s_1}) ;
\]

\[
\text{(last term } \mathcal{J}^{(\eta_j)}_{j_1,s_1} = \mathcal{J}^{(\eta_j)}_{j_1,s_1} \text{)} \& \left. s := s_5 = s_1 , S = 2s \right) ;
\]

\[
\hat{a}^{(2)}_{ss} (\mathcal{J}^{(\eta_j)}_{j_1,s_1}, \mathcal{J}^{(\eta_j)}_{j_1,s_1}) = |\hat{a}|_{s_1} (\mathcal{J}^{(\eta_j)}_{j_1,s_1}) \exp\left\{ i |\hat{a}|_{s_1} (\mathcal{J}^{(\eta_j)}_{j_1,s_1}) \right\} ;
\]

\[a\]

The dimension \( \mathcal{H} \) is given by \( \mathcal{H} = (\mu = \{e', l'\}) \times (s = \uparrow, \downarrow) \times (\eta_j = \pm) \times (N = t_N / \Delta t) \times (N_s = (L / \Delta x)^d ) \times 2 \times 2 \times N \times N_s ; \)

a factor of two for the electron-hole, spin and contour time metric degrees of freedom, respectively, times the number of time-like and discrete, spatial points defines the relevant dimension \( \mathcal{H} \) within \( SO(\mathcal{H}, \mathcal{H}) / U(\mathcal{H}) \otimes U(\mathcal{H}) \).
\[ d[\hat{\alpha}^{(k)}_{s}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}})] = \prod_{s=\uparrow, \downarrow, j_{1}=1,...,2N}^{S=2s} \prod_{s=s', j_{1}=1,...,2N}^{S=2s', S'=2s'} \left( \frac{d\hat{\alpha}^{(2)}_{ss}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \wedge d\hat{\alpha}^{(2)}_{s's'}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}})}{2i} \right)^{1/2} \] (1.19c)

In comparison to [16, 17, 20, 21, 32], we introduce a generalized gradient term \( \hat{\alpha}^{(k)}_{s}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \) within the determinant of (1.18) which consists of the adjoint operator action of the sum of a one-particle operator and a saddle point solution \( \langle \hat{\tilde{\alpha}}_{s}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \rangle \) of the density-related background self-energy.

\[ \left( \hat{\alpha}^{(k)}_{s}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \right)^{b_{a}} = \left[ \left( \exp \left\{ \ln(-\mathcal{F}[\hat{\tilde{\alpha}}_{s}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}})]) \right\} - 1 \right) \hat{\tilde{\alpha}}^{b'a'}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \right]^{b_{a}} \] (1.20)

The coset matrix \( \hat{\tilde{\alpha}}^{b_{a}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \) with locally Euclidean, 'Nambu' field increments \( \Delta \hat{\tilde{\alpha}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \), \( \Delta \hat{\tilde{\alpha}}^{\dagger}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \) in the off-diagonal blocks \( b \neq a \) is outlined in (1.21a) with the block diagonal, density-related terms \( \Delta \hat{\tilde{\alpha}}^{b_{a}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \) which depend on the local Euclidean fields of exciton-related quasi-particles.

\[ \Delta \hat{\tilde{\alpha}}^{b_{a}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) = \left( \begin{array}{cc} \Delta \hat{\tilde{\alpha}}^{11}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) & \Delta \hat{\tilde{\alpha}}^{12}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \\ \Delta \hat{\tilde{\alpha}}^{21}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) & \Delta \hat{\tilde{\alpha}}^{22}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \end{array} \right) \] (1.21a)

\[ \Delta \hat{\tilde{\alpha}}^{11}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) = -\Delta \hat{\tilde{\alpha}}^{22}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) = -\frac{1}{2} \sum_{k=0}^{3} (\hat{\tilde{\alpha}}^{(k)}_{s_{5}s_{1}})_{\mu_{5}\mu_{1}} \times \left\{ \left( \text{tanh} \left( \frac{2 \hat{\tilde{\alpha}}^{(k)}_{s_{5}s_{1}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}})}{2} \right) \right) - \text{tanh} \left( \frac{2 \hat{\tilde{\alpha}}^{(k)}_{s_{5}s_{1}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}})}{2} \right) \right\} \times \exp \left\{ -i \int_{\mathcal{T}^{(j_{5})}_{0}}^{\mathcal{T}^{(j_{5})}_{\mathcal{T}_{j_{5}}}} d\mathcal{T}^{(j_{5})}_{r} \left( \frac{2 \left[ \hat{\tilde{\alpha}}^{(k)}_{s_{5}s_{1}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \right]}{\sinh \left( 2 \left[ \hat{\tilde{\alpha}}^{(k)}_{s_{5}s_{1}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \right] \right)} \right) \frac{\partial \hat{\tilde{\alpha}}^{(k)}_{s_{5}s_{1}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}})}{\partial T^{(j_{5})}_{r}} \right\} \Delta \hat{\tilde{\alpha}}^{(k)}_{s_{5}s_{1}}(\mathcal{T}_{j_{5}}, \mathcal{T}_{j_{1}}) \right\} \] (1.21b)

The derived equations with various kinds of approximations can be applied for the investigation of self-induced transparency and the area theorem in the case of ultrashort coherent transients propagating in matter with reduced absorption [26, 30]; this can be further combined with holography and Fourier optics.
2 Representation of second quantized Hamiltonians with coherent states

2.1 The coherent state path integral for the time development

In this section we define the total Hamiltonian $\hat{H}_{\text{tot}}(\hat{\psi}^\dagger, \hat{\psi}; t)$ of second quantized Fermi operators $\hat{\psi}_{\mu,s}(\vec{x}_1)$ (2.1a) and transform to a coherent state path integral (2.1b) which is used for the four different HST’s in sections 3.1 to 3.3 and 4. We approximate the solid by a single pair of semiconductor-related electron-hole bands which are regarded by the indices $\mu, \nu$, the opposite charges $q_e, q_h$ (2.1b) and further spin degrees of freedom $s, s’ = \uparrow, \downarrow$ (2.1c) with projection labels $\alpha_\mu, \beta_\mu$ to be applied for the dipole moments. The corresponding anticommutators of Fermi operators $\hat{\psi}_{\mu,s}(\vec{x}_1), \hat{\psi}_{\nu,s'}(\vec{x}_2)$ are listed in (2.1a) with their various definitions (2.1b,2.1c) for the single ‘(e)lectron’, the single ‘(h)ole’ and the two spin degrees of freedom. The delta function $\delta^{(N_x)}_{\vec{x}_2,\vec{x}_1}$ denotes a Kronecker delta $\delta_{\vec{x}_2,\vec{x}_1}$ weighted by the total number $N_x = (\frac{L}{d})^d$ of space points in ‘d’ dimensions with system length ‘L’ and spatial grid intervals ‘$\Delta x$’. The total number $N_x = (\frac{L}{d})^d$ of space points can be applied as a parameter of expansion for the ‘HST’ transformed generating functions, provided that the density-related background fields do not vary considerably on the spatial grid with $N_x$ points in comparison to the ‘Nambu’-related terms. This parameter $N_x = (\frac{L}{d})^d$ of total space points is therefore similar to the parameter of matrix dimensions within random matrix theories. Spatial sums $\sum_{\vec{x}} \ldots = \int_{L^d} d^d \vec{x} \ldots$ are normalized by the system volume $L^d$ to dimensionless integrals, whose overall dimensions are determined by the physical dimension of the prevailing integrand, apart from the primed, unweighted case $\sum_{\vec{x}} \ldots$ of pure summation over numbered, spatial points. Double or multiple appearance of indices or spacetime points involve the standard convention of summations, except for indices or spacetime points set in parentheses

\[
\{\hat{\psi}_{\mu,s}(\vec{x}_1), \hat{\psi}^\dagger_{\nu,s'}(\vec{x}_2)\} = \delta_{\mu\nu} \delta_{ss'} \delta^{(N_x)}_{\vec{x}_1,\vec{x}_2}; \quad \{\hat{\psi}_{\mu,s}(\vec{x}_1), \hat{\psi}_{\nu,s'}(\vec{x}_2)\} = \{\hat{\psi}^\dagger_{\mu,s}(\vec{x}_1), \hat{\psi}^\dagger_{\nu,s'}(\vec{x}_2)\} = 0; \quad (2.1a)
\]

\[
\mu, \nu = \text{‘(e)lectron’, ‘(h)ole’}; \quad q_\mu, q_\nu : q_\epsilon = -1, q_\hbar = +1; \quad (2.1b)
\]

\[
s, s’ = \uparrow, \downarrow; \quad \alpha_\mu, \beta_\mu : \alpha_\epsilon = +1, \alpha_\hbar = 0; \quad \beta_\epsilon = 0, \beta_\hbar = +1. \quad (2.1c)
\]

The total Hamiltonian $\hat{H}_{\text{tot}}(\hat{\psi}^\dagger, \hat{\psi}; t)$ (2.2a) is given in terms of a bilinear, kinetic density part $\hat{H}_F(\hat{\psi}^\dagger, \hat{\psi})$ (2.2b) with general dispersion $\hat{E}_{\mu,s’}(\vec{x}_2, \vec{x}_1;s’)$, which can be approximated by relations (2.2c,2.3a), and also contains an anomalous part $\hat{H}_E(\hat{\psi}^\dagger, \hat{\psi}; t)$ (2.2d), which is driven by an electric field, coupled to a dipole moment $\hat{D}_{\nu,s’}(\vec{x}_2, \vec{x}_1)$; hence, one only creates electron-hole pairs in correspondence with charge conversation. Moreover, the general dipole moment $\hat{D}_{\nu,s’}(\vec{x}_2, \vec{x}_1)$ should be constrained to the two parallel spin directions $s = s’ = \uparrow, \downarrow$ in order to regard the spin one-property of the electromagnetic field with the two possible spin components $2s = 2s’ = s’ = \{+1, -1\}$. We therefore summarize this electron-hole generation by a rather general, time-dependent, driving dipole moment $\hat{D}_{\nu,s’}(\vec{x}_2, \vec{x}_1; t)$ (2.2c) with the specific dipole inter-band moment $\hat{D}_{\nu,s’}(\vec{x}_2, \vec{x}_1; t) \approx \delta_{ss’} \hat{D}_{\nu,\mu}(\vec{x}_2, \vec{x}_1; t) (2.3c)$, a spatially dependent band gap $E_\nu(\vec{x}; t)$ and an electric field of central frequency $\omega$ and envelope $E_\nu(\vec{x}; t)$. According to various kinds of possible dipole moments, a rather general kind is used with the type $\delta D_{\nu,s’}(\vec{x}_2, \vec{x}_1; t)$ and projection labels $\beta_\nu, \alpha_\mu$ for inter-band transitions which may be approximated by a constant dipole moment in the case of very ultrashort pulses (31). Apart from the Coulomb interaction $\hat{H}_{FF}(\hat{\psi}^\dagger, \hat{\psi})$ (2.2e) of quartic Fermi operators and Coulomb potential $V(\vec{x}_2, \vec{x}_1)$, we also consider a random potential $u(\vec{x}; t)$ with a Gaussian distribution (2.3d,2.3e) for disorder and noise in the semiconductor-related solid. After an ensemble average according to (2.3d,2.3e), one thus also achieves an anti-hermitian, dissipative, quartic interaction of Fermi fields with the spacetime variance $f(\vec{x}_2, t_2; \vec{x}_1, t_1)$ and energy scale $\’u_0’$

\[
\hat{H}_{\text{tot}}(\hat{\psi}^\dagger, \hat{\psi}; t) = \hat{H}_F(\hat{\psi}^\dagger, \hat{\psi}) + \hat{H}_E(\hat{\psi}^\dagger, \hat{\psi}; t) + \hat{H}_{FF}(\hat{\psi}^\dagger, \hat{\psi}) + \hat{H}_D(\hat{\psi}^\dagger, \hat{\psi}; t); \quad (2.2a)
\]
\[ \hat{H}_F(\hat{\psi}^\dagger, \hat{\psi}) = \sum_{\vec{x}_{1,2}} \hat{\psi}_{\mu,s}^\dagger(\vec{x}_2) \hat{\xi}_{\mu,s',s}(\vec{x}_2) \delta^{(N_e)}(\vec{x}_2, \vec{x}_1) \psi_{\mu,s}(\vec{x}_1) ; \]  
\[ \hat{H}_F(\hat{\psi}^\dagger, \hat{\psi}; t) = \sum_{\vec{x}_{1,2}} \left[ \alpha_\nu \beta_\mu \hat{\psi}_{\nu,s'}^\dagger(\vec{x}_2) \hat{D}_{\nu,s';\mu,s}(\vec{x}_2, \vec{x}_1; t) \hat{\psi}_{\mu,s}^\dagger(\vec{x}_1) + \beta_\nu \alpha_\mu \hat{\psi}_{\nu,s'}(\vec{x}_2) \hat{D}_{\nu,s';\mu,s}(\vec{x}_2, \vec{x}_1; t) \hat{\psi}_{\mu,s}(\vec{x}_1) \right] ; \]  
\[ \hat{H}_{FF}(\hat{\psi}^\dagger, \hat{\psi}) = \sum_{\vec{x}_{1,2}} \hat{\psi}_{\mu,s}^\dagger(\vec{x}_1) \hat{\psi}_{\nu,s'}^\dagger(\vec{x}_2) q_{\mu} q_{\nu} V(\vec{x}_2, \vec{x}_1) \hat{\psi}_{\nu,s'}(\vec{x}_2) \hat{\psi}_{\mu,s}(\vec{x}_1) ; \]  
\[ \hat{H}_D(\hat{\psi}^\dagger, \hat{\psi}; t) = \sum_{\vec{x}} \hat{\psi}_{\mu,s}(\vec{x}) q_{\mu} u(\vec{x}, t) \psi_{\mu,s}(\vec{x}) . \]  

We further extend the total Hamiltonian \( \hat{H}_{\text{tot}}(\hat{\psi}^\dagger, \hat{\psi}; t) \) (2.2a) by a source term \( \hat{\beta} \), defined in (2.4) for generating bilinear observables of Fermi fields, and introduce the corresponding coherent state path integral (2.4) on a non-equilibrium time contour which is taken into account by a 'plus' branch (2.5a) for forward propagation (lower line in (2.4)) and a 'minus' branch (2.5b) (upper line in (2.4)) for backward propagation. In order to include the changing sign in the exponent of the time-step development operators, a contour metric \( \eta_j \) (2.7) is defined in correspondence with a contour time \( T_j \) (2.6) which has equivalent values \( T_j = T_{2N-j} = t_j \) (2.8a), due to the separation of the sign with metric \( \eta_j = -\eta_{2N-j} \) (\( j \neq N \)). We accomplish a path integral by using the anti-commuting coherent states (2.8b) with 'electron-' 'hole' and spin degrees of freedom and further distinction of the prevailing plus-, minus-branch so that the labeling \( \chi_{\mu,s}(T_j, \vec{x}) \) with an additional sign \( \gamma(T_j) \) has to be applied for distinguishing the Grassmann-valued coherent states between their plus-, minus-branch of propagation:

\[ Z[\hat{\beta}] = \langle 0 | \exp \{ \frac{i}{\hbar} \Delta t \hat{H}_{\text{tot};\hat{\beta}}(\hat{\psi}^\dagger, \hat{\psi}; t_1) \} \cdot \ldots \cdot \exp \{ \frac{i}{\hbar} \Delta t \hat{H}_{\text{tot};\hat{\beta}}(\hat{\psi}^\dagger, \hat{\psi}; t_N) \} \times \exp \{ -\frac{i}{\hbar} \Delta t \hat{H}_{\text{tot};\hat{\beta}}(\hat{\psi}^\dagger, \hat{\psi}; t_1) \} \cdot \ldots \cdot \exp \{ -\frac{i}{\hbar} \Delta t \hat{H}_{\text{tot};\hat{\beta}}(\hat{\psi}^\dagger, \hat{\psi}; t_N) \} | 0 \rangle ; \]
2.2 Anomalous doubling of the one-particle part of the Hamiltonian

As one performs the additional ensemble average \((2.3d, 2.3e)\), one attains the ensemble averaged coherent state path integral \((2.10a, 2.10b)\), containing a dissipative noise part, and density pairs \(b(J^{(\eta_1)}, x_1)\) \((2.10c)\) for the quartic interaction of fields. An anti-hermitian epsilon term \((\varepsilon_+ > 0)\) is added for analytic, convergent properties of Green functions following after the integration over the Grassmann fields

\[
\overline{Z[\hat{\beta}]} = \int \prod_{j=0}^{2N} d\chi_{\mu,s}^*(\tilde{T}_{j}^{(\eta_j)}, \tilde{x}) d\chi_{\mu,s}(\tilde{T}_j^{(\eta_j)}, \tilde{x}) \times
\]

\[
\times \exp \left\{ -\sum_{j=0}^{2N} \chi_{\mu,s}^*(\tilde{T}_{j}^{(\eta_j)}, \tilde{x}) \chi_{\mu,s}(\tilde{T}_j^{(\eta_j)}, \tilde{x}) - \sum_{j=1}^{2N} \chi_{\mu,s}^*(\tilde{T}_{j}^{(\eta_j-1)}, \tilde{x}) \chi_{\mu,s}(\tilde{T}_{j-1}^{(\eta_j-1)}, \tilde{x}) \right\} \times
\]

\[
\times \exp \left\{ -\sum_{j=1}^{2N} \eta_j \frac{i}{\hbar} \left[ \chi_{\mu,s}^*(\tilde{T}_{j}^{(\eta_j)}, \tilde{x}_2), \chi_{\mu,s}(\tilde{T}_j^{(\eta_j-1)}, \tilde{x}_1) \right] \right\}.
\]

2.2 Anomalous doubling of the one-particle part of the Hamiltonian

Due to the over-completeness relation of coherent states and the normal ordering of the total Hamiltonian, we finally acquire the defining coherent state path integral \((2.9a, 2.9b)\) for the time development

\[
Z[\hat{\beta}] = \left\langle 0 \right| \prod_{j=1}^{2N} \exp \left\{ -\frac{i}{\hbar} \Delta t \eta_j \hat{H}_{\text{tot},j} \left( \hat{\psi}, \hat{\psi}^* ; \tilde{T}_{j+(\eta_j-1)/2} \right) \right\} | 0 \rangle \quad \text{(2.9a)}
\]

\[
Z[\hat{\beta}] = \int \prod_{j=0}^{2N} d\chi_{\mu,s}^*(\tilde{T}_{j}^{(\eta_j)}, \tilde{x}) d\chi_{\mu,s}(\tilde{T}_j^{(\eta_j)}, \tilde{x}) \times
\]

\[
\times \exp \left\{ -\sum_{j=0}^{2N} \chi_{\mu,s}^*(\tilde{T}_{j}^{(\eta_j)}, \tilde{x}) \chi_{\mu,s}(\tilde{T}_j^{(\eta_j)}, \tilde{x}) - \sum_{j=1}^{2N} \chi_{\mu,s}^*(\tilde{T}_{j}^{(\eta_j)}, \tilde{x}) \chi_{\mu,s}(\tilde{T}_{j-1}^{(\eta_j)}, \tilde{x}) \right\} \times
\]

\[
\times \prod_{j=1}^{2N} \exp \left\{ -\frac{i}{\hbar} \Delta t \eta_j \hat{H}_{\text{tot},j} \left( \chi^*(\tilde{T}_{j}^{(\eta_j)}), \chi(\tilde{T}_{j-1}^{(\eta_j-1)}); \tilde{T}_{j+(\eta_j-1)/2} \right) \right\}.
\]
\[ + \alpha \beta \chi^{s_{\mu}}_{\nu_{\alpha}}(J_{j}^{(\eta_{j})}, \vec{x}_{2}) \hat{D}_{\nu_{\alpha},\mu_{\beta}}(\vec{x}_{2}, \vec{x}_{1}; J_{j}+(\eta_{j}-1)/2) \chi^{s_{\mu}}_{\nu_{\alpha}}(J_{j}^{(\eta_{j})}, \vec{x}_{1}) + \\
+ \beta \alpha \chi^{s_{\mu}}_{\nu_{\alpha}}(J_{j}^{(\eta_{j}-1)}, \vec{x}_{2}) \hat{D}_{\nu_{\alpha},\mu_{\beta}}(\vec{x}_{2}, \vec{x}_{1}; J_{j}+(\eta_{j}-1)/2) \chi^{s_{\mu}}_{\nu_{\alpha}}(J_{j}^{(\eta_{j}-1)}, \vec{x}_{1}) \right\} \times \]
\[
\exp \left\{ - \sum_{j_{1,2}, j_{1,2}=0, 2N+1}^{2N} \sum_{\mu_{1,2}, s_{1,2}} \chi^{\mu_{1,2}}_{s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{2}) \eta_{j_{1}} \frac{\hbar}{2M} \hat{V}(\vec{x}_{2}, J_{j_{1,2}}^{(\eta_{j_{1,2}})}; \vec{x}_{1}, J_{j_{1,2}}^{(\eta_{j_{1,2}})}) \eta_{j_{1}} \chi^{\mu_{1,2}}_{s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \right\} \times \]
\[
\exp \left\{ - \sum_{j_{1,2}, j_{1,2}=0, 2N+1}^{2N} \sum_{\mu_{1,2}, s_{1,2}} \chi^{\mu_{1,2}}_{s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{2}) \eta_{j_{2}} \frac{\hbar}{2M} \hat{D}_{\mu_{1,2}, s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{2}; J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \eta_{j_{1}} \chi^{\mu_{1,2}}_{s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \right\} ; \]
\[
\hat{V}(\vec{x}_{2}, J_{j_{1,2}}^{(\eta_{j_{1,2}})}; \vec{x}_{1}, J_{j_{1,2}}^{(\eta_{j_{1,2}})}) = \eta_{j_{1}} \delta_{j_{2, j_{1}}} (\delta_{j_{2}, j_{1}}) \frac{\hbar}{M} \eta_{j_{2}} (\eta_{j_{2}}) (j_{1}, j_{2} = 1, \ldots, 2N) ; \]
\[
(2.10b) \quad \chi^{s_{\mu}}_{\nu_{\alpha}}(J_{j}^{(\eta_{j})}, \vec{x}_{1}) = \sum_{\mu_{1,2}, s_{1,2}} \chi^{\mu_{1,2}}_{s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \eta_{j_{2}} \chi^{\mu_{1,2}}_{s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) . \quad (2.10c) \]

Note, that we have also specified the source term for generating observables in the last line of (2.10a). However, aside from the density terms of \( \hat{H}_{B}(\hat{\psi}, \hat{\psi}) \), \( \hat{H}_{D}(\hat{\psi}, \hat{\psi}; t) \), and for \( \hat{H}_{F}(\hat{\psi}, \hat{\psi}) \) with pairwise creation and annihilation of electrons and holes so that an anomalous doubling has to be taken for the anti-commuting fields. This is illustrated in relations (2.11a)(2.11g) by introducing the anomalous doubled fields \( \Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \) (2.11a). We emphasize the time step shift from \( j_{1} = 1 \) to \( j_{1} \) between the upper \( \chi^{\mu_{1,2}}_{s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) (a = 1) \) and lower component \( \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) (a = 2) \) of the anomalous-doubled field \( \Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \) (2.11a), due to the normal ordering of the Hamilton operators. Furthermore, we separately define extensions (2.11b)(2.11c) without anomalous doubling for the end points \( j_{1} = 0 \) and \( j_{1} = 2N + 1 \) in order to consider the exact, proper sequence of time steps for the quantum problem (cf. Ref. 20). This extension has also to be regarded for the hermitian conjugation of \( j_{1} \) in (2.11d) with the peculiar time step between upper \( 'b = 1' \) and lower component \( 'b = 2' \), but in reversed order compared to (2.11a). The particular extension for the end points \( j_{2} = 0 \) and \( j_{2} = 2N + 1 \) is similarly given in Eqs. (2.11c)(2.11e), also with the absence of anomalous doubling as in (2.11f)(2.11g). The hermitian conjugated field \( \Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \) (2.11d) also follows by multiplying \( \Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \) (2.11a) with the 'Nambu'-related Pauli matrix \( (\hat{\gamma})^{a_{\mu_{1,2}, s_{1,2}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1})} \) and a subsequent transposition so that one has property (2.11g) which is used in removing the 'Nambu'-doubled anti-commuting fields by integration

\[
\Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) = \left( \begin{array}{c} \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \\ \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})}, \vec{x}_{1}) \end{array} \right) ; \quad (2.11a) \]
\[
\Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=1}), \vec{x}_{1}) = \left( \begin{array}{c} \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=1}), \vec{x}_{1}) \\ \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=1}), \vec{x}_{1}) \end{array} \right) ; \quad (2.11b) \]
\[
\Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=+1}), \vec{x}_{1}) = \left( \begin{array}{c} \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=+1}), \vec{x}_{1}) \\ \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=+1}), \vec{x}_{1}) \end{array} \right) ; \quad (2.11c) \]
\[
\Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=0}), \vec{x}_{1}) = \left( \begin{array}{c} \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=0}), \vec{x}_{1}) \\ \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=0}), \vec{x}_{1}) \end{array} \right) ; \quad (2.11d) \]
\[
\Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=+1}), \vec{x}_{1}) = \left( \begin{array}{c} \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=+1}), \vec{x}_{1}) \\ \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=+1}), \vec{x}_{1}) \end{array} \right) ; \quad (2.11e) \]
\[
\Xi^{a_{\mu_{1,2}, s_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=0}), \vec{x}_{1}) = \left( \begin{array}{c} \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=0}), \vec{x}_{1}) \\ \chi^{s_{\mu_{1,2}}}(J_{j_{1,2}}^{(\eta_{j_{1,2}})=0}), \vec{x}_{1}) \end{array} \right) ; \quad (2.11f) \]
\[
\Xi_{\mu_2,s_2}^{a,b}(\mathcal{F}^{(\nu_2)}_{j_2}; \vec{x}_2) = \left( (\hat{\tau}_1)^{ba} \Xi_{\mu_2,s_2}^{a}(\mathcal{F}^{(\nu_2)}_{j_2}; \vec{x}_2) \right)^{T}.
\]  

(2.11g)

As one applies the anti-commuting fields \(\Xi_{\mu_1,s_1}^{a}(\mathcal{F}^{(\nu_1)}_{j_1}; \vec{x}_1)\) \((2.11a)\), \(\Xi_{\mu_2,s_2}^{a,b}(\mathcal{F}^{(\nu_2)}_{j_2}; \vec{x}_2)\) \((2.11d)\) to the exponent in \((2.10a)\), one obtains relation \((2.12)\) with doubled one-particle part \(\hat{H}_{\nu_1,s_1,\mu_2,s_2}^{a}(\vec{x}_2, \mathcal{F}^{(\nu_2)}_{j_2}; \vec{x}_1, \mathcal{F}^{(\nu_1)}_{j_1})\) \((2.14a, 2.14d)\) which has to be combined with the 'Nambu' metric tensors \((2.13a, 2.13b)\) for the process of anomalous doubling of anti-commuting fields (cf. Ref. [20]).

\[
\sum_{\vec{x}_1,\mu_1} \left( \sum_{j_1=0}^{2N} \chi_{\mu_1,s_1}^{*}(\mathcal{F}^{(\nu_1)}_{j_1}; \vec{x}_1) \chi_{\mu_1,s_1}(\mathcal{F}^{(\nu_1)}_{j_1}; \vec{x}_1) + \sum_{j_2=1}^{2N} \chi_{\mu_1,s_2}^{*}(\mathcal{F}^{(\nu_2)}_{j_2}; \vec{x}_2) \chi_{\mu_1,s_2}(\mathcal{F}^{(\nu_2)}_{j_2}; \vec{x}_2) \right)
\tag{2.12}
\]

The combination of \((2.10a)\) with \((2.12)\) \((2.14d)\) results into the path integral \((2.15)\) which contains the one-particle part in an anomalous-doubled manner and the remaining interaction part of quadratic, density related...
pairs \( b(\mathcal{I}^{(qj)}_1, \bar{x}_1) \), each composed of fermionic field pairs for density terms

\[
\overline{Z[\hat{g}]} = \int \prod_{\bar{x}, \mu, s} \frac{2N}{\pi} d\chi^\ast_{\mu,s}(\mathcal{I}^{(q)}_1, \bar{x}) \, d\chi_{\mu,s}(\mathcal{I}^{(q)}_1, \bar{x}) \times \tag{2.15}
\]

\[
\times \exp \left\{ -\frac{1}{2} \sum_{\bar{x}_{1,2}} \sum_{j_{1,2}=0}^{2N} \sum_{\bar{x}_{1,2}} \sum_{j_{1,2}=0}^{2N+1} \sum_{\mu,s,s'} \Xi_{\mu,s}^b(\mathcal{I}^{(q)}_2, \bar{x}_2) \left[ \hat{\mu}_{s,s'}(\mathcal{I}^{(q)}_1, \bar{x}_1 ; \mathcal{I}^{(q)}_1, \bar{x}_1) \right] \times \right.
\]

\[
+ \left. \eta_{j_{2}} \hat{\mu}_{s,s'}(\mathcal{I}^{(q)}_2, \bar{x}_2 \mathcal{I}^{(q)}_1, \bar{x}_1 \mathcal{I}^{(q)}_1) \eta_{j_{1}} \right\} \sum_{\mu,s} \Xi_{\mu,s}(\mathcal{I}^{(q)}_1, \bar{x}_1) \right \} \times \exp \left\{ -\sum_{\bar{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} \right. b(\mathcal{I}^{(q)}_2, \bar{x}_2) \eta_{j_{2}} \left. \left( \frac{\Delta t}{\hbar} \right) \hat{\nu}(\bar{x}_2 \mathcal{I}^{(q)}_2, \bar{x}_1 \mathcal{I}^{(q)}_1) \eta_{j_{1}} b(\mathcal{I}^{(q)}_1, \bar{x}_1) \right \} = \tag{3.1a}
\]

\[
= \int d[\sigma(\mathcal{I}^{(q)}_1, \bar{x})] \exp \left\{ \frac{i}{\hbar} \sum_{j_{1,2}=1}^{2N} \sum_{j_{1,2}} \sigma(\mathcal{I}^{(q)}_2, \bar{x}_2) \eta_{j_{2}} \left( \frac{\Delta t}{\hbar} \right) \hat{\nu}(\bar{x}_2 \mathcal{I}^{(q)}_2, \bar{x}_1 \mathcal{I}^{(q)}_1) \eta_{j_{1}} \sigma(\mathcal{I}^{(q)}_1, \bar{x}_1) \right \} \times \exp \left\{ -i \frac{\Delta t}{\hbar} \sum_{j_{1,2}=1}^{2N} \sum_{j_{1,2}} \sum_{\bar{x}_{1,2}} \sum_{\bar{x}_{1,2}} \sum_{\mu,s,s'} b(\mathcal{I}^{(q)}_2, \bar{x}_2) \delta_{\mu,s,s'}(\mathcal{I}^{(q)}_2, \bar{x}_2) \eta_{j_{2}} \eta_{j_{1}} \sigma(\mathcal{I}^{(q)}_1, \bar{x}_1) \right \} ;
\]

\[
\sum_{j_{4}=1} \hat{\nu}(\bar{x}_2 \mathcal{I}^{(q)}_2, \bar{x}_4 \mathcal{I}^{(q)}_4) \eta_{j_{4}} \hat{\nu}^{-1}(\bar{x}_4 \mathcal{I}^{(q)}_4, \bar{x}_1 \mathcal{I}^{(q)}_1) = \delta_{j_{2},j_{1}} \eta_{j_{1}} \delta_{j_{4}N_{x}} (\mathcal{I}^{(q)}_1, \bar{x}_1) ; \tag{3.1b}
\]

\[
\sum_{\bar{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} \sum_{\bar{x}_{1,2}} b(\mathcal{I}^{(q)}_2, \bar{x}_2) \delta_{\mu,s,s'}(\mathcal{I}^{(q)}_2, \bar{x}_2) \delta_{j_{2},j_{1}} \eta_{j_{1}} \sigma(\mathcal{I}^{(q)}_1, \bar{x}_1) = \tag{3.1c}
\]

\[
= \sum_{\bar{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} \sum_{\mu,s,s'} \chi_{\mu,s,s'}(\mathcal{I}^{(q)}_2, \bar{x}_2) \delta_{\mu,s,s'}(\mathcal{I}^{(q)}_2, \bar{x}_2) \delta_{j_{2},j_{1}} \eta_{j_{1}} \eta_{j_{2}} \eta_{j_{3}} \sigma(\mathcal{I}^{(q)}_1, \bar{x}_1) \chi_{\mu,s,s'}(\mathcal{I}^{(q)}_1, \bar{x}_1) \]
As one applies the exponential form (3.2) of (3.1c), one finally acquires the coherent state path integral (3.3) with only bilinear, anomalous-doubled anti-commuting fields (2.11a, 2.11g) instead of the quartic interaction of Grassmann fields as in (2.15)

\[
\exp \left\{ -\frac{\lambda}{\hbar} \sum_{x,1,2} \sum_{j=1,2} \sum_{\mu=1}^{2N} \Xi^a_{\nu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2) \delta_{\eta_j} \nabla(\bar{x}_2, \tau^{(\eta_j)}_{x,j}; \bar{x}_1, \tau^{(\eta_j)}_{x,1}) \, \sigma(\tau^{(\eta_j)}_{x,1}, \bar{x}_1) \right\} = \int d[\sigma(\tau^{(\eta_j)}_{x,j}, \bar{x})] \exp \left\{ \frac{i \lambda}{\hbar} \sum_{x,1,2} \sum_{j=1,2} \sum_{\mu=1}^{2N} \sigma(\tau^{(\eta_j)}_{x,j}, \bar{x}_2) \delta_{\eta_j} \nabla(\bar{x}_2, \tau^{(\eta_j)}_{x,j}; \bar{x}_1, \tau^{(\eta_j)}_{x,1}) \, \sigma(\tau^{(\eta_j)}_{x,1}, \bar{x}_1) \right\} \times \exp \left\{ -\frac{1}{2} \sum_{x,1,2} \sum_{j=1,2} \sum_{\mu=1}^{2N} \Xi^a_{\nu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2) \delta_{\eta_j} \nabla(\bar{x}_2, \tau^{(\eta_j)}_{x,j}; \bar{x}_1, \tau^{(\eta_j)}_{x,1}) \, \sigma(\tau^{(\eta_j)}_{x,1}, \bar{x}_1) \right\};
\]

The bilinear Grassmann term in (3.3) is removed by the property of anti-commuting integrations where one has to use the identity (3.4) for an anti-symmetric matrix \( \tilde{M} \) of even-integer dimensions as a one-particle part. One can also state that the symmetric part of the matrix \( \tilde{M} \) cancels in the exponent of (3.4) so that one is only left with the anti-symmetric part of \( \tilde{M} \) in the resulting square root of the determinant \( (\det[\tilde{M}])^{1/2} \)

\[
\int d[\xi] \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^{2n} \xi_i^T \tilde{M} \xi_j \right\} = \left( \det[\tilde{M}] \right)^{1/2}; \quad \tilde{M} = -\tilde{M}^T.
\]

As we define the matrix \( \tilde{M}_{\mu,s',s;\nu,s;\mu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2; \tau^{(\eta_j)}_{x,1}, \bar{x}_1) \) (3.5a) and use property (2.11g) between the hermitian conjugated fields \( \Xi^a_{\mu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2) \) and their transposed form \((\tilde{\tau}^{(\eta_j)}_{x,j})^{ba}_{\mu,s,2} \Xi^a_{\mu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2)\)T, we can reduce the path integral (3.3) to (3.5b) with the square root of the determinant as exemplified in (3.4). Note that the Pauli matrix \( (\tilde{\tau}^{(\eta_j)}_{x,j})^{ba} \) appears as a redundant factor within the determinant and can be disregarded due to the overall even dimensions of the matrix \( \tilde{M}_{\mu,s',s;\nu,s;\mu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2; \tau^{(\eta_j)}_{x,1}, \bar{x}_1) \) so that the path integral (3.5b) is accomplished with the remaining, real- and even-valued self-energy variables \( \sigma(\tau^{(\eta_j)}_{x,1}, \bar{x}_1) \)

\[
\tilde{M}_{\mu,s',s;\nu,s;\mu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2; \tau^{(\eta_j)}_{x,1}, \bar{x}_1) = \tilde{b} b^{(ba)}_{\nu,s';\mu,s}(\bar{x}_2, \tau^{(\eta_j)}_{x,j}; \bar{x}_1, \tau^{(\eta_j)}_{x,1}) \tilde{\tau}^{(\eta_j)}_{x,j} + \eta_j \delta^{(ba)}_{\nu,s';\mu,s}(\bar{x}_2, \tau^{(\eta_j)}_{x,j}; \bar{x}_1, \tau^{(\eta_j)}_{x,1}) \Xi^a_{\nu,s';\mu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2) \Xi^a_{\nu,s';\mu,s}(\tau^{(\eta_j)}_{x,j}, \bar{x}_2)
\]
achieve proper solutions according to the total number of spatial points

\[
Z[\tilde{\varphi}] = \int d[\sigma(\mathcal{T}^{(ij)}, \vec{x})] \exp \left\{ \frac{i}{\hbar} \sum_{j_2,j_1=1}^{2N} \sum_{\vec{x}_{2,1}} \sigma(\mathcal{T}_{j_2}^{(ij)}, \vec{x}_2; \mathcal{T}_{j_1}^{(ij)}, \vec{x}_1) \eta_{j_2} \hat{\Psi}_{j_2}^{\dagger}(\vec{x}_2, \mathcal{T}_{j_2}^{(ij)}, \vec{x}_1) \right\} \times (3.5a)
\]

In general one could have problems to find proper solutions of (3.7,3.8); however, one can use a rotational or auxiliary, scalar self-energy variable \(\sigma_0(\mathcal{T}_{j_1}^{(ij)}, \vec{x})\) by the value \(N_x^{-1/2}\) in order to achieve proper solutions according to the total number of spatial points

\[
\frac{\sigma_0(\mathcal{T}_{j_1}^{(ij)}, \vec{x}_1)}{N_x^{1/2}} = -\frac{1}{2} \sum_{j_2=1}^{2N} \sum_{j_1}^{\mu,s} \left[ \hat{b}^{b'} \hat{a}^{a'}_{\mu,s} \sigma_0(\mathcal{T}_{j_2}^{(ij)}, \vec{x}_2; \mathcal{T}_{j_1}^{(ij)}, \vec{x}_1) \right]^{b'a'} \frac{1}{N_x^{1/2}} + \delta_{\mu,\mu_2} \delta_{s,\sigma_1} \eta_{j_3} \frac{i}{\hbar} \delta_{j_4,j_3} q_\mu \times \hat{S}^{b'a'} \left( \delta(\vec{x}_{2,1}) \sigma_0(\mathcal{T}_{j_2}^{(ij)}, \vec{x}_2; \mathcal{T}_{j_1}^{(ij)}, \vec{x}_1) / N_x^{3/2} \right) \nonumber
\]

Solutions of (3.6) follow from iteration with \(\sigma_0(\mathcal{T}_{j_1}^{(ij)}, \vec{x}_1) / N_x^{1/2}\) or continued fraction which involves the eigenvalue problem with the operator

\[
\sum_{j_1}^{\mu,s} \left[ \hat{b}^{b'} \hat{a}^{a'}_{\mu,s} \sigma_0(\mathcal{T}_{j_2}^{(ij)}, \vec{x}_2; \mathcal{T}_{j_1}^{(ij)}, \vec{x}_1) \right]^{b'a'} \frac{1}{N_x^{1/2}} + \delta_{\mu,\mu_2} \delta_{s,\sigma_1} \eta_{j_3} \frac{i}{\hbar} \delta_{j_4,j_3} q_\mu \times \hat{S}^{b'a'} \left( \delta(\vec{x}_{2,1}) \sigma_0(\mathcal{T}_{j_2}^{(ij)}, \vec{x}_2; \mathcal{T}_{j_1}^{(ij)}, \vec{x}_1) / N_x^{3/2} \right) \nonumber
\]

(3.7)

so that the first order variational equation (3.6) takes the form

\[
\frac{\sigma_0(\mathcal{T}_{j_1}^{(ij)}, \vec{x}_1)}{N_x^{1/2}} = -\frac{1}{2} \sum_{j_2}^{2N} \sum_{j_1}^{\mu,s} \sum_{M}^{a=1,2} \Psi_{M,\mu,s}^{a}(\mathcal{T}_{j_2}^{(ij)}, \vec{x}_2; \mathcal{T}_{j_1}^{(ij)}, \vec{x}_1) \frac{1}{E_M / N_x^{1/2}} \eta_{j_2} q_\mu \hat{S}^{a'a'} \hat{\Psi}_{j_2}^{(ij), \vec{x}_2} \left( \hat{\Psi}_{j_1}^{(ij), \vec{x}_1} \right).
\]

(3.8)

In general one could have problems to find proper solutions of (3.7,3.8); however, one can use a rotational or translational invariance in most cases so that a 2D or 3D problem simplifies to radial equations with angular momenta labels or also wave-vectors. The iterated terms of (3.6,3.7) yield a complex solution \(\sigma_0(\mathcal{T}_{j_1}^{(ij)}, \vec{x})\) whose imaginary part has to comply with the sign of the imaginary \((\epsilon > 0)\)-terms for a proper, stable propagation or the correct analytic convergence of Green functions. This implies a complex-valued eigenvalue problem (3.7) whose imaginary eigenvalues \(E_M\) have also to correspond to the sign of the infinitesimal, imaginary \((\epsilon > 0)\)-term. If the imaginary parts of the solution \(\sigma_0(\mathcal{T}_{j_1}^{(ij)}, \vec{x})\) or \(E_M\) take opposite sign as the imaginary \((\epsilon > 0)\)-term of the Green function, one acquires an instable range of parameters for the semiconductor-related solid which may have a physical origin or may be related to insufficient approximations with our simplified HST and auxiliary, scalar self-energy variable \(\sigma(\mathcal{T}_{j_1}^{(ij)}, \vec{x})\). This range of instable properties (also due to possible artifacts of our HST in this section) can be improved by the more detailed kinds of HST's described in sections 3.2 to 3.3.
3.2 Auxiliary matrix $\hat{\Sigma}_{\nu, s'; \mu, s}(T_{j_2}^{(n_j)}, \vec{x}_2; T_{j_1}^{(n_j)}, \vec{x}_1)$ as hermitian self-energy density

We extend the real self-energy variable $\sigma(T_{j}^{(n)}, \vec{x})$, replacing the sum $\sum_{\mu, s} \chi_{\mu, s}(T_{j}^{(n)}, \vec{x}) q_{\mu} \chi_{\mu, s}(T_{-j}^{(n-1)}, \vec{x})$, to a self-energy matrix of fermionic densities which is specified as a dyadic product (3.9a) of $\chi_{\mu_1, s_1}(T_{j_1}^{(n_{j_1}-1)}, \vec{x}_1)$ with the complex conjugate $\chi_{\mu_2, s_2}^*(T_{j_2}^{(n_{j_2})}, \vec{x}_2)$. The self-energy density $\hat{\Sigma}_{\nu, s'; \mu, s}(T_{j_2}^{(n_j)}, \vec{x}_2; T_{j_1}^{(n_j)}, \vec{x}_1)$ \((3.9b)\) has a hermitian symmetry where the hermitian conjugation involves the electron-hole and spin labels with inclusion of the time contour and coordinate variables. As we apply the operation of dyadic products to the quartic interaction of Fermi fields, one attains the interaction \((3.10)\) in terms of two, solely density-related matrices from the dyadic product \((3.9a)\), which allow for a similar HST as in section \(3.1\) but with matrices for densities in place of scalar variables representing sums of fermionic densities

\[
\hat{R}_{\mu_1, s_1; \mu_2, s_2} (T_{j_2}^{(n_{j_2})}, \vec{x}_2) \eta_{j_2} \eta_{j_1} (T_{j_1}^{(n_{j_1})}, \vec{x}_1) = \chi_{\mu_1, s_1} (T_{j_1}^{(n_{j_1}-1)}, \vec{x}_1) \otimes \chi_{\mu_2, s_2}^* (T_{j_2}^{(n_{j_2})}, \vec{x}_2); \quad (3.9a)
\]

\[
\hat{\Sigma}_{\mu_1, s_1; \mu_2, s_2} (T_{j_2}^{(n_{j_2})}, \vec{x}_2; T_{j_1}^{(n_{j_1})}, \vec{x}_1) = \left( \hat{\Sigma}_{\mu_1, s_1; \mu_2, s_2} (T_{j_2}^{(n_{j_2})}, \vec{x}_2; T_{j_1}^{(n_{j_1})}, \vec{x}_1) \right)^\dagger; \quad (3.9b)
\]

\[
\sum_{\vec{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} b(T_{j_2}^{(n_{j_2})}, \vec{x}_2) \eta_{j_2} \eta_{j_1} b(T_{j_1}^{(n_{j_1})}, \vec{x}_1) \hat{V}(\vec{x}_2, T_{j_2}^{(n_{j_2})}; \vec{x}_1, T_{j_1}^{(n_{j_1})}) = \sum_{\vec{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} \hat{V}(\vec{x}_2, T_{j_2}^{(n_{j_2})}; \vec{x}_1, T_{j_1}^{(n_{j_1})}) \times
\]

\[
\times \chi_{\mu_2, s_2}^* (T_{j_2}^{(n_{j_2})}, \vec{x}_2) \eta_{j_2} q_{\mu_2} \chi_{\mu_1, s_1} (T_{j_1}^{(n_{j_1}-1)}, \vec{x}_1) \eta_{j_1} q_{\mu_1} \chi_{\mu_1, s_1} (T_{j_1}^{(n_{j_1}-1)}, \vec{x}_1) = \]

\[
= - \sum_{\vec{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} \hat{V}(\vec{x}_2, T_{j_2}^{(n_{j_2})}; \vec{x}_1, T_{j_1}^{(n_{j_1})}) \times
\]

\[
\times \left[ \eta_{j_2} q_{\mu_2} \hat{R}_{\mu_2, s_2; \mu_1, s_1} (T_{j_2}^{(n_{j_2})}, \vec{x}_2; T_{j_1}^{(n_{j_1})}, \vec{x}_1) \eta_{j_1} q_{\mu_1} \hat{R}_{\mu_1, s_1; \mu_2, s_2} (T_{j_1}^{(n_{j_1})}, \vec{x}_1; T_{j_2}^{(n_{j_2})}, \vec{x}_2) \right].
\]

One has to start from a similar Gaussian identity with a hermitian self-energy matrix of densities as in the case of scalar self-energy variables $\sigma(T_{j}^{(n)}, \vec{x})$ in section 3.1, we then transform the quartic interaction of Grassmann fields to bilinear fields given as a hermitian dyadic product with linear coupling to the hermitian self-energy matrix in a trace relation of internal and time contour, coordinate variables

\[
\exp \left\{ -i \frac{\hbar}{\hbar} \sum_{\vec{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} b(T_{j_2}^{(n_{j_2})}, \vec{x}_2) \eta_{j_2} \eta_{j_1} b(T_{j_1}^{(n_{j_1})}, \vec{x}_1) \hat{V}(\vec{x}_2, T_{j_2}^{(n_{j_2})}; \vec{x}_1, T_{j_1}^{(n_{j_1})}) \right\} =
\]

\[
= \int d[\hat{\Omega}_{\nu, s'; \mu, s}(\vec{x}_2, T_{j_2}^{(n_{j_2})}; \vec{x}_1, T_{j_1}^{(n_{j_1})})] \times \exp \left\{ -i \frac{\hbar}{\hbar} \sum_{\vec{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} \hat{V}^{-1}(\vec{x}_2, T_{j_2}^{(n_{j_2})}; \vec{x}_1, T_{j_1}^{(n_{j_1})}) \times \right\}
\]

\[
\times \left[ \eta_{j_2} q_{\mu_2} \hat{\Sigma}_{\mu_2, s_2; \mu_1, s_1} (T_{j_2}^{(n_{j_2})}, \vec{x}_2; T_{j_1}^{(n_{j_1})}, \vec{x}_1) \eta_{j_1} q_{\mu_1} \hat{\Sigma}_{\mu_1, s_1; \mu_2, s_2} (T_{j_1}^{(n_{j_1})}, \vec{x}_1; T_{j_2}^{(n_{j_2})}, \vec{x}_2) \right] \times \right\}
\]

\[
\times \exp \left\{ i \frac{\hbar}{\hbar} \sum_{\vec{x}_{1,2}} \sum_{j_{1,2}=1}^{2N} \left[ \eta_{j_2} q_{\mu_2} \hat{\Sigma}_{\mu_2, s_2; \mu_1, s_1} (T_{j_2}^{(n_{j_2})}, \vec{x}_2; T_{j_1}^{(n_{j_1})}, \vec{x}_1) \eta_{j_1} q_{\mu_1} \hat{\Sigma}_{\mu_1, s_1; \mu_2, s_2} (T_{j_1}^{(n_{j_1})}, \vec{x}_1; T_{j_2}^{(n_{j_2})}, \vec{x}_2) \right] \right\}.
\]

Since the one-particle part $\hat{\Omega}_{\nu, s'; \mu, s}(\vec{x}_2, T_{j_2}^{(n_{j_2})}; \vec{x}_1, T_{j_1}^{(n_{j_1})}) \quad (2.14a)(2.14d)$ also contains anomalous terms with dipole moments, one has to perform the anomalous doubling within the coupling of $"(dyadic product)\times (hermitian self-energy matrix)"$; hence, we have to introduce two hermitian self-energy matrices $\hat{\Sigma}_{\nu, s'; \mu, s}(T_{j_2}^{(n_{j_2})}, \vec{x}_2; T_{j_1}^{(n_{j_1})}, \vec{x}_1),\]
(a = 1, 2) which are related by transposition with opposite sign (3.12a, 3.12b). One thus achieves relation (3.12c) with the anomalous-doubled Fermi fields $\hat{\Sigma}_{\nu,s}^{b}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2) \ldots \hat{\Sigma}_{\mu,s}^{a}(\mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1)$ and the two self-energy matrices which only comprise density terms without any 'Nambu' part

$$\hat{\Sigma}_{\nu,s'}^{aa}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) = \left( \hat{\Sigma}_{\nu,s'}^{aa}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) \right)^{\dagger} \; ;$$

(3.12a)

$$\hat{\Sigma}_{\nu,s'}^{22}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) = \left( \hat{\Sigma}_{\nu,s'}^{11}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) \right)^{T} \; ;$$

(3.12b)

$$\exp \left\{ \frac{\hbar}{N} \sum_{x_{1,2}} \sum_{j_{1,2}=1}^{2N} \left[ \eta_{j_2} q_{\mu_2} \hat{R}_{\mu_2,s_j,s_1}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) \right] \right\} = \exp \left\{ \frac{\hbar}{N} \sum_{x_{1,2}} \sum_{j_{1,2}=1}^{2N} \chi_{\mu_1,s_1}^{a}(\mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) \times \right.$$

(3.12c)

$$\times \eta_{j_1} q_{\mu_1} \hat{\Sigma}_{\mu_1,s_1}^{a}(\mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1; \mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2) \right\} = \exp \left\{ \frac{\hbar}{N} \sum_{x_{1,2}} \sum_{j_{1,2}=1}^{2N} \chi_{\mu_2,j_2}^{a}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2) \eta_{j_2} q_{\mu_2} \chi_{\mu_2,j_2}^{a}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2) \right\} = \exp \left\{ \frac{\hbar}{N} \sum_{x_{1,2}} \sum_{j_{1,2}=1}^{2N} \chi_{\mu,s}^{a}(\mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) \right.$$

After insertion of (3.11), (3.12c) into (2.15), we obtain the path integral (3.13a) whose 'Nambu'-doubled, anti-commuting fields are removed by integration according to the already given relation (3.4) in section 3.1. The resulting path integral (3.13b) only contains the hermitian self-energy density $\hat{\Sigma}_{\nu,s'}^{a}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1)$ as remaining quantum mechanical field degrees of freedom

$$\overline{Z[\mathcal{J}]} = \int d[\hat{\Sigma}_{\nu,s'}^{aa}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1)] \exp \left\{ - \frac{\hbar}{N} \sum_{x_{1,2}} \sum_{j_{1,2}=1}^{2N} \hat{V}(\mathcal{J}_{j_2}^{(q_{j_2})}, \vec{x}_2; \mathcal{J}_{j_1}^{(q_{j_1})}, \vec{x}_1) \right.$$

(3.13a)
3.3 Self-energy $\hat{\Sigma}^{ba}_{\nu,s';\mu,s}(\bar{T}^{(j_{2})},\bar{x}_{2};\bar{T}^{(j_{1})},\bar{x}_{1})$ of densities \(b = a\) and of anomalous parts \(b \neq a\)

$$\times \text{DET} \left( \frac{1}{N} \right) \left[ \hat{I}^{bb}_{\nu,s';\mu,s} \left( \bar{x}_2, \bar{T}^{(j_{2})}; \bar{x}_1, \bar{T}^{(j_{1})} \right) \hat{I}^{aa} + \eta_{j_{2}} \delta_{\nu,s';\mu,s} \left( \bar{T}^{(j_{2})}, \bar{x}_2; \bar{T}^{(j_{1})}, \bar{x}_1 \right) \right]^{1/2} + \eta_{j_{2}} q_{\nu} 2 \left( \frac{\mu}{\hbar} \right) \delta_{\nu,s';\mu,s} \left( \bar{T}^{(j_{2})}, \bar{x}_2; \bar{T}^{(j_{1})}, \bar{x}_1 \right) \eta_{j_{1}} + \eta_{j_{2}} q_{\nu} 2 \left( \frac{\mu}{\hbar} \right) \delta_{\nu,s';\mu,s} \left( \bar{T}^{(j_{2})}, \bar{x}_2; \bar{T}^{(j_{1})}, \bar{x}_1 \right) \eta_{j_{1}} + \frac{1}{N} \sum_{x_{1}}^{x_{1}'} \psi_{a}^{(a=1)}(\bar{T}^{(j_{1})}, \bar{x}_{1}) \psi_{b}^{(a=2)}(\bar{T}^{(j_{2})}, \bar{x}_{2}) \psi_{a}^{(a=2)}(\bar{T}^{(j_{1})}, \bar{x}_{1}) \psi_{b}^{(a=1)}(\bar{T}^{(j_{2})}, \bar{x}_{2}) \right] \right].$$

The final path integral (3.13b) consists of the matrix $\hat{M}^{(b=\nu)}_{\nu,s';\mu,s}(\bar{T}^{(j_{2})}, \bar{x}_{2}; \bar{T}^{(j_{1})}, \bar{x}_{1})$, whose anomalous terms \((b \neq a)\) are still restricted to those of the one-particle part $\hat{H}^{b=\nu}_{\nu,s';\mu,s}(\bar{x}_2, \bar{T}^{(j_{2})}; \bar{x}_1, \bar{T}^{(j_{1})})$. In sections 3.3 we generalize to self-energy matrices which also comprise the anomalous terms $\hat{\Sigma}^{ba}_{\nu,s';\mu,s}(\bar{T}^{(j_{2})}, \bar{x}_2; \bar{T}^{(j_{1})}, \bar{x}_1)$ in the off-diagonal 'Nambu' blocks \((b \neq a)\).

In analogy to section 3.1 we take the first order variation of (3.13b) so that one achieves the saddle point equation (3.15a) in terms of the self-energy density $\hat{\Sigma}^{(0)aa}_{\nu,s';\mu,s}(\bar{T}^{(j_{2})}, \bar{x}_{2}; \bar{T}^{(j_{1})}, \bar{x}_{1})$ whose solution is also determined by continued fraction and a similar eigenvalue problem (3.15b,3.15c) of complex-valued matrices in compliance with the imaginary \((\varepsilon > 0)\)-terms of stable propagating Green functions (relations (3.15a,3.15b) are free of any summations over spacetime coordinates and internal state labels, aside from relation (3.15b) with the weighted spatial sum $\sum_{x_{1}}^{x_{1}'}$ and contour time step summations)

$$\hat{\Sigma}^{(0)aa}_{\nu,s';\mu,s}(\bar{T}^{(j_{2})}, \bar{x}_{2}; \bar{T}^{(j_{1})}, \bar{x}_{1}) \equiv \frac{1}{2} \sum_{x_{1}}^{x_{1}'} \psi_{a}^{(a=1)}(\bar{T}^{(j_{1})}, \bar{x}_{1}) \psi_{b}^{(a=2)}(\bar{T}^{(j_{2})}, \bar{x}_{2}) \psi_{a}^{(a=2)}(\bar{T}^{(j_{1})}, \bar{x}_{1}) \psi_{b}^{(a=1)}(\bar{T}^{(j_{2})}, \bar{x}_{2}).$$

3.3 Self-energy $\hat{\Sigma}^{ba}_{\nu,s';\mu,s}(\bar{T}^{(j_{2})}, \bar{x}_{2}; \bar{T}^{(j_{1})}, \bar{x}_{1})$ of densities \(b = a\) and of anomalous parts \(b \neq a\)

Since the one-particle part $\hat{H}^{ba}_{\nu,s';\mu,s}(\bar{x}_2, \bar{T}^{(j_{2})}; \bar{x}_1, \bar{T}^{(j_{1})})$ contains anomalous terms, one has also to introduce 'Nambu' parts in the off-diagonal blocks \((b \neq a)\) of self-energy matrices. We therefore take the dyadic product (3.16) of the anomalous-doubled, anti-commuting fields $\Xi^{a}_{\mu,s}(\bar{T}^{(j_{1})}, \bar{x}_{1})$ and $\Xi^{b}_{\mu,s}(\bar{T}^{(j_{2})}, \bar{x}_{2})$ and consider anomalous-doubled, fermionic density pairs $\mathcal{B}(\bar{T}^{(j_{1})}, \bar{x}_{1})$ (3.17) with extended 'Nambu' metric tensor $\hat{S}^{ba}_{\mu,s}(\bar{T}^{(j_{1})}, \bar{x}_{1}; \bar{T}^{(j_{2})}, \bar{x}_{2})$ (3.18) which implies an overall, additional factor $\frac{1}{N}$. The combination of (3.16,3.18) transforms the quartic interaction of Fermi fields to a trace relation (3.19) of density pairs $\hat{S}^{ab}_{\mu,s;\mu',s'}(\bar{T}^{(j_{1})}, \bar{x}_{1}; \bar{T}^{(j_{2})}, \bar{x}_{2})$ (3.16) with inclusion of
'Nambu' parts which involve the additional trace summation \((a, b = 1, 2)\)

\[
\hat{R}_{\mu_1, s_1; \mu_2, s_2}^{ab} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = \Xi_{a, \mu_1, s_1}^{a} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1) \otimes \Xi_{b, \mu_2, s_2}^{b} (\mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = \\
\left( \chi_{\mu_1, s_1}^{a} (\mathcal{F}_{j_1}^{(\eta_j-1)}), \vec{x}_1) \right)^{a} \otimes \left( \chi_{\mu_2, s_2}^{b} (\mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) , \chi_{\mu_2, s_2}^{b} (\mathcal{F}_{j_2}^{(\eta_j-1)}, \vec{x}_2) \right)^{b};
\]

\[
b(\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1) = \frac{1}{2} \mathcal{B}(\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1) = \frac{1}{2} \sum_{\vec{x}_1, \mu, s} \Xi_{a, \mu_1, s_1}^{a} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1) \hat{S}_{\mu}^{a} \Xi_{\mu_1, s_1}^{a} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1);
\]

\[
\hat{S}_{a}^{b} = \hat{S}_{b}^{a} q_{\mu} ;
\]

\[
\sum_{\vec{x}_1, 2} \sum_{j_1, 2 = 1}^{2N} \mathcal{B}(\mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \eta_{j_1} \mathcal{B}(\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1) \hat{V}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_j)}; \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_j)}) = \sum_{\vec{x}_1, 2} \sum_{j_1, 2 = 1}^{2N} \hat{V}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_j)}; \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_j)}) \times \\
\times \frac{1}{4} \mathcal{B}(\mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \eta_{j_1} \mathcal{B}(\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1) = \frac{1}{4} \sum_{\vec{x}_1, 2} \sum_{j_1, 2 = 1}^{2N} \hat{V}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_j)}; \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_j)}) \times \\
\times \text{Tr}_{a,b} \left[ \hat{S}_{\mu_2}^{b} \eta_{j_1} \hat{R}_{\mu_1, s_1; \mu_2, s_2}^{ab} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \right].
\]

According to the dyadic product \((3.19)\) of anomalous-doubled, anti-commuting fields, we take into account 'Nambu' parts \((b \neq a)\) of an overall, hermitian self-energy matrix \(\Sigma_{\mu_1, s_1; \mu_2, s_2}^{ab} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2)\) \((3.20a)\) whose diagonal blocks \((3.20b)\); \((3.20c)\) are related by opposite sign and transposition and whose two, anti-symmetric 'Nambu' parts are related by hermitian conjugation \((3.20d)\); \((3.20e)\); \((3.20f)\) \((3.20g)\)

\[
\Sigma_{\mu_1, s_1; \mu_2, s_2}^{ab} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = \left( \Sigma_{\mu_1, s_1; \mu_2, s_2}^{11} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) + \Sigma_{\mu_1, s_1; \mu_2, s_2}^{12} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \right)^{ab} ;
\]

\[
\Sigma_{\mu_1, s_1; \mu_2, s_2}^{aa} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = \left( \Sigma_{\mu_1, s_1; \mu_2, s_2}^{11} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \right)^{a} ;
\]

\[
\Sigma_{\mu_1, s_1; \mu_2, s_2}^{22} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = \left( \Sigma_{\mu_1, s_1; \mu_2, s_2}^{11} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \right)^{a} ;
\]

\[
\Sigma_{\mu_1, s_1; \mu_2, s_2}^{21} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = \left( \Sigma_{\mu_1, s_1; \mu_2, s_2}^{11} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \right)^{a} ;
\]

\[
\Sigma_{\mu_1, s_1; \mu_2, s_2}^{12} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = \left( \Sigma_{\mu_1, s_1; \mu_2, s_2}^{11} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) \right)^{a} ;
\]

\[
\Sigma_{\mu_1, s_1; \mu_2, s_2}^{ab} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2) = 0 \quad \text{for} \quad (j_1 \text{or} j_2 = 0) 'or' \quad (j_1 \text{or} j_2 = 2N + 1) , \quad \text{(cf. (2.115 2.111))} ;
\]

As one applies the dyadic product \(\hat{R}_{\mu_1, s_1; \mu_2, s_2}^{ab} (\mathcal{F}_{j_1}^{(\eta_j)}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_j)}, \vec{x}_2)\) \((3.16)\); \((3.19)\) and the self-energy matrix \((3.20a)\); \((3.20f)\) with the 'Nambu' parts in the off-diagonal blocks \((a \neq b)\) to the original, quartic interaction of Fermi fields, one attains the HST \((3.21)\) with Gaussian term of self-energy matrices and a phase factor with linear
3.3 Self-energy \( \hat{\Sigma}_{\nu,s';\mu,s}^{\mu,b}(\mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1) \) of densities \( b = a' \) and of anomalous parts \( b \neq a' \)

coupling between dyadic product of doubled Fermi fields and the anomalous-doubled self-energy in the exponent

\[
\exp \left\{ -\frac{i}{\hbar} \sum_{\vec{x}_1, \vec{x}_2} \sum_{j_1 = 1}^{2N} b_2(\mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2) \eta_{j_2} \eta_{j_1} b(\mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1) \hat{V}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_{j_1})}) \right\} =
\]

\[
= \exp \left\{ \frac{4\pi}{\hbar} \sum_{\vec{x}_1, \vec{x}_2, j_1 = 1}^{2N} \hat{V}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_{j_2})}; \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_{j_1})}) \times \text{Tr}_{a,b} \left[ \hat{S}_{\mu_2}^{bb} \eta_{j_2} \hat{S}_{\mu_2,s_2;\mu_1,s_1}^{ba} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \times \hat{S}_{\mu_1}^{aa} \eta_{j_1} \hat{S}_{\mu_1,s_1;\mu_2,s_2}^{ab} \left( \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2 \right) \right] \right\} 
\]

\[
\times \exp \left\{ -\frac{4\pi}{\hbar} \sum_{\vec{x}_1, \vec{x}_2, j_1 = 1}^{2N} \hat{V}^{-1}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_{j_2})}; \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_{j_1})}) \times \text{Tr}_{a,b} \left[ \hat{S}_{\mu_2}^{bb} \eta_{j_2} \hat{S}_{\mu_2,s_2;\mu_1,s_1}^{ba} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \times \hat{S}_{\mu_1}^{aa} \eta_{j_1} \hat{S}_{\mu_1,s_1;\mu_2,s_2}^{ab} \left( \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2 \right) \right] \right\} 
\]

\[
\times \text{Tr}_{a,b} \left[ \hat{S}_{\mu_2}^{bb} \eta_{j_2} \hat{S}_{\mu_2,s_2;\mu_1,s_1}^{ba} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \hat{S}_{\mu_1}^{aa} \eta_{j_1} \hat{S}_{\mu_1,s_1;\mu_2,s_2}^{ab} \left( \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2 \right) \right] 
\]

\[
\text{Insertion of the HST (3.21) into the path integral (2.13) results into relation (3.22a) with additional Gaussian factor of anomalous-doubled self-energy matrices, but with only bilinear, anti-commuting fields. These are removed by integration according to relation (3.4) so that we finally achieve path integral (3.22b) with the square root of the determinant and the 'Nambu'-doubled self-energy matrix as remaining field degree of freedom.}
\]

\[
\overline{Z}[\tilde{Y}] = \int d[\hat{S}_{\nu,s';\mu,s}^{\mu,b}(\mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1)] \exp \left\{ -\frac{4\pi}{\hbar} \sum_{\vec{x}_1, \vec{x}_2, j_1 = 1}^{2N} \hat{V}^{-1}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_{j_2})}; \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_{j_1})}) \times \hat{S}_{\mu_2}^{bb} \eta_{j_2} \hat{S}_{\mu_2,s_2;\mu_1,s_1}^{ba} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \hat{S}_{\mu_1}^{aa} \eta_{j_1} \hat{S}_{\mu_1,s_1;\mu_2,s_2}^{ab} \left( \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2 \right) \right\} 
\]

\[
\times \left[ \int \prod_{\vec{x}_1, \vec{x}_2, j_1 = 0}^{2N} d\mathcal{X}_{\mu,s,s'}(\mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1) \hat{S}_{\mu_2}^{bb} \eta_{j_2} \hat{S}_{\nu,s';\mu,s}^{\mu,b} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \right] \times \left[ \int \prod_{\vec{x}_1, \vec{x}_2, j_1 = 0}^{2N} d\mathcal{X}_{\mu,s,s'}(\mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1) \hat{S}_{\mu_2}^{bb} \eta_{j_2} \hat{S}_{\nu,s';\mu,s}^{\mu,b} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \right] 
\]

\[
\overline{Z}[\tilde{Y}] = \int d[\hat{S}_{\nu,s';\mu,s}^{\mu,b}(\mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1)] \exp \left\{ -\frac{4\pi}{\hbar} \sum_{\vec{x}_1, \vec{x}_2, j_1 = 1}^{2N} \hat{V}^{-1}(\vec{x}_2, \mathcal{F}_{j_2}^{(\eta_{j_2})}; \vec{x}_1, \mathcal{F}_{j_1}^{(\eta_{j_1})}) \times \hat{S}_{\mu_2}^{bb} \eta_{j_2} \hat{S}_{\mu_2,s_2;\mu_1,s_1}^{ba} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \hat{S}_{\mu_1}^{aa} \eta_{j_1} \hat{S}_{\mu_1,s_1;\mu_2,s_2}^{ab} \left( \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1; \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2 \right) \right\} 
\]

\[
\times \text{DET} \left( \int \prod_{\vec{x}_1, \vec{x}_2, j_1 = 0}^{2N} d\mathcal{X}_{\mu,s,s'}(\mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1) \right) \frac{1}{\hbar} \}
\]

\[
+ \hat{S}_{\nu}^{bb} \eta_{j_2} \left( \frac{1}{\hbar} \right) \hat{S}_{\nu,s';\mu,s}^{\mu,b} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \hat{S}_{\mu}^{aa} \eta_{j_1} \left( \frac{1}{\hbar} \right) \hat{S}_{\nu,s';\mu,s}^{\mu,b} \left( \mathcal{F}_{j_2}^{(\eta_{j_2})}, \vec{x}_2; \mathcal{F}_{j_1}^{(\eta_{j_1})}, \vec{x}_1 \right) \right]^{1/2}. \]
The saddle point equation of (3.22a) and its related eigenvalue problem follow in analogy to sections 3.1, 3.2 and are therefore listed in brevity in subsequent equations (3.23a-3.23f) (relations 3.23a-3.23c, 3.23d, 3.23e, 3.23f) are free of any summations over space-contour-time coordinates and internal state labels; note, however, that (3.23d) implies the scaled $\frac{1}{N_x}$ spatial and contour time step summations)

$$\hat{M}^{(\gamma)(\beta)}_{\nu,s',\mu,s}(x_2; J_{J_1}, x_1) = \hat{\Phi}^{(n)}_{\nu,s',\mu,s}(x_2; J_{J_1}, x_1) \hat{\Phi}^{(n)}_{\nu,s',\mu,s}(x_2; J_{J_1}, x_1) \hat{\Phi}^{(n)}_{\nu,s',\mu,s}(x_2; J_{J_1}, x_1) \hat{\Phi}^{(n)}_{\nu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

$$\hat{S}^{(\gamma)(\beta)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \hat{S}^{(\gamma)(\beta)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \hat{S}^{(\gamma)(\beta)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \hat{S}^{(\gamma)(\beta)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

$$\Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

(3.23a)

$$\hat{V}^{-1}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

(3.23b)

$$\hat{V}^{-1}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

(3.23c)

$$\hat{V}^{-1}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

(3.23d)

$$\hat{V}^{-1}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

(3.23e)

$$\hat{V}^{-1}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1) \Psi^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$$

(3.23f)

4 Coset decomposition with density- and exciton-related parts

4.1 SSB with 'hinge' fields and anomalous-doubled parts

In section 3.3 the density and 'Nambu' parts are regarded within a single, total matrix, having the appropriate block structure (3.20a). In this section we aim at a further factorization of the total self-energy into block diagonal, density-related parts and coset matrices whose generators are composed of the anomalous-doubled field degrees of freedom. In order to separate the coset matrices with 'Nambu' generators from the densities in a HST, one has to introduce block diagonal self-energy densities $\hat{\Phi}^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$ (4.1a) in a SSB as invariant ground or vacuum state and a total self-energy matrix $\hat{\Phi}^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$ (4.1c) with anti-hermitian 'Nambu' parts for the factorization with coset matrices (cf. section 3 in [16]). The matrix $\hat{\Phi}^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$ (4.1c) has a similar structure (4.1d) as $\hat{\Phi}^{(n)}_{\mu,s',\mu,s}(x_2; J_{J_1}, x_1)$ (4.2a) in relations (3.20b,3.20c) of section 3.3, except for the inclusion of the imaginary factor 'i' in the off-diagonal blocks for the anti-hermitian property which is required for a factorization by a coset decomposition.
\[ \delta_{\mu_2, s_2, \mu_1, s_1}^{aa}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) = \left( \delta_{\mu_2, s_2, \mu_1, s_1}^{aa}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \right) \dagger; \]

\[ \delta_{\mu_2, s_2, \mu_1, s_1}^{22}(T_{j_2}^{(\eta); \bar{x}_2}, T_{j_1}^{(\eta)}; \bar{x}_1) = - \delta_{\mu_2, s_2, \mu_1, s_1}^{11}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \]

\[ \delta_{\mu_2, s_2, \mu_1, s_1}(\bar{T}_{j_2}^{\eta}; \bar{x}_2, \bar{T}_{j_1}^{\eta}; \bar{x}_1) = \left( \delta_{\mu_2, s_2, \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \right)^T; \]

\[ \delta_{\mu_2, s_2, \mu_1, s_1}(\bar{T}_{j_2}^{\eta}; \bar{x}_2, \bar{T}_{j_1}^{\eta}; \bar{x}_1) = - \delta_{\mu_2, s_2, \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \]

\[ \delta_{\mu_2, s_2, \mu_1, s_1}(\bar{T}_{j_2}^{\eta}; \bar{x}_2, \bar{T}_{j_1}^{\eta}; \bar{x}_1) = \left( \delta_{\mu_2, s_2, \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \right)^T; \]

\[ \delta_{\mu_2, s_2, \mu_1, s_1}(\bar{T}_{j_2}^{\eta}; \bar{x}_2, \bar{T}_{j_1}^{\eta}; \bar{x}_1) = - \delta_{\mu_2, s_2, \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \]

The coset decomposition of (4.1c) needs further, block diagonal self-energy matrices of densities or 'hinge' fields \( \delta_{\mu_2, s_2, \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \) with equivalent structure (4.2a,4.2b) as the matrices (4.1a,4.1b) in order to perform the factorization (4.3) with the coset matrices \( \tilde{T}_{\mu_2, s_2; \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \). The generators \( \tilde{Y}_{\mu_2, s_2; \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \) of latter coset matrices consist of two anti-symmetric subgenerators \( \tilde{X}_{\mu_2, s_2; \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) \) which describe the anomalous paired fermions of a SSB under inclusion of excitons.

\[ \delta_{\mu_2, s_2, \mu_1, s_1}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) = (\text{un-scaled, discrete sums} \bar{x}_{3,4}, j_{3,4} = 1, \ldots, 2N, \mu_{3,4} = e, h, s_{3,4} = \uparrow, \downarrow) = \]

\[ = \tilde{T}_{\mu_2, s_2; \mu_4, s_4}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_4}^{(\eta)}; \bar{x}_4) \delta_{\tilde{W}^{\mu_2, s_2; \mu_4, s_4}=a'}(\tilde{T}_{\mu_4, s_4}^{(\eta)}; \bar{x}_4, T_{j_3}^{(\eta)}; \bar{x}_3) \]

\[ \tilde{T}_{\mu_3, s_3}^{bb'}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) = \left[ \exp \left\{-\tilde{Y}_{\mu_2, s_2; \mu_4, s_4}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_4}^{(\eta)}; \bar{x}_4) \right\} \right]_{b' \mu_2, s_2, \mu_4, s_4} \]

\[ \tilde{T}_{\mu_2, s_2; \mu_4, s_4}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_4}^{(\eta)}; \bar{x}_4) \delta_{\tilde{W}^{\mu_2, s_2; \mu_4, s_4}=a'}(\tilde{T}_{\mu_4, s_4}^{(\eta)}; \bar{x}_4, T_{j_3}^{(\eta)}; \bar{x}_3) \]

\[ \tilde{T}_{\mu_3, s_3}^{bb'}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) = \left[ \exp \left\{-\tilde{Y}_{\mu_2, s_2; \mu_4, s_4}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_4}^{(\eta)}; \bar{x}_4) \right\} \right]_{b' \mu_2, s_2, \mu_4, s_4} \]

\[ \tilde{T}_{\mu_3, s_3}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_1}^{(\eta)}; \bar{x}_1) = \left( \tilde{T}_{\mu_2, s_2; \mu_4, s_4}(T_{j_2}^{(\eta)}; \bar{x}_2, T_{j_4}^{(\eta)}; \bar{x}_4) \right)^T \]
In analogy to [20] we accomplish the HST (4.5) by one half with the densities \( \hat{g}_{\mu_2,s_2;\mu_1,s_1}^{a\alpha}((T_j^{(a_j)}, \hat{x}_2; T_j^{(a_j)}, \hat{x}_1)) \) (4.1a) and by one half with the anomalous parts of \( \delta \Sigma_a^{\mu_2,s_2;\mu_1,s_1}(T_j^{(a_j)}, \hat{x}_2; T_j^{(a_j)}, \hat{x}_1) \) (4.1d,4.1e) or \( \delta \Sigma_a^{\mu_2,s_2;\mu_1,s_1}(T_j^{(a_j)}, \hat{x}_2; T_j^{(a_j)}, \hat{x}_1) \) (4.2a,4.2b) only have the final effect of normalized Gaussian factors (cf. [20]). The true HST for the SSB to coset degrees of freedom is therefore given by the integrations of the densities \( \hat{g}_{\mu_2,s_2;\mu_1,s_1}^{a\alpha}((T_j^{(a_j)}, \hat{x}_2; T_j^{(a_j)}, \hat{x}_1)) \) (4.1a) and by the integrations of the anti-hermitian related 'Nambu' parts of the total self-energy matrix, each contributing in half for the transformation of the quartic interaction of Fermi fields (\( \hat{S}_\mu^a = \hat{S}_\mu^a q_\mu \), cf. (3.18))

\[
\exp \left\{ - \frac{i}{\hbar} \sum_{i=1}^{N} \sum_{\hat{x}_1,\hat{x}_2} b(\mu_1^{(a_j)}, \hat{x}_2) \eta_{j_2} \eta_{j_1} b(\mu_1^{(a_j)}, \hat{x}_1) \right\} =
\exp \left\{ - \frac{i}{\hbar} \sum_{i=1}^{N} \sum_{\hat{x}_1,\hat{x}_2} \hat{g}_{\mu_2,s_2;\mu_1,s_1}^{a\alpha}((T_j^{(a_j)}, \hat{x}_2; T_j^{(a_j)}, \hat{x}_1)) \right\} \times \exp \left\{ - \frac{i}{\hbar} \sum_{i=1}^{N} \sum_{\hat{x}_1,\hat{x}_2} \hat{g}_{\mu_2,s_2;\mu_1,s_1}^{a\alpha}((T_j^{(a_j)}, \hat{x}_2; T_j^{(a_j)}, \hat{x}_1)) \right\}
\]

As we substitute (4.5) into the original path integral (2.15) and integrate according to relation (3.4) with the doubled anti-commuting fields (2.11a,2.11g), we attain the square root of the determinant and results into a modified Gaussian with a generating dipole term for the quadratic self-energy.
4.1 SSB with 'hinge' fields and anomalous-doubled parts

\[
\int d[\delta\Sigma^a_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)] \exp \left\{ \frac{i}{\hbar} \sum_{j_1,j_2=1}^{2N} \sum_{x_{1,2}} \hat{V}^{-1}(\bar{x}_2, T_{j_2}^{(\eta_2)}; \bar{x}_1, T_{j_1}^{(\eta_1)}) \times \right.
\]

\[
\times \text{Tr}_{a,b} \left[ \hat{\Sigma}^{a}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1) \hat{Q}^{a\mu_1}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1) \right] \}
\]

\[
\times \text{DET} \left( \frac{1}{N_x} \left[ \hat{b}^{ba}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1) \hat{P}^{ba}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1) \right] \right) \left[ \frac{1}{2} \right].
\]

The coset decomposition follows from the factorization \((4.3,4.4c)\) where one has to consider the integration measure from the coset metric tensor and the Vandermonde determinants of eigenvalues \([25]\) in order to replace the Euclidean integration variables \(d[\delta\Sigma^a_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)]\) \([16,17,20]\). A particular simple change of integration measure arises from \((4.7)\) as one diagonalizes the sub-generators \(\hat{X}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\), \(\hat{X}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.4c)\) within the total generator \(\hat{Y}^{ba}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.4b)\) of coset matrices \(\hat{Y}^{ba}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.4a,4.4c)\) and eigenvectors \(\hat{P}^{aa}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.9a,4.9f)\). We regard the anti-symmetry of \(\hat{X}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) and of its hermitian conjugate \(\hat{X}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.4c)\) by quaternion eigenvalues \(\hat{P}^{aa}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.9a,4.9f)\). Furthermore, we use quaternion matrix elements \(\hat{f}^{(k)}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.8c)\) for the sub-generators \(\hat{X}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.4c)\) of coset matrices \(\hat{Y}^{ba}_{\mu_2,s_2;\mu_1,s_1}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.4b)\). The correct number of independent field degrees of freedom is maintained by setting the quaternion-diagonal elements \(\hat{f}^{(k)}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.9c)\) as part of the total generator \((4.9d)\) for the 'eigenvector matrices' \(\hat{P}^{aa}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2; T_{j_1}^{(\eta_1)}, \bar{x}_1)\) \((4.9a,4.9f)\) identical to zero; these 2 × 2 quaternion degrees of freedom along the main diagonal are already contained within the anti-symmetric, quaternion eigenvalues \(\hat{X}_{\nu',s';\mu,s}(T_{j_2}^{(\eta_2)}, \bar{x}_2)\) \((4.8b)\) with Pauli matrix \((\hat{\tau}_2)_{\nu',s'}\) \((4.7)\)
The chosen kind of eigenvalues (4.7a-4.8b) and parametrization with quaternion matrices \( \hat{\sigma} \mu, s \) of neutral charge. According from a rigorous, mathematical point of view, one could also prefer eigenvalues (as 'hinge' fields for a SSB) is specified in relations (4.10a-4.10f) with the anomalous-doubled eigenvalues of electron-hole pairs leading to the two possible, total spin values \( S \). Further factorization of the block diagonal self-energy densities \( \hat{\sigma} \mu, s \) takes into account the strong Coulomb attraction between electrons and holes resulting into exciton quasi-particles of neutral charge. According from a rigorous, mathematical point of view, one could also prefer eigenvalues \( \hat{\sigma} \mu, s \) with separate eigenvalue parameters \( \hat{\sigma} \mu \) for the respective electron- \( \mu = e' \) and hole-band \( \mu = h' \) which are composed of opposite spin pairs due to \( \hat{\sigma} \mu, s \) for separate degrees of freedom of "electron\( \uparrow \)-electron\( \downarrow \)" and "hole\( \uparrow \)-hole\( \downarrow \)" pairs. This latter kind of parametrization of eigenvalues is more appropriate for superconductors with Cooper-pairs of equal kind of fermions caused by an attractive interaction, e.g. by phonons. However, corresponding to the strong Coulomb interaction in the presented case defined within section 2 the excitons are certainly the prevailing, appropriate kind of quasi-particle formation for the eigenvalues \( \hat{\sigma} \mu, s \) with parallel spins of electron-hole pairs leading to the two possible, total spin values \( S = 2 s \) for \( s = \pm \frac{1}{2} \) one might be astonished about the missing, total spin value \( S = 0 \) of electron-hole pairs, having opposite spin directions, within our chosen parametrization of eigenvalues; however, the direct generation of electron-hole pairs with opposite spins is negligible in comparison to the case with parallel spin directions according to the massless, spin-one property of the electromagnetic field with the only two possible spin components \( S = \pm 1 \) without lacking spin zero component (cf. (2.3c)).

Further factorization of the block diagonal self-energy densities \( \delta \hat{\sigma} \mu, s \) (as 'hinge' fields for a SSB) is specified in relations (4.10a,4.10b) with the anomalous-doubled eigenvalues \( \delta \hat{\sigma} \mu, s \) and 'rotation' matrices \( \hat{\sigma} \mu, s \) so that we regard the hermiticity of \( \delta \hat{\sigma} \mu, s \) and their relation (4.2b) of transposition. The generator \( \hat{\sigma} \mu, s \) of \( \delta \hat{\sigma} \mu, s \) consists of a hermitian matrix (4.10a), but with vanishing diagonal elements (4.10b).
4.1 SSB with 'hinge' fields and anomalous-doubled parts

\[
\delta \tilde{\Sigma}_{D;\nu,s',\mu,s}^{aa}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) =
= \tilde{Q}_{D,\nu,s',s;\mu,s}^{aa-1}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_3}^{(\eta_j)}, \xi_3) \left( \delta \tilde{\lambda}_{\mu,s;\nu,s}^{(\eta_j)}(\xi_{j_3}^{(\eta_j)}, \xi_3) - \delta \tilde{\lambda}_{\nu,s;\mu,s}^{(\eta_j)}(\xi_{j_3}^{(\eta_j)}, \xi_3) \right) \left( \tilde{Q}_{D;\mu,s;\nu,s}^{aa}(\xi_{j_3}^{(\eta_j)}, \xi_3; \xi_{j_1}^{(\eta_j)}, \xi_1) \right) ;
\]

\[
\delta \tilde{\lambda}_{\mu,s}^{(\eta_j)}(\xi_{j}^{(\eta_j)}, \xi) = \left\{ \begin{array}{ll}
-\delta \tilde{\lambda}_{\mu,s}^{(\eta_j)}(\xi_{j}^{(\eta_j)}, \xi) & \text{for } a = 1, \\
-\delta \tilde{\lambda}_{\nu,s}^{(\eta_j)}(\xi_{j}^{(\eta_j)}, \xi) & \text{for } a = 2
\end{array} \right. ;
\]

\[
\tilde{Q}_{D;\nu,s',\mu,s}^{a1}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) = \left[ \exp \left\{ \tilde{T}_{D;\nu,s',s;\mu,s}^{a1}(\xi_{j_4}^{(\eta_j)}, \xi_4; \xi_{j_3}^{(\eta_j)}, \xi_3) \right\} \right] \left( \tilde{T}_{D;\mu,s;\nu,s}^{a1}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) \right)^T =
= \left[ \exp \left\{ -\tilde{T}_{D;\nu,s',s;\mu,s}^{a1}(\xi_{j_4}^{(\eta_j)}, \xi_4; \xi_{j_3}^{(\eta_j)}, \xi_3) \right\} \right] \left( \tilde{T}_{D;\mu,s;\nu,s}^{a1}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) \right)^T;
\]

\[
\tilde{T}_{D;\mu,s;\nu,s}^{a1}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) = \left( \tilde{T}_{D;\mu,s;\nu,s}^{a1}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) \right)^T ;
\]

\[
\tilde{T}_{D;\mu,s;\nu,s}^{a1}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) \equiv 0 ; \ (\mu = e,h ; \ s = \uparrow, \downarrow) .
\]

In terms of the above parameters (4.7, 4.9), and (4.10a, 4.10f), it is straightforward to derive the Jacobi determinant for the change of integration measure from Euclidean integration variables \(d[\delta \tilde{\Sigma}_{D;\nu,s',\mu,s}^{ba}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)]\) to the coset part \(d[(\tilde{T}^{-1})^{ba}_{D;\nu,s',\mu,s}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)]\) under inclusion of the block diagonal self-energy densities \(d[\delta \tilde{\Sigma}_{D;\nu,s',\mu,s}^{ba}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)]\) and Vandermonde eigenvalue polynomial \(\text{Poly}(\delta \tilde{\lambda}_{\mu,s}^{(\eta_j)}, \xi)\). According to appendix A of Ref. [10], we can state the transformation of integration variables

\[
d[\delta \tilde{\Sigma}_{D;\nu,s',\mu,s}^{ba}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)] \Delta \prod_{s,s'=\uparrow,\downarrow,j=1,...,2N} \prod \left[ \delta \tilde{\Sigma}_{D;\nu,s',\mu,s}^{ba}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1) \right] =
= d[\delta \tilde{\Sigma}_{D;\nu,s',\mu,s}^{ba}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)] \times \text{Poly}(\delta \tilde{\lambda}_{\mu,s}^{(\eta_j)}, \xi) \times d[(\tilde{T}^{-1})^{ba}_{D;\nu,s',\mu,s}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)] ;
\]

\[
\text{Poly}(\delta \tilde{\lambda}_{\mu,s}^{(\eta_j)}, \xi) = \prod_{s=\uparrow,\downarrow,j=1,...,2N} \left( \delta \lambda_{e,s}(\xi_{j}^{(\eta_j)}, \xi) + \delta \lambda_{h,s}(\xi_{j}^{(\eta_j)}, \xi) \right)^2 \times
\]

\[
\prod_{s, s'=\uparrow,\downarrow,j=1,...,2N} \left( \delta \lambda_{e,s}(\xi_{j}^{(\eta_j)}, \xi) + \delta \lambda_{h,s}(\xi_{j}^{(\eta_j)}, \xi) \right)^2 \times \left( \delta \lambda_{e,s}(\xi_{j}^{(\eta_j)}, \xi) + \delta \lambda_{h,s}(\xi_{j}^{(\eta_j)}, \xi) \right)^2 \times \left( \delta \lambda_{e,s}(\xi_{j}^{(\eta_j)}, \xi) + \delta \lambda_{h,s}(\xi_{j}^{(\eta_j)}, \xi) \right)^2 \right)^{1/2} ;
\]

\[
d[(\tilde{T}^{-1})^{ba}_{D;\nu,s',\mu,s}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)] =
= \prod_{s=\uparrow,\downarrow,j=1,...,2N} \left( d[T^{(\eta_j)}_{D;\nu,s',\mu,s}(\xi_{j_2}^{(\eta_j)}, \xi_2; \xi_{j_1}^{(\eta_j)}, \xi_1)] \times \sinh(2(\tilde{T}_{S}(\xi_{j}^{(\eta_j)}, \xi))) \right) \times
\]
\[ Z[\hat{\mathcal{S}}] = \int d[\hat{s}^{11}]_{\nu',s',\mu,s}(\mathcal{S}^{(n)}_{j_2}, \bar{x}_2; \mathcal{S}^{(n)}_{j_1}, \bar{x}_1) \times \exp \left\{ -\frac{i}{\hbar} \sum_{j_1=1}^{2N} \sum_{j_2} \mathcal{S}^{-1}(\mathcal{S}^{(n)}_{j_2}, \bar{x}_2; \mathcal{S}^{(n)}_{j_1}, \bar{x}_1) \right\} \times \frac{1}{\sqrt{\det(\mathcal{F}(\mathcal{S}^{(n)}_{j_2}, \bar{x}_2); \mathcal{F}(\mathcal{S}^{(n)}_{j_1}, \bar{x}_1))}}. \]
4.2 Saddle point approximation with various kinds of gradient and $1/N_x$ expansions

The factorization into coset and density-related field degrees of freedom with corresponding factorization of the integration measure allows to separate the total path integral (4.13); corresponding to this coset decomposition, the anomalous-doubled fields of the generator (4.7-4.9) for the coset matrices are regarded to be 'immersed' in density-related self-energies of a background functional (4.17). This 'background averaging' (4.17) of the path integral (4.16) for the coset degrees of freedom is denoted by \( \langle \ldots \rangle_{g^{aa}} \)

\[
\overline{Z[\hat{g}]} = \left\langle \int d\left[ \left( \hat{T}^{-1} d\hat{T} \right)^{ba}_{\nu,s';\mu,s} (J_{j_2}^{(q_{j_2})}, \hat{x}_2; J_{j_1}^{(q_{j_1})}, \hat{x}_1) \right] \right\rangle_{g^{aa}} \left[ \hat{T}^{-1}; \hat{T}; \hat{H} \right] \times \left[ \hat{T}^{1/2}_{\nu,s';\mu,s} \right] = \left\langle \int d\left[ \left( \hat{T}^{-1} d\hat{T} \right)^{ba}_{\nu,s';\mu,s} (J_{j_2}^{(q_{j_2})}, \hat{x}_2; J_{j_1}^{(q_{j_1})}, \hat{x}_1) \right] \right\rangle_{g^{aa}} \left[ \hat{T}^{-1}; \hat{T}; \hat{H} \right] \times \exp \left\{ \frac{1}{2} \sum_{j_1,j_2=1}^{2N} \sum_{\nu,s'=\mu,s} \sum_{s=\uparrow,\downarrow} \sum_{x=0}^{N_x+1} \sum_{\mu=e,h} \sum_{x=0}^{N_x+1} \sum_{\mu=e,h} \sum_{x=0}^{N_x+1} \sum_{\mu=e,h} \right\} \times \left\{ \left( \text{path integral of coset field degrees of freedom} \right) \right\}_{g^{aa}}.
\]

(4.16)
= \int \left[ \hat{\mathcal{D}}_{\nu,s';\mu,s}(\hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1) \right] \text{Det} \left( \frac{1}{N_x} \hat{\mathcal{H}}[\hat{\mathcal{S}}, \hat{\mathcal{S}}] \right)^{11} \exp \left\{ -\frac{i}{\hbar} \sum_{\vec{x}_1, \mu_1, e,h} \sum_{\eta_1} \sum_{j=1}^{2N} \hat{\mathcal{V}}^{-1}(\vec{x}_2, \hat{T}^{(\nu \eta j)}_{2}; \vec{x}_1, \hat{T}^{(\nu \eta j)}_{1}) \times \left[ \eta_{2} q_{\mu_2} \hat{\mathcal{S}}^{11}_{\mu_2, s_2; \mu_1, s_1}(\hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1) \eta_{1} q_{\mu_1} \hat{\mathcal{S}}^{11}_{\mu_1, s_1; \mu_2, s_2}(\hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1; \hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2) \right] \right\} \times \left( \text{path integral of coset field degrees of freedom} \right).

Moreover, it is possible to derive from the background functional (4.17) of the self-energy densities a saddle point equation (4.18a); in consequence one can approximate the path integral (4.18) of the anomalous-doubled field variables by replacing the density-related field variables \( \hat{\mathcal{S}}^{aa}_{\nu,s';\mu,s}(\hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1) \) with definite, fixed functions \( \langle \hat{\mathcal{S}}^{aa}_{\nu,s';\mu,s}(\hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1) \rangle \). The latter saddle point solution results from a similar eigenvalue problem (4.18b) as in section 3.2, but without the off-diagonal term of the driving interaction which has been shifted into the functional \( \mathcal{S}[\hat{T}^{-1}; \hat{T}; \hat{\mathcal{H}}] \) (4.15), so that the computation of the saddle point equation can be reduced to the ‘11’ block of densities

\[
\hat{\mathcal{V}}^{-1}(\vec{x}_2, \hat{T}^{(\nu \eta j)}_{2}; \vec{x}_1, \hat{T}^{(\nu \eta j)}_{1}) \langle \hat{\mathcal{S}}^{11}_{\mu_2, s_2; \mu_1, s_1}(\hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1) \rangle = N_x \left[ \hat{\mathcal{S}}^{11}_{\mu_4, s_4; \mu_3, s_3}(\vec{x}_4, \hat{T}^{(\nu \eta j)}_{4}; \vec{x}_3, \hat{T}^{(\nu \eta j)}_{3}) + \left( 4.18a \right) \right.
\]

\[
+ \eta_{4} q_{\mu_4} \left( \hat{\mathcal{S}}^{11}_{\mu_4, s_4; \mu_3, s_3}(\vec{x}_4, \hat{T}^{(\nu \eta j)}_{4}; \vec{x}_3, \hat{T}^{(\nu \eta j)}_{3}) \right) \eta_{3} q_{\mu_3} \psi_{M, \mu_3}(\hat{T}^{(\nu \eta j)}_{3}, \vec{x}_3) = E_M \psi_{M, \mu_3}(\hat{T}^{(\nu \eta j)}_{3}, \vec{x}_3) ;
\]

\[
\text{(eigenstate label } M \text{ with complex eigenvalue } E_M) ;
\]

\[
\hat{\mathcal{V}}^{-1}(\vec{x}_2, \hat{T}^{(\nu \eta j)}_{2}; \vec{x}_1, \hat{T}^{(\nu \eta j)}_{1}) \langle \hat{T}^{(\nu \eta j)}_{\mu_2, s_2; \mu_1, s_1}(\hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1) \rangle = \sum_{M \in \text{eigenstates}} \frac{\psi_{M, \mu_2, s_2}(\hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2) \psi_{M, \mu_1, s_1}(\hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1)}{E_M} .
\]

\[
\text{(4.18c)}
\]

The saddle point approximation of (4.19) with (4.18a) even leads to a further simplification as one considers the relation between the block diagonal propagator \( \hat{\mathcal{H}}[\hat{\mathcal{S}}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1] \) and the self-energy density \( \hat{\mathcal{S}}^{aa}_{\nu,s';\mu,s} \) as consequence of (4.18a)

\[
N_x \left[ \frac{1}{N_x} \hat{\mathcal{H}}[\hat{\mathcal{S}}, \hat{\mathcal{S}}] \right]^{-1,b=a}_{\nu,s';\mu,s} \langle \hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1 \rangle = \hat{\mathcal{V}}^{-1}(\vec{x}_2, \hat{T}^{(\nu \eta j)}_{2}; \vec{x}_1, \hat{T}^{(\nu \eta j)}_{1}) \hat{\mathcal{S}}^{b=a}_{\nu,s';\mu,s} \langle \hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1 \rangle .
\]

\[
\text{(4.19)}
\]

After insertion of the saddle point solution \( \langle \hat{\mathcal{S}} \rangle \) from (4.18a) with inclusion of (4.19), we can remove the block diagonal propagator \( \langle \hat{\mathcal{H}}[\hat{\mathcal{S}}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1] \rangle^{-1:a} \) of purely density-related terms and obtain the path integral (4.20a) with the gradient term (4.20b) which simplifies the gradient expansion of the trace-log functional. In this manner one has succeeded into a path integral of exciton related coset matrices

\[
\bar{Z}^{[\vec{b}]} \approx \int d[\hat{T}^{-1} d\hat{T}]^{b=a}_{\nu,s';\mu,s} \langle \hat{T}^{(\nu \eta j)}_{2}, \vec{x}_2; \hat{T}^{(\nu \eta j)}_{1}, \vec{x}_1 \rangle \times \exp \left\{ \frac{1}{2} N_x \sum_{\vec{x}} \sum_{j=0}^{2N+1} \sum_{\mu=e,h}^{s=1} \right\).
\]

\[
\text{(4.20a)}
\]
4.2 Saddle point approximation with various kinds of gradient and $1/N_x$ expansions

\[ \times \text{Tr}_{a,b} \left[ \ln \left( \mathbf{1} + \hat{\partial}_T \hat{\mathcal{H}}[\delta, \langle \delta \rangle; \hat{T}^{-1}, \hat{T}] \left( \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \right)^{-1} + \hat{\lambda}[\hat{T}^{-1}, \hat{T}] \left( \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \right)^{-1} \right) \right]_{\mu,s;\mu,s}^{b=a} \left( \mathbf{1}; \hat{T}^{-1}, \hat{T}; \hat{\mathcal{H}} \right) \]

\[ \bar{Z}[\hat{\beta}] \approx \int d[\hat{T}^{-1} d\hat{T}]_{\nu,s;\mu,s}^{\mu,s} \left( \hat{T}_2^{(\eta_j)}, \hat{x}_2; \hat{T}_1^{(\eta_j)}, \hat{x}_1 \right) \mathbf{1} \left[ \hat{T}^{-1}; \hat{T} \right] \times \exp \left( \frac{1}{2} N_x \sum_{j=0}^{2N+1} \sum_{s=\uparrow,\downarrow} \sum_{\mu=e,h} \text{Tr}_{a,b} \left[ \ln \left( \mathbf{1} + \hat{\partial}_T \hat{\mathcal{H}}[\delta, \langle \delta \rangle; \hat{T}^{-1}, \hat{T}] \left( \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \right)^{-1} \right) \right]_{\mu,s;\mu,s}^{b=a} \left( \mathbf{1}; \hat{T}^{-1}, \hat{T}; \hat{\mathcal{H}} \right) \right) ; \quad (4.20b) \]

\[ \hat{\partial}_T \hat{\mathcal{H}}[\delta, \langle \delta \rangle; \hat{T}^{-1}, \hat{T}] \left( \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \right)^{-1} \approx \left[ \hat{T}^{-1} \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \hat{T} \left( \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \right)^{-1} - \mathbf{1} \right] \]

\[ \hat{\partial}_{\ln(-\hat{\mathcal{H}})} \left( \text{matrix} \right)^{ba} := \left[ \left( \exp \left( \left[ \ln(-\hat{\mathcal{H}}[\delta, \langle \delta \rangle]) \right] \right) \right) - \mathbf{1} \right] \left( \text{matrix} \right)^{b'a'} \]

\[ \hat{\partial}_{\ln(-\hat{\mathcal{H}})} \left( \text{matrix} \right)^{ba} := \left[ \left( \exp \left( \left[ \ln(-\hat{\mathcal{H}}[\delta, \langle \delta \rangle]) \right] \right) \right) - \mathbf{1} \right] \left( \text{matrix} \right)^{b'a'} \]

where the exponentiation $-\hat{\mathcal{H}}[\delta, \langle \delta \rangle] = \exp\{\ln(-\hat{\mathcal{H}}[\delta, \langle \delta \rangle])\}$ allows for a further possible expansion in terms of $[\ln(-\hat{\mathcal{H}}[\delta, \langle \delta \rangle])] \ldots$. Since the one-particle operators $\hat{\mathcal{H}}_{\mu,s;\mu,s}^{\alpha_\mu,a}(\hat{x}_2, \hat{T}_2^{(\eta_j)}, \hat{x}_1, \hat{T}_1^{(\eta_j)})$ contain the total unit operator (in fact $-\mathbf{1}^{-1}$, cf. (2.14a, 2.14b), therefore $\hat{\partial}_{\ln(-\hat{\mathcal{H}})}$ instead of $\hat{\partial}_{\ln(\hat{\mathcal{H}})}$ in (4.22)), one also achieves reasonable approximations and simplifications from the expansion of the logarithm by using $\ln(1-x) = -x - \frac{1}{2} x^2 - \frac{1}{3} x^3 - \ldots$

\[ \ln\left( \mathbf{1} + \hat{\partial}_T \hat{\mathcal{H}}[\delta, \langle \delta \rangle; \hat{T}^{-1}, \hat{T}] \left( \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \right)^{-1} + \hat{\lambda}[\hat{T}^{-1}, \hat{T}] \left( \hat{\mathcal{H}}[\delta, \langle \delta \rangle] \right)^{-1} \right) \]

\[ = \ln \left[ \delta_{\mu_\mu} \delta_{\delta_\delta} \delta_{\hat{\mathcal{H}}_{\nu,s;\mu,s}}^{(N_s)} \delta_{j_2,j_1} - \left( \delta_{\mu_\mu} \delta_{\delta_\delta} \delta_{\hat{\mathcal{H}}_{\nu,s;\mu,s}}^{(N_s)} \delta_{j_2+1,j_1} + i \frac{\eta_j_2}{\hat{T}} \left[ \delta_{j_2,j_1} \eta_j_2 \delta_{\mu_\nu} \left( \tilde{E}_{\mu,s}(\hat{x}_2) - i \eta_j_2 \delta_{\delta_\theta} \varepsilon_+ \right) \delta_{\hat{\mathcal{H}}_{\nu,s;\mu,s}}^{(N_s)} \right) \right] \right] \]

There are various possibilities to expand the determinants in (4.20a, 4.20b). One possibility refers to the parameter $N_x = (L/\Delta x)^d$ of total space points which allows to consider the self-energy $(\hat{\mathcal{H}})_{\alpha_\mu,a}(\hat{x}_2, \hat{T}_2^{(\eta_j)}, \hat{x}_1, \hat{T}_1^{(\eta_j)})$ as a background density provided that its spatial variation is negligible compared to that of the coset matrices. A further possibility follows from performing the total trace for the determinants in (4.20a, 4.20b) in a momentum basis of plane waves with a gradient expansion of spatial derivative or momentum operators [32]; in this manner one can derive effective Lagrangians of coset matrices with a minimum number of derivatives and effective coupling coefficients consisting of the saddle point solution. One of the guiding principles for such a gradient expansion has to include ‘Derrick’s theorem’ [33] which restricts the expansion to stable configurations under a minimum number of derivatives of coset matrices. Moreover, one can introduce a generalized gradient operator in the two following kinds
However, the appropriate expansion or approximation of the determinant in (4.20a,4.20b) relies on the prevailing phenomenon which may refer to universal fluctuations of a 'Random Matrix Theory', an effective expansion with second and fourth order gradients for Skyrme-like Lagrangians or extraction of relevant terms for a renormalization theory. Therefore, 'ultimately' valid expressions from expansions of the determinant are hardly accessible or universally applicable so that the prevailing approximation of terms has to be adapted to the corresponding physical phenomenon which has originally been defined as a coherent state path integral (2.9a, 2.10a) [6], a Dyson equation or further diagrammatic expressions [7]-[15].

5  TRANSFORMATION TO EUCLIDEAN PATH INTEGRATION VARIABLES

5.1  Transformation with the square root of the coset metric tensor

In the following we abbreviate the contour time variable \( \mathcal{F}_{j_i}^{(n_j)} \) and the spatial vector \( \mathbf{x}_i \) by \( \mathcal{F}_{j_i,\mathbf{x}_i}^{(n_j)} \) and also combine the total set of variables with electron-hole and spin indices into respective numbers '1', '2', ... for brevity

\[
\mathcal{F}_{j_i}^{(n_j)} = \mathcal{F}_{j_i,\mathbf{x}_i}^{(n_j)} ; \\
'1' ; '2' ; ... \triangleq '(\mathcal{F}_{j_1}^{(n_1)}, \mathbf{x}_1, \mu_1, s_1)' ; '(\mathcal{F}_{j_2}^{(n_2)}, \mathbf{x}_2, \mu_2, s_2)' ; ... .
\]

Section 1.2 contains a description of a nontrivial, multi-dimensional integral with the square root of a metric tensor as the Jacobi matrix. In order to convey these transformations to coset spaces as SO(\( \mathfrak{N}, \mathfrak{N} \)) / U(\( \mathfrak{N} \)), we identify the matrix combination \( \hat{T}^{-1}(\Delta \hat{T}) \) as the analogous square root of the metric tensor as in section 1.2.

The derivative symbol '\( \Delta \)' can denote a partial derivative of space '\( \partial_x = \frac{\partial}{\partial x} \)' or time '\( \partial_{\Delta x} = \frac{\partial}{\partial \Delta x} \)', a total derivative '\( d \)' with respect to time slices for the contour time ordering of path integration variables or a variation symbol '\( \delta \)' for deriving classical equations in the exponentials on the condition that the integration measure has been transformed to Euclidean variables as in this section. The derivative symbol '\( \Delta \)' can even refer to the matrix operations (4.21,4.22) for generalized derivatives (\( \partial_{\mathcal{F}_{j_i}^{(n_j)}} \hat{T} \), \( \partial_{(\mathcal{F}_{j_i}^{(n_j)}, \mathbf{x}_i)} \hat{T} \)) of the coset matrix (4.4a,4.4c). We achieve the transformation to Euclidean variables by using the property that \( \hat{T}^{-1}(\Delta \hat{T}) \) can be expressed as elements of the so(\( \mathfrak{N}, \mathfrak{N} \)) Lie algebra. This can be illustrated by consideration of a Lie group \( \hat{U}(\varphi^i) \) of matrices with Lie algebra generators \( \hat{t}_i \) and parameters \( \varphi^i \). As we take the variation \( \Delta \hat{U}(\varphi^i) = \hat{U}(\varphi^i + \Delta \varphi^i) - \hat{U}(\varphi^i) \) of \( \hat{U}(\varphi^i) \), we can apply the closed Lie algebra of generators \( \hat{t}_i \) in order to relate \( \hat{U}(\varphi^i + \Delta \varphi^i) \) to a rotation of \( \hat{U}(\varphi^i) \) by another Lie group element \( \hat{U}(\Delta f^j(\varphi^i, \Delta \varphi^i)) \) with function increments \( \Delta f^j(\varphi^i, \Delta \varphi^i) \) to be determined. Nevertheless, we can specify \( \hat{U}^{-1}(\varphi^i) (\Delta \hat{U}(\varphi^i)) \) in terms of the generators \( \hat{t}_j \)

\[
\hat{U}(\varphi^i) = \exp \{ \imath \varphi^i \hat{t}_i \} ; \quad \hat{U}(\varphi^i + \Delta \varphi^i) = \exp \{ \imath (\varphi^i + \Delta \varphi^i) \hat{t}_i \} = \hat{U}(\varphi^i) \hat{U}(\Delta f^j(\varphi^i, \Delta \varphi^i)) = \exp \{ \imath \varphi^i \hat{t}_i \} \exp \{ \imath \Delta f^j(\varphi^i, \Delta \varphi^i) \hat{t}_j \} ;
\]

\[
\hat{U}^{-1}(\varphi^i) (\Delta \hat{U}(\varphi^i)) = \hat{U}^{-1}(\varphi^i) (\hat{U}(\varphi^i + \Delta \varphi^i) - \hat{U}(\varphi^i)) = \imath \Delta f^j(\varphi^i, \Delta \varphi^i) \hat{t}_j = \imath \Delta \varphi^k \frac{\partial f^j(\varphi^i)}{\partial \varphi^k} \hat{t}_j .
\]

Application of (5.2a,5.2c) to \( \hat{T}^{-1}(\Delta \hat{T}) \) combined with an additional gauge transformation \( \hat{P} \ldots \hat{P}^{-1} \) therefore gives rise to the so(\( \mathfrak{N}, \mathfrak{N} \)) Lie algebra elements \( \Delta \hat{X}^{a\mathfrak{b}}(\varphi^i, \mu^i) \) [5.3a] with the block diagonal, density-related parts \( \Delta \hat{Y}^{11}(\varphi^i, \mu^i) \) [5.3d,5.3c] as dependent variables and off-diagonal, anomalous-doubled parts \( \Delta \hat{X}^{11}(\varphi^i, \mu^i) \) [5.3d,5.3c] as the independent variables which we also define by quaternion-valued matrix elements \( \Delta a_{\mathfrak{a}b}^{(\mathfrak{k})(\mathfrak{t})} \) \( \left( \mathcal{F}_{j_i,\mathbf{x}_i}^{(n_j, s_i)}, \mathcal{F}_{j_i,\mathbf{x}_i}^{(n_j, s_i)} \right) \). The block diagonal density parts consist of anti-hermitian sub-matrices.
5.1 Transformation with the square root of the coset metric tensor

\[ \tau \Delta \hat{g}(s', 1') \] which are functions of the coset variables \( \Delta \hat{X}(s', 1'), \Delta \hat{X}(s', 1') \) or their quaternion-valued correspondents \( \Delta \hat{\alpha}^{(k)}(\mu_{s,j',\hat{x}_k}; \mu_{s,j',\hat{x}_k}), \Delta \hat{\alpha}^{(k)}(\mu_{s,j',\hat{x}_k}; \mu_{s,j',\hat{x}_k}) \)

\[ \Delta \hat{\alpha}^{(k)}(s', 1') = - \left( \hat{P}(s', 1') \hat{T}^{-1}(s', 3') \left( \Delta \hat{T}(s', 2') \right) \right) \hat{P}^{-1}(s', 1') = 3 \sum_{k=0}^{3} (\hat{\tau}_k)_{\mu_{s,j',\hat{x}_k}} \Delta \hat{\alpha}^{(k)}(\mu_{s,j',\hat{x}_k}; \mu_{s,j',\hat{x}_k}) ; \] (5.3a)

\[ \Delta \hat{X}(s', 1') = \Delta \hat{X}_{\mu_{s,j',\hat{x}_k}}(\mu_{s,j',\hat{x}_k}; \mu_{s,j',\hat{x}_k}) = \Delta \hat{\alpha}(s', 1') = \Delta \hat{\alpha}(s', 1') = \Delta \hat{g}(s', 1') ; \] (5.3b)

\[ \Delta \hat{\alpha}^{(k)}(s', 1') = -\Delta \hat{\alpha}^{(k)}(s', 1') ; \] (5.3c)

\[ \Delta \hat{g}(s', 1') = \Delta \hat{g}(s', 1') = \Delta \hat{g}(s', 1') . \] (5.3d)

In order to calculate the dependence of \( \Delta \hat{\alpha}(s', 1') \) on the anomalous-doubled fields within the generator \( \hat{Y}(s', 1') \) (4.47, 4.91) of coset matrices \( \hat{T} \) (4.4a, 4.4c), we use the property

\[ \exp \{ B \} \delta(\exp \{ -B \}) = -\int_{0}^{1} dv \exp \{ v B \} \delta B \exp \{ -v B \} , \] (5.4)

and perform the derivative \( \Delta' \) of \( \hat{P} \hat{T}^{-1}(s', 1') \hat{P}^{-1} \) with its additional gauge transformation so that the eigenvalue decomposition of \( \hat{Y}(s', 1') \) into \( \hat{P}^{-1}(s', 3') \hat{Y}_{D}(s', 2') \hat{P}(s', 1') \) simplifies the expression considerably

\[ -\Delta \hat{\alpha}^{(k)}(s', 1') = \left( \hat{P}(s', 1') \hat{T}^{-1}(s', 3') \left( \Delta \hat{T}(s', 2') \right) \right) \hat{P}^{-1}(s', 1') = \left( \hat{P}(s', 1') \exp \{ \hat{Y}(s', 2') \} \left( \Delta \exp \{ -\hat{Y}(s', 2') \} \right) \hat{P}^{-1}(s', 1') \right) \]

\[ = -\int_{0}^{1} dv \left( \exp \{ v \hat{Y}_{D} \}(s', 4') \right) \left( \Delta \hat{Y}(s', 3') \right) \left( \exp \{ -v \hat{Y}_{D} \}(s', 3') \right) . \] (5.5)

Note, however, that we do not directly transform from \( \Delta \hat{Y}(s', 1') \) (5.6a) to \( \Delta \hat{\alpha}^{(k)}(s', 1') \) in (5.5), but from rotated variables \( \Delta \hat{Y}(s', 1') = \hat{P}(s', 3') \Delta \hat{Y}(s', 2') \hat{P}^{-1}(s', 1') \) (5.6b) to \( \Delta \hat{\alpha}^{(k)}(s', 1') \) which does not alter the integration measure due to the invariance of the trace (5.6c)

\[ \Delta \hat{Y}(s', 1') = \left( \begin{array}{cc} 0 & \Delta \hat{X}(s', 1') \\ \Delta \hat{X}^{-1}(s', 1') & 0 \end{array} \right) ; \quad \Delta \hat{X}(s', 1') = \sum_{k=0}^{3} (\hat{\tau}_k)_{\mu_{s,j',\hat{x}_k}} \hat{f}_{D,H}^{(k)}(\mu_{s,j',\hat{x}_k}; \mu_{s,j',\hat{x}_k}) ; \] (5.6a)

\[ \Delta \hat{Y}'(s', 1') = \hat{P}(s', 3') \Delta \hat{Y}(s', 2') \hat{P}^{-1}(s', 1') \ ; \quad \Delta \hat{X}'(s', 1') = \sum_{k=0}^{3} (\hat{\tau}_k)_{\mu_{s,j',\hat{x}_k}} \hat{f}_{D,H}^{(k)}(\mu_{s,j',\hat{x}_k}; \mu_{s,j',\hat{x}_k}) ; \] (5.6b)

\[ \text{Tr}_{a,b} \left[ \Delta \hat{Y}(s', 2') \right] \left( \Delta \hat{Y}'(s', 1') \right) = \text{Tr}_{a,b} \left[ \Delta \hat{Y}(s', 2') \right] \left( \Delta \hat{Y}'(s', 1') \right) . \] (5.6c)

It remains to compute the exponential \( \exp \{ \pm v \hat{Y}_{D} \}(s', 4') \) of the quaternion-eigenvalue matrix \( \hat{Y}_{D}(s', 2') \) (4.47, 4.91) with anti-symmetric element \( (\hat{\tau}_2)_{\nu_{s,j',\hat{x}_k}} \) and to integrate over \( v \in [0, 1] \) in (5.5) in order to relate the
'rotated' independent fields \((\Delta \tilde{Y}'(\tilde{z}'; \tilde{z}))\) \((5.6b)\) or \(\Delta f^{(k)}_{D;8s2}(\tau^{(n);j}_s, \tau^{(n);j}_x)\) to \(\Delta \hat{\omega}^{ba}(\tilde{y}; \tilde{u})\). This has to be accomplished for the various parts \(a, b = 1, 2\) of \(\Delta \hat{\omega}^{ba}(\tilde{y}; \tilde{u})\) \((5.3a, 5.3c)\) where the off-diagonal parts comprise the independent integration variables \(\Delta \tilde{\alpha}_{\mu_5, \mu_4; \mu_5, \mu_1, s}^{(k)}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\) \((5.3b, 5.3c)\) and where the dependent, block diagonal parts are to be determined by corresponding off-diagonal entries. We emphasize that the eigenvalue decomposition \(\tilde{Y}'(\tilde{z}'; \tilde{z}) = \tilde{P}^{-1}(\tilde{z}'; \tilde{z}) \tilde{Y}_{D}(\tilde{z}'; \tilde{z}) \tilde{P}(\tilde{z}'; \tilde{z})\) \((1.7, 4.9)\) diagonalizes the coset metric tensor in analogy to section \(\[2\]\) in a 'transferred sense' and yields straightforward, simplified expressions. Therefore, we simply list the various parts, following from \((5.5)\), and specify the various variables \(\Delta \tilde{\alpha}(\tilde{y}; \tilde{u})\), \(\Delta \tilde{\alpha}_{\mu_5, \mu_4; \mu_5, \mu_1, s}^{(k)}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\) \((5.3b, 5.3c)\) in terms of \(\Delta \tilde{f}^{(k)}_{D;8s1}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\) \((5.6b)\) or \(\Delta \tilde{f}^{(k)}_{D;8s1}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\) \((5.6a)\) and compute the respective change of integration measure so that we can attain Euclidean integration variables \(\Delta \tilde{\alpha}_{\mu_5, \mu_4; \mu_5, \mu_1, s}^{(k)}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\) under consideration of the integration measure \((4.12)\) for \(\Delta \tilde{f}^{(k)}_{D;8s1}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\) or \(\Delta \tilde{f}^{(k)}_{D;8s1}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\). The diagonal elements for the anomalous-doubled \((5.3b, 5.3c)\) and block-diagonal terms \((5.3d, 5.3e)\) are given in \((5.4, 5.7a)\) and \((5.9)\) with the respective change of integration measure \((5.8)\) which cancels the corresponding term in \((4.12)\) and results in Euclidean path integration fields \(d\tilde{\alpha}_{\mu_5, \mu_4; \mu_5, \mu_1, s}^{(2)}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x) \wedge d\tilde{\alpha}_{\mu_5, \mu_4; \mu_5, \mu_1, s}^{(2)*}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x)\)

\[
- \left(\tilde{P} \tilde{T}^{-1} (\Delta \tilde{T}) \tilde{P}^{-1}\right)^{11}_{\mu_5, \mu_4; \mu_1, s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) = - \left(\tilde{P} \tilde{T}^{-1} (\Delta \tilde{T}) \tilde{P}^{-1}\right)^{21\dagger}_{\mu_5, \mu_4; \mu_1, s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) = \Delta \tilde{\alpha}_{\mu_5, \mu_1, s}^{(k)}(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x) \quad (5.7a) \\
+ \left(\frac{1}{2} - \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \\
= \left(\frac{1}{2} + \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \\
= \left(\frac{1}{2} + \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \quad (5.7b) \\
= \left(\frac{1}{2} + \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \\
= \left(\frac{1}{2} + \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \quad (5.7c) \\
= \left(\frac{1}{2} + \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \\
= \left(\frac{1}{2} + \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \quad (5.8) \\
= \left(\frac{1}{2} + \frac{\sinh \left(\frac{2 \tilde{y}^{(n);j}_s}{\tilde{y}^{(n);j}_x}\right)}{\tilde{y}^{(n);j}_x}\right) \exp\{\pm 2 \phi_{\tilde{s}}(\tilde{y}^{(n);j}_x)\} \Delta \tilde{f}^{(k)*}_{D;8s} \left(\tilde{y}^{(n);j}_s, \tilde{y}^{(n);j}_x\right) \quad (5.9)
\]
\[
= \delta_{\mu_5, \mu_1} \left( \sinh \left( \mathcal{J}_5^{(\eta_1)} \right) \right)^2 \left[ \Delta \hat{f}^{(2)*}_{D;8,5} \left( \mathcal{J}_{j,x}^{(\eta_1)} ; \mathcal{J}_{j,x}^{(\eta_1)} \right) \exp \left\{ i \phi_5 \left( \mathcal{J}_{j,x}^{(\eta_1)} \right) \right\} - \Delta \hat{f}^{(2)}_{D;8,5} \left( \mathcal{J}_{j,x}^{(\eta_1)} ; \mathcal{J}_{j,x}^{(\eta_1)} \right) \exp \left\{ -i \phi_5 \left( \mathcal{J}_{j,x}^{(\eta_1)} \right) \right\} \right] = \\
= -\delta_{\mu_5, \mu_1} \frac{1}{2} \tanh \left( \mathcal{J}_5^{(\eta_1)} \right) \left[ \Delta \hat{a}^{(2)}_{a,a} \left( \mathcal{J}_{j,x}^{(\eta_1)} ; \mathcal{J}_{j,x}^{(\eta_1)} \right) \exp \left\{ -i \phi_5 \left( \mathcal{J}_{j,x}^{(\eta_1)} \right) \right\} - \Delta \hat{a}^{(2)*}_{a,a} \left( \mathcal{J}_{j,x}^{(\eta_1)} ; \mathcal{J}_{j,x}^{(\eta_1)} \right) \exp \left\{ i \phi_5 \left( \mathcal{J}_{j,x}^{(\eta_1)} \right) \right\} \right].
\]

Similarly, we calculate the off-diagonal matrix elements for anomalous-doublet (5.3d, 5.3e) and block diagonal parts (5.3d, 5.3e). Expressions are abbreviated by coefficients \(A('s'; 't'), B('s'; 't'), C('s'; 't'), D('s'; 't')\) in order to attain corresponding results from calculating and analyzing matrix terms in eqs. (5.3a, 5.3c, 5.3f, 5.6a, 5.6c).

Note that we again obtain the result of eqs. (5.7a, 5.9) with the diagonal elements, as one takes the limit process 's' → 't' of the spacetime labels \(J_{j_5,x_5} \to J_{j_1,x_1}\) with identical spins \(s_5 = s_1\) and \(2 \times 2\) quaternion elements \(\mu_5, \mu_1 = 'e'; 'i'\).

\[
\begin{align*}
('s'; 't') & \equiv \sum_{k=0}^{3} (\hat{\tau}_k)_{\mu_5, \mu_1} \Delta \hat{a}_{a_5, a_1} \left( J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} \right) ; \\
A('s'; 't') & = \frac{\left| J_{s_5, x_5}^{(s_5)} \right| \cosh \left( \mathcal{J}_{s_5}^{(s_5)} \right) - \left| J_{s_1, x_1}^{(s_1)} \right| \cosh \left( \mathcal{J}_{s_1}^{(s_1)} \right)}{\left| J_{s_5, x_5}^{(s_5)} \right| - \left| J_{s_1, x_1}^{(s_1)} \right|} ; \\
B('s'; 't') & = \frac{\left| J_{s_5, x_5}^{(s_5)} \right| \cosh \left( \mathcal{J}_{s_5}^{(s_5)} \right) - \left| J_{s_1, x_1}^{(s_1)} \right| \cosh \left( \mathcal{J}_{s_1}^{(s_1)} \right)}{\left| J_{s_5, x_5}^{(s_5)} \right| - \left| J_{s_1, x_1}^{(s_1)} \right|} ; \\
\Delta \hat{a}_{a_5, a_1}^{(0)} (J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} ) & = A('s'; 't') B('s'; 't') \quad \text{(5.10a)} \\
\Delta \hat{a}_{a_5, a_1}^{(0)*} (J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} ) & = A('s'; 't') B('s'; 't') \quad \text{(5.10b)} \\
\Delta \hat{f}^{(0)}_{D;5,8;5} (J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} ) & = A('s'; 't') B('s'; 't') \quad \text{(5.10c)} \\
\Delta \hat{a}_{a_5, a_1}^{(k)} (J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} ) & = A('s'; 't') B('s'; 't') \quad \text{(5.10d)} \\
\Delta \hat{f}^{(k)}_{D;5,8;5} (J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} ) & = A('s'; 't') B('s'; 't') \quad \text{(5.10e)} \\
\Delta \hat{a}_{a_5, a_1}^{(k)*} (J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} ) & = \frac{1}{A('s'; 't') - B('s'; 't')} \quad \text{(5.10f)} \\
\Delta \hat{f}^{(k)*}_{D;5,8;5} (J_{j_5, x_5}^{(s_5)} ; J_{j_1, x_1}^{(s_1)} ) & = \frac{1}{A('s'; 't') - B('s'; 't')} \quad \text{(5.10g)}
\end{align*}
\]
5.2 Eigenvalues of coset generators and their transformed, Euclidean correspondents

In the previous section 5.1 we have accomplished various relations (5.7a, 5.8, 5.10a, 5.11) between the rotated variables \( \Delta Y'(q'; \nu') \) = \( P(q'; \nu') \) \( \Delta \hat{Y}(q'; \nu') \) \( \hat{P}^{-1}(q'; \nu') \) or their quaternion representation with \( \Delta \hat{X}'(q'; \nu') = \sum_{k=0}^{\hat{D}} (\hat{r}_k)_{\mu_4 \mu_3} \Delta s_{\mu_4 \mu_3} (\hat{T}_{q_0}^{(q_0)}) (\hat{T}_{q_0}^{(q_0)}) \) and the Euclidean path integration variables \( \Delta \hat{a}^{(k)}(s_{\mu_4 \mu_3}) (\hat{T}_{q_0}^{(q_0)}) \); aside from the anomalous-doubled fields, we have also transformed the block diagonal density parts 5.9c, 5.12b, 5.12c as dependent variables on the off-diagonal terms. The eigenvalue terms \( \hat{Y}_D (q'; \nu') \) or \( \hat{T}_S (\hat{T}_{q_0}^{(q_0)}) \) of \( \hat{Y}(q'; \nu') \) 1.7(4.9f) have diagonalized the coset metric tensor and have simplified the resulting transformations between \( \Delta s_{\mu_4 \mu_3} (\hat{T}_{q_0}^{(q_0)}) \) and \( \Delta s_{\mu_4 \mu_3} (\hat{T}_{q_0}^{(q_0)}) \); therefore, it remains to determine the exciton-related eigenvalues \( \hat{F}_S (\hat{T}_{q_0}^{(q_0)}) \) 1.8b, 1.8c with absolute value \( |\hat{F}_S (\hat{T}_{q_0}^{(q_0)})| \) and phase \( \phi_S (\hat{T}_{q_0}^{(q_0)}) \) in terms of the Euclidean fields \( \hat{a}^{(k)}(s_{\mu_4 \mu_3}) (\hat{T}_{q_0}^{(q_0)}) \). Since the coherent state path integral has been derived from a contour time ordering of the time development operator, one has to apply the infinitesimal integration increment \( 'd' \) of the contour time in order to relate the Euclidean fields \( \hat{a}^{(k)}(s_{\mu_4 \mu_3}) \) to the eigenvalues \( \hat{F}_S (\hat{T}_{q_0}^{(q_0)}) \)

\[
\hat{F}_S (\hat{T}_{q_0}^{(q_0)}) = \int_{0}^{\hat{T}_{q_0}^{(q_0)}} d\hat{T}_{q_0}^{(q_0)} \frac{\partial \hat{F}_S (\hat{T}_{q_0}^{(q_0)})}{\partial \hat{T}_{q_0}^{(q_0)}} ; \quad \phi_S (\hat{T}_{q_0}^{(q_0)}) = \int_{0}^{\hat{T}_{q_0}^{(q_0)}} d\hat{T}_{q_0}^{(q_0)} \frac{\partial \phi_S (\hat{T}_{q_0}^{(q_0)})}{\partial \hat{T}_{q_0}^{(q_0)}} ; \quad (5.13)
\]

\[
\delta_{\mu_4 \mu_3 \nu_4 \nu_3} (\hat{T}_{q_0}^{(q_0)}, \hat{T}_{q_0}^{(q_0)}) = \delta_{\mu_4 \mu_3 \nu_4 \nu_3} (\hat{T}_{q_0}^{(q_0)}, \hat{T}_{q_0}^{(q_0)}) = \delta_{\mu_4 \mu_3 \nu_4 \nu_3} (\hat{T}_{q_0}^{(q_0)}, \hat{T}_{q_0}^{(q_0)}) \delta_{\mu_4 \mu_3 \nu_4 \nu_3} (\hat{T}_{q_0}^{(q_0)}, \hat{T}_{q_0}^{(q_0)}) . \quad (5.14)
\]

As we reconsider the relation 5.3 for \( 'd' \) of the contour time and expand \( d\hat{Y} = (d\hat{P}^{-1} \hat{Y}_D \hat{P}) \), we note the additional commutator \( [\hat{Y}_D, (d\hat{P} \hat{P}^{-1})] \) whose resulting diagonal quaternion elements in the off-diagonal parts can be chosen to vanish due to the gauge invariance of the integration measure with respect to \( \hat{P} \ldots \hat{P}^{-1} \).
After computation of \( (5.16a) \), we achieve for the various diagonal elements of \((5.16b-5.16e)\) following relations (cf. \((5.6c)\), compare appendix A)

\[
-(d\hat{z}_{ab}^{ab}(\tau'; \tau')) = \left( \hat{P} \hat{T}^{-1} \left( d\hat{T} \right) \hat{P}^{-1} \right)^{ab}(\tau'; \tau') = -\int_0^1 dv \left( e^{v \hat{Y}_D} \hat{P} \left( d\hat{P}^{-1} \hat{Y}_D \hat{P} \right) \hat{P}^{-1} e^{-v \hat{Y}_D} \right)^{ab}(\tau'; \tau'); \tag{5.15a}
\]

\[
0 = \left( (d\hat{P}) \hat{P}^{-1} \right) \left( \hat{Y}_D, (d\hat{P}) \hat{P}^{-1} \right) (\tau'; \tau'); \tag{5.15b}
\]

The chosen gauge \((5.15b)\) simplifies \((5.15a)\) and leads to the direct transformation \((5.16a)\) between the diagonal elements of the Euclidean path integration fields \(d\hat{z}_{ab}^{ab}(\tau'; \tau')\) and the eigenvalues \(d\hat{Y}_D\) where the integration of \(v \in [0, 1]\) and the calculation of \(\exp\{\pm v \hat{Y}_D\}\) becomes straightforward due to the restriction to diagonal matrix elements

\[
d\hat{z}_{ab}^{ab}(\tau'^{nj}; \tau^{nj}) = \int_0^1 dv \left( \exp\{v \hat{Y}_D\} \left( d\hat{Y}_D \right) \exp\{-v \hat{Y}_D\} \right)^{ab}(\tau'^{nj}; \tau^{nj}); \tag{5.16a}
\]

\[
d\hat{z}_{12}^{12}(\tau'^{nj}; \tau^{nj}) = (\tau_{2\mu})^{\bar{s}s} d\hat{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj}); \tag{5.16b}
\]

\[
d\hat{z}_{21}^{21}(\tau'^{nj}; \tau^{nj}) = (\tau_{2\mu})^{\bar{s}s} d\hat{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj}); \tag{5.16c}
\]

\[
d\hat{z}_{22}^{11}(\tau'^{nj}; \tau^{nj}) = \delta_{\mu\nu} d\hat{g}_{\mu\nu} (\tau'^{nj}; \tau^{nj}); \tag{5.16d}
\]

\[
d\hat{z}_{22}^{22}(\tau'^{nj}; \tau^{nj}) = -\delta_{\mu\nu} d\hat{g}_{\mu\nu} (\tau'^{nj}; \tau^{nj}). \tag{5.16e}
\]

After computation of \((5.16a)\), we achieve for the various diagonal elements of \((5.16b-5.16c)\) following relations

\[
d\hat{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj}) = d\left( |\bar{\alpha}_S(\tau^{nj})| \exp\{i \alpha_S(\tau^{nj})\} \right); \tag{5.17a}
\]

\[
d\bar{f}_S(\tau'^{nj}, \tau^{nj}) = \left( \frac{1}{2} - \frac{\sinh (2 |\bar{f}_S(\tau^{nj})|)}{2 |\bar{f}_S(\tau^{nj})|} \right) d\bar{f}_S(\tau^{nj}) \left( \frac{1}{2} + \frac{\sinh (2 |\bar{f}_S(\tau^{nj})|)}{2 |\bar{f}_S(\tau^{nj})|} \right); \tag{5.17b}
\]

\[
d\bar{g}_{\mu\nu} (\tau'^{nj}, \tau^{nj}) = -\left[ \left( d\bar{f}_S(\tau^{nj}) \right) \exp\{-i \phi_S(\tau^{nj})\} - (d\bar{f}_S(\tau^{nj})) \exp\{i \phi_S(\tau^{nj})\} \right] \left( \frac{\sinh (|\bar{f}_S(\tau^{nj})|)}{2 |\bar{f}_S(\tau^{nj})|} \right)^2; \tag{5.17c}
\]

\[
\bar{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj}) = |\bar{a}_S(\tau^{nj})| \exp\{i \bar{a}_S(\tau^{nj})\}; \tag{5.18a}
\]

\[
d\bar{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj}) = d\left( |\bar{a}_S(\tau^{nj})| \exp\{i \alpha_S(\tau^{nj})\} \right); \tag{5.18b}
\]

where we have also introduced the separation of the exciton-related, diagonal elements \(d\hat{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj}) = d(|\bar{a}_S(\tau^{nj})| \exp\{i \alpha_S(\tau^{nj})\})\) into absolute value \(|\bar{a}_S(\tau^{nj})|\) and phase \(\alpha_S(\tau^{nj})\). However, in order to compare real and imaginary parts of the transformations \((5.17a)(5.18b)\), we perform a phase rotation from \(d\hat{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj})\) to \(d\bar{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj})\) with absolute value \(|\bar{a}_S(\tau^{nj})|\) and corresponding phase \(\bar{a}_S(\tau^{nj})\)

\[
\bar{a}_{ss}^{(2)}(\tau'^{nj}; \tau^{nj}) = |\bar{a}_S(\tau^{nj})| \exp\{i \bar{a}_S(\tau^{nj})\}; \tag{5.19}
\]
Accordingly, we can replace the diagonal, Euclidean, pair condensate integration variables of eqs. (5.17a, 5.17b, 5.17c) by \( \tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)}) \), \( (+c.c.) \) and obtain new relations between the coset eigenvalues \( \tilde{F}_\Sigma(\tilde{\tau}_{j,x}^{(n)}) = |\tilde{F}_\Sigma(\tilde{\tau}_{j,x}^{(n)})| \exp\{i \phi_S(\tilde{\tau}_{j,x}^{(n)})\} \) and the new Euclidean elements \( \tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)}) = |\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})| \exp\{i \tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})\} \) (5.19). Note, that the absolute values transform as state variables whereas the phase transformations are path dependent, therefore, can only result from contour time integrals

\[
\frac{d|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})| + i|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{|\tilde{F}_\Sigma(\tilde{\tau}_{j,x}^{(n)})|} = \frac{i}{2} \frac{\sinh(2|\tilde{F}_\Sigma(\tilde{\tau}_{j,x}^{(n)})|)}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})} \frac{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})};
\]

\[
\phi_S(\tilde{\tau}_{j,x}^{(n)}) - \phi_S(\tilde{\tau}_{j,x}^{(n)}) = \int_{\tilde{\tau}_{j,x}^{(n)}}^{\tilde{\tau}_{j,x}^{(n)}} d\phi_S(\tilde{\tau}_{j,x}^{(n)}) - \int_{\tilde{\tau}_{j,x}^{(n)}}^{\tilde{\tau}_{j,x}^{(n)}} d\phi_S(\tilde{\tau}_{j,x}^{(n)}) = - \int_{\tilde{\tau}_{j,x}^{(n)}}^{\tilde{\tau}_{j,x}^{(n)}} \frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{\sinh\left(\frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right)} \frac{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})};
\]

\[
\hat{g}_{\mu,s,\mu,s}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)}) - \hat{g}_{\mu,s,\mu,s}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)}) = \exp\left\{ \frac{i}{2} \frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{\sinh\left(\frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right)} \frac{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right\};
\]

\[
\frac{d\tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)})}{d\tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)})} = \exp\left\{ \frac{i}{2} \frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{\sinh\left(\frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right)} \frac{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right\};
\]

\[
\sum_{\tilde{\tau}_{j,x}^{(n)}} \frac{d\tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)})}{d\tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)})} = \exp\left\{ \frac{i}{2} \frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{\sinh\left(\frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right)} \frac{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right\};
\]

Consequently, we have the final relations between absolute values (5.21a) and the phases (5.21b) where the Euclidean field variables \( \tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)}) = |\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})| \exp\{i \tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})\} \) have to replace the original diagonal elements \( \tilde{a}_{ss}^{(2)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)}) \) by subsequent relation (5.21c), due to the inclusion of the phase rotation (5.19)

\[
|\tilde{F}_\Sigma(\tilde{\tau}_{j,x}^{(n)})| = |\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|;
\]

\[
\phi_S(\tilde{\tau}_{j,x}^{(n)}) = \int_{\tilde{\tau}_{j,x}^{(n)}}^{\tilde{\tau}_{j,x}^{(n)}} d\phi_S(\tilde{\tau}_{j,x}^{(n)}) - \int_{\tilde{\tau}_{j,x}^{(n)}}^{\tilde{\tau}_{j,x}^{(n)}} d\phi_S(\tilde{\tau}_{j,x}^{(n)}) = \exp\left\{ \frac{i}{2} \frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{\sinh\left(\frac{2|\tilde{a}_S(\tilde{\tau}_{j,x}^{(n)})|}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right)} \frac{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}{d\phi_S(\tilde{\tau}_{j,x}^{(n)})}\right\}.
\]

5.3 Nonlinear sigma model with Euclidean coset fields SO(\( \mathfrak{g} \), \( \mathfrak{h} \))/U(\( \mathfrak{g} \)) \( \otimes \) U(\( \mathfrak{h} \))

It remains to collect and summarize the transformation to Euclidean field degrees of freedom and their relation to the eigenvalues of the coset generators following from sections 5.1, 5.2. According to these transformations, the final nonlinear sigma model, following from (4.20a, 4.20b), is determined to be

\[
\tilde{Z}[\tilde{g}] \approx \int \left[ \tilde{a}_{ss}^{(k)}(\tilde{\tau}_{j,x}^{(n)}; \tilde{\tau}_{j,x}^{(n)}) \right] \left[ \tilde{T}^{-1}(\tilde{a}) \tilde{T}(\tilde{a}) \tilde{T}^{-1}(\tilde{a}) \tilde{T}(\tilde{a}) \right] \times \exp\left\{ \frac{1}{2} \sum_{\tilde{x}} \sum_{j=0}^{2N+1} \sum_{\mu=e,h} \sum_{s=\uparrow,\downarrow} \times \text{Tr}_{a,b} \left[ \ln \left( \mathbf{1} - (\hat{\partial}_{\mu,s} \hat{g}_{a}^\mu) \hat{a}^\mu \right) \mathbf{1} \hat{a}^\mu \hat{a}^\mu \right] \right\},
\]
with the sub-functional $3\{\hat{T}^{-1}(\hat{a}); \hat{T}^{-1}(\hat{a}); \hat{H}\}$ (4.15) containing the driving laser field for creation of exciton quasi-particles in the off-diagonal, anomalous parts

$$3\{\hat{T}^{-1}; \hat{T}; \hat{H}\} = \int d[\delta\hat{\Sigma}_{D;\mu,s;\mu,s}; (\hat{T}^{(\eta_2)}, \hat{x}; \hat{T}^{(\eta_1)}, \hat{x})] \text{ Poly}(\delta\hat{\lambda}_{\mu,s}(\hat{T}^{(\eta_1)}, \hat{x})) \times \exp\left\{ \frac{i}{\hbar} \sum_{j=1,2}^{2N} \right\} \quad (5.23)$$

$$\times \hat{\psi}^{-1}(\hat{x}_2, \hat{T}^{(\eta_2)}, \hat{x}_1, \hat{T}) \times \text{Tr}_{a,b} \left[ \hat{S}^{ab}_{\mu_2;\mu_2,s_2;\eta_1} \hat{T}^{(\eta_2)}_{\mu_2;\mu_2,s_2;\eta_1} \hat{T}^{(\eta_1)}_{\mu_1;\mu_1,s_1;\eta_1} \hat{S}^{aa}_{\mu_1;\mu_1,s_1;\eta_1} \right] \times \hat{T}^{-1}_{\mu_3;\mu_3,s_3;\eta_1} \hat{T}^{(\eta_3)}_{\mu_3;\mu_3,s_3;\eta_1} \hat{T}^{(\eta_2)}_{\mu_2;\mu_2,s_2;\eta_2} \hat{T}^{(\eta_1)}_{\mu_1;\mu_1,s_1;\eta_1}$$

$$\times \text{Tr}_{a,b} \left[ \hat{S}^{ab}_{\mu_2;\mu_2,s_2;\eta_1} \hat{T}^{(\eta_2)}_{\mu_2;\mu_2,s_2;\eta_1} \hat{T}^{(\eta_1)}_{\mu_1;\mu_1,s_1;\eta_1} \hat{S}^{aa}_{\mu_1;\mu_1,s_1;\eta_1} \right] \times \hat{T}^{-1}_{\mu_3;\mu_3,s_3;\eta_1} \hat{T}^{(\eta_3)}_{\mu_3;\mu_3,s_3;\eta_1} \hat{T}^{(\eta_2)}_{\mu_2;\mu_2,s_2;\eta_2} \hat{T}^{(\eta_1)}_{\mu_1;\mu_1,s_1;\eta_1}$$

The increments of Euclidean path integration fields $\tilde{\Delta}(s'; t')$ (5.3a,5.3c) are specified by quaternion matrix elements $\Delta\hat{a}^{(k)}_{s_5 s_1}(\hat{T}^{(\eta_j)}, \hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)}, \hat{T}^{(\eta_j)})$ with the exception of the diagonal elements $\Delta\hat{a}^{(k)}_{s s}(\hat{T}^{(\eta_j)}, \hat{T}^{(\eta_j)})$ which incorporate a contour time dependent phase correction (5.19), denoted by the tilde $\tilde{\cdot}$ in $\tilde{\Delta}(s'; t')$

$$\tilde{\Delta}(s'; t') = -\Delta\hat{a}^T(s'; t') = \sum_{k=0}^{3} (\hat{\tau}_k)_{\mu_5;\mu_5} \Delta\hat{a}^{(k)}_{s_5 s_1}(\hat{T}^{(\eta_j)}, \hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)}, \hat{T}^{(\eta_j)}) +$$

$$+ (\hat{\tau}_2)_{\mu_5;\mu_5} \exp\left\{ \frac{i}{\hbar} \int_{\hat{T}^{(++)}}^{\hat{T}^{(--)}} d\hat{T}^{(qr)} \left( \frac{2}{\sinh(2|\Delta\hat{a}^{(qr)}_{s^{--}}|)} - 1 \right) \frac{\partial\Delta\hat{a}^{(qr)}_{s^{--}}(\hat{T}^{(qr)}; \hat{T}^{(qr)})}{\partial T^{(qr)}} \right\} \Delta\hat{a}^{(2)}_{s s}(\hat{T}^{(qr)}; \hat{T}^{(qr)});$$

$$\Delta\hat{a}^{(2)}_{s s}(\hat{T}^{(qr)}; \hat{T}^{(qr)}) = |\Delta\hat{a}^{(qr)}_{s^{--}}| \exp\{i \Delta\hat{a}^{(qr)}_{s^{--}}\};$$

$$d[\hat{a}^{(k)}_{s'; s}(\hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)})] = \prod_{s'=s+2}^{s=1} \prod_{j=1}^{s=2N} \left\{ \hat{x}_{j,0} \right\}$$

$$\times \prod_{k=0}^{3} \left\{ \sum_{s'=s+2}^{s=1} \prod_{j=1}^{s=2N} \left( \text{except : } (j' = j_2) \text{ & } (s' = S) \right) \times$$

$$\times \Delta\hat{a}^{(k)}_{s' s}(\hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)}; \hat{T}^{(\eta_j)}) \right\}^{1/2} \right\}.$$
we have introduced generalized gradient operators \((4.21,4.22)\), \((5.26,5.28)\) with the adjoint action of the logarithm of the density-related one-particle operator \((4.23)\)

\[
\hat{\delta}_{(4)} \text{(matrix)}^{ba} := \left[ \hat{H} \left[ \hat{\rho} \right]^{bb} \text{(matrix)}^{ba} \right] \hat{H} \left[ \hat{\rho} \right]^{aa} - \text{(matrix)}^{ba} ; \\
\hat{\delta}_{\ln(-\hat{H})} \text{(matrix)}^{ba} := \left[ \exp \left\{ \left[ \ln(-\hat{H} [\hat{\rho}]), \ldots \right] \right\} - 1 \right] \text{(matrix)}^{ba} ; \\
\left( \hat{\delta}_{\ln(-\hat{H})} \hat{\rho} \right)^{ba} \left( 3'; 2' \right) \left( 5'; 1' \right) := \left[ \exp \left\{ \left[ \ln(-\hat{H} [\hat{\rho}]), \ldots \right] \right\} - 1 \right] \hat{\rho}^{ba} \left( 3'; 2' \right) \left( 5'; 1' \right). 
\]

(5.26), (5.27), (5.28)

The logarithm of the density-related, block diagonal operator \(\hat{H} \left[ \hat{\rho} \right]\) allows to simplify the action of the adjoint operator \(a_{d(4)} \) in \((5.27,5.28)\) by expanding the logarithm around the unit value with the saddle point solution \(\langle \hat{\rho} \rangle^{aa}_{\nu,s',\mu,s} \left( \gamma_{j_2}^{(n_2)} ; \gamma_{j_1}^{(n_1)} \right)\) and kinetic energy operator of the one-particle part

\[
\ln(-\hat{H} [\hat{\rho}]) = \ln \left( -\hat{\delta}_{\mu,s',\mu,s} \left( \hat{\psi}_2, \gamma_{j_2}^{(n_2)} ; \hat{\psi}_1, \gamma_{j_1}^{(n_1)} \right) - q_v \eta_{j_2} \left( i \frac{\Delta}{\hbar} \right) \langle \hat{\rho} \rangle^{aa}_{\mu,s',\mu,s} \left( \gamma_{j_2}^{(n_2)} ; \psi_2, \gamma_{j_1}^{(n_1)} ; \hat{\psi}_1 \right) q_{\mu} \eta_{j_1} \right) = \\
\ln \left[ \hat{\delta}_{\mu,s',\mu,s} \left( \hat{\psi}_2, \gamma_{j_2}^{(n_2)} ; \hat{\psi}_1, \gamma_{j_1}^{(n_1)} \right) \right] \\
\left[ \exp \left\{ \left[ \ln(-\hat{H} [\hat{\rho}]), \ldots \right] \right\} - 1 \right] \left( \hat{\rho} \right)^{ba} \left( 3'; 2' \right) \left( 5'; 1' \right). 
\]

(5.29a), (5.29b), (5.29c)

The coset matrix \(\hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right)\) \((5.3a,5.3c)\), \((5.30a,5.30c)\) consists of the anomalous-doubled, exciton-related, antisymmetric matrices \(\Delta \hat{a} \left( \gamma_{j}^{(n)} \right)\), \(\Delta \hat{a} \left( \gamma_{j}^{(n)} \right)\) as the independent field degrees of freedom which prescribe the dependent, block diagonal, density-related parts \((5.30d,5.30e)\)

\[
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = - \left( \hat{P} \left( \gamma_{j}^{(n)} \right) \hat{T}^{\left( \gamma_{j}^{(n)} \right)} \left( \Delta \hat{T} \left( \gamma_{j}^{(n)} \right) \right) \right) \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{T} \left( \gamma_{j}^{(n)} \right) \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \sum_{k=0}^{3} \left( \hat{T}_{k} \right)_{\mu_5 \mu_1} \Delta \hat{a}^{(k)} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = - \Delta \hat{\delta}^{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) ; \\
\Delta \hat{T}_{\gamma_{j}^{(n)} ; \gamma_{j}^{(n)}} = - \left( \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \sum_{k=0}^{3} \left( \hat{T}_{k} \right)_{\mu_5 \mu_1} \Delta \hat{a}^{(k)} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = - \Delta \hat{\delta}^{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) ; \\
\Delta \hat{T}_{\gamma_{j}^{(n)} ; \gamma_{j}^{(n)}} = - \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \sum_{k=0}^{3} \left( \hat{T}_{k} \right)_{\mu_5 \mu_1} \Delta \hat{a}^{(k)} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = - \Delta \hat{\delta}^{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) ; \\
\Delta \hat{T}_{\gamma_{j}^{(n)} ; \gamma_{j}^{(n)}} = - \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = \sum_{k=0}^{3} \left( \hat{T}_{k} \right)_{\mu_5 \mu_1} \Delta \hat{a}^{(k)} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \\
\Delta \hat{\delta}^{ba} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) = - \Delta \hat{\delta}^{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) ; \\
\Delta \hat{T}_{\gamma_{j}^{(n)} ; \gamma_{j}^{(n)}} = - \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{T} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \left( \Delta \hat{P} \left( \gamma_{j}^{(n)} ; \gamma_{j}^{(n)} \right) \right) \
\]
\[ \Delta \hat{y}^{11}(\tau'; \tau) = -\Delta \hat{y}^{22, T}(\tau'; \tau) = -\frac{i}{2} \sum_{k=0}^{3} (\hat{\tau}_k \hat{\tau}_2)_{\mu_5 \mu_1} \times \]
\[ \times \left\{ \tanh \left( \frac{\hat{a}_{s_5}(T_{\tau_5}^{(n_5)}) + \hat{a}_{s_1}(T_{\tau_1}^{(n_1)})}{2} \right) - \tanh \left( \frac{\hat{a}_{s_5}(T_{\tau_5}^{(n_5)}) - \hat{a}_{s_1}(T_{\tau_1}^{(n_1)})}{2} \right) \right\} \times \]
\[ \times \exp \left\{ -i \int_{t_0}^{t_{j_1}} d\tau \frac{2 |\hat{a}_{s_1}(T_{\tau_1}^{(n_1)})|}{\sinh (2 |\hat{a}_{s_1}(T_{\tau_1}^{(n_1)})|)} \frac{\partial \hat{a}_{s_1}(T_{\tau_1}^{(n_1)})}{\partial \tau} \right\} \Delta \hat{a}_{s_5 s_1}(T_{\tau_s}^{(n_s)}; T_{\tau_1, \tau_1}^{(n_1)}) + \]
\[ - (-1)^k \left[ \tanh \left( \frac{\hat{a}_{s_5}(T_{\tau_5}^{(n_5)}) + \hat{a}_{s_1}(T_{\tau_1}^{(n_1)})}{2} \right) + \tanh \left( \frac{\hat{a}_{s_5}(T_{\tau_5}^{(n_5)}) - \hat{a}_{s_1}(T_{\tau_1}^{(n_1)})}{2} \right) \right] \times \]
\[ \times \exp \left\{ i \int_{t_0}^{t_{j_5}} d\tau \frac{2 |\hat{a}_{s_5}(T_{\tau_5}^{(n_5)})|}{\sinh (2 |\hat{a}_{s_5}(T_{\tau_5}^{(n_5)})|)} \frac{\partial \hat{a}_{s_5}(T_{\tau_5}^{(n_5)})}{\partial \tau} \right\} \Delta \hat{a}_{s_5 s_1}(T_{\tau_s}^{(n_s)}; T_{\tau_1, \tau_1}^{(n_1)}) + \]
\[ - i \delta_{\mu_5 \mu_1} \delta_{s_s s_1} \tanh (|\hat{a}_{s_5}(T_{\tau_5}^{(n_5)})|) |\hat{a}_{s_5}(T_{\tau_5}^{(n_5)})| \Delta \hat{a}_{s_5}(T_{\tau_5}^{(n_5)}) ; \]
\[ \text{(last term} \quad (\hat{a}_{s_5}^{(n_5)} := T_{\tau_s}^{(n_s)}; \tau_5 = T_{\tau_5}^{(n_5)})) \text{) and} \quad (\text{matrix}^\dagger = (\text{matrix})^{*T} ; \]
\[ \text{(cf. gauge fixing in eqs. (5.15a 5.15b) and appendix A)} . \]

The transformations to Euclidean fields yields corrections for the derivation of classical field equations by first order variation of the anomalous field parts \( \Delta \hat{a}(\tau'; \tau) \) and becomes even more important for higher order variation of fluctuation terms (cf. section 1.2). The derivations to Euclidean path integration fields within sections 5.1 5.2 therefore hint at the importance of the correct coset integration measure, originally included in the factorization of the total self-energy \( \delta \Sigma_{q_{\mu, s'; \mu, s}}(T_{\tau_2, \hat{x}_2, \hat{v}_2}^{(n_2)}; T_{\tau_1, \hat{x}_1, \hat{v}_1}^{(n_1)}) \) into block diagonal, density-related self-energies as hinge fields of a SSB and into anomalous-related, off-diagonal parts by using a coset decomposition.

6 Summary and conclusion

6.1 Observable quantities by differentiating with the generating source term

In sections 3.1 to 3.3 we have performed the three possible Gaussian transformations of increasing complexity in order to convert the quartic interaction of Fermi fields to even-valued self-energy variables and matrices. These path integration fields, following from the prevailing HST (3.1a 3.3), (3.1a 3.12c), (3.16 3.21) of sections 3.1 to 3.3, only take values in completely Euclidean spaces and flat integration measures so that the calculation of observables results from differentiation of the corresponding HST-transformed generating function (3.5a 3.5c), (3.13a 3.14), (3.22a 3.22b) with respect to the source field \( \hat{a}_{q_{\mu, s'; \mu, s}}(T_{\tau_2, \hat{x}_2, \hat{v}_2}^{(n_2)}; T_{\tau_1, \hat{x}_1, \hat{v}_1}^{(n_1)}) \) (2.10a). The latter source field allows to track the original observables, as combinations of anti-commuting fields in even number, to corresponding Green functions with the prevailing self-energy variable or matrix which can be restricted by the computation of a saddle point equation (3.6 3.8), 3.15a 3.15c, (3.23c 3.23i).

We outline the calculation for the more profound SSB case with the factorization into density- and anomalous-related parts by a coset decomposition according to sections 4 and 5. The density- and Nambu-related parts follow straightforwardly as one tracks the original observables in terms of fermionic coherent states to the independent, anomalous-related, locally Euclidean coset fields \( \hat{a}(\tau'; \tau) \) (5.30a 5.30c) by differentiation with respect.
6.1 Observable quantities by differentiating with the generating source term

\[ \frac{1}{N_x} \chi_{r,s}(\mathcal{J}_{j,x}^{(n)}) \chi_{r,s}(\mathcal{J}_{j,x}^{(n-1)}) \]  

\[ = -N_x \sum_{a,b=1,2} \frac{\partial Z[b]}{\partial \chi_{r,s}(\mathcal{J}_{j,x}^{(n)})} \]  

(6.1a)

\[ \frac{1}{N_x} \chi_{r,s}(\mathcal{J}_{j,x}^{(n)}) \chi_{r,s}(\mathcal{J}_{j,x}^{(n-1)}) \]  

\[ = -2N_x \frac{\partial Z[b]}{\partial \chi_{r,s}(\mathcal{J}_{j,x}^{(n)})} \]  

(6.1b)

\[ \frac{1}{N_x} \chi_{r,s}(\mathcal{J}_{j,x}^{(n)}) \chi_{r,s}(\mathcal{J}_{j,x}^{(n-1)}) \]  

\[ = -2N_x \frac{\partial Z[b]}{\partial \chi_{r,s}(\mathcal{J}_{j,x}^{(n)})} \]  

(6.1c)

\[ \frac{1}{N_x} \chi_{r,s}(\mathcal{J}_{j,x}^{(n)}) \chi_{r,s}(\mathcal{J}_{j,x}^{(n-1)}) \]  

\[ = -2N_x \frac{\partial Z[b]}{\partial \chi_{r,s}(\mathcal{J}_{j,x}^{(n)})} \]  

(6.1d)

Applying relations of sections 4 and 5, we can reduce and simplify the inverse of combined anomalous- and density-related parts in \[ \mathcal{J}_{j,x}^{(n)} \] so that one is reminded of the astonishing result that the exciton quasi-particles in the off-diagonal blocks of \[ \hat{\mathcal{J}}_{j,x}^{(n)} \] are multiplied by the one-particle operator \[ \hat{\mathcal{J}}_j \] in order to achieve the true exciton field

\[ \hat{\mathcal{J}}_{j,x}^{(n)} \]  

\[ = \hat{\mathcal{J}}_{j,x}^{(n)} \]  

(6.2)

In consequence, the 'Nambu'-doubled, locally Euclidean fields \[ \Delta \hat{\mathcal{J}}_j^{(n)} \] have to be regarded as exciton fields relatively to the density-related parts. Further restriction to solely diagonal elements \[ \hat{\mathcal{J}}_{j,x}^{(n)} \] of \[ \hat{\mathcal{J}}_j \] allows to derive and investigate classical soliton equations from first and higher order variations which are determined by the prevailing coset integration measure (cf. section 1.2). Therefore, we have pointed out and intensively examined the symmetries of the corresponding self-energies throughout the paper under inclusion of the precise time contour step ordering of the original time development operator. The derived equations of section 5.3 for the observables 6.1a and 6.2 can be used to study self-induced transparency of ultrashort coherent transients and the area theorem in combination with holography and selective Fourier optics for possible means of reduced absorption within matter. Possible extensions of the presented approaches of sections 3.1, 3.3, and 4.5 may include the quantization of the electromagnetic field which can also
be represented in terms of coherent states so that corresponding HSTs lead to analogous self-energies as for the quartic interaction of fermionic fields.
A  Gauge fixing for 'Nambu'-fields within the coset integration measure

In section 5.2 with eqs. (5.15a,5.15b) we have to require that the quaternionic diagonal, anti-symmetric matrix elements have to vanish in the gauge combination \((d\hat{P})^{-1})_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})\) of the block diagonal matrices \(\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})\) with their contour time derivatives \(d\hat{P}\)

\[
0 = \left( (d\hat{P})^{-1} \right)_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu}) .
\]  

(A.1)

This can be accomplished by a gauge transformation \((A.2)\) of \(\hat{P}(s';\nu')\) with a quaternion diagonal matrix \(\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})\) \((A.3, A.4)\) which has only non-vanishing matrix elements \(\hat{S}_{D,\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu}) \neq 0\) along the \(2 \times 2\) diagonals, just in opposite to \(\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})\) \((4.9a, 4.9b)\). These hermitian \(2 \times 2\) matrix elements \(\hat{S}_{D,\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})\) \((A.5)\) along the main diagonal have to depend on the off-diagonal parameters of the ladder operators in \(\hat{P}_{D}(s';\nu')\) and have to be chosen with suitable dependence in such a manner that the block diagonal, gauge transformed matrices \(\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu}) = 0\) \((A.1)\). One has to take into account the quaternion algebra in order to achieve \((A.7)\) for diagonal elements referring to the quaternion matrix elements with anti-symmetric element \((\hat{\tau}_2)_{\mu_5,\mu_1}\) of exciton quasi-particles

\[
\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu}) = \sum_{\nu'_5,\nu'_1} \hat{P}_{\mu_5,\nu'_5,\mu_1,\nu'_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})
\]

\[
\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu}) = \exp \left\{ \hat{T}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu}) \right\} (T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu}) .
\]  

(A.2)

(A.3)

(A.4)

(A.5)

\[
\left( (d\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1})^{-1} \right)(s';\nu') = \hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(s';\nu') \hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(s';\nu') .
\]

(A.6)

(A.7)

Note that there is no summation over the contour time and coordinate space labels \(T_{j,\hat{x}}^{(n)_j}\) and the spin label \('s'\) in relations \((A.2)(A.7)\), due to the 'diagonal' property of \(\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})\) with respect to these labels, apart from the electron-hole labels \('e', 'h'\) which restrict the degrees of freedom to the anti-symmetric, quaternion element \((\hat{\tau}_2)_{\mu_5,\mu_1}\). A corresponding description holds for relation \((A.6)\) with the quaternion diagonal matrix elements \(\hat{P}_{\mu_5,\nu_5,\mu_1,\nu_1}(T_{j,\hat{x}}^{(n)_j}; T_{\mu,\hat{x}}^{(n)_\mu})\), having only the quaternion elements \((\hat{\tau}_2)_{\mu_5,\mu_1}, (\hat{\tau}_2)_{\mu_2,\mu_1}, (\hat{\tau}_2)_{\mu_3,\mu_1}, (\hat{\tau}_2)_{\mu_4,\mu_1}\) for respective gauge matrices \(\hat{P}_D(s';\nu'), \hat{P}_D^{-1}(s';\nu', \nu')\).
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