An elementary representation of the higher-order Jacobi-type differential equation

Clemens Markett

Abstract

We investigate the differential equation for the Jacobi-type polynomials which are orthogonal on the interval $[-1, 1]$ with respect to the classical Jacobi measure and an additional point mass at one endpoint. This scale of higher-order equations was introduced by J. and R. Koekoek in 1999 essentially by using special function methods. In this paper, a completely elementary representation of the Jacobi-type differential operator of any even order is given. This enables us to trace the orthogonality relation of the Jacobi-type polynomials back to their differential equation. Moreover, we establish a new factorization of the Jacobi-type operator which gives rise to a recurrence relation with respect to the order of the equation.

Key words: orthogonal polynomials, higher-order linear differential equations, Jacobi-type equations, Jacobi-type polynomials, factorization.

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1 Introduction and main result

In 1999, J. and R. Koekoek [4] established a new class of higher-order linear differential equations satisfied by the generalized Jacobi polynomials $\{P_{n}^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$, $\alpha, \beta > -1$, $M, N \geq 0$. These function systems were introduced and studied by T. H. Koornwinder [6] as the orthogonal polynomials with respect to a linear combination of the Jacobi weight function $w_{\alpha,\beta}(x)$ and one or two delta functions at the endpoints of the interval $-1 \leq x \leq 1$.

$$w_{\alpha,\beta,M,N}(x) = w_{\alpha,\beta}(x) + M\delta(x+1) + N\delta(x-1),$$

$$w_{\alpha,\beta}(x) = h_{\alpha,\beta}^{-1}(1-x)^{\alpha}(1+x)^{\beta},$$

$$h_{\alpha,\beta} = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} dx = 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)/\Gamma(\alpha+\beta+2).$$

In the present paper we investigate the so-called Jacobi-type equation with one additional mass point in the weight function, i.e. with either $M$ or $N$ being positive. In terms of the classical Jacobi polynomials [2 Sec. 10.8]

$$P_{n}^{\alpha,\beta}(x) = \frac{(\alpha+1)_{n}}{n!}F_{1}(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2})$$

(1.2)
for $n \in \mathbb{N}_0 = \{0, 1, \cdots \}$, the Jacobi-type polynomials are given by, cf. [4,3],

$$P_{n}^{\alpha,\beta,M,N}(x) = P_{n}^{\alpha,\beta}(x) + MQ_{n}^{\alpha,\beta}(x) + NR_{n}^{\alpha,\beta}(x), \quad n \in \mathbb{N}_0, \quad M \cdot N = 0,$$

where, provided that $A_{n}^{\alpha,\beta} = (\alpha + 2)_{n-1} (\alpha + \beta + 2)_n / [2n!(\beta + 1)_n]$, 

$$Q_{n}^{\alpha,\beta}(x) = A_{n}^{\beta,\alpha}(x + 1)P_{n-1}^{\alpha,\beta+2}(x), \quad n \in \mathbb{N}, \quad Q_{0}^{\alpha,\beta}(x) = 0,$$

$$R_{n}^{\alpha,\beta}(x) = A_{n}^{\alpha,\beta}(x - 1)P_{n-1}^{\alpha+2,\beta}(x), \quad n \in \mathbb{N}, \quad R_{0}^{\alpha,\beta}(x) = 0.$$ 

By means of the well-known relationship $P_{n}^{\alpha,\beta}(x) = (-1)^n P_{n}^{\beta,\alpha}(-x)$, it follows that 

$$Q_{n}^{\alpha,\beta}(x) = (-1)^n R_{n}^{\beta,\alpha}(-x), \quad P_{n}^{\alpha,\beta,M,0}(x) = (-1)^n P_{n}^{\beta,\alpha,0,M}(-x), \quad n \in \mathbb{N}_0, \quad M > 0.$$ 

Hence it suffices to treat, for instance, the case $M = 0, N > 0$ in full detail. The corresponding results for $M > 0, N = 0$ then follow immediately.

For $\alpha \in \mathbb{N}_0$ and any $\beta > -1, N > 0$, J. and R. Koekoek [4] found that the Jacobi-type polynomials $\{P_{n}^{\alpha,\beta,0,N}(x)\}_{n=0}^{\infty}$ satisfy a linear differential equation of order $2\alpha + 4$ which, for our purpose, is conveniently described in the form 

$$N\{L_{2\alpha+4,x}^{\alpha,\beta} - \Lambda_{2\alpha+4,n}^{\alpha,\beta}\}y(x) + C_{\alpha,\beta}\{L_{2,x}^{\alpha,\beta} - \Lambda_{2,n}^{\alpha,\beta}\}y(x) = 0, \quad -1 < x < 1.$$ 

The crucial part of this equation consists in the higher-order differential expression 

$$L_{2\alpha+4,x}^{\alpha,\beta}y(x) = \sum_{i=1}^{2\alpha+4} d_{i}^{\alpha,\beta}(x) D_{x}^{i}y(x)$$

with coefficient functions involving, among others, a generalized hypergeometric sum,

$$d_{i}^{\alpha,\beta}(x) = -(\alpha + 2)!(\beta + 1)_{\alpha+2} \sum_{k=\text{max}(0,i-\alpha-3)}^{i-1} \frac{(-2)^{i}(\alpha + 3)_{i-1-k}(\beta + 3)_{i-1-k}}{(\alpha - 2)_{i-1-k}(\beta + 1)_{i-1-k}(i-1-k)!k!} \cdot F_{3}^{\alpha+3,\beta+i+2-k, \alpha+2}(\beta+i-k, i+1-k; 1)(\frac{x-1}{2})^{k+1}. \quad (1.9)$$

Throughout this paper, $D_{x}^{i} \equiv (D_{x})^{i}$ denotes the $i$-fold differentiation with respect to $x$. Notice that the highest coefficient function in the sum [1.8] simplifies to $d_{2\alpha+4}^{\alpha,\beta}(x) = (x^2 - 1)^{\alpha+2}$, see [2.4] below. Furthermore, the eigenvalue parameters and the coupling constant read 

$$\Lambda_{2\alpha+4,n}^{\alpha,\beta} = (n)_{\alpha+2}(n + \beta)_{\alpha+2}, \quad \Lambda_{2,n}^{\alpha,\beta} = n(n + \alpha + \beta + 1), \quad C_{\alpha,\beta} = (\alpha + 2)!(\beta + 1)_{\alpha+1} \quad (1.10)$$

In particular, when $N$ tends to zero, equation [1.7] reduces to the classical equation for the Jacobi polynomials $P_{n}^{\alpha,\beta}(x)$, based on the second-order differential operator

$$L_{2,x}^{\alpha,\beta}y(x) = \{(x^2 - 1)D_{x}^{2} + [\alpha - \beta + (\alpha + \beta + 2)x] D_{x}\}y(x) = (x-1)^{-\alpha}(x+1)^{-\beta} D_{x}[(x-1)^{\alpha+1}(x+1)^{\beta+1}D_{x}y(x)]. \quad (1.11)$$
For the lowest parameter value \( \alpha = 0 \), the Jacobi-type equation (1.7) belongs to the very few fourth-order differential equations with polynomial solutions, which were discovered by H. L. Krall [9] in 1940 and further investigated by A. M. Krall and L. L. Littlejohn [7], [8]. To our knowledge, it was also Littlejohn who explicitly determined the sixth- and eighth-order Jacobi-type equations for \( \alpha = 1 \) and \( \alpha = 2 \), both in normal and symmetric form. Later and more generally, Kwon, Littlejohn, and Yoon [10] characterized all orthogonal polynomial systems satisfying a finite order differential equation of spectral type and named them BKOPS after Bochner and Krall. Zhedanov [12] stated some necessary conditions for the polynomials \( P_n^{\alpha, \beta, M, N} (x) \) to belong to this class. In addition he gave a representation of the Jacobi-type differential operator in a form similar to (1.8), (1.9).

Almost simultaneously and quite differently to [4], Bavinck [1] used some operator theoretical arguments to present the Jacobi-type differential operator in the factorized form

\[
L_{2\alpha+4,x}^{\alpha, \beta} y(x) = \prod_{j=0}^{\alpha+1} \left\{ \left( L_{2,x}^{\alpha, \beta} - \frac{2(\alpha + 1)}{x - 1} + j(\alpha + \beta + 1 - j) \right) y(x) \right\},
\]

(1.12)

The main purpose of this paper is to establish the following elementary representation of the higher-order differential expression in equation (1.7).

**Theorem 1.1.** For any \( \alpha \in \mathbb{N}_0 \), \( \beta > -1 \), and for any sufficiently smooth function \( y(x) \),

\[
L_{2\alpha+4,x}^{\alpha, \beta} y(x) = \frac{x - 1}{(x + 1)^\beta} D_x^{\alpha+2} \left\{ (x + 1)^{\alpha+\beta+2} D_x^{\alpha+2} \left[ (x - 1)^{\alpha+1} y(x) \right] \right\}, -1 < x < 1.
\]

(1.13)

Consequently, the Jacobi-type polynomials \( P_n^{\alpha, \beta, 0, N} (x), n \in \mathbb{N}_0 \), arise as eigensolutions of the equation

\[
N(x - 1)^{\alpha+1} D_x^{\alpha+2} \left\{ (x + 1)^{\alpha+\beta+2} D_x^{\alpha+2} \left[ (x - 1)^{\alpha+1} y_n(x) \right] \right\} + C_{\alpha, \beta} D_x \left[ (x - 1)^{\alpha+1} (x + 1)^{\beta+1} D_x y(x) \right] = \Lambda_{2\alpha+4,n}^{\alpha, \beta, N}(x - 1)^{\alpha}(x + 1)^{\beta} y_n(x)
\]

(1.14)

with combined eigenvalue parameter

\[
\Lambda_{2\alpha+4,n}^{\alpha, \beta, N} = [N(n + 1)_{\alpha+1}(n + \beta)_{\alpha+1} + (2)_{\alpha+1}(\beta + 1)_{\alpha+1}] n(n + \alpha + \beta + 1).
\]

(1.15)

**Corollary 1.2.** For \( \beta \in \mathbb{N}_0 \) and any \( \alpha > -1 \), \( M > 0 \), the Jacobi-type orthogonal polynomials \( y_n(x) = P_n^{\alpha, \beta, M, 0} (x) = P_n^{\alpha, \beta} (x) + MQ_n^{\alpha, \beta} (x), n \in \mathbb{N}_0 \), satisfy the linear differential equation of order \( 2\beta + 4 \),

\[
M \{ \tilde{L}_{2\beta+4,x}^{\beta, \alpha} - \Lambda_{2\beta+4,n}^{\beta, \alpha} \} y_n(x) + C_{\beta, \alpha} \{ L_{2,x}^{\alpha, \beta} - \Lambda_{2,n}^{\alpha, \beta} \} y_n(x) = 0, -1 < x < 1,
\]

(1.16)

\[
\tilde{L}_{2\beta+4,x}^{\beta, \alpha} y_n(x) = \frac{x + 1}{(x - 1)^\alpha} D_x^{\beta+2} \left\{ (x - 1)^{\alpha+\beta+2} D_x^{\beta+2} \left[ (x + 1)^{\beta+1} y_n(x) \right] \right\}.
\]

(1.17)
Proceeding from the sum (1.8) defining $L_{2\alpha+4,-\xi}^{\alpha,\beta}$ and substitute $x = -\xi$ to obtain
\[ L_{2\beta+4,-\xi}^{\alpha,\beta} y_n(-\xi) = L_{2\beta+4,\xi}^{\alpha,\beta} y_n(\xi), \quad L_{2\alpha+\xi}^{\alpha,\beta} y_n(-\xi) = L_{2\alpha+\xi}^{\alpha,\beta} y_n(\xi), \quad L_{2,n}^{\alpha,\beta} = L_{2,n}^{\alpha,\beta}. \]

The proof of Theorem 1.1 is carried out in Section 2 by verifying the equivalence of the two representations (1.13) and (1.8), (1.9). Another, direct proof is postponed to Section 5.

In addition to gaining some deeper insight into the nature of the Jacobi-type equation, the new results reveal a number of nice properties that make them accessible for wider applications. In Section 3 we show that the Jacobi-type differential operator is symmetric with respect to the scalar product associated with the weight function $w_{\alpha,\beta,0,N}(x)$. This may lay foundation to a spectral theoretical treatment of the Jacobi-type equation (1.7). In particular, it follows that its polynomial solutions are mutually orthogonal in the respective space. Another interesting feature to be discussed in Section 3 is a new factorization of the differential operator $L_{2\alpha+4,x}^{\alpha,\beta}$ into a product of $\alpha + 2$ linear second-order differential expressions, which is distinct from Bavinck’s (1.12). This leads to a significant recurrence relation with respect to the order of the differential equation. Finally, in Section 6 we show how certain Jacobi-type equations are related to the equation for the symmetric ultraspherical-type polynomials $P_n^{\alpha,\alpha,N,N}(x)$, $\alpha \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, $N > 0$. Originally due to R. Koekoek [5], this equation was recently stated by the author [11] in a form analogous to (1.7) and (1.13), i.e.

\[ N\{L_{2\alpha+4,x}^{\alpha,\alpha} - \Lambda_{2\alpha+4,n}\} y(x) + C \{L_{2,x}^{\alpha,\alpha} - \Lambda_{2,n}^{\alpha}\} y(x) = 0, \quad -1 < x < 1. \tag{1.18} \]

The constant linking the two terms is given here by $C_{\alpha} = \frac{1}{2}(\alpha + 2)(2\alpha + 2)!$, while

\[ L_{2\alpha+4,x}^{\alpha,\alpha} y(x) = (x^2 - 1)D_x^{2\alpha+4}[(x^2 - 1)^{\alpha+1} y(x)], \quad \Lambda_{2\alpha+4,n} = (n - 1)_{2\alpha+4} \]
\[ L_{2,x}^{\alpha,\alpha} = \{(x^2 - 1)D_x^2 + 2(\alpha + 1)x D_x\} y(x), \quad \Lambda_{2,n}^{\alpha} = n(n + 2\alpha + 1). \tag{1.19} \]

## 2 Proof of Theorem 1.1

Proceeding from the sum (1.8) defining $L_{2\alpha+4,x}^{\alpha,\beta}$, we rewrite the coefficient functions $d_{\alpha,\beta}^{i,\alpha}(x)$, $1 \leq i \leq \alpha + 2$, as follows. Firstly, a repeated use of Vandermonde’s summation formula

\[ _2F_1(-n,p,q;1) = (q - p)_n/(q)_n, \quad q > 0, \quad n \in \mathbb{N}_0, \]

yields

\[ _3F_2 \left( \binom{-k,b,p}{c,q};1 \right) = \sum_{m=0}^{k} \frac{(-k)_m(b)_m}{m!(c)_m} \sum_{j=0}^{m} \frac{(-m)_j(q - p)_j}{j!(q)_j}, \]
\[ = \sum_{j=0}^{k} (-1)^j \frac{(-k)_j(b)_j(q - p)_j}{j!(c)_j(q)_j} \sum_{m=j}^{k} \frac{(j - k)_m - (b + j)_m - j}{(m-j)! (c + j)_m - j}, \]
\[ = \frac{1}{(c)_k} \sum_{m=0}^{k} (-1)^m \frac{(-k)_m(b)_m(b)_m - (q - p)_m}{m!(q)_m}. \]
and thus, choosing \( p = \alpha + \beta + 3 \) and \( q = \beta + i - k \),

\[
\begin{align*}
3F2\left( \frac{-k, \alpha + i + 2 - k, \alpha + \beta + 3}{i + 1 - k, \beta + i - k}; 1 \right) &= \frac{1}{(i + 1 - k)_k}, \\
\sum_{m=\max(0,i-\alpha-3)}^{\min(k,\alpha+1)} (-1)^m \frac{(-k)_m (-\alpha - 1)_m (\alpha + i + 2 - k)_{k-m} (i - k - \alpha - 3)_{k-m}}{m! (\beta + i - k)_{k-m}}
\end{align*}
\]

Inserting this expression into the right-hand side of (1.9) and observing that

\[
\begin{align*}
(\alpha + 3)_{i-1-k}(\alpha + i + 2 - k)_{k-m} &= (\alpha + 3)_{i-1-m} = (\alpha + i + 1 - m)!/(\alpha + 2)!
\end{align*}
\]

\[
\begin{align*}
(-\alpha - 2)_{i-1-k}(i - k - \alpha - 3)_{k-m} &= (-\alpha - 2)_{i-1-m} = (-1)^{i-1-m}(\alpha + 2)!/(\alpha + 3 - i + m)!
\end{align*}
\]

\[
\begin{align*}
(\beta + 1)_{i-1-k}(\beta + i - k)_{k-m} &= (\beta + 1)_{i-1-m}, (i - k)!/(i + 1 - k)_k = i!
\end{align*}
\]

we arrive at the double sum

\[
\begin{align*}
a^{\alpha,\beta}_i (x) &= A^{\alpha,\beta}_{2\alpha+4,i} \sum_{k=\max(0,i-\alpha-3)}^{\min(k,\alpha+1)} \frac{1}{(i - 1 - k)_k! k!} \left( \frac{x - 1}{2} \right)^{k+1} - \alpha - 1)_m (\alpha + 1 + i - m)! \\
&\quad \cdot \sum_{m=\max(0,i-\alpha-3)}^{\min(k,\alpha+1)} (-k)_m (-\alpha - 1)_m (\alpha + 1 + i - m)! \\
&\quad \cdot \sum_{m=\max(0,i-\alpha-3)}^{\min(k,\alpha+1)} (\beta + 1)_{i-1-m}(\alpha + 3 - i + m)! \\
&\quad \cdot \sum_{m=\max(0,i-\alpha-3)}^{\min(k,\alpha+1)} (\beta + 1)_{i-1-m}(\alpha + 3 - i + m)!
\end{align*}
\]

On the other hand, we expand the new representation (1.13) of \( L^{\alpha,\beta}_{2\alpha+4,x} \) in the form

\[
\frac{x - 1}{(x + 1)^\beta} D_x^{\alpha+2} \{(x + 1)^{\alpha+\beta+2} D_x^{\alpha+2}[(x - 1)^{\alpha+1} y(x)]\} = \sum_{i=1}^{2\alpha+4} e^{\alpha,\beta}_i (x) D^i_x y(x).
\]

So we have to verify that \( e^{\alpha,\beta}_i (x) = a^{\alpha,\beta}_i (x) \) for all \( 1 \leq i \leq 2\alpha + 4 \). To begin with, we see that

\[
\begin{align*}
\frac{x - 1}{(x + 1)^\beta} D_x^{\alpha+2} \{(x + 1)^{\alpha+\beta+2} D_x^{\alpha+2}[(x - 1)^{\alpha+1} y(x)]\} &= \frac{x - 1}{(x + 1)^\beta} D_x^{\alpha+2} \{(x + 1)^{\alpha+\beta+2} D_x^{\alpha+2}[(x - 1)^{\alpha+1} y(x)]\} \\
&= \sum_{s=1}^{\alpha+2} \binom{\alpha + 2}{s} \frac{(\alpha + 1)!}{(s - 1)!} \frac{x - 1}{(x + 1)^\beta} D_x^{\alpha+2} \{(x + 1)^{\alpha+\beta+2} (x - 1)^{s-1} D_x^s y(x)\} \\
&= \sum_{r=0}^{\alpha+2} \binom{\alpha + 2}{r} \frac{(\alpha + 1)!}{(s - 1)!} \frac{x - 1}{(x + 1)^\beta} D_x^{\alpha+2-r} \{(x + 1)^{\alpha+\beta+2} (x - 1)^{s-1} D_x^s y(x)\}. \\
\end{align*}
\]
Substituting \( s + r = i \), \( 0 \leq r \leq \alpha + 2 \), the index \( s \) ranges over \( i - \alpha - 2 \leq s \leq i \). Hence,

\[
e_{i}^{\alpha,\beta}(x) = \sum_{s = \max(1, i - \alpha - 2)}^{\min(\alpha + 2, i)} \left( \frac{\alpha + 2}{s} \right) \left( \frac{\alpha + 1}{i - s} \right) a_{s}^{\alpha,\beta,i}(x)
\]

with

\[
a_{s}^{\alpha,\beta,i}(x) = \frac{x - 1}{(x + 1)^{\beta}} D_{x}^{\alpha+2-i+s} [(x + 1)^{\alpha+\beta+2}(x - 1)^{s-1}] 
\]

\[
= \sum_{t = \max(0, \alpha + 3 - i)}^{\alpha + 2 - i + s} \left( \frac{\alpha + 2 - i + s}{t} \right) \frac{x - 1}{(x + 1)^{\beta}} D_{x}^{t} [(x + 1)^{\alpha+\beta+2}] D_{x}^{\alpha+2-i+s-t} [(x - 1)^{s-1}]
\]

Inserting the last sum into (2.2) and interchanging the order of summation we achieve

\[
e_{i}^{\alpha,\beta}(x) = \sum_{t = \max(0, \alpha + 3 - i)}^{\min(2\alpha + 4 - i, \alpha + 2)} b_{t}^{\alpha,\beta,i}(x) \frac{(\beta + 1)_{\alpha + 2}}{(\beta + 1)_{\alpha + 2 - t}} \frac{(\alpha + 1)! (x + 1)^{\alpha + 2 - t} (x - 1)^{i - \alpha - 2 + t}}{(i - \alpha - 3 + t)!},
\]

where the new inner sum, \( b_{t}^{\alpha,\beta,i}(x) \), simplifies to

\[
b_{t}^{\alpha,\beta,i}(x) = \sum_{s = t + i - \alpha - 2}^{\min(2\alpha + 4 - i, t, \alpha + 2 - t)} \left( \frac{\alpha + 2}{s} \right) \left( \frac{\alpha + 2 - i + s}{i - s} \right) \left( \frac{\alpha + 2 - t}{t} \right)
\]

\[
= \sum_{s = 0}^{\min(2\alpha + 4 - i, t, \alpha + 2 - t)} \frac{(\alpha + 2)!}{(s + t + i - \alpha - 2)!(2\alpha + 4 - i - t - s)!(\alpha + 2 - t - s)!s!t!}
\]

\[
= \left( \frac{\alpha + 2}{t} \right) \left( \frac{\alpha + 2}{t + i - \alpha - 2} \right) \frac{(\alpha + 3)_{\alpha + 2 - t}}{(t + i - \alpha - 1)_{\alpha + 2 - t}} = \left( \frac{\alpha + 2}{t} \right) \left( \frac{2\alpha + 4 - t}{i} \right).
\]

Moreover we use

\[
(x + 1)^{\alpha + 2 - t} (x - 1)^{i - \alpha - 2 + t} = 2^{t} \sum_{r=0}^{\alpha + 2 - t} \binom{\alpha + 2 - t}{r} \left( \frac{x - 1}{2} \right)^{i - r}
\]

\[
= 2^{t} \sum_{k=t+i-\alpha-3}^{i-1} \binom{\alpha + 2 - t}{i - 1 - k} \left( \frac{x - 1}{2} \right)^{k+1}
\]

and interchange the order of summation once more to obtain

\[
e_{i}^{\alpha,\beta}(x) = \frac{2^{i}}{i!} \sum_{k = \max(0, i - \alpha - 3)}^{i - 1} \frac{(\beta + 1)_{\alpha + 2} (\alpha + 2)!}{(i - 1 - k)! k!} \left( \frac{x - 1}{2} \right)^{k+1}.
\]

min(\( k + i + 3, 2\alpha + 4 - i \))

\[
\sum_{t = \max(0, \alpha + 3 - i)}^{\min(2\alpha + 4 - i, \alpha + 2 - t)} \frac{k!(2\alpha + 4 - i - t)!(\alpha + 1)!}{(\beta + 1)_{\alpha + 2 - t}! (2\alpha + 4 - i - t)!(i - \alpha - 3 + t)! (\alpha + 3 - i - t + k)!}.
\]
Again, by another index transformation \( t = m - i + \alpha + 3 \), the inner sum reduces to the same expression as in (2.1). This concludes the proof of Theorem 1.1.

Notice that for \( i = 2\alpha + 4 \), identity (2.1) and, even more directly, the equivalent identity (2.3) reduce to

\[
ed_{2\alpha+4}(x) = e_{2\alpha+4}(x) = (x - 1)^{\alpha+2}(x + 1)^{\alpha+2} = (x^2 - 1)^{\alpha+2}. \tag{2.4}
\]

3. The orthogonality relation of the eigensolutions of the Jacobi-type equation

The aim of this section is to show that for different eigenvalues \( \Lambda_{2\alpha+4,n}^{\alpha,\beta,N} \), \( n \in \mathbb{N}_0 \), the solutions of the Jacobi-type equation (1.7) are orthogonal with respect to the scalar product

\[
(f,g)_{w(\alpha,\beta,0,N)} = \int_{-1}^{1} f(x) g(x) w_{\alpha,\beta}(x) dx + N f(1) g(1) = 0, \quad f,g \in C[-1,1]. \tag{3.1}
\]

This, in turn, is a direct consequence of the following fundamental result.

**Theorem 3.1.** For \( \alpha \in \mathbb{N}_0, \beta > -1, \) and \( N > 0 \), the combined differential operator in equation (1.7),

\[
L_{2\alpha+4,x}^{\alpha,\beta,N} f = N L_{2\alpha+4,x}^{\alpha,\beta} f + C_{\alpha,\beta} L_{2,x}^{\alpha,\beta} f, \quad f \in C^{(2\alpha+4)}[-1,1], \tag{3.2}
\]

is symmetric with respect to the scalar product (3.1) by virtue of

\[
(L_{2\alpha+4,x}^{\alpha,\beta,N} f, g)_{w(\alpha,\beta,0,N)} = (f, L_{2\alpha+4,x}^{\alpha,\beta,N} g)_{w(\alpha,\beta,0,N)}, \quad f,g \in C^{(2\alpha+4)}[-1,1]. \tag{3.3}
\]

**Proposition 3.2.** For any function \( f,g \in C^{(2\alpha+4)}[-1,1] \) we define the two integrals

\[
S^\alpha(f,g) = h_{\alpha,\beta}^{-1} \int_{-1}^{1} D_x^{\alpha+2}[(x - 1)^{\alpha+1} f(x)] D_x^{\alpha+2}[(x - 1)^{\alpha+1} g(x)] (x + 1)^{\alpha+\beta+2} dx,
\]

\[
T(f,g) = h_{\alpha,\beta}^{-1} \int_{-1}^{1} f'(x) g'(x) (1 - x)^{\alpha+1} (1 + x)^{\beta+1} dx.
\]

Then the differential operator (3.2) has the properties

(i) \( (L_{2\alpha+4,x}^{\alpha,\beta} f, g)_{w(\alpha,\beta)} = S^\alpha(f,g) - 2(\alpha + 1) C_{\alpha,\beta} f'(1) g(1) \)

(ii) \( (L_{2,x}^{\alpha,\beta} f, g)_{w(\alpha,\beta)} = T(f,g) \)

(iii) \( L_{2\alpha+4,x}^{\alpha,\beta} f(x) \big|_{x=1} = 0, \quad L_{2,x}^{\alpha,\beta} f(x) \big|_{x=1} = 2(\alpha + 1) f'(1). \)
Proof. (i) In view of the representation (1.13) of \(L_{2α+4,2}^{\alpha,β}\) it follows by an \((α + 2)\)-fold integration by parts that

\[
\begin{align*}
    h_{α,β}(L_{2α+4,2}^{α,β}f,g)_{w(α,β)} &= (-1)^{α} \int_{-1}^{1} D_x^{α+2} \{ (x + 1)^{α+β+2}D_x^{α+2}[(x-1)^{α+1}f(x)] \} (x-1)^{α+1}g(x)dx \\
    &= \sum_{j=0}^{α+1} (-1)^{α+j} D_x^{α+1-j} \{ (x + 1)^{α+β+2}D_x^{α+2}[(x-1)^{α+1}f(x)] \} D_x^{j}[(x-1)^{α+1}g(x)]|_{x=-1}^{x=1} \\
    &+ \int_{-1}^{1} D_x^{α+2}[(x - 1)^{α+1}f(x)]D_x^{α+2}[(x-1)^{α+1}g(x)](x + 1)^{α+β+2}dx.
\end{align*}
\]

Here, all terms of the sum vanish up to the last one for \(j = α + 1\), evaluated at \(x = 1\). Hence,

\[
\begin{align*}
    (L_{2α+4,2}^{α,β}f,g)_{w(α,β)} &= S^α(f,g) - h_{α,β}^{-1}(x + 1)^{α+β+2}D_x^{α+2}[(x-1)^{α+1}f(x)]D_x^{α+2}[(x-1)^{α+1}g(x)]|_{x=1} \\
    &= S^α(f,g) - h_{α,β}^{-1}2^{α+β+2}(α + 2)!f'(1)(α + 1)!g(1) \\
    &= S^α(f,g) - 2(α + 1)C_{α,β}f'(1)g(1).
\end{align*}
\]

(ii) Employing the second representation of \(L_{2,x}^{α,β}\) in (1.11) we find, now by a simple integration by parts, that

\[
(L_{2,x}^{α,β}f,g)_{w(α,β)} = (-1)^{α}h_{α,β}^{-1} \int_{-1}^{1} D_x \{ (x - 1)^{α+1}(x + 1)^{β+1}f(x) \} g(x)dx = T(f,g).
\]

(iii) The required values of the two differential expressions at \(x = 1\) follow by definition (1.13) and (1.11), respectively.

**Proof of Theorem 3.1** In view of Proposition 3.2 and the symmetry relations \(S^α(f,g) = S^α(g,f)\) and \(T(f,g) = T(g,f)\), we obtain the required result (3.3), i.e.

\[
\begin{align*}
    (L_{2α+4,2}^{α,β,N}f,g)_{w(α,β,0,N)} &= \{ L_{2α+4,2}^{α,β,N}f,g \}_{w(α,β,0,N)} + N \{ N \ L_{2α+4,2}^{α,β,N}f,g \}_{w(α,β,0,N)} \\
    &= N S^α(f,g) - N 2(α + 1)C_{α,β}f'(1)g(1) + C_{α,β}T(f,g) + NC_{α,β}2(α + 1)f'(1)g(1) \\
    &= (f,L_{2α+4,2}^{α,β,N}g)_{w(α,β,0,N)}.
\end{align*}
\]

**Corollary 3.3.** For \(α \in \mathbb{N}_0, \ β > -1, \ N > 0\), the polynomial eigensolutions of equation (1.7), \(y_n(x) = P_n^{α,β,0,N}(x), n \in \mathbb{N}_0\), satisfy the orthogonality relation, for \(n \neq m\),

\[
(y_n, y_m)_{w(α,β,0,N)} = \int_{-1}^{1} y_n(x) y_m(x) w_{α,β}(x)dx + Ny_n(1)y_m(1) = 0. \tag{3.4}
\]
Proof. In view of equation (1.14) and the symmetry property stated in Theorem 3.1 we have
\[
\left(\Lambda_{2\alpha+4,n}^{\alpha,\beta,N} - \Lambda_{2\alpha+4,m}^{\alpha,\beta,N}\right) (y_n, y_m)_{u(\alpha,\beta,0,N)} = 0.
\]
Since the difference of the eigenvalues on the left-hand side do not vanish for \(n \neq m\), the assertion follows.

4 A new factorization of the Jacobi-type differential equation

Let us begin with a slightly different version of Bavinck’s factorization formula (1.12). Setting \(y(x) = (x - 1)u(x)\) and recalling the definition (1.11) we obtain
\[
L^{\alpha,\beta}_{2\alpha+4,x}[(x - 1)u(x)] = \prod_{j=0}^{\alpha+1} \left\{ L^{\alpha,\beta}_{2,x} - \frac{2(\alpha + 1)}{x - 1} + j(\alpha + \beta + 1 - j) \right\} [(x - 1)u(x)]
\]
\[
= (x - 1) \prod_{j=0}^{\alpha+1} \left\{ L^{\alpha+2,\beta}_{2,x} + (j + 1)(\alpha + \beta + 2 - j) \right\} u(x).
\]
(4.1)

Here, the second identity follows by successively applying, for any \(j = 0, 1, \ldots, \alpha + 1\),
\[
\{(x^2 - 1)D_x^2 + [\alpha - \beta + (\alpha + \beta + 2)x]D_x - \frac{2(\alpha + 1)}{x - 1} + j(\alpha + \beta + 1 - j)\} [(x - 1)u(x)]
\]
\[
= (x - 1)\{(x^2 - 1)D_x^2 + [\alpha - \beta + (\alpha + \beta + 4)x]D_x + (j + 1)(\alpha + \beta + 2 - j)\} u(x).
\]
(4.2)

Corollary 4.1. [1] (2.6) Let the functions \(R_{n,\alpha,\beta}(x), n \in \mathbb{N}\), be defined as in (1.5). Then
\[
L^{\alpha,\beta}_{2\alpha+4,x}R_{n,\alpha,\beta}(x) = \Lambda_{2\alpha+4,n}^{\alpha,\beta}R_{n,\alpha,\beta}(x), n \in \mathbb{N}.
\]
(4.3)

Proof. In view of formula (4.1) and the Jacobi equation with first parameter being increased to \(\alpha + 2\),
\[
L^{\alpha,\beta}_{2\alpha+4,x}P_{n,\alpha,\beta}^{\alpha,\beta}(x)
\]
\[
= A_{n,\alpha,\beta}^{\alpha,\beta}(x - 1) \prod_{j=0}^{\alpha+1} \left\{ L^{\alpha+2,\beta}_{2,x} + (j + 1)(\alpha + \beta + 2 - j) \right\} P_{n-1,\alpha+2,\beta}^{\alpha+2,\beta}(x)
\]
\[
= A_{n,\alpha,\beta}^{\alpha,\beta}(x - 1) \prod_{j=0}^{\alpha+1} \left\{ (n - 1)(n + \alpha + \beta + 2) + (j + 1)(\alpha + \beta + 2 - j) \right\} \cdot P_{n-1,\alpha+2,\beta}^{\alpha+2,\beta}(x)
\]
\[
= \prod_{j=0}^{\alpha+1} \{(n + j)(\alpha + \beta + 1 - j)\} \cdot A_{n,\alpha,\beta}^{\alpha,\beta}(x - 1)P_{n-1,\alpha+2,\beta}^{\alpha+2,\beta}(x)
\]
\[
= (n)_{\alpha+2}(n + \beta)A_{n,\alpha,\beta}^{\alpha,\beta}(x - 1)P_{n-1,\alpha+2,\beta}^{\alpha+2,\beta}(x) = \Lambda_{2\alpha+4,n}^{\alpha,\beta}P_{n,\alpha,\beta}^{\alpha,\beta}(x).
\]

\[\Box\]
Another elegant proof is solely based on our representation (1.13) of $L_n^{\alpha,\beta}$. In order to carry out the two higher-order differentiations $D_x^{\alpha+2}$ occurring there, we iteratively use the following two differentiation formulas which may be derived from standard properties of the Jacobi polynomials [2, Sec.10.8],

\[
D_x[(x-1)^\gamma P_n^{\gamma,\delta}(x)] = (n+\gamma)(x-1)^{\gamma-1}P_n^{\gamma-1,\delta+1}(x), \quad \gamma > 0, \quad \delta > -1, \quad (4.5)
\]
\[
D_x[(x+1)^{\delta} P_n^{\gamma,\delta}(x)] = (n+\delta)(x+1)^{\delta-1}P_n^{\gamma+1,\delta-1}(x), \quad \gamma > -1, \quad \delta > 0. \quad (4.6)
\]

Then we get, for any $n \in \mathbb{N}$,

\[
L_n^{\alpha,\beta} = \frac{x-1}{(x+1)^{\beta}} D_x^{\alpha+2} \left\{(x+1)^{\alpha+\beta+2} D_x^{\alpha+2} \left\{(x-1)^{\alpha+2} A_n^{\alpha,\beta} P_n^{\alpha+2,\beta}(x) \right\} \right\}
\]
\[
= A_n^{\alpha,\beta} \frac{x-1}{(x+1)^{\beta}} D_x^{\alpha+2} \left\{(x+1)^{\alpha+\beta+2} (n)_{\alpha+2} P_n^{\alpha+2,\beta}(x) \right\}
\]
\[
= A_n^{\alpha,\beta} (x-1)(n+\beta)_{\alpha+2} (n)_{\alpha+2} P_n^{\alpha+2,\beta}(x) = \Lambda_n^{\alpha,\beta} P_n^{\alpha,\beta}(x).
\]

The main purpose of this section is to present a factorization of the Jacobi-type differential operator which is distinct from (1.2) and more reminiscent of our non-commutative factorization of the symmetric ultraspherical-type equation [11, Thm. 4.1].

**Theorem 4.2.** For $\alpha \in \mathbb{N}_0$, $\beta > -1$, the Jacobi-type differential operator (1.13) can be factorized by

\[
L_n^{\alpha,\beta} y(x) = \prod_{j=0}^{\alpha+1} \{ L_{2,x}^{2j-1,\beta} - \frac{4j}{x-1} + j(j+\beta) \} y(x), \quad (4.8)
\]
\[
L_{2,x}^{2j-1,\beta} = (x^2 - 1) D_x^2 + [2j - \beta - 1 + (2j + \beta + 1) x] D_x, \quad j = 0, 1, \ldots, \alpha + 1.
\]

Here, the product $\prod_{j=0}^{\alpha+1}$ is understood as a successive application of each second-order operator to the respective function on its right-hand side, in the order from $j = 0$ to $j = \alpha + 1$.

**Proof.** Analogously to (4.1), (4.2), identity (4.3) is equivalent to

\[
L_n^{\alpha,\beta} [(x-1) u(x)] = (x-1) \prod_{j=0}^{\alpha+1} \{ L_{2,x}^{2j+1,\beta} + (j+1)(j+\beta+1) \} u(x). \quad (4.9)
\]

Moreover, it is not hard to see that

\[
(x+1)^{\beta} \{ L_{2,x}^{2j+1,\beta} + (j+1)(j+\beta+1) \} u(x)
\]
\[
= \{ L_{2,x}^{2j+1,-\beta} + (j+1)(j-\beta+1) \} [(x+1)^{\beta} u(x)]. \quad (4.10)
\]
So we have to verify that

$$(x + 1)^\beta (x - 1)^{-1} L_{2\alpha + 4, x}^\alpha ((x - 1)u(x))$$

$$= D_x^{\alpha + 2} \left\{ (x + 1)^{\alpha + 2} D_x^{\alpha + 2} [(x - 1)^{\alpha + 2} u(x)] \right\}$$

$$= \prod_{j=0}^{\alpha + 1} \left\{ L_{2, x}^{2j + 1} - \beta + (j + 1)(j - \beta + 1) \right\} [(x + 1)^{\beta} u(x)].$$

(4.11)

This, however, follows by successively applying the following recurrence relation. □

**Proposition 4.3.** For any $j \in \mathbb{N}_0$, the expression $u_j^\beta (x) = (x + 1)^{j+\beta} D_x^j [(x - 1)^j u(x)]$ satisfies

$$D_x^{j+1} u_j^\beta (x) = \left\{ L_{2, x}^{2j + 1} - \beta + (j + 1)(j - \beta + 1) \right\} D_x^j u_j^\beta (x).$$

(4.12)

**Proof.** Since

$$(x + 1)^{j+1} (x - 1)^j u(x) = (x^2 - 1) D_x u_j^\beta (x) - (j + \beta)(x - 1)u_j^\beta (x),$$

we have, recalling $L_{2, x}^{2j + 1} - \beta = (x^2 - 1) D_x^j + [2j + \beta + 1 + (2j - \beta + 3)x] D_x$, that

$$D_x^{j+1} u_j^\beta (x) = D_x^{j+1} \left\{ (x + 1)^{j+1} D_x^{j+1} [(x - 1)^j u(x)] \right\}$$

$$= D_x^{j+1} \left\{ (x + 1)^{j+1} (x - 1)^j u(x) \right\}$$

$$= D_x^{j+1} \left\{ (x^2 - 1) D_x D_x^j u_j^\beta (x) + (j + 1)(j + 1) D_x^j u_j^\beta (x) \right\}$$

$$= (x^2 - 1) D_x^{j+2} u_j^\beta (x) + (j + 1) 2x D_x^{j+1} u_j^\beta (x) + (j + 1) j D_x^j u_j^\beta (x)$$

$$+ [2j + \beta + 1 + (1 - \beta)x] D_x^{j+1} u_j^\beta (x) + (j + 1)(j - \beta + 1) D_x^j u_j^\beta (x)$$

$$= (x^2 - 1) D_x^{j+2} u_j^\beta (x) + [2j + \beta + 1 + (2j - \beta + 3)x] D_x^{j+1} u_j^\beta (x)$$

$$+ (j + 1)(j - \beta + 1) D_x^j u_j^\beta (x)$$

$$= \left\{ L_{2, x}^{2j + 1} - \beta + (j + 1)(j - \beta + 1) \right\} D_x^j u_j^\beta (x).$$

□

**Corollary 4.4.** For $\alpha \in \mathbb{N}_0$, $\beta > -1$, the Jacobi-type differential operator (1.13) satisfies the recurrence relation

$$L_{2\alpha + 4, x}^\alpha y(x) = \left\{ (x^2 - 1) D_x^2 + [2\alpha - \beta + 1 + (2\alpha + \beta + 3)x] D_x - \frac{4(\alpha + 1)}{x - 1} + (\alpha + 1)(\alpha + \beta + 1) \right\} L_{2\alpha + 4, x} y(x),$$

(4.13)

where, for $\alpha = 0$, the recurrence starts with the operator

$$L_{2, x}^{1\beta} y(x) = \left\{ (x^2 - 1) D_x^2 + (\beta + 1)(x - 1) D_x \right\} y(x).$$

(4.14)

Notice that the operator (4.14) is obtained as well, if we formally take $\alpha = -1$ in the definition (1.11) of the Jacobi differential operator $L_{2, x}^{\alpha, \beta}$. 

11
5 A direct verification of the Jacobi-type differential equation

Now we are in a position to prove, just by utilizing the representation (1.13) of $L_{2α+4}^{α,β}$, that the Jacobi-type polynomials $P_{n}^{α,β,0,N}(x) = P_{n}^{α,β}(x) + NR_{n}^{α,β}(x), n \in \mathbb{N}_0$, solve equation (1.7). By Corollary 4.1 and the classical Jacobi equation we already know that

$$\{L_{2α}^{α,β} - \Lambda_{2α+4}^{α,β}\}R_{n}^{α,β}(x) = 0, \quad \{L_{2}^{α,β} - \Lambda_{2}^{α,β}\}P_{n}^{α,β}(x) = 0.$$  

So it remains to show that

$$\{L_{2α+4}^{α,β} - \Lambda_{2α+4}^{α,β}\}P_{n}^{α,β}(x) + C_{α,β}\{L_{2}^{α,β} - \Lambda_{2}^{α,β}\}R_{n}^{α,β}(x) = 0, \quad -1 < x < 1. \quad (5.1)$$

This is obtained by combining the following two identities, of which the latter one turns out to be crucial here.

**Theorem 5.1.** For all $n \in \mathbb{N}$, there hold

(i) Applying identity (4.2) for $j = 0$ as well as the Jacobi equation with increased parameter $α + 2$, we find that

$$\{L_{2}^{α} - \frac{2(α + 1)}{x - 1} - \Lambda_{2}^{α}\}[(x - 1)P_{n+2}^{α+2,β}(x)] = (x - 1)\{L_{2}^{α+2} - \Lambda_{2}^{α+2}\}P_{n}^{α+2,β}(x) = 0.$$  

(ii) In order to carry out the differentiations in

$$L_{2α+4}^{α,β}P_{n}^{α,β}(x) = \frac{x - 1}{(x + 1)^{β}}D_{x}^{α+2}\{(x + 1)^{α+2}D_{x}^{α+2}[(x - 1)^{α+1}P_{n+1}^{α,β}(x)]\}, \quad (5.4)$$

we cannot use the formulas (4.5), (4.6) right away as in (4.7). Instead, we first adjust the parameters of the Jacobi polynomials by employing the well-known identities

$$P_{n}^{α,β}(x) = P_{n}^{α+1,β-1}(x) - P_{n-1}^{α+1,β}(x). \quad (5.5)$$
we arrive at

\[(2n + \alpha + \beta + 1)P_n^{\alpha,\beta}(x) = (n + \alpha + \beta + 1)P_n^{\alpha+1,\beta}(x) - (n + \beta)P_n^{\alpha+1,\beta}(x) \quad (5.6)\]

\[(2n + \alpha + \beta + 2)\frac{1 - x^2}{2}P_n^{\alpha+1,\beta}(x) = (n + \alpha + 1)P_n^{\alpha,\beta}(x) - (n + 1)P_n^{\alpha,\beta}(x) \quad (5.7)\]

\[D_x P_n^{\gamma,\delta}(x) = \frac{1}{2}(n + \gamma + \delta + 1)P_{n-1}^{\gamma+1,\delta+1}(x). \quad (5.8)\]

In fact, using (5.5), (4.6), (5.8), and (5.5) again, we obtain

\[D_x^{\alpha+2}[(x - 1)^{\alpha+1}P_n^{\alpha,\beta}(x)] \]
\[= D_x^{\alpha+2}[(x - 1)^{\alpha+1}P_n^{\alpha+1,\beta-1}(x)] - D_x^{\alpha+2}[(x - 1)^{\alpha+1}P_n^{\alpha+1,\beta}(x)] \]
\[= D_x[(n + 1)_{\alpha+1}P_n^{\alpha,\beta}(x)] - D_x[(n)_{\alpha+1}P_n^{0,\alpha+1,\beta}(x)] \]
\[= \frac{1}{2}(n + \alpha + \beta + 1)[(n + 1)_{\alpha+1}P_{n-1}^{1,\alpha+\beta+1}(x) - (n)_{\alpha+1}P_{n-2}^{1,\alpha+\beta+2}(x)] \]
\[= \frac{1}{2}(n + \alpha + \beta + 1)(n + 1)_{\alpha}[(n + \alpha + 1)P_{n-1}^{0,\alpha+\beta+2}(x) + (\alpha + 1)P_{n-2}^{1,\alpha+\beta+2}(x)]. \]

Inserting this last identity into the right-hand side of (5.4) and observing that, in view of (4.6),

\[D_x^{\alpha+2}[(x + 1)^{\alpha+\beta+2}P_n^{0,\alpha+\beta+2}(x)] = (n + \beta)_{\alpha+2}(x + 1)^{\beta}P_n^{0,\alpha+\beta+2}(x), \]
\[D_x^{\alpha+2}[(x + 1)^{\alpha+\beta+2}P_{n-1}^{0,\alpha+\beta+2}(x)] = (n + \beta - 1)_{\alpha+2}(x + 1)^{\beta}P_{n-2}^{0,\alpha+\beta+2}(x), \]

we arrive at

\[L_{2\alpha+4,\beta}^{\alpha,\beta}P_n^{\alpha,\beta}(x) = (n)_{\alpha+2}(n + \beta)_{\alpha+2} \cdot \]
\[\left\{ \frac{n + \alpha + \beta + 1}{n} x - 1 \frac{P^{\alpha,\beta}(x)}{2n-1} \right\} + \frac{(\alpha + 1)(n + \beta + 1)}{n(n + \alpha + 1)} \left\{ x - 1 \frac{P^{\alpha+3,\beta}(x)}{2n-2} \right\}. \]

Finally, due to (5.7) and (5.6), the two terms in curly brackets simplify to

\[\frac{n + \alpha + \beta + 1}{n} x - 1 \frac{P^{\alpha+2,\beta}(x)}{2n-1} = P^{\alpha,\beta}(x) - \frac{\alpha + 1}{n} P^{\alpha+1,\beta}(x), \]
\[\frac{(\alpha + 1)(n + \beta + 1)}{n(n + \alpha + 1)} x - 1 \frac{P^{\alpha+3,\beta}(x)}{2n-2} = \frac{\alpha + 1}{n} \left( P^{\alpha+1,\beta}(x) - \frac{\alpha + 2}{n + \alpha + 1} P^{\alpha+2,\beta}(x) \right). \]

This proves the required identity (5.3).

\[\square \]

6 Some relations between the equations of Jacobi- and ultraspherical-type

We first observe a surprisingly simple connection between the higher-order differential expressions (1.13) and (1.19).
**Theorem 6.1.** The symmetric ultraspherical-type operator \(L_{2\alpha+4,x}\) and the Jacobi-type operator \(L_{2\alpha+4,x}^{\alpha+2}\), both of the same order \(2\alpha+4, \alpha \in \mathbb{N}_0\), are linked to each other via

\[
L_{2\alpha+4,x}[(x+1)g(x)] = (x+1)L_{2\alpha+4,x}^{\alpha+2}y(x). \tag{6.1}
\]

**Proof.** For any smooth function \(\phi(x)\) and \(m \in \mathbb{N}_0\), we have

\[
D_x^m[(x+1)2^mD_x^m\phi(x)] = \sum_{k=0}^{m} \binom{m}{k} D_x^k[(x+1)^{2m}] D_x^{m-k}D_x^m\phi(x)
\]

\[
= (x+1)^m \sum_{k=0}^{m} \binom{m}{k} \binom{2m}{k} k! (x+1)^{m-k}D_x^{2m-k}\phi(x)
\]

\[
= (x+1)^m D_x^{2m}[(x+1)^m\phi(x)]. \tag{6.2}
\]

So, starting with the Jacobi-type differential expression on the right-hand side of (6.1) and choosing \(m = \alpha + 2\) and \(\phi(x) = (x-1)^{\alpha+1}y(x)\) in identity (6.2), we find that

\[
L_{2\alpha+4,x}^{\alpha+2}y(x) = (x-1)(x+1)^{-\alpha-2}D_x^{\alpha+2}[(x+1)^{2\alpha+4}D_x^{\alpha+2}\phi(x)]
\]

\[
= (x-1)D_x^{2\alpha+4}[(x+1)^{\alpha+2}(x-1)^{\alpha+1}y(x)]
\]

\[
= (x+1)^{-1}L_{2\alpha+4,x}[(x+1)y(x)].
\]

\[\square\]

**Remark 6.2.** Identity (6.1) follows as well by comparing the Jacobi-type factorization formula (4.1) in case \(\beta = \alpha + 2\) with the factorized ultraspherical-type differential expression given in [11, Sec.4-6]. Indeed,

\[
(x+1)L_{2\alpha+4,x}^{\alpha+2}[(x-1)u(x)] = (x^2-1) \prod_{j=0}^{\alpha+1} \left[ L_{2,x}^{\alpha+2,\alpha+2} + (j+1)(2\alpha+4-j) \right] u(x)
\]

\[
= (x^2-1) \prod_{j=0}^{\alpha+1} \left[ (x^2-1)D_x^2 + (2\alpha+6)x D_x + (j+1)(2\alpha+4-j) \right] u(x)
\]

\[
= (x^2-1)D_x^{2\alpha+4}[(x^2-1)^{\alpha+2}u(x)] = L_{2\alpha+4,x}[(x^2-1)u(x)].
\]

What the differential equations are concerned, a second, even more striking relationship is suggested by the two quadratic transformations due to Koornwinder [9] (4.6-7),

\[
P_{2n}^{\alpha,\alpha,N,N}(x) = p_n P_{2n}^{\alpha,-1/2,0,2N}(2x^2-1), \quad p_n = P_{2n}^{\alpha,\alpha,N,N}(1)/P_{2n}^{\alpha,-1/2,0,2N}(1), \tag{6.3}
\]

\[
P_{2n+1}^{\alpha,\alpha,N,N}(x) = q_n x P_{2n+1}^{\alpha,1/2,0,(4\alpha+6)N}(2x^2-1), \quad q_n = P_{2n+1}^{\alpha,\alpha,N,N}(1)/P_{2n+1}^{\alpha,1/2,0,(4\alpha+6)N}(1). \tag{6.4}
\]
**Theorem 6.3.** Let \( \alpha \in \mathbb{N}_0, \ N > 0 \).

(i) Considering the differential equation \((1.7)\) associated with the Jacobi-type polynomials \( y_n(x) = p_nP_{n+1}^{\alpha-1/2,0,2N}(x) \), \( n \in \mathbb{N}_0 \), the substitution \( x = 2\xi^2 - 1 \), \( 0 \leq \xi \leq 1 \), leads to the equation \((1.18)\) for the even ultraspheical-type polynomials

\[
u_n(\xi) := y_n(2\xi^2 - 1) = P_{2n}^{\alpha,\alpha,N,N}(\xi).
\]

(ii) Under the same substitution as in part (i) and a transformation of the dependent variable, the equation \((1.7)\) for the Jacobi-type polynomials \( y_n(x) = q_nP_{n+1/2}^{\alpha,1/2,0,4(\alpha+1)N}(x) \), \( n \in \mathbb{N}_0 \), reduces to the equation \((1.18)\) for the odd ultraspheical-type polynomials

\[
u_n(\xi) = y_n(2\xi^2 - 1) = P_{2n+1}^{\alpha,\alpha,N,N}(\xi).
\]

**Proof.** (i) We multiply equation \((1.7)\) in the case that \( y_n(x) = p_nP_{n+1}^{\alpha-1/2,0,2N}(x) \), i.e.

\[
2N\{L_{2\alpha+1/2}^{\alpha-1/2,1} - L_{2\alpha+1/2}^{\alpha-1/2,1}\}y(x) + C_{\alpha-1/2}\{L_{2\alpha}^{\alpha-1/2,1} - L_{2\alpha}^{\alpha-1/2,1}\}y_n(x) = 0, \ -1 < x < 1,
\]

by \( 2^{2\alpha+3} \) and observe that the constants in \((1.10)\) and \((1.17)\) are related to each other by

\[
2^{2\alpha+4}\Lambda_{2\alpha+1/2}^{\alpha-1/2} = \Lambda_{2\alpha+2}^{\alpha-1/2}, \ 2^{2\alpha+1}C_{\alpha-1/2} = C_{\alpha}, \ 4\Lambda_{2\alpha}^{\alpha-1/2} = \Lambda_{2\alpha}^{\alpha-1/2}.
\]

Concerning the two differential expressions in \((6.5)\) we formally replace \( D_x \) by \((4\xi)^{-1}D_\xi\) in view of the substitution \( x = 2\xi^2 - 1 \). Years ago, we used already the so-called Bessel derivate \( \delta_\xi = \xi^{-1}D_\xi \) in order to define the Bessel-type functions via a confluent limit of the Laguerre-type polynomials, see [3]. Now we can use the expansion formula for the iterated Bessel derivatives \( \delta_\xi^{m+1}, \ m \in \mathbb{N}_0 \), as stated in [3] (2.8)). A quite lengthy, but straightforward induction argument then shows that for all \( j = 0, \ldots, m \) and any smooth function \( \phi(\xi) \),

\[
\delta_\xi^j[\xi^{2m+1}\delta_\xi^{m+1}\phi(\xi)] = \sum_{k=j}^m \frac{(-2)^{k-m}(2m-k-j)!}{(m-k)!(k-j)!} \xi^{k-j} D_\xi^{k+j+1}\phi(\xi).
\]

Hence, for \( j = m \), we end up with the surprisingly simple identity needed in the following,

\[
\delta_\xi^m[\xi^{2m+1}\delta_\xi^{m+1}\phi(\xi)] = D_\xi^{2m+1}\phi(\xi), \ m \in \mathbb{N}_0.
\]

In fact, choosing \( m = \alpha + 1 \) and taking into account that \( \xi \delta_\xi^{\alpha+2} = D_\xi \delta_\xi^{\alpha+1} \) we have

\[
\xi \delta_\xi^{\alpha+2}[\xi^{2\alpha+3}\delta_\xi^{\alpha+2}\phi(\xi)] = D_\xi D_\xi^{2\alpha+3}\phi(\xi) = D_\xi^{2\alpha+4}\phi(\xi)
\]

and thus

\[
2^{2\alpha+4}L_{2\alpha+4,x}y_n(x) = 2^{2\alpha+4} \frac{x - 1}{(x + 1)^{1/2}} D_x^{\alpha+2}\{x^{1/2}(1+1)^{3/2}D_x^{\alpha+2}[y_n(x)]\}
\]

\[
= (\xi^2 - 1) \xi \delta_\xi^{\alpha+2}\{\xi^{2\alpha+3}\delta_\xi^{\alpha+2}[\xi^{2\alpha+1}y_n(2\xi^2 - 1)]\}
\]

\[
= (\xi^2 - 1) D_\xi^{2\alpha+4}\{[\xi^{2\alpha+1}u_n(\xi)]\}
\]

\[
= L_{2\alpha+4,\xi}u_n(\xi).
\]
Moreover, it is not hard to see that
\[
4L_{2,x}^{\alpha-1/2}y_n(x) = 4\{ (x^2 - 1)D_x^2 + [\alpha + 1/2 + (\alpha + 3/2)x] D_x \} y_n(x)
\]
\[
= \{ (\xi^2 - 1)D_\xi^2 + 2(\alpha + 1)\xi \; D_\xi \} y_n(2\xi^2 - 1)
\]
\[
= L_{2,\xi}^{\alpha,\alpha} u_n(\xi).
\]
Putting all parts together, we arrive at equation \((1.18)\) applied to \(u_n(\xi) = P_{2n}^{\alpha,\alpha,N,N}(\xi)\).

(ii) In case of the Jacobi-type polynomials \(y_n(x) = q_n P_{n}^{\alpha,1/2,0,(4\alpha+6)N}(x)\), equation \((1.7)\) reads
\[
(4\alpha + 6)N\{ L_{2\alpha+4,x}^{\alpha,1/2} - \Lambda_{2\alpha+4,n}^{\alpha,1/2} \} y_n(x) + C_{\alpha,1/2}\{ L_{2,x}^{\alpha,1/2} - \Lambda_{2,n}^{\alpha,1/2} \} y_n(x) = 0, \; -1 < x < 1. \quad (6.7)
\]
Here we multiply this equation by \(2^{2\alpha+4}(4\alpha + 6)^{-1}\xi, \; \xi = \sqrt{(x+1)/2}\), and use
\[
2^{2\alpha+4}\Lambda_{2\alpha+4,n}^{\alpha,1/2} = \Lambda_{2\alpha+4,2\alpha+1}, \; 2^{2\alpha+2}(4\alpha + 6)^{-1}C_{\alpha,1/2} = C_{\alpha}, \; 4\Lambda_{2,n}^{\alpha,1/2} = \Lambda_{2,2\alpha+1}^{\alpha,\alpha} - 2(\alpha + 1).
\]
Moreover, we find that
\[
4\xi \; L_{2,x}^{\alpha,1/2} y_n(x) = 4\xi \; \{ (x^2 - 1)D_x^2 + [\alpha - 1/2 + (\alpha + 5/2)x] D_x \} y_n(x)
\]
\[
= \{ 4L_{2,x}^{\alpha,-1/2} + 2(\alpha + 1) \}[\xi \; y_n(x)] = \{ L_{2,\xi}^{\alpha,\alpha} + 2(\alpha + 1) \} v_n(\xi)
\]
and therefore
\[
4\xi \; \{ L_{2,x}^{\alpha,1/2} - \Lambda_{2,n}^{\alpha,1/2} \} y_n(x) = \{ L_{2,\xi}^{\alpha,\alpha} - \Lambda_{2,2\alpha+1}^{\alpha,\alpha} \} v_n(\xi).
\]
Finally, we use identity \((6.6)\) again. With \(m = \alpha + 2\) and \(\psi(\xi) := D_\xi \phi(\xi)\) we obtain
\[
\delta_\xi^{\alpha+2} \{ \xi^{2\alpha+5} \delta_\xi^{\alpha+2} [\xi^{-1} \psi(\xi)] \} = \delta_\xi^{\alpha+2} \{ \xi^{2\alpha+5} \delta_\xi^{\alpha+3} \phi(\xi) \} = D_\xi^{2\alpha+5} \phi(\xi) = D_\xi^{2\alpha+4} \psi(\xi).
\]
Hence,
\[
2^{2\alpha+4}\xi \; L_{2\alpha+4,x}^{\alpha,1/2} y_n(x) = 2^{2\alpha+4} \frac{\xi(x-1)}{(x+1)^{1/2}} D_x^{\alpha+2}\{ (x+1)^{\alpha+5/2} D_x^{\alpha+2}[x^2 - 1] y_n(x) \}
\]
\[
= (\xi^2 - 1)\delta_\xi^{\alpha+2} \{ \xi^{2\alpha+5} \delta_\xi^{\alpha+2} [\xi^{-1}(\xi^2 - 1)^{\alpha+1} \xi \; y_n(2\xi^2 - 1)] \}
\]
\[
= (\xi^2 - 1)D_\xi^{2\alpha+4}[(\xi^2 - 1)^{\alpha+1} v_n(\xi)]
\]
\[
= L_{2\alpha+4,\xi} v_n(\xi).
\]
Combining all parts then yields equation \((1.18)\) applied to \(v_n(\xi) = P_{2n+1}^{\alpha,\alpha,N,N}(\xi)\). Notice that by symmetry, the resulting equation for the ultraspherical-type polynomials, both of even and odd degree, can easily be extended to the full range \(-1 \leq \xi \leq 1\).

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C. Markett, Lehrstuhl A für Mathematik, RWTH Aachen, 52056 Aachen, Germany; E-mail: markett@matha.rwth-aachen.de