DOUBLE AFFINE HECKE ALGEBRA IN LOGARITHMIC CONFORMAL FIELD THEORY

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ABSTRACT. We construct the representation of Double Affine Hecke Algebra whose symmetrization gives the center of the quantum group $\overline{U}_q\mathfrak{sl}(2)$ and by Kazhdan–Lusztig duality the Verlinde algebra of $(1, p)$ models of logarithmic conformal field theory.

1. INTRODUCTION

Recently, quantum group methods led (see the recent review [1]) to a progress in logarithmic conformal field theory [2]. For the $(1, p)$ models [3], an equivalence between representation categories of the chiral algebra and the quantum group $\overline{U}_q\mathfrak{sl}(2)$ was established [4] in the general framework of the Kazhdan–Lusztig duality [5]. Remarkably, the KL duality extends to an isomorphism between modular group representations on the quantum group center and on the space of generalized characters [6] of a $(1, p)$ model. Moreover, the Verlinde algebra of $(1, p)$ models [7, 8] (see also [9]) coincides [6] with the Grothendieck ring of $\overline{U}_q\mathfrak{sl}(2)$.

An unusual property of logarithmic conformal field theory is the nonsemisimplicity of the Verlinde algebra. However, this phenomenon does not look extraordinary in the Double affine Hecke algebra representation framework [11] of the Verlinde algebra classification. It leads to a natural conjecture [12] that the $(1, p)$ model Verlinde algebra can be realized in terms of a DAHA representation. Indeed, the representation of DAHA whose symmetrization gives the center of the quantum group $\overline{U}_q\mathfrak{sl}(2)$ and therefore the Verlinde algebra of $(1, p)$ models is identified in the present paper.

1.1. DAHA. We consider the symplest DAHA [11] generated by $X$, $Y$, and $T$ with the relations

\begin{align}
TXT &= X^{-1}, \quad TY^{-1}T = Y, \quad XY = qYXT^2, \\
(T - t^{\frac{1}{2}})(T + t^{-\frac{1}{2}}) &= 0, \quad t = q^2.
\end{align}

In the paper we fix the deformation parameter

\begin{align}
q &= e^{\frac{i\pi}{p}}, \quad q^\frac{1}{2} = e^{\frac{i\pi}{2p}},
\end{align}

where $p = 3, 4, 5, 6, \ldots$. We let $\mathcal{H}$ denote this algebra. The group $PSL(2, \mathbb{Z})$ acts by automorphisms on $\mathcal{H}$

\begin{align}
\tau_+: \quad Y \rightarrow q^{-1/2}XY, \quad X \rightarrow X, \quad T \rightarrow T
\end{align}
\[ \tau_- : X \to q^{1/2}YX, \quad Y \to Y, \quad T \to T \]

where
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \tau_+, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \to \tau_-.
\]

We note that the Fourier transform is given by
\[ \sigma : X \to Y^{-1}, \quad Y \to XT^2, \quad T \to T, \]
\[ \sigma = \tau_+ \tau_- \tau_+^{-1} = \tau_-^{-1} \tau_+ \tau_-^{-1}. \]

1.2. The representation. We consider a \(6p - 4\)-dimensional reducible but indecomposable representation \(Z\) of \(\mathcal{H}\). The representation \(Z\) contains the maximal subrepresentation \(V^{-2}\), which in notations of [11] is defined as the quotient \(V^{-2} = \mathcal{P}/(X^{2p} + X^{-2p} - 2)\), where \(\mathcal{P} = \mathbb{C}[X, X^{-1}]\) is the standard representation of \(\mathcal{H}\) in the Laurent polynomials. The \(2p - 4\)-dimensional irreducible quotient \(M = Z/V^{-2}\) is isomorphic to the representation \(V_{2p-4}\) from [11] given by the quotient \(\mathcal{P}/\epsilon_{-p+2}\), where \(\epsilon_{-p+2} = \prod_{j=2}^{p-1} (q^{-j}X - q^jX^{-1})\).

We note also that \(V^{-2}\) is also reducible and contains the maximal \(2p + 4\)-dimensional irreducible subrepresentation \(W\) and the quotient \(E = V^{-2}/W\) is isomorphic to \(M\).

In Sec. 2 we describe the representation \(Z\) by the explicit action of operators \(T, X\) and \(Y\) in a basis. Then we describe its structure and explicitly find the subrepresentation and quotients.

The representation \(Z\) bears a commutative associative multiplication, which is described in Sec. 1.5. The multiplication gives further the multiplication in the Verlinde algebra.

The \(PSL(2, \mathbb{Z})\) generators \(\sigma\) and \(\tau_+\) are realized as a conjugation with some operators \(S\) and \(v\) respectively, acting in \(Z\). The operator \(v\) acts by a multiplication with an element from \(Z\), which is abusing notation denoted by the same symbol \(v\). By analogy with [11] we call \(v\) the Gaussian element.

1.3. Symmetrization. The operator \(T\) has two different eigenvalues \(q\) and \(-q^{-1}\) in \(Z\). The eigenspace of \(T\) with the eigenvalue \(q\) is \(3p - 1\)-dimensional. We let \(T_q\) denote this eigenspace. In accordance with the general theory [11], \(T_q\) is an associative commutative algebra with multiplication induced by the multiplication in \(Z\) and at the same time a representation of \(SL(2, \mathbb{Z})\) induced by the \(PSL(2, \mathbb{Z})\)-action in \(Z\). The operators \(S, v, C = -(X + X^{-1})\) and \(H = -(Y + Y^{-1})\) have well defined restrictions on \(T_q\). Now we are ready to formulate the main result of the paper (the reader can find all needed quantum group definitions in [6]).

1.4. Theorem. \(\bullet T_q\) is isomorphic to the center of the \(\mathbb{C}_q\) algebra and as \(SL(2, \mathbb{Z})\) representation.
Under the isomorphism the eigenvectors of $C$ correspond to Radford images and eigenvectors of $H$ correspond to Drinfeld images of the characters of $\mathfrak{U}_q\mathfrak{sl}(2)$ irreducible representations.

The Gaussian element $v$ coincides with the ribbon element of $\mathfrak{U}_q\mathfrak{sl}(2)$.

In Sec. 4, we describe the subspace of $T$ with the eigenvalue $q$, and in Sec. 5 we give the proof of theorem 1.4. The notations in this part correspond to the same notations in [6].

1.5. Structure of $Z$. The very important information about the representation $Z$ is encoded in the spectra of operators $X$ and $Y$. These operators are not diagonalizable but both have Jordan blocks of dimension 2. In order to describe their Jordan structure we introduce two basises in which operators $X$ and $Y^{-1}$ have a Jordan form. We call the first basis the $X$-basis and the second one the $Y$-basis. Jordan forms of $X$ and $Y^{-1}$ coincide in $Z$.

1.5.1. $X$-basis. The representation $Z$ has the basis

\[(1.7)\]

\[w_1 \ldots w_{2p}, e_1, e_p, e_{p+1}, e_{2p}; \quad e_2 \ldots e_{p-1}, e_{p+2} \ldots e_{2p-1}; \quad m_2 \ldots m_{p-1}, m_{p+2} \ldots m_{2p-1}.\]

The subrepresentation $W$ is spanned by elements $w_1 \ldots w_{2p}, e_1, e_p, e_{p+1}$ and $e_{2p}$. The elements $e_2 \ldots e_{p-1}, e_{p+2} \ldots e_{2p-1}$ give a basis in $E$ (in $M$) under the canonical projection. In basis (1.7) we have

\[(1.8)\]

\[Xw_s = q^s w_s, \quad Xe_s = q^s (e_s + w_s), \quad Xm_s = q^s m_s.\]

We call (1.7) the $X$-basis.

1.5.2. The multiplication in $Z$. The representation $Z$ is endowed with a commutative associative multiplication, which is naturally written in basis (1.7) as

\[(1.9)\]

\[e_i e_j = \delta_{i,j} e_j, \quad e_i w_j = \delta_{i,j} w_j, \quad e_i m_j = \delta_{i,j} m_j, \quad w_i w_j = w_i m_j = m_i m_j = 0.\]

1.5.3. $Y$-basis. The representation $Z$ contains the basis

\[(1.10)\]

\[u_1 \ldots u_{2p}, f_1, f_p, f_{p+1}, f_{2p}; \quad f_2 \ldots f_{p-1}, f_{p+2} \ldots f_{2p-1}; \quad k_2 \ldots k_{p-1}, k_{p+2} \ldots k_{2p-1}\]

in which $Y^{-1}$ acts as follows

\[(1.11)\]

\[Y^{-1} u_s = q^s u_s, \quad Y^{-1} f_s = q^s (f_s + u_s), \quad Y^{-1} k_s = q^s k_s.\]

We call (1.10) the $Y$-basis.

In Subsec. 2.3 we find the $Y$-basis and give decompositions of elements from the $Y$-basis in the $X$-basis.
1.5.4. $PSL(2, \mathbb{Z})$-action. The operator $S$ maps the $X$-basis to the $Y$-basis

\begin{equation}
Sw_s = u_s, \quad Se_s = f_s, \quad Sm_s = k_s.
\end{equation}

In Subsec. 3.1, we establish properties of the $S$-operator. By a direct calculation, using the decompositions of the $Y$-basis in the $X$-basis, we check that this operator satisfies all relations (1.6) and $S^2 = qT^{-1}$.

In terms of the $X$-basis, the Gaussian element is

\begin{equation}
v = \sum_{s=1}^{2p} q^{-\frac{1}{2}(s^2-1)}e_s - w_1 + q^{-\frac{2}{2}}w_{p+1} + \left(\sum_{s=2}^{p-1} + \sum_{s=p+2}^{2p-1}\right) q^{-\frac{1}{2}(s^2-1)}((p-s)w_s + pm_s).
\end{equation}

The properties of this element are described in Subsec. 3.2.

In the end of Sec. 3, we prove the $PSL(2, \mathbb{Z})$ relations.

1.6. Notation. We introduce Chebyshov polynomials

\begin{equation}
U_s(x) = x^s - 1 + x^{-(s-3)} + \cdots + x^{-(s-3)} + x^{-(s-1)}.
\end{equation}

In what follows we often use the numbers

\begin{equation}
\{s\} = \frac{q^s + q^{-s}}{q - q^{-1}}, \quad [s] = \frac{q^s - q^{-s}}{q - q^{-1}},
\end{equation}

\begin{equation}
\omega_s = \frac{p\sqrt{2p}}{[s]^2}(-1)^{p+s+1}, \quad \xi_s = \frac{-(p-s)p\sqrt{2p}}{q^s - q^{-s}},
\end{equation}

\begin{equation}
[s, j] \equiv \begin{cases} s, & j = 0, 2p, \\ (-1)^{s-1}s, & j = p, \\ \frac{[s]}{[j]}, & j \ mod \ p \neq 0, \end{cases} \quad \{s, j\} \equiv \begin{cases} 0, & j \ mod \ p = 0, \\ \{sj\} \mod p, & \text{otherwise}. \end{cases}
\end{equation}

2. Representation

In this section, we recall the representation $V^{-2}$ [11] and then define the representation $Z$, which is an extension of $V^{-2}$. Then we find a Jordan basis for $Y$ in which $Y^{-1}$ acts by (1.11).

2.1. Polynomial representation $V^{-2}$. The representation $Z$ is an extension of the representation $V^{-2}$ from [11]. To describe $V^{-2}$ we recall the standard representation [11] of $\mathcal{H}$ in the space of Laurent polynomials $\mathcal{P} = \mathbb{C}[X, X^{-1}]$. The $\mathcal{H}$ generators act as follows

\begin{equation}
T \to t^{\frac{1}{2}} s + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{X^2 - 1}(s - 1), \quad t = q^k,
\end{equation}

\begin{equation}
Y \to -spT,
\end{equation}
where

\[ sf(X) = f(X^{-1}), \quad pf(X) = f(qX) \]

and \( X, X^{-1} \) act by multiplication. (We note that these formulas differ from \[11\] by the sign in the definition of \( Y \rightarrow s \rho T \).) The representation \( V^{-2} \) is the 4p-dimensional representation in the quotient space \( \mathcal{P}/(X^{2p} + X^{-2p} - 2) \).

**2.1.1. Proposition.**

- The operators \( X \) and \( Y \) have in \( V^{-2} \) eigenvalues \( q^s, s = 1 \ldots 2p \), each with multiplicity 2.
- The Jordan basis of \( X \) contains functions \( e_s \) and \( w_s \) for \( s = 1 \ldots 2p \).
- The Jordan basis of \( Y \) contains functions \( u_s \) for \( s = 1 \ldots 2p \) and \( k_s \) for \( s = 2 \ldots p - 1, p + 2 \ldots 2p - 1 \), and functions \( f_1, f_p, f_{p+1}, f_{2p} \).

The action of \( X \) and \( Y^{-1} \) in these basises is

\[
X w_s = q^s w_s, \quad X e_s = q^s (e_s + w_s),
\]

\[
Y^{-1} u_s = q^s u_s, \quad Y^{-1} f_s = q^s (f_s + u_s), \quad Y^{-1} k_s = q^s k_s.
\]

**Proof.** To describe the spectra of these operators, we introduce functions

\[
w_s = \frac{1}{4p^2} (X^{2p} - 1) \sum_{j=0}^{2p-1} q^{-sj} X^j, \quad s = 1 \ldots 2p,
\]

\[
e_s = \frac{1}{2p} + \frac{1}{4p^2} \sum_{j=1}^{2p-1} (2p - j) (q^{-sj} X^j + q^{sj} X^{-j}),
\]

and

\[
u_s = \frac{(-1)^s}{p \sqrt{2p}} \left( q^s U_{p-s}(X) + U_{p+s}(X) \right) + \frac{q^s (q^{-1} X) + U_{p+s}(q^{-1} X)}{2}, \quad s = 1 \ldots p,
\]

\[
u_{p+s} = \frac{(-1)^{p+s}}{p \sqrt{2p}} \left( q^{p+s} U_s(X) + U_{2p-s}(X) \right) + \frac{q^s (q^{-1} X) + U_{2p-s}(q^{-1} X)}{2},
\]

\[
k_s = \frac{(-1)^{s+1}}{p \sqrt{2p}} \left( q^s U_{p-s}(X) + q U_{p-s}(q^{-1} X) \right), \quad s = 2 \ldots p - 1
\]

\[
k_p+s = \frac{(-1)^{p+s}}{p \sqrt{2p}} \left( q^{p+s} U_s(X) + q U_s(q^{-1} X) \right),
\]

\[
f_s = \frac{1}{p \sqrt{2p}} \begin{cases} 
\frac{(-1)^{p+1} U_{2p}(X)}{2}, & s = p, \\
U_p(X), & s = 2p, \\
\frac{(-1)^p q X U_{2p}(X)}{2}, & s = p + 1, \\
-q X U_p(X), & s = 1.
\end{cases}
\]
Then, (2.4) is checked by a direct calculation. It is easy to see that \(1 = \sum_{s=1}^{2p} e_s\). Together with (2.4) this gives
\[
X^j = \sum_{s=1}^{2p} q^{e_j}(e_s + jw_s), \quad j = 0, \pm 1, \pm 2, \ldots
\]
i.e. functions \(e_s\) and \(w_s\) are linearly independent and form a basis in \(V^{-2}\).

(2.5) is also checked by a direct calculation using the following relations
\[
Y U_s(q^{-1}X) = (q^s + q^{-s}) U_s(q^{-1}X) - q^{-1} U_s(X),
Y U_s(X) = q U_s(q^{-1}X).
\]
The linear independence of these vectors is proved by the standard technic (See definition 2.5.4 and theorems 2.5.9 and 2.9.3 from [11]).

The representation \(V^{-2}\) is reducible. It has a \(2p + 4\)-dimensional subrepresentation \(W\) spanned by functions \(w_1 \ldots w_{2p}, e_1, e_p, e_{p+1}\) and \(e_{2p}\). The quotient \(\mathcal{E} = V^{-2}/W\) is isomorphic to \(V_{2p-4}\) from [11]. The representation \(V_{2p-4}\) is defined in [11] as the quotient \(V_{2p-4} = \mathcal{P}/\mathcal{E}_{-p+2}\), where \(\mathcal{E}_{-p+2} = \prod_{j=2}^{p-1} (q^{-j}X - q^jX^{-1})\).

Decomposition of any polynomial \(f(X)\) in the basis \(e_s, w_s\) is given by
\[
(2.10) \quad f(X) = \sum_{s=1}^{2p} \left( f(q^s)e_s + \left. \frac{X df(X)}{dX} \right|_{X=q^s} w_s \right).
\]
Using it, we check that \(u_1 \ldots u_{2p}, f_1, f_p, f_{p+1}\) and \(f_{2p}\) belong to \(W\) and therefore in \(W\) Jordan forms of \(X\) and \(Y^{-1}\) coincide. But in the whole \(V^{-2}\) they do not coincide, hence automorphism (1.6) cannot be realized as a conjugation. To recover this, we find an extension of \(V^{-2}\) to a \(6p - 4\)-dimensional representation \(\mathcal{Z}\) by adding vectors \(m_2, \ldots, m_{p-1}\) and \(m_{p+2}, \ldots, m_{2p-1}\). The action of \(X\) on them is \(X m_s = q^s m_s\). The whole \(\mathcal{Z}\) cannot be realized in a space of polynomials in 1 variable. We describe \(\mathcal{Z}\) in terms of an abstract vector space.

2.2. The representation \(\mathcal{Z}\) in the \(X\)-basis. We assume the following definition of \(\mathcal{Z}\). The representation \(\mathcal{Z}\) is a \(6p - 4\)-dimensional vector space with the basis consisting of \(4p\) vectors \(e_s\) and \(w_s\) with \(s = 1 \ldots 2p\), and \(2p - 4\) vectors \(m_s\) with \(s = 2 \ldots p - 1, p + 2 \ldots 2p - 1\). The action of \(\mathcal{H}\)-operators in this basis is defined by the formulas:
\[
(2.11) \quad Xw_s = q^s w_s, \quad Xe_s = q^s (e_s + w_s), \quad Xm_s = q^s m_s,
\]
\[
(2.12) \quad Tw_p = -q^{-1} w_p - (q - q^{-1}) e_p, \quad Tw_{2p} = -q^{-1} w_{2p} - (q - q^{-1}) e_{2p},
\]
\[
(2.13) \quad Tw_s = -q^{-s} \left[ \frac{s}{2p} \right] w_s - \left[ \frac{s - 1}{2p} \right] w_{2p-s}, \quad s \neq 0, p,
\]
\[
(2.14) \quad Te_p = q e_p, \quad Te_{2p} = q e_{2p},
\]
2.3.1. Proposition. 

\[ T s = \frac{2}{(q - q^{-1})[s]^2} (w_s - w_{2p-s}), \quad s \neq 0, p, \]

2.2.2. Proposition. 

\[ T m_{2p-1} = q m_{2p-1} - (q + q^{-1}) w_1, \]

\[ T m_{p-1} = q m_{p-1}, \]

\[ T m_s = -q^{-s} m_s + \frac{[s-1]}{[s]} m_{2p-s} \quad s = 2 \ldots p - 2, p + 2 \ldots 2p - 2, \]

\[ Y w_p = -q^{-1} w_{p+1} + (q - q^{-1}) e_{p+1}, \quad Y w_{2p} = -q^{-1} w_1 + (q - q^{-1}) e_1, \]

\[ Y w_s = -\frac{q^{-s}}{[s]} w_{2p-s+1} - \frac{[s-1]}{[s]} w_{s+1}, \quad s \neq 0, p, \]

\[ Y e_p = -q e_{p+1}, \quad Y e_{2p} = -q e_1, \]

We define a commutative associative multiplication in \( \mathbb{Z} \) by formulas (1.9).

2.2.2. Proposition. \( \mathbb{Z} \) is reducible. The 2p + 4-dimensional subspace

\[ \mathcal{W} \equiv \{ w_1 \ldots w_{2p}, e_1, e_p, e_{p+1}, e_{2p} \} \]

is invariant under the \( \mathcal{H} \)-action and is therefore a subrepresentation. The quotient is a direct sum: \( \mathbb{Z}/\mathcal{W} = \mathcal{E} \oplus \mathcal{M} \), where \( \mathcal{E} \equiv \{ e_2 \ldots e_{p-1}, e_{p+2} \ldots e_{2p-1} \} \) and \( \mathcal{M} \equiv \{ m_2 \ldots m_{p-1}, m_{p+2} \ldots m_{2p-1} \} \).

Proof. Immediately follows from (2.11)–(2.24). \hfill \Box

2.3. \( Y \)-basis. In this subsection we prove that the Jordan form of \( Y \) is (1.11).

2.3.1. Proposition. A Jordan basis of \( Y \) consists of 6p - 4 vectors: 4p vectors \( f_s, u_s \) for \( s = 1, \ldots, 2p \), and 2p - 4 vectors \( k_s \) for \( s = 2, \ldots, p - 1, p + 2 \ldots 2p - 1 \). The action of \( Y^{-1} \) on these vectors is given by (1.11).
Proof. We define in the $X$-basis the vectors

$$(2.26) u_s = \sum_{j=1}^{2p} u_{j,s}^{(w)} w_j + \sum_{j=1}^{2p} u_{j,s}^{(e)} e_j, \quad s = 1 \ldots 2p,$$

where coefficients are

$$u_{j,s}^{(w)} = \frac{(-1)^{s+j}}{\sqrt{2p}} \left( q^s \{s, j\} - q \{s, j-1\} \right), \quad j = 1 \ldots 2p;$$

$$u_{j,s}^{(e)} = (1)^s \frac{q}{\sqrt{2p}}; \quad u_{j,s}^{(e)} = (-1)^s \frac{q^s}{\sqrt{2p}};$$

$$u_{j,s} = (1)^{p+1} \frac{q}{\sqrt{2p}}; \quad u_{j,s} = (-1)^{p+1} \frac{q^s}{\sqrt{2p}};$$

$$u_{j,s}^{(e)} = 0, \quad j \neq 1, p, p + 1, 2p,$$

the vectors

$$(2.28) k_s = \sum_{j=1}^{2p} k_{j,s}^{(w)} w_j + \sum_{j=1}^{2p} k_{j,s}^{(e)} e_j, \quad s = 2 \ldots p - 1, p + 2 \ldots 2p - 1,$$

where coefficients are

$$k_{j,s}^{(w)} = \frac{(p - s)}{p} u_{j,s}^{(w)} - \frac{(-1)^{s+j}}{p\sqrt{2p}} \left( q^s \{s, j\} \{1, j\} - q \{s, j-1\} \{1, j-1\} \right),$$

$$k_{j,s}^{(e)} = (-1)^s + 1 \frac{q^s [s] + q(p - s)}{p\sqrt{2p}}, \quad k_{j,s}^{(m)} = (-1)^{p+1} \frac{q^s + q^s(p - s)}{p\sqrt{2p}};$$

$$k_{p+1,s}^{(e)} = (-1)^{p+1} \frac{q^s + q(p - s)}{p\sqrt{2p}}, \quad k_{j,s}^{(m)} = (-1)^{s+1} \frac{q^s + q^s(p - s)}{p\sqrt{2p}};$$

$$k_{j,s} = \frac{(1)^{s+j}}{p\sqrt{2p}} \left( q^s [s, j] - q [s, j-1] \right), \quad j \neq 1, p, p + 1, 2p,$$

and the vectors

$$(2.30) f_s = \sum_{j=1}^{2p} f_{j,s}^{(w)} w_j + \sum_{j=1}^{2p} f_{j,s}^{(e)} e_j + \left( \sum_{j=2}^{p-1} + \sum_{j=p+2}^{2p-1} \right) f_{j,s}^{(m)} m_j, \quad s = 1 \ldots 2p,$$

where coefficients are

$$f_{1,s}^{(w)} = \frac{2(-1)^s + 1}{(q - q^{-1})\sqrt{2p}}, \quad f_{p,s}^{(w)} = \frac{q(-1)^{p+1}[s]}{\sqrt{2p}}, \quad f_{p+1,s}^{(w)} = \frac{2(-1)^p q^2}{(q - q^{-1})\sqrt{2p}},$$

$$f_{1,j,s}^{(w)} = \frac{q(-1)^s [s]}{\sqrt{2p}},$$

$$(2.31) f_{j,s}^{(w)} = -p(p - j)k_{j,s}^{(e)} + \frac{(-1)^{s+j}}{\sqrt{2p}} \left( q^s [s, j-1] + q^s \{s, j\} \right), \quad j \neq 1, p, p + 1, 2p,$$

$$f_{p,s}^{(e)} = (-1)^{p+1} f_{j,s}^{(e)} = (-1)^p \frac{q^s}{\sqrt{2p}}, \quad f_{j,s}^{(e)} = 0, \quad j \neq 2, p,$$

$$f_{j,s}^{(m)} = -p^2 k_{j,s}^{(e)}, \quad j \neq 1, p, p + 1, 2p.$$
and the coefficient \( f_{j,s}^{(m)} \) in (2.30) is 0 for \( s = 1, p, p + 1, 2p \).

Then, (1.11) is checked by a simple calculation using formulas (2.19)–(2.24).

The linear independence of these vectors is proved in the following way. From the decompositions in the \( X \)-basis, we obtain that vectors \( u_s, f_1, f_p, f_{p+1}, f_{2p} \) belong to \( \mathcal{W} \), vectors \( k_s \) belong to \( \mathcal{W} + \mathcal{E} \), and vectors \( f_s \) with \( s = 2 \ldots p - 1, p + 2 \ldots 2p - 1 \) belong to \( \mathcal{W} + \mathcal{M} \). We recall, that under the isomorphism \( \mathcal{W} + \mathcal{E} \sim \mathcal{V}^{-2} \) vectors \( u_s, k_s, f_1, f_p, f_{p+1}, f_{2p} \) correspond to the linearly independent functions (2.7)–(2.9) in \( \mathcal{V}^{-2} \), and therefore the vectors are also linearly independent. In particular, vectors \( u_s, f_1, f_p, f_{p+1}, f_{2p} \) form a basis in \( \mathcal{W} \) and therefore images of the vectors \( k_s \) under the canonical projection to \( \mathcal{E} = (\mathcal{W} + \mathcal{E})/\mathcal{W} \) form a basis in \( \mathcal{E} \). We let abusing notations \( k_s \) denote these images. We recall that the isomorphism \( \mathcal{E} \sim \mathcal{M} \) maps vectors \( k_s \) (images under the canonical projections of \( k_s \in \mathcal{Z} \)) to \( f_s \) (images under the canonical projections of \( f_s \in \mathcal{Z} \)), and therefore \( f_s \) with \( s = 2 \ldots p - 1, p + 2 \ldots 2p - 1 \) are linearly independent. Thus, the linear independence of all vectors \( u, f, k \) is established. \( \square \)

3. \( PSL(2, \mathbb{Z}) \) ACTION IN \( \mathcal{Z} \)

In this section we define operators \( S \) and \( v \) and prove that they satisfy \( PSL(2, \mathbb{Z}) \) relations. Conjugations with operators \( S \) and \( v \) give automorphisms \( \sigma \) and \( \tau_+ \) respectively.

3.1. \( \sigma \). We define the \( S \)-operator that maps the \( X \)-basis to the \( Y \)-basis by formulas (1.12).

3.1.1. Proposition. \( S \) satisfies relations

\[
\begin{align*}
(3.1) & \quad SXS^{-1} = Y^{-1}, \\
(3.2) & \quad SYS^{-1} = XT^2, \\
(3.3) & \quad STS^{-1} = T, \\
(3.4) & \quad S^3 = qT^{-1}.
\end{align*}
\]

Proof. 

- (3.1) follows from the definition of \( S \).
- (3.4) follows from a direct calculation of \( TS^2 \)-action in the \( X \)-basis. We give a detailed calculation of \( TS^2 e_s \). The calculation of \( TS^2 w_s \) and \( TS^2 m_s \) is similar and is omitted. We check that \( TS^2 e_s = q e_s \). We begin with

\[
S^2 e_s = S f_s = S \left( \sum_{r=1}^{2p} f_{r,s}^{(w)} u_r + f_{p,s}^{(e)} e_p + f_{2p,s}^{(e)} e_{2p} + \left( \sum_{r=2}^{p} + \sum_{r=p+2}^{2p-1} \right) f_{r,s}^{(m)} m_r \right) = \]

\[
= f_{1,s}^{(w)} u_1 + f_{p,s}^{(w)} u_p + f_{p+1,s}^{(w)} u_{p+1} + f_{2p,s}^{(w)} u_{2p} + f_{p,s}^{(e)} e_p + f_{2p,s}^{(e)} e_{2p} + \]

\[
+ \left( \sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) f_{r,s}^{(m)} m_r = f_{r,s}^{(m)} m_r .
\]
Then we calculate coefficients in front of $e_j$ and $w_j$ in (3.5) using (2.26)-(2.31). This calculation is cumbersome and is given in Appendix A. The result of the calculation is

$$S^2 e_s = \frac{-q^{s+1}}{s} e_s + q \frac{[s - 1]}{s} e_{2p-s} + \frac{2(q^2 - 1)}{(q^s - q^{-s})^2} (w_s - w_{2p-s}), \quad s \neq p, 2p,$$

$$S^2 e_p = e_p, \quad S^2 e_{2p} = e_{2p}.$$

A simple calculation using (2.12)-(2.15) gives

$$TS^2 e_s = q e_s, \quad s = 1 \ldots 2p.$$

- (3.3) is checked as follows
  $$S^2 = q T^{-1} \Rightarrow ST = TS = q S^{-1} \Rightarrow STS^{-1} = T.$$

- (3.2) is checked as follows
  $$S X S^{-1} Y^{-1} \Rightarrow S X S^{-1} = Y \Rightarrow SY S^{-1} = S^2 X^{-1} S^{-2} \Rightarrow T^{-1} X^{-1} T \equiv XT^2.$$

3.2. $\tau_\pm$. The automorphism $\tau_\pm$ can be realized as a conjugation with the element $v \in \mathbb{Z}$ given by (1.13).

3.2.1. Proposition. For $v$ given by (1.13), the operator

$$\tau_+ (x) = v^{-1} x v, \quad \forall x \in \mathcal{H}$$

satisfies relations (1.4).

Proof. A direct calculation.

3.2.2. Proposition. The map

$$\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \rightarrow s, \quad \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \rightarrow v.$$

gives a $PSL(2, \mathbb{Z})$ action in $\mathcal{Z}$.

Proof. Relations (3.1)-(3.4) and (1.4) are sufficient to check the $PSL(2, \mathbb{Z})$ relations.

4. EIGENSPACE OF $T$ WITH EIGENVALUE $q$

In this section, we describe the representation of the symmetrized DAHA. In section 5, we prove that it is isomorphic to the centre of $\overline{U}_q \mathfrak{sl}(2)$.

We let $\mathcal{J}_q$ denote the eigenspace of $T$ with the eigenvalue $q$. It is $3p - 1$ dimensional. Operators $X$ and $Y$ have no well-defined restriction to $\mathcal{J}_q$ but "symmetrized" operators $C = -(X + X^{-1})$ and $H = -(Y + Y^{-1})$ have. Indeed, for a given $a \in \mathcal{J}_q$, we have

$$T(X + X^{-1}) a = (X^{-1}T^{-1} + TX^{-1}) a = (q^{-1} + T)X^{-1} a.$$
Thus, \((X + X^{-1})a \in \mathcal{T}_q\) and a similar calculation shows that \(H\) has the well-defined restriction to \(\mathcal{T}_q\) as well.

4.1. \(C\)-basis. The eigenvectors of \(C = -(X + X^{-1})\) are

\[
e_0 = e_p, \quad e_p = e_{2p}, \quad e_s = e_{p+s} + e_{p-s},
\]

\[
w^+_1 = \frac{1}{q-q^{-1}} m_{p-1}, \quad w^-_1 = \frac{1}{q-q^{-1}} (w_{p+1} - w_{p-1} - m_{p-1}),
\]

\[
w^+_s = \frac{[s]}{q-q^{-1}} (m_{p-s} + m_{p+s}), \quad w^-_s = \frac{[s]}{q-q^{-1}} (w_{p+s} - w_{p-s} - m_{p-s} - m_{p+s}),
\]

\[
w^+_{p-1} = \frac{1}{q-q^{-1}} (m_{2p-1} - w_1), \quad w^-_{p-1} = \frac{1}{q-q^{-1}} (w_{2p-1} - m_{2p-1}),
\]

\[
w_s = w^+_s + w^-_s = \frac{[s]}{q-q^{-1}} (w_{p+s} - w_{p-s}), \quad s = 1 \ldots p - 1.
\]

The action of \(C\) on them follows from (2.11)

\[
(4.2) \quad C e_0 = \mu_0 e_0, \quad C e_p = \mu_p e_p,
\]

\[
(4.3) \quad C e_s = \mu_s e_s + (q - q^{-1})^2 w_s, \quad s = 1 \ldots p - 1
\]

\[
(4.4) \quad C w^+_s = \mu_s w^+_s, \quad s = 1 \ldots p - 1,
\]

where

\[
(4.5) \quad \mu_s = q^s + q^{-s}, \quad 0 \leq s \leq p.
\]

The multiplication in \(\mathcal{T}_q\) is induced by (1.9)

\[
(4.6) \quad e_r w^+_s = \delta_{r,s} w^+_s, \quad e_r e_s = \delta_{r,s} e_s, \quad w^+_r w^+_s = 0.
\]

4.2. \(H\)-basis. The eigenvectors of \(H = -(Y + Y^{-1})\) are

\[
f_0 = f_p, \quad f_p = f_0, \quad f_s = f_{p+s} + f_{p-s},
\]

\[
u^+_1 = \frac{1}{q-q^{-1}} k_{p-1}, \quad v^-_1 = \frac{1}{q-q^{-1}} (u_{p+1} - u_{p-1} - k_{p-1}),
\]

\[
u^+_s = \frac{[s]}{q-q^{-1}} (k_{p-s} + k_{p+s}), \quad v^-_s = \frac{[s]}{q-q^{-1}} (u_{p+s} - u_{p-s} - k_{p-s} - k_{p+s}),
\]

\[
u^+_{p-1} = \frac{1}{q-q^{-1}} (k_{2p-1} - u_1), \quad v^-_{p-1} = \frac{1}{q-q^{-1}} (u_{2p-1} - k_{2p-1}),
\]

\[
u_s = v^+_s + v^-_s = \frac{[s]}{q-q^{-1}} (u_{p+s} - u_{p-s}), \quad s = 1 \ldots p - 1.
\]

The action of \(H\) on them follows from (1.11)

\[
(4.7) \quad H f_0 = \mu_0 f_0, \quad H f_p = \mu_p f_p,
\]

\[
(4.8) \quad H f_s = \mu_s f_s + (q - q^{-1})^2 u_s, \quad s = 1 \ldots p - 1,
\]

\[
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\]
where eigenvalues are given by (4.5).

4.3. $SL(2, \mathbb{Z})$ action. Operators $S$ and $v$ have well-defined restrictions to $T_q$. This endows $T_q$ with a representation of $SL(2, \mathbb{Z})$. In more detail, $S$-operator in $T_q$ satisfies

\begin{align}
S e_s = f_s, & \quad s = 0 \ldots p, \\
S u_s^\pm = u_s^\pm, & \quad s = 1 \ldots p - 1
\end{align}

(4.10)

and because $T = q$ in $T_q$, we have $S^2 = 1$. We note also that in $T_q$ relations (4.11) lead to

\begin{align}
S C S^{-1} = H.
\end{align}

5. Proof of Theorem 1.4

We note that $T_q$ and the center of $\overline{U}_q sl(2)$ from [6] have the same dimension equal to $3p - 1$. Then we identify $C$-basis ($H$-basis) in $T_q$ with the Radford images (Drinfeld images) of $q$-characters of irreducible representations

\begin{align}
\hat{\phi}^+(s) = \omega_s w_s^+, & \quad \hat{\phi}^-(s) = \omega_{p-s} w_{p-s}, \quad s = 1 \ldots p - 1, \\
\hat{\phi}^+(p) = p \sqrt{2p} e_p, & \quad \hat{\phi}^-(p) = (-1)^{p+1} p \sqrt{2p} e_0, \\
\chi^+(s) = \omega_s u_s^+, & \quad \chi^-(s) = \omega_{p-s} u_{p-s}, \quad s = 1 \ldots p - 1, \\
\chi^+(p) = p \sqrt{2p} f_p, & \quad \chi^-(p) = (-1)^{p+1} p \sqrt{2p} f_0.
\end{align}

(5.1)

This identification establishes an isomorphism between $T_q$ and the center of $\overline{U}_q sl(2)$ as associative commutative algebras.

Under the identification (5.1), $T_q$ coincides with the center of $\overline{U}_q sl(2)$ as the representation of $SL(2, \mathbb{Z})$. In particular, the relations $S(\chi^+(s)) = \hat{\phi}^+(s)$ for $s = 0 \ldots p$ in the center are parallel to the relations (4.10) in $T_q$. The Gaussian element $v$ in notations of [6]

\begin{align}
v = \sum_{s=0}^{p} (-1)^{s+1} q^{-\frac{1}{2}(s^2-1)} e_s + \sum_{s=1}^{p-1} (-1)^{p} q^{-\frac{1}{2}(s^2-1)} \frac{q^s - q^{-s}}{\sqrt{2p}} \hat{\phi}(s),
\end{align}

(5.2)

where $\hat{\phi}(s) = \frac{p-s}{p} \hat{\phi}^+(s) - \frac{s}{p} \hat{\phi}^-(p-s)$ for $1 \leq s \leq p-1$ coincides with the ribbon element of $\overline{U}_q sl(2)$.

6. Discussion

We identified the representation of DAHA that gives the Verlinde algebra of $(1, p)$ logarithmic conformal field models. The center of $\overline{U}_q sl(2)$ coincides with the symmetrization of $\mathcal{Z}$ and $C = -(X + X^{-1})$ coincides with the $\overline{U}_q sl(2)$ Casimir element. Probably the
whole representation $\mathbb{Z}$ can be realized in $\overline{U}_q\mathfrak{sl}(2)$ such that $X$ would be realized by a multiplication with a $\overline{U}_q\mathfrak{sl}(2)$ element.

Another interesting direction of investigations is to find a realization of $\mathcal{H}$ on $(1, p)$ logarithmic conformal field model conformal blocks. This can also be useful in boundary conformal field theories. The Ishibashi and Cardy boundary states can probably be identified with eigenvectors of operators $C = -(X + X^{-1})$ and $H = -(Y + Y^{-1})$ respectively.

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**Appendix A. Proof of $S^2 = qT^{-1}$**

We calculate coefficient in front of $e_j$ in (A.1) and coefficient in front of $w_j$ in (A.2).

**A.1. The coefficient in front of $e_j$.** The substitution of (2.26), (2.28), (2.30) in (3.5) gives the coefficient in front of $e_j$

\[
(A.1) \quad f_1^{(w)} u^{(e)}_{j,1} + f_{p,s}^{(w)} u^{(e)}_{j,p} + f_{p+1,s}^{(w)} u^{(e)}_{j,p+1} + f_{2p,s}^{(w)} u^{(e)}_{j,2p} + f_{p,s}^{(e)} + f_{p+1,s}^{(e)} + f_{2p,s}^{(e)} + f_{2p+1,s}^{(e)} + \left( \sum_{r=2}^{p-1} \sum_{r=p+2}^{2p-1} f_r^{(w)} u_{j,r}^{(e)} + f_r^{(e)} k_r^{(e)} \right)
\]

where all numbers $u$, $k$, $f$ are given in (2.27), (2.29), (2.31). A simplification of the underbraced expression gives

for $j \neq 1, p, p + 1, 2p$:

\[
A = \frac{(-1)^{s+j} q^2}{2p} \left( q^{r-1}[s, r-1][r, j] - [s, r-1][r, j-1] \right) + \frac{(-1)^{s+j} q^s}{2p} \left( q[s, r][r, j-1] - q^s[s, r][r, j] \right),
\]

for $j = 1$:

\[
A = \frac{(-1)^{s+1} q^2}{2p} \left( q^{r-1}[s, r-1][r, 1] - [s, r-1][r, 1] \right) + \frac{(-1)^{s+1} q^s}{2p} \left( -q[s, r] - q^s[s, r][r, 1] \right),
\]

for $j = p + 1$:

\[
A = \frac{(-1)^{s+p+1} q^2}{2p} \left( q^{r-1}[s, r-1][r, p+1] - [s + p, r - 1] \right) + \frac{(-1)^{s+p+1} q^s}{2p} \left( -q[s, r] - q^s[s, r][r, 1] \right),
\]
\[ A = \frac{(-1)^{s+p}q^p}{2p} \left( q^{-1}[s, r - 1] - [s, r - 1][r, p - 1] \right) + \frac{(-1)^{s+p}q^p}{2p} \left( q[s, r][r, p - 1] - q^r\{s + p, r\} \right). \]

For \( j = p \):

\[ A = \frac{(-1)^{s+p}q^p}{2p} \left( q^{-1}[s, r - 1] - [s, r - 1][r, p - 1] \right) + \frac{(-1)^{s+p}q^p}{2p} \left( q[s, r][r, p - 1] - q^r\{s + p, r\} \right). \]

For \( j = 2p \):

\[ A = \frac{(-1)^{s}q^2}{2p} \left( q^{-1}[s, r - 1] - [s, r - 1][r, 2p - 1] \right) + \frac{(-1)^{s}q^2}{2p} \left( q[s, r][2p - 1] + q^r\{s, r\} \right). \]

Then the summation in \( r \) of different terms in \( A \) is given by

\[
\sum_{r=2}^{p} \left( \sum_{r=p+2}^{2p-1} \right) q^{-1}[s, r - 1][r, j] = \left( p - \frac{1}{4}((s + j + 1 \mod 2p) + (s - j + 1 \mod 2p) + (s + j - 1 \mod 2p) + (s - j - 1 \mod 2p)) \right) \left( 1 + (-1)^{s+j} \right) + \frac{p}{(s - j \mod 2p)}, \]

\[
\sum_{r=2}^{p} \left( \sum_{r=p+2}^{2p-1} \right) [s, r - 1][r, j] = \left( 1 + (-1)^{s+j+1} \right) \left( p - \frac{1}{2}((s + j \mod 2p) + (s - j \mod 2p)) \right), \]

\[
\sum_{r=2}^{p} \left( \sum_{r=p+2}^{2p-1} \right) q^{-1}[s, r][r, j] = \frac{2p \left( \delta_{s, 2p-j} - \delta_{s, j} \right)}{q^j - q^{-j}} - q[s] \left( 1 + (-1)^{j+s} \right), \quad j \neq p, 2p, \]

\[
\sum_{r=1}^{2p} \left( \sum_{r=p+1}^{2p-1} \right) [s, r] = (1 - (-1)^{s})(p - (s \mod 2p)), \]

\[
\sum_{r=1}^{2p} \left( \sum_{r=p+1}^{2p-1} \right) q^{-1}[s, r] = (1 + (-1)^{s}) \left( p - \frac{(s + 1 \mod 2p) + ((s - 1 \mod 2p))}{2} \right), \]

\[
\sum_{r=1}^{2p} q^{-1}[s, r] = (1 + (-1)^{s}) \left( \frac{(s - 1 \mod 2p) - ((s + 1 \mod 2p))}{2} \right). \]
A.2. The coefficient in front of $w_j$. The substitution of (2.26), (2.28), (2.30) in (3.5) gives the coefficient in front of $w_j$

\[(A.2) \quad \begin{align*}
\frac{q^2(-1)^{s+j+1}}{p(q^{j-1} + q^{j+1})} & \left[ s, r-1 \right] [r-1, j-1] + \frac{(-1)^{s+j+1}}{p(q^{j-1} + q^{j-1})} q_r^{-1} [s, r-1] [r-1, j] + \\
& + \frac{q^s(-1)^{s+j}}{2p} \left( q^r \left\{ s, r \right\} \{r, j\} - q \left\{ s, r \right\} \{r, j-1\} + q^r\{s, r\}[r, j]\right) - q\{s, r\}[r, j-1]\left\{1, j-1\right\} - q [s, r] [r, j-1] \left\{1, j-1\right\}
\end{align*}\]

Then the simplification of (A.2) gives coefficients in (3.5) in front of $w_s$. Explicitly, the summation in $r$ of different terms in (A.3) is given by

\[(A.3) \quad \begin{align*}
\left( \sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) \left[ s, r-1 \right] [r-1, j] & = [s](1 - (-1)^{s+j}), \quad j \neq p, 2p, \\
\left( \sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) q_r^{-1} [s, r-1] [r-1, j] & = \frac{2p(\delta_{s,2p-j} - \delta_{s,j})}{q^j - q^{-j}} + q^{-1}[s](1 - (-1)^{j+s}, \quad j \neq p, 2p, \\
\left( \sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) q^r \left\{ s, r \right\} \{r, j\} & = \left\{ s, 1 \right\} \{1, j\} ((-1)^{s+j} - 1), \\
\left( \sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) q^r [s, r] [r, j] & = \frac{2p(\delta_{s,2p-j} - \delta_{s,j})}{q^j - q^{-j}} - q[s](1 - (-1)^{j+s}, \quad j \neq p, 2p, \\
\left( \sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) [s, r] [r, j] & = [s] ((-1)^{s+j} - 1), \quad j \neq p, 2p.
\end{align*}\]

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