GEOMETRY AND PHYSICS OF $Sp(3)/Sp(1)^3$  

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Abstract

The geometrical relations between the quarks that compose mesons and baryons are shown to be succinctly formulated as a complete flag over a vector space of three isospins, with the flag manifold (coset) $Sp(3)/Sp(1)^3$ conveying the interactions between the three linearly independent components. It is shown that curvature two-forms (or tensors) mediate the pair-wise additive interactions between quarks. The root space of the coset is shown to be isomorphic to that of $SU(3)$, suggesting similarities between the representations of the flag manifold and those of $SU(3)$.

Introduction: Flags and Flag Manifolds

Matter is built up from elementary particles: quarks comprise nucleons, nuclei and electrons make up atoms, atoms bond together to form molecules, which might comprise a crystal that is placed in an instrument that is located in a laboratory in a building on the campus of – the sequence can be continued to encompass as much of the world as one likes. This statement can be rendered abstractly: there exists a sequence of subsets of material objects, $S_1 \subset S_2 \subset S_3 \subset \cdots$, where $S_1 =$quarks, $S_2 =$nucleons, $S_3 =$nucleons + electrons = atoms, $S_4 =$molecules, etc., of some large set of objects. Since all matter is composed of the same building blocks, the elementary particles, each subset in the sequence is built up from the members of the preceding subset. There is also an implied geometry of spatial inclusion as one climbs the ladder of complexity. By imposing geometrical relations between objects we can do more than talk about abstract sets and subsets – there is a mathematical structure that accommodates these notions. A flag is a sequence of vector subspaces $V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n$ of a vector space $V_n$ of dimension determined by the space of interest, equivalent to truncating our sequence of objects at the desired level.
These elementary observations are ample reasons to be interested in flags and their associated flag manifolds.

In the standard definition, a flag is a sequence of vector spaces $V_i(K)$ over the real or complex field $K = \mathbb{R}, \mathbb{C}$ such that $V_i$ is a proper subset of $V_{i+1}$ for all $i$ up to the complete space $V_n$. (See Wikipedia for an introduction.) The standard definition can be extended to encompass the quaternion ring $K = \mathbb{H}$, which will be central to the theory presented here. One may define a natural action of a Lie group on the flag, and the flag manifold is constructed from cosets of the group in a manner to be discussed.

$SU(3)$ can be interpreted as a group acting on an abstract vector space $V_3(\mathbb{C})$ of quarks. Within the group itself there are no space-time coordinates, which implies that it aims at the underlying symmetries of a three-body object. These are intrinsic properties, which the group defines without relation to other objects. Thus all protons, for example, have identical intrinsic properties, independent of their location. The representations of the group are then revealed to enable the construction of a host of complex objects from simpler ones. Following in these footsteps, we will also not be using space-time coordinates in the flag structure. However, in exploring the mathematics of the flag manifold we will uncover many objects that bear directly on many-body physics.

It will emerge that some of the deeper aspects of quantum theory are natural consequences of the mathematical structure of the flag and the associated flag manifold; many quantum rules are straightforward consequences of the mathematics. The immediate goal is to apply the mathematics of the flag manifold to particle theory with an enlarged group, thereby laying the groundwork for extending the success of $SU(3)$ in the classification of elementary particles.

**Structural Preliminaries.** The mathematical structure introduced above has been formulated as a theory of interactions. In this theory, one is thinking about a state space of a many-body system, similar to that concept in statistical mechanics. States of the system are envisioned to transform into one another by the action of a group. This fits nicely into the flag structure, as there is a natural transitive group action on a flag, and by choosing the group to preserve a measure on the total space $V_n$, ...
the group is required to be compact. In the group context, a reducible representation corresponds to disconnected spaces, so one wants to think about systems that correspond to irreducible representations. Higher dimensional representations than the fundamental will be excited states, and the group can be imagined to move excitations from one component to another. This means that the elements of the group represent bosons while the vector space (or module) on which the group acts consists of fermions. But starting with the fundamental representation, subgroups of dimension compatible with the flag structure preserve subspaces. At the lowest level (apart from the null space), $V_1$ can be one-dimensional over $\mathbb{K}$, so we should consider subgroups that preserve each one-dimensional space. This starts at the level of elementary objects that are equivalent yet unique, much like the $x, y, z$ coordinates of three space are equivalent yet unique—they are linearly independent. One then envisions that composite particles are constructed as higher dimensional representations built from combinations of the elementary units, just as is done with $SU(3)$.

The spin (or isospin) of an elementary particle is incorporated at the outset in this description by choosing $\mathbb{K} = \mathbb{H}$. (Since we want a group structure, quaternions rather than Pauli matrices are used.) The point here is that, of the several properties of fundamental particles—mass, charge, and spin—only the last is intrinsically based on a group action. These ideas direct attention to the symplectic group $Sp(n)$, which is a compact topological space over $\mathbb{H}$. In the physical context the group acts on a Hilbert space, which is interpreted as the state space of an $n$-body system assembled into several parts conforming to a flag description. In application to fundamental particle systems, the cosets (complete flag manifolds) $Sp(k)/Sp(1)^k := Sp(k)/Sp(1) \times \cdots \times Sp(1)$ for small $k$ are of interest. The group $Sp(n)$ consists of matrices that are unitary over the quaternions: $Sp(n) \sim U(n, \mathbb{H})$. $Sp(n)$ is a compact form of $Sp(2n, \mathbb{C})$; its Lie algebra $\mathfrak{sp}(n) \subset \mathfrak{sp}(2n, \mathbb{C})$ is contained in that of the larger group. (Sources for group theory are refs. [2, 3, 4].)

The general setting for the theory is provided by a principal bundle,[5] generically written as $G(G/H, H)$, where $G = Sp(n)$ is the bundle space, $G/H$ is the base space, and for the restricted case under consideration here, $H = Sp(1)^k$ is the group of the
fiber. This subgroup leaves the components of the flag point-wise fixed in the vector space \( V_n(\mathbb{H}) \).

An extremely important aspect of a group action is that geodesics on a Lie group are left cosets of one parameter subgroups. A map from \( t \in \mathbb{R} \) to a coset is constructed by first selecting an \( x \in g \setminus h \), i.e. \( x \notin h \), where \( g \) is the Lie algebra of the group and \( h \) is the algebra of the fiber. The map \( \exp : t \to \exp(t\mathbf{x}) \), \( t \geq 0 \), is a geodesic through the origin in the base space \( G/H \), which introduces a global time coordinate.

The second aspect of the coset space structure involves the separation of the action of \( H \) and \( G/H \) on the Hilbert space. Define a function \( \Psi_\phi(x) \) in the representation space of \( G \) by

\[
\Psi_\phi(x) := \int \sigma(h)\phi(xh)dh
\]

where \( dh \) is normalized Haar measure on the group \( H \) and \( \phi(xh) \) is a map from \( G \) into a Hilbert space of dimension compatible with the representation \( \sigma(h) \); \( \sigma(ab) = \sigma(a)\sigma(b) \) for \( a, b \in H \); \( \sigma \) is a homomorphism. Given a left invariant Haar measure on \( H \) it follows that

\[
(1) \quad \Psi_\phi(x\eta) := \int \sigma(h)\phi(x\eta h)dh = \sigma(\eta^{-1})\Psi_\phi(x)
\]

for \( \eta \in H \). The \( L^2 \) measure \( \langle \Psi_\phi(x), \Psi_\phi(x) \rangle \) on the Hilbert space is thus invariant to \( \sigma(h) \). This encompasses linear combinations of the components of \( \Psi_\phi(x) \); in our case, \( \sigma(H) \) is represented as a diagonal matrix of \( Sp(1) \) factors in the fundamental representation. The quotient of \( Sp(k) \) by the stabilizer (isotropy) subgroup \( Sp(1)^k \) gives a coset that is invariant to the “orientation” of the one dimensional quaternion basis elements, and eq. (1) asserts the invariance of the measure \( \langle \Psi, \Psi \rangle \) to the choice of basis. Thus \( \sigma(H) \) plays the role of a gauge group, albeit different from the space-time dependent Weyl phase factors in scalar quantum theory.

Under the action of \( g \in G \), the coset \( xH \) is sent to \( g : xH \to yH \). A representation \( A_g \) acts by \( A_g\Psi(x) = \Psi(g^{-1}x) = \Psi(wh) = \sigma(h^{-1})\Psi(y) \). (See the parallel presentation for finite groups in Sec. V.3 of ref. [4].)

**Relation to Yang-Mills Theory.** A crucial example of a structure conforming to all of the above has long been known. Atiyah showed that the Yang-Mills
functional\[9, 10, 11]\ is minimized by the curvature two-form

\[ F = \frac{dq \wedge d\bar{q}}{(1 + q\bar{q})^2}, \]

where \(\bar{q}\) is the quaternion conjugate of \(q\). The geometrical structure that yields this curvature two-form is the Grassmannian \(Sp(2)/Sp(1)^2\).\[11, 12]\ One may interpret this coset space structure in the fundamental representation as the action of \(Sp(2)\) on a square-integrable, quaternion-valued Hilbert space \(V_2(\mathbb{H})\), with the subgroup \(Sp(1) \times Sp(1)\) acting on each \(V_1(\mathbb{H})\) separately, as was discussed above. The well-known Lie algebra isomorphism \(sp(1) \sim su(2) \sim so(3)\) then leads to the identification of the two \(Sp(1)\) components of the group \(Sp(1) \times Sp(1)\) of the fiber as spin (or gauge groups of isospin) degrees of freedom of single particle states. The coset, \(Sp(2)/Sp(1) \times Sp(1) \sim q\) consists of the instanton coordinates, and the action of the coset can be interpreted as a coupling of the two components of \(V_2(\mathbb{H})\); the two elementary or fundamental quaternions affect one another through the action of the coset. The curvature of the coset space is equivalent to an interaction or force between the particles, as conveyed by the curvature form above. The separation of the components of the group that is given by eq. \(\text{[11]}\) shows how the representation space of a principal bundle relates the gauge group of isotopic spin to the instanton content of the representation space.

The flag manifold structure can be extended to any \(Sp(n), n > 2,\) and partition of \(n:\)

\[ \{k_1, k_2, \cdots, k_m\}; \sum_i k_i = n, \]

with corresponding coset \(Sp(n)/Sp(k_1) \times Sp(k_2) \times \cdots \times Sp(k_m)\), consistent with the subspace decomposition, \(V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n\). For any decomposition, the metric on the coset space and the associated curvature two-forms or tensors are simply related, as will be shown.

Given this remarkable structure, and the encouraging confirmation from Yang-Mills theory, the next natural extension is to a system of three fundamental fermions, which should be described with \(Sp(3)/Sp(1)^3\). Knowing that \(SU(3)\) is a subgroup of \(Sp(3)\) is sufficient motivation to ask whether the larger group and coset might provide additional insight into the structure of mesons and baryons. Indeed it does;
we will be able to construct explicit functions to describe physical states from the
eigenspace of the Lie algebra of the coset.

The scope of the present work is limited to showing how composite particles may be
constructed from the eigenspaces of $Sp(n)/Sp(1)^n$ for $n = \{2, 3\}$. Most importantly,
it will be shown that the coset space $Sp(3)/Sp(1)^3$ has a rank two algebra, and the
corresponding root space is isomorphic to that of $SU(3)$. The primary objective is
tool development, and while a few comments about representations will be offered,
detailed assignments are not considered. Perhaps experts will find the preliminary
assignments sufficiently interesting to encourage their participation.

In the following, $\text{spin}$ will simply mean the quaternion content of the structures to
be developed; the reader may prefer to use $\text{isotopic spin}$ and the $Sp(1)$ fibers as $\text{gauge groups}$.
(However, our interpretation of these concepts may differ in some respects.)
The fact that there is an instanton in the theory means that this is not a relativistic
theory. The departure from relativity also relates back to the use of quaternions
rather than Pauli matrices to describe spin. There are mappings between compact
and hyperbolic spaces when restricted to a single interaction,[13] so connections with
relativity can be made.

**Algebraic Preliminaries.** A quaternion, $q$, is represented in the Hamiltonian basis
as $q = q_01 + q_1i + q_2j + q_3k : q_i \in \mathbb{R}, 0 \leq i \leq 3$, where the anti-commuting basis
elements of the algebra, $\{1, i, j, k\}$, satisfy $ii = jj = kk = ijk = -1$. The quaternion
algebra is associative and distributive, but not commutative (except for the
unit element $1$). A mixed scalar-vector notation, $q = q_01 + q$, where $q$ is the vec-
tor/imaginary part of the quaternion, is often handy for calculations. The product of
two quaternions, $a$ and $b$, is $ab = (a_0b_0 - a \cdot b)1 + a_0b + b_0a + a \times b$. The advantage of
the mixed notation is that standard vector operations, scalar and vector product as
used here, is useful shorthand. (There are hazards in the use of the scalar-vector nota-
tion. Written as a quaternion product, $rr = -r \cdot r + r \times r = -1$; note that the scalar
product contains a center dot: $r \cdot r = +1$.) The conjugate quaternion is $\bar{q} = q_0 - q$.
Using the product rule it is easy to show that $q\bar{q} = |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$, where $|q|
$ is the norm of the quaternion.

In many calculations to follow the norm of a quaternion factors from the problem
or is otherwise of secondary importance, so that only the unit part, $u$, of $q = |q|u$ is of
interest. The logarithm of a quaternion exits in the sense that a unit quaternion has an exponential form: \( u = \exp(v) \). Since \( u\bar{u} = \bar{u}u = 1 \), it follows that \( \bar{u} = u^{-1} \Rightarrow \bar{v} = -v \), signifying that \( v = \mathbf{v} \) is a purely imaginary quaternion. (A unit quaternion is isomorphic to the three sphere \( S^3 \), and the tangent space of \( S^3 \) is isomorphic to \( \mathbb{R}^3 \).) Expanding the exponential,

\[
\exp(\mathbf{v}) = 1 + \mathbf{v} - \frac{1}{2!}|\mathbf{v}|^2 \mathbf{1} - \frac{1}{3!}|\mathbf{v}|^3 \mathbf{v} + \frac{1}{4!}|\mathbf{v}|^4 \mathbf{1} + \cdots = \cos(|\mathbf{v}|)\mathbf{1} + |\mathbf{v}|^{-1}\sin(|\mathbf{v}|)\mathbf{v},
\]

which makes clear that the magnitude, \( |\mathbf{v}| \), of \( \mathbf{v} = |\mathbf{v}|\mathbf{r} \), also factors, so that a quaternion \( q \) may be expressed as \( q = |q|\exp(\chi\mathbf{r}) \), where \( \chi = |v| \). While there is no natural restriction on \( \chi \) when a quaternion acts as an operator, functions \( \Psi(u) \) that appear in a physical context may require periodicity conditions. It is also clear that \( \bar{u} = u^{-1} = \cos(\chi)\mathbf{1} - \sin(\chi)\mathbf{r} \). In the \( Sp(n) \) context, the representation \( q = |q|\exp(\mathbf{v}) \) recommends against identifying the component \( q_0 \mathbf{1} \) with a temporal variable, but there is more to this story that will emerge as the theory develops.

The Hamiltonian basis might also be described as the \( Sp(1) \)-basis or \( \mathbb{H} \) representation. There is also a well-known \( SU(2) \) basis or \( \mathbb{C}^2 \)-basis (and an inclusion \( \mathfrak{sp}(n) \subset \mathfrak{sp}(2n, \mathbb{C}) \) of Lie algebras), which enables one to identify isomorphic basis elements in the usual way:

\[
1 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{i} \sim \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{j} \sim \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \quad \mathbf{k} \sim \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}; \quad \mathbf{i} = \sqrt{-1}
\]

which gives a conventional form for a quaternion as

\[
q = \begin{bmatrix} q_0 + iq_3 & q_1 + iq_2 \\ -q_1 + iq_2 & q_0 - iq_3 \end{bmatrix} = \begin{bmatrix} \zeta_1 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{\zeta}_1 \end{bmatrix}.
\]

There are specific advantages to the use of both representations.

Many constructions to be encountered involve the trace operation over a product of quaternion matrices. In general, \( \text{tr}(AB) \neq \text{tr}(BA) \) for \( \{A, B\} \) compatible matrices over \( \mathbb{H} \). However, if just the real component is sought, the cyclic permutation rule is valid.

Within the \( SU(2) \) basis there is an operation that is extremely useful for computations. Define \( J \sim \mathbf{i} \); a small calculation shows that complex conjugation: \( a \rightarrow \bar{a} \) in
this basis is accomplished with

(3) \[ \bar{a} = J'aJ = -JaJ = JaJ' \]

this is just a rotation by \( \pi \) around the \( i \)-axis (see below). The quaternion conjugate in the matrix basis is \( a^* = \bar{a}' = J'a'J \), where \( a' \) is the transpose of \( a \). Use of the \( J \)-operator facilitates computation of derivatives in the \( SU(2) \) basis, while Hamilton’s scalar-vector notation is useful for algebraic calculations. (The use of \( \bar{a} \) to signify conjugation for both the \( Sp(1) \) and \( SU(2) \) representations has to be handled with care, as they are not equivalent. The equivalence is \( \bar{a}_H \sim a^*_C \); the context should make it clear which is intended. In the multi-dimensional case, say a matrix \( A \to A' \), both conjugation and transposition are intended, so the meaning of \( A' \) is unequivocal.)

There are three involution operations on quaternions that may be equivalent to the CPT operators of quantum theory. Working in the \( \mathbb{H} \)-basis, parity is clearly \( P : a \to \bar{a} \), as seen above. The other two are reversal of the identity component, \( I : a \to -\bar{a} \), and simple negation, \( N : a \to -a \). \( PIN \) in any order is the identity when operating on a simple quaternion. However, the operators are more interesting when acting on products, \( ab \). Now \( P : ab \to \bar{b}\bar{a} \neq P(a)P(b) = \bar{a}\bar{b} \). Similarly, \( I : ab \to -\bar{b}\bar{a} \neq I(a)I(b) = (-\bar{a})(-\bar{b}) = \bar{a}\bar{b} \), and \( N : ab = -ab \neq N(a)N(b) = (-a)(-b) = ab \), yet each squares to the identity, as one can easily prove. In addition to \( PIN \), the operators \( e_m = \{i, j, k\} \) acting by \( e_mae_m = -e_mae_m \) (one of which we’ve seen acting in the \( SU(2) \) representation as complex conjugation) is a rotation by \( \pi \) about the \( e_m \) axis. (For evaluating \( PIN \) in the \( SU(2) \)-basis, make the substitution \( \bar{x} \to x^* \).)

The quantum mechanical parity operator, \( P : x \to -x, x \in \mathbb{R}^3 \), is equivalent to conjugation: \( u \to \bar{u} \). The left, or right, action of the quaternion \( \exp(-2v) \) on \( u = \exp(v) \) gives the conjugate. However, care must be taken to distinguish this algebraic operator from the abstract parity operator \( P \), for which \( P(Pu) = P(\bar{u}) = u \implies P^2 = 1 \). Clearly \( \exp(-2v) \) does not square to the identity. The point is that conjugation can be realized by the multiplicative action of an appropriate quaternion residing in \( Sp(n) \). This provides explicit operators that execute transitions from a state with positive chirality to one of negative chirality and vice-versa. Since there is no notion of a direction of motion in this discussion, the word “helicity” is avoided.
GENERAL STRUCTURE OF THE EIGENVALUE PROBLEM ON $Sp(n)/Sp(1)^n$

The representations of the classical groups are well known. However, representations parameterized by cosets are apparently less well documented. Given eq. (1), and the desire to construct explicit representations to show how composite states are realized, we will use a direct, naïve approach to calculate eigenvalues and eigenvectors. This will show how the matrix elements of the group are related to the module on which the group acts, and it is anticipated that this will provide some insight into the relation between bosons and fermions.

As shown above, the parameterization of $Sp(n)$ that we will be working with is built on the coset $xH$ structure, such that an element $g \in Sp(n)$ is written as $g = xh$, where $x = \exp(x)$ and $h \in H$ is a diagonal matrix, all elements of which are unit quaternions. Since $gg^{-1} = gg^* = xhh^*x^* = xx^* = 1$, it follows that the Lie algebra $\mathfrak{r}$ of $x$ is skew-symmetric: $\mathfrak{r}^* = -\mathfrak{r}$. The diagonal elements of $\mathfrak{r}$ are identically zero, as they have been pulled into $\mathfrak{h}$, where $\exp(\mathfrak{h}) = H$. It is useful to introduce some notation. The fundamental representations of $Sp(n)/Sp(1)^n$ do not represent the whole group; clearly the maximal subgroup $H$ is excluded from the representation, and this implies that the rank of the "root" space of the coset is less than that of the whole group. Let $CF(n, \mathbb{H})$ denote the subgroup of $Sp(n)$ that is parameterized by the components of the coset, $Sp(n)/Sp(1)^n$, where $CF$ suggests Complete Flag. Since we are working exclusively in $\mathbb{H}$ or the isomorphic $\mathbb{R}^+ \times SU(2)$ presentations, this will be simply $CF(n)$.

The eigenvalue problems to be solved for $x \in Sp(n)/Sp(1)^n$ are $\Lambda = \tau^* x \tau = \tau^* (\exp(x)) \tau = \exp(\tau^* \mathfrak{r} \tau) = \exp(\lambda)$ since $\tau^* \tau = \tau \tau^* = 1$. Here $\{\Lambda, \lambda\}$ are diagonal matrices in either the $\mathbb{H}$ or $\mathbb{C}^2$ basis. One solves for eigenvalues in the algebra rather than the group. On selecting an $\mathfrak{r} \in \mathfrak{g}\setminus\mathfrak{h}$, geodesics are of the form $\exp(t \mathfrak{r})$, so that the maximal torus of the group is $\exp(t \lambda)$. These elementary statements about Lie groups are well known. Also well known is that higher dimensional irreducible representations are constructed from tensor products

$$\bigotimes_{i=1}^{m} x = x \otimes x \otimes \cdots \otimes x.$$
and that linear combinations of the elements of these products are classified by their symmetries with respect to interchanges of matrix elements. By relating the eigenvectors to the matrix elements, we will uncover symmetry relations between bosons and fermions.

**Case 1: \( Sp(2)/Sp(1) \)**

**Metric and Curvature.** The metric and curvature two-forms for the general Grassmannian \( Sp(k + n)/Sp(k) \times Sp(n) \) will be presented here, even though the case \( k + n = 2 \) is algebraically simpler. The restriction to \( Sp(2)/Sp(1)^2 \) will be presented at the end of this section.

A general matrix \( g \in Sp(k+n), gg^* = g^*g = 1 \) that is partitioned to be compatible with the subgroup \( Sp(k) \times Sp(n) \) is

\[
g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & X \\ -X^* & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}
\]

where the \( k \times n \) matrix \( X = BD^{-1} = -(A^*)^{-1}C^* \). The latter equality comes from the orthogonality \( g^*g = 1 \). Here \( X^* \) is the transpose conjugate of \( X \), a notation that covers both the \( \mathbb{H} \) and \( \mathbb{C}^2 \) bases as noted above. Orthogonality also yields \( 1 + XX^* = (AA^*)^{-1} \) and \( 1 + X^*X = (DD^*)^{-1} \). The eigenvalues of \( A \) and \( D \) are determined by the eigenvalues of \( X \). The invariance of \( AA^* \) and \( DD^* \) to the right action of \( h \in Sp(k) \times Sp(n) \):

\[
h = \begin{bmatrix} h_k & 0 \\ 0 & h_n \end{bmatrix}
\]

on \( \text{diag}(A, D) \) provides an explicit representation of the coset structure \( G/H \), i.e., \( A = (1 + XX^*)^{-1/2}h_k \) and \( D = (1 + X^*X)^{-1/2}h_n \), which is consistent with a count of real variables (a polar decomposition of \( A \) and \( D \) is implied here).

The action of \( g_1 \in Sp(k+n) \) on \( X \) is \( g_1 : X \rightarrow (A_1X + B_1)(C_1X + D_1)^{-1} \). From these relations one may construct the invariant metric on the coset space

\[
ds^2 = \text{tr}[(1 + XX^*)^{-1}dX(1 + X^*X)^{-1}dX^*].
\]
Making use of $dX = dB^{-1}BD^{-1}dDD^{-1} = (A^*)^{-1}(A^*dB+C^*dD)D^{-1}$ (note that the subscript on the blocks of $g_1$ have been dropped), the metric can also be written

$$ds^2 = \text{tr}[(A^*dB + C^*dD)(dB^*A + dD^*C)].$$

The metric on $Sp(k+n)$ is $\text{tr}(dg^*dg) = -\text{tr}(g^*dgg^*dg) = \text{tr}(\omega^*)$, where

$$\omega = g^*dg = \left[\begin{array}{cc} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22}\end{array}\right] = \left[\begin{array}{cc} \omega_{11} & \omega_{12} \\ -\omega^*_{12} & \omega_{22}\end{array}\right].$$

is skew-symmetric since $d(g^*g) = 0$. The metric on the coset space is simply the trace of the square of the off-diagonal block of $\omega$, i.e., $ds^2 = \text{tr}(\omega_{12}\omega^*_{12})$.

The left invariant differential form $\omega$ has an exterior derivative\cite{15,16}

$$d\omega = dg^* \land dg = -g^*dg \land g^*dg = -\omega \land \omega$$

which is the second Maurer-Cartan equation: $d\omega + \omega \land \omega = 0$. Writing this out in

$$d\omega + \omega \land \omega = \left[\begin{array}{c} d\omega_{11} + \omega_{11} \land \omega_{11} + \omega_{12} \land \omega_{21} \\ d\omega_{21} + \omega_{21} \land \omega_{11} + \omega_{22} \land \omega_{21} \end{array}\right] = 0$$

One can apply Cartan’s criterion: $d\omega_{\mu\mu} + \omega_{\mu\mu} \land \omega_{\mu\mu} = \Omega_{\mu\mu} \sim \Omega_\mu$ for the diagonal elements to define\cite{17} the curvature two-forms as

$$\Omega_1 = -\omega_{12} \land \omega_{21} = \omega_{12} \land \omega^*_{12}$$

$$\Omega_2 = -\omega_{21} \land \omega_{12} = \omega^*_{12} \land \omega_{12}.$$  

The curvature two-forms, or tensors, are determined by the matrix elements of the group. Curvature is equivalent to force – bosons are the physical carriers of force – therefore, the matrix elements represent bosons. This is consistent with the initial assertion that the group conveys interactions between subspaces of the flag. While calculated here for a Grassmannian, this will be shown to hold for flag manifolds in general.
The metric and curvature are easily specialized to the $Sp(2)/Sp(1)^2$ case. The curvature forms are particularly interesting. In this simple case, define the scalar-vector one-form as $\omega_{12} = \omega = w_0 \mathbf{1} + w$, so that

$$\begin{align*}
\Omega_1 &= \omega \wedge \bar{\omega} = -2w_0 \wedge w - w \wedge w \\
\Omega_2 &= \bar{\omega} \wedge \omega = +2w_0 \wedge w - w \wedge w
\end{align*}$$

which are anti-self-dual and self-dual two-forms, respectively, as is proved in Appendix 1. It makes no difference to the physics which is which. (The reader may want to map this into Atiyah’s representation, $q \sim X$, with use of the relations developed in the section on the metric.) The curvature forms sit on the diagonal, which means that they are associated with the individual components of $V_2(\mathbb{H})$. If one identifies $w_0$ with a time-like quantity (which makes an analogy with special relativity), the interaction between the two particles (Alice and Bob) moves forward as seen by Alice and backward as seen by Bob. Having entertained this thought, it is promptly dropped; there is a global time parameter that enters the picture, as claimed in “Structural Preliminaries”. Regardless of interpretation, the splitting of interactions into self-dual and anti-self-dual partners is a general phenomenon and will recur for three particles.

**Lie Algebra and Infinitesimal Generators.** A general matrix $x_q \in \mathfrak{x}$ in the Lie algebra $\mathfrak{x}$ of the coset $Sp(2)/Sp(1)^2$ is

$$x_q = \begin{bmatrix} 0 & q \\ -\bar{q} & 0 \end{bmatrix},$$

where $q$ is an arbitrary quaternion. The commutator $[x_a, x_b]$ is

$$[x_a, x_b] = \begin{bmatrix} -[a, b]^* & 0 \\ 0 & -[a^*, b]^* \end{bmatrix}$$

where $[a, b]^* = ab^* - ba^* = ab^* - (a^*)^* = 2\Im(ab^*)$. In scalar-vector notation the diagonal elements are

$$\begin{align*}
- [a, b]^* &= 2(a_0 \mathbf{b} - b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}), \\
- [a^*, b]^* &= 2(-a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})
\end{align*}$$
and note the sign difference in the terms containing the identity components of the respective quaternions, reflecting those of the curvature two-forms.

The infinitesimal generators of the algebra are known for Grassmannians, and displaying their commutators will unite the matrix and operator descriptions. Using the SU(2) basis, the matrix representation of the quaternion \( a \) appearing in eq. (2) is:

\[
(8) \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \zeta_1 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{\zeta}_1 \end{bmatrix}.
\]

The most convenient definition of the differential operator in the matrix representation is

\[
(9) \quad \partial \sim [\partial_{aa}] = \begin{bmatrix} \partial/\partial a_{11} & \partial/\partial a_{12} \\ \partial/\partial a_{21} & \partial/\partial a_{22} \end{bmatrix}
\]

such that \( \partial_{aa} a_{\beta b} = \delta_{\alpha\beta} \delta_{ab} \). Using different fonts for row and column indices is helpful for keeping track of terms when computing derivatives in multi-dimensional cases.

The Lie algebra of the Grassmannian \( Sp(n + k)/Sp(k) \times Sp(n) \) rendered in the SU(2)-basis might be denoted as \( \mathfrak{sp}(k,\mathbb{C}^2) \). It is parameterized by the elements \( a_{ab} \) of the \( 2k \times 2n \) Grassmannian matrix, and its generators are [1]:

\[
(10) \quad h_{\alpha\beta} = \Sigma_b (a_{ab} \partial_{\beta b} - (a_{ab} \partial_{\beta b})^*) ; \quad h_{\alpha\beta} \in \mathfrak{h} = \mathfrak{sp}(k);
\]

\[
(11) \quad H_{ab} = \Sigma_{\mu} (a_{\mu a} \partial_{\mu b} - (a_{\mu a} \partial_{\mu b})^*) ; \quad H_{ab} \in \mathfrak{h} = \mathfrak{sp}(n);
\]

\[
(12) \quad p_{aa} = \bar{\partial}_{aa} + \Sigma_{\mu b} a_{ab} a_{\mu b} \partial_{\mu b} ;
\]

the summation convention is not used in this paragraph. Note the remarkable fact that the generators within a subspace, the \( \mathfrak{h} \) and \( \mathfrak{h} \) components, are homogeneous operators, whereas those acting between subspaces, the \( p \) components, are inhomogeneous. This will be a key feature of the theory as it develops. For completeness in
this presentation, and to make a significant point later, the commutators are

\[
\begin{align*}
[(hJ)_{\alpha\beta}, (hJ)_{\mu\nu}] &= - J_{\alpha\mu} (hJ)_{\beta\nu} - J_{\alpha\nu} (hJ)_{\beta\mu} - J_{\beta\mu} (hJ)_{\alpha\nu} - J_{\beta\nu} (hJ)_{\alpha\mu} \\
[(HJ)_{ab}, (HJ)_{cd}] &= - J_{ac} (HJ)_{bd} - J_{ad} (HJ)_{bc} - J_{bc} (HJ)_{ad} - J_{bd} (HJ)_{ac} \\
[h_{\alpha\beta}, H_{ab}] &= 0 \\
[(hJ)_{\mu\nu}, p_{aa}] &= J_{\alpha\mu} p_{\nu a} + J_{\alpha\nu} p_{\mu a} \\
[(HJ)_{bc}, p_{aa}] &= J_{ab} p_{ac} + J_{ac} p_{ab} \\
[p_{aa}, p_{\beta b}] &= - J_{ab} (hJ)_{\alpha\beta} - J_{\alpha\beta} (HJ)_{ab} \\
[\bar{p}_{aa}, p_{\beta b}] &= \delta_{\alpha\beta} H_{ba} + \delta_{ab} h_{\beta a}
\end{align*}
\]

(13)

The \( J \)-matrix factors, see eq. (3), that are sprinkled throughout these equations make the symmetry of the commutation relations more apparent than they would be otherwise. (The \( 2k \times 2k \) \( J \)-factors with Greek indices are the tensor product of the \( k \)-dimensional identity with the \( i \) unit: \( J_{\mu\nu} \sim 1_k \otimes i \). \( J \)-factors with Roman indices are similar with dimension \( 2n \times 2n \).) These equations are easily specialized to the case at hand: \( n = k = 1 \).

For our present concern, the Grassmannian consists of a single quaternion, which enables a considerable simplification. Define the operators \( \eta_j = \zeta_j \partial / \partial \zeta_j - \bar{\zeta}_j \partial / \partial \bar{\zeta}_j = -i \partial / \partial \theta_j \) for \( \zeta_j = r_j \exp(i \theta_j) \): the components of the Cartan algebra from eqs. (10) and (11) are

\[
\begin{align*}
h_{11} &= \eta_1 + \eta_2 \\
h_{22} &= - h_{11} = - \eta_1 - \eta_2 \\
H_{11} &= \eta_1 - \eta_2 \\
H_{22} &= - H_{11} = - \eta_1 + \eta_2.
\end{align*}
\]

Substituting \( \phi_\pm = (1/2)(\theta_1 \pm \theta_2) \), and reverting to the quaternion basis, these are written succinctly as

\[
h_C = -k \partial / \partial \phi_+ \quad \text{and} \quad H_C = -k \partial / \partial \phi_-,
\]

where the subscript \( C \) is a reminder that these are in the Cartan algebra of the \( \mathfrak{sp}(1) \) generators. The eigenvectors of these operators are of the form \( \exp(mk \phi_\pm) \).
The point here is to show that the $\mathfrak{cf}(2)$ algebra contains two intertwined copies of the $\mathfrak{su}(2)$ algebra. Cartan algebras are critical to the analysis of representations of $CF(n)$ just as they are for any group.

**Eigenvalues and Eigenvectors.** It will be beneficial to start with this simple case to develop some algebraic tools. Select a $g \in Sp(2)$ and a corresponding $x \in \mathfrak{g} \sim \mathfrak{cf}(2)$ with an explicit representation as in eq. (5). To reinforce the previous section, an element $g$ in the fundamental representation of the group is parameterized by $g = \exp(x)h$, where

$$h = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$$

with $h_i \in Sp(1), 1 \leq i \leq 2$.

The diagonalization problem, $xr = r\lambda, r \subset \tau$, yields two simple equations

$$-r_1\lambda + qr_2 = 0$$

$$-\bar{q}r_1 - r_2\lambda = 0.$$ 

Multiply the first by $\bar{q}$ on the left and the second by $-\lambda$ on the right and add to get $r_2(|q|^2 + \lambda^2) = 0$, which follows because $|q|$ is a multiple of the identity and commutes with a quaternion. The non-trivial solution of this equation is $\lambda = \pm e_m|q|$, where $e_m$ is any one of the $\{i, j, k\}$ basis elements. In conformity with the usual convention in physics, where the quaternion basis element $k$ is associated with the diagonal (z-direction) in the $SU(2)$ representation, the choice $\lambda = \pm k|q|$ is made. Normalizing the eigenvector gives $r_2 = \pm q^{-1}r_1k|q|$. But $q^{-1} = \bar{q}/|q|^2$, so that $r_2 = \pm \bar{u}r_1k$, where $u = q/|q|$ is a unit quaternion.

It is clear that the norm $|q|$ factors from the eigenvalue/eigenvector equation, so that one is left with just the orthonormalization condition for the eigenvalues. A bit of experimentation with the algebra of this problem leads to a simpler structure if one writes $x$ as

$$x = |q| \begin{bmatrix} 0 & k\bar{u} \\ u\bar{k} & 0 \end{bmatrix}$$
Using this representation of the coset it quickly follows that the matrix
\[
\tau = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ u & -u \end{bmatrix} \in Sp(2),
\]
comprises the eigenvectors of \( x \) with eigenvalues \( \lambda_{\pm} = \pm |q|k \). (Since these eigenvalues are valid for all \( x \in \mathfrak{g} \), it is somewhat superfluous to instantiate with “select an \( x \in \mathfrak{g} \)”, so this distinction between an element of the algebra and the entire algebra will be dropped in subsequent equations.) The trick of representing the elements in the algebra in a particular way so as to facilitate the construction of the eigenvectors will also be used for the \( CF(3) \) case, where it will be seen to have deep physical significance.

The fact that one of the components of the vector space is trivial (the commutative basis element \( 1 \)) may explain the apparent mismatch in degrees of freedom between the formulation of the Yang-Mills theory for a single isospin and the \( Sp(2) \) context in which it resides. The identity component of \( \tau \) is trivial, and is effectively submerged in the Yang-Mills formulation. The visible isotopic spin content is in the \( u \) factors.

The most natural physical theory evolves the group along a geodesic with time \( t \), which recommends that one introduce a frequency \( \omega \) to write \( \omega t = |q| \), so that
\[
X = \exp(t\tau) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t)k\bar{u} \\ \sin(\omega t)u\bar{k} & \cos(\omega t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t)v \\ -\sin(\omega t)\bar{v} & \cos(\omega t) \end{bmatrix}.
\]

The presence of the unit (real) component in the eigenvectors is very interesting. An \( m \)-fold tensor product of \( X \in Sp(2)/Sp(1)^2 \), eq. (14), generates terms in ascending powers: \( 1, v, v^2, \cdots, v^m \) and their conjugates, together with trigonometric phase factors. These might be tentatively identified as primitive lepton states, for example, \( v \) for the electron, \( v^2 \) for muon, and \( v^3 \) for the \( \tau \) meson, . . . , with the conjugates being states of opposite chirality. However, these simple-minded assignments are probably not correct, as there is an important additional fact that has to be introduced, and which will make the physics much more interesting.

**Centralizer.** The centralizer of the torus consists of matrices that commute with the matrix of eigenvalues of the group; in this case the centralizer \( C \) consists of
matrices of the form

\[ C = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix} \]

with \( \nu_\alpha = \cos(\varphi_\alpha)1 + \sin(\varphi_\alpha)k \). These degrees of freedom exist in all representations of \( Sp(n) \), and will likely have a special place in particle theory.

To see what role the centralizer might play, and to build more tools to apply to assignments, requires further development of the theory, so rather than continue with \( CF(2) \) we turn attention to the more interesting three body problem to reveal yet more structure. In any case, a rigorous construction of representations will be a major but rewarding undertaking, and is left to the experts.

**Case 2: \( Sp(3)/Sp(1)^3 \)**

Given the success of \( SU(3) \) in organizing meson and baryon states, the hope is that \( CF(3) \), which includes spin degrees of freedom and apparently contains an \( SU(3) \) subgroup, will provide additional insight into the structure of these composite particles. This will be now be demonstrated.

**Metric and Curvature.** This case is the simplest example of a flag manifold that is not also a Grassmannian. To make the geometrical part of the presentation general, the metric and curvature for an arbitrary flag manifold will be presented, with specialization to \( Sp(3)/Sp(1)^3 \) left to the end.

Define a partition \( \{ k_1, k_2, \cdots, k_m \} \) of \( n \) such that \( \sum_{\mu} k_\mu = n \), and consider the flag manifold (coset space) \( Sp(n)/Sp(k_1) \times Sp(k_2) \times \cdots \times Sp(k_m) = Sp(n)/\bigotimes_{\mu} Sp(k_\mu) \). Corresponding to this partition, the left invariant one-form \( \omega \) is partitioned into block form

\[ \omega = g^*dg = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1m} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{m1} & \omega_{m2} & \cdots & \omega_{mm} \end{bmatrix} \]

where \( \omega_{\mu\nu} \) is a \( k_\mu \times k_\nu \) block. Just as for the Grassmannian, the scalar metric on the flag manifold is constructed from the squares of the blocks in the upper triangle as

\[ ds^2 = \sum_{1 \leq \mu \leq \nu \leq m} \text{tr}(\omega_{\mu\nu}\omega_{\mu\nu}^*). \]
The curvature two-forms are once more computed from the Maurer-Cartan equation just as was done for the Grassmannian, to give the curvature two-forms on the diagonal blocks

$$\Omega_\alpha = -\sum_{\mu \neq \alpha}^m \omega_{\alpha \mu} \wedge \omega_{\mu \alpha}.$$ 

The elements of the lower triangle in $\omega$ are the negative conjugates of those in the upper triangle, so that

$$\Omega_\mu = \sum_{\alpha < \mu} \omega_{\alpha \mu} \wedge \omega_{\alpha \mu}^* + \sum_{\alpha > \mu} \omega_{\alpha \mu}^* \wedge \omega_{\alpha \mu}.$$ 

This signifies that subspace or system $V_\mu : v \in V_\mu = \{v \in V_{i+1} | v \notin V_i\}$ interacts with all other subspaces in the flag. If the interaction of a single system with its surroundings is of interest, the components of the flag can be permuted so that the flag manifold reduces to a Grassmannian.

For the $Sp(3)/Sp(1)^3$ case at hand the symmetry of these equations is best displayed by making the change of notation $\omega_{12} = \omega_c, \omega_{13} = -\bar{\omega}_b, \omega_{23} = \omega_a$, and labeling the curvature two-forms with corresponding Greek letters to write

$$\Omega_\alpha = \bar{\omega}_b \wedge \omega_b + \omega_c \wedge \bar{\omega}_c$$

$$\Omega_\beta = \bar{\omega}_c \wedge \omega_c + \omega_a \wedge \bar{\omega}_a$$

$$\Omega_\gamma = \bar{\omega}_a \wedge \omega_a + \omega_b \wedge \bar{\omega}_b$$

This assignment of symbols reveals a beautiful symmetry; each particle sees the other two particles, one with identity-containing components running forward and the other running backward. The use of “forward” and “backward” is an arbitrary assignment of labels to the signs of these components of the curvature two-forms, as was discussed for $Sp(2)/Sp(1)^2$.

Cartan’s exterior algebra on the cotangent space makes the calculation of the curvature effortless. More elaborate machinery, exemplified by the Koszul formula,[7] is required for work in the tangent space. While I have not proved the correctness of eq. (15), Chern[17] did so for a Grassmannians over $\mathbb{R}$; his presentation easily extends to the flag manifold over $\mathbb{H}$. 
Generators. A general matrix $\mathbf{r}$ in the Lie algebra of the coset $Sp(3)/Sp(1)^3$ may be parameterized by

\begin{equation}
\mathbf{r} = \begin{bmatrix}
0 & c & -b^* \\
-c^* & 0 & a \\
-\bar{b} & -\bar{a} & 0
\end{bmatrix},
\end{equation}

where \{a, b, c\} are three linearly independent quaternions, represented for present purposes in the $SU(2)$ basis. In constructing generators, we will use the conventions in eqs. (8,9).

The infinitesimal operators have to follow the pattern established in eq. (10) for everything to be consistent. But since the $CF(3)$ representation is acting on $V_3$, we can use the analogy with $SO(3)$ acting on $\mathbb{R}^3$ to build the family of operators (using a somewhat inelegant $(xy)$ notation, but using the summation convention),

$$(xy)_{\alpha\beta} = x_{\alpha a} \partial / \partial y_{\beta a} - \bar{y}_{\beta a} \partial / \partial \bar{x}_{\alpha a}$$

with two others related by cyclic permutations from the set \{x, y, z\}. These generators have a nice symmetry property, as revealed by

$$(xy)_{\alpha\beta} = (xy)^*_{\beta\alpha},$$

which is written succinctly as $(yx) = -(xy)^*$. The first commutator to evaluate is

$$[(xy)_{\alpha\beta}, (xy)_{\mu\nu}] = [x_{\alpha a} \partial / \partial y_{\beta a} - \bar{y}_{\beta a} \partial / \partial \bar{x}_{\alpha a}, x_{\mu b} \partial / \partial y_{\nu b} - \bar{y}_{\nu b} \partial / \partial \bar{x}_{\mu b}]$$

$$= [y_{\nu b} \partial / \partial x_{\mu b}, x_{\alpha a} \partial / \partial y_{\beta a}] + [\bar{x}_{\mu b} \partial / \partial \bar{y}_{\nu b}, \bar{y}_{\beta a} \partial / \partial \bar{x}_{\alpha a}]$$

$$= \delta_{\alpha \mu} (yy)_{\nu \beta} - \delta_{\beta \nu} (xx)_{\alpha \mu}$$

where

$$(xx)_{\alpha\beta} = x_{\alpha a} \partial / \partial x_{\beta a} - \bar{x}_{\beta a} \partial / \partial \bar{x}_{\alpha a}$$

in obvious extension of the notation. So, $(xx), (yy)$ and $(zz)$ clearly belong to the diagonal blocks of the matrix of generators, and since $(rr)^* = -(rr)$, the diagonal elements are pure imaginary. Note that with our convention for labeling matrix elements, \(\partial x_{\alpha b} / \partial x_{\beta a} = \delta_{\alpha \beta} \delta_{ab}\) and \(\partial \bar{x}_{\alpha b} / \partial x_{\beta a} = J_{\alpha \beta} J_{ba}\), with the latter a result of the conjugation operation $\bar{x}_{ab} = J'_{\alpha \gamma} x_{\gamma \epsilon} J_{\epsilon b}$ in eq. (3).
Completing the list of non-trivial commutators (but not writing those obtained by cyclic permutations), it is not difficult to prove that

\[
[(xx)_{\alpha\beta}, (xx)_{\mu\nu}] = \delta_{\beta\mu} (xx)_{\alpha\nu} - \delta_{\alpha\nu} (xx)_{\mu\beta} - J_{\beta\nu} (xxJ)_{\alpha\nu} - J_{\alpha\mu} (Jxx)_{\beta\nu}
\]

\[
[(xy)_{\alpha\beta}, (xy)_{\mu\nu}] = J_{\beta\nu} (xxJ)_{\alpha\mu} - J_{\alpha\mu} (Jyy)_{\beta\nu}
\]

\[
[(xy)_{\alpha\beta}, (yz)_{\mu\nu}] = \delta_{\beta\mu} (xz)_{\alpha\nu}
\]

\[
[(xx)_{\alpha\beta}, (xy)_{\mu\nu}] = \delta_{\beta\mu} (xy)_{\alpha\nu} - J_{\alpha\mu} (Jxy)_{\beta\nu}
\]

Here \((Jxx)_{\alpha\beta} = J_{\alpha\gamma} (xx)_{\gamma\beta}\), and similarly for \((xxJ)\) and \((Jxy)\), are symmetrized versions of the operators. While messy, the first commutator can be shown to satisfy the usual relations for \(\mathfrak{su}(2)\). Together with conjugation, these are all the tools that are needed to construct the complete set of generators for the \(\mathfrak{e}_7(3)\) algebra. The last commutator can be used to show that \((xy)_{\alpha\beta}\) is a root vector, as are the other two operators \((yz)_{\alpha\beta}\) and \((zx)_{\alpha\beta}\). However, they are not linearly independent because any two generate the third. This implies a deep relation between the two-dimensional root spaces of \(CF(3)\) and \(SU(3)\), which will be explored later.

Given that the infinitesimal generators are homogeneous operators of degree zero, it follows that the irreducible representations of \(Sp(3)\) will be constructed from polynomials of degree \(m\) in the three parameters;

\[
\Psi_{ijk}(x, y, z) = x^i y^j z^k \pm \text{perm}; \quad i + j + k = m,
\]

where the permutations are over the other orders of the factors. Representations may include conjugates as well as basis elements, as will be seen. Permutations are essential because there is no physical reason for preferring one order of factors over another within a given symmetry class. Presumably one will be able to track the symmetries of the representations with Young diagrams, but that is not pursued here. We will return to consider the representations after solving for the eigenvalues.

**Diagonalization of \(Sp(3)/Sp(1)^3\)**

A fixed element \(\mathbf{r} = \mathbf{r}_3\) from the Lie algebra of \(Sp(3)/Sp(1)^3\) is parameterized by a matrix of the form in eq. (17), but now written in the \(\mathbb{H}\) basis. The eigenvectors
$t \subset \tau$ are the solutions of $tt - t\lambda = 0$, giving

$$
\begin{align*}
-t_1\lambda + ct_2 - bt_3 &= 0 \\
-\bar{c}t_1 - t_2\lambda + at_3 &= 0 \\
bt_1 - \bar{a}t_2 - t_3\lambda &= 0
\end{align*}
$$

(There should be no confusion between time $t$ and the eigenvectors denoted by the same symbol with subscript.) Multiply the first of these equations by $\bar{c}$ from the left and the second by $-\lambda$ on the right and add to eliminate $t_1$. Similarly, multiply the first by $b$ from the left and the last by $\lambda$ on the right and add to again eliminate $t_1$. This gives two equations from which, say, $t_3$ can be eliminated. Similar operations to eliminate $t_1$ and then $t_2$ give three equations, written symmetrically as

$$
\begin{align*}
&u_1(\lambda^3 + \lambda L^2) + (cab - \bar{cab})u_1 = 0 \\
&u_2(\lambda^3 + \lambda L^2) + (abc - \bar{abc})u_2 = 0 \\
&u_3(\lambda^3 + \lambda L^2) + (bca - \bar{bca})u_3 = 0
\end{align*}
$$

Here $L^2 = |a|^2 + |b|^2 + |c|^2$ and the unit quaternions, $u_i$, have been substituted for $t_i = |t_i|u_i$ since the norms of the $t_i$ cancel. These equations are obtained using only multiplication, inversion of diagonal matrices, addition and subtraction – no determinant was computed. The presentation is belabored to convey the care that has been taken with the non-commutative algebra.

The occurrence of three different versions of the characteristic polynomial is illustrative of the well-known fact that the determinant of a quaternion matrix is ill-defined – we have three different polynomials corresponding to different calculations of the term occupying the position corresponding to the determinant in the analogous problem over $\mathbb{R}$ or $\mathbb{C}$.

With our choice of basis for the eigenvalues of a quaternion matrix, $\lambda = \hat{\lambda}\mathbf{k}$, where $\hat{\lambda} \in \mathbb{R}$ is a scalar; substituting this expression in the equations gives

$$u_i(-\hat{\lambda}^3 + L^2\hat{\lambda})\mathbf{k} + d_iu_i = 0; \ 1 \leq i \leq 3$$
with the purely imaginary \( d_i \) defined in the obvious way from the three equations. Multiplying on the right with \( k\bar{u}_i \) gives

\[
(\lambda^2 - L^2)\lambda \mathbf{1} + d_i u_i k\bar{u}_i = 0.
\]

(19)

It is straightforward to see that \( yk\bar{y} \) has a vanishing identity component, since rotation of the “vector” part of a quaternion, in this case \( k \), by conjugation with a unit quaternion \( y \) does not generate an identity component. Now, since \( (\lambda^2 - L^2)\lambda \) is a scalar, \( d_i u_i k\bar{u}_i \) must also be scalar. Since \( u_i k\bar{u}_i \) is just the vector part of a quaternion, represent it as \( u_i^{(2)} \), such that the product \( d_i u_i k\bar{u}_i = -d_i \cdot u_i^{(2)} + d_i \times u_i^{(2)} \). This has to be a scalar, which forces \( u_i^{(2)} \) to be parallel to \( d_i \). Furthermore, \( u_i k\bar{u}_i \) is a unit quaternion, so that \( u_i^{(2)} = \pm d_i / |d_i| \) and \( d_i u_i k\bar{u}_i = \pm |d_i| \).

So, the \( u_i \) are defined by \( d_i \), which is the usual situation, but for the fact that this problem has three different “effective” determinants. Some thought to this puzzle leads to the idea that it can be turned around to define the elements of \( r \) in terms of the components of the eigenvectors, similar to what was done for the \( Sp(2)/Sp(1)^2 \) problem.

**The Simplification.** Given the \( Sp(2) \) example, in which the algebraic operations were simplified with a particular representation of the group algebra, a bit of experimentation with the \( Sp(3) \) problem just uncovered leads to the realization that a more convenient parameterization of the group algebra is

\[
r = L \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix} \begin{bmatrix} 0 & w_3 k & w_2 k \\ w_3 k & 0 & w_1 k \\ w_2 k & w_1 k & 0 \end{bmatrix} \begin{bmatrix} \bar{v}_1 & 0 & 0 \\ 0 & \bar{v}_2 & 0 \\ 0 & 0 & \bar{v}_3 \end{bmatrix} = LV(W \otimes k)V^*.
\]

where \( v_i, 1 \leq i \leq 3 \) are three linearly independent unit quaternions, and \( w_i \in \mathbb{R}^+ \): \( \Sigma_i w_i^2 = 1 \) because \( L^2 = |a|^2 + |b|^2 + |c|^2 \). The 12 real variables \( \{a, b, c\} \) in eq. (17) have been replaced by another 12 linearly independent variables. Nonetheless, \( r^* = -r \) as is required by the orthogonality condition.

Let \( M \) denote the matrix representation of \( r \) with elements \( m_{ij} = Lw_kv_jk\bar{v}_i \). Then \( \bar{m}_{ij} = Lw_kv_jk\bar{v}_i = -Lw_kv_jk\bar{v}_i = -m_{ji} \). So, it is legitimate to identify the matrix elements in eq. (17), the \( \{a, b, c\} \) parameters, with the nicely symmetric products

\[
a = Lw_1v_2k\bar{v}_3; \quad b = Lw_2v_3k\bar{v}_1; \quad c = Lw_3v_1k\bar{v}_2.
\]

(20)
The matrix that diagonalizes $\mathbf{r}$ by $\tau^* \mathbf{r} \tau = \lambda$ is of the form $\tau = V (R \otimes 1)$, where the elements of $R$ are scalars. The eigenvalue problem is reduced to

$$L(R \otimes 1)' V^* V (W \otimes k) V^* V (R \otimes 1) = (R \otimes 1)' (LW \otimes k) (R \otimes 1) = \hat{\lambda} \otimes k$$

where now

$$LR'WR = \hat{\lambda}$$

is a matrix problem over real variables, with $R \in SO(3)$. Note that use has been made of $(a \otimes b)(c \otimes d) = ac \otimes bd$, which is permitted because $R$ is a matrix of scalars.

It is easy to show that

$$X = \exp \mathbf{r} = V (R \otimes 1)[ \cos (\hat{\lambda}) 1 + \sin (\hat{\lambda}) k] (R \otimes 1)' V^*$$

The essential difference between the fundamental representations of the group and its algebra is that the eigenvalues of the algebra constitute the tangent space of the maximal torus of the group.

The eigenvalues of eq. (21) require solutions of

$$|W - \eta 1| = \det \begin{bmatrix} -\eta & w_3 & w_2 \\ w_3 & -\eta & w_1 \\ w_2 & w_1 & -\eta \end{bmatrix} = 0$$

or

$$\eta^3 - \eta - 2x = 0$$

where $\hat{\lambda} = L\eta$ and $x = w_1w_2w_3$. This simple equation captures all of eq. (19). This parameterization of $CF(3)$ might be construed as the origin of the three color variables of chromodynamics: $\{w_1, w_2, w_3\}$.

**Eigenvalues and Eigenvectors.** The discriminant of eq. (22) is $\Delta = 4(1 - 27x^2)$. Given that $\Sigma_i w_i^2 = 1$, the parameterization $[w_1, w_2, w_3] = [\cos \alpha, \sin \alpha \cos \beta, \sin \alpha \sin \beta]$, enables the evaluation $x = (1/4) \sin 2\alpha \sin \alpha \sin 2\beta$. Since $Lw_i$ is a non-negative definite norm of the corresponding quaternion element of $\mathbf{r}$, the angles $\alpha$ and $\beta$ are restricted to the first quadrant: $0 < \alpha, \beta \leq \pi/2$. If the discriminant $\Delta \geq 0$ there are three real roots, with multiple roots for the equality. The maximum value that $x$ attains is at $\sin 2\hat{\beta} = 1$, and the extrema of $\sin 2\alpha \sin \alpha$ occur at
2 \cos 2\hat{\alpha} \sin \hat{\alpha} + \sin 2\hat{\alpha} \cos \hat{\alpha} = \sin \hat{\alpha}(2 - 3 \sin^2 \hat{\alpha}) = 0. \text{ The minimum is at } \sin \hat{\alpha} = 0 \text{ and at the maximum } \sin \hat{\alpha} = \sqrt{2/3}. \text{ At the maximum, } \hat{x} = 1/3\sqrt{3}, \text{ proving that } \Delta \geq 0: \text{ the solutions of the cubic are real. At the maximum, } w_i = 1/\sqrt{3}, 1 \leq i \leq 3.

At the minimum at least one of the \(w_i = 0\) so that \(x = 0\); in this case \(\eta = \{0, \pm 1\}\), and the representation is reducible. The interesting aspect of this phenomenon is that \(A_1\) interacting with \(A_2\) interacting with \(A_3\) is not sufficient to hold three particles together – the interaction between \(A_1\) and \(A_3\) is also required to sustain the three particle state. This is the principle of detailed balance that we saw in the curvature two-forms.

The solutions of eq. (22) are obtained from the identity
\[ 4 \cos^3 \theta - 3 \cos \theta - \cos 3\theta = 0. \]
Set \(\eta = (2/\sqrt{3}) \cos \theta\), to find that
\[ \cos 3\theta = 3\sqrt{3} x \leq 1 \]
where the inequality follows from \(x \leq 1/3\sqrt{3}\). The solutions are \(\eta_{k+1} = (2/\sqrt{3}) \cos(\theta_0 + 2k\pi/3), 0 \leq k \leq 2\), where \(-\pi/6 < \theta_0 < \pi/6\). (The determinant vanishes at \(\theta_0 = \pm\pi/6\), which is excluded by the argument above.) The three solutions are alternatively written as
\[
\eta_1 = (2/\sqrt{3}) \cos \theta_0; \quad 1 < \eta_1 \leq 2/\sqrt{3} \\
\eta_2 = -(1/\sqrt{3}) \cos \theta_0 + \sin \theta_0; \quad -1/\sqrt{3} < \eta_2 < 0 \\
\eta_3 = -(1/\sqrt{3}) \cos \theta_0 - \sin \theta_0; \quad -1 < \eta_3 < -1/\sqrt{3}
\]
In this form the range of \(\theta_0\) may be restricted to \(0 \leq \theta_0 < 3\pi/6\), as \(\theta_0 < 0\) simply switches \(\eta_2 \leftrightarrow \eta_3\). At \(\theta_0 = 0\) there are two equal roots, and this simpler case is an easy calculation of eigenvectors.

The degenerate case \(x = w_1w_2w_3 = 1/3\sqrt{3}\) gives vanishing discriminant, and this is only possible if \(w = w_i = 1/\sqrt{3}, 1 \leq i \leq 3\). The eigenvalues are \(2/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}\), and a small calculation gives \(r_1 + r_2 + r_3 = 0\) for \(\eta = -1/\sqrt{3} = -w\), and
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$r_1 = r_2 = r_3 = w$ for $\eta = 2/\sqrt{3} = 2w$. The matrix of eigenvectors, is

$$R = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & -\gamma & \delta \\
\frac{1}{\sqrt{3}} & \delta & -\gamma
\end{bmatrix}; \quad \gamma = \frac{1}{2}(1 + 1/\sqrt{3}), \delta = \frac{1}{2}(1 - 1/\sqrt{3})$$

which can be multiplied on the right by any $M \in SO(2)$ that commutes with the matrix of eigenvalues.

The general case is a more interesting calculation. The eigenvectors for $\theta_0 \neq 0$ satisfy

$$\begin{bmatrix}
-\eta & w_3 & w_2 \\
w_3 & -\eta & w_1 \\
w_2 & w_1 & -\eta
\end{bmatrix}
\begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix} = 0$$

with indefinite values for the $w_i$. Calculations are facilitated by several useful identities, which can be derived from powers of $[\text{Tr}(\eta)] = 0$ and $\text{Tr}(\eta^k) = \text{Tr}(W^k)$. In addition to $\eta_1\eta_2\eta_3 = 2x$, the following are used extensively in the calculation of eigenvectors:

(23) \quad \text{Tr}(\eta^2) = 2; \text{Tr}(\eta^3) = 6x; \text{Tr}(\eta^4) = 2; \sum_{\alpha<\beta}\eta_\alpha\eta_\beta = -1; \sum_{\alpha<\beta}\eta_\alpha^2\eta_\beta^2 = 1

In solving for the $r_i$ it is convenient to define two combinations of the parameters:

$$b_i = \eta w_i + w_j w_k, \{i, j, k\} \text{ cyclic and } c_i = \eta^2 - w_i^2.$$ It is easy to prove that $b_i^2 = c_j c_k$ with use of $\eta^3 = \eta + 2x$. A bit of algebra yields the relations $r_i^2/r_j^2 = c_i/c_j$, so that normalization of the eigenvectors yields

$$\Sigma r_i^2 = 1 = [1 + c_2/c_3 + c_1/c_3] r_3^2.$$ This, together with obvious symmetry, yields the solutions

(24) \quad r_{i\mu}^2 = \frac{\eta_\mu^2 - w_i^2}{3\eta_\mu^2 - 1}, 1 \leq i, \mu \leq 3

and this is clearly column normalized since $\Sigma_i r_{i\mu}^2 = 1$. Proof that these components comprise a fundamental representation of $SO(3)$ is completed in Appendix 2.

The signs of the square roots in eq. (24) may be inferred from the eigenvalues. For $\eta_1$ all components of the eigenvectors are positive. For the negative eigenvalues,
$\eta_2$ and $\eta_3$, at least one component of the corresponding eigenvectors is negative. It appears that the general case will benefit from application of computer algebra; for now it suffices to observe that some components of $\tau$ are negative. It is more interesting to move on to explore the physical consequences of the relation between eigenvectors and the representations discussed above.

**Composite States**

Representations of isolated systems require a caveat. The fundamental idea of the flag is that systems are interrelated through the action of the flag manifold. Any representation of $CF(k)$ that is constructed in isolation will only capture intrinsic properties of the system, whereas extrinsic properties that derive from relations between a system and its surroundings are resolved in the larger question of how a $V_k$-system is imbedded in $V_{k+n}$ under the action of $Sp(k+n)/Sp(k) \times Sp(n)$. A few more comments on this aspect of the theory will be made later.

The infinitesimal generators of $\mathfrak{cf}(3)$ that have been developed are homogeneous operators, and it has been claimed that the irreducible representations are homogeneous polynomials of degree $m$ in three quaternion variables $\{x, y, z\}$. These will be constructed from terms of the form

$$\Psi(m_1, m_2, m_3) = x^{m_1}y^{m_2}z^{m_3} \pm \text{perm}: 0 \leq m_i \leq m; \Sigma_i m_i = m$$

where the permutations enable one to construct asymmetric, symmetric, and skew-symmetric states as appropriate to the choice of terms. As stated before, in quaternion products there is no physical reason for distinguishing between, say, $xy$ and $yx$. In this two-body case there are only symmetric and skew-symmetric representations to consider. Expanding on this observation, we will indicate how particle states might be constructed with two and three components.

The time-dependent representation of the coset $X \in CF(3)$ is constructed as

$$X = V(R \otimes 1)[\exp(\omega t \eta k)](R' \otimes 1)V^*$$

$$X = V[\cos(\omega t W_1) + \sin(\omega t W k)]V^*,$$

which shows how the maximal torus is sandwiched between the eigenvectors; this is a key feature of group theory in general.
The relation between bosons and fermions that is conveyed by the matrix structure of the fundamental representation means that a boson is written as a linear combination of products of fermions with the maximal torus. In higher dimensional representations, the bosons will be represented by products of these terms. The representation will evolve along geodesics as required by the parameterization in eq. (25). To execute a comprehensive program of assignments using these facts will require considerably more effort than can be accomplished within the scope of the present work. Nonetheless, it will be useful to indicate some promising directions with at least a few preliminary assignments and observations.

For displaying composite states it is beneficial to show how the quaternion components sort themselves out, and the scalar-vector notation is convenient for this. In anticipation of these products, some of which involve powers of a quaternion, an alternative is to use $x = |x|u, u\bar{u} = 1$, with

$$u = \cos(\chi)1 + \sin(\chi)\mathbf{r}; \quad \mathbf{r} = -\mathbf{r} \cdot \mathbf{r} = -1.$$  

Here are two useful theorems:

**Theorem 1:**

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

is the standard triple product from vector calculus;

**Theorem 2:**

$$u^k = \cos(k\chi)1 + \sin(k\chi)\mathbf{r}$$

is easily proved using eq. (26) with recursion.

Tensor products of two copies of $X$ in eq. (25) will generate three copies of 1 from $v_i\bar{v}_i$, three $\bar{v}_i v_j, i < j$, and three conjugates of the latter. In constructing product states, we can work in either the algebra or the group, but given that velocity, and hence dynamics, are functions on the tangent space of any manifold, it is more immediately appealing and algebraically simpler to work in the tangent space, even though, as stated previously, dynamics is not currently within reach. The simplification implies we are working in a neighborhood of the identity of the group. In the following the scalar magnitudes of the $\mathbb{H}$-valued functions will frequently not be of immediate concern, which is not to say that the magnitudes are not important. By introducing a frequency-like variable $\omega$ to partner with the time-like variable $t$ in
the description of geodesics, as in eq. (25), the global magnitude of the algebra was parameterized. The $\mathbb{H}$-algebra between units of the group algebra and the module will be our focus. Before starting a discussion of potential meson and baryon states, it will be useful to return to the $CF(2)$ case to offer some insights or conjectures that set the stage for further discussion.

**Leptons.** The simplest assumption that can be made is to identify a single (un-normalized) quaternion as a electron, $u \sim e$. Given this, the muon cannot be simply $u^2$ and the $\tau \sim u^3$, since no assignment of masses would make sense with this assignment, nor do the spin states look right. The next thing to try for a muon is to give $u$ a twist with a factor of $k$. This is appealing because the decay of $\mu^-$ is almost exclusively into $e^-\bar{\nu}_e\nu_\mu$. If the components of the centralizer are identified as neutrinos, a $ku \sim \mu$ state only needs another factor of $\pm k$ to produce an $e^-$. On the other hand, if we are to imagine this state in the context of a much larger $Sp(n)$, it could be produced by the action of a very particular boson on an electron: $(uk\bar{u})(u) \rightarrow uk$, would be interpreted as an electroweak interaction of a boson with an electron to produce a muon. (Concurrent with this, the conjugate boson acts elsewhere on the flag, but this is a topic for the future.) This is not intended to be an experimental method for producing muons; the simple product is only meant to illustrate an algebraic operation that executes a transformation from one state to another. To make an assignment for the $\tau$ meson requires more careful attention to magnitudes, as will be discussed later.

**Products of Two Quaternions.** To begin the development, select two quaternions from the three quaternions extracted from the infinitesimal generators, giving six possibilities, symmetric and skew-symmetric:

\[
\Psi_{\pm}(x, y) = (1/2)(xy \pm yx)
\]
\[
\Psi_{\pm}(y, z) = (1/2)(yz \pm zy)
\]
\[
\Psi_{\pm}(z, x) = (1/2)(zx \pm xz)
\]
where the factor of $1/2$ is a simple normalization. In addition to these, the conjugates of each of the three quaternions are available for composing states. This gives $4 \times 6$ potential states.

In addition to these states, we are allowed to construct symmetric and skew-symmetric states with inserted factors of $k$. For example,

$$\pi_{xy} = (1/2)(xky + \bar{y}k\bar{x})$$

has odd parity, since $P: \pi_{xy} = (1/2)(\bar{y}k\bar{x} + xky) = -\pi_{xy}$. Further, if one chooses $y = \bar{x}$, this state becomes $\pi_x = xk\bar{x}$. If we identify $-\pi_{xy}$ as an anti-$\pi$, then this combination of quaternions qualifies as its own anti-matter state. Now, does this state belong to the group algebra or the module? It has the right components to belong to the algebra, but it has the wrong symmetry under the PIN operators. A matrix element $m_{ij} = u_i k \bar{u}_j$ of the algebra transforms as follows:

$$P: m_{ij} \rightarrow -u_j k \bar{u}_i = -m_{ji}$$
$$I: m_{ij} \rightarrow u_j k \bar{u}_i = m_{ji}$$
$$N: m_{ij} \rightarrow -m_{ij}$$

However, $P: \pi_{xy} \rightarrow -\pi_{xy}$, which is the same as $N: \pi_{xy} \rightarrow -\pi_{xy}$, but $I: \pi_{xy} \rightarrow \pi_{xy}$ is equivalent to the identity operator, so $\pi_{xy}$ does not belong to the group algebra. This simple calculation appears to be useful for identifying and categorizing terms. Such aids are essential, because things have become quite complex, with many combinations possible; states constructed from an $a,b$ pair may contain many parity combinations as well as a $k$ factor.

In computing explicit terms of products using the scalar-vector notation it is not difficult to see that many different combinations of terms, particularly symmetric states, will contain identity components, i.e., terms with the basis element $1$. Using $\{a,b\}$ to be any pair of the $\{x,y,z\}$ fermions, the skew-symmetric combinations $(1/2)(ab - ba) = a \times b$ and $a k b$ do not contain a term in the identity basis element, but the latter is a function of the identity components $\{a_0, b_0\}$ of the basic quaternions. The reason for making a distinction between these two cases is the following:
Define a quaternion
\[ A(a, w) = A_0(a, w)\mathbf{1} + A_1(a, w)i + A_2(a, w)j + A_3(a, w)k = A_0\mathbf{1} + A \]
where \( a \) is a quaternion and \( w \) is any set of parameters not including \( a \), and also define the differential operator (see \( p \) above)
\[ \partial_a = \partial/\partial a_0 \mathbf{1} - (\partial/\partial a_1 i + \partial/\partial a_2 j + \partial/\partial a_3 k) = \partial/\partial a_0 \mathbf{1} - \nabla \]
This acts on \( \bar{A} \) to give
\[ \partial_a \bar{A} = (\partial A_0/\partial a_0 - \nabla \cdot A)\mathbf{1} - (\nabla A_0 + \partial A/\partial a_0) + \nabla \times A \]
which looks like \( \partial_a \bar{A} = \xi \mathbf{1} + \mathbf{E} + \mathbf{B} \), where \( \mathbf{E}, \mathbf{B} \) are the electric and magnetic fields of Maxwell theory. (Further pursuit of this with a Wick rotation of \( a_0 \) to \( it \) to get Maxwell’s equations does not appear to be fruitful. The quadratic piece of \( p \) is non-Euclidean, and may be omitted near the origin of the coset.) Now, the \( ab - ba \) state has neither an identity component nor does it contain \( a_0 \) and \( b_0 \), so the “apparent” \( \mathbf{E} \) vanishes, as consistent with a neutral particle. A calculation gives \( \partial_a (a \times b) = 2b \), so that this state has a magnetic moment. I do not know if this corresponds to any known meson state.

The symmetric state of \( \{a, b\} \), denoted \( \text{Sym}^2(a, b) \), is
\[ \text{Sym}^2(a, b) = (a_0b_0 - a \cdot b)\mathbf{1} + (a_0b + b_0a) \]
\[ \text{Sym}^2(a, a) = (a_0^2 - a \cdot a)\mathbf{1} + 2a_0a = a^2 = \cos(2\theta_a)\mathbf{1} + \sin(2\theta_a)u_a; \]
\[ a = \sin(\theta_a)u_a. \]
Applying the operator defined in eq. (27) to the first of these gives
\[ \partial_a \text{Sym}^2(a, b) = 4b_0\mathbf{1} + 2b \]
which appears to have an electric field but no magnetic moment. It is premature to attempt to quantize this prior to a more thorough analysis of the infinitesimal generators – these elementary calculations are merely intended to show how the various operators can be used to construct functions that have a classical interpretation.

**Products of Three Quaternions.** In constructing homogeneous products of the fundamental \( \{x, y, z\} \) quaternions, their scalar norms will factor and will be ignored
for the present. The unit quaternions will at first be given their meaning as the components of eigenvectors developed in the eigenvalue/eigenvector section.

**Symmetric Products.** Totally symmetric states of $k$ quaternions can be formalized in $\text{Sym}^k$, the symmetrized product of $v_\mu$, $1 \leq \mu \leq k$, with

$$k\text{Sym}^k(v_1, v_2, v_3, \cdots, v_k) = \sum_{i=1}^k v_i \text{Sym}^{k-1}(v_1, v_2, \cdots, \hat{v}_i, \cdots, v_k),$$

where $\hat{v}_i$ signifies that the term is missing. The first term in the sequence is $\text{Sym}^1(v_1) = v_1$. The first few symmetric states are:

$$\text{Sym}^2(v_1, v_2) = \frac{(v_1 v_2 + v_2 v_1)}{2},$$

consistent with the previous section

$$\text{Sym}^3(v_1, v_2, v_3) = [v_1 \text{Sym}^2(v_2, v_3) + v_2 \text{Sym}^2(v_1, v_3) + v_3 \text{Sym}^2(v_1, v_2)]/3$$

$$= [v_1(v_2 v_3 + v_3 v_2) + v_2(v_1 v_3 + v_3 v_1) + v_3(v_1 v_2 + v_2 v_1)]/6$$

A few explicit calculations will serve to show how the structure of symmetric and skew-symmetric products are different from one another. For this purpose the $\{a, b, c\}$ set (arbitrary labels, not bosons) is reclaimed to avoid multiple subscripts. The following may be readily verified:

$$\text{Sym}^3(a, b, c) = (a_0 b_0 c_0 - a_0 b \cdot c - b_0 a \cdot c - c_0 a \cdot b) \mathbf{1}$$

$$+ (b_0 c_0 - b \cdot c/3) a + (a_0 c_0 - a \cdot c/3) b + (a_0 b_0 - a \cdot b/3) c$$

$$s_3(a, a, b) = (a_0^2 b_0 - b_0 a \cdot a - 2a_0 a \cdot b) \mathbf{1} + (a_0^2 - a \cdot a/3) b + 2(a_0 b_0 - a \cdot b/3) a$$

$$a^3 = (a_0^3 - 3a_0 a \cdot a) \mathbf{1} + (3a_0^2 - a \cdot a)a = \cos(3\theta_a) \mathbf{1} + \sin(3\theta_a) \mathbf{u}_a$$

Note that the symmetric functions all have identity components that contain the identity components of their constituents as well as components from the vector parts. The function $s_3(a, a, b)$ has no symmetry with respect to interchange of its components, but is included here rather than separating it out. States composed with quarks having alternative chirality are not written down here, but they will be of interest.

**Skew-Symmetric Products.** The normalized skew-symmetric product, $\wedge^k$, of $k$ functions, $v_i$, $1 \leq i \leq k$, is

(28)  \quad n \wedge^k (v_1, v_2, v_3, \cdots, v_k) = \sum_{i=1}^n (-1)^{i-1} v_i \wedge^{n-1} (v_1, v_2, \cdots, \hat{v}_i, \cdots, v_k)$
By convention, $\wedge^k(v_1) = v_1$, and the first term of the sum on the right has a positive sign when the variables in the function are in sequential or lexical order. The first few skew-symmetric functions are

$$
\wedge^2(v_1, v_2) = (v_1 v_2 - v_2 v_1)/2 \\
\wedge^3(v_1, v_2, v_3) = [v_1(v_2 v_3 - v_3 v_2) - v_2(v_1 v_3 - v_3 v_1) + v_3(v_1 v_2 - v_2 v_1)]/6
$$

As before, revert to the $a, b, c$ notation to avoid multiple subscripts to get the simplified skew-symmetric states:

$$
\wedge^2(a, b) = a \times b, \text{ as before,} \\
\wedge^3(a, b, c) = -[a \cdot (b \times c)] + (1/3)[a_0(b \times c) + b_0(c \times a) + c_0(a \times b)], \\
\wedge^4(a, b, c, d) \equiv 0.
$$

There is no skew-symmetric state of four quaternions! The proof is contained in Appendix 3. The skew-symmetric state of three quaternions is very special in that it is a “complete” quaternion, whereas the skew-state for two quaternions lacks a scalar part and is “incomplete.”

One might augment these states of three quaternions with additional ones that are twisted with inserted $k$ factors. If this factor is included, the states begin to take on the character of linear combinations of products of fermions and bosons, which might be precursors to decomposition products. It is difficult to avoid speculating, as so many possibilities are open for consideration.

The assignment of states in the flag environment will likely not coincide with currently accepted $SU(3)$ assignments with $\{u, d, s\}$ quarks, simply because the vector spaces on which the $Sp(3)$ and $SU(3)$ groups act are different. For example, I have a strong suspicion that protons and neutrons correspond to $\wedge^3(v_1, v_2, v_3)$ and $\mathrm{Sym}^3(v_1, v_2, v_3)$, but this awaits further analysis.

**Why Six Quaternions?** The collision of two baryons, protons for example, entails a strong contact between and mixing of states with content $\psi(v_1, v_2, v_3)$ and $\psi(v_4, v_5, v_6)$. (The flag is currently silent on the dynamics of the collision.) We have to expand the basis to six dimensions, which is why there are six quarks. This enables the construction of states such as $\psi(v_1, v_2, v_4)$ and other combinations so as
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to cover states that are currently represented as, say, \( udc \). The mixing of these states also requires extension of the scheme to \( Sp(6)/Sp(1)^6 \).

Given that two protons are substates of a larger vector space, there is a small but non-vanishing probability that a third proton can become involved in a three-body collision. If that were to happen, a seventh and even up to ninth quark would become evident. A three body collision can only occur in highly concentrated counter-rotating proton beams, or in a collision with or of nuclei.

**Excited States vs. Baryon States.** The first excited state of, say, a symmetric two-particle state with ground state \( ab + ba \), is composed as \( a^2b + ab^2 + b^2a + ba^2 \), which is different from the three quaternion ground state \( a^2b + aba + ba^2 \). However, once normalized, these two different states should be comparable in some respects; for example, they might have closely similar masses (assuming that masses are assigned to the \( a, b \) states). The search for these close coincidences is part of the larger program of state assignments. Another example that previously appeared is \( \tau \sim u^3 \) as an excited state, which begs to be compared with a ground state of three identical quarks. Assigning a mass to a \( q \) with the replacement \( u \rightarrow q \) for the \( \tau \), a more appealing assignment is \( \tau \sim q\bar{q}q = |q|^2q \). Assuming the \( SU(3) \) classifications hold, there are a few \( \Delta \) states, \( uuu \) and \( ddd \), with masses around 1600-1800 MeV, as well the \( \Omega \sim sss \) (1672 MeV), that are similar to the mass of \( \tau \sim |q|^2q \) (1777 MeV), so give or take a few hundred MeV, this assignment for the \( \tau \) meson seems reasonable.

**Discussion**

The two and three particle examples that have been developed above show how the flag and flag manifold can be used to construct composite states. A few more quantum-physical principles will reinforce an understanding of how the theory can be exploited to comprehend additional properties of matter. To begin, the generators of the Lie algebra, \( \mathfrak{so}(3) \), can be shown to yield ladders of states having weights that are separated by integer values. Transitions in systems that increase or decrease by an integer quantum number are understood to result from absorption or emission of a boson. These transitions are taking place within the quark manifold, so the bosons
that create the transitions must have dynamic properties similar to the quarks themselves. The generators within a subgroup are homogeneous, whereas inhomogeneous operators couple disjoint subspaces with one another. This may help to explain the two different categories of forces in particle physics. One way to look at assignments is to give a bare quaternion the electron mass, and make the glue that holds the many particle states together carry most of the mass.

The overarching goal of the interaction theory is to incorporate the states found for small systems into ever larger flags, to build up descriptions of more complex states of matter. In doing this, the principles that have been uncovered here have to transcend small system bounds. Some of this has been seen in the curvature forms and the Grassmannian generators. The basic ideas are easily summarized: The action of the fundamental representation $g \in Sp(n)$ on the state space $V_n(\mathbb{H})$ by $g : V_n \to \hat{V}_n$ is a linear transformation of the space. Eigenvectors of this action are stationary states; one dimensional subspaces are quaternion valued fermions, which transform as fundamental representations of the $Sp(1) \sim SU(2)$ group. Any vector in the fundamental representation can be constructed as a linear combination of eigenvectors. Obviously the presentation of the eigenvectors in a diagonal matrix will not hold for $n > 3$, but the representation of the bosonic matrix elements as linear combinations of products of fermions will be true for any $CF(\{k_\mu\})$, where $\{k_\mu\}, \Sigma_\mu k_\mu = n$, is the partition of $n$ introduced previously. The relation between bosons and fermions for the $n = 2$ case is unique; for $n = 3$ we found bosons as twisted products of two fermions, the twist being the $k$-component. For larger $n$ the matrix elements of the flag manifold are linear combinations of binary twisted products of eigenvectors, so the relation between bosons and fermions carries over, it just becomes more complicated by linear combinations.

The crucial relation between $CF(3)$ and $SU(3)$ is that their root spaces are the same – the characteristic polynomials of the algebras, $|x - \lambda I| = 0$, are isomorphic. Thus, the success of $SU(3)$ in organizing elementary particle properties is expected to transfer to the $CF(3)$ representation. However, it may happen that the detailed assignments differ from one another, simply because the $V_3(\mathbb{H})$ vector space picture is not identical to that of the $\{u, d, s\}$ quarks. The extension to six quarks and more has been addressed but not pursued.
In building higher dimensional representations of the $CF(3)$ group and the module on which it acts, two constructions have to be addressed simultaneously. The states in the group will be linear combinations of products of terms of the form in eq. (20); each contains contributions from the maximal torus, the $k$ factors, making them bosons. The construction of geodesics in higher dimensional representations has not yet been addressed, and is left as a problem for the future. The interesting part of that problem is to learn how the eigenvectors of the higher dimensional representations acquire time dependence from components of the torus.

Embedding representations of $CF(3)$ in a larger dimension flag will require construction of asymmetric, symmetric or skew-symmetric module states that are functions of their one dimensional states, the latter of which are composites of fundamental states with their own internal symmetry. So we have to be careful with the language. Higher dimensional representations of a group are classified by their symmetries, which can be tracked with Young diagrams, so there are asymmetric, symmetric and skew-symmetric states of the composite particles, for example, the protons and neutrons comprising nuclei. Representations of, say, $Sp(3m)/Sp(3)^m$ can be decomposed into asymmetric, symmetric and skew-symmetric pieces, while each of the $Sp(3)$ subgroups can be further rendered into asymmetric, symmetric and skew-symmetric states of their quark constituents. It seems likely that the individual $Sp(3)$ states will retain their identities within the larger flag, but this is not a given. The complexity of nature is revealed in the many ways that the elementary pieces fit together.

An immediately attractive, but very messy, algebraic problem is to relate the matrix elements $\{a, b, c\}$ in eq. (20) to the components of the curvature tensors. That has not yet been done, as the goal has been tool development rather than a thorough analysis of all of the functions that have been developed. Further use of the tools, together with some stop-gap empirical parameter assignments, should be sufficient to build a description of particle states, both stable and exotic. To do this effectively will first require an analysis of the infinitesimal generators of $cf(3)$ to identify the linearly independent operators and their invariants. First amongst these is to identify the charge operator. The generators, coupled with the $PIN$ involution operators, will provide a basis for cataloguing states by their symmetries and quantum numbers.
A thorough classification of states in comparison with established meson and baryon assignments will be a major undertaking and is left to experts.

If the conjecture that $\text{Sym}^3(a, b, c)$ represents the neutron and $\wedge^3(a, b, c)$ the proton is correct, the near equality of their masses seems assured. But now the large difference between these masses and the exotic states that are constructed from $\{a, b, c\}$ is a problem. Perhaps the time dependence of the conjectured neutron and proton states is very different from other composites, signifying that the latter have energetic components that the former lack. When coupled to the external environment through $\text{Sp}(n+3)/\text{Sp}(3) \times \text{Sp}(n)$, it seems possible that the extrinsic properties of exotic states will differ from those of the stable states. To begin to address these questions, the $p$ operators cited in the Case 1 section can be turned around and written in terms of the components of the $(xy)$ operators developed for Case 2. This will enable one to couple these states into other components of a larger flag.

Acknowledgement

This work benefitted from conversations with Prof. John Sullivan of the University of Washington.

Appendix 1: Self-Dual and Anti-Self Dual Curvature Two-Forms

The two curvature two-forms, $\Omega_1 = \omega \wedge \bar{\omega}$ and $\Omega_2 = \bar{\omega} \wedge \omega$, are anti-self-dual and self-dual, respectively, as will be shown. Define the one-form $\omega = w_0 + w$ in scalar-vector notation, so that

$$\Omega_1 = \omega \wedge \bar{\omega} = -2w_0 \wedge w - w \wedge w$$
$$\Omega_2 = \bar{\omega} \wedge \omega = +2w_0 \wedge w - w \wedge w$$

with which it follows that

$$\Omega_1 \wedge \Omega_1 = +4w_0 \wedge w \wedge w \wedge w$$
$$\Omega_2 \wedge \Omega_2 = -4w_0 \wedge w \wedge w \wedge w.$$ 

Since $w = w_1i + w_2j + w_3k$, it is seen that the only terms that survive the triple exterior product, $w \wedge w \wedge w$, are those with $ijk = -1$ in some order. Ordering the quaternion basis in serial order in the triple product, and counting the permutations
of terms with their symmetries, both with respect to the exterior algebra and the quaternion algebra, one finds

\[ \Omega_1 = -24w_0 \wedge w_1 \wedge w_2 \wedge w_3 \]
\[ \Omega_2 = +24w_0 \wedge w_1 \wedge w_2 \wedge w_3 \]

proving that the curvature two-forms are anti-self-dual and self-dual, respectively. The reader will find the same result with use of the SU(2) basis representation.

The proof that \( \Omega_\mu = d\omega_\mu + \omega_\mu \wedge \omega_\mu \) is a tensor over \( \mathbb{H} \), i.e., \( \Omega_\mu \rightarrow h_\mu^* \Omega_\mu h_\mu \) (no sum) with a change of basis of \( V_n: e_n \rightarrow e_n H \) where \( H = \bigoplus \mu h_\mu \), preserves the stability subgroup and hence the flag structure, is the same as for the complex case. [16]

**Appendix 2: Proof of \( R \in SO(3) \)**

The proof that the matrix \( R = (r_{i\alpha}) \in SO(3) \) for the general case \( x \neq 1/3\sqrt{3} \) is completed here. In the text the normalization in columns was obvious. Normalization in rows requires

\[ \Sigma_\alpha r_{i\alpha}^2 = \Sigma_\alpha \eta_\alpha^2 - \frac{w_\alpha^2}{3\eta_\alpha^2 - 1} = 1, \quad 1 \leq i \leq 3 \]

which is proved with use of the relations in eq. (23). Two intermediates that arise in this calculation,

\[ \prod_\alpha (3\eta_\alpha^2 - 1) = 4(27x^2 - 1) \quad \text{and} \quad \sum_{\alpha<\beta} (3\eta_\alpha^2 - 1)(3\eta_\beta^2 - 1) = 0, \]

are easily proved. To prove orthogonality of the rows, begin with

\[ r_{i\alpha}^2 r_{j\alpha}^2 = \frac{c_{i\alpha}c_{j\alpha}}{(3\eta_\alpha^2 - 1)^2} = \frac{b_{ka}^2}{(3\eta_\alpha^2 - 1)^2} \]

where a Greek index has been added to \( b_i \) and \( c_i \) to associate each with the corresponding eigenvalue. The choice of the positive sign in the square root of eq. (29) gives

\[ \Sigma_\alpha b_{ka}^2 = \frac{b_{ka}^2}{3\eta_\alpha^2 - 1} = 0; \quad k \neq \{i, j\}, \quad i \neq j, \]
which is proved with use of the relations established above. The calculation of the column sum, \( \Sigma r_{i\alpha r_{i\beta}}; \alpha \neq \beta \) is done by squaring, so that

\[
(\Sigma r_{i\alpha r_{i\beta}})^2 = \Sigma r_{i\alpha}^2 r_{i\beta}^2 + 2 \Sigma_{i<j} r_{i\alpha} r_{j\alpha} r_{i\beta} r_{j\beta}
\]

The cross-terms are simplified since the numerator of \( r_{i\alpha} r_{j\beta} \) is \((c_{i\alpha} c_{j\beta})^{1/2} = \pm b_{k\alpha} \).

All of the denominators are the same, and with choice of the positive square root this simplifies to

\[
T_{\alpha\beta} = (3\eta_{\alpha}^2 - 1)(3\eta_{\beta}^2 - 1)(\Sigma r_{i\alpha r_{i\beta}})^2 = \Sigma (c_{i\alpha} c_{i\beta} + 2b_{i\alpha} b_{i\beta})
\]

\[
= \Sigma i [\eta_{\alpha}^2 \eta_{\beta} - w_i^2 (\eta_{\alpha}^2 + \eta_{\beta}^2) + w_i^4 + 2w_i^2 \eta_{\alpha} \eta_{\beta} + 2x(\eta_{\alpha} + \eta_{\beta}) + 2x^2 / w_i^2]
\]

\[
= 3\eta_{\alpha}^2 \eta_{\beta} - (\eta_{\alpha}^2 + \eta_{\beta}^2) + \Sigma i w_i^4 + 2\eta_{\alpha} \eta_{\beta} + 6x(\eta_{\alpha} + \eta_{\beta}) + 2x^2 \Sigma 1 / w_i^2
\]

The two remaining sums in \( T_{\alpha\beta} \) are very simple:

\[
\Sigma i (w_i^4 + 2x^2 / w_i^2) = \Sigma i w_i^4 + 2\Sigma_{i<j} w_i^2 w_j^2 = (\Sigma i w_i^2)^2 = 1,
\]

which gives

\[
T_{\alpha\beta} = 3\eta_{\alpha}^2 \eta_{\beta} - (\eta_{\alpha}^2 + \eta_{\beta}^2) + 2\eta_{\alpha} \eta_{\beta} + 6x(\eta_{\alpha} + \eta_{\beta}) + 1,
\]

\[
= 12x^2 / \eta_{\gamma}^2 - (2 - \eta_{\gamma}^2) + 4x / \eta_{\gamma} - 6x \eta_{\gamma} + 1,
\]

\[
\eta_{\gamma}^2 T_{\alpha\beta} = 12x^2 + \eta_{\gamma}(\eta_{\gamma} + 2x) + 4x \eta_{\gamma} - 6x(\eta_{\gamma} + 2x) - \eta_{\gamma}^2 = 0.
\]

This completes the proof of orthonormality of the matrix \( (r_{i\alpha}) \).

**APPENDIX 3: PROOF THAT \( \sigma_4(a, b, c, d) = 0 \)**

Simplification of eq. (28) gives

\[
24\sigma_4(a, b, c, d) = \{(a, b), [c, d]\} - \{[a, c], [b, d]\} + \{[a, d], [b, c]\}
\]

where \( \{x, y\} = xy + yx \) is the symmetrizer. Since \( [a, b] = 2a \times b \), it follows that

\[
3\sigma_4(a, b, c, d) = -(a \times b) \cdot (c \times d) + (a \times c) \cdot (b \times d) - (a \times d) \cdot (b \times c).
\]
The cross-products, e.g., $(a \times b) \times (c \times d) + (c \times d) \times (a \times b)$, vanish in the symmetrizers. Writing out just the $i \cdot i$ component of the three terms gives

$$\text{coeff}(i \cdot i) = - (a_2 b_3 - a_3 b_2)(c_2 d_3 - c_3 d_2)$$

$$+ (a_2 c_3 - a_3 c_2)(b_2 d_3 - b_3 d_2) - (a_2 d_3 - a_3 d_2)(b_2 c_3 - b_3 c_2)$$

$$= (-a_2 c_2 b_3 d_3 + a_2 d_2 b_3 c_3 + b_2 c_2 a_3 d_3 - b_2 d_2 a_3 c_3)$$

$$+ a_2 b_2 c_3 d_3 - a_2 d_2 b_3 c_3 - b_2 c_2 a_3 d_3 + c_2 d_2 a_3 b_3$$

$$- a_2 b_2 c_3 d_3 + a_2 c_2 b_3 d_3 + b_2 d_2 a_3 c_3 - c_2 d_2 a_3 b_3)$$

in which it is seen that all terms cancel in pairs. The remaining terms vanish by symmetry.

A much simpler proof is this: $\mathbb{R}^3$ does not admit four orthogonal vectors.

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