A NOTE ON REIDEMEISTER TORSION OF G-ANOSOV REPRESENTATIONS

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Abstract. This article considers G-Anosov representations of a fixed closed oriented Riemann surface Σ of genus at least 2. Here, G is the Lie group PSp(2n, R), PSO(n, n) or PSO(n, n + 1). It proves that Reidemeister torsion (R-torsion) associated to Σ with coefficients in the adjoint bundle representations of such representations is well-defined. Moreover, by using symplectic chain complex method, it establishes a novel formula for R-torsion of such representations in terms of the Atiyah-Bott-Goldman symplectic form corresponding to the Lie group G. Furthermore, it applies the results to Hitchin components, in particular, Teichmüller space.

1. Introduction

Throughout the paper, Σ is a closed Riemann surface of genus g ≥ 2. Teichmüller space Teich(Σ) of Σ is the space of isotopy classes of complex structures on Σ. It is a differentiable manifold and diffeomorphic to a ball of dimension 6g − 6. By the Uniformization Theorem, it can be interpreted as the isotopy classes of hyperbolic metrics on Σ, i.e. Riemannian metrics of constant Gaussian curvature (−1). It can also be interpreted as Hom_{df}(π₁(Σ), PSL(2, R)) discrete, faithful representations of the fundamental group π₁(Σ) of Σ to PSL(2, R). This is a connected component of the space Rep(π₁(Σ), PSL(2, R)) = Hom⁺(π₁(Σ), PSL(2, R))/PSL(2, R) of all reductive representations of π₁(Σ) to PSL(2, R).

In the paper [15], N. Hitchin proved the existence of an analogous component of Rep(π₁(Σ), G), where G is a split real-simply connected Lie group, such as PSL(n, R), PSp(2n, R), PO(n, n + 1), and PO(n, n). He called this component Teichmüller component but now it is called Hitchin component. He proved that Hitchin component of Hom(π₁(Σ), G)/G is diffeomorphic to \( \mathbb{R}^{(6g-6)\dim G} \). He also paused the problem about the geometric significance of this component.

We already mentioned the geometric significance of the Hitchin component for G = PSL(2, R). Namely, the hyperbolic structures on Σ. For G = PSL(3, R), S. Choi and W.M. Goldman proved that the Hitchin component is diffeomorphic to convex real projective structures on Σ [5]. F. Labourie introduced the notion of Anosov representations in his investigation of Hitchin component by dynamical system method [19], where he also proved that such representations are purely loxodromic, discrete, faithful, and irreducible.

The problem of giving a geometric interpretation of Hitchin components was completely solved by O. Guichard and A. Wienhard [14]. To be more precise, they proved that the Hitchin component of Hom(π₁(Σ), G)/G parametrizes the

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deformation space of $(G, X)$—structures on a compact manifold $M$. Here, if $X = \mathbb{R}P^{2n-1}$ then $G$ denotes $\text{PSL}(2n, \mathbb{R})$, $\text{PSp}(2n, \mathbb{R})$ when $n \geq 2$, $\text{PSO}(n, n)$ when $n \geq 3$ or if $X = F_{1,2n}(\mathbb{R}^{2n+1}) = \{ (D, H) \in \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^* : D \subset H \}$ then $G$ denotes $\text{PSL}(2n + 1, \mathbb{R})$ when $n \geq 1$, $\text{PSO}(n, n + 1)$ when $n \geq 2$ [14]. For details and more information, we refer the reader to [12, 13, 14].

In the present paper, we consider $G$-Anosov representations where the Lie group $G$ belongs to $\{ \text{PSp}(2n, \mathbb{R}), \text{PSO}(n, n), \text{PSO}(n, n + 1) \}$. We showed that the Reidemeister torsion of such representations is well-defined (Proposition 4.1). Furthermore, with the help of symplectic chain complex method, we establish a novel formula for R-torsion of such representation in terms of the Atiyah-Bott-Goldman symplectic form corresponding to the Lie group $G$ (Theorem 4.3).

2. The Reidemeister Torsion

For more information and the detailed proofs, we refer the reader to [24, 27, 31, 33, 34], and the references therein.

Suppose $C_* = (C_n \overset{d_n}{\rightarrow} C_{n-1} \rightarrow \cdots \rightarrow C_1 \overset{d_1}{\rightarrow} C_0 \rightarrow 0)$ is a chain complex of finite dimensional vector spaces over the field $\mathbb{R}$ of real numbers. Let $H_p(C_*) = Z_p(C_*)/B_p(C_*)$ denote the $p$-th homology group of $C_*$, $p = 0, \ldots, n$, where $B_p(C_*) = \text{Im} d_{p+1}$ and $Z_p(C_*) = \text{Ker} d_p$.

Assume that $c_p$, $b_p$, and $h_p$ are bases of $C_p$, $B_p(C_*)$, and $H_p(C_*)$, respectively, and that $\ell_p : H_p(C_*) \rightarrow Z_p(C_*)$, $s_p : B_p-1(C_*) \rightarrow C_p$ are sections of $Z_p(C_*) \rightarrow H_p(C_*)$, $C_p \rightarrow B_p-1(C_*)$, respectively, $p = 0, \ldots, n$. The definition of $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$ result the following short-exact sequences:

\begin{align*}
0 & \rightarrow Z_p(C_*) \hookrightarrow C_p \rightarrow B_p-1(C_*) \rightarrow 0, \\
0 & \rightarrow B_p(C_*) \hookrightarrow Z_p(C_*) \rightarrow H_p(C_*) \rightarrow 0.
\end{align*}

These short-exact sequences yield a new basis $b_p \sqcup \ell_p(h_p) \sqcup s_p (b_{p-1})$ of $C_p$.

The Reidemeister torsion of $C_*$ with respect to bases $\{c_p\}_{p=0}^n$, $\{h_p\}_{p=0}^n$ is defined by

$$T(C_*, \{c_p\}_{0}^n, \{h_p\}_{0}^n) = \prod_{p=0}^n |b_p \sqcup \ell_p(h_p) \sqcup s_p(b_{p-1}), c_p|^{(-1)^{p+1}},$$

where $|e_p, f_p|$ is the determinant of the change-base-matrix from $f_p$ to $e_p$.

The Reidemeister torsion $T(C_*, \{c_p\}_{0}^n, \{h_p\}_{0}^n)$ is independent of the bases $b_p$, sections $s_p, \ell_p$ [21]. If $c'_p, h'_p$ are also bases respectively for $C_p$, $H_p(C_*)$, then an easy computation results the following change-base-formula:

$$T(C_*, \{c'_p\}_{0}^n, \{h'_p\}_{0}^n) = \prod_{p=0}^n \left( \frac{[c'_p, c_p]}{[h'_p, h_p]} \right)^{(-1)^p} T(C_*, \{c_p\}_{0}^n, \{h_p\}_{0}^n).$$

If

$$0 \rightarrow A_* \overset{i_*}{\rightarrow} B_* \overset{\pi_*}{\rightarrow} D_* \rightarrow 0 \tag{2.4}$$

is a short-exact sequence of chain complexes, then we have the long-exact sequence of vector spaces of length $3n + 2$

$$\mathcal{H}_* : \cdots \rightarrow H_p(A_*) \overset{i_*}{\rightarrow} H_p(B_*) \overset{\pi_*}{\rightarrow} H_p(D_*) \overset{\delta_*}{\rightarrow} H_{p-1}(A_*) \rightarrow \cdots , \tag{2.5}$$

where $\mathcal{H}_{3p} = H_p(D_*)$, $\mathcal{H}_{3p+1} = H_p(A_*)$, and $\mathcal{H}_{3p+2} = H_p(B_*)$. 
The bases $h^A_p$, $h^B_p$ are clearly bases for $H_{3p}$, $H_{3p+1}$, and $H_{3p+2}$, respectively. Considering sequences (24) and (25), we have the following result of J. Milnor:

**Theorem 2.1.** (21) Let $c^A_p$, $c^B_p$, $c^D_p$, $h^A_p$, $h^B_p$, and $h^D_p$ be bases of $A_p$, $B_p$, $D_p$, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Let $c^A_p$, $c^B_p$, and $c^D_p$ be compatible in the sense that $[c^B_p, c^A_p \oplus c^D_p] = \pm 1$, where $\pi \left( c^D_p \right) = c^D_p$. Then,

$$
T(B_*, \{h^B_p\}_0, \{h^B_p\}_0) = T(A_*, \{c^A_p\}_0, \{h^A_p\}_0) \left(\pi(D_*, \{c^D_p\}_0, \{h^D_p\}_0) \times T(H_*, \{c^B_p\}_0, 0) \right).
$$

\[ Q.E.D. \]

Considering the short exact sequence

$$
0 \rightarrow A_* \xrightarrow{i} A_* \oplus D_* \xrightarrow{\pi} D_* \rightarrow 0,
$$

where for $p = 0, \ldots, n$, $i_p : A_p \rightarrow A_p \oplus D_p$ denotes the inclusion, $\pi_p : A_p \oplus D_p \rightarrow D_p$ denotes the projection, and the compatible bases $c^A_p$, $c^B_p \oplus c^D_p$, and $c^D_p$, where we consider the inclusion as a section of $\pi_p : A_p \oplus D_p \rightarrow D_p$, then by Theorem 2.1 we get:

**Lemma 2.2.** (30) If $A_*$, $D_*$ are two chain complexes, $c^A_p$, $c^B_p$, $h^A_p$, and $h^B_p$ are bases of $A_p$, $D_p$, $H_p(A_*)$, and $H_p(D_*)$, respectively, then

$$
T(A_* \oplus D_*, \{c^A_p \oplus c^D_p\}_0, \{h^A_p \oplus h^B_p\}_0) = T(A_*, \{c^A_p\}_0, \{h^A_p\}_0) \left(\pi(D_*, \{c^D_p\}_0, \{h^D_p\}_0) \right).
$$

\[ Q.E.D. \]

Note that one can split a general chain complex as a direct sum of an exact and a $\partial$–zero chain complexes. Moreover, Reidemeister torsion $T(C_*)$ of a general complex $C_*$ is as an element of $\otimes^n_{p=0}(\det(H_p(C_*)))^{(-1)^{p+1}}$, where $\det(H_p(C_*)) = \bigwedge^{\dim H_p(C_*)} H_p(C_*)$ denotes the top exterior power of $H_p(C_*)$, and $\det(H_p(C_*))^{-1}$ is the dual of $\det(H_p(C_*))$. We refer the reader [27, 36] for more information and the detailed proofs.

A symplectic chain complex is a chain complex of finite dimensional real vector spaces $C_* : 0 \rightarrow C_{2n} \xrightarrow{\partial_{2n}} C_{2n-1} \rightarrow \cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow 0$ of length $2n(n \text{ odd})$ together with for each $p = 0, \ldots, n$, a $\partial$–compatible anti-symmetric non-degenerate bilinear form $\omega_{p,2n-p} : C_p \times C_{2n-p} \rightarrow \mathbb{R}$. Namely,

$$
\omega_{p,2n-p} (\partial a, b) = (-1)^{p+1} \omega_{p+1,2n-(p+1)}(a, \partial b),
$$

$$
\omega_{p,2n-p}(a, b) = (-1)^p \omega_{2n-p,b}(b, a).
$$

Clearly, we have anti-symmetric and non-degenerate bilinear map $[\omega_{p,n-p}] : H_p(C_*) \times H_{n-p}(C_*) \rightarrow \mathbb{R}$ defined by $[\omega_{p,n-p}]([x], [y]) = \omega_{p,n-p}(x, y)$.

Suppose $C_*$ is a real-symplectic chain complex of length $2n$, and that $c_p$ is a basis of $C_p$, $p = 0, \ldots, 2n$. We say that the bases $c_p$ of $C_p$ and $c_{2n-p}$ of $C_{2n-p}$ are $\omega$–compatible, if the matrix of $\omega_{p,2n-p}$ in bases $c_p$, $c_{2n-p}$ is the $k \times k$ identity matrix $I_k \times k$ when $p \neq n$ and

$$
\begin{pmatrix}
 0 & I_k \\
-I_k & 0
\end{pmatrix}
$$

when $p = n$, where $k = \dim C_p = \dim C_{2n-p}$ and $2l = \dim C_n$.

Let us introduce the following notation used throughout the paper. Let $C_*$ be a real-symplectic chain complex and let $h^C_p$, $h^C_{2n-p}$ be bases of $H_p(C_*)$, $H_{2n-p}(C_*)$, respectively.
respectively. Then, $\Delta_{p,2n-p}(C_\ast)$ denotes the determinant of the matrix of the non-degenerate pairing $[\omega_{p,2n-p}]: H_p(C_\ast) \times H_{2n-p}(C_\ast) \to \mathbb{R}$ in bases $h_p^C, h_{p=0}^C$.

By an easy linear algebra argument, we have:

**Lemma 2.3.** ([31]) For a symplectic chain complex, there exist $\omega$-compatible bases. 

**Theorem 2.4.** ([24]) Suppose $C_\ast$ is a symplectic chain complex of length $2n$, and $c_p, h_p^C$ are bases of $C_p, H_p(C_\ast)$, respectively, $p = 0, \ldots, 2n$. Then, the following formula is valid:

$$T(C_\ast, \{c_p\}_{p=0}^{2n}, \{h_p^C\}_{p=0}^{2n}) = \prod_{p=0}^{n-1} \Delta_{p,2n-p}(C_\ast) (-1)^p \cdot \sqrt{\Delta_{n,n}(C_\ast)} (-1)^n.$$ 

We refer the reader to [27] for detailed proof and unexplained subjects. See also [28, 29, 30, 31] for further applications of Theorem 2.4.

### 3. Anosov Representations

Let $\Sigma$ be a closed oriented Riemann surface of genus at least 2, $h$ be a hyperbolic metric on $\Sigma$, $M = UT(\Sigma)$ be its unit tangent bundle of $\Sigma$ and $g_t$ be the geodesic flow for the hyperbolic metric $h$. Since the geodesic flow on the unit tangent bundle of a negatively curved manifold is Anosov [16], $g_t$ is an Anosov flow. To be more precise, there is a $g_t$-invariant splitting of the tangent bundle

$$T\Sigma = E^s \oplus E^u \oplus E^t.$$ 

Here,

- $E^t$ is a line bundle, which is tangent to the flow $g_t$,
- $E^u$ is expanding, namely, there are constant $A > 0, \alpha > 1$ such that for each $t \in \mathbb{R}$ and $v \in E^u$,

$$\|D_{g_t}(v)\| \geq A\alpha^t \|v\|,$$

- $E^s$ is contracting, in other words, there are constants $B > 0, 0 \leq \beta < 1$ such that for each $t \in \mathbb{R}$ and $v \in E^s$,

$$\|D_{g_t}(v)\| \leq B\beta^t \|v\|.$$ 

Let $\tilde{\Sigma}$ be the universal covering of $\Sigma$, $\tilde{M} = UT(\tilde{\Sigma})$ be the $\pi_1(\Sigma)$-cover of $M$ and let us also denote by $g_t$ the geodesic flow on $\tilde{M}$.

For a semi-simple Lie group $G$, let $(P^+, P^-)$ be a pair of opposite parabolic subgroups of $G$. We denote respectively the quotient spaces $G/P^+, G/P^-$, and $G/L$ by $\mathcal{F}^+, \mathcal{F}^-$, and $\mathcal{X}$, where $L = P^+ \cap P^-$. Note that considering the diagonal action of $G$ on $\mathcal{F}^+ \times \mathcal{F}^-$, $\mathcal{X}$ is the unique open $G$-orbit. This product structure induces two $G$-invariant distributions $E^+$ and $E^-$ on $\mathcal{X}$. To be more precise, $E^+_x = T_x \mathcal{F}^+$ and $E^-_x = T_x \mathcal{F}^-$, where $x = (x_+, x_-) \in \mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-$. Clearly, any $\mathcal{X}$-bundle can be equipped with two distributions which will be denoted by the same letters $E^+$ and $E^-$. 

Let $\varrho: \pi_1(\Sigma) \to G$ be a representation. By the diagonal action of $\pi_1(\Sigma), \tilde{M} \times \mathcal{X}$ is a $\pi_1(\Sigma)$-space. Here, $\pi_1(\Sigma)$ acts on $\tilde{M}$ as the deck transformation and the action of $\pi_1(\Sigma)$ on $\mathcal{X}$ by conjugation, via the representations $\varrho$. Thus, under this action

$$\mathcal{X}_\varrho := \tilde{M} \times_{\varrho} \mathcal{X} = (\tilde{M} \times \mathcal{X})/\pi_1(\Sigma).$$
Note that the projection of $\tilde{M} \times X$ onto $\tilde{M}$ descends to a map $X_{\phi} \to M$, which gives $\tilde{M} \times_{\phi} X$ the structure of a flat $X$-bundle over $M$.

Clearly, the geodesic flow $g_t$ can be lifted to a flow on $\tilde{M} \times X$ by defining

$$G_t(m, x) := (g_t m, x).$$

Let us also note that the resulting flow is invariant under the $\pi_1(\Sigma)$ action. Therefore, it defines a flow on $X_{\phi}$, which we denote by the same symbol $G_t$ lifting geodesic flow $g_t$.

We say that a representation $\pi_1(\Sigma) \to G$ is a $(P^+, P^-)$--Anosov, if the following two conditions are satisfied:

1) there is a section $\sigma : M \to X_{\phi}$ of the flat bundle $X_{\phi}$ which is flat along the flow lines, namely, the restriction of $\sigma$ to any geodesic leaf is flat,

2) the lifted action of the geodesic flow $g_t$ on $\sigma^* E^+, \sigma^* E^-$ is respectively expanding, contracting. More precisely, for some continuous family of norms on the fibers of the $\sigma^* E^+, \sigma^* E^-$ bundles, the expanding-contracting properties are fulfilled.

Let $\text{Rep}_{\text{Anosov}}(\pi_1(\Sigma), G)$ be the set of all Anosov representations. It was proved in [19] by F. Labourie that $\text{Rep}_{\text{Anosov}}(\pi_1(\Sigma), G)$ is open in $\text{Rep}(\pi_1(\Sigma), G)$. He also proved that every such representation $1$-1, discrete, irreducible, and purely loxodromic.

4. Main Theorems

Let $\Sigma$ be a closed oriented Riemann surface with genus $g \geq 2$, $\tilde{\Sigma}$ be the universal covering of $\Sigma$. Let $G \in \{\text{PSp}(2n, \mathbb{R})(n \geq 2), \text{PSO}(n, n+1)(n \geq 2), \text{PSO}(n, n+1)(n \geq 3)\}$ and $\mathcal{G}$ be the corresponding Lie algebra with the non-degenerate Killing form $B$.

For a representation $\varrho : \pi_1(\Sigma) \to G$, consider the associated adjoint bundle $E_{\varrho} = \tilde{\Sigma} \times G/ \sim$ over $\Sigma$. Here, for all $\gamma \in \pi_1(\Sigma)$, $(\gamma \cdot x, \gamma \cdot t) \sim (x, t)$, the action of $\gamma$ in the first component as a deck transformation and in the second component as conjugation by $\varrho(\gamma)$.

Suppose that $K$ is a cell-decomposition of $\Sigma$ so that the adjoint bundle $E_{\varrho}$ over $\Sigma$ is trivial over each cell. Let $\tilde{K}$ be the lift of $K$ to the universal covering $\tilde{\Sigma}$ of $\Sigma$. Let $\mathbb{Z}[\pi_1(\Sigma)] = \{\sum_{i=1}^p m_i \gamma_i : m_i \in \mathbb{Z}, \gamma_i \in \pi_1(\Sigma), p \in \mathbb{N}\}$ be the integral group ring. Let $C_*(K; \mathcal{G}_{\text{AdG}}) = C_*(\tilde{K}; \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{G}) = C_*(\tilde{K}; \mathbb{Z}) \otimes \mathcal{G}/ \sim$, where $\sigma \otimes t$ and all the elements in orbit $\{\gamma \cdot \sigma \otimes \gamma \cdot t ; \gamma \in \pi_1(\Sigma)\}$ are identified, and where $\pi_1(\Sigma)$ acts on $\tilde{\Sigma}$ by the deck transformation and the action of $\pi_1(\Sigma)$ on $\mathcal{G}$ is by conjugation.

We have

$$0 \to C_2(K; \mathcal{G}_{\text{AdG}}) \xrightarrow{\partial_2 \otimes \text{id}} C_1(K; \mathcal{G}_{\text{AdG}}) \xrightarrow{\partial_1 \otimes \text{id}} C_0(K; \mathcal{G}_{\text{AdG}}) \to 0.$$ 

Here, $\partial_p$ is the usual boundary operator. Let $H_*(K; \mathcal{G}_{\text{AdG}})$ denote the homologies of the above chain complex. The cochains $C^*(K; \mathcal{G}_{\text{AdG}})$ yield that $H^*(K; \mathcal{G}_{\text{AdG}})$. Here, $C^*(K; \mathcal{G}_{\text{AdG}})$ denotes the set of $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphisms from $C_*(\tilde{K}; \mathbb{Z})$ to $\mathcal{G}$. For more information, we refer the reader to [23, 27, 30], and the references therein.

We say that $\varrho : \pi_1(\Sigma) \to G$ is purely loxodromic, if for every non-trivial $\gamma \in \pi_1(\Sigma)$, the eigenvalues of $\varrho(\gamma)$ are real with multiplicity 1.
Suppose that \( \varrho : \pi_1(\Sigma) \to G \) is purely loxodromic. Consider chain complex

\[
0 \to C_2(K; \mathcal{G}_{Adg}) \xrightarrow{\partial_2 \otimes \text{id}} C_1(K; \mathcal{G}_{Adg}) \xrightarrow{\partial_1 \otimes \text{id}} C_0(K; \mathcal{G}_{Adg}) \to 0.
\]

Let \( c_p^j \) be the \( p \)-cells of \( K \) which gives us a \( \mathbb{Z} \)-basis for \( C_p(K; \mathbb{Z}) \). Fix a lift \( \tilde{c}_p^j \) of \( c_p^j \), \( j = 1, \ldots, m_p \). Then, \( c_p = \{ \tilde{c}_p^j \}_{j=1}^{m_p} \) is a \( \mathbb{Z}[\pi_1(\Sigma)] \)-basis for \( C_p(\tilde{K}; \mathbb{Z}) \). Let \( \mathcal{A} = \{ a_k \}_{k=1}^{\dim G} \) be an \( \mathbb{R} \)-basis of the semisimple Lie algebra \( \mathcal{G} \) so that the matrix of the Killing form \( B \) is the diagonal matrix \( \text{Diag}(1, \ldots, 1, -1, \ldots, -1) \), where \( p + r = \dim \mathcal{G} \). Such a basis is called \( B \)-orthonormal basis. Then, \( c_p = c_p \otimes_\varrho \mathcal{A} \) is an \( \mathbb{R} \)-basis for \( C_p(K; \mathcal{G}_{Adg}) \) and called a geometric basis for \( C_p(K; \mathcal{G}_{Adg}) \).

Let us assume that \( h_p \) is an \( \mathbb{R} \)-basis for \( H_p(K; \mathcal{G}_{Adg}) \) then \( T(C_*(K; \mathcal{G}_{Adg}), \{ c_p \otimes_\varrho \mathcal{A} \}_{j=0}^{p}, \{ h_p \}_{j=0}^{p}) \) is called the Reidemeister torsion of the triple \( K, \text{Ad} \circ \varrho \), and \( \{ h_p \}_{j=0}^{p} \) is a lift of \( \varrho \), and conjugacy class of \( \varrho \) will be explained below and for the independence of the cell-decomposition, the reader is referred to \[27\] Lemma 2.0.5.

**Proposition 4.1.** \( T(C_*(K; \mathcal{G}_{Adg}), \{ c_p \otimes_\varrho \mathcal{A} \}_{j=0}^{p}, \{ h_p \}_{j=0}^{p}) \) is independent of \( \mathcal{A} \), lifts \( \tilde{c}_p^j \), conjugacy class of \( \varrho \), and the cell-decomposition \( K \).

**Proof.** Let \( \mathcal{A'} \) be another \( B \)-orthonormal basis of \( \mathcal{G} \). From change-base-formula \[23\] of Reidemeister torsion it follows that

\[
\frac{T(C_*(K; \mathcal{G}_{Adg}), \{ c_p' \}_{j=0}^{p}, \{ h_p \}_{j=0}^{p})}{T(C_*(K; \mathcal{G}_{Adg}), \{ c_p \}_{j=0}^{p}, \{ h_p \}_{j=0}^{p})} = \det(T)^{-\chi(\Sigma)}.
\]

Here, \( c_p' = c_p \otimes_\varrho \mathcal{A'} \), \( T \) is the change-base-matrix from \( \mathcal{A'} \) to \( \mathcal{A} \), and \( \chi \) is the Euler characteristic.

Note that since \( \mathcal{A}, \mathcal{A'} \) are \( B \)-orthonormal bases of \( \mathcal{G} \), \( \det T = \pm 1 \). The independence of the Reidemeister torsion from \( B \)-orthonormal basis \( \mathcal{A} \) follows from the fact that the Euler-characteristic \( \chi(\Sigma) \) of \( \Sigma \) is even.

Next, let us fix \( \gamma \in \pi_1(\Sigma) \). Assume \( c_p' = \{ \tilde{c}_p^1 \cdot \gamma, \tilde{c}_p^2, \ldots, \tilde{c}_p^{m_p} \} \) is also a lift of \( \{ c_p^1, \ldots, c_p^{m_p} \} \), where only another lift of \( c_p^1 \) is considered and the others are kept the same. From the tensor product property it follows that \( \tilde{c}_p^1 \cdot \gamma \otimes t = \tilde{c}_p^1 \otimes \text{Ad}_{\varrho(\gamma)}(t) \).

By change-base-formula \[23\], we have

\[
\frac{T(C_*(K; \mathcal{G}_{Adg}), \{ c_p' \}_{j=0}^{p}, \{ h_p \}_{j=0}^{p})}{T(C_*(K; \mathcal{G}_{Adg}), \{ c_p \}_{j=0}^{p}, \{ h_p \}_{j=0}^{p})} = \det(A).
\]

Here, \( c_p = c_p \otimes_\varrho \mathcal{A}, c_p' = c_p' \otimes_\varrho \mathcal{A}, \) and \( A \) denotes the matrix of \( \text{Ad}_{\varrho(\gamma)} : \mathcal{G} \to \mathcal{G} \) with respect to basis \( \mathcal{A} \).

To compute the determinant of the matrix of \( \text{Ad}_{\varrho(\gamma)} \), let us consider the basis

\[
\mathcal{B}_{sp_{2n}(\mathbb{R})} = \left\{ \begin{array}{ll}
E_{ii} - E_{n+i,n+i}, & 1 \leq i \leq n, \\
E_{ij} - E_{n+j,n+i}, & 1 \leq i \neq j \leq n, \\
E_{i,n+i}, & 1 \leq i \leq n, \\
E_{n+i,i}, & 1 \leq i \leq n, \\
E_{i,n+j} + E_{j,n+i}, & 1 \leq i < j \leq n, \\
E_{n+i,j} + E_{n+j,i}, & 1 \leq i < j \leq n.
\end{array} \right\}
\]
\[
B_{\mathfrak{so}, n}(\mathbb{R}) = \begin{cases}
E_{ij} - E_{n+j, n+i}, & 1 \leq i \neq j \leq n, \\
E_{ii} - E_{n+i, n+i}, & 1 \leq i \leq n, \\
E_{i+n, j} - E_{j+n, i}, & 1 \leq i < j \leq n, \\
E_{n+i, j} - E_{n+j, i}, & 1 \leq i < j \leq n,
\end{cases}
\]
and
\[
B_{\mathfrak{so}, n+1}(\mathbb{R}) = \begin{cases}
E_{ii} - E_{n+i, n+i}, & 2 \leq i \leq n + 1, \\
E_{1,n+i+1} - E_{i+1,1}, & 1 \leq i \leq n, \\
E_{1,i+1} - E_{n+i+1,1}, & 1 \leq i \leq n, \\
E_{i+1,j+1} - E_{n+j+1, n+i+1}, & 1 \leq i \neq j \leq n, \\
E_{i+1,n+j+1} - E_{j+1,n+i+1}, & 1 \leq i \neq j \leq n, \\
E_{i+n+1,j+1} - E_{j+n+1,i+1}, & 1 \leq j \neq i \leq n,
\end{cases}
\]
of \( \mathfrak{sp}_{2n}(\mathbb{R}), \mathfrak{so}_{n}(\mathbb{R}), \) and \( \mathfrak{so}_{n,n+1}(\mathbb{R}) \), respectively. Here, \( E_{ij} \) denotes the matrix with 1 in the \( ij \) entry and 0 elsewhere.

By the assumption that \( \varrho \) is purely loxodromic, we have for each \( \gamma \in \pi_1(\Sigma) \) there is \( Q = Q(\gamma) \in G \) such that \( Q\varrho Q^{-1} = D = \text{Diag}(\lambda_1, \ldots, \lambda_m) \). Here, \( m \) is equal to \( 2n \) for \( G \in \{ \text{PSp}(2n, \mathbb{R}), \text{PSO}(n, n) \} \), and for \( G = \text{PSO}(n, n + 1) \), \( m = 2n + 1 \).

Note that by the spectral properties of such diagonalizable matrices we have \( D = \text{Diag}(\lambda_1, \ldots, \lambda_n, 1/\lambda_1, \ldots, 1/\lambda_n) \) for \( G \in \{ \text{PSp}(2n, \mathbb{R}), \text{PSO}(n, n) \} \) and \( D = \text{Diag}(1, \lambda_1, \ldots, \lambda_n, 1/\lambda_1, \ldots, 1/\lambda_n) \) for \( G = \text{PSO}(n, n + 1) \). Note also that
\[
DE_{ij}D^{-1} = \frac{\lambda_i}{\lambda_j} E_{ij}.
\]

From this it follows that the matrix of \( \text{Ad}_D \) in the basis \( B_{\mathfrak{sp}_{2n}}(\mathbb{R}) \) is the diagonal matrix with the diagonal entries
\[
\begin{cases}
1, & 1 \leq i \leq n, \\
\lambda_i/\lambda_j, & 1 \leq i \neq j \leq n, \\
\lambda_i^2, & 1 \leq i \leq n, \\
1/\lambda_i, & 1 \leq i \leq n, \\
\lambda_i \lambda_j, & 1 \leq i < j \leq n, \\
1/\lambda_j, & 1 \leq i \leq n, \\
\lambda_i/\lambda_j, & 1 \leq i < j \leq n,
\end{cases}
\]
in the basis \( B_{\mathfrak{so}_{n}}(\mathbb{R}) \) is the diagonal matrix with the diagonal entries
\[
\begin{cases}
\lambda_i/\lambda_j, & 1 \leq i \neq j \leq n, \\
1, & 1 \leq i \leq n, \\
\lambda_i \lambda_j, & 1 \leq i < j \leq n, \\
1/\lambda_i, & 1 \leq i < j \leq n,
\end{cases}
\]
in the basis \( B_{\mathfrak{so}_{n+1}}(\mathbb{R}) \) is the diagonal matrix with the diagonal entries
\[
\begin{cases}
1, & 2 \leq i \leq n + 1, \\
\lambda_{i+1}, & 1 \leq i \leq n, \\
1/\lambda_{i+1}, & 1 \leq i \leq n, \\
\lambda_{i+1}/\lambda_{j+1}, & 1 \leq i \neq j \leq n, \\
\lambda_{i+1} \lambda_{j+1}, & 1 \leq i < j \leq n, \\
1/\lambda_{i+1} \lambda_{j+1}, & 1 \leq j < i \leq n.
\end{cases}
\]

Note that the determinant of these diagonal matrices is 1. Hence, we proved the independence of the Reidemeister torsion from the lifts.
By the fact that the twisted chains and cochains for conjugate representations are isomorphic, we also have the independence of Reidemeister torsion from conjugacy class of \( \varrho \).

This is the end of proof Proposition 4.1. \( \square \)

Let us continue with the following well known result which will be used in the proof of our main theorem (Theorem 4.3). For the sake of completeness, we will also give the proof of this auxiliary result in details.

**Lemma 4.2.** Let \( f : V \times V \to \mathbb{R} \) be a non-degenerate, anti-symmetric bilinear map on the real vector space \( V \) of dimension \( 2n \). Let \( \{v_1, \ldots, v_{2n}\} \) be a basis of \( V \) and let \( \{v^1, \ldots, v^{2n}\} \) be the corresponding dual basis, namely \( v^i(v_j) = \delta_{ij} \). Let \( f^* : V^* \times V^* \to \mathbb{R} \) be the dual bilinear map of \( f \), which is defined by \( f^*(v^i, v^j) := f(v_i, v_j) \).

If \( G(f; \{v_1, \ldots, v_{2n}\}) \) denotes the Gram matrix of \( f \) in the basis \( \{v_1, \ldots, v_{2n}\} \), then \( G(f^*; \{v^1, \ldots, v^{2n}\})G(f; \{v_1, \ldots, v_{2n}\})^T = I_{2n \times 2n} \). Here, \( I_{2n \times 2n} \) is the \( n \times n \) identity matrix and \( ^T \) denotes the transpose of a matrix.

**Proof.** Let us first note that there is a symplectic basis \( \{e_1, \ldots, e_{2n}\} \) of \( V \) so that the Gram matrix \( G(f; \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}) \) and \( G(f^*; \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}) \) are both equal to \( \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix} \). Thus, we have

\[
G(f^*; \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\})G(f; \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\})^T = I_{2n \times 2n}.
\]

If \( L \) is the change-base matrix from basis \( \{e_1, \ldots, e_{2n}\} \) to \( \{v_1, \ldots, v_{2n}\} \) of \( V \) and if \( M \) is the change-base matrix from basis \( \{e_1, \ldots, e_{2n}\} \) to \( \{v^1, \ldots, v^{2n}\} \) of \( V^* \), then clearly we have \( LM^T = I_{2n \times 2n} \). Note also that

\[
G(f; \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}) = L^T G(f; \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}) L,
\]

\[
G(f^*; \{v^1, \ldots, v^n, v^{n+1}, \ldots, v^{2n}\}) = M^T G(f^*; \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}) M.
\]

From these it follows that \( G(f^*; \{v^1, \ldots, v^{2n}\})G(f; \{v_1, \ldots, v_{2n}\})^T = I_{2n \times 2n} \).

This finishes the proof of Lemma 4.2. \( \square \)

**Theorem 4.3.** Assume that \( \Sigma \) is a closed orientable surface of genus \( g \geq 2 \) and \( \varrho : \pi_1(\Sigma) \to G \) is an irreducible, purely loxodromic representation. Assume also \( K \) is a cell-decomposition of \( \Sigma \), \( c_p \) is the geometric bases of \( C_p(K; G_{Adoq}) \), \( p = 0, 1, 2 \), and \( h_1 \) is a basis for \( H_1(\Sigma; G_{Adoq}) \). Then, the following formula is valid:

\[
\mathbb{T}(C_*(K; G_{Adoq}), \{c_p\}_{p=0}^2, \{0, h_1, 0\}) = \sqrt{\text{det} \Omega_{\omega_B}},
\]

where \( \omega_B : H^1(\Sigma; G_{Adoq}) \times H^1(\Sigma; G_{Adoq}) \to H^2(\Sigma; \mathbb{R}) \xrightarrow{\mathbb{J}_\omega} \mathbb{R} \) is the Atiyah-Bott-Goldman symplectic form for the Lie group \( G \), \( \Omega_{\omega_B} \) is the matrix of \( \omega_B \) in the basis \( h^1 \), and where \( h^1 \) is the Poincaré dual basis of \( H^1(\Sigma; G_{Adoq}) \) corresponding to \( h_1 \) of \( H_1(\Sigma; G_{Adoq}) \).

**Proof.** By the invariance of the Cartan-Killing form \( B \) of \( G \) under conjugation, for \( k = 0, 1, 2 \), we have the non-degenerate form, the Kronecker pairing,

\[
\langle \cdot, \cdot \rangle : C^k(K; G_{Adoq}) \times C_k(K; G_{Adoq}) \to \mathbb{R}
\]

defined by \( B(t, \vartheta(\sigma)) \), \( \vartheta \in C^\infty(K; G_{Adoq}) \), \( \sigma \in C_k(K; G_{Adoq}) \). It is extended to

\[
\langle \cdot, \cdot \rangle : H^k(\Sigma; G_{Adoq}) \times H_k(\Sigma; G_{Adoq}) \to \mathbb{R}.
\]
Invariance of $B$ under the conjugation and the non-degeneracy of $B$ yield the cup product
\[ \sim_B: C^k(K; G_{Ado}) \times C^\ell(K; G_{Ado}) \to C^{k+\ell}(K; \mathbb{R}) \]
defined by $(\theta_k \sim_B \theta_\ell)(\sigma_{k+\ell}) = B(\theta_k((\sigma_{k+\ell})_{front})), \theta_\ell((\sigma_{k+\ell})_{back})$. Clearly, $\sim_B$ has the extension
\[ \sim_B: H^k(\Sigma; G_{Ado}) \times H^\ell(\Sigma; G_{Ado}) \to H^{k+\ell}(\Sigma; \mathbb{R}). \]

Let us denote by $K'$ the dual cell-decomposition of $\Sigma$ corresponding to the cell decomposition $K$. Suppose that cells $\sigma \in K$, $\sigma' \in K'$ meet at most once and also the diameter of each cell is less than, say, half of the injectivity radius of $\Sigma$. By the fact that the Reidemeister torsion is invariant under subdivision, this assumption is not loss of generality. Let $c'_p$ be the basis of $C_p(K'; \mathbb{Z})$ corresponding to the basis $c_p$ of $C_p(K; \mathbb{Z})$, and let $c'_p = c_p \otimes gA$ be the corresponding basis for $C_p(K'; G_{Ado})$.

We have the intersection form
\[ (\cdot, \cdot)_{k,2-k} : C_k(K; G_{Ado}) \times C_{2-k}(K'; G_{Ado}) \to \mathbb{R} \quad (4.1) \]
defined by
\[ (\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{k,2-k} = \sum_{\gamma \in \pi_1(\Sigma)} \sigma_1(\gamma \cdot \sigma_2) B(t_1, \gamma \cdot t_2), \]
where $\cdot, \cdot$ is the intersection number pairing. Note that $(\cdot, \cdot)_{k,2-k}$ are $\partial$-compatible because the intersection number pairing $\cdot$ is compatible with the usual boundary operator in the sense $(\partial \alpha, \beta) = (-1)^{|\alpha|}(\partial \beta)$, where $|\alpha|$ is the dimension of the cell $\alpha$. Since the intersection number form $\cdot, \cdot$ is anti-symmetric and $B$ is invariant under adjoint action, then $(\cdot, \cdot)_{k,2-k}$ is anti-symmetric.

By the independence of the twisted homologies from the cell-decomposition, we get the non-degenerate anti-symmetric form
\[ (\cdot, \cdot)_{k,2-k} : H_k(\Sigma; G_{Ado}) \times H_{2-k}(\Sigma; G_{Ado}) \to \mathbb{R}. \quad (4.2) \]
Combining the isomorphisms induced by the Kronecker pairing and the intersection form, we obtain the Poincaré duality isomorphisms
\[ PD : H_k(\Sigma; G_{Ado}) \cong H_{2-k}(\Sigma; G_{Ado})^* \cong H^{2-k}(\Sigma; G_{Ado}). \]
Thus, for $k = 0, 1, 2$, we have the following commutative diagram
\[
\begin{array}{ccc}
H^{2-k}(\Sigma; G_{Ado}) & \times & H^k(\Sigma; G_{Ado}) \\
\uparrow PD & & \uparrow PD \\
H_k(\Sigma; G_{Ado}) & \times & H_{2-k}(\Sigma; G_{Ado}) \\
\end{array} \quad (\cup_{k,2-k} \quad \mathbb{R}).
\]
Here, the isomorphism $\mathbb{R} \to H^2(\Sigma; \mathbb{R})$ sends 1 to the fundamental class of $H^2(\Sigma; \mathbb{R})$.

By the irreducibility of $g$, we have $H_0(\Sigma; G_{Ado})$, $H_2(\Sigma; G_{Ado})$, $H^0(\Sigma; G_{Ado})$, and $H^2(\Sigma; G_{Ado})$ are all zero. Hence,
\[
\begin{array}{ccc}
H^1(\Sigma; G_{Ado}) & \times & H^1(\Sigma; G_{Ado}) \\
\uparrow PD & & \uparrow PD \\
H_1(\Sigma; G_{Ado}) & \times & H_1(\Sigma; G_{Ado}) \\
\end{array} \quad (\cup_{1,1} \quad \mathbb{R}).
\quad (4.3)
\]
Recall that $\omega_B : H^1(\Sigma; G_{Ado}) \times H^1(\Sigma; G_{Ado}) \cong H^2(\Sigma; \mathbb{R}) \xrightarrow{f_B} \mathbb{R}$ is called the Atiyah-Bott-Goldman symplectic form for the Lie group $G$. Note also that from
the one that once composed with adjoint representation of $G$ is a sum of irreducible representations. 

\[
\begin{pmatrix}
0 & (\cdot, \cdot)_{1,1} \\
-(\cdot, \cdot)_{1,1} & 0
\end{pmatrix}
\]

and $(\cdot, \cdot)_{1,1}$ is the intersection form \((4.2)\) for $k = 1$. Then, from Lemma \[2.2\] Theorem \[2.4\] independence of the Reidemeister torsion from the cell-decomposition of $\Sigma$, and the fact that $D_\ast$ is a symplectic chain complex it follows that

\[
T(D_\ast, \{c_p \oplus c_p'\}_{p=0}^2, \{0 \oplus 0, h_1 \oplus h_1, 0 \oplus 0\}) = \sqrt{\Delta_{1,1}(D_\ast)} \cdot (4.4)
\]

Since the intersection form \((4.2)\) for $k = 1$ is non-degenerate, then equation \((4.4)\) becomes

\[
T(D_\ast, \{c_p \oplus c_p'\}_{p=0}^2, \{0 \oplus 0, h_1 \oplus h_1, 0 \oplus 0\}) = \Delta_{1,1}(C_\ast)^{(1)} \cdot (4.5)
\]

Let us consider the short-exact sequence

\[
0 \to C_\ast \to D_\ast = C_\ast \oplus C'_\ast \to C'_\ast \to 0.
\]

Here, $C_\ast \to D_\ast$ denotes the inclusion, $D_\ast \to C'_\ast$ denotes the projection. Clearly, the bases $c_p$ of $C_p, c_p \oplus c_p'$ of $D_\ast,$ and $c_p'$ of $C'_\ast$ are compatible. By Lemma \[2.2\] and the independence of the Reidemeister torsion from the cell-decomposition of $\Sigma$, we have

\[
T(D_\ast, \{c_p \oplus c_p'\}_{p=0}^2, \{0 \oplus 0, h_1 \oplus h_1, 0 \oplus 0\}) = \left( T(C_\ast, \{c_p\}_{p=0}^2, \{0, h_1, 0\}) \right)^2 \cdot (4.6)
\]

Thus, combining equations \((4.5)\) and \((4.6)\), we obtain

\[
T(C_\ast, \{c_p\}_{p=0}^2, \{0, h_1, 0\}) = \sqrt{\Delta_{1,1}(C_\ast)} \cdot (4.7)
\]

The fact that $-B$ is the dual of the intersection pairing $(\cdot, \cdot)_{1,1}$ and Lemma \[4.2\] yield that

\[
T(C_\ast, \{c_p\}_{p=0}^2, \{0, h_1, 0\}) = \sqrt{\det \Omega_B} \cdot (4.8)
\]

This concludes the proof of Theorem \[4.3\] \hfill \Box

**Corollary 4.4.** Since every Anosov representation is 1-1, discrete, irreducible, and purely loxodromic \[19\], then Theorem \[4.3\] also holds for Anosov representations.

5. Application: A Volume Element on Some Hitchin Components

For a closed oriented Riemann surface $\Sigma$ with genus $g > 1$ and a semi-simple Lie group $G$, let us denote by $\text{Hom}(\pi_1(\Sigma), G)$ the set of all homomorphisms from the fundamental group $\pi_1(\Sigma)$ of $\Sigma$ to $G$.

Let us consider the orbit space $\text{Hom}(\pi_1(\Sigma), G)/G$, where the action of $G$ on $\text{Hom}(\pi_1(\Sigma), G)$ by conjugation i.e. $g \cdot g(\gamma) = g g(\gamma) g^{-1}$, for $g \in G, \gamma \in \text{Hom}(\pi_1(\Sigma), G)$, and $\gamma \in \pi_1(\Sigma)$. It is well known that this is a real analytic variety. Moreover, for algebraic $G$, $\text{Hom}(\pi_1(\Sigma), G)/G$ is also algebraic. This orbit space is not necessarily Hausdorff (cf., e.g. \[10\]) but the space $\text{Rep}(\pi_1(\Sigma), G) = \text{Hom}^+(\pi_1(\Sigma), G)/G$ of all reductive representations of $\pi_1(\Sigma)$ in $G$ is Hausdorff. A reductive representation is the one that once composed with adjoint representation of $G$ on its Lie algebra $G$ is a sum of irreducible representations.
Teichmüller space Teich(Σ) of Σ is the space of isotopy classes of complex structures on Σ. A complex structure on Σ is a homotopy equivalence of a homeomorphism \( f : Σ \to S \). Here, \( S \) is a Riemann surface, and two such homeomorphisms \( f : Σ \to S, f' : Σ \to S' \) are said to be equivalent, if there exists a conformal diffeomorphism \( g : S \to S' \) so that \((f')^{-1} \circ g \circ f \) is isotopic to the identity map on \( Σ \).

One can lift a complex structure on \( Σ \) to a complex structure on the universal covering \( \tilde{Σ} \) of \( Σ \). By the Uniformization Theorem, \( \tilde{Σ} \) is biholomorphic to the upper half-plane \( \mathbb{H}^2 \subset \mathbb{C} \). It is well known that each biholomorphic homeomorphism of \( \mathbb{H}^2 \) is of the form \( f(z) = (az+b)/(cz+d) \) with \( a, b, c, d \in \mathbb{R}, ad-bc = 1 \). This yields a discrete, faithful homomorphism from \( π_1(Σ) \) to \( \text{PSL}(2, \mathbb{R}) \). This homomorphism is also well defined up to conjugation by the orientation preserving isometries of \( \mathbb{H}^2 \). Thus, one can identify \( \text{Teich}(Σ) \) with the Fricke space, i.e. the set \( \text{Rep}_\text{diff}(π_1(Σ), \text{PSL}(2, \mathbb{R})) \) of discrete faithful representations from \( π_1(Σ) \) to \( \text{PSL}(2, \mathbb{R}) \).

Fricke space is a connected component of \( \text{Rep}(π_1(Σ), \text{PSL}(2, \mathbb{R})) \). Openness follows from [35], closedness from [6, 24], and connectedness from the Uniformization Theorem together with the identification of \( \text{Teich}(Σ) \) as a cell.

For a finite cover \( G \) of \( \text{PSL}(2, \mathbb{R}) \), W. Goldman investigated the connected components of the representation space \( \text{Hom}(π_1(Σ), G)/G \) [11]. He proved that there exist \( 4g-3 \) connected components of \( \text{Hom}(π_1(Σ), \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R}) \). There exist two homeomorphic components, called Teichmüller spaces, which are homeomorphic to \( \mathbb{R} \times \chi(Σ) \dim \text{PSL}(2, \mathbb{R}) \).

For a split real form \( G \) of a semi-simple Lie group, N. Hitchin investigated the connected components of \( \text{Rep}(π_1(Σ), G) \) in [15] by using techniques of Higgs bundle. He proved that there exists an interesting connected component not detected by characteristic classes. He called it as Teichmüller component but it is called now Hitchin component.

A Hitchin component \( \text{Rep}_{\text{Hitchin}}(π_1(Σ), G) \) of \( \text{Rep}(π_1(Σ), G) \) is the connected component containing Fuchsian representations, i.e. representations of the form \( \varphi \circ \iota \), where \( \varphi : π_1(Σ) \to \text{PSL}(2, \mathbb{R}) \) is Fuchsian, \( \iota : \text{PSL}(2, \mathbb{R}) \to G \) is the representation corresponding to the 3-dimensional principal subgroup of \( B \). Kostant [17]. For \( G = \text{PSp}(2n, \mathbb{R}), \iota \) denotes the \( 2n \)-dimensional irreducible representation corresponding to symmetric power \( \text{Sym}^{2n-1}(\mathbb{R}^2) \).

This enables one to identify the Fricke space and thus \( \text{Teich}(Σ) \) by a subset of \( \text{Rep}(π_1(Σ), G) \). N. Hitchin proved in [15] that each Hitchin component is homeomorphic to a ball of dimension \( (6g-6) \dim G \). Recall that it was proved by F.Labourie in [19] that the set \( \text{Rep}_{\text{Hitchin}}(π_1(Σ), G) \) of Hitchin representations is a subset of \( \text{Rep}_{\text{Anosov}}(π_1(Σ), G) \).

Applying Theorem [4,3] and Corollary [4,4] we have the following result.

**Corollary 5.1.** Let \( Σ \) be a closed orientable surface of genus \( g \geq 2 \) and \( g \) be in \( \text{Rep}_{\text{Hitchin}}(π_1(Σ), G) \). Let \( K \) be a cell-decomposition of \( Σ \), \( c_p \) be the geometric bases of \( C_p(K; G_{\text{Ad}_{G}}) \), \( p = 0, 1, 2 \), and \( h_1 \) is a basis for \( H_1(Σ; G_{\text{Ad}_{G}}) \). Then, we have

\[
T(C_*(K; G_{\text{Ad}_{G}}), \{c_p\}_{p=0}^2, \{0, h_1, 0\}) = \sqrt{\det Ω_{\text{Ad}}}. \]

Moreover, it is a volume element on the Hitchin component \( \text{Rep}_{\text{Hitchin}}(π_1(Σ), G) \). Here, \( G \) is one of \( \{\text{PSp}(2n, \mathbb{R})(n \geq 2), \text{PSO}(n, n+1)n \geq 2, \text{PSO}(n, n+1)(n \geq 3)\} \) and \( G \) is the corresponding Lie algebra with the non-degenerate Killing form \( B \) and
\[ \omega_B : H^1(\Sigma; G_{Ado}) \times H^1(\Sigma; G_{Ado}) \xrightarrow{\sim} H^2(\Sigma; R) \xrightarrow{f_{\omega}} R \] is the Atiyah-Bott-Goldman symplectic form for \( G \).

**Proof.** From the fact that \( \varrho \) belongs to \( \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), G) \) it follows that it is discrete, faithful, irreducible, and purely loxodromic (cf. \([2,8,19]\)). The irreducibility yields that \( H^0(\Sigma, G_{Ado}) \) and \( H^2(\Sigma, G_{Ado}) \) are both zero. Hence, by Theorem \ref{thm:1} we obtain

\[ \mathbb{T}(C_*(K; G_{Ado}), \{ c_p \}_{p=0}^2, \{ 0, h_1, 0 \}) = \sqrt{\det \Omega_{\omega_B}}. \]

It is well known that \( H^1(\Sigma, G_{Ado}) \), \( H^1(\Sigma, G_{Ado}) \) can be identified respectively with the tangent space \( T_\varrho \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), G) \), cotangent space \( T^{*}_\varrho \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), G) \) of \( \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), G) \) (cf., e.g. \([10]\)). Recall also that Reidemeister torsion \( \mathbb{T}(A_\varrho) \) of a general chain complex \( A_\varrho \) of length \( n \) belongs to \( \otimes_{p=0}^n (\det(H_{\varrho}(A_\varrho))^{(-1)^{p+1}} \mathbb{R} \) (\([27,36]\)). Here, \( \det(H_{\varrho}(A_\varrho)) \) is the top exterior power \( \bigwedge^{\dim H_{\varrho}(A_\varrho)} H_{\varrho}(A_\varrho) \) of \( H_{\varrho}(A_\varrho) \) and \( \det(H_{\varrho}(A_\varrho))^{-1} \) is the dual of \( \det(H_{\varrho}(A_\varrho)) \). Thus, we get a volume element on the Hitchin component \( \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), G) \) of \( \text{Rep}(\pi_1(\Sigma), G) \).

Let us note that since \( \text{Teich}(\Sigma) \subset \text{Rep}_{\text{Hitchin}}(\pi_1(\Sigma), G) \), Corollary \ref{cor:5} is also valid for \( \text{Teich}(\Sigma) \) representations.

For the isomorphism \( T_\varrho \text{Teich}(\Sigma) \cong H^1(\Sigma; G_{Ado}) \), in \([10]\), Goldman proved that \( \omega_{\text{PSL}(2, \mathbb{R})} : H^1(\Sigma; G_{Ado}) \times H^1(\Sigma; G_{Ado}) \to \mathbb{R} \) and Weil-Petersson 2–form differ only by a constant multiple. More precisely,

\[ \omega_{\text{WP}} = -8\omega_{\text{PSL}(2, \mathbb{R})}. \]

Bonahon parametrized the Teichmüller space of \( \Sigma \) by using a maximal geodesic lamination \( \lambda \) on \( \Sigma \) \([1]\). Geodesic laminations are generalizations of deformation classes of simple closed curves on \( \Sigma \). More precisely, a geodesic lamination \( \lambda \) on the surface \( \Sigma \) is by definition a closed subset of \( \Sigma \) which can be decomposed into family of disjoint simple geodesics, possibly infinite, called its leaves. The geodesic lamination is maximal if it is maximal with respect to inclusion; this is equivalent to the property that the complement \( \Sigma - \lambda \) is union of finitely many triangles with vertices at infinity.

The real-analytical parametrization given by Bonahon identifies \( \text{Teich}(\Sigma) \) to an open convex cone in the vector space \( \mathcal{H}(\lambda, \mathbb{R}) \) of all transverse cocycles for \( \lambda \). In particular, at each \( \varrho \in \text{Teich}(\Sigma) \), the tangent space \( T_\varrho \text{Teich}(\Sigma) \) is now identified with \( \mathcal{H}(\lambda, \mathbb{R}) \), which is a real vector space of dimension \( 3|\chi(\Sigma)| \).

A transverse cocycle \( \sigma \) for \( \lambda \) on \( \Sigma \) is a real-valued function on the set of all arcs \( k \) transverse to (the leaves) of \( \lambda \) with the following properties:

- \( \sigma \) is finitely additive, i.e. \( \sigma(k) = \sigma(k_1) + \sigma(k_2) \), whenever the arc \( k \) transverse to \( \lambda \) is decomposed into two subarcs \( k_1, k_2 \) with disjoint interiors,
- \( \sigma \) is invariant under the homotopy of arcs transverse to \( \lambda \), i.e. \( \sigma(k) = \sigma(k') \) whenever the transverse arc \( k \) is deformed to arc \( k' \) by a family of arcs which are all transverse to the leaves of \( \lambda \).
The space $\mathcal{H}(\lambda; \mathbb{R})$ has also anti-symmetric bilinear form, namely the Thurston symplectic form $\omega_{\text{Thurston}}$. Let $\lambda$ be a maximal geodesic lamination on $\Sigma$ and $\Phi$ be a fattened train-track carrying the maximal geodesic lamination.

The Thurston symplectic form is the anti-symmetric bilinear form $\omega_{\text{Thurston}} : \mathcal{H}(\lambda; \mathbb{R}) \times \mathcal{H}(\lambda; \mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\omega_{\text{Thurston}}(\sigma_1, \sigma_2) = \frac{1}{2} \sum_s \det \begin{bmatrix} \sigma_1(e_s^{\text{left}}) & \sigma_1(e_s^{\text{right}}) \\ \sigma_2(e_s^{\text{left}}) & \sigma_2(e_s^{\text{right}}) \end{bmatrix},$$

where $\sigma_i(e) \in \mathbb{R}$ is the weight associated to the edge $e$ by the transverse cocycle $\sigma_i$. Note that, $\omega_{\text{Thurston}}$ is actually independent of the train-track $\Phi$.

It is proved in [26] that up to a multiplicative constant, $\omega_{\text{Thurston}}$ is the same as $\omega_{\text{PSL}(2, \mathbb{R})}$, and hence is in the same equivalence class of $\omega_{\text{WP}}$. More precisely, for the identification $T_e\text{Teich}(\Sigma) \cong \mathcal{H}(\lambda; \mathbb{R})$, the following is valid $\omega_{\text{PSL}(2, \mathbb{R})} = 2\omega_{\text{Thurston}}$.

As a final word on this study, Reidemeister torsion of $\varrho \in \text{Teich}(\Sigma)$ can be expressed in terms of $\omega_{\text{Thurston}}$.

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