STABILIZATION OF MULTIDIMENSIONAL WAVE EQUATION WITH LOCALLY BOUNDARY FRACTIONAL DISSIPATION LAW UNDER GEOMETRIC CONDITIONS

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Abstract. In this paper, we consider a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. First, combining a general criteria of Arendt and Batty with Holmgrens theorem we show the strong stability of our system in the absence of the compactness of the resolvent and without any additional geometric conditions. Next, we show that our system is not uniformly stable in general, since it is the case of the interval. Hence, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method. Indeed, by assuming that the boundary control region satisfy some geometric conditions and by using the exponential decay of the wave equation with a standard damping, we establish a polynomial energy decay rate for smooth solutions, which depends on the order of the fractional derivative.

1. Introduction. Let \( \Omega \) be an open bounded set of \( \mathbb{R}^d \), \( d \geq 2 \), with boundary \( \Gamma \) of class \( C^2 \), consisting of a clamped part \( \Gamma_0 \neq \emptyset \) and a rimmed part \( \Gamma_1 \neq \emptyset \) such that \( \Gamma_0 \cap \Gamma_1 = \emptyset \). We consider the multidimensional wave equation

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) &= 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \quad (1) \\
\frac{\partial u}{\partial t}(x,t) &= 0, \quad (x,t) \in \Gamma_0 \times \mathbb{R}^+, \quad (2) \\
\frac{\partial^\alpha}{\partial^\alpha t} u(x,t) + \gamma \partial^\alpha_{\alpha,\eta} u(x,t) &= 0, \quad (x,t) \in \Gamma_1 \times \mathbb{R}^+ 
\end{align*}
\]

where \( \nu \) is the unit outward normal vector along the boundary \( \Gamma_1 \) and \( \gamma \) is a positive constant involved in the boundary control. The notation \( \partial^\alpha_{\alpha,\eta} \) stands for the generalized Caputo’s fractional derivative see [14] of order \( \alpha \) with respect to the time variable and is defined by

\[
\partial^\alpha_{\alpha,\eta} \omega(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(x,s)ds, \quad 0 < \alpha < 1, \ \eta \geq 0.
\]

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The system (1)-(3) is considered with initial conditions
\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \]  

(4)

The fractional derivative operator of order \( \alpha, 0 < \alpha < 1, \) is defined by
\[ D^\alpha f(t) = \int_0^t \frac{(t - \tau)^{-\alpha}}{\Gamma(1 - \alpha)} df(t) d\tau. \]  

(5)

The fractional differentiation is inverse operation of fractional integration that is defined by
\[ I^\alpha f(t) = \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \quad 0 < \alpha < 1. \]  

(6)

From equations (5) and (6), clearly that
\[ D^\alpha f = I^{1-\alpha} D f. \]  

(7)

Now, we present marginally distinctive forms of (5) and (6). These exponentially modified fractional integro-differential operators and will be denoted as follows
\[ D^{\alpha,\eta} f(t) = \int_0^t \frac{(t - \tau)^{-\alpha} e^{-\eta(t-\tau)}}{\Gamma(1 - \alpha)} df(t) d\tau. \]  

(8)

and
\[ I^{\alpha,\eta} f(t) = \int_0^t \frac{(t - \tau)^{\alpha-1} e^{-\eta(t-\tau)}}{\Gamma(\alpha)} f(\tau) d\tau. \]  

(9)

Note that the two operators \( D^\alpha \) and \( D^{\alpha,\eta} \) differ just by their Kernels. \( D^{\alpha,\eta} \) is merely Caputo's fractional derivative operator, expect for its exponential factor. Thus, similar to identity (7), we do have
\[ D^{\alpha,\eta} f = I^{1-\alpha,\eta} D f. \]  

(10)

The order of our derivatives is between 0 and 1.

The boundary fractional damping of the type \( \partial_t^{\alpha,\eta} \) where \( 0 < \alpha < 1, \eta \geq 0 \) arising from the material property has been used in several applications such as in physical, chemical, biological, ecological phenomena. The order of our derivatives is between 0 and 1. Very little attention has been paid to this type of feedback. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels \( (t^{-\alpha}, 0 < \alpha < 1). \) This leads to substantial mathematical difficulties because all the previous methods developed for convolution terms with regular and/or integrable kernels are no longer valid.

In the last year, fractional differential equations have become popular among scientists in order to model various stable physical phenomena with a slow decay rate, say that are not uniformly stable (i.e., are not of exponential type).

It has been shown (see [24] and [26]) that, as \( \partial_t^{\alpha,\eta} \), the fractional derivative \( \partial_t^{\alpha} \) forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations. Boundary dissipations of fractional order or, in general, of convolution type are not only important from the theoretical point of view but also for applications. They naturally arise in physical, chemical, biological, ecological phenomena see for example [29], [33] and references therein. They are used to describe memory and hereditary properties of various materials and processes. For example, in viscoelasticity, due to the nature of the material micro-structure, both elastic solid and viscous fluid like response qualities are involved. Using Boltzmann assumption, we end up with a stress-strain relationship
defined by a time convolution. Viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [5]-[6]-[7] and [23]). In our case, the fractional dissipations may come from a viscoelastic surface of the beam or simply describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations (see [25] and [26]).

In [15], Zhang and Dai considered the multidimensional wave equation with boundary source term and fractional dissipation defined by

\[
\begin{align*}
\partial_t u(x,t) - \Delta u(x,t) &= 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\
u(x,t) &= 0, & (x,t) \in \Gamma_0 \times \mathbb{R}^+, \\
\partial_t \nu(x,t) + \partial_t^\alpha u(x,t) &= |u(x,t)|^{m-1}u(x,t), & (x,t) \in \Gamma_1 \times \mathbb{R}^+, \\
u(x,0) &= u_0(x), & x \in \Omega, \\
u_t(x,0) &= u_1(x), & x \in \Omega
\end{align*}
\]

where \( m > 1 \). They proved by Fourier transforms and the Hardy-Littelwood-Sobolev inequality the exponential stability for sufficiently large initial data.

In [2], Benaissa and al. considered the Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type defined by

\[
\begin{align*}
\varphi_{tt}(x,t) + \varphi_{xxxx}(x,t) &= 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\
\varphi(0,t) &= \varphi_x(0,t) = 0, & t \in \mathbb{R}^+, \\
\varphi_{xx}(L,t) &= 0, & t \in \mathbb{R}^+, \\
\varphi_{xxx}(L,t) &= \gamma \partial_t^\alpha \eta \varphi(L,t), & t \in \mathbb{R}^+
\end{align*}
\]

where \( 0 < \alpha < 1, \eta \geq 0 \) and \( \gamma > 0 \). They proved, under the condition \( \eta > 0 \), by a spectral analysis, the non uniform stability. On the other hand, for \( \eta > 0 \), they also proved that the energy of system (12) decay as time goes to infinity as \( t^{-\frac{1}{1-\alpha}} \).

In [25], B. Mbodje investigate the asymptotic behavior of solutions with the system

\[
\begin{align*}
\partial_t^2 u(x,t) - \partial_x^2 u(x,t) &= 0, & (x,t) \in (0,1) \times \mathbb{R}^+, \\
u(0,t) &= 0, & t \in \mathbb{R}^+, \\
\partial_x u(1,t) &= -\gamma \partial_t^\alpha \eta u(1,t), & t \in \mathbb{R}^+, \alpha, \eta \geq 0, \\
u(x,0) &= u_0(x), & x \in (0,1), \\
u_t(x,0) &= v_0(x), & x \in (0,1).
\end{align*}
\]

He proved that the associated semigroup is not exponentially stable, but only strongly asymptotically and the energy of this system will decay, as time goes to infinity, as \( t^{-\frac{1}{2}} \). In [3], Alabau-Boussouira and al. have studied the exponential and polynomial stability of a wave equation for boundary memory damping with singular kernels.

This paper is organized as follows: In Subsection 2.1, we reformulate the system (1)-(4) into an augmented system by coupling the wave equation with a suitable diffusion equation and we prove the well-posedness of our system by semigroup approach. In the Subsection 2.2, combining a general criteria of Arendt and Batty with Holmgren’s theorem we show that the strong stability of our system in the absence of the compactness of the resolvent and without any additional geometric conditions. In Subsection 2.3, We show that our system is not uniformly stable in general, since it is the case of the interval, more precisely we show that an
infinite number of eigenvalues approach the imaginary axis. In Section 3, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method. Indeed, by assuming that the boundary control region satisfy some geometric conditions (see Remark 3.1 below) and by using the exponential decay of the wave equation with a standard damping, we establish a polynomial energy decay for smooth solution as type $t^{-\frac{1}{1-\alpha}}$.

2. Well-posedness - strong stability and non uniform stability. In this section, we will study the strong stability of system (1)-(4) in the absence of the compactness of the resolvent and without any additional geometric conditions on the domain $\Omega$. First, we will study the existence, uniqueness and regularity of the solution of our system.

2.1. Augmented model and well-Posedness. Firstly, we reformulate system (1)-(4) into an augmented system and we generalize the Theorem 2 in [25] from dimension 1 to dimension $d$. For this aim, we need the following results

**Proposition 1.** Let $\mu$ be the function defined by

$$\mu(\xi) = |\xi|^\frac{2\alpha - d}{2}, \quad \xi \in \mathbb{R}^d \quad \text{and} \quad 0 < \alpha < 1.$$  

Then the relation between the 'input' $U$ and the 'output' $O$ of the following system

$$\partial_t \omega(x, t, \xi) + (|\xi|^2 + \eta) \omega(x, t, \xi) - U(x, t) \mu(\xi) = 0, \quad x \in \Gamma_1, \quad t > 0, \quad \xi \in \mathbb{R}^d, \quad \omega(x, 0, \xi) = 0, \quad x \in \Gamma_1, \quad \xi \in \mathbb{R}^d,$$

is given by

$$O(x, t) = \frac{2 \sin(\alpha \pi) \Gamma \left(\frac{d}{2} + 1\right)}{d\pi^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} \mu(\xi) \omega(x, t, \xi) d\xi, \quad x \in \Gamma_1, \quad t > 0, \quad \xi \in \mathbb{R}^d,$$

where $D^{\alpha, \eta}$ and $I^{1-\alpha, \eta}$ are given by (8) and (9) respectively.

**Proof.** We distinguish two cases:

**Step 1.** The case $\eta = 0$. Using (15) and (16), we get

$$\omega(x, t, \xi) = \int_0^t \mu(\xi) e^{-|\xi|^2(t-\tau)} U(x, \tau) d\tau.$$  

(19)

It follows, from (17), that

$$O(x, t) = \frac{2 \sin(\alpha \pi) \Gamma \left(\frac{d}{2} + 1\right)}{d\pi^{\frac{d+1}{2}}} \int_0^t U(x, \tau) \int_{\mathbb{R}^d} |\xi|^{2\alpha - d} e^{-|\xi|^2(t-\tau)} d\xi d\tau,$$

then

$$O(x, t) = \pi^{-1} \sin(\alpha \pi) \int_0^t \left(2 \int_0^{+\infty} \rho^{2\alpha - 1} e^{-\rho^2(t-\tau)} d\rho\right) U(x, \tau) d\tau.$$  

(20)

Using the fact $\pi^{-1} \sin(\alpha \pi) = \frac{1}{\Gamma(\alpha) \Gamma(1-\alpha)}$ in (20), we get (18).

**Step 2.** Suppose $\eta > 0$. By simply effecting the following change of function

$$\omega(x, t, \xi) = e^{-\eta \xi} \zeta(x, t, \xi)$$  

(21)

and using step 1, we get (18).
Now, using Proposition 1, system (1)-(4) may be recast into the following augmented model:

\[ u_{tt}(x, t) - \Delta u(x, t) = 0, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad (22) \]

\[ u(x, t) = 0, \quad (x, t) \in \Gamma_0 \times \mathbb{R}^+, \quad (23) \]

\[ \partial_t \omega(x, t, \xi) + (|\xi|^2 + \eta) \omega(x, t, \xi) - \mu(\xi) \partial_t u(x, t) = 0, \quad (x, t, \xi) \in \Gamma_1 \times \mathbb{R}^+ \times \mathbb{R}^d. \quad (24) \]

\[ \frac{\partial u}{\partial \nu}(x, t) + \gamma \int_{\mathbb{R}^d} \mu(\xi) \omega(x, t, \xi) d\xi = 0, \quad (x, t, \xi) \in \Gamma_1 \times \mathbb{R}^+ \times \mathbb{R}^d. \quad (25) \]

where \( \gamma \) is a positive constant, \( \eta \geq 0 \) and \( \kappa = \frac{2 \sin(\alpha \pi) \Gamma(\frac{d}{2} + 1)}{d \pi^{\frac{d}{2} + 1}} \). Finally, system (22)-(25) is considered with the following initial conditions

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \omega(x, 0, \xi) = 0, \quad x \in \Omega, \quad \xi \in \mathbb{R}^d. \quad (26) \]

Our main interest is the existence, uniqueness and regularity of the solution to this system. We define the Hilbert space

\[ \mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1 \times \mathbb{R}^d), \quad (27) \]

equipped with the following inner product

\[ \langle (u, v, \omega), (\tilde{u}, \tilde{v}, \tilde{\omega}) \rangle_{\mathcal{H}} = \int_{\Omega} (v(x) \tilde{v}(x) + \nabla u(x) \nabla \tilde{u}(x)) \, dx + \gamma \kappa \int_{\Gamma_1} \int_{\mathbb{R}^d} \omega(x, \xi) \tilde{\omega}(x, \xi) d\xi d\Gamma \]

where \( H^1_{\Gamma_0}(\Omega) \) is given by

\[ H^1_{\Gamma_0}(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0 \}. \]

The energy of the solution of system (22)-(26) is defined by:

\[ E(t) = \frac{1}{2} \| (u, u_t, \omega) \|^2_{\mathcal{H}}. \quad (28) \]

**Lemma 2.1.** Let \( U = (u, u_t, \omega) \) be a regular solution of problem (22)-(26). Then, the functional energy defined in equation (28) satisfies

\[ \frac{d}{dt} E(t) = -\gamma \kappa \int_{\Gamma_1} \int_{\mathbb{R}^d} (|\xi|^2 + \eta) |\omega(x, \xi, t)|^2 d\xi d\Gamma. \quad (29) \]

**Proof.** Multiplying equation (22) by \( \tilde{u}_t \), using integration by parts over \( \Omega \) and equation (25), we get

\[ \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} (|u_t|^2 + |\nabla u|^2) dx \right) + \gamma \kappa \Re \left( \int_{\Gamma_1} \tilde{u}_t \int_{\mathbb{R}^d} \mu(\xi) \omega(x, t, \xi) d\xi d\Gamma \right) = 0. \quad (30) \]

Multiplying equation (24) by \( \gamma \kappa \tilde{\omega}(x, t, \xi) \) and integrating over \( \Gamma_1 \times \mathbb{R}^d \), we get

\[ \frac{1}{2} \frac{d}{dt} \left( \gamma \kappa \int_{\Gamma_1} \int_{\mathbb{R}^d} |\omega|^2 d\xi \right) + \gamma \kappa \int_{\Gamma_1} \int_{\mathbb{R}^d} (|\xi|^2 + \eta) |\omega(x, t, \xi)|^2 d\xi d\Gamma = \gamma \kappa \Re \left( \int_{\Gamma_1} \tilde{u}_t \int_{\mathbb{R}^d} \mu(\xi) \tilde{\omega}(x, t, \xi) d\xi d\Gamma \right). \quad (31) \]

Combining equations (30)-(31), we obtain

\[ \frac{d}{dt} E(t) = -\gamma \kappa \int_{\Gamma_1} \int_{\mathbb{R}^d} (|\xi|^2 + \eta) |\omega(x, \xi, t)|^2 d\xi d\Gamma. \]
This completes the proof.

Hence, from Lemma 2.1, system (22)-(26) is dissipative in the sense that its energy is a non-increasing function of the time variable $t$. Now, we define the linear unbounded operator $A$ by

$$D(A) = \begin{cases} 
U = (u, v, \omega)^\top \in H; & \Delta u \in L^2(\Omega), \ v \in H^1_{\Gamma_0}(\Omega), \\
|\xi|\omega \in L^2(\Gamma_1 \times \mathbb{R}^d), & -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1, \mu(\xi)} \in L^2(\Gamma_1 \times \mathbb{R}^d), \\
\frac{\partial u}{\partial \nu}|_{\Gamma_1} = -\gamma \kappa \int_{\mathbb{R}^d} \mu(\xi)\omega(\cdot, \xi) \, d\xi
\end{cases}$$

and

$$A(u, v, \omega)^\top = (v, \Delta u, -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1, \mu(\xi)})^\top.$$  

By denoting $v = u_t$ and $U_0 = (u_0, v_0, w_0)^\top$, system (22)-(26) can be written as an abstract linear evolution equation on the space $H$

$$U_t = AU, \quad U(0) = U_0.$$  

It is known that operator $A$ is m-dissipative on $H$ and consequently, generates a $C_0$-semigroup of contractions $e^{tA}$ following Lumer-Phillips theorem (see [21, 30]). Then the solution to the evolution equation (33) admits the following representation:

$$U(t) = e^{tA}U_0, \quad t \geq 0,$$

which leads to the well-posedness of (33). Hence, semigroup theory allows to show the next existence and uniqueness results:

**Theorem 2.2.** For any initial data $U_0 \in H$, the problem (33) admits a unique weak solution

$$U(t) \in C^0(\mathbb{R}^+; H).$$

Moreover, if $U_0 \in D(A)$, then the problem (33) admits a unique strong solution

$$U(t) \in C^1(\mathbb{R}^+, H) \cap C^0(\mathbb{R}^+, D(A)).$$

2.2. **Strong stability of the system.** In this part, we study the strong stability of system (22)-(26) in the sense that its energy converges to zero when $t$ goes to infinity for all initial data in $H$. It is easy to see that the resolvent of $A$ is not compact, then the classical methods such as Lasalle’s invariance principle [34] or the spectrum decomposition theory of Benchimol [11] are not applicable in this case. We use then a general criteria of Arendt-Batty [4], following to which a $C_0$-semigroup of contractions $e^{tA}$ in a Banach space is strongly stable, if $A$ has no pure imaginary eigenvalues and $\sigma(A) \cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result

**Theorem 2.3.** Assume that $\eta \geq 0$. Then the $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is strongly stable on the energy space $H$, i.e., for any $U_0 \in H$, we have

$$\lim_{t \to +\infty} \|e^{tA}U_0\|_H = 0.$$  

For the proof of Theorem 2.3, we need the following lemmas:

**Lemma 2.4.** Assume that $\eta \geq 0$. Then, for all $\lambda \in \mathbb{R}$, we have

$$\ker (i\lambda I - A) = \{0\}.$$
Proof. Let $U \in D(A)$ and $\lambda \in \mathbb{R}$, such that
\[ AU = i\lambda U. \] (34)
Equivalently, we have
\[ v = i\lambda u, \] (35)
\[ \Delta u = i\lambda v, \] (36)
\[ -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1}\mu(\xi) = i\lambda \omega. \] (37)
Next, a straightforward computation gives
\[ \Re(AU, U)_H = -\gamma \kappa \int_{\Gamma_1} \int_{\mathbb{R}^d} (|\xi|^2 + \eta)|\omega|^2 d\xi d\Gamma. \] (38)
Then, using (34) and (38), we deduce that
\[ \omega = 0 \text{ a.e. in } \Gamma_1 \times \mathbb{R}^d. \] (39)
It follows, from (32) and (37), that
\[ \frac{\partial u}{\partial \nu} = 0 \text{ and } v = 0 \text{ on } \Gamma_1. \] (40)
Thus, by eliminating $v$, the system (35)-(37) implies that
\[ \lambda^2 u + \Delta u = 0 \text{ in } \Omega, \] (41)
\[ u = 0 \text{ on } \Gamma_0, \] (42)
\[ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1. \] (43)
Now we distinguish two cases:

**Case 1.** If $\lambda = 0$, then a straightforward computation gives $u = 0$ and consequently, $U = 0$.

**Case 2.** If $\lambda \neq 0$, then using (35) and (40) we deduce that $u = 0$ on $\Gamma_1$. Therefore, using Holmgren's theorem, we deduce that $u = 0$ and consequently, $U = 0$. This completes the proof.

**Lemma 2.5.** Assume that $\eta = 0$. Then, the operator $-A$ is not invertible and consequently $0 \in \sigma(A)$.

**Proof.** First, let $\varphi_k \in H^1_{\Gamma_0}(\Omega)$ be an eigenfunction of the following problem
\[
\begin{cases}
-\Delta \varphi_k = \mu_k^2 \varphi_k, & \text{in } \Omega, \\
\varphi_k = 0, & \text{on } \Gamma_0, \\
\frac{\partial \varphi_k}{\partial \nu} = 0, & \text{on } \Gamma_1
\end{cases}
\] (44)
such that
\[ ||\varphi_k||^2_{H^1_{\Gamma_0}(\Omega)} = \int_{\Omega} |\nabla \varphi_k|^2 dx. \]
Next, define the vector $F = (\varphi_k, 0, 0)^\top \in \mathcal{H}$. Assume that there exists $U = (u, v, w)^\top \in D(A)$ such that
\[ -AU = F. \]
It follows that
\[ v = -\varphi_k \text{ in } \Omega, \quad -|\xi|^2 \omega + \mu(\xi)v = 0 \text{ on } \Gamma_1 \times \mathbb{R}^d \] (45)
and
\[
\begin{cases}
\Delta u = 0, \text{ in } \Omega, \\
u = 0, \text{ on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} + \gamma \kappa \int_{\mathbb{R}^d} \mu(\xi) \omega(\cdot, \xi) d\xi = 0, \text{ on } \Gamma_1.
\end{cases}
\] (46)

Note that \( \varphi_k|_{\Gamma_1} \neq 0 \) because, if not a unique continuation result allows to deduce \( \varphi_k = 0 \) in \( \Omega \). Now, from (45), we deduce that \( \omega(\cdot, \xi) = |\xi|^{\frac{d-1}{2}} \varphi_k|_{\Gamma_1} \). We easily check that for \( \alpha \in [0, 1] \), the function \( \omega(x, \xi) \notin L^2(\Gamma_1 \times \mathbb{R}^d) \). So, the assumption of the existence of \( U \) is false and consequently the operator \( -A \) is not invertible. This completes the proof.

**Lemma 2.6.** Assume that \( (\eta > 0) \) or \( (\eta = 0 \text{ and } \lambda \in \mathbb{R}^*) \). Then, for any \( f \in L^2(\Omega) \), the following problem
\[
\begin{cases}
\lambda^2 u + \Delta u = f, \text{ in } \Omega, \\
u = 0, \text{ on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} + (\lambda^2 c_1 + i\lambda c_2)u = 0, \text{ on } \Gamma_1
\end{cases}
\] (47)

admits a unique solution \( u \in H^1_{\Gamma_0}(\Omega) \), where
\[
\begin{align*}
c_1(\lambda, \eta) &= \frac{\sqrt{2}}{\lambda^2 + (|\xi|^2 + \eta)^2} \int_{\mathbb{R}^d} \mu(\xi) \omega(\cdot, \xi) d\xi, \\
c_2(\lambda, \eta) &= \frac{\sqrt{2}}{\lambda^2 + (|\xi|^2 + \eta)^2} \int_{\mathbb{R}^d} \mu(\xi)(|\xi|^2 + \eta) d\xi.
\end{align*}
\] (48)

**Proof.** First, it is easy to check that, if \( (\eta > 0) \) and \( \lambda \in \mathbb{R} \) or \( (\eta = 0 \text{ and } \lambda \in \mathbb{R}^*) \), then, for \( \alpha \in [0, 1] \), the coefficients \( c_1(\lambda, \eta) \) and \( c_2(\lambda, \eta) \) are well defined. Moreover, if \( \eta > 0 \) and \( \lambda = 0 \) then, using Lax-Milgram’s theorem we deduce that system (47) admits a unique solution \( u \in H^1_{\Gamma_0}(\Omega) \). Now, assume that \( \eta \geq 0 \) and \( \lambda \in \mathbb{R}^* \) and let us consider the following problem
\[
\begin{cases}
-\Delta u = f, \text{ in } \Omega, \\
u = 0, \text{ on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} + (\lambda^2 c_1 + i\lambda c_2)u = 0, \text{ on } \Gamma_1
\end{cases}
\] (49)

Let \( u = u_1 + iu_2, f = f_1 + if_2 \) and we separate the real and the imaginary part of (49), we obtain
\[
\begin{align*}
-\Delta u_1 &= f_1 \text{ in } \Omega, \\
-\Delta u_2 &= f_2 \text{ in } \Omega, \\
u_1 = u_2 &= 0 \text{ on } \Gamma_0, \\
\frac{\partial u_1}{\partial \nu} + \lambda^2 c_1 u_1 - \lambda c_2 u_2 &= 0 \text{ on } \Gamma_1, \\
\frac{\partial u_2}{\partial \nu} + \lambda^2 c_1 u_2 + \lambda c_2 u_1 &= 0 \text{ on } \Gamma_1
\end{align*}
\] (50)

Next, we give a variational formulation of (50). For this aim, find \( (u_1, u_2) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega) \) such that
\[
a((u_1, u_2), (\varphi_1, \varphi_2)) = L((\varphi_1, \varphi_2)), \quad \forall (\varphi_1, \varphi_2) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega),
\] (51)
where
\[ a((u_1, u_2), (\varphi_1, \varphi_2)) = \int_{\Omega} (\nabla u_1 \nabla \varphi_1 + \nabla u_2 \nabla \varphi_2) \, dx + c_1 \int_{\Gamma_1} (\lambda^2 u_1 \varphi_1 + \lambda^2 u_2 \varphi_2) \, d\Gamma_1 + c_2 \int_{\Gamma_1} (\lambda u_1 \varphi_2 - \lambda u_2 \varphi_1) \, d\Gamma_1, \]
and
\[ L((\varphi_1, \varphi_2)) = \int_{\Omega} (f_1 \varphi_1 + f_2 \varphi_2) \, dx. \]

It is clear that the bilinear form \( a \) is continuous and coercive on the space \((H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega))^2\) and the linear form \( L \) is continuous on the space \( H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega) \). Consequently, by Lax-Milligram’s theorem, the variational problem (51) admits a unique solution \((u_1, u_2) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)\). By choosing appropriated test functions in (51), we see that \((u_1, u_2)\) satisfies (50) and therefore problem (49) admits a unique solution \( u \in H^1_{\Gamma_0}(\Omega) \). In addition, we have (see [19])
\[ \|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}. \]  
(52)

It follows, from the compactness of the embedding \( H^1_{\Gamma_0}(\Omega) \subset L^2(\Omega) \), that the inverse operator \((-\Delta)^{-1}\) defined in (49) is compact in \( L^2(\Omega) \). Then applying \((-\Delta)^{-1}\) to (47), we get
\[ (\lambda^2(-\Delta)^{-1} - I) u = (-\Delta)^{-1} f. \]  
(53)

In addition, the same computation in (41)-(43) shows that \( \ker(\lambda^2(-\Delta)^{-1} - I) = \{0\} \). Then, following Fredholm’s alternative (see [13]), the equation (53) admits a unique solution. This completes the proof.

\[ \square \]

**Lemma 2.7.** If \( \eta > 0 \), for all \( \lambda \in \mathbb{R} \), we have
\[ \text{R}(i\lambda I - \mathcal{A}) = \mathcal{H} \]
while if \( \eta = 0 \), for all \( \lambda \in \mathbb{R}^* \), we have
\[ \text{R}(i\lambda I - \mathcal{A}) = \mathcal{H}. \]

**Proof.** We give the proof in the case \( \eta > 0 \), the proof of the second statement is fully similar. Let \( \lambda \in \mathbb{R} \) and \( F = (f, g, h)^T \in \mathcal{H} \), then we look for \( U = (u, v, \omega)^T \in D(\mathcal{A}) \) solution of
\[ (i\lambda I - \mathcal{A}) U = F. \]  
(54)

Equivalently, we have\[ \begin{cases} 
\lambda \mu u - v &= f, \quad \text{in } \Omega, \\
i \mu v - \Delta u &= g, \quad \text{in } \Omega, \\
i \omega(\cdot, \xi) + (|\xi|^2 + \eta) \omega(\cdot, \xi) - \nu(\cdot, \mu(\xi)) &= h(\cdot, \xi), \quad \text{on } \Gamma_1 \times \mathbb{R}^d.
\end{cases} \]

As before, by eliminating \( v \) and \( \omega \) from the above system and using the fact that
\[ \partial_{\nu} u + \gamma \kappa \int_{\mathbb{R}^d} \mu(\xi) \omega(\cdot, \xi) \, d\xi = 0 \quad \text{on } \Gamma_1, \]
we get the following system:
\[ \begin{cases} 
\lambda^2 u + \Delta u &= -g - i\lambda f, \quad \text{in } \Omega, \\
u u &= 0, \quad \text{on } \Gamma_0, \\
\partial_{\nu} u + (\lambda^2 c_1 + i\lambda c_2) u &= -i\lambda c_1 f + c_2 f + I^1_h + I^2_h, \quad \text{on } \Gamma_1.
\end{cases} \]  
(55)
where \( c_1, c_2 \) is defined in equation (48) and \( I_1^h, I_2^h \) are given by
\[
\begin{align*}
I_1^h(x, \lambda, \eta) &= i\lambda \gamma \kappa \int_{\mathbb{R}^d} \frac{h(x, \xi)}{\lambda^2 + (|\xi|^2 + \eta)^2} d\xi \\
I_2^h(x, \lambda, \eta) &= -\gamma \kappa \int_{\mathbb{R}^d} \frac{h(x, \xi)\mu(\xi)}{\lambda^2 + (|\xi|^2 + \eta)^2} d\xi.
\end{align*}
\]

It is easy to see that, for \( h \in L^2(\Gamma_1 \times \mathbb{R}^d) \) and \( \alpha \in [0, 1] \), the integrals \( I_1^h, I_2^h \in L^2(\Gamma_1 \times \mathbb{R}^d) \). Next, let \( \theta \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \) such that \( \theta = 0 \), on \( \Gamma_1 \) and \( \frac{\partial \theta}{\partial \nu} = -i\lambda c_1 f + c_2 f + (\lambda^2 c_1 + i\lambda c_2) \varphi_h \in H^{\frac{1}{2}}(\Gamma_1) \) on \( \Gamma_1 \). (56)

Therefore, setting \( \chi = u - \theta \) in (55), we get
\[
\begin{align*}
\lambda^2 \chi + \Delta \chi &= -\lambda^2 \theta - \Delta \theta - g - i\lambda f \quad \text{in} \quad \Omega, \\
\chi &= 0 \quad \text{on} \quad \Gamma_0, \\
\frac{\partial \chi}{\partial \nu} + (\lambda^2 c_1 + i\lambda c_2) \chi &= 0 \quad \text{on} \quad \Gamma_1.
\end{align*}
\]

Using Lemma 2.6, problem (57) has a unique solution \( \chi \in H^1_{\Gamma_0}(\Omega) \) and therefore problem (55) has a unique solution \( u \in H^1_{\Gamma_0}(\Omega) \). By defining \( v = i\lambda u - f \) in \( \Omega \) and
\[
\omega(x, \xi) = \frac{h(x, \xi)}{i\lambda + |\xi|^2 + \eta} + \frac{i\lambda w_{1\xi}(x)\mu(\xi)}{i\lambda + |\xi|^2 + \eta} - \frac{f_{1\xi}(x)\mu(\xi)}{i\lambda + |\xi|^2 + \eta},
\]
we deduce that \( U = (u, v, \omega) \) belongs to \( D(A) \) and is solution of (54). This completes the proof.

**Proof of Theorem 2.3.** Following a general criteria of Arendt-Batty see [4], the \( C_0 \)-semigroup of contractions \( e^{tA} \) is strongly stable, if \( \sigma(A) \cap i\mathbb{R} \) is countable and no eigenvalue of \( A \) lies on the imaginary axis. First, from Lemma 2.4 we directly deduce that \( A \) has non pure imaginary eigenvalues. Next, using Lemmas 2.5 and 2.7, we conclude, with the help of the closed graph theorem of Banach, that \( \sigma(A) \cap i\mathbb{R} = \{0\} \) if \( \eta > 0 \) and \( \sigma(A) \cap i\mathbb{R} = \{0\} \) if \( \eta = 0 \). The proof is thus completed.

**Remark 1.** We mention [32] for a direct approach of the strong stability of Kirchhoff plates in the absence of compactness of the resolvent. □

### 2.3. Non uniform stability

The aim of this section is to show that system (22)-(26) is not uniformly \( (i.e. \) exponentially) stable in general since it is already the case for \( \Omega = (0, 1) \) as shown below. Our result is the following

**Theorem 2.8.** Assume that \( d = 1 \). The semigroup of contractions \( e^{tA} \) is not uniformly stable in the energy space \( H \).

This result is due to the fact that a subsequence of eigenvalues of \( A \) is close to the imaginary axis. For this aim, let \( \lambda \in \mathbb{C} \) and \( U = (u, v, \omega)^\top \in D(A) \) be such that \( AU = \lambda U \). Equivalently we have
\[
\begin{align*}
v &= \lambda u, \\
u_{xx} &= \lambda v, \\
-(|\xi|^2 + \eta)\omega + v(1)\mu(\xi) &= \lambda \omega.
\end{align*}
\]

Since \( A \) is dissipative, we study the asymptotic behavior of the large eigenvalues \( \lambda \) of \( A \) in the strip \( \alpha_0 \leq \Re(\lambda) \leq 0 \), for some \( \alpha_0 > 0 \) large enough. By eliminating \( v \)
and \( \omega \) from the above system and using the fact that
\[
ux(1) + \gamma \kappa \int_{\mathbb{R}} \mu(\xi) \omega(\xi) d\xi = 0,
\]
and Lemma 4.1 we get the following system:
\[
\begin{align*}
\lambda^2 u - u_{xx} &= 0, \\
u(0) &= 0, \\
ux(1) + \gamma \lambda (\lambda + \eta)^{\alpha-1} u(1) &= 0.
\end{align*}
\]
(58)

We have the following asymptotic behaviour

**Proposition 2.** There exist \( k_0 \in \mathbb{N}^* \) and a sequence \( (\lambda_k)_{|k| \geq k_0} \) of simple eigenvalues of \( A \) and satisfying the following asymptotic behavior:
\[
\lambda_k = i(k + \frac{1}{2})\pi + i \frac{\gamma \sin \left( \frac{\pi}{2} (1 - \alpha) \right)}{\pi^{1-\alpha} k^{1-\alpha}} - \frac{\gamma \cos \left( \frac{\pi}{2} (1 - \alpha) \right)}{\pi^{1-\alpha} k^{1-\alpha}} + o \left( \frac{1}{k^{1-\alpha}} \right),
\]
(59)

for \( k \) large enough.

**Proof.** The general solution of (58) is given by
\[
u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}.
\]
(60)

Thus the boundary conditions may be written as the following system
\[
M(\lambda)C(\lambda) = \begin{pmatrix} 1 \\
h_1(\lambda) e^\lambda \\
h_2(\lambda) e^{-\lambda} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
(61)

where
\[
h_1(\lambda) = \lambda + \gamma \lambda (\lambda + \eta)^{\alpha-1} \quad \text{and} \quad h_2(\lambda) = -\lambda + \gamma \lambda (\lambda + \eta)^{\alpha-1}.
\]

Hence a non-trivial solution \( u \) of system (58) exists if and only if the determinant of \( M(\lambda) \) vanishes. Set \( -\lambda f(\lambda) = \det M(\lambda) \), then we have
\[
f(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{(\lambda + \eta)^{1-\alpha}},
\]
(62)

where
\[
f_0(\lambda) = e^\lambda + e^{-\lambda} \quad \text{and} \quad f_1(\lambda) = \gamma \left( e^\lambda - e^{-\lambda} \right).
\]
(63)

Note that \( f_0 \) and \( f_1 \) remain bounded in the strip \( \alpha_0 \leq \Re(\lambda) \leq 0 \). It is easy to check that the roots of \( f_0 \) are given by
\[
\lambda_k^0 = i \mu_k, \quad k \in \mathbb{Z},
\]
(64)

where \( \mu_k = i \left( k + \frac{1}{2} \right) \pi \). Using Rouché’s theorem, we deduce that \( f(\lambda) \) admits an infinity of simple roots in the strip \( \alpha_0 \leq \Re(\lambda) \leq 0 \) denoted by \( \lambda_k \), with \( |k| \geq k_0 \), for \( k_0 \) large enough, such that
\[
\lambda_k = i \mu_k + \varepsilon_k \quad \text{where} \quad \lim_{|k| \to +\infty} \varepsilon_k = 0.
\]
(65)

Using (63), we get
\[
\begin{align*}
f_0(\lambda_k) &= 2i(-1)^k \varepsilon_k + O(\varepsilon_k^2), \\
f_1(\lambda_k) &= 2i\gamma (-1)^k + O(\varepsilon_k^2), \\
\frac{1}{(\lambda_k + \eta)^{1-\alpha}} &= \frac{\cos \left( \frac{\pi}{2} (1 - \alpha) \right)}{k^{1-\alpha} \pi^{1-\alpha}} - \frac{i \sin \left( \frac{\pi}{2} (1 - \alpha) \right)}{k^{1-\alpha} \pi^{1-\alpha}} + O \left( \frac{1}{k^{2-\alpha}} \right).
\end{align*}
\]
(66-68)
Next, by inserting (66)-(68) in the identity \( f(\lambda) = 0 \) and keeping only the terms of order \( k^{1-\alpha} \), we find after a simplification

\[
\varepsilon_k = -\frac{\gamma \cos \left( \frac{\pi}{2} (1 - \alpha) \right)}{k^{1-\alpha} \pi^{1-\alpha}} + i \frac{\gamma \sin \left( \frac{\pi}{2} (1 - \alpha) \right)}{k^{1-\alpha} \pi^{1-\alpha}} + o \left( \frac{1}{k^{1-\alpha}} \right). \tag{69}
\]

From equation (69), we have

\[
|k|^{1-\alpha} \Re(\lambda_k) \approx -\frac{\gamma \cos \left( \frac{\pi}{2} (1 - \alpha) \right)}{\pi^{1-\alpha}}.
\]

Inserting equation (69) in (65), we get the desired equation (59). This implies that the \( C_0 \)-semigroup of contractions \( e^{tA} \) is not uniformly stable in the energy space \( \mathcal{H} \).

**Numerical Validation.** By using Wolfram Mathematica, the asymptotic behavior \( \lambda_k \) in equation (59) can be numerically validated. For instance, with \( \alpha = 0.5, \eta = 1 \) and \( \gamma = 1 \) then from equation (59) we have

\[
\lim_{k \to +\infty} \sqrt{k} \Re(\lambda_k) = -\frac{\sqrt{2}}{2\pi} \approx -0.398942.
\]

The table below confirms this behavior.

| \( k \) | 400 | 500 | 600 | 700 | 800 | 900 | 1000 |
|-------|-----|-----|-----|-----|-----|-----|-------|
| \( \sqrt{k} \Re(\lambda_k) \) | -0.39874 | -0.398781 | -0.398808 | -0.398827 | -0.398842 | -0.398853 | -0.398862 |

3. **Polynomial stability under geometric control condition.** This section is devoted to the study of the polynomial stability of system (22)-(26) in the case \( \eta > 0 \) and under appropriated geometric conditions. For that purpose, we will use a frequency domain approach, namely we will use Theorem 2.4 of [12] (see also [9, 10, 20]) that we partially recall.

**Theorem 3.1.** Let \( (T(t))_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on a Hilbert space \( H \) with generator \( A \) such that \( i\mathbb{R} \subset \rho(A) \). Then for a fixed \( \ell > 0 \) the following conditions are equivalent

\[
\| (is - A)^{-1} \| = O(|s|^\ell), \ s \to \infty, \tag{70}
\]

\[
\| T(t)A^{-1} \| = O(t^{-1/\ell}), \ t \to \infty. \tag{71}
\]

As the condition \( i\mathbb{R} \subset \rho(A) \) was already checked in Theorem 2.3, it remains to prove that condition (70) holds. This is made with the help of a multiplier method under some geometric conditions on the boundary of the domain and by using the exponential decay of an auxiliary problem. Firstly, like as [1, 28], we consider the following auxiliary problem, namely the wave equation with standard boundary damping on \( \Gamma_1 \):

\[
\begin{aligned}
\varphi_{tt}(x,t) - \Delta \varphi(x,t) &= 0, \quad x \in \Omega, \quad t > 0, \\
\varphi(x,t) &= 0, \quad x \in \Gamma_0, \quad t > 0, \\
\partial_{\nu}\varphi(x,t) &= -\varphi_t(x,t), \quad x \in \Gamma_1, \quad t > 0.
\end{aligned} \tag{72}
\]

Define the auxiliary space \( \mathcal{H}_a = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \) and the auxiliary unbounded linear operator \( A_a \) by

\[
A_a(\varphi, \psi) = (\psi, \Delta \varphi)
\]

and

\[
D(A_a) = \left\{ \Phi = (\varphi, \psi) \in \mathcal{H}_a : \; \Delta \varphi \in L^2(\Omega); \; \psi \in H^1_{\Gamma_0}(\Omega); \; \frac{\partial \varphi}{\partial \nu} = -\psi \; \text{ on } \Gamma_1 \right\}.
\]
We then introduce the following condition:

\((H)\): the problem (72) is uniformly stable in the energy space \(H^1_{0,0}(\Omega) \times L^2(\Omega)\).

Secondly, we recall the Geometric Control condition \((GCC)\) introduced by Bardos, Lebeau and Rauch [8]:

**Definition 3.2.** We say that \(\Gamma\) satisfies the geometric condition named \((GCC)\), if every ray of geometrical optics, starting at any point \(x \in \Omega\) at time \(t = 0\), hits \(\Gamma_1\) in finite time \(T\).

![Figure 1](image.png)

**Figure 1.** Models explain the condition \((GCC)\) holds.

We also recall the multiplier control condition \((MGC)\) in the following definition:

**Definition 3.3.** We say that the multiplier control condition \((MGC)\) holds if there exist \(x_0 \in \mathbb{R}^d\) and a positive constant \(m_0 > 0\) such that

\[
  m \cdot \nu \leq 0 \text{ on } \Gamma_0 \quad \text{and} \quad m \cdot \nu \geq m_0 \text{ on } \Gamma_1,
\]

with \(m(x) = x - x_0\), for all \(x \in \mathbb{R}^d\).

**Remark 2.** In [8], Bardos and al. proved that \((H)\) holds if \(\Gamma\) is smooth (of class \(C^\infty\)), \(\Gamma_0 \cap \Gamma_1 = \emptyset\) and under the \((GCC)\) condition. For less regular domains, namely of class \(C^2\), \((H)\) holds if the vector field assumptions described in [17] (see (i),(ii),(iii) of Theorem 1 in [17]) hold. Moreover, in Theorem 1.2 of [18] the authors proved that \((H)\) holds for smooth domains under weaker geometric conditions than in [17] (without (ii) of Theorem 1). Finally, it is easy to see that the multiplier control condition \((MGC)\) implies that the vector field assumptions described in [17] are satisfied and therefore the condition \((H)\) holds if \((MGC)\) holds.

**Remark 3.** In Figure 1, We take an open arc \(\Upsilon\) in the boundary that contains a half-circumference and let \(P\) denote the midpoint of \(\Upsilon\). For \(\varepsilon\) sufficiently small denote \(\gamma_\varepsilon\) the closed arc centered at \(P\) with length less \(\varepsilon\). For a ray to miss \(\Upsilon \setminus P\) at must hit \(P\) as does the equilateral triangle with vertex \(P\). Let \(\theta\) denote the union of two open arcs centered respectively at the antipodal of \(P\) and one of the other vertices of the equilateral triangle. Let \(\Gamma_1 = (\Upsilon \cup \theta) \setminus \gamma_\varepsilon\) and \(\Gamma_0 = \partial \Omega \setminus \Gamma_1\), then the condition \((GCC)\) holds.

Next, we present the main result of this section.
Theorem 3.4. Assume that \( \eta > 0 \) and that the condition (H) holds. Then, for all initial data \( U_0 \in D(A) \), there exists a constant \( C > 0 \) independent of \( U_0 \), such that the energy of the strong solution \( U \) of (33) satisfies the following estimation

\[
E(t, U) \leq C \frac{1}{t^{1-\eta}} \|U_0\|_{D(A)}^2, \quad \forall t > 0.
\]

(73)

In particular, for \( U_0 \in H \), the energy \( E(t, U) \) converges to zero as \( t \) goes to infinity.

As announced in Theorem 3.1, by taking \( \ell = 2 - 2\alpha \), the polynomial energy decay (73) holds if the following conditions

\[
i \mathbb{R} \subset \rho(A)
\]

and

\[
\sup_{|\lambda| \in \mathbb{R}} \frac{1}{|\lambda|} \|(i\lambda I - A)^{-1}\| < +\infty
\]

are satisfied. Condition (H1) is already proved in Theorem 2.3. We will prove condition (H2) using an argument of contradiction. For this purpose, suppose that (H2) is false, then there exist a real sequence \( (\lambda_n) \), with \( |\lambda_n| \rightarrow +\infty \) and a sequence \( (U^n) \subset D(A) \), verifying the following conditions

\[
\|U^n\|_H = \|(u^n, v^n, \omega^n)\|_H = 1
\]

(74)

and

\[
\lambda_n^\ell (i\lambda_n - A)U^n = (f_1^n, f_2^n, f_3^n)^\top \rightarrow 0 \quad \text{in} \quad H.
\]

(75)

For simplicity, we drop the index \( n \). Detailing equation (75), we get

\[
\lambda^\ell (i\lambda u - v) = f_1 \rightarrow 0 \quad \text{in} \quad H^1_0(\Omega),
\]

(76)

\[
\lambda^\ell (i\lambda v - \Delta u) = f_2 \rightarrow 0 \quad \text{in} \quad L^2(\Omega),
\]

(77)

\[
\lambda^\ell ((i\lambda + |\xi|^2 + \eta)\omega - v|\Gamma, \mu(\xi)) = f_3 \rightarrow 0 \quad \text{in} \quad L^2(\Gamma_1 \times \mathbb{R}^d).
\]

(78)

Note that \( U \) is uniformly bounded in \( H \). Then, taking the inner product of (75) with \( U \) in \( H \), we get

\[
-\gamma \kappa \int_{\Gamma_1} \int_{\mathbb{R}^d} ((|\xi|^2 + \eta)|\omega|^2 d\xi d\Gamma = \Re ((i\lambda I - A)U, U)_H = o(\lambda^{-\ell}).
\]

(79)

Inserting equation (76) in (77), we get

\[
\lambda^2 u + \Delta u = -\frac{f_2}{\lambda^\ell} - i\frac{f_1}{\lambda^{\ell-1}}.
\]

(80)

Lemma 3.5. Assume that \( \eta > 0 \). Then the solution \( (u, v, w) \in D(A) \) of (76)-(78) satisfies the following asymptotic behavior estimation

\[
\|u\|_{L^2(\Omega)} = O(\lambda^{-1}),
\]

(81)

\[
\|\partial_v u\|_{L^2(\Gamma_1)} = o(\lambda^{-1+\alpha}),
\]

(82)

\[
\|u\|_{L^2(\Gamma_1)} = o(\lambda^{-1}).
\]

(83)

Proof. Using equations (74) and (76), we deduce directly the first estimation (81). Now, from the boundary condition

\[
\partial_v u(x) = -\gamma \kappa \int_{\mathbb{R}^d} \mu(\xi) \omega(x, \xi) d\xi, \quad x \in \Gamma_1,
\]

by integrating over \( \Gamma_1 \) and using Cauchy-Schwartz inequality, we get

\[
\int_{\Gamma_1} |\partial_v u|^2 d\Gamma \leq \gamma \kappa \left( \int_{\mathbb{R}^d} \frac{\mu^2(\xi)}{|\xi|^2 + \eta} d\xi \right) \int_{\Gamma_1} \int_{\mathbb{R}^d} (|\xi|^2 + \eta)|\omega(x, t)|^2 d\xi d\Gamma.
\]

(84)
Then, combining equation (79) and equation (84), we obtain the desired estimation (82). Finally, multiplying equation (78) by $(i\lambda + |\xi|^2 + \eta)^{-1}\mu(\xi)$, integrating over $\mathbb{R}^d$ with respect to the variable $\xi$ and applying Cauchy-Shwartz inequality, we obtain

\[ A_1 |v|_{\Gamma_1}(x) \leq A_2 \left( \int_{\mathbb{R}^d} (|\xi|^2 + \eta)|\omega(x, \xi)|^2d\xi \right)^{\frac{1}{2}} + \frac{1}{|\lambda|^2} A_3 \left( \int_{\mathbb{R}^d} |f_3(x, \xi)|^2d\xi \right)^{\frac{1}{2}} \]  

(85)

where

\[ \begin{aligned}
A_1 &= \int_{\mathbb{R}^d} \frac{|\mu(\xi)|^2}{(|\lambda| + |\xi|^2 + \eta)}d\xi, \\
A_2 &= \left( \int_{\mathbb{R}^d} \frac{|\mu(\xi)|^2}{|\xi|^2 + \eta}d\xi \right)^{\frac{1}{2}}, \\
A_3 &= \left( \int_{\mathbb{R}^d} \frac{|\mu(\xi)|^2}{(|\lambda| + |\xi|^2 + \eta)^2}d\xi \right)^{\frac{1}{2}}.
\end{aligned} \]

Using Holder inequality in equation (85), then integrating over $\Gamma_1$, we get

\[ \int_{\Gamma_1} |v(x)|^2d\Gamma \leq \frac{2A_2}{A_1} \int_{\Gamma_1} \int_{\mathbb{R}^d} (|\xi|^2 + \eta)|\omega(x, \xi)|^2d\xi d\Gamma + \frac{2A_3}{A_1^2 |\lambda|^2 \ell} \int_{\Gamma_1} \int_{\mathbb{R}^d} |f_3(x, \xi)|^2d\xi d\Gamma, \]

(86)

On the other hand, using Lemma 4.2, we have

\[ A_1 = c(|\lambda| + \eta)^{\alpha-1} \quad \text{and} \quad A_3 = \tilde{c}(|\lambda| + \eta)^{\frac{\alpha}{2}-1} \]

(87)

where $c$ and $\tilde{c}$ are two positive constants. Inserting equation (79) and (87) in equation (86) and using the fact that $\ell = 2 - 2\alpha$, we get

\[ \|v\|_{L^2(\Gamma_1)} = o(1). \]

(88)

It follows, from (76), that equation (83) holds. The proof has been completed. \( \square \)

**Lemma 3.6.** Assume that $\eta > 0$. Then the solution $(u, v, w) \in D(A)$ of (76)-(78) satisfies the following asymptotic behavior estimation

\[ \int_{\Omega} |\lambda u|^2dx - \int_{\Omega} |\nabla u|^2dx = o(\lambda^{-\ell}). \]

(89)

**Proof.** Multiplying equation (80) by $\bar{u}$, using Green formula and Lemma 3.5, we get equation (89). The proof has been completed. \( \square \)

**Lemma 3.7.** Assume that condition (H) holds and let $u$ be a solution of (80). Then, for any $\lambda \in \mathbb{R}$, the solution $\varphi_u \in H^1(\Omega)$ of system

\[ \begin{aligned}
-(\lambda^2 + \Delta)\varphi_u &= u & \text{in} & & \Omega, \\
\varphi_u &= 0 & \text{on} & & \Gamma_0, \\
\frac{\partial \varphi_u}{\partial \nu} + i\lambda \varphi_u &= 0 & \text{on} & & \Gamma_1,
\end{aligned} \]

(90)

satisfies the following estimate

\[ \|\lambda \varphi_u\|_{L^2(\Gamma_1)} + \|\nabla \varphi_u\|_{L^2(\Omega)} + \|\lambda \varphi_u\|_{\Omega} \lesssim \|u\|_{L^2(\Omega)}. \]

(91)
Proof. Following Huang [16] and Pruss [31], the exponential stability of system (72) implies that the resolvent of the auxiliary operator $A_a$ is uniformly bounded on the imaginary axis i.e. there exists $M > 0$ such that
\[
\|(i\lambda I - A_a)^{-1}\|_{L(H)} \leq M < +\infty, \tag{92}
\]
for all $\lambda \in \mathbb{R}$. Now, since $u \in L^2(\Omega)$, then the pair $(0, u)$ belongs to $H_a$, and from (92), there exists a unique $(\varphi_u, \psi_u) \in D(A_a)$ such that $(i\lambda I - A_a)(\varphi_u, \psi_u) = (0, u)^\top$ i.e.
\[
i\lambda \psi_u - \Delta \varphi_u = u, \tag{93}
i\lambda \varphi_u = \psi_u \tag{94}
\]
and such that
\[
\|((\varphi_u, \psi_u))_{H_a} \leq M\|u\|_{L^2(\Omega)}. \tag{95}
\]
From equations (93)-(95), we deduce that $\varphi_u$ is solution of (90) and we have
\[
\|\nabla \varphi_u\|_{L^2(\Omega)} + \|i\lambda \varphi_u\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}. \tag{96}
\]
Therefore, multiplying the first equation of (90) by $\lambda \varphi_u$ and using Green formula, we get
\[
-\lambda \int_{\Omega} |\lambda \varphi_u|^2 dx + \lambda \int_{\Omega} |\nabla \varphi_u|^2 dx + i \int_{\Gamma_1} |\lambda \varphi_u|^2 d\Gamma_1 = \lambda \int_{\Omega} u \varphi_u dx. \tag{97}
\]
By taking the imaginary part of equation (97) and using Cauchy-Schwartz inequality, we deduce, from (96), that
\[
\|\lambda \varphi_u\|_{L^2(\Gamma_1)} \lesssim \|u\|_{L^2(\Omega)}. \tag{98}
\]
Finally, Combining equations (96) and (98) we obtain the desired equation (91). The proof is thus complete.

Lemma 3.8. Assume that $\eta > 0$ and condition (H) holds. Then the solution $(u, v, w) \in D(\mathcal{A})$ of (76)-(78) satisfies the following asymptotic behavior estimation
\[
\int_{\Omega} |\lambda u|^2 dx = o(1). \tag{99}
\]

Proof. Multiplying equation (80) by $\varphi_u$ and using Green formula, we obtain
\[
\int_{\Omega} u(\lambda^2 \varphi_u + \Delta \varphi_u) dx + \int_{\Gamma_1} ((\partial_v u) \varphi_u - (\partial_v \varphi_u) u) d\Gamma = -\int_{\Omega} \left( f_2 + i \frac{\lambda f_1}{\lambda^2} \right) \varphi_u dx. \tag{100}
\]
It follows from (90) that
\[
\int_{\Omega} |u|^2 dx = \int_{\Gamma_1} ((\partial_v \varphi_u + i\lambda \varphi_u) u) d\Gamma + \int_{\Omega} \left( f_2 + i \frac{\lambda f_1}{\lambda^2} \right) \varphi_u dx. \tag{101}
\]
Firstly, using (91), (81) and the fact that $\|f_2\|_{L^2(\Omega)} = o(1)$, we get
\[
\int_{\Omega} f_2^2 \varphi_u dx = o(\lambda^{-2-\epsilon}). \tag{102}
\]
On the other hand, multiplying the first equation of (90) by $f_1$ and integrating, we get
\[
\int_{\Omega} \lambda^2 f_1 \varphi_u dx = \int_{\Omega} \nabla f_1 \cdot \nabla \varphi_u dx + i \int_{\Gamma_1} \lambda f_1 \varphi_u d\Gamma - \int_{\Omega} u f_1 dx. \tag{103}
\]
Hence, using (91), (81) and the fact that \( \| f_1 \|_{H^1_0(\Omega)} = o(1) \), we obtain, from (103) that
\[
\int_{\Omega} \lambda^2 f_1 \varphi_u \, dx = o(\lambda^{-1}).
\] (104)
Secondly, using (82), (83) and (91), we get
\[
\int_{\Gamma_1} (\partial_n u) \varphi_u \, d\Gamma = o(\lambda^{-3+\alpha}) \quad \text{and} \quad \int_{\Gamma_1} \lambda u \varphi_u \, d\Gamma = o(\lambda^{-2}).
\] (105)
Inserting equations (102), (104) and (105) in (101) and use the fact that \( \ell = 2 - 2\alpha \), we get
\[
\int_{\Omega} |\lambda u|^2 \, dx = o(1).
\] (106)
The proof is thus complete.

**Proof of Theorem 3.4.** Using (89) and (99), we get
\[
\int_{\Omega} |\nabla u|^2 \, dx = o(1).
\]
It follows, from (79) and (89), that \( \| U \|_H = o(1) \) which is a contradiction with (74). Consequently condition (H2) holds and the energy of smooth solution of system (22)-(25) decays polynomial to zero as \( t \) goes to infinity. Finally, using the density of the domain \( D(A) \) in \( H \), we can easily prove that the energy of weak solution of system (22)-(25) decays to zero as \( t \) goes to infinity. The proof has been completed.

**Conclusion.** We have studied the stabilization of multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. Non uniform stability is proved and a polynomial energy decay rate of type \( t^{-\frac{1}{1-\alpha}} \) is established. In view of the asymptotic behavior of the eigenvalues of the operator \( A \) (59), we deduce that the optimal energy decay rate is of type \( t^{-\frac{1}{1-\alpha}} \) (see [22, 27]). This question still be open.

4. **Appendix.** Let \( \mu \) be the function defined by
\[
\mu(\xi) = |\xi|^{2\alpha - d}, \quad \xi \in \mathbb{R}^d \quad \text{and} \quad 0 < \alpha < 1.
\] (107)

**Lemma 4.1.** Let \( \eta > 0 \). For any real number \( \lambda > -\eta \), we have
\[
\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + |\xi|^2} \, d\xi = \frac{\pi}{\sin(\alpha \pi)} (\lambda + \eta)^{\alpha - 1}.
\]

**Proof.** A direct computation gives
\[
\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + |\xi|^2} \, d\xi = \int_{0}^{+\infty} \frac{z^{\alpha - 1}}{\lambda + \eta + z} \, dz
\]
\[
= (\lambda + \eta)^{\alpha - 1} \int_{0}^{1} \frac{(1 - z)^{\alpha - 1}}{z^\alpha} \, dz
\]
\[
= (\lambda + \eta)^{\alpha - 1} \Gamma(1 - \alpha) \Gamma(\alpha)
\]
\[
= \frac{\pi}{\sin(\alpha \pi)} (\lambda + \eta)^{\alpha - 1}.
\]
\( \square \)
Lemma 4.2. Let $\eta > 0$, then we have

\[ A_1 = \int_{\mathbb{R}^d} \frac{\mu^2(\xi)}{|\lambda + \eta + |\xi|^2} d\xi = c (|\lambda + \eta)^{\alpha - 1} \]

and

\[ A_3 = \left( \int_{\mathbb{R}^d} \frac{\mu^2(\xi)}{(|\lambda + \eta + |\xi|^2)^2} d\xi \right)^{\frac{1}{2}} = \tilde{c} (|\lambda + \eta)^{\frac{\alpha}{2} - 1} \]

where

\[ c := \frac{d\pi^\frac{d}{2} + 1}{2\Gamma (\frac{d}{2} + 1) \sin(\alpha \pi)} \quad \text{and} \quad \tilde{c} := \left( \frac{d\pi^\frac{d}{2}}{2\Gamma (\frac{d}{2} + 1)} \int_1^{+\infty} \frac{(y - 1)^{\alpha - 1}}{y^2} dy \right)^{\frac{1}{2}}. \quad (108) \]

Proof. Firstly, we calculate $A_1$. Using the hyper-spherical coordinates and the fact that the Jacobian $J$ is defined by

\[ J = \rho^{d-1} \prod_{j=1}^{d-2} \sin^{d-1-j} (\phi_j), \]

we get

\[ A_1 = \int_0^{+\infty} \frac{\rho^{2\alpha - 1}}{|\lambda + \eta + \rho^2} \left( \prod_{j=1}^{d-2} \left( \int_0^{\pi} \sin^{d-1-j} (\phi_j) d\phi_j \right) \int_0^{2\pi} d\phi_{d-1} \right) d\rho. \]

On the other hand, it is easy to see that

\[ \prod_{j=1}^{d-2} \left( \int_0^{\pi} \sin^{d-1-j} (\phi_j) d\phi_j \right) \int_0^{2\pi} d\phi_{d-1} = \frac{d\pi^\frac{d}{2}}{\Gamma (\frac{d}{2} + 1)}. \]

This implies that

\[ A_1 = \frac{d\pi^\frac{d}{2}}{2\Gamma (\frac{d}{2} + 1)} \int_0^{+\infty} \frac{\rho^{2\alpha - 1}}{|\lambda + \eta + \rho^2} d\rho. \]

Consequently, using the same calculation used in Lemma 4.1, we obtain

\[ A_1 = c (|\lambda + \eta)^{\alpha - 1} \]

where $c$ is defined in (108).

Secondly, we calculate $A_3$. Using the same calculation of $A_1$, we obtain

\[ A_3 = \frac{d\pi^\frac{d}{2}}{2\Gamma (\frac{d}{2} + 1)} \int_0^{+\infty} \frac{\rho^{2\alpha - 1}}{(|\lambda + \eta + \rho^2)^2} d\rho. \]

Let $x = \rho^2$ in (109), we get

\[ A_3 = \frac{d\pi^\frac{d}{2}}{2\Gamma (\frac{d}{2} + 1)} \int_0^{+\infty} \frac{x^{\alpha - 1}}{(|\lambda + \eta + x)^2} dx. \quad (110) \]

By tacking $y = \frac{x}{|\lambda + \eta + 1}$ in (110), we deduce that

\[ A_3 = \frac{d\pi^\frac{d}{2}}{2\Gamma (\frac{d}{2} + 1)} I_y \]

where $I_y$ is defined by

\[ I_y = \int_1^{+\infty} \frac{(y - 1)^{\alpha - 1}}{y^2} dy. \]
In addition, it is easy to check that for $\alpha \in [0,1]$, the integral $I_y$ is well defined. Finally, from equation (110), we obtain

$$A_3 = \tilde{c} (|\lambda| + \eta)^{-\frac{\alpha}{2} - 1}$$

where $\tilde{c}$ is defined in equation (108).

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