On the Supremum of Some Random Dirichlet Polynomials

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Abstract

We study the average supremum of some random Dirichlet polynomials
\[ D_N(t) = \sum_{n=1}^{N} \varepsilon_n d(n) n^{-\sigma - it}, \]
where \((\varepsilon_n)\) is a sequence of independent Rademacher random variables, the weights \((d(n))\) satisfy some reasonable conditions and \(0 \leq \sigma \leq 1/2\). We use an approach based on methods of stochastic processes, in particular the metric entropy method developed in [8].

1 Introduction

Let \(\{d(n), n \geq 1\}\) be a sequence of real numbers. Consider the Dirichlet polynomials
\[ P(\sigma + it) = P_N(\sigma + it) = \sum_{n=1}^{N} d(n) n^{-\sigma - it}. \]
As is well-known, the abscissa of uniform convergence \(\sigma_u\) of the associated Dirichlet series \(\sum_{n=1}^{\infty} d(n) n^{-\sigma - it}\), which is defined by
\[ \sigma_u = \inf \left\{ \sigma : \sum_{n=1}^{\infty} d(n) n^{-\sigma - it} \text{ converges uniformly over } t \in \mathbb{R} \right\}, \]
satisfies the relation \(\sigma_u = \limsup_{N \to \infty} \frac{\log \sup_{t \in \mathbb{R}} |P_N(it)|}{\log N}\). And this gives a motivation to study of the supremum of the Dirichlet polynomials \(P_N(s)\) over lines \(\{s = \sigma + it, t \in \mathbb{R}\}\) (see for instance [6], [8] and the references therein).

A first basic reduction step allows to replace the Dirichlet polynomial by some relevant trigonometric polynomial. Introduce some necessary notation. Let \(2 = p_1 < p_2 < \ldots\) be the sequence of all primes. Let \(\pi(N)\) denote the number of prime numbers that are less or equal to \(N\). Now fix \(N\) and put \(\tau = \pi(N)\). If \(n = \prod_{j=1}^{\tau} p_j^{a_j(n)}\), we write \(a(n) = \{a_j(n), 1 \leq j \leq \tau\}\). Let also \(T = [0,1[= \mathbb{R}/\mathbb{Z}\) be the torus. Define for \(\bar{z} = (z_1, \ldots, z_\tau) \in T^\tau\)
\[ Q(\bar{z}) = \sum_{n=1}^{N} d(n) n^{-\sigma} e^{2i\pi \langle a(n), \bar{z} \rangle}. \]

H. Bohr’s observation (see e.g. [10]), based on Kronecker’s Theorem (see [4], Theorem 442, p.382) states that
\[ \sup_{t \in \mathbb{R}} |P(\sigma + it)| = \sup_{\bar{z} \in T^\tau} |Q(\bar{z})|. \]
The supremum properties of random Dirichlet polynomials and random Dirichlet series were investigated by Halász, Bayard, Konyagin, Queffélec, recently by the authors, and earlier, by Hartman, Clarke, Dvoretzky and Erdős, where random power series are also considered (see [8] for references).

Let $\varepsilon = \{\varepsilon_n, n \geq 1\}$ be (here and throughout the whole paper) a sequence of independent Rademacher random variables ($P\{\varepsilon_i = \pm 1\} = 1/2$) defined on a basic probability space $(\Omega, A, P)$. Consider the random Dirichlet polynomials

$$D(s) = \sum_{n=1}^{N} \varepsilon_n d(n)n^{-s}. \quad (2)$$

In the particular case $d(n) \equiv 1$, if $\sigma = 0$, the following result was proved by Halász (see [11],[12]): for some absolute constant $C$, and all integers $N \geq 2$

$$C^{-1} \frac{N}{\log N} \leq \mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} \varepsilon_n n^{-\sigma - it} \right| \leq C \frac{N}{\log N}. \quad (3)$$

In [11],[12] (see also [10] for a first result), Queffélec extended Halász’s result to the range of values $0 \leq \sigma < 1/2$; and provided a probabilistic proof of the original one, using Bernstein’s inequality for polynomials, properties of complex Gaussian processes and the sieve method introduced by Halász. He obtained that for some constant $C_\sigma$ depending on $\sigma$ only, and all integers $N \geq 2$

$$C_\sigma^{-1} \frac{N^{1-\sigma}}{\log N} \leq \mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} \varepsilon_n n^{-\sigma - it} \right| \leq C_\sigma \frac{N^{1-\sigma}}{\log N}. \quad (4)$$

This result was further extended in [8] for the polynomials

$$\sum_{\substack{P^+(n) \leq \theta, \ n \leq N}} \varepsilon_n n^{-\sigma - it}$$

(here and in what follows $P^+(n)$ denotes the largest prime divisor of $n$) with fine estimates depending on both parameters $\theta$ and $N$. For small $\theta$ the estimates of such kind are related to the construction of so called Rudin-Shapiro Dirichlet polynomials.

In this work, we show that the approach developed in [8] is sufficiently robust to allow the similar study for random Dirichlet polynomials with reasonable weights. We will also extend in a separate section the main result in [8] to the boundary case $\sigma = 1/2$.

We introduce now some characteristics of weights. Let

$$D_1(M) = \sum_{m=1}^{M} d(m); \quad \bar{D}_1(M) = \max_{1 \leq m \leq M} \frac{D_1(m)}{m},$$

and, similarly,

$$D_2(M) = \sum_{m=1}^{M} d(m)^2; \quad \bar{D}_2^2(M) = \max_{1 \leq m \leq M} \frac{D_2(m)}{m},$$

$$2$$
Obviously, we have $\tilde{D}_2(M) \geq \tilde{D}_1(M)$.

**Example 1.** Let $d(n)$ be the divisor function (number of divisors of integer $n$). Then $d$ is a multiplicative function and it is well-known that

$$\tilde{D}_1(M) \sim \log M, \quad \tilde{D}_2(M) \sim \log^{3/2} M.$$ 

**Example 2.** Let von Mangoldt function $\Lambda$ be defined as follows:

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, \ p \text{ is prime} \\ 0, & \text{else} \end{cases} \quad (5)$$

Then $\Lambda$ is neither additive nor multiplicative, and we have for any integers $k, j$ and any prime $p$ either $\Lambda(kp^j) = 0$, or $k = p^j$, hence $\Lambda(kp^j) = \Lambda(k) = \log p$.

Moreover, elementary calculations show that

$$\lim_{M \to \infty} \tilde{D}_1(M) = 1, \quad \tilde{D}_2(M) \sim \log M.$$ 

**Theorem 1** Let $0 \leq \sigma \leq 1/2$ and assume that

$$d(kp^j) \leq Cd(k)j^H \quad (6)$$

for some positive $C, H$, any positive integer $k, j$ and any prime $p$. Then there exists a constant $C$ depending on $d$ and $\sigma$ such that for any integer $N \geq 2$ it is true that

$$E \sup_{t \in \mathbb{R}} |D(\sigma + it)| \leq \frac{C N^{1-\sigma} \tilde{D}_2(N)}{\log N}.$$ \quad (7)

Moreover, if for some $b < b_* := (\sqrt{5} - 1)/4 \approx 0.31$

$$\tilde{D}_2(M) \leq CM^b,$$ \quad (8)

then

$$E \sup_{t \in \mathbb{R}} |D(\sigma + it)| \leq \frac{C N^{1-\sigma}}{\log N}.$$ 

**Remark 2** If one assumes $[8]$ with $b \in (b_*, 1/2)$, then our proof shows that

$$E \sup_{t \in \mathbb{R}} |D(\sigma + it)| \leq \frac{C N^{r-\sigma}}{(\log N)^{1/2}}, \quad r = r(b) = b + \frac{2b + 3}{4(1 + b)},$$

which is better than (7).

**Remark 3** If $d(n)$ is a multiplicative function, then condition (6) is satisfied iff

$$d(p^{r+j}) \leq Cd(p^r)j^H,$$

for some $C > 0, H > 0$ and any $j \geq 1, r \geq 0$. This last condition is satisfied for instance if $d(p) = O(1)$ and

$$\frac{d(p^{k+1})}{d(p^k)} \leq (1 + \frac{1}{k})^H, \quad k = 1, 2, \ldots$$

A multiplicative function being completely defined by its values $d(p^k)$, these ones can be prescribed arbitrarily. This observation shows that condition (6) is satisfied for a very large class of multiplicative functions.
2 Proof of Theorem 1

For proving the upper bound, we first operate the reduction to the study of a random polynomial $Q$ on the multidimensional torus by using (1). Next we use a decomposition $Q = Q_1 + Q_2$. The study of the supremum of the polynomial $Q_1$ is made by using the metric entropy method. The investigation of the supremum of the polynomial $Q_2$ first relies upon the contraction principle, reducing the study to the one of a complex valued Gaussian process, next via Slepian’s comparison Lemma, by a thorough study of the $L^2$-metric induced by this process.

The calculations are first provided for $\sigma < \frac{1}{2}$. At the end of the proof we comment on the case $\sigma = \frac{1}{2}$ for which some minor details are different from the generic case.

Introduce some notation. We can represent $[1, N]$ as the union of disjoint sets

$$E_j = \{2 \leq n \leq N : P^+(n) = p_j\}, \quad j = 1, \ldots, \tau.$$

For $\omega \in T^\tau$ we put

$$Q(\omega) = \sum_{j=1}^\tau \sum_{n \in E_j} \varepsilon_n d(n) n^{-\sigma} e^{2i\pi (\omega(n) \cdot \omega)}.$$

By (1) we have

$$\sup_{t \in \mathbb{R}} \sum_{j=1}^\tau \sum_{n \in E_j} \varepsilon_n d(n) n^{-\sigma - it} = \sup_{\omega \in T^\tau} |Q(\omega)|.$$

Let $1 \leq \nu < \tau$ be fixed. Write $Q = Q_1 + Q_2$ where

$$Q_1(\omega) = \sum_{p^+(n) \leq p_\nu} \varepsilon_n d(n) n^{-\sigma} e^{2i\pi (\omega(n) \cdot \omega)}$$

$$Q_2(\omega) = \sum_{p_\nu < p^+(n) \leq p_\tau} \varepsilon_n d(n) n^{-\sigma} e^{2i\pi (\omega(n) \cdot \omega)}.$$

First, evaluate the supremum of $Q_2$. Introduce the following random process

$$X^{\gamma}(\omega) = \sum_{\nu < j \leq \tau} \alpha_j \sum_{n \in E_j} \varepsilon_n d(n) n^{-\sigma} \beta_{n \omega_{P_j}}, \quad \gamma \in \Gamma,$$

where $\gamma = (\alpha_j)_{\nu < j \leq \tau}, (\beta_m)_{1 \leq m \leq N/2}$ and

$$\Gamma = \{ \gamma : |\alpha_j| \vee |\beta_m| \leq 1, \nu < j \leq \tau, 1 \leq m \leq N/2 \}.$$

Writing

$$Q_2(\omega) = \sum_{\nu < j \leq \tau} e^{2i\pi z_j} \sum_{n \in E_j} \varepsilon_n d(n) n^{-\sigma} e^{2i\pi \left\{ \sum_{k \neq j} a_k(n) z_k + [a_j(n) - 1] z_j \right\}}$$

$$= \sum_{\nu < j \leq \tau} e^{2i\pi z_j} \sum_{n \in E_j} \varepsilon_n d(n) n^{-\sigma} e^{2i\pi \left\{ \sum_k a_k(n) z_k \right\}}$$
and considering separately the imaginary and real parts of the exponents, it follows that $Q_2(z)$ can be written as the sum of four terms each being of the form

$$
\eta \sum_{\nu < j \leq \tau} \alpha_j \sum_{n \in E_j} \varepsilon_n d(n) n^{-\sigma} \beta \nu_j,
$$

where $\eta \in \{1, i, -i, -1\}$, and

$$
\alpha_j = \alpha_j(z) = \begin{cases} 
\cos(2\pi z_j), & \nu < j \leq \tau; \\
\sin(2\pi z_j), & \nu > j > 0.
\end{cases}
$$

$$
\beta_m = \beta_m(z) = \begin{cases} 
\cos(2\pi \sum_k a_k(m) z_k), & 1 \leq m \leq \frac{N}{2}; \\
\sin(2\pi \sum_k a_k(m) z_k), & \text{otherwise}.
\end{cases}
$$

Therefore, we obtain

$$
\sup_{z \in \Gamma} |Q_2(z)| \leq 4 \sup_{\gamma \in \Gamma} |X^\nu(\gamma)|.
$$

By the contraction principle ([5] p.16-17)

$$
E \sup_{z \in \Gamma} |Q_2(z)| \leq 4 \sqrt{\frac{\pi}{2}} E \sup_{\gamma \in \Gamma} |X(\gamma)|,
$$

where $\{X(\gamma), \gamma \in \Gamma\}$ is the same process as $X^\nu(\gamma)$ except that the Rademacher random variables $\varepsilon_n$ are replaced by independent $\mathcal{N}(0, 1)$ random variables $\mu_n$:

$$
X(\gamma) = \sum_{\nu < j \leq \tau} \alpha_j \sum_{n \in E_j} \mu_n d(n) n^{-\sigma} \beta \nu_j.
$$

The problem now reduces to estimating the supremum of the real valued Gaussian process $X$. Towards this aim, we examine the $L^2$-norm of its increments:

$$
\|X_\gamma - X_\nu\|^2_2 = \sum_{\nu < j \leq \tau} \sum_{n \in E_j} d(n)^2 n^{-2\sigma} [\alpha_j \beta \nu_j - \alpha'_j \beta'_\nu_j]^2 
\leq 2 \sum_{\nu < j \leq \tau} \sum_{n \in E_j} d(n)^2 n^{-2\sigma} ([\alpha_j - \alpha'_j]^2 + (\beta \nu_j - \beta'_\nu_j)^2],
$$

where we have used the identity

$$
\alpha_j \beta \nu_j - \alpha'_j \beta'_\nu_j = (\alpha_j - \alpha'_j) \beta \nu_j + (\beta \nu_j - \beta'_\nu_j) \alpha'_j.
$$

The "$\alpha$" component part is easily controlled as follows,

$$
\sum_{\nu < j \leq \tau} \sum_{n \in E_j} d(n)^2 n^{-2\sigma} (\alpha_j - \alpha'_j)^2 \leq C \sum_{\nu < j \leq \tau} (\alpha_j - \alpha'_j)^2 p_j^{-2\sigma} \sum_{m \leq N/p_j} \frac{d(m)^2}{m^{2\sigma}} 
\leq C \sum_{\nu < j \leq \tau} (\alpha_j - \alpha'_j)^2 \frac{N^{1-2\sigma} \mathcal{D}_2(N/p_j)}{p_j}, (9)
$$

since by Abel transformation

$$
\sum_{m \leq M} d(m)^2 m^{-2\sigma} \leq C \mathcal{D}_2(M) M^{1-2\sigma}. (10)
$$
For the "β" component part, we use $d(mp_j) \leq C d(m)$, which is a particular case of (13), and obtain

$$
\sum_{\nu < j \leq \tau} \sum_{n \in E_j} \frac{d(n)^2 (\beta_m - \beta'_m)^2}{n^{2\sigma}} \leq C \sum_{m \leq N/p_{\nu}} (\beta_m - \beta'_m)^2 \left( \sum_{m \leq N/p_{\nu}} \frac{d(m)^2}{(mp_j)^{2\sigma}} \right)
$$

$$
:= C \sum_{m \leq N/p_{\nu}} K_m^2 (\beta_m - \beta'_m)^2. \quad (11)
$$

Now we evaluate the coefficients $K_m$.

Take a unique $k \in (\nu, \tau]$ such that $N/p_k < m \leq N/p_{k-1}$. By using

$$
p_j \sim j \log j \quad (12)
$$

we have

$$
K_m^2 = \sum_{\nu < j \leq k-1 \atop \nu < j \leq k-1} d(m)^2 (mp_j)^{-2\sigma} \leq d(m)^2 m^{-2\sigma} \sum_{j \leq k-1} p_j^{-2\sigma}
$$

$$
\leq C d(m)^2 m^{-2\sigma} \sum_{j \leq k} (j \log j)^{-2\sigma} \leq C d(m)^2 m^{-2\sigma} \frac{k^{1-2\sigma}}{(\log k)^{2\sigma}}
$$

$$
\leq C d(m)^2 m^{-2\sigma} \frac{k}{p_k^{2\sigma}} \leq C m^{-2\sigma} d(m)^2 \frac{k}{(N/m)^{2\sigma}}
$$

$$
= C d(m)^2 \frac{k}{N^{2\sigma}}.
$$

Since $k \log k \leq C p_k \leq C \frac{N}{m}$, we have

$$
k \leq C \frac{N}{m} (\log \frac{N}{m})^{-1}.
$$

We arrive at

$$
K_m \leq C d(m) N^{-\sigma} \left( \frac{N}{m} \right)^{1/2} \left( \log \left( \frac{N}{m} \right) \right)^{-1/2}. \quad (13)
$$

By Abel transformation,

$$
\sum_{m \leq M} \left( \frac{N}{m} \right)^{1/2} \log \left( \frac{N}{m} \right)^{-1/2} d(m)
$$

$$
\leq D_1(M) \left( \frac{N}{M} \right)^{1/2} \log \left( \frac{N}{M} \right)^{-1/2} + C \sum_{m \leq M} \left( \frac{N}{m} \right)^{1/2} \log \left( \frac{N}{m} \right)^{-1/2} d_1(m)
$$

$$
\leq \tilde{D}_1(M) \left( (NM)^{1/2} \log \left( \frac{N}{M} \right) \right)^{-1/2} + C \sum_{m \leq M} \left( \frac{N}{m} \right)^{1/2} \log \left( \frac{N}{m} \right)^{-1/2} d_1(m)
$$

$$
\leq \tilde{D}_1(M) \left( (NM)^{1/2} \log \left( \frac{N}{M} \right) \right)^{-1/2} + C N \int_0^{M/N} u^{-1/2} \log(1/u)^{-1/2} du
$$

$$
\leq C \tilde{D}_1(M) \left( (NM)^{1/2} \log \left( \frac{N}{M} \right) \right)^{-1/2}.
$$

It follows that

$$
\sum_{m \leq N/p_{\nu}} K_m \leq C N^{-\sigma} \sum_{m \leq N/p_{\nu}} \left( \frac{N}{m} \right)^{1/2} \log \left( \frac{N}{m} \right)^{-1/2} d(m)
$$
\[
\leq \frac{CN^{1-\sigma} \bar{D}_1(N/p_\nu)}{\nu^{1/2} \log \nu}.
\]

Now define a second Gaussian process by putting for all \(\gamma \in \Gamma\)
\[
Y(\gamma) = \sum_{\nu<j \leq \tau} \left( \frac{\bar{D}_2(N/p_j) N^{1-2\sigma}}{p_j} \right)^{1/2} \alpha_j \xi_j' + \sum_{m \leq N/p_\nu} K_m \beta_m \xi_m' := Y'_\gamma + Y''_\gamma,
\]
where \(\xi_j', \xi_m''\) are independent \(\mathcal{N}(0,1)\) random variables. It follows from (9) and (11) that for some suitable constant \(C\), one has the comparison relations: for all \(\gamma, \gamma' \in \Gamma\),
\[
\|X_\gamma - X_{\gamma'}\|_2 \leq C\|Y_\gamma - Y_{\gamma'}\|_2.
\]
By virtue of the Slepian comparison lemma (see [7], Theorem 4 p.190), since
\(X_0 = Y_0 = 0\), we have
\[
E \sup_{\gamma \in \Gamma} |X_\gamma| \leq 2E \sup_{\gamma \in \Gamma} X_\gamma \leq 2CE \sup_{\gamma \in \Gamma} Y_\gamma \leq 2CE \sup_{\gamma \in \Gamma} |Y_\gamma|.
\]
It remains to evaluate the supremum of \(Y\). First of all,
\[
E \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq N^{1-\sigma} \sum_{\nu<j \leq \tau} p_j^{-1/2} \bar{D}_2(N/p_j).
\]
By using (12) we have
\[
\sum_{\nu<j \leq \tau} p_j^{-1/2} \leq \sum_{1<j \leq \tau} p_j^{-1/2} \leq \frac{C\tau^{1/2}}{(\log \tau)^{1/2}},
\]
thus
\[
E \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq C N^{1-\sigma} \bar{D}_2(N/p_\nu) \frac{\tau^{1/2}}{(\log \tau)^{1/2}} \leq C N^{1-\sigma} \bar{D}_2(N/p_\nu) \frac{\tau^{1/2}}{(\log \tau)^{1/2}} \leq C N^{1-\sigma} \bar{D}_2(N/p_\nu) \frac{\tau^{1/2}}{(\log \tau)^{1/2}}.
\]
Under assumption (5) we can get a better estimate by using the following lemma.

**Lemma 4** Let \(f(N) \leq cN^b, b < 1/2\). Then
\[
\sum_{1 \leq j \leq \tau(N)} p_j^{-1/2} f(N/p_j) \leq \frac{CN^{1/2}}{\log N},
\]
with \(C\) depending on \(c\) and \(b\).

**Proof.**
\[
\sum_{1 \leq j \leq \tau(N)} p_j^{-1/2} f(N/p_j) \leq c \sum_{1 \leq j \leq \tau(N)} p_j^{-1/2} (N/p_j)^b = cN^b \sum_{1 \leq j \leq \tau(N)} p_j^{-(1/2+b)} \leq C N^b \tau(N)^{1/2-b}(\log \tau(N))^{-1/2-b} \leq \frac{CN^{1/2}}{\log N}.
\]
By applying lemma to $f(N) = \tilde{D}_2(N)$ we see that (8) implies
\[ \mathbb{E} \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq \frac{C N^{1-\sigma}}{\log N}. \] (16)

To control the supremum of $Y''$, we use our estimates for the sums of $K_m$ and write that
\[ \mathbb{E} \sup_{\gamma \in \Gamma} |Y''(\gamma)| \leq \sum_{m \leq N/p} K_m \leq \frac{C N^{1-\sigma} \tilde{D}_1(N/p_\nu)}{\nu^{1/2} \log \nu}. \] (17)

Now, we turn to the supremum of $Q_1$. Introduce the auxiliary Gaussian process
\[ \Upsilon(z) = \sum_{n : P(n) \leq p_\nu} d(n) n^{-\sigma} \{ \vartheta_n \cos 2\pi (\varphi(n),z) + \varphi_n' \sin 2\pi (\varphi(n),z) \}, \quad z \in T^{\nu}, \]
where $\vartheta_j$, $\varphi_j'$ are independent $\mathcal{N}(0,1)$ random variables. By symmetrization (see e.g. Lemma 2.3 p. 269 in [9]),
\[ \mathbb{E} \sup_{z \in T^{\nu}} |Q_1(z)| \leq \sqrt{8\pi} \mathbb{E} \sup_{z \in T^{\nu}} |\Upsilon(z)|, \]
so that we are again led to evaluating the supremum of a real valued Gaussian process. For $z,z' \in T^{\nu}$ put $\| \Upsilon(z) - \Upsilon(z') \|_2 := \rho(z,z')$, and observe that
\[ \rho(z,z')^2 = 4 \sum_{n : P(n) \leq p_\nu} d(n)^2 \frac{d(n)^2}{n^{2\sigma}} \sin^2 (\pi (\varphi(n),z - z')) \]
\[ \leq 4\pi^2 \sum_{n : P(n) \leq p_\nu} \frac{d(n)^2}{n^{2\sigma}} |(\varphi(n),z - z')|^2 \]
\[ \leq 4\pi^2 \sum_{n : P(n) \leq p_\nu} d(n)^2 n^{-2\sigma} \left[ \sum_{j=1}^\nu |a_j(n)| |z_j - z'_j| \right]^2 \]
\[ = 4\pi^2 \sum_{n : P(n) \leq p_\nu} \sum_{j_1,j_2=1}^\nu |a_{j_1}(n) a_{j_2}(n)| |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| d(n)^2 n^{-2\sigma} \]
\[ \leq 4\pi^2 \sum_{j_1,j_2=1}^\nu |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1,b_2=1}^{\infty} b_1 b_2 \sum_{n \leq N, n a_{j_1}(n) = b_1, a_{j_2}(n) = b_2} d(n)^2 n^{-2\sigma} \]
\[ \leq C \sum_{j_1,j_2=1}^\nu \frac{b_1^{1+H} b_2^{1+H}}{P_{j_1} P_{j_2}} \sum_{k \leq N^{b_1 p_{j_1}} b_2^{-p_{j_2}}} d(k)^2 k^{-2\sigma}. \]

By Abel transformation argument (10), denoting $\Delta := \tilde{D}_2(N)$, we have
\[ \rho(z,z')^2 \leq C N^{1-2\sigma} \Delta^2 \sum_{j_1,j_2=1}^\nu |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1,b_2=1}^{\infty} \frac{b_1^{1+H} b_2^{1+H}}{P_{j_1}^2 P_{j_2}^{2\sigma}} \sum_{k \leq N^{b_1 p_{j_1}} b_2^{-p_{j_2}}} [p_{j_1} p_{j_2}]^{-1-2\sigma}. \]
Thus, every cell is a product of two cubes of different size and dimension. The necessary number of cells \( M \) is bounded as follows:

\[
M(\varepsilon) \leq \left( \frac{\log \log \nu}{\varepsilon} \right)^{[\nu^{1/2}]} \epsilon^{-([\nu^{1/2}])} = (1/\varepsilon)^{V} (\log \log \nu)^{[\nu^{1/2}]}.
\]

Let us now evaluate the distance \( \rho(z, z') \) for \( z, z' \) satisfying (19). By (19) we have

\[
\rho(z, z') \leq CN^{1/2-\sigma} \Delta \left\{ \sum_{j=1}^{\nu} |z_j - z'_j| \right\}.
\]

Thus, every cell is a product of two cubes of different size and dimension. The necessary number of cells \( M(\varepsilon) \) is bounded as follows

\[
M(\varepsilon) \leq \left( \frac{\log \log \nu}{\varepsilon} \right)^{[\nu^{1/2}]} \epsilon^{-([\nu^{1/2}])} = (1/\varepsilon)^{V} (\log \log \nu)^{[\nu^{1/2}]}.
\]

Now we explore the entropy properties of the metric space \((T^\nu, \rho)\). Towards this aim, take \( \varepsilon \in (0, 1) \) and cover \( T^\nu \) by rectangular cells so that if \( z \) and \( z' \) belong to the same cell we have

\[
|z_j - z'_j| \leq \varepsilon, \quad 1 \leq j \leq \nu^{1/2}, \quad \nu^{1/2} < j \leq \nu.
\]

Thus, every cell is a product of two cubes of different size and dimension. The necessary number of cells \( M(\varepsilon) \) is bounded as follows,

\[
M(\varepsilon) \leq \left( \frac{\log \log \nu}{\varepsilon} \right)^{[\nu^{1/2}]} \epsilon^{-([\nu^{1/2}])} = (1/\varepsilon)^{V} (\log \log \nu)^{[\nu^{1/2}]}.
\]
\[ \leq C \int_{\nu^{1/2}}^{\nu} \frac{du}{u \log u} \varepsilon = C(\log \log \nu - \log(\frac{\log \nu}{2})) \varepsilon = C(\log 2) \varepsilon. \]

Finally,
\[ \rho_3 \leq \left( \sum_{j=1}^{\nu} p_j^{-1} \right) \max_{j \leq \nu^{1/2}} |z_j - z'_j| \leq C \left( \sum_{j=1}^{\nu} (j \log j)^{-1} \right) \frac{\varepsilon}{\log \log \nu} \leq C \varepsilon. \]

By summing up three estimates, we have \( \rho(z, z') \leq C N^{1/2 - \sigma} \Delta \varepsilon \) which enables the evaluation of the metric entropy.

Let \( N(T', \rho, u) \) be the minimal number of balls of radius \( u \) that cover the space \( (T', d) \). We have
\[ \log N \left( T', \rho, C N^{1/2 - \sigma} \Delta \varepsilon \right) \leq \log M(\varepsilon) \leq \nu |\log \varepsilon| + \nu^{1/2} \cdot \log \log \log \nu. \]

Observe also that
\[ \| \Upsilon(z) \|_2 \leq C N^{1/2 - \sigma} \Delta, \quad z \in T'. \] (21)

Hence, \( D := \text{diam}(T', \rho) \leq C N^{1/2 - \sigma} \Delta \), and by the classical Dudley’s entropy theorem (see [7], Theorem 1 p.179), for any fixed \( z \in T' \)
\[ E \sup_{z' \in T'} | \Upsilon(z') - \Upsilon(z) | \leq C \int_0^D \left[ \log N(T', \rho, u) \right]^{1/2} du \]
\[ \leq C \int_0^{C N^{1/2 - \sigma} \Delta} \left[ \log N(T', \rho, u) \right]^{1/2} du \]
\[ = C N^{1/2 - \sigma} \Delta \int_0^1 \left[ \log N(T', \rho, C N^{1/2 - \sigma} \varepsilon) \right]^{1/2} d\varepsilon \]
\[ \leq C N^{1/2 - \sigma} \Delta \int_0^1 [\nu |\log \varepsilon| + \log \log \log \nu \cdot \nu^{1/2}]^{1/2} d\varepsilon \]
\[ \leq C N^{1/2 - \sigma} \Delta \nu^{1/2}. \]

Using again (21), we have
\[ E \sup_{z' \in T'} | \Upsilon(z') | \leq C N^{1/2 - \sigma} \Delta \nu^{1/2}. \]

The final stage of the proof provides the optimal choice of the parameter \( \nu \) balancing the quantities (15), (17), and (22). In the first version, we brutally replace by \( \Delta \) the quantities \( \tilde{D}_2(N/p_\nu) \), resp. \( \tilde{D}_1(N/p_\nu) \) in (15) and (17). By taking any \( \nu \) in the fairly vast range
\[ \frac{\log^2 N}{\log^2 \log N} \leq \nu \leq \frac{N}{\log^2 N}, \]
we conclude that the main term is (15) and thus obtain the first result of the theorem.

Moreover, assuming (5) to be verified, we can choose \( \nu = N^h \), where \( h = h(b) = \frac{1}{\log(8b + 1)} \) satisfies \( h < 1 \). Due to the choice of \( h \), we have
\[ b + h/2 = b((1 - h) - h)/2 = 4b^2 + 2b - 1 \]
\[ 4(8b + 1) < 0, \]
(23)
since \( b < \left( \sqrt{5} - 1 \right)/4 \). The estimate (10) already provides a good order. We now take care of the two remaining bounds. For (17) we have

\[
\frac{C N^{1-\sigma} \tilde{D}_1(N/p_\nu)}{\nu^{1/2} \log \nu} \leq \frac{C N^{1-\sigma}}{\log N} \cdot \frac{\tilde{D}_2(N/p_\nu)}{\nu^{1/2}} \leq \frac{C N^{1-\sigma}}{\log N} \cdot N^{b(1-h)-h/2},
\]

which is good by (23). Similarly, for (22) we have

\[
CN^{1/2-\sigma} \tilde{D}_2(N) \nu^{1/2} \leq \frac{C N^{1-\sigma}}{\log N} \left( \nu^{b+h/2-1/2} \cdot \log N \right),
\]

which is also good by (23). Therefore, we obtain the second result of the theorem.

Finally, let us give the necessary modifications for the exceptional case \( \sigma = 1/2 \). The first deviation of the proof occurs at (10) where we have this time

\[
\sum_{\nu \leq j \leq \tau} d(m)^2 m^{-1} \leq C \tilde{D}_2(M) \log M.
\]

After substitution in (9) this yields

\[
\sum_{\nu \leq j \leq \tau} \sum_{n \in E_j} (n^2 m^{-1}(\alpha_j - \alpha_j'))^2 \leq C \sum_{\nu \leq j \leq \tau} (\alpha_j - \alpha_j')^2 \frac{\tilde{D}_2(N/p_j) \log \nu}{p_j},
\]

Furthermore, at the place of (13) we have

\[
K_m = \frac{d(m)}{m^{1/2}} \left( \sum_{\nu \leq j \leq \tau} \frac{p_j^{-1}}{m^{1/2}} \right)^{1/2} \leq C \frac{d(m)}{m^{1/2}} \left( \log \log(N/m) \right)^{1/2}.
\]

which results in

\[
\sum_{m \leq N/p_\nu} K_m \leq C \sum_{m \leq N/p_\nu} \frac{d(m)}{m^{1/2}} \left( \log \log(N/m) \right)^{1/2} \leq C \frac{N^{1/2} \tilde{D}_1(N/p_\nu) \log \nu}{\nu^{1/2} \log \nu}.
\]

at the place of (14) and (17). Next, at the place of (14) we have

\[
\mathbb{E} \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq C \sum_{\nu \leq j \leq \tau} \frac{\tilde{D}_2(N/p_j) \log^{1/2}(N/p_j)}{p_j} \leq C \tilde{D}_2(N/p_\nu) \sum_{\nu \leq j \leq \tau} \frac{\log^{1/2}(N/p_j)}{p_j} \leq C \frac{N^{1/2} \tilde{D}_2(N/p_\nu)}{\log N},
\]

(28)
where at the last step we applied Lemma 4 to the logarithmic function. Moreover, under assumption (8) one applies Lemma 4 to the function $f(r) = r^b \log^{1/2} r$, and obtains a bound

$$\mathbb{E} \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq \frac{C N^{1/2}}{\log N},$$

which perfectly matches (16).

The evaluation of the process $Q_1$ goes along the same lines as before until we have to use (24) thus arriving to

$$\rho(z, z') \leq C \left( \log \frac{N}{\tau} \right)^{1/2} \Delta \left\{ \sum_{j=1}^{\nu} |z_j - z'_j| \sum_{b=1}^{\infty} b^{1+H} p_j^{-b} \right\}.$$  (30)

at the place of (18). One observes that only the constant is different: $(\log N)^{1/2}$ stands for $N^{1/2-\sigma}$. By carrying on this constant along the rest of the calculation, we arrive at

$$\mathbb{E} \sup_{z' \in T^\nu} |\Upsilon(z')| \leq C \left( \log N \right)^{1/2} \Delta \nu^{1/2}.$$  (31)

at the place of (22). The concluding part of the proof which deals with optimal balance of three expressions does not require any substantial change since the most sensible expression (15), resp. (28) is the same in both cases.

3 Other results

Fix some positive integer $\tau \leq \pi(N)$, and recall that $p_1 < p_2 < \ldots$ is the sequence of primes. Put

$$\mathcal{E}_\tau = \mathcal{E}_\tau(N) = \{2 \leq n \leq N : P^+(n) \leq p_\tau\}.$$  

Note that for $\mu = \pi(N)$ we have $\mathcal{E}_\mu = \{2, \ldots, N\}$. The $\mathcal{E}_\tau$-based Dirichlet polynomials were considered in [12] and [8]. In this section, we will establish the theorem below extending the main result of [8] (Theorem 1.1) to the boundary case $\sigma = 1/2$.

**Theorem 5** a) Upper bound. Then there exists a constant $C$ such that for any integer $N \geq 2$ it is true that

$$\mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathcal{E}_\tau} \varepsilon_n n^{-1/2-it} \right| \leq \begin{cases} C \left( \frac{\tau}{\log \tau} \log \frac{N}{p_\tau} \right)^{1/2}, & \text{if } (N \log \log N)^{1/2} \leq \tau \leq \frac{N}{\log N}, \\ C (N \log \log N)^{1/4}, & \text{if } \frac{(N \log \log N)^{1/2}}{\log N} \leq \tau \leq (N \log \log N)^{1/2}, \\ C (\log N \tau)^{1/2}, & \text{if } 1 \leq \tau \leq \frac{(N \log \log N)^{1/2}}{\log N}. \end{cases}$$

b) Lower bound. There exists a constant $c$ such that for every $N \geq 2$,

$$\mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathcal{E}_\tau} \varepsilon_n n^{-1/2-it} \right| \geq c \left( \frac{\tau}{\log \tau} \min \left\{ \log \frac{N}{p_\tau} ; \log p_\tau/2 \right\} \right)^{1/2}.$$  

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Proof of the upper bound. We start exactly as in the proof of Theorem 1 until (9) where one has
\[
\sum_{\nu < j \leq \tau} \sum_{n \in E_j} n^{-2\sigma} (\alpha_j - \alpha'_j)^2 \leq \sum_{\nu < j \leq \tau} (\alpha_j - \alpha'_j)^2 p_j^{-2\sigma} \sum_{m \leq N/p_j} m^{-2\sigma}.
\] (32)

When \(\sigma = 1/2\), it follows that
\[
\sum_{\nu < j \leq \tau} \sum_{n \in E_j} n^{-2\sigma} (\alpha_j - \alpha'_j)^2 \leq C \sum_{\nu < j \leq \tau} (\alpha_j - \alpha'_j)^2 p_j^{-1} \log_+ (N/p_j).
\] (33)

Next, in (11) one obtains
\[
\sum_{\nu < j \leq \tau} \sum_{n \in E_j} \frac{(\beta_n - \beta'_n)^2}{n^{2\sigma}} \leq \sum_{m \leq N/p} (\beta_m - \beta'_m)^2 \left( \sum_{\nu < j \leq \tau: m \leq N/p} \frac{1}{(mp_j)^{2\sigma}} \right) := \sum_{m \leq N/p} K_m^2 (\beta_m - \beta'_m)^2.
\] (34)

while with \(\sigma = 1/2\) we have
\[
K_m^2 = \sum_{\nu < j \leq \tau: m \leq N/p} (mp_j)^{-1} = m^{-1} \sum_{\nu < j \leq \tau: m \leq N/p} p_j^{-1}.
\]

The upper summation border is different in two cases (\(p_j \leq p \tau\) vs \(p_j \leq N/m\)). Therefore, we distinguish two cases.

1) \(m \leq N/p\). Here, \(N/m \geq p\) and the border \(j \leq \tau\) is crucial. We obtain
\[
K_m^2 = m^{-1} \sum_{\nu < j \leq \tau: 1 \leq j \leq \tau} p_j^{-1} \leq m^{-1} \sum_{1 \leq j \leq \tau} p_j^{-1} \leq C m^{-1} \log j^{-1} \leq C m^{-1} \log \log \tau.
\]

It follows that
\[
\sum_{m \leq N/p} K_m \leq C \sum_{m \leq N/p} m^{-1/2} (\log \log \tau)^{1/2} \leq C (N/p)^{1/2} (\log \log \tau)^{1/2}.
\]

2) \(N/p \leq m \leq N/p\). Here, the border \(p_j \leq N/m\) is crucial. We choose a positive integer \(k\) such that \(N/p_k \sim m\) and obtain
\[
K_m^2 \leq m^{-1} \sum_{j \leq k} p_j^{-1} \leq C m^{-1} \sum_{1 \leq j \leq k} (j \log j)^{-1} \leq C m^{-1} \log \log k
\]
\[
\sim C m^{-1} \log \log p_k \sim C m^{-1} \log \log (N/m).
\]

It follows
\[
K_m \leq C m^{-1/2} [\log \log (N/m)]^{1/2}.
\]

Hence,
\[
\sum_{m \leq N/p} K_m \leq C \sum_{m \leq N/p} m^{-1/2} [\log \log (N/m)]^{1/2}
\]
As in [8], we define a second Gaussian process by putting for all $\gamma \in \Gamma$

$$Y(\gamma) = \sum_{\nu<j \leq \tau} (p_j^{-1} \log(p_j))^{1/2} \alpha_j \xi'_j + \sum_{m \leq N/p_\nu} K_m \beta_m \xi''_m := Y'_\gamma + Y''_\gamma,$$

where $\xi'_j, \xi''_j$ are independent $\mathcal{N}(0, 1)$ random variables. It follows from (33) and (34) that for some suitable constant $C_{\sigma}$, one has the comparison relations: for all $\gamma, \gamma' \in \Gamma$,

$$\|X_\gamma - X_{\gamma'}\|_2 \leq C_{\sigma} \|Y_\gamma - Y_{\gamma'}\|_2.$$

Next, by virtue of the Slepian comparison lemma, since $X_0 = Y_0 = 0$, we have

$$E \sup_{\gamma \in \Gamma} |X_\gamma| \leq 2E \sup_{\gamma \in \Gamma} X_\gamma \leq 2C_{\sigma} E \sup_{\gamma \in \Gamma} Y_\gamma \leq 2C_{\sigma} E \sup_{\gamma \in \Gamma} |Y_\gamma|.$$

It remains to evaluate the supremum of $Y$. First of all,

$$E \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq \sum_{\nu<j \leq \tau} p_j^{-1/2} |\log(p_j)|^{1/2}.$$

We have

$$\sum_{\nu<j \leq \tau} p_j^{-1/2} |\log(p_j)|^{1/2} \leq C \sum_{1<j \leq \tau} (j \log j)^{-1/2} |\log(p_j)|^{1/2} \leq C \left( \frac{\tau}{\log \tau} \right)^{1/2} |\log(p_{\nu})|^{1/2}.$$

thus

$$E \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq C \frac{\tau^{1/2}}{(\log \tau)^{1/2}} |\log(p_{\nu})|^{1/2}. \quad (35)$$

To control the supremum of $Y''$, we use our estimates for the sums of $K_m$ and write that

$$E \sup_{\gamma \in \Gamma} |Y''(\gamma)| \leq \sum_{m \leq N/p_\nu} K_m \leq C \left( \frac{N}{p_{\nu}} \right)^{1/2} (\log \tau)^{1/2} + C \left( \frac{N}{p_{\nu}} \right)^{1/2} |\log(p_{\nu})|^{1/2}.$$

Since

$$\frac{\log \log \tau}{p_{\tau}} \leq \frac{\log \log \nu}{p_{\nu}} \sim \frac{\log \log p_{\nu}}{p_{\nu}},$$

only the second term is important and by

$$\log \log p_{\nu} \sim \log \log \nu$$

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we obtain
\[
\mathbb{E} \sup_{\gamma \in \Gamma} |Y''(\gamma)| \leq C \left( \frac{N}{p_\nu} \log \log \nu \right)^{1/2}.
\]

(36)

Next, as in [8], we turn to the supremum of \( Q_1 \). Towards this aim, introduce the auxiliary Gaussian process
\[
Y(z) = \sum_{p^+(n) \leq p_\nu} n^{-\sigma} \left\{ \vartheta_n \cos 2\pi (a(n), z) + \vartheta'_n \sin 2\pi (a(n), z) \right\}, \quad z \in \mathbb{T}^\nu,
\]
where \( \vartheta_n, \vartheta'_n \) are independent \( \mathcal{N}(0,1) \) random variables. By symmetrization, \( \mathbb{E} \sup_{z \in \mathbb{T}^\nu} |Q_1(z)| \leq \sqrt{8\pi} \mathbb{E} \sup_{z \in \mathbb{T}^\nu} \|Y(z)\|_2 \), so that we are again led to evaluating the supremum of a real valued Gaussian process. For \( z, z' \in \mathbb{T}^\nu \) put
\[
d(z, z') := \|Y(z) - Y(z')\|_2,
\]
and observe that
\[
d(z, z')^2 = 4 \sum_{n: p^+(n) \leq p_\nu} \frac{1}{n^{2\sigma}} \sin^2(\pi (a(n), z - z')) \]
\[
\leq 4\pi^2 \sum_{n: p^+(n) \leq p_\nu} \frac{1}{n^{2\sigma}} |(a(n), z - z')|^2 \]
\[
\leq 4\pi^2 \sum_{n: p^+(n) \leq p_\nu} n^{-2\sigma} \left[ \sum_{j=1}^{\nu} a_j(n) |z_j - z'_j| \right]^2 \]
\[
= 4\pi^2 \sum_{n: p^+(n) \leq p_\nu} \sum_{j_1, j_2=1}^{\nu} a_{j_1}(n) a_{j_2}(n) |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| n^{-2\sigma} \]
\[
= 4\pi^2 \sum_{j_1, j_2=1}^{\nu} \sum_{n: p^+(n) \leq p_\nu} a_{j_1}(n) a_{j_2}(n) |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| n^{-2\sigma} \]
\[
\leq 4\pi^2 \sum_{j_1, j_2=1}^{\nu} |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2=1}^{\infty} b_1 b_2 \sum_{n \leq N, a_{j_1}(n) = b_1, a_{j_2}(n) = b_2} n^{-2\sigma} \]
\[
\leq 4\pi^2 \sum_{j_1, j_2=1}^{\nu} |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2=1}^{\infty} b_1 b_2 p_{j_1}^{-2b_1\sigma} p_{j_2}^{-2b_2\sigma} \sum_{k \leq N, k \geq p_{j_1}^{-b_1} p_{j_2}^{-b_2}} k^{-2\sigma}.
\]

(37)

In the case \( \sigma = 1/2 \) we obtain
\[
d(z, z')^2 \leq 4\pi^2 \sum_{j_1, j_2=1}^{\nu} |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2=1}^{\infty} b_1 b_2 p_{j_1}^{-b_1} p_{j_2}^{-b_2} \log_+ \left( \frac{N}{p_{j_1}^{-b_1} p_{j_2}^{-b_2}} \right).
\]

We ignore the denominator in the logarithm and move \( \log N \) ahead:
\[
d(z, z')^2 \leq C \log N \left\{ \sum_{j=1}^{\nu} |z_j - z'_j| \sum_{b=1}^{\infty} b p_j^{-b} \right\}^2,
\]

(38)

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or, equivalently

\[
d(\mathbf{z}, \mathbf{z}') \leq C (\log N)^{1/2} \sum_{j=1}^{\nu} |z_j - z_j'| \sum_{b=1}^{\infty} b \rho_j^{-b}. \tag{39}
\]

The subsequent estimates are identical to those of \(\text{[8]}\) with \(N^{1-2\sigma}\) being replaced by \((\log N)^{1/2}\) until we arrive at

\[
E \sup_{z' \in T^{\nu}} |\mathcal{Y}(z')| \leq C_\sigma (\log N)^{1/2} \nu^{1/2}. \tag{40}
\]

The final stage of the proof provides the optimal choice of the parameter \(\nu\) balancing the quantities (35), (36), and (40). As suggests the Theorem’s claim, we consider three cases.

**Case 1.** \((N \log \log N)^{1/2} \leq \tau \leq \frac{N}{\log N} \).

In this case we choose

\[
\nu = \frac{(N \log \log N)^{1/2}}{\log N} \tag{41}
\]

thus balancing (36) and (40). Both terms yield then \((N \log \log N)^{1/4}\) which is dominated in this zone by the constant \((35)\). From \((35)\) we obtain the bound

\[
C_\sigma \left( \frac{\tau}{\log \tau} \log \left( N/p_\tau \right) \right)^{1/2}. \]

The correctness condition \(\nu \leq \tau\) is obvious.

**Case 2.** \(\frac{(N \log \log N)^{1/2}}{\log N} \leq \tau \leq (N \log \log N)^{1/2} \).

In this case we still choose \(\nu\) from (41) thus balancing (36) and (40) and getting the bound \((N \log \log N)^{1/4}\). The difference is that in this range the constant term \((35)\) is negligible. It follows that our total bound is \((N \log \log N)^{1/4}\). The correctness condition \(\nu \leq \tau\) is still obvious for the range under consideration.

**Case 3.** \(1 \leq \tau \leq \frac{(N \log \log N)^{1/2}}{\log N} \).

Here we just set \(\nu = \tau\). It means that we do not need the splitting of the polynomial in two parts. Formally, the quantities \((35)\) and \((36)\) are not necessary and we obtain the bound \(C_\sigma (\log N \tau)^{1/2}\) directly from (40).

The upper bound is proved completely.

**Proof of the lower bound.** We shall first collect some auxiliary useful results, valid for general Dirichlet polynomials. Next we will apply them to the considered setting. Let \(d = \{d_n, n \geq 1\}\) be a sequence of reals. Recall that by (1) we have

\[
\sup_{t \in \mathbb{R}} \left| \sum_{j=1}^{\tau} \sum_{n \in E_j} d_n \varepsilon_n n^{-\sigma - it} \right| = \sup_{\mathbf{z} \in \mathbb{T}^\tau} |Q(\mathbf{z})|.
\]

where

\[
Q(\mathbf{z}) = \sum_{j=1}^{\tau} \sum_{n \in E_j} d_n \varepsilon_n n^{-\sigma} e^{2\pi i \langle \mathbf{z}(n), \mathbf{w} \rangle}.
\]

Consider the subset \(Z\) of \(\mathbb{T}^\tau\) defined by

\[
Z = \{ \mathbf{z} = \{z_j, 1 \leq j \leq \tau\} : z_j = 0, \text{if } j \leq \tau/2, \text{and } z_j \in \{0, 1/2\}, \text{if } j \in (\tau/2, \tau) \}.
\]
Observe that the imaginary part of $Q$ vanishes on $\mathcal{Z}$, since for any $z \in \mathcal{Z}$ and any $n$ it is true that
\[ e^{2\pi i(\langle a(n), \bar{z} \rangle)} = \cos(2\pi(\langle a(n), z \rangle)) = (-1)^{\langle a(n), \bar{z} \rangle}. \]
Hence, $Q$ takes the following simple form on $\mathcal{Z}$
\[ Q(z) = \sum_{\tau/2 < j \leq \tau} \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n n^{-\sigma} (-1)^{\langle a(n), \bar{z} \rangle}. \]
This is no longer a trigonometric polynomial, but simply a finite rank Rademacher process.

For $j \in (\tau/2, \tau]$ define
\[ \mathcal{L}_j = \left\{ n = p_j \tilde{n} : \tilde{n} \leq \frac{N}{p_j} \text{ and } P^+(\tilde{n}) \leq p_{\tau/2} \right\}. \]
Since
\[ E_j \supset \mathcal{L}_j, \quad j = 1, \ldots, \tau, \]
the sets $\mathcal{L}_j$ are pairwise disjoint.

Put for $z \in \mathcal{Z}$,
\[ Q'(z) = \sum_{\tau/2 < j \leq \tau} \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n n^{-\sigma} (-1)^{\langle a(n), \bar{z} \rangle}. \]
We now recall a useful fact. ([8] Lemma 3.1)

**Lemma 6** Let $X = \{X_z, z \in \mathcal{Z}\}$ and $Y = \{Y_z, z \in \mathcal{Z}\}$ be two finite sets of random variables defined on a common probability space. We assume that $X$ and $Y$ are independent and that the random variables $Y_z$ are all centered. Then
\[ E \sup_{z \in \mathcal{Z}} |X_z + Y_z| \geq E \sup_{z \in \mathcal{Z}} |X_z|. \]
Clearly, since $\{Q(z) - Q'(z), z \in \mathcal{Z}\}$ and $\{Q'(z), z \in \mathcal{Z}\}$ are independent,
\[ E \sup_{z \in \mathcal{Z}} |Q(z)| \geq E \sup_{z \in \mathcal{Z}} |Q'(z)|. \]
We now proceed to a direct evaluation of $Q'(z)$ by proving

**Proposition 7** There exists a universal constant $c$ such that for any system of coefficients $(d_n)$
\[ c \sum_{\tau/2 < j \leq \tau} \left| \sum_{n \in \mathcal{L}_j} d_n^2 \right|^{1/2} \leq E \sup_{z \in \mathcal{Z}} |Q'(z)| \leq \sum_{\tau/2 < j \leq \tau} \left| \sum_{n \in \mathcal{L}_j} d_n^2 \right|^{1/2}. \]

**Corollary 8** If $(d_n)$ is a multiplicative system (namely $d_{nm} = d_n d_m$ if $n, m$ are coprimes), we have
\[ E \sup_{z \in \mathcal{Z}} |Q'(z)| \geq c \sum_{\tau/2 < j \leq \tau} d_{p_j} \left( \sum_{\tilde{n} \leq N/p_j} \left| \sum_{P^+(\tilde{n}) \leq p_{\tau/2}} d_{\tilde{n}}^2 \right|^{1/2} \right). \]
When $\sigma = 1/2$, we apply Corollary 3.3 with $d_n = n^{-1/2}$ which yields

$$E \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathbb{E}_r} \varepsilon_n n^{-1/2 - it} \right| \geq c \sum_{\tau/2 < j \leq \tau} p_j^{-1/2} \left( \sum_{n \leq N/p_j \atop p^+(n) \leq \tau/2} n^{-1} \right)^{1/2}$$

$$\geq c \frac{\tau}{2} p_r^{-1/2} \left( \sum_{n \leq \min\{N/p_r; p_r/2\}} n^{-1} \right)^{1/2}$$

$$\geq \left( \frac{c \tau}{\log \tau} \right)^{1/2} \left( \log_+ \min\{N/p_r; p_r/2\} \right)^{1/2}.$$ 

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