An even simpler understanding of quantum weak values

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ABSTRACT:
We explain the properties and clarify the meaning of quantum weak values using only the basic notions of elementary quantum mechanics.

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...And look not for answers where no answers can be found.

Bob Dylan

In a recent publication \cite{1} Qin and co-authors sought to provide a simplified understanding of the physics of the so-called weak measurements (for a recent review see \cite{2}). They formulated their discussion in the framework of the quantum Bayesian approach \cite{3}, and followed other authors \cite{4,5} in asserting that "anomalous" weak values (WV) may not occur in a purely classical context. One may wonder whether a yet more straightforward explanation of these properties could be obtained directly from the basic principles of quantum theory. In the following, we will provide such an explanation.

In quantum mechanics, e.g., in its field and many-body theory, but the quantum nature of the problem dictates that in order to evaluate $P^{φ←ψ}$ for the system to start in an initial state $\psi$ and end up, after some time, in a final state $φ$, the quantity of interest is often the probability $P^{φ←ψ}$, one must first obtain a complex valued transition probability amplitude $A^{φ←ψ}$, so that

$$P^{φ←ψ} = |A^{φ←ψ}|^2.$$  \hspace{1cm} (1)

Typically, an amplitude can be decomposed into various sub-amplitudes, corresponding to elementary processes, which all lead to the same outcome $φ$, $A^{φ←ψ} = \sum_{n} A_{n}^{φ←ψ}$. \hspace{1cm} (2)

For example, for a system of interacting particles, $A_{n}^{φ←ψ}$ could correspond to Feynman diagrams describing various scattering scenarios \cite{6}. The scenarios are "virtual", in the sense that only the probability amplitudes, and not the probabilities, can be ascribed to them individually. Together, virtual scenarios form a "real" pathway, connecting $ψ$ with $φ$, which the system will be seen as taking with the probability \cite{1}, if the experiment is repeated many times.

A simple illustration of the above is the Young’s double slit experiment, sketched in Fig.\textsuperscript{1a}. An electron starts at some location $(x,y)$, and ends up in a final position $(x',y')$, which it can reach through two holes made in the screen. There are two virtual pathways, passing through the holes 1 and 2, with the probability amplitudes $A_{1}^{(x',y')←(x,y)}$ and $A_{2}^{(x',y')←(x,y)}$, respectively. A well known feature of quantum description is the impossibility do decide which of the two routes was actually taken. Any attempt to accurately determine it, destroys the interference pattern, by changing the probability $P^{(x',y')←(x,y)}$ from $|A_{1}^{(x',y')←(x,y)} + A_{2}^{(x',y')←(x,y)}|^2$ to $|A_{1}^{(x',y')←(x,y)}|^2 + |A_{2}^{(x',y')←(x,y)}|^2$. If no such attempt is made, "one may not say that an electron goes either through hole 1 or hole 2" \cite{7}. The two virtual routes together form for the electron a single real pathway from $(x,y)$ to $(x',y)$. This is the uncertainty principle \cite{6}.

A further simplification of the double slit experiment, which brings us closer to issue of weak values, is shown in Fig.\textsuperscript{1b}. Let a system, consisting of spin 1/2, start in a state $|ψ⟩$ at $t = 0$, evolve with a Hamiltonian $\hat{H}$ until $t = T$, and then be observed in the final state $|φ⟩$. Choosing an arbitrary basis \{\{i\}, \{i\}′\} = δ\_ij, $i = 1,2$, and inserting the unity $\sum_{i=1}^{2} |i⟩⟨i|$ at $t = T/2$, we can write the transition amplitude $A^{φ←ψ} = ⟨φ|\exp(-i\hat{H}T)|ψ⟩$ as

$$A^{φ←ψ} = A_{1}^{φ←ψ} + A_{2}^{φ←ψ},$$  \hspace{1cm} (3)

where $(i = 1,2)$.

$$A_{i}^{φ←ψ} = ⟨φ|\exp(-i\hat{H}T/2)|i⟩⟨i|\exp(-i\hat{H}T/2)|ψ⟩.$$ \hspace{1cm} (4)

Unless one of the states $|i⟩$ coincides with $|ψ⟩$ or $|φ⟩$, we have an analogue of the double slit experiment, with $|i⟩$’s playing the role of the two holes, and $|φ⟩$ representing the final position of the electron. Our intention is to see whether the first "hole" was chosen by the system. To obtain a yes/no answer, we couple the spin to a von Neumann pointer at $t = T/2$, using the interaction Hamiltonian,

$$\hat{H}_{\text{int}} = -i\partial_{t}\hat{\Pi}_{1}\delta(t - T/2),$$  \hspace{1cm} (5)
If the experiment is repeated many times, each run of the experiment the pointer will be shifted by either 0, or 1. If the two terms in (6) can have a non-zero value, and in values. Thus the routes will be seen as travelled with the probabilities \( \omega_i, i = 1, 2 \)

\[
\omega_i^{\phi-\psi} = \lim_{N \to \infty} \frac{N_i}{N} = \frac{|A_i^{\phi-\psi}|^2}{|A_1^{\phi-\psi}|^2 + |A_2^{\phi-\psi}|^2}.
\]

With this we can determine the mean value of the number operator \( \hat{N}_1 \) simply by writing down 1 whenever the spin is seen to pass via the state \( |1\rangle \), and 0 when it passes via the state \( |2\rangle \), add up the results, and divide by the number of trials \( N \),

\[
\langle \hat{N}_1 \rangle^{\phi-\psi} = \frac{1 \times N_1 + 0 \times N_2}{N} = \omega_1^{\phi-\psi} = \langle f \rangle^{\phi-\psi}.
\]

We can number the routes as 1 and 2, and ask the "which route?" question, by measuring instead of \( \hat{N}_1 \) the "route number operator"

\[
\hat{n} = |1\rangle n_1 \langle 1| + |2\rangle n_2 \langle 2|,
\]

so that our accurate meter reads 1 or 2, depending on whether the system passes through the states \( |1\rangle \) or \( |2\rangle \). By linearity, the mean value of any operator with the eigenvalues \( B_i, \hat{B} = \sum_{i=1}^{2} |i\rangle B_i \langle i| \) must be given by

\[
\langle \hat{B} \rangle^{\phi-\psi} = \omega_1^{\phi-\psi} B_1 + \omega_2^{\phi-\psi} B_2,
\]

and the mean route number is \( \langle \hat{n} \rangle^{\phi-\psi} = \omega_1^{\phi-\psi} + 2 \omega_2^{\phi-\psi} \). We note that \( 1 \leq \langle \hat{n} \rangle^{\phi-\psi} \leq 2 \), and, from the position of \( \langle \hat{n} \rangle^{\phi-\psi} \) inside the interval, it is possible to decide which of the two routes is travelled more often.

So far, there has been little "quantum" in our attempt to answer the "which way?" question. Out of the original quantum system, we have "manufactured" a simple classical system, capable of reaching its final state by taking one of the two available paths at random (see Fig. 2a).

The measured operator is replaced by a functional on the paths, whose mean value is obtained by recording \( B_1 \) or \( B_2 \), depending on the path taken, summing all values, and dividing the result by the number of trials \( N \).

The quantum nature of the problem comes to light if one tries to answer the "which way?" question with the interference between the routes intact. We may try to use the same number to measure \( \hat{n} \) in equation (10), but this time making sure that no new "real" (as opposed to "virtual") routes are created for the system, and the transition is perturbed as little as possible. One way to achieve this is to make the meter highly inaccurate, by choosing \( G(f) \) so broad, that \( G(f - 1) \approx G(f - 2) \approx G(f) \), and equation (11) becomes \( \langle \hat{n} \rangle^{\phi-\psi} \approx G(f) A^{\phi-\psi} \). As a result, the pointer's readings become equally spread over the whole real axis. This is what we should expect from the
the projector $\hat{\Pi}$ for the first virtual route. It is easy to check that if the real first route is travelled in equation (7), the probability of interference present, the probability
\[ \alpha_i \]
where
\[ \omega_i \]
the mean shift of the pointer in the limit $\Delta f \rightarrow \infty$ creates two alternative pathways, \{1\} and \{2\}, which can be seen to take with the probabilities $\omega_i^{\phi+\psi}$ in equation (8); (b) in an inaccurate (weak) measurement there is a single real pathway, arising from the interference between the virtual paths \{1\} and \{2\}.

uncertainty principle, which suggests that number of the route taken by the spin, like the number of the slit taken in Fig.1a, remains indeterminate, provided the routes interfere. Indeed, according to Feynman [6], this "which way?" question cannot be answered, and our experiment gives the only answer possible under the circumstances, which must be "anything at all".

The mean pointer reading is, however, uniquely defined for any choice of the initial and final states. In an accurate measurement, the mean shift of an accurate meter cannot exceed the larger numerator of the ratio in (14) very small, while its denominator remains finite, in which case the mean shift of an inaccurate pointer can again be anything at all, for any choice of the operator $\hat{B}$.

always has a solution, so that for any given $|\psi\rangle$ it is always possible to find a $|\phi\rangle$, such that in equation (13) the complex valued quantity in the curly brackets takes any complex value $Z$. Thus, with all final states considered, the mean shift of an inaccurate pointer can again be anything at all, for any choice of the operator $\hat{B}$.

The mechanism, which allows $Z$ in equation (14) to take an arbitrary value is simple. The l.h.s. of (14) has the form of an average, computed with a distribution $\eta$, which can take complex values. What is more important, its real and imaginary parts do not have to have definite signs. For example, by choosing $|\psi\rangle$ and $|\phi\rangle$ to be nearly orthogonal, $\eta_1 \approx -\eta_2$, one can make the denominator of the ratio in (14) very small, while its numerator remains finite, in which case the mean shift of an inaccurate pointer will be very large. Note that this could never happen in an accurate measurement, since both $\omega_i^{\phi+\psi}$ in equation (11) are non-negative, and the mean shift of an accurate meter cannot exceed the larger of the eigenvalues $B_1$, nor be smaller than the smaller one. By the same token, such anomalously large values cannot occur in purely classical theories, operating only with non-negative probabilities, contrary to the suggestion made in the much criticised (see [11,12,13], and Refs.
therein) work by Ferrie and Combes [13].

In the title we have promised to provide a simple un-
derstanding of quantum weak values, a task we have
avoided mentioning so far. Above we have demonstrated
that a response of quantum system to probe by a par-
ticular weak interaction is formulated in terms of the
corresponding probability amplitudes. This could be an-
ticipated from a textbook on perturbation theory. We
have also checked that the results are in full agreement
with the uncertainty principle, and are, to a large degree,
dictated by it. Next we try to describe these results in
terms of the "weak values", as they were introduced in
[14], and find the origin of the controversy which follows
the subject.

For an accurate meter, we were able to evaluate the mean
shift of the pointer using (7), calculate independently
the mean value of the measured quantity in equation (11),
and find a perfect agreement between the two. For an inaccu-
rate (weak) meter, we can still evaluate the mean
shift in (12), but do not know how to calculate the mean
value of the projector in the presence of interference. One
possible course of action is to use the similarity between
Eqs. (7) and (12), and define its intrinsic mean to be
the complex valued quantity in the curly brackets in (12).
Although this probability amplitude already has a name,
we can follow [14] in re-branding it as a "weak value of
the projector \( \Pi_1 \) for a system pre- and post-selected in
the states \( |\psi\rangle \) and \( |\phi\rangle \)’, \( \phi(\Pi_1)|\psi\rangle \). The change is purely
cosmetic, and our result still reads

\[
\phi(\Pi_1)|\psi\rangle \equiv \text{the (relative) probability amplitude to reach } |\phi\rangle \text{ from } |\psi\rangle, \text{ via path } \{1\} \text{ in Fig.1b. (15)}
\]

\[
= \frac{\langle \phi | \exp(-iHT/2)\Pi_1 \exp(-iHT/2)|\psi \rangle}{\langle \phi | \exp(-iHT)|\psi \rangle}.
\]

This definition is readily extended to an arbitrary
operator \( \hat{B} \), whose "weak value" \( \phi(\hat{B})|\psi\rangle =
\frac{\langle \phi | \exp(-iHT/2)\hat{B} \exp(-iHT/2)|\psi \rangle}{\langle \phi | \exp(-iHT)|\psi \rangle} \), reduces to its original
definition [14] for \( \hat{H} = 0, \)

\[
\phi(\hat{B})|\psi\rangle = \frac{\langle \phi | \hat{B}|\psi \rangle}{\langle \phi | \psi \rangle}, (16)
\]

and is the sum of all such amplitudes, weighted by
the eigenvalues \( B_i \). The properties of probability amplitudes
are well known, some of them have been discussed above,
and there is nothing unusual about the weak values so far.
The main controversy stems from [14], and is of its au-
thor’s own making. The authors considered an inac-
curate measurement of the \( z \)-component of a spin 1/2,
pre- and post-selecting it in two nearly orthogonal states.
They found a mean pointer shift to be 100, which is
hardly surprising, given that the corresponding relative
probability amplitudes in equation [13] are large, since
the transition is unlikely and \( \sum_{i=1}^{2} A_i \phi_{i+1}|\psi\rangle \) is small. Yet in [13] this outcome is presented as an "usual" result
of a "usual" measuring procedure.

The two quoted adjectives should, in fact, be inter-
changed. The measuring procedure is hardly a usual
one, since in the chosen regime the meter ceases to
destroy interference between measured alternatives, which
according to Bohm [15] is likely to lead to "absurd re-
results". This is readily seen from our analysis, but may
be less clear in original approach used in [14], where the
authors chose to reduce the coupling to the pointer, in-
stead of broadening its initial state. The two methods are
equivalent, since scaling the pointers position \( f \rightarrow f/\gamma \)
is equivalent to multiplying \( H_{int} \) in equation (5) by \( \gamma \).
The result is, however, what one would expect. Above
we have shown that a weak measurement must be able
to produce all possible results, with mean shifts that are
large, small, negative and positive. This is necessary, if
we are to satisfy the uncertainty principle which forbids
such results. Instead of dwelling on the meaning of two interfering alternatives [16]. Indeed, if a group of experimenters decide to make
accurate measurements of the \( z \)-component of a spin 1/2,
using all possible initial and final states, they will be able
to agree that all readings are either 1/2 or \(-1/2\), and
draw further conclusions about the nature of the studied
system. If the same experiment is repeated with inaccu-
rate weak meters, the measured shifts will lie anywhere
on the real axis. Making the experimenters evaluate ac-
curately mean shifts for each choice of \( |\psi\rangle \) and \( |\phi\rangle \), would
not help either. The mean shifts will also lie everywhere
on the real axis, and the researches will be able to agree
only on that "anything is possible".

In summary, we note that the uncertainty principle has
elegantly frustrated our attempt to answer the "which
way?" question in the presence of interference. We
started with a theoretical notion of a probability ampli-
tude and employed a weak meter, hoping to gain further
insight into what happens when the alternatives interfere.
In the end, in this practical way, we arrived at noth-
ing more than the very probability amplitude we started
with.

Identification of the weak values with probability am-
plitudes has, however, the advantage of explaining most
of WV’s controversial properties, using only the notions
from the first chapter of Feynman’s textbook [6]. Firstly,
the existence of "anomalous" weak values, lying outside
the spectrum of the measured operator, is a rule, rather
then an exception. They are just as common as the "nor-
mal" weak values, which occur when all amplitudes have
the same sign, and coincide with the accurate mean val-
ues if only one of the amplitudes has a non-zero value.
From this it is clear that no"anomalous" mean values can
be found in a classical theory, where all physical prob-
abilities are non-negative. Secondly, the authors of [14]
have measured a difference between two large relative
amplitudes of opposite signs, where an accurate meas-
urement would give the difference between two probabilities,
and should not be surprised by the large result. In the
three-box case [17], involving three virtual paths with
the amplitudes \( A_1^{\phi_{i+1}}|\psi\rangle = A_2^{\phi_{i+2}}|\psi\rangle = -A_3^{\phi_{i+3}}|\psi\rangle = 1, \) simulta-
neous weak measurements of the projectors on the first
and the second paths both yield values of 1. This does not mean that the particle is "in two places at the same time", but simply confirms the relation between the amplitudes, already known to us. A similar simple analysis can be applied to explain other "surprising" results, obtained within the weak measurement formalism. Finally, Vaidman's observation [18] that "The weak value shifts exist if measured or not" comes out as trivial, given that the WV's are nothing but the probability amplitudes, or their combinations.

It has not been our intention to belittle the technological effort invested in experimental realisations of "weak measurements" [19], or their practical application in metrology [11]. In our view, such efforts can only be helped by clarifying the status of the measured quantities within the framework of elementary quantum mechanics.

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