Dual Logic Concepts based on Mathematical Morphology in Stratified Institutions: Applications to Spatial Reasoning

Marc Aiguier\textsuperscript{1} and Isabelle Bloch\textsuperscript{2}
1. MICS, CentraleSupelec, Université Paris Saclay, France
marc.aiguier@centralesupelec.fr
2. LTCI, Télécom ParisTech, Université Paris Saclay, Paris, France
isabelle.bloch@telecom-paristech.fr

Abstract
Several logical operators are defined as dual pairs, in different types of logics. Such dual pairs of operators also occur in other algebraic theories, such as mathematical morphology. Based on this observation, this paper proposes to define, at the abstract level of institutions, a pair of abstract dual and logical operators as morphological erosion and dilation. Standard quantifiers and modalities are then derived from these two abstract logical operators. These operators are studied both on sets of states and sets of models. To cope with the lack of explicit set of states in institutions, the proposed abstract logical dual operators are defined in an extension of institutions, the stratified institutions, which take into account the notion of open sentences, the satisfaction of which is parametrized by sets of states. A hint on the potential interest of the proposed framework for spatial reasoning is also provided.

Keywords: Stratified institutions, mathematical morphology, dual operators, dilation, erosion, states, spatial reasoning.

1 Introduction
There exists a profusion of logics but all of them satisfy the same structure defined by a syntax, a semantics and a calculus. Syntax gives both the language (signatures) and the formal rules that define well-formed formulas and theories. Semantics, so-called model theory, gives the mathematical meaning of all these syntactic notions, among others the rules that associate truth values to formulas. Finally, calculus, so-called proof theory, gives the inference rules that govern the reasoning and thus translate semantics into syntax as correctly as possible. To cope with the explosion of logics, a categorical abstract model-theory, the theory of institutions \cite{19, 20}, has been proposed, that generalizes
Barwise’s “Translation Axiom” \[6\]. Institutions then define both syntax and semantics of logics at an abstract level, independently of commitment to any particular logic. Later, institutions have been extended to propose a syntactic approach to truth \[18, 19, 24, 32\]. For the sake of generalization, in institutions signatures are simply defined as objects of a category and formulas built over signatures are simply required to form a set. All other contingencies such as inductive definition of formulas are not considered. However, the reasoning (both syntactic and semantic) is defined by induction on the structure of formulas. Indeed, usually, formulas are built from “atomic” formulas by applying iteratively operators such as connectives, quantifiers or modalities. What we can then observe is that most of these logical operators come through dual pairs (conjunction and disjunction $\land$ and $\lor$, quantifiers $\forall$ and $\exists$, modalities $\Box$ and $\Diamond$).

When looking at the algebraic properties of mathematical morphology \[12, 39\] on the one hand, and of all these dual operators on the other hand, several similarities can be shown, and suggest that links between institutions and mathematical morphology are worth to be investigated. This has already been done in the restricted framework of modal propositional logic \[8\]. In \[8\], it was then shown that modalities $\Box$ and $\Diamond$ can be defined as morphological erosion and dilation. The interest is, based on properties of morphological operators, that this leads to a set of axioms and inference rules which are de facto sound. In this paper, we propose to extend this work by defining, at the abstract level of institutions, a pair of abstract operators as morphological erosion and dilation. We will then show how to obtain standard quantifiers and modalities from these two abstract operators.

In mathematical morphology, erosion and dilation are operations that work on lattices, for instance on sets. Thus, they can be applied to formulas by identifying formulas with sets. We have two ways of doing this, either given a model $M$ identifying a formula $\varphi$ by the set of states $\eta$ that satisfy $\varphi$ and classically noted $M \models_\eta \varphi$, or identifying $\varphi$ by the set of models that satisfy it. As usual in logic, our abstract dual operators based on morphological erosion and dilation will be studied both on sets of states and sets of models. The problem is that institutions do not explicit, given a model $M$, its set of states. This is why we will define our abstract logical dual operators based on erosion and dilation in an extension of institutions, the stratified institutions \[3\]. Stratified institutions have been defined in \[3\] as an extension of institutions to take into account the notion of open sentences, the satisfaction of which is parametrized by sets of states. For instance, in first-order logic, the satisfaction is parametrized by the valuation of unbound variables, while in modal logics it is further parametrized by possible worlds. Hence, stratified institutions allow for a uniform treatment of such parametrizations of the satisfaction relation within the abstract setting of logics as institutions.

Another interest of the approach proposed in this paper is that mathematical morphology provides tools for spatial reasoning. Until now, mathematical morphology has been used mainly for quantitative representations of spatial relations, or semi-qualitative ones, in a fuzzy set framework (see e.g. \[9\]). For
qualitative spatial reasoning, several symbolic and logical approaches have been developed (see e.g. [1, 2, 30]), but mathematical morphology has not been much used in this context to our knowledge. In this paper, inspired by the work that was done in [8, 10, 12, 13] in the propositional and modal logic framework, we show how logical connectives based on morphological operators can be used for symbolic representations of spatial relations. Indeed, spatial relations are a main component of spatial reasoning [1], and several frameworks have been proposed to model spatial relations and reason about them in logical frameworks (see e.g. [7, 16, 17, 37, 41] for topological relations, [30, 33] for directional relations, [38] for constraint based techniques for topology, distances and directions, and [9, 10] for semi-qualitative representations in the framework of fuzzy sets). Since it is usual to introduce uncertainty in qualitative spatial reasoning, we propose to extend our abstract logical connectives based on erosion and dilation to the fuzzy case. This first requires to develop fuzzy reasoning in stratified institutions. Fuzzy (or many-valued) reasoning has an institutional semantics [20, 21]. The approach proposed here is substantially similar to that proposed in [20], although developed in stratified institutions.

The paper is organized as follows. Section 2 reviews some concepts, notations and terminology about institutions and stratified institutions which are used in this work. In Section 3 we propose to define abstractly the important concept of Boolean connectives, quantifiers, and fuzzy reasoning in stratified institutions. Section 4 introduces a new way to build dual operators from the notion of morphological erosion and dilation operators. We study two ways to build such dual operators. We first define them from morphological dilation and erosion of formulas based on a structuring element, and then as algebraic erosion and dilation over the lattice of formulas. This last point allows us to define modalities when they are interpreted topologically as algebraic erosion and dilation. Finally, in Section 5 we show how these modalities can be interpreted for abstract spatial reasoning using qualitative representations of spatial relationships derived from mathematical morphology.

2 Stratified institutions

The notions introduced here make use of basic notions of category theory (category, functors, natural transformations, etc.). We do not present these notions in these preliminaries, but interested readers may refer to textbooks such as [5, 31].

2.1 Institutions

Let us start by recalling the definition of institutions, over which stratified institutions are defined as an extension, by introducing the notion of states for models.

Definition 2.1 (Institution) An institution \( \mathcal{I} = (\text{Sig}, \text{Sen}, \text{Mod}, \models) \) consists of
• a category \( \text{Sig} \), objects of which are called signatures and are denoted \( \Sigma \),

• a functor \( \text{Sen} : \text{Sig} \to \text{Set} \) giving for each signature \( \Sigma \) a set \( \text{Sen}(\Sigma) \), elements of which are called sentences,

• a contravariant functor \( \text{Mod} : \text{Sig}^{\text{op}} \to \text{Cat} \) giving for each signature \( \Sigma \) a category, objects and arrows of which are called \( \Sigma \)-models and \( \Sigma \)-morphisms respectively, and

• a \( \text{Sig} \)-indexed family of relations \( \models_{\Sigma} \subseteq \text{Mod}(\Sigma) \times \text{Sen}(\Sigma) \) called satisfaction relation, such that the following property, called the satisfaction condition, holds:

\[
\forall \sigma : \Sigma \to \Sigma', \forall M' \in \text{Mod}(\Sigma'), \forall \varphi \in \text{Sen}(\Sigma),
\]

\[
M' \models_{\Sigma'} \text{Sen}(\sigma)(\varphi) \iff \text{Mod}(\sigma)(M') \models_{\Sigma} \varphi
\]

**Notation 2.2** The functor \( \text{Mod} \) can be extended to formulas. Hence, given a signature \( \Sigma \) and two formulas \( \varphi, \psi \in \text{Sen}(\Sigma) \), we note:

- \( \text{Mod}(\varphi) = \{ M \in \text{Mod}(\Sigma) \mid M \models_{\Sigma} \varphi \} \),
- \( \varphi \models \psi \iff \text{Mod}(\varphi) \subseteq \text{Mod}(\psi) \), and
- \( \varphi \equiv \psi \iff \text{Mod}(\varphi) = \text{Mod}(\psi) \).

**Example 2.3** The following examples of institutions are of particular importance both in computer science and in this paper. Many other examples can be found in the literature (e.g. [19, 26, 40]).

**Propositional Logic (PL)** The category of signatures is \( \text{Set} \), the category of sets and functions.

Given a signature \( P \), the set of \( P \)-sentences is the least set of sentences finitely built over propositional variables in \( P \) and Boolean connectives in \( \{ \neg, \lor, \land, \Rightarrow \} \). Given a signature morphism \( \sigma : P \to P' \), \( \text{Sen}(\sigma) \) translates \( P \)-formulas to \( P' \)-formulas by renaming propositional variables according to \( \sigma \).

Given a signature \( P \), the category of \( P \)-models is \( (\{0,1\}^P, \leq) \) such that 0 and 1 are the usual truth values, and \( \leq \) is a partial ordering such that \( \nu \leq \nu' \) iff \( \forall p \in P, \nu(p) \leq \nu'(p) \). Given a signature morphism \( \sigma : P \to P' \), the forgetful functor \( \text{Mod}(\sigma) \) maps a \( P' \)-model \( \nu' \) to the \( P \)-model \( \nu = \nu' \circ \sigma \).

Finally, satisfaction is the usual propositional satisfaction.

**Many-sorted First Order Logic (FOL)** Signatures are triplets \( (S, F, P) \) where \( S \) is a set of sorts, and \( F \) and \( P \) are sets of function and predicate names respectively, both with arities in \( S^* \times S \) and \( S^+ \) respectively.\(^2\) Signature morphisms \( \sigma : (S, F, P) \to (S', F', P') \) consist of three functions between

---

\(^1\)Standardly in category theory, \( \text{Sig}^{\text{op}} \) is the opposite of \( \text{Sig} \) by reversing morphisms.

\(^2\)\( S^+ \) is the set of all non-empty sequences of elements in \( S \) and \( S^* = S^+ \cup \{ \epsilon \} \) where \( \epsilon \) denotes the empty sequence.
sets of sorts, sets of functions and sets of predicates respectively, the last two preserving arities.

Given a signature $\Sigma = (S, F, P)$, the $\Sigma$-atoms are $p(t_1, \ldots, t_n)$, where $p : s_1 \times \ldots \times s_n \in P$ and $t_i \in T_F(X)_{s_i}$ ($1 \leq i \leq n$, $s_i \in S$). The set of $\Sigma$-sentences is the least set of formulas built over the set of $\Sigma$-atoms by finitely applying Boolean connectives in $\{\neg, \lor, \land, \Rightarrow\}$ and the quantifiers $\forall$ and $\exists$. Given a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, $\operatorname{Sen}(\sigma)$ is the mapping defined by renaming functions and predicates according to $\sigma$.

Given a signature $\Sigma = (S, F, P)$, a $\Sigma$-model $\mathcal{M}$ is a family $\mathcal{M} = (M_s)_{s \in S}$ of sets (one for every $s \in S$), each one equipped with a function $f^M : M_{s_1} \times \ldots \times M_{s_n} \rightarrow M_s$ for every $f : s_1 \times \ldots \times s_n \rightarrow s \in F$ and with a $n$-ary relation $p^M \subseteq M_{s_1} \times \ldots \times M_{s_n}$ for every $p : s_1 \times \ldots \times s_n \in P$. A model morphism $\mu : \mathcal{M} \rightarrow \mathcal{M}'$ is a mapping $\mu : M \rightarrow M'$ that preserves sorts (i.e. $\mu(M_s) \subseteq M'_s$ for each $s \in S$) such that for every $f : s_1 \times \ldots \times s_n \rightarrow s \in F$ and every $(a_1, \ldots, a_n) \in M_{s_1} \times \ldots \times M_{s_n}$, $\mu(f^M(a_1, \ldots, a_n)) = f^{M'}(\mu(a_1), \ldots, \mu(a_n))$, and for every $p : s_1 \times \ldots \times s_n \in P$ and every $(a_1, \ldots, a_n) \in M_{s_1} \times \ldots \times M_{s_n}$, $(a_1, \ldots, a_n) \in p^M \implies (\mu(a_1), \ldots, \mu(a_n)) \in p'^{M'}$.

Given a signature morphism $\sigma : \Sigma = (S, F, P) \rightarrow \Sigma' = (S', F', P')$ and a $\Sigma'$-model $\mathcal{M}'$, $\operatorname{Mod}(\sigma)(\mathcal{M}')$ is the $\Sigma$-model $\mathcal{M}$ defined for every $s \in S$ by $M_s = M'_s$, and for every function name $f \in F$ and every predicate name $p \in P$, by $f^\mathcal{M} = \sigma(f)^{\mathcal{M}'}$ and $p^\mathcal{M} = \sigma(p)^{\mathcal{M}'}$.

Finally, satisfaction is the usual first-order satisfaction.

**Modal Propositional Logic (MPL)** The category of signatures is the same as $\text{PL}$. For each set $P$, the $P$-sentences are formed from the elements of $P$ by closing under Boolean connectives and unary modal connectives $\Box$ (necessity) and $\Diamond$ (possibility). A model $(I, W, R)$ for a signature $P$, called Kripke model, consists of

- an index set $I$,
- a family $W = \{W^i\}_{i \in I}$ of “possible worlds”, which are functions from $P$ to $\{0, 1\}$ (or equivalently subsets of $P$),
- an “accessibility” relation $R \subseteq I \times I$.

A model homomorphism $h : (I, W, R) \rightarrow (I', W', R')$ consists of a function $h : I \rightarrow I'$ which preserves the accessibility relation, i.e. $\langle i, j \rangle \in R$ implies $\langle h(i), h(j) \rangle \in R'$, and such that $W^i_i \subseteq W'^{h(i)}_i$ for each $i \in I$. Given a signature morphism $\sigma : P \rightarrow P'$ and a $P'$-model $(I', W', R')$, $\operatorname{Mod}(\sigma)((I', W', R'))$ is the $P$-model $(I, W, R)$ such that $I = I'$, $R = R'$ and $W^i = \{\nu' \circ \sigma \mid \nu' \in W'^{h(i)}_i\}$ for each $i \in I$.

The satisfaction of $P$-sentences by the Kripke $P$-models, $(I, W, R) \models_P \varphi$, is defined by $(I, W, R) \models_i^1 \varphi$ for each $i \in I$, where $\models_i^1$ is defined by induction on the structure of the sentences as follows:

\footnote{$T_F(X)_{s_i}$ is the term algebra of sort $s$ built over $F$ with sorted variables in a given set $X$.}
Topological MPL (TMPL) \[\text{In MPL, the modalities } \square \text{ and } \Diamond \text{ are interpreted relationally (i.e. in Kripke models). Here, they will be interpreted topologically. Hence, the category of signatures and the functor } \text{Sen are the same as MPL. Conversely, given a signature } P, \text{ a } P\text{-model } M \text{ is a topological space } (X, \tau) \text{ equipped with a valuation function } \nu : P \to P(X). \text{ Such models are called topos-models. A model morphism } h : (X, \tau, \nu) \to (X', \tau', \nu') \text{ is a continuous mapping such that for every } p \in P, h(\nu(p)) \subseteq \nu'(p). \text{ Given a signature morphism } \sigma : P \to P' \text{ and a } P'\text{-model } (X', \tau', \nu'), \text{ Mod}(\sigma)((X', \tau', \nu')) \text{ is the } P\text{-model } (X, \tau, \nu) \text{ such that } X = X', \tau = \tau', \text{ and } \nu = \nu' \circ \sigma. \text{ The satisfaction of sentences by the topological models, } (X, \tau, \nu) \models_P \varphi, \text{ is defined by } (X, \tau, \nu) \models_P^\tau \varphi \text{ for each } x \in X, \text{ where } \models_P^\tau \text{ is defined by induction on the structure of the sentences as follows:}
\begin{itemize}
  \item \((I, W, R) \models_P p \text{ iff } p \in W^i \text{ for each } p \in P,
  \item \((I, W, R) \models_P \varphi_1 \land \varphi_2 \text{ iff } (I, W, R) \models_P \varphi_1 \text{ and } (I, W, R) \models_P \varphi_2; \text{ and similarly for the other Boolean connectives,}
  \item \((I, W, R) \models_P \square \varphi \text{ iff } (I, W, R) \models_P \varphi \text{ for each } j \text{ such that } (i, j) \in R, \text{ and}
  \item \Diamond \varphi \text{ is the same as } \neg \square \neg \varphi.
\end{itemize}

Hence, \square \text{ and } \Diamond \text{ are interpreted as both topological notions of interior and closure, respectively.}

Metric MPL (MMPL) \[\text{Here, modalities will be interpreted in a metric space.}
\text{The institution MMPL has the same signatures and sentences as MPL and TMPL. Conversely, given a signature } P, \text{ a } P\text{-model is a metric space } (X, d) \text{ equipped with a valuation function } \nu : P \to \mathcal{P}(X). \text{ Such models are called metric models. A model morphism } h : (X, d, \nu) \to (X', d', \nu') \text{ is a continuous mapping such that for every } p \in P, h(\nu(p)) \subseteq \nu'(p). \text{ Given a signature morphism } \sigma : P \to P' \text{ and a } P'\text{-model } (X', d', \nu'), \text{ Mod}(\sigma)((X', d', \nu')) \text{ is the } P\text{-model } (X, d, \nu) \text{ such that } X = X', d = d', \text{ and } \nu = \nu' \circ \sigma. \text{ The satisfaction of sentences by metric models } (X, d, \nu) \models_P \varphi \text{ is defined by } (X, d, \nu) \models_P^\tau \varphi \text{ for each } x \in X, \text{ where } \models_P^\tau \text{ is defined by induction on the structure of the sentences as follows:}
\begin{itemize}
  \item \((I, W, R) \models_P p \text{ iff } p \in W^i \text{ for each } p \in P,
  \item \((I, W, R) \models_P \varphi_1 \land \varphi_2 \text{ iff } (I, W, R) \models_P \varphi_1 \text{ and } (I, W, R) \models_P \varphi_2; \text{ and similarly for the other Boolean connectives,}
  \item \((I, W, R) \models_P \square \varphi \text{ iff } (I, W, R) \models_P \varphi \text{ for each } j \text{ such that } (i, j) \in R, \text{ and}
  \item \Diamond \varphi \text{ is the same as } \neg \square \neg \varphi.
\end{itemize}
\]
• basic sentences and Boolean connectives are satisfied stand ardly;
• \((X, d, \nu) \models \forall \varphi \iff \exists \varepsilon > 0, \forall y \in X, d(x, y) < \varepsilon \Rightarrow (X, d, \nu) \models \nu \varphi;\)
• \(\Diamond \varphi\) is the same as \(\neg \Box \neg \varphi\).

2.2 Stratified institutions

Stratified institutions refine institutions by introducing the notion of states for models. Hence, each model \(M\) is equipped with a set \([M]\), elements of which are called states, such as possible worlds for Kripke models.

The definition of stratified institutions given in Definition 2.4 slightly improves the original one in [3] by considering a concrete category to equip models with states rather than the category of sets. This is motivated by the different applications developed in this paper such as the extensions of stratified institutions to modalities or to qualitative spatial reasoning, which require to consider in the first case sets equipped with binary relations, and in the second one topological or metric spaces.

Definition 2.4 (Stratified institution) A stratified institution consists of:

• a category \(\text{Sig}\) of signatures;
• a sentence functor \(\text{Sen} : \text{Sig} \rightarrow \text{Set}\);
• a model functor \(\text{Mod} : \text{Sig}^{\text{op}} \rightarrow \text{Cat}\);
• a “stratification” \([\cdot]\) which consists of a functor \([\cdot]_{\Sigma} : \text{Mod}(\Sigma) \rightarrow C\) for each signature \(\Sigma \in \text{Sig}\) (states of models) where \(C\) is a concrete category (i.e. \(C\) is equipped with a faithful functor \(U : C \rightarrow \text{Set}\)), and a natural transformation \([\cdot]_{\sigma} : [\cdot]_{\Sigma} \rightarrow [\cdot]_{\Sigma'} \circ \text{Mod}(\sigma)\) for each signature morphism \(\sigma : \Sigma \rightarrow \Sigma'\) such that \(U([\cdot]_{\sigma} M')\) is surjective for each \(M' \in \text{Mod}(\Sigma')\) (and then by standard results in the category theory, \([\cdot]_{\sigma} M'\) is an epimorphism in \(C\)\).

To simplify the notations and when this does not raise ambiguities, we use in the rest of this paper the notation \([M]_{\Sigma}\), given a signature \(\Sigma\) and a model \(M \in \text{Mod}(\Sigma)\), to denote both the object in the concrete category \(C\) and the underlying set \(U([M]_{\Sigma})\). Similarly, given a signature morphism \(\sigma : \Sigma \rightarrow \Sigma'\) and a \(\Sigma'\)-model \(M'\), we will use the notation \([M']_{\sigma}\) to denote both the morphism \([M']_{\sigma} M\) in \(C\) and the mapping \(U([M']_{\sigma})\) in \(\text{Set}\);

• a satisfaction relation between models and sentences which is parametrized by model states, \(M \models_{\eta} \varphi\) where \(\eta \in [M]_{\Sigma}\) such that, \(\forall \sigma : \Sigma \rightarrow \Sigma', \forall M \in \text{Mod}(\Sigma'), \forall \eta \in [M]_{\Sigma}, \forall \varphi \in \text{Sen}(\Sigma)\), the two following properties are equivalent:

\[
1. \text{Mod}(\sigma)(M) \models [M]_{\sigma}(\eta) \varphi,
\]

\[5\]In many concrete categories of interest the converse is also true. However, this does not hold in general.
2. \( M \models^\Sigma_S, \text{Sen}(\sigma)(\varphi) \). 

Then, we can define for every \( \Sigma \in \text{Sig} \), the satisfaction relation \( \models^\Sigma_{\Sigma} \subseteq \text{Mod}(\Sigma) \times \text{Sen}(\Sigma) \) as follows:

\[
M \models^\Sigma \varphi \text{ if and only if } M \models^\eta_{\Sigma} \varphi \text{ for all } \eta \in [M]_{\Sigma}.
\]

**Notation 2.5** Given a signature \( \Sigma \in \text{Sig} \), a model \( M \in \text{Mod}(\Sigma) \) and a formula \( \varphi \in \text{Sen}(\Sigma) \), we note \([M]_{\Sigma}(\varphi) = \{ \eta \in [M]_{\Sigma} \mid M \models^\eta_{\Sigma} \varphi \}\).

**Example 2.6** PL is the stratified institution with Set as concrete category and \([\nu]_P = \{ 1 \} \) (1 is any singleton up to isomorphism) for each set \( P \) of propositional variables and each \( P \)-model \( \nu \).

**Example 2.7** (Internal stratification [3]) In any institution \( I = (\text{Sig}, \text{Sen}, \text{Mod}, \models) \), we can define the stratified institution, denoted \( \text{St}(I) = (\text{Sig}', \text{Sen}', \text{Mod}', \models) \), as follows:

- **Sig’** is the category, objects and morphisms of which are, respectively, quasi-representable signatures \( \chi : \Sigma \to \Sigma' \), \( \chi \) and pairs of base institution signature morphisms \( (\varphi : \Sigma \to \Sigma_1, \varphi' : \Sigma' \to \Sigma'_1) : (\chi : \Sigma \to \Sigma') \to (\chi_1 : \Sigma_1 \to \Sigma'_1) \) such that:

\[
\begin{array}{c c}
\Sigma & \xrightarrow{\chi} & \Sigma' \\
\varphi & \downarrow & \varphi' \\
\Sigma_1 & \xrightarrow{\chi_1} & \Sigma'_1
\end{array}
\]

is a weak amalgamation square [7].

- **Sen’ : Sig’ \to Set** is the functor that maps every \( \chi : \Sigma \to \Sigma' \) to \( \text{Sen}(\Sigma') \),

- **Mod’ : Sig’\text{op} \to \text{Cat** is the functor that maps \( \chi : \Sigma \to \Sigma' \) to \( \text{Mod}(\Sigma) \), and

- **\([\cdot]_\chi \)** is the \( \text{Sig’} \)-indexed family of functors \( \text{Fu}_\chi : \text{Mod}(\chi) \to \text{Set} \) that maps every \( \chi \)-model \( M \) to its set of states \( [M]_\chi = \{ M' \in \text{Mod}(\Sigma') \mid \text{Mod}(\chi)(M') = M \} \).

Given \( \chi : \Sigma \to \Sigma' \) and a \( \chi \)-model \( M \), for each state \( M' \in [M]_\chi \), we define the satisfaction of \( \varphi \in \text{Sen'}(\chi) \) by \( M \) at \( M' \), denoted \( M \models^M_{\chi} \varphi \), by:

\[
M \models^M_{\chi} \varphi \iff M' \models^\Sigma \varphi
\]

Finally, a \( \chi \)-model \( M \) satisfies \( \varphi \), denoted \( M \models^\chi \varphi \) if and only if \( M \models^M_{\chi} \varphi \) for every \( M' \in [M]_\chi \).

---

6 A signature morphism \( \chi : \Sigma \to \Sigma' \) is **quasi-representable** if and only if each model homomorphism \( h : \text{Mod}(\chi)(M') \to N \) has a unique \( \chi \)-expansion \( h' : M' \to N' \).

7 We refer the reader to [19] for a definition of weak amalgamation square.
$\text{St}(\mathcal{I})$ is a stratified institution where the concrete category is $\text{Set}$. Indeed, for each signature morphism $(\varphi, \varphi’ : (\chi : \Sigma \to \Sigma’) \to (\chi_1 : \Sigma_1 \to \Sigma_1’))$, the natural transformation $[\chi](\varphi, \varphi’)$ is defined by $[M](\varphi, \varphi’)(M’) = \text{Mod}(\varphi’)(M’)$ for each state $M’ \in [M]_{\chi}$. The definition of $[\chi]$ on model homomorphisms uses the quasi-representable property of $\chi$. The surjectivity of $[\chi](\varphi, \varphi’)$ is assured by the weak amalgamation property of the square defining $(\varphi, \varphi’)$.

**Example 2.8** MPL is the stratified institution where the concrete category is $\text{Graph}$, $[[((I, W, R))_P = (I, R)]$ for each set $P$ of propositional variables and each $P$-model $(I, W, R)$, and for each signature morphism $\sigma : P \to P’$ and each $P’$-model $(I’, W’, R’)$, $[[((I’, W’, R’))_\sigma$ is simply the identity morphism on $(I’, R’)$.

**Example 2.9** TMPL is a stratified institution which follows the same definition as MPL by replacing $[[((I, W, R))_P = (I, R)]$ by $[[((X, \tau, \nu))_P = (X, \tau)$.

Hence, the concrete category is the category of topological spaces $\text{Top}$.

**Proposition 2.10** (3) Any stratified institution is an institution.

(The proof of Proposition 2.10 is substantially similar to that given in [3].)

By this proposition, we will also denote by $\mathcal{I}$ the generic stratified institution $(\text{Sig, Sen, Mod, } [\cdot], =)$.

### 3 Internal logic and extension to fuzzy case

Here, we propose to define abstractly the important logic concepts of Boolean connectives, quantifiers, and fuzzy reasoning. By “abstractly” we mean independently of any stratified institution. Boolean connectives and quantifiers have already been defined internally to any institution [19]. But institutions only consider sentences (i.e. closed formulas), and the institution MPL does not have semantic negation, disjunction and implication connectives, as abstractly defined in institutions. Here, as the satisfaction of formulas is defined from model states, the standard Boolean connectives can be defined in stratified institutions more “finely” than in institutions, and allow stratified institutions such as MPL to have all standard Boolean connectives.

Fuzzy (or many-valued) reasoning has already received an institutional semantics [20, 21]. The approach proposed here is substantially similar to that proposed in [20] although defined in the framework of stratified institutions.

#### 3.1 Internal logic and quantifiers

Let $\mathcal{I}$ be a stratified institution. Let $\Sigma$ be a signature of $\mathcal{I}$. Let $M$ be a $\Sigma$-model. A $\Sigma$-sentence $\varphi’$ is in $M$ a

- **semantic negation** of $\varphi$ when $[M]_{\Sigma}(\varphi) = [M]_{\Sigma} \setminus [M]_{\Sigma}(\varphi)$;
- **semantic conjunction** of $\varphi_1$ and $\varphi_2$ when $[M]_{\Sigma}(\varphi’_1) = [M]_{\Sigma}(\varphi_1) \cap [M]_{\Sigma}(\varphi_2)$;
• **semantic disjunction** of \( \varphi_1 \) and \( \varphi_2 \) when \([M]_\Sigma(\varphi') = [M]_\Sigma(\varphi_1) \cup [M]_\Sigma(\varphi_2)\);

• **semantic implication** of \( \varphi_1 \) and \( \varphi_2 \) when \([M]_\Sigma(\varphi') = ([M]_\Sigma[M]_\Sigma(\varphi_1)) \cup [M]_\Sigma(\varphi_2)\).

A stratified institution \( \mathcal{I} \) has (semantic) negation when each \( \Sigma \)-formula has a negation in each \( \Sigma \)-model. It has (semantic) conjunction (respectively disjunction and implication) when any two \( \Sigma \)-formulas have a conjunction (respectively disjunction and implication) in each \( \Sigma \)-model. As usual, we note negation, conjunction, disjunction and implication by \( \neg \), \( \land \), \( \lor \) and \( \Rightarrow \), respectively. Unlike institutions that deal with sentences, stratified institutions such as MPL, MMPL and TMPL have now semantic negation, disjunction and implication.

In the same way, it is equally easy to introduce abstract quantifiers in stratified institutions by following the same construction as in the definition of internal stratification given in Example 2.7. Hence, let \( \mathcal{I} = (\text{Sig}, \text{Sen}, \text{Mod}, \llbracket \cdot \rrbracket, \vdash) \) be a stratified institution, let \( \chi : \Sigma \to \Sigma' \) be a signature morphism in \( \text{Sig} \) and let \( M \in \text{Mod}(\Sigma) \) be a model. Then, \( M \models_{\Sigma'} (\forall \chi) \varphi \) if and only if for every \( \Sigma' \)-model \( M' \) such that \( \text{Mod}(\chi)(M') = M \) and every state \( \eta' \in \llbracket M' \rrbracket_{\Sigma'} \) such that \( \llbracket M' \rrbracket_{\chi}(\eta') = \eta \) we have that \( M' \models_{\Sigma'} \varphi \). Existential quantification is defined dually by replacing “every model \( M' \)” and “every state \( \eta' \)” by “some model \( M' \)” and “some state \( \eta' \)” in the definition of universal quantification.

### 3.2 Fuzzy case

#### 3.2.1 Residuated lattice

The algebraic structures underlying fuzzy logic are usually residuated lattices. Residuated lattices generalize Boolean algebras for classical logic by considering a set of truth values which may contain more than two values.

**Definition 3.1 (Residuated lattice)** A residuated lattice \((L, \land, \lor, \otimes, \to, 0, 1)\) is:

- a bounded lattice \((L, \land, \lor, 0, 1)\) where \( \land \) and \( \lor \) are the supremum and infimum operators associated with a partial ordering \( \leq \), and 0 and 1 are the least and the greatest elements, respectively;

- \( \otimes \) and \( \to \) are binary operators such that:
  - \((L, \otimes, 1)\) is a monoid, that is, \( \otimes \) is a commutative and associative operation with the identity \( a \otimes 1 = a \);
  - \( \otimes \) is isotone in both arguments;
  - the operation \( \to \) is a residuation operation with respect to \( \otimes \), i.e.
    \[
    a \otimes b \leq c \text{ iff } a \leq b \to c
    \]
Most famous examples of residuated lattices are Goguen algebra and Luckasiewicz algebra, defined respectively as follows:

- **Goguen algebra.** \([0,1], \wedge, \vee, \otimes, \to, 0, 1\) where \(\otimes\) is the ordinary product of reals and
  \[
  a \to b = \begin{cases} 
  1 & \text{if } a \leq b \\
  b/a & \text{otherwise}
  \end{cases}
  \]

- **Luckasiewicz algebra.** \([0,1], \wedge, \vee, \otimes, \to, 0, 1\) where:
  \[
  a \otimes b = 0 (a + b - 1) \\
  a \to b = 1 (1 - a + b)
  \]

### 3.2.2 Institutional semantics

Let \(\mathcal{I} = (\text{Sig}, \text{Sen}, \text{Mod}, [\cdot], \models)\) be a stratified institution. Let \(\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)\) be a residuated lattice. We can consider that for every signature \(\Sigma\), the truth of \(\Sigma\)-formulas \(\varphi \in \text{Sen}(\Sigma)\) is a value in \(L\), i.e. for every \(\Sigma\)-model \(M \in \text{Mod}(\Sigma)\), \([M]_{\Sigma}(\varphi)\) is a fuzzy subset of \([M]_{\Sigma}\) over \(L\). Hence, whereas in \(\mathcal{I}\), the satisfaction relation \(M \models_{\Sigma} \varphi\) can be seen as a mapping from \([M]_{\Sigma}\) to \(\{0, 1\}\), in a fuzzy extension of \(\mathcal{I}\), \(M \models_{\Sigma} \varphi\) is a mapping from \([M]_{\Sigma}\) to \(L\). For every \(\eta \in [M]_{\Sigma}\), we will rather use the notation \((M \models_{\Sigma}^\eta \varphi)\) than \(M \models_{\Sigma} \varphi(\eta)\) to denote the value in \(L\) yielded by the mapping \(M \models_{\Sigma} \varphi\). Of course, to preserve the satisfaction condition, we have to impose the following equivalence: for each signature morphism \(\sigma : \Sigma \to \Sigma'\), every \(\Sigma'\)-model \(M'\), every \(\Sigma\)-formula \(\varphi\) and every \(\eta' \in [M']_{\Sigma}\),

\[
(M' \models_{\Sigma'}^\eta' \text{Sen}(\sigma)(\varphi)) = (\text{Mod}(\sigma)(M') \models_{\Sigma} [M']_{\Sigma}(\sigma(\eta')) \varphi)
\]

Standardly, Boolean connectives and quantifiers can be internally defined in any fuzzy extension of a stratified institution \(\mathcal{I}\). To give a meaning to negation, we suppose that \(\mathcal{L}\) is with complements (\(\bar{.}\)). Hence, a \(\Sigma\)-sentence \(\psi\) is, in a \(\Sigma\)-model \(M\), a

- **fuzzy semantic negation** of \(\varphi\) when for every \(\eta \in [M]_{\Sigma}\), \((M \models_{\Sigma}^\eta \psi) = (M \models_{\Sigma}^\eta \bar{\varphi})\);

- **fuzzy semantic conjunction** of \(\varphi_1\) and \(\varphi_2\) when for every \(\eta \in [M]_{\Sigma}\),
  \((M \models_{\Sigma}^\eta \psi) = (M \models_{\Sigma}^\eta \varphi_1) \wedge (M \models_{\Sigma}^\eta \varphi_2)\);

- **fuzzy semantic disjunction** of \(\varphi_1\) and \(\varphi_2\) when for every \(\eta \in [M]_{\Sigma}\),
  \((M \models_{\Sigma}^\eta \psi) = (M \models_{\Sigma}^\eta \varphi_1) \lor (M \models_{\Sigma}^\eta \varphi_2)\);

- **fuzzy semantic implication** of \(\varphi_1\) and \(\varphi_2\) when for every \(\eta \in [M]_{\Sigma}\),
  \((M \models_{\Sigma}^\eta \psi) = (M \models_{\Sigma}^\eta \varphi_1) \to (M \models_{\Sigma}^\eta \varphi_2)\).

The following connective \(\otimes\) is often added, the fuzzy semantics of which is:

\[
\forall \eta \in [M]_{\Sigma}; (M \models_{\Sigma}^\eta \varphi_1 \otimes \varphi_2) = ((M \models_{\Sigma}^\eta \varphi_1) \otimes (M \models_{\Sigma}^\eta \varphi_2)).
\]
First-order quantifiers can also be easily represented in a fuzzy way. Let \( \chi : \Sigma \to \Sigma' \) be a signature morphism in \( \text{Sig} \) and let \( M \in \text{Mod}(\Sigma) \) be a model. A \( \Sigma \)-sentence \( \varphi' \) is a (fuzzy semantic) universal \( \chi \)-quantification in \( M \) when for every \( \eta \in \llbracket M \rrbracket_\Sigma \), \( (M \models_\Sigma \varphi) = \wedge \{ (M' \models_\Sigma \varphi') | \text{Mod}(\chi)(M') = M \text{ and } [M']_{\chi}(\eta') = \eta \} \). Existential quantification is defined dually by replacing the infimum \( \wedge \) by the supremum \( \vee \). In Section 4.2.2, we will give a more general definition which allows us to extend to the fuzzy case a large family of dual logical operators such as modalities.

Fuzzy logics allow us to reason about formulas according to uncertainty. This leads to extend the satisfaction relation \( \models_\Sigma \) to a binary relation between models in \( \text{Mod}(\Sigma) \) and couples in \( \text{Sen}(\Sigma) \times L \) as follows:

\[
M \models_\Sigma (\varphi, l) \iff l \leq \bigwedge \{ (M \models_\Sigma \varphi) | \eta \in \llbracket M \rrbracket_\Sigma \}
\]

where \( \leq \) is the ordering defined on \( L \).

We have then the following result that proves that fuzzy extensions of stratified institutions are institutions.

**Proposition 3.2** For every signature morphism \( \sigma : \Sigma \to \Sigma' \), every \( \Sigma' \)-model \( M' \) and couple \( (\varphi, l) \in \text{Sen}(\Sigma) \times L \), we have:

\[
M' \models_{\Sigma'} (\text{Sen}(\sigma)(\varphi), l) \iff \text{Mod}(\sigma)(M') \models_{\Sigma} (\varphi, l).
\]

**Proof** By definition, we have that:

\[
(M' \models_{\Sigma'} \text{Sen}(\sigma)(\varphi)) = (\text{Mod}(\sigma)(M') \models_{\Sigma}\llbracket M' \rrbracket_{\sigma}(\varphi)).
\]

As \( \llbracket M' \rrbracket_{\sigma} \) is surjective, we also have that:

\[
\bigwedge \{ (M' \models_{\Sigma'} \text{Sen}(\sigma)(\varphi)) | \eta' \in \llbracket M' \rrbracket_{\Sigma'} \} = \bigwedge \{ (\text{Mod}(\sigma)(M') \models_{\Sigma} \varphi) | \eta \in \llbracket M \rrbracket_\Sigma \},
\]

and we can conclude that:

\[
l \leq \bigwedge \{ (M' \models_{\Sigma'} \text{Sen}(\sigma)(\varphi)) | \eta' \in \llbracket M' \rrbracket_{\Sigma'} \} \iff
\]

\[
l \leq \bigwedge \{ (\text{Mod}(\sigma)(M') \models_{\Sigma} \varphi) | \eta \in \llbracket M \rrbracket_\Sigma \}.
\]

\[\square\]

4 **Duality from morphological dilations and erosions in stratified institutions**

In this section, we show that mathematical morphology \cite{12, 39} can be used for defining systematically and uniformly the different logical concepts such as quantifiers and modalities. Indeed, we can observe that most of unary modalities and quantifiers have always a dual, and they commute with conjunction and disjunction. This then enables us to define such logic concepts via algebraic dilations and erosions based on the notion of adjunction.

In the rest of the paper, we consider a stratified institution \( \mathcal{I} \) which has conjunction, disjunction and negation.
4.1 Lattice of formulas

Let us first define the lattice \((Sen(\Sigma)/\equiv_M, \preceq_M)\) where \(M \in Mod(\Sigma)\). In the following, we consider only finite sets of formulas.

Let \(M \in Mod(\Sigma)\) be a model. Considering the inclusion on the power set \(\mathcal{P}([M]_\Sigma)\), the poset \((\mathcal{P}([M]_\Sigma), \subseteq)\) is a complete lattice. Similarly, a lattice is defined on the set \(Sen(\Sigma)/\equiv_M\), where \(Sen(\Sigma)/\equiv_M\) is the quotient space of \(Sen(\Sigma)\) by the equivalence relation \(\equiv_M\) defined by:

\[ \varphi \equiv_M \psi \iff [M]_\Sigma(\varphi) = [M]_\Sigma(\psi). \]

This lattice is \((Sen(\Sigma)/\equiv_M, \preceq_M)\) where \(\preceq_M\) is the partial ordering defined by:

\[ \varphi \preceq_M \psi \iff [M]_\Sigma(\varphi) \subseteq [M]_\Sigma(\psi). \]

Any finite subset \(\{\varphi_i\}\) of \(Sen(\Sigma)\) has as supremum \(\bigvee\{\varphi_i\}\) and infimum \(\bigwedge\{\varphi_i\}\), corresponding to union and intersection in the complete lattice \((\mathcal{P}([M]_\Sigma), \subseteq)\), and then, following the definitions given in Section 3.1, \(\bigvee\{\varphi_i\}\) and \(\bigwedge\{\varphi_i\}\) are the semantic disjunction \(\bigvee_i \varphi_i\) and semantic conjunction \(\bigwedge_i \varphi_i\) of the formulas in \(\{\varphi_i\}\), respectively. Hence, \((Sen(\Sigma)/\equiv_M, \preceq_M)\) is a bounded lattice. Greatest and least elements are respectively \(\top\) and \(\bot\), corresponding to equivalence classes of tautologies and antilogies.

4.2 Morphological dilations and erosions of formulas based on structuring elements

4.2.1 Definitions

The most abstract way to define dilation and erosion is as follows. Let \((L, \preceq)\) and \((L', \preceq')\) be two lattices. An algebraic dilation is an operator \(\delta : L \to L'\) that commutes with the supremum, and an erosion is an operator \(\varepsilon : L' \to L\) that commutes with the infimum. It follows that both operators are increasing, \(\delta\) preserves the least element \(\bot\), and \(\varepsilon\) preserves the greatest element \(\top\). Now, in the practice of mathematical morphology, morphological operators are often defined on sets (i.e. \(L\) and \(L'\) are the powersets or finite powersets of given sets \(E\) and \(E'\), and often \(E = E'\) and \(L = L'\)) through a structuring element designed in advance. Let us recall here the basic definitions of dilation and erosion \(D_B\) and \(E_B\) over sets, where \(B\) is a set called structuring element. Let \(X\) and \(B\) be two subsets of a set \(E\), endowed with a translation operator. The dilation and erosion of \(X\) by the structuring element \(B\), denoted respectively by \(D_B(X)\) and \(E_B(X)\), are defined as follows:

\[
D_B(X) = \{ x \in E \mid \hat{B} \cap X \neq \emptyset \}
\]

\[
E_B(X) = \{ x \in E \mid B_x \subseteq X \}
\]

where \(B_x\) denotes the translation of \(B\) at \(x\), and \(\hat{B}\) the symmetrical of \(B\) with respect to the origin of space.\(^8\) Similarly, the structuring element \(B\) can also be
seen as a binary relation on the set $E$ as follows: $(x, y) \in B \iff y \in B_x$ [12].

This is the way we will consider structuring elements in this paper.

The most important properties of dilation and erosion based on a structuring element are the following ones [12, 35, 39]:

- **Monotonicity:** if $X \subseteq Y$, then $D_B(X) \subseteq D_B(Y)$ and $E_B(X) \subseteq E_B(Y)$; if $B \subseteq B'$, then $D_B(X) \subseteq D_{B'}(X)$ and $E_B(X) \subseteq E_{B'}(X)$.

- If for every $x \in E$, $x \in B_x$ (and this condition is actually necessary and sufficient), then
  - $D_B$ is **extensive:** $X \subseteq D_B(X)$;
  - $E_B$ is **anti-extensive:** $E_B(X) \subseteq X$.

- **Commutativity:** $D_B(X \cup Y) = D_B(X) \cup D_B(Y)$ and $E_B(X \cap Y) = E_B(X) \cap E_B(Y)$.

- **Adjunction:** $X \subseteq E_B(Y) \iff D_B(X) \subseteq Y$.

- **Duality:** $E_B(X) = [D_B(X^C)]^C$ where $^C$ is the set-theoretical complementation.

Hence, $D_B$ and $E_B$ are particular cases of general algebraic dilation and erosion on the lattice $(\mathcal{P}(E), \subseteq)$.

In stratified institutions, given a $\Sigma$-model $M$, $[[M]]_{\Sigma}$ is an element of a concrete category $\mathcal{C}$. Therefore, let us suppose that for each model $M \in \text{Mod}(\Sigma)$, $[[M]]_{\Sigma}$ is equipped with a structuring element $B$ (i.e. $\forall \eta \in [[M]]_{\Sigma}, B_\eta \subseteq [[M]]_{\Sigma}$) which represents a relationship between states, i.e. $\eta' \in B_\eta$ iff $\eta'$ satisfies some relationship to $\eta$ (see the next section to have examples of structuring elements for given stratified institutions), and $\Bar{B}_\eta$ is defined by $\eta' \in \Bar{B}_\eta \iff \eta \in B_{\eta'}$. Drawing inspiration from Bloch & al. in [8, 13], dilation and erosion of a formula $\varphi \in \text{Sen}(\Sigma)$ then give rise to two formulas $D_B(\varphi)$ and $E_B(\varphi)$ satisfying the following equivalences:

\[
\begin{align*}
M \models_\Sigma D_B(\varphi) & \iff \Bar{B}_\eta \cap \left\{ \eta' \in [[M]]_{\Sigma} \mid M \models_\Sigma \varphi^{\eta'} \right\} \neq \emptyset \\
& \iff \exists \eta' \in \Bar{B}_\eta, M \models_\Sigma \varphi \\
& \iff \Bar{B}_\eta \cap [[M]]_{\Sigma}(\varphi) \neq \emptyset \\
M \models_\Sigma E_B(\varphi) & \iff B_\eta \subseteq \left\{ \eta' \in [[M]]_{\Sigma} \mid M \models_\Sigma \varphi^{\eta'} \right\} \\
& \iff \forall \eta' \in B_\eta, M \models_\Sigma \varphi^{\eta'} \\
& \iff B_\eta \subseteq [[M]]_{\Sigma}(\varphi)
\end{align*}
\]

\[\text{Let us recall that for simplicity in the notations we use } [[M]]_{\Sigma} \text{ to denote both the object in the concrete category } \mathcal{C} \text{ and the underlying set associated by the faithful functor } \mathcal{U}.\]
### 4.2.2 Extension to the fuzzy case

From our extension of stratified institutions to fuzzy reasoning, we can also define fuzzy dilation and erosion of formulas based on structuring elements. Several definitions of mathematical morphology on fuzzy sets with fuzzy structuring elements have been proposed in the literature, since the early work in [4, 14] (see e.g. [11, 15, 34] for reviews). Here, we follow the approach developed in [11] using conjunctions and implications in residuated lattices. Hence, given a \( \Sigma \)-model \( M \) with a structuring element \( B \) such that for every \( \eta \in [M]_{\Sigma} \), \( B_\eta \) is a fuzzy set, the dilation of a fuzzy formula by \( B \) is defined for every \( \eta \in [M]_{\Sigma} \) as follows:

\[
(M \models^\eta_{\Sigma} D_B(\varphi)) = \bigvee \{ \bar{B}_\eta(\eta') \otimes (M \models^\eta_{\Sigma} \varphi) \mid \eta' \in [M]_{\Sigma} \}.
\]

The erosion of a fuzzy formula by \( B \) is defined for every \( \eta \in [M]_{\Sigma} \) as follows:

\[
(M \models^\eta_{\Sigma} E_B(\varphi)) = \bigwedge \{ \bar{B}_\eta(\eta') \rightarrow (M \models^\eta_{\Sigma} \varphi) \mid \eta' \in [M]_{\Sigma} \}.
\]

If we note \( \mathcal{F}([M]_\Sigma) \) the set of all fuzzy sets on \([M]_\Sigma\), the couple \((\mathcal{F}([M]_\Sigma), \leq)\) where \( \leq \) denotes the fuzzy inclusion, is a complete lattice. Therefore, we can consider the lattice \((\text{Sen}(\Sigma)/\equiv_M, \preceq_M)\) where \( \equiv_M \) and \( \preceq_M \) are the fuzzy extensions of the two relations \( \equiv_M \) and \( \preceq_M \) defined in Section 4.1. Here again, it is easy to show that the fuzzy versions of \( D_B \) and \( E_B \) commute with union and intersection of fuzzy sets of states, respectively, i.e. for every \( \varphi_1, \varphi_2 \in \text{Sen}(\Sigma) \), we have:

- \( D_B(\varphi_1 \lor \varphi_2) \equiv_M D_B(\varphi_1) \lor D_B(\varphi_2) \),
- \( E_B(\varphi_1 \land \varphi_2) \equiv_M E_B(\varphi_1) \land E_B(\varphi_2) \),

and then, \( D_B \) and \( E_B \), interpreted in a fuzzy sets setting, are algebraic dilation and erosion, respectively. As for the crisp case, it is quite straightforward to show that these fuzzy dilation and erosion are monotonous, extensive and anti-extensive when \( \eta \in B_\eta \), and dual (resp. adjoint) if \( \otimes \) and \( \rightarrow \) are dual (resp. adjoint).

### 4.2.3 Examples

We show in this section that the two dual logical operators \( E_B \) and \( D_B \) can be instantiated to define both first-order quantifiers \( \forall, \exists \) and modalities \( \Box, \Diamond \). Moreover, from Section 4.2.2 all these operators can naturally be extended to fuzzy cases.

**First-order quantifiers** Let \( \text{St}(\text{Fol}) \) be the stratified institution of the first-order logic Let \( \chi : (S, F, P) \hookrightarrow (S, F \sqcup X, P) \) be a signature and let \( x \) be a variable in \( X \). For every \((S,F,P)\)-model \( M \), let us define the structuring element \( B^x_M \) as follows:

\[
\forall M' \in [M]_\Sigma, B_M^x = \{ M'' \in [M]_\Sigma \mid \forall y \neq x \in X, y^{M''} = y^{M'} \},
\]
i.e. the set of models identical to $M'$ on all variables except possibly $x$. This structuring element is symmetrical (i.e. $M'' \in B_{M'}^x \iff M' \in B_{M''}^x$) and contains the origin (i.e. $M' \in B_{M'}^x$).

We can then define the first-order quantifiers $\forall x$ and $\exists x$ as erosion and dilation from $B^x$ as follows:

$$\forall x. \varphi \equiv E_{B^x}(\varphi),$$
$$\exists x. \varphi \equiv D_{B^x}(\varphi).$$

More generally, in any internal stratification $St(I)$ of an institution $I$, both quantifiers $\forall \chi$ and $\exists \chi$ for a signature $\chi : \Sigma \to \Sigma'$ can be defined similarly. Indeed, for every $\chi$-model $M$, let us define the structuring element $B^\chi$ as follows:

$$\forall M' \in [M]_\chi, B^\chi_{M'} = [M]_\chi$$

Again, the structuring element is symmetrical and contains the origin, we then have:

$$\forall \chi. \varphi \equiv E_{B^\chi}(\varphi),$$
$$\exists \chi. \varphi \equiv D_{B^\chi}(\varphi).$$

**Modalities for Kripke models** Let $I$ be a stratified institution whose concrete category is $\text{Graph}$. Hence for each $\Sigma$-model $M$, $[M]_\Sigma$ is a directed graph $([M]_\Sigma, R_M)$. Obviously, this accessibility relation $R_M$ naturally leads to the structuring element $B$ defined as follows:

$$R_M(\eta, \eta') \iff \eta' \in B_\eta.$$ The modalities $\Box$ and $\Diamond$ are then defined as follows:

$$\Box \varphi \equiv E_B(\varphi),$$
$$\Diamond \varphi \equiv D_B(\varphi).$$

**4.2.4 Properties**

The following properties are the direct extensions of properties of dilation and erosion on sets to formulas.

- **Monotonicity**: if $\varphi \preceq_M \psi$, then $D_B(\varphi) \preceq_M D_B(\psi)$ and $E_B(\varphi) \preceq_M E_B(\psi)$.
- **Extensivity of dilation**: $\varphi \preceq_M D_B(\varphi)$ and **anti-extensivity of erosion**: $E_B(\varphi) \preceq_M \varphi$ if and only if for every $\eta \in [M]_\Sigma$, $\eta \in B_\eta$.
- **Adjunction**: $\varphi \preceq_M E_B(\psi) \iff D_B(\varphi) \preceq_M \psi$.
- **Commutativity with supremum or infimum**: $D_B(\varphi \lor \varphi_2) \equiv_M D_B(\varphi_1) \lor D_B(\varphi_2)$ and $E_B(\varphi_1 \land \varphi_2) \equiv_M E_B(\varphi_1) \land E_B(\varphi_2)$.

\[\text{Here, we consider the set } B \text{ to define dilation because the accessibility relation is not necessarily symmetrical.}\]
• **Duality:** $E_B(\varphi) \equiv_M \neg D_B(\neg \varphi)$.

It follows that $D_B$ and $E_B$ are respectively algebraic dilation and erosion over $(\text{Sen}(\Sigma)_{/\equiv_M}, \preceq_M)$, i.e. in $(\text{Sen}(\Sigma)_{/\equiv_M}, \preceq_M)$ $D_B$ and $E_B$ commute with supremum and infimum, respectively. Moreover, by a standard result of mathematical morphology [12], $E_B$ (respectively $D_B$) is the unique erosion (respectively the unique dilation) associated with $D_B$ (respectively $E_B$) by the adjunction property. From standard results of mathematical morphology and the adjunction property, we also have the following properties:

**Corollary 4.1**

- $E_B(\top) \equiv_M \top$
- $D_B(\bot) \equiv_M \bot$
- $\varphi \preceq_M E_B(D_B(\varphi))$
- $D_B(E_B(\varphi)) \preceq_M \varphi$
- $E_B(D_B(E_B(\varphi))) \equiv_M E_B(\varphi)$
- $D_B(E_B(D_B(\varphi))) \equiv_M D_B(\varphi)$
- $E_B(\varphi) \equiv_M \{ \psi \mid D_B(\psi) \preceq_M \varphi \}$
- $D_B(\varphi) \equiv_M \{ \psi \mid \varphi \preceq_M E_B(\psi) \}$

It follows that $E_B D_B$ (closing) and $D_B E_B$ (opening) are morphological filters (i.e. increasing and idempotent operators). Moreover, closing and opening are dual (i.e. $D_B(E_B(\varphi)) \equiv_M \neg E_B(D_B(\neg \varphi))$).

**Theorem 4.2** The following properties are satisfied by dilation and erosion of formulas. Note that now properties are expressed independently of a model $M$.

1. $E_B(\top) \equiv \top$ and $D_B(\bot) \equiv \bot$.
2. $\varphi \models E_B(\varphi)$.
3. If for every model $M \in \text{Mod}(\Sigma)$ and every $\eta \in [M]_{\Sigma}$, $\eta \in B_{\eta}$, then $\varphi \models D_B(\varphi)$ and $E_B(\varphi) \models \varphi$.
4. $D_B(\varphi \lor \psi) \equiv D_B(\varphi) \lor D_B(\psi)$ and $E_B(\varphi \land \psi) \equiv E_B(\varphi) \land E_B(\psi)$. Moreover, we have: $D_B(\varphi \land \psi) \models D_B(\varphi) \land D_B(\psi)$ and $E_B(\varphi) \lor E_B(\psi) \models E_B(\varphi \lor \psi)$.
5. $E_B(\varphi) \equiv \neg D_B(\neg \varphi)$, or dually $D_B(\varphi) \equiv \neg E_B(\neg \varphi)$.
6. If the stratified institution has implication, then
   - (a) $E_B(\varphi \Rightarrow \psi) \models E_B(\varphi) \Rightarrow E_B(\psi)$,
   - (b) $E_B(\varphi \Rightarrow D_B(\varphi)) \equiv \top$ if for every $M \in \text{Mod}(\Sigma)$ and every $\eta \in [M]_{\Sigma}$, $B_{\eta} \cap \bar{B}_{\eta} \neq \emptyset$. 
Proof

1. These first two properties are obvious to check.

2. Let $M \models \Sigma \varphi$. Let $\eta \in [M]_{\Sigma}$ and let $\eta' \in B_{\eta}$. By hypothesis, $M \models \eta' \varphi$, and then $M \models \eta'_{\Sigma} E_B(\varphi)$.

3. Let $M \models \Sigma \varphi$. Let $\eta \in [M]_{\Sigma}$. As $\eta \in B_{\eta}$, we directly deduce that $M \models \eta_{\Sigma} D_B(\varphi)$.

4. Let $M \in \text{Mod}(D_B(\varphi \lor \psi))$. This means that for every $\eta \in [M]_{\Sigma}$, there exists $\eta' \in B_{\eta}$ such that $M \models \eta' \varphi \lor \psi$, and then $M \models \eta'_{\Sigma} \varphi$ or $M \models \eta'_{\Sigma} \psi$. From this, we can directly conclude that $M \models \eta_{\Sigma} D_B(\varphi)$ or $M \models \eta_{\Sigma} D_B(\psi)$, i.e. $M \models \eta_{\Sigma} D_B(\varphi) \lor D_B(\psi)$.

5. Let $M \models \Sigma E_B(\varphi) \iff \forall \eta \in [M]_{\Sigma}, M \models \eta_{\Sigma} E_B(\varphi)$

6. (a) Let $M \models \Sigma E_B(\varphi \Rightarrow \psi)$. Let $\eta \in [M]_{\Sigma}$ such that $M \models \eta_{\Sigma} E_B(\varphi)$. Let $\eta' \in B_{\eta}$. By hypothesis, $M \models \eta' \varphi$, and then, as $M \models \Sigma E_B(\varphi \Rightarrow \psi)$, we also have that $M \models \eta'_{\Sigma} (\varphi \Rightarrow \psi)$, and $M \models \eta'_{\Sigma} \psi$.

(b) Let $\eta \in [M]_{\Sigma}$ such that $M \models \eta_{\Sigma} E_B(\varphi)$. Let $\eta' \in B_{\eta} \cap \hat{B}_{\eta}$ (by hypothesis this intersection is not empty). Then we have that $M \models \eta'_{\Sigma} \varphi$ since $\eta' \in B_{\eta}$, and then $M \models \eta'_{\Sigma} D_B(\varphi)$ since $\eta' \in \hat{B}_{\eta}$.
4.3 Dual logical operators as algebraic dilation and erosion

In this section, we provide an algebraic view of dual dilation and erosion, without referring to any structuring element over the set $[M]_\Sigma$.

4.3.1 Definition

Definition 4.3 (Algebraic erosion and dilation) Let $E$ and $D$ be two dual logical operators for $I$, i.e. $E$ and $D$ satisfy the equation:

$$\forall M \in Mod(\Sigma), \forall \varphi \in Sen(\Sigma), E(\varphi) \equiv_M \lnot D(\lnot \varphi)$$

We will say that $E$ and $D$ are algebraic erosion and dilation if they satisfy the two following equations: $\forall M \in Mod(\Sigma), \forall \varphi_1, \varphi_2 \in Sen(\Sigma)$,

1. $D(\varphi_1 \lor \varphi_2) \equiv_M D(\varphi_1) \lor D(\varphi_2)$;
2. $E(\varphi_1 \land \varphi_2) \equiv_M E(\varphi_1) \land E(\varphi_2)$.

By standard results of mathematical morphology, we then have the following properties:

Proposition 4.4

- **Monotonicity of $D$:** if $\varphi \preceq_M \psi$, then $D(\varphi) \preceq_M D(\psi)$;
- **Preservation of $\bot$ by $D$:** $D(\bot) \equiv_M \bot$;
- **Monotonicity of $E$:** if $\varphi \preceq_M \psi$, then $E(\varphi) \preceq_M E(\psi)$;
- **Preservation of $\top$ by $E$:** $E(\top) \equiv_M \top$;

Unlike dilation and erosion defined through structuring elements, the dual logical operators $E$ and $D$ defined as algebraic erosion and dilation do not form necessarily an adjunction (see Section 4.3.2 for an example) which is expressed, when it holds, as follows:

$$\forall \varphi, \psi \in Sen(\Sigma), D(\varphi) \preceq_M \psi \iff \varphi \preceq_M E(\psi)$$

When adjunction holds between $E$ and $D$, by standard results in mathematical morphology, the following properties are satisfied:

- $\varphi \preceq_M E(D(\varphi))$ (extensivity of $ED$);
• $D(E(\varphi)) \preceq_M \varphi$ (anti-extensivity of $DE$);
• $E(D(E(\varphi))) \equiv_M E(\varphi)$;
• $D(E(D(\varphi))) \equiv_M D(\varphi)$;
• $E(D(E(D(\varphi)))) \equiv_M E(D(\varphi))$;
• $D(E(D(E(\varphi)))) \equiv_M D(E(\varphi))$.

Some properties are preserved independently of a model $M$.

**Theorem 4.5** The following properties are satisfied by dilation and erosion of formulas:

• **Duality:** $D(\varphi) \equiv \neg E(\neg \varphi)$.

• **Commutativity:** $D(\varphi_1 \lor \varphi_2) \equiv D(\varphi_1) \lor D(\varphi_2)$ and $E(\varphi_1 \land \varphi_2) \equiv E(\varphi_1) \land E(\varphi_2)$.

• **Monotonicity:** if $\varphi \models \psi$, then $D(\varphi) \models D(\psi)$ and $E(\varphi) \models E(\psi)$.

• **Preservation:** $D(\bot) \equiv \bot$ and $E(\top) \equiv \top$.

**Proof** Duality, commutativity and preservation are direct consequences of the fact that $(\forall M \in \text{Mod}(\Sigma), \varphi \equiv_M \psi \Rightarrow \varphi \equiv \psi$. To prove monotonicity, let us suppose that $\varphi \models \psi$. Therefore, for every $M \in \text{Mod}(\varphi)$ we have that $M \models \varphi$ and $M \models \psi$, and then for every $\eta \in [M]_{\Sigma}$ we have $M \models^\eta \varphi$ and $M \models^\eta \psi$ i.e. $[M]_{\Sigma}(\varphi) = [M]_{\Sigma}(\psi) = [M]_{\Sigma}$, whence we conclude $\varphi \equiv_M \psi$. As $D$ is monotonous for $\equiv_M$, we then have that $D(\varphi) \equiv_M D(\psi)$. Hence, for every $\eta \in [M]_{\Sigma}$, we have that $M \models^\eta D(\varphi)$, and then $M \in \text{Mod}(D(\varphi))$. The reasoning for $E$ is similar. \(\square\)

### 4.3.2 Example: modalities for topos-models

When the modalities $\Box$ and $\Diamond$ are interpreted topologically, they cannot be expressed as erosion and dilation based on a structuring element. The reason is the heterogeneity of elements used to express $\Box$ and $\Diamond$ which quantify existentially over open sets and universally over elements in open sets. We might be tempted to define the modality $\Box$ by an erosion $E_B$ followed by a dilation $D_B$ (i.e. a morphological opening) where $B$ would be the structuring element defined as: $\forall \eta \in [M]_{\Sigma}, B_\eta = \bigcup \{O \in \tau \mid \eta \in O\}$ where $M = (X, \tau, \nu)$ is a topos-model. The problem is that in this case we would quantify universally on open sets and not existentially. However, we have seen that $[M]_{\Sigma}(\Box \varphi)$ and $[M]_{\Sigma}(\Diamond \varphi)$ define topological interior and closure of $[M]_{\Sigma}(\varphi)$. It is well known that interior and closure commute with intersection and union, respectively. Moreover, they are dual operators. Hence, $\Box$ and $\Diamond$ are algebraic erosion and dilation, respectively. Finally, $\Box$ is anti-extensive (and dually $\Diamond$ is extensive) for $\preceq_M$. Indeed, let $\eta \in [M]_{\Sigma}(\Box \varphi)$ be a state. This means that there exists an
open set \( O \in \tau \) such that \( \eta \in O \) and for every \( \eta' \in O, \eta' \in [M]_{\Sigma}(\varphi) \). Hence, we necessarily have that \( \eta \in [M]_{\Sigma}(\varphi) \). We can also easily show that \( \varphi \equiv \Box \varphi \)\(^\text{11}\) On the contrary, adjunction does not hold in general except under the (necessary and sufficient) condition that the underlying topology of topos-models satisfies that the closed sets defining formulas are precisely the open sets.

**Proposition 4.6** Let \( M = (X, \tau, \nu) \) be a topos-model over a signature \( \Sigma \). Then, we have: \( \forall \varphi, \psi \in \text{Sen}(\Sigma), \Diamond \varphi \preceq_M \psi \iff \varphi \preceq_M \Box \psi \) if and only if for every \( \varphi \in \text{Sen}(\Sigma) \), \([M]_{\Sigma}(\varphi)\) is a closed set of \( X \) is equivalent to \([M]_{\Sigma}(\varphi)\) is an open set of \( X \).

**Proof** \( \implies \): Let us assume that \( \forall \varphi, \psi \in \text{Sen}(\Sigma), \Diamond \varphi \preceq_M \psi \iff \varphi \preceq_M \Box \psi \). Let \( \varphi \) be a \( \Sigma \)-formula such that \([M]_{\Sigma}(\varphi)\) is a closed set. We then have that \([M]_{\Sigma}(\Diamond \varphi) = [M]_{\Sigma}(\varphi)\), and then \( \Diamond \varphi \preceq_M \varphi \). By applying the equivalence \( \Diamond \varphi \preceq_M \psi \iff \varphi \preceq_M \Box \psi \) to \( \psi = \varphi \), we obtain that \( \varphi \preceq_M \Box \varphi \). As \( \Box \) is anti-extensive, we can then conclude that \([M]_{\Sigma}(\varphi) = [M]_{\Sigma}(\Box \varphi)\), and then \([M]_{\Sigma}(\varphi)\) is open. Dually, applying this to the complement set allows us to conclude that all open sets of \( X \) are closed.

\( \impliedby \): Let us assume that the closed sets of \( X \) defining a formula are precisely the open sets of \( X \). Let \( \varphi \) and \( \psi \) be two formulas such that \( \Diamond \varphi \preceq_M \psi \). By monotonicity of \( \Box \), we have that \( \Box \Diamond \varphi \preceq_M \Box \psi \). Now, by definition of \( \Diamond \), \([M]_{\Sigma}(\Diamond \varphi)\) is open, and then closed by hypothesis. Hence, we have that \([M]_{\Sigma}(\Box \Diamond \varphi) = [M]_{\Sigma}(\Diamond \varphi)\). But, as \( \Diamond \) is extensive, we have that \( \varphi \preceq_M \Diamond \varphi \), whence we can conclude that \( \varphi \preceq_M \Box \psi \).

Conversely, if \( \varphi \preceq_M \Box \psi \), then by monotonicity of \( \Diamond \), we have that \( \Diamond \varphi \preceq_M \Diamond \Box \psi \). But, \([M]_{\Sigma}(\Diamond \Box \psi) = [M]_{\Sigma}(\Box \psi)\). By anti-extensivity of \( \Box \), we can directly conclude that \( \Diamond \varphi \preceq_M \psi \).

\( \Box \)

### 4.4 A sound and complete entailment system

In this section, we define the syntactic approach to truth for stratified institutions equipped with dual operators. This consists in establishing consequence relations \( \vdash \), called **proofs**, between set of formulas and formulas. The syntactic approach of truth is then complementary to the semantic one represented by the semantic consequence \( \models \). When we have that \( \vdash \subseteq \models \), the syntactic approach is said **sound** and when we have the opposite inclusion, it is said **complete**. To obtain the result of completeness, we need to consider that formulas are built inductively from “basic” formulas by applying iteratively Boolean connectives and a \( I \)-indexed family of dual operators \( E^i \) and \( D^i \) (resp. \( E^i_B \) and \( D^i_B \) when erosion and dilation are defined based on a structuring element \( B \)) for \( i \in I \).

\(^{11}\)Let us note that the equivalence \( \varphi \equiv E(\varphi) \) is satisfied by all logics for which the satisfaction of formulas of the form \( E(\varphi) \) requires that, for all models, the relation between states is reflexive, such as FOL, MPL with reflexive model, TMPL and MMPL. On the other hand, we do not have for every \( M \in \text{Mod}(\Sigma) \) that \( \varphi \preceq_M E(\varphi) \).
In Sections 3.1, 4.2 and 4.3 we have already given an abstract definition of Boolean connectives and of dual operators $E^i$ and $D^i$. It remains then to give an abstract definition of basic formulas.

**Definition 4.7 (Basic formulas)** A set of formulas $B \subseteq \text{Sen}(\Sigma)$ is basic if there exists a $\Sigma$-model $M_B \in \text{Mod}(\Sigma)$ and a state $\eta \in [M_B]_\Sigma$ such that for every $M \in \text{Mod}(\Sigma)$ and every $\eta' \in [M]_\Sigma$, $M \models_{\Sigma}^\eta B$ if and only if there exists a morphism $\mu_{\eta'} : M_B \rightarrow M$ such that $[\mu_{\eta'}]_\Sigma(\eta) = \eta'$.

$M_B$ and $\eta$ are called **basic model** and **basic state** for $B$, respectively.

The notion of basic formulas has been first defined in [19, 25] but in institutions, and then for sentences (i.e. closed formulas). Here, to take into account open formulas, the definition of basic formulas involves states.

**Proposition 4.8** Any set of atomic formulas in $\text{PL}$, $\text{FOL}$, $\text{MPL}$, $\text{TMPL}$ and $\text{MMPL}$ is basic.

**Proof**

**PL.** Let $P$ be a propositional signature. Let $B \subseteq P$. Let $M_B$ be the model that associates 1 to any $p \in B$ and 0 to any $p \in P \setminus B$. The choice of $\eta \in [M_B]_P$ is obvious because $[M_B]_P = 1$ (cf. Example 2.6).

Let $M \in \text{Mod}(P)$ such that $M \models_P B$. This means that for every $p \in B$, $M(p) = 1$ whence we can conclude that $M_B \leq M$ where $\leq$ is the partial ordering on models in $\text{Mod}(P)$. Conversely, let us suppose a morphism $\mu : M_B \rightarrow M$ (obviously, by the definition of models in $\text{PL}$, we have that $[\mu]_P(1) = 1$). By hypothesis, we have that $M_B \leq M$ whence we can directly conclude that for every $p \in B$, $M(p) = 1$.

**FOL.** Let $\Sigma = (S, F, P)$ be a signature. Let $B$ be a set of atomic formulas over a set of variables $X$. Let us denote $M_B$ the $\Sigma$-model defined by:

- $\forall s \in S, M_{B_s} = T_F(X)_s$;
- $\forall f : s_1 \times \ldots \times s_n \rightarrow s \in F, f^{M_B} : (t_1, \ldots, t_n) \mapsto f(t_1, \ldots, t_n)$;
- $\forall p : s_1 \times \ldots \times s_n \in P, p^{M_B} = \{(t_1, \ldots, t_n) | p(t_1, \ldots, t_n) \in B\}$.

Let us set $\eta$ the variable interpretation defined as $x \mapsto x$.

Let $M \in \text{Mod}(\Sigma)$ be a model and $\nu : X \rightarrow M$ be an interpretation such that $M \models_{\Sigma}^\nu B$. Therefore, we can define $\mu_{\nu} : \{x \mapsto \nu(x)\}$

$$f(t_1, \ldots, t_n) \mapsto f^M(\mu_\nu(t_1), \ldots, \mu_\nu(t_n))$$

which is a morphism. Obviously, we have that $[\mu_{\nu}]_{\Sigma}(\eta) = \nu$.

Conversely, let us suppose a morphism $\mu : M_B \rightarrow M$ such that $[\mu]_{\Sigma}(\eta) = \nu$. Let $p(t_1, \ldots, t_n) \in B$. As $[\mu]_{\Sigma}(\eta) = \nu$, for every $t \in T_F(X)$, we have that $\mu(t) = \nu(t)$, and then, as $\mu$ is a morphism, we can conclude that $\nu(t_1, \ldots, \nu(t_n)) \in p^M$.

**MPL.** Let $P$ be a propositional signature. Let $B$ be a subset of $P$. Let $M_B$ be the model defined by:
– \( I = \{1\} \) (any singleton);
– \( W^I = B \);
– \( R = \emptyset \).

Obviously, \( \eta = 1 \). Let \( M = (I', W', R') \) be a \( P \)-model and let \( i' \in I' \) be a state such that \( M \models_{i'} P \). Let us define the morphism \( \mu_{i'} : \{1\} \rightarrow i' \).

Obviously, we have that \( \llbracket \mu_{i'} \rrbracket P(1) = i' \).

Conversely, let us suppose a morphism \( \mu : MB \rightarrow M \) such that \( \llbracket \mu \rrbracket P(1) = i' \). As \( W^I \subseteq W'' \), we directly have that \( M \models_{i'} P \).

It is standard in modal logic to restrict the class of models to satisfy supplementary axioms. For instance, to satisfy \( \Box \varphi \Rightarrow \varphi \), models have to be reflexive (i.e. the accessibility relation is reflexive). In this case, the basic model \( MB \) is defined as previously except that \( R = \{(1, 1)\} \).

**TMPL.** Let \( P \) be a propositional signature. Let \( B \subseteq P \). Let us denote \( MB \) the \( P \)-model defined by:

– \( X = \{B\} \);
– \( \tau = \{\emptyset, \{B\}\} \) (the topology is both discrete and trivial);
– \( \nu : p \mapsto \begin{cases} \{B\} & \text{if } p \in B \\ \emptyset & \text{otherwise} \end{cases} \)

Let us set \( \eta = B \). Let \( M = (X', \tau', \nu') \) be a \( P \)-model and \( x \in X' \) such that \( M \models_{x} P \). Then, let us define the mapping \( \mu_{x} : B \rightarrow x \). Let us show that \( \mu_{x} \) is a morphism. First, let us show that it is continuous. Let \( O \in \tau' \) be an open set. Two possibilities can occur:

1. \( x \in O \). In this case, \( \mu_{x}^{-1}(O) = \{B\} \);
2. \( x \notin O \). In this case, \( \mu_{x}^{-1}(O) = \emptyset \).

In both cases, \( \mu_{x}^{-1}(O) \) is an open set, and then \( \mu_{x} \) is continuous. Let \( p \in P \). Here, two cases have to be considered:

1. \( p \in B \). As \( M \models_{x} P \), we have that \( x \in \nu'(p) \), and then \( \mu_{x}(\nu(p)) \subseteq \nu'(p) \);
2. \( p \notin B \). By definition of \( MB \), \( \nu(p) = \emptyset \), and then \( \mu_{x}(\nu(p)) = \emptyset \).

Conversely, let us suppose a morphism \( \mu : MB \rightarrow M \) such that \( \llbracket \mu \rrbracket P(B) = x \). Let \( p \in B \). As \( \mu \) is a morphism, we have that \( \mu(B) = x \in \nu'(p) \), and then \( M \models_{x} P \).

**MMPL.** The construction of the model \( MB \) for the logic **MMPL** is similar to that for **TMPL**, as from any metric space a topology can be induced.

Then, let us set the framework for this section.
**Framework:** we consider a stratified institution $I$ the functor $\text{Sen}$ of which has a subfunctor $\text{Sen}^{\text{base}} : \text{Sig} \to \text{Set}$ (i.e. $\text{Sen}^{\text{base}}(\Sigma) \subseteq \text{Sen}(\Sigma)$) such that for every signature $\Sigma \in \text{Sig}$:

- $\text{Sen}^{\text{base}}(\Sigma)$ is basic, and
- $\text{Sen}(\Sigma)$ is inductively defined from $\text{Sen}^{\text{base}}(\Sigma)$ by applying Boolean connectives in $\{\land, \lor, \Rightarrow, \neg\}$ and a $I$-indexed family of dual operators $E^i$ and $D^i$ (resp. $E^i_B$ and $D^i_B$ when erosion and dilation are defined over a structuring element $B$) such that for each $i \in I$, $E^i$ and $D^i$ are anti-extensive and extensive, respectively, and for all $\varphi \in \text{Sen}(\Sigma)$, $\varphi \models E^i(\varphi)$.

For all the examples of stratified institutions developed in this paper, we define the functor $\text{Sen}^{\text{base}}$ as the mapping which associates to any signature $\Sigma \in \text{Sig}$ the set of atomic formulas. In PL, the family of dual operators is indexed by the emptyset. In FOL, the family of dual operators is indexed by a set of variables $X$. Hence, in FOL, $E^x$ and $D^x$ are respectively $\forall x$ and $\exists x$. In MPL, TMPL, and MMPL, the family is indexed by any singleton as we only consider the couple of dual operators $\Box$ and $\Diamond$.

We have seen for all the examples where the dual operators $E^i$ and $D^i$ are erosion and dilation based on a structuring element $B$ that they are anti-extensive and extensive if for every model $M \in \text{Mod}(\Sigma)$ and for every state $\eta \in [[M]]_\Sigma$, we have $\eta \in B_\eta$. Hence, PL, as well as MPL when the category of models is restricted to reflexive models, meet all the requirements of our framework. This is the same for TMPL (and hence for MMPL) as $\Box$ and $\Diamond$ define topological interior and closure which are known to be anti-extensive and extensive (see Section 4.3.2).

Finally, from Property 2 in Theorem 4.2, the property $\varphi \models E^i(\varphi)$ is always satisfied when dual operators $E^i$ and $D^i$ are defined using a structuring element $B$, as in FOL and MPL. For TMPL (and then MMPL), we have also seen in Section 4.3.2 that this last property holds.

**Definition 4.9 (Tautology instance)** We call tautology instance any formula $\varphi \in \text{Sen}(\Sigma)$ such that there exists a propositional tautology $\psi$ (i.e. $\psi$ is a tautology in the logic PL) the propositional variables of which are among $\{p_1, \ldots, p_n\}$ and $n$ formulas $\varphi_i \in \text{Sen}(\Sigma)$ such that $\varphi$ is obtained by replacing in $\psi$ all the occurrences of $p_i$ by $\varphi_i$ for $i \in \{1, \ldots, n\}$.

What justifies such a definition is the following result:

**Proposition 4.10** Let $\psi$ be a propositional tautology the propositional variables of which are among $\{p_1, \ldots, p_n\}$. Let $\varphi_1, \ldots, \varphi_n \in \text{Sen}(\Sigma)$ be $n$ formulas. Then, the formula $\varphi$ in $\text{Sen}(\Sigma)$ obtained by replacing in $\psi$ all the occurrences of $p_i$ by $\varphi_i$ for $i \in \{1, \ldots, n\}$ is a tautology, i.e. for every $M \in \text{Mod}(\Sigma)$, $[[M]]_\Sigma(\varphi) = [[M]]_\Sigma$.

\[^{12}\text{In modal logic, the proof systems satisfying such a condition are said normal.}\]
Proof Let $M \in \text{Mod}(\Sigma)$ be a model. Let $\eta \in \llbracket M \rrbracket_\Sigma$ be a state. Let us define the propositional model $\nu$ in PL by:

$$
\nu : p_i \mapsto \begin{cases} 
1 & \text{if } M \models_\Sigma \varphi_i \\
0 & \text{otherwise}
\end{cases}
$$

By hypothesis, we have that $\nu \models \psi$, and then we can conclude that $M \models_\Sigma \varphi$. □

The proof of completeness that we present here follows Henkin’s method [28]. This method relies on the proof that every consistent set of formulas has a model. This relies on the deduction theorem which is known to fail for modal logics except under some conditions (see [27]). Here, we give a condition based on the notion of “invariant formula” that we define just below and which ensures the deduction theorem. This condition differs from that given in [27] in the sense that it is not about a restriction of the application of the inference rule Necessity (see below). As we will see later in this section, our condition will prove to be similar for MPL and TMPL (and then MMPL) to change the definition of $\Gamma \vdash_\Sigma \varphi$ into: $\Gamma \vdash_\Sigma \varphi$ if there exists a finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ such that $\vdash_\Sigma \varphi_1 \land \ldots \land \varphi_n \Rightarrow \varphi$ (so-called local derivation).

Definition 4.11 (Invariant formula) Let $\varphi \in \text{Sen}(\Sigma)$. $\varphi$ is said invariant if: $\forall i \in I, \forall M \in \text{Mod}(\Sigma), \varphi \preceq_M E^i(\varphi)$.

When $E^i$ and $D^i$ are erosion and dilation based on a structuring element $B$, it is easy to see that every formula $\varphi \in \text{Sen}(\Sigma)$ such that, for every $M \in \text{Mod}(\Sigma)$, $\llbracket M \rrbracket_\Sigma(\varphi)$ is equal to either $\llbracket M \rrbracket_\Sigma$ or $\emptyset$ is an invariant formula. Hence, in FOL, all closed formulas (i.e. without free (unbound) variables) are invariant, and in MPL, tautologies and antilogies are invariant formulas. It is easy to see that when an invariant formula is a tautology or an antilogy, then so is its negation. In TMPL (and then MMPL), all tautologies and antilogies are also invariant formulas.\(^{13}\)

Definition 4.12 (Formula instance) Let $\varphi, \varphi' \in \text{Sen}(\Sigma)$. The formula $\varphi'$ is an instance of $\varphi$ for $i \in I$ (i is the index set of the family of the dual operators $E^i$ and $D^i$) if for every $M \in \text{Mod}(\Sigma)$, $E^i(\varphi) \preceq_M \varphi'$.

Formula instance generalizes in stratified institution the concept of substitutions which are standard in first-order logics. Indeed, in FOL, given a formula $\varphi$, we have for every variable $x \in X$ that $\forall x. \varphi \Rightarrow \varphi(x/t)$ is a tautology where $t \in T_{F}(X)$ and $\varphi(x/t)$ is the formula obtained from $\varphi$ by substituting every free occurrence of $x$ by the term $t$. Of course, by the hypothesis that each $E^i$ is anti-extensive, $\varphi$ is always an instance of itself for $i \in I$.

We then consider the following Hilbert-system for the stratified institution $\mathcal{T}$.

\(^{13}\) Note that the name “invariant” was chosen since it also holds that $E^i(\varphi) \preceq_M \varphi$.\(^{25}\)
**Axioms:**
- **Tautologies:** all tautology instances;
- **Duality:** \( E^i(\varphi) \iff \neg D^i(\neg \varphi); \)
- **Distribution:** \( E^i(\varphi \Rightarrow \psi) \Rightarrow E^i(\varphi) \Rightarrow E^i(\psi) \) (this axiom is called the Kripke schema);
- **Instantiation:** \( E^i(\varphi) \Rightarrow \varphi' \) when \( \varphi' \) is an instance of \( \varphi \) for \( i \in I; \)
- **Invariability:** \( \varphi \Rightarrow E^i(\varphi) \) when \( \varphi \) is an invariant formula.

**Inference rules:**
- **Modus Ponens:** \( \varphi \Rightarrow \psi \frac{\varphi}{\psi}; \)
- **Necessity:** \( \frac{\varphi}{E^i(\varphi)}. \)

In modal logic, the inference rules and axioms given above define the system \( T. \) The systems \( S_4, B \) and \( S_5 \) can be obtained by adding respectively the axioms written in our framework as follows:

- \( E^i(\varphi) \Rightarrow E^i(E^i(\varphi)) \) (\( S_4 \)),
- \( \varphi \Rightarrow E^i(D^i(\varphi)) \) (\( B \)),
- \( D^i(\varphi) \Rightarrow E^i(D^i(\varphi)) \) (\( S_5 \)),

In contrast, by imposing the anti-extensivity property, the systems \( K \) and \( D \) of the modal logic are not taken into account here.

**Definition 4.13 (Derivation)** A formula \( \varphi \in \text{Sen}(\Sigma) \) is derivable from a set of assumptions \( \Gamma \subseteq \text{Sen}(\Sigma) \), written \( \Gamma \vdash_{\Sigma} \varphi \), if \( \varphi \in \Gamma \), or is one of the axioms, or follows from derivable formulas through applications of the inference rules.

Hence, the proof system for \( \mathcal{I} \) can be defined by the four following inference rules:

\[
\begin{align*}
&\frac{\varphi \in \Gamma}{\Gamma \vdash_{\Sigma} \varphi} \quad \frac{\text{Axiom}}{\Gamma \vdash_{\Sigma} \varphi} \\
&\frac{\Gamma \vdash_{\Sigma} \varphi; \Delta \vdash_{\Sigma} \psi}{\Gamma \cup \Delta \vdash_{\Sigma} \psi} \quad \frac{\Gamma \vdash_{\Sigma} \varphi}{\Gamma \vdash_{\Sigma} E^i(\varphi)}
\end{align*}
\]

These inference rules give rise to an entailment system \( \mathcal{I} \), i.e. a \( \text{Sig} \)-indexed family of binary relations \( \vdash_{\Sigma} \subseteq \mathcal{P}(\text{Sen}(\Sigma)) \times \text{Sen}(\Sigma) \). Standardly, the \( \text{Sig} \)-indexed family \( \{ \vdash_{\Sigma} \}_{\Sigma \in \text{Sig}} \) satisfies the following properties:

- **Transitivity** if \( \Gamma \vdash_{\Sigma} \Gamma' \) and \( \Gamma' \vdash_{\Sigma} \Gamma'' \), then \( \Gamma \vdash_{\Sigma} \Gamma''; \)
- **Monotonicity** if \( \Gamma \vdash_{\Sigma} \varphi \) and \( \Gamma \subseteq \Gamma' \), then \( \Gamma' \vdash_{\Sigma} \varphi; \)
- **Compacity** if \( \Gamma \vdash_{\Sigma} \varphi \), then there exists a finite subset \( \Gamma_0 \) of \( \Gamma \) such that \( \Gamma_0 \vdash_{\Sigma} \varphi; \)
- **Translation** \( \Gamma \vdash_{\Sigma} \varphi \), then there exists a finite subset \( \Gamma_0 \) of \( \Gamma \) such that \( \Gamma_0 \vdash_{\Sigma} \varphi; \)

This system is enough to infer other properties of \( E^i \) and \( D^i \) such as the commutativity of \( E^i \) (resp. \( D^i \)) with the infimum (resp. supremum). Moreover, by
the assumptions and the properties of dilation and erosion (see Sections 4.2.4 and 4.3), the proof system defined above is sound, i.e. if \( \Gamma \vdash \Sigma \varphi \), then \( \Gamma \models \Sigma \varphi \).

Finally, thanks to the condition of “invariability” for formulas, we get the deduction theorem.

**Proposition 4.14 (Deduction theorem)** Let \( \Gamma \subseteq \text{Sen}(\Sigma) \) be a set of assumptions. If \( \varphi \) is an invariant formula, then we have \( \Gamma \cup \{ \varphi \} \vdash \Sigma \psi \) if and only if \( \Gamma \vdash \Sigma \varphi \Rightarrow \psi \).

**Proof** The necessary condition is obvious and can be easily obtained by Modus Ponens. The sufficient condition is proved by induction on the given proof. The more difficult case is that where the last inference rule is Necessity. We then have that \( \Gamma \cup \{ \varphi \} \vdash \Sigma \psi \). This means that \( \Gamma \cup \{ \varphi \} \vdash \Sigma \psi \) previously in the proof, and then by the induction hypothesis we have that \( \Gamma \vdash \Sigma \varphi \Rightarrow \psi \). By Necessity, Distribution and Modus Ponens, we have that \( \Gamma \vdash \Sigma \psi \).

The following corollary justifies proof by reduction ad absurdum.

**Corollary 4.15** For every \( \Gamma \subseteq \text{Sen}(\Sigma) \) and \( \varphi \in \text{Sen}(\Sigma) \) such that \( \neg \varphi \) is an invariant formula, we have that \( \Gamma \vdash \varphi \) if and only if \( \Gamma \cup \{ \neg \varphi \} \) is inconsistent (i.e. for every formula \( \psi \in \text{Sen}(\Sigma) \), \( \Gamma \cup \{ \neg \varphi \} \vdash \Sigma \psi \) and \( \Gamma \cup \{ \neg \varphi \} \vdash \Sigma \neg \psi \)).

**Proof** The “\( \Rightarrow \)” part is obvious. Let us prove the “\( \Leftarrow \)” part. Let us suppose that \( \Gamma \cup \{ \neg \varphi \} \) is inconsistent. This then means that we have both \( \Gamma \cup \{ \neg \varphi \} \vdash \Sigma \varphi \) and \( \Gamma \cup \{ \neg \varphi \} \vdash \Sigma \neg \varphi \). As \( \neg \varphi \) is an invariant formula by Proposition 4.14, we can write that \( \Gamma \vdash \Sigma \neg \varphi \). The formula \( (\neg \varphi \Rightarrow \varphi) \Rightarrow \varphi \) is a tautology axiom, and then by transitivity, we can conclude that \( \Gamma \vdash \Sigma \varphi \).

**Definition 4.16 (Maximal Consistence)** A set of formulas \( \Gamma \subseteq \text{Sen}(\Sigma) \) is maximally consistent if it is consistent and there is no consistent set of formulas properly containing \( \Gamma \) (i.e. for each formula \( \varphi \in \text{Sen}(\Sigma) \), either \( \varphi \in \Gamma \) or \( \neg \varphi \in \Gamma \), but not both).

**Proposition 4.17** Let \( \Gamma \subseteq \text{Sen}(\Sigma) \) be a consistent set of formulas. There exists a maximally consistent set of formulas \( \Gamma \subseteq \text{Sen}(\Sigma) \) that contains \( \Gamma \).

**Proof** Let \( S = \{ \Gamma' \subseteq \text{Sen}(\Sigma) \mid \Gamma' \text{ is consistent and } \Gamma \subseteq \Gamma' \} \). The poset \( (S, \subseteq) \) is inductive. Therefore, by Zorn’s lemma, \( S \) has a maximal element \( \uparrow \Gamma \). By definition of \( S \), \( \uparrow \Gamma \) is consistent and contains \( \Gamma \). Moreover, it is maximal. Otherwise, there exists a formula \( \varphi \in \text{Sen}(\Sigma) \) such that \( \varphi \notin \uparrow \Gamma \). As \( \Gamma \) is maximal, this means that \( \Gamma \cup \{ \varphi \} \) is inconsistent, and then \( \Gamma \vdash \Sigma \neg \varphi \). As \( \uparrow \Gamma \) is maximal, we can conclude that \( \neg \varphi \in \Gamma \).

Proposition 4.17 is a quite direct generalization to stratified institutions of Lindenbaum’s Lemma. To obtain our result of completeness, we need to impose the following condition:
Assumption. For every basic set of formulas $B \subseteq \text{Sen}^{\text{base}}(\Sigma)$, there exists a basic model $M_B \in \text{Mod}(\Sigma)$ and a basic state $\eta \in |M_B|_\Sigma$ for $B$ such that for every $i \in I$ ($I$ is the index-set of the dual operators $E^i$ and $D^i$) and every $\varphi \in \text{Sen}(\Sigma)$, there exists a subset $\text{Inst}_i(\varphi)$ of instances of $\varphi$ for $i$ satisfying:

1. for every $\varphi' \in \text{Inst}_i(\varphi)$, $|\varphi'| \leq |\varphi|$ where $|\varphi|$ and $|\varphi'|$ are the numbers of Boolean connectives and dual operators in $\varphi$ and $\varphi'$, and

2. $(\forall \varphi' \in \text{Inst}_i(\varphi), M_B \models_{\Sigma} \varphi') \implies M_B \models_{\Sigma} E^i(\varphi)$.

Proposition 4.18 All the couples $(M_B, \eta)$ defined in the proof of Proposition 4.17 for PL, FOL, MPL, TMPL and MMPL satisfy such an assumption.

Proof The proof for PL is obvious because the set of dual operators is empty (except the conjunction and disjunction which are assumed in the definition of the logic). For MPL, TMPL and MMPL, as $\square$ is anti-extensive, for every $\varphi \in \text{Sen}(\Sigma)$, we can set $\text{Inst}_\square(\varphi) = \{\varphi\}$ (let us recall that the index set for dual operators is here represented by the singleton with the unique element 1).

The first condition of the assumption is obviously satisfied. Finally, as $\square$ is anti-extensive, the accessibility relation is reflexive, and then if $M_B \models_{\Sigma} \varphi$ in MPL (resp. $M_B \models_{\Sigma} \varphi$ in TMPL and MMPL), then we necessarily have that $M_B \models_{\Sigma} \square \varphi$ in MPL (resp. $M_B \models_{\Sigma} \square \varphi$ in TMPL and MMPL).

In FOL, given a variable $x$, let us set $\text{Inst}_x(\varphi) = \{\varphi(x/t) | t \in T_F(X)\}$. Obviously, the first condition of the assumption is satisfied. Finally, if we suppose that $M_B \models_{\Sigma} \varphi(x/t)$ for every $t \in T_F(X)$, then we have for each $\sigma : X \rightarrow T_F(X)$ such that for every $y \neq x \in X$, $\sigma(y) = y$ and $\sigma(x) = t$ that $M_B \models_{\Sigma} \varphi$, whence we can conclude that $M_B \models_{\Sigma} \forall x. \varphi$. \hfill \Box

Proposition 4.19 Let assume that the assumption is satisfied. Then, for every maximal consistent set of formulas $\Gamma \subseteq \text{Sen}(\Sigma)$, there exists a $\Sigma$-model $M$ and a state $\eta \in |M|_\Sigma$ such that $\Gamma = \{\varphi \mid M \models_{\Sigma} \varphi\}$.

Proof Let us denote $B = \Gamma \cap \text{Sen}^{\text{base}}(\Sigma)$. By definition of basic set of formulas, there exists a basic model $M_B$ and a state $\eta$ for $B$ that satisfy the assumption. Then, let us show by induction on the size of $\varphi$ that:

$\Gamma \vdash \varphi \iff M_B \models_{\Sigma} \varphi$

The cases of basic formulas and Boolean connectives are easily provable. Then, let $\varphi$ be of the form $E^i(\psi)$.

$(\Rightarrow)$ Let us suppose that $\Gamma \vdash E^i(\psi)$. By Modus Ponens with $\Gamma \vdash_{\Sigma} E^i(\psi) \Rightarrow \psi'$ (Instantiation) where $\psi' \in \text{Inst}_x(\psi)$, we then have that $\Gamma \vdash \psi'$. By the first condition of the assumption, we can apply the induction hypothesis on every $\psi' \in \text{Inst}_x(\psi)$, and the we have that $M_B \models_{\Sigma} \psi'$, whence by the second condition of the assumption, we can conclude that $M_B \models_{\Sigma} E^i(\psi)$.

$(\Leftarrow)$ Let us suppose that $M_B \models_{\Sigma} E^i(\psi)$. By anti-extensivity of $E^i$, we then have that $M_B \models_{\Sigma} \psi$. By the induction hypothesis, we have that $\Gamma \vdash \psi$, and then by Necessity, $\Gamma \vdash E^i(\psi)$. \hfill \Box
Theorem 4.20 (Completeness) Let assume that the assumption is satisfied. Then, for every \( \Gamma \subseteq \text{Sen}(\Sigma) \) and every \( \varphi \in \text{Sen}(\Sigma) \) such that \( \neg \varphi \) is an invariant formula, we have that:
\[
\Gamma \models \varphi \implies \Gamma \vdash \varphi
\]

Proof If \( \Gamma \not\vdash \varphi \), then \( \Gamma \cup \{\neg \varphi\} \) is consistent. By Proposition 4.17, there exists a maximal consistent set of formulas \( \Gamma' \) that extends \( \Gamma \), and then by Proposition 4.19, there exists a model \( M \) and a state \( \eta \in [M]_\Sigma \) such that \( M \models \eta \neg \varphi \), i.e. \( M \not\models \eta \varphi \).

\( \square \)

Corollary 4.21 The inference rules for \( \text{PL} \) is complete for any formulas. They are complete in \( \text{FOL} \) for every closed formulas, and in \( \text{MPL} \) and \( \text{TMPL} \) for tautologies (and then so is for \( \text{MMPL} \)).

We find the standard results of completeness, among other to \( \text{MPL} \) and \( \text{TMPL} \) (and then \( \text{MMPL} \)) where it is known that the completeness result holds for the local derivation (which amounts to demonstrate tautologies). More precisely, for \( \text{MPL} \), we have shown the completeness for the proof system known under the name \( T \) and its extensions \( S4, B \) and \( S5 \). On the contrary, as the anti-extensivity and extensivity properties of \( E^i \) and \( D^i \) are imposed (and then the accessibility relations are necessarily reflexive), the abstract proof given here cannot be instantiated to show the completeness result for the systems \( K \) and \( D \). For these two systems, we cannot use the model \( M_B \) defined for the logic \( \text{MPL} \) in the proof of Proposition 4.8 to prove their incompleteness. We have to consider the canonical model for which the set of states is the whole set of sets of maximally consistent formulas. The problem is that such a model has no equivalent for \( \text{PL} \) and \( \text{FOL} \). An open problem would be to see if there exists a general proof based on Henkin’s method which works both for logics with dual operators which are extensive and anti-extensive, and for logics with dual operators which are not.

Similar proofs of completeness have already been obtained in the framework of institutions but only for first-order logics \( \text{36, 25} \). In \( \text{36} \), the author follows Henkin’s method to prove his first-order completeness result while in \( \text{25} \), the authors use forcing methods to extend their first completeness result to infinitary first-order logics.

Here, we have extended these first results by unifying, in the framework of stratified institutions, a completeness proof which works both for \( \text{FOL} \) and the modal logics such as \( T, S4, B \) and \( S5 \), \( \text{TMPL} \) and \( \text{MMPL} \).

5 Towards applications in qualitative spatial reasoning

When dealing with qualitative spatial reasoning, spatial relationships are usually classified into topological, metric or directional relations \( \text{1, 29} \). In this section, we briefly show how such relations can be expressed in our framework.

29
5.1 Topological relationships

Topological approaches to qualitative spatial reasoning usually describe relationships between spatial regions. Two models have emerged to formalize topological spatial relations between spatial entities: RCC-8 [37] and 9-intersection [22, 23].

5.1.1 RCC-8

RCC-8 is a first-order theory based on a primitive connectedness relation $C$. From this binary relation $C$, many other binary relations can be defined, among which 8 were identified as being of particular importance, via the definition of a parthood predicate $P$ defined from $C$:

1. $DC(X, Y)$ which means that $X$ is disconnected from $Y$;
2. $EC(X, Y)$ which means that $X$ is externally connected to $Y$;
3. $PO(X, Y)$ which means that $X$ partially overlaps $Y$;
4. $TPP(X, Y)$ (resp. $TPPi(X, Y)$) which means that $X$ (resp. $Y$) is a tangential proper part of $Y$ (resp. $X$);
5. $NTPP(X, Y)$ (resp. $NTPPi(X, Y)$) which means that $X$ (resp. $Y$) is a non-tangential proper part of $Y$ (resp. $X$);
6. $EQ(X, Y)$ which means that $X$ is identical to $Y$.

Here, given a stratified institution $I = (\Sigma, \text{Sen}, \text{Mod}, [\cdot], \models)$ and a model $M \in \text{Mod}(\Sigma)$, the elements in $[M]_\Sigma$ are spatial entities, and then formulas are combinations of such entities. The model RCC-8 is a first-order theory which allows one to quantify on spatial entities. Following [2], we introduce the modality $U$ and its dual $A$ the semantics of which is as follows:

- $M \models_\Sigma U \varphi$ iff $\forall \eta' \in [M]_\Sigma, M \models_\Sigma \varphi$
- $M \models_\Sigma A \varphi$ iff $\exists \eta' \in [M]_\Sigma, M \models_\Sigma \varphi$

Using these primitives connectors, following [3], it is easy to define, independently of any stratified institution, simple relations such as inclusion, exclusion and intersection by using standard Boolean connectives in $\{\land, \lor, \Rightarrow, \neg\}$ and the modalities $U$ and $A$. Hence, the binary relations $C$, $DC$, $PO$ and $EQ$ can be expressed in our framework as follows, where $\varphi$ and $\psi$ are formulas that denote, respectively, the regions $X$ and $Y$:

- $C(X, Y)$: $A(\varphi \land \psi)$;
- $DC(X, Y)$: $U(\neg \varphi \lor \neg \psi)$;

\[\text{In [2], authors use } E. \text{ We prefer } A \text{ in order to avoid confusion with the notation for erosion.}\]
• **PO**(X,Y): \( A(\varphi \land \psi) \), \( A(\varphi \land \neg \psi) \), and \( A(\neg \varphi \land \psi) \);

• **EQ**(X,Y): \( \varphi \iff \psi \).

The other relations can benefit from the morphological operators. For this, we suppose that the stratified institution \( \mathcal{I} \) is equipped with two dual logical operators \( E \) and \( D \) defined as an erosion and a dilation on the lattice \( (\text{Sen}(\Sigma)/(\Sigma, \preceq)) \) for every signature \( \Sigma \) and every \( \Sigma \)-model \( M \) such that \( E \) and \( D \) are anti-extensive and extensive, respectively, for the binary relation \( \preceq_M \). To define adjacency (or external connection) \( \text{EC}(X,Y) \) between two regions \( X \) and \( Y \), we can then consider that these regions do not intersect but as soon as one of them is dilated, it has a non-empty intersection with the other one. This can be expressed as:

• **EC**(X,Y): \( \neg(\varphi \land \psi) \) and \( A(D(\varphi) \land \psi) \) and \( A(\varphi \land D(\psi)) \).

Now, the fact that a region \( X \) is a *tangential proper part of* a region \( Y \) (i.e. \( \text{TTPP}(X,Y) \)) can be expressed by the fact that \( X \) is included in \( Y \) but the dilation of \( X \) is not, i.e.:

• **TTPP**(X,Y): \( \varphi \Rightarrow \psi \) and \( A(D(\varphi) \land \neg \psi) \).

Similarly, the fact that a region \( X \) is a *non-tangential proper part of* a region \( Y \) (i.e. \( \text{NTPP}(X,Y) \)) can be expressed as:

• **NTPP**(X,Y): \( \varphi \Rightarrow \psi \) and \( \varphi \Rightarrow E(\psi) \) (or equivalently, \( D(\varphi) \Rightarrow \psi \)).

### 5.1.2 9-intersection

The 9-intersection model transforms the topological relationships between two spatial entities \( X \) and \( Y \) into a point-set topology problem. That is, the topological relations between two objects \( X \) and \( Y \) are defined in terms of the intersection of boundary, interior and exterior of \( X \) and \( Y \). Hence, the 9-intersection model captures the topological relation between two spatial entities \( X \) and \( Y \) based on the intersections of the three topological parts of \( X \) and those of \( Y \). These 3 \( \times \) 3 types of intersections are concisely represented by the 9-intersection matrix:

\[
\begin{pmatrix}
\delta X \cap \delta Y & \delta X \cap Y^\circ & \delta X \cap Y^- \\
X^\circ \cap \delta Y & X^\circ \cap Y^\circ & X^\circ \cap Y^- \\
X^- \cap \delta Y & X^- \cap Y^\circ & X^- \cap Y^-
\end{pmatrix}
\]

where \( \delta, \_^\circ \) and \( \_^- \) denote the interior, the exterior and the boundary, respectively.

For any stratified institution the model of which are topos-model, these 3 \( \times \) 3 types of intersections can be easily defined. Indeed, if we suppose that the two regions \( X \) and \( Y \) are denoted by the two formulas \( \varphi \) and \( \psi \), then

• their interior are \( \Box \varphi \) and \( \Box \psi \),

• the exterior are \( \neg \Box \varphi \) and \( \neg \Box \psi \), and

• their boundary are \( \varphi \land \neg \Box \varphi \) and \( \psi \land \neg \Box \psi \), and in our framework \( \Box \) and \( \Diamond \) are algebraic erosion and dilation, respectively.
5.2 Distances and directional relative position

Here, we assume a stratified institution \( \mathcal{I} \) such that

- either the category of states is the category of metric spaces \( \text{Met} \) and in this case \( \mathcal{I} \) is equipped with two logical operators \( E \) and \( D \) defined as erosion and dilation on the lattice \( (\text{Sen}(\Sigma)/\equiv_M, \preceq_M) \) for every signature \( \Sigma \) and every \( \Sigma \)-model \( M \) such that \( E \) and \( D \) are anti-extensive and extensive, respectively, for the binary relation \( \preceq_M \);

- or \( \mathcal{I} \) is equipped with two logical operators \( E \) and \( D \) defined as an erosion and dilation based on an elementary symmetrical structuring element \( B \).

In this last case, we can define a distance \( d \) that can take different forms depending on the considered spatial domain, as follows:

- \( \forall \eta, d(\eta, \eta) = 0 \);
- \( \forall \eta, \eta', \eta \neq \eta', d(\eta, \eta') = 1 \) iff \( \eta' \in B_\eta \);
- \( \forall \eta, \eta', d(\eta, \eta') = \inf_{\pi(\eta, \eta')} l(\pi) \), where \( \pi(\eta, \eta') \) is a path from \( \eta \) to \( \eta' \), i.e. a sequence \( \eta_0 = \eta, \eta_1, ... \eta_n = \eta' \) such that \( \forall i = 0, ... n-1, d(\eta_i, \eta_{i+1}) = 1 \), and \( l(\pi) \) is the length of the path (i.e. \( \pi = \eta_0, \eta_1, ... \eta_n \), \( l(\pi) = n = \sum_{i=0}^{n-1} d(\eta_i, \eta_{i+1}) \)).

By construction, \( d \) defines a metric.

In both cases, we can define a distance to a formula for every model \( M \in \text{Mod}(\Sigma) \) as done in the Euclidean space for a distance from a point to a compact set:

\[
    d(\eta, \varphi) = \inf_{M \models \Sigma, \varphi} d(\eta, \eta').
\]

Given two formulas \( \varphi \) and \( \varphi' \), their minimum \( d_{\min} \) and Hausdorff \( d_H \) distances can be derived as:

\[
    d_{\min}(\varphi, \varphi') = \inf_{M \models \Sigma, \varphi} d(\eta, \varphi),
\]

\[
    d_H(\varphi, \varphi') = \max \left( \sup_{M \models \Sigma, \varphi'} d(\eta, \varphi), \sup_{M \models \Sigma, \varphi'} d(\eta', \varphi) \right).
\]

As in the Euclidean case, these two distances can be conveniently expressed in terms of mathematical morphology. Details for the logic PL are given in [8]. Similarly, we have here:

\[
    d_{\min}(\varphi, \varphi') \leq n \iff A(D^n(\varphi) \land \varphi'),
\]

where \( D^0 \) is the identity mapping, \( D^1 = D \) and \( D^n = DD^{n-1} \) for \( n > 1 \), and:

\[
    d_H(\varphi, \varphi') \leq n \iff \varphi' \Rightarrow D^n_B(\varphi) \quad \text{and} \quad \varphi \Rightarrow D^n_B(\varphi').
\]
As an example of the potential use of such links between distances and dilation in spatial reasoning, let us consider the example in [8]. If we are looking at an object represented by $\psi$ in an area which is at a distance in an interval $[n_1, n_2]$ of a region represented by $\varphi$, this corresponds to a minimum distance greater than $n_1$ and to a Hausdorff distance less than $n_2$. Then we have to check the following relation:

$$\psi \Rightarrow \neg D^{n_1}(\varphi) \land D^{n_2}(\varphi).$$

This expresses in a symbolic way an imprecise knowledge about distances represented as an interval. If we consider a fuzzy interval, this extends directly by means of fuzzy dilation. These expressions show how we can convert distance information, which is usually defined in an analytical way, into algebraic expressions through mathematical morphology, and then into logical expressions through the proposed abstract dual operators based on dilation and erosion.

Directional relations can be defined in a similar way in the proposed framework, extending directly the PL case detailed in [8]. Here, $D^d$ denotes the dilation corresponding to a directional information in the direction $d$. Then assessing whether $\varphi'$ represents a region of space which is in direction $d$ with respect to the region represented by $\varphi$ amounts to check the following relation:

$$\varphi' \Rightarrow D^d(\varphi).$$

6 Conclusion

In this paper, we have shown that the abstract framework of stratified institutions allows for unified definitions of connectives, quantifiers and morphological operators. Morphological dilation and erosion are defined in this framework both algebraically as operators that commute with the supremum and infimum of the underlying lattices, and using structuring elements. The duality property is emphasized, as a common property of pairs of operators or modalities in several logics. The proposed abstract definitions and properties are then instantiated in different logics, such as propositional logic, first order logic, modal logics, fuzzy logics. Finally, they are used in qualitative spatial reasoning framework to define abstract topological, metric and directional relations. This is consistent with the common use of mathematical morphology to deal with spatial information.

Many perspectives are naturally occurring. First, the completeness result of this paper requires that the dual operators $E^i$ and $D^i$ are anti-extensive and extensive, respectively, which excludes the modal logics $D$ and $K$. As mentioned in Section 4.3, it would be interesting to see whether there exists a general proof based on Henkin’s method which works both for logics with dual operators which are extensive and anti-extensive, and for logics with dual operators which are not. Another interesting perspective would be to extend our general completeness result to the fuzzy setting. Finally, future work will aim at further exploring the spatial reasoning aspects. Moreover, theoretical results on complexity and tractability could be explored.
References

[1] M. Aiello, I. Pratt-Hartman, and J. van Benthem. *Handbook of Spatial Logics*. Springer-Verlag, 2007.

[2] M. Aiello and J. van Benthem. A modal walk through space. *Journal of Applied Non-Classical Logics*, 12(3-4):319–363, 2002.

[3] M. Aiguier and R. Diaconescu. Stratified institutions and elementary homomorphisms. *Information Processing Letters*, 103(5-13), 2007.

[4] B.-D. Baets. Idempotent Closing and Opening Operations in Fuzzy Mathematical Morphology. In *North American Fuzzy Information Processing Society (NAFIPS)*, pages 228–233. IEEE Computer Society, 1995.

[5] M. Barr and C. Wells. *Category Theory for Computing Science*. Prentice-Hall, 1990.

[6] J. Barwise. Axioms for abstract model theory. *Annals of Mathematical Logic*, 7:221–265, 1974.

[7] B. Bennett and I. Duntsch. *Handbook of Spatial Logics*, chapter Axioms, Algebras and Topology, pages 99–159. Springer-Verlag, 2007.

[8] I. Bloch. Modal Logics on Mathematical Morphology for Qualitative Spatial Reasoning. *Journal of Applied Non-Classical Logics*, 12(3-4):399–423, 2002.

[9] I. Bloch. Fuzzy Spatial Relationships for Image Processing and Interpretation: A Review. *Image and Vision Computing*, 23(2):89–110, 2005.

[10] I. Bloch. Spatial Reasoning under Imprecision using Fuzzy Set Theory, Formal Logics and Mathematical Morphology. *International Journal of Approximate Reasoning*, 41(2):77–95, 2006.

[11] I. Bloch. Duality vs. Adjunction for Fuzzy Mathematical Morphology and General Form of Fuzzy Erosions and Dilations. *Fuzzy Sets and Systems*, 160:1858–1867, 2009.

[12] I. Bloch, H. Heijmans, and C. Ronse. *Handbook of Spatial Logics*, chapter Mathematical Morphology, pages 857–947. Springer-Verlag, 2007.

[13] I. Bloch and J. Lang. Towards mathematical morpho-logics. In B. Bouchon-Meunier, J. Gutierrez-Rios, L. Magdalena, and R. Yager, editors, *Technologies for Constructing Intelligent Systems*, pages 367–380. Springer-Verlag, 2002.

[14] I. Bloch and H. Maitre. Constructing a Fuzzy Mathematical Morphology: Alternative ways. In *International Conference on Fuzzy Systems (FUZZIEEE)*, pages 1303–1308. IEEE Computer Society, 1993.
[15] I. Bloch and H. Maitre. Fuzzy Mathematical Morphology: A Comparative Study. *Pattern Recognition*, 28(9):1341–1387, 1995.

[16] E. Clementini and O.-D. Felice. Approximate Topological Relations. *International Journal of Approximate Reasoning*, 16:173–204, 1997.

[17] A. Cohn, B. Bennett, B. Gooday, and N.-M. Gotts. Representing and Reasoning with Qualitative Spatial Relations about Regions. In O. Stock, editor, *Spatial and Temporal Reasoning*, pages 97–134. Kluwer, 1997.

[18] R. Diaconescu. Proof systems for institutional logic. *Journal of Logic and Computation*, 16(3):339–357, 2006.

[19] R. Diaconescu. *Institution-independent Model Theory*. Universal Logic. Birkäuser, 2008.

[20] R. Diaconescu. Institutional semantics for many-valued logics. *Fuzzy Sets and Systems*, 218:32–52, 2013.

[21] R. Diaconescu. Graded consequence: an institution theoretic study. *Soft Computing*, 18(7):1247–1267, 2014.

[22] M.-J. Egenhofer. Reasoning about binary topological relations. In *Advances in Spatial Databases, Second International Symposium, SSD ’91*, volume 525 of *Lecture Notes in Computer Science*, pages 143–160. Springer, 1991.

[23] M.-J. Egenhofer and R. Franzosa. Point-set topological spatial relations. *International Journal of Geographical Information Systems*, 5:161–174, 1991.

[24] J. Fiadeiro and A. Sernadas. Structuring theories on consequence. In *Recent Trends in Algebraic Development Techniques*, volume 332 of *Lecture Notes in Computer Science*, pages 44–72. Springer, 1988.

[25] D. Gaina and M. Petria. Completeness by Forcing. *Journal of Logic and Computation*, 20(6):1165–1186, 2010.

[26] J.A. Goguen and R.-M. Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the ACM*, 39(1):95–146, 1992.

[27] R. Hakli and S. Negri. Does the deduction theorem fail for modal logic? *Synthese*, 187(3):849–867, 2012.

[28] L. Henkin. The completeness of the first-order functional calculus. *The Journal of Symbolic Logic*, 14(3):159–166, 1949.

[29] B. Kuipers. The Spatial Semantic Hierarchy. *Artificial Intelligence*, 119:191–233, 2000.

[30] G. Ligozat. *Qualitative Spatial and Temporal Reasoning*. Wiley, 2011.

[31] S. MacLane. *Categories for the Working Mathematician*. Springer-Verlag, 1971.
[32] J. Meseguer. General logics. In Logic Colloq.’87, pages 275–329. Holland, 1989.

[33] T. Mossakowski and R. Moratz. Relations between spatial calculi about directions and orientations. Journal of Artificial Intelligence Research, 54:277–308, 2015.

[34] M. Nachtegael and E. E. Kerre. Classical and Fuzzy Approaches towards Mathematical Morphology. In E. E. Kerre and M. Nachtegael, editors, Fuzzy Techniques in Image Processing, Studies in Fuzziness and Soft Computing, chapter 1, pages 3–57. Physica-Verlag, Springer, 2000.

[35] L. Najman and H. Talbot. Mathematical morphology: from theory to applications. ISTE-Wiley, June 2010.

[36] M. Petria. An institutional version of Godel’s completeness theorem. In Algebra and CoAlgebra in Computer-Science, volume 4624 of Lecture Notes in Computer Science, pages 409–425. Springer, 2007.

[37] D. Randell, Z. Cui, and A. Cohn. A Spatial Logic based on Regions and Connection. In International Conference on Principles of Knowledge Representation and Reasoning (KR), pages 165–176. AAAI Press, 1992.

[38] J. Renz and B. Nebel. Handbook of Spatial Logics, chapter Qualitative Spatial Reasoning Using Constraint Calculi, pages 161–215. Springer-Verlag, 2007.

[39] J. Serra. Image Analysis and Mathematical Morphology. Academic Press, 1982.

[40] A. Tarlecki. Algebraic Foundations of Systems Specification, chapter Institutions: An Abstract Framework for Formal Specifications. IFIP State-of-the-Art Reports. Springer-Verlag, 1999.

[41] J. van Benthem and G. Bezhanishvili. Handbook of Spatial Logics, chapter Modal Logics of Space, pages 217–298. Springer-Verlag, 2007.