Loss tolerant quantum state tomography by number-resolving measurements without approximate displacements

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We introduce a quantum state tomography technique to reconstruct the density matrix of an arbitrary quantum state by experimentally determining the Wigner function overlap, or the fidelity, between the unknown state and a set of tomographically complete coherent states. Each fidelity is determined by a parity measurement from number-resolving statistics on only a single mode of the output field after an interference between each probing coherent state and the unknown state. We demonstrate that the state can be accurately reconstructed following an arbitrary well-calibrated loss, and that this reconstruction is robust to noise. Unlike conventional continuous variable state tomography methods, our method utilizes computationally efficient semidefinite programming (SDP).

Characterization of quantum states plays an important role in discrete as well as continuous-variables (CV) quantum information processing [1–4]. The most common method to characterize a CV quantum state is to reconstruct its Wigner quasiprobability distribution with field quadrature measurements obtained from balanced homodyne detection (BHD) [5], which requires computationally intensive reconstruction algorithms [6]. In addition, the Wigner function can also be reconstructed by deconvoluting an experimentally measured Husimi Q function [7].

To alleviate the demands posed by reconstruction algorithms, a more direct method was proposed by Wallentowitz and Vogel [8] and by Banaszek and Woźniakiewicz [9]. It probes the Wigner function point-by-point by displacing the quantum state over the entire phase-space and reconstructing its Wigner function from the expectation value of the parity operator, which can be obtained from photon-number-resolving (PNR) measurements [10–13] or using atomic interferometry techniques for cavity QED systems [14, 15]. This method, however, does suffer from an inherent loss due to approximated displacement operations implemented using a highly unbalanced beamsplitter (BS) and a macroscopic Local Oscillator (LO) [16, 17]. Furthermore, the number of measurements required to fully probe Wigner functions of complicated structure, such as Schrödinger cat states or Gottesman-Kitaev-Preskill (GKP) states, might be demanding.

In addition, to compensate for detection inefficiencies and optical losses one needs to infer the true photon-number distribution from the experimentally measured distributions for each phase space coordinate of the Wigner function. Although there have been experimental implementations for this scheme in the presence of losses [11, 18, 19], it might become particularly cumbersome when probing the phase space densely. Compensating for these losses requires solving inversion problems that can potentially lead to numerical instabilities and an unphysical reconstructed state [20–22].

Recently, there has been significant interest in homodyne-like (HL) schemes, where the actual photon-number difference distribution is measured in place of macroscopic photocurrent differences. It has been proven and experimentally demonstrated that the pattern function technique, originally developed for macroscopic photocurrents [23], can also be applied to a HL schemes where the LO intensity is comparable to the quantum state under investigation [24]. The HL scheme can be tuned to the BHD method as one increases the amplitude of the LO to the regime where it can be treated as a classical field [25]. Although, the pattern functions can be calculated using numerically efficient algorithms [23], they are prone to instabilities in the presence of noise and low detection efficiencies [26].

In this work, we propose a technique to perform complete state tomography by interfering an unknown state with coherent state probes at a balanced BS and performing PNR measurements on one of the BS output modes, thus necessitating only a single PNR detector. Our method strikes a balance between the conventional BHD and the weak-field unbalanced homodyne by reconstructing the density matrix of the unknown state using computationally efficient semidefinite programming (SDP), which, unlike some reconstruction algorithms at times, always yields a positive reconstructed density matrix with unity trace. Our method is based on measuring the Wigner function overlaps which allows us to acquire nonlocal information about the quantum state as opposed to probing a single point in the unbalanced homodyne method. Therefore, the number of measurements is substantially reduced in the proposed scheme. Additionally, we show how this technique is made to be robust against experimental fluctuations and that known losses

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can be easily compensated using SDP that avoids potential numerical instabilities arising from analytic inversion. The loss reconstruction is applied to the entire density matrix at once, as opposed to each set of photon statistics, and it recovers non-classical features entirely erased by loss. The proposed technique is based on finding the overlap integral between the unknown state under investigation with known coherent states (or probes), which is visualized in Fig. 1a. In order to do that, we utilize two main results. First, the Wigner function at the origin of the phase space is essentially the expectation value of the parity operator [27], which has been used to perform quantum state tomography by directly reconstructing the Wigner function of the quantum states using photon-number-resolving (PNR) measurements [10–13, 28]. Second, since a coherent state is a pure state, the overlap integral of the Wigner functions can always be mathematically formulated as the fidelity, $F$, between the unknown state, $\rho$, and the coherent state, $\sigma$. Thus, we have

$$F := Tr[\rho \sigma] = 2\pi \int W_1(q,p)W_2(q,p)dqdp. \quad (1)$$

To experimentally measure the overlap integral, we utilize a scheme [29] which involves interfering the unknown state, $\rho_{in}$, with a known coherent state, $|\alpha^j\rangle\langle\alpha^j|$, at a balanced BS as shown in Fig 1b.

![Fig. 1: (a) Experimental schematic to measure the overlap integral. (b) Visualization of the Wigner function overlap.](image)

After the BS interaction, the Wigner function of one of the output modes is measured at the origin of the phase space via expectation value of parity operator. Note that the photon-number states are the eigenstates of the parity operator, which allows us to experimentally measure the parity operator from the PNR detections. The Wigner function at the origin after the BS directly gives the fidelity between the coherent state, $|\alpha^j\rangle\langle\alpha^j|$, and the state, $\rho_{in}$. To see that, we adopt the Heisenberg picture to determine the evolved quadratures under the BS interaction. Under the BS transformation, $q'_1 = tq_1 - rq_2$ and $p'_1 = tp_1 - rp_2$. Likewise, $q'_2 = rq_1 + tq_2$ and $p'_2 = rp_1 + tp_2$. The two-mode input state is written in the Wigner function representation as

$$W_{1,2}(x) = W_1(q_1,p_1)W_2(q_2,p_2). \quad (2)$$

Next, by using the evolved quadratures, one can write the Wigner function of the BS output as

$$W'_{1,2}(x') = W_1(tq_1' + rq_2', tp_1' + rp_2') \times W_2(-rq_1' + tq_2', -rp_1' + tp_2') \quad (3)$$

where $x, x'$ are column vectors consisting of quadratures corresponding to the input and output modes respectively. When $r = t = \frac{1}{\sqrt{2}}$, the amplitude of the Wigner function at the origin of phase space for mode 1 can be mathematically formulated as

$$\int W_1(q)p\rho_2(q)p\rho dqdp = \sqrt{2}\int W_1(q,p)W_2(q,p)dqdp, \quad (4)$$

which is proportional to Eq. (1). Thus, one can simply find the overlap integral between two unknown states. It is worth mentioning that with a highly unbalanced BS ($t \gg r$), the proposed method becomes unbalanced homodyne. In that case, the Wigner function overlap is given by the Wigner function of the quantum state at a certain phase space coordinate defined by $\beta = \frac{r}{t}\alpha$. We now use the formalism discussed above to perform the complete state tomography of an arbitrary quantum state. For a given single-mode quantum state, one can write the density matrix in the photon-number basis as

$$\rho = \sum_{n,n'=0}^{\infty} \rho_{n,n'} |n\rangle\langle n'|. \quad (5)$$

Complete characterization of $\rho$ requires determining $\rho_{n,n'}$. To do that, we choose a set of distinct coherent states, $|\alpha^j\rangle$. Using Eq. (4), we obtain the fidelity between $|\alpha^j\rangle$ and $\rho$, formulated as

$$F^j = Tr[|\alpha^j\rangle\langle\alpha^j|\rho] = \langle\alpha^j|\rho|\alpha^j\rangle. \quad (6)$$

Using Eq. (6) and the coherent state represented in the photon-number basis, $|\alpha^j\rangle = \sum_{m=0}^{\infty} c_m^j |m\rangle$, we get

$$F^j = \sum_{m'=0}^{\infty} c_{m'}^j \sum_{n,n'=0}^{\infty} \rho_{n,n'} |n\rangle\langle n'| \sum_{m=0}^{\infty} c_m^j |m\rangle. \quad (7)$$

Further simplification leads to

$$F^j = \sum_{n,m=0}^{\infty} c_n^j c_m^{j*} \rho_{n,m}. \quad (8)$$

Ideally, the sum over $n, m$ goes to infinity but for practical purposes one needs to truncate it at, say $n_0$, such that any terms $n, m > n_0$ do not significantly contribute to the sum. As a result, we have

$$F^j = \sum_{n,m=0}^{n_0} c_n^j c_m^{j*} \rho_{n,m}. \quad (9)$$
where $c_{m}c_{n}^{*} = e^{-|\alpha|^2} |\alpha^{m} \alpha^{n}\rangle\langle \alpha^{n} \alpha^{m}|$.

Furthermore, by using $N_{p} = (n_{0} + 1)^{2}$ coherent states, Eq. (9) can be written in the matrix form as

$$
\begin{pmatrix}
F(0) \\
F(1) \\
\vdots \\
F^{N_{p}}
\end{pmatrix} = 
\begin{pmatrix}
\rho_{0,0}^{0} & \rho_{0,1}^{0} & \cdots & \rho_{0,N_{p}}^{0} \\
\rho_{1,0}^{1} & \rho_{1,1}^{1} & \cdots & \rho_{1,N_{p}}^{1} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{N_{p},0}^{N_{p}} & \rho_{N_{p},1}^{N_{p}} & \cdots & \rho_{N_{p},N_{p}}^{N_{p}}
\end{pmatrix} 
\begin{pmatrix}
c_{0}^{0} & c_{1}^{0} & \cdots & c_{N_{p}}^{0} \\
c_{0}^{1} & c_{1}^{1} & \cdots & c_{N_{p}}^{1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{0}^{N_{p}} & c_{1}^{N_{p}} & \cdots & c_{N_{p}}^{N_{p}}
\end{pmatrix} 
\begin{pmatrix}
\rho_{0,0} \\
\rho_{0,1} \\
\vdots \\
\rho_{N_{p},N_{p}}
\end{pmatrix}
$$

Rewriting in the compact form as

$$
\mathbf{F} = \mathbf{C} \mathbf{M},
$$

where $\mathbf{F} \in \mathbb{R}^{(n_{0}+1)^{2}}$, $\mathbf{M} \in \mathbb{C}^{(n_{0}+1)^{2}}$ and $\mathbf{C} \in \mathbb{C}^{((n_{0}+1)^{2} \times (n_{0}+1)^{2})}$. Next, we can invert Eq. (10) to reconstruct $\mathbf{M}$. It can be achieved by solving the following semidefinite program.

$$
\text{Minimize} \quad ||\mathbf{F} - \mathbf{C} \mathbf{M}||_{2}
$$

Subject to $\quad \rho \geq 0$ and $\text{Tr}[\rho] = 1,
$$

where $||.||_{2}$ is the $l_{2}$ norm defined as $||V||_{2} = \sqrt{\sum |v_{i}|^{2}}$. Note that this kind of quadratic convex techniques have been extensively discussed in the context of quantum detector tomography [30–33]. The optimization problem is convex and can be efficiently solved for a guaranteed unique $\mathbf{M}$, and hence for the unknown state, $\rho$, using open source Python module CVXPY [34]. Although this method holds for general quantum states, we restrict our simulations to real-valued density matrices for numerical ease. However, we do show the reconstruction for complex-valued density matrices in the appendix I A.

All numerical simulations were performed in QuTip [35] where the Hilbert space for each optical mode was constructed in the Fock basis with a high enough dimensionality to ensure state probability amplitudes decayed to less than $10^{-7}$ before truncation. Under these parameters, the numerically efficient SDP algorithms converged in $\mathcal{O}(10^{-2})$ seconds on a 3GHz Intel i5 quad core processor with 16Gb RAM.

Our method is demonstrated in Fig. 2 for the example cases of the small amplitude cat state, $|\psi\rangle \propto |\alpha\rangle + |-\alpha\rangle$ where $\alpha = \sqrt{3}$, and a Gottesman-Kitaev-Preskill (GKP) state of mean photon number 5. These states were reconstructed using 400 different probing coherent states of 20 amplitude increments from $\beta = 0$ to $\beta = \sqrt{6}$ and 20 phase increments from 0 to $2\pi$, to achieve fidelities with the target states greater than 0.999 for the cat state, and a fidelity of 0.985 for the GKP state. Because the observer is assumed to have no prior knowledge of the state to be characterized, it is important to scan the entirety of phase space in question with different coherent states so as to have sufficient overlap between all portions of the state under test. If some prior knowledge of the state is obtained, then the probing coherent states can be restricted to a localized region of phase space near the unknown quantum state.

Loss is the primary source of imperfection in the state tomography. Methods to correct for loss-degraded photon number distributions when counting photons are known [11, 18, 22], but this requires performing a matrix inversion for each experimental data point. To decrease the computational load and generalize the correction of loss to arbitrary detection schemes, consider reconstructing the density matrix of an arbitrary quantum state, not necessarily pure, following a known loss. In this case, we have experimentally measured the loss degraded state, $\rho'$ instead of the full density matrix before the loss, $\rho$. Under a known loss, as modelled by a fictitious BS with transmission $\eta$, the relationship between density matrix elements as derived in appendix IB 2 is given as [21]

$$
\rho'_{m,m'} = \sum_{k=0}^{\infty} \rho_{(m+k),(m'+k)}A(m, m'; k)(1 - \eta)^{k} \eta^{\frac{m+m'}{2}},
$$

where $A(m, m'; k) = (\frac{m+k}{2})^{2} \lambda^{m+k}$. In practice, the sum over $k$ can be truncated to some value, $N_{\text{max}}$, beyond which the entries in the initial density matrix are negligible. We can then reformulate Eq. (12) as a series of $N_{\text{max}}$ linear maps from the $i^{th}$ diagonal of $\rho'$ to the $i^{th}$ diagonal of $\rho$, where the main diagonal is given by $i = 0$. Each of these linear maps, $\mathbf{M}^{(i)}$, is an upper triangular
The application of these constraints enforces physicality:

\[ M^{(i)}_{jk}(\eta) = \begin{cases} 0 & j > k \\ \sqrt{\frac{k}{(k-j)(k+j)}(1-\eta)^{(k-j)}\eta^{2+j}} & \text{otherwise} \end{cases} \] (13)

If we only consider the diagonal elements of \( \rho \), then \( M^{(0)} \) is exactly the process that describes a lossy photon number distribution; the inversion of which was first derived in Ref. 22. Since each \( M^{(i)}(\eta) \) is triangular with nonzero diagonal elements, the inverse mappings can be found by inverting the generalized Bernoulli transformation and are given by [21]

\[ M^{(i)}(\eta)^{-1} = M^{(i)}(\eta^{-1}). \] (14)

The existence of this inversion is due to the known well-defined statistical nature of loss channel, which makes it possible to perfectly reconstruct any \( \rho \) within a finite-dimensional Hilbert space when \( \eta \) and \( \rho' \) are precisely known [21]. However, the presence of any small deviations in an experimentally measured \( \rho' \) can lead to unphysically large or non-positive diagonal density matrix elements in the reconstruction of \( \rho \), which is similar to the possible numerical instabilities that arise when using pattern-functions [26]. These errors become pronounced due to the presence of any small deviations in an experimentally measured density matrix, which is similar to the possible numerical instabilities that arise when using pattern-functions [26]. These errors become pronounced due to the presence of any small deviations in an experimentally measured density matrix, which is similar to the possible numerical instabilities that arise when using pattern-functions [26].

Next, we utilize SDP defined in Eq. (16) to compensate for the calibrated losses and obtain the corrected density matrix, where we display the corresponding Wigner function and fidelities in Fig. 4b. We can clearly see that the finely resolvable features that are erased due to losses, including negativity of the Wigner function, are recovered in detail following the reconstruction. It is worth emphasizing that no information about the states was assumed before the characterization process, and we only assume well calibrated loss and coherent state probes. Note that this inversion technique employing SDP could also be used for other tomographic schemes. As long as losses are calibrated and the degraded density matrix is well-determined, the true physically valid density matrix can be recovered.
FIG. 4: Reconstruction of various states after 50% loss. We have $|\alpha| = \sqrt{3}$ for the coherent state and 
$\rho_{\text{mix}} = \frac{4}{5}|4\rangle\langle 4| + \frac{1}{5}|\beta\rangle\langle \beta|$ with $|\beta| = 2$. (a) shows the reconstructed Wigner functions of the target states after losses using our tomographic scheme and (b) shows the Wigner functions obtained by correcting the states in (a) under the known loss. All fidelity insets compare the depicted state with the target.

We now turn to test the robustness to inevitable experimental fluctuations. The stability of the solution of Eq. (10) is governed by matrix $C$, which by nature is ill-conditioned as verified by the large ratio of the largest and smallest singular values [36]. The ill-conditioning makes the solution extremely sensitive to experimental fluctuations, as a small variation in the measured PNR statistics (or the overlap integral) results in substantial changes in the reconstructed density matrix. However, these instabilities might be suppressed by adding regularization to the optimization problem. In this work, we use Tikhonov regularization [37] mathematically formulated as

$$\text{Minimize} \quad \| F - CM \|_2^2 + \gamma \| M \|_2^2$$

Subject to $\rho \geq 0$ and $\text{Tr}[\rho] = 1$, \hspace{1cm} (16)

where $\gamma \geq 0$ is known as regularization parameter. The inclusion of small but nonzero $\gamma$ in the presence of noise aids to enforce physicality of the reconstruction results by preventing runaway numerical solutions from diverging. We note that this procedure remains robust and state-independent in that allowing $\gamma$ to vary by an order of magnitude (from 0.001 to 0.01) negligibly impacts the state reconstruction in each case considered, as measured by the change in fidelity with the appropriate target state. Several other kinds of regularization techniques have been explored for state tomography, process tomography, and detector tomography [30, 32, 33, 38–40]. With the Tikhonov regularization, the optimization problem still remains convex quadratic, and hence can be solved efficiently.

We model our noise in the same spirit as [30, 32] by introducing artificial fluctuations in the amplitude and phase of the coherent states, $|\alpha^j\rangle$. The amplitude noise is sampled using a Gaussian distribution of zero mean and standard deviation of $\sigma = 2\%|\alpha|$, and likewise the phase noise is sampled from $\sigma \in [-1, 1]$ degrees using a Gaussian distribution with zero mean. To demonstrate the effect of experimental fluctuations, we performed the state tomography of an ideal 4-photon Fock state and the same statistical mixture as in Fig. 4 in the presence of 50% losses, where the varying probe coherent states now have both phase and amplitude noise. Numerical results are displayed in Fig. 5. We are able to identify the Wigner functions; however, we now see the appear-
 ance of noise-induced ripples as shown in Fig. 5a, which obfuscate the differences between the pure Fock state and the mixture. In order to suppress the fluctuation effects, we repeat the experiment \(N = 30\) times in order to obtain smooth Wigner functions as evident from Fig. 5b. We can clearly see the ripples become diminished, which demonstrates the rather intuitive result that performing multiple measurements in the presence of noise can improve resolution.

We note that quantum states with losses greater than 50\% can be accurately tomographed and compensated for loss even in the presence of noise as seen in the appendix 1C; however, increasing loss requires greater accuracy in the measured loss-degraded density matrix for the reconstruction to be valid. This is not prohibitive to the success and simply requires increasing the number of coherent state probes until the desired accuracy in the tomographic procedure is reached.

In conclusion, a method to perform a complete state tomography for an arbitrary quantum state is proposed using PNR measurements for one of the output ports of the conventional BHD based state tomography. The method proposed in this article has several main advantages over current methods for state characterization. First, we employ numerically efficient SDP which is made to be robust against inevitable experimental noise. We only assume a practical limitation on the energy of the state in order to truncate the Hilbert space which is better justified than having to discretize CV quadratures in BHD. By the virtue of SDP, our reconstruction guarantees the physical nature of the reconstructed state.

Second, the intensity of the LO is comparable to the quantum state under characterization as opposed to a macroscopic LO used with BHD or unbalanced homodyne scheme. Our approach does not suffer from an inherent decoherence caused by an approximated implementation of displacement operations used in unbalanced homodyne based state tomography. Note that inaccuracies in the displacement operations may become crucial when characterizing high energy states. Since the unbalanced homodyne method only measures the Wigner function at a single phase space coordinate, the number of measurements might be experimentally demanding to probe a Wigner function with complicated structure. In contrast, our method obtains nonlocal information by measuring the Wigner function overlap, which drastically reduces the number of probes used for the accurate complete state tomography.

Third, we use only a single PNR detector which reduces the detection resources used in the HL scheme and also eliminates the problems posed by unequal detector efficiencies in BHD.

Finally, we devised a computationally efficient method to correct for losses involved in the experiments which allows for the detection of lower energy states and helps to mitigate the errors caused by detector saturation. Our inversion scheme guarantees the physicality of the reconstructed state and is tolerant to numerical instabilities that are otherwise present in the inversion using the generalized Bernoulli transformation.

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I. APPENDIX

A. Complex Reconstruction

We demonstrate the tomography protocol for complex-valued density matrices displayed in Fig. 6. We perform the tomography with coherent state probes that range in amplitude in 20 steps from $|\beta| \in [0, \sqrt{3}]$ and 20 phases $\phi \in [0, 2\pi]$, for a coherent state denoted by complex variable, $\alpha = \sqrt{2}(i + 1)$ and a superposition of photon-number states with the complex probability amplitude. The Wigner functions are shown along with separate plots for the real and imaginary elements of the respective reconstructed density matrices, including an inset fidelity with the ideal states.

B. Loss Compensation

1. Photon-number distribution correction

In this section, we deal with losses due to imperfect detection in the state tomography process, which can be simply modeled by setting up a fictitious BS in front of a perfect detector. For a detector of quantum efficiency, $\eta$, we use a BS of transmission $\eta$ and reflection $1 - \eta$. Since our method solely requires photon-number distributions to perform the complete state tomography, we are interested in inferring the true photon-number distribution from the loss-degraded measured photon-number distribution denoted by $P_n$ and $P'_n$ respectively. For a detector of quantum efficiency, $\eta$, and no darkcount noise, the POVM element corresponding to $n$-photon detection event is given by

$$\Pi_n = \sum_{m=n}^{m_0} P(n|m)|m\rangle\langle m|,$$

(17)

where $P(n|m) = \binom{m}{n} \eta^n (1 - \eta)^{m-n}$ is the conditional probability of detecting $n$ photons if $m$ photons are incident to the detector and $m_0$ is the photon-number where the detector saturates. For a given input denoted by density matrix $\rho = \sum_{n,n'=0}^{\infty} c_{n'n}|n\rangle\langle n'|$, the probability of detecting $n$ photons is

$$P'_n = \text{Tr}[\rho \Pi_n].$$

(18)
FIG. 6: Reconstruction of states with complex-valued density matrices for (a) the coherent state, $|\alpha\rangle = \sqrt{2}(i + 1)$ and (b) the superposition $\frac{1}{\sqrt{2}}(|2\rangle - i|3\rangle)$. The inset fidelity is calculated between the reconstructed state and the ideal target state. (i) Reconstructed Wigner functions. (ii) Real elements of the density matrices. (iii) Imaginary elements of the density matrices.

Using Eq. (17) and Eq. (18), we get

$$P'_n = \sum_{m=n}^{m_0} P(n|m)c_{m,m} = \sum_{m=n}^{m_0} P(n|m)P_m,$$

where $c_{m,m} = P_m$ is the probability of having $m$ photons in the input state $\rho$ to the imperfect detector. From Eq. (19), one can see that the measured photon-number distribution, $P'_n$, is linearly related to true photon-number distribution, $P_m$. Furthermore, Eq. (19) can be rewritten for all detector outcomes in the matrix form as

$$P' = \Pi P,$$

where $P'$ and $P$ are column vectors of length $m_0 + 1$ representing measured and actual photon-number distribution. $\Pi$ is an upper triangular matrix of dimension $(m_0 + 1) \times (m_0 + 1)$ characterizing the photon-number resolving detector. Thus, by simply inverting Eq. (20), one can obtain the true photon-number statistics from the experimental data, which allows you to perform the complete state tomography using the method discussed in the main text. It is worth noting that for a perfect detector, i.e., $\eta = 1$, $\Pi$ is an identity matrix which means the measured photon-number distribution is essentially the true distribution.

2. Complete density matrix correction

We now wish to correct an arbitrary density matrix given a known loss. In this case, we have experimentally measured $\rho'$, but our goal is to reconstruct the density matrix before the loss, $\rho$. As shown in Fig. 7, this can be modeled by sending $\rho$ through a fictitious ‘loss beamsplitter’ with reflection and transmission coefficients of $r = \sqrt{1-\eta}$ and $t = \sqrt{\eta}$, where $\eta$ is the overall transmission efficiency.

The general quantum state density matrix before the loss can be written as

$$\rho = \sum_{n,n'=0}^{\infty} \rho_{n,n'}|n\rangle\langle n'|.$$
FIG. 7: Lossy channel.

If this state enters into the loss beamsplitter in mode \( a \) with vacuum in mode \( b \), then the mode operators transform in the Heisenberg picture according to

\[
\begin{align*}
\hat{a} &\rightarrow \sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{b}, \\
\hat{b} &\rightarrow -\sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{b},
\end{align*}
\]

yielding an output density matrix

\[
\rho_{\text{out}} = \sum_{n,n'=0}^{\infty} \rho_{n,n'} (t a^\dagger + r b^\dagger)^n (t a + r b)^{n'} \sqrt{n!} \langle 0 | a \rangle \langle 0 | a \rangle \langle 0 | b \rangle \langle 0 | b \rangle \langle 0 | a \rangle (t a + r b)^n \langle 0 | a \rangle
\]

(22)

Tracing out over mode \( b \) yields the final state after loss, which is given by

\[
\rho' = \text{Tr}_b[\rho_{\text{out}}] = \sum_{n,n'=0}^{\infty} \rho_{n,n'} \sum_{k=0}^{n} \sum_{k'=0}^{n'} A_{n,n',k,k'} \langle n-k | k' \rangle \delta_{k,k'},
\]

(23)

where

\[
A(n, n', k, k') = \sqrt{n \choose k} n' \choose k' r^{k+k'} t^{n+n'-k-k'}.
\]

Replacing the respective dummy variables \( n-k \) and \( n'-k \) with \( m \) and \( m' \) allows us to rearrange the expression and write a sum over the Fock components in order, which can be written as

\[
\rho' = \sum_{m,m',k=0}^{\infty} \rho_{(m+k),(m'+k)} A(m+k, m'+k, k, k) | m \rangle \langle m' |,
\]

(25)

where it is easy to see that each element of the density matrix after loss is related to the original state by

\[
\rho'_{m,m'} = \sum_{k=0}^{\infty} \rho_{(m+k),(m'+k)} \sqrt{m+k \choose k} m'+k \choose k r^{2k} t^{m+m'}.
\]

(26)

This can be viewed as a generalized Bernoulli distribution [21], so can be inverted to read

\[
\rho_{m,m'} = \sum_{k=0}^{\infty} \rho'_{(m+k),(m'+k)} \sqrt{m+k \choose k} m'+k \choose k (-1)^k (r/t) 2k t^{-m-m'}.
\]

(27)

In practice, the sum over \( k \) can be truncated to some value, \( N_{\text{max}} \), beyond which the entries in the initial density matrix are negligible. We can then reformulate Eq. 27 as a series of \( N_{\text{max}} \) linear maps from the \( i^{\text{th}} \) diagonal of \( \rho' \) to the \( i^{\text{th}} \) diagonal of \( \rho \), where the main diagonal is given by \( i = 0 \). Each of these linear maps, \( M^{(i)} \), is an upper triangular matrix of dimension \( N_{\text{max}} - i \times N_{\text{max}} - i \) with elements

\[
M^{(i)}_{jk}(\eta) = \begin{cases} 
0 & j > k \\
\sqrt{m+k \choose k} m'+k \choose k (-1)^k (r/t) 2k t^{-m-m'} & j = k
\end{cases}
\]

(28)
3. Equivalence of photon-number distributions

To show the equivalence of the photon-number distributions obtained by each configuration in Fig. 8, we adapt the approach originally introduced in [8]. Derived in [41], the photon-number probability distribution is given as

\[ P(n) = \langle : e^{-N} N^n : \rangle, \]  

(29)

where \( N = d^\dagger d \) is the photon-number operator of the detection mode and the expectation value is calculated over the initial states, and :: is the normal ordering. We first determine the detection mode in terms of input modes for the network on the left of Fig. 8. In the Heisenberg picture, the input mode denoted by annihilation operator, \( a \), evolves to

\[ \text{After first BS: } a \rightarrow \frac{a + b}{\sqrt{2}} \]  

\[ \text{After second BS: } \sqrt{\eta} \left(\frac{a + b}{\sqrt{2}}\right) + \sqrt{1 - \eta} c_v \]  

(31)

Since the input states for mode \( b \) and \( c_v \) are coherent and vacuum states respectively, the normal ordering allows to treat them as complex numbers. As a result, the effective photon-number operator is given by

\[ N^L = d^\dagger d, \]  

(32)

where the detection mode is

\[ d^L = \sqrt{\eta} \left(\frac{a + \beta}{\sqrt{2}}\right) \]  

(33)

Likewise, for the right network, we have

\[ \text{After first BS: } a \rightarrow \sqrt{\eta} a + \sqrt{1 - \eta} b_v \]  

\[ \text{After top BS: } b \rightarrow \sqrt{\eta} b + \sqrt{1 - \eta} c_v \]  

\[ \text{After balanced BS: } d = \frac{1}{\sqrt{2}} (\sqrt{\eta} a + \sqrt{1 - \eta} b_v + \sqrt{\eta} b + \sqrt{1 - \eta} c_v) \]  

(35)

where \( c_v, b_v \) are vacuum modes and \( b \) is a coherent state. We again utilize the fact that normal ordering allows coherent states to be represented by a complex number and the vacuum state can also be considered as a coherent state with zero amplitude. Thus, the detection mode can be further simplified as

\[ d^R = \sqrt{\eta} \left(\frac{a + \beta}{\sqrt{2}}\right). \]  

(36)
From Eq. (33) and Eq. (36), one can see that both networks have the same detection mode and thus produce the same photon-number distribution for a given quantum state under investigation.

C. High-loss compensation

If the state we are characterizing exists in a finite-dimensional Hilbert space, then losses of any magnitude can be corrected in principle. However, due to experimental noise and numerical instabilities in the reconstruction, loss compensation requires increasing accuracy on the tomography of $\rho'$ as losses increase. Furthermore, states with higher complexity also require a denser sampling of phase space for accurate reconstruction. This leads to a general rule that, while all finite representations of states can be reconstructed and compensated for losses, the number of measurements required to make the process accurate increases with both increasing losses and increasing state complexity.

Nonetheless, we demonstrate that the process is successful for high losses in the presence of noise without an undue requirement on the number of measurements to be made. For the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$, the reconstruction is displayed in Fig. 9 in the presence of a known 70% loss with phase and amplitude fluctuations on the coherent state probes. The tomography is performed with coherent states at 40 amplitudes and 40 phases, and the overall fidelity of the loss-compensated state with the target is $F = 0.978$. The state reconstruction can be improved by increasing the number of measurements to reduce the noise as discussed in the main text.

![Fig. 9: Reconstruction for the superposition $\frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$ after 70% loss in the presence of 0.5% intensity 0.5 degree phase fluctuations on the tomographic probes.](image)