Admissibility Condition and Nontrivial Indices on a Noncommutative Torus

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We study the index of the Ginsparg-Wilson Dirac operator on a noncommutative torus numerically. To do this, we first formulate an admissibility condition which suppresses the fluctuation of gauge fields. Assuming this condition, we generate gauge configurations randomly, and find various configurations with nontrivial indices. We show one example of configurations with index 1 explicitly. This result provides the first evidence that nontrivial indices can be naturally defined on the noncommutative torus by utilizing the Ginsparg-Wilson relation and the admissibility condition.

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Introduction

Noncommutative (NC) geometry [1] had attracted much attention recently since it appears from string theory in B_{µν} background [2], and also from matrix models [3, 4] as their background space-time. Matrix models are the most promising candidates to formulate the superstring theory nonperturbatively. The construction of configurations with nontrivial indices in finite NC geometries or matrix models has been one of important subjects not only from mathematical interests but also from physical points of view. For instance, to realize four dimensional chiral gauge theories, we need to perform the Kaluza-Klein compactification of extra dimensions with nontrivial indices. Topologically nontrivial configurations have been constructed on finite NC geometries such as fuzzy 2-sphere based on algebraic K-theory and projective modules in many papers, but the relation to indices of Dirac operators remains unclear in these formulations. We believe that the most suitable framework to discuss these problems is to utilize the Ginsparg-Wilson (GW) relation [5] developed in lattice gauge theory.

General prescription

In ref. [6], we proposed a general prescription to construct chirality and Dirac operators satisfying the GW relation and an index in general gauge field backgrounds on general finite NC geometries. The prescription proposed in ref. [6] is as follows. Let us introduce two hermitian chirality operators: one is a chirality operator γ, which is assumed to be independent of gauge fields, while the other is constructed in terms of a hermitian operator H as

\[ \hat{\gamma} \equiv \frac{H}{\sqrt{H^2}}, \quad H^\dagger = H. \]  

(1)

γ and \( \hat{\gamma} \) satisfy \( \gamma^2 = \hat{\gamma}^2 = 1 \). \( \hat{\gamma} \) depends on gauge fields through H. The Dirac operator \( D_{GW} \) is defined by

\[ 1 - \gamma \hat{\gamma} = f(a, \gamma)D_{GW}, \]  

(2)

where a is a small parameter. H and the function f must be properly chosen so that the \( D_{GW} \) is free of species doubling and behaves correctly in the commutative limit \( (a \to 0) \). \( D_{GW} \) satisfies the GW relation [5]:

\[ \gamma D_{GW} + D_{GW}\hat{\gamma} = 0. \]  

(3)

Therefore the fermionic action \( S_F = \text{tr}(\hat{\Psi} D_{GW} \Psi) \) is invariant under the modified chiral transformation [4, 5, 6]

\[ \delta \Psi = i\lambda \hat{\gamma} \Psi, \quad \delta \bar{\Psi} = i\bar{\Psi} \lambda \gamma. \]  

The Jacobian, however, is not invariant and has the form

\[ q(\lambda) = \frac{1}{2} \text{Tr}(\lambda \hat{\gamma} + \gamma \hat{\gamma}), \]  

(4)

where \( \text{Tr} \) is a trace of operators acting on matrices. This \( q(\lambda) \) is expected to provide a topological charge density, and the index for \( \lambda = 1 \).

An index theorem is given by

\[ \text{index}D_{GW} \equiv (n_+ - n_-) = \frac{1}{2} \text{Tr}(\gamma \hat{\gamma}), \]  

(5)

where \( n_\pm \) are numbers of zero eigenstates of \( D_{GW} \) with a positive (or negative) chirality (for either \( \gamma \) or \( \hat{\gamma} \)). This index theorem can be easily proven [6], as done in lattice gauge theory [10, 11]. The index defined in eq. (5) is invariant under small deformation of any parameters such as gauge configurations in the operator H. We note that \( \hat{\gamma} \) becomes singular when \( H \) has zero modes. When an eigenvalue of \( H \) crosses zero, the value of \( \text{Tr} \gamma \hat{\gamma} \) changes by two.

We may recall here that in lattice gauge theory the configuration space of gauge fields is topologically trivial if we do not impose any conditions on gauge fields. However, noting that physically interesting gauge fields are smooth, we can impose an admissibility condition [11, 12] on gauge fields. This condition suppresses the fluctuation of gauge fields, and consequently forms a topological structure composed of isolated islands in the configuration space. This condition can also exclude zero modes of \( H \) [14]. In ref. [6] we have thus expected that a similar mechanism would work also in finite NC geometries or matrix models, and that the index [6] could take various integers according to gauge configurations.
The index on fuzzy 2-sphere In ref.[6] we have provided a set of simplest chirality and Dirac operators on fuzzy 2-sphere, as a concrete example given by the prescription. The set in the absence of gauge fields corresponds to that constructed previously in ref.[13]. The nontrivial Jacobian is shown to have the correct form of the Chern character in the commutative limit. The index[14], however, cannot take nonzero integers.

The authors in ref.[16] applied projective modules to the index[14], so that it can take nonzero integers ±1. The modified index[16] which can take an arbitrary nonzero integer $m[3]$ is symbolically expressed as

$$\text{index}D_{GW} = \frac{1}{2} \text{Tr} \left\{ P^{(m)}[A^{(m)}] \left[ \gamma + \hat{\gamma} [A^{(m)}] \right] \right\} = m. \tag{6}$$

The gauge fields $A^{(m)}$ are determined dependent on $m$. $P^{(m)}$ is a projector to pick up a Hilbert space on which $\hat{\gamma}$ acts. The insertion of $P^{(m)}$ is necessary on fuzzy 2-sphere. The physical interpretation can be understood by considering its commutative theory. The configuration space, and the index (5) could take nonzero integers. This situation comes from the noncompactness of gauge fields on fuzzy 2-sphere.

In this letter we shall study the index (5) on a NC torus, where gauge fields are defined compactly as in lattice gauge theory. It is hence expected that if we formulate an admissibility condition on gauge fields, which can be written down so that zero modes of $H$ are excluded, results in providing just a vacuum sector with trivial configurations. This situation would improve the index numerically. To do this, we first formulate an admissibility condition.

**An admissibility condition on a NC torus** The gauge action on the NC torus is given by

$$S_G = N\beta \sum_{\mu > \nu} \text{tr} \left[ 1 - \frac{1}{2} (P_{\mu \nu} + P_{\nu \mu}) \right]. \tag{7}$$

$P_{\mu \nu}$ is the plaquette which is expressed as

$$P_{\mu \nu} = Z_{\mu \nu} V_{\mu} V_{\nu} V_{\mu}^\dagger, \tag{8}$$

where we have introduced $V_{\mu} = U_{\mu} \Gamma_{\mu}$ and $Z_{\mu \nu} = \exp(-i2\pi)\delta_{\mu \nu}$. This is the twisted version[22] of the Eguchi-Kawai model[23], which was shown to be a nonperturbative description of NC Yang-Mills theory[24,13].

For an admissibility condition on the NC torus, we assume the following gauge-invariant expression:

$$\|1 - P_{\mu \nu}\| < \eta_{\mu \nu} \quad \text{for all } \mu > \nu. \tag{9}$$

The norm $\|O\|$ of a matrix $O$ is defined as $\|O\| \equiv \sqrt{\text{max} \{ |O(a)| |O(b)| \}}$, $\eta_{\mu \nu}$ are some positive parameters which should be chosen appropriately.
The condition (9) implies $\|\nabla_{\mu}, \nabla_{\nu}\| < \eta_{\mu\nu}/a^2$. This is the bound on the field strength, which becomes irrelevant in the continuum limit. In this sense the condition (9) is physically natural. Applying arguments in refs.[12, 13] onto the NC torus, we can show that zero modes of $H$ are excluded, namely, the minimal eigenvalue of $H^2$ is larger than zero, if we choose $\eta_{\mu\nu}$ such that $\sum_{\mu, \nu} \eta_{\mu\nu} \leq \frac{1}{2} \{ 1 - (1 - m_0)^2 \}$. If we take $\eta_{\mu\nu} = \eta$, the inequality reads $\eta \leq \frac{2 - \sqrt{2}}{(d-1)} \{ 1 - (1 - m_0)^2 \} \equiv \eta_{\text{max}}$. In the following we set $\eta_{\mu\nu} = \eta_{\text{max}}$. It is noteworthy that the upper bound on $\|H\|$ can be shown as

$$\|H\| \leq |m_0 - d| + d. \quad (10)$$

**Nontrivial indices on a NC torus** We shall analyze the index (5) numerically under the admissibility condition (9) with $\eta_{\mu\nu} = \eta_{\text{max}}$. The index can be estimated by evaluating the eigenvalues of $H$. In fact the index is equal to half of the difference of the number of the positive eigenvalues and that of the negative ones. To evaluate the eigenvalues of $H$, we write down a matrix representation of $H$:

$$H \Rightarrow \begin{pmatrix} H_{\bar{\kappa}(1), \bar{\kappa}(1)} & H_{\bar{\kappa}(1), \bar{\kappa}(2)} & \cdots & H_{\bar{\kappa}(1), \bar{\kappa}(L_1)} \\ \vdots & H_{\bar{\kappa}(2), \bar{\kappa}(2)} & \cdots & H_{\bar{\kappa}(2), \bar{\kappa}(L_2)} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & H_{\bar{\kappa}(L_1), \bar{\kappa}(L_1)} \end{pmatrix}, \quad (11)$$

where the lower triangular components are abbreviated since $H$ is hermitian. Each component $H_{\bar{\kappa}, \bar{\kappa'}}$, where the superscripts of $\bar{k}$ and $\bar{k}'$ are omitted, is a $2^2 \times 2^2$ matrix spanned by spinor indices:

$$\frac{1}{N} \langle s | \text{tr} [\exp(-i k_{\mu} \cdot x_{\nu}) H \exp(i k'_{\mu} \cdot x_{\nu})] | s' \rangle$$

$$= \langle s | \gamma_{d+1} | s' \rangle \otimes \sum_{\mu} \left\{ \frac{1}{N^2} U_{\mu}(k - k') \exp \left( i \frac{1}{2} k'_{\mu} k_{\sigma} \theta_{\rho\sigma} + k'_{\mu} a \right) ight.$$  

$$+ \frac{1}{N^2} U_{\mu}^\dagger(k' - k) \exp \left( i \frac{1}{2} k'_{\mu} k_{\sigma} \theta_{\rho\sigma} - k_{\mu} a \right)$$  

$$+ (m_0 - 2) \delta_{k\bar{k}\bar{k}'} \right\} - \sum_{\mu} \langle s | \gamma_{d+1} \gamma_{\mu} | s' \rangle$$

$$\otimes \frac{1}{2N^2} \left( U_{\mu}(k - k') \exp \left( i \frac{1}{2} k'_{\mu} k_{\sigma} \theta_{\rho\sigma} + k'_{\mu} a \right) ight.$$  

$$- U_{\mu}^\dagger(k' - k) \exp \left( i \frac{1}{2} k'_{\mu} k_{\sigma} \theta_{\rho\sigma} - k_{\mu} a \right) \right\}. \quad (12)$$

In the first line $\text{tr}$ does not act on spinor indices, and $e^{ik\cdot x}$ is a plane wave. $\bar{m}$ is an integral vector in $\mathbb{Z}^d$ modulo $L$. $\tilde{x}$ is a formal expression of a hermitian NC coordinate satisfying $[\tilde{x}_{\mu}, \tilde{x}_{\nu}] = 0$. Here, $\theta_{\mu\nu}$ is a NC parameter. We leave the detail to ref.[20]. In the left-hand side of eq. (12) $s$ and $s'$ represent the spinor indices, and $\frac{1}{N}$ is a normalization factor.

Using the matrix representation of $H$, we shall look for configurations with nontrivial indices under the admissibility condition (9). We generate many configurations of $U_{\mu} = e^{i k\cdot x + \sum_{\mu, \nu} (2\pi m_{\mu, \nu} / a^2)}$, which is linked to the corresponding field $U_{\mu}(x) = e^{i k\cdot x + \sum_{\mu, \nu} (2\pi m_{\mu, \nu} / a^2)}$ through $U_{\mu}(k)$, by utilizing random numbers on a computer. Extracting configurations satisfying the admissibility condition (9), we evaluate the eigenvalues of $H$ for each configuration by using LAPACK. We thus obtain the index for each configuration. Then the configurations with nontrivial indices can be picked up. According to this prescription, we have analyzed the index on the simplest $d = 2$ dimensional NC torus for various $L (= N)$ with setting $r = m_0 = a = 1$. We have thus found various configurations with nontrivial indices for $L \geq 3$.

In the case of $L = 4$, for instance, we have discovered configurations with indices $0, \pm 1, \cdots, \pm 4$. One example of configurations with index 1 is exhibited in the left-center column of table I. In the right column thirty-two eigenvalues of $H$ are listed. The range of the eigenvalues is contained in the closed segment $[-3, 3]$, which is consistent with (10). The number of positive eigenvalues is seventeen, while that of negative ones is fifteen. We thus see that the index is given by 1. For our purpose here it is sufficient to find and show at least one configuration with a nontrivial index. We do not have to check whether the configuration minimizes the gauge action (7), since the admissibility condition which suppresses the fluctuation of gauge fields on a NC torus by following the construction of it in lattice gauge theory. Next we have numerically studied the index of the GW Dirac operator under the admissibility condition on the simplest $d = 2$ dimensional NC torus for various $L (= N)$. We have found various configurations with nontrivial indices. As a concrete example we have exhibited a configuration with index 1 in the case of $L = 4$. This is the first evidence which has confirmed that nontrivial indices can be naturally realized on the NC torus by utilizing the GW relation and the admissibility condition.

Further investigation of the index on the NC torus is necessary to understand the topological structure of gauge fields under the admissibility condition. In lattice gauge theory the admissibility condition in refs.[11, 12, 13] has been numerically analyzed.[28, 29, 30]. On the NC torus, the gauge action accommodated to the admissibility condition can be written down as $S_G = N \beta \sum_{\mu > \nu} \text{tr} \left[ \frac{1 - (P_{\mu\nu} + P_{\nu\mu})/2}{1 - ||1 - P_{\mu\nu}||} \right]$ if $||1 - P_{\mu\nu}|| < \eta_{\mu\nu}$, and $S_G = \infty$ otherwise, by following the construction of it in
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