Three-point function of semiclassical states at weak coupling

Ivan Kostov

Institut de Physique Théorique, CNRS—URA 2306, CEA-Saclay, F-91191 Gif-sur-Yvette, France

E-mail: Ivan.Kostov@cea.fr

Received 3 June 2012
Published 27 November 2012
Online at stacks.iop.org/JPhysA/45/494018

Abstract
We give the derivation of the previously announced analytic expression for the
correlation function of three heavy non-BPS operators in \( \mathcal{N} = 4 \) super-Yang–
Mills theory at weak coupling. The three operators belong to three different
\( \mathfrak{su}(2) \) sectors and are dual to three classical strings moving on the sphere. Our
computation is based on the reformulation of the problem in terms of the Bethe
ansatz for periodic XXX spin-1/2 chains. In these terms, the three operators are
described by long-wavelength excitations over the ferromagnetic vacuum, for
which the number of the overturned spins is a finite fraction of the length of
the chain, and the classical limit is known as the Sutherland limit. Technically,
our main result is a factorized operator expression for the scalar product of two
Bethe states. The derivation is based on a fermionic representation of Slavnov’s
determinant formula, and a subsequent bosonization.

This article is part of ‘Lattice models and integrability’, a special issue of
Journal of Physics A: Mathematical and Theoretical in honour of F Y Wu’s
80th birthday.

PACS numbers: 11.25.Tq, 11.25.Hf, 11.15.Kc, 11.30.Na

(1) Associate member of the Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria.

1. Introduction

During the last decade, starting with the pioneer paper by Minahan and Zarembo [1], a vast
integrable structure has been unveiled in the \( \mathcal{N} = 4 \) supersymmetric Yang–Mills (SYM)
theory and in its dual supersymmetric string theory in the \( \text{AdS}_5 \times S^5 \) spacetime. The integrable
structure, nicknamed by many integrability, was formulated in terms of an effective long range
spin chain having \( PSL(2, 2|4) \) symmetry. For a recent review, see [2].

The spectral problem, i.e. the problem of computing the anomalous dimensions of the
gauge-invariant states in SYM and their string counterparts, is nowadays considered as
conceptually solved. Spectacular progress was achieved in recent years in the computation of the gluon amplitudes [3, 4] and Wilson loops [5, 6]. The next step toward the complete solution is to compute the three-point function of gauge-invariant operators representing traces of products of fundamental fields. This is obviously an extremely hard problem, but the encouraging developments over the last several years [7–11] raise the hope that integrability can be used for this problem as well.

Of special interest are the correlation functions of one-trace operators in the classical limit when the length of the traces is very large. Such operators are dual to extended classical strings in the $\text{AdS}_5 \times S^5$ background. This correspondence allows us to approach the problem both at strong and weak coupling. The classical, or long-trace, operators are described in terms of algebraic curves, which appear as finite-gap solutions of the Bethe equations in the classical limit, or as classical solutions of the string sigma model [12–14] (see also the review [15]).

On the string theory side, the problem of computing the correlation function of operators dual to classical spinning string solutions was addressed by several authors [16–22]. The three-point function of such operators can be thought of as a classical tunneling amplitude. To compute it, one should find the appropriate Euclidean classical solution for a world sheet embedded in $\text{AdS}_5 \times S^5$ and having the topology of a sphere with three punctures. However, there is not yet a consensus among the active workers in the field about the criteria to distinguish the relevant classical solution, neither is there an unambiguous prescription about how to construct the vertex operators associated with the punctures. The only case where the complete answer is known is that of two heavy and one light operators [23, 17, 24].

Alternatively, one can start with three one-trace operators in the weakly coupled gauge theory, where the computation of the correlation functions is a well-defined problem. At tree level, it is sufficient to count all possible planar sets of Wick contractions between the three operators. It was pointed out by Roiban and Volkovich [8] that the calculation of correlation functions of gauge invariant operators reduces at the level of the spin chain to the calculation of the scalar products of states constructed out of B and C operators in the algebraic Bethe ansatz [25]. A systematic study of the case when the three operators belong to three different $\text{su}(2)$ sectors of the theory was presented by Escobedo et al [9]. The tree-level correlation function of three $\text{su}(2)$ operators was expressed in [9] in terms of scalar products of Bethe states in a periodic XXX spin chain with spin-1/2. The formalism developed in [9] was later applied to compute the correlation function of two heavy and one (or more) light operators [10, 26], and the comparison with the strong coupling results on the string theory side [23, 17, 24] showed a precise match. The limit of three heavy, or classical, operators was then obtained in [11] in the case when one of the operators is BPS type, that is, protected by the supersymmetry. The main result of [11] is an elegant analytic formula in the form of a contour integral. The derivation was based on the expansion formula for the scalar products of two generic Bethe states due to Korepin [27].

In this paper, we tackle the general case when neither of the three heavy operators is BPS type. We compute, at tree-level, the correlation function of three non-protected classical operators belonging to three different $\text{su}(2)$ sectors of the gauge theory. The principal object to be computed is the restricted scalar product of two Bethe states, in which part of the rapidities are frozen to a special value. Our result, which is a generalization of the main result of [11], was announced in the short note [28]. The analytic expression found in [28] is based on a factorization formula, which follows from the representation of the structure constant in terms of Slavnov-like determinants [29], proposed recently by Foda [30, 31].

The plan of the paper is as follows. In section 2, we review the basic notions about the XXX spin chain we are going to use, including the Gaudin norm, the Slavnov determinant formula for the scalar product and its restricted version. Following Foda [30], we will perform the
computations for the inhomogeneous spin chain characterized by a set of external parameters (impurities) associated with the sites of the chain. Such a deformation of the problem allows us to avoid ambiguous expressions containing poles and zeros on top of each other. In our approach, the homogeneous limit should be taken at the very end of the computation. In section 3, we derive a representation of the Slavnov determinant in terms of free chiral fermions, and then perform bosonization. As a side result, we obtain a novel and potentially useful representation of the Izergin determinant for the domain wall partition function (DWPF) of the six-vertex model. In the limit, when the number of Bethe roots tends to infinity, the bosonized expression decomposes into two computable factors, for which we find the analytic expression in section 4. In section 5, we derive the structure constant of three classical non-BPS operators in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM.

2. Inner product of Bethe states in the inhomogeneous XXX chain

2.1. The monodromy matrix

The local fluctuation variable in a XXX spin-1/2 chain can be in two states, $\downarrow$ and $\uparrow$, which can be thought of as a basis of a two-dimensional linear space $V$ on $\mathbb{C}$. The spin chain is characterized by an isotropic Hamiltonian:

$$H_{\text{XXX}} = -\sum_{m=1}^{L} \left( \sigma_{m}^{+} \sigma_{m+1}^{-} + \sigma_{m}^{-} \sigma_{m+1}^{+} + \frac{1}{2} \sigma_{m}^{x} \sigma_{m+1}^{x} \right).$$  \hspace{1cm} (2.1)

We will assume twisted periodic boundary conditions,

$$\sigma_{m+L} = \kappa_{\pm} \sigma_{m},$$  \hspace{1cm} (2.2)

which do not spoil the integrability, but allow us to have better control over the singularities. If the twist

$$\kappa \equiv \kappa_{-}/\kappa_{+} = e^{i\phi}$$  \hspace{1cm} (2.3)

is a pure phase, the twisted boundary conditions can be thought of as the effect of turning on a magnetic flux of strength $\phi$ [32, 33]. For us, $\kappa$ will be an unrestricted complex parameter.

In the framework of the algebraic Bethe ansatz [25], the spin chain is characterized by an $R$-matrix $R_{12}(u, v)$ acting in the tensor product $V_{1} \otimes V_{2}$ of two copies of the target space,

$$R_{12}(u, v) = u - v + iP_{12},$$  \hspace{1cm} (2.4)

where $P_{12}$ is the permutation operator [34]. We will consider the inhomogeneous XXX spin chain, characterized by background parameters (impurities) $\theta = [\theta_{1}, \ldots, \theta_{L}]$ associated with the $L$ sites of the chain. The twisted monodromy matrix $T_{a}(u) \in \text{End}(V_{a})$ is defined as the product of the $R$-matrices along the spin chain and a twist matrix $K = \begin{pmatrix} \kappa_{+} & 0 \\ 0 & \kappa_{-} \end{pmatrix}$,

$$T_{a}(u) \equiv KR_{1a}(u, \theta_{1} + \frac{1}{2}i)R_{2a}(u, \theta_{2} + \frac{1}{2}i)\cdots R_{La}(u, \theta_{L} + \frac{1}{2}i) = K \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \hspace{1cm} (2.5)$$

For the homogeneous XXX spin chain, all $\theta_{m}$ are equal to 0. The advantage of introducing the twist and the inhomogeneity parameters is that the expressions for some scalar products, which are ambiguous for $\theta_{m} = 0$ and $\kappa = 1$, becomes well defined for generic $\theta_{m}$ and $\kappa$.

The matrix elements $A, B, C$ and $D$ are operators in the Hilbert space $V = V_{1} \otimes \cdots \otimes V_{L}$ of the spin chain. The commutation relations between the elements of the monodromy matrix are determined by the relation:

$$R_{12}(u - v)T_{1}(u)T_{2}(v) = T_{2}(v)T_{1}(u)R_{12}(u - v),$$  \hspace{1cm} (2.6)
which follows from the Yang–Baxter equation for $R$. As a consequence of (2.6), the families of operators $B(u)$, $C(u)$, and the transfer matrices

$$T(u) \equiv \text{Tr}_a [T_a(u)] = A(u) + D(u),$$

are commuting

$$[B(u), B(v)] = [C(u), C(v)] = [T(u), T(v)] = 0.$$

### 2.2. The Hilbert space as a Fock space for the pseudo-particles (the magnons)

One can give the Hilbert space $V$ a structure of a Fock space generated by the action of the operators $B(u)$ on the pseudo-vacuum $\langle \uparrow^L | = \langle \uparrow \cdots \uparrow^L |$. The pseudo-vacuum is an eigenvector for the diagonal elements $A$ and $D$ and is annihilated by $C$:

$$A(u) | \uparrow^L \rangle = a(u) | \uparrow^L \rangle, \quad D(u) | \uparrow^L \rangle = d(u) | \uparrow^L \rangle, \quad C(u) | \uparrow^L \rangle = 0.$$

The dual Bethe states are generated by the action of the $C$-operators on the dual pseudo-vacuum $\langle \uparrow^L |$. The functions $a(u)$ and $d(u)$ depend on the representation of the algebra (2.6), while the $R$-matrix (2.4) is universal. For the inhomogeneous XXX spin-1/2 magnet, the functions $a(u)$ and $d(u)$ are given, according to (2.5), by

$$a(u) = \kappa + \prod_{m=1}^{L} \left( u - \theta_m + \frac{i}{2} \right),$$

$$d(u) = \kappa - \prod_{m=1}^{L} \left( u - \theta_m - \frac{i}{2} \right).$$

The algebraic construction of the Hilbert space does not use the particular form of the functions $a(u)$ and $d(u)$, which can be considered as free functional parameters [27].

The operator $B(u)$ can be viewed as a creation operator of a pseudo-particle (magnon) with momentum $p = \log \frac{2u + i}{2u - i}$. Similarly, the dual space of states is a closure of the linear span of all vectors of the form

$$| u \rangle = \prod_{j=1}^{N} B(u_j) | \uparrow^L \rangle.$$

Here and below we will use the shorthand notation

$$u = u_N = \{ u_1, \ldots, u_N \}.$$

The operator $B(u)$ can be viewed as a creation operator of a pseudo-particle (magnon) with momentum $p = \log \frac{2u + i}{2u - i}$. Similarly, the dual space of states is a closure of the linear span of all vectors of the form

$$\langle u | = (\uparrow^L) \prod_{j=1}^{N} C(u_j).$$

Such states are called called generic, or off-shell, Bethe states.

A Bethe state (2.12) which is also an eigenstate of the transfer matrix (2.7) is called a Bethe eigenstate. For a chain of length $L$, there are $2^L$ linearly independent on-shell states. Applying

---

2 In this case, one speaks of the generalized $SU(2)$ model.

3 We will indicate the cardinality $N$ only if this is required by the context.
repeatedly the RTT relations (2.6) for the upper triangular elements of the monodromy matrix, namely

\[
A(v)B(u) = \frac{u - v + i}{u - v} B(u)A(v) - \frac{i}{u - v} B(v)A(u),
\]

\[
D(v)B(u) = \frac{u - v - i}{u - v} B(u)A(v) + \frac{i}{u - v} B(v)D(u),
\]

one obtains [34] that the state (2.14) is a Bethe eigenstate if and only if the rapidities \( u = \{u_1, \ldots, u_N\} \) satisfy the (twisted) Bethe equations:

\[
d(u_j) \prod_{k=1}^{N} \frac{u_j - u_k + i}{u_j - u_k - i} = -1, \quad a = 1, \ldots, N. \tag{2.16}
\]

If the rapidities \( u \) are roots of the Bethe equations (2.16), we say that they are on shell. Since the XXX Hamiltonian is Hermitian, the set of roots \( u \) of a Bethe eigenstate must be invariant under complex conjugation. The corresponding eigenvalue \( T_a(u) \) of the transfer matrix is

\[
T_a(u) = a(u) \frac{Q_a(u - i)}{Q_a(u)} + d(u) \frac{Q_a(u + i)}{Q_a(u)}, \tag{2.17}
\]

where \( Q_a(u) \) is the Baxter polynomial

\[
Q_a(u) = \prod_{i=1}^{N} (u - u_i). \tag{2.18}
\]

The Bethe equations (2.16) can be considered as saddle-point equations for the Yang–Yang functional [35, 25], which we denote by \( \mathcal{L}_a \).

\[
\frac{\partial \mathcal{L}_a}{\partial u_a} = 2\pi i n_j \quad (a = 1, \ldots, N; \ n_j \in \mathbb{Z}). \tag{2.19}
\]

For the twisted periodic XXX\(_{1/2}\) spin chain, the Yang–Yang functional is given by

\[
\mathcal{L}_a = \sum_{j=1}^{N} \sum_{m=1}^{L} \left( u_j - \theta_m + \frac{i}{2} \right) \log \left( u_j - \theta_m + \frac{1}{2} \right) - \left( u_j - \theta_m - \frac{1}{2} \right) \log \left( u_j - \theta_m - \frac{1}{2} \right)
\]

\[
+ \sum_{j<k} [(u_j - u_k + i) \log(u_j - u_k + i) - (u_j - u_k - i) \log(u_j - u_k - i)]
\]

\[
+ i \phi \sum_{j=1}^{N} u_j. \tag{2.20}
\]

2.3. The pseudo-momentum

For each set of points \( u = \{u_j\}_{j=1}^{N} \), define the function \( p_a(u) \), called pseudo-momentum\(^4\)

\[
e^{2ip_a(u)} = \frac{d(u)}{a(u)} \frac{Q_a(u + i)}{Q_a(u - i)} = \kappa \frac{Q_a(u - i/2)}{Q_a(u + i/2)} \frac{Q_a(u + i)}{Q_a(u - i)}. \tag{2.21}
\]

Here and below we will use the notation (2.18) for products over rapidities. In terms of the pseudo-momentum, the Bethe equations (2.16) read

\[
e^{2ip_a(u)} = -1, \quad u \in u. \tag{2.22}
\]

The pseudo-momentum is determined by (2.21) modulo \( i\pi \). To characterize this function completely, it is necessary to specify a set of integers (mode numbers) \( \{n_j\}_{j=1}^{N} \), not necessarily different, so that \( p(u_j) = \pi n_j \).

\(^4\) There is no complete consensus about the terminology. The quantity we refer to as pseudo-momentum is related to the counting function \( Z(u) \) by \( e^{2ip_a(u)} = (-1)^{N + M} e^{-Z(u)} \).
2.4. Slavnov’s determinant formula for the inner product

The scalar product of two generic Bethe states can be computed from algebra (2.6), relations (2.9) and (2.10) and convention (0|0) = 1. In general, the scalar product ⟨u|v⟩ of two Bethe states is given by a double sum over partitions of the sets u = u_N and v = v_N, which becomes increasingly difficult to tackle when number of magnons N becomes large. A significant simplification occurs when one of the two sets of rapidities, say u, satisfies the Bethe equations (2.16). It was discovered by Slavnov [29] that in this case the scalar product is a determinant. This is true for all integrable models with A_1⁽¹⁾ type R-matrix.

Let the set u satisfy the Bethe equations (2.16). Then, ⟨u|v⟩ is an eigenvector for the transfer matrix with eigenvalue T_u(u), given by equation (2.17). One can prove [29, 36] that the scalar product with a generic Bethe state |v⟩ is proportional to the determinant of the matrix of the derivatives of T_u(u), evaluated at the points of v.

\[ \langle v|u \rangle = \prod_{j=1}^{N} a(v_j) d(u_j) \mathcal{J}_{u,v}; \quad (2.23) \]

\[ \mathcal{J}_{u,v} \stackrel{\text{def}}{=} \frac{1}{\prod_{j=1}^{N} a(v_j)} \frac{\det_{j,k} \frac{\partial}{\partial u_k} T_u(v_k)}{\det_{j,k} \frac{1}{a_j - a_k}}. \quad (2.24) \]

(Equations (2.23) and (2.24) are equivalent to equation (6.16) of [36].) The explicit expression for the matrix of the derivatives \( \partial_u T_u(v_k) \) is

\[ -\partial_u T_u(v_k) / \partial u_j = t(u_j - v_k) a(v_k) Q_u(v_k - i) Q_u(v_k) - t(v_k - u_j) d(v_k) Q_u(v_k + i) Q_u(v_k) \]

\[ = a(v_k) Q_u(v_k - i) Q_u(v_k) \Omega(u_j, v_k), \quad (2.25) \]

where the kernel \( \Omega(u, v) \) is defined by

\[ \Omega(u, v) = t(u - v) - e^{2i\theta(u) - i} t(v - u), \quad t(u) = \frac{1}{u} - \frac{1}{u + i}. \quad (2.26) \]

We can get rid of the factors \( a(v_j) d(u_j) \) in (2.23) by rescaling the annihilation and creation operators as

\[ B(u) \rightarrow \bar{B}(u) = -i B(u) / d(u), \quad C(u) \rightarrow \bar{C}(u) = -i C(u) / a(u). \quad (2.27) \]

We denote the rescaled inner product by ⟨u|v⟩:

\[ \langle v|u \rangle \equiv \langle \uparrow \rangle \prod_{j=1}^{N} \bar{C}(u_j) \prod_{k=1}^{N} \bar{B}(v_k) \langle \uparrow \rangle = \mathcal{J}_{u,v}. \quad (2.28) \]

Also, one can simplify the Slavnov expression (2.24) by noting that \( \prod_{k=1}^{N} Q_u(v_k - i) / Q_u(v_k) = \frac{\det_{j,k} (u_j - v_k + i) \Omega(u_j, v_k) \det_{j,k} (u_j - v_k + i)}{\det_{j,k} (u_j - v_k + i)} \). After that the inner product acquires a reasonably simple form

\[ \mathcal{J}_{u,v} = \frac{\det_{j,k} \Omega(u_j, v_k)}{\det_{j,k} a_j - a_k}. \quad (2.29) \]

2.5. The Gaudin norm

The Hermitian conjugation compatible with the scalar product (2.23) is \( \bar{C}(u) = -\bar{B}(u) \). Assuming that the sets u and v are invariant under complex conjugation, \( \bar{u} = u, \bar{v} = v \), and taking the limit \( v \rightarrow u \) in (2.29), one reproduces the determinant expression for the square
of the norm of a Bethe eigenstate conjectured by Gaudin \[37, 38\] and proved by Korepin in \[27\]. The square of the norm is proportional to the Hessian of the Yang–Yang functional. In our normalization,

\[
\langle \langle u | u \rangle \rangle = \det_{jk} \left( \frac{i \partial^2 Y_u}{\partial u_j \partial u_k} \right) \det_{jk} \left( 1 + \frac{1}{4} \right).
\]  \( (2.30) \)

The explicit expression for the matrix of the second derivatives is

\[
i \frac{\partial^2 Y_u}{\partial u_j \partial u_k} = \frac{2}{(u_j - u_k)^2 + 1} - \delta_{jk} \left( \sum_{l=1}^{N} \frac{2}{(u_j - u_l)^2 + 1} - \sum_{j=1}^{L} \frac{1}{(u_j - \theta_m)^2 + \frac{1}{4}} \right). \]  \( (2.31) \)

### 2.6. Non-highest weight states

The \( N \)-magnon states \( (2.12) \) are highest weight states with respect to the fully ordered state, pseudo-vacuum \(|↑L\rangle\). Each operator \( B(u_j) \) flips one spin down so that the third component of the spin of the state with \( N \) magnons built on the pseudo vacuum \(|0\rangle\) is \( S_z = \frac{1}{2} L - N \).

A complete system of states is obtained by acting with magnon creation operators on the vacuum descendant states \( S^- K |↑L\rangle \) with \( K \leq \frac{L}{2} \), which correspond to ferromagnetic vacua rotated away from the third axis. On the other hand, the components \( S^\pm \) of the total spin can be obtained by taking the infinite rapidity limit of the magnon creation and annihilation operators. In the normalization \( (2.27) \),

\[
B(u) \simeq \frac{i}{u} S^- \quad \text{and} \quad C(u) \simeq \frac{i}{u} S^+ \quad (u \to \infty). \]  \( (2.32) \)

The factor \( 1/u \) is obtained by comparing the large \( u \) asymptotics \( \langle ↑L | B^+(\tilde{u})B(u) | ↑L \rangle \sim L/u^2 \), which follows from \( (2.31) \), with the normalization of the spin operator \( \langle ↑L | S^+ S^- | ↑L \rangle = L \).

Therefore, an \( M \)-magnon state built upon a vacuum descendant can be obtained as the limit of a \( N \)-magnon state \( (2.12) \) with \( K = N - M \) of the rapidities sent to infinity:

\[
\langle ↑L | (S^+)^K \prod_{j=1}^{N-K} C(v_j) \prod_{l=1}^{N-K} B(u_l) (S^-)^K | ↑L \rangle = \langle ⟨ v_{N-K} \cup ∞^K | u_{N-K} \cup ∞^K \rangle \rangle, \]  \( (2.33) \)

where the limits should be taken sequentially according to the definition

\[
| u \cup ∞ \rangle \overset{\text{def}}{=} \lim_{u \to ∞} \frac{1}{u} | u \rangle = S^- | u_{N-1} \rangle. \]  \( (2.34) \)

### 2.7. Two-kink pseudo-vacua and restricted scalar products

In view of the applications to SYM, we are going to consider pseudo-vacua with two kinks at distance \( K \). In such a state, the first \( K \) spins are down and the rest of the \( L - K \) spins are oriented up:

\[
\langle \downarrow^K \uparrow^{L-K} \rangle = \{ \downarrow \downarrow K \downarrow \uparrow \uparrow \downarrow L-K \uparrow \}. \]  \( (2.35) \)

This state can be obtained by acting on the left pseudo-vacuum \( \langle ↑L \rangle \) with \( K \) annihilation operators \[30\]:

\[
\langle ↓^K \uparrow^{L-K} \rangle = \langle ↑L \rangle \prod_{j=1}^{K} C(\zeta_j) = \{ z \}. \]  \( (2.36) \)
with rapidities \( z_K = \{z_1, \ldots, z_K\} \) determined by the values of the inhomogeneity parameters associated with the first \( K \) sites:

\[
z_k \equiv \theta_k + \frac{i}{2} (k = 1, \ldots, K).
\]

Hence, the inner products with the left pseudo-vacuum given by (2.35) are evaluated by restricting \( K \) of the rapidities in the inner product (2.28):

\[
\langle \downarrow^K \prod_{j=1}^{N-K} C(v_j) \prod_{k=1}^{N} B(u_k) \uparrow^L | N-K \rangle = \langle \upsilon_{N-K} \cup z_K | u_N \rangle.
\]

(2.37)

The \textit{restricted inner product} (2.38) has been studied in [39, 40, 30, 31]. A Slavnov-type determinant expression was derived in [39, 40] discussed in the context of its application to SYM in [30, 31]. A nice statistical interpretation of the restricted scalar product as partition functions of the six-vertex model is given in [40, 31].

\subsection*{2.8. The Gaudin–Izergin determinant}

In the particular case \( K = L = N \), the restricted scalar product

\[
\mathcal{Z}_{u, z}((\downarrow)^N | \prod_{j=1}^{N} B(u_j) \uparrow^N) = \langle (\downarrow^N) | \prod_{j=1}^{N} C(z_j) \prod_{j=1}^{N} B(u_j) | \uparrow^N \rangle
\]

(2.39)

with \( z_j = \theta_j + i/2 \) evaluates the partition function of the six-vertex model with domain-wall boundary conditions (DWBC) on an \( N \times N \) square grid [27, 41]. With this specialization of the rapidities \( z = \{z_i\}_{i=1}^N \), the second term of the kernel \( \Omega(u, v) \), equation (2.26), vanishes at \( v = z_k \) and the Slavnov formula (2.29) gives

\[
\mathcal{Z}_{u, z} = \frac{\det_{jk} t(u_j - z_k)}{\det_{jk} u_j - z_k + i}, \quad t(u) = \frac{1}{u} = \frac{1}{u + i}.
\]

(2.40)

The determinant representation (2.40) of the DWBC partition function was obtained by Izergin [43, 44]. For the first time, the ratio of determinants (2.40) appeared in the works of Gaudin [42, 37] as the scalar product of two Bethe wavefunctions for a Bose gas with point-like interaction on an infinite line.

\section*{3. Operator factorization formula for the inner product}

\subsection*{3.1. Slavnov’s determinant as a fermionic Fock space expectation value}

The Slavnov determinant expression (2.29) for the scalar product can be formulated in terms of free fermions and represents a tau-function of the KP/Toda hierarchy [45–47]. For the Izergin determinant, this was shown and used in [48, 49, 31], see also the review paper [50]. The two sets of Toda times are related to the moments of the two sets of rapidities, \( u = \{u_1, \ldots, u_N\} \) and \( v = \{v_1, \ldots, v_N\} \). This is a very interesting direction which is worth exploring further.

The fermion representation we are going to use here is not the most natural one from the point of view of integrable hierarchies, but it reveals a hidden factorization property of the scalar product, which can be used to find an analytic expression in the thermodynamical limit \( N, L \to \infty \).
Introduce a chiral Neveu–Schwarz fermion living in the rapidity complex plane and having mode expansion:

$$
\psi(u) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r u^{-r - \frac{1}{2}}, \quad \bar{\psi}(u) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \bar{\psi}_r u^{-r - \frac{1}{2}}.
$$

(3.1)

The fermion modes are assumed to satisfy the anticommutation relations

$$
[\psi_r, \bar{\psi}_r]_+ = [\psi_r, \psi_r]_+ = 0, \quad [\psi_r, \bar{\psi}_r]_+ = \delta_{r,r'},
$$

(3.2)

and the left/right vacuum states are defined by

$$
\langle 0 | \psi^{-r} = \langle 0 | \bar{\psi} = 0 \quad \text{and} \quad \psi_r | 0 \rangle = \bar{\psi}_r | 0 \rangle = 0, \quad \text{for } r > 0.
$$

(3.3)

The operator $\bar{\psi}_r$ creates a particle (or annihilates a hole) with mode number $r$ and the operator $\psi_r$ annihilates a particle (or creates a hole) with mode number $r$. The particles carry charge 1, while the holes carry charge $-1$. The charge zero vacuum states (3.3) are obtained by filling the Dirac sea up to level zero. The left vacuum states with integer charge $\pm N$ are constructed as

$$
\langle N | = \begin{cases}
0 | \psi_{\frac{1}{2}} \cdots \psi_{N-\frac{1}{2}} & \text{if } N > 0, \\
0 | \psi_{-\frac{1}{2}} \cdots \psi_{-N+\frac{1}{2}} & \text{if } N < 0.
\end{cases}
$$

(3.4)

Any correlation function of the operators (3.1) is a determinant of two-point correlators

$$
\langle 0 | \psi(u) \bar{\psi}(v) | 0 \rangle = \frac{1}{u - v}.
$$

(3.5)

Obviously, the Slavnov kernel (2.26) can be represented as the correlation function of two fermionic operators, located at the points $u$ and $v$ of the rapidity plane,

$$
\Omega(u, v) = \langle 0 | \left[ \bar{\psi}(v) - e^{2i\rho(u)} \bar{\psi}(v + i) \right] [\psi(u) - \psi(u + i)] | 0 \rangle.
$$

(3.6)

The determinant of the matrix $\Omega(u_j, v_k)$ is equal to the correlation function of $N$ pairs of such operators, and the Slavnov inner product (2.29) can be given the following Fock space representation:

$$
\mathcal{S}_{u,v} = \frac{\langle 0 | \prod_{j=1}^{N} \left[ \psi(u_j) - \psi(u_j + i) \right] \prod_{k=1}^{N} \left[ \bar{\psi}(v_k) - e^{2i\rho(v_k)} \bar{\psi}(v_k + i) \right] | 0 \rangle}{\langle 0 | \prod_{j=1}^{N} \psi(u_j + i) \prod_{k=1}^{N} \bar{\psi}(v_k) | 0 \rangle}.
$$

(3.7)

Our aim is to rewrite (3.7) in a form convenient for taking the limit $N \to \infty$. For that, we first transform the denominator, using the Cauchy identity, to

$$
K_{u,v} = \left\langle 0 \prod_{j=1}^{N} \psi(u_j + i) \prod_{k=1}^{N} \bar{\psi}(v_k) | 0 \right\rangle = \frac{1}{\Delta_u \Delta_v}
$$

$$
= \pm \frac{\Delta_u \Delta_v}{\prod_{j,k=1}^{N} (u_j - v_k + i)}.
$$

(3.8)

Here and below we denote by $\Delta_w$ the Vandermonde determinant associated with the set of complex numbers $w = \{w_1, \ldots, w_N\}$,

$$
\Delta_w \equiv \prod_{j<k}^{N} (w_j - w_k).
$$

(3.9)
Then we represent the expectation value in the numerator of (3.7) as a product of difference operators acting on the Cauchy product (3.8). The result is

$$\mathcal{S}_{u,v} = \frac{1}{k_{u,v}} \prod_{j=1}^{N} \left(1 - e^{2i\rho_{j}(u_j)} e^{i\partial/\partial v_{j}} \right) (e^{-i\partial/\partial u_{j}} - 1) k_{u,v} \times 1, \quad (3.10)$$

where $e^{i\partial/\partial u}$ denotes the shift operator $u \to u + i$. We can commute the denominator of the Cauchy product to the left, using the relations

$$e^{-i\partial/\partial u_{j}} \prod_{k,l=1}^{N} \frac{1}{u_{k} - v_{l} + i} = \frac{Q_{v}(u_{j} + i)}{Q_{v}(u_{j})} \prod_{k,l=1}^{N} \frac{1}{u_{k} - v_{l} + i} e^{i\partial/\partial u_{j}},$$

$$e^{i\partial/\partial v_{j}} \prod_{k,l=1}^{N} \frac{1}{u_{k} - v_{l} + i} = \frac{Q_{u}(v_{j} - i)}{Q_{u}(v_{j})} \prod_{k,l=1}^{N} \frac{1}{u_{k} - v_{l} + i} e^{-i\partial/\partial v_{j}}, \quad (3.11)$$

until it cancels its inverse, and write (3.10) in a factorized operator form,

$$\mathcal{S}_{u,v} = (-1)^N \frac{1}{\Delta_{v}} \prod_{j=1}^{N} \left(1 - e^{2i\rho_{j}(u_j)} \frac{Q_{u}(v_{j} - i)}{Q_{u}(v_{j})} e^{i\partial/\partial v_{j}} \right) \Delta_{v}$$

$$\times \frac{1}{\Delta_{u}} \prod_{j=1}^{N} \left(1 - \frac{Q_{v}(u_{j} + i)}{Q_{v}(u_{j})} e^{-i\partial/\partial u_{j}} \right) \Delta_{u} \times 1. \quad (3.12)$$

The factorization is not complete because we must commute the operators $e^{i\partial/\partial v_{j}}$ to the right through the $Q$-functions in the second factor, which depend implicitly of the $v$-variables:

$$e^{i\partial/\partial v_{j}} \frac{Q_{v}(u_{j} + i)}{Q_{v}(u_{j})} = \left(1 - \frac{1}{(u_{j} - v_{k})^2 + 1} \right) \frac{Q_{v}(u_{j} + i)}{Q_{v}(u_{j})} e^{i\partial/\partial v_{j}}. \quad (3.13)$$

### 3.2. Operator factorization formula

The operator representation (3.12) of the inner product of an on-shell state $\{|u\rangle\}$ and an off-shell state $|v\rangle$ is the main tool we are going to use to investigate the classical limit of large $L$ and $N$. Here we will give it a more abstract formulation in terms of a pair of non-commuting functional variables, which become $c$-functions in the classical limit.

For any set of points $u = \{u_{j}\}_{j=1}^{N}$ in the complex plane and for any function $f(u)$, define the functional

$$\mathcal{F}_{u}^{\pm} [f] \overset{def}{=} \frac{1}{\Delta_{u}} \prod_{j=1}^{N} (1 - f(u_j) e^{i\partial/\partial u_{j}}) \Delta_{u}. \quad (3.14)$$

or, in terms of free fermions,

$$\mathcal{F}_{u}^{\pm} [f] = \frac{\langle N | \prod_{j=1}^{N} \left[ \tilde{\psi}(u_j) - f(u_j) \tilde{\psi}(u_j \pm i) \right] |0\rangle}{\langle N | \prod_{j=1}^{N} \tilde{\psi}(u_j) |0\rangle} \frac{\det_{jk} \left( u_{k}^{-1} - f(u_j) (u_j \pm i)^{k-1} \right)}{\det_{jk} u_{k}^{k-1}}. \quad (3.15)$$

The functionals $\mathcal{F}_{u}^{\pm} [f]$ are completely symmetric polynomials of the values of the function $f$ on the set $u$, having total degree $N$.

Assuming that $u \cap v = \emptyset$, we can write the rhs of (3.12) as a matrix element

$$\mathcal{S}_{u,v} = (-1)^N \langle v | \mathcal{F}_{u}^{\pm} [U] \mathcal{F}_{v}^{\pm} [V] | u \rangle \quad \text{for} \ u \cap v = \emptyset. \quad (3.16)$$
where the operator functions $U(v)$ and $V(u)$ satisfy the algebra

$$U(v)V(u) = V(v)U(u) \left(1 - \frac{1}{(u-v)^2 + 1}\right)$$

(3.17)

and act on the vectors $(v)$ and $[u]$ as

$$U(v)[u] = e^{2ipu(v)} \frac{Q_u(v-i)}{Q_u(v)} [u],$$
$$d(v) \frac{Q_u(v+i)}{Q_u(v)} [u].$$

(3.18)

In this way, the problem of evaluating the inner product reduces to the problem of evaluating the functionals $\langle 3 \rangle$. Moving (3) to infinity as well. Applying sequentially (3) the functionals (3) in this way.

3.3. Generalization to non-highest weight states

As discussed in section 2.6, the non-highest weight states can be obtained from the highest weight states by sending part of the rapidities to infinity:

$$[u \cup \infty] \triangleq \lim_{u_N \to \infty} \frac{[u_N]}{i} = \tilde{S}^{-1}[u_{N-1}]}.$$  

(3.19)

To evaluate the inner products of such states, we need to compute the result of sending $u_N$ to infinity in the functionals $\rho^f[u^f]$, assuming that the function $f^\pm(u)$ behaves at infinity as

$$f^\pm(u) \simeq e^{\pm iK/u}, \quad u \to \infty.$$  

(3.20)

From the definition (3.14) we find, taking into account that $\Delta u_\nu \simeq u_N^{-1} \Delta u_{N-1}$,

$$\rho^f_{u_{N-1} \cup \infty}[f^\pm] \triangleq \lim_{u_N \to \infty} \frac{u_N}{\Delta u_{N-1}} \rho^f_{u_N}[f^\pm]$$

$$= \lim_{u_N \to \infty} \frac{1}{\Delta u_{N-1}} u_N^{-1}(1 - e^{\pm iK/u_N} e^{\pm h/\Delta u_N}) u_N^{-1} \Delta u_{N-1}$$

$$= \pm (K_\pm - N + 1) \rho^f_{u_{N-1}[f^\pm]}.$$  

(3.21)

Applying this relation the necessary number of times, one can obtain the generalization of the operator factorization formula (3.16) to the case of an inner product of non-highest weight states.

3.4. The case when the Bethe eigenstate is a vacuum descendent

The inner product of Bethe state with a vacuum descendant has been computed in [11]. The scalar product in this case is obtained from equation (3.16) by sending all the roots from the set $u$ to infinity. In this limit, the sets $u$ and $v$ are infinitely separated, the commutator in (3.17) vanishes and (3.18) gives $U(v) = \frac{d(v)}{i\Delta v}$ and $V(u) \sim e^{\pm h/\Delta u}$. Applying sequentially (3.21) to all variables $u$, one obtains $\rho^f_{u_[\cup \infty}[V] = (-1)^N N！$, and (3.16) gives

$$\langle \langle 1 \mid \prod_{j=1}^N \tilde{C}(v_j) (\tilde{S}^-)^j |\tilde{L}^f \rangle = \rho^f_{u_[\cup \infty}[d/a].$$  

(3.22)

An alternative derivation of (3.22) can be found in [51, 52].

To evaluate the scalar product of two vacuum descendants, we have to send the set $v$ to infinity as well. Applying sequentially (3.21), with $K_- = L$, to all variables from the set $v$, one finds

$$\langle \langle 1 \mid (\tilde{S}^+)^j (\tilde{S}^-)^j |\tilde{L}^f \rangle = \rho^f_{u_[\cup \infty}[\rho^f_{u-[\cup \infty}[d/a].$$  

(3.23)

The second factor counts the number of ways to have $N$ reversed spins in a chain of length $L$.  

11
3.5. Restricted inner product

We want to compute the restricted scalar product (2.38),
\[ \langle v \cup z | u \rangle = \mathcal{S}_{u,v,z}, \quad \theta_L = \theta \cup (z - \frac{1}{r}) , \]
where \( v = v_{N-k}, z = z_{k}, u = u_{N}, \theta = \theta_{L-k} \). For that, we substitute \( Q_{\theta_L} = Q_{\theta} Q_{L-\theta} \), and \( Q_{\theta_L} = Q_{\theta} Q_{L} \) in the operator factorization formula (3.16). The result is
\[ \mathcal{S}_{u,v,z} = (-1)^{N} \frac{\mathcal{S}_{u}^{*}[U] \mathcal{S}_{u}^{-}[V]|u\rangle}{\langle v|u\rangle} , \]
where the operator functions \( U(v) \) and \( V(u) \) satisfy algebra (3.17) and act on the vectors \( |v\rangle \) and \( |u\rangle \) as
\[ U(v)|u\rangle = \frac{Q_{\theta}(v)}{Q_{\theta}(v + \frac{1}{r})} \frac{Q_{u}(v + i)}{Q_{u}(v)} |u\rangle , \]
\[ (v|V(u) = \frac{Q_{\theta}(u + i)}{Q_{\theta}(u)} \frac{Q_{u}(u + i)}{Q_{u}(v)} |v\rangle . \]
This trivial substitution shows the power of the operator formula (3.16). As a comparison, when evaluating the restricted scalar product using the original Slavnov determinant, one comes upon spurious singularities (poles and zeroes on top of each other), which require multiple use of l’Hôpital’s rule.

In the limit \( u, v \to \infty \), defined as in (3.21), one obtains
\[ \mathcal{S}_{\infty, \infty, \infty}^{\infty} = N! (N - K)! \left( \frac{L - K}{N} \right) \]
The second factor in (3.27) counts the number of non-equivalent ways to reverse \( M \) spins among the remaining \( L - K \) up-spins of the partially ordered pseudo-vacuum.

3.6. Gaudin–Izergin determinant and pDWPF

The Gaudin–Izergin determinant (2.40), or in terms of free fermions,
\[ \mathcal{S}_{u} = \frac{\langle 0| \prod_{j=1}^{N} \left[ \psi(u_{j}) - \psi(u_{j} + i) \right] \prod_{k=1}^{N} \bar{\psi}(z_{k}) |0\rangle}{\langle 0| \prod_{j=1}^{N} \psi(u_{j} + i) \prod_{k=1}^{N} \bar{\psi}(z_{k}) |0\rangle} = \frac{\langle 0| \prod_{j=1}^{N} \psi(u_{j}) \prod_{k=1}^{N} \bar{\psi}(z_{k}) ) |0\rangle}{\langle 0| \prod_{j=1}^{N} \psi(u_{j}) \prod_{k=1}^{N} \bar{\psi}(z_{k} - i) |0\rangle} \]
is expressed in a two-fold way in terms of the functionals (3.14),  
\[ \mathcal{S}_{u} = (-1)^{N} \left\{ \mathcal{S}_{u}^{*}[E_{u}^{+}] \quad \text{with} \quad E_{u}^{+}(u) = \frac{Q_{\theta}(u + i)}{Q_{\theta}(u)} , \right\} \]
\[ \left\{ \mathcal{S}_{u}^{*}[E_{u}^{-}] \quad \text{with} \quad E_{u}^{-}(z) = \frac{Q_{\theta}(z - i)}{Q_{\theta}(z)} . \right\} \]
The limit of \( \mathcal{S}_{u} \) when part of the rapidities \( u \) are sent to infinity was recently studied by Foda and Wheeler in [52] and given the name partial domain-wall partition function, or pDWPF. The pDWPF is computed by applying sequentially (3.21) to \( N - n \) of the variables, with the result
\[ \mathcal{S}_{u_{N-n},x_{0}} = (-1)^{n} (N - n)! \mathcal{S}_{u_{N-n}}^{*}[E_{x}^{+}], \quad E_{x}^{+}(u) = \frac{Q_{\theta}(u + i)}{Q_{\theta}(u)} . \]
A direct proof of equation (3.30) is presented in [52]. Similarly, one obtains

\[ \mathcal{A}_{u, z}^{+} = (-1)^{n}(N-n)! \mathcal{A}_{u}^{+}[E_{u}^{-}], \quad E_{u}^{-}(z) = \frac{Q_{u}(z+i)}{Q_{u}(z)}. \quad (3.31) \]

In terms of the fermion representation (3.28) of the DWPF, the limit in (3.30) results in replacing \( N-n \) fermions by an electric charge \( N-n \),

\[ \mathcal{A}_{u, N-n, z}^{-} = (N-n)! \frac{\langle N-n | \prod_{j=1}^{n} [\psi(u_j) - \psi(u_j + i)] \prod_{k=1}^{N} \tilde{\psi}(z_k) | 0 \rangle}{\langle N-n | \prod_{j=1}^{n} \psi(u_j + i) \prod_{k=1}^{N} \tilde{\psi}(z_k) | 0 \rangle}. \quad (3.32) \]

The expectation value on the r.h.s. of (3.32) is defined for any pair of non-negative integers \( N \) and \( n \), but it vanishes identically when \( n > N \). This yields a pair of identities

\[ \mathcal{A}_{u, N-n}^{+}[E_{z}^{\pm}] = \mathcal{A}_{u}^{+}[E_{z}^{-}] = 0 \quad \text{for} \quad E_{z}^{-} = Q_{z}(u \pm i)/Q_{z}(u) \quad \text{with} \quad N < n. \quad (3.33) \]

### 3.7. Properties of the functionals \( \mathcal{A}_{u}^{\pm}[f] \)

**Expansions.** The functionals \( \mathcal{A}_{u}^{\pm} \), defined by equation (3.14), are obviously completely symmetric polynomials of degree \( N \) of the variables \( f(u_1), \ldots, f(u_N) \). The coefficients of the polynomial are obtained by expanding the product in (3.14) as a sum of monomials labeled by all possible partitions of the set \( u \) into two disjoint subsets \( u' \) and \( u'' \), with \( u' \cup u'' = u \),

\[ \mathcal{A}_{u}^{\pm} = \sum_{u' \cup u'' = u} (-1)^{|u'|} \left( \prod_{u' \cap u'' = u''} f(u') \right) \frac{1}{\Delta_u} \prod_{u' \cap u''} e^{\frac{i\bar{k}_j}{\Delta_u}} \Delta_u. \quad (3.34) \]

Here, \( |u'| \) stands for the number of elements of the subset \( u' \). The last factor is evaluated as

\[ \frac{1}{\Delta_u} \left( \prod_{u' \cap u''} e^{\frac{i\bar{k}_j}{\Delta_u}} \right) \Delta_u = \prod_{u' \cap u''} \frac{u' - u'' \pm i}{u' - u''}. \quad (3.35) \]

Expansion (3.34) can be used as an alternative definition of the functional \( \mathcal{A}_{u}^{\pm}[f] \). For \( f = d/a \), this expansion was thoroughly studied by Gromov et al [11]. It was found in [11] that for constant function \( f(u) = \kappa \), the expansion (3.34) does not depend on the positions of the rapidities \( u \) and the functional \( \mathcal{A}_{u}^{\pm}[f] \) is given in this case by

\[ \mathcal{A}_{u}^{\pm}[\kappa] = (1-\kappa)^{N} = \exp \left( -N \sum_{n=1}^{\infty} \frac{\kappa^{n}}{n} \right). \quad (3.36) \]

**The linear term in \( f \) as a contour integral.** The linear term in \( f \) can be evaluated as a contour integral:

\[ \mathcal{A}_{u}^{\pm}[f] = 1 - \sum_{j=1}^{N} f(u_j) \prod_{k \neq j} \frac{u_j - u_k \pm i}{u_j - u_k} + O[f^2] \]

\[ = 1 \pm \oint_{A_u} \frac{du}{2\pi i} f(u) \frac{Q_u(u \pm i)}{Q_u(u)} + O[f^2]. \quad (3.37) \]

The integration contour \( A_u \) encircles all points of the set \( u \) and leaves outside the possible singularities of the function \( f(u) \).
Functional identities. Using the fermionic representation (3.15) and the fact that the fermion correlator is translation invariant, we transform

\[
\mathcal{A}_u^+ [f] = \frac{\langle N | \prod_{k=1}^{N} [\bar{\psi}(v_k - i) - f(v_k)\psi(v_k)] | 0 \rangle}{\langle N | \prod_{k=1}^{N} \bar{\psi}(v_k) | 0 \rangle}.
\]

\[
= (-1)^N \prod_{k=1}^{N} \left[ \frac{\psi(v_k) - \frac{1}{f(v_k)}\bar{\psi}(v_k - i)}{0} \right] | N \rangle.
\]

\[
= (-1)^N \prod_{k=1}^{N} f(v_k) \cdot f(v_N) \cdot \mathcal{A}_u^+ [1/f].
\]

Hence, \( \mathcal{A}_u^+ \) and \( \mathcal{A}_u^- \) satisfy the functional identities

\[
\mathcal{A}_u^\pm[1/f] = (-1)^N \frac{\mathcal{A}_u^\pm[1/f]}{\prod_{j=1}^{N} f(u_j)}
\]

as well as the compatibility condition

\[
\mathcal{A}_u^+(f) \mathcal{A}_u^-[1/f] = \mathcal{A}_u^+(f) \mathcal{A}_u^-[1/f].
\]

Using the functional identity (3.39), one can write representation (3.29) of the Gaudin–Izergin determinant as

\[
\mathcal{Z}_{u,z} = \left( \prod_{j,k=1}^{N} \frac{u_j - z_k + i}{u_j - z_k} \right) \mathcal{A}_u^+ [1/E_z^+], \quad \text{with} \quad E_z^+(u) = \frac{Q_z(u + i)}{Q_z(u)}.
\]

4. Classical limit

In this section, we will find the classical limit \( (L, N \gg 1) \), with \( \alpha = N/L \) finite\cite{12}, of the scalar product (2.29), using representation (3.16). In the condensed matter literature, the limit when each Bethe string has macroscopic number of particles has been studied by Sutherland\cite{33} and by Dhar and Shastry\cite{53}. In this regime, the roots \( u \) distribute themselves along a curve in the complex \( u \)-plane, symmetric about the real axis, with some linear density \( \rho_u(u) \sim 1 \)\cite{12}. Typically, the curve \( C_u \) consists of several non-overlapping connected components \( C_{u_1}, \ldots, C_{u_n} \), with the curve \( C_u \) containing \( N \) roots,

\[
C_u = \bigcup_{k=1}^{n} C_{u_k}, \quad N_1 + \cdots + N_n = N.
\]

We assume that the filling fractions \( \alpha_k = N_k/L \) associated with the cuts \( C_k \) remain finite when \( L \) and \( N_k \) tend to infinity \( (k = 1, \ldots, n) \). Then the endpoints of each curve are of order \( L \). We assume similar behavior for the rapidities \( v \). An example of distributions \( u \) and \( v \) with \( N = 50 \) having as a limit a curve with a single connected component \( (n = 1) \) is given in figure 1. The roots in the core of the curve are distributed with a density close to one, while the density vanishes continuously near the ends.

In the quasi-classical limit, the operator functional arguments \( U \) and \( V \) in (3.16) become c-functions and the scalar product factorizes to

\[
\mathcal{Z}_{u,v} = (-1)^N \mathcal{A}_u^+[\kappa e^{\lambda G_u - \lambda G_v}] \mathcal{A}_v^-[e^{\lambda G_v}],
\]

where

\[
G_u(u) = \partial_u \log Q_u(u), \quad G_v(u) = \partial_u \log Q_v(u), \quad G_\theta(u) = \partial_u \log Q_\theta(u)
\]

are the resolvents associated, respectively, with the sets \( u, v \) and \( \theta \).
Figure 1. An example of the distributions $u$ and $v$ for $N = 50$, each consisting of a single macroscopic Bethe string.

The resolvent $G_u(u)$ is a meromorphic function of $u$ with cuts along the curves $C^1_u$, \ldots, $C^n_u$ and asymptotics $N/u$ at infinity. The discontinuity across the cuts is proportional to the density $\rho_u(u)$. In the quasi-classical limit, the Bethe equations (2.22) impose a boundary condition on the pseudo-momentum on the cuts $C_u$.

The form of the curve $C_u$ and the density of the distribution of the roots is determined by the finite-zone solution constructed in [12], which we discuss briefly below. In the vicinity of each cut $C^k_u$, the pseudo-momentum

$$p(u) = G_u(u) - \frac{1}{2} G_\theta(u) + \frac{1}{2} \phi(\text{mod } \pi)$$

splits into a continuous part $\dot{p}(u)$, equal to the half-sum of the values of $p(u)$ on both sides of $C_u$ and a discontinuous part $\hat{p}(u)$, which is proportional to the density $\rho_u$:

$$\rho_u(u) = \dot{p}_u(u) + \hat{p}_u(u), \quad |\hat{p}_u(u)| = \pi \rho_u(u).$$

Since by assumption the set $u$ satisfies the Bethe equations, $T_u = 2 \cos \rho_u$ is analytic in $u$ and hence takes the same value on both sides of the cuts $C^k_u$. This yields the boundary condition

$$\sin \hat{p}_u \sin \dot{p} = 0$$

on the cuts. Along each cut $\rho_u > 0$, which implies $\hat{p}_u = 0$ (mod $\pi$), or

$$\dot{p}(u) = \pi n_k, \quad u \in C^k_u,$$

where $n_k$ is the mode number associated with the cut $C^k_u$.

The classical transfer matrix $T_u(u) = 2 \cos \rho_u(u)$ is an entire function. The branch points are zeros of $\Delta(u) \equiv T_u^2 - 4 = -4 \sin^2 \rho_u$. The forbidden zones $\Delta > 0$ are associated with the cuts of the pseudo-momentum $p(u)$ on the first sheet.

The typical situation in the homogeneous limit is when $\Delta(u)$ has a double zero at $u = \infty$, $2n$ simple zeroes $a_1, \hat{a}_1, \ldots, a_n, \hat{a}_n$, and infinitely many negative double zeros at $a_{-1}, a_{-2}, \ldots, a_{-k}, \ldots$ where $p(a_{-k}) = 2\pi n_{-k}$, accumulating at the point of essential singularity $u = 0$. The cuts are along the forbidden zones between $a_k$ and $\hat{a}_k$ ($k = 1, \ldots, n$). The derivative of pseudo-momentum $\dot{p}_u(u)$ as well as the exponential $e^{ip(u)}$ are defined on a Riemann
The cuts $C_u$ and $C_v$ and the integration contours $A_u$ and $A_v$ for the one-cut solution of figure 1.

surface associated with the hyper-elliptic complex curve

$$y^2 = \prod_{k=1}^{n} (u^2 - a_k^2).$$  
(4.7)

The values of $e^{ip(u)}$ on the first and the second sheets are related by

$$e^{ip_{1}(u)} = e^{-ip_{2}(u)}.$$  
(4.8)

The derivatives of the pseudo-momentum in $N_j$,

$$\partial N_j p(u) du = \omega_j(u), \quad j = 1, \ldots, n$$  
(4.9)

form a basis of holomorphic Abelian differentials on the complex curve:

$$\frac{1}{2\pi i} \oint_{A_{k}} \omega_j = \delta_{k,j}, \quad k, j = 1, \ldots, n.$$  
(4.10)

The cycle $A_k^u$ represents a closed contour on the Riemann surface which encircles the cut $C_u^k$ anti-clockwise, as shown in figure 2.

By the factorization formula (4.2), we reduced the computation of the scalar product in the quasi-classical limit to that of the functionals $A^{k}_{u}$. This last problem was solved in [11]. For the sake of self-consistency we give below a short derivation of their result.

4.1. Classical limit of the functionals $A^{k}_{u}[f]$

Assume that the rapidities $u$ are distributed along a contour $C_u$ in the complex $u$-plane with finite density. When the functional argument $f$ is small, by equation (3.37),

$$\log A^{k}_{u}[f] = \pm \oint_{A_{k}} \frac{dz}{2\pi} e^{q_{k}(z)} + O(f^2).$$  
(4.11)
where the integration contour $A_u$ encircles the open contour $C_u$ anticlockwise, and the function $q^\pm(u)$ is defined by

$$q^\pm(u) = -i \log[f(u)] \pm G_u(u).$$

(4.12)

On the other hand, the functional relations (3.39) allow us to determine the asymptotics for large $f$, which can be written again in the form of a contour integral. To see that we first express

$$\log \left[ (-1)^N \prod_{j=1}^{N} f(u_j) \right] = \oint_{A_u} \frac{du}{2\pi i} G_u(u) \left( \log[f(u)] + i\pi \right)$$

$$= \pm \oint_{A_u} \frac{du}{2\pi} \left[ \frac{1}{2} q^2_\pm(u) + i\pi q_\pm(u) \right].$$

(13.14)

Substituting (4.13) in (3.39), we find the large $f$ asymptotics

$$\log f^\pm[f] \simeq \oint_{A_u} \frac{du}{2\pi} \left( \frac{1}{2} q^2_\pm(u) + i\pi q_\pm(u) \right) + O(f^{-2}).$$

(4.14)

Taking into account the behavior of the solution at $f \to 0$ and $f \to \infty$, we will look for a solution of the form

$$\log f^\pm[f] = \oint_{A_u} \frac{du}{2\pi} F^\pm(e^{i\omega}(u)),$$

(4.15)

where $q(z)$ is defined by (4.12) and the meromorphic function $F(\omega)$ has asymptotics

$$F^\pm(\omega) \simeq \begin{cases} \pm \omega + O(\omega^2) & \text{if } |\omega| \ll 1, \\ \frac{1}{2} \log(-\omega)^2 \mp 1/\omega + O(1/\omega^2) & \text{if } |\omega| \gg 1. \end{cases}$$

(4.16)

The function $F(\omega)$ can be determined completely by comparing the ansatz (4.15) with the known exact solution (3.36) for $f$ constant. Assume that the function $F^\pm(\omega)$ is expanded in a Taylor series

$$F^\pm(\omega) = \sum_{n \geq 1} F^\pm_n \omega^n$$

(4.17)

in some vicinity of the point $\omega = 0$, and compute the r.h.s. of (4.15) for $q^\pm(u) = -i \log \kappa \pm G_u(u)$, which corresponds to $f(u) = \kappa$. The contour integral can be evaluated by expanding the contour to infinity, and we find

$$\sum_n F^\pm_n \oint \frac{du}{2\pi} e^{i\omega q^\pm(u)} = \sum_n F^\pm_n \oint \frac{du}{2\pi} \kappa^n \left( 1 \pm \frac{1}{n} N + \cdots \right)^n = \mp \sum_n F^\pm_n n N \kappa^n.$$  

(4.18)

Comparing (4.18) and (3.36), we find that $F_n^\pm = \pm 1/n^2$ and that the Taylor expansion (4.17) is that of the dilogarithm,

$$F^\pm(\omega) = \pm \sum_{n=1}^{\infty} \frac{\omega^n}{n^2} = \pm \text{Li}_2(\omega).$$

(4.19)

The asymptotic behavior (4.16) is satisfied thanks to the functional equation for the dilogarithm,

$$\text{Li}_2\left( \frac{1}{\omega} \right) = -\text{Li}_2(\omega) - \frac{\pi^2}{6} - \frac{1}{2} \log^2(-\omega).$$

(4.20)

Moreover, property (4.20) of the dilogarithm leads to a pair of functional equations for $\omega^\pm[f]$, which are the scaling limit of (3.39).
If the resolvent $G_u$ has several cuts on $C^1_u, \ldots, C^n_u$, then the integration contour in (4.15) splits into $n$ disjoint contours $A^1_u, \ldots, A^n_u$, and the functional $\mathcal{A}^\pm_u[f]$ is given in the classical limit by

$$
\mathcal{A}^\pm_u[f] \simeq \exp \left( \pm \oint_{A^\pm_u} \frac{du}{2\pi} \text{Li}_2 \left( f(u) \ e^{\pm iG_u(u)} \right) \right), \quad A_u = \bigcup_{k=1}^n A^k_u. \quad (4.21)
$$

The $k$th term grows as $\alpha_k L$, where $\alpha_k = N_k / L$ is the filling fraction associated with the cut $C^k_u$.

Finally, let us note that the functional identity (3.39), or equivalently, the property of the dilogarithm (4.20), leads to a second integral representation,

$$
\log \mathcal{A}^\pm_u[f] \simeq \pm \oint_{A^\pm_u} \frac{du}{2\pi} \text{Li}_2 \left( f^{-1}(u) \ e^{\mp iG_u(u)} \right) = \mp \oint_{A^\pm_u} \frac{du}{2\pi} \text{Li}_2 \left( f^{-1}(u) \ e^{\mp iG_u(u)} \right) + \oint_{A^\pm_u} \frac{du}{2\pi} G_u(u) \log f(u) + i\pi N. \quad (4.22)
$$

### 4.2. Classical limit of the Slavnov inner product $\mathcal{S}_{u,v}$

Substituting (4.22) in (4.2), we write the logarithm of the scalar product as a contour integral

$$
\log \mathcal{S}_{u,v} = i\pi N + \oint_{A_u} \frac{du}{2\pi} \text{Li}_2 \left( \kappa e^{iG_u(u)+iG_v(u)-iG_\theta(u)} - G_u(u) \log f(u) + i\pi N \right). \quad (4.23)
$$

The integration contours in general consists of several disjoint components, $A_u = \bigcup_{k=1}^n A^k_u$ and $A_v = \bigcup_{l=1}^m A^l_v$, where $A^k_u$ encircles the cut $C^k_u$ of the resolvent $G_u$ and $A^l_v$ encircles the cut $C^l_v$ of the resolvent $G_v$, as shown in figure 3.

The rhs of (4.23) can be reformulated in terms of a contour integral around the ensemble of the cuts of the function

$$
q(u) \overset{\text{def}}{=} G_u(u) + G_v(u) - G_\theta(u) + \phi \quad (4.24)
$$
on the physical sheet of its Riemann surface, with integrand depending only on \( q(u) \). This follows from the fact that the resolvent \( G_u \) satisfies on its cuts \( C_u^k \) the boundary condition (4.6),

\[
2G_u(u) - G_u(\bar{u}) + \phi = 2\pi n_k \quad \text{for} \quad u \in C_u^k.
\]

(Here \( \mathcal{G}_u \) is the half-sum of the values of the resolvent on both sides of \( C_u^k \) and \( n_k \in \mathbb{Z} \) is the corresponding mode number.) Hence, if \( q^{(1)} \) is the value of the function \( q(z) \) on the physical sheet, defined by (4.24), then the value of \( q(u) \) on the second sheet under the cut \( C_u^k \) is given by \( q^{(2)} = G_u + G_{\bar{u}} \). We conclude that the two integrals in (4.23) have the same integrand \( \text{Li}_2(e^{i|q(u)|}) \), but the contours \( A_u \) are placed on the second sheet of the Riemann surface of the function \( q(u) \). After pulling all connected components \( A_k^u \subset A_u \) up to the first sheet across the cuts \( C_u^k \), equation (4.23) takes the form

\[
\log \mathcal{S}_{u,v} = i\pi N + \oint_{A_u \cup A_v} \frac{du}{2\pi i} \text{Li}_2(e^{i|q(u)|}).
\]

(The minus sign is compensated by the change of the orientation of the contours \( A_u^k \) after they are moved to the first sheet.)

The integral along \( A_u^k \) is however ambiguous because the integrand has two logarithmic cuts which start at \( u = \infty \) on the second sheet, cross the cut \( C_u^k \) and end at two branch points on the first sheet. The ambiguity is resolved by deforming the contour \( A_u \) to a contour \( A_u^\infty \) which also encircles the point \( z = \infty \) on the second sheet.\(^5\) In the case of a one-cut solution, the contour \( A_u^\infty \) is depicted in figure 3.

In the limit \( u \to \infty \), equation (4.26) must reproduce the result of [11] about the scalar product of two general Bethe states and a vacuum descendant (equations (3.28) and (3.29) of [11]). This is indeed the case. In the limit \( u \to \infty \), the integration goes only along the contour \( C_u \) and the function \( q \) in the integrand is given in the homogeneous limit by \( q = G_v - \frac{L}{u} + \phi \).

### 4.3. Classical limit of the Gaudin norm

An expression for the square of the Gaudin norm in the classical limit can be formally obtained from (4.26) by taking \( G_v = G_u \). Here we assumed that the two sets of rapidities are invariant under complex conjugation, \( \tilde{u} = u \) and \( \tilde{v} = v \). When \( v \to u \), the integration contour \( A_u \) in (4.26) can be closed around \( C_u \) and \( C_v \) as in figure 4, and in the integrand one can replace \( q(z) \to 2p(z) \), where \( p(u) \) is the quasi-momentum (4.4). We find for the square of the Gaudin norm \( \langle u|u \rangle^2 = \mathcal{S}_{u,u} \) with

\[
\log \mathcal{S}_{u,u} = \oint_{C_u} \frac{du}{2\pi i} \text{Li}_2(e^{2i|p(u)|}).
\]

The term \( i\pi N \) is compensated by another such term which appears because of the Hermitian conjugation \( B^\dagger = -C \). Furthermore, when \( C_u^k = C_u^\bar{k} \), the two cuts on the second sheet end at the two branch points where \( 1 - e^{2ip_u} = 2ie^{ip_u} \sin p_u = 0 \) and \( \text{Li}_2(1 - e^{2ip_u}) \) has a logarithmic singularity \( \text{Li}_2 \sim \log(p_u) \log(\sin(p_u)) \). Therefore, there is no obstruction to placing the integration contour \( A_u \).

An expression of the Gaudin norm as a linear integral was derived in [11]. One can check, using the fact that \( p_u(z) = \pm i\pi p_v(z) \) on the two edges of the cut, that the contour integral (4.27) can be transformed into (twice) the linear integral in equation (2.15) of [11].

\(^5\) The author is indebted to Nikolay Gromov for performing a numerical test and for suggesting how to place the integration contours.
4.4. Classical limit of the restricted inner product

Here we evaluate the classical limit of the restricted scalar product (2.38), using the representation (3.25) and (3.26),

\[ \mathcal{J}_{u,v,z} = (-1)^N \oint_{\Gamma_{u,v}} \left[ \kappa e^{iG_u - iG_v - iG_z} \right] \cdot \left[ e^{iG_v + iG_z} \right], \]

and then use (4.21). We find

\[ \log \mathcal{J}_{u,v,z} = \log A_u \left[ e^{iG_z} \right] = -\int_{A_u} \frac{du}{2\pi} \text{Li}_2 \left( e^{iG_z(u)} - iG_u(u) \right). \]

In the second line we used the classical Bethe equation for \( \phi = 0 \)

\[ 2G_u(u) - G_u(u) - G_z(u) = 0 \mod 2\pi, \quad u \in C_u. \]

We see that the restricted scalar product \( \mathcal{J}_{u,v,z} \) is given by the same contour integral (4.26), where the integrand depends only on \( u, v \) and \( \theta \). The dependence on \( z \) is through the boundary condition on the cuts of the resolvent \( G_u \).

The classical limit of the Gaudin–Izergin determinant can be obtained directly from (3.29),

\[ \log \mathcal{Z}_{u,z} = \log A_u \left[ e^{iG_z} \right] = -\int_{A_u} \frac{du}{2\pi} \text{Li}_2 \left( e^{iG_z(u)} - iG_u(u) \right). \]

4.5. The derivatives in \( N_k \)

Knowing the hyper-elliptic Riemann surface (4.7), it is possible to compute the logarithmic derivatives of the scalar product with respect to the filling numbers \( N_k \). For that we use the fact that, according to (4.9), the derivatives of the pseudo-momentum form a basis of Abelian differentials for the Riemann surface.
Consider, following [9–11], the correlation function of three single-trace operators of the type $O^{\text{order}}$. The weights are chosen so that the operator $\omega_k(u)$ of the trace operators $O^{\text{order}}$ is an eigenstates of the dilatation operator with dimensions $\Delta_i$. The two-point function is given by choosing

$$O_1 \sim \text{Tr}[Z^{-N_1}X^{N_1} + \cdots]$$

$$O_2 \sim \text{Tr}[\bar{X}^{-N_2}\bar{X}^{N_2} + \cdots]$$

$$O_3 \sim \text{Tr}[Z^{-N_3}X^{N_3} + \cdots]$$

(5.1)

where the omitted terms are weighted products of the same constituents taken in different order. The weights are chosen so that the operator $O_i$ is an eigenstates of the dilatation operator with dimensions $\Delta_i$. The two-point and the three-point functions are determined, up to multiplicative factors, by the conformal invariance of the theory,

$$\langle O_i(x_i)O_j(x_j) \rangle = L_i \delta_{i,j} \frac{N'_i}{|x_1 - x_2|^2}\Delta_i,$$

(5.2)

$$\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle = \frac{L_1L_2L_3 \sqrt{N'_1N'_2N'_3}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3}|x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}|x_{31}|^{\Delta_3 + \Delta_1 - \Delta_2}} C_{123}(\lambda).$$

(5.3)

where $N'_i$ are arbitrary normalization factors. The origin of the factors $L_i$ is the cyclic symmetry of the trace operators $O_i$. The structure constant $C_{123}(\lambda)$ has perturbative expansion

$$N_i C_{123}(\lambda) = C_{123}^{(0)} + \lambda C_{123}^{(1)} + \cdots,$$

(5.4)

where $N_i$ is the number of colors and $\lambda$ is the 't Hooft coupling.

### 5.2. The structure constant in terms of scalar products of Bethe states

In order to compute the tree-level structure constant $C_{123}^{(0)}$ by the method of [9], one should know the wave functions at one loop. At one loop level, the operator $O_i$ is represented by a $N'_i$-magnon Bethe eigenstate with energy $\Delta_i$ of the XXX$_{1/2}$ spin chain of length $L_i$ ($i = 1, 2, 3$),

$$O_1 \rightarrow |u\rangle_{L_1}, \quad O_2 \rightarrow |v\rangle_{L_2}, \quad O_3 \rightarrow |w\rangle_{L_3},$$

(5.5)

with $u = \{u_1, \ldots, u_{N_1}\}, v = \{v_1, \ldots, v_{N_2}\}, w = \{w_1, \ldots, w_{N_3}\}$. A natural normalization of the two-point function is given by choosing

$$N'_1 = \langle uu \rangle, \quad N'_2 = \langle vv \rangle, \quad N'_3 = \langle ww \rangle.$$

(5.6)

6 Our normalization is slightly different from the normalization used in [9], namely $N^{\text{phys}}_i = N^{\text{phys}}_i/L_i$ and $C^{\text{tree}}_{123} = C_{123}^{\text{tree}}/\sqrt{L_1L_2L_3}$.

7 For simplicity, we restrict ourselves to the highest weight states. The generalization to arbitrary states is outlined in section 3.5.
Following [30], we deform the problem by introducing impurities with rapidities \( \theta^{(i)} = \{\theta_l^{(i)}\}_{l=1}^{L_i} \) at the sites of the \( i \)th spin chain \((i = 1, 2, 3)\). We denote the rapidities associated with the contractions between the operators \( O_i \) and \( O_j \) by \( \theta^{(ij)} \), so that \( \theta^{(1)} = \theta^{(12)} \cup \theta^{(13)} \), etc. Then the tree-level structure constant is given by the ratio [9, 30]

\[
C_{123}^{(0)} = \frac{(v \cup z(u)_{g^{(1)}}(z|w)_{g^{(1)}})}{\sqrt{(u|w)_{g^{(2)}}(v|w)_{g^{(2)}}}} \frac{\mathcal{Z}_{u,v,z}}{\mathcal{Z}_{w,z}} \frac{\mathcal{Z}_{1/2}}{\mathcal{Z}_{w,w}}^{1/2} \]

in the homogeneous limit \( \theta^{(12)} \to 0, \theta^{(13)} \equiv z - i/2 \to 0 \).

Let us sketch the derivation. At tree level, the structure coefficient is a sum over all possible ways to perform the Wick contractions between the fundamental fields and their conjugates. A non-zero result is obtained only if

\[
N_1 = N_2 + N_3
\]

and the number of contractions \( L_{ij} = \frac{1}{2}(L_i - L_j - L_k) \) between the operators \( O_i \) and \( O_j \) are

\[
L_{12} = L_1 - N_2, \quad L_{13} = N_3, \quad L_{23} = L_3 - N_3.
\]

The product of all free propagators in the contractions between \( O_i \) and \( O_j \) reproduce the factor \(|x_{ij}|^{-\Delta_i - \Delta_j + \Delta_k}\) in (5.3), with tree-level conformal dimensions \( \Delta_i = \Delta^{(0)}_i \).

By planarity, all \( Z \) fields in \( O_3 \) must contract with \( \bar{Z} \) fields in \( O_2 \) and all \( \bar{X} \) fields in \( O_3 \) must contract with \( X \) fields in \( O_1 \), as shown in figure 5. The contractions between \( O_2 \) and \( O_3 \) contribute a factor 1. The contractions between \( O_1 \) and \( O_3 \) contribute a factor equal to the
scalar product (recall that $L_{13} = N_3$)
\[
\langle \uparrow \uparrow \ldots \uparrow | C(w_1) \ldots C(w_{N_3}) | \downarrow \downarrow \ldots \downarrow \rangle_{\theta^{(13)}} = \mathcal{Z}_{u,z}.
\]
(5.10)
with
\[
z = \theta^{(13)} + i/2.
\]
(5.11)
The contractions between $O_1$ and $O_2$ contribute a factor
\[
\langle \uparrow \uparrow \ldots \uparrow | \prod_{j=1}^{N_1} C(u_j) | \downarrow \downarrow \ldots \downarrow \rangle_{\theta^{(12)}} = \mathcal{Z}_{u,v}.
\]
(5.12)
This scalar product is defined for a spin chain of length $L_1$ with inhomogeneity parameters $\theta^{(1)} = \theta^{(12)} \cup \theta^{(13)}$. The rapidities $u$ correspond to an on-shell Bethe state, while the rapidities $v$ satisfy the Bethe equation for another spin chain and thus correspond to an off-shell Bethe state. An interpretation of these factors in terms of six-vertex-model partition functions is given in [30].

5.3. The BPS limit

The structure coefficient should be normalized so that in the limit when all rapidities go to infinity it tends to the structure coefficient for three BPS fields [54],
\[
\lim_{u,v,w \to \infty} C^{(0)}_{123} = C^{BPS}_{123}.
\]
(5.13)
From (3.23) and (3.21), we obtain the expected result
\[
C^{BPS}_{123} = \frac{N_1! N_2! (L_{12}) \times N_3!}{\sqrt{(N_1!)^2 (L_{12}) \times (N_2!)^2 (L_{13}) \times (N_3!)^2 (L_{23})}}
\]
\[
= \frac{(L_{12}) (L_{13}) (L_{23})}{\sqrt{(L_{12}) (L_{13}) (L_{23})}}.
\]
(5.14)

5.4. The limit of three classical operators

Here we take the limit when the three fields become classical,
\[
L_i \to \infty \quad \text{with} \quad \alpha_i = \frac{N_i}{L_i} \quad \text{fixed} \quad (i = 1, 2, 3).
\]
(5.15)
using the results of section 4. The two factors in the numerator in (5.7) are evaluated using (4.31) and (4.29),
\[
\log \mathcal{Z}_{u,v,w} = i\pi N + \oint_{A_{u}\cup A_{v}} \frac{du}{2\pi} \text{Li}_2 \left( e^{i\theta_1 + i\theta_2 - i\theta_3} \right)
\]
\[
= i\pi N + \oint_{A_{u}\cup A_{v}} \frac{du}{2\pi} \text{Li}_2 \left( e^{i\theta_1 + i\theta_2 + i\theta_3} \right)
\]
(5.16)
\[
\log \mathcal{Z}_{u,v} = - \oint_{A_{u}} \frac{du}{2\pi} \text{Li}_2 \left( e^{i\theta_1 - i\theta_2 - i\theta_3} \right),
\]
(5.17)
where we introduced the three pseudo-momenta (for $\phi = 0$)

$$p_u = G_u - \frac{1}{2}G_{\theta^{(1)}}, \quad p_x = G_x - \frac{1}{2}G_{\theta^{(2)}}, \quad p_w = G_w - \frac{1}{2}G_{\theta^{(3)}},$$

and the norms in the denominator are evaluated using (4.27). Collecting all terms, we find

$$\log C_{123}^{(0)} \simeq \oint_{A_\infty^0} \frac{du}{2\pi} L_{12}(e^{i\theta_{(1)} u} + i\theta_{(2)} + i\theta_{(3)}) - \oint_{A_\infty} \frac{du}{2\pi} L_{12}(e^{i\theta_{(1)} u} - i\theta_{(2)} - i\theta_{(3)})$$

$$- \frac{1}{2} \int_{A_\infty} \frac{du}{2\pi} L_{12}(e^{2\theta_{(1)} u}) - \frac{1}{2} \int_{A_\infty} \frac{du}{2\pi} L_{12}(e^{2\theta_{(2)} u}) - \frac{1}{2} \int_{A_\infty} \frac{du}{2\pi} L_{12}(e^{2\theta_{(3)} u}).$$

The tree-level structure constant is obtained by setting all inhomogeneity parameters to zero:

$$\log C_{123}^{(0)} \simeq \oint_{A_\infty^0} \frac{du}{2\pi} L_{12}(e^{i\theta_{(1)} u} + i\theta_{(2)} + i\theta_{(3)}) - \oint_{A_\infty} \frac{du}{2\pi} L_{12}(e^{i\theta_{(1)} u} - i\theta_{(2)} - i\theta_{(3)})$$

$$- \frac{1}{2} \int_{A_\infty} \frac{du}{2\pi} L_{12}(e^{2\theta_{(1)} u}) - \frac{1}{2} \int_{A_\infty} \frac{du}{2\pi} L_{12}(e^{2\theta_{(2)} u}) - \frac{1}{2} \int_{A_\infty} \frac{du}{2\pi} L_{12}(e^{2\theta_{(3)} u}).$$

To make connection with the result of [11], one should send all $\theta$s to infinity, which is the same as taking $G_u = 0$ and neglecting the integration along $A_\infty^0$.

6. Conclusions and speculations

The principal result of this work is the operator factorization formula (3.16) for the scalar product and its classical limit (4.11)–(4.2). Using this result, we were able to write a compact expression for the correlation function of three non-BPS operators in maximally supersymmetric Yang–Mills theory in the classical limit when the traces become large. Our starting point was Foda’s determinant expression for the three-point structure constant [30]. The determinant formula of [30] was derived supposing that the three operators are deformed by a set of inhomogeneity parameters, whose values can be chosen at will. We computed the bosonized determinant expression for the inhomogeneous problem and took, as in [30], the homogeneous limit at the very end in order to avoid spurious singularities.

Another reason to treat the inhomogeneous problem is that this allows, as argued in [55, 56], the tree-level result to be extended to higher orders in $\lambda$. Gromov and Vieira [55] showed that knowing the tree level solution for $C_{123}^{(0)}$ in the presence of impurities, one can obtain the one-loop and the two-loop corrections by applying a special differential operator acting on the inhomogeneity parameters $\theta^{(1)}$, $\theta^{(2)}$ and $\theta^{(3)}$. Serban [56] proposed that this statement can be extended to all loops in the BDS model [57], i.e. when the dressing phase is not taken into account, and in the limit of large lengths $L_1$, $L_2$, $L_3$. The analysis of [56] leads to the prescription that the higher loop corrections can be taken into account only by modifying the pseudo-momenta. For example, the three-loop result for the structure constant is obtained by changing the pseudo-momenta $p_u$, $p_x$ and $p_w$ according to the three-loop Bethe ansatz equations [57].

According to [56], for finite value of the ’t Hooft coupling $\lambda$, the structure constant for three classical operators in the BDS model is given by (5.19) with a particular distribution of the $L_i$ inhomogeneity parameters$^8$ in the interval $[-2g, 2g]$, where $\lambda = 16\pi^2 g^2$. For this distribution the resolvents for the inhomogeneities associated with the three chains are given by

$$G_{\theta^{(i)}}(u) = \frac{L_i}{\sqrt{u^2 - 4g^2}} = \frac{L_i}{x} \frac{dx}{du} \quad (i = 1, 2, 3),$$

$^8$ This is the distribution of $L_i$ equal charges confined to the segment $[-2g, 2g]$ in the absence of the external potential.
where $x$ is the ‘Zhukovsky variable’ defined by
\[ u = x + \frac{g^2}{x}. \]  
(6.2)
This has the same effect as changing the vacuum eigenvalues $a(u)$ and $d(u)$ of the transfer matrix, equation (2.11), to
\[ a(u) = [x(u + \frac{i}{2})]^L, \quad d(u) = [x(u - \frac{i}{2})]^L. \]  
(6.3)
In the strong coupling regime, it is convenient to perform the change variable (6.2) in the contour integrals in (5.19). We should mention here that the replacement (6.3) as a possible way to take into account (some of) the loop corrections was previously discussed in [10].

Assuming that this conjecture is correct, the expression (5.19) with the choice (6.1) for the inhomogeneities will give the all-loops result for the structure constant for the BDS model in which the dressing phase is neglected. On the other hand, the effect of the dressing phase [58], in the limit of large $L_1, L_2, L_3$, is that the pseudo-momentum is modified as
\[ p(u) \rightarrow p^{\text{BES}}(u) = p(u) - i \log \sigma^{\text{BES}}(u). \]  
(6.4)
The fact that (5.19) depends only on the three quasi-momenta $p_u, p_v, p_w$ and the resolvents $G_{\theta\nu} = L_0 \partial_u \log x$ invites one to consider the possibility that the result at finite ’t Hooft coupling $\lambda$ is given in the classical limit again by (5.19), with the three pseudo-momenta modified according to (6.4).

It is natural to expect that a systematic approach to the correlation functions of heavy operators in the full field strength multiplet of $\mathcal{N} = 4$ SYM should be some extension of the algebraic curve method used in the spectral problem in [12–14]. In the case of three long-trace operators, the structure constant $C_{123}$ is expected to be described by the ensemble of three algebraic curves, associated with the pseudo-momenta $p_u, p_v, p_w$. In order to build a general algebraic curve formalism, one should learn how to compute more general inner products, at least in the classical limit. An interesting development in this direction was reported by Wheeler [60], who wrote determinant formulas analogous to (3.22) are obtained for the generalized model with $SU(3)$ symmetry.

Finally, it is certainly worth trying to adjust expression (5.19) for the non-compact rank-one sectors of the full symmetry $PSU(2, 2|4)$, such as $SL(2, \mathbb{R})$ and $SU(1, 1)$, where the integration contours should be placed along the real axis.

Acknowledgment

The author is obliged to S Alexandrov, O Foda, N Gromov, C Kristjansen, A Sever, D Serban, F Smirnov, P Vieira and K Zarembo for many useful discussions and comments, and to O Foda for a critical reading of the manuscript. Part of this work was done during the visit of the author to NORDITA in February 2012.

References

[1] Minahan J A and Zarembo K 2003 The Bethe-ansatz for $N = 4$ super Yang–Mills J. High Energy Phys. JHEP03(2003)013 hep-th/0212208
[2] Beisert N et al 2012 Review of AdS/CFT integrability: an overview Lett. Math. Phys. 99 3–32 (arXiv:1012.3982)
[3] Alday L F, Gaiotto D and Maldacena J 2011 Thermodynamic bubble ansatz J. High Energy Phys. JHEP09(2011)32 (arXiv:0911.4708)

9 Very recently, Janik and Laskos-Grabowski [59] showed that the algebraic curve formalism can be used to compute Wilson loops and the correlators between a Wilson loop and a local operator.
[4] Alday L F, Maldacena J, Sever A and Vieira P 2010 Y-system for scattering amplitudes J. Phys. A: Math. Theor. 43 485401 (arXiv:1002.2459)

[5] Drukker N 2012 Integrable Wilson loops arXiv:1203.1617

[6] Correa D, Maldacena J and Sever A 2012 The quark anti-quark potential and the cusp anomalous dimension from a TBA equation arXiv:1203.1913

[7] Okuyama K and Tseng L-S 2004 Three-point functions in $N=4$ SYM theory at one-loop J. High Energy Phys. JHEP08(2004)055 (arXiv:hep-th/0404190)

[8] Roiban R and Volovich A 2004 Yang–Mills correlation functions from integrable spin chains J. High Energy Phys. JHEP09(2004)32 (arXiv:hep-th/0407140)

[9] Escobedo J, Gromov N, Sever A and Vieira P 2010 Tailoring three-point functions and integrability arXiv:1012.2475

[10] Escobedo J, Gromov N, Sever A and Vieira P 2011 Tailoring three-point functions and integrability: II. Weak/strong coupling match arXiv:1104.5501

[11] Gromov N, Sever A and Vieira P 2011 Tailoring three-point functions and integrability: III. Classical tunneling arXiv:1111.2349

[12] Kazakov V, Marshakov A, Minaha J A and Zarembo K 2004 Classical/quantum integrability in AdS/CFT J. High Energy Phys. JHEP05(2004)024 hep-th/0402207

[13] Kazakov V and Zarembo K 2004 Classical/quantum integrability in non-compact sector of AdS/CFT J. High Energy Phys. JHEP10(2004)060 hep-th/0410105

[14] Beisert N, Kazakov V, Sakai K and Zarembo K 2006 The algebraic curve of classical superstrings on AdS(5) $S^5$ Commun. Math. Phys. 263 659–710 (arXiv:hep-th/0502226)

[15] Schafer-Nameki S 2010 Review of AdS/CFT integrability chapter II.4. the spectral curve arXiv:1012.3989

[16] Janik R A, Surowka P and Wereszczynski A 2010 On correlation functions of operators dual to classical spinning string states arXiv:1002.4613

[17] Zarembo K 2010 Holographic three-point functions of semiclassical states arXiv:1008.1059

[18] Buchbinder E and Tseytlin A 2010 On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT J. High Energy Phys. JHEP08(2010)057 (arXiv:1005.4516)

[19] Janik R A and Wereszczynski A 2011 Correlation functions of three heavy operators—the AdS contribution arXiv:1109.6262

[20] Buchbinder E I and Tseytlin A A 2012 Semiclassical correlators of three states with large $S$ charges in string theory in AdS$^5 \times S^5$ Phys. Rev. D 85 026001 (arXiv:1110.5621)

[21] Klose T and McLoughlin T 2011 A light-cone approach to three-point functions in AdS$^5 \times S^5$ arXiv:1106.0495

[22] Kazama Y and Komatsu S 2011 On holographic three point functions for GKP strings from integrability arXiv:1110.3949

[23] Costa M S, Monteiro R, Santos J E and Zoakos D 2010 On three-point correlation functions in the gauge/gravity duality J. High Energy Phys. 11 141 (arXiv:1008.1070)

[24] Roiban R and Tseytlin A 2008 On semiclassical computation of 3-point functions of closed string vertex operators in AdS$^5 \times S^5$ arXiv:1008.4921

[25] Faddeev L D, Sklyanin E K and Takhtajan L A 1979 The quantum inverse problem method 1 Theor. Math. Phys. 30 688–706

[26] Caetano J and Escobedo J 2011 On four-point functions and integrability in $N = 4$ SYM: from weak to strong coupling arXiv:1107.5580

[27] Korepin V E 1982 Calculation of norms of Bethe wave functions Commun. Math. Phys. 86 391–418 (arXiv:10.1007/BF01212176)

[28] Kostov I 2012 Classical limit of the three-point function from integrability arXiv:1203.6180

[29] Slavnov N A 1989 Calculation of scalar products of wave functions and form factors in the framework of the algebral Bethe ansatz Theor. Math. Phys. 79 502–8 (arXiv:10.1007/BF01016531)

[30] Foda O 2012 $N = 4$ SYM structure constants as determinants J. High Energy Phys. JHEP03(2012)96 (arXiv:1111.4663)

[31] Foda O and Wheeler M 2012 Slavnov determinants, Yang–Mills structure constants and discrete KP arXiv:1203.5621

[32] Byers N and Yang C N 1961 Theoretical considerations concerning quantized magnetic flux in superconducting cylinders Phys. Rev. Lett. 7 46–9

[33] Sutherland B 1995 Low-lying eigenstates of the one-dimensional Heisenberg ferromagnet for any magnetization and momentum Phys. Rev. Lett. 74 816–9

[34] Faddeev L D and Takhtajan L A 1984 Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model J. Sov. Math. 24 241–67
Yang C N and Yang C P 1966 One-dimensional chain of anisotropic spin–spin interactions: I. Proof of Bethe’s hypothesis for ground state in a finite system Phys. Rev. 150 321–7

Gaudin M 1983 La Fonction d’onde de Bethe (Paris: Masson)

Gaudin M, McCoy B M and Wu T T 1981 Normalization sum for the Bethe’s hypothesis wave functions of the Heisenberg–Ising chain Phys. Rev. D 23 417–9

Kitanine N, Maillet J M and Terras V 1999 Form factors of the XXZ Heisenberg spin-1/2 finite chain Nucl. Phys. B 554 647–78 (arXiv:math-ph/9807020)

Wheeler M 2011 An Izergin-korepin procedure for calculating scalar products in the six-vertex model Nucl. Phys. B 852 468–507 (arXiv:1104.2113)

Korepin V and Zinn-Justin P 2000 Inhomogeneous six-vertex model with domain wall boundary conditions and Bethe ansatz arXiv:nlin/0008030

Gaudin M 1971 Boze gas in one dimension: II. Orthogonality of the scattering states J. Math. Phys. 12 1677–80

Izergin A G 1987 Partition function of the six-vertex model in a finite volume Sov. Phys.—Dokl. 32 878

Izergin A G, Coker D A and Korepin V E 1992 Determinant formula for the six-vertex model J. Phys. A: Math. Gen. 25 4315–34

Sogo K 1993 Time-dependent orthogonal polynomials and theory of soliton—applications to matrix model, vertex model and level statistics J. Phys. Soc. Japan 62 1887–94

Foda O, Wheeler M and Zuparic M 2009 XXZ scalar products and KP Nucl. Phys. B 820 649–63 arXiv:0903.2611

Foda O and Schrader G 2010 XXZ scalar products, Miwa variables and discrete KP arXiv:1003.2524

Zinn-Justin P 2009 Six-vertex, loop and tiling models: integrability and combinatorics arXiv:0901.0665

Colomo F, Pronko A G and Zinn-Justin P 2010 Letter: the arctic curve of the domain wall six-vertex model in its antiferroelectric regime J. Stat. Mech. 3 L2 (arXiv:1001.2189)

Takasaki K 2010 KP and Toda tau functions in Bethe ansatz arXiv:1003.3071

Foda O and Wheeler M 2012 private communication

Foda O and Wheeler M Partial domain wall partition functions to appear

Dhar A and Shastry B Sriram 2000 Bloch walls and macroscopic string states in Bethe’s solution of the Heisenberg ferromagnetic linear chain Phys. Rev. Lett. 85 2813–6

Lee S, Minwalla S, Rangamani M and Seiberg N 1998 Three-point functions of chiral operators in D = 4, $5$, $7$: $\mathcal{CN} = 4$ SYM at large N arXiv:hep-th/9806074

Gromov N and Vieira P 2012 Quantum integrability for three-point functions arXiv:1202.4103

Serban D 2012 A note on the eigenvectors of long-range spin chains and their scalar products arXiv:1203.5842

Beisert N, Dippel V and Staudacher M 2004 A novel long range spin chain and planar $N = 4$ super Yang–Mills J. High Energy Phys. JHEP07(2004)075 hep-th/040501001

Beisert N, Eden B and Staudacher M 2007 Transcendentality and crossing J. Stat. Mech. 0701 021 (arXiv:hep-th/0610251)

Janik R A and Laskos-Grabowski P 2012 Surprises in the AdS algebraic curve constructions—Wilson loops and correlation functions arXiv:1203.4246

Wheeler M 2012 Scalar products in generalized models with SU(3)-symmetry arXiv:1204.2089