An analogue of the Riemann Hypothesis
via quantum walks

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Abstract

We consider an analogue of the well-known Riemann Hypothesis based on quantum walks on graphs with the help of the Konno-Sato theorem. Furthermore, we give some examples for complete, cycle, and star graphs.

Keywords: Zeta function, Riemann Hypothesis, Quantum walk, Grover walk, Konno-Sato theorem

1 Introduction

Inspired our recent work for a series of Zeta/Correspondence [6, 7, 8, 10, 11, 12, 13, 16] on the relation between zeta functions and some models such as random walk (RW) and quantum walk (QW), we consider an analogue of the well-known Riemann Hypothesis via QWs on graphs with the help of the Konno-Sato theorem. The QW is a quantum counterpart of the RW. The Konno-Sato theorem [15] treats the Grover walk which is one of the most well-investigated QWs. Concerning QW, see [14, 17, 19, 24], as for RW, see [18, 22], and as for the Riemann Hypothesis, see [3, 4, 23], for examples. More precisely, we introduce a new zeta function \( \Lambda_{QW}^G(s) = \det (\mathbf{M} - s(1-s)\mathbf{I}) \) for a suitable matrix \( \mathbf{M} \) on a simple connected graph \( G \) by using the Konno-Sato theorem, where \( s \) is a complex number and \( \mathbf{I} \) is the identity matrix. Here \( \Lambda_{QW}^G(s) \) corresponds the completed zeta function \( \Lambda(s) \). As for the detailed definition, see Section 3. Moreover, we prove that \( \Lambda_{QW}^G(s) \) satisfies the Functional Equation: \( \Lambda_{QW}^G(s) = \Lambda_{QW}^G(1-s) \) and the Riemann Hypothesis: if \( \rho \in \text{Zero}(\Lambda_{QW}^G) \), then \( \Re(\rho) = 1/2 \), where \( \text{Zero}(f) \) is the set of the zeros of \( f(s) \) and \( \Re(z) \) is the real part of \( z \). Remark that the original Riemann Hypothesis: if \( \rho \in \text{Zero}(\Lambda) \), then \( \Re(\rho) = 1/2 \) for the completed zeta function \( \Lambda(s) \). Equivalently, if \( \rho \in \text{Zero}(\zeta) \) with \( 0 < \Re(\rho) < 1 \), then \( \Re(\rho) = 1/2 \) for Riemann’s zeta function \( \zeta(s) = \sum_{n=1}^{\infty} 1/n^s \). The present manuscript is the first step of the study on a connection between the Riemann Hypothesis and the QW.

The rest of this paper is organized as follows. Section 2 gives a brief overview of the Konno-Sato theorem. In Section 3 we explain an analogue of the Riemann Hypothesis in our setting. Section 4 presents some examples for complete, cycle, and star graphs. Finally, Section 5 is devoted to conclusion.

2 Konno-Sato Theorem

First we introduce the following notation: \( \mathbb{Z} \) is the set of integers, \( \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\} \), \( \mathbb{R} \) is the set of real numbers, and \( \mathbb{C} \) is the set of complex numbers.

In this section, we briefly review the Konno-Sato theorem given by [15]. This theorem treats a relation for eigenvalues between QWs and RWs. More specifically, the Grover walk (which is QW determined by the Grover matrix) with flip-flop shift type (called F-type) and simple symmetric RW (whose walker jumps to each of its nearest neighbors with equal probability) on a graph. We assume that all graphs are simple.

Let \( G = (V(G), E(G)) \) be a connected graph (without multiple edges and loops) with the set \( V(G) \) of vertices and the set \( E(G) \) of unoriented edges \( uv \) joining two vertices \( u \) and \( v \). Moreover, let \( n = |V(G)| \) and \( m = |E(G)| \) be the number of vertices and edges of \( G \), respectively. For \( uv \in E(G) \), an arc \((u, v)\) is the oriented edge from \( u \) to \( v \). Let \( D(G) \) be the symmetric digraph corresponding to \( G \), i.e., \( D(G) = \{(u, v), (v, u) \mid uv \in E(G)\} \).
For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$. For $v \in V(G)$, the degree $\deg_G v = \deg v = d_v$ of $v$ is the number of vertices adjacent to $v$ in $G$. If $\deg_G v = k$ (constant) for each $v \in V(G)$, then $G$ is called $k$-regular. A path $P$ of length $n$ in $G$ is a sequence $P = (e_1, \ldots, e_n)$ of $n$ arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})$ $(1 \leq i \leq n - 1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, n$, then we write $P = (v_0, v_1, \ldots, v_{n-1}, v_n)$. Put $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, $P$ is called an $(o(P), t(P))$-path. We say that a path $P = (e_1, \ldots, e_n)$ has a backtracking if $e_{i+1} = e_i$ for some $i$ $(1 \leq i \leq n - 1)$. A $(v, w)$-path is called a $v$-cycle (or $v$-closed path) if $v = w$. Let $B^r$ be the cycle obtained by going $r$ times around a cycle $B$. Such a cycle is called a multiple of $B$. A cycle $C$ is reduced if both $C$ and $C^2$ have no backtracking. The Ihara zeta function of a graph $G$ is a function of a complex variable $u$ with $|u|$ sufficiently small, defined by

$$Z(G, u) = \exp \left( \sum_{r=1}^{\infty} \frac{N_r}{r} u^r \right),$$

where $N_r$ is the number of reduced cycles of length $r$ in $G$. Let $G$ be a simple connected graph with $n$ vertices $v_1, \ldots, v_n$. The adjacency matrix $A_n = [a_{ij}]$ is the $n \times n$ matrix such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. The following result was obtained by Ihara [5] and Bass [2].

**Theorem 1 (Ihara [5], Bass [2])** Let $G$ be a simple connected graph with $V(G) = \{v_1, \ldots, v_n\}$ and $m$ edges. Then we have

$$Z(G, u)^{-1} = (1 - u^2)^{-1} \det (I_n - uA_n + u^2(D_n - I_n)).$$

Here $\gamma$ is the Betti number of $G$ (i.e., $\gamma = m - n + 1$), $I_n$ is the $n \times n$ identity matrix, and $D_n = [d_{ij}]$ is the $n \times n$ diagonal matrix with $d_{ii} = \deg v_i$ and $d_{ij} = 0$ ($i \neq j$).

Let $G$ be a simple connected graph with $V(G) = \{v_1, \ldots, v_n\}$ and $m$ edges. Set $d_j = d_{v_j} = \deg v_j$ $(j = 1, \ldots, n)$. Then the $2m \times 2m$ Grover matrix $U_{2m} = [U_{ef}]_{e,f \in D(G)}$ of $G$ is defined by

$$U_{ef} = \begin{cases} 2/d_{t(f)} & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ 2/d_{t(f)} - 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$

The discrete-time QW with the Grover matrix $U_{2m}$ as a time evolution matrix is the Grover walk with F-type on $G$. Then the $n \times n$ matrix $P_n = [P_{uv}]_{u,v \in V(G)}$ is given by

$$P_{uv} = \begin{cases} 1/\deg_G u & \text{if } (u, v) \in D(G), \\ 0 & \text{otherwise}. \end{cases}$$

Note that the matrix $P_n$ is the transition probability matrix of the simple symmetric RW on $G$. We introduce the positive support $F^+ = [F_{ij}^+]$ of a real matrix $F = [F_{ij}]$ as follows:

$$F_{ij}^+ = \begin{cases} 1 & \text{if } F_{ij} > 0, \\ 0 & \text{otherwise}. \end{cases}$$

Ren et al. [20] showed that the Perron-Frobenius operator (or edge matrix) of a graph is the positive support $(U_{2m})^+$ of the transpose of its Grover matrix $U_{2m}$, i.e.,

$$Z(G, u)^{-1} = \det (I_{2m} - u(U_{2m})^+) = \det (I_{2m} - uU_{2m}^+).$$

The Ihara zeta function of a graph $G$ is just a zeta function on the positive support of the Grover matrix of $G$. That is, the Ihara zeta function corresponds to the positive support
version of the Grover walk (defined by the positive support of the Grover matrix $U_{2m}^+$) with F-type on $G$.

Now we propose another zeta function of a graph. Let $G$ be a simple connected graph with $m$ edges. Then we define a zeta function $Z(G, u)$ of $G$ satisfying

$$Z(G, u)^{-1} = \det(I_{2m} - uU_{2m}).$$

In other words, this zeta function corresponds to the Grover walk (defined by the Grover matrix $U_{2m}$) with F-type on $G$.

In this setting, Konno and Sato [15] presented the following result which is called the Konno-Sato theorem.

**Theorem 2 (Konno and Sato [15])** Let $G$ be a simple connected graph with $n$ vertices and $m$ edges. Then

$$Z(G, u)^{-1} = \det(I_{2m} - uU_{2m}) = (1 - u^2)^{m-n} \det((1 + u^2)I_n - 2uP_n).$$

(1)

If we take $u = 1/\lambda$, then Eq. (1) implies

$$\det(\lambda I_{2m} - U_{2m}) = (\lambda^2 - 1)^{m-n} \det((\lambda^2 + 1)I_n - 2\lambda P_n).$$

(2)

Furthermore, Eq. (2) can be rewritten as

$$\det(\lambda I_{2m} - U_{2m}) = (\lambda^2 - 1)^{m-n} \prod_{\lambda_{P_n} \in \text{Spec}(P_n)} (\lambda^2 + 1 - 2\lambda_{P_n}\lambda),$$

(3)

where $\text{Spec}(B)$ is the set of eigenvalues of a square matrix $B$. More precisely, we also use the following notation:

$$\text{Spec}(B) = \left\{ |\lambda_1|^{l_1}, |\lambda_2|^{l_2}, \ldots, |\lambda_k|^{l_k} \right\},$$

where $\lambda_j$ is the eigenvalue of $B$ and $l_j \in \mathbb{Z}_{\geq 0}$ is the multiplicity of $\lambda_j$ for $j = 1, 2, \ldots, k$. Set $|\text{Spec}(B)| = l_1 + l_2 + \cdots + l_k$. It follows from $\lambda^2 + 2\lambda_{P_n}\lambda = 0$ that $\lambda_{U_{2m}} \in \text{Spec}(U_{2m})$ is given by

$$\lambda_{U_{2m}} = \lambda_{P_n} \pm i \sqrt{1 - \lambda_{P_n}^2}. \quad (4)$$

Remark that $\lambda_{P_n} \in [-1, 1]$. Noting Eqs. (3) and (4), we introduce $\text{Spec}(U_{2m} : \text{RW})$ and $\text{Spec}(U_{2m} : \text{RW}^c)$ as follows:

$$\text{Spec}(U_{2m} : \text{RW}) = \left\{ \left[ \lambda_{P_n} + i \sqrt{1 - \lambda_{P_n}^2} \right], \left[ \lambda_{P_n} - i \sqrt{1 - \lambda_{P_n}^2} \right] : \lambda_{P_n} \in \text{Spec}(P_n) \right\},$$

$$\text{Spec}(U_{2m} : \text{RW}^c) = \left\{ [1]^{m-n}, [-1]^{m-n} \right\}. $$

When $m = n$, we let $\text{Spec}(U_{2m} : \text{RW}) = \emptyset$. Note that $|\text{Spec}(U_{2m} : \text{RW})| = 2n$ and $|\text{Spec}(U_{2m} : \text{RW}^c)| = 2|m - n|$. We should remark that $\text{Spec}(U_{2m} : \text{RW})$ corresponds to the eigenvalue of $P_n$ which is the transition probability matrix of the simple symmetric RW on $G$. On the other hand, $\text{Spec}(U_{2m} : \text{RW}^c)$ does not correspond to the RW, so superscript “c” (of RW) stands for “complement”. Therefore we obtain
Corollary 1 Let $G$ be a simple connected graph with $n$ vertices and $m$ edges.

(i) If $m > n$, then
\[ \text{Spec}(U_{2m}) = \text{Spec} (U_{2m} : \text{RW}) \cup \text{Spec} (U_{2m} : \text{RW}^c). \]

(ii) If $m = n$, then
\[ \text{Spec}(U_{2m}) = \text{Spec} (U_{2m} : \text{RW}). \]

(iii) If $m < n$, then
\[ \text{Spec}(U_{2m}) = \text{Spec} (U_{2m} : \text{RW}) \setminus \text{Spec} (U_{2m} : \text{RW}^c). \]

In Section 4 we will give some examples for each case. Since $U_{2m}$ is unitary, $\lambda \in \text{Spec}(U_{2m})$ satisfies $|\lambda| = 1$, so we put $\lambda = e^{i\theta}$ ($\theta \in [0, 2\pi)$). Thus we get
\[ \cos \theta = \frac{\lambda + \bar{\lambda}}{2}, \tag{5} \]
where $\bar{\lambda}$ is the complex conjugate of $\lambda \in \mathbb{C}$. This is called the Joukowsky transform. It follows from Eq. (5) and $\lambda - 1 = \bar{\lambda}$ that Eq. (2) becomes
\[ \det (\lambda I_{2m} - U_{2m}) = (\lambda^2 - 1)^{m-n} (2\lambda)^n \det (\cos \theta \cdot I_n - P_n). \]
Therefore we have a relation $\cos \theta \in \text{Spec}(P_n) \implies \lambda = e^{i\theta} \in \text{Spec}(U_{2m})$ which is sometimes called the spectral mapping theorem in the study of QW (see [21], for example). We will explain in a more detailed fashion. We set
\[ \text{Spec}(P_n) = \left\{ [\cos \theta_1]^{l_1}, [\cos \theta_2]^{l_2}, \ldots, [\cos \theta_p]^{l_p} \right\}, \]
where $|\text{Spec}(P_n)| = l_1 + l_2 + \cdots + l_p = n$ and $0 = \theta_1 < \theta_2 < \cdots < \theta_p < 2\pi$. Then we get
\[ \text{Spec} (U_{2m} : \text{RW}) = \left\{ [e^{i\theta_1}]^{l_1}, [e^{-i\theta_1}]^{l_1}, [e^{i\theta_2}]^{l_2}, [e^{-i\theta_2}]^{l_2}, \ldots, [e^{i\theta_p}]^{l_p}, [e^{-i\theta_p}]^{l_p} \right\}. \tag{6} \]
Remark that $|\text{Spec} (U_{2m} : \text{RW})| = 2 (l_1 + l_2 + \cdots + l_p) = 2n$.

3 Analogue of the Riemann Hypothesis

The Riemann zeta function for $s \in \mathbb{C}$ with $\Re(s) > 1$ is defined by
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \]
where $\Re(z)$ is the real part of $z \in \mathbb{C}$. It is known that $\zeta(s)$ has a meromorphic continuation to the entire complex plane. Moreover the completed zeta function $\Lambda(s)$ is given by
\[ \Lambda(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s), \]
where $\Gamma(z)$ is the gamma function (see [1], for example). Then $\Lambda(s)$ satisfies the **Functional Equation**:

$$\Lambda(s) = \Lambda(1 - s).$$

The **Riemann Hypothesis**, which is stated in terms of $\Lambda$, is the following:

$$\Re(\rho) = \frac{1}{2} \quad (\rho \in \text{Zero}(\Lambda)), $$

where

$$\text{Zero}(\Lambda) = \{ s \in \mathbb{C} : \Lambda(s) = 0 \}.$$ 

Concerning the Riemann Hypothesis, see [3, 4, 23], for instance. In this background, we want to find the following zeta function $\Lambda^Q_W(s)$ via the QW for a graph $G$ as a counterpart of completed zeta function $\Lambda(s)$ such as

(i) **Functional Equation**: 

$$\Lambda^Q_W(s) = \Lambda^Q_W(1 - s).$$

(ii) **Riemann Hypothesis**:

$$\Re(\rho) = \frac{1}{2} \quad (\rho \in \text{Zero}(\Lambda^Q_W)), $$

where

$$\text{Zero}(\Lambda^Q_W) = \{ s \in \mathbb{C} : \Lambda^Q_W(s) = 0 \}.$$ 

That is,

$$\text{if} \quad \Lambda^Q_W(\rho) = 0, \quad \text{then} \quad \Re(\rho) = \frac{1}{2}. $$

To do so, we use the Konno-Sato theorem as described below. We assume that $G$ is a simple connected graph with $n$ vertices and $m$ edges. Put $\lambda \in \text{Spec}(U_{2m})$, where $\lambda = e^{i\theta}$ with $\theta \in [0, 2\pi)$. In the complex plane, $\lambda$ corresponds to a point $(\cos \theta, \sin \theta)$. Let $\ell$ be the line through the points $(\cos \theta, \sin \theta)$ and $(1, 0)$. If we consider the complex plane as $xy$ plane, then the equation for line $\ell$ becomes

$$y = -\frac{\sin \theta}{1 - \cos \theta} (x - 1) \quad (\theta \in (0, 2\pi)). \quad (7)$$

When $x = 1/2$, Eq. (7) yields the value

$$y = \frac{1}{2} \cot \left( \frac{\theta}{2} \right),$$

for $\theta \in (0, 2\pi)$, so the point $(1/2, \cot(\theta/2)/2)$ lies on $\ell$. Thus, $\cot(\theta/2)/2$ is the $y$ coordinate of the intersection of $\ell$ and $x = 1/2$. Therefore, we want to relate $\lambda = e^{i\theta} \in \text{Spec}(U_{2m})$ with $\theta \in (0, 2\pi)$ to $\rho = \rho(\theta) \in \text{Zero}(\Lambda^Q_W)$ in the following way:

$$\rho = \rho(\theta) = \frac{1}{2} + \frac{i}{2} \cot \left( \frac{\theta}{2} \right) \quad (\theta \in (0, 2\pi)). \quad (8)$$
On the other hand, if $\theta = 0$, then we put
\[
\rho = \rho(0) = \frac{1}{2} + i \cdot (+\infty).
\] (9)
Noting Eqs. (6), (8) and (9), we introduce the following sets \(\text{Zero}(\Lambda_G^{\text{QW}} : \text{RW})\) and \(\text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}^c)\) corresponding to \(\text{Spec}(U_{2m} : \text{RW})\) and \(\text{Spec}(U_{2m} : \text{RW}^c)\), respectively.
\[
\text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}) = \left\{ \rho(\theta_1)^l_1, \rho(-\theta_1)^l_1, \rho(\theta_2)^l_2, \rho(-\theta_2)^l_2, \ldots, \rho(\theta_p)^l_p, \rho(-\theta_p)^l_p \right\}
\]
\[
= \left\{ \left[ \frac{1}{2} + \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q}, \left[ \frac{1}{2} - \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q} : q = 1, 2, \ldots, p \right\},
\]
\[
\text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}^c) = \left\{ \rho(0)^{|m-n|}, \rho(\pi)^{|m-n|} \right\} = \left\{ \left[ \frac{1}{2} + i \cdot (+\infty) \right]^{m-n}, \left[ \frac{1}{2} \right]^{m-n} \right\}.
\]
Remark that \(|\text{Zero}(\Lambda_G^{\text{QW}} : \text{RW})| = 2 (l_1 + l_2 + \cdots + l_p) = 2n\) and \(|\text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}^c)| = 2|m-n|\). Furthermore, we introduce \(\text{Zero}(\Lambda_G^{\text{QW}})\) as follows:
(i) If \(m > n\), then
\[
\text{Zero}(\Lambda_G^{\text{QW}}) = \text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}) \cup \text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}^c)
\]
\[
= \left\{ \left[ \frac{1}{2} + \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q}, \left[ \frac{1}{2} - \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q} : q = 1, 2, \ldots, p \right\}
\]
\[
\cup \left\{ \left[ \frac{1}{2} + i \cdot (+\infty) \right]^{m-n}, \left[ \frac{1}{2} \right]^{m-n} \right\}.
\]
(ii) If \(m = n\), then
\[
\text{Zero}(\Lambda_G^{\text{QW}}) = \text{Zero}(\Lambda_G^{\text{QW}} : \text{RW})
\]
\[
= \left\{ \left[ \frac{1}{2} + \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q}, \left[ \frac{1}{2} - \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q} : q = 1, 2, \ldots, p \right\}.
\]
(iii) If \(m < n\), then
\[
\text{Zero}(\Lambda_G^{\text{QW}}) = \text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}) \setminus \text{Zero}(\Lambda_G^{\text{QW}} : \text{RW}^c)
\]
\[
= \left\{ \left[ \frac{1}{2} + \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q}, \left[ \frac{1}{2} - \frac{i}{2} \cot \left( \frac{\theta_q}{2} \right) \right]^{l_q} : q = 1, 2, \ldots, p \right\}
\]
\[
\setminus \left\{ \left[ \frac{1}{2} + i \cdot (+\infty) \right]^{n-m}, \left[ \frac{1}{2} \right]^{n-m} \right\}.
\]
Noting Eq. (8), we define \(\Lambda_G^{\text{QW}}(s)\) by
\[
\Lambda_G^{\text{QW}}(s) = \det (M_n - s(1-s)I_n),
\] (10)
for a suitable $n \times n$ matrix $M_n$. By Eq. (10), we easily confirm that $\Lambda_G^{QW}(s)$ satisfies $\Lambda_G^{QW}(s) = \Lambda_G^{QW}(1 - s)$, i.e., Functional Equation. On the other hand, $\Lambda_G^{QW}(s)$ can be rewritten as

$$\Lambda_G^{QW}(s) = \prod_{\lambda_{M_n} \in \text{Spec}(M_n)} (\lambda_{M_n} - s(1 - s)). \quad (11)$$

We recall that it follows from Eq. (8) that any $\rho \in \text{Zero}(\Lambda_G^{QW})$ should be expressed as

$$\rho = \frac{1}{2} + i \frac{1}{2} \cot \left( \frac{\theta}{2} \right). \quad (12)$$

If so, then $\Lambda_G^{QW}(s)$ satisfies the Riemann Hypothesis, that is, if $\rho \in \text{Zero}(\Lambda_G^{QW})$, then $\Re(\rho) = 1/2$. Next noting Eq. (11), we need to confirm that $\rho \in \text{Zero}(\Lambda_G^{QW})$ with Eq. (12) satisfies

$$s^2 - s + \lambda_{M_n} = 0,$$

for $\lambda_{M_n} \in \text{Spec}(M_n)$. This is,

$$(s - \rho)(s - \overline{\rho}) = s^2 - s + \lambda_{M_n}.$$ 

So we have

$$\rho + \overline{\rho} = 1 \quad (13)$$

$$\rho\overline{\rho} = \lambda_{M_n}. \quad (14)$$

Then Eq. (13) comes from Eq. (12). Concerning Eq. (14), by using Eq. (12), we compute

$$\rho\overline{\rho} = \left( \frac{1}{2} + i \frac{1}{2} \cot \left( \frac{\theta}{2} \right) \right) \left( \frac{1}{2} - i \frac{1}{2} \cot \left( \frac{\theta}{2} \right) \right)$$

$$= \frac{1}{4} \left( 1 + \cot^2 \left( \frac{\theta}{2} \right) \right) = \frac{1}{2} \frac{1}{1 - \cos \theta}$$

$$= \frac{1}{2} \frac{1}{1 - \lambda_{P_n}},$$

where $\theta \in (0, 2\pi)$. Thus we have

$$\rho\overline{\rho} = \frac{1}{2} \frac{1}{1 - \lambda_{P_n}}. \quad (15)$$

Combining Eq. (14) with Eq. (15) yields

$$\lambda_{M_n} = \frac{1}{2} \left( 1 - \lambda_{P_n} \right)^{-1}. \quad (16)$$

Then Eq. (16) suggests that $M_n$ may be expressed as

$$M_n = \frac{1}{2} (I_n - P_n)^{-1}. \quad (17)$$

Therefore we define $\text{Spec}(M_n)$ by

$$\text{Spec}(M_n) = \left\{ \left[ \frac{1}{2} \frac{1}{1 - \cos \theta_1} \right] \lambda_1, \left[ \frac{1}{2} \frac{1}{1 - \cos \theta_2} \right] \lambda_2, \ldots, \left[ \frac{1}{2} \frac{1}{1 - \cos \theta_p} \right] \lambda_p \right\},$$
since
\[ \text{Spec}(P_n) = \left\{ [\cos\theta_1]^{l_1}, [\cos\theta_2]^{l_2}, \ldots, [\cos\theta_p]^{l_p} \right\}. \]

Remark that \( |\text{Spec}(M_n)| = l_1 + l_2 + \cdots + l_p = n \) and \( 0 = \theta_1 < \theta_2 < \cdots < \theta_p < 2\pi \). In particular, when \( \cos\theta_1 = 1 \) (i.e., \( \theta_1 = 0 \)), we put
\[ \frac{1}{2} \frac{1}{1 - \cos\theta_1} = +\infty. \]

This matrix “\( M_n \)” is exactly what we wanted to find. However, \((I_n - P_n)^{-1}\) does not exist, so we understand Eq. (17) “in the sense of the eigenvalue”, that is, as Eq. (16). In fact, we see that \( 0 \in \text{Spec}(I_n - P_n) \), since all one vector \( 1_n \) is the eigenvector corresponding to eigenvalue 0, i.e., \((I_n - P_n)1_n = 0 \cdot 1_n\). Thus \( I_n - P_n \) is not invertible. Therefore Eq. (17) implies that \( M_n \) does not exist. If \( G \) is \((q + 1)\)-regular, then we have the Laplacian \( \Delta_n \) as
\[ \Delta_n = D_n - A_n = (q + 1)(I_n - P_n). \] (18)

It follows from Eq. (18) that Eq. (17) becomes
\[ M_n = \frac{q + 1}{2} \Delta_n^{-1} \]
in the sense of the eigenvalue. We also remark that \( \Delta_n^{-1} \) does not exist. Then we obtain the following main result.

**Theorem 3** Let \( G \) be a simple connected graph with \( n \) vertices and \( m \) edges. Put
\[ \Lambda_G^{QW}(s) = \det (M_n - s(1-s)I_n), \]
where
\[ M_n = \frac{1}{2} (I_n - P_n)^{-1} \]
in the sense of the eigenvalue. Then \( \Lambda_G^{QW}(s) \) satisfies
(i) **Functional Equation:**
\[ \Lambda_G^{QW}(s) = \Lambda_G^{QW}(1-s). \]
(ii) **Riemann Hypothesis:**
\[ \Re(\rho) = \frac{1}{2} \text{ for any } \rho \in \text{Zero}(\Lambda_G^{QW}), \]
where \( \text{Zero}(\Lambda_G^{QW}) = \left\{ s \in \mathbb{C} : \Lambda_G^{QW}(s) = 0 \right\}. \)

4 **Example**

This section gives examples, i.e., (i) \( m > n \), (ii) \( m = n \), and (iii) \( m < n \) cases. The results here can be obtained by the direct computation and the Konno-Sato theorem.
(i) $m > n$ case. $G = K_n$ (complete graph) with $n$ vertices and $m = n(n-1)/2$ edges for $n \geq 4$. We should remark that if $n = 3$, then (ii) $m = n = 3$ case. If $n = 2$, then (iii) $m = 1 < n = 2$ case. Then we get

$$\text{Spec }(P_n) = \left\{ [1]^1, \left[-\frac{1}{n-1}\right]^{n-1}\right\},$$

$$\text{Spec }(U_{n(n-1)}) = \left\{ [1]^{(n(n-3)+4)/2}, [-1]^{n(n-3)/2}, \left[-\frac{1}{n-1} + i\sqrt{n(n-2)}\right]^{n-1}, \left[-\frac{1}{n-1} - i\sqrt{n(n-2)}\right]^{n-1}\right\}.$$  

Remark that we confirm that spectral mapping theorem holds such as

$$\mathbb{R}\left[\frac{-1 + i\sqrt{n(n-2)}}{n-1}\right] = \mathbb{R}\left[\frac{-1 - i\sqrt{n(n-2)}}{n-1}\right] = -\frac{1}{n-1}.$$  

Therefore we have

$$\text{Zero } \left(\Lambda_{K_n}^{QW}\right) = \left\{ \left[\frac{1}{2} + i \cdot (+\infty)\right]^2, \left[\frac{1}{2} + \frac{i}{2}\sqrt{1 - \frac{2}{n}}\right]^{n-1}, \left[\frac{1}{2} - \frac{i}{2}\sqrt{1 - \frac{2}{n}}\right]^{n-1}\right\}$$

$$\cup \left\{ \left[\frac{1}{2} + i \cdot (+\infty)\right]^{n(n-3)/2}, \left[\frac{1}{2}\right]^{n(n-3)/2}\right\}.$$  

Note that $m - n = n(n-3)/2$. Furthermore, we obtain

$$\text{Spec }(M_n) = \left\{ [+\infty]^1, \left[\frac{1}{2}(1 - \frac{1}{n})\right]^{n-1}\right\},$$  

for $n \geq 4$. Formally, if we take $n \to \infty$, then

$$\lim_{n \to \infty} \text{Spec }(M_n) = \left\{ [+\infty]^1, \left[\frac{1}{2}\right]^{+\infty}\right\}.$$  

(ii) $m = n$ case. $G = C_n$ (cycle graph) with $n$ vertices and $m = n$ edges. In this case, $2n \times 2n$ matrix $U_{2n}$ and $n \times n$ matrix $P_n$ are expressed as follows (see [7, 8], for example):

$$U_{2n} = \begin{bmatrix} O & P & O & \ldots & \ldots & O & Q \\ Q & O & P & O & \ldots & \ldots & O \\ O & Q & O & P & \ldots & \ldots & O \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ O & \ldots & O & Q & O & P & O \\ O & \ldots & \ldots & O & Q & O & P \\ P & O & \ldots & \ldots & O & Q & O \end{bmatrix}, \quad P_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 & 1 \\ 1 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 1 & 0 & 1 & 0 \\ 0 & \ldots & \ldots & 0 & 1 & 0 & 1 \\ 1 & 0 & \ldots & \ldots & 0 & 1 & 0 \end{bmatrix},$$  

where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
Let \( \xi_k = 2\pi k/n \) for \( k = 0, 1, \ldots, n - 1 \). Thus we get
\[
\text{Spec} (P_n) = \{ \cos \xi_k \} : k = 0, 1, \ldots, n - 1 \},
\]
\[
\text{Spec} (U_2n) = \{ e^{i\xi_k}, e^{-i\xi_k} \} : k = 0, 1, \ldots, n - 1 \}.
\]
Therefore we obtain
\[
\text{Zero} \left( \Lambda^{QW}_{C_n} \right) = \left\{ \left[ \frac{1}{2} + \frac{i}{2} \cot \left( \frac{\xi_k}{2} \right) \right]^{1}, \left[ \frac{1}{2} - \frac{i}{2} \cot \left( \frac{\xi_k}{2} \right) \right]^{1} : k = 0, 1, \ldots, n - 1 \right\}.
\]
Formally, if we take \( n \to \infty \), then
\[
\lim_{n \to \infty} \text{Zero} \left( \Lambda^{QW}_{C_n} \right) = \left\{ \left[ \frac{1}{2} + i \cdot \gamma \right]^{1} : \gamma \in \mathbb{R} \right\}.
\]
Moreover we get
\[
\text{Spec} (M_n) = \left\{ [+, \infty], \left[ \frac{1}{2}, 1 - \cos \xi_1 \right], \ldots, \left[ \frac{1}{2}, 1 - \cos \xi_{n-1} \right] \right\},
\]
since \( \xi_0 = 0 \) (i.e., \( \cos \xi_0 = 1 \)). From now on, we consider \( n = 3 \) and \( n = 4 \) cases. First we deal with \( n = 3 \). Then we have
\[
\text{Spec} (P_3) = \{ \cos (0 \cdot \pi/3), \cos (2 \cdot \pi/3), \cos (4 \cdot \pi/3) \} = \left\{ [1], \left[ -\frac{1}{2} \right] \right\},
\]
\[
\text{Spec} (U_6) = \left\{ e^{i(0 \cdot \pi/3)^2}, e^{i(2 \cdot \pi/3)^2}, e^{i(4 \cdot \pi/3)^2} \right\} = \left\{ [1]^2, \left[ -1 + i\sqrt{3} \right]^2, \left[ -1 - i\sqrt{3} \right]^2 \right\}.
\]
Thus we obtain
\[
\text{Zero} \left( \Lambda^{QW}_{C_3} \right) = \left\{ \left[ \frac{1}{2} + i \cdot (+\infty) \right]^2, \left[ \frac{1}{2} + i\frac{\sqrt{3}}{6} \right]^2, \left[ \frac{1}{2} - i\frac{\sqrt{3}}{6} \right]^2 \right\}.
\]
Note that
\[
\text{Spec} (M_3) = \left\{ [+, \infty]^2, \left[ \frac{1}{3} \right]^2 \right\}, \quad \text{Spec} (\Delta_3) = \left\{ [0]^1, [3]^2 \right\},
\]
where
\[
\Delta_3 = 2 (I_3 - P_3) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.
\]
So we confirm that \( M_3 = \Delta_3^{-1} \) does not exist. As in the case of \( n = 3 \), we treat \( n = 4 \) case. Then we get
\[
\text{Spec} (P_4) = \{ \cos (0 \cdot \pi/4), \cos (2 \cdot \pi/4), \cos (4 \cdot \pi/4), \cos (6 \cdot \pi/4) \} = \left\{ [1]^1, [0]^2, [-1]^1 \right\},
\]
\[
\text{Spec} (U_8) = \left\{ e^{i(0 \cdot \pi/4)^2}, e^{i(2 \cdot \pi/4)^2}, e^{i(4 \cdot \pi/4)^2}, e^{i(6 \cdot \pi/4)^2} \right\} = \left\{ [1]^2, [i]^2, [-1]^2, [-i]^2 \right\}.
\]
Therefore we have
\[
\text{Zero} \left( \Lambda_{QW}^{c,4} \right) = \left\{ \left[\frac{1}{2} + i \cdot (+\infty)\right]^{2}, \left[\frac{1}{2} + \frac{i}{2}\right]^{2}, \left[\frac{1}{2}\right]^{2}, \left[\frac{1}{2} - \frac{i}{2}\right]^{2} \right\}.
\]

Remark that
\[
\text{Spec} (\mathbf{M}_4) = \left\{ [+\infty], \left[\frac{1}{2}\right]^{2}, \left[\frac{1}{4}\right]^{1} \right\}, \quad \text{Spec}\ (\Delta_4) = \left\{ [0], [2]^{2}, [4]^{1} \right\},
\]

where
\[
\Delta_4 = 2 (\mathbf{I}_4 - \mathbf{P}_4) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.
\]

Thus we confirm that \( \mathbf{M}_4 = \Delta_4^{-1} \) does not exist.

(iii) \( m < n \) case. \( G = S_n \) (star graph) with \( n \) vertices and \( m = n - 1 \) edges. Remark that \( S_n \) is isomorphic to the complete bipartite graph \( K_{1,n-1} \). Then we obtain
\[
\text{Spec} (\mathbf{P}_n) = \left\{ [1]^{1}, [0]^{n-2}, [-1]^{1} \right\},
\]
\[
\text{Spec}\ (\mathbf{U}_{2(n-1)}) = \left\{ [1]^{2}, [i]^{n-2}, [-i]^{n-2}, [1]^{2} \right\} \setminus \left\{ [1], [-1]^{1} \right\}.
\]
Thus we have
\[
\text{Zero} \left( \Lambda_{QW}^{S_n} \right) = \left\{ \left[\frac{1}{2} + i \cdot (+\infty)\right]^{1}, \left[\frac{1}{2} + \frac{i}{2}\right]^{n-2}, \left[\frac{1}{2} - \frac{i}{2}\right]^{n-2}, \left[\frac{1}{2}\right]^{1} \right\}.
\]

Note that
\[
\text{Spec} (\mathbf{M}_n) = \left\{ [+\infty], \left[\frac{1}{2}\right]^{n-2}, \left[\frac{1}{4}\right]^{1} \right\}, \quad \text{Spec}\ (\Delta_n) = \left\{ [0], [2]^{n-2}, [4]^{1} \right\}.
\]

5 Conclusion

In this paper, we introduced a new zeta function \( \Lambda_{QW}^{G} (s) = \det (\mathbf{M}_n - s(1 - s)\mathbf{I}_n) \) for a suitable \( n \times n \) matrix \( \mathbf{M}_n \) on a simple connected graph \( G \) with \( n \) vertices via the QW by the help of the Konno-Sato theorem. In addition, we showed that \( \Lambda_{QW}^{G} (s) \) satisfies the Functional Equation: \( \Lambda_{QW}^{G} (s) = \Lambda_{QW}^{G} (1 - s) \) and the Riemann Hypothesis: if \( \rho \in \text{Zero}(\Lambda_{QW}^{G}) \), then \( \Re(\rho) = 1/2 \). The challenging problem, of course, is to prove the original Riemann Hypothesis: if \( \rho \in \text{Zero}(\Lambda) \), then \( \Re(\rho) = 1/2 \), by using our approach based on the QW. Moreover, one of the interesting problems might be to clarify the relation between our zeta function \( \Lambda_{QW}^{G} (s) \) and other zeta functions for a graph \( G \).

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