Abstract
Following an early observation of Ignatowsky, we present a derivation of the transformation rules between inertial systems making no other assumptions than the existence of the latter, and show that generically these rules are characterized by a constant of nature with the dimensions of an inverse velocity squared having the same value in all inertial frames. No independent postulate of the existence of an absolute velocity for light waves or other carriers of physical information is necessary. Aside from the usual Lorentz- and galilean transformations, our analysis also allows for the existence of four-dimensional Euclidian transformation rules between inertial systems.
1 Special Relativity

In his revolutionary article of 1905 on the *Electrodynamics of moving bodies* Einstein derived the theory of Special Relativity, in particular the Lorentz transformations, on the basis of two assumptions; the first assumption was the existence of inertial frames in which the laws of mechanics and electrodynamics always take the same form; and the second one was the postulate that the speed of light in vacuum, $c$, is the same in all these frames, independent of any relative motion\[1, 2\].

However, the second postulate is superfluous: special relativity holds independent of whether light propagates with the same speed in all inertial frames. This was already noticed in 1910 by Ignatowsky \[3\]. Indeed, although historically the invariance of the speed of light as observed in different inertial frames led to the discovery of the Lorentz transformations \[4, 5, 6\], from a modern perspective a spontaneous breaking of gauge invariance could have made photons massive without spoiling special relativity. At the time the fact that the speed of light does not reveal motion with respect to an electromagnetic aether, and that the Lorentz transformations imply that any such aether effects are unobservable, was considered of utmost importance. In hindsight it could have been otherwise, and electrodynamics is only a particular instance of a relativistic theory of fields and particles. Many other phenomena, from time dilation to gravitational waves or the behaviour of quarks and leptons with their color and weak interactions, are observed confirming special relativity as the theory of relations between inertial frames. Therefore it is instructive to reconsider the argument of Ignatowsky and discuss special relativity without taking recourse to Maxwell’s theory and the propagation of light.

In the following we show, that the mere assumption of the existence of inertial frames suffices to prove that there exists a constant of nature $a$ (by necessity taking the same value in all inertial systems) from which the existence of a universal invariant speed $c$ in all inertial frames can be inferred. This invariant speed does not a priori need to be interpreted as the speed of light, although it is an experimental fact that light does propagate at this speed. The nature of the relation between inertial frames is determined exclusively by the value of $a$, which can be zero, positive or negative, giving rise to either galilean transformations, Lorentz transformations or 4-dimensional euclidean rotations.

2 Inertial frames

In order to stress the generic nature of transformations between inertial reference frames to be derived, we will proceed by using a purely kinematical definition of an inertial frame: an inertial frame is characterized by the absence of fields of force: electromagnetic, gravitational or otherwise, and the property that all free non-interacting particles in such frames move uniformly on straight lines.

Such frames are not unique; any reference frame related to an inertial frame by a constant translation or rotation is an inertial frame as well. A more complicated question is, what the relative state of motion of inertial frames is allowed to be. This question we
Let $I$ and $I'$ be two inertial frames with associated time- and space co-ordinates. The co-ordinates in each frame being unique labels of space-time points, they must be related by well-defined functions

$$t' = f^0(t, x, y, z), \quad x' = f^1(t, x, y, z), \quad y' = f^2(t, x, y, z), \quad z' = f^3(t, x, y, z).$$

(1)

Now let $r_p(t) = r_p(0) + w_p t$ describe the path of a free particle $p$ in system $I$; the path of the particle with respect to the reference frame $I'$ then is $r'(t')$ such that it moves with a velocity

$$w'_p(t') = \frac{dr'_p}{dt'} = \frac{\partial_t f + w_p \cdot \nabla f}{\partial_t f^0 + w_p \cdot \nabla f^0} \bigg|_p.$$

(2)

By definition the frame $I'$ is an inertial frame if this velocity is constant for all and any possible particles $p$ with any constant velocities $w_p$. This implies that the gradients of the transformation functions (1) must be constant: $\partial_t f^a = \text{constant}$, $\nabla f^a = \text{constant}$, for $a = (0, 1, 2, 3)$. Therefore arbitrary inertial frames are related (in their region of overlap) by linear transformations between space-time co-ordinates $x^a = (t, x, y, z)$ and $x'^a = (t', x', y', z')$:

$$x'^a = x_0^a + \Lambda^a b x^b.$$

(3)

In particular this confirms that inertial frames can move with respect to each other with constant linear velocity, but not with any rotational motion. Indeed, in a rotating frame one is subject to pseudo-forces such as centrifugal forces, which disqualifies them from being inertial frames.

### 3 Velocity dependence of inertial-frame relations

Having established that inertial frames can be related only by constant rotations, and translations involving uniform motion, we now proceed to determine how these transformation depend on the relative velocity. As by a constant rotation we can always align inertial frames to have the same orientation (provided we restrict them to either all right-handed or left-handed ones, excluding reflections) and we can shift their spatial origins to coincide at times $t = t' = 0$ so as to imply $x_0^a = 0$, the transformations (3) reduce to the explicit form

$$t' = \gamma t + \kappa x, \quad x' = \beta t + \sigma x, \quad y' = y, \quad z' = z,$$

(4)

where $(\gamma, \kappa, \beta, \sigma)$ are constants, which may still depend on the relative velocity $v = (v, 0, 0)$. As both frames use the same units of length and time in laying out their respective co-ordinates, and we exclude reflections or time reversal, this transformation must have unit determinant:

$$\gamma \sigma - \kappa \beta = 1.$$

(5)
Now suppose a particle moves in frame $I$ with a velocity $\mathbf{w} = d\mathbf{x}/dt$. Then its velocity in the frame $I'$, using equation (2), is

\[
\begin{align*}
    w'^x &= \frac{\beta + \sigma w^x}{\gamma + \kappa w^x}, \\
    w'^y &= \frac{w^y}{\gamma + \kappa w^x}, \\
    w'^z &= \frac{w^z}{\gamma + \kappa w^x},
\end{align*}
\]

(6)

and by equation (5) the inverse takes the form

\[
\begin{align*}
    t &= \sigma t' - \kappa x', \\
    x &= -\beta t' + \gamma x', \\
    y &= y', \\
    z &= z',
\end{align*}
\]

(7)

such that

\[
\begin{align*}
    w^x &= \frac{-\beta + \gamma w'^x}{\sigma - \kappa w'^x}, \\
    w^y &= \frac{w'^y}{\sigma - \kappa w'^x}, \\
    w^z &= \frac{w'^z}{\sigma - \kappa w'^x}.
\end{align*}
\]

(8)

In particular, a particle at rest in $I'$: $\mathbf{w}' = 0$, moves in $I$ with a velocity $\mathbf{w} = (v, 0, 0)$ where

\[
v = -\frac{\beta}{\sigma}.
\]

(9)

Necessarily a particle at rest in $I$: $\mathbf{w} = 0$, then moves in $I'$ with the velocity $\mathbf{w}' = (-v, 0, 0)$ where

\[
-v = \frac{\beta}{\gamma}.
\]

(10)

Therefore the co-efficients of the transformations (4) and (7) have to satisfy the relations

\[
\sigma = \gamma, \quad \beta = -\gamma v.
\]

(11)

It then follows that

\[
x' = \gamma (x - vt), \quad x = \gamma (x' + vt'),
\]

(12)

related by reversing the velocity, as one might have anticipated.

The final information we need to completely fix the co-efficients of the transformation is to require the group property of transitivity of the transformations [8]: if a third frame $I''$, similarly oriented, moves with respect to $I'$ with velocity $(v', 0, 0)$, and its co-ordinates are related to those of $I'$ by a transformation

\[
\begin{align*}
    t'' &= \gamma' t' + \kappa' x', \\
    x'' &= \gamma' (x' - v't'), \\
    y'' &= y', \\
    z'' &= z',
\end{align*}
\]

(13)

then they can be expressed in terms of the co-ordinates of $I$ by

\[
\begin{align*}
    t'' &= \gamma'' t + \kappa'' x = \gamma' (\gamma t + \kappa x) + \kappa' \gamma (x - vt), \\
    x'' &= \gamma'' (x - v''t) = \gamma' (\gamma (x - vt) - v' (\gamma t + \kappa x)),
\end{align*}
\]

(14)

and of course $y'' = y' = y$ and $z'' = z' = z$. It directly follows that

\[
\gamma'' = \gamma (\gamma' - \kappa' v) = \gamma' (\gamma - \kappa v'), \quad \kappa'' = \gamma' \kappa + \kappa \gamma', \quad \gamma'' v'' = \gamma' (v + v').
\]

(15)
Now if \( v = 0 \), clearly the two frames \( I \) and \( I' \) coincide, and \( \gamma = 1, \kappa = 0 \); similarly if \( v' = 0 \) then \( I' \) and \( I'' \) coincide and \( \gamma' = 1, \kappa' = 0 \). However, in the non-trivial case \( v, v' \neq 0 \) and observing that \( \gamma, \gamma' \neq 0 \) because of the condition on the determinant (5), the first equation (15) implies that

\[
\frac{\kappa}{\gamma v} = \frac{\kappa'}{\gamma' v'} = -a,
\]

a universal constant, as the transformation \( I \to I' \) is completely independent of the transformation \( I' \to I'' \), and therefore the velocities \( v \) and \( v' \) can be chosen at will. Inserting this result back into the first equation (15): \( \gamma'' = \gamma \gamma' (1 + avv') \), the last equation (15) gives an equation for the composition of the velocities:

\[
v'' = \frac{v + v'}{1 + avv'}.
\]

Finally, using these results we find that

\[
\gamma''^2 (1 - av''^2) = [\gamma^2 (1 - av^2)] [\gamma'^2 (1 - av'^2)],
\]

and since the velocities \( v, v' \) can be chosen freely it follows that the transformation \( I \to I' \) between any two inertial frames requires

\[
\gamma = \frac{1}{\sqrt{1 - av^2}},
\]

where we have chosen the positive square root to guarantee that the times in the two frames run in the same direction. Combining the results, the complete most general form of the transformation between two equally oriented inertial frames is

\[
t' = \frac{t - avx}{\sqrt{1 - av^2}}, \quad x' = \frac{x - vt}{\sqrt{1 - av^2}}, \quad y' = y, \quad z' = z.
\]

### 4 Discussion

From the transformation rules (20) it is easy to verify that the space-time interval

\[
(\Delta \tau)^2 = (\Delta t)^2 - a [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]
\]

is invariant under these transformations. Here \( \Delta \tau \) is the proper time interval, measured by a clock in the rest frame. Clearly there are three different cases to consider, characterized by the value of the real universal constant \( a \), which can vanish, can be positive or can be negative. In the simplest case \( a = 0 \) the transformations (20) reduce to those of classical galilean relativity. In that case time intervals are always equal in all inertial frames. This certainly qualifies as a mathematically consistent solution, but for all we know not one that describes the correct physics.
The next case: \( a > 0 \), allows us to write
\[
a = \frac{1}{c^2},
\]
where \( c \) has the dimensions of a velocity; as \( a \) is a universal constant, so must be \( c \). Indeed, in this case we actually reproduce the standard Lorentz transformations, and the constant \( c \) can be identified with the speed of light; however, note that this identification is not one made \emph{a priori}, but it is an experimental result that the speed of light appears to be the same in all inertial reference frames \cite{9}. Indeed, it is easily verified that if the first velocity \( v = c \) in equation \eqref{17}, then also \( v'' = c \) independent of the velocity \( v' \). Note furthermore the well-known consequence of equation \eqref{17} that composing velocities less than \( c \) one can never equal or exceed the value \( c \); if \((v, v') < c\) then necessarily \( v'' < c \). Also the Lorentz transformations imply
\[
\Delta t = \frac{\Delta \tau}{\sqrt{1 - v^2/c^2}},
\]
and there is a time dilation between clocks in moving frames and in the rest frame.

Finally, if \( a < 0 \) we can similarly write
\[
a = -\frac{1}{c^2},
\]
where \( c \) is a universal constant with the dimensions of velocity. In this case the line element \eqref{21} is that of 4-dimensional euclidean geometry; more specifically, it implies that the four-velocity always lies on a fixed 4-dimensional sphere: if for \( a = (1, 2, 3, 4) \) we define \( x_a = (x, y, z, ct) \), then
\[
\sum_a u_a^2 = c^2.
\]
Therefore no four-velocity component can exceed \( c \). However, the relation between four-velocity and co-ordinate velocity \( v_i = dx_i/dt \), is
\[
\frac{u_i}{c} = \frac{v_i/c}{\sqrt{1 + v^2/c^2}}, \quad \frac{u_4}{c} = \frac{dt/d\tau}{\sqrt{1 + v^2/c^2}} = \frac{1}{\sqrt{1 + v^2/c^2}},
\]
and \( v_i \) is not restricted unlike the lorentzian case, although \( u_i \) is. Moreover we see, that instead of time dilation there is a time contraction between moving frames. Observe that eq.\eqref{26} can be inverted, expressing \( v_i \) in terms of \( u_i \):
\[
\frac{v_i}{c} = \frac{u_i/c}{\sqrt{1 - u^2/c^2}}
\]
It can be seen that in this case the role of the proper velocity and the relative velocity are reversed compared to their relation in the \( a > 0 \) case.

As \( a \) is a universal constant of nature, it can only have a single value. For deciding between the three cases, if one would not be aware of the frame-independence of the speed
of light already, it would be most natural to use the comparison of proper time and co-ordinate time; in particular, the fact that time-dilation is experimentally observed, for example in the life-time of unstable high-energy particles like muons in cosmic ray showers, decides in the favor of positive $a = 1/c^2 > 0$, with the Lorentz transformations providing the actual relations between inertial frames.

The attentive reader may have noticed that the space-time interval (21) is similar to that of a homogeneous and isotropic universe, with the difference that by requiring the universality if clocks and rulers here $a$ is fixed to a universal constant, whereas in the cosmological setting the scale factor can be a function of cosmic time. However, in the context of standard general-relativistic cosmology this is not interpreted as a time-dependent speed of light, but as a mismatch between physical distances and co-ordinate differences $(\Delta x, \Delta y, \Delta z)$.

In view of the role and interpretation of the constant $a$, which is more general than just a rewriting of Lorentz transformations, we consider $a$ more fundamental than the speed of light $c$ itself. Because of the central role of the relation (21) we propose to call it Minkowski’s constant [12].

\footnote{For an account of the historical development of special relativity, see [10] and references therein.}
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