LONG TIME DECAY OF LERAY SOLUTION OF 3D-NSE WITH
EXPONENTIAL DAMPING

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Abstract. We study the uniqueness, the continuity in $L^2$ and the large time
decay for the Leray solutions of the 3D incompressible Navier-Stokes equations
with nonlinear exponential damping term $a(e^{b|u|^4} - 1)u$, $(a, b > 0)$.

1. Introduction

In this paper, we investigate the questions of the existence, uniqueness and as-
swermptotic study of global weak solution to the modified incompressible Navier-Stokes
equations in $\mathbb{R}^3$

\[
\begin{aligned}
\partial_t u - \nu \Delta u + u.\nabla u + a(e^{b|u|^4} - 1)u &= -\nabla p \\
\text{div} u &= 0 \\
u0 = u^0(x) & \\
a, b > 0
\end{aligned}
\]

where $u = u(t, x) = (u_1, u_2, u_3)$, $p = p(t, x)$ denote respectively the unknown
velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, the
viscosity of fluid $\nu > 0$ and $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is the initial given velocity.
The damping is from the resistance to the motion of the flow. It describes various
physical situations such as porous media flow, drag or friction effects, and some
dissipative mechanisms (see [6, 7, 13, 14] and references therein). The fact that
\[\text{div} u = 0,\]
allows to write the term $(u.\nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$ in the
following form $\text{div} (u \otimes u) := (\text{div} (u_1 u), \text{div} (u_2 u), \text{div} (u_3 u))$. If the initial velocity
$u^0$ is quite regular, the divergence free condition determines the pressure $p$.
Without loss of generality and in order to simplify the proofs of our results, we
consider the viscosity unitary ($\nu = 1$).

The global existence of weak solution of initial value problem of classical incom-
pressible Navier-Stokes were proved by Leray and Hopf (see [12, 15]) long before.
Uniqueness remains an open problem for the dimensions $d \geq 3$.
The polynomial damping $a|u|^\beta - 1 u$ is studied in [9] by Cai and Jiu, where they
proved the global existence of weak solution in

\[L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)).\]

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The exponential damping \( a(e^{b|u|^2} - 1)u \) is studied in [2] by J. Benamer, where he proved the global existence of weak solution in

\[ L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b, \]

where \( \mathcal{E}_b = \{ f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} : \text{measurable, } (e^{b|f|^2} - 1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3) \} \).

The purpose of this paper is to study the well-posedness and the asymptotic study of the incompressible Navier-Stokes equations with exponential damping \( a(e^{b|u|^2} - 1)u \). We will show that the Cauchy problem (S) has a global solutions for any \( a, b \in (0, \infty) \). We apply the Friedrictch method to construct the approximate solutions and make more delicate estimates to proceed to compactness arguments. In particular, we obtain new more a priori estimates:

\[ \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a \int_0^t \|(e^{b|u(s)|^2} - 1)|u(s)|^2\|_{L^1} ds \leq \|u_0\|_{L^2}^2, \]

comparing with the Navier-Stokes equations, to guarantee that the solution \( u \) belongs to

\[ L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{F}_b, \]

where \( \mathcal{F}_b = \{ f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} \text{ measurable, } (e^{b|f|^2} - 1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3) \} \).

To prove the uniqueness we use an energy method and the approximate systems. The proof of the asymptotic study is based on a decomposition of the solution in high and low frequencies and the uniqueness of such solution in a well chosen time \( t_0 \).

In our case of exponential damping, we are trying to find more regularity of Leray solution in \( \cap_b L^p(\mathbb{R}^+, L^p(\mathbb{R}^3)) \). In particular, we give a new energy estimate. Our main result is the following:

**Theorem 1.1.**

Let \( u^i \in L^2(\mathbb{R}^3) \) be a divergence free vector fields, then there is a unique global solution of the system (S): \( u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b \). Moreover, for all \( t \geq 0 \)

\[ (1.1) \quad \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a \int_0^t \|(e^{b|u(s)|^2} - 1)|u(s)|^2\|_{L^1} ds \leq \|u_0\|_{L^2}^2. \]

Moreover, we have

\[ (1.2) \quad \lim_{t \to \infty} \sup \|u(t)\|_{L^2} = 0. \]

**Remark 1.2.**

(1) The new results in this theorem is the uniqueness of the global solution, the continuity of the solution in the \( L^2(\mathbb{R}^3) \) space and the asymptotic behavior at infinity.

(2) Generally, for \( r \geq 1 \) the following problem

\[
(P_r) \begin{cases}
\partial_t u - \nu \Delta u + u.\nabla u + a(e^{b|u|^2} - 1)u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) = u^0(x) & \text{in } \mathbb{R}^3, \\
a, b > 0
\end{cases}
\]
and by adapting the same proof of result of [2], we show the global existence of such a solution in $C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \cap \mathcal{E}_R^2$, where

$$
\mathcal{E}_R^2 = \{ f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} \text{ measurable}; (e^{bf(x) - 1})|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3) \}.
$$

Moreover, we get

$$
\|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla u(s)\|^2_{L^2} ds + 2a \int_0^t (e^{bf(s)}) |u(s)|^2 \|u\|^2_{L^2} ds \leq \|u_0\|^2_{L^2}.
$$

The asymptotic result (1.3) is true for all $r \geq \frac{7}{3}$, and the index $\frac{7}{3}$ is a critical technical condition (See (1.3)-(2.1)).

2. Notations and Preliminary Results

For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and $R > 0$, the Friedrich operator $J_R$ is defined by: $J_R(D)f = F^{-1}(\chi_{B_R}\tilde{f})$, where $B_R$ is the ball of center 0 and radius $R$. If $L^2(\mathbb{R}^3)$ denotes the space of divergence-free vector fields in $L^2(\mathbb{R}^3)$, the Leray projector $P : (L^2(\mathbb{R}^3))^3 \to (L^2(\mathbb{R}^3))^3$ is defined by:

$$
\mathcal{F}(Pf) = \hat{f}(\xi) - (\hat{f}(\xi) \frac{\xi}{|\xi|}) \frac{\xi}{|\xi|} = M(\xi)\hat{f}(\xi),
$$

where $M(\xi)$ is the matrix $(\delta_{k,\ell} - \frac{\xi_k \xi_\ell}{|\xi|^2})_{1 \leq k, \ell \leq 3}$. If $u \in \mathcal{S}(\mathbb{R}^3)^3$,

$$
P(u) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\delta_{kj} - \frac{\xi_k \xi_j}{|\xi|^2}) \hat{u}_j(\xi) e^{i\xi x} d\xi,
$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space. Define also the operator $A_R(D)$ on $L^2(\mathbb{R}^3)$ by:

$$
A_R(D)u = P J_R(D)u = F^{-1}(M(\xi)\chi_{B_R}(\xi)\hat{u}).
$$

To simplify the exposition of the main result, we first collect some preliminary results and we give some new technical lemmas.

**Proposition 2.1.** ([8])

Let $H$ be a Hilbert space.

1. The unit ball is weakly compact, that is: if $(x_n)$ is a bounded sequence in $H$, then there is a subsequence $(x_{n_k})$ such that

$$
(x_{n_k})|y| \to (x|y), \forall y \in H.
$$

2. If $x \in H$ and $(x_n)$ a bounded sequence in $H$ such that $\lim_{n \to \infty} (x_n|y) = (x|y)$, for all $y \in H$, then $\|x\| \leq \liminf_{n \to \infty} \|x_n\|$.

3. If $x \in H$ and $(x_n)$ a bounded sequence in $H$ such that $\lim_{n \to \infty} (x_n|y) = (x|y)$, for all $y \in H$ and $\limsup_{n \to \infty} \|x_n\| \leq \|x\|$, then $\lim_{n \to \infty} \|x_n - x\| = 0$.

We recall the following product law in the homogeneous Sobolev spaces:

**Lemma 2.2.** ([10])

Let $s_1$, $s_2$ be two real numbers and $d \in \mathbb{N}$.

1. If $s_1 < \frac{d}{2}$ and $s_1 + s_2 > 0$, there exists a constant $C_1 = C_1(d, s_1, s_2)$ such that: if $f, g \in \dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d)$, then $f, g \in \dot{H}^{s_1 + s_2 - \frac{d}{2}}(\mathbb{R}^d)$ and

$$
\|fg\|_{\dot{H}^{s_1 + s_2 - \frac{d}{2}}} \leq C_1(\|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}} \|g\|_{\dot{H}^{s_1}}).
$$
(2) If \(s_1, s_2 < \frac{d}{2}\) and \(s_1 + s_2 > 0\) there exists a constant \(C = C_2(d, s_1, s_2)\) such that: if \(f \in \dot{H}^{s_1}(\mathbb{R}^d)\) and \(g \in \dot{H}^{s_2}(\mathbb{R}^d)\), then \(f \cdot g \in \dot{H}^{s_1 + s_2 - \frac{d}{2}}(\mathbb{R}^d)\)

\[
\|fg\|_{H^{s_1 + s_2 - \frac{d}{2}}} \leq C_2 \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}.
\]

Lemma 2.3.

Let \(b > 0\) and \(d \in \mathbb{N}\). Then, for all \(x, y \in \mathbb{R}^d\), we have

(2.1) \[
\langle |x|^\beta, x - |y|^\beta, y, x - y \rangle \geq \frac{1}{2} (|x|^\beta + |y|^\beta) |x - y|^2,
\]

and, for \(r > 0\), we have

(2.2) \[
\langle (e^{b|x|^r} - 1)x - (e^{b|y|^r} - 1)y, x - y \rangle \geq \frac{1}{2} \left( (e^{b|x|^r} - 1) + (e^{b|y|^r} - 1) \right) |x - y|^2.
\]

Proof.

Suppose that \(|x| > |y| > 0\). For \(u > v > 0\), we have

(2.3) \[
2\langle ux - vy, x - y \rangle - (u + v)|x - y|^2 = (u - v)(|x|^2 - |y|^2) \geq 0.
\]

It suffices to take \(u = |x|^\beta\) and \(v = |y|^\beta\), we get the inequality (2.1).

Suppose that \(|x| > |y| > 0\). In use of the inequality (2.3) with \(u = (e^{b|x|^r} - 1)\) and \(v = (e^{b|y|^r} - 1)\), we get

\[
2((e^{b|x|^r} - 1)x - (e^{b|y|^r} - 1)y, x - y) - \left( (e^{b|x|^r} - 1) + (e^{b|y|^r} - 1) \right) |x - y|^2
\]

\[
= (e^{b|x|^r} - e^{b|y|^r}) |x - y|^2 \geq 0.
\]

This proves the inequality (2.2).

The following result is a generalization of Proposition 3.1 in [2].

Proposition 2.4.

Let \(v_1, v_2, v_3 \in [0, \infty), r_1, r_2, r_3 \in (0, \infty)\) and \(f^0 \in L^2_+(\mathbb{R}^3)\).

For \(n \in \mathbb{N}\), let \(F_n : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3\) be a measurable function in \(C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))\) such that

\[
A_n(D)F_n = F_n, \quad F_n(0, x) = A_n(D)f^0(x)
\]

and

\[
\begin{align*}
\tag{E1}
\partial_tF_n + \sum_{k=1}^{3} \nu_k |D_k|^{2r_k}F_n + A_n(D)\text{div} (F_n \otimes F_n) + A_n(D)h(|F_n|)F_n &= 0.
\end{align*}
\]

\[
\begin{align*}
\tag{E2}
\left\|F_n(t, .)\right\|_{L^2}^2 + 2 \sum_{k=1}^{3} \nu_k \int_0^t \|D_k |^{r_k}F_n(s, .)\|_{L^2}^2 ds \noalign{\smallskip} + a \int_0^t \|h(|F_n(s, .)|)F_n(s, .)\|_{L^1}^2 ds \leq \|f^0\|_{L^2}^2,
\end{align*}
\]

where \(h(z) = a(e^{bz} - 1)\), with \(r \geq 1\) and \(a, b > 0\). Then: for every \(\varepsilon > 0\) there is \(\delta = \delta(\varepsilon, a, b, v_1, v_2, v_3, r_1, r_2, r_3, \|f^0\|_{L^2}) > 0\) such that: for all \(t_1, t_2 \in \mathbb{R}^+\), we have

(2.4) \[
|t_2 - t_1| < \delta \implies \|F_n(t_2) - F_n(t_1)\|_{H^{-\alpha}} < \varepsilon, \quad \forall n \in \mathbb{N},
\]
We deduce that

\[ I = \frac{3}{1} \int_{t_1}^{t_2} \sum_{k=1}^{3} \nu_k \|D_k |^{2r_k} F_n(t)\|_{H^{-s_0}} dt. \]

Proof.
Integrate (E1) on the interval \([t_1, t_2] \subset \mathbb{R}^+\) and take the inner product in \(H^{-s_0}\), we get

\[
\|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} \leq \int_{t_1}^{t_2} \sum_{k=1}^{3} \nu_k \|D_k |^{2r_k} F_n(t)\|_{H^{-s_0}} dt
\]

\[
+ \int_{t_1}^{t_2} \|A_n(D) div (F_n \otimes F_n)(t)\|_{H^{-s_0}} dt
\]

\[
+ \int_{t_1}^{t_2} \|A_n(D) h(|F_n|) F_n(t)\|_{H^{-s_0}} dt.
\]

Let

\[
I_{1,n}(t_1, t_2) = \int_{t_1}^{t_2} \sum_{k=1}^{3} \nu_k \|D_k |^{2r_k} F_n(t)\|_{H^{-s_0}} dt,
\]

\[
I_{2,n}(t_1, t_2) = \int_{t_1}^{t_2} \|A_n(D) div (F_n \otimes F_n)(t)\|_{H^{-s_0}} dt,
\]

and

\[
I_{3,n}(t_1, t_2) = \int_{t_1}^{t_2} \|A_n(D) h(|F_n|) F_n(t)\|_{H^{-s_0}} dt.
\]

We have:

\[
(2.5) \quad I_{1,n}(t_1, t_2) \leq \sum_{k=1}^{3} \nu_k \int_{t_1}^{t_2} \|F_n(t)\|_{H^{2r_k-s_0}} dt
\]

\[
\leq \left( \sum_{k=1}^{3} \nu_k \right) \int_{t_1}^{t_2} \|F_n(t)\|_{L^2} dt
\]

\[
\leq \left( \sum_{k=1}^{3} \nu_k \right) \|f^0\|_{L^2} (t_2 - t_1).
\]

\[
I_{2,n}(t_1, t_2) = \int_{t_1}^{t_2} \|A_n(D) div (F_n \otimes F_n)(t)\|_{H^{-s_0}} dt \leq \int_{t_1}^{t_2} \|div (F_n \otimes F_n)(s)\|_{H^{-s}} dt
\]

\[
\leq \int_{t_1}^{t_2} \| (F_n \otimes F_n)(t)\|_{H^{-s_0+1}} dt \leq \int_{t_1}^{t_2} \| (F_n \otimes F_n)(t)\|_{H^{-s}} dt
\]

Recall that if \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is an integrable function, then for all \( s > \frac{3}{4} \),

\[
\|f\|_{H^{-s}}^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-s} |\hat{f}(\xi)|^2 d\xi \leq \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-s} d\xi \right) \| \hat{f} \|_{L^2}^2
\]

\[
\leq \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-s} d\xi \right) \| f \|_{L^3}^2.
\]

We deduce that

\[
(2.6) \quad \|f\|_{H^{-s}}^2 \leq \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-s} d\xi \right) \| f \|_{L^3}^2
\]
and there exists $C > 0$ such that

$$\begin{align*}
(2.7) \quad I_{2,n}(t_1, t_2) & \leq C \int_{t_1}^{t_2} \| (F_n \otimes F_n)(t) \|_{L^1} dt \
& \leq C(t_2 - t_1) \| f^0 \|_{L^2}^2.
\end{align*}$$

To estimate the integral $I_{3,n}(t_1, t_2)$, consider for $R > 1$ the sub-level sets:

$$X_n(R, t) = \{ x \in \mathbb{R}^3 : |F_n(t, x)| \leq R \}.$$

We remark that

$$(e^{b|F_n(t, x)|^2} - 1)|F_n(t, x)| \leq \left(\frac{e^{bR^2} - 1}{R}\right)|F_n(t, x)|^2, \forall x \in X_n(R, t).$$

Let $M(R) = \frac{e^{bR^2} - 1}{R}$. From (2.6), there exists $C_1, C_2, C_3 > 0$ such that

$$\begin{align*}
I_{3,n}(t_1, t_2) &= \int_{t_1}^{t_2} \| A_n(D)h(|F_n|)F_n(t) \|_{H^{-\alpha}} dt \
& \leq C_1 \int_{t_1}^{t_2} \| h(|F_n|)F_n(t) \|_{L^1} dt \
& \leq C_1 \int_{t_1}^{t_2} \int_{X_n(R, t)} h(|F_n|)|F_n| dx dt + C_1 \int_{t_1}^{t_2} \int_{X_n(R, t)^c} h(|F_n|)|F_n| dx dt \
& \leq C_2 \int_{t_1}^{t_2} \int_{X_n(R, t)} |F_n|^2 dx dt + \frac{C_1}{R} \int_{t_1}^{t_2} \int_{X_n(R, t)} h(|F_n|)|F_n|^2 dx dt \
& \leq C_2 M(R) \int_{t_1}^{t_2} \| F_n(t) \|_{L^2}^2 dt + \frac{C_3}{R} \int_{t_1}^{t_2} \| h(|F_n|)F_n \|_{L^1}^2 dt \
& \leq C_2 M(R) \| f^0 \|_{L^2}^2 (t_2 - t_1) + \frac{C_3}{R} \| f^0 \|_{L^2}^2.
\end{align*}$$

Hence

$$\begin{align*}
(2.8) \quad I_{3,n}(t_1, t_2) & \leq C_2 M(R) \| f^0 \|_{L^2}^2 (t_2 - t_1) + \frac{C_3}{R} \| f^0 \|_{L^2}^2.
\end{align*}$$

Now using the inequalities (2.8), (2.7) and (2.8), for $\varepsilon > 0$, consider $R$ such that $\frac{C_3}{R} \| f^0 \|_{L^2}^2 < \frac{\varepsilon}{4}$ and

$$0 < \delta < \min\left(\frac{\varepsilon}{4(C_2 M(R) \| f^0 \|_{L^2}^2 + 1)}, \frac{\varepsilon}{4(C \| f^0 \|_{L^2}^2 + 1)}, \frac{\varepsilon}{4(C_2 M(R) \| f^0 \|_{L^2}^2 + 1)}\right).$$

For such $\delta$, we get (2.4).

3. Proof of the Main Theorem

The proof is given in four steps:
3.1. Existence of Weak Solution.

In this step, we build approximate solutions of the system (S) inspired by the method used in [2, 10], hence we construct a global solution. For this, consider the approximate system with parameter $n \in \mathbb{N}$:

$$ (S_n) \begin{cases} \partial_t u - \Delta J_n u + J_n u.\nabla J_n u + a J_n[(e^{b|J_n u|^4} - 1)J_n u] = -\nabla p_n \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ p_n = (-\Delta)^{-1} \left( \text{div} J_n (J_n u.\nabla J_n u) + a \text{div} J_n [(e^{b|J_n u|^2} - 1)J_n u] \right) \\ \text{div} u = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,x) = J_n u^0(x) \text{ in } \mathbb{R}^3. \end{cases} $$

$J_n$ is the Friedrich operator defined in the second section.

- By Cauchy-Lipschitz Theorem, we obtain a unique solution $u_n \in C^1(\mathbb{R}^+, L^2_2(\mathbb{R}^3))$ of $(S_{2,n})$. Moreover, $J_n u_n = u_n$ such that

$$ (3.1) \quad \|u_n(t)\|_{L^2}^2 + 2 \int_0^t \|
abla u_n\|_{L^2}^2 + 2a \int_0^t \|\left(e^{b|u_n|^4} - 1\right)|u_n|^2\|_{L^1} \leq \|u^0\|_{L^2}^2. $$

- The sequence $(u_n)_n$ is bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3))$ and on $L^2(\mathbb{R}^+, H^1(\mathbb{R}^3))$. Using proposition [2.4] and the interpolation method, we deduce that the sequence $(u_n)_n$ is equicontinuous on $H^{-1}(\mathbb{R}^3)$.

- Let $(T_q)_q$ be a strictly increasing sequence such that $\lim_{q \to +\infty} T_q = \infty$. Consider a sequence of functions $(\theta_q)_q \in C^0_0(\mathbb{R}^3)$ such that

$$ \begin{cases} \theta_q(x) = 1, \text{ for } |x| \leq q + \frac{5}{4} \\ \theta_q(x) = 0, \text{ for } |x| \geq q + 2 \\ 0 \leq \theta_q \leq 1. \end{cases} $$

Using (3.1), the equicontinuity of the sequence $(u_n)_n$ on $H^{-1}(\mathbb{R}^3)$ and classical argument by combining Ascoli’s theorem and the Cantor diagonal process, there exists a subsequence $(u_{\varphi(n)})_n$ and $u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3))$ such that: for all $q \in \mathbb{N}$,

$$ (3.2) \quad \lim_{n \to +\infty} \|\theta_q(u_{\varphi(n)} - u)\|_{L^\infty([0,T], H^{-4})} = 0. $$

In particular, the sequence $(u_{\varphi(n)}(t))_n$ converges weakly in $L^2(\mathbb{R}^3)$ to $u(t)$ for all $t \geq 0$.

- Combining the above inequalities, we obtain:

$$ (3.3) \quad \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a \int_0^t \|\left(e^{b|u(s)|^4} - 1\right)|u(s)|^2\|_{L^1} ds \leq \|u^0\|_{L^2}^2. $$

for all $t \geq 0$.

- $u$ is a solution of the system (S).

3.2. Continuity of the Solution in $L^2$.

In this section, we give a simple proof of the continuity of the solution $u$ of the system $(S)$ and we prove also that $u \in C(\mathbb{R}^+, L^2(\mathbb{R}^3))$. The construction of the solution is based on the Friedrich approximation method. We point out that we
Using inequality (2.2), we get

\[ \limsup_{t \to 0} \| u(t) \|_{L^2} \leq \| u^0 \|_{L^2}. \]

Then, proposition 2.1 (3) implies that

\[ \limsup_{t \to 0} \| u(t) - u^0 \|_{L^2} = 0, \]

which ensures the continuity of \( u \) at 0.

- Consider the functions

\[ v_{n, \varepsilon}(t, \cdot) = u_{\varphi(n)}(t + \varepsilon, \cdot), \quad p_{n, \varepsilon}(t, \cdot) = p_{\varphi(n)}(t + \varepsilon, \cdot), \]

for \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). We have:

\[
\begin{align*}
\partial_t u_{\varphi(n)} - \Delta u_{\varphi(n)} + J_{\varphi(n)}(u_{\varphi(n)}, \nabla u_{\varphi(n)}) + a J_{\varphi(n)}(e^{b u_{\varphi(n)}|\varepsilon^4} - 1)u_{\varphi(n)} &= -\nabla p_{\varphi(n)} \\
\partial_t v_{n, \varepsilon} - \Delta v_{n, \varepsilon} + J_{\varphi(n)}(v_{n, \varepsilon}, \nabla v_{n, \varepsilon}) + a J_{\varphi(n)}(e^{b v_{n, \varepsilon}|\varepsilon^4} - 1)v_{n, \varepsilon} &= -\nabla p_{n, \varepsilon}
\end{align*}
\]

The function \( w_{n, \varepsilon} = u_{\varphi(n)} - v_{n, \varepsilon} \) fulfills the following:

\[
\partial_t w_{n, \varepsilon} - \Delta w_{n, \varepsilon} + a J_{\varphi(n)} \left( (e^{b u_{\varphi(n)}|\varepsilon^4} - 1)u_{\varphi(n)} - (e^{b v_{n, \varepsilon}|\varepsilon^4} - 1)v_{n, \varepsilon} \right) = -\nabla (p_{\varphi(n)} - p_{n, \varepsilon}) + J_{\varphi(n)}(w_{n, \varepsilon}, \nabla w_{n, \varepsilon}).
\]

Taking the scalar product in \( L^2(\mathbb{R}^3) \) with \( w_{n, \varepsilon} \) and using the properties \( \text{div} w_{n, \varepsilon} = 0 \) and \( \langle w_{n, \varepsilon}, \nabla w_{n, \varepsilon}, w_{n, \varepsilon} \rangle = 0 \), we get

\[
\frac{1}{2} \frac{d}{dt} \| w_{n, \varepsilon} \|^2_{L^2} + \| \nabla w_{n, \varepsilon} \|^2_{L^2} + a J_{\varphi(n)} \left( (e^{b u_{\varphi(n)}|\varepsilon^4} - 1)u_{\varphi(n)} - (e^{b v_{n, \varepsilon}|\varepsilon^4} - 1)v_{n, \varepsilon} \right) : w_{n, \varepsilon} = \langle w_{n, \varepsilon}, \nabla u_{\varphi(n)} \rangle : w_{n, \varepsilon} \mid_{L^2}.
\]

(3.4)

Using inequality (2.2), we get

\[
\langle J_{\varphi(n)}(e^{b u_{\varphi(n)}|\varepsilon^4} - 1)u_{\varphi(n)} - (e^{b v_{n, \varepsilon}|\varepsilon^4} - 1)v_{n, \varepsilon} \rangle : w_{n, \varepsilon} = \frac{1}{2} \int_{\mathbb{R}^3} \left( (e^{b u_{\varphi(n)}|\varepsilon^4} - 1)u_{\varphi(n)} - (e^{b v_{n, \varepsilon}|\varepsilon^4} - 1)v_{n, \varepsilon} \right) : w_{n, \varepsilon} \leq \frac{1}{2} \int_{\mathbb{R}^3} \left( (e^{b u_{\varphi(n)}|\varepsilon^4} - 1)u_{\varphi(n)} - (e^{b v_{n, \varepsilon}|\varepsilon^4} - 1)v_{n, \varepsilon} \right) |w_{n, \varepsilon}|^2 \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_{\varphi(n)}|^2 |w_{n, \varepsilon}|^2.
\]

\[
|\langle J_{\varphi(n)}(w_{n, \varepsilon}, \nabla u_{\varphi(n)}) \rangle : w_{n, \varepsilon} \rangle \mid_{L^2} \leq \frac{1}{2} \int_{\mathbb{R}^3} |w_{n, \varepsilon}| |u_{\varphi(n)}| \mid_{\nabla w_{n, \varepsilon}} \mid_{L^2} \leq \frac{1}{2} \int_{\mathbb{R}^3} |w_{n, \varepsilon}|^2 |u_{\varphi(n)}|^2 + \frac{1}{2} \mid_{\nabla w_{n, \varepsilon}}^2.\]
Again by using the elementary inequality $xy \leq \frac{ab}{8} x^2 + \frac{2}{ab} y^2$, for $x, y \geq 0$, we get

$$\langle J_{\varphi(n)}(w_{n,e}, \nabla u_{\varphi(n)}); w_{n,e} \rangle_{L^2} \leq \frac{ab}{8} \int_{\Omega} |u_{\varphi(n)}|^2 |w_{n,e}|^2 + \frac{2}{ab} \|w_{n,e}\|_{L^2}^2 + \frac{1}{2} \|\nabla w_{n,e}\|_{L^2}^2.$$ 

Combining the identity (2.2) and the inequality (3.4), we get

$$\frac{1}{2} \frac{d}{dt} \|w_{n,e}\|_{L^2}^2 + \frac{1}{2} \|\nabla w_{n,e}\|_{L^2}^2 \leq \frac{2}{ab} \|w_{n,e}\|_{L^2}^2.$$ 

By Gronwall Lemma, we get

$$\|w_{n,e}(t)\|_{L^2}^2 \leq \|w_{n,e}(0)\|_{L^2}^2 e^{\frac{4t}{ab}}.$$ 

But

$$\|u_{\varphi(n)}(t + \varepsilon) - u_{\varphi(n)}(t)\|_{L^2}^2 \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 e^{\frac{4t}{ab}}.$$ 

For $t_0 > 0$ and $\varepsilon \in (0, t_0)$, we have

$$\|u_{\varphi(n)}(t_0 + \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \exp \left(\frac{4t_0}{ab}\right),$$

$$\|u_{\varphi(n)}(t_0 - \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \exp \left(\frac{4t_0}{ab}\right).$$

So

$$\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 = \|J_{\varphi(n)} u_{\varphi(n)}(\varepsilon) - J_{\varphi(n)} u_{\varphi(n)}(0)\|_{L^2}^2$$

$$= \|\chi_{\varphi(n)}(u_{\varphi(n)} - \tilde{u}_0)\|_{L^2}^2$$

$$\leq \|u_{\varphi(n)}(\varepsilon) - u(0)\|_{L^2}^2$$

$$\leq 2 \|u(0)\|_{L^2}^2 - 2 \Re \langle u_{\varphi(n)}(\varepsilon), u(0)\rangle.$$ 

But

$$\lim_{t \to +\infty} \langle u_{\varphi(n)}(\varepsilon), u(0)\rangle = \langle u(\varepsilon), u(0)\rangle.$$ 

Hence

$$\liminf_{t \to +\infty} \|u_{\varphi(n)}(\varepsilon) - u(0)\|_{L^2}^2 \leq 2 \|u(0)\|_{L^2}^2 - 2 \Re \langle u(\varepsilon), u(0)\rangle.$$ 

Moreover, for all $q, N \in \mathbb{N}$

$$\|J_N \left(\theta_q(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0))\right)\|_{L^2}^2 \leq \|\theta_q(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2}^2 \leq 2 \|u(0)\|_{L^2}^2 - 2 \Re \langle u(\varepsilon), u(0)\rangle.$$ 

Using (3.2) we get, for $q$ big enough,

$$\|J_N \left(\theta_q(u(t_0 \pm \varepsilon) - u(t_0))\right)\|_{L^2}^2 \leq \liminf_{t \to +\infty} \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.$$ 

Then

$$\|J_N \left(\theta_q(u(t_0 \pm \varepsilon) - u(t_0))\right)\|_{L^2}^2 \leq 2 \left(\|u(0)\|_{L^2}^2 - \Re \langle u(\varepsilon), u(0)\rangle\right) \exp \left(\frac{4t_0}{ab}\right).$$ 

By applying the monotone convergence theorem in the order $N \to \infty$ and $q \to \infty$, we get

$$\|u(t_0 \pm \varepsilon) - u(t_0)\|_{L^2}^2 \leq 2 \left(\|u(0)\|_{L^2}^2 - \Re \langle u(\varepsilon), u(0)\rangle\right) \exp \left(\frac{4t_0}{ab}\right).$$ 

Using the continuity at 0 and make $\varepsilon \to 0$, we get the continuity at $t_0$. 

3.3. Uniqueness of the Solution.
Let \( u, v \) be two solutions of \((S)\) in the space
\[ C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{F}_\beta.\]
The function \( w = u - v \) satisfies the following:
\[
\partial_t w - \Delta w + a \left( (e^{b|u|^4} - 1)u - (e^{b|v|^2} - 1)v \right) = -\nabla(p - \tilde{p}) + w.\nabla w - w.\nabla u - u.\nabla w.\
\]
Taking the scalar product in \( L^2 \) with \( w \), we get
\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + a \left( (e^{b|u|^4} - 1)u - (e^{b|v|^2} - 1)v \right) ; w \rangle_{L^2} = -\langle w.\nabla u; w \rangle_{L^2}.
\]
The idea is to lower the term \( \left( (e^{b|u|^4} - 1)u - (e^{b|v|^2} - 1)v \right) ; w \rangle_{L^2} \) with the help of the Lemma 2.3 and then divide the term find into two equal pieces, one to absorb the nonlinear term and the other is used in the last inequality.

By using inequality (2.2), we get
\[
\langle \left( (e^{b|u|^4} - 1)u - (e^{b|v|^2} - 1)v \right) ; w \rangle_{L^2} \geq \frac{1}{2} \int_{\mathbb{R}^3} \left( (e^{b|u|^4} - 1) + (e^{b|v|^2} - 1) \right) |w|^2 \geq \frac{b}{2} \int_{\mathbb{R}^3} |u|^4 |w|^2.
\]
Moreover, we have
\[
|\langle w.\nabla u; w \rangle_{L^2}| = |\langle \text{div} (w \otimes u); w \rangle_{L^2}| = |\langle w \otimes u; \nabla w \rangle_{L^2}| \leq \int_{\mathbb{R}^3} |w|.|u|.|\nabla w| \leq \frac{1}{2} \int_{\mathbb{R}^3} |w|^2 |u|^2 + \frac{1}{2} \|\nabla w\|_{L^2}^2
\leq \frac{ab}{8} \int_{\mathbb{R}^3} |u|^4 |w|^2 + \frac{1}{2ab} \|w\|_{L^2}^2 + \frac{1}{2} \|\nabla w\|_{L^2}^2.
\]
Combining the above inequalities, we get
\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \frac{1}{2} \|\nabla w\|_{L^2}^2 + \frac{a}{4} \int_{\mathbb{R}^3} \left( (e^{b|u|^2} - 1) + (e^{b|v|^2} - 1) \right) |w|^2 \leq \frac{1}{2ab} \|w\|_{L^2}^2.
\]
and, Gronwall Lemma gives
\[
\|w\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 + \frac{a}{2} \int_0^t \int_{\mathbb{R}^3} \left( (e^{b|u|^4} - 1) + (e^{b|v|^2} - 1) \right) |w|^2 \leq \|w^0\|_{L^2}^2 e^{\frac{at}{2}}.
\]
As \( w^0 = 0 \), then \( w = 0 \) and \( u = v \). Which implies the uniqueness.

3.4. Asymptotic Study of the Global Solution.
In this subsection we prove the asymptotic behavior (1.2). For this we prove some preliminaries lemmas:

**Lemma 3.1.**
*If \( u \) is a global solution of \((S)\), then \( (e^{b|u|^4} - 1)u \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\).*
Proof
If \( X_1 = \{(t, x) : |u(t, x)| \leq 1\} \) and \( X_2 = \{(t, x) : |u(t, x)| > 1\} \), we have
\[
\int_0^\infty \|(e^{b|u(s,.)|^4} - 1)u(s,.)\|_{L^1} ds = K_1 + K_2
\]
where
\[
K_1 = \int_{X_1} (e^{b|u(s,.)|^4} - 1)|u(s, x)| dx ds \quad \text{and} \quad K_2 = \int_{X_2} (e^{b|u(s,.)|^4} - 1)|u(s, x)| dx ds.
\]

K_1 = \int_{X_1} (e^{b|u(s,.)|^4} - 1)|u(s, x)| dx ds
\[
= \int_{X_1} b|u(s, x)| \left( \frac{e^{b|u(s,.)|^4} - 1}{b|u(s, x)|^4} \right) |u(s, x)|^{\frac{4}{b}} dx ds
\leq b(e^b - 1) \int_0^\infty \int_{\mathbb{R}^3} |u(s, x)|^{\frac{4}{b}} dx ds \leq be^b \int_0^\infty \|u(s, .)\|^\frac{4}{b} ds.
\]

By using the Sobolev injection \( \dot{H}^{\frac{2}{b}}(\mathbb{R}^3) \hookrightarrow L^{\frac{4}{b}}(\mathbb{R}^3) \), we get
\[
(3.5) \quad K_1 \leq C \int_0^\infty \|u(s, .)\|^\frac{4}{b} ds.
\]

By interpolation inequality \( \|u(s)\|_{\dot{H}^{\frac{2}{b}}} \leq \|u(s)\|_{\dot{H}^0}^{\frac{2}{b}} \|u(s)\|_{\dot{H}^{\frac{2}{b}}}^{\frac{2}{b}} \), we obtain
\[
(3.6) \quad K_1 \leq C \int_0^\infty \|u(s, .)\|_{L^2}^{\frac{2}{b}} \|\nabla u(s)\|_{L^2}^{2} \leq C \|u^0\|_{L^2}^{\frac{2}{b}} \int_0^\infty \|\nabla u(s)\|_{L^2}^{2}.
\]

For the term \( K_2 \), we have
\[
K_2 = \int_{X_2} (e^{b|u(s,.)|^4} - 1)|u(s, x)| dx ds \leq \int_{\mathbb{R}^3} (e^{b|u(s,.)|^4} - 1)|u(s, x)|^2 dx ds.
\]
Hence
\[
\|(e^{b|u|^4} - 1)u\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^3)} \leq C \|u^0\|_{L^2}^{\frac{4}{b}} \int_0^\infty \|\nabla u(s, .)\|_{L^2}^2 ds + \int_0^\infty \int_{\mathbb{R}^3} (e^{b|u(s,.)|^4} - 1)|u(s, x)|^2 dx ds.
\]
Therefore \( (e^{b|u|^4} - 1)u \in L^1(\mathbb{R}^+ \times \mathbb{R}^3) \).

Lemma 3.2.
If \( u \) is a global solution of (1.1), then \( \lim_{t \to \infty} \|u(t)\|_{H^{-2}} = 0 \).

Proof
Let \( \varepsilon > 0 \). By the energy inequality (1.1) and Lemma 3.1, there exists \( t_0 \geq 0 \) such that
\[
\|\nabla u\|_{L^2([t_0, \infty) \times \mathbb{R}^3)} < \frac{\varepsilon}{4},
\]
(3.7)
\[
\|(e^{b|u|^4} - 1)u\|_{L^1([t_0, \infty) \times \mathbb{R}^3)} < \frac{\varepsilon}{4}.
\]
(3.8)
By the Duhamel formula, 
\[ v(t, x) = e^{\Delta t}v^0(x) + f(t, x) + g(t, x), \]
where
\[ f(t, x) = -\int_0^t e^{(s-t)\Delta} F \div (v \otimes v)(s, x) \, ds \]
and
\[ g(t, x) = -\int_0^t e^{(s-t)\Delta} F \div (e^{b|v_0|^2 - 1})v(s, x) \, ds. \]

By the Dominated Convergence Theorem, we have: \( \lim_{t \to \infty} \|e^{t\Delta}v^0\|_{L^2} = 0 \) and hence \( \lim_{t \to \infty} \|e^{t\Delta}v^0\|_{H^{-2}} = 0 \). Moreover, we have
\[
\|f(t)\|_{H^{-2}}^2 \leq \int_{\mathbb{R}^3} |\xi|^{-1} \left( \int_0^t e^{-(s-t)}|\xi|^2 \left| F \div (v \otimes v)(s, \xi) \right| ds \right)^2 d\xi 
\leq \int_{\mathbb{R}^3} |\xi| \left( \int_0^t e^{-(s-t)}|\xi|^2 \left| F(v \otimes v)(s, \xi) \right| ds \right)^2 d\xi.
\]
As
\[
\left( \int_0^t e^{-(s-t)}|\xi|^2 \left| F(v \otimes v)(s, \xi) \right| ds \right)^2 \leq \left( \int_0^t e^{-2(s-t)}|\xi|^2 ds \right) \int_0^t |F(v \otimes v)(s, \xi)|^2 ds 
\leq |\xi|^{-2} \int_0^t |F(v \otimes v)(s, \xi)|^2 ds,
\]
we obtain
\[
\|f(t)\|_{H^{-2}}^2 \leq \int_{\mathbb{R}^3} |\xi|^{-1} \int_0^t |F(v \otimes v)(s, \xi)|^2 ds d\xi 
\leq \int_0^t \int_{\mathbb{R}^3} |\xi|^{-1} |(v \otimes v)(s, \xi)|^2 d\xi ds = \int_0^t \|v \otimes v(s)\|_{H^{-2}}^2 ds.
\]
Using the product law in homogeneous Sobolev spaces, with \( s_1 = 0, s_2 = 1 \), we get
\[
\|f(t)\|_{H^{-2}}^2 \leq C \int_0^t \|v(s)\|_{L^2}^2 \|\nabla v(s)\|_{L^2}^2 ds.
\]
Using inequalities (3.7) and (3.8), we get
\[
\|f(t)\|^2_{H^{-2}} dt \leq C\|u_0\|^2_{L^2} \int_0^t \|\nabla u(t_0 + s)\|^2_{L^2} ds
\leq C\|u_0\|^2_{L^2} \int_0^\infty \|\nabla u(t_0 + s)\|^2_{L^2} ds
\leq C\|u_0\|^2_{L^2} \int_0^\infty \|\nabla u(s)\|^2_{L^2} ds
\leq C\|u_0\|^2_{L^2} \frac{\varepsilon^2}{9C\|u_0\|^2_{L^2} + 1},
\]
which implies
\[
\|f(t)\|_{H^{-2}} < \frac{\varepsilon}{3}, \forall t \geq 0.
\]
For an estimation of \(\|g(t)\|_{H^{-2}}\) and by using (2.6) with \(s = 2\), we get
\[
\|g(t)\|^2_{H^{-2}} dt \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-2} \left( \int_0^t e^{-(t-s)}|\xi|^2 |\mathcal{F}(e^{b|\xi|^4} - 1)v(s, \xi)| ds \right)^2 d\xi
\leq C \left( \int_0^t \| (e^{b|\xi|^4} - 1)v(s)\|^2_{L^1(\mathbb{R}^3)} ds \right)^2
\leq C \| (e^{b|\xi|^4} - 1)v\|^2_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)},
\]
where \(C = \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-2} d\xi\). Also by using inequality (3.8), we get
\[
\|g(t)\|^2_{H^{-2}} dt \leq C \| (e^{b|u(t_0+)|^4} - 1)u(t_0 + .)\|^2_{L^1(\mathbb{R}^+ \times \mathbb{R}^3)}
\leq C \| (e^{b|u|^4} - 1)u\|^2_{L^1([t_0, \infty) \times \mathbb{R}^3)} \leq C \frac{\varepsilon^2}{9C},
\]
which implies that \(\|g(t)\|_{H^{-2}} < \frac{\varepsilon}{3}, \forall t \geq 0\).
Combining the above inequalities, we obtain
\[
\lim_{t \to \infty} \|u(t)\|_{H^{-2}} = 0.
\]

**Lemma 3.3.**
If \(u\) is a global solution of (5), then \(\lim_{t \to \infty} \|u(t)\|_{L^2} = 0\).

**Proof**
We have \(u = w_1 + w_2\), where
\[
w_1 = 1_{|D|<1}u = \mathcal{F}^{-1}(1_{|\xi|<1} \hat{u}), \quad w_2 = 1_{|D|\geq1}u = \mathcal{F}^{-1}(1_{|\xi|\geq1} \hat{u}).
\]
By the second step, we get
\[
\|w_1(t)\|_{L^2} = c_0\|w_1(t)\|_{H^0} \leq 2c_0\|w_1(t)\|_{H^{-2}} \leq 2\|u(t)\|_{H^{-2}},
\]
which implies
\[
\lim_{t \to \infty} \|w_1(t)\|_{L^2} = 0.
\]
Let \(\varepsilon > 0\). There is a time \(t_1 > 0\) such that
\[
\|w_1(t)\|_{L^2} < \frac{\varepsilon}{2}, \forall t \geq t_1.
\]
We have
\[
\int_{t_1}^{\infty} \|w_2(t)\|_{L^2}^2 \, dt \leq \int_{t_1}^{\infty} \|\nabla w_2(t)\|_{L^2}^2 \, dt \leq \int_{t_1}^{\infty} \|\nabla u(t)\|_{L^2}^2 \, dt < \infty.
\]
As \( t \to \epsilon \), \( \|w_2(t)\|_{L^2} \) is continuous, then there is a time \( t_2 \geq t_1 \) such that
\[
\|w_2(t_2)\|_{L^2} < \frac{\epsilon}{2}.
\]
Particularly
\[
\|u(t_2)\|_{L^2}^2 = \|w_1(t_2)\|_{L^2}^2 + \|w_2(t_2)\|_{L^2}^2 < \frac{\epsilon^2}{2}.
\]
By using the following energy estimate
\[
\|u(t)\|_{L^2}^2 + 2 \int_{t_2}^{t} \|\nabla u(s)\|_{L^2}^2 \, ds + 2a \int_{t_2}^{t} (e^{B|u(s)|^2 - 1}) |u(s)|^2 \, ds \leq \|u(t_2)\|_{L^2}^2, \quad \forall t \geq t_2,
\]
we get
\[
\|u(t)\|_{L^2} < \epsilon, \quad \forall t \geq t_2,
\]
and the proof is completed.

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