Classical theory for non-perturbative dynamics in hot QCD

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I present an effective classical theory which describes the non-perturbative real-time dynamics in hot gauge theories and has the potential for numerical implementation.

I. INTRODUCTION

At high temperature, the naïve perturbation theory is well known to break down at soft momenta because of large collective effects. During the last decades, there has been a considerable effort toward the understanding of such effects and their resummation into effective perturbative expansions, with a better infrared behaviour. (See Refs. [1,2] for some recent reviews and more references.) In particular, it has been recognized, by Braaten and Pisarski [3], that the gauge-invariant resummation of the collective phenomena at the soft scale gT involves the whole series of “hard thermal loops” (HTL’s): these are one-loop corrections to the amplitudes of the soft (k ≲ gT ≪ T) fields due to their scattering off the hard (k ∼ T) plasma particles. (See also Refs. [4,5].)

As well known, however, infrared (IR) problems persist even after resumming the HTL’s: in general, they are associated with (quasi)static magnetic gluons, or photons, which are not screened by the collective phenomena at the scale gT.

In Abelian theories, we encounter only mass-shell divergences, as, for example, in the calculation of the fermion damping rate [6]: the divergences occur because a charged particle which is nearly on-shell can emit or absorb an arbitrary number of soft, (quasi)static, magnetic photons (the latter are virtual quanta, so they actually describe collisions in the plasma). The kinematics is very similar to the old “IR catastrophe” of zero-temperature QED [7], and, as a matter of facts, the divergences can be indeed removed [8,9] by an all-orders resummation of the soft photons, à la Bloch-Nordsieck [10].

In QCD, the IR problems are more complicated, because of the mutual interactions of the magnetic gluons [6]. Perturbation theory breaks down at the scale g^2T because of IR divergences. Specifically, the usual connection between the powers of the coupling constant and the number of loops is lost: starting with some given order in g (e.g., O(g^6) in the case of the pressure [11]), one finds diagrams with any number of loops which contribute at the same order. In some cases, perturbation theory breaks down already at leading order: this is what happens for genuinely non-perturbative phenomena like the static magnetic screening [12] or the anomalous violation of the baryon number in the hot electroweak theory [13,14] (see also below).

For static quantities like the pressure, there exists a well defined procedure for going beyond the perturbation theory: this is lattice QCD at finite temperature [15], possibly supplemented by dimensional reduction [16,17]. But the standard lattice simulations, as formulated in imaginary time, are not well suited for computing real-time correlation functions, like the baryon number violation alluded to before. It is my purpose in this paper to describe a semi-classical approximation which correctly reproduces the non-perturbative real-time dynamics to lowest order in g and has the potential for numerical implementation (a concise presentation of this method has been previously given in Ref. [18]).

II. THE CLASSICAL APPROXIMATION

Let me first explain why one expects the classical approximation to be appropriate for the problem at hand.

The basic idea is not new (see e.g. [13]): the non-perturbative phenomena are associated with long wavelength (λ ≳ 1/g^2 T) magnetic fields, which, because of the Bose-Einstein enhancement, have large occupation numbers:

\[ N_0(k) \equiv \frac{1}{e^{\hbar k/T} - 1} \simeq \frac{T}{k} \gg 1 \quad \text{for} \quad k \ll T, \quad (2.1) \]

and should therefore exhibit a classical behaviour. To see this in a simple way, let us put back Planck’s constant \( \hbar \) in Eq. (2.1) and compute the average energy per mode in thermal equilibrium:

\[ \varepsilon(k) = \frac{\hbar k}{e^{\hbar k/T} - 1} \simeq T \quad \text{as} \quad \hbar k \ll T. \quad (2.2) \]

As \( \hbar \to 0 \), we recover the classical equipartition theorem, as expected. But Eq. (2.2) also shows that the relevant inequality is \( \hbar k \ll T \), so that the classical limit (\( \hbar \to 0 \))
0 at fixed $k$ and $T$) is actually equivalent to the soft momentum limit ($k \to 0$ at fixed $\hbar$ and $T$).

This observation is useful since we know how to perform real-time lattice simulations for a classical thermal field theory: All we have to do is to solve the classical equations of motion for given initial conditions, and then average over the classical phase space with the Boltzmann weight $\exp(-\beta H)$. Since the initial conditions (say $\phi(x)$ and $\dot{\phi}(x)$ for a scalar theory) depend only on the spatial coordinate $x$, the phase space integration is actually a three-dimensional functional integral, which can be implemented on a lattice in the standard way [19].

On the other hand, the classical approximation is certainly not appropriate at high momenta, $k \gtrsim T$, and this sets limits on its applicability (see below).

To be more specific, let me consider classical Yang-Mills theory at finite temperature. In the temporal gauge $A_{t0} = 0$, the independent degrees of freedom are the vector potentials $A_i^a(x)$ and the electric fields (“the canonically conjugate momenta”) $E_i^a(x)$, with the Hamiltonian:

$$H_{YM} = \frac{1}{2} \int d^3x \{E_a \cdot E_a + B_a \cdot B_a \}.$$  \hspace{1cm} (2.3)

In terms of these, the classical equations of motion read:

$$\partial_0 A_i^a = -E_i^a, \quad \partial_0 E_i^a = \epsilon_{ijk} (D_j B_k)^a,$$ \hspace{1cm} (2.4)

together with Gauss’ law which in this gauge must be imposed as a constraint:

$$\mathbf{D} \cdot \mathbf{E} = 0.$$ \hspace{1cm} (2.5)

The thermal phase space is defined by the initial conditions for Eqs. (2.4), which I denote with calligraphic letters: $\mathcal{E}_i^a(x)$ and $\mathcal{A}_i^a(x)$. Then, the canonical partition function reads:

$$Z_{YM} = \int \mathcal{D}\mathcal{E}_i^a(x) \mathcal{D}\mathcal{A}_i^a(x) \delta(\mathbf{D} \cdot \mathbf{E}) \exp(-\beta H_{YM}),$$ \hspace{1cm} (2.6)

where $H_{YM}$ is expressed in terms of the initial fields, cf. Eq. (2.3). The classical thermal correlation functions can be obtained from the following generating functional:

$$Z_{YM}[J_i^a] = \int \mathcal{D}\mathcal{E}_i^a \mathcal{D}\mathcal{A}_i^a \delta(\mathbf{D} \cdot \mathbf{E}) \times \exp \left\{ -\beta H_{YM} + \int d^4x J_i^a(x) A_i^a(x) \right\},$$ \hspace{1cm} (2.7)

where $A_i^a(x)$ is the solution to Eqs. (2.4) with the initial conditions $\{\mathcal{E}_i^a, \mathcal{A}_i^a\}$. For instance, the free ($g = 0$) two-point correlation is obtained as:

$$D_{ij}^{0\,cl}(k) = (\delta_{ij} - \hat{k}_i \hat{k}_j) \rho_0(k_0, k) N_{cl}(k_0),$$

$$\rho_0(k_0, k) \equiv 2\pi \epsilon(k_0) \delta(k_0^2 - k^2), \quad N_{cl}(k_0) \equiv \frac{T}{k_0},$$ \hspace{1cm} (2.8)

which is simply the soft (or classical) limit of the corresponding Wightman functions $\mathcal{D}_{ij}(k) = D_{ij}^{0\,cl}(k)$ or $D_{ij}^{\neq}(k) = D_{ij}^{0\,cl}(k)$. Other thermal expectation values are defined similarly. For instance, the study of the baryon number violation at high temperature [13] requires the quantity $\langle (\Delta B(t))^2 \rangle$, where

$$\Delta B(t) \propto \int_0^t dt_0 \int d^3x F_{\mu\nu}^a F_{\mu\nu}^a,$$ \hspace{1cm} (2.9)

(with $F_{\mu\nu}^a \equiv (1/2)\varepsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}^a$) is proportional to the change in the Chern-Simons number. And actually there have been attempts to compute $\langle (\Delta B(t))^2 \rangle$ via lattice simulations of the equations above [19].

Recall, however, that the classical description is justified only at soft momenta (cf. Eqs. (2.1) and (2.2)), which greatly restricts its applicability. In general, equations like (2.6) or (2.7) will be afflicted with ultraviolet (UV) divergences which occur because of the replacement of the correct, quantum distribution $N_0(k) = 1/(e^{\beta k} - 1)$ with the classical one $N_{cl}(k) = T/k$ (cf. Eq. (2.7)). Recall, for instance, the famous “ultraviolet catastrophe” of Rayleigh and Jeans: The classical estimate for the energy density of the black body radiation, namely ($\Lambda$ is an ad-hoc UV cutoff):

$$E_{cl}/V = \int \frac{d^3k}{(2\pi)^3} \frac{k}{e^{\beta k} - 1} = \frac{\pi^2 T^4}{30},$$ \hspace{1cm} (2.10)

is obviously wrong (since UV divergent), in contrast to the quantum result:

$$E/V = \int \frac{d^3k}{(2\pi)^3} \frac{k}{e^{\beta k} - 1} = \frac{\pi^2 T^4}{30},$$ \hspace{1cm} (2.11)

which is finite because the large momenta $k \gg T$ are exponentially suppressed by the Bose-Einstein thermal distribution.

Similarly, the Debye screening mass is linearly UV divergent when computed in the classical theory. Recall, indeed, the correct, quantum, expression for the Debye mass to leading order in $g$ [2]:

$$m_D^2 = -\frac{g^2 C_A}{\pi^2} \int_0^\infty dk k^2 \frac{dN_0}{dk} = \frac{g^2 C_A T^2}{3},$$ \hspace{1cm} (2.12)

which is replaced by:

$$m_D^2(\Lambda) = -\frac{g^2 C_A}{\pi^2} \int_0^\Lambda dk k^2 \frac{dN_{cl}}{dk} = \frac{g^2 C_A T \Lambda}{\pi^2},$$ \hspace{1cm} (2.13)

where the UV cutoff $\Lambda$ has been introduced as an upper momentum cutoff.

\footnote{Recall the definition of the Wightman functions $\mathcal{D}^\neq(k) \equiv \rho(k)[1 + N_0(k_0)]$ and $D^{\neq}(k) \equiv \rho(k)N_0(k_0)$.}
Thus, the classical approximation, as formulated above, makes sense only for quantities which (at least, to some approximation) are not sensitive to the hard \((k \sim T)\) plasma modes. Such quantities exist indeed: the magnetic screening mass is an example. More generally, all the static, non-perturbative, correlation functions of the magnetic fields are correctly reproduced by the classical Yang-Mills theory, to lowest order in \(g\). (In higher orders, the quantum corrections to the thermal distribution start to play a role.)

The easiest way to see this is to note that, in its magnetic sector, the classical Yang-Mills theory is the same as three-dimensional QCD, a theory which is well known to describe the leading IR behaviour at high-\(T\) (see, e.g., [12,17]). Indeed, Eq. (2.6) can be rewritten as:

\[
Z_{YM} = \int \mathcal{D}A^a_0 \mathcal{D}A_i^a \exp \left\{ -\frac{\beta}{2} \int d^3x \left( B_i^a B_i^a + (D_i A_0)^2 \right) \right\}, \tag{2.14}
\]

where the \(A_0^a\) components of the gauge fields have been reintroduced as Lagrange multipliers to enforce Gauss’ law. As anticipated, the magnetic sector in Eq. (2.14) is equivalent to three-dimensional QCD with dimensionful coupling constant \(g_3^2 = g^2 T\). Incidentally, this shows that the “dimensional reduction” at finite temperature (cf. [16,17] and Refs. therein) is a reflection of the classical character of the leading IR dynamics. Thus, the various lattice investigations of QCD in the literature (e.g., the calculation of the magnetic mass [20], or of the non-perturbative correction, of \(O(g^6)\), to the pressure [21]) can be also interpreted as classical calculations.

On the other hand, we shall shortly see that the classical Yang-Mills theory is not appropriate for the time-dependent non-perturbative correlations, which remain sensitive to the hard plasma modes. Accordingly, one expects [14,22] the aforementioned classical studies of the baryon number violation [14] not to be reliable.

### III. THE ROLE OF LANDAU DAMPING

The reason for such a failure is that the time-dependent magnetic fields are sensitive to the hard thermal modes, via Landau damping [1]. To see this, consider the propagator of a soft magnetic gluon in the HTL approximation (that is, after resumming the effects of the hard modes to leading order in \(g\)). This reads \(D_{ij}(\omega, k) = (\delta_{ij} - \hat{k}_i \hat{k}_j) \Delta_T(\omega, k)\), where

\[
\Delta_T(\omega, k) \equiv -\frac{1}{\omega^2 - k^2 - \Pi_T(\omega, k)}, \tag{3.1}
\]

\(3\)The retarded prescription is implicit here: \(\omega \equiv \omega^+\).

and \(\Pi_T(\omega, k)\) is the corresponding HTL.

In the static limit, \(\Pi_T(0, k) = 0\), and there is no sensitivity to the hard modes. More generally, all the HTL corrections vanish in the static limit, except for the term yielding Debye screening [12].

For non-zero \(\omega\), \(\Pi_T(\omega, k)\) is parametrically of the order \(m_D^2(\omega/k)\), which is of \(O(g^2 T^2)\) if \(\omega \sim k\): that is, for large enough frequencies, the time-dependent magnetic fields are screened as efficiently as the electric ones, and decouple from the non-perturbative physics.

As \(\omega \to 0\), the polarization tensor is dominated by its imaginary part, which describes the absorption of the soft gluon by a hard thermal particle (Landau damping):

\[
\Pi_T(\omega \ll k) \simeq -i \frac{\pi}{4} m_D^2 \frac{\omega}{k}. \tag{3.2}
\]

Accordingly, the magnetic propagator at \(\omega \ll k\) reads:

\[
\Delta_T(\omega \ll k) \simeq \frac{1}{k^2 - i (\pi \omega/4k) m_D^2}, \tag{3.3}
\]

which strongly suggests that non-perturbative phenomena can be associated only with frequencies which are low enough: \(\omega \lesssim k^3/m_D^2\). For \(k \sim g^2 T\), this implies \(\omega \lesssim g^4 T\). That is, the non-perturbative physics involves very soft and quasistatic magnetic fields, with spatial momenta of \(O(g^2 T)\) and even softer frequencies, of \(O(g^4 T)\) [4]. This conclusion is further supported by the power counting analysis in Ref. [4].

Note that the inclusion of the hard modes effects (in the form of the HTL) was essential for the previous analysis. In particular, the typical time scale for non-perturbative phenomena \(\tau \sim m_D^2/k^3 \sim 1/(g^4 T)\) proportional to the Debye mass [4], a quantity which is dominated by the hard modes (cf. Eq. (2.12)). We conclude that the dynamics of the time-dependent, non-perturbative, magnetic fields is strongly influenced by their scattering off the hard, quantum, modes, and cannot be described within the classical Yang-Mills theory.

But this does not mean that the classical approximation by itself is totally useless: we just have to be careful to apply it to the soft fields alone. That is, we have to treat hard and soft modes separately [22]: The hard \((k \sim T)\) modes are quantum but perturbative, so they can be explicitly integrated over in perturbation theory, to obtain an effective theory for the soft \((k \lesssim g T)\) modes. The resulting effective theory is non-perturbative, but for it the classical approximation is valid, so it can be studied numerically on a classical, three-dimensional, lattice.

An essential step in this strategy is to clearly distinguish between hard and soft degrees of freedom (to avoid overcounting, and to provide an ultraviolet cutoff to the effective theory for the soft modes). Loosely speaking, this requires an intermediate scale \(\Lambda\), with \(g T \ll \Lambda \ll T\), which should act as an infrared cutoff for the hard modes and as an ultraviolet cutoff for the soft ones, and which should cancel in the calculation of physical quantities.

To lowest order in \(g\), the effective theory for the soft modes is essentially known: this is the HTL-resummed...
theory by Braaten and Pisarski [11], which includes all the one-loop amplitudes with soft external lines and hard loop momenta. And actually it has been already proposed by Bödeker, McLerran and Smilga [23] to use the HTL effective theory as a classical theory for non-perturbative calculations in real time. However, the practical implementation of this suggestion met with several technical difficulties: (i) There is no intermediate scale $\Lambda$ in the original formulation of the HTL theory: indeed, this was not necessary for perturbative calculations, where one can avoid overcounting by adding and subtracting the HTL Lagrangian to, and respectively, from, the tree-level Lagrangian [3]. (ii) The HTL Lagrangian is non-local [3], so it is a priori not obvious how to construct the associated classical thermal field theory (cf. the discussion in Sec. II).

Actually, problem (ii) above can be overcome by using one of the local formulations of the HTL theory [3]. In what follows, I shall use the formulation in terms of kinetic equations, as developed in Refs. [3] (see also [3]). This has the advantage to clearly emphasize the physical content of the HTL’s (see also [3]). Besides, we shall see later that problem (i) above — namely, the implementation of the intermediate scale $\Lambda$ — can also be solved in the framework of the kinetic theory [3].

IV. KINETIC EQUATIONS FOR HTL’S

I consider a purely Yang-Mills plasma in thermal equilibrium at a temperature $T$ which is high enough for the coupling constant to be small: $g \ll 1$. The typical excitations are “hard” gluons, with momenta $k \approx T$. Such gluons can develop a collective behaviour over a typical space-time scale $\sim 1/gT$, which is large as compared to the mean interparticle distance $\sim 1/T$. This results in long-wavelength ($\lambda \sim 1/gT \gg 1/T$) colour oscillations which are most economically described in terms of kinetic equations [3]. In these equations, the hard ($k \sim T$) gluons are represented by their average colour density $\delta N_a(k,x)$ to which couple the soft (i.e., long-wavelength) colour fields $A_\mu^a(x)$. The relevant equations read:

\[
(D_\nu F^{\nu\mu})_a(x) = 2gC_A \int \frac{d^3k}{(2\pi)^3} v^\mu \delta N_a(k,x),
\]

\[
(v \cdot D_x)_{ab} \delta N_b(k,x) = -g \mathbf{v} \cdot \mathbf{E}_a(x) \frac{dN_0}{dk},
\]

where $D_\mu = \partial_\mu + igA^a_\mu T_a$ is the covariant derivative, $E^a_\mu \equiv F^a_\mu$ is the chromoelectric field, and $v^\mu \equiv (1, \mathbf{v})$, with $v = k/k$ denoting the velocity of the hard particles ($k = |k|$), so that $|\mathbf{v}| = 1$.

The Yang-Mills equation above involves the colour current induced by the hard particles:

\[
J_0^\mu(x) = 2gC_A \int \frac{d^3k}{(2\pi)^3} v^\mu \delta N_a(k,x).
\]

By solving the second Eq. (4.1) (which may be seen as a non-Abelian generalization of the familiar Vlasov equation [32]), we can express this current in terms of the gauge fields $A_\mu^a$, and thus obtain an effective Yang-Mills equation which involves the soft fields alone:

\[
D_\nu F^{\nu\mu} = m_\Delta^2 \int \frac{d\Omega}{4\pi} v^\mu W_a(x,\mathbf{v}).
\]

Here, the angular integral $\int d\Omega$ runs over the orientations of $\mathbf{v}$, and $m_\Delta$ is the Debye mass, Eq. (2.12).

Eq. (4.3) describes the propagation of soft colour fields in the high-$T$ plasma. The hard particles are not explicit anymore, since they have been integrated to yield the induced current in the r.h.s. By expanding this current in powers of the gauge fields one generates all the HTL’s of Braaten and Pisarski. However, because of the non-local structure of this current (note the covariant derivative in the denominator), Eq. (4.3) is not very convenient for the construction of the thermal partition function [3].

To this aim, it is preferable to go one step back and replace Eq. (4.3) with the coupled system (4.1) of local equations. There is a price to be payed for that: in addition to the gauge fields $A_\mu^a(x)$, the local description in Eq. (4.1) also involves the average colour density $\delta N_a(k,x)$, which can be seen as an “auxiliary field”. Still, when working with a local theory, we are in a better position to look for a Hamiltonian formulation, as I discuss now.

V. THE CLASSICAL EFFECTIVE THEORY

The first step is to recognize, on the second Eq. (4.3), that the $\mathbf{v}$ and $k$-dependence can be factorized in $\delta N^a(k,x)$ by writing:

\[
\delta N^a(x,k) = -gW_a(x,\mathbf{v}) \frac{dN_0}{dk}.
\]

The new functions $W^a(x,\mathbf{v})$ satisfy the equation:

\[
(v \cdot D_x)_{ab} W^b(x,\mathbf{v}) = \mathbf{v} \cdot \mathbf{E}_a(x),
\]

which is independent of $k$ since the hard particles move at the speed of light: $|\mathbf{v}| = 1$. By using Eq. (5.1), the induced current can be written as:

\[
J_0^\mu(x) = m_\Delta^2 \int \frac{d\Omega}{4\pi} v^\mu W_a(x,\mathbf{v}),
\]

where the radial integration has been explicitly worked out to yield the Debye mass (cf. Eq. (2.12)).

The Hamiltonian formulation of the effective theory [3] involves the auxiliary fields $W_a(x,\mathbf{v})$ together with the soft gauge fields $A_\mu^a(x)$. [Note that a different Hamiltonian formulation of the HTL theory has been previously given by Nair [4], in terms of some other auxiliary fields.] In the temporal gauge $A_0^a = 0$, the independent
degrees of freedom are \( E_i^a \), \( A_i^a \) and \( W^a \), and the corresponding equations of motion can be read from Eqs. (4.1), (4.2) and (5.3) above. Specifically:

\[
E_i^a = -\partial_0 A_i^a,
\]
\[
-\partial_0 E_i^a + \epsilon_{ijk}(D_j B_k)^a = m^2_D \int \frac{d\Omega}{4\pi} v_i W^a(x, v),
\]
\[
(\partial_0 + v \cdot D)_{ab} W_b = v \cdot E^a,
\]
(5.4)

together with the constraint expressing Gauss’ law:

\[
G^a(x) \equiv (D \cdot E)^a + m^2_D \int \frac{d\Omega}{4\pi} W^a(x, v) = 0. 
\]
(5.5)

Note that Eqs. (5.4) are not in canonical form: this is already obvious from the fact that we have an odd number of equations. Still, it can be verified that these equations are conserved: the corresponding, conserved energy functional has the gauge-invariant expression [1]:

\[
H = \frac{1}{2} \int d^3x \left\{ E_a \cdot E_a + B_a \cdot B_a + m^2_D \int \frac{d\Omega}{4\pi} W_a(x, v) W_a(x, v) \right\}. 
\]
(5.6)

Moreover, in the gauge \( A_0^a = 0 \), the functional (5.6) acts also as a Hamiltonian, that is, as a generator of the time evolution. To see this, we need the following generalized Poisson brackets, as first introduced by Nair [1]:

\[
\{ E_i^a(x), A_j^b(y) \} = -\delta^{ab}\delta_{ij}\delta^{(3)}(x - y),
\]
\[
\{ E_i^a(x), W^b(y, v) \} = v_i \delta^{ab}\delta^{(3)}(x - y),
\]
\[
m^2_D \{ W^a(x, v), W^b(y, v') \} = \left( gf^{abc}W^c + (v \cdot D_v)^{ab} \right) \delta^{(3)}(x - y)\delta(v, v'). 
\]
(5.7)

Here, \( \delta(v, v') \) is the delta function on the unit sphere:

\[
\int \frac{d\Omega}{4\pi} \delta(v, v') f(v) = f(v'), 
\]
(5.8)

and all the other Poisson brackets are assumed to vanish. We also assume standard properties for such brackets, namely antisymmetry, bilinearity and Leibniz identity. It is then easy to verify that (a) the Poisson brackets (5.7) satisfy the Jacobi identity (as necessary for consistency) and (b) the equations of motion (5.4) follow as canonical equations for the Hamiltonian (5.6). For instance, \( \partial_0 W^a = \{ H, W^a \} \), and similarly for \( E_i^a \) and \( A_i^a \).

Note also that the Hamiltonian in Eq. (5.3) is remarkably simple: the piece involving the auxiliary fields \( W^a \) is simply quadratic! In fact, all the dynamical complications are hidden in the non-trivial Poisson brackets (5.7).

VI. THE THERMAL PARTITION FUNCTION

We are now in position to write down the classical partition function and compute (generally time-dependent) thermal expectation values. The classical phase-space is defined by the initial conditions to Eqs. (6.4), and the thermal weight is provided by the Hamiltonian in Eq. (6.6). The correlation functions of the magnetic fields \( A_i^a \) are then generated by (compare to Eq. (2.7)):

\[
Z_{cl}[J_i^a] = \int D\mathcal{E}_i^a D\mathcal{A}_i^a D\mathcal{W}^a \delta(\mathcal{G}^a) \times \exp \left\{ -\beta H + \int d^4x J_i^a(x) A_i^a(x) \right\}, 
\]
(6.1)

where \( A_i^a(x) \) is the solution to Eqs. (6.4) with the initial conditions \( \{ \mathcal{E}_i^a, \mathcal{A}_i^a, \mathcal{W}^a \} \) (that is, \( E_i^a(t_0, x) = \mathcal{E}_i^a(x) \), etc., with arbitrary \( t_0 \)), and \( \mathcal{G}^a \) and \( H \) are expressed in terms of the initial fields (cf. Eqs. (2.3) and (5.6)).

It can be verified [18] that the phase-space measure \( D\mathcal{E}_i^a D\mathcal{A}_i^a D\mathcal{W}^a \) in Eq. (6.1) is invariant under the time evolution described by Eqs. (6.4), so that \( Z_{cl}[J] \) is independent of the (arbitrary) initial time \( t_0 \), as it should. (This point is not trivial because of the non-canonical structure of the equations of motion.)

Furthermore, strictly speaking, Eq. (6.1) is not well-defined without an UV cutoff (recall the discussion in Secs. II and III). This will be discussed in the next two sections. But before doing that, it might be illuminating to consider Eq. (6.1) in some particular cases.

For instance, for \( J = 0 \), it yields the same thermodynamic potential as the leading-order dimensional reduction [16,17]. Specifically (compare to Eq. (2.14)):

\[
Z_{cl} = \int D\mathcal{A}_0^a D\mathcal{A}_i^a \times \exp \left\{ -\beta \int d^4x \left( \mathcal{B}^2 + (D_v\mathcal{A}_0)^2 + m^2_D \mathcal{A}_0^2 \right) \right\}, 
\]
(6.2)

where the fields \( \mathcal{A}_0^a \) have been introduced as Lagrange multipliers, and the (Gaussian) functional integrals over \( \mathcal{E}_i^a \) and \( \mathcal{W}^a \) have been explicitly performed. In particular, the integral over the auxiliary fields \( \mathcal{W}^a \) has correctly generated the Debye screening mass for the electric fields.

Consider also the Abelian limit, where the classical theory can be exactly solved. Physically, this describes the propagation of soft electromagnetic waves in a hot QED plasma, with the fermions integrated over (in the HTL approximation). In the present approximation, the Abelian theory is quadratic, and does not require any UV cutoff. This is consistent with the fact that, in QED, soft and hard degrees of freedom correspond to different kind of fields (the hard fields are fermions, while the soft ones are photons), so, unlike QCD, there is no danger of overcounting.

A straightforward, if lengthy, calculation yields (see the Appendix for details):

\[
Z_{cl}[J_i] = \exp \left\{ -\frac{1}{2} \int d^4x d^4y J_i(x) D_i^{cl}(x - y) J_j(y) \right\}, 
\]
(6.3)

with (compare to Eq. (2.3)).
\[ D^\text{cl}_{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \frac{T}{k_0} \rho_{ij}(k). \]  

(6.4)

Here, \( \rho_{ij}(k) \) is the photon spectral density in the HTL approximation and in the temporal gauge:

\[ \rho_{ij}(k_0, k) = (\delta_{ij} - \hat{k}_i \hat{k}_j) \rho_T(k_0, k) + \frac{k_i k_j}{k_0^2} \rho_L(k_0, k), \]  

(6.5)

with the (transverse and longitudinal) spectral functions \( \rho_{T,L} \) defined as standard \[3\]. They include both delta functions for on-shell quasiparticles and off-shell pieces corresponding to Landau damping. For instance (cf. Eq. (3.1)):

\[ \rho_T(k_0, k) \equiv 2 \text{Im} \Delta_T(k_0 + i\eta, k) = 2\pi \epsilon(k_0) z_T(k) \delta(k_0^2 - \omega_T^2(k)) + \beta_T(k_0, k), \]  

(6.6)

where \( z_T(k) \) denotes the residue of the (time-like) poles at \( k_0 = \pm \omega_T(k) \), and \( \beta_T(k_0, k) \propto \text{Im} \Pi_T(k_0, k) \) has support only at space-like momenta, \( k^2 < k_0^2 \).

More generally, in QCD, \( D^\text{cl} \) will enter as a propagator in the perturbative expansion of Eq. (6.1).

VII. THE HARD-SOFT MATCHING

As already mentioned, Eq. (6.1) must be supplemented with an UV cutoff \( \Lambda \) which is meant to separate between hard and soft gluons: \( gT \ll \Lambda \ll T \). Correspondingly, the effective Hamiltonian must include some \( \Lambda \)-dependent counterterms, chosen so as to cancel the dependence on \( \Lambda \) in the calculation of physical quantities (a cancellation to be referred to as “matching”). The matching can be done only to one-loop order (since the HTL theory is already an one-loop approximation). But this is enough to remove the most dangerous, linear, UV divergences of the classical theory (see below).

For more clarity, consider first a very simple cutoff procedure: the UV cutoff \( \Lambda \) is implemented as an upper momentum cutoff on the loop integrals of the classical theory\[6\]. (This would be just fine if we were interested in perturbative calculations.) For matching purposes, we need the \( \Lambda \)-dependent corrections to the soft amplitudes, to one-loop order in the effective theory. Such corrections are easy to isolate\[18\]: they are precisely the “HTL’s” of the classical theory. Namely, they differ from the usual HTL’s only by the replacement of the correct Debye mass \( m^2_D(\Lambda) \), Eq. (2.12), with the \( \Lambda \)-dependent classical “Debye mass” \( m^2_D(\Lambda) \), Eq. (2.13).

In this case, the matching is very easy: just replace the physical Debye mass \( m^2_D \) in the definition of the effective theory (that is, in all the equations in Secs. V and VI) with the following \( \Lambda \)-dependent parameter:

\[ m^2_D(\Lambda) \equiv m^2_D - m^2_{cl}(\Lambda) = \frac{g^2 C_A T^2}{3} \left( 1 - \frac{3}{\pi^2} \frac{\Lambda}{T} \right). \]  

(7.1)

Then, to one-loop order, the effective theory generates the correct soft amplitudes, that is, the usual HTL’s with total strength \( m^2_D(\Lambda) + m^2_{cl}(\Lambda) = m^2_D \).

For non-perturbative calculations, however, one cannot use a simple momentum cutoff, as above, but one rather needs an UV cutoff which can be also implemented on the lattice. This cannot be the lattice spacing itself: indeed, a finite (and relatively large) lattice spacing \( a \sim 1/\Lambda \gg 1/T \) will introduce lattice artifacts (essentially, the loss of rotational symmetry) which will make impossible the matching with the hard sector\[23\].

A first solution to this problem have been proposed in Ref. \[18\]: there, a smooth cutoff has been introduced in the classical theory by replacing, in Eq. (6.4),

\[ \text{Tr} B^i B^i \longrightarrow \text{Tr} B^i f \left( \frac{D^2}{\Lambda^2} \right) B^i, \]  

(7.2)

where \( f(z) = 1 + z^2 \) and the trace refers to color indices. In principle, such a higher-derivative cutoff can be indeed implemented on the lattice, as discussed in Ref. \[24\]. Then, the matching can be done (at least for the non-perturbative correlation functions) by adding a \( \Lambda \)-dependent counterterm to the Debye mass, as in Eq. (7.1)\[18\]. In practice, however, this might be quite tedious, as it requires highly improved lattice Hamiltonians\[24\]. Fortunately, there exists a simpler way to perform classical non-perturbative calculations without running into ultraviolet problems. This will be presented in the next section.

VIII. THE EASY WAY

Note first that, within the effective theory, it is easy to distinguish between the gluon fields which are responsible for the non-perturbative IR phenomena and those which are responsible for the leading UV behaviour:

The non-perturbative phenomena are generated by soft and quasistatic magnetic gluons (cf. Sec. III), which are described by the off-shell piece \( = \beta_T \) of the gluon propagator (cf. Eqs. (6.4) and (6.5)).

The UV behaviour, on the other hand, is determined by the relatively hard \( (k \sim \Lambda) \) magnetic gluons, which are essentially on-shell (and, actually, even on the tree-level mass shell). Indeed, for momenta \( k \sim \Lambda \gg m_D \), the HTL corrections are negligible and the classical propagator in Eq. (6.4) reduces to its tree-level expression, Eq. (6.8). In fact, the “classical HTL’s” mentioned in Sec. VII (in relation with the matching problem) have been computed\[23\] by using the free propagator \( D^\text{cl}_0(k) \) for the hard \( (k \sim \Lambda) \) gluons around the loop.

\[4\] By “loops”, I mean here, of course, the classical, thermal, loops, as generated in Eq. (6.4) by the non-linearities of the classical theory.
This suggests that one can avoid the UV problems of the classical theory by subtracting (in a way to be specified shortly) the tree-level piece \( \rho_0(k) \), Eq. (2.8), from the magnetic spectral density \( \rho_T(k) \), Eq. (6.6).

To understand how this can be achieved in practice, recall from Sec. VI that the magnetic two-point function \( D_T^2(k) \equiv (T/k_0)\rho_T(k) \) is obtained by averaging the classical solution over the initial conditions \( \{ \mathcal{A}^i, \xi^i, W \} \) at the (arbitrary) initial time \( t_0 \). By using the equations in the Appendix, it can be verified that, as \( t_0 \to -\infty \), (i) the off-shell piece of the propagator, proportional to \( \beta_T \), is generated by the integral over the colour fluctuations \( \mathcal{W}_a \), while (ii) the on-shell piece arises from the integration over the initial colour fields \( \mathcal{A}^a_0 \) and \( \xi^a_0 \).

This is physically intuitive: (i) the off-shell piece describes Landau damping, so it should arise from the coupling of the soft fields to the long-range colour fluctuations \( \mathcal{W}_a \) of the hard particles; (ii) at high momenta \( k \sim \Lambda \gg m_p \), the one-shell piece goes into the tree-level propagator \( D_0(k) \), which is the same as in the purely Yang-Mills theory (cf. Eqs. (2.7) and (2.8)), and is therefore independent of the \( \mathcal{W}_a \’s \).

This can be also verified by a rapid calculation (see the Appendix for more details): To this aim, it is enough to consider the linearized (or Abelian) version of the effective theory, since this is what determines the propagator. The corresponding equation for the transverse field \( A_T^i(x) \) reads, for \( t_0 > t_0 \) (cf. Eq. (A.14)):

\[
(\partial_t^2 - \nabla^2)A_T^i + \int_{t_0}^{\infty} d^4y \Pi_T(x-y)A_T^i(y) = \xi_T^i(x),
\]

where \( \xi^i \) is determined by the initial fluctuations \( \mathcal{W}(x, v) \), cf. Eq. (A.3):

\[
\xi^i(x) = m_D^2 \int \frac{d\Omega}{4\pi} v^i \mathcal{W}(x - v(x_0 - t_0), v),
\]

If \( t_0 \to -\infty \), Eq. (8.1) can be easily solved by Fourier transform. For vanishing initial conditions, \( A_T^i = \xi_T^i = 0 \), the solution reads:

\[
A_T^i(k) = \Delta_T(k) \xi_T^i(k),
\]

where \( \Delta_T(k) \) is the retarded magnetic propagator in Eq. (6.1). The correlation function:

\[
(A_T^i(k)A_T^j(p)) = (2\pi)^4 \delta^{(4)}(p + k) D_T^{ij}(k)
\]

can then be obtained by performing the functional average over the \( \mathcal{W}_a \’s \). Equivalently, note that, according to Eqs. (6.1) and (6.3), the \( \mathcal{W}_a \’s \) can be seen as Gaussian random variables with the local correlation function:

\[
\langle \mathcal{W}(x, v) \mathcal{W}(y, v') \rangle = (T/m_D^2)^2 \delta^{(3)}(x - y) \delta(v, v'),
\]

This immediately implies:

\[
\langle \xi^i(x)\xi^j(y) \rangle = m_D^2 T \int \frac{d\Omega}{4\pi} v^i v^j \delta^{(3)}(x - y - v(x_0 - y_0)),
\]

and therefore, after some simple algebra,

\[
\tilde{D}_T^2(\omega, k) = -2 (T/\omega) |\Delta_T(k)|^2 \text{Im} \Pi_T(\omega, k) \equiv (T/\omega) \beta_T(\omega, k),
\]

where in writing the last equation I have recognized the imaginary part of the HTL polarization tensor [1].

\[
\text{Im} \Pi_T(\omega, k) = -\pi \omega m_D^2 \int \frac{d\Omega}{4\pi} v^i v^j \delta(\omega - v \cdot k),
\]

as anticipated, this has been generated here by averaging over the initial fluctuations \( \mathcal{W} \), while keeping constant the initial fields \( \mathcal{A}_T \) and \( \xi_T \).

The above discussion suggests the following strategy for non-perturbative calculations in the classical effective theory: The (fully non-linear) equations of motion (5.4) should be solved with the following initial conditions at \( t = t_0 \):

\[
\mathcal{A}^a(x) = 0, \quad \xi^a_T(x) = 0, \quad \mathcal{W}^a(x, v),
\]

\[
\frac{ik\xi^a_L}{T}(x) = m_D^2 \int \frac{d\Omega}{4\pi} \mathcal{W}^a(k, v).
\]

That is, the initial colour fluctuations \( \mathcal{W}^a \) are arbitrary, but all the initial gauge fields are taken to be zero, except for the longitudinal component of the electric field which is fixed by Gauss’ law (6.3). The system is let to evolve for a long time (so as to simulate the limit \( t_0 \to -\infty \)), and then the correlation function of interest (e.g., Eq. (2.9)) is evaluated along the classical solution. This yields the correlation function as a functional of the \( \mathcal{W}_a \’s \), say \( \Omega[\mathcal{W}] \) which is then averaged as follows (below, \( N \equiv 1/\langle 1 \rangle \)):

\[
\langle \Omega \rangle = N \int D\xi_L D\mathcal{W} \delta(\mathcal{G}) \Omega[\mathcal{W}]
\]

\[
\times \exp \left\{ -\frac{\beta}{2} \int d^4x \left( \xi_L^2 + m_D^2 \int \frac{d\Omega}{4\pi} \mathcal{W}^2 \right) \right\}.
\]

Clearly, this equation involves only a restricted phase-space integration (compare to Eq. (6.1)), in the sense that the initial transverse fields are kept constant (and equal to zero). But this is precisely what we need in order to generate the correct IR physics, without being sensitive to the hard classical modes.

More precisely, the perturbative expansion of Eq. (8.10) will involve the reduced transverse propagator \( \tilde{D}_T^2 \), Eq. (6.7), rather than the full classical propagator in Eq. (6.4). The one-loop classical amplitudes constructed with \( \tilde{D}_T^2 \) will be UV finite, since dominated by soft loop momenta \( k \lesssim gT \). For instance, the tadpole diagram constructed from \( \tilde{D}_T^2 \) reads:

\[
\tilde{\Pi}_{tp} \propto \int \frac{d^4k}{(2\pi)^4} \tilde{D}_T^2(k) = T \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \beta_T(\omega, k)
\]

\[
= T \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{k^2} - \frac{2T(k)}{\omega_T^2(k)} \right),
\]

\[
\tilde{\Pi}_{tp} \propto \int \frac{d^4k}{(2\pi)^4} \tilde{D}_T^2(k) = T \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \beta_T(\omega, k)
\]

\[
= T \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{k^2} - \frac{2T(k)}{\omega_T^2(k)} \right),
\]
which is indeed finite (and of $O(m_D T)$) since the integrand in the last equation behaves like $m_D^2/k^4$ at momenta $k \gg g T$ \cite{10}. (To obtain this last equation, I have used the sum rule (A.21) for $\rho_T/\omega$, together with Eq. (6.4).) This should be contrasted with the usual (classical) tadpole, as generated by the full propagator $D^2_T$, which is linearly UV divergent:

$$\Pi_\tau \propto \int \frac{d^4k}{(2\pi)^4} D^2_T(k) = T \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi\omega} \rho_T(\omega, k) = T \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \sim \Lambda T.$$ \hspace{1cm} (8.12)

In general, all the “classical HTL’s” of Sec. VII will be identically zero when computed with the reduced propagator $D^2_T$, so there is no need for matching.

On the other hand, the reduced propagator generates the correct infrared structure, as determined by the off-shell, soft ($k \sim g^2 T$) and quasistatic ($\omega \lesssim g^2 T$) magnetic gluons. In fact, the spectral weight of the non-perturbative gluons is concentrated at low frequencies $\omega \ll k$, so it is entirely contained in $D^2_T$. To see this, recall the low frequency behaviour of $\beta_T/\omega$ (cf. Eq. (3.3)):

$$\frac{\beta_T(\omega \ll k)}{\omega} \sim \frac{\pi}{2} \frac{m_D^2 k}{k^6 + (\pi m_D^2 \omega/4)^2}.$$ \hspace{1cm} (8.13)

For $k \ll m_D$, this is strongly peaked at $\omega = 0$, with a width $\Delta \omega \simeq k^3/m_D^2$. (See Fig. 1 for an illustration.) For $k \sim g^2 T$, this yields $\omega \lesssim g^4 T$. In particular, as $k \to 0$, $\beta_T(\omega, k)/\omega \to (2\pi/k^2)\delta(\omega)$, which is just another way to see the “dimensional reduction” \cite{10}.

**FIG. 1.** The quantity $\rho_T(\omega, k)/\omega$ as a function of $\omega$ for $k = 0.1 m_D$. All the quantities are made adimensional by multiplying them by appropriate powers of $m_D$.

**IX. CONCLUSIONS**

I have presented here a classical field theory for hot QCD which describes the non-perturbative infrared physics to lowest order in $g$ and is well suited for numerical simulations on a lattice. This is an effective theory for the soft gluons (with momenta $k \lesssim g T$), as obtained after integrating out the hard ones ($k \sim T$) in the HTL approximation (i.e., to leading order in $g$). The classical theory has a built-in UV cutoff $\Lambda$ (with $g T \ll \Lambda \ll T$) which can be introduced as a higher-derivative cutoff for lattice purposes. In practice, there is a simpler way to avoid UV problems, which is to restrict the phase-space averaging to the colour fluctuations of the hard gluons (the “auxiliary fields” $W_{\mu}(x, v)$).

The classical theory allows for three-dimensional lattice simulations of the real-time dynamics. In this sense, it appears as a generalization of the (leading-order) dimensional reduction (DR), to which it reduces in the static limit (cf. Eq. (6.3)). However — and unlike DR which, at least in principle, can be pushed to arbitrarily high orders \cite{16,17} — the present procedure is not systematic. That is, one cannot improve the classical theory by integrating out the hard modes, e.g., to next-to-leading order in $g$, not only because the latter operation would be technically involved, but also because such an improvement would not be consistent with the validity of the classical approximation. Different to say, the quantum corrections to the dynamics of the soft fields will generally enter on the same footing as the hard-field corrections to the HTL’s. Besides, the whole construction of the classical theory as given above has relied in a very non-trivial way on the specific structure and symmetries of the HTL’s.

Still, the effective theory formulated as above is accurate enough to allow for leading-order calculations (e.g., on a lattice) of non-perturbative real-time correlations like the hot baryon number violation rate \cite{13,14,19,22}, the quasiparticle damping rates \cite{8,10}, or some transport coefficients (e.g., the bulk viscosity \cite{23}).

The effective theory should be further compared to the method developed in Refs. \cite{22}, where classical coloured particles have been introduced to simulate the HTL’s on the lattice (see also Refs. \cite{4,27}). The present formalism has the advantage to avoid the UV problem of the classical theory, thus allowing for the continuum limit $a \to 0$ to be eventually taken. Moreover, this formalism should also include, as a special limit, the effective theory recently proposed by Bödeker \cite{27}. Therefore, by performing accurate lattice calculations in the present formalism, it should be possible to improve over the calculation in Ref. \cite{22}, and eventually distinguish between the various theoretical predictions \cite{13,14,27} for the hot baryon number violation rate.

**Appendix A**

Below, I shall derive Eqs. (6.3)–(6.4) for the thermal partition function in QED. The first step is to solve the Abelian version of Eqs. (5.4), namely $(E^i = -A^i)$: 
\[ \left( \partial_t^2 - \nabla^2 \right) \delta^{ij} + \partial_i \partial_j \right) A^j(x) = m_D^2 \int \frac{d\Omega}{4\pi} v^j W(x, v), \]

\[ \left( \partial_t + v \cdot \nabla \right) W(x, v) = v \cdot E(x), \quad (A.1) \]

with the initial conditions at \( t_0 = 0 \):

\[ A^i(0, x) = \mathcal{A}^i(x), \quad \dot{A}^i(0, x) = -\mathcal{E}^i(x), \]

\[ W(0, x, v) = \mathcal{W}(x, v), \quad (A.2) \]

which are constrained by Gauss' law:

\[ \nabla \cdot \mathcal{E} - m_D^2 \int \frac{d\Omega}{4\pi} W(x, v) = 0. \quad (A.3) \]

To completely fix the gauge, I further restrict the initial fields to be transverse: \( \nabla \cdot \mathcal{A} = 0 \).

We thus have a linear initial value problem which can be solved by Laplace transform, in the standard way. (See, e.g., Ref. [2] for a similar problem.)

Consider first the Vlasov equation (i.e., the second equation (A.2)) for \( W(x, v) \). Its solution is conveniently written as:

\[ W = W_{ind} + W_{fl}, \quad (A.4) \]

where \( W_{ind} \) is the piece induced by the gauge fields, i.e., the solution to the Vlasov equation with zero initial condition \( W_{ind}(0, x, v) = 0 \), while \( W_{fl} \) satisfies the homogeneous equation:

\[ \left( \partial_t + v \cdot \nabla \right) W_{fl} = 0, \quad (A.5) \]

with the initial condition \( W_{fl}(0, x, v) = \mathcal{W}(x, v) \). It immediately follows that:

\[ W_{fl}(x, v) = \mathcal{W}(x - v \cdot x_0, v), \quad (A.6) \]

which is referred to as “the fluctuating piece” since it is determined by the initial charge fluctuations \( \mathcal{W} \). Also, for \( x_0 > t_0 \):

\[ W_{ind}(x, v) = \int d^4 y \theta(y_0) G_R(x, y|v) \left( \mathcal{W}(y) - \mathcal{E}(y) \cdot v \right), \quad (A.7) \]

where \( G_R(x, y|v) \) is the retarded Green’s function of the line derivative \( \partial_t + v \cdot \nabla \):

\[ G_R(x, y|v) = \theta(x_0 - y_0) \delta^{(3)}(x - y - v(x_0 - y_0)). \quad (A.8) \]

Eq. (A.4) implies a similar decomposition for the current \( j^i \):

\[ j^i = j^i_{ind} + \xi^i, \quad (A.9) \]

where \( \xi^i \) is the fluctuating piece:

\[ \xi^i(x) = m_D^2 \int \frac{d\Omega}{4\pi} v^i W(x - v x_0, v), \quad (A.9) \]

and \( j^i_{ind} \) is the current determined by \( W_{ind} \):

\[ j^i_{ind}(x) \equiv m_D^2 \int \frac{d\Omega}{4\pi} v^i W_{ind}(x, v) \]

\[ = -\int d^4 y \theta(y_0) \Pi^{ij}(x - y) A^j(y) \]

\[ + m_D^2 \int \frac{d\Omega}{4\pi} v^i v^j A^j(x - v x_0), \quad (A.10) \]

where in writing the second line I have used Eq. (A.3) together with an integration by parts, and I have identified the (retarded) polarization tensor in the HTL approximation:

\[ \Pi^{ij}(x - y) = m_D^2 \int \frac{d\Omega}{4\pi} v^i v^j \partial_0 G_R(x, y|v) \]

\[ = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (y-x)} \Pi^{ij}(k), \quad (A.11) \]

with the familiar HTL expression:

\[ \Pi^{ij}(k) = m_D^2 \partial_0 \int \frac{d\Omega}{4\pi} \frac{v^i v^j}{k_0 - v \cdot k + i\epsilon}. \quad (A.12) \]

The Maxwell equation (i.e., the first equation (A.3)) reads then (for \( x_0 > t_0 \)):

\[ \left[ \left( \partial_t^2 - \nabla^2 \right) \delta^{ij} + \partial_i \partial_j \right] A^j(x) = j^i_{ind}(x) + \xi^i(x), \quad (A.13) \]

and can be most easily solved by going to the three-momentum space and then separating the Fourier modes into transverse and longitudinal components:

\[ A^i(t, x) = \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot x} A^i(t, k), \]

\[ A^i(t, k) = A_T^i(t, k) + k^i A_L(t, k), \quad (A.14) \]

where \( k \cdot A_T = 0 \). A similar decomposition holds for the initial fields \( A^i \) and \( \mathcal{E}^i \), with \( A_L = 0 \) (gauge fixing) and \( \mathcal{E}_L \) as determined by Gauss’ law (cf. Eq. (A.3)):

\[ ik \mathcal{E}_L(k) = m_D^2 \int \frac{d\Omega}{4\pi} \mathcal{W}(k, v). \quad (A.15) \]

Let me consider the transverse sector in more detail. Eqs. (A.13) and (A.10) yield:

\[ \left( \partial_t^2 + k^2 \right) A_T^i(t) + \int dt' \theta(t') \Pi_T(t - t') A_T^j(t') - \alpha^i(t) = \xi_T^i(t), \quad (A.16) \]

where the \( k \)-dependence is implicit, \( \xi_T^i(t, k) \) is the transverse projection of \( \xi^i \), Eq. (A.9), and

\[ \alpha^i(t, k) \equiv m_D^2 \int \frac{d\Omega}{4\pi} e^{-ik \cdot x_0} A_L^i(k), \quad (A.17) \]

which is transverse as well. \( k \cdot \alpha = 0 \) (N.B. \( A^i \equiv A_T^i \)).

The solution to Eq. (A.16) corresponding to the initial conditions \( \xi_T^i(0, k) = 0 \) can be written as in Eq. (A.9):

\[ A_T(t, k) = A_T^{ind}(t, k) + A_T^{fl}(t, k), \quad (A.18) \]

where \( A_T^{ind} \) is the piece determined by the initial electromagnetic fields \( \{ A_T^i, \mathcal{E}_T \} \), while \( A_T^{fl} \) is generated by the \( \mathcal{W}'s \) (via \( \xi_T^i \)). Specifically:

\[ A_T^{ind}(t, k) = \int \frac{d\omega}{2\pi} \rho_T(\omega) \left\{ k^2 A_T^i \frac{\cos \omega t}{\omega} - \mathcal{E}_T \sin \omega t \right\} \quad (A.19) \]
and, respectively,

\[ A_T^I(t, k) = \int dt' \theta(t') \Delta_T(t - t') \xi_T(t'), \quad (A.20) \]

where \( \Delta_T \) is the (retarded) transverse propagator (cf. Eq. (B.1)), with spectral density \( \rho_T \) (cf. Eq. (6.6)).

Eqs. (A.18) to (A.20) express the transverse solution \( A_T(t, k) \) as an explicit function of the initial conditions \( \xi_T, \xi_T' \) (recall Eq. (A.9)). The fluctuating piece in Eq. (A.20) is the solution to the inhomogeneous equation (A.16) with zero initial conditions: \( \bar{A}_T' = \bar{E}_T' = 0 \). The induced piece, Eq. (A.19), satisfies the homogeneous equation (i.e., Eq. (A.16) with \( \xi_T' = 0 \)) with the proper initial conditions \( \{A_T, \bar{E}_T\} \). To verify the latter, use the following sum-rules [8,2]:

\[ \int \frac{d\omega}{2\pi} \omega \rho_T(\omega) = 1 = k^2 \int \frac{d\omega}{2\pi} \rho_T(\omega), \tag{A.21} \]

The next step is to perform the average over the initial conditions, as expressed by the following functional integral (cf. Eq. (6.1)):

\[ Z_{cl}[J_L] = \int \mathcal{D}\xi_T \mathcal{D}A_T \mathcal{D}W(\delta(\mathcal{G})) e^{-\beta H_L + \int d^4x J_L A_T} \]

\[ = Z_{ind}[J_L] \times Z_f^T[J_L], \tag{A.22} \]

where (recall that \( A_L = 0 \)):

\[ Z_{ind}[J_L] = \int \mathcal{D}\xi_T \mathcal{D}A_T e^{-\beta H_T + \int d^4x J_L A_T^i}, \]

\[ Z_f^T[J_L] = \int \mathcal{D}L \mathcal{D}W(\delta(\mathcal{G})) e^{-\beta H_L + \int d^4x J_L \cdot A_T^i}, \tag{A.23} \]

and I have denoted:

\[ H_T \equiv \frac{1}{2} \int d^3x \left( \mathcal{E}_T^2 + \mathcal{B}^2 \right) \]

\[ H_L \equiv \frac{1}{2} \int d^3x \left( \mathcal{E}_L^2 + m_D^2 \int \frac{d\Omega}{4\pi} \mathcal{W}^2 \right), \tag{A.24} \]

with \( \mathcal{H} = H_T + H_L \).

The first integral in Eq. (A.22) is straightforward and yields (cf. Eq. (A.19)):

\[ Z_{ind}[J_L] = \exp \left\{ \frac{T}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d\omega d\omega'}{2\pi} J_T(\omega) \cdot J_T^*(\omega') \int \frac{d\Omega}{4\pi} \right\} \times \left( 1 + \frac{k^2}{\omega \omega'} \right) \rho_T(\omega, k) \rho_T^*(\omega', k). \tag{A.25} \]

The second integral involves \( A_T^{fi} \) which is proportional to \( \mathcal{W} \). Specifically, Eqs. (A.19) and (A.20) imply:

\[ A_T^{fi}(\omega, k) = -2m_D^2 \int \frac{d\Omega}{4\pi} v_T \mathcal{W}(k, v) \times \text{Im} \left( \frac{\Delta_T^{ii}(\omega, k)}{\omega - v \cdot k + i\eta} \right), \tag{A.26} \]

By using this expression, together with the identity:

\[ \int \mathcal{D}E \mathcal{D}W(\delta(\mathcal{G})) e^{-\beta H_L + \int d^4x J_L A_L} = \exp \left\{ \frac{T}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d\Omega}{4\pi} \frac{\rho_T(k_0, k)}{k_0} \mathcal{J}_L(k) \mathcal{J}_L^*(k) \right\}. \tag{A.27} \]

one eventually obtains:

\[ Z_{cl}[J_L] = \int \mathcal{D}E \mathcal{D}W(\delta(\mathcal{G})) e^{-\beta H_L + \int d^4x J_L A_L}, \tag{A.29} \]

where \( A_L \equiv A_L^{fi} \) is the solution of the longitudinal equation of motion, and is a purely fluctuating field: that is, it is fully determined by the initial charge fluctuations \( \mathcal{W} \) (since \( A_L = 0 \) and \( \mathcal{E}_L \) is fixed by Gauss’ law (A.13)). A calculation similar to the one above eventually yields:

\[ Z_{cl}[J_L] = \int \mathcal{D}E \mathcal{D}W(\delta(\mathcal{G})) e^{-\beta H_L + \int d^4x J_L A_L^i}, \tag{A.30} \]

Note that the would-be pole at \( k_0 \to 0 \) (as typical for the temporal gauge) is actually innocuous in the classical theory, where the current conservation \( \partial_\mu J^\mu = 0 \) ensures that the longitudinal current also vanishes as \( k_0 \to 0 \):

\[ J_L(k_0, k) = (k_0/k) J_0(k_0, k). \]

Together, Eqs. (A.28) and (A.30) provide the Abelian partition function presented in Eqs. (6.3)-(6.4).

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