Derivation of isothermal quantum fluid equations with Fermi-Dirac and Bose-Einstein statistics

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Abstract

By using the quantum maximum entropy principle we formally derive, from a underlying kinetic description, isothermal (hydrodynamic and diffusive) quantum fluid equations for particles with Fermi-Dirac and Bose-Einstein statistics. A semiclassical expansion of the quantum fluid equations, up to $O(h^2)$-terms, leads to classical fluid equations with statistics-dependent quantum corrections, including a modified Bohm potential. The Maxwell-Boltzmann limit and the zero temperature limit are eventually discussed.

1 Introduction and main results

The theory of quantum fluid equations originates from a seminal paper by E. Madelung [25], who discovered that Schrödinger equation can be put in a hydrodynamic form (the Madelung equations, see Eq. (4.16)). These equations have the form of an irrotational, compressible and isothermal Euler system with an additional term, of order $h^2$, interpreted as a “quantum potential” or a “quantum pressure”. This was later named Bohm potential after D. Bohm, who based on it his celebrated, although controversial, interpretation of quantum mechanics [4, 5, 12].

Besides their undoubted theoretical importance, quantum fluid equations have become very interesting also for applications, in particular to semiconductor devices modeling [18]. Indeed, the fluid description of a quantum system has many practical advantages. Not only it provides a description in terms of macroscopic variables with a direct physical interpretation (such as density, current, temperature) but it is also amenable to semiclassical approximations, usually leading to fluid equations in a quasi-classical form (classical fluid equations with quantum corrections). Such quasi-classical form is particularly suited for modeling purposes since it can be easily “contaminated” with phenomenological elements (boundary conditions, external couplings and so on), that would be very difficult to incorporate within a purely quantum mechanical framework.
Madelung equations describe the evolution of a pure (i.e. non-statistical) quantum state and (being basically equivalent to Schrödinger equation [17]) are formally closed. However, the most interesting case for applications is usually that of statistical systems, in which any description in terms of a finite number of macroscopic moments is, in general, not closed. Then, analogously to what happens in classical statistical mechanics, a central problem in the theory of quantum fluids is the closure of the moment equations. A commonly accepted solution to this problem is furnished by the quantum version, due to P. Degond and C. Ringhofer [11], of the *maximum entropy principle*, a well-known paradigm from information theory, widely used in classical statistical mechanics and thermodynamics [23] as well as in many other disciplines (e.g. signal analysis). By using the quantum maximum entropy principle (QMEP), various kind of quantum fluid models have been deduced: drift-diffusion and energy transport [10], SHE-model [6], isothermal hydrodynamic [9, 21], non-isothermal hydrodynamic [22], viscous hydrodynamic (Navier-Stokes) [7], spin (or pseudo-spin) drift-diffusion [2, 3] and hydrodynamic [33]. Many other references can be found in Refs. [18, 19].

Although the QMEP was originally stated for a general convex entropy functional [11], in all the quoted references explicit models are deduced only for Maxwell-Boltzmann statistics. In the present paper we consider the QMEP for an entropy function that incorporates different particle statistics (Fermi-Dirac, Maxwell-Boltzmann, Bose-Einstein, see Eq. (2.33)). Then, we derive the corresponding isothermal fluid equations (drift-diffusion and hydrodynamic) and compute their explicit semiclassical expansions up to $O(h^2)$ terms.

To our knowledge, in the framework of QMEP, Fermi-Dirac and Bose-Einstein statistics have been so far considered in Refs. [27, 28], where a hierarchy of moment equations, in the spirit of extended thermodynamics, is derived. Explicit (or partially explicit) semiclassical equations, with $O(h^2)$ terms, are computed for the first levels of the hierarchy (including the isothermal equations considered in the present paper). However, having assumed that quantum corrections only depend on the density and its derivatives, some terms are missing that depend on the derivatives of the current (namely, the rotational terms of order $h^2$ that appear instead in our Eqs. (3.19), (4.5) and (4.12)). Also Ref. [20] is worth to be mentioned, where a hierarchy of diffusive moment equations with Fermi-Dirac statistics is derived in the semiclassical limit (no $O(h^2)$ corrections).

Our derivation starts from a quantum kinetic level, represented by a one-particle, $d$-dimensional, Wigner equation [14, 30, 32], with a BGK collisional term (see Eq. (2.13)) that relaxes the system to a local equilibrium. The local equilibrium Wigner function $w_{\text{eq}}$ is assumed to be given by the QMEP which, generally speaking, stipulates that the local equilibrium maximizes an entropy functional under the constraint that some of its macroscopic moments are given. Which moments are constrained depends on which kind of fluid equations we are interested in. In our case, the moments are the particle density $n$, for the diffusive equations, and, in addition, the current $J = (J_1, \ldots, J_d)$ (or, equivalently, the velocity $u = J/n$) for the isothermal hydrodynamic equations. We choose an entropy functional (see Eq. (2.29)) that contains the information on the particle statistics and also accounts for the fixed

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1Reference [9] is a partial exception, because the fully-quantum model is deduced for a generic entropy. However, the semiclassical expansion assumes Boltzmann entropy.
equilibrium temperature (and, therefore, is a free-energy rather than an entropy). As we shall see in Subsec. 2.4, a solution $w_{eq}$ of the constrained minimization problem can be formally written and depends on $d + 1$ Lagrange multipliers, $A$ and $B = (B_1, \ldots, B_d)$, which are implicitly related to the moments $n$ and $J$ because of the constraints. After a suitable scaling of the Wigner-BGK equation, we identify two dimensionless parameters: a scaled relaxation time $\alpha$ and a scaled Planck constant $\epsilon$. Then, in the limit of vanishing $\alpha$ (i.e., basically, assuming that the system has relaxed to the state $w_{eq}$) we can deduce moment equations for $n$ (diffusive) and for $n$ and $J$ (hydrodynamic), where the extra moments are expressed in function of $A$ and $B$. Therefore, the moments equations are formally closed because of the constraint relations. The fully-quantum models obtained in this way, given by Eqs. (2.37) and (2.38), are very implicit and, therefore, it is reasonable to look for approximated, but explicit, equations. In particular, we look for the semiclassical approximation of Eqs. (2.37) and (2.38), by assuming $\epsilon$ small.

The semiclassical approximation of the quantum fluid models requires the semiclassical expansions of $w_{eq}$, which is particularly natural in the Wigner-Weyl-Moyal formalism [14, 32]. This expansion, which involves an interesting application of the Moyal calculus, as well as the computation of a variety of integrals of Fermi and Bose type (see Appendix A), is carried out explicitly up to order $O(\epsilon^2)$ and leads to the main result of the paper, represented by Theorems 3.1 and 3.2. The semiclassical equations that we obtain, Eqs. (3.19) and (3.20), are the generalization of the semiclassical diffusive and hydrodynamic equations derived in Refs. [9, 10, 21] for Maxwell-Boltzmann statistics. They have the form of their classical counterparts (compressible isothermal Euler system and drift-diffusion equations) with quantum corrections (terms of order $O(\epsilon^2)$) as well as corrections coming from the particle statistics. In particular, a term corresponding to a modified Bohm potential can be identified (see Eq. (3.18)). It is worth remarking that in the Bose-Einstein case, and dimension $d \geq 3$, our equations are only valid by assuming that the fluid is entirely in the non-condensate phase or, equivalently, that the temperature is uniformly supercritical (see Proposition 3.2 and Remark 3.2).

In the last part of the paper we perform a formal analysis of specific physical regimes, where Eqs. (3.19) and (3.20) take particular forms. We discuss the case of irrotational fluids, the Maxwell-Boltzmann limit (recovering the equations derived in Refs. [9, 10, 21]), and the vanishing-temperature limit. The latter is particularly interesting in the Fermi-Dirac case (the only one in which Eqs. (3.19) and (3.20) have a regular behavior as $T \to 0$) and leads to Eqs. (4.12) and (4.13), describing a so-called “completely degenerate fluid”. These equations contain power-law diffusive and rotational terms, and a limit Bohm potential which differs from the usual one by just a dimension-dependent constant factor. The $T \to 0$ limit in the Maxwell-Boltzmann and Bose-Einstein cases shows a singular behavior and reasonable results can only be given for the Bose-Einstein case with $d \leq 2$ and assuming that the fluid is irrotational.

The outline of the paper is as follows. Section 2 is devoted to the derivation of the quantum fluid equations: in Subsecs. 2.1 and 2.2 we introduce the kinetic Wigner-BGK equation and its hydrodynamic and diffusive scalings; in Subsec. 2.3 we derive the equations
for the moments \( n \) and \( J \) and, in Subsec. 2.4, we perform their formal closure by using the QMEP. Section 3 is devoted to the semiclassical approximation of the quantum fluid models derived in the preceding section: in Subsec. 3.1 we compute the semiclassical expansion of the local equilibrium Wigner function and, in particular, of the Lagrange multipliers \( A \) and \( B \) as functions of the moments \( n \) and \( J \); in Subsec. 3.2 the expansion of \( A \) and \( B \) is used to derive the semiclassical equations (3.19) and (3.20). Finally, Sec. 4 is devoted to the analysis of the above-mentioned particular regimes: the irrotational fluid (Subsec. 4.1), the Maxwell-Boltzmann limit (Subsec. 4.2) and the zero-temperature limit (Subsec. 4.3). Some technical material has been placed in two appendices: Appendix A contains generalities about Fermi and Bose integrals as well as the computation of related integrals that have been encountered in the paper; Appendix B contains some postponed proofs.

## 2 Quantum fluid equations

In this section we derive fully quantum fluid-dynamic equations of two types: isothermal hydrodynamic equations and diffusive equations. Such derivations, and the structure of the resulting equations, do not differ significantly from what is already well known in literature \[9, 18, 19\]. The specific role played by statistics (and, therefore, the novelty of the present work) will be more explicit in the semiclassical expansion of the equations, which will be carried out in the remainder of the paper.

### 2.1 The Wigner-BGK equation

The starting point of our derivation is the kinetic description of a one-particle quantum statistical state, given in terms of one-particle Wigner functions \[30, 32\]. Let us now briefly recall the basic definitions and properties.

A mixed (statistical) one-particle quantum state for an ensemble of scalar particles in \( \mathbb{R}^d \) (where \( d = 1, 2 \) or 3 are the interesting values, e.g. for nano-electronic applications), is described by a density operator \( \varrho \), i.e. a bounded non-negative operator with unit trace, acting on \( L^2(\mathbb{R}^d, \mathbb{C}) \). The associated Wigner function, \( w = w(x, p) \), \((x, p) \in \mathbb{R}^{2d}\), is given by the inverse Weyl quantization of \( \varrho \),

\[
w = \text{Op}_h^{-1}(\varrho),
\]

where the Weyl quantization of a phase-space function (a “symbol”) \( a = a(x, p) \) is the operator \( \text{Op}_h(a) \) formally defined by

\[
[\text{Op}_h(a)\psi](x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2}, p\right) \psi(y) e^{i(x-y)p/\hbar} \, dy \, dp.
\]

The Wigner function has a more direct definition as the “Wigner transform”

\[
w(x, p) = \int_{\mathbb{R}^d} \varrho\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right) e^{-ip\cdot\xi/\hbar} \, d\xi,
\]

of the density matrix \( \varrho(x, y) \), i.e. (with a little abuse of notation) the integral kernel of the density operator \( \varrho \).
The operator product translated at the level of symbols leads to the definition of Moyal (or “twisted”) product
\[ a \# b = \text{Op}^{-1}_h (\text{Op}_h(a) \text{Op}_h(b)), \] which possesses the formal semiclassical expansion
\[ a \# b = \sum_{k=0}^{\infty} \hbar^k a \#_k b, \]
\[ a \#_k b = \frac{1}{(2i)^k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\alpha|}}{\alpha! \beta!} \left( \nabla_{x}^\alpha \nabla_{p}^\beta a \right) \left( \nabla_{p}^\alpha \nabla_{x}^\beta b \right). \] In particular, \( \#_0 \) is the usual function product, \( a \#_0 b = ab \), and \( \#_1 \) is the Poisson bracket
\[ a \#_1 b = \frac{i}{2} \sum_{k=1}^{d} \left( \frac{\partial a}{\partial x_k} \frac{\partial b}{\partial p_k} - \frac{\partial a}{\partial p_k} \frac{\partial b}{\partial x_k} \right). \]

The dynamics of the time-dependent Wigner function \( w(t) = w(x,p,t) \) can be immediately deduced from the dynamics of the corresponding density operator \( \varrho(t) \), i.e. from the von Neumann equation (Schrödinger equation for mixed states)
\[ i \hbar \frac{\partial}{\partial t} \varrho(t) = [H, \varrho(t)] := H \varrho(t) - \varrho(t) H, \]
where \( H \) denotes the Hamiltonian operator of the system. If \( h = \text{Op}^{-1}_h(H) \) is the symbol of \( H \), then, from Eqs. (2.7) and (2.4) we obtain the “Wigner equation”
\[ i \hbar \frac{\partial}{\partial t} w(t) = \{ h, w(t) \}_\# := h \# w(t) - w(t) \# h. \]

Taking \( h \) as the standard hamiltonian symbol
\[ h(x,p) = \frac{|p|^2}{2m} + V(x) \] (where \( m \) is the particle effective mass and \( V \) is a one-particle potential), the Wigner equation (2.8) can be written in the more explicit form
\[ \frac{\partial}{\partial t} w(t) + \frac{p}{m} \cdot \nabla_x w(t) + \Theta_h[V]w(t) = 0, \]
where \( \Theta_h[V]w(t) = \frac{i}{\hbar} \{ V, w(t) \}_\# \) is given by
\[ \left[ \Theta_h[V]w(t) \right](x,p) = \frac{i}{\hbar} \int_{\mathbb{R}^{2d}} \left[ V \left( x + \frac{\xi}{2} x - \frac{\xi}{2} \right) + V \left( x + \frac{\xi}{2} x - \frac{\xi}{2} \right) \right] e^{i \xi \cdot (p' - p)/\hbar} w(x', p', t) \frac{d\xi}{(2\pi\hbar)^d}. \]

One of the most interesting properties of the Wigner function is that its moments have a direct physical interpretation in terms of macroscopic fluid quantities, which makes Wigner
functions an ideal tool for the derivation of quantum fluid equations. In this paper we shall write equations, in different fluid regimes, for the first 1 + $d$ moments: the density

$$n(x,t) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} w(x,p,t) \, dp = \rho(x,x,t)$$  \hspace{1cm} (2.11)

and the $d$ components of the current

$$J_k(x,t) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} p_k w(x,p,t) \, dp = \hbar^2 i \left( \frac{\partial \rho}{\partial x_k} - \frac{\partial \rho}{\partial y_k} \right) \rho(x,x,t),$$  \hspace{1cm} (2.12)

(where the corresponding expressions in terms of the time-dependent density matrix $\rho(x,y,t)$ have also been shown).

**Remark 2.1** Our choice of defining the Wigner function as the inverse Weyl quantization of the density operator implies that $w$ is a dimensionless quantity (this is apparent from Eq. (2.3), recalling that the density matrix $\rho(x,y)$ has the physical dimensions of a number density in position space). However, the usual “physical” definition of Wigner function \cite{30,32} requires an extra factor $1/(2\pi\hbar)^d$, so that the physical Wigner function has the dimensions of a number density in phase space. This is the reason of the factor $1/(2\pi\hbar)^d$ appearing in Eqs. (2.11) and (2.12).

Now, by definition, a system is driven to a fluid regime by collisions. Following Degond, Ringhofer and Méhats \cite{10,11}, we endow the Wigner equation (2.10) with a collision mechanism of BGK type \cite{1}

$$\frac{\partial w}{\partial t} + \frac{p}{m} \cdot \nabla_x w + \Theta[V] w = \frac{1}{\tau} (w_{eq}[w] - w).$$ \hspace{1cm} (2.13)

Here, $\tau$ is a typical relaxation time and $w_{eq}[w]$ is a Wigner function which represent the local equilibrium state reached by the system because of collisions. As we shall see in the following, the central point of the whole derivation is that $w_{eq}[w]$ is assumed to be the maximizer of a suitable quantum entropy functional, subject to the constraint of sharing certain moments with $w$ (namely, $n$ and $J$ in the isothermal hydrodynamic case, and $n$ in the diffusive case). We remark that the one-particle potential $V$ accounts for other kinds of interactions, including mean-field Poisson or Hartree-like \cite{27,28} potentials.

### 2.2 Scaling the Wigner-BGK equation

In order to write Eq. (2.13) in the hydrodynamic and diffusive scalings, let us introduce a reference length $x_0$, time $t_0$ and energy $E_0$. Reference temperature and momentum are naturally related to $E_0$ by

$$k_B T_0 = E_0, \hspace{1cm} \frac{p_0^2}{m} = E_0,$$

where $k_B$ is the Boltzmann constant. Then, in Eq. (2.13) we switch to dimensionless quantities

$$x \to x_0 x, \hspace{1cm} t \to t_0 t, \hspace{1cm} p \to p_0 p, \hspace{1cm} V \to E_0 V,$$
(for the sake of simplicity the new dimensionless variables are denoted by the same symbols as the old ones), which yields

\[
\frac{1}{t_0} \frac{\partial w}{\partial t} + \frac{p_0}{m x_0} \mathbf{p} \cdot \nabla_x + \frac{E_0}{x_0 p_0} \Theta_{\varphi_0}[V] = \frac{1}{\tau} (w_{\text{eq}}[w] - w).
\]

Note that we have not to rescale the Wigner functions \( w \) and \( w_{\text{eq}} \) because we are already using dimensionless Wigner functions (see Remark 2.1). We rewrite the last equation by introducing the semiclassical parameter

\[
\epsilon = \frac{\hbar}{x_0 p_0}
\]

and the energy time scale

\[
t_E = \frac{m x_0}{p_0}
\]

(i.e. the order of time for a particle of kinetic energy \( E_0 \) to travel a distance \( x_0 \)), obtaining:

\[
\frac{1}{t_0} \frac{\partial w}{\partial t} + \frac{1}{t_E} \mathbf{p} \cdot \nabla_x w + \frac{1}{t_E} \Theta_{\epsilon}[V]w = \frac{1}{\tau} (w_{\text{eq}}[w] - w).
\]

(2.15)

Now, two different scaling assumptions, corresponding to different fluid regimes, can be made.

**Hydrodynamic regime.** In this regime the system is observed on the time-scale \( t_E \) and collisions are assumed to act on a much shorter time-scale; then we put

\[
\alpha := \frac{\tau}{t_E} \ll 1, \quad t_0 = t_E.
\]

The corresponding Wigner-BGK equation takes therefore the hydrodynamic scaling form:

\[
\alpha \frac{\partial w}{\partial t} + \alpha \mathbf{p} \cdot \nabla_x w + \alpha \Theta_{\epsilon}[V]w = w_{\text{eq}}[w] - w.
\]

(2.17)

**Diffusive regime.** In this regime the collisions are still assumed to act on a time-scale much shorter than \( t_E \), but the system is observed on a time-scale much larger than \( t_E \); then we put

\[
\alpha := \frac{\tau}{t_E} \ll 1, \quad \frac{t_E}{t_0} = \alpha
\]

(2.18)

(so that \( t_0 = t_E^2/\tau \)). The corresponding Wigner-BGK equation takes in this case the diffusive scaling form:

\[
\alpha^2 \frac{\partial w}{\partial t} + \alpha \mathbf{p} \cdot \nabla_x w + \alpha \Theta_{\epsilon}[V]w = w_{\text{eq}}[w] - w.
\]

(2.19)

Note, in both cases, the presence of two dimensionless parameters: \( \epsilon \) and \( \alpha \). In the remainder of this section we shall deal with the fluid asymptotics, \( \alpha \rightarrow 0 \), leaving \( \epsilon \) untouched; then, in the following sections, we shall work on the semiclassical expansion of the fluid equations for small \( \epsilon \).

**Remark 2.2** In the new dimensionless variables, all the identities involving Weyl quantization and Moyal product are obtained from the original ones by the formal substitution \( \hbar \mapsto \epsilon \).
2.3 Derivation of quantum fluid equations

First of all, let us introduce the short notation

\[ \langle w \rangle(x, t) := \int_{\mathbb{R}^d} w(x, p, t) \, dp. \]

If \( w \) is the Wigner function of our particle system, we are going to write down equations for the moments \( n = \langle w \rangle \) and \( J = \langle pw \rangle \), but we have to keep in mind that the true density and current are \( N_0n \) and \( p_0N_0J \), where

\[ N_0 = \left( \frac{p_0}{2\pi\hbar} \right)^d = \left( \frac{mk_BT_0}{(2\pi\hbar)^2} \right)^{d/2} \]  

(2.20)

(see Eqs. (2.11) and (2.12), and Remark 2.1).

2.3.1 Isothermal quantum hydrodynamic equations

In order to derive isothermal hydrodynamic equations from Eq. (2.17), we have to assume that collisions conserve the number of particles and their momentum while making the system relax towards a local equilibrium state \( w_{eq}[w] \) (to be completely described later on) at a constant temperature \( T_{ext} \). Thus, we have to impose on \( w_{eq}[w] \) the moment constraints

\[ \langle w_{eq}[w] \rangle = n = \langle w \rangle, \quad \langle p_iw_{eq}[w] \rangle = J_i = \langle p_iw \rangle, \]  

(2.21)

for \( i = 1, \ldots, d \). Let now \( w_\alpha \) be solution of Eq. (2.17) and assume that the limit \( w_\alpha \rightarrow w_0 \) for \( \alpha \rightarrow 0 \) exists with finite moments \( n = \langle w_0 \rangle \) and \( J = \langle pw_0 \rangle \). Then, from (2.17) and (2.21) we get \( w_0 = w_{eq}[w_0] \). Taking the moments of both sides of Eq. (2.17) and letting \( \alpha \rightarrow 0 \) we obtain

\[ \frac{\partial}{\partial t} \langle w_{eq}[w_0] \rangle + \frac{\partial}{\partial x_i} \langle p_iw_{eq}[w_0] \rangle + \langle \Theta_\epsilon[V]w_{eq}[w_0] \rangle = 0, \]

\[ \frac{\partial}{\partial t} \langle p_iw_{eq}[w_0] \rangle + \frac{\partial}{\partial x_j} \langle p_ip_jw_{eq}[w_0] \rangle + \langle p_i\Theta_\epsilon[V]w_{eq}[w_0] \rangle = 0, \]

where the summation convention on repeated indices has been assumed. From the semiclassical expansion of the potential operator,

\[ \Theta_\epsilon[V] = \frac{i}{\epsilon} \{ V, w(t) \}_{\#} = \frac{i}{\epsilon} (V\#w - w\#V) \]

\[ = -\sum_{k=0}^{\infty} (-1)^k \left( \frac{\epsilon}{2} \right)^{2k} \sum_{|\alpha|=2k+1} \nabla_x^{\alpha} V \nabla_p^{\alpha} w \]  

(2.22)

(where (2.5) was used, see also Remark 2.2), we immediately obtain

\[ \langle p_i\Theta_\epsilon[V]w_{eq}[w_0] \rangle = \langle w_{eq}[w_0] \rangle \frac{\partial V}{\partial x_i} = n \frac{\partial V}{\partial x_i}, \]  

(2.23)

and, then, the moment equations read as follows:

\[ \frac{\partial n}{\partial t} + \frac{\partial J_i}{\partial x_i} = 0 \]

\[ \frac{\partial J_i}{\partial t} + \frac{\partial}{\partial x_j} \langle p_ip_jw_{eq}[w_0] \rangle + n \frac{\partial}{\partial x_i} V = 0. \]  

(2.24)
If \( w_{eq}[w_0] \) can be uniquely specified as a function of \( n \) and \( J \), from the constraints (2.21), then the system (2.24) is formally closed.

### 2.3.2 Quantum diffusive equations

Diffusive equations can be obtained from Eq. (2.19) by using the “Chapman-Enskog” method. We now only assume that collisions conserve the number of particles, which leads to the unique constraint

\[
\langle w_{eq}[w] \rangle = n = \langle w \rangle. \tag{2.25}
\]

Then, we assume that the solution \( w_\alpha \) of Eq. (2.19), for \( \alpha \to 0 \), has a limit \( w_\alpha \to w_0 \) with finite density \( n = \langle w_0 \rangle \). Letting \( \alpha \to 0 \) in Eq. (2.19) we still obtain \( w_0 = w_{eq}[w_0] \) but, contrarily to the previous case, the equation for the density

\[
\alpha \frac{\partial}{\partial t} \langle w_\alpha \rangle + \frac{\partial}{\partial x_i} \langle p_i w_\alpha \rangle = 0 \tag{2.26}
\]

only gives, in the limit, the condition

\[
\langle pw_{eq}[w_0] \rangle = 0, \tag{2.27}
\]

i.e. the equilibrium state carries no current. The diffusive equations must be sought at next order of the Chapman-Enskog expansion

\[
w_\alpha = w_{eq}[w_\alpha] + \alpha w_1.
\]

Substituting this ansatz into Eq. (2.19), and letting \( \alpha \to 0 \), yields

\[
w_1 = - (p \cdot \nabla x + \Theta_e[V]) w_{eq}[w_0]
\]

and, therefore, from Eqs. (2.26) and (2.28), we obtain the diffusive equation

\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial x_i} \left( J_i + n \frac{\partial V}{\partial x_i} \right), \quad J_i = \frac{\partial}{\partial x_j} \langle p_i p_j w_{eq}[w_0] \rangle. \tag{2.28}
\]

Once again, if \( w_{eq}[w_0] \) can be uniquely specified as a function of \( n \) from the constraint (2.25), then Eq. (2.28) is formally closed.

The “closure” of Eqs. (2.24) and (2.28) will be the central issue of the remainder of the paper.

### 2.4 Maximum entropy closure

Following Refs. [10, 11], we assume that the local equilibrium state \( w_{eq}[w] \) satisfies a quantum maximum entropy principle (QMEP), which basically states that \( w_{eq}[w] \) is the most probable state compatible with the information we have about it. In our case, such information is:

1. the temperature has a constant value \( T_{ext} \);
2. collisions conserve the number of particles;
3. Collisions conserve also the momentum in the hydrodynamic regime.

Point 1 implies that \( w_{eq}[w] \) should be a minimizer of the free-energy (rather than a maximizer of the entropy) \(^{[10]}\). Points 2 and 3 imply that \( w_{eq}[w] \) is subject to the constraints \((2.21)\) in the hydrodynamic case, or to the single constraint \((2.25)\) in the diffusive case.

Let \( s : \mathbb{R}^+ \rightarrow \mathbb{R} \) be a regular, convex, function and \( \rho \) a density operator. We can define the \textit{von Neumann entropy} \(^{[29]}\) of the state \( \rho \) as \( \text{Tr}\{ -s(\rho) \} \), where \( s(\rho) \) is given by the functional calculus on self-adjoint operators and \( \text{Tr} \) denotes the operator trace. It is worth remarking that we are using dimensionless quantities (see Subsection \( 2.2 \)); the dimensional definition of entropy would be \( \text{Tr}\{ -k_B s(\rho/N_0) \} \), where \( N_0 \) is given by \((2.20)\). The corresponding free-energy at temperature \( T_{\text{ext}} \) is

\[
E(\rho) = \text{Tr}\{ H\rho + T s(\rho) \},
\]

where \( H \) is the (scaled) Hamiltonian and

\[
T = \frac{T_{\text{ext}}}{T_0} = \frac{k_B T_{\text{ext}}}{E_0}
\]

is the scaled external temperature. The Wigner-Weyl correspondence (see Subsection \( 2.1 \)) allows us to define the free-energy of a Wigner function \( w \) simply as \( E(\text{Op}_x(w)) \).

All this considered, we shall assume the local-equilibrium Wigner function \( w_{eq}[w] \) to be solution of the following constrained minimization problem.

\textbf{Problem 2.1} Let \( n = \langle w \rangle \) and \( J = \langle pw \rangle \). Find a Wigner function \( w_{eq}[w] \) that minimizes the functional \( E(\text{Op}_x(f)) \) among all Wigner functions \( f \) that satisfy

\[
\begin{align*}
\langle f \rangle &= n, \\
\langle pf \rangle &= J \quad (\text{hydrodynamic case})
\end{align*}
\]

and

\[
\begin{align*}
\langle f \rangle &= n, \\
\langle pf \rangle &= J \quad (\text{diffusive case})
\end{align*}
\]

In Ref. \([11]\) is formally proven the following necessary condition\(^2\) for \( w_{eq}[w] \).

\textbf{Theorem 2.1} A necessary condition for \( w_{eq}[w] \) to be a solution of Problem \( 2.1 \) is that \( d + 1 \) Lagrange multipliers \( A \) and \( B = (B_1, \ldots, B_d) \), functions of \( x \) and \( t \), exist such that

\[
w_{eq}[w] = G_{A,B},
\]

where

\[
G_{A,B} = \text{Op}_x^{-1} \left\{ (s')^{-1} \left( \text{Op}_x \left( \frac{h_{A,B}}{T} \right) \right) \right\},
\]

and

\[
\begin{align*}
h_{A,B}(x,p,t) &= \left| p - B(x,t) \right|^2 - A(x,t), \\
& \quad (\text{hydrodynamic case})
\end{align*}
\]

\[
\begin{align*}
h_{A,B}(x,p,t) &= \frac{|p|^2}{2} - A(x,t), \\
& \quad (\text{diffusive case})
\end{align*}
\]

\(^2\) A rigorous proof of existence and uniqueness of the constrained minimization problem has been recently obtained by Méhats and Pinaud \([26]\) for the one-dimensional case with periodic boundary conditions.
Of course, the Weyl quantization $\text{Op}_p$ acts on functions of $x$ and $p$ (see Subsec. 2.1), $t$ being just a parameter. Note that the Lagrange multipliers $A, B_1, \ldots, B_d$ furnish the necessary degrees of freedom to satisfy the constraints. Note also that the hydrodynamic case contains the diffusive as a particular case corresponding to $B = 0$. This fact allows us to treat the two cases at once, the latter being simply obtained by taking $B = 0$. The Lagrange multiplier $A$ is the so-called chemical potential.

So far, $s$ is a generic entropy function (minus the entropy, to be precise). A further piece of information, namely the statistics of indistinguishable particles, can be inserted by choosing a suitable form of $s$. In this paper we consider a typical family of entropy functions, dependent on the real parameter $\lambda$, of the form

$$s(f) = f \log f + \lambda^{-1}(1 - \lambda f) \log(1 - \lambda f)$$

(2.33)

(where, for $\lambda = 0$, $s(f) = f \log f - f$ has to be intended as a limit). For such $s$ we have

$$\left( s' \right)^{-1}(h) = \frac{1}{e^h + \lambda}$$

and, then, Eq. (2.31) is specialized in

$$\mathcal{G}_{A,B} = \text{Op}_p^{-1} \left\{ \left[ \exp \frac{\text{Op}_p \langle h_{A,B} \rangle}{T} \right] + \lambda \right\}^{-1}.$$  

(2.35)

As usual, the parameter $\lambda$ has been introduced in order to consider different cases at once, the most important being of course:

$$\lambda = \begin{cases} 
1, & \text{Fermi-Dirac (FD) statistics,} \\
0, & \text{Maxwell-Boltzmann (MB) statistics,} \\
-1, & \text{Bose-Einstein (BE) statistics.}
\end{cases}$$

The quantum hydrodynamic/diffusive equations (2.24)/(2.28) can now be formally closed by assuming that the local equilibrium Wigner function $\text{w}_{\text{eq}}[w]$ is given by the QMEP. Then, according to Theorem 2.1 $\text{w}_{\text{eq}}[w] = \mathcal{G}_{A,B}$, where the Lagrange multipliers $A$ and $B$ are related to the moments $n = \langle w \rangle$ and $J = \langle pw \rangle$ through the constraints (2.21)/(2.25). Then, in Eqs. (2.21) and (2.28) the extra moment $\langle p_i p_j w_{\text{eq}}[w_0] \rangle = \langle p_i p_j \mathcal{G}_{A,B} \rangle$ can be viewed (at least in principle) as a function of $n = \langle w_0 \rangle$ and $J = \langle p w_0 \rangle$, which means that the equations are closed.

The following Proposition is proven in Ref. [9] for MB entropy, using the density-operator formalism. The proof given there is indeed independent on the choice of the entropy function and so the result is certainly valid also in the present case. However, we decided to give a proof (in Appendix), just because it may be interesting to see how it looks like in the Wigner formalism.\(^\text{3}\)

\(^{3}\)The proof given in Ref. [9], however, is still more general because the density-operator formalism covers the case of a system confined in a domain $\Omega \in \mathbb{R}^d$, while the Wigner formalism is valid only in the whole-space case.
Proposition 2.1 Let $\mathcal{G}_{A,B}$ be given by Eq. (2.35) with $\langle \mathcal{G}_{A,B} \rangle = n$ and $\langle p \mathcal{G}_{A,B} \rangle = J$; then:

$$\frac{\partial}{\partial x_j} \langle p_i p_j \mathcal{G}_{A,B} \rangle = \frac{\partial}{\partial x_j} (J_i B_j) + (J_j - n B_j) \frac{\partial B_j}{\partial x_i} + n \frac{\partial A}{\partial x_i}. \quad (2.36)$$

Proof See Appendix B.1. $\square$

Equation (2.36) represent the formal closure of Eqs. (2.24) and (2.28), obtained by taking $w_{eq}[w_0] = \mathcal{G}_{A,B}$. Then, we can conclude this section by summarizing the fully-quantum hydrodynamic and diffusive models with FD, MB or BE statistics.

Isothermal quantum hydrodynamic model

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial J_i}{\partial x_i} &= 0 \\
\frac{\partial J_i}{\partial t} + \frac{\partial}{\partial x_j} (J_i B_j) + (J_j - n B_j) \frac{\partial B_j}{\partial x_i} + n \frac{\partial}{\partial x_i} (A + V) &= 0.
\end{align*}
\quad (2.37a)
\]

\[
\begin{align*}
\langle \mathcal{G}_{A,B} \rangle &= n, \\
\langle p \mathcal{G}_{A,B} \rangle &= J, \quad (2.37b)
\end{align*}
\]

\[
\text{Op}_\epsilon(\mathcal{G}_{A,B}) = \left[ \exp \left( \text{Op}_\epsilon \left( \frac{|p - B|^2}{2T} - \frac{A}{T} \right) \right) + \lambda \right]^{-1}. \quad (2.37c)
\]

Quantum diffusive model

\[
\begin{align*}
\frac{\partial n}{\partial t} &= \frac{\partial}{\partial x_i} \left( n \frac{\partial}{\partial x_i} (A + V) \right), \\
\langle g_A \rangle &= n, \quad (2.38b)
\end{align*}
\]

\[
\text{Op}_\epsilon(g_A) = \left[ \exp \left( \text{Op}_\epsilon \left( \frac{|p|^2}{2T} - \frac{A}{T} \right) \right) + \lambda \right]^{-1}. \quad (2.38c)
\]

Note that the two models are constituted by the moment equations and an implicit relation linking the Lagrange multipliers to the moments. Such structure is rather involved, because it implies solving Eq. (2.37b) or Eq. (2.38b) for $A$ and $B$, where $\mathcal{G}_{A,B}$ is a complicated object defined in terms of back and forth Weyl quantization. Although models of this kind are amenable to numerical implementation [15, 16], it is certainly convenient to look for approximated, but more explicit, models. In particular, it is natural to look for a semiclassical ($\epsilon \ll 1$) approximation of systems (2.37) and (2.38). This is what the remainder of the paper will be devoted to.

\footnote{In the Bose-Einstein case, these models are only suited to describe the non-condensate phase; we shall discuss this point later on, in the semiclassical framework (see Proposition 3.2 and Remark 4.2).}
3 Semiclassical approximation of the fluid models

In this section we perform a formal semiclassical expansion of the quantum hydrodynamic and diffusive models (2.37) and (2.38) up to order $\epsilon^2$. We shall derive explicit expressions of the Lagrange multipliers $A$ and $B = (B_1, \ldots, B_d)$ as functions of the unknown moments $n$ and $J = (J_1, \ldots, J_d)$ neglecting terms of order higher than $\epsilon^2$ (as we shall see, the neglected terms will be actually of order $\epsilon^4$). Once such expressions are obtained, they can be substituted for $A$ and $B$ in Eqs. (2.37a)/(2.38a), yielding therefore semiclassical hydrodynamic/diffusive equations.

3.1 Semiclassical expansion of the Lagrange multipliers

The first step is the computation of the semiclassical expansion of $G_{A,B}$, the local equilibrium Wigner function given by Eqs. (2.35) and (2.32). We recall that the hydrodynamic and the diffusive cases can be unified at this stage, since the latter corresponds to the special case $B = 0$. Then, throughout this subsection, we shall work with the most general equilibrium function $G_{A,B}$.

In order to avoid cumbersome notations, let us simply denote $G_{A,B}$ by $G$ and define

$$h := \frac{h_{A,B}}{T} = \frac{|p - B|^2}{2T} - \frac{A}{T}$$

(3.1)

(not to be confused with the symbol $h$ used for original Hamiltonian (2.9)). The following proposition, based on a simple remark, is actually the key point for the computation of the semiclassical expansion.

Proposition 3.1 Let us temporarily assume that $A$ and $B$ do not depend on $\epsilon$ and consider the formal semiclassical expansion of $G \equiv G_{A,B}$:

$$G = G_0 + \epsilon G_1 + \epsilon^2 G_2 + \cdots.$$  

Moreover, let $\mathcal{E}xp(h) = \text{Op}^{-1}_\epsilon[\exp(\text{Op}_\epsilon(h))]$ be the “quantum exponential” (so that $G = \mathcal{E}xp(-h)$, for $\lambda = 0$) and let

$$\mathcal{E}xp(h) = \mathcal{E}xp_0(h) + \epsilon \mathcal{E}xp_1(h) + \epsilon^2 \mathcal{E}xp_2(h) + \cdots$$

(3.3)

be its formal semiclassical expansion. Then,

$$G_0 = \frac{1}{e^h + \lambda},$$

(3.4a)

$$G_{2n+1} = 0, \quad n \geq 0,$$

(3.4b)

$$G_{2n} = -\sum_{m=0}^{n-1} \sum_{k+l+m=n} \frac{\mathcal{E}xp_{2k}(h) \#_{2l} G_{2m}}{e^h + \lambda}, \quad n \geq 1$$

(3.4c)

where $\#_{2l}$ are the even terms of the (scaled) Moyal product expansion (2.5).
**Proof** Let \( H = \text{Op}_\epsilon(h) \) and \( G = \text{Op}_\epsilon(\mathcal{G}) = (e^H + \lambda)^{-1} \). Then, from the relation
\[
(e^H + \lambda)G = G(e^H + \lambda) = I
\]
and the definition of the Moyal product \( [22] \), we get
\[
(\text{Exp}(h) + \lambda) \# G = G \# (\text{Exp}(h) + \lambda) = 1,
\]
that is
\[
\frac{(\text{Exp}(h) + \lambda) \# G + G \# (\text{Exp}(h) + \lambda)}{2} = 1.
\]
Now we substitute in this identity the semiclassical expansions of \( \text{Exp}(h) \), \( G \) and \( \# \), and use the following facts:

(i) \( \text{Exp}_n(h) = 0 \) for odd \( n \) (see e.g. Ref. [10]);

(ii) \( \#_n \) is symmetric for even \( n \) and antisymmetric for odd \( n \) (see Eq. (2.5)).

After the substitution, equating the coefficients of equal powers of \( \epsilon \) yields \( (e^H + \lambda)G_0 = 1 \) (i.e. Eq. (3.4a)) and
\[
\sum_{2k+2\ell+m'=n} \text{Exp}_{2k}(h) \#_{2\ell} G_{m'} = 0 \tag{3.5}
\]
for \( n \geq 1 \). For \( n = 1 \), we immediately get \( G_1 = 0 \). If \( n > 1 \) is odd, the sum in (3.5) has only terms \( m' \leq n \) with \( m' \) odd and then, by induction, we can conclude that \( G_n = 0 \), which proves Eq. (3.4b). Hence, only terms with even \( m' \) survive and, therefore, we put \( m' = 2m \) in (3.5), which can be rewritten as follows:
\[
\sum_{k+\ell+m=n} \text{Exp}_{2k}(h) \#_{2\ell} G_{2m} = 0, \quad n \geq 1.
\]
Isolating the term with \( m = n \), and using \( \text{Exp}_0(h) = e^h \), yields Eq. (3.4c). □

Equation (3.4c) recursively relates the terms of the semiclassical expansion of \( \mathcal{G} \) to \( \mathcal{G}_0 \) (which is a “classical” distribution) and to the terms of the expansion of \( \text{Exp}(h) \), which are well known in literature (see e.g. Ref. [10]).

**Remark 3.1** The expansion of \( \text{Exp}(h) \) can obtained as follows. For \( \beta \geq 0 \) let \( F(\beta) = e^{\beta H} \), with \( H = \text{Op}_\epsilon(h) \). Then \( F(\beta) \) satisfies the semigroup equation
\[
F'(\beta) = HF(\beta), \quad F(0) = I,
\]
and, therefore, \( f(\beta) = \text{Op}_\epsilon^{-1}(F(\beta)) = \text{Exp}(\beta h) \) satisfies
\[
f'(\beta) = h\# f(\beta), \quad f(0) = 1,
\]
whose solution can be easily expanded at the different orders in \( \epsilon \), yielding the expansion of \( \text{Exp}(h) \) for \( \beta = 1 \) (see Ref. [10] for details). A similar procedure could be used to get the
expansion of $G$ from an evolution equation. Indeed, if we now define $F(\beta) = (e^{\beta H} + \lambda)^{-1}$, it is not difficult to see that $F$ satisfies the nonlinear semigroup equation

$$F'(\beta) = -HF(\beta)(I - \lambda F(\beta)), \quad F(0) = (1 + \lambda)^{-1}I,$$

and $f(\beta) = \text{Op}_{\epsilon}^{-1}(F(\beta))$ satisfies

$$f'(\beta) = -\hbar f(\lambda f(\beta)) # (1 - \lambda f(\beta)), \quad f(0) = (1 + \lambda)^{-1},$$

whose semiclassical expansion yields the terms $G_n$ for $\beta = 1$. Note that in the MB case, $\lambda = 0$, we get the equation for $\text{Exp}(\hbar)$. Note also that in the Bose-Einstein case, $\lambda = -1$ the initial datum is singular. Of course, using Eqs. (3.4) is a much simpler approach (if the expansion of $\text{Exp}(\hbar)$ is known) but the differential approach may be of some interest, e.g. from the numerical point of view.

The expansion (3.4) will now be used to obtain semiclassically approximated expressions for $A$ and $B$ as functions of $n$ and $J$ from the constraints (2.37b), that we rewrite here:

$$\langle G \rangle = n, \quad \langle p G \rangle = J. \quad (3.6)$$

We remark that the expansion (3.4) refers to $G$ as a function of $\epsilon$, because we provisionally assumed $A$ and $B$ of order 1. We have now to consider that $A$ and $B$ have an expansion in powers of $\epsilon$, which is determined from the constraint equations according to the following lemma.

**Lemma 3.1** Let $A$ and $B$ be solutions of the constraint system (3.6). Then, they can be formally expanded as follows:

$$A = A^{(0)} + \epsilon^2 A^{(2)} + O(\epsilon^4), \quad B = B^{(0)} + \epsilon^2 B^{(2)} + O(\epsilon^4), \quad (3.7)$$

where $A^{(0)}$, $A^{(2)}$, $B^{(0)}$ and $B^{(2)}$ satisfy the following system:

$$\begin{pmatrix}
\langle G_0 \rangle \\
\langle p_i G_0 \rangle
\end{pmatrix}_{(A^{(0)}, B^{(0)})} = \begin{pmatrix}
n \\
J_i
\end{pmatrix} \quad (3.8a)$$

$$\begin{pmatrix}
\frac{\partial \langle G_0 \rangle}{\partial A} & \frac{\partial \langle G_0 \rangle}{\partial p_i} \\
\frac{\partial \langle p_i G_0 \rangle}{\partial A} & \frac{\partial \langle p_i G_0 \rangle}{\partial p_j}
\end{pmatrix}_{(A^{(0)}, B^{(0)})} \begin{pmatrix}
A^{(2)} \\
B^{(2)}
\end{pmatrix} = - \begin{pmatrix}
\langle G_2 \rangle \\
\langle p_i G_2 \rangle
\end{pmatrix}_{(A^{(0)}, B^{(0)})} \quad (3.8b)$$

where $G_0$ and $G_2$ are given by Proposition 3.1 and the subscript $(A^{(0)}, B^{(0)})$ means that the expression has to be evaluated in $A = A^{(0)}$ and $B = B^{(0)}$.

**Proof** See Appendix B.2.

Equations (3.8) involve moments of the functions $G_0$ and $G_2$. Such moments will be explicitly expressed in terms of the functions

$$\phi_s(z) := -\frac{1}{\lambda} \text{Li}_s(-\lambda e^z)$$
(Li_s denoting the polylogarithm function of order s [24]), which are extensively described in Appendix A.

We begin with the computation of \( A^{(0)} \) and \( B^{(0)} \), which are determined by Eq. (3.8a) alone.

**Proposition 3.2** Let \( n_d := (2\pi T)^{\frac{d}{2}} \) and assume

\[
0 < n < \begin{cases} 
\frac{n_d \zeta(\frac{d}{2})}{|\lambda|}, & \text{if } \lambda < 0 \text{ and } d \geq 3, \\
\infty, & \text{otherwise}
\end{cases} 
\]  

(3.9)

(where \( \zeta \) is the Riemann zeta function). Then, the solution of system (3.8a) is

\[
A^{(0)} = T \phi_{\frac{d}{2}}^{-1}\left(\frac{n}{n_d}\right), \quad B^{(0)}_i = u_i,
\]

(3.10)

where \( u = J/n \), and \( \phi_{\frac{d}{2}}^{-1} \) is the inverse of the function \( \phi_{\frac{d}{2}} \) (Definition A.1).

**Proof** Since

\[
G_0 = \left( e^{\frac{-p^2}{2} - \frac{A}{T} + \lambda} \right)^{-1},
\]

by using Eq. (A.3) we obtain

\[
\langle G_0 \rangle = n_d \phi_{\frac{d}{2}}\left(\frac{A}{T}\right), \quad \langle p_i G_0 \rangle = B_i \langle G_0 \rangle.
\]

(3.11)

Then, from Eq. (3.8a) we immediately get \( B^{(0)}_i = J_i/n = u_i \), while for \( A^{(0)} \) we have to solve the equation

\[
\phi_{\frac{d}{2}}\left(\frac{A^{(0)}}{T}\right) = \frac{n}{n_d}.
\]

Now, \( \phi_{\frac{d}{2}}(z) \) is an increasing function of \( z \) (as it is apparent from Eq. (A.3)), and ranges from 0 to \( +\infty \) unless \( \lambda < 0 \) and \( d \geq 3 \), in which case it reaches a maximum value \( \zeta(\frac{d}{2})/|\lambda| \) as \( z \to 0^- \) (this follows from Eq. (A.1)). Thus, in the assumption (3.9), the above equation can be uniquely solved and we can write \( A^{(0)} = T \phi_{\frac{d}{2}}^{-1}\left(\frac{n}{n_d}\right) \). □

**Remark 3.2** The condition \( n < n_d \zeta(\frac{d}{2})/|\lambda| \), for \( \lambda < 0 \) and \( d \geq 3 \), reflects the fact that, at dimension 3 or higher, BE statistics is able to “accommodate” only a limited number of particles. The exceeding particles, according to Bose-Einstein theory (and to experiments as well), are expected to fall in the fundamental state, giving rise to the Bose-Einstein condensate. Since \( n_d = (2\pi T)^{\frac{d}{2}} \), we have that the particles are all non-condensate if \( T \) is above the critical temperature

\[
T_c = \frac{1}{2\pi} \left( \frac{|\lambda| n}{\zeta(\frac{d}{2})} \right)^{2/d}.
\]
or, using the physical density \( N_0 n \) (with \( N_0 \) given by (2.20)),

\[
T_c = \frac{2\pi \hbar^2}{mk_B} \left( \frac{|\lambda| n}{\zeta \left( \frac{d}{2} \right)} \right)^{2/d}.
\]

Our discussion is therefore limited to the non-condensate, or supercritical, phase. The full description of a quantum fluid equation with BE statistics would require a coupling between the non-condensate and the condensate phases, which is matter for future work.

Next, we compute \( A^{(2)} \) and \( B^{(2)} \) from Eq. (3.8b). This involves the computation of \( \langle G_2 \rangle \) and \( \langle p_i G_2 \rangle \), which is done in next lemma.

**Lemma 3.2** The moments \( \langle G_2 \rangle \) and \( \langle p_i G_2 \rangle \), for generic Lagrange multipliers \( A \) and \( B \), are given by

\[
\langle G_2 \rangle = \frac{n_d}{24T^2} \left[ 2\Delta A - \frac{\partial B_j}{\partial x_k} \left( \frac{\partial B_j}{\partial x_k} - \frac{\partial B_k}{\partial x_j} \right) \right] \phi_{2-2}^{\frac{n}{2-1}} \left( \frac{A}{T} \right) + \frac{n_d}{24T^3} \nabla A^2 \phi_{2-3}^{\frac{n}{2-3}} \left( \frac{A}{T} \right),
\]

\[
\langle p_i G_2 \rangle = B_i \langle G_2 \rangle + \frac{n_d}{12T} \frac{\partial}{\partial x_j} \left[ \left( \frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j} \right) \phi_{2-1}^{\frac{n}{2-1}} \left( \frac{A}{T} \right) \right],
\]

with \( n_d = (2\pi T)^{\frac{d}{2}} \).

**Proof** See Appendix [B.3].

Thanks to Lemma 3.2, we are now ready to compute the second-order terms in the semiclassical expansion (3.7) of the Lagrange multipliers, which are obtained from system (3.8b).

**Proposition 3.3** The solution of system (3.8b) is

\[
A^{(2)} = \frac{1}{24T} \frac{\partial u_j}{\partial x_k} \left( \frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} \right) \phi_{2-2}^{\frac{n}{2-1}} (n)
\]

\[
- \frac{1}{24} \left[ 2\Delta A^{(0)} (n) \phi_{2-2}^{\frac{n}{2-1}} (n) + \left| \frac{\nabla A^{(0)} (n)}{T} \right|^2 \phi_{2-3}^{\frac{n}{2-1}} (n) \right],
\]

\[
B_i^{(2)} = \frac{n_d}{12Tn} \frac{\partial}{\partial x_j} \left[ \left( \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) \phi_{2-1}^{\frac{n}{2-1}} (n) \right],
\]

where \( A^{(0)} (n) \) is given by Eq. (3.10), \( u = J/n, n_d = (2\pi T)^{\frac{d}{2}} \) and

\[
\phi_{s}^{\frac{n}{d}} (n) = \phi_{s} \left( \frac{A^{(0)} (n)}{T} \right) = \phi_{s} \left( \frac{\phi_{2-1}^{\frac{n}{2-1}} (n)}{n_d} \right).
\]
Proof What we have to do is solving the linear equation (3.8b) for the unknowns $A^{(2)}$ and $B^{(2)}$, the expressions of $A^{(0)}$ and $B^{(0)}$ being given by (3.10). The derivatives of $\langle G_0 \rangle$ and $\langle p_j G_0 \rangle$ with respect to $A$ and $B$ are easily obtained from Eq. (3.11). Evaluating the resulting expressions in $A = A^{(0)}$ and $B = B^{(0)}$ we obtain

\[
\left( \frac{\partial \langle G_0 \rangle}{\partial A} \quad \frac{\partial \langle G_0 \rangle}{\partial B} \right)_{(A^{(0)},B^{(0)})} = \frac{n_d}{T} \left( \begin{array}{cc} \phi_0^{0} - 1 & 0 \\ u_i \phi_0^{0} - 1 & \delta_{ij} Tn/n_d \end{array} \right),
\]

where $\phi_s = \phi_s(n)$ is given by (3.14), and we used the fact that $\phi_0^{0} = n/n_d$. The inverse matrix is easily computed to be

\[
\left[ \frac{n_d}{T} \left( \begin{array}{cc} \phi_0^{0} - 1 & 0 \\ u_i \phi_0^{0} - 1 & \delta_{ij} Tn/n_d \end{array} \right) \right]^{-1} = \left( T/(n_d \phi_0^{0} - 1) \quad 0 \\ -u_i/n \quad \delta_{ij}/n \right)
\]

and then

\[
\left( \begin{array}{c} A^{(2)} \\ B_i^{(2)} \end{array} \right) = \left( -T/(n_d \phi_0^{0} - 1) \quad 0 \\ u_i/n \quad -\delta_{ij}/n \right) \left( \begin{array}{c} \langle G_2 \rangle \\ \langle p_j G_2 \rangle \end{array} \right)_{(A^{(0)},B^{(0)})},
\]

where $\langle G_2 \rangle_{(A^{(0)},B^{(0)})}$ and $\langle p_j G_2 \rangle_{(A^{(0)},B^{(0)})}$ are obtained by substituting $A^{(0)}$ and $B^{(0)}$ for $A$ and $B$ in the expressions (3.12). This immediately yields Eqs. (3.13).

By using the derivation rules

\[
\nabla A^{(0)}(n) = \frac{n_d}{n_d \phi_0^{0} - 1(n)} \nabla n, \quad \nabla \phi_0^{0}(n) = \phi_0^{0} - 1(n) \frac{n_d}{n_d \phi_0^{0} - 1(n)} \nabla n,
\]

it is readily seen that the term between square brackets in Eq. (3.13a) can be given the more explicit form

\[
\frac{2\Delta A^{(0)}}{T} \frac{\phi_0^{0} - 2}{\phi_0^{0} - 1} + \frac{|\nabla A^{(0)}|^2}{T^2} \frac{\phi_0^{0} - 3}{\phi_0^{0} - 1} = 2\Delta n \frac{\phi_0^{0} - 2}{n_d (\phi_0^{0} - 1)^2} + \frac{|\nabla n|^2}{n_d^2} \left( \frac{\phi_0^{0} - 3}{(\phi_0^{0} - 1)^3} - 2 \frac{(\phi_0^{0} - 2)^2}{(\phi_0^{0} - 1)^4} \right)
\]

(where the arguments $n$ have been omitted). As we shall see in Sec. 4.2, this term can be identified as a modified Bohm potential, since it gives the usual “statistical” Bohm potential in the MB limit $\lambda \to 0$.

3.2 Semiclassical fluid equations

The semiclassical expansion of the Lagrange multipliers, found in the previous section, can now be substituted in Eqs. (2.37a) and (2.38a) to obtain semiclassical hydrodynamic and drift-diffusion equations.

In order to do that, let us consider the term (2.37a) that appear in both equations (with $B = 0$ in the diffusive case) and contains the Lagrange multipliers. Let us rewrite it, by
using the velocity variable \( u = J/n \), and expand it according to (3.1). Taking account that \( B^{(0)} = u \) (Proposition 3.2), we obtain
\[
\frac{\partial}{\partial x_j}(nu_i B_j) + n(u_j - B_j) \frac{\partial B_j}{\partial x_i} + n \frac{\partial A}{\partial x_j} (nu_i u_j) + n \frac{\partial}{\partial x_i} A^{(0)}
\]
\[
+ \epsilon^2 \left( n B_j^{(2)} R_{ij} + u_i \frac{\partial}{\partial x_j} (n B_j^{(2)}) + n \frac{\partial}{\partial x_i} A^{(2)} \right) + O(\epsilon^4),
\]
where we introduced the notation
\[
R_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}
\]
for the velocity curl tensor. Now, using Eq. (3.15) we can write
\[
\frac{\partial}{\partial x_i} A^{(0)} = \frac{T}{n_d \phi_s^{(2)}} \frac{\partial n}{\partial x_i}.
\]
Moreover, from Eqs. (3.13) and (3.16) we obtain
\[
n B_j^{(2)} R_{ij} + u_i \frac{\partial}{\partial x_j} (n B_j^{(2)}) + n \frac{\partial}{\partial x_i} A^{(2)} =
\]
\[
n d R_{ij} \frac{\partial}{\partial x_k} \left( R_{kj} \phi_s^{(2)} \right) + n \frac{\partial}{\partial x_i} Q(n)
\]
where we used the identities
\[
R_{jk} \frac{\partial u_j}{\partial x_k} = \frac{1}{2} R_{jk} R_{jk}, \quad \frac{\partial^2}{\partial x_j \partial x_k} \left( R_{kj} \phi_s^{(2)} \right) = 0,
\]
and introduced the notation
\[
Q(n) = -\frac{1}{24} \left[ \frac{2 \Delta n}{n_d} \phi_s^{(2)} - \frac{2}{\phi_s^{(2)}} \right] + \frac{1}{48T} \frac{\partial n}{\partial x_i} \left( R_{jk} \phi_s^{(2)} \phi_s^{(2)} \phi_s^{(2)} \phi_s^{(2)} \right) = 0
\]
for the modified Bohm potential. Hence, we can state the main results of this section.

**Theorem 3.1** Assume that condition (3.9) is satisfied for all times. Then, neglecting terms of order \( O(\epsilon^4) \), the isothermal quantum hydrodynamic model (2.37) admits the following, formal, approximation
\[
\begin{cases}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} (nu_i) = 0 \\
\frac{\partial}{\partial t} (nu_i) + \frac{\partial}{\partial x_j} (nu_i u_j) + n \frac{\partial V}{\partial x_i} + Tn \frac{n \partial}{n \phi_s^{(2)} \partial x_i} + \epsilon^2 n \frac{\partial Q}{\partial x_i} \\
+ \epsilon^2 n d R_{ij} \frac{\partial}{\partial x_k} \left( R_{kj} \phi_s^{(2)} \right) + \epsilon^2 n \frac{\partial}{48T} \frac{\partial}{\partial x_i} \left( R_{jk} \phi_s^{(2)} \phi_s^{(2)} \phi_s^{(2)} \phi_s^{(2)} \right) = 0,
\end{cases}
\]
where \( \phi_s^{(2)} = \phi_s^{(2)}(n) \) is given by Eq. (3.14), \( R_{ij} \) is given by Eq. (3.17), \( Q = Q(n) \) is given by Eq. (3.18) and \( n_d = (2\pi T)^{\frac{1}{2}} \).
Theorem 3.2 Assume that condition (3.9) is satisfied at all times. Then, neglecting terms of order \(O(\epsilon^4)\), the quantum diffusive model (2.38) admits the following, formal, approximation

\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial x_i} \left( \frac{T n}{n_d \phi_s^{0}} \frac{\partial n}{\partial x_i} + n \frac{\partial V}{\partial x_i} + \epsilon^2 n \frac{\partial Q}{\partial x_i} \right),
\]

(3.20)

where \(\phi_s^0 = \phi_s^0(n)\) is given by Eq. (3.14), \(Q = Q(n)\) is given by Eq. (3.18) and \(n_d = (2\pi T)^{\frac{d}{2}}\).

Equations (3.19) and (3.20) are the generalized version of the semiclassical hydrodynamic and diffusive equations derived for MB statistics in Refs. [9, 21] and [10] (see also Subsec. 4.2). Equation (3.20) with \(\epsilon = 0\) has been derived in Ref. [20] (where \(\lambda > 0\) is assumed, although this does not affects the form of the equation). A simplified version of Eq. (3.19) has been derived in Ref. [27], where the terms of order \(\epsilon^2\) that depend on \(R\) are missing.

4 Analysis of particular regimes

In this section we investigate the form taken by the semiclassical hydrodynamic and diffusive equations, Eqs. (3.19) and (3.20), in some specific physical regime. In particular, we shall consider the irrotational regime, the Maxwell-Boltzmann limit, \(\lambda \rightarrow 0\), and the zero temperature limit, \(T \rightarrow 0\). Let us stress the fact that all statement and proofs are purely formal.

4.1 The irrotational fluid

First of all, let us look at the form taken by the hydrodynamic equations (3.19) when initial data are irrotational (\(R = 0\)). The following proposition and its proof are similar to those of an analogous result given in Ref. [9] for the fully-quantum equations with general convex entropy. On the other hand, our statement can be a bit more precise because we are dealing with the simpler case of semiclassical approximation.

Proposition 4.1 Let \((n, u)\) be a smooth solution of Eq. (3.19) such that \(B^{(2)} \in W^{1, \infty}(\mathbb{R}^d)\) at all times, and assume that the fluid is initially irrotational, i.e. \(R = 0\) at \(t = 0\). Then the fluid remains irrotational at all times and, therefore, \(n\) and \(u\) satisfy

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} (nu_i) &= 0 \\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_j}{\partial x_i} + \frac{\partial V}{\partial x_i} + \frac{T}{n_d \phi_s^{0}} \frac{\partial n}{\partial x_i} + \epsilon^2 \frac{\partial Q}{\partial x_i} &= 0.
\end{align*}
\]

(4.1)

Proof By using the identity

\[
\frac{\partial}{\partial t} (nu_i) + \frac{\partial}{\partial x_j} (nu_i u_j) = n \left( \frac{\partial u_i}{\partial t} + R_{ij} u_j + u_j \frac{\partial u_i}{\partial x_i} \right)
\]

(4.2)
(where the continuity equation, i.e. the first of (3.19), was used), the second equation of (3.19) can be rewritten as follows:

\[
\frac{\partial u_i}{\partial t} + R_{ij} u_j + \frac{\epsilon^2 n_d R_{ij}}{12 T n} \frac{\partial}{\partial x_l} \left( R_{ij} \phi^0_{\frac{d}{2}-1} \right) \\
+ \frac{\partial}{\partial x_i} \left( \frac{1}{2} |u|^2 + V + T \phi^{-1}_d \left( \frac{n}{n_d} \right) + \epsilon^2 Q + \frac{\epsilon^2 R_{jk} R_{jk}}{48 T} \phi^0_{\frac{d}{2}-2} \right) = 0.
\]

Then, the following equation for \( R_{ik} = \frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \) is obtained:

\[
\frac{\partial}{\partial t} R_{ik} + \frac{\partial}{\partial x_k} \left( R_{ij} \theta_j \right) - \frac{\partial}{\partial x_i} \left( R_{kj} \theta_j \right) = 0,
\]

where

\[
\theta_j := u_j + \frac{\epsilon^2 n_d}{12 T n} \frac{\partial}{\partial x_l} \left( R_{ij} \phi^0_{\frac{d}{2}-1} \right) = B^{(0)}_j + \epsilon^2 B^{(2)}_j,
\]

and also (from a direct calculation using the definition of \( R_{ij} \))

\[
\frac{\partial}{\partial t} R_{ik} + \theta_j \frac{\partial}{\partial x_j} R_{ik} + R_{ij} \frac{\partial}{\partial x_k} \theta_j - R_{kj} \frac{\partial}{\partial x_i} \theta_j = 0.
\]

Multiplying both sides by \( R_{ik} \), summing over \( i \) and \( k \), and integrating over \( x \in \mathbb{R}^d \) yields

\[
\frac{d}{dt} \int_{\mathbb{R}^d} |R|^2 \, dx = \int_{\mathbb{R}^d} |R|^2 \, \text{div} \theta \, dx + 4 \int_{\mathbb{R}^d} R^2 : \nabla \theta \, dx,
\]

where \( |R|^2 = \sum_{i,k} R_{ik}^2 \) and \( R^2 : \nabla \theta = R_{ik} R_{kj} \frac{\partial}{\partial x_l} \theta_j \). From our assumptions on \( u \) and \( B^{(2)} \) we have that \( \|\nabla \theta\|_{\infty} \) is finite (and continuous) in time, and we can write

\[
\frac{d}{dt} \int_{\mathbb{R}^d} |R|^2 \, dx \leq C \|\nabla \theta\|_{\infty} \int_{\mathbb{R}^d} |R|^2 \, dx
\]

for some constant \( C > 0 \). Since \( R = 0 \) at \( t = 0 \), by the Gronwall lemma we obtain that \( R = 0 \) at all times and then, using again the identity (4.2), Eq. (3.19) reduces to Eq. (4.1). \( \square \)

We remark that an important class of irrotational initial data is that of pure states. In fact, it is easy to show that the velocity field

\[
u = \frac{\epsilon}{2 i |\psi|^2} \left( \overline{\psi} \nabla \psi - \psi \nabla \overline{\psi} \right),
\]

associated to the pure state represented by the wave function \( \psi(x) \) (see Eq. (2.12)), has \( R = 0 \). Of course, Proposition 4.1 is not saying that an initially pure state will remain pure, but just that it will remain irrotational. In Ref. [9] it is proven that a pure state remains pure in the limit \( T \to 0 \) and assuming MB statistics, in which case the fully-quantum system (2.37) reduces to Madelung equations (4.16) (that are the hydrodynamic form of Schrödinger equation [25]). We stress that the fully-quantum system (2.37) that possesses a limit for \( T \to 0 \) and not the semiclassical equations Eq. (3.19), which behave singularly in this limit for MB statistics. We shall discuss this point with more details in Subsec. 4.3.
4.2 The Maxwell-Boltzmann limit

The MB limit of Eqs. (3.19) and (3.20) is obtained by letting $\lambda \to 0$. We remark that the parameter $\lambda$ is hidden in the functions $\phi^0_s$, that are defined by (3.14) and (A.2). Then, from property (A.4a) we immediately obtain

$$\lim_{\lambda \to 0} \phi^0_s(n) = \frac{n}{n_d},$$

(4.3)

for any $s \in \mathbb{R}$. In particular, as far as the modified Bohm potential is concerned (see Eq. (3.18)), we obtain

$$\lim_{\lambda \to 0} \epsilon^2 Q(n) = -\frac{\epsilon^2}{24} \left( \frac{2\Delta n}{n} - \frac{|\nabla n|^2}{n^2} \right) = -\frac{\epsilon^2}{6} \Delta \sqrt{n},$$

(4.4)

that is the usual (statistical) Bohm potential [10].

**Remark 4.1** What is commonly termed “Bohm potential” [4, 5] is the quantum potential appearing in Madelung equations (4.16), namely

$$V_B(n) = -\frac{\epsilon^2}{2} \frac{\Delta \sqrt{n}}{\sqrt{n}}.$$

This is a “pure-state” Bohm potential, which differs for a factor $1/3$ from what we termed “statistical” Bohm potential. i.e. (4.4). The latter arises naturally from the quantum entropy principle. Which form the Bohm potential should have in quantum fluid equations is a long-standing debate, see e.g. Ref. [13] and references therein.

As far as the rotational terms are concerned, i.e. the terms of Eq. (3.19) that depend on the velocity curl tensor $R$, we obtain

$$\lim_{\lambda \to 0} \left[ \frac{\epsilon^2 n_d R_{ij}}{12T} \frac{\partial}{\partial x_k} \left( R_{kj} \phi^0_{\frac{\Phi}{\Phi} - 1} \right) + \frac{\epsilon^2 n}{48T} \frac{\partial}{\partial x_i} \left( R_{jk} R_{jk} \phi^0_{\frac{\Phi}{\Phi} - 2} \right) \right] = \frac{\epsilon^2 R_{ij}}{12T} \frac{\partial}{\partial x_k} (R_{kj}) + \frac{\epsilon^2 n}{48T} \frac{\partial}{\partial x_i} (R_{jk} R_{jk}).$$

The last expression can be simplified by considering the identity

$$\frac{\partial}{\partial x_k} (n R_{ij} R_{kj}) = R_{ij} \frac{\partial}{\partial x_k} (n R_{kj}) + \frac{n}{4} \frac{\partial}{\partial x_i} (R_{jk} R_{jk})$$

(recall that we sum over the repeated indices $j$ and $k$), so that

$$\frac{\epsilon^2 R_{ij}}{12T} \frac{\partial}{\partial x_k} (R_{kj}) + \frac{\epsilon^2 n}{48T} \frac{\partial}{\partial x_i} (R_{jk} R_{jk}) = \frac{\epsilon^2}{12T} \frac{\partial}{\partial x_k} (n R_{ij} R_{kj}),$$

where $\frac{\partial}{\partial x_k} (n R_{ij} R_{kj})$ is the expression in components of $\text{div}(n R R^T)$ (this is the form in which the rotational terms are written in Ref. [24]). All this considered we can state the following.
Proposition 4.2 In the Maxwell-Boltzmann limit, \( \lambda \to 0 \), the semiclassical hydrodynamic and diffusive equations (3.19) and (3.20) take, respectively, the form

\[
\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} (nu_i) &= 0 \\
\frac{\partial}{\partial t} (nu_i) + \frac{\partial}{\partial x_j} (nu_i u_j) + n \frac{\partial V}{\partial x_i} + T \frac{\partial n}{\partial x_i} \\
&- \frac{\epsilon^2}{6} n \frac{\partial}{\partial x_i} \frac{\Delta \sqrt{n}}{\sqrt{n}} + \frac{\epsilon^2}{12T} \frac{\partial}{\partial x_k} (n R_{ij} R_{kj}) &= 0,
\end{aligned}
\]

(4.5)

and

\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial x_i} \left( T \frac{\partial n}{\partial x_i} + n \frac{\partial V}{\partial x_i} - \frac{\epsilon^2}{6} n \frac{\partial}{\partial x_i} \frac{\Delta \sqrt{n}}{\sqrt{n}} \right).
\]

(4.6)

The isothermal hydrodynamic equations (4.5) were first derived in Refs. [21] and [9]. In the latter, the rotational term is expressed in the equivalent\(^5\) form (in the three-dimensional case)

\[
\frac{\epsilon^2}{12T} \text{div} (n R R^T) = \frac{\epsilon^2}{12T} \omega \times (\nabla \times (n \omega)) + \frac{\epsilon^2}{24T} n \nabla |\omega|^2,
\]

where \( \omega = \nabla \times u \). The diffusive equation (4.6) was first derived in Ref. [10].

We finally remark that, as it can be easily deduced from (A.4b), the MB limit can be equivalently obtained by fixing \( \lambda \neq 0 \) and letting \( T \to +\infty \).

4.3 The zero-temperature limit

The behavior of Eqs. (3.19) and (3.20) in the limit \( T \to 0 \) depends dramatically on the sign of \( \lambda \). For this reason we divide the analysis of such limit in the three reference cases \( \lambda = 1 \) (FD), \( \lambda = 0 \) (MB) and \( \lambda = -1 \) (BE).

In view of the following discussion, it is convenient to extend property (A.4c) to all positive \( \lambda \). For \( z \in \mathbb{R} \) and \( s \in \mathbb{R} \), let us denote by \( F_s(z) := -\text{Li}_s(-e^z) \) the function \( \phi_s(z) \) for \( \lambda = 1 \). Then, assuming \( \lambda > 0 \), from property (A.4c) we obtain

\[
\phi_s(z) = \frac{1}{\lambda} F_s(z + \log \lambda) \sim \frac{(z + \log \lambda)^s}{\lambda \Gamma(s + 1)}, \quad \text{as } z \to +\infty,
\]

(4.7)

where \( s \neq -1, -2, \ldots \), and \( f \sim g \) means \( f/g \to 1 \). Then, recalling that \( n_d = (2\pi T)^{d/2} \), we have

\[
\phi_s^{-1} \left( \frac{n}{n_d} \right) \sim \left( \frac{\lambda \Gamma \left( \frac{d}{2} + 1 \right) n}{n_d} \right)^{\frac{d}{2}} - \log \lambda, \quad \text{as } T \to 0.
\]

(4.8)

Hence, recalling definition (3.14) and combining (4.7) with (4.8) we obtain

\[
\phi_0^0(n) \sim \frac{\lambda^{\frac{d}{4} - 1}}{\Gamma(s + 1)} \left( \frac{\Gamma \left( \frac{d}{4} + 1 \right) n}{n_d} \right)^{\frac{d}{4}}, \quad \text{as } T \to 0,
\]

(4.9)

\(^5\)Actually, in Ref. [9] the factor \( 1/T \) seems to be missing.
which holds for \( \lambda > 0 \) and \( s \neq -1, -2, \ldots \). This formula can be used to obtain the formal asymptotics for \( T \to 0 \) of the various temperature-dependent terms that appear in Eqs. (3.19) and (3.20). In particular, we have

\[
\frac{T}{n_d \phi^0_{d-1}} \to \lambda^{\frac{d}{2}} \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi} n^{\frac{d-2}{d}},
\]

(4.10a)

\[
\frac{1}{n_d (\phi^0_{d-1})^2} \to \frac{(d-2)}{d} \frac{1}{n},
\]

(4.10b)

\[
\frac{1}{n_d^2 (\phi^0_{d-1})^3} \to \frac{(d-4)(d-2)}{d^2} \frac{1}{n^2},
\]

(4.10c)

\[
\frac{1}{T} \phi^0_{d-1} \to \frac{d-2}{2} \frac{2\pi n^{-\frac{d}{2}}}{\lambda^{\frac{d}{2}} \Gamma\left(\frac{d}{2} + 1\right)},
\]

(4.10d)

as \( T \to 0 \) (note that the right-hand sides do not depend on \( T \)). From (4.10b), (4.10c) and (3.18) we get the interesting limit

\[
Q(n) \to -\frac{1}{6} \frac{(d-2)}{d} \frac{\Delta \sqrt{n}}{\sqrt{n}}, \quad \text{as } T \to 0,
\]

(4.11)

which is independent on \( \lambda \) too.

**4.3.1 FD case**

Let us first of all consider the limit \( T \to 0 \) of Eqs. (3.19) and (3.20) assuming FD statistics. This is the richest case since, as Eqs. (4.10) and (4.11) show, for \( \lambda > 0 \) the behavior of the semiclassical fluid equations is regular as temperature goes to 0 (this is not the case for MB and BE statistics, as we shall see next). Then, it is enough to set \( \lambda = 1 \) in Eqs. (4.10) and (4.11) to obtain the following.

**Proposition 4.3** Let \( \lambda = 1 \). Then, in the limit \( T \to 0 \) (also known as completely degenerate limit) the semiclassical hydrodynamic and diffusive equations (3.19) and (3.20) take, respectively, the form

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} (nu_i) &= 0, \\
\frac{\partial}{\partial t} (nu_i) + \frac{\partial}{\partial x_j} (nu_i u_j) + n \frac{\partial V}{\partial x_i} + \gamma_1 \frac{\partial}{\partial x_i} n^{\frac{d-2}{d}} - \epsilon^2 \gamma_2 n \frac{\partial}{\partial x_i} \frac{\Delta \sqrt{n}}{\sqrt{n}} \\
&\quad + \epsilon^2 \gamma_3 R_{ij} \frac{\partial}{\partial x_k} \left( R_{kj} n^{\frac{d-2}{d}} \right) + \epsilon^2 \gamma_4 n \frac{\partial}{\partial x_i} \frac{R_{jk} R_{jk}}{n^2} = 0,
\end{align*}
\]

(4.12)

and

\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial x_i} \left( \gamma_1 \frac{\partial}{\partial x_i} n^{\frac{d-2}{d}} + n \frac{\partial V}{\partial x_i} - \epsilon^2 \gamma_2 n \frac{\partial}{\partial x_i} \frac{\Delta \sqrt{n}}{\sqrt{n}} \right),
\]

(4.13)
where
\[
\gamma_1 = \frac{1}{2\pi} \frac{d}{d+2} \left( \frac{d}{2} \right)^{\frac{2-d}{2}} \Gamma \left( \frac{d}{2} \right), \quad \gamma_2 = \frac{d-2}{6d},
\]
\[
\gamma_3 = \frac{d\pi}{12\Gamma \left( \frac{d}{2} + 1 \right)^{\frac{3}{2}}}, \quad \gamma_4 = \frac{(d-2)d}{48\Gamma \left( \frac{d}{2} + 1 \right)^{\frac{3}{2}}},
\]
We remark the particularly simple form of the limit Bohm potential: it is just the (statistical) Bohm potential multiplied by the factor \( \frac{d-2}{d} \). Noticeably, it vanishes for \( d = 2 \) and changes sign for \( d = 1 \). Note that for \( d = 2 \) also the coefficient \( \gamma_4 \) vanishes.

Equations (4.12) and (4.13) with \( R = 0 \) and \( d = 3 \) have been obtained in Ref. [27] (where also “weakly degenerate” and “strongly degenerate” limits are considered). Equation (4.13) with \( \epsilon = 0 \) and \( d = 3 \) has been obtained in Ref. [20] (where also energy-transport equations are considered, which however reduce to the diffusive equation for \( T \to 0 \)). Further references to degenerate fluid models can be found in Refs. [20, 27, 28].

4.3.2 MB case

Maxwell-Boltzmann statistics, for the \( T \to 0 \) limit, is a very singular case. Indeed, looking at Eqs. (4.3) and (4.9), we notice that, at fixed \( n \), the function \( \phi^n_{\epsilon-k} (n) \) (where \( k = 1, 2, 3 \) are the relevant cases) behaves like \( T^{-\frac{d}{2}} \) for \( \lambda \to 0 \) and \( T > 0 \), and behaves like \( \lambda^{-\frac{d}{2}} T^{k-\frac{d}{2}} \) for \( T \to 0 \) and \( \lambda > 0 \). Then, the two limits \( \lambda \to 0 \) and \( T \to 0 \) are somehow incompatible, and the corresponding behavior of the fluid equations depends on how the point \((0, 0)\) is approached in the parameter space \((\lambda, T)\).

Let us consider the two paths: \( \lambda \to 0 \) followed by \( T \to 0 \), and \( T \to 0 \) followed \( \lambda \to 0 \). The first path corresponds to starting from the MB equations (4.5) and (4.6), and then letting \( T \to 0 \); the second path corresponds to using first the asymptotic identities (4.10) and (4.11) in Eqs. (3.19) and (3.20), and then letting \( \lambda \to 0 \). In both cases the rotational terms are singular and, therefore, the limit is only compatible with irrotational solutions (see Proposition 4.1). Moreover, in both cases, the diffusive term \( T_n \frac{\partial n}{\partial x_i} \) vanishes asymptotically. Thus, assuming \( R = 0 \) (see Eq. (4.1)), we obtain from both paths equations of the form
\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} (nu_i) &= 0 \\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_j}{\partial x_i} + \frac{\partial V}{\partial x_i} - \gamma \frac{\epsilon^2}{6} \frac{\partial}{\partial x_i} \Delta \sqrt{n} &= 0,
\end{align*}
\]
and
\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial x_i} \left( n \frac{\partial V}{\partial x_i} - \gamma \frac{\epsilon^2}{6n} \frac{\partial}{\partial x_i} \Delta \sqrt{n} \right),
\]
but the coefficient \( \gamma \) changes: it is 1 for the first path and \( \frac{d-2}{d} \) for the second path.

As already mentioned, the correct point of view is probably that of Ref. [9], where the \( T \to 0 \) limit for MB statistics is discussed for the fully-quantum hydrodynamic equations (here
represented by Eqs. (2.37) and it is proven that such limit yields the Madelung equations:

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} (nu_i) &= 0, \\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_j}{\partial x_i} + \frac{\partial V}{\partial x_i} - \frac{\epsilon^2}{12} \frac{\partial}{\partial x_i} \Delta \sqrt{n} \sqrt{n} &= 0.
\end{align*}
\]

Equations (4.16) were first derived by E. Madelung [25], and can be easily obtained from Schrödinger equation by writing the wave function as \( \psi = \sqrt{n} e^{iS/\epsilon} \) and then putting \( u = \nabla S \).

4.3.3 BE case

The discussion of the \( T \to 0 \) limit for Bose-Einstein statistics, \( \lambda = -1 \), is strongly dimension-dependent and we shall examine three cases, \( d \geq 3 \), \( d = 2 \) and \( d = 1 \), separately.

For \( d \geq 3 \), the condition (3.9) is never satisfied in the limit of vanishing temperature (physically speaking, all particles will be in the condensate phase) and then such limit is a nonsense in our framework, because we are only considering a completely non-condensate fluid (see Remark 3.2). As it is well known, the mathematical description of the bose-Einstein condensate should be given in terms of a nonlinear Schrödinger equation [8].

For \( d = 2 \), the condensation does not occur and we can let \( T \) go to 0 in the semiclassical equations (3.19) and (3.20). As usual, we have to examine the asymptotic behavior of the functions \( \phi_d^0(n) \). This requires the inversion of \( \phi_d^0(\Delta) \) which, for \( \lambda = -1 \) and \( d = 2 \), is given by [24]

\[
\phi_1(z) = \text{Li}_1(e^z) = - \log(1 - e^z).
\]

Recalling that \( n_2 = 2\pi T \), we have

\[
\phi_1^{-1} \left( \frac{n}{2\pi T} \right) = \log \left( 1 - e^{-\pi T} \right) \sim -e^{-\pi T}, \quad \text{as } T \to 0.
\]

Now, since the relevant values of \( s \) for the present case are \( s = 0 \), \( s = -1 \) and \( s = -2 \) (i.e. \( \frac{d}{2} - 1 \), \( \frac{d}{2} - 2 \) and \( \frac{d}{2} - 3 \)), we can use (A.4d) and conclude that

\[
\phi_d^0(n) \sim \Gamma(1 - s) e^{\frac{(1-s)n}{\pi T}}, \quad \text{as } T \to 0.
\]

By using (4.17) in Eqs. (3.19) and (3.20) it is readily seen that in order to obtain a finite limit we have to assume \( R = 0 \) and to rescale the density as

\[
\bar{n} = \frac{n}{2\pi T},
\]

(4.18)

Then, it is not difficult to prove the following.

**Proposition 4.4** Let \( \lambda = -1 \) and \( d = 2 \), and assume \( R = 0 \). Then, in the limit \( T \to 0 \), from the semiclassical hydrodynamic and diffusive equations (3.19) and (3.20) we formally obtain, respectively, the equations

\[
\begin{align*}
\frac{\partial \bar{n}}{\partial t} + \frac{\partial}{\partial x_i} (\bar{n} u_i) &= 0, \\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_j}{\partial x_i} + \frac{\partial V}{\partial x_i} - \frac{\epsilon^2}{12} \frac{\partial}{\partial x_i} \Delta \bar{n} &= 0,
\end{align*}
\]

(4.19)
and
\[
\frac{\partial \tilde{n}}{\partial t} = \frac{\partial}{\partial x_i} \left( \tilde{n} \frac{\partial V}{\partial x_i} - \frac{e^2}{12} \tilde{n} \frac{\partial}{\partial x_i} \Delta \tilde{n} \right)
\]
(4.20)
for the rescaled density \(\tilde{n}\).

Note that in this case we have found a “degenerate” form of the limit Bohm potential, which reduces to a Laplacian.

Let us finally examine the case \(d = 1\), which does not admit condensation as well. From (A.4d) we have
\[
\phi_1(z) \sim \sqrt{\pi} \sqrt{-z}, \quad \text{as } z \to 0^-,
\]
and then, recalling that \(n_1 = \sqrt{2\pi T}\), we can write
\[
\phi^{-1}_1 \left( \frac{n}{\sqrt{2\pi T}} \right) \sim -\frac{\pi}{n^2}, \quad \text{as } T \to 0.
\]
We obtain, therefore,
\[
\phi^0_2(n) \sim \Gamma(1-s) \left( \frac{n^2}{2\pi^2 T} \right)^{1-s}, \quad \text{as } T \to 0,
\]
(4.21)
where now the relevant values are \(s = -\frac{1}{2}\), \(s = -\frac{3}{2}\) and \(s = -\frac{5}{2}\). By using (4.21), and recalling that \(R = 0\) in the present one-dimensional case, we see that the limit behavior of Eqs. (3.19) and (3.20) is non-singular, and the following is readily proven.

**Proposition 4.5** Let \(\lambda = -1\) and \(d = 1\). Then, in the limit \(T \to 0\), the semiclassical hydrodynamic and diffusive equations (3.19) and (3.20) take, respectively, the form
\[
\begin{cases}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial V}{\partial x} - \frac{e^2}{2} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{n}} \frac{\partial^2 \sqrt{n}}{\partial x^2} \right) = 0,
\end{cases}
\]
(4.22)
and
\[
\frac{\partial n}{\partial t} = \frac{\partial}{\partial x} \left( n \frac{\partial V}{\partial x} - \frac{e^2}{2} n \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{n}} \frac{\partial^2 \sqrt{n}}{\partial x^2} \right) \right).
\]
(4.23)
Note that (4.22) are the standard one-dimensional Madelung equations.

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A Moments of Fermi and Bose distributions and related integrals

We recall that the polylogarithm of order $s$, with $s \in \mathbb{R}$, is defined in the complex unit disc by the power series
\[
\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| < 1,
\]
and can be analytically continued to a larger domain (depending on $s$). To our purposes it will be enough to know that $\text{Li}_s(z)$ is always well defined, real-valued and regular for $z \in (-\infty, 1)$, and
\[
\lim_{z \to 1^{-}} \text{Li}_s(z) = \begin{cases} 
\zeta(s), & \text{if } s > 1, \\
+\infty, & \text{if } s \leq 1,
\end{cases}
\]
where $\zeta$ is the Riemann zeta function. The polylogarithms are strictly connected with the moments of FD and BE distributions.

**Definition A.1** For $\lambda e^z > -1$, $\lambda \neq 0$, and $s \in \mathbb{R}$ we define
\[
\phi_s(z) = -\frac{1}{\lambda} \text{Li}_s(-\lambda e^z),
\]
where $\text{Li}_s$ is the polylogarithm of order $s$. From known identities $[23]$ we have that, for $s > 0$, the above definition is equivalent to
\[
\phi_s(z) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}}{e^{t-z} + \lambda} dt \quad (s > 0)
\]
(known as Fermi integral).

Note that $\phi_s(z)$ is defined for all $z \in \mathbb{R}$ if $\lambda > 0$, and for $z < -\log |\lambda|$ if $\lambda < 0$. In particular, in the FD case, $\lambda = 1$, $\phi_s(z)$ is defined on the whole real line while in the BE case, $\lambda = -1$, only for $z < 0$.

The following properties of the functions $\phi_s$ can be easily deduced from the properties of polylogarithms (see e.g. Refs. [24, 31]):
\[
\begin{align*}
\lim_{\lambda \to 0} \phi_s(z) &= e^z, & \text{for } z \in \mathbb{R} \text{ and } s \in \mathbb{R}; \quad \text{(A.4a)} \\
\phi_s(z) &\sim e^z, & \text{for } z \to -\infty, \text{ and } s \in \mathbb{R}; \quad \text{(A.4b)} \\
\phi_s(z) &\sim \frac{z^s}{\Gamma(s+1)}, & \text{for } z \to +\infty, \lambda = 1 \text{ and } s \neq -1, -2, \ldots; \quad \text{(A.4c)} \\
\phi_s(z) &\sim \Gamma(1-s)(-z)^{s-1}, & \text{for } z \to 0^-, \lambda = -1 \text{ and } s < 1; \quad \text{(A.4d)} \\
\frac{d}{dz} \phi_s(z) &= \phi_{s-1}(z), & \text{for } \lambda e^z > -1 \text{ and } s \in \mathbb{R}. \quad \text{(A.4e)}
\end{align*}
\]
Here, “$f(x) \sim g(x)$ for $x \to y$” means $\lim_{x \to y} f(x)/g(x) = 1$.

Starting from the identity (A.3), we shall now compute explicit expressions, in terms of the functions $\phi_s$, of all the kinds of integrals that have been encountered throughout this paper.
Lemma A.1  For \( \lambda e^z > -1 \), \( k = 1, 2, 3, \ldots \) and \( s > 0 \), let us consider the integrals

\[
I_s^k(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{(e^{t-z} + \lambda)^k} dt.
\]

(A.5)

Then, \( I_s^1 \) is given by Eq. (A.3) and higher values of \( k \) are recursively obtained by

\[
I_s^{k+1}(z) = \frac{1}{\lambda} \left( I_s^k(z) - \frac{1}{k} \frac{dI_s^k}{dz}(z) \right).
\]

(A.6)

In particular (omitting the argument \( z \)),

\[
\begin{align*}
I_s^1 &= \phi_s \\
I_s^2 &= \lambda^{-1} (\phi_s - \phi_{s-1}) \\
I_s^3 &= \lambda^{-2} \left( \phi_s - \frac{3}{2} \phi_{s-1} + \frac{1}{2} \phi_{s-2} \right) \\
I_s^4 &= \lambda^{-3} \left( \phi_s - \frac{11}{6} \phi_{s-1} + \frac{1}{2} \phi_{s-2} - \frac{1}{6} \phi_{s-3} \right).
\end{align*}
\]

(A.7)

Proof  The recursive formula (A.6) follows immediately from a formal derivation of \( I_s^k(z) \) with respect to \( z \); Eq. (A.7) follows from (A.3) and (A.6) by using the property (A.4e). □

Equation (A.7) suggests that we can look for an expression for \( I_s^k \) of this kind:

\[
I_s^k = \frac{1}{\lambda^{k-1}} \sum_{j=0}^{k-1} c_j^k \phi_{s-j}, \quad k \geq 1,
\]

(A.8)

where \( c_j^k \) are numerical coefficients (independent on \( s \)) to be determined. Since

\[
\frac{dI_s^k}{dz}(z) = I_s^{k-1}(z)
\]

(A.9)

(as it is apparent from Lemma (A.1), from the recursive relation (A.6) we can write the equivalent relation

\[
I_s^{k+1} = \frac{1}{\lambda} \left( I_s^k - \frac{1}{k} I_s^{k-1} \right).
\]

(A.10)

Inserting (A.8) into (A.10) yields

\[
\sum_{j=0}^{k} c_j^{k+1} \phi_{s-j} = \sum_{j=0}^{k-1} c_j^k \phi_{s-j} - \frac{1}{k} \sum_{j=1}^{k} c_j^{k-1} \phi_{s-j}, \quad k \geq 2.
\]

Equating the coefficients of \( \phi_s \) \((j = 0)\) we obtain \( c_0^{k+1} = c_0^k \), and then (since \( c_0^1 = 1 \), as follows form (A.8) with \( k = 1 \))

\[
c_0^k = 1, \quad k \geq 1;
\]

(A.11a)

equating the coefficients of \( \phi_{s-k} \) \((j = k)\) we obtain \( c_k^{k+1} = -\frac{1}{k} c_k^{k-1} \), and then

\[
c_k^{k-1} = \frac{(-1)^k}{(k-1)!}, \quad k \geq 1;
\]

(A.11b)
finally, equating the coefficients of \( \phi_{s-j} \), with \( 1 \leq j \leq k-1 \), we obtain
\[
c_{j}^{k+1} = c_{j}^{k} - \frac{1}{k} c_{j-1}^{k}, \quad k \geq 1, \quad 1 \leq j \leq k-1.
\] (A.11c)

By using the recursive relations (A.11a)–(A.11c) one can easily generate all the coefficients of the expansion (A.8).

**Proposition A.1** Let \( I_{k}^{i}(z) \) be given as in the previous lemma. Then,
\[
\int_{\mathbb{R}^{d}} \frac{1}{\left| e^{i \frac{z^{2}}{2T} - z} \right|^{k}} dp = n_{d} I_{k}^{\frac{d}{2}}(z), \quad (A.12a)
\]
\[
\int_{\mathbb{R}^{d}} \frac{p_{i}p_{j}}{\left| e^{i \frac{z^{2}}{2T} - z} \right|^{k}} dp = \delta_{ij} n_{d} T I_{k}^{\frac{d}{2}+1}(z), \quad (A.12b)
\]
\[
\int_{\mathbb{R}^{d}} \frac{p_{i}p_{j}p_{k}p_{l}}{\left| e^{i \frac{z^{2}}{2T} - z} \right|^{k}} dp = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) n_{d} T^{2} I_{k}^{\frac{d}{2}+2}(z), \quad (A.12c)
\]
where, as usual, \( n_{d} := (2\pi T)^{\frac{d}{2}} \).

**Proof** By using polar coordinates it is easily shown that
\[
\int_{\mathbb{R}^{d}} \frac{|p|^{2m}}{\left| e^{i \frac{z^{2}}{2T} - z} \right|^{k}} dp = n_{d} (2T)^{m} \Gamma \left( \frac{d}{2} + m \right) \Gamma \left( \frac{d}{2} \right) I_{k}^{\frac{d}{2}+m}(z). \quad (A.13)
\]
This formula immediately yields Eq. (A.12a) and, by obvious symmetry considerations, Eq. (A.12b). The derivation of Eq. (A.12c) requires more explanations. We first show that
\[
J_{d}(z) := \int_{\mathbb{R}^{d}} \frac{p_{i}^{4}}{\left| e^{i \frac{z^{2}}{2T} - z} \right|^{k}} dp = 3 n_{d} T^{2} I_{k}^{\frac{d}{2}+2}(z) \quad (A.14)
\]
(which is of course independent on \( i \)). We proceed by double induction on \( d \). The cases \( d = 1, 2 \) can be easily verified by direct computations. Then, assuming (A.14) to be valid for \( d \), for \( d + 2 \) we can write
\[
J_{d+2}(z) = \int_{\mathbb{R}^{2}} J_{d} \left( z - |q|^{2}/2T \right) dq = 3 n_{d} 2\pi T^{2} \int_{0}^{\infty} I_{k}^{\frac{d}{2}+2} \left( z - \rho^{2}/2T \right) \rho dp.
\]
From Eq. (A.9) we obtain therefore
\[
J_{d+2}(z) = 3 n_{d} 2\pi T^{2} I_{k}^{\frac{d}{2}+3} \left( z - \rho^{2}/2T \right) \bigg|_{0}^{\infty} = 3 n_{d} 2\pi T^{2} I_{k}^{\frac{d}{2}+2+2}(z),
\]
which proves (A.14) by induction. On the other hand, Eq. (A.13) yields
\[
\int_{\mathbb{R}^{d}} \frac{|p|^{4}}{\left| e^{i \frac{z^{2}}{2T} - z} \right|^{k}} dp = d(d + 2) n_{d} T^{2} I_{k}^{\frac{d}{2}+m}(z)
\]
and then, using \( |p|^{4} = (\sum_{i} p_{i}^{2})^{2} = \sum_{i} p_{i}^{4} + \sum_{i \neq j} p_{i}^{2} p_{j}^{2} \) (and symmetry considerations), we obtain
\[
\int_{\mathbb{R}^{d}} \frac{p_{i}^{2} p_{j}^{2}}{\left| e^{i \frac{z^{2}}{2T} - z} \right|^{k}} dp = n_{d} T^{2} I_{k}^{\frac{d}{2}+2}(z). \quad (A.15)
\]
for \( i \neq j \). The two cases, (A.14) and (A.15), are summarized by (A.12c). \( \square \)
B Postponed proofs

B.1 Proof of Proposition 2.1

According to the short notations adopted from Subsec. 3.1 on, let us denote $G_{A,B}$ by $G$ and $\frac{1}{\hbar} h_{A,B}$ by $h$ (definition (3.1)). Recalling, moreover, the formalism introduced in Sec. 2.1 we can write

$$p_j \frac{\partial G}{\partial x_j} = \frac{i}{\hbar} \left\{ \frac{1}{2} |p|^2, G \right\} = \frac{i}{\hbar} \left\{ T h + p_j B_j - \frac{1}{2} |B|^2 + A, G \right\}$$

$$= \frac{i}{\hbar} \{ p_j B_j, G \}_\# + \frac{i}{\hbar} \left\{ A - \frac{1}{2} |B|^2, G \right\}_\# = \frac{i}{\hbar} \{ p_j B_j, G \}_\# + \Theta_\epsilon \left[ A - \frac{1}{2} |B|^2 \right] G,$$

where we used the fact that $\text{Op}_\epsilon(G)$ is, by definition (2.35), a function of $\text{Op}_\epsilon(h)$ and then $\{ h, G \}_\# = 0$, because it is the inverse Weyl quantization of a vanishing commutator. Since, from a direct computation,

$$\frac{i}{\hbar} \{ p_j B_j, G \}_\# = -p_j \frac{\partial B_j}{\partial x_k} \frac{\partial G}{\partial p_k} + B_j \frac{\partial G}{\partial x_j},$$

then from the previous identity we have

$$\frac{\partial}{\partial x_j} \langle p_j p_j G \rangle = - \frac{\partial B_j}{\partial x_k} \left\langle p_k p_j \frac{\partial G}{\partial p_k} \right\rangle + B_j \frac{\partial}{\partial x_j} \langle p_j G \rangle + \left\langle p_j \Theta_\epsilon \left[ A - \frac{1}{2} |B|^2 \right] G \right\rangle$$

$$= \frac{\partial B_j}{\partial x_i} \langle p_j G \rangle + \frac{\partial B_j}{\partial x_j} \langle p_i G \rangle + B_j \frac{\partial}{\partial x_j} \langle p_i G \rangle + \langle G \rangle \frac{\partial}{\partial x_i} \left( A - \frac{1}{2} |B|^2 \right)$$

$$= \frac{\partial B_j}{\partial x_i} J_j + \frac{\partial B_j}{\partial x_j} J_i + B_j \frac{\partial J_i}{\partial x_j} + n \left( \frac{\partial A}{\partial x_i} - B_j \frac{\partial B_j}{\partial x_i} \right),$$

(\text{where Eq. (2.23) was used}), which yields Eq. (2.36).

\[ \square \]

B.2 Proof of Lemma 3.1

Lemma 3.1 follows from elementary manipulations of formal Taylor expansions. In order to shorten the notations, let us introduce the $(d+1)$-dimensional vectors

$$m := (n, J_1, \ldots, J_d),$$

$$\mu = (A, B_1, \ldots, B_d),$$

$$f = f(\mu) = \langle G \rangle, \langle p_1 G \rangle, \ldots, \langle p_d G \rangle,$$

$$f^{(k)} = f^{(k)}(\mu) = \langle G_k \rangle, \langle p_1 G_k \rangle, \ldots, \langle p_d G_k \rangle.$$ 

The constraint system (3.6), in these notations, reads as follows:

$$f(\mu) = m. \quad (B.1)$$

Note that $f$ has a double dependence on $\epsilon$: one is direct (which leads to the expansion (3.2), i.e. to the terms $f^{(k)}), and the other is through the Lagrange multipliers, which are expanded.
as $\mu = \mu^{(0)} + \epsilon \mu^{(1)} + \epsilon^2 \mu^{(2)} + \cdots$. Then, we regard $f$ as $f(\epsilon, \mu(\epsilon))$, whose Taylor expansion at $\epsilon = 0$ can be written in this way:

$$f_i(\epsilon, \mu(\epsilon)) = f_i^{(0)}(\mu^{(0)}) + \epsilon \frac{\partial f_i^{(0)}}{\partial \mu_j}(\mu^{(0)}) \mu_j^{(1)}$$

$$+ \epsilon^2 \left[ \frac{1}{2} \frac{\partial^2 f_i^{(0)}}{\partial \mu_j \partial \mu_k}(\mu^{(0)}) \mu_j^{(1)} \mu_k^{(1)} + \frac{\partial f_i^{(0)}}{\partial \mu_j}(\mu^{(0)}) \mu_j^{(2)} + f_i^{(2)}(\mu^{(0)}) \right] + \cdots.$$  

(where we took into account that $f^{(1)} = 0$, from (3.4b), and, for the sake brevity, the third-order term was not shown). Since the moments $m_i$ do not depend on $\epsilon$, the constraint equation (B.1) is expanded as follows:

$$f_i^{(0)}(\mu^{(0)}) = m_i$$

$$\frac{\partial f_i^{(0)}}{\partial \mu_j}(\mu^{(0)}) \mu_j^{(1)} = 0$$

$$\frac{1}{2} \frac{\partial^2 f_i^{(0)}}{\partial \mu_j \partial \mu_k}(\mu^{(0)}) \mu_j^{(1)} \mu_k^{(1)} + \frac{\partial f_i^{(0)}}{\partial \mu_j}(\mu^{(0)}) \mu_j^{(2)} + f_i^{(2)}(\mu^{(0)}) = 0$$

$$\cdots$$

The first equation is Eq. (3.8a), the second one implies $\mu^{(1)} = 0$ (i.e. $A^{(1)} = B^{(1)} = 0$), the third one is Eq. (3.8b) and, finally, the fourth one (which has been omitted for brevity) implies $\mu^{(3)} = 0$ (i.e. $A^{(3)} = B^{(3)} = 0$).

**B.3 Proof of Lemma 3.2**

By using Eq. (3.4c) with $n = 1$ we obtain

$$G_2 = -G_0 \exp_2(h) + G_0 \#_2 e^h.$$  

(B.2)

The first term contains $\exp_2(h)$, whose explicit expression can be taken from Eq. (5.14) of Ref. [10] and reads as follows:

$$\exp_2(h) = -\frac{e^h}{8} (X_{ij} P_j - S_{ij} S_{ji} + \frac{1}{3} X_{ij} P_i P_j - \frac{2}{3} S_{ij} P_i X_j + \frac{1}{3} P_{ij} X_i X_j),$$  

(B.3)

where we introduced the following notations:

$$X_i := \frac{\partial h}{\partial x_i} = -\frac{1}{T} \left[ (p - B)_k \frac{\partial B_k}{\partial x_i} + \frac{\partial A}{\partial x_i} \right]$$

$$P_i := \frac{\partial h}{\partial p_i} = \frac{1}{T} (p - B)_i$$

$$X_{ij} := \frac{\partial^2 h}{\partial x_i \partial x_j} = \frac{1}{T} \left[ \frac{\partial B_k}{\partial x_i} \frac{\partial B_k}{\partial x_j} - (p - B)_k \frac{\partial^2 B_k}{\partial x_i \partial x_j} - \frac{\partial^2 A}{\partial x_i \partial x_j} \right]$$

$$P_{ij} := \frac{\partial^2 h}{\partial p_i \partial p_j} = \frac{1}{T} \delta_{ij}$$

$$S_{ij} := \frac{\partial^2 h}{\partial x_i \partial p_j} = -\frac{1}{T} \frac{\partial B_i}{\partial x_i}.$$  

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we recall that $h = \frac{1}{2} h_{A,B}$ is given by (3.1). The other term, $\mathcal{G}_0 \# 2 e^h$, is a second-order Moyal product, given by (2.5). Using the notations just introduced, we obtain:

$$\mathcal{G}_0 \# 2 e^h = -\frac{1}{4} \left( \frac{1}{2} \frac{\partial^2 \mathcal{G}_0}{\partial x_i \partial x_j} \frac{\partial^2 e^h}{\partial p_i \partial p_j} - \frac{\partial^2 \mathcal{G}_0}{\partial x_i \partial p_j} \frac{\partial^2 e^h}{\partial p_i \partial x_j} + \frac{1}{2} \frac{\partial^2 \mathcal{G}_0}{\partial p_i \partial p_j} \frac{\partial^2 e^h}{\partial x_i \partial x_j} \right)$$

$$= -\frac{e^{kh}}{8} \left[ (2F_2 X_i X_j - F_1 (X_{ij} + X_i X_j)) (P_{ij} + P_i P_j) \right]$$

$$- 2 (2F_2 X_i P_j - F_1 (S_{ij} + X_i P_j)) (S_{ij} + P_i X_j)$$

$$+ (2F_2 P_i P_j - F_1 (P_{ij} + P_i P_j)) (X_{ij} + X_i X_j) \right],$$

where, for $k \geq 0$, we define.

$$F_k = \frac{e^{kh}}{(e^h + \lambda)^{k+1}}.$$  \hspace{1cm} (B.4)

Putting together the two terms we obtain the following expression for $\mathcal{G}_2$:

$$\mathcal{G}_2 = \frac{1}{8} (X_{ij} P_{ij} - S_{ij} S_{ji}) (F_1 - 2F_2)$$

$$+ \frac{1}{8} (X_{ij} P_{ij} - 2S_{ij} P_i X_j + P_{ij} X_i X_j) \left( \frac{1}{3} F_1 - 2F_2 + 2F_3 \right).$$  \hspace{1cm} (B.5)

Note that in the Maxwell-Boltzmann case, $\lambda = 0$, we have $F_k = e^{-h}$ for all $k \geq 0$ and, therefore,

$$F_1 - 2F_2 = -e^{-h}, \quad \frac{1}{3} F_1 - 2F_2 + 2F_3 = \frac{1}{3} e^{-h}.$$  \hspace{1cm} (B.6)

Then, in this case, Eq. (B.5) reduces (as it should) to $\mathcal{E}r p_2 (-h)$, which is given by (B.3) with the suitable changes of sign.

Now, defining $q = p - B$, taking the moments $\langle \mathcal{G}_2 \rangle$ and $\langle q_i \mathcal{G}_2 \rangle$ from Eq. (B.5), and taking account of vanishing integrals of odd functions, we get

$$\langle \mathcal{G}_2 \rangle = \frac{1}{8T^2} \left[ \frac{\partial B_j}{\partial x_k} \left( \frac{\partial B_j}{\partial x_k} - \frac{\partial B_k}{\partial x_j} \right) - \Delta A \right] \langle F_1 - 2F_2 \rangle$$

$$+ \frac{1}{8T^3} \left[ \frac{\partial B_j}{\partial x_k} \left( \frac{\partial B_j}{\partial x_l} - 2 \frac{\partial B_l}{\partial x_j} \right) - \frac{\partial A}{\partial x_k \partial x_l} + \frac{\partial B_k}{\partial x_j} \frac{\partial B_l}{\partial x_j} \right] \langle q_i q_j \left( \frac{1}{3} F_1 - 2F_2 + 2F_3 \right) \rangle$$

$$+ \frac{1}{8T^3} \langle \nabla A \rangle^2 \langle F_1 - 2F_2 + 2F_3 \rangle$$  \hspace{1cm} (B.6)

and

$$\langle q_i \mathcal{G}_2 \rangle = -\frac{1}{8T^2} \Delta B_j \langle q_i q_j \left( F_1 - 2F_2 \right) \rangle$$

$$- \frac{1}{4T^3} \frac{\partial A}{\partial x_k} \left( \frac{\partial B_k}{\partial x_j} - \frac{\partial B_j}{\partial x_k} \right) \langle q_i q_j \left( \frac{1}{3} F_1 - 2F_2 + 2F_3 \right) \rangle$$

$$- \frac{1}{8T^3} \frac{\partial^2 B_l}{\partial x_j \partial x_k} \langle q_i q_j q_k q_l \left( \frac{1}{3} F_1 - 2F_2 + 2F_3 \right) \rangle.$$  \hspace{1cm} (B.7)

The moments of functions $F_k$ can be reduced to integrals of type $I_k^s$ (see Lemma A.1) by using

$$F_k = \frac{1}{e^h + \lambda} \left( 1 - \frac{\lambda}{e^h + \lambda} \right)^k = \sum_{j=0}^{k} \binom{k}{j} \frac{(-\lambda)^j}{(e^h + \lambda)^{j+1}}.$$  \hspace{1cm} (B.8)
Recalling that
\[ h = \frac{|q|^2}{2T} - \frac{A}{T}, \]
from (B.8) and (A.12) we obtain
\[ \langle F_1 - 2F_2 \rangle = -n_d \phi_{\frac{q}{2}-2} \left( \frac{A}{T} \right) \]
\[ \langle q_i q_j (F_1 - 2F_2) \rangle = -\delta_{ij} n_d T \phi_{\frac{q}{2}-1} \left( \frac{A}{T} \right) \]
\[ \langle \frac{1}{3} F_1 - 2F_2 + 2F_3 \rangle = \frac{1}{3} n_d \phi_{\frac{q}{2}-3} \left( \frac{A}{T} \right) \]
\[ \langle q_i q_j \left( \frac{1}{3} F_1 - 2F_2 + 2F_3 \right) \rangle = \frac{1}{3} \delta_{ij} n_d T \phi_{\frac{q}{2}-2} \left( \frac{A}{T} \right) \]
\[ \langle q_i q_j q_k q_l \left( \frac{1}{3} F_1 - 2F_2 + 2F_3 \right) \rangle = \frac{1}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) n_d T^2 \phi_{\frac{q}{2}-1} \left( \frac{A}{T} \right). \]

Then, by using \( \langle p_i \mathcal{G}_2 \rangle = B_i \langle \mathcal{G}_2 \rangle + \langle q_i \mathcal{G}_2 \rangle \) and the identity
\[ \frac{1}{T} \frac{\partial A}{\partial x_k} \phi_{\frac{q}{2}-2} \left( \frac{A}{T} \right) = \frac{\partial}{\partial x_k} \phi_{\frac{q}{2}-1} \left( \frac{A}{T} \right) \]
(from property (A.4e)), Eqs. (3.12) are easily obtained from the expressions (B.6) and (B.7).

\[ \square \]

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