On the existence of a torsor structure for Galois covers

Mohamed Saïdi

Abstract

Let $R$ be a complete discrete valuation ring with residue characteristic $p > 0$. And let $f : Y \to X$ be a finite Galois cover between flat and normal formal $R$-schemes of finite type with Galois group $G$ which is étale above the generic fibre of $X$ and such that the special fibre of $Y$ is reduced. Assume that the special fibre of both $X$ and $Y$ are geometrically integral. If $G$ is cyclic of order $p$ then it is well known that $f$ has the structure of a torsor under a finite and flat $R$-group scheme of rank $p$ (in the unequal characteristic case one has to assume the existence of the $p$-th roots of unity in the ground field). In this note we give an example of a Galois cover as above with Galois group a cyclic group of order $p^2$ and such that $f$ doesn’t have the structure of a torsor under a finite and flat $R$-group scheme of rank $p^2$ in the case where $R$ has equal characteristic $p$ (in the example one can even choose $X$ to be smooth). Such examples are certainly known to some experts but, to the best of my knowledge, do not appear in the literature. Also it is not surprising that such examples also exist in unequal characteristic.

I. Artin-Schreier-Witt theory of $p^n$-cyclic covers in characteristic $p$.

1.1. In this section we review the Artin-Schreier-Witt theory (first developed in [W]) which provides explicit equations describing cyclic covers of degree $p^n$ in characteristic $p$. We refer the reader to the modern treatment of the theory as in [D-G].

Let $X$ be a scheme of characteristic $p$ and denote by $X_{\text{et}}$ the étale site on $X$. Let $n > 0$ be an integer. We denote by $W_{n,X}$ (or simply $W_n$ if there is no confusion) the sheaf of Witt vectors of length $n$ on $X_{\text{et}}$ (cf. [D-G], chapitre 5, 1). In the sequel any addition or substraction of Witt vectors will mean the addition and substraction in the sense of Witt theory. We denote by $F$ the Frobenius endomorphism of $W_{n,X}$ which is locally defined by $F(x_1, x_2, \ldots, x_n) = (x_1^p, x_2^p, \ldots, x_n^p)$, for a Witt vector $(x_1, x_2, \ldots, x_n)$ of length $n$, and by $\text{Id}$ the identity automorphism of $W_{n,X}$. The following sequence is exact on $X_{\text{et}}$:

\[ 0 \to (\mathbb{Z}/p^n\mathbb{Z})_X \xrightarrow{i_n} W_{n,X} \xrightarrow{F-\text{Id}} W_{n,X} \to 0 \]

where $(\mathbb{Z}/p^n\mathbb{Z})_X$ denotes the constant sheaf $(\mathbb{Z}/p^n\mathbb{Z})$ on $X_{\text{et}}$ and $i_n$ is the natural monomorphism which applies $1 \in \mathbb{Z}/p^n\mathbb{Z}$ to $1 \in W_n$ (cf [G-D], chapitre 5, 5.4). From the long
cohomology exact sequence associated to (1) one deduces the following exact sequence:

\[(2) \quad W_{n,X}(X) \xrightarrow{F-\text{Id}} W_{n,X}(X) \to H^1_{\text{et}}(X, \mathbb{Z}/p^n\mathbb{Z}) \to H^1_{\text{et}}(X, W_n) \xrightarrow{F-\text{Id}} H^1_{\text{et}}(X, W_n)\]

Assume that $X = \text{Spec} \, A$ is affine in which case we have $H^1_{\text{et}}(\text{Spec} \, A, W_n) = 0$ and hence an isomorphism: $H^1_{\text{et}}(\text{Spec} \, A, \mathbb{Z}/p^n\mathbb{Z}) \cong W_{n,A}(A)/(F-I)(W_{n,A}(A))$. This isomorphism has the following interpretation: to an étale $\mathbb{Z}/p^n\mathbb{Z}$-torsor $f : Y \to X = \text{Spec} \, A$ above $X$ corresponds a Witt vector $(a_1, a_2, ..., a_n) \in W_{n,A}(A)$ of length $n$ which is uniquely determined modulo addition of elements of the form $b_1, b_2, ..., b_n$. Moreover the equations $F.(x_1, x_2, ..., x_n) - (x_1, x_2, ..., x_n) = (a_1, a_2, ..., a_n)$ where the $x_i$ are indeterminates are equations for the torsor $f$. More precisely there is a canonical factorisation of $f$ as $Y = Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} ... \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 := X$ where each $Y_i = \text{Spec} \, B_i$ is affine and $f_i : Y_i := \text{Spec} \, B_i \to f_{i-1} := \text{Spec} \, B_{i-1}$ is the étale $\mathbb{Z}/p\mathbb{Z}$-torsor corresponding to the algebra extension $B_{i+1} := B_i[x_i]$. In the general case (where $X$ is not necessarily affine) the above equations provide local equations for an étale $\mathbb{Z}/p^n\mathbb{Z}$-torsor in characteristic $p$.

1.2. Examples. In what follows $X$ is a scheme of characteristic $p$.

1.2.1. $\mathbb{Z}/p\mathbb{Z}$-Torsors. Let $f : Y \to X$ be an étale $\mathbb{Z}/p\mathbb{Z}$-torsor. Then locally $f$ is given by an equation $x^p - x = a$ where $a$ is a regular function on $X$ which is uniquely defined up to addition of elements of the form $b^p - b$ for some regular function $b$.

1.2.2. $\mathbb{Z}/p^2\mathbb{Z}$-Torsors. Let $f : Y \to X$ be an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor. Then we have a canonical factorisation of $f$ as: $Y = Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X$ where $f_2$ and $f_1$ are étale $\mathbb{Z}/p\mathbb{Z}$-torsors. The torsor $f$ is locally given by equations of the form:

$$F.(x_1, x_2) - (x_1, x_2) := (x_1^p - x_1, x_2^p - x_2 - p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} x_1^k (-x_1)^{p-k}) = (a_1, a_2)$$

for some regular functions $a_1$ and $a_2$ on $X$ and the Witt vector $(a_1, a_2)$ is uniquely determined up to addition (in the Witt theory) of vectors of the form:

$$(b_1^p, b_2^p) - (b_1, b_2) := (b_1^p - b_1, b_2^p - b_2 - p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} b_1^k (-b_1)^{p-k})$$

Thus locally the torsor $f_1$ is defined by an equation:

$$x_1^p - x_1 = a_1$$

and $f_2$ by an equation:

$$x_2^p - x_2 = a_2 + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} x_1^k (-x_1)^{p-k}$$
Moreover if we replace the vector \((a_1, a_2)\) by the vector \((a_1, a_2) + (b_1^p, b_2^p) - (b_1, b_2)\) the above equations are replaced by:
\[
x_1^p - x_1 = a_1 + b_1^p - b_1
\]
and:
\[
x_2^p - x_2 = a_2 + b_2^p - b_2 + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} x_1^{pk} (-x_1)^{p-k} - p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} b_1^{pk} (-b_1)^{p-k} - p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} (b_1^p - b_1)^k (a_1)^{p-k}
\]
respectively.

II. The group schemes \(\mathcal{M}_n\) (cf. also [M], 3.2).

In all this paragraph we use the following notations: \(R\) is a complete discrete valuation ring of equal characteristic \(p\) with residue field \(k\) and fraction field \(K := \text{Fr} R\). We denote by \(\pi\) a uniformising parameter of \(R\).

Let \(n \geq 0\) be an integer and let \(G_{a,R} = \text{Spec} R[X]\) be the additive group scheme over \(R\). The map:
\[
\phi_n : G_{a,R} \to G_{a,R}
\]
given by:
\[
X \to X^p - \pi^{(p-1)n} X
\]
is an isogeny of group schemes. The kernel of \(\phi_n\) is denoted by \(\mathcal{M}_{n,R} := \mathcal{M}_n\). We have \(\mathcal{M}_n := \text{Spec} R[X]/(X^p - \pi^{(p-1)n} X)\), and \(\mathcal{M}_n\) is a finite and flat \(R\)-group scheme of rank \(p\). Further the following sequence is exact in the fppf topology:
\[
(3) \quad 0 \to \mathcal{M}_n \to G_{a,R} \xrightarrow{\phi_n} G_{a,R} \to 0
\]

If \(n = 0\) then the sequence (3) is the Artin-Schreir sequence which is exact in the étale topology and \(\mathcal{M}_0\) is the étale constant group scheme \((\mathbb{Z}/p\mathbb{Z})_R\). If \(n > 0\) the sequence (3) has a generic fibre which is isomorphic to the étale Artin-Schreier sequence and a special fibre isomorphic to the radicial exact sequence:
\[
(4) \quad 0 \to \alpha_{p,k} \to G_{a,k} \xrightarrow{x^p} G_{a,k} \to 0
\]

Thus if \(n > 0\) the group scheme \(\mathcal{M}_n\) has a generic fibre which is étale isomorphic to \((\mathbb{Z}/p\mathbb{Z})_K\) and its special fibre is isomorphic to the infinitesimal group scheme \(\alpha_{p,k}\). Let \(X\) be an \(R\)-scheme. The sequence (3) induces a long cohomology exact sequence:
The cohomology group $H^1_{fppf}(X, \mathcal{M}_n)$ classifies the isomorphism classes of fppf-torsors with group $\mathcal{M}_n$ above $X$. The above sequence allows the following description of $\mathcal{M}_n$-torsors: locally a torsor $f : Y \to X$ under the group scheme $\mathcal{M}_n$ is given by an equation $T^p - \pi (p-1)^n T = a$ where $T$ is an indeterminate and $a$ is a regular function on $X$ which is uniquely defined up to addition of elements of the form $b^p - \pi (p-1)^n b$ for some regular function $b$. In particular if $H^1_{fppf}(X, \mathbb{G}_{a,R}) = 0$ (e.g. if $X$ is affine) then an $\mathcal{M}_n$-torsor above $X$ is globally defined by an equation as above.

III. The example.

In all this paragraph we use the same notations as in II. Let $X := \text{Spf } A$ be a formal smooth affine $R$-scheme with geometrically connected fibres (the assumption on $X$ being smooth is not essential for the example). Consider the cyclic $p^2$-cover $f : Y \to X$ given generically by the equation:

$$(T^p_1, T^p_2) - (T_1, T_2) = (\pi^{-pm} a_1, a_2)$$

where $m = pm'$ is a positive integer divisible by $p$ and $a_1$ and $a_2$ are elements of $A$ such that the image $\bar{a}_1$ of $a_1$ modulo $\pi$ is not a $p$-power. Then the generic fibre $f_K : Y_K \to X_K$ of $f$ is an étale $\mathbb{Z}/p^2 \mathbb{Z}$ torsor. Further the finite cover $f$ factorises canonically as $f_1 \circ f_2$ where $f_2 : Y \to Y_1$ and $f_1 : Y_1 \to Y$ are $p$-cyclic covers. The finite cover $f_1$ is given generically by the equation:

$$T^p_1 - T_1 = a_1 \pi^{-pm}$$

which can be transformed after making the change of variables $\tilde{T}_1 := \pi^m T_1$ to:

$$\tilde{T}^p_1 - \pi^m (p-1) \tilde{T}_1 = a_1$$

which is a defining equation for the cover $f_1$. In particular we see that $f_1$ has the structure of a torsor under the $R$-group scheme $\mathcal{M}_m$. Its special fibre is the $\alpha_p$-torsor $f_{1,k} : Y_{1,k} \to X_{1,k}$ given by the equation $\tilde{t}_1^p = \bar{a}_1$ where $\tilde{t}_1$ is the image of $\tilde{T}_1$ modulo $\pi$.

The cover $f_2$ is generically given by the equation:

$$T^p_2 - T_2 = a_2 + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} T^p_1 (-T_1)^{p-k}$$

and after adapting this to the change of variables $\tilde{T}_1 := \pi^m T_1$ we get the equation:

$$T^p_2 - T_2 = a_2 + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} \tilde{T}^p_1 (-\tilde{T}_1)^{p-k} \pi^{-m(pk+p-k)}$$
which can be transformed after the change of variables $\tilde{T}_2 := \pi^{\tilde{m}}T_2$ where $\tilde{m} := m'(p(p - 1) + 1)$ to:

$$\tilde{T}_2^p - \pi^{\tilde{m}(p-1)}\tilde{T}_2 = \pi^{m(p+1)}a_2 + p^{-1}\sum_{k=1}^{p-1} \binom{p}{k} \tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{m(p-k)}.$$

The above equation is an integral equation defining the cover $f_2$. In particular one sees that $f_2$ has the structure of a torsor under the $R$-group scheme $\mathcal{M}_{\tilde{m}}$. Its special fibre is the $\alpha_p$-torsor $f_{2,k}: Y_{2,k} \rightarrow X_{2,k}$ given by the equation $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$ where $\tilde{t}_2$ is the image of modulo $\pi$. In particular the special fibre $Y_k$ of $Y$ is reduced. Although both $f_1$ and $f_2$ have the structure of a torsor the next proposition shows that the cover $f$ doesn’t.

3.1. Proposition. With the same notations as above and assume that $p > 2$. Then the finite cover $f$ doesn’t have the structure of a torsor under a finite and flat $R$-group scheme of rank $p^2$.

Proof. Let $b_1 \in A$. Then the equations: $(S_1^p, S_2^p) - (S_1, S_2) = (\pi^{-m}a_1, a_2) + (\pi^{-p}b_1, 0) - (\pi^{-m}b_1, 0)$ are also defining equations for the torsor $f_K$. The cover $f$ is then given by the equations (1):

$$\tilde{T}_1^p - \pi^{m(p-1)}\tilde{T}_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{m(p-1)}\tilde{T}_2 = \pi^{m}a_2 + p^{-1}\sum_{k=1}^{p-1} \binom{p}{k} \tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{m(p-k)}.$$

And $f$ is also given by the equations (2):

$$\tilde{S}_1^p - \pi^{m(p-1)}\tilde{S}_1 = a_1 + b_1^p - \pi^{m(p-1)}b_1$$

and:

$$S_2^p - S_2 = a_2 + p^{-1}(\sum_{k=1}^{p-1} \binom{p}{k} \tilde{S}_1^{pk}(-\tilde{S}_1)^{p-k}\pi^{-m(p-k)} - \sum_{k=1}^{p-1} \binom{p}{k} b_1^{pk}(-b_1)^{p-k}\pi^{-m(p-k)}).$$

By reducing modulo $\pi$ both equations (1) and (2) we obtain equations (1)’ and (2)’ for the cover $f_K$ on the level of special fibres. Suppose that $f$ has the structure of a torsor under a finite and flat $R$-group scheme $G_R$ of rank $p^2$. Then the special fibre $f_K$ of $f$ is a torsor under the special fibre $G_K$ of $G$ which is a group scheme of rank $p^2$ and an extension of $\alpha_p$ by $\alpha_p$. In particular both equations (1)’ and (2)’ that we obtain for $f_K$ must define the same $\alpha_p$-torsor $f_{1,k}$ and $f_{2,k}$. This is the case for $f_{1,k}$ since both equations are $\tilde{t}_1^p = \tilde{a}_1$ and $\tilde{S}_1^p = \tilde{a}_1 + \tilde{b}_1^p$ but we will see that this is not the case for $f_{2,k}$ if $p > 2$. Indeed one can see after some (easy) computations that the equations we obtain for $f_{2,k}$ are: $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$ and $\tilde{S}_2^p = -\tilde{S}_1^{p(p-1)+1} + \tilde{b}_1^{p(p-1)+1} + \sum_{k=1}^{p-1} \tilde{b}_1^{p(k-1)+1}(\tilde{S}_1 - \tilde{b}_1)^{p-k} + b_1^{pk}(\tilde{S}_1 - \tilde{b}_1)^{p(p-k)+1})$, where $\tilde{S}_2$ is the image of $\pi^{\tilde{m}}S_2$ modulo $\pi$, which gives
\( \tilde{s}_2 = -\tilde{s}_1^{p(p-1)+1} + \tilde{b}_1^{p(p-1)+1} + ((\tilde{s}_1 - \tilde{b}_1)^{p-1} + \tilde{b}_1^{p-1}) \sum_{k=1}^{p-1} \tilde{b}_1^{p(k-1)+1} (\tilde{s}_1 - \tilde{b}_1)^{p(p-k-1)+1} \). An easy verification shows that these two equations define non isomorphic \( \alpha_p \)-torsors (although they define isomorphic covers). If for example \( p = 3 \) then the two equations are: \( \tilde{t}_2^3 = -\tilde{t}_1^7 \) and \( \tilde{s}_2^3 = -\tilde{s}_1^7 + \tilde{b}_1^7 + ((\tilde{s}_1 - \tilde{b}_1)^2 - \tilde{b}_1^2) (\tilde{b}_1 (\tilde{s}_1 - \tilde{b}_1)^4 + \tilde{b}_1^4 (\tilde{s}_1 - \tilde{b}_1)) \). Using that \( \tilde{s}_1 = \tilde{t}_1 + \tilde{b}_1 \) we get \( \tilde{s}_2^3 = -(\tilde{t}_1 + \tilde{b}_1)^7 + \tilde{b}_1^7 + (\tilde{t}_1^2 - \tilde{b}_1^2) (\tilde{b}_1 \tilde{t}_1^4 + \tilde{b}_1^4 \tilde{t}_1) \) thus \( \tilde{s}_2^3 = -\tilde{t}_1^7 - 34\tilde{t}_1^3 \tilde{b}_1^4 - 8\tilde{t}_1 \tilde{b}_1^6 = -\tilde{t}_1^7 + 2\tilde{t}_1^3 \tilde{b}_1^4 + \tilde{t}_1 \tilde{b}_1^6 \) and since \( 2\tilde{t}_1^3 \tilde{b}_1^4 + \tilde{t}_1 \tilde{b}_1^6 = 2\tilde{a}_1 \tilde{b}_1^4 + \tilde{t}_1 \tilde{b}_1^6 \) is not a cube we deduce that the two equations do not define the same \( \alpha_3 \)-torsor.

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