ADJOINT ORBITS OF THE JACOBI GROUP

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Abstract. In this article, we study adjoint orbits of the Jacobi group, and in particular describe nilpotent orbits explicitly.

1. Introduction

It is known that if $G$ is a real reductive Lie group, there are only finitely many nilpotent $G$-orbits and that there is the so-called Kostant-Sekiguchi correspondence between the set of all adjoint nilpotent $G$-orbits in the Lie algebra $g$ of $G$ and the set of all $K_C$-orbits in $p_C$, where $K_C$ is the complexification of a maximal compact subgroup $K$ of $G$ and $g_C = k_C + p_C$ is the Cartan decomposition of the complexification of $g_C$ of $g$ (cf. [9, 10, 11, 12]).

In this paper, we consider the Jacobi group $G_J = SL(2, \mathbb{R}) \ltimes H_{\mathbb{R}}$, where $H_{\mathbb{R}}$ is the 2-step nilpotent Lie group with the following multiplication law $(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda \mu' - \lambda' \mu)$.

The Jacobi group $G_J$ is a non-reductive Lie group endowed with the following multiplication

$$(M, (\lambda, \mu, \kappa)) \cdot (M', (\lambda', \mu', \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu', \kappa + \kappa' + \lambda \mu' - \lambda' \mu)).$$

Let $\mathbb{H}$ be the Poincaré upper half plane. Then $G_J$ acts on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}$ transitively by

$$(M, (\lambda, \mu, \kappa)) \cdot (\tau, z) = (M < \tau >, (z + \lambda \tau + \mu)(c \tau + d)^{-1}),$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, $(\lambda, \mu, \kappa) \in H_{\mathbb{R}}$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ and

$$M < \tau > = (a \tau + b)(c \tau + d)^{-1}.$$
where $K = SO(2)$ is a maximal compact subgroup of $SL(2, \mathbb{R})$. Then the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}$ is biholomorphic to the Hermitian-Kähler homogeneous space $G^J/K^J$ via
\[
G^J/K^J \rightarrow \mathbb{H} \times \mathbb{C} \\
gK^J \rightarrow g \cdot (i, 0), \quad g \in G^J.
\] (1.4)

Therefore the Jacobi group $G^J$ plays an important role in number theory (e.g. theory of Jacobi forms) [4, 13, 14, 15, 16, 17, 18, 27, 29, 32, 33], algebraic geometry [20, 22, 29, 31], complex geometry [23, 24, 25, 26, 29], representation theory [2, 28, 30] and mathematical physics [1, 8].

In this paper, we study the adjoint orbits of $G^J$, and in particular, we calculate the adjoint nilpotent orbits of $G^J$ explicitly. We show that unlike the case of a reductive Lie group, there are uncountably many nilpotent $G^J$-orbits.

This paper is organized as follows. In Section 2, we review the Kostant-Sekiguchi correspondence for a reductive real Lie group and adjoint orbits of $SL(2, \mathbb{R})$. In Section 3, we study the adjoint orbits of $G^J$ in the Lie algebra $g^J$. We describe the set of nilpotent orbits of $G^J$ and the set of nilpotent orbits of $K^J_C$ in $p^J_C$ explicitly. Here $K^J_C$ is the complexification of $K^J$ and $g^J_C = \mathfrak{k}^J_C + p^J_C$ is the decomposition of the complexification $g^J_C$ of $g^J$.

Notations: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. We denote by $\mathbb{R}^+$ and $\mathbb{C}^+$ the set of nonzero real numbers and the set of nonzero complex numbers respectively. We denote by $\mathbb{Z}^+$ (resp. $\mathbb{Z}_{\geq 0}$) the set of all positive (resp. nonnegative) integers, by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring $F$. For any $M \in F^{(k,l)}$, $^tM$ denotes the transpose matrix of $M$. We denote the identity matrix of degree $n$ by $I_n$.

2. The Kostant-Sekiguchi Correspondence

In this section, we review the Kostant-Sekiguchi correspondence for a reductive real Lie group and adjoint orbits of $SL(2, \mathbb{R})$ (cf. [5, 7, 9, 10, 11, 12]). Let $G$ be a real reductive group with Lie algebra $\mathfrak{g}$, and let $K$ be a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with the associated Cartan involution $\theta$. Let $\mathfrak{g}_C = \mathfrak{t}_C \oplus \mathfrak{p}_C$ denote the complexification of $\mathfrak{g}$, and let $\sigma$ be the associated complex conjugation. Let $G_C$ and $K_C$ denote the complexifications of $G$ and $K$ with the Lie algebras $\mathfrak{g}_C$ and $\mathfrak{k}_C$, respectively.

J. Sekiguchi and B. Kostant established a bijection between the set of all nilpotent $G$-orbits in $\mathfrak{g}$ and the set of all nilpotent $K_C$-orbit in $\mathfrak{p}_C$.

Definition 2.1. Let $L$ denote $\mathfrak{g}$ or $\mathfrak{g}_C$.

1. An ordered triple $\{Z_1, Z_2, Z_3\}$ of elements in $L$ is said to be an $\mathfrak{sl}_2$-triple if
\[
[Z_1, Z_2] = 2Z_2, \quad [Z_1, Z_3] = -2Z_3, \quad [Z_2, Z_3] = Z_1.
\] (2.1)
(2) Two \(sl_2\)-triples \(\{Z_1, Z_2, Z_3\}\) and \(\{Z'_1, Z'_2, Z'_3\}\) in \(L\) are said to be \textit{conjugate under a subgroup} \(W\) of \(L\) if there exists an element \(w \in W\) such that \(Z_i = w \cdot Z'_i (= w \cdot Z'_iw^{-1})\) for \(i = 1, 2, 3\).

To describe the Kostant-Sekiguchi correspondence, it is necessary to consider the following classes of \(sl_2\)-triples.

**Definition 2.2** (Kostant-Sekiguchi triples).

1. An \(sl_2\)-triple \(\{H, E, F\}\) in \(g\) is said to be a \(KS\)-triple in \(g\) if \(\theta(E) = -F\).
2. An \(sl_2\)-triple \(\{x, e, f\}\) in \(g_C\) is said to be a normal if \(x \in k_C\) and \(e, f \in \mathfrak{p}_C\).
3. A normal \(sl_2\)-triple \(\{x, e, f\}\) in \(g_C\) is said to be a \(KS\)-triple in \(g_C\) if \(f = \sigma(e)\).

**Theorem 2.3.** Let \(G\) be a real reductive group with Lie algebra \(g\), and let \(K\) be a maximal compact subgroup of \(G\) with Lie algebra \(k\). Let \(g_C = k_C \oplus \mathfrak{p}_C\) denote the complexification of \(g\). Let \(K_C\) denote the complexification of \(K\) with the Lie algebra \(k_C\). The following sets (1)-(6) are in natural one-to-one correspondence:

1. Nilpotent \(G\)-orbits in \(g\).
2. \(G\)-conjugacy classes of \(sl_2\)-triples in \(g\).
3. \(K\)-conjugacy classes of \(KS\)-triples in \(g\).
4. \(K\)-conjugacy classes of \(KS\)-triples in \(g_C\).
5. \(K_C\)-conjugacy classes of normal \(sl_2\)-triples in \(g_C\).
6. Nilpotent \(K_C\)-orbits in \(p_C\).

The correspondence between (1) and (6) is the Kostant-Sekiguchi correspondence.

We refer to [5, 6, 10, 21] for more details on Theorem 2.3.

**Remark 2.4.** With the notations as in Theorem 2.3, M. Vergne [11] proved that if \(O\) is a real nilpotent orbit in \(g\), then there exists a canonical \(K\)-equivariant diffeomorphism of \(O\) onto the nilpotent \(K_C\)-orbit in \(p_C\) associated to \(O\) via the Kostant-Sekiguchi correspondence. (cf. [12] p.206)

**Example.** We let \(G = SL(2, \mathbb{R})\) and let \(K = SO(2)\) be a maximal compact subgroup of \(G\). The Lie algebra \(g\) of \(G\) is given by

\[
g = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}.
\]

We put

\[
X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Then the set \(\{X, Y, Z\}\) forms a basis for \(g\). We define an element \(F(x, y, z) \in g\) by

\[
F(x, y, z) := xX + yY + zZ = \begin{pmatrix} x & y + z \\ y - z & -x \end{pmatrix}.
\] (2.2)
Then we have the relations
\[ X^2 + Y^2 - Z^2 = 3I_2, \quad [X, Y] = 2Z, \quad [X, Z] = 2Y, \quad [Y, Z] = -2X. \] (2.3)

It is easy to see that \( X \) and \( Y \) are hyperbolic elements and \( Z \) is an elliptic element. For a nonzero real number \( \alpha \), the \( G \)-orbit of \( \alpha X \) is represented by the one-sheeted hyperboloid
\[ x^2 + y^2 - z^2 = \alpha^2. \] (2.4)

The \( G \)-orbit of \( \alpha Y (\alpha \in \mathbb{R}^\times) \) is also represented by the hyperboloid (2.4). The \( G \)-orbit of \( \alpha Z (\alpha \in \mathbb{R}^\times) \) is represented by two-sheeted hyperboloids
\[ x^2 + y^2 - z^2 = -\alpha^2. \] (2.5)

Since
\[ F(x, y, z)^2 = (x^2 + y^2 - z^2) \cdot I_2, \]
we have for any \( k \in \mathbb{Z}^+ \),
\[ F(x, y, z)^{2k} = (x^2 + y^2 - z^2)^k \cdot I_2. \]
Thus we see that \( F(x, y, z) \) is nilpotent if and only if \( x^2 + y^2 - z^2 = 0 \). Therefore the set \( \mathcal{N}_\mathbb{R} \) of all nilpotent elements in \( \mathfrak{g} \) is given by
\[ \mathcal{N}_\mathbb{R} = \left\{ F(x, y, z) = \begin{pmatrix} x & y + z \\ y - z & -x \end{pmatrix} \mid x^2 + y^2 - z^2 = 0 \right\}. \] (2.6)

We put
\[ S = \frac{1}{2}(Y + Z) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T = \frac{1}{2}(Y - Z) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \] (2.7)

Obviously \( S \) and \( T \) are nilpotent elements in \( \mathcal{N}_\mathbb{R} \) and they satisfy
\[ [X, S] = 2S, \quad [X, T] = -2T, \quad [S, T] = X \] (2.8)
and
\[ \theta(X) = -X, \quad \theta(S) = -T, \quad \theta(T) = -S. \] (2.9)

Here \( \theta \) is the Cartan involution defined by \( \theta(g) = -g \) for \( g \) in \( \mathfrak{g} \).

According to equations (2.8) and (2.9), \( \{X, S, T\} \) and \( \{-X, -S, -T\} \) are KS-triples in \( \mathfrak{g} \).

The \( G \)-orbit of \( \alpha S (\alpha \in \mathbb{R}^\times) \) is represented by the cone
\[ x^2 + y^2 - z^2 = 0, \quad (x, y, z) \neq (0, 0, 0) \] (2.10)
depending on the sign of \( \alpha \).

If \( \alpha > 0 \), the \( G \)-orbit of \( \alpha S \) is characterized by the one-sheeted cone
\[ x^2 + y^2 - z^2 = 0, \quad z > 0. \] (2.11)

If \( \alpha < 0 \), the \( G \)-orbit of \( \alpha S \) is characterized by the one-sheeted cone
\[ x^2 + y^2 - z^2 = 0, \quad z < 0. \] (2.12)

The \( G \)-orbits of \( \alpha T (\alpha > 0) \) are characterized by the one-sheeted cone (2.12) and the \( G \)-orbits of \( \alpha T (\alpha < 0) \) are characterized by the one-sheeted cone (2.11).

We define the \( G \)-orbits \( \mathcal{N}_\mathbb{R}^+ \) and \( \mathcal{N}_\mathbb{R}^- \) by
\[ \mathcal{N}_\mathbb{R}^+ = G \cdot S \quad \text{and} \quad \mathcal{N}_\mathbb{R}^- = G \cdot T. \] (2.13)

Then we obtain
According to (2.4), (2.5) and (2.14), we see that there are infinitely many hyperbolic orbits and elliptic orbits, and on the other hand there are only three nilpotent orbits in \( g \).

Let

\[
K_C = SO(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \bigg| a^2 + b^2 = 1, \ a, b \in \mathbb{C} \right\}
\]

be the complexification of \( K \). The complexification \( g_C \) of \( g \) has the Cartan decomposition

\[
g_C = k_C + p_C,
\]

where

\[
k_C = \left\{ \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} \bigg| z \in \mathbb{C} \right\},
\]

and

\[
p_C = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \bigg| x, y \in \mathbb{C} \right\}.
\]

The set \( N_\theta \) of all nilpotent elements in \( p_C \) is given by

\[
N_\theta = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \in p_C \bigg| x^2 + y^2 = 0 \right\} \subset p_C.
\]

We note that \( K_C \) acts on \( N_\theta \).

We put

\[
H_\theta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X_\theta = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad Y_\theta = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.
\]

Then they satisfy

\[
H_\theta \in k_C, \quad X_\theta, Y_\theta \in p_C
\]

and

\[
[H_\theta, Y_\theta] = 2Y_\theta, \quad [H_\theta, X_\theta] = -2X_\theta, \quad [Y_\theta, X_\theta] = H_\theta.
\]

We note that \( K_C \) acts on \( N_\theta \).

According to equation (2.17), (2.18) and (2.19), \( \{H_\theta, Y_\theta, X_\theta\} \) and \( \{-H_\theta, -Y_\theta, -X_\theta\} \) are KS-triples in \( g_C \). Moreover, two KS-triples \( \{X, S, T\} \) and \( \{H_\theta, Y_\theta, X_\theta\} \) satisfy the following conditions:

\[
H_\theta = i(S - T), \quad Y_\theta = \frac{1}{2}(S + T + iX), \quad X_\theta = \frac{1}{2}(S + T - iX).
\]

Now we define the \( K_C \)-orbits \( N_\theta^+ \) and \( N_\theta^- \) by

\[
N_\theta^+ = K_C \cdot X_\theta \quad \text{and} \quad N_\theta^- = K_C \cdot Y_\theta.
\]

Then we see that

\[
N_\theta = N_\theta^+ \cup \{0\} \cup N_\theta^-.
\]

The \( K_C \)-orbit \( N_\theta^+ \) is characterized by the straight line
\[ y = ix, \quad x \in \mathbb{C} - \{0\}. \quad (2.23) \]

On the other hand, the \( K_C \)-orbit \( N_{\theta}^- \) is characterized by the straight line
\[ y = -ix, \quad x \in \mathbb{C} - \{0\}. \quad (2.24) \]

It is easily seen that the \( K_C \)-orbits of \( \alpha H_\theta (\alpha \in \mathbb{C}^*) \) are represented by complex hyperboloids and that there are infinitely many hyperbolic and elliptic orbits in \( g_C \). However there are only three nilpotent orbits in \( p_C \) which are \( N_{\theta}^+ \), \( \{0\} \) and \( N_{\theta}^- \).

The Kostant-Sekiguchi correspondence between the \( G \)-nilpotent orbits in \( N_R \) and the \( K_C \)-nilpotent orbits in \( N_{\theta} \) is given by
\[ N_{\theta}^+ \mapsto N_{\theta}^-, \quad \{0\} \mapsto \{0\}, \quad N_{\theta}^- \mapsto N_{\theta}^+. \quad (2.25) \]

3. Adjoint Orbits of the Jacobi Group

In this section, we compute the adjoint orbits for the Jacobi group. We observe that the Jacobi group \( G^J \) is embedded in the symplectic group \( Sp(4, \mathbb{R}) \) via

\[ (M, (\lambda, \mu, \kappa)) \mapsto \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.1) \]

where \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \). The Lie algebra \( g^J \) of \( G^J \) is given by
\[ g^J = \{(X, (p, q, r)) \mid X \in g, \ p, q, r \in \mathbb{R}\} \quad (3.2) \]

with the bracket
\[ [(X_1, (p_1, q_1, r_1)), (X_2, (p_2, q_2, r_2))] = (\tilde{X}, (\tilde{p}, \tilde{q}, \tilde{r})), \quad (3.3) \]

where
\[ X_1 = \begin{pmatrix} x_1 & y_1 \\ z_1 & -x_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & -x_2 \end{pmatrix} \in sl(2, \mathbb{R}) \]

and
\[ \tilde{X} = X_1X_2 - X_2X_1, \]
\[ \tilde{p} = p_1x_2 + q_1z_2 - p_2x_1 - q_2z_1, \]
\[ \tilde{q} = q_2x_1 + p_1y_2 - q_1x_2 - p_2y_1, \]
\[ \tilde{r} = 2(p_1q_2 - p_2q_1). \]

Indeed, an element \((X, (p, q, r))\) in \( g^J \) with \( X = \begin{pmatrix} x & y + z \\ y - z & -x \end{pmatrix} \in sl(2, \mathbb{R}) \) may be identified with the matrix
\[ G(x, y, z, p, q, r) := \begin{pmatrix} x & 0 & y + z & q \\ p & 0 & q & r \\ y - z & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.4) \]
Lemma 3.1. If \( G(x, y, z, p, q, r) \) is an element in \( g^J \) given by (3.4), then for a positive integer \( k \in \mathbb{Z}^+ \),
\[
G(x, y, z, p, q, r)^{2k} = (x^2 + y^2 - z^2)^{k-1}G(x, y, z, p, q, r)^2.
\] (3.5)

Proof. By the Cayley-Hamilton theorem or a direct computation, we obtain
\[
G(x, y, z, p, q, r)^4 = (x^2 + y^2 - z^2)G(x, y, z, p, q, r)^2.
\] (3.6)

The formula (3.5) follows immediately from (3.6). □

According to Lemma 3.1, the set \( N_R^J \) of all nilpotent elements in \( g^J \) is given by
\[
N_R^J = \{ G(x, y, z, p, q, r) \in g^J \mid x^2 + y^2 - z^2 = 0 \}.
\] (3.7)

We have the adjoint action of \( G^J \) on \( g^J \) given by
\[
g \cdot X = \text{Ad}(g)X = gXg^{-1}, \quad g \in G^J, \quad X \in N_R^J.
\] (3.8)

According to (3.1), we may write \( g = (M, (\lambda, \mu, \kappa)) \in G^J \) with \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \) as
\[
g = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\] (3.9)

Then the inverse of \( g \) is given by
\[
g^{-1} = \begin{pmatrix} d & 0 & -b & -\mu \\ a\mu - d\lambda & 1 & b\lambda - a\mu & -\kappa \\ -c & 0 & a & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\] (3.10)

Lemma 3.2. If \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu, \kappa) \) is an element of \( G^J \), then the action of \( g \) on \( G(x, y, z, p, q, r) \) is given by
\[
g \cdot G(x, y, z, p, q, r) = G(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}),
\] (3.11)

where
\[
\begin{align*}
\tilde{x} & = (ad + bc)x - ac(y + z) + bd(y - z), \\
\tilde{y} + \tilde{z} & = -2abx + a^2(y + z) - b^2(y - z), \\
\tilde{y} - \tilde{z} & = 2cdx - c^2(y + z) + d^2(y - z), \\
\tilde{p} & = d \{ \lambda x + \mu(y - z) + p \} + c \{ \mu x - \lambda(y + z) - q \}, \\
\tilde{q} & = -b \{ \lambda x + \mu(y - z) + p \} - a \{ \mu x - \lambda(y + z) - q \}, \\
\tilde{r} & = -2\lambda \mu x + \lambda^2(y + z) - \mu^2(y - z) - 2p\mu + 2q\lambda + r.
\end{align*}
\]

In particular, \( N_R^J \) is stable under the action of \( G^J \).
Proof. The proof of the first part follows from a direct computation. Let \( G(x, y, z, p, q, r) \) be an element of \( N^J_{\mathbb{R}} \). Since

\[
\tilde{x}^2 + \tilde{y}^2 - \tilde{z}^2 = (ad - be)^2(x^2 + y^2 - z^2) = 0,
\]

we see that \( g \cdot N^J_{\mathbb{R}} \subset N^J_{\mathbb{R}} \) for all \( g \in G^J \). \( \square \)

We set

\[
\begin{align*}
X^J &= G(1, 0, 0, 0, 0, 0), \\
Y^J &= G(0, 1, 0, 0, 0, 0), \\
Z^J &= G(0, 0, 1, 0, 0, 0), \\
P^J &= G(0, 0, 0, 1, 0, 0), \\
Q^J &= G(0, 0, 0, 0, 1, 0), \\
R^J &= G(0, 0, 0, 0, 0, 1).
\end{align*}
\]

Obviously the set \{ \( X^J, Y^J, Z^J, P^J, Q^J, R^J \) \} forms a basis for \( \mathfrak{g}^J \). So we have

\[
G(x, y, z, p, q, r) = xX^J + yY^J + zZ^J + pP^J + qQ^J + rR^J.
\]

We note that \( X^J \) and \( Y^J \) are hyperbolic elements, \( Z^J \) is an elliptic element and \( P^J, Q^J, R^J \) are nilpotent elements.

Let \( \Pi(G(x, y, z, p, q, r)) \) denote the \( G^J \)-orbit of \( G(x, y, z, p, q, r) \). According to Lemma 3.2, we can get the following lemma.

**Lemma 3.3.** Let \( \alpha \in \mathbb{R} \) be a fixed nonzero real number. Then

\[
\begin{align*}
\Pi(\alpha X^J) &= \{ G(x, y, z, p, q, r) \in \mathfrak{g}^J \mid x^2 + y^2 - z^2 = \alpha^2 \} \quad \text{(3.12)} \\
\Pi(\alpha Y^J) &= \{ G(x, y, z, p, q, r) \in \mathfrak{g}^J \mid f(x, y, z, p, q) = \alpha^2 r \} \quad \text{(3.13)} \\
\Pi(\alpha Z^J) &= \{ G(x, y, z, p, q, r) \in \mathfrak{g}^J \mid x^2 + y^2 - z^2 = -\alpha^2 \} \quad \text{(3.14)} \\
\Pi(\alpha P^J) &= \{ G(0, 0, 0, p, q, r) \in \mathfrak{g}^J \mid pq \neq 0 \} \quad \text{(3.15)} \\
\Pi(\alpha R^J) &= \alpha R^J.
\end{align*}
\]

where

\[
f(x, y, z, p, q) = 2pqx - p^2(y + z) + q^2(y - z). \quad \text{(3.16)}
\]
Proof. By Lemma 3.2,
\[
\Pi(\alpha X^J) = \Bigg\{ \alpha G(x, y, z, p, q, r) \in g^J : \begin{array}{l}
x = ad + bc, \; y + z = -2ab, \; y - z = 2cd, \\
p = d\lambda + c\mu, \; q = -b\lambda - a\mu, \; r = -2\lambda\mu,
\end{array} \Bigg\},
\]
\[
\Pi(\alpha Y^J) = \Bigg\{ \alpha G(x, y, z, p, q, r) \in g^J : \begin{array}{l}
x = -ac + bd, \; y + z = a^2 - b^2, \; y - z = c^2 + d^2, \\
p = -c\lambda + d\mu, \; q = a\lambda - b\mu, \; r = \lambda^2 - \mu^2,
\end{array} \Bigg\},
\]
\[
\Pi(\alpha Z^J) = \Bigg\{ \alpha G(x, y, z, p, q, r) \in g^J : \begin{array}{l}
x = -ac - bd, \; y + z = a^2 + b^2, \; y - z = c^2 - d^2, \\
p = -c\lambda - d\mu, \; q = a\lambda + b\mu, \; r = \lambda^2 + \mu^2,
\end{array} \Bigg\},
\]
\[
\Pi(\alpha P^J) = \{ \alpha G(0, 0, 0, p, q, r) \in g^J : p = d, \; q = -b, \; r = -2\mu, \; b, d, \mu \in \mathbb{R} \},
\]
\[
\Pi(\alpha Q^J) = \{ \alpha G(0, 0, 0, p, q, r) \in g^J : p = -c, \; q = a, \; r = 2\lambda, \; a, c, \lambda \in \mathbb{R} \},
\]
\[
\Pi(\alpha R^J) = \alpha R^J.
\]
\]

We define the nilpotent elements \( S^J \) and \( T^J \) by
\[
S^J = \frac{1}{2}(Y^J + Z^J) \quad \text{and} \quad T^J = \frac{1}{2}(Y^J - Z^J).
\] (3.17)

Lemma 3.4. Let \( \alpha \in \mathbb{R} \) be a fixed nonzero real number. Then \( \alpha G(0, 1, 1, p, q, r) \) and \( \alpha G(0, 1, 1, p, 0, r - \frac{1}{2}q^2) \) lie in the same \( G^J \)-orbit in \( g^J \).

Proof. If \( g = (I_2, (-q/2, 0, 0)) \), then the action of \( g \) on \( G(0, 1, 1, p, q, r) \) is given by
\[
g \cdot G(0, 1, 1, p, q, r) = G\left(0, 1, 1, p, 0, r - \frac{1}{2}q^2\right).
\]
\]

Lemma 3.5. Let \( \alpha, \beta \in \mathbb{R} \) be fixed nonzero real numbers. Then
\[
\Pi(\alpha S^J) = \Bigg\{ G(x, y, z, p, q, r) \in g^J : \begin{array}{l}
x^2 + y^2 - z^2 = 0, \; z/\alpha > 0, \\
r \text{ is dependant on } x, y, z, p, q.
\end{array} \Bigg\},
\] (3.18)
\[
\Pi(\alpha T^J) = \Bigg\{ G(x, y, z, p, q, r) \in g^J : \begin{array}{l}
x^2 + y^2 - z^2 = 0, \; z/\alpha < 0, \\
r \text{ is dependant on } x, y, z, p, q.
\end{array} \Bigg\},
\] (3.19)
\[
= \Pi(\alpha(-S^J)),
\]
\[
\Pi(\alpha(S^J + \beta P^J)) = \Bigg\{ G(x, y, z, p, q, r) \in g^J : \begin{array}{l}
x^2 + y^2 - z^2 = 0, \; z/\alpha > 0, \\
f(x, y, z, p, q) = -\alpha^2\beta^2
\end{array} \Bigg\},
\] (3.20)
\[
= \Pi(\alpha|\beta|^{2/3}(S^J + P^J))
\]

where
\[
f(x, y, z, p, q) = 2pqx - p^2(y + z) + q^2(y - z).
\]
Proof. By Lemma 3.2, 

\[ \Pi(\alpha S^J) = \begin{cases} 
\alpha G(x, y, z, p, q, r) \in \mathfrak{g}^J & \begin{cases} x = -ac, \ y + z = a^2, \ y - z = -c^2, \\
p = -c\lambda, \ q = a\lambda, \ r = \lambda^2,
\end{cases} \\
ad - bc = 1, \ a, b, c, d, \lambda, \mu \in \mathbb{R}
\end{cases}, \]

\[ \Pi(\alpha T^J) = \begin{cases} 
\alpha G(x, y, z, p, q, r) \in \mathfrak{g}^J & \begin{cases} x = bd, \ y + z = -b^2, \ y - z = -d^2,
\end{cases} \\
p = d\mu, \ q = -b\mu, \ r = -\mu^2,
\end{cases} \\
ad - bc = 1, \ a, b, c, d, \lambda, \mu \in \mathbb{R}
\]

\[ \Pi(\alpha(S^J + \beta P^J)) = \begin{cases} 
\alpha G(x, y, z, p, q, r) \in \mathfrak{g}^J & \begin{cases} x = -ac, \ y + z = a^2, \ y - z = -c^2,
\end{cases} \\
p = -c\lambda + \beta d, \ q = a\lambda - \beta b, \ r = \lambda^2 - 2\beta\mu,
\end{cases} \\
ad - bc = 1, \ a, b, c, d, \lambda, \mu \in \mathbb{R}. \quad \Box
\]

Now we can prove the following theorem.

**Theorem 3.6.** We have a disjoint union

\[ \mathcal{N}_R^J = \{0\} \bigcup \Pi(S^J) \bigcup \Pi(T^J) \bigcup \Pi(P^J) \bigcup \left( \bigcup_{\alpha \in \mathbb{R}^*} \{ \Pi(\alpha R^J) \} \right) \]

\[ \bigcup \left( \bigcup_{\alpha \in \mathbb{R}^*} \{ \Pi(S^J + \alpha R^J) \} \right) \bigcup \left( \bigcup_{\alpha \in \mathbb{R}^*} \{ \Pi(\alpha(S^J + P^J)) \} \right). \]

In particular, there are infinitely many nilpotent \(G^J\)-orbits in \(\mathcal{N}_R^J \subset \mathfrak{g}^J\).

Proof. By (3.7), It is easily checked that 

\[ \mathcal{N}_R^J = \{(X, (p, q, r)) \mid X \in \mathcal{N}_R, \ p, q, r \in \mathbb{R} \}, \]

\[ = \mathcal{N}^{J,0}_R \cup \mathcal{N}^{J,+}_R \cup \mathcal{N}^{J,-}_R. \]

Here

\[ \mathcal{N}^{J,0}_R = \{(X, (p, q, r)) \mid X \in \{0\}, \ p, q, r \in \mathbb{R} \}, \]

\[ \mathcal{N}^{J,+}_R = \{(X, (p, q, r)) \mid X \in \mathcal{N}^+_R, \ p, q, r \in \mathbb{R} \}, \]

\[ \mathcal{N}^{J,-}_R = \{(X, (p, q, r)) \mid X \in \mathcal{N}^-_R, \ p, q, r \in \mathbb{R} \}. \]

According to Lemma 3.3–3.5,

\[ \mathcal{N}^{J,0}_R = \bigcup_{p, q, r \in \mathbb{R}} \Pi(G(0, 0, 0, p, q, r)), \]

\[ = \Pi(P^J) \bigcup \{0\} \bigcup \left( \bigcup_{\alpha \in \mathbb{R}^*} \{ \Pi(\alpha R^J) \} \right). \]

\[ \mathcal{N}^{J,+}_R \cup \mathcal{N}^{J,-}_R = \bigcup_{p, q, r \in \mathbb{R}} \Pi(\alpha G(0, 1, 1, p, q, r)), \]

\[ = \bigcup_{\alpha \in \{\pm 1\}} (\Pi(\alpha S^J + pP^J) + \Pi(r R^J)), \]

\[ = \Pi(S^J) \bigcup \Pi(T^J) \bigcup \left( \bigcup_{\alpha \in \mathbb{R}^*} \{ \Pi(S^J + \alpha R^J) \} \right) \bigcup \left( \bigcup_{\alpha \in \mathbb{R}^*} \{ \Pi(\alpha(S^J + P^J)) \} \right). \]

(3.24)
Clearly, the set in (3.23) and the set in (3.24) are disjoint union. Hence, we obtain the formula (3.21).

For \( x, y, p, q \in \mathbb{C} \), we set

\[
H(x, y, p, q) := \begin{pmatrix}
x & 0 & y & q \\
p & 0 & q & 0 \\
y & 0 & -x & -p \\
0 & 0 & 0 & 0
\end{pmatrix}.
\] (3.25)

Let \( \mathfrak{p}_C^J \) be the vector space consisting of all \( H(x, y, p, q) \) \((x, y, p, q \in \mathbb{C})\). Let \( \mathfrak{g}_C^J \) be the complexification of \( \mathfrak{g}^J \). Then we have the direct sum

\[
\mathfrak{g}_C^J = \mathfrak{t}_C^J + \mathfrak{p}_C^J,
\] (3.26)

where

\[
\mathfrak{t}_C^J = \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, (0, 0, \kappa) \right\} \mid x, \kappa \in \mathbb{C}
\]

is the complexification of the Lie algebra \( \mathfrak{t}^J \) of \( K^J \). Let \( \mathcal{N}_\theta^J \) be the set of all nilpotent elements in \( \mathfrak{p}_C^J \). Then

\[
\mathcal{N}_\theta^J = \left\{ H(x, y, p, q) \in \mathfrak{p}_C^J \mid x^2 + y^2 = 0 \right\}.
\] (3.27)

Indeed, (3.26) follows from the fact that

\[
H(x, y, p, q)^{2k} = (x^2 + y^2)^{k-1} H(x, y, p, q)^2 \quad \text{for all } k \in \mathbb{Z}^+.
\] (3.28)

We set

\[
\begin{align*}
X_{\theta}^J & = \frac{1}{2} H(-i, 1, 0, 0), \\
Y_{\theta}^J & = \frac{1}{2} H(i, 1, 0, 0), \\
P_{\theta}^J & = H(0, 0, 1, 0), \\
Q_{\theta}^J & = H(0, 0, 0, 1).
\end{align*}
\]

Obviously the set \( \{X_{\theta}^J, Y_{\theta}^J, P_{\theta}^J, Q_{\theta}^J\} \) forms a basis for a complex vector space \( \mathfrak{p}_C^J \).

**Proposition 3.7.** \( K_C^J \) acts on \( \mathfrak{p}_C^J \) preserving \( \mathcal{N}_\theta^J \).

**Proof.** An element \( k^J \) of \( K_C^J \) is of the form

\[
k^J = \begin{pmatrix}
a & 0 & b & 0 \\
0 & 1 & 0 & \kappa \\
-b & 0 & a & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad a, b, \kappa \in \mathbb{C}, \quad a^2 + b^2 = 1.
\] (3.29)

We obtain

\[
k^J \cdot H(x, y, p, q) = H(x^*, y^*, p^*, q^*),
\]
where
\[
x^* = (a^2 - b^2)x + 2a y,
\]
\[
y^* = -2ab x + (a^2 - b^2)y,
\]
\[
p^* = ap + bq,
\]
\[
q^* = aq - bp.
\]
(3.30)

If \( H(x, y, p, q) \in \mathcal{N}^J_\theta \), then
\[
(x^*)^2 + (y^*)^2 = (a^2 + b^2)^2(x^2 + y^2) = 0.
\]
Therefore \( k^J \cdot \mathcal{N}^J_\theta \subset \mathcal{N}^J_\theta \) for each element \( k^J \in K^J_\theta \).

We define the \( K^J_\theta \)-orbits \( \mathcal{N}^{J+}_\theta \) and \( \mathcal{N}^{J-}_\theta \) by
\[
\mathcal{N}^{J+}_\theta = K^J_\theta \cdot X^J_\theta \quad \text{and} \quad \mathcal{N}^{J-}_\theta = K^J_\theta \cdot Y^J_\theta.
\]
It is easily checked that \( \mathcal{N}^{J+}_\theta \) and \( \mathcal{N}^{J-}_\theta \) are given by
\[
\mathcal{N}^{J+}_\theta = \left\{ H(x, y, 0, 0) \in \mathfrak{g}^J_\theta \bigg| \begin{array}{l}
x = -ia^2 + 2ab + ib^2, \\
y = a^2 + 2iab - b^2 \\
a^2 + b^2 = 1, a, b \in \mathbb{C}
\end{array} \right\},
\]
\[
= \left\{ H(ix, 0, 0) \big| x \neq 0, x \in \mathbb{C} \right\},
\]
and
\[
\mathcal{N}^{J-}_\theta = \left\{ H(x, y, 0, 0) \in \mathfrak{g}^J_\theta \bigg| \begin{array}{l}
x = ia^2 + 2ab - ib^2, \\
y = a^2 - 2iab - b^2 \\
a^2 + b^2 = 1, a, b \in \mathbb{C}
\end{array} \right\},
\]
\[
= \left\{ H(x, -ix, 0, 0) \big| x \neq 0, x \in \mathbb{C} \right\}.
(3.31)
\]

We define, for a nonzero complex number \( \delta \in \mathbb{C} \),
\[
\mathcal{N}^{J+}P_\theta(\delta) = K^J_\theta \cdot \delta P^J_\theta \quad \text{and} \quad \mathcal{N}^{J+}Q_\theta(\delta) = K^J_\theta \cdot \delta Q^J_\theta.
\]
Then
\[
\mathcal{N}^{J+}P_\theta(\delta) = \mathcal{N}^{J+}Q_\theta(\delta) = \{ H(0, 0, p, q) \mid p^2 + q^2 = \delta^2, p, q \in \mathbb{C} \}. \quad (3.32)
\]

**Lemma 3.8.** \( H(x, y, \delta, 0) \) and \( H(\tilde{x}, \tilde{y}, \delta, 0) \) lie in the same \( K^J_\theta \)-orbit in \( \mathfrak{p}^J_\theta \) if and only if \( \tilde{x} = x, \tilde{y} = y \).

**Proof.** By (3.30), we have \( a = 1 \) and \( b = 0 \). Hence \( \tilde{x} = x, \tilde{y} = y \).

**Lemma 3.9.** Let \( (x, y) \in \mathbb{C}^2 \) with \( (x, y) \neq (0, 0) \). Then \( H(x, y, \delta, 0) \) and \( H(x, y, \tilde{\delta}, 0) \) lie in the same \( K^J_\theta \)-orbit in \( \mathfrak{p}^J_\theta \) if and only if \( \tilde{\delta} = \pm \delta \).

**Proof.** According to (3.30), \( b = 0 \) and so \( a = \pm 1 \). Since \( \tilde{\delta} = a\delta, \tilde{\delta} = \pm \delta \).

**Lemma 3.10.** Suppose \( H(x, y, p, q) \in \mathcal{N}^J_\theta \) with \( y = \xi x \), where \( \xi = i \) or \( -i \). If \( H(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}) \) is in the \( K^J_\theta \)-orbit of \( H(x, y, p, q) \) in \( \mathfrak{p}^J_\theta \), then \( \tilde{y} = \xi \tilde{x} \).

**Proof.** According to (3.30), we obtain
\[
\tilde{x} = (a + \xi b)^2 x, \quad \tilde{y} = \xi(a + \xi b)^2 x.
\]
Hence \( \tilde{y} = \xi \tilde{x} \).
Lemma 3.11. Let $x, \delta \in \mathbb{C}$ with $x \neq 0$. We denote by $N^J_\theta^+(x, \delta)$ and $N^J_\theta^-(x, \delta)$ the $K_C$-orbits of $H(x, ix, \delta, 0)$ and $H(x, -ix, \delta, 0)$ respectively. Then $N^J_\theta^+(x, \delta)$ and $N^J_\theta^-(x, \delta)$ are given by

$$N^J_\theta^+(x, \delta) = \{H(z, iz, p, q) \mid z, p, q \in \mathbb{C}, \ p^2 + q^2 = \delta^2 \}$$

and

$$N^J_\theta^-(x, \delta) = \{H(z, -iz, p, q) \mid z, p, q \in \mathbb{C}, \ p^2 + q^2 = \delta^2 \}.$$  \hspace{1cm} (3.34)

Proof. If $H(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q})$ is an element of $N^J_\theta^+(x, \delta)$, then there exist $a, b \in \mathbb{C}$ with $a^2 + b^2 = 1$ satisfying

$$\tilde{x} = (a + ib)^2 x, \quad \tilde{y} = i(a + ib)^2 x, \quad \tilde{p} = a\delta, \quad \tilde{q} = -b\delta.$$  \hspace{1cm} (3.35)

Thus $\tilde{y} = i\tilde{x}$, $\tilde{p}^2 + \tilde{q}^2 = (a^2 + b^2)\delta^2 = \delta^2$. Hence we obtain the formula (3.34). In a similar way, we get the formula (3.35). \hfill \Box

According to (3.31) – (3.35), Lemma 3.8 – Lemma 3.11, we obtain the following theorem.

Theorem 3.12. We have the following disjoint union

$$N^J_\theta = \{0\} \cup N^J_\theta^+ \cup N^J_\theta^- \cup \left( \bigcup_{\delta \in \mathbb{C}^* / \{\pm 1\}} N^J_\theta^P (\delta) \right)$$

$$\cup \left( \bigcup_{\delta \in \mathbb{C}^*} N^J_\theta^+(x, \delta) \right) \cup \left( \bigcup_{\delta \in \mathbb{C}^*} N^J_\theta^-(x, \delta) \right).$$

In particular, there are infinitely many nilpotent $K^J_C$-orbits in $N^J_\theta \subset p^J_C$.

Remark 3.13. It is known that if $G$ is a real reductive Lie group, there are only finitely many nilpotent orbits and that there is the so-called Kostant-Sekiguchi correspondence between the set of all nilpotent $G$-orbits in $\mathfrak{g}$ and the set of all nilpotent $K_C$-orbits in $p_C$, where $K_C$ is the complexification of a maximal compact subgroup $K$ of $G$ and $\mathfrak{g}_C = \mathfrak{k}_C + p_C$ is the Cartan decomposition of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$ (cf. [9, 10, 11, 12]). We refer to [3] for adjoint orbits of complex semisimple groups.

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