Stochastic comparisons between the extreme claim amounts from two heterogeneous portfolios in the case of transmuted-G model

Hossein Nadeb, Hamzeh Torabi, Ali Dolati
Department of Statistics, Yazd University, Yazd, Iran,

Abstract

Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent non-negative random variables belong to the transmuted-G model and let $Y_i = I_{p_i}X_{\lambda_i}$, $i = 1, \ldots, n$, where $I_{p_1}, \ldots, I_{p_n}$ are independent Bernoulli random variables independent of $X_{\lambda_i}$’s, with $\mathbb{E}[I_{p_i}] = p_i$, $i = 1, \ldots, n$. In actuarial sciences, $Y_i$ corresponds to the claim amount in a portfolio of risks. In this paper we compare the smallest and the largest claim amounts of two sets of independent portfolios belonging to the transmuted-G model, in the sense of usual stochastic order, hazard rate order and dispersive order, when the variables in one set have the parameters $\lambda_1, \ldots, \lambda_n$ and the variables in the other set have the parameters $\lambda_1^*, \ldots, \lambda_n^*$. For illustration we apply the results to the transmuted-G exponential and the transmuted-G Weibull models.

Keywords Largest claim amount, Majorization, Smallest claim amount, Stochastic ordering.

1 Introduction

Annual premium is the amount paid by the policyholder as the cost of the insurance cover being purchased. Indeed, it is the primary cost to the policyholder for assigning the risk to the insurer which depends on the type of insurance. Determination of the annual premium is one of the important problem in insurance analysis. For this purpose, the smallest and the largest claim amounts play an important role in providing useful information. An attractive problem for the actuaries is expressing preferences between random future gains or losses (Barmalzan et al. (2017)). For this purpose, stochastic orderings are very helpful. Stochastic orderings have been extensively used in some areas of sciences such as management science, financial economics, insurance, actuarial science, operation research, reliability theory, queuing theory and survival analysis. For more details on stochastic orderings we refer to Müller and Stoyan (2002), Shaked and Shanthikumar (2007) and Li and Li (2013). The transmuted-G (TG) model, which introduced by Mirhossaini and Dolati (2008) and Shaw and Buckley (2009), is an attractive model for constructing new flexible distributions. Let $F$ be an absolutely continuous distribution function with the corresponding
survival function \( \bar{F} \). The random variables \( X_\lambda \) said to belong to the TG model with the baseline distribution function \( F \), if \( X_\lambda \) has the distribution function

\[
F_{X_\lambda}(x) = F(x)(1 + \lambda \bar{F}(x)),
\]

where \(-1 \leq \lambda \leq 1\). We use the notion \( X_\lambda \sim TG(\lambda) \) for the transmuted-G model.

Several distributions have been generalized by this transmuting approach in the literature. Some of them are the transmuted Weibull distribution by Aryal and Tsokos (2011), the transmuted Maxwell distribution by Iriarte and Astorga (2014), the transmuted linear exponential distribution by Tian et al. (2014), the transmuted log-logistic distribution by Granzotto and Louzada (2015), the transmuted Dagum distribution by Elbatal and Aryal (2015), the transmuted Erlang-truncated exponential distribution by Okorie et al. (2016), the transmuted exponentiated Weibull geometric distribution by Saboor et al. (2016), the transmuted exponential Pareto distribution by Al-Babtain (2017), the transmuted two-parameter Lindley distribution by Kemaloglu and Yilmaz (2017) and the transmuted Birnbaum-Saunders distribution by Bourguignon et al. (2017).

The problem of stochastic comparisons of some quantities such as the number of claims, the aggregate claim amounts, the smallest and the largest claim amounts in two portfolios, have been considered by many researchers in literature; see, e.g., Karlin and Novikoff (1963), Ma (2000), Frostig (2001), Hu and Ruan (2004), Denuit and Frostig (2006), Khaledi and Ahmadi (2008), Zhang and Zhao (2015), Barmalzan et al. (2015), Li and Li (2016), Barmalzan and Najafabadi (2015), Barmalzan et al. (2016), Barmalzan et al. (2017) and Balakrishnan et al. (2018). Flexibility of the transmuted-G model is a good property to assuming this model as the distribution of severities in insurance. Motivated by the extensive applications of the transmuted-G family to make flexible models from a given baseline distribution, in this paper we study stochastic comparisons between the extreme claim amounts from two heterogeneous portfolios in the case of transmuted-G model. To be exact, suppose that \( X_\lambda \) denotes the total random severities of a policyholder in an insurance period, and let \( I_p \) be a Bernoulli random variable associated with \( X_\lambda \), such that \( I_p = 1 \) whenever the policyholder makes random claim amounts \( X_\lambda \) and \( I_p = 0 \) whenever does not make a claim. In this notation, \( Y = I_p X_\lambda \) is the claim amount in a portfolio of risks. Consider two sets of heterogeneous portfolios \( X_{\lambda_1}, \ldots, X_{\lambda_n} \) and \( X_{\lambda'_1}, \ldots, X_{\lambda'_n} \) belonging to the TG model and let \( Y_i = I_{p_i} X_{\lambda_i} \) and \( Y_i^* = I_{p_i'} X_{\lambda'_i} \), \( i = 1, \ldots, n \), where \( I_{p_i} \) independent of \( X_{\lambda_i} \) and \( I_{p_i'} \) independent of \( X_{\lambda'_i} \) are independent Bernoulli random variables with \( E[I_{p_i}] = p_i \) and \( E[I_{p_i'}] = p_i' \). Let \( Y_{1:n} = \min(Y_1, \ldots, Y_n) \), \( Y_{1:n}^* = \min(Y_1^*, \ldots, Y_n^*) \), \( Y_{n:n} = \max(Y_1, \ldots, Y_n) \) and \( Y_{n:n}^* = \max(Y_1^*, \ldots, Y_n^*) \) be the smallest and the largest claim amounts, arise from \( Y_1, \ldots, Y_n \) and \( Y_1^*, \ldots, Y_n^* \). In this paper we compare \( Y_{1:n} \) and \( Y_{1:n}^* \) in the sense of the usual stochastic order, hazard rate order and dispersive order and \( Y_{n:n} \) and \( Y_{n:n}^* \) in the sense of the usual stochastic order and hazard rate order. For illustration we apply the results to the transmuted-G exponential and the transmuted-G Weibull models. The rest of the paper is organized as follows. In Section 2, we recall
some definitions and lemmas which will be used in the sequel. In Section 2 stochastic comparisons of the largest claim amounts from two heterogeneous portfolios of risks in a transmuted-G model in the sense of the usual stochastic ordering and reversed hazard rate ordering are discussed. In Section 3 stochastic comparisons of the smallest claim amounts from two heterogeneous portfolios of risks in a transmuted-G model in the sense of the usual stochastic ordering, hazard rate ordering and dispersive ordering are discussed. In Section 5 we consider the transmuted-G exponential and the transmuted-G Weibull models for illustration of the established results.

2 The basic definitions and some prerequisites

In this section, we recall some notions of stochastic orderings, majorization, weakly majorization and related orderings and some useful lemmas which are helpful to prove the main results. Throughout the paper, we use the notations $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_{++} = (0, +\infty)$. The term increasing (decreasing) is used for monotone nondecreasing (nonincreasing). Let $X$ and $Y$ be two non-negative random variables with the respective distribution functions $F$ and $G$, the density functions $f$ and $g$, the survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, the right continuous inverses $F^{-1}$ and $G^{-1}$, the hazard rate functions $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, and the reversed hazard rate functions $\tilde{r}_X = f/F$ and $\tilde{r}_Y = g/G$.

**Definition 2.1.** $X$ is said to be smaller than $Y$ in the

(i) usual stochastic ordering, denoted by $X \preceq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in \mathbb{R}$,

(ii) hazard rate ordering, denoted by $X \preceq_{hr} Y$, if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in \mathbb{R}$, or $r_Y(x) \leq r_X(x)$ for all $x \in \mathbb{R}$,

(iii) reversed hazard rate ordering, denoted by $X \preceq_{rh} Y$, if $G(x)/F(x)$ is increasing in $x \in \mathbb{R}_+$, or $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$ for all $x \in \mathbb{R}_+$,

(iv) dispersive ordering, denoted by $X \preceq_{disp} Y$, if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 \leq \alpha \leq \beta \leq 1$.

We know that the hazard rate and reversed hazard rate orderings imply the usual stochastic ordering.

**Lemma 2.1** (Shaked and Shanthikumar (2007), Theorem 3.B.20). Let $X$ and $Y$ be two non-negative random variables. If $X \preceq_{hr} Y$ and $X$ or $Y$ is decreasing failure rate (DFR), then $X \preceq_{disp} Y$.

For a comprehensive discussion on various stochastic orderings, we refer to Li and Li (2013) and Shaked and Shanthikumar (2007).

We also need the concept of majorization of vectors and matrices and the Schur-convexity and Schur-concavity of functions. For a comprehensive discussion of these topics we refer to Marshall
et al. (2011). We use the notation $x_1 \leq x_2 \leq \ldots \leq x_n$ to denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

**Definition 2.2.** The vector $\mathbf{x}$ is said to be

(i) weakly submajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \preceq_w \mathbf{y}$) if $\sum_{i=1}^{n} x(i) \leq \sum_{i=1}^{n} y(i)$ for all $j = 1, \ldots, n$,

(ii) weakly supermajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \succeq_w \mathbf{y}$) if $\sum_{i=1}^{j} x(i) \geq \sum_{i=1}^{j} y(i)$ for all $j = 1, \ldots, n$,

(iii) majorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \preceq_m \mathbf{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x(i) \geq \sum_{i=1}^{j} y(i)$ for all $j = 1, \ldots, n - 1$.

**Definition 2.3.** A real valued function $\varphi$ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathcal{A}$ if

$$\mathbf{x} \preceq \mathbf{y} \quad \text{on} \quad \mathcal{A} \implies \varphi(\mathbf{x}) \leq (\geq) \varphi(\mathbf{y}).$$

**Lemma 2.2** (Marshall et al. (2011), Theorem 3.A.4). Let $\mathcal{A} \subseteq \mathbb{R}$ be an open interval and let $l : \mathcal{A}^n \to \mathbb{R}$ be continuously differentiable. $l$ is Schur-convex (Schur-concave) on $\mathcal{A}^n$ if and only if, $l$ is symmetric on $\mathcal{A}^n$ and for all $i \neq j$, $$(x_i - x_j) \left( \frac{\partial l(x)}{\partial x_i} - \frac{\partial l(x)}{\partial x_j} \right) \geq (\leq) 0, \quad \text{for all} \quad \mathbf{x} \in \mathcal{A}^n.$$ \[\text{Lemma 2.3} \quad (\text{Marshall et al. (2011), Theorem 3.A.8}). \quad \text{For a function } l \text{ on } \mathcal{A} \subseteq \mathbb{R}^n, \mathbf{x} \preceq_w \mathbf{y} \implies l(\mathbf{x}) \leq (\geq) l(\mathbf{y}) \text{ if and only if it is increasing (decreasing) and Schur-convex (Schur-concave) on } \mathcal{A}. \]

In the following we recall the concepts of $T$-transform matrix and chain majorization of matrices. We refer to Marshall et al. (2011) for more details.

**Definition 2.4.** A square matrix is called a

(i) permutation matrix if each row and each column has a single unit, and all other entries are zero,

(ii) $T$-transform matrix if it is of the form $T_\omega = \omega I_n + (1 - \omega)\Pi$, where $0 \leq \omega \leq 1$, $I_n$ is an $n \times n$ identity matrix and $\Pi$ is a permutation matrix that just interchanges two coordinates.

Two $T$-transform matrices said to have the same structure if their permutation matrices are identical; otherwise they said to have different structures.

In the following definition, we recall a multivariate majorization notion which will be used in the sequel.
Definition 2.5. Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ be two $m \times n$ matrices. Then $A$ is said to be chain majorized by $B$, denoted by $A \ll B$, if there exists a finite set of $n \times n$ $T$-transform matrices $T_{\omega_1}, \ldots, T_{\omega_k}$ such that $A = BT_{\omega_1} \times \ldots \times T_{\omega_k}$.

For $i = 1, \ldots, m$, let $a_i^R$ and $b_i^R$, denote the $i$th row of $A$ and $B$, respectively. Then we have

\[ A \ll B \implies a_i^R \preceq_w b_i^R \implies a_i^R \preceq w b_i^R \implies \prod_{j=1}^{n} b_{ij} \leq \prod_{j=1}^{n} a_{ij}, \]

where the last consequence holds whenever $a_i^R, b_i^R \in \mathbb{R}^n_{++}$. Let

\[ S_n = \left\{ (x, y) = \begin{bmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{bmatrix} : -1 \leq x_i \leq 1, y_i > 0, \text{ and } (x_i - x_j)(y_i - y_j) \leq 0, \right\}, \]

\[ i, j = 1, \ldots, n \right\}. \]

We recall the following lemmas similar to the lemmas in Balakrishnan et al. (2015), which their proofs are very similar to the proofs of lemmas in Balakrishnan et al. (2015). So, the proofs are omitted for simplicity.

Lemma 2.4. A differentiable function $\varphi : \mathbb{R}^4 \to \mathbb{R}_+$ satisfies

\[ \varphi(A) \leq \varphi(B) \text{ for all } A, B \text{ such that } B \in S_2, \text{ and } A \ll B \] (1)

if and only if

(i) $\varphi(B) = \varphi(B\Pi)$ for all permutation matrices $\Pi$, and all $B \in S_2$; and

(ii) $\sum_{i=1}^{2} (b_{ik} - b_{ij})(\varphi_{ik}(B) - \varphi_{ij}(B)) \geq 0$ for all $j, k = 1, 2$, and all $B \in S_2$, where $\varphi_{ij}(B) = \frac{\partial \varphi(B)}{\partial b_{ij}}$.

Lemma 2.5. Let $\Psi : \mathbb{R}^2 \to \mathbb{R}_+$ be a differentiable function, and let the function $\nu_n : \mathbb{R}^{2n} \to \mathbb{R}_+$ defined by

\[ \nu_n(B) = \prod_{i=1}^{n} \Psi(b_{1i}, b_{2i}). \]

If $\nu_2$ satisfies (1), then, for $B \in S_n$, and $A = BT_\omega$, we have $\nu_n(A) \leq \nu_n(B)$.

Lemma 2.6. Let $k$ be a function defined by

\[ k(x, y, z) := \frac{x}{1 - xyz}. \]

Then,

(i) $k$ is increasing in $x$,

(ii) $k$ is increasing in $y$, when $z \geq 0$.  

5
3 Results for the largest claim amounts

It is clear that the random variables $Y_i = I_{p_i} X_{\lambda_i}$, $i = 1, \ldots, n$, are discrete-continuous, which are equal to zero with the probability $1 - p_i$, and $X_{\lambda_i}$ with the probability $p_i$, $i = 1, \ldots, n$. The distribution function and the reversed hazard rate function of $Y_{n:n}$, the largest claim amount, are given by

$$G_{Y_{n:n}}(x) = \prod_{i=1}^{n} \left(1 - p_i \bar{F}(x) \left(1 - \lambda_i F(x)\right)\right), \quad x \geq 0. \quad (2)$$

and

$$\tilde{G}_{Y_{n:n}}(x) = \sum_{i=1}^{n} \frac{f(x)(1 + \lambda_i(1 - 2F(x))) p_i}{1 - p_i F(x)(1 - \lambda_i F(x))} I_{[x>0]} + I_{[x=0]}, \quad (3)$$

respectively; where $I_A$ denotes the indicator function. Similarly, the distribution function and the reversed hazard rate function of $Y_{n:n}^*$ is the same as in (2) and (3) upon replacing $\lambda_i$ by $\lambda_i^*$ and $p_i$ by $p_i^*$, $i = 1, \ldots, n$, respectively.

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolio of risks, in the sense of the usual stochastic ordering via matrix majorization.

Theorem 3.1. Let $X_{\lambda_1}, X_{\lambda_2} (X_{\lambda_1}', X_{\lambda_2}')$ be independent non-negative random variables with $X_{\lambda_i} \sim TG(\lambda_i)$ ($X_{\lambda_1}' \sim TG(\lambda_1')$), $i = 1, 2$. Further, suppose that $I_{p_1}, I_{p_2} (I_{p_1}', I_{p_2}')$ is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s ($X_{\lambda_i}'$’s), with $E[I_{p_i}] = p_i$ ($E[I_{p_i}'] = p_i'$), $i = 1, 2$. Let $h : [0,1] \rightarrow I \subset \mathbb{R}_+$ be a differentiable and strictly increasing concave function on $[0,1]$ with the non-zero derivative. Then for $(\lambda, h(p)) \in S_2$, we have

$$\begin{bmatrix} \lambda_1^* & \lambda_2^* \\ h(p_1^*) & h(p_2^*) \end{bmatrix} \ll \begin{bmatrix} \lambda_1 & \lambda_2 \\ h(p_1) & h(p_2) \end{bmatrix} \implies Y_{2:2}^* \leq_{st} Y_{2:2}.$$

Proof. In view of (2), the distribution function of $Y_{2:2}$ can be rewritten as

$$G_{Y_{2:2}}(x) = \prod_{i=1}^{2} \left(1 - h^{-1}(u_i) \bar{F}(x) \left(1 - \lambda_i F(x)\right)\right), \quad x \geq 0,$$

where $h^{-1}$ is the inverse of the function $h$, and $u_i = h(p_i)$, $i = 1, 2$. For fixed $x \geq 0$, we have to show that the function $G_{Y_{2:2}}(x)$ satisfies the conditions of Lemma 2.4. Clearly, the condition (i) is satisfied. To check the condition (ii), consider the function $\rho$ given by

$$\rho(\lambda, u) = \rho_1(\lambda, u) + \rho_2(\lambda, u), \quad (4)$$

where

$$\rho_1(\lambda, u) = (u_1 - u_2) \left(\frac{\partial G_{Y_{2:2}}(x)}{\partial u_1} - \frac{\partial G_{Y_{2:2}}(x)}{\partial u_2}\right),$$

and

$$\rho_2(\lambda, u) = (\lambda_1 - \lambda_2) \left(\frac{\partial G_{Y_{2:2}}(x)}{\partial \lambda_1} - \frac{\partial G_{Y_{2:2}}(x)}{\partial \lambda_2}\right).$$
The partial derivatives of \( G_{Y_2,2}(x) \) with respect to \( u_i \) and \( \lambda_i \) are given by

\[
\frac{\partial G_{Y_2,2}(x)}{\partial u_i} = -\frac{(1 - \lambda_i F(x)) \frac{\partial h^{-1}(u_i)}{\partial u_i}}{1 - h^{-1}(u_i) F(x) (1 - \lambda_i F(x))} \bar{F}(x) G_{Y_2,2}(x),
\]

and

\[
\frac{\partial G_{Y_2,2}(x)}{\partial \lambda_i} = \frac{h^{-1}(u_i)}{1 - h^{-1}(u_i) F(x) (1 - \lambda_i F(x))} F(x) \bar{F}(x) G_{Y_2,2}(x).
\]

Thus

\[
\rho_1(\lambda, u) = -(u_1 - u_2) \bar{F}(x) G_{Y_2,2}(x) \left( \eta_1(\lambda_1, u_1) \frac{\partial h^{-1}(u_1)}{\partial u_1} - \eta_1(\lambda_2, u_2) \frac{\partial h^{-1}(u_2)}{\partial u_2} \right),
\]

where, \( \eta_1(\lambda, u) = k \left( 1 - \lambda F(x), h^{-1}(u), \bar{F}(x) \right) \), and \( k \) is the function defined in Lemma 2.6. The assumption \((\lambda, u) \in S_2\) implies that \((\lambda_1 - \lambda_2)(u_1 - u_2) \leq 0\) or equivalently, \( \lambda_1 \leq \lambda_2 \) and \( u_1 \geq u_2 \), or \( \lambda_1 \geq \lambda_2 \) and \( u_1 \leq u_2 \). We only state the proof for the case \( \lambda_1 \leq \lambda_2 \) and \( u_1 \geq u_2 \). The other case is analogously proven. Since \( h \) is strictly increasing and concave then \( h^{-1} \) is strictly increasing and convex. The convexity of \( h^{-1} \) implies that

\[
0 \leq \frac{\partial h^{-1}(u_2)}{\partial u_2} \leq \frac{\partial h^{-1}(u_1)}{\partial u_1}.
\]

In view of Lemma 2.6 the function \( \eta_1 \) is decreasing in \( \lambda \) and increasing in \( u \), so that

\[
0 \leq \eta_1(\lambda_2, u_2) \leq \eta_1(\lambda_1, u_2) \leq \eta_1(\lambda_1, u_1),
\]

which implies that

\[
\rho_1(\lambda, u) \leq 0. \quad (5)
\]

On the other hand,

\[
\rho_2(\lambda, u) = (\lambda_1 - \lambda_2) F(x) \bar{F}(x) G_{Y_2,2}(x) \left( \eta_2(\lambda_1, u_1) - \eta_2(\lambda_2, u_2) \right),
\]

where, \( \eta_2(\lambda, u) = k \left( h^{-1}(u), 1 - \lambda F(x), \bar{F}(x) \right) \). By a similar argument the function \( \eta_2 \) is decreasing in \( \lambda \) and increasing in \( u \) and

\[
\eta_2(\lambda_2, u_2) \leq \eta_2(\lambda_1, u_2) \leq \eta_2(\lambda_1, u_1),
\]

which implies that

\[
\rho_2(\lambda, u) \leq 0. \quad (6)
\]

By using the inequalities (4), (5) and (6), we have that

\[
\rho(\lambda, u) \leq 0,
\]

and the function \( G_{Y_2,2}(x) \) satisfies the condition (ii) of Lemma 2.4. Now Lemma 2.4 and the condition \((\lambda^*, u^*) \ll (\lambda, u)\) implies that

\[
G_{Y_2,2}(x) \leq G_{Y_2,2}^*(x),
\]

which is the required result. □
The following result provides a lower bound for the survival function of the largest claim amount based on a heterogeneous portfolio of risks in terms of the survival function of largest claim amounts based on a homogeneous portfolio of risks.

**Corollary 3.1.** Let \( \bar{\lambda} = \frac{1}{2}(\lambda_1 + \lambda_2) \) and \( \overline{h(p)} = \frac{1}{2}(h(p_1) + h(p_2)) \). Under the conditions of Theorem 3.4 we have

\[
\bar{G}_{Y_{2,2}}(x) \geq 1 - \left( 1 - h^{-1}(\overline{h(p)}) \overline{F}(x) \left( 1 - \bar{\lambda} F(x) \right) \right)^2.
\]

**Proof.** It is clear that

\[
\begin{bmatrix} \bar{\lambda} & \bar{\lambda} \\ h(\overline{p}) & h(\overline{p}) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ h(p_1) & h(p_2) \end{bmatrix} T_{0.5}. \text{ Thus we have } \begin{bmatrix} \bar{\lambda} & \bar{\lambda} \\ h(\overline{p}) & h(\overline{p}) \end{bmatrix} \ll \begin{bmatrix} \lambda_1 & \lambda_2 \\ h(p_1) & h(p_2) \end{bmatrix}.
\]

Now Theorem 2.4 gives the required result. \( \square \)

The following result generalizes the result of Theorem 2.4 for an arbitrary number of random variables.

**Theorem 3.2.** Let \( X_{\lambda_1}, \ldots, X_{\lambda_n} (X_{\lambda_1'}, \ldots, X_{\lambda_n'}) \) be independent non-negative random variables with \( X_{\lambda_i} \sim TG(\lambda_i) (X_{\lambda_1'} \sim TG(\lambda_1')) \), \( i = 1, \ldots, n \). Further, suppose that \( I_{p_1}, \ldots, I_{p_n} (I_{p_1'}, \ldots, I_{p_n'}) \) is a set of independent Bernoulli random variables, independent of the \( X_{\lambda_i}'s (X_{\lambda_i'}'s) \), with \( E[I_{p_i}] = p_i \) \((E[I_{p_i'}] = p_i')\), \( i = 1, \ldots, n \). Let \( h : [0, 1] \to I \subset \mathbb{R}_+ \) be a differentiable and strictly increasing concave function on \([0, 1]\), with non-zero derivative. Then for \((\lambda, h(p)) \in S_n\), we have

\[
\begin{bmatrix} \lambda_1^* & \ldots & \lambda_n^* \\ h(p_1^*) & \ldots & h(p_n^*) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \ldots & \lambda_n \\ h(p_1) & \ldots & h(p_n) \end{bmatrix} T_{\omega} \Rightarrow Y_{n:n}^{*} \leq_{st} Y_{n:n}.
\]

**Proof.** Using Lemma 2.5 and Theorem 3.1 we immediately obtain the required result. \( \square \)

According to Balakrishnan et al. (2015), a finite product of \( T \)-transform matrices with the same structure is also a \( T \)-transform matrix. Thus the following result is a direct consequence of Theorem 3.2.

**Corollary 3.2.** Under the assumptions of Theorem 3.2 for \((\lambda, h(p)) \in S_n\), we have

\[
\begin{bmatrix} \lambda_1^* & \ldots & \lambda_n^* \\ h(p_1^*) & \ldots & h(p_n^*) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \ldots & \lambda_n \\ h(p_1) & \ldots & h(p_n) \end{bmatrix} T_{\omega_1} \ldots T_{\omega_m} \Rightarrow Y_{n:n}^{*} \leq_{st} Y_{n:n},
\]

where \( T_{\omega_i}, i = 1, \ldots, m \), have the same structure.

The following corollary provides a result for the case where the \( T \)-transform matrices have different structures.

**Corollary 3.3.** Under the assumptions of Theorem 3.2 for \((\lambda, h(p)) \in S_n\), and \((\lambda, h(p))T_{\omega_1}, \ldots, T_{\omega_i} \in S_n\), for \( i = 1, \ldots, m - 1 \), where \( m \geq 2 \), we have

\[
\begin{bmatrix} \lambda_1^* & \ldots & \lambda_n^* \\ h(p_1^*) & \ldots & h(p_n^*) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \ldots & \lambda_n \\ h(p_1) & \ldots & h(p_n) \end{bmatrix} T_{\omega_1} \ldots T_{\omega_m} \Rightarrow Y_{n:n}^{*} \leq_{st} Y_{n:n},
\]
Proof. Using Theorem 3.2 consecutively, the desired result is immediately obtained.

The following result deals with the comparison of the largest claim amounts in a homogeneous portfolio of risks, in the sense of the reversed hazard rate ordering via weakly majorization.

**Theorem 3.3.** Under the assumptions of Theorem 3.2 with $\lambda_i = \lambda^*_i = \lambda$, for $i = 1, \ldots, n$, we have

$$\left(h(p_1^*), \ldots, h(p_n^*)\right) \preceq_w \left(h(p_1), \ldots, h(p_n)\right) \implies Y^*_{n:n} \lesssim_{\text{rh}} Y_{n:n}.$$  

*Proof.* According to (3), the reversed hazard rate function of $Y_{n:n}$ can be rewritten as

$$\tilde{r}_{Y_{n:n}}(x) = \sum_{i=1}^{n} \frac{f(x)\left(1 + \lambda(1 - 2F(x))\right)h^{-1}(u_i)}{1 - h^{-1}(u_i)F(x)(1 - \lambda F(x))} I_{x>0} + I_{x=0},$$

where, $u_i = h(p_i), i = 1, \ldots, n$. First, consider $x = 0$. In this case, $\tilde{r}_{Y_{n:n}}(0) = \tilde{r}_{Y^{*}_{n:n}}(0) = 1$, and the desired result is obvious. Now, consider $x > 0$. Using Lemma 2.2, it is enough to show that the function $\tilde{r}_{Y_{n:n}}(x)$ is Schur-convex and increasing in $u_i$'s. The partial derivatives of $\tilde{r}_{Y_{n:n}}(x)$ with respect to $u_i$ is given by

$$\frac{\partial \tilde{r}_{Y_{n:n}}(x)}{\partial u_i} = f(x)\left(1 + \lambda(1 - 2F(x))\right)\frac{\partial h^{-1}(u_i)}{\partial u_i} \left(1 - h^{-1}(u_i)F(x)(1 - \lambda F(x))\right)^2 \geq 0.$$

Thus, $\tilde{r}_{Y_{n:n}}(x)$ is increasing in each $u_i$. To prove the Schur-convexity of $\tilde{r}_{Y_{n:n}}(x)$, from Lemma 2.2, it is enough to show that for $i \neq j$,

$$(u_i - u_j)\left(\frac{\partial \tilde{r}_{Y_{n:n}}(x)}{\partial u_i} - \frac{\partial \tilde{r}_{Y_{n:n}}(x)}{\partial u_j}\right) \geq 0,$$

that is, for $i \neq j$,

$$(u_i - u_j)f(x)\left(1 + \lambda(1 - 2F(x))\right) \times \left(\frac{\partial h^{-1}(u_i)}{\partial u_i} \left(1 - h^{-1}(u_i)F(x)(1 - \lambda F(x))\right)^2 - \frac{\partial h^{-1}(u_j)}{\partial u_j} \left(1 - h^{-1}(u_j)F(x)(1 - \lambda F(x))\right)^2\right) \geq 0.$$  

Since $h$ is increasing and concave, then $h^{-1}$ is increasing and convex. Thus, the inequality immediately holds.

---

4 Results for the smallest claim amounts

It can be easily seen that the survival function and the hazard rate function of $Y_{1:n}$, the smallest claim amount, are given by

$$\tilde{G}_{Y_{1:n}}(x) = \left(\prod_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \left(\tilde{F}(x)(1 - \lambda_i F(1 - \lambda F(x))\right), \quad x \geq 0,$$  

(7)
Thus by Lemma 2.3, it is enough to show that the function \( \bar{X}_i \) is a set of independent Bernoulli random variables, independent of the \( X_i \)’s, with \( \text{E}[I_{p_i}] = p_i \) (\( E[I_{p_i}^*] = p_i^* \)), \( i = 1, \ldots, n \). Then, we have

\[
\prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i, \quad (\lambda_1, \ldots, \lambda_n) \preceq_w (\lambda_1^*, \ldots, \lambda_n^*) \implies Y_{1:n}^* \leq_{st} Y_{1:n}.
\]

**Proof.** Assume that \( \prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i \). Now using (7), the required result holds if \( X_{1:n}^* \leq_{st} X_{1:n} \), where \( X_{1:n}^* \) and \( X_{1:n} \) are the smallest order statistics of \( (X_{\lambda_1}, \ldots, X_{\lambda_n}) \) and \( (X_{\lambda_1}, \ldots, X_{\lambda_n}) \), respectively.

The survival function of \( X_{1:n} \) is given by

\[
\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n} \left( \bar{F}(x)(1 - \lambda_i F(x)) \right), \quad x \geq 0.
\]

Thus by Lemma 2.3 it is enough to show that the function \( \bar{F}_{X_{1:n}}(x) \) is Schur-concave and decreasing in \( \lambda_i \)’s. The partial derivative of \( \bar{F}_{X_{1:n}}(x) \) with respect to \( \lambda_i \) is given by

\[
\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \lambda_i} = \frac{F(x)\bar{F}_{X_{1:n}}(x)}{1 - \lambda_i F(x)} \leq 0.
\]

Thus \( \bar{F}_{X_{1:n}}(x) \) is decreasing in each \( \lambda_i \). To prove the Schur-concavity of \( \bar{F}_{X_{1:n}}(x) \), from Lemma 2.2 it is enough to show that for \( i \neq j \),

\[
(\lambda_i - \lambda_j) \left( \frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \lambda_i} - \frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \lambda_j} \right) \leq 0,
\]

that is, for \( i \neq j \),

\[
-(\lambda_i - \lambda_j)F(x)\bar{F}_{X_{1:n}}(x) \left( \frac{1}{1 - \lambda_i F(x)} - \frac{1}{1 - \lambda_j F(x)} \right) \leq 0,
\]

which is immediately concluded. \( \square \)

The following result provides a lower bound for the survival function of the smallest claim amount based on a heterogeneous portfolio of risks in terms of the survival function of smallest claim amounts based on a homogeneous portfolio of risks.
Corollary 4.1. Under the assumption of Theorem 4.1

\[ \bar{G}_{Y_{1:n}}(x) \geq \left( \tilde{p} \bar{F}(x) \left(1 - \tilde{\lambda} \bar{F}(x)\right) \right)^n, \]

where \( \prod_{i=1}^{n} p_i = \tilde{p}^n \) and \( \tilde{\lambda} = \max \left( \frac{1}{2}(1 + \lambda_1), \ldots, \frac{1}{2}(1 + \lambda_n) \right) \).

Proof. It is clear that

\( (\lambda_1, \ldots, \lambda_n) \preceq_w \left( \frac{1}{2}(1 + \lambda_1), \ldots, \frac{1}{2}(1 + \lambda_n) \right) \preceq_w (\tilde{\lambda}, \ldots, \tilde{\lambda}). \)

These assumptions satisfy the conditions of Theorem 4.1, which implies the result.

The following result shows that under the same conditions of Theorem 4.1, a stronger result also holds.

Theorem 4.2. Under the assumptions of Theorem 4.1, we have

\[ \prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i, \quad (\lambda_1^*, \ldots, \lambda_n^*) \preceq_w \lambda_{1:n} \implies Y_{1:n}^* \leq_{hr} Y_{1:n}. \]

Proof. According to (7), we have

\[ \bar{G}_{Y_{1:n}}(x) = \left( \prod_{i=1}^{n} p_i \right) \frac{\bar{F}_{X_{1:n}}(x)}{\bar{F}_{X_{1:n}^*}(x)}, \quad x \geq 0. \]

We have to show that \( \frac{\bar{G}_{Y_{1:n}}(x)}{\bar{G}_{Y_{1:n}^*}(x)} \) is increasing in \( x \), which holds if

\[ 1 = \lim_{x \to 0} \frac{\bar{G}_{Y_{1:n}}(x)}{\bar{G}_{Y_{1:n}^*}(x)} \leq \frac{\bar{G}_{Y_{1:n}}(0)}{\bar{G}_{Y_{1:n}^*}(0)} = \frac{\prod_{i=1}^{n} p_i}{\prod_{i=1}^{n} p_i^*}, \]

and \( \frac{\bar{F}_{X_{1:n}^*}(x)}{\bar{F}_{X_{1:n}}(x)} \) is increasing in \( x \). Since Inequality (9) holds according to the assumptions, it is enough to show that \( X_{1:n}^* \leq_{hr} X_{1:n} \) or equivalently \( r_{X_{1:n}}(x) \leq r_{X_{1:n}^*}(x) \), for \( x \geq 0 \). The hazard rate function of \( X_{1:n} \) is given by

\[ r_{X_{1:n}}(x) = \sum_{i=1}^{n} \frac{1 + \lambda_i (1 - 2F(x))}{1 - \lambda_i F(x)} r(x), \quad x \geq 0. \]

Thus by Lemma 2.3 it is enough to show that the function \( r_{X_{1:n}}(x) \) is Schur-convex and increasing in \( \lambda_i \)'s. The partial derivative of \( r_{X_{1:n}}(x) \) with respect to \( \lambda_i \) is given by

\[ \frac{\partial r_{X_{1:n}}(x)}{\partial \lambda_i} = \frac{\bar{F}(x)r(x)}{(1 - \lambda_i F(x))^2} \geq 0. \]
Thus $r_{X_{1:n}}(x)$ is increasing in each $\lambda_i$. To prove the Schur-convexity of $r_{X_{1:n}}(x)$, from Lemma 2.2, it is enough to show that for $i \neq j$,

$$(\lambda_i - \lambda_j) \left( \frac{\partial r_{X_{1:n}}(x)}{\partial \lambda_i} - \frac{\partial r_{X_{1:n}}(x)}{\partial \lambda_j} \right) \geq 0,$$

that is, for $i \neq j$,

$$(\lambda_i - \lambda_j) \bar{F}(x) r(x) \left( \frac{1}{(1 - \lambda_i F(x))^2} - \frac{1}{(1 - \lambda_j F(x))^2} \right) \geq 0,$$

where, the inequality is immediately concluded.

The following result deals with the comparison of the smallest claim amounts in two portfolios of risks, in the sense of the dispersive ordering via majorization.

**Theorem 4.3.** Under the assumptions of Theorem 4.1, if $F$ is DFR, $0 \leq \lambda_i^* \leq 1$, $i = 1, \ldots, n$, and $f(0) \leq \frac{1 - \prod_{i=1}^{n} p_i^*}{\sum_{i=1}^{n} (1 + \lambda_i^*)}$, then we have

$$\prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i, \ (\lambda_1, \ldots, \lambda_n) \preceq_w (\lambda_1^*, \ldots, \lambda_n^*) \implies Y_{1:n}^* \leq_{\text{disp}} Y_{1:n}.$$

**Proof.** By Theorem 4.2, we have that $Y_{1:n}^* \leq_{hr} Y_{1:n}$. According to Mirhossaini et al. (2011), $F$ is DFR and $0 \leq \lambda_i^* \leq 1$, $i = 1, \ldots, n$, imply that the first terms of (8) is decreasing in $x > 0$. Therefore,

$$1 - \prod_{i=1}^{n} p_i^* = r_{Y_{1:n}^*}(0) \geq \lim_{x \to 0^+} r_{Y_{1:n}^*}(x) = \sum_{i=1}^{n} (1 + \lambda_i^*) r(0) = \sum_{i=1}^{n} (1 + \lambda_i^*) f(0),$$

implies that $r_{Y_{1:n}^*}(x)$ is decreasing in $x \geq 0$ and $Y_{1:n}^*$ is DFR. Thus, Lemma 2.1 completes the proof.

Note that under the assumptions of Theorem 4.3, we can conclude that the variance of $Y_{1:n}^*$ is equal or less than the variance of $Y_{1:n}$.

**5 Application**

In this section, we provide some special cases for illustration of some results of the paper for $n = 3$.

**5.1 Transmuted-G exponential distribution**

Suppose that the baseline distribution in transmuted-G model is exponential distribution with mean $\theta$. Here this distribution is denoted by TE($\lambda, \theta$). For more details on this distribution, we refer to Mirhossaini and Dolati (2008).
• Let $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3}$ be independent random variables with $X_{\lambda_i} \sim \text{TE}(\lambda_i, 0.5)$ ($X_{\lambda_i^*} \sim \text{TE}(\lambda_i^*, 0.5)$), $i = 1, 2, 3$. Further, suppose that $I_{p_1}, I_{p_2}, I_{p_3}$ (and $I_{p_1^*}, I_{p_2^*}, I_{p_3^*}$) is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s (and $X_{\lambda_i^*}$’s), with $E[I_{p_i}] = p_i$ ($E[I_{p_i^*}] = p_i^*$), $i = 1, 2, 3$. Also, suppose that $(\lambda_1, \lambda_2, \lambda_3) = (-0.7, 0.8, -0.9)$, $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (-0.1806, 0.0896, -0.7090)$, $(p_1, p_2, p_3) = (0.4, 0.2, 0.7)$, and $(p_1^*, p_2^*, p_3^*) = (0.4345, 0.3698, 0.4711)$. Take $h(p) = \log(2+p)$ and the $T$-transform matrices with the different structures as

$$
t_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix}, \quad t_{0.3} = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0 & 1 & 0 \\ 0.7 & 0 & 0.3 \end{bmatrix} \quad \text{and} \quad t_{0.6} = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

It can be easily verified that $(\lambda, h(p))$, $(\lambda, h(p))t_{0.9}$, and $(\lambda, h(p))t_{0.9}t_{0.3}$ are in $S_3$, and

$$(\lambda^*, h(p^*)) = (\lambda, h(p))t_{0.9}t_{0.3}t_{0.6}.$$ 

Thus, Corollary 3.3 implies $Y_{3.3}^{*} \leq_{st} Y_{3.3}$.

• Let $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3}$ be independent random variables with $X_{\lambda_i} \sim \text{TE}(\lambda_i, 2)$ ($X_{\lambda_i^*} \sim \text{TE}(\lambda_i^*, 2)$), $i = 1, 2, 3$. Further, suppose that $I_{p_1}, I_{p_2}, I_{p_3}$ (and $I_{p_1^*}, I_{p_2^*}, I_{p_3^*}$) is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s (and $X_{\lambda_i^*}$’s), with $E[I_{p_i}] = p_i$ ($E[I_{p_i^*}] = p_i^*$), $i = 1, 2, 3$. Also, suppose that $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 0.3, -0.6)$, $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0.5, -0.3, 0.1)$, $(p_1, p_2, p_3) = (0.5, 0.3, 0.7)$, and $(p_1^*, p_2^*, p_3^*) = (0.3, 0.9, 0.1)$. It can be easily verified that the conditions of Theorem 4.1 hold and so we can conclude that $Y_{1.3}^{*} \leq_{st} Y_{1.3}$.

Figure 1 (top panels) represents the survival functions of $Y_{1.3}$, $Y_{1.3}^{*}$, $Y_{3.3}$ and $Y_{3.3}^{*}$ for the transmuted exponential distribution.

5.2 Transmuted-G Weibull distribution

Suppose that the baseline distribution in transmuted-G model is Weibull distribution with shape parameter $\alpha$ and scale parameter $\beta$. Here this distribution is denoted by $\text{TW}(\lambda, \alpha, \beta)$. For more details on this distribution, we refer to Aryal and Tsokos (2011) and Khan et al. (2017).

• Let $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3}$ be independent random variables with $X_{\lambda_i} \sim \text{TW}(\lambda_i, 0.3, 1.5)$ ($X_{\lambda_i^*} \sim \text{TW}(\lambda_i^*, 0.3, 1.5)$), $i = 1, 2, 3$. Further, suppose that $I_{p_1}, I_{p_2}, I_{p_3}$ (and $I_{p_1^*}, I_{p_2^*}, I_{p_3^*}$) is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s (and $X_{\lambda_i^*}$’s), with $E[I_{p_i}] = p_i$ ($E[I_{p_i^*}] = p_i^*$), $i = 1, 2, 3$. Also, suppose that $(\lambda_1, \lambda_2, \lambda_3) = (0.7, 0.3, -0.9)$, $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0.1544, -0.5464, 0.4920)$, $(p_1, p_2, p_3) = (0.1, 0.4, 0.8)$, and $(p_1^*, p_2^*, p_3^*) = (0.3506, 0.6295, 0.2124)$. Take $h(p) = \frac{5p+2}{p+1}$ and the $T$-transform matrices with the different structures as

$$
t_{0.1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0.9 \\ 0 & 0.9 & 0.1 \end{bmatrix}, \quad t_{0.4} = \begin{bmatrix} 0.4 & 0 & 0.6 \\ 0 & 1 & 0 \\ 0.6 & 0 & 0.4 \end{bmatrix} \quad \text{and} \quad t_{0.8} = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

It can be easily verified that $(\lambda, h(p))$, $(\lambda, h(p))t_{0.1}$ and $(\lambda, h(p))t_{0.1}t_{0.4}$ are in $S_3$, and

$$(\lambda^*, h(p^*)) = (\lambda, h(p))t_{0.1}t_{0.4}t_{0.8}.$$ 

Thus, Corollary 3.3 implies $Y_{3.3}^{*} \leq_{st} Y_{3.3}$. 

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• Let \( X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_3} (X_{\lambda_1^*}, X_{\lambda_2^*}, X_{\lambda_3^*}) \) be independent random variables with \( X_{\lambda_i} \sim TW(\lambda_i, 2, 0.6) \) \( (X_{\lambda_i^*} \sim TW(\lambda_i^*, 2, 0.6)), i = 1, 2, 3. \) Further, suppose that \( I_{p_1}, I_{p_2}, I_{p_3} (I_{p_1^*}, I_{p_2^*}, I_{p_3^*}) \) is a set of independent Bernoulli random variables, independent of the \( X_{\lambda_i}'s \) \( (X_{\lambda_i^*}'s), \) with \( E[I_{p_i}] = p_i \) \( (E[I_{p_i^*}] = p_i^*), i = 1, 2, 3. \) Also, suppose \( (\lambda_1, \lambda_2, \lambda_3) = (0.3, 0.7, 0.5), (\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0.8, 0.4, 0.5), (p_1, p_2, p_3) = (0.6, 0.3, 0.2), \) and \( (p_1^*, p_2^*, p_3^*) = (0.4, 0.5, 0.1). \) It can be easily verified that the conditions of Theorem 4.1 hold. Thus we can conclude that \( Y_{1:3}^* \leq_{st} Y_{1:3}. \)

Figure 1 (bottom panels) represents survival functions of \( Y_{1:3}, Y_{1:3}^*, Y_{3:3} \) and \( Y_{3:3}^* \) for the transmuted Weibull distribution.

**Conclusion**

In this paper, under some certain conditions, we discussed stochastic comparisons between the largest claim amounts in the sense of usual stochastic ordering and reversed hazard rate ordering
and stochastic comparisons between the smallest claim amounts in the sense of usual stochastic ordering, hazard rate ordering and dispersive ordering in transmuted-G model. However, we applied some established results for two special cases of transmuted-G model, such as the transmuted exponential distribution and the transmuted Weibull distribution. It is very important to mention that the conditions of the most established results do not depend on the baseline distribution properties.

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