ON SOME RESTRICTED INEQUALITIES FOR THE ITERATED HARDY-TYPE OPERATOR INVOLVING SUPREMA AND THEIR APPLICATIONS

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ABSTRACT. In this paper we characterize the inequality

\[
\left( \int_0^\infty \left( \int_0^\infty [T_{a,b} f^+(\tau)]^p \, d\tau \right)^{\frac{\phi}{p}} \, w(\lambda) \, d\lambda \right)^{\frac{1}{\phi}} \leq C \left( \int_0^\infty \left( \int_0^\infty [f^+(\tau)]^p \, d\tau \right)^{\frac{\phi}{p}} \, v(\lambda) \, d\lambda \right)^{\frac{1}{\phi}}
\]

for \( 1 < m < p < r < q < \infty \) or \( 1 < m \leq r < \min\{p, q\} < \infty \), where \( w \) and \( v \) are weight functions on \((0, \infty)\). The inequality is required to hold with some positive constant \( C \) for all measurable functions defined on measure space \((\mathbb{R}^n, dx)\). Here \( f^+ \) is the non-increasing rearrangement of a measurable function \( f \) defined on \( \mathbb{R}^n \) and \( T_{a,b} \) is the iterated Hardy-type operator involving suprema, which is defined for a measurable non-negative function \( f \) on \((0, \infty)\) by

\[
(T_{a,b} f)(t) := \sup_{t < \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s) b(s) \, ds, \quad t \in (0, \infty),
\]

where \( u \) and \( b \) are two weight functions on \((0, \infty)\) such that \( u \) is continuous on \((0, \infty)\) and the function \( B(t) := \int_0^t b(s) \, ds \) satisfies \( 0 < B(t) < \infty \) for every \( t \in (0, \infty)\).

At the end of the paper, as an application of obtained results, we calculate the norm of the generalized maximal operator \( M_{\phi, \Lambda^w(b)} \), defined with \( 0 < \alpha < \infty \) and functions \( b, \phi : (0, \infty) \to (0, \infty) \) for all measurable functions \( f \) on \( \mathbb{R}^n \) by

\[
M_{\phi, \Lambda^w(b)} f(x) := \frac{\|f\chi_Q\|_{\Lambda^w(b)}}{\phi(Q)}, \quad x \in \mathbb{R}^n,
\]

from \( \Gamma(p_1, m_1, v) \) into \( \Gamma(p_2, m_2, w) \). Here \( \Lambda^w(b) \) and \( \Gamma(p, m, w) \) are the classical and generalized Lorentz spaces, defined as a set of all measurable functions \( f \) defined on \( \mathbb{R}^n \) for which

\[
\|f\|_{\Lambda^w(b)} = \left( \int_0^\infty [f^+(s)]^p b(s) \, ds \right)^{\frac{1}{p}} < \infty \quad \text{and} \quad \|f\|_{\Gamma(p, m, w)} = \left( \int_0^\infty \left( \int_0^x [f^+(\tau)]^p \, d\tau \right)^{\frac{1}{p}} \, v(s) \, ds \right)^{\frac{1}{p}} < \infty,
\]

respectively.

1. Introduction

Let \( (\mathcal{R}, \mu) \) be a \( \sigma \)-finite non-atomic measure space. Denote by \( \mathcal{M}(\mathcal{R}) \) the set of all \( \mu \)-measurable functions on \( \mathcal{R} \) and \( \mathcal{M}_0(\mathcal{R}) \) the class of functions in \( \mathcal{M}(\mathcal{R}) \) that are finite \( \mu \)-a.e. on \( \mathcal{R} \). The symbol \( \mathcal{M}^+(\mathcal{R}) \) stands for the collection of all \( f \in \mathcal{M}(\mathcal{R}) \) which are non-negative on \( \mathcal{R} \).

The non-increasing rearrangement \( f^+ \) of \( f \in \mathcal{M}_0(\mathcal{R}) \) is given by

\[
f^+(t) = \inf \{ \lambda \geq 0 : \mu(\{x \in \mathcal{R} : |f(x)| > \lambda\}) \leq t \}, \quad t \in (0, \mu(\mathcal{R})).
\]

The maximal non-increasing rearrangement of \( f \) is defined as follows

\[
f^{**}(t) := \frac{1}{t} \int_0^t f^+(\tau) \, d\tau, \quad t \in (0, \mu(\mathcal{R})).
\]

Most of the functions which we shall deal with will be defined on \( \mathbb{R}^n \) or \((0, \infty)\). In this case, \( (\mathcal{R}, \mu) \) is \( \mathbb{R}^n \) or \((0, \infty)\) endowed with the \( n \)-dimensional Lebesgue measure or the one-dimensional Lebesgue measure, respectively. We shall write just \( \mathcal{M}^+ \) instead of \( \mathcal{M}^+(0, \infty) \).

Let \( \Omega \) be any measurable subset of \( \mathbb{R}^n \), \( n \geq 1 \). The family of all weight functions (also called just weights) on \( \Omega \), that is, locally integrable non-negative functions on \( \Omega \), denoted by \( \mathcal{W}(\Omega) \).

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For \( p \in (0, \infty) \) and \( w \in \mathfrak{W}^+(\Omega) \), we define the functional \( \| \cdot \|_{p,w,\Omega} \) on \( \mathfrak{W}(\Omega) \) by
\[
\|f\|_{p,w,\Omega} := \begin{cases} 
\left( \int_{\Omega} |f(x)|^p w(x) \, dx \right)^{1/p} & \text{if } p < \infty, \\
\text{ess sup}_\Omega |f(x)| w(x) & \text{if } p = \infty.
\end{cases}
\]

If, in addition, \( w \in \mathcal{W}(\Omega) \), then the weighted Lebesgue space \( L^p(w, \Omega) \) is given by
\[
L^p(w, \Omega) = \{ f \in \mathfrak{W}(\Omega) : \|f\|_{p,w,\Omega} < \infty \}
\]
and it is equipped with the quasi-norm \( \| \cdot \|_{p,w,\Omega} \).

When \( w \equiv 1 \) on \( \Omega \), we write simply \( L^p(\Omega) \) and \( \| \cdot \|_{p,\Omega} \) instead of \( L^p(w, \Omega) \) and \( \| \cdot \|_{p,w,\Omega} \), respectively.

Quite many familiar function spaces can be defined using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let \( p \in (0, \infty) \) and \( w \in \mathcal{W}(0, \mu(\mathcal{R})) \). Then the classical Lorentz spaces \( \Lambda^p(w) \) and \( \Gamma^p(w) \) consist of all functions \( f \in \mathfrak{W}(\mathcal{R}) \) for which
\[
\|f\|_{\Lambda^p(w)} := \left( \int_0^{\mu(\mathcal{R})} [f^+(s)]^p w(s) \, ds \right)^{1/p} < \infty \quad \text{and} \quad \|f\|_{\Gamma^p(w)} := \left( \int_0^{\mu(\mathcal{R})} [f^{**}(s)]^p w(s) \, ds \right)^{1/p} < \infty,
\]
respectively. For more information about the Lorentz \( \Lambda \) and \( \Gamma \) see e.g. [3] and the references therein.

The study of particular problems in the regularity theory of PDE’s led to the definition of spaces involving inner integral means involving powers of the non-increasing rearrangements of functions. The generalized Lorentz \( \text{GI}(p,m,v)(\mathcal{R}, \mu) \) space (denoted simply by \( \text{GI}(p,m,v) \)), introduced and studied in [12] and [13], is defined as the collection of all \( g \in \mathfrak{W}(\mathcal{R}) \) such that
\[
\|g\|_{\text{GI}(p,m,v)} = \left( \int_0^{\mu(\mathcal{R})} \left( \int_0^x [g^*(\tau)]^p \, d\tau \right)^{1/p} v(x) \, dx \right)^{1/p} < \infty,
\]
where \( m, p \in (0, \infty) \), \( v \in \mathcal{W}(0, \mu(\mathcal{R})) \).

The spaces \( \text{GI}(p,m,v) \) cover several types of important function spaces and have plenty of applications. For example, when \( \mu(\mathcal{R}) = \infty \), \( p = 1 \), \( m > 1 \) and \( v(t) = t^{-m} w(t) \), \( t \in (0, \infty) \), where \( w \) is another weight on \( (0, \infty) \), then \( \text{GI}(p,m,v) \) reduces the spaces \( \Gamma^m(w) \). Another important example is obtained when \( \mu(\mathcal{R}) = 1 \), \( m = 1 \), \( p \in (1, \infty) \) and \( v(t) = t^{-1} (\log(2/t))^{-1/p} \) for \( t \in (0, 1) \). In this case the space \( \text{GI}(p,m,v) \) coincides with the small Lebesgue space, which was originally studied by Fiorenza in [9]. In the same paper it was proved that this space is the associate space of the grand Lebesgue space introduced in [30] in connection with integrability properties of Jacobians. Subsequently, Fiorenza and Karadzhov in [11] derived an equivalent form of the norm in the small Lebesgue space written in the form of the norm in the \( \text{GI}(p,m,v) \) space with the above mentioned parameters and weight. Note that the condition \( \int_{(0,1)} t^{m/p} v(t) \, dt < \infty \) ensures \( \text{GI}(p,m,v) \) to be a quasi-Banach function space when \( \mathcal{R} \subset \mathbb{R}^n \) with \( \mu(\mathcal{R}) = 1 \) (cf. [10]). Recently, the connection of the \( \text{GI}(p,m,v) \) space with some well-known function spaces have been studied in [1]. In the present paper we take \( (\mathbb{R}^n, dx) \) as underlying measure space and use the notation \( \text{GI}(p,m,v)(\mathbb{R}^n, dx) \).

The study on maximal operators occupies an important place in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, [14, 27–29, 53, 54, 56]).

The main example is the Hardy-Littlewood maximal function which is defined for locally integrable functions \( f \) on \( \mathbb{R}^n \) by
\[
(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n,
\]
where the supremum is taken over all cubes \( Q \) containing \( x \). By a cube, we mean an open cube with sides parallel to the coordinate axes.

Another important example is the fractional maximal operator, \( M_\gamma \), \( \gamma \in (0, n) \), defined for locally integrable functions \( f \) on \( \mathbb{R}^n \) by
\[
(M_\gamma f)(x) := \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n.
\]
One more example is the fractional maximal operator $M_{s,\gamma,\alpha}$ defined in [7] for all measurable functions $f$ on $\mathbb{R}^n$ by
\[
(M_{s,\gamma,\alpha}f)(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_s}{\|x\|_{s/(n-\gamma)\bar{A}}}, \quad x \in \mathbb{R}^n.
\]
Here $s \in (0, \infty)$, $\gamma \in [0, n)$, $\bar{A} = (A_0, A_\infty) \in \mathbb{R}^2$ and
\[
f^\alpha(t) := (1 + |\log t|)^{A_0} \chi_{[0,1]}(t) + (1 + |\log t|)^{A_\infty} \chi_{[1,\infty)}(t), \quad t \in (0, \infty).
\]
Recall that the following equivalency holds:
\[
(M_{s,\gamma,\alpha}f)(x) \approx \sup_{Q \ni x} \frac{\|f\chi_Q\|_s}{\|Q^{-(n-\gamma)/(sn)} f^\alpha(\|Q\|)\|_{\ell^\infty}}, \quad x \in \mathbb{R}^n.
\]
Hence, if $s = 1$, $\gamma = 0$ and $\bar{A} = (0, 0)$, then $M_{s,\gamma,\alpha}$ is equivalent to $M$. If $s = 1$, $\gamma \in (0, n)$ and $\bar{A} = (0, 0)$, then $M_{s,\gamma,\alpha}$ is equivalent to $M_{\gamma}$. Moreover, if $s = 1$, $\gamma \in (0, n)$ and $\bar{A} \in \mathbb{R}^2$, then $M_{s,\gamma,\alpha}$ is the fractional maximal operator which corresponds to potentials with logarithmic smoothness treated in [39, 40]. In particular, if $\gamma = 0$, then $M_{1,\gamma,\alpha}$ is the maximal operator of purely logarithmic order.

Given $p$ and $q$, $0 < p, q < \infty$, let $M_{p,q}$ denote the maximal operator associated to the Lorentz $L^{p,q}$ spaces defined for all measurable function $f$ on $\mathbb{R}^n$ by
\[
M_{p,q}f(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_{p,q}}{\|x\|_{p,q}},
\]
where $\|\cdot\|_{p,q}$ is the usual Lorentz norm
\[
\|f\|_{p,q} := \left( \int_0^\infty \left[ \tau^{1/p} f^+(\tau) \right]^q \frac{d\tau}{\tau} \right)^{1/q}.
\]
This operator was introduced by Stein in [52] in order to obtain certain endpoint results in differentiation theory. The operator $M_{p,q}$ has been also considered by other authors, for instance see [2, 33, 34, 38, 42].

Let $0 < \alpha < \infty$, $b \in \mathcal{W}(0, \infty)$ and $\phi : (0, \infty) \rightarrow (0, \infty)$. Recall the definition of the generalized maximal function introduced in [36] and denoted for all measurable function $f$ on $\mathbb{R}^n$ by
\[
(1.2) \quad M_{\phi,\Lambda}(f)(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_{\Lambda}}{\phi(\|Q\|)}, \quad x \in \mathbb{R}^n.
\]
Obviously, $M_{\phi,\Lambda}(f) = M$, where $M$ is the Hardy-Littlewood maximal operator, when $\alpha = 1$, $b \equiv 1$ and $\phi(t) = t$ ($t > 0$). Note that $M_{\phi,\Lambda}(f) = M_{\gamma}$, where $M_{\gamma}$ is the fractional maximal operator, when $\alpha = 1$, $b \equiv 1$ and $\phi(t) = t^{1-\gamma/n}$ ($t > 0$) with $0 < \gamma < n$. Moreover, $M_{\phi,\Lambda}(f) = M_{s,\gamma,\alpha}$, when $\alpha = s$, $b \equiv 1$ and $\phi(t) = t^{(n-\gamma)/(sn)} f^\alpha(t)$ ($t > 0$) with $0 < \gamma < n$ and $\bar{A} = (A_0, A_\infty) \in \mathbb{R}^2$. It is worth also to mention that $M_{\phi,\Lambda}(f) = M_{p,q}$, when $\alpha = q$, $b(t) = t^{q/p-1}$ and $\phi(t) = t^{1/p}$ ($t > 0$).

The boundedness of $M_{\phi,\Lambda}(f)$ between classical Lorentz spaces $\Lambda$ was completely characterized in [36]. In view of importance of generalized Lorentz spaces $\Lambda$ in our opinion it will be interesting to obtain a characterization of the boundedness of this maximal function between Lorentz $\text{GT}^\prime$ spaces.

The iterated Hardy-type operator involving suprema $T_{u,b}$ is defined for non-negative measurable function $g$ on the interval $(0, \infty)$ by
\[
(T_{u,b}g)(t) := \sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(y)b(y)dy, \quad t \in (0, \infty),
\]
where $u$ and $b$ are two weight functions on $(0, \infty)$ such that $u$ is continuous on $(0, \infty)$ and the function $B(t) := \int_0^t b(s)ds$ satisfies $0 < B(t) < \infty$ for every $t \in (0, \infty)$. Such operators have been found indispensable in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds (cf. [31]). They constitute a very useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding (cf. [41, 43]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [4, 5, 8, 48].

The aim of the paper is to characterize the following restricted inequality for $T_{u,b}$:
\[
(1.3) \quad \left( \int_0^\infty \left( \int_0^x [T_{u,b}f^+(t)]^p dt \right)^{\frac{r}{p}} w(x)dx \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty \left( \int_0^x [f^+(\tau)]^p d\tau \right)^{\frac{q}{p}} v(x)dx \right)^{\frac{1}{q}}.
\]
Here $m, p, q, r$ are positive real numbers and $w, v$ are weight functions on $(0, \infty)$. 
A function $\phi : (0, \infty) \to (0, \infty)$ is said to satisfy the $\Delta_2$-condition, denoted $\phi \in \Delta_2$, if for some $C > 0$

$$\phi(2t) \le C \phi(t) \quad \text{for all } 0 < t < \infty,$$

A function $\phi : (0, \infty) \to (0, \infty)$ is said to be quasi-increasing, if for some $C > 0$

$$\phi(t_1) \le C \phi(t_2),$$

whenever $0 < t_1 \le t_2 < \infty$.

A function $\phi : (0, \infty) \to (0, \infty)$ is said to satisfy the $Q_r$-condition, $0 < r < \infty$, denoted $\phi \in Q_r(0, \infty)$, if for some constant $C > 0$

$$\phi\left(\sum_{i=1}^{n} t_i\right) \le C \left(\sum_{i=1}^{n} \phi(t_i)^r\right)^{1/r},$$

for every finite set of non-negative real numbers $\{t_1, \ldots, t_n\}$.

Motivation for studying inequality (1.3) comes directly from the following equivalency statement.

**Theorem 1.1.** Let $0 < p_1, p_2 < \infty$, $0 < m_1, m_2 < \infty$, $0 < \alpha \le r < \infty$ and $\nu, \omega \in W(0, \infty)$. Assume that $\phi \in Q_r(0, \infty)$ is a quasi-increasing function. Suppose that $b \in W(0, \infty)$ is such that $0 < B(t) < \infty$ for all $t > 0$, $B(\infty) = \infty$ and $B(t)t^{\alpha/r}$ is quasi-increasing. Then the inequality

$$(1.4) \quad \left(\int_0^\infty \left(\int_0^\tau \left[(M_{\phi,\Delta}^\alpha(b)f)^\ast(t)\right]^{p_2} w(x) dx\right)^{\frac{m_2}{m_1}} dt\right)^{m_1/m_2} \le C \left(\int_0^\infty \left(\int_0^\tau \left[f^{\ast}(t)\right]^{p_1} \nu(x) dx\right)^{\frac{m_1}{p_1}} v(x) dx\right)^{1/p_1}$$

holds for all $f \in \mathcal{M}(\mathbb{R}^n)$ if and only if the inequality

$$(1.5) \quad \left(\int_0^\infty \left(\int_0^\tau \left[T_{B/\phi,\nu,b}h^\ast(t)\right]^{p_2} w(x) dx\right)^{\frac{m_2}{m_1}} dt\right)^{m_1/m_2} \le C \left(\int_0^\infty \left(\int_0^\tau \left[h^\ast(t)\right]^{p_1} \nu(x) dx\right)^{\frac{m_1}{p_1}} v(x) dx\right)^{1/p_1}$$

holds for all $h \in \mathcal{M}(\mathbb{R}^n)$.

The method used for solution of inequality (1.3) is based on the combination of the duality techniques with the formula

$$\sup_{g \in \mathcal{M}} \int_0^\infty g(x)w(x) dx = \int_0^\infty f(x) \left(\sup_{t \ge x} w(t)\right) dx$$

from [50], which holds for $f, w \in \mathcal{M}^+(0, \infty)$. On the other hand, it uses estimates of optimal constants in iterated Hardy-type inequalities, “gluing” lemmas, which allows to reduce the problem to using of the boundedness of weighted iterated Hardy-type operators involving suprema from weighted Lebesgue spaces into weighted Cesàro function spaces. Detailed information on materials that are used in the proofs of the main results is given in the following section.

It should be noted that the method developed in the present paper allows to solve inequality (1.3) in the case $0 < m \le 1$, $0 < p < \infty$ or $1 < m < \infty$, $0 < p \le 1$, as well. However, we are able to solve the inequality under the restrictions $1 < m < p \le r < q < \infty$ or $1 < m \le r \le \min\{p, q\} < \infty$. For the remaining values of parameters the conditions that characterize the weighted iterated Hardy-type inequalities contains more complicated expressions and the approach used in our paper needs an improvement.

Using duality principle puts restriction $r \le q$ on parameters. But for $r = q$ inequality (1.3) is a special case of the inequality

$$(1.5) \quad \left(\int_0^\infty [T_{a,b}f^\ast(x)]^q w(x) dx\right)^{\frac{1}{q}} \le C \left(\int_0^\infty \left(\int_0^\tau [f^{\ast}(t)]^{p} dt\right)^{p/q} v(x) dx\right)^{1/p},$$

which we are going to investigate in our forthcoming studies.

Throughout the paper, we always denote by $C$ a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript such as $C_1$ does not change in different occurrences. By $a \le b$, we mean that $a \le \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \le b$ and $b \le a$, we write $a \approx b$ and say that $a$ and $b$ are equivalent.

As usual, we put $0 \cdot \infty = 0$, $\infty/0 = 0$ and $0/0 = 0$. If $p \in [1, +\infty]$, we define $p'$ by $1/p + 1/p' = 1$.

The paper is organized as follows. We start with formulations of background material in Section 2. In Section 3 we present solution of the restricted inequality. Finally, in Section 4 we calculate the norm of generalized maximal function between Lorentz $G_t^\ast$ spaces.
2. Background material

In this section we collect background material that will be used in the proofs of the main theorems.

We begin with the following characterization of the norm of the associate space of \( G(p, m, v) \) given in \([26]\).

Recall that the associate space \( G(p, m, v) \) of \( G(p, m, v) \) is defined as the collection of all functions \( g \in \mathcal{M}(\mathbb{R}^p) \) such that

\[
\|g\|_{G(p, m, v)} = \sup_{\|f\|_{G(p, m, v)} \leq 1} \int_0^\infty f^*(t)g^*(t)dt < \infty.
\]

As it is mentioned in \([26]\), it is reasonable to adopt a general assumption that \( p, m \) and \( v \) are such that

\[
\int_0^\infty v(s)s^{\frac{m}{p}}ds + \int_t^\infty v(s)ds < \infty, \quad t \in (0, \infty),
\]

because if this requirement is not satisfied, then \( G(p, m, v) = \{0\} \).

Under the assumption (2.1), we denote

\[
v_0(t) := t^{\frac{m}{p}-1}\int_0^t v(s)s^{\frac{m}{p}}ds \int_t^\infty v(s)ds, \quad t \in (0, \infty),
\]

and

\[
v_1(t) := \int_0^t v(s)s^{\frac{m}{p}}ds + t^{\frac{m}{p}} \int_0^\infty v(s)ds, \quad t \in (0, \infty).
\]

Moreover, we assume that a weight \( v \) is non-degenerate (with respect to the power function \( t^{m/p} \)), that is,

\[
\int_0^1 v(s)ds = \int_1^\infty v(s)s^{\frac{m}{p}}ds = \infty.
\]

We denote the set of all weight functions satisfying conditions (2.1) and (2.4) by \( \mathcal{W}_{m,p}(0,\infty) \).

**Theorem 2.1.** \([26, \text{Theorem 1.1}]\) Assume that \( 1 < m, p < \infty \) and \( v \in \mathcal{W}_{m,p}(0,\infty) \). Then

\[
\|g\|_{G(p, m, v)} \approx \left( \int_0^\infty \left( \int_0^\infty g^*(s)(s)^pds \right)^{\frac{m}{p}} \left( \frac{t^{\frac{m}{p}}v_0(t)}{v_1(t)^{m+1}} \right)^{\frac{1}{p'}} \right)^{\frac{1}{m}}.
\]

We recall the following well-known duality principle in weighted Lebesgue spaces.

**Theorem 2.2.** Let \( p > 1, f \in \mathcal{M}^+(0,\infty) \) and \( w \in \mathcal{W}(0,\infty) \). Then

\[
\left( \int_0^\infty f(t)^p w(t)dt \right)^\frac{1}{p} = \sup_{h \in \mathcal{M}^+} \frac{\int_0^\infty f(t)h(t)dt}{\left( \int_0^\infty h(t)^{p'} w(t)^{1-p'} dt \right)^{\frac{1}{p'}}}.
\]

We will use the following statement.

**Theorem 2.3.** \([50, \text{Theorem 2.1}]\) Suppose \( f, w \in \mathcal{M}^+ \). Then

\[
\sup_{g \in \mathcal{M}^+, \|g\|_{\mathcal{M}^+} \leq \|f\|_{\mathcal{M}^+}} \int_0^\infty g(x)w(x)dx = \int_0^\infty f(x)\left( \sup_{t \geq x} w(t) \right)dx.
\]

We will apply the following ”gluing” lemma.

**Lemma 2.4.** \([23, \text{Lemma 2.7}]\) Let \( \alpha \) and \( \beta \) be positive numbers. Suppose that \( g, h \in \mathcal{M}^+ \) and \( a \in \mathcal{W}(0,\infty) \) is non-decreasing. Then

\[
\text{ess sup}_{x \in (0,\infty)} \left( \int_0^\infty \left( \frac{a(x)}{a(x) + a(t)} \right)^\beta g(t)dt \right)^\frac{1}{\beta} \left( \int_0^\infty \left( \frac{a(t)}{a(x) + a(t)} \right)^\alpha h(t)dt \right)^\frac{1}{\alpha} \approx \text{ess sup}_{x \in (0,\infty)} \left( \int_0^\infty g(t)dt \right)^\frac{1}{\beta} \left( \int_x^\infty h(t)dt \right)^\frac{1}{\alpha} + \text{ess sup}_{x \in (0,\infty)} \left( \int_x^\infty a(t)^{-\beta} g(t)dt \right)^\frac{1}{\beta} \left( \int_0^x a(t)^\alpha h(t)dt \right)^\frac{1}{\alpha}.
\]

We recall the following ”an integration by parts” formula.
Theorem 2.5. [37, Theorem 2.1] Let $\alpha > 0$. Let $g$ be a non-negative function on $(0, \infty)$ such that $0 < \int_0^t g < \infty$ for all $t \in (0, \infty)$ and let $f$ be a non-negative non-increasing right-continuous function on $(0, \infty)$. Then

$$A_1 := \int_0^\infty \left( \int_0^t g^\alpha \right) g(t) |f(t) - \lim_{t \to +\infty} f(t)| \, dt < \infty \quad \iff \quad A_2 := \int_{(0, \infty)} \left( \int_0^t g \right)^{\alpha+1} \, d[-f(t)] < \infty.$$  

Moreover, $A_1 \approx A_2$.

Investigation of weighted iterated Hardy-type inequalities started with studying of the inequality

$$(2.5) \quad \left( \int_0^\infty \left( \int_0^t h(y) \, dy \right)^m u(s) \, ds \right)^{\frac{1}{m}} w(t) \, dt \leq C \left( \int_0^\infty h(t)^p v(t) \, dt \right)^{\frac{1}{p}}, \quad h \in \mathcal{W}^+. $$

Note that inequality (2.5) have been considered in the case $m = 1$ in [6] (see also [15]), where the result was presented without proof, and in the case $p = 1$ in [16] and [51], where the special type of weight function $v$ was considered. Recall that the inequality has been completely characterized in [19] and [20] in the case $0 < m < \infty$, $0 < q \leq \infty$, $1 \leq p < \infty$ by using discretization and anti-discretization methods. Another approach to get the characterization of inequalities (2.5) was presented in [44]. The characterization of the inequality can be reduced to the characterization of the weighted Hardy inequality on the cones of non-increasing functions (see, [21] and [22]). Different approach to solve iterated Hardy-type inequalities has been given in [35].

As it was mentioned in [21] the characterization of “dual” inequality

$$(2.6) \quad \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^t h(y) \, dy \right)^m u(s) \, ds \right)^{\frac{1}{m}} w(t) \, dt \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty h(t)^p v(t) \, dt \right)^{\frac{1}{p}}, \quad h \in \mathcal{W}^+. $$

can be easily obtained from the solutions of inequality (2.5), which was presented in [17].

Theorem 2.6. [17, Theorem 2.9, (a) and (c)] Let $p, q, m \in (1, \infty)$ and $u$, $w$, $v$ be weights on $(0, \infty)$. Assume that the following non-degeneracy conditions are satisfied:

- $u$ is strictly positive, \( \int_0^\infty u(s) \, ds < \infty \) for all $t \in (0, \infty)$, \( \int_0^\infty u(s) \, ds = \infty \),

- \( \int_0^t w(s) \, ds < \infty \) and \( \int_0^\infty w(s) \left( \int_s^t u(y) \, dy \right)^{\frac{m}{p}} \, ds < \infty \) for all $t \in (0, \infty)$,

- \( \int_0^1 w(s) \left( \int_s^\infty u(y) \, dy \right)^{\frac{m}{p}} \, ds = \infty \) and \( \int_1^\infty w(s) \, ds = \infty \).

Let

$$C = \sup_{h \in \mathcal{W}^+} \left( \frac{\int_0^\infty \left( \int_0^\infty \left( \int_0^t h(y) \, dy \right)^m u(s) \, ds \right)^{\frac{1}{m}} w(t) \, dt}{\int_0^\infty h(t)^p v(t) \, dt} \right)^{\frac{1}{p}}. $$

(a) If $p \leq m$ and $p \leq q$, then $C \approx D_1 + D_2$, where

$$D_1 = \sup_{t \in (0, \infty)} \left( \int_0^t w(s) \, ds \right)^{\frac{1}{q}} \left( \int_0^\infty u(s) \, ds \right)^{\frac{1}{p}} \left( \int_0^t v(\tau)^{1-p'} \, d\tau \right)^{\frac{1}{p'}}$$

and

$$D_2 = \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^\infty u(y) \, dy \right)^{\frac{m}{p}} w(s) \, ds \right)^{\frac{1}{q}} \left( \int_0^\infty v(s)^{1-p'} \, ds \right)^{\frac{1}{p'q'}}.$$

(b) If $m < p$ and $p \leq q$, then $C \approx D_2 + D_3$, where

$$D_3 = \sup_{t \in (0, \infty)} \left( \int_0^t w(s) \, ds \right)^{\frac{1}{q}} \left( \int_0^t \left( \int_s^\infty u(y) \, dy \right)^{\frac{m}{p}} w(s) \, ds \right)^{\frac{1}{q}} \left( \int_0^\infty v(\tau)^{1-p'} \, d\tau \right)^{\frac{m}{pm}} \left( \int_0^\infty v(s)^{1-p'} \, ds \right)^{\frac{m}{pq}}.$$

Another pair of “dual” weighted iterated Hardy-type inequalities are

$$(2.7) \quad \left( \int_0^\infty \left( \int_0^t h(y) \, dy \right)^m u(s) \, ds \right)^{\frac{1}{m}} w(t) \, dt \leq C \left( \int_0^\infty h(t)^p v(t) \, dt \right)^{\frac{1}{p}}, \quad h \in \mathcal{W}^+.$$
and
\begin{equation}
\left( \int_0^\infty \left( \int_0^t \left( \int_s^t h(y) \, dy \right)^m u(s) \, ds \right)^{\frac{q}{m}} \, w(t) \, dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty \left( \int_0^t h(t)^p \, v(t) \, dt \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+.
\end{equation}
Both of them were characterized in [21] by so-called “flipped” conditions. The “classical” conditions ensuring the validity of (2.7) was recently presented in [32].

**Theorem 2.7.** [32, Theorem 1.1, (a) and (c)] Let $p, q, m \in (1, \infty)$ and $u, v$ be weights such that the pair $(u, v)$ is admissible with respect to $(m, q)$, that is,
\[ 0 < \int_0^\infty \left( \int_0^t u(y) \, dy \right)^{\frac{q}{m}} w(s) \, ds < \infty, \quad t \in (0, \infty). \]
Let
\[ C = \sup_{h \in \mathcal{M}^+} \frac{\left( \int_0^\infty \left( \int_0^t \left( \int_s^t h(y) \, dy \right)^m u(s) \, ds \right)^{\frac{q}{m}} \, w(t) \, dt \right)^{\frac{1}{q}}}{\left( \int_0^\infty h(t)^p \, v(t) \, dt \right)^{\frac{1}{p}}}. \]
(a) If $p \leq m$ and $p \leq q$, then $C \approx E_1$, where
\[ E_1 = \sup_{t \in (0, \infty)} \left( \int_0^t w(s) \left( \int_0^t u(y) \, dy \right)^{\frac{q}{m}} ds \right)^{\frac{1}{q}} \left( \int_0^\infty v(s)^{1-p'} \, ds \right)^{\frac{1}{p'}}. \]
(b) If $m < p$ and $p \leq q$, then $C \approx E_1 + E_2$, where
\[ E_2 = \sup_{t \in (0, \infty)} \left( \int_0^t w(s) \, ds \right)^{\frac{1}{q}} \left( \int_0^\infty \left( \int_0^t u(y) \, dy \right)^{\frac{q}{m}} u(s) \left( \int_0^\infty v(z)^{1-p'} \, dz \right)^{\frac{m-1}{p-m}} \, ds \right)^{\frac{1}{p' q}}. \]
We need solutions of inequalities
\[ \left( \int_0^\infty \left( \int_0^t \left( \sup_{s \leq r} u(r) \left( \int_s^r f(y) \, dy \right) \right)^q a(s) \, ds \right)^{\frac{1}{q}} \, w(x) \, dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f(s)^p v(s) \, ds \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}^+(t, \infty) \]
and
\[ \left( \int_0^\infty \left( \int_{s \leq t} \left( \int_t^\infty f(z) \, dz \right) \right)^q a(s) \, ds \right)^{\frac{1}{q}} \, w(y) \, dy \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f(s)^p v(s) \, ds \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}^+(t, \infty) \]
where $t$ is a given point in $[0, \infty)$ and $1 < p < \infty$ and $u \in \mathcal{W}(t, \infty) \cap C(t, \infty)$ and $a, v \in \mathcal{W}(t, \infty)$.

The proofs of the following two statements are straightforward and capacious, but for the convenience of the reader we give the complete proof of both of them.

**Theorem 2.8.** Let $1 < p, q < \infty$. Given $t \geq 0$ assume that $u \in \mathcal{W}(t, \infty) \cap C(t, \infty)$ and $a, v \in \mathcal{W}(t, \infty)$. Moreover, assume that $0 < \int_0^\infty v(\tau)^{1-p'} \, d\tau < \infty$, $x > t$.

- If $p \leq q$, then
\[ \sup_{f \in \mathcal{M}^+(t, \infty)} \left( \int_t^\infty \left( \int_t^\infty \left( \sup_{s \leq r} u(r) \left( \int_s^r f(y) \, dy \right) \right)^q a(s) \, ds \right)^{\frac{1}{q}} \, w(x) \, dx \right)^{\frac{1}{q}} \approx \sup_{y \in (t, \infty)} \left( \int_t^\infty v(x)^{1-p'} \left( \int_x^\infty \left( \sup_{s \leq r} u(r) \left( \int_s^r a(s) \right) \right)^q \, dx \right)^{\frac{1}{p'}} \left( \int_y^\infty w(z) \, dz \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \]
\[ + \sup_{y \in (t, \infty)} \left( \int_t^\infty v(x)^{1-p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^\infty \left( \sup_{s \leq r} u(r) \left( \int_s^r a(s) \right) \right)^q \, w(z) \, dz \right)^{\frac{1}{q}} \]
\[ + \sup_{y \in (t, \infty)} \left( \int_{[1, \infty)} d \left( - \sup_{y \leq \tau} u(\tau)^p \left( \int_{\tau}^\infty v(s)^{1-p'} \, ds \right) \right)^{\frac{1}{p}} \left( \int_y^\infty a \right)^q \, w(z) \, dz \right)^{\frac{1}{q}}. \]
\[\begin{align*}
+ \sup_{x \in (t, \infty)} \left( \int_t^x \left( \int_t^y a \, d \left( -u(t) \rho' \left( \int_t^\tau v(s)^{1-\rho'} \, ds \right) \right) \right) \frac{1}{\rho} \left( \int_x^\infty w(z) \, dz \right)^{\frac{1}{\rho}} \\
+ \left( \int_t^\infty \left( \int_t^\infty a \right)^q w(z) \, dz \right)^{\frac{1}{q}} \lim_{x \to 0^+} \left( \sup_{x \in [t, \infty)} \left( \int_t^\tau v(s)^{1-\rho'} \, ds \right) \right)^{\frac{1}{\rho'}}.
\end{align*}\]

- If \(q < p\), then
\[
\sup_{f \in \mathbb{M}^+(t, \infty)} \left( \int_t^\infty f(x)^p v(x) \, dx \right)^{\frac{1}{p}} \approx \left( \int_t^\infty \left( \int_t^y v(x)^{1-\rho'} \, dx \right) \frac{\mu_{p-1}}{p-q} v(y)^{1-\rho'} \left( \int_y^\infty \left( \sup_{x \leq \tau} u(a) \right) w(z) \, dz \right)^{\frac{p}{p-q}} \, dy \right)^{\frac{p}{p-q}}
\]
\[
+ \left( \int_t^\infty \left( \int_t^y \left( \sup_{x \leq \tau} u(a) \right) \rho' \left( \int_t^\tau v(x)^{1-\rho'} \, dx \right) \right) \left( \int_y^\infty w(z) \, dz \right)^{\frac{p}{p-q}} w(y) \, dy \right)^{\frac{p}{p-q}}
\]
\[
+ \left( \int_t^\infty \left( \int_x^\infty d \left( -u(t) \rho' \left( \int_t^\tau v(s)^{1-\rho'} \, ds \right) \right) \right) \frac{\mu_{p-1}}{p-q} \left( \int_t^x \left( \int_t^z \rho' \right) w(z) \, dz \right)^{\frac{q}{q-q}} \left( \int_t^x a \right)^q w(x) \, dx \right)^{\frac{p}{p-q}}
\]
\[
+ \left( \int_t^\infty \left( \int_x^\infty a \right)^q w(z) \, dz \right)^{\frac{1}{q}} \lim_{x \to 0^+} \left( \sup_{x \leq \tau} u(a) \right) \left( \int_t^\tau v(s)^{1-\rho'} \, ds \right)^{\frac{1}{\rho'}}.
\]

**Proof.** The statement was formulated in [37, Theorem 3.1] for \(t = 0\). Assume that \(t > 0\). The proof uses multiple changes of variables of the type \(x + t = y\) several times, which we will not specify at any step exactly.

It is easy to see that, given a point \(t \in (0, \infty)\), the inequality

\[(2.9) \quad \left( \int_t^\infty \left( \int_t^x \left( \sup_{x \leq \tau} u(t) \left( \int_t^\tau f(y) \, dy \right) \right) a(s) \, ds \right)^q w(x) \, dx \right)^{\frac{1}{q}} \leq C \left( \int_t^\infty f(x)^p v(x) \, dx \right)^{\frac{1}{p}}, \quad f \in \mathbb{M}^+(t, \infty)
\]

is equivalent to the inequality

\[(2.10) \quad \left( \int_0^\infty \left( \int_t^x \left( \sup_{x \leq \tau} u(t) \left( \int_t^\tau f(y) \, dy \right) \right) a_t(s) \, ds \right)^q w_t(x) \, dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f(s)^p v_t(s) \, ds \right)^{\frac{1}{p}}, \quad f \in \mathbb{M}^+(0, \infty),
\]

where the weight functions \(u_t, w_t, v_t\) and \(a_t\) are defined for \(x > 0\) as follows:

\[u_t(x) := u(x + t), \quad w_t(x) := w(x + t), \quad v_t(x) := v(x + t), \quad a_t(x) := a(x + t).
\]

Indeed: Applying changes of variables, we get that
\[
\int_t^\infty \left( \int_t^x \left( \sup_{x \leq \tau} u(t) \left( \int_t^\tau f(z) \, dz \right) \right) a(s) \, ds \right)^q w(x) \, dx
\]
\[
= \int_t^\infty \left( \int_t^x \left( \sup_{x \leq \tau} u(t) \left( \int_0^{\tau-t} f(z + t) \, dz \right) \right) a(s) \, ds \right)^q w(x) \, dx
\]
\[
= \int_t^\infty \left( \int_t^x \left( \sup_{s+t \leq \tau-t} u(t+t) \left( \int_0^{\tau-t} f(z + t) \, dz \right) \right) a(s) \, ds \right)^q w(x) \, dx
\]
\[
= \int_t^\infty \left( \int_t^x \left( \sup_{s+t \leq \tau} u(t+t) \left( \int_0^{\tau} f(z + t) \, dz \right) \right) a(s+t) \, ds \right)^q w(x) \, dx
\]
\[
= \int_0^\infty \left( \int_0^x \left( \sup_{s+t \leq \tau} u(t+t) \left( \int_0^{\tau} f(z + t) \, dz \right) \right) a(s+t) \, ds \right)^q w(x+t) \, dx
\]
\[
q = \int_0^\infty \left( \int_0^\infty \left( \sup_{\tau \leq t} u_\tau(\tau) \left( \int_0^\tau f(z + t) \, dz \right) \right) a(t) \, ds \right)^q \, w_t(x) \, dx
\]
and
\[
\int_t^\infty f(s)^p v(s) \, ds = \int_0^\infty f(s + t)^p v(s + t) \, ds = \int_0^\infty f(s + t)^p v_t(s) \, ds.
\]
Hence inequality (2.9) can be rewritten as follows:
\[
\left( \int_0^\infty \left( \sup_{\tau \leq t} u_\tau(\tau) \left( \int_0^\tau f(z + t) \, dz \right) a_t(s) \, ds \right)^q \, w_t(y) \, dy \right)^{\frac{q}{p}} \leq C \left( \int_0^\infty f(s + t)^p v_t(s) \, ds \right)^{\frac{1}{p}}, \quad f \in M^+(t, \infty).
\]
It remains to note that the latter is equivalent to inequality (2.10).
Let \( p \leq q \). By [37, Theorem 3.1], we have for any \( t \in (0, \infty) \) that
\[
\sup_{f \in M^+(0, \infty)} \frac{\left( \int_0^\infty \left( \sup_{\tau \leq t} u_\tau(\tau) \left( \int_0^\tau f(y) \, dy \right) a_t(s) \, ds \right)^q \, w_t(x) \, dx \right)^{\frac{1}{q}}}{\left( \int_0^\infty f(s)^p v_t(s) \, ds \right)^{\frac{1}{p}}}
\approx \sup_{y \in (0, \infty)} \left( \int_0^y v_t(x)^{1-p'} \left( \int_x^y \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a_t(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^\infty w_t(z) \, dz \right)^{\frac{1}{q}}
+ \sup_{y \in (0, \infty)} \left( \int_0^y v_t(x)^{1-p'} \left( \int_x^y \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a_t(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^\infty w_t(z) \, dz \right)^{\frac{1}{q}}
+ \sup_{x \in (0, \infty)} \left( \int_0^x d_t \left( \sup_{\tau \leq t} u_\tau(\tau) \left( \int_0^\tau v_t(s)^{1-p} \, ds \right) \right) \right)^{\frac{1}{p'}} \left( \int_x^\infty A_t(z) v_t(z) \, dz \right)^{\frac{1}{q}}
+ \left( \int_0^x A_t(z)^q v_t(z) \, dz \right)^{\frac{1}{q}} \lim_{x \to \infty} \sup_{\tau \leq t} u_\tau(\tau) \left( \int_0^\tau v_t(s)^{1-p} \, ds \right)^{\frac{1}{p'}}.
\]
Since
\[
\sup_{y \in (0, \infty)} \left( \int_0^y v_t(x)^{1-p'} \left( \int_x^y \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a_t(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^\infty w_t(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x + t)^{1-p'} \left( \int_x^y \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s + t) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^\infty w(z + t) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x + t)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z + t) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
= \sup_{y \in (0, \infty)} \left( \int_0^y v(x)^{1-p'} \left( \int_x^{y+t} \left( \sup_{\tau \leq t} u_\tau(\tau) \right) a(s) \, ds \right)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_y^{y+t} w(z) \, dz \right)^{\frac{1}{q}}
\[
\begin{align*}
&= \sup_{y \in (0, \infty)} \left( \int_{y}^{y+t} v(x)^{1-p'} \, dx \right)^{1-p'} \left( \int_{y}^{\infty} \left( \int_{s}^{\infty} \left( \sup_{\tau \leq s} u(\tau) \right) a(s) \, ds \right)^{q} w(z+t) \, dz \right)^{1/q} \\
&= \sup_{y \in (0, \infty)} \left( \int_{y}^{y+t} v(x)^{1-p'} \, dx \right)^{1-p'} \left( \int_{y}^{\infty} \left( \int_{s}^{\infty} \left( \sup_{\tau \leq s} u(\tau) \right) a(s) \, ds \right)^{q} w(z) \, dz \right)^{1/q} \\
&= \sup_{y \in (0, \infty)} \left( \int_{y}^{y+t} v(x)^{1-p'} \, dx \right)^{1-p'} \left( \int_{y}^{\infty} \left( \int_{s}^{\infty} \left( \sup_{\tau \leq s} u(\tau) \right) a(s) \, ds \right)^{q} w(z) \, dz \right)^{1/q}
\end{align*}
\]

\[
\begin{align*}
&\sup_{x \in (0, \infty)} \left( \int_{0}^{x} d\left( -\sup_{y \leq x} u(\tau)^{p'} \left( \int_{0}^{\tau} v(s)^{1-p'} \, ds \right) \right) \right)^{1/p'} \left( \int_{0}^{\infty} A(z)^{q} w(z) \, dz \right)^{1/q} \\
&= \sup_{x \in (0, \infty)} \left( \int_{0}^{x} d\left( -\sup_{y \leq x} u(\tau)^{p'} \left( \int_{0}^{\tau} v(s)^{1-p'} \, ds \right) \right) \right)^{1/p'} \left( \int_{0}^{\infty} A(z)^{q} w(z) \, dz \right)^{1/q} \\
&= \sup_{x \in (0, \infty)} \left( \int_{0}^{x} d\left( -\sup_{y \leq x} u(\tau)^{p'} \left( \int_{0}^{\tau} v(s)^{1-p'} \, ds \right) \right) \right)^{1/p'} \left( \int_{0}^{\infty} A(z)^{q} w(z) \, dz \right)^{1/q}
\end{align*}
\]
we arrive at

\[
\sup_{f \in \mathbb{R}^n(t, \infty)} \left( \int_0^\infty f(s) \rho v(s) ds \right)^{\frac{1}{\rho}} \left( \int_0^y v(x)^{1-\rho'} \left( \int_x^y \left( \sup_{x \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w(x) dx \right)^{\frac{\rho'}{\rho}} \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
\approx \sup_{y \in (t, \infty)} \left( \int_t^y v(x)^{1-\rho'} \left( \int_x^y \left( \sup_{x \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w(x) dx \right)^{\frac{\rho'}{\rho}} \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
+ \sup_{y \in (t, \infty)} \left( \int_t^y a(x)^{p-1} w(x) dx \right)^{\frac{p-1}{p}} \left( \int_y^\infty \left( \sup_{x \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w(x) dx \right)^{\frac{\rho'}{\rho}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
+ \sup_{x \in (t, \infty)} \left( \int_t^x a(x)^{p-1} w(x) dx \right)^{\frac{p-1}{p}} \left( \int_x^\infty \left( \sup_{x \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w(x) dx \right)^{\frac{\rho'}{\rho}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
+ \left( \int_0^\infty \left( \int_0^x a(x)^{p-1} w(x) dx \right)^{\frac{p-1}{p}} \lim_{x \to \infty} \left( \sup_{x \leq \tau} u(\tau) \left( \int_t^x v(s)^{1-\rho'} ds \right)^{\frac{\rho'}{\rho}} \right) \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]

Let \( q < p \). By [37, Theorem 3.1], we have for any \( t \in (0, \infty) \) that

\[
\sup_{f \in \mathbb{R}^n(0, \infty)} \left( \int_0^\infty f(s)^p v_1(s) ds \right)^{\frac{1}{p}} \left( \int_0^y v_1(x)^{1-p'} \left( \int_x^y \left( \sup_{x \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w_1(x) dx \right)^{\frac{\rho'}{\rho}} \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w_1(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
\approx \left( \int_0^\infty \left( \int_0^y v_1(x)^{1-p'} dx \right)^{\frac{p-1}{p-q}} v_1(y)^{1-p'} \left( \int_y^\infty \left( \sup_{y \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w_1(y) dy \right)^{\frac{\rho'}{\rho}} \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w_1(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
+ \left( \int_0^\infty \left( \int_0^y v_1(x)^{1-p'} \left( \int_x^y \left( \sup_{x \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w_1(x) dx \right)^{\frac{\rho'}{\rho}} \right)^{\frac{1}{\rho'}} \left( \int_x^\infty \left( \sup_{x \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w_1(x) dx \right)^{\frac{\rho'}{\rho}} \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w_1(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
+ \left( \int_0^\infty \left( \int_0^x a(x)^{p-1} w_1(x) dx \right)^{\frac{p-1}{p}} \lim_{x \to \infty} \left( \sup_{x \leq \tau} u(\tau) \left( \int_0^x v_1(s)^{1-\rho'} ds \right)^{\frac{\rho'}{\rho}} \right) \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w_1(z) dz \right)^{\frac{1}{\rho'}}
\]

Since

\[
\left( \int_0^\infty \left( \int_0^y v_1(x)^{1-p'} dx \right)^{\frac{p-1}{p-q}} v_1(y)^{1-p'} \left( \int_y^\infty \left( \sup_{y \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w_1(y) dy \right)^{\frac{\rho'}{\rho}} \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w_1(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
= \left( \int_0^\infty \left( \int_0^y v(x+t)^{1-p'} dx \right)^{\frac{p-1}{p-q}} v(y+t)^{1-p'} \left( \int_y^\infty \left( \sup_{y \leq \tau} u(\tau) a(s+t) ds \right)^{\frac{q}{\rho'}} w(z+t) dz \right)^{\frac{\rho'}{\rho}} dy \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
= \left( \int_0^\infty \left( \int_y^{y+t} v(x)^{1-p'} dx \right)^{\frac{p-1}{p-q}} v(y+t)^{1-p'} \left( \int_y^{y+t} \left( \sup_{y \leq \tau} u(\tau) a(s+t) ds \right)^{\frac{q}{\rho'}} w(z+t) dz \right)^{\frac{\rho'}{\rho}} dy \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]

\[
= \left( \int_0^\infty \left( \int_y^{y+t} v(x)^{1-p'} dx \right)^{\frac{p-1}{p-q}} v(y+t)^{1-p'} \left( \int_y^{y+t} \left( \sup_{y \leq \tau} u(\tau) a(s) ds \right)^{\frac{q}{\rho'}} w(z+t) dz \right)^{\frac{\rho'}{\rho}} dy \right)^{\frac{1}{\rho'}} \left( \int_0^\infty w(z) dz \right)^{\frac{1}{\rho'}}
\]
\[
\begin{align*}
&= \left( \int_0^\infty \left( \int_{y+t}^{y+q} v(x)^{1-p'} dx \right)^{\frac{q(p-1)}{p-q}} v(y+t)^{1-p'} \left( \int_0^\infty \left( \int_{s+t}^{\infty} (\sup_{s\leq r} u(r)) a(s) ds \right)^{q} w(z) dz \right)^{\frac{p-q}{p}} dy \right)^{\frac{p-q}{p}} \\
&= \left( \int_0^\infty \left( \int_{y}^{\infty} v(x)^{1-p'} dx \right)^{\frac{q(p-1)}{p-q}} v(y)^{1-p'} \left( \int_y^\infty \left( \int_{s}^{\infty} (\sup_{s\leq r} u(r)) a(s) ds \right)^{q} w(z) dz \right)^{\frac{p-q}{p}} dy \right)^{\frac{p-q}{p}} ,
\end{align*}
\]

\[
\left( \int_0^\infty \left( \int_{0}^{y} v(x)^{1-p'} \left( \int_x^{\infty} (\sup_{y\leq r} u(r)) a(s) ds \right)^{p'} dx \right)^{\frac{q(p-1)}{p-q}} \left( \int_y^\infty \left( \int_{y \leq t}^{\infty} w_t(z) dz \right)^{\frac{p-q}{p}} \right) w(y) dy \right)^{\frac{p-q}{p}} 
\]

\[
\left( \int_0^\infty \left( \int_{0}^{y} \left( \int_{y}^{\infty} v(x)^{1-p'} dx \right)^{\frac{q(p-1)}{p-q}} v(y)^{1-p'} \left( \int_y^\infty \left( \int_{s}^{\infty} (\sup_{s\leq r} u(r)) a(s) ds \right)^{q} w(z) dz \right)^{\frac{p-q}{p}} dy \right)^{\frac{p-q}{p}} \right)^{\frac{p-q}{p}},
\]

\[
\left( \int_0^\infty \left( \int_{0,x}^y \left( \int_{y}^{\infty} v(x)^{1-p'} dx \right)^{\frac{q(p-1)}{p-q}} \left( \int_{0,x}^y \left( \int_{0}^{x} \left( \int_{0}^{y} v(x)^{1-p'} dx \right)^{\frac{q(p-1)}{p-q}} v(y)^{1-p'} \left( \int_y^\infty \left( \int_{s}^{\infty} (\sup_{s\leq r} u(r)) a(s) ds \right)^{q} w(z) dz \right)^{\frac{p-q}{p}} dy \right)^{\frac{p-q}{p}} \right)^{\frac{p-q}{p}} \right)^{\frac{p-q}{p}} \right)^{\frac{p-q}{p}} 
\]
we arrive at

\[
\sup_{f \in \mathcal{W}(t, \infty)} \left( \int_t^\infty \left( \sup_{s \leq \tau} u(\tau) \left( \int_\tau^\infty f(y) dy \right) a(s) ds \right)^q w(x) dx \right)^{\frac{1}{q}}
\]

\[
\left( \int_t^\infty f(s)^p v(s) ds \right)^{\frac{1}{p}}
\]

\[
\approx \left( \int_t^\infty \left( \int_t^y v(x)^{1-p'} dx \right)^{\frac{p-1}{p}} v(y)^{-p'} \left( \int_y^\infty \left( \sup_{s \leq \tau} a(s) ds \right)^q w(z) dz \right)^{\frac{p}{p-q}} dy \right)^{\frac{p-q}{p}}
\]

\[
+ \left( \int_t^\infty \left( \int_t^y \left( \sup_{s \leq \tau} a(s) ds \right)^p dx \right)^{\frac{q-p}{q}} \left( \int_y^\infty w(z) dz \right)^{\frac{p}{p-q}} w(y) dy \right)^{\frac{p-q}{p}}
\]

\[
+ \left( \int_t^\infty \left( \int_y^\infty d(\sup_{s \leq \tau} \mathcal{U}(s)) \left( \int_\tau^\infty v(y) dy \right)^{\frac{p}{p-q}} \left( \int_y^\infty a(z) w(z) dz \right)^{\frac{p}{p-q}} \left( \int_\tau^\infty a(z) w(z) dz \right)^{\frac{p}{p-q}} \right)^{\frac{p}{p-q}}
\]

\[
+ \left( \int_t^\infty \left( \int_y^\infty a(z) w(z) dz \right)^{\frac{q}{q-p}} \lim_{s \to \infty} \left( \sup_{s \leq \tau} \mathcal{U}(s) \left( \int_\tau^\infty v(y) dy \right)^{\frac{p}{p-q}} \right)^{\frac{p}{p-q}} \right)^{\frac{p}{p-q}}
\]

The proof is completed. \(\square\)

**Theorem 2.9.** Let \(1 < p, q < \infty\). Given \(t \geq 0\) assume that \(u \in \mathcal{W}(t, \infty) \cap \mathcal{C}(t, \infty)\) and \(a, v, w \in \mathcal{W}(t, \infty)\). Moreover, assume that \(0 < \int_x^\infty v(\tau)^{1-p'} d\tau < \infty, x > t\). Denote by

\[
\mathcal{V}(x) := \left( \int_x^\infty v^{1-p'} \right)^{\frac{2-p}{p}} v(x)^{1-p'}, \quad \mathcal{U}(x) := u(x) \left( \int_x^\infty v^{1-p'} \right)^{\frac{1}{p}}, \quad x \in (0, \infty).
\]

- If \(p \leq q\), then

\[
\sup_{f \in \mathcal{W}(t, \infty)} \left( \int_t^\infty \left( \sup_{s \leq \tau} u(\tau) \left( \int_\tau^\infty f(z) dz \right) a(s) ds \right)^q w(y) dy \right)^{\frac{1}{q}}
\]

\[
\left( \int_t^\infty f(s)^p v(s) ds \right)^{\frac{1}{p}}
\]

\[
\approx \sup_{y \in (t, \infty)} \left( \int_t^y \mathcal{V}(x) dx \right)^{\frac{2-p}{p}} \left( \int_y^\infty \left( \sup_{s \leq \tau} \mathcal{U}(s) a(s) ds \right)^p w(x) dx \right)^{\frac{1}{p}}
\]

\[
+ \sup_{y \in (t, \infty)} \left( \int_t^y \mathcal{V}(x) dx \right)^{\frac{q-p}{q}} \left( \int_y^\infty \left( \sup_{s \leq \tau} \mathcal{U}(s) a(s) ds \right)^q w(z) dz \right)^{\frac{1}{q}}
\]

\[
+ \sup_{x \in (t, \infty)} \left( \int_t^x a(z) d(\sup_{s \leq \tau} \mathcal{U}(s)) \left( \int_\tau^\infty \mathcal{V}(y) dy \right)^{\frac{p}{p-q}} \left( \int_x^\infty a(z) w(y) dy \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
\]

\[
+ \sup_{x \in (t, \infty)} \left( \int_t^x a(z) w(z) dz \right)^{\frac{1}{q}} \lim_{s \to \infty} \left( \sup_{s \leq \tau} \mathcal{U}(s) \left( \int_\tau^\infty \mathcal{V}(y) dy \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_t^x v(s)^{1-p'} ds \right)^{\frac{-p}{q-p-1}} \left( \int_t^x \left( \sup_{s \leq \tau} a(s) ds \right)^q w(x) dx \right)^{\frac{1}{q}}.
\]
If $q < p$, then

$$\sup_{f \in \mathcal{M}^+(t, \infty)} \left( \int_t^\infty \left( \int_s^\infty u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy \right)^{\frac{1}{q}} \leq C \left( \int_t^\infty f(s)^p \, v(s) \, ds \right)^{\frac{1}{p}},$$

$$f \in \mathcal{M}^+(t, \infty)$$

is equivalent to the inequality

$$\left( \int_t^\infty \left( \int_s^\infty u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy \right)^{\frac{1}{q}} \leq C \left( \int_t^\infty f(s)^p \, v(s) \, ds \right)^{\frac{1}{p}}.$$

Proof. The statement was formulated in [37, Theorem 3.3] for $t = 0$. Note that, given a point $t \in (0, \infty)$, the inequality

(2.11) $$\left( \int_t^\infty \left( \int_s^\infty u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy \right)^{\frac{1}{q}} \leq C \left( \int_t^\infty f(s)^p \, v(s) \, ds \right)^{\frac{1}{p}},$$

$$f \in \mathcal{M}^+(t, \infty)$$

Indeed: Applying changes of variables, we get that

$$\int_t^\infty \left( \int_s^\infty u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy$$

$$= \int_t^\infty \left( \int_s^\infty u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy$$

$$= \int_t^\infty \left( \int_s^{\tau} u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy$$

$$= \int_t^\infty \left( \int_s^{\tau} u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy$$

$$= \int_t^\infty \left( \int_s^{\tau} u(\tau) \left( \int_{t \wedge \tau}^\infty f(z) \, dz \right) a(s) \, ds \right)^q \, w(y) \, dy$$

and

$$\int_t^\infty f(s)^p \, v(s) \, ds = \int_0^\infty f(s+t)^p \, v(s+t) \, ds = \int_0^\infty f(s+t)^p \, v(s+t) \, ds.$$
Hence inequality (2.11) can be rewritten as follows:

\[
\left( \int_0^\infty \left( \int_0^t \left( \sup_{s \leq \tau} u_t(\tau) \left( \int_\tau^\infty f(z+tdz) \right) a_t(s) ds \right)^q w_t(y) dy \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty f(s+t)^p v_t(s) ds \right)^{\frac{1}{p}}, \quad f \in \mathcal{W}^+(t, \infty).
\]

It remains to note that the latter is equivalent to inequality (2.12).

Let \( p \leq q \). By [37, Theorem 3.3], we have for any \( t \in (0, \infty) \) that

\[
\sup_{f \in \mathcal{W}^+(0, \infty)} \left( \int_0^\infty f(s)^p v_t(s) ds \right)^{\frac{1}{p}} \geq \left( \int_0^\infty \left( \sup_{x \in (0, \infty)} \left( \int_0^y \Psi_t(x) x^{p-1} dw_t(z) \right)^{\frac{1}{p}} \left( \int_x^\infty a_t(s) ds \right)^q w_t(y) dy \right)^{\frac{1}{q}} dx \right)^{\frac{1}{p}}.
\]

where

\[
\Psi_t(x) := \int_0^x a_t(z) dz, \quad \psi_t(x) := \left( \int_x^\infty v_t(s) s^{1-p'} ds \right)^{-\frac{1}{p'}} v_t(x)^{1-p'}, \quad \Psi_t(x) := \left( \int_x^\infty v_t(s) s^{1-p'} ds \right)^{-\frac{1}{p'}}, \quad x \in (0, \infty).
\]

Using changes of variables, we get for any \( x > 0 \) that

\[
A_t(x) = \int_0^x a_t(z) dz = \int_x^{x+t} a_t(z) dz,
\]

\[
\psi_t(x) = \left( \int_x^\infty v(s+t)^{1-p'} ds \right)^{-\frac{1}{p'}} v(x+t)^{1-p'} = \left( \int_x^{x+t} v(s)^{1-p'} ds \right)^{-\frac{1}{p'}} v(x)^{1-p'},
\]

and

\[
\Psi_t(x) = \left( \int_x^{x+t} v(s)^{1-p'} ds \right)^{-\frac{1}{p'}},
\]

Since

\[
\sup_{y \geq 0} \left( \int_y^\infty \psi_t(x) x^{p-1} dw_t(z) \right)^{\frac{1}{p}} \left( \int_x^\infty a_t(s) ds \right)^q w_t(y) dy \left( \int_0^\infty w_t(y) dy \right)^{\frac{1}{q}} = \sup_{y \geq 0} \left( \int_y^\infty v(x+t) \left( \int_x^{x+t} a_t(s) ds \right)^q w(x+t) dy \right)^{\frac{1}{q}} \left( \int_0^\infty w(x+t) dx \right)^{\frac{1}{q}} = \sup_{y \geq 0} \left( \int_y^\infty v(x+t) \left( \int_x^{x+t} a_t(s) ds \right)^q w(x+t) dx \right)^{\frac{1}{q}} \left( \int_0^\infty w(x+t) dx \right)^{\frac{1}{q}}.
\]
\[
\begin{align*}
&= \sup_{y \in (0, \infty)} \left( \int_y^y V(x + t) \left( \int_{s+t}^{y+t} \left( \sup_{s \leq \tau} U(\tau) a(s) d\tau \right) \right)^{p'} \left( \int_y^{x+\infty} w(x + t) \, dx \right)^{-\frac{1}{q'}} \right) \\
&= \sup_{y \in (0, \infty)} \left( \int_t^{y+t} V(x) \left( \int_x^{y+t} \left( \sup_{s \leq \tau} U(\tau) a(s) d\tau \right) \right)^{p'} \left( \int_y^{x+\infty} w(x) \, dx \right)^{-\frac{1}{q'}} \right) \\
&= \sup_{y \in (t, \infty)} \left( \int_t^{y+t} V(x) \left( \int_x^{y+t} \left( \sup_{s \leq \tau} U(\tau) a(s) d\tau \right) \right)^{p'} \left( \int_y^{x+\infty} w(x) \, dx \right)^{-\frac{1}{q'}} \right),
\end{align*}
\]
\[
\begin{align*}
&= \sup_{y \in (0, \infty)} \left( \int_0^y \left( \int_y^y \left( \sup_{s \leq \tau} U(\tau + t) a(s + t) d\tau \right) \right)^{q} \left( \int_0^{1+\infty} w(z + t) \, dz \right)^{-\frac{1}{q'}} \right) \\
&= \sup_{y \in (0, \infty)} \left( \int_0^y \left( \int_y^y \left( \sup_{s \leq \tau} U(\tau) a(s) d\tau \right) \right)^{q} \left( \int_0^{1+\infty} w(z) \, dz \right)^{-\frac{1}{q'}} \right) \\
&= \sup_{y \in (t, \infty)} \left( \int_t^{y+t} \left( \int_x^{y+t} \left( \sup_{s \leq \tau} U(\tau) a(s) d\tau \right) \right)^{q} \left( \int_y^{x+\infty} w(x) \, dx \right)^{-\frac{1}{q'}} \right).
\end{align*}
\]
ON SOME RESTRICTED INEQUALITIES FOR $T_{ab}$ AND THEIR APPLICATIONS

\[
= \sup_{x \in (0, \infty)} \left( \int_{(0, \infty)} \left( \int_{t}^{x} a^{p'} d(-\sup_{s \leq t} \mathcal{U}(\tau)^{p'} \left( \int_{t}^{\tau} \mathcal{V}(y) dy \right)) \right)^{\frac{1}{p'}} \left( \int_{x}^{\infty} w(z) dz \right)^{\frac{1}{q'}} \right) \]

\[
= \sup_{x \in (0, \infty)} \left( \int_{(t, x]} \left( \int_{t}^{x} a^{p'} d(-\sup_{s \leq t} \mathcal{U}(\tau)^{p'} \left( \int_{t}^{\tau} \mathcal{V}(y) dy \right)) \right)^{\frac{1}{p'}} \left( \int_{x}^{\infty} w(z) dz \right)^{\frac{1}{q'}} \right) \]

\[
= \sup_{x \in (t, \infty)} \left( \int_{(t, x]} \left( \int_{t}^{x} a^{p'} d(-\sup_{s \leq t} \mathcal{U}(\tau)^{p'} \left( \int_{t}^{\tau} \mathcal{V}(y) dy \right)) \right)^{\frac{1}{p'}} \left( \int_{x}^{\infty} w(z) dz \right)^{\frac{1}{q'}} \right) ,
\]

combining, we arrive at

\[
\sup_{f \in \mathcal{W}^{*}(t, \infty)} \frac{\left( \int_{t}^{x} \left( \sup_{s \leq t} u(\tau) \left( \int_{t}^{x} f(z) dz \right) \right) a(s) ds \right)^{q} w(x) dy}{\left( \int_{t}^{x} f(s)^{p} v(s) ds \right)^{\frac{1}{q'}}} \approx \sup_{y \in (t, \infty)} \left( \int_{t}^{x} \mathcal{V}(x) \left( \int_{x}^{\infty} \left( \sup_{s \leq t} \mathcal{U}(\tau) a(s) ds \right)^{p'} \right) dx \right)^{\frac{1}{p'}} \left( \int_{x}^{\infty} w(x) dx \right)^{\frac{1}{q'}} \]

\[
+ \sup_{y \in (t, \infty)} \left( \int_{t}^{x} \mathcal{V}(x) dx \right)^{\frac{1}{p'}} \left( \int_{y}^{\infty} \left( \int_{y}^{x} \left( \sup_{s \leq t} \mathcal{U}(\tau) a(s) ds \right)^{q} w(z) dz \right) \right)^{\frac{1}{q'}} \]

\[
+ \sup_{x \in (t, \infty)} \left( \int_{t}^{x} d(-\sup_{s \leq t} \mathcal{U}(\tau)^{p'} \left( \int_{t}^{\tau} \mathcal{V}(y) dy \right)) \right)^{\frac{1}{p'}} \left( \int_{x}^{\infty} \left( \int_{s}^{\infty} a^{q} w(y) dy \right)^{\frac{1}{q'}} \right),
\]
Let $q < p$. By [37, Theorem 3.3], we have for any $t \in (0, \infty)$ that

$$
\sup_{f \in \mathcal{W}^1(0, \infty)} \left( \int_0^\infty \left( \int_0^t \left( \sup_{s \leq t} u_t(\tau) \left( \int_0^\infty f(s) \, ds \right) \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}}
$$

\[ = \left( \int_0^\infty \left( \int_0^t \Psi_t(x)^{-p} \psi_t(x) \left( \int_0^\infty \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_0^\infty \left( \int_0^t y \left( \sup_{s \leq t} u_t(\tau) \Psi_t(\tau)^2 \right) a_t(s) \, ds \right)^q \right) w_t(y) \, dy \right)^{\frac{1}{q}} \]
ON SOME RESTRICTED INEQUALITIES FOR $T_{ab}$ AND THEIR APPLICATIONS

\[ \left( \int_0^\infty \left( \int_0^\infty V(x+t) \left( \int_x^\infty \left( \sup_{s \leq \tau} U(\tau) a(s+t) ds \right)^p dx \right) \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z+t) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ = \left( \int_0^\infty \left( \int_0^\infty V(x+t) \left( \int_x^\infty \left( \sup_{s \leq \tau} U(\tau) a(s) ds \right) \right)^p dx \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z+t) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ = \left( \int_0^\infty \left( \int_0^\infty V(x) \left( \int_x^\infty \left( \sup_{s \leq \tau} U(\tau) a(s) ds \right) \right)^p dx \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ = \left( \int_t^\infty \left( \int_t^\infty V(x) \left( \int_x^\infty \left( \sup_{s \leq \tau} U(\tau) a(s) ds \right) \right)^p dx \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ \left( \int_0^\infty \left( \int_0^\infty d \left( -\left( \sup_{s \leq \tau} U(\tau) a(s) ds \right)^p \right) \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ = \left( \int_0^\infty \left( \int_0^\infty d \left( -\left( \sup_{s \leq \tau} U(\tau) a(s) ds \right)^p \right) \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ = \left( \int_0^\infty \left( \int_0^\infty d \left( -\left( \sup_{s \leq \tau} U(\tau) a(s) ds \right)^p \right) \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ = \left( \int_0^\infty \left( \int_0^\infty d \left( -\left( \sup_{s \leq \tau} U(\tau) a(s) ds \right)^p \right) \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ = \left( \int_0^\infty \left( \int_0^\infty d \left( -\left( \sup_{s \leq \tau} U(\tau) a(s) ds \right)^p \right) \right) \right)^{\frac{q}{p}} \cdot \left( \int_y^\infty \left( \int_0^\infty w(z) dz \right)^{\frac{p}{q}} w(y) dy \right)^{\frac{q}{p}} \]

\[ \sup_{f \in IP^p(t,\infty)} \left( \int_0^\infty \left( \sup_{s \leq \tau} u(\tau) \left( \int_0^\infty f(z) dz \right) \right) \left( \int_0^\infty f(s)^p v(s) ds \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \]

\[ \approx \left( \int_t^\infty \left( \int_t^\infty V(x) dx \right)^{\frac{p+1}{p}} V(y) \left( \int_y^\infty \left( \int_x^\infty \left( \sup_{s \leq \tau} U(\tau) a(s) ds \right)^q w(z) dz \right) \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \]

\[ + \left( \int_t^\infty \left( \int_t^\infty V(x) \left( \int_x^\infty \left( \sup_{s \leq \tau} U(\tau) a(s) ds \right) \right)^p dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \]
By duality, on using Fubini’s Theorem, we have that

\[
\lim_{s \to \infty} \sup_{s \leq T} \left( \int_{t}^{\infty} \left( \int_{t}^{x} a^{-1} \, dy \right) \right)^{\frac{p}{(p-1)q}} \left( \int_{t}^{\infty} w(y) \, dy \right)^{\frac{q}{p}} = \left( \int_{t}^{\infty} w(y) \, dy \right)^{\frac{q}{p}}.
\]

The proof is completed.

\[\square\]

3. Characterization of the restricted inequality

We start this section with some historical remarks concerning restricted inequalities related to the operator \( T_{a,b} \).

Note that the inequality

\[
\|T_{u,b}f\|_{q,W,(0,\infty)} \leq C\|f\|_{p,W,(0,\infty)}, \quad f \in \mathcal{W}^{1-}(0,\infty)
\]

was characterized in [24, Theorem 3.5] under condition

\[
\sup_{t \in (0,\infty)} \frac{u(t)}{B(t)} \int_{0}^{t} b(\tau) \, d\tau < \infty.
\]

However, the case when \( 0 < p \leq 1 < q < \infty \) was not considered in [24]. It is also worth to mention that in the case when \( 1 < p < \infty, 0 < q < \infty, q \neq 1 \) [24, Theorem 3.5] contains only discrete condition. In [25] the new reduction theorem was obtained when \( 0 < p \leq 1 \), and this technique allowed to characterize inequality (3.1) when \( b \equiv 1 \), and in the case when \( 0 < q < p \leq 1 \), [25] contains only discrete condition. The complete characterizations of inequality (3.1) for \( 0 < q \leq \infty, 0 < p \leq \infty \) were given in [18] and [36]. Using the results in [44–47], another characterization of (3.1) was obtained in [55] and [49].

Now we present characterization of inequality (1.3).

Denote the best constant in inequality (1.3) by \( K \), that is,

\[
K := \sup_{f \in \mathbb{R}^+} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} [T_{u,b}f^+(t)] \, dt \right)^{\frac{q}{p}} \, \varphi(x) \, dx \right)^{\frac{1}{q}} = \sup_{f \in \mathbb{R}^+} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} [f^+(t)] \, dt \right)^{\frac{q}{p}} \, \varphi(x) \, dx \right)^{\frac{1}{q}}.
\]

The following two reduction lemmas hold true.

**Lemma 3.1.** Let \( 1 < r < q < \infty, 0 < p \leq \infty, 0 < m < \infty \) and \( b \in \mathcal{W}(0,\infty) \) be such that the function \( B(t) \) satisfies \( 0 < B(t) < \infty \) for every \( t \in (0,\infty) \). Assume that \( u \in \mathcal{W}(0,\infty) \cap C(0,\infty) \) and \( v, w \in \mathcal{W}(0,\infty) \). Then

\[
K = \sup_{g \in \mathbb{R}^+} \frac{1}{\|g\|_{\mathcal{W},0,\infty}^{\frac{1}{q}}} \sup_{h \in \mathbb{R}^+} \frac{1}{\|\varphi\|_{\mathcal{W},0,\infty}^{\frac{1}{q}}} \frac{\int_{0}^{\infty} f^+(y) b(y) \int_{0}^{\infty} \varphi(x) \frac{1}{\varphi(x)} \, dx \, dy}{\|f\|_{\mathcal{W},0,\infty}^{\frac{1}{q}} \varphi(x) \, dx}.
\]

**Proof.** By duality, on using Fubini’s Theorem, we have that

\[
K = \sup_{f \in \mathbb{R}^+} \frac{1}{\|f\|_{\mathcal{W},0,\infty}^{\frac{1}{q}}} \left( \sup_{g \in \mathbb{R}^+} \frac{1}{\|g\|_{\mathcal{W},0,\infty}^{\frac{1}{q}}} \int_{0}^{\infty} \left( \int_{0}^{\infty} [T_{u,b}f^+(t)] \, dt \right) g(x) \, dx \right)^{\frac{1}{q}}.
\]
On using Theorem 2.3, we arrive at

\[ K = \sup_{f \in \mathfrak{N}^r} \left\{ \frac{1}{\|f\|_{\mathfrak{L}(p,m,v)}} \sup_{g \in \mathfrak{N}^r} \sup_{h \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \frac{1}{\|g\|_{\frac{1}{r} \mathfrak{N}^r_{\infty}\left(\frac{q}{q-r},w\right),\infty}(0,\infty)} \sup_{s \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \sup_{\varphi \in \mathfrak{W}(0,\infty) \cap \mathcal{C}(0,\infty)} \int_0^\infty \left( \left( \int_0^\infty \varphi(s) \frac{B(s)}{s} ds \right) v_{2}(t) dt \right)^{\frac{1}{q}} \right\}. \]

By Fubini’s Theorem, interchanging the suprema, we arrive at

\[ K = \sup_{f \in \mathfrak{N}^r} \left\{ \frac{1}{\|f\|_{\mathfrak{L}(p,m,v)}} \sup_{g \in \mathfrak{N}^r} \sup_{h \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \frac{1}{\|g\|_{\frac{1}{r} \mathfrak{N}^r_{\infty}\left(\frac{q}{q-r},w\right),\infty}(0,\infty)} \sup_{s \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \sup_{\varphi \in \mathfrak{W}(0,\infty) \cap \mathcal{C}(0,\infty)} \frac{1}{\|\varphi\|_{\frac{1}{r} \mathfrak{N}^r_{\infty}\left(\frac{q}{q-r},w\right),\infty}(0,\infty)} \int_0^\infty \left( \left( \int_0^\infty \varphi(s) \frac{B(s)}{s} ds \right) v_{2}(t) dt \right)^{\frac{1}{q}} \right\}. \]

By applying (3.2), where \( v_0 \) and \( v_1 \) are defined by (2.2) and (2.3), respectively. Then

\[ K \approx A + B, \]

where

\[ A := \sup_{g \in \mathfrak{N}^r} \left\{ \frac{1}{\|g\|_{\frac{1}{r} \mathfrak{N}^r_{\infty}\left(\frac{q}{q-r},w\right),\infty}(0,\infty)} \sup_{h \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \sup_{s \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \left( \int_0^\infty \left( \int_t^\infty \varphi(s) \frac{B(s)}{s} ds \right) \left( \int_0^\infty \frac{v_{2}(t) dt}{v_{2}(t) dt} \right)^{\frac{1}{q}} \right) \right\}. \]

and

\[ B := \sup_{g \in \mathfrak{N}^r} \left\{ \frac{1}{\|g\|_{\frac{1}{r} \mathfrak{N}^r_{\infty}\left(\frac{q}{q-r},w\right),\infty}(0,\infty)} \sup_{h \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \sup_{s \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \left( \int_0^\infty \left( \int_t^\infty \varphi(s) \frac{B(s)}{s} ds \right) \left( \int_0^\infty \frac{v_{2}(t) dt}{v_{2}(t) dt} \right)^{\frac{1}{q}} \right) \right\}. \]

**Proof.** By Lemma 3.1 and Theorem 2.1, we have that

\[ K \approx \sup_{g \in \mathfrak{N}^r} \left\{ \frac{1}{\|g\|_{\frac{1}{r} \mathfrak{N}^r_{\infty}\left(\frac{q}{q-r},w\right),\infty}(0,\infty)} \sup_{h \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \sup_{s \in \mathcal{L}^r_{\infty}((0,\infty),\mathcal{F})} \left( \int_0^\infty \left( \int_t^\infty \varphi(s) \frac{B(s)}{s} ds \right) \left( \int_0^\infty \frac{v_{2}(t) dt}{v_{2}(t) dt} \right)^{\frac{1}{q}} \right) \right\}. \]

Since

\[ \int_s^t b(y) \left( \int_y^\infty \varphi(x) \frac{u(x)}{B(x)} dx \right) dy = \frac{1}{s} \int_0^s b(y) \left( \int_y^\infty \varphi(x) \frac{u(x)}{B(x)} dx \right) dy + \frac{1}{s} \int_0^s b(y) dy \int_s^\infty \varphi(x) \frac{u(x)}{B(x)} dx, \]
we arrive at

\[ K \approx \sup_{g \in \mathfrak{M}^+} \frac{1}{\|g\|_{(0, \infty), \mathfrak{M}^+}^T} \sup_{h \colon \int_0^h f \leq \int_0^\infty (f^\prime)^\prime g} \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) \frac{u(x)}{B(x)} dx \right)^\prime ds \right)^\frac{1}{\nu} v_2(t) dt \right)^\frac{1}{\nu} + \sup_{g \in \mathfrak{M}^+} \frac{1}{\|g\|_{(0, \infty), \mathfrak{M}^+}^T} \sup_{h \colon \int_0^h f \leq \int_0^\infty (f^\prime)^\prime g} \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) dx \right)^\prime ds \right)^\frac{1}{\nu} v_2(t) dt \right)^\frac{1}{\nu} .

Observe that the inequality

\[ \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) dx \right)^\prime \frac{u(x)}{B(x)} dx \right)^\prime ds \right)^\frac{1}{\nu} v_2(t) dt \leq c \left( \int_0^\infty \varphi(x)^\prime h(x)^{1-\prime} dx \right)^\frac{1}{\nu} \]

holds true for all \( h \in \mathfrak{M}^+ \) if and only if the inequality

\[ \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) dx \right)^\prime \frac{B(s)}{B(x)} dx \right)^\prime ds \right)^\frac{1}{\nu} v_2(t) dt \leq c \left( \int_0^\infty \varphi(x)^\prime \frac{B(x)}{u(x)} h(x)^{1-\prime} dx \right)^\frac{1}{\nu} \]

holds for all \( h \in \mathfrak{M}^+ \).

Similarly, the inequality

\[ \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) dx \right)^\prime ds \right)^\frac{1}{\nu} v_2(t) dt \right)^\frac{1}{\nu} \leq c \left( \int_0^\infty \varphi(x)^\prime h(x)^{1-\prime} dx \right)^\frac{1}{\nu} \]

holds true for all \( h \in \mathfrak{M}^+ \) if and only if the inequality

\[ \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) dx \right)^\prime \frac{1}{s^\nu} ds \right)^\frac{1}{\nu} v_2(t) dt \right)^\frac{1}{\nu} \leq c \left( \int_0^\infty \varphi(x)^\prime \frac{1}{u(x)} h(x)^{1-\prime} dx \right)^\frac{1}{\nu} \]

holds for all \( h \in \mathfrak{M}^+ \).

Thus,

\[ K \approx \sup_{g \in \mathfrak{M}^+} \frac{1}{\|g\|_{(0, \infty), \mathfrak{M}^+}^T} \sup_{h \colon \int_0^h f \leq \int_0^\infty (f^\prime)^\prime g} \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) dx \right)^\prime \frac{B(s)}{B(x)} dx \right)^\prime ds \right)^\frac{1}{\nu} v_2(t) dt \]

\[ + \sup_{g \in \mathfrak{M}^+} \frac{1}{\|g\|_{(0, \infty), \mathfrak{M}^+}^T} \sup_{h \colon \int_0^h f \leq \int_0^\infty (f^\prime)^\prime g} \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty \varphi(x) dx \right)^\prime \frac{1}{s^\nu} ds \right)^\frac{1}{\nu} v_2(t) dt \right)^\frac{1}{\nu} = A + B \]

holds.

\[ \square \]

**Theorem 3.3.** Let \( 1 < m < p \leq r < q < \infty \) and \( b \in \mathcal{W}(0, \infty) \cap \mathfrak{M}^+((0, \infty); \downarrow) \) be such that the function \( B(t) \) satisfies \( 0 < B(t) < \infty \) for every \( t \in (0, \infty) \). Suppose that \( u \in \mathcal{W}(0, \infty) \cap C(0, \infty) \), \( v \in \mathcal{W}_{m,p}(0, \infty) \) and \( w \in \mathcal{W}(0, \infty) \). Suppose
that

\[ 0 < \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds < \infty, \quad t \in (0, \infty), \]

where \( v_2 \) is defined by (3.2). Then

\[ K \approx \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}} = \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{u(\tau)}{B(\tau)} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} w(y) \,dy \right)^{\frac{1}{\rho'}}. \]

Interchanging the suprema, by duality, we get that

\[ \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \approx \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} w(y) \,dy \right)^{\frac{1}{\rho'}}. \]

We estimate \( A \). By Theorem 2.7, (a), we have that

\[ A \approx \sup_{t \in (0, \infty)} \frac{1}{\|g\|^{\frac{1}{\rho'}}} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}} = \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}}. \]

Interchanging the suprema, we get that

\[ A \approx \sup_{t \in (0, \infty)} \frac{1}{\|g\|^{\frac{1}{\rho'}}} \sup_{h \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}} \sup_{h \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}}. \]

By Theorem 2.3, we have that

\[ A \approx \sup_{t \in (0, \infty)} \frac{1}{\|g\|^{\frac{1}{\rho'}}} \sup_{h \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}} \sup_{h \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}}. \]

Applying Fubini’s Theorem, interchanging the suprema, by duality, we get that

\[ A \approx \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}} \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \,dy \right)^{\frac{\mu}{\rho'}} v_2(s) \,ds \right)^{\frac{1}{\mu'}}. \]
Since
\[
\sup_{x \leq t} \chi_{(t,\infty)}(\tau) \left( \frac{u(\tau)}{B(\tau)} \right)^r = \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r,
\]
when \(0 < x \leq t\), we get that
\[
A \approx \sup_{t \in (0,\infty)} \left( \int_0^t \left( \int_s^\infty \frac{B(y)}{y} \right)^{\rho^*} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left( \int_0^t \left( \int_0^\infty \left( \sup_{x \leq \tau} \chi_{(t,\infty)}(\tau) \right) dx \right)^{\frac{q}{2}} w(y) dy \right)^{\frac{1}{q}}
+ \sup_{t \in (0,\infty)} \left( \int_0^t \left( \int_s^\infty \frac{B(y)}{y} \right)^{\rho^*} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left( \int_t^\infty \left( \int_0^\infty \left( \sup_{t \leq \tau} \chi_{(t,\infty)}(\tau) \right) dx \right)^{\frac{q}{2}} w(y) dy \right)^{\frac{1}{q}}
+ \sup_{t \in (0,\infty)} \left( \int_0^t \left( \int_s^\infty \frac{B(y)}{y} \right)^{\rho^*} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left( \int_t^\infty \left( \int_t^\infty \left( \sup_{t \leq \tau} \chi_{(t,\infty)}(\tau) \right) dx \right)^{\frac{q}{2}} w(y) dy \right)^{\frac{1}{q}}
+ \sup_{t \in (0,\infty)} \left( \int_0^t \left( \int_s^\infty \frac{B(y)}{y} \right)^{\rho^*} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left( \int_t^\infty \left( \int_t^\infty \left( \sup_{t \leq \tau} \chi_{(t,\infty)}(\tau) \right) dx \right)^{\frac{q}{2}} w(y) dy \right)^{\frac{1}{q}}.
\]
To estimate \(B\), by Theorem 2.6, (a), we have that
\[
\sup_{\varphi \in \mathcal{Y}^r} \frac{1}{\|\varphi\|_{L^q,r,1-r,\rho^*,(0,\infty)}} \left( \int_0^\infty \left( \int_0^\infty \varphi(x) dx \right)^{\rho^*} \frac{dx}{s^{\rho^*}} \right)^{\frac{1}{\rho^*}} v_2(t) dt \approx \sup_{t \in (0,\infty)} \left( \int_0^t v_2(\tau) d\tau \right)^{\frac{1}{\rho^*}} \left( \int_0^\infty \frac{dx}{s^{\rho^*}} \right)^{\frac{1}{\rho^*}} \left( \int_0^\infty h(x) u(x)^r dx \right)^{\frac{1}{\rho^*}}
+ \sup_{t \in (0,\infty)} \left( \int_t^\infty \left( \int_s^\infty \frac{dy}{y^{\rho^*}} \right)^{\frac{1}{\rho^*}} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left( \int_0^t h(x) u(x)^r dx \right)^{\frac{1}{\rho^*}}
+ \sup_{t \in (0,\infty)} \left( \int_t^\infty \left( \frac{1}{s + t} \right)^{\frac{1}{\rho^*}} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left( \int_0^t h(x) u(x)^r dx \right)^{\frac{1}{\rho^*}}.
\]
Hence, interchanging the suprema, we get that
\[
B \approx \sup_{g \in \mathcal{Y}^r} \frac{1}{\|g\|_{L^q,r,1-r,\rho^*,(0,\infty)}} \sup_{h \in (0,\infty)} \left( \int_0^\infty \left( \frac{1}{s + t} \right)^{\frac{1}{\rho^*}} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left( \int_0^t h(x) u(x)^r dx \right)^{\frac{1}{\rho^*}}
= \sup_{g \in \mathcal{Y}^r} \frac{1}{\|g\|_{L^q,r,1-r,\rho^*,(0,\infty)}} \sup_{t \in (0,\infty)} \left( \int_0^\infty \left( \frac{1}{s + t} \right)^{\frac{1}{\rho^*}} v_2(s) ds \right)^{\frac{1}{\rho^*}} \left\{ \sup_{h \in (0,\infty)} \left( \int_0^t h(x) u(x)^r \chi_{(0,\infty)}(x) dx \right) \right\}^{\frac{1}{\rho^*}}.
\]
By Theorem 2.3, we have that
\[
B \approx \sup_{g \in \mathbb{R}^+} \frac{1}{||g||_{\frac{1}{r}}} \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \frac{1}{s + t} \right)^{\frac{m}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \int_x^\infty g \left( \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) \right) dx \right)^{\frac{1}{r}} \right)^{\frac{1}{q}}.
\]

By Fubini’s Theorem, interchanging the suprema, by duality we get that
\[
B \approx \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \frac{1}{s + t} \right)^{\frac{m}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \int_0^\infty \left( \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) \right) dx \right)^{\frac{1}{r}} \right)^{\frac{1}{q}} \sup_{g \in \mathbb{R}^+} \frac{1}{||g||_{\frac{1}{r}}} \left( \sup_{t \in (0, \infty)} \int_0^\infty g(y) \int_0^\infty \left( \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) \right) dx \right)^{\frac{1}{q}} dy.
\]

Since
\[
\sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) = \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau)
\]
for $0 < x \leq t$, we arrive at
\[
B \approx \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \frac{1}{s + t} \right)^{\frac{m}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \int_0^\infty \left( \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) \right) dx \right)^{\frac{1}{r}} \right)^{\frac{1}{q}} \left( \int_0^\infty (w(y) dy)^{\frac{1}{q}} \right)^{\frac{1}{r}} + \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \frac{1}{s + t} \right)^{\frac{m}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \int_0^\infty (w(y) dy)^{\frac{1}{q}} \right)^{\frac{1}{r}} \right)^{\frac{1}{q}}.
\]

Combining the estimates for $A$ and $B$, we obtain that
\[
K \approx \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{p_x} dy \right)^{\frac{1}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) \right) dx \right)^{\frac{1}{r}} \left( \int_0^\infty (w(y) dy)^{\frac{1}{q}} \right)^{\frac{1}{r}} + \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{p_x} dy \right)^{\frac{1}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \int_0^\infty (w(y) dy)^{\frac{1}{q}} \right)^{\frac{1}{r}} \right)^{\frac{1}{q}} + \sup_{t \in (0, \infty)} \left( \int_0^t \left( \frac{1}{s + t} \right)^{\frac{m}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) \right) dx \right)^{\frac{1}{r}} \left( \int_0^\infty (w(y) dy)^{\frac{1}{q}} \right)^{\frac{1}{r}} + \sup_{t \in (0, \infty)} \left( \int_0^t \left( \frac{1}{s + t} \right)^{\frac{m}{r}} v_2(s) ds \right)^{\frac{1}{r}} \left( \int_0^\infty \left( \sup_{x \leq \tau} (\tau)^{p_x} \chi_{(0, \tau)}(\tau) \right) dx \right)^{\frac{1}{r}} \left( \int_0^\infty (w(y) dy)^{\frac{1}{q}} \right)^{\frac{1}{r}}.
\]

\[\square\]

**Theorem 3.4.** Let $1 < m \leq r < \min\{p, q\} < \infty$ and $b \in \mathcal{W}(0, \infty) \cap \mathcal{M}^+((0, \infty); \downarrow)$ be such that the function $B(t)$ satisfies $0 < B(t) < \infty$ for every $t \in (0, \infty)$. Assume that $u \in \mathcal{W}(0, \infty) \cap \mathcal{C}(0, \infty)$, $v \in \mathcal{W}_{m,p}(0, \infty)$ and $w \in \mathcal{W}(0, \infty)$. Suppose that
\[
0 < \int_0^\infty \left( \int_s^t \left( \frac{B(y)}{y} \right)^{p_x} dy \right)^{\frac{1}{r}} v_2(s) ds < \infty, \quad t \in (0, \infty),
\]
\[
\int_0^t \frac{s^{m\left(1-\frac{1}{2}\right)} v_0(s)}{v_1(s)^{m+1}} ds < \infty, \quad \int_t^{\infty} \frac{s^{m\left(1-\frac{1}{2}\right)} v_0(s)}{v_1(s)^{m+1}} ds < \infty, \quad t \in (0, \infty),
\]
and
\[
\int_0^1 \frac{s^{m\left(1-\frac{1}{2}\right)} v_0(s)}{v_1(s)^{m+1}} ds = \int_1^{\infty} \frac{s^{m\left(1-\frac{1}{2}\right)} v_0(s)}{v_1(s)^{m+1}} ds = \infty,
\]
where $v_0$, $v_1$ and $v_2$ are defined by (2.2), (2.3) and (3.2), respectively. Denote by
\[
V_2(t) := \left( \int_0^\infty \frac{s^{\frac{m}{r}} v_0(s)}{v_1(s)^{m+1}} ds \right)^{\frac{1}{r}}, \quad V_3(t) := \left( \int_0^\infty \left( \frac{t}{y + t} \right)^{\frac{m}{r}} v_2(y) dy \right)^{\frac{1}{r}}, \quad 0 < t < \infty,
\]
\[ \mathcal{B}(t, s) = \left( \int_t^s \left( \frac{B(y)}{y} \right)^{\rho'} \frac{dy}{y} \right) \frac{\mu(t)}{\mu(t) + \rho} \left( \frac{B(s)}{s} \right)^{\rho'}, \quad 0 < t < s < \infty, \]

and

\[ \mathcal{K}(\tau, t) = u(\tau)(\tau + t)^{-\frac{\tau}{\mu(t)}}, \quad 0 < \tau, t < \infty. \]

(i) If \( p \leq q \), then

\[ K \approx \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \frac{dy}{y} \right) \frac{\mu'(t)}{\mu'(t) + \rho} v_2(s) ds \right)^\frac{1}{\rho} \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \right) \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \frac{dy}{y} \right) \frac{\mu'(t)}{\mu'(t) + \rho} v_2(s) ds \right)^\frac{1}{\rho} \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \sup_{x \in (t, \infty)} \left( \int_x^\infty \mathcal{B}(t, y) \left( \int_y^\infty \left( \frac{u(t)}{B(t)} \right)' ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \int_y^\infty w(y) dy \right)^\frac{1}{\rho} \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \sup_{x \in (t, \infty)} \left( \int_x^\infty \mathcal{B}(t, y) \left( \int_y^\infty \left( \frac{u(t)}{B(t)} \right)' ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \int_y^\infty w(y) dy \right)^\frac{1}{\rho} \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \sup_{x \in (t, \infty)} \left( \int_x^\infty \mathcal{B}(t, y) \left( \int_y^\infty \left( \frac{u(t)}{B(t)} \right)' ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \int_y^\infty w(y) dy \right)^\frac{1}{\rho} \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_0^{\infty} \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \left( \sup_{x \in (t, \infty)} \frac{u(t)}{B(t)} \left( \int_x^\infty \mathcal{B}(t, y) \left( \int_y^\infty \left( \frac{u(t)}{B(t)} \right)' ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \int_y^\infty w(y) dy \right)^\frac{1}{\rho} \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_0^{\infty} \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \left( \sup_{x \in (t, \infty)} \frac{u(t)}{B(t)} \left( \int_x^\infty \mathcal{B}(t, y) \left( \int_y^\infty \left( \frac{u(t)}{B(t)} \right)' ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \int_y^\infty w(y) dy \right)^\frac{1}{\rho} \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_0^{\infty} \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \left( \sup_{x \in (t, \infty)} \frac{u(t)}{B(t)} \left( \int_x^\infty \mathcal{B}(t, y) \left( \int_y^\infty \left( \frac{u(t)}{B(t)} \right)' ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \int_y^\infty w(y) dy \right)^\frac{1}{\rho} \right)^\frac{1}{\rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_0^{\infty} \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \left( \sup_{x \in (t, \infty)} \frac{u(t)}{B(t)} \left( \int_x^\infty \mathcal{B}(t, y) \left( \int_y^\infty \left( \frac{u(t)}{B(t)} \right)' ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \int_y^\infty w(y) dy \right)^\frac{1}{\rho} \right)^\frac{1}{\rho} \]

(ii) If \( q < p \), then

\[ K \approx \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \frac{dy}{y} \right) \frac{\mu'(t)}{\mu'(t) + \rho} v_2(s) ds \right)^\frac{1}{\rho} \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \right)^\frac{\mu(t)}{\mu(t) + \rho} \]

\[ + \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^t \left( \frac{B(y)}{y} \right)^{\rho'} \frac{dy}{y} \right) \frac{\mu'(t)}{\mu'(t) + \rho} v_2(s) ds \right)^\frac{1}{\rho} \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \right)^\frac{\mu(t)}{\mu(t) + \rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_t^\tau \mathcal{B}(t, s) ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \right)^\frac{\mu(t)}{\mu(t) + \rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_t^\tau \mathcal{B}(t, s) ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \right)^\frac{\mu(t)}{\mu(t) + \rho} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_t^\tau \mathcal{B}(t, s) ds \right)^\frac{\mu(t)}{\mu(t) + \rho} \left( \sup_{t \leq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{\frac{\tau}{\mu(t)}} w(y) dy \right)^\frac{1}{\rho} \right)^\frac{\mu(t)}{\mu(t) + \rho} \]
ON SOME RESTRICTED INEQUALITIES FOR $T_{ab}$ AND THEIR APPLICATIONS

\begin{align*}
+ \sup_{r \in (0, \infty)} V_2(t) \left( \int_t^\infty \left( \int_t^\tau \mathcal{B}(t, s) ds \right)^{\frac{\rho + 1}{p - \rho}} \mathcal{B}(t, \tau) \left( \int_\tau^\infty \left( \int_s^\infty \frac{u(y)}{B(y)} dy \right)^{\frac{\rho'}{p'}} ds \right)^{\frac{m}{p'}} \frac{w(\tau) d\tau}{\tau^\frac{1}{q}} \right) \frac{\tau^{\frac{1}{q}}}{\tau^\frac{1}{q}} \\
+ \sup_{r \in (0, \infty)} V_2(t) \left( \int_t^\infty \left( \int_t^\tau \mathcal{B}(t, s) \left( \int_s^\infty \sup_{y \leq t} \frac{u(y)}{B(y)} dy \right)^{\frac{\rho'}{p'}} ds \right)^{\frac{m}{p'}} \frac{w(x) dx}{x^\frac{1}{q}} \right) \frac{\tau^{\frac{1}{q}}}{\tau^\frac{1}{q}} \\
+ \sup_{r \in (0, \infty)} V_2(t) \left( \int_t^\infty \left( \int_t^\tau \mathcal{B}(t, s) \int_s^\infty \left( \int_t^\tau \mathcal{B}(t, s) ds \right)^{\frac{m}{p'}} \frac{w(x) dx}{x^\frac{1}{q}} \right) \frac{\tau^{\frac{1}{q}}}{\tau^\frac{1}{q}} \right) \frac{\tau^{\frac{1}{q}}}{\tau^\frac{1}{q}} \\
+ \sup_{r \in (0, \infty)} V_2(t) \left( \int_t^\infty \left( \int_t^\tau \mathcal{B}(t, s) \int_s^\infty \left( \int_t^\tau \mathcal{B}(t, s) ds \right)^{\frac{m}{p'}} \frac{w(x) dx}{x^\frac{1}{q}} \right) \frac{\tau^{\frac{1}{q}}}{\tau^\frac{1}{q}} \right) \frac{\tau^{\frac{1}{q}}}{\tau^\frac{1}{q}} \\
\end{align*}

Proof. Recall that, by Lemma 3.2, we know that

$K \approx A + B.$

We estimate $A$. By Theorem 2.7, (b), we have that

\[
\sup_{\varphi \in \Omega^*} \left( \int_0^\infty \left( \int_s^\infty \varphi(x) dx \right)^{\rho'} \left( \frac{B(s)}{s} \right)^{\rho'} ds \right)^{\frac{m}{p'}} \frac{v_2(t) dt}{t^\frac{1}{q}}
\]

\[
\approx \sup_{r \in (0, \infty)} \left( \int_0^\infty \left( \int_s^\infty \frac{B(y)}{y} dy \right)^{\rho'} v_2(s) ds \right)^{\frac{m}{p'}} \left( \int_0^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^{\rho'} dx \right)^{\frac{1}{p'}}
\]

\[
+ \sup_{r \in (0, \infty)} \left( \int_0^\infty \left( \frac{B(s)}{s} \right)^{\rho'} \frac{v_2(s) ds}{s^{\rho'}} \right)^{\frac{m}{p'}} \left( \int_0^\infty \frac{h(x)}{B(x)} \left( \frac{u(x)}{B(x)} \right)^{\rho'} dx \right)^{\frac{1}{p'}} ds
\]

\[
= I + II.
\]

Consequently, we get that

$A \approx \sup_{g \in \Omega^*} \left( \int_0^\infty \left( \int_0^h \left( \frac{B(y)}{y} \right)^{\rho'} v_2(s) ds \right)^{\frac{m}{p'}} \left( \int_0^\infty \frac{h(x)}{B(x)} \left( \frac{u(x)}{B(x)} \right)^{\rho'} dx \right)^{\frac{1}{p'}} \frac{[I + II] ds}{s^{\rho'}} \right)^{\frac{1}{p'}}
$

It has been already shown in the proof of Theorem 3.3 that

$A_1 \approx \sup_{r \in (0, \infty)} \left( \int_0^\infty \left( \int_0^\infty \left( \frac{B(y)}{y} \right)^{\rho'} v_2(s) ds \right)^{\frac{m}{p'}} \left( \int_0^\infty \frac{h(x)}{B(x)} \left( \frac{u(x)}{B(x)} \right)^{\rho'} dx \right)^{\frac{1}{p'}} \frac{[I + II] ds}{s^{\rho'}} \right)^{\frac{1}{p'}}$
\[ + \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_0^r \frac{B(y)}{y} \right)^p dy \right)^{\frac{1}{p}} \int_t^\infty \left( \int_t^\infty \left( \sup_{x \leq r} \frac{u(\tau)}{B(\tau)} \right) dx \right)^{\frac{1}{p}} w(y) dy \right]^{\frac{1}{p}}. \]

Interchanging the suprema, we get that
\[
A_2 = \sup_{g \in \mathbb{R}^+} \frac{1}{\|g\|^{\frac{1}{p}}} \sup_{t \in (0, \infty)} V_2(t) \left( \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \right)^{\frac{p}{n}} ds \left( \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \right)^{\frac{1}{r}}. \]

By duality (recall that \( p/(p-r) = (p/r)' \)), we have that
\[
\left( \int_t^\infty B(t, s) \left( \int_s^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \right)^{\frac{p}{n}} ds \right)^{\frac{1}{r}} = \left( \sup_{\psi \in \mathbb{R}^+(t, \infty)} \frac{\int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \psi(s) B(t, s) ds}{\int_t^\infty \psi(s) \left( \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \right)^{\frac{1}{r}} B(t, s) ds} \right)^{\frac{1}{r}}. \]

Thus
\[
A_2 = \sup_{g \in \mathbb{R}^+} \frac{1}{\|g\|^{\frac{1}{p}}} \sup_{t \in (0, \infty)} V_2(t) \left( \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \right)^{\frac{p}{n}} ds \left( \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \right)^{\frac{1}{r}}. \]

By Fubini’s Theorem, interchanging the suprema, we get that
\[
A_2 = \sup_{g \in \mathbb{R}^+} \frac{1}{\|g\|^{\frac{1}{p}}} \sup_{t \in (0, \infty)} V_2(t) \left( \sup_{\psi \in \mathbb{R}^+(t, \infty)} \int_t^\infty \psi(s) B(t, s) ds \right)^{\frac{1}{r}} \sup_{h \in \mathbb{R}^+} \left( \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \right)^{\frac{p}{n}} \int_t^\infty \psi(s) B(t, s) ds dx \right)^{\frac{1}{r}}. \]

By Theorem 2.3, we have that
\[
\sup_{h \in \mathbb{R}^+} \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \int_t^\infty \psi(s) B(t, s) ds dx = \sup_{h \in \mathbb{R}^+} \int_0^h \chi(t, \infty) \left( \frac{u(x)}{B(x)} \right)^r dx \int_t^\infty \psi(s) B(t, s) ds dx
\]
\[
= \int_0^\infty \left( \int_0^h g(z) dz \right) \sup_{x \leq t} \chi(t, \infty) \left( \frac{u(x)}{B(x)} \right)^r \int_t^\infty \psi(s) B(t, s) ds ds dx
\]
\[
= \int_t^\infty \left( \int_x^\infty g(z) dz \right) \sup_{x \leq t} \chi(t, \infty) \left( \frac{u(x)}{B(x)} \right)^r \int_t^\infty \psi(s) B(t, s) ds ds dx
\]
\[
+ \int_t^\infty \left( \int_x^\infty g(z) dz \right) \sup_{x \leq t} \chi(t, \infty) \left( \frac{u(x)}{B(x)} \right)^r \int_t^\infty \psi(s) B(t, s) ds ds dx.
\]

Since
\[
\sup_{x \leq t} \chi(t, \infty) \left( \frac{u(x)}{B(x)} \right)^r \int_t^\infty \psi(s) B(t, s) ds = \sup_{x \leq t} \left( \frac{u(x)}{B(x)} \right)^r \int_t^\infty \psi(s) B(t, s) ds,
\]
when \( 0 < x \leq t \), by Fubini’s Theorem, we get that
\[
\sup_{h \in \mathbb{R}^+} \int_t^\infty h(x) \left( \frac{u(x)}{B(x)} \right)^r dx \int_t^\infty \psi(s) B(t, s) ds dx
\]
Interchanging the suprema, we get that

\[ = \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau \left( \int_x^\infty g(z) dz \right) dx \]

\[ + \int_t^\tau \left( \int_x^\infty g(z) dz \right) \sup_{x \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds d\tau \]

\[ = \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) dz \]

\[ + \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) \frac{z}{z + t} dz \]

\[ \approx \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) \frac{z}{z + t} dz \]

Thus

\[ A_2 \approx \sup_{t \leq \tau} V_2(t) \left( \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) \frac{z}{z + t} dz \right)^\frac{1}{r} \]

\[ + \sup_{t \leq \tau} V_2(t) \left( \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) \frac{z}{z + t} dz \right)^\frac{1}{r} \]

Interchanging the suprema, we get that

\[ A_2 \approx \sup_{t \leq \tau} V_2(t) \left( \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) \frac{z}{z + t} dz \right)^\frac{1}{r} \]

\[ + \sup_{t \leq \tau} V_2(t) \left( \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) \frac{z}{z + t} dz \right)^\frac{1}{r} \]

\[ \approx \sup_{t \leq \tau} V_2(t) \left( \sup_{t \leq \tau} \left( \frac{u(\tau)}{B(\tau)} \right)^r \int_t^\tau \psi(s) B(t, s) ds \int_0^\tau g(z) \frac{z}{z + t} dz \right)^\frac{1}{r} \]

By duality, we arrive at

\[ A_2 \approx \sup_{t \leq \tau} V_2(t) \left( \int_t^\tau \psi(s) B(t, s) ds \right)^\frac{1}{r} \left( \int_0^\tau \left( \frac{v(t)}{y + t} \right)^2 w(y) dy \right)^\frac{1}{r} \]
\[ + \sup_{t \in (0, \infty)} V_2(t) \left\{ \sup_{\psi \in \mathcal{W}^1(t, \infty)} \left( \int_{t^c}^\infty \left( \sup_{s \leq t} \frac{u(t)}{B(t)} \right) \int_s^\infty \psi(s) B(t, s) d s d x \right)^{\frac{q}{p}} w(z) d z \right\}^{\frac{1}{q-1}}. \]

(i) Let \( p \leq q \). By Theorem 2.8, we have for any \( t \in (0, \infty) \) that

\[
\sup_{\psi \in \mathcal{W}^1(t, \infty)} \left( \int_{t^c}^\infty \left( \sup_{s \leq t} \frac{u(t)}{B(t)} \right) \int_s^\infty \psi(s) B(t, s) d s d x \right)^{\frac{q}{p}} w(z) d z \]

\[ \approx \sup_{x \in (t, \infty)} \left( \int_t^x B(t, y) \left( \int_y^\infty \sup_{s \leq t} \left( \frac{u(t)}{B(t)} \right)^p d s \right)^{\frac{p}{p-1}} w(y) d y \right)^{\frac{1}{q-1}} \]

\[ + \sup_{x \in (t, \infty)} \left( \int_t^x B(t, y) \left( \int_y^\infty \sup_{s \leq t} \left( \frac{u(t)}{B(t)} \right)^p d s \right)^{\frac{p}{p-1}} w(y) d y \right)^{\frac{1}{q-1}} \]

Consequently,

\[ A_2 \approx \sup_{t \in (0, \infty)} V_2(t) \sup_{t \leq t} \left( \int_t^\infty B(t, y) \left( \int_y^\infty \sup_{s \leq t} \left( \frac{u(t)}{B(t)} \right)^p d s \right)^{\frac{p}{p-1}} w(y) d y \right)^{\frac{1}{q-1}} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \sup_{x \in (t, \infty)} \left( \int_t^x B(t, y) \left( \int_y^\infty \sup_{s \leq t} \left( \frac{u(t)}{B(t)} \right)^p d s \right)^{\frac{p}{p-1}} w(y) d y \right)^{\frac{1}{q-1}} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \sup_{x \in (t, \infty)} \left( \int_t^x B(t, y) \left( \int_y^\infty \sup_{s \leq t} \left( \frac{u(t)}{B(t)} \right)^p d s \right)^{\frac{p}{p-1}} w(y) d y \right)^{\frac{1}{q-1}} \]

\[ + \sup_{t \in (0, \infty)} V_2(t) \left( \int_t^\infty \left( \int_0^\infty \left( \frac{B(y)}{y} \right)^p v_2(s) d s \right)^{\frac{1}{p}} w(y) d y \right)^{\frac{1}{q-1}} \]

Combining, we arrive at

\[ A \approx \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_0^\infty \left( \frac{B(y)}{y} \right)^p v_2(s) d s \right)^{\frac{1}{p}} w(y) d y \right)^{\frac{1}{q-1}} \]

\[ + \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_0^\infty \left( \frac{B(y)}{y} \right)^p d y \right)^{\frac{1}{p}} v_2(s) d s \right)^{\frac{1}{p}} \left( \int_t^\infty \left( \int_0^\infty \left( \frac{u(t)}{B(t)} \right)^p d x \right)^{\frac{q}{p}} w(y) d y \right)^{\frac{1}{q-1}} \]

\[ + \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_0^\infty \left( \frac{B(y)}{y} \right)^p d y \right)^{\frac{1}{p}} v_2(s) d s \right)^{\frac{1}{p}} \left( \int_t^\infty \left( \int_0^\infty \left( \frac{u(t)}{B(t)} \right)^p d x \right)^{\frac{q}{p}} w(y) d y \right)^{\frac{1}{q-1}} \]

\[ + \sup_{t \in (0, \infty)} \left( \int_t^\infty \left( \int_0^\infty \left( \frac{u(t)}{B(t)} \right)^p d x \right)^{\frac{q}{p}} w(y) d y \right)^{\frac{1}{q-1}} \]
ON SOME RESTRICTED INEQUALITIES FOR $T_{ab}$ AND THEIR APPLICATIONS

+ $\sup_{t \in (0, \infty)} V_2(t) \sup_{x \in (0, \infty)} \left( \int_0^x B(t, y) dy \right)^{\frac{p}{p'}} \left( \int_x^\infty \left( \int_x^{\infty} \left( \sup_{s \leq t} \frac{u(\tau)}{B(\tau)} \right) ds \right)^{\frac{q}{q'}} w(y) dy \right)^{\frac{1}{q'}}$

+ $\sup_{t \in (0, \infty)} V_2(t) \sup_{x \in (0, \infty)} \left( \int_{[x, \infty)} d \left( - \sup_{y \geq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\tau B(t, y) dy \right) \right)^{\frac{p'}{p-q}} \left( \int_x^\infty w(y) dy \right)^{\frac{1}{q'}} \left( \int_{x}^{\infty} w(y) dy \right)^{\frac{1}{q'}}$

+ $\sup_{t \in (0, \infty)} V_2(t) \left( \int_x^\infty \left( y - t \right)^{\frac{q}{q'}} w(y) dy \right)^{\frac{1}{q'}} \lim_{x \to \infty} \left( \sup_{x \leq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\tau B(t, y) dy \right)^{\frac{p}{p'}} \right)^{\frac{1}{p'}}.$

(ii) Let $q < p$. By Theorem 2.8, we get for any $t \in (0, \infty)$ that

\[
\sup_{\psi \in \mathcal{W}'(t, \infty)} \left( \int_0^\infty \left( \int_0^\tau \psi(s) B(t, s) ds dx \right)^{\frac{p}{p'}} \left( \int_0^\infty \psi(s) B(t, s) ds \right)^{\frac{q}{q'}} w(z) dz \right)^{\frac{1}{q'}}
\]

\[
\approx \left( \int_0^\infty \left( \int_0^\tau B(t, s) ds \right)^{\frac{p}{p'}} \left( \int_0^\tau \left( \int_0^\tau \sup_{s \leq \tau} \frac{u(\tau)}{B(\tau)} ds \right)^{\frac{q}{q'}} w(z) dz \right)^{\frac{q}{q'}} \right)^{\frac{1}{q'}}
\]

+ $\left( \int_0^\infty \left( \int_0^\tau B(t, s) \left( \int_0^\tau \sup_{z \leq \tau} \frac{u(\tau)}{B(\tau)} ds \right) \right)^{\frac{q}{q'}} \left( \int_0^\tau w(x) dx \right)^{\frac{q}{q'}} \right)^{\frac{1}{q'}}$

+ $\left( \int_0^\infty \left( \int_0^\tau \left( \int[x, \infty) d \left( - \sup_{y \geq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\tau B(t, s) ds \right) \right)^{\frac{p'}{p-q}} \left( \int_0^\tau w(x) dx \right)^{\frac{q}{q'}} \right)^{\frac{1}{q'}} \right)^{\frac{1}{q'}}$

Thus

\[
A_2 \approx \sup_{t \in (0, \infty)} V_2(t) \sup_{t \geq \tau} \left( \int_0^\tau B(t, s) ds \right)^{\frac{p}{p'}} \left( \int_0^\infty \left( \frac{zt}{y + t} \right)^{\frac{q}{q'}} w(y) dy \right)^{\frac{1}{q'}}
\]

+ $\sup_{t \in (0, \infty)} V_2(t) \left( \int_0^\infty \left( \int_0^\tau B(t, s) ds \right)^{\frac{p}{p-q}} \left( \int_0^\tau \sup_{z \leq \tau} \frac{u(\tau)}{B(\tau)} ds \right)^{\frac{q}{q'}} \left( \int_0^\tau w(x) dx \right)^{\frac{q}{q'}} \right)^{\frac{1}{q'}}$

+ $\sup_{t \in (0, \infty)} V_2(t) \left( \int_0^\infty \left( \int_0^\tau B(t, s) \left( \int_0^\tau \sup_{z \leq \tau} \frac{u(\tau)}{B(\tau)} ds \right) \right)^{\frac{q}{q'}} \left( \int_0^\tau w(x) dx \right)^{\frac{q}{q'}} \right)^{\frac{1}{q'}}$

+ $\sup_{t \in (0, \infty)} V_2(t) \left( \int_0^\infty \left( \int_0^\tau B(t, s) ds \right)^{\frac{p}{p-q}} \left( \int_0^\tau \left( \int_0^\tau B(t, s) ds \right)^{\frac{q}{q'}} \left( \int_0^\tau \sup_{z \leq \tau} \frac{u(\tau)}{B(\tau)} ds \right)^{\frac{q}{q'}} \right)^{\frac{1}{q'}} \right)^{\frac{1}{q'}}$

+ $\sup_{t \in (0, \infty)} V_2(t) \left( \int_0^\infty \left( \int_0^\tau \left( y - t \right)^{\frac{q}{q'}} w(y) dy \right)^{\frac{1}{q'}} \lim_{x \to \infty} \left( \sup_{x \leq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\tau B(t, s) ds \right)^{\frac{p}{p'}} \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}}.$

Combining, we arrive at

\[
A \approx \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_0^\tau B(t, s) ds \right)^{\frac{p}{p'}} \left( \sup_{t \geq \tau} \frac{u(\tau)}{B(\tau)} \left( \int_0^\infty \left( \frac{zt}{y + t} \right)^{\frac{q}{q'}} w(y) dy \right)^{\frac{1}{q'}} \right)^{\frac{1}{q'}}
\]
+ \sup_{\tau \leq t} \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

+ \sup_{\tau \leq t} V_{2}(t) \left( \int_{\tau}^{t} \left( \int_{s}^{\tau} B(t,s) ds \right)^{\frac{p}{p-r}} \left( \int_{s}^{\tau} \left( \sup_{t \leq \tau} \left( \frac{f(t)}{B(t)} \right) \right) \right)^{\frac{1}{p-r}} \right) w(y) dy \right)^{\frac{1}{p}}

Now we estimate B. By Theorem 2.6, (b), and integrating by parts, we have that

\[
\sup_{\varphi \in \mathbb{R}^{n}} \frac{1}{\|\varphi\|_{L^{p}, L^{1-r'}, (0, \infty)}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

\[
\approx \sup_{\tau \leq t} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

\[
+ \sup_{\tau \leq t} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

\[
\approx \sup_{\tau \leq t} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

\[
+ \sup_{\tau \leq t} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

Since

\[
\left( \int_{0}^{s} h(t) dt \right)^{\frac{p}{p-r'}} = \int_{0}^{s} d \left( \int_{0}^{s} h(t) dt \right)^{\frac{p}{p-r'}} \approx \int_{0}^{s} \left( \int_{0}^{s} h(t) dt \right)^{\frac{p}{p-r'}} h(s) u(s)^{r} ds
\]

then we get that

\[
\sup_{\varphi \in \mathbb{R}^{n}} \frac{1}{\|\varphi\|_{L^{p}, L^{1-r'}, (0, \infty)}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

\[
\approx \sup_{\tau \leq t} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

\[
+ \sup_{\tau \leq t} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]

By Lemma 2.4, we arrive at

\[
\sup_{\varphi \in \mathbb{R}^{n}} \frac{1}{\|\varphi\|_{L^{p}, L^{1-r'}, (0, \infty)}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \int_{0}^{s} \varphi(x) dx \right)^{\frac{p}{p-r'}} \right) \left( \int_{0}^{s} f(t) dt \right)^{\frac{p}{p-r'}} \right)^{\frac{1}{p}}
\]
Thus, we have that

\[ \sup_{t \in (0, \infty)} \left( \int_{0}^{\infty} \left( \frac{1}{s+t} \right)^{\frac{p}{r}} v_{2}(s) \, ds \right)^{\frac{p}{r}} \left( \int_{0}^{\infty} \left( \frac{1}{s+t} \right)^{\frac{q}{r}} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} h(s)u(s)^{r} \, ds \right)^{\frac{p}{q}} \]

Similarly, if

\[ \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} \, d\left(-\left(\frac{1}{s+t}\right)^{\frac{p}{q}}\right) < \infty \]

for some \( t > 0 \), then

\[ \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} h(s)u(s)^{r} \, ds \leq \frac{p-r}{p} \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} \, d\left(-\left(\frac{1}{s+t}\right)^{\frac{p}{q}}\right). \]

So, we can write

\[ \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} h(s)u(s)^{r} \, ds = \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} \, d\left(-\left(\frac{1}{s+t}\right)^{\frac{p}{q}}\right). \]

We obtain that

\[ \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} h(s)u(s)^{r} \, ds \approx -\int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} \left(\frac{1}{s+t}\right)^{\frac{p}{q}} \, ds \]

Thus, we have that

\[ \sup_{t \in (0, \infty)} \left( \int_{0}^{\infty} \left( \int_{t'}^{t} \int_{0}^{\infty} \varphi(x)u(x) \, dx \right)^{\frac{p}{q}} v_{2}(t) \, dt \right)^{\frac{q}{p}} \]

Consequently, we get that

\[ B \approx \sup_{g \in \mathbb{R}^{+}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} \left(\frac{1}{s+t}\right)^{\frac{p}{q}} \, ds \right)^{\frac{q}{p}}. \]

Interchanging suprema, by duality (since \( p > r \)), we arrive at

\[ B \approx \sup_{g \in \mathbb{R}^{+}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} hu' \, ds \right)^{\frac{p}{q}} \left(\frac{1}{s+t}\right)^{\frac{p}{q}} \, ds \right)^{\frac{q}{p}}. \]
Applying Fubini Theorem, by Theorem 2.3, we have that

$$B \approx \sup_{g \in \mathbb{R}^+} \frac{1}{\|g\|^{\frac{1}{p}}} \sup_{t \in (0, \infty)} V_3(t) \left\{ \sup_{\varphi \in \mathbb{R}^+ (0, \infty)} \frac{1}{\|g\|^{\frac{1}{p}}} \sup_{t \in (0, \infty)} V_3(t) \left( \int_0^\infty g \sup_{s \leq t} u(\tau)^\frac{q}{p} \left( \int_\tau^\infty \psi d\tau \right) \right)^{\frac{1}{q}} \right\} \frac{1}{\|g\|^{\frac{1}{p}}}$$

Again, by Fubini Theorem, and interchanging suprema, we have that

$$B \approx \sup_{t \in (0, \infty)} V_3(t) \left\{ \frac{1}{\|g\|^{\frac{1}{p}}} \sup_{\varphi \in \mathbb{R}^+ (0, \infty)} \left( \int_0^\infty \psi(s)^\frac{p}{p} \left( 1 + t \right)^\frac{\varphi}{p} \right) \right\} \frac{1}{\|g\|^{\frac{1}{p}}} \left( \int_0^\infty V_3(t) \left( \int_0^\infty g \sup_{s \leq t} u(\tau)^\frac{q}{p} \left( \int_\tau^\infty \psi d\tau \right) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

Applying duality principle yields the following estimate:

$$B \approx \sup_{t \in (0, \infty)} V_3(t) \left\{ \frac{1}{\|g\|^{\frac{1}{p}}} \sup_{\varphi \in \mathbb{R}^+ (0, \infty)} \left( \int_0^\infty \psi(s)^\frac{p}{p} \left( 1 + t \right)^\frac{\varphi}{p} \right) \right\} \frac{1}{\|g\|^{\frac{1}{p}}} \left( \int_0^\infty V_3(t) \left( \int_0^\infty g \sup_{s \leq t} u(\tau)^\frac{q}{p} \left( \int_\tau^\infty \psi d\tau \right) \right)^{\frac{1}{q}} \right)$$

Here we apply Theorem 2.9.

(i) Let $p \leq q$. Then we have

$$B \approx \sup_{t \in (0, \infty)} V_3(t) \left\{ \left( \int_0^\infty \sup_{s \in (0, \infty)} \left( \int_0^s (y+t)^\frac{\mu(3-2p)}{\mu(2p-3p+3)} \left( \int_0^y \left( \sup_{s \leq \tau} K(\tau, t)^\frac{p}{p} \right) d\tau \right)^{\frac{p}{p}} \right)^{\frac{p}{p}} \right)^{\frac{1}{q}} \right\} \frac{1}{\|g\|^{\frac{1}{p}}} \left( \int_0^\infty \left( \int_0^\infty g \sup_{s \leq t} u(\tau)^\frac{q}{p} \left( \int_\tau^\infty \psi d\tau \right) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

(ii) Let $q < p$. Then we have

$$B \approx \sup_{t \in (0, \infty)} V_3(t) \left\{ \left( \int_0^\infty \sup_{s \in (0, \infty)} \left( \int_0^s (y+t)^\frac{\mu(3-2p)}{\mu(2p-3p+3)} \left( \int_0^y \left( \sup_{s \leq \tau} K(\tau, t)^\frac{p}{p} \right) d\tau \right)^{\frac{p}{p}} \right)^{\frac{p}{p}} \right)^{\frac{1}{q}} \right\} \frac{1}{\|g\|^{\frac{1}{p}}} \left( \int_0^\infty \left( \int_0^\infty g \sup_{s \leq t} u(\tau)^\frac{q}{p} \left( \int_\tau^\infty \psi d\tau \right) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$
Proof of Theorem 1.1:

Assume that the inequality
\[
\left( \int_0^\infty \left( \int_0^x \left( \sup_{t \leq s} (M_{\phi, \Lambda^\alpha(b)} f)^*(t) \right)^{p_2} \, dt \right)^{\frac{m_2}{p_2}} w(x) \, dx \right)^{\frac{1}{m_2}} \leq C \left( \int_0^\infty \left( \int_0^x \left( f^*(y) \right)^{p_1} \, dy \right)^{\frac{m_1}{p_1}} v(x) \, dx \right)^{\frac{1}{m_1}}
\]
holds for all \( f \in \mathcal{M}^{\text{rad}, 1}(\mathbb{R}^n) \) with constant \( C > 0 \) independent of \( g \) and \( t \) (cf. \cite[Lemma 3.12]{36}).

Thus the inequality
\[
\left( \int_0^\infty \left( \int_0^x \sup_{t \leq s} (g^*(y))^{p_2} b(y) \, dy \right)^{\frac{m_2}{p_2}} w(x) \, dx \right)^{\frac{1}{m_2}} \leq C \left( \int_0^\infty \left( \int_0^x \left( g^*(y) \right)^{p_1} \, dy \right)^{\frac{m_1}{p_1}} v(x) \, dx \right)^{\frac{1}{m_1}}
\]
holds for all \( g \in \mathcal{M}^{\text{rad}, 1}(\mathbb{R}^n) \), which evidently can be rewritten as follows
\[
\left( \int_0^\infty \left( \int_0^x \left( T_{B_1}[g^*, b^*] \right) \right)^{\frac{m_2}{p_2}} w(x) \, dx \right)^{\frac{1}{m_2}} \leq C \left( \int_0^\infty \left( \int_0^x \left( h^*(y) \right)^{\frac{p_1}{m_1}} \, dy \right)^{\frac{m_1}{p_1}} v(x) \, dx \right)^{\frac{1}{m_1}}.
\]
Since for any \( h \in \mathcal{M}(\mathbb{R}^n) \) there exists \( g \in \mathcal{M}^{\text{rad}, 1}(\mathbb{R}^n) \) such that \( g^* = h^* \), then the inequality
\[
\left( \int_0^\infty \left( \int_0^x \left( T_{B_1}[g^*, h^*] \right) \right)^{\frac{m_2}{p_2}} w(x) \, dx \right)^{\frac{1}{m_2}} \leq C \left( \int_0^\infty \left( \int_0^x \left( h^*(y) \right)^{\frac{p_1}{m_1}} \, dy \right)^{\frac{m_1}{p_1}} v(x) \, dx \right)^{\frac{1}{m_1}}
\]
holds for all \( h \in \mathcal{M}(\mathbb{R}^n) \), as well.

Now assume that the inequality
\[
\left( \int_0^\infty \left( \int_0^x \sup_{t \leq s} (h^*(y))^{p_2} b(y) \, dy \right)^{\frac{m_2}{p_2}} w(x) \, dx \right)^{\frac{1}{m_2}} \leq C \left( \int_0^\infty \left( \int_0^x \left( h^*(y) \right)^{p_1} \, dy \right)^{\frac{m_1}{p_1}} v(x) \, dx \right)^{\frac{1}{m_1}}
\]
holds for all \( h \in \mathcal{M}(\mathbb{R}^n) \).

Obviously, the last inequality is equivalent to the inequality
\[
\left( \int_0^\infty \left( \int_0^x \sup_{t \leq s} (f^*(y))^{p_2} b(y) \, dy \right)^{\frac{m_2}{p_2}} w(x) \, dx \right)^{\frac{1}{m_2}} \leq C \left( \int_0^\infty \left( \int_0^x \left( f^*(y) \right)^{p_1} \, dy \right)^{\frac{m_1}{p_1}} v(x) \, dx \right)^{\frac{1}{m_1}}
\]
for all \( f \in \mathcal{M}(\mathbb{R}^n) \).
Recall that the inequality
\[(M_{\phi, \Lambda^\alpha(b)}f)^*(t) \leq C \sup_{\tau \leq t} \phi(\tau)^{-1} \left( \int_0^\tau [f^*(y)]^\rho b(y) dy \right)^{\frac{1}{\rho}}\]
holds for all \(f \in \mathcal{M}(\mathbb{R}^d)\) (cf. [36, Corollary 3.6]).
Consequently, the inequality
\[\left( \int_0^\infty \left( \int_0^\infty [(M_{\phi, \Lambda^\alpha(b)}f)^*(t)]^{p_2} dt \right)^{\frac{m_2}{p_2}} w(x) dx \right)^{\frac{1}{m_2}} \leq C \left( \int_0^\infty \left( \int_0^\tau [f^*(\tau)]^{p_1} d\tau \right)^{\frac{m_1}{p_1}} v(x) dx \right)^{\frac{1}{m_1}}\]
holds for all \(f \in \mathcal{M}(\mathbb{R}^d)\), as well.

The proof is completed. \(\square\)

Denote by
\[(4.1) \quad \hat{\nu}_0(t) := \frac{m_{1\min}}{m_1} \int_0^t v(s) s^{\frac{m_1}{m_1}} ds \int_t^\infty v(s) ds, \quad t \in (0, \infty),\]
\[(4.2) \quad \hat{\nu}_1(t) := \int_0^t v(s) s^{\frac{m_1}{m_1}} ds + \frac{m_{1\min}}{m_1} \int_t^\infty v(s) ds, \quad t \in (0, \infty),\]
and
\[(4.3) \quad \hat{\nu}_2(t) := \frac{m_{2\min}}{m_2 \min_{p_1, p_2}} \hat{\nu}_0(t), \quad t \in (0, \infty).\]

**Theorem 4.1.** Let \(0 < \alpha < m_1 < p_1 \leq p_2 < m_2 < \infty, \alpha \leq r < \infty \) and \(b \in \mathcal{W}(0, \infty) \cap \mathcal{M}^+((0, \infty); \downarrow)\) be such that the function \(B(t)\) satisfies \(0 < B(t) < \infty\) for every \(t \in (0, \infty)\), \(B \in \Delta_2\), \(B(\infty) = \infty\) and \(B(t) t^{p_2/r}\) is quasi-increasing. Moreover, let \(\phi \in \mathcal{W}(0, \infty) \cap \mathcal{C}(0, \infty)\) be such that \(\phi \in \mathcal{Q}_r(0, \infty)\) is a quasi-increasing function. Assume that \(v \in \mathcal{W}_{m_1, p_1}(0, \infty)\) and \(w \in \mathcal{W}(0, \infty)\). Suppose that

\[0 < \int_0^\infty \left( \int_t^\infty \frac{(B(y))^{\frac{p_1}{p_1 - r}}}{y^{\frac{m_1}{m_1 - p_1/r}}} \hat{\nu}_2(s) ds \right) dy < \infty, \quad t \in (0, \infty),\]

where \(\hat{\nu}_2\) is defined by (4.3). Then

\[
\|M_{\phi, \Lambda^\alpha(b)}\|_{\mathcal{G}(p_1, m_1, 1, \nu) \to \mathcal{G}(p_2, m_2, w)} \\
\approx \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_s^\infty \left( \frac{B(y)}{y} \right)^{\frac{p_1}{p_1 - r}} \hat{\nu}_2(s) ds \right) dy \right)^{\frac{1}{p_1}} \sup_{\tau \leq t} \frac{1}{\phi(\tau)} \left( \int_0^{\infty} \left( \frac{y}{y + t} \right)^{\frac{m_2}{p_2}} w(y) dy \right)^{\frac{1}{m_2}}
\]
\[
+ \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \int_s^\infty \left( \frac{B(y)}{y} \right)^{\frac{p_1}{p_1 - r}} \hat{\nu}_2(s) ds \right) dy \right)^{\frac{1}{p_1}} \left( \int_t^\infty \left( \sup_{\tau \leq \tau} \left( \frac{B(\tau)}{\phi(\tau)^{r/2}} \right) \right) w(y) dy \right)^{\frac{1}{m_2}}
\]
\[
+ \sup_{t \in (0, \infty)} \left( \int_0^\infty \left( \frac{1}{s + t} \right)^{\frac{m_2}{p_2}} \hat{\nu}_2(s) ds \right) \left( \int_0^\infty \left( \frac{B(\tau)}{\phi(\tau)^{r/2}} \right) w(y) dy \right)^{\frac{1}{m_2}}.
\]

**Proof.** The statement immediately follows by Theorems 1.1 and 3.3. \(\square\)

**Theorem 4.2.** Let \(0 < \alpha < m_1 \leq p_2 < \min\{p_1, m_2\} < \infty, \alpha \leq r < \infty \) and \(b \in \mathcal{W}(0, \infty) \cap \mathcal{M}^+((0, \infty); \downarrow)\) be such that the function \(B(t)\) satisfies \(0 < B(t) < \infty\) for every \(t \in (0, \infty)\), \(B \in \Delta_2\), \(B(\infty) = \infty\) and \(B(t) t^{p_2/r}\) is quasi-increasing. Moreover, let \(\phi \in \mathcal{W}(0, \infty) \cap \mathcal{C}(0, \infty)\) be such that \(\phi \in \mathcal{Q}_r(0, \infty)\) is a quasi-increasing function. Assume that \(v \in \mathcal{W}_{m_1, p_1}(0, \infty)\) and \(w \in \mathcal{W}(0, \infty)\). Suppose that

\[0 < \int_0^\infty \left( \int_t^\infty \frac{(B(y))^{\frac{p_1}{p_1 - r}}}{y^{\frac{m_1}{m_1 - p_1/r}}} \hat{\nu}_2(s) ds \right) dy < \infty, \quad t \in (0, \infty),\]
\[
\int_0^\infty s^{\frac{m_1(p_1-1)}{p_1^2(m_1^2-1)}} v_0(s)v_1(s)^{\frac{2m_1-2}{m_1^2-1}} \, ds < \infty, \quad \int_t^\infty s^{\frac{m_1(p_1-2\alpha)}{p_1^2(m_1^2-1)}} v_0(s)v_1(s)^{\frac{2m_1-2\alpha}{m_1^2-1}} \, ds < \infty, \quad t \in (0, \infty),
\]
and
\[
\int_1^1 s^{\frac{m_1(p_1-2\alpha)}{p_1^2(m_1^2-1)}} v_0(s)v_1(s)^{\frac{2m_1-2\alpha}{m_1^2-1}} \, ds = \int_1^\infty s^{\frac{m_1(p_1-2\alpha)}{p_1^2(m_1^2-1)}} v_0(s)v_1(s)^{\frac{2m_1-2\alpha}{m_1^2-1}} \, ds = \infty,
\]
where \(v_0, v_1 \) and \(\bar{v}_2\) are defined by (4.1), (4.2) and (4.3), respectively. Denote by
\[
\bar{V}_2(t) := \left(\int_0^t \left(\int_0^s \left(\int_0^\infty \frac{t}{s+t} \bar{v}_2(y) \, dy\right) \frac{p_1}{p_1^2} \, ds\right) \frac{m_1-1}{m_1^2-1} \frac{1}{\phi(t)} \right), \quad 0 < t < \infty,
\]
and
\[
\bar{B}(t,s) = \left(\int_t^\infty \frac{B(y)}{y} \frac{p_1}{p_1^2} \, dy\right) \frac{1}{\phi(t)}, \quad 0 < s < t < \infty,
\]
and if \(p_1 \leq m_2\), then
\[
||M_{\phi,\Lambda^{\alpha}(\beta)}||_{\Gamma(p_1,m_1,v)} \rightarrow \Gamma(p_2,m_2,w)
\]
\[
\begin{align*}
&+ \left( \int_0^\infty y^{m_2} w(y) dy \right)^{\frac{1}{m_2}} 
\sup_{t \in (0, \infty)} \mathcal{V}_3(t) \lim_{s \to \infty} \left( \sup_{s \leq \tau} \mathcal{K}(\tau, t) \left( \int_0^\tau (y + t)^{m_1(3p_2 - 2p_1)} y^{m_1(3p_1 - 2p_2)} dy \right) \right)^{\frac{p_1 - p_2}{p_1 p_2}} \\
&+ \sup_{t \in (0, \infty)} \mathcal{V}_3(t) t^{\frac{p_2}{p_1 p_2}} (\int_0^\infty \left( \int_0^\tau \sup_{s \leq \tau} \mathcal{K}(\tau, t)^{p_2} ds \right)^{\frac{m_1}{p_2}} w(x) dx)^{\frac{1}{m_2}}.
\end{align*}
\]
(ii) If \( m_2 < p_1 \), then

\[
\|M_{\phi, \Lambda^2(b)}\|_{\text{GT}(p_1, m_1, y) \to \text{GT}(p_2, m_2, w)} \\
\approx \sup_{t \in (0, \infty)} \left( \int_0^\tau \left( \int_s^\tau \frac{B(y)}{y} \right)^{\frac{p_2}{p_1 p_2}} dy \right)^{\frac{m_1(3p_1 - 2p_2)}{m_1 p_1 - m_2 p_2}} \mathcal{V}_2(s) ds \left( \sup_{s \leq \tau} \frac{1}{\phi(\tau)} \right)^{\frac{p_1}{m_1 m_2}} \left( \int_0^\infty \left( \frac{y t}{y + t} \right)^{m_2} w(y) dy \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_2(t) \left( \int_0^\tau \left( \int_0^\tau B(s) ds \right)^{\frac{p_2}{p_1 p_2}} \mathcal{B}(t, s) \left( \int_0^\tau \left( \sup_{s \leq \tau} \frac{1}{\phi(\tau)} \right)^{p_2} dx \right)^{\frac{m_2}{p_2}} w(z) dz \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_2(t) \left( \int_0^\tau \left( \int_0^\tau B(t, s) ds \right)^{\frac{p_2}{p_1 p_2}} \mathcal{B}(t, \tau) \left( \int_0^\tau \left( \sup_{s \leq \tau} \frac{1}{\phi(\tau)} \right)^{p_2} dx \right)^{\frac{m_2}{p_2}} w(z) dz \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_2(t) \left( \int_0^\tau \left( \int_0^\tau \mathcal{B}(t, s) \left( \int_0^\tau \mathcal{B}(t, s) ds \right) \left( \int_0^\tau \frac{1}{\phi(\tau)} \left( \int_0^\tau \mathcal{B}(t, s) ds \right)^{\frac{1}{m_2}} \right)^{\frac{m_2}{p_2}} \right)^{\frac{1}{m_2}} w(z) dz \right)^{\frac{1}{m_2}} \\
\times \left( \int_0^\tau (z - t)^{\frac{m_2}{p_2}} w(z) dz \right)^{\frac{m_1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_2(t) \left( \int_0^\tau \left( \int_0^\tau (y - t)^{\frac{p_1}{p_2}} d \left( \sup_{s \leq \tau} \frac{1}{\phi(\tau)} \left( \int_0^\tau \mathcal{B}(t, s) ds \right)^{\frac{1}{m_2}} \right) \right)^{\frac{m_2}{p_2}} \right)^{\frac{1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_3(t) \left( \int_0^\tau \left( \int_0^\tau (x + t)^{\frac{p_1}{p_2}} dx \right)^{\frac{m_2}{p_2}} \right)^{\frac{1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_3(t) \left( \int_0^\tau \left( \sup_{y \leq \tau} \mathcal{K}(\tau, t)^{p_2} dx \right)^{\frac{m_2}{p_2}} \right)^{\frac{1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_3(t) \left( \sup_{y \leq \tau} \mathcal{K}(\tau, t)^{p_2} \left( \int_0^\tau (z + t)^{\frac{p_1}{p_2}} dx \right)^{\frac{m_2}{p_2}} \right)^{\frac{1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
\times \left( \int_0^\tau \left( \int_0^\tau (z - t)^{\frac{m_2}{p_2}} w(z) dz \right)^{\frac{m_1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_3(t) \left( \int_0^\tau \left( \int_0^\tau \left( \sup_{y \leq \tau} \mathcal{K}(\tau, t)^{p_2} dx \right)^{\frac{m_2}{p_2}} \right)^{\frac{1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
\times \left( \int_0^\tau \left( \int_0^\tau (z - t)^{\frac{m_2}{p_2}} w(z) dz \right)^{\frac{m_1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
+ \sup_{t \in (0, \infty)} \mathcal{V}_3(t) \left( \int_0^\tau \left( \int_0^\tau \left( \sup_{y \leq \tau} \mathcal{K}(\tau, t)^{p_2} dx \right)^{\frac{m_2}{p_2}} \right)^{\frac{1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}} \\
\times \left( \int_0^\tau \left( \int_0^\tau (z - t)^{\frac{m_2}{p_2}} w(z) dz \right)^{\frac{m_1}{m_2}} w(x) dx \right)^{\frac{1}{m_2}}.
\]

\[ \begin{align*}
&\sup_{t \in (0, \infty)} \widetilde{V}_3(t) \left( \int_0^\infty \left( \int_0^x y^{p_1/p_2} \left( -\left( \sup_{y \leq t} \tilde{K}(y, t) \frac{y^{p_1/p_2} \left( \int_0^T (z + t)^{k_1/p_1} \frac{p_1\left(3p_2 - 2p_1\right)\left(1/p_1 - 1/p_2\right)}{p_2\left(1 - m_2\right)} \right)}{\left(\int_0^x (w(z))^{m_2/1 - m_2} w(x) dx\right)^{1/m_2}} \right) \right) dx \right)^{\frac{p_2}{1 - p_2}},
&\left( \int_0^\infty \frac{w(x)}{x} dx \right)^{p_2/2}
\end{align*} \]

\[ + \left( \int_0^\infty y^{p_1/p_2} w(y) dy \right)^{\frac{1}{p_2}} \sup_{t \in (0, \infty)} \widetilde{V}_3(t) \lim_{s \to \infty} \left( \sup_{s \leq T} \tilde{K}(s, t) \left( \int_0^T (y + t)^{k_1/p_1} \frac{p_1\left(3p_2 - 2p_1\right)\left(1/p_1 - 1/p_2\right)}{p_2\left(1 - m_2\right)} \right) \right) \left( \int_0^x \left( \int_0^x (w(z))^{m_2/1 - m_2} w(x) dx\right)^{1/m_2} \right)^{p_2/2}.
\]

Proof. The statement immediately follows by Theorems 1.1 and 3.4. \( \Box \)

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