We consider two dimensional $U(N)$ QCD on the cylinder with a timelike Wilson line in an arbitrary representation. We show that the theory is equivalent to $N$ fermions with internal degrees of freedom which interact among themselves with a generalized Sutherland-type interaction. By evaluating the expectation value of the Wilson line in the original theory we explicitly find the spectrum and degeneracies of these particle systems.
There has been recent interest in two dimensional QCD, mainly in understanding its string like properties [1-4]. One of the salient features of this theory is that pure gauge $U(N)$ theory is equivalent to $N$ free fermions on a circle [5,6]. In this letter we consider the modification to this theory when an immobile color source is added to the theory. As far as the path integral goes, this is equivalent to inserting a timelike Wilson line in the theory. We will show that the effect of this source is to turn this into a theory of interacting fermions with internal degrees of freedom that transform under a representation of some group $SU(n)$, where $n$ is determined by the representation of the Wilson line. If its Young tableau consists of $n$ rows then one can choose the representations of the internal quantum numbers to be symmetric representations of $SU(n)$. One can also choose the internal quantum numbers to transform under antisymmetric representations of $SU(m)$, where $m$ is the number of columns in the Young tableau. We will then derive the spectrum of this interacting fermion theory, with its degeneracies, by comparing with the expectation value of the Wilson loop in the original theory.

Consider $U(N)$ QCD$_2$ on the cylinder. Let $L$ be the circumference of the cylinder and $T$ be its length in the time direction. Suppose that there is a Wilson line along the time direction with spatial position $x = 0$ in the representation $R$ of the group $U(N)$. Hence this theory is described by having an immobile color source at position $x = 0$.

Let us then consider the action for the continuum theory

$$\frac{1}{4} \int d^2x \, \text{tr} \, F_{\mu\nu}F^{\mu\nu} + \int dt \bar{\psi} \left( i\partial_t - gA_0(x = 0) + M \right) \psi$$

(1)

where the $T_R^n$ are the generators of the group in representation $R$, $\psi$ ($\bar{\psi}$) is the annihilation (creation) operator for the heavy color source and $M$ is its mass.

The euclidean path integral over the heavy fermion can be explicitly evaluated. It can be easily obtained, however, by noticing that in the proper fermion basis the theory consists of $d_R$ independent fermions with energies $M - i\omega_n$, where $\exp(i\omega_n)$ are the eigenvalues of the timelike Wilson loop in the representation $R$ (notice
that the gauge term remains imaginary even in euclidean time). Integrating out \( \psi \), then, will give the partition function for the fermions which is

\[
Z = \prod_{n=1}^{dR} \left( 1 + e^{-TM+i\omega_n} \right) = 1 + e^{-TM} \sum_{n=1}^{dR} e^{i\omega_n} + \mathcal{O}(e^{-2TM})
\]

To lowest order we have pure QCD\(_2\), while to order \( e^{-TM} \) we obtain the trace of the Wilson loop. Isolating the \( \mathcal{O}(\exp(-TM)) \) term in the full partition function will then give us the theory with an insertion of a timelike Wilson loop in representation \( R \).

In the canonical formulation, the Hamiltonian in the gauge \( A_0 = 0 \) is

\[
H = \frac{1}{2} \int_0^L dx \text{tr} F_{01}^2 + M \bar{\psi} \psi = \frac{1}{2} \int_0^L dx \text{tr} \dot{A}_1^2 + M \bar{\psi} \psi
\]

with the overdot denoting a time derivative. The \( A_0 \) equation of motion is now the constraint

\[
D_1 F_{10} = \partial_1 \dot{A}_1 + ig[A_1, \dot{A}_1] = gK \delta(x - L + \epsilon),
\]

where

\[
K = \sum_{a=1}^{N^2-1} K^a \tau^a, \quad K^a = \bar{\psi} T^a_R \psi
\]

and \( \tau^a \) are the \( SU(N) \) generators in the fundamental representation. For later convenience, we have actually moved the source over to the left by a small amount \( \epsilon \).

We can now proceed in a manner similar to that in [5]. Define a new variable
\[ V(x), \]
\[ V(x) = W^x_0 \dot{A}_1(x) W^L_x, \]  
(6)

where
\[ W^b_a = \text{Pe}^{ig \int_a^b dx A_1}. \]  
(7)

Then (4) can be written as
\[ \partial_1 V(x) = gW^L_0 \varepsilon \delta(x - L) + \varepsilon \delta(x - L + \varepsilon). \]  
(8)

Thus \( V(x) \) is constant until it reaches \( x = L - \varepsilon \), at which point it jumps by \( gW^L_0 \varepsilon \delta(x - L) \), which in the limit \( \varepsilon \to 0 \) becomes \( gWK \), where \( W = W^L_0 \). Hence \( V(0) = V(L) + gWK \), which implies that
\[ [W, \dot{A}_1(0)] = gWK, \]  
(9)

where we have used the periodicity of \( A_1 \) in \( x \).

From the definitions (6) and (7), we find the relation
\[ \dot{W} = ig \int_0^L dx W^x_0 \dot{A}_1(x) W^L_x = ig \int_0^L dx V(x), \]  
(10)

and therefore using (8) and (9) and letting \( \varepsilon \to 0 \), one finds
\[ \dot{W} = igL \dot{A}_1(0)W. \]  
(11)

Equations (9) and (11) then imply that
\[ [W, \dot{W}] = ig^2LK. \]  
(12)
Because $V(x) = V(0)$ for $0 \leq x < L - \epsilon$, $\dot{A}_1(x)$ satisfies

$$\dot{A}_1(x) = W_0^x \dot{A}_1(0) W_0^x.$$  \hspace{1cm} (13)

Thus, using this relation along with (11), we can rewrite the Hamiltonian in (3) as

$$H = -\frac{1}{2g^2L} \text{tr}(W^{-1}\dot{W})^2 + M\bar{\psi}\psi.$$  \hspace{1cm} (14)

The operators $K^a$ provide a realization of the $SU(N)$ algebra in terms of fermionic oscillators, which decomposes into irreducible representations. The lowest one is the singlet (corresponding to the fermionic vacuum). For this irrep the fermion mass term in (14) (which is a casimir of $SU(N)$) vanishes. If the gauge group is $U(N)$, with the $U(1)$ coupling given by $g/N$, then (14) becomes the Hamiltonian for the one-dimensional unitary matrix model. The right-hand side of (12) also vanishes. We thus recover the one-dimensional matrix model and constraint equivalent to QCD$_2$ [5]. This corresponds to the $\mathcal{O}(1)$ term in (2). The next highest irrep contained in $K$ is $R$. For this irrep the fermion mass term equals $M$ and thus it corresponds to the order $\mathcal{O}((\exp(-TM))$ term in (2). This is, then, the part corresponding to the Wilson loop insertion, and in this case (12) carries the representation $R$.

We note here that we could have quantized $\psi$ as a bosonic field (this is allowed in $0 + 1$ dimensions). In that case, the partition function (2) would be an infinite series in $\exp(-TM)$ and, correspondingly, $K$ would contain an infinite tower of representations of $SU(N)$. The first two terms, however, would be the same as in the fermionic case and we would obtain the same result.

Ignoring the constant $M$, thus, we again have the same matrix model Hamiltonian. Unlike, however, the case in [5], the physical degrees of freedom are not merely the diagonal components of $W$, since the commutator of $W$ with $\dot{W}$ does not vanish. The commutator in (12) is, in fact, the generator of unitary transformations of $W$ and (12) tells us that the angular degrees of freedom of $W$ are
in an “angular momentum” state determined by the representation $R$ of $K$. To isolate the relevant degrees of freedom, let us rewrite $W$ as $W = U\Lambda U^\dagger$, where $\Lambda$ is diagonal. Then the constraint (12) leads to the equation

$$2U^\dagger \dot{U} - \Lambda U^\dagger \dot{U} \Lambda - \Lambda^\dagger U^\dagger \dot{U} \Lambda = -ig^2 LJ,$$

(15)

where $J = U^\dagger KU$. Letting $\Omega = U^\dagger \dot{U}$, (15) becomes

$$\Omega_{ij} \left(2 - e^{i(\theta_i - \theta_j)} - e^{-i(\theta_i - \theta_j)}\right) = -ig^2 LJ_{ij},$$

$$\Omega_{ij} = -ig^2 L \frac{J_{ij}}{4 \sin^2(\theta_i - \theta_j)/2}$$

(16)

where $e^{i\theta_i}$ are the eigenvalues of $\Lambda$. Under these substitutions, the Hamiltonian becomes

$$H = \frac{1}{2g^2 L} \sum_i \dot{\theta}_i^2 + \frac{g^2 L}{2} \sum_{i \neq j} \frac{J_{ij}J_{ji}}{4 \sin^2(\theta_i - \theta_j)/2}.$$  

(17)

Hence this is a theory of fermions on the circle interacting through two-body forces which depend on $J$. The fermionization is achieved through the Jacobian factor of the change of variables from $W$ to $(U, \Lambda)$. This introduces the Vandermonde determinant in the wavefunction of the states, which in the unitary matrix case reads

$$\Delta = \prod_{i<j} \sin \frac{\theta_i - \theta_j}{2}.$$  

(18)

Each factor in (18) is antiperiodic on the circle. Thus, if $N$ is even the fermions have antiperiodic boundary conditions. Likewise, if $N$ is odd they have periodic boundary conditions. This can be understood in terms of transporting a fermion once around the circle, passing by $N - 1$ other fermions along the way and therefore picking up $N - 1$ minus signs.

Notice that $K$ commutes with the Hamiltonian and is a constant of the motion. The interaction term in (17) on the other hand involves $J = U^\dagger KU$ rather
than $K$. As an operator, $J$ also obeys the $SU(N)$ algebra and carries the same representation $R$ as $K$. In fact, using (12), we see that $K$ acting on physical states generates left-rotations of $U$ and thus unitary rotations of $W$ while $J$ generates right-rotations of $U$. It is not, however, a constant of the motion since it does not commute with the Hamiltonian (17).

The form of the Hamiltonian (17) is reminiscent of the Sutherland model [7], but the coefficients of the interaction terms are particle, color and time dependent, apparently making a solution impossible. But as it turns out, there are some residual constraints which will allow us to further reduce these terms. In fact, (12), and hence (16), do not actually contain all of the constraints in the theory. This is because there are constraints for the diagonal components of $J$, implied by (4), which do not show up in the commutation relations. What is missing is the analog of Gauss’ law for the $U(1)$ components of the theory. In the diagonal basis, Gauss’ law (4) states that the charges for each of the diagonal generators must be zero, since space is compact. This means that the Wilson loop must carry no $U(1)$ charge, but also that $J$ satisfies on physical states

$$J_{ii}|\text{phys}>=0 \quad \text{for all } i.$$ \hspace{1cm} (19)

(Alternatively, the diagonal components of $J$ generate diagonal right-rotations of $U$ which is a redundancy of the parametrization $(U, \Lambda)$ since it gives the same $W$, and (19) expresses the fact that physical states are independent of this redundancy.) One implication of this is that $R$ must be restricted to $SU(N)$ representations with $mN$ boxes in the Young tableau, where $m$ is an integer, since these are the only representations that have states where all diagonal charges are zero. This restriction also agrees with the requirement that the Wilson loop must have zero $Z_N$ charge, else its expectation value vanishes.

To establish the description of the above theory as a system of fermions with internal degrees of freedom we proceed using two different methods. The first involves constructing $SU(N)$ representations out of bosonic creation operators.
Let these operators be given by $a_i^\dagger I$, where $i$ is an index that runs from 1 to $N$ and $I$ is an index that runs from 1 to $n$. The generators can then be expressed as

$$J^a = \sum_I a_i^\dagger I \tau^a_{ij} a_j^I.$$  \hfill (20)

$SU(N)$ representations are then constructed by acting with the $a_i^\dagger I$ on the vacuum state, so a typical state looks like

$$a_1^\dagger a_2^\dagger \ldots a_N^\dagger |0\rangle.$$

The constraint (19) that the diagonal charges all be zero means that the only allowed states are those where the number of creation operators for each lower index $i$ is the same for all $i$. Therefore, each lower index $i$, which corresponds to a particular fermion, has an associated state made up of the $n$ different creation operators $a_i^\dagger I$. This state necessarily transforms as a symmetric representation of $SU(n)$. If there are $m$ creation operators per lower index, then the representation is the $m$-fold symmetric representation. Hence, the problem can be thought of as a system of interacting fermions with internal degrees of freedom which transform in the $m$-fold symmetric representation of $SU(n)$.

In terms of the creation and annihilation operators, the currents $J_{ij}$ are given by

$$J_{ij} = \sum_I a_i^\dagger I a_j^I,$$  \hfill (21)

hence the symmetrized product $J_{ij} J_{ji}$ is given by

$$\frac{1}{2}(J_{ij} J_{ji} + J_{ji} J_{ij}) = \sum_{I \neq J} a_i^\dagger I a_j^\dagger J a_j^I a_i^I + \sum_I a_i^\dagger I a_j^\dagger I a_j^I a_i^I + \frac{1}{2} \sum_I (a_i^\dagger I a_j^I + a_j^I a_i^I).$$  \hfill (22)

Clearly, the first sum in (22) is twice the sum over all nondiagonal generators in the product $L_i^a L_j^a$, where $L^a$ are generators of $SU(n)$. The next sum contains the
diagonal generators as well as an overall constant. To find their contributions, note that a useful basis for the diagonal components is

\[ M^k = \frac{1}{\sqrt{2k(k+1)}} \left( \sum_{l=1}^{k} a^\dagger_l a^l - ka^\dagger_{k+1} a^{k+1} \right). \]

(23)

Then using the relations

\[ \sum_{k=1}^{n-1} \frac{1}{2k(k+1)} = \frac{(n-1)}{2n}, \]

(24)

and

\[ -\frac{j}{2j(j+1)} + \sum_{k=j+1}^{n-1} \frac{1}{2k(k+1)} = -\frac{1}{2n}, \]

(25)

one finds that

\[ 2 \sum_{k=1}^{n-1} M_k^k M_j^k = \frac{n-1}{n} \sum_{I=1}^{n} n_I^l n_j^I - \frac{1}{n} \sum_{I \neq J} n_I^l n_J^I \]

\[ = \sum_{I} n_I^l n_j^I - \frac{1}{n} \sum_{I} n_I^l \sum_{I} n_j^I, \]

(26)

where \( n_I^l \) is the number operator \( a_I^\dagger a_I^l \). Plugging (26) into (22), one finds that

\[ \frac{1}{2}(J_{ij}J_{ji} + J_{ji}J_{ij}) = m + m^2/n + 2L_i^a L_j^a \]

\[ = \frac{2C_{2m}}{(n-1)} + 2L_i^a L_j^a, \]

(27)

where \( C_{2m} \) is the quadratic casimir for the \( m \)-fold symmetric representation of \( SU(n) \). Hence the complete Hamiltonian is

\[ H = -\frac{g^2 L}{2} \sum_{i} \frac{\partial^2}{\partial \theta_i^2} + \frac{g^2 L}{2} \sum_{i \neq j} \frac{2C_{2m} / (n-1) + 2L_i^a L_j^a}{4 \sin^2(\theta_i - \theta_j)/2}. \]

(28)

The particular representation of the Wilson lines determines how many sets of bosonic operators we should choose. If the Young tableau has \( n \) rows, then it is
necessary to choose at least $n$ sets of such operators. Of course we could also build other representations of $SU(N)$ from these operators, including ones with fewer than $n$ rows. The thing to notice is that the total $SU(n)$ operator $L^a = \sum_i L_i^a$ commutes with the Hamiltonian. Hence the representation of $SU(N)$ is determined by the particular irrep of $SU(n)$ chosen for $L^a$. Since the total operator $L^a$ is in the tensor product of $N$ $m$-fold symmetric irreps for $SU(n)$, to each irrep in the decomposition of this product corresponds an irrep of $SU(N)$, found by taking the Young tableau of the $SU(n)$ irrep and adding to the left enough columns of length $n$ so as to make the total number of boxes equal to $mN$.

It is instructive to examine a couple of simple examples. Consider, for instance, representations with $N$ boxes, all in the first or second row. Thus we choose $n = 2$ sets of bosonic operators. Since $m = 1$, each fermion carries the degrees of freedom of a doublet under $SU(2)$. The only casimir is of course the total spin squared, and since we have $N$ fermions, the possible total spin states range from $N/2$ down to $1/2$ or $0$, depending on whether $N$ is odd or even. The states with maximum total spin correspond to the totally symmetric representation, and hence has all $N$ boxes in the first row. The states with total spin $N/2 - l$ are in the representation with $l$ of its boxes in the second row. The smallest possible spin has the maximum allowed value in the second row, which is $N/2$ ($(N - 1)/2$) for $N$ even (odd). As another example consider the totally symmetric representation with $mN$ boxes. As was shown in [8], this gives rise to the Sutherland model with integral coefficient. In this case we choose $n = 1$ oscillator and thus there are no internal degrees of freedom. Putting $L = 0$ in (27) we recover the standard coefficient $m(m + 1)$ of the Sutherland term. The connection of QCD$_2$ and the Sutherland system was also recently analyzed in [9].

The other way to construct states is to use fermionic operators $b^+_I$. Hence the generators are

$$J^a = \sum_I b^+_I \tau^a_I b_I^I. \tag{29}$$

One can then proceed as before. In this case, each fermion will transform in an
antisymmetric representation of $SU(n)$. If the Wilson line has $n$ columns, then it is necessary to choose at least $n$ sets of $b_i^I$ operators. Finally, the Hamiltonian is given as

$$H = -\frac{g^2 L}{2} \sum_i \frac{\partial^2}{\partial \theta_i^2} + \frac{g^2 L}{2} \sum_{i \neq j} \frac{2\tilde{C}_{2m} / (n + 1) - 2L_i^a L_j^a}{4 \sin^2(\theta_i - \theta_j) / 2},$$

(30)

where $\tilde{C}_{2m}$ is the quadratic casimir for the $m$-fold antisymmetric representation of $SU(n)$. The fact that the problem can be expressed either way leads to some interesting relations between theories, since for a particular representation we can describe the theory either by the number of rows or by the number of columns.

It should be stressed that the above theories are fermionic with respect to total particle exchange (position and internal). This is because there is yet a discrete remnant of parametrization invariance, namely the one where $U$ changes by a component that interchanges two eigenvalues and $\Lambda$ changes accordingly. The first transformation exchanges the internal degrees of freedom of two particles while the second exchanges their position. Due to the Vandermonde (18), the wavefunction picks up a minus sign under these transformations and is thus fermionic.

We now come to one of the main points of this paper. So far we have reduced QCD$_2$ with a Wilson loop to a theory of interacting particles but we have not solved it. By using, however, an alternative reduction of the initial theory we will explicitly evaluate its spectrum. We will merely consider the Wilson loop to be spacelike, as we have the freedom to do. The time evolution then is the one of a free theory, with a particular weight introduced to initial and final states due to the Wilson loop. The result for the partition function (as also calculated by Migdal and later by Rusakov [10,11] using the heat kernel action) is

$$\sum_{R'} \int dU \chi_{R'}(U) \chi_{R'}(U^\dagger) \chi_R(U) e^{-g^2 L T C_{2 R'}},$$

(31)

where the sum is over all representations of $U(N)$, $\chi_R(U)$ is the character for the group element $U$ in representation $R$ and $C_{2R'}$ is the quadratic casimir for $R'$. The
first two characters are the wavefunctions of the initial and final states while the
third is the Wilson loop insertion.

The integral over $U$ of the characters gives an integer $D(R, R')$ which mea-
sures how many times the representation $R'$ is contained in the the tensor product
$R \times R'$. Thus a representation contributes to the path integral in (31) only if the
tensor product $R \times R'$ contains the representation $R'$. This immediately implies
that $R$ must have no $U(1)$ charge in order for the path integral to be nonzero.
Moreover, the number of boxes in the Young tableau that describes this represen-
tation must be a multiple of $N$ in order for there to exist representations $R'$ that
satisfy $R' \in R \times R'$, recovering once more this condition. The casimiris appearing
in the exponent of (31), on the other hand, correspond to the energy eigenvalues
of $N$ free nonrelativistic particles. Thus we conclude that the spectrum of this
theory is identical to the spectrum of a free fermion theory, but with degeneracies
$D(R, R')$ determined by the particular representation $R$.

Again it is instructive to work out explicitly the case where $R$ is the $m$-fold
completely symmetric representation which corresponds to the Sutherland model.
Using standard Young tableau rules, we find that $R'$ can be contained in the
product $R \times R'$ at most once, and that will happen if the rows $n_i$ of $R'$ satisfy the
condition

$$n_i \geq n_{i+1} + m.$$  \hspace{1cm} (32)

Writing $n_i = k_i + (m - 1)(N - i)$, we see that $k_i$ must satisfy $k_{i+1} > k_i$. The second
Casimir then is

$$C_{2R'} = \sum_i \frac{1}{2}(n_i + N - i)^2 = \sum_i \frac{1}{2}k_i^2 + m \sum_{i<j} |k_i - k_j| + m^2 \frac{N(N - 1)(2N - 1)}{12} \hspace{1cm} (33)$$

which is indeed the spectrum of the Sutherland model in the “fermionic” parametriza-
tion of the momenta [7]. It is clear from (32) that Sutherland particles can be
thought of as particles whose momenta must be at least $m + 1$ quanta away from
each other, putting forth the description of this model as free fermions with an enhanced exclusion principle.

The above theories then are solvable generalizations of the Sutherland model. If the number of boxes for the representation $R$ is $N$ then $m = 1$ and the fermions transform in the fundamental representation of $SU(n)$. In this case, the operator that exchanges the internal quantum numbers is given by

$$\sigma_{ij} = \frac{1}{n} + 2\tau_i^a \tau_j^a,$$

hence we can reexpress the Hamiltonians in (28) or (30) in terms of exchange operators. Such a theory has been previously examined [12-14]. In particular, in [14] this theory was shown to be integrable for a generic range of coefficients for the interaction term. For the higher representations, however, it is not possible to describe $L_i^a L_j^a$ in terms of an operator that exchanges all of their internal quantum numbers. But clearly, such a theory must be integrable, at least for a certain choice of coefficients, since its spectrum can be explicitly found. It is therefore a challenge to find a proof of integrability for these particular theories with generic coefficients.

Another point to consider is that, in general, while the spectra for the Sutherland-like models are known, the correlation functions have proven to be difficult to compute. It is possible that QCD$_2$ might provide a useful means for studying this problem.

Acknowledgements: The research of J.A.M. was supported in part by D.O.E. grant DE-AS05-85ER-40518.
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