Thermodynamics of charged AdS black holes in rainbow gravity

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Abstract

In this paper, we discuss the charged anti-de Sitter(AdS) black holes where metric depends on the energy of a probe. By the Heisenberg uncertainty principle and the modified dispersion relation, we achieve deformed temperature. Moreover, in rainbow gravity we calculate the heat capacity in a fixed charge and discuss the thermal stability. We also achieve similar behaviour with this system of the liquid-gas system in extending phase space (including $P$ and $r$) and study its critical behavior with the pressure given by the cosmological constant and with a fixed black hole charge $Q$. Furthermore, we study the Gibbs function and find characteristic swallow tail behavior, which indicates there is the phase transition. We also find there is a special value about the the mass of test particle which would make the black hole zero temperature and diverging heat capacity with fixed charge.

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I. INTRODUCTION

It is known Lorentz symmetry is one of most important symmetries in nature, however, some researches indicate the Lorentz symmetry might be violated in the ultraviolet limit \[1–5\]. Since the standard energy-momentum dispersion relation relates to the Lorentz symmetry, the deformation of Lorentz symmetry would impel the modification of energy-momentum dispersion relation. In fact, some calculations in loop quantum gravity have showed the dispersion relations may be deformed. Meanwhile, based on the deformed energy-momentum dispersion relation the double special relativity has arisen \[6, 7\]. In this theory, in addition to the velocity of light being the maximum velocity attainable there is another constant for maximum energy scale in nature which is the Planck energy $E_P$. It gives different picture for the special relativity in microcosmic physics. The theory has been generalized to curved spacetime by Joao Magueijo and Lee Smolin \[8\] and is called gravity’s rainbow. In this theory, the geometry of spacetime depends on the energy of the test particle and observers of different energy would see different geometry of spacetime. Hence, a family of energy-dependent metrics named as rainbow metrics will describe the geometry of spacetime, which is different from general gravity theory. Based on the non-linear of Lorentz transformation, the energy-momentum dispersion relation rewrites as

$$E^2 f^2(E/E_P) - p^2 g^2(E/E_P) = m^2, \quad (1)$$

where $E_P$ is the Planck energy. The rainbow functions $f(E/E_P)$ and $g(E/E_P)$ are required to satisfy

$$\lim_{E/E_P \to 0} f(E/E_P) = 1, \quad \lim_{E/E_P \to 0} g(E/E_P) = 1. \quad (2)$$

In this case, the deformed energy moment relationship Eq.(1) will go back to classical one when the energy of the test particle is much lower than $E_P$. Due to this
energy-dependent modification to the dispersion relation, the metric $h(E)$ in gravity’s rainbow will be rewritten as

$$h(E) = \eta^{ab}e_a(E) \otimes e_b(E),$$

(3)

here the energy dependence of the frame fields can be written as

$$e_0(E) = \frac{1}{f(E/E_P)} \tilde{e}_0, \quad e_i(E) = \frac{1}{g(E/E_P)} \tilde{e}_i,$$

(4)

where the tilde quantities refer to the energy-independent frame fields. This then leads to a one-parameter Einstein equation

$$G_{\mu\nu}(E/E_P) + \Lambda(E/E_P)g_{\mu\nu}(E/E_P) = 8\pi G(E/E_P)T_{\mu\nu}(E/E_P),$$

(5)

where $G_{\mu\nu}(E/E_P)$ and $T_{\mu\nu}(E/E_P)$ is energy-dependent Einstein tensor and energy-momentum tensor, $\Lambda(E/E_P)$ and $G(E/E_P)$ is an energy-dependent Newton constant and cosmological constant. Generally, lots of forms of rainbow function have been discussed in literatures, in this paper we will mainly employ following rainbow function

$$f(E/E_P) = 1, \quad g(E/E_P) = \sqrt{1 - \eta(E/E_P)^n}.$$

(6)

which has been widely used in Refs [10–18].

Recently, some literatures have also studied the Schwarzschild black holes, Schwarzschild AdS black holes and Reissner-Nordstrom black holes in rainbow gravity [19–21]. In this paper, we will extend the discussion of the Ref. [15] for charged AdS black holes in rainbow gravity. Compared with Ref. [15], we will further study the charged AdS black holes in rainbow gravity and discuss the critical behavior and phase transition.

The paper is organized as follows. In the next section, by using the Heisenberg Uncertainty Principle(HUP) and the modified dispersion relation, we obtain deformed
temperature, we also calculate heat capacity with a fixed charge and discuss the thermal stability. In Sec.III, we find the charged AdS black holes have similar behaviour with the liquid-gas system with the pressure given by the cosmological constant while we treat the black holes charge $Q$ as a fixed external parameter, not a thermodynamic variable. We also calculate the Gibbs free energy and find characteristic swallow tail behavior. Finally, the conclusion and discussion will be offered in Sec.IV.

II. THE THERMAL STABILITY

In rainbow gravity the line element of the modified charged AdS black holes can be described as [15]

$$ds^2 = -\frac{N}{f^2} dt^2 + \frac{1}{Ng^2} dr^2 + \frac{r^2}{g^2} d\Omega^2,$$

(7)

where

$$N = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2},$$

(8)

here, $-\frac{3}{l^2} = \Lambda$ which $\Lambda$ is cosmological constant. Because all energy dependence in the energy-independent coordinates must be in the rainbow functions $f$ and $g$, $N$ is independent on the energy of test particle [8]. In gravity’s rainbow, the modified temperature related to the standard temperature $T_0$ was [15]

$$T = -\frac{1}{4\pi} \lim_{r \to r_+} \sqrt{-g^{11}} (g^{00})' = \frac{g(E/E_P)}{f(E/E_P)} T_0,$$

(9)

here, $r_+$ is event horizon.

In gravity’s rainbow, although the metric depends on the energy of test particle, the usual HUP still holds [19]. The uncertainty of momentum near the horizon can be described as [22] $p = \Delta p \sim \frac{1}{r_+}$. So when we choose $n = 2$ in rainbow function, through Eq.(1) and Eq.(6) we can get

$$g = \sqrt{1 - \eta G_0 m^2} \sqrt{\frac{r_+^2}{r_+^2 + \eta G_0}},$$

(10)
here, $G_0 = 1/E^2_P$, $m$ is the mass of test particle and $\eta$ is a constant parameter. When the test particle is photon, $m = 0$.

When $f = g = 1$, the standard temperature was given as

$$T_0 = \frac{1}{4\pi} \left( \frac{1}{r_+} + \frac{3r_+}{l^2} - \frac{Q^2}{r_+^3} \right).$$

(11)

So when using the Eq.(10) and $f = 1$, we can get the temperature of charged AdS black holes in rainbow gravity

$$T = gT_0 = \frac{1}{4\pi k} \sqrt{\frac{r_+^2}{r_+^2 + \eta G_0}} \left( \frac{1}{r_+} + \frac{3r_+}{l^2} - \frac{Q^2}{r_+^3} \right),$$

(12)

here $k = 1/\sqrt{1 - \eta G_0 m^2}$. It is easy to find $T = T_0$ when $\eta = 0$. The Eq.(12) shows there are two solution when $T = 0$, one corresponds to extreme black hole, another corresponds to $m^2 = \frac{1}{\eta G}$. In terms of the second solution, we will give some discussions. The result indicates the temperature of black holes completely depends on the mass of the test particle when the black holes keep with fixed mass, charge and anti-de Sitter radius. The bigger the mass of test particle is, the smaller the temperature of black holes is. When $m^2 = \frac{1}{\eta G}$, the temperature keeps zero. Generally, due to gravity’s rainbow a minimum radius is given with respect to black holes when the temperature tends to zero, however, the paper shows all black holes can keep zero temperature when the test particle mass approaches a value, such as, $m^2 = \frac{1}{\eta G}$. But due to $m \ll M_P$ in general condition, so it may be difficult to test the phenomenon with zero temperature about black holes.

In generally, the thermal stability can be determined by the heat capacity, the thought also be used to the systems of black holes [17]. In other words, the positive heat capacity for black holes corresponds to a stable state and the negative heat capacity for black holes corresponds to unstable state. In following discussion, we will focus on the heat capacity to discuss the stability of black holes. We can calculate
Figure 1. $C_Q - r_+$ diagram of charged AdS black holes in the rainbow gravity. It corresponds to $l = 6$. We have set $Q = 1, \eta = 1, m = 0$.

Figure 2. $C_Q - r_+$ diagram of charged AdS black holes in the rainbow gravity. It corresponds to $l = 7.05$. We have set $Q = 1, \eta = 1, m = 0$. 
Figure 3. $C_Q - r_+$ diagram of charged AdS black holes in the rainbow gravity. It corresponds to $l = 8$, right one corresponds to an extending part of near $r_+ = 1$. We have set $\eta = 1, Q = 1, m = 0$.

the mass of charged AdS black holes through $N = 0$ as

$$M = \frac{1}{2} (r_+ + \frac{Q^2}{r_+} + \frac{r_+^3}{l^2}).$$

(13)

The modified entropy can also be computed from the first law $dM = TdS$ with the modified temperature as

$$S = \int \frac{dM}{T} = \pi k r_+ \sqrt{r_+^2 + \eta G_0} + \pi k \eta G_0 \ln(r_+ + \sqrt{r_+^2 + \eta G_0}).$$

(14)

Note that the next leading order is logarithmic as $S \approx \pi r_+^2 + \frac{1}{2} \pi \eta G_0 \ln(4r_+^2)$, which is similar to the quantum correction in Refs[24–28]. With $A = 4\pi r_+^2$ we can achieve $S \approx \frac{4}{4} + \frac{1}{2} \pi \eta G_0 \ln(\frac{A}{\pi})$. We can find the result is in agreement with standard entropy $S = A/4$ when $\eta = 0$ which is standard condition.

The heat capacity with a fixed charge can be calculated as

$$C_Q = T \frac{dS}{dT} = \left( \frac{\partial M/\partial r_+}{\partial T/\partial r_+} \right)$$

$$= 2\pi k \frac{-Q^2l^2r_+^2 + l^2 r_+^4 + 3r_+^6)(r_+^2 + \eta G_0)^{3/2}}{3r_+^2 + (6\eta G_0 - l^2)r_+^3 + 3Q^2l^2 r_+^4 + 2\eta G_0 Q^2 l^2 r_+^6}$$

(15)
which shows that $C_Q$ reduces to standard condition [23] with $\eta = 0$. It is easy to find the heat capacity is diverging when $m^2 = \frac{1}{G}$. Generally, when the temperature tends to zero, the heat capacity also tends to zero. However, our paper show a different and anomalous phenomenon. It may be explained as the black holes could not be observed when test particle mass encounters $m^2 = \frac{1}{G}$. It is lucky the phenomenon is just a observation effect, but the result gives us a way to test the theory of rainbow gravity.

The numerical methods indicate there is special point corresponding to one divergent point of heat capacity with $l = l_c$, there are two divergent points when $l > l_c$ and no divergent point with $l < l_c$ which have been described in Fig.1, Fig.2, Fig.3. Through Eq. 9, we get the different expression compared with Ref. [15]. The Fig.1 shows there is a continuous phase and doesn’t appear phase transition with $l < l_c$. In the Fig.2 there is one divergent point with $l = l_c$, it shows two stable phases for $C_Q > 0$, phase 1 and phase 2, which individually represents a phase of large black hole(LBH) and a small black hole(SBH). In Fig.3, we can find there are three phases and two divergent points with $l > l_c$. Phase 1 experiences a continuous process from an unstable phase for $C_Q < 0$ to a stable phase for $C_Q > 0$; phase 2 is a pure unstable phase for $C_Q < 0$; and phase 3 is a stable phase for $C_Q > 0$. It is easy to find phase 1 represents the phase of SBH and phase 3 represents the phase of LBH. However, there is a special unstable phase 2 between phase 1 and phase 3. It indicates if the system experiences from phase 3 to phase 1, the system must experience a medium unstable state. The state may be explained as an exotic quark-gluon plasma with negative heat capacity [29, 30].
Surprisingly, although rainbow functions modify the $\Lambda(E/E_P)$ term, they don’t affect thermodynamical pressure related to the cosmological constant \cite{17, 31}. So we can take the following relation

\[ P = -\frac{\Lambda(0)}{8\pi} = \frac{3}{8\pi l^2}. \]  

(16)

Figure 4. $P-r_+$ diagram of charged AdS black holes in the rainbow gravity. The temperature of isotherms decreases from top to bottom. The $P_0(1)$ line corresponds to the ideal gas one-phase behaviour for $T > T_c$, the critical isotherm $T = T_c = 0.0358, r_c = 2.7739, P_c = 0.0036$ is denoted by the $P_0(2)$ line, the lowest two lines correspond to the temperature smaller than the critical temperature. We have set $Q = 1, \eta = 1, Q = 1, m = 0$.  

Since David Kubiznak and Robert B. Mann have showed the critical behaviour of
charged-AdS black holes and completed the analogy of this system with the liquid-gas system [23], in what follows we will study whether the critical behavior of the charged AdS black holes system in rainbow gravity was kept. Using Eq.(12) and Eq.(16) in extended phase space, we can get

$$P = \frac{k}{2} \sqrt{\frac{r_+^2 + \eta G_0}{r_+^4}} T - \frac{1}{8\pi r_+^2} \frac{1}{r_+^2} + \frac{1}{8\pi} \frac{Q^2}{r_+^4}. \quad (17)$$

We will follow the method discussed by Ref.[17] to study the critical behaviour. The critical point is obtained from

$$\frac{\partial P}{\partial r_+} = 0, \quad \frac{\partial^2 P}{\partial r_+^2} = 0, \quad (18)$$

which leads to

$$r_c = \sqrt{\frac{2^{4/3} \eta G_0 Q^2 + 2^{4/3} Q^4 + 2 Q^2 (x + y)^{1/3} + 2^{2/3} (x + y)^{2/3}}{(x + y)^{1/3}}}$$

$$T_c = \frac{1}{2\pi k} \frac{r_c^2 - 2 Q^2}{r_c^4 + 2 \eta G_0 r_c^2} \sqrt{r_c^2 + \eta G_0}$$

$$P_c = \frac{r_c^4 - 3 Q^2 r_c^2 - 2 \eta G_0 Q^2}{8\pi r_c^4 (r_c^2 + 2 \eta G_0)}, \quad (19)$$

here $x = \eta G_0 Q^2 (\eta G_0 + Q^2)$, $y = Q^2 (\eta G_0 + Q^2) (\eta G_0 + 2 Q^2)$. We can achieve

$$\frac{P_c r_c}{T_c} = k \frac{r_c^4 - 3 Q^2 r_c^2 - 2 \eta G_0 Q^2}{4 r_c^2 (r_c^2 - 2 Q^2) \sqrt{r_c^2 + \eta G_0}}. \quad (20)$$

which shows the critical ratio was deformed due to the existence of rainbow gravity. It is notable that Eq.(20) will back to the usual universal ratio with $\eta = 0$. Generally, in charged AdS black holes the pressure and temperature were demanded as positive real value. From Eq.(19), we can find with $P_c > 0$ and $T_c > 0$

$$r_c^4 - 3 Q^2 r_c^2 - 2 \eta G_0 Q^2 > 0, r_c > \sqrt{2Q}. \quad (21)$$

It indicates there is a restriction between $Q$ and $\eta$ when there are the positive real critical values.
The $P - r_+$ diagram has been described in Fig. 4. From Fig. 4, we can find that charged AdS black holes in rainbow gravity have an analogy with the Van der Waals system and have a first-order phase transition with $T < T_c$. Namely, when considering rainbow gravity with the form of Eq. (6), the behavior like Van der Waals system also reserve.

Figure 5. Gibbs free energy of charged AdS black holes in rainbow gravity. The blue line $G_0(3)$ corresponds to critical pressure $P = P_c \approx 0.0024$, the line $G_0(4)$ corresponds to pressure $P > P_c$, and the other corresponds to pressure $P < P_c$. We have set $Q = 1, \eta = 1, Q = 1, m = 0$.

We also can discuss the phase transition from the Gibbs free energy, through Ref. [14, 32] the black hole mass is then identified with the enthalpy, rather than the internal energy, so the Gibbs free energy for fixed charge in the rainbow gravity will
be
\[
G = H - TS \\
= \frac{1}{2} \left( r_+ + \frac{Q^2}{r_+} + \frac{r_+^3}{l^2} \right) \\
- kT \left( \pi r_+ \sqrt{r_+^2 + \eta G_0} + \pi \eta G_0 \ln(r_+ + \sqrt{r_+^2 + \eta G_0}) \right).
\]

(22)

It has been showed in the Fig.5. Since the picture of $G$ demonstrates the characteristic swallow tail behaviour, there is a first order transition in the system.

IV. CONCLUSION

In this paper, we have discussed the charged AdS black holes with the test particle with mass $m$. By the modified dispersion relation and HUP, we got deformed temperature in charged AdS black holes. We have found that the temperature is just a rescaling for different mass of test particle. We have discussed the divergence about the heat capacity with a fixed charge. It indicates the numbers of phase will be changing as the $l$ takes different values. When $(l = l_c)$, there is only one divergent point about heat capacity; with $l > l_c$, we have found there are two divergent points and three phases including two stable phases and one unstable phase. In particular, we have finished an analogy between the charged AdS black holes in the rainbow gravity in extending phase space and the liquid-gas system. We have also studied $P - r_+$ critical behavior about the charged AdS black holes in the rainbow gravity. The consequence shows there is the Van der Waals like behavior in the rainbow gravity when $\eta$ and $Q$ coincide with Eq.(21). But the rainbow function deform the consequence of critical pressure, temperature and radius. At last, we have discussed the Gibbs free energy and have obtained characteristic ‘swallow tail’ behaviour that can be explained first-order phase transition.

We also find the temperature tends to zero and heat capacity with a fixed charge
tends to diverging with $m^2 = \frac{1}{ηG}$. Those results indicate there is a special value of mass about test particle encountered $m^2 = \frac{1}{ηG}$ which would test zero temperature and diverging heat capacity. Those results also indicate whether there is a black hole remnant depends on how to choose rainbow function $f$ and $g$ and HUP. In generally, when we choose $E \sim \frac{1}{r_v}$, the black hole remnant may exit. But when we choose $p \sim \frac{1}{r_v}$, the result may be different.

**CONFLICTS OF INTEREST**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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