Mass and 3-metrics of Non-negative Scalar Curvature

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Abstract

Physicists believe, with some justification, that there should be a correspondence between familiar properties of Newtonian gravity and properties of solutions of the Einstein equations. The Positive Mass Theorem (PMT), first proved over twenty years ago [45, 53], is a remarkable testament to this faith. However, fundamental mathematical questions concerning mass in general relativity remain, associated with the definition and properties of quasi-local mass. Central themes are the structure of metrics with non-negative scalar curvature, and the role played by minimal area 2-spheres (black holes).

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1. Positive Mass Theorem

The Positive Mass Theorem provides a good example of “the unreasonable effectiveness of physics in mathematics1”. The need to define mass in general relativity is motivated directly by the physics imperative to establish a correspondence between general relativity and classical Newtonian gravity. Already difficulties arise: although the vacuum Einstein equations $\text{Ric}_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0$ for the Lorentz metric $g_{\alpha\beta}$ suggest (by analogy with the wave equation, for example) that a mass (energy) which includes contributions from the gravitational field, should be built from the first derivatives of the field $g_{\alpha\beta}$, it is clear that this is incompatible with coordinate invariance.

The Schwarzschild vacuum spacetime metric, for $r > \max(0, 2M)$,

$$ds^2 = -(1 - 2M/r) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1.1)$$

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1with apologies to Eugene Wigner [52].
provides an important clue, since the parameter $m \in \mathbb{R}$ governs the behaviour of timelike geodesics and may be regarded as the total mass. Note that $m > 0$ ensures the boundary $r = 2m$ is smooth and totally geodesic in the hypersurfaces $t = \text{const}$.

A Riemannian 3-manifold $(M, g)$ is said to be asymptotically flat if $M \setminus K \cong \mathbb{R}^3 \setminus B_1(0)$ for some compact $K$, and $M$ admits a metric $\tilde{g}$ which is flat outside $K$, and the metric components $g_{ij}$ in the induced rectangular coordinates satisfy

$$|g_{ij} - \tilde{g}_{ij}| = O(r^{-1}), \quad |\partial_k g_{ij}| = O(r^{-2}), \quad |\partial_k \partial_l g_{ij}| = O(r^{-3}).$$

(1.2)

The total mass of $(M, g)$ is defined informally by

$$m_{\text{ADM}} = \frac{1}{16\pi} \int_{S^2(\infty)} (\partial_k g_{ij} - \partial_j g_{ii}) dS_j. \quad (1.3)$$

If the scalar curvature $R(g) \in L^1(M)$ then $m_{\text{ADM}}$ is well-defined, independent of the choices of rectangular coordinates and of exhaustion of $M$ used to define $\int_{S^2(\infty)}$ — see [3, 15, 37] for weaker decay and smoothness assumptions.

For simplicity, the discussion here is restricted to $C^\infty$ Riemannian 3-dimensional geometry. This corresponds to the case of time-symmetric initial data: $(M, g)$ is a totally geodesic spacelike hypersurface in a Lorentzian manifold, and we can identify the local matter (equivalently, energy) density with the scalar curvature $R(g) \geq 0$. This simplification entails a small loss of generality: most, but not all, of the results we describe have been extended to general asymptotically flat space-time initial data $(M, g, K)$, where $K_{ij}$ is the second fundamental form of a spacelike hypersurface $M$. Some results also generalize to the closely related Bondi mass, which measures mass and gravitational radiation flux near null infinity, and to mass on asymptotically hyperbolic and anti-deSitter spaces cf. [51, 16], but these involve additional complications which we will not discuss here.

The Positive Mass Theorem (PMT) in its simplest form is

**Theorem 1** Suppose $(M, g)$ is a complete asymptotically flat 3-manifold with non-negative scalar curvature $R(g) \geq 0$. Then $m_{\text{ADM}} \geq 0$, and $m_{\text{ADM}} = 0$ iff $(M, g) = (\mathbb{R}^3, \delta)$.

The rigidity conclusion in the case $m_{\text{ADM}} = 0$ shows that $m_{\text{ADM}} > 0$ for $(M, g)$ scalar flat (“matter-free”) but non-flat, so $m_{\text{ADM}}$ does provide a measure of the gravitational field.

Three distinct approaches have been successfully used to prove the PMT: with stable minimal surfaces [45, 46]; with spinors [53, 36] and the Schrödinger-Lichnerowicz identity [48, 35]; and using the Geroch foliation condition [23, 30]. A number of other approaches have produced partial results: using spacetime geodesics [42]; a nonlinear elliptic system for a distinguished orthonormal frame [49, 18]; and alternative foliation conditions [22, 33, 15]. The connection between these approaches remains mysterious; the only discernable common thread is mean curvature, and this is quite tenuous.

The application of the positive mass theorem to resolve the Yamabe conjecture [14, 34] is well known. Less well known is the proof of the uniqueness of the
Schwarzschild spacetime amongst static metrics with smooth black hole boundary [13], which we briefly outline.

A static spacetime is a Lorentzian 4-manifold with a hypersurface-orthogonal timelike Killing vector. With $V$ denoting the length of the Killing vector, the metric $g$ on the spacelike hypersurface satisfies the static equations

$$R_{\text{g}} = V^{-1} \nabla^2 V,$$

$$\Delta_g V = 0.$$  \hfill (1.4)

Smoothness implies the boundary set $\Sigma = \{V = 0\}$ is totally geodesic; analyticity of $g, V$ can be used to show the asymptotic expansions

$$g_{ij} = (1 + 2m/r)\delta_{ij} + O(r^{-2}),$$

$$V = 1 - m/r + O(r^{-2}),$$

as $r \to \infty$ for some constant $m \in \mathbb{R}$. The metrics $g^\pm = \phi_\pm^4 g$ where $\phi_\pm = (1 \pm V)/2$ both have $R(g^\pm) = 0$, and $g^+$ is asymptotically flat with vanishing ADM mass, and $g^-$ is a (smooth) metric on a compact manifold. Gluing two copies of $(M, g)$ along the totally geodesic boundary $\Sigma$ and conformally changing to $\tilde{g} = \tilde{\phi}^4 g$ where $\tilde{\phi} = \phi_\pm$ on the two copies of $M$, gives a complete AF manifold with $R(\tilde{g}) = 0$ and vanishing mass. The PMT shows $(M, \tilde{g})$ is flat and it follows without difficulty that $(M, g)$ is Schwarzschild. This extends previous results [31, 43] which required the boundary to be connected.

2. Penrose conjecture

A boundary component $\Sigma$ with mean curvature $H = 0$ is called a black hole or horizon, since if $(M, g)$ is a totally geodesic hypersurface then $\Sigma$ is a trapped surface and hence, by the Penrose singularity theorem [26], lies within an event horizon and is destined to encounter geodesic incompleteness in the predictable future.

The spatial Schwarzschild metric $g = \frac{dr^2}{1 - 2m/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$ with $m < 0$ shows that the completeness condition in the PMT is important, but it can be weakened to allow horizon boundary components of $M$. This follows immediately from the minimal surface argument [45]; or by an extension to the Witten argument [22], imposing one of the boundary conditions

$$\psi = \pm c \psi \text{ on } \Sigma = \partial M,$$  \hfill (2.1)

on the spinor field $\psi$, where $c = \gamma^a \gamma^0$ satisfies $c^2 = 1$. An interesting extension is obtained by imposing the spectral boundary condition

$$P_+ \psi = 0 \text{ on } \Sigma$$  \hfill (2.2)

where $P_+$ is the projection onto the subspace of positive eigenspinors of the induced Dirac operator $D_\Sigma$. Using the remarkable Hijazi-Bär estimate [28, 2]

$$|\lambda| \geq \sqrt{4\pi/|\Sigma|},$$  \hfill (2.3)

for the eigenvalues of $D_\Sigma$ when $\Sigma \simeq S^2$, Herzlich showed [27]
Theorem 2 If \((M, g)\) is asymptotically flat with \(R(g) \geq 0\) and boundary \(\Sigma \simeq S^2\) with mean curvature satisfying
\[
H_\Sigma \leq 2/r
\]
where \(r = \sqrt{|\Sigma|/4\pi}\), then \(m_{\text{ADM}} \geq 0\), with equality iff \((M, g) = (\mathbb{R}^3 \setminus B(r), \delta)\).

The proof starts with the Riemannian form of the Schrödinger-Lichnerowicz-Witten identity \([48, 35, 53]\)
\[
\int_M \left( |\nabla \psi|^2 + \frac{1}{4} R(g) |\psi|^2 - |D\psi|^2 \right) dv_M = 4\pi |\psi_\infty|^2 m_{\text{ADM}} + \int_\Sigma \mu(\psi),
\]
where \(\mu(\psi)\) is the Nester-Witten form \([38]\)
\[
\mu(\psi) = \langle \psi, (D_\Sigma + \frac{1}{2} H_\Sigma) \psi \rangle dv_\Sigma.
\]
The boundary condition \(P^+ \psi|_\Sigma = 0\) is elliptic and it can be shown \([8]\) there is a spinor on \(M\) satisfying \(D\psi = 0\) with boundary conditions \(\psi \to \psi_\infty \neq 0\) as \(r \to \infty\) and \([2.2]\) on \(\Sigma\). It follows from \([2.3]\) and \([2.2]\) that \(\langle \psi, (D_\Sigma + \frac{1}{2} H_\Sigma) \psi \rangle \leq \left( \frac{1}{2} H_\Sigma - |\lambda^-| \right) |\psi|^2 \leq 0\) and the result follows.

Observe that in each case, equality leads to flat \(\mathbb{R}^3\). An elegant physical argument lead Penrose to conjecture an analogous inequality, but which distinguishes the Schwarzschild metric instead \([10]\), see also \([24]\).

Conjecture 3 (Penrose) If \((M, g)\) satisfies the conditions of the PMT, except that \(\partial M = \Sigma\) is compact with vanishing mean curvature and such that \(\Sigma\) is the “outermost” closed minimal surface in \(M\), then
\[
m_{\text{ADM}} \geq \sqrt{|\Sigma|/16\pi},
\]
with equality only for the Schwarzschild metric.

A closed minimal surface is said to be an outermost horizon or outer-minimizing horizon if \(M\) contains no least area surfaces homologous to \(\Sigma\) in the asymptotic region exterior to \(\Sigma\). The outermost condition is essential, since examples of non-negative scalar curvature manifolds can be constructed by forming the connected sum of \(M\) and large spheres by arbitrarily small and large necks.

The Penrose conjecture has been established by Huisken and Ilmanen \([29, 30]\) using a variational level set formulation of the inverse mean curvature flow \([23]\), and by Bray \([12]\) by a very interesting conformal deformation argument. Bray’s proof is more general since it takes into account contributions from all the connected components of the boundary.

3. Quasi-local mass

Thus it is natural to consider \(\sqrt{|\Sigma|/16\pi}\) as the mass of a black hole (minimal surface) \(\Sigma\). More generally, the correspondence with Newtonian gravity suggests that any bounded region \((\Omega, g)\) should have a quasi-local mass, which measures both
the matter density (represented in this case by the scalar curvature $R(g) \geq 0$), and some contribution from the gravitational field. The rather satisfactory positivity properties of the total mass, as established by the PMT, motivate the properties we might expect such a geometric mass to possess \[20, 14, 7\].

1. (non-negativity) $m_{QL}(\Omega) \geq 0$;
2. (rigidity/strict positivity) $m_{QL}(\Omega) = 0$ if and only if $(\Omega, g)$ is flat;
3. (monotonicity) $m_{QL}(\Omega_1) \leq m_{QL}(\Omega_2)$ whenever $\Omega_1 \subset \Omega_2$, where it is understood that the inclusion is a metric isometry;
4. (spherical mass) $m_{QL}$ should agree with the spherical mass, for spherically symmetric regions;
5. (ADM limit) $m_{QL}$ should be asymptotic to the ADM mass;
6. (black hole limit) $m_{QL}$ should agree with the black hole mass \[2.7\].

Many candidates have been proposed for quasi-local mass (see for example \[10\] for a comparison of some definitions), the most significant being that of Hawking \[25\],

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \oint_{\Sigma} H^2\right) \quad (3.1)$$

where $\Sigma = \partial \Omega$. This equals $m$ for standard spheres in Schwarzschild. Although $m_H \leq 0$ for surfaces in $\mathbb{R}^3$, it was shown in \[14\] that $m_H(\Sigma) \geq 0$ for a stable constant mean curvature 2-sphere $\Sigma$ in a 3-manifold of non-negative scalar curvature. Thus for such “round” spheres, $m_H$ is nonegative, and the black hole limit condition is trivially satisfied. However the remaining properties, in particular rigidity and monotonicity, are rather problematic. Although the twistorially-defined Penrose quasi-local mass \[14\] is well-behaved in special cases \[60\], it is defined unambiguously only for surfaces arising from embedding into a conformally flat spacetime, and even then numerical experiments \[11\] strongly suggest that monotonicity is violated.

In fact, of the various proposals for $m_{QL}$, only the definitions of \[14, 5, 19\] are known to satisfy positivity. Dougan and Mason \[19\] show that the integral $\int_{\Sigma} \mu(\psi)$ of the Nester-Witten 2-form \[26\] is positive for spinor fields $\psi$ on $\Sigma$ which satisfy a certain elliptic system on $\Sigma$. However, Bergqvist \[9\] shows that positivity holds under much weaker conditions on $\psi$, and there are many variant definitions with similar properties. It would be useful to understand these DM-style definitions better, and in particular whether any satisfy monotonicity.

Monotonicity and ADM-compatibility imply $m_{QL}(\Omega) \leq m_{ADM}(M, g)$ for any region $\Omega$ embedded isometrically in an $(M, g)$ satisfying (as always) the PMT conditions. This motivates the following definition \[4, 30\]

**Definition 4** Let $\mathcal{PM}$ denote the set of all asymptotically flat 3-manifolds $(M, g)$ of non-negative scalar curvature, with boundary which if non-empty, consists of compact outermost horizons, and such that $(M, g)$ has no other horizons. For any bounded open connected region $(\Omega, g)$, let $\mathcal{PM}(\Omega)$ be the set of $(M, g) \in \mathcal{PM}$ such that $\Omega$ embeds isometrically into $M$, and define

$$m_{QL}(\Omega) = \inf\{m_{ADM}(M, g) : (M, g) \in \mathcal{PM}(\Omega)\}. \quad (3.2)$$

We say that $M$ satisfying these conditions is an admissible extension of $\Omega$. 
The horizon condition serves to exclude examples which hide $\Omega$ inside an arbitrarily small neck, which would force the infimum to zero. This is a refinement \[30\] of the original definition \[4\], which prohibited horizons altogether.

Clearly $m_{QL}(\Omega)$ is well-defined and finite, once the region $\Omega$ admits just one admissible extension. The PMT with horizon boundary implies non-negativity, and monotonicity follows directly. Strict positivity of $m_{QL}$ was established in \[30\], with the slightly weaker rigidity conclusion that if $m_{QL}(\Omega) = 0$ then $\Omega$ is locally flat. Agreement with the spherical mass, and the ADM limit condition, follows also from \[30\]. Bray’s results imply that $m_{QL}(\Omega)$ agrees with the black hole mass in the limit as $\Omega$ shrinks down to a black hole. In addition, $m_{QL}(\Omega) \leq m_{ADM}(M)$ for any admissible extension $M$, so $m_{QL}$ is the optimal quasi-local mass definition with respect to this condition.

The optimal form of the horizon condition remains conjectural. Bray has suggested an alternative condition, that $\Omega$ be a “strictly minimizing hull” \[30\] in $M$, so $\Sigma = \partial \Omega$ has the least area amongst all enclosing surfaces in the exterior. In this case we say $\Sigma$ is outer minimizing, and denote by $\tilde{m}_{QL}(\Omega)$ the quasilocal mass function defined by restricting admissible extensions to those $M$ in which $\Sigma$ is outer minimizing. For this modified definition the Penrose inequality \[30, 12\] applies to show that if $\partial \Omega$ embeds into the Schwarzschild 3-manifold with the same induced metric and mean curvature (cf. (4.1), (4.2)) and encloses the horizon, then $m_{QL}(\Omega) = m$. It is not clear how to establish this natural result for the unmodified definition $m_{QL}(\Omega)$.

4. Static metrics

Although in many respects the definition of $m_{QL}$ is quite satisfactory, it is not constructive, and thus it is important to determine computational methods. The key is the following \[4\]

Conjecture 5 The infimum in $m_{QL}$ is realised by a 3-metric agreeing with $\Omega$ in the interior, static \[1.4\] in the exterior region, and such that the metric is Lipschitz-continuous across the matching surface $\Sigma$, and the mean curvatures of the two sides agree along $\Sigma$.

A similar conjecture for the space-time generalisation of the quasi-local mass, asserts that the exterior metric is stationary, i.e. admits a timelike Killing field \[4, 7\].

As motivation for this conjecture, note first that if $R(g) > 0$ in some region, then a conformal factor $\phi$ can be found such that $\phi^4 g$ has less mass and $R(\phi^4 g) \geq 0$. Thus a mass-minimizing metric for \[4\], if such a metric exists, must have vanishing scalar curvature. Now if the linearization $DR(g) h = \delta g \delta g h - \Delta tr_g h - Ric \cdot h$ is surjective then $g$ admits a variation which produces positive scalar curvature. The formal obstruction to surjectivity is non-trivial $\ker DR(g)^*$, which leads to the static metric equations \[1.4\]. Corvino \[17\] shows that if $\ker DR(g)^*$ is trivial in $U \subset M$ then there are compactly supported metric variations in $U$ which increase the scalar curvature. This gives
Theorem 6: If \((M, g)\) realizes the infimum in Definition 4, then there is a \(V \in C^\infty(M \setminus \Omega)\) such that \(g, V\) satisfy the static metric equations (1.4) in \(M \setminus \Omega\).

This suggests a computational algorithm for determining \(m_{QL}(\Omega)\): find an asymptotically flat static metric with boundary geometry matching that of \(\partial \Omega\). To determine the appropriate boundary conditions, recall the second variation formula for the area of the leaves of a foliation labelled by \(r\):

\[
R(g) = 2D_nH - |II|^2 - H^2 + 2K - 2\lambda^{-1}\Delta_r\lambda
\]

where \(II, H, K\) are respectively the second fundamental form, mean curvature and Gauss curvature of the leaves, \(\lambda\) is the lapse function, \(n = \lambda^{-1}\partial_r\) is the normal vector and \(\Delta_r\) is the Laplacian on the leaves. Our conventions give \(H = -D_n(\log \sqrt{\det g_r})\) where \(g_r\) is the volume element of the leaves. This shows that \(R(g)\) will be defined distributionally across a matching surface as a bounded function if

\[
g|_{\partial \Omega} = g|_{\Sigma}, \quad H|_{\partial \Omega} = H_{\Sigma}. \tag{4.2}
\]

Conjecture 7: \((\Omega, g)\) determines a unique static asymptotically flat manifold \((S, g)\) with boundary \(\Sigma \simeq \partial \Omega\) satisfying (4.2).

If true, this would give a prime candidate for the minimal mass extension. It is known (Pengzi Miao, private communication) that the boundary conditions (4.2) are elliptic for (1.4).

It is tempting to conjecture that mass-minimizing sequences for \(m_{QL}\) should converge to a static metric. For example, [3, Theorem 5.2] shows that a sequence of metrics \(g_k\), close in the weighted Sobolev space \(W^{2,q}_{r}\), \(q > 3, r > 1/2\), to the flat metric \(\delta\) on \(\mathbb{R}^3\) and such that \(m_{ADM}(g_k) \to 0\), converges strongly to \(\delta\) in \(W^{1,2}\). Similar results, under rather different size conditions, are given in [21], and a discussion of the general “weak compactness” conjecture may be found in [30].

5. Estimating quasi-local mass

To estimate \(m_{QL}\) from above, it suffices to construct admissible extensions — metrics with non-negative scalar curvature and satisfying (4.2). These boundary conditions exclude the usual conformal method. Instead, metrics in quasi-spherical form

\[
g = u^2 dr^2 + (r d\theta + \beta^1 dr)^2 + (r \sin \theta d\varphi + \beta^2 dr)^2 \tag{5.1}
\]

satisfy a parabolic equation for \(u\) on \(S^2\) evolving in the radial direction, when \(R(g) = 0\), with \(\beta^1, \beta^2\) freely specifiable. Since the metric 2-spheres \(S^2_r\) have mean curvature \(H_r = (2 - \text{div} g_r \beta)/ur > 0\), (5.1) provides admissible extensions for \(\partial \Omega = S^2_r\) with mean curvature \(H > 0\). The underlying parabolic equation derives from (4.1), and has been generalized to non-spherical foliations in [19]. As an application, choosing \(\beta = 0\) we can show
Theorem 8 Suppose $\partial \Omega = S^2$ metrically, with $H \geq 0$. Then
\[ m_{QL}(\Omega) \leq \frac{1}{2} r(1 - \frac{1}{4} r^2 \min_{\partial \Omega} H^2). \] (5.2)

This bound is sharp when $\Omega$ is a flat ball or a Schwarzschild horizon.

Finding lower bounds for $m_{QL}(\Omega)$ is more difficult. Bray’s definition of inner mass \cite[p243]{12} gives a lower bound, but for $\tilde{m}_{QL}(\Omega)$. The difficulty here as above lies in showing that a horizon inside $\Omega$ remains outermost when the inner region is glued to a general exterior region $M_{ext} \subset M \in PM(\Omega)$. This follows easily when $\Sigma = \partial \Omega$ is outer-minimizing in $M_{ext}$, as guaranteed by the definition for $\tilde{m}_{QL}(\Omega)$.

On physical grounds one expects that if “too much” matter is compressed into region which is “too small”, then a black hole must be present. The geometric challenge lies in making this heuristic statement precise, and the only result in this direction has been \cite{7}, which gives quantitative measures which guarantee the existence of a black hole. An observation by Walter Simon (private communication) is thus very interesting: if $m_{QL}(\Omega) = 1$ (say) and $\Omega$ embeds isometrically into a complete asymptotically flat manifold $M$ without boundary and with non-negative scalar curvature, and such that $m_{ADM}(M) < 1$, then $M$ must have a horizon. This reinforces the importance of finding good lower bounds for $m_{QL}$, since the existence of a horizon in a similar situation with $\tilde{m}_{QL}$ does not follow.

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240 Robert Bartnik

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