Sound waves in strongly coupled non-conformal gauge theory plasma

Paolo Benincasa\textsuperscript{1}, Alex Buchel\textsuperscript{1,2} and Andrei O. Starinets\textsuperscript{2}

\textsuperscript{1}Department of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B7, Canada

\textsuperscript{2}Perimeter Institute for Theoretical Physics
Waterloo, Ontario N2J 2W9, Canada

Abstract

Using gauge theory/gravity duality we study sound wave propagation in strongly coupled non-conformal gauge theory plasma. We compute the speed of sound and the bulk viscosity of $\mathcal{N} = 2^*$ supersymmetric $SU(N_c)$ Yang-Mills plasma at a temperature much larger than the mass scale of the theory in the limit of large $N_c$ and large 't Hooft coupling. The speed of sound is computed both from the equation of state and the hydrodynamic pole in the stress-energy tensor two-point correlation function. Both computations lead to the same result. Bulk viscosity is determined by computing the attenuation constant of the sound wave mode.

July 2005
\section{Introduction and summary}

The conjectured duality between gauge theory and string theory \cite{1, 2, 3, 4} is useful for insights into the dynamics of gauge theories. For strongly coupled finite-temperature gauge theories in particular, the duality provides an effective description in terms of a supergravity background involving black holes or black branes. A prescription for computing the Minkowski space correlation functions in the gauge/gravity correspondence \cite{5, 6} makes it possible to study the real-time near-equilibrium processes (e.g. diffusion and sound propagation) in thermal gauge theories at strong coupling \cite{7, 8, 9, 10, 11, 12, 13, 14, 15}. Computation of transport coefficients in gauge theories whose gravity or string theory duals are currently known may shed light on the true values of those coefficients in QCD, and be of interest for hydrodynamic models used in theoretical interpretation of elliptic flows measured in heavy ion collision experiments at RHIC \cite{16, 17, 18}. This expectation is supported by an intriguing observation \cite{13, 15, 19, 20} that the ratio of the shear viscosity to the entropy density in a strongly coupled gauge theory plasma in the regime described by a gravity dual is universal for all such theories and equal to $1/4\pi$. (Finite \textquoteleft t Hooft coupling corrections to the shear viscosity of $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ gauge theory plasma in the limit of infinite $N_c$ were computed in \cite{14}.)

Previously, it was shown \cite{10} that the dual supergravity computations reproduce the expected dispersion relation for sound waves in the strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory plasma

$$\omega(q) = v_s q - i \frac{\Gamma}{2} q^2 + O(q^3), \quad (1.1)$$

where

$$v_s = \left( \frac{\partial P}{\partial E} \right)^{1/2} \quad (1.2)$$

is the speed of sound, $P$ is pressure and $E$ is the volume energy density. The attenuation constant $\Gamma$ depends on shear and bulk viscosities $\eta$ and $\zeta$,

$$\Gamma = \frac{1}{E + P} \left( \zeta + \frac{4}{3} \eta \right). \quad (1.3)$$

For homogeneous systems with zero chemical potential $E + P = s T$, where $s$ is the volume entropy density. Conformal symmetry of the $\mathcal{N} = 4$ gauge theory ensures that

$$v_s = \frac{1}{\sqrt{3}}, \quad \zeta = 0. \quad (1.4)$$
Indeed, in conformal theories the trace of the stress-energy tensor vanishes, implying the relation between $E$ and $P$ of the form $E = 3P$, from which the speed of sound follows.

Non-conformal gauge theories\(^1\), and QCD in particular, are expected to have non-vanishing bulk viscosity and the speed of sound different from the one given in Eq. (1.4). Lattice QCD results for the equation of state $E = P(E)$ suggest that $v_s \approx 1/\sqrt{3} \approx 0.577$ for $T \approx 2T_c$, where $T_c \approx 173$ MeV is the temperature of the deconfining phase transition in QCD. When $T \to T_c$ from above, the speed of sound decreases rather sharply (see Fig. 11 in [21], [22], [23], and references therein). To the best of our knowledge, no results were previously available for the bulk viscosity of non-conformal four-dimensional gauge theories\(^2\).

In this paper we take a next step toward understanding transport phenomena in four-dimensional gauge theories at strong coupling. Specifically, we study sound wave propagation in mass-deformed $\mathcal{N} = 4$ $SU(N_c)$ gauge theory plasma. In the language of four-dimensional $\mathcal{N} = 1$ supersymmetry, $\mathcal{N} = 4$ gauge theory contains a vector multiplet $V$ and three chiral multiplets $\Phi, Q, \tilde{Q}$, all in the adjoint representation of the gauge group. Consider the mass deformation of the $\mathcal{N} = 4$ theory, where all bosonic components of the chiral multiples $Q, \tilde{Q}$ receive the same mass $m_b$, and all their fermion components receive the same mass $m_f$. Generically, $m_f \neq m_b$, and the supersymmetry is completely broken. When $m_b = m_f = m$ (and at zero temperature), the mass-deformed theory enjoys the enhanced $\mathcal{N} = 2$ supersymmetry with $V, \Phi$ forming an $\mathcal{N} = 2$ vector multiplet and $Q, \tilde{Q}$ forming a hypermultiplet. In this case, besides the usual gauge-invariant kinetic terms for these fields\(^3\), the theory has additional interaction and hypermultiplet mass terms given by the superpotential

$$W = \frac{2\sqrt{2}}{g_{YM}^2} \text{Tr} \left( [Q, \tilde{Q}] \Phi \right) + \frac{m}{g_{YM}^2} \left( \text{Tr} Q^2 + \text{Tr} \tilde{Q}^2 \right).$$  \hspace{1cm} (1.5)

---

\(^1\)One should add at once that throughout the paper terms "conformal" and "non-conformal" refer to the corresponding property of a theory at zero temperature. Conformal invariance of $\mathcal{N} = 4$ SYM is obviously broken at finite temperature, but we still refer to it as "conformal theory" meaning that the only scale in the theory is the temperature itself.

\(^2\)Perturbative results for the bulk viscosity and other kinetic coefficients in thermal quantum field theory of a scalar field were reported in [24]. Bulk viscosity of Little String Theory at Hagedorn temperature was recently computed in [25] using string (gravity) dual description. The speed of sound of a four-dimensional (non-conformal) cascading gauge theory was computed in [26], also from the dual gravitational description.

\(^3\)The classical Kähler potential is normalized according to $2/g_{YM}^2 \text{Tr}[\Phi^2 + Q\tilde{Q} + \bar{Q}\tilde{Q}]$. 

3
The mass deformation of $\mathcal{N} = 4$ Yang-Mills theory described above is known as the $\mathcal{N} = 2^*$ gauge theory. This theory has an exact nonperturbative solution found in [27]. Moreover, in the regime of a large 't Hooft coupling, $g_{YM}^2 N_c \gg 1$, the theory has an explicit supergravity dual description known as the Pilch-Warner (PW) flow [28]. Thus this model provides an explicit example of the gauge theory/gravity duality where some aspects of the non-conformal dynamics at strong coupling can be quantitatively understood on both gauge theory and gravity sides, and compared. (The results were found to agree [29, 30].) Given that the $\mathcal{N} = 2^*$ gauge theory is non-conformal, and at strong coupling has a well-understood dual supergravity representation, it appears that its finite-temperature version should be a good laboratory for studying transport coefficients in non-conformal gauge theories.

The supergravity dual to finite-temperature strongly coupled $\mathcal{N} = 2^*$ gauge theory was considered in [31]. It was shown that the singularity-free nonextremal deformation of the full ten-dimensional supergravity background of Pilch and Warner [28] allows for a consistent Kaluza-Klein reduction to five dimensions. These nonextremal geometries are characterized by three independent parameters, i.e., the temperature and the coefficients of the non-normalizable modes of two scalar fields $\alpha$ and $\chi$ (see Section 2 for more details). Using the standard gauge/gravity correspondence, from the asymptotic behavior of the scalars the coefficient of the non-normalizable mode of $\alpha$-scalar was identified with the coefficient of the mass dimension-two operator in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, i.e., the bosonic mass term $m_b$; the corresponding coefficient of the $\chi$-scalar is identified with the fermionic mass term $m_f$. The supersymmetric Pilch-Warner flow [28] constraints those coefficients so that $m_b = m_f$. However, as emphasized in [31], on the supergravity side $m_b$ and $m_f$ are independent, and for $m_b \neq m_f$ correspond to the mass deformation of $\mathcal{N} = 4$ which completely breaks supersymmetry. In the high temperature limits $m_b/T \ll 1$ (with $m_f \ll m_b$) or $m_f/T \ll 1$ (with $m_b \ll m_f$), the background black brane geometry was constructed analytically. (For generic values of $m_b$, $m_f$, the background supergravity fields satisfy a certain system of coupled nonlinear ODEs [31], which in principle can be solved numerically.)

The holographic renormalization of this theory was discussed in [12], where it was also explicitly checked that for arbitrary values of $m_b$, $m_f$ in the limit of large $N_c$ and large 't Hooft coupling the ratio of the shear viscosity to the entropy density in the theory equals $1/4\pi$, in agreement with universality [13, 15, 19, 20].

Our goal in this paper is to compute the parameters of the dispersion relation (1.1)
for the $\mathcal{N} = 2^*$ plasma. Since the ratio $\eta/s$ is known, this will allow us to determine the speed of sound and the ratio of bulk to shear viscosity in the $\mathcal{N} = 2^*$ theory. We will be able to do it only in the high-temperature regime where the metric of the gravity dual is known analytically, leaving the investigation of the full parameter space to future work. The dispersion relation (1.1) appears as a pole of the thermal two-point function of certain components of the stress-energy tensor in the hydrodynamic approximation, i.e. in the regime where energy and momentum are small in comparison with the inverse thermal wavelength ($\omega/T \ll 1$, $q/T \ll 1$). Equivalently, Eq. (1.1) can be computed as the lowest (hydrodynamic) quasinormal frequency of a certain class of gravitational perturbations of the dual supergravity background [32].

Though the general framework for studying sound wave propagation in strongly coupled gauge theory plasma from the supergravity perspective is known [10, 32], the application of this procedure to a non-conformal gauge theory is technically quite challenging. The main difficulty stems from the fact that unlike the (dual) shear mode graviton fluctuations, the (dual) sound wave graviton fluctuations do not decouple from supergravity matter fluctuations. The interactions between various fluctuations and their background coupling appear to be gauge-theory specific. As a result, we do not expect speed of sound or bulk viscosity to exhibit any universality property similar to the one displayed by the shear viscosity in the supergravity approximation. In fact, we find that the speed of sound and its attenuation do depend on the mass parameters of the $\mathcal{N} = 2^*$ gauge theory.

Let us summarize our results. For the speed of sound and the ratio of the shear to bulk viscosity we find, respectively,

$$v_s = \frac{1}{\sqrt{3}} \left( 1 - \left[ \frac{\Gamma\left(\frac{3}{4}\right)}{3\pi^4} \right]^4 \left( \frac{m_f}{T} \right)^2 - \frac{1}{18\pi^4} \left( \frac{m_b}{T} \right)^4 + \cdots \right),$$  \hspace{1cm} (1.6)

$$\frac{\zeta}{\eta} = \beta_f^\Gamma \left[ \frac{\Gamma\left(\frac{3}{4}\right)}{3\pi^4} \right]^4 \left( \frac{m_f}{T} \right)^2 + \frac{\beta_b^\Gamma}{432\pi^2} \left( \frac{m_b}{T} \right)^4 + \cdots,$$  \hspace{1cm} (1.7)

where $\beta_f^\Gamma \approx 0.9672$, $\beta_b^\Gamma \approx 8.001$, and the ellipses denote higher order terms in $m_f/T$ and $m_b/T$. From the dependence (1.6), (1.7) it follows that at least in the high temperature regime the ratio of bulk viscosity to shear viscosity is proportional to the deviation of the speed of sound squared from its value in conformal theory,

$$\frac{\zeta}{\eta} \approx -\kappa \left( v_s^2 - \frac{1}{3} \right),$$  \hspace{1cm} (1.8)
where\(^4\) \(\kappa = 3\pi \beta^f / 2 \approx 4.558\) for \(m_b = 0\), and \(\kappa = \pi^2 \beta_b^f / 16 \approx 4.935\) for \(m_f = 0\). (Note that the result (1.8) appears to disagree with the estimates \(\zeta \sim \eta (v_s^2 - 1/3)^2\) \([33, 34]\), later criticized in [24].)

The paper is organized as follows. In Section 2 we review the non-extremal (finite-temperature) generalization of the Pilch-Warner flow. We discuss holographic renormalization and thermodynamics of \(\mathcal{N} = 2^*\) gauge theory, and determine its equation of state to leading order in \(m_b/T \ll 1\), \(m_f/T \ll 1\) in Section 3. The equation of state determines the speed of sound (1.6). Computation of the sound attenuation constant requires evaluation of the two-point correlation function of the stress-energy tensor, which is a subject of Section 4. The computation follows the general strategy of [32], albeit with some technical novelties. We keep the parameters \(m_f/T\) and \(m_b/T\) arbitrary until the very end, when we substitute into equations the analytic solution for the metric valid in the high-temperature regime only. Alas, even in that regime we have to resort to numerical methods when computing the bulk viscosity. Having found the sound wave pole of the stress-energy tensor correlator, we confirm the result (1.6) for the speed of sound, and obtain the ratio of shear to bulk viscosity (1.7) for the strongly coupled \(\mathcal{N} = 2^*\) gauge theory in the high temperature regime. Our conclusions are presented in Section 6. Details of the holographic renormalization are discussed in Appendix A. Some technical details appear in Appendixes B–F.

2 Non-extremal \(\mathcal{N} = 2^*\) geometry

The supergravity background dual to a finite temperature \(\mathcal{N} = 2^*\) gauge theory [31] is a deformation of the original \(AdS_5 \times S^5\) geometry induced by a pair of scalars \(\alpha\) and \(\chi\) of the five-dimensional gauge supergravity. (At zero temperature, such a deformation was constructed by Pilch and Warner [28].) According to the general scenario of a holographic RG flow [35], [36], the asymptotic boundary behavior of the supergravity scalars is related to the bosonic and fermionic mass parameters of the relevant operators inducing the RG flow in the boundary gauge theory. The action of the five-dimensional

\(^4\)In general \(\kappa = \kappa(\lambda)\) is a function of the ratio \(\lambda \equiv m_b/m_f\) which we were able to compute in two limits \(\lambda \to 0\) and \(\lambda \to \infty\). Assuming \(\kappa(\lambda)\) to be a smooth monotonic function, we find that it varies by \(\sim 8\%\) over the whole range \(\lambda \in [0, +\infty)\). Additionally, for both finite (and small) \(m_b/T\), \(m_f/T\) we verified scaling (1.8) by explicit fits.
The five-dimensional gauged supergravity is
\[ S = \int_{M_5} d\xi^5 \sqrt{-g} \ L_5 \]
\[ = \frac{1}{4\pi G_5} \int_{M_5} d\xi^5 \sqrt{-g} \left[ \frac{1}{4} R - 3(\partial \alpha)^2 - (\partial \chi)^2 - P \right], \]  
(2.1)
where the potential\(^5\)
\[ P = \frac{1}{16} \left[ \frac{1}{3} \left( \frac{\partial W}{\partial \alpha} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 \right] - \frac{1}{3} W^2 \]  
(2.2)
is a function of \( \alpha, \chi \) determined by the superpotential
\[ W = -e^{-2\alpha} - \frac{1}{2} e^{4\alpha} \cosh(2\chi). \]  
(2.3)
The five-dimensional Newton’s constant is
\[ G_5 \equiv \frac{G_{10}}{2^5 \ vol_{S^5}} = \frac{4\pi}{N^2}. \]  
(2.4)
The action (2.1) yields the Einstein equations
\[ R_{\mu\nu} = 12 \partial_\mu \alpha \partial_\nu \alpha + 4 \partial_\mu \chi \partial_\nu \chi + \frac{4}{3} g_{\mu\nu} P, \]  
(2.5)
as well as the equations for the scalars
\[ \square \alpha = \frac{1}{6} \frac{\partial P}{\partial \alpha}, \quad \square \chi = \frac{1}{2} \frac{\partial P}{\partial \chi}. \]  
(2.6)
To construct a finite-temperature version of the Pilch-Warner flow, one chooses an ansatz for the metric respecting rotational but not the Lorentzian invariance\(^6\)
\[ ds_5^2 = -c_1^2(r) \ dt^2 + c_2^2(r) \left( dx_1^2 + dx_2^2 + dx_3^2 \right) + dr^2. \]  
(2.7)
The equations of motion for the background become
\[ \alpha'' + \alpha' \left( \ln c_1c_2^2 \right)' - \frac{1}{6} \frac{\partial P}{\partial \alpha} = 0, \]
\[ \chi'' + \chi' \left( \ln c_1c_2^2 \right)' - \frac{1}{2} \frac{\partial P}{\partial \chi} = 0, \]
\[ c_1'' + c_1' \left( \ln c_1^2 \right)' + \frac{4}{3} c_1 P = 0, \]
\[ c_2'' + c_2' \left( \ln c_1c_2^2 \right)' + \frac{4}{3} c_2 P = 0, \]  
(2.8)
\(^5\)We set the five-dimensional gauged supergravity coupling to one. This corresponds to setting the radius \( L \) of the five-dimensional sphere in the undeformed metric to 2.
\(^6\)The full ten-dimensional metric is given by Eq. (4.12) in [31].
where the prime denotes the derivative with respect to the radial coordinate $r$. In addition, there is a first-order constraint
\[(\alpha')^2 + \frac{1}{3} (\chi')^2 - \frac{1}{3} P - \frac{1}{2}(\ln c_2)'(\ln c_1 c_2)' = 0. \tag{2.9}\]

A convenient choice of the radial coordinate is
\[x(r) = \frac{c_1}{c_2}, \quad x \in [0, 1]. \tag{2.10}\]

With the new coordinate, the black brane’s horizon is at $x = 0$ while the boundary of the asymptotically $AdS_5$ space-time is at $x = 1$. The background equations of motion (2.8) become
\[c_2'' + 4c_2 (\alpha')^2 - \frac{1}{x} c_2' - \frac{5}{c_2} (c_2')^2 + \frac{4}{3} c_2 (\chi')^2 = 0, \tag{2.11}\]
\[\alpha'' + \frac{1}{x} \alpha' - \frac{1}{12 P c_2^2 x} \left[6(\alpha')^2 c_2^2 x + 2(\chi')^2 c_2^2 x - 3c_2' + 6(c_2')^2 x\right] \frac{\partial P}{\partial \alpha} = 0, \tag{2.11}\]
\[\chi'' + \frac{1}{x} \chi' - \frac{1}{4 P c_2^2 x} \left[6(\alpha')^2 c_2^2 x + 2(\chi')^2 c_2^2 x - 3c_2' + 6(c_2')^2 x\right] \frac{\partial P}{\partial \chi} = 0, \tag{2.11}\]

where the prime now denotes the derivative with respect to $x$. A physical RG flow should correspond to the background geometry with a regular horizon. To ensure regularity, it is necessary to impose the following boundary conditions at the horizon,
\[x \to 0_+ : \quad \left\{ \alpha(x), \chi(x), c_2(x) \right\} \longrightarrow \left\{ \delta_1, \delta_2, \delta_3 \right\}, \tag{2.12}\]

where $\delta_i$ are constants. In addition, the condition $\delta_3 > 0$ guarantees the absence of a naked singularity in the bulk.

The boundary conditions at $x = 1$ are determined from the requirement that the solution should approach the $AdS_5$ geometry as $x \to 1$ _-_:
\[x \to 1_- : \quad \left\{ \alpha(x), \chi(x), c_2(x) \right\} \longrightarrow \left\{ 0, 0, \propto (1 - x^2)^{-1/4} \right\}. \tag{2.13}\]

The three supergravity parameters $\delta_i$ uniquely determine a non-singular RG flow in the dual gauge theory. As we review shortly, they are unambiguously related to the three physical parameters in the gauge theory: the temperature $T$, and the bosonic and fermionic masses $m_b, m_f$ of the $\mathcal{N} = 2^*$ hypermultiplet components.

General analytical solution of the system (2.11) with the boundary conditions (2.12), (2.13) is unknown\(^7\). However, it is possible to find an analytical solution in the regime of high temperatures.

\(^7\)One can study the system numerically, see Fig. 2 of [31].
2.1 The high temperature Pilch-Warner flow

Differential equations (2.11) describing finite temperature PW renormalization group flow admit a perturbative analytical solution at high temperature [31]. The appropriate expansion parameters are

\[ \delta_1 \propto \left( \frac{m_b}{T} \right)^2 \ll 1, \quad \delta_2 \propto \frac{m_f}{T} \ll 1. \tag{2.14} \]

Introducing a function \( A(x) \) by

\[ c_2 \equiv e^A, \tag{2.15} \]

we have [31]

\[ A(x) = \ln \delta_3 - \frac{1}{4} \ln(1 - x^2) + \delta_1^2 A_1(x) + \delta_2^2 A_2(x), \]

\[ \alpha(x) = \delta_1 \alpha_1(x), \]

\[ \chi(x) = \delta_2 \chi_2(x), \tag{2.16} \]

where

\[ \alpha_1 = (1 - x^2)^{1/2} \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x^2 \right), \tag{2.17} \]

\[ \chi_2 = (1 - x^2)^{3/4} \, _2F_1 \left( \frac{3}{4}, \frac{3}{4}; 1; x^2 \right), \tag{2.18} \]

\[ A_1 = -4 \int_0^x \frac{zd\zeta}{(1 - \zeta^2)^2} \left( \gamma_1 + \int_0^\zeta d\eta \left( \frac{\partial \alpha_1}{\partial \eta} \right)^2 \frac{(1 - \eta^2)^2}{\eta} \right), \]

\[ A_2 = -4 \int_0^x \frac{zd\zeta}{(1 - \zeta^2)^2} \left( \gamma_2 + \int_0^\zeta d\eta \left( \frac{\partial \chi_2}{\partial \eta} \right)^2 \frac{(1 - \eta^2)^2}{\eta} \right). \tag{2.19} \]

The constants \( \gamma_i \) were fine-tuned to satisfy the boundary conditions [31]:

\[ \gamma_1 = \frac{8 - \pi^2}{2\pi^2}, \quad \gamma_2 = \frac{8 - 3\pi}{8\pi}. \tag{2.20} \]

The parameters \( \delta_i \) are related to the parameters \( m_b, m_f, T \) of the dual gauge theory via

\[ \delta_1 = -\frac{1}{24\pi} \left( \frac{m_b}{T} \right)^2, \]

\[ \delta_2 = \left[ \frac{\Gamma \left( \frac{3}{4} \right)}{2\pi^{3/2}} \right]^2 \frac{m_f}{T}, \tag{2.21} \]

\[ 2\pi T = \delta_3 \left( 1 + \frac{16}{\pi^2} \delta_1^2 + \frac{4}{3\pi} \delta_2^2 \right). \]
Given the solution (2.16), \( c_2 \) is found from Eq. (2.15), and \( c_1 = xc_2 \). The transition to the original radial variable \( r \) can be made by using the constraint equation (2.9). At ultra high temperatures \( \delta_1 \to 0, \delta_2 \to 0, \) the conformal symmetry in the gauge theory is restored, and one recovers the usual near-extremal black three-brane metric

\[
d s_5^2 = (2\pi T)^2 (1 - x^2)^{-1/2} \left( -x^2 dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \frac{dx_4^2}{1 - x^2} \right),
\]

(2.22)
describing a gravity dual to a finite-temperature \( \mathcal{N} = 4 \) SYM in flat space.

3 \( \mathcal{N} = 2^* \) SYM equation of state and the speed of sound

To determine the equation of state of \( \mathcal{N} = 2^* \) gauge theory, one needs to compute energy and pressure, given by the corresponding one-point functions of the stress-energy tensor. In computing the one-point functions, one has to deal with divergences at the boundary of the asymptotically AdS space which are related to UV divergences in the gauge theory. The method addressing those issues is known as the holographic renormalization [37, 38, 39, 40]. Some details of the holographic renormalization for the \( \mathcal{N} = 2^* \) gauge theory are given in Appendix A. The method works for arbitrary values of \( m_b/T, m_f/T, \) once the solution to Eqs. (2.8) is known. In the high-temperature limit, energy density and pressure can be computed explicitly

\[
E = \frac{3}{8} \pi^2 N^2 T^4 \left[ 1 + \frac{64}{\pi^2} \left( \ln(\pi T) - 1 \right) \delta_1^2 - \frac{8}{3\pi} \delta_1^2 \right],
\]

\[
P = \frac{1}{8} \pi^2 N^2 T^4 \left[ 1 - \frac{192}{\pi^2} \ln(\pi T) \delta_1^2 - \frac{8}{\pi} \delta_1^2 \right],
\]

(3.1)

where \( \delta_1, \delta_2 \) are given by Eqs. (2.21). One can independently compute the entropy density of the non-extremal Pilch-Warner geometry [31],

\[
s = \frac{1}{2} \pi^2 N^2 T^3 \left( 1 - \frac{48}{\pi^2} \delta_1^2 - \frac{4}{\pi} \delta_1^2 \right),
\]

(3.2)

and verify that the thermodynamic relation,

\[
E - Ts = -P,
\]

(3.3)
is satisfied. Alternatively, the free energy can be computed as a renormalized Euclidean action. Then it can be shown [12] that the free energy density \( \mathcal{F} = -P \) obeys \( \mathcal{F} = \mathcal{E} - \)
Ts for arbitrary mass deformation parameters $m_b/T$, $m_f/T$. Finally, using Eq. (2.21) it can be verified that
\[ d\mathcal{E} = T ds . \] (3.4)
These checks demonstrate that the $\mathcal{N} = 2^*$ thermodynamics is unambiguously and correctly determined from gravity.

We can now evaluate leading correction to the speed of sound in $\mathcal{N} = 2^*$ gauge theory plasma at temperatures much larger than the conformal symmetry breaking scales $m_f$, $m_b$. Using Eqs. (3.1), (2.21) we find
\[ v_s^2 = \frac{\partial P}{\partial \mathcal{E}} = \frac{\partial P}{\partial T} \left( \frac{\partial \mathcal{E}}{\partial T} \right)^{-1} = \frac{1}{3} \left( 1 - \frac{64}{\pi^2} \delta_1^2 + \frac{8}{3\pi} \delta_2^2 \right) + \cdots , \] (3.5)
where ellipses denote higher order terms in $m_f/T$ and $m_b/T$. Substituting $\delta_1$, $\delta_2$ from Eqs. (2.21), we arrive at Eq. (1.6). In the next Section we confirm the result (3.5) by evaluating the two-point correlation function of the stress-energy tensor in the sound mode channel and identifying the pole corresponding to the sound wave propagation. In addition to confirming Eq. (3.5), this will allow us to compute the sound wave attenuation constant and thus the bulk viscosity.

### 4 Sound attenuation in $\mathcal{N} = 2^*$ plasma

#### 4.1 Correlation functions from supergravity

We calculate the poles of the two-point function of the stress-energy tensor of the $\mathcal{N} = 2^*$ theory from gravity following the general scheme outlined in [32]. Up to a certain index structure, the generic thermal two-point function of stress-energy tensor is determined by five scalar functions. In the hydrodynamic approximation, one of these functions contains a pole at $\omega = \omega(q)$ given by the dispersion relation (1.1) and corresponding to the sound wave propagation in $\mathcal{N} = 2^*$ plasma. On the gravity side, the five functions characterizing the correlator correspond to five gauge-invariant combinations of the fluctuations of the gravitational background. The functions are determined by the ratios of the connection coefficients of ODEs satisfied by the gauge-invariant variables. Moreover, if one is interested in poles rather than the full correlators, it is sufficient to compute the quasinormal spectrum of the corresponding gauge-invariant fluctuation. This approach is illustrated in [32] by taking $\mathcal{N} = 4$ SYM as an example. For a conformal theory such as $\mathcal{N} = 4$ SYM, the number of independent functions
determining the correlator (and thus the number of independent gauge-invariant variables on the gravity side of the duality) is three. In the $\mathcal{N} = 2^*$ case, the situation is technically more complicated, since we need to take into account fluctuations of the two background scalars. These matter fluctuations do not affect the "scalar" and the "shear" channels\(^8\), entering only the sound channel. In the sound channel, this will lead to a system of three coupled ODEs for three gauge-invariant variables mixing gravitational and scalar fluctuations. The lowest or "hydrodynamic" quasinormal frequency\(^9\) in the spectrum of the mode corresponding to the sound wave gives the dispersion relation (1.1) from which the attenuation constant and thus the bulk viscosity can be read off.

In this Section, we derive the equations for the gauge-invariant variables for a generic finite-temperature Pilch-Warner flow, i.e. without making any simplifying assumptions about the parameters of the flow. Then we solve those equations in the regime $m_b/T \ll 1, m_f/T \ll 1$, where the background is known explicitly. Solving the equations involves numerical integration. The final result is given by Eqs. (1.6), (1.7).

### 4.2 Fluctuations of the non-extremal Pilch-Warner geometry

Consider fluctuations of the background geometry

$$
\begin{align*}
    g_{\mu\nu} &\to g_{\mu\nu}^{BG} + h_{\mu\nu}, \\
    \alpha &\to \alpha_{BG} + \phi, \\
    \chi &\to \chi_{BG} + \psi,
\end{align*}
$$

(4.1)

where $g_{\mu\nu}^{BG}$ (more precisely, $c_1^{BG}$, $c_2^{BG}$), $\alpha_{BG}$, $\chi_{BG}$ are the solutions of the equations of motion (2.8), (2.9). To simplify notations, in the following we use $c_1$, $c_2$ to denote the background values of these fields, omitting the label "BG".

For convenience, we partially fix the gauge by requiring

$$
h_{tr} = h_{x_3r} = h_{rr} = 0.
$$

(4.2)

This gauge-fixing is not essential, since we switch to gauge-invariant variables shortly, but it makes the equations at the intermediate stage less cumbersome. We orient the coordinate system in such a way that the $x_3$ axis is directed along the spatial momentum, and assume that all the fluctuations depend only on $t, x_3, r$. The dependence of

---

\(^8\)See [32] for classification of fluctuations.

\(^9\)Gapless frequency with the property $\lim_{q \to 0} \omega(q) = 0$ required by hydrodynamics.
all variables on time and on the spatial coordinate is of the form $\propto e^{-i\omega t +iqx_3}$, so the only non-trivial dependence is on the radial coordinate $r$.

The fluctuations can be classified according to their transformation properties with respect to the $O(2)$ rotational symmetry in the $x_1 - x_2$ plane [8], [32]. The set of fluctuations corresponding to the sound wave mode consists of

$$h_{tt}, h_{aa} \equiv h_{x_1 x_1} + h_{x_2 x_2}, h_{tx_3}, h_{x_3 x_3}, \phi, \psi.$$  \hspace{1cm} (4.3)

Due to the $O(2)$ symmetry all other components of $h_{\mu\nu}$ can be consistently set to zero to linear order. It will be convenient to introduce new variables $H_{tt}, H_{tz}, H_{aa}, H_{zz}$ by rescaling

$$h_{tt} = c_1^2 H_{tt}, \quad h_{tz} = c_2^2 H_{tz}, \quad h_{aa} = c_2^2 H_{aa}, \quad h_{zz} = c_2^2 H_{zz}.$$  \hspace{1cm} (4.4)

We also use $H_{ii} \equiv H_{aa} + H_{zz}$. Expanding Eqs. (2.8), (2.9) to linear order in fluctuations, we obtain the coupled system of second-order ODEs

$$H''_{tt} + H'_{tt} \left(\ln \frac{c_1^2 c_2^3}{c_1^3}\right)' - H''_{ii} \left(\ln c_1\right)' - \frac{1}{c_1^2} \left(\omega^2 H_{ii} + \omega^2 H_{tt} + 2\omega q H_{tz}\right)$$

$$- \frac{8}{3} \left(\frac{\partial P}{\partial \alpha} \phi + \frac{\partial P}{\partial \chi} \psi\right) = 0,$$  \hspace{1cm} (4.5)

$$H''_{tz} + H'_{tz} \left(\ln \frac{c_2^5}{c_1}\right)' + \frac{1}{c_1^2} \omega q H_{aa} = 0,$$  \hspace{1cm} (4.6)

$$H''_{aa} + H'_{aa} \left(\ln c_1 c_2^5\right)' + (H'_{zz} - H'_{tt}) \left(\ln c_2\right)' + \frac{1}{c_2^4} \left(\omega^2 - q^2 \frac{c_1^2}{c_2^2}\right) H_{aa}$$

$$+ \frac{16}{3} \left(\frac{\partial P}{\partial \alpha} \phi + \frac{\partial P}{\partial \chi} \psi\right) = 0,$$  \hspace{1cm} (4.7)

$$H''_{zz} + H'_{zz} \left(\ln c_1 c_2^4\right)' + (H'_{aa} - H'_{tt}) \left(\ln c_2\right)' + \frac{1}{c_2^4} \left[\omega^2 H_{zz} + 2\omega q H_{tz}\right.$$

$$+ q^2 \frac{c_1^2}{c_2^2} (H_{tt} - H_{aa})]\right] + \frac{8}{3} \left(\frac{\partial P}{\partial \alpha} \phi + \frac{\partial P}{\partial \chi} \psi\right) = 0,$$  \hspace{1cm} (4.8)

$$\phi'' + \phi' \left(\ln c_1 c_2^3\right)' + \frac{1}{2} \alpha''_{BC} (H_{ii} - H_{tt})' + \frac{1}{c_1^2} \left(\omega^2 - q^2 \frac{c_1^2}{c_2^2}\right) \phi$$

$$- \frac{1}{6} \left(\frac{\partial^2 P}{\partial \alpha^2} \phi + \frac{\partial^2 P}{\partial \alpha \partial \chi} \psi\right) = 0,$$  \hspace{1cm} (4.9)
\[
\psi'' + \psi' \left( \ln c_1 c_2^3 \right)' + \frac{1}{2} \chi_{BG}' (H_{ii} - H_{tt})' + \frac{1}{c_1^2} \left( \omega^2 - q^2 c_1^2 c_2^2 \right) \psi - \frac{1}{2} \left( \frac{\partial^2 P}{\partial \alpha \partial \chi} \phi + \frac{\partial^2 P}{\partial \chi^2} \psi \right) = 0, \tag{4.10}
\]

where the prime denotes the derivative with respect to \(r\), and all the derivatives of the potential \(P\) are evaluated in the background geometry. In addition, there are three first-order equations

\[
\omega \left[ H_{ii}' + \left( \ln \frac{c_2}{c_1} \right)' H_{ii} \right] + q \left[ H_{tz}' + 2 \left( \ln \frac{c_2}{c_1} \right)' H_{tz} \right] + 8 \omega \left( 3 \alpha_{BG}' \phi + \chi_{BG}' \psi \right) = 0, \tag{4.11}
\]

\[
q \left[ H_{tt}' - \left( \ln \frac{c_2}{c_1} \right)' H_{tt} \right] + \frac{c_2^2}{c_1^2} \omega H_{tz}' - q H_{aa} - 8q \left( 3 \alpha_{BG}' \phi + \chi_{BG}' \psi \right) = 0, \tag{4.12}
\]

\[
\left( \ln c_1 c_2^3 \right)' H_{ii}' - \left( \ln c_2^3 \right)' H_{tt}' + \frac{1}{c_1^2} \left[ \omega^2 H_{ii} + 2 \omega q H_{tz} + q^2 \frac{c_1^2}{c_2^2} (H_{tt} - H_{aa}) \right] + 4 \left( \frac{\partial P}{\partial \alpha} \phi + \frac{\partial P}{\partial \chi} \psi \right) - 8 \left( 3 \alpha_{BG}' \phi + \chi_{BG}' \psi' \right) = 0. \tag{4.13}
\]

### 4.3 Gauge-invariant variables

A convenient way to deal with the fluctuation equations is to introduce gauge-invariant variables [32]. (Such an approach has long been used in cosmology [41], [42] and in studying black hole fluctuations [43].) Under the infinitesimal diffeomorphisms

\[
x^\mu \rightarrow x^\mu + \xi^\mu \tag{4.14}
\]

the metric and the scalar field fluctuations transform as

\[
g_{\mu \nu} \rightarrow g_{\mu \nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu,
\]

\[
\phi \rightarrow \phi - \nabla^\lambda \alpha_{BG} \xi_\lambda,
\]

\[
\psi \rightarrow \psi - \nabla^\lambda \chi_{BG} \xi_\lambda,
\]

where the covariant derivatives are computed in the background metric. One finds the following linear combinations of fluctuations which are invariant under the diffeomorphisms (4.15)

\[
Z_H = 4q \omega H_{tz} + 2 H_{zz} - H_{aa} \left( 1 - \frac{q^2 c_1' c_2}{\omega^2 c_1^2 c_2^2} \right) + 2q^2 \frac{c_1^2}{c_2^2} H_{tt},
\]

\[
Z_\phi = \phi - \frac{\alpha_{BG}}{\left( \ln c_2^3 \right)'} H_{aa}, \tag{4.16}
\]

\[
Z_\psi = \psi - \frac{\chi_{BG}}{\left( \ln c_2^3 \right)'} H_{aa}.
\]

14
Using Eqs. (4.5)-(4.13), one finds the new variables $Z_H$, $Z_\phi$, $Z_\psi$ satisfy the following system of coupled equations

\begin{align}
A_H Z''_H + B_H Z'_H + C_H Z_H + D_H Z_\phi + E_H Z_\psi &= 0, \quad (4.17a) \\
A_\phi Z''_\phi + B_\phi Z'_\phi + C_\phi Z_\phi + D_\phi Z_\psi + E_\phi Z'_H + F_\phi Z_H &= 0, \quad (4.17b) \\
A_\psi Z''_\psi + B_\psi Z'_\psi + C_\psi Z_\psi + D_\psi Z_\phi + E_\psi Z'_H + F_\psi Z_H &= 0, \quad (4.17c)
\end{align}

where the coefficients depend on the background values $c_1, c_2, \alpha_{BG}, \chi_{BG}$. (The coefficients are given explicitly in Appendixes B, C, D.) Eqs. (4.17) describe fluctuations of the background for arbitrary values of the deformation parameters $m_b/T$, $m_f/T$.

The analysis of Eqs. (4.17) is simplified by switching to the new radial coordinate (2.10). Asymptotic behavior of the solutions to Eqs. (4.17) near the horizon, $x \to 0_+$, corresponds to waves incoming to the horizon and outgoing from it, i.e. for each of the gauge-invariant variables we have $Z_H, Z_\phi, Z_\psi \propto x^{\pm i \omega}$. We are interested in the lowest quasinormal frequency of the “sound wave” variable $Z_H$. To ensure that this frequency is indeed the hydrodynamic dispersion relation (1.1) appearing as the pole in the retarded two-point function of the stress-energy tensor of the $\mathcal{N} = 2^*$ SYM, one has to impose the following boundary conditions [32]

- the incoming wave boundary condition on all fields at the horizon: $Z_H, Z_\phi, Z_\psi \propto x^{-i \omega}$ as $x \to 0_+$;
- Dirichlet condition on $Z_H$ at the boundary $x = 0$: $Z_H(0) = 0$.

The incoming boundary condition on physical modes implies that

$$Z_H(x) = x^{-i \omega} \tilde{Z}_H(x), \quad Z_\phi(x) = x^{-i \omega} \tilde{Z}_\phi(x), \quad Z_\psi(x) = x^{-i \omega} \tilde{Z}_\psi(x), \quad (4.18)$$

where $\tilde{Z}_H, \tilde{Z}_\phi, \tilde{Z}_\psi$ are are regular functions at the horizon. Without the loss of generality the integration constant can be fixed as

$$\left. \tilde{Z}_H \right|_{x \to 0_+} = 1. \quad (4.19)$$

Then the dispersion relation (1.1) is determined by the Dirichlet condition at the boundary [32]

$$\left. \tilde{Z}_H(x) \right|_{x \to 1_-} = 0. \quad (4.20)$$

Following [44], [8], a solution to Eqs. (4.17) can in principle be found in the hydrodynamic approximation as a series in small $\omega, q$ (more precisely, $\omega/T \ll 1, \, q/T \ll 1$),
provided the background values \( c_1, c_2, \alpha_{BG}, \chi_{BG} \) are known explicitly. Here we consider the high-temperature limit (2.14) discussed in Section 2.1, and expand all fields in series for \( \delta_1 \ll 1, \delta_2 \ll 1 \). Accordingly, we introduce

\[
\begin{align*}
\tilde{Z}_H &= \left( Z_0^0 + \delta_1^2 Z_1^0 + \delta_2^2 Z_2^0 \right) + i q \left( Z_0^1 + \delta_1^2 Z_1^1 + \delta_2^2 Z_2^1 \right), \\
\tilde{Z}_\phi &= \delta_1 \left( Z_0^0 + i q Z_1^0 \right), \\
\tilde{Z}_\psi &= \delta_2 \left( Z_0^0 + i q Z_1^0 \right),
\end{align*}
\]

(4.21)

where the upper index refers to either the leading, \( \propto q^0 \), or to the next-to-leading, \( \propto q^1 \) order in the hydrodynamic approximation, and the lower index keeps track of the bosonic, \( \delta_1 \), or fermionic, \( \delta_2 \), mass deformation parameter. Eqs. (4.18), (4.21) represent a perturbative solution of Eqs. (4.17) to first order in \( \omega, q \), and to leading nontrivial order in \( \delta_1, \delta_2 \). From (4.19), the boundary conditions at the horizon are

\[
\begin{align*}
Z_0^0 \bigg|_{x \to 0^+} &= 1, & Z_1^0 \bigg|_{x \to 0^+} &= 0, & Z_2^0 \bigg|_{x \to 0^+} &= 0, \\
Z_0^1 \bigg|_{x \to 0^+} &= 0, & Z_1^0 \bigg|_{x \to 0^+} &= 0, & Z_2^1 \bigg|_{x \to 0^+} &= 0.
\end{align*}
\]

(4.22)

The Dirichlet condition at the boundary (4.20) becomes

\[
\begin{align*}
Z_0^0 \bigg|_{x \to 1^-} &= 0, & Z_1^0 \bigg|_{x \to 1^-} &= 0, & Z_2^0 \bigg|_{x \to 1^-} &= 0, \\
Z_0^1 \bigg|_{x \to 1^-} &= 0, & Z_1^1 \bigg|_{x \to 1^-} &= 0, & Z_2^1 \bigg|_{x \to 1^-} &= 0.
\end{align*}
\]

(4.23)

We also find it convenient to parametrize the frequency as

\[
\omega = \frac{q}{\sqrt{3}} \left( 1 + \beta_v^v \delta_1^2 + \beta_v^s \delta_2^2 \right) - \frac{i q^2}{3} \left( 1 + \beta^v \delta_1^2 + \beta^s \delta_2^2 \right).
\]

(4.24)

In the absence of mass deformation (\( \delta_1 = \delta_2 = 0 \)), Eq. (4.24) reduces to the sound wave dispersion relation for the \( \mathcal{N} = 4 \) SYM plasma [10]. The parametrization (4.24) reflects our expectations that the conformal \( \mathcal{N} = 4 \) SYM dispersion relation will be modified by corrections proportional to the mass deformation parameters \( \delta_1, \delta_2 \). Our goal is to determine the coefficients \( \beta_v^v, \beta_v^s, \beta^v, \beta^s \) by requiring that the perturbative solution (4.21) should satisfy the boundary conditions (4.22), (4.23).
Using the high-temperature non-extremal Pilch-Warner background (2.16), parameterizations (4.21) and (4.24), and rewriting Eqs. (4.17) in the radial coordinate (2.10), we obtain three sets of ODEs describing, correspondingly:

- the pure $\mathcal{N} = 4$ physical sound wave mode ($\delta_1 = 0, \delta_2 = 0$);
- corrections to pure $\mathcal{N} = 4$ physical sound wave mode due to the bosonic mass deformation ($\delta_1 \neq 0, \delta_2 = 0$);
- corrections to pure $\mathcal{N} = 4$ physical sound wave mode due to fermionic mass deformation ($\delta_1 = 0, \delta_2 \neq 0$).

In the remaining part of this subsection we derive equations corresponding to each of these three sets.

### 4.3.1 Sound wave quasinormal mode for $\mathcal{N} = 4$ SYM

Setting $\delta_1 = \delta_2 = 0$ in Eqs. (4.17), (4.21), (4.24) leads to the following equations

$$x(x^2 + 1) \frac{d^2 Z_0^0}{dx^2} + (1 - 3x^2) \frac{dZ_0^0}{dx} + 4xZ_0^0 = 0,$$

(4.25)

$$x(x^2 + 1)^2 \frac{d^2 Z_1^0}{dx^2} - (x^2 + 1)(3x^2 - 1) \frac{dZ_1^0}{dx} + 4x(x^2 + 1) Z_0^1 \bigg(\frac{2}{\sqrt{3}} (x^2 - 1) \frac{dZ_0^0}{dx} + 4 \frac{\sqrt{3}}{x(x^2 - 1)} Z_0^0 \bigg) = 0.$$

(4.26)

The general solution of Eq. (4.25) is

$$Z_0^0 = C_1 (1 - x^2) + C_2 \left[(x^2 - 1) \ln x - 2 \right],$$

(4.27)

where $C_1, C_2$ are integration constants. The condition of regularity at the horizon and the boundary condition (4.22) lead to

$$Z_0^0 = 1 - x^2.$$

(4.28)

Notice that the boundary condition (4.23) is automatically satisfied, as it should be, since (4.24) with $\delta_1 = 0, \delta_2 = 0$ is the correct quasinormal frequency of the gravitational fluctuation in the sound wave channel for the background dual to pure $\mathcal{N} = 4$ SYM plasma. Given the solution (4.28), the general solution to Eq. (4.26) reads

$$Z_0^1 = C_3 (1 - x^2) + C_4 \left[(x^2 - 1) \ln x - 2 \right],$$

(4.29)

where $C_3, C_4$ are integration constants. Imposing regularity at the horizon and the boundary condition (4.22) gives

$$Z_0^1 = 0.$$

(4.30)
Again, the boundary condition (4.23) at \( x = 0 \) is automatically satisfied as a result of the parametrization (4.24).

### 4.3.2 Bosonic mass deformation of the \( N = 4 \) sound wave mode

Turning on the bosonic mass deformation parameter \( \delta_1 \) (while keeping \( \delta_2 = 0 \)) and using the zeroth-order solutions (4.28), (4.30), we find from Eqs. (4.17), (4.21), (4.24)

\[
3x^3(x^2 - 1)^2 \frac{d^2 Z_0^0}{dx^2} + 3x^2(x^2 - 1)^2 \frac{dZ_0^0}{dx} + 3x^3 Z_0^0 + 2(x^2 - 1)^2 \frac{d\alpha_1}{dx} \\
- x(x^2 - 1) \alpha_1 = 0, \tag{4.31}
\]

\[
x^2(x^4 - 1) \frac{d^2 Z_1^0}{dx^2} - x(x^2 - 1)(3x^2 - 1) \frac{dZ_1^0}{dx} + 4x^2(x^2 - 1) Z_1^0 \\
+ 192x^2 \left[ 4x(1 - x^2) \frac{d\alpha_1}{dx} + (1 + x^2) \alpha_1 \right] Z_0^0 + 16x(x^2 - 1)^3 \frac{dA_1}{dx} \\
+ 32(x^4 - 1)(x^2 - 1)^2 \left( \frac{d\alpha_1}{dx} \right)^2 + 8x^2(x^2 - 1) \beta_1^v = 0, \tag{4.32}
\]

\[
3x^3(x^2 - 1)^2 \frac{d^2 Z_0^1}{dx^2} + 3x^2(x^2 - 1)^2 \frac{dZ_0^1}{dx} + 3x^3 Z_0^1 \\
- 2\sqrt{3}x^2(x^2 - 1)^2 \frac{dZ_0^0}{dx} - \sqrt{3}(x^2 - 1) \left[ 2(x^2 - 1) \frac{d\alpha_1}{dx} - x \alpha_1 \right] = 0, \tag{4.33}
\]

\[
x^2(x^4 - 1)(1 + x^2) \frac{d^2 Z_1^1}{dx^2} - x(x^4 - 1)(3x^2 - 1) \frac{dZ_1^1}{dx} + 4x^2(x^4 - 1) Z_1^1 \\
- 192x^2(1 + x^2) \left[ 4x(x^2 - 1) \frac{d\alpha_1}{dx} - (x^2 + 1) \alpha_1 \right] Z_1^0 \\
- \frac{2}{\sqrt{3}}x(x^2 - 1)^3 \frac{dZ_1^0}{dx} + \frac{4}{\sqrt{3}}x^2(x^2 - 1)^2 Z_0^0 \\
- 128\sqrt{3}x^2 \left[ 2x(x^2 - 1)(3x^2 + 1) \frac{d\alpha_1}{dx} - (1 + x^2)^2 \alpha_1 \right] Z_0^0 \\
- \frac{32}{\sqrt{3}}(1 + x^2)^2(x^2 - 1)^3 \left( \frac{d\alpha_1}{dx} \right)^2 - \frac{16}{\sqrt{3}}x(x^2 + 3)(x^2 - 1)^3 \frac{dA_1}{dx} \\
- \frac{8}{\sqrt{3}}(x^2 + 3)(x^2 - 1)x^2 \beta_1^v - \frac{8}{\sqrt{3}}x^2(x^4 - 1) \beta_1^v = 0, \tag{4.34}
\]

where functions \( A_1(x), \alpha_1(x) \) are given by Eqs. (2.17), (2.19).
4.3.3 Fermionic mass deformation of the $N = 4$ sound wave mode

Similarly, turning on the fermionic mass deformation parameter $\delta_2$ and leaving $\delta_1 = 0$ we get

$$12x^3(x^2 - 1)^2 \frac{d^2 Z_0^0}{dx^2} + 12x^2(x^2 - 1)^2 \frac{dZ_0^0}{dx} + 9x^3 Z_0^0 + 8(x^2 - 1)^2 \frac{d\chi_2}{dx}$$

$$- 3x(x^2 - 1) \chi_2 = 0 ,$$

$$3x^2(x^4 - 1) \frac{d^2 Z_0^0}{dx^2} - 3x(x^2 - 1)(3x^2 - 1) \frac{dZ_0^0}{dx} + 12x^2(x^2 - 1) Z_2^0$$

$$- 48x^2 \left[ 16x(x^2 - 1) \frac{d\chi_2}{dx} - 3(1 + x^2) \chi_2 \right] Z_0^0$$

$$+ 32(x^4 - 1)(x^2 - 1)^2 \left( \frac{d\chi_2}{dx} \right)^2 + 48x(x^2 - 1)^3 \frac{dA_2}{dx} + 24x^2(x^2 - 1) \beta_2^\psi = 0 ,$$

$$12x^3(x^2 - 1)^2 \frac{d^2 Z_1^1}{dx^2} + 24x^4(x^2 - 1) \frac{dZ_1^1}{dx} + 9x^3 Z_1^1$$

$$- 8\sqrt{3}x^2(x^2 - 1)^2 \frac{dZ_1^0}{dx} - \sqrt{3}(x^2 - 1) \left[ 8(x^2 - 1) \frac{d\chi_2}{dx} - 3x \chi_2 \right] = 0 ,$$

$$3x^2(x^4 - 1)(1 + x^2) \frac{d^2 Z_2^1}{dx^2} - 3x(x^4 - 1)(3x^2 - 1) \frac{dZ_2^1}{dx} + 12x^2(x^4 - 1) Z_2^1$$

$$- 48x^2(1 + x^2) \left[ 16x(x^2 - 1) \frac{d\chi_2}{dx} - 3(1 + x^2)\chi_2 \right] Z_1^0$$

$$- 2\sqrt{3}x(x^2 - 1)^3 \frac{dZ_2^0}{dx} + 4\sqrt{3}x^2(x^2 - 1)^2 Z_2^0$$

$$- 32\sqrt{3}x^2 \left[ 8x(x^2 - 1)(3x^2 + 1) \frac{d\chi_2}{dx} - 3\chi_2(1 + x^2)^2 \right] Z_1^0$$

$$- \frac{32}{\sqrt{3}}(1 + x^2)^3(x^2 - 1) \left( \frac{d\chi_2}{dx} \right)^2 - 16\sqrt{3}x(x^2 + 3)(x^2 - 1)^3 \frac{dA_2}{dx}$$

$$- 8\sqrt{3}(x^2 + 3)(x^2 - 1)x^2 \beta_2^\psi - 8\sqrt{3}x^2(x^4 - 1) \beta_2^\Gamma = 0 ,$$

where functions $A_2(x), \chi_2(x)$ are given by Eqs. (2.18), (2.19).

5 Solving the fluctuation equations

In this Section, we provide some details on solving the boundary value problems for the bosonic and fermionic mass deformations of the $N = 4$ SYM sound wave mode, discussing in particular the numerical techniques involved. We start with solving Eqs. (4.35)–(4.38) subject to the boundary conditions (4.22)–(4.23).
5.1 Speed of sound and attenuation constant to $O(\delta_2^2)$ in $\mathcal{N} = 2^*$ plasma

Here we solve Eqs. (4.35)–(4.38) subject to the boundary conditions (4.22)–(4.23). Notice that the coefficient $\beta_2^v$ can be determined by imposing the boundary condition (4.23) on the perturbation mode $Z_2^0$. The coefficient $\beta_2^{\Gamma}$ can then be extracted by solving for the mode $Z_2^1$ subject to the boundary condition (4.23).

For the purposes of numerical analysis it will be convenient to redefine the radial coordinate by introducing

$$y \equiv \frac{x^2}{1 - x^2}.$$  

Near the horizon ($x = 0$) we have $y = x^2 + O(x^4)$, while the boundary ($x = 1$) is pushed to $y \to +\infty$. The computation proceeds in four steps:

- First, we solve Eq. (4.35). Applying the arguments of [32] to Eq. (4.35), we find that the appropriate boundary condition on $Z_2^0 \psi$ at $y \to \infty$ is

$$Z_2^0 \psi \sim y^{-3/4} \quad \text{as} \quad y \to \infty.$$  

Eq. (5.2), along with the requirement of regularity at the horizon, uniquely determines the solution $Z_2^0(x)$.

- Second, we solve Eq. (4.36). The solution is an analytic expression involving integrals of the solution $Z_2^0(x)$ constructed in Step 1. Again, the regularity at the horizon plus the horizon boundary condition (4.22) uniquely determine $Z_2^0(x)$. The coefficient $\beta_2^v$ is evaluated numerically after imposing the boundary condition (4.23).

- Third, we solve Eq. (4.37). The boundary condition

$$Z_2^1 \psi \sim y^{-3/4} \quad \text{as} \quad y \to \infty$$  

uniquely determines the solution.

- Finally, we solve Eq. (4.38). The regularity at the horizon and the horizon boundary condition (4.22) uniquely determine $Z_2^1(x)$. Then the coefficient $\beta_2^{\Gamma}$ is determined numerically by imposing the boundary condition (4.23).

Having outlined the four-step approach, we now provide more details on each of the steps involved.

5.1.1 Step 1

Asymptotic behavior near the boundary of the general solution to Eq. (4.35) regular at the horizon is given by

$$Z_2^0 = A_2^0 y^{-1/4} + \cdots + B_2^0 y^{-3/4} + \cdots,$$  

where $A_2^0$ and $B_2^0$ are constants determined by boundary conditions.

20
where $A^0_\psi$, $B^0_\psi$ are the connection coefficients of the ODE. Rescaling the dependent variable as

$$Z^0_\psi \equiv (1 + y)^{-3/4} g_\psi(y),$$

we find that the new function $g_\psi(y)$ satisfies the following differential equation

$$\frac{d^2 g_\psi}{dy^2} + \frac{y + 2}{2y(1 + y)} \frac{dg_\psi}{dy} - \frac{9g_\psi}{16y(1 + y)^2} - \frac{3}{512y(1 + y)^2} \, _2F_1 \left( \frac{7}{4}, \frac{7}{4}; 3; \frac{y}{1 + y} \right) = 0.$$  

(5.6)

Imposing the regularity condition at the horizon, one constructs a power series solution near $y = 0$

$$g_\psi = g^0_\psi + \left( \frac{9}{16} g^0_\psi + \frac{3}{512} \right) y + \left( -\frac{135}{1024} g^0_\psi + \frac{1}{8192} \right) y^2 + \mathcal{O}(y^3).$$

(5.7)

The integration constant $g^0_\psi$ is fixed by requiring that, as Eq. (5.2) suggests,

$$g_\psi \sim \mathcal{O}(1) \quad \text{as} \quad y \to \infty.$$  

(5.8)

For the numerical analysis, we had constructed the power series solution (5.7) to order $\mathcal{O}(y^{19})$. Then, for a given $g^0_\psi$, we numerically integrated $g_\psi$ from some small initial value $y = y_{in}$, using the constructed power series solution to set the initial values of $g_\psi(y_{in})$, $g'_\psi(y_{in})$. (This is necessary as the differential equation has a singularity at $y = 0$.) We verified that the final numerical result is insensitive to the choice of $y_{in}$ as long as $y_{in}$ is sufficiently small (we used $y_{in} = 10^{-15}$). Then we applied a “shooting” method to determine $g^0_\psi$. The “shooting” method is convenient in view of the asymptotic behavior (5.4) near the boundary: unless $g^0_\psi$ is fine-tuned appropriately, for large values of $y$, $g_\psi(y)$ would diverge, $g_\psi \propto y^{1/2}$ as $y \to \infty$. We thus find

$$g^0_\psi \approx -0.02083333(4),$$

(5.9)

where the brackets indicate an error in the corresponding digit. We conjecture that the exact result is

$$g^0_\psi = -\frac{1}{48}.$$  

(5.10)

(Note that a formal analytical solution to Eq. (5.6) is available. It allows to express $g^0_\psi$ in terms of a certain definite integral. Numerical evaluation of the integral confirms the result (5.10).)
5.1.2 Step 2

The solution to Eq. (4.36) regular at the horizon and satisfying the boundary condition (4.22) is

$$Z_2^0(x) = \frac{8}{3}(x^2 - 1) T_{Z_2^0}^a(x) - \frac{8}{3} \left[(x^2 - 1) \ln x - 2\right] T_{Z_2^0}^b(x), \quad (5.11)$$

where

$$T_{Z_2^0}^a(x) = -\frac{3x^2(x^2 + 3 + 2 \ln x)}{4(1 + x^2)^2} \beta_2^u + \int_0^x dz \frac{(z^2 - 1) \ln z - 2}{z(z^4 - 1)(z^2 + 1)^2} \times$$

$$\left\{ 6z^2 \left[ -16z(z^2 - 1) \frac{d\chi_2}{dz} + 3(z^2 + 1) \chi_2 \right] Z_0^0(z) \right. \right. \right.$$

$$\left. \left. + 4(z^4 - 1)(z^2 - 1)^2 \left( \frac{d\chi_2}{dz} \right)^2 + 6z(z^2 - 1)^3 \frac{dA_2}{dz} \right \}, \quad (5.12)$$

and

$$T_{Z_2^0}^b(x) = -\frac{3x^2}{2(1 + x^2)^2} \beta_2^v + \int_0^x dz \frac{1}{z(z^2 + 1)^3} \times$$

$$\left\{ 6z^2 \left[ -16z(z^2 - 1) \frac{d\chi_2}{dz} + 3(z^2 + 1) \chi_2 \right] Z_0^0(z) \right. \right. \right.$$

$$\left. \left. + 4(z^4 - 1)(z^2 - 1)^2 \left( \frac{d\chi_2}{dz} \right)^2 + 6z(z^2 - 1)^3 \frac{dA_2}{dz} \right \}. \quad (5.13)$$

We explicitly verified that

$$\lim_{x \to 1^-} T_{Z_2^0}^a(x) = O(1). \quad (5.14)$$

Thus the boundary condition (4.23) becomes

$$\lim_{x \to 1^-} T_{Z_2^0}^b(x) = 0. \quad (5.15)$$

Numerically solving Eq. (5.15) for $\beta_2^v$, we find

$$\beta_2^v \approx -\frac{4}{3\pi} \times 0.9999(5), \quad (5.16)$$

where we factored out the value $-4/3\pi$ for the coefficient $\beta_2^v$ obtained earlier from the equation of state (see Eq. (3.5)).
5.1.3 Step 3

The general solution to Eq. (4.37) has the form

\[ Z_1^\psi \sin \left( \frac{\sqrt{3}}{2} \arctanh x \right) \times \left\{ C_1 - \frac{1}{6} \int_0^x dz \frac{\cos \left( \frac{\sqrt{3}}{2} \arctanh z \right)}{z^3} \times \right. \]

\[ \left( 8z^2(z^2 - 1) \frac{dZ_0^\psi}{dz} + 8(z^2 - 1) \frac{d\chi_2}{dz} - 3z \chi_2 \right) \right\} \]

\[ + \cos \left( \frac{\sqrt{3}}{2} \arctanh x \right) \times \left\{ C_2 - \frac{1}{6} \int_0^x dz \frac{\sin \left( \frac{\sqrt{3}}{2} \arctanh z \right)}{z^3} \times \right. \]

\[ \left( 8z^2(z^2 - 1) \frac{dZ_0^\psi}{dz} + 8(z^2 - 1) \frac{d\chi_2}{dz} - 3z \chi_2 \right) \right\}, \tag{5.17} \]

where \( C_1, C_2 \) are integration constants. For generic values of these constants we have asymptotically

\[ Z_1^\psi \sim \sin \left( \frac{\sqrt{3}}{2} \arctanh x \right) \left( A_s^\psi (1 + \cdots) + B_s^\psi (1 - x)^{3/4} + \cdots \right) \]

\[ + \cos \left( \frac{\sqrt{3}}{2} \arctanh x \right) \left( A_c^\psi (1 + \cdots) + B_c^\psi (1 - x)^{3/4} + \cdots \right) \tag{5.18} \]

where \( A_s^\psi, B_s^\psi, A_c^\psi, B_c^\psi \) are the ODE connection coefficients. The integration constants \( C_1, C_2 \) should be chosen in such a way that the boundary conditions for the matter fields \( A_s^\psi = 0, A_c^\psi = 0 \) are satisfied.

To make numerical analysis more convenient, we introduce functions \( F_s(x), F_c(x) \) by

\[ Z_1^\psi(x) = (1 - x^2)^{3/4} \left\{ \sin \left( \frac{\sqrt{3}}{2} \arctanh x \right) F_s(x) + \cos \left( \frac{\sqrt{3}}{2} \arctanh x \right) F_c(x) \right\}. \tag{5.19} \]

Redefining the radial coordinate as in Eq. (5.1) and using Eq. (4.37) we find that \( F_s(x), F_c(x) \) satisfy the equations

\[ \frac{dF_s}{dy} - \frac{3}{4(1+y)} F_s = \]

\[ - \frac{\cos \left( \frac{\sqrt{3}}{2} \arctanh y \right)}{y^{1/2}(1+y)^{3/2}} \left[ \frac{3}{128} {}_2F_1 \left( \frac{7}{4}, \frac{7}{4}; \frac{3}{1+y} \right) + g_\psi - \frac{4}{3}(y + 1) \frac{dg_\psi}{dy} \right], \tag{5.20} \]

\[ 23 \]
\[
\frac{dF_c}{dy} - \frac{3}{4(1 + y)} F_c = \sin \left( \frac{\sqrt{3}}{2} \arctanh y \right) \left[ \frac{3}{128} {}_2F_1 \left( \frac{7}{4}, \frac{7}{4}; \frac{3}{1 + y} \right) + g_\psi - \frac{4}{3} (y + 1) \frac{dg_\psi}{dy} \right].
\]

(5.21)

Asymptotics of the solutions \(F_s(x), F_c(x)\) for \(y \to \infty\) are

\[
F_s \to \mathcal{O}(1) + \mathcal{O} \left( y^{3/4} \right), \\
F_c \to \mathcal{O}(1) + \mathcal{O} \left( y^{3/4} \right).
\]

(5.22)

Accordingly, the initial conditions for the first-order ODEs (5.20) and (5.21) should be chosen in such a way that the coefficients of the leading asymptotics in (5.22) vanish.

(This choice guarantees that the matter fluctuations of the mode \(Z_\psi^1\) do not change the fermionic mass parameter of the dual gauge theory.)

Near the horizon, power series solutions to Eqs. (5.20), (5.21) are

\[
F_s = f_s^0 + \left( -\frac{1}{2} g_\psi^0 - \frac{1}{32} \right) y^{1/2} + \frac{3}{4} f_s^0 y + \mathcal{O}(y^{3/2}), \\
F_c = f_c^0 + \left( \frac{\sqrt{3}}{128} + \frac{3}{4} f_c^0 + \frac{\sqrt{3}}{8} g_\psi^0 \right) y + \left( -\frac{3}{32} f_c^0 + \frac{11\sqrt{3}}{8192} - \frac{37\sqrt{3}}{1536} g_\psi^0 \right) y^2 + \mathcal{O}(y^3),
\]

(5.23)

(5.24)

where \(f_s^0, f_c^0\) are integration constants, and \(g_\psi^0\) is chosen as in Eq. (5.10). We use a "shooting" method to determine \(f_s^0, f_c^0\): these initial values should be tuned to ensure that \(F_c, F_s\) remain finite in the limit \(y \to \infty\). We find

\[
f_s^0 \approx 0.01964015(5), \\
f_c^0 \approx -0.01743333(5),
\]

(5.25)

where the brackets symbolize that there is an error in the corresponding digit.

5.1.4 Step 4

The solution to Eq. (4.38) regular at the horizon and obeying the boundary condition (4.22) reads

\[
Z_2^1(x) = \frac{2(x^2 - 1)}{3^{3/2}} T_{Z_2^1}^a(x) + \frac{4}{3^{3/2}} T_{Z_2^1}^b(x) + \frac{2}{3^{3/2}} (x^2 - 1) \ln x \ T_{Z_2^1}^c(x),
\]

(5.26)

where the functions \(T_{Z_2^1}^a(x), T_{Z_2^1}^b(x), T_{Z_2^1}^c(x)\) are given explicitly in Appendix E. We verified that

\[
\lim_{x \to 1^-} T_{Z_2^1}^a = \mathcal{O}(1), \quad \lim_{x \to 1^-} T_{Z_2^1}^c = \mathcal{O}(1),
\]

(5.27)
and thus the boundary condition (4.23) translates into the equation

$$\lim_{x \to 1^-} \frac{T^b}{Z^2} = 0.$$  \hfill (5.28)

To determine the coefficient $\beta^\Gamma_2$, we solve Eq. (5.28) numerically using the value of $\beta^\Gamma_2$ computed in (5.16). We find

$$\beta^\Gamma_2 \equiv \beta^\Gamma_j \approx 0.9672(1).$$  \hfill (5.29)

### 5.2 Speed of sound and attenuation constant to $O(\delta^2_N)$ in $N = 2^*\text{ plasma}$

We now turn to solving Eqs. (4.31)–(4.34) subject to boundary conditions (4.22)-(4.23). The computation is essentially identical to the one in the previous Section, thus we only highlight the main steps.

#### 5.2.1 Step 1

Using the radial coordinate defined by Eq. (5.1), we find that the asymptotic behavior near the boundary of the general solution to Eq. (4.31) is given by

$$Z^0_\phi = A^0_\phi y^{-1/2} + \cdots + B^0_\phi y^{-1/2} \log y + \cdots,$$  \hfill (5.30)

where $A^0_\phi$, $B^0_\phi$ are the connection coefficients of the differential equation. Introducing a new function $g_\phi(y)$ by

$$Z^0_\phi \equiv (1 + y)^{-1/2} g_\phi(y),$$  \hfill (5.31)

we obtain an inhomogeneous differential equation for $g_\phi$:

$$\frac{d^2 g_\phi}{dy^2} + \frac{1}{y} \frac{dg_\phi}{dy} - \frac{1}{4y(1+y)^2} g_\phi - \frac{1}{96y(1+y)^2} \ _2F_1 \left( \frac{3}{2}, \frac{3}{2}; 3; \frac{y}{1+y} \right) = 0.$$  \hfill (5.32)

Near the horizon one can construct a series solution parametrized by the integration constant $g^0_\phi$ of the solution to the homogeneous equation

$$g_\phi = g^0_\phi + \left( \frac{1}{96} + \frac{1}{4} g^0_\phi \right) y + \left( -\frac{1}{384} - \frac{7}{64} g^0_\phi \right) y^2 + O(y^3).$$  \hfill (5.33)

The integration constant $g^0_\phi$ is fixed by requiring that

$$g_\phi \sim O(1) \quad \text{as} \quad y \to \infty.$$  \hfill (5.34)
Using the “shooting” method we obtain
\[ g_\phi^0 \approx -0.08333333(3). \] (5.35)

We conjecture that the exact result is
\[ g_\phi^0 = -\frac{1}{12}. \] (5.36)

(Note that a formal analytical solution to Eq. (5.32) is available. It allows to express \( g_\phi^0 \) in terms of a certain definite integral. Numerical evaluation of the integral confirms the result (5.36).)

5.2.2 Step 2

The solution to Eq. (4.32) regular at the horizon and satisfying the boundary condition (4.22) has the form
\[ Z_1^0(x) = 8(x^2 - 1) T_{Z_1^0}^{a}(x) - 8 \left[(x^2 - 1) \ln x - 2\right] T_{Z_1^0}^{b}(x), \] (5.37)

where
\[ T_{Z_1^0}^{a}(x) = -\frac{x^2(3 + x^2 + 2 \ln x)}{4(1 + x^2)^2} \beta_1^v + \int_0^x dz \frac{(z^2 - 1) \ln z - 2}{z(z^4 - 1)(1 + z^2)^2} \times \left\{ 24z^2 \left[ -4z(z^2 - 1) \frac{d\alpha_1}{dz} + (z^2 + 1) \alpha_1 \right] Z_\phi^0 \right. \]
\[ + 4(z^4 - 1)(z^2 - 1)^2 \left( \frac{d\chi_2}{dz} \right)^2 + 6z(z^2 - 1)^2 \frac{dA_2}{dz} \left\} \right. \] (5.38)

\[ T_{Z_1^0}^{b}(x) = -\frac{x^2}{2(1 + x^2)^2} \beta_1^v + \int_0^x dz \frac{1}{z(z^2 + 1)^3} \times \left\{ 24z^2 \left[ -4z(z^2 - 1) \frac{d\alpha_1}{dz} + (z^2 + 1) \alpha_1 \right] Z_\phi^0 \right. \]
\[ + 4(z^4 - 1)(z^2 - 1)^2 \left( \frac{d\chi_2}{dz} \right)^2 + 6z(z^2 - 1)^2 \frac{dA_2}{dz} \left\} \right. \] (5.39)

We have verified that
\[ \lim_{x \to 1^-} T_{Z_1^0}^{a}(x) = O(1), \] (5.40)

and thus the boundary condition (4.23) translates into the equation
\[ \lim_{x \to 1^-} T_{Z_1^0}^{b}(x) = 0. \] (5.41)
Solving Eq. (5.15) numerically for $\beta_1^\nu$ we find

$$\beta_1^\nu \approx -\frac{32}{\pi^2} \times 1.00000(1), \quad (5.42)$$

where we factored out the value $-32/\pi^2$ obtained from thermodynamics (see Eq. (3.5)).

5.2.3 Step 3

Asymptotics of the general solution to Eq. (4.33) near the boundary $y \to \infty$ is given by Eq. (5.30) with different coefficients $A^1_\phi, B^1_\phi$. Rescaling the dependent variable

$$Z^1_\phi = (1 + y)^{-1/2} G_\phi(y), \quad (5.43)$$

we find that the function $G_\phi$ satisfies the following differential equation

$$\begin{align*}
\frac{d^2 G_\phi}{dy^2} + \frac{1}{y} \frac{dG_\phi}{dy} - \frac{1}{4y(1+y)^2} G_\phi + \frac{\sqrt{3}}{96y(1+y)^2} \, _2F_1 \left( \frac{3}{2}, \frac{3}{2}; 3; \frac{y}{1+y} \right) \\
- \frac{1}{2\sqrt{3}y(1+y)^2} \left[ 2(y+1) \frac{dg_\phi}{dy} - g_\phi \right] = 0. \quad (5.44)
\end{align*}$$

One can construct a series solution regular near $y = 0$

$$G_\phi = G^0_\phi + \left( -\frac{\sqrt{3}}{144} - \frac{\sqrt{3}}{12} g^0_\phi + \frac{1}{4} G^0_\phi \right) y + \mathcal{O}(y^2). \quad (5.45)$$

The integration constant $G^0_\phi$ is fixed by requiring that

$$G_\phi \sim \mathcal{O}(1) \quad \text{as} \quad y \to \infty. \quad (5.46)$$

Using the “shooting” method and the value of the constant $g^0_\phi$ given by Eq. (5.36) we find

$$G^0_\phi \approx 0.059(0). \quad (5.47)$$

5.2.4 Step 4

The solution to Eq. (4.34) regular at the horizon and obeying the boundary condition (4.22) reads

$$Z^1_1(x) = \frac{2(x^2 - 1)}{3^{1/2}} T^a_{Z^1_1}(x) + \frac{4}{3^{1/2}} T^b_{Z^1_1}(x) + \frac{2(x^2 - 1) \ln x}{3^{1/2}} T^c_{Z^1_1}(x), \quad (5.48)$$
where the functions $T^a_{Z_1}(x), T^b_{Z_1}(x), T^c_{Z_1}(x)$ are given explicitly in Appendix F. We checked that
\[ \lim_{x \to 1^{-}} T^a_{Z_1} \sim \mathcal{O}(1), \quad \lim_{x \to 1^{-}} T^c_{Z_1} \sim \mathcal{O}(1). \] (5.49)
In view of Eq. (5.49), the condition (4.23) becomes
\[ \lim_{x \to 1^{-}} T^b_{Z_1} = 0. \] (5.50)
To obtain the coefficient $\beta^{\Gamma}_1$, we solve Eq. (5.50) numerically using the value of $\beta^{\eta}_1$ given by Eq. (5.42). We find
\[ \beta^{\Gamma}_1 \equiv \beta^{\Gamma}_b \approx 8.001(8). \] (5.51)
This completes our computation of the coefficients $\beta^{\eta}_1, \beta^{\eta}_2, \beta^{\Gamma}_1, \beta^{\Gamma}_2$: they are given, respectively, by Eqs. (5.42), (5.16), (5.51), (5.29).

6 Conclusion

In this paper, we considered the problem of computing the speed of sound and the bulk viscosity of $\mathcal{N} = 2^*$ supersymmetric $SU(N_c)$ gauge theory in the limit of large 't Hooft coupling and large $N_c$, using the approach of gauge theory/gravity duality. The computation can be done explicitly in the high temperature regime, i.e. at a temperature much larger than the mass scale $m_b$ and $m_f$ of the bosonic and fermionic components of the chiral multiplets, where the metric of the dual gravitational background is known. Our results for the speed of sound and the bulk viscosity computed in that regime are summarized in Eqs. (1.6), (1.7), (1.8). It would be interesting to extend the computation to the full parameter space of the theory as well as to other theories with non-vanishing bulk viscosity. It would also be interesting to compare our results with a perturbative calculation of bulk viscosity in a finite-temperature gauge theory at weak coupling.

Acknowledgments

AB would like to thank O. Aharony for valuable discussions. A.O.S. would like to thank P. Kovtun and D. T. Son for helpful conversations, and L. G. Yaffe for correspondence. Research at Perimeter Institute is supported in part by funds from NSERC of Canada. AB gratefully acknowledges support by NSERC Discovery grant.
A Energy density and pressure in $\mathcal{N} = 2^*$ gauge theory

Energy density and pressure of $\mathcal{N} = 2^*$ SYM theory on the boundary $\partial M_5$ with the metric (2.7) can be related to the renormalized stress-energy tensor one-point function as follows

\[ \mathcal{E} = \sqrt{\sigma} N_{\Sigma} u^\mu u^\nu \langle T_{\mu\nu} \rangle, \quad (A.1) \]

\[ P = \sqrt{\sigma} N_{\Sigma} \langle T_{xx_1} \rangle \gamma^{x_1 x_1}, \quad (A.2) \]

where $u^\mu$ is the unit normal vector to a spacelike hypersurface $\Sigma$ in $\partial M_5$, $\sigma$ is the determinant of the induced metric on $\Sigma$, and $N_{\Sigma}$ is the norm of the timelike Killing vector in the metric (2.7). The renormalized stress-energy tensor correlation functions are determined from the boundary gravitational action (with the appropriate counterterms added) in the procedure known as the holographic renormalization. Holographic renormalization of $\mathcal{N} = 2^*$ gauge theory on a constant curvature manifold was studied in [12]. Using the results for the renormalized stress-energy tensor one-point functions [12], one finds

\[ \mathcal{E} = \frac{1}{2064\pi G_5} e^\xi \left( 18\beta - 9\hat{\rho}^2_{11} - 12\hat{\chi}^2_0 \hat{\chi}_{10} + 36\hat{\rho}_{11} \hat{\rho}_{10} - 16\hat{\chi}^4_0 \xi + 36\hat{\rho}^2_{11} \xi \right), \quad (A.3) \]

\[ P = \frac{1}{2064\pi G_5} e^\xi \left( 6\beta + 9\hat{\rho}^2_{11} + 12\hat{\chi}^2_0 \hat{\chi}_{10} - 36\hat{\rho}_{11} \hat{\rho}_{10} + 16\hat{\chi}^4_0 \xi - 36\hat{\rho}^2_{11} \xi \right), \quad (A.4) \]

where the parameters $\beta, \xi, \hat{\rho}_{10}, \hat{\rho}_{11}, \hat{\chi}_0, \hat{\chi}_{10}$ are related to physical masses and the temperature for generic values of $m_b/T$ and $m_f/T$ (see Section 6.4 of [31]). In the limit $m_b/T \ll 1$ and $m_f/T \ll 1$, matching the high temperature Pilch-Warner flow background (2.16) with the general UV asymptotics of the supergravity fields we find [31]

\[ \beta = 2, \quad e^\xi = 2^{1/2} \pi T \left( 1 - \frac{12}{\pi^2} \delta_1^2 - \frac{1}{\pi^2} \delta_2^2 \right), \]

\[ \hat{\rho}_{10} = \frac{4\ln 2}{\pi} \delta_1, \quad \hat{\rho}_{11} = -\frac{8}{\pi} \delta_1, \quad (A.5) \]

\[ \hat{\chi}_0 = -\frac{\sqrt{2\pi}}{\left[ \Gamma \left( \frac{3}{4} \right) \right]^2} \delta_2, \quad \hat{\chi}_{10} = -\frac{2}{\pi^2} \left[ \Gamma \left( \frac{3}{4} \right) \right]^4 \] .

Using Eq. (A.5), from Eqs. (A.3), (A.4) we obtain (to quadratic order in $\delta_1, \delta_2$) the energy density and the pressure given in Eqs. (3.1).
B Coefficients of Eq. (4.17a)

\[ A_H(x) = 3\omega^2 c_2^2 c_2^2 c_1^2 \left(-c_2 c_1 q^2 c'_1 - 2c_2 c_1^2 q + 3\omega^2 c_2^2 c_2^2\right), \]

\[ B_H(x) = \omega^2 c_2 c_1 \left(27\omega^2 c_2 c_1 c_2^3 - 42c_1^2 q^2 c_2 + 9c_2 c_1^2 c_2^2 c_1^2 - 8c_2^3 c_1^2 q^2 c_2^2 p + 8c_2^3 q^2 c_2^3 c_1^2 - 3c_2^2 c_1^2 q^2 c_2^2 c_1^2 + 9\omega^2 c_2^3 c_2^2 c_1^2\right), \]

\[ C_H(x) = \omega^2 \left(-3\omega^2 c_2^3 c_2^2 q^2 c_1 c'_1 + 9\omega^4 c_2^2 c_2^2 - 8q^2 c_1^2 c_2^2 c_2^2 p + 36q^2 c_1^3 c_2^3 c_1 c_2 \right. \]

\[-24q^2 c_1^4 c_2^4 + 16q^2 c_1^3 c_2^3 c_1 c_2 p - 8q^2 c_1^2 c_2^2 c_1^2 p + 3c_1^3 c_2^3 c_1 c_2 + 6c_2^3 c_1^2 q^4 \]

\[-12q^2 c_1^3 c_2^3 c_2 - 15\omega^2 c_2^3 c_2^2 q^2 c_2\right), \]

\[ D_H(x) = 16q^2 c_1^2 \left(-c_1 c'_2 + c'_1 c_2\right) \left(24c_1 \omega^2 (\alpha^{BG})' p c_2^3 + 36c_2 \omega^2 c_2^2 (\alpha^{BG})' c'_1 + 3c_2^3 c_1 \omega^2 c_2^2 \frac{\partial p}{\partial \alpha}\right. \]

\[-c_2 q^2 c_1^2 \frac{\partial p}{\partial \alpha} c'_1 - 36 c_2 c_1^2 c_2^2 \omega^2 (\alpha^{BG})' - 24 c_2^2 c_2^3 c_1^3 p (\alpha^{BG})' - 2q^2 c_2^3 c_1^2 \frac{\partial p}{\partial \alpha}\right), \]

\[ E_H(x) = 16q^2 c_1^2 \left(-c_1 c'_2 + c'_1 c_2\right) \left(8c_1 \omega^2 \chi'_{BG} p c_2^3 + 12\omega^2 c_2^2 \chi'_{BG} c'_1 c_2 - 3c_2^3 c_1 \omega^2 c_2^2 \frac{\partial p}{\partial \chi} \right. \]

\[-c_2 q^2 c_1^2 \frac{\partial p}{\partial \chi} c'_1 - 8c_2 q^2 c_1^3 p \chi'_{BG} - 12c_1 c_2^2 \omega^2 \chi'_{BG} c_2 - 2q^2 c_2^3 c_1^2 \frac{\partial p}{\partial \chi}\right). \]

The prime denotes the derivative with respect to \( r \).
C Coefficients of Eq. (4.17b)

\[ A_\phi = 12c_2^2c_1^2c_2' \left( -c_2c_1q^2c_1' - 2c_2^2c_1^2q^2 + 3\omega^2c_2c_2^2 \right), \]

\[ B_\phi = 12c_1c_2'c_2 \left( -c_2c_1q^2c_1' - 2c_2^2c_1^2q^2 + 3\omega^2c_2c_2^2 \right) \left( 3c_1c_2' + c_1c_2 \right), \]

\[ C_\phi = -6c_2^3\omega^2c_2c_2'q^2c_1 - 192c_2^5\omega^2c_2^3(\alpha_{BG})^2 \mathcal{P} - 3c_2^4\omega^2c_2c_2^2 \frac{\partial^2 \mathcal{P}}{\partial \alpha^2} \]

\[ - 30c_2^3\omega^2c_2^2 q^2 c_1 + 6q^4c_2^3c_1c_2 + 2q^2c_2^2c_2^2 \frac{\partial^2 \mathcal{P}}{\partial \alpha^2} - 48c_2^5\omega^2c_2'c_{BG}c_2^2 \frac{\partial \mathcal{P}}{\partial \alpha} ^2 \]

\[ + q^2c_2^3c_2'c_2^2 \frac{\partial^2 \mathcal{P}}{\partial \alpha^2} c_1 + 40q^2c_2^3c_2c_{BG}c_2^3 \frac{\partial \mathcal{P}}{\partial \alpha} c_1 + 12q^4c_2^2c_1^4 \]

\[ + 18c_2^2c_2^2\omega^4 + 192c_2^4c_2^4c_{BG}\mathcal{P} , \]

\[ D_\phi = -2c_2^2c_2c_2' \frac{\partial^2 \mathcal{P}}{\partial \alpha \partial \chi} c_2'q^2 + 24c_2^2c_2'c_{BG} \frac{\partial \mathcal{P}}{\partial \chi} \omega^2 + 3c_2^2c_2^2 \frac{\partial^2 \mathcal{P}}{\partial \alpha \partial \chi} \omega^2 \]

\[ - 2c_2^2c_2c_2' \frac{\partial^2 \mathcal{P}}{\partial \alpha \partial \chi} q^2 + 8c_2^2c_2\omega^2 \chi_{BG} \frac{\partial \mathcal{P}}{\partial \alpha} + 64c_2c_{BG}^2 \omega^2 \chi_{BG} \mathcal{P} \]

\[ - 16c_2c_2c_2^2c_{BG} \frac{\partial \mathcal{P}}{\partial \chi} q^2 - 64c_2^2c_2c_{BG}^2q^2 \chi_{BG} \mathcal{P} - 8c_2c_2^2c_2^2 \chi_{BG}^2 \frac{\partial \mathcal{P}}{\partial \alpha} \]

\[ - 8c_2^2c_2^2c_{BG} \frac{\partial \mathcal{P}}{\partial \chi} c_1q^2 \right), \]

\[ E_\phi = -c_2^2\omega^2 \left( 8\alpha_{BG} \mathcal{P}c_2 + c_2^2 \frac{\partial \mathcal{P}}{\partial \alpha} \right), \]

\[ F_\phi = c_1c_2^2\omega^2 \left( 8\alpha_{BG} \mathcal{P}c_2 + c_2^2 \frac{\partial \mathcal{P}}{\partial \alpha} \right) \left( -c_1c_2' + c_1c_2 \right). \]

The prime denotes the derivative with respect to \( r \).
D Coefficients of Eq. (4.17c)

\[ A_\psi = 12c_2^2c_1'c_2\left(-c_2c_1q^2c_1' - 2c_2'c_1q^2 + 3\omega^2c_2c_1^2\right), \]

\[ B_\psi = 12c_1'c_2\left(-c_2c_1q^2c_1' - 2c_2'c_1q^2 + 3\omega^2c_2c_1^2\right)
     \left(3c_1c_2' + c_1c_2\right), \]

\[ C_\psi = 2\left(3q^2c_2^3c_2'c_2\frac{\partial^2 p}{\partial \chi^2}c_1 + 6q^2c_2^2c_1c_1'o_c - 30c_2^2\omega^2c_2^2q^2c_1 - 64c_2^2\omega^2c_2^2(\chi'_{BG})^2P
     \right. \]

\[ \left. - 48c_2^2\omega^2c_2'c_2'c_2'c_1\frac{\partial p}{\partial \chi} - 6c_2^2\omega^2c_2'c_1'q^2c_1 + 6q^2c_1c_1c_2'(c_2')^2\frac{\partial^2 p}{\partial \chi^2} + 64q^2c_2^4(\chi'_{BG})^2P
     \right. \]

\[ + 8q^2c_1c_2c_4c_2'c_1\frac{\partial p}{\partial \chi} + 40q^2c_2c_2c_2c_2\frac{\partial p}{\partial \chi} - 9c_2^2\omega^2c_2c_1^2\frac{\partial^2 p}{\partial \chi^2} + 18c_2^2c_4^2\omega^4
     \right. \]

\[ + 12q^4(c_2')^2c_1^4, \]

\[ D_\psi = -2c_2^2c_1\left(-6c_2c_1\frac{\partial^2 p}{\partial \alpha \partial \chi}q^2 + 9c_2c_2\frac{\partial^2 p}{\partial \alpha \partial \chi} - 192c_2^2c_1c_2\omega^2c_2c_1c_1'q^2c_1c_1'o_c - 192c_2c_1c_2c_2c_1c_1'o_c^2c_2c_2c_2c_2\frac{\partial^2 p}{\partial \alpha ^2}c_1c_1'o_c^2c_2c_2c_2c_2\frac{\partial p}{\partial \alpha} \right. \]

\[ \left. + 3c_2c_2c_2c_2\frac{\partial^2 p}{\partial \alpha \partial \chi}c_1c_1'o_c - 72c_2c_2c_2c_2\frac{\partial p}{\partial \chi} + 72c_2c_2c_2c_2\frac{\partial p}{\partial \chi}c_1c_1'o_c \right) \]

\[ + 24c_2^3c_2\omega^2c_2'c_2'c_2'c_2\frac{\partial p}{\partial \alpha} \right), \]

\[ E_\psi = -c_2^2c_1\omega^2\left(3\frac{\partial p}{\partial \chi}c_2' + 8c_2c_1c_1'o_c \right), \]

\[ F_\psi = -c_2^2c_1\omega^2\left(3\frac{\partial p}{\partial \chi}c_2' + 8c_2c_1c_1'o_c \right)(c_1c_2' - c_1c_2). \]

The prime denotes the derivative with respect to \( r \).
E  Structure functions of the solution (5.26)

\[ T^a_{Z_2}(x) = \frac{3x^2(3 + 2 \ln x + x^2)}{(1 + x^2)^2} \beta_2^\Gamma + \left[ \frac{2x^2(x^2 + 3)^2}{(1 + x^2)^3} - \ln x - \ln(x^2 + 1) \right] \beta_2^\nu + \int_0^x dz \frac{1}{z(z^4 - 1)(1 + z^2)^3} \times \\
\left\{ 24\sqrt{3}z^2(1 + z^2) \left[ -16z(z^2 - 1) \frac{d\chi_2}{dz} + 3(z^2 + 1) \chi_2 \right] Z^1_\psi(z) \right. \\
- 3z(z^2 - 1)^3 \frac{dZ_0^2}{dz} + 6z^2(z^2 - 1)^2 Z^0_2 \\
\left. + 48z^2 \left[ -8z(3z^2 + 1)(z^2 - 1) \frac{d\chi_2}{dz} + 3(1 + z^2)^2 \chi_2 \right] Z^0_\psi(z) \right. \\
- 24z(z^2 + 3)(z^2 - 1)^3 \frac{dA_2}{dz} - 16(1 + z^2)^2(z^2 - 1)^3 \left( \frac{d\chi_2}{dz} \right)^2 \right\}.
\]

\[ T^b_{Z_2}(x) = \frac{6x^2}{(1 + x^2)^2} \beta_2^\Gamma + \frac{2x^2(9 + 6x^2 + x^4)}{(1 + x^2)^3} \beta_2^\nu + \int_0^x dz \frac{1}{z(1 + z^2)^4} \times \\
\left\{ 24\sqrt{3}z^2(1 + z^2) \left[ -16z(z^2 - 1) \frac{d\chi_2}{dz} + 3(z^2 + 1) \chi_2 \right] Z^1_\psi \\
- 3z(z^2 - 1)^3 \frac{dZ_0^2}{dz} + 6z^2(z^2 - 1)^2 Z^0_2 \\
+ 48z^2 \left[ -8z(3z^2 + 1)(z^2 - 1) \frac{d\chi_2}{dz} + 3(1 + z^2)^2 \chi_2 \right] Z^0_\psi \\
- 24z(z^2 + 3)(z^2 - 1)^3 \frac{dA_2}{dz} - 16(1 + z^2)^2(z^2 - 1)^3 \left( \frac{d\chi_2}{dz} \right)^2 \right\}.
\]

\[ T^c_{Z_2}(x) = -\frac{6x^2}{(1 + x^2)^2} \beta_2^\Gamma - \frac{2x^2(9 + 6x^2 + x^4)}{(1 + x^2)^3} \beta_2^\nu - \int_0^x dz \frac{1}{z(1 + z^2)^4} \times \\
\left\{ 24\sqrt{3}z^2(1 + z^2) \left[ -16z(z^2 - 1) \frac{d\chi_2}{dz} + 3(z^2 + 1) \chi_2 \right] Z^1_\psi \\
- 3z(z^2 - 1)^3 \frac{dZ_0^2}{dz} + 6z^2(z^2 - 1)^2 Z^0_2 \\
+ 48z^2 \left[ -8z(3z^2 + 1)(z^2 - 1) \frac{d\chi_2}{dz} + 3(1 + z^2)^2 \chi_2 \right] Z^0_\psi \\
- 24z(z^2 + 3)(z^2 - 1)^3 \frac{dA_2}{dz} - 16(1 + z^2)^2(z^2 - 1)^3 \left( \frac{d\chi_2}{dz} \right)^2 \right\}.
\]
F Structure functions of the solution (5.48)

\[ \mathcal{F}_Z^a(x) = \frac{x^2(3 + 2 \ln x + x^2)}{(1 + x^2)^2} \beta_1^p + \left( \frac{2x^2(x^2 + 3)^2}{3(1 + x^2)^3} \ln x - \frac{1}{3} \ln(x^2 + 1) + \frac{4x^2(7 + 7x^2 + 2x^4)}{3(1 + x^2)^3} \right) \beta_1^p + \int_0^x dz \frac{(z^2 - 1) \ln z - 2}{z(z^2 - 1)(1 + z^2)^3} \times \right. \\
\left. \left\{ 96\sqrt{3}z^2(1 + z^2) \left[ -4z(z^2 - 1) \frac{d\alpha_1}{dz} + (z^2 + 1) \alpha_1 \right] Z_\phi^1 \\
- z(z^2 - 1)^3 \frac{dZ_0}{dz} + 2z^2(z^2 - 1)^2 Z_1^0 \\
+ 192z^2 \left[ -2z(3z^2 + 1)(z^2 - 1) \frac{d\alpha_1}{dz} + (1 + z^2)^2 \alpha_1 \right] Z_\phi^0 \\
- 8z(z^2 + 3)(z^2 - 1)^3 \frac{dA_1}{dz} - 16(1 + z^2)^2(z^2 - 1)^3 \left( \frac{d\alpha_1}{dz} \right)^2 \right\}, \right. \\
\left. \mathcal{F}_Z^b(x) = \frac{2x^2(3 + 2 \ln x + x^2)}{(1 + x^2)^2} \beta_1^p + \frac{2x^2(9 + 6x^2 + x^4)}{3(1 + x^2)^3} \beta_1^p + \int_0^x dz \frac{1}{z(1 + z^2)^4} \times \\
\left. \left\{ 96\sqrt{3}z^2(1 + z^2) \left[ -4z(z^2 - 1) \frac{d\alpha_1}{dz} + (z^2 + 1) \alpha_1 \right] Z_\phi^1 \\
- z(z^2 - 1)^3 \frac{dZ_0}{dz} + 2z^2(z^2 - 1)^2 Z_1^0 \\
+ 192z^2 \left[ -2z(3z^2 + 1)(z^2 - 1) \frac{d\alpha_1}{dz} + (1 + z^2)^2 \alpha_1 \right] Z_\phi^0 \\
- 8z(z^2 + 3)(z^2 - 1)^3 \frac{dA_1}{dz} - 16(1 + z^2)^2(z^2 - 1)^3 \left( \frac{d\alpha_1}{dz} \right)^2 \right\}, \right. \\
\left. \mathcal{F}_Z^c(x) = -\frac{2x^2}{(1 + x^2)^2} \beta_1^p - \frac{2x^2(9 + 6x^2 + x^4)}{3(1 + x^2)^3} \beta_1^p - \int_0^x dz \frac{1}{z(1 + z^2)^4} \times \\
\left. \left\{ 96\sqrt{3}z^2(1 + z^2) \left[ -4z(z^2 - 1) \frac{d\alpha_1}{dz} + (z^2 + 1) \alpha_1 \right] Z_\phi^1 \\
- z(z^2 - 1)^3 \frac{dZ_0}{dz} + 2z^2(z^2 - 1)^2 Z_1^0 \\
+ 192z^2 \left[ -2z(3z^2 + 1)(z^2 - 1) \frac{d\alpha_1}{dz} + (1 + z^2)^2 \alpha_1 \right] Z_\phi^0 \\
- 8z(z^2 + 3)(z^2 - 1)^3 \frac{dA_1}{dz} - 16(1 + z^2)^2(z^2 - 1)^3 \left( \frac{d\alpha_1}{dz} \right)^2 \right\}. \right. \]
References

[1] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large $N$ field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[5] D. T. Son and A. O. Starinets, “Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications,” JHEP 0209, 042 (2002) [arXiv:hep-th/0205051].

[6] C. P. Herzog and D. T. Son, “Schwinger-Keldysh propagators from AdS/CFT correspondence,” JHEP 0303, 046 (2003) [arXiv:hep-th/0212072].

[7] G. Policastro, D. T. Son and A. O. Starinets, “The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma,” Phys. Rev. Lett. 87, 081601 (2001) [arXiv:hep-th/0104066].

[8] G. Policastro, D. T. Son and A. O. Starinets, “From AdS/CFT correspondence to hydrodynamics,” JHEP 0209, 043 (2002) [arXiv:hep-th/0205052].

[9] C. P. Herzog, “The hydrodynamics of M-theory,” JHEP 0212, 026 (2002) [arXiv:hep-th/0210126].

[10] G. Policastro, D. T. Son and A. O. Starinets, “From AdS/CFT correspondence to hydrodynamics. II: Sound waves,” JHEP 0212, 054 (2002) [arXiv:hep-th/0210220].

[11] C. P. Herzog, “The sound of M-theory,” Phys. Rev. D 68, 024013 (2003) [arXiv:hep-th/0302086].
[12] A. Buchel, “N = 2* hydrodynamics,” Nucl. Phys. B 708, 451 (2005) [arXiv:hep-th/0406200].

[13] P. Kovtun, D. T. Son and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” Phys. Rev. Lett. 94, 111601 (2005) [arXiv:hep-th/0405231].

[14] A. Buchel, J. T. Liu and A. O. Starinets, “Coupling constant dependence of the shear viscosity in N = 4 supersymmetric Yang-Mills theory,” Nucl. Phys. B 707, 56 (2005) [arXiv:hep-th/0406264].

[15] A. Buchel, “On universality of stress-energy tensor correlation functions in supergravity,” Phys. Lett. B 609, 392 (2005) [arXiv:hep-th/0408095].

[16] D. Teaney, “Effect of shear viscosity on spectra, elliptic flow, and Hanbury Brown-Twiss radii,” Phys. Rev. C 68, 034913 (2003).

[17] E. Shuryak, “Why does the quark gluon plasma at RHIC behave as a nearly ideal fluid?,” Prog. Part. Nucl. Phys. 53, 273 (2004) [arXiv:hep-ph/0312227].

[18] D. Molnar and M. Gyulassy, “Saturation of elliptic flow at RHIC: Results from the covariant elastic parton cascade model MPC,” Nucl. Phys. A 697, 495 (2002) [Erratum-ibid. A 703, 893 (2002)] [arXiv:nucl-th/0104073].

[19] P. Kovtun, D. T. Son and A. O. Starinets, “Holography and hydrodynamics: Diffusion on stretched horizons,” JHEP 0310, 064 (2003) [arXiv:hep-th/0309213].

[20] A. Buchel and J. T. Liu, “Universality of the shear viscosity in supergravity,” Phys. Rev. Lett. 93, 090602 (2004) [arXiv:hep-th/0311175].

[21] F. Karsch and E. Laermann, “Thermodynamics and in-medium hadron properties from lattice QCD,” arXiv:hep-lat/0305025.

[22] R. V. Gavai, S. Gupta and S. Mukherjee, “The speed of sound and specific heat in the QCD plasma: Hydrodynamics, fluctuations and conformal symmetry,” Phys. Rev. D 71, 074013 (2005) [arXiv:hep-lat/0412036].

[23] U. W. Heinz, “Equation of state and collective dynamics,” arXiv:nucl-th/0504011.
[24] S. Jeon and L. G. Yaffe, “From Quantum Field Theory to Hydrodynamics: Transport Coefficients and Effective Kinetic Theory,” Phys. Rev. D 53, 5799 (1996) [arXiv:hep-ph/9512263].

[25] A. Parnachev and A. Starinets, “The silence of the little strings,” arXiv:hep-th/0506144.

[26] O. Aharony, A. Buchel and A. Yarom, “Holographic renormalization of cascading gauge theories,” arXiv:hep-th/0506002.

[27] R. Donagi and E. Witten, “Supersymmetric Yang-Mills Theory And Integrable Systems,” Nucl. Phys. B 460, 299 (1996) [arXiv:hep-th/9510101].

[28] K. Pilch and N. P. Warner, “N = 2 supersymmetric RG flows and the IIB dilaton,” Nucl. Phys. B 594, 209 (2001) [arXiv:hep-th/0004063].

[29] A. Buchel, A. W. Peet and J. Polchinski, “Gauge dual and noncommutative extension of an N = 2 supergravity solution,” Phys. Rev. D 63, 044009 (2001) [arXiv:hep-th/0008076].

[30] N. J. Evans, C. V. Johnson and M. Petrini, “The enhancon and N = 2 gauge theory/gravity RG flows,” JHEP 0010, 022 (2000) [arXiv:hep-th/0008081].

[31] A. Buchel and J. T. Liu, “Thermodynamics of the N = 2* flow,” JHEP 0311, 031 (2003) [arXiv:hep-th/0305064].

[32] P. K. Kovtun and A. O. Starinets, “Quasinormal modes and holography,” arXiv:hep-th/0506184.

[33] A. Hosoya, M. a. Sakagami and M. Takao, “Nonequilibrium Thermodynamics In Field Theory: Transport Coefficients,” Annals Phys. 154, 229 (1984).

[34] R. Horsley and W. Schoenmaker, “Quantum Field Theories Out Of Thermal Equilibrium. 1. General Considerations,” Nucl. Phys. B 280, 716 (1987).

[35] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, “Novel local CFT and exact results on perturbations of N = 4 super Yang-Mills from AdS dynamics,” JHEP 9812, 022 (1998) [arXiv:hep-th/9810126].
[36] J. Distler and F. Zamora, “Non-supersymmetric conformal field theories from stable anti-de Sitter spaces,” Adv. Theor. Math. Phys. 2, 1405 (1999) [arXiv:hep-th/9810206].

[37] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP 9807, 023 (1998) [arXiv:hep-th/9806087].

[38] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. 208, 413 (1999) [arXiv:hep-th/9902121].

[39] M. Bianchi, D. Z. Freedman and K. Skenderis, “How to go with an RG flow,” JHEP 0108, 041 (2001) [arXiv:hep-th/0105276].

[40] M. Bianchi, D. Z. Freedman and K. Skenderis, Nucl. Phys. B 631, 159 (2002) [arXiv:hep-th/0112119].

[41] J. M. Bardeen, “Gauge Invariant Cosmological Perturbations,” Phys. Rev. D 22 (1980) 1882.

[42] J. M. Bardeen, P. J. Steinhardt and M. S. Turner, “Spontaneous Creation Of Almost Scale - Free Density Perturbations In An Inflationary Universe,” Phys. Rev. D 28, 679 (1983).

[43] H. Kodama and A. Ishibashi, “A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions,” Prog. Theor. Phys. 110, 701 (2003) [arXiv:hep-th/0305147].

[44] G. Policastro and A. Starinets, “On the absorption by near-extremal black branes,” Nucl. Phys. B 610, 117 (2001) [arXiv:hep-th/0104065].