Electronic sound modes and plasmons in hydrodynamic two-dimensional metals

Andrew Lucas\textsuperscript{a} and Sankar Das Sarma\textsuperscript{b}
\textsuperscript{a} Department of Physics, Stanford University, Stanford, CA 94305, USA
\textsuperscript{b} Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, MD 20742 USA

ajlucas@stanford.edu

Abstract: Using an analytically tractable kinetic model of a two dimensional Fermi liquid of electrons, we characterize the crossovers between zero sound, first sound and plasmons. For experimentally realized Fermi liquids in a hydrodynamic limit, both zero and first sound waves are essentially replaced by plasmons. The plasmon dispersion relation is robust against hydrodynamic effects, up to acquiring the viscous-limited decay rate of a first sound wave in the hydrodynamic limit. We discuss implications for experiments in clean two dimensional electron gases.

1 Introduction

1.1 Summary of Results

2 The Boltzmann Equation

3 A Solvable Toy Model

4 Conclusion

Acknowledgements

References

1 Introduction

Advances in the quality of crystal growth have led to strong experimental evidence for the hydrodynamic flows of electrons in solid state systems [1, 2, 3, 4]; see [5] for a recent review. Recent theoretical work has clarified the signatures of hydrodynamic behavior on correlated electron flow in a diverse set of
materials [6, 7, 8, 9, 10, 11, 12, 13, 14], including those where the textbook Navier-Stokes equations are not applicable.

The condition for an electron liquid to obey hydrodynamics was discussed in the literature a long time ago [15, 16, 17]. The key condition is that electron-electron (or equivalently, inter-particle) collision rate must be rapid enough to bring about local thermal equilibrium. This necessary condition boils down (for an electron liquid, the subject of interest in the current paper) to the electron-electron interaction induced scattering rate being larger than the momentum relaxation scattering rates associated with electron-impurity and/or electron-phonon scattering rates. Leaving aside phonon scattering, which is usually important only at rather high temperatures, this implies that electron-electron scattering should be stronger than electron-impurity scattering. Since electron-electron scattering rates are typically \( \propto T^2 \) in a Fermi liquid, it may seem that at some finite temperature, all metals should manifest hydrodynamic behavior. This turns out to be untrue, and typically in normal 3D metals (even in the cleanest possible scenario) electron-electron scattering rate is rather weak, and the hydrodynamic condition is never satisfied. On the other hand, graphene and ultra-clean 2D GaAs-based systems do satisfy the hydrodynamic condition at moderate temperatures.

An immediate consequence of hydrodynamics is the existence of an electronic sound mode (the so-called “first sound”) where the long wavelength energy dispersion of the collective mode is linear in wave number: \( \omega \propto k \). This collective electronic sound mode has never been observed in any 2D electron liquid (either graphene or 2D semiconductor systems). Even an indirect observation of sound waves through the hydrodynamic Dyakonov-Shur instability [18] has never been cleanly observed, despite many years of experimental efforts [19, 20]. Instead, the observed collective mode is the usual long wavelength plasmon mode, with \( \omega \propto \sqrt{k} \) due to the long-range nature of the Coulomb interaction. As we will see, this may be understood as an appropriate ‘zero sound’ mode in the collisionless regime. This casts a shadow on the theoretically proposed hydrodynamic descriptions of strongly interacting electron liquid, as a fluid should have a sound mode. Our work aims to clearly resolve this conundrum. Appropriately incorporating long-range Coulomb interactions into a hydrodynamic and kinetic model, we show that hydrodynamics in 2D metals is consistent with a long wavelength collective mode dispersing as \( \omega \propto \sqrt{k} \). For purely short-range interactions, there is indeed a linear-in-wavenumber first sound mode in the hydrodynamic regime (as observed, for example, in normal He-3 [21]), but Coulomb interaction modifies this dispersion to a square-root in wavenumber dependence at long wavelength.

1.1 Summary of Results

In this paper, we present a simple and analytically tractable kinetic theory model for the dynamics of a two dimensional Fermi liquid. We account for both Landau’s Fermi liquid interaction function, and the long-range nature of the Coulomb interaction, as appropriate for two-dimensional metals. We observe that in a typical Fermi liquid with strong electron-electron interactions, both the first and zero sound mode are replaced by a plasmon mode which persists (at least) until the breakdown of the kinetic theory. The main effects of hydrodynamics on this conventional plasmon mode are in the decay rate of the excitation, which is more challenging to experimentally measure. Furthermore, the higher-order wave-number corrections to the collective mode dispersions are also affected by hydrodynamics, and can be captured in our model (the leading-order dispersion is fixed by the Coulomb interaction to be a plasmonic \( \sqrt{k} \)).

Let us be more quantitative. We study a toy model of an isotropic two dimensional Fermi liquid with a single electron-electron scattering rate \( \gamma \), electron-impurity (momentum-relaxing) scattering rate \( \gamma_{\text{imp}} \leq \gamma \), and effective interaction constant \( \alpha \). We consider spinless fermions, in order to focus our discussion the salient features of plasmon physics. The generalization to the spinful case is straightforward. In a Fermi liquid \( \gamma \sim T^2 \) (up to logarithms [22, 23, 24, 25]). We also take the only non-vanishing Landau parameter of Fermi liquid theory to be \( F_0 \), for the sake of simplicity (although a generalization including more Landau
parameters is straightforward, but cumbersome). As we have assumed spinless fermions, \( F_0 \) corresponds to a Landau parameter in the symmetric channel. First, let us assume the absence of long-range Coulomb interactions (which can be experimentally achieved by placing gates very close to the sample to screen out the long-range part of the interaction). If \( \gamma_{\text{imp}} = 0 \), then we find a simple crossover between hydrodynamic first sound waves, with dispersion relation

\[
\omega = \pm \sqrt{\frac{1 + F_0}{2} vk - \frac{v^2}{8\gamma} k^2} + O(k^3),
\]

valid when \( |\omega| \ll \gamma \), and collisionless zero sound when \( |\omega| \gg \gamma \):

\[
\omega = \pm \frac{1 + F_0}{\sqrt{1 + 2F_0}} vk - i\gamma \frac{1 + F_0}{(1 + 2F_0)^2} + O\left(\frac{1}{k}\right).
\]

Whenever \( F_0 > 0 \), the speed of zero sound \( v_{s0} \) is always faster than the speed of first sound \( v_{s1} \):

\[
1 \leq \frac{v_{s0}}{v_{s1}} = \sqrt{\frac{2 + 2F_0}{1 + 2F_0}} \leq \sqrt{2}.
\]

So the most important difference between these two sound waves (beyond their physical interpretation) is the decay rate: \( \text{Im}(\omega) \sim T^{-2} \) in the hydrodynamic limit, while \( \text{Im}(\omega) \sim T^2 \) in the collisionless limit. These results are qualitatively similar to the well-understood theory of first and zero sound waves in He-3, which were experimentally observed a long time ago [21].

If \( \gamma_{\text{imp}} \neq 0 \), then sound waves and plasmons are destroyed at long wavelengths: impurity collisions act as a cutoff, overdamping these ballistically propagating modes at low enough frequency. The two sound modes merge as \( k \to 0 \), and form a purely diffusive mode associated with the Ohmic diffusion of charge:

\[
\omega = -i \frac{v_{s1}^2}{\gamma_{\text{imp}}} k^2
\]

as well as a gapped mode associated with momentum relaxation: \( \omega \approx -i\gamma_{\text{imp}} \). Eq. (4) defines the well-known diffusion pole of the disordered electron Green’s function. When \( v_{s1} k \gtrsim \gamma_{\text{imp}} \), we instead find the dispersion relation of sound waves described above, but with the decay rates approximately shifted by

\[
\text{Im}(\omega; \gamma_{\text{imp}} \neq 0) \approx \text{Im}(\omega; \gamma_{\text{imp}} = 0) - \frac{\gamma_{\text{imp}}}{2}.
\]

For zero sound waves, the shift is slightly more complicated – see Eq. (47) below.

Our most important result is that when we include long-range Coulomb interactions, we can approximately replace

\[
F_0 \to F_0 + \frac{2\pi \alpha}{\lambda_F |k|},
\]

with \( \lambda_F \) the Fermi wavelength. This may be construed as an effective generalization to a Landau-Silin theory applicable to 2D metals, as compared with a 2D Fermi liquid theory appropriate for neutral Fermi liquids. Eq. (6) can also be understood from the RPA approximation [26]. The \( 1/k \) dependence in Eq. (6) is the precise 2D Coulomb interaction behavior for the \( 1/r \) long-range potential. We then find that first sound (1) is replaced by

\[
\omega \approx \pm \sqrt{\frac{\pi \alpha v^2}{\lambda_F} |k| - \frac{v^2}{8\gamma} k^2} + O\left(k^3\right),
\]

and the collisionless zero sound is replaced by

\[
\omega \approx \pm \sqrt{\frac{\pi \alpha v^2}{\lambda_F} |k| - i\gamma \frac{\lambda_F |k|}{8\pi \alpha}}.
\]
The quantitative prefactor of Re(ω) in both of the above equations is identical to the well known plasmon dispersion relation [27, 28, 29]. The main results of this paper are twofold. First, we derive the unexpected Im(ω) ∼ |k| scaling for the decay rate of plasmons in a collisionless regime. Second, we show that the crossover between the “hydrodynamic plasmon” (7) and the “collisionless plasmon” (8) is when

\[ k \sim \frac{\lambda F \gamma^2}{\alpha v^2}. \]  

This crossover occurs at a much smaller wave number than naively expected (k ∼ γ/v). The breakdown of these equations, and the replacement of plasmons with conventional first or zero sound modes, occurs when

\[ \lambda F k \gtrsim \frac{2\pi \alpha}{\max(1, F_0)} \]  

which is generally outside the regime of validity of kinetic theory. Since \( \alpha \gtrsim 1 \) is indeed required to observe hydrodynamic behavior in a two-dimensional Fermi liquid at present, it is not possible to observe either a first or zero sound mode: the long-range Coulomb interactions will destroy conventional sound waves at all wavelengths, and the only observable long wavelength collective mode would be the usual plasmon mode, which indeed seems to be the generic experimental situation. Thus, in contrast to the 3D neutral Fermi liquid He-3, the hydrodynamic behavior in 2D Coulomb Fermi systems may not necessarily manifest long wavelength sound modes.

These results clarify previous literature relating plasmons to sound waves in Fermi liquids. In particular, the hydrodynamic and collisionless plasmon are essentially the same mode at long wavelengths in a Fermi liquid, albeit with distinct decay mechanisms (and different subleading wave-number corrections). A crossover between plasmons and first sound will only be observable in nearly charge neutral (non-degenerate) systems, such as the Dirac fluid in graphene [24, 30, 31, 32].

2 The Boltzmann Equation

In order to derive these results, we first review the kinetic theory of a two dimensional Fermi liquid with a circular Fermi surface. Within linear response, the most important changes to the distribution function \( f \) occur at the Fermi surface:

\[ f \approx f_{eq} + \delta f \equiv \Theta(\mu - \epsilon(p)) + \delta(\mu - \epsilon(p)) \Phi(x, \theta, t). \]  

Here \( \tan \theta = p_y/p_x \) denotes the angle on the circular Fermi surface. In what follows, we will denote with \( p_F \) the Fermi momentum \( \epsilon(p_F) = \mu \), and also define the Fermi velocity

\[ v \equiv \epsilon'(p_F). \]

The quantum Boltzmann equation, neglecting spin, reads

\[ \frac{\partial f}{\partial t} + \frac{\partial \epsilon}{\partial p} \cdot \frac{\partial f}{\partial x} - \frac{\partial \epsilon}{\partial x} \cdot \frac{\partial f}{\partial p} = C[f] \]  

\( C[\Phi] \) is the collision integral, and \( \epsilon \) denotes the energy of a quasiparticle, accounting for both the bare band structure \( \epsilon_0(p) \), Landau’s short-range interaction function \( \mathcal{E} \), and the long-range part of the Coulomb interaction, arising from fluctuations in the Fermi surface:

\[ \epsilon(x, p) = \epsilon_0(p) + \mathcal{E}[f(x, p)] + \frac{e^2}{4\pi \epsilon} \int \frac{d^2 y d^2 q}{(2\pi \hbar)^2} \frac{f(x, p)(f(y, q) - f_{eq})}{|x - y|}. \]
We have treated Coulomb interactions here in the conventional self-consistent Vlasov approximation. The interaction function is

\[ E[\Phi] \approx \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} F(\theta - \phi) \Phi(\phi), \]  

with \( F \) an even function. Combining Eqs. (13) and (14), we obtain

\[ \frac{\partial \delta f}{\partial t} + \frac{\partial \epsilon_0}{\partial p} \frac{\partial \delta f}{\partial x} - \frac{\partial f_{\text{eq}}}{\partial p} \frac{\partial \epsilon}{\partial x} = C[f], \]  

which leads to

\[ \frac{\partial \Phi}{\partial t} + v \cdot \left[ \frac{\partial \Phi}{\partial x} + \frac{\partial \epsilon}{\partial x} \right] = \delta C[\Phi] \]  

where \( v = \partial \epsilon_0/\partial p \) and \( \delta C[\Phi] \) denotes a linearized collision integral, which will be a local in \( x \), but nonlocal in \( \theta \), linear expression in \( \Phi \).

In this paper we are interested in studying propagating waves. So without loss of generality we may look for solutions to (17) of the form

\[ \Phi(x, \theta, t) = \sum_{n \in \mathbb{Z}} a_n e^{i(k \cdot x - \omega t + n\theta)}. \]  

Writing

\[ F(\phi) = \sum_{k \in \mathbb{Z}} F_n e^{i n\phi}, \]  

with \( F_n = F_{-n} \), we find

\[ \frac{\partial E[\Phi]}{\partial x} = i k \sum_{m,n} \int \frac{d\phi}{2\pi} F_m e^{im(\theta - \phi)} a_n e^{i(k \cdot x - \omega t + n\phi)} = i k \sum_n \alpha_n e^{i(k \cdot x - \omega t + n\phi)} \]  

Taking the spatial Fourier transform of the long-range Coulomb interaction, we find

\[ \frac{e^2}{4\pi \epsilon} \int d^2x \frac{\partial}{\partial x} \int d^2y d^2q \frac{\delta(\mu - \epsilon(q)) \Phi(y, \theta_q, t)}{|x - y|} = i k \frac{\alpha}{\lambda_F |k|} a_0 e^{i(k \cdot x - \omega t)} \]  

where

\[ \alpha \equiv \frac{e^2}{4\pi \epsilon \hbar v_F} \]  

is the effective fine structure constant, and \( \lambda_F = 2\pi \hbar /p_F \) is the Fermi wavelength. Finally, similarly to (20), rotational invariance demands that

\[ \delta C[\Phi] = - \sum_n \gamma_n a_n e^{i(k \cdot x - \omega t + n\phi)} \]  

with \( \gamma_n \geq 0 \) also required by the second law of thermodynamics. Combining (17), (20), (21) and (23), and choosing \( k = k\hat{x} \), we obtain

\[ -i \omega a_n + \frac{ikv}{2}(a_{n+1} + a_{n-1}) + \frac{ikv}{2} \left( F_{n+1}a_{n+1} + F_{n-1}a_{n-1} \right) + ik \frac{\pi \nu \alpha}{\lambda_F |k|} a_0 \delta_{|n|,1} = -\gamma_n a_n. \]  

This infinite set of algebraic equations governs the normal modes of our kinetic theory.
A key observation is that the replacement (6) “removes” the third term of (24). In other words, long-range Coulomb interactions are a $k$-dependent $F_0$. Since $k$ is not dynamical in (24), we can therefore solve (24) without explicitly accounting for long-range Coulomb interactions, and then include them at the end of the calculation through (6). Indeed, in the section that follows, we will set $\alpha = 0$ and characterize the zero-to-first sound crossover of this model, which is akin to solving for the collective modes in the short-range interacting 2d Fermi liquids.

Let us also stress that the regime of validity of the Boltzmann equation is $k\lambda_F \ll 1$. Therefore, unless $\alpha$ is parametrically small, the long-range Coulomb interactions will dominate the dynamics of the $\pm 1$ harmonics. This is why we stated that plasmons destroy sound waves in typical Fermi liquids, in the introduction.

### 3 A Solvable Toy Model

To find the exact solutions advertised in the introduction, we must now choose a simple model for $F_n$ and $\gamma_n$. A simple solvable model is the relaxation time model of [10, 11, 12]:

$$
\gamma_n = \begin{cases} 
0 & n = 0 \\
\gamma_{\text{imp}} & |n| = 1 \\
\gamma & |n| \geq 2
\end{cases}.
$$

(25)

While this model is not microscopically realistic when only two-body collisions are important [33, 34], it captures many non-trivial features of the hydrodynamic to ballistic crossover, and for our purposes this will suffice. We will also consider the simplest possible non-trivial choice of $F_n$:

$$
F_n = \begin{cases} 
0 & n \neq 0 \\
F_0 & n = 0
\end{cases}.
$$

(26)

These assumptions can be relaxed at the expense of losing some of the analytic tractability. For a more microscopic model of $F_n$ and $\gamma_n$, it would be straightforward to numerically compute the normal modes of (24).

The solvability of (24), in the model (25) and (26), comes from the following observation. For $|n| \geq 2$, we have the generic equations

$$
(\gamma - i\omega)a_n + \frac{ikv}{2}(a_{n-1} + a_{n+1}) = 0.
$$

(27)

Let us look for solutions of the form $a_n = \lambda^{n-1}a_1$ for $n > 1$. (27) implies

$$
\lambda^2 + \frac{2(\gamma - i\omega)}{ikv}\lambda + 1 = 0.
$$

(28)

This is solved by

$$
\lambda = -i\frac{\gamma - i\omega}{kv}\left(\pm \sqrt{1 + \left(\frac{kv}{\gamma - i\omega}\right)^2 - 1}\right).
$$

(29)

It is important to keep in mind that $|\lambda| \leq 1$ is required in order for a normal mode to exist (be normalizable).

Now, we look for the spectrum of normal modes. The equations (24) are analogous to the solution of tight-binding models in one dimension, with scattering off of a defect near the origin. Looking for solutions with $\lambda = e^{-i\phi}$, which heuristically corresponds to a $\delta$ function fluctuation of the Fermi surface: $\Phi(\theta) \sim \delta(\theta - \phi)$, we expect a continuum of normal modes with

$$
\omega = -i\gamma + kv\cos\phi
$$

(30)
for (almost) any real $0 < \phi \leq 2\pi$. It remains to satisfy the boundary conditions $a_{\pm 1} = 0$ ($a_0 = 0$ then trivially follows). This can be done as follows. With the exception of $\phi = 0, \pi$, there are two $\phi$ which have the same value of $\omega$, but different $\lambda$. We add these two modes together, with suitably chosen constant prefactors so that $a_{\pm 1}$ vanishes. The final result is

$$a_n = \begin{cases} e^{i(n-1)\phi} - e^{i(1-n)\phi} & n > 1 \\ 0 & n \leq 1 \end{cases},$$

and a similar mode with $a_n \neq 0$ only for $n < -1$.

Now let us look for the remaining modes, which do not obey (30). These are analogous to the “bound states” of the tight-binding model and will have $a_0$ and/or $a_{\pm 1}$ non-vanishing. The equations for $a_0$ and $a_{\pm 1}$ are

$$-i\omega a_0 + \frac{ikv}{2}(a_1 + a_{-1}) = 0,$$

$$(\gamma_{\text{imp}} - i\omega)a_{\pm 1} + \frac{ikv}{2}(a_0 + a_{\pm 2}) + \frac{ikvF_0}{2}a_0 = 0. \quad (32b)$$

First, we look for a mode which has $a_0 = 0$, but $a_{\pm 1} \neq 0$. (32a) implies that $a_1 = -a_{-1}$. So let us solve just for the modes with $n > 0$. Making the ansatz $a_n = \lambda^{n-1}a_1$, we find

$$i \left( \omega + i\gamma_{\text{imp}} - \frac{\lambda}{2}kv \right) a_1 = 0. \quad (33)$$

We find that (29) and (33) are exactly solved by

$$\omega = -i\gamma_{\text{imp}} - \frac{v^2}{4(\gamma - \gamma_{\text{imp}})}k^2. \quad (34)$$

This is the hydrodynamic shear diffusion mode, which has become gapped by momentum relaxation. In order for the diffusion constant to be positive, and for the theory to be stable, we require that $\gamma > \gamma_{\text{imp}}$. Also, note that this mode disappears into the continuum of normal modes once $|\lambda| = 1$. We find that this occurs when $\omega = -i\gamma$, or when $|k|v = 2(\gamma - \gamma_{\text{imp}})$. Finally, observe that if $\gamma_{\text{imp}} = 0$, (34) is nothing more than the conventional hydrodynamic diffusive mode associated with transverse momentum. The diffusion constant and the attenuation constant of first sound (1) are related by viscous hydrodynamics [10, 11].

The final mode has $a_0 \neq 0$, and $a_1 = a_{-1}$. We find the equations

$$\omega a_0 = kva_1, \quad (35a)$$

$$\left( \omega + i\gamma_{\text{imp}} - \frac{\lambda}{2}kv \right) a_1 = \frac{k(v + F_0)}{2}a_0, \quad (35b)$$

or

$$\omega + i\gamma_{\text{imp}} - \frac{\lambda}{2}kv = \frac{k^2v^2(1+F_0)}{2\omega}. \quad (36)$$

(29) and (36) do not have a simple analytic solution for all $\omega$. Let us instead focus on the two asymptotic limits of interest for large and small $k$. We start with the hydrodynamic limit of small $k$, where we find from (29) that

$$\lambda \approx -\frac{ikv}{2(\gamma - i\omega(k = 0))}. \quad (37)$$

In the special case $\gamma_{\text{imp}} = 0$, then $\omega(k = 0) = 0$ and we find (1). This is the dispersion relation for a first sound wave. Using the fact that the dynamical viscosity of this toy model is [10, 11, 12]

$$\nu = \frac{v^2}{4\gamma}. \quad (38)$$
we see that the decay rate of sound is consistent with the hydrodynamic coefficients of this model.

Upon setting $\gamma \sim T^2$, one then finds from (38) that the viscosity of the Fermi liquid is proportional to $T^{-2}$. In fact, the viscosity of a Fermi liquid is modified by factors of $\log(T/T_F)$ [23]. These additional logarithmic factors are associated with the breakdown of the relaxation time approximation in a two-dimensional Fermi liquid. More generally, $\nu = v^2/4\gamma_2$, with $\gamma_2$ defined in (23), and the logarithms observed in the viscosity are due to $\gamma_2$. Further discussion of the breakdown of the relaxation time approximation can be found in [33, 34].

In the case $\gamma_{\text{imp}} \neq 0$, a bit more care is required. We expect that in the $k \to 0$ limit, there is one gapped mode with $\omega = -i\gamma_{\text{imp}} + \cdots$, and one diffusive gapless mode. If $\gamma_{\text{imp}} \ll \gamma$, then $\gamma - i\omega(k = 0) \approx \gamma$ in both cases and (1) generalizes to

$$\omega \approx \pm \sqrt{\frac{v(v + F_0)}{2}} k^2 - \left(\frac{\gamma_{\text{imp}}}{2} + \frac{v^2}{8\gamma k^2}\right)^2 - i\left(\frac{\gamma_{\text{imp}}}{2} + \frac{v^2}{8\gamma k^2}\right) + \cdots$$  \hspace{1cm} (39)

If $\gamma_{\text{imp}}$ is comparable to $\gamma$, then (39) cannot be trusted. There will be no appreciable first sound mode, and instead we find the following low $k$ expansion of a diffusive mode and a gapped mode:

$$\omega = -i\gamma_{\text{imp}} - \frac{v^2}{2} \left[ \frac{1}{\gamma - \gamma_{\text{imp}}} - \frac{1 + F_0}{\gamma_{\text{imp}}} \right] k^2 + O(k^4).$$  \hspace{1cm} (40a)

At large $k$, we find zero sound waves. Making the ansatz that $\omega \approx v_0 k + \zeta + O(k^{-1})$, we find that

$$\lambda \approx \frac{v_0 - \sqrt{v_0^2 - v^2}}{v} \left[ 1 - \frac{i\gamma + \alpha}{\sqrt{v_0^2 - v^2} k^2} + O\left(\frac{1}{k^2}\right) \right].$$  \hspace{1cm} (42)

To solve for the speed of zero sound waves, $v_0$, we can neglect $\zeta$. (36) gives

$$v_0 \left( v_0 + \sqrt{v_0^2 - v^2} \right) = v^2 (1 + F_0),$$  \hspace{1cm} (43)

which is solved by

$$v_0 = \pm \frac{v}{\sqrt{1 + 2F_0}}.$$  \hspace{1cm} (44)

Next, we can perturbatively solve (36) for $\zeta$, and we find

$$\zeta = -i \frac{(\gamma + 2F_0\gamma_{\text{imp}})(1 + F_0)}{(1 + 2F_0)^2}.$$  \hspace{1cm} (45)

The decay rate of zero sound modes is thus approximately set by $\gamma$, but is smaller whenever $F_0 \neq 0$. Combining (44) and (45) we obtain (2), when $\gamma_{\text{imp}} = 0$.

We can qualitatively estimate the crossover between the zero and first sound waves by asking when $|\omega| \sim \gamma$. This occurs when

$$k \sim \frac{\gamma}{v\sqrt{1 + 2F_0}}.$$  \hspace{1cm} (46)

As emphasized previously, we are most interested in theories where (the effective) $F_0 \gg 1$. In such systems, we find that the hydrodynamic regime – as measured by the presence of first sound – is pushed to much longer length scales than naively anticipated.
If $\gamma_{\text{imp}} \neq 0$, then we find that the zero sound decay rate is approximately shifted by
\[
\text{Im}(\omega; \gamma_{\text{imp}} \neq 0) \approx \text{Im}(\omega; \gamma_{\text{imp}} = 0) - \gamma_{\text{imp}} \frac{2\mathcal{F}_0 (1 + \mathcal{F}_0)}{(1 + 2\mathcal{F}_0)^2}.
\]
(47)

In the limit $\mathcal{F}_0 \gg 1$, the zero sound decay rate remains approximately offset by (5), analogous to first sound waves. When $\mathcal{F}_0$ is small, the decay rate of first sound waves is, interestingly, nearly independent of momentum relaxation. We emphasize that this is a theoretical point of interest, as in a real electronic Fermi liquid, the long-range Coulomb interactions always make the effective $\mathcal{F}_0 \gg 1$.

## 4 Conclusion

We have described an analytically solvable toy model for the crossover between sound waves and plasmons in a strongly interacting two dimensional Fermi liquid. At long wavelengths, the conventional plasmon mode arising from the long-range Coulomb interactions overtakes both the first sound and zero sound modes of simple two dimensional Fermi liquids. We have further computed the decay rates of plasmons at both short and long wavelengths, and shown that signatures of hydrodynamics in the plasmonic decay are severely limited. The transition out of the hydrodynamic regime occurs (for plasmons) at a parametrically long length scale (9). Given that the plasmonic decay rate is also affected by impurities, this means that the electron-impurity scattering rate needs not simply obey $\gamma_{\text{imp}} \ll \gamma$, but
\[
\gamma_{\text{imp}} \ll \gamma^3 \left( \frac{\lambda F}{v_F} \right)^2,
\]
(48)
in order for impurity scattering to not dominate the decay of plasmons. While $\gamma_{\text{imp}} \ll \gamma$ is achievable in 2DEGs, this stronger constraint may not be.

The absence of a sound-like collective mode in 2D electron liquids is completely consistent with the system being in the hydrodynamic regime since, as we show explicitly, the long-range Coulomb interaction modifies all collective modes (both zero and first sound modes of neutral short-range interacting models) to being plasmon-like, with $\omega \propto \sqrt{k}$ at long wavelengths. Hydrodynamic effects do manifest themselves in the effective plasmon decay rate and in the higher-order wavenumber corrections to the mode dispersion, but these are not easy effects to measure experimentally. We have thus resolved the conundrum of why strongly interacting and ultraclean 2D Fermi liquids (e.g. graphene, high-mobility 2D GaAs systems) only manifest plasma type long wavelength collective mode dispersion. This is an essential effect of the long-range nature of Coulomb interaction – the sound modes are no longer linear in wavenumber in a Coulomb system. The same behavior should also apply to 3D Coulomb Fermi liquids where the hydrodynamic zero and first sound modes will in fact develop a mass gap because of the long-range nature of the 3D Coulomb interaction, but we know of no examples of 3D Coulomb Fermi liquids crossing over to the hydrodynamic regime (3D metals are always in the collisionless regime). Several 2D Coulomb Fermi liquid systems, however, are expected to be in the hydrodynamic regime at moderate temperatures (e.g. monolayer and bilayer graphene, high-mobility 2D GaAs systems), and we predict that the only observable collective modes in these systems will always manifest the square-root in wavenumber long wavelength 2D plasmon dispersion, no matter how deep they are in the hydrodynamic regime (i.e. no matter how strong the quasiparticle collision rate is). Our work also indicates that the easiest way to see the hydrodynamic sound mode in a 2D system would be to screen out the long-range part of the inter-particle Coulomb interaction by putting a parallel metal gate close to the 2D layer. In the hydrodynamic regime this will lead to a linearly dispersing sound wave whose speed of sound is sensitive to the properties of the gate. Sound waves in 2D can also be observed in systems with explicitly short-ranged interactions, such as thin films of normal He-3.
Many previous works have attempted to include the interplay of both electron-electron interactions \([35, 36, 37]\) and impurities \([38]\) in the plasmon dispersion relation of a weakly interacting metal. Our work provides a simple model, treating impurity scattering and interactions on an equal footing all the way from the collisionless to the hydrodynamic regime. In particular, the viscous plasmonic decay rate given in (7) may be much larger than the plasmon decay rate due to electron-phonon scattering \([37]\). More recent work on plasmons in the hydrodynamic regime includes \([39, 40, 41]\). Our key findings, including the novel collisionless dispersion relation (8) and the unexpectedly early crossover from hydrodynamic to collisionless plasmons at the length scale (9), are not contained in these earlier works. Our work also explains why the collective modes of 2D Fermi liquids which are expected to be in a hydrodynamic regime, including doped graphene \([42]\) and 2D semiconductor systems, have exhibited no hints of hydrodynamics.

We should also make some comments on the implication of our work in the context of the widely used ‘hydrodynamic dielectric function’ approach to collective modes in metals and semiconductors, particularly in calculating the response of finite systems, including surfaces, interfaces and inhomogeneous electron systems, to external electromagnetic fields. Some representative, but by no means exhaustive, references are \([43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]\). This theory is extensively used in the emergent field of nanoplasmonics, where collective electronic response properties of ultrasmall systems are studied for potential technological applications. This heuristic approach, dating back to Bloch in 1933 \([43]\), treats the metal (or doped semiconductor) as a hydrodynamic fluid rather uncritically without any consideration for whether the system is or is not in the collision-dominated regime. In fact, most 3D metals and doped semiconductors are not in the collision-dominated regime since the electron-electron scattering rate is invariably weaker than electron-impurity and electron-phonon scattering rates at low and high temperatures, respectively. But the simplicity of the hydrodynamic approach makes it particularly attractive for response calculations as a dynamical generalization of the static Thomas-Fermi approximation, with the fluid pressure term approximated by a parametrized ansatz so as to give the correct plasma dispersion (e.g. as obtained by the random phase approximation) in the bulk, up to second order in wave number. Such a hydrodynamic response theory has been used extensively for calculating surface and interface plasma modes of finite electron systems in many situations, mainly because of its manifest simplicity and tractability: the bulk hydrodynamic dielectric function has a simple finite frequency pole with a well-defined analytic form for the wave number dependence of the pole. Our work shows that such hydrodynamic response theories, while being easy to implement numerically, are not rigorous from a fundamental microscopic perspective. In particular, a simple ansatz for the fluid pressure applicable for all frequencies in the response calculation is not justifiable. The dissipative response of the system changes qualitatively between the collision dominated regime at low frequency (where hydrodynamics is valid) to a collisionless ballistic theory at high frequency where hydrodynamics manifestly fails. Our work also shows that, although the leading order collective mode is always defined by the standard plasma frequency in Coulomb systems (independent of collision-dominated hydrodynamic or collisionless ballistic regime), all higher-order dispersion corrections depend explicitly on whether the system is or is not in the hydrodynamic regime (analogous to the difference between zero sound and first sound). Since most of these hydrodynamic theories of electron response are focused on obtaining the collective mode dispersion beyond the long wavelength limit (i.e. the so-called ‘non-local’ effects), we caution against taking the quantitative predictions of these theories too literally. In addition, our work shows that the nature of the collective mode damping depends crucially on whether the system is the collision-dominated hydrodynamic or collisionless ballistic regime, another subtlety not considered in these hydrodynamic response theories.
Acknowledgements

AL was supported by the Gordon and Betty Moore Foundation’s EPiQS Initiative through Grant GBMF4302. SDS was supported by Laboratory for Physical Sciences.

References

[1] D. A. Bandurin et al. “Negative local resistance due to viscous electron backflow in graphene”, Science 351 1055 (2016), arXiv:1509.04165.

[2] J. Crossno et al. “Observation of the Dirac fluid and the breakdown of the Wiedemann-Franz law in graphene”, Science 351 1058 (2016), arXiv:1509.04713.

[3] P. J. W. Moll, P. Kushwaha, N. Nandi, B. Schmidt, and A. P. Mackenzie. “Evidence for hydrodynamic electron flow in PdCoO$_2$”, Science 351 1061 (2016), arXiv:1509.05691.

[4] R. Krishna Kumar et al. “Super-ballistic flow of viscous electron fluid through graphene constrictions”, Nature Physics 13 1182 (2017), arXiv:1703.06672.

[5] A. Lucas and K. C. Fong. “Hydrodynamics of electrons in graphene”, arXiv:1710.08425.

[6] A. V. Andreev, S. A. Kivelson, and B. Spivak. “Hydrodynamic description of transport in strongly correlated electron systems”, Physical Review Letters 106 256804 (2011), arXiv:1011.3068.

[7] A. Lucas, J. Crossno, K. C. Fong, P. Kim, and S. Sachdev. “Transport in inhomogeneous quantum critical fluids and in the Dirac fluid in graphene”, Physical Review B93 075426 (2016), arXiv:1510.01738.

[8] I. Torre, A. Tomadin, A. K. Geim, and M. Polini. “Non-local transport and the hydrodynamic shear viscosity in graphene”, Physical Review B92 165433 (2016), arXiv:1508.00363.

[9] L. Levitov and G. Falkovich. “Electron viscosity, current vortices and negative nonlocal resistance in graphene”, Nature Physics 12 672 (2016), arXiv:1508.00836.

[10] H. Guo, E. Ilseven, G. Falkovich, and L. Levitov. “Higher-than-ballistic conduction of viscous electron flows”, Proceedings of the National Academy of Sciences 114 3068 (2017), arXiv:1607.07269.

[11] A. Lucas. “Stokes paradox in electronic Fermi liquids”, Physical Review B95 115425 (2017), arXiv:1612.00856.

[12] H. Guo, E. Ilseven, G. Falkovich, and L. Levitov. “Stokes paradox, back reflections and interaction-enhanced conductance”, arXiv:1612.09239.

[13] A. Lucas and S. A. Hartnoll. “Kinetic theory of transport for inhomogeneous electron fluids”, arXiv:1706.04621.

[14] A. Lucas. “Kinetic theory of electronic transport in random magnetic fields”, arXiv:1710.11141.

[15] D. Pines and P. Nozières. The Theory of Quantum Liquids, Volume I (W. A. Benjamin, 1966).

[16] E.M. Lifshitz and L. P. Pitaevskii. Statistical Physics Part 2 (Butterworth Heinemann, 1980).
[17] G. F. Giuliani and G. Vignale. Quantum Theory of the Electron Liquid (Cambridge University Press, 2008).

[18] M. Dyakonov and M. Shur. “Shallow water analogy for a ballistic field effect transistor: New mechanism of plasma wave generation by dc current”, Physical Review Letters 71 2465 (1993).

[19] R. Tauk et al. “Plasma wave detection of terahertz radiation by silicon field effects transistors: responsivity and noise equivalent power”, Applied Physics Letters 89 253511 (2006).

[20] V. Giliberti, A. Di Gaspare, E. Giovine, M. Ortolani, L. Sorba, G. Biasiol, V. V. Popov, D. V. Fateev, and F. Evangelisti. “Downconversion of terahertz radiation due to intrinsic hydrodynamic nonlinearity of a two-dimensional electron plasma”, Physical Review B91 165313 (2015).

[21] W. R. Abel, A. C. Anderson, and J. C. Wheatley. “Propagation of zero sound in liquid He$^3$ at low temperatures”, Physical Review Letters 17 74 (1966).

[22] L. Zheng and S. Das Sarma. “Coulomb scattering lifetime of a two-dimensional electron gas”, Physical Review B53 9964 (1996), arXiv:cond-mat/9602066.

[23] D. S. Novikov. “Viscosity of a two-dimensional Fermi liquid”, arXiv:cond-mat/0603184.

[24] Q. Li and S. Das Sarma. “Finite temperature inelastic mean free path and quasiparticle lifetime in graphene”, Physical Review B87 085406 (2013), arXiv:1211.6430.

[25] A. Principi, G. Vignale, M. Carrega, and M. Polini. “Bulk and shear viscosities of the 2D electron liquid in a doped graphene sheet”, Physical Review B93 125410 (2016), arXiv:1506.06030.

[26] G. Zala, B. N. Narozhny, and I. L. Aleiner. “Interaction corrections at intermediate temperatures: Longitudinal conductivity and kinetic equation”, Physical Review B64 214204 (2001), arXiv:cond-mat/0105406.

[27] T. Ando, A. B. Fowler, and F. Stern. “Electronic properties of two-dimensional systems”, Reviews of Modern Physics 54 437 (1982).

[28] E. H. Hwang and S. Das Sarma. “Dielectric function, screening and plasmons in two-dimensional graphene”, Physical Review B75 205418 (2007), arXiv:cond-mat/0610561.

[29] S. Das Sarma and E. H. Hwang. “Collective modes of the massless Dirac plasma”, Physical Review Letters 102 206412 (2009), arXiv:0902.3822.

[30] T. V. Phan, J. C. W. Song, and L. S. Levitov. “Ballistic heat transfer and energy waves in an electron system”, arXiv:1306.4972.

[31] Z. Sun, D. N. Basov, and M. M. Fogler. “Adiabatic amplification of plasmons and demons in 2d systems”, Physical Review Letters 117 076805 (2016), arXiv:1601.02722.

[32] A. Lucas. “Sound waves and resonances in electron-hole plasma”, Physical Review B93 245153 (2016), arXiv:1604.03955.

[33] P. Ledwith, H. Guo, and L. Levitov. “Fermion collisions in two dimensions”, arXiv:1708.01915.

[34] P. Ledwith, H. Guo, A. V. Shytov, and L. Levitov. “Head-on collisions and scale-dependent viscosity in two-dimensional electron systems”, arXiv:1708.02376.
[35] A. Czachor, A. Holas, S. R. Sharma, and K. S. Singwi. “Dynamical correlations in a two-dimensional electron gas: First-order perturbation theory”, Physical Review B25 2144 (1982).

[36] E. H. Hwang and S. Das Sarma. “Plasmon dispersion in dilute 2D electron systems: Quantum-Classical and Wigner Crystal-Electron Liquid Crossover”, Physical Review B64 165409 (2001), arXiv:cond-mat/0102057.

[37] E. G. Mishchenko, M. Yu. Reizer, and L. I. Glazman. “Plasmon attenuation and optical conductivity of a two-dimensional electron gas”, Physical Review B69 195302 (2004), arXiv:cond-mat/0312684.

[38] S. Das Sarma and E. H. Hwang. “Dynamical response of a one dimensional quantum wire electron system”, Physical Review B54 1936 (1996), arXiv:cond-mat/9602157.

[39] D. N. Basov, M. M. Fogler, A. Lanzara, F. Wang, and Y. Zhang. “Colloquium: Graphene spectroscopy”, Reviews of Modern Physics 86 959 (2014), arXiv:1407.6721.

[40] U. Briskot, M. Schütt, I. V. Gornyi, M. Titov, B. N. Narozhny, and A. D. Mirlin. “Collision-dominated nonlinear hydrodynamics in graphene”, Physical Review B92 115426 (2015), arXiv:1507.08946.

[41] D. Svintsov. “Hydrodynamic-to-ballistic crossover in Dirac fluid”, arXiv:1710.05054.

[42] S. Das Sarma, E. H. Hwang, and W-K. Tse. “Many-body interaction effects in doped and undoped graphene: Fermi liquid versus non-Fermi liquid”, Physical Review B75 121406 (2007), arXiv:cond-mat/0610581.

[43] F. Bloch. “Bremsvermögen von atomen mit mehreren elektronen”, Zeitschrift für Physik 81 363 (1933).

[44] A. J. Bennett. “Influence of the electron charge distribution on surface-plasmon dispersion”, Physical Review B1 203 (1970).

[45] J. Heinrichs. “Hydrodynamic theory of surface-plasmon dispersion”, Physical Review B7 3487 (1973).

[46] A. L. Fetter. “Electrodynamics of a layered electron gas. I. Single layer”, Annals of Physics 81 367 (1973).

[47] A. Eguiluz, S. C. Ying, and J. J. Quinn. “Influence of the electron density profile on surface plasmons in a hydrodynamic model”, Physical Review B11 2118 (1975).

[48] G. Barton. “Some surface effects in the hydrodynamic model of metals”, Reports on Progress in Physics 42 963 (1979).

[49] S. Das Sarma and J. J. Quinn. “Hydrodynamic model of linear response for a jellium surface: Nonretarded limit”, Physical Review B20 4872 (1979).

[50] W. L. Schiach and C. Schwartz. “Phonons at metal surfaces”, Physical Review B25 7365 (1982).

[51] S. Das Sarma. “Electrodynamical response of a bounded electron gas in hydrodynamic formalism: Theory and applications”, Physical Review B26 6559 (1982).

[52] S. Das Sarma. “Nonlocal theory for surface-plasmon excitation in simple metals”, Physical Review B36 3026 (1987).
[53] Y. Luo, A. I. Fernandez-Dominguez, A. Wiener, S. A. Maier, and J. B. Pendry. “Surface plasmons and nonlocality: a simple model”, Physical Review Letters 111 093901 (2013), arXiv:1308.1708.

[54] C. Ciraci and F. Della Sala. “Quantum hydrodynamic theory for plasmonics: impact of the electron density tail”, Physical Review B93 205405 (2016), arXiv:1601.01584.