THE UNIQUENESS QUESTION IN THE MULTIDIMENSIONAL MOMENT PROBLEM WITH APPLICATIONS TO PHASE SPACE OBSERVABLES

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Abstract. The theory of holomorphic functions of several complex variables is applied in proving a multidimensional variant of a theorem involving an exponential boundedness criterion for the classical moment problem. A theorem of Petersen concerning the relation between the multidimensional and one-dimensional moment problems is extended for half-lines and compact subsets of the real line \( \mathbb{R} \). These results are used to solve the moment problem for the quantum phase space observables generated by the number states.

Keywords: Multidimensional moment problem, exponentially bounded measures, phase space observables.

1. Introduction and notations

The need to regard quantum observables as positive normalized operator measures, as opposed to the more traditional spectral measure approach, motivates the study of the moment operators of such observables, and in particular raises the question of the uniqueness of the observable given its moment operators. The spectral theorem for self-adjoint operators suffices to exhaust these problems in the case of spectral measures; in particular, the first moment of a spectral measure already determines it uniquely.

An important class of quantum observables that are not spectral measures consists of certain phase space observables. These have proved highly useful in several branches of quantum physics, including quantum communication and information theory, quantum optics and quantum measurement theory. Especially the possibility of experimental implementation of such observables by modern technology has drawn a lot of attention to their study.

The original motivation for the research reported in this paper came from the desire to shed light on the problem of the moment operators of phase space observables. This is intimately connected with the general multidimensional moment problem, whose study in our presentation occupies Sections 2 and 3. The choice of material in this part is basically dictated by applications to phase space observables, though not all the results are strictly needed in the sequel.

We denote as usual \(|x| = (x_1^2 + \cdots + x_n^2)^{1/2}\) for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). If \( K \) is a nonempty Borel subset of \( \mathbb{R}^n \), we let \( \mathcal{B}(K) \) denote its Borel \( \sigma \)-algebra and let \( \mathcal{M}_n^*(K) \)
be the set of all measures \( \mu : B(K) \to [0, \infty) \) satisfying \( \int_K ||x||^{2k} d\mu(x) < \infty \) for \( k = 0, 1, 2, \ldots \). We write \( \mathbb{N}_0 := \{0, 1, 2, \ldots \} \). For \( \mu \in \mathcal{M}_n^*(K) \) and \( k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n \), we define the moment \( c_k(\mu) \) as follows:

\[
c_k(\mu) = c_{k_1,\ldots,k_n}(\mu) := \int_K x_1^{k_1} \cdots x_n^{k_n} d\mu(x_1, \ldots, x_n).
\]

The multidimensional moment problem is to find conditions on a (multi)sequence \( (c_k)_{k \in \mathbb{N}_0^n} \) under which there exists a measure \( \mu \) on \( B(K) \) such that \( c_k = c_k(\mu) \) for all \( k \in \mathbb{N}_0^n \). It is known that a measure \( \mu \) need not be uniquely determined by its moment sequence \( (c_k)_{k \in \mathbb{N}_0^n} \).

For \( \mu \in \mathcal{M}_n^*(K) \), we denote \( V[K,\mu] := \{ \nu \in \mathcal{M}_n^*(K) : c_k(\nu) = c_k(\mu) \text{ for all } k \in \mathbb{N}_0^n \} \).

We say that \( \mu \) is determined on \( K \) by its moment sequence if \( V[K,\mu] \) is a singleton. In this situation we also say that \( \mu \) is determinate (on \( K \)).

As usual, for \( p \geq 1 \), \( L^p(K,\mu) \) will denote the space of all (equivalence classes of) Borel functions \( f : K \to \mathbb{R} \) satisfying \( \int_K |f(x)|^p d\mu(x) < \infty \). Let \( \mathcal{P}_n \) denote the set of all polynomials in \( x_1, \ldots, x_n \) (or also the set of their restrictions to a subset of \( \mathbb{R}^n \) clear from the context).

In Section 2 we prove a multidimensional generalization of an exponential boundedness criterion in the classical moment problem, a result involving [6, Theorem 6]. Section 3 extends the results of Petersen [22] on the relation between the multidimensional and the one-dimensional moment problems. Section 4 introduces the phase space observables, Section 5 investigates their moment operators, and Section 6 shows the uniqueness of the number state generated phase space observables in view of their moment sequences. In the final sections the same results are obtained using the Cartesian margins (Section 7) and the polar margins (Section 8) of the phase space observables.

2. Uniqueness in the multidimensional moment problem: exponentially bounded measures

The theorem of this section is a multidimensional generalization of [3, Theorem 6]. Our proof also resembles that of [3] (which according to the authors is inspired by [13]), but in our multidimensional case the theory of holomorphic functions of several complex variables is used. It is worth noting that even before [13], a closely related proof was given in 1950 in the Russian original of [3, 4], see [4, p. 25–26]. We call exponentially bounded the type of measures appearing in the next result.

**Theorem 2.1.** Let \( \mu : B(\mathbb{R}^n) \to [0, \infty) \) be a measure such that

\[
(1) \quad \int_{\mathbb{R}^n} e^{a||x||} d\mu(x) < \infty
\]

for some \( a > 0 \). Then for any \( p \geq 1 \) the set \( \mathcal{P}_n \) of real polynomials in \( n \) variables is dense in \( L^p(\mathbb{R}^n,\mu) \).
Proof. Since \( L^q(\mathbb{R}^n, \mu) \) for \( \frac{1}{p} + \frac{1}{q} = 1 \) is the dual of \( L^p(\mathbb{R}^n, \mu) \), in view of the Hahn-Banach theorem it suffices to show that if \( f \in L^q(\mathbb{R}^n, \mu) \) is such that

\[
\int_{\mathbb{R}^n} x^k f(x) \, d\mu(x) = 0
\]

for every multi-index \( k \in \mathbb{N}_0^n \), then \( f(x) = 0 \) a.e. (Note that by (4) and the Hölder inequality, the integral in (2) exists.) We denote

\[
A = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |\text{Im} \, z_j| < \frac{a}{2\sqrt{n}p} \text{ for all } j = 1, \ldots, n\}.
\]

If \((z_1, \ldots, z_n) \in A\), using the Schwarz inequality we get

\[
\left| \exp(-i \sum_{j=1}^n z_j x_j)^p \right| \left| \exp(\frac{a}{2p} \|x\|) \right|^p = \left[ \exp(\sum_{j=1}^n \text{Im} \, z_j x_j) \exp(\frac{a}{2p} \|x\|) \right]^p \leq \left[ \exp(\sqrt{n} \frac{a}{2\sqrt{n}p} \|x\|) \exp(\frac{a}{2p} \|x\|) \right]^p = e^{a\|x\|},
\]

and so by the Hölder inequality

\[
\int_{\mathbb{R}^n} \left| \exp(-i \sum_{j=1}^n z_j x_j) f(x) e^{\frac{a}{2p}\|x\|} \right| \, d\mu(x) < \infty.
\]

We may thus define \( F: A \to \mathbb{C} \) by the formula

\[
F(z_1, \ldots, z_n) = \int_{\mathbb{R}^n} \exp(-i \sum_{j=1}^n z_j x_j) f(x) \, d\mu(x),
\]

and using (3) we also find by induction that

\[
\frac{\partial^k}{\partial z_{j_1} \cdots \partial z_{j_k}} F(z_1, \ldots, z_n) = \int_{\mathbb{R}^n} (-i)^k x_{j_1} \cdots x_{j_k} \exp(-i \sum_{j=1}^n z_j x_j) f(x) \, d\mu(x)
\]

for all \((z_1, \ldots, z_n) \in A\). (Differentiation under the integral sign is allowed in view of a standard argument \cite[p.282]{21} based on the dominated convergence theorem and a general mean value theorem \cite[p.103]{21}.) Since \( F \) is in each variable separately complex differentiable, it is in \( A \) a holomorphic function of \( n \) complex variables. (This is so by Hartogs’s theorem but the more elementary Osgood lemma in \cite[p. 2]{18} suffices here, as it is easy to show using the dominated convergence theorem that \( F \) is continuous.) Since each \( \frac{\partial^k}{\partial z_{j_1} \cdots \partial z_{j_k}} F(0, \ldots, 0) = 0 \) by assumption, the coefficients of the power series expansion of \( F \) about the origin vanish \cite[p. 3]{18}, and so by the identity theorem \cite[p. 6]{18}, \( F \) is identically zero in \( A \). In particular, for the Fourier–Stieltjes transform of the complex measure \( f \cdot \mu \) we have

\[
\int_{\mathbb{R}^n} \exp(-i \sum_{j=1}^n u_j x_j) f(x) \, d\mu(x) = 0.
\]
for all \((u_1, \ldots, u_n) \in \mathbb{R}^n\), and so \(f \cdot \mu = 0\), that is, \(f\) vanishes \(\mu\) a.e. \(\square\)

**Corollary 2.2.** An exponentially bounded measure \(\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty)\) is ultradeterminate in the sense of \([16]\) and thus determinate.

**Proof.** See \([16, \text{pp. 61, 58}]\) \(\square\)

### 3. Marginal measures and the uniqueness question

In view of the importance of the final conclusion of Corollary 2.2 for our applications, we develop in this section an alternative proof, which does not depend on the theory of holomorphic functions of several complex variables. Some steps on the way have independent interest.

**Lemma 3.1.** Let \(\mu\) be a Borel measure which is determinate on \(K \subseteq \mathbb{R}\), where \(K\) is one of the following possibilities:

(i) \(K = [a, \infty),\ a \in \mathbb{R}\);
(ii) \(K = (-\infty, b],\ b \in \mathbb{R}\);
(iii) \(K\) is a compact subset in \(\mathbb{R}\).

Then the set of all polynomials \(\mathcal{P}_1\) is dense in \(L^2(K, \mu)\).

**Proof.** (i) If \(K = [0, \infty)\), then by \([4, \text{Corollary 3.9}]\), \(\mathcal{P}_1\) is dense in \(L^2([0, \infty), \mu)\).

Let now \(K = [a, \infty)\), and let \(T_a : [0, \infty) \to [a, \infty)\) be the translation defined by \(T(u) := u + a\). Put \(\mu_a(X) := \mu \circ T_a(X)\) for any Borel subset \(X\) of \([0, \infty)\). Then \(\mu_a\) is a Borel measure on \([0, \infty)\). Let \(c_k := \int_a^\infty t^k d\mu(t)\) and \(\tilde{c}_k := \int_0^\infty u^k d\mu(u)\). Then \(c_k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i}\tilde{c}_i\) and \(\tilde{c}_k := \sum_{i=0}^{k} \binom{k}{i} (-1)^{n-i} a^{k-i} c_i\). Therefore, \(\mu\) is determinate on \([a, \infty)\) if and only if \(\mu_a\) is determinate on \([0, \infty)\).

(ii) This case can be proved by modifying the previous argument.

(iii) Let \(K\) be a compact set in \(\mathbb{R}\). Then in view of the Weierstrass approximation theorem, \(\mathcal{P}_1\) is dense in \(L^2(K, \mu)\). \(\square\)

The following result generalizes \([22]\), Theorem 3.

**Theorem 3.2.** Let \(K = K_1 \times \cdots \times K_n\), where each \(K_i\) is either \(\mathbb{R}\) or a nonempty subset of \(\mathbb{R}\) satisfying one of the conditions (i)–(iii) in \(\text{Lemma 3.1}\). Let \(\pi_i : K \to K_i\) be the \(i\)-th projection of \(K\) onto \(K_i\), \(i = 1, \ldots, n\). Then a measure \(\mu \in \mathcal{M}_1(K)\) is determinate whenever all the projection measures \(\mu \circ \pi_i^{-1}\) are determinate on \(K_i\) for \(i = 1, \ldots, n\).

**Proof.** We only sketch the proof, the omitted details being essentially the same as in the proof of Theorem 3 in \([22]\). Let \(\nu \in V[K, \mu]\). Since by assumption the measures \(X \mapsto \mu(\pi_i^{-1}(X))\) are determinate, we get \(\nu \circ \pi_i^{-1} = \mu \circ \pi_i^{-1}\). For any closed set \(K \subseteq \mathbb{R}\), let \(C_c(K)\) denote the space of continuous real functions on \(K\) with compact support. For any real functions \(f_i\) on \(K_i\), \(i = 1, \ldots, n\), let \(f_1 \otimes \cdots \otimes f_n\) be defined by \(f_1 \otimes \cdots \otimes f_n(x_1, \ldots, x_n) := f_1(x_1) \cdots f_n(x_n)\). Using the density results
of Lemma 3.1 we may show, following the argument of [22] referred to above, that, given \( \epsilon > 0 \), for each \( f_i \in C_c(K_i), i = 1, \cdots, n \) we can find \( n \) polynomials \( p_i \in P_1, i = 1, \ldots, n \), such that

\[
\int_K |f_1 \otimes \cdots \otimes f_n - p_1 \otimes \cdots \otimes p_n|d\nu < \epsilon.
\]

For \( f : K \to \mathbb{R} \), define the extension \( f^K \) of \( f \) to \( \mathbb{R}^n \) via \( f^K(x) = f(x) \) if \( x \in K \), otherwise \( f^K(x) = 0 \). Let \( \mu^K \) be the extension of \( \mu \) to \( \mathcal{B}(\mathbb{R}^n) \) given by \( \mu^K(E) := \mu(E \cap K) \), \( E \in \mathcal{B}(\mathbb{R}^n) \). Then \( f \in L^1(K, \mu) \) if and only if \( f^K \in L^1(\mathbb{R}^n, \mu^K) \), and if this is the case, then \( \int_K f d\mu = \int_{\mathbb{R}^n} f d\mu^K \). It is known that the set \( \{ f_1 \otimes \cdots \otimes f_n : f_i \in C_c(\mathbb{R}) \} \) is dense in \( L^1(\mathbb{R}^n, \mu^K) \). For \( f \in C_c(\mathbb{R}) \), the restriction \( f|_{K_i} \) of \( f \) to \( K_i \) gives a function in \( C_c(K_i) \) for any \( i = 1, \ldots, n \). Let \( f \in L^1(K, \nu) \), so that \( f^K \in L^1(\mathbb{R}^n, \nu^K) \). For a given \( \eta > 0 \) we can find \( f_1, \ldots, f_n \in C_c(\mathbb{R}) \) such that \( \int_{\mathbb{R}^n} |f^K - f_1 \otimes \cdots \otimes f_n|d\nu^K < \eta \). Then \( \int_K |f - f_1 \otimes \cdots \otimes f_n \chi_K|d\nu = \int_{\mathbb{R}^n} |f^K - f_1 \otimes \cdots \otimes f_n|d\nu^K < \eta \), which proves that the set \( \{ f_1 \chi_{K_1} \otimes \cdots \otimes f_n \chi_{K_n} : f_i \in C_c(\mathbb{R}) \} \) is dense in \( L^1(K, \nu) \). From (2.3) it easily follows that \( P_n \) is dense in \( L^1(K, \nu) \), and therefore by a theorem of Douglas [12], \( \nu \) is an extremal point of the convex set \( V[K, \mu] \). Since this is true for any \( \nu \in V[K, \mu] \), the set \( V[K, \mu] \) has to be a singleton.

**Corollary 3.3.** Let \( \mu \) be an exponentially bounded Borel measure on \( K = K_1 \times \cdots \times K_n \), with \( K_i \) as in Theorem 3.2, that is,

\[
\int_K e^{a||x||}d\mu(x) < \infty,
\]

for some \( a > 0 \). Then all the moments \( c_k(\mu), k \in \mathbb{N}_0^n \), exist and are finite, and the measure \( \mu \) is determinate.

**Proof.** By assumption, it is clear that all the multidimensional moments \( \int_K x^k d\mu(x) \) exist and are finite. Fix \( i = 1, \ldots, n \). Then

\[
\int_{\mathbb{R}^n} e^{a|x|}d\mu \circ \pi_i^{-1}(t) = \int_K e^{a|x_i|}d\mu(x_1, \ldots, x_n) \leq \int_K e^{a||x||}d\mu(x_1, \ldots, x_n) < \infty.
\]

Using [13, Theorem II.5.2], or [3, Theorem 6], we see that \( \mu \circ \pi_i^{-1} \) is determinate for \( i = 1, \ldots, n \). By Theorem 3.2, \( \mu \) is determinate. \( \square \)

4. Phase space observables

Consider a phase space observable \( A^T \), defined by a state operator \( T \), a positive trace one operator, by means of the weakly defined integral

\[
A^T(Z) := \frac{1}{\pi} \int_Z \mathcal{D}z T \mathcal{D}z^* d\lambda(z), \ Z \in \mathcal{B}(\mathbb{C}),
\]
where $\lambda$ is the Lebesgue measure on the complex plane $\mathbb{C}$, and $D_z = e^{za^*-\overline{za}}$, $z \in \mathbb{C}$, is the unitary shift operator associated with the ladder operators $a = \sum_{n \geq 0} \sqrt{n+1}|n\rangle \langle n+1|$ and $a^* = \sum_{n \geq 0} \sqrt{n+1}|n+1\rangle \langle n|$ of an orthonormal basis $\{|n\rangle\}_{n=0}^{\infty}$, called the number basis, of a complex separable Hilbert space $\mathcal{H}$. Let $A^T_{\psi,\varphi}$ denote the complex measure $Z \mapsto \langle \psi | A^T(Z) \varphi \rangle$ defined by the (positive normalized) operator measure $A^T$ and the vectors $\psi, \varphi \in \mathcal{H}$, and let $\mathcal{L}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$.

The moment operators of the operator measure $A^T$ are the linear operators

$$A^T[m,n] := \int_{\mathbb{C}} z^m \overline{z}^n \, dA^T(z),$$

each defined on the linear subspaces

$$\mathcal{D}[m,n] = \{ \varphi \in \mathcal{H} \mid z \mapsto z^m \overline{z}^n \text{ is integrable w.r.t. } A^T_{\psi,\varphi} \text{ for all } \psi \in \mathcal{H} \},$$

and satisfying, for any $\varphi \in \mathcal{D}[m,n], \psi \in \mathcal{H},$

$$\langle \psi | A^T[m,n] \varphi \rangle = \int_{\mathbb{C}} z^m \overline{z}^n \, dA^T_{\psi,\varphi}(z).$$

We say that the operator measure $A^T$ is determinate if it is uniquely determined by its moment operators $A^T[m,n], m,n \geq 0$.

In a previous article [14] we have investigated the moment problem for the polar coordinate $(\mathbb{C} \ni z = |z|e^{i\theta}, |z| \in [0, \infty), \theta \in [0, 2\pi))$ marginal measures of the phase space observables $A^{(s)}$ associated with the number states $|s\rangle$, $s \in \mathbb{N}_0$. The operator measures

$$\mathcal{B}([0, \infty)) \ni R \mapsto A^{(s)}(R \times [0, 2\pi)) \in \mathcal{L}(\mathcal{H}),$$
$$\mathcal{B}([0, 2\pi)) \ni X \mapsto A^{(s)}([0, \infty) \times X) \in \mathcal{L}(\mathcal{H})$$

were shown to be determinate. Here we investigate the moment problem for the phase space observables $A^{(s)}$.

**Remark 4.1.** The complex moment problem of the measures $A^{(s)}_{\psi,\varphi} : \mathcal{B}(\mathbb{C}) \to [0, 1]$ is here interpreted as the $\mathbb{R}^2$-moment problem of $A^{(s)}_{\psi,\varphi} : \mathcal{B}(\mathbb{R}^2) \to [0, 1]$ via the identification $z = x + iy$. In [23, Appendix] the one-to-one correspondence of the complex and the two-dimensional moment sequences $\int_{\mathbb{C}} z^m \overline{z}^n A^{(s)}_{\psi,\varphi}(d\lambda(z))$ and $\int_{\mathbb{R}^2} x^m y^n A^{(s)}_{\psi,\varphi}(dxdy)$, with $z = x + iy$, has been demonstrated.

5. **On the moment operators of $A^{(s)}$**

To determine the moment operators $A^{(s)}[m,n], m,n \geq 0$, of a phase space observable $A^{(s)}$ defined by a number state $|s\rangle$, we first observe that for any $m,n$, and for any number states $|k\rangle, |l\rangle$, the integral

$$\langle k | A^{(s)}[m,n] | l \rangle = \int_{\mathbb{C}} z^m \overline{z}^n \, dA^{(s)}_{|k\rangle,|l\rangle}(z)$$
exists and is finite. Indeed, by a direct computation one gets

\[
\int_{\mathbb{C}} |z|^{m+n} dA^{(s)}_{[k,l]}(z) = \frac{1}{\pi} \int_{\mathbb{C}} |z|^{m+n} \langle k \mid D_z s \rangle \langle s \mid D_z^* l \rangle \, d\lambda(z) = \\
\delta_{k,l} \sum_{r=0}^{\infty} \sum_{r'=0}^{\infty} a(s, k, r) a(s, k, r') \int_0^\infty e^{-|z|^2} |z|^{m+n+2(s+k-r-r')} 2|z|d|z| < \infty,
\]

where

\[
a(s, k, r) = (-1)^{s-r} \binom{s}{r} \sqrt{k!/(k-r)!},
\]

and where \([k, s]\) denotes the minimum of \(k\) and \(s\) \([\ref{20}].\)

We recall from \([\ref{20}, \text{Lemma A.2}]\) that by the positivity of the operator measure \(A^T\), that is, by the fact that any \(A^T(Z), Z \in \mathcal{B}(\mathbb{C})\), is a positive operator, the domain \(D[m, n]\) of \(A^T[m, n]\) contains as a subspace the set

\[
\tilde{D}[m, n] = \{ \varphi \in \mathcal{H} \mid z \mapsto |z|^{2(m+n)} \text{ is integrable w.r.t. } A^T_{\varphi, \varphi} \}.
\]

Since \(A^T\) is not projection valued, the set \(\tilde{D}[m, n]\) could be a proper subset of \(D[m, n]\).

The above result \((\ref{7})\) shows that for any \(A^{(s)}\)

\[
\text{lin} \{ |k\rangle \mid k \in \mathbb{N}_0 \} \subset \tilde{D}[m, n] \subset D[m, n],
\]

showing that all the moment operators \(A^{(s)}[m, n]\) are densely defined. Denoting

\[
\overline{A^{(s)}[m, n]} := \int_{\mathbb{C}} z^m \mathcal{C} dA^{(s)}(z)
\]

we observe that \(\overline{A^{(s)}[m, n]} = A^{(s)}[n, n]\), as well as \(\overline{A^{(s)}[n, m]} = A^{(s)}[m, n]\). Therefore, using \([\ref{20}, \text{Lemma A.4}]\), we see that the adjoint of \(A^{(s)}[m, n]\), resp. \(A^{(s)}[n, m]\), is an extension of \(\overline{A^{(s)}[m, n]}\), resp. \(\overline{A^{(s)}[n, m]}\), that is,

\[
A^{(s)}[m, n] \subseteq A^{(s)}[n, m]^*,
\]

\[
A^{(s)}[n, m] \subseteq A^{(s)}[m, n]^*.
\]

The matrix elements of the operators \(A^{(s)}[m, n]\) in the number basis can easily be computed, and we get \([\ref{20}]\):

\[
\langle k \mid A^{(s)}[m, n] |l \rangle = \frac{1}{\pi} \sum_{r=0}^{\infty} \sum_{r'=0}^{\infty} a(s, k, r) a(s, l, r') I(m, n, s, k, l, r, r')
\]

where

\[
I(m, n, s, k, l, r, r') = \int_{\mathbb{C}} e^{-|z|^2} z^{m+k+s+r-r'} z^{s+l+s-r-r'} d\lambda(z) = 0, \text{ whenever } k + m \neq l + n,
\]

\[
= \pi (m + s + k - r - r')!, \text{ for } k + m = l + n.
\]
Therefore, 
\[
\langle k \mid A^{(s)}[m, n]|l \rangle = 0, \quad \text{for } k + m \neq l + n,
\]
\[
= \frac{1}{s!} \sum_{r=0}^{[k, s]} \sum_{r'=0}^{[m-n+k, s]} a(s, k, r)a(s, m - n + k, r')(m + s + k - r - r')!,
\]
for \( k + m = l + n. \)

It seems difficult to determine the explicit form of the operators \( A^{(s)}[m, n] \). However, it is known \([20]\) that
\[
A^{(s)}[n, n] = \sum_{i, j=0}^{n} a_{ij} s^{n-j} N^i, \quad a_{ij} \text{ integers, } \mathcal{D}(A^{(s)}[n, n]) = \mathcal{D}(N^n), \quad N = a^*a,
\]
\[
A^*[n, 0] = a^n, \quad A^*[0, n] = (a^*)^n, \quad \mathcal{D}(A^*[n, 0]) = \mathcal{D}(A^*[0, n]) = \mathcal{D}(a^n).
\]

For \( |s\rangle = |0\rangle \) one may quickly confirm that
\[
\langle k \mid A^{(0)}[m, n]|l \rangle = \langle k \mid a^m(a^*)^n|l \rangle
\]
for any \( m, n \in \mathbb{N}_0 \), and for any number states \( |k\rangle, |l\rangle \). Moreover, one easily shows that \( \mathcal{D}(a^m(a^*)^n) \subseteq \mathcal{D}[m, n] \) and that, actually, \( A^{(0)}[m, n] \) extends the operator \( a^m(a^*)^n \), which, together with the above relations \([8, 14]\) shows that
\[
A^{(0)}[m, n] = a^m(a^*)^n.
\]

The possibility of obtaining the operators \( a^m(a^*)^n \) from the ”diagonal coherent state representation” \( \frac{1}{\pi} \int_{\mathbb{C}} z^n \overline{z}^m |z\rangle \langle z| d\lambda(z) \) was perhaps first noticed by Sudarshan \([24]\). The papers \([10, 11]\) are further elaborations on the related ‘phase space quantization methods’. From the point of view of the theory of operator integrals these pioneering papers amounted to showing that \( a^m(a^*)^n \subset A^{(0)}[m, n] \).

6. The uniqueness of \( A^{(s)} \)

We show next that the phase space observable \( A^{(s)} \) is uniquely determined by its moment operators \( A^{(s)}[m, n], m, n \in \mathbb{N}_0 \). In other words, if \( E : \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{L}(\mathcal{H}) \) is another normalised positive operator measure such that its moment operators equal those of \( A^{(s)} \), that is, \( E[m, n] = A^{(s)}[m, n] \) for all \( m, n \in \mathbb{N}_0 \), then \( E = A^{(s)} \). Actually, the equality \( E = A^{(s)} \) already follows if the moment operators of \( E \) agree with those of \( A^{(s)} \) on a dense subset.

Let \( |k\rangle, |l\rangle \) be any two number states and consider the complex measure \( A^{(s)}[|k\rangle, |l\rangle] \). Its values are
\[
A^{(s)}[|k\rangle, |l\rangle](Z) = \frac{1}{\pi} \int_Z \langle k \mid D_{z}s \rangle \langle s \mid D_{z}^*l \rangle \ d\lambda(z) = \frac{1}{\pi s!} \sum_{r=0}^{[k, s]} \sum_{r'=0}^{[l, s]} a(s, k, r)a(s, l, r') \int_Z e^{-|z|^2} z^{s+k-r-r'} \overline{z}^{s+l-r-r'} \ d\lambda(z).
\]
In particular, for each $|k\rangle$ the probability measure $A_{[k],|k\rangle}^{(s)}$ has the density

$$
\frac{1}{\pi} \frac{1}{s!} \sum_{r=0}^{[k,s]} \sum_{r'=0}^{[k,s]} a(s, k, r) a(s, k, r') e^{-|z|^2} |z|^{2(s+k-r-r')} d\lambda(z)
$$

with respect to the Lebesgue measure $\lambda$. But then for any $a \in \mathbb{R}$,

$$
\int_\mathbb{C} e^{a|z|} dA_{[k],|k\rangle}^{(s)} = \frac{1}{\pi} \frac{1}{s!} \sum_{r=0}^{[k,s]} \sum_{r'=0}^{[k,s]} a(s, k, r) a(s, k, r') \int_\mathbb{C} e^{a|z|} e^{-|z|^2} |z|^{2(s+k-r-r')} d\lambda(z)
$$

$$
= \frac{1}{s!} \sum_{r=0}^{[k,s]} \sum_{r'=0}^{[k,s]} a(s, k, r) a(s, k, r') \int_0^{\infty} e^{a|z|} e^{-|z|^2} |z|^{2(s+k-r-r')} 2|z| d|z|
$$

$$
= \frac{1}{s!} \sum_{r=0}^{[k,s]} \sum_{r'=0}^{[k,s]} a(s, k, r) a(s, k, r') e^{(a/2)^2} \int_0^{\infty} e^{-(|z|/2)^2} |z|^{2(s+k-r-r')} 2|z| d|z| < \infty.
$$

By Corollary 2.2 each $A_{[k],|k\rangle}^{(s)}$ is determinate, that is, $|V(\mathbb{C}, A_{[k],|k\rangle}^{(s)})| = 1$.

For any $|k\rangle, |l\rangle, k \neq l, c \in \mathbb{C}, |c| = 1$, we also have

$$
\int_\mathbb{C} e^{a|z|} dA_{[l]+c|k\rangle,|l\rangle+c|k\rangle}^{(s)} = \int_\mathbb{C} e^{a|z|} dA_{[l],|l\rangle}^{(s)} + \int_\mathbb{C} e^{a|z|} dA_{[k],|k\rangle}^{(s)} < \infty,
$$

since for instance

$$
\int_\mathbb{C} e^{a|z|} dA_{[k],|l\rangle}^{(s)} = 0.
$$

Thus all the measures $A_{[l]+c|k\rangle,|l\rangle+c|k\rangle}^{(s)}$ are determinate.

Assume now that $E: B(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ is another operator measure for which $E[m, n] = A^{(s)}[m, n]$ on $\text{lin} \{ |k\rangle \mid k \in \mathbb{N}_0 \}$. Using the polarization identity we get for all number states $|k\rangle$ and $|l\rangle$,

$$
E_{[k],|l\rangle} = \frac{1}{4} \sum_{r=0}^{3} i^r E_{[l]+i^r|k\rangle,|l\rangle+i^r|k\rangle}
$$

$$
= \frac{1}{4} \sum_{r=0}^{3} i^r A_{[l]+i^r|k\rangle,|l\rangle+i^r|k\rangle}^{(s)} = A_{[k],|l\rangle}^{(s)}.
$$

This shows that $E = A^{(s)}$, that is, the phase space observable $A^{(s)}$ defined by the number state $|s\rangle, s \in \mathbb{N}_0$, is determinate.

7. THE UNIQUENESS OF $A^{(s)}$ THROUGH ITS CARTESIAN MARGINS

Using Theorem 3 of Petersen [22], the uniqueness $A^{(s)}$ may also be obtained from the determinacy of its Cartesian marginal measures. We shall demonstrate this result next.
To facilitate the calculations, we pass to the $L^2(\mathbb{R})$-realization of the phase space observables $A^{(s)}$. Let $W: \mathcal{H} \to L^2(\mathbb{R})$ be the unitary mapping for which $W(|n\rangle) = f_n$, $n \in \mathbb{N}_0$, where $f_n$ is the $n$-th Hermite function,

$$
\begin{align*}
    f_n(x) &= N_n e^{-x^2/2} H_n(x), \quad x \in \mathbb{R}, \\
    N_n &= (\sqrt{\pi} 2^n n!)^{-1/2}, \\
    H_n(x) &= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad x \in \mathbb{R}.
\end{align*}
$$

When we identify $\mathbb{C}$ with $\mathbb{R}^2$, and write $z = \frac{q+i p}{\sqrt{2}}$, the phase space observable $A^{(s)}$, defined by $f_s$, gets the form

$$
A^{(s)}(Z) = \frac{1}{2\pi} \int_Z |e^{-iqP+ipQ} f_s \rangle \langle e^{-iqP+ipQ} f_s| \ dq \ dp,
$$

with $(Q, P)$ being the Schrödinger pair on $L^2(\mathbb{R})$ [17]. The Cartesian marginal measures of $A^{(s)}$ are known to be the unsharp position $E^{(Q,s)}$ and the unsharp momentum $E^{(P,s)}$, with

$$
\begin{align*}
    E^{(Q,s)}(X) &= (\chi_X \ast |f_s|^2)(Q), \quad X \in \mathcal{B}(\mathbb{R}), \\
    E^{(P,s)}(Y) &= (\chi_Y \ast |\hat{f}_s|^2)(P), \quad Y \in \mathcal{B}(\mathbb{R}),
\end{align*}
$$

respectively, where $\chi_X \ast |f_s|^2$ is the convolution of the characteristic function $\chi_X$ with the density function $|f_s|^2$, and $\hat{f}_s$ is the Fourier transform of $f_s$, see, e.g. [11, Theorem 3.4.1].

Let $\varphi \in L^2(\mathbb{R})$ be a unit vector, and consider the probability measure $A^{(s)}_{\varphi,\varphi}$. Its Cartesian marginal probability measures are $E^{(Q,s)}_{\varphi,\varphi}$ and $E^{(P,s)}_{\varphi,\varphi}$, respectively. Clearly, they are absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}$. Let $g^{(Q,s)}_{\varphi,\varphi}$ be the Radon-Nikodym derivative of $E^{(Q,s)}_{\varphi,\varphi}$ with respect to $dq$. We assume now that $\varphi \in C_0^\infty(\mathbb{R})$ so that we may take

$$
g^{(Q,s)}_{\varphi,\varphi}(x) = \int_\mathbb{R} |f_s(x-q)|^2 |\varphi(q)|^2 \ dq.
$$

Let $\text{supp} \ q \subseteq [a, b]$, $M \in [0, \infty)$, be such that $|\varphi(q)|^2 \leq M$ for all $x \in [a, b]$, and let $|q| \leq C$, $a \leq q \leq b$. Then

$$
\begin{align*}
g^{(Q,s)}_{\varphi,\varphi}(x) &= N_s^2 \int_a^b e^{-(x-q)^2} H_s(x-q)^2 |\varphi(q)|^2 \ dq \\
&\leq M N_s^2 e^{-x^2} \int_a^b e^{-q^2+2qy} H_s(x-q)^2 \ dq \\
&\leq M N_s^2 e^{-x^2} e^{2C|x|} p_{2s}(x),
\end{align*}
$$

where $p_{2s}(x)$ is the $2s$-th Hermite polynomial.
where $p_{2s}$ is a polynomial of $x$ of degree $2s$. But then for any $a > 0$,
\[
\int_{\mathbb{R}} e^{a|x|} e^{-x^2} e^{2C|x|} p_{2s}(x) \, dx < \infty,
\]
which shows that the probability measure $E^{(Q,s)}_{\varphi,\varphi}$ is exponentially bounded. Therefore, all the moments of the probability measure $E^{(Q,s)}_{\varphi,\varphi}$ are finite and the measure $E^{(Q,s)}_{\varphi,\varphi}$ is determinate for each unit vector $\varphi \in C^\infty_0(\mathbb{R})$. Similarly, any probability measure $E^{(P,s)}_{\varphi,\varphi}$, $\varphi \in C^\infty_0(\mathbb{R})$, $\| \varphi \| = 1$, is determinate, so that, by [22, Theorem 3], or by Theorem 3.2, any phase space probability measure $A^{(s)}_{\varphi,\varphi}$, $\varphi \in C^\infty_0(\mathbb{R})$, $\| \varphi \| = 1$, is determinate.

**Remark 7.1.** Not all the probability measures $E^{(Q,s)}_{\varphi,\varphi}$, resp. $E^{(P,s)}_{\varphi,\varphi}$, $f \in L^2(\mathbb{R}), \| f \| = 1$, can be determinate since the moment operators of $E^{(Q,s)}_{\varphi,\varphi}$ are unbounded operators. Since the phase space observable $A^{(s)}$ is known to be informationally complete [3, 8] (that is, for any two state operators $T, U$, if $A^{(s)}_T = A^{(s)}_U$, then $T = U$), it also follows that if $\psi$ and $\varphi$ are two different vector states such that $E^{(Q,s)}_{\psi,\psi} = E^{(Q,s)}_{\varphi,\varphi}$ and $E^{(P,s)}_{\psi,\psi} = E^{(P,s)}_{\varphi,\varphi}$, then the measures $E^{(Q,s)}_{\psi,\psi}$ and $E^{(P,s)}_{\psi,\psi}$ cannot both be determinate, since otherwise also $A^{(s)}_{\psi,\psi}$ (as well as $A^{(s)}_{\varphi,\varphi}$) would be determinate, with the implication that the states $| \psi \rangle \langle \psi |$ and $| \varphi \rangle \langle \varphi |$ would be the same, which need not be the case, see, e.g. [3, Sect. 2.3].

Assume now that $E : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(L^2(\mathbb{R}))$ is another positive operator measure such that $E_{\varphi,\varphi} = A^{(s)}_{\varphi,\varphi}$ for all $\varphi \in C^\infty_0(\mathbb{R})$, $\| \varphi \| = 1$. Let $\psi$ be any unit vector of $L^2(\mathbb{R})$. Since $C^\infty_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, $\psi$ either is in $C^\infty_0(\mathbb{R})$ or a limit of a sequence of vectors $\varphi_n \in C^\infty_0(\mathbb{R})$. Let $\psi = \lim \varphi_n$. Then $\lim E_{\varphi_n,\varphi_n}(Z) = E_{\psi,\psi}(Z)$ as well as $\lim E_{\psi,\psi}(Z) = A^{(s)}_{\psi,\psi}(Z)$ uniformly for $Z \in \mathcal{B}(\mathbb{C})$, which implies that $E_{\varphi_n,\varphi_n} \to E_{\psi,\psi}$ and $E_{\varphi_n,\varphi_n} \to A^{(s)}_{\psi,\psi}$ in the total variation norm [13, p. 97]. Therefore, $E_{\psi,\psi} = A^{(s)}_{\psi,\psi}$ for any unit vector $\psi \in L^2(\mathbb{R})$. By the polarization identity, the operator measures $E$ and $A^{(s)}$ are the same. To conclude, we have established the following result.

**Corollary 7.2.** Let $A^{(s)}$ be the phase space observable defined by the number state $| s \rangle = W^{-1} f_s$, $s \in N_0$. The moment operators $A^{(s)}[m,n]$ are densely defined, $W^{-1}(C^\infty_0(\mathbb{R})) \subset \mathcal{D}[m,n] \subseteq \mathcal{D}[m,n]$, and the observable $A^{(s)}$ is uniquely determined by the restrictions of its moment operators to $W^{-1}(C^\infty_0(\mathbb{R}))$.

8. **The uniqueness of $A^{(s)}$ in terms of its polar coordinate margins**

In addition to the complex moments - which, as we have seen, essentially amount to the real moments in terms of the Cartesian representation - of a phase space observable, it is illuminating to consider the real moments in terms of the polar coordinate representation. This we do next making use of the generalization of Petersen’s result expounded in Theorem 3.2.
Consider the phase space observable $A^{(s)}$ and its polar coordinate moment operators $\int_0^\infty \int_0^{2\pi} r^n \theta^m \, dA^{(s)}(re^{i\theta})$. The polar coordinate marginal measures are

\[ B([0, \infty)) \ni R \mapsto A^{(s)}(R \times [0, 2\pi)) \in \mathcal{L}(\mathcal{H}), \]

\[ B([0, 2\pi)) \ni X \mapsto A^{(s)}([0, \infty) \times X) \in \mathcal{L}(\mathcal{H}), \]

the second of them being compactly supported and thus determinate. In [14, Section 5] it was shown that also the radial margin of $A^{(s)}$ is uniquely determined by its (unbounded self-adjoint) moment operators. Therefore, by Theorem 3.2, we may conclude that the phase space observable $A^{(s)}$ is uniquely determined also by its polar coordinate moment operators $\int_0^\infty \int_0^{2\pi} r^n \theta^m \, dA^{(s)}(re^{i\theta}), n, m \in \mathbb{N}_0$. The same conclusion can also be obtained from [7, Theorem 3.6] concerning rotation invariant moment problem. We do not pursue to determine the moment operators $\int_0^\infty \int_0^{2\pi} r^n \theta^m \, dA^{(s)}(re^{i\theta})$, since their physical relevance is less direct.

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