Text mixing shapes the anatomy of rank-frequency distributions:
A modern Zipfian mechanics for natural language

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Natural languages are full of rules and exceptions. One of the most famous quantitative rules is Zipf’s law which states that the frequency of occurrence of a word is approximately inversely proportional to its rank. Though this ‘law’ of ranks has been found to hold across disparate texts and forms of data, analyses of increasingly large corpora over the last 15 years have revealed the existence of two scaling regimes. These regimes have thus far been explained by a hypothesis suggesting a separability of languages into core and non-core lexica. Here, we present and defend an alternative hypothesis, that the two scaling regimes result from the act of aggregating texts. We observe that text mixing leads to an effective decay of word introduction, which we show provides accurate predictions of the location and severity of breaks in scaling. Upon examining large corpora from 10 languages, we find emphatic empirical support for the universality of our claim.

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ZIPF’S LAW AND (NON) UNIVERSALITY

Given some collection of distinct kinds of objects occurring with frequency \( f \) and associated rank \( r \) according to decreasing frequency, Zipf’s law is said to be fulfilled when ranks and frequencies are approximately inversely proportional:

\[
f(r) \sim r^{-\theta},
\]

(1)
typically with \( \theta \approx 1 \). Though Zipf’s functional form has been found to be a reasonable one for disparate forms of data, ranging from frequencies of words to sizes of cities in Zipf’s original work [1, 2], its lack of total universality in application to natural languages is now widely acknowledged [3–8].

Recently it was suggested [3, 4] that large corpora exhibit two scaling regimes (delineated by some \( b > 0 \)):

\[
f(r) \sim \begin{cases} r^{-\theta}, & : r \leq b \\ r^{-\gamma}, & : r > b \\ \end{cases},
\]

(2)
the first being that of Zipf (\( \theta = 1 \)) and the second distinctly more variable [4]. (though generally \( \gamma > 1 \)). Ferrer and Solé hypothesized in [3] that these two regimes reflected a division of natural languages into two lexical subsets—the kernel (core) and unlimited (non-core) lexica.

With this inspiration, we focus on understanding the effects of combining texts of varying heterogeneity—a process we refer to as “text mixing”.

We observe that in all studies finding dual scalings that the texts analyzed are of mixed origin, that is, they are not derived from a single author, or even a single topic. Montemurro indicated in [4] that combining heterogeneous texts could generate effects that shield investigators from the true underlying nature of this second scaling regime:

To resolve the behavior of those [high rank] words we need a significant increase in volume of data, probably exceeding the length of any conceivable single text. Still, at the same time it is desirable to maintain as high a degree of homogeneity in the texts as possible, in the hope of revealing a more complex phenomenology than that simply originating from a bulk average of a wide range of disparate sources.

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Data: The complete historical romances of Georg Ebers.
STOCHASTIC MODELS

In the years following Zipf’s original work, various stochastic models have been proposed for the generation of natural language vocabularies. The first of these was that proposed by Simon [9], and based on Yule’s model of evolution [10]. This work is a powerful companion to understanding Zipf’s empirical work, and can be seen as the natural antecedent of the rich-gets-richer models [11, 12] for growing networks that have interested the complex systems community over recent years. Indeed, perhaps the most important piece we may draw from Simon’s model is that a rich-gets-richer mechanism is a reasonable one for the growth of a vocabulary.

An important limitation of Simon’s model is that it is only capable of producing a single scaling regime, which, as we know is an incomplete picture. Furthermore, the scalings accessible via the Simon model were strictly less severe than the ‘universal’ $\theta = 1$ exponent. So, if one assumes the Simon model as truth, with a fixed word introduction rate $\alpha_0$, Zipf’s exponent should be variable and necessarily less than 1, though empirically found indistinguishable from 1, that is $\theta = 1 - \alpha_0$, with $\alpha_0 \ll 1$ [9].

Recently, a modification to Simon’s model was proposed in which two types of words could be produced—core and non-core words [4]. As a built-in feature of the core/non-core vocabulary (CNCV) model, the size of the core set of words was prescribed to be finite, while the non-core was allowed to expand indefinitely. Aside from introducing two classes of words, the most important distinction of this model from its predecessor was a rule for the decay in the rate of introduction of new words, $\alpha$. Along with producing the CNCV model they showed that when $\alpha$ decays as a power-law with exponent $-\mu$, of the number of unique words, $n$, the relationship between $\mu$ and the lower rank-frequency exponent, $\gamma$, is a difference of $\theta$, i.e.,

$$\alpha(n) = \alpha_0 \cdot n^{-\mu} \Rightarrow f(r) \sim r^{-(\theta + \mu)}, \quad (3)$$

with $\gamma = \theta + \mu$ [5]. The distinction between word types provided a means for postponing the point at which their power law decay would occur, thereby generating two scaling regimes. We note that the severity of the second scaling was only contingent upon the existence of a decay in the rate of introduction of new words, and that this decay was imposed, rather than the result of the existence of two word types. We are therefore drawn to find an explicit mechanism capable of producing power-law decaying word introduction rates, and hence multiple scaling regimes.

FIG. 2: (A) Direct comparison of $b$ with $N_{avg}$. Note that many corpora fall quite close to the line $N_{avg} = b$ (green dashed line). Of those the three corpora that most extremely deviate (red points), two are the smallest and least refined (Greek and Swedish), and the other (Finnish) represents the only non-Indo-European language. Further, all three have relatively shallow second scalings. (B) Bar-plot of $b$ and $N_{avg}$ showing how for each corpus, the two numbers are quite close in log-space.

FIG. 3: Rank-frequency plots of corpora defined by the deciles of the distribution of text sizes in the English Project Gutenberg eBooks collection. Corpora presented are colored (red to blue) according to decile (1–10). Note that the regressed breaks in scaling (green points), $b$, and corpus average text sizes (colored points), $N_{avg}$, both increase with decile, indicating that the location of the English scaling break is not universal property of the language, but rather a product of the texts in the corpus. Looking closer at the relationship between $N_{avg}$ with $b$, we plot the two against one another in the upper right inset, where the green dashed line indicates the line $b = N_{avg}$. There, we note that the extreme deciles are the most biased and hence show a weaker agreement of $N_{avg}$ with $b$. 

FIG.
As we have described, the CNCV model offers a means by which one can obtain a second scaling. The model is, like Simon’s, framed as a model of the generation of a vocabulary. However, we are led to question whether lower scalings are a product of vocabulary generation or an artifact of an interaction between disparate texts. Suppose a collection of texts, $C = \{T_1, ..., T_k\}$, is read sequentially, and that each has rank-frequency distribution of Zipf/Simon form. Indeed, we find that idealized (see Materials and Methods) Zipf/Simon rank-frequency distributions, once combined, result in a corpus having multiple scaling regimes (see Fig. 1). Though each individual vocabulary might have been created without a decay of word introduction, an overlap in the words they use have it seem as though the appearance of new words is rarer by the time the later texts are read. If one reads the texts repeatedly and in permuted orders, the resulting decay in the rate of word introduction likely does not evince itself until the mean text size (mean number of unique words per text) is reached, but certainly not before the minimum text size is reached.

In the following, we run text mixing experiments that measure decay in rates of word introduction directly attributable to mixing texts to predict lower scalings in composite distributions. As we read out texts (in some order) let $m$ be the volume of words observed at any point, and $n_m$ be the number of distinct words in the volume $m$, which we will refer to as the vocabulary size of the growing text. To exhibit the effects of text mixing we contrast the vocabulary size of the growing text with the vocabulary size of the memoryless text, $N_m$, where we “forget” the words read in all previous texts and continuing counting appearances of words that were initial in their text (regardless of appearances in previous texts). From $n_m$ and $N_m$ we then have two proxies for the word introduction rate, one for the growing text $\alpha_m = n_m/m$ and one for the memoryless text $A_m = N_m/m$. We may consider $\alpha_m$ to be the word introduction rate of the composite (which includes mixing effects), and $A_m$ to be the word introduction rate of the individual texts (excluding mixing effects).

There are many conceivable mechanisms that lead to a power-law decay in the rate of word introduction. To measure the severity of scaling breaks we do not need to know the true values of the word introduction rates, but instead just their scalings. So, to determine the extent to which text mixing generates word introduction decay, we isolate the portion of the scaling that results from mixing by measuring $\alpha_m/A_m$, the portion of word introduction remaining after mixing texts. Note that since $n_m \leq N_m$, one has $\alpha_m \leq A_m$, and hence $\alpha_m/A_m \leq 1$ for all $m$. Hence, this normalized rate behaves as a non-constant only when mixing ensues, and so any decay measured via $\alpha_m/A_m$ implies the presence and is the direct consequence of text mixing. For example, consider the two excerpts from Charles Dickens’ “A Tale of Two Cities”,

| Language | $N_{books}$ | $N_{char}$ | $N_{min}$ | $N_{ave}$ | $b$ | $N_{max}$ | $N_{corp}$ |
|----------|-------------|------------|-----------|-----------|----|-----------|-----------|
| en       | 19,793      | 46         | 5         | 5,899.3   | 7,477| 219,990   | 2,836,900 |
| fr       | 1,360       | 44         | 395       | 8,300.7   | 9,884| 26,171    | 528,314   |
| fi       | 505         | 31         | 1,144     | 8,572.6   | 3,899| 31,623    | 811,742   |
| nl       | 434         | 48         | 133       | 6,747.1   | 5,154| 82,246    | 443,816   |
| pt       | 375         | 38         | 203       | 4,675.8   | 6,813| 17,818    | 246,497   |
| de       | 327         | 30         | 153       | 7,554.9   | 7,137| 113,089   | 2,047     |
| es       | 223         | 34         | 406       | 8,735.1   | 10,355| 29,452    | 8,872.6   |
| it       | 194         | 29         | 1,083     | 9,388.7   | 9,435| 258,500   | 7,137     |
| sv       | 56          | 34         | 1,389     | 7,499.8   | 4,279| 18,726    | 123,806   |
| el       | 42          | 35         | 2,047     | 6,414.7   | 15,739| 110,940   | 7,499.8   |

TABLE I: Table of information concerning the data used from the Project Gutenberg eBooks database. For each language we record the number of books ($N_{books}$); the number of characters ($N_{char}$), which we take to be the number of letters (including diacritics and ligatures); the minimum text size ($N_{min}$); the maximum text size ($N_{max}$); and the total corpus size ($N_{corp}$). For reference, we additionally record the regressed point of scaling break, $b$. 

FIG. 4: Box plots of the base ten logarithm vocabulary sizes of the texts contained in the 10 Project Gutenberg corpora studied. Center bars indicate means and whiskers extend to most extremal values up to 1.5 times the I.Q.R. length, whereupon more extremal values are plotted as points designated “outliers”. Note that the three most extreme values are reference texts in English, German, and Dutch.
FIG. 5: The results of text mixing experiments for the largest corpus from the Project Gutenberg eBooks collection—English. The main axes show the empirical frequency-data distribution (black points), \( f \), and the model determined by text mixing (red points), \( \hat{f} \). The measured lower and upper exponents, \( \gamma \) and \( \theta \), are depicted in the lower-right and upper-left respectively, with triangles indicating the measured slopes. We also present gray boxes in the main axes to highlight the different mixing regimes, marked by \( \hat{N}_{\text{char}} \), \( N_{\text{min}} \), \( N_{\text{avg}} \), and \( N_{\text{max}} \) (see Tab. I for a full description of these quantities). Note how \( N_{\text{avg}} \) frequently lies quite close to \( b \), the regressed point of scaling break (depicted in green). The lower left inset shows the point-wise squared error \( (f(r) - \hat{f}(r))^2 \), whose sum is minimized in the transformation of \( \hat{f} \) into \( f \). Note here that the squared error tends to drop off at or near each language’s alphabet size, indicating the possibility that \( \hat{N}_{\text{char}} \)’s marks or influences the point of stabilization of the Zipf/Simon regime. The upper right inset shows the untransformed form of word introduction, \( \alpha \), and the decay exponent \( \mu \), depicted by the regressed slope (red dashed line).

Takes as texts:

\[
T_1 : (\text{it, was, the, best, of, times, it, was, the, worst, of, times,}) \quad \text{and} \\
T_2 : (\text{it, was, the, age, of, wisdom, it, was, the, age, of, foolishness,})
\]

Supposing we read \( T_1 \) first, the sequence of words is:

\[
(T_1, T_2) : (\text{it, was, the, best, of, times, it, was, the, worst, of, times, it, was, the, age, of, wisdom, it, was, the, age, of, foolishness,})
\]

where we have highlighted initial (growing text) word appearances in red. The corresponding sequences of values, \( m, n_m, \hat{N}_m, \alpha_m, A_m \) and \( \alpha_m/A_m \), are then

\[
m : (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24) \\
n_m : (1, 2, 3, 4, 5, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7) \\
N_m : (1, 2, 3, 4, 5, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7) \\
\alpha_m : (1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7) \\
A_m : (1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7) \\
\alpha_m/A_m : (1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7)
\]

where we only record those values of \( \alpha_m/A_m \) when a new word is observed (highlighted in red) in the growing text. Since \( \alpha_m/A_m \) will be the only quantity used in the measurement of word introduction decay, we relax the notation, and simply write \( \alpha \) for \( \alpha_m/A_m \) and \( n \) for \( n_m \) in what follows.

To test the effects of text mixing, we not only observe the word introduction rate \( \alpha(n) \), but consider its ability to predict the scalings of rank-frequency distributions. To do this, we first require an estimate, \( b \), of the point of scaling break. Working with the log-centers of the plateaux the rank-frequency distribution (see Materials and Methods for more details), this estimation is done in log-space by applying a two-line least-squares regression, constrained by intersection at the point of scaling break. With \( b \), we note that by design, the data for \( \alpha(n) \) are aligned with \( f(r) \)—both have domain \( \{1, ..., N_{\text{corp}}\} \) (where \( N_{\text{corp}} \) is the vocabulary size of the corpus). Further, since the theory has \( \gamma = \theta + \mu \), we may also observe that \( \alpha(n) \cdot n^{-\theta} \), need only be scaled by a constant to produce a model

\[
\hat{f}(n) = C \cdot \alpha(n) \cdot n^{-\theta},
\]

for \( f \). To determine optimal model parameters, for each \( \theta \in \{0.75, 0.751, ..., 1.25\} \) we find the value \( C \) such that \( \hat{f}(b) = f(b) \). We then accept the \( \theta \) and \( C \) for the optimal as the pair with smallest sum of squares error.

MATERIALS AND METHODS

In our experiments we work with a subset of the Project Gutenberg eBooks collection. We collected those texts which were annotated sufficiently well to
allow for the removal of meta-data as well as for the parsing of authorship, title, and language. All together, this resulted in the inclusion of 23,309 books from across ten languages (broken down in Tab. I).

To determine an estimate for $b$, we regress in log-space on the log-centers of the plateaux of the rank-frequency distribution. This estimation is done by applying a two-line least-squares regression, constrained by intersection at the point of scaling break. Given data points $(x, y)$, and a point of break, $x_b$, we solve for the model

$$
\hat{y} = \begin{cases} 
\beta_1 + \beta_2 x, & x \leq x_b \\
\beta_3 + \beta_4 x, & x > x_b 
\end{cases},
$$

(5)

constrained by $\beta_1 + \beta_2 x_b = \beta_3 + \beta_4 x_b$, through standard minimization of the sum of squares error. We compute this regression for 100 log-spaced points, $x_b$, across $10^2$ to $10^5$. Performing these 100 regressions, we accept the value $b$ with smallest SSE.

To estimate $\mu$ we perform common least squares linear regression on the log-transformed data over the region $[b, N_{\text{corps}}]$. Since $b$ is an estimate of the point of scaling break, and not necessarily the point at which mixing-derived decay becomes clear, the regression of $\mu$ reflects this. In particular, for those corpora where $N_{\text{avg}}$ and $b$ disagree the most, the regression of $\mu$ is least reliable.

Computation of $\alpha(n)$ involves running many realizations of the text mixing procedure, randomizing the order in which the texts are read. To ensure that our measurements are accurate, we adhere to a heuristic—that the number of text mixing runs be no less than $10 \times N_{\text{books}}$ for the given corpus. The final values used is in our experiments are computed as an average of $\alpha_m / A_m$. However, we note that $\alpha_m / A_m = n_m / N_m$, where $n_m$ ranges with rank: $n_m = 1, 2, 3, \ldots, N_{\text{corps}}$. So, the only quantities that vary across runs that are necessary to compute $\alpha(n)$ are the $N_m$. Hence we take the average as $\alpha(n_m) = n_m / \langle N_m \rangle$, which is in fact the harmonic mean of the $\alpha(n_m)$ (the true mean for rates).

**RESULTS AND DISCUSSION**

To understand our results we define $N_{\text{min}}$, $N_{\text{avg}}$ and $N_{\text{max}}$ as the minimum, average, and maximum text sizes (by numbers of unique words) respectively (see Tab. I). These three values obviate four text mixing regimes:

- $n < N_{\text{min}}$: Zipf/Simon (no mixing)
- $N_{\text{min}} \leq n \leq N_{\text{avg}}$: initial (minimal mixing)
- $N_{\text{avg}} \leq n \leq N_{\text{max}}$: crossover (partial mixing)
- $n > N_{\text{max}}$: terminal (full mixing)

In the Zipf/Simon regime we expect the result of an unperturbed Simon model, though because mixing is also minimal over the initial regime we expect that behavior over the first two regimes to more or less be consistent. Once in the crossover regime, words will on average have appeared under the effects of text mixing and so there is the expectation that $N_{\text{avg}}$ will mark the macroscopically observable change in behavior, or scaling break of the rank-frequency distribution, i.e., we expect $b \approx N_{\text{avg}}$. Plotting the two against one another, we see this relationship holds across the majority of corpora (see Fig. 3A). Finally, over the terminal regime, all words will appear in the presence of mixing, and so this regime exhibit the sta-
FIG. 7: The results of text mixing experiments for the nine smaller corpora analyzed (See Fig. 5 for a full description).

We observe that the models defined by text mixing, \( \hat{f} \), produce excellent predictions of the rank-frequency distributions (Main axes, Figs. 5, 7, and center panel of 6), which is made quite clear by plotting point-wise squared error (lower-left insets, Figs. 5, 7, and center panel of 6). For each corpus we see a broad range of ranks beginning not far before \( 10^2 \), and extending far into the second scaling where the error is quite low (disregarding the effect of the finite-size plateaux). Looking closer, we note that this point generally appears at or near each language’s number of characters, \( N_{\text{char}} \) (blue dashed line, lower-left insets). Hence, we consider the possibility of its origin in a subordinate rich-gets-richer process. It was shown in [10] that random typing models poorly represent word construction processes—even when characters are “typed” from empirical distributions. It therefore seems plausible for a preferential selection process to have driven word construction, and continue influencing bilized second scaling, characterized by the decay parameter \( \mu \) (upper right insets, Figs. 5, 7, and 6).
the type of words with highest frequency (which generally possess fewer characters [11][2]). However, given the word standardizations of most languages, it seems highly unlikely that individual characters are selected by modern language practitioners. In other words, since the vast majority of semantic units are presently selected on the scales of words or phrases, the current selection format of characters is likely subordinate, leaving character choices highly dependent on one another. Such a subordinate selection process could be at the root of our heuristic observation, that the language’s number of characters (see Table 1). N_char, marks the point of stabilization for the Zipf/Simon regime.

Interestingly, the slope $\theta$ is frequently measured to lie outside the Simon-productive range, $(0, 1)$. Therefore, we are left to conclude that individually, the texts are subject to internally-derived decay in word introduction rates, i.e., the underlying rank-frequency distributions are not of pure Zipf/Simon form (as we suggest in other work [8], and explore in Figs. [3] and [1]), but, instead, subject to internally-derived decay. Though we do not exhaustively investigate the occurrence of internally-derived decay in the rates of word introduction across the Project Gutenberg eBooks dataset, it seems quite possible that all of the texts parsed are subject to mixing effects, whether from non-original annotation by the Project Gutenberg e-Text editors, or simply just the mixing of chapters. This of course would require that mixing effects be of low-impact in the cases generally considered strong examples Zipf’s law.

We also notice a different behavior (all of which is appropriately captured by the experiment) in the English data set. Here, we find a relatively shallow lower scaling ($\gamma \approx 1.65$), but notice that this appears to be one of two lower scalings. For English, the crossover regime exhibits a consistently steeper scaling that dies away in the terminal regime. Though we have no certain explanation for this behavior, part of what makes the English collection so different from the others is the sheer number of texts (see Table 1). Further, we see that the distribution of text vocabulary sizes in the English collection is rife with small outliers, and by far possesses the single largest text (see Fig. [3]). English is also well-known for its willingness to adopt foreign words, which may lead to an increased rate of appearance of low-count loan words. Regardless of the reasons for this difference with English, we find that text mixing captures the shape of both lower scaling regimes, and so both are well explained by the text mixing model.

In light of the results presented, we take time to consider the validity of the core language hypothesis. We see significant variation in both the location and severity of scaling breaks across languages. Further, upon sampling the English corpus by deciles, we’ve observed that the regressed point of scaling break, $b$, is in fact not stationary (see Fig. [3]). We take this as indication of the lack of validity of the core language hypothesis, as a language’s core should exhibit a strong consistency of size. Moreover, languages closely related via a common ancestor should likewise exhibit this consistency, but notably two of the languages most closely related in the study, Spanish and Portuguese, present a large difference in $b$, (10,355 for Spanish, and 6,813 for Portuguese—see Table 1). It may then be more reasonable to consider a language core as a collection of words necessary for basic description, but not overlapping in use or meaning. However, such a core lexicon must then be determined by native practitioners, and not a scaling regime of a rank-frequency distribution. Alternatively, one could consider a corpus-core by it’s collection of words common to it’s texts. However, such a “common core” would be entirely dependent on the composition of the corpus, and hence not a universal property of a language proper.

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