ON AN ANALYTIC VERSION OF LAZARD’S ISOMORPHISM

GEORG TAMME

Abstract. We prove a comparison theorem between locally analytic group cohomology and Lie algebra cohomology for locally analytic representations of a Lie group over a nonarchimedean field of characteristic 0. The proof is similar to that of van-Ést’s isomorphism and uses only a minimum of functional analysis.

Contents

Introduction 1
Notations and conventions 3
1. Locally analytic group cohomology 3
2. Locally analytic representations 5
3. Differential forms and Lie algebra cohomology 7
4. Differential forms and locally analytic group cohomology 9
5. Explicit description of the comparison map 11
Appendix: The Poincaré lemma 12
References 15

Introduction

In his seminal paper [Laz65] Lazard established two basic theorems concerning the cohomology of a compact $\mathbb{Q}_p$-analytic Lie group $G$ with Lie algebra $\mathfrak{g}$. Firstly, if $V$ is a finite dimensional $\mathbb{Q}_p$-vector space with continuous $G$-action, the natural map from locally analytic group cohomology $H^*_\mathrm{an}(G,V)$, defined in terms of locally analytic cochains, to continuous group cohomology $H^*_\mathrm{cont}(G,V)$ is an isomorphism. Secondly, there is a natural isomorphism between the direct limit $\colim_{G' \subset G} H^*_\mathrm{cont}(G',V)$, where $G'$ runs through the system of open subgroups of $G$, and the Lie algebra cohomology $H^*(\mathfrak{g}, V)$. Hence, combining both, there is a natural isomorphism

$$\colim_{G' \subset G \text{ open}} H^*_\mathrm{an}(G', V) \cong H^*(\mathfrak{g}, V).$$

These results play an important role in arithmetic geometry, in particular in the theory of Galois representations, or in the study of $p$-adic regulators [HK11].

Date: August 20, 2014.
The author is supported by the CRC 1085 Higher Invariants (Universität Regensburg).
At least for certain Lie groups, integral and $K$-analytic versions have been obtained by Huber, Kings, and Naumann [HKN11], when $K$ is a finite extension of $\mathbb{Q}_p$. The proofs are based on Lazard’s original argument via continuous group cohomology, and are not easily accessible. A somewhat simplified proof has been given by Lechner [Lec12] using formal group cohomology.

On the other hand, the situation for a real Lie group $G$ is much more transparent. The analogous result is van Est’s isomorphism $H^\ast_d(G,V) \cong H^\ast(\mathfrak{g},K;V)$, which relates differentiable group cohomology with relative Lie algebra cohomology for a maximal compact subgroup $K \subseteq G$. Its proof is based on the following observations: The quotient $G/K$ is contractible, hence the de Rham complex $\Omega^\ast(G/K,V)$ with coefficients in a $G$-representation $V$ is a resolution of $V$. Moreover, for any $k$, the space $\Omega^k(G/K,V)$ is $G$-acyclic. Hence, $H^\ast_d(G,V)$ is computed by the $G$-invariants of the complex $\Omega^\ast(G/K,V)$, which is precisely the Chevalley-Eilenberg complex computing relative Lie algebra cohomology $H^\ast(\mathfrak{g},K;V)$.

It is a natural question whether a similar argument works in the nonarchimedean situation. In this note, we show that this is indeed the case. This gives a direct proof of the isomorphism (1) and generalizes it with respect to the ground field and the coefficients:

**Theorem.** Let $K$ be a nonarchimedean field of characteristic 0. Let $G$ be a locally $K$-analytic Lie group and $G \to \text{Aut}(V)$ a locally analytic representation on a barrelled locally convex $K$-vector space. Denote by $\mathfrak{g}$ the $K$-Lie algebra of $G$. Then there are natural isomorphisms

$$\colim_{G' \subseteq G \text{ open}} H^\ast_{\text{an}}(G',V) \cong H^\ast(\mathfrak{g},V),$$

where the colimit is taken over the system of open subgroups of $G$.

The rough argument is as follows: The de Rham complex $\Omega^\ast(G,V)$ is a resolution of the locally constant $V$-valued functions on $G$. As in the real case, each $\Omega^k(G,V)$ is $G$-acyclic, hence the cohomology of the locally constant $V$-valued functions on $G$ is isomorphic to the Lie algebra cohomology $H^\ast(\mathfrak{g},V)$ (see Sections 3 and 4 for precise results). The Theorem then follows by taking the direct limit over the open subgroups of $G$.

The proof also shows that, for compact $G$, one can recover the locally analytic group cohomology from the Lie algebra cohomology as the invariants under the natural $G$-action: $H^\ast_{\text{an}}(G,V) \cong H^\ast(\mathfrak{g},V)^G$ (see Corollary 11).

Moreover, we describe the comparison map between locally analytic group cohomology and Lie algebra cohomology explicitly on the level of complexes: It is given by differentiating locally analytic cocycles at 1 (see Section 5).

In order to apply usual arguments from homological algebra, we show, following Flach [Fla08], that the locally analytic cochain cohomology groups can be interpreted as derived functors of the global section functor on a topos $BG$ (Sections 1 and 2). The nice feature of this is that it gives a quick proof of the main results which requires only a minimum of functional analysis.

An alternative approach to the cohomology of locally analytic representations of Lie groups over finite extensions of $\mathbb{Q}_p$ is due to Kohlhaase [Koh11]. It is based on relative homological algebra. He obtains similar results under
an additional assumption on the group, which, as he proves, is fulfilled in many cases. The cohomology groups he defines are finer than ours in the sense that they themselves carry a locally convex topology. In contrast to the groups we use, they do not always coincide with the cohomology groups defined in terms of locally analytic cochains.

**Notations and conventions.** For the whole paper, we let $K$ be a nonarchimedean field of characteristic 0, i.e., $K$ is equipped with a nontrivial nonarchimedean absolute value $|.|$ such that $K$ is complete for the topology defined by $|.|$. By a manifold we will always mean a paracompact locally $K$-analytic manifold. Note that, by [Sch11, Cor. 18.8], any locally $K$-analytic Lie group is paracompact. For manifolds $X,Y$, we denote by $C^m(X,Y)$ the set of locally $K$-analytic maps from $X$ to $Y$. We will refer to them simply as analytic maps.

1. **Locally analytic group cohomology**

In this section, we describe the topos-theoretic approach to locally analytic group cohomology. We refer to [Fla08] for the case of continuous cohomology.

Denote by $La$ the category of manifolds. We will sometimes use the fact that every manifold is even strictly paracompact, i.e., every open covering can be refined by a disjoint one, since we assumed manifolds to be paracompact [Sch11, Prop. 8.7]. We let $Sh(La)$ be the category of sheaves on $La$ with respect to the topology generated by open coverings. For this topology, every representable presheaf is a sheaf, hence we have the Yoneda embedding $y: La \to Sh(La)$.

Let $G$ be a Lie group. Then $y(G)$ is a group object in $Sh(La)$. The category of sheaves with a $y(G)$-action is a topos [SGA4 I Prop. 8.7]. We let $Sh(La)$ be the category of sheaves on $La$ with respect to the topology generated by open coverings. For this topology, every representable presheaf is a sheaf, hence we have the Yoneda embedding $y: La \to Sh(La)$.

Let $G$ be a Lie group. Then $y(G)$ is a group object in $Sh(La)$. The category of sheaves with a $y(G)$-action is a topos [SGA4 I Prop. 8.7], called the classifying topos of $y(G)$. It will be denoted by $BG$. We denote its global section functor by $\Gamma: BG \to Set$, $\Gamma(F) = Hom_{BG}(*,F) = F(*)^G$. Similarly, if $X$ is an object of $BG$, we denote by $\Gamma(X,-) = Hom_{BG}(X,-)$ the functor of sections over $X$.

**Definition 2.** Let $A$ be an abelian group object of $BG$. Then we define

$$H^i(G,A) := (R^i\Gamma)(A).$$

In the next section we will explain how to associate an abelian sheaf in $BG$ to any locally analytic representation of $G$.

We want to describe these cohomology groups in terms of a concrete complex, similar to locally analytic cochains. We begin with some general considerations.

Let $T$ be a topos, and let $G$ be a group object in $T$. For objects $A,B$ of $BG$ the internal hom $Hom_{BG}(A,B)$ is given as follows: The underlying object of $T$ is $Hom_T(A,B)$ and the action of $G$ is given by the formula $(g\phi)(a) = g(\phi(g^{-1}a))$.

\[^1\text{More precisely, we assume the existence of universes and only consider manifolds which are elements of a given universe } \mathcal{U}. \text{ Then } Sh(La) \text{ and } BG \text{ are } \mathcal{V}\text{-topoi for a universe } \mathcal{V} \text{ with } \mathcal{U} \in \mathcal{V}.\]
Denote by \( i: * \to \mathcal{G} \) the morphism from the trivial group in \( T \) to \( \mathcal{G} \). It induces a geometric morphism of topoi (see [SGA4\( \text{Exp. IV, 4.5} \)])

\[
i: T \cong B* \to BG.
\]

The left adjoint \( i* \) simply forgets the \( \mathcal{G} \)-action. The right adjoint is given by \( i_*(\mathcal{F}) = \mathcal{H}om_{BG}(\mathcal{G}, \mathcal{F}) \) where \( \mathcal{G} \) is \( \mathcal{G} \) with its natural left action, viewed as an object of \( BG \), and \( \mathcal{F} \) is viewed as object of \( BG \) with trivial \( \mathcal{G} \)-action. The functor \( i* \) also has a left adjoint \( i_! \) given by \( \mathcal{F} \mapsto \mathcal{G} \times \mathcal{F} \) with \( \mathcal{G} \)-action via the first factor.

For an object \( A \in BG \), we denote by \( A^\sharp \) the object of \( BG \) with the same underlying object in \( T \) and trivial \( \mathcal{G} \)-action.

**Lemma 3.** For \( A, B \in BG \) we have

\[
\mathcal{H}om_{BG}(\mathcal{G} \times A, B) \cong i_* \mathcal{H}om_{T}(i^* A, i^* B).
\]

**Proof.** Let \( \mathcal{X} \) be an object of \( BG \). Then we have natural isomorphisms

\[
\mathcal{H}om_{BG}(\mathcal{X}, i_* \mathcal{H}om_{T}(i^* A, i^* B)) \cong \mathcal{H}om_{T}(i^* \mathcal{X}, \mathcal{H}om_{T}(i^* A, i^* B))
\]

\[
\cong \mathcal{H}om_{T}(i^* \mathcal{X} \times A, i_* B)
\]

\[
\cong \mathcal{H}om_{BG}(i_! i^* \mathcal{X} \times A, B)
\]

\[
\cong \mathcal{H}om_{BG}(\mathcal{G} \times (\mathcal{X} \times A)^\sharp, B)
\]

\[
\cong \mathcal{H}om_{BG}(\mathcal{G} \times \mathcal{X} \times A, B)
\]

\[
\cong \mathcal{H}om_{BG}(\mathcal{X}, \mathcal{H}om_{BG}(\mathcal{G} \times A, B))
\]

where we used the isomorphism \( \mathcal{G} \times (\mathcal{X} \times A)^\sharp \cong \mathcal{G} \times \mathcal{X} \times A \) given by \( (pr_1, \text{action}) \). This implies the lemma. \( \square \)

We now consider the case \( T = \mathcal{H}om_{Sh(La)} \), \( \mathcal{G} = y(G) \). For a sheaf \( \mathcal{F} \) on \( La \), the sheaf underlying \( i_* \mathcal{F} \) is, by the above, given by \( X \mapsto \mathcal{H}om_{Sh(La)}(y(G), \mathcal{F})(X) \cong \mathcal{F}(G \times X) \) (Yoneda lemma).

**Lemma 4.** The functor \( i_* \) from abelian sheaves on \( La \) to abelian group objects in \( BG \) is exact.

**Proof.** Since \( i_* \) is a right adjoint, it is left exact. Consider an epimorphism \( A \to B \) of abelian sheaves on \( La \). Since \( G \times X \) is strictly paracompact the functor of sections over \( G \times X \) is exact and \( A(G \times X) \to B(G \times X) \) is an epimorphism of abelian groups. From this we deduce that \( i_* A \to i_* B \) is an epimorphism. \( \square \)

**Corollary 5.** For any abelian sheaf \( A \) on \( La \), we have

\[
H^i(G, i_* A) \cong \begin{cases} A(*) & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}
\]

**Proof.** Since the left adjoint \( i^* \) is exact, \( i_* \) sends injectives to injectives. Since \( i_* \) is exact and \( \Gamma \circ i_*(-) = \mathcal{H}om_{BG}(*, i_*(-)) \cong \mathcal{H}om_{Sh(La)}(*, -) \) we see that

\[
H^i(G, i_* A) \cong H^i_{Sh(La)}(*, A) \cong \begin{cases} A(*) & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}
\]

\( \square \)
We let $E_q G$ be the simplicial manifold given in degree $p$ by $E_p G := G^{p+1}$, and $\phi^*: E_q G \to E_p G$, for $\phi: \{0 < \cdots < p\} \to \{0 < \cdots < q\}$, given by $(g_0, \ldots, g_q) \mapsto (g_0(\phi), \ldots, g_\phi(p))$. The group $G$ acts on $E_q G$ via diagonal left multiplication. We denote the simplicial object of $BG$ given by $y(E_0 G)$ equipped with its the diagonal $y(G)$-action by $iE_0 G$. For an abelian group object $A$ in $BG$, the degree-wise sections over $iE_0 G$ form a cosimplicial abelian group $\Gamma(iE_0 G, A)$.

**Proposition 6.** Let $A$ be an abelian group object of $BG$. Then

$$H^*(G, A) \cong H^*(\Gamma(iE_0 G, A)).$$

**Proof.** The projection $y(G) \to *$ is an epimorphism in $BG$. The Čech nerve of this morphism is precisely $iE_0 G$. We thus have a quasi-isomorphism

$$A \cong \mathcal{H}om_{BG}(*, A) \xrightarrow{\cong} \mathcal{H}om_{BG}(iE_0 G, A).$$

Using Lemma 3 and Corollary 5 we see that the complex on the right hand side consists of $\Gamma$-acyclic objects. We conclude using $\Gamma(\mathcal{H}om_{BG}(iE_0 G, A)) = \mathcal{H}om_{BG}(*, \mathcal{H}om_{BG}(iE_0 G, A)) \cong \mathcal{H}om_{BG}(iE_0 G, A) = \Gamma(iE_0 G, A).$ □

## 2. Locally analytic representations

In this section we associate to a locally analytic representation $V$ of $G$ an abelian group object $V$ in $BG$. The main result is that the cohomology of $V$ on $BG$ is isomorphic to the cohomology of $G$ with coefficients in $V$ defined in terms of analytic cochains (Proposition 8). We begin by recalling some basic notions about locally analytic representations.

Let $V$ be a locally convex separated $K$-vector space. A BH-space for $V$ is a continuous inclusion of a separated Banach space $W \hookrightarrow V$. Let $X$ be a manifold. A function $f: X \to V$ is called analytic, if every $x \in X$ admits a neighborhood $U$, a chart $\psi: U \xrightarrow{\cong} B_\epsilon(0) \subseteq K^n$, and a BH-space $W \hookrightarrow V$ such that $f \circ \psi^{-1}$ factors as $B_\epsilon(0) \to W \hookrightarrow V$ where the first map is given by a convergent power series (see [FdL99, Def. 2.1.7]). Here $B_\epsilon(0)$ denotes the closed ball of radius $\epsilon$ around $0$ in $K^n$. Since a finite sum of BH-spaces for $V$ is again a BH-space for $V$, the analytic functions $X \to V$ form a vector space denoted by $C^{an}(X, V)$. More precisely, $C^{an}(X, V)$ is even a module over the algebra of analytic functions $C^{an}(X, K)$ [FdL99, Kor. 2.4.4]. For varying $X$, this is a sheaf on $\text{La}$ denoted by $C^{an}(\cdot, V)$.

A locally analytic representation of the Lie group $G$ on $V$ is a topological representation, i.e., an action of $G$ on $V$ by continuous automorphisms such that all orbit maps $G \to V, g \mapsto gv$, are analytic.

To any locally analytic representation $V$ of $G$ we want to associate an object of $BG$. If $f: X \to V$ and $\rho: X \to G$ are analytic, then the pointwise product $\rho f: X \to V$ may not necessarily be analytic. Hence we cannot simply take $C^{an}(\cdot, V)$ as sheaf. Instead, we have to consider the subsheaf of $C^{an}(\cdot, V)$ of admissible functions as we now explain.

Let $G \to \text{Aut}(V)$ be a topological representation. We call an analytic function $f: X \to V$ admissible, if the map $\hat{f}: G \times X \to V, (g, x) \mapsto gf(x)$, is analytic. Note that $\hat{f}$ is analytic iff its restriction $\hat{f}|_{U \times X}$ for some open
subset \( U \subset G \) is analytic. Indeed, if this is the case, then for any \( h \in G \) the restriction \( \hat{f}|_{hU \times X} \) is equal to the composition
\[
(hU \times X) \xrightarrow{(g,x) \mapsto (h^{-1}g,x)} U \times X \xrightarrow{f|_{U \times X}} V \xrightarrow{h} V,
\]
where the first two maps are analytic, and the last one is continuous and linear. We define
\[
\mathcal{V}(X) := C^{an}(X, V) := \{ f \in C^{an}(X, V) \mid f \text{ is admissible} \}.
\]
This is a \( C^{an}(X, K) \)-submodule of \( C^{an}(X, V) \). We claim that \( \mathcal{V} \) is a subsheaf of \( C^{an}(\cdot, V) \) and that the point-wise multiplication by \( G \)-valued analytic maps defines an action of \( g(G) \) on \( \mathcal{V} \). We will henceforth view \( \mathcal{V} \) as an abelian group object of \( BG \).

**Proof.** If \( f \in \mathcal{V}(X) \) and \( \phi : Y \to X \) is an analytic map between manifolds, then \( f \circ \phi \) is analytic. Moreover, \( \hat{f} \circ \hat{\phi} = \hat{f} \circ (\id_G \times \phi) \) is analytic as well, hence \( f \circ \phi \) is admissible. Thus \( \mathcal{V} \) is a presheaf. Since admissibility is a local property, \( \mathcal{V} \) is a sheaf.

Now let \( \rho : X \to G \) be an analytic map. We define \( \rho f \) by \( (\rho f)(x) := \rho(x)f(x) \). We have to show that \( \rho f \) is analytic and admissible. But this is clear since \( \rho f \) equals the composition \( X \xrightarrow{(\rho, \id_X)} G \times X \xrightarrow{f} V \) and \( \rho f \) equals the composition \( G \times X \xrightarrow{(g,x) \mapsto (\rho g)(x)} G \times X \xrightarrow{f} V \).

**Remark.** If \( V \) is a Banach space and \( G \to \Aut(V) \) a locally analytic representation then every analytic function \( X \to V \) is admissible. This follows from [FdL99, Kor. 3.1.9].

**Definition 7.** For a locally analytic representation of \( G \) on \( V \) and \( i \geq 0 \) we define the locally analytic group cohomology of \( G \) with coefficients in \( V \) as
\[
H^{an}_{\ast}(G, V) := H^{\ast}(G, \mathcal{V}).
\]

Recall that a homogeneous analytic \( p \)-cochain of \( G \) with coefficients in \( V \) is an analytic function \( f : E_pG \to V \) which is \( G \)-equivariant, i.e., satisfies \( f(gg_0\ldots g_0, \ldots, g_p) = gff(g_0\ldots g_p) \). We denote the complex of homogeneous analytic cochains by \( C^{an}_{\mathbb{G}}(E_pG, V) \). Its differential is induced by the simplicial structure of \( E_pG \).

**Proposition 8.** The cohomology \( H^{an}_{\ast}(G, V) \) is isomorphic to the cohomology of the complex of homogeneous analytic cochains \( C^{an}_{\mathbb{G}}(E_pG, V) \).

**Proof.** By Proposition 6 we have \( H^{\ast}(G, V) \cong H^{\ast}(\Gamma(E_pG, V)) \). Using the Yoneda lemma we see that a section in \( \Gamma(\xi E_pG, V) = \Hom_{BG}(\xi E_pG, V) \) is just an admissible function \( f : E_pG \to V \) such that
\[
\begin{array}{ccc}
G \times E_pG & \xrightarrow{\id \times f} & G \times V \\
\downarrow \text{diagonal multiplication} & & \downarrow \text{action} \\
E_pG & \xrightarrow{f} & V
\end{array}
\]
commutes, i.e., a \( G \)-equivariant admissible function \( E_pG \to V \).

To prove the claim, it suffices to show that, vice versa, every \( G \)-equivariant analytic function \( f : E_pG \to V \) is admissible. But, by the \( G \)-equivariance, \( \hat{f} \)
Lemma 9. This is a well-defined $y(G)$-action.
Proof. We have to show that $\rho\omega$ is analytic and admissible. Consider the analytic maps $\tilde{\rho}: G \times X \times G \to G \times X \times G$, $(h, x, g) \mapsto (h\rho(x), x, \rho(x)^{-1}g)$ and $i_1: X \times G \to G \times X \times G$, $(x, g) \mapsto (1, x, g)$. Then $\omega = i_1^*\rho^*\omega$, hence $\rho\omega$ is analytic. Similarly, we have $\tilde{\rho}\omega = \rho^*\tilde{\omega}$, hence $\rho\omega$ is admissible. □

We thus consider $\Omega^k_{ad}(G, V)$ as an abelian group object in $BG$. We want to show that it is acyclic. Write $\tilde{V} := Hom(\Lambda^k g, V)$. The adjoint action of $G$ on $g$ and the given action of $G$ on $V$ induce a natural $G$-action on $\tilde{V}$.

**Lemma 10.** This representation of $G$ on $\tilde{V}$ is locally analytic. We have an isomorphism

$$\Omega^k_{ad}(G, V) \cong \text{Hom}_{BG}(\tilde{y}(G), \tilde{V}) \cong i_* i^* \tilde{V}.$$  

Proof. Let $Y$ be a manifold. We claim that a function $f: Y \to \tilde{V}$ is admissible if and only if the function $f_x: Y \to V, y \mapsto f(y)(x)$, is admissible for every $x \in \Lambda^k g$. Taking $Y$ to be a point this implies the first assertion of the lemma.

To prove the claim, assume first that $f$ is admissible. We have $\hat{f}_x(g, y) = g(f(y)(x)) = (g(f(y))(gx) = \hat{f}(g, y)(gx)$. The function $\hat{f}$ is analytic by assumption and so is $g \mapsto gx$. Since the evaluation $\tilde{V} \times \Lambda^k g \to V$ is continuous and bilinear, and since $\Lambda^k g$ is finite dimensional, [Edl99, Satz 2.4.3] implies that $\hat{f}_x$ is analytic.

To see the converse, let $x_1, \ldots, x_N$ be a basis of $\Lambda^k g$ and $x_1^*, \ldots, x_N^*$ the dual basis of $\Lambda^k g^\vee$. We can write $f$ as a sum $f = \sum_{i=1}^N f_{x_i} x_i^*$ with $f_{x_i}$ admissible. Then $\hat{f}(g, y) = \sum_i \hat{f}_{x_i}(g, y) g(x_i^*)$ and by loc. cit. again, $\hat{f}$ is analytic.

We now prove the second assertion of the lemma. For any manifold $X$, right translations by elements of $G$ induce a trivialization of the vertical tangent bundle $T(X \times G/X) \cong (X \times G) \times g$. This gives a natural isomorphism of vector spaces

$$\Omega^k(X, \tilde{V}) \cong C^m(X \times G, \tilde{V}).$$

Using the above claim one sees that this isomorphism restricts to an isomorphism

$$\Omega^k_{ad}(X \times G, V) \cong C^m(X \times G, \tilde{V}).$$

Under this isomorphism, the $y(G)(X)$-action on the left-hand side corresponds to the action on the right-hand side induced by left translations on $X \times G$ and the action on $\tilde{V}$ mentioned above. Using the isomorphism $C^m(X \times G, V) \cong \text{Hom}_{BG}(\tilde{y}(G), \tilde{V})(X)$, this gives the first isomorphism stated in the Lemma. The second follows immediately from Lemma 3. □

**Corollary 11.** We have

$$H^i(G, \Omega^k_{ad}(G, V)) \cong \begin{cases} \text{Hom}_{K}(\Lambda^k g, V) & \text{if } i = 0, \\ 0 & \text{else}. \end{cases}$$

Proof. By Lemma 10 and Corollary 3 the higher cohomology groups vanish, and

$$H^0(G, \Omega^k_{ad}(G, V)) \cong \tilde{V}(*) = \text{Hom}_{K}(\Lambda^k g, V).$$  

□
Explicitly, this isomorphism is given by evaluating a form at 1 ∈ G.

The differential d of the complex Ω^*(G, V) is compatible with the y(G)-action. Hence we can view Ω^*(G, V) as a complex in BG and we can compute its hypercohomology.

We now assume that V is barrelled, i.e., that every closed convex absorbing subset is open (see [Sch02, §6]). Differentiating the orbit maps g ↦ gv induces an action of the Lie algebra g on V [FdL99, Sätze 3.1.3, 3.1.7].

**Corollary 13.** We have natural isomorphisms

\[ H^i(G, \Omega^*_\text{ad}(G, V)) \cong H^i(\mathfrak{g}, V) \]

where the right-hand side is Lie algebra cohomology.

**Proof.** Corollary [11] gives an isomorphism

\[ H^i(G, \Omega^*_\text{ad}(G, V)) \cong H^i(\text{Hom}_K(\bigwedge^* \mathfrak{g}, V)), \]

where the differential on Hom_K(\bigwedge^* \mathfrak{g}, V) is induced from the de Rham differential via [12]. This is precisely the Chevalley-Eilenberg complex computing Lie algebra cohomology. \( \square \)

4. DIFFERENTIAL FORMS AND LOCALLY ANALYTIC GROUP COHOMOLOGY

As before, we fix a locally analytic representation G → Aut(V). In this section we use the Poincaré lemma to compare the hypercohomology of the complex of V-valued admissible forms with locally analytic group cohomology, and we give the proof of the Theorem announced in the introduction.

Fix a manifold Y. A function f : Y × X → V will be called locally constant along Y, if, for every (y, x) ∈ Y × X, there exist open neighborhoods Y′ ⊆ Y of y and X′ ⊆ X of x such that f|_{Y′×X′} factors through the projection Y′ × X′ → X′. We define

\[
C^\text{lc}_\text{ad}(Y, V)(X) := \{ f ∈ C^\text{ad}(X × Y, V) \mid f \text{ is locally constant along } Y \}.
\]

It is easy to see that X ↦ C^\text{lc}_\text{ad}(Y, V)(X) defines a sheaf on La.

**Proposition 14.** The inclusion in degree 0

\[ C^\text{lc}_\text{ad}(Y, V) → \Omega^*_\text{ad}(Y, V) \]

is a quasi-isomorphism.

**Proof.** The map clearly induces an isomorphism on H^0, and it remains to show that H^k(Ω^*_\text{ad}(Y, V)) = 0 for k > 0.

Let X be a manifold, and let ω be a closed form in Ω^k(X × Y/X, V). We will show that there is an η ∈ Ω^{k-1}(X × Y/X, V) such that dη = ω. Since all manifolds are strictly paracompact, it is enough to construct such an η locally on X and Y.

The rest of the proof uses some results and notations from the Appendix.

It can be skipped on first reading. Since d\tilde{ω} = \tilde{dω} = 0, the form \tilde{ω} ∈ Ω^k(G × X × Y/G × X, V) is closed. Replacing G be a small open neighborhood of 1 and using local charts, we may assume that there are multiradii δ ∈ R^m_+, ε ∈ R^n such that G × X ⊆ B_δ(0) ⊆ K^m, Y ⊆ B_ε(0) ⊆ K^n, and a BH-space W → V such that \tilde{ω} is given by a power series in F_δ(Ω^k(W)) (cf. (19)). Choose a multiradius \epsilon' < \epsilon. The homotopy operator h: Ω^k(W) → Ω^{k-1}(W)
given by lemma 20 induces an operator \( h : F^k_\delta(\Omega^k_x(W)) \to F^k_\delta(\Omega^{k-1}_x(W)) \).

We define \( \tilde{\eta} := h(\omega) \). Hence \( \tilde{\eta} \) represents a relative analytic \( k-1 \)-form on \( G \times X \times Y' \) with an open subset \( Y' \subset Y \). Since \( \omega \) is closed, we have \( d\tilde{\eta} = \omega |_{G \times X \times Y'} \).

For \( g \in G \), let \( i_g : X \times Y \to G \times X \times Y \) (and similarly with \( Y \) replaced by \( Y' \)) be the inclusion \((x,y) \mapsto (g,x,y)\). We set \( \eta := i^*_g\tilde{\eta} \). Clearly, \( d\eta = i^*_g d\tilde{\eta} = i^*_g \omega = \omega \). To prove that \( \eta \) is admissible, we show that \( \tilde{\eta} = \eta \).

Let \( \Phi_g : V \to V \) be the continuous automorphism given by the action of \( g \). We have to check that \( i^*_g \tilde{\eta} = \Phi_g \circ \eta \). By restriction, \( \Phi_g \) induces a continuous isomorphism of BH-spaces \( W \to g(W) \) (more precisely, we view \( W \) as a linear subspace of \( V \) and let \( g(W) \) be its image under the action of \( g \in G \) with Banach space structure induced from \( W \) via the linear isomorphism \( \Phi_g|_W : W \xrightarrow{\cong} g(W) \)).

We have
\[
\begin{align*}
i^*_g \tilde{\eta} &= i^*_g(h(\omega)) \quad \text{(by definition)} \\
&= h(i^*_g \omega) \quad \text{(using (18) with } \Phi = h : \Omega_x^k(W) \to \Omega^{k-1}_x(W)) \\
&= h(\Phi_g \circ \omega) \quad \text{(definition of } \omega) \\
&= \Phi_g \circ h(\omega) \quad \text{(Lemma 21 for } \Phi = \Phi_g : W \to g(W)) \\
&= \Phi_g \circ \eta \quad \text{(since } h(\omega) = h(i^*_g \omega) = i^*_g h(\omega) = i^*_g \tilde{\eta} = \eta).
\end{align*}
\]

\( \Box \)

Proof of the Theorem. The sheaf \( \underline{\mathcal{C}}_{\text{ad}}^{\text{lc}}(G,V) \) carries a natural \( y(G) \)-action induced by left translations on \( G \) and the given action on \( V \). By Proposition 14 and Corollary 13 we have isomorphisms
\[
H^*(G,\underline{\mathcal{C}}_{\text{ad}}^{\text{lc}}(G,V)) \cong H^*(G,\Omega^*_x(\underline{\mathcal{C}}_{\text{ad}}^{\text{ad}}(G,V))) \cong H^*(\mathfrak{g},V).
\]

As in the proof of Proposition 8, Proposition 9 implies that \( H^*(G,\underline{\mathcal{C}}_{\text{ad}}^{\text{lc}}(G,V)) \) is the cohomology of the complex \( C^\infty_G(G \times E_G V) \) of \( G \)-equivariant analytic functions \( G \times E_G V \) that are locally constant along the first factor.

Since the open subgroups \( G' \subseteq G \) form a fundamental system of neighborhoods of \( 1 \in G \) (see [Sch11, Lemma 18.7]), we have an isomorphism
\[
\colim_{G' \subseteq G \text{ open}} C^\infty_{G'}(E_{G'} V) \cong \colim_{G' \subseteq G \text{ open}} C^{\text{lc}}_{G'}(G' \times E_{G'} V).
\]

Because taking the colimit over a directed system is exact, we see that
\[
\colim_{G' \subseteq G} H^*_\text{an}(G',V) \to \colim_{G' \subseteq G} H^*(G',\underline{\mathcal{C}}^{\text{lc}}(G',V))
\]
is an isomorphism. Since the isomorphisms (15) are compatible with the restriction to open subgroups, the claim follows. \( \Box \)

There is an additional action of \( G \) on \( \underline{\mathcal{C}}_{\text{ad}}^{\text{lc}}(G,V) \) and on \( \Omega^*_x(\underline{\mathcal{C}}_{\text{ad}}^{\text{ad}}(G,V)) \) induced by right translations on \( G \). This action is compatible with the given \( y(G) \)-action. It induces a \( G \)-action on the cohomology groups. Via the isomorphism (15), this corresponds to the \( G \)-action on \( H^*(\mathfrak{g},V) \) induced by the adjoint action on \( \mathfrak{g} \) and left multiplication on \( V \).

Corollary 16. If \( G \) is compact, there is a natural isomorphism
\[
H^*_\text{an}(G,V) \cong H^*(\mathfrak{g},V)^G.
\]
Proof. Since $G$ is compact, every open subgroup is of finite index and contains an open normal subgroup. If $X$ is a compact manifold, every function in $C^\text{lc}(G,V)(X)$ factors through $G/H \times X$ for some open normal subgroup $H \leq G$. Thus – using the notation from the previous proof –

$$C^\text{lc}_G(G \times E_\bullet G,V) = \colim_{H \leq G \text{ open}} C^\text{an}_G(G/H \times E_\bullet G,V).$$

Since the colimit over a directed system is exact, this induces an isomorphism $H^*(g,V) \cong \colim_{H \leq G} H^*(C^\text{an}_G(G/H \times E_\bullet G,V)).$ Since each quotient $G/H$ is finite, and taking invariants under a finite group is an exact functor on $K$-vector spaces with an action by that group, we get

$$H^*(g,V)^G \cong \colim_{H \leq G} H^*(C^\text{an}_G(G/H \times E_\bullet G,V))^G/H$$

$$\cong \colim_{H \leq G} H^*(C^\text{an}_G(G/H \times E_\bullet G,V)^{G/H})$$

$$\cong \colim_{H \leq G} H^*(C^\text{an}_G(E_\bullet G,V)) \cong H^*_\text{an}(G,V).$$

\[ \Box \]

5. Explicit description of the comparison map

We want to describe an explicit map of complexes which induces the comparison map $H^*_\text{an}(G,V) \to H^*(g,V)$. Recall that $H^*_\text{an}(G,V)$ is computed by the complex of homogeneous locally analytic cochains $C^\text{an}_G(E_\bullet G,V)$, and that $H^*(g,V)$ is computed by the complex of $G$-invariant admissible differential forms $\Omega^*_\text{ad}(G,V)^G$.

For integers $p \geq 0$ and $0 \leq i \leq p$, we denote by $d_i$ the partial exterior derivative in the direction of the $(i+1)$-th factor of the product $E_{p,G} = G^{p+1}$. We denote by $\Delta_p: G \to E_{p,G}$ the diagonal map. For $f \in C^\text{an}(E_{p,G},V)$ we set

$$\Phi(f) := \Delta^*_p(d_1d_2\ldots d_pf) \in \Omega^p(G,V).$$

**Proposition 17.** The map $\Phi$ induces a morphism of complexes $C^\text{an}_G(E_\bullet G,V) \to \Omega^*_\text{ad}(G,V)^G$, which on cohomology groups agrees with the comparison map $H^*_\text{an}(G,V) \to H^*(g,V)$.

**Remark.** Let us consider the case of trivial coefficients. The space of functions $C^\text{an}(E_{p,G},K)$ has topological generators the form $f_0 \otimes \cdots \otimes f_p$ with $f_i \in C^\text{an}(G,K)$. For such a function we have

$$\Phi(f_0 \otimes \cdots \otimes f_p) = f_0 df_1 \ldots df_p.$$  

Baring in mind that the identification $\Omega^*_G(V)^G \cong \text{Hom}(\wedge^*_K g,K)$ is given by evaluating a differential form at $1 \in G$, one sees that, under this identification, our map $\Phi$ coincides with the map $\Psi$ defined in [HK11] Def. 3.4.5.

**Proof.** From the proof of Proposition 6 we have the acyclic resolution $V \cong \mathcal{H}om_{BG}(E_\bullet G,V)$. For a manifold $X$ we have

$$\mathcal{H}om_{BG}(E_\bullet G,V)(X) = C^\text{ad}(X \times E_\bullet G,V)$$

with $y(G)$-action induced from left translations on $E_\bullet G$ and the action on $V$. We define $\Phi: C^\text{ad}(X \times E_{p,G},V) \to \Omega^p_{\text{ad}}(X \times G/X,V)$ by the same formula as above. We claim that this gives a morphism of complexes $\Phi: \mathcal{H}om_{BG}(E_\bullet G,V) \to \Omega^*_\text{ad}(G,V)$ in $BG$. 

Proof of the claim. One checks without difficulty that $\Phi$ is equivariant for the $y(G)$-action. Now consider $f \in C^{ad}(E_p G \times X, V)$. Recall the face maps $\partial_i : E_{p+1} G \to E_p G, (g_0, \ldots, g_{p+1}) \mapsto (g_0, \ldots, \hat{g}_i, \ldots, g_{p+1})$. The differential of the complex $C^{ad}(E_{\bullet} G \times X, V)$ maps $f$ to

$$\sum_{i=0}^{p+1} (-1)^i \partial_i^* f.$$ 

Since $\partial_i^* f$ is constant along the $(i+1)$-th factor $G$, we have $d_i(\partial_i^* f) = 0$. Since the partial derivatives commute up to sign, it follows that

$$\Phi(\sum_{i=0}^{p+1} (-1)^i \partial_i^* f) = \Phi(\partial_0^* f) = \Delta^*_p (d_1 \ldots d_{p+1} (\partial_0^* f)).$$

We thus have a commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\sim} & \Omega^*_{ad}(G, V) \\
\downarrow \Phi & & \downarrow \Phi \\
\Omega^*_{ad}(G, V) & & \Omega^*_{ad}(G, V)
\end{array}$$

where the complexes on the right-hand side consist of acyclic sheaves. The proposition now follows by taking global sections.

Appendix: The Poincaré lemma

Let $W$ be a $K$-Banach space with norm $\| \cdot \|$. For a multiradius $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}_+^n$ we denote the space of $\epsilon$-convergent power series in $n$ variables $x = (x_1, \ldots, x_n)$ with coefficients in $W$ by $F_\epsilon(W)$:

$$F_\epsilon(W) := \left\{ \sum_{I \in \mathbb{N}_0^n} a_I x^I \mid a_I \in W, \|a_I\| \epsilon^I \xrightarrow{I \to \infty} 0 \right\}$$

Equipped with the norm $\| \sum_I a_I x^I \|_\epsilon := \max_I \|a_I\| \epsilon^I$, this is again a Banach space.

Let $\Phi : W \to W'$ be a continuous linear map between Banach spaces. It induces a continuous linear map $F_\epsilon(W) \to F_\epsilon(W')$. Let $B_\epsilon(0) \subset K^n$ be the closed ball of radius $\epsilon$ around $0$. For any $x \in B_\epsilon(0)$ we have the evaluation at $x$, written $i_x^* : F_\epsilon(W) \to W$ and similarly for $W'$. Since $\Phi$ is continuous
the diagram

\[
\begin{array}{c}
F_\epsilon(W) \xrightarrow{\phi} F_\epsilon(W') \\
\downarrow i^*_x \quad \downarrow i^*_x \\
W \xrightarrow{\phi} W'
\end{array}
\]

(18)

commutes.

For \( q \geq 0 \) we denote by \( \Omega^q_\epsilon(W) \) the space of \( \epsilon \)-convergent \( W \)-valued \( q \)-forms in \( n \) variables:

\[
\Omega^q_\epsilon(W) := \bigwedge^q_K (K^n)^\vee \otimes_K F_\epsilon(W).
\]

Since \( \bigwedge^q_K (K^n)^\vee \) is a finite dimensional \( K \)-vector space, this is again a Banach space. The usual differential defines a continuous linear map \( d: \Omega^q_\epsilon(W) \to \Omega^{q+1}_\epsilon(W) \).

There is natural injection \( \Omega^q_\epsilon(W) \hookrightarrow \Omega^q(B_\epsilon(0),W) \) into the space of locally analytic \( W \)-valued \( q \)-forms. It is compatible with the differential. More generally, if \( \delta \in R^m_+ \) is a second multiradius, we can identify \( \delta \)-convergent power series with coefficients in \( \Omega^q_\epsilon(W) \) with relative \( W \)-valued forms:

\[
F_\delta(\Omega^q_\epsilon(W)) \hookrightarrow \Omega^q(B_\delta(0) \times B_\epsilon(0)/B_\delta(0),W).
\]

(19)

On the other hand, every relative \( q \)-form is in the image of (19) after shrinking \( \delta \) and \( \epsilon \) appropriately.

If \( \epsilon' \in R^m_+ \) is a multiradius which is component-wise strictly smaller than \( \epsilon \), written \( \epsilon' < \epsilon \), then there is a continuous inclusion \( i: \Omega^q_\epsilon(W) \to \Omega^q_{\epsilon'}(W) \).

**Lemma 20** (Poincaré lemma). Let \( \epsilon' < \epsilon \) and \( q > 0 \). Then there exists a bounded linear map

\[
h: \Omega^q_\epsilon(W) \to \Omega^{q-1}_{\epsilon'}(W)
\]

such that \( d \circ h + h \circ d = i \).

**Proof.** We have

\[
\Omega^q_\epsilon(W) = \bigoplus_{1 \leq k_1 < \ldots < k_q \leq n} F_\epsilon(W) dx_{k_1} \ldots dx_{k_q}.
\]

Set \( C := \max(\epsilon_i/\epsilon'_i) \). By assumption we have \( C > 1 \). Hence, for integers \( N \gg 0 \), we have \( |1/(N + q)| \leq C^N \). We define

\[
h(x^I dx_{k_1} \ldots dx_{k_q}) := \frac{1}{|I| + q} \sum_{\alpha=1}^q (-1)^{\alpha-1} x^I + e_{k_\alpha} dx_{k_1} \ldots \hat{dx}_{k_\alpha} \ldots dx_{k_q};
\]

and

\[
h \left( \sum a_I x^I dx_{k_1} \ldots dx_{k_q} \right) := \sum a_I h(x^I dx_{k_1} \ldots dx_{k_q}).
\]

Since

\[
\left\| \frac{a_I}{|I| + q} \right\| \epsilon'^I \leq \|a_I\| C^{|I|} \epsilon'^I \leq \|a_I\| \epsilon'^I \text{ for } |I| \gg 0
\]

it follows that the power series \( \sum_I \frac{a_I}{|I| + q} x^I + e_{k_\alpha} \) is \( \epsilon' \)-convergent, whence that \( h \) is well defined, and also that \( h \) is a bounded linear operator.
By continuity, it is now enough to check the equality $dh + hd = i$ on monomials $x^I dx_{k_1} \ldots dx_{k_q}$. Relabeling the coordinates, we may moreover assume that $(k_1, \ldots, k_q) = (1, \ldots, q)$. We have

\[
dh(x^I dx \ldots dx_q) = d\left(\frac{1}{|I| + q} \sum_{\alpha = 1}^{q} (-1)^{\alpha-1} x^{I+e_\alpha} dx_1 \ldots dx_\alpha \ldots dx_q\right)
\]

\[
= \left(\frac{1}{|I| + q} \sum_{\alpha = 1}^{q} (i_\alpha + 1) x^I dx_1 \ldots dx_q\right) + \frac{1}{|I| + q} \sum_{\alpha = 1}^{q} \sum_{\beta = q+1}^{n} (-1)^{\alpha-1} i_\beta x^I + e_\alpha - e_\beta dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q dx_\beta
\]

and

\[
hd(x^I dx \ldots dx_q) = h\left((-1)^{q} \sum_{\beta = q+1}^{n} i_\beta x^I - e_\beta dx_1 \ldots dx_q dx_\beta\right)
\]

\[
= \frac{(-1)^{q}}{|I| + q} \sum_{\alpha = 1}^{q} \sum_{\beta = q+1}^{n} (-1)^{\alpha-1} i_\beta x^I + e_\alpha - e_\beta dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q dx_\beta + \frac{(-1)^{q}}{|I| + q} \sum_{\beta = q+1}^{n} (-1)^{q} i_\beta x^I dx_1 \ldots dx_q
\]

\[
= \frac{1}{|I| + q} \sum_{\alpha = 1}^{q} \sum_{\beta = q+1}^{n} (-1)^{\alpha+q-1} i_\beta x^I + e_\alpha - e_\beta dx_1 \ldots \widehat{dx_\alpha} \ldots dx_q dx_\beta + \frac{\sum_{\beta = q+1}^{n} i_\beta}{|I| + q} x^I dx_1 \ldots dx_q
\]

Thus, $(dh + hd)(x^I dx_1 \ldots dx_k) = x^I dx_1 \ldots dx_k$. This finishes the proof of the lemma.  

**Lemma 21.** Let $\Phi: W \to W'$ be a bounded linear map between Banach spaces. It induces a map $\Omega^q(W) \to \Omega^q(W')$, denoted by the same symbol. For $q > 0$ and $\epsilon' < \epsilon$, the diagram

\[
\begin{array}{ccc}
\Omega^q(W) & \xrightarrow{h} & \Omega^{q-1}(W) \\
\Phi \downarrow & & \Phi \downarrow \\
\Omega^q(W') & \xrightarrow{h} & \Omega^{q-1}(W')
\end{array}
\]

commutes.

**Proof.** This follows directly from the definitions.  

ON AN ANALYTIC VERSION OF LAZARD’S ISOMORPHISM

References

[FdL99] Christian Tobias Féaux de Lacroix, Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf endlich dimensionalen Vektorräumen über einem p-adischen Körper, Schriftenreihe Math. Inst. Univ. Münster 3. Ser., vol. 23, Univ. Münster, 1999, pp. x+111. MR 1691735 (2000k:22021)

[Fla08] M. Flach, Cohomology of topological groups with applications to the Weil group, Compos. Math. 144 (2008), no. 3, 633–656. MR 2422342 (2009f:14033)

[HK11] Annette Huber and Guido Kings, A p-adic analogue of the Borel regulator and the Bloch-Kato exponential map, J. Inst. Math. Jussieu 10 (2011), no. 1, 149–190. MR 2749574 (2012a:19010)

[HKN11] Annette Huber, Guido Kings, and Niko Naumann, Some complements to the Lazard isomorphism, Compos. Math. 147 (2011), no. 1, 235–262. MR 2771131 (2012d:22016)

[Koh11] Jan Kohlhaase, The cohomology of locally analytic representations, J. Reine Angew. Math. 651 (2011), 187–240. MR 2774315

[Laz65] Michel Lazard, Groupes analytiques p-adiques, Inst. Hautes Études Sci. Publ. Math. (1965), no. 26, 389–603. MR 0209286 (35 #188)

[Lec12] Sabine Lechner, A comparison of locally analytic group cohomology and Lie algebra cohomology for p-adic Lie groups, arXiv:1201.4550, 2012.

[Sch02] Peter Schneider, Nonarchimedean functional analysis, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. MR 1869547 (2003a:46106)

[Sch11] _____, p-adic Lie groups, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 344, Springer, Heidelberg, 2011. MR 2810332 (2012h:22010)

[SGA4$_1$] Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos, Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. MR 0354652 (50 #7130)

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: georg.tamme@ur.de
URL: mathematik.uni-r.de/tamme