Thermodynamic extremality relations in massive gravity *

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Abstract: A universal relation between the leading correction to the entropy and extremality was proposed in the work of Goon and Penco. In this paper, we extend this work to massive gravity and investigate thermodynamic extremality relations in a topologically higher-dimensional black hole. A rescaled cosmological constant is added to the action of the massive gravity as a perturbative correction. This correction modifies the extremality bound of the black hole and leads to shifts in the mass, entropy, etc. Regarding the cosmological constant as a variable related to pressure, we obtain the thermodynamic extremality relations between the mass and entropy, pressure, charge, and parameters $c_i$ by accurate calculations. Finally, these relations are verified by a triple product identity, which shows that the universal relation exists in black holes.

Keywords: thermodynamic extremality relations, massive gravity, perturbative corrections

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I. INTRODUCTION

The string landscapes formed by effective quantum field theories are broad and complex. However, there are some theories that appear to be self-consistent but are not compatible with string theory. Thus, the swampland program was proposed [1-4]. Its aim is to find the subset of infinite space in effective field theories that arises at low energies from quantum gravity theories with specific constraints. These constraints were first proposed in [1]. As one of the constraints, the weak gravity conjecture (WGC) has attracted much attention. It asserts that, for the lightest charged particle along the direction of a basis vector in charge space, the charge-to-mass ratio is larger than those for extremal black holes [2]. This conjecture shows that extremal black holes are allowed to decay.

A proof of the WGC is that it is mathematically equivalent to a certain property of a black hole entropy. In [5], the authors introduced the higher-derivative operators to the action to compute the shift in the entropy. Using these operators, the extremality condition of the black hole is modified, and the mass and entropy are shifted. These authors derived the relation between the ratio of charge-to-mass and the entropy shift, $q/m - 1 \propto \Delta S$, where $\Delta S > 0$. The charge-to-mass ratio approaches unity asymptotically with increasing mass. Thus, a large extremal black hole is unstable and decays to a smaller extremal black hole with charge-to-mass ratios greater than unity. This phenomenon satisfies the requirement of the WGC. Subsequently, WGC behavior was found in a four-dimensional rotating dyonic black hole and other spacetimes [6, 7]. Other studies of the WGC have been reported in [8-23]; see also the references therein.

In a recent study [24], Goon and Penco derived a universal extremality relation using perturbative corrections to the free energy of generic thermodynamic systems. This relation takes the form

$$\frac{\partial M_{\text{ext}}(\vec{Q}, \epsilon)}{\partial \epsilon} = \lim_{M \rightarrow M_{\text{ext}}(\vec{Q}, \epsilon)} -T \left( \frac{\partial S(M, \vec{Q}, \epsilon)}{\partial \epsilon} \right)_{M, \vec{Q}},$$

(1)

where $M_{\text{ext}}(\vec{Q}, \epsilon)$ and $S(M, \vec{Q}, \epsilon)$ are the extremal mass and entropy, respectively. Both of them are $\epsilon$-dependent, and $\epsilon$ is a control parameter for the free energy. $\vec{Q}$ are additional quantities in thermodynamic systems, other than the mass. The above relation can be interpreted as a comparison between states in the classical and corrected theories. Meanwhile, an approximation relation $\Delta M_{\text{ext}}(\vec{Q}) \approx -T_0 (M, \vec{Q}) \Delta S(M, \vec{Q})$ was proposed, where $\Delta M_{\text{ext}}(\vec{Q})$ and $\Delta S(M, \vec{Q})$ are the leading order corrections to the extremal bound and to the entropy of a state with fixed mass and $\vec{Q}$, respectively. $M_0$ is the mass in the...
extremal case without corrections. The result shows that
the mass of the perturbed extremal black hole is less than
that of the unperturbed one with the same quantum num-
bers, if $\Delta S > 0$, which implies that the perturbation
decreases the mass of the extremal black hole. Therefore,
WGC-like behavior exists in the extremal black hole. In
particular, the Goon-Penco relation (1) was verified in an
AdS-Reissner-Nordström black hole by rescaling the
cosmological constant as a perturbative correction. The
approximation relation was also verified using higher-deri-
"vable operators introduced in the action.

To further explore the WGC behavior and the Goon-
Penco relation, researchers have studied the thermody-
namic corrections in specific spacetimes by introducing
higher-derivative operators or perturbative parameters
[25, 26]. The Goon-Penco relation was confirmed, and
other extremality relations were obtained. In [25], Cre-
monini et al. computed the four-derivative corrections to
thermodynamic quantities in the higher-dimensional AdS-Reissner-Nordström black hole and found the ex-
tremality relation between the mass and charge,

$$\lim_{M \to 0} \left( \frac{\partial M}{\partial \epsilon} \right)_{Q,T} = \lim_{M \to 0} \Phi \left( \frac{\partial Q}{\partial \epsilon} \right)_{M,T}.$$  (2)

Extending this work to rotating anti-de Sitter spacetimes,
Liu et al. derived the extremality relation between the
mass and angular momentum in the BTZ and Kerr anti-de
Sitter spacetimes [26],

$$\left( \frac{\partial M_{\text{ext}}}{\partial \epsilon} \right)_{JJ} = \lim_{M \to M_{\text{ext}}} -\Omega \left( \frac{\partial J}{\partial \epsilon} \right)_{M,S,J}.$$  (3)

Relations (2) and (3) are extensions of the Goon-Penco
relation (1). These relations will shed light on theories of
quantum gravity.

In this paper, we extend the work of [24] to massive
gravity and investigate the extremality relations between
the mass and pressure, entropy, charge, and parameters
$c_i$ of a charged topological black hole in higher-dimen-
sional spacetime. Einstein’s UV completeness requires
that GR be modified to meet physical descriptions in the
high energy region. Massive gravity is a straightforward
modification to GR. We introduce a perturbative correc-
tion by adding a rescaled cosmological constant to the ac-
tion of massive gravity. This scenario is different from
that in [24], where the cosmological constant was
directly rescaled in the action and consistent with that in
[26]. In our investigation, the cosmological constant is
regarded as a variable related to pressure [27-31]. Its con-
gugate quantity is a thermodynamic volume. The black
hole mass is naturally interpreted as an enthalpy. The first
reason for this is that the cosmological constant, as a vari-
able, can reconcile the inconsistency between the first law
of thermodynamics of black holes and the Smarr relation,
derived from the scaling method. The second reason is
that physical constants, such as the gauge coupling con-
stants, Newtonian constant, or cosmological constant,
which are vacuum expectation values, are not fixed and
vary in the more fundamental theories [32].

The rest of this paper is organized as follows. In the
next section, the solution of the higher-dimensional black
hole in massive gravity is given, and its thermodynamic
properties are discussed. In section III, we introduce a
perturbative correction to the action and derive the ex-
tremality relations between the mass and pressure, en-
tropy, charge, and parameters $c_i$. Section IV is devoted
to our discussion and conclusion.

II. BLACK HOLE SOLUTION IN MASSIVE
GRAVITY

The action for an $(n+2)$-dimensional massive gravity
is [33]

$$S = \frac{1}{16\pi} \int dx^{n+2} \sqrt{-g} \left\{ R + \frac{n(n+1)}{r^2} - \frac{F^2}{4} + \sum_{i=1}^4 c_i u_i(g,f) \right\},$$  (4)

where the terms including $m^2$ represent the massive po-
tential associated with the graviton mass, $f$ is a fixed sym-
metric tensor called the reference metric, $c_i$ are constants,
and $u_i$ are symmetric polynomials of the eigenvalues of
the $(n+2) \times (n+2)$ matrix $K_{ij} = \sqrt{f^{ij}} g_{ij}$:

$$u_1 = [K], \quad u_2 = [K]^2 - [K^2],$$
$$u_3 = [K]^3 - 3[K][K^2]^2 + 3[K^3],$$
$$u_4 = [K]^4 - 6[K]^2[K^2]^2 + 8[K^2][K^3] + 3[K^4]^2 - 6[K^4].$$  (5)

The square root in $K$ denotes $(\sqrt{\mathcal{A}})^i_j = A^i_{j\mu}$ and
$[K] = K_{\mu}^\nu$.

The solution of the charged black hole with the space-
time metric and reference metric is given by [34]

$$\begin{aligned}
&ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 h_{ij} dx^i dx^j, \\
f_{\mu\nu} = \text{diag}(0,0,c_0^2 h_{ij}),
\end{aligned}$$

(6)  (7)

where

$$f(r) = k + \frac{r^2}{l} - \frac{16\pi M}{n\Omega_2 r^{n-1}} - \frac{(16\pi Q)^2}{2n(n-1)\Omega_2 r^{2n-1}} - \frac{c_0 c_1 m^2}{n} + \frac{c_0^2 c_2 m^2}{r} + \frac{(n-1)c_0^2 c_3 m^2}{r^2},$$

(8)

\[025108-2\]
\[ \dot{S} \] is related to the cosmological constant \( \Lambda \) as 
\[ \dot{S} = -\frac{n(n+1)}{2\lambda} \]. \( M \) and \( Q \) are the mass and charge of the \( M \) black hole, respectively. \( \Omega_0 \) is the volume spanned by coordinates \( x^i \), and \( c_0 \) is a positive integral constant. \( h_i dx^i dx^j \) is the line element for an Einstein space with the constant curvature \( n(n-1)k \), \( k = 1, 0, \) or \(-1\) denotes spherical, Ricci flat, or hyperbolic topology black hole horizons, respectively. The thermodynamics in the extended phase space of massive gravity have been studied in [35-41]. The event horizon \( r_+ \) is determined by \( f(r) = 0 \). A general formula for the Hawking temperature can be given as 
\[ T = \frac{k}{2\pi}, \] 
where \( k = \frac{1}{2\pi} \lim_{r \to r_+} \sqrt{\frac{-g^{11} \partial ln(-g_{00})}{\partial r}} \) is the surface gravity. For this black hole, the Hawking temperature is

\[ T = \frac{f'(r_+)}{4\pi} \left[ \frac{(n+1)r^2}{l^2} + \frac{(16\lambda Q)^2}{2n\Omega^2 r_{+}^{2(n-1)}} + c_0c_1 m^2 r_+ \right] \]
\[ + \frac{(n-1)c_0^2 c_3 m^2}{r_+^2} + \frac{(n-1)(n-2)c_1^3 c_3 m^2}{r_+^2} \]
\[ + \frac{(n-1)(n-2)(n-3)c_0^4 c_4 m^2}{r_+^2} \]. \( (9) \)

The mass expressed by the horizon radius and charge is

\[ M = \frac{n\Omega_0 r_{+}^{n-1}}{16\pi} \left[ k + \frac{r_+^2}{l^2} + \frac{(16\lambda Q)^2}{2n\Omega^2 r_{+}^{2(n-1)}} \right] \]
\[ + \frac{c_0c_1 m^2 r_+}{n} + \frac{c_0^2 c_3 m^2}{r_+^2} + \frac{(n-1)c_1^3 c_3 m^2}{r_+^2} \]
\[ + \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r_+^2} \]. \( (10) \)

The cosmological constant was seen as a fixed constant in the past. In this paper, it is regarded as a variable related to pressure, \( P = \frac{\Delta \zeta}{8\pi} = \frac{n(n+1)}{16\pi l^2} \), and its conjugate quantity is a thermodynamic volume \( V \). The entropy, volume, and electric potential at the event horizon are given by

\[ S = \frac{\Omega_0 r_{+}^n}{4}, \quad V = \frac{n\Omega_0 r_{+}^{n-1}}{n+1}, \quad \Phi_e = \frac{16\lambda Q}{(n-1)\Omega_0 r_{+}^{n-1}} \]. \( (11) \)

respectively. Because of the appearance of pressure, the mass is no longer interpreted as the internal energy but as an enthalpy. \( c_1, c_2, c_3 \), and \( c_4 \) are seen as extensive parameters for the mass. Their conjugate quantities are

\[ \Phi_1 = \frac{\Omega_0 c_0 m^2 r_{+}^n}{16\pi}, \quad \Phi_2 = \frac{n\Omega_0 c_0^2 m^2 r_{+}^{n-1}}{16\pi}, \quad \Phi_3 = \frac{n(n-1)\Omega_0 c_1^3 m^2 r_{+}^{n-2}}{16\pi}, \quad \Phi_4 = \frac{n(n-1)(n-2)\Omega_0 c_0^4 c_3 m^2}{16\pi} \]. \( (12) \)

respectively. It is easy to verify that these thermodynamic quantities obey the first law of thermodynamics,

\[ dM = TdS + VdP + \Phi_e dQ + \sum_{i=1}^{4} \Phi_i dc_i. \] \( (13) \)

When the cosmological constant is fixed, the term \( VdP \) disappears, and the mass is interpreted as the internal energy. When a perturbative correction is introduced, the related thermodynamic quantities are shifted, which is discussed in the next section.

### III. Extremality Relations in Massive Gravity

In this section, we derive the extremality relations between the mass and entropy, charge, pressure, and parameters \( c_i \) by adding a rescaled cosmological constant to the action as a perturbative correction. The rescaled parameter is \( \epsilon \). Here, the black hole is designated as an extremal one.

We first introduce the correction

\[ \Delta S = \frac{1}{16\pi} \int dx^{n+2} \sqrt{-g} \frac{n(n+1)\epsilon}{l^2}, \] \( (14) \)

to the action (4). The corrected action is \( S + \Delta S \). The action (4) is recovered when \( \epsilon = 0 \). A black hole solution is obtained from the corrected action and takes the form of Eqs. (6) and (8), but there is a shift. With the correction, the Hawking temperature is also shifted, and it is given by

\[ T = \frac{1}{4\pi r_+} \left[ \frac{(n+1)r_+^2}{l^2} + \frac{(n-1)r_+^2}{l^2} + \frac{(16\lambda Q)^2}{2n\Omega^2 r_{+}^{2(n-1)}} + c_0c_1 m^2 r_+ \right] \]
\[ + \frac{(n-1)c_0^2 c_3 m^2}{r_+^2} + \frac{(n-1)(n-2)c_1^3 c_3 m^2}{r_+^2} \]
\[ + \frac{(n-1)(n-2)(n-3)c_0^4 c_4 m^2}{r_+^2} \]. \( (15) \)

The corrected mass is
\[ M = \frac{n\Omega nr_{n-1}^2}{16\pi} \left[ \frac{r_F^2 \epsilon}{2} + k \right] + \frac{(16\pi Q)^2}{2n(n-1)\Omega r_{n-1}^4} + \frac{c_0 c_1 m^2 r_+}{n} + \frac{c_0 c_2 m^2}{r_+^2} + \frac{(n-1)c_3 c_0 m^2}{r_+^3} + \frac{(n-1)(n-2)c_4 c_0 m^2}{r_+^4} + \frac{377775}{r_+^5} \]. \tag{16}

which is a function of parameters \( r_+, \epsilon, Q, l, c_1, c_2, c_3, \) and \( c_4 \). Our interest is focused on the thermodynamic extremality relation. The Hawking temperature (15) in the extremal case is zero, which leads to a solution \( r_+ = r_+(\epsilon) \). Inserting this solution into the above equation yields an expression regarding the mass, \( M_{\text{ext}} = M_{\text{ext}}(\epsilon) \). Carrying out the differential on \( M_{\text{ext}}(\epsilon) \), we have

\[ \left( \frac{\partial M_{\text{ext}}}{\partial \epsilon} \right)_{Q,l,c_1,c_2,c_3,c_4} = \frac{n\Omega nr_{n-1}^2}{16\pi \epsilon^2}. \tag{17} \]

Because the expression of the differential expressed by \( \epsilon \) is very complex, we adopted the expression of \( r_+ \) in the above derivation. In fact, this relation can also be derived by the following calculation. For convenience, we use \( c \) to denote all parameters \( Q, l, c_1, c_2, c_3, c_4 \), except for \( r_+ \) and \( \epsilon \). From Eq. (16), the differential of \( M \) to \( \epsilon \) is obtained as follows:

\[
\left( \frac{\partial M}{\partial \epsilon} \right)_{c} = \left( \frac{\partial M}{\partial r_{n-1}} \right)_{c} \left( \frac{\partial r_{n-1}}{\partial \epsilon} \right)_{c} + \left( \frac{\partial M}{\partial \epsilon} \right)_{c,c}. \\
\left( \frac{\partial M}{\partial \epsilon} \right)_{c,c} = \frac{1}{4} \frac{T \Omega n r_{n-1}^2}{16\pi} \left[ \frac{3(n-1)c_3 c_0 m^2}{r_+^4} + \frac{4(n-1)(n-2)c_4 c_0 m^2}{r_+^5} \right]. \tag{18}
\]

To evaluate the value, we insert the expression of the mass into the above equation and obtain

\[
\left( \frac{\partial \epsilon}{\partial S} \right)_{M,Q,l,c_1,c_2,c_3,c_4} = \frac{4\pi^2}{n\Omega n r_{n-1}^2} \left[ -\frac{(n-1)k}{r_+^2} + \frac{(n+1)(1+\epsilon)}{r_+^2} + \frac{(16\pi Q)^2}{2n(2n+1)\Omega r_{n-1}^4} - \frac{c_3 c_0 m^2}{r_+^4} - \frac{(n-1)(n-2)c_4 c_0 m^2}{r_+^5} \right]. \tag{20}
\]

Combining the inverse of the above differential with the expression of the temperature given in Eq. (15), we have

\[
T \left( \frac{\partial S}{\partial \epsilon} \right)_{M,Q,l,c_1,c_2,c_3,c_4} = \frac{n\Omega n r_{n-1}^2}{16\pi l^2}. \tag{22}
\]

In the extremal case, the first term in the last line of the above equation disappears, and the mass can be rewritten as \( M = M_{\text{ext}} \). Therefore, Eq. (17) is readily recovered.

The entropy \( S \), pressure \( P \), charge \( Q \), \( c_1 \), \( c_2 \), \( c_3 \), and \( c_4 \) are usually regarded as a complete set of extensive parameters for the mass. Their conjugate quantities can be derived from the mass and take the same form as those given in section II, except for the temperature and volume. We first verify the extremality relation between the mass and entropy.

The expression for \( \epsilon \) is obtained from Eq. (16) and takes the form

\[
\epsilon = \left[ \frac{16\pi M}{n\Omega n r_{n-1}^2} - \frac{k}{r_+^2} + \frac{(16\pi Q)^2}{2n(2n+1)\Omega r_{n-1}^4} - \frac{c_3 c_0 m^2}{r_+^4} - \frac{2c_4 c_0 m^2}{r_+^5} \right] r_+^2 - 1. \tag{19}
\]

Using the relation between the entropy and horizon radius given in Eq. (11), the above equation is a function \( \epsilon(S) \), and \( \frac{\partial r_+}{\partial S} = \frac{4}{n\Omega n r_{n-1}^2} \). Carrying out the differential calculation on this function yields

\[
\left( \frac{\partial \epsilon}{\partial S} \right)_{M,Q,l,c_1,c_2,c_3,c_4} = \frac{4\pi^2}{n\Omega n r_{n-1}^2} \left[ -\frac{(n-1)k}{r_+^2} + \frac{(n+1)(1+\epsilon)}{r_+^2} + \frac{(16\pi Q)^2}{2n(2n+1)\Omega r_{n-1}^4} - \frac{c_3 c_0 m^2}{r_+^4} - \frac{(n-1)(n-2)c_4 c_0 m^2}{r_+^5} \right]. \tag{21}
\]

Compared with relation (17), it is easy to see that

\[
\lim_{M \to M_{\text{ext}}} T \left( \frac{\partial S}{\partial \epsilon} \right)_{M,Q,l,c_1,c_2,c_3,c_4} = \lim_{M \to M_{\text{ext}}} -T \left( \frac{\partial S}{\partial \epsilon} \right)_{M,Q,l,c_1,c_2,c_3,c_4}, \tag{23}
\]

where \( S \) is a function of \( M, Q, l, c_1, c_2, c_3, c_4, \) and \( \epsilon \).
Therefore, the Goon-Penco relation is verified in the higher-dimensional black hole.

In this paper, the cosmological constant is regarded as a variable related to pressure. The entropy, pressure, charge, $c_1$, $c_2$, $c_3$, and $c_4$ are usually regarded as extensive parameters for the mass. Because the entropy satisfies the thermodynamic extremality relation, it is natural to ask whether other extensive quantities also satisfy corresponding relations. The goal of the following investigation is to determine these relations. Let us first derive the extremality relation between the mass and pressure. The pressure can be expressed by the constant $P$ as $P = \frac{n(n+1)}{16\pi l^2}$. Then, $\frac{\partial P}{\partial l} = -\frac{16\pi l^2}{n(n+1)}$. Using Eqs. (16) and (19), we get the differential of $\epsilon$ with respect to the pressure,

$$
\left( \frac{\partial \epsilon}{\partial P} \right)_{M, r_+, Q, c_1, c_2, c_3, c_4} = -\frac{16\pi l^2(1+\epsilon)}{n(n+1)}.
$$

(24)

The perturbation parameter $\epsilon$ exists in the above differential relation as an explicit function. The reason for this is that the perturbation correction is introduced by adding the rescaled cosmological constant to the action, and this constant is related to the pressure. Because of the shift in the mass, the thermodynamic volume is also shifted, and its expression is different from that given in Eq. (11). The volume is

$$
V = \frac{\epsilon + 1}{n+1} \Omega_n r_+^{n+1}.
$$

(25)

Using Eq. (25) and the inverse of the differential of $\epsilon$ to $P$ yields

$$
V \left( \frac{\partial P}{\partial \epsilon} \right)_{M, r_+, Q, c_1, c_2, c_3, c_4} = -\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}.
$$

(26)

Comparing the above equation with Eq. (17), we obtain the extremality relation between the mass and pressure,

$$
\left( \frac{\partial M_{\text{ext}}}{\partial \epsilon} \right)_{Q, r_+, c_1, c_2, c_3, c_4} = \lim_{M \to M_{\text{au}}} -V \left( \frac{\partial P}{\partial \epsilon} \right)_{M, r_+, Q, c_1, c_2, c_3, c_4},
$$

(27)

where $P$ is a function of $M, r_+, Q, c_1, c_2, c_3, c_4$, and $\epsilon$. This relation is an extension of the Goon-Penco relation.

We continue to investigate the extremality relation between the mass and charge. The calculation process is similar. From Eq. (19), the differential of $\epsilon$ with respect to $Q$ takes the form

$$
\left( \frac{\partial \epsilon}{\partial Q} \right)_{M, r_+, c_1, c_2, c_3, c_4} = -\frac{16\pi^2 Q^2}{n(n-1)\Omega_n r_+^{2n}}.
$$

(28)

Multiplying the electric potential $\Phi_e = \frac{16\pi Q}{(n-1)\Omega_n r_+^{n-1}}$ by the inverse of the above differential yields

$$
\Phi_e \left( \frac{\partial Q}{\partial \epsilon} \right)_{M, r_+, Q, c_1, c_2, c_3, c_4} = \frac{-n\Omega_n r_+^{n+1}}{16\pi l^2}.
$$

(29)

Obviously, there is a minus sign difference between Eqs. (17) and (29). Therefore,

$$
\left( \frac{\partial M_{\text{ext}}}{\partial \epsilon} \right)_{Q, r_+, c_1, c_2, c_3, c_4} = \lim_{M \to M_{\text{au}}} -\Phi_e \left( \frac{\partial Q}{\partial \epsilon} \right)_{M, r_+, Q, c_1, c_2, c_3, c_4},
$$

(30)

which is the extremality relation between the mass and charge. Now, $Q$ is a function of $M, r_+, l, c_1, c_2, c_3, c_4$, and $\epsilon$. This relation is also an extension of the Goon-Penco relation.

For the extremality relations between the mass and parameters $c_1, c_2, c_3$, and $c_4$, the calculations are parallel. Their differential relations are

$$
\left( \frac{\partial \epsilon}{\partial c_1} \right)_{M, r_+, Q, c_2, c_3, c_4} = \frac{c_0 m_0^2 l^2}{nr_+},
$$

(31)

$$
\left( \frac{\partial \epsilon}{\partial c_2} \right)_{M, r_+, Q, c_1, c_3, c_4} = \frac{c_2 m_0^2 l^2}{r_+^2},
$$

(32)

$$
\left( \frac{\partial \epsilon}{\partial c_3} \right)_{M, r_+, Q, c_1, c_2, c_4} = \frac{(n-1)c_3 m_0^2 l^2}{r_+^3},
$$

(33)

$$
\left( \frac{\partial \epsilon}{\partial c_4} \right)_{M, r_+, Q, c_1, c_2, c_3} = \frac{(n-1)(n-2)c_4 m_0^2 l^2}{r_+^4}.
$$

(34)

The conjugate quantities of $c_1, c_2, c_3$, and $c_4$ are

$$
\Phi_1 = \frac{\Omega_n c_0 m_0^2 r_+^n}{16\pi},
$$

$$
\Phi_2 = \frac{n\Omega_n c_0^2 m_0^2 r_+^{n-1}}{16\pi},
$$

$$
\Phi_3 = \frac{n(n-1)\Omega_n c_0^3 m_0^2 r_+^{n-2}}{16\pi},
$$

$$
\Phi_4 = \frac{n(n-1)(n-2)\Omega_n c_0^4 m_0^2 r_+^{n-3}}{16\pi},
$$

respectively. Using these quantities, it is not difficult to obtain
Thus, the extremality relations between the mass and extensive parameters $c_i$ are
\[
\left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4} = \lim_{M \to M_{\text{ext}}} -\Phi_1\left(\frac{\partial c_1}{\partial \epsilon}\right)_{M,r+Q,l,c_1,c_2,c_3,c_4} = \left(\frac{\partial c_4}{\partial \epsilon}\right)_{M,r+Q,l,c_1,c_2,c_3,c_4} = -\frac{n\Omega_l r^{n+1}}{16\pi l^2}.
\]
(35)
where $i, j, k, u = 1, 2, 3, 4$, and $i \neq j \neq k \neq u$. Therefore, the Goon-Penco relation is extended to the case of the extensive parameters $c_i$ of the higher-dimensional black hole.

In the above investigation, the thermodynamic extremality relations between the mass and entropy, pressure, charge, and parameters $c_i$ were obtained by accurate calculations. They are expressed as Eqs. (22), (27), (30), and (36), respectively. The values of these relations are equal. In fact, these relations can be derived uniformly using the triple product identity
\[
\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T} \left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,T} \left(\frac{\partial \epsilon}{\partial M}\right)_{M,X^i} = -1,
\]
(37)
which yields
\[
\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T} = -\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T}\left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,T} = -\Phi_1\left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,T}.
\]
(38)
In the above derivation, $\frac{\partial M}{\partial X^i}$ were identified as $\Phi_1$, which are the conjugate quantities of $X^i$. Here, $X^i$ are chosen as $S, Q, P, c_1, c_2, c_3$, and $c_4$, and $M$ and $T$ are the corrected mass and temperature given in (16) and (15), respectively. In the extremal case, $T \to 0$ and $M \to M_{\text{ext}}$. The above relation becomes

\[\left(\frac{\partial M_{\text{ext}}}{\partial \epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4} = \lim_{M \to M_{\text{ext}}} -\Phi_1\left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,X^i}.
\]
(39)

\[\frac{n\Omega_l r^{n+1}}{16\pi l^2}.
\]

where $X^i \neq X^j$, and $X^i$ are parameters $S, Q, P, c_1, c_2, c_3$, or $c_4$, except for $X^i$. This relation implies that the universal extremality relation exists in black holes. The relation (39) is easily reduced to (22), (27), (30), and (36) when $X^i$ are the entropy, charge, parameters $c_i$, and pressure, respectively. In the calculation, because of the shift in the mass, the expression of the volume $V = \frac{\epsilon + 1}{\Omega_l r^{n+1}}$ is different from that given in Eq. (11). In [26], the authors derived the extremality relation between the mass and angular momentum in BTZ and Kerr anti-de Sitter space-times and suggested that a general formula of the extremality relation existed in black holes. Our result provides verification of this conjecture.

\[\text{IV. CONCLUSION}
\]

In this paper, we extended the work of Goon and Penco to massive gravity and investigated the thermodynamic extremality relations in a higher-dimensional black hole. The extremality relations between the mass and pressure, entropy, charge, and parameters $c_i$ were derived by accurate calculations. The values of these extremality relations are equal, which may be due to the first law of thermodynamics. In the calculation, the cosmological constant was treated as a variable related to pressure. A perturbative correction was introduced by adding the rescaled cosmological constant to the action, but this addition does not affect the form of the extremality relation between the mass and pressure.

\[\text{References}
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