Embedding irreducible connected sets

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Abstract

We show that every connected set \(X\) which is irreducible between two points \(a\) and \(b\) embeds into the Hilbert cube in a way that \(X\) is irreducible between \(a\) and \(b\) for every point \(c\) in the closure of \(X\). Also, a connected set \(X\) is indecomposable if and only if for every compactum \(Y \supseteq X\) and \(a \in X\) there are two points \(b,c \in X\) such that \(X \cup \{b,c\}\) is irreducible between every two points from \(\{a,b,c\}\). Following the proofs, we illustrate a cube embedding of the main example from On indecomposability of \(\beta X\). We prove that the example embeds into the plane.

1 Introduction

The present paper is the author’s fourth in a series on connected sets and irreducibility: [8], [9], [10].

The first half of this paper concerns one and two point irreducible extensions of connected separable metric spaces. Recall that if \(X\) is connected and \(p,q \in X\), then \(X\) is reducible between \(p\) and \(q\) if there is a connected \(C \subseteq X\) with \(\{p,q\} \subseteq C\) and \(\overline{C} \neq X\). If there is no such \(C\), then \(X\) is irreducible between \(p\) and \(q\).

For a connected set \(X \cup \{p\}\) to be irreducible between two points \(x \in X\) and \(p\), it is clearly necessary that \(p\) miss the closure of every proper component of \(X\) which contains \(x\). By \(C\) is a proper component of \(X\), we mean there is a proper closed \(A \subseteq X\) such that \(C\) is a connected component of \(A\). This condition, however, is not sufficient. Example 1 of [9] describes a connected set \(X \subseteq \mathbb{R}^3\) such that every proper

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[8] features three mutually non-homeomorphic widely-connected plane sets, including a completely metrizable example. [9] examines indecomposability of the Stone-Čech compactification, and gives several relevant examples which will be explored further in this paper. [10] classifies one-to-one images of \([0, \infty)\) and \((\infty, -\infty)\) which are equal to maximal continuum-wise connected subsets (i.e. composants) of continua.
component of $X$ is degenerate, and $X \cup \{0,0,0\}$ is reducible.\(^2\) Nevertheless, $X$ embeds into the Hilbert cube (a natural habitat) so that all reasonable one-point extensions are irreducible.

**Theorem 1.** For every connected set $X$ and $x \in X$, there is a homeomorphic embedding $e : X \hookrightarrow [0,1]^\omega$ such that $e[X] \cup \{p\}$ is irreducible between $e(x)$ and $p$ for every point $p \in e[X]$, which is not in the closure of any proper component of $e[X]$ containing $e(x)$.

Further,

**Theorem 2.** If $X$ is irreducible between two points $a$ and $b$, there is a homeomorphic embedding $e : X \hookrightarrow [0,1]^\omega$ such that $e[X] \cup \{c\}$ is irreducible between $e(a)$ and $e(b)$ for every $c \in e[X]$.

We will see that $\dim(e[X]) = \dim(X)$ is possible in Theorems 1 and 2.

In contrast with Theorem 2, there is an irreducible $X$ such that the entire $e[X]$ is reducible (between every two points of $e[X]$) for every embedding $e$. We present a simple example to this effect in Section 5.

Following the proofs of Theorems 1 and 2, we generalize an elementary result from continuum theory. A continuum $X$ is indecomposable if and only if there are three points $\{a,b,c\} \subseteq X$ such that $X$ is irreducible between every two points from $\{a,b,c\}$ ([12] Cor. 11.20, also [7] §48 VI Theorem 7*).

**Theorem 3.** A connected separable metric space $X$ is indecomposable if and only if for every compactum $Y \supseteq X$ and $a \in X$ there are two points $b, c \in X$ such that $X \cup \{b, c\}$ is irreducible between every two points from $\{a, b, c\}$.

Here, $X$ is *indecomposable* means that $X$ cannot be written as the union of two proper components. This is equivalent to saying $X$ is the only closed connected subset of $X$ with non-void interior ([7] §48 V Theorem 2).

In the second half of the paper we describe a connected set $\hat{W} \subseteq [0,1]^3$ which has the same remarkable properties as $\hat{W}$ from Example 4 of [9]. Namely, $\hat{W}$ is irreducible between every two of its points, but every continuum enclosing $\hat{W}$ is reducible between every two points of $\hat{W}$. We indicate why $\hat{W}$ has said properties in Section 4.2, after its construction in Section 4.1. The construction technique here is analogous to the one in [9], but is more geometric in nature. For all practical purposes $\hat{W}$ is an embedding of $\hat{W}_*$, hence the title of Section 4: “Embedding a special irreducible set”. Planarity of $\hat{W}$ is established in Section 4.3.

We conclude with some remarks about $\leq 2$-point compactifications.

## 2 Component ordinals

In this section $X$ is a separable metric space. We introduce the notion of a *component ordinal*, which will facilitate the proofs of Theorems 1 through 3.

\(^2\)This $X$ was denoted $q[W]$ in [9]. The mapping $q[W]$ was a homeomorphic embedding of the widely-connected plane set $W = W[X_3]$ from [8].

D.S. Lipham
For any point \(x\) and space \(X \ni x\), let \(C(x, X)\) denote the connected component of \(x\) in \(X\). That is, \(C(x, X) = \bigcup\{C \subseteq X : C\) is connected and \(x \in C\}\). When \(X\) is clear from context, we will simply write \(C(x)\) for \(C(x, X)\).

**Lemma 1.** For every \(x \in X\) there is a decreasing collection of closed sets \(\{A^\alpha : \alpha < \omega_1\}\) such that \(A^0 = X\), \((\bigcap_{\beta < \alpha} A^\beta) \setminus A^\alpha\) is closed for each \(\alpha < \omega_1\), and \(\bigcap_{\alpha < \gamma} A^\alpha = C(x)\) for some \(\gamma < \omega_1\).

**Proof.** Set \(A_0 = X\).

Suppose \(0 < \alpha < \omega_1\) and \(A^\beta\) has been defined for all \(\beta < \alpha\). If \(X^\alpha := \bigcap_{\beta < \alpha} A^\beta\) is not connected, let \(A^\alpha \ni x\) be a relatively clopen proper subset of \(X^\alpha\). Otherwise, put \(A^\alpha = X^\alpha\).

The open \((X \setminus C(x))\)-cover \(\{X \setminus A^\alpha : \alpha < \omega_1\}\) has a countable subcover.\(^3\) This means \((A^\alpha)_{\alpha < \omega_1}\) stabilizes, i.e. \(X^\gamma\) is connected for some \(\gamma < \omega_1\).

Apparently \(X^\gamma \subseteq C(x)\) by maximality of \(C(x)\). It remains to show \(C(x) \subseteq X^\gamma\). Well, if \(C(x) \notin X^\gamma\) then there is a least \(\beta < \gamma\) such that \(C(x) \notin A^\beta\). Then \(\beta > 0\), \(C(x) \subseteq X^\beta\), and the closed sets \(A^\beta\) and \(X^\beta \setminus A^\beta\) partition \(C(x)\). This is absurd because \(C(x)\) is connected.

Whenever \(x\) is a point in a space \(X\), we let \(\text{ord}(x, X)\) be the least (countable) ordinal \(\gamma > 0\) such that \(X^\gamma = C(x)\), considering all \(\omega_1\)-sequences \((A^\alpha)\) from Lemma 1. Observe that in general, \(\text{ord}\) is either 1, a limit, or the successor of a limit. Also, \(C(x, X)\) is equal to the quasi-component of \(x\) in \(X\) if and only if \(\text{ord}(x, X) \in \{1, \omega\}\).

Let us take a moment to calculate \(\text{ord}\) for a fixed point \(x = \langle 1/2, 1/2\rangle\) in a few different subspaces of \([0, 1]^2\) (see Figure 1). In each space the component of \(x\) is \(\{x\}\), but \(\text{ord}(x)\) varies.

\[
\begin{align*}
\text{ord}(x, \{x\}) &= \text{ord}(x, \{x\} \cup (2 \times [0, 1])) = 1 \\
\text{ord}(x, [X \cap ([0, 1/2] \times [0, 1])) \cup \{x\}) &= \omega \\
\text{ord}(x, [X \cap ([0, 1/2] \times [0, 1])) \cup \{x, \langle 1/2, 1/2\rangle\}) &= \omega + 1 \\
\text{ord}(x, [X \cap ([0, 1/2] \times [0, 1])) \cup \{x, \langle 1/2, 1/2\rangle\}) &= \omega + \omega \\
\text{ord}(x, [X \cap ([0, 1/2] \times [0, 1])) \cup \{x, \langle 1/2, 1/2\rangle\}) &= \omega + \omega \\
\text{ord}(x, X) &= \omega + \omega + 1
\end{align*}
\]

Figure 1: \(X \subseteq [0, 1]^2\) and the ordinal of \(x\) in subspaces of \(X\)

**Lemma 2.** Let \(\{A^\beta : \beta < \gamma := \text{ord}(x, X)\}\) be given for a point \(x \in X\) by Lemma 1. If \(X \cup \{p\}\) is a topological extension of \(X\), and \(p \in C(x, X \cup \{p\})\setminus C(x)\), then there exists \(\beta < \gamma\) such that \(p \in A^\beta \cap X^\beta \setminus A^\beta\).

\(^3\)Use the fact that \((X \setminus \bigcap_{\alpha < \omega_1} A^\alpha)\) has a countable basis. Actually, the property that \(X\) has no strictly decreasing \(\omega_1\)-sequence of closed subsets is equivalent to \(X\) being hereditarily Lindelöf, which is true under our assumption that \(X\) is separable and metrizable.
3 Proofs of Theorems

Throughout, suppose $X$ is a connected separable metric space.

Fix a basis $\{U_n : n < \omega \}$ for $X$ with each $U_n \neq \emptyset$.

For each $n < \omega$ and $x \in X \setminus U_n$, let $\{A^\alpha_n(x) : \alpha < \omega_1\}$ be given by Lemma 1, with $A^0_n = X \setminus U_n$, $X^\alpha_n = \bigcap_{\beta < \alpha} A^\beta_n$, and

$$
X^\alpha_n \setminus (x, X \setminus U_n) = C_n(x) := C(x, X \setminus U_n).
$$

If $x \in U_n$ then put $\text{ord}(x, X \setminus U_n) = 0$ and $C_n(x) = A^\alpha_n(x) = \emptyset$ for all $\alpha < \omega_1$.

3.1 Proof of Theorem 1

Since $\gamma := \sup \{\text{ord}(x, X \setminus U_n) : n < \omega \}$ is countable, there exists $e : X \hookrightarrow [0, 1]^\omega$ such that $\overline{e[A^\alpha_n]} \cap \overline{e[X^\beta_n \setminus A^\alpha_n]} = \emptyset$ for every $n < \omega$ and $\beta < \gamma$. If $p \in e[X] \cup n < \omega e[C_n(x)]$, then $p \notin \bigcup n < \omega C(e(x), e[X \setminus U_n] \cup \{p\})$ by Lemma 2, so $e[X] \cup \{p\}$ is irreducible between $e(x)$ and $p$.

3.2 Proof of Theorem 2

Let $a$ and $b$ be such that $X$ is irreducible between them. Put

$$
\xi = \sup \{\text{ord}(x, X \setminus U_n) : \langle x, n \rangle \in \{a, b\} \times \omega \},
$$

and construct $e : X \hookrightarrow [0, 1]^\omega$ so that

\begin{align}
(3.2.1) & \quad e[\overline{A^\beta_n(x)}] \cap e[\overline{X^\beta_n \setminus A^\beta_n(x)}] = \emptyset \text{ for all } \langle x, n, \beta \rangle \in \{a, b\} \times \omega \times \xi; \text{ and} \\
(3.2.2) & \quad e[C_n(a)] \cap e[C_n(b)] = \emptyset \text{ for all } n < \omega.
\end{align}

Now let $c \in e[X]$. For any $n < \omega$, we have

$$
C(e(a), e[X \setminus U_n] \cup \{c\}) \cap C(e(b), e[X \setminus U_n] \cup \{c\}) = \emptyset
$$

by Lemma 2. Thus $e[X] \cup \{c\}$ is irreducible between $e(a)$ and $e(b)$.

Remark. In Theorems 1 and 2 we can obtain $\dim(e[X]) = \dim(X)$.

For if $\Phi$ is a countable family of continuous maps of $X$ into $[0, 1]$, then $X$ has a $\Phi$-compactification $\overline{e[X]}$ (i.e. each member of $\Phi$ continuously extends to $\overline{e[X]}$ with $\dim(\overline{e[X]}) = \dim(X)$. This is Problem 1.7.C in [4], and is based on Engelking’s work
in [3]. To get $e[A_n^\beta] \cap e[X_n^{\beta}\setminus A_n^\beta] = \emptyset$ in the proof of Theorem 1, simply include a map $\varphi \in \Phi$ such that $\varphi[A_n^\beta] = 0$ and $\varphi[X_n^{\beta}\setminus A_n^\beta] = 1$. Likewise, $\Phi$ can guarantee (3.2.1) and (3.2.2) in the proof of Theorem 2.

### 3.3 Proof of Theorem 3

The condition is sufficient. If $X$ decomposes into two proper components $H$ and $K$, then two of any three points \( \{a, b, c\} \subseteq X \) must belong to $H$ or $K$, and thus $X \cup \{a, b, c\}$ fails to have the three-point irreducible property.

The condition is necessary. Suppose $X$ is indecomposable.

**Case 1:** $X$ is reducible. Fix $a \in X$. Let $\gamma = \sup\{\text{ord}(a, X\setminus U_n) : n < \omega\}$. We may select \( b, c \in X \setminus \bigcup \{ [A_n^\beta(a) \cap X_n^{\beta}(a) \setminus A_n^\beta(a)] \cup C_n(a) : (n, \beta) \in \omega \times \gamma\} \),

since the sets $A_n^\beta(a) \cap X_n^{\beta}(a) \setminus A_n^\beta(a)$ and $C_n(a)$ are closed & nowhere dense in $X$. Then by Lemma 2, $X \cup \{b, c\}$ is irreducible between $a$ and $b$, and $a$ and $c$. It follows that $X \cup \{b, c\}$ is irreducible between $b$ and $c$. For suppose $D$ is a proper closed connected subset of $X \cup \{b, c\}$ containing $b$ and $c$. Since $X$ is reducible and $D \cap X \neq \emptyset$, there is a closed connected $C \subseteq X$ with $a \in C$ and $C \cap D \neq \emptyset$. By indecomposability of $X$, $C$ is nowhere dense in $X$, whence $C \cup D \neq X$. Thus, the connected set $C \cup D$ violates the previously established irreducibility.

**Case 2:** $X$ is irreducible. Let $a$ and $b$ be such that $X$ is irreducible between them. Let $\xi = \sup\{\text{ord}(x, X\setminus U_n) : (x, n) \in \{a, b\} \times \omega\}$. Choose

$$c \in X \setminus \bigcup \{ [A_n^\beta(x) \cap X_n^{\beta}(x) \setminus A_n^\beta(x)] \cup C_n(x) : (x, n, \beta) \in \{a, b\} \times \omega \times \xi\}. $$

Then $X \cup \{c\}$ is irreducible between $a$ and $c$, and $b$ and $c$. It follows easily that $X \cup \{c\}$ is also irreducible between $a$ and $b$. \( \square \)

### 3.4 Questions

**Question 1.** Let $X$ be indecomposable and reducible. If $p$ is a limit point of $X$ not in the closure of any proper component of $X$, then is $X \cup \{p\}$ necessarily irreducible?

**Question 2.** Let $f : [0, \infty) \to \mathbb{R}^2$ be a one-to-one mapping of the half-line into the plane, with range $X := f[0, \infty)$. If $f[n, \infty) = X$ for each $n < \omega$, then is $X \cup \{p\}$ irreducible for every point $p \in X\setminus X$?

### 4 Embedding a special irreducible set

Here we will construct the set $\hat{W}$ that was outlined in Section 1.

The set $\hat{W}$ will be connected and irreducible between every two of its points. The term for these properties is *widely-connected*. One may see that a connected set $W$ is widely-connected if and only if every subset of $W$ is either connected or hereditarily
disconnected (a set is *hereditarily disconnected* provided each of its connected subsets is degenerate).

Bernstein subsets of certain continua, such as the dyadic solenoid and bucket-handle, are widely-connected.\(^4\) Other examples, given in [13] and [8], are still less complicated than the one we are about to describe. But, like the Bernstein sets, they fail to have the desired additional property 4.2.iii (item iii in Section 4.2). \(\hat{W}\) was the first example with that property.

### 4.1 Construction of \(\hat{W}\)

Let \(S \subseteq [0,1]^2\) be the quinary (middle two-fifths) double bucket-handle continuum depicted in Figure 2(a).\(^5\) For other illustrations of \(S\), see [6] and [7] §48 V.

![Figure 2: The double bucket-handle continuum and subsets](image)

Let \(W \supseteq S \cap ([0,1] \times \{\frac{1}{2}\})\) be a widely-connected subset of \(S\).

For visualization purposes, construct \(W\) as in [8]: \(S\setminus([0,1] \times \{\frac{1}{2}\})\) is the union of a countable discrete collection of sets homeomorphic to \(C \times \mathbb{R}\), \(C\) being the middle two-fifths Cantor set. Refine each of these sets to a copy of the punctured Knaster-Kuratowski fan. So if \(c\) is an endpoint of an interval removed during the construction of \(C\), then the part of each \(\{c\} \times \mathbb{R}\) remaining is \(\{c\} \times \mathbb{Q}\); the other fibers become \(\{c\} \times (\mathbb{R}\setminus\mathbb{Q})\). To complete the construction of \(W\), restore \(S \cap ([0,1] \times \{\frac{1}{2}\})\).\(^6\)

Let \(W_0 = W \cap [0,\frac{1}{2}]\) and \(W_1 = W \cap [\frac{1}{2},1]\) be the lower and upper ‘halves’ of \(W\) depicted in Figure 2(b). So \(W = W_0 \cup W_1\) and \(W_0 \cap W_1 = S \cap ([0,1] \times \{\frac{1}{2}\})\).

The indecomposable \(S\) has two accessible composants (maximal arcwise-connected subsets) \(P_0\) and \(P_1\), indicated in Figure 2(c). Each is a one-to-one continuous image of the half-line \([0,\infty)\) and is dense in \(S\). Identifying their ‘endpoints’ \(\langle 0,0\rangle\) and \(\langle 1,1\rangle\)

\(^4\)For a Bernstein subset \(B\) of a continuum \(Y\) to be connected, every separator in \(Y\) should be uncountable. For \(B\) to be *widely-connected*, it helps to know that every non-degenerate non-dense connected subset of \(Y\) contains a non-degenerate continuum. That way, the only proper components of \(B\) are degenerate. Solenoids and bucket-handle continua satisfy all of these criteria since they have uncountably many composants, and all of their proper subcontinua are arcs.

\(^5\)Parts of \(S\) will play the roles of both \(K\) and the orbit in \(2^\omega\) from the construction in [9].

\(^6\)The horizontal middle two-fifths Cantor set \(S \cap ([0,1] \times \{\frac{1}{2}\})\) replaces the diagonal middle-thirds Cantor sets \(\Delta\) and \(\Delta'\) from [8] and [9], respectively.
produces an indecomposable continuum $\hat{S} := S/\{\langle 0,0 \rangle, \langle 1,1 \rangle \}$ which embeds into the plane $[0, 1]^2 \times \{1/2\}$ (Figure 3) and has composant $\hat{P} := (P_0 \cup P_1)/\{\langle 0,0 \rangle, \langle 1,1 \rangle \}$.

Let $f : (-\infty, \infty) \to \hat{P}$ be a one-to-one mapping onto $\hat{P} \subseteq [0, 1]^2 \times \{1/2\}$ such that $f \upharpoonright [0, \infty)$ maps onto $P_0$ and $f \upharpoonright (-\infty, 0]$ maps onto $P_1$. The point $f(0)$ joining the two original composants is indicated in Figure 3.

Let $(r_n) \in (-\infty, \infty)^\mathbb{Z}$ be the increasing $\mathbb{Z}$-enumeration of numbers in $(-\infty, \infty)$ such that $r_0 = 0$ and $f[r_{n-1}, r_n] \perp f[r_n, r_{n+1}]$ for $n \neq 0$. In words, the points $f(r_n)$, $n \neq 0$, are the ninety-degree ‘corners’ of $\hat{P}$.

For each $n \geq 0$, let

$$k_n = \|f(r_n) - f(r_{n+1})\|,$$

$$W^n = \{\langle k_n \cdot x, y \rangle : \langle x, y \rangle \in W_0 \},$$

$$R^n = \{\langle x, y, z \rangle \in [0, 1]^3 : \langle x, y, 1/2 \rangle \in f[r_n, r_{n+1}] \text{ and } z \in [0, 1/2]\}.$$ 

Then $W^n$ is a copy of $W_0$ horizontally scaled by factor $k_n$ (the length of $f[r_n, r_{n+1}]$), and $R^n$ is a rectangular region extending below $f[r_n, r_{n+1}]$. For a given $n \geq 0$ there are two rigid transformations

$$T^n : [0, k_n] \times [0, 1/2] \to R^n$$

that map $[0, k_n] \times \{1/2\}$ onto $f[r_n, r_{n+1}]$. One transformation maps $\langle 0, 1/2 \rangle$ and $\langle k_n, 1/2 \rangle$ to $f(r_n)$ and $f(r_{n+1})$, respectively, while the other maps these points in the opposite way. By inductively choosing the ‘correct’ $T^n$, we can ensure that whenever $f[r_m, r_{m+1}]$ and $f[r_k, r_{k+1}]$ are parallel ($0 \leq m < n$), $T^n[W^n]$ and $T^m[W^m]$ are similarly aligned. Specifically, $T^n(\langle 0, 1/2 \rangle)$ and $T^m(\langle 0, 1/2 \rangle)$ always lie on the same side of the line $\ell \subseteq [0, 1]^2 \times \{1/2\}$ that bisects the segments $f[r_m, r_{m+1}]$ and $f[r_k, r_{k+1}]$, and the points $T^n(\langle k_n, 1/2 \rangle)$ and $T^m(\langle k_n, 1/2 \rangle)$ lie on the other side of $\ell$. See Fig. 4.

Likewise, use rigid transformations $T^n$, $n < 0$, to attach horizontally scaled copies of $W_1$ to $P_1$. Construct each $T^n$ so that $T^n[W^n \cap ([0, 1] \times \{1/2\})]$ is the middle two-fifths Cantor set in $f[r_n, r_{n+1}]$, and $T^n[W^n]$ spans the rectangular region $R^n := f[r_n, r_{n+1}] \times [1/2, 1)$. In the beginning, $T^{-1}[W^{-1}]$ and $T^{-2}[W^{-2}]$ should be inversely aligned with respect to $T^0[W^0]$ and $T^1[W^1]$ (just as $W_1$ and $W_0$ are aligned in $W$).
Then, choose the other $T^n \,(n \leq -2)$ so that the orientations of parallel $T^n[W^n]$ and $T^m[W^m]$ are the same. This forces all parallel $T^n[W^n]$ and $T^m[W^m] \,(n \geq 0 \, \text{and} \, m < 0)$ to be aligned as in $W$. See Figure 6.

Put

$$\hat{W} = \bigcup \{T^n[W^n] : n \in \mathbb{Z}\}.$$  

Figure 4: Attachment of parallel copies of $W_0$

4.2 Properties of $\hat{W}$

Verification of $\hat{W}$’s properties will be kept fairly brief. The reader will be referred to [9] for supporting claims and proofs.

i. Since $W$ is connected and $\partial W_0 = \partial W_1 = S \cap ([0, 1] \times \{1/2\})$ is compact, Claims 1 and 2 in [9] apply with $X' = W$ and $X'_i = W_i$. They show the following.
First, every clopen subset of the Cantor set
\[ \mathcal{C} := (\{0\} \cup \{1/n : n = 1, 2, 3, \ldots\}) \times [S \cap ([0, 1] \times \{1/2\})] \]
is eventually a product of clopen sets. That is, if \( C \) is a clopen subset of \( \mathcal{C} \) then there is a sufficiently large positive integer \( m \) such that
\[ C \cap ([0, 1/m] \times S \cap ([0, 1] \times \{1/2\})) = [0, 1/m] \times \{s \in S \cap ([0, 1] \times \{1/2\}) : <1/m, s> \in C\}. \]

Next, suppose \( i \in \{0, 1\} \) and let \( \mathfrak{W} \supseteq \mathcal{C} \) be the set
\[ (\{0\} \times W_i) \cup (\{1/n : n = 1, 2, 3, \ldots\} \times W_{1-i}). \]
If \( A \) is a clopen subset of \( \mathfrak{W} \) such that \( A \cap (\{0\} \times W_i) \neq \emptyset \), then, letting \( m \) be as above (for \( C = A \cap \mathcal{C} \)), we find that \([A \cap (\{0\} \times W_i)] \cup [A \cap (\{1/m\} \times W_{1-i})]\) projects onto a clopen subset of \( \{0\} \times W \). Since \( W \) is connected, \( (\{0\} \times W_i) \subseteq A \). This argument shows that \( \{0\} \times W_i \) is contained in a quasi-component of \( \mathfrak{W} \).

Back in \( \hat{W} \), density of the \( P_i \) and alignment of the \( T^n[W^n] \) and \( T^m[W^m] \) now imply that each \( T^n[W^n] \) is contained in a quasi-component of \( \hat{W} \) (each half of \( W \) in \( \hat{W} \) is the “limit” of a sequence of complementary halves which are properly oriented, as was the case in \( \mathfrak{W} \)). It follows that \( \hat{W} \) has only one quasi-component, i.e. \( \hat{W} \) is connected (Claim 3 of [9]).

ii. The closure of \( \hat{W} \) is an indecomposable continuum by the facts that \( \hat{S} \) is indecomposable and \( S \) is irreducible between \( \{0\} \times [0, 1] \) and \( \{1\} \times [0, 1] \) (see the proof of Claim 4 in [9]).

Moreover, if \( L \) is a proper subcontinuum of \( cl_{[0,1]} \hat{W} \), then its orthogonal projection into \( \hat{P} \) is contained in an arc. This means for every proper component \( X \subseteq \hat{W} \) there exists \( m \in \mathbb{N} \) such that \( X \subseteq \bigcup \{T^n[W^n] : |n| \leq m\} \). Each \( T^n[W^n] \) is hereditarily disconnected, as are the joining sets
\[ (\{f(r_n)\} \times [0, 1]) \cap (T^{n-1}[W^{n-1}] \cup T^n[W^n]). \]
Thus \( \bigcup \{T^n[W^n] : |n| \leq m\} \) decomposes into a Cantor set of hereditarily disconnected fibers. It follows that \( X \) is degenerate, establishing the widely-connected property of \( \hat{W} \).

iii. Every continuum containing \( \hat{W} \) is reducible between every two points of \( \hat{W} \). Indeed, let \( Y \supseteq \hat{W} \) be any continuum, and let \( p, q \in \hat{W} \). There exists \( m < \omega \) such that \( \{p, q\} \subseteq \bigcup \{T^n[W^n] : |n| \leq m\} \). By the reasoning in part i, as well as the fact that the quasi-components of closed subsets of \( Y \) are connected, \( \bigcup \{T^n[W^n] : |n| \leq m\} \) is contained in a proper subcontinuum of \( Y \). So \( Y \) is reducible between \( p \) and \( q \). For further details, see the proof of Claim 5 in [9].

\[ ^7 \text{For certain } W, \text{ it is possible that } (\{f(r_n)\} \times [0, 1]) \cap (T^{n-1}[W^{n-1}] \cup T^n[W^n]) \text{ contains an arc. But with the 'Knaster-Kuratowski' type } W, \ \hat{W} \cap (\{f(r_n)\} \times [0, 1]) \approx \mathbb{Q}. \]
4.3 Plane embedding

At the 2018 Spring Topology & Dynamical Systems Conference, Jernej Činč asked: *Does $\hat{W}$ embed into the plane?* Here we will argue that the answer is *yes*. This shows the example is minimal in some sense.

First of all, notice that a slight alteration to $\hat{W}$ makes its closure a chainable continuum. Let $P$ be any composant of $\hat{S}$ other than $P_0$ and $P_1$. Then $P$ is a one-to-one continuous image of $(-\infty, \infty)$, and may be used instead of $\hat{P}$ to construct widely-connected set $\hat{W} \cong W$. To see that $\text{cl}_{\mathbb{R}^3} \hat{W}$ is chainable, use chainability of the center template $\hat{S} \times \{1/2\}$ together with chainability of the vertical copies of $\hat{S}$ in $\text{cl}_{\mathbb{R}^3} \hat{W}$. Every chainable continuum is planar, so $\hat{W}$ embeds into the plane.

The continuum $\text{cl}_{[0,1]} \hat{W}$ is not chainable, but it is circle-like. Such a continuum does not automatically embed into the plane, but Bing ([1] Theorem 4) proved: If $X$ is a circle-like continuum and there is a sequence $\mathcal{C}_0, \mathcal{C}_1, \ldots$ of circular chains covering $X$ such that $\text{mesh}(\mathcal{C}_n) \to 0$ as $n \to \infty$, and $\mathcal{C}_{n+1}$ circles $\mathcal{C}_n$ exactly once, then $X$ homeomorphically embeds into the plane. According to the definitions in Bing’s paper, $X = \hat{S}$ is “circulable” in the appropriate manner. Using chainability of the vertical copies of $\hat{S}$ in $\text{cl}_{[0,1]} \hat{W}$, one can see that $\text{cl}_{[0,1]} \hat{W}$ also satisfies the hypotheses of Bing’s theorem.

5 Conclusion

There is an irreducible connected plane set every compactification of which is reducible. See Figure 7. The set $\mathcal{J} := ([1/2, 0] \times 2) \cup \hat{S} \setminus \{0\} \times (0, 1]$ is irreducible between its two left-most points $\langle -1/2, 0 \rangle$ and $\langle -1/2, 1 \rangle$, but every continuum enclosing $\mathcal{J}$ has just one composant. This example demonstrates the need for the strong hypothesis in Question (D) of [9] (if its answer is to be *yes*).

Of course, $\mathcal{J}$ contains the indecomposable connexe $S \setminus \{0\} \times (0, 1]$.

**Question 3.** *Does every hereditarily decomposable irreducible connected set densely embed into an irreducible continuum?*

Let us end with a brief discussion of $\leq 2$-point compactifications. The one-point compactification of a locally compact irreducible connected set may be reducible. For an indecomposable example, see the remark following Example 1 in [9]. Decomposable (hereditarily or non-hereditarily) examples exist as well.

$[0, 1]$ is the only locally connected & locally compact connected set whose one-point compactification is irreducible. For suppose $X$ and its one-point compactification $\alpha X := X \cup \{x\}$ have the stated properties. By Theorem 4.1 in [5], $\alpha X$ is locally connected. Every locally connected irreducible continuum is homeomorphic
Embedding irreducible sets

to $[0, 1]$ (cf. [7]), so $\alpha X \simeq [0, 1]$. Since $X$ is connected, $\infty$ corresponds to an endpoint of $[0, 1]$. Thus $X \simeq [0, 1]$.

The same line of reasoning will show that $\mathbb{R}$ is the only locally connected & locally compact connected set with an irreducible two-point compactification.

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