Quasi-local structure of $p$-form theory

Dariusz Chruściński
Institute of Physics, Nicholas Copernicus University
ul. Grudziądzka 5/7, 87-100 Toruń, Poland

Abstract

We show that the Hamiltonian dynamics of the self-interacting, abelian $p$-form theory in $D = 2p + 2$ dimensional space-time gives rise to the quasi-local structure. Roughly speaking, it means that the field energy is localized but on closed $2p$-dimensional surfaces (quasi-localised). From the mathematical point of view this approach is implied by the boundary value problem for the corresponding field equations. Various boundary problems, e.g. Dirichlet or Neumann, lead to different Hamiltonian dynamics. Physics seems to prefer gauge-invariant, positively defined Hamiltonians which turn out to be quasi-local. Our approach is closely related with the standard two-potential formulation and enables one to generate e.g. duality transformations in a perfectly local way (but with respect to a new set of nonlocal variables). Moreover, the form of the quantization condition displays very similar structure to that of the symplectic form of the underlying $p$-form theory expressed in the quasi-local language.

Keywords: $p$-form theory, duality invariance, Hamiltonian dynamics
PACS numbers: 11.15-q, 11.10Kk, 10.10Lm

1 Introduction

One of the most important idea of modern physics is locality. It is strongly related with relativity and quantum mechanics and plays a central role in relativistic (classical and quantum) field theories. Let us cite only two very influential books: physics is simple when analyzed locally [1] and the role of fields is to implement the principle of locality [2]. It should be stressed, therefore, from the very beginning that we are not going to discuss nonlocal theories. The abelian $p$-form theory is a simple generalization of an ordinary electrodynamics in 4-dimensional Minkowski space-time $\mathcal{M}^4$ where the electromagnetic field potential 1-form $A_\mu$ is replaced by a $p$-form in $D$-dimensional space-time [3], [4]. This theory is perfectly local, i.e. it is defined via the local Lagrangian.

The motivations to study $p$-form theory are already discussed in [3]. Recently the new input comes with electric-magnetic duality [5], [6], [7]. It was observed long ago [3] that the duality symmetry for the standard Maxwell electrodynamics in four dimensional Minkowski space-time (i.e. $p = 1$ theory) is generated by the nonlocal generator (its physical interpretation as a chirality operator was discussed in [9]), i.e. it is nonlocal functional of the electromagnetic field. Therefore, the nonlocality enters into the game in a very natural way. We shall see that the above mentioned nonlocality is closely related with the Hamiltonian description of the field dynamics.

To define the Hamiltonian evolution one splits the entire space-time into space and time and then formulates the initial value problem. But in field theory one has to specify also the boundary condition for the fields. Very often one assumes that all fields do vanish at spatial infinity and simply forgets about this problem. It should be stressed, however, that even if the boundary values vanish numerically they do not vanish functionally, i.e. they are necessary in the proper definition

*E-mail: darch@phys.uni.torun.pl
of the functional phase space of the dynamical problem. This is typical for the problems with infinitely many degrees of freedom. Boundary value problem is not only a mathematical problem. It also does belong to physics. Different boundary problems lead to different Hamiltonians, i.e. different definitions of the field energy, e.g. energies defined via canonical and symmetric energy-momentum tensors. Now, in the standard (i.e. $p = 1$) electrodynamics the “canonical” energy, which is neither gauge-invariant nor positively defined, is related to the boundary value problem for the scalar potential $A_0$. On the other hand the “symmetric energy” (defined by the symmetric energy-momentum tensor), which is perfectly gauge-invariant and positively defined, is related to the control of the electric and magnetic fluxes on the boundary [10], [11], [12], [13]. Therefore, it distinguishes a new set of electromagnetical variables $Q^1$ and $Q^2$ consistent with the boundary problem. Together with the canonically conjugated momenta $\Pi_1$ and $\Pi_2$ they encode the entire gauge-invariant information about the electromagnetic field $F = dA$, i.e. knowing $Q$’s and $\Pi$’s one may uniquely reconstruct $F$ [10]. Actually, it was shown long ago by Debye [14] that Maxwell theory could be described in terms of two complex functions (so called Debye potentials). It turns out that this formulation is very well suited to describe e.g. radiative phenomena [15]. Our $Q$’s and $\Pi$’s (they may be rearranged into complex $Q$ and $\Pi$) are closely related to Debye potentials. They solve the Gauss constraint and, therefore, they reduce the symplectic form in the space of Cauchy data for the field dynamics. However, they are nonlocal functions of the electromagnetic fields $D$ and $B$. The nonlocality is of the very special structure and the Hamiltonian generating the dynamics defines a quasi-local functional, i.e. performing an integration over angle variables one obtains perfectly local functional.

Now, in the abelian self-interacting $p$-form theory in $D = 2p + 2$ dimensional space-time one may perform the similar analysis [16]: instead of two complex functions $Q$ and $\Pi$, the dynamical information about a $p$-form electromagnetic fields $D$ and $B$ is now encoded into two complex $(p-1)$– forms. In the present paper we relate the quasi-local picture implied by these $(p-1)$–forms with the proper definition of the Hamiltonian dynamics for a $p$-form theory. Moreover, we show that this formulation is perfectly suited for the description of the duality symmetry, i.e. the duality rotations (for odd $p$) are generated locally in terms of $Q$ and $\Pi$. We show that the canonical generator has the following form:

$$\int Q^1\Pi_2 - Q^2\Pi_1 .$$

(1.1)

It is evident that this approach is closely related to the two-potential formulation [7], [17] (see Appendix D).

It is well known that there is a crucial difference between theories with different parities of $p$, e.g. for even $p$ a theory can not be duality invariant. Now, it was observed only recently [7] that the quantization condition for $(p-1)$–brane dyons crucially depends upon $p$, namely

$$e_1g_2 + (-1)^pe_2g_1 = nh ,$$

(1.2)

with integer $n$ ($h$ is the Planck constant). It turns out that the symplectic form of a $p$-form theory written in terms of $Q$ and $\Pi$ has very similar structure

$$\Omega_p = \int \delta\Pi_1 \wedge \delta Q^1 + (-1)^{p+1}\delta\Pi_2 \wedge \delta Q^2 ,$$

(1.3)

therefore, there is a striking correspondence between the form of the quantization condition (1.2) and the structure of symplectic form (1.3). This correspondence is universal, i.e. it holds for any gauge-invariant, self-interacting theory.

The paper is organized as follows: we remind the quasi-local structure of standard (1-form) electrodynamics in Section 2. This is the prototype of the $p$-form theory for odd $p$. Then in
Section 3 we make the generalization for \( p = 2 \) which is the prototype for even \( p \). The general case (i.e. an arbitrary \( p \)) is discussed in Appendices B and C. In Section 4 we describe the gauge-invariant coupling of \( p \)-form electrodynamics to the charged matter and the Hamiltonian structure of the interacting theory. The details of notation are clarified in Appendix A.

2 1-form theory in \( D = 4 \)

2.1 Generating formula

Let us consider a 1-form theory defined by the Lagrangian \( \mathcal{L} = \mathcal{L}(A, \partial A) \). Field dynamics of this theory may be written in terms of the following generating formula (see Appendix A for details of notation):

\[
- \delta \mathcal{L} = \partial_\nu (\mathcal{G}^{\nu \mu} \delta A_\mu) = (\partial_\nu \mathcal{G}^{\nu \mu}) \delta A_\mu + \mathcal{G}^{\nu \mu} \delta (\partial_\nu A_\mu).
\]

(2.1)

The formula (2.1) implies the following definition of “momenta”:

\[
\mathcal{G}^{\nu \mu} = -2 \frac{\partial \mathcal{L}}{\partial F_{\nu \mu}}.
\]

(2.2)

Moreover, (2.1) generates dynamical (in general nonlinear) field equations

\[
\partial_\nu \mathcal{G}^{\nu \mu} = -\mathcal{J}^\mu,
\]

(2.3)

where the external 1-form current reads:

\[
\mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial A_\mu}.
\]

(2.4)

Let us start with a source free theory, i.e. \( \mathcal{J} = 0 \). We shall study the \( p \)-form electromagnetism coupled to a charged matter in section 4. To obtain the Hamiltonian description of the field dynamics let us integrate equation (2.1) over a 3-dimensional volume \( V \) contained in the constant-time hyperplane \( \Sigma \):

\[
- \delta \int_V \mathcal{L} = \int_V \partial_\nu (\mathcal{G}^{0 \nu} \delta A_0) + \int_{\partial V} \mathcal{G}^{\perp \mu} \delta A_\mu,
\]

(2.5)

where \( \perp \) denotes the component orthogonal to the 2-dimensional boundary \( \partial V \). To simplify our notation let us introduce the spherical coordinates on \( \Sigma \):

\[
x^3 = r, \quad x^4 = \varphi_A ; \quad A = 1, 2,
\]

(2.6)

where \( \varphi_1, \varphi_2 \) denote spherical angles (usually one writes \( \varphi_1 = \varphi \) and \( \varphi_2 = \Theta \)). To enumerate angles we shall use capital letters \( A, B, C, \ldots \). The Euclidean metric tensor is diagonal

\[
g_{11} = r^2, \quad g_{22} = r^2 \sin \varphi_2, \quad g_{rr} = 1,
\]

(2.7)

and the volume form \( \Lambda_1 = \sqrt{\det(g_{\mu \nu})} = r^2 \sin \varphi_2 \). Let \( V \) be a 3-ball with a finite radius \( R \). In such a coordinate system the formula (2.5) takes the following form:

\[
\delta \int_V \mathcal{L} = - \int_V \partial_0 (\mathcal{D}^i \delta A_i) + \int_{\partial V} \mathcal{D}^r \delta A_0 - \int_{\partial V} \mathcal{G}^{B} \delta A_B,
\]

(2.8)

where

\[
\mathcal{D}_i = \mathcal{G}_{i0}
\]

(2.9)
denotes the 1-form electric induction density on \( \Sigma \). Now, performing the Legendre transformation between induction 1-form \( D^i \) and \( \dot{A}_i \) one obtains the following Hamiltonian formula:

\[
- \delta H_{can} = - \int_V \left( \dot{D}^i \delta A_i - \dot{A}_i \delta D^i \right) + \int_{\partial V} \mathcal{D}^r \delta A_0 - \int_{\partial V} \mathcal{G}^{rB} \delta A_B ,
\]

(2.10)

where the canonical Hamiltonian

\[
H_{can} = \int_V \left( - D^i \dot{A}_i - \mathcal{L} \right) .
\]

(2.11)

Equation (2.10) generates an infinite-dimensional Hamiltonian system in the phase space \( P_p = (D^i, A_i) \) fulfilling Dirichlet boundary conditions for the 1-form potential \( A_i \): \( A_0|\partial V \) and \( A|\partial V \). From the mathematical point of view this is the missing part of the definition of the functional space. The Hamiltonian structure of a general nonlinear 1-form electrodynamics described above is mathematically well defined, i.e. a mixed Cauchy problem (Cauchy data given on \( \Sigma \) and Dirichlet data given on \( \partial V \times \mathbb{R} \)) has a unique solution (modulo gauge transformations which reduce to the identity on \( \partial V \times \mathbb{R} \)).

There is, however, another way to describe the Hamiltonian evolution of fields in the region \( V \). Let us perform the Legendre transformation between \( D^r \) and \( A_0 |\partial V \). One obtains:

\[
- \delta H_{sym} = - \int_V \left( \dot{D}^i \delta A_i - \dot{A}_i \delta D^i \right) - \int_{\partial V} A_0 \delta D^r - \int_{\partial V} \mathcal{G}^{rB} \delta A_B ,
\]

(2.12)

where the new “symmetric” Hamiltonian

\[
H_{sym} = H_{can} + \int_{\partial V} \mathcal{D}^r A_0 .
\]

(2.13)

Observe, that formula (2.12) defines the Hamiltonian evolution but on a different phase space. In order to kill boundary terms in (2.12) one has to control on \( \partial V \): \( D^r \) (instead of \( A_0 \)) and \( A_B \). We stress that from the mathematical point of view both descriptions are equally good and an additional physical argument has to be given if we want to choose one of them as more fundamental.

### 2.2 Canonical vs. symmetric energy

Now, let us discuss the relation between \( H_{can} \) and \( H_{sym} \) defined by (2.11) and (2.13) respectively. One has:

\[
H_{sym} = H_{can} + \int_{\partial V} \mathcal{D}^r A_0 = H + \int_V \partial_k \left( \mathcal{D}^k A_0 \right) = \int_V \left\{ - \mathcal{D}^i \dot{A}_i - \mathcal{L} + \left( A_0 \partial_k \mathcal{D}^k + \mathcal{D}^k \partial_k A_0 \right) \right\} = \int_V \left( \dot{D}^i E^i - \mathcal{L} \right) ,
\]

(2.14)

where the 1-form electric field is defined by

\[
E_i = F_{i0} = \partial_i A_0 .
\]

(2.15)

Therefore, \( H_{sym} \) is related to \( \mathcal{L} \) via different Legendre transformation (compare (2.11) with (2.14)). Contrary to \( H_{can} \), \( H_{sym} \) is perfectly gauge-invariant. It is evident that \( H_{sym} \) is defined via the symmetric energy-momentum tensor:

\[
T_{sym}^{\mu\nu} = F^{\mu\lambda} \mathcal{G}_\lambda^{\nu} + g^{\mu\nu} \mathcal{L} ,
\]

(2.16)

whereas \( H_{can} \) via the canonical one:

\[
T_{can}^{\mu\nu} = (\partial^\mu A^\lambda) \mathcal{G}_\lambda^{\nu} + g^{\mu\nu} \mathcal{L} ,
\]

(2.17)
i.e. $H_{\text{sym}} = \int_V T_{\text{sym}}^{00}$ and $H_{\text{can}} = \int_V T_{\text{can}}^{00}$. Therefore, the “symmetric energy” $H_{\text{sym}}$ is gauge-invariant and positively defined, e.g. for the 1-form Maxwell theory one has

$$H_{\text{sym}}^{\text{Maxwell}} = \frac{1}{2} \int (D^i D_i + B^i B_i).$$

On the other hand, the “canonical energy” $H_{\text{can}}$ is neither positively defined nor gauge-invariant. These properties show that the Hamiltonian evolution based on $H_{\text{sym}}$ is more natural from the physical point of view than the one based on $H_{\text{can}}$ (see also discussion in [12]).

### 2.3 Reduction of the generating formula

Now, it turns out that the formula (2.12) may be considerably simplified. Any geometrical object on a 3-dimensional hyperplane $\Sigma$ may be decomposed into the radial and tangential (i.e. tangential to any sphere $S^2(r)$) components, e.g. a 1-form gauge potential $A_i$ decomposes into the radial $A_r$ and tangential $A_A$. Now, any 1-form on $S^2(r)$ may be further decomposed into “longitudinal” and “transversal” parts:

$$A_A = \nabla_A u + \epsilon_{AB} \nabla^B v,$$  

where both $u$ and $v$ are scalar functions on $S^2(r)$. Now, using (2.18) and integrating by parts one gets:

$$\int_V (D^i \delta A_i - \dot{A}_i \delta D^i) = \int_V \left\{ (\dot{D}^r \delta A_r - \dot{A}_r \delta D^r) + \left[ (\partial_r \dot{D}^r) \delta u - \dot{u} \delta (\partial_r D^r) \right] - \epsilon_{AB} \left[ (\nabla^B \dot{D}^A) \delta v - \dot{v} \delta (\nabla^B D^A) \right] \right\},$$  

(2.19)

where we have used the Gauss law

$$\nabla_A D^A = -\partial_r D^r.$$  

(2.20)

Moreover, due to (2.18)

$$\int_{\partial V} \mathcal{G}^{ra} \delta A_A = - \int_{\partial V} \left\{ -\dot{D}^r \delta u + \left( \epsilon_{AB} \nabla^B \mathcal{G}^{ra} \right) \delta v \right\}.$$  

(2.21)

In deriving (2.21) we have used

$$\nabla_A \mathcal{G}^{Ar} = -\dot{D}^r,$$  

(2.22)

which follows from the field equations $\nabla_A \mathcal{G}^{Ar} + \partial_0 \mathcal{G}^{0r} = 0$. Now, taking into account (2.19) and (2.21) the generating formula (2.12) may be rewritten in the following way:

$$-\delta H_{\text{sym}} = - \int_V \left\{ [\dot{D}^r (A_r - \partial_r u) - (\dot{A}_r - \partial_r \dot{u})] \delta D^r \right\} - \left[ \epsilon_{AB} (\nabla^B \dot{D}^A) \delta v - \dot{v} \delta (\epsilon_{AB} \nabla^B D^A) \right] + \int_{\partial V} \left\{ (A_0 - \dot{u}) \delta D^r - \left( \epsilon_{AB} \nabla^B \mathcal{G}^{ra} \right) \delta v \right\}.$$  

(2.23)

Note, that although $A_r$, $A_0$ and $u$ are manifestly gauge-dependent, the combinations $A_r - \partial_r u$ and $A_0 - \partial_0 u$ are gauge-invariant. To simplify our consideration we choose the special gauge $u \equiv 0$, i.e. a 1-form $A_A$ on $S^2(r)$ is purely transversal. This condition, due to (2.18), may be equivalently rewritten as

$$\nabla_A A^A = 0.$$  

(2.24)
Assuming (2.24) one may show [16]

\[ \Delta_0 A^r = r^2 \epsilon^{AB} \nabla_B B_A , \]

(2.25)

where

\[ \Delta_0 = r^2 \nabla_A \nabla^A \]

(2.26)

denotes the 2-dimensional Laplacian on \( S^2(1) \), i.e. the 2-dim. Laplace-Beltrami operator on scalar functions (0-forms). Moreover,

\[ B^r = \epsilon^{AB} \nabla_A A_B = -r^{-2} \Delta_0 v . \]

(2.27)

Since \( \Delta_0 \) is invertible in the source free theory [10] the formula (2.23) may be rewritten as follows:

\[
- \delta H_{sym} = - \int_V \left\{ \left[ (r \dot{D}^r) \delta \left( r \Delta_0^{-1} \epsilon_{AB} \nabla^B B^A \right) - \left( r \Delta_0^{-1} \epsilon_{AB} \nabla^B \dot{B}^A \right) \delta (r \dot{D}^r) \right] + \left( r \Delta_0^{-1} \epsilon_{AB} \nabla^B D^A \right) \delta (r B^r) \right\} \\
- \int_{\partial V} \left\{ (r^{-1} A_0) \delta (r D^r) + \left( \Delta_0^{-1} \epsilon_{AB} \nabla^B G^{rA} \right) \delta (r B^r) \right\} .
\]

(2.28)

Now, introducing the following set of variables

\[ Q^1 = r D^r , \]

(2.29)

\[ Q^2 = r B^r , \]

(2.30)

\[ \Pi_1 = r \Delta_0^{-1} \epsilon^{AB} \nabla_B A_A , \]

(2.31)

\[ \Pi_2 = -r \Delta_0^{-1} \epsilon^{AB} \nabla_B D_A , \]

(2.32)

eq. (2.28) simplifies to

\[
- \delta H_{sym} = \int_V \Lambda_1 \left\{ \left( \dot{A}^1 \delta Q_1 - \dot{Q}_1 \delta \Pi^1 \right) + \left( \dot{A}^2 \delta Q_2 - \dot{Q}_2 \delta \Pi^2 \right) \right\} \\
+ \int_{\partial V} \Lambda_1 \left( \chi^1 \delta Q_1 + \chi^2 \delta Q_2 \right) ,
\]

(2.33)

where we introduced the boundary momenta:

\[ \chi_1 = -\frac{1}{r} A_0 , \]

(2.34)

\[ \chi_2 = -r \Delta_0^{-1} \epsilon_{AB} \nabla^B G^{rA} . \]

(2.35)

Tensor \( G^{\mu
u} \) is defined by \( G^{\mu
u} = \Lambda_1 G^{\mu
u} \), and, therefore, \( D^i = \Lambda_1 D^i \). Note, that

\[ \chi^l = \frac{\delta H_{sym}}{\delta (\partial_\mu Q_l)} , \quad l = 1, 2 . \]

(2.36)

For a Maxwell theory one obtains

\[
H^{Maxwell}_{sym} = \frac{1}{2} \int_V \Lambda_1 \sum_{l=1}^2 \left\{ \frac{1}{r^2} Q_l Q_l - \frac{1}{r^2} \partial_\nu (r Q_l) \Delta_0^{-1} \partial_\nu (r Q_l) - \Pi_l \delta_0 \Pi^l \right\} ,
\]

(2.37)

and, therefore

\[ \chi^l = \frac{1}{r} \Delta_0^{-1} \partial_\nu (r Q_l) , \quad l = 1, 2 , \]

(2.38)

have perfectly symmetric form.
2.4 Canonical symmetries

The symplectic form $\int \delta D^k \wedge \delta A_k$ rewritten in terms of $Q$’s and $\Pi$’s have the following form [10], [16]:

$$
\Omega = \operatorname{Im} \int \Lambda_1 \delta \Pi \wedge \delta \overline{Q},
$$

(2.39)

where we introduced a complex notation

$$
Q = Q^1 + iQ^2,
$$

(2.40)

$$
\Pi = i(\Pi_1 + i\Pi_2).
$$

(2.41)

The form (2.39) is invariant under the following set of $\mathbb{R}$-linear transformations:

$$
Q \rightarrow e^{i\alpha} Q,
$$

(2.42)

$$
Q \rightarrow \cosh \alpha Q + i \sinh \alpha \overline{Q},
$$

(2.43)

$$
Q \rightarrow \cosh \lambda Q + \sinh \lambda \overline{Q},
$$

(2.44)

and the same rules for $\Pi$. It is easy to see that these transformations form the group $SO(2,1)$. In terms of $D$ and $B$, (2.42)–(2.44) have more familiar form:

(2.42) corresponds to orthogonal $SO(2)$ duality rotations:

$$
D \rightarrow D \cos \alpha - B \sin \alpha,
$$

(2.45)

$$
B \rightarrow D \sin \alpha + B \cos \alpha,
$$

(2.46)

(2.43) corresponds to hyperbolic $SO(1,1)$ rotations:

$$
D \rightarrow D \cosh \alpha + B \sinh \alpha,
$$

(2.47)

$$
B \rightarrow D \sinh \alpha + B \cosh \alpha,
$$

(2.48)

(2.44) corresponds to scaling transformations:

$$
D \rightarrow e^\lambda D,
$$

$$
B \rightarrow e^{-\lambda} B.
$$

(2.49)

The canonical generators corresponding to (2.42)–(2.44) have the following form:

$$
G_1 = \int \Lambda_1 (Q^2 \Pi_1 - Q^1 \Pi_2) = \operatorname{Re} \int \Lambda_1 (\Pi \overline{Q}),
$$

(2.48)

$$
G_2 = -\int \Lambda_1 (Q^2 \Pi_1 + Q^1 \Pi_2) = \operatorname{Re} \int \Lambda_1 (\Pi Q),
$$

(2.49)

$$
G_3 = \int \Lambda_1 (Q^1 \Pi_1 - Q^2 \Pi_2) = \operatorname{Im} \int \Lambda_1 (\Pi Q).
$$

(2.50)

Note, that for the duality invariant theory $G_1$ defined in (2.48) is constant in time. Its physical interpretation was clarified in [9]. Obviously, $G_1$, $G_2$ and $G_3$ rewritten in terms of $D$ and $B$ are highly nonlocal functionals of the fields [10], [8].

2.5 Summary

The reduced variables $(Q_l, \Pi^l)$ play the role of generalized positions and momenta for an electromagnetic field. They are perfectly gauge-invariant and contain the entire (gauge-invariant) information about $D$ and $B$. Let us note that $Q$’s and $\Pi$’s are nonlocal functions of $D$ and $B$. The nonlocality
enters via the operations on each sphere $S^2(r)$, i.e. via the operator $\Delta_0^{-1}$. On the other hand the operations in the radial direction do not produce any nonlocality.

The Hamiltonian generating the dynamics is perfectly local in $D$ and $B$ but is nonlocal in $Q$'s and $\Pi$'s. The field functional with the above described nonlocality we shall call quasi-local. Note, that generators $G_i$ are perfectly local in reduced variables.

The “symmetric” Hamiltonian dynamics is defined by the Dirichlet boundary conditions for positions $Q_l$. On the other hand the “canonical” formula (2.12) is defined by the Dirichlet boundary condition for $\chi$ and $Q_2$. Note, however, that in the Maxwell case

$$\int_{\partial V} \Lambda_1 Q_1 \delta \chi = \int_{\partial V} \Lambda_\perp \left( \Delta_0^{-1} Q_1 \right) \delta \partial_r (r^2 D^\nu) = \int_{\partial V} r \left( \Delta_0^{-1} Q_1 \right) \delta \left( \partial_r D^\nu \right),$$

i.e. a Dirichlet condition $\chi |_{\partial V}$ is equivalent to the Neumann condition $\partial_r D^\nu |_{\partial V}$.

3 2-form theory in $D = 6$

3.1 Generating formula

Now, consider a 2-form theory defined by the Lagrangian $L = L(A, \partial A)$. Field dynamics of this theory may be written in terms of the following generating formula:

$$-\delta L = \partial_\nu (G^{\nu \mu \lambda} \delta A_{\mu \lambda}) = (\partial_\nu G^{\nu \mu \lambda}) \delta A_{\mu \lambda} + G^{\nu \mu \lambda} \delta (\partial_\nu A_{\mu \lambda}).$$

(3.1)

The formula (3.1) implies the following definition of “momenta”:

$$G^{\mu \nu \lambda} = -3! \frac{\partial L}{\partial F_{\mu \nu \lambda}}.$$

(3.2)

Moreover, (3.1) generates dynamical (in general nonlinear) field equations

$$\partial_\nu G^{\nu \mu \lambda} = -J^\mu \lambda,$$

(3.3)

where the external 2-form current reads:

$$J^\mu \lambda = 2 \frac{\partial L}{\partial A_{\mu \lambda}}.$$

(3.4)

In the present section we consider only $J = 0$ (for $J \neq 0$ see section 4.) To obtain the Hamiltonian description of the field dynamics let us integrate equation (3.1) over a 5-dimensional volume $V$ contained in the constant-time hyperplane $\Sigma$:

$$-\delta \int_V L = \int_V \partial_0 (G^{0ij} \delta A_{ij}) + \int_{\partial V} G^{\perp \mu \nu} \delta A_{\mu \nu},$$

(3.5)

where $\perp$ denotes the component orthogonal to the 4-dimensional boundary $\partial V$. To simplify our notation let us introduce the spherical coordinates on $\Sigma$:

$$x^5 = r, \quad x^A = \varphi_A; \quad A = 1, 2, 3, 4,$$

(3.6)

where $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ denote spherical angles (to enumerate angles we shall use capital letters $A, B, C, \ldots$). The Euclidean metric on $\Sigma$ reads:

$$g_{11} = r^2 \sin^2 \varphi_2 \sin^2 \varphi_3 \sin^2 \varphi_4, \quad g_{22} = r^2 \sin^2 \varphi_3 \sin^2 \varphi_4,$$

$$g_{33} = r^2 \sin^2 \varphi_4, \quad g_{44} = r^2, \quad g_{55} = g_{rr} = 1,$$

(3.7)

and the corresponding volume form

$$\Lambda_2 = \sqrt{\det(g_{ij})} = r^4 \sin \varphi_2 \sin^2 \varphi_3 \sin^3 \varphi_4.$$

(3.8)
Let $V$ be a 5-dim. ball with a finite radius $R$. In such a coordinate system the formula (3.5) takes the following form:

$$
\delta \int_V \mathcal{L} = \int_V \partial_0 (D_{ij} \delta A_{ij}) - \int_{\partial V} 2 \mathcal{D}^{rA} \delta A_{0A} - \int_{\partial V} G^{rAB} \delta A_{AB},
$$

(3.9)

where

$$
D_{ij} = G_{ij0}
$$

(3.10)
denotes the 2-form electric induction density. Now, performing the Legendre transformation between induction 2-form $D_{ij}$ and $\dot{A}_{ij}$ one obtains the following Hamiltonian formula:

$$
- \delta H_{\text{can}} = \int_V \left( \dot{D}^{ij} \delta A_{ij} - \dot{A}_{ij} \delta D^{ij} \right) - \int_{\partial V} 2 \mathcal{D}^{rA} \delta A_{0A} - \int_{\partial V} G^{rAB} \delta A_{AB},
$$

(3.11)

where the canonical Hamiltonian

$$
H_{\text{can}} = \int_V \left( D^{ij} \dot{A}_{ij} - \mathcal{L} \right).
$$

(3.12)

Equation (3.11) generates an infinite-dimensional Hamiltonian system in the phase space $P_2 = (D^{ij}, A_{ij})$ fulfilling Dirichlet boundary conditions for the 2-form potential $A_{ij}$: $A_{0A} |_{\partial V}$ and $A_{AB} |_{\partial V}$. From the mathematical point of view this is the missing part of the definition of the functional space. The Hamiltonian structure of a general nonlinear 2-form electrodynamics described above is mathematically well defined, i.e. a mixed Cauchy problem (Cauchy data given on $\Sigma$ and Dirichlet data given on $\partial V \times \mathbb{R}$) has a unique solution (modulo gauge transformations which reduce to the identity on $\partial V \times \mathbb{R}$).

Note the difference in signs between corresponding formulae of the present section and that of section 2. This difference follows from the difference between corresponding symplectic structures $\Omega$. For 1-form theory one has

$$
\Omega_1 = \int_V \delta G^{0i} \wedge \delta A_i = + \int_V \delta D^i \wedge \delta A_i,
$$

(3.13)

whereas for 2-form theory

$$
\Omega_2 = \int_V \delta G^{0ij} \wedge \delta A_{ij} = - \int_V \delta D^{ij} \wedge \delta A_{ij},
$$

(3.14)

Now, in analogy to (2.12) we pass to another Hamiltonian description of the field evolution in the finite region $V$. Let us perform the Legendre transformation between $D^{rA}$ and $A_{0A}$ at the boundary $\partial V$. One obtains:

$$
- \delta H_{\text{sym}} = \int_V \left( \dot{D}^{ij} \delta A_{ij} - \dot{A}_{ij} \delta D^{ij} \right) + \int_{\partial V} 2 A_{0A} \delta D^{rA} - \int_{\partial V} G^{rAB} \delta A_{AB},
$$

(3.15)

where the new “symmetric” Hamiltonian

$$
H_{\text{sym}} = H_{\text{can}} - \int_{\partial V} 2 \mathcal{D}^{rA} A_{0A}.
$$

(3.16)

Observe, that formula (3.15) defines the Hamiltonian evolution but on a different phase space. In order to kill boundary terms in (3.13) one has to control on $\partial V$: $D^{rA}$ (instead of $A_{0A}$) and $A_{AB}$. We stress that from the mathematical point of view both descriptions are equally good and an additional physical argument has to be given if we want to choose one of them as more fundamental.
3.2 Canonical vs. symmetric energy

The relation between $H_{\text{can}}$ and $H_{\text{sym}}$ is exactly the same as in $p = 1$ case:

\[
H_{\text{sym}} = H_{\text{can}} - \int_{\partial V} 2 \mathcal{D}^A A_0 A = H_{\text{can}} - \int_V 2 \partial_k \left( \mathcal{D}^{ki} A_{0i} \right)
\]

\[
= \int_{\mathcal{V}} \left\{ \mathcal{D}^{ij} A_{ij} - \mathcal{L} + 2 \left( A_{0i} \partial_k \mathcal{D}^{ki} + \mathcal{D}^{ki} \partial_k A_{0i} \right) \right\}
\]

\[
= \int_{\mathcal{V}} \left( \frac{1}{2} \mathcal{D}^{ij} E_{ij} - \mathcal{L} \right), \tag{3.17}
\]

where the 2-form electric field is defined by

\[
E_{ij} = F_{ij0} = \partial_i [A_j^0]. \tag{3.18}
\]

Therefore, $H_{\text{sym}} = \int T_{\text{sym}}^{00}$ and $H_{\text{can}} = \int T_{\text{can}}^{00}$ with

\[
T_{\text{sym}}^{\mu\nu} = \frac{1}{2} F^\mu\lambda\sigma G^\nu_{\lambda\sigma} + g^{\mu\nu} \mathcal{L}, \tag{3.19}
\]

and

\[
T_{\text{can}}^{\mu\nu} = (\partial^\mu A^\lambda_\sigma) G^\nu_{\lambda\sigma} + g^{\mu\nu} \mathcal{L}. \tag{3.20}
\]

In the 2-form Maxwell theory the “symmetric energy” (gauge-invariant and positively defined) reads:

\[
H_{\text{sym}}^{\text{Maxwell}} = \frac{1}{4} \int (D^{ij} D_{ij} + B^{ij} B_{ij}).
\]

3.3 Reduction of the generating formula

Now, in analogy to (2.18) let us make the following decomposition:

\[
A_{AB} = \nabla_{[A} u_{B]} + \epsilon_{ABCD} \nabla^C v^D, \tag{3.21}
\]

where $\nabla_A$ denotes a covariant derivative on each $S^4(r)$ defined by the induced metric $g_{AB}$ and $\epsilon_{ABCD}$ stands for the Lévi-Civita tensor density such that $\epsilon_{1234} = \Lambda_2$. Both $u^A$ and $v^A$ are 1-forms on $S^4(r)$. Using (3.21) and integrating by parts one gets:

\[
\int_{\mathcal{V}} \left( \mathcal{D}^{ij} \delta A_{ij} - \mathcal{A}_{ij} \mathcal{D}^{ij} \right) = \int_{\mathcal{V}} \left\{ 2 \left( \mathcal{D}^{rA} \delta A_{rA} - \mathcal{A}_{rA} \delta \mathcal{D}^{rA} \right) + 2 \left[ (\partial_r \mathcal{D}^{rA}) \delta u_A - \dot{u}_A \delta (\partial_r \mathcal{D}^{rA}) \right] - \epsilon_{ABCD} \left[ (\nabla^C \mathcal{D}^{rA}) \delta v_A - \dot{v}_A \delta (\nabla^C \mathcal{D}^{rA}) \right] \right\}, \tag{3.22}
\]

where we have used the Gauss law

\[
\nabla_A D^{AB} = -\partial_r D^{rB}. \tag{3.23}
\]

Moreover, due to (3.21)

\[
\int_{\partial V} G^{rAB} \delta A_{AB} = -\int_{\partial V} \left\{ -2 \mathcal{D}^{rA} \delta u_A + \left( \epsilon_{ABCD} \nabla^C G^{rAB} \right) \delta v^A \right\}. \tag{3.24}
\]

In deriving (3.24) we have used

\[
\nabla_A G^{ABr} = -\mathcal{D}^r B, \tag{3.25}
\]
which follows from the field equations $\nabla_A G^{ABr} + \partial_0 G^{0Br} = 0$. Now, taking into account (3.22) and (3.24) the generating formula (3.13) may be rewritten in the following way:

$$- \delta \mathcal{H}_{\text{sym}} = \int_V \left\{ \left[ \delta D^A \delta (2 A_{rA} - 2 \partial_r u_A) - (2 \dot{A}_{rA} - 2 \partial_r \dot{u}_A) \delta D^A \right] - \left( \epsilon_{ABCD} \nabla^C \dot{D}^{AB} \right) \delta v^D - \dot{v}^D \delta (\epsilon_{ABCD} \nabla^C D^{AB}) \right\} + \int_{\partial V} \left\{ (2 A_{0A} - 2 \dot{u}_A) \delta D^A + \int_{\partial V} (\epsilon_{ABCD} \nabla^C G^{rAB}) \delta v^D \right\}. \quad (3.26)$$

Note, that although $A_{rA}$, $A_{0A}$ and $u_A$ are manifestly gauge-dependent, the combinations $A_{rA} - \partial_r u_A$ and $A_{0A} - \partial_0 u_A$ are gauge-invariant. To simplify our consideration we choose the special gauge $u \equiv 0$, i.e. a 2-form $A_{AB}$ on $S^4(r)$ is purely transversal. This condition, due to (3.21), may be equivalently rewritten as

$$\nabla_A A_{AB} = 0. \quad (3.27)$$

But now, contrary to the $p = 1$ case, we have an additional covector field on $S^4(r)$, namely $A_{rA}$. For this covector we choose an analogous gauge condition, i.e.

$$\nabla_A A^r_A = 0. \quad (3.28)$$

Assuming (3.27) and (3.28) one may show [16]

$$\Delta_1 A^r_A = \frac{r^2}{4} \epsilon^{ABCD} \nabla_C B_{AB}, \quad (3.29)$$

where

$$\Delta_1 = r^2 \nabla_A \nabla^A - 3, \quad (3.30)$$

equals to the Laplace-Beltrami operator on co-exact 1-forms on $S^4(1)$ [16]. Moreover, in analogy to (3.27) one has [16]

$$B^{rA} = -2r^{-2} \Delta_1 v^A, \quad (3.31)$$

and, therefore, the formula (3.26) simplifies to

$$- \delta \mathcal{H}_{\text{sym}} = \frac{1}{2} \int_V \left\{ - \left( r \dot{D}^r D \right) \delta \left( r \Delta_1^{-1} \epsilon_{ABCD} \nabla^C B^{AB} \right) - \left( r \Delta_1^{-1} \epsilon_{ABCD} \nabla^C \dot{B}^{AB} \right) \delta (r D^r D) \right\} + \left\{ \left( r \Delta_1^{-1} \epsilon_{ABCD} \nabla^C \dot{D}^{AB} \right) \delta (r B^r D) - \left( r \dot{B}^r D \right) \delta \left( r \Delta_1^{-1} \epsilon_{ABCD} \nabla^C B^{AB} \right) \right\} + \int_{\partial V} \left\{ (2r^{-1} A_{0A}) \delta (r D^r A) - \left( \frac{1}{2} r \Delta_1^{-1} \epsilon_{ABCD} \nabla^C G^{rAB} \right) \delta (r B^r D) \right\}. \quad (3.32)$$

Now, introducing the following set of variables

$$Q_1^A = r D^r A, \quad (3.33)$$

$$Q_2^A = r B^r A, \quad (3.34)$$

$$\Pi_1^D = \frac{r}{2} \Delta_1^{-1} \epsilon_{ABCD} \nabla^C B^{AB}, \quad (3.35)$$

$$\Pi_2^D = -\frac{r}{2} \Delta_1^{-1} \epsilon_{ABCD} \nabla^C D^{AB}, \quad (3.36)$$

eq (3.32) simplifies to

$$- \delta \mathcal{H}_{\text{sym}} = \int_V \Lambda_2 \left\{ \left( \Pi_1^D \delta Q_1^A - \dot{Q}_1^A \delta \Pi_1^A \right) - \left( \Pi_2^A \delta Q_2^A - \dot{Q}_2^A \delta \Pi_2^A \right) \right\} + \int_{\partial V} \Lambda_2 \left( \chi_1^A \delta Q_1^A + \chi_2^A \delta Q_2^A \right), \quad (3.37)$$

11
where we introduced the boundary momenta:

\[
\chi^1_A = \frac{2}{r} A_0 A, \quad (3.38)
\]

\[
\chi^2_D = -\frac{r}{2} \Delta_1^{-1} \epsilon_{ABCD} \nabla^C G^{rA}. \quad (3.39)
\]

In (3.37) we defined

\[
Q^l_A := g^{AB} Q^B_l, \quad \Pi^l_A := g^{AB} \Pi^B_l. \quad (3.40)
\]

Note the crucial difference between (3.37) and (2.33): the sign “+” in (2.33) is replaced by “−” in (3.37).

For a Maxwell theory one obtains

\[
H^\text{Maxwell sym} = \frac{1}{4} \int_V \Lambda^2 \sum_{l=1}^2 \left\{ \frac{1}{r^2} Q^l_A Q^l_A - \frac{1}{r^4} \partial_r (r^3 Q^l_A) \Delta_1^{-1} \partial_r (r Q^l_A) - \Pi^l_A \Delta_1 \Pi^l_A \right\} \quad (3.41)
\]

and, therefore

\[
\chi^l_A = \frac{1}{r^3} \Delta_1^{-1} \partial_r (r^3 Q^l_A), \quad l = 1, 2. \quad (3.42)
\]

### 3.4 Canonical symmetries

The symplectic form \( -\int \delta D^{ij} \wedge \delta A_{ij} \) rewritten in terms of \( Q \)'s and \( \Pi \)'s have the following form [16]:

\[
\Omega = \text{Im} \int \Lambda^2 \delta \Pi^A \wedge \delta Q_A, \quad (3.43)
\]

where we introduced a complex notation

\[
Q_A = Q^1_A + i Q^2_A, \quad (3.44)
\]

\[
\Pi^A = i (\Pi^1_A + i \Pi^2_A). \quad (3.45)
\]

The form (3.43) contrary to (2.39) is invariant only under the following transformations:

\[
Q_A \rightarrow \cosh \lambda Q_A + \sinh \lambda \overline{Q}_A, \quad (3.46)
\]

and the same rule for \( \Pi^A \). It is easy to see that these transformations form the group \( SO(1, 1) \). In terms of \( D^{ij} \) and \( B^{ij} \), (3.46) reads:

\[
D^{ij} \rightarrow e^\lambda D^{ij}, \quad B^{ij} \rightarrow e^{-\lambda} B^{ij}. \quad (3.47)
\]

The canonical generator corresponding to (3.44) has the following form:

\[
G_4 = -\int \Lambda^2 (Q^1_A \Pi^1_A + Q^2_A \Pi^2_A) = \text{Im} \int \Lambda^2 (\Pi^A \overline{Q}_A). \quad (3.48)
\]
3.5 Summary

Contrary to the $p = 1$ case the reduced variables $(Q^A_l, \Pi^l_A)$ do not solve completely the Gauss constraints $\partial_i D^{ij} = \partial_i B^{ij} = 0$. They fulfill the following additional conditions [16]:

$$\nabla_A Q^A_l = \nabla^A \Pi^l_A = 0, \quad l = 1, 2.$$  

(3.49)

In the geometric language it means that $\star Q^A_l$ and $\star \Pi^l_A$ are closed 3-forms on $S^4(r)$ ($\star$ denotes the Hodge dual defined via $\epsilon^{ABCD}$). They are gauge-invariant and contain the entire information about 2-forms $D^{ij}$ and $B^{ij}$. The dynamics is generated by the quasi-local functional of $Q$’s and $\Pi$’s.

The “symmetric” dynamics defined by (3.37) corresponds to the Dirichlet boundary condition for positions $Q_l$ whereas the “canonical” dynamics corresponds to the Dirichlet conditions for $\chi^1$ and $Q_2$. But Dirichlet condition for $\chi^1_A$ is equivalent to the Neumann condition for $\partial_r D^r_A$

$$\int_{\partial V} \Lambda_2 Q_1^A \delta \chi^1_A = \int_{\partial V} r \Delta^{-1}_1 Q_1^A \delta (\partial_r D^r_A).$$  

(3.50)

4 Coupling to the charged matter

In the present Section we study the coupling of $p$-form electrodynamics to the charged matter. We present parallel discussion for $p = 1$ and $p = 2$. The general case is presented in Appendix C.

4.1 $p = 1$

Consider a 1-form electromagnetism interacting with the charged matter field $\Phi$ (for simplicity let $\Phi$ be a complex scalar field). In the presence of charged matter the Lagrangian generating formula (2.1) has to be replaced by:

$$-\delta L = \partial_\nu (G^{\nu\mu} \delta A_\mu + \mathcal{P}^{\nu} \delta \Phi),$$  

(4.1)

where the matter “momentum”

$$\mathcal{P}^{\nu} = -\frac{\partial L}{\partial (\partial_\nu \Phi)}.$$  

(4.2)

Because $L$ should define a gauge-invariant theory let us assume that there is a group of gauge transformations $U_\Lambda$ parameterized by a a function (0-form) $\Lambda$ acting in the following way: $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ and $\Phi \rightarrow U_\Lambda(\Phi)$.

Now, the target space of the matter field $\Phi$ may be reparameterized $\Phi = (\varphi, U)$ in such a way that, a parameter $U$ is gauge invariant and $\varphi$ is the phase undergoing the following gauge transformation: $\varphi \rightarrow \varphi + \Lambda$. For the scalar (complex) field one has: $U := |\Phi|$ and the $\varphi = \text{Arg} \Phi$. Therefore, the matter part in (4.1) may be rewritten as follows:

$$\mathcal{P}^{\nu} \delta \Phi = J^{\nu} \delta \varphi + p^{\nu} \delta U.$$  

(4.3)

Gauge invariance of the theory means that the gauge dependent quantities, i.e. $A_\mu$ and $\varphi$, enter into $L$ via the gauge-invariant combinations only:

$$L = L(F_{\mu\nu}, D_\mu \varphi, U, \partial_\mu U),$$  

(4.4)

where

$$D_\mu \varphi := \partial_\mu \varphi - A_\mu.$$  

(4.5)
denotes a covariant derivative of \( \varphi \). This implies, that the momentum \( J^\mu \) canonically conjugated to \( \varphi \) is equal to the electric current

\[
J^\mu = - \frac{\partial L}{\partial (\partial_\mu \varphi)} = \frac{\partial L}{\partial A_\mu} = - \partial_\nu G^{\nu \mu} .
\]  

(4.6)

Now, instead of (2.8) one has

\[
- \delta \int_V L = \int_V \partial_0 \left( D^i \delta A_i + \rho \delta \varphi + p^0 \delta U \right) + \int_{\partial V} \left( - D^r \delta A_0 + G^{r B} \delta A_B + J^r \delta \varphi + p^r \delta U \right) ,
\]

(4.7)

with \( \rho := J^0 \). Performing the set of Lagrange transformations between: 1) \( D^k \) and \( \dot{A}_k \), 2) \( \rho \) and \( \dot{\varphi} \), 3) \( \pi := p^0 \) and \( \dot{U} \) in the volume \( V \), and between \( D^r \) and \( A_0 \) at the boundary \( \partial V \), one obtains the following generalization of (2.12):

\[
- \delta H_{\text{sym}} = - \int_V \left\{ \left( \dot{D}^i \delta A_i - \dot{A}_i \delta D^i \right) + \left( \dot{\rho} \delta \varphi - \dot{\varphi} \delta \rho \right) + \left( \dot{\pi} \delta U - \dot{U} \delta \pi \right) \right\} \\
- \int_{\partial V} \left\{ \dot{A}_0 \delta D^r + G^{r B} \delta A_B + J^r \delta \varphi + p^r \delta U \right\} ,
\]

(4.8)

where the “symmetric” Hamiltonian of the interacting electromagnetic field and the charged matter represented by \( \Phi \) reads:

\[
H_{\text{sym}} = \int_V \left( - \dot{D}^i \dot{A}_i - \rho \dot{\varphi} - \pi \dot{U} - L + \partial_k (A_0 D^k) \right) .
\]

(4.9)

Now, using

\[
\partial_k D^k = \rho ,
\]

implied by (4.10), one gets the following formula for \( H_{\text{sym}} \):

\[
H_{\text{sym}} = \int_V \left( D^i \dot{E}_i - \rho D_0 \varphi - \pi \dot{U} - L \right) .
\]

(4.11)

Note, that the gauge-dependent phase \( \varphi \) enters into \( H_{\text{sym}} \) via the gauge-invariant combination \( D_0 \varphi \) only. Moreover, due to (4.10), we may rewrite the dynamical part for \( \varphi \) in (4.8) as follows:

\[
\int_V \left( \dot{\rho} \delta \varphi - \dot{\varphi} \delta \rho \right) = \int_V \left( - \dot{D}^k \delta (\partial_k \varphi) + (\partial_k \dot{\varphi}) \delta D^k \right) + \int_{\partial V} \left( \dot{\varphi} \delta \varphi - \dot{\varphi} \delta D^r \right) .
\]

(4.12)

Now, the \( \dot{D}^r \) at the boundary may be easily eliminated by the field equations (4.6)

\[
\dot{D}^r = - \partial_0 G^0 = \partial_\mu G^{\mu r} - \partial_A G^{A r} = - J^r + \partial_A G^{A r} .
\]

(4.13)

Introducing a hydrodynamical variables:

\[
V_\mu := - D_\mu \varphi ,
\]

(4.14)

we may rewrite finally (4.8) as follows:

\[
- \delta H_{\text{sym}} = - \int_V \left\{ \left( \dot{D}^i \delta V_i - \dot{V}_i \delta D^i \right) + \left( \dot{\pi} \delta U - \dot{U} \delta \pi \right) \right\} \\
- \int_{\partial V} \left\{ \dot{V}_0 \delta D^r + G^{r B} \delta V_B + p^r \delta U \right\} ,
\]

(4.15)

i.e. (4.15) has exactly the same form as (2.12) with \( A_\mu \) replaced by the gauge-invariant \( V_\mu \) and supplemented by the gauge-invariant canonical pair \( (U, \pi) \) together with the boundary momentum \( p^r \).
4.2 $p = 2$

Now, consider a 2-form electromagnetism interacting with the charged matter field $\Phi_\mu$ (for simplicity let $\Phi_\mu$ be a complex vector field). In the presence of charged matter the Lagrangian generating formula (3.1) has to be replaced by:

$$-\delta \mathcal{L} = \partial_\nu (G^{\nu\mu\lambda} \delta A_{\mu\lambda} + \mathcal{P}^{\nu\mu} \delta \Phi_\mu),$$

(4.16)

where the matter "momentum"

$$\mathcal{P}^{\nu\mu} = -2 \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi_\mu)}.$$  

(4.17)

Because $\mathcal{L}$ should define a gauge-invariant theory let us assume that there is a group of gauge transformations $U_\Lambda$ parameterized by a a 1-form $\Lambda$ acting in the following way: $A \rightarrow A + d\Lambda$ and $\Phi \rightarrow U_\Lambda(\Phi)$.

Now, the target space of the matter field $\Phi_\mu$ may be reparameterized $\Phi_\mu = (\varphi_\mu, U_\mu)$ in such a way that a 1-form $U_\mu$ is gauge invariant and a 1-form $\varphi_\mu$ is the phase undergoing the following gauge transformation: $\varphi \rightarrow \varphi + \Lambda$. For the vector (complex) field one has: $U_\mu := |\Phi_\mu|$ and $\varphi_\mu = \text{Arg} \Phi_\mu$.

Therefore, the matter part in (4.16) may be rewritten as follows:

$$\mathcal{P}^{\nu\mu} \delta \Phi_\mu = J^{\nu\mu} \delta \varphi_\mu + p^{\nu\mu} \delta U_\mu.$$  

(4.18)

Gauge invariance of the theory means that the gauge dependent quantities, i.e. $A_{\mu\nu}$ and $\varphi_\mu$, enter into $\mathcal{L}$ via the gauge-invariant combinations only:

$$\mathcal{L} = \mathcal{L}(F_{\mu\nu}, D_\mu \varphi_\nu, U_\mu, \partial_\mu U_\nu),$$

(4.19)

where

$$D_\mu \varphi_\nu := \frac{1}{2} \partial_\mu \varphi_\nu - A_{\mu\nu}$$

(4.20)

denotes a “covariant derivative” of $\varphi_\mu$. This implies, that the momentum $J^{\mu\lambda}$ canonically conjugated to $\varphi_\lambda$ is equal to the electric current

$$J^{\mu\lambda} = -2 \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_\lambda)} = 2 \frac{\partial \mathcal{L}}{\partial A_{\mu\lambda}} = -\partial_\lambda G^{\mu\nu\lambda}.$$  

(4.21)

Now, instead of (3.15) one has

$$-\delta \int_V \mathcal{L} = \int_\Sigma \partial_0 \left( -D^{ij} \delta A_{ij} - \rho^k \delta \varphi_k + \pi^k \delta U_k \right)$$

$$+ \int_{\partial V} \left( 2D^{rA} \delta A_{0A} + G^{rAB} \delta A_{AB} + \rho^r \delta \varphi_0 + J^{rA} \delta \varphi_A - \pi^r \delta U_0 + p^r A \delta U_A \right),$$

(4.22)

with $\rho^k := J^{k0}$ (it defines a 1-form charge density on 5-dim. hyperplane $\Sigma$) and $\pi^k := p^{0k}$. Now, to pass to the Hamiltonian picture one has to perform the following Legendre transformations between: 1) $D$ and $\dot{A}$, 2) $\rho$ and $\dot{\varphi}$, 3) $\pi$ and $\dot{U}$ in the volume $V$, and between 4) $D^r$ and $A_0$, 5) $\rho^r$ and $\varphi_0$ and 6) $\pi^r$ and $U_0$ at the boundary $\partial V$. One obtains the following generalization of (3.15):

$$-\delta \mathcal{H}_{sym} = \int_V \left\{ \left( D^{ij} \delta A_{ij} - \dot{A}_{ij} D^{ij} \right) + \left( \rho^k \delta \varphi_k - \varphi_k \delta \rho^k \right) - \left( \pi^k \delta U_k - \dot{U}_k \delta \pi^k \right) \right\}$$

$$- \int_{\partial V} \left\{ -2A_{0A} \delta D^{rA} + G^{rAB} \delta A_{AB} - \varphi_0 \delta \rho^r + J^{rA} \delta \varphi_A + U_0 \delta \pi^r + p^r A \delta U_A \right\},$$

(4.23)
where the “symmetric” Hamiltonian of the interacting electromagnetic field and the charged matter represented by $\Phi^\mu$ reads:

$$
H_{\text{sym}} = \int_V \left\{ D^{ij} \dot{A}_{ij} + \rho^k (\dot{\varphi}_k - \partial_k \varphi_0) - \pi^k \dot{U}_k - \partial_k \left( 2A_{0i}D^{ki} - U_0 \pi^k \right) - L \right\},
$$

where we have used $\partial_k \rho^k = 0$. Now, using

$$
\partial_i D^{ik} = \rho^k,
$$

one gets the following formula for $H_{\text{sym}}$:

$$
H_{\text{sym}} = \int_V \left( \frac{1}{2} D^{ij} E_{ij} + 2\rho^k D_{0i} \varphi_k - \pi^k \dot{U}_k - L + \partial_k (\pi^k U_0) \right).
$$

Note, that the gauge-dependent phase $\varphi_\mu$ enters into $H_{\text{sym}}$ via the gauge-invariant combination $D_{0i} \varphi_k$. Moreover, due to (4.25), we may rewrite the dynamical part for $\varphi_\mu$ in (4.23) as follows:

$$
\int_V \left( \dot{\rho}^k \delta \varphi_k - \dot{\varphi}_k \delta \rho^k \right) = \int_V \left( -\dot{D}^{ik} \delta (\partial_i \varphi_k) + (\partial_i \dot{\varphi}_k) \delta D^{ik} \right) + \int_{\partial V} \left( \dot{D}^{rA} \delta \varphi_A - \dot{\varphi}_A \delta D^{rA} \right).
$$

Now, the term $\dot{D}^{rA}$ at the boundary may be easily eliminated by the field equations (4.21)

$$
\dot{D}^{rA} = J^{rA} + \partial_B G^{rAB}.
$$

Introducing hydrodynamical variables:

$$
V_{\mu \nu} := -D_{[\mu} \varphi_{\nu]},
$$

we may rewrite finally (4.23) as follows:

$$
-\delta H_{\text{sym}} = \int_V \left\{ \left( \dot{D}^{ij} \delta V_{ij} - \dot{V}_{ij} \delta D^{ij} \right) - \left( \pi^k \delta \dot{U}_k - U_0 \delta \pi^k \right) \right\}
$$

$$
- \int_{\partial V} \left\{ -2V_{0A} \delta D^{rA} + G^{rAB} \delta V_{AB} - U_0 \delta \pi^r + \rho^{rA} \delta U_A \right\},
$$

i.e. (4.13) has exactly the same form as (3.15) with $A_{\mu \nu}$ replaced by the gauge-invariant 2-form $V_{\mu \nu}$ and supplemented by the gauge-invariant canonical pair $(U_k, \pi^k)$ together with the boundary momenta $U_0$ and $\rho^{rA}$. All gauge-dependent terms dropped out.

**Appendices**

**A Notation**

Consider a $p$-form potential $A$ defined in the $D = 2p + 2$ dimensional Minkowski space-time $\mathcal{M}^{2p+2}$ with the signature of the metric tensor $(-, +, ..., +)$. The corresponding field tensor is defined as a $(p + 1)$-form by $F = dA$:

$$
F_{\mu_1 ... \mu_{p+1}} = \partial_{[\mu_1} A_{\mu_2 ... \mu_{p+1}]} ,
$$

where the antisymmetrization is defined by $X_{[kl]} := X_{kl} - X_{lk}$. Having a Lagrangian $\mathcal{L}$ of the theory one defines another $(p + 1)$-form $G$ as follows:

$$
G^{\mu_1 ... \mu_{p+1}} = -(p + 1)! \frac{\partial \mathcal{L}}{\partial F^{\mu_1 ... \mu_{p+1}}} .
$$
Now one may define the electric and magnetic intensities and inductions in the obvious way:

\[
E_{i_1\ldots i_p} = F_{i_1\ldots i_p 0} ,
\]

\[
B_{i_1\ldots i_p} = \frac{1}{(p+1)!} \epsilon_{i_1\ldots i_p j_1\ldots j_{p+1}} F^{j_1\ldots j_{p+1}} ,
\]

\[
D_{i_1\ldots i_p} = \mathcal{G}_{i_1\ldots i_p 0} ,
\]

\[
H_{i_1\ldots i_p} = \frac{1}{(p+1)!} \epsilon_{i_1\ldots i_p j_1\ldots j_{p+1}} \mathcal{G}^{j_1\ldots j_{p+1}} ,
\]

where the indices \(i_1, i_2, \ldots, j_1, j_2, \ldots\) run from 1 up to \(2p + 1\) and \(\epsilon_{i_1 i_2 \ldots i_{2p+1}}\) is the Lévi-Civita tensor in \(2p + 1\) dimensional Euclidean space, i.e. a space-like hyperplane \(\Sigma\) in the Minkowski space-time. The field equations are given by the Bianchi identities \(dF = 0\), or in components

\[
\partial_\lambda F_{\mu_1\ldots \mu_{p+1}} = 0 ,
\]

and the true dynamical equations \(d \star \mathcal{G} = 0\), or equivalently

\[
\partial_\lambda \star \mathcal{G}_{\mu_1\ldots \mu_{p+1}} = 0 ,
\]

where the Hodge star operation in \(\mathcal{M}^{2p+2}\) is defined by:

\[
\star X_{\mu_1\ldots \mu_{p+1}} = \frac{1}{(p+1)!} \eta_{\mu_1\ldots \mu_{p+1} \nu_1\ldots \nu_{p+1}} X_{\nu_1\ldots \nu_{p+1}}
\]

and \(\eta_{\mu_1\mu_2 \ldots \mu_{2p+2}}\) is the covariantly constant volume form in the Minkowski space-time. Note, that \(\epsilon_{i_1 i_2 \ldots i_{2p+1}} := \eta_{0 i_1 \ldots i_{2p+1}}\). In terms of electric and magnetic fields defined in (A.3)–(A.6) the field equations (A.7)–(A.8) have the following form:

\[
\partial_0 B_{i_1\ldots i_p} = (-1)^p \frac{1}{p!} \epsilon_{i_1\ldots i_p k j_1\ldots j_p} \nabla_k E_{j_1\ldots j_p} ,
\]

\[
\nabla_i B_{i_1\ldots i_p} = 0 ,
\]

\[
\partial_0 D_{i_1\ldots i_p} = \frac{1}{p!} \epsilon_{i_1\ldots i_p k j_1\ldots j_p} \nabla_k H_{j_1\ldots j_p} ,
\]

\[
\nabla_i D_{i_1\ldots i_p} = 0 ,
\]

where \(\nabla_k\) denotes the covariant derivative on \(\Sigma\) compatible with the metric \(g_{kl}\) induced from \(\mathcal{M}^{2p+2}\). The Lévi-Civita tensor density satisfies \(\epsilon_{2\ldots 2p+1} = \sqrt{g}\), with \(g = \det(g_{kl})\).

**B General p-form theory without matter**

**B.1 Generating formula**

For an arbitrary \(p\) the formulae (2.4) and (3.1) generalize to:

\[
- \delta \mathcal{L} = (\partial_\nu \mathcal{G}^{\nu \mu_1\ldots \mu_p} \delta A_{\mu_1\ldots \mu_p}) = (\partial_\nu \mathcal{G}^{\nu \mu_1\ldots \mu_p} \delta A_{\mu_1\ldots \mu_p} + \mathcal{G}^{\nu \mu_1\ldots \mu_p} \delta (\partial_\nu A_{\mu_1\ldots \mu_p}) .
\]

The formula (B.1) implies the following definition of “momenta”:

\[
\mathcal{G}^{\mu_1\ldots \mu_{p+1}} = -(p+1)! \frac{\partial \mathcal{L}}{\partial F_{\mu_1\ldots \mu_{p+1}}} .
\]

Moreover, (B.1) generates dynamical (in general nonlinear) field equations

\[
\partial_\nu \mathcal{G}^{\nu \mu_1\ldots \mu_p} = - J^{\mu_1\ldots \mu_p} ,
\]
where the external $p$-form current reads:

$$ J^{\mu_1...\mu_p} = p! \frac{\partial L}{\partial A_{\mu_1...\mu_p}}. \tag{B.4} $$

Let us start with $J = 0$ and discuss a general $p$-form charged matter in Appendix C. To obtain the Hamiltonian description of the field dynamics let us integrate equation (B.1) over a $(2p+1)$-dimensional volume $V$ contained in the constant-time hyperplane $\Sigma$:

$$ - \delta \int_V L = \int_V \partial_0 (G^{0i_1...i_p} \delta A_{i_1...i_p}) + \int_{\partial V} G^{\mu_1...\mu_p} \delta A_{\mu_1...\mu_p}, \tag{B.5} $$

where $\perp$ denotes the component orthogonal to the $2p$-dimensional boundary $\partial V$. To simplify our notation let us introduce the spherical coordinates on $\Sigma$:

$$ x^{2p+1} = r, \quad x^A = \varphi_A; \quad A = 1,2,...,2p, \tag{B.6} $$

where $\varphi_1, \varphi_2, ..., \varphi_{2p}$ denote spherical angles (to enumerate angles we shall use capital letters $A, B, C, ...$). The metric tensor $g_{ij}$ is diagonal and has the following form:

$$ g_{11} = r^2 \sin^2 \varphi_2 \sin^2 \varphi_3 ... \sin^2 \varphi_{2p}, $$

$$ g_{22} = r^2 \sin^2 \varphi_3 \sin^2 \varphi_4 ... \sin^2 \varphi_{2p}, $$

$$ \vdots $$

$$ g_{2p-1,2p-1} = r^2 \sin^2 \varphi_{2p-1} \sin^2 \varphi_{2p}, $$

$$ g_{2p,2p} = r^2 \sin^2 \varphi_{2p}, $$

$$ g_{rr} = r^2. \tag{B.7} $$

Therefore, the volume form

$$ \Lambda_p = \sqrt{\det(g_{ij})} = r^{2p} \sin \varphi_2 \sin^2 \varphi_3 ... \sin^{2p-2} \varphi_{2p-1} \sin^{2p-1} \varphi_{2p}. \tag{B.8} $$

Let $V$ be a $(2p+1)$-dim. ball with a finite radius $R$. In such a coordinate system the formula (B.3) takes the following form:

$$ \delta \int_V L = (-1)^p \int_V \partial_0 (D^{i_1...i_p} \delta A_{i_1...i_p}) - (-1)^p \int_{\partial V} D^{rA_1...A_p} \delta A_{0A_1...A_p} $$

$$ - \int_{\partial V} g^{rB_1...B_p} \delta A_{B_1...B_p}, \tag{B.9} $$

where

$$ D_{i_1...i_p} = G_{i_1...i_p} \tag{B.10} $$

denotes the $p$-form electric induction density. Now, performing the Legendre transformation between induction $p$-form $D^{i_1...i_p}$ and $\dot{A}_{i_1...i_p}$ one obtains the following Hamiltonian formula:

$$ - \delta H_{\text{can}} = (-1)^p \int_V \left( \dot{D}^{i_1...i_p} \delta A_{i_1...i_p} - \dot{A}_{i_1...i_p} \delta D^{i_1...i_p} \right) - (-1)^p \int_{\partial V} D^{rA_1...A_p} \delta A_{0A_1...A_p} $$

$$ - \int_{\partial V} g^{rB_1...B_p} \delta A_{B_1...B_p}, \tag{B.11} $$

where the canonical Hamiltonian

$$ H_{\text{can}} = \int_V \left( (-1)^p D^{i_1...i_p} \dot{A}_{i_1...i_p} - L \right). \tag{B.12} $$
Equation (B.11) generates an infinite-dimensional Hamiltonian system in the phase space \( P_p = (\mathcal{D}^{i_1...i_p}, A_{i_1...i_p}) \) fulfilling Dirichlet boundary conditions for the \( p \)-form potential \( A_{i_1...i_p} \). From the mathematical point of view this is the missing part of the definition of the functional space. The Hamiltonian structure of a general nonlinear \( p \)-form electrodynamics described above is mathematically well defined, i.e. a mixed Cauchy problem (Cauchy data given on \( \Sigma \) and Dirichlet data given on \( \partial V \times \mathbb{R} \)) has a unique solution (modulo gauge transformations which reduce to the identity on \( \partial V \times \mathbb{R} \)).

The presence of a \( p \)-dependent sign \((-1)^p\) follows from the \( p \)-dependence of the corresponding symplectic form:

\[
\Omega_p = \int_V \delta G^{01...i_p} \wedge \delta A_{i_1...i_p} = (-1)^{p+1} \int_V \delta \mathcal{D}^{i_1...i_p} \wedge \delta A_{i_1...i_p} .
\]

(B.13)

There is, however, another way to describe the Hamiltonian evolution of fields in the region \( V \). Let us perform the Legendre transformation between \( \mathcal{D}^{rA_2...A_p} \) and \( A_{0A_2...A_p} \) at the boundary \( \partial V \). One obtains:

\[
-\delta \mathcal{H}_{sym} = (-1)^p \int_{\partial V} (\mathcal{D}^{i_1...i_p} \delta A_{i_1...i_p} - \dot{A}_{i_1...i_p} \delta \mathcal{D}^{i_1...i_p}) + (-1)^p \int_{\partial V} pA_{0A_2...A_p} \delta \mathcal{D}^{rA_2...A_p} \]

\[
- \int_{\partial V} \mathcal{G}^{B_1...B_p} \delta A_{B_1...B_p} ,
\]

where the new “symmetric” Hamiltonian

\[
\mathcal{H}_{sym} = \mathcal{H}_{can} - (-1)^p \int_{\partial V} p \mathcal{D}^{rA_2...A_p} A_{0A_2...A_p} .
\]

(B.14)

Observe, that formula (B.14) defines the Hamiltonian evolution but on a different phase space. In order to kill boundary terms in (B.11) one has to control on \( \partial V \): \( \mathcal{D}^{rA_2...A_p} \) (instead of \( A_{0A_2...A_p} \)) and \( A_{B_1...B_p} \). We stress that from the mathematical point of view both descriptions are equally good and an additional physical argument has to be given if we want to choose one of them as more fundamental.

### B.2 Canonical vs. symmetric energy

Now, let us discuss the relation between \( \mathcal{H}_{can} \) and \( \mathcal{H}_{sym} \) defined by (B.12) and (B.15) respectively. One has:

\[
\mathcal{H}_{sym} = \mathcal{H}_{can} - (-1)^p \int_{\partial V} p \mathcal{D}^{rA_2...A_p} A_{0A_2...A_p} = \mathcal{H} - (-1)^p \int_V p \partial_k (\mathcal{D}^{k_2...i_p} A_{0i_2...i_p})
\]

\[
= \int_V \left\{ (-1)^p \mathcal{D}^{i_1...i_p} \dot{A}_{i_1...i_p} - \mathcal{L} + (-1)^p p \left( A_{0i_2...i_p} \partial_k \mathcal{D}^{k_2...i_p} + \mathcal{D}^{i_1...i_p} \partial_k A_{0i_2...i_p} \right) \right\}
\]

\[
= \int_V \left( \frac{1}{p!} \mathcal{D}^{i_1...i_p} E_{i_1...i_p} - \mathcal{L} \right) ,
\]

(B.16)

where the \( p \)-form electric field is defined by

\[
E_{i_1...i_p} = F_{i_1...i_p0} = \partial_{[i_1} A_{i_2...i_p]0} .
\]

(B.17)

Therefore, \( \mathcal{H}_{sym} = \int_V T^{00}_{sym} \) and \( \mathcal{H}_{p} = \int_V T^{00}_{can} \), where

\[
T^{\mu\nu}_{sym} = \frac{1}{p!} F^{\mu_{i_1...i_p}} G^{\nu_{i_1...i_p}} + g^{\mu\nu} \mathcal{L} ,
\]

(B.18)

\[
T^{\mu\nu}_{can} = \partial^\mu A^{\nu_{i_1...i_p}} G^{\nu_{i_1...i_p}} + \partial^\nu A^{\mu_{i_1...i_p}} G^{\mu_{i_1...i_p}} + g^{\mu\nu} \mathcal{L} .
\]

(B.19)

Obviously, for the Maxwell theory one has:

\[
\mathcal{H}_{sym}^{Maxwell} = \frac{1}{2p!} \int_V \left( \mathcal{D}^{i_1...i_p} D_{i_1...i_p} + B^{i_1...i_p} B_{i_1...i_p} \right) .
\]

(B.20)
B.3 Reduction of the generating formula

Any geometrical object on \((2p + 1)\)-dimensional hyperplane \(\Sigma\) may be decomposed into the radial and tangential components, e.g. a \(p\)-form gauge potential \(A_{i_1...i_p}\) decomposes into the radial \(A_{rA_2...A_p}\) and tangential \(A_{A_1...A_p}\). On each sphere \(2(p-1)\)-dimensional sphere \(S^{2p}(r)\), \(A_{rA_2...A_p}\) defines a \((p-1)\)-form whereas \(A_{A_1...A_p}\) a \(p\)-form. Now, any \(p\)-form on \(S^{2p}(r)\) may be further decomposed into “longitudinal” and “transversal” parts:

\[
A_{A_1...A_p} = \nabla \left[ A_{r} u_{A_2...A_p} \right] + \epsilon_{A_1...A_p B_1...B_p} \nabla B_1 u_{B_2...B_p},
\]

(B.21)

where \(\epsilon_{A_1...A_p B_1...B_p}\) denotes the Lévi-Civita tensor density on \(S^{2p}(r)\) such that \(\epsilon_{12...2p} = \Lambda_p\). Both \(u\) and \(v\) are \((p-1)\)-forms on \(S^{2p}(r)\). Now, using (B.21) and integrating by parts one gets:

\[
\int_V \left( \delta A_{i_1...i_p} - \hat{A}_{i_1...i_p} \delta \mathcal{D}^{i_1...i_p} \right) = \int_V \left\{ \left( \delta \mathcal{D}^{rA_2...A_p} \right) \delta u_{A_2...A_p} - \hat{u}_{A_2...A_p} \delta \left( \partial_r \mathcal{D}^{rA_2...A_p} \right) \right\} + \epsilon_{A_1...A_p B_1...B_p} \left\{ \nabla B_1 \mathcal{D}^{A_1...A_p} \right\} \delta v_{B_2...B_p} - \hat{v}_{B_2...B_p} \delta \left( \nabla B_1 \mathcal{D}^{A_1...A_p} \right),
\]

(B.22)

where we have used the Gauss law

\[
\nabla A_1 \mathcal{D}^{A_1...A_p} = -\partial_r \mathcal{D}^{rA_2...A_p}.
\]

Moreover, due to (B.21)

\[
\int_{\partial V} \mathcal{G}^{rA_1...A_p} \delta A_{A_1...A_p} = \int_{\partial V} \left\{ \left( -1 \right)^p p! \nabla B_1 \mathcal{D}^{rA_1...A_p} \delta v_{B_2...B_p} - \hat{v}_{B_2...B_p} \delta \left( \nabla B_1 \mathcal{D}^{rA_1...A_p} \right) \right\}.
\]

(B.24)

In deriving (B.24) we have used

\[
\nabla A_1 \mathcal{G}^{A_1...A_p r} = - \mathcal{D}^{rA_2...A_p},
\]

(B.25)

which follows from the field equations \(\nabla A_1 \mathcal{G}^{A_1...A_p r} + \partial_0 \mathcal{G}^{0A_2...A_p r} = 0\). Now, taking into account (B.22) and (B.24) the generating formula (B.14) may be rewritten in the following way:

\[
- \delta H_{sym} = \left( -1 \right)^p \int_V \left\{ \left[ \delta \mathcal{D}^{rA_2...A_p} \delta (p A_{r} A_2...A_p - p \partial_r u_{A_2...A_p}) \right] - \left[ \left( \epsilon_{A_1...A_p B_1...B_p} \nabla B_1 \mathcal{D}^{A_1...A_p} \right) \delta u_{B_2...B_p} \right] - \hat{u}_{B_2...B_p} \delta \left( \epsilon_{A_1...A_p B_1...B_p} \nabla B_1 \mathcal{D}^{A_1...A_p} \right) \right\} + \left( -1 \right)^p \int_{\partial V} \left( p A_{0A_2...A_p} - p! \hat{u}_{A_2...A_p} \right) \delta \mathcal{D}^{rA_2...A_p} + \int_{\partial V} \left( \epsilon_{A_1...A_p B_1...B_p} \nabla B_1 \mathcal{G}^{A_1...A_p} \right) \delta v_{B_2...B_p}.
\]

(B.26)

Note, that although \(A_r A_2...A_p\), \(A_{0A_2...A_p}\) and \(u_{A_2...A_p}\) are manifestly gauge-dependent, the combinations \(p A_{r} A_2...A_p - p \partial_r u_{A_2...A_p}\) and \(p A_{0A_2...A_p} - p! \partial_0 u_{A_2...A_p}\) are gauge-invariant. To simplify our consideration we choose the special gauge \(u \equiv 0\), i.e. a \(p\)-form \(A_{A_1...A_p}\) on \(S^{2p}(r)\) is purely transversal. This condition, due to (B.21), may be equivalently rewritten as

\[
\nabla A_1 \mathcal{G}^{A_1...A_p r} = 0.
\]

(B.27)
Let us choose the same condition for the radial part

\[ \nabla A_2 A^r A_2 \ldots A_p = 0. \]  

(B.28)

Assuming (B.27) and (B.28) one may show [16]

\[ \Delta_{p-1} A^r B_2 \ldots B_p = (-1)^{p+1} \frac{r^2}{pp!} \epsilon A_1 \ldots A_p B_1 \ldots B_p \nabla B_1 B_2 A_1 \ldots A_p, \]  

(B.29)

where

\[ \Delta_{p-1} = (p-1)! \left[ r^2 \nabla_A \nabla^A - (p^2 - 1) \right] \]  

(B.30)

equals to the Laplace-Beltrami operator on co-exact \((p-1)\)-forms on \(S^{2p}(1)\) [16]. In the same way

\[ B^r A_2 \ldots A_p = - \frac{p!}{r^2} \Delta_{p-1} v^r A_2 \ldots A_p. \]  

(B.31)

Finally, introducing

\[ Q_1^{A_2 \ldots A_p} = D^r A_2 \ldots A_p, \]

(B.32)

\[ Q_2^{A_2 \ldots A_p} = B^r A_2 \ldots A_p, \]

(B.33)

\[ \Pi^1_{B_2 \ldots B_p} = \frac{r}{p!} \Delta_{p-1} \left( \epsilon A_1 \ldots A_p B_1 \ldots B_p \nabla B_1 B_2 A_1 \ldots A_p \right), \]

(B.34)

\[ \Pi^2_{B_2 \ldots B_p} = - \frac{r}{p!} \Delta_{p-1} \left( \epsilon A_1 \ldots A_p B_1 \ldots B_p \nabla B_1 B_2 A_1 \ldots A_p \right), \]

(B.35)

the formula (B.26) simplifies to

\[- \delta H_{sym} = \int_V \Lambda_p \left\{ \left( \Pi^1_{A_2 \ldots A_p} \delta Q_1^{A_2 \ldots A_p} - \dot{Q}_1^{A_2 \ldots A_p} \delta \Pi^1_{A_2 \ldots A_p} \right) \right. \]

\[ + \left. (-1)^{p+1} \left( \Pi^2_{A_2 \ldots A_p} \delta Q_2^{A_2 \ldots A_p} - \dot{Q}_2^{A_2 \ldots A_p} \delta \Pi^2_{A_2 \ldots A_p} \right) \right\} \]

\[ + \int_{\partial V} \Lambda_p \left( \chi_{A_2 \ldots A_p} \delta Q_1^{A_2 \ldots A_p} + \chi_{A_2 \ldots A_p} \delta Q_1^{A_2 \ldots A_p} \right), \]  

(B.36)

where we introduced the boundary momenta:

\[ \chi_{A_2 \ldots A_p} = (-1)^p \frac{p}{r} A_0 A_2 \ldots A_p, \]

(B.37)

\[ \chi_{B_2 \ldots B_p} = - \frac{r}{p!} \Delta_{p-1} \epsilon A_1 \ldots A_p B_1 \ldots B_p \nabla B_1 G^r A_2 \ldots A_p. \]  

(B.38)

In the formula (B.36) we have introduced:

\[ Q_l^{A_2 \ldots A_p} := g A_2 B_2 \ldots g A_p B_p Q_l^{B_2 \ldots B_p}, \]

(B.39)

\[ \Pi^l_{A_2 \ldots A_p} := g A_2 B_2 \ldots g A_p B_p \Pi^l_{B_2 \ldots B_p}, \]

(B.40)

for \(l = 1, 2\). For the Maxwell theory

\[ H_{sym}^{Maxwell} = \frac{1}{2(p-1)!} \int_V \Lambda_p \sum_{l=1}^2 \left\{ \frac{1}{r^2} Q_l^{A_2 \ldots A_p} Q_l A_2 \ldots A_p - \Pi^l A_2 \ldots A_p \Delta_{p-1} \Pi^l A_2 \ldots A_p \right\} \]

\[ - \frac{1}{r^{2p-1}} \partial_r \left( r^{2p-1} Q_l A_2 \ldots A_p \right) \Delta_{p-1} \partial_r \left( r Q_l A_2 \ldots A_p \right), \]  

(B.41)

and, therefore, the boundary momenta read:

\[ \chi^l_{A_2 \ldots A_p} = \frac{1}{r^{2p-1}} \Delta_{p-1} \partial_r \left( r^{2p-1} Q_l A_2 \ldots A_p \right), \quad l = 1, 2. \]  

(B.42)
B.4 Summary

The quasi-local reduced variables \((Q_l^{A_2\ldots A_p}, \Pi_l^{A_2\ldots A_p})\) fulfill the following conditions \([10]\):

\[
\nabla A_2 Q_l^{A_2\ldots A_p} = \nabla^A A_2\Pi_l^{A_2\ldots A_p} = 0 , \quad l = 1, 2 ,
\]

which follow from the Gauss laws. In the geometric language it means that \(\ast Q_l\) and \(\ast\Pi_l\) are closed \((p + 1)\)-forms on \(S^{2p}(r)\) (\(\ast\) denotes the Hodge dual defined via \(\epsilon^{A_1\ldots A_pB_1\ldots B_p}\)). They are gauge-invariant and contain the entire information about \(p\)-forms \(D\) and \(B\).

The “symmetric” dynamics defined by \((B.36)\) corresponds to the Dirichlet boundary condition for positions \(Q_l\) whereas the “canonical” dynamics corresponds to the Dirichlet conditions for \(\chi^1_{A_2\ldots A_p}\) and \(Q_2\). But Dirichlet condition for \(\chi^1_A\) is equivalent to the Neumann condition for \(\partial_r D^r A_2\ldots A_p\)

\[
\int_{\partial V} \Lambda_p Q_1^{A_2\ldots A_p} \delta \chi^1_{A_2\ldots A_p} = \int_{\partial V} r \Delta_{p-1}^{-1} Q_1^{A_2\ldots A_p} \delta (\partial_r D^r A_2\ldots A_p) .
\]

(C.44)

\[8.4\] General \(p\)-form theory with matter

Now, consider a \(p\)-form electromagnetism interacting with the charged matter field \(\Phi\) (for simplicity let \(\Phi\) be a complex \((p - 1)\)-form). In the presence of charged matter the Lagrangian generating formula \([19]\) has to be replaced by:

\[
\delta \mathcal{L} = \partial_{\rho} (G_{\mu_1\mu_2\ldots \mu_p} \delta A_{\mu_1\mu_2} + \mathcal{P}_{\mu_1\mu_2\ldots \mu_p} \delta \Phi_{\mu_2\ldots \mu_p}) ,
\]

(C.1)

where the matter “momentum”

\[
\mathcal{P}_{\mu_1\mu_2\ldots \mu_p} = -p! \frac{\partial \mathcal{L}}{\partial (\partial_{[\mu_1} \Phi_{\mu_2\ldots \mu_p]})} .
\]

(C.2)

Because \(\mathcal{L}\) should define a gauge-invariant theory let us assume that there is a group of gauge transformations \(U_\Lambda\) parameterized by a \(p\)-form \(\Lambda\) acting in the following way: \(A \to A + d\Lambda\) and \(\Phi \to U_\Lambda(\Phi)\).

Now, the target space of the matter field \(\Phi\) may be reparameterized \(\Phi = (\varphi, U)\) in such a way that a \((p - 1)\)-form \(U\) is gauge invariant and a \((p - 1)\)-form \(\varphi\) is the phase undergoing the following gauge transformation: \(\varphi \to \varphi + \Lambda\). For the (complex) \((p - 1)\)-form one has: \(U_{\mu_1\ldots \mu_{p - 1}} := [\Phi_{\mu_1\ldots \mu_{p - 1}}]\) and \(\varphi_{\mu_1\ldots \mu_{p - 1}} = \text{Arg} \Phi_{\mu_1\ldots \mu_{p - 1}}\). Therefore, the matter part in \((C.3)\) may be rewritten as follows:

\[
\mathcal{P}_{\mu_1\mu_2\ldots \mu_p} \delta \Phi_{\mu_2\ldots \mu_p} = J_{\mu_1\ldots \mu_p} \delta \varphi_{\mu_2\ldots \mu_p} + p^{\mu_1\ldots \mu_p} \delta U_{\mu_2\ldots \mu_p} .
\]

(C.3)

Gauge invariance of the theory means that the gauge dependent quantities, i.e. \(A\) and \(\varphi\), enter into \(\mathcal{L}\) via the gauge-invariant combinations only:

\[
\mathcal{L} = \mathcal{L}(F_{\mu_1\ldots \mu_{p + 1}}, D_{\rho} \varphi_{\mu_1\ldots \mu_{p - 1}}, U_{\mu_1\ldots \mu_{p - 1}}, \partial_{\rho} U_{\mu_1\ldots \mu_{p - 1}}) ,
\]

(C.4)

where

\[
D_{\rho} \varphi_{\mu_1\ldots \mu_{p - 1}} := \frac{1}{p !} \partial_{[\rho} \varphi_{\mu_1\ldots \mu_{p - 1}]} - A_{\nu\mu_1\ldots \mu_{p - 1}}
\]

(C.5)

denotes a covariant derivative of \(\varphi_{\mu_1\ldots \mu_{p - 1}}\). This implies, that the momentum \(J_{\mu_1\ldots \mu_p}\) canonically conjugated to \(\varphi_{\mu_1\ldots \mu_p}\) is equal to the electric current

\[
J_{\mu_1\ldots \mu_p} = -p ! \frac{\partial \mathcal{L}}{\partial (\partial_{[\mu_1} \varphi_{\mu_2\ldots \mu_p]})} = p ! \frac{\partial \mathcal{L}}{\partial A_{\mu_1\ldots \mu_p}} = -\partial_{\rho} G_{\mu_1\ldots \mu_p} .
\]

(C.6)
Now, instead of (B.9) one has
\[
\delta \int_V \mathcal{L} = \int_V \partial_0 \left\{ (-1)^p D^{i_1 \ldots i_p} \delta A_{i_1 \ldots i_p} + (-1)^p \rho^{i_1 \ldots i_{p-1}} \delta \varphi_{i_1 \ldots i_{p-1}} - \pi^{i_1 \ldots i_{p-1}} \delta U_{i_1 \ldots i_{p-1}} \right\} \\
- \int_{\partial V} \left\{ (-1)^p D_{\rho} A_{i_1 \ldots i_p} + \rho^{A_{i_1 \ldots i_p}} \delta A_{i_1 \ldots i_p} + (-1)^p (p-1) \pi^{A_{i_1 \ldots i_p}} \delta \varphi_{A_{i_1 \ldots i_p}} + (\partial_0 A_{i_1 \ldots i_p}) \delta \varphi_{i_1 \ldots i_{p-1}} \right\},
\]
with \( \rho^{i_1 \ldots i_{p-1}} := J^{i_1 \ldots i_{p-1}} \) (it defines a \((p-1)\)-form charge density on \((2p+1)\)-dim. hyperplane \( \Sigma \)) and \( \pi^{i_1 \ldots i_{p-1}} := \rho^{B_{i_1 \ldots i_{p-1}}} \). Now, to pass to the Hamiltonian picture one has to perform the following Legendre transformations between: 1) \( D \) and \( \dot{A} \), 2) \( \rho \) and \( \dot{\varphi} \), 3) \( \pi \) and \( \dot{U} \) in the volume \( V \), and between 4) \( D^r \) and \( A_0 \), 5) \( \rho^r \) and \( \varphi_0 \) and 6) \( \pi^r \) and \( U_0 \) at the boundary \( \partial V \). One obtains the following generalization of (B.14):
\[
- \delta \mathcal{H}_{sym} = \int_V \left\{ (-1)^p \left( \dot{D}^{i_1 \ldots i_p} \delta A_{i_1 \ldots i_p} - \dot{A}_{i_1 \ldots i_p} \delta D^{i_1 \ldots i_p} \right) \\
+ \left( (-1)^p \left( \dot{\rho}^{i_1 \ldots i_{p-1}} \delta \varphi_{i_1 \ldots i_{p-1}} - \dot{\varphi}_{i_1 \ldots i_{p-1}} \delta \rho^{i_1 \ldots i_{p-1}} \right) \\
- \left( \dot{\pi}^{i_1 \ldots i_{p-1}} \delta U_{i_1 \ldots i_{p-1}} - \dot{U}_{i_1 \ldots i_{p-1}} \delta \pi^{i_1 \ldots i_{p-1}} \right) \right\} \\
- \int_{\partial V} \left\{ (-1)^p A_{i_1 \ldots i_p} \delta D^r A_{i_1 \ldots i_p} + G^r A_{i_1 \ldots i_p} \delta A_{i_1 \ldots i_p} + (-1)^p (p-1) \varphi_{A_{i_1 \ldots i_p}} \delta \rho^r A_{i_1 \ldots i_p} \\
- \left( J^r A_{i_1 \ldots i_p} \delta \varphi_{A_{i_1 \ldots i_p}} - (p-1) U_{A_{i_1 \ldots i_p}} \delta \pi^r A_{i_1 \ldots i_p} - p^r A_{i_1 \ldots i_p} \delta U_{A_{i_1 \ldots i_p}} \right) \right\},
\]
where the “symmetric” Hamiltonian of the interacting electromagnetic field and the charged matter represented by \( \Phi \) reads:
\[
\mathcal{H}_{sym} = \int_V \left\{ (-1)^p D^{i_1 \ldots i_p} \dot{A}_{i_1 \ldots i_p} + (\partial_0 \pi^{i_1 \ldots i_{p-1}} \delta U_{i_1 \ldots i_{p-1}} - \pi^{i_1 \ldots i_{p-1}} \delta U_{i_1 \ldots i_{p-1}} - \mathcal{L} \\
- \partial_0 \left[ (-1)^p \rho^{i_1 \ldots i_{p-1}} \delta \varphi_{i_1 \ldots i_{p-1}} \right] \right\},
\]
Now, using
\[
\partial_0 D^{i_1 \ldots i_p} = \rho^{i_1 \ldots i_p}, \tag{C.10}
\]
one gets the following formula for \( \mathcal{H}_{sym} \):
\[
\mathcal{H}_{sym} = \int_V \left\{ \frac{1}{p!} D^{i_1 \ldots i_p} E_{i_1 \ldots i_p} + (\partial_0 \pi^{i_1 \ldots i_{p-1}} \delta U_{i_1 \ldots i_{p-1}} - \pi^{i_1 \ldots i_{p-1}} \delta U_{i_1 \ldots i_{p-1}} - \mathcal{L} \\
+ \partial_0 \left[ (p-1) \pi^{i_1 \ldots i_p} \delta U_{i_1 \ldots i_p} \right] \right\}. \tag{C.11}
\]
Moreover, due to (C.10), we may rewrite the dynamical part for \( \varphi \) in (C.8) as follows:
\[
\int_V \left( \dot{\rho}^{i_1 \ldots i_p} \delta \varphi_{i_1 \ldots i_p} - \dot{\varphi}_{i_1 \ldots i_p} \delta \rho^{i_1 \ldots i_p} \right) = \int_V \left( \dot{D}^{i_1 \ldots i_p} \delta \varphi_{i_1 \ldots i_p} + (\partial_0 \pi^{i_1 \ldots i_p}) \delta D^{i_1 \ldots i_p} \\
+ \int_{\partial V} \left( \dot{D}^r A_{i_1 \ldots i_p} \delta \varphi_{A_{i_1 \ldots i_p}} - \dot{\varphi}_{A_{i_1 \ldots i_p}} \delta D^r A_{i_1 \ldots i_p} \right). \tag{C.12}
\]
Now, the term \( \dot{D}^r A_{i_1 \ldots i_p} \) at the boundary may be easily eliminated by the field equations (C.6)
\[
\dot{D}^r A_{i_1 \ldots i_p} = (-1)^p (J^r A_{i_1 \ldots i_p} - \partial_A G^r A_{i_1 \ldots A_{i_1 \ldots i_p}}). \tag{C.13}
\]
Introducing hydrodynamical variables:
\[
V_{\mu_1 \mu_2 \ldots \mu_p} := -D_{\mu_1} \varphi_{\mu_2 \ldots \mu_p}, \tag{C.14}
\]
we may rewrite finally (C.8) as follows:

\[- \delta H_{\text{sym}} = \int_V \left\{ \left( -1 \right)^p \left( \delta D^{i_1 \ldots i_p} V_{i_1 \ldots i_p} - \delta \dot{V}_{i_1 \ldots i_p} \right) + \left( -1 \right)^{p+1} \delta \pi_{i_1 \ldots i_p} \right\} \]

\[- \int_{\partial V} \left\{ \left( -1 \right)^p V_{0 A_2 \ldots A_p} \delta D^{r A_2 \ldots A_p} + G^{r A_1 \ldots A_p} \delta V_{A_1 \ldots A_p} - \left( -1 \right)^{p+1} \pi_{r A_2 \ldots A_p} \right\} \],

\[\text{(C.15)}\]

i.e. \[(C.15)\] has exactly the same form as \[(B.14)\] with \(A\) replaced by the gauge-invariant \(p\)-form \(V\) and supplemented by the gauge-invariant canonical pair of \((p-1)\)-forms \((U, \pi)\) together with the boundary momenta: \((p-2)\)-form \(U_0\) and \((p-1)\)-form \(p^r\) on \(\partial V\). All gauge-dependent terms dropped out.

\section{D \ 2 potentials vs. reduced variables}

Let us introduce a second \(p\)-form gauge potential \(Z\) on \(\Sigma\) such that

\[D^{i_1 \ldots i_p} = \epsilon^{i_1 \ldots i_p j_1 \ldots j_p} \partial_{k} Z_{j_1 \ldots j_p}.\]  \[\text{(D.1)}\]

Assuming for \(Z\) the same gauge conditions as for \(A\), i.e.

\[\nabla A_1 Z^{A_1 \ldots A_p} = 0,\]
\[\nabla A_2 Z^{r A_2 \ldots A_p} = 0,\]

we have in analogy to \[(B.29)\]

\[\Delta_{p-1} Z^{r B_2 \ldots B_p} = \left( -1 \right)^{p+1} \frac{r^2}{p!} \epsilon^{A_1 \ldots A_p B_1 \ldots B_p} \nabla B_1 D^{A_1 \ldots A_p}.\]  \[\text{(D.4)}\]

Therefore, taking into account \[(B.34)-(B.35)\] one has:

\[\Pi^1_{B_2 \ldots B_p} = \left( -1 \right)^{p+1} \frac{r}{p} A_{r B_2 \ldots B_p},\]
\[\Pi^2_{B_2 \ldots B_p} = \left( -1 \right)^{p} \frac{r}{p} Z_{r B_2 \ldots B_p},\]

i.e. the entire gauge-invariant information about two \(p\)-forms \(Z\) and \(A\) on \(\Sigma\) is encoded into two complex \((p-1)\)-forms \(Q\) and \(\Pi\) on each \(S^{2p}(r)\).

\section*{Acknowledgements}

This work was partially supported by the KBN Grant no 2 P03A 047 15.
References

[1] C. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Co., San Francisco (1973).

[2] R. Haag, *Local Quantum Physics*, Springer-Verlag, Berlin (1996).

[3] C. Teitelboim, Phys. Lett. B 167 (1986) 63, 69; M. Henneaux, C. Teitelboim, Foundations of Physics 16 (1986) 593.

[4] R. Nepomechie, Phys. Rev. D 31 (1985) 1921.

[5] G. W. Gibbons, D. A. Rasheed, Nucl. Phys. B 454 (1995) 18.

[6] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, Phys. Lett. B 400 (1997) 80.

[7] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, Nucl. Phys. B 520 (1998) 179.

[8] S. Deser and C. Teitelboim, Phys. Rev. D 13 (1976) 1592.

[9] I. Bialynicki-Birula, *Nonlinear Electrodynamics: Variations on a Theme by Born and Infeld*, in *Quantization Theory of Particles and Fields*, (ed. B. Jancewicz and J. Lukierski, World-Scientific, 1983).

[10] J. Jezierski and J. Kijowski, Gen. Rel. Grav. 22 (1990) 1284.

[11] J. Kijowski and D. Chruściński, Gen. Rel. Grav. 27 (1995) 267.

[12] J. Kijowski, Gen. Rel. Grav. 29 (1997) 307.

[13] D. Chruściński, Rep. Math. Phys. 41 (1997) 13.

[14] P. Debye, *Dissertation*, Munich, 1908.

[15] H. S. Green and E. Wolf, Proc. Phys. Soc. A 66 (1953) 1129; C. J. Bouwkamp and H. B. G. Casimir, Physica 20 (1954) 539; C. H. Papas, *Theory of Electromagnetic Wave Propagation*, McGraw-Hill Book Company, 1965.

[16] D. Chruściński, Rep. Math. Phys. 45 (2000) 121.

[17] J. H. Schwarz and A. Sen, Nucl. Phys. B 411 (1994) 35.

[18] D. Chruściński, *Strong field limit of the Born-Infeld p-form electrodynamics*, hep-th/0005213.