A NOTE ON TIME-OPTIMAL PATHS ON PERTURBED SPHEROID

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ABSTRACT. We consider Zermelo’s problem of navigation on a spheroid in the presence of space-dependent perturbation \( W \) determined by a weak velocity vector field, \( |W|_h < 1 \). The approach is purely geometric with application of Finsler metric of Randers type making use of the corresponding optimal control represented by a time-minimal ship’s heading \( \varphi(t) \) (a steering direction). A detailed exposition including investigation of the navigational quantities is provided under a rotational vector field. This demonstrates, in particular, a preservation of the optimal control \( \varphi(t) \) of the time-efficient trajectories in the presence and absence of acting perturbation. Such navigational treatment of the problem leads to some simple relations between the background Riemannian and the resulting Finsler geodesics, thought of the deformed Riemannian paths. Also, we show some connections with Clairaut’s relation and a collision problem. The study is illustrated with an example considered on an oblate ellipsoid.

1. Problem and motivation. To begin with, we set up a landscape for our exposition which is created by navigation data \( (h, W) \), namely a background Riemannian metric \( h \) and acting vector field \( W \).

1.1. Why a spheroid? A spheroid is widely applied as a geometric model in navigation as well as geodesy, cartography, mechanics and engineering. Due to the combined effects of gravity and rotation the Earth’s shape is often approximated by an oblate spheroid. In particular, cartographic and geodetic systems for the Earth are based on a reference ellipsoid, for instance the current World Geodetic System. Satellite data have provided new measurements to define the best Earth-fitting ellipsoid and for relating existing coordinate systems to the Earth’s centre of mass. It is the approximate shape of many planets and celestial bodies and the quickly-spinning stars. Several moons of the Solar System approximate prolate spheroids in shape, though they are actually triaxial ellipsoids. The most common shapes for the density distribution of protons and neutrons in an atomic nucleus are spherical and spheroidal, where the polar axis is assumed to be the spin axis or direction of rotation.

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the spin angular momentum vector. Also, the current algorithms of the accurate and global navigational calculations correspond to the spheroidal geometric models (see [13, 25, 30, 31, 32, 22]). Furthermore, making use of the fact that a spheroid is the surface of revolution we observe some connections of our study to Clairaut’s relation and a collision problem. Recall that Clairaut’s theorem is a general law giving the surface gravity on a viscous rotating ellipsoid in equilibrium under the action of its gravitational field and centrifugal force. In particular, this synthesized physical and geodetic model of the Earth as an oblate spheroid.

1.2. Towards Zermelo’s problem and choice of perturbation. The most natural motion which is related to a spheroid is rotation. The surface itself is an ellipsoid of revolution. Thus, the choice of a rotational perturbation refers to various physical phenomena in the nature as well as the applications in engineering, in particular. When modelling such rigid motion in an embedded space, for example thought of fluid or gas, one can consider instead an object remaining at rest with action of simulated relative vector field acting on its spheroidal surface. Among the optimal trajectories with respect to different criteria and related to the connections (transfers) between the points of a space, the time-minimal ones in the presence of perturbing vector field $W$ are of special interest, e.g., in navigational analysis of the flows. Thus, it is natural to refer here to the Zermelo navigation problem [34, 35].

Our objective is to consider the problem on the ellipsoid by means of Finsler geometry (for more details, see, e.g., [6, 12]) in the context of the optimal control. Thus the key role plays the investigation of a ship’s heading (a steering direction), i.e., the angle $\varphi(t)$ which a vector of a true velocity (a.k.a. self-velocity, “through the water”) forms with a fixed direction, e.g. north, east. In dimension two and three, with the Euclidean background metric the implicit formulae for the optimal headings were given by E. Zermelo (1931) when the problem was formalized initially in [34, 35]. Shortly after, the research was followed in detailed analysis with application of the Hamiltonian formalism by C. Carathéodory [11] as well as, among others, T. Levi-Civita [23], K. Arrow [4], A. De Mira Fernandes [14], B. Mania [24]. Next, in optimal control the problem started to be approached with application of Pontryagin’s maximum principle. However, this note deals with the recent Finslerian treatment of the problem. The purely geometric analysis simplifies if we observe that it suffices to find the locally optimal solution on the tangent space. In other words, we aim to find the deformation of the unperturbed minimum-time trajectories represented by the Riemannian geodesics on a spheroidal “sea” $(\Sigma^2, h)$ and the corresponding “courses to steer” such that a ship following the resulting path completes its travel in the least time under the action of distributed “wind” $W$.

Note that a 2-level system in quantum mechanics (basically a spin degree of freedom with two states, up or down) can be viewed as a sphere (see [9, 7]). This also makes it attractive for illustrative purposes. The quantum mechanical theory mainly deals with higher-dimensional complex projective spaces, i.e., systems with more levels, which are not spheres any more. Nevertheless, the quantum mechanical wind in a 2-level system is simply an underlying rotation around a fixed axis. So, it is in line to a particular case of the current scenario, that is, with a flattening of a spheroid equal to 0. In other words, $a := 1$ in the below setting. One may observe the close analogies or convergence between the control of two-level systems, low-dimensional real navigation and at the same time space-time geometry (see [15]). Although it was necessary to provide some longer computations by means of Finsler
geometry in order to present the obtained solutions, we will also show that the relation between the corresponding time-optimal paths in the presence and absence of applied perturbation is in fact simple. By our approach to the Zermelo problem some purely geometric considerations on deformation of Riemannian geodesics leading to the resulting Randers geodesics can be expressed in the sense of the direct (inverse) geodetic problem applied to navigation in windy conditions, that is, without (with) counteracting perturbation \( W \), respectively (cf. [30, 18]). Such alternative point of view focusing on some essential navigational parameters, e.g. drift, heading, true velocity, resulting speed, is useful and provides some illustrative interpretations of the corresponding results.

As the computations in Finsler geometry can be very complicated, it is convenient to find the most efficient way in order to solve Zermelo’s problem. In this respect, one can observe that the Hamiltonian function associated with the Zermelo metric has a simple expression (see [36]). Namely, it is the addition of the Hamiltonian function of the Riemannian metric \( h \) and the one determined by the vector field \( W \). Note that the Randers metrics generated by the Euclidean \( (\mathbb{S}^n, h) \) and the infinitesimal isometries through the Zermelo navigation were initially introduced by Katok [17] and later investigated by Ziller [36] in the context of Hamiltonian systems [27, 7]. Next, the study was referred and extended by C. Robles in [27]. Remark that a rotational perturbation applied below in Section 3 in case of the 2-sphere had led to the significant results in purely geometric studies of the geodesics’ flows (see [36, 27]). It is also worth of mentioning that by rotation of a spheroid one can consider the case in which the wind is modeled by the Killing vector field. Thus, after adapting to the current setting the time-optimal paths could be described using the unit speed geodesics with application of Theorem 2 from [27] as done in the case of the 2-sphere in the presence of weak perturbation. However, our approach to the problem in this note differs, making use of the navigational investigation. Also, in order to verify the consistency of the results we compare the solution to an alternative treatment coming from Lagrangian. Consequently, we obtained the same resulting minimum-time trajectories including the corresponding optimal control.

2. Time-optimal paths of unperturbed spheroid. Let a pair \((M, h)\) be a Riemannian manifold, where \( h = h_{ij} dx^i \otimes dx^j \) is a Riemannian metric and the corresponding norm-squared of tangent vectors \( y \in T_x M \) is denoted by \( |y|_h^2 = h_{ij} y^i y^j = h(y, y) \). The unit tangent sphere in each \( T_x M \) consists of all tangent vectors \( u \) such that \( h(u, u) = 1 \); \( x = (x^1, \ldots, x^n) \). We assume that a ship proceeds with the constant speed relative to a perturbation, \( |u|_h = 1 \), and the problem of time-efficient navigation is treated for the case of a mild wind \( W \), namely \( |W|_h < 1 \) everywhere on \( M \). The time-optimal paths on Riemannian manifold in unperturbed scenario are represented by the geodesics of the corresponding Riemannian metric. Let \( \Sigma^2 \) be an ellipsoid embedded in the Euclidean space \( \mathbb{R}^3 \) referred to Cartesian coordinates \((x^i)\), with axes \( 2r, 2r, 2ar \). The parametrization of \( \Sigma^2 \) in the spherical coordinate system \((\rho, \phi, \theta)\) which we apply in the paper yields \( x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = ar \cos \theta \), where the azimuth \( \phi \in [0, 2\pi) \) and the inclination (colatitude) \( \theta \in [0, \pi] \). The parameter \( a > 0 \) determines the shape of an ellipsoid and as a consequence the flow of the geodesics on a spheroid which can be oblate \((0 < a < 1)\) or prolate \((a > 1)\). We equip the ellipsoid with a Riemannian metric induced by the Euclidean metric of \( \mathbb{R}^3 \). Computing the square distance \( ds^2 \) between two points \((x^i)\) and \((x^i + dx^i)\) of \( \Sigma^2 \) expressed in the terms of the spherical
coordinates \((\phi, \theta)\) on \(\Sigma^2\) leads to
\[
dx^2 = r^2 \sin^2 \theta (d\phi)^2 + r^2 (\cos^2 \theta + a^2 \sin^2 \theta)(d\theta)^2.
\]
Hence, the background Riemannian metric \(h = h_{ij} dx^i dx^j\) is given by
\[
h_{11} = (r \sin \theta)^2, \quad h_{22} = r^2 (\cos^2 \theta + a^2 \sin^2 \theta), \quad h_{12} = h_{21} = 0. \tag{1}
\]
Computing the nonzero Christoffel symbols of the first and second kind yields
\[
\Gamma_{111} = \Gamma_{211} = \frac{1}{2} r^2 \sin 2\theta, \quad \Gamma_{222} = \frac{1}{3} (a^2 - 1) r^2 \sin 2\theta \quad \text{and} \quad \Gamma_{12} = \cot \theta, \quad \Gamma_{21} = \frac{-\sin 2\theta}{2(a^2 \sin^2 \theta + \cos^2 \theta)} \quad \text{for more details, see Appendix.}
\]
For abbreviation, we continue to write \(\varepsilon\) for \(\cos^2 \theta + a^2 \sin^2 \theta\). We thus get the geodesic equations of \(\Sigma^2\) in the following form
\[
\begin{align*}
\ddot{\phi} \sin \theta + 2\dot{\phi} \dot{\theta} \cos \theta &= 0 \\
2\varepsilon \dot{\theta} + [(a^2 - 1) \dot{\theta}^2 - \dot{\phi}^2] \sin 2\theta &= 0,
\end{align*}
\tag{2}
\]
where the dots indicate derivatives with respect to \(t\). From now on, we assume that \(r := 1\) so the ellipsoid has the semiaxes \((1, 1, a)\). Without loss of generality we shall consider the oblate spheroid with \(a := \frac{3}{4}\) in the presented example.

Keeping in mind the assumption that the self-speed of a ship is unit, the form of the initial conditions in the case of the time-optimal solutions before perturbation become \(\phi(0) = \phi_0 \in [0, 2\pi), \theta(0) = \theta_0 \in (0, \pi)\) and
\[
\dot{\phi}(0) = u, \quad \dot{\theta}(0) = \pm \sqrt{\frac{1 - u^2 \sin^2 \theta_0}{\varepsilon_0}} \tag{3}
\]
or
\[
\dot{\phi}(0) = \pm \frac{1}{\sin \theta_0} \sqrt{1 - \varepsilon_0 u^2}, \quad \dot{\theta}(0) = v, \tag{4}
\]
where \(\varepsilon_0 = \cos^2 \theta_0 + a^2 \sin^2 \theta_0\). For instance, if a ship commences the passage from any point of the spheroid’s equator, i.e. \(\theta_0 = \frac{\pi}{2}\), then the condition which determines the coordinates of an arbitrary tangent vector is simplified to \(u^2 + a^2 v^2 = 1\). Let a starting point on \(\Sigma^2\) be determined by \(\phi_0 = 0\) and \(\theta_0 = \frac{\pi}{2}\) to show the flow of unperturbed Riemannian geodesics. Note that because of the form of the set spherical coordinates, if \(\phi(t) < 0\) then a path runs west (clockwise in a “top view”, i.e., from north pole), and if \(\theta(t) < 0\) then the path runs north. Similarly, for the positive values it goes east and south, respectively. In Figure 1 we show the Riemannian geodesics on the spheroid with \(a := \frac{3}{4}\) starting from \((0, \frac{\pi}{2})\), with the increments \(\Delta \phi_0 = \frac{\pi}{8}\) and their solutions \(x(t), y(t), z(t)\) for \((1, 0, 0) \in \mathbb{R}^3, t \leq 3\).

Due to flattening of a spheroid \(\Sigma^2\) the flows of the background geodesics differ clearly from those of closed Riemannian great circles of \(\mathbb{S}^2\), namely, \(\Sigma^2\) with \(a := 1\) intersecting in the antipodal points on the equator. This is also seen in Figure 1.

3. Velocity vector field of perturbation. To begin, we note that in general the most complete form of navigation data \(W\) in the Zermelo navigation problem involves a space- and time-dependent vector field of an arbitrary magnitude. Then the general form of perturbation \(W\) can be given as follows
\[
W = \frac{\partial}{\partial t} + W^i(t, x^i) \frac{\partial}{\partial x^i}.
\]
Remark that another effective way of solving the problem makes use of space-time geometry including a moving frame (cf. [9, 7, 8]). However, in the purely geometric approach under consideration we restrict our study to a space-dependent velocity vector field. Acting wind may have a rotational effect as well as a translational effect on a ship and this effect depends on a ship’s heading. Our aim is to consider the stationary wind, so this makes in fact the special case of Zermelo’s problem. We introduce the perturbing vector field in the form \( \tilde{W}(x,y,z) = \tilde{W}_1(x,y,z) \frac{\partial}{\partial x} + \tilde{W}_2(x,y,z) \frac{\partial}{\partial y} + \tilde{W}_3(x,y,z) \frac{\partial}{\partial z} \) which acts on spheroid \( \Sigma^2 \) seen as embedded in \( \mathbb{R}^3 \).

After transformation the form of \( W \) in the new base \((\phi, \theta)\) becomes

\[
W(\phi, \theta) = W^1(\phi, \theta) \frac{\partial}{\partial \phi} + W^2(\phi, \theta) \frac{\partial}{\partial \theta}.
\]

The condition \(|W|_h < 1\), where

\[
|W(\phi, \theta)|_h = \sqrt{(W^1(\phi, \theta) \sin \theta)^2 + (W^2(\phi, \theta))^2} + \varepsilon (W^3(\phi, \theta))^2,
\]

ensures that the resulting Randers metric \( F \) mentioned below will be strongly convex. This is the necessary condition as we study the problem via the construction of the Finsler metric (see [6, 12] for more details). Now, we apply the perturbation \( \tilde{W} = cy \frac{\partial}{\partial x} - cx \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial z} \) which determines a circulation of an angular speed \( c \), where \( c \) is a constant. Thus, we can say that \( W \) refers to rigid motions of the spheroid and describes a rotation about the \( z \)-axis. Let \(|c| < 1\). Then, finally \( W \) is given by

\[
W(\phi, \theta) = -c \frac{\partial}{\partial \phi} \quad \text{with} \quad |W(\phi, \theta)|_h = |c| \sin \theta < 1.
\]

Consequently, \(|W(\phi, \theta)|_h \in [0, c]\) and \(|W(\phi, \frac{\pi}{2})|_{\max} = c\) in the case of perturbing east wind \((c > 0)\) flowing along the ellipsoid’s equator. Without loss of generality let us consider east wind (clockwise when looking at the equator from the north pole of \( \Sigma^2 \)). In the example which follows below we shall take \( c := \frac{5}{7} \). Figure 2 shows the contour plot and the graph of the norm \(|W|_h\) of the acting mild perturbation, that is, the applied infinitesimal rotation.

What about the case where \(|W|_h \geq 1\)? First, note that originally the solutions to the problem with the Euclidean background as well as the Hamiltonian approach were not restricted by wind “force”, i.e., the norm, and they included the cases where \(|W| \geq 1\). In the further extensions of the Zermelo navigation in Finsler geometry the Kropina metric has been applied for stronger, i.e., critical, perturbation, \(|W|_h = \)
Figure 2. The contour plot and the graph of the norm $|W|_h$ in the case of the infinitesimal rotation as the acting mild perturbation, with $c := \frac{1}{2}$.

$f := 1 \ [20, 33]$. Also, a strong wind with $|W|_h > 1$ has been investigated recently, making use of a Lorentzian metric \[10\].

An additional reason for choosing a rotational vector field is its simpler form after transformation of the coordinates what makes the complex derivations easier and allows us to provide some symbolic computations to the final explicit form. In general, the computations in Finsler geometry are laborious, for instance, the spray coefficients even in dimension 2 what can be seen in this note, e.g., Appendix. Therefore, some numerical computing which regards implementations is also necessary and useful.

4. Time-optimal paths under mild perturbation. Under the influence of $W$ the paths of the shortest time are no longer the geodesics of the Riemannian metric $h := (\delta_{ij})$. Instead, they are the geodesics of the Finsler metric $F \ [6, 12]$. Z. Shen showed \[12, 29\] that the same phenomenon as in $\mathbb{R}^2$ holds for arbitrary Riemannian backgrounds in all dimensions. The following theorem lets us formally to combine the Randers geodesics with the optimality condition.

Theorem 4.1. \[6\] A strongly convex Finsler metric $F$ is of Randers type if and only if it solves the Zermelo navigation problem on a Riemannian manifold $(M, h)$, under the influence of a wind $W$ which satisfies $h(W,W) < 1$. Also, $F$ is Riemannian if and only if $W = 0$.

This means that Randers metrics may be identified with solutions to the navigation problem on Riemannian manifolds. This navigation structure establishes a bijection between Randers spaces $(M, F = \alpha + \beta)$ and pairs $(h, W)$ of Riemannian metrics $h$ and vector fields $W$, such that $|W|_h < 1$, on the manifold $M$. Recently we have generalized Theorem 4.1 slightly extending and developing the conformal solutions by introducing a conformal factor, thought of an additional degree of freedom in the Zermelo problem (cf. \[18, 15\]). In other words, this plays the role of varying in magnitude self-velocity of a navigating ship in the model. Also, note that basing on Theorem 4.1 we also developed the solutions to the generalized navigation problem on Hermitian manifolds in complex Finsler geometry with application of complex Randers \[2\] and complex Kropina \[3\] metric. In what follows, we shall
apply the resulting metric $F$ which correlates to the background geodesics on the ellipsoid $\Sigma^2$.

4.1. The resulting Randers metric $F$. A bijective relation between Finsler spaces of Randers type and pairs $(h, W)$ of Riemannian metrics $h = h_{ij}y^iy^j$ and acting vector fields $W$ on the manifold $M$ is a key. The resulting Randers metric is composed of the new Riemannian metric and 1-form [6], and is given by

$$F(x, y) = \sqrt{\frac{|h(W, y)|^2 + |y|^2}{\lambda} - \frac{h(W, y)}{\lambda}},$$

(8)

where $W_i = h_{ij}W^j$ and $\lambda = 1 - W^iW_i$. Hence,

$$F(x, y) = \sqrt{\frac{(h_{ij}W^i y^j)^2 + |y|^2 (1 - |W|^2) - h_{ij}W^i y^j}{1 - |W|^2}}.$$

(9)

The formula expresses the length of any tangent vector $y \in T_xM$, and so the minimum time of travel between two points on $M$.

Being in dimension two, after the coordinates’ transformation we denote the position $(x^1, x^2)$ by $(\phi, \theta)$, and expand arbitrary tangent vectors $u\frac{\partial}{\partial \phi} + v\frac{\partial}{\partial \theta}$ at $(x^1, x^2)$ as $(\phi; u, v)$ or $u\frac{\partial}{\partial \phi} + v\frac{\partial}{\partial \phi}$. Thus, adopting the notations in two dimensional case yields

$$F(\phi, \theta; u, v) = \sqrt{\frac{u^2h_{11} + v^2h_{22} - (uW^2 - vW^1)^2}{h_{11}h_{22} - uW^1h_{11} - vW^2h_{22}}}.$$

(10)

where $W^i = W^i(\phi, \theta)$. The norm of function $F$ measures travel time on $\Sigma^2$. In our problem the perturbed background Riemannian metric $h$ is the induced Euclidean metric on a spheroid $\Sigma^2$. From (10) we obtain the Randers metric $F = \alpha + \beta$ on the spheroid $(1, 1, a)$, with

$$\alpha(\phi, \theta; u, v) = \sqrt{\frac{u^2 \sin^2 \theta + \varepsilon [v^2 - (uW^2(\phi, \theta) - vW^1(\phi, \theta))^2 \sin^2 \theta]}{1 - (W^1(\phi, \theta) \sin \theta)^2 - \varepsilon (W^2(\phi, \theta))^2}},$$

(11)

$$\beta(\phi, \theta; u, v) = -\frac{uW^1(\phi, \theta) \sin^2 \theta + \varepsilon vW^2(\phi, \theta)}{1 - (W^1(\phi, \theta) \sin \theta)^2 - \varepsilon (W^2(\phi, \theta))^2}.$$

(12)

For $W(\phi, \theta) = -c\frac{\partial}{\partial \phi}$ the form of the resulting metric becomes

$$F(\phi, \theta; u, v) = \sqrt{\frac{v^2(1 - c^2 \sin^2 \theta) + u^2 \sin^2 \theta + cu \sin \theta}{1 - c^2 \sin^2 \theta}}.$$

(13)

Next, we use the partial derivatives of $L = \frac{1}{2}F^2$ to obtain the spray coefficients what comes below and consequently the final geodesic equations. Their solutions will determine the time-optimal paths locally, keeping in mind the fact that in general geodesics are not length-minimizing between any two points.
4.2. Equations of time-minimal paths. A spray on \( M \) is a smooth vector field on \( TM_0 = TM \setminus \{ 0 \} \) locally expressed in the standard form

\[
G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},
\]

where \( G^i = G^i(x, y) \) are the local functions on \( TM_0 \) satisfying \( G^i(x, \hat{x}y) = \hat{\lambda}^2 G^i (x, y) \), where \( \hat{\lambda} > 0 \). The spray is induced by \( F \) and the spray coefficients \( G^i \) of \( G \) given by [12]

\[
G^i = \frac{1}{4} g^{il} \left\{ \left[ F^2 \right]_{x^i y^l} (y^k - [F^2] x^l) \right\}
\]

are the spray coefficients of \( F \). In two dimensional case we express the spray coefficients \( G^1 := G(\phi, \theta; u, v) \) and \( G^2 := H(\phi, \theta; u, v) \), after adopting the spherical coordinates \((\phi, \theta)\), by

\[
G(\phi, \theta; u, v) = \frac{\partial^2 L}{\partial \phi^2} \frac{\partial L}{\partial \phi} - \frac{\partial L}{\partial \phi} \frac{\partial^2 L}{\partial \phi \partial v} - \frac{\partial L}{\partial \theta} \left( \frac{\partial^2 L}{\partial \phi \partial v} - \frac{\partial^2 L}{\partial \theta \partial u} \right),
\]

\[
H(\phi, \theta; u, v) = \frac{\partial^2 L}{\partial \theta^2} \frac{\partial L}{\partial \phi} - \frac{\partial L}{\partial \phi} \frac{\partial^2 L}{\partial \theta \partial u} - \frac{\partial L}{\partial \theta} \left( \frac{\partial^2 L}{\partial \phi \partial u} - \frac{\partial^2 L}{\partial \theta \partial u} \right).
\]

For a standard local coordinate system \((x^i, y^i)\) in \( TM_0 \) the geodesic equation for Finsler metric is expressed in the general form

\[
y^i + 2G^i(x, y) = 0.
\]

The solution curves in the problem are found by working out the geodesics of \( F \) given by (11) and (12). We apply the above formulae to compute the spray coefficients of the resulting Randers metric for the perturbation (7) (see Appendix). However, first, we obtain the spray coefficients \( G^1, H^1 \) of the new Riemannian term \( \alpha \) of \( F \) for acting infinitesimal rotation. The result, together with the corresponding Figure 28 presenting the geodesics of the new Riemannian metric \( \alpha \), is also included in Appendix. We can confirm by this note that investigations involving Randers spaces are in general difficult and finding solutions to the geodesic equations is laborious and not straightforward; cf. [9, 12]. As it takes a while if one computes some expressions manually, we created some programmes with use of Wolfram Mathematica 10.3-11.0 to generate the graphs and provide some numerical calculations when the complete symbolic ones cannot be obtained. The numerical schemes can give useful information studying the geometric properties of obtained solutions as is shown in the attached graphs.

The form of the initial conditions including the optimal control \( \varphi \) under perturbing vector field become \( \phi(0) = \phi_0 \in [0, 2\pi), \theta(0) = \theta_0 \in (0, \pi) \), and for the first derivative

\[
\dot{\phi}(0) = W^1(\phi_0, \theta_0) + \frac{\cos \varphi_0}{\sin \theta_0}, \quad \dot{\theta}(0) = W^2(\phi_0, \theta_0) - \frac{\sin \varphi_0}{\sqrt{\varepsilon_0}},
\]

where \( \varphi = \varphi(t) \) is the angle measured counterclockwise which the vector of the self-velocity forms with a parallel defined by a colatitude (inclination) \( \theta; \varepsilon_0 = \cos^2 \theta_0 + a^2 \sin^2 \theta_0 \). The relations (19) have been derived by direct consideration of the angular equations of motion including the angular representation of the components of a ship’s self-velocity and the background Riemannian metric. For more details in this regard, see Appendix.
When the families of the time-optimal paths coming from the same fixed point on the spheroid are considered, \( \varphi_0 \) plays the role of the parameter which rotates the unit tangent vector of unperturbed Riemannian geodesic. To visualize this, we set the increments, e.g., \( \Delta \varphi_0 = \pi/8 \). Then there are \( \left\lfloor \frac{2\pi}{\Delta \varphi_0} \right\rfloor \) corresponding geodesics presented in the included figures, where the square brackets mean an integer part. The time-efficient paths on the spheroid starting from \((0, \pi/2)\), with \( a := \frac{3}{4} \), \( \Delta \varphi_0 = \pi/8 \), \( t \leq 3 \) and divided into time segments \( t \in [0, 1) \) - red, \( t \in [1, 2) \) - blue, \( t \in [2, 3] \) - purple are presented in Figure 3.

Increasing time, one can also see that the resulting paths create transpolar and circumpolar flows on the spheroid. In Figure 4 we present the transpolar (blue) and circumpolar (red) Randers geodesics starting from \((\phi_0, \theta_0) = (0, \pi/2)\) in the presence of the rotational perturbation, with \( c := \frac{5}{7} \), the increments \( \Delta \varphi_0 = \pi/4 \), \( t \leq 50 \). Note that the transpolar Randers geodesics appear only if and the optimal heading is constant and \( \varphi \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \), so when a ship proceeds orthogonally to the rotational wind in the entire passage. Then a relative trajectory determined by a self-velocity \( \mathbf{u} \) traces the Riemannian geodesic which is a meridian. Thus, we have a special scenario, namely single-heading \((\varphi(t) = const.)\) Zermelo solution with varying in direction.
and magnitude resulting velocity $v$, i.e., the tangent vector to the Randers geodesic. Such situation corresponds to an analogous Euclidean (planar) example which, in particular, was emphasized by C. Carathéodory (cf. § 283 in [11]) in the initial research of the problem. More recently, other 2D simulations with the planar river-type perturbations by means of Finsler geometry have been presented in [19] in our parallel study on applications of the solutions of the Zermelo problem to search models in the context of their refinements.

Remark 1. The transpolar solutions to the Zermelo navigation problem in windy conditions given by (7) exist only with a constant optimal heading (a.k.a. a single-heading Zermelo solution, piecewise) such that $\varphi \in \{ \pi/2, 3\pi/2 \}$. Namely, if a ship proceeds orthogonally to the perturbation in the entire passage, i.e., $u \perp W$, so it follows a meridian in the motion relative to the “water” on $\Sigma^2$.

4.2.1. Consistency of the results. Although it is not our objective in this note to focus on the variational approach to the problem, we show briefly just the final equivalent system (20), under introduced assumptions, for comparison of the efficiency of the computations as well as confirmation that the solutions coming from both approaches are consistent. This includes the kinematic equations of motion (cf. (19) and Appendix) and the optimality condition which we obtained starting from the Euler-Lagrange equations. The result is

\[
\begin{align*}
\dot{\phi} &= \frac{\cos \varphi}{\sin \theta} - c \\
\dot{\theta} &= -\frac{\sin \varphi}{\sqrt{\varepsilon}} \\
\dot{\varphi} &= -\frac{\cos \theta \cos \varphi}{\sqrt{\varepsilon} \sin \theta}.
\end{align*}
\]

(20)

One can check that the last system gives the same Zermelo trajectories as presented above via Finsler geometry. Clearly, the efficiency in the sense of simplicity or, on the other hand, complexity of the computations, makes a significant difference, that is, with a considerable advantage of the non-Finslerian treatment (20). In addition, from the above worked out results one may test the computational times in the context of the implementations based on both approaches, for example in the potential applications in real-time computing. Remark that regarding some navigational applications it would be more convenient to adopt the orientation of the control $\varphi$ such that it is represented by the angle with respect to north determined by the ellipsoid’s meridians or to convert obtained heading to the corresponding azimuth.

4.3. Isochrones and comparison of $h$-, $\alpha$-, $F$- geodesics. The unit circle of $h$ in each tangent plane represents the destinations which are reached in one unit of time in the absence of background wind $W$. This circle is translated rigidly due to the action of the perturbation. Thus, the resulting indicatrices of $F$ are off-centered in comparison to the initial indicatrix and represent the loci of unit time destinations in windy conditions. The isochrones created by the vector field (7) with $c := \frac{2}{3}$ on the ellipsoid for $t \in \{1, 2, 3, 4\}$ are shown in Figures 5 and 6.

In the presence of background wind the Riemannian metric $h$ no longer gives the travel time along vectors, but a new metric $F$ on $T\Sigma^2$. Also, in Riemannian geometry two geodesics which pass through a common point in opposite directions necessarily trace the same curve. All reversible (symmetric) Finsler metrics have
Figure 5. The $F$-isochrones in the presence of the perturbing infinitesimal rotation, with $c := \frac{5}{7}$ for $t = 1$ (blue), $t = 2$ (red), $t = 3$ (purple), $t = 4$ (magenta) and the starting point in $(\phi_0, \theta_0) := (0, \frac{\pi}{2})$.

This property. However, the phenomenon does not extend to nonreversible (non-symmetric) settings; for more details, see, e.g., [27].

Figure 6. The $F$-isochrones on the spheroid $\Sigma^2$ under the perturbing infinitesimal rotation (7), with $c := \frac{5}{7}$, $t = \{1 \text{ (blue)}, 2 \text{ (red)}, 3 \text{ (purple)}, 4 \text{ (magenta)}\}$ and the starting point $(0, \frac{\pi}{2})$. Right: “top” view.

Much relevant information can be obtained directly from comparisons of the corresponding geodesics’ flows. In Figure 7 we can observe the solutions $x(t)$-blue, $y(t)$-red, $z(t)$-black in the absence (dashed) and in the presence (solid) of perturbation, with $\Delta \phi_0 = \frac{\pi}{8}$, $t \leq 3$ and the starting point $(1, 0, 0) \in \mathbb{R}^3$. The comparisons of the background Riemannian $h$- (blue) to the corresponding perturbed new Riemannian $\alpha$- (green; see also Appendix) and the resulting Randers $F$-geodesics (red) are shown in Figure 9. One can also check whether the background Riemannian geodesic passes through or omits the fixed points of the flow of the time-optimal paths (the Randers geodesics), for example, the ellipsoid’s poles (cf. Remark 1). For Randers metric $F$ expressed in terms of Riemannian metric $h$ and a vector field $W$ the relations between the spray coefficients of the metrics $F$, $\alpha$ and $h$ can be found in [6, 12].

4.4. Deformation of Riemannian paths, optimal control and velocities. A drift angle introduced below can represent a measure of the directional deformation of the Riemannian geodesics, thought of the minimum-time paths at calm sea ($W =$
Figure 7. Comparing the solutions $x(t)$ - blue, $y(t)$ - red, $z(t)$ - black in the absence (dashed) and the presence (solid) of the wind (7), with $\Delta \varphi_0 = \frac{\pi}{8}$ and the starting point $(1,0,0) \in \mathbb{R}^3$; $t \leq 3$.

Figure 8. The comparison of the perturbed (red) and unperturbed (blue) time-efficient paths starting from $(0, \frac{\pi}{2})$, with $\Delta \varphi_0 = \frac{\pi}{4}$, $t \leq 1$ (left) and $\Delta \varphi_0 = \frac{\pi}{8}$, $t \leq 3$ (middle and “top” view on the right).

Due to action of $W$. We mention here that the connection between Randers geodesic preserving the optimal heading and its Riemannian generator will be shown on further reading, i.e., Corollary 1.

4.4.1. Drift $\Psi$. We consider an angle $\Psi$ which shows the relation between the time-optimal trajectory represented by the resulting angle $\Phi$ (a.k.a. “course over ground”) and the corresponding optimal control $\varphi$, i.e., a heading (a.k.a. “true course”). $\Psi$ determines the angular difference between $\Phi$ and $\varphi$, so between the resulting and self-velocities, $v$ and $u$, respectively, in the sense of direction. This also yields the variation between the corresponding optimal trajectories in the presence and absence of perturbation $W$. Thus, $\Psi$ shows the effect of acting perturbation on the Zermelo navigation of a ship. In this sense we can say that $\Psi$ measures a directional deformation of the background Riemannian geodesics $h$ which due to the action of $W$ become the corresponding Finsler geodesics. Such angle denoted by $\Psi$ we will call a drift. Observe that $|\Psi| < \frac{\pi}{2}$, if $|W|_h < 1$. We apply the convention such that $\Psi$ is positive, if the perturbation pushes a navigating ship anticlockwise and negative, if it is perturbed clockwise on $\Sigma^2$, i.e., $\Psi = \varphi - \Phi$. By analogy, this corresponds to a perturbation taken into consideration in real marine navigation like a wind or a current (a stream). Then $\Psi$ is positive, if $W$ pushes a ship to starboard
side and negative, if to port side (the nautical terms for right and left side, respectively). However, remark that in navigation the angles referring to the absolute directions are taken clockwise from north, i.e., a meridian.

Making use of a dot product with reference to the velocities $\mathbf{u}$ and $\mathbf{v}$ yields

$$\cos |\Psi| = \frac{1}{(|\mathbf{u}|_h|\mathbf{v}|_h)^{-1}} (h_{11}u_\phi v_\phi + h_{22}u_\theta v_\theta).$$

Recall that $(\dot{\phi}_h \sin \theta_h)^2 + (\dot{\theta}_h)^2(\cos^2 \theta_h + a^2 \sin^2 \theta_h) = 1$ and observe that $\dot{\theta}_h = \dot{\theta}_F - W^2$, $\dot{\phi}_h = \dot{\phi}_F - W^1$, where $(\phi_h, \theta_h)$ stand for the solutions to the system of the Riemannian geodesics given by (2). Then, a drift $\Psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ referring to a time-optimal path on a spheroid $\Sigma^2$ with the semiaxes $(1, 1, a)$, $a > 0$, under the influence of a mild perturbation $W(\phi, \theta)$ with $|W(\phi, \theta)|_h < 1$, is given by

$$\cos \Psi = \frac{\dot{\phi} (\dot{\phi} - W^1(\phi, \theta)) \sin^2 \theta + \varepsilon \dot{\theta} (\dot{\theta} - W^2(\phi, \theta))}{\sqrt{\eta^2 + \varepsilon \dot{\theta}^2}},$$

where $(\phi(t), \theta(t))$ determine the geodesics of $F$ shown by (11) and (12); for abbreviation, $\eta = \dot{\phi} \sin \theta$. In fact, formula (21) can be used for any (non-optimal) resulting trajectory which comes from unperturbed path and acting vector field on an arbitrary spheroid. In particular, the time-optimal one, where the optimality refers to the applied Randers metric $F$. Equivalently, applying the law of cosines in the tangent plane $TS^2$ yields $|W|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos |\Psi|$. One sees immediately that $\cos \Psi = \frac{\lambda + |\mathbf{v}|^2}{2|\mathbf{v}|_h}$. Hence, this allows us to prove that

$$\cos \Psi = \frac{1 + \left[\dot{\phi}^2 - (W^1(\phi, \theta))^2\right] \sin^2 \theta + \varepsilon \left[\dot{\theta}^2 - (W^2(\phi, \theta))^2\right]}{2\sqrt{\eta^2 + \varepsilon \dot{\theta}^2}}. \quad (22)$$

Let us observe that $\Psi$ is positive, if $\varepsilon \dot{\theta} > 0$ and negative, if $\varepsilon \dot{\theta} < 0$ in the case of the perturbing infinitesimal rotation (7) applied in the example, $|c| < 1$. The result is

$$\Psi = \text{sgn}(\varepsilon \dot{\theta}) \arccos \left(\frac{\dot{\phi}(\dot{\phi} + c) \sin^2 \theta + \varepsilon \dot{\theta}^2}{\sqrt{\eta^2 + \varepsilon \dot{\theta}^2}}\right). \quad (23)$$

If we admit stronger windy conditions, namely $|W|_h \geq 1$, then $\Psi \in [-\pi, \pi]$. The investigations of $\varphi$ are therefore of interest. By direct consideration of the triangle of the linear velocities in $TS^2$ and basing on the Randers geodesics of $F$ given by (11) and (12) we find the optimal control $\varphi$ on the perturbed rotational ellipsoid.

4.4.2. Optimal control $\varphi$. The optimal control $\varphi$ on a spheroid $\Sigma^2$ of the semiaxes $(1, 1, a)$, $a > 0$, under the influence of an arbitrary mild perturbation $W(\phi, \theta)$ with $|W(\phi, \theta)|_h < 1$, is given by

$$\tan \varphi = \left(\frac{W^2(\phi, \theta) - \dot{\theta}}{\dot{\phi} - W^1(\phi, \theta)}\right) \frac{\sqrt{\varepsilon}}{\sin \theta}. \quad (24)$$

Formula (24) includes both wind components and excludes the solutions where $|\varphi(t)| = \frac{\pi}{2}$. Actually, having the optimal path in hand which is determined by the
Randers geodesics \((\phi, \theta)\), it is sufficient to apply shorter relations which bring the solutions for the entire range of \(\varphi \in [0, 2\pi)\), simply
\[
\cos \varphi = \left(\dot{\phi} - W^1(\phi, \theta)\right) \sin \theta \quad \text{or} \quad \sin \varphi = \left(W^2(\phi, \theta) - \dot{\theta}\right) \sqrt{\varepsilon}, \tag{25}
\]
where \((\phi(t), \theta(t))\) determine the geodesics of \(F\) defined by (11) and (12). Computing the control by the inverse function we take into account the quadrant in which the argument lies. It is sufficient to apply (25) with the wind components zeroed to obtain the optimal resulting angle \(\Phi\) ("course over ground"). So, we thus use the components of the tangent vector to the Randers geodesic which stand for the linear components of the resulting velocity. This makes it clear that, if \(W = 0\) then \(\Phi = \varphi\) and \(\Psi = 0\). Both angles are measured counterclockwise from a spheroid’s parallel of latitude determined by the corresponding colatitude (inclination) \(\theta\). A simple conversion can give the corresponding courses expressed as in navigation, namely taken clockwise from a meridian (north) like an azimuth.

4.4.3. Linear and angular velocities. The angular velocities \(\omega_\phi, \omega_\theta\) referring to the time-efficient paths determined by \(\phi(t), \theta(t)\) are related by \(\omega_\phi = \dot{\phi}, \omega_\theta = \dot{\theta}\). The corresponding components of the linear resulting velocity are represented by \(v_\phi = \dot{\phi} \sin \theta, v_\theta = \dot{\theta} \sqrt{\varepsilon}\) in the case of the unperturbed as well as perturbed scenario, where the pairs \((\phi, \theta)\) stand for the solutions to the corresponding Riemannian and Randers geodesic equations, respectively. Comparing the resulting speed, i.e.,
\[
|v|_h = \sqrt{\eta^2 + \varepsilon \theta^2} \tag{26}
\]
given as a function of time to the unit self-speed leads to the relevant information on when a perturbation acts against or with the navigating ship, that is, decreases or increases the resulting speed \(|v|_h\). Consequently, this influences the total time of travel. Substituting (19) in (26) and rearranging terms, we are thus led to the relation for \(|v(\varphi_0)|_h\). Square of the resulting speed given as the function of the initial control \(\varphi_0\), with departing point \((\phi_0, \theta_0)\) and in the presence of perturbation \(W\) on the spheroid \(\Sigma^2\) reads
\[
|v(\varphi_0)|_h^2 = \left(1 + (W^1(\phi_0, \theta_0) \sin \theta_0)^2 + 2W^1(\phi_0, \theta_0) \sin \theta_0 \cos \varphi_0ight.
\]
\[\left.\quad + \varepsilon_0 (W^2(\phi_0, \theta_0))^2 - 2 \sqrt{\varepsilon_0} W^2(\phi_0, \theta_0) \sin \varphi_0. \right) \tag{27}
\]

Figure 9. The corresponding background \(h\)-Riemannian (blue), new \(\alpha\)-Riemannian (green) and \(F\)-Randers (red) geodesics starting from \((0, \frac{\pi}{2})\), with \(\Delta \varphi_0 = \frac{\pi}{4}, t \leq 1\) (left) and with \(\Delta \varphi_0 = \frac{\pi}{8}, t \leq 3\) (middle: “side” view and right: “top” view).
With fixed initial steering one may find the spheroid’s parallel $\theta$ for which the speed is extreme or alternatively search for the corresponding heading when commencing the passage from a fixed point on $\Sigma^2$. One sees immediately that for $\tilde{W}(x, y, z) = (-y, x, z) \mapsto W(\phi, \theta) = (-c, 0)$ the resulting speed equals

$$|v(\phi_0)|_h = \sqrt{(c \sin \theta_0)^2 - 2c \sin \theta_0 \cos \phi + 1}$$

$$\theta_0 = \frac{\pi}{2} \begin{cases} 1 - c & (\min), \quad \phi_0 := 0 \\ 1 + c & (\max), \quad \phi_0 := \pi \end{cases} \quad (28)$$

The upper and lower limits of $|v_0|_h$ differ by at most $2|c|$. In the below presented example the initial resulting speed (black) as the function of the initial control angle $\phi_0 \in [0, 2\pi)$ is shown in Figure 15 as well as both, the angular and linear, speed changes during the ship’s passage in Figure 16.

4.5. Relation between Riemannian and Randers geodesics. Now, we ask which path a ship follows, if a navigator did not observe a weak current or mild breeze and so (s)he neglects the action of perturbation. Thus, (s)he still keeps the same heading as on the time-optimal track in the absence of perturbation. This is called a passive navigation which refers to first (direct) geodetic problem applied to navigation in windy conditions, i.e., without counteracting perturbation. Surprisingly, the ship still sails the time-optimal route, however represented by the corresponding Randers geodesic. This means it is sufficient to keep the same true course (heading) like there is no wind in order to proceed a time-optimal path in the presence of perturbation. Obviously, if the corresponding Riemannian and Finslerian paths do not intersect then such passive navigation brings a ship to another destination, i.e., position $(\phi, \theta)$ on $\Sigma^2$, which is different from the determined in the potential boundary conditions. However, then a navigator can follow in fact another corresponding Riemannian geodesic in motion related to the flowing surrounding sea (not the fixed ground) in order to arrive at desired destination in the least time.

It is not a common scenario from the point of view of optimal control and navigation as the most wanted knowledge in the Zermelo problem since its genesis is to obtain the optimal controls, so $\phi(t)$ in the two-dimensional case. Formalizing the findings we get

**Corollary 1.** Let $\phi$ and $(\phi, \theta)$, together with the lower indices $F$, $h$, be the control (the heading) of a ship and the points of the corresponding Randers and Riemannian geodesics in the presence and absence of rotational wind $W$ given by (7), respectively, on a spheroid $\Sigma^2$. Then, we have

$$\forall t \quad \phi_F(t) = \phi_h(t) \quad \text{and} \quad \phi_F(t) = \left\{ \begin{array}{ll} \phi_h(t) - ct & \text{for} \quad \phi_h(t) \geq ct \\ \phi_h(t) - ct + 2k\pi & \text{for} \quad \phi_h(t) < ct \end{array} \right. \quad (29)$$

or

$$\forall t \quad \phi_F(t) = \pi - \phi_h(t) \quad \text{and} \quad \phi_F(t) = \left\{ \begin{array}{ll} -\phi_h(t) - ct + 2\pi & \text{for} \quad \phi_h(t) \leq 2\pi - ct \\ -\phi_h(t) - ct + 2(k + 1)\pi & \text{for} \quad \phi_h(t) > 2\pi - ct \end{array} \right. \quad (30)$$

where $k = \left[ \frac{ct}{2\pi} \right] + 1$ and the square brackets mean the integer part. In other words, in particular, $h$-Riemannian geodesic on a spheroid, under a weak rotational wind, generates two $F$-Randers geodesics such that one preserves the optimal control and another has the supplementary optimal control, and vice versa.
Proof. Since \( \forall t \ \theta_F(t) = \theta_h(t) \) the first two relations seem to be the straightforward consequences of the equations of the rotational motion. Note that the motion is arbitrary so far, that is, not necessarily time-optimal. The equations of motion on \( \Sigma^2 \) are worked out in Appendix. By \( \dot{\theta} = W^2(\phi, \theta) - \frac{4\sin \phi}{\sqrt{1 - \sin^2 \theta}} \) applied to both, Randers and Riemannian, trajectories we get that \( \sin \phi_F = \sin \phi_h \) what leads to the forms of the first above relations referring to the control, i.e., the ship’s heading. This together with another equation of motion, i.e., \( \dot{\phi} = W^1(\phi, \theta) + \frac{\cos \phi}{\sin \theta} \) gives after integration the relations between the azimuths \( \phi_F \) and \( \phi_h \), without taking into consideration the periodicity. Clearly, if \( W^1 := c = \text{const.} \), so as in (7), then \( \int W^1(\phi, \theta) dt := -ct; W^2 := 0 \). Therefore, we have

\[
\begin{align*}
\varphi_F &= \varphi_h \\
\phi_F &= \phi_h - ct + C_1 \quad \text{or} \quad \phi_F &= -\phi_h - ct + C_2 \\
\theta_F &= \theta_h
\end{align*}
\]

where \( C_1, C_2 \) are the constants of integrations which vanish for both paths have the same starting point, with \( t = 0 \). In order to see that the motions are in fact optimal we check the optimality condition including the perturbation given by (7). In this regard, we make use of (20), i.e., \( \dot{\phi} = -\frac{\cos \theta \cos \varphi}{\sqrt{1 - \sin^2 \theta}} \). Both systems satisfy the optimality condition, so indeed the trajectories are time-minimal. Remark that alternatively one may also start the proof with the comparison of both, Riemannian and Randers, geodesic equations, which by Theorem 4.1 include the optimality and then obtain the relation \( \theta_F = \theta_h \) which we started with.

Finally, the coordinates \( \phi, \theta \) are spherical, so we include periodicity due to their limited range, i.e., \( \phi_F, \phi_h \in [0, 2\pi) \). This is done only for the first set if \( \phi_h(t) < ct \). If \( \phi_h(t) \geq ct \), then \( 0 \leq \phi_F(t) = \phi_h(t) - ct < 2\pi \). Similarly, in the second case only for \( \phi_h(t) > 2\pi - ct \) Note that \( -\phi_h \equiv 2\pi - \phi_h \). If \( \phi_h(t) \leq 2\pi - ct \), then also \( 0 \leq \phi_F(t) < 2\pi \). This is complemented by adding the term \( 2k\pi \), where \( k = \left[ \frac{ct}{2\pi} \right] + 1 \). Likewise, by the analogous considerations for the Randers geodesic with the supplementary heading we can indicate another Riemannian geodesic with the same optimal control. Such Riemannian geodesic also generates both Randers geodesics as the first one, so with the headings \( \pi - \varphi, \pi - (\pi - \varphi) = \varphi \). Thus, there are the pairs of Randers and Riemannian geodesics on \( \Sigma^2 \) which generate each other, with the same and the supplementary optimal controls, i.e., \( \varphi, \pi - \varphi \). \( \square \)

In addition, we give some interpretations of the above statements for the case of the Randers geodesic preserving the optimal control in windy conditions. Imagine two ships commence their travels from the same position on \( \Sigma^2 \). Then, one ship follows the Riemannian geodesic in the absence of perturbation and another the corresponding Randers geodesic in the presence of perturbation \( W \) given by (7). Consequently, both ships have the same colatitude \( \theta(t) \) in each moment \( t \), so they are positioned on the same parallel. What is interesting, both ships have the same optimal control for every \( t \), so the same headings in the entire passages. Therefore, the deformation of the time-optimal path due to action of the perturbation preserves the optimal control.

Also, the measure of directional deformation represented by a drift \( \Psi \) discussed in Section 4.4.1 can show the angular difference between the resulting courses on both trajectories. Moreover, the coordinates \( \phi \) of both geodesics are related explicitly as described above. This means that by the obtained simple relation we can generate
the Finsler (Randers) geodesics, thought of the time-optimal trajectories in the presence of perturbation, from the corresponding Riemannian geodesic, thought of a time-optimal trajectory in the absence of perturbation. With this knowledge in hand, we can solve different scenarios of the Zermelo navigation problem avoiding complex computations in purely geometric approach. Moreover, this relation can lead to the solution of the problem even simpler than via some Hamiltonians or system (20). For more general and deeper discussion on the Randers geodesics generated from the Riemannian geodesics by rotation, with the purely geometric approach, we refer the reader to [27].

4.5.1. Connection with Clairaut’s relation. The Riemannian geodesics on a surface of revolution can be characterized by Clairaut’s relation, which essentially says that the geodesics are curves of fixed angular momentum. Clearly, a spheroid is a surface of revolution, so Clairaut’s relation remains valid for its background $h$-geodesics. Recall that this classical result relates the distance from a point of $h$-geodesic to the axis of rotation and the angle between a tangent vector and a parallel (or a meridian). In other words, this expresses conservation of angular momentum about the axis of revolution when a particle slides along a $h$-geodesic in the absence of forces different from those that keep it on the surface (see, e.g., [26, p. 228]. A quick look at above findings including Corollary 1 leads to Corollary 2.

Let $\varphi$ be the optimal control (the heading) of a ship following the Randers geodesic, i.e., the time-minimal path, on a spheroid $\Sigma^2$ of the semiaxes $(1,1,a)$, under a rotational perturbation given by (7). Then

$$\frac{a \sin \theta}{\sqrt{\varepsilon}} |\cos \varphi| = \text{const.}$$

(31)

**Proof.** A radius of spheroid’s parallel of colatitude $\theta$ is given by $r_\theta = \frac{a}{\sqrt{\alpha^2 + \cos^2 \theta}} = \frac{a \sin \theta}{\sqrt{\varepsilon}}$, where $\varepsilon = \cos^2 \theta + a^2 \sin^2 \theta$. From Corollary 1 we know that the direction of the velocity $\mathbf{u}$ of a ship following the Randers geodesic preserving the optimal heading $\varphi(t)$ for every $t$ is equal to the direction of the tangent vector to the corresponding Riemannian $h$-geodesic. Taking into consideration that a spheroid is a rotational surface, we make use of Clairaut’s relation applied to $\Sigma^2$. Consequently, $r_\theta |\cos \varphi| = \text{const.}$, where $\varphi \in [0, 2\pi)$, which leads to (31). Note that the relation is valid for both Randers geodesics generated from the same Riemannian geodesic as mentioned in Corollary 1, i.e., with the preserved and supplementary optimal control, since $|\cos(\pi - \varphi)| = |\cos \varphi|$.

4.5.2. Collision problem. Following the above interpretation now we aim to pay some attention to collision problem (in the navigational sense) of two imaginary ships (objects) sailing the corresponding Riemannian and Randers geodesics on the same Riemannian sea $(M,h)$. If we consider the compact set, i.e., a spheroid, as the model of sea and both geodesics intersect, then one may ask whether the two ships will collide. To be precise, a collision means that

$$\exists t : \phi_F(t) = \phi_h(t) \quad \land \quad \theta_F(t) = \theta_h(t),$$

(32)

so that both ships arrive at the same point on $\Sigma^2$ in the same time. First, what about the potential collisions in the poles where the azimuth $\phi$ is arbitrary? To make the coordinates unique, one can use the convention that in these cases the arbitrary coordinates are equal to zero. Note that
Remark 2. A collision implies an intersection of the trajectories, but the converse is not true.

Remark that the collision problem (or anticollision, i.e., avoiding collision) in the navigational sense, for example, with reference to the satellites in motion on the spherical orbits or real marine ships navigating at full speed in a fog (using in reality the local Euclidean models) are non-trivial but well-known mathematical problems with the real-life applications; see [1]. However, here we combine this subject with the Zermelo problem. This means that anticollision with the special type of motions, i.e., time-minimal with the perturbing vector field involved, is of interest (cf. [16]). Now, recalling Corollary 1 in the current context yields

Corollary 3. A collision between the Riemannian geodesic and the generated Randers geodesic on $\Sigma^2$ occurs if

- $\phi_h < ct$ for the Randers geodesic preserving the optimal control,
- $\phi_h > 2\pi - ct$ for the Randers geodesic having the supplementary optimal control (the initial situation, with $t = 0$, we don't consider as a collision en route).

Hence, if $\phi_h(t) \geq ct$, then by (32) we get that $\phi_F(t) = \phi_h(t)$ if and only if $t = 0$, so the starting point for both paths. Therefore, recalling Corollary 1 we obtain the time of collision $t_{col}$ between the Riemannian geodesic and the Randers geodesic preserving the optimal control, i.e.

$$t_{col} = \frac{2k\pi}{c}, \quad \text{where} \quad k = \left\lceil \frac{ct}{2\pi} \right\rceil + 1. \quad (33)$$

The transpolar Randers geodesics pass through a pole of $\Sigma^2$, where the colatitude $\theta_F$ is equal to 0 or $\pi$. By Remark 1 we know that then the optimal control $\varphi(t)$ is constant and $u \perp W$. Therefore, the corresponding Riemannian $h$-geodesic is a meridian. Clearly, a Riemannian geodesic of $\Sigma^2$ which reaches a pole must be a meridian. By (20) and since $\forall t \theta_F(t) = \theta_h(t)$, there are the collisions in the poles, if time of ship voyage is sufficient to reach a pole. Briefly, there is always a collision in a pole, if the Riemannian (or Randers) ship arrives there. Observe also that the relative trajectories of the corresponding Riemannain and Finslerian ships with reference to each other are the arcs of the parallels of a spheroid. Summarizing, we thus get

Corollary 4. The transpolar solutions to the Zermelo navigation problem on $\Sigma^2$ in the presence of the rotational wind (7) have the collisions with the corresponding Riemannian $h$-geodesics which are the meridians. The collisions occur then in the poles of a spheroid.

In general, the problem becomes more advanced when the set is compact, e.g., $\Sigma^2$. Therefore, opposite to the Euclidean model used in the algorithms of different navigational anticollision systems, also the maximal distance between objects in motion becomes of interest. Here we touch the collision aspect referring to the trajectories coming from the Zermelo problem, so depending on the navigation data $(h, W)$. This generates some interesting questions in case of a spheroid, in particular. For instance, what $W$ causes that the Randers and Riemannian ships will occur in the antipodal points in the same time? Then, is there any periodicity and are the corresponding geodesics on $\Sigma^2$ closed (cf. [27])? Similarly, one can extend the problem and study the collision aspect of other Zermelo’s solutions which do not preserve the optimal control or the properties of the perturbing vector fields which
deform the Riemannian geodesics such that a ship following the resulting Finsler geodesic will avoid any collision with \( n \geq 2 \) other ships. This can be proposed for another research. In the below example we also added some remarks on the collision problem and the intersection of \( F^- \) and \( h^- \)geodesics including some visualisations in Figures 20 - 24.

Together with the introduced directional measure of deformation of the Riemannian geodesics, i.e., the drift \( \Psi \), one can also consider the linear measure of deformation which can be represented by the \( h^- \)geodesic distance between the corresponding Randers and Riemannian ships (geodesics). For simplicity, this can also be done by the colatitudinal distance \( d_\theta \) (so also the particular loxodromic distance), i.e., the length of shorter arc of the parallel on which both ships are positioned, since the second coordinates \( \theta \) are always equal. Recall from Corollary 2 that the radius of spheroid’s parallel of colatitude \( \theta \) is given by \( r_\theta = \frac{a}{\sqrt{a^2 + \cot^2 \theta}} = \frac{a \sin \theta}{\sqrt{\epsilon}} \). Thus, we get

\[
d_\theta(t) = \begin{cases} 
\tilde{d}_\theta & \text{if } \tilde{d}_\theta \leq \pi \\
2\pi - \tilde{d}_\theta & \text{if } \pi < \tilde{d}_\theta < 2\pi 
\end{cases},
\]

where \( \tilde{d}_\theta = \frac{a|\phi_F - \phi_h| \sin \theta}{\sqrt{\epsilon}} \); \( \phi_F, \phi_h \in [0, 2\pi) \). In the below example regarding the collision problem, for simplicity and without loss of generality we also make use of the Euclidean distance.

We close this section by mentioning that anticollision plays a very significant role, in particular, in the safety of marine and air navigation \([1, 16]\). During the motions of many ships at the same time in the neighbourhood of own ship a navigator aims to avoid collisions and pass other ships at distance not less than allowed, that is, the minimum safe distance. Thus, if there is no collision situation, then it is of interest to ask about the distance at the closest point of approach (abbreviated CPA) and time to the closest point of approach (abbreviated TCPA). So, practically the future dangerous situations occur, if both parameters together (CPA and TCPA) are exceeded, i.e., less than the minimum values and \( TCPA > 0 \). Note that in real navigation a marine radar with automatic radar plotting aid (abbreviated ARPA) capability can create tracks using radar contacts. Such anti-collision system, however with the flat Euclidean background, can calculate the tracked object’s course, speed and CPA, thereby knowing, if there is a danger of collision with the other ship or landmass.

5. **Example with oblate spheroid.** Now, we discuss the concrete example of the time-optimal trajectories with preserved optimal control on an oblate spheroid. We proceed assuming as above for the families of the geodesics that the initial point is determined by the coordinates \((\phi_0, \theta_0) := (0, \frac{\pi}{2}) \). Without loss of generality we can consider the oblate ellipsoid and fix the flattening by, e.g., \( a := \frac{3}{4} \).

5.1. **Optimal trajectories, courses to steer and speeds.** By (2) the background Riemannian geodesic equations of \( \Sigma^2 \) become

\[
\begin{align*}
\ddot{\phi} \sin \theta + 2\dot{\theta} \dot{\phi} \cos \theta &= 0 \\
\ddot{\theta} - \dot{\phi} \left( 7\dot{\theta}^2 + 16\dot{\phi}^2 \right) \sin 2\theta &= 0 
\end{align*}
\]

where \( \dot{\phi} = (7 \cos 2\theta + 25)^{-1} \). The initial conditions are complemented by \( \dot{\phi}(0) = \frac{1}{2} \), \( \dot{\theta}(0) = -\frac{2}{\sqrt{3}} \), so \( \left\| \left( \frac{1}{2}, -\frac{2}{\sqrt{3}} \right) \right\|_h = 1 \), since the ship true speed is constant (unit) at all
times in the standard formulation of Zermelo’s problem (cf. [18] in the generalized version). Thus, the ship commences the travel with the initial heading equal to the resulting course (“over ground”), i.e., \( \phi_0 = \Phi_0 = 60^\circ \) at unit speed “through the water”. This corresponds to the heading (true course) 030° ≈ NEbN (northeast by north) in the rhumb system with respect to true north on a compass rose applied to the oblate spheroid \( \Sigma^2 \). The Riemannian geodesic departing from \((0, \pi)\) and its planar \(xy\)-projection, with \( t \leq 25 \) are presented in Figure 10.

The final system of the Randers geodesic equations, thought of the time-optimal paths in windy conditions, is given by (43) and (44) in Appendix. To solve the system the initial conditions are then set according to (19), i.e., \( \phi_0 = 0, \theta_0 = \frac{\pi}{2}, \dot{\phi}_0 = -\frac{3}{14}, \dot{\theta}_0 = -\frac{2}{\sqrt{3}} \). The resulting geodesic creates the circumpolar time-efficient path on the ellipsoid, since \( 0 < \theta(t) < \pi \) for any \( t \). We leave it to the reader to find out what colatitude \( \theta \) (so the parallels) determines the northern (southern) boundary of this trajectory on \( \Sigma^2 \). One can immediately observe that then \( \Phi(t) = 0 \), so \( W \) and \( v \) are collinear. To be precise, the ship proceeds exactly “against the wind” then and \( |W|_h = 1 - |W|_h \). Much relevant information can be obtained directly from the graphs. Thus, we present the figure including the comparison of the background Riemannian geodesic and the corresponding Randers geodesic, with \( t \leq 7 \) and their \( xy\)-projections, with \( t \leq 20 \) in Figure 12 as well as both paths, with \( t \leq 25 \) in
Figure 13. The solutions in the absence (dashed) and in the presence (solid) of the perturbation in the base \((x, y, z)\) and the corresponding polar plot of the solutions in the base \((\phi, \theta)\), with \(t \leq 7\) are shown in Figure 14.

We can observe easily that the coordinates \(z(t)\) and \(\theta(t)\) of both paths do not differ under the action of the applied perturbation. The same holds for the meridian components of the linear velocities \(v_\theta\). This is the simple consequence of the fact that \(\tilde{W}_3(x, y, z) = 0\) which implies \(W^2(\phi, \theta) = 0\). The effect of the perturbing wind \(W(\phi, \theta) = -\frac{5}{7} \frac{\partial}{\partial \phi}\) causes that the time-optimal path in windy conditions is modified anticlockwise at the beginning as shown in Figure 11.

In addition, we also present the parametric plots of the solutions \((t \leq 7, \text{left})\), their first derivatives \((t \leq 3, \text{middle})\) and second derivatives \((t \leq 5\times\frac{3}{5}, \text{right})\), without (dashed) and with (solid) the action of the vector field (7) in Figure 18. The perturbation causes that the resulting course, i.e., the direction of the tangent vector to the Randers geodesic, at the departure \(\Phi_0 \approx 104^\circ\), while the corresponding optimal control \(\varphi_0 = 60^\circ\). Hence, the initial drift is negative, i.e., \(\Psi_0 \approx -44^\circ\).

Due to the fact that a ship commences the travel against the perturbation the resulting speed decreases. The graph of the linear and angular speeds in the absence (dashed) and presence (solid) of perturbation are presented in Figure 16. This shows the translation of the graphs referring to the angular speeds \(\dot{\phi}(t)\) by the vector \(\left[-\frac{5}{7}, 0\right]\) as the result of \(W^1(\phi, \theta) = -c = \text{const.}\) The resulting speed equals \(|v_0|_h = \frac{\sqrt{39}}{7} \approx 0.892\) at the departure. It is also lower than the ship self-speed (“through the water”) \(|u(t)|_h\) during the entire voyage, namely \(|v(t)|_h < 1\) for every \(t\). For comparison, if the ship starts the travel “with the current”, changing the sign of the component \(u\), i.e., \(\dot{\phi}(0) = -\frac{1}{2}\), then the optimal heading is \(\varphi_0 = 120^\circ\) what corresponds to \(330^\circ \approx \text{NWbN}\) (northwest by north) in the rhumb system with respect to true north on the spheroid’s compass rose. Recall here Corollary 1 and notice that this scenario gives the second Randers geodesic generated from the same Riemannian geodesic, i.e., with the supplementary optimal control, \(\varphi_0 := \pi - \frac{\pi}{3}\). Also, the tangent vector to the time-optimal path \((u, v) = (-\frac{17}{14}, -\frac{2}{\sqrt{3}})\)
Figure 13. Comparing the corresponding geodesics in the absence (blue, $h$-Riemannian) and in the presence (red, $F$-Randers) of the perturbing vector field $(7)$, with $c := \frac{5}{7}$. Right: “top” view; $t \leq 25$.

Figure 14. The Cartesian solutions in the absence (dashed) and presence (solid) of the applied rotational perturbation $(7)$ in the base $(x, y, z)$ (left) and the corresponding polar plot of the spherical solutions in the base $(\phi, \theta)$ (right); $t \leq 7$.

Figure 15. The initial resulting speed $|v_0|_h$ (black) as the function of the initial control angle $\phi_0 \in [0, 2\pi)$.

what causes that the resulting direction (“course over ground”) $\Phi_0 \approx 144.5^\circ$, so the drift $\Psi_0 \approx -24.5^\circ$. Consequently, by $(27)$ the resulting speed increases, i.e., $|v_0|_h = \frac{\sqrt{109}}{7} \approx 1.491 > 1 = |u|_h$.

We observe that the changes of $\varphi(t)$ are the same for both scenarios, in the absence and presence of perturbation, while the variations of $\Phi(t)$ differ. This is
confirmed by the investigations graphically shown in the figures. Namely, the resulting course (dashed blue) in Figure 17 and the optimal control (black) in Figure 19 overlap. Thus, the deformation preserves the optimal control as stated in Corollary 1. This means that there is the continuous nonzero wind effect generating the drift \( \Psi(t) \neq 0 \) in almost entire passage (excluding \( t : \varphi(t) = \Phi(t) \)). Thus, we let the perturbation to deflect our Riemannian route ("passive navigation"). This conclusion also arises clearly from (23) applied to the example. A different task would be, if the ship reacts continuously to neutralize the effect of the perturbation ("active navigation") and adjusts \( \varphi(t) \) so that it follows the preset (optimal) track "over ground" determined by \( \Phi(t) \). The time-optimal resulting course \( \Phi \) with and without the perturbation in the Cartesian graph and the corresponding polar plot are presented in Figure 17. These curves arise from (25). The control (the steering course) \( \varphi(t) \) according to (25) guarantees that the ship proceeds the Zermelo’s path on \( \Sigma^2 \). Finally, by (23) we thus obtain the graph of the drift angle \( \Psi(t) \) which is compared to the optimal control \( \varphi(t) \) and the optimal resulting course \( \Phi(t) \) in the perturbed scenario. This is shown in Figure 19. We observe that \(|\Psi(t)| \in [0, \approx 43.9^\circ] \), the

\[ \begin{align*}
\text{Figure 16.} \quad & \text{The linear (on the left) and angular speeds (on the right) as the functions of time, in the absence (dashed) and in the presence (solid) of the perturbation (7); the resulting linear speed is shown in black; } t \leq 7. \\
\text{Figure 17.} \quad & \text{The resulting time-optimal steering angle } \Phi(t) \text{ ("course over ground") without (dashed blue) and with (solid red) the wind (7) in the Cartesian plot (left) and the corresponding polar plot (right); } t \leq 7. 
\end{align*} \]
ship is drifted to port side (counterclockwise) and starboard side (clockwise) alternately during the voyage on the perturbed ellipsoidal sea $\Sigma^2$. Also, remark that by Corollary 2 the optimal course to steer $\varphi$ can be obtained making use of Clairaut’s relation (31).

5.2. Collision aspect. A separate subsection we dedicate to the collision problem (or anticollision) in the navigational sense. In general, two ships starting from the same position and following the Riemannian and Finsler geodesics can collide or pass each other clearly. What about the example? First, we obtain some quick, straightforward and useful information from analysis of the included graphs. A look at Figure 11 leads to the initial observation saying that the obtained Randers geodesic is circumpolar. If so, then the collisions in the poles like the transpolar geodesics is excluded due to Remark 4. Clearly, a ship does not sail with the constant control $|\varphi| = \pi/2$, since $\varphi_0 = \pi/3$ and $\varphi(t) \neq \text{const}$.

From the polar plot of the solutions presented in Figure 14 one might conclude that there is no collision for the graphs referring to $\theta(t)$ do not intersect (the blue curves). Recall that time is expressed by the radius in the polar plots. So, both ships have the colatitude (inclination) different in time. Indeed, there is a clear situation, i.e., no collision, however just for $t \leq 7$. What about further travel? If we increase time of motion up to, e.g., $t = 20$, then we observe two intersections of the trajectories (Riemannian - dashed blue and Randers - solid blue) in this period.
This is shown by the polar plot in Figure 20. Clearly, this means that the collisions exist. For simplicity, instead of spheroidal (h-geodesic) distance it is sufficient to calculate the Euclidean distance with the rectangular coordinates between the ships in their motions on the spheroid $\Sigma^2$ as shown in Figure 21. Theoretically, the maximum distance is $2r := 2$ (indicated by the green horizontal line), if the ships are positioned in the antipodal points of the equator. Obviously, the corresponding $h$-distance measured on the surface of the spheroid via a pole differs, but it doesn’t change the location of both ships in this case. On the other hand, the minimum distance (indicated by the red horizontal line) is 0 which means collision. In fact, both boundaries are achieved in the example, so the distance ranges all possible values between the “Riemannian ship” and the “Finslerian ship” on the spheroidal sea in windy conditions. This implies that $\exists t : \phi_F(t) = \phi_h(t)$.

Figure 21. The distance (Euclidean) between the “Riemannian ship” and “Finslerian ship”, with $t \leq 15$ (left) and $t \leq 50$ (right). The red horizontal line indicates collision and the green one - maximum distance, so the ships are positioned in the antipodal points of the spheroid’s equator. The time of the first collision is $t_{col1} = \frac{14\pi}{5} \approx 8.8$. 

The first collision (also, the seventh intersection) of the corresponding time-minimal trajectories, i.e., the background Riemannian (blue) and the Randers preserving the optimal control (red); $t \leq \frac{14\pi}{5} \approx 8.8$.

Next, we recall Corollary 1 in order to say more about the time and position of the first collision. From Remark 2 we know that a collision implies intersection of the trajectories but not vice versa. We thus get

$$\phi_F(t) = \phi_h(t) - ct + 2k\pi, \quad k := 1,$$

so

$$\phi_F(t) = \phi_h(t) - \frac{5}{7}t + 2\pi.$$ 

Hence, by (32) we obtain the time of the first collision of the corresponding Riemannian and Randers geodesic preserving the optimal control, that is

$$t_{col1} = \frac{14\pi}{5} \approx 8.8,$$

and the position is given by the coordinates $(\phi(t_{col1}), \theta(t_{col1})) = (\phi_{col1}, \theta_{col1}) \approx (146^\circ, 136^\circ)$. Note that the first collision is the seventh intersection of the trajectories at the same time since the beginning of the travel. This situation is illustrated in Figure 22. However, the first intersection of both geodesics after starting the travel ($t > 0$) occurs earlier, i.e., for $t \approx 1.66$ in position $(\phi, \theta) \approx (45^\circ, 36^\circ)$. The “Riemannian ship” arrives there first, i.e., with $t \approx 1$. The “Randers ship” achieves the position after $\Delta t \approx 0.65$ while the “Riemannian ship” is already on different meridian and, of course, on the same parallel, since we have $\forall \ t \ \theta_F(t) = \theta_h(t)$. This is presented in Figure 23 and the investigation of the coordinates for this scenario in Figure 24.

In addition, having in mind also the second relation in Corollary 1 we ask what about the collision situations between the ships following both Randers geodesics generated from the same Riemannian geodesic as well as the Riemannian geodesic and the Randers geodesic having the supplementary optimal control? Briefly, the first collision (and the first intersection at the same time) of both Randers paths occurs with $t \approx 3.3$ in position $(\phi, \theta) \approx (45^\circ, 128^\circ)$. This is shown in Figure 26. The first collision and also the first intersection of those trajectories: the Riemannian
Figure 23. The first intersection (no collision) of the Riemannian (blue) and Randers (red) geodesics coming from the starting point in \((0, \frac{\pi}{2})\); with \(t \leq 1\) (solid), \(t \leq 1.66\) (dashed). Right: “top” view.

Figure 24. The spherical coordinates \(\phi\) (blue), \(\theta\) (red) of the corresponding Riemannian (dashed, \(t \leq 1\)) and Randers (solid, \(t \leq 1.66\)) geodesics coming from the starting point in \((0, \frac{\pi}{2})\) till the first intersection (no collision) in the Cartesian (left) and the polar plot (right; time is expressed by length of the radius). The azimuth (longitude) of the first intersection \(\phi \approx 45^\circ\) is marked by the black line.

generator and the Randers one having the supplementary optimal heading, i.e., 
\(\varphi_F(t) = \pi - \varphi_h(t) := 120^\circ\), occurs with \(t \approx 2.06\) in position \((\phi, \theta) \approx (138^\circ, 53^\circ)\). This is presented in Figure 27. Also, all three solutions to the Zermelo problem (with and without the perturbation) are presented together in Figure 25 (left) as well as the distances between the ships following these three time-optimal paths (right); \(t \leq 15\). The distance equal to 0 (marked by the red horizontal line) means collision. On the other hand, the green horizontal line stands for the maximum possible Euclidean distance between two ships, so when they are positioned in the antipodal points of the equator.
Figure 25. Left: the Riemannian geodesic (blue) and its two generated Randers geodesics: with the preserved optimal control (red, “Randers_1”) and the supplementary optimal control (black, “Randers_2”). Right: the corresponding distances (Euclidean) between the ships: “Riemannian-Randers_1” (purple), “Riemannian-Randers_2” (dashed black) and “Randers_1-Randers_2” (blue); $t \leq 15$. The red horizontal line indicates collision and the green one the maximum distance, i.e., the ships are located in the antipodal points of the spheroid’s equator.

Figure 26. The first collision (also, the first intersection) in $(\phi, \theta) \approx (45^\circ, 128^\circ)$ of the time-minimal trajectories generated from the same Riemannian geodesic (blue), i.e. the Randers geodesic with the preserved optimal control (red) and with the supplementary optimal control (black). On the right, “top view”; $t \leq 3.3$.

6. Conclusions. The Zermelo navigation problem-based investigation including the optimal control, i.e., the heading (the course to steer) and the drift angle can be applied to an arbitrary oblate ($0 < a < 1$) or prolate ($a > 1$) ellipsoid $\Sigma^2$ as well as the 2-sphere ($a = 1$) in the presence of perturbing mild velocity vector field $W$, $|W|_h < 1$. In contrast to the variational approach the Finslerian solutions
require the restriction of a strong convexity which arises from the assumption of applied Proposition 1.1 [6] and in addition provides more complicated computations. The theorem (also, its recent generalized version, cf. [18, 2]) establishes the direct relation between the Randers geodesics and the time-optimal paths as the solutions to the navigation problem. Solving the system of the Randers geodesics of $F$ given by (11) and (12) or applying the formula for $\varphi$ yield the time-optimal control of a ship on the perturbed spheroid. In that way it also gives rise to obtain the spheroidal analogue of the classical formula of Zermelo for the optimal heading which implicitly solved the problem of finding the shortest time paths with the Euclidean background in $\mathbb{R}^2$ and $\mathbb{R}^3$ (cf. [34, 35, 11]).

In order to complement the presented solution of the Zermelo problem it would be reasonable to consider also the perturbation which satisfies $|W|_h \geq 1$. Note, however, that in general, if we introduce stronger wind globally defined on a manifold $M$, then we must have in mind the topological restrictions coming from the Poincaré-Hopf theorem (the Hairy ball theorem) [33, 20]. Namely, it follows that for any compact regular 2-dimensional manifold (the 2$\pi$-sphere) with non-zero Euler characteristic, so in particular $\Sigma^2$, there is no non-vanishing continuous tangent vector field. Also, the general obstacle which appears is to obtain the explicit solutions for the computational analyses involving Randers spaces are not straightforward. This is the significant disadvantage regarding the efficiency of computations and implementations in comparison to other approaches by Hamiltonian, making use of the Killing vector field, Lagrangian, space-time geometry including a moving frame (cf. [27, 8, 28, 21]). Nevertheless, we aimed to focus on the counterfactual solution to the problem via Finsler geometry including the investigation of some navigational quantities. Thus, we combined the Zermelo problem with the passive (active) navigation, i.e., without (with) counteracting the perturbation $W$, which refers to the direct (inverse) geodetic problem applied in navigation, respectively.
Supporting the research and the intuition by some numerical computations followed by the relevant schemes gave us useful information on the geometric properties of the time-efficient paths including the optimal control. This, followed by the symbolic expressions, paid our attention to some simple relations, e.g., Corollary 1 and 2, between the Riemannian and Randers geodesics, thought of locally the time-minimal trajectories in the absence and presence of applied perturbation, respectively. Consequently, making use of the obtained relations the efficiency of the solution of the Zermelo problem increases in the presented setting. In addition, by applying Clairaut’s relation to the problem on a spheroid $\Sigma^2$ one can deduce the azimuths of the unperturbed minimum-time paths, so the optimal control of a ship in windy conditions following the Randers geodesic route.

We conclude by noting that a number of non-Riemannian Randers metrics via the Zermelo problem that are either Einstein or Ricci-constant which include other surfaces of revolution as the examples, with Riemannian-Einstein navigation data $(h, W)$ can be found in Section 4 of [5]. Such research as well as our study presented in this note can also be considered wider. Namely, with generalization of the Zermelo navigation like the recent contributions which admit the extended navigation data $(h, W, \frac{|u|}{h})$; cf. [18, 20, 15, 3, 2]. This means that we can then introduce a space-dependent speed $|u| = f(x)$ in purely geometric solution or space- and/or time-dependent $|u| = f(x, t)$ with $W = W(x, t)$ in other approaches which include time-dependence; $x \in M$.

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Appendix A. Derivation of some geometric quantities.

**A1. The Christoffel symbols.** Computing the nonzero Christoffel symbols of the first kind, $\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial h_{ij}}{\partial x^k} + \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right)$, and the second kind, $\Gamma_{kj} = \frac{1}{2} (a^2 - 1) \varepsilon \sin^2 \theta$, with reference to the background Riemannian $h$-metric given by (1), yields

$$\Gamma_{121} = \Gamma_{211} = \frac{1}{2} r^2 \sin 2\theta, \quad \Gamma_{222} = \frac{1}{2} (a^2 - 1) r^2 \sin 2\theta,$$

and

$$\Gamma_{11}^1 = \cot \theta, \quad \Gamma_{11}^2 = -\frac{1}{2\varepsilon} \sin 2\theta, \quad \Gamma_{22}^2 = \frac{(a^2 - 1)}{2\varepsilon} \sin 2\theta,$$

where $\varepsilon = \cos^2 \theta + a^2 \sin^2 \theta$.

**A2. The new Riemannian metric $\alpha$.** Making use of the formulae (16) and (17) we compute the spray coefficients $G_\alpha, H_\alpha$ of the new Riemannian term $\alpha$ of the resulting metric $F$, with acting infinitesimal rotation given by (7). The result is

$$G_\alpha(\phi, \theta; u, v) = quv \left( c^2 + \csc^2 \theta \right) \cot \theta,$$

(36)
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$$H_{\alpha}(\phi, \theta; u, v) = \frac{\varrho^2 \cot \theta \csc^4 \theta \left[ c^2 (\zeta v^2 + u^2) \cos 2\theta - \zeta v^2 (c^2 - 2) - (c^2 + 2) u^2 \right]}{4 \left( a^2 + \cot^2 \theta \right)},$$  \hspace{1cm} (37)

where $\varrho = (\csc^2 \theta - c^2)^{-1}$ and $\zeta = c^2 + a^2 - 1$. Hence, the geodesic equations referring to the new Riemannian metric $\alpha$ become

$$\begin{cases}
\ddot{\phi} \sin \theta + 2 \dot{\phi} \dot{\theta} \left( c^2 + \csc^2 \theta \right) \cos \theta = 0, \\
\varrho^2 \left[ \zeta (c^2 \cos 2\theta - c^2 + 2) \ddot{\phi}^2 + (c^2 \cos 2\theta - c^2 - 2) \dot{\phi}^2 \right] \cot \theta \csc^4 \theta \cdot \\
+ 2 \left( a^2 + \cot^2 \theta \right) \ddot{\theta} = 0 \hspace{1cm} (38)
\end{cases}$$

The geodesics of the new Riemannian metric $\alpha$ starting from $(0, \frac{\pi}{2})$ in the increments $\Delta \phi_0 = \frac{\pi}{8}$ and $\Delta \theta_0 = \frac{\pi}{8}$ and $t \leq 3$ are presented in Figure 28. In the example plugging also the constant $c$ which determines the wind “force” and rearranging terms we are thus led to the relation for $L = \frac{1}{2}E^2$, that is

$$L(\phi, \theta, u, v) =$$

$$\frac{49}{8} \varrho^2 \left( \sqrt{-4 \left( 98u^2 - 71v^2 \right) \cos 2\theta + 392u^2 + \frac{175}{8} v^2 \cos 4\theta + \frac{3825v^2}{8} + 20u \sin^2 \theta} \right)^2,$$  \hspace{1cm} (39)

where $\varrho = (25 \cos 2\theta + 73)^{-1}$. Then, by (38) we obtain the concrete geodesic equations for the new Riemannian metric $\alpha$, namely

$$\begin{cases}
0 = \frac{\ddot{\phi}}{\dot{\phi}} + \frac{2 \dot{\phi} \left( 49 \csc^2 \theta + 25 \right) \cot \theta}{49 \csc^2 \theta - 25}, \\
0 = \ddot{\theta} + 2 \dot{\theta} \ddot{\phi}^2 \left[ 25 \left( 784\dot{\phi}^2 + 57\dot{\theta}^2 \right) \cos 2\theta - 96432\dot{\phi}^2 + 4161\dot{\theta}^2 \right] \sin 2\theta \hspace{1cm} (40)
\end{cases}$$

**Figure 28.** The geodesics of the new Riemannian metric $\alpha$ starting from $(0, \frac{\pi}{2})$, with the increments $\Delta \phi_0 = \frac{\pi}{8}$ (16 curves); $t \leq 3$. 
A3. Spray coefficients of the resulting Randers metric $F$. We present the spray coefficients $G$ and $H$ of the resulting metric (13) which induce the Randers geodesic equations for $W(\phi, \theta) = -c \frac{\partial}{\partial \phi}$. For abbreviation, we write

$$\mu = a^2 v^2 + u^2,$$
$$\psi = \sqrt{\left( \mu - c^2 a^2 v^2 \sin^2 \theta \right) \sin^2 \theta + \left( 1 - c^2 \sin^2 \theta \right) v^2 \cos^2 \theta},$$
$$\tau = 3 cu^2 \psi - ca^2 v^2 \psi + 3mu,$$
$$\nu = c^2 \cos 2\theta - c^2 + 2.$$

Hence, by (16) and (17) we obtain

$$G(\phi, \theta; u, v) = \frac{\partial v (cv + u) (\psi \csc^2 \theta + cu)^3 \cot \theta}{c^3 u (u^2 - 3a^2 v^2) + \mu \psi \csc^4 \theta + cr \csc^2 \theta + \frac{1}{4} \nu v^2 (\psi + 3cu \sin^2 \theta) \cot^2 \theta \csc^4 \theta},$$

$$H(\phi, \theta; u, v) = \frac{(\psi + cu \sin^2 \theta)^3 \sin 2\theta \left[c^4 (2a^2 - 1) v^2 \sin^4 \theta - c^2 \sin^2 \theta \right] - \left((3a^2 - 2) v^2 + u^2\right) - 2cu \psi + c^2 v^2 (c^2 \sin^2 \theta - 1) \cos^2 \theta + a^2 v^2 - u^2 - v^2}{\varepsilon \left(c^2 \sin^2 \theta - 1\right)^2 \left[\nu \sin^2 \theta \left(25 \cos^2 \theta - 6c \sin^2 \theta\right) - 4 \sin^2 \theta \left(c^3 u (u^2 - 3a^2 v^2) \sin^4 \theta + \mu \psi + cr \sin^2 \theta\right)\right]}.$$ (42)

Substituting the constants $a := \frac{3}{4}$, $c := \frac{5}{7}$ and rearranging terms, by (18) we are thus led to the final system of the Randers geodesic equations in the example, thought of the time-optimal paths in windy conditions (7), given by (43) and (44). Namely, abbreviating

$$\tau = \sqrt{\hat{\theta}^2(4544 \cos 2\theta + 350 \cos 4\theta + 7650) + 12544 \hat{\phi}^2 \sin^2 \theta},$$

the result is

$$\ddot{\phi} + \frac{\left(5\tau + 784\phi\right) \hat{\theta} \cot \theta \csc^2 \theta}{8(49 \csc^2 \theta - 25)} = 0,$$ (43)

$$\ddot{\theta} - \frac{\left(\tau + 80\phi \sin^2 \theta\right)^3 \left[392 \dot{\phi} \left(5\tau + \dot{\phi}(492 - 100 \cos 2\theta)\right)\right] + \hat{\theta}^2(53950 \cos 2\theta) + 4375 \cos 4\theta + 87303) \sin 2\theta}{2(7 \cos 2\theta + 25)(25 \cos 2\theta + 73)^2 \left[\tau + 240 \hat{\phi} \sin^2 \theta\right]} = 0.$$ (44)

A4. The form of the initial conditions on $\Sigma^2$. The relations in (19) have been derived by the direct consideration of the equations of the rotational motion on $\Sigma^2$ including the angular representation of the components of the ship velocity. The angular velocities $\omega_{\phi_h}, \omega_{\theta_h}$ referring to the Riemannian geodesics are related by $\omega_{\phi_h} = \dot{\phi}_h, \omega_{\theta_h} = \dot{\theta}_h$, and $\omega_{\phi_F}, \omega_{\theta_F}$ referring to the Randers geodesics by $\omega_{\phi_F} = \dot{\phi}_F, \omega_{\theta_F} = \dot{\theta}_F$. The pairs $(\phi_h, \theta_h)$ and $(\phi_F, \theta_F)$ stand for the solutions of the systems of
the time-optimal paths in the absence and presence of perturbation $W$, respectively. The components of the linear velocity are given by $u_{\phi_h} = \left| \left( \dot{\phi}_h(t), 0 \right) \right|_h = \dot{\phi}(t) \sin \theta_h$, $u_{\theta_h} = \left| \left( 0, \dot{\theta}_h(t) \right) \right|_h = \dot{\theta}_h(t) \sqrt{\varepsilon}$. On the other hand, making use of the heading $\varphi$, we have $u_{\phi_h} = |u|_h \cos \varphi$, $u_{\theta_h} = |u|_h \sin \varphi$. Comparing the above expressions and recalling that $|u|_h = 1 = \text{const}$, we have $\omega_{\phi_h} = \dot{\phi}_h = \frac{\cos \varphi}{\sin \theta_h}$, $\omega_{\theta_h} = \dot{\theta}_h = -\frac{\sin \varphi}{\sqrt{\varepsilon}}$. So, $u = \left( \dot{\phi}_h, \dot{\theta}_h \right) = \left( \frac{\cos \varphi}{\sin \theta_h}, -\frac{\sin \varphi}{\sqrt{\varepsilon}} \right)$. Note that the minus sign in front of the second coordinate $\dot{\theta}$ appears due to the applied orientation (the parametrization of $\Sigma^2$ given in Section 2). By $\omega_{\phi_F} = W^1 + \omega_{\phi_h}$ and $\omega_{\theta_F} = W^2 + \omega_{\theta_h}$ we thus get the form of the initial conditions given in (19). Namely, the starting point, i.e., $\phi(0) = \phi_0 \in [0, 2\pi)$, $\theta(0) = \theta_0 \in (0, \pi)$, and the tangent vector

$$
\dot{\phi}(0) = W^1(\phi_0, \theta_0) + \frac{\cos \varphi_0}{\sin \theta_0}, \quad \dot{\theta}(0) = W^2(\phi_0, \theta_0) - \frac{\sin \varphi_0}{\sqrt{\varepsilon_0}},
$$

where $\varphi = \varphi(t)$ is the angle (the heading) measured counterclockwise which $u$ (in other words, the tangent vector to the corresponding background Riemannian $h$-geodesic) forms with a parallel of the spheroid $\Sigma^2$ determined by the colatitude (the inclination) $\theta$; $\varepsilon_0 = \cos^2 \theta_0 + a^2 \sin^2 \theta_0$.

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