$C^{2,\alpha}$ REGULARITY OF FREE BOUNDARIES IN PLANAR OPTIMAL PARTIAL TRANSPORTATION

SHIBING CHEN, JIAKUN LIU, AND XU-JIA WANG

Abstract. As announced in [7], in this paper we establish the $C^{2,\alpha}$ regularity for free boundary in the optimal transport problem in dimension two. The main ingredient is to prove the uniform obliqueness at the free boundary, for which we adopt some techniques from [7]. The regularity in high dimensions is under investigation.

1. INTRODUCTION

Let $\Omega, \Omega^*$ be two disjoint, convex domains associated with densities $f$ and $g$ respectively. Let $c = \frac{1}{2} |x - y|^2$ be the cost function. Let $m$ be a positive number satisfying

$$m \leq \min \{ \int_{\Omega} f, \int_{\Omega^*} g \}. \quad (1.1)$$

The optimal partial transport problem asks what is the optimal transport plan that minimises the cost transporting mass $m$ from $(\Omega, f)$ to $(\Omega^*, g)$. A transport plan is described as a non-negative, finite Borel measure $\gamma$ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\gamma(A \times \mathbb{R}^n) \leq \int_A f(x) dx, \quad \gamma(\mathbb{R}^n \times A) \leq \int_A g(x) dx$$

for any Borel set $A$. An optimal transport plan minimises the following functional

$$\gamma \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) d\gamma(x, y). \quad (1.2)$$

In a remarkable paper [5], Caffarelli and McCann proved the existence and uniqueness of solutions to the optimal partial transport problem, they showed that the mass in $\Omega$ is either fixed or transported entirely to $\Omega^*$ and they called the portion transported the active region $U$. Then, they go further to show that the free boundary $\partial U \cap \Omega$ is $C^{1,\alpha}$ under the assumptions that $\Omega$ and $\Omega^*$ are both strictly convex and disjoint and that the densities are bounded from below and above. When the domains $\Omega$ and $\Omega^*$ are allowed to have overlap, in an important work [10][11] Figalli proved that away from the common region $\Omega \cap \Omega^*$, the free boundary is locally $C^1$, and this result was later improved by Indrei [12] to a local $C^{1,\alpha}$ regularity result away from the common region and up to a relatively closed singular set.

2000 Mathematics Subject Classification. 35J96, 35J25, 35B65.
This work was supported by ARC FL130100118 and ARC DP170100929.
In a recent work by the first two authors [6], we removed the strict convexity condition on the domains to get the $C^{1,\alpha}$ regularity of the free boundary. However, the higher order regularity of free boundary turns out to be a difficult problem, and it remains widely open so far. The only known result in this direction was proved in [6], where the higher order regularity of free boundary was shown assuming the domains are far away from each other.

Recall that for the complete transport problem, namely $m = \|f\|_1 = \|g\|_1$, the optimal transport plan is characterised by a convex potential function $u$ in $\Omega$, which satisfies a Monge-Ampère equation with the natural boundary condition $Du(\Omega) = \Omega^\ast$. When $\Omega, \Omega^\ast$ are convex, Caffarelli [3] obtained that $u \in C^{1,\alpha}(\Omega)$ for bounded densities. When $\Omega, \Omega^\ast$ are uniformly convex, $u \in C^{2,\alpha}(\Omega)$ was obtained by Delanoë [9], Urbas [14] for smooth densities, and by Caffarelli [4] for Hölder continuous densities. Recently, in [7] we reduced the uniform convexity assumption to convexity and obtained $u \in C^{2,\alpha}(\Omega)$ (see also [8, 13] for dimension two case). Note that the above global regularity theory cannot be applied directly to the partial transport problem since $U$ and $V$ generally fail to be convex.

In this paper, we consider the optimal partial transport between planar convex domains. Our main result is the following theorem.

**Theorem 1.1.** Assume $\Omega$ is a bounded convex domain, and $\Omega^\ast$ is a $C^2$ uniformly convex domain. Suppose $0 < f \in C^{\alpha}(\Omega), 0 < g \in C^\alpha(\Omega^\ast)$. Suppose $m$ satisfies (1.1) and $U \subset \Omega, V \subset \Omega^\ast$ are the active regions. Suppose $\Omega$ and $\Omega^\ast$ are separated by a hyperplane. Then the free boundary $\partial U \cap \Omega$ is $C^{2,\alpha}$.

This paper is organised as follows. In §2 we introduce some useful notations and results in the optimal partial transport problem. In §3 we established the obliqueness property, which is the key of the proof of the main result. In §4 we show that the potential function is $C^{1,1-\epsilon}$ up to the free boundary. In the last section §5 we use perturbation method to prove Theorem 1.1.

## 2. Preliminaries and notations

In the following, we will always assume the densities $1/\lambda < f, g < \lambda$ for some positive constant $\lambda$. For a fixed $m$ satisfying (1.1), it is shown in [5] that $\gamma_m$, the minimiser of (1.2), is characterised by

$$\gamma_m := (Id \times T_m)\#f_m = (T_m^{-1} \times Id)\#g_m,$$

where $T_m$ is the optimal transport map from the active domain $U \subset \Omega$ to the active target $V \subset \Omega^\ast$, the functions $f_m = f \chi_U$ and $g_m = g \chi_V$. Indeed, it is proved in [5] that $T_m = Du$ for some convex potential function $u$ solving

$$(Du)\#(f_m + (g - g_m)) = g,$$
and by the interior regularity and strict convexity of $u$ [2], one has

$$(2.3) \quad Du : U \to V \text{ is a } C^\alpha \text{ homeomorphism between active interiors.}$$

Similarly, $T_m^{-1} = Dv$, for some convex function $v$ solving

$$(2.3) \quad (Dv)_\gamma ((f - f_m) + g_m) = f,$$

with a convex target $\Omega$. By [2, Lemma 2] we can extend $u, v$ globally to $\mathbb{R}^n$ as follows

$$(2.4) \quad \tilde{u}(x) = \sup \{ L(x) : L \text{ affine, support of } u \text{ at some } x_0 \in (\Omega^* \setminus V) \cup U \},$$

$$(2.5) \quad \tilde{v}(x) = \sup \{ L(x) : L \text{ affine, support of } v \text{ at some } x_0 \in (\Omega \setminus U) \cup V \}.$$

For brevity, we still denote by $\tilde{u}, \tilde{v}$ the extensions $\tilde{u}, \tilde{v}$. Let $v^*(x) := \sup_{y \in \mathbb{R}^n} \{ x \cdot y - v(y) \}$, for $x \in \bar{\Omega}$

$$u^*(y) := \sup_{x \in \mathbb{R}^n} \{ y \cdot x - u(x) \}, \text{ for } y \in \bar{\Omega}^*$$

be the standard Legendre transforms of $u, v$. The following two facts are very important for our argument:

1. $u(x) = v^*(x)$ for any $x \in U$, $v(y) = u^*(y)$ for any $y \in V$,
2. $Dv(x) = x$ for a.e $x \in \Omega \setminus U$, hence $v = \frac{1}{2} |x|^2 + C$ on each connect component of $\Omega \setminus \bar{U}$. Similarly, $u = \frac{1}{2} |x|^2 + C$ on each connect component of $\Omega^* \setminus \bar{V}$.

Then, $u, v$ are globally Lipschitz convex solutions of

$$(2.6) \quad C_1(\chi_{\Omega^* \setminus V} + \chi_U) \leq \det D_{ij}u \leq C_2(\chi_{\Omega^* \setminus V} + \chi_U),$$

$$(2.7) \quad C_1(\chi_{\Omega \setminus U} + \chi_V) \leq \det D_{ij}v \leq C_2(\chi_{\Omega \setminus U} + \chi_V),$$

in the sense of Alexandrov, where $C_1, C_2$ are positive constants depending on the upper and lower bounds of $f, g$.

In general, given a convex function $v : \mathbb{R}^n \to (-\infty, \infty]$ we define its associated Monge-Ampère measure $M_v$ on $\mathbb{R}^n$ by

$$(2.8) \quad M_v(B) := \text{Vol}[\partial_v(B)]$$

for every Borel set $B \subset \mathbb{R}^n$. If $v$ is smooth and strictly convex, then

$$M_v(B) = \int_B \det[D^2v(x)] \, dx.$$

The inequality $[2.7]$ is interpreted in the above measure sense, namely $\det D_{ij}v = f$ if

$$M_v(B) = \int_B f$$

for every Borel set $B \subset \mathbb{R}^n$. Hence, $[2.7]$ implies that the Monge-Ampère measure $M_v$ is actually supported and bounded on $(\Omega \setminus U) \cup V$. 

Next, we recall the interior ball condition obtained in [5], which will be useful in our subsequent analysis.

**Lemma 2.1.** Let \( x \in U \) and \( y = Du(x) \), then
\[
\Omega \cap B_{|x-y|}(y) \subset U.
\]
Likewise, let \( y \in V \) and \( x = Dv(y) \), then
\[
\Omega^* \cap B_{|x-y|}(x) \subset V.
\]

When \( u \) is \( C^1 \) up to the free boundary \( \partial U \cap \Omega \), one can see that [5] the unit inner normal of \( \partial U \cap \Omega \) is given by
\[
\nu(x) = \frac{Du(x) - x}{|Du(x) - x|}, \quad \text{at } x \in \partial U \cap \Omega.
\]
Hence, the regularity of \( u \) up to the free boundary \( \partial U \cap \Omega \) implies the regularity of the free boundary itself.

Useful elements in investigating the convexity and regularity of the convex function \( v \) on the boundary are the centred sections and sub-level sets, see [3, 4].

**Definition 2.1.** Let \( v \) be the above convex function, extended in (2.5). Let \( y_0 \in V \) and \( h > 0 \) small. We denote
\[
S^c_h[v](y_0) := \{ y \in \mathbb{R}^n : v(y) < v(y_0) + (y - y_0) \cdot x + h \}
\]
as the centred section of \( v \) with height \( h \), where \( x \in \mathbb{R}^n \) is chosen such that the centre of mass of \( S^c_h[v](y_0) \) is \( y_0 \). Also, we denote
\[
S_h[v](y_0) := \{ y \in V : v(y) < \ell_{y_0}(y) + h \}
\]
as the sub-level set of \( v \) with height \( h \), where \( \ell_{y_0} \) is a support function of \( v \) at \( y_0 \).

We recall the following results proved in [5] for strictly convex domains and in [6] for general convex domains, which will be useful in the proof of Theorem 1.1.

**Theorem 2.1.** Suppose \( \Omega, \Omega^*, U, V, f, g \) satisfy the same conditions as in Theorem 1.1. Denote \( \mathcal{F} = \partial U \cap \Omega \). Then
1) \( u \) restricted to \( U \cup \mathcal{F} \) is \( C^{1,\alpha} \), and hence \( \mathcal{F} \) is \( C^{1,\alpha} \).
2) There exists a neighborhood \( N \) of \( \mathcal{F} \) such that \( v \) is strictly convex in \( Du(N) \cap V \).
3) (free boundary maps to fixed boundary) \( Du(\mathcal{F}) \subset \partial V \setminus \partial V \cap \Omega^* \).

Given a point \( x \in \mathcal{F} \), denote \( y = Du(x) \). Without loss of generality we may assume \( x = y = 0 \) and \( \Omega^* \subset \{ x_2 \geq 0 \} \). Indeed, in [6] the authors show that \( S^c_h[v] \cap \Omega^* \subset V \) for
h sufficiently small, and $S^c_h[v] \cap \Omega = \emptyset$, where we use $S^c_h[v]$ to denote $S^c_h[v](0)$ for short. Moreover,

(2.12) \[ v(y) \geq C|y|\beta \text{ for } y \in \bar{V} \text{ near } 0, \text{ for some constant } \beta > 1. \]

Now, we recall two lemmas from [4, Theorem 3.1, Theorem 4.1]

Lemma 2.2 (Uniform density). \[ \frac{|S^c_h[v] \cap V|}{|S^c_h[v]|} \geq \delta \text{ for some universal constant } \delta > 0. \]

Lemma 2.3 (Tangential $C^{1,1-\epsilon}$ for $v$). For any $\epsilon > 0$ there exists a constant $C_\epsilon$ such that $B_{C_\epsilon h^{1+\epsilon}} \cap \{x_2 = 0\} \subset S^c_h[v] \cap \{x_2 = 0\}$ for $h$ small.

Remark 2.1. In [4], the uniform density property (for the two dimensional case) was proved assuming the domains are convex. In our case, we consider the optimal transport between $U$ and $V$, where $U$ is locally convex near 0, but $V$ may be not locally convex near 0. However, the same proof in [4] still works in our case. By checking the proof of [4, Theorem 3.1], one can see that we only need to use the Lipschitz property of $\partial U$ near 0, which is ensured by the $C^{1,\alpha}$ regularity of $F$.

Remark 2.2. A direct corollary of Lemma 2.2 is that

\[ |S_h[v]| \approx |S^c_h[v] \cap V| \approx |S^c_h[v]| \approx h^{\frac{\alpha}{2}}. \]

Moreover, if one of $S^c_h[v]$ and $S_h[v]$ is normalised, then the other one is also normalised.

Remark 2.3. If $v$ is strictly convex up to the boundary, we actually have an equivalency relation between its sub-level sets $S_h[v](y_0)$ and centred sections $S^c_h[v](y_0)$, that is for all small $h > 0$,

(2.13) \[ S_{b^{-1}h}^c[v](y_0) \cap V \subset S_h[v](y_0) \subset S^c_h[v](y_0) \cap V, \]

where $b \geq 1$ is a constant independent of $h$. For the proof of (2.13), we refer the reader to [4] and [7, Lemma 2.2]. Then, by Lemma 2.3 we have that $B_{C_\epsilon h^{1+\epsilon}} \cap \partial V \subset S_h[v]$ for $h$ small.

3. Obliqueness

For any given $z = (z_1, z_2) \in F$, suppose $Du(z) = y \in \partial V \setminus \partial V \cap \Omega^\circ$. Denote by $\nu_U(z), \nu_V(y)$ the unit inner normals of $U, V$ at $z, y$, respectively. Without loss of generality, we may assume $z = 0$.

Proposition 3.1. $\nu_U(z) \cdot \nu_V(y) > 0$.

In the following, we suppose the obliqueness fails, then up to a rotation of coordinates, we may assume $\nu_U(0) = e_2, \nu_V(y) = e_1$. By (2.9), we may assume $y = re_2$ for some $r > 0$. Assume $F = \{x_2 = \rho(x_1)\}$ for some function $\rho$ locally near 0. Since $\partial V$ is uniformly convex
near 0, we may assume \( \partial V = \{ y_1 = \rho^*(y_2 - r) \} \) near \( y \), with \( \rho^*(t) = at^2 + o(t^2) \) for some constant \( a > 0 \).

**Lemma 3.1.** \( \rho(x_1) \geq 0 \) for \( x_1 < 0 \).

*Proof.* Suppose not, then there exists a point \( -se_1 \in U \) for some \( s > 0 \). Then, \( (-se_1 - 0) \cdot (Du(-se_1) - Du(0)) < 0 \) which contradicts to the monotonicity of convex function \( u \). Therefore the conclusion of the lemma holds. \( \square \)

Next, we characterise the asymptotic behaviour of \( \rho(x_1) \) for \( x_1 \) negative and close to 0.

**Lemma 3.2.** \( \rho(x_1) = \frac{1}{2r} x_1^2 + o(x_1^2) \) for \( x_1 < 0 \) and close to 0.

*Proof.* First, by the interior ball property, \( F \) is below the ball centred at \( y \) with radius \( r \). Therefore \( \rho(x_1) \leq \frac{1}{2r} x_1^2 + o(x_1^2) \). Hence we only need to prove \( \rho(x_1) \geq \frac{1}{2r} x_1^2 + o(x_1^2) \) for \( x_1 \) negative and close to 0.

Given a point \( q = (q_1, \rho(q_1)) \) with \( q_1 < 0 \) and \( |q_1| \) small, we denote \( p = Du(q) \in \partial V \setminus \partial V \cap \Omega^* \). Denote by \( se_2 \) the intersection of the segment \( pq \) and the \( x_2 \) axis. By monotonicity we have \( \nu_V(q) \cdot \nu_V(p) = \frac{p - q}{|p - q|} \cdot \nu_V(p) \geq 0 \). Therefore, the segment \( pq \) only touches \( \Omega^* \) at \( p \). Hence, \( s \geq r \).

Now, we must have \( |p - q| \leq |p - 0| \), since otherwise we have that the ball centred at \( p \) with radius \( |p - q| \) will contain 0 as an interior point, and then the interior ball property forces 0 to be an interior point of \( U \) which is impossible. Hence we have

\[
|p - q|^2 = |p_2 - \rho(q_1)|^2 + (p_1 + |q_1|)^2 \leq |p|^2 = p_1^2 + p_2^2. \tag{3.1}
\]

A straightforward computation shows that

\[ \rho(q_1) \geq \frac{1}{2p_2} q_1^2. \]

By continuity of \( Du \) we see that \( p_2 \) converges to \( r \) as \( q_1 \) converges to 0, hence \( p_2 = r + o(1) \) as \( q_1 \to 0 \). Therefore

\[ \rho(q_1) \geq \frac{1}{2r + o(1)} q_1^2 \geq \frac{1}{2r} q_1^2 + o(q_1^2). \]

\( \square \)

**Lemma 3.3.** For \( x \in V \), close to \( y \), we have

\[ v_2(x) = Dv(x) \cdot e_2 \geq -Cx_1^2, \]

where \( C \) depends only on the \( \text{dist}(\Omega, \Omega^*) \).

*Proof.* Denote \( p = Dv(x) \). We must have \( |p - x| \leq |x - 0| = |x| \), since otherwise the ball with centre \( x \) and radius \( |x - p| \) will contain 0 as an interior point, then by interior ball
property that 0 is an interior point of $U$ which is impossible. Therefore, if $p_2 < 0$ we have
\[
|p_2| \leq |x| - x_2 = \sqrt{x_1^2 + x_2^2} - x_2 \leq \frac{1}{x_2} x_1^2
\]
for $x_1 > 0$ small. Hence, $v_2(x) \geq -\frac{1}{x_2} x_1^2$ as expected. \hfill \Box

Now, by a translate of coordinates, we may assume $y = 0$. By subtracting a constant, we may also assume $u(0) = v(0) = 0$ and $Du(0) = Dv(0) = 0$. Let $p = (p_1, p_2), \xi = (\xi_1, \xi_2) \in \partial S_h[v]$ be the points such that
\[
(3.2) \quad p_2 = \sup \{x_2 : x = (x_1, x_2) \in S_h[v]\}
\]
and
\[
\xi_2 = \inf \{x_2 : x = (x_1, x_2) \in S_h[v]\}
\]
Let $q = (q_1, q_2)$ be the intersection of $x_1$-axis and $\partial S_h[v]$.

By the same proof of [7, Lemma 4.1] we have

**Lemma 3.4.** $p_2 \geq C|\xi_2|$ for some universal constant $C$.

**Remark 3.1.** Suppose $z = (z_1, z_2) \in S_h[v] \cap \{x_2 \geq 0\}$. Let $q$ be as above. By Lemma 2.2 and Lemma 2.3 we have that $h \approx |S_h[v]| \gtrsim h^{\frac{1}{2} + \epsilon} z_1$. Hence, $z_1 \leq C h^{\frac{1}{2} - \epsilon}$ for $\epsilon > 0$ as small as we want. Then, by (2.12) we also have that $z_2 \leq C h^{1/\beta}$. Then, by Lemma 3.3 we have that $v_2(z_1 e_1) \geq -C \xi_2^2 \geq -C h^{1/2}$. By convexity of $v$ we have that
\[
v(z) \geq v(z_1 e_1) + v_2(z_1 e_1) z_2 \geq v(z_1 e_1) - C h^{1-2\epsilon+1/\beta}.
\]
Therefore $v(z_1 e_1) \leq v(p) + C h^{1-2\epsilon+1/\beta} < 2h$ for $h$ small. Hence, by convexity we have $v(\frac{1}{2} z_1 e_1) < h$, which implies
\[
q_1 \geq \frac{1}{2} z_1.
\]
Using Lemma 3.4 and (3.3), by convexity we can normalise $S_h[v]$ using the following transformation
\[
\bar{x}_1 = \frac{1}{q_1} x_1,
\]
\[
\bar{x}_2 = \frac{1}{p_2} x_2,
\]
and
\[
\bar{v}_h(\bar{x}) = \frac{1}{h} v(q_1 \bar{x}_1, p_2 \bar{x}_2).
\]
Denote
\[
(3.4) \quad A_h = \begin{pmatrix} \frac{1}{q_1} & 0 \\ 0 & \frac{1}{p_2} \end{pmatrix}.
\]
Then $A_h S_h^c[v] \approx B_1(0)$. Note that by Remark 2.2 we also have $A_h S_h[v] \approx B_1(z)$ for some point $z$. 
Then, we prove the following key lemma

**Lemma 3.5.** $p_2 \approx h^{1/3}, q_1 \approx h^{2/3}$.

**Proof.** First we show that $p_2 \gtrsim h^{1/3}$. Otherwise, suppose there exists a sequence of $h \to 0$ such that $p_2, h^{1/3} \to 0$. Then, since $h \approx |S_h[v]| \approx q_1 p_2$, we have that $q_1 / h^{1/3} \to \infty$. Then up to a subsequence, $\bar{v}_h, A_h(S_h^i[v])$ converge to some $v_0, S_0$ as $h \to 0$. Note that locally near 0, $\partial V = \{x_1 = \rho^s(x_2)\}$ for some convex function $0 \leq \rho^s(x_2) \leq C|x_2|^2$. Hence after the transformation we have $A_h(\partial V) = \{x_1 = \frac{1}{q_1} \rho^s(p_2 x_2)\}$ locally near 0. Since $\frac{1}{q_1} \rho^s(p_2 x_2) \leq C \frac{h^2}{q_1} x_2 \to 0$ as $h \to 0$ we see that $A_h(\partial V)$ becomes flatter and flatter as $h \to 0$. In the limit $\partial S_0$ contains a segment $(-se_2, se_2)$ on $x_2$-axis.

Now, since $D \bar{v}_h(x) = \frac{1}{h} A^{-1}_h Dv(\bar{x})$, we see that $D \bar{v}_h(A_h(S_h^i[v] \cap \partial V))$ is on $D F^{-1}(\mathcal{F})$. Recall that $\mathcal{F} = \{x_2 = \rho^s(x_1)\}$ near 0, moreover $\rho^s(x_1) = \frac{1}{2\pi} x_1^2$ for $x_1 < 0$. Then $\frac{1}{h} A^{-1}_h(\mathcal{F}) = \{x_2 = \frac{1}{h} \rho^s(h x_1)\}$ locally near 0. Note that we have $\frac{1}{h} \rho^s(h x_1) \leq C \frac{h^2}{q_1} / h \to 0$ as $h \to 0$. Therefore $D \bar{v}_h(\partial S_0)$ is on the negative $x_1$-axis for $0 < t < s$, namely $D \bar{v}_h(\partial S_0) \cdot e_2 = 0$ for $0 < t < s$. Hence, $v(t e_2) = v_0(0) = 0$ for $0 < t < s$, which contradicts to the strict convexity of $v_0$.

Then, we show the opposite direction, namely, $p_2 \lesssim h^{1/3}$. Suppose not, then there exists a sequence $h \to 0$, such that $p_2 \gg h^{1/3}$. Then, since $V$ is uniformly convex near 0, we have that $p_1 \gg h^{2/3}$. Now, by (3.3) we have $q_1 \geq \frac{1}{2} p_1 \gg h^{1/3}$. Therefore $|S_h[v]| \geq q_1 p_2 \gg h$, which contradicts to Remark 2.2.

**Remark 3.2.** By Lemma 3.5, the transformation $A_h$ in Remark 3.1 can be chosen as

$$A_h = \begin{pmatrix} h^{-\frac{2}{3}}, & 0 \\ 0, & h^{-\frac{1}{3}} \end{pmatrix}.$$  

**Lemma 3.6.** There exists a universal constant $K$ such that

$$D := \{x = (x_1, x_2) : -\frac{1}{K} h^{1/3} \leq x_1 \leq 0; \rho(x_1) \leq x_2 \leq \frac{1}{K} h^{2/3}\} \subset Dv(S_h^i[v]).$$

**Proof.** First, let $p$ be the point defined as in (3.2). Let $se_2 = Dv(p)$. Then, $s \geq \frac{h}{p_2} \approx h^{2/3} \approx p_2$. $u(se_2) = v^s(se_2) = se_2 \cdot p - v(p) = sp_2 - h \leq Cs^{3/2}$. Since $Du(U) \subset \{x_1 \geq 0\}$, we see that $u$ is increasing in $e_1$ direction. Therefore, $u(x_1, x_2) \leq u(0, x_2) \leq C x_2^{3/2}$ for $x_1 < 0$. Hence, by convexity we have $0 \leq u_2(x) \leq C x_2^{1/2} \leq \frac{C}{K} h^{1/3}$ for $x \in D$.

Now, since $u_1(x_2) = p_1 \leq C h^{2/3} \leq C s$, by convexity we have $u_1(x) \leq u_1(x_2 e_2) \leq \frac{C}{K} h^{2/3}$ for $x \in D$. Therefore, by choose $K$ large enough we have that $Du(D) \subset S_h^i[v].$

Let $V_h = A_h(V), U_h = \frac{1}{h} A^{-1}_h(U)$. Then,

$$\partial U_h = \{x_2 = h^{-2/3} \rho(h^{1/3} x_1)\}$$

and

$$\partial V_h = \{x_1 = h^{-2/3} \rho^s(h^{1/3} x_2) = ax_2^2 + o(1)x_2^2\},$$
where $o(1) \to 0$ as $h \to 0$. Denote $v_h(x) = \frac{1}{h} v(A_h^{-1} x)$, $\bar{S}_h = A_h(S_0^h[v])$. Then up to a subsequence, we may assume $U_h, V_h, \rho_h, \rho_h^*, \bar{S}_h, v_h$ converge to $U_0, V_0, \rho_0, \rho_0^*, S_0, v_0$. Note that $\rho_0(x_1) = \frac{1}{2\pi} x_1^2$ for $x_1 < 0$, and $\rho_0^*(x_2) = ax_2^2$. Moreover,

$$(3.6) \quad \det D^2 v_0 = \chi_{S_0 \cap v_0}.$$

By Lemma 3.6 we also have

$$(3.7) \quad B_{\delta_0} \cap \{ x : x_2 \geq \rho_0(x_1), x_1 \leq 0 \} \subset Dv_0(S_0),$$

for some universal constant $\delta_0$. Now, observe that $Dv_0$ is the optimal transport map from $S_0 \cap V_0$ to $Dv_0(S_0)$. Note that $\partial(S_0 \cap V_0)$ is smooth and uniformly convex near 0. For any $x \in \partial S_0 \cap \partial V_0$ with $x_2 > 0$, we have $Dv_0(x) \in \{ x : x_1 < 0 \} \cap \partial U_0$ and $\partial(Dv_0(S_0))$ is locally smooth and uniformly convex. Therefore, by the localised version of Caffarelli’s $C^{2,\alpha}$ result in [4] we have that $v_0$ is smooth up to the part of boundary $\{ x : x_1 = ax_2^2, x_2 > 0 \}$.

We need one more lemma to proceed.

**Lemma 3.7.** $Dv_0(x) \cdot e_2 \geq 0$ for any $x \in S_h[v_0]$.

**Proof.** Recall that $v_h(x_1, x_2) = \frac{1}{h} v(h^{2/3} x_1, h^{1/3} x_2)$. Hence

$$Dv_h(x_1, x_2) \cdot e_2 = h^{-2/3} Dv(h^{2/3} x_1, h^{1/3} x_2) \cdot e_2.$$

By Lemma 3.3 we have that $Dv_h(x_1, x_2) \cdot e_2 \geq -Ch^{-2/3} h^{4/3} x_1^2 = -Ch^{2/3} x_1^2$. Let $h \to 0$, we have $Dv_0(x) \cdot e_2 \geq 0$. \hfill $\square$

Now, we may use the method developed in [7] to prove Proposition 3.1. For reader’s convenience we include the details here. In the following for simplicity of notations we will use $v$ to denote $v_0$ in the limit profile. Let $p = (p_1, p_2) \in \partial S_h[v]$ be the point such that

$$(3.8) \quad p_2 = \sup \{ x_2 : x = (x_1, x_2) \in S_h[v] \},$$

then following the same proof of Lemma 3.5 we have that $p_2 \approx h^{1/3}, p_1 \approx h^{2/3}$. Hence $v(p) \approx p_2^2$. Denote by $\bar{p}$ such point of $\partial S_{2h}[v]$. Then, by convexity and Lemma 3.7 we have that $0 \leq v_2(p) \leq Ch^{2/3} \leq Cp_2^2$.

Introduce the function

$$(3.9) \quad w(x) := v_2 + v - x_2 v_2.$$

Then define the following function

$$w(t) = \inf \{ w(x_1, t) : x_1 > \rho_0^*(t) \}, \quad 0 < t < 1.$$ 

By the above discussion we have

**Lemma 3.8.** $0 \leq w(t) \leq Ct^2$ for $0 < t < \delta_0$. 
Proof. First, \( w = (1 - x_2)v_2 + v \geq 0 \). Let \( p \) be defined as \( \ref{eqn:w_opt} \). Then \( w(p_2) \leq w(p) \leq C t^3 + C t^2 \).

Lemma 3.9. For \( t \) small, the minimum of \( w(\cdot, t) \) is attained in the interior of \( S_0 \cap V_0 \).

Proof. Recall that \( v \) is smooth up to the boundary \( S_0 \cap \partial V_0 \). \( S_0 \cap \partial V_0 = \{ x_1 = \rho_0(x_2) = ax_2^2 \} \) locally near 0, and for \( x = (x_1, x_2) \in S_0 \cap \partial V_0 \) with \( x_2 > 0 \) we have that
\[
Dv(x) \in \{ x = (x_1, x_2): x_2 = \rho(x_1) = \frac{1}{2r} x_1^2, x_1 < 0 \}.
\]
Hence we have
\[
v_2(\rho^*(t), t) = \rho(v_1(\rho^*(t), t))
\]
for \( t < 0 \) and close to 0. Differentiating the above equation we have
\[
v_2 \left( (\rho^*)' - \rho' \right) = \rho'(\rho^*)' v_{11} - v_{22}.
\]
Since \( (\rho^*(t))' > 0, \rho'(v_1(\rho^*(t), t)) < 0 \) for \( t > 0 \), and \( v_{11} > 0, v_{22} > 0 \) we have that \( v_{21} < 0 \) for \( t > 0 \). Hence, for \( x = (\rho^*(x_2), x_2) \) with \( x_2 > 0 \) we have that \( w_1 = (1 - x_2)v_{21} + v_1 < 0 \).

Lemma 3.10. \( w(t) \) is concave in \( (0, \delta_0) \).

Proof. If \( w \) is not concave, then there is an affine function \( L(t) \) such that the set \( \{ t \in (0, \delta_0): w(t) < L(t) \} \) is an set compactly contained in \( (0, \delta_0) \). Extend \( L \) to an affine function \( \hat{L} \) defined in \( \mathbb{R}^2 \), such that \( \hat{L}(s,t) = L(t) \). Then we can make
\[
\{ x \in S_0 \cap U_0 : x_2 \in (0, \delta_0), \text{ and } w(x) < \hat{L}(x) \} \subset U_0.
\]
Note that by Lemma 3.9 we can always achieve \( \ref{eqn:w_opt} \). Since \( v^{ij} w_{ij} = 0 \), we reach a contradiction by the maximum principle.

Proof of Proposition. \ref{prop:obliqueness} Suppose the obliqueness fails. By Lemma 3.8 and Lemma 3.10 we have that \( w(t) \) is concave in \( (0, \delta_0) \) and satisfies \( 0 \leq w(t) \leq C t^2 \). Combine this with the fact that \( w(t) \to 0 \) as \( t \to 0 \), it follows \( w(t) = 0 \) for \( t \in (0, \delta_0) \), this is impossible, since \( w(x) = (1 - x_2)v_2 + v \geq v \) and by strict convexity of \( v \) we have that \( v(x) \geq \eta(t) > 0 \).

4. \( C^{1,1-\epsilon} \) regularity

For any \( x_0 \in \mathcal{F} \), in this section we show that \( u \) is pointwise \( C^{1,1-\epsilon} \) at \( x_0 \) for \( \epsilon > 0 \) as small as we want. Denote by \( y_0 = Du(x_0) \in \partial V \setminus \partial V \cap \Omega^\circ \). We now consider the optimal transport between \( U \) and \( V \), without loss of generality, we assume \( x_0 = y_0 = 0 \). By
Proposition 3.1, up to an affine transformation, we may assume \( \nu_U(0) = \nu_V(0) = e_2 \). We also have \( \partial U = \{ x_2 = \rho(x_1) \}, \partial V = \{ x_2 = \rho^*(x_1) \} \) near 0. Note that
\[
-C|x_1|^{1+\alpha} \leq \rho(x_1) \leq C|x_1|^2
\]
by the \( C^{1,\alpha} \) regularity of \( F \) and the interior ball property, and that
\[
0 \leq \rho^*(x_1) \leq C|x_1|^2.
\]

**Lemma 4.1.** There exists a positive constant \( r_0 \) such that \( u(x) \geq C|x_1|^{2+\epsilon} \) for \( x \in U \cap B_{r_0} \).

We first prove the following two estimates.

**Proof.** By Lemma 2.3, we have that \( v(t, \rho^*(t)) \leq C t^{2-\epsilon} \) for \( \epsilon \) as small as we want, where \( C \) depends on \( \epsilon \). Then
\[
u(t) = \sup_{y \in V} \{ x \cdot y - v(y) \} \geq x \cdot (t, \rho^*(t)) - C|t|^{2-\epsilon} \geq x_1 t - C|x_2|^2 - C|t|^{2-\epsilon}.
\]
Choosing \( t = \frac{1}{2C^2} |x_1|^{3\epsilon} x_1 \), we have that \( u \geq C|x_1|^{2+3\epsilon} \). The desired estimate follows since we can choose \( \epsilon \) as small as we want. \( \square \)

**Lemma 4.2.** There exists a positive constant \( r_0 \) such that \( u(te^2) \leq C t^{2-\epsilon} \) for \( 0 \leq t \leq r_0 \).

**Proof.** Let \( q \in \partial S_h[v] \) be a point such that
\[
q_2 = \sup \{ x_2 : x \in S_h[v] \}.
\]
Note that by Lemma 2.2 and Lemma 2.3, we have \( q_2 \leq C h^2 = Ch^{\frac{1}{2} - \epsilon} \). Denote \( p = Du(q) \). Then \( p_2 = |p| \geq C \frac{h}{q_2} \geq h^{1+\epsilon} \). Hence
\[
u(p) = q \cdot Dv(q) - v(q) = q_2 p_2 - v(q) \leq p_2^{2-2\epsilon}.
\]
Therefore, \( u(te^2) \leq C t^{2-2\epsilon} \) for \( \epsilon \) as small as we want. \( \square \)

Next, we prove a uniform density property for \( u \).

**Lemma 4.3.** \( S_{h_0}^c[u] \) converges to \( \{0\} \) as \( h \to 0 \). Moreover, there exists a universal constant \( h_0 > 0 \) such that \( \frac{|S_{h_0}^c[u] \cap U|}{|S_{h_0}^c[u]|} \geq \delta_0 \) for some universal constant \( \delta_0 \).
Proof. Let \( z = se_{2} \) be the intersection of \( S_{h}^{c}[u] \) and the \( x_{2} \)-axis. Since \( v(x) \geq C|x|^{\beta} \) for \( x \in \tilde{V} \) and near 0 for some constant \( \beta > 1 \), we have that \( u(x) < C|x|^{\delta} \) near 0, where \( \delta = \frac{\beta}{\beta-1} \). Hence,

\[
(4.3) \quad s \geq h^{\frac{1}{\delta}}.
\]

By Lemma 4.1 we also have that

\[
(4.4) \quad S_{h}^{c}[u] \cap U \subset S_{Ch}[u] \subset \{ x : -Ch^{\frac{1}{2}} - \epsilon < x_{1} < Ch^{\frac{1}{2}} - \epsilon \}
\]

By (4.3), (4.4) and the strict convexity of \( u \) in \( U \), and the fact that \( S_{h}^{c}[u] \) is balanced around 0, we conclude that \( S_{h}^{c}[u] \) converges to \( \{0\} \) as \( h \to 0 \).

Now, since \( \rho(x_{1}) \leq C|x_{1}|^{2} \), we have that

\[
(4.5) \quad S_{h}^{c} \cap \{ x : x_{2} \geq Ch^{1-2\epsilon} \} \subset U.
\]

We have

\[
(4.6) \quad \frac{h^{1-2\epsilon}}{s} \to 0 \quad \text{as} \quad h \to 0,
\]

provided we choose \( \epsilon \) sufficiently small. By (4.5), (4.6), the fact that \( S_{h}^{c}[u] \) is convex and balanced around 0, we conclude that \( \frac{|S_{h}^{c}[u] \cap U|}{|S_{h}^{c}[u]|} \geq \delta_{0} \) for some universal constant \( \delta_{0} \). □

By the proof of Lemma 4.3 we also have \( |\frac{1}{2}S_{h}^{c}[u] \cap U| \geq |S_{h}^{c}[u] \cap U| \). Using this doubling property, the uniform density property and the Alexandrov estimates [4] Corollary 2.1] we have

**Corollary 4.1.** \( |S_{h}^{c}[u]| \approx |S_{h}^{c}[u] \cap U| \approx h. \)

Let \( z \) be as in the proof of Lemma 4.3. Then, by Lemma 4.3, Corollary 4.1 and (4.4) we have that

\[
(4.7) \quad z_{2} \geq Ch^{\frac{1}{2} + \epsilon}.
\]

Now, we write

\[
S_{h}^{c}[u] \approx E := \{ x \in \mathbb{R}^{2} : \frac{(x_{1} - kx_{2})^{2}}{a^{2}} + \frac{x_{2}^{2}}{b^{2}} \leq 1 \}.
\]

From the above discussion, we have

\[
(4.8) \quad a \lesssim h^{\frac{1}{2}} - \epsilon, \quad b \gtrsim h^{\frac{1}{2}} + \epsilon.
\]

Moreover,

\[
(4.9) \quad |k| \lesssim \frac{a}{z_{2}} \leq h^{-2\epsilon}.
\]

Let \( A_{h} \) be the affine transformation normalising \( S_{h}^{c}[u] \), namely

\[
\hat{x}_{1} = \frac{x_{1} - kx_{2}}{a}, \quad \hat{x}_{2} = \frac{x_{2}}{b}
\]

such that \( A_{h}E = B_{1} \).
By (4.8), (4.9) and (4.11) it is straightforward to check that \( A_h(S_h^c[u] \cap U) \) converges to a line segment on \( x_1 \)-axis as \( h \to 0 \). Hence \( |D \setminus (A_hU)| \to 0 \) as \( h \to 0 \), where

\[
D := A_h S_h^c \cap \{ x_2 \geq 0 \}.
\]

Then, following the same proof of [3, Lemma 4.1] we conclude

**Lemma 4.4.** For any \( \epsilon > 0 \), there exists a constant \( C > 0 \) such that

\[
B_{Ch^{\frac{1}{2}+\epsilon}} \cap \{ x_2 = 0 \} \subset S_h^c[u].
\]

**Corollary 4.2.** \( C|x|^{2+\epsilon} \leq u(x) \leq C_2|x|^{2-\epsilon} \) for \( x \in U \).

**Proof.** By Lemma 4.4, Corollary 4.1 and (4.7) we have that

\[
B_{Ch^{\frac{1}{2}+\epsilon}} \cap U \subset S_h^c[u] \cap U \subset B_{Ch^{\frac{1}{2}-\epsilon}} \cap U.
\]

Then, we have

\[
(4.10) \quad B_{Ch^{\frac{1}{2}+\epsilon}} \cap U \subset S_h^c[u] \cap U \subset S_h[u]
\]

which implies that \( u(x) \leq C|x|^{2-\epsilon} \) for \( x \in U \).

Let \( q \in \partial S_h[u] \) be the point such that \( q_2 = \sup \{ x_2 : x \in S_h[u] \} \). Since \( S_h^c \subset S_h[u] \), by (4.7) we have that \( q_2 \geq Ch^{\frac{1}{2}+\epsilon} \). By (4.4) and (4.11) we have that \( \tilde{D} := S_h[u] \cap \{ x_2 \geq Ch^{1-2\epsilon} \} \subset U \). Then \( \frac{1}{C} \leq \det D^2 u \leq C \) in \( D \) and \( 0 \leq u \leq h \) on \( \tilde{D} \). By Alexandrov estimate we have that \( |\tilde{D}| \leq Ch \). By (4.10) we also have \( |\tilde{D}| \geq Ch^{\frac{1}{2}+\epsilon} q_2 \). Hence,

\[
(4.11) \quad q_2 \leq Ch^{1-\frac{1}{2}-\epsilon} = Ch^{\frac{1}{2}-\epsilon}.
\]

By (4.11) and Lemma 4.1 we have \( S_h[u] \subset B_{Ch^{\frac{1}{2}-\epsilon}} \cap U \) which implies \( u(x) \geq C|x|^{2+\epsilon} \) for \( x \in U \). \( \square \)

It follows from the above Lemma that

\[
(4.12) \quad u \in C^{1,1-\epsilon}(B_{\delta_0} \cap U), \text{ for some universal constant } \delta_0.
\]

5. \( C^{2,\alpha} \) Regularity

In this section, we adopt the method developed in [2] to prove the \( C^{2,\alpha} \) regularity of \( u \) up to the free boundary \( F \). First we construct an approximate solution of \( u \) in \( S_h[u] \) as follows. Denote

\[
D^+_h = S_h[u] \cap \{ x_2 \geq h^{1-3\epsilon} \}.
\]

When \( h > 0 \) is sufficiently small, we have \( D^+_h \subset U \). Let \( D^-_h \) be the reflection of \( D^+_h \) with respect to the hyperplane \( \{ x_2 = h^{1-3\epsilon} \} \). Denote \( D_h = D^+_h \cup D^-_h \). By the property that
Divide $0 \leq \tau$ for some $\tau$ with respect to $|x|$. Then
\begin{equation}
(5.1)
\end{equation}

Our proof relies on the following lemma.

**Lemma 5.1.** Assume that
\[
\frac{f(x)}{g(Du(x))} - 1 \lesssim h^\tau \quad \text{in} \quad D_h.
\]
Then we have
\[
\|u - w\|_{L^\infty(D_h \cap U)} \lesssim h^{1+\tau'}
\]
for some $\tau' \in (0, \tau)$.

**Proof.** Divide $\partial D_h^+ = C_1 \cup C_2$ into two parts, where $C_1 \subset \{x_n > h^{1-3\epsilon}\}$ and $C_2 \subset \{x_n = h^{1-3\epsilon}\}$. On $C_1$ we have $u = w$. On $C_2$, by symmetry we have $D_w = 0$. We claim that $0 \leq D_2 u \leq C_1 h^{1-4\epsilon}$ on $C_2$, for any given small $\epsilon > 0$.

To see this, for any $x = (x', x_n) \in C_2$, let $z = (x', \rho(x'))$ be the point on $F$. Since $Du(\partial U) \subset \partial V$ and $u \in C^{1,1-\epsilon}(B_{\delta_0} \cap \overline{U})$, for any $\epsilon \in (0, 1)$, it is straightforward to compute that $|D_2 u(z)| \leq C h^{2(1/2-\epsilon)(1-\epsilon)}$. On the other hand $|D_2 u(x) - D_2 u(z)| \leq C h^{1/2-\epsilon}(1+\alpha)$. Hence $0 \leq D_2 u(x) \leq C_1 h^{2+1/2+3\alpha}$, provided $\epsilon$ is sufficiently small.

Let
\[
\tilde{w} = (1 - h^\tau)^{1/n} w - (1 - h^\tau)^{1/n} h + h,
\]
and
\[
\tilde{w} = (1 + h^\tau)^{1/n} w - (1 + h^\tau)^{1/n} h + C_1(x_n - C h^{1/2-\epsilon}) h^{1/2+1/2+3\alpha}.
\]
Then
\[
\det D^2 \tilde{w} \leq \det D^2 u \leq \det D^2 \tilde{w} \quad \text{in} \quad D_h^+,
\]
\[
\tilde{w} \leq u = \tilde{w} = h \quad \text{on} \quad C_1,
\]
\[
D_2 \tilde{w} = 0 < D_2 u < D_2 \tilde{w} \quad \text{on} \quad C_2.
\]
By comparison principle, we have $\tilde{w} \geq u \geq \tilde{w}$ in $D_h^+$.

Since $h > 0$ is small, $\tau' = \min\{\alpha, \tau\} < 1/2$, and $\epsilon > 0$ is small, we obtain
\begin{equation}
(5.2)
|u - w| \leq C h^{1+\tau'} \quad \text{in} \quad D_h^+.
\end{equation}

Next, we estimate $|u - w|$ in $D_h \cap U$. For $x = (x_1, x_2) \in D_h^+ \cap U$, let $z = (x_1, 2h^{1-3\epsilon} - x_2) \in D_h^+$. Then $|x - z| \leq C h^{1-3\epsilon}$. From (5.2), $|u(z) - w(z)| \leq C h^{1+\tau'}$. Since $w$ is symmetric with respect to $\{x_2 = h^{1-3\epsilon}\}$, we have $w(x) = w(z)$. Since $u \in C^{1,1-\epsilon}(B_{\delta_0} \cap \overline{U})$, we obtain
\[
|u(x) - u(z)| \leq \|Du\|_{L^\infty(D_h)} |x - z| \leq C h^{3/2-4\epsilon}.
\]
Therefore, for the given constant $\tau'$,
\[ |u(x) - w(x)| \leq |u(x) - u(z)| + |u(z) - w(z)| \leq Ch^{1+\tau'}.
\]
Combining with (5.2) we thus obtain the desired $L^\infty$ estimate
\[ |u - w| \leq Ch^{1+\tau'} \quad \text{in} \quad D_h \cap U. \]

Proof of Theorem 1.1. Once we have Lemma 5.1, we can prove that $u \in C^{2,\alpha'}(\overline{B}_0 \cap U)$ following the same line of the proof Theorem 1.1 in [7, Section 6]. Then, since $\nu_U(x) = \frac{Du(x) - x}{|Du(x) - x|}$ for $x \in F$, it follows that $\nu$ is $C^{1,\alpha'}$ along $F$, namely, $F$ is $C^{2,\alpha'}$.

Remark 5.1. By using the strategy in this paper and the approximation technique developed in [7, Section 4.3], in Theorem 1.1 the uniform convexity condition on the domain $\Omega^*$ can be reduced to the usual convexity. Moreover, the obliqueness can also be proved when $\partial \Omega^*$ is only $C^{1,\alpha}$. Note that in the proof of obliqueness, $v_0$ satisfies $\det D^2 v_0 = \chi_{V_0}$ in $\mathbb{R}^n$, and $Dv_0(\mathbb{R}^n)$ is a convex set. We would also like to point out that the methods in §4 and §5 also work for general dimensions. Namely, if at some point $x \in F$ we have $\nu_U(x) \cdot \nu_V(Du(x)) > 0$, then the free boundary is $C^{2,\alpha}$ in a neighborhood of $x$.

References

1. L. A. Caffarelli. Some regularity properties of solutions of Monge Ampère equation. *Comm. Pure Appl. Math.* 44 (1991), 965–969.
2. L. A. Caffarelli. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.* 5 (1992), 99–104.
3. L. A. Caffarelli. Boundary regularity of maps with convex potentials. *Comm. Pure Appl. Math.* 45 (1992), 1141–1151.
4. L. A. Caffarelli. Boundary regularity of maps with convex potentials. II. *Ann. of Math.* 144 (1996), 453–496.
5. L.A. Caffarelli and R.J. McCann, Free boundaries in optimal transport and Monge-Ampère obstacle problems. *Ann. of Math.* 171 (2010), 673–730.
6. S. Chen, J. Liu. Regularity of free boundaries in optimal transportation. Preprint, available at arXiv.
7. S. Chen; J. Liu and X.-J. Wang, Global regularity for the Monge-Ampère equation with natural boundary condition, submitted. Available at [arXiv:1802.07518](https://arxiv.org/abs/1802.07518)
8. S. Chen; J. Liu and X.-J. Wang, Boundary regularity for the second boundary-value problem of Monge-Ampère equations in dimension two, submitted. Available at [arXiv:1806.09482](https://arxiv.org/abs/1806.09482)
9. Ph. Delanoë, Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator. *Ann. Inst. Henri Poincaré, Analyse Non Linéaire*, 8 (1991), 443–457.
10. A. Figalli, A note on the regularity of the free boundaries in the optimal partial transport problem. *Rend. Circ. Mat. Palermo*, 58 (2009), no. 2, 283-286.
11. A. Figalli, The optimal partial transport problem. *Arch. Ration. Mech. Anal.*, 195 (2010), 533–560.
12. E. Indrei, Free boundary regularity in the optimal partial transport problem. *J. Funct. Anal.*, 264 (2013), no. 11, 2497–2528.
13. O. Savin and H. Yu, Regularity of optimal transport between planar convex domains, available at [arXiv:1806.06252](https://arxiv.org/abs/1806.06252) to appear in *Duke Math. J.*
14. J. Urbas, On the second boundary value problem of Monge-Ampère type. *J. Reine Angew. Math.*, 487 (1997), 115–124.
E-mail address: chenshibing1982@hotmail.com

School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, AUSTRALIA
E-mail address: jiakun1@uow.edu.au

Centre for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200, AUSTRALIA
E-mail address: Xu-Jia.Wang@anu.edu.au