Knizhnik-Zamolodchikov-Bernard equations connected with the eight-vertex model

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Abstract

Using quasiclassical limit of Baxter’s 8 - vertex R - matrix, an elliptic generalization of the Knizhnik-Zamolodchikov equation is constructed. Via Off-Shell Bethe ansatz an integrable representation for this equation is obtained. It is shown that there exists a gauge transformation connecting this equation with Knizhnik-Zamolodchikov-Bernard equation for SU(2)-WZNW model on torus.

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1 Introduction

Conformal Field Theory (CFT) describes the critical behavior of two-dimensional statistical systems, many of which are integrable. Usually correlation functions are governed by linear differential equations which are derived from degeneracy relations of the underlying symmetry algebra (see e.g. [1], [2], [3], [4]). In integrable models of statistical mechanics [5], the main object is the $R$-matrix which depends on a spectral parameter $\lambda$ and acts on tensor product $V \otimes V$ for some vector space $V$. The main condition on $R$, leading to integrability is the Yang-Baxter equation

$$R_{12}(\lambda)R_{13}(\lambda + \mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda + \mu)R_{12}(\lambda)$$  \hspace{1cm} (1)

It is assumed that (1) is defined on threefold tensor product $V^1 \otimes V^2 \otimes V^3$. Here $V^1$, $V^2$, $V^3$ are some copies of $V$ and lower indices of $R$ show the components of tensor product on which $R$-matrix acts non trivially. The Yang-Baxter equation implies the commutativity of transfer matrices constructed out of $R$ for two arbitrarily chosen values of spectral parameter $\lambda$. If $R$ has extra dependence on some (Plank-like) parameter $\eta$ so that $R = 1 + \eta r + o(\eta^2)$, as $\eta \to 0$, then the classical $r$-matrix obeys the classical Yang-Baxter equation

$$[r_{12}(\lambda), r_{13}(\lambda + \mu) + r_{23}(\mu)] + [r_{13}(\lambda + \mu), r_{23}(\mu)] = 0$$  \hspace{1cm} (2)

This equation has the following relation with conformal field theory. In the skew-symmetric case $r_{21}(-\lambda) = -r_{12}(\lambda)$, it is the compatibility condition for the system of linear differential equations

$$\frac{d\Psi}{d\lambda_i} = \sum_{j \neq i} r_{ij}(\lambda_i - \lambda_j)\Psi$$  \hspace{1cm} (3)

for a function $\Psi(\lambda_1, \lambda_2, \cdots, \lambda_N)$ with values in $V^1 \otimes \cdots \otimes V^N$. In the rational case [7], very simple skew-symmetric solutions are known: $r(\lambda) = C/\lambda$, where $C \in g \otimes g$ is a symmetric invariant tensor of a finite dimensional Lie algebra $g$ acting on a representation space $V$. Then the corresponding system of differential
equations (2) is the Knizhnik-Zamolodchikov (KZ) equation for the conformal blocks of the Wess-Zumino-Novikov-Witten (WZNW) model on the sphere [2]. Usually the exactly integrable homogeneous vertex model and its connection with the conformal field theory [3] or quantum field theory [9] has been considered. Here we consider the inhomogeneous vertex model, where to each vertex we associate two parameters: the global spectral parameter \( \lambda \) and the disorder parameter \( z \). The vertex weight matrix \( \mathcal{R} \) depends on \( \lambda - z \). Hence the transfer matrix of the vertex model now depends on the disorder parameters \( z_i, i = 1, \ldots, N \).

If we have some rational solution of Yang-Baxter equation (1) and the transfer matrix \( T(\lambda|z) \), then by construction of the algebraic Bethe ansatz [10] we have an equation

\[
T(\lambda|z)\Phi(\lambda_1, \ldots, \lambda_n|z) = \Lambda(\lambda, \lambda_1, \ldots, \lambda_n|z)\Phi(\lambda_1, \ldots, \lambda_n|z) - \sum_{\alpha=1}^{n} \frac{F_\alpha \Phi^\alpha}{\lambda - \lambda_\alpha} \tag{4}
\]

where

\[
\Phi(\lambda_1, \ldots, \lambda_n|z) = \Phi(\lambda_1, \ldots, \lambda_n|z_1, \ldots, z_N)
\]
is the Bethe wave vector,

\[
\Phi^\alpha = \Phi(\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda, \lambda_{\alpha-1}, \ldots, \lambda_n|z_1, \ldots, z_N)
\]

and \( F_\alpha(\lambda_1, \ldots, \lambda_n|z_1, \ldots, z_N), \Lambda(\lambda, \lambda_1, \ldots, \lambda_n|z_1, \ldots, z_N) \) are some \( c \)-valued functions. In the Bethe ansatz we impose the condition \( F_\alpha = 0 \). Under this condition the wave vector \( \Phi \) becomes an eigenvector and \( \Lambda \) an eigenvalue of transfer matrix \( T(\lambda|z) \). In general case, when the condition \( F_\alpha = 0 \) doesn’t imposed, following [11]-[14] we’ll refer to (4) as Off-Shell Bethe Ansatz Equation (OSBAE). Using OSBAE (4) in quasi classical limit \( \eta \rightarrow 0 \), in [11]-[14] the solution of the Knizhnik-Zamolodchikov equation for the \( N \)-point correlation function \( \Psi(z_1, \ldots, z_N) \) in WZNW theory [2]

\[
\kappa \frac{\partial \Psi}{\partial z_k} = \sum_{i \neq k}^{N} \frac{t_k^a \otimes t_i^a}{z_k - z_i} \Psi \tag{5}
\]
has been constructed. In quasi classical limit \( \hat{H}_k \) transforms into OSBAE for the Gaudin non-local hamiltonians \( H_k, k = 1, \ldots, N \):

\[
H_k = \sum_{i \neq k}^N t_k^a \otimes t_i^a \frac{z_k - z_i}{z_k - z_i},
\]

where as in (6) \( t_i^a, a = 1, \ldots, \text{dim}(g) \), represent the generators of the simple Lie algebra \( g \) and act non-trivially in the representation spaces \( V_i, i = 1, \ldots, N \). The vector-valued function \( \Psi(z_1, \ldots, z_N) \) is the holomorphic part (or conformal block) of the \( N \)-point correlation function in WZNW theory.

The solution of (5) has the form

\[
\Psi(z_1, \ldots, z_N) = \oint \cdots \oint \mathcal{X}(\lambda_1, \ldots, \lambda_n | z) \phi(\lambda_1, \ldots, \lambda_n | z) d\lambda_1 \cdots d\lambda_n
\]

(7)

where \( \phi(\lambda_1, \ldots, \lambda_n | z) \) is the semiclassical limit of the Bethe vector \( \Phi(\lambda_1, \ldots, \lambda_n | z) \) and in fact it is the Bethe wave vector for Gaudin magnets (6), but off mass shell. The scalar function \( \mathcal{X}(\lambda_1, \ldots, \lambda_n | z) \) is constructed from the semiclassical limit of the \( \Lambda(\lambda = z_k; \lambda_1, \ldots, \lambda_n | z) \) and \( F_\alpha \) (for more details see [11] - [14]).

This representation of the \( N \)-point correlation function in WZNW theory shows an intriguing connection between the inhomogeneous vertex models and the WZNW theory. A deeper understanding of this fine structure may provide us with more complete knowledge about exact integrability and conformal field theory in two-dimensions.

In this paper we consider OSBAE for the inhomogeneous eight-vertex model and corresponding \( XYZ \) Gaudin magnet, which provide the solution of an elliptic generalization of the Knizhnik-Zamolodchikov equation

\[
\kappa(\frac{\partial}{\partial z_i} - \Gamma_i)\Psi = \sum_{j \neq i} r_{ij}(z_i - z_j)\Psi - \frac{\partial}{\partial w}(\Sigma_j \Psi),
\]

(8)

where \( r_{ij}(\lambda) \) is the elliptic quasi classical \( \mathcal{r} \)-matrix. \( \Sigma_i, \Gamma_i \) are some operators acting non trivially only on \( i \)-th component of \( N \)-fold tensor product \( \mathbb{C}^2 \otimes \cdots \mathbb{C}^2 \), where the vector \( \Psi \) takes its values. The exact form of \( \Sigma_i, \Gamma_i \) will be specified in section 6.

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The distinguishing feature of elliptic case is that the vector $\Psi$ depends not only on $z_1,\ldots,z_N$ as in trigonometric and rational cases, but also on elliptic moduli and on some auxiliary parameter $w$, the meaning of which will be clarified in the main text.

In Section 2, we present the BAE and OSBAE for the inhomogeneous eight-vertex model and its quasi classical limit spin-$1/2$ XYZ Gaudin model. In Section 3, we construct EKZ equation (8) and its solution. In Section 4, we discuss the connection of EKZ equation and the Knizhnik-Zamolodchikov-Bernard equation for $su(2)$ WZNW theory on torus and finally we left the last section to some remarks and conclusions.

2 Algebraic Bethe ansatz for the inhomogeneous eight-vertex model

In this section we modify Algebraic Bethe Ansatz (ABE) technique for 8-vertex model developed in [10] in order to adjust it to the inhomogeneous case.

The inhomogeneous eight-vertex model is parameterized by an anisotropy parameter $\eta$, elliptic modulus $\tau$ and the shifts of the spectral parameter (inhomogeneity parameters $z_i$).

The algebraic Bethe ansatz solution of the inhomogeneous eight-vertex model is closely related to the homogeneous case. As we’ll be clear from the further consideration, the only modification one have to do, is local shifting of the spectral parameter $\lambda \rightarrow \lambda - z_i$ and auxiliary parameters $s,t \rightarrow s - z_i, t + z_i$. Note one nonessential difference from the conventions of [10], that we label the sites in a horizontal row of lattice from left to right (as in [6]). We define a local transition
matrix \( L_i(\lambda - z_i) \) by

\[
L_i(\lambda - z_i) = \sum_{a=1}^{4} w_a \sigma^a \otimes \sigma^a = \begin{pmatrix}
w_4 \sigma_4^4 + w_3 \sigma_3^3 & w_1 \sigma_1^1 - iw_2 \sigma_2^2 \\
w_1 \sigma_1^1 + iw_2 \sigma_2^2 & w_4 \sigma_4^4 - w_3 \sigma_3^3
\end{pmatrix}
\]  

(9)

where the \( \sigma^i \) are the spin-1/2 Pauli matrices, which in the basis

\[
e^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]  

(10)

have the standard form

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]  

(11)

and \( \sigma^4 = I \) it is the 2×2 identity matrix. For the inhomogeneous case the coefficients are parameterized as

\[
w_4 + w_3 = \rho(\lambda - z_i) \Theta(2\eta) \Theta(\lambda - z_i - \eta) H(\lambda - z_i + \eta)
\]

\[
w_4 - w_3 = \rho(\lambda - z_i) \Theta(2\eta) H(\lambda - z_i - \eta) \Theta(\lambda - z_i + \eta)
\]

\[
w_1 + w_2 = \rho(\lambda - z_i) H(2\eta) \Theta(\lambda - z_i - \eta) \Theta(\lambda - z_i + \eta)
\]

\[
w_1 - w_2 = \rho(\lambda - z_i) H(2\eta) H(\lambda - z_i - \eta) H(\lambda - z_i + \eta)
\]  

(12)

where \( \Theta(\lambda) \) and \( H(\lambda) \) are the Jacobi theta-functions,

\[
g(\lambda) = H(\lambda) \Theta(\lambda)
\]  

(13)

and we have introduced the normalizing function

\[
\rho(\lambda) = \frac{g(K)}{\Theta(2\eta)g(K + \eta)g(\lambda)}
\]  

(14)

in order to fix Boltzmann weight \( w_4 \equiv 1 \) (\( K \) is the half-period of Jacobi function \( \Theta(\lambda) \)).

The matrix \( L_i(\lambda - z_i) \) satisfies the Yang-Baxter relation

\[
R(\lambda, \mu)L_i(\lambda - z_i) \otimes L_i(\mu - z_i) = L_i(\mu - z_i) \otimes L_i(\lambda - z_i)R(\lambda, \mu)
\]  

(15)
where the $\mathcal{R}$-matrix has the form

$$
\mathcal{R}(\lambda, \mu) = \begin{pmatrix}
    a & 0 & 0 & d \\
    0 & b & c & 0 \\
    0 & c & b & 0 \\
    d & 0 & 0 & a
\end{pmatrix}
$$

with

$$
a(\lambda, \mu) = \Theta(2\eta) \Theta(\lambda - \mu - \eta) H(\lambda - \mu + \eta)
$$

$$
b(\lambda, \mu) = \Theta(2\eta) H(\lambda - \mu - \eta) \Theta(\lambda - \mu + \eta)
$$

$$
c(\lambda, \mu) = H(2\eta) \Theta(\lambda - \mu - \eta) \Theta(\lambda - \mu + \eta)
$$

$$
d(\lambda, \mu) = H(2\eta) H(\lambda - \mu - \eta) H(\lambda - \mu + \eta)
$$

The operator matrix

$$
\mathcal{T}_N(\lambda|z) = \mathcal{L}_1(\lambda - z_1)\mathcal{L}_2(\lambda - z_2) \cdots \mathcal{L}_N(\lambda - z_N)
$$

$$
= \begin{pmatrix}
    A_N(\lambda|z) & B_N(\lambda|z) \\
    C_N(\lambda|z) & D_N(\lambda|z)
\end{pmatrix}
$$

is called the monodromy matrix and the operator

$$
T_N(\lambda|z) = A_N(\lambda|z) + D_N(\lambda|z)
$$

is called the transfer matrix. In this text we will use a compact notation for the arguments with the shifted spectral parameter: $(\lambda|z) = (\lambda - z_1, \ldots, \lambda - z_N)$.

In contrast to the trigonometric and rational cases, the eight-vertex model does not have a unique vacuum vector in its local state space. We follow [10] to overcome this difficulty by introducing a family of gauge transformations with parameters $s$ and $t$ as follows:

$$
\mathcal{L}_i(\lambda - z_i) \rightarrow \mathcal{L}_i(\lambda, z_i; s, t) = 
$$
where matrix $L_i$ has the properties:

From the local formulae (21-23) one can derive that matrix elements of (24) satisfy in the simplest possible way. Following [5], [10] we choose its lower left element. Again, in contrast to the case of the six-vertex model, the local vacuum $\omega_i^l$, independent of $\lambda$ that is annihilated for all $\lambda$ by $\omega_l$, has a local vacuum, $\omega_i^l$, where $\omega_i^l$ is not an eigenvector of $\alpha_i^l$ and $\delta_i^l$. So, the matrices $M_k$ must also be chosen in such a way, that the diagonal elements of $L_i^l$ have a local vacuum, $\omega_i^l$, independent of $\lambda$. Corresponding local vacuum

$$\omega_i^l = H(s + 2l + i)\eta - \eta + z_i)e_i^+ + \Theta(s + 2(l + i)\eta - \eta + z_i)e_i^-$$

has the properties:

$$\begin{align*}
\alpha_i^l(\lambda, z_i; s, t)\omega_i^l &= \rho(\lambda - z_i)\Theta(0)g(\lambda - z_i + \eta)\omega_i^{l+1}
\delta_i^l(\lambda, z_i; s, t)\omega_i^l &= \Theta(0)g(\lambda - z_i - \eta)\omega_i^{l-1}
\gamma_i^l(\lambda, z_i; s, t)\omega_i^l &= 0
\end{align*}$$

(23)

Now let us define the set of gauge transformed monodromy matrices

$$T_N^l(\lambda|z; s, t) = L_1^l(\lambda, z_1; s, t)L_2^l(\lambda, z_2; s, t)\cdots L_N^l(\lambda, z_N; s, t)$$

(24)

From the local formulae (21-23) one can derive that matrix elements of (24) satisfy the relations

$$\begin{align*}
A_N^l(\lambda|z; s, t)\Omega_N^l &= \prod_{i=1}^N [\rho(\lambda - z_i)\Theta(0)g(\lambda - z_i + \eta)]\Omega_{N+1}^l
B_N^l(\lambda|z; s, t)\Omega_N^l &= \prod_{i=1}^N [\rho(\lambda - z_i)\Theta(0)g(\lambda - z_i - \eta)]\Omega_{N-1}^l
C_N^l(\lambda|z; s, t)\Omega_N^l &= 0
\end{align*}$$

(25)
Where the set of generating vectors \( \{ \Omega^l_N \}_{l=-\infty}^{l=\infty} \) is defined by

\[
\Omega^l_N = \omega_1^l \otimes \omega_2^l \otimes \cdots \otimes \omega_N^l
\]  

(26)

It is useful to introduce a collection of generalized monodromy matrices

\[
\mathcal{T}_{k,l}(\lambda|z; s, t) = M_k^{-1}(\lambda|z; s, t) \mathcal{T}_N(\lambda|z) M_l(\lambda|z; s, t)
\]

(27)

where

\[
k, l = -\infty, \cdots, \infty
\]

The monodromy matrix \( \mathcal{T}_N^l(\lambda|z; s, t) \) can be written in the new notation as \( \mathcal{T}_{N+l,l}(\lambda|z; s, t) \) and for all values of \( l \)

\[
T(\lambda|z) = A_N(\lambda|z) + D_N(\lambda|z) = A_{l,l}(\lambda|z; s, t) + D_{l,l}(\lambda|z; s, t),
\]

(28)

It turns out from the relation

\[
\mathcal{R}(\lambda, \mu) \mathcal{T}_N(\lambda|z) \otimes \mathcal{T}_N(\mu|z) = \mathcal{T}_N(\mu|z) \otimes \mathcal{T}_N(\lambda|z) \mathcal{R}(\lambda, \mu)
\]

(29)

that the permutation relations for \( A_N, B_N, C_N \) and \( D_N \), as in the homogeneous case, lead to simple relations for \( A_{k,l}, B_{k,l}, C_{k,l} \) and \( D_{k,l} \):

\[
B_{k,l-1}(\lambda|z; s, t) B_{k-1,l}(\mu|z; s, t) = B_{k,l-1}(\mu|z; s, t) B_{k-1,l}(\lambda|z; s, t),
\]

(30)

\[
A_{k,l}(\lambda|z; s, t) B_{k-1,l+1}(\mu|z; s, t) = \alpha(\lambda, \mu) B_{k,l+2}(\mu|z; s, t) A_{k-1,l+1}(\lambda|z; s, t)
\]

\[+ \beta_{l+1}(\lambda, \mu) B_{k,l+2}(\lambda|z; s, t) A_{k-1,l+1}(\mu|z; s, t)
\]

(31)

\[
D_{k,l}(\lambda|z; s, t) B_{k-1,l+1}(\mu|z; s, t) = \alpha(\mu, \lambda) B_{k-2,l}(\mu|z; s, t) D_{k-1,l+1}(\lambda|z; s, t)
\]

\[- \beta_{k-1}(\lambda, \mu) B_{k-2,l}(\lambda|z; s, t) D_{k-1,l+1}(\mu|z; s, t)
\]

(32)

where

\[
\alpha(\lambda, \mu) = \frac{g(\lambda - \mu - 2\eta)}{g(\lambda - \mu)}, \quad \beta_l(\lambda, \mu) = \frac{g(2\eta)g(w_l + \lambda - \mu)}{g(\lambda - \mu)g(w_l)}
\]

(33)
Next, let us consider the Bethe vectors

\[ \Psi_t(\lambda_1, \lambda_2, \cdots, \lambda_n) = B_{t-1,t+1}(\lambda_1) \cdots B_{t-n,t+n}(\lambda_n) \Omega_{N}^{i-n} \]  \hspace{1cm} (34)

where \( n = N/2 \) (\( N \) even).

The Bethe vectors (34) are not eigenstates of the transfer matrix (28), but satisfy the relations

\[ T(\lambda|z)\Psi_t(\lambda_1, \cdots, \lambda_n) = \sum_{\alpha=1}^{n} \left[ \Lambda_1^\alpha(\lambda|z; \lambda_1, \cdots, \lambda_n) \Psi_{t+1}(\lambda_1, \cdots, \lambda_n) + \Lambda_2(\lambda|z; \lambda_1, \cdots, \lambda_n) \Psi_{t-1}(\lambda_1, \cdots, \lambda_n) \right] \]  \hspace{1cm} (35)

where

\[ \Lambda_1(\lambda|z; \lambda_1, \cdots, \lambda_n) = \prod_{i=1}^{N} \left[ \rho(\lambda - z_i) \Theta(0) g(\lambda - z_i + \eta) \right] \prod_{\alpha=1}^{n} \alpha(\lambda, \lambda_\alpha) \]
\[ \Lambda_2(\lambda|z; \lambda_1, \cdots, \lambda_n) = \prod_{i=1}^{N} \left[ \rho(\lambda - z_i) \Theta(0) g(\lambda - z_i - \eta) \right] \prod_{\alpha=1}^{n} \alpha(\lambda_\alpha, \lambda) \]  \hspace{1cm} (36)

\[ \Lambda_1^\alpha(\lambda|z; \lambda_1, \cdots, \lambda_n) = \beta_{t+1}(\lambda, \lambda_\alpha) \prod_{i=1}^{N} \left[ \rho(\lambda_\alpha - z_i) \Theta(0) g(\lambda_\alpha - z_i + \eta) \right] \prod_{\alpha=1}^{n} \alpha(\lambda_\alpha, \lambda_\beta) \]
\[ \Lambda_2^\alpha(\lambda|z; \lambda_1, \cdots, \lambda_n) = -\beta_{t-1}(\lambda, \lambda_\alpha) \prod_{i=1}^{N} \left[ \rho(\lambda_\alpha - z_i) \Theta(0) g(\lambda_\alpha - z_i - \eta) \right] \prod_{\alpha=1}^{n} \alpha(\lambda_\beta, \lambda_\alpha) \]

and

\[ \Psi_t^{(\alpha)}(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda, \lambda_{n+1}, \cdots, \lambda_n) \]  \hspace{1cm} (37)

The equation (35) will be referred as OSBAE for 8-vertex model. It is easy to show that \( \sum_{t=-\infty}^{\infty} \omega^t \Psi_t \) is an eigenstate of the \( T \) with eigenvalue

\[ \Lambda = \omega \Lambda_1(\lambda|z; \lambda_1, \cdots, \lambda_n) + \omega^{-1} \Lambda_2(\lambda|z; \lambda_1, \cdots, \lambda_n) \]  \hspace{1cm} (38)

provided that the \( \lambda_j \) satisfy the Bethe ansatz equations

\[ \prod_{i=1}^{N} \frac{g(\lambda_\alpha - z_i + \eta)}{g(\lambda_\alpha - z_i - \eta)} = \omega^{2} \prod_{\beta=1}^{n} \frac{\alpha(\lambda_\beta, \lambda_\alpha)}{\alpha(\lambda_\alpha, \lambda_\beta)} \]  \hspace{1cm} (39)
where $\omega = \exp(2\pi i \theta)$, with $0 \leq \theta \leq 1$. But, in what follows, we’ll not consider
the diagonalization problem of transfer matrix $T$ and therefore we’ll not impose
Bethe ansatz equations (39) on the parameters $\lambda_\alpha$. We’ll use the quasi classical
limit $\eta \to 0$ of (35) to solve some system of linear differential equation, which is an
elliptic generalization of Knizhnik-Zamolodchikov equation.

3  Semiclassical limit of the OSBAE and the XYZ
Gaudin magnets

By semiclassical limit one understands the expansion of the vertex weight $R(\lambda, \eta)$
around the point $\eta_0$, such that $R(\lambda, \eta_0) = I \otimes I [1].$ One can parameterize $\eta$, such
that $\eta_0 = 0$. In this subsection we examine the asymptotic behavior of the OSBAE
(35) when $\eta \to 0$. From the equation (12) we obtain the asymptotic behavior of
the coefficients $w_i$:

$$
\begin{align*}
  w_1(\lambda - z_i) &= \eta J_1^i(\lambda) + o(\eta^2) \\
  w_2(\lambda - z_i) &= \eta J_2^i(\lambda) + o(\eta^2) \\
  w_3(\lambda - z_i) &= \eta J_3^i(\lambda) + o(\eta^2),
\end{align*}
$$

(40)

where

$$
\begin{align*}
  J_1^i &= \frac{1 + k \text{sn}^2(\lambda - z_i)}{\text{sn}(\lambda - z_i)}, \\
  J_2^i &= \frac{1 - k \text{sn}^2(\lambda - z_i)}{\text{sn}(\lambda - z_i)}, \\
  J_3^i &= \frac{\text{cn}(\lambda - z_i) \text{dn}(\lambda - z_i)}{\text{sn}(\lambda - z_i)}.
\end{align*}
$$

(41)
Here we have used the following theta-function relations

\[
\frac{d}{d\lambda} \text{sn} \lambda = \text{cn} \lambda \text{dn} \lambda, \quad \frac{d}{d\lambda} \text{cn} \lambda = -\text{sn} \lambda \text{dn} \lambda
\]

\[
\frac{d}{d\lambda} \text{dn} \lambda = -k^2 \text{sn} \lambda \text{cn} \lambda
\]

(42)

\[
\text{sn}^2 \lambda + \text{cn}^2 \lambda = 1, \quad k^2 \text{sn}^2 \lambda + \text{dn}^2 \lambda = 1
\]

Simple calculations give us the following \(\eta\)-expansion for the transfer matrix (19)

\[
T_N(\lambda|z) = A_N(\lambda|z) + D_N(\lambda|z) \equiv 2 + 2\eta^2 T_N^{(2)} + o(\eta^3),
\]

\[
T_N^{(2)} = \sum_{i<j} \left\{ J_{ij}^1 \sigma_i^1 \otimes \sigma_j^1 + J_{ij}^2 \sigma_i^2 \otimes \sigma_j^2 + J_{ij}^3 \sigma_i^3 \otimes \sigma_j^3 \right\}.
\]

(43)

As usual \(T_N^{(2)}(\lambda|z)\) can be viewed as generating (operator) function for the Gaudin magnet hamiltonians \(H_i\)

\[
H_i = \text{res}_{\lambda \to z_i} T_N = \sum_{j \neq i} \left\{ J_{ij}^1 \sigma_i^1 \otimes \sigma_j^1 + J_{ij}^2 \sigma_i^2 \otimes \sigma_j^2 + J_{ij}^3 \sigma_i^3 \otimes \sigma_j^3 \right\},
\]

(44)

where \(J_{ij}^a = J_{ij}^a(z_i)\). Here we observe that in the limit \(k \to 0\), these expressions degenerate to the trigonometric expressions presented in reference [13]. To find quasi classical version of OSBAE (35) we need also consider the quasi classical limit of the \(B_{k,l}(\lambda|z; s, t)\) operators and the global vacuum states \(\Omega^I_N\). From (27) using (40), (41) we get the following semiclassical expansion for the \(B_{k,l}(\lambda|z; s, t)\) operators and for the semiclassical vacuum states

\[
B_{k,l}(\lambda|z; s, t) = \eta^n B_{k,l}^{(1)}(\lambda|z; s, t) + o(\eta^{n+1}),
\]

\[
B_{k,l}^{(1)}(\lambda|z; s, t) = \frac{g'(0)g(t - \lambda - K)}{g^2(w)g(u + \lambda)} (l - k) + \]

\[
+ \frac{g(K)}{2g(w)g(u + \lambda)} \sum_{i=1}^N \left\{ [H^2(t - \lambda) - \Theta^2(t - \lambda)] J_i^1 \sigma_i^1 + \right.
\]

\[
+ i[H^2(t - \lambda) + \Theta^2(t - \lambda)] J_i^2 \sigma_i^2 + 2H(t - \lambda)\Theta(t - \lambda) J_i^2 \sigma_i^3 \right\}
\]

(45)
\[\Omega_N^{(l)} = \Omega_N + o(\eta),\]
\[\Omega_N = \otimes_{i=1}^N \left\{ H(s + z_i)e_i^+ + \Theta(s + z_i)e_i^- \right\} \]  (46)

Taking the residues in the pole \(\lambda = z_i\) of (33), we get
\[H_i\Phi = (\partial_w + h_i)\Phi - 2\sum_{\alpha=1}^n (\partial_w + f_\alpha)\Phi_{\alpha,i},\]  (47)

where

\[h_i = \frac{1}{2} \sum_{j \neq i}^N \log' g(z_i - z_j) - \sum_{\alpha=1}^n \log' g(z_i - \lambda_\alpha)\]  (48)

\[f_\alpha = -\sum_{\beta \neq \alpha}^n \log' g(\lambda_\alpha - \lambda_\beta) + \frac{1}{2} \sum_{j=1}^N \log' g(\lambda_\alpha - z_j),\]  (49)

\[\Phi = B_{l-1,l+1}^{(1)}(\lambda_1) \cdots B_{l-n,l+n}^{(1)}(\lambda_n) \Omega\]
\[\Phi_{\alpha,i} = B_{l-1,l+1}^{(1)}(\lambda_1) \cdots B_{l-n,l+n}^{(1)}(\lambda_{\alpha-1}) B_{\alpha,i} \times\]
\[\times B_{l-1,a,l+1+\alpha}(\lambda_{\alpha+1}) \cdots B_{l-a,l+n}^{(1)}(\lambda_n) \Omega,\]  (51)

\[B_{\alpha,i} = \text{res}_{\lambda = z_i} B_{l-1,a,l+\alpha}^{(1)} \frac{g'(0)g(z_i - \lambda_\alpha + w)}{g(w)g(\lambda_\alpha - z_i)}.\]  (52)

In (47) have used the relation
\[\Psi_{l\pm 1}(\lambda|z; s, t) - \Psi_l(\lambda|z; s, t) = \pm 2\eta^{n+1}\partial_w \Phi(\lambda|z; s, t) + o(\eta^{n+2}),\]  (53)

which is a direct consequence of the remarkable fact, that
\[\Psi_{l\pm 1}(\lambda|z; s, t) = \Psi_l(\lambda|z; s \pm 2\eta, t \pm 2\eta).\]  (54)

One may use the equations (47)-(52) to obtain the algebraic Bethe ansatz solution for the spin-1/2 XYZ Gaudin model. Indeed, using the periodicity of \(\Phi, \Phi_{\alpha,i}\) over the parameter \(w\) with period \(4K\), it is easy to see, that Fourier component \(\Phi_l\) of the semi classical Bethe wave vector \(\Phi\)
\[\Phi_l = \int_{-2K}^{2K} e^{\frac{inlw}{2K}} \Phi dw\]  (55)
is the eigenvector of Gaudin hamiltonian \( H_i \) with eigenvalue \( h_i + i\pi l/2K \), provided that the Bethe ansatz equation

\[
f_\alpha + i\frac{\pi l}{2K} = 0
\]

for the parameters \( \lambda_\alpha \) are valid. But our purpose in this article is to obtain solutions for an elliptic generalization of KZ equation, and therefore in what follows, we’ll use (47) without imposing the condition (56).

4 Elliptic Knizhnik-Zamolodchikov equation

The off-shell Bethe ansatz equations corresponding to rational and trigonometric Gaudin magnets provide solution for the following differential equation [11]-[14], [15], [16]

\[
\kappa \frac{\partial \Psi}{\partial z_i} = \sum_{j \neq i}^N r_{i,j}(z_i - z_j)\Psi
\]

(57)

Where \( r_{i,j}(\lambda) \) is trigonometric or rational solution of quasi classical Yang-Baxter equation and operator \( H_i = \sum_{j \neq i}^N r_{i,j}(z_i - z_j) \) coincides with Gaudin magnet Hamiltonian. Recall, that self consistency condition of (57) is the quasi classical Yang-Baxter equation for \( r_{i,j}(\lambda) \). Of course, elliptic \( r_{i,j}(\lambda) \) also obeys the quasi classical Yang-Baxter equation, but corresponding differential equation (57) can not be solved via off-shell Bethe ansatz method. The main reason is the fact, that the quasi vacuum \( \Omega_N \) of the Gaudin elliptic magnet (144), essentially depends on inhomogeneities \( z_1, \cdots z_N \) and auxiliary parameter \( s \) (see (46)). The elliptic generalization of (57) is considered in [21] but till now, an integral representation of this differential equation is not constructed. In this section via off-shell Bethe ansatz method we are constructing the solution for the following modification of (57) which we’ll call the Elliptic Knizhnik-Zamolodchikov (EKZ) equation:

\[
\kappa \nabla_j \Psi = H_j \Psi - \frac{\partial}{\partial w}(\Sigma_j \Psi),
\]

(58)
where

\[ \nabla_j = \frac{\partial}{\partial z_j} - \Gamma_j, \]

(59)

The operators \( \Gamma_j \) and \( \Sigma_j \) act on two dimensional space \( V_j \) and have the following matrix elements

\[
\begin{align*}
\Gamma_{11} &= \frac{1}{2} \log' g(w) + \frac{1}{2} \log' g(z_j + u); \\
\Gamma_{12} &= \frac{g'(0)g(w - u - z_j)}{2g(w)g(z_j + u)}, \\
\Gamma_{21} &= -\frac{g'(0)g(w + u + z_j)}{2g(w)g(z_j + u)}, \\
\Gamma_{22} &= -\frac{1}{2} \log' g(w) + \frac{1}{2} \log' g(z_j + u),
\end{align*}
\]

(60)

\[
\begin{align*}
\Sigma_{11} &= g(w + K)g(u + z_j - K) / g(w)g(u + z_j); \\
\Sigma_{12} &= \frac{g(K)H(s + z_j)H(t - z_j)}{g(w)g(u + z_j)}, \\
\Sigma_{21} &= -\frac{g(K)\Theta(s + z_j)\Theta(t - z_j)}{g(w)g(u + z_j)}; \\
\Sigma_{22} &= -\frac{g(w + K)g(u + z_j - K)}{g(w)g(u + z_j)}.
\end{align*}
\]

(61)

The solution of the EKZ equation (58) is given by the multiple integral

\[ \Psi = \oint \cdots \oint X(\lambda|z)\Phi(\lambda|z;s,t) d\lambda_1 \cdots d\lambda_n, \]

(62)

where \( \Phi \) is the quasi classical Bethe wave vector (50) and \( X(\lambda|z) \) is a solution of the following self consistent system of differential equations

\[
\kappa \partial_{z_j} X = h_j X, \quad (63)
\]

\[
\kappa \partial_{\lambda_\alpha} X = -2f_\alpha X. \quad (64)
\]

It is easy to see that solution is given by

\[ X(\lambda|z) = \prod_{j<i}^N g(z_j - z_i)^{1/2\kappa} \prod_{\alpha<\beta}^n g(\lambda_\alpha - \lambda_\beta)^{2/\kappa} \prod_{j,\gamma} g(z_j - \lambda_\gamma)^{-1/\kappa}, \]

(65)

Integration in (62) is over closed, homologically nontrivial circles of the set \( \mathbb{C}^n \setminus \mathbb{D} \subset \mathbb{C}^n \), where \( (\lambda_1, \cdots, \lambda_n) \in \mathbb{D} \) if and only if if \( \lambda_\alpha = z_j \) or \( \lambda_\alpha = \lambda_\beta \) or \( \lambda_\alpha = u \) for some \( \alpha, \beta \in \{1, 2, \cdots, n\} \) and \( j \in \{1, 2, \cdots, N\} \). The proof, that \( \Psi(z_1, \cdots, z_N|s,t) \) given by (62) indeed is solution of the EKZ equation (58) is straightforward, if one takes into account OSBA equation (47) and the relations

\[ \partial_{z_j} \Phi = \Gamma_j \Phi - \sum_{\alpha=1}^n \partial_{\lambda_\alpha} \Phi_{\alpha,j}, \]

(66)
\[ \sum_j \Phi = \Phi - 2 \sum_{\alpha=1}^{n} \Phi_{\alpha,j}. \]  

(67)

The identities can be proved directly, but it is easier first to perform gauge transformation \( \Phi \to G\Phi \), where

\[ G = G_1 \otimes \cdots \otimes G_N \]  

(68)

and

\[ G_i = \begin{pmatrix} H(s - z_i) & H(t - z_i) \\ \Theta(s + z_i) & \Theta(t - z_i) \end{pmatrix} \]  

(69)

This will be done in the next section.

5 Elliptic Knizhnik-Zamolodchikov equation and KZB equation

In this section we are going to show, that under some appropriately chosen gauge transformation Elliptic KZ equation transforms into KZB equation. Recall that KZB equation is linear differential equation for N-point correlation function of WZNW model on torus. \( \Gamma_j \) does not depend on \( z_i \), \( i \neq j \), and acts non trivially only on \( j \)-th component of tensor product, therefore the connection \( \Gamma \) is flat

\[ [\nabla_j, \nabla_i] = \frac{\partial}{\partial z_j} \Gamma_i - \frac{\partial}{\partial z_i} \Gamma_j - [\Gamma_j, \Gamma_i] = 0 \]  

(70)

This means that there exists some gauge transformation \( G(z|s, t) \) such that

\[ \Gamma_j^G = G^{-1} \Gamma_j G + \left( \partial_j G^{-1} \right) G \equiv 0 \]  

(71)

Now it is not difficult to see that \( G(z|s, t) \), which satisfies above relation (71) is given by (4), (69). In other words we have

\[ G^{-1} \nabla_j G = \frac{\partial}{\partial z_j} \]  

(72)
To find gauge transformed version of EKZ, we also need expressions for $G^{-1}H_jG$ and $G^{-1}\Sigma_jG$. By direct calculation we have

$$G^{-1}H_jG = \Omega_j + \left(G^{-1}\partial_u G - \frac{N}{2} \log' g(w)\right) \sigma_j^3$$  \hspace{1cm} (73)

$$G^{-1}\Sigma_jG = \sigma_j^3,$$  \hspace{1cm} (74)

Where

$$\Omega_j = \sum_{k \neq j}^N \left[\frac{1}{2} \log' g(z_{jk}) \sigma_j^3 \otimes \sigma_k^3 + \frac{g'(0)g(w + z_{jk})}{g(w)g(z_{jk})} \sigma_j^- \otimes \sigma_k^+ + \frac{g'(0)g(w - z_{jk})}{g(w)g(z_{jk})} \sigma_j^+ \otimes \sigma_k^- \right].$$  \hspace{1cm} (75)

Using (72)-(75) one easily obtains that $\phi \equiv G^{-1}\Psi$ obeys the following differential equation

$$\kappa \frac{\partial}{\partial z_j} \phi = \Omega_j \phi - \sigma_j^3 \partial_u \phi$$  \hspace{1cm} (76)

which is nothing but the KZB equation in special case of $sl(2,c)$ algebra, when all representations are taken fundamental. Now let us perform gauge transformation on the solution of EKZ equation (62). For $G^{-1}B_{k,l}(\lambda)G$ one obtains

$$G^{-1}B_{k,l}(\lambda)G = \frac{g'(0)g(t - \lambda - K)}{g(w)g(u + \lambda)} \left[(l - k) - \sum_{i=1}^N \sigma_i^2 \right] + \sum_{i=1}^N \frac{2g'(0)g(w - \lambda + z_i)}{g(w)g(\lambda - z_i)} \sigma_i^- \equiv \beta_{l-k}(\lambda),$$  \hspace{1cm} (77)

where $\sigma^\pm = (\sigma^1 \pm \sigma^2)/2$. Similarly acting on $\Omega$ by the operator $G^{-1}$ we obtain

$$G^{-1}\Omega = e_1^+ \otimes e_2^+ \otimes \cdots \otimes e_N^+ \equiv |0 >.$$  \hspace{1cm} (78)

Combining (50), (77) and (78) we get the following expression for the gauge transformed Bethe wave vector

$$G^{-1}\Phi = \beta_2(\lambda_1)\beta_4(\lambda_2) \cdots \beta_{2n}(\lambda_n)|0 > = \tilde{\beta}(\lambda_1)\tilde{\beta}(\lambda_2) \cdots \tilde{\beta}(\lambda_n)|0 >,$$  \hspace{1cm} (79)
where we have introduced the notation $\hat{\beta}(\lambda)$ for the second term in (77)

$$\hat{\beta}(\lambda) = \sum_{i=1}^{N} \frac{2g'(0)g(w - \lambda + z_i)}{g(w)g(\lambda - z_i)} \sigma_i. \quad (80)$$

The second equality in (79) follows from the fact, that first term in (77) does not give any contribution. So, finally for the solution of KZB equation (76) we have

$$\phi = \oint \cdots \oint \mathcal{X}(\lambda | z) \hat{\beta}(\lambda_1) \cdots \hat{\beta}(\lambda_n) |0 \rangle \ d\lambda_1 \cdots d\lambda_n, \quad (81)$$

which is in complete agreement with the solution of KZB given by Felder and Varchenko [20].

6 Conclusion

In this paper an elliptic generalization of the Knizhnik-Zamolodchikov equation is constructed. The main constructing block in this differential equation is the quasi classical limit of the Baxters 8-vertex R-matrix, or in other words the elliptic Gaudin Hamiltonian. An integral representation for the solution of this differential equation is found, using off-shell Bethe ansatz method. It is shown that above mentioned differential equation is connected with KZB equation via a gauge transformation. It is worth noting that gauge transformation is the quasi classical analogue of Baxter’s famous 8-vertex-RSOS correspondence.

Acknowledgments In the course of this work we benefited from useful discussion with A.A.Belavin, R. Flume and one of us (H.B.) with A. P. Veselov, M. Karowski, M. Schmidt. It is HB’s pleasure to thank the Departamento de Física da Universidade Federal de São Carlos for hospitality and to FAPESP, Fundação de Amparo a Pesquisa do Estado de São Paulo, for financial support.

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