Abstract. We prove that Bi-Zassenhaus loop algebras are finitely presented up to central and second central elements. In particular, we show an explicit finite presentation for a Lie algebra whose quotient over its second centre is isomorphic to a Bi-Zassenhaus loop algebra.

1. Introduction

The family of simple modular Lie algebras nowadays known as Hamiltonian algebras was discovered in the fifties by Albert and Frank [AF55] and it was named after them. By definition, they are graded over a non-cyclic elementary abelian $p$-group. Shalev [Sha94] noticed that they also have a cyclic grading with a non-singular derivation cycling over this grading. This property allowed him to build an infinite-dimensional loop algebra starting from the original simple one. The new structure is of maximal class, i.e. it is $N$-graded with all homogeneous components of dimension one except the first one which has dimension two and generates the entire algebra. These new algebras are known as Albert-Frank-Shalev (AFS, for short) algebras.

In the following years there has been a growing interest in these algebras. Caranti, Mattarei and Newman developed new techniques to construct more algebras [CMN97] and later Caranti and Newman achieved a classification theorem for odd characteristic fields of definition [CN00]. The remarkable result they reached yields that every infinite-dimensional $N$-graded Lie algebras of maximal class generated by its first homogeneous component can be built starting from an AFS-algebra. To get the claim they need a theorem proved by Carrara in [Car98, Car01], which states that every AFS algebra is uniquely determined by a suitable finite-dimensional quotient. This result was obtained as a corollary of a more general property she discovered: AFS-algebras are finitely presented up to central and second central elements. A similar classification for characteristic two is still valid [Jur05], but in this case another family of algebras is involved, the Bi-Zassenhaus loop algebras ($B_l$ for short). The loop construction process for them is exactly the same as for Albert-Frank-Shalev algebras, but the simple finite-dimensional algebra at the beginning is different. Here we show that the property described above for AFS-algebras is satisfied by $B_l$-algebras, too. The fact that $B_l$-algebras are uniquely determined by a suitable finite-dimensional quotient again follows as a corollary. In particular, the proven result is the following

**Theorem.** For every Bi-Zassenhaus loop algebra $B_l(g, h)$, there exists a finitely presented graded Lie algebra $M(g, h)$ such that $M(g, h)/Z_2(M(g, h)) \cong B_l(g, h)$.

The author is a member of UMI and INdAM-GNSAGA.
The construction of the algebra $M$ by means of cohomological arguments is shown in [Jur04]. Since the centre of $M$ is infinite-dimensional, a group-theoretical result of B.H. Neumann [Rob93, pp. 52-53] transferred to Lie algebra shows that this implies that $B_l$ itself is not finitely presented. The relations in the presentation of the algebra $M$ are retrieved by expanding several suitable generalized Jacobi identities, thus machine computations took a main role throughout the work. These computations were performed by the software $p$-Quotient Program [HNO97] developed at the Australian National University in Canberra. Nevertheless, all the proofs are independent from such calculations.

2. Preliminaries

A graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i ,$$

defined over a field of positive characteristic $p$ and generated by $L_1$ is said to be of maximal class if $\dim(L_1) = 2$ and $\dim(L_i) \leq 1$ for $i > 1$. Its elements are written left-normed and in exponential form:

$$[uv^n] = [[[uv]v]v \cdots v] .$$

The two-step centralizers $C_i$ of $L$ are defined as the one-dimensional subspaces $C_{L_1}(L_i)$ of $L_1$ centralizing the homogeneous components $L_i$ when $i > 1$, while $C_1 = C_2$ is formally assumed. Following the classical notation, we define the element $y$ by means of $C_2 = \mathbb{F}y$ and we choose another element $x \in L_1 \setminus \mathbb{F}y$. The pair $\{x, y\}$ is thus a set of generators for $L_1$. Let $C_k$ the first occurrence of a two-step centralizer different from $\mathbb{F}y$. When the class of $L$ is larger than $k + 1$, all the two-step centralizers coincide with $C_2$ apart from isolated occurrences of different subspaces, so it is possible to define a constituent of $L$ as a subsequence $(C_{i_1}, \ldots, C_{i_j})$ where all centralizers coincide with $C_2$ but the last one, and either $i_1 = 1$ or $C_{i_1} \neq C_2$. Homogeneous elements lying in class $i_1 - 1$ and $j$ are respectively said to be at the beginning and at the end of the constituent and they are not centralized by $C_2$.

A graded Lie algebras of maximal class is determined by its sequence of two-step centralizers, and, when only two of them are distinct, by its sequence of constituent lengths. The first constituent of a graded Lie algebra of maximal class whose class is larger than $k + 2$ is always of length $2q$ for some power $q = p^h$ called its parameter, and all other constituents can only be short (i.e. of length $q$) or long ($2q$) or intermediate ($2q - p^s$, where $0 \leq s \leq h - 1$). The standard result [CMN97, Prop. 5.6] holds also when central and second central elements occur, as shown in [Car01]: for a (large enough) algebra of parameter $q$,

$$(CL) \quad \text{the only possible constituent lengths are } 2q \text{ and } 2q - 2^s, \ 0 \leq s \leq h .$$

From now on, unless explicitly stated, we will assume $p = 2$. This allows us to ignore signs safely so that the generalized Jacobi identity reads as follows in the two
The generalized Jacobi identity can be more effectively used by introducing the element 

\[ [v^i w^j u^k] = \sum_{i=0}^{\lambda} \binom{\lambda}{i} [v^i u^j w^{\lambda-i}] = \sum_{i=0}^{\lambda} \binom{\lambda}{i} [v^i w^{\lambda-i} u^i]. \]

The generalized Jacobi identity can be more effectively used by introducing the element \( z = x + y \): if \( 0 \neq v \in L_i \), for some \( i > 1 \), and if \( C_i, C_{i+1}, \ldots, C_{i+n-1} \in \{ \mathbb{F}x, \mathbb{F}y \} \), then for every non-zero commutator \([v x_1 x_2 \ldots x_n]\) with \( x_i \in \{x, y\} \) we have that \([v x_1 x_2 \ldots x_n] = [v z^n]\).

The Bi-Zassenhaus loop algebras \( B_1(g, h) \) form a family of elementary objects among infinite-dimensional graded Lie algebras of maximal class. They are defined when the characteristic of the underlying field is two and \((h, g-1) \in \mathbb{N} \times \mathbb{N}\). If exponential notation is employed in order to indicate consecutive occurrences of patterns, the sequence of constituent lengths of the algebra \( B_3(g, h) \) reads as

\[ 2q, 2q - 1, (2q)^{-1}, (2q - 1)^2 \] \( \infty \),

where \( q = 2^h \) and \( \eta = 2^{g-1} \). Thus in a Bi-algebra the only intermediate constituents are those of maximal length \( 2q - 1 \) and there are no short constituents. Suppose \( M = \bigoplus_{i=1}^{\infty} M_i \) is a graded Lie algebra such that \( M/Z_2(M) \) is a graded Lie algebra of maximal class. We define a constituent of \( M \) as a constituent of \( M/Z_2(M) \), ignoring the central or second central elements of \( M \).

The notation \( w^{-c} \) will be used for an homogeneous element such that \([w^{-c}x^c] = w\).

Then we define the elements

\[ v_n = [yx^{2q-1}(yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2})^n] \],

which lie in the homogeneous components of indices \( 2q + dn \) where \( d = 2^{g+h+1}-2 \) is the dimension of the simple algebra \( B(g, h) \) and \( n \) ranges over the non-negative integers.

When the two parameters \( a, b \) lie in the intervals \( 2 \leq a \leq h+1 \) and \( h+2 \leq b \leq g+h \), we can define the following elements, consistently with the notation in \[Jur04\]:

\[
\begin{align*}
\theta^1_n &= [v_n x] \\
\theta^a_n &= [v_n y x^{2q-2^{h+2-a-1}} y] \\
\theta^b_n &= [v_n y x^{2q-2} (yx^{2q-1})^{\eta-2^{g+h+1-b}} y x^{2q-2} y] \\
\theta^\omega_n &= [\theta^1_{2n+1} y] \\
\end{align*}
\]

\[ (2q + 1 + dn) \]

\[ (4q - 2^{h+2-a} + 1 + dn) \]

\[ (2q + 3 - 2^{g+h+1-b}) + 1 + dn \]

\[ (q + 2 + d(2n + 1)) \]

The number in parentheses is the weight of the element, i.e. the index \( i \geq 1 \) of the homogeneous component \( M_i \) in which it lies. Finally, we define the shorthands

\[ \mu_{n,i} = \left\{ \begin{array}{ll}
[v_n y x^{2q-3}] & \text{for } i = 1, \\
[v_n y x^{2q-2} (yx^{2q-1})^{i-2} y x^{2q-2}] & \text{otherwise}.
\end{array} \right. \]

Note that \([\mu_0, \eta-2^{g+h+1+b} + 2y] = \theta^b_0 \) when \( h + 2 \leq b \leq g + h \).
3. The finite presentation

Fix a particular \( B_l \) algebra \( L \) and then find a finite set \( R' \) of homogeneous relations such that \( M' = \langle x, y : R' \rangle \) is a presentation of a suitable \( m \)-dimensional Lie algebra such that \( M' / \mathbb{Z}_2(M') \) is isomorphic to a graded quotient of \( L \) and remove the relations of degree \( m + 1 \). What obtained will be shown to be a finitely presented, infinite-dimensional graded Lie algebra \( M \) such that \( M / \mathbb{Z}_2(M) \cong L \). Adding to \( R' \) the set of relations used to annihilate the central and second central elements, a graded Lie algebra of maximal class \( M \) gets defined, isomorphic to \( L \) and uniquely determined by a suitable finite-dimensional quotient \( M' \). When all the homogeneous relations that define the algebra \( L \) up to central and second central elements up to class \( m = 2q(\eta+2) \) are added to \( R' \), then the resulting algebra \( M \) starts as a graded Lie algebra of maximal class with initial segment of constituent lengths \( 2q, 2q-1, 2q^{-1}, 2q-1 \) and \( C_i \in \{ \mathbb{F}x, \mathbb{F}y \} \) for every \( 1 \leq i \leq m \), but most of the \( m-2 \) relations that in every class \( 2 \leq i \leq m \) set the two-step centralizer \( C_i \) are actually redundant. Thus the proof of the main theorem reduces to show that the following \( q + h + \eta \) relations

\[
\begin{align*}
[yx^{2j+1}y] &= 0, & 0 \leq j &\leq q-2, \\
[\theta^i_0] &= 0, & 1 \leq t &\leq g + h \text{ or } t = \omega, \\
[\mu_{0,t+2y}] &= 0, & 0 \leq t &\leq \eta - 2 \text{ and } \eta - t \neq 2^\alpha, \text{ for } 1 \leq \alpha \leq g - 1.
\end{align*}
\]

defining the set \( R \) are sufficient to give a presentation for \( M(g, h) \).

The first set of \( q \) relations defines the parameter \( q = 2^h \):

\[
yx^{2j+1}y = 0 \quad \text{for} \quad 0 \leq j \leq q - 2, \quad [yx^{2q+1}] = 0.
\]

This makes all homogeneous components have the same two-step centralizer up to weight \( 2q \). Moreover, they force all further constituents to be short, long or intermediate. The next relation

\[
[yx^{2q-1}yx^{q-1}yx] = 0
\]

states that the second constituent is not short: in particular, this implies that no more short constituents or two-step centralizers other than the first two will be involved. A standard argument shows that the second constituent cannot be long, so it can be only intermediate. The following \( h - 1 \) relations

\[
[yx^{2q-1}yx^{2q-2s-1}yx] = 0 \quad \text{for} \quad 1 \leq s \leq h - 1
\]

establish its length as the maximal possible \( 2q - 1 \). So the algebra is not inflated and starts moving on the branch of the limit algebra \( \text{AFS}(h, h+1, \infty, 2) \): furthermore, we know that from now on its sequence of constituent lengths will contain only long and maximal intermediate constituents. By the classification we know that the number of long constituents before two intermediate ones is a power of two minus one or minus two. The remaining \( \eta \) relations determine the algebra as belonging to the \( B_l \) family rather than to the AFS one:

\[
[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^t yx^{2q-2}yx(x)] = 0 \quad \text{for} \quad 0 \leq t \leq \eta - 2, \\
[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^\eta - 1 yx^{2q-1}yx] = 0,
\]

where the \((x)\) appears only when \( \eta - t \) is a power of two.
4. The proof: the structure

We will prove that the sequence of constituent lengths of \((M, R)\) and \(L\) are the same and that the two algebras differ only in central or second central elements; in particular, we claim that the elements \(\theta_n^q\) are the only central and second central elements that may occur: the construction shown in \([\text{Jur04}]\) completes the picture. For clarity, the expansion of the following Jacobi identities which are not immediate is carried out in Appendix A.

4.1. Using the first group of relations. The first homogeneous component is obviously two-dimensional:

\[ M_1 = \langle x, y \rangle. \]

The first group of relations written above shows the structure of \(M\) up to class \(2q+1\):

\[ M_i = \langle [yx^{i-1}] \rangle \quad \text{for} \quad 2 \leq i \leq 2q, \quad M_{2q+1} = \langle [yx^{2q-1}y], [yx^{2q}] = \theta_0^q \rangle. \]

In fact, the elements \([yx^jy]\) for \(0 \leq j \leq 2q - 2\)
vanish either by the relations in \(R\) (in the case \(j\) odd) or by the following inductive argument (when \(j\) is even):

\[ 0 = [[yx^i][yx^j]] = \left(\frac{i}{2}\right)[yx^jy]. \]

Finally, the relation in class \(2q + 2\)

\[ [yx^{2q+1}] = [\theta_0^q x] = 0 \]

and the expansions

\[ 0 = [yx^{2q-2}[xy]] = [yx^{2q-1}yy], \]

\[ 0 = [[yx^q][yx^q]] = \left(\frac{q}{q-1}\right)[yx^{2q-1}y] + \left(\frac{q}{q}\right)[yx^{2q}y] = [\theta_0^q y] \]

show that the first constituent of \(M\) has length \(2q\), that \(C_{2q} = \mathbb{F}x\) and that \(\theta_0^1\) is central:

\[ M_{2q+2} = \langle [yx^{2q-1}y, x] \rangle. \]

4.2. The constituent lengths. The relations just obtained are those needed to prove that property \((\mathcal{C}L)\) holds in \(M\). As a direct consequence, if \(v\) is not centralized by \(y\), then

\[ [v, yx^k] = 0 \quad \text{when} \quad k \neq 2q - 2^a - 1 \quad \text{for} \quad 0 \leq s \leq h. \]

By using this equation, it is not difficult to show that the second constituent of \(M\) is of maximal intermediate length \(2q - 1\), that its last element is centralized by \(x\) and that the elements \(\theta_0^a\) are central, for \(2 \leq a \leq h + 1\):

\[ M_{2q+i} = \begin{cases} \langle [yx^{2q-1}yx^{i-1}] \rangle, & \text{for} \quad 3 \leq i \leq 2q - 1, \\ \langle yx^{2q-1}yx^{i-2}y \rangle = \theta_0^q & i = 2q - 2^{h+2-a} + 1, 2 \leq a \leq h + 1, \\ \langle [yx^{2q-1}yx^{i-1}] \rangle, & \text{otherwise} \end{cases}. \]

\[ M_{4q} = \langle [yx^{2q-1}yx^{2q-2}y] \rangle. \]
Since two consecutive values of $i$ cannot both be of the type $2q - 2^{h+2-a} - 1$, the thesis follows from the fact that every element

$$[yx^{2q-1}yx^i]$$

vanishes by the property $(CL)$ when $0 \leq j \leq 2q - 3$ and $j \neq 2q - 2^s - 1$ for $1 \leq s \leq h$ and otherwise, by the relation

$$[yx^{2q-1}yx^i] = [\theta_0^{h+2-s}x] = 0$$

and the expansions

$$0 = [yx^{2q-1}yx^i[yxy]] = [yx^{2q-1}yx^jyy] = [\theta_0^{h+2-s}y],$$

$$0 = [[yx^{2q-1}]yx^{2q-1}] = \left(\begin{array}{c} 2q - 1 \\ 0 \end{array}\right) [yx^{2q-1}yx^{2q-1}].$$

4.3. The number of long constituents. Now we deal with the remaining structure of $M$ up to class $m = 2q(\eta + 2)$, by showing that when $i$ runs between zero and $\eta - 1$ and $k$ between zero and $2q - 1$, with $k \neq 2q - 1$ for $i = \eta - 1$, we have that

$$M_{4q+2q+i+k} = \begin{cases} 
\langle [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^iyx^{2q-1}] \rangle & \text{for } k = 2q - 1, \\
[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^iyx^{2q-2}] & \text{for } i = \eta - 2^q + h - 1, \text{ and } h + 2 \leq b \leq g + h, \\
\langle [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^iyx^k] \rangle & \text{otherwise}.
\end{cases}$$

To prove the above result, we consider separately some cases:

- when $k = 0$
  - and $i > 0$ (the case $i = 0$ has been already dealt with in the previous subsection), first of all we have the relation $[\theta_0^1x] = [yx^{2q+1}] = 0$:

$$0 = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i-1}yx^{2q-3}yx^i] = \left(\begin{array}{c} 2q + 1 \\ 1 \end{array}\right) [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^iyx^{2q-1}];$$

  - in particular, if $i = \eta - 2^q + h - 1$, for $h + 2 \leq b \leq g + h$, then the previous homogeneous component has dimension two; in this case, to show that $\theta_0^b$ is central, we can use the relation

$$[\theta_0^b] = 0$$

and the standard expansion

$$0 = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i-1}yx^{2q-3}yx^i] = [\theta_0^by];$$

- when $k = 2q - 2^s$, for $1 \leq s \leq h$,
  - and $i = 0$, we can use the following Jacobi expansion:

$$0 = \left[[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i-1}yx^{2q-2}yx^{2q-1}y]\right] = [yx^{2q-1}yx^{2q-2}yx^{k+1}y];$$

  - and $i > 0$, by using the relation $[\theta_0^{a+1}x] = 0$ in the following identity, which is the case $n = 0$ of $(8)$:

$$0 = (\mu_{0,i}[\theta_0^{a+1}] = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^iyx^{k+1}y].)$$
• when \( k = 2q - 1 \) and \( i \leq \eta - 2 \), the thesis explicitly follows by the relation
\[
[\mu_{0, i+2} y] = 0,
\]
when \( i \neq \eta - 2^{q+h+1-b} \), while otherwise there is nothing to prove since the homogeneous component is two-dimensional.

• by proposition (CL) in the remaining cases.

So we have proved that \( M \) has the same structure of \( L \) up to class \( m \), apart from some central elements.

4.4. From the quotient algebra to the whole algebra. Now we have to prove that the finite-dimensional quotient we built determines \( M \) as a \( B_L \)-algebra, apart from the central and second central elements \( \theta_{n}^{1} \), for any integer \( n \geq 1 \).

The last equation in the previous section shows that
\[
M_{2q+d} = \langle v_1 \rangle ;
\]
by induction the homogeneous component in class \( 2q + dn \) is one-dimensional too, generated by the element
\[
M_{2q+dn} = \langle v_n \rangle .
\]

In what follows we describe the structure of the next \( d \) homogeneous components of \( M \), i.e. of an entire period of the algebra. The homogeneous component in class \( 2q + 1 + dn \) is two-dimensional:
\[
M_{2q+1+dn} = \langle [v_n x] = \theta_{n}^{1}, [v_n y]\rangle ,
\]
where \( \theta_{n}^{1} \) can be central or second central, since in the next class
\[
M_{2q+2+dn} = \begin{cases} 
\langle [v_n y x] \rangle \\
\langle [v_n x y], [v_n x y] = \theta_{n}^{1} \rangle 
\end{cases}
\]
when \( n \) is even ,
\[
\langle [v_n x y], [v_n x y] = \theta_{n}^{1} \rangle
\]
when \( n \) is odd .

In fact, in addition to the standard identities
\[
0 = [v_{n}^{-1} y x y] = [v_{n} y y]
\]
and the following one derived by the relation \( [\theta_{0}^{1} y] = [y x^{2q+1}] = 0 \)
\[
0 = [v_{n-1}^{-1} y x^{2q-2} (x^{2q-1})^{n-2} y x^{2q-2} [\theta_{0}^{1} x]]
\]
\[
= \left( 2q + 1 \right) [v_{n-1}^{-1} y x^{2q-2} (x^{2q-1})^{n-1} y x^{2q}] = [\theta_{n}^{1} x] = [v_{n} x x] ,
\]
when \( n = 2s \) is even we have the equation
\[
[v_n x y] = [\theta_{n}^{1} y] = 0 ,
\]
given by the expansion
\[
(2) \quad [[v_{s}^{-1(q-1)}]][[v_{s}^{-1(q-1)}]] = 0 ,
\]
that shows the centrality of \( \theta_{n}^{1} \) in those cases.

Now we consider class \( 2q + 3 + dn \): if \( n \geq 1 \) is odd, first we have the equation
\[
0 = [v_n [y x y]] = [\theta_{\omega n}^{1} y] = [v_n x y y] ;
\]
then, if \( n = 1 \) we have the relation
\[
0 = [\theta_{0}^{1} x] = [v_n x y x] ,
\]
otherwise we can use the expansion
\[
(3) \quad 0 = [v_{n-2}y x^{2q-2}(y x^{2q-1})^n y x^{2q-2} (\theta_0^x)] = [\theta_n^{\omega} x] = [v_n y x] ,
\]
to show that the element \( \theta_n^{\omega} x \) is central. Finally, if \( q = 2 \) the element \([v_n y x]\) is just \( \theta_n^2 \), while when \( q > 2 \) the relation \([y x x y]\) = 0 holds, so we can expand
\[
0 = [v_n^{-2}[y x x y]] = \left( \frac{3}{2} \right) [v_n y x y] = [v_n y x] .
\]

Summarizing, we have
\[
M_{2q+3+dn} = \begin{cases} 
\langle [v_n y x], [v_n y x y] = \theta_n^2 \rangle & \text{when } q = 2 , \\
\langle [v_n y x] \rangle & \text{otherwise} .
\end{cases}
\]

Now we focus the attention on the homogeneous components up to class \( 4q + dn \).
If \( q > 2 \), then \( 2q + 3 + dn < 4q - 1 + dn \) so we can study what happens in classes \( 2q + 1 + k + dn \), where \( 3 \leq k \leq 2q - 3 \). We claim that
\[
M_{2q+1+k+dn} = \begin{cases} 
\langle [v_n y x^k], [v_n y x^h y] = \theta_n^a \rangle & \text{when } k = 2q - 2^{h+2-a} \\
\langle [v_n y x^k] \rangle & \text{for some } 2 \leq a \leq h + 1 , \\
\langle [v_n y x^k] \rangle & \text{otherwise} .
\end{cases}
\]

If \( k = 2q - 2^{h+2-a} \) for some \( 2 \leq a \leq h + 1 \), by hypothesis \( M_{2q+1+k+dn} \) is generated by both \([v_n y x^k]\) and \([v_n y x^h y]\); since \( h + 2 - a > 0 \) the number \( k + 1 \) cannot be of the form \( 2q - 2^{h+2-a'} \) for any \( 2 \leq a' \leq h + 1 \), so \([v_n y x^k y] = 0 \) by \((\mathcal{CL})\); moreover, the following equation \( (4) \)
\[
(4) \quad 0 = [v_{n-1} y x^{2q-2}(y x^{2q-1})^n y x^{2q-2} (\theta_0^x)] = [\theta_n^a x] ,
\]
and the standard expansion
\[
0 = [v_n y x^{2q-2} y x^{2q-2} (\theta_0^x)] = [\theta_n^a y]
\]
show that the element \([v_n y x^h y] = \theta_n^a\) is central.

The remaining case is covered by the proposition \((\mathcal{CL})\).

Now look at the class \( 4q - 1 + dn \), which in the case \( q = 2 \) (i.e. \( h = 1 \)) coincide with the class \( 2q + 3 + dn \):
\[
M_{4q-1+dn} = \langle [v_n y x^{2q-2}], [v_n y x^{2q-3} y] = \theta_n^{h+1} \rangle .
\]
The centrality of the element \( \theta_n^{h+1} \) has been already proven; moreover, the expansion
\[
(5) \quad 0 = [\Xi_n \Xi_n] = [v_n y x^{2q-1}]
\]
where
\[
\Xi_n = \begin{cases} 
\{ v_s \} & \text{for } n = 2s , \\
[v_s y x^{2q-2}(y x^{2q-1})^{\frac{a+1}{2}}] & \text{for } n = 2s + 1 ,
\end{cases}
\]
shows that
\[
M_{4q+dn} = \langle [v_n y x^{2q-2} y] \rangle = \langle [v_n y x^{2q-2}] \rangle .
\]
Finally, the last homogeneous components are generated as follows, where $i$ runs between zero and $\eta - 1$ and $k$ between zero and $2q - 1$, with $k \neq 2q - 2$ for $i = \eta - 1$:

$$M_{4q+2q\eta+k+dn} = \begin{cases} 
\langle [v_n y x^{2q-2}(yx^{2q-1})i y x^{2q-1}] \rangle, & \text{for } k = 2q - 1, \\
[v_n y x^{2q-2}(yx^{2q-1})^i y x^{2q-2}y] = \theta_{i}^h, & \text{for } \eta - 2^{g+h+1-b}, \\
\langle [v_n y x^{2q-2}(yx^{2q-1})y x^k] \rangle, & \text{for } h + 2 \leq b \leq g + h, \\
\text{otherwise}. & 
\end{cases}$$

To prove this result, we distinguish some cases.

- when $k = 0$ and $1 \leq i \leq \eta - 1$ (the case $i = 0$ has already been proven), then first we have that

$$0 = [v_n y x^{2q-2}(yx^{2q-1})^i \theta_0^i] = \begin{pmatrix} 2q \\ 0 \end{pmatrix} [v_n y x^{2q-2}(yx^{2q-1})^i y x^{2q-1} x].$$

In particular, in the case $i = \eta - 2^{g+h+1-b}$ for some integer $h + 2 \leq b \leq g + h$, two more identities are required, since $M_{4q+2q\eta+dn}$ is two-dimensional: the former reads as

$$0 = [v_n y x^{2q-2}(yx^{2q-1})^i y x^{2q-3}[y x y]] = [v_n y x^{2q-2}(yx^{2q-1})^i y x^{2q-2} y y] = [\theta_{i}^h y],$$

while the latter uses the relation $[\theta_{i}^h x] = 0$:

(6) $0 = [v_n y x^{2q-2}(yx^{2q-1})^{\eta-2 y} y x^{2q-2} \theta_{i}^h b] = [v_n y x^{2q-2}(yx^{2q-1})^i y x^{2q-2} y x] = [\theta_{i}^h x]$;

- when $k = 2q - 2^s$ for $1 \leq s \leq h$, and $i = 0$, then the relation $[\mu_{0,2} y] = 0$ can be used:

(7) $0 = [v_n y x^{2q-2}(yx^{2q-1})^{\eta-2 y} y x^{2q-2} \mu_{0,2} y] = [v_n y x^{2q-2} y x^{k-1} y];$

- and $i \geq 1$, we can use the relation $[\theta_{i}^{s+1} x] = 0$:

(8) $0 = [\mu_{0,i}] [\theta_{i}^{s+1} x] = [v_n y x^{2q-2}(yx^{2q-1})^i x x^{k-1} y]$,

- when $k = 2q - 1$,

- and $i = \eta - 2^{g+h+1-b}$ for some $h + 2 \leq b \leq g + h$, then $M_{4q+2q\eta+2q-1+dn}$ is two-dimensional;

- and $i$ is not one of the above values, then define $\lambda$ as the exponent of the highest power of two dividing $i + 1$ and use the relation $[\mu_{0, i + 2^\lambda + 1} y] = 0$ as follows:

(9) $0 = [v_n y x^{2q-2}(yx^{2q-1})^{\eta-2^\lambda y} x^{2q-2} \mu_{0, i + 2^\lambda + 1} y] = [v_n y x^{2q-2}(yx^{2q-1})^i x x^{2q-2}];$

- when $k$ is not one of the above values, then proposition (CC) proves the claim.

Now to conclude look at the homogeneous component occurring for $k = 2q - 2$ and $i = \eta - 1$:

$$M_{4q+2q(\eta-1)+2q-2+dn} = M_{2q+d(n+1)} = \langle [v_n y x^{2q-2}(yx^{2q-1})^{\eta-1} y x^{2q-2}] \rangle = \langle [v_{n+1}] \rangle.$$
ACKNOWLEDGEMENT

The author is grateful for his help to A. Caranti, advisor of the doctoral dissertation \[\text{Jur98}\] this work is based on. He is also grateful to M.F. Newman and S. Mattarei for making useful suggestions and reading various versions of this paper and to an anonymous referee for her/his precious help in indicating improvements and signaling mistakes.

REFERENCES

\[\text{AF55}\] A. A. Albert and M. S. Frank, Simple Lie algebras of characteristic \(p\), Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954–55), 117–139.

\[\text{Car98}\] C. Carrara, (Finite) presentations of loop algebras of Albert-Frank Lie algebras, Ph.D. thesis, Trento, 1998.

\[\text{Car01}\] ______, (Finite) presentations of the Albert-Frank-Shalev Lie algebras, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 4 (2001), no. 2, 391–427.

\[\text{CMN97}\] A. Caranti, S. Mattarei, and M. F. Newman, Graded Lie algebras of maximal class, Trans. Amer. Math. Soc. 349 (1997), no. 10, 4021–4051.

\[\text{CN00}\] A. Caranti and M. F. Newman, Graded Lie algebras of maximal class, II, J. Algebra 229 (2000), 750–784.

\[\text{HNO97}\] G. Havas, M. F. Newman, and E. A. O’Brien, ANU \(p\)-Quotient Program (version 1.4), written in \(C\), available as a share library with GAP and as part of Magma, or from \(http://www.maths.anu.edu.au/services/ftp.html\), School of Mathematical Sciences, Canberra, 1997.

\[\text{Jur98}\] G. Jurman, On graded Lie algebras in characteristic two, Ph.D. thesis, Trento, 1998.

\[\text{Jur04}\] ______, A family of simple Lie algebras in characteristic two, J. Algebra 271 (2004), no. 2, 454–481.

\[\text{Jur05}\] ______, Graded Lie algebras of maximal class, III, J. Algebra 284 (2005), no. 2, 435–461.

\[\text{KW89}\] D. E. Knuth and H. S. Wilf, The power of a prime that divides a generalized binomial coefficient, J. Reine Angew. Math. 396 (1989), 212–219.

\[\text{Luc78}\] É. Lucas, Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier, Bull. Soc. Math. France 6 (1878), 49–54.

\[\text{Mat06}\] S. Mattarei, On a special congruence of Carlitz, Integers 6 (2006), A09.

\[\text{Rob93}\] D. J. S. Robinson, A course in the theory of groups, Springer-Verlag, New York, 1993.

\[\text{Sha94}\] A. Shalev, Simple Lie algebras and Lie algebras of maximal class, Arch. Math. (Basel) 63 (1994), no. 4, 297–301.

APPENDIX A. THE EXPANSIONS

A.0. Computing tools. Lucas’ Theorem \[\text{KW89} \text{Luc78}\] will be used several times: if \(a = \sum_{i=0}^{n} a_i \cdot 2^i\) and \(b = \sum_{i=0}^{n} b_i \cdot 2^i\) are the 2-adic expansions of two integers, then

\[
\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \pmod{2}.
\]

For instance, as a consequence of the formula above, we have that

\[
\binom{2^w - 1}{i} \equiv 1 \pmod{2} \quad \forall \, 0 \leq i \leq 2^w - 1.
\]
The following identity (I) will be be useful in simplifying some evaluations: if $Q$ is a power of 2, then

\[(Q - 1) s + r \equiv \binom{r}{k} \pmod{2} \quad 0 \leq r, k \leq Q - 2 .\]

To prove the identity (I), consider for $z > 0$ the power sum of the elements of the field $\mathbb{F}_Q$:

$$\sum_{\alpha \in \mathbb{F}_Q^*} \alpha^z \equiv \begin{cases} 1 \pmod{2} & \text{if } Q - 1 | z, \\ 0 \pmod{2} & \text{otherwise}. \end{cases}$$

Then, for $n = (Q - 1)s + r > 0$ we have that

$$\sum_{\alpha \in \mathbb{F}_Q^*} (1 + \alpha)^n \alpha^{-k} = \sum_{\alpha \in \mathbb{F}_Q^*} \sum_{i=0}^{n} \binom{n}{i} \alpha^{i-k} = \sum_{i=0}^{n} \binom{n}{i} \sum_{\alpha \in \mathbb{F}_Q^*} \alpha^{i-k} = \sum_{j=0}^{s} \left( (Q - 1) j + k \right) .$$

Since $\alpha^n = \alpha^r$ in $\mathbb{F}_Q$, the leftmost term in the above equation only depends on $r = n \pmod{Q - 1}$: thus, this is true for the rightmost term too, and then identity (I) follows. A more general statement, generalization of a congruence originally shown by Glaisher in 1899, is proven as Prop. 6 in [Mat06] by multisection of series.

A.1. Expansion of eq. (1). The first equation we deal with is (1), whose expansion gives

\[
0 = \left[ y x^{2q-1} y x^{q-2^s-1} \right] y x^{2q-1} y x^{q-2^s-1} \equiv \left( 3 q - 2^{s-1} - 1 \right) \left( q + 2^{s-1} - 1 \right) .
\]

since the binomial coefficient is equivalent to

\[
\binom{3 q - 2^{s-1} - 1}{q + 2^{s-1} - 1} \equiv \binom{2}{1} \binom{q - 2^{s-1} - 1}{2^{s-1} - 1} \equiv 0 \pmod{2} .
\]
A.2. Expansion of eq. (2). Expanding from the back:

\[
0 = [[v_{s-1}y^x2^q-2(yx2^q-1)^{\eta-1}yx^{q-1}] [v_{s-1}y^x2^q-2(yx2^q-1)^{\eta-1}yx^{q-1}] \\
= [[v_{s-1}y^x2^q-2(yx2^q-1)^{\eta-1}yx^{q-1}] [y^zdz+q]] \\
= [v_{s-1}y^x2^q-2(yx2^q-1)^{\eta-1}yx^{q-1}yz+q] \\
+ [v_{s-1}y^x2^q-2(yx2^q-1)^{\eta-1}yx^{q-1}dz+q] \\
\sum_{j=0}^{2^g-1} \sum_{i=1}^{s-1} \left( (2^{g+h+1} - 2)s + 2^h \right) \\
\sum_{j=0}^{2^g-1} \sum_{i=1}^{s-1} \left( (2^{g+h+1} - 1)s + 2^h \right) \equiv 0 \pmod{2},
\]

In fact, the second and the last term in the above coefficient are equivalent to zero modulo two in view of Lucas’ Theorem, since their denominator is odd while the numerator is even. The remaining terms can be rewritten in terms of \( g \) and \( h \) as follows:

\[
\begin{align*}
\left( \begin{array}{c}
    ds + q \\
    2qi
\end{array} \right) &= \left( \begin{array}{c}
    (2^{g+h+1} - 2)s + 2^h \\
    2^{h+1}i
\end{array} \right) \\
\left( \begin{array}{c}
    ds + q \\
    dj - 2 + 2q(\eta + 2)
\end{array} \right) &= \left( \begin{array}{c}
    ds + q \\
    (2^{g+h+1} - 2)(j + 1) + 2^{h+1}
\end{array} \right) \\
\left( \begin{array}{c}
    ds + q \\
    dj - 2 + 2q(\eta + 2 + i)
\end{array} \right) &= \left( \begin{array}{c}
    ds + q \\
    (2^{g+h+1} - 2)(j + 1) + 2^{h+1}(i + 1)
\end{array} \right),
\end{align*}
\]

and thus arranged in an unique sum that can be shown to vanish by using Lucas’ Theorem and identity (I):

\[
\sum_{j=0}^{2^g-1} \sum_{i=1}^{s-1} \left( (2^{g+h+1} - 2)s + 2^h \right) \equiv \sum_{j=0}^{2^g-1} \sum_{i=1}^{s-1} \left( (2^{g+h+1} - 1)s + 2^{h-1} \right) \equiv 0 \pmod{2},
\]

since \( 2^{h-1} < 2^h i \) for \( i \) ranging between 1 and \( 2^g - 1 \).

A.3. Expansion of eq. (3). Then we have the equation (3), needed to prove the centrality of \( \theta_n^{\omega_1} \): note that, in this case, the relation can be conveniently written as

\[
\theta_n^{\omega_1} [x] = yx^{2^q-1}yx^{2^q-2}yx^{2^q-1}yx^{2^q-1}yx = \theta_n^{\omega_1} [y^2\eta(\eta+1)+2q-2yx],
\]
since the element $[yz^{2q(\eta+1)}+2q-3yx.x]$ is non zero. The expansion begins as:

$$0 = [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}y^{2q-2}[\theta_t x]]$$
$$= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q-2}(yx^{2q-1})^{\eta-1}y^{2q-1}yx]]$$
$$= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$+ [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$+ [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$+ [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$= a_1 \cdot [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$+ a_0 \cdot [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-1}yx]]$$
$$+ [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$+ (b + c_0) \cdot [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-1}yx]]$$
$$+ (b + c_1) \cdot [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-1}yx]]$$
$$= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-1}yx]]$$
$$+ [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$+ [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-2}yx]]$$
$$= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[y^{2q(\eta+1)+2q-4}yx]]$$

where, for $t = 0, 1$, we have

$$a_t = \left(\frac{2q(\eta + 1) + 2q - 2}{t}\right) + \left(\frac{2q(\eta + 1) + 2q - 2}{2q - 1 + t}\right)$$
$$+ \left(\frac{2q(\eta + 1) + 2q - 2}{2q + 2q - 2 + t}\right) + \sum_{i=1}^{\eta-1} \left(\frac{2q(\eta + 1) + 2q - 2}{2q(i + 1) + 2q - 2 + t}\right)$$
$$b = \left(\frac{2q(\eta + 1) + 2q - 2}{2q - 2}\right) + \left(\frac{2q(\eta + 1) + 2q - 2}{2q + 2q - 3}\right) + \sum_{i=1}^{\eta-1} \left(\frac{2q(\eta + 1) + 2q - 2}{2q(i + 1) + 2q - 3}\right)$$
$$c_t = \left(\frac{2q(\eta + 1) + 2q - 2}{2q - 4 + t}\right).$$

The identity follows by applying Lucas’ Theorem to the above binomial coefficients, since for $0 \leq a \leq \eta + 1$ and $1 \leq b \leq 2q$ we get

$$\left(\frac{2q(\eta + 1) + 2q - 2}{2qa + 2q - b}\right) \equiv \left(\frac{\eta + 1}{a}\right) \left(\frac{2q - 2}{2q - b}\right) \equiv \left(\frac{2^a}{a}\right) \left(\frac{2q - 2}{2q - b}\right) \pmod{2},$$

which does not vanish only when $a = 0, \eta + 1$ and $b = 2, 4, 2q$. 
A.5. Expansion of eq. (4). To prove equation (4), let $a = h + 2 - s$, for $1 \leq s \leq h$ and expand as follows:

\[
0 = [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} [\theta_0^{h+2-s} x]]
\]
\[
= [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} [y x^{2q-1} y x^{2q-2} - y x]]
\]
\[
= [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} [y z^{4q-2s} x]]
\]
\[
= [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} [y z^{4q-2s} x]]
\]
\[
+ [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} x [y z^{4q-2s}]]
\]
\[
= \left( \binom{4q - 2s}{1} + \binom{4q - 2s}{2q} \right) + \left( \binom{4q - 2s}{2q + 2q - 2s} \right)
\]
\[
+ \left( \binom{4q - 2s}{2q - 1} \right) + \left( \binom{4q - 2s}{2q - 2s - 1} \right)
\]
\[
+ [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} y z^{4q-2s} x]
\]
\[
+ [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} y z^{4q-2s} x]
\]
\[
= \left( \binom{4q - 2s}{1} + \binom{4q - 2s}{2q} \right) + \left( \binom{4q - 2s}{2q + 2q - 2s} \right)
\]
\[
+ \left( \binom{4q - 2s}{2q - 1} \right) + \left( \binom{4q - 2s}{2q - 2s - 1} \right)
\]
\[
+ [v_n y x^{2q-2} (y x^{2q-1})^{n-2} y x^{2q-2} y z^{4q-2s}]
\]
\[
= [v_n y x^{2q-2 - 1} x] + [v_n y x^{2q-2 - 1} y x] + [v_n y x^{2q-2 - 1} x]
\]
\[
= [v_n y x^{2q-2 - 1} y x]
\]
\[
= [\theta_1^{h+2-s} x]
\]

since the binomial coefficients which are not immediate can be evaluated by means of Lucas' Theorem as follows:

\[
\binom{4q - 2s}{2q} \equiv \binom{4q - 2s}{2q + 2q - 2s} \equiv 1 \pmod{2},
\]
\[
\binom{4q - 2s}{2q - 1} \equiv \binom{4q - 2s}{2q + 2q - 2s - 1} \equiv 0 \pmod{2}.
\]

A.5. Expansion of eq. (5). First we expand in the case $n$ even:

\[
0 = [v_s v_s]
\]
\[
= [v_s [y z^{2q-1 + ds}]]
\]
\[
= [v_s z^{2q-1 + ds} y] + \left( \sum_{l=0}^{s-1} \binom{2q - 1 + ds}{dl} \right) + \left( \sum_{j=0}^{n-1} \binom{2q - 1 + ds}{2q - 1 + 2qj + dl} \right)
\]
\[
+ \left( \binom{2q - 1 + ds}{ds} \right) \cdot [v_s z^{2q+ds}]
\]
\[
= [v_n y x^{2q-1}].
\]
In fact, the involved binomial coefficients can be rewritten first as

\[
\binom{2q - 1 + ds}{0} + \sum_{l=1}^{s} \left( \binom{2q - 1 + ds}{dl} + \sum_{j=0}^{\eta-1} \left( \binom{2q - 1 + ds}{2q - 1 + 2qj + d(l - 1)} \right) \right),
\]

and then, by using binomial properties, as

\[
1 + \sum_{l=1}^{s} \left( \binom{2q - 1 + ds}{dl} + \binom{2q - 1 + ds}{d(s - l + 1)} + \sum_{j=1}^{n-1} \left( \binom{2q - 1 + ds}{d(s - l + 1) - 2qj} \right) \right);
\]

but now,

\[
\sum_{l=1}^{s} \binom{2q - 1 + ds}{dl} = \sum_{l=1}^{s} \left( \binom{2q - 1 + ds}{d(s - l + 1)} \right);
\]

and so the remaining coefficient is just

\[
1 + \sum_{l=1}^{s} \sum_{j=1}^{\eta-1} \left( \binom{2q - 1 + ds}{d(s - l + 1) - 2qj} \right) = 1 + \sum_{l=1}^{s} \sum_{j=1}^{\eta-1} \left( \binom{2q - 1 + ds}{dl - 2qj} \right).
\]

Now, by Lucas’ Theorem and (I), one gets

\[
\sum_{l=1}^{s} \sum_{j=1}^{\eta-1} \binom{2q - 1 + ds}{dl - 2qj} \equiv \sum_{l=1}^{s} \sum_{j=1}^{2^{s-2}} \left( \binom{(2^{g+h+1} - 2)s + 2^{h+1} - 1}{2^{g+h+1} - 2l - 2^{h+1}j} \right) \pmod{2}
\]

\[
\equiv \sum_{l=1}^{s} \sum_{j=1}^{2^{s-2}} \left( \binom{(2^{g+h+1} - 2)s + 2^{h+1} - 2}{2^{g+h+1} - 2l - 2^{h+1}j} \right) \pmod{2}
\]

\[
\equiv \sum_{l=1}^{s} \sum_{j=1}^{2^{s-2}} \left( \binom{(2^{g+h} - 1)s + 2^{h} - 1}{2^{g+h} - 1l - 2^{h}j} \right) \pmod{2}
\]

\[
\equiv \sum_{l=1}^{s} \sum_{j=1}^{2^{s-2}} \binom{(2^{g+h} - 1)s + 2^{h} - 1}{(2^{g+h} - 1)l + (2^{g+h} - 2^{h}j - 1)} \pmod{2}
\]

\[
\equiv 0 \pmod{2},
\]

since \(2^{h} - 1 < 2^{g+h} - 2^{h}j - 1\) when \(j\) ranges between 1 and \(2^{g} - 2\).
Then we expand equation (5) in the $n$ odd case:

\[ 0 = \left[ v_n y x^{2q-2} (y x^{2q-1})^{\frac{n-1}{2}} \right] [v_s y x^{2q-2} (y x^{2q-1})^{\frac{n-1}{2}}] \]

\[ = \left[ v_s y x^{2q-2} (y x^{2q-1})^{\frac{n-1}{2}} \right] [y z^{2q(\frac{n+1}{2}) + 2q - 2 + ds}] \]

\[ = \left[ v_s y x^{2q-2} (y x^{2q-1})^{\frac{n-1}{2}} \right] z^{2q(\frac{n+1}{2}) + 2q - 2 + ds} \]

\[ + [v_s y x^{2q-2} (y x^{2q-1})^{\frac{n-1}{2}} \right] z^{2q(\frac{n+1}{2}) + 2q - 1 + ds} \cdot \left( \sum_{j=0}^{\frac{n-1}{2}} \left( 2q \left( \frac{n+1}{2} \right) + 2q - 2 + d \right) \right) \]

\[ + \sum_{l=0}^{s-1} \left( \sum_{j=0}^{\frac{n-1}{2}} \left( 2q \left( \frac{n+1}{2} \right) + 2q - 2 + ds \right) \right) \]

\[ + \sum_{l=0}^{s-1} \sum_{j=0}^{\frac{n-1}{2}} \left( 2q \left( \frac{n+1}{2} \right) + 2q - 2 + dl \right) \]

\[ = \left[ v_s y x^{2q-2} (y x^{2q-1})^{\frac{n-1}{2}} \right] z^{2q(\frac{n+1}{2}) + 2q - 2 + ds} \]

\[ + S \cdot \left[ v_s y x^{2q-2} (y x^{2q-1})^{\frac{n-1}{2}} \right] z^{2q(\frac{n+1}{2}) + 2q - 1 + ds} \]

\[ = \left[ v_n y x^{2q-1} \right]. \]

The second and the last binomial coefficients vanish since they have an even numerator and an odd denominator. The sum $S$ of the two remaining terms can be arranged as
follows, by using Lucas’ Theorem:

\[
S = \sum_{j=0}^{\eta - 1} \left( \frac{2q \left( \frac{\eta + 1}{2} \right) + 2q - 2 + ds}{2qj} \right) + \sum_{l=0}^{s-1} \sum_{j=0}^{\eta - 1} \left( \frac{2q \left( \frac{\eta + 1}{2} \right) + 2q - 2 + ds}{2q \left( \frac{\eta + 1}{2} + j \right)} + 2q - 2 + dl \right)
\]

\[
= \sum_{j=0}^{2^g - 1} \left( \frac{(2g+h+1 - 2)s + (2g+h + 2h+1 - 2)}{2h+1j} \right)
\]

\[
+ \sum_{l=0}^{s-1} \sum_{j=0}^{2^g - 2} \left( \frac{(2g+h+1 - 2)s + 2g+h + 2h+1 - 2}{2g+h+1(2g+h+1 - 2)l + 2g+h + 2h+1(j + l) - 2} \right)
\]

\[
= \sum_{j=0}^{2^g - 1} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{2h} \right)
\]

\[
+ \sum_{l=0}^{s-1} \sum_{j=0}^{2^g - 2} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{2g+h-1(l + 2g+h-1 + 2h(j + 1) - 1)} \right)
\]

\[
= \sum_{j=0}^{2^g - 1} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{2h} \right)
\]

\[
+ \sum_{l=0}^{s-1} \sum_{j=0}^{2^g - 1-2} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{2g+h-1(l + 2g+h-1 + 2h(j + 1) - 1)} \right)
\]

\[
+ \sum_{l=0}^{s-1} \sum_{j=2^g - 1-1}^{2^g - 2} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{2g+h-1(l + 2g+h-1 + 2h(j + 1) - 1)} \right)
\]

\[
= \sum_{j=0}^{2^g - 1} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{2h} \right)
\]

\[
+ \sum_{l=0}^{s-1} \sum_{j=0}^{2^g - 1-2} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{2g+h - 1(l + 2g+h-1 + 2h(j + 1) - 1)} \right)
\]

\[
+ \sum_{l=0}^{s-1} \sum_{j=0}^{2^g - 1-1} \left( \frac{(2g+h - 1)s + 2g+h-1 + 2h - 1}{(2g+h - 1)(l + 1) + 2h} \right)
\]
The first term can be grouped into the last one, and then $S$ can be evaluated by using the property $(\mathcal{I})$:

$$S = \sum_{l=0}^{s-1} \sum_{j=0}^{2^s-1-2} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h(j+1) - 1} \right) + \sum_{l=0}^{s-1} \sum_{j=0}^{2^s-1-1} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h} \right)$$

$$+ \sum_{l=0}^{s} \sum_{j=0}^{2^s-1-2} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h(j+1) - 1} \right) + \sum_{l=0}^{s} \sum_{j=0}^{2^s-1-1} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h} \right)$$

$$= \sum_{l=0}^{s-1} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h} \right) - \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h} \right)$$

$$+ \sum_{l=0}^{s} \sum_{j=0}^{2^s-1-2} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h(j+1) - 1} \right) + \sum_{l=0}^{s} \sum_{j=0}^{2^s-1-1} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^{g+h-1} + 2^h} \right)$$

$$\equiv \left( \frac{2^{g+h-1} + 2^h - 1}{2^{g+h-1} + 2^h - 1} \right) - 1 + 0 + \sum_{j=0}^{2^{s-1}-1} \left( \frac{(2^{g+h}-1)s + 2^{g+h-1} + 2^h - 1}{(2^{g+h}-1)l + 2^h} \right) \pmod{2}$$

$$\equiv \sum_{j=0}^{2^{g-1}-1} \left( \frac{2^{g+h-1} + 2^h - 1}{2^h j} \right) \pmod{2}$$

$$\equiv 1 + \sum_{j=1}^{2^{g-1}-1} \left( \frac{2^{g+h-1}}{2^h j} \right) \left( \frac{2^h - 1}{0} \right) \pmod{2}$$

$$\equiv 1 \pmod{2},$$

and thus the thesis.
A.6. Expansion of eq. (6). Then we proceed with (6), where \( i = 2^{g+h+1-b} \):

\[
0 = [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}[\theta_{0}^b x]] \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}[yx^{2q-1}y^{x^{2q-2}(yx^{2q-1})y^{x^{2q-2}}y^{x^{2q-2}}x]} \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}[yx^{2q-2}2+2q(i+2) x]} \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}[yx^{2q-2}2+2q(i+2)]x] \\
+ [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}[yx^{2q-2}2+2q(i+2)]x] \\
= r_1 \cdot [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}}2q(i+2) y,x] \\
+ [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-1}y^{x^{2q-2}}2q(i+2)] y \\
+ r_0 \cdot [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-1}y^{x^{2q-2}}2q(i+2)] y \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-1}y^{x^{2q-2}}2q(i+2)] y + [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-1}y^{x^{2q-2}}2q(i+2)] y \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-1}y^{x^{2q-2}}2q(i+2)] y \\
= [\theta_{0}^b x] ,
\]

where for \( t = 0, 1 \)

\[
r_t = \left(\frac{2q(i+2)+2q-2}{t}\right) + \left(\frac{2q(i+2)+2q-2}{2q+2q-2+t}\right) + \left(\frac{2q(i+2)+2q-2}{2q+2q-2+t}\right) \\
+ \sum_{j=1}^{i} \left(\frac{2q(i+2)+2q-2}{2q(j+1)+2q-2+t}\right) + \left(\frac{2q(i+2)+2q-2}{2q(i+2)+2q-3+t}\right) .
\]

Since the binomial coefficients whose denominator is odd vanish, the non-zero and non-trivial binomial coefficients reduce to

\[
\left(\frac{2q(i+2)+2q-2}{2q}\right) \equiv \left(\frac{i+2}{1}\right) = \eta - 2^{g+h+1-b} + 2 \equiv 1 \pmod{2} , \\
\left(\frac{2q(i+2)+2q-2}{2q(j+1)+2q-2}\right) \equiv \left(\frac{i+2}{j+1}\right) \pmod{2} ,
\]

and thus the identity follows.

A.7. Expansion of eq. (7).

\[
0 = [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}[\mu_{0,2} y]] \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}[yx^{2q-1}y^{x^{2q-2}2}y^{x^{2q-2}}y]] \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}[y^{x^{2q-2}2}y^{x^{2q-2}}y^{x^{2q-2}}y^{x^{2q-2}}y^{x^{2q-2}}y^{x^{2q-2}}y]] \\
= A \cdot [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}z^{6q-2}] \\
= [v_{n-1}yx^{2q-2}(yx^{2q-1})\eta^{-2}y^{x^{2q-2}2}y^{x^{2q-2}}y^{x^{2q-2}}y^{x^{2q-2}}y^{x^{2q-2}}y^{x^{2q-2}}y^{x^{2q-2}}y] \\
= [v_{n}yx^{2q-2}y^{x^{k-1}y}] ,
\]

for

\[
A = \left(\begin{array}{c}
\frac{6q-3}{2} \\
\frac{6q-3}{2q+2^s-1} \\
\frac{6q-3}{4q+2^s-2}
\end{array}\right) .
\]
In fact, the evaluation of the coefficient \( A \) can be done as usual by Lucas’ Theorem first

\[
\binom{6q - 3}{2^s} \equiv \binom{4q}{0} \binom{2q - 3}{2^s} \equiv \binom{2q - 3}{2^s} \quad (\text{mod } 2)
\]

\[
\binom{6q - 3}{2q + 2^s - 1} \equiv \binom{4q}{0} \binom{2q - 3}{2^s - 1} \equiv 1 \cdot 0 \cdot \binom{2q - 3}{2^s - 1} \equiv 0 \quad (\text{mod } 2)
\]

\[
\binom{6q - 3}{4q + 2^s - 2} \equiv \binom{4q}{2q - 3} \binom{2q - 3}{2^s - 2} \equiv \binom{2q - 3}{2^s - 2} \quad (\text{mod } 2)
\]

and then by using some elementary properties

\[
\binom{2q - 3}{2^s} + \binom{2q - 3}{2^s - 2} \equiv \binom{2q - 3}{2^s} + 2 \binom{2q - 3}{2^s - 1} + \binom{2q - 3}{2^s - 2} \quad (\text{mod } 2)
\]

\[
= \left( \binom{2q - 3}{2^s} + \binom{2q - 3}{2^s - 1} + \binom{2q - 3}{2^s - 2} \right)
\]

\[
= \binom{2q - 2}{2^s} + \binom{2q - 2}{2^s - 1}
\]

\[
= \binom{2q - 1}{2^s}
\]

\[
\equiv 1 \quad (\text{mod } 2)
\]

A.8. Expansion of eq. (8).

\[
0 = \left[ \mu_{n,i} [\theta_{0}^{k+1} x] \right]
\]

\[
= \left[ \mu_{n,i} [y x^{2q-1} y x^{2q-2} - (s+1)] \right]
\]

\[
= \left[ \mu_{n,i} [y z^{4q-2h+1-s} x] \right]
\]

\[
= \left[ \mu_{n,i} [y z^{4q-2h+1-s} x] + [\mu_{n,i} x] [y z^{4q-2h+1-s}] \right]
\]

\[
= [\mu_{n,i} z^{4q-2h+1-s+1} x] \cdot \left( \binom{4q - 2^{h+1-s}}{2q - 1} \right) \left( \binom{4q - 2^{h+1-s}}{2q} \right)
\]

\[
+ [\mu_{n,i} x z^{4q-2h+1-s}] \cdot \left( \binom{4q - 2^{h+1-s}}{0} \right) \left( \binom{4q - 2^{h+1-s}}{2q} \right)
\]

\[
= [\mu_{n,i} x z^{4q-2h+1-s}] \cdot \left( \binom{4q - 2^{h+1-s}}{0} \right) \left( \binom{4q - 2^{h+1-s}}{2q} \right)
\]

\[
\vdots
\]

\[
= [v_{n}, y x^{2q-2} (y x^{2q-1})^i y x^{k-1}] ,
\]

since via Lucas’ Theorem the involved binomial coefficients can be evaluated as follows, where \( \alpha, \beta \in \{0, 1\} \)

\[
\binom{4q - 2^{h+1-s}}{2q \alpha + \beta} \equiv \binom{1}{\alpha} \binom{2q - 2^{h+1-s}}{\beta} \equiv (1 - \beta) \quad (\text{mod } 2)
\]
A.9. Expansion of eq. \([9]\). Consider the index \(0 \leq i \leq \eta - 3\) and let \(\lambda\) be the exponent of the higher power of two dividing \(i + 1\), hence \(i + 1 = 2^\lambda \cdot r\) where \(\lambda \geq 0\) (with equality when \(i\) is even) and \(r\) is positive odd integer. Due to the bounds on \(i\), we have

\[
r \leq \left\lfloor \frac{2^g - 2}{2^\lambda} \right\rfloor \leq 2^{g-\lambda} - 1,
\]

where the first becomes an equality if and only if \(i = \eta - 2^h+1-b\) for some \(h+2 \leq b \leq g + h\), or, equivalently, \(i = 2^g - 2^\gamma - 1\) for some \(1 \leq \gamma \leq g - 1\) so that \(\lambda = \gamma\). When we are not in the above case, then

\[
i + 2^\lambda - 1 = 2^\lambda \cdot (r + 1) - 2 < 2^\lambda \cdot (2^\gamma - 1 + 1) - 2 \leq 2^g - 2^\lambda - 2 \leq \eta - 2.
\]

Moreover, if \(i\) is even, then \(\lambda = 0\) and then \(i + 2^\lambda - 1\) is even, too; otherwise, if \(i\) is odd, then \(\lambda \geq 1\), and then \(i + 2^\lambda - 1\) is still even. But \(\eta - 2^\alpha\) is always even, for \(1 \leq \alpha \leq g - 1\), hence \(\eta - (i + 2^\lambda - 1)\) is never \(2^\alpha\), for \(1 \leq \alpha \leq g - 1\). Then \([\mu_{0,i+2^\lambda+1}y] = 0\) and we can expand the Jacobi identity

\[
0 = \left[v_{n-1}yx^{2q-2}(yx^{2q-1})\eta-1-2^\lambda yx^{2q-2}[\mu_{0,i+2^\lambda+1}y]\right]
\]

\[
= \left[v_{n-1}yx^{2q-2}(yx^{2q-1})\eta-1-2^\lambda yx^{2q-2}(yz^{2q(i+1+2^\lambda)+2q-3}y)\right]
\]

\[
= \left[v_{n-1}yx^{2q-2}(yx^{2q-1})\eta-1-2^\lambda yx^{2q-2}(yz^{2q(i+1+2^\lambda)+2q-3}y)\right]
\]

\[
= \left(v_{n-1}yx^{2q-2}(yx^{2q-1})\eta-1-2^\lambda yx^{2q-2}(yz^{2q(i+1+2^\lambda)+2q-3}y)\right)
\]

\[
= \left(\sum_{j=0}^{2^\lambda-1} \left(\frac{2q(i + 1 + 2^\lambda) + 2q - 3}{2qj + 1}\right) + \left(\frac{2q(i + 1 + 2^\lambda) + 2q - 3}{2q \cdot 2^\lambda}\right)\right)
\]

\[
= \left(\sum_{j=0}^{i} \left(\frac{2q(i + 1 + 2^\lambda) + 2q - 3}{2q(2^\lambda + j) + 2q - 1}\right)\right)
\]

As usual, the terms in the last sum vanish since \(2q - 1 > 2q - 3\), while the remaining terms can be rewritten together as follows:

\[
\sum_{j=0}^{2^\lambda} \left(\frac{2q(i + 1 + 2^\lambda)}{2qj}\right) \equiv \sum_{j=0}^{2^\lambda} \left(\frac{2^\lambda \cdot r + 2^\lambda}{j}\right)
\]

\[
\equiv 1 + \sum_{j=1}^{2^\lambda-1} \left(\frac{2^\lambda \cdot (r + 1)}{j}\right) + \left(\frac{r + 1}{1}\right)
\]

\[
\equiv 1 + 0 + r + 1
\]

\[
\equiv 1 \pmod{2},
\]

since \(r\) is odd.