New self-dual additive $\mathbb{F}_4$-codes constructed from circulant graphs

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Abstract
In order to construct quantum $[[n, 0, d]]$ codes for $(n, d) = (56, 15)$, $(57, 15)$, $(58, 16)$, $(63, 16)$, $(67, 17)$, $(70, 18)$, $(71, 18)$, $(79, 19)$, $(83, 20)$, $(87, 20)$, $(89, 21)$, $(95, 20)$, we construct self-dual additive $\mathbb{F}_4$-codes of length $n$ and minimum weight $d$ from circulant graphs. The quantum codes with these parameters are constructed for the first time.

1 Introduction

Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ be the finite field with four elements, where $\bar{\omega} = \omega^2 = \omega + 1$. An additive $\mathbb{F}_4$-code of length $n$ is an additive subgroup of $\mathbb{F}_4^n$. An element of $C$ is called a codeword of $C$. An additive $(n, 2^k)$ $\mathbb{F}_4$-code is an additive $\mathbb{F}_4$-code of length $n$ with $2^k$ codewords. The (Hamming) weight of a codeword $x$ of $C$ is the number of non-zero components of $x$. The minimum non-zero weight of all codewords in $C$ is called the minimum weight of $C$.

Let $C$ be an additive $\mathbb{F}_4$-code of length $n$. The symplectic dual code $C^*$ of $C$ is defined as $\{x \in \mathbb{F}_4^n \mid x \ast y = 0 \text{ for all } y \in C\}$ under the trace inner product:

$$x \ast y = \sum_{i=1}^{n}(x_i y_i^2 + x_i^2 y_i)$$

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for \( x = (x_1, x_2, \ldots, x_n), \ y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}_4^n \). An additive \( \mathbb{F}_4 \)-code \( C \) is called (symplectic) self-orthogonal (resp. self-dual) if \( C \subset C^* \) (resp. \( C = C^* \)).

Calderbank, Rains, Shor and Sloane [3] gave the following useful method for constructing quantum codes from self-orthogonal additive \( \mathbb{F}_4 \)-codes (see [3] for more details on quantum codes). A self-orthogonal additive \((n, 2^{n-k})\) \( \mathbb{F}_4 \)-code \( C \) such that there is no element of weight less than \( d \) in \( C^* \setminus C \), gives a quantum \([n, k, d]_4\) code, where \( k \neq 0 \). In addition, a self-dual additive \( \mathbb{F}_4 \)-code of length \( n \) and minimum weight \( d \) gives a quantum \([n, 0, d]_4\) code. Let \( d_{\text{max}}(n, k) \) denote the maximum integer \( d \) such that a quantum \([n, k, d]_4\) code exists. It is a fundamental problem to determine the value \( d_{\text{max}}(n, k) \) for a given \((n, k)\). A table on \( d_{\text{max}}(n, k) \) is given in [3, Table III] for \( n \leq 30 \), and an extended table is available online [5].

In this note, we construct self-dual additive \( \mathbb{F}_4 \)-codes of length \( n \) and minimum weight \( d \) for

\[
(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17), \\
(70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20).
\]

These codes are obtained from adjacency matrices of some circulant graphs. The above self-dual additive \( \mathbb{F}_4 \)-codes allow us to construct quantum \([n, 0, d]_4\) codes for the \((n, d)\) given in (1). These quantum codes improve the previously known lower bounds on \( d_{\text{max}}(n, 0) \) for the above \( n \).

The data of these new quantum codes has already been included in [5]. All computer calculations in this note were performed using MAGMA [1].

## 2 Self-dual additive \( \mathbb{F}_4 \)-codes from circulant graphs

A graph \( \Gamma \) consists of a finite set \( V \) of vertices together with a set of edges, where an edge is a subset of \( V \) of cardinality 2. All graphs in this note are simple, that is, graphs are undirected without loops and multiple edges. The adjacency matrix of a graph \( \Gamma \) with \( V = \{x_1, x_2, \ldots, x_v\} \) is a \( v \times v \) matrix \( A_{\Gamma} = (a_{ij}) \), where \( a_{ij} = a_{ji} = 1 \) if \( \{x_i, x_j\} \) is an edge and \( a_{ij} = 0 \) otherwise. Let \( \Gamma \) be a graph and let \( A_{\Gamma} \) be the adjacency matrix of \( \Gamma \). Let \( C(\Gamma) \) denote the additive \( \mathbb{F}_4 \)-code generated by the rows of \( A_{\Gamma} + \omega I \), where \( I \) denotes the identity matrix. Then \( C(\Gamma) \) is a self-dual additive \( \mathbb{F}_4 \)-code [4].
Two additive $\mathbb{F}_4$-codes $C_1$ and $C_2$ of length $n$ are equivalent if there is a map from $S_3^n \times S_n$ sending $C_1$ onto $C_2$, where the symmetric group $S_n$ acts on the set of the $n$ coordinates and each copy of the symmetric group $S_3$ permutes the non-zero elements $1, \omega, \bar{\omega}$ of the field in the respective coordinate. For any self-dual additive $\mathbb{F}_4$-code $C$, it was shown in [4, Theorem 6] that there is a graph $\Gamma$ such that $C(\Gamma)$ is equivalent to $C$. Using this characterization, all self-dual additive $\mathbb{F}_4$-codes were classified for lengths up to 12 [4, Section 5].

An $n \times n$ matrix is circulant if it has the following form:

$$
M = \begin{pmatrix}
  r_1 & r_2 & \cdots & r_{n-1} & r_n \\
  r_n & r_1 & \cdots & r_{n-2} & r_{n-1} \\
  r_{n-1} & r_n & \cdots & r_{n-3} & r_{n-2} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  r_2 & r_3 & \cdots & r_n & r_1
\end{pmatrix}.
$$

(2)

Trivially, the matrix $M$ is fully determined by its first row $(r_1, r_2, \ldots, r_n)$. A graph is called circulant if it has a circulant adjacency matrix. For a circulant adjacency matrix of the form (2), we have

$$
r_1 = 0 \quad \text{and} \quad r_i = r_{n+2-i} \quad \text{for} \quad i = 2, \ldots, \lfloor n/2 \rfloor.
$$

(3)

Circulant graphs and their applications have been widely studied (see [7] for a recent survey on this subject). For example, it is known that the number of non-isomorphic circulant graphs is known for orders up to 47 (see the sequence A049287 in [8]). In this note, we concentrate on self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ generated by the rows of $A_\Gamma + \omega I$, where $A_\Gamma$ are the adjacency matrices of circulant graphs $\Gamma$. These codes were studied, for example, in [6] and [9].

3 New self-dual additive $\mathbb{F}_4$-codes and quantum codes from circulant graphs

3.1 Lengths up to 50

Throughout this section, let $\Gamma$ denote a circulant graph with adjacency matrix $A_\Gamma$. Let $C(\Gamma)$ denote the self-dual additive $\mathbb{F}_4$-code generated by the
rows of $A_\Gamma + \omega I$. Let $d_{\text{max}}^\Gamma(n)$ denote the maximum integer $d$ such that a self-dual additive $\mathbb{F}_4$-code $C(\Gamma)$ of length $n$ and minimum weight $d$ exists. Varbanov [9] gave a classification of self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ for lengths $n = 13, 14, \ldots, 29, 31, 32, 33$ and determined the values $d_{\text{max}}^\Gamma(n)$ for lengths up to 33.

Table 1: Self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ of lengths $n = 34, 35, \ldots, 50$

| $n$ | $d_{\text{max}}^\Gamma(n)$ | Support of the first row of $A_{\Gamma_n}$ | $d_{\text{max}}(n, 0)$ |
|-----|--------------------------|------------------------------------------|------------------------|
| 34  | 10                       | 2, 3, 6, 8, 9, 27, 28, 30, 33, 34        | 10–12                  |
| 35  | 10                       | 2, 4, 6, 7, 10, 27, 30, 31, 33, 35       | 11–13                  |
| 36  | 11                       | 2, 3, 4, 5, 7, 9, 13, 14, 24, 25, 29, 31, 33, 34, 35, 36 | 12–14                  |
| 37  | 11                       | 5, 6, 7, 9, 11, 12, 27, 28, 30, 32, 33, 34 | 11–14                  |
| 38  | 12                       | 2, 3, 5, 7, 10, 11, 20, 29, 30, 33, 35, 37, 38 | 12–14                  |
| 39  | 11                       | 2, 4, 5, 6, 7, 10, 11, 30, 31, 34, 35, 36, 37, 39 | 11–14                  |
| 40  | 12                       | 2, 3, 5, 8, 10, 21, 32, 34, 37, 39, 40 | 12–14                  |
| 41  | 12                       | 2, 3, 4, 5, 6, 10, 11, 13, 30, 32, 33, 37, 38, 39, 40, 41, 42 | 12–15                  |
| 42  | 12                       | 2, 3, 13, 15, 16, 18, 21, 22, 23, 26, 28, 29, 31, 41, 42 | 12–16                  |
| 43  | 12                       | 3, 4, 7, 9, 10, 12, 33, 35, 36, 38, 41, 42 | 13–16                  |
| 44  | 14                       | 4, 5, 8, 10, 13, 17, 18, 21, 23, 25, 28, 29, 33, 36, 38, 41, 42 | 14–16                  |
| 45  | 13                       | 2, 4, 5, 9, 10, 12, 14, 15, 17, 18, 20, 27, 29, 30, 32, 33, 35, 37, 38, 42, 43, 45 | 13–16                  |
| 46  | 14                       | 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 24, 29, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 43, 44 | 14–16                  |
| 47  | 13                       | 4, 8, 11, 13, 14, 15, 34, 35, 36, 38, 41, 45 | 13–17                  |
| 48  | 14                       | 3, 4, 5, 10, 12, 14, 15, 16, 25, 34, 35, 36, 38, 40, 45, 46, 47 | 14–18                  |
| 49  | 13                       | 4, 5, 7, 8, 9, 10, 13, 14, 17, 37, 38, 41, 42, 43, 44, 46, 47 | 13–18                  |
| 50  | 14                       | 3, 7, 8, 9, 11, 12, 13, 17, 20, 22, 24, 25, 26, 27, 28, 30, 32, 35, 39, 40, 41, 43, 44, 45, 49 | 14–18                  |

For lengths $n = 13, 14, \ldots, 50$, by exhaustive search, we determined the largest minimum weights $d_{\text{max}}^\Gamma(n)$. In Table II for lengths $n = 34, 35, \ldots, 50$, we list $d_{\text{max}}^\Gamma(n)$ and an example of a self-dual additive $\mathbb{F}_4$-code $C(\Gamma_n)$ having minimum weight $d_{\text{max}}^\Gamma(n)$, where the support of the first row of the circulant adjacency matrix $A_{\Gamma_n}$ is given. Our present state of knowledge about the upper bound $d_{\text{max}}(n, 0)$ on the minimum distance is also listed in the table. For most lengths, the self-dual additive $\mathbb{F}_4$-codes give quantum $[[n, 0, d]]$ codes such that $d = d_{\text{max}}(n, 0)$ or $d$ attains the currently known lower bound on $d_{\text{max}}(n, 0)$; three exceptions (lengths 35, 36 and 43) are typeset in italics.

Note that $d_{\text{max}}^\Gamma(36) = 11$. For lengths 34, 35 and 36, self-dual additive
Table 2: Weight distribution of $C(\Gamma_{36})$

| $i$ | $A_i$ | $i$ | $A_i$ | $i$ | $A_i$ | $i$ | $A_i$ |
|-----|-------|-----|-------|-----|-------|-----|-------|
| 0   | 1     | 17  | 16145280 | 24  | 5144050296 | 31  | 3388564144 |
| 11  | 1584  | 18  | 51147440 | 25  | 7408053504 | 32  | 1588252581  |
| 12  | 9936  | 19  | 145391760| 26  | 9402473952 | 33  | 577571712   |
| 13  | 52992 | 20  | 370815624| 27  | 10446604880| 34  | 152925552   |
| 14  | 265392| 21  | 847669248| 28  | 10073332800| 35  | 26213616    |
| 15  | 1168032| 22 | 1733647968| 29  | 8336897280 | 36  | 2179688     |
| 16  | 4578786| 23 | 3165414336| 30  | 5836058352 |       |        |

$\mathbb{F}_4$-codes $C(\Gamma)$ with minimum weight 10 were constructed in [9]. For length 36, we found a self-dual additive $\mathbb{F}_4$-code $C(\Gamma_{36})$ of length 36 and minimum weight 11 (see Table 1). The weight distribution of the code $C(\Gamma_{36})$ is listed in Table 2, where $A_i$ denotes the number of codewords of weight $i$.

**Proposition 1.** The largest minimum weight $d_{\text{max}}^\Omega(36)$ among all self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ of length 36 from circulant graphs is 11.

A self-dual additive $\mathbb{F}_4$-code is called Type II if it is even. It is known that a Type II additive $\mathbb{F}_4$-code must have even length. A self-dual additive $\mathbb{F}_4$-code, which is not Type II, is called Type I. Although the following proposition is somewhat trivial, we give a proof for completeness.

**Proposition 2.** Let $C(\Gamma)$ be the self-dual additive $\mathbb{F}_4$-code of even length $n$ generated by the rows of $A_\Gamma + \omega I$, where $A_\Gamma$ is circulant. Let $S$ be the support of the first row of $A_\Gamma$. Then $C(\Gamma)$ is Type II if and only if $n/2 + 1 \in S$.

**Proof.** It was shown in [4, Theorem 15] that the codes $C(\Gamma)$ are Type II if and only if all the vertices of $\Gamma$ have odd degree. For a circulant graph $\Gamma$, the degree of the vertices is constant and equals the size of the support $S$ of the first row of $A_\Gamma$. From (3) it follows that the size of the support $S$ is odd if and only if $r_{n/2+1} = 1$, i.e., $n/2 + 1 \in S$. 

Note that (3) also implies that the size of the support $S$ of the first row of $A_\gamma$ is always even when $n$ is odd, i.e., self-dual codes of odd length from circulant graphs cannot be Type II.

By Proposition 2, the codes $C(\Gamma_n)$ ($n = 38, 40, 42, 44, 46, 48, 50$) are Type II. In addition, the other codes in Table 1 are Type I. Let $d_{\text{max},i}^\Omega(n)$
denote the maximum integer $d$ such that a Type I additive $\mathbb{F}_4$-code $C(\Gamma)$ of length $n$ and minimum weight $d$ exists. By exhaustive search, we verified that $d_{\text{max},I}(44) = d_{\text{max}}^I(44) - 2$, $d_{\text{max},I}(n) = d_{\text{max}}^I(n) - 1$ ($n = 38, 40, 46, 48$) and $d_{\text{max},I}(n) = d_{\text{max}}^I(n)$ ($n = 42, 50$). For $(n,d) = (42,12)$ and $(50,14)$, we list an example of Type I additive $\mathbb{F}_4$-code $C(\Gamma')$ of length $n$ and minimum weight $d$, where the support of the first row of the circulant adjacency matrix $A_{\Gamma'_n}$ is given in Table 3.

Table 3: Type I additive $\mathbb{F}_4$-codes $C(\Gamma'_n)$ of lengths 42, 50

| $n$ | $d$ | Support of the first row of $A_{\Gamma'_n}$ |
|-----|-----|------------------------------------------|
| 42  | 12  | 2, 3, 5, 6, 8, 11, 12, 13, 31, 32, 33, 36, 38, 39, 41, 42 |
| 50  | 14  | 5, 6, 7, 9, 10, 11, 12, 20, 32, 40, 41, 42, 43, 45, 46, 47 |

3.2 Sporadic lengths $n \geq 51$

For lengths $n \geq 51$, by non-exhaustive search, we tried to find self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ with large minimum weight, where $\Gamma$ is a circulant graph. By this method, we found new self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ of length $n$ and minimum weight $d$ for

$$(n,d) = (56,15), (57,15), (58,16), (63,16), (67,17),$$
$$(70,18), (71,18), (79,19), (83,20), (87,20), (89,21), (95,20).$$

For each self-dual additive $\mathbb{F}_4$-code $C(\Gamma_n)$, the support of the first row of the circulant adjacency matrix $A_{\Gamma_n}$ is listed in Table 4. Additionally, for $n = 51, \ldots, 55, 59, 60, 64, 65, 66, 69, 72, \ldots, 78, 81, 82, 84, 88, 94, 100$, we found self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ from circulant graphs matching the known lower bound on the minimum distance of quantum codes $[[n,0,d]]$. For the remaining lengths, our non-exhaustive computer search failed to discover a self-dual additive $\mathbb{F}_4$-code from a circulant graph matching the known lower bound.

For the codes $C(\Gamma_n)$ ($n = 56, 57, 58, 63, 67, 70, 71, 79$), we give in Table 5 part of the weight distribution. Due to the computational complexity, we calculated the number $A_i$ of codewords of weight $i$ for only $i = 15, 16, \ldots, 19$. As some basic properties of the graphs $\Gamma_n$, we give in Table 6 the valency.
Table 4: New self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$

| Code    | Support of the first row of $A_{\Gamma_n}$ |
|---------|-------------------------------------------|
| $C(\Gamma_{56})$ | 2, 3, 7, 8, 12, 14, 15, 16, 17, 20, 22, 26, 28, 30, 32, 36, 38, 41, 42, 43, 44, 46, 50, 51, 55, 56 |
| $C(\Gamma_{57})$ | 7, 8, 10, 12, 17, 18, 22, 23, 24, 35, 36, 37, 41, 42, 47, 49, 51, 52 |
| $C(\Gamma_{58})$ | 2, 3, 7, 10, 13, 14, 15, 17, 21, 25, 27, 29, 30, 31, 33, 35, 39, 43, 45, 46, 47, 50, 53, 57, 58 |
| $C(\Gamma_{63})$ | 2, 5, 6, 9, 13, 14, 15, 16, 17, 19, 46, 48, 49, 50, 51, 52, 56, 59, 60, 63 |
| $C(\Gamma_{67})$ | 4, 5, 6, 11, 12, 14, 15, 16, 17, 18, 21, 25, 26, 27, 28, 30, 39, 41, 42, 43, 44, 48, 51, 52, 53, 54, 55, 57, 58, 63, 64, 65 |
| $C(\Gamma_{70})$ | 2, 6, 7, 8, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 28, 29, 30, 32, 33, 35, 36, 37, 39, 40, 42, 43, 44, 48, 49, 50, 51, 52, 53, 55, 57, 58, 59, 60, 61, 64, 65, 66, 70 |
| $C(\Gamma_{71})$ | 2, 3, 5, 11, 12, 15, 17, 20, 23, 26, 27, 28, 31, 34, 35, 38, 39, 42, 45, 46, 47, 50, 53, 56, 58, 61, 62, 68, 70, 71 |
| $C(\Gamma_{79})$ | 3, 4, 5, 7, 9, 11, 14, 19, 20, 21, 22, 23, 24, 27, 28, 30, 31, 32, 33, 34, 36, 38, 41, 44, 47, 49, 51, 52, 53, 54, 55, 57, 58, 60, 61, 62, 63, 64, 65, 66, 71, 74, 77, 79 |
| $C(\Gamma_{83})$ | 7, 11, 12, 13, 14, 15, 20, 23, 24, 25, 27, 28, 29, 30, 31, 34, 35, 37, 40, 41, 42, 47, 48, 49, 52, 54, 55, 58, 59, 60, 61, 62, 64, 65, 66, 69, 74, 75, 76, 77, 78, 82 |
| $C(\Gamma_{87})$ | 3, 4, 7, 10, 13, 15, 18, 19, 21, 23, 25, 26, 30, 32, 34, 35, 37, 39, 40, 45, 46, 51, 52, 54, 56, 57, 59, 61, 65, 66, 68, 70, 72, 73, 76, 77, 81, 84, 87, 88 |
| $C(\Gamma_{89})$ | 4, 5, 6, 11, 12, 14, 15, 18, 19, 26, 27, 28, 30, 31, 32, 33, 34, 35, 36, 38, 40, 42, 43, 45, 47, 50, 52, 54, 55, 57, 59, 61, 62, 63, 64, 65, 66, 67, 69, 70, 71, 78, 79, 82, 83, 85, 86, 91, 92, 93 |
Table 5: Number $A_i$ of codewords of weight $i$ ($i = 15, 16, \ldots, 19$)

| Code            | $d$ | $A_{15}$ | $A_{16}$ | $A_{17}$ | $A_{18}$ | $A_{19}$ |
|-----------------|-----|----------|----------|----------|----------|----------|
| $C(\Gamma_{56})$ | 15  | 4 032    | 25 508   | 173 264  | 1 124 648| 6 839 224|
| $C(\Gamma_{57})$ | 15  | 1 938    | 18 126   | 120 783  | 838 451  | 5 093 409|
| $C(\Gamma_{58})$ | 16  | 24 882   | 1 205 240| 0        | 0        | 0        |
| $C(\Gamma_{59})$ | 16  | 2 142    | 12 726   | 113 568  | 757 575  | 0        |
| $C(\Gamma_{67})$ | 17  | 2 278    | 23 785   | 193 429  | 0        | 0        |
| $C(\Gamma_{68})$ | 18  | 15 260   | 0        | 0        | 0        | 0        |
| $C(\Gamma_{70})$ | 18  | 6 745    | 43 949   | 0        | 0        | 0        |
| $C(\Gamma_{79})$ | 19  | 1 343    | 0        | 0        | 0        | 0        |

$k(\Gamma_n)$, the diameter $d(\Gamma_n)$, the girth $g(\Gamma_n)$, the size $\omega(\Gamma_n)$ of the maximum clique and the order $|\text{Aut}(\Gamma_n)|$ of the automorphism group. With the exception of $n = 53$, the automorphism group is the dihedral group on $n$ points of order $2n$. Note, however, that the notion of equivalence for graphs and codes are different, i.e., the graph invariants are not preserved with respect to code equivalence \cite{2}. By Proposition \cite{2} the codes $C(\Gamma_{58})$ and $C(\Gamma_{70})$ are Type II.

Finally, by the method in \cite{3}, the existence of our self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ yields the following:

**Theorem 3.** There are a quantum $[[n, 0, d]]$ codes for

$$(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17),$$

$$(70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20).$$

The above quantum $[[n, 0, d]]$ codes improve the previously known lower bounds on $d_{\text{max}}(n, 0)$ ($n = 56, 57, 58, 63, 67, 70, 71, 79, 87, 89$). More precisely,
Table 6: Properties of the graphs $\Gamma_n$

| Graph $\Gamma_n$ | $d_{\text{min}}(C(\Gamma_n))$ | $k(\Gamma_n)$ | $d(\Gamma_n)$ | $g(\Gamma_n)$ | $\omega(\Gamma_n)$ | $|\text{Aut}(\Gamma_n)|$ |
|----------------|------------------|----------------|----------------|--------------|----------------|-----------------|
| $\Gamma_{51}$  | 14               | 26             | 3              | 6            | 1378           | 102             |
| $\Gamma_{52}$  | 14               | 16             | 3              | 3            | 4              | 104             |
| $\Gamma_{53}$  | 16               | 29             | 3              | 8            | 108            | 112             |
| $\Gamma_{54}$  | 16               | 18             | 3              | 5            | 110            | 114             |
| $\Gamma_{55}$  | 16               | 30             | 3              | 4            | 116            | 118             |
| $\Gamma_{56}$  | 16               | 31             | 3              | 6            | 120            | 126             |
| $\Gamma_{57}$  | 16               | 33             | 3              | 6            | 130            | 132             |
| $\Gamma_{58}$  | 16               | 20             | 3              | 6            | 134            | 138             |
| $\Gamma_{59}$  | 16               | 43             | 2              | 8            | 140            | 144             |
| $\Gamma_{60}$  | 16               | 30             | 3              | 6            | 142            | 146             |
| $\Gamma_{61}$  | 16               | 7              | 2              | 8            | 150            | 152             |
| $\Gamma_{62}$  | 17               | 32             | 3              | 6            | 154            | 158             |
| $\Gamma_{63}$  | 17               | 38             | 3              | 7            | 160            | 164             |
| $\Gamma_{64}$  | 18               | 45             | 2              | 10           | 168            | 170             |
| $\Gamma_{65}$  | 18               | 30             | 3              | 6            | 170            | 172             |
| $\Gamma_{66}$  | 18               | 33             | 3              | 6            | 172            | 174             |
| $\Gamma_{67}$  | 18               | 37             | 3              | 8            | 174            | 176             |
| $\Gamma_{68}$  | 19               | 42             | 3              | 6            | 176            | 180             |
| $\Gamma_{69}$  | 19               | 40             | 3              | 7            | 178            | 182             |
| $\Gamma_{70}$  | 20               | 43             | 2              | 6            | 184            | 186             |
| $\Gamma_{71}$  | 20               | 46             | 2              | 3            | 186            | 190             |
| $\Gamma_{72}$  | 20               | 25             | 2              | 6            | 190            | 194             |
| $\Gamma_{73}$  | 20               | 42             | 2              | 7            | 194            | 198             |
| $\Gamma_{74}$  | 20               | 37             | 2              | 6            | 198            | 200             |
| $\Gamma_{75}$  | 20               | 44             | 2              | 10           | 200            | 202             |
| $\Gamma_{76}$  | 20               | 50             | 2              | 7            | 202            | 206             |
| $\Gamma_{77}$  | 20               | 50             | 2              | 7            | 206            | 210             |
| $\Gamma_{78}$  | 20               | 50             | 2              | 7            | 210            | 214             |
| $\Gamma_{79}$  | 20               | 50             | 2              | 7            | 214            | 218             |
| $\Gamma_{80}$  | 20               | 50             | 2              | 7            | 218            | 222             |
| $\Gamma_{81}$  | 21               | 40             | 2              | 6            | 222            | 226             |
| $\Gamma_{82}$  | 21               | 40             | 2              | 6            | 226            | 230             |
| $\Gamma_{83}$  | 21               | 40             | 2              | 6            | 230            | 234             |
| $\Gamma_{84}$  | 21               | 40             | 2              | 6            | 234            | 238             |
| $\Gamma_{85}$  | 21               | 40             | 2              | 6            | 238            | 242             |
| $\Gamma_{86}$  | 21               | 40             | 2              | 6            | 242            | 246             |
| $\Gamma_{87}$  | 21               | 40             | 2              | 6            | 246            | 250             |
| $\Gamma_{88}$  | 21               | 40             | 2              | 6            | 250            | 254             |
| $\Gamma_{89}$  | 21               | 40             | 2              | 6            | 254            | 258             |
| $\Gamma_{90}$  | 21               | 40             | 2              | 6            | 258            | 262             |
| $\Gamma_{91}$  | 21               | 40             | 2              | 6            | 262            | 266             |
| $\Gamma_{92}$  | 21               | 40             | 2              | 6            | 266            | 270             |
| $\Gamma_{93}$  | 21               | 40             | 2              | 6            | 270            | 274             |
| $\Gamma_{94}$  | 21               | 40             | 2              | 6            | 274            | 278             |
| $\Gamma_{95}$  | 21               | 40             | 2              | 6            | 278            | 282             |
| $\Gamma_{96}$  | 21               | 40             | 2              | 6            | 282            | 286             |
| $\Gamma_{97}$  | 21               | 40             | 2              | 6            | 286            | 290             |
| $\Gamma_{98}$  | 21               | 40             | 2              | 6            | 290            | 294             |
| $\Gamma_{99}$  | 21               | 40             | 2              | 6            | 294            | 298             |
| $\Gamma_{100}$ | 21               | 40             | 2              | 6            | 298            | 302             |
we give our present state of knowledge about $d_{\text{max}}(n,0)$ [5]:

\[
\begin{align*}
15 & \leq d_{\text{max}}(56,0) \leq 20, \\
16 & \leq d_{\text{max}}(58,0) \leq 20, \\
17 & \leq d_{\text{max}}(67,0) \leq 24, \\
18 & \leq d_{\text{max}}(70,0) \leq 24, \\
19 & \leq d_{\text{max}}(79,0) \leq 28, \\
20 & \leq d_{\text{max}}(87,0) \leq 30, \\
21 & \leq d_{\text{max}}(89,0) \leq 31, \\
20 & \leq d_{\text{max}}(95,0) \leq 33.
\end{align*}
\]

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