Do we understand near-forward elastic scattering up to TeV energies?  

Claude Bourrely
Aix-Marseille Université, Département de Physique, Faculté des Sciences, site de Luminy, 13288 Marseille, Cedex 09, France

Jacques Soffer
Department of Physics, Temple University, Philadelphia, PA 19122-6082, USA

Tai Tsun WU
Gordon McKay Laboratory Harvard University Cambridge, MA 02138, USA
and
Theoretical Physics Division, CERN, 1211 Geneva 23, Switzerland

Abstract

In 1970, on purely theoretical grounds, all total hadronic total cross sections were predicted to increase without limit for higher and higher energies. This was contrary to the conventional belief at that time. In 1978, an accurate phenomenological model was formulated for the case of proton-proton and antiproton-proton interactions. The parameters for this model were slightly improved in 1984 using the additional available experimental data. Since then, for thirty years these parameters have not changed. This development, including especially the difficult task of formulating this phenomenological model and the comparison of the predictions of this model with later experimental results, is summarized.

Key words: elastic hadronic reactions, pomeron.

PACS numbers: 13.85.Dz,11.80Fv,25.40.Cm,25.45.De,25.80.Dj, 12.40.Ee,14.65.Bt,25.10.+s

¹Contribution to the special issue of the International Journal of Modern Physics A on "Elastic and diffractive scattering" coordinated by Christophe Royon
1 Introduction

In the 1958 International Conference on High Energy Physics, Oppenheimer [1] gave the concluding remarks. He said, among other points:

“There are areas where we know very little – extremely high energy collisions, for instance – where little can be done by anyone.”

Physics is basically an experimental science. At that time over half a century ago when Oppenheimer gave his speech, there were no high-energy accelerator where the collisions can be called even remotely "extremely high energy", in the sense that the incident particles are extremely relativistic in the center-of-mass system.

This situation began to change about a decade later: the 200-GeV proton accelerator was to be built at National Accelerator Laboratory, now called the Fermilab, and the Intersecting Storage Ring (ISR) was to be built at CERN. At the laboratory energy of 200 GeV, the center-of-mass energy of each proton is about ten times the proton mass (times the velocity of light squared), which is extremely relativistic. The proton energy of ISR is even higher. That these accelerators would be soon available gave motivation, indeed urgency, to study theoretically the Oppenheimer challenge of understanding extremely high energy collisions.

A main role of theoretical physics is to give quantitative predictions that can be either confirmed or refuted by later experiments. In this way our understanding of physics can be deepened.

What quantity should be studied first at such extremely high energies? At any energy, the overall property of a collision process is provided by the total cross section, which is the sum of the integrated cross sections of all possible scattering and production processes. In turn, the cross sections of these various processes are necessarily affected by the behavior of the total cross section.

Motivated by the Oppenheimer challenge and the accelerators to be available soon, we embarked on the task of studying theoretically the proton-proton total cross section at the energies of these accelerators and beyond.

In the nineteen sixties, there were two distinct schools of thought on the high-energy scattering of strongly interacting particles: the droplet model of Yang and collaborators [2] and the Regge pole model [3]. The droplet model has numerous successes for many processes at high energies, and is based on the following observations from experimental data: the elastic differential cross sections appear to approach limiting values as the incident energy $E \to \infty$, above about 300 MeV excitation energy, the nucleon has many excited states and the large-angle proton-proton elastic cross section drops spectacularly with energy. In contrast, the Regge pole model deals mostly with scattering processes where some quantum number is exchanged. The intuition obtained from these two models guided our thinking on total cross sections at high energies.

Shortly after Professor Yang developed the droplet model for hadron-hadron collisions, the question was asked: What are the experimental evidences for or against the existence of limiting values for total cross section? By studying the
experimental data on the ratio

\[ \rho = \frac{\text{Real part of } pp \rightarrow pp \text{ in the forward direction}}{\text{Imaginary part of } pp \rightarrow pp \text{ in the forward direction}} \]

in the case of proton-proton elastic scattering and using dispersion relation, there was a weak and preliminary indication that there may not be such a limiting value in this case. This weak indication actually played an important role in our decision in the nineteen sixties to study the total cross section at high energies. This episode has been described in [4] and we will come back to it in Section 3. In the above expression, the amplitude for \( pp \rightarrow pp \) does not include the contribution from the Coulomb interaction, (See Eq. (10) in Section 3).

For the purpose of this study, we began by writing down a list of the basic features of the interaction between elementary particles, features that are valid not only at high energies but at all energies:
- three spatial dimensions and one time dimension
- relativistic kinematics
- unitarity and
- particle production.
Even though each of these four features may be considered to be "trivial", it is nevertheless not easy to have a model with all these four features. The simplest way to have all features is to have a relativistic quantum field theory.

As emphasized by Yang and Mills [5, 6], "... the usual principle of invariance under isotopic spin rotation is not consistent with the concept of localized fields." This inconsistency is avoided in theories with a gauge invariance of the second kind. For this reason, the decision was made to study the high-energy behavior of a relativistic quantum gauge field theory in 3+1 dimensions. At that time in the nineteen sixties, there was no choice because the only such theory that was understood was the Abelian case. It has turned out that this was most fortunate: even now nearly half a century later, the high-energy behavior of non-Abelian Yang-Mills theory remains unknown, even for the simplest case with the gauge group \( SU(2) \).

The theoretical investigation of the high-energy behavior, with fixed transverse momentum, for this Abelian relativistic quantum gauge field theory began in the late nineteen sixties. The main theoretical result, obtained in 1970, is that the total cross section increases without limit for higher and higher energies; this result is then interpreted to imply that the total cross sections of strongly interaction particles all have to increase this way [7]. This is in agreement with the preliminary indication from the experimental measurements on the ratio \( \rho \) mentioned above. This theoretical development, together with some of the early phenomenology has been describe in detail [8]. This book [8] is already over quarter of a century old; reference [9] gives a later summary of the situation.
2 Basic Idea of Phenomenology

It is an interesting and important theoretical statement that hadronic total cross sections all increase without bound for higher and higher energies. Nevertheless, such a statement by itself is not of great help for us to gain a better understanding of nature.

It is the foremost purpose of theoretical physics to make quantitative predictions. The most important predictions are those that open up new directions of investigation and are verified accurately by subsequent experiments; they allow us to have a novel way of understanding nature. The next type of important predictions are those that open up new directions but are refuted by subsequent experiments; they tell us that the novel view needs suitable modification.\(^2\)

In the present context of scattering processes at extremely high energies, the step after the asymptotic calculation of the perturbation series for the Abelian gauge theory is therefore to develop a quantitatively accurate phenomenology. What does this mean? Since the internal structure of a proton is exceedingly complicated and not understood quantitatively from first principles, this phenomenology must satisfy the following two conditions:

On the one hand, the proton-proton and proton-antiproton total cross sections must increase without bound at very high energies, and

On the other hand, this phenomenology must give a reasonable description of these two cross sections at the relatively high-energy region of the existing experimental data.

The development of such a phenomenology requires a great deal of physical intuition and is perhaps the most difficult step for the present approach to scattering at extremely high energies, an attempt to response to the Oppenheimer challenge. More precisely, the difficulty was as follows. While the field-theoretic calculation\(^8\)\(^7\) leading to the prediction of increasing total cross section played a central role, it was physical intuition that determined which field theoretic results should be incorporated into the phenomenology and which ones must be rejected.

Here are the two most important and difficult decisions for the development of our phenomenology, the first one making use of one of the results from field theory, while the second one purposely contradicting another result from field theory.

(I) A choice must be made of the variable for describing the increasing total cross sections. For this purpose, the Regge language is most helpful. It was known that the leading Regge singularity should be above 1, but the question is: what is the nature of this singularity? The possible choices are:

- moving Regge pole, as in the usual Regge theory
- fixed Regge pole
- moving Regge cut or
- fixed Regge cut.

\(^2\)To the best of the knowledge of the authors, this wisdom is due to Niels Bohr
Our initial inclination was to follow the prevailing thinking in the nineteen sixties and to use a moving Regge pole. However, our intuitive feeling was that the leading singularity being a Regge pole is intimately related to the nature of the underlying quantum field theory. Regge poles are obtained from the summation of the ladder diagrams in $\phi^3$ theory, which is a super renormalizable field theory. In general, this property of being super renormalizable is connected with the ladder-like diagrams being described by a Fredholm integral equation, and this is the origin of the moving Regge pole. In contrast, quantum gauge theories in four dimensions are never super renormalizable, and the explicit calculation for the Abelian case gives a fixed Regge cut. This was our reason in 1978 to use a fixed Regge cut above 1 \cite{10}. The other two cases of a fixed Regge pole and a moving Regge cut were considered too unnatural to be used for our phenomenology.

(II) We also need to make a decision how to describe the absorption in the scattering of strongly interacting particles. The specific issue is: when the impact distance between the two incident particles is small, what is the amount of absorption that should be incorporated into the phenomenology?

In the field-theoretic calculation for the quantum Abelian gauge field theory, this absorption is only partial even at extremely high energies. Since this lack of complete absorption has a rather complicated origin, a major issue was whether this property was likely to hold for the interactions between strongly interacting particles. This was an agonizing choice, and we finally decided to go against this field-theoretic result and take the absorption in this limit to be complete. The basic reason for this choice was that it seemed to be against physical intuition that, in the language of Yang and collaborators, some of the "stuff" is absorbed while others not \cite{2}.

It has been gratifying that our phenomenology has worked out well after these two decisions based on our physical intuition. While we have to make a number of other choices to complete our phenomenology, the above two are the most difficult and far-reaching ones.

To describe the experimental data taken at the relatively low energies available to experiments forty years ago, a new model was proposed in 1978 \cite{10}, including Regge backgrounds. Besides those pertaining to the Regge terms, there is a total of six parameters for $pp$ and $\bar{p}p$ elastic scattering. From the overall fit to the existing data in 1978, the values of these six parameters \cite{10} are given in the left column of Table 1.

Six years later, when there were significantly more experimental data at high energies, the overall fit was repeated \cite{12}. The revised values of these six parameters are given in the right column of Table 1. It should be emphasized that, in these six years from 1978 to 1984, the expressions used to describe the model is not altered at all; these formulas are given explicitly later in this Section 2. It is interesting to compare the two columns of values in Table 1: the six values have not changed much due to the additional information. This implies that this new model, sometimes referred to as the BSW Model, is quite robust. There has been no further change of
these parameter values in the thirty years since.

Both for the energies of the present-day colliders and for the purpose of studying the asymptotic behavior of the model at high energies, all the Regge backgrounds can be neglected. The BSW model is given by the following matrix element for elastic scattering

\[ \mathcal{M}(s, \Delta) = \frac{is}{2\pi} \int d\mathbf{x}_\perp e^{-i\Delta \cdot \mathbf{x}_\perp} D(s, \mathbf{x}_\perp) , \]  

(1)

where \( s \) is the square of the center-of-mass energy, \( \Delta \) is the momentum transfer, \( \mathbf{x}_\perp \) is the impact parameter and all spin variables have been omitted. For this model we use for the opacity

\[ D(s, \mathbf{x}_\perp) = 1 - e^{-\Omega(s, x_\perp^2)} , \]  

(2)

with

\[ \Omega(s, x_\perp^2) = S(s) F(x_\perp^2) , \]  

(3)

where \( x_\perp \equiv |\mathbf{x}_\perp| \). The function \( S(s) \) is given by the complex symmetric expression, obtained from the high energy behavior of quantum field theory [8, 7]

\[ S(s) = \frac{s^c}{(\ln s)^c} + \frac{u^c}{(\ln u)^c} , \]  

(4)

with \( s \) and \( u \) in units of GeV\(^2\), where \( u \) is the third Mandelstam variable [11]. In this Eq. (4), \( c \) and \( c' \) are two dimensionless constants given in Table 1. That they are constants implies that the Pomeron is a fixed Regge cut as discussed above. For the asymptotic behavior at high energy and modest momentum transfers, we have to a good approximation

\[ \ln u = \ln s - i\pi , \]  

(5)

so that

\[ S(s) = \frac{s^c}{(\ln s)^c} + \frac{s^c e^{-i\pi c}}{(\ln s - i\pi)^c} . \]  

(6)

Because \( F \) depends on \( x_\perp \) only through \( x_\perp^2 \), the Fourier transform in Eq. (1) simplifies to

\[ \mathcal{M}(s, \Delta) = is \int_0^\infty dx_\perp x_\perp J_0(x_\perp \Delta) \left[ 1 - e^{-S(s)F(x_\perp^2)} \right] , \]  

(7)

where \( \Delta \equiv |\Delta| \). The function \( F(x_\perp^2) \) is taken to be related to the electromagnetic form factor \( G(t) \) of the proton, where \( t = -\Delta^2 \) is the Mandelstam variable for the square of the momentum transfer. Specifically, \( F(x_\perp^2) \) is defined as in [10] via its Fourier transform \( \tilde{F}(t) \) by

\[ \tilde{F}(t) = f[G(t)]^2 \frac{a^2 + t}{a^2 - t} , \]  

(8)

with

\[ G(t) = \frac{1}{(1 - t/m_1^2)(1 - t/m_2^2)} . \]  

(9)
The remaining four parameters of the model, \( f, a, m_1 \) and \( m_2 \), are given in Table 1. In the next section we will recall some of the early successes of our approach at the CERN \( \bar{p}p \) collider and at the FNAL Tevatron and the next section will be devoted to a preliminary discussion of the situation at the Large Hadron Collider.

We define the ratio of the real to imaginary parts of the forward amplitude, mentioned earlier in the introduction, \( \rho(s) = \frac{\text{Re} \mathcal{M}(s,t=0)}{\text{Im} \mathcal{M}(s,t=0)} \), the total cross section \( \sigma_{\text{tot}}(s) = (4\pi/s)\text{Im} \mathcal{M}(s,t=0) \), the differential cross section \( d\sigma(s,t)/dt = \frac{2}{\pi}|\mathcal{M}(s,t)|^2 \), and the integrated elastic cross section \( \sigma_{\text{el}}(s) = \int dt \frac{d\sigma(s,t)}{dt} \). One important feature of the BSW model is, as a consequence of Eq. (6), the fact that the phase of the amplitude is built in. Therefore real and imaginary parts of the amplitude cannot be chosen independently and we will show now how to test them.

| Year | 1979 | 1984 |
|------|------|------|
| \( c \)  | 0.151 | 0.167 |
| \( c' \) | 0.756 | 0.748 |
| \( m_1 \) | 0.619 | 0.586 |
| \( m_2 \) | 1.587 | 1.704 |
| \( f \) | 8.125 | 7.115 |
| \( a \) | 2.257 | 1.953 |

Table 1: Pomeron fitted parameters for \( pp(\bar{p}p) \). Comparison of the 1979 and 1984 solutions.

\section{Early successes}

Before showing some early successes of our phenomenology we would like to come back to an important observation which gave the first hint against the possibility that the total cross section would remain constant at very high energies. This is indeed related to the behavior of \( \rho(s) \) which is shown in Fig. 1. The forward scattering amplitude is expected to satisfy a dispersion relation. This means that the real part of this amplitude can be written as an integral over the imaginary part, which is essentially the total cross section. If the \( pp \) total cross section does approach a finite limit, then this ratio \( \rho(s) \) must approach zero at high energies. In the mid-sixties the experimentally measured values of \( \rho(s) \) in the low energy region, below \( \sqrt{s} \sim 10\text{GeV} \), were negative and increasing toward zero. However, the rate of increase seemed to have a tendency to overshoot to become positive and this was the first indication for a possible increasing total cross section at high energies. Indeed positive \( \rho(s) \) values were later obtained as displayed in Fig. 1, which also shows the BSW results and predictions up to high energies.
Figure 1: The ratio $\rho(s) = \frac{\text{Re} \, M(s,t=0)}{\text{Im} \, M(s,t=0)}$ for $pp$ (black points) and $\bar{p}p$ (open points) elastic scattering and the BSW predictions up to high energies (Taken from Ref. [12]).

Let us now turn to some successful BSW predictions for near-forward $\bar{p}p$ elastic scattering at the FNAL and CERN colliders energies. For this kinematic region near the forward direction, one must consider the contribution of the Coulomb amplitude $a^C$, so Eq. (1) is replaced by

$$\mathcal{M}(s, \Delta) \pm a^C_{\pm}(-\Delta^2),$$

the upper sign is for $\bar{p}p$ while the lower one is for $pp$ scattering and

$$a^C_{\pm}(t) = 2\alpha s \frac{G^2(t)}{|t|} \exp[\mp i\alpha \phi(t)],$$

where $\alpha$ is the fine structure constant, $\phi(t)$ is the West-Yennie phase [13] given by $\phi(t) = \ln(t_0/|t|) - \gamma$, with $t_0 = 0.08\text{GeV}^2$ and $\gamma \sim 0.577$ is the Euler constant.

At the FNAL-Tevatron, the E710 experiment running at $\sqrt{s}=1.8\text{TeV}$, has obtained $\sigma_{\text{tot}} = 72.8 \pm 3.1 \text{ mb}$ and $\sigma_{\text{el}}/\sigma_{\text{tot}} = 0.23 \pm 0.012$ [14], whereas the BSW predictions are 74.8 mb and 0.230 respectively. They were also able to extract the following $\rho$ value, $\rho = 0.140 \pm 0.069$ [15]. This important measurement is in agreement with the BSW prediction, but has unfortunately little significance because of its large error. These data are reported in Fig. 2 together with the results of the CDF experiment.
at two different Tevatron energies $\sqrt{s}=1.8\text{TeV}$ and $\sqrt{s}=546\text{GeV}$ [16] and the results of UA(4) at the CERN $\bar{p}p$ collider at $\sqrt{s}=541\text{GeV}$ [17]. At $\sqrt{s}=1.8\text{TeV}$ CDF found $\sigma_{\text{tot}} = 80.03 \pm 2.24 \text{ mb}$ and $\sigma_{\text{el}}/\sigma_{\text{tot}} = 0.246 \pm 0.004$, at variance with the E710 results. However at $\sqrt{s}=546\text{GeV}$, the CDF results $\sigma_{\text{tot}} = 61.26 \pm 0.93 \text{ mb}$ and

\[
\sigma_{\text{el}}/\sigma_{\text{tot}} = 0.210 \pm 0.002
\]

agree well with those of UA(4), $\sigma_{\text{tot}} = 63.0 \pm 2.1 \text{ mb}$ and $\sigma_{\text{el}}/\sigma_{\text{tot}} = 0.208 \pm 0.007$. The UA(4) experiment has obtained a very precise value for the parameter $\rho$, $\rho = 0.135 \pm 0.015$, from the measurement of $d\sigma/dt$ in the Coulomb-nuclear interference region [18], as shown in Fig. 3. One notices the rapid
rise of the cross section in the very low $t$ region and the remarquable agreement with the BSW prediction. The BSW model predicts the correct $\rho(s)$ which appears to have a flat energy dependence in the high energy region (see Fig. 1) and for $s \to \infty$, one expects $\rho(s) \to 0$. Another specific feature of the BSW model is the fact that it incorporates the theory of expanding protons [8, 7], with the physical consequence that the ratio $\sigma_{el}/\sigma_{tot}$ increase with energy. This is precisely in agreement with the data and when $s \to \infty$ one expects $\sigma_{el}/\sigma_{tot} \to 1/2$, which is the black disk limit.

Finally we show in Fig. 4 the $t$-dependence of the elastic cross section measured by the E710 experiment, which again confirms the BSW prediction. It may be worth emphasizing that in this $t$ domain, the $t$-behavior is definitely not a straight line.

Before moving to the LHC energy it is worth mentioning another independent test of the BSW amplitude, by means of the analyzing power $A_N$, in $pp$ elastic scattering near the very forward direction. In addition to the non-flip component Eq. (11), the Coulomb amplitude has also a single-flip component $a_C^5$, which involves the proton anomalous magnetic moment. In this kinematic region, the CNI region, $A_N$ results only from the interference of $a_C^5$, which is purely real, with the imaginary part of the hadronic non-flip amplitude, if one assumes that there is no contribution from the single-flip hadronic amplitude. This is what we have done in the calculation of the curve displayed in Fig. 5 compared to some new data from STAR [23]. It confirms the absence of single-flip hadronic amplitude and the right determination of $\text{Im} \mathcal{M}(s, t)$ in the CNI region.

Figure 4: $d\sigma/dt$ for near-forward $\bar{p}p$ elastic scattering at $\sqrt{s} = 1.8\text{TeV}$ data from Ref. [14], the curve is the BSW prediction [21] (Taken from Ref. [22]).
Figure 5: The analyzing power $A_N$ versus $t$ at RHIC energy. The data from Ref. [23] are in excellent agreement with the BSW prediction.

4 The LHC energy

At the moment, the situation with the experimental data at the energies of the Large Hadron Collider is not completely clear. Here is a description of the present data.

There are two experiments measuring the proton-proton total cross section: TOTEM associated with the CMS detector and the ALFA associated with the ATLAS Collaboration. The center-of-mass energy chosen by both experiments are 7 TeV.

The ALFA has not published yet any result from their measurements, although such publications are expected in the very near future. Thus the discussion here has to be limited to those from TOTEM. The total proton-proton cross section given by TOTEM is $\sigma_{tot}(TOTEM) = (98.0 \pm 2.5) \text{ mb}$, which is 1.7 $\sigma$ above the BSW prediction at 7 TeV. It should be emphasized that the BSW parameters, shown in Table 1, was determined in 1984 and has not changed in thirty years.
As usual, the total cross section is obtained by extrapolating the differential cross section to the forward direction. It is therefore of interest to compare the TOTEM measurement of the proton-proton differential cross section directly with the BSW prediction as discussed in Sec. 2. Such a comparison is shown in Fig. 6.

The meaning of this comparison of Fig. 6 is not straightforward and has not yet been completely clarified, and we look forward to future developments, especially at even higher energies of LHC.

Figure 6: $d\sigma/dt$ for near-forward $pp$ elastic scattering. The data are from TOTEM [24] and the curve is the BSW prediction.

5 Conclusion

The basic idea for the development of the present phenomenology has been described in Sec. 2. On the one hand, it is essential to incorporate suitably chosen results from field theory; however, some other results from field theory have to be purposely contradicted in the phenomenology. Physics has played an essential role in the choices, and is largely responsible for the success in the many predictions of the present phenomenology, which is entirely unchanged in nearly thirty years.

When the present phenomenology was first worked out, the highest center-of-mass energy of available experimental data was 62 GeV. As shown in Sec. 3, the predictions of this phenomenology are in good agreement with later experimental data up to the center-of-mass energy of 1.96 TeV. This is an increase of energy by a factor of more than thirty – an extraordinary success as discussed in the previous sections.
The conclusion is therefore reached that there is a good understanding of proton-proton elastic scattering near the forward direction. It will be of interest to push this phenomenology to higher energies by comparing its predictions with the data from the Large Hadron Collider at the center-of-mass energy of 7 TeV. Some such comparison has already been presented in Sec. 4, and some more data are expected from the ALFA Collaboration in the near future. It is even more interesting to compare the present predictions with the data after an upgrade of the center-of-energy of the Large Hadron Collider next year to 13 or 14 TeV, both for the differential elastic cross section and the parameter \( \rho \) discussed above, whose relevance of its measurement has been strongly emphasized \[25\].

It may be of some importance to add the following comment to the present development on the increasing total cross section. It is the production of relatively low-energy particles in the center-of-mass system that leads to this increasing total cross section; such production processes are usually referred to as "pionization". Also it was known from the beginning \[8,7\] that, at extremely high energies, half of this increase in the total cross section is due to the integrated elastic cross section. The obvious question, already raised more than forty years ago, is: what processes are responsible for the other half of this increase at extremely high energies?

This question has been answered recently through the application of geometrical optics to production processes \[26\]: the same processes, i.e., pionization, are not only responsible for the increasing total cross section, but are also responsible for the other half of the increasing total cross section at extremely high energies – an answer that we find to be philosophically satisfying.

**Acknowledgments**

We thank André Martin and Maurice Haguenauer for several fruitful discussions.

**References**

[1] J.R. Oppenheimer, Proc. 1958 Annual International Conference on High Energy Physics at CERN, ed. B. Ferretti, p.291.

[2] T.T. Wu and C.N. Yang, Phys. Rev. **137**, B708 (1965). N. Byers and C.N. Yang, Phys. Rev. **142**, 967 (1966); T.T. Chou and C.N. Yang, Phys. Rev. Lett. **20**, 1213 (1968), Phys. Rev. **170**, 1591 (1968) and **175**, 1832 (1968).

[3] T. Regge, Nuovo Cimento **14**, 951 (1959) and 18, 947 (1960).

[4] T.T. Wu, Proceedings of the Conference in Honor of C.N. Yang’s 85th Birthday, ed M.-L. Ge, C. H. Oh, and K.K. Phua, World Scientific (2008), p.112.

[5] C.N. Yang and R.L. Mills, Phys. Rev. **95**, 631 (1954).
[6] C.N. Yang and R.L. Mills, Phys. Rev. 96, 191 (1954).
[7] H. Cheng and T.T. Wu, Phys. Rev. Lett. 24, 1456 (1970).
[8] H. Cheng and T.T. Wu, Expanding Protons: Scattering at High Energies, M.I.T. Press, Cambridge, MA (1987).
[9] T. T. Wu, Chapter 4.3.4 in Scattering – Scattering and Inverse Scattering in Pure and Applied Science, ed E. Pike and P. Sabatier, Academic Press, London, 2002. Vol. 2, p.1582.
[10] C. Bourrely, J. Soffer and T.T. Wu, Phys. Rev. D 19, 3249 (1979).
[11] S. Mandelstam, Phys. Rev. 112, 1344 (1958).
[12] C. Bourrely, J. Soffer and T.T. Wu, Nucl. Phys. B 247, 15 (1984).
[13] G.B. West and D.R. Yennie, Phys. Rev. 172, 1413 (1968).
[14] N.A. Amos et al., Phys. Rev. Lett. 63, 2784 (1989); preprint FERMILAB-Pub-90/96-E (May 1990).
[15] N.A. Amos et al., Phys. Rev. Lett. 68, 2433 (1992).
[16] F. Abe et al., Phys. Rev. D 50, 5550 (1994).
[17] C. Augier et al., Phys. Lett. B 344, 451 (1995).
[18] C. Augier et al., Phys. Lett. B 316, 448 (1993).
[19] C. Bourrely, J. Soffer and T.T. Wu, Phys. Letters. B 196, 237 (1987).
[20] C. Bourrely, J. Soffer and T.T. Wu, Phys. Letters B 315, 195 (1993).
[21] C. Bourrely, J. Soffer and T.T. Wu, Z. Phys. C 37, 369 (1988); erratum ibid., 53, 538 (1992).
[22] C. Bourrely, J. Soffer and T.T. Wu, Phys. Letters B 252, 287 (1990).
[23] L. Adamczyk et al., Phys. Lett. B 719, 62 (2013).
[24] G. Antchev et al., Europhys. Lett. 101, 21002 (2013); ibid 21003 and 21004.
[25] C. Bourrely, N.N. Khuri, A. Martin, J. Soffer and T.T. Wu, Proceedings EDS-2005, The Giod Publishers, Vietnam, pp. 41-44 (2006).
[26] R. Gastmans, S.L. Wu and T.T. Wu, Phys. Letters B 720, 205 (2013).