On the evolution of higher order fluxes in non-equilibrium thermodynamics

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The connection between the balance structure of the evolution equations of higher order fluxes and different forms of the entropy current is investigated on the example of rigid heat conductors. Compatibility conditions of the theories are given. Thermodynamic closure relations are derived.

Keywords: Extended Thermodynamics; Irreversible Thermodynamics with dynamic variables; Balance laws; Higher order fluxes; Onsager linear equations; Closure relations; Second sound propagation; Phonon gas hydrodynamics; 4-field model; 9-field model.

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I. INTRODUCTION

In the last two decades two different thermodynamic theories have been applied in modeling fast non-equilibrium phenomena: the Extended Thermodynamics [1, 2, 3] and the Irreversible Thermodynamics with dynamic variables [4, 5]. In the following investigations instead of referring directly to persons we sometimes distinct theories of Extended Thermodynamics (Rational Extended Thermodynamics [1], Extended Irreversible Thermodynamics [2, 3] and Wave Approach of Thermodynamics [6, 7]).

Extended Thermodynamics (ET) lies on the kinetic theory of gases and postulates suitable balance equations for dissipative fluxes. In this way a hierarchical system in which the flux at step $n$ becomes the wanted field at step $n+1$, may be obtained. Furthermore, the evolutionary systems are required to be symmetric hyperbolic so that finite speed of propagation of disturbances and well posedness of the Cauchy problem are guaranteed [1].

Irreversible Thermodynamics (IT) enlarges the basic state space by introducing certain additional dynamic variables whose evolution is ruled either by ordinary differential equations (local theory) [4, 8] or by partial differential equations (weakly nonlocal theory) [9, 10]. In such a framework the parabolic models are also allowed in that, once their solutions are interpreted in the light of the experimental measurements, they also lead to finite speed of propagation [11]. ET is a special case of IT with internal variables, where the form of the evolution equations is restricted and the role of the internal variables is fixed.

In the present paper we compare these two approaches by deriving conditions under which they lead to the same system of equations. To achieve that task we apply the classical Onsager analysis of Second Law [4, 12].

We point out the central role of the entropy current in the two theories. In Rational Extended Thermodynamics the entropy current is a derived quantity. Its form is restricted by the balance structure of the evolution equations through the Second Law (see e.g. [1], page 24 or page 60). The restriction can be expressed by the following differential expression:

$$d j_i = \frac{\partial s}{\partial a} d j^a_i \quad (1.1)$$

Here $j_i$ is the $i$th component of the entropy current, $s$ is the entropy density, $a$ and $j^a$ denotes the densities their currents in the corresponding balances $\dot{a} + j^a_i = 0$, respectively. Here the entropy current appears as a potential on the space of the currents of the extensive quantities. Considering that the currents of the extensives $j^a$ are also constitutive quantities, the above expression constrains considerably their form. The surprisingly restrictive constraints were proved to be compatible with some kinetic theory models. As a price some of the higher order currents were considered as constitutive quantities instead of being variables.
On the other hand in IT with dynamic variables, the starting point is the following form of the entropy current

\[ j_i = \frac{\partial s}{\partial \sigma} f^i. \]  

(II.2)

This form was suggested as a direct generalization of the classical heat flux over temperature expression \( j_i = q_i / T \).

The entropy current (II.2) is the consequence of the balance structure of the evolution equations. On the other hand, from any given form of the entropy current one can determine, or at least restrict, the form of the evolution equations of the dynamic variables. In Extended Irreversible Thermodynamics one can meet both forms, Jou, Casas-Vázquez and Lebon seem to hesitate on the advantages and disadvantages of the two suggestions (compare \( \text{[1], [2], p. 138 and [2], p. 56-58, [14]}. \n
In the following we will investigate the compatibility of the Verhás form of the entropy current with the balance structure of the evolution equations in a general and highly nonlinear case. We give conditions when the balance structure and the entropy current are compatible. Our investigations show that if we do not insist strictly to the balance structure then the momentum hierarchy can be closed easily on the phenomenological level.

In section 2, after considering a rigid body at rest and out of local equilibrium, we specify the basic balance equations together with the main constitutive assumptions. In section 3 we obtain the evolution equations for the dynamic variables. In section 4 the conditions are pointed out under which these equations yield a 4 moments model and a 13 moments model of rigid heat conductor of Extended Thermodynamics. In section 5 we consider the propagation of heat waves at low temperatures. We reinspect some extended thermodynamic models and determine the conditions under which they can be recovered in the framework of the presented theory. A concluding discussion of the obtained results is developed in section 6.

II. BALANCE EQUATIONS AND SECOND LAW OF THERMODYNAMICS

As a simple but representative example let us consider a rigid heat conductor at rest and let us start from the following local balance equation of the internal energy

\[ \dot{\varepsilon} + q_i \dot{x}_i = 0, \]  

(II.1)

where \( \varepsilon \) is the density of internal energy, \( q_i \) \( i = 1, 2, 3 \) are the components of the heat flux \( \mathbf{q} \), \( \dot{f} \equiv \frac{\partial f}{\partial t}, \) \( f_{\dot{r}} \equiv \frac{\partial f}{\partial x_i} \), \( x_i \) \( i = 1, 2, 3 \) are the Cartesian coordinates of the points of the body and the Einstein convention of summation over the repeated indices has been applied. The local balance of entropy is given by

\[ \dot{s} + j_{\dot{s}i} = \sigma_s, \]  

(II.2)

with \( s \) standing for the entropy density, \( j_i \) \( i = 1, 2, 3 \) for the components of the entropy current \( \mathbf{j} \) and \( \sigma_s \) for the density of entropy production. Second Law of Thermodynamics forces \( \sigma_s \) to be nonnegative. The only equilibrium variable will be the internal energy \( \varepsilon \), while the first dynamic variable is supposed to be the heat flux itself. Such an assumption, following by the celebrated Cattaneo’s equation \( [17] \), is included within the basic postulates of both theories under examination (see \( \text{[1], Chapter 1, and [2], p. 56-58, [14]}. \n
In order to derive such a system let us calculate \( \sigma_s \) according to the classical procedures of Irreversible Thermodynamics. Along with Gyarmati and Verhás \( \text{[9], [10]} \), we represent the entropy function, out of local equilibrium, as

\[ s(e, q_i, \Phi_{ij}) = s_0(e) + \frac{1}{2} m_{ij} q_i q_j + \frac{1}{2} n_{ijkl} \Phi_{ij} \Phi_{kl}. \]  

(II.4)

In \( \text{[11-14]} \), \( s_0(e) \) is the local equilibrium entropy while the matrices of thermodynamic inductivities \( m_{ij} \) and \( n_{ijkl} \) are constitutive functions depending on the basic fields \( e, q_i \) and \( \Phi_{ij} \). Moreover, we assume that the local equilibrium is
globally thermodynamically stable and there are no phase boundaries at the non-equilibrium part of the state space. Therefore, due to the principle of maximum entropy at the equilibrium, \( m_{ij} \) and \( n_{ijkl} \) are negative definite. As the antisymmetric part of the inductivities \( m \) and \( n \) do not contribute to the entropy we assume that they are symmetric as follows:

\[
m_{ij} = m_{ji}, \quad n_{ijkl} = n_{klij},
\]

(II.5)

Finally, along with Verhás [13, 14, 18] we represent the entropy current as

\[
\dot{j}_i = \frac{\partial s}{\partial e} q_i + \frac{\partial s}{\partial q_j} \Phi_{ji}.
\]

(II.6)

Let us observe that the previous form of the entropy current is not the most general one and some extensions are also possible (see for instance [19]). Due to (II.4), \( \dot{j}_i \) specialize to

\[
\dot{j}_i = \frac{\partial s}{\partial e} q_i + \frac{1}{2} A_{hk} q_h q_k q_i + \frac{1}{2} B_{hklm} \Phi_{hk} \Phi_{lm} q_i + \frac{1}{2} C_{hk} q_h \Phi_{ji} + \frac{1}{2} D_{hklm} \Phi_{hk} \Phi_{lm} \Phi_{ji},
\]

(II.7)

and

\[
A_{hk} \equiv \frac{\partial m_{hk}}{\partial e}; \quad B_{hklm} \equiv \frac{\partial n_{hklm}}{\partial e}; \quad C_{hk} \equiv \frac{\partial m_{hk}}{\partial q_j}; \quad D_{hklm} \equiv \frac{\partial n_{hklm}}{\partial q_j}.
\]

In the next section we will use (II.4) and (II.7) in order to calculate the most important quantity related to this procedure, namely, the density of entropy production. Let us observe the special dependence of vector \( \dot{j} \) on the components \( q_i \) and \( \Phi_{ij} \) of the basic field. Due to (II.7) the scalar \( \dot{j}_i \) does not contain the divergence of any third order tensor to be interpreted as the flux of \( \Phi \). As far as the form of the evolution equations is concerned, such a property results to be crucial.

III. BILINEAR FORM OF THE ENTROPY PRODUCTION

Our aim here is to prove that the entropy production \( \sigma_s \) can be put in the classical bilinear form of Irreversible Thermodynamics [12]. Some straightforward calculations show that, due to (II.4) and (II.7) the function \( \sigma_s \) can be written as follows

\[
\sigma_s = \left[ \frac{\partial^2 s}{\partial e^2} c_{ij} + \frac{1}{2} A_{ik} q_l q_k + A_{hk} q_h q_{k'i} + \frac{1}{2} A_{h'k'i} q_h q_k + \right. \\
+ \frac{1}{2} B_{hklm} \Phi_{hk} \Phi_{lm} + \frac{1}{2} (B_{hklm} + B_{lnhk}) \Phi_{hk'} \Phi_{lm} + \frac{1}{2} C_{kl'} m q_k \Phi_{lm} + \\
+ \frac{1}{2} C_{hl'} m q_k \Phi_{lm} + \frac{1}{2} m_{hl'} \Phi_{hl} + m_{h'i} \Phi_{h'i} + \\
+ \frac{1}{2} A_{ij} q_j e + \frac{1}{2} C_{ijk} q_j q_k + m_{ij} q_j + \frac{1}{2} F_{ijkl} \Phi_{kl} q_j \left. \right] q_i + \left[ \frac{1}{2} B_{hklm} \Phi_{lm} q_{i'l'} + \\
+ m_{h'p} q_{i'k} + \frac{1}{2} D_{hklmp} \Phi_{lm} \Phi_{pi} + D_{hklmp} \Phi_{lm} \Phi_{pi} + \\
+ \frac{1}{2} D_{hklmp} \Phi_{lm} \Phi_{pi} + \frac{1}{2} B_{hklj} \Phi_{ij} e + \frac{1}{2} D_{hklj} \Phi_{ij} e + \frac{1}{2} D_{hklj} \Phi_{ij} e +
\]
\[ + \frac{1}{2} G_{hklpq} \dot{\Phi}_{jl} \dot{\Phi}_{pq} + n_{hklm} \dot{\Phi}_{lm} \] \quad (\text{III.1})

In deriving (III.1) we have put
\[ F_{ijkl} = \frac{\partial m_{ij}}{\partial \Phi_{kl}}, \quad G_{ijklpq} = \frac{\partial n_{ijkl}}{\partial \Phi_{pq}}. \]

A direct inspection of (III.1) shows that each term stands for the combination of a relaxation and a transport. The quantities in the square brackets are regarded as process forces while their coefficients \( q_i \) and \( \Phi_{ij} \) as thermodynamic fluxes. Because the expression (III.1) of the entropy production is of the usual bilinear form, let us apply Onsager’s linear equations. In this way we get the evolution equations we are looking for, i.e.
\[ [q]_i = L_{ik} q_k + M_{ikl} \Phi_{kl}, \quad \text{(III.2)} \]
\[ [\Phi]_{hk} = P_{hkl} q_k + Q_{hkl} \Phi_{ij}, \quad \text{(III.3)} \]
where the symbols \( [q]_i \) and \( [\Phi]_{hk} \) stand for the coefficients of \( q_i \) and \( \Phi_{ij} \) respectively and the tensor functions \( L, M, P, Q \) are constitutive functions depending on the basic fields. With these conditions the above form is the solution of the entropy inequality.

It is important to remark here, that the classical Coleman-Noll \[20\] and Liu \[21\] procedures for the exploitation of Second Law are not suitable to obtain the equations above since these techniques lead to thermodynamic restrictions on the constitutive equations which are not in the form of phenomenological evolution equations. After applying these exploitation techniques one can get a restricted form of the dissipation inequality. The Onsagerian approach could start here, with the identification of the fluxes and the forces. Indeed, the essence of the Onsager approach is the strict distinction between the undetermined constitutive quantities (currents) and given functions of the constitutive space (forces). With a suitable distinction and some continuity requirements one can give the general solution of the dissipation inequality in the form of dynamic equations \[22\].

Here, in this paper we have chosen a more direct approach with assuming suitable specific forms of the entropy \( \text{(II.4)} \) and the entropy current \( \text{(II.6)} \). The resulted entropy production makes possible to derive explicit evolution equations for the fluxes. The procedure allows to close the system and this is due to the Second Law.

The expressions (III.2) and (III.3) represent a system of 12 differential equations which, together with the balance of energy \( \text{(II.1)} \), allows - in principle - the determination of the 13 unknown functions \( e, q_i, \Phi_{ij} \). However it is not in the balance form and hence, to achieve that task, some additional assumptions seem to be necessary. This subject will be investigated next.

IV. BALANCE FORM FOR EVOLUTIONARY SYSTEMS

Up to now we did not restrict the inductivity tensors \( m \) and \( n \) at all. Let us assume now, that they are constant and non-degenerate. Let us remark, that non-degeneracy is not trivial, since there exist real materials for which it is not guaranteed. A classical example is given in \[23\], where an electric circuit described by dynamic variables is considered.

For the sake of simplicity, let us assume that the inductivities take the form
\[ m = -mI, \quad n = -nI, \quad \text{(IV.1)} \]
where \( m \) and \( n \) are positive real coefficients and \( I \) means the unitary tensor in the corresponding tensorial space. Finally, according to arguments from the kinetic theory of rigid heat conductors, we assume that \( \Phi \) is symmetric \[1, 2, 16\]. Under the simplification above the equations (III.2) and (III.3) yield
\[ -\dot{q}_i - \Phi_{ik} q_k = \frac{1}{m} \left( L_{ik} q_k + M_{ikl} \Phi_{kl} - \frac{\partial^2 s_0}{\partial e^2} c_i \right), \quad \text{(IV.2)} \]
\[ -\dot{\Phi}_{ij} = \frac{1}{n} \left( P_{ijk} q_k + Q_{ijkl} \Phi_{kl} \right) + \frac{m}{n} q_i q_j. \quad \text{(IV.3)} \]

Here we denoted the symmetries of the constitutive functions, too. From (IV.2), (IV.3) results in that the evolution of \( q \) is ruled by a set of balance laws with gradient dependent source terms while tensor \( \Phi \) obeys a set of ordinary differential equations (with a given \( q \)). The system (III.1), (IV.2) and (IV.3) may be regarded as the four moments
theory for rigid heat conductors of Extended Thermodynamics \[16\] provided the equations \[(IV.3)\] are interpreted as suitable closure conditions \[1,2\]. This point of view is quite different from that often applied in the current literature, where the closure relations are deduced either by different assumptions inside the kinetic theory \[1,24\], by a maximum entropy formalism \[27\] or by some algebraic considerations \[26\]. To our opinion it is remarkable that closure relations have been obtained in a fully thermodynamic framework.

Let us prove now that the 10 moments theory for rigid heat conductors may be obtained by a model with 27 dynamic variables. Therefore in the following we assume that the introduced dynamic variables are symmetric. In this way we can compare our results with those of the kinetic theory. We assume the constitutive equation \[(IV.3)\] takes the form

\[ F = F^*(e, q_i, \Phi_{[ij]}, \Psi_{[ijkl]}), \]  

where \(\Phi_{[ij]}\) is a symmetric second order tensor and \(\Psi_{[ijkl]}\) denotes a component of a fully symmetric third order tensor \(\Psi\) which enters the basic field. Here and in the following the bracketed indices. As a consequence we can write the entropy density as follows

\[ s(e, q_i, \Phi_{[ij]}, \Psi_{[ijkl]}) = s_0(e) + \frac{1}{2} m_{ij} q_i q_j + \frac{1}{2} n_{ijkl} \Phi_{[ij]} \Phi_{[kl]} + \frac{1}{2} r_{ijklpq} \Psi_{[ijkl]} \Psi_{[lpq]}, \]  

where the meaning of the negative definite tensor \(r\) is plausible. Here, in addition to the symmetry properties \[(II.5)\] we can see, that \(n_{ijkl} = n_{jikl}\) and \(n_{ijkl} = n_{ijlk}\) that we could denote as \(n_{[[ij][kl]]}\). Similarly, the symmetry properties of \(r\) could be denoted by \(r_{[ijkl][lpq]}\). In the following we omit the notation of the symmetries \(m, n, r\) that we could denote as \(r_{[ijkl][lpq]}\).

Due to \[(IV.5)\] and \[(IV.6)\] the entropy production takes the form

\[ \sigma_s = \left[ \frac{\partial^2 s_0}{\partial e^2} \gamma_i + m_{hi} \Phi_{[ij]} - m_{ik} \dot{q}_k \right] \dot{q}_i + \left[ m_{hp} q_j \dot{p}_k + n_{hkll} \Psi_{[kl]} + n_{hklm} \dot{\Phi}_{[lk]} \right] \Phi_{[hk]} + \left[ n_{ijhm} \Psi_{[kl]} + r_{ijklpq} \dot{\Psi}_{[lpq]} \right] \Psi_{[ijkl]}, \]  

Again, due to \[(IV.7)\] the matrices \(m, n, r\) are symmetric (and hence invertible) and we are allowed to suppose that \(m\) and \(n\) take the form \[(IV.3)\] while

\[ r = -r I \]  

with \(r\) positive and \(I\) unitary. Finally we get

\[ -\dot{q}_i - \Phi_{[ij]} \gamma_j = \frac{1}{m} \left( L_{ik} \dot{q}_k + M_{ijkl} \Phi_{[kl]} + E_{[ijkl]} \Psi_{[ijkl]} - \frac{\partial^2 s_0}{\partial e^2} \gamma_i \right), \]  

\[ -\dot{\Phi}_{[ij]} - \Psi_{[ijkl]} r_k = \frac{1}{n} \left( P_{[ij]} \dot{q}_p + Q_{[ij][kl]} \Phi_{[kl]} + R_{[ijkl][klm]} \Psi_{[klm]} \right) + \frac{m}{n} \dot{q}_i \]  

\[ -\dot{\Psi}_{[ijkl]} = \frac{1}{r} \left( S_{[ijkl]} \dot{q}_t + T_{[ijkl][klm]} \Phi_{[klm]} + Z_{[ijkl][lmn]} \Psi_{[lmn]} \right) + \frac{n}{r} \dot{\Phi}_{[ij]} r_k. \]  

Here \(E, R, S, T, Z\) are Onsagerian conductivity tensors defined on the thermodynamic state space. In this way we get the 10 moments theory by \[(IV.3), (IV.9), (IV.10)\] together with the closure relation \[(IV.11)\]. It is worth noticing that the previous procedure of the extension of the basic fields can be continued analogously. That represents a general method to obtain extended thermodynamic systems together with suitable closure relations.
V. EXTENDED THERMODYNAMICS OF SECOND SOUND

Second sound, i.e. thermal wave propagation, is a typical low temperature phenomenon which can be observed, for instance, in dielectric crystals such as Sodium Fluoride (NaF) and Bismuth (Bi) [27, 28]. Its appropriate description requires an extension of the classical Fourier’s theory leading to the paradox of an infinite speed of propagation of thermal disturbances [17]. From the microscopic point of view heat transport at low temperature is modelled through the phonon gas hydrodynamics [29, 30]. Phonons are quasi-particles which obey the Bose-Einstein statistics. In a solid crystal at low temperature they form a rarefied gas whose kinetic equation can be derived similarly to that of an ordinary gas. Phonons may interact among themselves and with lattice imperfections through two different types of processes:

i) Normal-(N) processes, that conserve the phonon momentum;

ii) Resistive-(R) processes, in which the phonon momentum is not conserved.

The frequencies $\nu_N$ and $\nu_R$ of normal and resistive processes determine the characteristic relaxation times

$$\tau_N = \frac{1}{\nu_N}$$

and

$$\tau_R = \frac{1}{\nu_R}.$$ Diffusive processes take over when there are many more R-processes than N-processes. If instead there are only few R-processes and many more N-processes, then a wave like energy transport may occur. This is said second sound propagation. Finally, there is a third mechanism of energy transport due to the presence of ballistic phonons travelling through the crystal without any interaction. The three different mechanisms of energy transport described above can be represented at a different level of approximation in the framework of Extended Thermodynamics. To achieve this goal let us suppose the dynamic variables are the first moment $p_i$, ($i = 1, 2, 3$) and the momentum flux $N_{[ij]}$, ($i, j = 1, 2, 3$). By phonon gas hydrodynamics [30] follows that the functions $p_i$ are connected to the heat flux by the relation $q_i = c^2 p_i$ where $c$ means the Debye’s phonon velocity. Moreover, from kinetic theory we apply the interrelation of the traces of consecutive currents, which in our case is expressed by the definition $N_{[ii]} = e$. Therefore it is convenient to decompose the momentum flux $N_{[ij]}$ into an isotropic part and a deviatoric part according to the equation

$$N_{[ij]} = \frac{1}{3} e \delta_{ij} + N_{<ij>},$$

where $N_{<ij>}$ is symmetric and traceless. Due to the decomposition above the energy density coincides with the trace of $N_{[ij]}$. As a consequence, we have only nine independent thermodynamic variables (9-field model) whose evolution is controlled by the balance laws

$$\dot{p}_i + N_{[ik]k} = P_i,$$  

$$N_{[ij]} + J_{[ij]k} k = P_{[ij]}.$$  

Here $P_i$ and $P_{[ij]}$ are suitable production terms while $J_{[ij]k}$ represents the flux of $N_{[ij]}$, to be assigned through a closure relation. Under suitable constitutive assumptions on $J_{[ij]k}$ and $P_{[ij]}$, the balance of energy

$$\dot{e} + c^2 p_i v_i = 0$$

can be obtained by taking the trace of (V.3). Later on we illustrate such a property in more details by reinspecting two typical examples often used in the applications.

4-field model

This model holds true under the approximation $\tau_N = 0$ and describes second sound effects without taking care for N-processes. It is controlled by the equations

$$\dot{e} + c^2 p_i v_i = 0,$$

$$\dot{p}_i + \frac{1}{3} e \delta_{ii} = - \frac{1}{\tau_R} p_i.$$  

If one assumes $\Phi_{ij} = c^2 \frac{1}{3} e \delta_{ij}$ then the system above is recovered by (IV.2) and (IV.3). Moreover, if in (IV.2) we assume the isotropy of the constitutive equations

$$M_{ikl} = 0, \quad L_{ij} = \frac{m}{\tau_R} \delta_{ij},$$

the constitutive equations for the 4-field model are recovered.
then we get

\[
\dot{p}_i + \left(\frac{1}{3} - \frac{1}{mc^2} \frac{\partial^2 s_0}{\partial e^2}\right) e_i = -\frac{1}{\tau_R} p_i, \tag{V.8}
\]

On the other hand, due to the high speed of phonons \(42 \times 10^4 \text{cm/sec}^{-1}\) in NaF \(31\), the constant \(mc^2\) is very big. Hence, the coefficient \(\frac{1}{mc^2} \frac{\partial^2 s_0}{\partial e^2}\) in (V.8) can be neglected whenever the absolute temperature \(T = \left(\frac{\partial s}{\partial e}\right)^{-1}\) has no jumps, i.e. whenever the Lax conditions for shock wave formation \(31, 32\) are not fulfilled. In such a case (V.8) reduces exactly to (V.6). Further assuming isotropy and some more we get

\[
P_{[ij]k} = 0, \quad Q_{[ij][kl]} = 0, \quad \frac{m}{n} = c^2, \tag{V.9}
\]

and taking into account the decomposition \(p_{[ij]} = \frac{1}{3} p_{k} \delta_{ij} + p_{<i'j>}\), with \(p_{<i'j>}\) symmetric and traceless, equation (IV.3) can be rewritten as follows

\[
\frac{1}{3}\dot{\epsilon}_{ij} + \frac{1}{3} p_{k} \delta_{ij} + c^2 p_{<i'j>} = 0. \tag{V.10}
\]

Finally, taking the trace of (V.10) we get the balance of energy (V.5).

**9-field model**

Such an approximation holds for small \(\tau_N\) and arbitrary \(\tau_R\). It is able to describe second sound taking into account the effects due to the propagation of ballistic phonons. Again, the calculated values of wave speeds are qualitatively but not quantitatively in agreement with the measured ones. The resulting system of equations is

\[
\dot{e} + c^2 \dot{p}_{i'i} = 0, \tag{V.11}
\]

\[
\dot{p}_i + \frac{1}{3} e_i + N_{<ij>} = -\frac{1}{\tau_R} p_i, \tag{V.12}
\]

\[
\dot{N}_{<ij>} + c^2 \frac{2}{3} p_{<i'j>} = -\frac{1}{\tau} N_{<ij>}, \tag{V.13}
\]

where

\[
\frac{1}{\tau} = \frac{1}{\tau_R} + \frac{1}{\tau_N} \tag{V.14}
\]

is the total collision frequency. Let us observe that equation (V.13) is not in the balance form since there is no flux of \(N_{<ij>}\) inside. Hence, nothing prevents to regard it as a closure relation and use (IV.2) and (IV.3) to derive the governing system of equations. Then, let us make the identification

\[
\Phi_{[ij]} = \frac{c^2}{3} \epsilon_{ij} + c^2 N_{<ij>}, \tag{V.15}
\]

and let us assume isotropic constitutive equations

\[
M_{[kl]} = 0, \quad L_{ij} = \frac{m}{\tau_R} \delta_{ij}, \quad P_{[ij]k} = 0, \quad Q_{[ij][kl]} = -\frac{1}{\tau} \delta_{ij} \delta_{kl} + \frac{1}{\tau} \delta_{ik} \delta_{jl} \frac{m}{n} = c^2. \tag{V.16}
\]

Here \(Q_{[ij][kl]}\) is a constant isotropic tensor. There are only two material quantities because of the symmetries of \(Q_{[ij][kl]}\). Here we require that \(\frac{1}{\tau} = \frac{1}{\tau'}\). Then by (IV.2) and (IV.3) we get

\[
\dot{p}_i + \left(\frac{1}{3} - \frac{1}{mc^2} \frac{\partial^2 s_0}{\partial e^2}\right) e_i + N_{<ij>} = -\frac{1}{\tau_R} p_i, \tag{V.17}
\]
\[
\frac{1}{3} \delta_{ij} + \dot{N}_{\langle ij \rangle} + \frac{1}{3} p_{k'k} \delta_{ij} + c^2 p_{\langle i'j \rangle} = -\frac{1}{\tau} N_{\langle ij \rangle}. \quad \text{(V.18)}
\]

The trace of (V.18) yields the balance of the energy (V.11). As a consequence, due to (V.11) the last equation reduces to
\[
\dot{N}_{\langle ij \rangle} + c^2 p_{\langle i'j \rangle} = -\frac{1}{\tau} N_{\langle ij \rangle}. \quad \text{(V.19)}
\]

Also in this case in the absence of shocks (V.17) reduces to (V.12) while (V.19) differs from (V.13) only for the coefficient in front of \( p_{\langle i'j \rangle} \). Such a discrepancy is of no concern at that level since the 9-field model gives only qualitative agreement with the experiments [16].

VI. CONCLUSIONS

We reinspected the problem of thermal wave propagation and proved that the balance structure can be recovered with the Verhás form of the entropy current (II.7) if the thermodynamic inductivity coefficients \( m_{ij} \) and \( n_{ijkl} \) are constant in the general entropy function (II.4). The results above prove that the balance form of evolution equations is compatible with the mathematical structure of classical IT. The balance structure was obtained only for a subset of the independent thermodynamic variables. In fact, the highest order fluxes are controlled by ordinary differential equations, here regarded as closure relations.

For our further investigations we have assumed some interrelations of the consecutive currents (V.1), to be compatible to the moment interpretation of the kinetic theory. With assuming isotropic material we have arrived to the expected system of equations of the 4 and 9 moment theory of rigid heat conductors, thereby demonstrating the compatibility of the two approaches. Let us observe the minor, but important additional assumptions (e.g. \( \tau = \tau' \)).

Nowadays in ET the balance form of the evolution equations is a well motivated assumption from the kinetic theory and the entropy current is a derived quantity with a restrictive potential structure (I.1). Moreover, the derived restrictions can be exploited predictively in different particular cases (e.g. for monatomic gases [1], p.52-60. We do not know related calculations for rigid heat conductors). In ET one of the important problems is to give a reasonable closure of the arising hierarchy.

However, the applications of the phenomenological theory in solids and fluids (see e.g. [4]) and recent generalizations of the principles to incorporate nonlocality are far beyond the validity of the kinetic theory. That was our most important motivation of the recent investigations. In this paper we have shown that with suitable constitutive assumptions on the form of the entropy current and on the entropy one can recover the balance structure. We have demonstrated that one can give a kind of natural closure of the hierarchy.

On the other hand, the conditions of using all the currents as state variables and to restrict ourselves to constant inductivities looks like too restrictive. However, to assume balance form evolution equations, where the current is a constitutive function is a weakly nonlocal theory from our point of view. E.g.
\[
\dot{q}_i = -\Phi_{ik;k}(e, q_i) = f(e, q_i, e_{i'}, q_{i'j}).
\]

Calculation of the derivatives in the divergence of \( \Phi \) gives that the form of the evolution equation, defined by the function \( f \), depends on the space derivatives of the state variables, too. Therefore, for a systematic investigation of the interrelation of the evolution equations and the entropy current without imposing the balance structure one should enlarge the basic thermodynamic space. Hence, one should weaken an other basic assumption in ET, the locality, by including the gradients of the state variables in the constitutive state space [10, 33]. A study of the effects of nonlocality on the evolution of higher order fluxes is developed in [34].

Simplifying the arguments that we wanted to collide here we can say that although from the point of view of kinetic theory the specific form of the entropy current is a restriction, from a phenomenological point of view the balance structure looks like too restrictive.

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