Crystal Interpretation of 
Kerov-Kirillov-Reshetikhin Bijection 
Atsuo Kuniba, Masato Okado, Reiho Sakamoto, 
Taichiro Takagi and Yasuhiko Yamada 

Abstract: The Kerov-Kirillov-Reshetikhin (KKR) bijection is the crux in proving fermionic formulas. It is defined by a combinatorial algorithm on rigged configurations and highest paths. We reformulate the KKR bijection as a vertex operator by purely using combinatorial R in crystal base theory. The result is viewed as a nested Bethe ansatz at q = 0 as well as the direct and the inverse scattering (Gel’fand-Levitan) map in the associated soliton cellular automaton.

1. Introduction

Among many approaches to quantum integrable systems, Bethe ansatz [1] stands as a most efficient tool. In the context of solvable lattice models [2], it produces eigenvectors of transfer matrices from solutions of Bethe equations. Beside exact evaluation of physical quantities, Bethe ansatz has brought a number of applications also in representation theory and combinatorics. A prominent example is the fermionic formula [3, 4], which grew out of the completeness problem, a certain counting of Bethe vectors under string hypotheses. With the advent of the corner transfer matrix [2] and the crystal base theory [5], the fermionic formulas have been reformulated as the so called $X = M$ conjecture for any affine Lie algebra [6, 7]. See [8] for the current status of the conjecture.

The proof of the fermionic formula for $A_n^{(1)}$ [3, 4] may be viewed as a combinatorial version of Bethe ansatz. As a substitute of solutions to Bethe equations, the combinatorial object called rigged configuration (RC) is introduced. It is an $n$-tuple of Young diagrams (configuration) each row of which is assigned with an integer (rigging) obeying a selection rule. The Bethe vectors are replaced by Littlewood-Richardson tableaux, or equivalently, highest paths. The latter are $A_n$ highest weight elements in $B_{\mu_1} \otimes \cdots \otimes B_{\mu_m}$, where $B_l$ is the $A_n^{(1)}$ crystal of the $l$-fold symmetric tensor representation corresponding to a “local spin”. Under these setting, Bethe ansatz should produce highest paths from rigged configurations and vice versa. What achieves this and thereby proves the fermionic formula is the celebrated Kerov-Kirillov-Reshetikhin (KKR) bijection. It is defined by a purely combinatorial algorithm on rigged configurations and highest paths, whose meaning however has remained rather mysterious.

The purpose of this paper is to clarify the representation theoretical origin of the KKR bijection in the light of crystal base theory and associated soliton cellular automata [9]. Let $p$ be the highest path that corresponds to a rigged configuration under the KKR bijection. Our main Theorem 2.2 asserts that

\[ p = \Phi_1 C_1 \Phi_2 C_2 \cdots \Phi_n C_n (p^{(n)}), \]

Here $p^{(n)}$ is a trivial vacuum path and $\Phi_1 C_1 \Phi_2 C_2 \cdots \Phi_n C_n$ is a vertex operator in the sense explained below. Recall that the $A_n^{(1)}$ crystal $B_l$ consists of length $l$ row semistandard tableaux with letters 1, 2, \ldots, $n + 1$. As a set the affine crystal is given by $\text{Aff}(B_l) = \{ b[d] \mid b \in B_l, d \in \mathbb{Z} \}$ by assigning the mode $d$ to $B_l$. The isomorphism $\text{Aff}(B_l) \otimes \text{Aff}(B_m) \simeq \text{Aff}(B_m) \otimes \text{Aff}(B_l)$ is called the combinatorial $R$ [10, 11]. We will
deal with the nested family of algebras \( A_n^{(1)} \supset A_{n-1}^{(1)} \supset \cdots \supset A_0^{(1)} \) and the associated crystals:

\[
B_t = B_t^{\geq 1} \supset B_t^{\geq 2} \supset \cdots \supset B_t^{\geq n+1},
\]

where the superscript means the restriction on tableau letters. Given a rigged configuration \((1.1)\), the right hand side of \((1.1)\) is determined by using the combinatorial \( R \) for the crystals \((1.2)\) alone. In fact \( p(a) = \Phi_{n+1}C_{a+1} \cdots \Phi_nC_n(p^{(n)}) \) becomes a highest path in \( B_{\mu_1}^{\geq a+1} \otimes \cdots \otimes B_{\mu_n}^{\geq a+1} \) with respect to \( A_{n-a} \). Here \( \mu = \mu^{(a)} \) is the \( a \)-th Young diagram in the rigged configuration. In particular \( p^{(n)} \) is trivially fixed. The map \( C_a \) sends \( p^{(a)} \) to \( \text{Aff}(B_{\mu_1}^{\geq a+1}) \otimes \cdots \otimes \text{Aff}(B_{\mu_n}^{\geq a+1}) \) by assigning modes based on the rigging with certain normal ordering afterwards. Then the map \( \Phi_a \) creates a highest path \( p^{(a-1)} \) in \( B_{\lambda_1}^{\geq a} \otimes \cdots \otimes B_{\lambda_n}^{\geq a} \) (\( \lambda = \mu^{(a-1)} \)) by using the data \( C_a(p^{(a)}) \) and the natural embedding \( B_t^{\geq a+1} \hookrightarrow B_t^{\geq a} \). Thus the composition \( \Phi_1C_1\Phi_2C_2 \cdots \Phi_nC_n \) grows the trivial \( A_0 \) path \( p^{(n)} \). Then an \( A_n \) highest path by gradually taking the rigged configuration into account. Such a usage of the family \( A_n^{(1)} \supset A_{n-1}^{(1)} \supset \cdots \supset A_0^{(1)} \) is the typical strategy in the nested Bethe ansatz. In fact our formula \((1.1)\) is a crystal theoretical formulation, i.e., \( q = 0 \) analogue, of Schultz’s construction of Bethe vectors \cite{12}.

The result \((1.1)\) admits a further interpretation in the realm of soliton theory, which we shall now explain. Consider the crystal \( B_{\mu_1} \otimes \cdots \otimes B_{\mu_m} \), where we formally take \( m \to \infty \) and impose the boundary condition \( p_j = \frac{1}{1 \cdots 1} \in B_{\mu_j} \) for \( j \gg 1 \) on its elements \( p = p_1 \otimes p_2 \otimes \cdots \). Then the system can be endowed with a commuting family of time evolutions and behaves as a soliton cellular automaton. It was invented as the box-ball system \cite{13, 14} and subsequently reformulated by the crystal theory \cite{15, 16, 9}. Here is a typical time evolution pattern when \( \mu_j = 1 \) for all \( j \). (We omit \( \otimes \) and write \( \frac{1}{1} \) simply as 1, etc.)

\[
\begin{array}{cccc}
t = 0 : & 11112222111111111111111111111111111111111111 \cdots \\
t = 1 : & 111111111111111111111111111111111111111111111 \cdots \\
t = 2 : & 111111111111111111111111111111111111111111111 \cdots \\
t = 3 : & 111111111111111111111111111111111111111111111 \cdots \\
t = 4 : & 111111111111111111111111111111111111111111111 \cdots \\
t = 5 : & 111111111111111111111111111111111111111111111 \cdots \\
t = 6 : & 111111111111111111111111111111111111111111111 \cdots \\
t = 7 : & 111111111111111111111111111111111111111111111 \cdots 
\end{array}
\]

The three solitons with amplitudes 4, 3 and 1 regain the original amplitudes after the collision while interchanging the internal degrees of freedom. The dynamics is governed by the combinatorial \( R \) and as a result, highest paths remain highest. In the above example, all the paths are highest indeed. (The paths regarded as words are lattice permutations.) Thus it is natural to ask: what kind of time evolutions does the soliton cellular automaton induce on the rigged configurations? Our answer to the question is summarized in the following table.

| Bethe ansatz             | Crystal base theory               | Soliton cellular automaton            |
|--------------------------|-----------------------------------|---------------------------------------|
| rigged configuration     | \( \text{Aff}(B_{\lambda_1}) \otimes \cdots \otimes \text{Aff}(B_{\lambda_n}) \)      | action-angle variable                  |
| KKR bijection            | vertex operator                    | (inverse) scattering map               |
We show that under time evolutions the configuration remains unchanged while the rigging flows linearly. In this sense the rigged configurations are action-angle variables of the associated soliton cellular automaton. The first Young diagram \( \mu^{(1)} \) in the configuration gives the list of amplitudes of solitons. For example, \( t = 4 \) state in the above pattern coincides with \( p \) in Example 2.9 (apart from redundant 1’s) for which \( \mu^{(1)} = (4, 3, 1) \) indeed. The meaning of the other Young diagrams is understood similarly along the nested family of cellular automata explained in Section 2.7.

Finally to interpret the formula (1.1), write it as \( p = \Phi_1 C_1(p^{(1)}) \). In the above example, one has \( p^{(1)} = \begin{array}{ccc} 2 & 2 & 2 \\ 2 & 3 & 3 \end{array} \otimes \begin{array}{c} 4 \end{array} \) and this is nothing but the list of incoming solitons. The map \( C_1 \) assigns \( p^{(1)} \) with the modes in affine crystals that encode the positions of solitons. Thus \( C_1(p^{(1)}) \) is the scattering data of the soliton cellular automaton. Then \( \Phi_1 \) plays the role of a vertex operator to create solitons over the “vacuum path” \( 111 \ldots \) by injecting the scattering data \( C_1(p^{(1)}) \) by using combinatorial \( R \). Again these constructions work inductively along the nested family of crystals for \( A^{(1)}_n \supset A^{(1)}_{n-1} \supset \cdots \supset A^{(1)}_{0} \). Thus the composition \( \Phi_1 C_1 \cdots \Phi_n C_n \) is the inverse scattering map that reproduces solitons from the scattering data.

The layout of the paper is as follows. In Section 2, we treat the \( A^{(1)}_n \) case. A proof of Theorem 2.2 will be presented in [17]. Section 2.7 includes the solution of the direct scattering problem, namely, a method to determine the rigged configuration from a given highest path by only using combinatorial \( R \) and the KKR bijection for \( \widehat{sl}_2 \). It is a crystal theoretical separation of variables. In Section 3, conjectures similar to (1.1) are presented for the bijection in the other non-exceptional affine algebras [18, 19]. They all possess a similar feature to the \( A^{(1)}_n \) case, but the items as \( \Phi_n, C_n \) need individual descriptions. We do this from Sections 4 to 8. Appendix A is a brief exposition of the crystal base theory. Further applications of the present results to fermionic formulas and soliton cellular automata will be given elsewhere.

2. \( A^{(1)}_n \) case

2.1. Rigged configurations. Consider the data of the form

\[
(\mu^{(0)}, (\mu^{(1)}, J^{(1)}), \ldots, (\mu^{(n)}, J^{(n)})),
\]

where \( \mu^{(a)} = (\mu_1^{(a)}, \ldots, \mu_{I_a}^{(a)}) \) is a partition and \( J^{(a)} = (J_1^{(a)}, \ldots, J_{l_a}^{(a)}) \in (\mathbb{Z}_{\geq 0})^{l_a} \). Set

\[
E_j^{(a)} = \sum_{i=1}^{l_a} \min(j, \mu_i^{(a)}).
\]

Define the vacancy numbers by

\[
p_j^{(a)} = E_j^{(a-1)} - 2E_j^{(a)} + E_j^{(a+1)} \quad (1 \leq a \leq n),
\]

where \( E_j^{(n+1)} = 0 \). The data (2.1) is called a rigged configuration if the following conditions are satisfied for all \( 1 \leq a \leq n \) and \( j \in \mathbb{Z}_{\geq 0} \):

\[
0 \leq J_1^{(a)} \leq J_{i+1}^{(a)} \leq \cdots \leq J_l^{(a)} \leq p_j^{(a)} \quad \text{if } \{i, i+1, \ldots, l\} = \{k \mid \mu_k^{(a)} = j\}.
\]

The array of partitions \( \mu^{(0)}, \ldots, \mu^{(n)} \) is called a configuration and the nonnegative integers \( J_1^{(a)}, \ldots, J_{l_a}^{(a)} \) are called rigging. If \( \mu^{(0)} \) is represented as a Young diagram, the vacancy number \( p_j^{(a)} \) is assigned to each “cliff” of width \( j \). \( J^{(a)} \) may be viewed as assigning
to the cliff a partition whose parts are at most \( p_j^{(a)} \). Here is an example of the \( A_3^{(1)} \) rigged configuration.

\[
\begin{array}{cccc}
\mu^{(0)} & \mu^{(1)} & \mu^{(2)} & \mu^{(3)} \\
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

(2.5)

The vacancy number is written on the left of the Young diagrams and the rigging is attached to each row. For a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \), we let \( \text{RC}(\lambda) \) denote the set of rigged configurations (2.1) with \( \mu^{(0)} = \lambda \).

2.2. Crystals. We recapitulate basic facts on the \( A_n^{(1)} \) crystal \( B_l \). For a general background see Appendix A. The \( B_l \) is the crystal base of the \( l \)-fold symmetric tensor representation. As the set it is given by

(2.6) \( B_l = \{ x = (x_1, \ldots, x_{n+1}) \in (\mathbb{Z}_{\geq 0})^{n+1} \mid x_1 + \cdots + x_{n+1} = l \} \).

The Kashiwara operators act as \( \tilde{e}_i(x) = x' \), \( \tilde{f}_i(x) = x'' \) with \( x'_i = x_j + \delta_{i,j} - \delta_{i,j+1} \) and \( x''_i = x_j - \delta_{i,j} + \delta_{i,j+1} \). Here indices are in \( \mathbb{Z}_{n+1} \) and \( x' \) and \( x'' \) are to be understood as 0 unless they belong to \( (\mathbb{Z}_{\geq 0})^{n+1} \). The combinatorial \( R : \text{Aff}(B_l) \otimes \text{Aff}(B_m) \rightarrow \text{Aff}(B_{m+n}) \) has the form \( R : x[d] \otimes y[e] \mapsto y[e - H(x \otimes y)] \otimes x[d + H(x \otimes y)] \) with

(2.7) \( \tilde{x}_i = x_i + Q_i(x, y) - Q_{i-1}(x, y) \), \( \tilde{y}_i = y_i + Q_{i-1}(x, y) - Q_i(x, y) \),

(2.8) \( Q_i(x, y) = \min \left\{ \sum_{j=1}^{k-1} x_{i+j} + \sum_{j=k+1}^{n+1} y_{i+j} \mid 1 \leq k \leq n+1 \right\} \),

(2.9) \( H(x \otimes y) = \min(l, m) - Q_0(x, y) \).

The energy function \( H \) here is normalized so that \( 0 \leq H \leq \min(l, m) \) and coincides with the “winding number” [11]. The element \( x = (x_1, \ldots, x_{n+1}) \) is also denoted by a row shape semistandard tableau of length \( l \) containing the letter \( i \) \( x_i \) times and \( x[d] \in \text{Aff}(B_l) \) by the tableau with index \( d \). For example in \( A_3^{(1)} \), the following stand for the same relation under \( R \):

\[
(1, 2, 0, 1)[5] \otimes (1, 0, 1, 0)[9] \simeq (0, 1, 0, 1)[8] \otimes (2, 1, 1, 0)[6],
\]

\[
\begin{array}{c}
1224 \otimes 13 \otimes 9 \simeq 24 \otimes 1123 \otimes 0.
\end{array}
\]

To save the space we use the notation:

(2.10) \( a^l = [... ... a] \in B_l \).

The relation \( x \otimes y \simeq \tilde{y} \otimes \tilde{x} \) is depicted as

(2.11) \( \begin{array}{c}
y \\
\tilde{y}
\end{array} \begin{array}{c}
x \begin{array}{c}
\tilde{x}
\end{array}
\end{array} \quad \text{or} \quad \begin{array}{c}
\tilde{y}
\end{array} \begin{array}{c}
y \begin{array}{c}
\tilde{x}
\end{array}
\end{array} \)

Setting

(2.12) \( B_l^{\geq a+1} = \{ (x_1, \ldots, x_{n+1}) \in B_l \mid x_1 = \cdots = x_a = 0 \} \quad (0 \leq a \leq n) \),
we have
\begin{equation}
B_l = B_{l-1}^{\geq 1} \supset B_{l-1}^{\geq 2} \supset \cdots \supset B_{l-1}^{\geq n+1} = \{(n+1)^l\}
\end{equation}
as sets. We will need to consider the crystals not only for \(A_n^{(1)}\) but also for the nested family \(A_0^{(1)}, A_1^{(1)}, \ldots, A_{n-1}^{(1)}\). In such a circumstance we realize the crystal \(B_l\) for \(A^{(1)}_{n-a}\) \((0 \leq a \leq n)\) on the set \(B_{l-1}^{\geq a+1}\) with the Kashiwara operators \(\tilde{e}_i, \tilde{f}_i\) \((a \leq i \leq n)\). In this convention the highest element with respect to \(A_{n-a}\) is \((a+1)^l\) in \(B_{l-1}^{\geq a+1}\).

Let
\begin{equation}
P_+ (\lambda) = \{ p \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_k} \mid \tilde{e}_i p = 0, \ 1 \leq i \leq n \}.
\end{equation}
be the set of highest elements (paths) with respect to \(A_n\). The bijection [3, 4] between \(\text{RC}(\lambda)\) and the Littlewood-Richardson tableaux is translated to the one between \(\text{RC}(\lambda)\) and \(P_+ (\lambda)\). We call the resulting map the KKR bijection. See [8] for a recent review. It sends the rigged configuration (2.5) to
\begin{equation}
\begin{array}{c}
111 \otimes 22 \otimes 3 \otimes 1 \otimes 4 \otimes 2 \otimes 3
\end{array}
\end{equation}

2.3. **Normal ordering.** For an element \(b_1 [d_1] \otimes \cdots \otimes b_m [d_m] \in \text{Aff}(B_{l_1}) \otimes \cdots \otimes \text{Aff}(B_{l_m})\) we call the number \(d_i\) the \(i\)-th mode. By using the combinatorial \(R\), tensor products can be reordered and the modes are changed accordingly. Given an element \(s \in \text{Aff}(B_{l_1}) \otimes \cdots \otimes \text{Aff}(B_{l_m})\), define \(S_m\) to be the set of such reordering as
\begin{equation}
S_m = \{ s' \in \biguplus_{\sigma \in \mathcal{S}_m} \text{Aff}(B_{\sigma(1)}) \otimes \cdots \otimes \text{Aff}(B_{\sigma(m)}) \mid s' \simeq s \},
\end{equation}
where \(\biguplus'\) means the disjoint union over \(\sigma \in \mathcal{S}_m\) satisfying \(\sigma(i) < \sigma(j)\) for any \(i, j\) such that \(i < j\) and \(l_i = l_j\). The cardinality of \(S_m\) is \(m!\) if \(l_1, \ldots, l_m\) are distinct. For \(i = 2, \ldots, m\), let \(S_{i-1}\) be the subset of \(S_i\) having the maximal \(i\)-th mode. Then we have
\begin{equation}
\emptyset \neq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m.
\end{equation}
We call the elements of \(S_1\) **normal ordered forms** of \(s\). In general the normal ordered form \(b_1 [d_1] \otimes \cdots \otimes b_m [d_m]\) is not unique but the mode sequence \(d_1, \ldots, d_m\) is unique by the definition. For type \(A_n^{(1)}\), one has \(d_1 \leq \cdots \leq d_m\). An element of \(S_1\) is denoted by \(s^1\).

In the context of soliton cellular automaton explained in Section 1, the restriction on the union \(\biguplus'\) in (2.16) reflects the fact that solitons of equal velocity (=amplitude) do not collide with each other.

**Example 2.1.** Take \(s = 1223 \otimes 34 \otimes 1\) for \(A_3^{(1)}\).

\(S_3 = \{ 1223 \otimes 34 \otimes 1, 23 \otimes 1223 \otimes 1, 23 \otimes 3 \otimes 1223, 23 \otimes 4 \otimes 1123, 3 \otimes 24 \otimes 1123, 3 \otimes 1224 \otimes 3, 1223 \otimes 4 \otimes 13, 1223 \otimes 4 \otimes 13 \}\),
\(S_2 = S_1 = \{ 3 \otimes 1224 \otimes 13, 1223 \otimes 4 \otimes 13 \} \).

2.4. **Maps \(C_1, \ldots, C_n.** Pick the color \(a\) part \((\mu^{(a)}, J^{(a)})\) of the rigged configuration. Here we simply write it as \((\mu, J)\). Namely \(\mu = (\mu_1, \ldots, \mu_m)\) is a partition and \(J = (J_i)\), where \(J_i\) is the rigging attached to the \(i\)-th row in \(\mu\) of length \(\mu_i\). For \(1 \leq a \leq n\),
let $B_i = B_{2a+1}^{2a+1}$ be the $A_{n-a}^{(1)}$ crystal in the sense explained around (2.13). Define the map $C_a$ among the $A_{n-a}^{(1)}$ crystals by

$$C_a : B_{\mu_1} \otimes \cdots \otimes B_{\mu_m} \rightarrow \text{Aff}(B_{\mu_1}) \otimes \cdots \otimes \text{Aff}(B_{\mu_m}) : (1 \leq a \leq n)$$
(2.18)

$$b_1 \otimes \cdots \otimes b_m \mapsto :b_1[d_1] \otimes \cdots \otimes b_m[d_m]:$$
(2.19)

$$d_i = J_i + \sum_{0 \leq k < i} H(b_k \otimes b_i^{(k+1)}), \quad b_0 = (a+1)^{\mu_1}.$$ Here $b_i^{(j)} \in B_{\mu_j} (j \leq i)$ is defined by bringing $b_i$ to the left by the combinatorial $R$ as

$$b_j \otimes \cdots \otimes b_{i-1} \otimes b_i \simeq b_i^{(j)} \otimes (\cdots)$$
under the isomorphism $(B_{\mu_j} \otimes \cdots \otimes B_{\mu_1}) \otimes B_{\mu_i} \simeq B_{\mu_i} \otimes (B_{\mu_j} \otimes \cdots \otimes B_{\mu_1})$.

The map $C_n$ involves "$A_0^{(1)}$ crystal" $B_i^{2n+1} = \{(n+1)!\}$. The following suffices to define $C_n$:

$$(n+1)! \otimes (n+1)^m \simeq (n+1)^m \otimes (n+1)!$$
$$H((n+1)! \otimes (n+1)^m) = \min(l, m).$$

Since the normal ordering in (2.18) is not unique, $C_a$ is actually multi-valued in general. Here we mean by $C_a(\cdot)$ to pick any one of the normal ordered forms.

### 2.5. Maps $\Phi_1, \ldots, \Phi_n$.

Pick the color $a$ and $a-1$ parts of the configuration and denote them simply by $\mu^{(a)} = (\mu_1, \ldots, \mu_m)$ and $\mu^{(a-1)} = (\lambda_1, \ldots, \lambda_k)$. Set $B_i = B_{2a+1}^{2a+1}$ and $B_i' = B_{2a}^{2a}$. We define the map $\Phi_a$ from the normal ordered elements in $A_{n-a}^{(1)}$ affine crystals to $A_{n-a+1}^{(1)}$ crystals:

$$\Phi_a : \text{Aff}(B_{\mu_1}) \otimes \cdots \otimes \text{Aff}(B_{\mu_m}) : \rightarrow B'_{\lambda_1} \otimes \cdots \otimes B'_{\lambda_k} \quad (1 \leq a \leq n)$$
(2.22)

$$b_1[d_1] \otimes \cdots \otimes b_m[d_m] \mapsto c_1 \otimes \cdots \otimes c_k.$$ Since $b_1[d_1] \otimes \cdots \otimes b_m[d_m]$ is normal ordered, we know that $d_1 \leq \cdots \leq d_m$. We suppose further that $d_1 \geq 0$, which is the case in the actual use later. Then the image $c_1 \otimes \cdots \otimes c_k$ is determined by the following relation under the isomorphism of $A_{n-a+1}^{(1)}$ crystals: (We write $T_a^d = \left[ \begin{array}{c} a \\ d \end{array} \right] \in (B_{2a}^{2a}) \otimes d$ for short.)

$$\left( T_a^{d_1} \otimes b_1 \otimes T_a^{d_2-d_1} \otimes b_2 \otimes \cdots \otimes T_a^{d_m-d_{m-1}} \otimes b_m \right) \otimes (a^{\lambda_1} \otimes a^{\lambda_2} \otimes \cdots \otimes a^{\lambda_k})$$
$$\simeq (c_1 \otimes \cdots \otimes c_k) \otimes \text{tail},$$
(2.23)

Here we are regarding $b_i \in B_{\mu_i} = B_{2a+1}^{2a+1}$ as an element of $B_{\mu_i}' = B_{2a}^{2a}$ by the natural embedding (2.13) as sets. The tail part has the same structure as $(T_a^{d_1} \otimes b_1 \otimes T_a^{d_2-d_1} \otimes \cdots \otimes b_m)$ on the left hand side. In the actual use, it turns out to be $(T_a^{d_1} \otimes a^{\mu_1} \otimes T_a^{d_2-d_1} \otimes \cdots \otimes a^{\mu_m})$ containing the letter $a$ only. (This fact will not be used.)

To obtain $c_1 \otimes \cdots \otimes c_k$ using (2.23), one applies the combinatorial $R$ on $B_i' \otimes B_i'$ to carry $(T_a^{d_1} \otimes b_1 \otimes \cdots \otimes T_a^{d_m-d_{m-1}} \otimes b_m)$ through $(a^{\lambda_1} \otimes a^{\lambda_2} \otimes \cdots \otimes a^{\lambda_k})$ to the right. The procedure is depicted as
2.6. Main theorem. Define the $A_{n-1}^{(1)}$ crystal element

\begin{equation}
 p^{(n)} = (n+1)^{\mu_1^{(n)}(a)} \otimes \cdots \otimes (n+1)^{\mu_{m}^{(n)}(a)}. \tag{2.24}
\end{equation}

**Theorem 2.2.** The image $p$ of the rigged configuration $(\mu^{(0)}, (\mu^{(1)}, J^{(1)}), \ldots, (\mu^{(n)}, J^{(n)}))$ under the KKR bijection is given by

\begin{equation}
 p = \Phi_1 C_1 \Phi_2 C_2 \cdots \Phi_n C_n(p^{(n)}). \tag{2.25}
\end{equation}

The theorem asserts that $\Phi_1 C_1 \Phi_2 C_2 \cdots \Phi_n C_n(p^{(n)})$ is independent of the choices of the possibly non-unique normal ordered forms when applying the maps $C_1, \ldots, C_n$. A proof is presented in [17].

Set

\begin{equation}
 p^{(a)} = \Phi_{a+1} C_{a+1} \cdots \Phi_n C_n(p^{(n)}) \quad (0 \leq a \leq n-1), \tag{2.26}
\end{equation}

which belongs to the $A_{n-a}^{(1)}$ crystal $B_{\mu_{a}^{(n)}}^{(n)\geq a+1} \otimes \cdots \otimes B_{\mu_{m}^{(n)}}^{(n)\geq a+1}$. Thus $p$ in (2.25) is $p^{(0)}$.

**Corollary 2.3.** For $0 \leq a \leq n-1$, $p^{(a)}$ coincides with the image of the truncated rigged configuration $(\mu^{(a)}, (\mu^{(a+1)}, J^{(a+1)}), \ldots, (\mu^{(n)}, J^{(n)}))$ under the KKR bijection.

**Remark 2.4.** In a rigged configuration (2.1), one may treat $\mu^{(a)}$ as a composition rather than a partition. In that case, the condition (2.4) should read $0 \leq J_{i_1}^{(a)} \leq \cdots \leq J_{i_m}^{(a)} \leq p_j^{(a)}$ if \{i_1 < \cdots < i_m\} = \{k \mid \mu_k^{(a)} = j\}. The other necessary change is only to redefine $b_0$ in (2.19) as $b_0 = (a+1)^{\max\{\mu_1, \ldots, \mu_m\}}$. Actually, $b_0 = (a+1)^M$ with any $M \geq \max\{\mu_1, \ldots, \mu_m\}$ leads to the same $d_i$ in (2.19). The same remark applies also for the other algebras treated in this paper.

**Example 2.5.** Consider the $A_3^{(1)}$ rigged configuration (2.5). According to (2.24) we set

\begin{equation*}
 p^{(3)} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \quad C_3(p^{(3)}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\end{equation*}
We use (2.23) to find \( p^{(2)} = \Phi_3 C_3(p^{(3)}) \). It amounts to calculating \( T_1^3 \otimes 4 \otimes (33 \otimes 3) \). This is done as

\[
\begin{array}{c|c|c}
33 & 3 & 3 \\
34 & 3 & 3 \\
33 & 4 & 3 \\
\end{array}
\]

Thus we have

\[
p^{(2)} = 33 \otimes 4, \quad C_2(p^{(2)}) = 3 \otimes 34 3
\]

where we have used \( H(33 \otimes 4) = 0, H(33 \otimes 33) = 2 \) and \( H((33 \otimes 3) = 1 \). To find \( p^{(1)} = \Phi_1 C_1(p^{(2)}) \), we calculate \( T_1^3 \otimes 3 \otimes T_2^3 \otimes 34 \otimes (22 \otimes 22 \otimes 2 \otimes 2) \).

\[
\begin{array}{c|c|c|c|c|c}
22 & 22 & 22 & 22 & 22 & 22 \\
34 & 34 & 22 & 22 & 2 & 2 \\
34 & 34 & 22 & 22 & 2 & 2 \\
22 & 22 & 34 & 2 & 2 & 2 \\
22 & 22 & 34 & 2 & 2 & 2 \\
\end{array}
\]

Thus we find

\[
p^{(1)} = 22 \otimes 23 \otimes 4 \otimes 3, \quad C_1(p^{(1)}) = : 22_2 \otimes 23_3 \otimes 4_1 \otimes 3_3 :
\]

There are three normal ordered forms \( 2_1 \otimes 3_2 \otimes 22_3 \otimes 34_4, 2_1 \otimes 23_2 \otimes 2_3 \otimes 3_4 \) and \( 2_1 \otimes 23_2 \otimes 24_3 \otimes 3_4 \). Any one of them can be chosen as \( C_1(p^{(1)}) \). We illustrate the derivation of \( p = \Phi_1 C_1(p^{(1)}) \) along the first one. According to (2.23) we
calculate $\mathcal{T}_1^1 \otimes 2 \otimes \mathcal{T}_1^1 \otimes 3 \otimes \mathcal{T}_1^1 \otimes 22 \otimes 34 \otimes (111 \otimes 11 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1)$.

The bottom line yields the path $p = \Phi_1 C_1(p^{(1)})$ in agreement with (2.15).

2.7. Finding rigged configuration from highest path. In this subsection we consider a map from highest paths to the corresponding RCs by only using crystal isomorphisms and the KKR bijection for the $\hat{sl}_2$ case. The original idea for this has already arisen in [20]. To do this, we need not only crystals $B_l$ but also $B_{2,l}$, two-row tableaux. It is known that the set of semistandard tableaux of $k \times l$ rectangular shape $B_{k,l}$ admits the $U'_q(A^{(1)}_n)$-crystal structure and there is a generalized KKR bijection from the highest elements in $B_{k1,l1} \otimes \cdots \otimes B_{km,lm}$ to RCs [21]. For a quick study of these matters we recommend [8]. (Note that the order of the tensor products of crystals is opposite from ours.) In particular, the readers are recommended to see Definition 4.5 to understand the operations $lh, ls, lb, \delta, i, j$ used in the proofs below.

Let $u_{k,l}$ be the highest element of $B_{k,l}$. $u_{k,l}$ is the $k \times l$ tableau whose entries in the $j$-th row are all $j$ for $1 \leq j \leq k$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we define $B_\lambda = B_{1,\lambda_1} \otimes \cdots \otimes B_{1,\lambda_k}$ and $B_{\lambda}^{(2)} = B_{2,\lambda_1} \otimes \cdots \otimes B_{2,\lambda_k}$. In this subsection we do not always write $\mu^{(0)}$ for the RC even when it corresponds to a path in $P_+(\mu^{(0)})$.

**Proposition 2.6.** If an element $p$ of $P_+(\lambda)$ corresponds to the RC $((\mu^{(1)}, J^{(1)}), \ldots, (\mu^{(n)}, J^{(n)}))$ under the bijection, then $u_{k,l} \otimes p$ corresponds to $((\mu^{(1)}, J^{(1)}), \ldots, (\mu^{(k)}, J^{(k)}), \ldots, (\mu^{(n)}, J^{(n)}))$, where $J^{(k)} = J^{(k)} + \min(l, \mu^{(k)}_i)$.

**Proof.** Let $p_j^{(a)}$ (resp. $\bar{p}_j^{(a)}$) be the vacancy number for $p$ (resp. $u_{k,l} \otimes p$). Note that $\bar{p}_j^{(a)} - p_j^{(a)} = \min(l, j)\delta_{a,k}$. Noticing this fact the proof goes by induction on $|\lambda|$ using operations $ls, ll, i, \delta$. 

**Theorem 2.7.** Let $p$ be an element of $P_+(\lambda)$ and let $((\mu^{(1)}, J^{(1)}), \ldots, (\mu^{(n)}, J^{(n)}))$ be the corresponding RC. We assume that $p$ has $(u_{1,1}^{1,1})^{\otimes L}$ at the right end with sufficiently
large $L$. Set $\mu = \mu^{(1)} = (\mu_1, \ldots, \mu_l)$. Consider $(u^{2,\mu_1} \otimes \cdots \otimes u^{2,\mu_l}) \otimes p$ and switch the order by the combinatorial $R$:

$$B^{(2)}_\mu \otimes B_\lambda \quad (u^{2,\mu_1} \otimes \cdots \otimes u^{2,\mu_l}) \otimes p \quad \rightarrow \quad B_\lambda \otimes B^{(2)}_\mu \quad \rightarrow \quad \tilde{p} \otimes (b_1 \otimes \cdots \otimes b_l).$$

Then we have the following.

1. $\tilde{p}$ is an element of $\mathcal{P}_+(\lambda)$ that does not contain letters greater than 2.
2. $\tilde{p}$ corresponds to the RC $(\mu^{(1)}, J^{(1)})$.
3. The letters in the first row of $b_j$ are all 1 for any $j$.
4. $b_1 \otimes \cdots \otimes b_l$ is an element of $\mathcal{P}_+(\mu)$ that corresponds to the RC $((\mu^{(2)}, J^{(2)}), \ldots, (\mu^{(n)}, J^{(n)}))$. Here $b_2$ is the second row of $b_j$.

Proof. Let $RC = ((\mu^{(1)}, J^{(1)}), (\mu^{(2)}, J^{(2)}), \ldots, (\mu^{(n)}, J^{(n)}))$ be the RC corresponding to $p$. From Proposition 2.6 the RC corresponding to $(u^{2,\mu_1} \otimes \cdots \otimes u^{2,\mu_l}) \otimes p$ is given by

$$\overline{RC} = ((\mu^{(1)}, J^{(1)}), (\mu^{(2)}, \tilde{j}^{(2)}), \ldots, (\mu^{(n)}, J^{(n)}))$$

where $\tilde{j}^{(2)} = J^{(2)} + \sum_{k=1}^{l} \min(\mu_k, \mu_i^{(2)})$. Since the RC does not change by the application of the combinatorial $R$ (see [21] Lemma 8.5), the RC corresponding to $\tilde{p} \otimes (b_1 \otimes \cdots \otimes b_l)$ is also given by $\overline{RC}$. Note the following facts.

(i) A string in $\overline{RC}$ is singular, if and only if the corresponding string in RC is singular.
(ii) There is no singular string in $(\mu^{(1)}, J^{(1)})$.

We now apply the procedures $ls$, $lb$, $lh$, $i$, $j$, $\delta$ successively on $RC$ to obtain $b_1 \otimes \cdots \otimes b_l$ and $\tilde{p}$. By applying $ls$, $i$, one can assume $b_l \in B^{2,1}$. By applying $lb$, $j$, $lh$, $\delta$, we then obtain $b_i = \begin{array}{c} \lambda \\ a \end{array}$ $(a \geq 2)$. Notice that this process is parallel to applying $ls, i, lh, \delta$ to the RC

$$\overline{RC} = (\mu, (\mu^{(2)}, J^{(2)}), \ldots, (\mu^{(n)}, J^{(n)})).$$

Namely, if $b_i = \begin{array}{c} \lambda \\ 1 \end{array}$ then the first letter to be obtained from $\overline{RC}$ is $a - 1$. By doing these procedures successively until we obtain $b_1, \ldots, b_l$, we can obtain (3)(4). $(b_1 \otimes \cdots \otimes b_l \in \mathcal{P}_+(\mu)$ follows from the fact that it corresponds to an admissible RC.)

These procedures continue until we finish removing letters corresponding to $B^{(2)}_\mu$. We then see $\tilde{p}$ corresponds to $(\mu^{(1)}, J^{(1)})$ and therefore has no letter greater than 2. $\tilde{p} \in \mathcal{P}_+(\lambda)$ follows from the simple fact on crystals that if $b_1 \otimes b_2$ is a highest element, then so is $b_1$.

Let $p$ be an element of $\mathcal{P}_+(\lambda)$. In view of the above theorem we consider the following procedure for $p$.

1. Supply $p$ with sufficiently many $u^{1,1}$ on the right.
2. Compute the image of the map

$$L \rightarrow B^{(2,1)}_\mu \otimes B_\lambda \quad \rightarrow \quad \tilde{p} \otimes p'.$$

$L$ should be chosen so large that $\tilde{p}$ does not contain letters greater than 2.
3. Compute the RC $(\mu, J)$ corresponding to $\tilde{p}$.
(4) Cut the first rows of $p'$ (for their entries are all 1), decrease each letter by 1 and denote it by $p''$. Compute the image of the map
\[
B \mu \otimes (B^{1,1})^\otimes L \longrightarrow (B^{1,1})^\otimes L \otimes B \mu
\]
\[
(u^{1,\mu_1} \otimes \cdots \otimes u^{1,\mu_l}) \otimes p'' \longrightarrow u \otimes p'''.
\]
(In fact, $u = (u^{1,1})^\otimes L$)

(5) Replace $p$ with $p'''$ and repeat (1)-(4) until $p'''$ has no letter greater than 2.

This procedure is shown to give the RC corresponding to $p$ by Proposition 2.6, Theorem 5.7 of [20], Theorem 2.7 and the following lemma, that can easily be obtained from [22].

**Lemma 2.8.** Let $b \in B^{2,l}, b' \in B^{2,m}$. Suppose the letters of the first rows of $b, b'$ are all 1. Let $\tilde{b}' \otimes \tilde{b}$ be the image of $b \otimes b'$ under the map $B^{2,l} \otimes B^{2,m} \rightarrow B^{2,m} \otimes B^{2,l}$. Then the letters of the first rows of $\tilde{b}'', \tilde{b}$ are again all 1, and their second rows are given by the image of $b_2 \otimes b_2'$ under $B^{1,l} \otimes B^{1,m} \rightarrow B^{1,m} \otimes B^{1,l}$, where $b_2, b_2'$ are the second rows of $b, b'$.

**Example 2.9.** $n = 3$. We abbreviate $\otimes$ to $\cdot$ and draw frames of tableaux only for $B^{2,l}$.

\[
p = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \in B^{13}.
\]

By $(B^{2,l})^\otimes B_{(1l)} \rightarrow B_{(1l)} \otimes (B^{2,l})^\otimes$, $(u^{2,1})^\otimes \otimes (p \otimes (u^{1,1})^\otimes)$ maps to $\tilde{p} \otimes p'$ where

\[
\tilde{p} = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdots 1 \cdot 2 \cdot 2 \cdot 2, \quad p' = \begin{array}{cccc}
1 & 3 & 1 & 4 \\
3 & 1 & 3 & 1
\end{array}
\]

The RC for $\tilde{p}$ is $(\mu, J) = ((4, 1, 3, 1), (0, 1, 4))$, and $p'' = 2 \cdot 2 \cdot 3$. By $B_\mu \otimes (B^{1,1})^\otimes \rightarrow (B^{1,1})^\otimes \otimes B_\mu$, $(u^{1,3} \otimes u^{1,1}) \otimes p''$ maps to $1^\otimes \otimes p'''$ where

\[
p''' = 1111 \cdot 122 \cdot 3.
\]

Redefine $p = p'''$ and repeat the procedure. We obtain

\[
(2.27) \quad \tilde{p} = 1111 \cdot 122 \cdot 1 \cdot 2, \quad p' = \begin{array}{cc}
1 & 3 \\
3 & 1
\end{array}
\]

(2.28) \quad $(\mu, J) = ((2, 1), (0, 0)), \quad p'' = 2, \quad p''' = 11 \cdot 2$.

The RC for this $p'''$ is $(1, 0)$. Thus we find the RC for the original $p$ as

\[
\begin{array}{ccc}
\mu^{(1)} & \mu^{(2)} & \mu^{(3)} \\
0 & 0 & 0 \\
4 & 1 & 0
\end{array}
\]

3. Conjectures on other types

There is a bijection between the set of rigged configurations $RC(\lambda)$ and the highest paths (2.6) for $g_n = B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_{2n-1}$, $A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$. It has been established in [18, 19] by a combinatorial algorithm similar to the $A^{(1)}_n$ case. Here we state conjectures analogous to (2.25).

Precise specifications of the data $RC(\lambda)$, $C_\alpha, \Phi_\alpha, \rho^{(n)}$ and $\tilde{p}^{(n)}$ will be given in subsequent sections. In particular for $g_n = A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$, we utilize the embedding $\iota$ into $C^{(1)}_n$ specified in (8.5) and (8.8). The relevant combinatorial $R$ for these algebras.
has been obtained in [23, 24] by an insertion scheme. A piecewise linear formula is also available for \( D_n^{(1)} \) in [25] and the other \( g_n \) using the embedding described in [26].

Conjecture 3.1. Given a rigged configuration \((\mu, J)\) as in (2.1), let \( p \in \mathcal{P}^+ (\lambda) \) be the classically highest path (2.14) that corresponds to it under the bijection in [18, 19]. Then the following formulas are valid:

\[
\begin{align*}
(3.1) & \quad p = \Phi_1 C_1 \cdots \Phi_{n-2} C_{n-3} (\Phi_{n-1} C_{n-2} (p^{(n)}) + \Phi_n C_n (\bar{p}^{(n)})) \quad \text{for} \quad g_n = D_n^{(1)}, \\
(3.2) & \quad p = \Phi_1 C_1 \cdots \Phi_n C_n (p^{(n)}) \quad \text{for} \quad g_n = A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}.
\end{align*}
\]

For \( g_n = A_{2n-1}^{(2)} \) and \( D_n^{(2)} \), let \( p' \) be the above \( \Phi_1 C_1 \cdots \Phi_n C_n (p^{(n)}) \) for \( C_n^{(1)} \) corresponding to the rigged configuration \( \iota ((\mu, J)) \). Then \( \iota^{-1} (p') \) exists and is equal to \( p \).

For \( g_n = A_{2n-1}^{(2)} \) and \( C_n^{(1)} \), define the \( g_{n-a} \) path \( p^{(a)} = \Phi_{a+1} C_{a+1} \cdots \Phi_n C_n (\bar{p}^{(n)}) \) for \( 0 \leq a \leq n-1 \). Similarly for \( g_n = D_n^{(1)} \), set \( p^{(a)} = \Phi_{a+1} C_{a+1} \cdots \Phi_{n-2} C_{n-2} (\Phi_{n-1} C_{n-1} (p^{(n)}) + \Phi_n C_n (\bar{p}^{(n)})) \) for \( 0 \leq a \leq n-2 \). Under Conjecture 3.1, \( p^{(a)} \) coincides with the image of the truncated rigged configuration \( (\mu^{(a)}, (\mu^{(a+1)}, J^{(a+1)}), \ldots, (\mu^{(n)}, J^{(n)})) \) under the bijection in [18, 19].

4. \( D_n^{(1)} \) CASE

4.1. Rigged configurations. Consider the data of the form (2.1) and define \( E_j^{(a)} \) as in (2.2). The vacancy number is specified by (2.3) for \( 1 \leq a \leq n-3 \) and

\[
\begin{align*}
(4.1) & \quad p_j^{(n-2)} = E_j^{(n-3)} + E_j^{(n-1)} + E_j^{(n)} - 2E_j^{(n-2)}, \\
& \quad p_j^{(a)} = E_j^{(n-2)} - 2E_j^{(a)} \quad (a = n-1, n).
\end{align*}
\]

The data (2.1) is called a \( D_n^{(1)} \) rigged configuration if (2.4) is satisfied. It is depicted as in \( A_n^{(1)} \) case by the Young diagrams with rigging. The following is an example of \( D_4^{(1)} \) rigged configuration.

\[
\begin{array}{cccccc}
\mu^{(0)} & \mu^{(1)} & \mu^{(2)} & \mu^{(3)} & \mu^{(4)} \\
(1^3) & 0 & 0 & 0 & 0 \ & \end{array}
\]

4.2. Crystals.

\[
(4.3) \quad B_l = \{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in (\mathbb{Z}_{\geq 0})^{2n} \mid \sum_{i=1}^n (x_i + \bar{x}_i) = l, x_n \bar{x}_n = 0 \}.
\]

\[
(4.4) \quad B_{l_i}^{\geq a+1} = \{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in B_l \mid x_i = \bar{x}_i = 0 \ (1 \leq i \leq a) \}
\]

for \( 0 \leq a \leq n-2 \). Then

\[
B_l = B_{l_1}^{\geq 1} \supset B_{l_2}^{\geq 2} \supset \cdots \supset B_{l_{n-1}}^{\geq n-1}
\]

as sets. We regard \( B_{l_1}^{\geq a+1} \) as \( D_n^{(1)} \) crystal and represent the elements by length \( l \) semistandard row tableaux over the alphabet \( a+1 < \cdots < n < \bar{n} < \cdots < a+1 \). (Actually the letter \( n \) and \( \bar{n} \) do not coexist in a tableau.) We use the notation (2.10).
Then the highest element in $B_l^{2n+1}$ with respect to $D_{n-a}$ is $(a+1)^l$. The rigged configuration (4.2) corresponds to

$$\begin{array}{ll}
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 2 & 1 & 3 & 2 \\
\end{array}
\end{array} \in P_+((1^9)).$$  

### 4.3. $C_{n-1}(p^{(n)})$ and $C_n(\bar{p}^{(n)})$

Set

$$p^{(n)} = n^{\mu_i^{(n-1)}} \cdots n^{\mu_1^{(n-1)}},$$

$$\bar{p}^{(n)} = \bar{n}^{\mu_i^{(n)}} \cdots \bar{n}^{\mu_1^{(n)}},$$

where $n^i$ means the unique element of $A_0^{(1)}$ crystal $B_l$ realized by the tableau with letter $n$ only. (See (2.10).) $\bar{n}^i$ stands for the same object in another copy of $A_0^{(1)}$ crystal $B_l$. Then $C_{n-1}(p^{(n)})$ is defined from (2.18) and (2.19) by setting $(\mu, J) = (\mu^{(n-1)}, J^{(n-1)})$, $b_l = n^{\mu_i}(i \geq 1)$, $b_0 = n^{\mu_1}$ and regarding $B_l$ there as the $A_0^{(1)}$ crystal with letter $n$ only. Namely, $C_{n-1}(p^{(n)}) := n^{\mu_1}[d_1] \cdots \otimes n^{\mu_n}[d_m]$; where the normal ordering :: is done along the $A_0^{(1)}$ combinatorial $R$ (2.21) with $n+1$ replaced by $n$. Similarly, $C_n(\bar{p}^{(n)})$ is defined by setting $(\mu, J) = (\mu^{(n)}, J^{(n)})$, $b_l = \bar{n}^{\mu_i}(i \geq 1)$ and $b_0 = \bar{n}^{\mu_1}$ in (2.18) and (2.19). As the result $C_{n-1}(p^{(n)})$ and $C_n(\bar{p}^{(n)})$ are elements of the two independent copies of $A_0^{(1)}$ affine crystals realized with tableaux with letters $n$ only or $\bar{n}$ only. Since the energy function in (2.21) is positive, the $i$-th mode $d_i$ in the normal ordered form satisfies $0 \leq d_1 \leq \cdots \leq d_m$.

### 4.4. $\Phi_{n-1}C_{n-1}(p^{(n)})$, $\Phi_nC_n(\bar{p}^{(n)})$ and their superposition.

Let us write as $(\lambda_1, \ldots, \lambda_k) = (\mu_1^{(n-2)}, \ldots, \mu_{k-1}^{(n-2)})$ and $C_{n-1}(p^{(n)}) = n^{\mu_1}[d_1] \cdots \otimes n^{\mu_k}[d_m]$. Let $B'_l = \{(x_{n-1}, x_n) \in (\mathbb{Z}_{\geq 0})^2 | x_{n-1} + x_n = l\}$ be the $A_1^{(1)}$ crystal with tableau letters $n - 1, n$, where $n^l = \bar{n} \cdots \bar{n}$ is identified as the lowest weight element $(0, l)^l$. We define $\Phi_{n-1}C_{n-1}(p^{(n)}) \in B'_l$ to be $c_1 \otimes \cdots \otimes c_k$ determined by (2.23) by setting $b_l = n^{\mu_i}$ and $a = n - 1$.

$\Phi_nC_n(\bar{p}^{(n)})$ is defined similarly by replacing the letter $n$ by $\bar{n}$. To be precise, we reset the meaning of $\mu_i$ and $d_i$ by saying that $C_n(\bar{p}^{(n)}) = \bar{n}^{\mu_1}[d_1] \cdots \otimes \bar{n}^{\mu_k}[d_m]$. Let $B''_l = \{(x_{n-1}, \bar{x}_n) \in (\mathbb{Z}_{\geq 0})^2 | x_{n-1} + \bar{x}_n = l\}$ be the $A_1^{(1)}$ crystal with tableau letters $n - 1, \bar{n}$, where $\bar{n}^l = \bar{n} \cdots \bar{n}$ is identified as the lowest weight element $(0, l)^\bar{n}$. We keep the meaning $\lambda = \mu^{(n-2)}$ as before. Then $\Phi_nC_n(\bar{p}^{(n)}) \in B''_l$ is defined to be $c_1 \otimes \cdots \otimes c_k$ determined by (2.23) by setting $b_l = \bar{n}^{\mu_i}$ and $a = n - 1$.

We introduce the “superposition” $+$ of the two copies of $A_1^{(1)}$ crystals $B'_l$ and $B''_l$:

$$+ : B'_l \times B''_l \rightarrow D^{(1)}_l \quad (x_{n-1}, x_n) \times (\bar{y}_{n-1}, \bar{y}_n) \mapsto (0, \ldots, 0, z_{n-1}, z_n, \bar{z}_n, \bar{z}_{n-1}, 0, \ldots, 0),$$

$$z_{n-1} = \min(x_{n-1}, y_{n-1}), \quad z_n = \max(0, y_n - x_n), \quad \bar{z}_n = \max(0, \bar{y}_n - x_n), \quad \bar{z}_{n-1} = \min(x_n, \bar{y}_n).$$

In terms of the tableaux, this means to “superpose” the letters $n - 1, n$ from $B'_l$ and $n - 1, \bar{n}$ from $B''_l$ as $(n - 1, n - 1) \mapsto n - 1, (n, n - 1) \mapsto n, (n - 1, \bar{n}) \mapsto \bar{n}$ and $(n, \bar{n}) \mapsto \bar{n} - 1$. For example in $n = 4$ case, one has $[334] + [344] = [343]$. Note that the conditions $z_n \bar{z}_n = 0$ and $z_{n-1} + z_n + \bar{z}_n + \bar{z}_{n-1} = l$ are satisfied. Under these
conditions, the map (4.8) is invertible. Suppose we have
\[ \Phi_{n-1}C_{n-1}(p^{(n)}) = b'_1 \otimes \cdots \otimes b'_k \in B'_{\lambda_1} \otimes \cdots \otimes B'_{\lambda_k}, \]
\[ \Phi_{n}C_{n}(\tilde{p}^{(n)}) = b''_1 \otimes \cdots \otimes b''_k \in B''_{\lambda_1} \otimes \cdots \otimes B''_{\lambda_k}, \]
where \( \lambda = \mu^{(n-2)} \) as before. Then we specify their superposition as
\[ (4.9) \Phi_{n-1}C_{n-1}(p^{(n)}) + \Phi_{n}C_{n}(\tilde{p}^{(n)}) = (b'_1 + b''_1) \otimes \cdots \otimes (b'_k + b''_k) \in B_{\lambda_1}^{n-1} \otimes \cdots \otimes B_{\lambda_k}^{n-1}. \]

4.5. Maps \( C_1, \ldots, C_{n-2} \). For \( 1 \leq a \leq n - 2 \), \( C_a \) are given by (2.18) and (2.19) provided that \( B_i \) is regarded as the \( D^{(1)}_{n-a} \) crystal \( B_{\lambda}^{n-a+1} \). The normal ordered form \( b[d_1] \otimes \cdots \otimes b_m[d_m] \) is defined in the same way as the \( A^{(1)}_n \) case described in Section 2.3. A new feature is that the energy function can be negative, hence \( d_1 \leq \cdots \leq d_m \) is no longer valid in general.

4.6. Maps \( \Phi_1, \ldots, \Phi_{n-2} \). For \( 1 \leq a \leq n - 2 \), we define the map \( \Phi_a \) by (2.22) and (2.23) by regarding \( B_i = B_i^{\geq a+1} \) and \( B'_i = B_i^{\geq a} \) as the \( D^{(1)}_{n-a} \) and \( D^{(1)}_{n-a+1} \) crystals, provided \( d_1 \leq \cdots \leq d_m \) holds among the modes of the normal ordered element \( b[d_1] \otimes \cdots \otimes b_m[d_m] \). In contrast to the \( A^{(1)}_n \) case, this is no longer valid in general since the energy function can be negative. \( (d_1 \geq 0 \text{ can be assured}) \) To make sense of \( T_{\lambda}^{d} = \emptyset^{\otimes d} \) for negative \( d \) in (2.23), we make the regularization explained in the sequel.

In (2.23), take the largest \( i \) such that \( d_i > d_{i+1} \). Then its left hand side looks as
\[ (4.10) \quad (T_{\lambda}^{d_1} \otimes \cdots \otimes b_i \otimes T_{\lambda}^{\delta} \otimes b_{i+1} \otimes T_{\lambda}^{d_2-d_{i+1}} \otimes \cdots \otimes b_m) \otimes (a^{\lambda_1} \otimes a^{\lambda_2} \otimes \cdots \otimes a^{\lambda_k}), \]
which is regarded as an element in the tensor product of \( D^{(1)}_{n-a+1} \) crystal \( B_i^{\geq a} \)'s. We have set \( \delta = d_i - d_{i+1} > 0 \). There is no problem in carrying the components \( (b_{i+1} \otimes \cdots \otimes b_m) \) through \( (a^{\lambda_1} \otimes \cdots \otimes a^{\lambda_k}) \) to the right to find \( p_j \)'s in \( (b_{i+1} \otimes \cdots \otimes b_m) \otimes (a^{\lambda_1} \otimes \cdots \otimes a^{\lambda_k}) \simeq (p_1 \otimes \cdots \otimes p_k) \otimes \text{tail} \). Thus we are to define \( T_{\lambda}^{\delta} \otimes (p_1 \otimes \cdots \otimes p_k) \otimes \text{tail} \). Actually we declare that \( b_i \otimes T_{\lambda}^{\delta} \otimes (p_1 \otimes \cdots \otimes p_k) \otimes \text{tail} \) is to be understood as
\[ (4.11) \quad \tilde{b}_i \otimes (\tilde{p}_1 \otimes \cdots \otimes \tilde{p}_k) \otimes \text{tail}, \]
where \( \tilde{p}_1 \otimes \cdots \otimes \tilde{p}_k \) is determined by sending \( T_{\lambda}^{\delta} \) from the right to the left by the \( D^{(1)}_{n-a+1} \) combinatorial \( R \):
\[ (4.12) \quad (p_1 \otimes \cdots \otimes p_k) \otimes \tilde{\emptyset}^{\otimes \delta} \simeq \text{head} \otimes (\tilde{p}_1 \otimes \cdots \otimes \tilde{p}_k). \]

In the actual use, head = \( \emptyset_{a+1}^{\otimes \delta} \) seems valid, but this is not necessary for our definition. To specify \( \tilde{b}_i \) in (4.11), we use the expression \( \tilde{b}_i = (x_1, \ldots, x_a, x_{a+1}, \ldots, \bar{x}_1) \in B_{\bar{\lambda}}^{\geq a+1} \). Then \( \tilde{b}_i \) is given by modifying it as (\( x_{a+1}, \bar{x}_a = 0 \) → (\( x_{a+1} + \delta, \delta \)) leaving the other coordinates unchanged. It is a part of our conjecture that \( x_{a+1} \geq \delta \) is always valid as the result of normal ordering hence \( \tilde{b}_i \) is well defined.

This regularization of \( b_i \otimes T_{\lambda}^{d_1-d_{i+1}} \) with \( d_i > d_{i+1} \) can be done successively also in the part \( (T_{\lambda}^{d_1} \otimes b_1 \otimes \cdots \otimes b_{a-1} \otimes T_{\lambda}^{d_i-d_{i+1}}) \) in (4.10), which leads to the (conjectural) definition of our \( \Phi_a \). The change \( b_i \rightarrow \tilde{b}_i \) is analogous to the pair annihilation and creation in the \( D^{(1)}_n \) automaton described in [26].
4.7. Example. Consider the rigged configuration (4.2). We have \( p^{(4)} = \begin{array}{c} 44 \\ 33 \\ 33 \\ 34 \\ 33 \\ 33 \\ 4 \\ 3 \\ 4 \end{array} \) and \( \bar{p}^{(4)} = \begin{array}{c} 44 \\ 33 \\ 33 \\ 34 \\ 33 \\ 33 \\ 4 \\ 3 \\ 4 \end{array} \). From

\[
\begin{array}{c}
\Phi_3 C_3(\bar{p}^{(4)}) = 44 \\
\Phi_4 C_4(p^{(4)}) = \bar{4} \bar{4}
\end{array}
\]

we have \( \Phi_3 C_3(p^{(4)}) = 33 \otimes 44 \). Similarly, \( \Phi_4 C_4(\bar{p}^{(4)}) = 33 \otimes 44 \). Thus the superposition is determined as \( p^{(2)} = \Phi_3 C_3(p^{(4)}) + \Phi_4 C_4(\bar{p}^{(4)}) = 33 \otimes 33 \). Since

\[
\begin{array}{c}
\Phi_4 33 \otimes 33 = 33 \otimes 33 \\
\Phi_3 33 \otimes 33 = 33 \otimes 33 \\
\end{array}
\]

in \( D^{(1)}_2 \) crystal, we obtain \( C_4(p^{(2)}) = 33 \otimes 33 \). Next to find \( p^{(1)} = \Phi_2 C_2(p^{(2)}) \), we compute \( T_2^2 \otimes 33 \otimes T_2^{-1} \otimes 33 \otimes (222 \otimes 222) \) in \( D^{(1)}_3 \) crystal as follows:

\[
\begin{array}{c}
\otimes \otimes \otimes \otimes \\
\Phi_2 \Phi_1 = \Phi_1 \Phi_2 \\
\end{array}
\]

Here the symbol \( < \) in the horizontal line signifies \( T_2^{-1} \) where the combinatorial \( R \) acts from NE to SW. (Our usual convention is from NW to SE as in (2.11).) The change 33 into 32 due to the pair annihilation and creation is indicated by \( \sim \). We have \( p^{(1)} = 2222 \otimes 322 \). Under the \( D^{(1)}_4 \) combinatorial \( R \), one has

\[
\begin{array}{c}
2222 \otimes 322 = 2222 \otimes 322 \\
\end{array}
\]

Therefore the both sides are normal ordered forms and we choose the left hand side as \( C_1(p^{(1)}) \). Finally to obtain \( p = \Phi_1 C_1(p^{(1)}) \), we compute \( T_1^4 \otimes 2222 \otimes T_1^{-2} \otimes 322 \otimes (1) \) in \( D^{(1)}_4 \) crystal. It is calculated by applying \( T_1^4 \otimes \) to the output of
the following diagram (\(1^{10}\) has been replaced with \(\otimes 6\) below):

\[
\begin{array}{cccccccc}
& & & & & & & \\
322 & 132 & 111 & 111 & 111 & 111 & 111 \\
& 2 & 2 & 3 & 1 & 1 & 1 & 1 \\
& 2 & 3 & 1 & 1 & 1 & 1 & 1 \\
& 2 & 3 & 1 & 1 & 1 & 1 & 1 \\
& 2 & 3 & 1 & 1 & 1 & 1 & 1 \\
2222 & 22\overline{I} & 23\overline{I} & 123\overline{I} & 1123 & 1112 & 1111 & 1111 \\
\end{array}
\]

Thus we get \(p = 11112\overline{1}\overline{3}\overline{2}\) in agreement with (4.5).

5. \(A_{2n-1}^{(2)}\) CASE

The \(A_{2n-1}^{(2)}\) case can be reduced to \(D_{n+1}^{(1)}\) by dropping one term in the superposition in (3.1).

5.1. Rigged configurations. For the data (2.1) define \(E_j^{(a)}\) as in (2.2). The vacancy numbers are specified by (2.3) for \(1 \leq a \leq n - 2\) and

\[
\begin{align*}
E_j^{(n-1)} &= E_j^{(n-2)} - 2E_j^{(n-1)} + 2E_j^{(n)}, \\
E_j^{(n)} &= E_j^{(n-1)} - 2E_j^{(n)}. \\
\end{align*}
\]

(5.1)

The data (2.1) is called an \(A_{2n-1}^{(2)}\) rigged configuration if the condition (2.4) is satisfied.

The following is an example of \(A_{5}^{(2)}\) rigged configuration.

\[
\begin{array}{cccc}
\mu^{(0)} & \mu^{(1)} & \mu^{(2)} & \mu^{(3)} \\
(1^7) & 3 & 3 & 0 & 0 \\
& 3 & 1 & 0 & 0 \\
& 1 & 0 & 0 & 0 \\
\end{array}
\]

(5.2)

5.2. Crystals.

\(B_l = \{(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in (\mathbb{Z}_{\geq 0})^{2n} \mid \sum_{i=1}^{n} (x_i + \bar{x}_i) = l\},\)

\(B_l^{\geq a+1} = \{(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in B_l \mid x_i = \bar{x}_i = 0 (1 \leq i \leq a)\} (0 \leq a \leq n - 1).\)

\(B_l = B_l^{1+1} \supset B_l^{2+2} \supset \cdots \supset B_l^{\geq n}\)

as sets. We regard \(B_l^{\geq a+1}\) as \(A_{2n-2a+1}^{(2)}\) crystal and represent the elements by length \(l\) semistandard row tableaux over the alphabet \(a + 1 \cdots n \bar{n} \cdots \bar{a} + 1\). We use the notation (2.10). Then the highest element in \(B_l^{\geq a+1}\) with respect to the classical part \(C_{n-a}\) is \((a + 1)^l\). The rigged configuration (5.2) corresponds to the following element in \(P_+((1^7))\):

\[
\begin{array}{cccccccc}
& & & & & & & \\
1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\
\end{array}
\]

(5.3)
5.3. Maps $C_1, \ldots, C_n$. For $1 \leq a \leq n - 1$, $C_a$ is defined by (2.18) and (2.19) provided that $B_l$ is regarded as the $A_n^{(2)}$ crystal $B_l^{2a+1}$.

We define $\tilde{p}^{(n)}$ and $C_n(\tilde{p}^{(n)})$ in the same way as in Section 4.3. The latter is obtained by assigning the modes to the former and normal ordering by using the $A_n^{(2)}$ combinatorial $R$ (2.21) with letter $\tilde{n}$.

5.4. Maps $\Phi_1, \ldots, \Phi_n$. For $1 \leq a \leq n - 1$, we define the map $\Phi_a$ by (2.22) and (2.23) by regarding $B_l = B^{2a+1}$ and $B_l' = B^{2a}$ as the $A_n^{(2)}$ and $A_{2a+1}$ crystals. Since the energy function can be negative we employ the same regularization as $D_n^{(1)}$ explained in Section 4.6.

$\Phi_nC_n(\bar{p}(n))$ is defined similarly to Section 4.4 by changing the letter $n-1$ to $n$. To be precise, let $C_n(\tilde{p}^{(n)}) = \tilde{n}\mu_1|d_1|\cdots|\tilde{n}\mu_m|d_m|$. Regard $B_{\tilde{n}\lambda}^{2n} = \{(0, \ldots, 0, x_n, \bar{x}_n, 0, \ldots, 0) \in (\mathbb{Z}_{\geq 0})^2 \mid x_n + \bar{x}_n = l\}$ as the $A_1^{(1)}$ crystal with tableau letters $n, \bar{n}$, where $\tilde{n} = \tilde{n}_1\cdots\tilde{n}_l$ is identified as the lowest weight element. Set $\lambda = \mu^{(n-1)}$. Then $\Phi_nC_n(\tilde{p}^{(n)}) \in B_{\lambda_1}^{\tilde{n}\mu_1} \cdots B_{\lambda_k}^{\tilde{n}\mu_k}$ is defined to be $c_1 \otimes \cdots \otimes c_k$ determined by (2.23) with $b_i = \tilde{n}\mu_i$ and $a = n$.

5.5. Example. Consider the rigged configuration (5.2). We have $\tilde{p}^{(3)} = [33]_2$ and $C_3(\tilde{p}^{(3)}) = [33]_2$. To find $\Phi_3C_3(\tilde{p}^{(3)})$, we calculate $T_3^2 \otimes [33]_2 \otimes ([33]_2 \otimes [33]_2)$ as

Thus we get $p^{(3)} = \Phi_3C_3(\tilde{p}^{(3)}) = [33]_2 \otimes [33]_2$. In view of $H([33]_2 \otimes [33]_2) = -2$, we have $C_2(p^{(2)}) = [33]_2 \otimes [33]_2$. To find $\Phi_2C_2(p^{(2)})$, we calculate $T_2^2 \otimes [33]_2 \otimes T_2^{-2} \otimes [33]_2 \otimes$
Thus we get \( p^{(1)} = \Phi_2 C_2(p^{(2)}) = \bigotimes 2 \otimes \bigotimes \bar{2}. \) Assigning the mode and normal ordering, we find \( C_1(p^{(1)}) = 22 \otimes 2 \otimes 2 \otimes \bigotimes \bigotimes \). Finally to find \( \Phi_1 C_1(p^{(1)}) \) we compute \( T_1^2 \otimes 2 \otimes T_1^2 \otimes \bigotimes \bigotimes \) by using the \( A_5^{(2)} \) combinatorial \( R \) on letters 1, 2, 3, 2, 1. It is by given applying \( T_1^2 \otimes \) to the output of the following diagram (\( \bigotimes \) has been replaced with \( \bigotimes \) below):

Therefore we obtain \( \Phi_1 C_1(p^{(1)}) = 11211 \) in agreement with (5.3).

6. \( B_n^{(1)} \) case

6.1. Rigged configurations. Consider the data of the form (2.1), where \( \mu^{(a)} \) with \( a < n \) and \( 2\mu^{(n)} \) are partitions. Thus \( \mu^{(n)} \in \mathbb{Z}/2. \) \( J^{(a)} = (J_1^{(a)}, \ldots, J_n^{(a)}) \) is taken from \( (\mathbb{Z}_{\geq 0})^n \) for all \( 1 \leq a \leq n. \) Using \( E_j^{(a)} \) in (2.2), we define the vacancy numbers
by (2.3) with $1 \leq a \leq n - 2$, $j \in \mathbb{Z}_{>0}$ and
\[
\phi_j^{(n-1)} = E_j^{(n-2)} - 2E_j^{(n-1)} + 2E_j^{(n)} \quad (j \in \mathbb{Z}_{>0}),
\phi_j^{(n)} = 2E_j^{(n-1)} - 4E_j^{(n)} \quad (j \in \mathbb{Z}_{>0}/2).
\]

All the vacancy numbers are integers even though $E_j^{(n)} \in \mathbb{Z}/2$. The data (2.1) is called a $B_1^{(1)}$ rigged configuration if (2.4) is satisfied, where $j \in \mathbb{Z}_{>0}$ for $1 \leq a \leq n - 1$ and $j \in \mathbb{Z}_{>0}/2$ for $a = n$.

The following is an example of $B_2^{(1)}$ rigged configuration.

\[
\begin{array}{ccc}
\mu^{(0)} & \mu^{(1)} & \mu^{(2)} \\
(1^7) & & \\
0 & 2 & 0 \\
2 & 0 & 1 \\
2 & 2 & 2
\end{array}
\]

We have depicted $\mu^{(2)}$ by a Young diagram consisting of $1 \times \frac{1}{2}$ elementary blocks.

### 6.2. Crystals.

$B_l = \{(x_1, \ldots, x_n, x_0, \bar{x}, \ldots, \bar{x}) \in (\mathbb{Z}_{>0})^{2n+1} \mid x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l, x_0 = 0, 1\}$,

$B_l^{2a+1} = \{(x_1, \ldots, x_n, x_0, \bar{x}, \ldots, \bar{x}) \in B_l \mid x_i = \bar{x}_i = 0 (1 \leq i \leq a)\} \quad (0 \leq a \leq n - 1)$.

$B_l = B_l^{21} \supset B_l^{22} \supset \cdots \supset B_l^{2n}$

as sets. We regard $B_l^{2a+1}$ as $B_n^{(1)}$ crystal and represent the elements by length $l$ semistandard row tableaux over the alphabet $a + 1 < \cdots < n < 0 < \bar{n} \cdots < a + 1$. We use the notation (2.10). Then the highest element in $B_l^{2a+1}$ with respect to the classical part $B_n^{(a)}$ is $(a+1)^l$.

The $B_2^{(1)}$ rigged configuration (6.1) corresponds to

\[
\begin{array}{ccccccc}
1 & \otimes & 1 & \otimes & 2 & \otimes & 2 & \otimes & 0 & \otimes & 0 & \otimes & 0
\end{array} \in \mathcal{P}_+((1^7)).
\]

### 6.3. Maps $C_1, \ldots, C_n$.

For $1 \leq a \leq n - 1$, the map $C_a$ is specified by (2.18) and (2.19) provided that $B_l$ is regarded as the $B_n^{(1)}$ crystal $B_l^{2a+1}$.

Define $\tilde{p}^{(n)}$ and $C_n(\tilde{p}^{(n)})$ in the same way as in Section 4.3 by replacing $\mu_i^{(n)}$ with $2\mu_i^{(n)}$.

### 6.4. Maps $\Phi_1, \ldots, \Phi_n$.

For $1 \leq a \leq n - 1$, we define the map $\Phi_a$ by (2.22) and (2.23) by regarding $B_l = B_l^{2a+1}$ and $B_{l}' = B_l^{2a}$ as the $B_n^{(1)}$ and $B_n^{(a)}$ crystals. Since the energy function can be negative we employ the same regularization as $D_n^{(1)}$ explained in Section 4.6.

To define $\Phi_nC_n(\tilde{p}^{(n)})$, we put $b_l[d_1] \otimes \cdots \otimes b_m[d_m] = C_n(\tilde{p}^{(n)})$ and $(\lambda_1, \ldots, \lambda_k) = 2\mu^{(n-1)}$ in (2.22) regarding $B_l$ as the $A_0^{(1)}$ crystal with letter $\bar{n}$ only and $B_l'$ as the $A_1^{(1)}$ crystal with letters $n, \bar{n}$. By using the resulting $c_1 \otimes \cdots \otimes c_k$ in (2.22), we construct $\Phi_nC_n(\tilde{p}^{(n)})$ as

\[
\Phi_nC_n(\tilde{p}^{(n)}) = \tilde{c}_1 \otimes \cdots \otimes \tilde{c}_k \in B_{n_1}^{2n} \otimes \cdots \otimes B_{n_k}^{2n}.
\]
Here for $B_l'$ with even $l$, $\tilde{c}_j$ is determined from $c_j$ as

$$\tilde{c}_j = \left\lfloor \frac{x_n}{2} \right\rfloor, \quad \tilde{z}_n = \left\lceil \frac{x_n}{2} \right\rceil, \quad z_0 = l - z_n - \tilde{z}_n,$$

where $[x]$ denotes the largest integer not exceeding $x$.

6.5. **Example.** Consider the $B_2^{(1)}$ rigged configuration (6.1). Doubling the width of $\mu^{(2)}$, we have

$$\tilde{p}^{(2)} = 22 \otimes 2, \quad C_2(\tilde{p}^{(2)}) = \xi^{(2)} \otimes 2 \otimes 2 := 2 \otimes 2 \otimes 2.$$  

To find $\Phi_2 C_2(\tilde{p}^{(2)})$ we first need $T_1^{(1)} \otimes 2 \otimes T_2^{(1)} \otimes 2 \otimes (2222 \otimes 22 \otimes 22 \otimes 22)$. Under $\tilde{\cdot}$, the elements 2222 and 22 turn into 22 and 0. Thus we get

$$p^{(1)} = \Phi_2 C_2(\tilde{p}^{(2)}) = 22 \otimes 0 \otimes 0 \otimes 0,$$

which is an element of $B_1^{(1)}$ crystal with letters 2, 0, 2. In view of $22 \otimes 0 \simeq 2 \otimes 20$ and $H(22 \otimes 0) = H(0 \otimes 0) = 0$, we have $C_1(p^{(1)}) = 22 \otimes 0 \otimes 0 \otimes 0$. $\Phi_1 C_1(p^{(1)})$ is derived from $T_1^{(1)} \otimes 22 \otimes T_1^{(1)} \otimes 0^{(1)} \otimes (1^{(2)} \otimes 1^{(2)})$. Since the combinatorial $R$ is the identity on $B_1 \otimes B_1$, the part $T_1^{(1)} \otimes 0^{(1)}$ just pushes 1 to the right. $\Phi_1 C_1(p^{(1)})$ is obtained by putting 1 in front of the output of the diagram:

$$\begin{array}{cccccccccccc}
22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 & 22 \\
12 & 10 & 10 & 10 & 10 & 10 & 10 & 11 & 11 & 11 & 11 & 11 & 11 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}$$

Thus we get $\Phi_1 C_1(p^{(1)}) = 1122000$ in agreement with (6.2).
7. \( \mathcal{C}^{(1)}_n \) case

7.1. Rigged configurations. Consider the data of the form (2.1) and define \( E_j^{(a)} \) as in (2.2). The vacancy number is specified by (2.3) for \( 1 \leq a \leq n-1 \) and

\[
E_j^{(a)} = E_j^{(a-1)} - E_j^{(n)}.
\]

The data (2.1) is called a \( \mathcal{C}^{(1)}_n \) rigged configuration if \( \mu^{(n)}_i \in 2\mathbb{Z} \) and (2.4) is satisfied. It is depicted as in \( A_n^{(1)} \) case by the Young diagrams with rigging. The following is an example of \( \mathcal{C}^{(1)}_2 \) rigged configuration.

\[
\begin{array}{c|c|c}
\mu^{(0)} & \mu^{(1)} & \mu^{(2)} \\
\hline
(1^{10}) & 0 & 2 \\
0 & 1 & 1 \\
6 & 2 & 0 \\
\end{array}
\]

7.2. Crystals.

\[
B_l = \{(x_0, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_0) \in (\mathbb{Z}_{\geq 0})^{2n+2} | \sum_{i=0}^{n} (x_i + \bar{x}_i) = l, x_0 = \bar{x}_0\}.
\]

\[
B_l^{\geq a} = \{(x_0, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_0) \in B_l | x_i = \bar{x}_i = 0 (0 \leq i \leq a-1), x_a = \bar{x}_a \}
\]

for \( 0 \leq a \leq n \). Then

\[
B_l = B_l^{0} \supset B_l^{1} \supset \cdots \supset B_l^{2n} = \left\{ \{0, \ldots, 0, \frac{l}{2}, \frac{l}{2}, 0, \ldots, 0\} | l: \text{odd} \right\}
\]

as sets. We regard \( B_l^{\geq a} \) as \( \mathcal{C}^{(1)}_{n-a} \) crystal and represent the elements by length \( l \) semistandard row tableaux over the alphabet \( a < \cdots < n < \bar{n} < \cdots < \bar{a} \). For example, for \( n = 3, a = 2 \), the highest elements in \( B_5^{\geq 2} \) with respect to \( \mathcal{C}_{n-a} \) are

\[
\begin{array}{c|c|c}
33333 & 23332 & 22322 \\
\end{array}
\]

The rigged configuration (7.2) corresponds to

\[
1 \otimes 1 \otimes 1 \otimes 2 \otimes 2 \otimes 2 \otimes 1 \otimes 1 \otimes 1 \in \mathcal{P}_+((1^{10})).
\]

7.3. Maps \( \mathcal{C}_1, \ldots, \mathcal{C}_n \). Let \( B_l = B_l^{\geq a} \) be the \( \mathcal{C}^{(1)}_{n-a} \) crystal. Then the map \( \mathcal{C}_a \) with \( a < n \) is defined in the same way as in Section 2.4. Since the energy function is nonnegative, one has \( 0 \leq d_1 \leq \cdots \leq d_m \) for normal ordered forms \( b_1[d_1] \otimes \cdots \otimes b_m[d_m] \). Define the \( \mathcal{C}^{(1)}_0 \) crystal element

\[
\bar{p}_l^{(n)} = (n\bar{n})^{\mu^{(n)}_l}/2 \otimes \cdots \otimes (n\bar{n})^{\mu^{(n)}_l}/2,
\]

where \( (n\bar{n})^{l/2} \) denotes the unique element in \( B_l^{\geq n} \) with even \( l \) in (7.5). (Recall that \( \mu^{(n)}_l \) is even for \( \mathcal{C}^{(1)}_n \) rigged configuration.) Then \( \mathcal{C}_n(\bar{p}_l^{(n)}) \) is obtained by assigning the mode in (2.18) as \( d_i = J_l + \mu_l/2 \) instead of (2.19). For normal ordering we need the \( \mathcal{C}^{(1)}_0 \) combinatorial \( R \). It is formally specified by saying that \( H = 0 \) on \( B_l \otimes B_m \) and \( b[d] \otimes b'[d'] \simeq b'[d'] \otimes b[d] \).

7.4. Maps \( \Phi_1, \ldots, \Phi_n \). The map \( \Phi_a \) is defined in the same way as in Section 2.5 if \( B_l \) and \( B'_l \) there are replaced by \( B_l^{\geq a} \) and \( B_l^{\geq a-1} \), respectively.
7.5. **Example.** We start from the $C_2^{(1)}$ rigged configuration (7.2). Then

$$\bar{p}^{(2)} = \begin{array}{cc} 2222 & 22 \end{array}, \quad C_2(\bar{p}^{(2)}) = \begin{array}{cc} 22 & 22 \end{array}.$$ 

In order to find $\Phi_2C_2(\bar{p}^{(2)})$ we are to compute

$$T_2^1 \otimes \begin{array}{cc} 22 \end{array} \otimes T_2^3 \otimes \begin{array}{cc} 2222 \end{array} \otimes (\begin{array}{cc} 2222 \otimes 222 \otimes 2 \end{array})$$

for $C_1^{(1)}$ crystal with letters 1, 2, $\bar{2}$, $\bar{1}$. From the diagram

```
2222 2222 2222 2222
2222 2222 2222 2222
2 2 2 2
1221 222 222 222
2 2 2 2
1121 222 222 222
2 2 2 2
1221 222 222 222
2 2 2 2
2222 1221 222 222
2 2 2 2
1221 1221 222 222
```

we get $p^{(1)} = \Phi_2C_2(\bar{p}^{(2)}) = \begin{array}{cc} 1221 \otimes 1221 \otimes 2 \end{array}$. Assigning the mode to $p^{(1)}$ we have

$C_1(p^{(1)}) = : \begin{array}{cc} 1221 \end{array} \otimes \begin{array}{cc} 1221 \end{array} \otimes \begin{array}{cc} 2 \end{array} :$, where the normal ordering is to be done along the $C_1^{(1)}$ crystal with letters 1, 2, $\bar{2}$, $\bar{1}$. There are four normal ordered forms:

$$\begin{array}{cc} 2 \end{array} \otimes \begin{array}{cc} 222 \end{array} \otimes \begin{array}{cc} 111 \end{array} \otimes \begin{array}{cc} 1 \end{array}, \quad \begin{array}{cc} 2 \end{array} \otimes \begin{array}{cc} 222 \end{array} \otimes \begin{array}{cc} 122 \end{array} \otimes \begin{array}{cc} 1 \end{array}$$

$$\begin{array}{cc} 121 \end{array} \otimes \begin{array}{cc} 2 \end{array} \otimes \begin{array}{cc} 111 \end{array} \otimes \begin{array}{cc} 1 \end{array}, \quad \begin{array}{cc} 121 \end{array} \otimes \begin{array}{cc} 2 \end{array} \otimes \begin{array}{cc} 111 \end{array} \otimes \begin{array}{cc} 1 \end{array}$$

We pick the first one as $C_1(p^{(1)})$. Then $\Phi_1C_1(p^{(1)})$ is calculated from

$$T_1^3 \otimes \begin{array}{cc} 2 \end{array} \otimes \begin{array}{cc} 222 \end{array} \otimes T_1^1 \otimes \begin{array}{cc} 1111 \end{array} \otimes (\begin{array}{cc} 1 \end{array} \otimes \begin{array}{cc} 10 \end{array})$$
for $C^{(1)}_2$ crystal with letters 0, 1, 2, 3, 4, 5. It is given by applying $T_3$ to the output of the following diagram (10 has been replaced with 7 below):

Thus we find $\Phi_1 C_1 \Phi_2 C_2(\vec{p}^{(2)}) = 11122221 \bar{1}$ in agreement with (7.6).

8. $A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$ cases

In this section we treat the algebras $g_n = A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$ simultaneously. They are reduced to the $C^{(1)}_n$ case.

8.1. Rigged configurations. Consider the data of the form (2.1) and define $E_j^{(a)}$ as in (2.2). The vacancy number is specified by (2.3) for $1 \leq a \leq n-1$ and

\begin{align}
(p_j^{(a)}) &= E_j^{(a-1)} - E_j^{(a)} & \text{for } A^{(2)}_{2n}, \\
(p_j^{(a)}) &= 2E_j^{(a-1)} - 2E_j^{(a)} & \text{for } D^{(2)}_{n+1}.
\end{align}

The data (2.1) is called a $g_n$ rigged configuration if (2.4) is satisfied. It is depicted as in $A^{(1)}_n$ case by the Young diagrams with rigging. For example

\begin{align}
\mu^{(0)} & \quad \mu^{(1)} & \quad \mu^{(2)} \\
(1^8) & \quad 3 \quad 2 & \quad 0 \\
& & \quad 0 \\
& & \quad 0
\end{align}

is an $A^{(2)}_4$ rigged configuration, and

\begin{align}
\mu^{(0)} & \quad \mu^{(1)} & \quad \mu^{(2)} \\
(1^{10}) & \quad 5 \quad 2 & \quad 1 \\
& & \quad 2 \\
& & \quad 3 \\
& & \quad 0
\end{align}

is a $D^{(2)}_3$ rigged configuration. Let $RC(\lambda)$ be the set of $g_n$ rigged configurations (2.1) with $\mu^{(0)} = \lambda$. One has the embedding:

\begin{align}
\iota : RC(\lambda) & \quad \rightarrow \quad RC(2\lambda) & \quad \text{for } C^{(1)}_n \\
(\lambda, (\mu^{(1)}, J^{(1)}), \ldots, (\mu^{(n)}, J^{(n)})) & \quad \rightarrow \quad \begin{cases} 
(2\lambda, (2\mu^{(1)}, 2J^{(1)}), \ldots, (2\mu^{(n)}, 2J^{(n)})) & \text{for } A^{(2)}_{2n}, \\
(2\lambda, (2\mu^{(1)}, 2J^{(1)}), \ldots, (2\mu^{(n)}, J^{(n)})) & \text{for } D^{(2)}_{n+1}.
\end{cases}
\end{align}
where for an array \( \lambda = (\lambda_i) \), \( 2\lambda \) means \((2\lambda_i) \). In the \( D_n^{(2)} \) case, all the rigging except the \( n \)-th one \( J^{(n)} \) are doubled.

8.2. Crystals. For \( A_{2n}^{(2)} \),

\[
(8.6) \quad B_l = \{(x_1, \ldots, x_n, x_\emptyset, \bar{x}_n, \ldots, \bar{x}_1) \in (\mathbb{Z}_{\geq 0})^{2n+1} \mid x_\emptyset + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \}.
\]

For \( D_{n+1}^{(2)} \),

\[
(8.7) \quad B_l = \{(x_1, \ldots, x_n, x_\emptyset, \bar{x}_n, \ldots, \bar{x}_1) \in (\mathbb{Z}_{\geq 0})^{2n+2} \mid x_\emptyset + x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l, x_0 = 0, 1 \}.
\]

There is an embedding \( (B_l \text{ for } g_n \rightarrow (B_{2l} \text{ for } C_n^{(1)}) \) as sets, which will also be denoted by \( \iota \):

\[
(8.8) \quad \iota(x) = \begin{cases} (x_\emptyset, 2x_1, \ldots, 2x_n, 2\bar{x}_n, \ldots, 2\bar{x}_1, x_\emptyset) & \text{for } A_{2n}^{(2)}, \\ (x_\emptyset, 2x_1, \ldots, 2x_n-1, 2x_n + x_\emptyset, 2\bar{x}_n + x_\emptyset, 2\bar{x}_n-1, \ldots, 2\bar{x}_1, x_\emptyset) & \text{for } D_{n+1}^{(2)}. \end{cases}
\]

Here \( x = (x_1, \ldots, \bar{x}_1) \in B_l \) is specified as in (8.6) and (8.7). When applying \( \iota^{-1} \), the tableau letters are halved (in case the image exists). In particular, a pair of 0 and \( \emptyset \) turns into \( \emptyset \). As for \( n \) and \( \bar{n} \), the change is described by (6.4). We extend the map \( \iota \) to the tensor product of \( g_n \) crystals by \( \iota(p_1 \otimes \cdots \otimes p_k) = \iota(p_1) \otimes \cdots \otimes \iota(p_k) \).

The \( A_4^{(2)} \) rigged configuration (8.3) corresponds to

\[
(8.9) \quad \begin{array}{cccccccc} 1 \otimes 1 \otimes \emptyset \otimes 1 \otimes 1 \otimes \emptyset \otimes 2 \otimes 2 \end{array} \in \mathcal{P}_+(1^8)).
\]

The \( D_3^{(2)} \) rigged configuration (8.4) corresponds to

\[
(8.10) \quad \begin{array}{cccccccc} 1 \otimes \emptyset \otimes 2 \otimes 1 \otimes 0 \otimes 1 \otimes 2 \otimes 2 \otimes 0 \otimes 0 \end{array} \in \mathcal{P}_+(1^{10})).
\]

8.3. Example. Consider the \( A_4^{(2)} \) rigged configuration (8.3). According to (8.5) we double everything horizontally to get

\[
(8.11) \quad \begin{array}{cccccccc} \mu^{(0)} & & & & & & & \mu^{(1)} \\
| & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
| & 6 & 4 & 2 & 2 & 2 & 2 & 2 \\
& & & & & & & \mu^{(2)} \\
| & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
| & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & &
\end{array}
\]

We regard this as a \( C_2^{(1)} \) rigged configuration. The vacancy numbers are determined by (7.1) which is the same as (8.1). As the result they are all doubled as well. From (7.7) we set

\[
\tilde{p}^{(2)} = \begin{array}{cccccccc} \emptyset & 222222 & \emptyset & \emptyset \end{array}, \quad C_2(\tilde{p}^{(2)}) = \begin{array}{cccccccc} \emptyset & 222222 & \emptyset & \emptyset \end{array}. 
\]

Then \( p^{(1)} := \Phi_2 C_2(\tilde{p}^{(2)}) \) is determined as

\[
p^{(1)} = \begin{array}{cccccccc} 11122111 \otimes 12211 \end{array}, \quad C_1(p^{(1)}) = \begin{array}{cccccccc} 11122111 \otimes 12211 \end{array}. 
\]

\[
p = \Phi_1 C_1(p^{(1)}) \text{ reads}
\]

\[
p = \begin{array}{cccccccc} 11 \otimes 11 \otimes \emptyset \otimes 11 \otimes 11 \otimes 00 \otimes 22 \otimes 22 \end{array}.
\]

Finally applying \( \iota^{-1} \) in (8.8), we find \( \iota^{-1}(p) = 1 \emptyset 1 \emptyset 0 2 2 2 \), reproducing (8.9).
Next we consider the $D_3^{(2)}$ rigged configuration (8.4). According to (8.5) we double everything horizontally except the rigging attached to $\mu^{(2)}$:

\[
\begin{array}{ccc}
\mu^{(0)} & \mu^{(1)} & \mu^{(2)} \\
0 & 0 & 0 \\
2 & 2 & 2 \\
10 & 4 & 2 \\
\end{array}
\]

(8.12)

We regard this as a $C_2^{(1)}$ rigged configuration. The vacancy numbers determined by (7.1) instead of (8.2) have been doubled except $p_j^{(2)}$’s which remain unchanged. From (7.7) we set

\[
\tilde{p}^{(2)} = \text{pictograph} \quad \tilde{C}_2(p^{(2)}) = \text{pictograph}.
\]

Then $p^{(1)} = \Phi_2 C_2(p^{(2)})$ is determined as

\[
p^{(1)} = \text{pictograph}.
\]

There are two normal ordered forms for $C_1(p^{(1)})$:

\[
\text{pictograph, pictograph, pictograph}.
\]

Both of them lead to

\[
p = \Phi_1 C_1(p^{(1)}) = \text{pictograph}.
\]

Thus we obtain $\epsilon^{-1}(p) = \text{pictograph}$, in agreement with (8.10).

**APPENDIX A. CRYSTALS AND COMBINATORIAL R**

The crystals $B_1$ used in the main text are crystal bases of irreducible finite-dimensional representations of a quantum affine algebra $U_q(\mathfrak{g})$. Let us recall basic facts on them following [5, 10, 27].

Let $P$ be the weight lattice, $\{\alpha_i\}_{0 \leq i \leq n}$ the simple roots, and $\{\Lambda_i\}_{0 \leq i \leq n}$ the fundamental weights of $\mathfrak{g}$. A crystal $B$ is a finite set with weight decomposition $B = \cup_{\lambda \in P} B_{\lambda}$. The Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ ($i = 0, 1, \cdots, n$) act on $B$ as $\tilde{e}_i : B_{\lambda} \rightarrow B_{\lambda + \alpha_i} \cup \{0\}$, $\tilde{f}_i : B_{\lambda} \rightarrow B_{\lambda - \alpha_i} \cup \{0\}$. In particular, these operators are nilpotent. By definition, we have $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$. For any $b \in B$, set $\varepsilon_i(b) = \max \{m \geq 0 \mid \tilde{e}_i^m b \neq 0\}$ and $\varphi_i(b) = \max \{m \geq 0 \mid \tilde{f}_i^m b \neq 0\}$. Then we have the weight $\text{wt} b$ of $b$ by $\text{wt} b = \sum_{i=0}^n (\varphi_i(b) - \varepsilon_i(b)) \Lambda_i$.

For two crystals $B$ and $B'$, one can define the tensor product $B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}$. The operators $\tilde{e}_i, \tilde{f}_i$ act on $B \otimes B'$ by

\[
\tilde{e}_i(b \otimes b') = \begin{cases} 
\tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\
\tilde{e}_i b \otimes \hat{b}' & \text{if } \varphi_i(b) < \varepsilon_i(b'),
\end{cases}
\]

\[
\tilde{f}_i(b \otimes b') = \begin{cases} 
\hat{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\
\tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'),
\end{cases}
\]

Here $0 \otimes b'$ and $b \otimes 0$ are understood as 0. For crystals we are considering, there exists a unique isomorphism $B \otimes B' \simeq B' \otimes B$, i.e. a unique map which commutes with the action of Kashiwara operators. In particular, it preserves the weight.

For a crystal $B$ we define its affinization $\text{Aff}(B) = \{b[d] \mid d \in \mathbb{Z}, b \in B\}$ by $\tilde{e}_i(b[d]) = (\hat{e}_i b)[d - \delta_{i0}]$ and $\tilde{f}_i(b[d]) = (\hat{f}_i b)[d + \delta_{i0}]$. $(b[d])$ here corresponds to $T^{-d} a f(b)$ in [10].
The crystal isomorphism $B \otimes B' \cong B' \otimes B$ is lifted up to a map $\text{Aff}(B) \otimes \text{Aff}(B') \cong \text{Aff}(B') \otimes \text{Aff}(B)$ called the combinatorial $R$. It has the following form:

$$R : \text{Aff}(B) \otimes \text{Aff}(B') \longrightarrow \text{Aff}(B') \otimes \text{Aff}(B)$$

$$b[d] \otimes b'[d'] \longmapsto \tilde{b}'[d'] - H[b \otimes b'] \otimes \tilde{b}[d + H(b \otimes b')] ,$$

where $b \otimes b' \mapsto \tilde{b}' \otimes \tilde{b}$ under the isomorphism $B \otimes B' \cong B' \otimes B$. $H(b \otimes b')$ is called the energy function and determined up to an additive constant by

$$H(\tilde{b}(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0, \ \varphi_0(b) \geq \varepsilon_0(b'), \ \varphi_0(b') \geq \varepsilon_0(\tilde{b}), \\ H(b \otimes b') - 1 & \text{if } i = 0, \ \varphi_0(b) < \varepsilon_0(b'), \ \varphi_0(b') < \varepsilon_0(\tilde{b}), \\ H(b \otimes b') & \text{otherwise}. \end{cases}$$

**Proposition A.1** (Yang-Baxter equation). The following equation holds on $\text{Aff}(B) \otimes \text{Aff}(B') \otimes \text{Aff}(B'')$:

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R).$$

We often write the map $R$ simply by $\sim$. The combinatorial $R$ is naturally restricted to $B \otimes B'$.

In the main text we are concerned about the crystal $B_l$ corresponding to the $l$-fold symmetric fusion of the vector representation. We normalize the energy function so that

$$(A.1) \quad \max \{H(b \otimes c) \mid b \otimes c \in B_l \otimes B_m\} = \min(l, m).$$

Under this convention one has

$$(A.2) \quad \min \{H(b \otimes c) \mid b \otimes c \in B_l \otimes B_m\} = \begin{cases} 0 & \text{if } g_n = A_n^{(1)}, C_n^{(1)}, \\ -\min(l, m) & \text{if } g_n \neq A_n^{(1)}, C_n^{(1)}. \end{cases}$$

When $l = m$, the combinatorial $R$ becomes the identity map on $B_l \otimes B_l$ but still acts non-trivially as $R(x[d] \otimes y[e]) = x[e - H(x \otimes y)] \otimes y[d + H(x \otimes y)]$.

**Acknowledgments.** The authors thank Anne Schilling and Mark Shimozono for kind interest and comments. A.K. and Y.Y. are supported by Grants-in-Aid for Scientific Research JSPS No.15540363 and No.17340047, respectively. R.S. is grateful to Miki Wadati for warm encouragement during the study.

**References**

[1] H. A. Bethe, Zur Theorie der Metalle, I. Eigenwerte und Eigenfunktionen der linearen Atomkette, Z. Physik 71 (1931) 205–231.

[2] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London (1982).

[3] S. V. Kerov, A. N. Kirillov and N. Yu. Reshetikhin, Combinatorics, the Bethe ansatz and representations of the symmetric group, J. Soviet Math. 41 (1988) 916–924.

[4] A. N. Kirillov and N. Yu. Reshetikhin, The Bethe ansatz and the combinatorics of Young tableaux, J. Soviet Math. 41 (1988) 925–955.

[5] M. Kashiwara, On crystal bases of the $q$-analogue of universal enveloping algebras. Duke Math. J. 63 (1991) 465–516.

[6] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Remarks on fermionic formula, Contemp. Math. 248 (AMS 1999) 243–291.

[7] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Z. Tsuboi: Paths, Crystals and Fermionic Formulas, Prog. in Math. Phys. 23 (2002) 205–272.

[8] A. Schilling, X=M Theorem: Fermionic formulas and rigged configurations under review preprint, math.QA/0512161.

[9] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Scattering rules in soliton cellular automata associated with crystal bases, Contemporary Math. 297 (2002) 151–182.
[10] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Affine crystals and vertex models, Int. J. Mod. Phys. A 7 (suppl. 1A), (1992) 449–484.

[11] A. Nakayashiki and Y. Yamada, Kostka polynomials and energy functions in solvable lattice models, Selecta Mathematica, New Ser. 3 (1997) 547–599.

[12] C. L. Schultz, Eigenvectors of the multicomponent generalization of the six-vertex model, Physica. A122 (1983) 71–88.

[13] D. Takahashi and J. Satsuma, A soliton cellular automaton, J. Phys. Soc. Jpn. 59 (1990) 3514–3519.

[14] D. Takahashi, On some soliton systems defined by using boxes and balls, Proceedings of the International Symposium on Nonlinear Theory and Its Applications (NOLTA ’93), (1993) 555–558.

[15] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi and T. Tokihiro, The $A^{(1)}_M$ automata related to crystals of symmetric tensors, J. Math. Phys. 42 (2001) 274–308.

[16] K. Fukuda, M. Okado, Y. Yamada, Energy functions in box ball systems, Int. J. Mod. Phys. A 15 (2000) 1379–1392.

[17] R. Sakamoto, Crystal interpretation of Kerov-Kirillov-Reshetikhin bijection II. Proof for $sl_3$ Case, preprint.

[18] M. Okado, A. Schilling and M. Shimozono, A crystal to rigged configuration bijection for nonexceptional affine algebras, Algebraic Combinatorics and Quantum Groups, Ed by N. Jing, World Scientific (2003), 85–124.

[19] A. Schilling and M. Shimozono, X=M for symmetric powers, J. Alg. 295 (2006) 562–610.

[20] T. Takagi, Separation of colour degree of freedom from dynamics in a soliton cellular automaton, J. Phys. A 38 (2005) 1961–1976.

[21] A. N. Kirillov, A. Schilling and M. Shimozono, A bijection between Littlewood-Richardson tableaux and rigged configurations, Selecta Math. 8 (2002) 67–135.

[22] M. Shimozono, Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties, J. Algebraic Combin. 15 (2002) 151–187.

[23] G. Hatayama, A. Kuniba, M. Okado and T. Takagi, Combinatorial $R$ matrices for a family of crystals: $C^{(1)}_n$ and $A^{(2)}_{2n-1}$ cases, Prog. in Math. 191 (2000) 105–139.

[24] G. Hatayama, A. Kuniba, M. Okado and T. Takagi, Combinatorial $R$ matrices for a family of crystals: $B^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_{2n}$, and $D^{(2)}_{n+1}$ cases, J. Alg. 247 (2002) 577–615.

[25] A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Geometric crystal and tropical $R$ for $D^{(1)}_n$, Int. Math. Res. Notices 48 (2003) 2565–2620.

[26] A. Kuniba, T. Takagi and A. Takenouchi, Factorization, reduction and embedding in integrable cellular automata, J. Phys. A. 37 (2004) 1691-1709.

[27] S-J. Kang, M. Kashiwara and K. C. Misra, Crystal bases of Verma modules for quantum affine Lie algebras, Compositio Math. 92 (1994) 299–325.

Atsuo Kuniba:
Institute of Physics, Graduate School of Arts and Sciences, University of Tokyo, Komaba, Tokyo 153-8902, Japan
atsuo@gokutan.c.u-tokyo.ac.jp

Masato Okado:
Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Osaka 560-8531, Japan
okado@sigmath.es.osaka-u.ac.jp

Reiho Sakamoto:
Department of Physics, Graduate School of Science, University of Tokyo, Hongo, Tokyo 113-0033, Japan
reiko@monet.phys.s.u-tokyo.ac.jp

Taichiro Takagi:
Department of Applied Physics, National Defense Academy, Kanagawa 239-8686, Japan
takagi@nda.ac.jp

Yasuhiko Yamada:
Department of Mathematics, Faculty of Science, Kobe University, Hyogo 657-8501, Japan
yamaday@math.kobe-u.ac.jp