Gauge Invariance of the Muonium-Antimuonium Oscillation Time Scale and Limits on Right-Handed Neutrino Masses

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Abstract

The gauge invariance of the muonium-antimuonium (M ¯M) oscillation time scale is explicitly demonstrated in the Standard Model modified only by the inclusion of singlet right-handed neutrinos and allowing for general renormalizable interactions. The see-saw mechanism is exploited resulting in three light Majorana neutrinos and three heavy Majorana neutrinos with mass scale MR ≫ MW. The leading order matrix element contribution to the M ¯M oscillation process is computed in Rξ gauge and shown to be ξ independent thereby establishing the gauge invariance to this order. Present experimental limits resulting from the non-observation of the oscillation process sets a lower limit on MR roughly of order 600 GeV.

1 Introduction

Muonium (M) is the Coulombic bound state of an electron and an antimuon (e−µ+), while antimuonium (M ¯) is the Coulombic bound state of a positron and a muon (e+µ−). It was suggested roughly 50 years ago[1] that there may be a spontaneous conversion between muonium and antimuonium which would violate the individul electron and muon number conservation laws by two units. Such a muonium-antimuonium oscillation is totally forbidden within the Standard Model. Hence, its observation will be a clear signal of physics beyond the Standard Model. Since the initial suggestion, experimental searches have been conducted[2]-[5] and a variety of theoretical models have been proposed which could give rise to such a muonium-antimuonium conversion. These include interactions which could be mediated by (a) a doubly charged Higgs boson Δ++[6, 7], which is contained in a left-right symmetric model, (b) massive Majorana neutrinos[8, 9], or (c) the τ-sneutrino in an R-parity violation supersymmetric model[10].

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In this paper, we focus on a modified Standard Model which includes singlet right-handed neutrinos. There is now compelling evidence of the existence of neutrino oscillations from the experimental study of atmospheric and solar neutrinos[11]-[15]. That implies nonzero neutrino masses and mixing matrix elements. The size and nature of the neutrino mass and the associated mixing is still an open question subject to experimental determination and theoretical speculation [16]-[18]. One simple neutrino mass model is obtained by modifying the Standard Model by including singlet right-handed neutrinos and allowing for a general mass matrix for neutrinos. Left-handed neutrinos along with their charged leptonic partners are components of $SU(2)_L$ doublets and experience the weak interaction while any right-handed neutrinos are completely neutral under the Standard Model gauge group. The see-saw mechanism[19]-[23] provides a natural explanation of the smallness of the three light Majorana neutrino masses, while ensuring that the other three Majorana neutrinos are heavy. Such a model could also lead to the muonium-antimuonium oscillation process. In order for there to be a nontrivial mixing between muonium and antimuonium, the individual electron and muon number conservation must be violated. Such a situation results provided the neutrinos are massive particles which mix amongst the various generations. This criterion can be met by the modified Standard Model and the $e^−\mu^+$ and $e^+\mu^−$ states could indeed mix.

2 Neutrino masses and mixings

The leptonic Yukawa interactions with the Higgs scalar doublet in the modified Standard Model take the form

$$L^{\phi}_{int} = -\frac{g}{\sqrt{2}M_W}\phi^−(\sum_{a,b=1}^{3} L_{Ra} \ell_{(0)}^{ab} \ell_{(0)}^{ab} - \ell_{La}^{ab} \nu_{(0)}^{abR}) + H.C. \quad (1)$$

Here $\ell_{La}^{(0)}$ and $\nu_{La}^{(0)}$ are respectively the charged lepton and its associated neutrino partner of the $SU(2)_L$ doublet, while $\nu_{Ra}^{(0)}$ is the right-handed neutrino singlet. The superscript zero indicates weak interaction eigenstates so that the leptonic charged current interaction is

$$L^{W}_{int} = -\frac{g}{\sqrt{2}}W^{-\mu}\sum_{a=1}^{3} \ell_{La}^{(0)} \gamma_{\mu} \nu_{La}^{(0)} + H.C. \quad (2)$$

After spontaneous symmetry breaking, the mass term for the charged leptons takes the form

$$L^{\ell}_{mass} = -\sum_{a,b=1}^{3} [\ell_{Ra}^{(0)} m^{\ell}_{ab} \ell_{(0)}^{ab} + \ell_{La}^{(0)} m^{\ell}_{ba} \ell_{(0)}^{ab}] \quad (3)$$
where $m^\ell$ is a $3 \times 3$ mass matrix. To diagonalize this matrix, one performs the biunitary transformation

$$ m^\ell = A^R m^\ell_{\text{diag}} (A^L)^\dagger $$

(4)

where $A^R$ and $A^L$ are $3 \times 3$ unitary matrices and $m^\ell_{\text{diag}}$ is a diagonal matrix whose entries are the charged lepton masses. To implement this basis change, the charged lepton fields participating in the weak interaction are rewritten in terms of the mass diagonal fields as

$$ \ell^{(0)}_{La} = \sum_{b=1}^{3} A^L_{ab} \ell_{Lb}, \hspace{1em} \ell^{(0)}_{Ra} = \sum_{b=1}^{3} A^R_{ab} \ell_{Rb} $$

(5)

So doing the mass term reads

$$ L^\ell_{\text{mass}} = -\frac{3}{2} \sum_{a=1}^{3} m^\ell_{a} [\ell_{Ra} \ell_{Lb} + \ell_{La} \ell_{Rb}] $$

(6)

A general neutrino mass term resulting from renormalizable interactions takes the form

$$ L^{\nu}_{\text{mass}} = -\frac{1}{2} \left( \begin{array}{c} (\nu^{(0)}_L)^c \\ \nu^{(0)}_R \end{array} \right) \left( \begin{array}{cc} 0 & (m^D)^T \\ m^D & m^R \end{array} \right) \left( \begin{array}{c} (\nu^{(0)}_L)^c \\ \nu^{(0)}_R \end{array} \right)^T + H.C. $$

(7)

Note that the upper left $3 \times 3$ block in the neutrino mass matrix is set to zero. This block matrix involves only left-handed neutrinos and in the (modified) Standard Model its generation requires a nonrenormalizable mass dimension-five operator. Consequently such a term will be ignored.

For three generations of neutrinos, the six mass eigenvalues, $m^{\nu}_{\nu A}$, are obtained from the diagonalization of the $6 \times 6$ matrix

$$ M^{\nu} = \left( \begin{array}{cc} 0 & (m^D)^T \\ m^D & m^R \end{array} \right) $$

(8)

Since $M^{\nu}$ is symmetric, it can be diagonalized by a single unitary $6 \times 6$ matrix, $U$, as

$$ M^{\nu}_{\text{diag}} = U^T M^{\nu} U. $$

(9)

This diagonalization is implemented via the basis change on the original neutrino fields organized as the 6 dimensional column vector

$$ N^{(0)}_L = \left( \begin{array}{c} \nu^{(0)}_L \\ (\nu^{(0)}_R)^c \end{array} \right), \hspace{1em} N^{(0)}_R = \left( \begin{array}{c} (\nu^{(0)}_L)^c \\ \nu^{(0)}_R \end{array} \right) $$

(10)

to the new neutrino fields defined as

$$ N^{(0)}_L = U N_L, \hspace{1em} N^{(0)}_R = U^* N_R $$

(11)
where
\[ N_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix}, \quad N_R = \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix}. \quad (12) \]

The neutrino mass term then takes the form
\[ \mathcal{L}_{\text{mass}}^\nu = -\frac{1}{2} \sum_{A=1}^{6} m_{\nu A} [\nu_A^T C \nu_A + \overline{\nu_A C \nu_A}^T] = -\sum_{A=1}^{6} m_{\nu A} \overline{\nu_A} \nu_A^c, \quad (13) \]

where \( m_{\nu A} \) are the Majorana neutrino masses.

Since a nonzero Majorana mass matrix \( m^R \) does not require \( SU(2)_L \times U(1) \) symmetry breaking, it is naturally characterized by a much larger scale, \( M_R \), than the elements of the matrix \( m^D \) whose nontrivial values do require \( SU(2)_L \times U(1) \) symmetry breaking and are thus expected to be somewhere of the order of the charged lepton mass to the W mass. Thus one can take the elements of \( m^D \), characterized by a scale \( m_D \), to be much less than \( M_R \), the scale of the elements of \( m^R \). One then finds on diagonalization of the \( 6 \times 6 \) neutrino mass matrix that three of the eigenvalues are crudely given by
\[ m_{\nu a} \sim \frac{m_D^2}{M_R} \ll m_D, \quad a = 1, 2, 3, \quad (14) \]
while the other three eigenvalues are roughly
\[ m_{\nu i} \sim M_R, \quad i = 4, 5, 6. \quad (15) \]

This constitutes the so called see-saw mechanism\[19]-[23] and provides a natural explanation of the smallness of the three light neutrino masses. Moreover, the elements of the mixing matrix are characterized by an \( M_R \) mass dependence
\[ U_{ab} \sim O(1), \quad a, b = 1, 2, 3 \]
\[ U_{ij} \sim O(1), \quad i, j = 4, 5, 6 \]
\[ U_{ia} \sim U_{ai} \sim O\left(\frac{m_D}{M_R}\right), \quad a = 1, 2, 3, i = 4, 5, 6. \quad (16) \]

Since the charged lepton mixing matrix is independent of \( M_R \), one finds that elements of the mixing matrix appearing in the charged current have the \( M_R \) mass dependence
\[ V_{ab} \sim O(1), \quad a, b = 1, 2, 3 \]
\[ V_{ai} \sim O\left(\frac{m_D}{M_R}\right), \quad a = 1, 2, 3, \quad i = 4, 5, 6 \quad (17) \]

Inserting the transformations (5) and (11) in the interaction terms (1) and (2), and taking into account the mass matrix transformations (4) and (9), one obtains the explicit interactions of
charged bosons with the leptons in their mass diagonal basis as

$$L^W_{\text{int}} = -\frac{g}{\sqrt{2}} W^- \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\ell}_a \gamma_\mu V_{aA} \nu_A - \frac{g}{\sqrt{2}} W^+ \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\nu}_A V^*_{aA} \gamma_\mu \ell_a$$

(18)

$$L^\phi_{\text{int}} = -\frac{g}{\sqrt{2} M_W} \phi^- \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\ell}_a V_{aA} \left( m_{la} \frac{1 - \gamma_5}{2} - m_{\nu A} \frac{1 + \gamma_5}{2} \right) \nu_A$$

$$-\frac{g}{\sqrt{2} M_W} \phi^+ \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\nu}_A V^*_{aA} \left( m_{la} \frac{1 + \gamma_5}{2} - m_{\nu A} \frac{1 - \gamma_5}{2} \right) \ell_a$$

(19)

where

$$V_{aA} = \sum_{b=1}^{3} (A_L^{-1})_{ab} U_{bA}$$

(20)

Note that the mixing matrix $V_{aA}$ satisfies the identities[24]:

$$\sum_{A=1}^{6} V_{aA} V_{bA}^* = \delta_{ab}$$

(21)

$$\sum_{A=1}^{6} V_{aA} V_{bA} m_{\nu A} = 0$$

(22)

Identity (21) stems from the unitarity of matrices $A_L$ and $U$, while identity (22) is a consequence of the particular form of the neutrino mass matrix. In particular, it requires the Majorana mass term of the left-handed neutrinos be set to zero. A detailed proof of this later identity is provided in the Appendix.

### 3 The gauge invariant T-matrix elements

The lowest order Feynman diagrams accounting for muonium and antimuonium mixing are displayed in Fig.1. We shall consistently employ the $R_\xi$ gauge. The gauge invariance of the T-matrix element will be demonstrated by establishing its $\xi$ independence. In Fig.1, there are two neutrinos in the intermediate state for each graph while every wavy line represents either a W boson or an $R_\xi$ gauge charged erstwhile Nambu-Goldstone boson.
Fig. 1 Feynman graphs contributing to the muonium-antimuonium mixing. Each wavy line is either a W boson or an $R_\xi$ gauge charged Nambu-Goldstone boson.

Note that in unitary gauge ($\xi \to \infty$), the W boson propagator takes the form $\frac{-i}{p^2-M_W^2+i\epsilon}[g_{\mu\nu} - \frac{p_\mu p_\nu}{M_W^2}]$. A theory with such a propagator has very bad power counting convergence properties. As it turns out, the unitary gauge power counting divergent pieces in the $W$ vector box diagrams vanish, as they must, after application of properties (21) and (22). Hence, when we calculate the T-matrix elements in $R_\xi$ gauge, we will also apply properties (21) and (22) to establish the cancellation of the various terms in this case.

As it turns out, graph (a) gives the same contribution as (b), as do graphs (c) and (d). Hence, we need only discuss the gauge invariant T-matrix elements of the graphs (a) and (c). Fig. 2 details explicitly the 4 separate graphs which are represented by the single graph in Fig. 1.
Fig. 2  Feynman graphs of type (a) in $R_\xi$ gauge.

A straightforward application of the $R_\xi$ gauge Feynman rules\cite{25} to the above graphs yields the T-matrix elements

\begin{align}
T_{a1} &= -\frac{g^4}{64\pi^2M_W^2} [\bar{\mu}(3)\gamma_{\mu}\frac{1}{2} e(2)][\bar{\mu}(4)\gamma_{\mu}\frac{1}{2} e(1)] \sum_{A=1}^{6} \sum_{B=1}^{6} (V_{\mu A} V_{e A}^*) (V_{\mu B} V_{e B}^*) \\
&\cdot \int_0^\infty dt \frac{x_A x_B}{(t + x_A) (t + x_B)(t + 1)} \cdot \left\{ 1 + \frac{2(\xi - 1)}{t + \xi} \cdot t + \frac{(\xi - 1)^2}{4(t + \xi)^2} \cdot t^2 \right\} \\
T_{a2} = T_{a3} &= -\frac{g^4}{64\pi^2M_W^2} [\bar{\mu}(3)\gamma_{\mu}\frac{1}{2} e(2)][\bar{\mu}(4)\gamma_{\mu}\frac{1}{2} e(1)] \sum_{A=1}^{6} \sum_{B=1}^{6} (V_{\mu A} V_{e A}^*) (V_{\mu B} V_{e B}^*) \\
&\cdot \int_0^\infty dt \frac{x_A x_B}{(t + x_A) (t + x_B)(t + 1)} \cdot \left\{ t + \frac{\xi - 1}{4(t + \xi)} \cdot t^2 \right\} \\
T_{a4} &= -\frac{g^4}{64\pi^2M_W^2} [\bar{\mu}(3)\gamma_{\mu}\frac{1}{2} e(2)][\bar{\mu}(4)\gamma_{\mu}\frac{1}{2} e(1)] \sum_{A=1}^{6} \sum_{B=1}^{6} (V_{\mu A} V_{e A}^*) (V_{\mu B} V_{e B}^*)
\end{align}
\[ \int_0^\infty dt \left[ \frac{x_A x_B}{(t + x_A)(t + x_B)(t + \xi)^2} \cdot \frac{t^2}{4} \right] \]  

(25)

where \( \bar{\mu}(3) = \bar{\mu}(p_3, s_3) \), \( \bar{\mu}(4) = \bar{\mu}(p_4, s_4) \), \( e(1) = e(p_1, s_1) \) and \( e(2) = e(p_2, s_2) \) are the spinors of the muons and electrons and \( x_A = \frac{m^2_A}{M_W^2} \), \( A = 1, \ldots, 6 \). Note that in obtaining these results, we already applied properties (21) and (22) to eliminate various self-cancelling terms. As such the integrals in (25)-(27) are finite even in the \( \xi \to \infty \) limit.

In order to discuss the \( \xi \) dependence in a manifest way, we rewrite these T-matrix elements as

**T - matrix elements in \( R_\xi \) gauge**

\[ T_{a1} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt \mathcal{A}(x_A, x_B, t) \cdot \left( h(t) \cdot \frac{1}{(t + \xi)^2} - 2g(t) \cdot \frac{1}{t + \xi} + f(t) \right) \]

\[ T_{a2} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt \mathcal{A}(x_A, x_B, t) \cdot \left( -h(t) \cdot \frac{1}{(t + \xi)^2} + g(t) \cdot \frac{1}{t + \xi} \right) \]

\[ T_{a3} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt \mathcal{A}(x_A, x_B, t) \cdot \left( -h(t) \cdot \frac{1}{(t + \xi)^2} + g(t) \cdot \frac{1}{t + \xi} \right) \]

\[ T_{a4} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt \mathcal{A}(x_A, x_B, t) \cdot \left( h(t) \cdot \frac{1}{(t + \xi)^2} \right) \]

where

\[ \mathcal{A}(x_A, x_B, t) = \frac{\gamma^4}{64\pi^2 M_W^2} \frac{[\bar{\mu}(3) \gamma^\mu - \frac{1 - \gamma_5}{2} e(2)][\bar{\mu}(4) \gamma^\mu - \frac{1 - \gamma_5}{2} e(1)](V_{\mu A} V^*_{e A})(V_{\mu B} V^*_{e B})}{(t + x_A)(t + x_B)} \]

(26)

with

\[ h(t) = \frac{t^2}{4}, \quad g(t) = \frac{t + \frac{t^2}{4}}{t + 1} \quad \text{and} \quad f(t) = \frac{1 + 2t + \frac{t^2}{4}}{(t + 1)^2} \]

(27)
Note that the \( \frac{1}{(t+\xi)} \) terms from the second and third graphs totally cancel against the ones from the first and fourth graphs, while the \( \frac{1}{t} \) terms from the second and third graphs exactly cancel the one from the first graph. All \( \xi \) dependent contributions thus vanish and the only remaining piece is the term containing \( f(t) \) from the first graph, which is \( \xi \) independent. Hence, we have the gauge invariant T-matrix element for graphs of type (a)

\[
T_a = -\frac{g^4}{64\pi^2 M_W^4} [\tilde{\mu}(3)\gamma_\mu \frac{1-\gamma_5}{2} e(2)] [\tilde{\mu}(4)\gamma^\mu \frac{1-\gamma_5}{2} e(1)] \sum_{A=1}^{6} \sum_{B=1}^{6} (V_{\mu A} V^{\ast}_{eA})(V_{\mu B} V^{\ast}_{eB}) x_A x_B \\
\cdot \int_{0}^{\infty} dt \frac{1 + 2t + \frac{r^2}{4}}{(t + x_A)(t + x_B)(t + 1)^2} \\
= -\frac{G_F^2 M_W^2}{8\pi^2} [\tilde{\mu}(3)\gamma_\mu (1-\gamma_5)e(2)] [\tilde{\mu}(4)\gamma^\mu (1-\gamma_5)e(1)] \sum_{A=1}^{6} (V_{\mu A} V^{\ast}_{eA})^2 S(x_A) \\
+ \sum_{A,B=1, A\neq B}^{6} (V_{\mu A} V^{\ast}_{eA})(V_{\mu B} V^{\ast}_{eB}) T(x_A, x_B) 
\] (28)

Here we have introduced the Fermi scale

\[
\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}
\] (29)

along with the Inami-Lim\cite{26} function

\[
S(x_A) = \frac{x^3 - 11x^2 + 4x}{4(1-x)^2} - \frac{3x^3}{2(1-x)^3} \ln(x) 
\] (30)

We have also defined

\[
T(x_A, x_B) = x_A x_B \left( \frac{J(x_A) - J(x_B)}{x_A - x_B} \right) = T(x_B, x_A) 
\] (31)

with

\[
J(x) = \frac{x^2 - 8x + 4}{4(1-x)^2} \ln(x) - \frac{3}{4} \frac{1}{(1-x)} 
\] (32)

In a similar manner, the graph (c) in Fig.1 represents the following four graphs:
Fig. 3  Feynman graphs of type (c)

The T-matrix elements of the above four graphs are

\[ T \text{ - matrix elements in } R_\xi \text{ gauge} \]

\[ T_{c1} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt B(x_A, x_B, t) \cdot \left( t \cdot \frac{1}{(t + \xi)^2} - 2 \cdot \frac{1}{t + \xi} + \tilde{f}(x_A, x_B, t) \right) \]

\[ T_{c2} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt B(x_A, x_B, t) \cdot \left( -t \cdot \frac{1}{(t + \xi)^2} + \frac{1}{t + \xi} \right) \]
\[ T_{c3} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt B(x_A, x_B, t) \cdot \left( -t \cdot \frac{1}{(t + \xi)^2} + \frac{1}{t + \xi} \right) \]

\[ T_{c4} = \sum_{A=1}^{6} \sum_{B=1}^{6} \int_0^\infty dt B(x_A, x_B, t) \cdot \left( t \cdot \frac{1}{(t + \xi)^2} \right) \]

where

\[ B(x_A, x_B, t) = \frac{g^4}{64\pi^2 M_W^2} \frac{\mu(3)\gamma\mu}{2} \frac{1 - \gamma_5 e(2)}{2} \frac{\mu(4)\gamma\mu}{2} \frac{1 - \gamma_5 e(1)}{2} (V_{\mu A})^2 (V_{\nu B})^2 \]

\[ \cdot \frac{x_A x_B}{m_A \cdot m_B} \cdot \frac{\sqrt{x_A x_B}}{(t + x_A)(t + x_B)} \]

and

\[ \tilde{f}(x_A, x_B, t) = \frac{4t + x_A x_B(t + 2)}{(t + 1)^2} \]

In a similar fashion to the case for the graphs of type (a), all the \( \xi \) dependent terms again cancel against each other leaving only the \( \xi \) independent \( \tilde{f}(x_A, x_B, t) \) term. Thus the type (c) T-matrix element is gauge invariant and is given by

\[ T_c = \frac{G_F^2 M_W^2}{8\pi^2} [\bar{\mu}(3)\gamma\mu] [\bar{\mu}(4)\gamma\mu] \sum_{A=1}^{6} (V_{\mu A})^2 \sum_{B=1}^{6} (V_{\nu B})^2 \]

\[ \frac{\sqrt{x_A x_B}}{2} \cdot \int_0^\infty dt \left\{ \frac{4t + x_A x_B(t + 2)}{(t + x_A)(t + x_B)(t + 1)^2} \right\} \]

If \( x_A = x_B \), the relevant integral is

\[ I(x_A) = \frac{\sqrt{x_A x_B}}{2} \cdot \int_0^\infty dt \frac{4t + x_A x_B(t + 2)}{(t + x_A)(t + x_B)(t + 1)^2} \]

\[ = \frac{(x_A - 4)x_A}{(x_A - 1)^2} \ln x_A \]

while for \( x_A \neq x_B \), it takes the form

\[ K(x_A, x_B) = \frac{\sqrt{x_A x_B}}{2} \cdot \int_0^\infty dt \frac{4t + x_A x_B(t + 2)}{(t + x_A)(t + x_B)(t + 1)^2} \]

\[ = \frac{L(x_A, x_B) - L(x_B, x_A)}{x_A - x_B} \]
with

\[ L(x_A, x_B) = \frac{4 - x_A x_B}{2(x_A - 1)} + \frac{x_A(2x_B - x_A x_B - 4)}{2(x_A - 1)^2} \ln x_A \]  

The T-matrix element of graph (c) is thus secured as

\[ T_c = \frac{G^2 M_w^2}{8\pi^2} [\bar{\mu}(3)\gamma^\mu(1 - \gamma_5)e(2)] [\bar{\mu}(4)\gamma_\mu(1 - \gamma_5)e(1)] \]

\[ \left\{ \sum_{A=1}^6 (V_{\mu A} V_{e A})^2 I(x_A) + \sum_{A,B=1;A\neq B}^6 (V_{\mu A}^*)^2 (V_{e B}^*)^2 K(x_A, x_B) \right\} \]  

Combining the various contributions, the T-matrix element can be reproduced using the gauge invariant effective Lagrangian given by:

\[ \mathcal{L}_{\text{eff}} = \frac{G_{\bar{\mu}M}}{\sqrt{2}} [\bar{\mu}\gamma^\mu(1 - \gamma_5)e][\bar{\mu}\gamma_\mu(1 - \gamma_5)e] \]  

where

\[ \frac{G_{\bar{\mu}M}}{\sqrt{2}} = \frac{G^2 M_w^2}{16\pi^2} \left[ \sum_{A=1}^6 (V_{\mu A} V_{e A})^2 S(x_A) + \sum_{A,B=1;A\neq B}^6 (V_{\mu A} V_{e A})^* (V_{\mu B} V_{e B})^* T(x_A, x_B) \right] 

- \sum_{A=1}^6 (V_{\mu A} V_{e A})^2 I(x_A) - \sum_{A,B=1;A\neq B}^6 (V_{\mu A})^2 (V_{e B}^*)^2 K(x_A, x_B) \right] \]

\[ = -\frac{G^2 M_w^2}{16\pi^2} \left[ \sum_{A=1}^6 (V_{\mu A} V_{e A})^2 \left( S(x_A) - I(x_A) \right) \right. 

+ \left. \sum_{A,B=1;A\neq B}^6 \left( (V_{\mu A} V_{e A}) (V_{\mu B} V_{e B})^* T(x_A, x_B) - (V_{\mu A})^2 (V_{e B}^*)^2 K(x_A, x_B) \right) \right] \]

4 Limit on \( M_R \)

Muonium (antimuonium) is a nonrelativistic Coulombic bound state of an electron and an antimuon (positron and muon). The nontrivial mixing between the muonium (|\( M > \)) and antimuonium (|\( \bar{M} > \)) states is encapsulated in the effective Lagrangian of Eq. (40) and leads to the mass diagonal states given by the linear combinations

\[ |M_\pm > = \frac{1}{\sqrt{2(1 + |\epsilon|^2)}} [(1 + \epsilon)|M > \pm (1 - \epsilon)|\bar{M} >] \]

where

\[ \epsilon = \frac{\sqrt{M_{MM}^2} - \sqrt{M_{\bar{M}\bar{M}}}}{\sqrt{M_{MM}^2} + \sqrt{M_{\bar{M}\bar{M}}}} \]
\[ \mathcal{M}_{MM} = \frac{< M | - \int d^3 r L_{\text{eff}} | \bar{M} >}{\sqrt{< M | M > < \bar{M} | M >}}, \quad \mathcal{M}_{\bar{M}M} = \frac{< \bar{M} | - \int d^3 r L_{\text{eff}} | M >}{\sqrt{< \bar{M} | \bar{M} > < M | \bar{M} >}} \]  

(44)

Since the neutrino sector is expected to be CP violating, these will be independent, complex matrix elements. If the neutrino sector conserves CP, with \(| M >\) and \(| \bar{M} >\) conjugate states, then \(\mathcal{M}_{MM} = \mathcal{M}_{\bar{M}M}\) and \(\epsilon = 0\). In general, the magnitude of the mass splitting between the two mass eigenstates is

\[ |\Delta M| = 2 \left| \text{Re} \sqrt{\mathcal{M}_{MM} \mathcal{M}_{\bar{M}M}} \right| \]  

(45)

Since muonium and antimuonium are linear combinations of the mass diagonal states, an initially prepared muonium or antimuonium state will undergo oscillations into one another as a function of time. The muonium-antimuonium oscillation time scale, \(\tau_{\bar{M}M}\), is given by

\[ \frac{1}{\tau_{\bar{M}M}} = |\Delta M|. \]  

(46)

We would like to evaluate \(|\Delta M|\) in the nonrelativistic limit. A nonrelativistic reduction of the effective Lagrangian of Eq. (40) produces the local, complex effective potential

\[ V_{\text{eff}}(r) = 8 \frac{G_{\bar{M}M}}{\sqrt{2}} \delta^3(r) \]  

(47)

Taking the muonium (anitmuonium) to be in their respective Coulombic ground states, \(\phi_{100}(r) = \frac{1}{\sqrt{\pi a_{\bar{MM}}^3}} e^{-r/a_{\bar{MM}}}\), where \(a_{\bar{MM}} = \frac{1}{m_{\text{red}} a}\) is the muonium Bohr radius with \(m_{\text{red}} = \frac{m_e m_\mu}{m_e + m_\mu} \approx m_e\) the reduced mass of muonium, it follows that

\[ \frac{1}{\tau_{\bar{M}M}} = 2 \int d^3 r \phi^*_{100}(r) \text{Re} V_{\text{eff}}(r) \phi_{100} = 16 \frac{|\text{Re} G_{\bar{M}M}|}{\sqrt{2}} \frac{1}{a_{\bar{MM}}^3} \]  

(48)

Thus we secure an oscillation time scale

\[ \frac{1}{\tau_{\bar{M}M}} \approx \frac{16 |\text{Re} G_{\bar{M}M}|}{\pi} \frac{m_e^3 a^3}{\sqrt{2}} \]  

(49)

The present experimental limit[5] on the non-observation of muonium-antimuonium oscillation translates into the bound \(|\text{Re} G_{\bar{M}M}| \leq 3.0 \times 10^{-3} G_F\) where \(G_F \approx 1.16 \times 10^{-5} \text{GeV}^{-2}\) is the Fermi scale. This limit can then be used to construct a crude lower bound on \(M_R\). For the case when the neutrino masses arise from a see-saw mechanism and taking \(m_D\) to be of order \(M_W\), the \(M_R\) dependence of \(G_{\bar{M}M}\) is obtained from Eq. (41) as:

Case 1: \(|\text{Re} G_{\bar{M}M}| \sim \frac{G^2 \alpha^4 M_W^4}{M_R^2} \ln \frac{M_R}{M_W}, \quad A = 1, 2, 3, \quad B = 1, 2, 3\)
Case 2:  $|\text{Re}G_{MM}| \sim \frac{G^2_\tau M^4_W}{M^2_R} \ln \frac{M_R}{M_W}$,  \( A = 4, 5, 6 \),  \( B = 4, 5, 6 \)

Case 3:  $|\text{Re}G_{MM}| \sim \frac{G^2_\tau M^6_W}{M^4_R} \ln \frac{M_R}{M_W}$,  \( A = 1, 2, 3 \),  \( B = 4, 5, 6 \)  \( (50) \)

Case 1 and case 2 give the same order $M_R$ dependence, while case 3 is suppressed by an additional factor of $\frac{M^2_W}{M^2_R}$. Hence, the term $\frac{G^2_\tau M^4_W}{M^2_R} \ln \frac{M_R}{M_W}$ gives the dominant contribution. We then roughly calculate a bound of $M_R$ as

$$\frac{G^2_\tau M^4_W}{M^2_R} \ln \frac{M_R}{M_W} \leq 3.0 \times 10^{-3} G_F$$  \( (51) \)

which has also been obtained in reference [9]. Using $M_W \approx 80.4 \text{ GeV}$ and $G_F = 1.166 \times 10^{-5} \text{GeV}^{-2}$, we finally secure

$$M_R \geq 6 \times 10^2 \text{GeV}$$  \( (52) \)

Note that this is just a rough estimate since we are retaining only the dependence on $M_R$ while neglecting all numerical dependence on the mixing angles and CP violating phases in $V_{aa}$. 

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Appendix: Proof of Identity (22)

Using the definition of the mixing matrix \( V_{aA} = \sum_{c=1}^{3} (A_L^{-1})_{ac} U_{cA} \), one can write

\[
\sum_{A=1}^{6} V_{aA} V_{bA} m_{vA} = \sum_{A=1}^{6} \left( \sum_{c=1}^{3} (A_L^{-1})_{ac} U_{cA} \right) \cdot \left( \sum_{d=1}^{3} (A_L^{-1})_{bd} U_{dA} \right) \cdot m_{vA} \\
= \sum_{c=1}^{3} \sum_{d=1}^{3} (A_L^{-1})_{ac} \left( \sum_{A=1}^{6} U_{cAm_{vA}U_{dA}} \right) (A_L^{-1})_{bd}
\]

(53)

where \( m_{vA} \) are the diagonal elements of matrix \( M'_{\text{diag}} \),

\[
m_{vA} = (M'_{\text{diag}})_{AA}.
\]

Consequently, we can express \( \sum_{A=1}^{6} U_{cAm_{vA}U_{dA}} \) as a product of matrices and equation (53) takes the form

\[
\sum_{A=1}^{6} V_{aA} V_{bA} m_{vA} = \sum_{c=1}^{3} \sum_{d=1}^{3} (A_L^{-1})_{ac} \left( U M'_{\text{diag}} U^T \right)_{cd} (A_L^{-1})_{bd}
\]

(55)

Using equation (9), \( M'_{\text{diag}} = U^T M' U \), it follows that

\[
M'^* = U M'_{\text{diag}} U^T
\]

(56)

Substituting this result back into eq. (55) then gives

\[
\sum_{A=1}^{6} V_{aA} V_{bA} m_{vA} = \sum_{c=1}^{3} \sum_{d=1}^{3} (A_L^{-1})_{ac} (M'^*)_{cd} (A_L^{-1})_{bd}
\]

(57)

where

\[
M'^* = \begin{pmatrix}
0 & (m^D)^* \\
(m^D)^* & m^R*
\end{pmatrix}
\]

(58)

Since \( c \) and \( d \) both run from 1 to 3, \( (M'^*)_{cd} \) are the elements of the upper left \( 3 \times 3 \) block of matrix (58), which is zero. Hence, we secure the identity

\[
\sum_{A=1}^{6} V_{aA} V_{bA} m_{vA} = 0
\]

(59)

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1Here the unitary gauge is employed. Note that contrary to the claim there, the graphs of figures (c) and (d) do not cancel and the corrected result is provided in the current paper.

2Here the Feynman gauge is used to evaluate the $T$ matrix elements. The current paper reproduces these results.
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