Super-resolution Geometry Processing Technology for Ill-sampled Astronomical Images

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Abstract. In order to improve the cell sensitivity or to satisfy the needs of a large field of view, a large size CCD pixel is usually chosen as detection unit in optical imaging systems. However, this methodology cannot meet the Nyquist sampling theorem, and thus generate ill-sampling images. In other words, the geometric resolution of images in optical diffraction limited systems is directly restricted by the size of CCD pixel. In this paper, a carefully designed optical mask is implemented to ensure loseless images before CCD sampling. By applying spatial spectral filtering technology, we can acquire images with appropriate resolution. The method presented in this paper significantly abates the resolution decline due to ill-sampling. By mathematical deduction and simulation, the geometric super resolution images can be achieved.

1. Introduction

Optics systems are generally diffraction limited systems restricted by optical aperture. The system diffraction capacity is usually defined by the non-distorted image of the object through optical system. In actual case, limited by various factors such as the light transmission medium, target motion, atmospheric turbulence, CCD cell size, noises and so on, the image obtained by the optical system is blurry, and it cannot achieve diffraction capabilities.

According to Nadar [1], the strict sense of super-resolution is defined as: the resolving power that exceeds diffraction limitation of the optical system. In particular, when the optical transfer function (OTF) is at zero position, by mathematical or physical method, image that contains spatial frequencies is constructed. In many practical systems, in order to improve signal noise ratio in target detection and increase cell sensitivity, or to meet the needs of a large field of view, usually a larger size CCD pixel is chosen as a detection unit. This can not meet the Nyquist sampling theorem, and it belongs to ill-sampling category, in which the image resolution is serious declined. This paper mainly deals CCD pixel resolution degradation caused by ill-sampling, and achieves super-resolution processing, such super-resolution can be defined as the geometric super resolution [2-9].

In this paper, the mathematical characteristic of optical imaging is analyzed. By using spectral filtering mask optical technology, the problem that CCD pixel size does not meet resolution in Nyquist sampling theorem is improved. The processing result and the effect by simulation are finally shown.
2. Optical Resolution Description
The optical imaging system collects the optical signal of the target, and the signal converges on an imaging unit by an electro-optical conversion to form an image. The imaging element of single pixel size is $\Delta x$. According to Rayleigh theorem, image resolution of the optical imaging system is determined by the system focal length $f$ and the effective aperture $D$, the diffraction limit is $1.22\lambda f / D$, the angular resolution of the target is $\delta \phi = 1.22\lambda / D$, the effect is shown in Figure 1.

![Figure 1](image1.png)

**Figure 1.** A schematic diagram of the optical imaging system resolution

Assuming that the object distance is $R >> f$, the object plane of the target line resolution limit can be expressed as $(\delta x)_{\text{diff}} = 1.22\lambda R / D$. The image resolution of the optical imaging system is determined by $\Delta x$, then the actual line resolution calculated by the sample image can be expressed as $(\delta x)_{\text{geo}} = \Delta x R / f$. In most optical system, $\Delta x > 1.22\lambda f / D$, the imaging pixel size has become an important restricted factor for the resolution. For example, when $\Delta x = 30\mu m$, $f = 9m$, $R = 300km$, $D = 0.9m$, $(\delta x)_{\text{geo}} = 1m$, while the optical system limit line resolution to the target is $(\delta x)_{\text{diff}} \approx 0.25m$, the difference is about four times.

3. Optical Imaging Theory

3.1. Imaging Model
In simplified diffraction limited optical system, the imaging process is the target light from object space to the image space through the optical system. Typical imaging system is a telescope or a camera, as shown in Figure 2.

![Figure 2](image2.png)

**Figure 2.** Simplified model of diffraction limited optical system

The light reflected from the target surface enters the camera lens, and converges to the imaging focal plane. This can be expressed as:

$$g(x_i, y_i) = \iiint f(x_0, y_0) h(x_i - x_0, y_i - y_0) dx_0 dy_0$$  \hspace{1cm} (1)
Where, \((x_0, y_0)\) represents the object plane spatial coordinates, \((x_i, y_i)\) represents the imaging plane spatial coordinates, \((\xi, \eta)\) represent the pupil plane spatial coordinates, \(f(x_i, y_i)\) is the goal of the distribution function in the object space, \(g(x_i, y_i)\) is the goal of the distribution function, \(h(x_i-x_0, y_i-y_0)\) is the impulse response function (Point Spread Function).

According to Fourier optics principle and the impulse response function expression, the impulse response function is the function that represents the relative positional relationship between object space and image space, with the nature of space invariance, and it is determined by the pupil function of the system. Most optical systems consider impulse response function as time invariant, and diffraction limited system is a linear space-invariant system.

By Fraunhofer diffraction theory (Fraunhofer diffraction), the system impulse response function is represented as follows:

\[
h(x_i, y_i) = \frac{e^{i\lambda d_i}}{i\lambda d_i} \exp\left\{i\pi \left(\frac{u d_i}{\lambda d_i} (x_i^2 + y_i^2)\right)\right\} \int \int \exp\left\{-i2\pi \left(\frac{\xi x_i + \eta y_i}{\lambda d_i}\right)\right\} d\xi d\eta
\]  

(2)

Where, \(k = 2\pi / \lambda\). \(d_i\) is focal length.

Since the actual imaging processing is discrete digital signal processing, the discrete Fourier transform can be used to express the latter half of Eq.(2) as follows:

\[
P_{u,v} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} p_{m,n} \exp\{-j2\pi(Mu + \nu N/M\}) = DFT(p_{m,n})
\]  

(3)

\[
h(u d_x, v d_y) = \frac{e^{i\lambda d_i}}{i\lambda d_i} \exp\left\{i\pi \left(\frac{u d_x}{\lambda d_i} (u d_x)^2 + (v d_y)^2\right)\right\} DFT[p(m d_{\xi}, n d_{\eta})]
\]  

(4)

Where, \(d_x, d_y\) are the pixel sizes, \(u, v\) are number of pixels, \(d_{\xi}, d_{\eta}\) are the minimum interval for the pupil space, \(m, n\) are pupil spatial position number, \(M, N\) are the numbers of divisions for the pupil, \(d_i\) is the system focal length.

The above equation can be simplified as follows:

\[
h(u d_x, v d_y) = w \cdot DFT[p(m d_{\xi}, n d_{\eta})]
\]  

(5)

Eq. (1) is a circular convolution form, according to the convolution theorem, it can be transformed into the frequency domain as:

\[
G(f_{\xi}, f_{\eta}) = H(f_{\xi}, f_{\eta}) \cdot F(f_{\xi}, f_{\eta})
\]  

(6)

Where, \(f_{\xi}, f_{\eta}\) is the spatial frequency.

The formula \(d_x = \frac{\lambda d_i}{M d_{\xi}}\) can be converted to a regular expression, where focal length represented by \(f\), aperture diameter represented by \(D\), and the smallest pixel size represented by \(d_x = \frac{\lambda f}{D}\). The above derivation is based on the coherent imaging system. In practical incoherent imaging optical
system, in order to avoid the phase aliasing, according to the sampling theorem, the CCD pixel resolution is equivalent to the diffraction limit of the optical system when pixel size is selected as \( d_x/2 \).

4. Spatial Filter Design

With the analysis in Section 3, the process of digital discrete sampling in optical imaging system is clarified. To solve the problem that sampling pixel size is too large, the binary encoded optical mask is added to the front of the imaging. The sampling interval of optical mask meets the sampling theorem, and satisfies a certain relationship with the pixel size, which ensures CCD image frequency information is not lost.

Assume that optical encoder mask is expressed as:

\[
\tilde{M}(f_x, f_y) = \sum_{m} \sum_{n} \delta(f_x - mp, f_y - nq)
\]

(7)

Where, \( p, q \) are the frequency domain sampling interval, and \( m, n \) are sampling numbers.

Actually the optical encoder mask is a two-dimensional discrete sampling pulse function. When the input bandwidth of the target is \( (2f_{cx}, 2f_{cy}) \), then the total number of samples is \( M = 2f_{cx}/p, N = 2f_{cy}/q \).

The optical image through optical encoder mask is:

\[
\tilde{O}(f_x, f_y) = G(f_x, f_y)\tilde{M}(f_x, f_y)
\]

(8)

The above equation can be inversely Fourier transformed to get the time domain image as:

\[
O(x_i, y_i) = \phi^{-1}[\tilde{O}(f_x, f_y)] = \phi^{-1}[G(f_x, f_y)\sum_{m} \sum_{n} \delta(f_x - mp, f_y - nq)]
\]

(9)

Optical images are acquired by CCD pixels sampling with relative larger pixels, and it is ill-sampled. The mathematical expression of CCD sampling is:

\[
CCD(x_i, y_i) = \sum_{u} \sum_{v} \delta(x_i - u\Delta x, y_i - v\Delta y)
\]

(10)

Where \( \Delta x, \Delta y \) are the CCD pixel sizes.

The sampled image time-domain expression is:

\[
S(x_i, y_i) = O(x_i, y_i)\cdot CCD(x_i, y_i)
\]

\[
= \phi^{-1}[G(f_x, f_y)\sum_{m} \sum_{n} \delta(f_x - mp, f_y - nq)\sum_{u} \sum_{v} \delta(x_i - u\Delta x, y_i - v\Delta y)]
\]

(11)

The relationship between optical mask interval period \( p \) and CCD sampling interval \( (\Delta x, \Delta y) \) is set to:

\[
\Delta x = 1/(f_{cx} + p/2), \Delta y = 1/(f_{cy} + q/2)
\]

The Fourier transform of Eq. (11) is:

\[
\tilde{S}(f_x, f_y) = \phi \phi^{-1}[G(f_x, f_y)\sum_{m} \sum_{n} \delta(f_x - mp, f_y - nq)\sum_{u} \sum_{v} \delta(x_i - u\Delta x, y_i - v\Delta y)]
\]

(12)

According to the Fourier nature of periodic pulse function:

\[
\phi \delta \sum_{u} \sum_{v} \delta(x_i - u\Delta x, y_i - v\Delta y) = \sum_{u} \sum_{v} \delta(f_x - u(f_{cx} + p/2), f_y - v(f_{cy} + q/2))
\]

(13)

According to the Fourier nature of convolution, Eq. (12) can be rewritten as follows:

\[
\tilde{S}(f_x, f_y) = (G(f_x, f_y)\sum_{m} \sum_{n} \delta(f_x - mp, f_y - nq)) \otimes \sum_{u} \sum_{v} \delta(f_x - u(f_{cx} + p/2), f_y - v(f_{cy} + q/2))
\]

(14)
According to the definition of convolution, Eq. (14) becomes:

\[
\tilde{S}(f_x, f_y) = \sum_m \sum_n G[f_x - u(f_{cx} + p/2), f_y - v(f_{cy} + q/2)]
\]

\[
\sum_m \sum_n \delta[f_x - mp - u(f_{cx} + p/2), f_y - nq - v(f_{cy} + q/2)]
\]

Eq. (15)

In order to obtain an image signal without aliasing, spatial frequency filter should be applied, and the mathematical characterization of filter and coding should be equivalent. After processing, the image spectrum can be obtained as follows:

\[
\tilde{R}(f_x, f_y) = \tilde{S}(f_x, f_y) \sum_k \sum_l \delta[f_x - kp, f_y - lq]
\]

\[
= \sum_m \sum_n G[f_x - uf_{cx} - up/2, f_y - vf_{cy} - vq/2] \sum_m \sum_n \delta[f_x - uf_{cx} - up/2 - mp, f_y - vf_{cy} - vq/2 - np]
\]

\[
\sum_k \sum_l \delta[f_x - kp, f_y - lq]
\]

Eq. (16)

Eq. (16) contains two Dirac function, and the first function samples at the position of the second function. Eq. (16) can be simplified as:

\[
\tilde{R}(f_x, f_y) = \tilde{S}(f_x, f_y) \sum_k \sum_l \delta[f_x - kp, f_y - lq]
\]

\[
= \sum_m \sum_n G[f_x - uf_{cx} - up/2, f_y - vf_{cy} - vq/2] \sum_m \sum_n \delta[f_x - uf_{cx} - up/2 - mp, f_y - vf_{cy} - vq/2 - np]
\]

\[
\sum_k \sum_l \delta[f_x - kp, f_y - lq]
\]

Eq. (17)

According to Eq. (17), if \( u, v \) are even numbers, the meaning of the above formula is a couple of copies of information in function \( G[f_x - uf_{cx} - up/2, f_y - vf_{cy} - vq/2] \). In order to obtain the copy of function center position, \( u, v \) are set to 0, then the target spectrum information can be acquired by the following equation:

\[
\tilde{R}(f_x, f_y) = \sum_m \sum_n G(f_x, f_y) \delta(f_x - mp, f_y - np)
\]

\[
= \sum_m \sum_n G(f_x, f_y) \delta(f_x - mp, f_y - np)
\]

Eq. (18)

Spectrum sample is determined by encoder interval \( p \), and the target original information can be obtained by \( \tilde{R}(f_x, f_y) \).

Wiener filtering technique can be used to Eq. (18), and images with better resolution can be obtained.

5. Simulation result
We make simulation of the above procedure and get satisfied result. We use the camera with resolution of 64×64, a single pixel size of 24µm×24µm, and optical encoder mask resolution is 128×128. Figure 3 and Figure 4 show the final effect.

Figure 3. Original input image
Figure 4. Optical mask plate scale
From the comparison between Figure 5 and Figure 6, we can see that the resolution increases four times.

**6. Conclusion**

This paper analyses the resolution of ill-sampled image. The feasibility of optical masks method and spatial filtering method is theoretically deduced. The simulation results verify the correctness of proposed method and show satisfied effect. The paper provides referenced technical route to improve resolution for ill-sampled images.

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