The distribution of $G$-Weyl CM fields and the Colmez conjecture

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Abstract

Let $G$ be a transitive subgroup of $S_d$, and let $E$ be a CM field of degree $2d$ with a maximal totally real $G$-field. If the Galois group of the Galois closure of $E$ is isomorphic to the wreath product $C_2 \wr G$, then we say that $E$ is a $G$-Weyl CM field. Let $N_{2d}^{\text{Weyl}}(X, G)$ count the $G$-Weyl CM fields $E$ of degree $2d$ with discriminant $|d_E| \leq X$, and define

$$ N_{2d}^{\text{Weyl}}(X) = \sum_{G \leq S_d} N_{2d}^{\text{Weyl}}(X, G). $$

Further, let $N_{2d}^{\text{cm}}(X)$ count the CM fields $E$ of degree $2d$ with discriminant $|d_E| \leq X$. Assuming a weak form of the upper bound in Malle’s conjecture which is known to be true in many cases, and that at least one totally real degree $d$ $G$-field exists, we prove that

$$ \frac{N_{2d}^{\text{Weyl}}(X, G)}{N_{2d}^{\text{cm}}(X)} = C(d, G) + O(X^{-\alpha(d, G)}) $$

and

$$ \frac{N_{2d}^{\text{Weyl}}(X)}{N_{2d}^{\text{cm}}(X)} = 1 + O(X^{-\beta(d)}) \quad (0.1) $$

for some explicit positive constants $C(d, G), \alpha(d, G),$ and $\beta(d)$. We apply these results to study the Colmez conjecture. More precisely, using the (recently proved) averaged Colmez conjecture, we show that the Colmez conjecture is true for $G$-Weyl CM fields. Then, combined with (0.1) we conclude that the Colmez conjecture is true for an asymptotic density of $100\%$ of CM fields of degree $2d$; in other words, the Colmez conjecture is true for a random CM field.

1 Introduction and statement of results

The distribution of number fields with prescribed Galois group has been studied extensively over the last two decades, spurred in part by very precise conjectures of Malle [27, 28].
for the asymptotic growth of the corresponding counting functions. In this paper, we will study this distribution problem for CM fields.

Recall that a CM field $E$ of degree $2d$ is a totally imaginary quadratic extension of a totally real field $F$ of degree $d$ over $\mathbb{Q}$. Let $E^c$ be the Galois closure of $E$ and $S_d$ be the symmetric group. Then the Galois group $\text{Gal}(E^c/\mathbb{Q})$ embeds as a subgroup of the wreath product $C_2 \wr S_d$. In their study of special points on Shimura varieties, Chai and Oort [9] defined $E$ to be a Weyl CM field if $\text{Gal}(E^c/\mathbb{Q}) \cong C_2 \wr S_d$. The Weyl CM fields are associated to special CM points on the moduli space of principally polarized abelian varieties of dimension $d$ called Weyl CM points.

Now, let $G$ be the transitive subgroup of $S_d$ such that $\text{Gal}(F^c/\mathbb{Q}) \cong G$. Then $\text{Gal}(E^c/\mathbb{Q})$ embeds as a subgroup of the wreath product $C_2 \wr G$ (see Proposition 3.2). This is a refinement of the above mentioned embedding into $C_2 \wr S_d$. We define $E$ to be a $G$-Weyl CM field if $\text{Gal}(E^c/\mathbb{Q}) \cong C_2 \wr G$. In other words, $E^c$ has maximal possible Galois group, subject to the restriction that its maximal totally real subfield $F^c$ have Galois group $G$. With this terminology, a Weyl CM field in the sense of Chai and Oort is an $S_d$-Weyl CM field.

Assuming a weak form of the upper bound in Malle’s conjecture which is known to be true in many cases (see Hypothesis 1.2 and Corollary 1.4), we will prove an asymptotic formula with a power-saving error term for the number of $G$-Weyl CM fields $E$ of degree $2d$ with discriminant $|d_E| \leq X$. We also assume that there exists at least one totally real degree $d$ field $F$ with $\text{Gal}(F^c/\mathbb{Q}) \cong G$, to ensure that we are not counting the empty set.

Our asymptotic formula implies that for any fixed choice of transitive subgroup $G \leq S_d$, a positive proportion of CM fields of degree $2d$ are $G$-Weyl, and moreover, that as $G$ ranges over the transitive subgroups of $S_d$, these fields collectively comprise an asymptotic density of 100% of CM fields of degree $2d$ (see Theorem 1.11). It will also be seen that each subfield $F$ contributes a positive proportion to the asymptotics.

Remark 1.1 In [33, p. 5], Oort suggests that it is likely that “most CM fields are $[S_d]$-Weyl CM fields”; but this heuristic doesn’t hold here. For example, the $S_3$-Weyl CM fields comprise approximately 31% of CM fields of degree 6, while the $C_3$-Weyl CM fields comprise approximately 69% of CM fields of degree 6. Similarly, the $S_4$-Weyl CM fields comprise approximately 20% of CM fields of degree 8, while the $D_4$-Weyl CM fields comprise approximately 48% of CM fields of degree 8. See Table 1 for these statistics.

Roughly speaking, this is because the asymptotics are ‘explained’ more by the quadratic extensions $E/F$ than by the degree $d$ extensions $F/\mathbb{Q}$. Similar phenomena can be seen in the work of Klüners [25] on which ours is based, as well as subsequent works such as [1, 2, 30].

Our approach to counting $G$-Weyl CM fields is based on work of Klüners [25], which established asymptotics for the counting function of number fields with Galois group $C_2 \wr G$, without signature conditions. We will adapt Klüners’ work to handle the signature conditions needed to count CM fields. We also give power-saving error terms which incorporate recent progress on non-trivial bounds for 2-torsion in class groups of number fields [8] and subconvexity bounds for ray class $L$–functions of totally real fields [18], and determine the weakest form of the upper bound in Malle’s conjecture needed for our results.

We next discuss the connection between the distribution of $G$-Weyl CM fields and a conjecture of Colmez [12] which relates the Faltings height of a CM abelian variety to
logarithmic derivatives of Artin $L$–functions at $s = 0$. In fact, this paper was motivated in part by our effort to answer the following:

**Question.** Is the Colmez conjecture true for a random CM field?

To address this problem, the first two authors [4, Section 1.2] developed a plan to study the Colmez conjecture from an arithmetic statistical point of view. Using the averaged Colmez conjecture, which was proved independently by Andreatta, Goren, Howard, and Madapusi Pera [3], and Yuan and Zhang [43], the first two authors [4, Theorem 1.4] showed that the Colmez conjecture is true for $S_d$-Weyl CM fields. Then, they [4, Theorem 1.9] applied work of Cohen, Diaz y Diaz, and Olivier [10] to conclude that 100% of quartic CM fields are $S_2$-Weyl, and consequently, satisfy the Colmez conjecture.

Due to the well known difficulties which arise when counting number fields with Galois group $S_d$, this line of attack seemed limited initially to CM fields of small degree. Here we overcome these difficulties by first using the averaged Colmez conjecture and the Galois theory of CM fields to prove that the Colmez conjecture is true for any $G$-Weyl CM field. Then, combined with our distribution results, we will conclude (conditional on Hypothesis 1.2) that the Colmez conjecture is true for 100% of CM fields of degree $2d$; in other words, the Colmez conjecture is true for a random CM field (see Theorem 1.18). Moreover, given the pairs $(d, G)$ for which Hypothesis 1.2 is known unconditionally, we will produce infinitely many density-one families of non-abelian CM fields which satisfy the Colmez conjecture (see the results of Sect. 1.2).

### 1.1 The distribution of $G$-Weyl CM fields

In order to state our distribution results, we first define some of the counting functions that will be used throughout the paper (all number fields are counted up to $\mathbb{Q}$-isomorphism).

- Let $N_d(X, G)$ count all number fields $K$ of degree $d$ and discriminant $|d_K| \leq X$ with $\text{Gal}(K^c/\mathbb{Q}) \cong G$. Here $G$ is a transitive subgroup of $S_d$, and the isomorphism is of permutation groups where $\text{Gal}(K^c/\mathbb{Q})$ acts on the homomorphisms $K \hookrightarrow K^c$, as in [38].
- Let $N_{cm}^{2d}(X, G)$ count all CM fields $E$ of degree $2d$ and discriminant $|d_E| \leq X$ which have a maximal totally real subfield $F$ with $\text{Gal}(F^c/\mathbb{Q}) \cong G$.
- Let $N_{2d}^{\text{Weyl}}(X, G)$ count all CM fields $E$ of degree $2d$ and discriminant $|d_E| \leq X$ which are $G$-Weyl.
- Let
  $$N_{2d}^{\text{Weyl}}(X) := \sum_{G \leq S_d} N_{2d}^{\text{Weyl}}(X, G).$$
- Let $N_{2d}^{\text{cm}}(X)$ count all CM fields $E$ of degree $2d$ and discriminant $|d_E| \leq X$.

In [27], Malle gave conjectural bounds for $N_d(X, G)$. If $g \in S_d$, the *index* of $g$ is defined by

$$\text{ind}(g) := d - \#([1, \ldots, d]/(g)).$$

Let

$$\text{ind}(G) := \min\{\text{ind}(g) : 1 \neq g \in G\}$$
and define the constant $0 < a(G) := \text{ind}(G)^{-1} \leq 1$. Malle conjectured that for any $\epsilon > 0$, there exist positive constants $c_1(G), c_2(G, \epsilon)$ such that

$$c_1(G)X^{a(G)} \leq N_d(X, G) \leq c_2(G, \epsilon)X^{a(G)+\epsilon}.$$  \hfill (1.1)

Malle [28] later refined this and gave a precise conjectural asymptotic formula for $N_d(X, G)$ as $X \to \infty$.

For our purposes, we only need an upper bound for $N_d(X, G)$ with an exponent which is much weaker than what is predicted by (1.1). This exponent will depend on bounds for 2-torsion in class groups.

For a number field $K$, let $\text{Cl}(K)[2]$ be the 2-torsion subgroup of the ideal class group. Let $\delta_d \geq 0$ be a variable such that

$$|\text{Cl}(K)[2]| \ll d_K^{\delta_d + \epsilon}$$  \hfill (1.2)

for all number fields $K$ of degree $d$. By the Brauer-Siegel theorem, the bound (1.2) holds with $\delta_d = 1/2$. Any bound (1.2) with $0 < \delta_d < 1/2$ is called a non-trivial bound, and $\delta_d = 0$ is the conjectured optimal bound.

If $d = 2$, then it is a classical result that (1.2) holds with $\delta_2 = 0$. The first non-trivial bounds in (1.2) for $d \geq 3$ were recently proved by Bhargava, Shankar, Taniguchi, Thorne, Tsimerman, and Zhang [8]. In particular, they proved that if $d = 3, 4$, then (1.2) holds with $\delta_d = 0.2784$, and if $d \geq 5$, then (1.2) holds with $\delta_d = 1/2 - 1/2d$.

With the variable $\delta_d$ as in (1.2), we state the following weak form of the upper bound in Malle’s conjecture (1.1).

**Hypothesis 1.2** For a fixed pair $(d, G)$ and $0 \leq \delta_d \leq 1/2$ satisfying (1.2), we have

$$N_d(X, G) \ll X^{M(G)}$$  \hfill (1.3)

for some $M(G) > 0$ such that

$$\delta_d + M(G) < 2$$

Moreover, there exists at least one totally real field $K$ of degree $d$ with $\text{Gal}(K^c/\mathbb{Q}) \cong G$.

Our first result gives an asymptotic formula with a power-saving error term for the density of those CM fields counted by $N_{2d}^{\text{cm}}(X, G)$ which are $G$-Weyl.

**Theorem 1.3** Assume that Hypothesis 1.2 is true for $(d, G)$. Then

$$\frac{N_{2d}^{\text{Weyl}}(X, G)}{N_{2d}^{\text{cm}}(X, G)} = 1 + O_{d,G,\epsilon}(X^{-C_1(\delta_d, M(G))+\epsilon}),$$  \hfill (1.4)

where

$$C_1(\delta_d, M(G)) := \begin{cases} 1/2, & \text{if } \delta_d + M(G) \leq 1 \\ 1 - \frac{\delta_d + M(G)}{2}, & \text{if } 1 < \delta_d + M(G) < 2 \end{cases}$$  \hfill (1.5)
The Hypothesis 1.2 is known in many cases, including work appearing in the following papers on the Malle conjectures: \cite{1, 2, 5–7, 10, 11, 14, 19, 22, 23, 25, 26, 29, 30, 37}. For convenience, we have summarized some of the more easily stated results in Table 2 of Sect. 4. (See also Remark 1.8.)

Given these known cases of Hypothesis 1.2, we get the following unconditional results.

**Corollary 1.4** The asymptotic formula (1.4) holds unconditionally for the following pairs \((d, G)\):

- Any \((d, G)\) with \(G\) abelian.
- Any \((d, G)\) with \(d = \ell\) prime and \(G = D_\ell\) dihedral.
- Any \((d, G)\) with \(G\) nilpotent.
- Any \((d, G)\) with \(d \geq 5\) and \(|G| = d\), for which at least one totally real degree \(d\) \(G\)-extension exists.
- Any \((d, G)\) with \(d \leq 5\).
- Any \((d, G)\) with \(d = k|A|\) \((k = 3, 4, 5)\) and \(G = S_k \times A\) with \(A\) abelian.
- Any \((2d, C_2 \wr G)\), when \((d, G)\) is on this list.

**Remark 1.5** In the 4th bullet of Corollary 1.4, the condition \(|G| = d\) is equivalent to all number fields counted by \(N_d(X, G)\) being Galois over \(\mathbb{Q}\). This case follows from Ellenberg and Venkatesh \cite[Proposition 1.3]{19}.

**Remark 1.6** The Malle conjecture can be formulated more generally for degree \(d\) extensions \(L/K\) of any global field \(K\) with \(N_{\mathbb{Q}/K}(\delta_L/K) < X\) and Galois group \(\text{Gal}(L/K) \cong G\) for a transitive subgroup \(G \leq S_d\) (here \(\delta_L/K\) is the relative discriminant). In this setting, Ellenberg, Tran, and Westerland \cite{20} recently proved the upper bound in Malle’s Conjecture when \(K = \mathbb{F}_q(t)\) is the rational function field.

**Example 1.7** If \((d, G) = (5, S_5)\), then Hypothesis 1.2 is true for the pair \((\delta_5, M(S_5)) = (2/5, 1)\). Since \(C_1(2/5, 1) = 3/10\), we have

\[
\frac{N_{10}^{\text{Weyl}}(X, S_5)}{N_{10}^{\text{cm}}(X, S_5)} = 1 + O_\epsilon(X^{-\frac{1}{m+\epsilon}}).
\]

**Remark 1.8** Work of Mehta \cite{30} and Alberts \cite{1} establishes Corollary 1.4 for additional infinite families, including Frobenius groups with abelian kernel, and many solvable groups.

The list above is not intended to be exhaustive, and additional cases could likely be proved with modest effort. For example, as Wang explained to us, her method can handle additional cases such as \(d = 9, G = S_3 \times S_3\). Moreover, Alberts, Lemke Oliver, Wang, and Wood have informed us of forthcoming work \cite{2} handling many more cases of this general type.

**Remark 1.9** Assuming a sufficiently strong value for the exponent \(\delta_d\) appearing in the 2-torsion bound (1.2), the asymptotic formula (1.4) is known for some further pairs \((d, G)\); see Table 3.

Our next goal is to give an asymptotic formula with a power-saving error term for the density of those CM fields counted by \(N_{2d}^{\text{cm}}(X)\) which are \(G\)-Weyl. This will involve the subconvexity problem for a certain family of Hecke \(L\)-functions for totally real fields.
Let $F$ be a totally real field of degree $d$. Let $c$ be an integral ideal of $F$ dividing 2, and let $c_{\infty} \subset m_{\infty}$ be a subset of the set $m_{\infty}$ of real places of $F$. Suppose that $\chi$ is a primitive character of the ray class group $Cl_{c_{\infty}}(F)$ modulo $c_{\infty}$. The $L$–function of $\chi$ is defined by

$$L_F(\chi, s) := \prod_p (1 - \chi(p)N_F/Q(p)^{-s})^{-1}, \quad \text{Re}(s) > 1.$$  

The completed $L$–function is defined by (see e.g. [21, p. 129])

$$\Lambda_F(\chi, s) := q(\chi)^{s/2} \gamma(\chi, s) L_F(\chi, s),$$

where $q(\chi) := d_F N_F/Q(c^2)$ and

$$\gamma(\chi, s) := \pi^{-ds/2} \Gamma\left(\frac{s}{2}\right)^{d/c_{\infty}} \Gamma\left(\frac{s + 1}{2}\right)^{d/c_{\infty}}.$$  

The completed $L$–function satisfies the functional equation

$$\Lambda_F(\chi, s) = \epsilon(\chi) \Lambda_F(\chi, 1 - s),$$

where the root number $\epsilon(\chi)$ is a complex number of modulus 1 which can be written explicitly as a normalized Gauss sum for $\chi$. Given this data, we calculate the analytic conductor of $L_F(\chi, s)$ as (a slightly weaker version of [21, eq. (5.7)])

$$q(F, \chi, s) = d_F N_F/Q(c^2)(|s| + 4)^d.$$  

Let $\delta' \geq 0$ be a variable such that

$$\left(\frac{s - 1}{s + 1}\right)^{\alpha(\chi)} L_F(\chi, s) \ll_{\delta,d} q(F, \chi, s)^{\delta'(1 - \sigma) + \epsilon}, \quad 1/2 \leq \sigma := \text{Re}(s) \leq 1 + \epsilon, \quad (1.6)$$

where $\alpha(\chi) = 1$ if $\chi$ is the trivial character and $\alpha(\chi) = 0$ if $\chi$ is non-trivial. The bound (1.6) holds when $\delta' = 1/2$ (the convexity bound). Any bound (1.6) with $0 < \delta' < 1/2$ is called a subconvexity bound, and $\delta' = 0$ is the Lindelöf hypothesis. For more details concerning these facts, see [21, Chapter 5].

**Remark 1.10** A subconvexity bound of the form (1.6) is known for $L_F(\chi, s)$ under certain conditions on the character $\chi$; see the discussion following Hypothesis A.2 on p. 881 of [18].

Now, let $D_{G}^{cm}(\chi)$ be the Dirichlet series which enumerates all fields counted by $N_{G}^{cm}(X, G)$ (see (2.3)). In Theorem 2.2, we will prove that if Hypothesis 1.2 is true for $(d, G)$ and $0 \leq \delta' \leq 1/2$ satisfies (1.6), then $D_{G}^{cm}(s)$ has a meromorphic continuation to a half-plane $\text{Re}(s) > \alpha$ for some $\alpha < 1$ (depending on $\delta_d, M(G)$ and $\delta'$) with only a single (simple) pole at $s = 1$. Moreover, the residue of $D_{G}^{cm}(s)$ at $s = 1$ is given by the convergent series

$$r_d(G) := \sum_{F \in \mathcal{F}_G^{\bot}} \frac{\operatorname{Res}_{s=1} \zeta_F(s)}{2^d d^2 \xi_F(2)} > 0, \quad (1.7)$$
where

\[ \mathcal{F}_G^+ := \{ F/Q : F \text{ totally real of degree } d, \ \text{Gal}(F^e/Q) \cong G \}. \]

Using properties of the Dirichlet series \( D_{cm}^G(s) \) and an upper bound for the number of CM fields counted by \( N_{2d}^{cm}(X) \) which are not G-Weyl for any transitive subgroup \( G \leq S_d \), we will prove the following asymptotic formulas with power-saving error terms.

**Theorem 1.11** Assume that Hypothesis 1.2 is true for every pair \( (d, G) \) where \( G \) ranges over all transitive subgroups \( G \leq S_d \). Moreover, assume that \( 0 \leq \delta' \leq 1/2 \) satisfies (1.6). Then for any such \( G \leq S_d \), we have

\[
\frac{N_{2d}^{Weyl}(X, G_0)}{N_{2d}^{cm}(X)} = \frac{r_d(G_0)}{\sum_{G \leq S_d} r_d(G)} + O_{d,G_0,\epsilon}(X^{-C_2(\delta_d, M(G_0), \delta')} + \epsilon),
\]

(1.8)

and

\[
\frac{N_{2d}^{Weyl}(X)}{N_{2d}^{cm}(X)} = 1 + O_{d,\epsilon}(X^{-C_3(\delta_d, \delta')} + \epsilon),
\]

(1.9)

where \( C_2(\delta_d, M(G), \delta') > 0 \) and \( C_3(\delta_d, \delta') > 0 \) are explicit constants defined in (2.20) and (2.21), respectively.

In Table 1 we give numerical computations for the residue \( r_d(G) \), and hence for the relative density of G-Weyl CM fields, for each transitive \( G \leq S_d \) with \( d \leq 5 \). We computed these by summing the series of (1.7) over the first \( n \) fields \( F \in \mathcal{F}_G^+ \), for \( n \) listed in the table. The basic field data was downloaded from the website lmfdb.org [35], and the remaining computations, including the \( L \)-function computations in (1.7), were handled with PARI/GP [36]. The (short) PARI/GP source code with which we put these computations together may be downloaded at the third author’s website.\(^1\)

From (1.7) we see that the residues are (very) approximately given by \( 2^{-d} \sum_F d_F^{-2} \). Assuming Malle’s conjecture (1.1), the series converge relatively rapidly; and indeed it is known that \( N_d(X, G) \ll X \) for all \( (d, G) \) listed in the table. With some effort, it should be possible to explicitly bound the error in our residue computations below; numerics suggest that these values are likely to be accurate within approximately \( \pm 1 \) in the least significant digit listed.

We observe: each totally real \( F \) contributes a positive proportion to its respective residue, with those of smallest discriminant making the largest contribution; also, the residues are decreasing with \( d \) – a pattern which should persist, in light of lower bounds on \( d_F \) which are exponential in \( d \) [32].

**Example 1.12** Let \( (d, G) \) be any pair with \( d = 5 \). Then Hypothesis 1.2 is true for the pair \( (\delta_5, M(G)) = (2/5, 1) \). If we take \( \delta' = 1/2 \), then we may take

\[
C_1(\delta_5, M(G)) = \frac{3}{10}, \quad \alpha(\delta_5, M(G), \delta') = \frac{19}{25}, \quad \beta(\delta_5, \delta') = \frac{17}{20},
\]

\[
C_2(\delta_d, M(G), \delta') = \frac{3}{20}, \quad C_3(\delta_d, \delta') = \frac{3}{20},
\]

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\(^1\)https://thornef.github.io
Table 1 Values of $r_d(G)$ for $d \leq 5$

| $d$ | $G$ | Number of fields | Minimal discriminant | Residue | Proportion in (1.8) |
|-----|-----|------------------|----------------------|---------|-------------------|
| 2   | $C_2$ | 100,000         | 5                    | 0.009856 | –                 |
| 3   | $C_3$ | 25,000          | 49                   | $3.30 \times 10^{-5}$ | 0.69     |
|     | $S_3$ | 107             | 49                   | $2.29 \times 10^{-5}$ | 0.31     |
|     | $C_4$ | 24,893          | 148                  | $1.01 \times 10^{-5}$ | –        |
| 4   | $C_4$ | 25,000          | 725                  | $1.24 \times 10^{-7}$ | –        |
|     | $V_4$ | 75              | 1125                 | $2.41 \times 10^{-8}$ | 0.19     |
|     | $D_4$ | 289             | 1600                 | $1.56 \times 10^{-8}$ | 0.13     |
|     | $A_4$ | 8147            | 725                  | $5.9 \times 10^{-8}$ | 0.48     |
|     | $S_4$ | 45              | 26569                | $9.3 \times 10^{-11}$ | 0.0008   |
| 5   | $C_5$ | 25,000          | 14641                | $1.05 \times 10^{-10}$ | –        |
|     | $D_5$ | 5               | 14641                | $3.08 \times 10^{-11}$ | 0.29     |
|     | $F_5$ | 28              | 160801               | $4.24 \times 10^{-13}$ | 0.003    |
|     | $A_5$ | 15              | 2382032              | $9 \times 10^{-15}$ | 0.00009  |
|     | $S_5$ | 21              | 3104644              | $5 \times 10^{-15}$ | 0.00005  |

these constants being defined in (1.5), (2.4), (2.11), (2.20), and (2.21) respectively. We conclude that

$$\frac{N_{W}^{\text{Weyl}}(X)}{N_{\text{cm}}^{\text{Weyl}}(X)} = 1 + O_{\epsilon}(X^{-\frac{1}{4}+\epsilon}).$$

Moreover, if we assume the conjectured optimal bound in (1.2) and the Lindelöf hypothesis in (1.6) (so that Hypothesis 1.2 is true for the pair $(\delta_5, M(G)) = (0, 1)$ and $\delta' = 0$), then

$$\frac{N_{W}^{\text{Weyl}}(X)}{N_{\text{cm}}^{\text{Weyl}}(X)} = 1 + O_{\epsilon}(X^{-\frac{1}{4}+\epsilon}).$$

Since Hypothesis 1.2 is known for every pair $(d, G)$ with $d \leq 5$ (see Corollary 1.4), we get the following unconditional result.

**Corollary 1.13** If $d \leq 5$, then (1.8) and (1.9) hold unconditionally. In particular, if $d \leq 5$, then the set

$$\bigcup_{G \leq S_d} \{G\text{-Weyl CM fields of degree } 2d\}$$

of all Weyl CM fields of degree $2d$ comprises 100% of all CM fields of degree $2d$.

1.2 Application to the Colmez conjecture

In this section we give some applications of our counting results to the Colmez conjecture.

The following result shows that the Colmez conjecture is true for $G$-Weyl CM fields.

**Theorem 1.14** If $E$ is a $G$-Weyl CM field, then the Colmez conjecture is true for $E$. In particular, if $X$ is an abelian variety of dimension $d$ with complex multiplication by a $G$-Weyl CM field $E$ of degree $2d$ with maximal totally real subfield $F$, then the Faltings height of $X$ is given by

$$h_{\text{Fal}}(X) = -\frac{1}{2} \frac{L'(\chi_{E/F}, 0)}{L(\chi_{E/F}, 0)} - \frac{1}{4} \log \left( \frac{|d_E|}{d_F} \right) - \frac{d}{2} \log(2\pi),$$

(1.10)
where $L(\chi_{E/F}, s)$ is the $L$–function of the Hecke character $\chi_{E/F}$ associated to the quadratic extension $E/F$.

**Remark 1.15** As discussed, in [4, Theorem 1.4] the first two authors proved that the Colmez conjecture is true for $S_d$-Weyl CM fields. Theorem 1.14 extends this result to any $G$-Weyl CM field.

Note also that if $E$ is a $G$-Weyl CM field of degree $2d \geq 4$, then $E/Q$ is non-abelian; see Remark 3.1. Hence Theorem 1.14 establishes many new cases of the Colmez conjecture for non-abelian CM fields.

With Theorem 1.14 in hand, we can now apply our counting results to the Colmez conjecture.

First, we have the following result which is an immediate consequence of Theorems 1.14 and 1.3.

**Theorem 1.16** Assume that Hypothesis 1.2 is true for the pair $(d, G)$. Then the Colmez conjecture is true for $100\%$ of CM fields $E$ of degree $2d$ which have a maximal totally real subfield $F$ with Galois group $\text{Gal}(F^c/Q) \cong G$.

Next, by combining Theorem 1.16 with Corollary 1.4, we get the following unconditional result.

**Corollary 1.17** If $(d, G)$ is any of the pairs in Corollary 1.4, then the Colmez conjecture is true for $100\%$ of CM fields $E$ of degree $2d$ which have a maximal totally real subfield $F$ with Galois group $\text{Gal}(F^c/Q) \cong G$.

Similarly, the following result is an immediate consequence of Theorems 1.14 and 1.11.

**Theorem 1.18** Assume that Hypothesis 1.2 is true for every pair $(d, G)$ where $G$ ranges over all transitive subgroups $G \leq S_d$. Then the Colmez conjecture is true for $100\%$ of CM fields of degree $2d$.

Finally, since Hypothesis 1.2 holds for every pair $(d, G)$ with $d \leq 5$ (as observed previously), we get the following unconditional result.

**Corollary 1.19** If $d \leq 5$, then the Colmez conjecture is true for $100\%$ of CM fields of degree $2d$.

**Remark 1.20** The Colmez conjecture is now known to be true for quartic CM fields, sextic CM fields, and many degree 10 CM fields (see e.g. [42]).

We conclude by briefly summarizing some results on the Colmez conjecture. Colmez [12] proved his conjecture for abelian CM fields, up to an error term which was eliminated by Obus [31]. Yang [39–41] proved the Colmez conjecture for a large class of quartic CM fields, including the first non-abelian cases. The averaged Colmez conjecture [3, 43] made it possible to deduce many new cases of the Colmez conjecture. For example, the first two authors [4] proved that if $F$ is any totally real number field of degree $d \geq 3$, then there are infinitely many effectively constructible, positive density sets of CM extensions $E/F$ such that $E/Q$ is non-abelian and the Colmez conjecture is true for $E$. Yang and Yin [42] proved that if $E$ is a CM field of the form $E = FK$ where $K = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic
field and \( \text{Gal}(F^c/\mathbb{Q}) \) is isomorphic to either \( S_d \) or \( A_d \), then the Colmez conjecture is true for \( E \). This follows from a more refined result of these authors \([42]\) which shows that the Colmez conjecture is true for CM types \( \Phi \) of \( E = FK \) of signature \((d - 1, 1)\). Parenti \([34]\) proved that if \( E \) is a CM field of the form \( E = FK \) and \( \text{Gal}(F^c/\mathbb{Q}) \cong \text{PSL}_2(\mathbb{F}_q) \) then the Colmez conjecture is true for \( E \).

2 Proof of Theorems 1.3 and 1.11

In this section we prove Theorems 1.3 and 1.11, following closely Klüners’s Dirichlet series approach \([25]\).

Let \( N_{c_2}^\text{cm}(X, G) \) count all CM fields \( E \) of degree 2 and discriminant \(|d_E| \leq X\) which have a maximal totally real subfield \( F \) with \( \text{Gal}(F^c/\mathbb{Q}) \cong G \).

Let \( N_{2d}^\text{Weyl}(X, G) \) count the subset of all CM fields counted by \( N_{c_2}^\text{cm}(X, G) \) which are not of \( G \)-Weyl type.

We first establish asymptotics for \( N_{c_2}^\text{cm}(X, G) \), and then give upper bounds for \( N_{2d}^\text{Weyl}(X, G) \).

2.1 Asymptotics for \( N_{c_2}^\text{cm}(X, G) \)

We begin by establishing an asymptotic formula with a power-saving error term for \( N_{c_2}^\text{cm}(X, G) \). The key is a theorem of Cohen, Diaz y Diaz, and Olivier \([10]\) which expresses the Dirichlet series enumerating all quadratic extensions of a number field as a linear combination of Hecke \( L \)-functions. We will use a version of their result which incorporates signature conditions.

Fix a totally real field \( F \) of degree \( d \) and define the Dirichlet series

\[
D_{F, C_2}(s) := \sum_{[E:F]=2} \frac{1}{N_F/\mathbb{Q}(\mathfrak{d}_{E/F})^s} = \sum_{n=1}^\infty \frac{a^-(n)}{n^s}, \quad \text{Re}(s) > 1
\]

where the sum is over all totally imaginary quadratic extensions \( E/F \), \( \mathfrak{d}_{E/F} \) is the relative discriminant, and

\[
a^-(n) := \#\{E/F \text{ totally imaginary quadratic, } N_{F/\mathbb{Q}}(\mathfrak{d}_{E/F}) = n\}.
\]

The following is a special case of \([10, \text{Theorem 3.11}]\), applied to the totally real field \( F \) (which has signature \((d, 0)\)) and with \( m_\infty \) equal to the set of real places of \( F \) (which are all ramified in the imaginary quadratic extension \( E/F \)).

**Theorem 2.1** ([10]) For \( \text{Re}(s) > 1 \) we have

\[
D_{F, C_2}(s) = \frac{1}{\xi_F(2s)} \sum_{\epsilon_\infty \subseteq m_\infty} \sum_{\epsilon \subseteq \epsilon_\infty} \frac{(-1)^{|\epsilon_\infty|}}{2^{\epsilon_\infty}} N_F/\mathbb{Q}(2/\epsilon)^{1-2s} \sum_{\chi \in Q(\text{Cl}_{\epsilon_\infty}^2 \epsilon_\infty)(F)} L_F(\chi, s),
\]

where \( \epsilon \) runs over all integral ideals of \( F \) dividing 2, \( \epsilon_\infty \) runs over all subsets of the set of real places \( m_\infty \) of \( F \), \( \chi \) runs over all quadratic characters \( Q(\text{Cl}_{\epsilon_\infty}^2 \epsilon_\infty)(F) \) of the ray class group \( \text{Cl}_{\epsilon_\infty}^2 \epsilon_\infty(F) \) modulo \( \epsilon_\infty \), and \( L_F(\chi, s) \) is the \( L \)-function of \( \chi \).

The following result establishes some important analytic properties of the Dirichlet series \( D_{F, C_2}(s) \) (this is analogous to \([25, \text{Theorem 5}]\)).
Theorem 2.2  Assume that $0 \leq \delta, \delta' \leq 1/2$ satisfy (1.2) and (1.6), respectively. Then the Dirichlet series $D_{-F,C2}(s)$ has a meromorphic continuation to $\text{Re}(s) > 1/2$ with only a single (simple) pole at $s = 1$ with residue
\[ R_d(F) := \frac{\text{Res}_{s=1} \zeta_F(s)}{2^d \zeta_F(2)} > 0. \]

Moreover, the function
\[ g_F(s) := D_{-F,C2}(s) - \frac{R_d(F)}{s-1} \]
is analytic for $\sigma := \text{Re}(s) > 1/2$ and satisfies the bound
\[ g_F(\sigma + it) \ll_{\epsilon, d} d_F^{\delta(1-\sigma)+\delta_d+\epsilon} (1+|t|)^{(1-\sigma)+\epsilon} (\sigma - 1/2)^d, \quad 1/2 < \sigma \leq 1 + \epsilon. \quad (2.1) \]

Proof Let $\chi_{0,c} \in \mathbb{Q}(\text{Cl}_{c}e_{\infty}(F))$ be the trivial character and write
\[ D_{-F,C2}(s) = A(s) + B(s), \quad (2.2) \]

where
\[ A(s) := \frac{1}{\zeta_F(2s)} \sum_{c_{\infty} \subseteq m_{\infty}} \frac{(-1)^{|c_{\infty}|}}{2^{|c_{\infty}|}} \sum_{c|2} N_{F/Q}(2/c)^{1-2s} L(\chi_{0,c}, s) \]

and
\[ B(s) := \frac{1}{\zeta_F(2s)} \sum_{c_{\infty} \subseteq m_{\infty}} \sum_{c|2} \frac{(-1)^{|c_{\infty}|}}{2^{|c_{\infty}|}} N_{F/Q}(2/c)^{1-2s} \sum_{\substack{\chi \in \mathbb{Q}(\text{Cl}_{c}e_{\infty}(F)) \\ \chi \neq \chi_{0,c}}} L_F(\chi, s). \]

The $L$–function
\[ L_F(\chi_{0,c}, s) = \zeta_F(s) \prod_{p|c} (1 - N_{F/Q}(p)^{-s}) \]

extends to a meromorphic function on $\mathbb{C}$ with only a single (simple) pole at $s = 1$, and if $\chi \neq \chi_{0,c}$ the $L$–function $L_F(\chi, s)$ extends to an analytic function on $\mathbb{C}$. Hence, by (2.2) the Dirichlet series $D_{-F,C2}(s)$ extends to a meromorphic function on $\sigma > 1/2$ with only a single (simple) pole at $s = 1$ with residue
\[ R_d(F) = \text{Res}_{s=1} A(s) = S \cdot \text{Res}_{s=1} \zeta_F(s), \]

where
\[ S := \frac{1}{\zeta_F(2)} \sum_{c_{\infty} \subseteq m_{\infty}} \frac{(-1)^{|c_{\infty}|}}{2^{|c_{\infty}|}} \sum_{c|2} N_{F/Q}(2/c)^{-1} \prod_{p|c} (1 - N_{F/Q}(p)^{-1}). \]

As in [10, Sections 3.3 and 3.4], we may compute that
\[ \sum_{c|2} N_{F/Q}(2/c)^{-1} \prod_{p|c} (1 - N_{F/Q}(p)^{-1}) = 1 \]
and
\[ \sum_{\kappa \subset m_\infty} \frac{(-1)^{|\kappa|}}{2^{|\kappa|}} = \frac{1}{2^{|m_\infty|}} = \frac{1}{2^d}. \]

Therefore, we have \( S = 2^{-d} \) and
\[ R_d(F) = \frac{\text{Res}_{s=1} \zeta_F(s)}{2^d \zeta_F(2)}. \]

By the preceding facts, the function \( g_F(s) := D_{F, C_0}(s) - R_d(F)/(s - 1) \) is analytic for \( \sigma > 1/2 \). Hence, it remains to establish the bound (2.1).

By (1.6), we have the bound
\[ (s - 1)\zeta_F(s) \ll_{\epsilon, d} q(F, \chi_0, s)^\delta'(1-\sigma)+\epsilon (|s| + 1) \]
\[ \ll_{\epsilon, d} (d_F(|s| + 4)^d)^{\delta'(1-\sigma)+\epsilon} (|s| + 1), \quad 1/2 < \sigma \leq 1 + \epsilon, \]
and we also have the bound
\[ \frac{1}{\zeta_F(2s)} \ll (\sigma - 1/2)^d, \quad 1/2 < \sigma \leq 1 + \epsilon \]
(the implied constant is uniform in \( F \)). These bounds yield the estimate
\[ (s - 1)A(s) - R_d(F) \ll_{\epsilon, d} (\sigma - 1/2)^{-d} (d_F(|s| + 4)^d)^{\delta'(1-\sigma)+\epsilon} (|s| + 1), \quad 1/2 < \sigma \leq 1 + \epsilon, \]
and thus with \( f(s) := A(s) - R_d(F)/(s - 1) \) the estimate
\[ f(\sigma + it) \ll_{\epsilon, d} (\sigma - 1/2)^{-d} d_F^{\delta'(1-\sigma)+\epsilon} (1 + |t|)^{d(\delta'(1-\sigma)+\epsilon)}, 
\quad 1/2 < \sigma \leq 1 + \epsilon. \]

Next observe that if the bound (1.6) holds for some \( 0 \leq \delta' \leq 1/2 \), then it also holds (with the same \( \delta' \)) if \( \chi \) is imprimitive, since the \( L \)-functions of an imprimitive and primitive character differ by a finite Euler product, uniformly bounded above and below by \( O(1) \) in the strip \( 1/2 < \sigma \leq 1 + \epsilon \). This was used for principal characters above, and for \( \chi \in Q(C_{\epsilon, \infty}(F)) \) with \( \chi \neq \chi_{c,0} \), we have
\[ L_F(\chi, s) \ll_{\epsilon, d} q(F, \chi, s)^{\delta'(1-\sigma)+2\epsilon} \]
\[ \ll_{\epsilon, d} (d_F(|s| + 4)^d)^{\delta'(1-\sigma)+2\epsilon}, \quad 1/2 < \sigma \leq 1 + \epsilon \]
with \( N_{F/Q}(\epsilon^2) = O(1) \) for all allowable \( \epsilon \). Also, from (1.2) we have the bound
\[ |Q(C_{\epsilon, \infty}(F))| = |C_{\epsilon, \infty}(F)[2]| \ll_{\epsilon} |C(F)[2]| \ll_{\epsilon, d} d_F^{\delta_d+\epsilon}. \]

Then arguing as above we get
\[ B(\sigma + it) \ll_{\epsilon, d} (\sigma - 1/2)^{-d} d_F^{\delta'(1-\sigma)+\epsilon} (1 + |t|)^{d(\delta'(1-\sigma)+\epsilon)} d_F^{\delta_d+\epsilon}, 
\quad 1/2 < \sigma \leq 1 + \epsilon. \]

Finally, since \( g_F(s) = f(s) + B(s) \), we have
\[ g_F(\sigma + it) \ll_{\epsilon, d} d_F^{\delta'(1-\sigma)+\delta_d+2\epsilon} \frac{(1 + |t|)^{d(\delta'(1-\sigma)+\epsilon)}}{\sigma - 1/2} d_F^{\delta_d+\epsilon}, \quad 1/2 < \sigma \leq 1 + \epsilon. \]
Now, given a pair \((d, G)\), let
\[
\mathcal{F}^+_G := \{F/\mathbb{Q} : F \text{ totally real of degree } d, \, \text{Gal}(F^c/\mathbb{Q}) \cong G\}.
\]

Define the Dirichlet series
\[
D^\text{cm}_G(s) := \sum_{F \in \mathcal{F}^+_G} \sum_{[E:F]=2} \frac{1}{d_E^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]
where the inner sum is over all totally imaginary quadratic extensions \(E/F\) and
\[
a(n) := \#\{(E, F) : F \in \mathcal{F}^+_G, E/F \text{ totally imaginary quadratic}, \, |d_E| = n\}.
\]

Clearly, the Dirichlet series \(D^\text{cm}_G(s)\) enumerates all CM fields counted by \(N^\text{cm}_{2d}(X, G)\). Using the relation
\[
|d_E| = d_F^2 N_{F/\mathbb{Q}}(d_{E/F}),
\]
we have
\[
D^\text{cm}_G(s) = \sum_{F \in \mathcal{F}^+_G} \frac{1}{d_F^2} \sum_{[E:F]=2} \frac{1}{N_{F/\mathbb{Q}}(d_{E/F})^s} = \sum_{F \in \mathcal{F}^+_G} \frac{D^-_{F,G}(s)}{d_F^{2s}}.
\]

**Theorem 2.3**  Assume that Hypothesis 1.2 is true for \((d, G)\) and \(\delta'\) satisfies \((1.6)\). Then the Dirichlet series \(D^\text{cm}_G(s)\) has a meromorphic continuation to the half-plane \(\text{Re}(s) > \alpha\) where
\[
\alpha = \alpha(\delta_d, M(G), \delta') := \max \left\{ \frac{\delta_d + \delta' + M(G)}{\delta' + 2}, \, \frac{M(G)}{2} \right\} < 1,
\]
with only a single (simple) pole at \(s = 1\) given by the convergent series
\[
r_d(G) := \sum_{F \in \mathcal{F}^+_G} \frac{R_d(F)}{d_F^2} = \sum_{F \in \mathcal{F}^+_G} \frac{\text{Res}_{s=1} \xi_F(s)}{2^d d_F^2 \xi_F(2)} > 0.
\]

Moreover, for \(\sigma := \text{Re}(s) \in [\alpha + \epsilon, 1 + \epsilon]\) and \(|t| > 1\), the Dirichlet series \(D^\text{cm}_G(s)\) satisfies the bound
\[
D^\text{cm}_G(\sigma + it) \ll_{\epsilon,d,G} (1 + |t|)^{\delta'/2} e^{\sigma - 1/2 + \epsilon}. \tag{2.5}
\]

**Proof**  Write
\[
D^\text{cm}_G(s) = g(s) + \frac{1}{s-1} h(s), \tag{2.6}
\]
where
\[
g(s) := \sum_{F \in \mathcal{F}^+_G} \frac{g_F(s)}{d_F^{2s}} \quad \text{and} \quad h(s) := \sum_{F \in \mathcal{F}^+_G} \frac{R_d(F)}{d_F^{2s}}.
\]
Using the estimate (2.1), we have
\[ g(s) \ll_{\epsilon,d} (1 + |t|)^{\delta'(1-\sigma)+\epsilon} \sum_{F \in \mathcal{F}_G^+} d_F^{\delta'(1-\sigma)+\delta_d-2\sigma+\epsilon}, \quad 1/2 < \sigma \leq 1 + \epsilon. \]

Hence, the absolute convergence of the series \( g(s) \) is guaranteed by the convergence of the series
\[ \sum_{F \in \mathcal{F}_G^+} d_F^{\delta'(1-\sigma)+\delta_d-2\sigma+\epsilon}. \tag{2.7} \]

Divide the sum over \( F \) into intervals \( N < d_F \leq 2N \) and let \( N \) range over the integer powers of 2. Then using the estimate (1.3), we see that (2.7) converges whenever the series
\[ \sum_N N^{\delta'(1-\sigma)+\delta_d-2\sigma+M(G)+\epsilon} \tag{2.8} \]
converges. The series (2.8) converges whenever the exponent is negative, i.e., whenever \( \sigma > \alpha_1 \) with
\[ \alpha_1 := \frac{\delta_d + \delta' + M(G)}{\delta' + 2} \]
(for an appropriate choice of \( \epsilon > 0 \)), uniformly for \( \sigma \geq \alpha_1 + \epsilon \). The condition \( \alpha_1 < 1 \) is equivalent to the condition \( \delta_d + M(G) < 2 \). Therefore, we see that \( g(s) \) is analytic for \( \sigma > \alpha_1 \) with \( \alpha_1 < 1 \), and that \( g(s) \) satisfies the bound
\[ g(\sigma + it) \ll_{\epsilon,d,G} (1 + |t|)^{\delta'(1-\sigma)+\epsilon} \frac{1}{(\sigma - 1/2)^d}, \quad \alpha_1 + \epsilon \leq \sigma \leq 1 + \epsilon. \tag{2.9} \]

Next, using the estimate \( R_d(F) \ll_{\epsilon,d} d_F^\epsilon \) we have
\[ h(s) \ll_{\epsilon,d} \sum_{F \in \mathcal{F}_G^+} d_F^{-2\sigma+\epsilon}. \]

Then a similar argument using the estimate (1.3) shows that the series \( h(s) \) converges for \( \sigma > \alpha_2 \) with \( \alpha_2 := M(G)/2 \). The condition \( \alpha_2 < 1 \) is ensured by \( \delta_d + M(G) < 2 \). Therefore, we see that \( (s-1)^{-1}h(s) \) is meromorphic for \( \sigma > \alpha_2 \) with \( \alpha_2 < 1 \) with only a single (simple) pole at \( s = 1 \) with residue
\[ r_d(G) := \sum_{F \in \mathcal{F}_G^+} \frac{R_d(F)}{d_F^\epsilon}, \]
and that \( (s-1)^{-1}h(s) \) satisfies the bound
\[ \frac{1}{s-1}h(s) \ll_{\epsilon,d,G} 1, \quad \sigma \geq \alpha_2 + \epsilon, \quad |t| > 1. \tag{2.10} \]

Finally, from (2.6) we conclude that \( D_G^{\text{cm}}(s) \) has a meromorphic continuation to \( \sigma > \alpha := \max(\alpha_1, \alpha_2) \) with \( \alpha < 1 \) with only a single (simple) pole at \( s = 1 \) with residue \( r_d(G) \). Moreover, from the estimates (2.9) and (2.10) we see that \( D_G^{\text{cm}}(s) \) satisfies the bound
\[ D_G^{\text{cm}}(\sigma + it) \ll_{\epsilon,d,G} (1 + |t|)^{\delta'(1-\sigma)+\epsilon} \frac{1}{(\sigma - 1/2)^d}, \quad \alpha + \epsilon \leq \sigma \leq 1 + \epsilon, \quad |t| > 1. \]
Theorem 2.4
(i) Under the assumptions of Theorem 2.3, we have
\[ N_{2d}^{cm}(X, G) = r_d(G)X + O_{d, \epsilon}(X^{\beta(\delta_d, M(G), \delta') + \epsilon}) \]
where
\[ \beta(\delta_d, M(G), \delta') := 1 - \frac{1 - \alpha}{1 + d\delta'(1 - \alpha)} \]
with \( \alpha = \alpha(\delta_d, M(G), \delta') < 1 \) defined by (2.4).
(ii) If Hypothesis 1.2 is true for every pair \((d, G)\) where \( G \) ranges over all transitive subgroups \( G \leq S_d \), then
\[ N_{2d}^{cm}(X) = \left( \sum_{G \leq S_d} r_d(G) \right) X + O_{d, \epsilon}(X^{\beta(\delta_d, \delta') + \epsilon}) \]
where
\[ \beta(\delta_d, \delta') := \max_{G \leq S_d} \beta(\delta_d, M(G), \delta') < 1. \] (2.11)

Proof Fix a smooth function \( \phi : [0, 1] \rightarrow [0, 1] \) with \( \phi(0) = 1 \) and \( \phi(1) = 0 \). Then, for each \( Y > 1 \), define
\[ \phi_Y(t) := \begin{cases} 1, & \text{if } t \in [0, 1]; \\ \phi(Y(t - 1)), & \text{if } t \in [1, 1 + Y^{-1}]; \\ 0, & \text{if } t \geq 1 + Y^{-1}. \end{cases} \]
Let
\[ \hat{\phi}_Y(s) := \int_0^\infty \phi_Y(t)t^{s-1}dt, \quad \text{Re}(s) > 0 \]
be the Mellin transform of \( \phi_Y \). Integrating by parts \( A \geq 1 \) times yields the estimate
\[ \hat{\phi}_Y(s) \ll Y^{-1} \left( \frac{Y}{1 + |t|} \right)^A, \] (2.12)
valid for all \( s \) in any fixed vertical strip \( \sigma_0 \leq \text{Re}(s) \leq \sigma_1 \) with \( \sigma_0 > 0 \), and also valid for all real numbers \( A \geq 1 \) by interpolation.

By construction, and then by Mellin inversion, we have
\[ N_{2d}^{cm}(X, G) = \sum_{n=1}^{X} a(n) \leq \sum_{n=1}^{\infty} a(n)\phi_Y \left( \frac{n}{X} \right) = \frac{1}{2\pi i} \int_{(1+\epsilon)} D_{G}^{cm}(s)\hat{\phi}_Y(s)X^s ds. \]
From the estimate (2.5), we see that
\[ D_{G}^{cm}(s) \ll (1 + |t|)^{d\delta'(1-\sigma)+\epsilon} \] (2.13)
in any vertical strip $1/2 < \alpha + \eta < \sigma \leq 1 + \epsilon, |t| > 1$, where the implied constant depends on $\epsilon, d, \eta$. Then using the estimates (2.12) and (2.13), we may shift the contour to $\text{Re}(s) = \alpha'$ with $\alpha < \alpha' < 1$ to get

$$\frac{1}{2\pi i} \int_{(\alpha')} D_{G}^{cm}(s) \hat{\phi}_Y(s) X^\epsilon ds = \hat{\phi}_Y(1) r_d(G)X + \frac{1}{2\pi i} \int_{(\alpha')} D_{G}^{cm}(s) \hat{\phi}_Y(s) X^\epsilon ds.$$  

For any $A \geq 1$ we have the estimate

$$\frac{1}{2\pi i} \int_{(\alpha')} D_{G}^{cm}(s) \hat{\phi}_Y(s) X^\epsilon ds \ll X^{\alpha'} Y^{-1} \int_{\mathbb{R}} (1 + |t|)^{d\delta'(1-\alpha')} + \epsilon \left( \frac{Y}{1 + |t|} \right)^A dt.$$  

Choose $A = d\delta'(1 - \alpha') + 1 + 2\epsilon$. Then

$$\frac{1}{2\pi i} \int_{(\alpha')} D_{G}^{cm}(s) \hat{\phi}_Y(s) X^\epsilon ds \ll X^{\alpha'} Y^{-1 - \delta'(1-\alpha')} + 2\epsilon.$$  

Since $\hat{\phi}_Y(1) = 1 + O(Y^{-1})$, we have

$$\hat{\phi}_Y(1) r_d(G)X = r_d(G)X + O(XY^{-1}).$$  

Then putting things together, and replacing $2\epsilon$ by $\epsilon$, we get

$$N_{2d}^{cm}(X, G) \leq \sum_{n=1}^{\infty} a(n) \phi_Y \left( \frac{n}{X} \right) = r_d(G)X + O(XY^{-1}) + O(X^{\alpha'} Y^{-1 - \delta'(1-\alpha')} + \epsilon). \quad (2.14)$$  

Similarly, we have

$$N_{2d}^{cm}(X, G) \geq \sum_{n=1}^{\infty} a(n) \phi_Y \left( \frac{n}{X/(1 + Y^{-1})} \right),$$

for which the same estimate in (2.14) also holds (since we may interchange $X$ and $X/(1 + Y^{-1})$ in (2.14), within the error terms given there), so that in fact we have

$$N_{2d}^{cm}(X, G) = r_d(G)X + O(XY^{-1}) + O(X^{\alpha'} Y^{-1 - \delta'(1-\alpha')} + \epsilon).$$  

We optimize (apart from epsilon factors) by choosing $\alpha' = \alpha + \epsilon$ and $Y = X^{\frac{1}{\nu + d\delta'(1 - \alpha')}}$, so as to obtain for each $\epsilon > 0$ that

$$N_{2d}^{cm}(X, G) = r_d(G)X + O_{\epsilon}(X^{\beta(\delta_d, M(G), \delta') + \epsilon}),$$

where

$$\beta(\delta_d, M(G), \delta') := 1 - \frac{1 - \alpha}{1 + d\delta'(1 - \alpha')}.$$  

This proves part (i). Part (ii) follows by summing the asymptotic formula in (i) over all transitive subgroups $G \leq S_d$. \qed
2.2 Upper bounds for $N_{2d}^{-\text{Weyl}}(X, G)$

Let $K$ be a number field of degree $d$ with $\text{Gal}(K^c/\mathbb{Q}) \cong G$, and let $L$ be a quadratic extension of $K$. Then $\text{Gal}(L^c/\mathbb{Q})$ embeds as a subgroup of the wreath product $C_2 \wr G$ (see Proposition 3.2). Clearly, we have

$$N_{2d}^{-\text{Weyl}}(X, G) \leq Y(X, G),$$

where

$$Y(X, G) := \#\{L/K : \text{Gal}(L^c/\mathbb{Q}) \ncong C_2 \wr G, \text{Gal}(K^c/\mathbb{Q}) \cong G, [L : K] = 2, |d_L| \leq X\}.$$ (2.15)

Therefore, it suffices to give an upper bound for $Y(X, G)$.

The extensions counted by $Y(X, G)$ are distinguished by the following fact: for each prime $p$ unramified in $K/\mathbb{Q}$ but ramified in $L/K$ (so that $p | d_L$), we must in fact have $p^2 | d_L$ (see [25, Lemma 4]).

Let $K_G(X^{1/2}) := \{K/\mathbb{Q} : \text{Gal}(K^c/\mathbb{Q}) \cong G, |d_K| \leq X^{1/2}\}.$

As in [25, p. 9-10], we have the bound

$$Y(X, G) \leq \sum_{K \in K_G(X^{1/2})} O_{c,d} \left( \frac{X^{\frac{1}{2}+\epsilon}}{|d_K| \left|\text{Cl}(K)[2]\right|} \right).$$ (2.16)

We briefly recall the proof. Each $L$ counted in (2.15) satisfies

$$d_L = d_K^2 \cdot N_{K/\mathbb{Q}}(\mathcal{D}_{L/K})$$

with $N_{K/\mathbb{Q}}(\mathcal{D}_{L/K}) = ab^2$, where $a$ is only divisible by primes dividing $d_K$. Since each such prime can only divide $a$ with bounded multiplicity, the problem is reduced to proving (for each positive integer $n$) that the number of quadratic extensions $L/K$ with $N_{K/\mathbb{Q}}(\mathcal{D}_{L/K}) = n$ is $O_{d,e}(\left|\text{Cl}(K)[2]\right| n^e)$, and this is done by bounding the 2-torsion in the relevant ray class group.

Continuing then, applying the bound (1.2) to (2.16) gives

$$N_{2d}^{-\text{Weyl}}(X, G) \ll_{\epsilon,d} X^{\frac{1}{2}+\epsilon} \sum_{K \in K_G(X^{1/2})} |d_K|^{-1+\delta_d}.$$ (2.17)

Again, divide the sum over $K$ into intervals with $N < |d_K| \leq 2N$ and let $N$ range over the integer powers of 2. Then applying the estimate (1.3) gives

$$N_{2d}^{-\text{Weyl}}(X, G) \ll_{\epsilon,d} X^{\frac{1}{2}+\epsilon} \sum_{N} N^{-1+\delta_d+M(G)}.$$ (2.18)

If $\delta_d + M(G) \leq 1$ then

$$N_{2d}^{-\text{Weyl}}(X, G) \ll_{\epsilon,d} X^{\frac{1}{4}+\epsilon},$$ (2.17)

while if $\delta_d + M(G) > 1$ then

$$N_{2d}^{-\text{Weyl}}(X, G) \ll_{\epsilon,d} X^{\frac{1}{4}+\frac{1+\delta_d+M(G)}{2}+\epsilon}.$$ (2.18)

The exponent in (2.18) is less than 1 (for an appropriate choice of $\epsilon > 0$) provided that $\delta_d + M(G) < 2$. 

2.3 Proof of Theorem 1.3

Using Theorem 2.4 and estimates (2.17) and (2.18), we have

\[
\frac{N^{\text{Weyl}}_{2d}(X, G)}{N^{\text{cm}}_{2d}(X, G)} = \frac{N^{\text{cm}}_{2d}(X, G) - N_{2d}^{-\text{Weyl}}(X, G)}{N^{\text{cm}}_{2d}(X, G)} = 1 + O_{d, G, \epsilon}(X^{-C_1(\delta_d, M(G)) + \epsilon}),
\]

where

\[
C_1(\delta_d, M(G)) := \begin{cases} 
1 / 2, & \text{if } \delta_d + M(G) \leq 1 \\
1 - \frac{\delta_d + M(G)}{2}, & \text{if } 1 < \delta_d + M(G) < 2.
\end{cases}
\]

This proves Theorem 1.3.

2.4 Proof of Theorem 1.11

As above we have

\[
\frac{N^{\text{Weyl}}_{2d}(X, G)}{N^{\text{cm}}_{2d}(X, G)} = \frac{N^{\text{cm}}_{2d}(X, G) - N_{2d}^{-\text{Weyl}}(X, G)}{N^{\text{cm}}_{2d}(X, G)} = 1 + O_{d, G, \epsilon}(X^{-C_1(\delta_d, M(G)) + \epsilon}),
\]

Also by Theorem 2.4 we have

\[
\frac{N^{\text{cm}}_{2d}(X, G)}{N^{\text{cm}}_{2d}(X)} = \frac{r_d(G)}{\sum_{G \leq S_d} r_d(G)} + O_{d, \epsilon}(X^{-1 + \beta(\delta_d, \delta')})
\]

so that

\[
\frac{N^{\text{Weyl}}_{2d}(X, G)}{N^{\text{cm}}_{2d}(X)} = \frac{r_d(G)}{\sum_{G \leq S_d} r_d(G)} + O_{d, G, \epsilon}(X^{-C_2(\delta_d, M(G), \delta') + \epsilon}),
\]

where

\[
C_2(\delta_d, M(G), \delta') := \min\{C_1(\delta_d, M(G)), 1 - \beta(\delta_d, \delta')\} > 0.
\]

This proves (1.8). To prove (1.9), we sum over all \( G \leq S_d \) in (2.19) to get

\[
\frac{N^{\text{Weyl}}_{2d}(X)}{N^{\text{cm}}_{2d}(X)} = 1 + O_{d, \epsilon}(X^{-C_3(\delta_d, \delta') + \epsilon}),
\]

where

\[
C_3(\delta_d, \delta') := \min_{G \leq S_d} C_2(\delta_d, M(G), \delta') > 0.
\]

This proves Theorem 1.11.

3 Proof of Theorem 1.14

In this section we review some facts about wreath products of groups, discuss the structure of Galois groups of CM fields, and prove Theorem 1.14.
3.1 Wreath products
We begin by reviewing some facts about wreath products of groups (see e.g. [15]). Let $H$ and $K$ be groups, and suppose that $\theta : H \rightarrow \text{Aut}(K)$ is a homomorphism where we write $\theta(h) = \theta_h$. This gives a (left) group action of $H$ on $K$ defined by $(h, k) \mapsto \theta_h(k)$. Recall that the semidirect product of $K$ and $H$ with respect to $\theta$ is the group

$$K \rtimes_{\theta} H := \{(k, h) \mid k \in K, h \in H\},$$

where the group operation is defined by

$$(k_1, h_1)(k_2, h_2) := (k_1\theta_{h_1}(k_2), h_1h_2).$$

When understood, we suppress $\theta$ in our notation for the semidirect product.

Now, let $\Omega$ be an arbitrary set, and let $K^\Omega$ denote the set of all functions $f : \Omega \rightarrow K$. Pointwise multiplication of functions gives $K^\Omega$ the structure of a group. A (left) group action of $H$ on $\Omega$ gives a homomorphism

$$\theta : H \rightarrow \text{Aut}(K^\Omega)$$

defined by $\theta_h(f)(\omega) := f(h^{-1} \cdot \omega)$ for every $\omega \in \Omega$ and every $f \in K^\Omega$. In turn, this gives a (left) group action of $H$ on $K^\Omega$ defined by $(h, f) \mapsto \theta_h(f)$. The wreath product of $K$ and $H$ with respect to $\theta$ is defined by

$$K \wr_{\theta} H := K^\Omega \rtimes_{\theta} H.$$

When the set $\Omega = \{\omega_1, \ldots, \omega_n\}$ is finite, it is customary to identify $K^\Omega$ with the direct product $K^n$ via the isomorphism $f \mapsto (f(\omega_1), \ldots, f(\omega_n))$. In particular, if $\Omega = \{1, \ldots, n\}$ and $H \leq S_n$ is a group of permutations, then we have a (left) group action of $H$ on $\Omega$ in the usual way, and the corresponding action of $H$ on $K^n$ is by permutation of the components, i.e., if $\sigma \in H \leq S_n$ and $x = (x_1, \ldots, x_n) \in K^n$, then

$$\sigma \circ x := (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}),$$

and in this case we write $K \wr H$ instead of $K \wr_{\{1, \ldots, n\}} H$.

Now, let $G \leq S_d$ be a transitive subgroup. Then the wreath product $C_2 \wr G$ from the introduction is given by

$$C_2 \wr G = C_2 \wr_{\{1, \ldots, d\}} G = C_2^d \rtimes G,$$

and we have a short exact sequence

$$1 \longrightarrow C_2^d \longrightarrow C_2 \wr G \longrightarrow G \longrightarrow 1.$$

The elements of $C_2 \wr G$ take the form $(x, \sigma)$ where $x = (x_1, \ldots, x_d) \in C_2^d$ and $\sigma \in G$ is a permutation of the set $\{1, \ldots, d\}$, with multiplication given by

$$(x, \sigma)(y, \tau) = (x(\sigma \circ y), \sigma \tau).$$
where

\[ \sigma \circ y := (y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(d)}). \]

Finally, observe that if \( d \geq 2 \), then \( C_2 \wr G \) is noncommutative; for example, choose \( x = y = (-1, 1, \ldots, 1), \tau = \text{id} \), and any \( \sigma \) such that \( \sigma^{-1}(-1) = d \).

**Remark 3.1** As a consequence of the preceding discussion, if \( E \) is a \( G \)-Weyl CM field of degree \( 2d \geq 4 \) then \( E/\mathbb{Q} \) is non-abelian.

### 3.2 Embedding Galois groups into wreath products

**Proposition 3.2** Let \( K \) be a number field of degree \( d \) with \( \text{Gal}(K^c/\mathbb{Q}) \cong G \leq S_d \), and let \( L \) be a quadratic extension of \( K \). Then \( \text{Gal}(L^c/\mathbb{Q}) \) embeds as a subgroup of the wreath product \( C_2 \wr G \).

**Proof** Choose \( \alpha_1 \in K \) such that \( L = K(\sqrt{\alpha_1}) \) and let \( \alpha_1, \ldots, \alpha_d \) be its conjugates, so that \( L^c = K^c(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_d}) \). For each \( g \in \text{Gal}(L^c/\mathbb{Q}) \) and \( i \in \{1, \ldots, d\} \), we have

\[ g(\alpha_i) = \alpha_j, \quad g(\sqrt{\alpha_i}) = \pm \sqrt{\alpha_j}, \quad \text{for some } j \in \{1, \ldots, d\} \quad \text{and choice of sign } \pm. \]

We define a function

\[ \phi : \text{Gal}(L^c/\mathbb{Q}) \to C_2 \wr G = \{ \pm 1 \}^d \rtimes G \]

\[ g \mapsto (x_g, \sigma_g), \]

where (matching \((3.1)\)) \( \sigma_g(i) = j \) and \( x_g := (x_g(1), \ldots, x_g(d)) \in \{ \pm 1 \}^d \) is the vector whose \( j \)-th component is given by

\[ x_g(j) := \frac{g(\sqrt{\alpha_i})}{\pm \sqrt{\alpha_j}} = \frac{g(\sqrt{\sigma_g^{-1}(\alpha_i)})}{\sqrt{\alpha_j}}. \]

In particular, we have

\[ g(\sqrt{\alpha_i}) = x_g(\sigma_g(i)) \sqrt{\alpha_g(\sigma_g(i))} \]

for \( i \in \{1, \ldots, d\} \).

The data of \( x_g \) and \( \sigma_g \) determines \( g(\sqrt{\alpha_i}) \) for each \( i \), and hence it determines \( g \), so that \( \phi \) is injective.

We next prove that \( \phi \) is a homomorphism. Let \( g, h \in \text{Gal}(L^c/\mathbb{Q}) \). Then by definition of the wreath product, we have

\[ \phi(gh) = \phi(g)\phi(h) \iff (x_{gh}, \sigma_{gh}) = (x_g(\sigma_g \circ x_h), \sigma_g \sigma_h) \]

where

\[ \sigma_g \circ x_h := (x_h(\sigma_g^{-1}(1)), \ldots, x_h(\sigma_g^{-1}(d))). \]

By the isomorphism \( \text{Gal}(K^c/\mathbb{Q}) \cong G \), we have \( \sigma_{gh} = \sigma_g \sigma_h \). Thus, it remains to prove that

\[ x_{gh} = x_g(\sigma_g \circ x_h). \]
Since the $\sigma_{gh}(i)$-th component of $\sigma_g \circ x_h$ is given by
\[ x_{h\sigma_g^{-1}(\sigma_{gh}(i))} = x_{h\sigma_{gh}(i)}, \]
we see that (3.2) is equivalent to
\[ x_{gh\sigma_g(i)} = x_{g\sigma_{gh}(i)}x_{h\sigma_g(i)} \]
for $i \in \{1, \ldots, d\}$. We compute
\[
(gh)(\sqrt{\alpha_i}) = g(h(\sqrt{\alpha_i}))
= g(x_{h\sigma_g(i)}\sqrt{\alpha_{\sigma_g(i)}})
= x_{h\sigma_g(i)}g(\sqrt{\alpha_{\sigma_g(i)}})
= x_{h\sigma_g(i)}x_{g\sigma_{gh}(i)}(\sqrt{\alpha_{\sigma_g(i)}})
= x_{h\sigma_g(i)}x_{g\sigma_{gh}(i)}\sqrt{\alpha_{\sigma_{gh}(i)}},
\]
and thus
\[ x_{gh\sigma_g(i)} := \frac{(gh)(\sqrt{\alpha_i})}{\sqrt{\alpha_{\sigma_{gh}(i)}}} = x_{h\sigma_g(i)}x_{g\sigma_{gh}(i)}. \]
This completes the proof. \(\Box\)

3.3 Galois groups of CM fields

Let $E$ be a CM field of degree $2d$ with maximal totally real subfield $F$. Further, let $G$ be the transitive subgroup of $S_d$ with $\text{Gal}(F^c/Q) \cong G$.

Choose $\alpha_1 \in F$ such that $E = F(\sqrt{\alpha_1})$ and let $\alpha_1, \ldots, \alpha_d$ be its conjugates, so that $E^c = F^c(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_d})$. For each $g \in \text{Gal}(E^c/F^c)$ and $i \in \{1, \ldots, n\}$, we have
\[ g(\alpha_i) = \alpha_i, \quad g(\sqrt{\alpha_i}) = \pm \sqrt{\alpha_i} \]
for some choice of sign $\pm$. We define a function
\[ \psi : \text{Gal}(E^c/F^c) \longrightarrow \{ \pm 1 \}^d \]
\[ g \longmapsto y_g, \]
where $y_g := (y_{g,1}, \ldots, y_{g,d}) \in \{ \pm 1 \}^d$ is the vector whose $i$-th component is given by
\[ y_{g,i} := \frac{g(\sqrt{\alpha_i})}{\sqrt{\alpha_i}}. \]

Arguing as in the proof of Proposition 3.2, we see that $\psi$ is an injective homomorphism. In particular, this implies that $\text{Gal}(E^c/F^c) \cong C_2^v$ for some $1 \leq v \leq d$.

Now, by Galois theory we have the short exact sequence
\[ 1 \longrightarrow \text{Gal}(E^c/F^c) \longrightarrow \text{Gal}(E^c/Q) \longrightarrow \text{Gal}(F^c/Q) \longrightarrow 1. \]
(3.3)

This yields a short exact sequence
\[ 1 \longrightarrow C_2^v \longrightarrow \text{Gal}(E^c/Q) \longrightarrow G \longrightarrow 1 \]
(3.4)
called the imprimitivity sequence for $\text{Gal}(E^c/Q)$ (see [16, p. 4]).
3.4 CM types and the Colmez conjecture
Let $E$ be a CM field of degree $2d$. The $2d$ embeddings of $E$ occur in complex conjugate pairs. A CM type $\Phi$ for $E$ is a collection of $d$ embeddings $\{\sigma_1, \ldots, \sigma_d\}$ consisting of one choice of embedding $\sigma_i$ from each complex conjugate pair. There are $2^d$ CM types $\Phi$.

Let $\Phi(E)$ denote the set of CM types for $E$. Then the group $\text{Gal}(E^c/\mathbb{Q})$ acts on $\Phi(E)$ by composition.

In [4, Proposition 5.1], it is shown as a consequence of the averaged Colmez conjecture proved in [3, 43] that if the action of $\text{Gal}(E^c/\mathbb{Q})$ on $\Phi(E)$ is transitive, then the Colmez conjecture is true for $E$ and takes the form (1.10).

With these facts in hand, we can now prove Theorem 1.14.

3.5 Proof of Theorem 1.14
Let $E$ be a $G$-Weyl CM field. By the preceding discussion, to prove that the Colmez conjecture is true for $E$, it suffices to show that the action of $\text{Gal}(E^c/Q)$ on $\Phi(E)$ is transitive. In fact, we will show that the action of $\text{Gal}(E^c/F^c)$ on $\Phi(E)$ is transitive.

Given a CM type $\Phi \in \Phi(E)$, let $G_{\Phi} \leq \text{Gal}(E^c/F^c)$ be the stabilizer of $\Phi$. By the orbit-stabilizer theorem,

$$|\text{Gal}(E^c/F^c) : \Phi| = \frac{|\text{Gal}(E^c/F^c)|}{|G_{\Phi}|}.$$  \tag{3.5}

Since $E$ is a $G$-Weyl CM field, then

$$\text{Gal}(E^c/Q) \cong C_2 \wr G = C_2^d \rtimes G,$$

and in particular, we have $|\text{Gal}(E^c/Q)| = 2^d|G|$. On the other hand, by the imprimitivity sequence (3.4) we have $G \cong \text{Gal}(E^c/Q)/C_2^d$, so that $|\text{Gal}(E^c/Q)| = 2^d|G|$. Hence $v = d$, and it follows that $|\text{Gal}(E^c/F^c)| = 2^d$.

Now, let $g \in \text{Gal}(E^c/F^c)$ and write $\Phi = \{\sigma_1, \ldots, \sigma_d\}$. If $g$ is non-trivial, then there is an $1 \leq i \leq d$ such that $g \circ \sigma_i = \overline{\sigma_i} \notin \Phi$. Hence $g$ does not fix $\Phi$, and so $G_{\Phi}$ is trivial.

By (3.5), the preceding facts imply that $|\text{Gal}(E^c/F^c) : \Phi| = 2^d$. Since there are $2^d$ CM types $\Phi$, it follows that the action of $\text{Gal}(E^c/F^c)$ on $\Phi(E)$ is transitive. This completes the proof. \qed

3.6 The reflex degree of a $G$-Weyl CM field
Let $E$ be a CM field and $\Phi$ be a CM type for $E$. The reflex field associated to the CM pair $(E, \Phi)$ is the field

$$E_{\Phi} := \mathbb{Q}(\{\text{Tr}_{\Phi}(a) \mid a \in E\})$$

where

$$\text{Tr}_{\Phi}(a) := \sum_{\phi \in \Phi} \phi(a)$$

is the type trace of $a \in E$. We have the following identity relating the size of the $\text{Gal}(E^c/Q)$-orbit of $\Phi$ to the degree of the reflex field $E_{\Phi}$ over $\mathbb{Q}$ (see [4, Proposition 6.3]),

$$|\text{Gal}(E^c/Q) : \Phi| = [E_{\Phi} : \mathbb{Q}].$$  \tag{3.6}
Table 2  General pairs \((d, G)\) for which Hypothesis 1.2 holds

| \((d, G)\) | Reference | Upper bound \(N_d(X, G) \ll X^{M(G)}\) |
|------------|-----------|---------------------------------|
| \(d \geq 1\) and \(G\) abelian | [26] | \(x^{\frac{1}{2}}\) for some \(\ell\) the smallest prime divisor of \(|G|\) |
| \(d = \ell\) prime, \(G = D_\ell\) | [11, 24] | \(x^{\frac{1}{2}}\) |
| \(d \geq 1\) and \(G\) nilpotent | [1, 23] | \(x^{\frac{3}{2}}\) |
| \(d \geq 5\) and \(|G| = d\) (if at least one totally real field exists) | [19] | \(x^{\frac{1}{2}}\) |
| \(d = 4,\) any \(G \leq S_4\) transitive | [5, 10] | \(x^1\) |
| \(d = 5,\) any \(G \leq S_5\) transitive | [7] | \(x^1\) |
| \(d = k|A|, G = S_k \times A k = 3, 4, 5,\) any \(A\) abelian | [29, 37] | \(x^{1/|A|}\) |

In the proof of Theorem 1.14, we showed that if \(E\) is a \(G\)-Weyl CM field, then \(|\text{Gal}(E^c/\mathbb{Q}) \cdot \Phi| = 2^d\). Hence by (3.6) the reflex degree \(|E_\Phi : \mathbb{Q}| = 2^d\), and in particular, \(G\)-Weyl CM fields have maximal reflex degree.

The framework of [4] was to prove that the Colmez conjecture is true for CM fields with maximal reflex degree, and then to explicitly construct positive density sets of CM fields with this property. As we have seen, the \(G\)-Weyl CM fields provide a rich supply of such fields. On the other hand, it is important to observe that there are CM fields with maximal reflex degree which are not \(G\)-Weyl. In [16, p. 23], examples of such CM fields are given with \(d = 4\) and \(G = C_2^2, C_4, D_4\) and \(S_4\).

4 Some known cases of Hypothesis 1.2

In this section, we give a table which lists some known cases of Hypothesis 1.2. We also give a table that lists cases of Hypothesis 1.2 which would follow from a sufficiently strong 2-torsion exponent \(\delta_d\).

For \(d \geq 6\), the lists are extracted from the tables in [17]; in particular, as Dummit notes, the labeling of the transitive subgroups is the standard one originally given by Conway, Hulpke, and McKay [13]. For simplicity, when summarizing results in the tables, we sometimes state upper bounds which are weaker than what is known.

For a transitive subgroup \(G \leq S_d\), Table 2 gives a list of general pairs \((d, G)\) for which Hypothesis 1.2 is known to hold. In each case, the upper bound in the Malle conjecture (1.1) is known, and we may take \(\delta_d = 1/2\). The table does not necessarily contain a complete list of all known results, and it should be possible to obtain additional cases of Hypothesis 1.2. Among other possibilities, the methods developed in Mehta [30], Alberts [1], Wang [37], and Alberts, Lemke Oliver, Wang, and Wood [2] are particularly applicable here.

We also note that when \(G\) satisfies Hypothesis 1.2, so does \(C_2 : G\) by the argument of Klüners [25] which we are adapting.

Remark 4.1  As observed previously, the condition \(|G| = d\) is equivalent to all number fields counted by \(N_d(X, G)\) being Galois over \(\mathbb{Q}\). This case follows from [19, Proposition 1.3].

Table 3 lists specific pairs \((d, G)\) with \(d = 6, 7\), for which an upper bound \(N_d(X, G) \ll X^{M(G)}\) is known for some \(M(G) < 2\), but such that \(\delta_d + M(G) > 2\). The last column lists a range of 2-torsion exponents which would suffice for \(\delta_d + M(G) < 2\) to hold.
Table 3 Specific pairs \((d,G)\) for which Hypothesis 1.2 holds for any 2-torsion exponent \(\delta_d\) in the specified range.

| Label | Order of group | Isomorphic to | Upper bound \(N_d(X,G) \ll X^{(M(G)}\) | Range of \(\delta_d\) |
|-------|---------------|--------------|---------------------------------|----------------|
| Transitive subgroups of \(S_6\) satisfying \(N_d(X,G) \ll X^{(M(G)}\) with \(|M(G)| < 2 (d = 6)\) |
| 6T5 | 18 | \(F_{18}\) | \(X^{7/4++}\) | \(\delta_6 \leq \frac{1}{4}\) |
| 6T12 | 60 | \(A_5\) | \(X^{11/5++}\) | \(\delta_6 \leq \frac{1}{5}\) |
| 6T14 | 120 | \(S_5\) | \(X^{19/10++}\) | \(\delta_6 \leq \frac{1}{10}\) |
| 6T15 | 360 | \(A_6\) | \(X^{19/10++}\) | \(\delta_6 \leq \frac{1}{10}\) |
| Transitive subgroups of \(S_7\) satisfying \(N_d(X,G) \ll X^{(M(G)}\) with \(|M(G)| < 2 (d = 7)\) |
| 7T2 | 14 | \(D_7\) | \(X^{19/12++}\) | \(\delta_7 \leq \frac{5}{12}\) |
| 7T3 | 21 | \(F_{21}\) | \(X^{7/4++}\) | \(\delta_7 \leq \frac{4}{7}\) |
| 7T5 | 168 | \(PSL_2(F_7)\) | \(X^{11/6++}\) | \(\delta_7 \leq \frac{6}{11}\) |

The upper bounds were obtained by Dummit [17], and some totally real fields for each \((d,G)\) are enumerated on lmfdb.org [35].

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Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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