Conservation laws for a mathematical model of HIV transmission

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Abstract

A theorem due to Nail H. Ibragimov (2007) provides a connection between symmetries and conservation laws for arbitrary differential equations. The theorem is valid for any system of differential equations provided that the number of equations is equal to the number of dependent variables. In this paper we use the theorem to determine conservation laws for a nonlinear system of ordinary differential equations that represents a mathematical model for HIV transmission.

KEY WORDS: Lie symmetry analysis, Adjoint equation, Noether symmetry, Nonlinear equations, Lagrangian, Conservation laws.

1 Introduction

Ibragimov [9] provides a general theorem on conservation laws by means of which conservation laws can be constructed for an arbitrary system of differential equations admitting Lie symmetries, provided the number of equations is equal to the number of dependent variables. Ibragimov’s theorem is an extension of the classical theorem of Noether [3] in that the latter theorem does not require existence of a Lagrangian. In fact, unlike Noether’s theorem, Ibragimov’s theorem allows one to associate a conservation law to every Lie point symmetry admitted by a given arbitrary system of differential equations. Applications that one comes across involving the use of admitted Lie point symmetries to construct conservation laws via either the classical Noether’s theorem or Ibragimov’s theorem are often limited to scalar partial differential equations [13, 2, 8, 7, 12, 14, 6, 14].

The authors were unable to find any applications involving systems of first-order ordinary differential equations, for instance. In this connection the application reported in this paper adds to the repertoire of nontrivial applications of Ibragimov’s theorem.

We consider a nonlinear system of three first-order ordinary differential equations that arise from a model formulated by Anderson [11] which describes the transmission of HIV/AIDS in male homosexual/bisexual cohorts. A particular case of the model trans-
lates into a coupled nonlinear system of first-order ordinary differential equations [15]:

\[ F_1(t, u, u(1)) = u'_1 + \frac{\beta cu_1 u^2}{u^1 + u^2 + u^3} + \mu u'_1 = 0 \]
\[ F_2(t, u, u(1)) = u'_2 - \frac{\beta cu_1 u^2}{u^1 + u^2 + u^3} + (\nu + \mu) u^2 = 0 \]
\[ F_3(t, u, u(1)) = u'_3 - \nu u^2 + (\mu + \beta c) u^3 = 0, \]

where \( u(1) \) denotes the first-order partial derivatives \( u'_1, u'_2 \) and \( u'_3 \). This system is typical of models that are formulated to mimic dynamics in epidemiology. Such systems are often highly nonlinear and difficult to analyze. In their seminal work, Torrisi and Nucci [15] perform Lie symmetry analysis on (1.1). They determine a solvable Lie algebra admitted by the system and exploit this to find a solution (albeit in quadrature form) of the system. In this article we extend the Lie symmetry analysis of (1.1) by generating conservation laws for the system via Ibragimov’s new conservation theorem [9].

Conservation laws of physical systems are fundamental to our understanding of the system being studied. Apart from having a direct physical interpretation, conservation laws may be essential in studying the integrability of the system. For example, in the numerical integration of partial differential equations conservation laws help to control numerical errors in that they describe the properties of the system that do not change. On the whole conservation laws play an important role in the analysis of basic properties of the solutions. Therefore, the construction of conservation laws is one of the most important applications of symmetries to physical systems [10, 5].

The rest of this paper is organised as follows. We present elements of Lie symmetry analysis of differential equations in Section 2. An overview of the theorems of Noether and Ibragimov for constructing conservation laws via admitted symmetries is provided in Section 3. Conservation Laws of Anderson’s HIV model are constructed in Section 4. In Section 4 we discuss the results and give concluding remarks.

## 2 Preliminaries

Let us consider an \( r \)-th-order \((r \geq 1)\) system of differential equations with \( m \) dependent variables \( u = (u^1, u^2, \ldots, u^m) \) and \( n \) independent variable \( x = (x^1, x^2, \ldots, x^n) \), \( u = u(x) \),

\[ F_\alpha(x, u, \ldots, u(r)) = 0, \quad \alpha = 1, \ldots, m, \]

where \( u_{(k)} \) denotes the collection \( \{u_{\alpha}^k\} \) of \( k \)-th-order derivatives, \( k \geq 1 \). Suppose that (2.1) admits a one parameter Lie group of point transformations

\[ \tilde{x}^i = x^i + \varepsilon \xi^i(x, u) + O(\varepsilon^2), \]
\[ \tilde{u}^\alpha = u^\alpha + \varepsilon \eta^\alpha(x, u) + O(\varepsilon^2), \]

where \( \varepsilon \) is a real parameter; \( \xi^i \) and \( \eta^\alpha \) are given smooth functions. Invariance of (2.1) under (2.2) is conveniently expressed in terms of the infinitesimal generator of (2.2) is

\[ X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \]

(2.3)
where the usual convention of summation over repeated indices is adopted \[5\]. In fact this convention is adopted in all subsequent expressions. We say that (2.2) is a symmetry of (2.1) if and only if for all \(\alpha = 1, \ldots, m\)

\[
X^{(r)}[F_\alpha(x, u, \ldots, u^{(r)})] \bigg|_{2.1} = 0, \tag{2.4}
\]

where \(X^{(r)}\) is the \(r\)th extension of (2.3) defined by

\[
X^{(r)} = \xi^i(x, u)\partial_{x^i} + \eta^\alpha(x, u)\partial_{u^\alpha} + \zeta^\alpha_i(x, u, u^{(1)})\partial_{u^\alpha_i} + \zeta^\alpha_{i_{1i2}}(x, u, u^{(2)})\partial_{u^\alpha_{i_{1i2}}} + \cdots + \zeta^\alpha_{i_{1i2}\ldots i_s(x, u, u^{(1)}, \ldots, u^{(r)})\partial_{u^\alpha_{i_{1i2}\ldots i_s}}}, \tag{2.5}
\]

with the explicit formulas for the extended infinitesimal coefficients given recursively by

\[
\zeta^\alpha_i = D_i (\eta^\alpha) - u^\alpha_j D_i (\xi^j), \quad i = 1, 2, \ldots, n, \tag{2.6}
\]

\[
\zeta^\alpha_{i_{1i2}\ldots i_s} = D_{i_s} (\zeta^\alpha_{i_{1i2}\ldots i_{s-1}}) - u^\alpha_{j_{i_{1i2}\ldots i_{s-1}}} D_{i_s} (\xi^j), \quad s > 1, \tag{2.7}
\]

where \(D_i\) is the total derivative operator with respect to \(x^i\) defined by

\[
D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u_j^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots + u_{i_{1i2}\ldots i_{n}}^\alpha \frac{\partial}{\partial u_{i_{1i2}\ldots i_{n}}^\alpha} + \cdots. \tag{2.8}
\]

Note that in terms of the total derivative operator the derivatives of \(u^\alpha\) with respect to \(x_i\) are

\[
u^\alpha_i = D_i (u^\alpha), \quad u^\alpha_{ij} = D_i (u^\alpha_j) - D_i D_j (u^\alpha), \quad \ldots
\]

A conserved vector of (2.1) is an \(n\)-tuple

\[
C = (C^1(x, u, \ldots, u^{(r-1)}), \ldots, C^n(x, u, \ldots, u^{(r-1)}) \tag{2.9}
\]

such that

\[
D_i (C^i) = 0 \tag{2.10}
\]

on the solution space of (2.1). The expression (2.10) is a conservation law of (2.1).

3 The connection between conservation laws and admitted symmetries

A fundamental relationship between symmetries and conservation laws is provided by Noether’s theorem \[3\], which states that for Euler-Lagrange systems of differential equations, to each Noether symmetry associated with the Lagrangian there corresponds a conservation law which can be determined explicitly by a formula \[5\]. Noether’s theorem therefore reduces the search for conservation laws to a search for Noether symmetries. However, the dependence upon knowledge of a suitable Lagrangian to exploit Noether’s theorem diminishes the applicability of the theorem significantly. Ibragimov’s theorem \[9\] extends the application of Noether’s theorem by providing for the association of a
conservation law to every symmetry of a system of differential equations, albeit with
the proviso that the number of equations in the system equals the number of dependent
variables and that the given system be considered together with the associated adjoint
system. The rest of this section introduces the essential elements of Noether’s theorem
and Ibragimov’s theorem.

3.1 Noether’s theorem

Consider a system of differential equations identical with Euler-Lagrange equations
\[ \frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i \left( \frac{\partial L}{\partial u_i^\alpha} \right) = 0, \quad \alpha = 1, \ldots, m, \] (3.1)
arising from the variational integral
\[ \int L(x, u, \ldots, u_r) dx \] (3.2)
taken over an arbitrary \( n \)-dimensional domain in the space of the independent vari-
ables \( x = (x^1, \ldots, x^n) \). The Lagrangian \( L(x, u, \ldots, u_r) \) involves \( x \) and the dependent
variables \( u = (u^1, \ldots, u^n), u = u(x) \), together with their derivatives \( u_i \).

Let the system of differential equations (2.1) admit a continuous group \( G \) with a gen-
erator (2.3). Noether’s theorem states that if the variational integral (3.2) is invariant
under the group \( G \), then the vector field \( C = (C^1, \ldots, C^n) \) defined by
\[
C^i = L\xi^i + \left( \eta^\alpha - \xi^i u_j^\alpha \right) \left[ \frac{\partial L}{\partial u_i^\alpha} - D_j \left( \frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_k D_j \left( \frac{\partial L}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\
+ D_j \left( \eta^\alpha - \xi^i u_j^\alpha \right) \left[ \frac{\partial L}{\partial u_{ij}^\alpha} - D_k \left( \frac{\partial L}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\
+ D_j D_k \left( \eta^\alpha - \xi^i u_j^\alpha \right) \left[ \frac{\partial L}{\partial u_{ijk}^\alpha} - \cdots \right] + \cdots \] (3.3)
provides a conservation law for the Euler-Lagrange equations (3.1). Noether’s theo-
rem states that if the Invariance of the variational integral (3.2) is invariant under the group \( G \) is
established via the infinitesimal test for invariance,
\[ X (L) + LD_i \left( \xi^i \right) = 0, \] (3.4)
where the appropriate prolongation of \( X \) is understood.
3.2 Extension of Noether’s theorem: Conservation Laws via Ibragimov’s theorem [9]

Consider a system of $r$th-order differential equations defined in (2.1). We introduce the differential functions

$$F^*_\alpha(x, u, v, \ldots, u^{(r)}, v^{(r)}) = \delta(v^\beta F_\beta), \quad \alpha = 1, \ldots, m$$

(3.5)

where $v = (v^1, \ldots, v^m)$ are new dependent variables, $v = v(x)$, and

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum (-1)^i D_i, \ldots, D_r \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_r}}, \quad \alpha = 1, \ldots, m.$$  

(3.6)

The system of adjoint equations to (2.1) is defined by

$$F^*_\alpha(x, u, v, \ldots, u^{(r)}, v^{(r)}) = 0, \quad \alpha = 1, \ldots, m.$$  

(3.7)

We now have that the simultaneous system consisting of the $r$th-order differential equations (2.1) considered together with its adjoint equation (3.7) has a Lagrangian defined by

$$L = v^\beta F_\beta(x, u, \ldots, u^{(r)}).$$  

(3.8)

Furthermore, the adjoint system (3.7) inherits the symmetries of the system (2.1) in the sense that if the system (2.1) admits a point transformation group with a generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha},$$  

(3.9)

then the adjoint system (3.7) admits the operator (3.9) extended to the variables $v^\alpha$ by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_v \frac{\partial}{\partial v^\alpha},$$  

(3.10)

with coefficients $\eta^\alpha_v = \eta^\alpha_v(x, u, v, \ldots)$ chosen in such a way that $Y$ satisfies the infinitesimal test for invariance of the variational integral associated with (3.8), i.e,

$$Y (L) + LD_i (\xi^i) = 0,$$  

(3.11)

where the generator $Y$ is prolonged appropriately to the $r$th derivatives $u^{(r)}$ and $v^{(r)}$. It turns out that

$$\eta^\alpha_v = - \left[ \lambda^\alpha_\beta v^\beta + v^\alpha D_i (\xi^i) \right], \quad \alpha = 1, \ldots, m.$$  

(3.12)

with $\lambda^\alpha_\beta$ defined by the invariance condition (2.4) in the form

$$X (F_\alpha) = \lambda^\alpha_\beta F_\beta, \quad \alpha = 1, \ldots, m,$$  

(3.13)

where the prolongation of $X$ to all derivatives involved in Lagrangian the system (2.1) is understood.
Noether’s theorem is now employed to furnish the conserved vector \( C^i = (C^1, \ldots, C^m) \), with \( C_i \) defined by

\[
C^i = L \xi^i + (\eta^\alpha - \xi^i u_j^\alpha) \left[ \frac{\partial L}{\partial u_i^\alpha} - D_j \left( \frac{\partial L}{\partial u_j^\alpha} \right) + D_j D_k \left( \frac{\partial L}{\partial u_j^\alpha} \right) - \cdots \right] \\
+ D_j (\eta^\alpha - \xi^i u_j^\alpha) \left[ \frac{\partial L}{\partial u_j^\alpha} - D_k \left( \frac{\partial L}{\partial u_k^\alpha} \right) + \cdots \right] \\
+ D_j D_k (\eta^\alpha - \xi^i u_j^\alpha) \left[ \frac{\partial L}{\partial u_k^\alpha} - \cdots \right] + \cdots 
\]

(3.14)

where \( u^\alpha = (u^1, \ldots, u^m, v^1, \ldots, v^m) \).

### 4 Conservation Laws of Anderson’s HIV model

For the problem at hand, (1.1), we have a system of three dependent variables \( u = (u^1, u^2, u^3) \) and one independent variable \( t \), where \( u = u(t) \). This system is considered together with the corresponding adjoint system of equations which is constructed as outlined in Section 3. Let \( v = (v^1, v^2, v^3) \) be the new dependent variables, \( v = v(t) \). According to (3.7) the adjoint system is given by

\[
F^*_\alpha(t, u, v, u^{(1)}, v^{(1)}) = \delta \frac{(v^\beta F^\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, 3, 
\]

(4.1)

where \( \delta/\delta u^\alpha \) is the Euler-Lagrange operator defined by (3.6), which for the problem being considered reduces to

\[
\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_t \frac{\partial}{\partial u^\alpha}, \quad \alpha = 1, 2, 3, 
\]

(4.2)

where \( D_t \) is the total derivative operator with respect to \( t \) defined by

\[
D_t = \frac{\partial}{\partial t} + u^1 \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^2} + u^3 \frac{\partial}{\partial u^3} + \cdots. 
\]

Thus (4.1) translates into the following adjoint system of equations:

\[
\begin{align*}
\frac{dv^1}{dt} &= v^1 \left( \frac{u^2 \beta \delta}{u^1 + u^2 + u^3} - \frac{u^1 u^2 \beta \delta}{(u^1 + u^2 + u^3)^2} + \mu \right) - \frac{v^2 u^2 (u^2 + u^3) \beta \delta}{(u^1 + u^2 + u^3)^2}, \\
\frac{dv^2}{dt} &= v^2 \left( \mu + \nu - \frac{u^1 (u^1 + u^3) \beta \delta}{(u^1 + u^2 + u^3)^2} \right) + \frac{v^1 u^1 (u^1 + u^3) \beta \delta}{(u^1 + u^2 + u^3)^2} - v^3 \nu, \\
\frac{dv^3}{dt} &= v^3 \alpha - \frac{v^1 u^1 u^2 \beta \delta}{(u^1 + u^2 + u^3)^2} + \frac{v^2 u^1 u^2 \beta \delta}{(u^1 + u^2 + u^3)^2}.
\end{align*}
\]

(4.3)
According to the infinitesimal condition for invariance (2.4) the system \( \text{(1.1)} \) admits a symmetry group with the infinitesimal generator

\[
X = \xi(t, u^1, u^2, u^3)\partial_t + \eta^1(t, u^1, u^2, u^3)\partial_{u^1} + \eta^2(t, u^1, u^2, u^3)\partial_{u^2} + \eta^3(t, u^1, u^2, u^3)\partial_{u^3}
\]

if and only if

\[
X^{(1)}F_1|_{\text{(1.1)}} = 0, \quad X^{(1)}F_2|_{\text{(1.1)}} = 0, \quad X^{(1)}F_3|_{\text{(1.1)}} = 0,
\]

(4.4)

where

\[
X^{(1)} = \xi\partial_t + \eta^1\partial_{u^1} + \eta^2\partial_{u^2} + \eta^3\partial_{u^3} + \zeta^1\partial_{u^1} + \zeta^2\partial_{u^2} + \zeta^3\partial_{u^3}
\]

(4.5)

with

\[
\zeta^1 = D_t(\eta^1) - u^1_t D_t(\xi), \quad \zeta^2 = D_t(\eta^2) - u^2_t D_t(\xi), \quad \zeta^3 = D_t(\eta^3) - u^3_t D_t(\xi).
\]

(4.6) - (4.8)

After making an ansatz on the form of the operator \( X \) (assuming that the functions \( \xi, \eta^1 \) and \( \eta^2 \) are polynomials of second degree on \( u^1, u^2 \) and \( u^3 \)) the solution of the equations (4.4), after lengthy analysis, leads to a three-dimensional Lie symmetry algebra \( L_3 = \langle X_1, X_2, X_3 \rangle \) admitted by \( \text{(1.1)} \) with the following basis operators [15]:

\[
X_1 = \partial_t, \quad X_2 = u^1\partial_{u^1} + u^2\partial_{u^2} + u^3\partial_{u^3}, \quad X_3 = e^{-(\mu+\nu)t}\partial_{u^2} + \frac{u^1 + u^3}{u^2} e^{-(\mu+\nu)t}\partial_{u^3}.
\]

(4.9)

We shall find conservation laws for the simultaneous system \( \text{(1.1)} \) and \( \text{(4.3)} \) using each of the symmetries in \( \text{(4.9)} \), extended suitably to the adjoint variables \( v^1, v^2 \) and \( v^3 \). According to (3.8) the system \( \text{(1.1)} \) and \( \text{(4.3)} \) considered together has the Lagrangian

\[
L = v^\beta F_\beta = v^1 \left[ u^1_t - \frac{\beta cu^1 u^2}{u^1 + u^2 + u^3} + \mu u^1 \right] - v^2 \left[ u^2_t - \frac{\beta cu^1 u^2}{u^1 + u^2 + u^3} + (\nu + \mu) u^2 \right] + v^3 \left[ u^3_t - \nu u^2 + (\mu + \beta c) u^3 \right].
\]

(4.10)

Each of the symmetries in \( \text{(4.9)} \) has the form

\[
X = \xi\partial_t + \eta^1\partial_{u^1} + \eta^2\partial_{u^2} + \eta^3\partial_{u^3}
\]

(4.11)

and needs to be extended appropriately to the operator

\[
Y = X + \eta^1_\alpha \partial_{u^1} + \eta^2_\alpha \partial_{u^2} + \eta^3_\alpha \partial_{u^3},
\]

(4.12)

to cater for the adjoint differential variables. It turns out that \( \text{(4.12)} \) is admitted by the adjoint system \( \text{(4.3)} \) provided the infinitesimal coefficients \( \eta^\alpha_\alpha \) in the operator are prescribed as follows [9]:

\[
\eta^\alpha_\alpha = - [X^\alpha_\beta v^\beta + v^\alpha D_t(\xi)], \quad \alpha = 1, 2, 3.
\]

(4.13)

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where $\lambda^\alpha_\beta$ are defined by the invariance condition \((3.13)\). Taking $X_1$ from \((4.9)\) and extending it once to $X^{(1)}_1 = X_1$, condition \((3.13)\) leads to a set of equations,

\[
\begin{align*}
X^{(1)}_1 F_1 &= \lambda_1^1 F_1 + \lambda_1^2 F_2 + \lambda_3^1 F_3 \\
X^{(1)}_1 F_2 &= \lambda_1^2 F_1 + \lambda_2^2 F_2 + \lambda_3^2 F_3 \\
X^{(1)}_1 F_3 &= \lambda_1^3 F_1 + \lambda_2^3 F_2 + \lambda_3^3 F_3
\end{align*}
\]  \((4.14-4.16)\)

which must be solved for $\lambda^\alpha_\beta$. Clearly $X^{(1)}_1 F_\alpha = 0$ for all $\alpha = 1, 2, 3$. It follows, therefore, that $\lambda^\alpha_\beta = 0$ for all $\alpha$ and $\beta$. Furthermore, $\xi = 1$ in the generator $X_1$, which leads us to the conclusion that in this case $\eta^\alpha_* = 0$ for all $\alpha = 1, 2, 3$. Therefore

\[
Y_1 = X_1 = \partial_t.
\]  \((4.17)\)

For $X_2$ and $X_3$, the coefficient $\xi$ equals zero, which means that the second term in \((3.12)\) vanishes. Proceeding as we did for $X_1$, we determine $\lambda^\alpha_\beta$ in each of the two cases and obtain $Y_2$ and $Y_3$, via the desired extensions of $X_2$ and $X_3$ respectively:

\[
\begin{align*}
Y_2 &= u^1 \partial_{u^1} + u^2 \partial_{u^2} + u^3 \partial_{u^3} - v^1 \partial_{v^1} - v^2 \partial_{v^2} - v^3 \partial_{v^3}, \\
Y_3 &= e^{-(\mu + \nu)t} \left[ \partial_{u^2} + \frac{u^1 + u^3}{u^2} \partial_{u^3} - c \left( \frac{1}{u^2} \partial_{u^1} - \frac{u^1 + u^3}{(u^2)^2} \partial_{v^1} + \frac{1}{u^2} \partial_{v^3} \right) \right].
\end{align*}
\]  \((4.18-4.19)\)

By applying Noether’s theorem to each of the generators $Y_i$ with the associated Lagrangian \((4.10)\) we wish to find the corresponding conservation laws. Let us rename the dependent variables $v^1$, $v^2$ and $v^3$ of the adjoint system as $u^4$, $u^5$ and $u^6$, respectively, so that each of the generators \((4.17)\), \((4.18)\) and \((4.19)\) is in the form

\[
Y = \xi(t, u) \frac{\partial}{\partial t} + \eta^\alpha(t, u) \frac{\partial}{\partial u^\alpha},
\]  \((4.20)\)

where

\[
u = (u^1, u^2, u^3, v^1, v^2, v^3) = (u^1, u^2, u^3, u^4, u^5, u^6).
\]

According to \((2.10)\) and \((3.14)\), the conservation law corresponding to \((4.20)\) is

\[
D_t \left\{ \xi L + (\eta^\alpha - \xi u^\alpha) \left( \frac{\partial L}{\partial u^\alpha} - D_t \frac{\partial L}{\partial u^\alpha} \right) \right\} = 0.
\]  \((4.21)\)

We therefore obtain the following conservation laws:

\[
D_t \left\{ \frac{u^1 u^2 \beta \delta (v^1 - v^2)}{u^1 + u^2 + u^3} + v^1 u^1 \mu + v^2 u^2 (\mu + \nu) + v^3 (u^3 \alpha - u^2 \nu) \right\} = 0
\]

\[
D_t \left\{ v^1 u^1 + v^2 u^2 + v^3 u^3 \right\} = 0
\]  \((4.22)\)

\[
D_t \left\{ e^{-(\mu + \nu)t} \left[ v^2 + \frac{v^3 (u^1 + u^3)}{(u^1)^2} \right] \right\} = 0,
\]

corresponding to each of the extended symmetries $Y_1$, $Y_2$ and $Y_3$, respectively, where $v = (v^1, v^2, v^3)$ is any solution of the adjoint system of equations \((3.14)\).
5 Concluding remarks

In this paper we have considered a nontrivial nonlinear system of first-order ordinary differential equations arising from a mathematical model formulated by Anderson [11] to describe the transmission of HIV/AIDS in male homosexual/bisexual cohorts. We have applied Ibragimov’s theorem and constructed conservation laws of the system considered together with the associated adjoint system. We have, however, not attempted to attach physical meaning to the adjoint system (4.3) and/or the constructed conservation laws (4.22). We defer this to possible future work on the model (1.1). The application reported in this paper is an instructive application of Ibragimov’s new theorem and may be used to construct conservation laws in other settings involving systems of first-order ordinary differential equations.

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