Precise comparison of theory and new experiment for the Casimir force leads to stronger constraints on thermal quantum effects and long-range interactions

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Abstract

We report an improved dynamic determination of the Casimir pressure $P_{\text{expt}}$ between two plane plates obtained using a micromachined torsional oscillator. The main improvements in the current experiment are a significant suppression of the surface roughness of the Au layers deposited on the interacting surfaces, and a decrease by a factor of 1.7 (down to 0.6 nm) in the experimental error in the measurement of the absolute separation. A metrological analysis of all data for $P_{\text{expt}}$ from 15 sets of measurements permitted us to determine both the random and systematic errors, and to find the total experimental error in $P_{\text{expt}}$ as a function of separation at the 95% confidence level. In contrast to all previous experiments on the Casimir effect, where a small relative error was achieved only at the shortest separation, our smallest experimental error ($\sim 0.5\%$) is achieved over a wide separation range. The theoretical Casimir pressures $P_{\text{theor}}$ in the experimental configuration were calculated by the use of four theoretical approaches suggested in the literature based on the Lifshitz formula at nonzero temperature. All corrections to the Casimir force due to grain structure of the overlying metal layers (including the variation of optical data and patch potentials), surface roughness (including non-multiplicative and diffraction-type effects), and nonlocal effects, were calculated or estimated. The maximum value of the roughness correction, achieved at the shortest separation of 160 nm, is equal to only 0.65\% of the Casimir pressure. All theoretical
errors, including those introduced by the proximity force theorem, finite size of the plate area, and uncertainties in the experimental separations, were analyzed and metrologically combined to obtain the total theoretical error at the 95% confidence level. Finally, the confidence interval for \((P_{\text{theor}} - P_{\text{expt}})\) was obtained as a function of separation. Our measurements are found to be consistent with two theoretical approaches utilizing the plasma model and the surface impedance over the entire measurement region from 160 nm to 750 nm. Two other approaches to the thermal Casimir force, utilizing the Drude model or a special prescription for the determination of the zero-frequency contribution to the Lifshitz formula, are excluded on the basis of our measurements at the 99% and 95% confidence levels, respectively. Finally, constraints on Yukawa-type hypothetical interactions are strengthened by up to a factor of 20 in a wide interaction range.

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1. Introduction

Recent advances in the experimental and theoretical investigation of the Casimir effect suggest that this field of research is important for both fundamental and applied physics. The Casimir force, which acts between two uncharged plates in vacuum, is a purely quantum effect caused by the alteration of the zero-point oscillations of the electromagnetic field due to boundaries [1]. The modern stage in Casimir research originated in a number of measurements of the Casimir force between metals (see [2,3,4,5,6,7,8,9,10,11] and review [12]). Additional motivation for studying the Casimir force comes from promising applications of these results in both fundamental physics (for constraining hypothetical long-range interactions [11,13,14,15,16,17]), and in nanotechnology [18,19]. Recently the Casimir-Polder force has attracted attention [20,21] in connection with experiments on quantum reflection and Bose-Einstein condensation.

As often happens, experimental progress placed more stringent demands upon theoretical calculations of the Casimir force. At present, to achieve an agreement between theory and experiment at the level of 1% of the measured force, special care must be taken to account for all relevant factors, such as the finite conductivity of the metallic boundaries, the effect of thermal corrections, surface roughness, grain structure of the covering layers, and patch potentials.

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These are reviewed in [12], and more recently in [22] which is devoted to the application of these questions to Casimir force measurements between gold surfaces by means of an atomic force microscope [6].

The basic theoretical formalism for the calculation of the van der Waals and Casimir forces between real materials is given by Lifshitz theory [23,24,25]. Beginning in 2000, several research groups applied this theory to the case of real metals at nonzero temperature and four different approaches were proposed. In the first approach [26] the zero-frequency contribution to the Lifshitz formula was determined by the use of the Drude dielectric function. The second approach [27] modified the zero-frequency term of the Lifshitz formula to make it the same for ideal and real metals. In the third approach [28,29] the Lifshitz formula was combined with the free electron plasma model neglecting relaxation. Finally, in [30] a fourth approach was proposed in which the Lifshitz formula at nonzero temperature is rederived using the Leontovich surface impedance boundary conditions rather than the bulk dielectric permittivity. Within this last approach [30,31], the zero-frequency term of the Lifshitz formula is determined by the form of the Leontovich impedance valid near the characteristic frequency. Detailed discussions of the advantages and disadvantages of each of proposed approaches can be found in [32,33,34,35,36,37,38,39,40,41,42]. These discussions are part of a broader context connected with the fundamental nature of the zero-point oscillations and their relation to basic physical principles such as the laws of thermodynamics. In particular, it has been shown [31,33] that the first and second theoretical approaches to the description of the thermal Casimir force between real metals lead to a violation of the third law of thermodynamics (the Nernst heat theorem) in the case of perfect crystals. The attempt made in [37,38] to avoid this conclusion in the framework of the first approach required the introduction of defects and impurities to the crystal, which cannot be considered as satisfactory since the perfect crystal forms the basis of the theory of electron-phonon interactions. It follows that any formalism applied to perfect crystals must satisfy the third law of thermodynamics, which the formulations of the first approach do not.

We consider now a comparison of the different theoretical approaches with experiment. It should be noted that the third and fourth approaches predict small thermal corrections to the Casimir force at short separations of order 100 nm, in qualitative agreement with the case of an ideal metal, whereas the first and second approaches lead to thermal corrections which are many times greater [32]. However, taking into account the small magnitude of the Casimir force, reliably distinguishing among the different predicted thermal corrections presents a serious challenge to experimental physics. In this regard it is important that the first approach (see [26] and also [34,35,36,37]) disagrees significantly [41,42] with the most famous modern measurement of the Casimir force between Au surfaces by means of a torsion pendulum [2]. A net deviation
from zero temperature Casimir force is predicted by the first approach to be about 25% at a separation $a = 1000$ nm. It was not, however, observed [2,41,42] despite the fact that the experimental uncertainty was less than 10%.

The most precise and accurate measurement of the Casimir force to date was made between two dissimilar metals (Cu and Au) by means of a microelectromechanical torsional oscillator [10,11]. In that experiment the Casimir force between a plate and a sphere was measured statically, and the effective Casimir pressure between two parallel plates was determined dynamically, with an absolute error of $\approx 0.6$ mPa at 95% confidence. The estimated theoretical uncertainty in [11] was approximately 1% of the Casimir pressure over the entire measurement range. This was, however, a purely theoretical error which did not take into account errors in the theoretical pressure induced by uncertainties in the experimental determination of the sphere-plate separations. (Recall that a micromachined oscillator was first used in [18] to demonstrate a micromechanical device driven entirely by the Casimir force.) The experimental data in [11] were not precise enough to observe the traditional thermal corrections of the third and fourth approaches. However, the comparison of these data with theory demonstrates that the alternative thermal corrections to the Casimir pressure, as predicted in the first [26] and second [27] approaches, are excluded by experiment within a wide separation region. The good agreement between theory (based on the surface impedance approach) and experiment was used in [11] to obtain stronger constraints on hypothetical long-range interactions predicted by many extensions to the Standard Model of fundamental interactions.

In this paper we present the results of an improved experimental determination of the Casimir pressure reported in [11], and of a refined comparison of these new results with all four theoretical approaches. The following modifications in the experimental setup allowed us to obtain results exceeding those of [11] in both reliability and conclusiveness. The main improvement is associated with a considerable decrease of the surface roughness on both the plate and the sphere. In [11] the highest roughness peaks were as much as $\approx 100$ nm in height on both bodies. In the improved experiment the maximum heights of the roughness peaks are 11.06 nm on the sphere and 20.63 nm on the plate. An improvement in detection sensitivity, together with a reduction of the coupling between the micromachined oscillator and the environment, yielded measurements at smaller separations between the test bodies (160 nm instead of 260 nm). Another important improvement was the reduction of the error in determining the absolute separation between the sphere and the top of the metal-coated plate from 1 nm in [11] to 0.6 nm. A special experimental study was performed to ensure the linearity of the microelectromechanical torsional oscillator. In addition Au was used as a covering layer on both the plate and the sphere (instead of Cu and Au, respectively, as in [11]), which permits a more reliable theoretical interpretation of the experimental results.
The refined theory includes a complete analysis of the finite conductivity, surface roughness, and thermal corrections to the Casimir pressure. In contrast to [11], the finite conductivity corrections are analyzed by taking into account the grain structure of the covering layers, possible sample-to-sample variations of the tabulated optical data, and the contribution of nonlocal effects. The roughness analysis incorporates non-multiplicative, diffraction-type and correlation effects. It is shown that in the present experiment the maximum value of the roughness correction, which is achieved at a separation of 160 nm, is only 0.65% of the Casimir pressure. Thermal Casimir pressures are computed in the framework of all four theoretical approaches.

A special metrological analysis has been performed separately for the experimental data and for the results of the theoretical computations. The random, systematic, and total experimental errors are found as functions of separation, taking into account the distribution laws for each error at the 95% confidence level. A careful comparison between experiment and theory has been carried out taking into account all experimental and theoretical errors. This comparison is based on rigorous metrological criteria which do not use the concept of the root-mean-square deviation between theory and experiment over the entire separation range, a procedure that had been criticized in literature [7]. As a result, the alternative thermal corrections to the Casimir pressure, as proposed in the first [26] and second [27] approaches, are excluded by experiment, whereas the third [28,29] and fourth [30,31] approaches are shown to be consistent with experiment. These results confirm those obtained earlier in [11]. In this paper we also improve the constraints on hypothetical long-range interactions obtained in [11]. By combining the results of this paper and [11], the previously known constraints are strengthened by a factor of up to 20 in a wide interaction region around 0.1 µm.

The present paper is organized as follows. In Section 2 the experimental configuration and sample characterization are briefly described. Section 3 contains the calibration procedure and the experimental results for the Casimir pressure. In Section 4 the metrological analysis of these results is performed and the experimental errors are found at 95% confidence. In Section 5 we calculate the Casimir pressure between Au layers in the framework of the four approaches described above taking into account different kinds of corrections. Section 6 is devoted to a metrological analysis of the theoretical uncertainties. In Section 7 all four theoretical approaches to the thermal Casimir force are compared with the experimental data. The level of agreement between experiment and theory is used to obtain stronger constraints on hypothetical long-range interactions in Section 8. We conclude in Section 9 with a discussion of our results.
2. Improved experimental setup and sample characterization

2.1. Details of the improved experiment

The general scheme of the experimental setup used to determine the Casimir interaction between two bodies by means of a micromachined oscillator was described previously in [10,11]. One of the new features of the present experiment is the fact that here the Casimir attraction is measured between two layers of Au, specifically, a Au-coated sapphire sphere and a Au-coated polysilicon microelectromechanical torsional oscillator (MTO). To improve the force sensitivity, the vertical separation between the sphere and the plate was changed harmonically with time, leading to a measurement of the $z$-derivative of the Casimir force. As described in [11], this is equivalent to measuring the Casimir force per unit area, or the Casimir pressure, for a configuration of two parallel plates (see below). The MTO presents a low coupling with the environment, since it is less sensitive to modes that involve a displacement of the center of mass. The miniaturization also allows us to achieve a large quality factor $Q \sim 8000$.

The experimental arrangement is shown schematically in Fig. 1. The MTO is made of a 3.5 $\mu$m thick, $500 \times 500 \mu$m$^2$ heavily doped polysilicon plate suspended at two opposite points by serpentine springs. Two independently contacted polysilicon electrodes located under the plate are used to induce an oscillation on the plate at the required resonant frequency of the MTO. A 80 $\mu$m wide ribbon at the edge of the plate was coated with 10 nm of Pt followed by 150 nm of Au. The Au layer constitutes one of the metals used in the measurement of the Casimir interaction. The separation between the two Au layers is given by

$$z = z_{\text{meas}} - D - b \theta.$$  \hspace{1cm} (1)

In this expression $z_{\text{meas}}$ is the separation between the end of the cleaved fiber and the platform, as interferometrically measured with the absolute error $\Delta z_{\text{meas}} = 0.2$ nm. $D = D_1 + D_2$, where $D_2$ represents the separation between the platform and the top of the Au-coated plate when the interaction between the sphere and the plate is negligible (i.e., $\theta = 0$), and $D_1$ is the distance between the end of the bottom of the cleaved fiber and the external surface of the Au film on the Au-coated sphere (see Fig. 1). The lever arm $b = (210 \pm 3) \mu$m is determined optically. The value of $\theta$ is determined by measuring the difference in capacitance between the plate and the right and left electrodes $C_{r,l} = C_{\text{right}} - C_{\text{left}}$. In all reported cases $\theta \leq 10^{-5}$ rad. A field effect transistor (FET), the first stage in the amplifier circuit, was placed as close as possible to the oscillator to minimize the effect of parasitic capacitances. The
smallest angular deviation that can be detected in this configuration is given by

$$\Delta \theta \approx \frac{\Delta C_{r,l} 2d_g}{C_o w'} \approx \frac{\Delta V}{V_i} \frac{C_T 2d_g}{C_o w'} \sim 10^{-9}\sqrt{\nu} \text{rad/Hz},$$

(2)

where $\Delta C_{r,l}$ is the minimum detectable change in $C_{r,l}$, and $C_o \approx \epsilon_o w w'/d_g$ is the capacitance between the oscillator and each electrode when no forces are present ($\epsilon_o$ is the permittivity of free space). The dimensions of the electrodes are $w = 250 \mu m$, $w' = 190 \mu m$, and $d_g = 2 \mu m$ is the separation between the plate and the electrodes. $C_T \approx 20 \text{pF} \gg C_o$ is the parasitic capacitance of the measurement circuit (determined by the capacitance of the FET transistor plus parasitic capacitances to ground), $\Delta V \sim 10\sqrt{\nu} \text{nV/Hz}^{1/2}$ is the input noise of the amplifier, $\nu$ is the bandwidth of the measurement, and $V_i (i = 1, 2) \sim 1 \text{ V}$ is the DC potential used to correct for initial asymmetries in the circuit and to linearize its response. In obtaining Eq. (2) parallel plate approximations, and the fact that the maximum angular deviation $\theta_{max} \ll 1$, have been used. In the relevant (100 – 5000) Hz range, actual measurements of $\Delta C_{r,l}$ coincide within 15% with those predicted by Eq. (2).

A sphere with a nominal radius $R = 150 \mu m$ was coated with a 10 nm layer of Ti followed by a 200 nm layer of Au. For both the plate of the MTO and the sphere, vacuum was not interrupted in between depositions. Deposition induced asymmetries in the sphere were found to be smaller than 10 nm, the resolution of the scanning electron microscope (SEM) used to characterize them. The entire setup (MTO and fiber-sphere) was rigidly mounted onto a can which was hung inside a vacuum chamber by means of soft springs. A built-in magnetic damping vibration isolation helped in reducing the coupling with the main resonance of the building, occurring at 7 Hz. Finally, the vacuum chamber was, in turn, mounted onto an air table. This combination of vibration isolation systems yielded peak-to-peak vibrations with $\Delta z_{pp} < 0.02 \text{ nm}$ (the detection limit in our accelerometer) for frequencies above 100 Hz. To reduce the damping in the MTO, the vacuum chamber was initially evacuated to a pressure less than $10^{-5} \text{ torr}$. In order to reduce external vibrations, the chamber was then closed, the pumping station removed, and the vacuum lines disconnected. The pressure was kept low during the experiment by means of a chemical pump, made of a cold ($T = 77\text{K}$) activated carbon trap located inside the vacuum chamber. A pressure $\lesssim 10^{-5} \text{ torr}$ was maintained with this setup for the length of each experimental run. When necessary, the vacuum chamber was re-evacuated between runs.
2.2. Sample characterization

Before performing the Casimir interaction measurements, the sample was characterized using an atomic force microscope (AFM). This serves the double purpose of determining the roughness of the sample and also, by repeating the AFM study after the Casimir measurements, to check that the sample was not modified during the measurement. The AFM images were obtained for both the plate and the sphere. The results are shown in Fig. 2. It is not possible for us to ensure that the region shown in Fig. 2a and Fig. 2b (before and after the measurements, respectively) is exactly the one giving the dominant contribution to the interaction with the sphere. We note, however, that we imaged with the AFM a region of \((100 \times 100) \mu m^2\), located at the center of the Au-coated region of the plate, and Fig. 2a represents just a fraction of this region. The remainder of the investigated region, however, shows a very similar behavior, with identical topography except for two dust particles observed. For the sphere it is easier to find the interacting area since the fiber itself provides a preferential direction. Figs. 2c and 2d show AFM images of the bottom of the sphere, obtained before and after the experiment. As in the case of the plate, there are no discernible modifications in the topography. It is worth mentioning that a spherical surface of radius \(R\) was first subtracted from the data shown in Fig. 2c and Fig. 2d. For all cases shown in Fig. 2 an extra planarization was performed to account for the small non-linearities of the \(z\)-axis of the AFM’s translational stage. The extra planarization for both the plate and sphere had \(|\Delta z| < 0.5 \text{ nm}\). In Fig. 3, typical cross-sections of the images of Figs. 2b and 2d at \(y = \text{const}\) are shown for the plate (a) and the sphere (b). Results obtained either before the Casimir measurements or at \(x = \text{const}\) are quite similar. Fig. 3 permits us to estimate the correlation length of surface roughness (see Section 5.4).

These AFM images were used to characterize the roughness of the sample, which has to be taken into account when comparing the measured Casimir interaction with the theoretical predictions. As was done in [11], the roughness is represented by the fraction \(v_i\) of the sample with height \(h_i\). Data resulting from the AFM images of Figs. 2b and 2d are presented in Figs. 4a and 4b, respectively. The heights \(h_i\) are plotted along the vertical axis as a function of the fraction \(w_i\) of the total surface area having height \(h < h_i\). The width of each horizontal step is equal to the fraction of the total area \(v_i\) with heights \(h_i \leq h < h_{i+1}\) (1 \(\leq i \leq K\), where \(K = 105\) for the plate and \(K = 112\) for the sphere; in both cases \(h_1 = 0\)). Evidently \(w_i = v_1 + v_2 + \ldots + v_i\) and \(w_K = 1\). The data of Figs. 4a and 4b will be used in Section 5.4 for the calculation of the roughness corrections to the Casimir pressure. As was noted in the Introduction, the surface roughness was considerably decreased compared with [11]. (The highest peaks here, \(h_{112} = 11.06 \text{ nm}\) on the sphere and \(h_{105} = 20.65 \text{ nm}\) on the plate, are much lower than the highest peak.
3. Experimental results for the Casimir pressure

3.1. System calibration

The calibration of the system was performed in an analogous manner to that reported in [11]. The electrostatic force $F_{el}(z)$ between the plate and the sphere was measured as a function of the separation $z$ for $z > 3 \mu m$, where the Casimir force is smaller than 0.1% of the total force. This measurement was repeated for different potential differences between the plate and the sphere. For a given potential difference, $F_{el}(z)$ between a sphere and an infinite plane is given by

$$F_{el} = 2\pi \varepsilon_0 (V_{Au} - V_0)^2 \sum_{n=1}^{\infty} \frac{\coth(u) - n \coth(nu)}{\sinh(nu)}.$$  \hspace{1cm} (3)

Here $V_{Au}$ is the voltage applied to the sphere, $V_0$ is the residual potential difference between the metallic layers when they are both grounded, and $\cosh u = [1 + z/R]$. Fig. 5 shows the dependence of $\theta$ (and hence the electrostatic force) on the applied voltage $V_{Au}$. Adjusting $V_{Au}$ in Eq. (3) we found that $F_{el} = 0$ N within the experimental uncertainty for $V_0 = (17.5 \pm 0.1) \text{ mV}$. $V_0$ represents the difference in work functions between the Au layer on the plate and the sphere, along the path that closes the electric circuit between them. This value was observed to be constant for $z$ in the (0.15–5) $\mu m$ range, and it did not vary when measured over different locations in the Au layer on the plate. Note that $F_{el}$ is measured between the zero roughness levels on a plate and a sphere relative to which the mean values of roughness profiles are zero (see also Section 5.4). As a result, the absolute separations from Eq. (1) are also determined between the zero roughness levels. For this reason, there is no systematic error due to roughness in the measurements of separations in addition to uncertainties discussed in Section 4.1 below.

A set of 120 curves of $F_{el}(z)$ was then used to fit for the parameter $D = D_1 + D_2$ (the error on $D$ was originally estimated to be no smaller than 50 nm). To check the stability of the system, a subset of these curves (typically 30 of them) was repeated after each set of the separation dependence measurements, $P(z)$, of the Casimir interaction. Three curves for different $\Delta V$ are shown in Fig. 6. The local $R$ in the configuration when the sphere and the plane are interacting, and the proportionality factor $k$ between the measured $\Delta C$ and $\theta$, are also obtained through the measured $F_{el}(z)$. Finally, the following values of all three
parameters were determined: \( D = (9349.7 \pm 0.5) \text{ nm} \), \( R = (148.7 \pm 0.2) \text{ \( \mu \text{m} \)} \), and \( k = (50455 \pm 7) \text{ N/F} \). The determination of \( D \) and the measurement of \( \theta \) then yield \( z \) when \( z_{\text{meas}} \) is measured interferometrically.

3.2. Casimir pressure measurements

As in [10,11], the most sensitive measurement of the Casimir interaction arises from measuring the change of the angular resonant frequency \( \omega_r \) of the oscillator in the presence of vacuum fluctuations. In this situation, \( \omega_r \) is given by [10,11,18]

\[
\omega_r^2 = \omega_0^2 \left[ 1 - \frac{b^2}{I \omega_0^2} \frac{\partial F}{\partial z} \right],
\]

(4)

where \( \omega_0 = 2\pi \times 702.92 \text{ Hz} \) is the natural angular resonant frequency of the MTO, \( F \) is the Casimir force between a plate and a sphere, \( I \) is the moment of inertia of the MTO, and \( b^2/I = (1.2579 \pm 0.0006) \text{ \( \mu \text{g}^{-1} \)} \) was obtained from the electrostatic interaction \( F_{el} \). Using the proximity force theorem, we arrive at the Casimir pressure between the two plates

\[
P(z) = -\frac{1}{2\pi R} \frac{\partial F(z)}{\partial z}.
\]

(5)

The resonant frequency of the MTO at a separation \( z \) [and, hence, \( P(z) \)] was measured by changing the separation between the sphere and the plate as

\[
\tilde{z}(t) = z + A_z \cos(\omega_r t),
\]

(6)

where \( A_z/z \ll 1 \). For appropriately small values of \( A_z \) Eq. (4) is recovered. In reality, however, this is only an approximation, and the actual equation of motion for the MTO becomes non-linear [18]. For the determination of \( \omega_r \) to be experimentally relevant, the value of \( A_z \) has to be sufficiently small such that the effect of non-linearities is smaller than \( \Delta \omega_r \), the error in the measurement of \( \omega_r \). On the other hand, \( A_z \) should be as large as possible to minimize \( \Delta \omega_r \). The empirical approach we used in this paper is the following: At a separation \( z \) we determined \( \tilde{\omega}_r \) by measuring the spectral response of \( C_{r,l} \) under the influence of thermodynamic noise. These measurements were performed with an integration time \( \tau = 500 \text{ s} \). Subsequently a motion described by Eq. (6) was induced with \( \omega = \omega_r \) fixed by means of a phase-lock-loop circuit. This was done while increasing the value of \( A_z \) until a decrease in \( \Delta \omega_r \) was observed, while at the same time holding \( \omega_r \) fixed at the value obtained by measuring the spectral response of the MTO. These results, shown in Fig. 7, allowed us
to define the maximum value of $A_z$ to be used at different $z$. The minimum value used was $A_z = 1.2$ nm at $z = 160$ nm.

The absolute error of $\omega_r$, $\Delta \omega_r$, was found to be $2\pi \times 6$ mHz for an integration time of 10 s. This is approximately a factor of 1.7 smaller than in our previous work [11], the main reasons being a better decoupling of the apparatus from the environmental noise, and an improvement in the amplifier to measure $C_{r,t}$.

The Casimir pressure $P(z)$ was measured 15 times over different positions on the sample within a separation region $z = (160 - 750)$ nm, and one of the data sets is shown in Fig. 8. Each point in the figure was obtained with an integration time of 10 s.

4. Metrological analysis of experimental errors

4.1. Random errors

We start from the determination of the absolute error $\Delta z$ in the separation $z$ measured between the zero roughness levels of Au films on the plate and on the sphere. According to Eq. (1), $\Delta z$ is determined by the absolute errors $\Delta z_{\text{meas}} = 0.2$ nm and $\Delta D = 0.5$ nm. (We note from Eq. (2) that with $\Delta \theta \approx 10^{-9} \sqrt{\nu} \text{rad/Hz}^{1/2}$ and $\theta \leq 10^{-5}$ rad [11], the third term in the right-hand side of Eq. (1) makes a negligible contribution to $\Delta z$.) Both $z_{\text{meas}}$ and $D$ are assumed to be distributed uniformly within the limits of their respective absolute errors (note that the use of other distributions would decrease $\Delta z$).

The absolute error of the quantity obtained by the composition of the uniform distributions is given by [44]

$$\Delta z = \min \left[ (\Delta z_{\text{meas}} + \Delta D), k_\beta \sqrt{(\Delta z_{\text{meas}})^2 + (\Delta D)^2} \right],$$

where the correction factor $k_\beta$ depends on the chosen confidence level and the number of composed quantities (two in our case). Here and below we will use the confidence level $\beta = 95\%$ which results in $k_\beta = 1.1$ [44]. Substituting this into Eq. (7) we conclude that the absolute separations are measured with an error $\Delta z = 0.6$ nm.

As mentioned in Section 3.2, the Casimir pressure was measured 15 times over the 160 nm to 750 nm separation region (with 288 to 293 points in each set of measurements). It is not trivial, however, to take advantage of the fifteen-fold repetition of the measurement in order to decrease the random error and to narrow the confidence interval. The difficulty is that in the measurement
procedure employed the separation step between any two neighboring points was not uniform, even in one set of measurements, and additionally was quite different in all sets of the data. This leads to an absence of even a few points (let alone fifteen) taken at the same separation which could thus be used for averaging in a statistical analysis.

The usual way to deal with such data is the following: The entire separation range under consideration (from 160 nm to 750 nm) is divided into partial subintervals of length $2\Delta z = 1.2$ nm each. All points from each set of measurements are plotted together on this interval and, as a result, each subinterval $j$ contains a group of a few points $n_j$ (in our case from 3 to 13). Inside each subinterval all points can be considered as equivalent because the value of the absolute separation is distributed uniformly within the limits of $2\Delta z$.

We note that the experimental points used should be checked for the presence of so-called “outlying” results. For this purpose inside of each subinterval $j$ it is necessary to consider the quantity

$$T_j = \frac{1}{s_{P_j}} \max |P_{j,i} - \bar{P}_j|,$$(8)

where $P_{j,i}$ is the value of the Casimir pressure at point number $i$ of the subinterval $j$ ($1 \leq i \leq n_j$, maximum is taken with respect to $i$). The mean and the variance of the pressure are respectively defined as

$$\bar{P}_j = \frac{1}{n_j} \sum_{i=1}^{n_j} P_{j,i}, \quad s_{P_j}^2 = \frac{1}{n_j - 1} \sum_{i=1}^{n_j} (P_{j,i} - \bar{P}_j)^2.$$(9)

If the inequality $T_j > T_{n_j,1-\beta}$ is satisfied, where $T_{n_j,1-\beta}$ are tabulated quantities, then the subinterval $j$ contains an outlying result, which should be rejected with a confidence probability $\beta$ [44,45].

The 15 available sets of measurements were analyzed using this criterion and one set of measurements was found to be outlying. The subintervals containing points of this set satisfy the above inequality with high probability $\beta$. As an example, at separations 170 nm, 174 nm, 180 nm and 250 nm the probabilities that the points of the rejected set are outlying are 80%, 98%, 95%, and 98%, respectively. For this reason in the subsequent analysis we only consider data from 14 sets of measurements which were confirmed not to be outlying.

Direct calculation shows that data from these 14 measurement sets consist of groups (belonging to the neighboring subintervals) which are uniform in mean values. This implies that the mean Casimir pressures $\bar{P}_j$ change smoothly in going from group (subinterval) $j$ to group (subinterval) $j + 1$. The variance,
however, turned out not to be uniform in going from one subinterval to another. For this case we have utilized the theory of repeated measurements to implement a special procedure [46,47]. At each separation distance $z_0$, in order to find the uniform variance of a mean, it is necessary to consider not only one subinterval (which covers the value of $z_0$) but also several neighboring subintervals to the left and to the right of $z_0$ (in our case usually 4 or 5) as well. The specific number of such subintervals $N$ is determined from the requirement that the variance of the mean computed over several subintervals be uniform in going to the next set of subintervals.

Mathematically the confidence interval at some fixed separation $z_0$ is found in the following way. We calculate the variance of the mean of the Casimir pressure in each subinterval $j (1 \leq j \leq N)$ chosen around $z_0$,

$$s^2_{\bar{P}_j} = \frac{1}{n_j(n_j - 1)} \sum_{i=1}^{n_j} (P_{j,i} - \bar{P}_j)^2 = \frac{s^2_{\bar{P}_j}}{n_j}. \quad (10)$$

Then the variance of the mean at a point $z_0$ is found via the equation [46,47]

$$s^2_{\bar{P}}(z_0) = \max \left[ N \sum_{j=1}^{N} \lambda_j^2 s^2_{\bar{P}_j} \right], \quad (11)$$

where $\lambda_j$ are the influence coefficients. The maximum is taken over two sets of influence coefficients, $\lambda_j = 1/N$, and

$$\lambda_j = \frac{1}{c_j \sum_{k=1}^{N} c_k^{-1}}, \quad (12)$$

with the constants $c_k$ determined from

$$s^2_{\bar{P}_1} : s^2_{\bar{P}_2} : \ldots : s^2_{\bar{P}_N} = c_1 : c_2 : \ldots : c_N.$$

The confidence interval at a confidence probability $\beta$

$$[\bar{P}_j(z_0) - \Delta^{\text{rand}} P^{\text{expt}}(z_0), \bar{P}_j(z_0) + \Delta^{\text{rand}} P^{\text{expt}}(z_0)] \quad (13)$$

can then be found where

$$\Delta^{\text{rand}} P^{\text{expt}}(z_0) = s_{\bar{P}}(z_0) t_{(1+\beta)/2}(\min n_j - 1). \quad (14)$$

Here $t_p(f)$ is obtained from the tabulated values for the Student’s $t$ distribution (see, e.g., [48,49]).
The calculated results for $\Delta^\text{rand} P^\text{expt}(z)$ over the entire interval (160–750) nm at $\beta = 0.95$ confidence are shown in Fig. 9a by the long-dashed line. As is seen from the figure, as the separation decreases the random error rapidly increases. This is a consequence of Eqs. (10) and (14) and the decrease of $n_j$ for subintervals $j$ at shortest separations.

4.2. Systematic errors

We turn now to a discussion of systematic errors in the Casimir pressure measurements. By convention we call two of the errors in our experiment systematic which are in fact not constant, but can vary within certain limits, and can be described by random quantities with a uniform distribution. The first of these is the sphere radius with the absolute error $\Delta R = 0.2 \mu$m, and the second is the angular resonant frequency of the oscillator $\omega_r$, with the absolute error $\Delta \omega_r = 2\pi \times 6$ mHz (Section 3.2).

The determination of the Casimir pressure from Eqs. (4) and (5) is in fact an indirect measurement. However, the error of $P^\text{expt}(z)$ should be expressed in terms of the errors of the directly measured quantities. In our case $P^\text{expt}(z)$ is the ratio of two directly measured quantities, and for this reason it is convenient to use the relative errors (which we denote by $\delta$), rather than the absolute errors. Eq. (7) (written in terms of the relative errors) can then be used once more [44] leading to

$$\delta^\text{syst} P^\text{expt}(z) = \min \left[ (\delta \omega_r + \delta R), 1.1 \sqrt{(\delta \omega_r)^2 + (\delta R)^2} \right], \quad (15)$$

where the 95% confidence level has been chosen. Recall that the quantities $\omega_o$ and $b^2/(I \omega_o^2)$ are determined so precisely that their uncertainties do not contribute to the overall error. It should be noted that the first term on the right-hand side of Eq. (15) determines the total result at large separations of about 450 nm or more. At small separations $\delta^\text{syst} P^\text{expt}$ is given by the second term on the right-hand side of Eq. (15).

The absolute systematic error in the Casimir pressure measurements $\Delta^\text{syst} P^\text{expt}(z) = |P^\text{expt}(z)| \delta^\text{syst} P^\text{expt}(z)$, computed from Eq. (15), is shown in Fig. 9a by the short-dashed line. It is seen that at separations $z > 350$ nm the systematic error is slightly greater than the random error. At the shortest separations the systematic error is negligible.
4.3. **Total experimental error**

To obtain the total experimental error in the Casimir pressure measurements, it is necessary to combine the random and systematic errors which are described by different distributions (normal or Student and uniform in our case). In fact, there are different methods in the literature for combining random and systematic errors [44]. A convenient method in practical applications [50], is based on consideration of the ratio

\[ r(z) = \frac{\Delta_{\text{syst}} P_{\text{expt}}(z)}{s_P(z)}. \] (16)

At distances where the inequality \( r(z) < 0.8 \) is satisfied, the systematic error can be neglected and the total error of the Casimir pressure measurements at 95% confidence level is given by

\[ \Delta_{\text{tot}} P_{\text{expt}}(z) = \Delta_{\text{rand}} P_{\text{expt}}(z). \] (17)

At distances where \( r(z) > 8 \), the random error can be neglected and the total error at the same 95% confidence level is

\[ \Delta_{\text{tot}} P_{\text{expt}}(z) = \Delta_{\text{syst}} P_{\text{expt}}(z). \] (18)

In the intermediate region \( 0.8 \leq r(z) \leq 8 \), it is recommended [50] that one use the expression

\[ \Delta_{\text{tot}} P_{\text{expt}}(z) = k_\beta(r) \left[ \Delta_{\text{rand}} P_{\text{expt}}(z) + \Delta_{\text{syst}} P_{\text{expt}}(z) \right]. \] (19)

At 95% confidence level the tabulated coefficient \( k_\beta(r) \) varies between 0.71 and 0.81. (For practical purposes it suffices to use \( k_\beta(r) = 0.8 \) [50].)

The total error on the measurements of the Casimir pressure in the separation region (200–750) nm is shown in Fig. 9b by the solid line. For the sake of convenience, the random and systematic errors are shown in the same figure by the long-dashed and short-dashed lines, respectively, on a larger scale than in Fig. 9a. At separations (160–200) nm the total error coincides with the random one, i.e. is given by the long-dashed line in Fig. 9a. As can be seen in Fig. 9b, at separations \( z \geq 400 \) nm the total experimental error is almost constant and varies in the interval (0.43, 0.41) mPa.

In Fig. 10, the total relative experimental error

\[ \delta_{\text{tot}} P_{\text{expt}}(z) = \Delta_{\text{tot}} P_{\text{expt}}(z) / |P_{\text{expt}}(z)| \]
is plotted versus separation. It is seen that the total relative error is almost constant (varies between 0.55% and 0.60%) in a wide separation region from 170 nm to 300 nm. This contrasts with previous experiments on the Casimir force, where the smallest relative error (1.75% at 95% confidence [22] in the most precise and accurate AFM experiment [6]) was achieved only at the shortest separation.

5. Different theoretical approaches to the Casimir pressure between real metals at nonzero temperature

5.1. Lifshitz formula combined with the surface impedance or with the plasma model

The dynamic determination of the Casimir pressure, as described in Section 3.2, was carried out using a Au-coated sphere over a Au-coated plate. Since the thicknesses of both metal coatings were greater than the plasma wavelength of Au, \( \lambda_p = 137 \) nm, one can calculate the Casimir pressure as if the sphere and plate were composed of solid Au [12]. In the case of two infinite plates (the correction due to finite sizes of the plates is negligible, see Section 6) the result at temperature \( T \) is given by the Lifshitz formula [23,24,25]

\[
P(z) = -\frac{k_B T}{\pi} \sum_{l=0}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} q_l \times \left\{ \left[ r_{\parallel}^{-2}(\xi_l, k_{\perp}) e^{2q_l z} - 1 \right]^{-1} + \left[ r_{\perp}^{-2}(\xi_l, k_{\perp}) e^{2q_l z} - 1 \right]^{-1} \right\}.
\]

Here \( k_{\perp} \) is the wave vector component in the plane of the plates, \( k_{\perp} = |k_{\perp}| \), \( q_l^2 = k_{\perp}^2 + \xi_l^2 / c^2 \), \( \xi_l = 2\pi k_B T l / \hbar \) are the Matsubara frequencies, \( k_B \) is the Boltzmann constant, and \( r_{\parallel,\perp} \) are the reflection coefficients for two independent polarization states computed for the imaginary frequencies \( \omega_l = i\xi_l \). The prime in Eq. (20) refers to the inclusion of a factor 1/2 for the term with \( l = 0 \).

Eq. (20) has been derived in the framework of many different formalisms, and can hence be considered as the firmly established foundation for the description of the van der Waals and Casimir pressures between real materials with finite conductivity at nonzero temperature. However, the explicit expressions for the reflection coefficients \( r_{\parallel,\perp} \) are less certain. Traditionally, following Lifshitz [23,24,25], they were expressed in terms of the dielectric permittivity \( \varepsilon(\omega) \)

\[
r_{\parallel,l}(\xi_l, k_{\perp}) = \left[ \frac{k_l + \varepsilon(i\xi_l)q_l}{k_l - \varepsilon(i\xi_l)q_l} \right]^{-2}, \quad r_{\perp,l}(\xi_l, k_{\perp}) = \left[ \frac{k_l + q_l}{k_l - q_l} \right]^{-2},
\]

(21)
where \( k_2^2 = k_\perp^2 + \varepsilon(i\xi_l)\xi_l^2/c^2 \). For real metals at nonzero temperature and separations greater than the plasma wavelength the reflection coefficients can also be represented in terms of the surface impedance \( Z \) [30]

\[
\begin{align*}
    r_{\parallel}^{-2}(\xi_l, k_\perp) &= \left[ \frac{Z(i\xi_l)\xi_l + cq_l}{Z(i\xi_l)\xi_l - cq_l} \right]^2, \\
    r_{\perp}^{-2}(\xi_l, k_\perp) &= \left[ \frac{Z(i\xi_l)cq_l + \xi_l}{Z(i\xi_l)cq_l - \xi_l} \right]^2.
\end{align*}
\] (22)

At \( T = 0 \) the Lifshitz formula with reflection coefficients given by Eq. (22) has long been known [51]. On the real frequency axis, the surface-impedance reflection coefficients, Eq. (22), are commonly used in the analysis of optical properties of metals [52]. We note that Eq. (22) contains the Leontovich impedance which does not depend on polarization or transverse momentum [31]. If the dielectric permittivity depends only on frequency (as admitted in the Lifshitz theory) the Leontovich impedance is given by \( Z(\omega) = 1/\sqrt{\varepsilon(\omega)} \). However, the Leontovich impedance and corresponding boundary conditions on the surface of metal are more general since they still hold, for instance, in the frequency domain of the anomalous skin effect where it is impossible to describe the metal in terms of \( \varepsilon(\omega) \) due to the spatial nonuniformity of the electromagnetic field [52].

It is important to note that expressions (21) and (22) are not equivalent. They would only become equivalent if the Leontovich impedance \( Z(\omega) \) in (22) is replaced by the so-called exact impedances which depend on both polarization and transverse momentum,

\[
\begin{align*}
    Z_{\parallel}(\omega, k_\perp) &= \frac{1}{\omega\varepsilon(\omega)} \sqrt{\omega^2\varepsilon(\omega) - c^2k_\perp^2}, \\
    Z_{\perp}(\omega, k_\perp) &= \frac{\omega}{\sqrt{\omega^2\varepsilon(\omega) - c^2k_\perp^2}},
\end{align*}
\] (23)

where \( \omega \) and \( k_\perp \) are independent and not constrained by the mass-shell equation \( k_\perp^2 + k_3^2 = \omega^2/c^2 \) valid for real photons [31] (\( k_3 \) is the wave vector component perpendicular to the plane of the plates). Within the fourth (impedance) approach to the thermal Casimir force it is postulated that the reflection properties of virtual photons on a metal boundary are the same as for real photons, i.e., \( \omega \) and \( k_\perp \) in Eq. (23) must be constrained by the mass-shell equation. Under this additional condition it then follows that the reflection coefficients calculated with the exact impedances (23) coincide precisely with (22), expressed in terms of the Leontovich impedance, at zero frequency and are approximately equal to (22) with a very high accuracy at all other frequencies (see [31] for more details). The use of reflection coefficients (22) removes the contradictions with thermodynamics inherent in the first approach to the thermal Casimir force [26,34,35,36,37] which uses the coefficients (21) in combination with the Drude dielectric function (see Introduction).

Within the experimental separations (160–750) nm the characteristic angular
frequency of the Casimir effect $\xi_c = c/(2\pi)$ lies within the region of infrared optics. In the fourth (impedance) approach (see Introduction) the following calculation procedure is used to compute the values of the Casimir pressure. First, using the tabulated optical data for Au [53], the dielectric permittivity along the imaginary frequency axis is computed

$$\varepsilon(i\xi) = 1 + \frac{2}{\pi} \int_0^\infty \frac{\omega \text{Im} \varepsilon(\omega)}{\omega^2 + \xi^2} d\omega. \quad (24)$$

$\text{Im} \varepsilon(\omega)$ is found from the tabulated complex refractive index extending for Au from 0.125 eV to 10000 eV [53]. To calculate $\varepsilon$ at all contributing Matsubara frequencies with a sufficient precision, it is necessary to extrapolate the available tabulated data for frequencies below 0.125 eV. This is usually done (see, e.g., [54,55]) using the imaginary part of the Drude dielectric permittivity

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}. \quad (25)$$

where $\omega_p$ is the plasma frequency and $\gamma$ is the relaxation frequency (for Au $\omega_p = 9.0$ eV and $\gamma = 0.035$ eV; $1$ eV $= 1.51927 \times 10^{15}$ rad/s). In all computations the following values of fundamental constants, recommended by the National Institute of Standards and Technology, were used: $c = 2.997924 \times 10^8$ m/s, $\hbar = 1.05457 \times 10^{-34}$ J $\cdot$ s, $k_B = 1.38065 \times 10^{-23}$ J/K, and $e = 1.602176 \times 10^{-19}$ C.

Once $\varepsilon(i\xi)$ has been found from Eq. (24), the values of the impedance for the imaginary Matsubara frequencies with $l \geq 1$ are calculated as $Z(i\xi) = 1/\sqrt{\varepsilon(i\xi)}$. The resulting values are only slightly different from those computed from the analytical representation for the surface impedance in the free electron plasma model [given by Eq. (25) with $\gamma = 0$],

$$Z(i\xi) = \frac{\xi}{\sqrt{\omega_p^2 + \xi^2}}. \quad (26)$$

The reason for minor differences is that the impedance from Eq. (26) is purely imaginary when evaluated on a real frequency axis. Eq. (26) ignores the small real part of the impedance in the region of infrared optics arising, for instance, from electron-electron scattering. This real part is, of course, contained in the tabulated optical data.

In the fourth approach [30,40], the value of the impedance at zero Matsubara frequency is obtained by extrapolation from the region of characteristic frequencies (infrared optics in our case). As a result, the behavior $Z(i\xi) \approx \xi/\omega_p$.
when $\xi \to 0$ is obtained from Eq. (26), which from Eq. (22) leads to
\[
\begin{align*}
r_{\parallel}^{-2}(0, k_{\perp}) &= 1, \\
r_{\perp}^{-2}(0, k_{\perp}) &= \left(\frac{ck_{\perp} + \omega_p}{ck_{\perp} - \omega_p}\right)^2.
\end{align*}
\tag{27}
\]

Using the resulting values of $Z(i\xi_l)$, the Casimir pressure $P_4(z)$ (using the fourth approach) was computed from Eqs. (20), (22), (24), (27) at all experimental separations for all 14 sets of measurements at $T = 300$ K. As an illustration, several results are included in Table 1 (columns 2–4). In column 2 the contribution of the zero-frequency term of the Lifshitz formula $P_4^{(l=0)}(z)$ is presented. Column 3 contains the contribution of all terms $P_4^{(l>1)}(z)$ with $l \geq 1$ in the Casimir pressure. Column 4 presents the values of the total pressure $P_4(z)$ (the sum of the contributions from the two previous columns).

It is useful to compare the results, obtained by the impedance approach, with those from [28,29], obtained by the substitution of the plasma model dielectric function [Eq. (25) with $\gamma = 0$] into the Lifshitz formula (20) with reflection coefficients (21) (the third approach, see Introduction). In this case at zero Matsubara frequency, one finds in place of Eq. (27)
\[
\begin{align*}
r_{\parallel,L}^{-2}(0, k_{\perp}) &= 1, \\
r_{\perp,L}^{-2}(0, k_{\perp}) &= \left(\frac{ck_{\perp} + \sqrt{c^2k_{\perp}^2 + \omega_p^2}}{ck_{\perp} - \sqrt{c^2k_{\perp}^2 + \omega_p^2}}\right)^2.
\end{align*}
\tag{28}
\]

The calculated results for the contribution of the term $P_3^{(l=0)}(z)$ with $l = 0$ in the Casimir pressure, the terms $P_3^{(l>1)}(z)$ with $l \geq 1$, and the total pressure $P_3(z)$, are presented in columns 5, 6, 7 of Table 1, respectively. In column 8, for comparison, the contribution of all terms with $l \geq 1$ is calculated using the tabulated optical data and Lifshitz reflection coefficients, Eq. (21), expressed in terms of the dielectric permittivity. This column should be compared with column 3 where the tabulated optical data were used in combination with the impedance reflection coefficients from Eq. (22).

As can be seen from Table 1 (columns 4 and 7), at the shortest separation $z = 160$ nm the plasma model gives smaller magnitudes of the Casimir pressure than the impedance approach by approximately 2.6%. The same holds at zero temperature [54] where the situation is quite clear and there are no conflicts among the different approaches. This underestimation decreases to only 0.7% with increasing separation until $z = 250$ nm. At greater separations the differences among the data in columns 4 and 7 quickly decrease. The above differences at short separations are explained by the fact that the plasma model does not include the internal photoelectric effect (interband transitions) and other processes taken into account by the tabulated optical data. On the other hand, if we compare columns 3 and 8 of Table 1, it is
seen that the differences are less than 0.42% (the greatest difference being at
the shortest separation \( z = 160 \) nm). It follows that the impedance reflection
coefficients from Eq. (22) lead to practically the same results as the Lifshitz
coefficients given by Eq. (21).

5.2. **Lifshitz formula with zero-frequency term determined by the Drude model
or by special prescription**

As was noted in the Introduction, the first and second alternative approaches
to the description of the thermal Casimir force between real metals lead to pre-
dictions which differ from those in Section 5.1. Here we compute the Casimir
pressure in the framework of these approaches as they were formulated by
their authors.

We start with the first approach [26] in which the Lifshitz formula (20) and
the reflection coefficients (21) were used to calculate the Casimir pressure
between Au plates, taking into account the effects of nonzero temperature
and finite conductivity. The values of the dielectric permittivity at imaginary
Matsubara frequencies \( \varepsilon(i\xi_l) \) were obtained in the same way as was described
in Section 5.1. In other words, tabulated optical data, extrapolated in the
low frequency range by the imaginary part of the Drude dielectric function,
were used to obtain \( \varepsilon(i\xi_l) \) by means of the dispersion relation (24). Up to
this point the approach of [26] almost coincides with the impedance approach.
(As was demonstrated in Section 5.1, column 8 of Table 1, the use of the
Lifshitz reflection coefficients instead of the impedance coefficients leads to
only minor differences at all Matsubara frequencies with \( l \geq 1 \).) The major
difference in the approach in [26] is the value used for the zero-frequency term
of the Lifshitz formula. Instead of Eq. (27), the reflection coefficients at zero
Matsubara frequency were obtained in [26] by the substitution of the Drude
dielectric function (25) into Eq. (21) with a result

\[
\begin{align*}
    r_{\|,L}(0, k_{\perp}) &= 1, \\
    r_{\perp, L}(0, k_{\perp}) &= 0.
\end{align*}
\]  

(29)

The approach of [26] was supported in [34,35] (see also [36,37]).

Using the calculational procedure of [26,34,35], the Casimir pressure \( P_1(z) \)
obtained from the first approach was computed at all experimental separations
at \( T = 300 \) K. Several results are presented in Table 2. In columns 2, 3 the
zero-frequency term \( P_1^{(l=0)}(z) \) and the total Casimir pressure \( P_1(z) \) are given,
respectively. (Recall that the contributions of the Matsubara frequencies with
\( l \geq 1 \) were already included in column 8 of Table 1.)

We next turn our attention to the second alternative approach to the thermal
Casimir force suggested in [27]. This approach also used the Lifshitz formula (20) with the reflection coefficients (21) to calculate the Casimir pressure between the plates made of real metals at nonzero temperature. The dielectric permittivity $\varepsilon(i\xi)$ was found in the same way as in [26]. The major difference between the approach [27] and those used in Section 5.1, or in [26], is a new alternative for the zero-frequency term of the Lifshitz formula. Instead of Eqs. (27), (28) or (29), it was postulated that for real metals both reflection coefficients at zero Matsubara frequency should be the same as for ideal metals, i.e.
\[
\begin{align*}
    r_\parallel^2(0, k_\perp) &= 1, \\
    r_\perp^2(0, k_\perp) &= 1.
\end{align*}
\]  
(30)

Using this postulate, the Casimir pressure $P_2(z)$ obtained from the second approach was computed from Eqs. (20), (21) at all experimental separations at $T = 300$ K. As an example, several calculated results are presented in columns 4, 5 in Table 2, where the zero-frequency term $P_2^{l=0}(z)$ and the total Casimir pressure $P_2(z)$, obtained from Eqs. (30) and (20), are given, respectively.

Comparing column 4 of Table 1 with column 3 of Table 2, we conclude that the first approach leads to smaller magnitudes of the Casimir pressure than the impedance approach at all separations. As an illustration, the differences of the theoretical predictions given by these two approaches change from 2.4% of the Casimir pressure at $z = 200$ nm to 7.4% at $z = 500$ nm. On the contrary, if we compare column 4 of Table 1 with column 5 of Table 2, we conclude that the second approach predicts larger magnitudes for the Casimir pressure than does the impedance approach. The differences of the theoretical predictions given by these two approaches change slowly from 2.5% of the Casimir pressure at $z = 200$ nm to 2.2% at $z = 500$ nm. It should be emphasized that the calculated results in Table 2 are in agreement with those obtained by the authors of [26,34,35,36,37] (columns 2, 3) and [27] (columns 4, 5), where the first and second approaches, respectively, were developed.

5.3. Variation of optical tabulated data and effects of spatial nonlocality

To ensure that the theoretical approaches which use tabulated optical data for the complex refractive index, lead to sufficiently precise results, one should check that small sample-to-sample variations of the data due to, for example, size of the grains and impurities are not problematic. This was verified in [22] for the experiment of [6]. It was found that at $z = 62$ nm the possible variations of the optical data due to grain size lead to a 0.8% decrease of reflectivity, and to a decrease of less than 0.5% in the Casimir force. In our case the sizes of grains vary between 25 nm and 300 nm (with a mean value of about 150 nm), and the characteristic wavelength at $z = 160$ nm is $\lambda_c =$
\[ \frac{2\pi c}{\xi_c} = 4\pi z \approx 2010 \text{ nm}. \] According to [56], even for grains as small as 45 nm in size (the largest studied in that paper) the reflectivity at \( \lambda \sim 2000 \text{ nm} \) is only 0.6% less than that calculated from tabulated data. Taking into account that the mean size of crystallite grains in our case is several times larger than 45 nm, we can conclude that the variations of the optical data may lead to a decrease in the magnitude of the Casimir pressure much lower than 0.5%, even at the shortest separation considered.

One more effect to discuss, related to the finite conductivity corrections, is the probable contribution from the effects of spatial nonlocality (i.e., wave vector dependence of the dielectric function). These effects influence the Casimir force magnitude in the region of the anomalous skin effect at large separations, \( z > 2.36 \mu \text{m} \) for Au [30], a region not relevant for our experiment. Another, short-separation, region where nonlocality may contribute to the van der Waals force, is \( z < \lambda_p/(4\pi) \approx 10.9 \text{ nm} \) which corresponds to characteristic frequencies \( \xi_c > \omega_p \). At such high frequencies the spatial dispersion modifies the frequencies of surface plasmons. In our experiment, however, the effects of nonlocality do not contribute to the Casimir pressure. This can be understood by noting that the largest characteristic frequency here, calculated at \( z = 160 \text{ nm} \), is \( \xi_c = 9.4 \times 10^{14} \text{ rad/s} \), which is almost 15 times smaller than \( \omega_p = 1.37 \times 10^{16} \text{ rad/s} \). A direct calculation shows that the Matsubara frequencies \( \xi_0, \xi_1, \ldots, \xi_{35} \) contribute 99.96% of the total Casimir pressure at \( z = 160 \text{ nm} \), and even \( \xi_{35} = 8.61 \times 10^{15} \text{ rad/s} < \omega_p \). We note in passing that the contribution of the nonlocal effects for Au of about 2% at a separation \( z = \lambda_p = 137 \text{ nm} \), claimed in [57], is in error, as was recognized by the authors in a recent Erratum [58]. (In fact the formalism of this reference leads to a correction of only 0.2% due to spatial nonlocality at \( z = \lambda_p \); the confusion arises from a notation \( L/\lambda_p \) on the \( x \)-axis in Fig. 2 of [57] instead of \( 2\pi L/\lambda_p \).) Furthermore, in [57] a spatially nonlocal dielectric function was used in the frequency region of infrared optics where it is in fact local [52,59]. As was specifically demonstrated in detail in [60], the short-separation effects of spatial nonlocality connected with the surface-plasmon charge fluctuations contribute considerably to the force at separations of the order of \( v_F/\omega_p \sim 1 \text{ Å} \), where \( v_F \) is the Fermi velocity. It was explicitly proved in [60] that for much larger distances \( z \gg v_F/\omega_p \) (in fact for \( z \) larger than several nanometers) the usual local Lifshitz formula is applicable. This is in agreement with our conclusion that at separations \( z \geq 160 \text{ nm} \) there is no noticeable contribution from the effects of nonlocality to the Casimir pressure.

5.4. Casimir pressures taking account of surface roughness

At separations below 1 \( \mu \text{m} \) surface roughness corrections may contribute from a fraction of a percent up to 20% of the measured force in different experi-
ments [12]. For this reason they should be carefully taken into account in precision computations of the Casimir pressure. The contribution of roughness is calculated with the help of the AFM characterization of both surfaces (see Section 2.2). As is seen in Figs. 2 and 3, the roughness is mostly represented by stochastically distributed distortions with a characteristic lateral size of approximately $l \approx (500 - 600) \text{ nm}$. This size is larger by a factor of $\approx 3$ than $z$ at the shortest separations (160–200 nm), where the roughness corrections may give a noticeable contribution to the Casimir pressure in our experiment. The influence of roughness can be calculated by the additive method of geometrical averaging [12]. As shown in [61], under the condition $z < l$ the additive method gives approximately the same results as more detailed methods. Moreover, the correction due to nonadditivity can be simply estimated.

There are two modifications to the additive method. The first one is based on the use of the Casimir pressures $P_i(z)$, found in the above four theoretical approaches as calculated in Sections 5.1, 5.2. Let us first determine the zero roughness levels $H_{o}^{p,s}$ on a plate and a sphere, respectively, relative to which the mean values of the functions, describing roughness, are zero (note that the experimental separations are measured between the zero roughness levels [12]):

$$
\sum_{i=1}^{K} (H_{o}^{p,s} - h_{i}^{p,s}) v_{i}^{p,s} = 0. \quad (31)
$$

Here the quantities $v_{i}^{p,s}$ and $h_{i}^{p,s}$ are taken from Figs. 4a and 4b for the plate and the sphere, respectively ($K$ was defined in Section 2.2). To find the Casimir pressure taking into account all three corrections, the values $P_i(z)$ should be geometrically averaged over all possible separations between the rough surfaces, weighted with the probability of each separation [3,11,12]

$$
P_{i}^{\text{theor}}(z) = \sum_{k=1}^{105} \sum_{j=1}^{112} v_{k}^{p} v_{j}^{s} P_{i} \left( z + H_{o}^{p} + H_{o}^{s} - h_{k}^{p} - h_{j}^{s} \right). \quad (32)
$$

Note that this expression does not reduce to a simple multiplication of the correction factors due to nonzero temperature and finite conductivity on the one hand, and due to surface roughness on the other, but takes into account their combined (nonmultiplicative) effect.

The second one is a simpler additive method. It was used for calculating the corrections due to stochastic surface roughness in [62]. According to this method, the Casimir pressure between two ideal metal parallel plates covered
by stochastic roughness with variances $\delta_{st}^p$ and $\delta_{st}^s$ is given by

$$P_r(z) = \eta_r P_o(z), \quad \eta_r = 1 + 10 \left[ \left( \frac{\delta_{st}^p}{z} \right)^2 + \left( \frac{\delta_{st}^s}{z} \right)^2 \right], \quad (33)$$

where $P_o(z) = -\pi^2 \hbar c / (240 z^4)$ is the Casimir pressure between perfectly shaped parallel plates. The variances describing the stochastic roughness are found from the formula

$$\left( \delta_{st}^{p,s} \right)^2 = K \sum_{i=1}^{K} (H_{0}^{p,s} - h_{1}^{p,s})^2 V_{i}^{p,s}. \quad (34)$$

Using data from Figs. 4a, 4b, one obtains the values for variances $\delta_{st}^p = 4.06 \text{ nm}$ and $\delta_{st}^s = 1.91 \text{ nm}$. The smallness of these variances justifies the neglect of the fourth order terms on the right-hand side of Eq. (33) (the fourth order contributes less than 0.01% even at the shortest separation $z = 160 \text{ nm}$). We can now obtain the complete theoretical result, including the corrections due to surface roughness, nonzero temperature and finite conductivity, by the use of a multiplicative procedure

$$P_{\text{theor}}^i(z) = \eta_r P_{i}^o(z). \quad (35)$$

It is interesting to compare the theoretical results for the Casimir pressure taking into account all corrections contained in Eq. (32), and the simpler expression in Eq. (35). As an example, at the shortest separation $z = 160 \text{ nm}$, Eq. (32) gives $P_{\text{theor}}^4 = -1.1515 \text{ Pa}$ and Eq. (35) results in $P_{\text{theor}}^4 = -1.1531 \text{ Pa}$, a difference only 0.14% of the Casimir pressure. At $z = 200 \text{ nm}$ the complete Casimir pressures are $P_{\text{theor}}^4 = -0.51143 \text{ Pa}$ and $P_{\text{theor}}^4 = -0.51185 \text{ Pa}$ computed from Eqs. (32) and (35), respectively. In this case the difference is even smaller: 0.08% of the Casimir pressure. For larger separations the role of the nonmultiplicative effects decreases and one may use Eq. (35) to calculate the theoretical values of the Casimir pressure with sufficiently small uncertainty. The same relative sizes of roughness corrections are obtained for the other theoretical approaches.

Note that the role of the surface roughness in the improved experiment is rather modest. Using the data of Tables 1 and 2 we find that at $z = 160 \text{ nm}$ roughness contributes 0.65%, and at a separation $z = 200 \text{ nm}$ only 0.42% of the Casimir pressure.

It is important to bear in mind that both Eqs. (32) and (35) are based on the approximation and do not take into account diffraction-type effects arising in the case of periodic roughness with small periods $l < z$ [61], or stochastic...
roughness with small correlation length [63]. To obtain an estimate for the upper limit of the contribution of diffraction-type effects we follow [22]. For this purpose, the bound on the roughness correlation length is found to be $l_{\text{corr}} \geq 600 \, \text{nm}$, where 600 nm is the lateral size of the largest cluster on the surface. We next consider periodic roughness with a period $l_{\text{corr}} = 600 \, \text{nm}$ using the fact that diffraction-type effects are greater for a periodic function with a period $l_{\text{corr}}$ than for a random function with a correlation length $l_{\text{corr}}$. The diffraction-type effects for a periodic function can be computed in the framework of the functional approach of [61]. At separations $z = 160 \, \text{nm}$ and $z = 300 \, \text{nm}$ one obtains $z/l_{\text{corr}} = 0.27$ and $0.318$, respectively. The coefficient $\tilde{c}_{\text{corr}}$ in the expression

$$\eta_{\text{corr}}^{r} = 1 + 10\tilde{c}_{\text{corr}} \left[ \left( \frac{\delta_{\text{st}}^p}{z} \right)^2 + \left( \frac{\delta_{\text{st}}^s}{z} \right)^2 \right],$$

(36)

taking diffraction-type effects into account, is related to that plotted in Fig. 2 of [61] by the equation

$$\tilde{c}_{\text{corr}} = c_{\text{corr}} - \frac{1}{5} \frac{\partial c_{\text{corr}}}{\partial z}.$$  

(37)

(In [61] the results for the Casimir energy between plates are presented, not those for the pressure.) Using the data of Fig. 2 from paper [61] and Eq. (37), we find $\tilde{c}_{\text{corr}} \approx 1$ and $\tilde{c}_{\text{corr}} \approx 1.18$ at separations $z = 160 \, \text{nm}$ and $z = 300 \, \text{nm}$, respectively. Hence at $z = 160 \, \text{nm}$ there is no noticeable contribution from correlation effects to the roughness correction. From Eqs. (33) and (36) it follows that at $z = 300 \, \text{nm}$ $\eta_r = 1.0022$ and $\eta_{\text{corr}}^{r} = 1.0026$, leading to a contribution from correlation effects to the roughness correction of less than 0.04%. Because of this, the correlation effects of roughness in the experiment under consideration can be ignored.

5.5. Contribution of patch potentials

In the experimental configuration described in Sections 2, 3, the electrostatic force due to the residual potential difference between the plate and the sphere is negligible. It is conceivable, however, that spatial variations of the surface potentials due to grains of polycrystalline metal film (the so called “patch potentials”) simulate the Casimir force [64]. Here we use the general results of [64] in order to demonstrate that the patch effect is negligible in our experiment. According to [64], for a configuration of two parallel plates the electric
pressure due to random variations in patch potentials is given by

\[ P_{\text{patch}}(z) = -\frac{2\varepsilon_0\sigma_v^2}{k_{\text{max}}^2 - k_{\text{min}}^2} \int_{k_{\text{min}}}^{k_{\text{max}}} \frac{k^3}{\sinh^2 k z} dk. \]  

(38)

Here \( \sigma_v \) is the variance of the potential distribution, and \( k_{\text{max}} \) (\( k_{\text{min}} \)) are the magnitudes of the extremal wavevectors corresponding to minimal (maximal) sizes of grains. Using the work functions of Au for different crystallographic surface orientations (\( V_1 = 5.47 \text{ eV}, V_2 = 5.37 \text{ eV}, \) and \( V_3 = 5.31 \text{ eV} \)), and assuming equal areas of respective crystallographic planes, we obtain

\[ \sigma_v^2 = \frac{1}{2} \sum_{i=1}^{3} (V_i - \bar{V})^2 \approx 6528.64 \text{ mV}^2. \]  

(39)

As was mentioned in Section 5.3, the extremal sizes of grains in the Au layers covering the test bodies are \( \lambda_{\text{min}} \approx 25 \text{ nm} \) and \( \lambda_{\text{max}} = 300 \text{ nm} \). This results in \( k_{\text{max}} = 0.251 \text{ nm}^{-1} \) and \( k_{\text{min}} = 0.0209 \text{ nm}^{-1} \). Substituting these values into Eq. (38) yields the additional pressures due to the patch electric forces. At the shortest separations \( z = 160 \text{ nm} \) and \( z = 170 \text{ nm} \) the patch pressures are \( P_{\text{patch}} = 0.42 \text{ mPa} \) and 0.25 mPa, respectively. In comparison to the Casimir pressures at the same separations, the relative contributions of the patch effect to the pressure are 0.037% and 0.027%, respectively, and further decrease with increasing \( z \). It follows that patch effects do not play any significant role in our Casimir pressure measurements by means of a micromachined oscillator.

6. Metrological analysis of theoretical errors

As it is evident from the foregoing, the theoretical computation of the Casimir pressure with all relevant corrections is a rather complicated procedure requiring a variety of data. Bearing in mind the eventual comparison of theory with experiment, it is important to analyze all uncertainties introduced at different stages of the computations.

According to Eq. (5), the derivative of the Casimir force between a plate and a sphere with respect to separation is related to the Casimir pressure between two plane plates by the use of the proximity force theorem. It is well known that this “theorem” is approximate and its relative error \( \delta_{\text{pft}} P_{\text{theor}} \) is less than \( z/R \) [65,66,67]. This error is separation-dependent and increases from 0.107% at a separation \( z = 160 \text{ nm} \) to 0.5% at \( z = 750 \text{ nm} \) (see the short-dashed line in Fig. 11).
The computation of the Casimir pressure taking into account nonzero temperature and finite conductivity in Sections 5.1, 5.2 was based on the use of tabulated optical data for Au (approaches 1, 2 and 4). The primary source of the theoretical errors is the sample to sample variation of these data. As was shown in Section 5.3, in the experiment under consideration the variation of the optical data leads to an uncertainty in the magnitude of the Casimir pressure which is much smaller than 0.5%. To be conservative, we admit an uncertainty in the computation of the finite conductivity corrections as large as \( \delta_c P_i^{\text{theor}} = 0.5\% \) over the entire measurement range. In fact, this error also includes all differences arising when one uses the Lifshitz reflection coefficients (21) instead of impedance coefficients (22) and vice versa (see Section 5.1). The theoretical error \( \delta_c \) is shown in Fig. 11 by the long-dashed line.

The other uncertainties discussed above, such as the contribution of nonlocality to the effect of finite conductivity (Section 5.3), the diffraction-type contributions to the effect of surface roughness (Section 5.4), and the correction to the Casimir pressure due to the patch potentials (Section 5.5), were shown to be far smaller than the previous two discussed above, \( \delta_{pft} P_i^{\text{theor}} \) and \( \delta_c P_i^{\text{theor}} \). For this reason they can all be neglected. One further correction to the Casimir pressure, arising from the finite size of plates, is also negligible. As was shown in [11], in the plate-sphere configuration of a micromachined oscillator this correction is less than 0.04\% at \( z = 500 \text{ nm} \), becoming smaller as the separation decreases. But even this correction does not contribute in our case of two equivalent parallel plates because its derivative with respect to separation vanishes. In fact, to obtain the correction due to the finiteness of the plates, one should consider corrections of higher order in \( z/R \) for a sphere above a plate. As a result, at least an extra power of the small parameter \( z/R \) appears in the correction, making it completely insignificant in the error analysis.

Let us now determine the theoretical error resulting from combining the two major errors \( \delta_{pft} P_i^{\text{theor}} \) and \( \delta_c P_i^{\text{theor}} \). Both random quantities are described by the same distribution law which is close to a uniform distribution. For this reason the method of [44], already used in Eqs. (7), (15), can be applied once more, giving

\[
\delta_o P_i^{\text{theor}}(z) = \min \left[ \left( \delta_{pft} P_i^{\text{theor}} + \delta_c P_i^{\text{theor}} \right), \right. \\
1.1 \sqrt{\left( \delta_{pft} P_i^{\text{theor}} \right)^2 + \left( \delta_c P_i^{\text{theor}} \right)^2} \right].
\]  

As in Eqs. (7) and (15), the 95\% confidence level is chosen. Note that in this case the minimum is achieved using the second term on the right-hand side of Eq. (40).

Bearing in mind the ensuing comparison of theory with experiment, it is perti-
nent to consider the error in the theoretical pressures resulting from the experimental error in the determination of separation distances $\Delta z$ [68]. Although this error is not purely theoretical (theoretical pressures by themselves can be calculated at any exact separation distance), it is of great importance when one computes the theoretical pressure at an experimental point defined with an error $\Delta z$. Using the main theoretical dependence of the Casimir pressure on the inverse fourth power of the separation one obtains $\delta_z P_{\text{theor}} = 4\Delta z/z$, where $\Delta z = 0.6$ nm at 95% confidence [see Eq. (7)]. The value of $\delta_z P_{\text{theor}}$ changes from 1.5% at $z = 160$ nm to 0.32% at $z = 750$ nm (see the dotted line in Fig. 11). Thus, this error is the primary one (especially at short separations) arising in the comparison of theory with experiment.

In [22] an additional fitting procedure was proposed in order to decrease the error in the determination of absolute separations down to $\Delta z = 0.15$ nm. For this purpose, for each set of measurements, $z$-values of all points were simultaneously changed within the limits of their joint absolute error in order to minimize the root-mean-square deviation between theory and experiment. This procedure, which is quite reasonable at very short separations ($z \geq 60$ nm in [22]), is not applied in our conservative metrological analysis because the method of least squares is not well-adapted for cases when both the argument (separation distance) and function (pressure) are determined with comparable relative errors.

We are now in a position to determine the total theoretical error of the Casimir pressure computations. To do this, we combine the errors determined above, $\delta_o P_{\text{theor}}$ and $\delta_z P_{\text{theor}}$. To use the analogy with Section 4.3, the error $\delta_o P_{\text{theor}}$ can be considered as “random” (in the sense that it is not described by a uniform distribution), and the error $\delta_z P_{\text{theor}}$ can be likened to “systematic” (as described by a uniform distribution). We then find ourselves in the regime of applicability of Eq. (19), and at 95% confidence, we obtain the final result

$$\delta^{\text{tot}} P_{\text{theor}}(z) = 0.8 \left[ \delta_z P_{\text{theor}}(z) + \delta_o P_{\text{theor}}(z) \right].$$  \hspace{1cm} (41)

This result is represented by the solid line in Fig. 11. It is seen that the total theoretical error decreases from 1.65% to 1% when the separation increases from 160 nm to 380 nm. With further increase of separation the total theoretical error decreases more slowly from 1% to 0.9%.
7. Comparison of theory and experiment and stronger constraints on thermal effects

7.1. Measure of agreement between theory and experiment

In all previous experiments on the Casimir effect [2,3,4,5,6,7,8,9,10,11,12] the agreement between theory and experiment was discussed in terms of the root-mean-square deviation between experimental and theoretical values of the force over the entire range of separation distances. In [7] this approach was criticized as inadequate when the force decreases rapidly as the separation distance increases, although no alternative approach was suggested. Here we compare experiment and theory with the help of rigorous metrological criteria, not based on the concept of root-mean-square deviation.

From Sections 4.3 and 6, we have at our disposal two independently obtained confidence intervals at the same confidence probability of 95% (the first for experiment and the second for theory). Both confidence intervals were determined by the application of a common statistical procedure, and the resulting errors for both experiment and theory can be described by random variables with a common distribution law. With this in mind, we consider the new variable \( P_{\text{theor}}(z) - P_{\text{expt}}(z) \) and determine the total absolute error on this quantity, and hence the confidence interval at a confidence probability 95%, using the same composition law as in Eqs. (7) and (15), i.e.

\[
\Delta^{\text{tot}} \left[ P_{\text{theor}}(z) - P_{\text{expt}}(z) \right] = \min \left\{ \left[ \Delta^{\text{tot}} P_{\text{expt}}(z) + \Delta^{\text{tot}} P_{\text{theor}}(z) \right], 1.1 \sqrt{\left[ \Delta^{\text{tot}} P_{\text{expt}}(z) \right]^2 + \left[ \Delta^{\text{tot}} P_{\text{theor}}(z) \right]^2} \right\} .
\]

\( \Delta^{\text{tot}} P_{\text{expt}}(z) \) is given by the solid line in Fig. 9b, and \( \Delta^{\text{tot}} P_{\text{theor}}(z) = P_{\text{theor}} \delta^{\text{tot}} P_{\text{theor}} \), where \( \delta^{\text{tot}} P_{\text{theor}} \) is shown by the solid line in Fig. 11. Note that the total absolute theoretical error is almost independent of the theoretical approach. For this reason we omit the index \( i = 1, 2, 3, 4 \) in the notation for a confidence interval. The minimum here is achieved using the second contribution from the right-hand side of Eq. (42). We also point out that to be conservative we use the value \( k_\beta = 1.1 \), as would describe the composition of two uniform distributions (for other distributions the value of \( k_\beta \) can be somewhat smaller).

The confidence interval for the quantity \( [P_{\text{theor}}(z) - P_{\text{expt}}(z)] \) at 95% confidence probability is given by

\[
\left[ -\Delta^{\text{tot}} \left( P_{\text{theor}}(z) - P_{\text{expt}}(z) \right), \Delta^{\text{tot}} \left( P_{\text{theor}}(z) - P_{\text{expt}}(z) \right) \right] .
\]
The meaning of the confidence interval (43) is that, if the theory is in agreement with experimental data, the mean value of \[ P_{\text{theor}}(z) - P_{\text{expt}}(z) \] must belong to this interval with a 95% probability. This criterion can be used for comparison of the above four theoretical approaches with experiment.

7.2. Comparison of different theoretical approaches with experiment

We start from the approach based on the surface impedance (the fourth approach). The solid lines in Fig. 12 exhibit the confidence interval (43) obtained from Eq. (42) within the separation ranges (160–250) nm (a) and (250–750) nm (b). In the same figure the differences of the theoretical (based on the surface impedance) and experimental Casimir pressures are plotted for one typical set of 14 measurements. It is evident that the true value of the quantity under consideration \[ P_{\text{theor}} - P_{\text{expt}} \] (which can only be achieved for a complete correct theory and an infinite number of measurements) is zero. As an example, for the measurement set plotted in Fig. 12, \( \langle P_{\text{theor}} - P_{\text{expt}} \rangle = 0.008 \) mPa, where the averaging was taken over all 288 points belonging to this set. If only points at separations \( z \geq 250 \) nm are considered (242 points), one obtains \( \langle P_{\text{theor}} - P_{\text{expt}} \rangle = 0.09 \) mPa. These values should be compared with the half-width of the confidence interval which decreases from 27.7 mPa at \( z = 160 \) nm to 0.46 mPa at \( z = 750 \) nm. Remarkably, only 9 of 288 points (i.e. 3.1%) fall outside the confidence interval in Fig. 12.

In Fig. 13a,b the same confidence interval (43) is shown once more and the points \[ P_{\text{theor}} - P_{\text{expt}} \] are plotted utilizing all 14 sets of measurements. For all 4066 points in Fig. 13 one has \( \langle P_{\text{theor}} - P_{\text{expt}} \rangle = 0.11 \) mPa. If only separations \( z \geq 250 \) nm are considered (3442 points) the result is \( \langle P_{\text{theor}} - P_{\text{expt}} \rangle = 0.12 \) mPa. Although these values are less than the half-width of the confidence interval, they are not sufficiently informative. In the case of Fig. 13, where all points are plotted, it is much more important to consider the local values of the quantity \( \Delta P_i(z) \equiv \langle P_{\text{theor}}(z) - P_{\text{expt}}(z) \rangle \), where the averaging is done over the vicinity of a point \( z \), in comparison with the local half-width of the confidence interval. Performing the averaging over the intervals \( (z_i - 3 \) nm, \( z_i + 3 \) nm) (so that each contains approximately 43 points) at different \( z = z_i \) one obtains the values for the mean differences between the theoretical (the fourth approach) and experimental Casimir pressures included in column 3 of Table 3. For comparison, the values of the half-width of a confidence interval are included in column 2. Columns 4–6 are discussed below.

Comparing columns 2 and 3 of Table 3 we notice that all mean values of the differences between theoretical and experimental Casimir pressures are well within the half-width of the confidence interval. The number of separate points which fall outside of the confidence interval in Fig. 13 is 207, i.e. 5.09%
of the total number of points from the fourteen sets of measurements. This is in accordance with expectations from the value of the confidence probability. To conclude, the experimental data of Section 3.2 are proved to be in excellent agreement with the theoretical computations of the Casimir pressure in the framework of the impedance approach (see also [41,42] arriving to the result in agreement with this approach).

Quantitatively the agreement between theory and experiment can be characterized by the quantity

\[
\delta_{\text{tot}}(z) = \frac{\Delta_{\text{tot}} \left[ P_{\text{theor}}(z) - P_{\text{expt}}(z) \right]}{|P_{4\text{theor}}(z)|}. \tag{44}
\]

Importantly, in the experiment under consideration \(\delta_{\text{tot}}(z)\) depends only slightly on \(z\) in a wide separation range, decreasing (with the fourth theoretical approach) from \(\delta_{\text{tot}}(z) = 1.9\%\) at \(z = 170\) nm to \(\delta_{\text{tot}}(z) = 1.4\%\) within the 270 nm \(\leq z \leq 370\) nm interval, and then increasing to \(\delta_{\text{tot}}(z) = 1.8\%\) at \(z = 420\) nm. The largest values of \(\delta_{\text{tot}}\) at \(z < 170\) nm and \(z > 420\) nm are \(\delta_{\text{tot}}(z) = 2.4\%\) at \(z = 160\) nm and \(\delta_{\text{tot}}(z) = 13\%\) at \(z = 750\) nm. Thus, the Casimir pressure measurement by means of a micromachined oscillator is the first Casimir effect experiment where metrological agreement between theory and experiment at the 1.5% level has been achieved at 95% confidence in a wide separation region.

The results of a comparison between experiment and the third theoretical approach, based on the plasma dielectric function, are presented in column 4 of Table 3. Comparing column 4 with column 2, one can conclude that the third approach is also consistent with the experimental data because all mean values of the Casimir pressures belong to the confidence interval. At the same time, it is seen that at the shortest separations (170, 250) nm this approach does not agree with experiment as well as does the impedance approach. The reasons for this were discussed in Section 5.1 (see also [54]).

We next compare the first theoretical approach with the experiment. In Fig. 14a, we plot the differences of the Casimir pressures \([P_{1\text{theor}} - P_{\text{expt}}]\) versus separation \(z\) (where \(P_{1\text{theor}}\) was computed in the approach of [26,34,35] as explained above) for all 14 sets of measurements (i.e., in the same way as in Fig. 13 for the surface impedance approach). The solid line gives the half-width of the confidence interval \(\Delta_{\text{tot}} \left( P_{1\text{theor}} - P_{\text{expt}} \right) \approx \Delta_{\text{tot}} \left( P_{4\text{theor}} - P_{\text{expt}} \right)\) at 95% confidence (note that there are practically no points below the \(z\)-axis). As is seen from Fig. 14a, in a wide separation region 230 nm \(\leq z \leq 500\) nm all points \([P_{1\text{theor}} - P_{\text{expt}}]\) fall outside the confidence interval. It follows that the theoretical prediction of the first approach is excluded by experiment at 95% confidence.
In column 5 of Table 3 the mean values of the differences between theoretical (obtained from the first approach) and experimental Casimir pressures are presented at different separations from 170 nm to 700 nm using the data from all 14 sets of measurements. The comparison with the half-width of the confidence interval (column 2) shows that at all separations the mean differences of the Casimir pressures fall outside the confidence interval. It follows that the first theoretical approach is excluded by the experiment at 95% confidence level within an even wider separation region, 170 nm ≤ z ≤ 700 nm.

In fact, the above confidence interval (43) was obtained in a rather conservative manner. The comparison of data from columns 2 and 5 of Table 3 shows that even if the confidence interval were to be widened to achieve 99% confidence probability, the quantities \( \langle P_{\text{theor}}^1 - P_{\text{expt}} \rangle \) would still remain outside this interval within some separation region. To make this argument quantitative we calculate the half-width of a new confidence interval from the equality

\[
\frac{\Delta_{\text{tot}}^\text{0.99} \left( P_{\text{theor}}^1 - P_{\text{expt}} \right)}{\Delta_{\text{tot}}^\text{0.95} \left( P_{\text{theor}}^1 - P_{\text{expt}} \right)} = \frac{t_{[1+0.99]/2}(2)}{t_{[1+0.95]/2}(2)} \approx 2.31.
\]  

(Being conservative, we preserve only two degrees of freedom, i.e. the minimum value from the analysis of Section 4.1). Using the data from column 2 in Table 3 for \( \Delta_{\text{tot}}^\text{0.95} \equiv \Delta_{\text{tot}}^\text{0.99} \left( P_{\text{theor}}^1 - P_{\text{expt}} \right) \approx 3.67 \text{ mPa}, 1.46 \text{ mPa}, \) and 1.13 mPa at separations \( z = 300 \text{ nm}, 400 \text{ nm}, \) and \( 500 \text{ nm}, \) respectively.

Comparing these results with the values of \( \langle P_{\text{theor}}^1 - P_{\text{expt}} \rangle \) from column 5 in the same table, one concludes that the quantities \( \langle P_{\text{theor}}^1 - P_{\text{expt}} \rangle \) still fall outside the new confidence interval in the separation region \( 300 \text{ nm} \leq z \leq 500 \text{ nm}. \)

We conclude that the first theoretical approach to the thermal Casimir force is excluded experimentally with 99% confidence.

Finally, we compare the second theoretical approach [27] with the experimental data. In Fig. 14b, we plot the differences of the Casimir pressures \( \left[ P_{\text{theor}}^2 - P_{\text{expt}} \right] \) versus separation \( z \) for all 14 sets of measurements, as in Figs. 13 and 14a. The solid lines show the half-width of the confidence interval \( \Delta_{\text{tot}}^\text{0.95} \left( P_{\text{theor}}^2 - P_{\text{expt}} \right) \approx \Delta_{\text{tot}}^\text{0.99} \left( P_{\text{theor}}^1 - P_{\text{expt}} \right) \) at 95% confidence (there are no points above the z-axis in the interval \( 160 \text{ nm} \leq z \leq 420 \text{ nm} \) in the approach of [27]). As is seen from Fig. 14b, within the separation region \( 160 \text{ nm} \leq z \leq 350 \text{ nm} \) almost all points \( \left[ P_{\text{theor}}^2 - P_{\text{expt}} \right] \) fall outside the confidence interval. It follows that the second theoretical approach is excluded experimentally at 95% confidence.

This conclusion is confirmed by considering the mean values \( \langle P_{\text{theor}}^2 - P_{\text{expt}} \rangle \) obtained from the results from all 14 sets of measurements. In column 6 of Table 3 the mean differences \( \langle P_{\text{theor}}^2 - P_{\text{expt}} \rangle \) are presented at different separations from 170 nm to 700 nm using the data from all 14 sets of measurements.
The comparison of these data with the half-width of the confidence interval in column 2 shows that at separations $170 \text{ nm} \leq z \leq 350 \text{ nm}$ the mean differences of the theoretical and experimental Casimir pressures fall outside the confidence interval. This confirms the conclusion that the second theoretical approach is excluded by the results of the Casimir pressure measurements using a micromachined oscillator at 95% confidence.

8. Stronger constraints on long-range interactions

As mentioned in the Introduction, measurements of the Casimir force between metals have been successfully used to obtain stronger constraints on hypothetical long-range interactions. Such interactions have long been predicted in elementary particle physics (see literature cited in [11]). They can arise from the exchange of light elementary particles (scalar axions, graviphotons, dilatons, and moduli among others [69,70]) predicted by many extensions of the Standard Model, and as a consequence of extra-dimensional theories with low compactification scales [71]. In most cases the potential energy between two point masses $m_1$ and $m_2$ separated by a distance $r$ is given by the usual Newtonian potential with a Yukawa correction

$$V(r) = -\frac{G m_1 m_2}{r} \left(1 + \alpha e^{-r/\lambda}\right), \quad (46)$$

where $G$ is the gravitational constant, $\alpha$ is a dimensionless constant characterizing the strength of the Yukawa force, and $\lambda$ is its interaction range.

There have been many proposals to constrain $\alpha$ and $\lambda$ over sub-$\mu$m interaction ranges from van der Waals and Casimir force measurements between macroscopic bodies (see references on the subject in [72]). During the last few years the previously known constraints on $(\alpha, \lambda)$ were strengthened by up to 4500 times from modern Casimir force measurements between metals [13,14,15,16,17]. In [11] an additional strengthening by up to a factor of 11 was achieved from the first precise Casimir pressure measurement by means of a micromachined oscillator [10,11].

The results presented above from improved measurements of the Casimir pressure compared with different theoretical approaches allow us to improve the constraints found in [11]. Previous limits have been strengthened by up to a factor of 2, and their application range widened, while increasing their reliability. This has been made possible by the use of the above metrological analysis which confirmed the agreement between experiment and two traditional theoretical approaches within a wide separation range. (In previous experiments constraints were usually obtained at some fixed separation distance.)
As in [11], the Newtonian gravitational force between the plate and the sphere of a micromachined oscillator, and the equivalent gravitational pressure between two plates are negligible. To calculate the equivalent Yukawa pressure between two plates, one needs the detailed structure of the sphere and plate materials. In the experiment under consideration an Al$_2$O$_3$ sphere of density $\rho_{AlO} = 4.1 \times 10^3$ kg/m$^3$ was coated with a layer of Ti of thickness $\Delta_{Ti} = 10$ nm with $\rho_{Ti} = 4.51 \times 10^3$ kg/m$^3$, and a layer of Au of thickness $\Delta_{Au} = 200$ nm with $\rho_{Au} = 19.28 \times 10^3$ kg/m$^3$. The Si plate of density $\rho_{Si} = 2.33 \times 10^3$ kg/m$^3$ was first coated with a layer of Pt of thickness $\Delta_{Pt} = 10$ nm, with $\rho_{Pt} = 21.47 \times 10^3$ kg/m$^3$, and then with a layer of Au of thickness $\Delta_{Au}^{Pt} = 150$ nm. Considering that the conditions $z, \lambda \ll R, L$ are satisfied (where $L = 3.5 \mu$m is the thickness of the plate), the equivalent hypothetical pressure between two parallel plates with the above layer structure is [11]

$$P^{hyp}(z) = -2\pi G \alpha \lambda^2 e^{-z/\lambda} \times \left[ \rho_{Au} - (\rho_{Au} - \rho_{Ti}) e^{-\Delta_{Au}/\lambda} - (\rho_{Ti} - \rho_{AlO}) e^{-(\Delta_{Au} + \Delta_{Ti})/\lambda} \right] \times \left[ \rho_{Au} - (\rho_{Au} - \rho_{Pt}) e^{-\Delta_{Au}^{Pt}/\lambda} - (\rho_{Pt} - \rho_{Si}) e^{-(\Delta_{Au}^{Pt} + \Delta_{Pt})/\lambda} \right].$$

(47)

Note that surface roughness, which was substantially reduced in this experiment, cannot significantly affect the magnitude of a hypothetical force with an interaction range of about 100 nm.

We can now obtain constraints on the hypothetical Yukawa pressure from the agreement between our measurements of the Casimir pressure and theory at 95% confidence. According to our results, no deviations between traditional theories and experiment were observed, i.e., the hypothetical pressure should be less than or equal to the half-width of the confidence interval

$$|P^{hyp}(z)| \leq \Delta^{tot} \left[ P^{theor}(z) - P^{expt}(z) \right].$$

(48)

Note that Eq. (48) is metrologically reliable, since it does not use the root-mean-square deviation as a measure of agreement between theory and experiment to obtain constraints on hypothetical interactions. It is also worthwhile to note that the Yukawa-type hypothetical interaction depends on the separation distance quite differently than the thermal Casimir pressures of the first and second approaches, which were excluded experimentally in Section 7.2. Because of this the mimicry of one phenomenon by another is extremely unlikely.

The numerical analysis of Eqs. (47) and (48) leads to the conclusion that the strongest constraints are obtained within the interaction region $40$ nm $\leq \lambda \leq 370$ nm from the measurement data at separations from $z = 210$ nm (where
the half-width of the confidence interval is 6.89 mPa) to \( z = 450 \) nm (where the half-width is equal to 0.53 mPa). In fact, with the increase of \( \lambda \), the strongest constraints on \( \alpha \) are obtained from the measurement data at larger separations. The resulting constraints on \( \alpha \) are plotted in Fig. 15 for different values of \( \lambda \) (line 1a). In the same figure constraints from earlier experiments are also shown. These were obtained from previous experiments with a micromachined oscillator (line 1b), from old measurements of the Casimir force between dielectrics [12] (line 2), from Casimir force measurements by means of a torsion pendulum [2,13] (line 3), and by the use of an atomic force microscope [6,16] (line 4). In all cases the region in the \((\alpha, \lambda)\) plane above the line is excluded, and below the line is allowed by the experimental results. Note that the constraints from our experiment, line 1a, are found at 95% confidence. For the previous constraints, given by lines 1b, 2, 3, and 4, the confidence levels were not determined.

As is seen from Fig. 15, the present experiment leads to the strongest constraints in the interaction range \( 40 \) nm \( \leq \lambda \leq 370 \) nm which is wider than in the experiment of [11]. Comparing the constraints from lines 2 and 4, the largest improvement (by a factor of 20) is achieved at \( \lambda \approx 150 \) nm. If we compare the present results with those of [11], the largest improvement, by a factor of 2.2, is achieved at \( \lambda \approx 316 \) nm. It should be noted that for the first time the new constraints, given by line 1a, completely fill in the gap between the modern constraints obtained by the atomic force microscope (line 4) and those obtained using a torsion pendulum (line 3). Furthermore, as was noted above, the constraints of line 1a are determined at the 95% confidence level, which makes them the most reliable constraints obtained to date from measurements of the Casimir force.

9. Conclusions and discussion

In this paper we have presented the results of an improved experiment on the dynamical determination of the Casimir pressure between two plane parallel Au coated plates using a micromachined oscillator. (We recall that the pressure between two plane plates was inferred from the Casimir force between a plate and a sphere using the proximity force theorem.) Many improvements in the previously performed experiment of [11] were made yielding a large dividend in precision, accuracy and reliability of the results. The chief advantages of the new experiment lie in a great improvement of the surface roughness, and a decrease by a factor of 1.7 in the error in determining of the absolute separations. The metrological analysis of all experimental errors was performed at 95% confidence. In doing so, the data were analyzed for the presence of outliers measurements. This permitted us to find the total experimental error as a function of separation for subsequent comparison of experiment and theory.
To utilize all the advantages of the improved measurements, the Casimir pressures between two Au plates were calculated in the framework of four theoretical approaches proposed in literature based on the Lifshitz formula at nonzero temperature, with the reflection coefficients expressed in terms of both the surface impedance and the dielectric function. The complete optical data for Au were utilized to take into account both the finite conductivity and the thermal corrections to the Casimir force. In doing so, many relevant factors and properties of Au films were computed or estimated, such as the variation of the tabulated optical data due to the grain structure of a metal, and the influence of spatial nonlocality and patch potentials. The surface roughness was carefully taken into account including the role of nonmultiplicative and diffraction-type effects. The metrological analysis of all theoretical errors was performed at 95% confidence (including the error resulting from use of the proximity force theorem, finiteness of the plate area, and errors in the determination of the surface separations). As a result, the total theoretical error (independent of the experimental one) was combined using the metrological criteria.

To compare different theoretical approaches and experiment, we replaced the root-mean-square deviation previously used in literature, by a rigorous procedure for the composition of the experimental and theoretical errors at 95% confidence. As a result, a confidence interval was found for the random variables $[P_{i\text{theor}}(z) - P_{\text{expt}}(z)]$, where $i = 1, 2, 3$ and 4 denotes the different theoretical approaches described in Sections 1, 5. It was demonstrated that with the impedance approach and the plasma model the mean values $\langle P_{3,4\text{theor}}(z) - P_{\text{expt}}(z)\rangle$, computed at all separations, fall inside the confidence interval. As a consequence, these approaches are consistent with experiment at 95% confidence. In terms of the relative uncertainty, there is agreement between each of them and experiment at the level (1.4–1.9)% when the separation changes in the interval $170 \text{ nm} \leq z \leq 420 \text{ nm}$. This is the best agreement achieved to date at 95% confidence compared to any other experiment measuring the Casimir force. Despite the fact that the thermal Casimir force computed within the impedance approach or the plasma model was found to be consistent with experiment, the thermal correction by itself has not yet been directly measured. According to these approaches, at $T = 300 \text{ K}$ this correction is small, in qualitative agreement with case of ideal metals, and can be readily measured in the near future by means of proposed experiments [73,74].

The experimental results were used as a test for the alternative approaches to the thermal Casimir force which predict large thermal corrections at short separations. The above metrological comparison between experiment and theory was repeated with the other values $P_{1\text{theor}}(z)$, computed in accordance with the first approach [26,34,35]. It was found that at all separations from $z = 170 \text{ nm}$ to $z = 700 \text{ nm}$ the mean values $\langle P_{1\text{theor}}(z) - P_{\text{expt}}(z)\rangle$ fall outside the confidence interval. Moreover, at separations $300 \text{ nm} \leq z \leq 500 \text{ nm}$ the mean
Casimir pressures computed in the first approach fall outside the even wider 99% confidence interval. This enabled us to conclude that the first theoretical approach to the thermal Casimir force is excluded by our measurements at 99% confidence.

The same metrological comparison was carried out with respect to the second approach [27]. The Casimir pressures $P_{2}^{\text{theor}}(z)$ were computed and compared with experiment. It was shown that the mean values $\langle P_{2}^{\text{theor}}(z) - P^{\text{expt}}(z) \rangle$ fall outside the confidence interval for separations $170 \text{ nm} \leq z \leq 350 \text{ nm}$. Thus, the second theoretical approach is also excluded experimentally at 95% confidence.

Finally, the strong agreement between experiment and traditional theory of the Casimir force was used to obtain more stringent constraints on the hypothetical Yukawa-type interactions predicted in high energy physics. Using the results of two experiments (this one and of [11]) the constraints on $\alpha$ as a function of $\lambda$ were strengthened by up to a factor of 20 within a wide interaction range.

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Table 1
Casimir pressures and different contributions to them. See text for further discussion.

| Separation (z nm) | Impedance approach | Plasma model |
|-------------------|--------------------|--------------|
|                   | $P_4^{(l=0)}$ (Pa) | $P_4^{(l≥1)}$ (Pa) | $P_1$ (Pa) | $P_3^{(l=0)}$ (Pa) | $P_3^{(l≥1)}$ (Pa) | $P_3$ (Pa) | $P_L^{(l≥1)}$ (Pa) |
| 160               | -0.0715            | -1.0726       | -1.1441    | -0.0719           | -1.0430         | -1.1149    | -1.07715          |
| 200               | -0.03834           | -0.47096      | -0.5093    | -0.03847          | -0.46335        | -0.5018    | -0.4724           |
| 250               | -0.02046           | -0.20421      | -0.2247    | -0.02050          | -0.20260        | -0.2231    | -0.2046           |
| 300               | -0.01220           | -0.10214      | -0.1143    | -0.01222          | -0.10182        | -0.1140    | -0.1023           |
| 350               | -0.007858          | -0.056424     | -0.06428   | -0.007866         | -0.056425       | -0.06429   | -0.05648          |
| 400               | -0.005358          | -0.033547     | -0.03890   | -0.005362         | -0.033616       | -0.03898   | -0.03357          |
| 450               | -0.003817          | -0.021105     | -0.02492   | -0.003820         | -0.021178       | -0.02500   | -0.02111          |
| 500               | -0.002816          | -0.013887     | -0.01670   | -0.002817         | -0.013948       | -0.01676   | -0.01389          |
| 600               | -0.001659          | -0.006665     | -0.008324  | -0.001660         | -0.006703       | -0.008363  | -0.006667         |
| 700               | -0.001059          | -0.003546     | -0.004605  | -0.001059         | -0.003569       | -0.004628  | -0.003547         |

Table 2
Casimir pressures and zero-frequency contributions to them in the alternative approaches to the thermal Casimir force.

| Separation (z nm) | Approach 1 [26,34,35] | Approach 2 [27] |
|-------------------|------------------------|-----------------|
|                   | $P_1^{(l=0)}$ (Pa)     | $P_1$ (Pa)      | $P_2^{(l=0)}$ (Pa) | $P_2$ (Pa)    |
| 160               | -0.04836              | -1.1255         | -0.0967           | -1.1739       |
| 200               | -0.0247               | -0.4971         | -0.0495           | -0.5219       |
| 250               | -0.0127               | -0.2173         | -0.0254           | -0.2300       |
| 300               | -0.00734              | -0.1096         | -0.0147           | -0.1170       |
| 350               | -0.00462              | -0.06110        | -0.00924          | -0.06572      |
| 400               | -0.00310              | -0.03667        | -0.00619          | -0.03976      |
| 450               | -0.00217              | -0.02329        | -0.00434          | -0.02546      |
| 500               | -0.00158              | -0.01547        | -0.00317          | -0.01706      |
| 600               | -0.000917             | -0.007584       | -0.001835         | -0.008502     |
| 700               | -0.000577             | -0.004124       | -0.001155         | -0.004702     |
Table 3
Mean values of differences between the theoretical and experimental Casimir pressures. See the text for the definition of $\Delta \bar{P}_i$.

| $z$ (nm) | $\Delta^{\text{tot}} [P_{\text{theor}}(z) - P_{\text{expt}}(z)]$ | $\Delta \bar{P}_4$ (mPa) | $\Delta \bar{P}_3$ (mPa) | $\Delta \bar{P}_1$ (mPa) | $\Delta \bar{P}_2$ (mPa) |
|---------|-------------------------------------------------|----------------|----------------|----------------|----------------|
| 170     | 17.2                                            | 2.01           | 13.0           | 18.8           | -21.8          |
| 180     | 13.4                                            | -0.74          | 7.54           | 14.4           | -19.8          |
| 200     | 8.59                                            | -1.21          | 5.3            | 11.0           | -13.9          |
| 250     | 3.34                                            | -0.31          | 1.3            | 7.09           | -5.66          |
| 300     | 1.59                                            | 0.34           | 0.6            | 5.07           | -2.28          |
| 350     | 0.89                                            | 0.38           | 0.39           | 3.58           | -1.05          |
| 400     | 0.63                                            | 0.28           | 0.20           | 2.59           | -0.62          |
| 500     | 0.49                                            | 0.11           | 0.05           | 1.37           | -0.22          |
| 600     | 0.46                                            | 0.08           | 0.04           | 0.82           | -0.09          |
| 700     | 0.46                                            | 0.02           | -0.01          | 0.51           | -0.07          |

Fig. 1. Schematic diagram showing the experimental setup. The different distances are defined in the text.

Fig. 2. AFM images of the Au films deposited on the MTO and on the sphere. Topographies on the different panels are indicated. $10 \times 10 \mu \text{m}^2$ images of the film on the MTO before the experiment (a) and after it (b) are provided. Figures (c) and (d) provide similar information for the sphere, except that a $5 \times 5 \mu \text{m}^2$ area is shown.

Fig. 3. Typical cross sections of the atomic force microscope images of the Au coatings on the plate (a) and on the sphere (b). $h$ denotes the height of the surface above the reference level defined in the text, and is plotted against the lateral position $x$.

Fig. 4. Topographic heights $h_i$ on the sample as a function of the total sample area with heights $h < h_{i+1}$ for the film deposited on the MTO (a) and the film deposited on the sphere (b). For both films the differences observed when analyzing different regions in the sample are negligible.

Fig. 5. Dependence of the angular deviation $\theta$ as a function of the applied voltage to the sphere. Data obtained at five different separations $z$ between the metallic layers are shown. Data have been displaced vertically for the sake of clarity.

Fig. 6. Electrostatic force $F_{el}$ as a function of separation $z$ for $V_{Au} - V_0 = 0.22 \text{ V}$, $0.25 \text{ V} 0.35 \text{ V}$. For simplicity, the $z$-axis has already been corrected for $D$ and $\theta$. The lines are fits to the data using Eq. (3).
Fig. 7. Change in resonant frequency $\Delta f_r = (\omega_r - \bar{\omega}_r)/(2\pi)$ as a function of amplitude of motion of the MTO, obtained at a separation $z = 300$ nm. The non-linear behavior becomes dominant at large amplitudes, increasing the uncertainty in the frequency. Frequency shifts are measured with respect to the one obtained using the thermodynamic noise as a driving force, see text. Inset: Resonance curve obtained under the conditions indicated by the arrow.

Fig. 8. Absolute value of the parallel plate Casimir pressure $P_{\text{expt}}$ as a function of separation $z$. The separation has been corrected taking into account $D$ and $\theta$.

Fig. 9. The random $\Delta^{\text{rand}} P_{\text{expt}}$ (long-dashed lines), systematic $\Delta^{\text{syst}} P_{\text{expt}}$ (short-dashed lines) and total $\Delta^{\text{tot}} P_{\text{expt}}$ (solid line) absolute experimental errors of the Casimir pressure measurements versus separation within the intervals $160 \text{ nm} \leq z \leq 750 \text{ nm}$ (a) and $160 \text{ nm} \leq z \leq 750 \text{ nm}$ (b).

Fig. 10. The total relative error of the Casimir pressure measurements versus separation.

Fig. 11. The relative theoretical errors of the Casimir pressure computed in different approaches due to use of the proximity force theorem (short-dashed line), variation of the tabulated optical data (long-dashed line), due to the experimental uncertainties of absolute separations (dotted line), and total theoretical relative error (solid line) versus separation.

Fig. 12. The 95% confidence intervals (solid lines) and differences $[P^{\text{theor}}_4 - P_{\text{expt}}]$ (dots) versus separation within the intervals $160 \text{ nm} \leq z \leq 250 \text{ nm}$ (a) and $250 \text{ nm} \leq z \leq 750 \text{ nm}$ (b) for one set of measurements computed in the impedance approach.

Fig. 13. The 95% confidence intervals (solid lines) and differences $[P^{\text{theor}}_4 - P_{\text{expt}}]$ (dots) versus separation within the intervals $160 \text{ nm} \leq z \leq 250 \text{ nm}$ (a) and $250 \text{ nm} \leq z \leq 750 \text{ nm}$ (b) for all fourteen sets of measurements computed in the impedance approach.

Fig. 14. The 95% confidence intervals (solid lines) and differences $[P^{\text{theor}}_{1/2} - P_{\text{expt}}]$ (dots) versus separation for theoretical approaches 1 [26,34,35] (a) and 2 [27] (b) incorporating different models of alternative thermal corrections. All fourteen sets of measurements as in Fig. 13 were used.

Fig. 15. Constraints on the strength of the Yukawa interaction $\alpha$ versus interaction range $\lambda$. Line 1a is obtained in this paper, line 1b is from [11], line 2 follows from old Casimir force measurements between dielectrics [12]. Lines 3 and 4 are obtained from Casimir force measurements by using a torsion pendulum [2,13], and by means of an atomic force microscope [6,16], respectively. The region of $(\alpha, \lambda)$ plane above each line is excluded and below each line is allowed.
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