QUANTIZING BRAIDS AND OTHER MATHEMATICAL OBJECTS: THE GENERAL QUANTIZATION PROCEDURE

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Abstract. Extending the methods from our previous work on quantum knots and quantum graphs, we describe a general procedure for quantizing a large class of mathematical structures which includes, for example, knots, graphs, groups, algebraic varieties, categories, topological spaces, geometric spaces, and more. This procedure is different from that normally found in quantum topology. We then demonstrate the power of this method by using it to quantize braids.

This general method produces a blueprint of a quantum system which is physically implementable in the same sense that Shor’s quantum factoring algorithm is physically implementable. Mathematical invariants become objects that are physically observable.

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1. Introduction

Extending the methods found in previous work \cite{12,13} on quantum knots and quantum graphs, we describe a general procedure for quantizing a large class of mathematical structures which includes, for example, knots, graphs, groups, algebraic varieties, categories, topological spaces, geometric spaces, and more. This procedure is different from that normally found in quantum topology. We then demonstrate the power of this method by using it to quantize braids.

We should also mention that this general method produces a blueprint of a quantum system which is physically implementable in the same sense that Shor’s quantum factoring algorithm is physically implementable. Moreover, mathematical invariants become objects that are physically observable.

The above mentioned general quantization procedure consists of two steps:

**Step 1.** Mathematical construction of a motif system $S$, and
**Step 2.** Mathematical construction of a quantum motif system $Q$ from the system $S$.

**Caveat.** The term ”motif” used in this paper should not be confused with the use of the term ”motive” (a.k.a., ”motif”) found in algebraic geometry.

2. Part I. A General Procedure for Quantizing Mathematical Structures

We now outline a general procedure for quantizing mathematical structures. One useful advantage to this quantization procedure is that the resulting system is a multipartite quantum system, a property that is of central importance in quantum computation, particularly in regard to the design of quantum algorithms. In a later section of this paper, we illustrate this quantization procedure by using it to quantize braids. Examples of the application of this quantization procedure to knots, graphs, and algebraic structures can be found in \cite{12,13,8}.

2.1. Stage 1. Construction of a motif system $S_n$.

Let

$$T = \{t_0, t_1, \ldots, t_{\ell-1}\}$$

be a finite set of symbols, with a distinguished element $t_0$, called the trivial symbol, and with a linear ordering denoted by ‘$<$’. Let

$$T^{\times N}$$

be the the $N$-fold cartesian product of $T$ with an induced LEX ordering also denoted by ‘$<$’, and let $S(n)$ be the group of all permutations of $T^{\times N}$. For positive integers $N$ and $N'$ ($N < N'$), let

$$t : T^{\times N} \rightarrow T^{\times N'}$$
be the injection defined by
\[ (t_j(0), t_j(1), t_j(2), \ldots, t_j(N-1)) \mapsto \left( (t_j(0), t_j(1), t_j(2), \ldots, t_j(N-1)), t_0, t_0, \ldots, t_0 \right) \]

Next, let
\[ N_0 < N_1 < N_2 < \ldots \]
be a monotone strictly increasing infinite sequence of positive integers.

For each positive integer \( n \geq 0 \), let \( M^{(n)} \) be a subset of \( T^{\times N_n} \) such that \( \iota \left( M^{(n)} \right) \) lies in \( M^{(n+1)} \), i.e., \( \iota \left( M^{(n)} \right) \subset M^{(n+1)} \). Moreover, for each non-negative integer \( n \), let \( A(n) \) be a subgroup of the permutation group \( S(N_n) \) having \( M^{(n)} \) as an invariant subset, and such that the injection \( \iota : T^{\times N_n} \rightarrow T^{\times N_{n+1}} \) induces a monomorphism \( \iota : A(N_n) \rightarrow A(N_{n+1}) \), also denoted by \( \iota \).

We define a \textbf{motif system} \( S_n = S \left( M^{(n)}, A(n) \right) \) of \textbf{order} \( n \) as the pair \( \left( M^{(n)}, A(n) \right) \), where \( M^{(n)} \) is called the \textbf{set of motifs}, and where \( A(n) \) is called the \textbf{ambient group}.

Finally, we define a \textbf{nested motif system} \( S_\ast = S_\ast \left( M^{(\ast)}, A(\ast) \right) \) as the following sequence of sets, groups, injections, and monomorphisms:
\[ S_1 \left( M^{(1)}, A(1) \right) \xrightarrow{\iota} S_2 \left( M^{(2)}, A(2) \right) \xrightarrow{\iota} \cdots \xrightarrow{\iota} S_n \left( M^{(n)}, A(n) \right) \xrightarrow{\iota} \cdots \]

\textbf{Remark 1.} \textit{There is also one more symbolic motif system that is often of use, the direct limit motif system defined by}
\[ S_\infty \left( M^{(\infty)}, A(\infty) \right) = \varinjlim S_\ast \left( M^{(\ast)}, A(\ast) \right), \]
where \( \varinjlim \) denotes the direct limit.

\section*{2.2. Stage 2. Motif equivalence and motif invariants.}

Let \( S_n = S \left( M^{(n)}, A(n) \right) \) be a motif system of order \( n \).

Two motifs \( m_1 \) and \( m_2 \) of the set \( M^{(n)} \) are said to be of the \textbf{same \( n \)-motif type}, written
\[ m_1 \sim_n m_2, \]
if there exists an element \( g \) of the ambient group \( A(n) \) which takes \( m_1 \) to \( m_2 \), i.e., such that
\[ gm_1 = m_2. \]

The motifs \( m_1 \) and \( m_2 \) are said to be of the \textbf{same motif type}, written
\[ m_1 \sim m_2, \]
if there exists a non-negative integer \( k \) such that
\[ \iota^k m_1 \sim_{n+k} \iota^k m_2. \]
We now wish to answer the question:

**Question.** What is meant by a motif invariant?

**Definition 1.** Let $S_n = S \left( M^{(n)}, A(n) \right)$ be a motif system, and let $\mathbb{D}$ be some yet to be chosen mathematical domain. By an \textbf{n-motif invariant} $I^{(n)}$, we mean a map

$$I^{(n)} : M^{(n)} \rightarrow \mathbb{D}$$

such that, when two motifs $m_1$ and $m_2$ are of the same $n$-type, i.e.,

$$m_1 \sim_n m_2,$$

then their respective invariants must be equal, i.e.,

$$I^{(n)}(m_1) = I^{(n)}(m_2).$$

In other words, $I^{(n)} : M^{(n)} \rightarrow \mathbb{D}$ is a map that is invariant under the action of the ambient group $A(n)$, i.e.,

$$I^{(n)}(m) = I^{(n)}(gm)$$

for all elements of $g$ in $A(n)$.

2.3. \textbf{Stage 3. Construction of the corresponding quantum motif systems $Q_n$.}

We now use the nested motif system $S_*$ to construct a nested sequence of quantum motif systems $Q_*$.

For each non-negative $n$, the corresponding \textbf{n-th order quantum motif system}

$$Q_n = Q \left( M^{(n)}, A(n) \right)$$

consists of a Hilbert space $M^{(n)}$, called the \textbf{quantum motif space}, and a group $A(n)$, also called the \textbf{ambient group}. The quantum motif space $M^{(n)}$ and the ambient group $A(n)$ are defined as follows:

- The \textbf{quantum motif space} $M^{(n)}$ is the Hilbert space with orthonormal basis

$$\left\{ |m\rangle : m \in M^{(n)} \right\}.$$

The elements of $M^{(n)}$ are called \textbf{quantum motifs}.

- The \textbf{ambient group} $A(n)$ is the unitary group acting on the Hilbert space $M^{(n)}$ consisting of all linear transformations of the form

$$\left\{ \tilde{g} : M^{(n)} \rightarrow M^{(n)} : g \in A(n) \right\},$$

where $\tilde{g}$ is the linear transformation defined by

$$\begin{align*}
\tilde{g} : M^{(n)} & \rightarrow M^{(n)} \\
|m\rangle & \mapsto |gm\rangle
\end{align*}$$

Since each element $g$ in $A(n)$ is a permutation, each $\tilde{g}$ permutes the orthonormal basis $\left\{ |m\rangle : m \in M^{(n)} \right\}$ of $M^{(n)}$. Hence, $\tilde{g}$ is automatically
a unitary transformation. It follows that $A(n)$ and $A(n)$ are isomorphic
as groups. We will often abuse notation by denoting $\tilde{g}$ by $g$, and $A(n)$ by
$A(n)$.

Next, for each non-negative integer $n$, let

$$\iota : M(n) \longrightarrow M(n + 1)$$

and

$$\iota : A(n) \longrightarrow A(n + 1)$$

respectively denote the Hilbert space monomorphism and the group monomorphism
induced by the injection

$$\iota : M(n) \longrightarrow M(n + 1)$$

and the group monomorphism

$$\iota : A(n) \longrightarrow A(n + 1).$$

Finally, we define the \textbf{nested quantum motif system} $Q_\ast = Q_\ast \left( M(\ast), A(\ast) \right)$
as the following sequence of Hilbert spaces, groups, Hilbert space monomorphisms,
and group monomorphisms:

$$Q_1 \left( M^{(1)}, A(1) \right) \xrightarrow{\iota_1} Q_2 \left( M^{(2)}, A(2) \right) \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_n} Q_n \left( M^{(n)}, A(n) \right) \xrightarrow{\iota_n} \cdots$$

\textbf{Remark 2.} We should also mention one other quantum motif system that can be
useful, namely, the \textbf{quantum direct limit motif system} defined by

$$Q_\infty = Q_\infty \left( M(\infty), A(\infty) \right) = \lim_{\longrightarrow} Q_\ast \left( M(\ast), A(\ast) \right),$$

where $\lim_{\longrightarrow}$ denotes the direct limit. This quantum system is often also physically
implementable.

\textbf{2.4. Stage 4. Quantum motif equivalence.} Let $Q_n = Q \left( M^{(n)}, A(n) \right)$ be a
quantum motif system of order $n$.

Two quantum motifs $|\psi_1\rangle$ and $|\psi_2\rangle$ of the Hilbert space $M^{(n)}$ are said to be of the \textbf{same n-motif type}, written

$$|\psi_1\rangle \sim_n |\psi_2\rangle,$$

if there exists an element $g$ of the ambient group $A(n)$ which takes $|\psi_1\rangle$ to $|\psi_2\rangle$, i.e., such that

$$g |\psi_1\rangle = |\psi_2\rangle.$$ The quantum motifs $|\psi_1\rangle$ and $|\psi_2\rangle$ are said to be of the \textbf{same motif type}, written

$$|\psi_1\rangle \sim |\psi_2\rangle,$$

if there exists a non-negative integer $m$ such that

$$\iota^m |\psi_1\rangle \sim_{n+m} \iota^m |\psi_2\rangle.$$
2.5. Stage 5. Motif invariants as quantum observables.

We consider the following question:

**Question:** What do we mean by a physically observable quantum motif invariant?

We answer this question with a definition.

**Definition 2.** Let \( Q_n = Q \left( M^{(n)}, A(n) \right) \) be a quantum motif system of order \( n \), and let \( \Omega \) be an observable, i.e., a Hermitian operator on the Hilbert space \( M^{(n)} \) of quantum motifs. Then \( \Omega \) is a **quantum motif \( n \)-invariant** provided \( \Omega \) is left invariant under the big adjoint action of the ambient group \( A(n) \), i.e., provided

\[
U \Omega U^{-1} = \Omega
\]

for all \( U \) in \( A(n) \).

**Proposition 1.** If

\[
I^{(n)} : M^{(n)} \rightarrow \mathbb{R}
\]

is a real valued \( n \)-motif invariant, then

\[
\Omega = \sum_{m \in M^{(n)}} I^{(n)}(m) |m\rangle \langle m|
\]

is a quantum motif observable which is a quantum motif \( n \)-invariant.

Much more can be said about this topic. For a more in-depth discussion of this issue, we refer the reader to [12, 13].

3. Part II. Quantizing Braids

We now illustrate the quantization procedure defined above by using it to quantize braids.

3.1. Stage 1. The set of braid mosaics \( \mathbb{B}^{(n, \ell)} \).

For each integer \( n \geq 2 \), let \( T^{(n)} \) denote the following set of the \( 2n - 1 \) symbols

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|
as indicated in the table given below:

| \( b_{-(n-1)} \) | \( \cdots \) | \( b_{-2} \) | \( b_{-1} \) | \( b_0 = 1 \) | \( b_1 \) | \( b_2 \) | \( \cdots \) | \( b_{n-1} \) |
|------------------|--------|--------|--------|--------|--------|--------|--------|--------|

**Definition 3.** An \((n, \ell)\)-braid mosaic \( \beta \) is defined as a sequence of \( n \)-stranded braid tiles

\[
\beta = b_{j(1)} b_{j(2)} \cdots b_{j(\ell)}
\]

of length \( \ell \). We let \( \mathbb{B}^{(n,\ell)} \) denote the set of all \((n, \ell)\)-braid mosaics.

An example of a \((3,8)\)-braid mosaic is given below

\[
1 \quad b_{-1} \quad b_1 \quad b_2 \quad 1 \quad 1 \quad b_{-1} \quad b_2
\]

The \((3,8)\)-braid mosaic \( \beta = 1b_{-1}b_1b_21b_{-1}b_2 \)

**Remark 3.** Please note that the set of all \((n, \ell)\)-braid mosaics \( \mathbb{B}^{(n,\ell)} \) is a finite set of cardinality \((2n - 1)^\ell\).

3.2. Stage 1 (Cont.) Braid mosaic moves.

**Definition 4.** Let \( \ell' \) and \( \ell \) be positive integers such that \( \ell' \leq \ell \). An \((n, \ell')\)-braid mosaic \( \gamma \) is said to be \((n, \ell')\)-braid submosaic of an \((n, \ell)\)-braid mosaic \( \beta \) provided \( \gamma \) is a subsequence of consecutive tiles of \( \beta \). The \((n, \ell')\)-braid submosaic \( \gamma \) is said to be at position \( p \) in \( \beta \) if the first (leftmost) tile of \( \gamma \) is the \( p \)-th tile of \( \beta \) from the left. We denote the \((n, \ell')\)-braid submosaic \( \gamma \) of \( \beta \) at location \( p \) by \( \gamma = \beta^{p:\ell'} \).

**Remark 4.** The number of \((n, \ell')\)-braid submosaics of an \((n, \ell)\)-braid mosaic \( \beta \) is \( \ell - \ell' + 1 \).

Two examples of braid submosaics of the \((3,8)\)-braid mosaic \( \beta = 1b_{-1}b_1b_21b_{-1}b_2 \) are given above are:

The \((3,3)\)-braid submosaic \( \beta^{2:3} \) of \( \beta \) at position 2

The \((3,4)\)-braid submosaic \( \beta^{5:4} \) of \( \beta \) at position 5
Definition 5. Let $\ell'$ and $\ell$ be positive integers such that $\ell' \leq \ell$. For any two $(n, \ell')$-braid mosaics $\gamma$ and $\gamma'$, we define the $\ell'$-braid mosaic move at location $p$ on the set of all $(n, \ell)$-braid mosaics $B^{(n, \ell)}$, denoted by

\[
\gamma \leftrightarrow_{\ell'} \gamma',
\]

as the map defined by

\[
\left( \gamma \leftrightarrow_{\ell'} \gamma' \right)(\beta) = \begin{cases} 
\beta \text{ with } \beta^{p,\ell'} \text{ replaced by } \gamma' & \text{if } \beta^{p,\ell'} = \gamma \\
\beta \text{ with } \beta^{p,\ell'} \text{ replaced by } \gamma & \text{if } \beta^{p,\ell'} = \gamma' \\
\beta & \text{otherwise}
\end{cases}
\]

As an example, consider the 2-braid mosaic move $\gamma \leftrightarrow_{\ell'} \gamma'$ at position 3 defined by

\[
\gamma \leftrightarrow_{\ell'} \gamma' = \quad \text{Braid Submosics Switched}
\]

Then

\[
\left( \gamma \leftrightarrow_{\ell'} \gamma' \right)(\text{Braid Submosics Switched}) = \quad \text{Braid Submosics Switched}
\]

The following proposition is an almost immediate consequence of the definition of a braid move.

Proposition 2. Each braid move is a permutation on the set $B^{(n, \ell)}$ of $(n, \ell)$-braid mosaics. In fact, it is a permutation which is a product of disjoint transpositions.

3.3. Stage 1. (Cont.) Planar isotopy moves.

Our next objective is to translate all the standard topological moves on braids into braid mosaic moves. To accomplish this, we must first note that there are two types of standard topological moves, i.e., those which do not change the topological type of the braid projection, called planar isotopy moves, and those which do change the topological type of the braid projection but not of the braid itself, called Reidemeister moves.

We begin with the planar isotopy moves.
**Definition 6.** For braid mosaics, there are two types planar isotopy moves, i.e., types $P_1$ and $P_2$, which are defined below as:

| 1bi $\xrightarrow{P_1}$ bi1 for $0 < |i| < n$ |
|---|
| **Definition of a type $P_1$ planar isotopy move** |

and

| $b_ib_j \xleftarrow{P_2} b_jb_i$ for $0 < |i|, |j| < n$ and $||i| - |j|| > 1$ |
|---|
| **Definition of a type $P_2$ planar isotopy move** |

**Example 1.** Examples of $P_1$ and $P_2$ moves are respectively given below:

\[\text{An example of a $P_1$ move.}\]

\[\text{An example of a $P_2$ move.}\]

**Remark 5.** The number of $P_1$ and $P_2$ moves are respectively $2(n-1)(\ell-1)$ and $(n-1)(2n-6)(\ell-1)$.

3.4. Stage 1. (Cont.) Reidemeister moves.

There are two types of topological moves, i.e., $R_2$ and $R_3$.

**Definition 7.** The Reidemeister $R_2$ moves are defined as

\[b_i b_{i-1} \xleftarrow{\lambda} b_{i-1}b_i\text{ for } 0 < |i| < n\]

where $0 < |i| < n$

**Example 2.** An example of a Reidemeister 2 move is given below

\[\text{An example of a Reidemeister 2 move.}\]

**Remark 6.** The number of $R_2$ moves is $2(n-1)(\ell-1)$
Definition 8. The Reidemeister $R_3$ moves are defined for $0 < |i| < n$, and given below:

\[
\begin{align*}
    b_i b_{i+1} b_{-(i+1)} b_{-(i+1)} & \xleftarrow{\lambda} 1^6 \\
    b_i b_{i+1} b_{i-1} b_{-(i+1)} & \xleftarrow{\lambda} b_{i+1} 1^4 \\
    b_i b_{i+1} b_{-(i+1)} & \xleftarrow{\lambda} b_{i+1} b_i^2 \\
    b_i b_{i+1} b_i & \xleftarrow{\lambda} b_{i+1} b_i b_{i+1} \\
    b_i b_{i+1} b_i^2 & \xleftarrow{\lambda} b_{i+1} b_i b_{i+1} b_{-i} \\
    b_i 1^4 & \xleftarrow{\lambda} b_{i+1} b_i b_{i+1} b_{-i} b_{-(i+1)} \\
    1^6 & \xleftarrow{\lambda} b_{i+1} b_i b_{i+1} b_{-i} b_{-(i+1)} b_{-i} 
\end{align*}
\]

Example 3. Two examples of Reidemeister $R_3$ are given below:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example1.png} \\
\includegraphics[width=0.2\textwidth]{example2.png} \\
\includegraphics[width=0.2\textwidth]{example3.png} \\
\includegraphics[width=0.2\textwidth]{example4.png}
\end{array}
\]

Remark 7. The number of Reidemeister 3 moves $R_3$ is given by

\[
\# R_3 \text{ Moves} = \begin{cases} 
    n(n-2)(6\ell-21) & \text{if } \ell \geq 6 \\
    n(n-2)(5\ell-16) & \text{if } \ell = 5 \\
    n(n-2)(3\ell-8) & \text{if } \ell = 4 \\
    n(n-2)(\ell-2) & \text{if } \ell = 3 \\
    0 & \text{if } \ell < 3 
\end{cases}
\]

3.5. Stage 1. (Cont.) The ambient group $A(n, \ell)$ and the braid mosaic system $B_{n, \ast}$.

At this point, we can define what is meant by the ambient group and the resulting braid mosaic system.

We begin reminding the reader of a fact noted earlier in this paper, namely the fact that each braid move is a permutation on the set $\mathbb{B}^{(n, \ell)}$ of $(n, \ell)$-braid mosaics.
Thus, since planar isotopy and Reidemeister moves are permutations, we can make the following definition:

**Definition 9.** We define the \((n, \ell)\)-braid mosaic ambient group \(A(n, \ell)\) as the group of all permutations on the set \(B(n, \ell)\) of \((n, \ell)\)-braid mosaics generated by \((n, \ell)\)-braid planar isotopy and Reidemeister moves.

We need one more definition, before we can move to the objective of this section.

**Definition 10.** We define the braid mosaic injection \(\iota: B(n, \ell) \to B(n, \ell + 1)\) as the map \(\beta \mapsto \iota \beta_1\) for each \((n, \ell)\)-braid mosaic in \(B(n, \ell)\). It immediately follows that the braid mosaic injection induces a monomorphism \(\iota: A(n, \ell) \to A(n, \ell + 1)\) from the \((n, \ell)\)-braid ambient group \(A(n, \ell)\) to the \((n, \ell + 1)\)-braid ambient group \(A(n, \ell + 1)\). This monomorphism is called the braid mosaic monomorphism.

**Definition 11.** We define an braid system \(B_{n, \ell} = (B(n, \ell), A(n, \ell))\) of order \((n, \ell)\) as the pair \((B(n, \ell), A(n, \ell))\), where \(B(n, \ell)\) is called the set of \((n, \ell)\)-braid mosaics, and where \(A(n, \ell)\) is called the ambient group. Finally, we define a nested motif system \(B_{n, *} = (B(n, *), A(n, *))\) as the following sequence of sets, groups, injections, and monomorphisms:

\[
\begin{align*}
B \left( B(n, 1), A(n, 1) \right) & \xrightarrow{\iota} B \left( B(n, 2), A(n, 2) \right) \xrightarrow{\iota} \cdots \xrightarrow{\iota} B \left( B(n, 2), A(n, 2) \right) \xrightarrow{\iota} \cdots
\end{align*}
\]

### 3.6. Stage 2. Braid mosaic type and braid mosaic invariants.

Our next objective is to define what it means for two braid mosaics to represent the same topological braid.

Two braid mosaics \(\beta_1\) and \(\beta_2\) of the set \(B(n, \ell)\) are said to be of the same n-braid mosaic type, written \(\beta_1 \sim_n \beta_2\), if there exists an element \(g\) of the ambient group \(A(n, \ell)\) which takes \(\beta_1\) to \(\beta_2\), i.e., such that \(g\beta_1 = \beta_2\). The braid mosaics \(\beta_1\) and \(\beta_2\) are said to be of the same braid mosaic type, written \(\beta_1 \sim \beta_2\), if there exists a non-negative integer \(k\) such that \(\iota^k \beta_1 \sim \iota^k \beta_2\).
We now wish to answer the question:

**Question.** What is meant by a braid mosaic invariant?

**Definition 12.** Let $B_{n,\ell} = B\left(\mathbb{B}^{(n,\ell)}, A(n)\right)$ be a braid system, and let $\mathbb{D}$ be some yet to be chosen mathematical domain. By an $n$-**braid mosaic invariant** $I^{(n)}$, we mean a map

$$I^{(n)} : \mathbb{B}^{(n,\ell)} \rightarrow \mathbb{D}$$

such that, when two braid mosaics $\beta_1$ and $\beta_2$ are of the same $n$-type, i.e., when

$$\beta_1 \sim_n \beta_2,$$

then their respective invariants must be equal, i.e.,

$$I^{(n)}(\beta_1) = I^{(n)}(\beta_2).$$

In other words, $I^{(n)} : \mathbb{B}^{(n,\ell)} \rightarrow \mathbb{D}$ is a map that is invariant under the action of the ambient group $A(n)$, i.e.,

$$I^{(n)}(\beta) = I^{(n)}(g\beta)$$

for all elements of $g$ in $A(n)$.

3.7. Stage 3. Construction of the corresponding quantum braid system.

We now use the nested braid mosaic system $B_{n,*}$ to construct a nested sequence of quantum braid mosaic systems $Q_{n,*}$.

For pair of non-negative integers $n$ and $\ell$ the corresponding $(n,\ell)$-**th order quantum braid system** $Q_{n,\ell} = Q\left(\mathcal{B}^{(n,\ell)}, A(n,\ell)\right)$ consists of a Hilbert space $\mathcal{B}^{(n,\ell)}$, called the **quantum mosaic space**, and a group $A(n,\ell)$, also called the **ambient group**. The quantum motif space $\mathcal{B}^{(n,\ell)}$ and the ambient group $A(n,\ell)$ are defined as follows:

- The **quantum motif space** $\mathcal{B}^{(n,\ell)}$ is the Hilbert space with orthonormal basis

  $$\left\{ |\beta\rangle : \beta \in \mathcal{B}^{(n,\ell)} \right\}.$$  

  The elements of $\mathcal{B}^{(n,\ell)}$ are called **quantum braids**.

- The **ambient group** $A(n,\ell)$ is the unitary group acting on the Hilbert space $\mathcal{B}^{(n,\ell)}$ consisting of all linear transformations of the form

  $$\left\{ \tilde{g} : \mathcal{B}^{(n,\ell)} \rightarrow \mathcal{B}^{(n,\ell)} : g \in A(n,\ell) \right\},$$

  where $\tilde{g}$ is the linear transformation defined by

  $$\tilde{g} : \mathcal{B}^{(n,\ell)} \rightarrow \mathcal{B}^{(n,\ell)}$$

  $$|\beta\rangle \rightarrow |g\beta\rangle.$$
Since each element \( g \) in \( A(n, \ell) \) is a permutation, each \( \tilde{g} \) permutes the orthonormal basis \( \{ |\beta\rangle : \beta \in B(n, \ell) \} \) of \( B(n, \ell) \). Hence, \( \tilde{g} \) is automatically a unitary transformation. It follows that \( A(n, \ell) \) and \( A(n, \ell) \) are isomorphic as groups. We will often abuse notation by denoting \( \tilde{g} \) by \( g \), and \( A(n, \ell) \) by \( A(n, \ell) \).

Next, for each pair of non-negative integers \( n \) and \( \ell \), let
\[
\iota : B(n, \ell) \rightarrow B(n+1, \ell)
\]
and
\[
\iota : A(n, \ell) \rightarrow A(n+1, \ell)
\]
respectively denote the Hilbert space monomorphism and the group monomorphism induced by the injection
\[
\iota : B(n, \ell) \rightarrow B(n+1, \ell)
\]
and the group monomorphism
\[
\iota : A(n, \ell) \rightarrow A(n+1, \ell).
\]

Finally, we define the nested quantum braid system \( Q_{n,\ast} = Q_{n,\ast} \left( B(n,\ast), A(n,\ast) \right) \) as the following sequence of Hilbert spaces, groups, Hilbert space monomorphisms, and group monomorphisms:
\[
Q_{1,\ell} \left( B(1, \ell), A(1, \ell) \right) \overset{\iota}{\rightarrow} Q_{2,\ell} \left( B(2, \ell), A(2, \ell) \right) \overset{\iota}{\rightarrow} \cdots \overset{\iota}{\rightarrow} Q_{n,\ell} \left( B(n, \ell), A(n, \ell) \right) \overset{\iota}{\rightarrow} \cdots
\]

3.8. **Stage 4. Quantum braid equivalence.** Let \( Q_{n,\ell} = Q \left( B(n, \ell), A(n, \ell) \right) \) be a quantum motif system of order \((n, \ell)\).

Two quantum braids \( |\psi_1\rangle \) and \( |\psi_2\rangle \) of the Hilbert space \( B(n, \ell) \) are said to be of the same \((n, \ell)\)-braid type, written
\[
|\psi_1\rangle \sim_n |\psi_2\rangle,
\]
if there exists an element \( g \) of the ambient group \( A(n, \ell) \) which takes \( |\psi_1\rangle \) to \( |\psi_2\rangle \), i.e., such that
\[
g |\psi_1\rangle = |\psi_2\rangle.
\]
The quantum motifs \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are said to be of the same braid type, written
\[
|\psi_1\rangle \sim |\psi_2\rangle,
\]
if there exists a non-negative integer \( m \) such that
\[
\iota^m |\psi_1\rangle \sim_{n+m} \iota^m |\psi_2\rangle.
\]
3.9. Stage 5. Quantum braid invariants as quantum observables.

We consider the following question:

**Question:** What do we mean by a physically observable quantum braid invariant?

We answer this question with a definition.

**Definition 13.** Let $Q_{n,\ell} = Q(B^{(n,\ell)}, A(n,\ell))$ be a quantum braid system of order $(n,\ell)$, and let $\Omega$ be an observable, i.e., a Hermitian operator on the Hilbert space $B^{(n,\ell)}$ of quantum braids. Then $\Omega$ is a quantum braid $(n,\ell)$-invariant provided $\Omega$ is left invariant under the big adjoint action of the ambient group $A(n,\ell)$, i.e., provided

$$U\Omega U^{-1} = \Omega$$

for all $U$ in $A(n,\ell)$.

**Proposition 3.** If

$$I^{(n)} : B^{(n,\ell)} \rightarrow \mathbb{R}$$

is a real valued $(n,\ell)$-braid invariant, then

$$\Omega = \sum_{\beta \in B^{(n,\ell)}} I^{(n,\ell)}(\beta) |\beta\rangle\langle \beta|$$

is a quantum motif observable which is a quantum motif $(n,\ell)$-invariant.

4. Conclusion

Much more can be said about this topic. For more examples of the application of the quantization procedure discussed in this paper, we refer the reader to [12, 13, 8, 2]. For knot theory and the braid group, we refer the reader to [3, 16, 4, 11, 1, 10]; for topological quantum computation, [5, 6, 7, 9, 17, 19]; and for quantum computation and information, [18, 14, 15].

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