A-HYPERGEOMETRIC SYSTEMS THAT COME FROM GEOMETRY

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ABSTRACT. In recent work, Beukers characterized $A$-hypergeometric systems having a full set of algebraic solutions. He accomplished this by (1) determining which $A$-hypergeometric systems have a full set of solutions modulo $p$ for almost all primes $p$ and (2) showing that these systems come from geometry. He then applied a fundamental theorem of N. Katz, which says that such systems have a full set of algebraic solutions. In this paper we establish some connections between nonresonant $A$-hypergeometric systems and de Rham-type complexes, which leads to a determination of which $A$-hypergeometric systems come from geometry. We do not use the fact that the system is irreducible or find integral formulas for its solutions.

1. Introduction

Let $A = \{a^{(1)}, \ldots, a^{(N)}\} \subseteq \mathbb{Z}^n$ with $a^{(j)} = (a_1^{(j)}, \ldots, a_n^{(j)})$. We shall assume throughout this paper that these lattice points generate $\mathbb{Z}^n$ as abelian group. Let $L$ be the corresponding lattice of relations,

$$L = \left\{(l_1, \ldots, l_N) \in \mathbb{Z}^N \mid \sum_{j=1}^N l_j a^{(j)} = 0\right\},$$

and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$. The $A$-hypergeometric system is the system of partial differential equations in the variables $\lambda_1, \ldots, \lambda_N$ consisting of the operators (we write $\partial_j$ for $\partial/\partial \lambda_j$)

$$\Box_l = \prod_{l_j > 0} \partial_j^{l_j} - \prod_{l_j < 0} \partial_j^{-l_j}$$

for all $l \in L$ and the operators

$$Z_{i,\alpha} = \sum_{j=1}^N a_i^{(j)} \lambda_j \partial_j - \alpha_i$$

for $i = 1, \ldots, n$. We denote by $D = \mathbb{C}(\lambda_1, \ldots, \lambda_N, \partial_1, \ldots, \partial_N)$ the ring of differential operators in the $\lambda_j$. The associated hypergeometric $D$-module is

$$\mathcal{M}_\alpha = D \left/ \left( \sum_{l \in L} D \Box_l + \sum_{i=1}^n D Z_{i,\alpha} \right) \right..$$

Let $\mathbb{C}[\lambda] = \mathbb{C}[\lambda_1, \ldots, \lambda_N]$ be the polynomial ring in $N$ variables and let $X$ be a smooth variety over $\mathbb{C}[\lambda]$. Let $G$ be a finite group acting on $X/\mathbb{C}[\lambda]$. Then
G acts on the relative de Rham cohomology groups \( H^i_{\text{DR}}(X/\mathbb{C}[\lambda]) \). For an irreducible representation \( \chi \) of \( G \), let \( H^i_{\text{DR}}(X/\mathbb{C}[\lambda])^\chi \) denote the \( \chi \)-isotypic component of \( H^i_{\text{DR}}(X/\mathbb{C}[\lambda]) \), i.e., \( H^i_{\text{DR}}(X/\mathbb{C}[\lambda])^\chi \) is the sum of all \( G \)-submodules of \( H^i_{\text{DR}}(X/\mathbb{C}[\lambda]) \) that are isomorphic to \( \chi \). The \( H^i_{\text{DR}}(X/\mathbb{C}[\lambda])^\chi \) are \( \mathcal{D} \)-modules via the Gauss-Manin connection. We say that a \( \mathcal{D} \)-module \( \mathcal{M} \) comes from geometry if it is isomorphic to \( H^i_{\text{DR}}(X/\mathbb{C}[\lambda])^\chi \) for some \( (X/\mathbb{C}[\lambda], G, \chi) \).

The main idea of this paper is to show that the \( \mathcal{M}_\alpha \) are isomorphic as \( \mathcal{D} \)-modules to certain cohomology groups that arise in algebraic geometry. Let \( R' = \mathbb{C}[\lambda][x_1^{\pm1}, \ldots, x_n^{\pm1}] \), the coordinate ring of the \( n \)-torus \( \mathbb{T}^n \) over \( \mathbb{C}[\lambda] \). Put

\[
(1.1) \quad f = \sum_{j=1}^N \lambda_j x^{a(j)} \in R'
\]

and let \( \Omega^k_{R'/\mathbb{C}[\lambda]} \) denote the module of relative \( k \)-forms. We use the set

\[
\left\{ \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \mid 1 \leq i_1 < \cdots < i_k \leq n \right\}
\]

as basis for \( \Omega^k_{R'/\mathbb{C}[\lambda]} \) as \( R' \)-module. The map \( \nabla_\alpha : \Omega^k_{R'/\mathbb{C}[\lambda]} \to \Omega^{k+1}_{R'/\mathbb{C}[\lambda]} \) given by

\[
(1.2) \quad \nabla_\alpha (\omega) = d\omega + \sum_{k=1}^n \alpha_k \frac{dx_k}{x_i} \wedge \omega + df \wedge \omega
\]

defines a complex \( (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha) \). In terms of the above basis, we have for \( \xi \in R' \)

\[
(1.3) \quad D_{i,\alpha} = x_i \frac{\partial}{\partial x_i} + \alpha_i + x_i \frac{\partial f}{\partial x_i}.
\]

Formally,

\[
\nabla_\alpha = \frac{1}{x^\alpha \exp f} \circ \partial \circ x^\alpha \exp f,
\]

so for any derivation \( \partial \in \text{Der}_{\mathbb{C}}(\mathbb{C}[\lambda]) \) the operator

\[
D_\partial = \frac{1}{x^\alpha \exp f} \circ \partial \circ x^\alpha \exp f = \partial + f^\partial
\]
on the \( \Omega^k_{R'/\mathbb{C}[\lambda]} \) commutes with \( \nabla_\alpha \) (where \( f^\partial \) denotes the polynomial obtained from \( f \) by applying \( \partial \) to its coefficients). This defines an action of \( \text{Der}_{\mathbb{C}}(\mathbb{C}[\lambda]) \) on \( (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha) \). In particular, \( \partial_j \) acts as \( D_j = \partial_j + x^\alpha \partial \). This action extends to an action of \( \mathcal{D} \) on \( (\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha) \), which makes the \( H^i(\Omega_{R'/\mathbb{C}[\lambda]}^\bullet, \nabla_\alpha) \) into \( \mathcal{D} \)-modules.

Let \( C(A) \subseteq \mathbb{R}^n \) be the real cone generated by \( A \) and let \( \ell_1, \ldots, \ell_s \in \mathbb{Z}[u_1, \ldots, u_n] \) be homogeneous linear forms defining the codimension-one faces of \( C(A) \), normalized so that the coefficients of each \( \ell_i \) are relatively prime and so that \( \ell_i \geq 0 \) on \( C(A) \) for each \( i \). We say that \( \alpha \) is nonresonant for \( A \) if \( \ell_i(\alpha) \notin \mathbb{Z} \) for all \( i \). Note that \( \alpha \) is nonresonant for \( A \) if and only if \( \alpha + u \) is nonresonant for \( A \) for all \( u \in \mathbb{Z}^n \).

(If \( \alpha \in \mathbb{R}^n \), this is equivalent to saying that no proper face of \( C(A) \) contains a point of \( \alpha + \mathbb{Z}^n \).)
Theorem 1.4. If $\alpha$ is nonresonant for $A$, then $\mathcal{M}_\alpha \cong H^n(\Omega^\bullet_{R'/C[\lambda]} \nabla_\alpha)$ as $\mathcal{D}$-modules.

Remark. It is straightforward to check that for $u \in \mathbb{Z}^n$ multiplication by $x^u$ defines an isomorphism of complexes of $\mathcal{D}$-modules

$$x^u : (\Omega^\bullet_{R'/C[\lambda]} \nabla_\alpha) \rightarrow (\Omega^\bullet_{R'/C[\lambda]} \nabla_\alpha)$$

(its inverse is multiplication by $x^{-u}$). Thus when $\alpha$ is nonresonant the $\mathcal{M}_{\alpha+u}$ for $u \in \mathbb{Z}^n$ are all isomorphic as $\mathcal{D}$-modules.

Consider the special case $f = x_n g$, where $g \in C[\lambda][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $\mathbb{T}^{n-1}$ be the $(n-1)$-torus over $C[\lambda]$ with coordinates $x_1, \ldots, x_{n-1}$. Let $U \subseteq \mathbb{T}^{n-1}$ be the open set where $g$ is nonvanishing and let $\Omega^k_U/C[\lambda]$ be the module of relative $k$-forms over the ring of regular functions on $U$. The map $\nabla_\alpha : \Omega^k_U/C[\lambda] \rightarrow \Omega^{k+1}_U/C[\lambda]$ defined by

$$\nabla_\alpha(\omega) = d\omega + \sum_{i=1}^{n-1} \alpha_i \frac{dx_i}{x_i} \wedge \omega - \alpha_n \frac{dg}{g} \wedge \omega$$

defines a complex $(\Omega^\bullet_{U/C[\lambda]} \nabla_\alpha)$. Formally we have

$$\nabla_\alpha = \frac{g^{\alpha_n}}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \circ d \circ \frac{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}{g^{\alpha_n}}$$

so if we define for $\partial \in \text{Der}_C(C[\lambda])$

$$\nabla_\alpha = \frac{g^{\alpha_n}}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \circ \partial \circ \frac{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}{g^{\alpha_n}} = \partial - \alpha_n \frac{g^\partial}{g},$$

we get an action of $\text{Der}_C(C[\lambda])$ on this complex. The action extends to an action of $\mathcal{D}$, making $(\Omega^\bullet_{U/C[\lambda]} \nabla_\alpha)$ into a complex of $\mathcal{D}$-modules. Note that for $u \in \mathbb{Z}^n$, multiplication by $x_1^{u_1} \cdots x_{n-1}^{u_{n-1}} / g^{u_n}$ defines an isomorphism of complexes of $\mathcal{D}$-modules

$$\frac{x_1^{u_1} \cdots x_{n-1}^{u_{n-1}}}{g^{u_n}} : (\Omega^\bullet_{U/C[\lambda]} \nabla_\alpha) \rightarrow (\Omega^\bullet_{U/C[\lambda]} \nabla_\alpha).$$

Theorem 1.5. Suppose $f = x_n g(x_1, \ldots, x_{n-1})$ and $\alpha$ is nonresonant for $A$. For all $i$ there are $\mathcal{D}$-module isomorphisms

$$H^i(\Omega^\bullet_{R'/C[\lambda]} \nabla_\alpha) \cong H^{i-1}(\Omega^\bullet_{U/C[\lambda]} \nabla_\alpha).$$

When $\alpha \in \mathbb{Q}^n$, it is well known that the $H^i(\Omega^\bullet_{U/C[\lambda]} \nabla_\alpha)$ come from geometry. (We sketch the proof of this fact in Section 4.) From Theorems 1.4 and 1.5, we then get the following result.

Corollary 1.6. Suppose $f = x_n g(x_1, \ldots, x_{n-1})$ and $\alpha$ is nonresonant for $A$. There is an isomorphism of $\mathcal{D}$-modules $\mathcal{M}_\alpha \cong H^{n-1}(\Omega^\bullet_{U/C[\lambda]} \nabla_\alpha)$. If in addition $\alpha \in \mathbb{Q}^n$, then $\mathcal{M}_\alpha$ comes from geometry.

The proofs of Theorems 1.4 and 1.5 are based on ideas from [1] and [3]. Those papers in turn are related to earlier work of Dwork, Dwork-Loeser, and N. Katz. We refer the reader to the introductions of [1] and [3] for more details on the connections with that earlier work.
2. Proof of Theorem 1.4

It is straightforward to check that the \( Z_{i,\alpha} \) commute with one another and that

\[
\square_l \circ Z_{i,\alpha} = Z_{i,\beta} \circ \square_l,
\]

where \( \beta = \alpha + \sum_{l_j > 0} l_j a^{(j)} \) (\( = \alpha - \sum_{l_j < 0} l_j a^{(j)} \) since \( l \in L \)). It follows that right multiplication by \( Z_{i,\alpha} \) maps the left ideal \( \sum_{l \in L} D \square_l \) into itself. If we put \( P = D/\sum_{l \in L} D \square_l \), then right multiplication by the \( Z_{i,\alpha} \) is a family of commuting endomorphisms of \( P \) as left \( D \)-module. Let \( C^* \) be the cohomological Koszul complex on \( P \) defined by the \( Z_{i,\alpha} \). Concretely,

\[
C^k = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} P e_{i_1} \wedge \cdots \wedge e_{i_k},
\]

where the \( e_{i} \) are formal symbols satisfying \( e_{i} \wedge e_{j} = -e_{j} \wedge e_{i} \) and the boundary operator \( \delta_{\alpha} : C^k \to C^{k+1} \) is defined by additivity and the formula (for \( \sigma \in \mathcal{P} \))

\[
\delta_{\alpha}(\sigma e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{i=1}^{n} \sigma Z_{i,\alpha} e_{i} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}.
\]

One obtains a complex of left \( D \)-modules \( (C^*, \delta_{\alpha}) \) for which

\[
H^n(C^*, \delta_{\alpha}) = M_{\alpha}.
\]

Let \( R = \mathbb{C}[\lambda][x^{a^{(1)}}, \ldots, x^{a^{(N)}}] \), a subring of \( R' \) which is also a \( D \)-submodule of \( R' \), and let

\[
\Omega^k_R(\log) = \left\{ \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi_{i_1 \cdots i_k} \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \mid \xi_{i_1 \cdots i_k} \in R \right\} \subseteq \Omega^k_{R'/\mathbb{C}[\lambda]}.
\]

By (1.2) and (1.3), one has \( \nabla_{\alpha}(\Omega^k_R(\log)) \subseteq \Omega^{k+1}_R(\log) \), so this defines a subcomplex \( (\Omega^*_R(\log), \nabla_{\alpha}) \) of \( (\Omega^*_R/\mathbb{C}[\lambda], \nabla_{\alpha}) \).

The \( \mathbb{C}[\lambda] \)-module homomorphism \( \phi : \mathcal{P} \to R \) defined by

\[
\phi(\partial^{a_1} \cdots \partial^{a_N}_1) = x^{\sum_{j=1}^{N} a_j^{(j)}}
\]

is an isomorphism of \( D \)-modules by [1] Theorem 4.4]. It extends to a map \( \phi : C^k \to \Omega^k_R(\log) \) by additivity and the formula

\[
\phi(\sigma e_{i_1} \wedge \cdots \wedge e_{i_k}) = \phi(\sigma) \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}.
\]

By [2] Corollary 2.4, this is an isomorphism of complexes of \( D \)-modules:

\[
\phi : (C^*, \delta_{\alpha}) \cong (\Omega^*_R(\log), \nabla_{\alpha}).
\]

By (2.1) and (2.2), Theorem 1.4 is a consequence of the following result.

Proposition 2.3. If \( \alpha \) is nonresonant for \( A \), then the inclusion map

\[
(\Omega^*_R(\log), \nabla_{\alpha}) \hookrightarrow (\Omega^*_R/\mathbb{C}[\lambda], \nabla_{\alpha})
\]

is a quasi-isomorphism of complexes of \( D \)-modules.

We state and prove a generalization of Proposition 2.3. Let \( U \subseteq \mathbb{Z}^n \) be a nonempty subset satisfying the condition:

\[
(2.4) \quad \text{if } u \in U, \text{ then } u + a^{(j)} \in U \text{ for } j = 1, \ldots, N.
\]
If we denote by $M_U$ the free $\mathbb{C}[\lambda]$-module with basis $\{x^u \mid u \in U\}$, then (2.4) implies that $M_U$ is both an $R$-submodule and a $D$-submodule of $R'$. Note that $R' = M_{Z^n}$ and that $R = M_{U_0}$ where $U_0 = \{\sum_{j=1}^N c_j a^{(j)} \mid c_j \in \mathbb{Z}_{\geq 0}\}$. Let $\Omega^k_{M_U}(\log) \subseteq \Omega^k_{R'/\mathbb{C}[\lambda]}$ be the subset

$$\Omega^k_{M_U}(\log) = \left\{ \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi_{i_1\cdots i_k} \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \mid \xi_{i_1\cdots i_k} \in M_U \right\}.$$  

By (1.2) and (1.3), we have $\varnothing \subseteq (\Omega^k_{M_U}(\log))$, so we get a subcomplex $(\Omega^*_{M_U}(\log), \nabla_M)$ of $(\Omega^*_{R'/\mathbb{C}[\lambda]}, \nabla)$. Proposition 2.3 is the special case $U = U_0$ of the following more general result.

**Proposition 2.5.** If $\alpha$ is nonresonant for $A$ and $U$ satisfies (2.4), then the inclusion

$$ (\Omega^*_{M_U}(\log), \nabla_M) \hookrightarrow (\Omega^*_{R'/\mathbb{C}[\lambda]}, \nabla) $$

is a quasi-isomorphism of complexes of $D$-modules.

As in Section 1, let $\ell_1, \ldots, \ell_s \in \mathbb{Z}[u_1, \ldots, u_n]$ be the normalized homogeneous linear forms defining the codimension-one faces of $C(A)$ and put $S = \{1, \ldots, s\}$. For $v = (v_1, \ldots, v_s) \in \mathbb{Z}^s$ and a subset $T \subseteq S$, put

$$W(v, T) = \{u \in \mathbb{Z}^n \mid \ell_i(u) \geq v_i \text{ for all } i \in T\}.$$  

Since $\ell_i(a^i_{\lambda^j}) \geq 0$ for all $i, j$, the set $W(v, T)$ satisfies (2.4). By [1, Lemma 3.12], there exists $v$ such that $W(v, S) \subseteq U_0$. It follows that if $U$ satisfies (2.4), then there exists $v$ such that $W(v, S) \subseteq U$.

**Lemma 2.6.** If $\alpha$ is nonresonant for $A$, $U$ satisfies (2.4), and $W(v, T) \subseteq U$, then the inclusion

$$ (\Omega^*_{M_{W(v, T)}}(\log), \nabla_M) \hookrightarrow (\Omega^*_{M_U}(\log), \nabla_M) $$

is a quasi-isomorphism of complexes of $D$-modules.

Since $W(v, T) \subseteq \mathbb{Z}^n$ for all $v, T$, Lemma 2.6 implies that the inclusion

$$ (\Omega^*_{M_{W(v, T')}}(\log), \nabla_M) \hookrightarrow (\Omega^*_{R'/\mathbb{C}[\lambda]}, \nabla) $$

is a quasi-isomorphism for all $v, T$. The quasi-isomorphisms (2.7) and (2.8) imply Proposition 2.5.

**Proof of Lemma 2.6.** For $T' \subseteq T$, let

$$U(v, T') = \{u \in U \mid \ell_i(u) \geq v_i \text{ for all } i \in T'\}.$$  

Note that $U(v, \emptyset) = U$ and that $U(v, T) = W(v, T)$ (since $W(v, T) \subseteq U$). By induction, it thus suffices to show that if $T' \subseteq T$ and $T'' = T' \cup \{t\}$ with $t \in T \setminus T'$, then the inclusion

$$ (\Omega^*_{M_{U(v, T'')}}(\log), \nabla_M) \hookrightarrow (\Omega^*_{M_{U(v, T')}}(\log), \nabla_M) $$

is a quasi-isomorphism. Let $Q^*$ be the quotient complex

$$Q^* = \Omega^*_{M_{U(v, T')}}(\log)/\Omega^*_{M_{U(v, T'')}}(\log).$$  

We show that (2.9) is a quasi-isomorphism by showing that $H^k(Q^*) = 0$ for all $k$.  

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*This text is a transcription of the natural language content of the document.*
We define a filtration \( \{ F_p \}_p \) on the complex \( \Omega^\bullet_{MU(s, T')} (\log) \). For \( p \leq v_1 \), let \( F^k_p \cdot \Omega^\bullet_{MU(s, T')} (\log) \) be the \( C[\lambda] \)-submodule spanned by differential forms
\[
(2.10) \quad x^u \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}
\]
satisfying \( \ell_t(u) \geq p \). By (1.2) and (1.3), \( \nabla_\alpha \) respects this filtration. We also denote by \( F_p \) the induced filtrations on the subcomplex \( \Omega^\bullet_{MU(s, T')} (\log) \) and the quotient complex \( Q^\bullet \). Note that since
\[
F_{v_1} \cdot \Omega^\bullet_{MU(s, T')} (\log) = \Omega^\bullet_{MU(s, T')} (\log)
\]
we have \( F_{v_1} \cdot Q^\bullet = 0 \). To show that \( H^k(Q^\bullet) = 0 \) for all \( k \), it suffices to show that \( H^k(\text{gr}_p Q^\bullet) = 0 \) for all \( k \) and \( p \), where \( \text{gr}_p Q^\bullet \) denotes the \( p \)-th graded piece of the associated graded of the filtration \( F_p \).

Write \( \ell_t(u) = \sum_{i=1}^n c_i u_i \in Z[u] \). Define \( \rho : \Omega^k_{MU(s, T')} (\log) \to \Omega^{k-1}_{MU(s, T')} (\log) \) to be the \( C[\lambda] \)-module homomorphism satisfying
\[
\rho \left( x^u \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \right) = x^u \sum_{j=1}^k (-1)^{j-1} c_{i_j} \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_{j-1}}}{x_{i_{j-1}}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}.
\]
It is straightforward to check that
\[
(\nabla_\alpha \circ \rho + \rho \circ \nabla_\alpha) \left( x^u \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \right) = \left( \sum_{i=1}^n c_i D_{i, \alpha}(x^u) \right) \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}},
\]
and a calculation using (1.1) and (1.3) shows that
\[
(2.11) \quad \sum_{i=1}^n c_i D_{i, \alpha}(x^u) = \ell_t(\alpha + u) x^u + \sum_{j=1}^N \lambda_j \ell_t(\alpha^{(j)}) x^u + a^{(j)}.
\]
Suppose that the form \( (2.10) \) lies in \( F_p \setminus F_{p+1} \). Then \( \ell_t(\alpha + u) = \ell_t(\alpha) + p \) and if \( \ell_t(\alpha^{(j)}) \neq 0 \), then \( \ell_t(\alpha + a^{(j)}) > p \). It follows that the second term on the right-hand side of (2.11) lies in \( F_{p+1} \), so on the associated graded complex the induced map
\[
(\nabla_\alpha \circ \rho + \rho \circ \nabla_\alpha) : \text{gr}_p Q^k \to \text{gr}_p Q^k
\]
is just multiplication by \( \ell_t(\alpha) + p \). Since \( \alpha \) is nonresonant for \( A \), \( \ell_t(\alpha) + p \neq 0 \). Thus multiplication by a nonzero constant is homotopic to the zero map, which implies that \( H^k(\text{gr}_p Q^\bullet) = 0 \) for all \( k \).

3. Proof of Theorem 1.5

In this section we assume that \( f = x_n g(x_1, \ldots, x_{n-1}) \). By the Remark following Theorem 1.4, we may assume that if \( \alpha_n \in Z \), then \( \alpha_n \geq 1 \). Let \( R_+ = \mathbb{C}[\lambda][x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}, x_n] \). By Proposition 2.5, the inclusion
\[
(\Omega^\bullet_{R_+} (\log), \nabla_\alpha) \hookrightarrow (\Omega^\bullet_{R'/\mathbb{C}[\lambda]} \nabla_\alpha)
\]
is a quasi-isomorphism of complexes of \( D \)-modules. Theorem 1.5 is then a consequence of the following result.

**Proposition 3.1.** If \( \alpha_n \not\in Z_{\leq 0} \), then there is a quasi-isoermorphism of complexes of \( D \)-modules
\[
(\Omega^\bullet_{R_+} (\log), \nabla_\alpha) \to (\Omega^\bullet_{U/\mathbb{C}[\lambda]} [-1], \tilde{\nabla_\alpha}).
\]
Proof: We regard $\Omega^*_R (\log)$ as the total complex associated to a certain two-row double complex: let

$$\Omega^{k,0} = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n - 1} R^*_+ \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}},$$

and let

$$\Omega^{k,1} = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n - 1} R^*_+ \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \wedge \frac{dx_n}{x_n}.$$

Let $\partial_h : \Omega^{k,i} \to \Omega^{k+1,i}$ be the map

$$\partial_h(\omega) = d'\omega + \sum_{i=1}^{n-1} \alpha_i \frac{dx_i}{x_i} \wedge \omega + x_n(d'g \wedge \omega),$$

where $d'$ is exterior differentiation relative to the variable $x_1, \ldots, x_n-1$, and let $\partial_v : \Omega^{k,0} \to \Omega^{k,1}$ be the map

$$\partial_v(\omega) = d''\omega + \alpha_n \frac{dx_n}{x_n} \wedge \omega + g dx_n \wedge \omega,$$

where $d''$ is exterior differentiation relative to the variable $x_n$. Since $\partial_v$ is injective, the canonical projection $\Omega^{k}_R (\log) \to \Omega^{k-1,1}$ induces a quasi-isomorphism

$$(\Omega^{k}_R (\log), \nabla_\alpha) \to ((\Omega^{k,1}/\partial_v \Omega^{k,0})[-1], \partial_h)$$

(see [13, Appendix B]).

To complete the proof of Proposition 3.1, we define a quasi-isomorphism of complexes between $(\Omega^{k,1}/\partial_v \Omega^{k,0}, \partial_h)$ and $(\Omega^{k}_{U/\mathbb{C}[\lambda]}, \nabla_\alpha)$. Define $\gamma : \Omega^{k,1} \to \Omega^{k}_{U/\mathbb{C}[\lambda]}$ to be the $\mathbb{C}[\lambda]$-module homomorphism satisfying

$$\gamma \left( \frac{dx_{i_1}}{x_{i_1}} \cdot \cdots \cdot \frac{dx_{i_k}}{x_{i_k}} \cdot \frac{dx_n}{x_n} \right) = \left(-1\right)^{u_n (\alpha_n)_{u_n} x_{i_1}^{u_1} \cdots x_{i_{k-1}}^{u_{k-1}} x_{i_k}^{u_k}} \left( \frac{dx_{i_1}}{x_{i_1}} \cdot \cdots \cdot \frac{dx_{i_k}}{x_{i_k}} \right),$$

where $(\alpha_n)_{u_n} = (\alpha_n + 1) \cdots (\alpha_n + u_n - 1)$. It is straightforward to check that $\gamma$ commutes with boundary operators, hence defines a homomorphism of complexes from $\Omega^{k,1}$ to $\Omega^{k}_{U/\mathbb{C}[\lambda]}$. The hypothesis that $\alpha_n \not\in \mathbb{Z}_{\leq 0}$ implies that $\gamma$ is surjective, and it is straightforward to check that $\ker(\gamma : \Omega^{k,1} \to \Omega^{k}_{U/\mathbb{C}[\lambda]}) = 0$. It remains only to show that

$$\ker(\gamma) = \Omega^{k,0} = \partial_v \Omega^{k,0}.$$

Let $\xi \in \Omega^{k,1}$ satisfy $\gamma(\xi) = 0$. Write

$$\xi = \sum_{i=m}^M x_{i}^i \xi_i \frac{dx_n}{x_n},$$

where $\xi_i \in \Omega^{k,0}$ is in the $\mathbb{C}[\lambda]$-span of the forms

$$(3.3) \quad x_{i_1}^{u_{i_1}} \cdots x_{i_{k-1}}^{u_{i_{k-1}}} \frac{dx_{i_k}}{x_{i_k}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}}, \quad 1 \leq i_1 < \cdots < i_k \leq n - 1.$$
Since \((\alpha_n)_M \neq 0\), this equation implies \(\xi_M = g\eta\), where \(\eta\) is in the \(\mathbb{C}[\lambda]\)-span of the forms (3.3). It follows that

\[
\xi = (-1)^k \partial_v (x_n^M \eta) - (\alpha_n + M - 1) x_n^{M-1} \eta \frac{dx_n}{x_n} + \sum_{m=1}^{M-1} x_n^i \xi_i \frac{dx_n}{x_n}.
\]

By induction we are reduced to the case \(\xi = x_n^m \xi_m \frac{dx_n}{x_n}\). But in this case

\[
0 = \gamma(\xi) = (-1)^m (\alpha_n)_m \xi_m / g^m,
\]

so \(\xi_m = 0\). \(\square\)

If \(\alpha_n \notin \mathbb{Z}\), there is no need to introduce the complex \(\Omega^\bullet_{R_1}(\log)\).

**Proposition 3.4.** If \(\alpha_n \notin \mathbb{Z}\), then there is a quasi-isomorphism of complexes of \(\mathcal{D}\)-modules

\[
(\Omega^\bullet_{R^/\mathbb{C}[\lambda]}, \nabla_\alpha) \to (\Omega^\bullet_{U^/\mathbb{C}[\lambda]}[-1], \tilde{\nabla}_\alpha).
\]

**Sketch of proof.** One proceeds as in the proof of Proposition 3.1 with the complex \(\Omega^\bullet_{R^/\mathbb{C}[\lambda]}(\log)\) replaced by \(\Omega^\bullet_{U^/\mathbb{C}[\lambda]}\). One defines \(\gamma\) as before with the understanding that for \(u_n < 0\)

\[
(\alpha_n)_{u_n} = ((\alpha_n - 1)(\alpha_n - 2) \cdots (\alpha_n + u_n))^{-1}.
\]

The proof then proceeds unchanged. (See \([3, \text{Lemma 2.5}]\) for the details in a similar situation.) \(\square\)

4. APPLICATION OF N. KATZ’S RESULTS

We begin by sketching the proof that the \(H^i(\Omega^\bullet_{U^/\mathbb{C}[\lambda]}, \tilde{\nabla}_\alpha)\) come from geometry. Let \(X \subseteq \mathbb{T}^n_{\mathbb{C}[\lambda]}\) be the hypersurface \(x_n^D g(x_1^D, \ldots, x_{n-1}^D) - 1 = 0\), where \(D \in \mathbb{Z}_{>0}\). Since \(\Omega^\bullet_{X^/\mathbb{C}[\lambda]}\) is a free module with basis \(\{dx_i\}_{i=1}^{n-1}\), \(\mathcal{D}\) acts on global \(i\)-forms by acting on their coefficients relative to exterior powers of this basis. The group \((\mu_D)^n\) acts on \(X\) and its relative de Rham complex \((\Omega^\bullet_{X^/\mathbb{C}[\lambda]}, d)\). The irreducible representations \(\chi\) of \((\mu_D)^n\) can be indexed by \(n\)-tuples \((a_1, \ldots, a_n)\), \(0 \leq a_i < D\), so that if \(\chi\) corresponds to \((a_1, \ldots, a_n)\), then there is an isomorphism of complexes of \(\mathcal{D}\)-modules \((\Omega^\bullet_{X^/\mathbb{C}[\lambda]}, d)\chi \cong (\Omega^\bullet_{U^/\mathbb{C}[\lambda]}, \tilde{\nabla}_\alpha)\), where \(\alpha = (a_1/D, \ldots, a_n/D)\). The first assertion of Corollary 1.6 then implies that

\[
\mathcal{M}_\alpha \cong H^{n-1}_{\text{DR}}(X/\mathbb{C}[\lambda])^\chi,
\]

which establishes the second assertion of Corollary 1.6.

Now suppose that \(C = \text{Spec}(A)\) is a smooth connected curve over \(\mathbb{C}\) and \(\phi: C \to \mathbb{A}^N = \text{Spec}(\mathbb{C}[\lambda])\) is a morphism. Let \(X'\) be the pullback of \(X\) to a variety over \(C\), i.e., \(X' = A \otimes_{\mathbb{C}[\lambda]} X\). Then \(X'\) is the hypersurface in \(\mathbb{T}^n_{A}\) defined by the equation \(x_n^D g^\phi(x_1^D, \ldots, x_{n-1}^D) - 1 = 0\), where \(g^\phi \in A[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]\) is the Laurent polynomial obtained from \(g\) by applying \(\phi\) to its coefficients (by abuse of notation, we also denote by \(\phi\) the homomorphism \(\mathbb{C}[\lambda] \to A\) corresponding to \(\phi: C \to \mathbb{A}^N\)). The varieties \(X\) and \(X'\) are smooth affine schemes whose de Rham cohomology can be computed as the cohomology of the complex of global sections of the de Rham complex. By the right-exactness of tensor products, one has

\[
\phi^*(H^{n-1}_{\text{DR}}(X/\mathbb{C}[\lambda])) \cong H^{n-1}_{\text{DR}}(X'/A).
\]
It follows from [9] Section 14] that $H^{n-1}_{\text{DR}}(X'/A)$ has regular singular points and quasi-unipotent local monodromy at infinity (i.e., at all points of the quotient field of $A$). Therefore $H^{n-1}_{\text{DR}}(X'/\mathbb{C}(\lambda))$ has regular singular points and quasi-unipotent local monodromy at infinity (in the sense of [11] Section VIII). Equation (4.1) then implies that $\mathcal{M}_\alpha$ has regular singular points and quasi-unipotent local monodromy at infinity.

To apply the results of [10], we observe that the results of this paper are valid when one replaces $\mathbb{C}[\lambda]$ by $\mathbb{C}(\lambda)$. Let $\mathcal{D}$ denote the ring of differential operators with coefficients in $\mathbb{C}(\lambda)$ and define

$$\mathcal{M}_\alpha = \mathcal{D}/\left(\sum_{i \in L} \mathcal{D} \partial_i + \sum_{i=1}^n \mathcal{D} \zeta_i, \alpha\right).$$

Put

$$\bar{R} = \mathbb{C}(\lambda)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

the coordinate ring of the $n$-torus $\mathbb{T}_{\mathbb{C}(\lambda)}$. The proof of Theorem 1.4 establishes the following result.

**Proposition 4.3.** If $\alpha$ is nonresonant for $A$, then $\bar{M}_\alpha \cong H^n(\Omega^\bullet_{\mathcal{R}/\mathbb{C}(\lambda)}, \nabla)_{\lambda}$ as $\mathcal{D}$-modules.

In the situation of Theorem 1.5, let $U \subseteq \mathbb{T}_{\mathbb{C}(\lambda)}^n$ be the open set where $g$ is nonvanishing. Then we have the following result.

**Proposition 4.4.** Suppose $f = x_n g(x_1, \ldots, x_{n-1})$ and $\alpha$ is nonresonant for $A$. For all $i$ there are $\mathcal{D}$-module isomorphisms

$$H^i(\Omega^\bullet_{\mathcal{R}/\mathbb{C}(\lambda)}, \nabla)_{\lambda} \cong H^{i-1}(\Omega^\bullet_{\bar{U}/\mathbb{C}(\lambda)}, \nabla).$$

Combining these propositions gives the following result.

**Corollary 4.5.** Suppose $f = x_n g(x_1, \ldots, x_{n-1})$ and $\alpha$ is nonresonant for $A$. There is an isomorphism of $\mathcal{D}$-modules $\mathcal{M}_\alpha \cong H^{n-1}_{\text{DR}}(\Omega^\bullet_{\bar{U}/\mathbb{C}(\lambda)}, \nabla)_{\lambda}$. If in addition $\alpha \in \mathbb{Q}^n$, then $\mathcal{M}_\alpha$ comes from geometry.

Explicitly, letting $\bar{X} \subseteq \mathbb{T}_{\mathbb{C}(\lambda)}^n$ be the hypersurface $x_n^p g(x_1^p, \ldots, x_{n-1}^p) - 1 = 0$, we have (corresponding to Equation (4.1))

$$\mathcal{M}_\alpha \cong H^{n-1}_{\text{DR}}(\bar{X}/\mathbb{C}(\lambda))^\chi.$$

By [10] Theorem 5.7 $H^{n-1}_{\text{DR}}(\bar{X}/\mathbb{C}(\lambda))^\chi$ has a full set of polynomial solutions modulo $p$ for almost all primes $p$ if and only if it has a full set of algebraic solutions. Note that the solution sets of $\mathcal{M}_\alpha$ and $\bar{M}_\alpha$ in the algebraic closure of $\mathbb{C}(\lambda)$ are identical. From Equation (4.6), we then get the following result.

**Corollary 4.7.** Suppose $f = x_n g(x_1, \ldots, x_{n-1})$ and $\alpha \in \mathbb{Q}^n$ is nonresonant for $A$. The hypergeometric $\mathcal{D}$-module $\mathcal{M}_\alpha$ has a full set of polynomial solutions modulo $p$ for almost all primes $p$ if and only if it has a full set of algebraic solutions.

**References**

[1] Adolphson, Alan. Hypergeometric functions and rings generated by monomials. Duke Math. J. **73** (1994), no. 2, 269–290.

[2] Adolphson, Alan. Higher solutions of hypergeometric systems and Dwork cohomology. Rend. Sem. Mat. Univ. Padova **101** (1999), 179–190.
[3] Adolphson, Alan; Sperber, Steven. On twisted de Rham cohomology. Nagoya Math. J. **146** (1997), 55–81.

[4] Beukers, Frits. Algebraic $A$-hypergeometric functions. Invent. Math. **180** (2010), no. 3, 589–610.

[5] Dwork, Bernard. Generalized hypergeometric functions. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990.

[6] Dwork, B.; Loeser, F. Hypergeometric series. Japan. J. Math. (N.S.) **19** (1993), no. 1, 81–129.

[7] Katz, Nicholas. Thesis (1966), Princeton University.

[8] Katz, Nicholas. On the differential equations satisfied by period matrices. Inst. Hautes Études Sci. Publ. Math. No. 35 (1968), 223–258.

[9] Katz, Nicholas. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. Inst. Hautes Études Sci. Publ. Math. No. 39 (1970), 175–232.

[10] Katz, Nicholas. Algebraic solutions of differential equations ($p$-curvature and the Hodge filtration). Invent. Math. **18** (1972), 1–118.

[11] Katz, Nicholas. A conjecture in the arithmetic theory of differential equations. Bull. Soc. Math. France **110** (1982), no. 2, 203–239.

[12] Katz, Nicholas M. Corrections to: “A conjecture in the arithmetic theory of differential equations”. Bull. Soc. Math. France **110** (1982), no. 3, 347–348.

[13] Matsumura, Hideyuki. Commutative ring theory. Translated from the Japanese by M. Reid. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1986.

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