How large is the contribution of cosmic web to $\Omega_\Lambda$?
A preliminary study on a novel inhomogenous model

Stefano Viaggiu
Dipartimento di Matematica, Università ”Tor Vergata” Roma,
Via della Ricerca Scientifica, 1
Rome, Italy 00133,
viaggiu@axp.mat.uniroma2.it

Marco Montuori
SMC-ISC-CNR and Dipartimento di Fisica,
Università ”La Sapienza” Roma,
Ple. Aldo Moro 2
00185, Rome, Italy,
marco.montuori@roma1.infn.it,

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Abstract

The distribution of matter in the universe shows a complex pattern, formed by cluster of galaxies, voids and filaments denoted as cosmic web. Different approaches have been proposed to model such structure in the framework of the general relativity. Recently, one of us has proposed a generalization ($\Lambda FB$ model) of the Fractal Bubble model, proposed by Wiltshire, which accounts for such large scale structure. The $\Lambda FB$ model is an evolution of FB model and includes in a consistent way a description of inhomogeneous matter distribution and a $\Lambda$ term. Here we analyze the $\Lambda FB$ model focusing on the relation between cosmological parameters. The main result is the consistency of $\Lambda$CDM model values for $\Omega_{\Lambda 0}$ ($\approx 0.7$) and $\Omega_{k 0}$ ($\vert \Omega_{k 0} \vert \approx 0.01$) with a large fraction of voids. This allows to quantify to which extent the inhomogeneous structure could account for $\Lambda$ constant consistently with standard values of the other cosmological parameters.

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1 Introduction

The model of universe which currently gives the best fit of the available astrophysical observations is the $\Lambda$ Cold Dark Matter model (ACDM). The
model is based on an exact solution of equation of General Relativity which assumes homogeneity and isotropy with a F-L metric. Quite soon, however, observational evidences required the addition of two main artefacts to the theory: the dark matter and the dark energy. Rotation curve of spiral galaxies, velocity dispersion of galaxies in galaxy clusters, cluster mass estimate from hot intra-cluster gas emission, gravitational lensing from galaxy cluster, large scale structure formation from CMB tiny fluctuations required an addition of dark matter. The observation of an accelerated expansion of the universe required the addition of the dark energy. According to the best fit model, dark matter should account for \( \approx 23\% \) of the total energy density, while the dark energy for \( \approx 73\% \) \cite{1, 2, 3}. Notwithstanding, the nature of both remains unknown and is amongst the deepest problems of modern physics. For these reason in the past decade, several authors \cite{4}-\cite{29} have explored alternatives to the ΛCDM model.

The discovery of a lumpier universe than expected offered a possibility in this respect. The universe appears indeed organized as a cosmic web, which can be described as an interconnected network: spherical clusters form the nodes and are joined by elongated filaments defining 2D sheets. Recent analysis have shown that voids in the network could fill between \( \approx 40\% - 70\% \) of the space (accordingly to definition and measure of the voids) and have a continuous distribution of sizes depending on the galaxy sample selected \cite{30, 31, 32, 33}. Such observed inhomogeneities are usually described as a first order perturbation with respect to the homogeneous and isotropic exact solution. This formulation has a limited application, since it is valid for small perturbation \( \delta \rho / \rho \) and cannot describe the lumpy universe at low redshift. Nevertheless, it was soon clear, mainly through the formulation of the Buchert formalism, that such an inhomogeneous matter distribution could indeed mimic the presence of dark energy \cite{34, 35, 36, 37, 38}. Since then, some authors explored this possibility to the extreme consequence to explain the whole amount of dark energy on the basis of inhomogeneous distribution. Our personal opinion is that the requested existence of Gpc scale voids makes these models questionable. From the other side, the standard perturbative approach of ΛCDM cannot answer to the question of the effect of small scale strong fluctuations on larger scale. In this context, an interesting line appears the exploration of cosmological models including the largely accepted cosmological constant paradigm and at the same time a non perturbative approach to the observed inhomogeneities. This issue is a very recent one (see \cite{39}) and so far limited to the not so realistic (even useful) case of spherical symmetry. Going beyond such studies and toward a more realistic model of matter distribution in the universe is the target of the present paper. The elaboration of a similar model will allow, for example, to study in a non perturbative way the percentage of dark energy that can be explained in terms of the observed inhomogeneities or the role of voids in the process of structure formation (see for example \cite{40}) in a more complete
way for the presence both of dark energy and voids in the model. More in
general, it would be possible to explore the effect of the observed inho-
genieties onto cosmic scale data, as barionic acoustic oscillations and cosmic
radiation.

Our baseline is the so-called 'fractal bubble' FB or 'timescape' cosmology
[22, 23]. It is based on the Buchert average scheme [18] and introduces a
non-uniform time flow with two scales corresponding to the voids and walls
of a schematic lattice cosmic structure. In the FB model the apparent
acceleration is explained in terms of different rate of clocks located in walls
(our point of observation) as compared to the clocks of voids. The viability
of the FB or Timescape model as an alternative to the ΛCDM model is still
matter of debate [24, 25]. In any case, the FB model is the first one which
describes a schematic cosmic web in a non perturbative way and without
invoking particular symmetries. This interesting feature has prompted one
of us to propose an inhomogeneous model based on the same partitioning
within the Buchert average scheme, but including a Λ ≠ 0 and an uniform
time flow [41]. This model, hereafter ΛFB model, allows to investigate, in
a non perturbative way, the effects of large scale spatial inhomogeneities
and non-vanishing curvature on the cosmological parameters. The present
preliminary study is devoted to such aim. The scheme of the paper is the
following: in section 2 we write the relevant equations of the model. We
obtain a new interesting equation involving all the parameters of the model
evaluated at the present epoch, which is analysed in sections 3 and 4. In
section 5 we 'dress' the cosmological parameters and finally, we report our
conclusions in Section 6.

2 Basic equations

The ΛFB model [41] incorporates the observed present inhomogeneities
within the cosmological constant paradigm. The starting point is the FB
model introduced by Wiltshire [22, 23]. This model considers a matter dis-
bution in the universe as a regular network formed by walls and voids.
The ΛFB model assumes the same partitioning of the two scale FB model
[23]: regions called 'finite infinity' (F_I) which are the set of timelike bound-
aries of compact disjoint domains I, with a zero average expansion and a
positive one outside, i.e. < θ >_I = 0. The F_I regions are within 'wall
regions' whose metric is, on average:

\[ ds^2_w = -dt^2 + a_w^2 \left[ d\eta_w^2 + \eta_w^2 d\Omega^2 \right]. \tag{1} \]

Our position is in F_I. The regions complementary to the walls with respect
to the particle horizon are called voids and are expanding with an average
hyperbolic metric given by:

\[ ds^2_v = -dt^2 + a_v^2 \left[ d\eta_v^2 + \sinh^2(\eta_v) d\Omega^2 \right]. \tag{2} \]
Note that, contrary to the FB model, in our model the time flow is isotropic. This is due to the presence of a non vanishing cosmological constant, which is absent in the FB model since it intends to explain it as an effect of inhomogeneity. The isotropic flow has the useful feature to avoid the shortcomings related to the junction conditions of the original FB model (see [42]). In any case, a nontrivial phenomenological lapse function involves new physics relating to gravitational energy which is not yet widely accepted, and we will therefore consider the commonly accepted alternative that time flows uniformly.

The Hubble parameters in walls and voids are respectively \( H_w = \frac{\dot{a}_w}{a_w^3} \), \( H_v = \frac{\dot{a}_v}{a_v^3} \). An important assumption of the ΛFB model (and FB one) is the existence of a scale of homogeneity with the scale factor \( a(t) \).

The average on the whole volume of the particle horizon \( V = a^3 V_i \) is:

\[
a^3 = f_{wi} a_w^3 + f_{vi} a_v^3, \quad f_{wi} + f_{vi} = 1,
\]

where \( f_{wi} \) and \( f_{vi} \) are the fractions of walls and voids at the time \( t = t_i \). We choose the time \( t_i \) (the initial time) as the recombination time in order to compare the model with available observational data. In general the following relations are valid for any time:

\[
f_v(t) + f_w(t) = 1, \quad f_w = f_{wi} \frac{a_w^3}{a^3}, \quad f_v = f_{vi} \frac{a_v^3}{a^3}.
\]

The Hubble rate \( H \) at the scale of homogeneity is:

\[
H = f_w H_w + f_v H_v, \quad H = I_w H_w = I_v H_v.
\]

and the density parameters are:

\[
\Omega_m = \frac{8\pi <\rho>}{3H^2}, \quad \Omega_k = -\frac{<R>}{6H^2}, \quad \Omega_Q = -\frac{Q}{6H^2}, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2},
\]

where \( Q \) is the kinematic backreaction and \( R \) is the spatial curvature. The Buchert equations are:

\[
\Omega_m + \Omega_\Lambda + \Omega_k + \Omega_Q = 1, \quad (6\Omega + 2 <R>) \dot{a} + a \left[ \dot{Q} + <\dot{R}> \right] = 0.
\]

Equation (7) can be easily manipulated to obtain the final equations for the model (for more details see [41]):

\[
(1 - f_v) \frac{\dot{a}}{a} - \frac{\dot{f}_v}{3} = \sqrt{\Omega_m \frac{a_f^3}{a^3} \left( \frac{1 - \epsilon_i}{\Omega_F} \right) \left( 1 - f_v \right)},
\]

\[
\frac{\dot{a}}{a} + \frac{\dot{f}_v}{3f_v} = a_0 H_0 \sqrt{\Omega_k \frac{a_f^2}{a^2} + \Omega_\Lambda \frac{a_f^2}{a^2} + \Omega_m \frac{a_f^2}{a^2} \left( 1 + \frac{\epsilon_i - 1}{\Omega_F} \right)},
\]

4
where $\epsilon_i$ is a integration constant. A first integral of eqs (8, 9) is:

$$
\frac{(1 - \epsilon_i)I^2_0 \Omega_m}{(1 - f_v)} = \Omega_F = e^{-f_v \frac{\Lambda}{H_w} dt}.
$$

(10)

From equation (8) (see [41]) we can solve for $a_w$ and $\Omega_F$:

$$
a_w = a_{w0} \sinh \frac{3}{2} \left( \sqrt{3} \frac{\Lambda}{t} \right), \quad \Omega_F = \left( \frac{\cosh \left( \sqrt{\frac{3}{4}} \frac{t}{t_i} \right)}{\cosh \left( \sqrt{\frac{3}{4}} t \right)} \right)^2.
$$

(11)

We are now able to derive an useful and interesting formula. By combining equations (8) and (9) and evaluating them at the present epoch $t_0$ we obtain:

$$
1 = \sqrt{\Omega_{m0}(1 - f_{v0})} \Omega_F(1 - \epsilon_i) + \sqrt{\Omega_{k0} + \Omega_{\Lambda0} + \Omega_{m0} \left( 1 + \epsilon_i - 1 \Omega_{F0} \right)},
$$

(12)

The equation (12) constrains the density parameters and the fraction of voids at the present time $t_0$. The equation was already present in [41]. Now we intend to go a step beyond, expressing the constant $\Omega_{F0}$ in terms of density parameters. Integrating equation (8) we get:

$$
(1 - f_v)^{\frac{3}{2}} a = a_0 \left( \Omega_{m0}(1 - \epsilon_i) \right)^{\frac{1}{2}} \sinh \frac{3}{2} \left( \sqrt{\frac{3}{2}} \frac{t}{t_i} \right).
$$

(13)

It is should be noticed that in the equation (13), according to [22] and without loss of generality, we have set to zero the integration constant. This choice constrains the integration constant in the integration of the equation (9) (see [22]). Moreover, since the current estimation of $\Lambda$ is very small and $t_i \approx 3.77 \cdot 10^{-5}$ Gyr, we can set with good approximation $\cosh \frac{3}{2} \left( \sqrt{\frac{3}{2}} t \right) \simeq 1$.

Evaluating equation (13) at the present epoch, we get:

$$
(1 - f_v)^{\frac{3}{2}} a = a_0 \left( \frac{\Omega_{m0}(1 - \epsilon_i)}{\Omega_{\Lambda0}} \right)^{\frac{1}{2}} \sinh \frac{3}{2} \left( \sqrt{\frac{3}{2}} \frac{t}{t_i} \right).
$$

(14)

and by putting (14) in (12) we have:

$$
1 = \sqrt{(1 - f_{v0})[\Omega_{m0}(1 - \epsilon_i) + \Omega_{\Lambda0}(1 - f_{v0})] + \sqrt{f_{v0} \Omega_{k0} + \Omega_{m0} \epsilon_i + f_{v0} \Omega_{\Lambda0}}.}
$$

(15)

The formula (15) was not present in [41]; it involves all the parameters of the $\Lambda$FB model and has been obtained by using all the relevant equations.
Moreover, it gives a compatibility equation for cosmological parameters that are averaged quantities in principle measurable. It should be noted that this equation is not perturbative. It allows to study the effects of the inhomogeneities without ruling out the dark energy and avoiding the shortcomings of the perturbation theory when applied to large inhomogeneities.

The cosmological parameters present in (15) are different from those of the \( \Lambda CDM \) model. The concordance model is based on an exact solution of Einstein’s equations and the corresponding cosmological parameters are local quantities. The homogeneity and isotropy of the solution implies that the cosmological parameters are the same in any spatial point for any fixed comoving time \( t \). As the timescape model, also the \( \Lambda FB \) one is obtained within the Buchert formalism, where the cosmological parameters are averaged non local quantities. In this formalism for a given scalar \( \psi \) at a fixed cosmological time \( t \), the average on the whole particle horizon is:

\[
<\psi(t)> = f_w(t)\psi_w(t) + f_v(t)\psi_v(t),
\]

where the subscripts \( v \) and \( w \) refer obviously to the values of \( \psi \) in voids and walls.

Astrophysical data refer to the local value of \( \psi \) that can be generally different from its averaged one \( <\psi> \). If we consider the universe as a cosmic web, these averaged parameters are physical quantities obtained from an average procedure over a given astrophysical sample containing voids and walls. Moreover, \( \psi_w(t) \) and \( \psi_v(t) \) are mean values calculated separately in walls and voids.

In any case, the model has a scale of statistical homogeneity, beyond which it has a Friedmann metric evolving with \( a(t) \) given by eq.(3). By consequence, the cosmological parameters obtained as averaged quantities at the scale of homogeneity can be straightforward compared with the corresponding values of the standard concordance model. For this reason, we expect that the numerical values for the cosmological parameters of the \( \Lambda FB \) model are similar to those of the standard cosmological model.

As a further consideration on the \( \Lambda FB \) model, note that the formalism introduced allows easily to compute the limit for \( \Omega_{k0} \rightarrow 0 \) in (8)-(9). At this aim we should just change the hyperbolic metric (2) with parabolic void metric:

\[
ds^2_v = -dt^2 + a_v^2 \left[ d\eta_v^2 + \eta_v^2 d\Omega^2 \right].
\]

This case is an interesting one: if \( \Omega_{k0} = 0 \) and \( \Omega_{m0} + \Omega_{\Lambda0} > 1 \), equation (17) implies that \( \Omega_Q \leq 0 \). The latter means that the flat void region described by metric (17) should be an underdensity. Equation (17) is relevant only to compute the distance-redshift relation, i.e. when considering data on the light cone. Then, we can study equation (15) also in the limit \( \Omega_{k0} = 0 \). In this case, we get the \( \Lambda CDM \) model by setting \( f_{v0} = \epsilon_i = 0 \); in another
words, at odds with FB model, the ΛFB contains the ΛCDM model for the
aforementioned choice of the parameters.

3 Constraints on the parameter $\epsilon_i$

From equation (15) and (19) we have (for $t = t_i$):

$$\epsilon_i = 1 - \frac{(1 - f_{vi})}{I_{wi}^2 \Omega_{mi}}$$

$$I_{wi} = 1 - f_{vi} + f_{vi} H_{vi} \frac{H_{wi}}{H_{wi}} = \frac{H_i}{H_{wi}}. \quad (18)$$

In the original FB model, the parameter $I_w$ is interpreted as a phenomeno-
logical lapse function and we have $I_{wi} = 1, \Omega_{mi} \simeq 1, \epsilon_i << 1$. By conse-
quence, in the FB model the parameter $\epsilon_i$ has no role. On the contrary, in
our model (ΛFB), $I_w$ is not a lapse function but simply a measure of the
ratio $\frac{H_i}{H_{wi}}$ or in another words of the ratio between the expansion rate of
voids and walls. By consequence, $\epsilon_i$ is only constrained to be $\leq 1$, from the
existence of II member of eq. (15). It is also possible to express, by means of
(15), $\epsilon_i$ in terms of the other current cosmological parameters. We have

$$\epsilon_i = \frac{-b \pm 4 \Omega_{m0} \sqrt{\Delta}}{2a} \quad \quad \quad (19)$$

$$b = -2 \Omega_{m0} \Omega_{\Lambda 0} + 2 f_{v0} \Omega_{m0} \Omega_{k0} + 4 f_{v0} \Omega_{m0} \Omega_{\Lambda 0} - 4 f_{v0} \Omega_{m0} -$$

$$-2 \Omega_{m0}^2 + 2 f_{v0} \Omega_{m0}^2 + 2 \Omega_{m0}, \quad \Delta = f_{v0} (\Omega_{k0} + \Omega_{m0} + \Omega_{\Lambda 0} - 1)(1 - f_{v0}).$$

Care must be taken to avoid spurious solutions. In particular, for $f_{v0} \leq 0.1$
we generally have one root, the greater one in (19). The existence of the
solution requires $\Omega_{m0} + \Omega_{\Lambda 0} + \Omega_{k0} \geq 1$, which implies for equation 7 $\Omega_Q \leq 0$, i.e. a non positive backreaction. This is a check of the consistency of
the model, since the partition chosen with voids with an average negative
curvature should imply a non positive backreaction, as it is.

4 Constraints on the cosmological parameters of
the ΛFB model

4.1 The current curvature $\Omega_{k0}$

In this section we discuss the constraints on the cosmological parameters
we can get from the observational data. Equation (15) can be solved with
respect to $\Omega_{k0}$ and we get:

$$\Omega_{k0} = \frac{\left[1 - \sqrt{(1 - f_{v0})(\Omega_{m0}(1 - \epsilon_i) + \Omega_{\Lambda 0}(1 - f_{v0}))}\right]^2}{f_{v0}} - f_{v0} \Omega_{\Lambda 0} - \Omega_{m0} \epsilon_i. \quad (20)$$
Current estimates of $\Omega_{k0}$ come from CMB and SZ effect data (see [43, 44, 45]). Such measures can be interpreted in the standard framework ($\Lambda$CDM) or in the void models (as the FB, the present $\Lambda$FB, etc...). In the $\Lambda$CDM model, which assumes a constant spatial curvature at fixed time, the observations imply $|\Omega_{k0}| < 0.01$. Remember that within the Buchert scheme, this limit can also be taken into account, provided that it is intended as a mean quantity. Such small value is also consistent with the inflationary scenario. On the contrary in void models, the curvature acts as an effective dark energy. For this reason and in order to fit the observational data, void models require a quite large curvature. Such large value has been claimed consistent to the observation of big voids in the large scale structure. Unfortunately such scenario has two additional implications: a low value for the Hubble constant and/or the existence of a giant void around our location (see [46]). Both of them appear quite implausible. The present model $\Lambda$FB on the contrary, since incorporates voids and dark energy, gives the possibility to explore the compatibility between dark energy, large voids, a standard range of values for $\Omega_{m0} \in [0.25, 0.35]$ and a small value for $\Omega_{k0}$ in agreement with the WMAP constraint. From the formula (19) we see that a relevant volume void fraction $f_{v0}$ is compatible with a rather small value of $\Omega_{k0}$. For examples considering $\Omega_{\Lambda 0} = 0.7$ and $\Omega_{m0} = 0.3$ we have:

- $f_{v0} = 0.7$ for $\Omega_{k0} = 0.0039$, $\epsilon_i = 0.5$
- $f_{v0} = 0.5$ for $\Omega_{k0} = 0.001$, $\epsilon_i = 0.16$
- $f_{v0} = 0.2$ for $\Omega_{k0} = 0.01$, $\epsilon_i = 0.46$, $\epsilon_i = -0.07$

Note that can exist values for $f_{v0}$ obtained with two values for $\epsilon_i$. Decreasing $\Omega_{\Lambda 0}$ in order to get a similar value for $f_{v0}$, it requires a larger $\Omega_{k0}$, in agreement with the inhomogeneous models where dark energy is mimicked by the curvature.

Note that for the same range of parameter values, the backreaction term is $\Omega_Q \approx -0.06$ and thus remains relatively small. Larger values for $\Omega_Q$ can be obtained with a small $\Omega_{k0}$ and a reasonable void fraction $f_{v0}$, for a smaller $\Omega_{\Lambda 0}$, but a larger $\Omega_{m0}$ value. It is worth to stress that our result is not a validation of the standard value for $\Omega_{k0}$. In fact, the current standard constraint $|\Omega_{k0}| \leq 0.01$ is obtained within the Friedmann paradigm. A larger value $\Omega_{k0} \approx 1$ can fit as well the experimental data (CMB and BAO), but it is at odds with inflationary paradigm. Our result is indeed a falsification of a common claim, i.e. large density contrast $|\delta \rho/\rho| \approx 1$ due to voids necessarily implies a large curvature contrast $|\delta R/R| \approx 1$. In our model a large fraction of voids is compatible with a small spatial curvature and a standard value for cosmological constant.
4.2 Current volume voids fraction $f_{v0}$

An interesting feature of the ΛFB model is the constraints on the present day volume void fraction $f_{v0}$ from the values of the other cosmological parameters. By posing the condition $\Omega_{k0} \geq 0$, $\Omega_{m0} > -2\Omega_{A0} + 2\sqrt{\Omega_{A0}}$ (which is always satisfied) and neglecting spurious solutions we have

$$f_{v0} \geq X_0, \quad X_0 = \frac{-b + 4\sqrt{\Delta}}{2a},$$  \hspace{1cm} (21)

where

$$a = 4\Omega_{A0}^2 + 4\Omega_{m0}^2 + 4\Omega_{A0}\Omega_{m0} + 4\Omega_{A0}\Omega_{k0} + 4\Omega_{m0}\Omega_{k0},$$

$$b = -4\Omega_{m0}\epsilon_i + 4\Omega_{m0}\Omega_{A0}\epsilon_i + 2\Omega_{m0}^2\epsilon_i - 2\Omega_{m0}^2 - 6\Omega_{m0}\Omega_{A0} + 4\Omega_{A0} + 2\Omega_{m0} - 4\Omega_{A0}^2,$$

$$\Delta = (\Omega_{A0} + \Omega_{m0} + \Omega_{k0} - 1)(\Omega_{m0}\Omega_{A0} + \Omega_{A0}^2 + \Omega_{A0}\Omega_{k0} - \Omega_{A0} + \Omega_{m0}^2\epsilon_i(1 - \epsilon_i) + \Omega_{m0}\Omega_{k0}(1 - \epsilon_i)).$$

Note that, if $\Omega_{m0} + \Omega_{A0} < 1$, no limitations arise for the actual volume void fraction $f_{v0}$. For $\Omega_{m0} + \Omega_{A0} = 1$ we obtain $X_0 = \epsilon_i$.

Finally, note that in the case $\Omega_{k0} = 0$ (for all times), the inequality (21) becomes an equality. As a result, in this case we can obtain the standard concordance model ($f_{v0} = 0$) or a model with voids with a flat Friedmann metrics and $f_{v0} = X_0$, provided that $\Omega_{m0} + \Omega_{A0} > 1$.

We study now the equation (15) in terms of the allowed current volume voids fraction $f_{v0}$. The equation can be solved in terms of $f_{v0}$ with some little algebra. Care must be taken for the possible appearance of spurious solutions, by solving it graphically (i.e. equation (15)) and eliminating the latter ones. Then expliciting equation (15) in terms of $f_{v0}$ we get:

$$f_{v0} = \frac{-b \pm 4\sqrt{\Delta}}{2a},$$  \hspace{1cm} (22)

where

$$\Delta = (\Omega_{A0} + \Omega_{m0} + \Omega_{k0} - 1)(\Omega_{m0}\Omega_{A0} + \Omega_{A0}^2 + \Omega_{A0}\Omega_{k0} - \Omega_{A0} + \Omega_{m0}^2\epsilon_i(1 - \epsilon_i) + \Omega_{m0}\Omega_{k0}(1 - \epsilon_i)).$$

Note that for $\Omega_{A0} + \Omega_{m0} > 1$ at least a solution is always present, while for $\Omega_{A0} + \Omega_{m0} < 1$ we can have no solutions. In any case for $\Omega_{m0} + \Omega_{A0} + \Omega_{k0} < 1$, since $\Omega_Q < 0$, according to equation (7), no solutions are available. For an example of possible values, see the table (1). Quite generally, we can have two possible values for $f_{v0}$:

- $f_{v01} \ll 1$
- $f_{v02} \approx 0.1$ or greater

This implies that if the initial fraction of voids is:
Table 1: Allowed actual volume voids fraction for ΛFB model.

| $\Omega_{\Lambda 0}$ | $\Omega_{m0}$ | $\Omega_{k0}$ | $\epsilon_i$ | $f_{v01}$ | $f_{v02}$ |
|----------------------|----------------|--------------|--------------|-----------|-----------|
| 0.7                  | 0.28           | 0.01         | 0            | no        | no        |
| 0.7                  | 0.28           | 0.02         | 0            | no        | 0.066     |
| 0.7                  | 0.28           | 0.021        | 0            | 0.03      | 0.14      |
| 0.7                  | 0.28           | 0.022        | 0            | 0.023     | 0.18      |
| 0.7                  | 0.29           | 0.018        | 0            | 0.0026    | 0.32      |
| 0.7                  | 0.29           | 0.018        | 0.01         | 0.004     | 0.33      |
| 0.7                  | 0.30           | 0.001        | 0            | no        | 0.043     |
| 0.7                  | 0.30           | 0.01         | 0            | no        | 0.32      |
| 0.7                  | 0.30           | 0.02         | 0            | no        | 0.51      |
| 0.7                  | 0.30           | 0.01         | 0.01         | 0.0002    | 0.34      |
| 0.7                  | 0.30           | 0.01         | 0.1          | 0.016     | 0.44      |
| 0.7                  | 0.305          | 0.0007       | 0            | no        | 0.18      |
| 0.7                  | 0.305          | 0.01         | 0            | no        | 0.4       |
| 0.7                  | 0.31           | 0.01         | 0            | no        | 0.47      |
| 0.7                  | 0.31           | 0.01         | 0.01         | no        | 0.48      |
| 0.7                  | 0.31           | 0.01         | 0.15         | 0.014     | 0.62      |

• $f_{vi} < f_{v01}$ then at $t = t_0$ $f_{v0} = f_{v01}$

• $f_{vi} > f_{v01}$ then at $t = t_0$ $f_{v0} = f_{v02}$

Considering the value for $f_{vi}$ estimated from WMAP data ($\approx \in [10^{-5}, 10^{-2}]$) and the values of $f_{v01}$ and $f_{v02}$, both of the previous cases could be fulfilled. According to the model, starting from a fraction of voids $f_{vi}$ within WMAP constraints, we could get at the present time $t_0$ both $f_{v01}$ and $f_{v02}$. Which one is a question of the precise value of $f_{vi}$ with respect to the $f_{v01}$. The dependence from the other parameters is quite complex. In fact from the table (1) is apparent that for a choice $\Omega_{A 0} \simeq 0.7$, $\Omega_{k0} \leq 0.02$ and $0 \leq \epsilon_i \leq 0.15$, changing $\Omega_{m0}$ from 0.28 to 0.31 produces a variation of $f_{v0}$ from $\sim 0.0002$ up to $\sim 0.6$. In the AFB, contrary to the FB one, the final fraction of voids $f_{v0}$ is dependent from the initial one $f_{vi}$. This happens because in the AFB model no tracker solutions are available.

4.3 Current cosmological constant $\Omega_{A 0}$

As mentioned, both FB and AFB models collapsing regions are within walls; for this reason the model has $\Omega_{k0} \geq 0$ and we get:

$$\Omega_{A 0} \leq G$$

$$G = \frac{\epsilon_i \Omega_{m0} - 2f_{v0} \epsilon_i \Omega_{m0} + 1 - 2\Omega_{m0}f_{v0}^2 - 2f_{v0} + 3f_{v0} \Omega_{m0} - \Omega_{m0} + 2f_{v0}^2 - 2\sqrt{Q}}{(1 - 2f_{v0})^2},$$

10
\[ Q = f_v(1 - f_v)[2f_v^2\Omega_{m0} - f_v^2 + f_v\Omega_{m0} + \epsilon\Omega_{m0}(1 - 2f_v)]. \]

Note that \( Q > 0 \) for all values allowed of \( f_v, \Omega_{m0} \). We conclude this subsection by analysing the formula (15) from the point of view of \( \Omega_{\Lambda 0} \). This is in our opinion the main point of the present study, since it allows to estimate the fraction of \( \Omega_{\Lambda 0} \) due to the inhomogeneities in the large scale structure.

By solving (15) with respect to \( \Omega_{\Lambda 0} \) we obtain

\[
\Omega_{\Lambda 0} = \frac{-b - 4\sqrt{f_v(1 - f_v)\sqrt{\Delta}}}{2a},
\]

\[
a = 1 + 4f_v^2 - 4f_v,
\]

\[
b = -6f_v^2\Omega_{m0} + 4f_v - 2 + 4f_v\Omega_{m0}\epsilon_i - 4f_v^2 + 2\Omega_{m0} + 4f_v\Omega_{m0} + +4f_v^2\Omega_{k0} - 2f_v\Omega_{k0} - 2\Omega_{m0}\epsilon_i,
\]

\[
\Delta = 2f_v^2\Omega_{m0} - f_v^2 + f_v - f_v\Omega_{m0} + \epsilon_i\Omega_{m0}(1 - 2f_v) + 2f_v\Omega_{k0} - 3f_v\Omega_{k0} + \Omega_{k0}.
\]

To our knowledge this is the first equation which allows to compute the behaviour of \( \Omega_{\Lambda 0} \) versus in particular \( f_v \) without imposing simple assumptions (as spherical or special symmetries) on inhomogeneities. First of all, from WMAP data we put a maximum allowed mean value for \( \Omega_{k0} \approx 0.01 \). For \( \Omega_{m0} \) we consider range of values \((0.2, 0.35)\) (see for example [47]). In fig 1 we plot \( \Omega_{\Lambda 0} \) vs. \( f_v, \Omega_{m0} \) with \( \Omega_{k0} = 0.01 \) and \( \{\epsilon_i\} = \{0, 0.1, 0.9\} \). As apparent from the figures, a set of standard \( \Lambda \)CDM values for \( \Omega_{\Lambda 0}, \Omega_{m0}, \Omega_{k0} \)
Figure 2: $\Omega_{\Lambda 0}$ vs. $\Omega_{m0}$ and $f_{v_0}$. $\epsilon_i = 0.1$. For the isocurves see caption fig1.

Figure 3: $\Omega_{\Lambda 0}$ vs. $\Omega_{m0}$ and $f_{v_0}$. $\epsilon_i = 0.9$. For the isocurves see caption fig1.

is compatible with a quite large fraction of voids (roughly up to 0.8). In particular, we see that the maximum value $\Omega_{m0} \simeq 0.35$ together with the maximum allowed mean value for $\Omega_{k0}$ predicted by WMAP ($\simeq 0.01$) is compatible with the minimum value $\Omega_{\Lambda 0} \simeq 0.65$ and a large fraction of voids (up to $\simeq 0.8$). If this were the case, spatial inhomogeneities could account
for a maximum percentage $\approx 10\%$ with respect to the concordance value $\Omega_{A0} \simeq 0.73$. Moreover, note that for $\epsilon = 0$ or $\sim 0.1$ we have that huge values for $\Omega_{A0} > 0.8$ can be obtained with $\Omega_{m0} \in [0.25, 0.35]$ together with a huge value for $f_{v0} > 0.8$. Conversely, for high $\epsilon \sim 0.9$ we have that with $\Omega_{m0} \in [0.25, 0.35]$, $\Omega_{A0} \simeq 0.73$ can be obtained for $f_{v0} \simeq 0.1$. This can help to understand the interesting role of $\epsilon$ in the $\Lambda$FB model. Finally, we consider the case with $\epsilon = \Omega_{k0} = 0$ where a situation close to figure 1 arises. Note that the concordance model is regained by further setting $f_{v0} = 0$. As explained at the end of section 2, this case corresponds to the physical situation where both walls and voids have mean zero spatial curvature at any time but where voids represent an underdensity. Hence, voids expand faster than walls and as a result a non trivial volume void fraction $f_v(t)$ with $f_{v0} \neq 0$ emerges as the case with $\Omega_{k0} \neq 0$.

5 Age of the universe compatible with the $\Lambda$FB model

An important test in cosmology is given by the age of the universe. The age of globular cluster implies the universe should be certainly older than $> 12\text{Gyr}$. In this section we give an estimation of the age of the universe predicted by $\Lambda$FB model. First of all, from equation (13) evaluated at present time $t_0$ we get:

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_{A0}}} \ln \left( \frac{\Omega_{A0}}{\Omega_{m0}(1-\epsilon_i)} + \frac{\Omega_{A0}}{\Omega_{m0}(1-\epsilon_i)} + 1 \right)$$

(25)

where $H_0$ is the present cosmological constant at the scale of homogeneity. A consistent value for $t_0 (> 12\text{Gyr})$ is always possible with a large volume fraction of voids (for some numerical examples see table 2).

Let us give an example: let us consider $t_0 \geq 13\text{ Gyr}$, a restrictive value for $\epsilon_i = 0$, a large amount of dark energy ($\Omega_{A0} \in [0.67, 0.75]$) and a relatively small curvature, i.e. $\Omega_{k0} \leq 0.08$. For $H_0$ we consider the range $(55 - 75)\text{km/s/Mpc}$, since a smaller value appears in contrast with the actual data. From eq. (25) we get a reasonable $t_0$ with a void fraction $f_{v0} \approx (40 - 50)\%$ and $H_0 \approx (55 - 60)\text{km/s/Mpc}$. Increasing $H_0$ up to $\approx 70\text{km/s/Mpc}$ and considering $\epsilon_i << 1$, we got an age $t_0 \geq 13\text{Gyr}$ with $f_{v0} \approx \in (0, 0.3)$. This shows how the model can account for a reasonable choice of the cosmological parameters (age, $H_0$, curvature, etc..) with a fraction of voids as large as $30\%$. An even greater fraction of voids, $f_{v0} > 0.3$, and $H_0 \approx 70\text{km/s/Mpc}$ require a larger value of $\epsilon_i$ and more precisely $\epsilon_i \sim f_{v0}$. 

13
6 Dressing cosmological parameters and the distance angular function

One of the issue of the Buchert averaging scheme is how to relate volume average quantities which are non-local to observable ones, which are local. In contrast to many approaches to the Buchert equations which usually neglect this fundamental issue and following Wiltshire [23, 41], we match the radial null section of the wall metric (1), rewritten as

$$\text{ds}^2_w = -dt^2 + a_w^2 \left[ d\eta^2_w + \eta_w^2 d\Omega^2 \right]$$

(26)

and the metric at the scale of homogeneity given by

$$\text{ds}^2 = -dt^2 + a^2 d\eta^2 + A(t, \eta) d\Omega^2,$$

(27)

with the metric at the scale of homogeneity given by

$$A(t, \eta) \text{ is an area function satisfying } 4\pi \int_0^{\eta_H} A d\eta = a^2 V_i(\eta_H),$$

where $\eta_H$ is the particle horizon radius. Practically, the wall observer must dress the cosmological parameter and not simply to relate the volume average scale.

| $\Omega_{\Lambda 0}$ | $\Omega_{m 0}$ | $f_{r 0}$ | $H_0$ | $t_0$ |
|---------------------|---------------|-----------|-------|-------|
| 0.67                | 0.26          | 0.4       | 55    | 15.1  |
| 0.67                | 0.26          | 0.4       | 60    | 13.9  |
| 0.67                | 0.26          | 0.4       | 55    | 14.1  |
| 0.67                | 0.31          | 0.4       | 55    | 14.2  |
| 0.67                | 0.31          | 0.4       | 60    | 13.0  |
| 0.67                | 0.31          | 0.5       | 55    | 13.2  |
| 0.67                | 0.34          | 0.4       | 55    | 13.6  |
| 0.70                | 0.27          | 0.4       | 55    | 14.9  |
| 0.70                | 0.27          | 0.4       | 60    | 13.6  |
| 0.70                | 0.27          | 0.5       | 55    | 13.9  |
| 0.70                | 0.30          | 0.4       | 55    | 14.3  |
| 0.70                | 0.30          | 0.4       | 60    | 13.1  |
| 0.70                | 0.30          | 0.5       | 55    | 13.3  |
| 0.71                | 0.26          | 0.4       | 55    | 15.0  |
| 0.71                | 0.26          | 0.4       | 60    | 13.8  |
| 0.71                | 0.26          | 0.5       | 55    | 14.0  |
| 0.73                | 0.26          | 0.4       | 60    | 13.7  |
| 0.73                | 0.26          | 0.5       | 55    | 14.0  |
| 0.75                | 0.26          | 0.4       | 60    | 13.7  |

Table 2: Age of the universe compatible with $\Lambda$FB model.
factor to the observed redshift \( z \). The dressed wall geometry is:

\[
ds^2_{w} = -dt^2 + a^2 \left[ d\eta^2 + (1 - f_v)\frac{2}{3} f_w \eta_w d\Omega^2 \right],
\]

with

\[
d\eta_w = \frac{f_w}{(1 - f_v)^{\frac{1}{2}}} d\eta, \quad \eta = \int_t^{t_0} \frac{dt}{a}.
\]

It is by means of the metric (28) that the wall observer, in galaxies, measures the distance-redshift function \( d_L(z) \). The angular-distance relation \( d_A(t) \) (remember that \( d_L = (1 + z)^2 d_A \)) is:

\[
d_A(z) = \frac{a_0}{(1 + z)^{\frac{1}{2}}} \eta_w, \quad 1 + z = \frac{a_0}{a},
\]

\[
\eta_w = (1 - f_v)^{\frac{1}{2}} \int_t^{t_0} \frac{dt}{(1 - f_v)^{\frac{1}{2}} a}.
\]

Using eq. (29), expression (30) becomes:

\[
d_A(t) = \sinh^{\frac{3}{2}} \left( \frac{3}{2} H_0 \sqrt{\Omega_\Lambda_0} t \right) \int_t^{t_0} \frac{dt}{\sinh^{\frac{3}{2}} \left( \frac{3}{2} H_0 \sqrt{\Omega_\Lambda_0} t \right)},
\]

where \( t_0 \) is given by eq. (25). We can see that \( d_A \) has the same functional form versus the time \( t \) both for \( \Lambda \)CDM and \( \Lambda \)FB models. The only difference is the expression of the function \( t_0 \). Obviously the expression for \( d_A \) changes with respect to the concordance model when expressed in terms of the redshift \( z \). To express \( t(z) \) along the past null cone, we must to know the function \( a(t) \) which, for the \( \Lambda \)FB model can be obtained by integrating numerically the equations [39].

7 Conclusions

We have presented a preliminary study of the recent \( \Lambda \)FB cosmological model. An explicit formula relating all the cosmological parameters of the \( \Lambda \)FB model is obtained and analyzed. The relevant feature of the \( \Lambda \)FB model is the presence of the cosmological constant and spatial inhomogeneities without spherical symmetry [39]. By consequence, the standard \( \Lambda \)CDM model can be recovered with a suitable choice of the parameters. The model allows to analyse the departures from the standard cosmological model without invoking perturbation theory.

The first result of the present study is the consistency of a large volume voids fraction (\( > 0.1 \)) with a small spatial curvature, even within WMAP constraint (\( |\Omega_{k0}| \leq 0.01 \)). This falsifies the argument often used in the literature to rule out dark energy, i.e. a large fraction of voids observed
necessarily implies a large negative value for the curvature. The second result is the important role (absent in the FB model) of the initial volume void fraction $f_{vi}$. In fact, for a reasonable range of values for the cosmological parameters $\Omega_{m0}, \Omega_{k0}, \Omega_{\Lambda0}$ and dependent on the parameter $\epsilon_i$, the model generally provides two possible values for the current volume void fraction: $f_{v01} \ll 1$ and $f_{v02} \geq 0.1$. If the initial value $f_{vi} < f_{v01}$, then the universe evolves up to a final volume void fraction $f_{v01}$, while for $f_{vi} > f_{v01}$ the system evolves up to the second root $f_{v02}$. As it is evident from the table (1), the model is quite sensitive to small variations of cosmological parameters. Finally, we analysed the formula (24) giving the exact value for $\Omega_{\Lambda0}$ in terms of the other current cosmological parameters. Setting the maximum value allowed for $\Omega_{k0} \simeq 0.01$ and $\Omega_{m0} \simeq 0.35$, we get (see figures [1][2][3]) that the lower value for $\Omega_{\Lambda0} \simeq 0.65$ is compatible with a volume void fraction $\in (0, 0.8)$. Hence, considering a total amount of dark energy 73% predicted by the $\Lambda$CDM model, the inhomogeneities could account for a maximum percentage $\approx 10\%$ of $\Omega_{\Lambda0}$.

As a final consideration, note that also in the $\Lambda$FB model it is possible to add clock effects present in [22]. This can be done simply by considering $I_w$ as a lapse function together with $\epsilon \ll 1 \sim f_{vi}$ (see [11]). Equations (8), (9) and formula (15) are left unchanged. The changes are in the dressing procedure, i.e. equations (26)-(31). Moreover, in the case of non vanishing lapse function the cosmological parameters must be 'dressed' [22, 23]. As an example, if clock effects are present, $\Omega_m$ is the bare volume-average density parameter while the measured density parameter in walls $\Omega_{mw}$ is given by $\Omega_{mw} = I_w^3 \Omega_m$. This is a preliminary parametric study on the relation between cosmological parameters at the present time. In a future paper we intend to study the complex problem to fit the observational data by integrating numerically the model field equations.

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