DUALITY THEOREMS FOR COINVARIANT SUBSPACES OF $H^1$

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Abstract. Let $\theta$ be an inner function satisfying the connected level set condition of B. Cohn, and let $K^1_\theta$ be the shift-coinvariant subspace of the Hardy space $H^1$ generated by $\theta$. We describe the dual space to $K^1_\theta$ in terms of a bounded mean oscillation with respect to the Clark measure $\sigma_\alpha$ of $\theta$. Namely, we prove that $(K^1_\theta \cap zH^1)^* = \text{BMO}(\sigma_\alpha)$. The result implies a two-sided estimate for the operator norm of a finite Hankel matrix of size $n \times n$ via $\text{BMO}(\mu_2^n)$-norm of its standard symbol, where $\mu_2^n$ is the Haar measure on the group $\{\xi \in \mathbb{C} : \xi^{2n} = 1\}$.

1. Introduction

A bounded analytic function $\theta$ in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is called inner if $|\theta(z)| = 1$ for almost all points $z$ on the unit circle $\mathbb{T}$ in the sense of angular boundary values. With every inner function $\theta$ we associate the shift-coinvariant subspace $K^p_\theta$ of the Hardy space $H^p$,

$$K^p_\theta = H^p \cap \overline{z\theta H^p}, \quad 1 \leq p \leq \infty.$$ (1)

As usual, functions in $H^p$ are identified with their angular boundary values on the unit circle $\mathbb{T}$; formula (1) means that $f \in K^p_\theta$ if $f \in H^p$ and there is $g \in H^p$ such that $f(z) = \overline{z}\theta(z)g(z)$ for almost all points $z \in \mathbb{T}$. An inner function $\theta$ is said to be one-component if its sublevel set $\Omega_\delta = \{z \in \mathbb{D} : |\theta(z)| < \delta\}$ is connected for a positive number $\delta < 1$. This class of inner functions was introduced by B. Cohn [12] in 1982. It is very useful in studying Carleson-type embeddings $K^p_\theta \hookrightarrow L^p(\mu)$ and Riesz bases of reproducing kernels in $K^p_\theta$, see [3, 5, 7, 12, 13, 17, 26] for results and further references.

In this paper we describe the dual space to the space $K^1_\theta$ generated by a one-component inner function $\theta$. Our main result is the following formula:

$$(K^1_\theta \cap zH^1)^* = \text{BMO}(\sigma_\alpha),$$ (2)

where $\sigma_\alpha$ denotes the Clark measure of the inner function $\theta$. Below we state this result formally and apply it to the boundedness problem for truncated Hankel operators.
1.1. Clark measures of one-component inner functions. Let \( \theta \) be a non-constant inner function in the open unit disk \( \mathbb{D} \). For each complex number \( \alpha \) of unit modulus the function \( \text{Re} \left( \frac{\alpha + \theta}{\alpha - \theta} \right) \) is positive and harmonic in \( \mathbb{D} \). Hence there exists the unique positive Borel measure \( \sigma_\alpha \) supported on the unit circle \( T \) such that

\[
\text{Re} \frac{\alpha + \theta(z)}{\alpha - \theta(z)} = \int_T \frac{1 - |z|^2}{1 - \overline{\xi} z} \, d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.
\]

The measures \( \{\sigma_\alpha\}_{|\alpha|=1} \) are usually referred to as Clark measures of the inner function \( \theta \) due to seminal work \([11]\) of D. N. Clark where their close connection to rank-one perturbations of singular unitary operators was discovered. For a modern exposition of this topic and subsequent results see survey \([21]\).

Each Clark measure \( \sigma_\alpha \) of an inner function \( \theta \) is singular with respect to the Lebesgue measure on the unit circle \( T \). Conversely if \( \mu \) is a finite positive Borel measure supported on \( T \) and \( |\alpha| = 1 \), then there exists the unique inner function \( \theta \) satisfying (3) with \( \sigma_\alpha = \mu \). Thus, there is one-to-one correspondence between inner functions in the unit disk \( \mathbb{D} \) and singular measures on the unit circle \( T \). It was unknown which singular measures on \( T \) correspond to the Clark measures of one-component inner functions. We fill this gap in Theorem 1 below.

For every Borel measure \( \mu \) on the unit circle \( T \) denote by \( a(\mu) \) the set of isolated atoms of \( \mu \). Then the set \( \rho(\mu) = \text{supp} \mu \setminus a(\mu) \) consists of accumulating points in the support \( \text{supp} \mu \) of \( \mu \). We will say that an atom \( \xi \in a(\mu) \) has two neighbours in \( a(\mu) \) if there is an open arc \( (\xi_-, \xi_+) \) of the unit circle \( T \) with endpoints \( \xi_\pm \in a(\mu) \) such that \( \xi \) is the only point in \( (\xi_- , \xi_+) \cap \text{supp} \mu \) by \( m \) we will denote the Lebesgue measure on \( T \) normalized so that \( m(T) = 1 \).

**Theorem 1.** Let \( |\alpha| = 1 \). The following conditions are necessary and sufficient for a Borel measure \( \mu \) to be the Clark measure \( \sigma_\alpha \) of a one-component inner function:

(a) \( \mu \) is a discrete measure on \( T \) with isolated atoms, \( m(\text{supp} \mu) = 0 \), every atom \( \xi \in a(\mu) \) has two neighbours \( \xi_\pm \) in \( a(\mu) \), and every connected component of \( T \setminus \rho(\mu) \) contains atoms of \( \mu \);

(b) \( A_\mu |\xi - \xi_\pm| \leq \mu(\xi) \leq B_\mu |\xi - \xi_\pm| \) for all \( \xi \in a(\mu) \) and some \( A_\mu > 0 \), \( B_\mu < \infty \);

(c) the discrete Hilbert transform \( (H_\mu 1)(z) = \int_{T \setminus \{z\}} \frac{d\mu(\xi)}{1 - \xi z} \) is bounded on \( a(\mu) \): we have \( |(H_\mu 1)(z)| \leq C_\mu \) for all \( z \in a(\mu) \).

The necessity of conditions (a) and (b) in Theorem 1 is well-known. I would like to thank A. D. Baranov who tell me the fact that condition (c) is necessary as well. The proof of sufficiency part in Theorem 1 relies on a characterization of one-component inner functions in terms of their derivatives which is due to A. B. Aleksandrov \([3]\).

1.2. The main result. Having a description of the Clark measures of one-component inner functions, we now turn back to formula (2). For a measure \( \mu \) with properties (a) – (c) define the space \( \text{BMO}(\mu) \) by

\[
\text{BMO}(\mu) = \left\{ b \in L^1(\mu) : \|b\|_{\mu,*} = \sup_{\Delta} \frac{1}{\mu(\Delta)} \int_{\Delta} |b - \langle b \rangle_{\Delta, \mu}| \, d\mu < \infty \right\},
\]

where \( \Delta \) runs over all arcs of \( T \) with non-zero mass \( \mu(\Delta) \) and \( \langle b \rangle_{\Delta, \mu} = \frac{1}{\mu(\Delta)} \int_{\Delta} b \, d\mu \) is the standard integral mean of \( b \) on \( \Delta \). The following theorem is the main result of the paper.
Theorem 2. Let \( \theta \) be a one-component inner function and let \( \sigma_\alpha \) be its Clark measure. We have \((K^1_0 \cap zH^1)^* = \text{BMO}(\sigma_\alpha)\). That is, for every continuous linear functional \( \Phi \) on \( K^1_0 \cap zH^1 \) there exists a function \( b \in \text{BMO}(\sigma_\alpha) \) such that \( \Phi = \Phi_b \), where
\[
\Phi_b : F \mapsto \int_T Fb \, d\sigma_\alpha, \quad F \in K^1_0 \cap zH^\infty.
\] (4)

Conversely, for every function \( b \in \text{BMO}(\sigma_\alpha) \) the functional \( \Phi_b \) is the densely defined continuous linear functional on \( K^1_0 \cap zH^1 \) with norm comparable to \( \|b\|_{\sigma_\alpha^*} \).

Every measure \( \mu \) with properties (a), (b) from Theorem 1 generates the doubling metric space \((\text{supp} \mu, \cdot, \cdot, \mu)\) in the sense of R. Coifman and G. Weiss [14]. For such measures \( \mu \) we have \( H_{a\xi}^1(\mu)^* = \text{BMO}(\mu) \), where \( H_{a\xi}^1(\mu) \) is the corresponding atomic Hardy space,
\[
H_{a\xi}^1(\mu) = \left\{ \sum_k \lambda_k a_k : a_k \text{ are } \mu\text{-atoms}, \sum_k |\lambda_k| < \infty \right\}.
\] (5)

By a \( \mu \)-atom we mean a complex-valued function \( a \in L^\infty(\mu) \) supported on an arc \( \Delta \) of \( \mathbb{T} \), with \( \|a\|_{L^\infty(\mu)} \leq 1/\mu(\Delta) \), and such that \( \langle a \rangle_{\Delta, \mu} = 0 \). The norm of \( f \in H_{a\xi}^1(\mu) \) is the infimum of \( \sum_k |\lambda_k| \) over all possible representations \( f = \sum_k \lambda_k a_k \) of \( f \) as a sum of \( \mu \)-atoms. We see from Theorem 1 that Theorem 2 admits the following equivalent reformulation.

Theorem 2’. Let \( \mu \) be a measure with properties (a) – (c). Then \( f \in H_{a\xi}^1(\mu) \) if and only if \( f \) admits the analytic continuation to the unit open disk \( \mathbb{D} \) as a function \( F \in K^1_0 \cap zH^1 \), where \( \theta \) is the inner function with the Clark measure \( \sigma_\alpha = \mu \). Moreover, such a function \( F \) is unique and the norms \( \|f\|_{H_{a\xi}^1(\mu)}, \|F\|_{L^1(\mathbb{T})} \) are comparable.

For the counting measure \( \mu = \delta_z \) on the set of integers \( \mathbb{Z} \), Theorem 2 follows from the results by C. Eoff [15], S. Boza and M. Carro [8]. They proved that \( f \in H_{a\xi}^1(\mathbb{Z}) \) if and only if \( f \) admits the analytic continuation to the complex plane \( \mathbb{C} \) as a function from the Paley-Wiener space \( PW^{-1}_{[0,2\pi]} \). It seems difficult to adapt the technique of [8] (where convolution operators were used to relate \( H_{a\xi}^1(\mathbb{Z}) \) and \( \text{Re} H^1(\mathbb{R}) \)) for the general measures \( \mu \) with properties (a) – (c). Instead we give a complex-analytic proof based on the Cauchy-type formula
\[
\int_\Delta F(\xi) \, d\sigma_\alpha(\xi) = \oint_F \frac{F(z)/z}{1 - \bar{\theta}(z)} \, dz,
\] (6)
where \( \Delta \) is an arc of \( \mathbb{T} \), \( \Gamma \) is a simple closed contour in \( \mathbb{C} \) which intersects \( \mathbb{T} \) at the endpoints of \( \Delta \), and \( F \in K^1_0 \cap zH^1 \). Once we have a good estimate for the function \( \frac{F(z)/z}{1 - \bar{\theta}(z)} \) on \( \Gamma \), formula (6) gives us an upper bound for the mean \( \langle F \rangle_{\Delta, \sigma_\alpha} \) on the arc \( \Delta \). Then we can use a standard Calderón-Zigmund decomposition to obtain the representation of \( F \) as a sum of atoms with respect to the measure \( \sigma_\alpha \). The idea of using a contour integration is taken from the classical proof of atomic decomposition of \( \text{Re}(zH^1) \), where the contour \( \Gamma \) comes from the Lusin-Privalov construction. In our situation we have to modify this construction so that the contour \( \Gamma \) does not approach the subsets of the unit disk \( \mathbb{D} \) where the function \( |\alpha - \theta| \) is small.
1.3. **Truncated Hankel operators.** One of important applications of the classical Fefferman duality theorem is the boundedness criterium for Hankel operators on the Hardy space $H^2$. Theorem [1] yields a similar criterium for truncations of Hankel operators to coinvariant subspaces of $H^2$.

Let $\theta$ be an inner function and let $K_{\theta}^2$ be the corresponding coinvariant subspace of the Hardy space $H^2$. Denote by $P_{\theta}$ the orthogonal projection in $L^2(\mathbb{T})$ to the subspace $\mathbb{Z}K_{\theta}^2 = \{ f \in L^2(\mathbb{T}) : f = zg, \ g \in K_{\theta}^2 \}$. The truncated Hankel operator with symbol $\varphi \in L^2(\mathbb{T})$ is the densely defined operator $\Gamma_{\varphi} : K_{\theta}^2 \to \mathbb{Z}K_{\theta}^2$,

$$
\Gamma_{\varphi} : f \mapsto P_{\theta}(\varphi f), \quad f \in K_{\theta}^\infty.
$$

The symbol $\varphi$ of $\Gamma_{\varphi}$ is not unique. However, it is easy to check that every truncated Hankel operator on $K_{\theta}^2$ has the unique "standard" symbol $\varphi \in K_{\theta}^2 \cap \pi H^2$, which plays the same role as the antianalytic symbol of a Hankel operator on $H^2$.

Two special cases of truncated Hankel operators are of traditional interest in the operator theory. If $\theta = z^n$, then the operators defined by (6) are classical Hankel matrices of size $n \times n$. Indeed, in this situation the space $K_{\theta}^2$ consists of analytic polynomials of degree at most $n - 1$ and the entries of the matrix of $\Gamma_{\varphi}$ in the standard bases of $K_{\theta}^2$ and $\mathbb{Z}K_{\theta}^2$ depend only on the difference $k - l$: we have $(\Gamma_{\varphi}z^k, \mathbb{Z}z^{l+1}) = \hat{\varphi}(-k - l - 1)$ for $0 \leq k, l \leq n - 1$. Similarly, for the inner function $\theta_a : z \mapsto e^{i\alpha z}$ in the upper half-plane $\mathbb{C}_+$, the corresponding coinvariant subspace $K_{\theta_a}^2$ of the Hardy space $H^2(\mathbb{C}_+)$ can be identified with the Paley-Wiener space $PW_{[0, a]}^2$, truncated Hankel operators on $PW_{[0, a]}^2$ are unitarily equivalent to the Wiener-Hopf convolution operators on the interval $[0, a]$, see [22,23].

The question for which symbols $\varphi \in L^2(\mathbb{T})$ the truncated Hankel operator $\Gamma_{\varphi}$ is bounded on $K_{\theta}^2$ (and how to estimate its operator norm in terms of $\varphi$) admits several equivalent reformulations. It has been studied in [5,6,9,18,23,24], see the discussion in Section 4. Most of known results are Nehary-type theorems: under certain restrictions they affirm the existence of a bounded symbol for a bounded truncated Hankel/Toeplitz operator with control of the norms. Until now, the only BMO-type criterium for truncated Hankel operators was known. In 2011, M. Carlsson [9] proved that a Hankel operator $\Gamma_{\varphi}$ on $PW_{[0, a]}^2$ with standard symbol $\varphi$ is bounded if and only if the sequence $\{ \varphi(n) \}_{n \in \mathbb{Z}}$ lies in the space $\text{BMO}(\mathbb{Z})$. Recall that we have $PW_{[0, a]}^2 = K_{\theta_a}^2$ for the special one-component inner function $\theta_a : z \mapsto e^{i\alpha z}$ in the upper half-plane $\mathbb{C}_+$. The counting measure $\delta_z$ on $\mathbb{Z}$ can be regarded as the Clark measure $\nu_1$ for the inner function $\theta_a^2$ (for every inner function $\theta$ the Clark measures of $\theta^2$ will be denoted by $\nu_\alpha$; from (3) we see that $\nu_\alpha = (\sigma_\alpha + \sigma_{-\alpha})/2, \ |\alpha| = 1$). Therefore the folowing result is a generalization of the criterium by M. Carlsson.

**Theorem 3.** Let $\theta$ be a one-component inner function, and let $\nu_\alpha$ be the Clark measure of the inner function $\theta^2$. The truncated Hankel operator $\Gamma_{\varphi} : K_{\theta}^2 \to \mathbb{Z}K_{\theta}^2$ with standard symbol $\varphi$ is bounded if and only if $\varphi \in \text{BMO}(\nu_\alpha)$. Moreover, we have

$$
\| \varphi \|_{\nu_\alpha} \leq \| \Gamma_{\varphi} \| \leq c_2 \| \varphi \|_{\nu_\alpha},
$$

for some constants $c_1, c_2$ depending only on the inner function $\theta$.

Similarly, one can describe compact truncated Hankel operators in terms of their standard symbols: we have $\Gamma_{\varphi} \in S_\infty$ if and only if $\varphi \in VMO(\nu_\alpha)$, see Section 4.
Theorem 3 for the inner function \( \theta = z^n \) yields the following interesting corollary for finite Hankel matrices.

**Corollary 1.** Let \( \Gamma = (\gamma_{j+k})_{0 \leq k, j \leq n-1} \) be a Hankel matrix of size \( n \times n \); consider its standard symbol \( \varphi = \gamma_0 \bar{z} + \gamma_1 z + \ldots + \gamma_{2n-2} \bar{z}^{2n-1} \). We have

\[
c_1 \| \varphi \|_{\mathcal{M}_n} \leq \| \Gamma \| \leq c_2 \| \varphi \|_{\mathcal{M}_n},
\]

(9)

where the constants \( c_1, c_2 \) do not depend on \( n \) and \( \mu_{2n} = \frac{1}{2\pi} \sum \delta_{z^n} \) is the Haar measure on the group \( \{ \xi \in \mathbb{C} : \xi^{2n} = 1 \} \).

Corollary 1 implies the boundedness criterion for the standard Hankel operators on \( H^2 \). Recall that the Hankel operator \( H_\varphi : H^2 \to \overline{zH^2} \) with symbol \( \varphi \in L^2(\mathbb{T}) \) is densely defined by

\[
H_\varphi : f \mapsto P_-(\varphi f), \quad f \in H^\infty,
\]

where \( P_- \) denotes the orthogonal projection in \( L^2(\mathbb{T}) \) to \( \overline{zH^2} \). It follows from the classical Fefferman duality theorem that \( H_\varphi \) is bounded if and only if its antianalytic symbol \( P_- \varphi \) lies in \( BMO(\mathbb{T}) \). Moreover, the operator norm of \( H_\varphi \) is comparable to \( \| P_- \varphi \| \), the norm of \( P_- \varphi \) in \( BMO(\mathbb{T}) \). Taking the limit in (9) as \( n \to \infty \) one can prove the estimate \( c_1 \| \varphi \|_\varphi \leq \| H_\varphi \| \leq c_2 \| \varphi \|_\varphi \) for every antianalytic polynomial \( \varphi \).

This is already sufficient to obtain the general version of the boundedness criterion for Hankel operators on \( H^2 \), see details in Section 4.

2. Proof of Theorem 1

2.1. Preliminaries. Given an inner function \( \theta \), denote by \( \rho(\theta) \) its boundary spectrum, that is, the set of points \( \zeta \in \mathbb{T} \) such that \( \liminf_{z \to \zeta, z \in \mathbb{D}} |\theta(z)| = 0 \). In this paper we always assume that \( \rho(\theta) \neq \mathbb{T} \), because this is so for one-component inner functions and for functions satisfying condition (a) in Theorem 1 (see Lemma 2.1 below). As is well-known, the function \( \theta \) admits the analytic continuation from the open unit disk \( \mathbb{D} \) to the open domain \( \mathbb{D} \cup G_\theta \), where \( G_\theta = (\mathbb{T} \setminus \rho(\theta)) \cup \{ z : |z| > 1, \theta(1/\bar{z}) \neq 0 \} \). The analytic continuation is given by

\[
\theta(z) = \frac{1}{\theta(1/\bar{z})}, \quad z \in G_\theta.
\]

(10)

Moreover, \( \mathbb{D} \cup G_\theta \) is the maximal domain to which \( \theta \) can be extended analytically. We need the following known lemma.

**Lemma 2.1.** Let \( \theta \) be an inner function with the Clark measure \( \sigma_\alpha, |\alpha| = 1 \). Then \( \rho(\theta) = \rho(\sigma_\alpha) \). A point \( z \in \mathbb{T} \setminus \rho(\theta) \) belongs to \( \text{supp} \sigma_\alpha \) if and only if \( \theta(z) = \alpha \). Moreover, in the latter case we have \( z \in \rho(\sigma_\alpha) \) and \( \sigma_\alpha \{ z \} = |\theta'(z)|^{-1} \).

**Proof.** As is easy to see from formula (3), we have

\[
\frac{\bar{\alpha} + \theta(z)}{\alpha - \theta(z)} = \int \frac{1 + \bar{\xi}z}{1 - \xi z} d\sigma_\alpha(\xi) + i \frac{\bar{\alpha} + \theta(0)}{\alpha - \theta(0)}, \quad z \in \mathbb{D} \cup G_\theta.
\]

(11)

Since \( \theta \) is analytic on \( \mathbb{D} \cup G_\theta \), a point \( z \in \mathbb{T} \setminus \rho(\theta) \) belongs to \( \text{supp} \sigma_\alpha \) if and only if \( \theta(z) = \alpha \), and in the latter case there is no other points of \( \text{supp} \sigma_\alpha \) in a small neighbourhood of \( z \). Hence \( z \in \rho(\sigma_\alpha) \) and we see from (11) that

\[
\sigma_\alpha \{ z \} = (\bar{\alpha} \theta'(z))^{-1} = |\theta'(z)|^{-1}.
\]

It follows that \( \mathbb{T} \setminus \rho(\theta) \subset \mathbb{T} \setminus \rho(\sigma_\alpha) \). For every \( z \in \mathbb{T} \setminus \rho(\sigma_\alpha) \) either \( z \) is an isolated atom of \( \sigma_\alpha \) or \( z \notin \text{supp} \sigma_\alpha \). In both cases formula (11) shows that the function
\( \theta \) admits the analytic continuation from \( \mathbb{D} \) to a small neighbourhood of \( z \). Hence \( z \in \mathbb{T} \setminus \rho(\theta) \) and we have \( \rho(\theta) = \rho(\sigma_\alpha) \).

The following result is in [3], see Theorem 1.11 and Remark 2 after its proof.

**Theorem (A. B. Aleksandrov).** An inner function \( \theta \) is one-component if and only if it satisfies the following conditions:

(A1) \( m(\rho(\theta)) = 0 \) and \( |\theta'| \) is unbounded on every open arc \( \Delta \subset \mathbb{T} \setminus \rho(\theta) \) such that \( \Delta \cap \rho(\theta) \neq \emptyset \);

(A2) \( \theta \) satisfies the estimate \( |\theta''(\xi)| \leq C|\theta'(\xi)|^2 \) for all \( \xi \in \mathbb{T} \setminus \rho(\theta) \).

2.2. **Proof of Theorem 1.** Essentially, we will show that conditions (a) \(-\) (c) in Theorem 1 are equivalent to conditions (A1), (A2) above.

**Necessity.** Let \( \theta \) be a one-component inner function and let \( \sigma_\alpha \) be its Clark measure. By Lemma 2.1 we have \( \rho(\theta) = \rho(\sigma_\alpha) \). It was proved in [3] that \( m(\rho(\theta)) = 0 \) and \( \rho(\sigma_\alpha) = 0 \). Hence \( \sigma_\alpha \) is a discrete measure with isolated atoms and we have \( m(\supp \sigma_\alpha) = 0 \). Let \( \Delta \) be a connected component of the set \( \mathbb{T} \setminus \rho(\sigma_\alpha) = \mathbb{T} \setminus \rho(\theta) \).

By property (A1) the argument of \( \theta \) on \( \Delta \) is a monotonic function unbounded near both endpoints of \( \Delta \). It follows that the arc \( \Delta \) contains infinitely many points \( \xi_k \) such that \( \theta(\xi_k) = \alpha \). Enumerate these points clockwise by integer numbers. We see from Lemma 2.1 that \( \xi_k \in a(\sigma_\alpha) \) for all \( k \in \mathbb{Z} \) and every atom \( \xi_k \) has two neighbours \( \xi_{k-1}, \xi_{k+1} \). This shows that the measure \( \sigma_\alpha \) satisfies condition (a). The fact that \( \sigma_\alpha \) satisfies condition (b) follows from Lemma 5.1 of [7]. Now check condition (c). Fix an atom \( \xi_0 \in a(\sigma_\alpha) \). From (11) we see that

\[
\frac{1}{1 - a\theta(z)} = \int_{\mathbb{T}} \frac{d\sigma_\alpha(\xi)}{1 - \xi z} + c_\alpha, \quad z \in \mathbb{D} \cup G_\theta,
\]

where \( c_\alpha = a\theta(0)/(1 - a\theta(0)) \). Hence,

\[
(H \sigma_\alpha)(\xi_0) + c_\alpha = \lim_{z \to \xi_0} \frac{\xi_0}{1 - \xi z} \left( \frac{1}{1 - a\theta(z)} \frac{\sigma_\alpha(\xi_0)}{1 - \xi_0 z} \right).
\]

Consider the analytic function \( k_{\xi_0} : z \mapsto \frac{1 - a\theta(z)}{1 - \xi_0 z} \) on the domain \( \mathbb{D} \cup G_\theta \). We have

\[
(H \sigma_\alpha, 1)(\xi_0) + c_\alpha = \lim_{z \to \xi_0} \frac{1}{1 - \xi_0 z} \left( \frac{1}{k_{\xi_0}(z)} \frac{1}{k_{\xi_0}(\xi_0)} \right) = \xi_0 k_{\xi_0}(\xi_0) \frac{\alpha\theta''(\xi_0)}{2|\theta'(\xi_0)|^2}.
\]

From here and the estimate in (A2) we see that \( H \sigma_\alpha, 1 \) is bounded on \( a(\sigma_\alpha) \). Surprisingly simple relation (13) between the discrete Hilbert transform \( H \sigma_\alpha, 1 \) and the inner function \( \theta \) is the key observation in the proof.

**Sufficiency.** Let \( \mu \) be a measure with properties (a) \(-\) (c). Construct the inner function \( \theta \) with the Clark measure \( \sigma_\alpha = \mu \). To prove that \( \theta \) is a one-component inner function we will check conditions (A1) and (A2).

By Lemma 2.1 we have \( \rho(\theta) = \rho(\sigma_\alpha) \). Hence \( m(\rho(\theta)) = 0 \) by property (a) of the measure \( \sigma_\alpha \). Let \( \Delta \) be an open arc of \( \mathbb{T} \) such that \( \Delta \subset \mathbb{T} \setminus \rho(\theta) \) and \( \Delta \cap \rho(\theta) \neq \emptyset \). Then it follows from property (a) of the measure \( \sigma_\alpha \) that \( \Delta \) contains infinitely many atoms of \( \sigma_\alpha \). Since \( \sigma_\alpha \) is finite and \( \sigma_\alpha(\xi) = |\theta'(\xi)|^{-1} \) for every \( \xi \in a(\sigma_\alpha) \), the function \( |\theta'| \) cannot be bounded on \( \Delta \). This gives us condition (A1).
Condition (A2) is more delicate. To check it we need the following lemma.

**Lemma 2.2.** Assume that the Clark measure $\sigma_\alpha$ of an inner function $\theta$ has properties (a)–(c). Then there exists a number $\kappa > 0$ such that for every $\xi \in a(\sigma_\alpha)$ the set $D_\xi(\kappa) = \{ z \in \mathbb{C} : |\xi - z| \leq \kappa \sigma_\alpha(\xi) \}$ is contained in $\mathbb{D} \cup G_{\theta}$ and we have

$$
\frac{1}{2\sigma_\alpha(\xi)} \leq \left| \frac{\alpha - \theta(z)}{\xi - z} \right| \leq \frac{2}{\sigma_\alpha(\xi)}
$$

(14)

for all $z \in D_\xi(\kappa)$.

**Proof.** Pick an atom $\xi_0 \in a(\sigma_\alpha)$ and rewrite formula (12) in the following form:

$$
\frac{1}{1 - \bar{\alpha} \theta(z)} = \int_{T \setminus \{ \xi_0 \}} \frac{d\sigma_\alpha(\xi)}{1 - \xi z} + \frac{\sigma_\alpha(\xi_0)}{1 - \xi_0 z} + c_\alpha, \quad z \in \mathbb{D} \cup G_{\theta}.
$$

We have

$$
\left| \int_{T \setminus \{ \xi_0 \}} \frac{d\sigma_\alpha(\xi)}{1 - \xi z} \right| \leq |(H_{\sigma_\alpha}(1)(\xi_0)| + \int_{T \setminus \{ \xi_0 \}} \frac{|\xi_0 - z| d\sigma_\alpha(\xi)}{|\xi - z| \cdot |\xi - \xi_0|}.
$$

By property (c), $|(H_{\sigma_\alpha}(1)(\xi_0)| \leq C_{\sigma_\alpha}$. Put $\kappa^* = (2B_{\sigma_\alpha})^{-1}$. For $\xi \in a(\sigma_\alpha) \setminus \{ \xi_0 \}$ and $z \in D_{\xi_0}(\kappa^*)$ we have $|\xi_0 - z| \leq \kappa^* \sigma_\alpha(\xi_0) \leq |\xi_0 - \xi|/2$ by property (b) of the measure $\sigma_\alpha$, which gives us the inequality $|\xi - z| \geq |\xi - \xi_0| - |\xi_0 - z| \geq \frac{1}{2} |\xi - \xi_0|$. It follows that for $z \in D_{\xi_0}(\kappa^*)$ we have

$$
\int_{T \setminus \{ \xi_0 \}} \frac{|z - \xi_0| d\sigma_\alpha(\xi)}{|\xi - z| \cdot |\xi - \xi_0|} \leq 2 \kappa^* \sigma_\alpha(\xi_0) \int_{T \setminus \{ \xi_0 \}} \frac{d\sigma_\alpha(\xi)}{|\xi - \xi_0|^2}.
$$

Denote by $\Delta$ the closed arc of $T$ with endpoints $\xi_{0 \pm} \in a(\sigma_\alpha)$. Using property (b), we obtain the estimate

$$
\int_{T \setminus \{ \xi_0 \}} \frac{d\sigma_\alpha(\xi)}{|\xi - \xi_0|^2} \leq \int_{T \setminus \Delta} \frac{d\sigma_\alpha(\xi)}{|\xi - \xi_0|^2} + \frac{2}{A_{\alpha}^2 \sigma_\alpha(\xi_0)}
$$

$$
\leq 2\pi B_{\sigma_\alpha} \int_{T \setminus \Delta} \frac{d\sigma(\xi)}{|\xi - \xi_0|^2} + \frac{2}{A_{\alpha}^2 \sigma_\alpha(\xi_0)}
$$

(16)

where $C_1$ is a constant depending only on the measure $\sigma_\alpha$. We now see from (15) that

$$
\frac{1}{1 - \bar{\alpha} \theta(z)} = \frac{\sigma_\alpha(\xi_0)}{1 - \xi_0 z} + f_{\xi_0}(z), \quad z \in D_{\xi_0}(\kappa_1) \cap (\mathbb{D} \cup G_{\theta}),
$$

(17)

where the function $|f_{\xi_0}|$ is bounded by the constant $C_2 = 2\kappa^* C_1 + C_{\sigma_\alpha} + |c_\alpha|$. Take a number $\kappa \leq \kappa^*$ such that $C_2 \leq (2\kappa)^{-1}$. We have $D_{\xi_0}(\kappa) \subset D_{\xi_0}(\kappa^*)$ and

$$
|f_{\xi_0}(z)| \leq \frac{1}{2} \left| \frac{\sigma_\alpha(\xi_0)}{1 - \xi_0 z} \right|
$$

for all $z \in D_{\xi_0}(\kappa) \cap (\mathbb{D} \cup G_{\theta})$. From here and (17) we get on $D_{\xi_0}(\kappa) \cap (\mathbb{D} \cup G_{\theta})$ the estimate

$$
\frac{1}{2} \left| \frac{\sigma_\alpha(\xi_0)}{1 - \xi_0 z} \right| \leq \frac{1}{1 - \bar{\alpha} \theta(z)} \leq \frac{2}{1 - \bar{\alpha} \theta(z)} \leq \frac{\sigma_\alpha(\xi_0)}{1 - \xi_0 z}
$$

which shows that $\theta$ admits the analytic continuation to a neighbourhood of $D_{\xi_0}(\kappa)$ (that is, $D_{\xi_0}(\kappa) \subset \mathbb{D} \cup G_{\theta}$) and proves formula (1) for points $z \in D_{\xi_0}(\kappa)$. Since
our choice of the number $\kappa$ is uniform with respect to $\xi_0 \in a(\sigma_\alpha)$, the lemma is proved. □

**Notation.** In what follows we write $E_1 \lesssim E_2$ (correspondingly, $E_1 \gtrsim E_2$) for two expressions $E_1, E_2$ to mean that there is a positive constant $c_0$ depending only on the inner function $\theta$ such that $E_1 \leq c_0 E_2$ (correspondingly, $c_0 E_1 \geq E_2$). We will write $E_1 \asymp E_2$ if $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$.

We are ready to complete the proof of Theorem 1. Differentiating (12) we get

\[
\frac{\tilde{\alpha} \theta'(z)}{1 - \tilde{\alpha} \theta'(z)} = \int_T \frac{\xi d\sigma_\alpha(\xi)}{(1 - \xi z)^2},
\]

\[
\frac{\tilde{\alpha} \theta''(z)}{1 - \tilde{\alpha} \theta'(z)} + \frac{2\tilde{\alpha}^2 \theta'(z)^2}{(1 - \tilde{\alpha} \theta'(z))^2} = 2 \int_T \frac{\xi^2 d\sigma_\alpha(\xi)}{(1 - \xi z)^3}.
\]

\[\text{(18)}\]

Pick a point $\xi_0 \in a(\sigma_\alpha)$. Let $D_{\xi_0}(\kappa)$ be the set from Lemma 2.2. Denote by $\partial D_{\xi_0}(\kappa)$ the boundary of $D_{\xi_0}(\kappa)$. By formula (14), $|\alpha - \theta(z)| \geq \kappa/2$ on $\partial D_{\xi_0}(\kappa)$. Arguing as in the Lemma 2.2 from (18) we obtain the estimates

\[
|\theta'(z)| \lesssim |\sigma_\alpha(\xi_0)| \frac{1}{1 - \xi \xi_0 z^2} + \int_{T \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{|1 - \xi \xi_0 z^2|} \lesssim \frac{1}{\sigma_\alpha(\xi_0)},
\]

\[
|\theta''(z)| \lesssim |\theta'(z)|^2 + |\sigma_\alpha(\xi_0)| \frac{1}{1 - \xi \xi_0 z^3} + \int_{T \setminus \{\xi_0\}} \frac{d\sigma_\alpha(\xi)}{|1 - \xi \xi_0 z^3|} \lesssim \frac{1}{\sigma_\alpha(\xi_0)^2}
\]

\[\text{(19)}\]

for all $z \in \partial D_{\xi_0}(\kappa)$. By the maximum principle we have $|\theta''(z)| \lesssim 1/\sigma_\alpha(\xi_0)^2$ for all points $z \in D_{\xi_0}(\kappa)$. On the unit circle $T$ we have

\[
\frac{\xi}{1 - \xi^2} = \frac{-\xi}{|1 - \xi^2|^2}.
\]

From here and formula (18) we get for $z \in D_{\xi_0}(\kappa) \cap T$ the estimate

\[
|\theta'(z)| = |\sigma_\alpha(\xi_0)| \frac{1 - \tilde{\alpha} \theta'(z)}{1 - \xi \xi_0 z^2} + \int_{T \setminus \{\xi_0\}} \left|1 - \tilde{\alpha} \theta'(z)\right|^2 \frac{d\sigma_\alpha(\xi)}{|1 - \xi \xi_0 z|^2} \lesssim \frac{1}{\sigma_\alpha(\xi_0)}.
\]

\[\text{(20)}\]

Combining (19) and (20) we see that $|\theta''/\theta'| \ll 1$ on $D_{\xi_0}(\kappa) \cap T$. It remains to obtain the same estimate for points $z \in T \setminus \rho(\theta)$ that do not belong to the union of the sets $D_z(\kappa), z \in a(\sigma_\alpha)$, from Lemma 2.2. Take such a point $z_0$. We claim that $|\theta(z_0) - \alpha| \geq \kappa/2$. Indeed, assume the converse and find the connected component $\Delta$ of the set $\{\xi \in T \setminus \rho(\theta) : |\theta(\xi) - \alpha| < \kappa/2\}$ containing the point $z_0$. Since the argument of the inner function $\theta$ is monotonic on $\Delta$, there exists a point $\xi \in \Delta$ such that $\theta(\xi) = \alpha$. By Lemma 2.2 we have $\xi \in a(\sigma_\alpha)$. Next, from (14) we see that $|\alpha - \theta(z)| \geq \kappa/2$ for both points in $T \cap \partial D_{\xi_0}(\kappa)$. Since $\Delta$ is connected this yields the inclusion $\Delta \subset D_\xi(\kappa)$ which gives us the contradiction with $z_0 \notin D_\xi(\kappa)$. Thus, we proved the inequality $|\theta(z_0) - \alpha| \geq \kappa/2$. Let $\xi_0$ be the nearest point to $z_0$ in $a(\sigma_\alpha)$. We have $\kappa \sigma_\alpha(\xi_0) \lesssim |\xi_0 - z| \leq B_{\sigma_\alpha} \sigma_\alpha(\xi_0)$. These two estimates imply (19) and (20) for $z = z_0$ and $\xi_0 = \xi_0$. It follows that $|\theta''(z_0)/\theta'(z_0)^2| \lesssim 1$ and $\theta$ satisfies condition (A2). □

**Remark.** Lemma 5.1 in [7] and formula (13) show that for every one-component inner function $\theta$ there exist positive constants $A_\theta, B_\theta, C_\theta$ such that $A_\theta \leq A_{\sigma_\alpha}, B_{\sigma_\alpha} \leq B_\theta, C_{\sigma_\alpha} \leq C_\theta$ for all Clark measures $\sigma_\alpha$ of $\theta$. Also, it follows from Lemma 5.1 in [7] that $|\theta'(z)| \asymp 1/\sigma_\alpha(\xi)$ for all $\xi \in a(\sigma_\alpha)$ and all $z \in T$ between the neighbours $\xi_\pm$ of $\xi$ in $a(\sigma_\alpha)$. In particular, we have $\sigma_\beta(\Delta) \asymp \sigma_\alpha(\Delta) \asymp m(\Delta)$ for all $\beta$
with $|\beta| = 1$ and all arcs $\Delta$ of the unit circle $\mathbb{T}$ containing at least two atoms of the measure $\sigma_\alpha$.

3. Proofs of Theorem 2 and Theorem 21

We first prove Theorem 2. The following result is classical, for the proof see [11] or Chapter 9 in [10].

**Theorem.** (D. N. Clark) Let $\theta$ be an inner function and let $\sigma_\alpha$ be its Clark measure. The natural embedding $V_\alpha : K^2_\theta \hookrightarrow L^2(\sigma_\alpha)$ defined on the reproducing kernels of the space $K^2_\theta$ by $V_\alpha \left( \frac{1 - \overline{\theta}(z)\theta(z)}{1 - \overline{\theta}(\lambda)\theta(\lambda)} \right) = 1 - \frac{\overline{\theta}(z)}{1 - \overline{\theta}(\lambda)}$ can be extended to the whole space $K^2_\theta$ as the unitary operator from $K^2_\theta$ to $L^2(\sigma_\alpha)$. For every $f \in L^2(\sigma_\alpha)$ the function

$$F(z) = \int_{\mathbb{T}} f(\xi) \frac{1 - \overline{\theta}(\xi)}{1 - \overline{\theta}(z)} d\sigma_\alpha(\xi)$$

(21)

in the unit disk $\mathbb{D}$ belongs to $K^2_\theta$ and $V_\alpha F = f$ as elements in $L^2(\sigma_\alpha)$.

It worth be mentioned that A. G. Poltoratski [22] established the existence of angular boundary values $\sigma_\alpha$-almost everywhere on $\mathbb{T}$ for all functions in the space $K^2_\theta$, thus proving that the unitary operator $V_\alpha$ in Clark theorem acts as the natural embedding on the whole space $K^2_\theta$. In our situation this follows from a very simple argument, see Lemma 3.3 in Section 3.2.

The embedding $V_\alpha : K^p_\theta \hookrightarrow L^p(\sigma_\alpha)$ defined on the linear span of the reproducing kernels of $K^p_\theta$ might be unbounded for $1 \leq p < 2$ and might have the unbounded inverse $V^{-1}_\alpha : L^p(\sigma_\alpha) \hookrightarrow K^p_\theta$ for $2 < p \leq \infty$, see Section 3 in [21]. However, the situation is ideal for the one-component inner functions $\theta$, as following results show:

- $V_\alpha K^p_\theta \subseteq L^p(\sigma_\alpha)$ for $1 < p < \infty$ – A. L. Volberg, S. R. Treil [26];
- $V_\alpha K^p_\theta = L^p(\sigma_\alpha)$ for $1 < p < \infty$ – A. B. Aleksandrov [2];
- $V_\alpha K^p_\theta \subseteq L^p(\sigma_\alpha)$ for $0 < p < 1$ – A. B. Aleksandrov [3].

Theorem 21 says that $V_\alpha K^p_\theta = H^p_{\mathrm{id}}(\sigma_\alpha)$ for every one-component inner function $\theta$. We are ready to prove its easy part – the inclusion $V_\alpha K^p_\theta \supseteq H^p_{\mathrm{id}}(\sigma_\alpha)$.

3.1. Proof of the part “$\Rightarrow$” in Theorem 21

Let $\mu$ be a measure on the unit circle $\mathbb{T}$ with properties (a) – (c). Take a complex number $\alpha$ of unit modulus and construct the one-component inner function $\theta$ with the Clark measure $\sigma_\alpha = \mu$. We want to show that every function $f \in H^p_{\mathrm{id}}(\sigma_\alpha)$ admits the analytic continuation to the open unit disk $\mathbb{D}$ as a function $F \in K^1_\theta \cap \mathbb{D}H^1$ with $\|F\|_{L^1(\mathbb{T})} \lesssim \|f\|_{H^p_{\mathrm{id}}(\sigma_\alpha)}$. At first, assume that $f$ is a $\sigma_\alpha$-atom supported on an arc $\Delta \subseteq \mathbb{T}$ with center $\xi_\alpha$. Then $f \in L^2(\sigma_\alpha)$ and the function $F$ in formula (21) lies in the space $K^2_\theta \subseteq K^1_\theta$ by Clark theorem. Since $\int_{\mathbb{T}} f d\sigma_\alpha = 0$, we have $F(0) = 0$. Moreover, we see from Lemma 2.1 that $F(\xi) = f(\xi)$ for all $\xi \in \alpha(\sigma_\alpha)$. Let us check that the norm of $F$ in $L^1(\mathbb{T})$ is bounded by a constant depending only on the inner function $\theta$. By Aleksandrov desintegration theorem (see [11] or Section 9.4 in [10]), we have

$$\int_{\mathbb{T}} |F| dm = \int_{\mathbb{T}} \int_{\mathbb{T}} |V_\beta F(\xi)| d\sigma_\beta(\xi) dm(\beta).$$

(22)

Fix a complex number $\beta \neq \alpha$ of unit modulus. We claim that $\|V_\beta F\|_{L^1(\sigma_\beta)} \lesssim 1$. Denote by $2\Delta$ the arc of $\mathbb{T}$ with center $\xi_\alpha$ such that $m(2\Delta) = 2m(\Delta)$ (in the case
where \( m(\Delta) \geq 1/2 \) put \( 2\Delta = T \). Break the integral \( \int_T |V_\beta F| \, d\sigma_\beta \) into two parts,

\[
\int_T |V_\beta F(\xi)| \, d\sigma_\beta(\xi) = \int_{2\Delta} |V_\beta F(\xi)| \, d\sigma_\beta(\xi) + \int_{T \setminus 2\Delta} |V_\beta F(\xi)| \, d\sigma_\beta(\xi). \tag{23}
\]

By Clark theorem we have \( \|V_\beta F\|_{L^2(\sigma_\beta)} = \|F\|_{L^2(T)} = \|V_\alpha F\|_{L^2(\sigma_\alpha)} \). Moreover, we have \( \|V_\alpha F\|_{L^2(\sigma_\alpha)} \leq 1/\sqrt{\sigma_\alpha(\Delta)} \) because the function \( V_\alpha F = f \) is a \( \sigma_\alpha \)-atom supported on the arc \( \Delta \). This yields the inequality

\[
\int_{2\Delta} |V_\beta F(\xi)| \, d\sigma_\beta(\xi) \leq \sqrt{\sigma_\beta(2\Delta)} : \|V_\beta F\|_{L^2(\sigma_\beta)} \leq \sqrt{\sigma_\beta(2\Delta)/\sigma_\alpha(\Delta)}. \tag{24}
\]

Note that the \( \Delta \) contains at least two points in \( a(\sigma_\alpha) \) because \( f \) has zero \( \sigma_\alpha \)-mean on \( \Delta \). Hence \( \sigma_\alpha(\Delta) \approx m(\Delta) \) and \( \sigma_\beta(2\Delta) \approx m(2\Delta) \), see remark after the proof of Theorem 1. This shows that \( \int_{2\Delta} |V_\beta F(\xi)| \, d\sigma_\beta(\xi) \lesssim 1 \). Let us now estimate the second term in (23). Take a point \( z \in a(\sigma_\beta) \setminus 2\Delta \). Using the fact that \( f \) is a \( \sigma_\alpha \)-atom we obtain the estimate

\[
|V_\beta F(z)| = \left| \int_\Delta f(\xi) \frac{1 - \bar{\alpha} \beta}{1 - \xi z} \, d\sigma_\alpha(\xi) \right| \\
= \left| \int_\Delta f(\xi) \left( \frac{1 - \bar{\alpha} \beta}{1 - \xi z} - \frac{1 - \bar{\alpha} \beta}{1 - \xi z} \right) \, d\sigma_\alpha(\xi) \right| \\
\leq 2 \int_\Delta |f(\xi)| \frac{\xi - \xi_c}{(1 - \xi z)(1 - \xi_c z)} \, d\sigma_\alpha(\xi) \tag{25}
\]

\[
\leq \frac{2\pi m(\Delta)}{|z - \xi_c|^2} \cdot \sup_{\xi \in \Delta} \left| \frac{z - \xi_c}{z - \xi} \right| \cdot \int_\Delta |f(\xi)| \, d\sigma_\alpha(\xi) \\
\leq \frac{4\pi m(\Delta)}{|z - \xi_c|^2}.
\]

From here we get

\[
\int_{T \setminus 2\Delta} |V_\beta F(z)| \lesssim m(2\Delta) \cdot \int_{T \setminus 2\Delta} \frac{d\sigma_\beta(z)}{|z - \xi_c|^2} \lesssim 1. \tag{26}
\]

Hence the norm of \( F \) in \( L^1(T) \) is bounded by a constant depending only on \( \theta \). Now take an arbitrary function \( f \in H^{1}_{d}(\sigma_{\alpha}) \) and consider its representation \( f = \sum k \lambda_k f_k \), where \( f_k \) are \( \sigma_\alpha \)-atoms and \( \sum_k |\lambda_k| \leq 2 \|f\|_{H^{1}_{d}(\sigma_{\alpha})} \). Let \( F_k \) be the functions in \( K^2_\beta \) such that \( V_\alpha F_k = f_k \). Then the sum \( \sum k \lambda_k F_k \) converges absolutely in \( L^1(T) \) to a function \( F \in K^1_\beta \) and we have \( \|F\|_{L^1(T)} \lesssim \|f\|_{H^{1}_{d}(\sigma_{\alpha})} \). From formula (21) we get

\[
F(z) = \sum k \lambda_k \int_T f_k(\xi) \frac{1 - \bar{\alpha} \theta(z)}{1 - \xi z} \, d\sigma_\alpha(\xi) = \int_T f(\xi) \frac{1 - \bar{\alpha} \theta(z)}{1 - \xi z} \, d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.
\]

Since \( f \in L^1(T) \), this formula determines the analytic continuation of \( F \) to the domain \( \mathbb{D} \cup G_\theta \). From Lemma 2.1 we see that \( F(\xi) = f(\xi) \) for all \( \xi \in a(\sigma_{\alpha}) \). \( \square \)

3.2. Preliminaries for the proof of the part “\( \Leftarrow \)” in Theorem 2. Let \( \theta \) be a one-component inner function with the Clark measure \( \sigma_{\alpha} \). Introduce positive constants \( A_{\sigma_{\alpha}}, B_{\sigma_{\alpha}} \) such that

\[
\hat{A}_{\sigma_{\alpha}} m[\xi, \xi_{\pm}] \leq \sigma_{\alpha}\{\xi\} \leq \hat{B}_{\sigma_{\alpha}} m[\xi, \xi_{\pm}], \quad \xi \in a(\sigma_{\alpha}).
\]

Here \([\xi, \xi_{\pm}]\) are the closed arcs of \( T \) with endpoints \( \xi, \xi_{\pm} \in a(\sigma_{\alpha}) \) such that the corresponding open arcs \((\xi, \xi_{\pm})\) do not intersect \( \text{supp} \sigma_{\alpha} \). Take a positive
number $\kappa \leq (2\tilde{B}_{\sigma_\alpha})^{-1}$ for which estimate (14) holds true. Denote by $D_{\sigma_\alpha}(\kappa)$ the union of the sets $D_\xi(\kappa)$, $\xi \in a(\sigma_\alpha)$ from Lemma 2.2.

**Lemma 3.1.** For every arc $\Delta$ of $T$ containing at least one atom of the measure $\sigma_\alpha$ we have $m(\Delta) \leq (2/\tilde{A}_{\sigma_\alpha})\sigma_\alpha(\Delta)$. If $\Delta$ contains two or more atoms of $\sigma_\alpha$, we have $\sigma_\alpha(\Delta) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(\kappa))$. In particular, the sets $D_\xi(\kappa)$, $\xi \in a(\sigma_\alpha)$ are disjoint.

**Proof.** It is sufficient to prove the statement in the case where $\Delta$ contains only finite number of atoms of $\sigma_\alpha$. Enumerate the atoms clockwise: $\xi_1, \ldots, \xi_n$. Find the neighbours of $\xi_1, \xi_n$ in $a(\sigma_\alpha) \setminus \Delta$ and denote them by $\xi_0$ and $\xi_{n+1}$ correspondingly. We have
\[
m(\Delta) \leq \sum_{k=0}^n m[\xi_k, \xi_{k+1}] \leq \frac{2}{\tilde{A}_{\sigma_\alpha}} \sum_{k=1}^n \sigma_\alpha{\xi_k} = \frac{2}{\tilde{A}_{\sigma_\alpha}}\sigma_\alpha(\Delta).
\]
In the case where $n \geq 2$ we have
\[
\sigma_\alpha(\Delta) = \sum_{k=1}^n \sigma_\alpha{\xi_k} \leq 2\tilde{B}_{\sigma_\alpha}m(\Delta) \leq 2\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(\kappa)) + \tilde{B}_{\sigma_\alpha}K\sigma_\alpha(\Delta).
\]
Now use the assumption $\kappa \leq (2\tilde{B}_{\sigma_\alpha})^{-1}$ and get $\sigma_\alpha(\Delta) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(\kappa))$. □

**Lemma 3.2.** There exists $\varepsilon > 0$ such that $|\alpha - \theta(z)| \geq \varepsilon$ for all $z \in \mathbb{D} \setminus D_{\sigma_\alpha}(\kappa)$.

**Proof.** Let $\delta \in (0, 1)$ be a number such that the set $\Omega_\delta = \{z \in \mathbb{D} : |\theta(z)| < 1/\delta\}$ is connected. The set
\[
\Omega_\delta^{\varepsilon} = \{z \in \mathbb{D} \cup G_\theta : \delta < |\theta(z)| < 1/\delta\}
\]
is at most countable union of the open connected components, $O_k$. It was proved by B. Cohn [12] that the restriction of the inner function $\theta$ to each of the sets $O_k$ is a covering map from $O_k$ to the ring $R_\delta = \{z \in \mathbb{C} : \delta < |z| < 1/\delta\}$. Take a positive number $\varepsilon < \min(\kappa/2, 1 - \delta)$. We claim that every connected component $E$ of the set $L_\varepsilon = \{z \in \mathbb{D} \cup G_\theta : |\alpha - \theta(z)| < \varepsilon\}$ contains an atom of $\sigma_\alpha$. Indeed, we have $E \subset O_k$ for some index $k$ because $L_\varepsilon \subset \Omega_\delta^{\varepsilon}$. Since $\theta$ is a covering map from $O_k$ to $R_\delta$, there exists a number $\varepsilon_1$ (which can be taken to be less than $\varepsilon$) such that the preimage of $\{\xi : |\alpha - \xi| < \varepsilon_1\}$ under $\theta$ on $O_k$ is at most countable union of the open disjoint sets $O_{km} \subset O_k$ and $\theta$ is a homeomorphism from $O_{km}$ to $\{\xi : |\xi - \alpha| < \varepsilon_1\}$ for every $m$. By the minimum principle, $\inf_{z \in E} |\theta(z) - \alpha| = 0$. It follows that $E \cap O_{km} \neq \emptyset$ for some index $m$. Since $E$ is connected and $|\theta - \alpha| < \varepsilon_1 < \varepsilon$ on $O_{km}$, we have $O_{km} \subset E$. But every set $O_{km}$ contains the unique point $\xi$ with $\theta(\xi) = \alpha$. By Lemma 2.1 $\xi \in a(\sigma_\alpha)$ and thus $E \cap a(\sigma_\alpha) \neq \emptyset$. To prove the lemma it is sufficient to show that $E \subset D_\xi(\kappa)$. For every $z \in E \cap D_\xi(\kappa)$ we get from (14) the estimate
\[
|\xi - z| \leq 2|\alpha - \theta(z)||\alpha_\sigma{\xi} \leq 2\varepsilon\sigma_\alpha{\xi}.
\]
Hence $E$ does not intersect the circle $\{z \in \mathbb{C} : |z - \xi| = r\sigma_\alpha{\xi}\}$ for every $r \in (2\varepsilon, \kappa)$. Since the set $E$ is connected this yields the desired inclusion $E \subset D_\xi(\kappa)$. □

**Lemma 3.3.** Let $\theta$ be an inner function with $\rho(\theta) \neq T$. Then every function in $K^1_{\theta}$ admits the analytic continuation from the unit disk $\mathbb{D}$ to the domain $\mathbb{D} \cup G_\theta$. Consequently, if $\sigma_\alpha(\rho(\theta)) = 0$ for a Clark measure $\sigma_\alpha$ of $\theta$, then every function in $K^1_{\theta}$ has a trace on the set $\mathbb{D} \cup G_\theta$ of full measure $\sigma_\alpha$. 
Proof. For every function $F \in K^1_{\theta}$ we have $\bar{\theta}F \in \bar{z}H^1$ on $T$. Hence,
\begin{equation}
F(z) = \int_T F(\xi) \frac{1 - \theta(z)\bar{\theta}(\xi)}{1 - z\bar{\xi}}
dn(\xi), \quad z \in \mathbb{D}.
\end{equation} (27)

Extend the inner function $\theta$ to the domain $\mathbb{D} \cup G_\theta$ by formula (11). The right hand side of (27) then determines the analytic continuation of the function $F$ to $\mathbb{D} \cup G_\theta$.

By Lemma 2.3 we have $\rho(\sigma_{\alpha}) = \rho(\theta)$ which completes the proof. \qed

Lemma 3.4. Let $\theta$ be an inner function and let $G \in K^1_{\theta} \cap zH^1$. Then there exist functions $G_1, G_2 \in K^1_{\theta} \cap zH^1$ such that $G = G_1 + iG_2$ and $G_{1,2} = \bar{\theta}G_{1,2}$ on $T \setminus \rho(\theta)$. Moreover, we have $\|G_{1,2}\|_{L^1(T)} \leq \|G\|_{L^1(T)}$.

Proof. Consider the function $\tilde{G} = \bar{\theta}G$ on the unit circle $T$. We have
\[ \tilde{G} \in \bar{\theta}(zH^1 \cap z\theta H^1) = \bar{z}\theta H^1 \cap zH^1 = K^1_{\theta} \cap zH^1. \]

This shows that $G$ can be continued to the open unit disk $\mathbb{D}$ as a function from the space $K^1_{\theta} \cap zH^1$. Now put $G_1 = (G + \tilde{G})/2$, $G_2 = (G - \tilde{G})/2i$ and obtain the desired representation. \qed

3.3. Proof of the part “$\Leftarrow$” in Theorem 2

Let $\mu$ be a measure on the unit circle $T$ with properties (a) – (c) and let $|\alpha| = 1$. Consider the one-component inner function $\theta$ with the Clark measure $\sigma_{\alpha} = \mu$. Take a function $F \in K^1_{\theta} \cap zH^1$.

By Lemma 3.3 $F$ is analytic on the domain $\mathbb{D} \cup G_\theta$. Denote by $f$ its trace on the set $a(\sigma_{\alpha}) \subset \mathbb{D} \cup G_\theta$ of full measure $\sigma_{\alpha}$. Our aim is to prove that $f \in H^1_a(\sigma_{\alpha})$ and $\|f\|_{H^1_a(\sigma_{\alpha})} \lesssim \|F\|_{L^1(T)}$. At first, assume that $F \in K^2_{\theta} \cap zH^2$ and $F = \theta F$ on $T \setminus \rho(\theta)$.

We will need the following modification of the Lusin-Privalov construction (see Section III.D in [16] for the standard one). Consider the non-tangential maximal function of $F$,
\[ F^*(\xi) = \sup_{z \in \Lambda_\xi} |F(z)|, \quad \xi \in T, \]
where $\Lambda_\xi$ denotes the convex hull of the set $\{\xi\} \cup \{z \in \mathbb{D} : |z| \leq 1/\sqrt{2}\}$. Put
\[ S_F(\lambda) = \mathbb{D} \setminus \{z \in \mathbb{D} : z \in \Lambda_\xi \text{ for some } \xi \in T \text{ with } F^*(\xi) < \lambda\}. \]

Let $D_{\sigma_{\alpha}}(k)$ be the set defined at the beginning of Section 3.2. By Lemma 3.2 we have $|\alpha - \theta| \geq \varepsilon$ on $\mathbb{D} \setminus D_{\sigma_{\alpha}}(k)$. Denote by $R_F(\lambda)$ the union of those connected components of the set $S_F(\lambda) \cup D_{\sigma_{\alpha}}(k)$ for which we have $E \cap S_F(\lambda) \neq \emptyset$ and $E \cap D_{\sigma_{\alpha}}(k) \neq \emptyset$. The sets $R_F(\lambda)$ are closed and have the following properties:

1. If $\lambda_1 < \lambda_2$, then $R_F(\lambda_2) \subset R_F(\lambda_1)$;
2. $|F(z)| \leq \lambda$ for $\sigma_{\alpha}$-almost all points $z \in T \setminus R_F(\lambda)$;
3. $|F(z)| \leq \lambda$ and $|\alpha - \theta(z)| \geq \varepsilon$ for $z \in \partial R_F(\lambda) \cap \mathbb{D}$.

More special properties of the sets $R_F(\lambda)$ are collected in the following lemma.

Lemma 3.5. Let $E$ be a connected component of the set $R_F(\lambda)$. Put $\gamma = \partial E \cap \mathbb{D}$ and $\Delta = \partial E \cap T$. There exist constants $c_4, c_5, c_6$ depending only on $\theta$ such that

1. $\gamma$ is a rectifiable curve with length $|\gamma| \leq c_4 \sigma_{\alpha}(\Delta)$;
2. $\sigma_{\alpha}(\Delta) \leq c_5 m(\Delta \cap S_F(\lambda))$ if $E$ contains at least two atoms of $\sigma_{\alpha}$;
3. $\int_{\Delta} f \, d\sigma_{\alpha} \leq c_6 \lambda$.

One can take $c_4 = 40/\hat{A}_{\sigma_{\alpha}}$, $c_5 = 4\hat{B}_{\sigma_{\alpha}}$, $c_6 = 60/(\varepsilon \hat{A}_{\sigma_{\alpha}})$.\]
Proof. By the construction and Lemma 3.1 we have
\[ |\gamma| \leq \sqrt{2 + \pi/2} |\Delta| \leq 20m(\Delta) \leq 40\sigma_\alpha(\Delta)/\tilde{A}_{\alpha}. \]
In the case where the arc \( \Delta \) contains at least two atoms of the measure \( \sigma_\alpha \), Lemma 3.1 gives us the estimate
\[ \sigma_\alpha(\Delta) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \setminus D_{\sigma_\alpha}(k)) \leq 4\tilde{B}_{\sigma_\alpha}m(\Delta \cap S_F(\lambda)). \]
Let us check property (6). At first, assume that \( \gamma \cap \rho(\theta) = \emptyset \). Then we have \( \gamma \cap \text{supp} \sigma_\alpha = \emptyset \) by the construction. For \( z \in \mathbb{C} \) with \( |z| \geq 1 \) denote \( z^* = 1/\bar{z} \) and put \( \gamma^* = \{ z \in \mathbb{C} : z^* \in \gamma \} \). The set \( \Gamma = \gamma \cup \gamma^* \) is a rectifiable curve in \( \mathbb{C} \) with length \( |\Gamma| \leq 3|\gamma| \). Let us check that
\[ \left| \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} \right| \leq 2\varepsilon^{-1}\lambda, \quad z \in \Gamma \cap (\mathbb{D} \cup G_\theta). \tag{28} \]
For \( z \in \gamma \) we have \( |z| \geq 1/\sqrt{2} \), \( |F| \leq \lambda, |\alpha - \theta| \geq \varepsilon \) and therefore (28) holds. The function \( z \mapsto F(z^*)/\theta(z^*) \) is analytic on the interior of \( G_\theta \) and coincides with the function \( F \) on \( G_\theta \cap \mathbb{T} = \mathbb{T} \setminus \rho(\theta) \) (recall that \( F \) admits the analytic continuation to the domain \( \mathbb{D} \cup G_\theta \) by Lemma 3.3 and \( F = \theta F \) on \( \mathbb{T} \setminus \rho(\theta) \) by the assumption). By the uniqueness of the analytic continuation we have \( F(z) = \bar{F}(z^*)/\theta(z^*) \) for all \( z \in G_\theta \). Now take a point \( z \in G_\theta \) and compute
\[ \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} = \frac{z^*F(z^*)}{\bar{\theta}(z^*)} = \frac{z^*F(z^*)}{\theta(z^*) - \alpha}. \]
This yields estimate (28) for \( z \in \gamma^* \cap G_\theta \). Next, we claim that
\[ \int f(z) d\sigma_\alpha(z) = -\frac{1}{2\pi i} \oint \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} dz. \tag{29} \]
Indeed, using formula (21) for the function \( F/z \in K_\theta^2 \) we obtain
\[ \oint \frac{F(z)/z}{1 - \bar{\alpha}\theta(z)} dz = \oint \frac{1}{1 - \bar{\alpha}\theta(z)} \left( \oint f(\xi) \frac{1 - \bar{\alpha}(z)}{1 - \bar{\alpha}(\xi)} d\sigma_\alpha(\xi) \right) dz d\sigma_\alpha(z) \]
\[ = \oint f(\xi) \oint \frac{1}{1 - \bar{\xi}/z} d\sigma_\alpha(\xi) = -2\pi i \oint f(\xi) \chi_\Delta(\xi) d\sigma_\alpha(\xi), \tag{30} \]
where \( \chi_\Delta \) denotes the indicator of the set \( \Delta \). Note that change of the order of integration is possible because \( \Gamma \cap \text{supp} \sigma_\alpha = \emptyset \) and therefore all integrals in (30) are absolutely convergent. We now see from (28) and (29) that
\[ \left| \int f(\xi) d\sigma_\alpha(\xi) \right| \leq (\pi\varepsilon)^{-1} \lambda |\Gamma| \leq 3(\pi\varepsilon)^{-1} \lambda |\gamma| \leq 3(\pi\varepsilon)^{-1} c_\lambda \cdot \lambda \cdot \sigma_\alpha(\Delta). \]
This gives us property (5) in the case where \( \gamma \cap \rho(\theta) = \emptyset \). The general case can be reduced to just considered one by a small perturbation of the contour \( \gamma \); use the fact that \( f \in L^2(\sigma_\alpha) \) by Clark theorem and property (a) of the measure \( \sigma_\alpha \) from from Theorem 1.

Lemma 3.5 is the key argument in the proof of Theorem 2. The rest of the proof is a standard Calderón-Zigmund decomposition. We will follow the exposition in
Section VII.E of [16]. For each $\lambda > 0$ the set $\Delta_F^k(\lambda) = R_F(\lambda) \cap \mathbb{T}$ is a union of closed disjoint arcs $\Delta_F^k(\lambda), \Delta_F(\lambda) = \bigcup_{k \in I_2} \Delta_F^k(\lambda)$. Consider the functions

$$G_\lambda = \begin{cases} f, & \xi \in \mathbb{T} \setminus \Delta_F^k(\lambda), \\ (f)_{\Delta_F^k(\lambda), \sigma}, & \xi \in \Delta_F^k(\lambda), \end{cases} \quad B_\lambda = \begin{cases} f - (f)_{\Delta_F^k(\lambda), \sigma}, & \xi \in \Delta_F(\lambda). \end{cases}$$

By Lemma 3.5 we have $|G_\lambda| \leq c_6 \sigma$-almost everywhere on $\mathbb{T}$. The function $B_\lambda$ has zero $\sigma$-mean on each arc $\Delta_F^k(\lambda), k \in I_\lambda$. For every integer $n \in \mathbb{Z}$ set $g_n = G_{2^n}$ and $b_n = B_{2^n}$. Fix a number $N_0 \in \mathbb{Z}$ such that

$$2^{N_0} < \inf_{|z| \leq 1/\sqrt{2}} |F(z)| \leq 2^{N_0 + 1}.$$

Note that $\Delta_F(2^{N_0}) = \mathbb{T}$. By formula (21),

$$g_{N_0} = \frac{1}{\sigma(\mathbb{T})} \int f \, d\sigma = \frac{F(0)}{\sigma(\mathbb{T})(1 - \alpha(0))} = 0.$$

Since $f$ is finite at each point $\xi \in a(\sigma)$ we have $f(\xi) = g_N(\xi)$ for every sufficiently big number $N$. Hence

$$f(\xi) = \sum_{n=N_0}^{\infty} (g_{n+1}(\xi) - g_n(\xi)), \quad \xi \in a(\sigma), \quad (31)$$

where the sum converges pointwise (in fact, only finite number of summands in (31) are non-zero for every $\xi \in a(\sigma)$). Note that $f = b_n + g_n$ and $g_{n+1} - g_n = b_n - b_{n+1}$ for all $n \geq N_0$. Let $I_{2^n}$ be the set of indexes $k \in I_{2^n}$ such that the set $\Delta_F^k(2^n)$ contains at least two atoms of the measure $\sigma$. The function $g_{n+1} - g_n$ vanishes $\sigma$-almost everywhere on each of the sets $\Delta_F^k(2^n), k \in I_{2^n} \setminus I_{2^{n+1}}$. Indeed, for such index $k$ we have by the construction. Hence $g_n(\xi) = g_{n+1}(\xi) = f(\xi)$ because the $\sigma$-mean of $f$ on any arc containing the only point $\xi \in a(\sigma)$ equals $f(\xi)$. Define

$$\tilde{a}_{n,k} = \chi_{\Delta_F^k(2^n)}(b_n - b_{n+1}), \quad n \geq N_0, \quad k \in I_{2^n}.$$

where $\chi_{\Delta_F^k(2^n)}$ is the indicator of the set $\Delta_F^k(2^n)$. The functions $\tilde{a}_{n,k}$ have zero $\sigma$-mean on $\mathbb{T}$. Indeed, let $I$ denote the set of indexes $m$ such that $\Delta_F^m(2^n+1) \subset \Delta_F^k(2^n)$ (note that $\Delta_F^m(2^n+1) \subset \Delta_F^k(2^n)$ by property (1) of the sets $R_F(\lambda)$). Then

$$\int_{\tilde{a}_{n,k}} \int_{\Delta_F^k(2^n)} (b_n - b_{n+1}) \, d\sigma = - \sum_{m \in I} \int_{\Delta_F^m(2^n+1)} b_{n+1} \, d\sigma = 0.$$

Also, we have $|\tilde{a}_{n,k}| \leq |g_n| + |g_{n+1}| \leq 3c_6 \cdot 2^n$ on $\mathbb{T}$ for every $n \geq N_0$ and $k \in I_{2^n}$. Now put

$$a_{n,k} = \frac{\tilde{a}_{n,k}}{3c_6 \cdot 2^n \cdot \sigma(\Delta_F^k(2^n))}, \quad n \geq N_0, \quad k \in I_{2^n},$$

and observe that $a_{n,k}$ are atoms with respect to the measure $\sigma$. It follows from formula (31) that

$$f(\xi) = \sum_{n=N_0}^{\infty} \sum_{k \in I_{2^n}} \lambda_{n,k} a_{n,k}(\xi), \quad \xi \in a(\sigma), \quad (32)$$

where $\lambda_{n,k} = 3c_6 \cdot 2^n \cdot \sigma(\Delta_F^k(2^n))$ and the sum is convergent pointwise. Since the set $a(\sigma)$ has full measure $\sigma$, it remains to check that

$$\sum_{n \geq N_0} \sum_{k \in I_{2^n}} \lambda_{n,k} \lesssim \|F\|_{L^1(\mathbb{T})}. \quad (33)$$
By Lemma 3.5 we have
\[ \sigma_\alpha(\Delta^k_F(2^n)) \leq c_5 m(\Delta^k_F(2^n) \cap S_F(2^n)) \]
for every \( n \geq N_0 \) and \( k \in I_{2^n} \). Hence,
\[
\sum_{n \geq N_0} \sum_{k \in I_{2^n}} m(\Delta^k_F(2^n)) \leq c_5 \sum_{n \geq N_0} \sum_{k \in I_{2^n}} m(\Delta^k_F(2^n) \cap S_F(2^n)) \leq c_5 \sum_{n \geq N_0} m(S_F(2^n) \cap T) = c_5 \sum_{n \geq N_0} m(\{ \xi \in T : F^*(\xi) \geq 2^n \}).
\]
The last sum does not exceed
\[
\sum_{n \geq N_0} \sum_{l \geq 0} 2^n m(\{ \xi \in T : 2^{n+l} \leq F^*(\xi) < 2^{n+l+1} \}) \leq \sum_{l \geq 0} m(\{ \xi \in T : 2^{N_0+l} \leq F^*(\xi) < 2^{N_0+l+1} \}) \sum_{k=N_0}^l 2^{N_0+k} \leq 2\|F^*\|_{L^1(T)} \leq 2M\|F\|_{L^1(T)}.
\]
where \( M \) denotes the norm of the maximal operator \( F \mapsto F^* \) on \( H^1 \). Thus, inequality (33) holds with the constant \( 6c_5c_6M \) and formula (32) gives us the atomic decomposition of the trace \( f \) provided \( F \in K^2_\theta \cap zH^2 \) and \( F = \theta F \). Now consider arbitrary function \( F \in K^1_\theta \cap zH^1 \) with the trace \( f \) on the set \( a(\sigma_\alpha) \). Since \( K^2_\theta \cap zH^2 \) is the dense subset of \( K^1_\theta \cap zH^1 \) in norm of \( L^1(T) \) one can find functions \( F_k \in K^2_\theta \cap zH^2 \) such that \( F = \sum_k F_k \) and \( \|F\|_{L^1(T)} \geq 1/2 \| \sum_k \|F_k\|_{L^1(T)} \). Let \( G_{1,k}, G_{2,k} \) be the functions from Lemma 3.3 for \( G = F_k \) and let \( g_{1,k}, g_{2,k} \) be their traces on \( a(\sigma_\alpha) \). We have \( f(\xi) = \sum g_{1,k}(\xi) + i \sum g_{2,k}(\xi) \) for every \( \xi \in a(\sigma_\alpha) \), see formula (24). It follows from the first part of the proof that \( f \) admits the atomic decomposition with respect to the measure \( \sigma_\alpha \) and we have \( \|f\|_H^2(\sigma_\alpha) \leq 24c_5c_6M\|F\|_{L^1(T)} \). \( \square \)

3.4. Proof of Theorem 2

Since \( (\text{supp} \sigma_\alpha, \| \cdot \|, \sigma_\alpha) \) is the doubling metric space, Theorem 2 and Theorem B in [14] imply Theorem 2. To make the paper more self-contained, we give a proof of this implication.

**Proof.** Let \( \theta \) be a one-component inner function. We first remark that the integral in formula (31) is correctly defined for \( F \in K^\infty_\theta \) and \( b \in \text{BMO}(\sigma_\alpha) \). Indeed, by Lemma 2.1 and Lemma 3.3 every function \( F \in K^1_\theta \) has the trace \( f \) on the set \( a(\sigma_\alpha) \) of full measure \( \sigma_\alpha \). If \( F \in K^1_\theta \cap zH^\infty \), then \( f \in L^\infty(\sigma_\alpha) \). Since \( \text{BMO}(\sigma_\alpha) \subset L^1(T) \) the integral in formula (31) converges absolutely.

Consider a continuous linear functional \( \Phi \) on \( K^1_\theta \cap zH^1 \). Since \( K^2_\theta \subset K^1_\theta \) and \( K^2_\theta \cap zH^2 \) is the Hilbert space there exists a function \( G \in K^2_\theta \cap zH^2 \) such that \( \Phi(F) = \int_T F \overline{G} \ dm \) for all \( F \in K^2_\theta \cap zH^2 \). Denote by \( b \) the restriction of the function \( G \) to the set \( a(\sigma_\alpha) \) of full measure \( \sigma_\alpha \). By Clark theorem, we have \( b \in L^2(\sigma_\alpha) \). Let us prove that \( b \in \text{BMO}(\sigma_\alpha) \). For every function \( F \in K^2_\theta \cap zH^2 \) we have
\[
\Phi(F) = \int_T F \overline{G} \ dm = \int_T F b \ dm = \int_T F b d\sigma_\alpha = \Phi_b(F),
\]
where use again Clark theorem. Take an arc \( \Delta \) of \( T \) and consider the function
\[
a_0 \in L^\infty(\sigma_\alpha) \text{ such that } |a_0| = 1, a_0(b - \langle b \rangle_{\Delta, \sigma_\alpha}) = |a_0(b - \langle b \rangle_{\Delta, \sigma_\alpha})| \text{ } \sigma_\alpha\text{-almost}
\]
everywhere on $\Delta$ and $a = 0$ $\sigma_\alpha$-everywhere off $\Delta$. Denote by $\chi_{\Delta}$ the indicator of the set $\Delta$. The function
\[ a = \frac{1}{2\sigma_\alpha(\Delta)}(a_0 - \langle a_0 \rangle_{\Delta, \sigma_\alpha})\chi_{\Delta} \]
is an atom with respect to the measure $\sigma_\alpha$ and we have
\[ \int_T ab \, d\sigma_\alpha = \int_{\Delta} a(b - \langle b \rangle_{\Delta, \sigma_\alpha}) \, d\sigma_\alpha = \frac{1}{2\sigma_\alpha(\Delta)} \int_{\Delta} |b - \langle b \rangle_{\Delta, \sigma_\alpha}| \, d\sigma_\alpha. \] (35)

By Theorem 24 the function $a$ can be continued analytically to $\mathbb{D}$ as a function $F_a \in K_\theta^1 \cap zH^1$ with $\|F_a\|_{L^1(\mathbb{T})} \lesssim 1$. Since $a \in L^2(\sigma_\alpha)$, we have $F_a \in K_\theta^2 \cap zH^2$ by Clark theorem. Now it follows from (34) and (35) that $\|b\|_{\sigma_\alpha} \lesssim \|F_b\|$. Conversely, take a function $b \in \text{BMO}(\sigma_\alpha)$ and consider the functional $\Phi_b$ densely defined on $K_\theta^1 \cap zH^1$ by formula (33). For every $\sigma_\alpha$-atom $a$ supported on an arc $\Delta$ we have
\[ \left| \int_T ab \, d\sigma_\alpha \right| = \left| \int_{\Delta} a(b - \langle b \rangle_{\Delta, \sigma_\alpha}) \, d\sigma_\alpha \right| \leq \frac{1}{\sigma_\alpha(\Delta)} \int_{\Delta} \|b - \langle b \rangle_{\Delta, \sigma_\alpha}\| \, d\sigma_\alpha. \] (36)

This shows that the functional $f \mapsto \int_T fb \, d\sigma_\alpha$ is continuous on $H^1_{\sigma_\alpha}(\sigma_\alpha)$. By Theorem 24 the restriction of every function $F \in K_\theta^1 \cap zH^1$ to $a(\sigma_\alpha)$ belongs to $H^1_{\sigma_\alpha}(\sigma_\alpha)$ and $\|F\|_{H^1_{\sigma_\alpha}(\sigma_\alpha)} \lesssim \|F\|_{L^1(\mathbb{T})}$. Hence the functional $\Phi_b$ is continuous on $K_\theta^1 \cap zH^1$ and we see from (36) that $\|\Phi_b\| \lesssim \|b\|_{\sigma_\alpha}$. □

4. Truncated Hankel and Toeplitz operators

Let $\theta$ be an inner function. Denote by $P_\theta$ the orthogonal projection in $L^2(\mathbb{T})$ to the subspace $K_\theta^2$. The truncated Toeplitz operator $A_\psi : K_\theta^2 \to K_\theta^2$ with symbol $\psi \in L^2(\mathbb{T})$ is densely defined by
\[ A_\psi : f \mapsto P_\theta(\psi f), \quad f \in K_\theta^\infty. \]

Truncated Toeplitz and Hankel operators are closely related. Indeed, the antilinear isometry $g \mapsto \overline{\theta g}$ on $L^2(\mathbb{T})$ preserves the subspace $K_\theta^2$ and for every $f, g \in K_\theta^\infty$ we have
\[ (A_\psi f, g) = (\psi f, g) = (\Gamma_{\theta \psi} f, \overline{g_1}), \quad g_1 = \overline{\theta g}. \] (37)

This shows that the operators $A_\psi, \Gamma_{\theta \psi}$ are bounded (compact, of trace class, etc.) or not simultaneously and $\|A_\psi\| = \|\Gamma_{\theta \psi}\|$. Below we briefly discuss some results related to the boundedness problem for truncated Toeplitz operators.

We will say that the truncated Toeplitz operator $A_\psi$ has a bounded symbol $\psi_1$ if $A_\psi = A_{\psi_1}$ for a function $\psi_1 \in L^\infty(\mathbb{T})$. It can be shown all symbols of the zero truncated Toeplitz operator on $K_\theta^2$ have the form $\theta g_1 + \theta g_2$, where $g_1, g_2 \in H^2$, see 25. Hence the operator $A_\psi : K_\theta^2 \to K_\theta^2$ has a bounded symbol if and only if the set $\psi + \overline{\theta H^2} + \theta H^2$ contains a bounded function on $\mathbb{T}$. Clearly, every truncated Toeplitz operator with bounded symbol is bounded. The following question arises: does every bounded truncated Toeplitz operator have a bounded symbol?
4.1. **Analytic symbols.** In 1967, D. Sarason [24] described the commutant \( \{ S_θ \}' \) of the restricted shift operator \( S_θ : f \mapsto P_θ(zf) \) on \( K_θ^2 \). He proved that a bounded operator \( A \) on \( K_θ^2 \) commutes with \( S_θ \) if and only if there exists a function \( ψ \in H^∞ \) such that \( A = A_ψ \). Moreover, we have \( \| A_ψ \| = \text{dist}_{H^∞}(ψ, θH^∞) \) and one can choose the function \( ψ \) so that \( \| A \| = \| ψ \|_{H^∞} \). This well-known theorem yields a boundedness criterion for truncated Toeplitz operators with analytic symbols. Indeed, for every \( ψ \in H^2 \) and \( f \in K_θ^2 \) we have \( A_ψ S_θ f = S_θ A_ψ f \). Hence the operator \( A_ψ \) is bounded if and only if \( A_ψ \in \{ S_θ \}' \) which is equivalent to the existence of a function \( ψ_1 \in H^∞ \) such that \( A_ψ = A_ψ_1 \) (in other words, we have \( ψ + θh \in H^∞ \) for some \( h \in H^2 \)). The equality \( \| A_ψ \| = \text{dist}_{H^∞}(ψ, θH^∞) \) for \( ψ \in H^∞ \) leads to a short proof for the Nevanlinna-Pick interpolation theorem and its generalization, see [24].

It was observed by N. K. Nikolskii that many problems for truncated Toeplitz operators with analytic symbols can be easily reduced to the problems for usual Hankel operators on \( H^2 \). The reduction is based on the fact that for every \( ψ \in H^2 \) the operator \( θA_ψ P_θ \) from \( H^2 \) to \( \overline{θH^2} \) coincides with the Hankel operator \( H_θ ψ \). In particular, the operator \( A_ψ \) is bounded (compact, of trace class, etc.) if and only if so is the operator \( H_θ ψ \). Since Hankel operators on \( H^2 \) are well studied this observation immediately yields consequences for truncated Toeplitz operators. As an example, the operator \( A_ψ \) on \( K_θ^2 \) with symbol \( ψ \in H^2 \) is compact if and only if \( θψ \in C(T) + H^2 \), where \( C(T) \) denotes the algebra of continuous functions on the unit circle \( T \). For more information see Lecture 8 in [19] and Section 1.2 in [20].

4.2. **General symbols.** Until recently, a little was known about truncated Toeplitz operators with general symbols in \( L^2(T) \). For such operators the boundedness problem is more complicated.

In 1987, R. Rochberg [23] proved that every bounded Toeplitz operator on the Paley-Wiener space \( \text{PW}^2_{[−a, a]} \) has a bounded symbol. Using the Fourier transform, he reduced the general case of the problem to consideration of the Toeplitz operators on \( \text{PW}^2_{[0, a]} \) with analytic symbols. Recently, M. Carlsson [11] use a result from [24] to prove the boundedness criterion for Toeplitz and Hankel operators on \( \text{PW}^2_{[−a, a]} \) in terms of \( \text{BMO}(\overline{Z}) \), see Section [1].

Every finite Toeplitz matrix \( A \) clearly have bounded symbols. However, the question concerning the best possible constant \( c_A \) in the inequality

\[
\inf \{ \| ψ \|_{L^∞(T)} : A_ψ = A \} \leq c_A \cdot \| A \|
\]

is nontrivial. In 2001, M. Bakonyi and D. Timotin proved that \( c_A \leq 2 \) for every self-adjoint finite Toeplitz matrix \( A \). As a corollary, we have \( c_A \leq 4 \) for a general finite Toeplitz matrix \( A \) that was improved to \( c_A \leq 3 \) by L. N. Nikolskaya and Yu. B. Farforovskaya [15] in 2003. Next, in 2007 D. Sarason [25] compute \( c_A = \pi/2 \) for \( A = (\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}) \) and proved that \( c_A \leq \pi/2 \) for every \( 2 \times 2 \) self-adjoint Toeplitz matrix \( A \). In paper [27] A. L. Volberg discuss several approaches to the dual version of the problem of determining \( \sup_A c_A \) over all finite Toeplitz matrices \( A \), which can be formulated in terms of weak factorizations of analytic polynomials.

In 2010, A. D. Baranov, I. Chalendar, E. Fricain, J. Mashreghi, and D. Timotin [6] constructed an inner function \( θ \) and a bounded truncated Toeplitz operator \( A \) on \( K_θ^2 \) that has no bounded symbols. Shortly after that in [6] appeared a
description of coinvariant subspaces $K^2_\theta$ on which every bounded truncated Toeplitz operator has a bounded symbol. The proof in [5] is based on a duality relation between the space of all bounded truncated Toeplitz operators on $K^2_\theta$ and a special function space. With help of formula (37) it is easy to reformulate the results of [5] for truncated Hankel operators. We do this below as a preparation to the proof of Theorem 3.

### 4.3. Duality for truncated Hankel operators

Let $\theta$ be an inner function. Consider the linear space

$$Y_\theta = \left\{ \sum_{k=0}^{\infty} x_k y_k, \ x_k \in K^2_\theta, \ y_k \in z K^2_{\bar{\theta}}, \ \sum_{k=0}^{\infty} \|x_k\|_{L^2(\mathbb{T})} \|y_k\|_{L^2(\mathbb{T})} < \infty \right\}.$$  

As is easy to see, we have $Y_\theta \subset K^1_{\theta z} \cap z H^1$. Define the norm in $Y_\theta$ by

$$\|h\|_{Y_\theta} = \inf \left\{ \sum_{k=0}^{\infty} \|x_k\|_{L^2(\mathbb{T})} \|y_k\|_{L^2(\mathbb{T})} : \ h = \sum_{k=0}^{\infty} x_k y_k, \ x_k \in K^2_\theta, \ y_k \in z K^2_{\bar{\theta}} \right\}.$$  

With this norm $Y_\theta$ is a Banach space. Denote by $H_\theta$ the linear space of all bounded truncated Hankel operators acting from $K^2_\theta$ to $z K^2_{\bar{\theta}}$. It follows from Theorem 4.2 in [25] that $H_\theta$ is closed in the weak operator topology. Hence $H_\theta$ is the Banach space under the standard operator norm and moreover it has a predual space. It follows from Theorem 2.3 of [5] that $Y_\theta^* = H_\theta$. That is, for every continuous linear functional $\Psi$ on $Y_\theta$ there exists the unique operator $\Gamma \in H_\theta$ such that $\Psi = \Psi_\Gamma$, where

$$\Psi_\Gamma : h \mapsto \sum_{k=0}^{\infty} (\Gamma x_k, y_k), \quad h \in Y_\theta, \quad h = \sum_{k=0}^{\infty} x_k y_k.$$  

(38)

Conversely, for every operator $\Gamma \in H_\theta$ the mapping $\Psi_\Gamma$ is the correctly defined continuous linear functional on the space $Y_\theta$ and we have $\|\Psi_\Gamma\| = \|\Gamma\|$.

With help of the equality $Y_\theta^* = H_\theta$ the boundedness problem for truncated Hankel operators can be reformulated in terms of function theory. Indeed, now it is easy to see from Hahn-Banach theorem that every bounded truncated Hankel operator on $K^2_\theta$ has a bounded symbol if and only if $Y_\theta$ is a closed subspace of $L^1(\mathbb{T})$, in which case $Y_\theta$ coincides with $K^1_{\theta z} \cap z H^1$ as set, see details in [5]. Note that if $Y_\theta = K^1_{\theta z} \cap z H^1$ as set, then the the norms $\|\cdot\|_{Y_\theta}$ and $\|\cdot\|_{L^1(\mathbb{T})}$ are equivalent on $Y_\theta$. It was proved in [5] that $Y_\theta = K^1_{\theta z} \cap z H^1$ for every one-component inner function $\theta$.

Thus, we see from the results of [5] and Theorem 2 that for every one-component inner function $\theta$ we have

$$Y_\theta^* = H_\theta, \quad Y_\theta = K^1_{\theta z} \cap z H^1, \quad (K^1_{\theta z} \cap z H^1)^* = \text{BMO}(\nu_\alpha),$$

where $\nu_\alpha$ is the Clark measure of the inner function $\theta^2$. It remains to combine this relations to obtain Theorem 3.

### 4.4. Proof of Theorem 3

Let $\theta$ be a one-component inner function and let $\Gamma_\theta$ be a truncated Hankel operator on $K^2_\theta$ with standard symbol $\varphi \in K^2_{\theta z} \cap z H^2$; we do not
assume now that the operator $\Gamma_\varphi$ is bounded. For every function $h = \sum_{k=0}^\infty x_k y_k$ in $Y_\theta \cap L^\infty(\mathbb{T})$ we have

$$\Psi_{\Gamma_\varphi}(h) = \sum_{k=0}^\infty (\Gamma_\varphi x_k, y_k) = \int_{\mathbb{T}} \varphi \sum_{k=0}^\infty x_k y_k \, dm = \int_{\mathbb{T}} h \varphi \, dm = \int_{\mathbb{T}} h \varphi \, d\nu_\alpha,$$

(39)

where the last equality follows from Clark theorem for the inner function $\theta^2$. We see that $\Psi_{\Gamma_\varphi}$ coincides on $Y_\theta \cap L^\infty(\mathbb{T})$ with the functional

$$\Phi_\varphi : h \mapsto \int_\mathbb{T} h \varphi \, d\nu_\alpha, \quad h \in K^1_{\theta^2} \cap zH^1.$$

Since the inner function $\theta^2$ is one-component the Banach spaces $Y_\theta$ and $K^1_{\theta^2} \cap zH^1$ coincide as sets and their norms are equivalent. It follows that the densely defined functionals $\Psi_{\Gamma_\varphi} : Y_\theta \rightarrow \mathbb{C}$ and $\Phi_\varphi : K^1_{\theta^2} \cap zH^1 \rightarrow \mathbb{C}$ are bounded or not simultaneously and $\|\Psi_{\Gamma_\varphi}\| \asymp \|\Phi_\varphi\|$, where the constants involved depend only on $\theta$. By Theorem 2 for the inner function $\theta^2$ the functional $\Phi_\varphi$ is bounded if and only if $\varphi \in \text{BMO}(\nu_\alpha)$, and in the latter case we have $\|\Phi_\varphi\| \asymp \|\varphi\|_{\nu_\alpha}$

Now result follows from the equality $\|\Gamma_\varphi\| = \|\Psi_{\Gamma_\varphi}\|$. □

4.5. Compact truncated Hankel operators. Let $\mu$ be a measure on $\mathbb{T}$ with properties $(a) - (c)$. For every $b \in \text{BMO}(\mu)$ define

$$M_\epsilon(b) = \sup \left\{ \frac{1}{\mu(\Delta)} \int_{\Delta} |b - \langle b \rangle_{\Delta, \mu}| \, d\mu, \quad \Delta \text{ is an arc of } \mathbb{T} \text{ with } 0 < \mu(\Delta) \leq \epsilon \right\}.$$  

Consider the space \text{VMO}(\mu) = $\{b \in \text{BMO}(\mu) : \lim_{\epsilon \to 0} M_\epsilon(b) = 0\}$ of functions of vanishing mean oscillation with respect to the measure $\mu$. It can be shown that VMO($\mu$) is the closure in BMO($\mu$) of the set of all finitely supported sequences.

**Proposition 4.1.** Let $\theta$ be a one-component inner function, and let $\nu_\alpha$ be the Clark measure of the inner function $\theta^2$. The truncated Hankel operator $\Gamma_\varphi : K^1_{\theta^2} \rightarrow zK^2_{\theta}$ with standard symbol $\varphi$ is compact if and only if $\varphi \in \text{VMO}(\nu_\alpha)$.

**Proof.** It follows from Theorem 2.3 of [5] that $(H_\theta \cap S_\infty)^* = Y_\theta$, where $S_\infty$ denotes the ideal of all compact operators acting from $K^1_{\theta^2}$ to $zK^2_{\theta}$. Hence a bounded truncated Hankel operator $\Gamma$ on $K^1_{\theta^2}$ is compact if and only if the functional $\Psi_{\Gamma}$ in (38) is continuous in the weak* topology on $Y_\theta$. Let $\varphi$ be the standard symbol of the operator $\Gamma_\varphi$. By Corollary 2.5 in [5] and Theorem 2 we have

$$Y_\theta = K^1_{\theta^2} \cap zH^1, \quad V_\alpha(K^1_{\theta^2} \cap zH^1) = H^1_{\text{af}}(\nu_\alpha).$$

From formula (38) we see that the operator $\Gamma_\varphi$ is compact if and only if the restriction of $\varphi$ to $a(\nu_\alpha)$ generates the weak* continuous functional $\Phi_\varphi : f \mapsto \int f \varphi \, d\nu_\alpha$ on the space $H^1_{\text{af}}(\nu_\alpha)$. For any doubling measure $\mu$ we have VMO($\mu$)* = $H^1_{\text{af}}(\mu)$, see Theorem 4.1 in [14]. It follows that $\Gamma_\varphi \in S_\infty$ if and only if $\varphi \in \text{VMO}(\nu_\alpha)$. □

4.6. Functions in $K^2_{\theta}$ of bounded mean oscillation. Theorem 3 provides the following description of functions in $K^2_{\theta} \cap \text{BMO}(\mathbb{T})$.

**Proposition 4.2.** Let $\theta$ be a one-component inner function and let $\varphi \in K^2_{\theta}$. Then we have $\varphi \in K^2_{\theta} \cap \text{BMO}(\mathbb{T})$ if and only if $\varphi \in \text{BMO}(\nu_\alpha)$, where $\nu_\alpha$ is the Clark measure of the inner function $\theta^2$.  


Proof. A function \( \varphi \in H^2 \) belongs to the space \( \text{BMO}(\mathbb{T}) \) if and only if the Hankel operator \( H_{\varphi} : H^2 \to \mathbb{C} \) is bounded, see Theorem 1.2 in Chapter 1 of [20]. Assume that \( \varphi \in K_\theta^2 \) and consider the truncated Hankel operator \( \Gamma_{\varphi} : K_\theta^2 \to zK_\theta^2 \). For every function \( f \in K_\theta^\infty \) we have \( \varphi f \in \theta H^2 \). Hence,

\[
H_{\varphi} f = P_\theta(\varphi f) = P_\theta(\delta f) = \Gamma_{\varphi} f, \quad f \in K_\theta^\infty.
\]

Also, \( H_{\varphi} f = 0 \) for all \( f \in \theta H^\infty \). Therefore the operators \( H_{\varphi} \) and \( \Gamma_{\varphi} \) are bounded or not simultaneously and \( \|H_{\varphi}\| = \|\Gamma_{\varphi}\| \). Now the result follows from Theorem 3. \( \square \)

4.7. Finite Hankel and Toeplitz matrices. Let \( \Gamma = (\gamma_{j+k})_{0 \leq j, k \leq n-1} \) be a Hankel matrix of size \( n \times n \). Associate with \( \Gamma \) the antianalytic polynomial

\[
\varphi = \gamma_0 z + \gamma_1 z^2 + \ldots + \gamma_{n-2} z^{2n-1}.
\]

For the inner function \( \theta_n = z^n \) the space \( K_\theta^2 \) consists of analytic polynomials of degree at most \( n - 1 \). Consider the truncated Hankel operator \( \Gamma_{\varphi} : K_\theta^2 \to zK_\theta^2 \),

\[
\Gamma_{\varphi} : f \mapsto P_{\theta_n}(\varphi f), \quad f \in K_\theta^2.
\]

We have \( (\Gamma_{z^j} z^k) = \gamma_{j+k} \) for every \( 0 \leq j, k \leq n-1 \). It follows that the matrix \( \Gamma \) as the operator on \( C^n \) is unitary equivalent to the operator \( \Gamma_{\varphi} \). Analogously, the Toeplitz matrix \( A = (\alpha_{j+k})_{0 \leq j, k \leq n-1} \) is unitarily equivalent to the truncated Toeplitz operator \( A_{\psi} : K_\theta^2 \to zK_\theta^2 \) with symbol

\[
\psi = \alpha_{-(n-1)} z^{n-1} + \ldots + \alpha_{n-1} z^{n-1}.
\]

If moreover \( \alpha_m = \gamma_{(n-1)-m} \) for all \( m \in \mathbb{Z} \) with \( |m| \leq n-1 \), then \( \varphi = \theta_n \psi \) and we have \( \|\Gamma\| = \|A\| \) by formula (37). Consider the measure

\[
\mu_{2n} = \frac{1}{2n} \sum \delta_{z^{2n}}
\]

equally distributed at the roots of identity of order \( 2n \): sup \( \mu = \{ \xi \in \mathbb{T} : \xi^{2n} = 1 \} \).

Let \( c_{1,n} \), \( c_{2,n} \) be the best possible constants in the inequality

\[
c_{1,n} \|\varphi\|_{L_1^\infty} \leq \|\Gamma_{\varphi}\| \leq c_{2,n} \|\varphi\|_{L_1^\infty},
\]

where \( \Gamma_{\varphi} \) runs over all truncated Hankel operators on \( K_\theta^2 \), \( \varphi \) is the standard symbol of \( \Gamma_{\varphi} \). Corollary 1 of Theorem 3 claims that the sequences \( \{c_{1,n}\}_{n \geq 1} \) and \( \{c_{2,n}\}_{n \geq 1} \) are bounded. We prove this below.

Proof of Corollary 1 We may assume that \( n \geq 2 \). It follows from Lemma 2.3 that \( \mu_{2n} \) is the Clark measure \( \nu_1 \) of the inner function \( \theta_n^2 = z^{2n} \). This allows us to estimate the constants in formula (10) using the proofs of Theorem 2 and Theorem 3. Denote

\[
d_1' = \sup \{ \|h\|_{L_1^\infty(\mathbb{T})}, \ h \in K_\theta^2 \cap zH^1, \ \|h\|_{L_1^\infty(\mu_{2n})} \leq 1 \};
\]

\[
n_2' = \sup \{ \|h\|_{V_{\theta_n}}, \ h \in K_\theta^2 \cap zH^1, \ \|h\|_{L_1^\infty(\mathbb{T})} \leq 1 \}.
\]

Let \( \Gamma_{\varphi} : K_\theta^2 \to zK_\theta^2 \) be a truncated Hankel operator with standard symbol \( \varphi \). Consider the functional \( \Psi : h \mapsto \int_{\mathbb{T}} h \varphi \ d\mu_{2n} \) on the Banach space \( V_{\theta_n} \). From formula (35) and the equality \( \|\Psi\| = \|\Gamma_{\varphi}\| \) (see Section 4.3) we obtain

\[
\|\varphi\|_{L_1^\infty} \leq 2 \sup \{ |\Psi(h)|, \ h \in V_{\theta_n}, \ \|h\|_{L_1^\infty(\mu_{2n})} \leq 1 \} \leq 2d_1' \sup \{ |\Psi(h)|, \ h \in V_{\theta_n}, \ \|h\|_{L_1^\infty(\mathbb{T})} \leq 1 \} \leq 2d_1' d_2' \|\Gamma_{\varphi}\|.
\]
Hence, \(c_{1,n}^{-1} \leq 2d'_n \cdot d''_n\). It follows from the results of Nikol'skaya and Farforovskaya [13] that \(d''_n \leq 3\), see also Section 1.2 in [27]. To estimate the constant \(d'_n\) assume that the restriction of \(f \in K_{\theta_{\mu_2}}^1 \cap zH^1\) to \(a(\mu_{2n})\) is a \(\mu_{2n}\)-atom supported on a closed arc \(\Delta\) of the unit circle \(\mathbb{T}\) with center \(\xi_c\) and endpoints in \(a(\mu_{2n})\). Let \(\{\nu_{\beta}^n\}_{n=1}^\infty\) be the family of the Clark measures of the inner function \(\theta_{\mu_2};\) we have \(\nu_1^n = \mu_{2n}\). Combining formulas (24) and (25) in the proof of Theorem 2, we obtain

\[
\int_{\mathbb{T}\setminus\{\xi_0\}} \frac{d\mu_{2n}(\xi)}{|\xi - \xi_0|^2} \leq 2\pi \int_{\mathbb{T}\setminus\Delta} \frac{d\mu_{2n}(\xi)}{|\xi - \xi_0|^2} + \frac{64}{\mu_{2n}|\xi_0^2|/2}.
\]

Observe that \(\nu_1^n(\Delta) \geq m(\Delta)\) and \(\nu_1^n(2\Delta) \leq 3m(\Delta)\). Let \(\xi_1, \xi_2\) be the nearest points to \(\xi_c\) in \(a(\nu_{\beta}^n \setminus 2\Delta)\). Then \(|\xi_c - \xi_{1,2}| \geq \text{diam}(2\Delta)/2 \geq m(\Delta)\) and we have

\[
\int_{\mathbb{T}\setminus\Delta} \frac{d\nu_1^n(z)}{|z - \xi_c|^2} \leq \int_{\mathbb{T}\setminus\Delta} \frac{dm(z)}{|z - \xi_c|^2} + \frac{1}{2n} \left( \frac{1}{|\xi_c - \xi_1|^2} + \frac{1}{|\xi_c - \xi_2|^2} \right).
\]

Hence, \(\|f\|_{L^1(\mathbb{T})} \leq \sqrt{\pi} + 4\pi < 15\). This gives us \(d'_n < 15\) and \(c_{1,n}^{-1} < 90\).

Let us turn to the second inequality in (40). As before, from formula (36) we obtain

\[
\|G_n\| = \sup \{|\Psi(h)|, \|h\|_{Y_{n-1}} \leq 1\} \leq D''_n \sup \{|\Psi(h)|, \|h\|_{L^1(\mathbb{T})} \leq 1\} \leq D'_n D''_n \|\varphi\|_{\mu_{2n}},
\]

where

\[
D'_n = \sup \{|h|_{H^{1}(\mu_{2n})}, h \in K_{\theta_{\mu_2}}^1 \cap zH^1, \|h\|_{L^1(\mathbb{T})} \leq 1\};
\]

\[
D''_n = \sup \{|h|_{Y_{n-1}}, h \in K_{\theta_{\mu_2}}^1 \cap zH^1, \|h\|_{Y_{n-1}} \leq 1\}.
\]

By the Cauchy-Schwarz inequality, \(D''_n \leq 1\). In the proof of Theorem 2 we have seen that \(D'_n \leq 24c_5 c_0^5M\), where \(M\) is the norm of the non-tangential maximal operator \(F \mapsto F^*\) on \(H^1\) and \(c_5, c_0\) from Lemma 3.3 for the inner function \(\theta = \theta_n\). Since \(A_{\mu_{2n}} = B_{\mu_{2n}} = 1\), we have \(D'_n \leq 24 \cdot 4 \cdot 60 \cdot M \cdot \varepsilon_n^{-1}\), where \(\varepsilon_n\) stands for the parameter \(\varepsilon\) in Lemma 3.2 for \(\theta = \theta_n\). Next, since the sublevel set \(\Omega_3\) of \(\theta_n\) is connected for every \(\delta > 0\), the proof of Lemma 3.2 shows that one can take \(\varepsilon_n = \kappa_n/2\), where \(\kappa_n \leq \kappa_n^* = (2B_{\mu_{2n}})^{-1} = 1/2\) is chosen so that estimate (13) holds for \(\theta = \theta_n^2\), \(\kappa = \kappa_n\). It remains to show that \(\inf_n \kappa_n > 0\). For this aim it is sufficient to prove that the functions \(f_{\xi_0,n} = f_{\xi_0}\) in formula (17) for \(\theta = \theta_n\) are bounded uniformly in \(n\). By formula (13), \(C_{\mu_{2n}} \leq 1/2\). Next, for every pair of atoms \(\xi, \xi_0 \in a(\mu_{2n})\) and for all \(z \in D_{\xi_0}(\kappa_n)\) we have \(|\xi - z| \geq |\xi - \xi_0|/2\).

Since \(\inf_n A_{\mu_{2n}} = A_{\mu_2} = \frac{1}{4\pi^2}\) and \(B_{\mu_{2n}} \leq B_{\mu_{2n}} = 1\) we see that estimate (16) for \(\sigma_n = \mu_{2n}\) takes the following form:

\[
\int_{\mathbb{T}\setminus\{\xi_0\}} \frac{d\mu_{2n}(\xi)}{|\xi - \xi_0|^2} \leq 2\pi \int_{\mathbb{T}\setminus\Delta} \frac{d\mu_{2n}(\xi)}{|\xi - \xi_0|^2} + \frac{64}{\mu_{2n}|\xi_0^2|/2}.
\]

\[
\leq \left( \frac{64 + \pi^2}{4} \right) \frac{1}{\mu_{2n}|\xi_0^2|/2}.
\]
It follows that $|f_\delta| < 70$ on $D_\delta$ and estimate (14) holds for $\theta = \theta_n$ with any constant $\kappa_n \leq \kappa_n^*$ such that $70 \leq (2\kappa_n)^{-1}$. In particular, one can take $\kappa_n = 1/140$ for all $n \geq 2$. We now see that the constants $c_{2,n}$ are bounded: $c_{2,n} \leq 24 \cdot 4 \cdot 60 \cdot 280 \cdot M < 10^7 M$.

**Corollary 2.** Let $A = (\alpha_{j-k})_{0 \leq k, j \leq n-1}$ be a Toeplitz matrix of size $n \times n$; consider its standard symbol $\psi = \alpha_{-(n-1)} z^{n-1} + \ldots + \alpha_{-1} z^{n-1}$. We have

$$c_1 \|z^n \psi\|_{\mu^2_n} \leq \|A\| \leq c_2 \|z^n \psi\|_{\mu^2_n},$$

where the constants $c_1, c_2$ do not depend on $n$.

The author failed to find a simple argument allowing obtain Corollary [1] from the BMO-criterion for the boundedness of Hankel operators on $H^2$. The inverse implication is quite elementary.

**Proposition 4.3.** Let $\varphi \in \overline{zH^2}$. The Hankel operator $H_\varphi : H^2 \to \overline{zH^2}$ is bounded if and only if $\varphi \in \text{BMO}(\mathbb{T})$. Moreover we have $c_1 \|\varphi\|_* \leq \|H_\varphi\| \leq c_2 \|\varphi\|_*$ with constants $c_1, c_2$ from Corollary [1].

**Proof.** Let $H_\varphi : H^2 \to \overline{zH^2}$ be a bounded Hankel operator on $H^2$ with symbol $\varphi \in \overline{zH^2}$. Then there are finite-rank Hankel operators $H_{\varphi_n}, \varphi_n \in K^2_{\theta_n} \cap zH^2$, such that $H_\varphi$ is the limit of $H_{\varphi_n}$ in the weak* operator topology. Moreover one can choose $H_{\varphi_n}$ so that $\sup_n \|H_{\varphi_n}\| \leq \|H_\varphi\|$. For every $n \geq 1$ and $k \geq n$ the operator norm of the Hankel operator $H_{\varphi_n}$ is equal to the operator norm of the truncated Hankel operator on $K^2_{\theta_n}$ with symbol $\varphi_n$, where $\theta_k = z^k$. Since $\|\varphi_n\|_* = \lim_{k \to \infty} \|\varphi_n\|_{\nu^2_k}$ we see from Corollary [1] that

$$c_1 \|\varphi\|_* \leq \|H_{\varphi_n}\| \leq c_2 \|\varphi\|_*,$$

It follows that $c_1 \sup_n \|\varphi_n\|_* \leq \|H_\varphi\|$. Since $H_{\varphi_n}$ tend to $H_\varphi$ in the weak* operator topology we have $\lim_{n \to \infty} \int_T p \varphi_n dm = \int_T p \varphi dm$ for every trigonometric polynomial $p$. It is well-known that $H^2_{\text{at}}(\mathbb{T})^* = \text{BMO}(\mathbb{T})$ (it worth be mentioned that this fact is much more easier than the Fefferman theorem on $\text{Re}(zH^1)^* = \text{BMO}(\mathbb{T})$ which is generally used in the proof of the boundedness criterion for Hankel operators). Since trigonometric polynomials are dense in $\text{BMO}(\mathbb{T})$ in the weak* topology generated by $H^2_{\text{at}}(\mathbb{T})$, we have $\varphi \in \text{BMO}(\mathbb{T})$ and $c_1 \|\varphi\|_* \leq \|H_\varphi\|$. Now let $\varphi \in \overline{zH^2} \cap \text{BMO}(\mathbb{T})$. Then there are functions $\varphi_n \in K^2_{\theta_n} \cap zH^2$ which tend to $\varphi$ in the weak* topology of $\text{BMO}(\mathbb{T})$ and such that $\sup_n \|\varphi_n\|_* \leq \|\varphi\|_*$. From (45) we see that $\|H_{\varphi_n}\| \leq c_2 \|\varphi\|_*$ for the corresponding Hankel operators $H_{\varphi_n}$. Since $L^2(\mathbb{T}) \subset H^2_{\text{at}}(\mathbb{T})$ the functions $\varphi_n$ converge to $\varphi$ weakly in $L^2(\mathbb{T})$. Hence for every pair of analytic polynomials $p_1, p_2$ we have $\lim_{n \to \infty} (H_{\varphi_n} p_1, \overline{p_2}) = (H_\varphi p_1, \overline{p_2})$. It follows that the operators $H_{\varphi_n}$ converge to the operator $H_\varphi$ in the weak operator topology and we have $\|H_\varphi\| \leq c_2 \|\varphi\|_*$. 

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