ON A PROBLEM OF
PIATETSKI-SHAPIRO AND SHAFAREVICH

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INTRODUCTION

The aim of this note is to revisit classical articles on the algebraic approach to uniformization and automorphic functions by Piatetski-Shapiro and Shafarevich. See [S] (with comments and additional references), [P], and [Pa-S, Sect. 4.1]; similar ideas were developed by Shimura independently of [S].

In classical geometric theory of automorphic forms, one considers a bounded symmetric domain $V \subset \mathbb{C}^n$ and study automorphic forms with respect to a discrete subgroup $\Gamma \subset Aut(V)$ in the group of all complex analytic automorphisms of $V$. Together with Siegel [Si, vol. III, Sect. 6.1, 6.2], one can ask about the scope of the classical theory.

From the geometric point of view, the short answer is a classification of the least complicated varieties of general type, namely, nonsingular projective algebraic varieties of general type with large and residually finite fundamental group as well as their noncompact counterparts.

Throughout the note, the ground field $k = \mathbb{C}$. After preliminaries, we consider several important examples.

In the last section, we prove our main theorem that provides a partial solution of a problem proposed by by Piatetski-Shapiro and Shafarevich in [S, Introduction], [P], and [Pa-S, p. 83], namely, whether the existence of proalgebraic quasi-homogeneous coverings of general type is the characteristic property of algebraic varieties whose universal coverings are bounded symmetric domains.

1. Preliminaries

1.1. All proalgebraic varieties will be assumed finite-dimensional, irreducible, and normal. Recall that a proalgebraic variety can be represented as a projective limit

$$Y = \varprojlim X_\alpha$$

of a countable set of algebraic varieties $X_\alpha$, where $f_{\alpha,\beta} : X_\alpha \to X_\beta$ are integral morphisms [S, Sect. 4, Proposition 2]. A point $x \in Y$ is called interior if the corresponding maximal ideal $m_x$ has a finite number of generators or, equivalently, there exists an index $\gamma$ such that all morphisms $X_\alpha \to X_\gamma$ are unramified at $x_\gamma$, where $x_\gamma$ is the projection of $x$ to $X_\gamma$ [S, Sect. 4, Definition 4, Proposition 3]. We also call a point $x_\alpha \in X_\alpha$ interior if the point $x \in Y$ lying over it is interior.

Unless stated otherwise, throughout the note $Y$ is not an algebraic variety.
1.2. We assume the set of interior points of $Y$, denoted by $U_Y$, is not empty, nonsingular, and $\pi_1^{alg}(U_Y) = \{1\}$. Then there exists an index $\gamma$ such that for $\alpha \geq \gamma$ all interior points of $X_\alpha$, denoted by $U_\alpha$, are nonsingular. In particular, $U_\alpha$ is a complex manifold.

In the sequel, we also assume $\pi_1(U_\alpha)$ is residually finite for every index $\alpha$, and large [Kol]. The scheme $U_Y$ has another topology, so-called pro-etale topology, so that the projections $U_Y \rightarrow U_\alpha$ are continuous in the etale topology on $U_\alpha$. Let $\tilde{U}$ denote the universal topological covering of $U_\alpha$. Then $\tilde{U}$ is also a complex manifold.

We get a natural topological embedding $\tilde{U} \hookrightarrow U_Y$, where $U_Y$ is equipped with its pro-etale topology. Our $\tilde{U}$ is also equipped with a Zariski-type topology induced by the Zariski topology on $U_Y$.

1.3. Let $K$ be a field of finite transcendence degree $n$ over $k$. We denote by $D^\ell(K)$ the set of all regular $n$-differentials of $K$ of weight $\ell$ [S, Sect. 5]. Let $D = \sum D^\ell(K)$ be the graded algebra of differentials. If $D$ is irreducible and integrally closed, and all its elements are integral over a certain homogeneous subalgebra of finite type, then $Y = \text{Proj} D$ is a projective proalgebraic variety [S, Sect. 4, Definition 3].

Let $k(Y)$ denote the field of rational function on $Y$. Clearly always $k(Y) \subseteq K$. Let $K_\alpha$ denote the canonical bundle on $U_\alpha \subset X_\alpha$, for a cofinal system of indexes $\alpha$.

Definition 1.3.1. The field $K$ is called a field of general type (in the sense of Piatetski-Shapiro and Shafarevich) if $k(Y) = K$ and there is an integer $m > 0$ such that each $K^m_\alpha$ and its global sections define an embedding in the corresponding finite-dimensional projective space.

2. Examples

Example 1. Let $V \subset \mathbb{C}^n$ be a bounded domain, and $\Gamma$ be a fixed point free discrete subgroup of $\text{Aut}(V)$ such that $V/\Gamma$ is a compact complex manifold. Let $k(\Gamma) = \bigcup \Delta_i$, where $\Delta_i$ runs over subgroups of finite index in $\Gamma$ ($\bigcap \Delta_i = 1$). Employing Poincaré series, it was shown that each $V/\Delta_i$ is a nonsingular projective variety with ample canonical bundle, and it is the absolute minimal model (Shioda) of its field of rational functions. Therefore

$$\lim \frac{V_\alpha}{\Delta_i} \simeq \text{Proj} D,$$

and all points of $\text{Proj} D$ are interior.

Example 2(i). Let $V \subset \mathbb{C}^n$ be a bounded symmetric domain and $\Gamma \subset \text{Aut}(V)$ be a fixed point free arithmetic subgroup in the group of all complex analytic automorphisms of $V$. Let $k(\Gamma) = \bigcup \Delta_\alpha$, where $\Delta_\alpha$ runs over neat subgroups of finite index in $\Gamma$ ($\bigcap \Delta_\alpha = 1$). Then each $V/\Delta_\alpha$ is a nonsingular subvariety of its Baily-Borel compactification (see, e.g., [M, Prop 3.4]).

According to Tai, each $V/\Delta_\alpha$ is a variety of general type provided $\Delta_\alpha$ is sufficiently small. The field $k(\Gamma)$ has abundance of regular $n$-differentials. Namely, the regular $n$-differentials on $V/\Delta_\alpha$ that have at worst logarithmic poles along the boundary of $V/\Delta_\alpha$ generate the homogeneous coordinate ring of the Baily-Borel compactification $\overline{V/\Delta_\alpha}$ [M, Prop 4.2]. Further, it follows from Mumford’s proof of Tai’s theorem that these differentials are regular differentials of $k(\Gamma)$ because each $\overline{V/\Delta_\alpha}$ has a tower of coverings universally ramified over its boundary (for details, see [M, Sect. 4]).
Example 2(ii). Let \( V \subset \mathbb{C}^{3g-3}, g \geq 2 \), be the Teichmuller moduli space of curves of genus \( g \). Let \( \Gamma \subset \Gamma_g \) be a sufficiently small subgroup of the mapping class group \( \Gamma_g \) that acts freely on \( U \). This example is similar to Example 2.2(i) as was suggested in [M, Sect. 4]. One can apply a more recent result of Looijenga [L] who finally established that the mapping class group has the key property, namely, *local universal ramification over the boundary*, needed to show that \( k(\Gamma) \) has abundance of regular \( n \)-differentials as in Example 2(i).

Example 3. Let \( X \) be a nonsingular \( n \)-dimensional projective variety with *ample* canonical bundle \( K_X \), and *large* and *residually finite* fundamental group, and without general elliptic curvilinear sections. Assume the universal covering \( \tilde{V} \) of \( X \) has a \( q \)-*Bergman* metric for an integer \( q \). Hence \( V \) is a bounded domain [T3, Corollary]. Thus we are in the situation of Example 1.

Example 4. (See Remark 4.3 by Campana in [T1].) Let \( X \) be a sufficiently ample divisor in an Abelian variety of dimension at least 3. The universal covering \( \tilde{V} \) of \( X \) is not a bounded domain, and the fundamental group \( \pi_1(X) \) is an Abelian group of rank at least 3, in particular, amenable. The canonical bundle on \( X \) is ample. If \( \dim A = 3 \) then \( \text{Aut}(V)^0 = \{1\} \) according to Nadel [N].

Since \( V \) is Stein, \( X \) is an absolute minimal model of its field of rational functions according to [Kob, (6.3.21)] (generalizing earlier partial results by Igusa and Shioda).

3. A characterization of algebraic varieties whose universal coverings are bounded symmetric domains

3.1. The problem “whether the existence of quasi-homogeneous proalgebraic coverings of general type (in the sense of Piatetski-Shapiro and Shafarevich) is the characteristic property of algebraic varieties whose universal coverings are bounded symmetric domains” is stated in [S, Introduction], [P], [Pa-S, pp. 82-83]. Recall

Definition 3.1.1 [S, Sect. 6]. A proalgebraic variety \( Y \) is called *quasi-homogeneous* if the set of its boundary points is different from \( Y \) and is closed, and the orbit of every interior point relative to the group \( \text{Aut}(Y) \) of all automorphisms of \( Y \) is everywhere dense (in Zariski topology) in \( Y \).

If \( Y \) is quasi-homogeneous then the set of its nonsingular points coincides with the set of interior points [S, Sect. 6, Proposition 2].

3.2. Let \( Y = \lim \leftarrow X_\alpha \) be a projective proalgebraic variety. We keep the notation and assumptions of the preliminaries.

Let \( k(Y) = \bigcup k(X_\alpha) \) where \( k(Y) \) and \( k(X_\alpha) \) are the fields of rational functions on \( Y \) and \( X_\alpha \), respectively. We assume \( Y = \text{Proj} \sum D^\alpha(k(Y)) \). Recall that \( k(X_\alpha) \) is the field of rational functions on \( U_\alpha = \tilde{U}/\Delta_\alpha \) for all \( \alpha \) greater than a fixed sufficiently large index \( \gamma \), where \( \Delta_\alpha \) is a subgroup in the group \( \text{Aut}(\tilde{U}) \) of complex analytic automorphisms of \( \tilde{U} \). Let

\[
\text{Comm}(Y) := \text{Comm}(\Delta_\gamma) \subset \text{Aut}(\tilde{U})
\]

denote the *commensurability* subgroup.

Let \( G \) denote the identity component of the closure of \( \text{Comm}(Y) \) in \( \text{Aut}(\tilde{U}) \). Given a point \( x \in \tilde{U} \), let \( G/B_x \) denote the orbit of \( G \) through \( x \) where \( B_x \subset G \) is
the stabilizer of $x$. The definition as well as non-vanishing of the cohomology group $H^{\text{top}}(G/B_x)$ is discussed in [H, p. 890] and [Kos, Theorem 13.1]. We observe that $G$ acts effectively on $G/B_x$, and $B_x$ is reductive in $G$ (see [H, (1.1)]).

3.3. What follows is a global version of [FK, pp. 5-15, Proposition I.1.6, Proposition I.2.1]. We replaced functions by sheaves on manifolds. Classically, embeddings in infinite-dimensional projective spaces were considered by Bochner, Calabi [C], and in articles by Kobayashi (see, e.g., [Kob, Chap. 4.10]).

We keep the notation and assumptions of (1.1) - (1.3). Assume $K^m_{\alpha}$ and its global sections define the embedding of $U_\alpha$ in the corresponding finite-dimensional projective space, and $k(Y)$ is of general type (as in Definition 1.3.1 with the same $m$). As in [T1, T2], we get the corresponding section

$$B := B_{U, K^m}(z, w)$$

holomorphic in $z$ and antiholomorphic in $w$, where $K$ is the canonical bundle on $\tilde{U}$. Further, $B(z, z) > 0$ and $\log B(z, z)$ is strictly plurisubharmonic. A priori, $B$ is not a Hermitian kernel of positive type [FK, p. 8, p. 12].

We denote by $B^o$ the restriction of $B$ to the orbit $G/B_x$. Clearly, a Hilbert subspace of a Hilbert space with a reproducing kernel also has a reproducing kernel. So $B^o$ is a reproducing kernel of a unique Hilbert space $H$ of holomorphic sections. Indeed, that is true locally in a Euclidean neighborhood of any point of $G/B_x$ hence globally because $G/B_x$ is homogeneous.

We consider the projective space $P(H^*)$ with its Fubini-Study metric. In local coordinates, the evaluation at a point $Q \in G/B_x$, $e_Q : f \mapsto f(Q)$, is a continuous linear functional on $H$. We get a natural map

$$\Upsilon : G/B_x \longrightarrow P(H^*).$$

By the assumption, $\Upsilon$ is an embedding. Let $\text{Aut}(G/B_x)$ denote the group of complex analytic automorphisms of $G/B_x$. All elements of $\text{Aut}(G/B_x)$ act as collineations of $P(H^*)$ (compare [FK, p. 15]).

Remark 3.4. We are in the situation of the standard Bergman metric. Hence $G/B_x$ “appears” to be a bounded domain. On the other hand, sections like $B$, above, exist in a more general situation because locally in Euclidean neighborhoods we have convergence like in [T2]. However, we get a metric that may not be the Bergman metric.

Theorem 3.5. Let $Y$ be a projective proalgebraic variety of general type. We keep the notation and assumptions of (1.1) – (1.3) and (3.2) – (3.3). Let $x \in \tilde{U}$ be a point such that the orbit of $G$ through $x$ is not zero-dimensional and is Zariski dense in $\tilde{U}$. Then $\tilde{U}$ is a bounded symmetric domain.

Proof. Let $G/B_x$ denote the orbit through $x \in \tilde{U}$. Then it has a complex homogeneous structure induced by its embedding in $P(H^*)$. We will apply classical results of Borel, Koszul, and Hano.

By assumptions, this complex structure is Kahler homogeneous. Further, the Ricci form as well as Koszul’s canonical form are non-degenerate (see [H, Sect. 6, p. 893]), and $H^{\text{top}}(G/B_x) \neq 0$. The latter follows from [H, p. 890] and [Kos, Theorem 13.1] because of our embedding $G/B_x \subset P(H^*)$. Clearly, $G$ acts effectively
on $G/B_x$. Hence we may apply a theorem of Hano [H, Theorem 3 and p. 893]. It follows $G$ is a semisimple group.

According to Borel [B, Theorem 4], $G/B_x$ is simply connected. Furthermore, $G/B_x$ is a bounded symmetric domain since the corresponding fundamental group is large, i.e., $G/B_x$ contains no compact analytic subsets.

For an appropriate discrete subgroup $\Delta_\alpha$, $\Delta_\alpha \backslash G/B_x$ is a locally symmetric space and we may consider its Satake compactification (Satake, Baily, Borel, Piatetski-Shapiro). It follows that its closure in $X_\alpha$ is an algebraic subvariety by a theorem of Chow.

Since the orbit is Zariski dense in $\tilde{U}$, we get $\tilde{U}$ is a bounded symmetric domain.

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