\( \widehat{su}(3)_k \) fusion coefficients

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Abstract

A closed and explicit formula for all \( \widehat{su}(3)_k \) fusion coefficients is presented which, in the limit \( k \to \infty \), turns into a simple and compact expression for the \( su(3) \) tensor product coefficients. The derivation is based on a new diagrammatic method which gives directly both tensor product and fusion coefficients.
Fusion rules in WZNW models can be calculated rather efficiently from tensor product coefficients by means of the affine Weyl group [1,2,3]. However, closed and explicit formulae for fusion coefficients have been obtained previously only in the $su(2)$ case. Here we display an explicit expression for the $\hat{su}(3)_k$ fusion coefficients. This result is the culmination of a number of recent works in which various combinatorial methods have been devised for the computation of $\hat{su}(3)_k$ fusion rules [4,5,6]. As an off-shoot of our analysis, we derive a compact expression for ordinary $su(3)$ tensor product coefficients. Our result follows from a new diagrammatic method to compute $su(3)$ tensor product coefficients, which is tailor-made for the analysis of fusion rules.

Let us first recall what fusion rules are. Consider a Kac-Moody algebra $\hat{g}$ at level $k$, whose finite Lie algebra is $g$. Let $\hat{\lambda}$ be the highest weight of an integrable representation of $\hat{g}$ at level $k$ and denote its finite part by $\lambda$, which is then a highest weight of $g$. $\hat{\lambda}$ is completely specified by $\lambda$ and $k$. A fusion rule is the decomposition of a product of two integrable representations of $\hat{g}_k$ into a sum of integrable representations. The fusion coefficient $N^{(k)}_{\lambda\mu\nu}$ gives the number of times $C\hat{\nu}$ (C stands for conjugation) occurs in the product $\hat{\lambda} \times \hat{\mu}$. In the limit $k \to \infty$, the fusion coefficient $N^{(k)}_{\lambda\mu\nu}$ reduces to the ordinary tensor product coefficient, denoted by $N_{\lambda\mu\nu}$ (i.e., $N^{(\infty)}_{\lambda\mu\nu} \equiv N_{\lambda\mu\nu}$). Recall that $N_{\lambda\mu\nu}$ gives the number of times the scalar representation occurs in the triple tensor product $\lambda \otimes \mu \otimes \nu$.

The concept of fusion rules originates from two dimensional conformal field theory. In this context the formal product $\times$ is related to the radially ordered operator product of primary fields [7,8]. For the WZNW model whose spectrum generating algebra is $\hat{g}_k$, the primary fields are in one-to-one correspondence with integrable representations of $\hat{g}_k$ [9,10].

Let us recall the explicit expression for the $\hat{su}(2)_k$ fusion coefficients [10,11]:

$$
\hat{su}(2)_k : N^{(k)}_{\lambda\mu\nu} = \begin{cases} 
1 & \text{if } |\lambda_1 - \mu_1| \leq \nu_1 \leq \min\{\lambda_1 + \mu_1, 2k - \lambda_1 - \mu_1\} \\
& \text{and } \frac{1}{2}(\lambda_1 + \mu_1 + \nu_1) \in \mathbb{Z}_+ \\
0 & \text{otherwise}
\end{cases}
$$

(1)

Here $\lambda_1$ stands for the Dynkin label of $\lambda$ (i.e., $\lambda = \lambda_1 \omega^1$, where $\omega^1$ is the fundamental root of $su(2)$) and $\mathbb{Z}_+$ denotes the set of non-negative integers. In this case, it is clear that the fusion coefficients are truncated tensor product coefficients, where the degree of truncation is fixed by the level $k$. More precisely, if a coupling $\lambda, \mu, \nu$ is allowed in the
finite case (which means that $|\lambda_1 - \mu_1| \leq \nu_1 \leq \lambda_1 + \mu_1$ and $\frac{1}{2}(\lambda_1 + \mu_1 + \nu_1) \in \mathbb{Z}_+$), it also exists in the affine case as long as $k \geq \frac{1}{2}(\lambda_1 + \mu_1 + \nu_1)$. There is thus a threshold level $k_0$, below which the affine coupling is absent and above which it is always present. Here clearly $k_0 = \frac{1}{2}(\lambda_1 + \mu_1 + \nu_1)$. For $\widehat{su}(2)_k$, it is obvious that the fusion coefficients $N^{(k)}_{\lambda\mu\nu}$ are completely fixed by the tensor product coefficients $N_{\lambda\mu\nu}$ and $k_0$:

$$\widehat{su}(2)_k : N^{(k)}_{\lambda\mu\nu} = \begin{cases} N_{\lambda\mu\nu} & \text{if } k \geq k_0 = \frac{1}{2}(\lambda_1 + \mu_1 + \nu_1) \\ 0 & \text{otherwise} \end{cases}$$

(2)

It has been conjectured [12] that for any Kac-Moody algebra $\hat{g}_k$, the fusion coefficients $N^{(k)}_{\lambda\mu\nu}$ are uniquely determined from $N_{\lambda\mu\nu}$ and the minimum level $k_0$ at which the various couplings first appear. Although for $su(2)$ $N_{\lambda\mu\nu}$ can only be 0 or 1, in general it can be greater than one. Therefore, to the triplet $(\lambda, \mu, \nu)$ there correspond $N_{\lambda\mu\nu}$ distinct couplings, hence $N_{\lambda\mu\nu}$ values of $k_0$, one for each distinct coupling. Let us denote these by $k^{(i)}_0$, $i = 1, ..., N_{\lambda\mu\nu}$, implementing in this notation the natural ordering $k^{(i)}_0 \leq k^{(i+1)}_0$. Then the precise conjectured relationship between $N^{(k)}_{\lambda\mu\nu}$ and the set $\{N_{\lambda\mu\nu}, k^{(i)}_0 (\lambda, \mu, \nu)\}$, is

$$N^{(k)}_{\lambda\mu\nu} = \begin{cases} \max(i) & \text{such that } k \geq k^{(i)}_0 \text{ and } N_{\lambda\mu\nu} \neq 0 \\ 0 & \text{if } k < k^{(1)}_0 \text{ or } N_{\lambda\mu\nu} = 0. \end{cases}$$

(3)

The conjecture has been proved rigorously for $su(3)$ [4].

We now state our result in the $\widehat{su}(3)_k$ case. At first we present a closed expression for $N_{\lambda\mu\nu}$ which appears to be new. Given three $su(3)$ integrable weights $\lambda = (\lambda_1, \lambda_2)$ ($\lambda_{1,2}$ are the Dynkin labels of $\lambda$), $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$, the number of scalars contained in the tensor product $\lambda \otimes \mu \otimes \nu$ is

$$N_{\lambda\mu\nu} = (k^{\text{max}}_0 - k^{\text{min}}_0 + 1)\delta$$

(4)

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3 More precisely, in [12] the existence of $k_0$ is conjectured.

4 Although at first sight, this result may appear to be a direct consequence of the Gepner-Witten depth rule [10], a closer analysis demonstrates that this is not the case except for $su(2)$ [6].
where
\[ k_{0}^{\text{min}} = \max(\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2, A - \min(\lambda_1, \mu_1, \nu_1), B - \min(\lambda_2, \mu_2, \nu_2)) \]
\[ k_{0}^{\text{max}} = \min(A, B) \]
\[ A = \frac{1}{3}[2(\lambda_1 + \mu_1 + \nu_1) + \lambda_2 + \mu_2 + \nu_2] = (\lambda + \mu + \nu, \omega^1) \]
\[ B = \frac{1}{3}[\lambda_1 + \mu_1 + \nu_1 + 2(\lambda_2 + \mu_2 + \nu_2)] = (\lambda + \mu + \nu, \omega^2) \]
\[ \delta = \begin{cases} 
1 & \text{if } k_{0}^{\text{max}} \geq k_{0}^{\text{min}} \text{ and } A, B \in \mathbb{Z}_+ \\
0 & \text{otherwise}
\end{cases} \]

The fusion coefficients \( N_{\lambda \mu \nu}^{(k)} \) are given by (3) with the above \( N_{\lambda \mu \nu} \) and the following values of \( k_{0}^{(i)} \):
\[ \{k_{0}^{(i)}\}(\lambda, \mu, \nu) = \{k_{0}^{\text{min}}, k_{0}^{\text{min}} + 1, ..., k_{0}^{\text{max}}\} \]

where \( k_{0}^{\text{min}} \) and \( k_{0}^{\text{max}} \) are given in (5). That
\[ k_{0}^{\text{min}} \geq \max(\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2) \]

simply reflects the fact that the affine extension of the three weights \( \lambda, \mu, \nu \) must be integrable.

These results can be derived directly using the following diagrammatic approach. As it will be shown below, \( N_{\lambda \mu \nu} \) is equal to the number of distinct (‘bird feet’ type) diagrams of the form
\[ \cdot \alpha_1 \quad \alpha_2 \cdot \\
\cdot q \cdot \beta_1 \quad \beta_2 \cdot \\
\cdot \gamma_1 \quad \gamma_2 \cdot \\
p \cdot \]

where \( p, q, \alpha_i, \beta_i, \gamma_i \) are non-negative integers, constrained by the relations
\[ p + \alpha_1 = \lambda_1 \quad q + \alpha_2 = \lambda_2 \]
\[ p + \beta_1 = \mu_1 \quad q + \beta_2 = \mu_2 \]
\[ p + \gamma_1 = \nu_1 \quad q + \gamma_2 = \nu_2 \]
\[ \begin{align*}
\alpha_1 &\leq \beta_2 + \gamma_2 & \beta_1 &\leq \alpha_2 + \gamma_2 & \gamma_1 &\leq \alpha_2 + \beta_2 \\
\alpha_2 &\leq \beta_1 + \gamma_1 & \beta_2 &\leq \alpha_1 + \gamma_1 & \gamma_2 &\leq \alpha_1 + \beta_1
\end{align*} \] (9)

\[ \alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 \]

(In its complete form, this diagram must appear with three lines joining \( p \) to \( \alpha_1, \beta_1, \gamma_1 \) and similarly on the other side.) Distinct diagrams describe distinct couplings. To each diagram one associates the minimum level \( k_0 \)

\[ k_0 = p + q + \alpha_1 + \beta_1 + \gamma_1 = p + q + \alpha_2 + \beta_2 + \gamma_2 \] (10)

The result (4) is a direct consequence of this construction. On the other hand the close relation between (8) and (9), the generating function for tensor product coefficients and that for fusion coefficients, implies (10) directly. Before we turn to an explicit derivation of these results, we give an example.

Let us calculate the multiplicity and the corresponding set \( \{k_0^{(i)}\} \) for the triplet \( \lambda = (10, 13), \mu = (11, 15), \nu = (19, 9) \). One finds

\[ k_0^{\text{max}} = \min\{39, 38\} = 38 \]
\[ k_0^{\text{min}} = \max\{23, 26, 28, 38 - 10, 39 - 9\} = 29 \]

so that

\[ N_{\lambda\mu\nu} = 10 \quad \text{and} \quad \{k_0^{(i)}\} = \{29, 30, \ldots, 38\} \]

One has for instance \( N_{\lambda\mu\nu}^{(32)} = 4 \). The corresponding diagrams are

\[
\begin{array}{ccc}
\cdot 9 - q & 13 - q \\
1 + q & \cdot 10 - q & 15 - q \\
\cdot 18 - q & 9 - q \\
\end{array}
\]

for \( 0 \leq q \leq 9 \)

This illustrates clearly the power of formula (4) for computing \( su(3) \) tensor product coefficients.
The derivation of (4)-(6) relies on the generating function for tensor product coefficients and the Berenstein-Zelevinsky (BZ) triangles. We introduce both concepts presently.

The idea of the generating function for tensor product coefficients is based on the observation that a generic coupling can be decomposed into a finite number of elementary couplings [13]. For \( su(3) \) there are eight such elementary couplings:

\[
\begin{align*}
E_1 &= (1,0)(0,1)(0,0) & E_2 &= (1,0)(0,0)(1,0) & E_3 &= (0,0)(1,0)(0,1) \\
E_4 &= (0,1)(0,0)(0,0) & E_5 &= (0,1)(0,0)(1,0) & E_6 &= (0,0)(0,1)(1,0) \\
E_7 &= (1,0)(1,0)(0,0) & E_8 &= (0,1)(0,1)(0,1)
\end{align*}
\]

(assuming the ordering \( \lambda \mu \nu \)). From a general product of the form

\[
E_1^a E_2^b E_3^c E_4^d E_5^e E_6^f E_7^g E_8^h
\]

one can read off the Dynkin labels of \( \lambda, \mu \) and \( \nu \) to be

\[
\begin{align*}
\lambda_1 &= a + b + g & \mu_1 &= c + d + g & \nu_1 &= e + f + g \\
\lambda_2 &= d + e + h & \mu_2 &= a + f + h & \nu_2 &= b + c + h
\end{align*}
\]

Notice that to the triplet \((1,1)(1,1)(1,1)\) there corresponds three couplings: \( E_1 E_3 E_5, E_2 E_4 E_6 \) and \( E_7 E_8 \). But are these all distinct? To answer one needs an explicit basis for the couplings. In turns out that in any basis, two of these couplings are identical (an explicit basis is described in the next paragraph, where this result is also illustrated). Whether \( E_1 E_3 E_5 = E_2 E_4 E_6 \), \( E_1 E_3 E_5 = E_7 E_8 \) or \( E_2 E_4 E_6 = E_7 E_8 \) is however dependent upon an explicit choice of basis. The invariant result is that there are only two distinct couplings (of course it is well known that \( N_{(1,1)(1,1)(1,1)} = 2 ! \)). This illustrates the fact that there are redundancies (called syzygies) in the decomposition into elementary couplings. Therefore particular products of elementary couplings must be forbidden. For instance, in the \( su(3) \) case, within the basis for which \( E_7 E_8 = E_1 E_3 E_5 \), one must forbid either \( E_7 E_8 \) or \( E_1 E_3 E_5 \). With this proviso, the decomposition of a general coupling into elementary ones is unique. Given a complete set of elementary couplings and a choice of forbidden couplings, one can construct in a systematic way the generating function for tensor product coefficients [14]. This is all the information required about generating functions for our present purpose.

A particularly interesting basis for the couplings is provided by BZ triangles [15]. They are defined as follows
where the $a_i$’s are non-negative integers fixed by the conditions

\[
\begin{align*}
    a_1 + a_2 &= \lambda_1 & a_4 + a_5 &= \mu_1 & a_7 + a_8 &= \nu_1 \\
    a_3 + a_4 &= \lambda_2 & a_6 + a_7 &= \mu_2 & a_9 + a_1 &= \nu_2 \\
    a_2 + a_3 &= a_6 + a_8 \\
    a_3 + a_5 &= a_9 + a_8 \\
    a_5 + a_6 &= a_2 + a_9
\end{align*}
\]

(14)

Given a triplet of $su(3)$ integrable weights $(\lambda, \mu, \nu)$, the number of triangles which can be constructed in this way is equal to $N_{\lambda\mu\nu}$. The triangles associated to $su(3)$ elementary couplings are

\[
E_1 = (1, 0)(0, 1)(0, 0) \quad E_2 = (1, 0)(0, 0)(0, 1) \quad E_3 = (0, 0)(1, 0)(0, 1)
\]

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
E_4 = (0, 1)(1, 0)(0, 0) \quad E_5 = (0, 1)(0, 0)(1, 0) \quad E_6 = (0, 0)(0, 1)(1, 0)
\]

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
E_7 = (1, 0)(1, 0)(1, 0) \quad E_8 = (0, 1)(0, 1)(0, 1)
\]

\[
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

(15)
The triangle associated to a non elementary coupling is simply the sum of the triangles of the elementary couplings in its decomposition. In this way one readily gets

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0
\end{array}
\]

(16)

Notice that if in the definition of the triangle we had interchanged the role of \(\mu\) and \(\nu\), we would have found instead \(E_1 E_3 E_5 \neq E_2 E_4 E_6 = E_7 E_8\).

Generating function for fusion coefficients have been introduced in [12] (see also [16]). Where it was conjectured that generically there is at least one choice of forbidden couplings for which the generating function of fusion coefficients is related in a simple way to that of tensor product coefficients. In that case, the value of \(k_0\) of a coupling which decomposes as \(\prod_i E_i^e\) reads

\[
k_0 = \sum_i e_i k_0(E_i)
\]

(17)

where \(k_0(E_i)\) is the minimum level at which the elementary coupling \(E_i\) first appears. For \(su(3)\), all \(k_0(E_i)\) are one. Furthermore, in order to reproduce fusion coefficients from (17) it turns out that \(E_7 E_8\) must \textit{not} be forbidden (e.g., one must forbid \(E_1 E_3 E_5\) if the basis is such that \(E_1 E_3 E_5 = E_7 E_8\)) [12]. As shown in [6] (see also [5]) the value of \(k_0\) of a given coupling is actually encoded in its BZ triangle and reads

\[
k_0 = \max\{a_1 + \mu_1 + \mu_2, a_4 + \nu_1 + \nu_2, a_7 + \lambda_1 + \lambda_2\}
\]

(18)

The analysis of fusion rules from BZ triangles is somewhat complicated (cf. the need to take a maximum over three terms) by the fact that in this basis the syzygy is not \(k_0\)-homogeneous, i.e., \(E_1 E_3 E_5\) (for which \(\sum k_0(E_i) = 3\) is equal to \(E_7 E_8\) (with \(\sum k_0(E_i) = 2\)). One would expect that a \(k_0\)-homogeneous basis would be better adapted to the study of fusion rules. This motivates us to look for a basis in which \(E_1 E_3 E_5 = E_2 E_4 E_6\).

To get a hint on how one should proceed, let us return to (13) and see how Dynkin labels get split off in the case of the BZ triangles. The comparison of (13) and (14) yields

\[
\begin{align*}
a_1 &= b \\
a_2 &= a + g \\
a_3 &= e + h \\
a_4 &= d \\
a_5 &= g + c \\
a_6 &= a + h \\
a_7 &= f \\
a_8 &= e + g \\
a_9 &= h + c
\end{align*}
\]

(19)
from which the hexagon relations \((a_2 + a_3 = a_6 + a_8, \text{ etc})\) follow automatically. Therefore this particular way of breaking Dynkin labels in parts ensures that \(E_1 E_3 E_5 = E_7 E_8\). It is clear that in order to get \(E_1 E_3 E_5 = E_2 E_4 E_6\) different combinations must be considered. It is not difficult to see that one actually needs to combine the integer in (13) according to:

\[
\begin{align*}
p = g \\
q = h \\
\alpha_1 &= a + b \\
\beta_1 &= d + c \\
\gamma_1 &= e + f \\
\alpha_2 &= d + e \\
\beta_2 &= a + f \\
\gamma_2 &= b + c
\end{align*}
\]  

(20)

This leads directly to the diagram (8) and the condition specified in (9). In this basis the elementary couplings are

\[
\begin{align*}
E_1 &= (1,0)(0,1)(0,0) \\
E_2 &= (1,0)(0,0)(0,1) \\
E_3 &= (0,0)(1,0)(0,1) \\
E_4 &= (0,1)(1,0)(0,0) \\
E_5 &= (0,1)(0,0)(1,0) \\
E_6 &= (0,0)(0,1)(1,0) \\
E_7 &= (1,0)(1,0)(1,0) \\
E_8 &= (0,1)(0,1)(0,1)
\end{align*}
\]

(21)

Now here comes the great simplifying feature of a \(k_0\)-homogeneous basis. In that case the \(k_0\) value of a given coupling can be computed without caring about which specific couplings have to be forbidden since any choice gives the same result! In other words, the \(k_0\) value of a general product of the form (12) can be computed by simply summing all the exponents (because \(k_0(E_i) = 1\) for all \(i\)) without bothering about the potential non uniqueness of the decomposition:

\[
k_0 = a + b + c + d + e + f + g + h
\]

(22)
Rewriting this result in terms of the data of the diagram (8), one gets directly (10).

Now the only thing we have to do is to extract the full set of conditions ensuring the existence of the diagram (8). From the last condition in (9) one has

\[ p = A - B + q \]  

(23)

where \( A \) and \( B \) are defined in (4). In terms of \( p \) or \( q \), the six inequalities in (9) require that

\[ p, q \leq \min(A, B) - \max(\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2) \]  

(24)

Furthermore since each entry in (8) must be a positive integer or zero, one must have

\[ p \leq \min(\lambda_1, \mu_1, \nu_1) \]
\[ q \leq \min(\lambda_2, \mu_2, \nu_2) \]  

(25)

(in addition to the obvious condition that \( A, B \in \mathbb{Z}_+ \)). Suppose first that \( A \geq B \), so that \( q \geq 0 \) and \( p \geq A - B \). Rewriting the first inequality in (25) in terms of \( q \) and taking into account its other two upper bounds, one finds that

\[ 0 \leq q \leq \min(B - \max(\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2), \lambda_2, \mu_2, \nu_2, \min(\lambda_1, \mu_1, \nu_1) - (A - B)) \]  

(26)

In terms of \( q \), \( k_0 \) as given in (10) becomes

\[ k_0 = B - q \]  

(27)

The number of distinct diagrams (8) one can draw (for a fixed triplet \((\lambda, \mu, \nu)\)) is thus equal to the number of values that \( q \) can take according to (26). This implies directly

\[ N_{\lambda\mu\nu} = q^{max} + 1 \]  

(28)

(Notice that if the minimum on the r.h.s. of (26) is negative, there is no allowed diagram and \( N_{\lambda\mu\nu} = 0 \)). \( N_{\lambda\mu\nu} \) can clearly be rewritten under the form

\[ N_{\lambda\mu\nu} = k_0^{max} - k_0^{min} + 1 \]  

(29)

with \( k_0^{max} = B \) (when \( A \geq B \)) and \( k_0^{min} \) given in (5). The requirement of positivity on \( q \) can be rephrased as the condition \( k_0^{max} \geq k_0^{min} \).
A similar analysis for $B \geq A$ (for which $p \geq 0$ and $q \geq B - A$), where now

$$k_0 = A - p$$

shows that $k_0^{\text{max}} = A$ while $k_0^{\text{min}}$ is the same as before. The two expressions obtained for $k_0^{\text{max}}$ are then equivalent to that given in (5). Now with either (27) or (30) one sees directly that to each distinct coupling associated to a given triplet $(\lambda, \mu, \nu)$, there corresponds a distinct value of $k_0^{(i)}$ and that all these values satisfy $k_0^{(i+1)} = k_0^{(i)} + 1$. This completes the proof of (5) and (6).

Let us conclude with some comments. Given a triplet $(\lambda, \mu, \nu)$, the coupling – described by a diagram of the form (8) – with largest value of $k_0$, is easily found to be that with either $q = 0$ (if $A \geq B$) or $p = 0$ (if $B \geq A$). All other couplings are obtained from it by repeated addition of the “diagram”

$$\Gamma = \begin{array}{ccc}
-1 & & -1 \\
1 & \cdot & -1 \\
\cdot & 1 & -1
\end{array} \quad (31)$$

In terms of elementary couplings, $\Gamma$ can be written as $E_7E_8(E_2E_4E_6)^{-1}$ and it contributes $-1$ to $k_0$. The situation is analogous for the BZ triangles. In that case, the triangle for the coupling with smallest value of $k_0$ corresponds to the one for which at least one corner is zero. Which of $a_1$, $a_4$, $a_7$ is zero for the triangle of smallest $k_0$, is fully determined by the following expression

$$\begin{align*}
a_1 &= \frac{1}{3} \max(0, \beta, (\beta - \gamma)) \\
a_4 &= \frac{1}{3} \max(0, \gamma, (\gamma - \beta)) \\
a_7 &= \frac{1}{3} \max(0, -\beta, -\gamma)
\end{align*} \quad (32)$$

where

$$\begin{align*}
\beta &= 3\lambda_1 - 3\mu_2 + 3(B - A) \\
\gamma &= 3\lambda_2 - 3\nu_1 + 3(A - B)
\end{align*}$$

From this coupling, with smallest possible value of $k_0$, all others can be obtained by repeated addition of the “triangle”:

$$\Omega = \begin{array}{ccc}
1 & & \\
-1 & -1 & \cdot \\
\cdot & 1 & -1 \\
1 & -1 & -1 & 1
\end{array} \quad (33)$$
which decomposes as $E_2E_4E_6(E_7E_8)^{-1}$ and whose value of $k_0$ is +1 (cf., eq. (18)). The number of copies of $\Omega$ which can be added is limited by the constraint that each entry of the BZ triangle must be a positive integer or zero. This gives directly

$$N_{\lambda\mu
u} = \min(a_2, a_3, a_5, a_6, a_8, a_9) + 1 \quad (34)$$

where the entries refer to the specific BZ triangle for which $\min(a_1, a_4, a_7) = 0$ for $(\lambda, \mu, \nu)$ fixed. It is straightforward to check the full equivalence of this expression with (4). When written at length, (34) gives $N_{\lambda\mu
u} - 1$ as a minimum over 18 terms which contain each Dynkin label separately. Thus for a triplet including one weight with a vanishing Dynkin label, $N_{\lambda\mu\nu}$ is necessarily one, if non-zero.

The result (4)-(6) could thus have been derived entirely within the BZ triangle basis. However the derivation is somewhat simpler in terms of the diagrams (8) and part of the interest of the present derivation lies in the fact that it displays a new basis for $su(3)$ couplings.

Let us point out that the product $N^{(k)}_{\lambda\mu\nu} N^{(k+1)}_{\lambda'\mu'\nu'}$ gives directly the fusion rules for the unitary minimal $W_3$ models, in which a primary field is described by a pair of integrable affine weights $(\hat{\lambda}, \hat{\mu})$, respectively at level $k$ and $k + 1$ [17].

We expect to report elsewhere on the $su(N)$ generalization of this work. Along that vein, Cummins [18] has informed us that using symmetric function techniques it is possible to show that

$$k^{max}_0 \leq \min[(\lambda + \mu + \nu, 1), (\lambda + \mu + \nu, 1)] \quad (35)$$

for general $su(N)_k$ fusion rules. This is of course consistent with our $su(3)$ expression for $k^{max}_0$ in (4).

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