A common misconception about *LIGO* detectors of gravitational waves

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**Abstract**

A common misconception about laser interferometric detectors of gravitational waves purports that, because the wavelength of laser light and the length of an interferometer’s arm are both stretched by a gravitational wave, no effect should be visible, invoking an analogy with cosmological redshift in an expanding universe. The issue is clarified with the help of a direct calculation.

Keywords: gravitational waves
1 Introduction

One of the most significant advances in experimental gravitation has been the development of giant laser interferometers for the detection of gravitational waves such as the LIGO [1] and VIRGO [2] projects. The LISA project [3] is even more ambitious and plans the construction of a similar interferometer in space, with arms of size $\sim 5 \times 10^6$ km, an interplanetary distance. The recent coming online of the LIGO detectors with actual data runs has brought the laser interferometric detectors to the forefront of current gravitational research. There is an objection as to how an interferometric detector works that recurs often, and persists as a misconception, among physics students and professionals alike. The objection is: “Given that the gravitational field stretches both the interferometer arm and the wavelength of laser light propagating along it, why is the effect of a gravitational wave detectable? After all, the same situation occurs in cosmology, when the expansion of space stretches all distances and the wavelength of light alike, causing cosmological redshift”.

The immediate answer to this objection is that the calculation of the phase shift $\Delta \phi$ between the laser beams of a laser interferometer produces a result that is gauge-independent, while the interpretation in terms of stretching of an interferometer’s arm and of the wavelength of light depends on the gauge adopted, and only gauge-independent results are acceptable in physics. However, this is truly an indirect answer and it may be preferable to provide a direct argument in the same gauge (TT gauge) used for the above-mentioned interpretation of the workings of LIGO.

We could find three instances in the literature in which this issue is raised:
1) A paper by Saulson [4] addresses the issue and provides a qualitative intuitive explanation.
2) In a Cal Tech lecture course on gravitational waves available on the internet [5], K.S. Thorne addresses the issue reporting that this is the most common question asked about laser interferometers, and he provides a qualitative answer: “Does the wavelength of the light in the gravitational wave get stretched and squeezed the same manner as these mirrors move back and forth? ... The answer is no, the spacetime curvature influences the light in a different manner that it influences the mirror separations ... the influence on the light is negligible and it is only the mirrors that get moved back and forth and the light’s wavelength does not get changed at all ...”. However, substantiating Thorne’s answer with a clear mathematical argument is not entirely trivial, as is shown in Sec. 3.
3) In a recent paper [6] the issue is raised again, together with the purported analogy with the cosmological situation. The author discusses an analogy between the gauge freedom of general relativity and the Aharonov-Bohm effect in quantum mechanics. The message is that in both situations gauge-dependent quantities appear in the equations
describing the physics but the final physical results calculated are gauge-independent. This is again the answer that at the end the only physical quantity measured (the phase shift $\Delta \phi$ between two laser beams in an interferometric detector) is gauge independent.

Saulson’s pedagogical paper [4] is clear and very physical but the argument presented is qualitative — the “stretching at a different rate” of the light wavelength and of an interferometer’s arm can be shown explicitly. On the other hand, the argument proposed in Ref. [6] requires the use of the Aharonov-Bohm effect foreign to classical relativity and truly unnecessary to understand laser interferometers. The answer provided is indirect and does not specifically address the question of “why there is a net effect if the wavelength of light and the interferometer’s arm are both stretched?” It would be more gratifying if a direct argument were provided showing how the (proper) length of an interferometer’s arm and the (proper) wavelength of laser light are “stretched at different rates” by a gravitational wave, which is what we set out to do in this paper.

\section{Laser interferometers}

Before we proceed to discuss a reply to the objection stated above, we summarize in this section the basics of laser interferometric detectors of gravitational waves [7, 8], which we need for reference in the following.

Consider a laser interferometer consisting of two perpendicular arms aligned along the $x$ and $y$ axis, respectively. In the absence of gravitational waves the two arms have exactly equal lengths $L$. A beam splitter is located at the origin $x = y = 0$ and two mirrors are placed at the opposite end of each arm, at $x = L$ and $y = L$, respectively. A monochromatic laser beam passing through the beam splitter is divided into two beams that propagate along the $x$ and $y$-arms, are reflected by the mirrors, and travel back to the beam splitter where they are collected and compared, thus detecting any phase shift that may have occurred during the travel. If a gravitational wave impinges on the interferometer, it will cause a phase shift $\Delta \phi$ between the two beams.

The gravitational wave is described as a small perturbation of the Minkowski metric $\eta_{\mu\nu}$. The spacetime metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} ,$$

where $\eta_{\mu\nu} = \text{diag} (-1, 1, 1, 1)$ and $|h_{\mu\nu}| \ll 1$ in an asymptotically Cartesian coordinate system. The transverse-traceless (TT) gauge is most often used in introductions to laser interferometric detectors. In this gauge $h_{0\mu} = h_{\mu\mu} = 0$ and the unperturbed metric is used to raise and lower indices. The only nonzero components of $h_{\mu\nu}$ in this gauge are $h_{xx} = -h_{yy}$ and $h_{xy} = h_{yx}$, corresponding to two independent polarizations of the gravitational wave. Only first order quantities in the metric perturbations $h_{\mu\nu}$...
and their derivatives are considered because of the smallness of these quantities in any physical situation of interest. For simplicity we consider a gravitational wave with a single polarization traveling along the z-axis perpendicular to the interferometer’s arms, perfectly reflecting mirrors, and a single reflection of each laser beam. The geodesic deviation equation rules the evolution of the proper distances $x^i$ along the $x$ and $y$-axis:

$$\ddot{x}^i = R^i_{\ 00j} x^j, \quad (2.2)$$

where $i, j = 1, 2$ and $R^\mu_{\nu\alpha\beta}$ is the Riemann tensor, which is most conveniently calculated in the TT gauge yielding [7]

$$\delta \ddot{x}^i = -\frac{1}{2} h^{(TT)}_{ij} x^j. \quad (2.3)$$

A further assumption almost always used in the literature on ground-based interferometric detectors is that the wavelength $\lambda_{gw}$ of the gravitational wave is much larger than the size of the interferometer, $\lambda_{gw} \gg L$, thus simplifying the integration of eq. (2.3) to

$$\delta x = \frac{h_{xx}}{2} x, \quad (2.4)$$

$$\delta y = \frac{h_{yy}}{2} y. \quad (2.5)$$

The difference in the variation of the proper lengths of the interferometer’s arms when a gravitational wave impinges along the z-axis with polarization $h_{xx} = -h_{yy} = h_+(t)$ is, to first order,

$$\delta l(t) = \delta x(t) - \delta y(t) = L h_+(t) \quad (2.6)$$

and the phase difference between the two beams collected at the origin is

$$\Delta \phi = 2\pi \frac{\delta l}{\lambda} = 2\pi \frac{L}{\lambda} h_+(t) \quad (2.7)$$

where $\lambda$ is the wavelength of the monochromatic laser light.

### 3 Variation of the wavelength of laser light in the $x$-arm

In this section we compute explicitly the variation of the wavelength of laser light propagating along one arm (say, the $x$-arm) of the laser interferometer, and then we compare the result with the variation of the proper length of this arm. It turns out that the two
quantities are different and this result shows that the objection “the wavelength and the arm are stretched by the same amount as in the expansion of the universe” is really unfounded. The TT gauge commonly used to calculate and interpret the effect of the gravitational wave is employed.

The variation of the wavelength of monochromatic laser light in the \( x \)-arm of the interferometer can be derived by considering the equation of null geodesics

\[
\frac{dk^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}k^\rho k^\sigma = 0,
\]  

(3.1)

where \( k^\mu = dx^\mu/d\tau \) is the geometric tangent to the null geodesic and \( \tau \) is an affine parameter along it. The photon four-momentum is \( p^\mu = \omega k^\mu = (\omega, \vec{k}) \), where \( \omega \) and \( \vec{k} \) are the angular frequency and the three-dimensional wave vector, respectively. The Christoffel symbols are

\[
\Gamma^\mu_{\rho\sigma} = \frac{1}{2} \eta^{\rho\alpha} \left( h_{\alpha\rho,\sigma} + h_{\alpha\sigma,\rho} - h_{\rho\sigma,\alpha} \right) + O(h^2).
\]  

(3.2)

The unperturbed laser beam travels along the \( x \)-axis with four-tangent \( k^\mu(0) = \delta^0\mu + \delta^1\mu \) while the actual (perturbed) four-tangent is \( k^\mu = k^\mu(0) + \delta k^\mu \) with \( \delta k^\mu = O(h) \). To first order, the deflections \( \delta k^\mu \) satisfy the equation

\[
\frac{d(\delta k^\mu)}{d\tau} = -\frac{1}{2} \eta^{\mu\alpha} \left( h_{\alpha\rho,\sigma} + h_{\alpha\sigma,\rho} - h_{\rho\sigma,\alpha} \right) k^\rho(0) k^\sigma(0),
\]  

(3.3)

where the product

\[
k^\rho k^\sigma = k^\rho(0) k^\sigma(0) + O(h) = \left( \delta^0\rho\delta^0\sigma + \delta^0\rho\delta^1\sigma + \delta^1\rho\delta^0\sigma + \delta^1\rho\delta^1\sigma \right) + O(h)
\]  

(3.4)

can be taken to zero order — including first order corrections only contributes second order terms to the deflections. Integration of eq. (3.3) with respect to \( x \) between the beam splitter \((x = 0)\) and the mirror \((x = L)\) along the interferometer’s arm yields the total deflection between the injection of the beam and its reflection at the mirror

\[
\delta k^\mu = -\int_0^L dx \left( h^\mu_{\rho,\sigma} - \frac{1}{2} h_{\rho\sigma} \right) \left( \delta^0\rho\delta^0\sigma + \delta^0\rho\delta^1\sigma + \delta^1\rho\delta^0\sigma + \delta^1\rho\delta^1\sigma \right) + O(h^2)
\]

\[= -\int_0^L dx \left( h^\mu_{0,0} + h^\mu_{1,0} + h^\mu_{0,1} + h^\mu_{1,1} \right) \]

\[+ \frac{1}{2} \int_0^L dx \left( h_{00} + 2h_{01} + h_{11} \right) + O(h^2).
\]  

(3.5)
In eq. (3.5) the integration with respect to the affine parameter $\tau$ has been replaced by an integration with respect to $x$, introducing only a second order error. In TT gauge $h_{0\mu} = 0$ and therefore the first three terms in the first integral vanish. The remaining term $h_{\mu 1, 1}$ contributes a boundary term $h_{11}(x = 0) - h_{11}(x = L)$. Since gravitational waves of wavelength $\lambda_{gw} \gg L$ are usually considered in laser interferometers the spatial variation of the gravitational wave in the interferometer’s arms is negligible and this term vanishes as well. In TT gauge we are left with

$$
\delta k^\mu = \frac{1}{2} \int_0^L dx \ h_{11, \mu} + O(h^2).
$$

(3.6)

The deflection of light traveling between the beam splitter at $x = 0$ and the detector at the same place (after reflection of the signal at $x = L$) is given by

$$
\delta k^\mu = \frac{1}{2} \int_0^{2L} dt \ h_{11}^{\mu} + O(h^2)
$$

(3.7)

by taking into account the fact that for the unperturbed laser beam traveling along the $x$-arm it is $x = t$ between emission and reflection, and $x = L - t$ between reflection and detection (the presence of the gravitational wave introduces corrections that only give second order terms in the deflections). Under the assumption $\lambda_{gw} \gg L$ the spatial dependence of $h_{\alpha \beta}$ can be neglected and $h_{\alpha \beta}(t, \vec{x}) \simeq h_{\alpha \beta}(t)$, therefore

$$
\delta k^\mu = \frac{\delta 0^\mu}{2} \left[ h_{11}(t = 2L) - h_{11}(t = 0) \right] + O(h^2).
$$

(3.8)

Therefore, in the approximation used, the laser photons do not suffer spatial deflections to first order. This fact is usually tacitly assumed in the standard presentations of how laser interferometers work (see [8] for a discussion) but it appears explicitly in the approach used here.

In general, the angular frequency of the light measured by an observer with four-velocity $u^\mu$ is $\omega = -p^\mu u_\mu$. This quantity is a scalar and therefore is gauge-invariant. Let $u_{0}^\mu$ be the four-velocity in the absence of gravitational waves and $u^\mu = u_{(0)}^\mu + \delta u^\mu$ the perturbed four-velocity. Then

$$
\omega = -g_{\mu \nu} p^\mu u^\nu = -\eta_{\mu \nu} p_{(0)}^\mu u_{(0)}^\nu - \eta_{\mu \nu} \left( p_{(0)}^\mu \delta u^\nu + u_{(0)}^\mu \delta p^\nu \right) - \eta_{\mu \nu} p_{(0)}^\mu u_{(0)}^\nu + O(h^2).
$$

(3.9)

In particular, by taking as the “observer” the half-transparent mirror at the origin, the unperturbed four-velocity is $u_{(0)}^\mu = \delta 0^\mu$ and $\delta u^\alpha = 0$. In fact, in TT gauge the coordinate locations of freely falling bodies are unaffected by the gravitational wave, although their
proper separations do vary. Then, upon detection at the beam splitter, the angular frequency of the laser light is

\[ \omega = \omega_0 - \eta_{\mu\nu} \delta^{\mu}_{\nu} \delta p^\nu - \omega_0 h_{\mu\nu} \left( \delta^{0\mu} \delta^{0\nu} + \delta^{1\mu} \delta^{1\nu} \right) + O(h^2) = \omega_0 + \delta p^0 + O(h^2) \]  \tag{3.10}

and the percent frequency shift, which is gauge-invariant, is

\[ \frac{\delta \omega}{\omega_0} = \frac{\omega - \omega_0}{\omega_0} = \delta p^0 = \frac{h_{11}(t = 0) - h_{11}(t = 2L)}{2}. \]  \tag{3.11}

In the approximation \( \lambda_{gw} \gg L \) the temporal variation of \( h_{11}(t) \) during the short time \( \approx 2L \) it takes for the light to travel to the mirror and back is negligible and \( \delta \lambda \approx 0 \) in this approximation. Now we want to compare the variation of the proper length of the interferometer arm with the variation in wavelength, and the way to do this correctly is to compare the former with the percent variation in the proper (or physical) wavelength of light \( \lambda_{phys} \equiv \sqrt{g_{11}} \lambda \). This is analogous to what is normally done in an expanding Friedmann-Lemaitre-Robertson-Walker spacetime with curvature index \( K = \pm 1 \) or 0, described by the line element

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \]  \tag{3.12}

in comoving coordinates \((t, r, \theta, \varphi)\). In this case one can consider the wavelength of light \( \lambda \) in comoving coordinates but what is physically relevant is the comoving (or physical) wavelength \( \lambda_{phys} \equiv a(t) \lambda \) which changes with the comoving time \( t \). Similarly, in the interferometer’s arm, the physical distance between mirror and beam splitter is not the coordinate distance \( L \) (which is constant in TT gauge) but rather \( \sqrt{g_{11}} L = \left[ 1 + \frac{h_{11}}{2} + O(h^2) \right] L \) and the physical wavelength is \( \lambda_{phys} = \sqrt{g_{11}} \lambda \). The percent variation of proper wavelength however coincides with the percent variation of coordinate wavelength,

\[ \frac{\delta \lambda_{phys}}{\lambda_{phys}} = \frac{\sqrt{g_{11}} \delta \lambda}{\sqrt{g_{11}} \lambda} = \frac{\delta \lambda}{\lambda} \]  \tag{3.13}

and therefore is also zero to first order in the approximation \( \lambda_{gw} \gg L \).

4 Discussion

The percent variation of the proper length of the interferometer’s \( x \)-arm is given by eq. (2.4) as

\[ \frac{\delta x(t)}{L} = \frac{1}{2} h_+(t). \]  \tag{4.1}
In the approximation $\lambda_{gw} \gg L$ the time dependence disappears and

$$\frac{\delta x}{L} = \frac{h_+(t = 0)}{2},$$  \hspace{1cm} (4.2)

which is different from zero, while the percent variation of proper wavelength $\delta \lambda_{phys}/\lambda_{phys}$ for the laser light traveling in this arm is zero in this approximation.

Therefore the objection that “all lengths are stretched at the same rate by the gravitational wave” and based on the analogy with the expanding three-space of cosmology, is incorrect. The gravitational wave “treats in a different way” the wavelength of light and the length of the interferometer’s arm. Physically, the interferometer works by measuring the differential stretching of the $x$ and $y$ arms while the high frequency light wave essentially experiences no inhomogeneities in the “medium” in which it propagates — the gravitational wave — because the wavelength $\lambda_{gw}$ of the gravitational wave is so much larger than the wavelength of light. This conclusion agrees with Thorne’s qualitative answer to the objection [5]. Technological issues aside, laser interferometers such as those of LIGO and VIRGO can indeed detect gravitational waves.

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