Relativistic Wave Equations on the lattice: an operational perspective

Nelson Faustino

Center of Mathematics, Computation and Cognition, UFABC

nelson.faustino@ufabc.edu.br

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1. Some ideas on semigroups
   - Evolution type problems on the lattice
   - Heat propagators
   - Wave propagators

2. Relativistic wave equations
   - An umbral calculus perspective
   - Chebyshev expansions
   - Factorization approach vs Cauchy-Kovaleskaya extension
Model Problem

Determine $\psi$ on $h\mathbb{Z}^n \times T$ s.t

\[
\begin{cases}
L_t^* L_t \psi(x, t) = \Delta_h \psi(x, t) - m^2 \psi(x, t) & \text{for } (x, t) \in h\mathbb{Z}^n \times T \\
\psi(x, 0) = \phi_0(x) & \text{for } x \in h\mathbb{Z}^n \\
L_t \psi(x, 0) = \phi_1(x) & \text{for } x \in h\mathbb{Z}^n
\end{cases}
\]

associated to the discrete Laplacian $\Delta_h$ on $h\mathbb{Z}^n$, and a mass term $m$.

1. **Differential-difference evolution problem** $L_t = \partial_t$ and $T = [0, \infty)$. 
2. **Difference-difference evolution problem**

   \[L_t \psi(x, t) = \frac{\psi(x, t + \frac{\tau}{2}) - \psi(x, t - \frac{\tau}{2})}{\tau}\]

   and

   $T = \{\frac{k\tau}{2} : k = 0, 1, 2, \ldots\}$. 
3. **General procedure** $L_t$ is a shift-invariant operator with respect to the translation semigroup $\{\exp (\tau \partial_t)\}_{t \geq 0}$ ($\tau > 0$) and $T \subseteq [0, \infty)$. 
Second Order Evolution Problem

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1. Differential-difference evolution problem $L_t = \partial_t$ and $T = [0, \infty)$.
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\[
L_t \psi(x, t) = \frac{1}{\tau} \left( \psi(x, t + \frac{\tau}{2}) - \psi(x, t - \frac{\tau}{2}) \right)
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and $T = \left\{ \frac{k\tau}{2} : k = 0, 1, 2, \ldots \right\}$.

3. General procedure $L_t$ is a shift-invariant operator with respect to the translation semigroup $\{\exp(\tau \partial_t)\}_{t \geq 0}$ ($\tau > 0$) and $T \subseteq [0, \infty)$.
Differential-Difference Heat type equation

Determine $\psi$ on $h\mathbb{Z}^n \times [0, \infty)$ s.t

$$\begin{cases}
\partial_t \psi(x, t) = \Delta_h \psi(x, t) - m^2 \psi(x, t) & \text{for } (x, t) \in h\mathbb{Z}^n \times [0, \infty) \\
\psi(x, 0) = \phi_0(x) & \text{for } x \in h\mathbb{Z}^n
\end{cases}$$

Solution:

$$\psi(x, t) = \exp(-m^2 t) \exp(t \Delta_h) \phi_0(x)$$

solves the above equation.
Heat semigroup approach
On the Fourier domain

Discrete Fourier Transform

\[
(F_h g)(\xi) = \begin{cases} 
\frac{h^n}{(2\pi)^n} \sum_{x \in h\mathbb{Z}^n} g(x) \exp(ix \cdot \xi) & \text{for } \xi \in Q_h \\
0 & \text{for } \xi \in \mathbb{R}^n \setminus Q_h.
\end{cases}
\]

Notice that \( Q_h = \left(-\frac{\pi}{h}, \frac{\pi}{h}\right)^n \) may be identified with the \( n \)-torus \( \mathbb{R}^n / \frac{2\pi}{h} \mathbb{Z}^n \) so that \( F_h \) may also be defined under periodic boundary conditions.

Properties:

\[ F_h : l_2(h\mathbb{Z}^n) \to L_2(Q_h) \] is an isometry.

\[
(F_h^{-1} g)(\xi) = \frac{1}{(2\pi)^n} \int_{Q_h} g(x) \exp(-ix \cdot \xi) d\xi.
\]
Discrete Fourier Transform

\[
(\mathcal{F}_h g)(\xi) = \begin{cases} 
\frac{h^n}{(2\pi)^{n/2}} \sum_{x \in h\mathbb{Z}^n} g(x) \exp(i x \cdot \xi) & \text{for} \quad \xi \in Q_h \\
0 & \text{for} \quad \xi \in \mathbb{R}^n \setminus Q_h.
\end{cases}
\]

Notice that \( Q_h = (-\frac{\pi}{h}, \frac{\pi}{h})^n \) may be identified with the \( n \)-torus \( \mathbb{R}^n / \frac{2\pi}{h} \mathbb{Z}^n \) so that \( \mathcal{F}_h \) may also be defined under periodic boundary conditions.

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**Properties:**

- \( \mathcal{F}_h : \ell_2(h\mathbb{Z}^n) \to L_2(Q_h) \) is an isometry.

- \( (\mathcal{F}_h^{-1} g)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{Q_h} g(x) \exp(-ix \cdot \xi) \, dx \).
The solution $\psi(x, t) = \exp(-m^2 t) \exp(t \Delta_h) \phi_0(x)$ may be represented as

$$\psi(x, t) = (\mathcal{F}_h^{-1} \hat{\psi}_h(\cdot, t))(x) = \frac{1}{(2\pi)^n} \int_{Q_h} \hat{\psi}_h(\xi, t) \exp(-ix \cdot \xi) \, d\xi,$$

whereby $\hat{\psi}_h$ is a solution of

$$\begin{cases}
\partial_t^2 \hat{\psi}_h(\xi, t) = (-d_h^2 + m^2) \hat{\psi}_h(\xi, t) & \text{for } (\xi, t) \in Q_h \times [0, \infty) \\
\hat{\psi}_h(\xi, 0) = \hat{\phi}_{0h}(\xi) & \text{for } \xi \in Q_h \\
\nabla_\tau \hat{\psi}_h(\xi, 0) = \hat{\phi}_{1h}(\xi) & \text{for } \xi \in Q_h
\end{cases}$$

Notice that $\mathcal{F}_h(\Delta_h \psi(x, t)) = -d_h^2 \mathcal{F}_h \psi(x, t)$, with

$$d_h^2 = \frac{4}{h^2} \sum_{j=1}^n \sin^2 \left( \frac{h \xi_j}{2} \right).$$
From the set of identities $d_h^2 = \frac{2n}{h^2} - \frac{2}{h^2} \sum_{j=1}^{n} \cos(h\xi_j)$ and

$$\int_{-\pi}^{\pi} \exp(-iks) \exp(u \cos s) \, ds = 2\pi I_k(u)$$

it follows straightforwardly that

$$W_h(x, t) := \frac{1}{(2\pi)^{n/2}} \int_{Q_h} \exp \left( \frac{4t}{h^2} d_h^2 \right) \exp(-ix \cdot \xi) \, d\xi$$

$$= \frac{(2\pi)^{n/2}}{h^n} \exp \left( -\frac{2nt}{h^2} \right) \prod_{j=1}^{n} I_{\frac{x_j}{h}} \left( \frac{2t}{h^2} \right)$$

Moreover, $\psi(x, t) = \exp(-m^2t) \exp(t\Delta_h)\phi_0(x)$ may be expanded in terms of the following **convolution formula**, involving the modified Bessel functions $I_k(u)$:

$$\psi(x, t) = \sum_{y \in h\mathbb{Z}^n} h^n \exp(-m^2t) W_h(x - y, t) \phi_0(y).$$
Differential-Difference Klein-Gordon problem
A Duhamel type formula

\[
\begin{align*}
\partial_t^2 \psi(x, t) &= \Delta_h \psi(x, t) - m^2 \psi(x, t) \quad \text{for} \quad (x, t) \in h\mathbb{Z}^n \times [0, \infty) \\
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\end{align*}
\]

**Formal solution:**

\[
\psi(x, t) = \cosh \left( t \sqrt{\Delta_h - m^2} \right) \phi_0(x) + \frac{\sinh \left( t \sqrt{\Delta_h - m^2} \right)}{\sqrt{\Delta_h - m^2}} \phi_1(x)
\]

**On the Fourier domain:**

\[
\hat{\psi}_h(\xi, t) = \cos \left( t \sqrt{d_h^2 + m^2} \right) \hat{\phi}_0(\xi) + \frac{\sin \left( t \sqrt{d_h^2 + m^2} \right)}{\sqrt{d_h^2 + m^2}} \hat{\phi}_1(\xi)
\]
Relativistic Wave Equations on the lattice

Some ideas on semigroups
Evolution type problems on the lattice
Heat propagators
Wave propagators

Differential-Difference Klein-Gordon problem
A Duhamel type formula

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\begin{align*}
\frac{\partial^2}{\partial t^2} \psi(x, t) &= \Delta_h \psi(x, t) - m^2 \psi(x, t) & \text{for } (x, t) \in h\mathbb{Z}^n \times [0, \infty) \\
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- **On the Fourier domain:**

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\hat{\psi}_h(\xi, t) = \cos\left( t \sqrt{d_h^2 + m^2} \right) \hat{\phi}_0(h) + \frac{\sin\left( t \sqrt{d_h^2 + m^2} \right)}{\sqrt{d_h^2 + m^2}} \hat{\phi}_1(h)
\]
Differential-Difference Klein-Gordon problem
The Laplace Transform Technique

Formal solution on the Fourier domain

\[ \hat{\psi}_h(\xi, t) = \cos \left( t \sqrt{d_h^2 + m^2} \right) \hat{\phi}_0 h(x) + \frac{\sin \left( t \sqrt{d_h^2 + m^2} \right)}{\sqrt{d_h^2 + m^2}} \hat{\phi}_1 h(x) \]

- Fourier + Inverse of Laplace Transform:

\[ \hat{\psi}_h(\xi, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s}{s^2 + d_h^2 + m^2} \hat{\phi}_0 h(\xi) e^{st} ds + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s^2 + d_h^2 + m^2} \hat{\phi}_1 h(\xi) e^{st} ds \]

- Remark: Fundamental solutions of \( \frac{1}{s}(-\Delta_h + m^2 + s^2) \) resp. \(-\Delta_h + m^2 + s^2\) are encoded on the above summation formula (Watson type integrals from the book \textit{A Treatise On The Theory Of Bessel Functions}).
Formal solution on the Fourier domain

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Formal solution on the Fourier domain

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Hypergeometric series representation

\[
\cosh \left( t \sqrt{\Delta_h^2 - m^2} \right) = \, _0F_1 \left( \frac{1}{2}; \frac{t^2}{4} (\Delta_h^2 - m^2) \right)
\]

\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\, _1F_1 \left( -1; \frac{1}{2}; -s^{-1} \right)}{s} \exp \left( \frac{st^2}{4} (\Delta_h - m^2) \right) ds
\]

\[
\sinh \left( t \sqrt{\Delta_h^2 - m^2} \right) \sqrt{\Delta_h^2 - m^2} = t \, _0F_1 \left( \frac{3}{2}; \frac{t^2}{4} (\Delta_h^2 - m^2) \right)
\]

\[
= \frac{t}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\, _1F_1 \left( -1; \frac{3}{2}; -s^{-1} \right)}{s} \exp \left( \frac{st^2}{4} (\Delta_h - m^2) \right) ds
\]

Remark: On the above formulae the 'discrete' heat semigroup is involved [as for the case of the 'discrete' Poisson semigroup \( \exp \left( -t \sqrt{\Delta_h^2 - m^2} \right) \)].
For further comparisons see, for instance, the following references:

- **Tao, T. (2006).** *Nonlinear dispersive equations: local and global analysis* (No. 106). American Mathematical Soc.

- **Baaske, F., Bernstein, S., De Ridder, H., & Sommen, F. (2014).** *On solutions of a discretized heat equation in discrete Clifford analysis*. Journal of Difference Equations and Applications, 20(2), 271-295.

- **Ciaurri, Ó., Gillespie, T. A., Roncal, L., Torrea, J. L., & Varona, J. L. (2017).** *Harmonic analysis associated with a discrete Laplacian*. Journal d'Analyse Mathématique, 132(1), 109–131.
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Further Generalizations
Some Remarks

Model Problem

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L_t \psi(x, 0) &= \phi_1(x) \quad \text{for } x \in h\mathbb{Z}^n
\end{align*}
\]

associated to the discrete Laplacian $\Delta_h$ on $h\mathbb{Z}^n$, and a mass term $m$.

Umbral calculus correspondence: In case where $L_t^* = L_t$ (self-adjoint type operator) satisfies $L_t = \ell (\partial_t)$, where $\ell (\partial_t)$ denotes a Taylor series expansion in terms of $\partial_t$, the solution is uniquely determined by the operational formula

\[
\psi(x, t) = \cosh(t \ell^{-1}(\sqrt{\Delta_h - m^2}))\phi_0(x) + \frac{\sinh(t \ell^{-1}(\sqrt{\Delta_h - m^2}))}{\sqrt{\Delta_h - m^2}} \phi_1(x).
\]
Central difference operator

\[
L_t \psi(x, t) := \frac{\psi(x, t + \frac{\tau}{2}) - \psi(x, t - \frac{\tau}{2})}{\tau} = \frac{2}{\tau} \sinh \left( \frac{\tau}{2} \partial_t \right) \psi(x, t).
\]

**Remark:** \( \ell^{-1}(s) = \frac{2}{\tau} \sinh^{-1} \left( \frac{\tau}{2} s \right) \)

**Operational formula:**

\[
\psi(x, t) = \cosh \left( \frac{2t}{\tau} \sinh^{-1} \left( \frac{\tau}{2} \sqrt{\Delta_h - m^2} \right) \right) \phi_0(x) + \\
\sinh \left( \frac{2t}{\tau} \sinh^{-1} \left( \frac{\tau}{2} \sqrt{\Delta_h - m^2} \right) \right) \frac{\phi_1(x)}{\sqrt{\Delta_h - m^2}}.
\]
Operational formula on the Fourier domain

\[ \hat{\psi}_h(\xi, t) = \cos \left( \frac{2t}{\tau} \sin^{-1} \left( \frac{\tau}{2} \sqrt{d^2_h + m^2} \right) \right) \hat{\phi}_{0h}(\xi) + \]
\[ + \frac{\sin \left( \frac{2t}{\tau} \sin^{-1} \left( \frac{\tau}{2} \sqrt{d^2_h + m^2} \right) \right)}{\sqrt{d^2_h + m^2}} \hat{\phi}_{1h}(\xi). \]

Connection with the Chebyshev polynomials \( T_k(z) \) & \( U_{k-1}(z) \): From the property \( 2 \sin^{-1}(z) = \cos^{-1}(1 - 2z^2) \) (0 ≤ \( z \) ≤ 1): 

\[ \hat{\psi}_h(\xi, t) = T_{\frac{t}{\tau}} \left( 1 - \frac{\tau^2}{2} (d^2_h + m^2) \right) \hat{\phi}_{0h}(\xi) + \]
\[ + U_{\frac{t}{\tau}-1} \left( 1 - \frac{\tau^2}{2} (d^2_h + m^2) \right) \sqrt{1 - \left( 1 - \frac{\tau^2}{2} (d^2_h + m^2) \right)^2} \frac{1}{\sqrt{d^2_h + m^2}} \hat{\phi}_{1h}(\xi). \]
Sketch of some ongoing results

Let us now consider the following evolution problem

\[
\begin{aligned}
\frac{\psi(x,t+\tau)+\psi(x,t-\tau)-2\psi(x,t)}{\tau^2} &= (\Delta_h - m^2)\psi(x,t) \quad , (x, t) \in h\mathbb{Z}^n \times \\
\psi(x,0) &= \phi_0(x) \quad , x \in h\mathbb{Z}^n \\
L_t\psi(x,0) &= \phi_1(x) \quad , x \in h\mathbb{Z}^n
\end{aligned}
\]

Connection with the Cauchy-Kovaleskaya extension: The factorization property \( \Delta_h - m^2 = -(D_h - m\gamma)^2 \) involving the finite difference Dirac operator with mass term \( m > 0 \), and chiral term \( \gamma(D_h^2 = -\Delta_h, \gamma^2 = +1 \& \gamma D_h + D_h \gamma = 0) \) assures that

\[
\psi(x, t) = \exp\left(t\varepsilon \ell^{-1}(D_h - m\gamma)\right)\phi_0(x)
\]

is a solution of the above problem whenever

1. \( \varepsilon^2 = +1, \gamma\varepsilon + \varepsilon\gamma = 0 \& \gamma D_h + D_h \gamma = 0 \)
2. \( L_t = \ell(\partial_t) \& L_t^2 \psi(x, t) = \frac{\psi(x,t+\tau)+\psi(x,t-\tau)-2\psi(x,t)}{\tau^2} \).
3. \( \phi_1(x) = (D_h - m\gamma)\phi_0(x) \).
Sketch of some ongoing results

Let us now consider the following evolution problem

\[
\begin{aligned}
\psi(x,t+\tau)+\psi(x,t-\tau)-2\psi(x,t) \\
\quad = (\Delta_h - m^2)\psi(x,t) \quad , \quad (x,t) \in h\mathbb{Z}^n \times h\mathbb{Z}^n
\end{aligned}
\]

\[
\psi(x,0) = \phi_0(x) \quad , \quad x \in h\mathbb{Z}^n
\]

\[
L_t\psi(x,0) = \phi_1(x) \quad , \quad x \in h\mathbb{Z}^n
\]

**Connection with the Cauchy-Kovaleskaya extension:** The factorization property \( \Delta_h - m^2 = -(D_h - m\gamma)^2 \) involving the finite difference Dirac operator with mass term \( m > 0 \), and chiral term \( \gamma \) \((D_h^2 = -\Delta_h, \gamma^2 = +1 & \gamma D_h + D_h \gamma = 0)\) assures that

\[
\psi(x,t) = \exp \left( t\varepsilon \ell^{-1}(D_h - m\gamma) \right) \phi_0(x)
\]

is a solution of the above problem whenever

1. \( \varepsilon^2 = +1, \gamma\varepsilon + \varepsilon\gamma = 0 & \gamma D_h + D_h \gamma = 0 \)
2. \( L_t = \ell(\partial_t) \) & \( L_t^2 \psi(x, t) = \frac{\psi(x,t+\tau)+\psi(x,t-\tau)-2\psi(x,t)}{\tau^2} \).
3. \( \phi_1(x) = (D_h - m\gamma)\phi_0(x) \).
Sketch of some ongoing results

Let us now consider the following evolution problem

\[
\begin{aligned}
\frac{\psi(x,t+\tau)+\psi(x,t-\tau) - 2\psi(x,t)}{\tau^2} &= (\Delta_h - m^2)\psi(x,t), \quad (x, t) \in h\mathbb{Z}^n \times h\mathbb{Z}^n \\
\psi(x,0) &= \phi_0(x), \quad x \in h\mathbb{Z}^n \\
L_t\psi(x,0) &= \phi_1(x), \quad x \in h\mathbb{Z}^n
\end{aligned}
\]

**Connection with the Cauchy-Kovaleskaya extension:** The factorization property \(\Delta_h - m^2 = -(D_h - m\gamma)^2\) involving the finite difference Dirac operator with mass term \(m > 0\), and chiral term \(\gamma\) \((D_h^2 = -\Delta_h, \gamma^2 = +1 \& \gamma D_h + D_h\gamma = 0)\) assures that

\[
\psi(x,t) = \exp\left(t\epsilon \ell^{-1}(D_h - m\gamma)\right) \phi_0(x)
\]

is a solution of the above problem whenever

1. \(\epsilon^2 = +1, \gamma \epsilon + \epsilon \gamma = 0 \& \gamma D_h + D_h\gamma = 0\)
2. \(L_t = \ell(\partial_t) \& L_t^2 \psi(x,t) = \frac{\psi(x,t+\tau)+\psi(x,t-\tau) - 2\psi(x,t)}{\tau^2}\)
3. \(\phi_1(x) = (D_h - m\gamma)\phi_0(x)\).
For further comparisons see, for instance, the following references:

**Constales, D., & De Ridder, H. (2014).** *A Compact Cauchy-Kovalevskaya Extension Formula in Discrete Clifford Analysis*. Advances in Applied Clifford Algebras, 24(4), 1005-1010.

**Faustino, N. (2014).** *Classes of hypercomplex polynomials of discrete variable based on the quasi-monomiality principle*. Applied Mathematics and Computation, 247, 607-622.

**Faustino, N. (2016).** *Solutions for the Klein-Gordon and Dirac equations on the lattice based on Chebyshev polynomials*. Complex Analysis and Operator Theory, 10(2), 379-399.
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Faustino, N. (2016). *Solutions for the Klein-Gordon and Dirac equations on the lattice based on Chebyshev polynomials*. Complex Analysis and Operator Theory, 10(2), 379-399.
Mathematics has the completely false reputation of yielding infallible conclusions. Its infallibility is nothing but identity. Two times two is not four, but it is just two times two, and that is what we call four for short. But four is nothing new at all. And thus it goes on and on in its conclusions, except that in the higher formulas the identity fades out of sight.

Johann Wolfgang von Goethe