Abstract. We present the study of two 3-brane system embedded in a 5-dimensional space-time in which the fifth dimension is compactified on a $S^1/Z_2$ orbifold. Assuming isotropic, homogeneous, and static branes, it can be shown that the dynamics of one brane is dominated by the other one when the metric coefficients have a particular form. We study the resulting cosmologies when one brane is dominated by a given single-fluid component.

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INTRODUCTION

In the last years, there has been a great expectation about braneworld scenario as an alternative proposal to explain some unsolved problems in particle physics and cosmology, including the hierarchy and the dark matter-energy problems. It is in this context that diverse braneworld models have emerged in an attempt to put a final point to these problems.

Superstring and M-theory suggest that we may live in a world that has more than three spatial dimensions [1]. Because only three of these dimensions are presently observable, one has to explain why the others are hidden from detection. One such explanation is the so-called Kaluza-Klein (KK) compactification (see [2] and [3] for a review), according to which the extra dimensions are very small, probably with a size of the order of the Planck length. As a consequence, modes that have momentum in the directions of the extra dimensions are excited at currently inaccessible energies.

In 1998, N. Arkani-Hamed, S. Dimopoulus and G. R. Dvali (ADD) [4] pointed out that the extra dimensions are not necessarily small, and may even be in the scale of millimeters. This model assumes that the Standard Model fields are confined to a three dimensional surface (a 3-brane) embedded in a larger dimensional background spacetime (bulk) where the gravitational field is free to propagate. Additional fields may live only on the brane or in the bulk, provided that their current undetectability is consistent with experimental bounds [5].

An alternative approach was proposed by L. Randall and R. Sundrum (RS) [6], which will be hereafter referred to as RS1 model (see [7] for a general treatment of two 3-branes in a RS setup). The bulk in this model is 5-dimensional, with the extra dimension being compactified on an $S_1/Z_2$ orbifold i.e. the extra dimension $y$ is periodic, and its
ends points are identified. In such a setting, the bulk necessarily contains two 3-branes, located, respectively, at the fixed points \( y = 0 \) and \( y = y_c \). The brane at \( y = 0 \) is usually called hidden (or Planck) brane, and the one at \( y = y_c \) is called visible (or TeV) brane.

It is then natural to ask if the evolution of one brane is related to the other one (just as in the RS1 model but in general for any metric). Binetruy et al [8] have showed that there exists a relationship between the energy density for the two branes. The goal of this paper is to generalize these relations found by Binetruy et al, and to analyze their cosmological evolution. As an example, we examine a single fluid component for the brane universes in this paper.

**MATHEMATICAL BACKGROUND**

We begin with the most general 5-dimensional metric in which flat branes lie in a homogeneous and isotropic subspace (located at \( y = 0 \) and \( y = y_c \))

\[
ds^2 = -n^2(\tau, y)dt^2 + a^2(\tau, y)\delta_{ij}dx^idx^j + b^2(\tau, y)dy^2. \tag{1}
\]

The imposed symmetries: \((x^\mu, y) \to (x^\mu, -y)\) (reflection) and \((x^\mu, y) \to (x^\mu, y + 2my_c)\) \( m = 1, 2, \ldots \) (compactification), demand that each one of the metric coefficients, \(a(t, y), n(t, y),\) and \(b(t, y)\), is subjected to the following conditions [8]:

\[
\begin{align*}
F(t, y) &= F(t, |y|), \quad (2) \\
[F']_0 &= 2F'|_{y=0+}, \quad (3) \\
[F']_c &= -2F'|_{y=y_c-}, \quad (4) \\
F''(t, y) &= F'' + [F']_0\delta(y) + [F']_c\delta(y-y_c). \quad (5)
\end{align*}
\]

In the above equations, \(F(t, y)\) represents any of the metric coefficients, the prime denotes derivative with respect to \( y \), the square brackets denote the jump in the first derivative at \( y = 0 \) and at \( y = y_c \). Eq. (5) is obtained if we demand that \(|y'| = 1\), and \(|y|'' = 2\delta(y) - 2\delta(y - y_c)\) in the interval \([0, y_c]\). We define \(F'' \equiv \frac{a^2F(t, |y|)}{dy^2}\). It is necessary here to recall that the subindex 0 will be used for quantities valued at \( y = 0 \); likewise, the subindex \( c \) will be used for quantities valued at \( y = y_c \).

The five-dimensional Einstein equations, \(\tilde{G}_{AB} = \kappa_5^2 \tilde{T}_{AB}\) for metric (1) are

\[
\begin{align*}
\tilde{G}_{00} &= 3\left\{\frac{\dot{a}}{a} \left(\frac{\dot{a} + b}{b}\right) - \frac{n^2}{b^2} \left[\frac{a''}{a} + \frac{a'}{a} \left(\frac{a'}{a} - \frac{b'}{b}\right)\right]\right\}, \quad (6) \\
\tilde{G}_{ij} &= \frac{a^2}{b^2} \delta_{ij} \left\{\frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} + \frac{2\dot{n}}{n}\right) - \frac{b'}{b} \left(\frac{n'}{n} + \frac{2\dot{a}}{a} + \frac{2a''}{a} + \frac{n''}{n}\right)\right\} + \frac{a^2}{b^2} \delta_{ij} \left\{\frac{\dot{a}}{a} \left(-\frac{\dot{a}}{a} + \frac{2\dot{n}}{n}\right) - \frac{\dot{b}}{b} \left(-\frac{2\dot{a}}{a} + \frac{n}{n}\right) - \frac{b}{b}\right\}, \quad (7) \\
\tilde{G}_{05} &= 3\left(\frac{\dot{a}n'}{a} + \frac{b\dot{a}}{a} - \frac{\dot{a}'}{a}\right), \quad (8)
\end{align*}
\]
\[ \mathcal{G}_{55} = 3 \left\{ \frac{a'^2}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b^2}{n^2} \left[ \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \left( \frac{\ddot{n}}{n} - \frac{\dot{n}}{n} \right) \right] \right\}. \]  

(10)

For the moment we consider the energy-momentum tensor associated with the brane in an empty bulk

\[ \tilde{T}_A^B = \frac{\delta(y)}{b_0} \text{diag}(-\rho_0, p_0, p_0, p_0, 0) + \frac{\delta(y-y_0)}{b_c} \text{diag}(-\rho_c, p_c, p_c, p_c, 0), \]  

(11)

i.e., only the brane contributions to the energy-momentum tensor at \( y = 0 \) and \( y = y_c \) are taken into account, this leads us, using the Bianchi identity, \( \nabla_A \tilde{G}^A_B = 0 \), to an equation of conservation for the energy density of the form

\[ \dot{\rho}_0 + 3(p_0 + \rho_0) \frac{\dot{a}_0}{a_0} = 0. \]  

(12)

According to the Israel junction conditions, we need to describe the presence of an energy density in terms of a discontinuity in the metric across the origin in the extra coordinate. In this way, using Eqs. (5), and (11) in the \((0,0)\) and \((i,j)\) components of the Einstein tensor valued at \( y = 0 \), it is not difficult to find the following relations for the metric coefficients:

\[ \left[ \frac{a'}{a} \right]_0 = -\frac{\kappa^2(5)}{3} \rho_0, \quad \left[ \frac{n'}{n} \right]_0 = \frac{\kappa^2(5)}{3} (3p_0 + 2\rho_0). \]  

(13)

This implies that the jump in the first derivative of the metric coefficients is proportional to the energy density across the origin. Similarly, at \( y = y_c \), we have

\[ \left[ \frac{a'}{a} \right]_c = -\frac{\kappa^2(5)}{3} \rho_c, \quad \left[ \frac{n'}{n} \right]_c = \frac{\kappa^2(5)}{3} (3p_c + 2\rho_c). \]  

(14)

To be consistent with the Friedmann-Robertson-Walker (FRW) metric at the origin, we demand that \( n_0 = n(t, y = 0) = 1 \), and that \( \dot{n}_0 = 0 \). With the aid of Eqs. (13), and (14), in the \((5,5)\) component of Einstein tensor, together with the conditions (4) and (5), we obtain the modified Friedmann equation,

\[ \frac{\dot{a}_0^2}{a_0^2} + \frac{\dot{a}_0}{a_0} = -\frac{\kappa^4(5)}{36} \rho_0 (\rho_0 + 3p_0). \]  

(15)

For a single component dominated universe with an equation of state \( p_0 = \omega_0 \rho_0 \), we get

\[ \kappa^2(5) \rho_0(t) = \frac{6}{(3+3\omega_0)t}, \]  

(16)

which is valid only for \( \omega \neq -1 \), whereas for \( \omega = -1 \) the energy density remains constant in time. The last two equations are fundamentally different from the standard Friedmann equation in that the latter depends linearly in the energy density, whereas in Eq. (15) the dependence is on the square of the energy density.
CONNECTION BETWEEN THE TWO BRANES

Let us begin with an ansatz for the metric coefficient \(a(t, |y|)\) such that it depends on two time dependent parameters, \(\lambda(t)\) and \(\alpha(t)\), as

\[
a(t, |y|) = \alpha f(\lambda |y|).
\]

Function \(f\), and its derivative \(f'\), are well defined at \([0, y_c]\), actually \(f(0) = 1\). In this way,

\[
a(t, |y|) = a_0 f(\lambda |y|).
\]

According to Eqs. (4) and (5), we obtain

\[
[a']_0 = 2\lambda a_0 f', \quad [a']_c = -2\lambda a_0 f',
\]

where \(f' \equiv \frac{df(\lambda |y|)}{d(\lambda |y|)}\). From Eqs. (13), (14), and (19), we find

\[
\kappa^2_{(5)} \rho_c = -\frac{b_0 f'_c}{b_c g'_c g_c} \kappa^2_{(5)} \rho_0.
\]

In the last equation, \(f_c\) and \(f'_c\) are functions of \(\lambda\), and consequently, functions of \(\rho_0\), namely,

\[
\lambda = -\frac{b_0}{6 f_0} \kappa^2_{(5)} \rho_0.
\]

These last two results show that there exists a connection between the two branes just in the form of topological constraints. Now, we are interested in the connection between the equations of state in each brane. In analogy to Eq. (20), but now for the ansatz

\[
n(t, |y|) = g(\beta |y|),
\]

where \(g(0) = 1\), and using Eqs. (13), and (14), we find

\[
\kappa^2_{(5)} \rho_c(2 + 3\omega_c) = -\frac{b_0 g'_c}{b_c g_0 g_c} \kappa^2_{(5)} \rho_0(2 + 3\omega_0),
\]

where \(g' \equiv \frac{dg(\beta |y|)}{d(\beta |y|)}\). In the former equation, both \(g_c\) and \(g'_c\) are functions of \(\beta\), where

\[
\beta = \frac{b_0}{6g'_0} \kappa^2_{(5)} \rho_0(2 + 3\omega_0).
\]

Combining Eq. (23) with Eq. (20), we find that \(\omega_c\) is connected to \(\omega_0\) and \(\rho_0\), namely,

\[
(2 + 3\omega_c) = \frac{f_c g'_c f'_0}{f'_c g_c g'_0}(2 + 3\omega_0)).
\]

This last equation gives us the relationship between the equations of state valued at the position of each brane. Eqs. (20), and (25), are the main results of this paper.
As an example, let us take $b = 1$, and
\[ a(t, |y|) = a_0(t)e^{\lambda|y|}, \quad n(t, |y|) = e^{\beta|y|}. \] (26)
We can see that $f'_0 = g'_0 = 1$, $f_c = f'_c$ and $g_c = g'_c$, and therefore $\omega_0 = \omega_c$. When $\lambda = \beta < 0$, which corresponds to Anti de Sitter bulk, we have $\omega_0 = -1$ and it is precisely the case of the RS setup [6].

**CONCLUSIONS**

In this work we have showed that, in a two-brane setup, there exists a relationship between the equation of state of the two brane universes. We considered a $S_1/Z_2$ compactification, and the metric was such that we recover the FRW setup at the location $y = y_0$. Our results are of general application, and we were able to recover the Randall-Sundrum cosmology under simple assumptions.

More involved models can be analyzed, in which the dynamics of the TeV brane can match that of the standard cosmology. This is work under progress that we expect to present elsewhere.

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