Canonical coordinates with tame estimates for the defocusing NLS equation on the circle

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Abstract: In a case study for integrable PDEs, we construct real analytic, canonical coordinates for the defocusing NLS equation on the circle, specifically tailored to the needs in perturbation theory. They are defined in neighbourhoods of families of finite-dimensional invariant tori and are shown to satisfy together with their derivatives tame estimates. When expressed in these coordinates, the defocusing NLS Hamiltonian is in normal form up to order three.

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1 Introduction

In form of a case study for integrable PDEs (iPDEs), the goal of this paper is to construct canonical coordinates for the defocusing NLS (dNLS) equation, specifically tailored to the needs in perturbation theory. We consider the dNLS equation in one space dimension

$$i\partial_t u = -\partial_x^2 u + 2|u|^2 u, \quad x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$$

(1.1)

on the Sobolev space $H^s_{\mathbb{C}} \equiv H^s(\mathbb{T}, \mathbb{C})$ of complex-valued functions on $\mathbb{T}$, whose distributional derivatives up to order $s \in \mathbb{Z}_{\geq 0}$ are in $L^2(\mathbb{T}, \mathbb{C})$. Equation (1.1) can be viewed as a Hamiltonian PDE, obtained by restricting the Hamiltonian system on the phase space...
\[ H^s_c := H^s_c \times H^s_c \] with Poisson bracket and Hamiltonian given by

\[
\{F, G\}(u, v) = -i \int_0^1 (\partial_u F \partial_v G - \partial_v F \partial_u G) \, dx,
\]

\[
H^{\text{als}}(u, v) = \int_0^1 (\partial_u u \partial_x v + u^2 v^2) \, dx
\] (1.2)

to the real subspace \( H^s_c \) consisting of elements \((u, v)\) with \( v = \bar{u} \). Here \( F, G \) are \( C^1 \)-smooth complex-valued functionals on \( H^s_c \) with sufficiently regular \( L^2 \)-gradients.

Equation (1.1) can then be rewritten as \( \partial_t u = -i \partial_v H^{\text{als}} \big|_{v=\bar{u}} \). The dNLS equation is an iPDE and according to [10], admits global Birkhoff coordinates on \( H^s_c \) with \( s \in \mathbb{Z}_{\geq 0} \). To state the main results of this paper we first need to describe these coordinates in more detail: for any \( s \in \mathbb{Z}_{\geq 0} \), let

\[
h^s_c \equiv h^s(\mathbb{Z}, \mathbb{C}) := \{ x = (x_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C} : \|x\|_s < +\infty \}, \quad \|x\|_s := \sum_{n \in \mathbb{Z}} (n)^{2|s|} |x_n|^2, \quad \langle n \rangle := \max\{1, |n|\},
\]

\[
h^s := h^s(\mathbb{Z}, \mathbb{R}) := \{ (x_n)_{n \in \mathbb{Z}} \in h^s_c : x_n \in \mathbb{R} \quad \forall n \in \mathbb{Z} \}\]

and

\[
h^s_c := h^s_c \times h^s_c, \quad h^s := h^s \times h^s.
\]

The Sobolev space \( H^s_c \) can then be described by

\[
H^s_c = \left\{ u = \sum_{n \in \mathbb{Z}} u_n e^{2\pi i n x} : (u_n)_{n \in \mathbb{Z}} \in h^s_c \right\}, \quad \|u\|_s := \|(u_n)_{n \in \mathbb{Z}}\|_s.
\]

Furthermore let

\[
\ell^{1, 2} \equiv \ell^{1, 2}(\mathbb{Z}, \mathbb{R}) := \left\{ x = (x_n)_{n \in \mathbb{Z}} \subseteq \mathbb{R} : \|x\|_{1, 2} := \sum_{n \in \mathbb{Z}} (n)^2 |x_n| < +\infty \right\},
\]

\[
\ell^{1, 2}_+ := \{ (x_n)_{n \in \mathbb{Z}} \in \ell^{1, 2} : x_n \geq 0, \quad \forall n \in \mathbb{Z} \}\]

and define the following version \( F_{\text{als}} \) of the Fourier transform, introduced in [10],

\[
F_{\text{als}} : H^0_c \to h^0_c, \quad (u, v) \mapsto \left( -\frac{1}{\sqrt{2}} (u_{-n} + v_n), -\frac{i}{\sqrt{2}} (u_{-n} - v_n) \right), \quad (1.3)
\]

where \( u_n \) denotes the \( n \)th Fourier coefficient of \( u \), \( u_n := \int_0^1 u(x) e^{-2\pi i n x} \, dx \). Note that for \( v = \bar{u} \), one has \( v_n = \bar{u}_{-n} \) for any \( n \in \mathbb{Z} \), implying that

\[
F_{\text{als}}(u, \bar{u}) = ( -\sqrt{2} \text{Re}(u_{-n}), \sqrt{2} \text{Im}(u_{-n}) ).
\]
The inverse of $F_{\text{nls}}$ is then given by

$$F_{\text{nls}}^{-1} : h_c^0 \to H_c^0, \quad (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}} \mapsto \left(-\frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} (x_n-i y_n) e^{2\pi i nx}, -\frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} (x_n+i y_n) e^{2\pi i nx}\right).$$

Finally we recall that a possibly nonlinear map $F : U \to Y$ of a subset $U$ of a Banach space $X$ into another Banach space $Y$ is said to be bounded if $F(V)$ is bounded for any bounded subset $V$ in $U$.

**Theorem 1.1** ([10, 14]). There exists a neighbourhood $\mathcal{W}$ of $H_c^0$ in $H_c^0$ and an analytic map

$$\Phi_{\text{nls}} : \mathcal{W} \to h_c^0, \quad (u, v) \mapsto (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}}$$

with $\Phi_{\text{nls}}(0) = 0$ such that the following holds:

(B1) For any $s \in \mathbb{Z}_{\geq 0}$, $\Phi_{\text{nls}}(H_c^s) \subseteq h_c^s$ and $\Phi_{\text{nls}} : H_c^s \to h_c^s$ is a real analytic diffeomorphism.

(B2) The map $\Phi_{\text{nls}}$ is canonical, meaning that on $\mathcal{W}$, $\{x_n, y_n\} = -1$ and all the other brackets between coordinate functions vanish.

(B3) The Hamiltonian $H_{\text{nls}} := H_{\text{nls}} \circ (\Phi_{\text{nls}})^{-1}$, defined on $h_c^s$, is a function of the actions $I_n := (x_n^2 + y_n^2)/2, n \in \mathbb{Z}$, only and $H_{\text{nls}} : \ell^1_{+} \to \mathbb{R}, I \mapsto H_{\text{nls}}(I)$ is real analytic.

(B4) The differential $d_0 \Phi_{\text{nls}}$ of $\Phi_{\text{nls}}$ at 0 is the Fourier transform $F_{\text{nls}}$ defined in (1.3).

(B5) The nonlinear parts $A_{\text{nls}} := \Phi_{\text{nls}} - F_{\text{nls}}$ of $\Phi_{\text{nls}}$ and $B_{\text{nls}} := \Psi_{\text{nls}} - F_{\text{nls}}^{-1}$ of $\Psi_{\text{nls}} := (\Phi_{\text{nls}})^{-1}$ are one smoothing, meaning that for any $s \in \mathbb{Z}_{\geq 1}$,

$$A_{\text{nls}} : H_r^s \to h_r^{s+1} \quad \text{and} \quad B_{\text{nls}} : h_r^s \to H_r^{s+1}$$

are real analytic and bounded.

The maps $\Phi_{\text{nls}}, \Psi_{\text{nls}}$ are referred to as Birkhoff maps and the coordinates $((x_n, y_n))_{n \in \mathbb{Z}}$ as Birkhoff coordinates for the dNLS equation.

Birkhoff coordinates are a tool to study perturbations of the dNLS equation far away from the equilibrium. In particular, in [2] they were used to show the existence of finite-dimensional invariant tori of large size for Hamiltonian perturbations of this equation, involving no derivatives of $u$. So far, no such results have been obtained for
perturbations involving $\partial_x u$ (and possibly $\partial_x^2 u$) — see [2, 4–7, 9, 12, 17] for results on perturbations of the dNLS equation on the circle obtained so far. In view of the recent results in [1] concerning the existence of small quasi-periodic solutions of quasi-linear Hamiltonian perturbations of the Karteweg de Vries (KdV) equation and our results in [2] described above, we expect that Hamiltonian perturbations of the dNLS equation, involving $\partial_x u$ (and possibly $\partial_x^2 u$), also admit large quasi-periodic solutions, also referred to as multi-solitons. For this purpose, the scheme developed in [2] has to be considerably refined. In particular, canonical coordinates are needed which together with their derivatives satisfy tame estimates. In [19], such estimates were derived for $\Phi^{\text{nls}} : H^0_t \to h^0_r$ on the real subspaces $H^s_r$ and for its inverse $\Psi^{\text{nls}} : h^0_r \to H^0_t$ on the real subspaces $h^s_r$ where $s \in \mathbb{Z}_{\geq 2}$. But so far they are not available for their derivatives. In this paper, we prove how to use the Birkhoff coordinates to construct near bounded, integrable, finite-dimensional subsystems (iSS) of the dNLS equation, local canonical coordinates so that they satisfy tame estimates and the dNLS Hamiltonian, when expressed in these coordinates, is in normal form up to order three—see Theorem 1.2 for a precise statement.

In future work, we will use these coordinates as a starting point for applying a KAM scheme to reduce certain linear operators with tame estimates, which come up in the Nash Moser iteration, to operators with constant coefficients. Recently, such schemes have been further developed in significant ways. In the context of the dNLS equation, results of this type in [3] will be particularly relevant.

To state our main result, we need to introduce some more notation. For any $S \subseteq \mathbb{Z}$ with $|S| < +\infty$, let $S^\perp := \mathbb{Z} \setminus S$. By a slight abuse of notation, we identify $h^s_c$ with $\mathbb{C}^S \times \mathbb{C}^S \times h^s_{1c}$ and $h^s_r$ with $\mathbb{R}^S \times \mathbb{R}^S \times h^s_{1r}$ where

$$ h^s_{1c} := h^s(S^\perp, \mathbb{C}) \times h^s(S^\perp, \mathbb{C}), \quad h^s_{1r} := h^s(S^\perp, \mathbb{R}) \times h^s(S^\perp, \mathbb{R}). $$

Accordingly, an element $z \in h^0_r$ is written as

$$ z = (z_S, z_\perp), \quad z_S = (x_j)_{j \in S}, \quad z_\perp = (y_j)_{j \in S^\perp}, $$

and as norm we choose $\|z\|_s := \|z_S\| + \|z_\perp\|_s$ where

$$ \|z_S\|_s := \|z_S\|_0 := \left( \sum_{j \in S} |x_j|^2 + |y_j|^2 \right)^{\frac{1}{2}}, \quad \|z_\perp\|_s := \left( \sum_{j \in S^\perp} (|x_j|^2 + |y_j|^2) \right)^{\frac{1}{2}}. $$

Furthermore, we introduce the bilinear form

$$ (z_\perp, z_\perp')_r := \sum_{j \in S^\perp} x_j x'_j + y_j y'_j, \quad z_\perp = (x_\perp, y_\perp), \quad z'_\perp = (x'_\perp, y'_\perp) \in h_{1c} $$

(1.4)
and write the sequence of actions $I = (I_k)_{k \in \mathbb{Z}}$ as $(I_S, I_\perp)$ where

$$I_S := (I_k)_{k \in S}, \quad I_\perp := (I_k)_{k \in S^\perp}, \quad I_k \equiv I_k(z) = \frac{|z_k|^2}{2} = \frac{x_k^2 + y_k^2}{2}, \quad \forall k \in \mathbb{Z}.$$ 

Finally, we introduce the dNLS frequencies

$$\omega_n^{\text{nls}}(I) := \partial_k H^{\text{nls}}(I), \quad k \in \mathbb{Z}. \quad (1.5)$$

They satisfy asymptotics of the form $\omega_k(I) = 4k^2\pi^2 + O(1)$ as $k \to \pm \infty$. More precisely, the map

$$\ell^1,2^+ \to \ell^\infty, \quad (I_k)_{k \in \mathbb{Z}} \mapsto (\omega_n^{\text{nls}}(I) - 4\pi^2 n^2)_{n \in \mathbb{Z}}$$

is real analytic and bounded—see Proposition 5.3 in Section 5.2. The main result of this paper is the following one.

**Theorem 1.2.** Let $S \subseteq \mathbb{Z}$ be finite. For any compact subset $\mathcal{K} \subseteq \mathbb{R}^S \times \mathbb{R}^S$, there exists an open, bounded, complex neighbourhood $\mathcal{V} \subseteq h^0_c$ of $\mathcal{K} \times \{0\}$ and a bounded analytic map

$$\Psi : \mathcal{V} \to H^0_c, \quad (z_n)_{n \in \mathbb{Z}} \mapsto w$$

so that the following holds:

(C1) For any $s \in \mathbb{Z}_{\geq 0}$, $\Psi(\mathcal{V} \cap h^s_r) \subseteq H^s_r$ and $\Psi : \mathcal{V} \cap h^s_r \to H^s_r$ is a real analytic diffeomorphism onto its image.

(C2) $\Psi$ is canonical, meaning that on $\Psi(\mathcal{V} \cap h^0_r)$, $\{x_n, y_n\} = -1$ for any $n \in \mathbb{Z}$, whereas all the other brackets between coordinate functions vanish.

(C3) The transformation $\Psi$ is related to $\Psi^{\text{nls}} = (\Phi^{\text{nls}})^{-1}$ by

$$\Psi |_{\mathcal{K} \times \{0\}} = \Psi^{\text{nls}} |_{\mathcal{K} \times \{0\}}, \quad d\Psi(z) = d\Psi^{\text{nls}}(z), \quad \forall z \in \mathcal{K} \times \{0\}.$$

(C4) The Hamiltonian $\mathcal{H} := \mathcal{H}^{\text{nls}} \circ \Psi$, defined on $\mathcal{V} \cap h^1_r$, is in normal form up to order three. More precisely,

$$\mathcal{H}(z) = H^{\text{nls}}(I_S, 0) + \sum_{n \in S^\perp} \omega_n^{\text{nls}}(I_S, 0)I_n(z) + \mathcal{P}_3(z), \quad (1.6)$$

where the Hamiltonian $\mathcal{P}_3 : \mathcal{V} \cap h^1_r \to \mathbb{R}$ is real analytic. Furthermore, $\mathcal{P}_3$ satisfies the following tame estimates: for any $s \in \mathbb{Z}_{\geq 0}$, $z \in \mathcal{V} \cap h^s_r$, $\hat{z} \in h^s_c$,

$$\|\nabla \mathcal{P}_3(z)\|_s \lesssim_s \|z\|_s \|z\perp\|_0, \quad \|d\nabla \mathcal{P}_3(z)(\hat{z})\|_s \lesssim_s \|z\|_s \|\hat{z}\|_0 + \|z\perp\|_0 \|\hat{z}\|_s \quad (1.7)$$
and for any \( k \in \mathbb{Z}_{\geq 2}, \hat{z}_1, \ldots, \hat{z}_k \in h^s_c, \)

\[
\|d^k \nabla \mathcal{P}_3(z)[\hat{z}_1, \ldots, \hat{z}_k]\|_s \lesssim_s \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z\|_s \prod_{j=1}^k \|\hat{z}_j\|_0.
\]

Here, the meaning of \( \lesssim_s \) is the standard one. So, for example, \( \|\nabla \mathcal{P}_3(z)\|_s \lesssim_s \|z\|_s \|z\|_0 \) says that there exists a constant \( C \equiv C(s) > 0 \) so that

\[
\|\nabla \mathcal{P}_3(z)\|_s \leq C \|z\|_s \|z\|_0, \quad \forall z \in \mathcal{V} \cap h^s_r.
\]

(C5) The nonlinear maps \( B := \Psi - F_{\text{nls}}^{-1} : \mathcal{V} \cap h^0_r \rightarrow H^0_r \) and \( A := \Psi^{-1} - F_{\text{nls}} : \Psi(\mathcal{V}) \cap H^0_r \rightarrow h^0_r \) are real analytic maps and so is

\[
A : \mathcal{V} \cap h^0_r \rightarrow \mathcal{L}(H^0_c, h^0_c), \quad z \mapsto A(z) := \text{d}\Psi(z)^{-1} - F_{\text{nls}}.
\]

On \( \mathcal{V} \cap h^0_r \), the maps \( B \) and \( A \) satisfy the following estimates: for any \( z \in \mathcal{V} \cap h^0_r, k \in \mathbb{Z}_{\geq 1}, \hat{z}_1, \ldots, \hat{z}_k \in h^0_c, \) and \( \hat{w} \in H^0_c, \)

\[
\|B(z)\|_0 \lesssim 1, \quad \|d^k B(z)[\hat{z}_1, \ldots, \hat{z}_k]\|_0 \lesssim_k \prod_{j=1}^k \|\hat{z}_j\|_0,
\]

\[
\|A(z)[\hat{w}]\|_0 \lesssim \|\hat{w}\|_0, \quad \|d^k(A(z)[\hat{w}])[\hat{z}_1, \ldots, \hat{z}_k]\|_0 \lesssim_k \|\hat{w}\|_0 \prod_{j=1}^k \|\hat{z}_j\|_0.
\]

Furthermore, \( B \) is one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1}, B : \mathcal{V} \cap h^s_r \rightarrow H^{s+1}_r \) is real analytic, and satisfies the following tame estimates: for any \( k \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V} \cap h^s_r, \) and \( \hat{z}_1, \ldots, \hat{z}_k \in h^s_c, \)

\[
\|B(z)\|_{s+1} \lesssim_s 1 + \|z\|_s,
\]

\[
\|d^k B(z)[\hat{z}_1, \ldots, \hat{z}_k]\|_{s+1} \lesssim_{s,k} \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z\|_s \prod_{j=1}^k \|\hat{z}_j\|_0.
\]

Similarly, the maps \( A \) and \( \hat{A} \) are one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1}, A : \Psi(\mathcal{V} \cap h^s_r) \rightarrow h^{s+1}_r \) and \( \hat{A} : \mathcal{V} \cap h^s_r \rightarrow \mathcal{L}(H^s_c, h^{s+1}_c) \) are real analytic. Moreover, \( \hat{A} \) satisfies the following tame estimates: for any \( z \in \mathcal{V} \cap h^s_r, \hat{w} \in H^s_c, \)

\[
\|\hat{A}(z)[\hat{w}]\|_{s+1} \lesssim_s \|z\|_s \|\hat{w}\|_0 + \|\hat{w}\|_s.
\]
and for any \( k \in \mathbb{Z}_{\geq 1}, \hat{z}_1, \ldots, \hat{z}_k \in h^c_c, \)

\[
\|d^k(A(z)(\hat{w}))[\hat{z}_1, \ldots, \hat{z}_k]\|_{s+1} \lesssim_{s,k} \left( \|z_{\perp}\|_s \|\hat{w}\|_0 + \|\hat{w}\|_s \right) \prod_{j=1}^k \|\hat{z}_j\|_0 \\
+ \|\hat{w}\|_0 \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0.
\]

**Remark 1.1.** In Theorem 1.2, apart from being compact, no further assumptions on \( \mathcal{K} \) are being made. In particular, \( \mathcal{K} \) may contain the equilibrium point 0 in which case \( \mathcal{K} \) does not admit action-angle coordinates. In subsequent work, the estimates for \( A(z) = d\Psi(z)^{-1} - F_{\text{nls}} \) will be used to study perturbations of the dNLS equation. Since such estimates are not needed for \( A(\Psi(z)) \), we have not included them in Theorem 1.2. □

**Remark 1.2.** In Section 7 we present additional results about the map \( \Psi \). In particular we study the restrictions of \( \Psi \) to \( \mathcal{V} \cap h^0_{r,1} \) and \( \mathcal{V} \cap h^0_{r,2} \) where \( h^0_{r,1} \) and \( h^0_{r,2} \) are the dNLS invariant subspaces, corresponding via the Birkhoff map \( \Phi_{\text{nls}} \) to potentials \( \phi \in H^0_c \) which are even and, respectively, odd. □

**Outline of the construction of \( \Psi \):** Let \( \mathcal{V} \) be of the form \( \mathcal{V} = \mathcal{V}_S \times \mathcal{V}_{\perp} \subset h^0_c \) where \( \mathcal{V}_S \) is a bounded, open neighbourhood of \( \mathcal{K} \) in \( C^S \times C^S \) and \( \mathcal{V}_{\perp} \) an open ball in \( h^0_{\perp,c} \), centred at \( \{0\} \). By Theorem 1.1, \( \mathcal{V}_S \) and \( \mathcal{V}_{\perp} \) can be chosen so that the Birkhoff map \( \Psi_{\text{nls}} \) is defined on \( \mathcal{V} \) and all the estimates of \( \Psi_{\text{nls}} \) and its derivatives used in the sequel are uniform on \( \mathcal{V} \). The canonical map \( \Psi \) is then defined to be the composition \( \Psi := \Psi_L \circ \Psi_C \) where \( \Psi_L \) is the Taylor expansion of \( \Psi_{\text{nls}} \) of order one in the normal directions \( z_{\perp} \) around \( (z_S, 0) \),

\[
\Psi_L(z_S, z_{\perp}) := \Psi_{\text{nls}}(z_S, 0) + d\Psi_{\text{nls}}(z_S, 0)(0, z_{\perp}),
\]

and \( \Psi_C \), referred to as symplectic corrector, is chosen so that \( \Psi_L \circ \Psi_C \) becomes symplectic and satisfies the claimed tame estimates.

In his pioneering work [16], Kuksin presents a general scheme for proving KAM-type theorems for semilinear Hamiltonian perturbations of iPDEs in one space dimension, such as the KdV or the sine Gordon (sG) equations, which possess a Lax pair formulation and admit finite-dimensional integrable subsystems, foliated by invariant tori. One of the key elements of his work is a normal form theory for such PDEs. Expanding on work of Krichever [15], Kuksin considers bounded iSS of such an iPDE which admit action-angle coordinates. In the case of the KdV and the sG equations, the angle variables are given by
the celebrated Its Matveev formulas. These action-angle coordinates are complemented by infinitely many coordinates whose construction is based on a set of time periodic solutions, referred to as Floquet solutions, of the PDE obtained by linearizing iPDE along solutions in $i\mathbb{S}$. The resulting coordinate transformation, denoted in [16] by $\Phi$, is typically not symplectic and to obtain canonical coordinates, an additional coordinate transformation needs to be applied. In [16], Kuksin constructs such a transformation, which he denotes by $\phi$, using arguments of Moser and Weinstein in the given infinite dimensional setup—see [16, Lemma 1.4 and Section 7.1]. To construct the map $\Psi_C$ we follow the same scheme of proof. Actually, the following result holds.

**Theorem 1.3.** Assume that in addition to the assumptions made in Theorem 1.2, the set $\mathcal{K}$ is contained in $\mathbb{R}^S \times \mathbb{R}^S$. Then, up to normalizations and natural identifications, $\Psi_L$ coincides with the map $\Phi$, obtained by applying the scheme of construction in [16] to the dNLS equation. As a consequence, so does $\Psi = \Psi_L \circ \Psi_C$ with $\Phi \circ \phi$.

Since Birkhoff coordinates provide a concise, self-contained, and efficient framework for proving Theorem 1.2—in particular the claimed tame estimates, the main goal of our study—Theorem 1.3 also provides in the case of the dNLS equation a valuable alternative for proving the normal form result for this equation, obtained by applying the scheme of proof in [16]. Note also that the assumptions on $\mathcal{K}$ in Theorem 1.2 are slightly weaker than the ones made in the setup of [16].

**Organization:** The maps $\Psi_L$ and $\Psi_C$ are introduced and studied in Sections 3 and 4, respectively, after a short Section 2, describing the Hamiltonian setup. In Section 5, we prove Theorem 1.2: in Section 5.1, we show that the composition $\Psi = \Psi_L \circ \Psi_C$ satisfies the analytic properties, stated in Theorem 1.2, and in the subsequent Section 5.2, the expansion of the dNLS Hamiltonian in the new coordinates is computed up to order three. In Section 5.3 we summarize the proof of Theorem 1.2. Finally, in Section 6 we prove Theorem 1.3 and in Section 7 results, concerning the restriction of $\Psi$ to subsets, satisfying symmetry conditions. In Appendix 1, we recall an infinite-dimensional version of the Poincaré Lemma, needed in Section 4 (cf. from [16], [18]).

**Notation:** For any $C^1$ map $F : h_c^0 \to X$ with $X$ being a Banach space, we denote by $d_\perp F(z)$ the differential of $F$ at $z$ with respect to the variable $z_\perp$,

$$
d_\perp F(z)[\tilde{z}_\perp] = \sum_{j \in S_\perp} \tilde{\mathbf{x}}_j \partial_{\mathbf{x}_j} F(z) + \tilde{\mathbf{y}}_j \partial_{\mathbf{y}_j} F(z), \quad \tilde{z}_\perp := (\tilde{\mathbf{x}}_j)_{j \in S_\perp}, (\tilde{\mathbf{y}}_j)_{j \in S_\perp} \in h_c^0.
$$
where for any $j \in S^\perp$, $\partial_{x_j} F$, $\partial_{y_j} F \in X$ denote the partial derivatives of $F$ with respect to the variables $x_j$ respectively $y_j$. Similarly, we define the gradient with respect to the variable $z_\perp$ as

$$\nabla_\perp F := (\partial_{x_j} F)_{j \in S^\perp}, (\partial_{y_j} F)_{j \in S^\perp}.$$ 

The gradient of $F$ with respect to $z_S$ is denoted by

$$\nabla_S F := (\partial_{x_j} F)_{j \in S}, (\partial_{y_j} F)_{j \in S}$$

and the differential of $F$ at $z$ with respect to $z_S$ by $d_S F(z)$,

$$d_S F(z) [\widehat{z}_S] = \sum_{j \in S} \widehat{x}_j \partial_{x_j} F(z) + \widehat{y}_j \partial_{y_j} F(z), \quad \widehat{z}_S := (\widehat{x}_j)_{j \in S}, (\widehat{y}_j)_{j \in S} \in \mathbb{C}^S \times \mathbb{C}^S.$$ 

For the partial derivatives of $F$ with respect to $z_j$, $j \in S$, we use the multi-index notation and write for any $\alpha, \beta \in \mathbb{Z}_{\geq 0}^S$

$$\partial^\alpha_\beta F := \left( \prod_{j \in S} \partial_{x_j}^{\alpha_j} \partial_{y_j}^{\beta_j} \right) F.$$ 

If not stated otherwise, $K$ denotes a compact subset of $\mathbb{R}^S \times \mathbb{R}^S$ and $V$ an open, bounded neighbourhood of $K \times \{0\}$ in $h_0^c$ of the form $V_S \times V_\perp$ where $V_\perp$ is a ball in $h_{1c}$, centred at 0. We write $V_\perp(\delta)$ to indicate that the radius of the ball $V_\perp$ is $\delta > 0$. Finally, we frequently will use the symbols $\lesssim, \lesssim_s, \ldots$ to express that a quantity is bounded by another one up to a constant which is “universal,” respectively, depends only on the Sobolev index $s$. For example, given two real valued functionals $A, B$ on $V$ we write $A \lesssim_s B$ if there is a constant $C \equiv C(s)$ so that $A(z) \leq CB(z)$ for any $z \in V \cap h_0^c$.

2 Hamiltonian Setup

In this preliminary section we discuss the Hamiltonian setup, introduced in Section 1, in more detail and introduce some additional notations.

The Hamiltonian vector field associated with a sufficiently smooth functional $\mathcal{F} : H_0^c \rightarrow \mathbb{C}$ and the Poisson bracket (1.2) on $H_0^c$ is denoted by

$$X_\mathcal{F} = i \overline{\partial} \mathcal{F}, \quad \nabla \mathcal{F} := (\nabla_u \mathcal{F}, \nabla_v \mathcal{F}),$$
where \( \nabla_u F, \nabla_v F \) denote the \( L^2 \) gradients with respect to \( u \) and \( v \), namely
\[
\text{d}F[(\hat{u},0)] = \int_T \nabla_u F \hat{u} \, dx, \quad \text{d}F[(0,\hat{v})] = \int_T \nabla_v F \hat{v} \, dx,
\]
\[
\mathbb{J} := \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix} : H^0_c \to H^0_c, \tag{2.1}
\]
and \( \text{Id} : H^0_c \to H^0_c \) is the identity operator. Furthermore we introduce the non-degenerate bilinear form
\[
\langle \cdot, \cdot \rangle_r : H^0_c \times H^0_c \to \mathbb{C}
\]
defined for any \( w = (u,v), w' = (u',v') \in H^0_c \) by
\[
\langle w, w' \rangle_r := \int_T u(x) u'(x) \, dx + \int_T v(x) v'(x) \, dx. \tag{2.2}
\]
The subscript \( r \) indicates that in the latter integrals, no complex conjugation appears. The Poisson bracket (1.2) then reads
\[
\{ F, G \} = \langle \nabla F, i\mathbb{J} \nabla G \rangle_r
\]
and the symplectic form, associated with it, is the two form
\[
\Lambda[\hat{w}, \hat{w}'] := -i \langle \mathbb{J}^{-1} \hat{w}, \hat{w}' \rangle_r = i \langle \mathbb{J} \hat{w}, \hat{w}' \rangle_r = i \int_T (\hat{u} \hat{v}' - \hat{v} \hat{u}') \, dx,
\]
\[
\forall \hat{w} = (\hat{u}, \hat{v}), \hat{w}' = (\hat{u}', \hat{v}') \in H^0_c. \tag{2.3}
\]
For any sufficiently smooth functionals \( F, G : H^0_c \to \mathbb{C} \), one has
\[
\Lambda(X_F, X_G) = \{ F, G \}.
\]
In terms of the Fourier coefficients of \( \hat{w} \) and \( \hat{w}' \), \( \Lambda[\hat{w}, \hat{w}'] \) can be expressed as
\[
\Lambda[\hat{w}, \hat{w}'] = i \sum_{k \in \mathbb{Z}} (\hat{u}_k \hat{v}'_{-k} - \hat{v}_{-k} \hat{u}'_k)
\]
and hence \( \Lambda \) can be conveniently written as
\[
\Lambda = i \sum_{k \in \mathbb{Z}} du_k \wedge dv_{-k},
\]
where
\[(du_k \wedge dv_{-k})[(\hat{u}, \hat{v}), (\hat{u}', \hat{v}')] = \hat{u}_k \hat{v}'_{-k} - \hat{v}_{-k} \hat{u}_k, \quad \forall k \in \mathbb{Z}.\]

In addition, we define the one form \(\lambda\) on \(H^0_c\) as
\[\lambda \equiv \lambda(w) = i \sum_{k \in \mathbb{Z}} u_k dv_{-k}.\]

Its action on a function \(\hat{w} = (\hat{u}, \hat{v}) \in H^0_c\) is given by
\[\lambda[\hat{w}] = i \int_T u(x) \hat{v}(x) \, dx = i \sum_{k \in \mathbb{Z}} u_k \hat{v}_{-k}.\]

The exterior differential of \(\lambda\), defined by \(d\lambda = i \sum_{k \in \mathbb{Z}} du_k \wedge dv_{-k}\), thus satisfies \(d\lambda = \Lambda\).

The Poisson bracket on the model space \(h^0_c\) is determined by defining it for the coordinate functions,
\[\{x_n, y_m\}_M = -\delta_{nm}, \quad \{y_n, x_m\}_M = \delta_{nm}, \quad \{x_n, x_m\}_M = 0, \quad \{y_n, y_m\}_M = 0, \quad \forall n, m \in \mathbb{Z}.\]

By a slight abuse of terminology in connection with the definition (1.4), we also denote by \((\cdot, \cdot)_r\) the non-degenerate bilinear form \((\cdot, \cdot)_r : h^0_c \times h^0_c \to \mathbb{C}\)
\[(z, z')_r := x \cdot x' + y \cdot y', \quad \forall z = (x, y), z' = (x', y') \in h^0_c \quad (2.4)\]
where \(x \cdot x' := \sum_{k \in \mathbb{Z}} x_k x'_k\). Given two sufficiently smooth functionals \(F, G : h^0_c \to \mathbb{C}\), one has
\[\{F, G\}_M = -\sum_k \left( \partial_{x_k} F \partial_{y_k} G - \partial_{y_k} F \partial_{x_k} G \right) = (\nabla F, J \nabla G)_r\]

where
\[J := \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix} : h^0_c \to h^0_c, \quad (2.5)\]

\(I_d : h^0_c \to h^0_c\) is the identity operator and
\[\nabla F = (\nabla_x F, \nabla_y F), \quad \nabla_x F = (\partial_{x_k} F)_{k \in \mathbb{Z}}, \quad \nabla_y F = (\partial_{y_k} F)_{k \in \mathbb{Z}}.\]
The Hamiltonian vector field $X_F$ of $F : h_c^0 \to \mathbb{C}$, corresponding to the Poisson bracket $\{\cdot, \cdot\}_M$, is then given by

$$X_F = J \nabla F$$ \hspace{1cm} (2.6)

and the symplectic form $\Lambda_M$, associated with it, by

$$\Lambda_M[\hat{z}, \hat{z}'] := (J^{-1}\hat{z}, \hat{z}')_r = \hat{y} \cdot \hat{x}' - \hat{x} \cdot \hat{y}', \hspace{0.5cm} \forall \hat{z} = (\hat{x}, \hat{y}), \hat{z}' = (\hat{x}', \hat{y}') \in h_c^0. \hspace{1cm} (2.7)$$

Note that

$$\Lambda_M = - \sum_{k \in \mathbb{Z}} dx_k \wedge dy_k$$

where as above, for any $k \in \mathbb{Z}$, the two form $dx_k \wedge dy_k$ is defined as

$$(dx_k \wedge dy_k)[(\hat{x}, \hat{y}), (\hat{x}', \hat{y}')] = \hat{x}_k \hat{y}'_k - \hat{y}_k \hat{x}'_k.$$

Then

$$\Lambda_M(X_F, X_G) = (\nabla F, J \nabla G)_r = \{F, G\}_M.$$ 

The one form associated with $\Lambda_M$ is defined as

$$\lambda_M \equiv \lambda_M(z) := \sum_{k \in \mathbb{Z}} y_k dx_k. \hspace{1cm} (2.8)$$

Its action on a vector $\hat{z} = (\hat{x}, \hat{y}) \in h_c^0$ is given by

$$\lambda_M[\hat{z}] = \sum_{k \in \mathbb{Z}} y_k \hat{x}_k.$$

The exterior differential of $\lambda_M$ then satisfies $d\lambda_M = \Lambda_M$.

3 The Map $\Psi_L$

In this section, we study the map $\Psi_L$ introduced in (1.8). In particular, we prove tame estimates and one smoothing properties for $\Psi_L$. First we introduce some more notations. Denote by $\Pi_S$ and $\Pi_\perp$ the standard projections

$$\Pi_S : (\mathbb{C}^S \times \mathbb{C}^S) \times h_{0,c}^0 \to (\mathbb{C}^S \times \mathbb{C}^S) \times \{0\}, \hspace{0.5cm} z \equiv (z_S, z_\perp) \mapsto (z_S, 0) \hspace{1cm} (3.1)$$
\[ \Pi_\perp : (\mathbb{C}^S \times \mathbb{C}^S) \times h^0_{\perp c} \to \{0\} \times h^0_{\perp c}, \quad z = (z_S, z_\perp) \mapsto (0, z_\perp). \tag{3.2} \]

The formula (1.8) for \( \Psi_L(z) \) with \( z = (z_S, z_\perp) \) then reads

\[ \Psi_L(z) = \Psi_{\text{nls}}(\Pi_S z) + d_\perp \Psi_{\text{nls}}(\Pi_S z)[z_\perp]. \tag{3.3} \]

For a quite explicit formula for \( d_\perp \Psi_{\text{nls}}(\Pi_S z)[z_\perp] \), we refer to Appendix 2. The map \( \Psi_L \) is defined on

\[ \mathcal{V}^{\text{max}} := \mathcal{V}^{\text{max}}_S \times h^0_{\perp c}, \quad \mathcal{V}^{\text{max}}_S := \Pi_S \Phi_{\text{nls}}(\mathcal{W}), \]

where \( \mathcal{W} \subseteq H^0_c \) is the domain of definition of the Birkhoff map \( \Phi_{\text{nls}} \) of Theorem 1.1. Note that

\[ \mathbb{R}^S \times \mathbb{R}^S \subseteq \mathcal{V}^{\text{max}} \subseteq \mathbb{C}^S \times \mathbb{C}^S, \quad h^0_r \subset \mathcal{V}^{\text{max}} \subset h^0_c, \quad \Psi_L(0) = 0. \]

Furthermore, the differential \( d\Psi_L(z) \) of \( \Psi_L \) at \( z = (z_S, z_\perp) \in \mathcal{V}^{\text{max}} \) applied to a vector \( \hat{z} = (\hat{z}_S, \hat{z}_\perp) \in h^0_c \) is given by

\[
d\Psi_L(z)[\hat{z}, \hat{z}_\perp] = d_S \Psi_{\text{nls}}(\Pi_S z)[\hat{z}]) + d_\perp \Psi_{\text{nls}}(\Pi_S z)[\hat{z}_\perp] + d_S (d_\perp \Psi_{\text{nls}}(\Pi_S z)[z_\perp])[\hat{z}_S]
\]
\[
= d\Psi_{\text{nls}}(\Pi_S z)[\hat{z}] + d^2 \Psi_{\text{nls}}(\Pi_S z)[\Pi_S \hat{z}, \Pi_\perp z]. \tag{3.4}
\]
\[
d\Psi_L(z)[\hat{z}, \hat{z}_\perp] = d\Psi_{\text{nls}}(\Pi_S z)[\hat{z}] + d^2 \Psi_{\text{nls}}(\Pi_S z)[\Pi_S \hat{z}, \Pi_\perp z]. \tag{3.5}
\]

The latter expression is independent of \( \Pi_\perp \hat{z} \) and that by Theorem 1.1, \( d\Psi_L(0) = d\Psi_{\text{nls}}(0) = F_{\text{nls}}^{-1} \). First we establish the following auxiliary results.

**Lemma 3.1.** (i) The map \( \Psi_L : \mathcal{V}^{\text{max}} \to H^0_c \) is analytic and for any \( s \in \mathbb{Z}_{\geq 0} \), the restriction \( \Psi_L |_{h^s_c} : h^s_c \to H^s_r \) is real analytic. Furthermore, for any \( z_S \in \mathbb{R}^S \times \mathbb{R}^S \) and any \( s \in \mathbb{Z}_{\geq 0} \), \( d\Psi_L(z_S, 0) : h^s_c \to H^s_c \) is a linear isomorphism.

(ii) For any compact subset \( \mathcal{K} \subseteq \mathbb{R}^S \times \mathbb{R}^S \), there exists a ball \( \mathcal{V}_\perp \) in \( h^0_{\perp r} \), centred at 0, so that the restriction \( \Psi_L : \mathcal{K} \times \mathcal{V}_\perp \to H^0_r \) is one to one. Furthermore, after shrinking the radius of the ball \( \mathcal{V}_\perp \), if necessary, the map \( \Psi_L : \mathcal{K} \times \mathcal{V}_\perp \to H^0_r \) is a local diffeomorphism. \( \square \)

**Proof.** (i) The claimed analyticity follows from the definition of \( \Psi_L \) and the corresponding properties of \( \Psi_{\text{nls}} \), stated in Theorem 1.1. Concerning the statement on the differential \( d\Psi_L(z_S, 0) \), note that by (3.5), \( d\Psi_L(z_S, 0) = d\Psi_{\text{nls}}(z_S, 0) \) and hence by Theorem 1.1, \( d\Psi_L(z_S, 0) : h^s_c \to H^s_c \) is a linear isomorphism for any \( s \in \mathbb{Z}_{\geq 0} \).
(ii) Let $\mathcal{K} \subseteq \mathbb{R}^S \times \mathbb{R}^S$ be a given compact subset. Assume that there exists no ball $\mathcal{V}_\perp$ in $h^0_{\mathcal{L}_r}$, centred at 0, so that $\Psi_L|_{\mathcal{K} \times \mathcal{V}_\perp}$ is one to one. Then there exist two sequences $z^{(j)} = (z^{(j)}_S, z^{(j)}_\perp)$, $j \geq 1$, and $\tilde{z}^{(j)} = (\tilde{z}^{(j)}_S, \tilde{z}^{(j)}_\perp)$, $j \geq 1$, in $\mathcal{K} \times h^0_{\mathcal{L}_r}$ such that for any $j \geq 1$

$$
\begin{align*}
  z^{(j)} &\neq \tilde{z}^{(j)}, \\
  \Psi_L(z^{(j)}) &\neq \Psi_L(\tilde{z}^{(j)}), \\
  \lim_{j \to \infty} z^{(j)} &\neq \lim_{j \to \infty} \tilde{z}^{(j)} = 0.
\end{align*}
$$

Since by assumption $\mathcal{K}$ is compact, there exist subsequences of $(z^{(j)})_{j \geq 1}, (\tilde{z}^{(j)})_{j \geq 1}$, denoted for simplicity in the same way, such that $(z^{(j)}_S)_{j \geq 1}, (\tilde{z}^{(j)}_S)_{j \geq 1}$ converge. Denote their limits by $z^{(\infty)}_S$ and $\tilde{z}^{(\infty)}_S$, respectively. Then

$$
\lim_{j \to \infty} z^{(j)} = (z^{(\infty)}_S, 0), \quad \lim_{j \to \infty} \tilde{z}^{(j)} = (\tilde{z}^{(\infty)}_S, 0)
$$

are elements in $\mathcal{K} \times \{0\}$. By the continuity of $\Psi_L$, one has $\Psi_L(z^{(\infty)}_S, 0) = \Psi_L(\tilde{z}^{(\infty)}_S, 0)$ and since $\Psi_L$ and $\Psi^{\mathrm{nls}}$ coincide on $\mathcal{V}^{\mathrm{max}}_S \times \{0\}$ it then follows from Theorem 1.1 that $z^{(\infty)}_S = \tilde{z}^{(\infty)}_S$. By item (i) and the local inversion theorem one then concludes that in contradiction to our assumption, $z^{(j)} = \tilde{z}^{(j)}$ for $j$ sufficiently large. This proves the first part of item (ii). Since according to item (i), for any given $z_S \in \mathcal{K}$, $d\Psi_L(z_S, 0) : h^0_{\mathcal{L}_c} \to H^0_{\mathcal{L}_c}$ is a linear isomorphism, $d\Psi_L(z)$ is such an operator for $z$ in a whole neighbourhood of $(z_S, 0)$. Using that $\mathcal{K}$ is compact it then follows that after shrinking the radius of the ball $\mathcal{V}_\perp$, if necessary, $\Psi_L : \mathcal{K} \times \mathcal{V}_\perp \to H^0_{\mathcal{L}_r}$ is a local diffeomorphism. □

**Proposition 3.1.** For any compact subset $\mathcal{K} \subseteq \mathbb{R}^S \times \mathbb{R}^S$ there exists an open complex neighbourhood $\mathcal{V}$ of $\mathcal{K} \times \{0\}$ in $h^0_{\mathcal{L}_c}$ of the form $\mathcal{V}_S \times \mathcal{V}_\perp$ where $\overline{\mathcal{V}}_S$ is compact with $\overline{\mathcal{V}}_S \subseteq \mathcal{V}^{\mathrm{max}}_S$ and $\mathcal{V}_\perp \subset h^0_{\mathcal{L}_c}$ is an open ball, centred at 0, so that the restriction of $\Psi_L$ to $\mathcal{V}$ has the following properties:

(L1) $\Psi_L$ is analytic on $\mathcal{V}$ and

$$
\psi_L|_{\mathcal{V}_S \times \{0\}} = \psi^{\mathrm{nls}}|_{\mathcal{V}_S \times \{0\}}, \quad d\psi_L(z_S, 0) = d\psi^{\mathrm{nls}}(z_S, 0), \quad \forall z_S \in \mathcal{V}_S.
$$

Furthermore, $\Psi_L : \mathcal{V} \cap h^0_{\mathcal{L}_c} \to H^0_{\mathcal{L}_c}$ is a real analytic diffeomorphism onto its image.

(L2) The map $B_L := \psi_L - F^{\mathrm{nls}}_S : \mathcal{V} \to H^0_{\mathcal{L}_c}$ is analytic and one smoothing. More precisely, the analytic map $B_L$ is given by

$$
B_L(z) = B^{\mathrm{nls}}(\Pi_S z) + d_\perp B^{\mathrm{nls}}(\Pi_S z)[z_\perp]
$$

with $B^{\mathrm{nls}}$ being the map introduced in Theorem 1.1, and for any $s \in \mathbb{Z}_{\geq 1}$, $B_L : \mathcal{V} \cap h^s_{\mathcal{L}_c} \to H^{s+1}_{\mathcal{L}_c}$ is real analytic. Furthermore

$$
d_\perp B_L(z) = d_\perp B^{\mathrm{nls}}(\Pi_S z), \quad d^2_\perp B_L(z) = 0, \quad \forall z \in \mathcal{V}
$$
and for any \( z \in \mathcal{V} \cap h^0_r, \alpha, \beta \in \mathbb{Z}^2_{\leq 0}, \)
\[
\| \partial_{S}^{\alpha, \beta} B_L(z) \|_0 \lesssim_{\alpha, \beta} 1, \quad \| \partial_{S}^{\alpha, \beta} d_{L} B_L(z) [\widehat{Z}_{\perp}] \|_0 \lesssim_{\alpha, \beta} \| \widehat{Z}_{\perp} \|_0, \quad \forall \widehat{Z}_{\perp} \in h^0_{1,c} \quad (3.8)
\]
and for any \( s \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V} \cap h^z_s, \)
\[
\| \partial_{S}^{\alpha, \beta} B_L(z) \|_{s+1} \lesssim_{\alpha, \beta} 1 + \| z \|_s, \quad \| \partial_{S}^{\alpha, \beta} d_{L} B_L(z) [\widehat{Z}_{\perp}] \|_{s+1} \lesssim_{\alpha, \beta} \| \widehat{Z}_{\perp} \|_s, \quad \forall \widehat{Z}_{\perp} \in h^z_{1,c}. \quad (3.9)
\]

(L3) For any \( s \in \mathbb{Z}_{\geq 1}, \) the restriction \( \varPsi_L |_{\mathcal{V} \cap h^z_r} \) is a map \( \mathcal{V} \cap h^z_r \to H^z_r \) which is a real analytic diffeomorphism onto its image.

(L4) The map \( A_L := \varPsi_L^{-1} - F_{nls} : \varPsi_L(\mathcal{V}) \to h^0_\ell \) is analytic and one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1}, A_L : \varPsi_L(\mathcal{V}) \cap H^s_r \to h^{s+1}_r \) is real analytic.

\[ \square \]

**Remark 3.1.** For convenience, in the sequel, we always choose \( \mathcal{V}_{\perp} \) to be a ball of radius smaller than one.

**Proof.** Choose \( \mathcal{V}_{\ast} \) to be an open bounded neighbourhood of \( \mathcal{K} \) in \( \mathbb{C}^S \times \mathbb{C}^S \) so that \( \overline{\mathcal{V}}_{\ast} \subseteq \mathcal{V}_{\ast} \) and let \( \mathcal{V}_{\perp} \) be an open ball in \( h^0_{1,c}, \) centred at 0, so that item (ii) of Lemma 3.1 applies to \( \mathcal{V} := \mathcal{V}_{\ast} \times \mathcal{V}_{\perp}, \) implying that \( \varPsi_L : \mathcal{V} \cap h^0_r \to H^0_r \) is one to one and a local diffeomorphism. The identities (3.6) hold by the definition of \( \varPsi_L \) and the analyticity of \( \varPsi_L, \) stated in (L1), follows by Lemma 3.1(i). One then concludes that
\[
\varPsi_L : \mathcal{V} \cap h^0_r \to H^0_r
\]
is a real analytic diffeomorphism on to its image. (L2) follows from the definition of \( \varPsi_L, \) Theorem 1.1, the compactness of \( \overline{\mathcal{V}}_{\ast}, \) and standard estimates in Sobolev spaces. Concerning (L3), first note that by Theorem 1.1, for any \( s \in \mathbb{Z}_{\geq 1}, \) the restriction \( \varPsi_L |_{\mathcal{V} \cap h^z_r} \) is a map with values in \( H^z_r \) and as such real analytic. By item (L1), \( \varPsi_L |_{\mathcal{V} \cap h^z_r} \) is one to one and so is its differential \( d\varPsi_L(z) : h^z_\ell \to H^z_\ell \) at any point \( z \in \mathcal{V} \cap h^z_r. \) Since by (L2) the map \( B_L \) is one smoothing, \( d\varPsi_L(z) : h^z_\ell \to H^z_\ell \) is Fredholm and hence a linear isomorphism, implying that \( \varPsi_L : \mathcal{V} \cap h^z_r \to H^z_r \) is a real analytic diffeomorphism on to its image. Finally, item (L4) follows from (L3) and Theorem 1.1.

Whereas the tame estimates (3.9) for \( B_L \) are an immediate consequence of the definition of \( \varPsi_L, \) Theorem 1.1 and the compactness of \( \overline{\mathcal{V}}_{\ast}, \) this is not so for \( A_L. \) Actually, for the applications in perturbation theory considered in subsequent work, we only need to derive tame estimates for
\[
\mathcal{A}_L : \mathcal{V} \cap h^0_r \to \mathcal{L}(H^0_\ell, h^0_\ell), \quad z \mapsto \mathcal{A}_L(z) := d\mathcal{A}_L(\varPsi_L(z)) = d\varPsi_L(z)^{-1} - F_{nls} \quad (3.10)
\]
with $\mathcal{V}$ denoting the neighbourhood of $\mathcal{K} \times \{0\}$ of Proposition 3.1. By formula (3.5), for any $z \in \mathcal{V} \cap h_r^0$, the operator $d\Psi_L(z) \in L(h_c^0, H_c^0)$ can be written as

$$d\Psi_L(z) = T(z) + R(z), \quad T(z) := d\Psi^{\text{nls}}(\Pi_S z)$$  \hspace{1cm} (3.11)

with $R(z) \in L(h_c^0, H_c^0)$ given by

$$R(z) : h_c^0 \rightarrow H_c^0, \quad \hat{z} \mapsto R(z)[\hat{z}] := d^2 \Psi^{\text{nls}}(\Pi_S z)[\Pi_S \hat{z}, \Pi_L z] = d^2 B^{\text{nls}}(\Pi_S z)[\Pi_S \hat{z}, \Pi_L z].$$  \hspace{1cm} (3.12)

Since by Theorem 1.1, respectively Proposition 3.1, the operators $T(z), d\Psi_L(z) : h_c^0 \rightarrow H_c^0$ are invertible, so is $T(z)^{-1} d\Psi_L(z) = \text{Id} + T(z)^{-1} R(z)$, implying that

$$d\Psi_L(z)^{-1} = (\text{Id} + T(z)^{-1} R(z))^{-1} T(z)^{-1}, = T(z)^{-1} - T(z)^{-1} R(z) S(z),$$  \hspace{1cm} (3.13)

where

$$S(z) := (\text{Id} + T(z)^{-1} R(z))^{-1} T(z)^{-1} \in L(H_c^0, h_c^0).$$  \hspace{1cm} (3.14)

Furthermore, by Theorem 1.1

$$T(z)^{-1} = (d\Psi^{\text{nls}}(\Pi_S z))^{-1} = d\Phi^{\text{nls}}(\Pi_S z) = F_{\text{nls}} + dA_{\text{nls}}(\Psi^{\text{nls}}(\Pi_S z)).$$

Altogether, it follows that for any $z \in \mathcal{V} \cap h_r^0$, the operator $A_L(z) = d\Psi_L(z)^{-1} - F_{\text{nls}} : h_c^0 \rightarrow H_c^0$ can be written as

$$A_L(z) = dA_{\text{nls}}(\Psi^{\text{nls}}(\Pi_S z)) - T(z)^{-1} R(z) S(z).$$  \hspace{1cm} (3.15)

Finally we note that by (L4) of Proposition 3.1, $A_L = dA_L \circ \Psi_L$ is one smoothing. More precisely, for any $s \in \mathbb{Z}_{\geq 1}$, the restriction of $A_L$ to $\mathcal{V} \cap h_r^s$ is a real analytic map,

$$A_L : \mathcal{V} \cap h_r^s \rightarrow L(H_c^s, h_c^{s+1}), \quad z \mapsto A_L(z).$$

**Proposition 3.2** (Tame estimates for $A_L$). After shrinking, if necessary, the radius of the ball $\mathcal{V}_\perp$ in $\mathcal{V} = \mathcal{V}_S \times \mathcal{V}_\perp$ of Proposition 3.1, the map $A_L$ satisfies for any $z \in \mathcal{V} \cap h_r^0$, $\hat{w} \in h_c^0$,

$$\|A_L(z)[\hat{w}]\|_0 \lesssim \|\hat{w}\|_0.$$
and for any \( k \in \mathbb{Z}_{\geq 1}, \hat{z}_1, \ldots, \hat{z}_k \in h^0_c, \)

\[
\| d^k(A_L(z)[\hat{w}]) [\hat{z}_1, \ldots, \hat{z}_k] \|_0 \lesssim_k \| \hat{w} \|_0 \prod_{j=1}^{k} \| \hat{z}_j \|_0.
\]

Furthermore, for any \( s \in \mathbb{Z}_{\geq 1}, z \in V \cap h^s_r, \hat{w} \in H^s_c, \)

\[
\| A_L(z)[\hat{w}] \|_{s+1} \lesssim_s \| z \|_s \| \hat{w} \|_s + \| \hat{w} \|_s \tag{3.16}
\]

and for any \( k \geq 1, \hat{z}_1, \ldots, \hat{z}_k \in h^s_c, \)

\[
\| d^k(A_L(z)[\hat{w}]) [\hat{z}_1, \ldots, \hat{z}_k] \|_{s+1} \lesssim_{s,k} (\| z \|_s \| \hat{w} \|_s + \| \hat{w} \|_s) \prod_{j=1}^{k} \| \hat{z}_j \|_0 + \| \hat{w} \|_0 \sum_{j=1}^{k} \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_0. \tag{3.17}
\]

**Proof.** First we prove estimate (3.16). The starting point is formula (3.15) for \( A_L(z) \). The two terms \( dA^{nls}(\Psi^{nls}(\Pi S z)) \) and \( T(z)^{-1}R(z)S(z) \) are estimated separately. By Theorem 1.1, \( \{\Psi^{nls}(\Pi S z) | z \in V \cap h^0_c\} \) is a relatively compact subset of \( H^s_c \) for any \( s \in \mathbb{Z}_{\geq 0}, \) and \( A^{nls}, B^{nls} \) are one smoothing maps. It implies that for any \( s \in \mathbb{Z}_{\geq 1}, \)

\[
\| dA^{nls}(\Psi^{nls}(\Pi S z))[\hat{w}] \|_{s+1} \lesssim_s \| \hat{w} \|_s, \quad \forall z \in V \cap h^0_c, \quad \forall \hat{w} \in H^s_c. \tag{3.18}
\]

Since \( \| \Pi S \hat{z} \|_s \lesssim \| \Pi S \hat{z} \|_0 \) for any \( z \in h^0_c, \) the linear operator \( R(z), \) defined in (3.12), satisfies

\[
\| R(z)[\hat{z}] \|_{s+1} \lesssim_s \| z \|_s \| \Pi S \hat{z} \|_s \lesssim_s \| z \|_s \| \Pi S \hat{z} \|_0 \quad \forall z \in V \cap h^0_c, \quad \forall \hat{z} \in h^0_c. \tag{3.19}
\]

Furthermore, also by Theorem 1.1, one has for any \( s \in \mathbb{Z}_{\geq 0}, \)

\[
\| T(z)^{-1}[\hat{w}] \|_s \lesssim_s \| \hat{w} \|_s, \quad \forall z \in V \cap h^0_c, \quad \forall \hat{w} \in H^s_c. \tag{3.20}
\]
Combining (3.18–3.20), formula (3.15) leads to the estimate

$$\| A_L(z)[\widehat{w}]\|_{s+1} \leq \| dA^{nls}(\Psi^{nls}(\Pi_S z))[\widehat{w}]\|_{s+1} + \| T(z)^{-1}R(z)S(z)[\widehat{w}]\|_{s+1} \lesssim_s \| \widehat{w}\|_s + \| z\|_s \| \Pi_S S(z)[\widehat{w}]\|_0. \quad (3.21)$$

It remains to estimate $\| S(z)[\widehat{w}]\|_0$. Recall that by (3.14), $S(z) = (\text{Id} + T(z)^{-1}R(z))^{-1}T(z)^{-1}$. By Theorem 1.1 there exists $C_0 > 0$ so that

$$\| T(z)^{-1}R(z)[\widehat{z}]\|_0 \leq C_0 \| z\|_0 \| \Pi_S \widehat{z}\|_0 \quad \forall z \in V \cap h^0_r, \quad \forall \widehat{z} \in h^0_c. \quad (3.22)$$

Shrinking the radius of the ball $V_\perp$ in $h^0_{1c}$, if necessary, so that $C_0 \| z\|_0 \leq 1/2$ for any $z_\perp \in V_\perp$, the Neumann series of the operator $(\text{Id} + T(z)^{-1}R(z))^{-1}$ absolutely converges in $L(h^0_c, h^0_c)$ and the operator norm of $(\text{Id} + T(z)^{-1}R(z))^{-1}$ in $L(h^0_c, h^0_c)$ is bounded by 2. Hence

$$\| S(z)[\widehat{w}]\|_0 \lesssim_s \| \widehat{w}\|_s, \quad \forall z \in V \cap h^0_r, \quad \forall \widehat{z} \in H^0_c, \quad (3.23)$$

implying together with (3.21) the claimed estimate (3.16).

Finally let us prove the estimate (3.17) for the derivatives of $A_L(z)$. By formula (3.15) for any $k, s \in \mathbb{Z}_{\geq 1}$, $z \in V \cap h^k_r$, $\widehat{w} \in H^k_c$, and $\widehat{z}_1, \ldots, \widehat{z}_k \in h^k_c$,

$$\| d^k(A_L(z)[\widehat{w}])[\widehat{z}_1, \ldots, \widehat{z}_k]\|_{s+1} \leq \| d^k(dA^{nls}(\Psi^{nls}(\Pi_S z))[\widehat{w}])[\widehat{z}_1, \ldots, \widehat{z}_k]\|_{s+1} \quad + \quad \| d^k(T(z)^{-1}R(z)S(z)[\widehat{w}])[\widehat{z}_1, \ldots, \widehat{z}_k]\|_{s+1}. \quad (3.24)$$

By Theorem 1.1, one concludes that

$$\| d^k(dA^{nls}(\Psi_L(\Pi_S z))[\widehat{w}])[\widehat{z}_1, \ldots, \widehat{z}_k]\|_{s+1} \lesssim_{s,k} \| \widehat{w}\|_s \prod_{j=1}^k \| \widehat{z}_j\|_0. \quad (3.25)$$

Furthermore

$$\| d^k(T(z)^{-1}[\widehat{w}])[\widehat{z}_1, \ldots, \widehat{z}_k]\|_s \lesssim_{s,k} \| \widehat{w}\|_s \prod_{j=1}^k \| \widehat{z}_j\|_0, \quad (3.26)$$

and

$$\| d^k(R(z)[\widehat{z}])[\widehat{z}_1, \ldots, \widehat{z}_k]\|_{s+1} \lesssim_{s,k} \| z\|_s \| \widehat{z}\|_0 \prod_{j=1}^k \| \widehat{z}_j\|_0 + \| \widehat{z}\|_0 \sum_{j=1}^k \| \widehat{z}_j\|_s \prod_{i \neq j} \| \widehat{z}_i\|_0. \quad (3.27)$$
and

\[ \|d^k(S(z)[\hat{\mathcal{W}}])[\hat{z}_1, \ldots, \hat{z}_k]\|_0 \lesssim_s \|\hat{\mathcal{W}}\|_0 \prod_{j=1}^k \|\hat{z}_j\|_0. \tag{3.28} \]

Combining the estimates (3.26–3.28) and using the product rule implies that

\[ \|d^k(T(z)^{-1}R(z)S(z)[\hat{\mathcal{W}}])[\hat{z}_1, \ldots, \hat{z}_k]\|_{s+1} \lesssim_{s,k} \]

\[ (\|z_\perp\|_s \|\hat{\mathcal{W}}\|_0 + \|\hat{\mathcal{W}}\|_s) \prod_{j=1}^k \|\hat{z}_j\|_0 + \|\hat{\mathcal{W}}\|_0 \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0. \tag{3.29} \]

The three estimates (3.24), (3.25), (3.29) together yield (3.17). \( \square \)

In the remaining part of this section we describe the pullback \( \Psi_L^* \Lambda \) by \( \Psi_L \) of the standard symplectic form \( \Lambda \) on \( H^0_r \), introduced in (2.3). It turns out that \( \Psi_L^* \Lambda \) is not the symplectic form \( \Lambda_M \) of (2.7), making it necessary to construct the symplectic corrector \( \Psi_C \) (see Section 4).

Given a bounded linear operator \( \mathcal{P} : h^0_c \to h^0_c \), its transpose \( \mathcal{P}^t : h^0_c \to h^0_c \) is defined to be the operator determined by

\[ (\mathcal{P}[\hat{z}], \hat{z}')_r = (\hat{z}, \mathcal{P}^t[\hat{z}'])_r, \quad \forall \hat{z}, \hat{z}' \in h^0_c, \tag{3.30} \]

where the bilinear form \( \langle \cdot, \cdot \rangle \) on \( h^0_c \) is defined in (2.4). Similarly, for a bounded linear operator \( \mathcal{Q} : h^0_c \to H^0_c \), we denote its transpose by \( \mathcal{Q}^t : H^0_c \to h^0_c \), determined by

\[ (\mathcal{Q}[\hat{z}], \hat{\mathcal{W}})_r = (\hat{z}, \mathcal{Q}^t[\hat{\mathcal{W}}])_r, \quad \forall \hat{z}, \hat{\mathcal{W}} \in h^0_c, \hat{\mathcal{W}} \in H^0_c. \tag{3.31} \]

where the bilinear form \( \langle \cdot, \cdot \rangle \) on \( H^0_c \) is the one introduced in (2.2). We now compute the pullback \( \Psi_L^* \Lambda(z) \) at \( z = (z_S, z_\perp) \in h^0_c \) applied to \( \hat{z} = (\hat{z}_S, \hat{z}_\perp) \), \( \hat{z}' = (\hat{z}'_S, \hat{z}'_\perp) \). By the definition of the pullback and the one of \( \Lambda \) in (2.3) we have

\[ \Psi_L^* \Lambda(z)[\hat{z}, \hat{z}'] = \Lambda(\Psi_L(z))[d\Psi_L(z)[\hat{z}], d\Psi_L(z)[\hat{z}']] = i\{\mathcal{J}d\Psi_L(z)[\hat{z}], d\Psi_L(z)[\hat{z}']\}_r. \tag{3.32} \]

By formula (3.5) for \( d\Psi_L(z) \),

\[ d\Psi_L(z)[\hat{z}] = d\Psi^{\text{nls}}(P_S z)[\hat{z}] + d_S(d_{\perp} \Psi^{\text{nls}}(P_S z)[z_\perp])[\hat{z}_S], \]

one gets

\[ \Psi_L^* \Lambda(z)[\hat{z}, \hat{z}'] = (I) + (II) + (III) + (IV), \tag{3.33} \]
where

\[
(I) := i \left[ \langle d\Psi^{\text{nl}}(\Pi_S z)[\widehat{z}], d\Psi^{\text{nl}}(\Pi_S z)[\widehat{z}] \rangle_r, \right. \\
\left. \right] = ((\Psi^{\text{nl}})^* \Lambda)(\Pi_S z)[\widehat{z}, \widehat{z}'], \quad (3.34)
\]

\[
(II) := i \left[ \langle d\Psi^{\text{nl}}(\Pi_S z)[\widehat{z}], d_S(\Delta \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_\perp] \rangle_r. \right. \\
\]

Writing \(d\Psi^{\text{nl}}(\Pi_S z)[\widehat{z}]\) as \(d_S \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_S] + d_\perp \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_\perp]\) one gets

\[
(II) = i \left[ \langle d_S \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_S], d_S(\Delta \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_S] \rangle_r, \\
+ i \left[ \langle d_\perp \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_\perp], d_S(\Delta \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_\perp] \rangle_r. \right. \right. \\
\]

(3.35)

Similarly one has

\[
(III) := i \left[ \langle d_S(d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_S], d\Psi^{\text{nl}}(\Pi_S z)[\widehat{z}] \rangle_r, \\
\left. \right] \right] = i \left[ \langle d_S(d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_S], d_S \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_S] \rangle_r, \\
+ i \left[ \langle d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp], \right. \\
\left. d_\perp \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_\perp] \rangle_r. \right. \right. \\
\]

(3.36)

and finally

\[
(IV) := i \left[ \langle d_S(d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_S], d_S(d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_S] \rangle_r, \\
\left. \right] \right] \right] = i \left[ \langle d_S(d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_S], d_\perp \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_\perp] \rangle_r. \right. \right. \\
\]

(3.37)

Since by Theorem 1.1, \(\Psi^{\text{nl}}\) is symplectic, one has \((\Psi^{\text{nl}})^* \Lambda = \Lambda_M\). Hence for any \(z \in \mathcal{V}\), \(\Psi_L^* \Lambda(z)\) can be written as

\[
\Psi_L^* \Lambda (z) = \Lambda_M + \Lambda_L (z), \quad \Lambda_L(z)[\widehat{z}, \widehat{z}'] := (L(z)[\widehat{z}], \widehat{z}')_r, \quad (3.39)
\]

where \(L(z) : \mathbb{C}^S \times \mathbb{C}^S \times h_{\perp 0}^0 \to \mathbb{C}^S \times \mathbb{C}^S \times h_{\perp 0}^0\) is the linear operator of the form

\[
L(z) = \begin{pmatrix}
L_S^S(z) & L_S^L(z) \\
L_L^S(z) & 0
\end{pmatrix}. \quad (3.40)
\]

By the computations above, \(L_S^S(z) : \mathbb{C}^S \times \mathbb{C}^S \to \mathbb{C}^S \times \mathbb{C}^S\), \(L_L^S(z) : h_{\perp 0}^0 \to \mathbb{C}^S \times \mathbb{C}^S\), and \(L_S^L(z) : \mathbb{C}^S \times \mathbb{C}^S \to h_{\perp 0}^0\) are the linear operators defined by \((z \in \mathcal{V} \cap h_{\perp 0}^0, \widehat{z}_S \in \mathbb{C}^S \times \mathbb{C}^S, \widehat{z}_\perp \in h_{\perp 0}^0)\)

\[
L_S^S(z)[\widehat{z}_S] := i \left( \langle \left[ d_S \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_S], \partial_{x_j} d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp] \rangle_{\mathcal{S}}, \right. \\
\left. \partial_{x_j} d_S \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_S], \rangle_{\mathcal{S}} \rangle_{\mathcal{S}}, \right. \\
+ i \left( \langle \left[ d_S(d_\perp \Psi^{\text{nl}}(\Pi_S z)[z_\perp])[\widehat{z}_S], \partial_{x_j} \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_\perp] \rangle_{\mathcal{S}}, \right. \\
\left. \partial_{x_j} d_S \Psi^{\text{nl}}(\Pi_S z)[\widehat{z}_\perp], \rangle_{\mathcal{S}} \rangle_{\mathcal{S}}, \right. \\
\right).
\]
and similarly

\[ L^\perp_S(z)[\widehat{z}_\perp] := i \left( \left( \| d_{S} \psi_{\text{nls}}^{\perp}(\Pi_{S} z)[z_{\perp}] [\widehat{z}_\perp], \partial_{\psi} d_{S} \psi_{\text{nls}}^{\perp}(\Pi_{S} z)[z_{\perp}], j \in S \right) \right), \]

\[ L^\perp_S(z)[\widehat{Z}_S] := i \left( \left( \| d_{S} \psi_{\text{nls}}^{\perp}(\Pi_{S} z)[z_{\perp}] [\widehat{Z}_S], \partial_{\psi} d_{S} \psi_{\text{nls}}^{\perp}(\Pi_{S} z)[z_{\perp}], j \in S \right) \right). \]

The operator valued map \( z \mapsto L(z) \) has the following properties:

**Lemma 3.2.** The map \( L : \mathcal{V} \cap h^0_c \to L(h^0_c, h^0_c), z \mapsto L(z) \) is real analytic. For any \( z \in \mathcal{V} \cap h^0_c, \widehat{z} \in h^0_c, \)

\[ \| L(z)[\widehat{z}] \|_0 \lesssim \| z \|_0 \| \widehat{z} \|_0 \]

and for any \( k \in \mathbb{Z}_{\geq 1}, \widehat{z}_1, \ldots, \widehat{z}_k \in h^0_c, \)

\[ \| d^k(L(z)[\widehat{z}])[\widehat{z}_1, \ldots, \widehat{z}_k] \|_0 \lesssim_k \| z \|_0 \prod_{j=1}^k \| \widehat{Z}_j \|_0. \]

Furthermore, the map \( L \) is one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1}, L : \mathcal{V} \cap h^s_c \to L(h^s_c, h^{s+1}_c), z \mapsto L(z) \) is real analytic and satisfies the following estimates: for any \( z \in \mathcal{V} \cap h^s_c, \widehat{z} \in h^s_c, \)

\[ \| L(z)[\widehat{z}] \|_{s+1} \lesssim_s \| z \|_s \| \widehat{z} \|_0 \]

and for any \( k \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V} \cap h^s_c, \widehat{z}_1, \ldots, \widehat{z}_k \in h^s_c, \)

\[ \| d^k(L(z)[\widehat{z}])[\widehat{z}_1, \ldots, \widehat{z}_k] \|_{s+1} \lesssim_{s,k} \| z \|_0 \sum_{j=1}^k \| \widehat{Z}_j \|_s \prod_{i \neq j} \| Z_i \|_0 + \| \widehat{z} \|_0 \| z \|_s \prod_{j=1}^k \| \widehat{Z}_j \|_0. \]

In particular, \( L(z) = 0 \) for any \( z \in \mathcal{V} \cap h^0_c \) with \( z_\perp = 0 \). Finally, \( L(z) = -L(z)^i \) or, more explicitly, for any \( z \in \mathcal{V} \cap h^*_r, \)

\[ L^\perp_S(z)^i = -L^\perp_S(z), \quad L^\perp_S(z)^i = -L^\perp_S(z), \quad L^\perp_S(z)^i = -L^\perp_S(z). \]
Proof. The analyticity of $L$ follows by Theorem 1.1, using again that $d_s d_{\perp} \psi_{\text{als}} = d_s d_{\perp} B_{\text{als}}$. Since $J^f = -J$, one reads off from the expressions (3.41–3.43) that (3.46) holds. The estimates (3.44) and (3.45) follow from Theorem 1.1 by differentiating the expressions in the definitions of $L^S_{\perp}(z)$, $L^\perp_{\perp}(z)$, and $L^S_{\perp}(z)$ with respect to $z$. ■

4 The Symplectic Corrector $\Psi_C$

In this section we construct the coordinate transformation $\Psi_C$ on $\mathcal{V} \cap h^0_c$ so that the composition $\Psi_L \circ \Psi_C$ is symplectic. As mentioned in the introduction, we follow Kuksin’s scheme of proof in [16], which uses arguments of Moser and Weinstein in the given infinite-dimensional setup. The map $\Psi_C$ will be defined as the time-one flow of an appropriately chosen non-autonomous vector field. In the sequel, $\mathcal{V}$ denotes the neighbourhood of $\mathcal{K} \times 0$, given by Propositions 3.1 and 3.2.

For any $z \in \mathcal{V}$ define the following two- and one-forms on $h^0_c$:

$$\Lambda_0 := \Lambda_M, \quad \Lambda_1(z) := \Psi^*_L \Lambda(z) = \Lambda_M + \Lambda_L(z),$$

$$\lambda_0 := \lambda_M, \quad \lambda_1(z) := \Psi^*_L \lambda(z).$$

4.0 Analysis of the two-form $\Lambda_1(z)$

Note that $d\lambda_i = \Lambda_i$, $i = 0, 1$, and

$$\Lambda_1 - \Lambda_0 = \Lambda_L = d(\lambda_1 - \lambda_0).$$

In particular, the two-form $\Lambda_L$ is closed. By (2.7), (3.39) one has

$$\Lambda_1(z)[\widehat{z}, \widehat{z}'] = (L_1(z)[\widehat{z}], \widehat{z}')_r, \quad L_1(z) := J^{-1} + L(z).$$

For any $\tau \in [0, 1]$, define the two-form $\Lambda_\tau = \Lambda_\tau(z)$,

$$\Lambda_\tau := \tau \Lambda_1 + (1 - \tau) \Lambda_0,$$

which can be written as

$$\Lambda_\tau(z)[\widehat{z}, \widehat{z}'] = (L_\tau(z)[\widehat{z}], \widehat{z}')_r, \quad L_\tau(z) = J^{-1} + \tau L(z).$$

It turns out that for any $\tau \in [0, 1]$ and $z \in \mathcal{V} \cap h^0_r$, the map $L_\tau(z)$ is invertible and one smoothing. More precisely, the following holds:

Lemma 4.1. After shrinking the ball $\mathcal{V} \subset h^0_{1C}$ in $\mathcal{V} = \mathcal{V}_S \times \mathcal{V}_{\perp}$, if necessary, one has that for any $s \in \mathbb{Z}_{\geq 0}$, $z \in \mathcal{V} \cap h^0_r$, and $\tau \in [0, 1]$, the operator $L_\tau(z) : h^s_c \to h^s_c$ is invertible and
for any \( k \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V} \cap h_r^0, \hat{z}, \hat{z}_1, \ldots, \hat{z}_k \in h_c^0, \)

\[
(\mathcal{L}_r(z)^{-1} - J)[\hat{z}]_0 \lesssim \|z\|_0 \|\hat{z}\|_0, \quad \|d^k(\mathcal{L}_r(z)^{-1}[\hat{z}])[\hat{z}_1, \ldots, \hat{z}_k]\|_0 \lesssim \|\hat{z}\|_0 \prod_{j=1}^k \|\hat{z}_j\|_0.
\]

Moreover for any \( s \in \mathbb{Z}_{\geq 1} \) and \( \tau \in [0, 1] \), the map

\[
\mathcal{L}_r^{-1} - J : \mathcal{V} \cap h_r^s \to \mathcal{L}(h_c^s, h_c^{s+1}), \quad z \mapsto \mathcal{L}_r(z)^{-1} - J
\]

is real analytic and the following tame estimates hold: for any \( k \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V} \cap h_r^s, \hat{z}, \hat{z}_1, \ldots, \hat{z}_k \in h_c^s, \)

\[
\|(\mathcal{L}_r(z)^{-1} - J)[\hat{z}]\|_{s+1} \lesssim_s \|z\|_s \|\hat{z}\|_0
\]

\[
\|d^k(\mathcal{L}_r(z)^{-1}[\hat{z}])[\hat{z}_1, \ldots, \hat{z}_k]\|_{s+1} \lesssim_{s, k} \|\hat{z}\|_0 \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|\hat{z}\|_0 \|z\|_s \prod_{j=1}^k \|\hat{z}_j\|_0. \quad \square
\]

**Proof.** For any \( \tau \in [0, 1] \), we write

\[
\mathcal{L}_r(z) = J^{-1}(\text{Id} + L_r(z)), \quad L_r(z) := \tau JL(z).
\]

By (3.40) and Theorem 1.1, the operator \( L_r(z) \) satisfies the estimate \( \|L_r(z)[\hat{z}]\|_0 \leq C_0\|z\|_0 \|\hat{z}\|_0 \), for any \( z \in \mathcal{V} \cap h_r^0 \) and \( \hat{z} \in h_c^0 \) for some constant \( C_0 > 0 \). By shrinking the ball \( \mathcal{V}_\perp \), if necessary, one has that for any \( z_\perp \in \mathcal{V}_\perp \), \( C_0\|z_\perp\|_0 \leq 1/2 \), implying that the operator \( \mathcal{L}_r(z) \) is invertible and its inverse \( \mathcal{L}_r(z)^{-1} \) is given by the Neumann series

\[
\mathcal{L}_r(z)^{-1} = J + \sum_{n \geq 1} (-1)^n L_r(z)^n J.
\]

By Lemma 3.2, for any \( s, n \in \mathbb{Z}_{\geq 1} \) and \( \tau \in [0, 1] \), one has

\[
\|L_r(z)^n J[\hat{z}]\|_{s+1} \leq C(s)\|z_\perp\|_s \|L_r(z)^{n-1} J[\hat{z}]\|_0 \leq C(s)C_0\|z_\perp\|_0 \|z_\perp\|_s \|\hat{z}\|_0
\]

(4.7)

for some constant \( C(s) > 0 \). Since \( C_0\|z_\perp\|_0 \leq 1/2 \), one gets

\[
\|(\mathcal{L}_r(z)^{-1} - J)[\hat{z}]\|_{s+1} \lesssim_s \|z_\perp\|_s \|\hat{z}\|_0.
\]

The estimates for the derivatives \( d^k(\mathcal{L}_r(z)^{-1}[\hat{z}]) \) follow by differentiating the expression (4.6) with respect to \( z \) and applying the estimates for \( d^k(L(z)[\hat{z}]) \) of Lemma 3.2. \( \blacksquare \)
Since by (4.3), the two-form $\Lambda_L = \Lambda_1 - \Lambda_0$ is closed and by Lemma 3.2, for any $z \in \mathcal{V} \cap h^0_\epsilon$, $\Lambda_L(\Pi_S z) = 0$, we can apply Lemma A.1 in Appendix 1. It says that the one-form

$$\lambda_L(z)[\widehat{z}] := \int_0^1 \Lambda_L(z, t\widehat{z})[(0, z_\perp), (\widehat{z}_S, t\widehat{z}_\perp)] \, dt$$

(4.8)
satisfies $d\lambda_L = \Lambda_L$. By (3.39), (3.40), the one-form $\lambda_L(z)$ can be written as

$$\lambda_L(z)[\widehat{z}] = \int_0^1 (L(z, t\widehat{z})(0, z_\perp), (\widehat{z}_S, t\widehat{z}_\perp)) \, dt = \int_0^1 L_S^1(z, t\widehat{z})[z_\perp] \cdot \widehat{z}_S \, dt.$$ 

Moreover, using that by (3.42), $L_S^1(z, t\widehat{z}) = tL_S^1(z, z_\perp)$, it turns out that

$$\lambda_L(z)[\widehat{z}] = (E(z), \widehat{z})_r, \quad E(z) := (E_S(z), 0) \in \mathbb{C}^S \times \mathbb{C}^S \times h^0_{\perp c},$$

(4.9)

where

$$E_S(z) := \frac{1}{2} L_S^1(z)[z_\perp] = \frac{i}{2} \left( \left[ \partial_{\lambda} \Psi^{nls}(\Pi_S z)[z_\perp], \partial_{\lambda j} \Psi^{nls}(\Pi_S z)[z_\perp] \right]_{\lambda \in S} \right).$$

(4.10)

One of the features of $\lambda_L(z)$ is that it is quadratic in $z_\perp$. In more detail, we have the following

**Lemma 4.2.** For any $s \in \mathbb{Z}_{\geq 0}$, the map $E : \mathcal{V} \cap h^0_\epsilon \to h^0_\epsilon$ is real analytic and satisfies the following tame estimates: for any $z \in \mathcal{V} \cap h^0_\epsilon$, $\widehat{z} \in h^0_\epsilon$,

$$\|E(z)\|_s \lesssim_s \|z_\perp\|^2_0, \quad \|dE(z)[\widehat{z}]\|_s \lesssim_s \|z_\perp\|_0 \|\widehat{z}\|_0,$$

and any $k \geq 2$, $\widehat{z}_1, \ldots, \widehat{z}_k \in h^0_\epsilon$,

$$\|d^k E(z)[\widehat{z}_1, \ldots, \widehat{z}_k]\|_s \lesssim_s k \prod_{j=1}^k \|\widehat{z}_j\|_0. \quad \square$$

**Proof.** The lemma follows by the properties of the map $\Psi^{nls}$, stated in Theorem 1.1, and the fact that $E = \Pi_S E$, $\|\Pi_S z\|_s \lesssim_s \|z\|_0$ for any vector $z \in h^0_\epsilon$, and $\mathcal{V}_\perp \subset h^0_{\perp c}$ is a ball of radius smaller than 1. \qed
4.1 Outline of the construction of $\Psi_C$

Following arguments of Moser and Weinstein, our candidate for $\Psi_C$ is $\Psi_{X^0}^0$ where $X \equiv X(z, \tau) \in h^0_r$ is a non-autonomous vector field with well-defined flow $\Psi_{X^0}^{0,\tau}, 0 \leq \tau_0, \tau \leq 1$, so that $(\Psi_{X^0}^{0,\tau})^* \Lambda_1 = \Lambda_0$. Here $z \in V$ and the flow is normalized by $\Psi_{X^0}^{0,\tau_0}(z) = z$. To see how to choose $X(z, \tau)$, consider the pullback of the two-form $\Lambda_\tau$ by $\Psi_{X^0,\tau}^0$. Since $(\Psi_{X^0}^{0,\tau})^* = Id$, one has $(\Psi_{X^0}^{0,0})^* \Lambda_0 = \Lambda_0$. The desired identity $(\Psi_{X^0,\tau}^0)^* \Lambda_1 = \Lambda_0$ then follows provided that $(\Psi_{X^0,\tau}^0)^* \Lambda_\tau$ is independent of $\tau$, that is, $\partial_\tau((\Psi_{X^0,\tau}^0)^* \Lambda_\tau) = 0$. Since $\partial_\tau \Lambda_\tau = \Lambda_1 - \Lambda_0 = d\lambda_L$, it turns out that the latter identity holds if $\lambda_L + \Lambda_\tau[X(\cdot, \tau), \cdot] = 0$. When expressed in terms of the bilinear form $(\cdot, \cdot)$ and taking into account the representation (4.5) of $\Lambda_\tau$ and (4.9) of $\lambda_L$, the latter identity reads

$$
(E(z), \hat{z})_r + (L_\tau(z)[X(z, \tau)], \hat{z})_r = 0. \tag{4.11}
$$

We choose the vector field $X(z, \tau)$ so that (4.11) is satisfied.

4.2 Vector field $X(z, \tau)$ and its flow

Motivated by (4.11), the non-autonomous vector field $X(z, \tau)$ is defined by

$$
X(z, \tau) := -L_\tau(z)^{-1}E(z), \quad z \in V \times V_\perp, \quad \tau \in [0, 1]. \tag{4.12}
$$

Lemmata 4.1 and 4.2 lead to the following

**Lemma 4.3.** The vector field $X : (V \cap h^0_r) \times [0, 1] \to h^0_r$ is real analytic and one smoothing, meaning that for any $s \in \mathbb{Z}_{\geq 1}$

$$
X : (V \cap h^s_r) \times [0, 1] \to h^{s+1}_r
$$

is real analytic. In addition, the following tame estimates hold: for any $\tau \in [0, 1], z \in V \cap h^0_r, \hat{z} \in h^0_c$,

$$
\|X(z, \tau)\|_0 \lesssim \|z\|_0^2, \quad \|dX(z, \tau)[\hat{z}]\|_0 \lesssim \|z\|_0 \|\hat{z}\|_0 \tag{4.13}
$$

and for any $k \geq 2, \hat{z}_1, \ldots, \hat{z}_k \in h^0_c$,

$$
\|d^kX(z, \tau)[\hat{z}_1, \ldots, \hat{z}_k]\|_0 \lesssim_k \prod_{j=1}^k \|\hat{z}_j\|_0.
$$
Moreover, for any \( s \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V} \cap h^*_r, \hat{z} \in h^*_c, \)
\[
\|X(z, \tau)\|_{s+1} \lesssim_s \|z\|_s \|z\|_0, \quad \|dX(z, \tau)[\hat{z}]\|_{s+1} \lesssim_s \|z\|_0 \|\hat{z}\|_s + \|z\|_s \|\hat{z}\|_0
\]
(4.14)
and for any \( k \geq 2, \hat{z}_1, \ldots, \hat{z}_k \in h^*_c, \)
\[
\|d^k X(z, \tau)[\hat{z}_1, \ldots, \hat{z}_k]\|_{s+1} \lesssim_{s,k} \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z\|_s \prod_{j=1}^k \|\hat{z}_j\|_0.
\]

**Proof.** The lemma follows from Lemmata 4.1 and 4.2. ■

We now want to study the flow of the non-autonomous differential equation
\[
\partial_{\tau} z = X(z, \tau). \tag{4.15}
\]
Recall that for any \( r > 0, \) we denote by \( \mathcal{V}_r \) the ball in \( h^0_{r,c} \) of radius \( r, \) centred at \( 0, \) and for any \( \tau_0, \tau \in [0, 1] \) by \( \Psi_X^{\tau_0, \tau} \) the flow map of the differential equation (4.15), satisfying \( \Psi_X^{\tau_0, \tau}(z) = z. \) By a standard contraction argument, there exists an open neighbourhood \( \mathcal{V}_s \subseteq \mathcal{V}_s \subseteq \mathcal{V}_s \) of \( \mathcal{K} \) in \( \mathbb{C}^S \times \mathbb{C}^S \) and \( \delta > 0 \) with \( \mathcal{V}_s(2\delta) \subseteq \mathcal{V}_s \) such that for any \( \tau, \tau_0 \in [0, 1] \)
\[
\Psi_X^{\tau_0, \tau} : \mathcal{V}_s \cap h^0_r \to \mathcal{V}_{2s} \cap h^0_r, \quad \Psi_s := \mathcal{V}_s \times \mathcal{V}_s(\delta), \quad \Psi_{2s} := \mathcal{V}_s \times \mathcal{V}_s(2\delta)
\]
is well defined and real analytic. In the next lemma we state the smoothing estimates for \( \Psi_X^{\tau_0, \tau} - \text{id} \) where \( \text{id} \) denotes the identity map on \( \mathcal{V}_s \cap h^0_r. \)

**Lemma 4.4.** By choosing \( 0 < \delta < 1 \) smaller, if necessary, it follows that for any \( \tau, \tau_0 \in [0, 1], \) the map \( \Psi_X^{\tau_0, \tau} - \text{id} : \mathcal{V}_s \cap h^0_r \to h^0_r \) is one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1}, \)
the map
\[
\Psi_X^{\tau_0, \tau} - \text{id} : \mathcal{V}_s \cap h^0_r \to h^{s+1}_r
\]
is real analytic. Furthermore, the following tame estimates hold: for any \( z \in \mathcal{V}_s \cap h^0_r, \hat{z} \in h^0_c, \)
\[
\|\Psi_X^{\tau_0, \tau}(z) - z\|_0 \lesssim \|z\|_0^2, \quad \|(d\Psi_X^{\tau_0, \tau}(z) - \text{id})[\hat{z}]\|_0 \lesssim \|z\|_0 \|\hat{z}\|_0
\]
(4.17)
and for any \( k \geq 2, \hat{z}_1, \ldots, \hat{z}_k \in h^0_c, \)
\[
\|d^k \Psi_X^{\tau_0, \tau}(z)[\hat{z}_1, \ldots, \hat{z}_k]\|_0 \lesssim_k \prod_{j=1}^k \|\hat{z}_j\|_0
\]
whereas for any \( s \in \mathbb{Z}_{\geq 1}, \), \( z \in \mathcal{V}_s \cap h^s_{r}, \) \( \hat{z} \in h^s_{r}, \)

\[
\| \Psi^{s+1}_r(z) - z \|_{s+1} \lesssim_s \| z \|_s \| z \|_0, \quad \| (d\Psi^{s+1}_r(z) - \text{Id})[\hat{z}] \|_{s+1} \lesssim_s \| z \|_0 \| \hat{z} \|_s + \| z \|_s \| \hat{z} \|_0. \tag{4.18}
\]

and for any \( k \geq 2, \hat{z}_1, \ldots, \hat{z}_k \in h^s_{r}, \)

\[
\| d^k \Psi^{s+1}_r(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_{s+1} \lesssim_s \sum_{j=1}^k \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_0 + \| z \|_s \prod_{j=1}^k \| \hat{z}_j \|_0. \tag{4.19}
\]

**Proof.** For any \( \tau_0, \tau \in [0, 1] \) and \( z \in \mathcal{V}_s \cap h^s_{r}, \) the flow \( \Psi^{s+1}_r(z) \) satisfies the integral equation

\[
\Psi^{s+1}_r(z) = z + \int_{\tau_0}^{\tau} X(\Psi^{s+1}_r(z), t) \, dt. \tag{4.20}
\]

In view of the estimate (4.14) of the vector field \( X(z, \tau), \) we first estimate \( \| \Pi_{\perp} \Psi^{s+1}_r(z) \|_s \)

for \( z \in \mathcal{V}_s \cap h^s_{r} \) with \( s \in \mathbb{Z}_{\geq 0}. \) Applying the operator \( \Pi_{\perp} \) to both sides of the identity (4.19), one gets

\[
\Pi_{\perp} \Psi^{s+1}_r(z) = \Pi_{\perp} z + \int_{\tau_0}^{\tau} \Pi_{\perp} X(\Psi^{s+1}_r(z), t) \, dt.
\]

By Lemma 4.3, for any \( \tau, \tau_0 \in [0, 1] \), one has

\[
\| \Pi_{\perp} \Psi^{s+1}_r(z) \|_s \leq \| z \|_s + C(s) \left| \int_{\tau_0}^{\tau} \| \Pi_{\perp} \Psi^{s+1}_r(z) \|_s \| \Pi_{\perp} \Psi^{s+1}_r(z) \|_0 \, dt \right| \tag{4.21}
\]

for some constant \( C(s) > 0, \) only depending on \( s. \) Then by shrinking \( \delta > 0, \) if necessary, so that for \( z_{\perp} \in \mathcal{V}_{\perp}(\delta), \) we have \( \sup_{\tau_0, \tau \in [0, 1]} \| \Pi_{\perp} \Psi^{s+1}_r(z) \|_0 \leq 1, \) the above estimate becomes

\[
\| \Pi_{\perp} \Psi^{s+1}_r(z) \|_s \leq \| z_{\perp} \|_s + C(s) \left| \int_{\tau_0}^{\tau} \| \Pi_{\perp} \Psi^{s+1}_r(z) \|_s \, dt \right|. \tag{4.22}
\]

By the Gronwall inequality one then gets

\[
\sup_{\tau_0, \tau \in [0, 1]} \| \Pi_{\perp} \Psi^{s+1}_r(z) \|_s \lesssim_s \| z_{\perp} \|_s, \quad \forall z \in \mathcal{V}_s \cap h^s_{r}. \tag{4.23}
\]

Now let us prove (4.18). By (4.19), using again Lemma 4.3, one gets for any \( s \in \mathbb{Z}_{\geq 1}, \tau_0, \tau \in [0, 1], \) and \( z \in \mathcal{V}_s \cap h^s_{r}\)

\[
\| \Psi^{s+1}_r(z) - z \|_{s+1} \lesssim_s \| z \|_s \| z \|_0. \tag{4.24}
\]

\[
\| (d\Psi^{s+1}_r(z) - \text{Id})[\hat{z}] \|_{s+1} \lesssim_s \| z \|_0 \| \hat{z} \|_s + \| z \|_s \| \hat{z} \|_0. \tag{4.25}
\]

\[
\| d^k \Psi^{s+1}_r(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_{s+1} \lesssim_s \sum_{j=1}^k \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_0 + \| z \|_s \prod_{j=1}^k \| \hat{z}_j \|_0. \tag{4.26}
\]
which is the first claimed inequality in (4.18). To prove the one for the differential $d\Psi_{X}^{0,\tau} - \text{Id}$, differentiate (4.19) with respect to $z$. Using the chain rule one gets
\[
d\Psi_{X}^{0,\tau}(z)[\hat{z}] = \hat{z} + \int_{t_0}^{\tau} dX(\Psi_{X}^{0,\tau}(z), t)[d\Psi_{X}^{0,\tau}(z)[\hat{z}]] \, dt. \tag{4.24}
\]

By applying the estimates of $dX(\cdot, \tau)$ of Lemma 4.3, it follows that for any $s \in Z_{\geq 0}$ there is a constant $C(s) > 0$ such that
\[
\|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{s} \\
\leq \|\hat{z}\|_{s} + C(s) \left| \int_{t_0}^{\tau} \left( \|\Pi_{\perp}\Psi_{X}^{0,\tau}(z)\|_{s} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{0} + \|\Pi_{\perp}\Psi_{X}^{0,\tau}(z)\|_{0} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{s} \right) dt \right| \tag{4.22} \\
\leq \|\hat{z}\|_{s} + C(1) \left( \int_{t_0}^{\tau} \left( \|z_{\perp}\|_{s} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{0} + \|z_{\perp}\|_{0} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{s} \right) dt \right). \tag{4.25}
\]

for some constant $C_1(s) > C(s) > 0$. For $s = 0$, using that $\|z_{\perp}\|_0 \leq \delta < 1$, (4.25) becomes
\[
\|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{0} \leq \|\hat{z}\|_{0} + 2C_1(0) \left( \int_{t_0}^{\tau} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{0} dt \right)
\]
and hence by the Gronwall inequality
\[
\|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{0} \lesssim \|\hat{z}\|_{0}.
\]

For $s \in Z_{\geq 1}$, substitute the latter estimate into (4.25) to get, again using that $\|z_{\perp}\|_{0} < \delta < 1$
\[
\|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{s} \leq \|\hat{z}\|_{s} + C(2(s)) \|z_{\perp}\|_{s} \|\hat{z}\|_{0} + C(2(s)) \left( \int_{t_0}^{\tau} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{s} dt \right) \tag{4.26}
\]
for some constant $C_2(s) > C_1(s)$. Then using again the Gronwall inequality one concludes that for any $0 \leq \tau_0 \leq 1$,
\[
\sup_{t \in [0,1]} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{s} \lesssim_{s} \|\hat{z}\|_{s} + \|z_{\perp}\|_{s} \|\hat{z}\|_{0}. \tag{4.27}
\]

We are now ready to prove the second estimate in (4.18). By (4.24) and the smoothing estimates on $dX(\cdot, \tau)$ of Lemma 4.3, one gets that for any $s \in Z_{\geq 1}$, $0 \leq \tau_0 \leq 1$,
\[
\|(d\Psi_{X}^{0,\tau}(z) - \text{Id})[\hat{z}]\|_{s+1} \lesssim_{s} \sup_{t \in [0,1]} \|\Pi_{\perp}\Psi_{X}^{0,\tau}(z)\|_{s} \sup_{t \in [0,1]} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{0} \\
+ \sup_{t \in [0,1]} \|\Pi_{\perp}\Psi_{X}^{0,\tau}(z)\|_{0} \sup_{t \in [0,1]} \|d\Psi_{X}^{0,\tau}(z)[\hat{z}]\|_{s} \tag{4.22,4.27}
\]
\[
\lesssim_{s} \|z_{\perp}\|_{s} \|\hat{z}\|_{0} + \|z_{\perp}\|_{0} \|\hat{z}\|_{s}.\]
where we used again that $\|z\|_0 < \delta < 1$. Hence the claimed estimate for $d\Psi_{X}^{0,\tau}(z) - \mathrm{Id}$ in (4.18) is established. The estimates for the higher-order derivatives $d^k\Psi_{X}^{0,\tau}$, $k \geq 2$, follow by similar arguments, differentiating $k$-times the equation (4.19) with respect to $z$. ■

4.3 Definition of $\Psi_C$ and its properties

Our candidate for the symplectic corrector is the time-one flow map of $X(z, \tau)$,

$$
\Psi_C := \Psi_{X}^{0,1} : \mathcal{V}_0 \cap h_0^D \to h_0^D. \tag{4.28}
$$

Clearly, $\Psi_C$ is one to one and its inverse is given by the backward flow of the PDE (4.15), namely $\Psi_C^{-1} = \Psi_{X}^{1,0}$. Hence the maps $\Psi_C^\pm$ satisfy the estimates stated in Lemma 4.4. Furthermore, recall that for any $\tau \in [0, 1]$, the two-form $\Lambda_\tau$ admits the representation (4.5). Then the following Darboux lemma holds.

**Proposition 4.1.** The map $\Psi_C$ is a symplectic corrector, that is, for any $z \in \mathcal{V}_0 \cap h_0^D$, $\Psi_C^* \Lambda_1(z) = \Lambda_0$. ■

**Proof.** For any $\tau \in [0, 1]$, consider the two-form $(\Psi_{X}^{0,\tau})^* \Lambda_\tau$. Since $\Psi_{X}^{0,0} = \mathrm{Id}$, one has $(\Psi_{X}^{0,0})^* \Lambda_0 = \Lambda_0$ and hence it suffices to prove that the map $\tau \mapsto (\Psi_{X}^{0,\tau})^* \Lambda_\tau$ is constant or, equivalently,

$$
\partial_\tau ((\Psi_{X}^{0,\tau})^* \Lambda_\tau) = 0, \quad \forall \tau \in [0, 1].
$$

By Cartan’s identity (see, e.g., Lemma 1.2 in [16]) and the fact that $\Lambda_\tau$ is closed, it follows that

$$
\partial_\tau ((\Psi_{X}^{0,\tau})^* \Lambda_\tau) = (\Psi_{X}^{0,\tau})^* (\partial_\tau \Lambda_\tau + d(\Lambda_{\tau}[X(\cdot, \tau), \cdot])).
$$

Since $\partial_\tau \Lambda_{\tau} = \Lambda_1 - \Lambda_0 = \Lambda_L$ and $\Lambda_L = d\lambda_L$, it remains to prove that

$$
d(\lambda_L + \Lambda_{\tau}[X(\cdot, \tau), \cdot]) = 0.
$$

By (4.5), (4.9), (4.12), one has for any $\tau \in [0, 1], z \in \mathcal{V}_0 \cap h_0^D$, and $\hat{z} \in h_0^0$

$$
\lambda_L(z)[\hat{z}] + \Lambda_{\tau}[X(z, \tau), \hat{z}] = (E(z), \hat{z})_\tau - (\mathcal{L}_{\tau}(z)\mathcal{L}_{\tau}(z)^{-1}E(z), \hat{z})_\tau = 0.
$$

It means that

$$
\lambda_L + \Lambda_{\tau}[X(\cdot, \tau), \cdot] = 0, \quad \forall \tau \in [0, 1],
$$

proving the proposition. ■
As a consequence of Lemma 4.4 we get the following.

**Corollary 4.1.** (i) For any \( s \in \mathbb{Z}_{\geq 0} \), the map \( \Psi_C : \mathcal{V}_s^r \cap h_r^s \to h_r^s \) is a real analytic diffeomorphism onto its image and its nonlinear part is one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1} \), the map \( B_C := \Psi_C - \iota d : \mathcal{V}_s^r \cap h_r^s \to h_r^{s+1} \) is real analytic. Furthermore, \( B_C \) satisfies the following tame estimates: for any \( z \in \mathcal{V}_s^r \cap h_r^s, \tilde{z} \in h_r^0 \),

\[
\|B_C(z)\|_0 \lesssim \|z\|_0^2, \quad \|dB_C(z)[\tilde{z}]\|_0 \lesssim \|z\|_0\|\tilde{z}\|_0
\]

and for any \( k \geq 2, \tilde{z}_1, \ldots, \tilde{z}_k \in h_r^0 \),

\[
\|d^kB_C(z)[\tilde{z}_1, \ldots, \tilde{z}_k]\|_0 \lesssim_k \prod_{j=1}^k \|\tilde{z}_j\|_0,
\]

whereas for any \( s \in \mathbb{Z}_{\geq 1} \), \( z \in \mathcal{V}_s^r \cap h_r^s, \tilde{z} \in h_r^s \),

\[
\|B_C(z)\|_{s+1} \lesssim_s \|z\|_s\|z\|_0, \quad \|dB_C(z)[\tilde{z}]\|_{s+1} \lesssim_s \|z\|_0\|\tilde{z}\|_s + \|z\|_s\|\tilde{z}\|_0
\]

and for any \( k \geq 2, \tilde{z}, \tilde{z}_1, \ldots, \tilde{z}_k \in h_r^s \),

\[
\|d^kB_C(z)[\tilde{z}_1, \ldots, \tilde{z}_k]\|_{s+1} \lesssim_{s,k} \sum_{j=1}^k \|\tilde{z}_j\|_s \prod_{i \neq j} \|\tilde{z}_i\|_0 + \|z\|_s \prod_{j=1}^k \|\tilde{z}_j\|_0.
\]

(ii) The map \( A_C := \Psi_C^{-1} - \iota d : \Psi_C(\mathcal{V}_s^r) \cap h_r^0 \to h_r^0 \) is real analytic and satisfies the following tame estimates: for any \( z \in \Psi_C(\mathcal{V}_s^r) \cap h_r^0, \tilde{z} \in h_r^0 \),

\[
\|A_C(z)\|_0 \lesssim \|z\|_0^2, \quad \|dA_C(z)[\tilde{z}]\|_0 \lesssim \|z\|_0\|\tilde{z}\|_0
\]

and for any \( k \geq 2, \tilde{z}_1, \ldots, \tilde{z}_k \in h_r^0 \),

\[
\|d^kA_C(z)[\tilde{z}_1, \ldots, \tilde{z}_k]\|_0 \lesssim_k \prod_{j=1}^k \|\tilde{z}_j\|_0.
\]

Furthermore, for any \( s \in \mathbb{Z}_{\geq 1} \), \( A_C : \Psi_C(\mathcal{V}_s^r) \cap h_r^s \to h_r^{s+1} \) is real analytic and satisfies the following tame estimates: for any \( z \in \Psi_C(\mathcal{V}_s^r) \cap h_r^s, \tilde{z} \in h_r^s \),

\[
\|A_C(z)\|_{s+1} \lesssim_s \|z\|_s\|z\|_0, \quad \|dA_C(z)[\tilde{z}]\|_{s+1} \lesssim_s \|z\|_0\|\tilde{z}\|_s + \|z\|_s\|\tilde{z}\|_0
\]
and for any \( k \geq 2, \hat{z}_1, \ldots, \hat{z}_k \in h^s_c, \)

\[
\|d^k A_C(z)[\hat{z}_1, \ldots, \hat{z}_k]\|_{s+1} \lesssim_{s,k} \sum_{j=1}^{k} \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^{k} \|\hat{z}_j\|_0. \tag*{\Box}
\]

**Proof.** The claimed results are a special case of Lemma 4.4, since \( \Psi_c = \Psi^{0,1}_c \) and \( \Psi^{1,0}_c. \)

An immediate consequence of Corollary 4.1 is the following result, needed in Section 5.2.

**Corollary 4.2.** The Taylor expansion of the map \( B_C = \Psi_c - id \) around \( \Pi_s z \) up to order three is of the form

\[
B_C(z) = B^C_2(z) + B^C_3(z), \quad z \in \mathcal{V}'_s \cap h^0_r,
\]

where

\[
B^C_2(z) := \frac{1}{2} d^2 B_C(\Pi_s z)[\Pi_{\perp} z, \Pi_{\perp} z] \quad (4.29)
\]

and \( B^C_3(z) \) is the Taylor remainder term

\[
B^C_3(z) := \frac{1}{2} \int_0^1 (1-t)^2 d^3 B_C(\Pi_s z + t\Pi_s z)[\Pi_{\perp} z, \Pi_{\perp} z, \Pi_{\perp} z] \, dt. \quad (4.30)
\]

The maps \( B^C_i : \mathcal{V}'_s \cap h^0_r \to h^0_c, \quad i = 2, 3, \) are real analytic and \( B^C_i \) satisfies the following estimates: for any \( z \in \mathcal{V}'_s \cap h^0_r, \hat{z}, \hat{z}_1, \hat{z}_2 \in h^0_c, \)

\[
\|B^C_i(z)\|_0 \lesssim \|z_\perp\|_0^3, \quad \|dB^C_i(z)[\hat{z}]\|_0 \lesssim \|z_\perp\|_0^2 \|\hat{z}\|_0, \quad \|d^2 B^C_i(z)[\hat{z}_1, \hat{z}_2]\|_0 \lesssim \|z_\perp\|_0 \|\hat{z}_1\|_0 \|\hat{z}_2\|_0
\]

and for any \( k \geq 3, \hat{z}_1, \ldots, \hat{z}_k \in h^0_c, \)

\[
\|d^k B^C_3(z)[\hat{z}_1, \ldots, \hat{z}_k]\|_0 \lesssim_k \prod_{j=1}^{k} \|\hat{z}_j\|_0.
\]

Furthermore, for any \( s \in \mathbb{Z}_{\geq 1}, B^C_i : \mathcal{V}'_s \cap h^s_r \to h^{s+1}_r, \quad i = 2, 3, \) are real analytic and \( B^C_i \) satisfies the following tame estimates: for any \( z \in \mathcal{V}'_s \cap h^s_r, \hat{z}, \hat{z}_1, \hat{z}_2 \in h^s_c, \)

\[
\|B^C_i(z)\|_{s+1} \lesssim_s \|z_\perp\|_s \|z_\perp\|_s^2, \quad \|dB^C_i(z)[\hat{z}]\|_{s+1} \lesssim_s \|z_\perp\|_0 \|\hat{z}\|_s + \|z_\perp\|_s \|z_\perp\|_0 \|\hat{z}\|_0,
\]

\[
\|d^2 B^C_i(z)[\hat{z}_1, \hat{z}_2]\|_{s+1} \lesssim_s \|z_\perp\|_0 (\|\hat{z}_1\|_0 \|\hat{z}_2\|_s + \|\hat{z}_1\|_s \|\hat{z}_2\|_0) + \|z_\perp\|_s \|\hat{z}_1\|_0 \|\hat{z}_2\|_0
\]
and for any \( k \geq 3, \hat{z}, \hat{z}_1, \ldots, \hat{z}_k \in h^s_c, \)

\[
\| d^k B^C_3(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_{s+1} \lesssim_{s,k} \sum_{j=1}^{k} \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_s + \| z_\perp \|_s \prod_{j=1}^{k} \| \hat{z}_j \|_0.
\]

\( \square \)

**Proof.** By Corollary 4.1, \( B_c(\Pi s z) = 0 \) and \( dB_c(\Pi s z) = 0 \). Thus \( B_c(z) = B^C_2(z) + B^C_3(z) \) is the Taylor expansion of \( B_c \) around \( \Pi s z \) with Taylor remainder term given by (4.30). The claimed analyticity and tame estimates follow from Corollary 4.1. \( \square \)

5 Proof of Theorem 1.2

In this section we prove Theorem 1.2. First we introduce and discuss our new canonical coordinates and then express the Hamiltonian of the dNLS equation in the new coordinates.

5.1 New canonical coordinates

Our candidate of the canonical transformation is the map

\[
\Psi := \Psi_L \circ \Psi_c : \mathcal{V}_\delta \rightarrow H^0_c,
\]

where \( \mathcal{V}_\delta \) is the neighbourhood introduced in (4.16).

**Proposition 5.1.** By shrinking \( 0 < \delta < 1 \), if necessary, it follows that for any \( s \in \mathbb{Z}_{\geq 0}, \)

\( \Psi : \mathcal{V}_\delta \cap h^s_r \rightarrow H^0_r \) is a real analytic symplectic diffeomorphism onto its image with the property that its nonlinear part \( B := \Psi - F^{-1}_{\text{nl}} : \mathcal{V}_\delta \cap h^0_r \rightarrow H^0_r \) satisfies the following estimates: for any \( k \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V}_\delta \cap h^0_r, \hat{z}_1, \ldots, \hat{z}_k \in h^0_c, \)

\[
\| B(z) \|_0 \lesssim 1, \quad \| d^k B(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_0 \lesssim_{s,k} \prod_{j=1}^{k} \| \hat{z}_j \|_0.
\]

Furthermore, \( B \) is one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1}, \) the map \( B : \mathcal{V}_\delta \cap h^s_r \rightarrow H^{s+1}_r \) is real analytic, and it satisfies the following tame estimates: for any \( k \in \mathbb{Z}_{\geq 1}, z \in \mathcal{V}_\delta \cap h^s_r, \)

\[
\| B(z) \|_{s+1} \lesssim_{s} 1 + \| z_\perp \|_s, \quad \| d^k B(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_{s+1} \lesssim_{s,k} \sum_{j=1}^{k} \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_s + \| z_\perp \|_s \prod_{j=1}^{k} \| \hat{z}_j \|_0. \]
\( \square \)
Proof. By Proposition 3.1 and Corollary 4.1 one has that for any \( s \in \mathbb{Z}_{\geq 0} \), the map \( \Psi = \Psi_L \circ \Psi_C : \mathcal{V}'_s \cap \mathcal{H}^s_r \to \mathcal{H}^s_r \) is real analytic and

\[
\Psi^* \Lambda = (\Psi_L \circ \Psi_C)^* \Lambda = \Psi_C^* \Psi_L^* \Lambda \overset{(4.1)}{=} \Psi_C^* \Lambda_1 \overset{\text{Proposition 4.1}}{=} \Lambda_0 \overset{(4.1)}{=} \Lambda_M, \tag{5.2}
\]

implying that \( \Psi \) is symplectic. Recalling that \( \Psi_L = F_{nls}^{-1} + B_L \) (see (3.7)) and using that, by Corollary 4.1, \( \Psi_C = \iota d + B_C \), a direct calculation shows that for any \( z \in \mathcal{V}'_s \cap \mathcal{H}^0_r \)

\[
B(z) = \Psi(z) - F_{nls}^{-1}(z) = F_{nls}^{-1}(B_C(z)) + B_L(\Psi_C(z)). \tag{5.3}
\]

The claimed estimates for \( B \) then follow from the estimates of Proposition 3.1 and the ones of Corollary 4.1. \( \blacksquare \)

Substituting formula (3.7) for \( B_L \) one gets

\[
\Psi(z) = F_{nls}^{-1}(z) + F_{nls}^{-1}(B_C(z)) + B_{nls}(\Pi_S z + \Pi_B B_C(z)) + d_L B_{nls}(\Pi_S z + \Pi_B B_C(z))[z_\perp + \pi_B B_C(z)], \tag{5.4}
\]

where according to Corollary 4.2,

\[
B_C(z) = \frac{1}{2} d^2 B_{C}(\Pi_S z)[\Pi_\perp z, \Pi_\perp z] + \frac{1}{2} \int_0^1 (1 - t^2) d^3 B_{C}(\Pi_S z + t \Pi_B B_C(z))[\Pi_\perp z, \Pi_\perp z, \Pi_\perp z] dt.
\]

Next, we state and prove the one smoothing property and tame estimates for the map

\[
\mathcal{A}(z) := d\Psi(z)^{-1} - F_{nls}, \quad z \in \mathcal{V}'_s \cap \mathcal{H}^0_r. \tag{5.5}
\]

By the chain rule,

\[
d\Psi(z)^{-1} = d\Psi_C(z)^{-1} (d\Psi_L(\Psi_C(z)))^{-1}. \tag{5.6}
\]

By Corollary 4.1,

\[
d\Psi_C(z)^{-1} = d\Psi_C^{-1}(\Psi_C(z)) = \text{Id} + d\mathcal{A}_C(\Psi_C(z)),
\]

and that by (3.10), \( d\Psi_L(z)^{-1} = F_{nls} + \mathcal{A}_L(z) \). Hence (5.6) can be written as

\[
d\Psi(z)^{-1} = F_{nls} + \mathcal{A}(z), \quad \mathcal{A}(z) := \mathcal{A}_L(\Psi_C(z)) + d\mathcal{A}_C(\Psi_C(z)) d\Psi_L(\Psi_C(z))^{-1}. \tag{5.7}
\]
Proposition 5.2 (Tame estimates for $A$). For any $s \in \mathbb{Z}_{\geq 1}$, the map $A : \mathcal{V}_{s} \cap h_{s}^{\circ} \to \mathcal{L}(H_{c}^{s}, h_{c}^{s+1})$ is real analytic and satisfies the following tame estimates: for any $z \in \mathcal{V}_{s} \cap h_{s}^{0}$, $\hat{w} \in H_{c}^{0}$,

$$\|A(z)[\hat{w}]\|_{0} \lesssim \|\hat{w}\|_{0}$$

and for any $k \geq 1$, $\hat{z}_{1}, \ldots, \hat{z}_{k} \in h_{c}^{0}$,

$$\|d^{k}(A(z)[\hat{w}])[\hat{z}_{1}, \ldots, \hat{z}_{k}]\|_{0} \lesssim_k \|\hat{w}\|_{0} \prod_{j=1}^{k} \|\hat{z}_{j}\|_{0}.$$ 

Moreover, for any $s \in \mathbb{Z}_{\geq 1}$, $z \in \mathcal{V}_{s} \cap h_{s}^{c}$, $w \in H_{s}^{c}$,

$$\|A(z)[\hat{w}]\|_{s+1} \lesssim_{s} \|z\|_{s} \|\hat{w}\|_{0} + \|\hat{w}\|_{s}$$

and for any $k \geq 1$, $\hat{z}_{1}, \ldots, \hat{z}_{k} \in h_{c}^{c}$,

$$\|d^{k}(A(z)[\hat{w}])[\hat{z}_{1}, \ldots, \hat{z}_{k}]\|_{s+1} \lesssim_{s,k} \left(\|z\|_{s} \|\hat{w}\|_{0} + \|\hat{w}\|_{s}\right) \prod_{j=1}^{k} \|\hat{z}_{j}\|_{0} + \|\hat{w}\|_{0} \sum_{j=1}^{k} \|\hat{z}_{j}\|_{s} \prod_{i \neq j} \|\hat{z}_{i}\|_{0}.$$ 

\[ \Box \]

Proof. The claimed estimates for $A$ follow by Lemma 3.2 and Corollary 4.1 by the chain and product rules.

\[ \blacksquare \]

5.2 The dNLS Hamiltonian in new coordinates

In this subsection we prove the expansion of $H_{\text{nls}} \circ \Psi$, stated in (C3) of Theorem 1.2. Recall from (1.2) that the Hamiltonian of the dNLS equation is given by

$$H_{\text{nls}}(w) = \int_{0}^{1} (\partial_{x} u \partial_{x} v + u^{2} v^{2}) \, dx, \quad w = (u, v) \in H_{r}^{1}.$$ 

By Theorem 1.1, $H_{\text{nls}} := H_{\text{nls}} \circ \Psi_{\text{nls}}$ only depends on the actions. By a slight abuse of notation we write

$$H_{\text{nls}} = H_{\text{nls}}(I), \quad I = (I_{k})_{k \in \mathbb{Z}} \in \ell_{+}^{1,2}, \quad I_{k} \equiv I_{k}(z) = |z_{k}|^{2}/2 = (x_{k}^{2} + y_{k}^{2})/2 \quad \forall k \in \mathbb{Z} \quad (5.8)$$

and denote by $\omega_{k}^{\text{nls}}(I)$ the dNLS frequencies,

$$\omega_{k}^{\text{nls}}(I) := \partial_{I_{k}} H_{\text{nls}}(I), \quad k \in \mathbb{Z}. \quad (5.9)$$
The properties of the frequency map $I \mapsto \omega(I) := (\omega_k(I))_{k \in \mathbb{Z}}$, needed in the sequel, are summarized in the following.

**Proposition 5.3. (dNLS frequencies)** The map

$$\ell^{1,2}_+ \rightarrow \ell^\infty, \ (I_k)_{k \in \mathbb{Z}} \mapsto (\omega_n^{\text{nls}}(I) - 4\pi^2 n^2)_{n \in \mathbb{Z}} \tag{5.10}$$

is real analytic and bounded.

**Proof.** See, for example, Theorem 3.2 in [2]. □

With the notation introduced above, the $L^2$-gradient $\nabla H^{\text{nls}}(z)$ is then given by

$$\nabla H^{\text{nls}}(z) = \nabla (\Omega^{\text{nls}}[I] z), \quad z \in h^1_r, \ I \equiv I(z) = (I_n(z))_{n \in \mathbb{Z}},$$

where for any $I \in \ell^{1,2}_+$, $\Omega^{\text{nls}}(I) : h^1_r \rightarrow h^{-1}_r$ is the diagonal operator

$$\Omega^{\text{nls}}(I) := \begin{pmatrix} \text{diag}_{k \in \mathbb{Z}} \omega_k^{\text{nls}}(I) & 0 \\ 0 & \text{diag}_{k \in \mathbb{Z}} \omega_k^{\text{nls}}(I) \end{pmatrix}. \tag{5.11}$$

Further note that since $H^{\text{nls}}(z) = H^{\text{nls}}(\Psi^{\text{nls}}(z))$ one has by the chain rule

$$\Omega^{\text{nls}}[I] z = \nabla H^{\text{nls}}(z) = (d \Psi^{\text{nls}}(z))^t \nabla H^{\text{nls}}(\Psi^{\text{nls}}(z)), \quad \forall z \in \mathcal{V} \cap h^1_r, \tag{5.12}$$

where $\mathcal{V}$ is the neighbourhood of $h^0_\mathbb{C}$ in $h^0_\mathbb{C}$ of Theorem 1.1, $\mathcal{V} = \Phi^{\text{nls}}(\mathcal{V})$. For later use we record that (5.12), evaluated at $z$ with $z = \Pi_S z$, reads

$$\Omega^{\text{nls}}(I_S, 0)[\Pi_S z] = (d \Psi^{\text{nls}}(\Pi_S z))^t \nabla H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_S z))$$

implying that

$$\Pi_S (d \Psi^{\text{nls}}(\Pi_S z))^t \nabla H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_S z)) = 0. \tag{5.13}$$

The equations of motion, associated with the Hamiltonian $H^{\text{nls}}$ are given by

$$\partial_t z = J\Omega^{\text{nls}}[I] z, \quad J = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}. \tag{5.14}$$
According to the splitting $z = (z_S, z_\perp) \in \mathbb{C}^S \times \mathbb{C}^S \times \mathfrak{h}_L^0$, we can decompose the equation (5.14) as

$$\begin{aligned}
\partial_t z_S &= J\Omega_{S}^{\text{nls}}(I)[z_S] \\
\partial_t z_\perp &= J\Omega_{\perp}^{\text{nls}}(I)[z_\perp],
\end{aligned}$$

(5.15)

where

$$\Omega_{S}^{\text{nls}}(I) := \begin{pmatrix}
\text{diag}_{k \in \mathbb{S}} \alpha_k^{\text{nls}}(I) & 0 \\
0 & \text{diag}_{k \in \mathbb{S}} \alpha_k^{\text{nls}}(I)
\end{pmatrix},
\Omega_{\perp}^{\text{nls}}(I) := \begin{pmatrix}
\text{diag}_{k \in \mathbb{S}} \alpha_k^{\text{nls}}(I) & 0 \\
0 & \text{diag}_{k \in \mathbb{S}} \alpha_k^{\text{nls}}(I)
\end{pmatrix}. $$

(5.16)

Similarly, by a slight abuse of terminology, we identify $I = (I_k)_{k \in \mathbb{Z}}$ with $(I_S, I_\perp)$,

$$I = (I_S, I_\perp), \quad I_S := (I_k)_{k \in \mathbb{S}}, \quad I_\perp := (I_k)_{k \in \mathbb{S} \perp}. $$

(5.17)

Although the frequencies $\omega_k(I)$ are functions of all the action variables $I_n$, $n \in \mathbb{Z}$, the system (5.15) decouples since the action variables are invariant in time and depend only on the initial data. Now let us assume that $z(t) = (z_S(t), 0)$ is a solution of (5.15) with initial data $z(0) = (z_S^{(0)}, 0)$ and consider the equation obtained from (5.15) by linearizing it along $(z_S(t), 0)$ with initial data given by $\mathcal{Z}^{(0)} = (0, \mathcal{Z}_\perp^{(0)})$ and $\mathcal{Z}_\perp^{(0)} \in \mathfrak{h}_{L_r}^1$ and denote by $\mathcal{Z}(t)$ the corresponding solution which evolves in $\mathfrak{h}_L^1$. By a straightforward computation one verifies that the differential of $\Omega^{\text{nls}}(I)$ at $(z_S^{(0)}, 0)$ in direction $(0, \mathcal{Z}_\perp^{(0)})$ vanishes, implying that $\mathcal{Z}(t) = (0, \mathcal{Z}_\perp(t))$ where $\mathcal{Z}_\perp(t)$ is the solution of

$$\partial_t \mathcal{Z}_\perp(t) = J\Omega_\perp(I_S, 0)[\mathcal{Z}_\perp(t)], \quad \mathcal{Z}_\perp(0) = \mathcal{Z}_\perp^{(0)}. $$

(5.18)

Since by Theorem 1.1, $\Psi^{\text{nls}} : \mathfrak{h}_L^1 \to \mathcal{S}$ is symplectic it follows that

$$\mathcal{W}(t) := \text{d}\Psi^{\text{nls}}(z_S(t), 0)([0, \mathcal{Z}_\perp(t)]) = \text{d}_\perp \Psi^{\text{nls}}(z_S(t), 0)[\mathcal{Z}_\perp(t)] $$

(5.19)

is a solution of the equation obtained by linearizing the dNLS equation along $\Psi^{\text{nls}}(z_S(t), 0)$. More precisely,

$$\partial_t \mathcal{W}(t) = i\text{d}\nabla \mathcal{H}^{\text{nls}}(\Psi^{\text{nls}}(z_S(t), 0))[\mathcal{W}(t)], \quad \mathcal{W}(0) = \text{d} \Psi^{\text{nls}}(z_S^{(0)}, 0)([0, \mathcal{Z}_\perp^{(0)})]. $$

(5.20)

On the other hand, by differentiating formula (5.19) with respect to $t$, one gets

$$\begin{aligned}
\partial_t \mathcal{W}(t) &= \text{d}_\perp \Psi^{\text{nls}}(z_S(t), 0)[\partial_t \mathcal{Z}_\perp(t)] + \text{d}_S \left( \text{d}_\perp \Psi^{\text{nls}}(z_S(t), 0)[\mathcal{Z}_\perp(t)] \right) [\partial_t z_S(t)] \\
&= \text{d}_\perp \Psi^{\text{nls}}(z_S(t), 0)[J\Omega^{\text{nls}}(I_S, 0)[\mathcal{Z}_\perp(t)] + \text{d}_S \left( \text{d}_\perp \Psi^{\text{nls}}(z_S(t), 0)[\mathcal{Z}_\perp(t)] \right) [J\Omega^{\text{nls}}(I_S, 0)z_S(t)].
\end{aligned} $$

(5.21)
Comparing (5.20) and (5.21) one gets

\[
i\partial \mathcal{V}^{nls}(\Psi^{nls}(z_S(t),0)[d_{\perp} \Psi^{nls}(z_S(t),0)\tilde{z}_{\perp}] = d_{\perp} \Psi^{nls}(z_S(t),0)[J \Omega^{nls}(I_S,0)\tilde{z}_{\perp}]
+ d_S(d_{\perp} \Psi^{nls}(z_S(t),0)[\tilde{z}_{\perp}])[J \Omega^{nls}_S(I_S,0)z_S],
\]

where \( \Psi^{nls} \) is symplectic, one has

\[
(\partial \mathcal{V}^{nls}(\Psi^{nls}(z_S,0))^{-1}i\partial \mathcal{V}^{nls}(\Psi^{nls}(z_S,0))[d_{\perp} \Psi^{nls}(z_S,0)\tilde{z}_{\perp}] = d_{\perp} \Psi^{nls}(z_S,0)(0,\tilde{z}_{\perp})
\]

and hence (5.24) reads

\[
\mathcal{V}^{nls}(I_S,0)[(0,\tilde{z}_{\perp})] = (d\Psi^{nls}(z_S,0))^{t}d\mathcal{V}^{nls}(\Psi^{nls}(z_S,0))d\Psi^{nls}(z_S,0)[(0,\tilde{z}_{\perp})] - \mathcal{R}^{(1)}(z_S)[\tilde{z}_{\perp}],
\]

where \( \mathcal{R}^{(1)}(z_S) : h_{lc}^{0} \rightarrow h_{c}^{0} \) is the bounded linear operator, defined by

\[
\mathcal{R}^{(1)}(z_S)[\tilde{z}_{\perp}] := d\Psi^{nls}(z_S,0)^{t}i\partial \mathcal{V}^{nls}(\Psi^{nls}(z_S,0)[\tilde{z}_{\perp}])[J \Omega^{nls}_S(I_S,0)z_S].
\]

For later use we record the following estimates for \( \mathcal{R}^{(1)}(z_S) \).

**Lemma 5.1.** The map \( \nu_S \cap (\mathbb{R}^S \times \mathbb{R}^S) \rightarrow \mathcal{L}(h_{lc}^{0}, h_{c}^{0}), z_S \mapsto \mathcal{R}^{(1)}(z_S) \) is real analytic and bounded. Moreover it is one smoothing, meaning that for any \( s \in \mathbb{Z}_{\geq 1} \), \( \nu_S \cap (\mathbb{R}^S \times \mathbb{R}^S) \rightarrow \mathcal{L}(h_{lc}^{s}, h_{c}^{s+1}), z_S \mapsto \mathcal{R}^{(1)}(z_S) \) is real analytic. Furthermore, for any \( s \in \mathbb{Z}_{\geq 1}, \alpha, \beta \in \mathbb{Z}_{\geq 0}, z_S \in \nu_S \cap (\mathbb{R}^S \times \mathbb{R}^S), \)

\[
\|\partial_{S}^{\alpha,\beta} \mathcal{R}^{(1)}(z_S)\|_{\mathcal{L}(h_{lc}^{s}, h_{c}^{s+1})} \lesssim_{s, \alpha, \beta} 1.
\]

**Proof.** By Theorem 1.1, \( \Psi^{nls} = F_{nls}^{-1} + B^{nls} \) and hence \( d_S(d_{\perp} \Psi^{nls}(z_S,0)[\tilde{z}_{\perp}]) = d_S(d_{\perp} B^{nls}(z_S,0)[\tilde{z}_{\perp}]) \). The claimed statements then follow from Theorem 1.1. \( \blacksquare \)
We also need to record some properties of the operator $\Omega_{\perp}^{\text{nls}}(I)$ for $I = (I_S, 0)$. Write
\[
\Omega_{\perp}^{\text{nls}}(I_S, 0) = D_{\perp} + \Omega_{\perp}^{(0)}(I_S, 0),
\]
where
\[
D_{\perp} := \begin{pmatrix} \text{diag}_{n \in S_{\perp}}(2\pi n) & 0 \\ 0 & \text{diag}_{n \in S_{\perp}}(2\pi n) \end{pmatrix},
\]
and
\[
\Omega_{\perp}^{(0)}(I_S, 0) := \begin{pmatrix} \text{diag}_{n \in S_{\perp}}(\omega_n(I_S, 0) - 4\pi^2 n^2) & 0 \\ 0 & \text{diag}_{n \in S_{\perp}}(\omega_n(I_S, 0) - 4\pi^2 n^2) \end{pmatrix}.
\]

**Lemma 5.2.** For any $s \in \mathbb{Z}_{\geq 0}$, the map $V_s \cap (\mathbb{R}^S \times \mathbb{R}^S) \rightarrow \mathcal{L}(h_{\perp c}, h_{\perp c}^s), z_S \mapsto \Omega_{\perp}^{(0)}(I_S(z_S), 0)$ is real analytic and bounded. □

**Proof.** The lemma is a straightforward application of Proposition 5.3, since for any $\alpha, \beta \in \mathbb{Z}_{\geq 0}^S$
\[
\sup_{n \in S_{\perp}} \| \partial_{\alpha, \beta}^{\alpha, \beta} (\omega_n(I_S, 0) - 4\pi^2 n^2) \| \lesssim_{\alpha, \beta} 1
\]
and
\[
\| \partial_{\alpha, \beta}^{\alpha, \beta} \Omega_{\perp}^{(0)}(I_S, 0) \|_{\mathcal{L}(h_{\perp c}, h_{\perp c}^s)} \lesssim \sup_{n \in S_{\perp}} \| \partial_{\alpha, \beta}^{\alpha, \beta} (\omega_n(I_S, 0) - 4\pi^2 n^2) \| \lesssim_{\alpha, \beta} 1
\]
uniformly on $V_s \cap (\mathbb{R}^S \times \mathbb{R}^S)$. □

After this preliminary discussion, we can now study the transformed Hamiltonian $\mathcal{H}_{\text{nls}}^{\Psi_2} \circ \Psi$ where $\Psi = \Psi_L \circ \Psi_C$ is the symplectic transformation introduced in Section 5.1. We split the analysis into two parts. First we expand $\mathcal{H}^{(1)} := \mathcal{H}_{\text{nls}}^{\Psi_2} \circ \Psi_L$ and then we analyse $\mathcal{H}^{(2)} = \mathcal{H}^{(1)} \circ \Psi_C$.

### 5.2.1 Expansion of $\mathcal{H}_{\text{nls}}^{\Psi_2} \circ \Psi_L$

To expand $\mathcal{H}_{\text{nls}}^{\Psi_2} \circ \Psi_L$, it is useful to write $\mathcal{H}_{\text{nls}}$ in the form
\[
\mathcal{H}_{\text{nls}}(w) = \mathcal{H}_{2}^{\text{nls}}(w) + \mathcal{H}_{4}^{\text{nls}}(w),
\]
where

\[ H_{2}^{\text{nls}}(w) := \frac{1}{2} \langle D_{2} w, w \rangle, \quad H_{4}^{\text{nls}}(w) := \int_{T} u^{2} v^{2} \, dx, \tag{5.31} \]

and the operator \( D_{2} \) is defined as

\[ D_{2} := \begin{pmatrix} 0 & -\partial_{xx} \\ -\partial_{xx} & 0 \end{pmatrix}. \]

Note that \( D_{2} = D_{2}^{t} \). The Hamiltonian equations associated with (5.30) can be written as

\[ \partial_{t} w = i \mathbb{J} \nabla H^{\text{nls}}(w), \quad \mathbb{J} = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad \nabla H^{\text{nls}} = (\nabla_u H^{\text{nls}}, \nabla_v H^{\text{nls}}), \tag{5.32} \]

where

\[ \nabla H^{\text{nls}}(w) = D_{2} w + \nabla H_{4}^{\text{nls}}(w), \quad d\nabla H^{\text{nls}}(w) = D_{2} + d\nabla H_{4}^{\text{nls}}(w). \tag{5.33} \]

The Taylor expansion of \( H^{\text{nls}} \) around \( \Psi^{\text{nls}}(\Pi_{S} z) \) up to order three reads

\[ H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z) + w) = H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z)) + \langle \nabla H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z)), w \rangle_{r} \]

\[ + \frac{1}{2} (d\nabla H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z)) [w], w)_{r} + T_{3}^{(1)}(z_{S}, w), \tag{5.34} \]

where \( T_{3}^{(1)}(z_{S}, w) \) is the Taylor remainder term of order three, given by

\[ T_{3}^{(1)}(z_{S}, w) := \frac{1}{2} \int_{0}^{1} (1 - t)^{2} d^{3} H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z) + tw)[w, w, w] \, dt \]

\[ \overset{(5.30),(5.31)}{=} \frac{1}{2} \int_{0}^{1} (1 - t)^{2} d^{3} H_{4}^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z) + tw)[w, w, w] \, dt. \tag{5.35} \]

For later use we record that the third derivative of \( H_{4}^{\text{nls}} \) at \( w_{0} = (u_{0}, v_{0}) \in H_{1}^{1} \) in direction \( w = (u, v) \) in \( H_{1}^{1} \) can be computed as

\[ d^{3} H_{4}^{\text{nls}}(w_{0})[w, w, w] = 12 \int_{0}^{1} (u_{0}uv^{2} + u^{2}v_{0}v) \, dx. \tag{5.36} \]

Substituting for \( w \) the function \( d_{\perp} \Psi^{\text{nls}}(\Pi_{S} z)[z_{\perp}] (=d_{\perp} \Psi^{\text{nls}}(\Pi_{S} z)[\Pi_{\perp} z]) \) and taking into account that by (3.3), \( \Psi_{L}(z) = \Psi^{\text{nls}}(\Pi_{S} z) + d_{\perp} \Psi^{\text{nls}}(\Pi_{S} z)[z_{\perp}] \) yields

\[ H^{(1)}(z) = H^{\text{nls}}(\Psi_{L}(z)) = H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z)) + \langle \nabla H^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S} z)), d\Psi^{\text{nls}}(\Pi_{S} z)[\Pi_{\perp} z] \rangle_{r} \]
Writing the right-hand side of the latter identity in a more convenient form one gets

\[ \mathcal{H}^{(1)}(z) = \mathcal{H}^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S}z)) + (\Pi_{\perp}(d\Psi^{\text{nls}}(\Pi_{S}z))' \nabla \mathcal{H}^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S}z)), \Pi_{\perp}z), \Pi_{\perp}z)' + \vartheta^{(1)}(z, d\Psi^{\text{nls}}(\Pi_{S}z)[\Pi_{\perp}z]). \] (5.37)

Recall that \( H^{\text{nls}} = \mathcal{H}^{\text{nls}} \circ \Psi^{\text{nls}} \). Hence by Theorem 1.1 one gets

\[ \mathcal{H}^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S}z)) = H^{\text{nls}}(I_{S}, 0). \] (5.38)

Furthermore by (5.13),

\[ \Pi_{\perp}(d\Psi^{\text{nls}}(\Pi_{S}z))' \nabla \mathcal{H}^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S}z)) = 0. \] (5.39)

Next, the term in (5.37), which is quadratic in \( z_{\perp} \), can be written as

\[ \frac{1}{2}(\Pi_{\perp}(d\Psi^{\text{nls}}(\Pi_{S}z))' \nabla \mathcal{H}^{\text{nls}}(\Psi^{\text{nls}}(\Pi_{S}z))d\Psi^{\text{nls}}(\Pi_{S}z)[\Pi_{\perp}z], \Pi_{\perp}z)' \]

\[ \overset{5.25}{=} \frac{1}{2}(\Omega^{\text{nls}}(I_{S}, 0)[\Pi_{\perp}z], \Pi_{\perp}z)' + \frac{1}{2}(\mathcal{R}^{(1)}(z_{S})[z_{\perp}], z_{\perp}), \] (5.40)

Substituting (5.38–5.40) into (5.37) then yields

\[ \mathcal{H}^{(1)}(z) = H^{\text{nls}}(I_{S}, 0) + \frac{1}{2}(\Omega^{\text{nls}}(I_{S}, 0)[z_{\perp}], z_{\perp}) + \mathcal{P}^{(1)}_{2}(z) + \mathcal{P}^{(1)}_{3}(z), \] (5.41)

where

\[ \mathcal{P}^{(1)}_{2}(z) := \frac{1}{2}(\mathcal{R}^{(1)}(z_{S})[z_{\perp}], z_{\perp}), \quad \mathcal{P}^{(1)}_{3}(z) := \vartheta^{(1)}(z_{S}, d\Psi^{\text{nls}}(\Pi_{S}z)[\Pi_{\perp}z]). \] (5.42)

**Lemma 5.3.** (i) For any \( s \in \mathbb{Z}_{\geq 0} \), \( \mathcal{P}^{(1)}_{2} : \mathcal{V} \cap h^{s}_{r} \to \mathbb{R} \) is real analytic and the following estimates hold: for any \( s \in \mathbb{Z}_{\geq 0} \), \( z \in \mathcal{V} \cap h^{s}_{r} \),

\[ \| \nabla \mathcal{P}^{(1)}_{2}(z) \|_{s} \lesssim_{s} \| z \|_{s} \]
and for any $k \in \mathbb{Z}_{\geq 1}, \hat{z}_1, \ldots, \hat{z}_k \in h_c^s$,

$$
\| d^k \nabla P_2^{(1)}(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_{s, k} \lesssim_s \sum_{j=1}^k \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_0 + \| z_{\perp} \|_s \prod_{j=1}^k \| \hat{z}_j \|_0.
$$

(ii) For any $s \in \mathbb{Z}_{\geq 0}$, $P_3^{(1)} : V \cap h_c^s \to \mathbb{R}$ is real analytic and the following estimates hold: for any $s \in \mathbb{Z}_{\geq 0}$, $z \in V \cap h_c^s$, $\hat{z} \in h_c^s$,

$$
\| \nabla P_3^{(1)}(z)[\hat{z}] \|_s \lesssim_s \| z_{\perp} \|_s \| \hat{z} \|_0,
\| d \nabla P_3^{(1)}(z)[\hat{z}] \|_s \lesssim_s \| z_{\perp} \|_s \| \hat{z} \|_0 + \| z_{\perp} \|_0 \| \hat{z} \|_s
$$

and for any $k \in \mathbb{Z}_{\geq 2}, \hat{z}_1, \ldots, \hat{z}_k \in h_c^s$,

$$
\| d^k \nabla P_3^{(1)}(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_{s, k} \lesssim_s \sum_{j=1}^k \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_0 + \| z_{\perp} \|_s \prod_{j=1}^k \| \hat{z}_j \|_0.
$$

\[ \square \]

**Proof.** Item (i) follows from Lemma 5.1 and item (ii) from (5.42), (5.35), (5.36), and Theorem 1.1.

5.2.2 Expansion of $H^{(2)} := H^{(1)} \circ \Psi_c$

To study the expansion of the composition $H^{(2)} = H^{(1)} \circ \Psi_c$ of the Hamiltonian $H^{(1)}$ with the symplectic corrector $\Psi_c$, constructed in Section 4, we separately expand the compositions of the terms on the right-hand side of the identity (5.41) with $\Psi_c$. In addition to the projectors $\Pi_S, \Pi_{\perp}$, defined in (3.1), (3.2), we also introduce the following versions of them,

$$
\pi_S : \mathbb{C}^S \times \mathbb{C}^S \times h_{1c}^0 \to \mathbb{C}^S \times \mathbb{C}^S, \quad z = (z_S, z_{\perp}) \to z_S,
$$

$$
\pi_{\perp} : \mathbb{C}^S \times \mathbb{C}^S \times h_{1c}^0 \to h_{1c}^0, \quad z = (z_S, z_{\perp}) \to z_{\perp}.
$$

5.2.2.1 Term $H^{\text{nls}}(I_S, 0)$. It is convenient to define

$$
H^{\text{nls}}(I_S) := H^{\text{nls}}(I_S, 0),
$$

where we recall that by (5.8), (5.17)

$$
I_S = I_S(z_S) = \left( \frac{1}{2} (x_j^2 + y_j^2) \right)_{j \in S}, \quad z_S = ((x_j)_{j \in S}, (y_j)_{j \in S}) \in \mathbb{R}^S \times \mathbb{R}^S.
$$

By Corollaries 4.1, 4.2 $\Psi_c(z)$, defined for $z \in V_S', h_c^0$, is of the form $\Psi_c(z) = z + B_c(z) = z + B_2^C(z) + B_4^C(z)$. Hence the Taylor expansion of $h^{\text{nls}}(\pi_S \Psi_c(z))$ around $z_S$ reads

$$
h^{\text{nls}}(\pi_S \Psi_c(z)) = h^{\text{nls}}(z_S) + \nabla_S h^{\text{nls}}(z_S) \cdot \pi_S B_2^C(z) + \mathcal{P}_3^{(2a)}(z),
$$

(5.46)
where \( \mathcal{P}_3^{(2a)}(z) \) is the Taylor remainder term of order three, given by
\[
\mathcal{P}_3^{(2a)}(z) := \nabla_s h^{\text{nl}}(z_s) \cdot \pi_s B^C_3(z) + \int_0^1 (1 - t) \, ds \nabla_s h^{\text{nl}}(z_s + t \pi_s B_C(z)) [\pi_s B_C(z)] \cdot \pi_s B_C(z) \, dt.
\]
(5.47)

In the next lemma we provide estimates for the Hamiltonian \( \mathcal{P}_3^{(2a)}(z) \).

**Lemma 5.4.** For any \( s \in \mathbb{Z}_{>0} \), \( \mathcal{P}_3^{(2a)} \circ \Psi_C : \mathcal{V}_s \cap h_s^r \to \mathbb{R} \) is real analytic. Furthermore, \( \nabla \mathcal{P}_3^{(2a)} \) satisfies the following tame estimates: for any \( s \in \mathbb{Z}_{>0} \), \( z \in \mathcal{V}_s \cap h_s^r \), \( \mathcal{Z} \in h_s^r \),
\[
\| \nabla \mathcal{P}_3^{(2a)}(z) \|_s \lesssim_s \| z \|_s \| z \|_0,
\]
and for any \( k \in \mathbb{Z}_{>2}, \mathcal{Z}_1, \ldots, \mathcal{Z}_k \in h_s^r \),
\[
\| d^k \nabla \mathcal{P}_3^{(2a)}(z)[\mathcal{Z}_1, \ldots, \mathcal{Z}_k] \|_s \lesssim_{s,k} \sum_{j=1}^k \| \mathcal{Z}_j \|_s \prod_{i \neq j} \| \mathcal{Z}_j \|_0 + \| z \|_s \prod_{j=1}^k \| \mathcal{Z}_j \|_0.
\]
□

**Proof.** The lemma follows by differentiating \( \mathcal{P}_3^{(2a)} \) and applying the estimates of Corollaries 4.1 and 4.2. ■

**5.2.2.2 Term** \( \mathcal{H}_3(z) := \frac{1}{2} (\Omega^{\text{nl}}_\perp(I_s, 0) z_\perp, z_\perp)_r \). To begin with let us point out that the expansion of the composition of the term \( \mathcal{H}_3(z) \) with the transformation \( \Psi_C \) needs special care. To explain this in more detail, write \( 2 \mathcal{H}_3(z) \) in the form
\[
(\Omega^{\text{nl}}_\perp(I_s, 0) z_\perp, z_\perp)_r = (D^2_\perp z_\perp, z_\perp)_r + (\Omega^{(0)}_\perp(I_s, 0) z_\perp, z_\perp)_r,
\]
where \( D_\perp \) is the diagonal operator defined in (5.28). When composed with \( \Psi_C = \iota d + B_C \), the term \( (D^2_\perp z_\perp, z_\perp)_r \) becomes
\[
(D^2_\perp [z_\perp + \pi_\perp B_C(z)], z_\perp + \pi_\perp B_C(z))_r = (D^2_\perp [z_\perp], z_\perp)_r + (D^2_\perp [z_\perp], \pi_\perp B_C(z))_r + (D^2_\perp [\pi_\perp B_C(z)], z_\perp)_r + (D^2_\perp [\pi_\perp B_C(z)], \pi_\perp B_C(z))_r,
\]
(5.48)
where \( \pi_\perp \) is defined in (5.44). By (4.29) and (4.30), it then follows that the difference
\[
\frac{1}{2} (D^2_\perp [z_\perp + \pi_\perp B_C(z)], z_\perp + \pi_\perp B_C(z))_r - \frac{1}{2} (D^2_\perp [z_\perp], z_\perp)_r
\]
belongs to the error term \( \mathcal{P}_3(z) \) in Theorem 1.2. Since \( B_C \) is only one smoothing, the two terms
\[
(D^2_\perp [z_\perp], \pi_\perp B_C(z))_r, \quad (D^2_\perp [\pi_\perp B_C(z)], z_\perp)_r
\]
could prevent that \( \mathcal{P}_3 \) satisfies the estimates (1.7), stated in Theorem 1.2.
To proceed, recall that $\Psi_{C}^{0,1}$ where $\Psi_{X}^{0,\tau}$ is the flow map, defined in (4.16). We have

$$\mathcal{H}_{\Omega}(\Psi_{C}(z)) = \mathcal{H}_{\Omega}(z) + P_{3}^{(2b)}(z), \quad P_{3}^{(2b)}(z) := \mathcal{H}_{\Omega}(\Psi_{C}(z)) - \mathcal{H}_{\Omega}(z).$$

(5.49)

Using the mean value theorem and recalling (4.16), one has

$$P_{3}(2b)_{3}(z) = \int_{0}^{1} P_{\Omega}(\Psi_{X}^{0,\tau}(z), \tau) \, d\tau,$$  

(5.50)

where for any $\tau \in [0, 1]$, the Hamiltonian $P_{\Omega}(z, \tau)$ is defined by

$$P_{\Omega}(z, \tau) := (\nabla \mathcal{H}_{\Omega}(z), X(z, \tau)).$$

(5.51)

One has that

$$\left(\nabla \mathcal{H}_{\Omega}(z), X(z, \tau)\right)_{r} = \frac{1}{2} \nabla S \mathcal{H}_{\Omega}(z) \cdot \pi S X(z, \tau) + \left(\Omega_{\perp}^{\text{nls}}(I_{S}, 0) z_{\perp}, \pi_{\perp} X(z, \tau)\right)_{r}.$$  

(5.52)

By (4.12), the vector field $X(z, \tau)$ was chosen to be

$$X(z, \tau) = -L_{r}(z)^{-1}E(z),$$

where $E(z)$ is given by (4.10) and $L_{r}(z)^{-1}$ by the Neumann series (4.6) in Lemma 4.1. Hence

$$X(z, \tau) = -L_{r}(z)^{-1}E(z) = -JE(z) - \sum_{n \geq 1} (-1)^{n} \tau^{n} (JL(z))^{n}JE(z)$$

$$= -JE(z) + \tau JL(z) \sum_{n \geq 0} (-1)^{n} \tau^{n} (JL(z))^{n}JE(z)$$

$$= -JE(z) + \tau JL(z) X(z, \tau).$$

(5.53)

Since $E = \Pi_{S}E$ and $J^{t} = -J$, the last term in (5.52) becomes

$$\left(\Omega_{\perp}^{\text{nls}}(I_{S}, 0) z_{\perp}, \pi_{\perp} X(z, \tau)\right)_{r} = \left(\Omega_{\perp}^{\text{nls}}(I_{S}, 0) z_{\perp}, \pi_{\perp} \tau JL(z) X(z, \tau)\right)_{r}$$

$$= -\tau \left(J \Omega_{\perp}^{\text{nls}}(I_{S}, 0) z_{\perp}, \pi_{\perp} L(z) X(z, \tau)\right)_{r}.$$  

(5.54)

By (3.40), the component $L_{\perp}^{1}(z)$ of $L(z)$ vanishes. Hence using the projections introduced in (5.43) and (5.44), one has

$$\pi_{\perp} L(z) X(z, \tau) = L_{\perp}^{S}(z) \pi_{S} X(z, \tau).$$
Substituting the latter expression into (5.54) then leads to

\[
\left( \Omega_{\perp}^{\text{nls}}(I, 0)z, \pi_{\perp}X(z, \tau) \right) = -\tau \left( J\Omega_{\perp}^{\text{nls}}(I, 0)z, L_{\perp}^{\text{nls}}(z)\pi_{\perp}X(z, \tau) \right)
\]

\[
= -\tau \left( L_{\perp}^{\text{nls}}(z)^{\dagger}J\Omega_{\perp}^{\text{nls}}(I, 0)z \cdot \pi_{\perp}X(z, \tau) \right)
\]

\[
= \tau \left( L_{\perp}^{\text{nls}}(z)J\Omega_{\perp}^{\text{nls}}(I, 0)z \cdot \pi_{\perp}X(z, \tau) \right). \tag{5.55}
\]

By the definition (3.42),

\[
L_{\perp}^{\text{nls}}(z)J\Omega_{\perp}^{\text{nls}}(I, 0)z = \left( \left( i \int d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [J\Omega_{\perp}^{\text{nls}}(I, 0)z], \partial_{\perp} d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right)_{j \in S} \right).
\tag{5.56}
\]

Let us take a closer look at the expression

\[
d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [J\Omega_{\perp}^{\text{nls}}(I, 0)z] = d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [J\Omega_{\perp}^{\text{nls}}(I, 0)](0, z_{\perp})].
\]

Substituting for \( J\Omega_{\perp}^{\text{nls}}(I, 0) \) the right-hand side of the identity (5.25), one gets

\[
d\Psi_{\text{nls}}(\Pi_{\perp}z) [J\Omega_{\perp}^{\text{nls}}(I, 0)](0, z_{\perp})] = d\Psi_{\text{nls}}(\Pi_{\perp}z) \left( J\left( d\Psi_{\text{nls}}(\Pi_{\perp}z) \right)^{\dagger}d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) d\Psi_{\text{nls}}(\Pi_{\perp}z) [0, \hat{z}_{\perp}] \right) \]

\[
- d\Psi_{\text{nls}}(\Pi_{\perp}z) J\mathcal{R}^{(1)}(z_{\perp})[\hat{z}_{\perp}].
\]

The first term on the right-hand side of the latter identity can be simplified. Since \( \Psi_{\text{nls}} \) is symplectic,

\[
d\Psi_{\text{nls}}(\Pi_{\perp}z) J(d\Psi_{\text{nls}}(\Pi_{\perp}z))^{\dagger} = i_{\Pi_{\perp}z},
\]

one has

\[
d\Psi_{\text{nls}}(\Pi_{\perp}z) J(d\Psi_{\text{nls}}(\Pi_{\perp}z))^{\dagger}d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) d\Psi_{\text{nls}}(\Pi_{\perp}z) [0, \hat{z}_{\perp}] \right) = i \int d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right] = i \int d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right] \]

\[
= i \int d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right] = i \int d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right]. \tag{5.57}
\]

Combining the above identities, the component \( i \int d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [J\Omega_{\perp}^{\text{nls}}(I, 0)z], \partial_{\perp} d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right) \) on the right-hand side of (5.56) becomes, for \( j \) arbitrary,

\[
i \int d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [J\Omega_{\perp}^{\text{nls}}(I, 0)z], \partial_{\perp} d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right] = i \int d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right] \]

\[
- i \int d\nabla H_{\text{nls}}(\Psi_{\text{nls}}(\Pi_{\perp}z) J\mathcal{R}^{(1)}(z_{\perp})[z_{\perp}], \partial_{\perp} d_{\perp} \Psi_{\text{nls}}(\Pi_{\perp}z) [z_{\perp}] \right]. \tag{5.58}
\]
which in view of $d\nabla H_{\text{als}}^n(w) = D_2 + d\nabla H_{\text{als}}^n(\phi)$ (cf. (5.33)) leads to
\[
\begin{align*}
& i\{\partial_1 \psi_{\text{als}}(\Pi_S z)|J\Omega_{\perp}^n(I_S,0)z,\partial_x \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r \\
& = \{D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|0,\partial_x \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r \\
& + \{d\nabla H_{\text{als}}^n(\psi_{\text{als}}(\Pi_S z)) \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}, \partial_x \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r \\
& - i\{\partial_1 \psi_{\text{als}}(\Pi_S z) J R^{(1)}(z), \partial_x \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r. \tag{5.59}
\end{align*}
\]

Since $D_2 = D_2^\perp$, the first term on the right-hand side on the latter identity can be written as
\[
\begin{align*}
\{D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}, \partial_x \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r &= \frac{1}{2} \partial_x \{D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}, \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r \\
& = \frac{1}{2} \partial_x \{D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|0,0\}_r, \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r \\
& = \frac{1}{2} \partial_x \{(d\psi_{\text{als}}(\Pi_S z))^t D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|0,0\}_r, (0,z)\}_r, \tag{5.60}
\end{align*}
\]
which can be further transformed as follows: using $D_2 = d\nabla H_{\text{als}} - d\nabla H_{\text{als}}^n$ (cf. (5.33)) and taking into account that by (5.25),
\[
(d\psi_{\text{als}}(z,0))^t d\nabla H_{\text{als}}(\psi_{\text{als}}(z,0)) d\psi_{\text{als}}(z,0)|0,0\} = \Omega_{\text{als}}(I_S,0)|0,0\} + R^{(1)}(z)
\]
one is lead to
\[
\begin{align*}
& \frac{1}{2} \partial_x \{(d\psi_{\text{als}}(\Pi_S z))^t D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|0,0\}_r, (0,z)\}_r = \frac{1}{2} \partial_x \{(d\psi_{\text{als}}(\Pi_S z))^t D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|0,0\}_r, (0,z)\}_r \\
& + \frac{1}{2} \partial_x \{(d\psi_{\text{als}}(\Pi_S z))^t D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|0,0\}_r, (0,z)\}_r \tag{5.61}
\end{align*}
\]
Let us analyse $\partial_x \{(d\psi_{\text{als}}(\Pi_S z))^t D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|0,0\}_r, (0,z)\}_r = (\partial_x \Omega_{\text{als}}(I_S,0) z, z)_r$ in more detail. Substituting for $\Omega_{\text{als}}(I_S,0)$ the expression $D_2^2 + \Omega_{\text{als}}(I_S,0)$ (cf. (5.27)) and using that $(\partial_x D_2^2 z, z)_r = 0$ for any $j \in S$, one concludes that

\[
(\partial_x \Omega_{\text{als}}(I_S,0) z, z)_r = (\partial_x \Omega_{\text{als}}(I_S,0) z, z)_r, \quad \forall j \in S.
\]

The above identities then imply that (5.60) becomes
\[
\begin{align*}
\{D_2 \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}, \partial_x \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r &= \frac{1}{2} (\partial_x \Omega_{\text{als}}(I_S,0) z, z)_r \\
& + \frac{1}{2} \{(\partial_x R^{(1)}(w) z, (0,z)) = \frac{1}{2} \partial_x \{(d\nabla H_{\text{als}}(\psi_{\text{als}}(\Pi_S z)) \partial_1 \psi_{\text{als}}(\Pi_S z)|z\}_r, (0,z)\}_r. \tag{5.61}
\end{align*}
\]
With (5.60) and (5.61), the identity (5.59) becomes

\[
i(j_{\perp} \psi_{\text{nls}}(\Pi S z)[J \Omega_{\perp}^{\text{nls}}(I_S, 0) z_{\perp}], \partial_{x_j} d_{\perp} \psi_{\text{nls}}(\Pi S z)[z_{\perp}]_r = (R_{x_j}(z_S)[z_{\perp}], z_{\perp})_r, \tag{5.62}
\]

where for any \( j \in S \), \( R_{x_j}(z_S) : h_{\perp}^{0} \rightarrow h_{\perp}^{0} \) is the linear operator defined by

\[
\frac{1}{2} \partial_{x_j} \Omega_{\perp}^{(0)}(I_S, 0) + \frac{1}{2} \pi_{\perp} \partial_{x_j} \mathcal{R}^{(1)}(z_S) - \frac{1}{2} \partial_{x_j} \left((d_{\perp} \psi_{\text{nls}}(z_S, 0))^t d\nabla \mathcal{H}_{4}^{\text{nls}}(\psi_{\text{nls}}(\Pi S z)) d_{\perp} \psi_{\text{nls}}(\Pi S z)\right) \\
+ \left(\partial_{x_j} d_{\perp} \psi_{\text{nls}}(\Pi S z)\right)^t d\nabla \mathcal{H}_{4}^{\text{nls}}(\psi_{\text{nls}}(\Pi S z)) d_{\perp} \psi_{\text{nls}}(\Pi S z) \\
- i \left(\partial_{x_j} d_{\perp} \psi_{\text{nls}}(\Pi S z)\right)^t j d\psi_{\text{nls}}(\Pi S z) J \mathcal{R}^{(1)}(z_S). \tag{5.63}
\]

Arguing similarly as above one obtains

\[
i(j_{\perp} \psi_{\text{nls}}(\Pi S z)[J \Omega_{\perp}^{\text{nls}}(I_S, 0) z_{\perp}], \partial_{y_j} d_{\perp} \psi_{\text{nls}}(\Pi S z)[z_{\perp}]_r = (R_{y_j}(z_S)[z_{\perp}], z_{\perp})_r, \tag{5.64}
\]

where \( R_{y_j}(z_S) : h_{\perp}^{0} \rightarrow h_{\perp}^{0} \) is given by

\[
\frac{1}{2} \partial_{y_j} \Omega_{\perp}^{(0)}(I_S, 0) + \frac{1}{2} \pi_{\perp} \partial_{y_j} \mathcal{R}^{(1)}(z_S) - \frac{1}{2} \partial_{y_j} \left((d_{\perp} \psi_{\text{nls}}(z_S, 0))^t d\nabla \mathcal{H}_{4}^{\text{nls}}(\psi_{\text{nls}}(\Pi S z)) d_{\perp} \psi_{\text{nls}}(\Pi S z)\right) \\
+ \left(\partial_{y_j} d_{\perp} \psi_{\text{nls}}(\Pi S z)\right)^t d\nabla \mathcal{H}_{4}^{\text{nls}}(\psi_{\text{nls}}(\Pi S z)) d_{\perp} \psi_{\text{nls}}(\Pi S z) - i \left(\partial_{y_j} d_{\perp} \psi_{\text{nls}}(\Pi S z)\right)^t j d\psi_{\text{nls}}(\Pi S z) J \mathcal{R}^{(1)}(z_S). \tag{5.65}
\]

In the next lemma we state estimates for the operators \( R_{x_j}(z_S) \) and \( R_{y_j}(z_S) \).

**Lemma 5.5.** For any \( j \in S \) and \( s \in \mathbb{Z}_{\geq 0} \), the maps

\[
R_{x_j} : \psi_{S} \cap (\mathbb{R}^S \times \mathbb{R}^S) \rightarrow \mathcal{L}(h_{\perp}^{s}, h_{\perp}^{s}), \quad z_S \mapsto R_{x_j}(z_S), \\
R_{y_j} : \psi_{S} \cap (\mathbb{R}^S \times \mathbb{R}^S) \rightarrow \mathcal{L}(h_{\perp}^{s}, h_{\perp}^{s}), \quad z_S \mapsto R_{y_j}(z_S)
\]

are real analytic and bounded. Furthermore, for any \( \alpha, \beta \in \mathbb{Z}_{\geq 0}^S \),

\[
\|\partial_{S}^{\alpha, \beta} R_{x_j}(z_S)\|_{\mathcal{L}(h_{\perp}^{s}, h_{\perp}^{s})}, \quad \|\partial_{S}^{\alpha, \beta} R_{y_j}(z_S)\|_{\mathcal{L}(h_{\perp}^{s}, h_{\perp}^{s})} \lesssim_{s, \alpha, \beta} 1.
\]

**Proof.** The lemma follows from Theorem 1.1 and Lemmata 5.1, 5.2. \( \square \)

Finally, by (5.51), (5.52), (5.55), (5.56), (5.62), (5.64) and writing

\[
\pi_S X(z, \tau) = \left((X_{j,+}(z, \tau))_{j \in S}, (X_{j,-}(z, \tau))_{j \in S}\right) \in \mathbb{R}^S \times \mathbb{R}^S
\]
one sees that the Hamiltonian $\mathcal{P}_3(z, \tau)$, defined by (5.51), can be written in the form

$$\frac{1}{2} \nabla H_3(z) \cdot \pi_3 X(z, \tau) + \sum_{j \in S} X_{j,+}(z, \tau) (R_{x_j}(z) [z_\perp], z_\perp), \quad \sum_{j \in S} X_{j,-}(z, \tau) (R_{y_j}(z) [z_\perp], z_\perp).$$

(5.66)

In the next lemma we state estimates for the Hamiltonian $\mathcal{P}_3^{(2b)}$, defined in (5.50).

**Lemma 5.6.** For any $s \in \mathbb{Z}_{\geq 0}$, the Hamiltonian $\mathcal{P}_3^{(2b)} : \mathcal{V}_s' \cap h^s_r \to \mathbb{R}$ is real analytic. Moreover, it satisfies the following tame estimates: for any $s \in \mathbb{Z}_{\geq 0}$, $z \in \mathcal{V}_s' \cap h^s_r$, $\hat{z}_1, \hat{z}_2 \in h^s_r$,

$$\|\nabla \mathcal{P}_3^{(2b)}(z)(\hat{z}_1, \hat{z}_2)\|_s \lesssim_s \|z_\perp\|_s \|z_\perp\|_0^2,$$

$$\|d^2 \nabla \mathcal{P}_3^{(2b)}(z)(\hat{z}_1, \hat{z}_2)\|_s \lesssim_s \|z_\perp\|_s \|z_\perp\|_0 \|\hat{z}_1\|_0 + \|z_\perp\|_0 \|\hat{z}_1\|_0 \|\hat{z}_2\|_s,$$

and for any $k \in \mathbb{Z}_{\geq 3}$, $\hat{z}_1, \ldots, \hat{z}_k \in h^s_r$,

$$\|d^k \nabla \mathcal{P}_3^{(2b)}(z)(\hat{z}_1, \ldots, \hat{z}_k)\|_{s+1} \lesssim_{s,k} \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^k \|\hat{z}_j\|_0. \quad \square$$

**Proof.** The lemma follows by (5.50), (5.66), and Lemmata 4.3, 4.4, 5.2, 5.5. □

5.2.2.3 Term $\mathcal{P}_2^{(1)}$. Recall that the Hamiltonian $\mathcal{P}_2^{(1)}$ was introduced in (5.42). For $z \in \mathcal{V}_s' \cap h^s_r$ one has $\Psi_C(z) = z + B_C(z)$ and hence the Taylor expansion of $\mathcal{P}_2^{(1)}(\Psi_C(z))$ around $z$ reads

$$\mathcal{P}_2^{(1)}(\Psi_C(z)) = \mathcal{P}_2^{(1)}(z) + \mathcal{P}_3^{(2c)}(z), \quad \mathcal{P}_3^{(2c)}(z) := \int_0^1 (\nabla \mathcal{P}_2^{(1)}(z + tB_C(z)), B_C(z))_r \ dt. \quad (5.67)$$

The following lemma holds:

**Lemma 5.7.** For any $s \in \mathbb{Z}_{\geq 0}$, the Hamiltonian $\mathcal{P}_2^{(1)} \circ \Psi_C : \mathcal{V}_s' \cap h^s_r \to \mathbb{R}$ is real analytic. Moreover, the Hamiltonian $\mathcal{P}_3^{(2c)}$, defined in (5.67), satisfies the following tame estimates: for any $s \in \mathbb{Z}_{\geq 0}$, $z \in \mathcal{V}_s' \cap h^s_r$, $\hat{z} \in h^s_r$,

$$\|\nabla \mathcal{P}_3^{(2c)}(z)\|_s \lesssim_s \|z_\perp\|_s \|z_\perp\|_0,$$

$$\|d\nabla \mathcal{P}_3^{(2c)}(z)(\hat{z})\|_s \lesssim_s \|z_\perp\|_s \|\hat{z}\|_0 + \|z_\perp\|_0 \|\hat{z}\|_s,$$

and for any $k \in \mathbb{Z}_{\geq 2}$, $\hat{z}_1, \ldots, \hat{z}_k \in h^s_r$,

$$\|d^k \nabla \mathcal{P}_3^{(2c)}(z)(\hat{z}_1, \ldots, \hat{z}_k)\|_s \lesssim_{s,k} \sum_{j=1}^k \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z_\perp\|_s \prod_{j=1}^k \|\hat{z}_j\|_0. \quad \square$$
Proof. The lemma follows by differentiating \( P_3^{(2c)} \) and applying Corollary 4.1 and Lemma 5.3(i).

5.2.2.4 Term \( P_3^{(1)} \). By (5.42), \( P_3^{(1)} \) is given by \( T_3^{(1)}(z_S, d\Psi^{(1)}(\Pi_S z)[\Pi_{\perp} z]) \) where \( T_3^{(1)} \) is the Taylor remainder term of order three, introduced in (5.35). Using the estimates of \( P_3^{(1)} \) of Lemma 5.3(ii), the Hamiltonian \( P_3^{(1)} \circ \Psi_C \) can be estimated as follows:

Lemma 5.8. For any \( s \in \mathbb{Z}_{\geq 0}, P_3^{(1)} \circ \Psi_C : V_h^s \cap h_r^s \rightarrow \mathbb{R} \) is real analytic. Moreover, the following tame estimates hold: for any \( s \in \mathbb{Z}_{\geq 1}, z \in V_h^s \cap h_r^s, \hat{z} \in h_r^s, \)

\[
\| \nabla (P_3^{(1)} \circ \Psi_C)(z) \|_s \lesssim_s \| z_{\perp} \|_s \| z_{\parallel} \|_0, \quad \| d\nabla (P_3^{(1)} \circ \Psi_C)(z)[\hat{z}] \|_s \lesssim_s \| z_{\perp} \|_s \| \hat{z} \|_0 + \| z_{\parallel} \|_0 \| \hat{z} \|_s,
\]

and for any \( k \in \mathbb{Z}_{\geq 2}, \hat{z}_1, \ldots, \hat{z}_k \in h_r^s, \)

\[
\| d^k \nabla (P_3^{(1)} \circ \Psi_C)(z)[\hat{z}_1, \ldots, \hat{z}_k] \|_s \lesssim_{s,k} \sum_{j=1}^k \| \hat{z}_j \|_s \prod_{i \neq j} \| \hat{z}_i \|_0 + \| z_{\perp} \|_s \prod_{j=1}^k \| \hat{z}_j \|_0.
\]

Proof. The lemma follows by differentiating the Hamiltonian \( P_3^{(1)} \circ \Psi_C \) and applying Corollary 4.1 and Lemma 5.3(ii).

By (5.41), (5.46), (5.49), (5.67) one gets that the Hamiltonian \( H^{(2)} := H^{(1)} \circ \Psi_C = H^{\text{nls}} \circ \Psi_{C} \circ \Psi_C \) has the form

\[
H^{(2)}(z) = H^{\text{nls}}(I_S, 0) + \frac{1}{2} (\Omega^{\text{nls}}(I_S, 0)[z_{\perp}, z_{\perp}]_r + P_2(z) + P_3(z), \quad (5.68)
\]

where for any \( z \in V_h^s \cap h_r^0, \)

\[
P_2(z) := \nabla_S h^{\text{nls}}(z_S) \cdot \pi_S B_2^C(z) + P_2^{(1)}(z), \quad (5.69)
\]

\[
P_3(z) := P_3^{(2a)}(z) + P_3^{(2b)}(z) + P_3^{(2c)}(z) + P_3^{(1)}(\Psi_C(z)). \quad (5.70)
\]

Note that \( P_2 \) is quadratic with respect to \( z_{\perp} \), whereas \( P_3 \) is a remainder term of order three in \( z_{\perp} \). Being quadratic with respect to \( z_{\perp} \), \( P_2 \) can be written as

\[
P_2(z) = \frac{1}{2} (d_{\perp}(\nabla_{\perp} P_2(\Pi_S z))[z_{\perp}, z_{\perp}]_r.
\]

We prove the following

Lemma 5.9. The Hamiltonian \( P_2 \) vanishes on \( V_h^s \cap h_r^0. \) 

\]
Proof. By Corollary 4.1, \( \Psi_C(\Pi S Z) = \Pi S Z \) and \( d\Psi_C(\Pi S Z) = 1d \). Hence by the chain rule and formula (3.5), the map \( \Psi = \Psi_L \circ \Psi_C \) satisfies

\[
d\Psi(\Pi S Z) = d\Psi_L(\Pi S Z) = d\Psi_{nls}(\Pi S Z).
\] (5.71)

Recall that we denoted by \( \hat{\nu}(t) \) the solution of equation (5.20), obtained by linearizing the dNLS equation along \( \nu(t) = \Psi_{nls}(\Pi S Z(t)) \) with initial data \( \hat{\nu}(0) = d\Psi_{nls}(\Pi S Z(t))(0, \hat{z}_{\perp}^{(0)}) \) and by \( \hat{z}(t) = (0, \hat{z}_{\perp}(t)) \) the one of the equations obtained by linearizing the dNLS equation, expressed in Birkhoff coordinates (cf. (5.14)), along \( (z_{S}(t), 0) = \Pi S Z(t) \) with initial data \( (0, \hat{z}_{\perp}^{(0)}) \). Since \( \Psi_{nls} \) is symplectic, \( \hat{\nu}(t) = d\Psi_{nls}(\Pi S Z(t))[\hat{z}(t)] \). We remark that \( (z_{S}(t), 0) = \Pi S Z(t) \) is also a solution of the Hamiltonian equation \( \partial_t Z^{(2)} = J\nabla H^{(2)}(Z^{(2)}) \) with \( H^{(2)} \) given by (5.68). Denote by \( \hat{z}_{\perp}(t) = (0, \hat{z}_{\perp}^{(2)}(t)) \) the solution of the equation obtained by linearizing \( \partial_t Z^{(2)} = J\nabla H^{(2)}(Z^{(2)}) \) along \( \Pi S Z(t) \) with the same initial data \( (0, \hat{z}_{\perp}^{(0)}) \) as above. Since \( \Psi \) is symplectic, \( \hat{\nu}(t) = d\Psi(\Pi S Z(t))[\hat{z}(t)] \), implying together with \( d\Psi(\Pi S Z) = d\Psi_{nls}(\Pi S Z) \) (cf. (5.71) above) that \( \hat{z}_{\perp}^{(2)}(t) = \hat{z}(t) \) for any \( t \). By (5.18), \( \hat{z}_{\perp}(t) \) satisfies

\[
\partial_{t} \hat{z}_{\perp}(t) = J\Omega_{nls}^{\perp}(I_{S}, 0)[\hat{z}_{\perp}(t)]
\] (5.72)

whereas by (5.68), one has

\[
\partial_{t} \hat{z}_{\perp}^{(2)}(t) = Jd_{\perp} \nabla H^{(2)}(\Pi S Z(t))[z_{\perp}^{(2)}(t)] = J\Omega_{nls}^{\perp}(I_{S}, 0)[\hat{z}_{\perp}^{(2)}(t)] + Jd_{\perp} \nabla_{\perp} \hat{P}_{2}(\Pi S Z(t))[\hat{z}_{\perp}(t)].
\] (5.73)

In particular, it follows that \( d_{\perp} \nabla_{\perp} \hat{P}_{2}(\Pi S Z(0))[\hat{z}_{\perp}^{(0)}] = 0 \). Since \( \hat{P}_{2}(z) \) is quadratic in \( z_{\perp} \) and the initial data \( z_{S}(0) \in \pi_{S}(V_{\delta}^{\perp} \cap h_{r}^{0}) \), \( \hat{z}_{\perp}^{(0)} \in h_{r,c}^{0} \) are arbitrary, it follows that \( \hat{P}_{2}(z) = 0 \) for any \( z \in V_{\delta}^{\perp} \cap h_{r}^{0} \), which proves the claimed statement.

As a consequence of Lemma 5.9, formula (5.68) becomes

\[
H^{(2)}(z) = H_{nls}^{\perp}(I_{S}, 0) + \frac{1}{2} \big( \Omega_{nls}^{\perp}(I_{S}, 0)[z_{\perp}], z_{\perp} \big) + \hat{P}_{2}(z).
\] (5.74)

The Hamiltonian \( \hat{P}_{3} \), introduced in (5.70), satisfies the following tame estimates.

**Lemma 5.10 (Tame estimates of \( \hat{P}_{3} \)).** For any \( s \in \mathbb{Z}_{\geq 0} \), the Hamiltonian \( \hat{P}_{3} : V_{\delta}^{\perp} \cap h_{r}^{0} \to \mathbb{R} \) is real analytic and satisfies the following tame estimates: for any \( z \in V_{\delta}^{\perp} \cap h_{r}^{0} \), \( \hat{z} \in h_{r,c}^{0} \),

\[
\| \nabla \hat{P}_{3}(z) \|_{s} \lesssim_{s} \| z_{\perp} \|_{s} \| z_{\perp} \|_{0}, \quad \| d\nabla \hat{P}_{3}(z)[\hat{z}] \|_{s} \lesssim_{s} \| z_{\perp} \|_{s} \| \hat{z} \|_{0} + \| z_{\perp} \|_{0} \| \hat{z} \|_{s}
\]
and for any $k \in \mathbb{Z}_{\geq 2}$, $\hat{z}_1, \ldots, \hat{z}_k \in h^*_s$, 
\[
\|d^k \nabla P_3(z) [\hat{z}_1, \ldots, \hat{z}_k]\|_s \lesssim_s \sum_{j=1}^{k} \|\hat{z}_j\|_s \prod_{i \neq j} \|\hat{z}_i\|_0 + \|z_{\perp}\|_s \prod_{j=1}^{k} \|\hat{z}_j\|_0.
\]
\[\square\]

**Proof.** The claimed statements follow from Lemmata 5.4 and 5.6–5.8. \[\blacksquare\]

5.3 Summary of the proof of Theorem 1.2

Theorem 1.2 is a direct consequence of Propositions 5.1, 5.2, formula (5.74), and Lemma 5.10. The neighbourhood $V$ in the statement of the theorem is given by $V'_\delta$, introduced in (4.16).

6 Proof of Theorem 1.3

Within this proof, it is convenient to use complex Birkhoff coordinates, given by $\zeta_n := (x_n - i y_n)/\sqrt{2}$, $n \in \mathbb{Z}$. A solution $z(t) = (x(t), y(t))$ of the dNLS equation in Birkhoff coordinates then satisfies the equations

\[
\partial_t \zeta_n = -i \omega_{nls}^{\psi_1} \zeta_n, \quad n \in \mathbb{Z},
\]

where

\[
\omega_{nls}^{\psi_1} \equiv \omega_{nls}^{\psi_1}(I_S, I_\perp) = \partial_{i \ell} H_{nls}^{\psi_1}(I_S, I_\perp).
\]

Linearize (6.1) at a solution $\zeta(t)$ of the form $(\zeta_S(t), 0)$. For initial data of the form $\hat{\zeta}(0) = (0, \hat{\zeta}_\perp(0))$, the corresponding solution $\hat{\zeta}(t) = (\hat{\zeta}_S(t), \hat{\zeta}_\perp(t))$ of the linearized equation satisfies

\[
\hat{\zeta}_S(t) \equiv 0, \quad \partial_t \hat{\zeta}_n(t) = -i \omega_{nls}^{\psi_1}(I_S, 0) \hat{\zeta}_n(t), \quad n \in S_\perp.
\]

The latter equation is reduced to constant coefficients and hence

\[
\hat{\zeta}_\perp(t) = (e^{-i \omega_{nls}^{\psi_1}(I_S, 0) t} \hat{\zeta}_n(0))_{n \in S_\perp}.
\]

Since $\Psi_{nls}$ is symplectic, the solution of the equation, obtained by linearizing the dNLS equation along $\Psi_{nls}(z_S(t), 0)$, with initial data $d\Psi_{nls}^{\psi_1}(0, \hat{\zeta}_\perp(0))$, is given by

\[
\hat{\omega}(t) = d\Psi_{nls}^{\psi_1}(z_S(t), 0)[0, \hat{\zeta}_\perp(t)].
\]
We now consider the special solutions
\[ \hat{\xi}^{\pm j}(t) = e^{\pm i\omega j(t)} \hat{\xi}^{\pm j}(0), \quad j \in S_\perp, \]
corresponding to the initial data
\[ \hat{\xi}^{\pm j}(0) = (e^{(1,j)} \pm i e^{(2,j)})/\sqrt{2}, \quad e^{(1,j)} = ((\delta_{nj})_{n \in \mathbb{Z}}, 0), \quad e^{(2,j)} = (0, (\delta_{nj})_{n \in \mathbb{Z}}). \]

These solutions are periodic in time and that \( d\Psi^{\text{nls}}(z(t))[\hat{\xi}^{\pm j}(t)] \) can be written as
\[ \hat{\nu}^{\pm j}(t) = e^{\pm i\omega j(t)} d\Psi^{\text{nls}}(z(t), 0)[\hat{\xi}^{\pm j}(0)]. \]

In the terminology of [16], \( \hat{\nu}^{+ j}(t) \), \( \hat{\nu}^{- j}(t) \), \( j \in S_\perp \), are Floquet solutions with Floquet exponents \( \pm \omega_j(I_S, 0) \). By Theorem 1.1 one then concludes that up to normalizations (cf. Appendix 2) and natural identifications (such as the identifications of action angle with Birkhoff coordinates), the map \( \Phi_1 \), obtained by applying the scheme of construction of [16] to the dNLS equation, coincides with the map
\[ \mathbb{R}^S \times \mathbb{R}^S \to \mathcal{L}(H^s_{\perp r}, H^s_{r}), \quad z_S \mapsto d\Psi^{\text{nls}}(z_S, 0) \big|_{H^s_{\perp r}}. \]

Since according to [16], the map \( \Phi(z) \) can be chosen of the form \( (z_S, 0) + \Phi_1(z) \) and since the symplectic corrector \( \Psi_C \) is constructed following the scheme in [16], one concludes that again up to normalizations and natural identifications, \( \Psi = \Psi_L \circ \Psi_C \) coincides with the map \( \Phi \circ \phi \) obtained by applying the scheme of [16] to the dNLS equation.

**Remark 6.1.** In the terminology of [16], the system of the Floquet exponents \( \pm \omega_j(I_S, 0) \), \( j \in S_\perp \), is nonresonant—see for example [2] where the relevant properties of the dNLS frequencies are discussed.

**7 Restrictions of \( \Psi \)**

In this section we present results concerning potentials \( \varphi \in H^s_r \) which are even, odd, or real valued. To describe them, introduce the operator \( T : H^0(\mathbb{T}, \mathbb{C}) \to H^0(\mathbb{T}, \mathbb{C}) \) where for any \( u \in H^0(\mathbb{T}, \mathbb{C}) \), \( T(u) \) is given by
\[ T(u)(x) := u(-x), \quad x \in \mathbb{R} \text{ a.e.} \]

(Here and in the sequel we identify an element in \( H^0(\mathbb{T}, \mathbb{C}) \) with a representative \( f : \mathbb{R} \to \mathbb{C} \) of its lift, obtained by extending \( f : [0, 1) \to \mathbb{C} \) periodically in \( x \) to \( \mathbb{R} \), \( f(x+n) = f(x), n \in \mathbb{Z} \). If this element is in \( H^s(\mathbb{T}, \mathbb{C}) \) with \( s \in \mathbb{Z}_{\geq 1} \), then \( f \) will be chosen to be the representative
of period 1 which is in \( C^{s-1}(\mathbb{R}, \mathbb{C}) \). Let \( T_1, T_2, T_3 : H^s_r(\mathbb{T}, \mathbb{C}) \to H^s_r(\mathbb{T}, \mathbb{C}), s \in \mathbb{Z}_{\geq 0} \), denote the involutions,

\[
T_1(u, v) := (T(u), T(v)), \quad T_2(u, v) := -(T(u), T(v)), \quad T_3(u, v) := (v, u)
\]

and \( H^s_{r,j} \) the following subspaces of \( H^s_r \),

\[
H^s_{r,1} := \{(u, \bar{u}) \in H^s_r : T(u) = u, \}
\]

\[
H^s_{r,2} := \{(u, \bar{u}) \in H^s_r : T(u) = -u, \}
\]

and

\[
H^s_{r,3} := \{(u, \bar{u}) \in H^s_r : u \text{ real valued}. \}
\]

For any \( 1 \leq j \leq 3 \) and \( s \in \mathbb{Z}_{\geq 0} \), \( T_j(u, v) = (u, v) \) on \( H^s_{r,j} \). It is straightforward to verify that \( \mathcal{H}^{\text{nls}} \) is left invariant by \( T_j \),

\[
\mathcal{H}^{\text{nls}}(T_j u) = \mathcal{H}^{\text{nls}}(u) \quad \forall u \in H^1_r, \; 1 \leq j \leq 3,
\]

(7.1)

\( T_1, T_2 \) are canonical, and hence the subspaces \( H^s_{r,1}, H^s_{r,2} \) are symplectic. In contrast, \( T_3 \) is not canonical and the subspace \( H^s_{r,3} \) Lagrangian. To describe how the involutions \( T_j \) act on Birkhoff coordinates, we define the operator \( \bar{T} : h^0_r \to h^0_r \), defined for \( x = (x_k)_{k \in \mathbb{Z}} \in h^0_r \) by \( (\bar{T}x)_k := x_{-k}, k \in \mathbb{Z}, \) and introduce the involutions \( \bar{T}_j \) on \( h^s_r, s \in \mathbb{Z}_{\geq 0}, \) given by

\[
\bar{T}_1(x, y) := (\bar{T}(x), \bar{T}(y)), \quad \bar{T}_2(x, y) := -(\bar{T}(x), \bar{T}(y)), \quad \bar{T}_3(x, y) := (\bar{T}(x), -\bar{T}(y))
\]

as well as the subspaces \( h^s_{r,j} \) of \( h^s_r \), defined by

\[
h^s_{r,j} := \{(x, y) \in h^s_r : \bar{T}_j(x, y) = (x, y)\}.
\]

For any \( 1 \leq j \leq 3 \) and \( s \in \mathbb{Z}_{\geq 0} \), it follows from Theorem 1.2 in [8] that \( \Phi^{\text{nls}} \circ T_j = \bar{T}_j \circ \Phi^{\text{nls}} \) on \( H^s_r \) implying that on \( h^s_r \),

\[
\Psi^{\text{nls}} \circ \bar{T}_j = T_j \circ \Psi^{\text{nls}}.
\]

(7.2)

Since elements in the subspaces \( H^s_{r,j} \) and \( h^s_{r,j} \) are kept fixed by the corresponding involutions introduced above, one then concludes from Theorem 1.1 that for any such \( j \) and \( s \), \( \Psi^{\text{nls}} : h^s_{r,j} \to H^s_{r,j} \) is a real analytic diffeomorphism. Furthermore, by (7.1) and (7.2), and
the fact that $I(\tilde{T}_j(z)) = \tilde{T}(I(z))$ for any $1 \leq j \leq 3$, it follows that $H^\text{nls}(\tilde{T}(I)) = H^\text{nls}(I)$ for any $I \equiv I(z) = ((x_k^2 + y_k^2)/2)_{k \in \mathbb{Z}}$ with $z = ((x_k)_{k \in \mathbb{Z}}, (y_k)_{k \in \mathbb{Z}}) \in h_1^r$. Thus for any $k \in \mathbb{Z}$,

$$
\omega_k^\text{nls}(\tilde{T}(I)) = (\partial_k H^\text{nls})(\tilde{T}(I)) = \partial_{l-k}(H^\text{nls}(\tilde{T}(I))) = \partial_{l-k}(H^\text{nls}(I)) = \omega_{-k}(I). \quad (7.3)
$$

In particular, $h_{r,1}^s$ and $h_{r,2}^s$ are left invariant by the dNLS flow (in Birkhoff coordinates). We remark that $h_{r,3}^s$ is left invariant by the flow of the defocusing mKdV equation (cf. e.g., [13]). Furthermore note that for any $u \in H^0(\mathbb{T}, \mathbb{C})$, the Fourier coefficients $(T(u))_n$, $n \in \mathbb{Z}$, of $T(u)$ satisfy

$$(T(u))_n = u_{-n} = (\tilde{T}((u_k)_{k \in \mathbb{Z}}))_n.$$  

The following proposition will be applied in subsequent work:

**Proposition 7.1.** In addition to the setup of Theorem 1.2, assume that $S \subset \mathbb{Z}$ is symmetric, $S = -S$, and $1 \leq j \leq 3$ and that the complex neighbourhood $\mathcal{V} \subset h_0^c$ of Theorem 1.2 is invariant with respect to $\tilde{T}_j$, $\tilde{T}_j(\mathcal{V}) = \mathcal{V}$. Then for any $s \in \mathbb{Z}_{\geq 0}$,

$$\Psi \circ \tilde{T}_j = T_j \circ \Psi \quad \text{on} \quad \mathcal{V} \cap h_s^r.$$  

As a consequence $\Psi : \mathcal{V} \cap h_{r,j}^s \to H_{r,j}^s$ is a real analytic diffeomorphism on to its image. Furthermore, on $\mathcal{V} \cap h_1^r$, the Hamiltonian $\mathcal{H} = H^\text{nls} \circ \Psi$ is invariant under $\tilde{T}_j$, $\mathcal{H} \circ \tilde{T}_j = \mathcal{H}$, and in the expansion (1.6),

$$\mathcal{H}(z) = H^\text{nls}(I_S, 0) + \sum_{n \in S^+} \omega_n^\text{nls}(I_S, 0)I_n(z) + \mathcal{P}_3(z),$$

the three terms on the right-hand side, when restricted to $\mathcal{V} \cap h_1^r$, are in view of (7.3) invariant under $\tilde{T}_j$. Since in the case $1 \leq j \leq 2$, $\tilde{T}_j$ is canonical, it then follows that the Hamiltonian vector fields $X_\mathcal{H}$ and $X_{\mathcal{P}_3}$ (cf. (2.6)) satisfy on $\mathcal{V} \cap h_1^r$ and for $1 \leq j \leq 2$

$$X_\mathcal{H}(\tilde{T}_j(z)) = \tilde{T}_jX_\mathcal{H}(z), \quad X_{\mathcal{P}_3}(\tilde{T}_j(z)) = \tilde{T}_jX_{\mathcal{P}_3}(z). \quad (7.4)$$

It implies that for $s \in \mathbb{Z}_{\geq 1}$,

$$X_\mathcal{H} : \mathcal{V} \cap h_{r,j}^{s+2} \to h_{r,j}^s, \quad X_{\mathcal{P}_3} : \mathcal{V} \cap h_{r,j}^s \to h_{r,j}^s. \quad (7.5)$$

**Proof.** Since the proofs for $j = 1, 2,$ and $3$ are similar, we concentrate on the case $j = 1$ only. Recall that $\Psi = \Psi_L \circ \Psi_C$ where the maps $\Psi_L$ and $\Psi_C$ were introduced in Sections
Theorem 3 and 4, respectively. In the latter section, the neighbourhood \( V \) in the statement of Theorem 1.2 is actually denoted by \( V'_v \) (cf. Section 5.3). By (4.16), \( V'_v \) is contained in the neighbourhood \( V \), which was introduced in Section 3. In the course of this proof we use the notation established in these two sections.

By (1.8), \( \Psi_L(z_S, z_\perp) = \Psi_{nls}(z_S, 0) + d\Psi_{nls}(z_S, 0)[0, z_\perp] \) for any \( z = (z_S, z_\perp) \) in \( V \). Since \( S = -S \), \( T_1(V) = V \), (7.2) applies to \( \Psi_{nls}(z_S, 0) \) and \( d\Psi_{nls}(z_S, 0)[0, z_\perp] \) and hence

\[
T_1 \circ \Psi_L = \Psi_L \circ \tilde{T}_1. \tag{7.6}
\]

Next we show that \( \tilde{T}_1 \circ \Psi_c = \Psi_c \circ \tilde{T}_1 \). Recall that \( \Psi_c \), defined on \( V'_v \cap h^0_\nu \) by (4.28), is given by the time-one flow of the non-autonomous vector field \( X(z, \tau) = -L_\tau(z)^{-1}E(z) \), introduced in (4.12). Here \( 0 \leq \tau \leq 1 \) and for any \( z \in V \cap h^0_\nu \), the operators \( L_\tau(z) = J^{-1} + \tau L(z) : C^\infty \to C^\infty \) and \( L(z) : C^\infty \to C^\infty \) are defined in (4.5), respectively (3.39), and the element \( E(z) = (E_S(z), 0) \in C^\infty \) in (4.9) and (4.10). In a first step we prove that

\[
L(\tilde{T}_1 z) \circ \tilde{T}_1 = \tilde{T}_1 \circ L(z) \quad \forall z \in V \cap h^0_\nu. \tag{7.7}
\]

Recall from (3.40) that \( L(z) \) is of the form

\[
L(z) = \begin{pmatrix}
L^S_\perp(z) & L^\perp_\parallel(z) \\
L^\parallel_\perp(z) & 0
\end{pmatrix},
\]

where the operators \( L^S_\perp(z) \), \( L^\perp_\parallel(z) \), and \( L^\parallel_\perp(z) \) are defined in (3.41), (3.42), and (3.43), respectively. It is to show that for any \( z = (z_S, z_\perp) \in V \cap h^0_\nu \) and \( \tilde{z} = (\tilde{z}_S, \tilde{z}_\perp) \) in \( h^0_\nu \), \( L^S_\perp(z)[\tilde{z}_S], L^\perp_\parallel(z)[\tilde{z}_\perp], \) and \( L^\parallel_\perp(z)[\tilde{z}_\perp] \) satisfy the symmetry conditions required for \( L(\tilde{T}_1 z)[\tilde{T}_1 \tilde{z}] = \tilde{T}_1(L(z)[\tilde{z}] \) to hold. Since the arguments for each of the vectors \( L^S_\perp(z)[\tilde{z}_S], L^\perp_\parallel(z)[\tilde{z}_\perp], \) and \( L^\parallel_\perp(z)[\tilde{z}_\perp] \) are similar, we consider only \( L^S_\perp(z)[\tilde{z}_S] \). By (3.43), it is given by

\[
L^S_\perp(z)[\tilde{z}_S] = i \left( \left( \left\langle j \right. d_S(d_\perp \Psi_{nls}(\Pi_S z)[z_\perp]) \right. \right. \left. \left| \tilde{z}_S \right. \left. \right. \left. \right. \left. \right. \right) \frac{\partial_j \Psi_{nls}(\Pi_S z)}{j \in S^\perp} \right). \]

The two components of \( L^S_\perp(z)[\tilde{z}_S] \) can be analysed in the same way, so it suffices to look at the first one. Write \( z = (x, y) \). Further introduce \( G^{(1)}(x, y)[\tilde{z}_S] := j d_S(d_\perp \Psi_{nls}(\Pi_S z)[z_\perp]) \tilde{z}_S \) and \( G^{(2)}(x, y) := \Psi_{nls}(\Pi_S z) \), which are both elements in \( H^0_\nu \), and define for any \( j \in S^\perp \)

\[
a_j := (G^{(1)}(x, y), \partial_j G^{(2)}(x, y)), \quad b_j := (G^{(1)}(\tilde{T}_1(x, y))[\tilde{T}_1 \tilde{z}_S], (\partial_j G^{(2)}) (\tilde{T}_1(x, y))).
\]
We show that \( b_j = a_{-j} \) for any \( j \in S^\perp \). Indeed, by [8] one has

\[
G^{(1)}(\vec{T}_1(x,y))[\vec{T}_1\vec{z}_S] = T_1(G^{(1)}(x,y))[\vec{z}_S], \quad G^{(2)}(\vec{T}_1(x,y)) = T_1(G^{(2)}(x,y)). \tag{7.8}
\]

Apply \( \partial_{x_j} \) to both sides of the latter identity to get

\[
(\partial_{x_j} G^{(2)})(\vec{T}_1(x,y)) = \partial_{x_j} (G^{(2)}(\vec{T}_1(x,y))) = \partial_{x_j} T_1(G^{(2)}(x,y)) = T_1(\partial_{x_j} G^{(2)}(x,y)).
\]

When combined with (7.8), one then concludes that for any \( j \in S^\perp \),

\[
b_j = \langle T_1(G^{(1)}(x,y))[\vec{z}_S], T_1(\partial_{x_j} G^{(2)}(x,y)) \rangle_r = \langle G^{(1)}(x,y)[\vec{z}_S], \partial_{x_j} G^{(2)}(x,y) \rangle_r = a_{-j}.
\]

We thus have verified (7.7). Since \( \mathcal{L}_r(z) = J^{-1} + \tau L(z), 0 \leq \tau \leq 1 \), with \( J^{-1} = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix} \), \( \mathcal{L}_r(z) \) is invertible (cf. Lemma 4.1), \( \tilde{T}_1^{-1} = \tilde{T}_1 \), and \( \tilde{T}_1(\mathcal{V} \cap h^0_r) = \mathcal{V} \cap h^0_r \) (by assumption), one then also has \( \mathcal{L}_r^{-1}(\tilde{T}_1 z) \circ \tilde{T}_1 = \tilde{T}_1 \circ \mathcal{L}_r^{-1}(z) \) for any \( z \in \mathcal{V} \cap h^0_r \). Furthermore, \( E(z) = (\frac{1}{2} L^*_S(z)[z],0) \) (cf. (4.10)) satisfies \( E(\tilde{T}_1 z) = \tilde{T}_1 E(z) \). Altogether we conclude that the vectorfield \( X(z,\tau) = -\mathcal{L}_r(z)^{-1} E(z) \) (cf. (4.12)) has the property that for any \( z \in \mathcal{V} \cap h^0_r \) and \( 0 \leq \tau \leq 1 \), \( X(\tilde{T}_1 z, \tau) = \tilde{T}_1 X(z, \tau) \). Hence by Lemma 4.4, \( \Psi_c \), given by the time-one flow of the vector field \( X(z,\tau) \), satisfies \( \tilde{T}_1 \circ \Psi_c = \Psi_c \circ \tilde{T}_1 \) on \( \mathcal{V}' \cap h^0_r \). When combined with the identity (7.6), we therefore have proved that \( T_1 \circ \Psi = \Psi \circ \tilde{T}_1 \) on \( \mathcal{V}' \cap h^0_r \). Clearly, the corresponding identity also holds on \( \mathcal{V}' \cap h^0_s \) for any \( s \in \mathbb{Z}_{\geq 1} \). Concerning (7.4), note that in view of the definition (2.6) of a Hamiltonian vector field,

\[
X_{\tau_1}(\tilde{T}_j(z)) = J(\nabla H)(\tilde{T}_j(z)) = J\tilde{T}_j(\nabla H(\tilde{T}_j(z))) = J\tilde{T}_j(\nabla H(z)) = \tilde{T}_j J(\nabla H(z)) = \tilde{T}_j X_{\tau_1}(z).
\]

A similar computation shows that \( X_{\tau_3}(\tilde{T}_j(z)) = \tilde{T}_j X_{\tau_3}(z) \). The remaining statements of the proposition are an immediate consequence of the proved identities.

\[ \blacksquare \]

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**Appendix 1: A Version of the Poincaré Lemma**

We follow the general approach of [18, Chapter V], and restrict to the finite-dimensional setup as the extension to infinite dimension is straightforward by restriction, see
[16, Lemma 1.1]. Let $E = \mathbb{R}^n$ and denote by $L^r_a(E)$ the space of multilinear continuous alternating forms of degree $0 \leq r \leq n$. Let $U \subseteq E$ be an open nonempty set and consider

$$\omega : U \to L^r_a(E).$$

For any $z \in U$, denote by

$$\omega(z)[\xi_1, \ldots, \xi_r] \in \mathbb{R}$$

the value of $\omega(z)$ when evaluated at $\xi_1, \ldots, \xi_r \in E$. Similarly, if $\xi_j = \xi_j(z) \in E, j = 1, \ldots, r$, are vector fields on $U$, then we denote by $\omega(\xi_1, \ldots, \xi_r)$ the function

$$U \to \mathbb{R}, \ z \mapsto \omega(z)[\xi_1(z), \ldots, \xi_r(z)].$$

Furthermore, we denote by $\omega'(z) \cdot \xi, \xi \in E$, the alternating $r$-form

$$\partial_\varepsilon \big|_{\varepsilon = 0} \omega(z + \varepsilon \xi) \in L^r_a(E). \tag{A.1}$$

The exterior differential $d\omega$ of $\omega$, evaluated at $z \in U, \xi_1, \ldots, \xi_{r+1} \in E$, is then given by the formula

$$\sum_{j=1}^{r+1} (-1)^{j+1} \omega'(z) \cdot \xi_j[\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_{r+1}], \tag{A.2}$$

also referred to as Cartan’s formula. Let us now consider the case where

$$E = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \ n = n_1 + n_2, \ n_2 \geq 1,$$

$$U = U_1 \times U_2 \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

and $U_2$ is a ball in $\mathbb{R}^{n_2}$ centred at 0. We denote the elements of $U$ by $z = (x, y)$ and the ones of $E$ by $\xi = (v, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For any $r$-form $\omega$ on $U$, denote by $\omega_c$ the $(r-1)$-form on $U$, obtained by the cone construction: for any $x \in U_1, y \in U_2, v_1, \ldots, v_{r-1} \in \mathbb{R}^{n_1}$, and $w_1, \ldots, w_{r-1} \in \mathbb{R}^{n_2},$

$$\omega_c(x, y)[(v_1, w_1), \ldots, (v_{r-1}, w_{r-1})] = \int_0^1 \omega(x, ty)[(0, y), (v_1, tw_1), \ldots, (v_{r-1}, tw_{r-1})] \, dt. \tag{A.3}$$

Since $U_2$ is a ball in $\mathbb{R}^{n_2}$, centred at 0, for any $0 \leq t \leq 1, (x, ty)$ is in $U_1 \times U_2$ and hence $\omega(x, ty)$ in (A.3) is well defined.
Lemma A.1 (Poincaré lemma). Assume that \( \omega \) is a \( r \)-form on \( U = U_1 \times U_2 \), with \( 1 \leq r \leq n \) and \( n_2 \geq 1 \), satisfying

\[
\omega(x, 0)[(v_1, 0), \ldots, (v_r, 0)] = 0, \quad \forall \ x \in U_1, \quad \forall \ v_1, \ldots, v_r \in \mathbb{R}^{n_1}.
\] (A.4)

Then

\[
d(\omega_c) + (d\omega)_c = \omega.
\] (A.5)

In particular, if in addition \( \omega \) is closed, \( d\omega = 0 \), then \( d(\omega_c) = \omega \). \qed

Appendix 2: Formulas for \( d\Psi^{\text{dNLS}}(z_S, 0)[(0, z_\perp)] \)

Note that for \( z = (z_S, z_\perp) \) with \( z_S \in \mathbb{R}^S \times \mathbb{R}^S \) and \( z_\perp = ((x_j)_{j \in S_\perp}, (y_j)_{j \in S_\perp}) \in h_{l, r}^0 \),

\[
d\Psi^{\text{dNLS}}(z_S, 0)[(0, z_\perp)] = \sum_{j \in S_\perp} x_j d\Psi^{\text{dNLS}}(z_S, 0)[e^{(1,j)}] + \sum_{j \in S_\perp} y_j d\Psi^{\text{dNLS}}(z_S, 0)[e^{(2,j)}],
\]

where for any \( j \in S_\perp \),

\[
e^{(1,j)} = ((\delta_{nj})_{n \in \mathbb{Z}}, 0), \quad e^{(2,j)} = (0, (\delta_{nj})_{n \in \mathbb{Z}}).
\]

It turns out that for \( j \in S_\perp \), \( d\Psi^{\text{dNLS}}(z_S, 0)[e^{(1,j)}] \) and \( d\Psi^{\text{dNLS}}(z_S, 0)[e^{(2,j)}] \) can be computed quite explicitly. Consider the Hamiltonian equation with Hamiltonian given by the coordinate function \( x_j \), \( \partial_t w = i\partial_x x_j \), and denote by \( w(t) \) its solution with initial data \( w(0) = \Psi^{\text{dNLS}}(z_S, 0) \). Then \( z(t) := \Phi^{\text{dNLS}}(w(t)) \) solves

\[
\partial_t z = d\Phi^{\text{dNLS}}(w(t)) \partial_t w(t) = d\Phi^{\text{dNLS}}(w(t)) i\partial_x x_j.
\] (B.1)

Since by Theorem 1.1, \( \Phi^{\text{dNLS}} \) is symplectic, one has \( \partial_t z = J e^{(1,j)} = e^{(2,j)} \). When combined with (B.1) it implies that \( d\Psi^{\text{dNLS}}(z(t))[e^{(2,j)}] = i\partial_x x_j \). Similarly, one derives the corresponding identity for the coordinate function \( y_j \). When evaluated at \( t = 0 \) we then obtain

\[
d\Psi^{\text{dNLS}}(z_S, 0)[e^{(2,j)}] = i\partial_x x_j = (-i\partial_y x_j, i\partial_x x_j), \quad d\Psi^{\text{dNLS}}(z_S, 0)[e^{(1,j)}] = i\partial_y y_j = (-i\partial_y y_j, i\partial_x y_j).
\]

By the definition of \( x_j, y_j \) in [10, p. 113], one has for a potential \( w \in H^0_l \) with Birkhoff coordinates \( (z_S, 0) \) (referred to as \( S \)-gap potential)

\[
x_j = \frac{\xi_j}{\sqrt{\delta}}(e^{i\delta_j \delta_j} + e^{-i\delta_j \delta_j}), \quad y_j = \frac{\xi_j}{\sqrt{\delta} i}(e^{i\delta_j \delta_j} - e^{-i\delta_j \delta_j}),
\]
where \( z_j^+ = \gamma_j e^{+i\eta_j} \) if \( \gamma_j \neq 0 \) and \( z_j^+ = 0 \) otherwise. We refer to [10] for the definitions of \( \xi_j, \eta_j, \) and \( \beta_j \). Since \( w \) is assumed to be a S–gap potential, it follows that for any \( j \in S^\perp, \)

\[
\partial x_j = \frac{\xi_j}{\sqrt{8}} (e^{i\beta_j} \partial z_j^+ + e^{-i\beta_j} \partial z_j^-), \quad \partial y_j = \frac{\xi_j}{\sqrt{8i}} (e^{i\beta_j} \partial z_j^+ - e^{-i\beta_j} \partial z_j^-),
\]

where by formula (17.3) in [10],

\[
\partial z_j^\pm = 2(\partial \tau_j - \partial \mu_j) \pm \left( i2\delta(\mu_j)\partial \phi_j + 2\phi_j (i\partial \delta |^{\lambda=\mu_j} + i\delta(\mu_j)\partial \mu_j) \right).
\]

We refer to [10] for the definitions of the various quantities as well as for formulas of the gradients in the latter expression. Each of the two components of these gradients are shown to be a linear combination of quadratic expressions in the entries of the fundamental solution \( M = M(x, \lambda) \) of the Zakharov Shabat operator

\[
L := \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}, \quad w = (u, v) = (u, \bar{u}).
\]

In fact, in [11], it has been proved that

\[
\partial z_j^\pm = (K_j^2 \pm iH_j^2, K_j^1 \pm iH_j^1)^2
\]

where

\[
H_j = (H_{j1}, H_{j2}) = \frac{1}{\|M_1 + M_2\|_{L^2}} (M_1 + M_2) |^{\lambda=\mu_j}
\]

denotes the \( L^2 \)-normalized eigenfunction of \( L \) for the Dirichlet eigenvalue \( \mu_j \), \( M_1, M_2 \) are the two columns of \( M \), and \( K_j = (K_{j1}, K_{j2}) \) is the \( L^2 \)-normalized solution of \( LF = \mu_j F \), which is \( L^2 \)-orthogonal to \( H_j \) and satisfies the additional normalization condition

\[
-\imath (K_{j1}(0) - K_{j2}(0)) > 0.
\]

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