The canonical structure of the first-order Einstein–Hilbert action with a flat background

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Abstract
It has been shown that the canonical structure of the first-order Einstein–Hilbert (1EH) action involves three generations of constraints and that these can be used to find the generator of a gauge transformation which leaves the action invariant; this transformation is a diffeomorphism with field-dependent gauge function while on shell. In this paper we examine the relationship between the canonical structure of this action and that of the first-order spin-2 (1S2) action, which is the weak field limit of the Einstein–Hilbert action. We find that the weak field limit of the Possion brackets algebra of first class constraints associated with the 1EH action is not that of the 1S2 action.

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1. Introduction

An analysis of the first-order Einstein–Hilbert (1EH) action \cite{1–3} using the Dirac constraint formalism \cite{4, 5} reveals the presence of primary, secondary and tertiary constraints. (In an earlier constraint analysis \cite{6, 7} of this action, no tertiary constraints arose as equations of motion that are in fact secondary first class constraints were used to eliminate fields from this action.)

The presence of first class tertiary constraints is required to derive a generator of the diffeomorphism transformation. In addition, second class secondary constraints \cite{2} yield unusual ghost contributions to the measure of the path integral used to quantize the 1EH action.

These peculiar features motivate us to examine more clearly the relationship between the 1EH action and the action derived by making a weak field expansion of the metric in the 1EH action about a flat background—the well-known first-order spin-2 (1S2) action. We find that
there is not an obvious connection between the tertiary constraints in these two actions and that the Poisson brackets algebra of the constraints in the 1EH action does not, in the weak field limit, reduce to the algebra arising in the 1S2 action.

2. The constraint structure

The 1EH Lagrangian can be written in the form,

\[ L_{\text{EH}} = h^{\mu \nu} \left( G_{\mu \nu, \lambda}^{\text{EH}} + \frac{1}{d-1} G_{\sigma \mu}^{\text{EH}} G_{\sigma \nu}^{\text{EH}} - G_{\sigma \mu}^{\text{EH}} G_{\lambda \nu}^{\text{EH}} \right) \]  

(1)

in \( d \) dimensions, where \( h^{\mu \nu} \) and \( G_{\mu \nu}^{\text{EH}} \) are related to the metric \( g_{\mu \nu} \) and the affine connection \( \Gamma_{\mu \nu}^{\lambda} \) by

\[ h_{\mu \nu} = \sqrt{-g} g_{\mu \nu} \quad \text{and} \quad G_{\mu \nu}^{\text{EH}} = \Gamma_{\mu \nu}^{\lambda} - \frac{1}{2} \left( \delta_{\mu}^{\lambda} \Gamma_{\sigma \nu}^{\sigma} + \delta_{\nu}^{\lambda} \Gamma_{\sigma \mu}^{\sigma} - \delta_{\mu \nu}^{\lambda} \right). \]

By making a weak field expansion

\[ h^{\mu \nu}(x) = \eta^{\mu \nu} + f^{\mu \nu}(x) \]

(2)

(\( \eta^{\mu \nu} = \text{diag}(-++,+,+) \)) and keeping only those terms in \( L_{\text{EH}} \) that are bilinear in the fields, one arrives at the first-order Lagrangian for a free spin-2 field,

\[ L_{\text{S2}} = f^{\mu \nu} F_{\mu \nu, \lambda}^{\text{S2}} + \eta^{\mu \nu} \left( \frac{1}{d-1} F_{\sigma \mu}^{\text{S2}} F_{\sigma \nu}^{\text{S2}} - F_{\sigma \mu}^{\text{S2}} F_{\sigma \nu}^{\text{S2}} \right) \]

(3)

The canonical analysis of both \( L_{\text{EH}} \) and \( L_{\text{S2}} \) appears in [1] (see also [3] for \( L_{\text{EH}} \)). This is most easily done for \( L_{\text{EH}} \) if one makes the following change of variables

\[ h = h^{00}, \quad h^i = h^{0i}, \quad H^{ij} = hh^{ij} - h^i h^j \quad (H^{ij} H_{jk} = \delta^{ij}_k) \]

(4-6)

\[ G_{00}^0 = - \left[ \Pi + \frac{\Pi^{ij}}{h} (H^{ij} + h^i h^j) \right] \]

(7)

\[ G_{0i}^0 = - \frac{1}{2} [\Pi_i - 2 \Pi_{ij} h^j] \]

(8)

\[ G_{ij}^0 = -h \Pi_{ij} \]

(9)

\[ G_{jk}^i = -\xi^i_{jk} \]

(10)

\[ G_{ij}^0 = - \frac{1}{2} \left[ \zeta^i_{jk} - \frac{2}{h} \xi^i_{jk} h^k + \bar{h} \delta^i_{jk} \right] \]

(11)

\[ G_{00}^0 = - \left[ \xi^0 + \frac{1}{h} \xi^0_{jk} h_{jk} \right] \]

(12)

One then finds primary constraints that satisfy \( \Pi, \Pi_i \) and \( \Pi_{ij} \) as the canonical momenta conjugate to \( h, h^i \) and \( H^{ij} \) respectively. There are subsequently the secondary first class constraints

\[ \chi_i = h^i_{,i} - h \Pi_i \]

(13)

and

\[ \chi = h^i_{,i} + h \Pi \]

(14)

as well as the secondary second class constraints

\[ \bar{\zeta}_{ij} = \frac{2}{h} \left( \lambda^i_j - \frac{1}{d-1} \delta^i_j \lambda^k_k \right) \]

(15)

\[ \xi^i_{jk} = -\frac{1}{2} (M^{-1})^i_{jk} \epsilon_{mn} \sigma^m_{kn} \]

(16)
where

\[
\lambda_i^j = h_i^j - \frac{1}{2} h^i \Pi_j - H^{jk} \Pi_k
\]

(17)

\[
\sigma_i^j = \frac{1}{h} H_i^j - \frac{1}{h} H_i^k \Pi_k + \frac{1}{2(d-1)h} \left( \delta_i^j H^{kl} + \delta_i^k H^{lj} \right) (\Pi_k - 2h^m \Pi_{lm})
\]

\[
+ \frac{1}{h} (h^i H^{jp} + h^j H^{ip}) \Pi_{ij}
\]

(18)

and

\[
(M^{-1})_{ij} \ell_m = -\frac{h}{2} \left( (H_{ij} \delta_i^k \delta_m^k + H_{mj} \delta_i^k \delta_m^k + H_{ik} \delta_m^k \delta_m^k + H_{im} \delta_i^k \delta_m^k) + \frac{2}{d-2} (H^{ik} H_{lm} \ell_m) - H^{ik} (H_{ij} H_{mj} + H_{ik} H_{ij}) \right).
\]

(19)

Once the second class constraints of equations (15), (16) have been eliminated through introduction of the Dirac bracket (DB) the canonical Hamiltonian takes the form

\[
\mathcal{H}_c = \frac{1}{h} (\tau + h^i \tau_i) + F(\chi, \chi_i)
\]

(20)

where \( F \) is a function, all of whose terms are at least linear in \( \chi \) or \( \chi_i \), and

\[
\tau = H_i^j - \frac{1}{2} H_{im}^{ij} H_{mj} - \frac{1}{4} H^{ij} H_{mn} H_{ij}
\]

\[
- \frac{1}{4(d-1)} H^{ij} H_{kl} H_{ij} H_{kl} + H^{ij} H^{kl} (\Pi_{ik} \Pi_{jl} - \Pi_{ik} \Pi_{jl})
\]

(21)

and

\[
\tau_i = -2 (H^{mn} \Pi_m)_{,i} + H^{mn} \Pi_{mn,i} + (H^{mn} \Pi_{mn})_{,i}.
\]

(22)

Since we have the DB algebra

\[
[\chi_i, \chi_j] = \chi_i
\]

(23)

\[
[\tau_i, \chi](\chi) = \tau_i (\chi)
\]

\[
[\tau_i, \chi](\chi) = \left( -\tau_i (\chi) \delta_i^j + \tau_j (\chi) \delta_i^j \right) \delta^{d-1} (\chi - \chi)
\]

(24a)

\[
[\tau (\chi), \tau (\chi)] = \left( \delta_i^j H^{ij} (\chi) \tau_i (\chi) - \delta_i^j H^{ij} (\chi) \tau_i (\chi) \right) \delta^{d-1} (\chi - \chi)
\]

(24b)

and

\[
[\tau (\chi), \tau (\chi)] = \left( -\delta_i^j \tau (\chi) + \delta_i^j \tau (\chi) \right) \delta^{d-1} (\chi - \chi),
\]

(24c)

we see that there are no further constraints and that \((\chi, \chi_i)\) are secondary first class constraints and \((\tau, \tau_i)\) are tertiary first class constraints.

We will now contrast this constraint structure with that which follows from \( L_{S2} \) in equation (3). In [1] the canonical structure of this 1S2 action was performed, however the variables used there are distinct from those used to analyze \( L_{EH} \) (summarized above). In order to effect a comparison between the canonical structures of \( L_{EH} \) and \( L_{S2} \), we make use of the variables,

\[
F_0^i = \Pi_{ij}, \quad F_0^0 = -\frac{1}{2} \Pi_i, \quad F_0^0 = -(\Pi + \Pi_{jj})
\]

(25a–c)

\[
f = 1 + h \quad f^i = h^i \quad f^{ij} = h \delta^{ij} - H^{ij}
\]

(26a–c)

\[
F_j^i = -\xi_j^i, \quad F_0^0 = -\xi_0^0 + \xi_j^j, \quad F_0^i = \frac{1}{2} \left[ -\xi_j^j + \xi_0^i \right] \quad (\text{where } \xi_i^0 = 0).
\]

(27a–c)

The variables appearing on the right side of equations (25)–(27) are the weak field limit of those in equations (4)–(12) when using the weak limit of equation (2).
With the change of variable of equations (25)–(27) we find that \( L_{S2} \) becomes

\[
L_{S2} = \Pi h_{i0} + \Pi_i h_{i0} + \Pi_j H^j_{i0} + \frac{d-2}{d-1} \left[ \left( \Pi + \Pi_i \right)^2 - \frac{1}{4} \Pi_i \Pi_i \right] + \frac{1}{d-1} \left( h_{ij} - \Pi_j \right) + \frac{\Pi_j h_{ij}}{d-1} + \frac{1}{4} \Pi_j \xi^j_i
\]

where

\[
\xi^j_i = -\delta^j_i \Pi_i - H^j_{ik} + \frac{1}{2(d-1)} \left( \delta^i_k \Pi_k + \delta^i_\ell \Pi_\ell \right) + \frac{1}{d-1} \delta^i_k \xi^j_k - \delta^j_i \xi^i_k.
\]

(28)

The equations of motion for \( \bar{\xi}_j^i \) and \( \xi^j_i \) clearly constitute a set of secondary second class constraints. If these equations are used to eliminate \( \bar{\xi}_j^i \) and \( \xi^j_i \) from \( L_{S2} \), then one can immediately see that the canonical Hamiltonian is given by

\[
H_{S2} = \frac{d-2}{d-1}(\Pi + \Pi_i)^2 - t_\chi - \bar{\xi}_j^i \chi^j_i + \frac{d-2}{d-1} h_{ij} h_{ij} + 2 \Pi_i h_{ij}
\]

\[
- \frac{2}{d-1} H_{ij} \Pi_{ij} + \Pi_i \Pi_{ij} - \frac{1}{d-1} \Pi_i \Pi_{jj} + \frac{1}{4} H_{ij}^k H_{kj}^j
\]

\[
- \frac{1}{4(d-2)} H_{ij}^k H_{kj}^j - \frac{1}{2} H_{ij}^k H_{kj}^i - H_{ij}^i \Pi_i,
\]

(29)

where

\[
\chi = h_{ij} - \Pi_i \Pi_i + \Pi_i j_i.
\]

(30a, b)

We see that the momenta conjugate to \( t \) and \( \bar{\xi}_j^i \) vanish; these primary first class constraints lead to the secondary constraints \( \chi = \chi_j = 0 \). If in equation (29) we express \( \Pi_i \) and \( \Pi_i j_i \) in terms of \( \chi \) and \( \chi_i j_i \), then those terms in \( H_{S2} \) that are independent of \( \chi \) and \( \chi_i j_i \) are

\[
H_{S2}^{(ii)} = (\Pi_i j_i - \Pi_{ij} \Pi_{ij}) + 2 \left( \Pi_{ij} h_{ij} - \Pi_{ij} h_{ij} \right) + H_{ij}^i h_{ij}
\]

\[
- \frac{1}{2} \left( H_{ij}^k H_{kj}^i - H_{ij}^k H_{kj}^j - \frac{1}{2(d-1)} H_{ij}^k H_{kj}^i \right).
\]

(31)

We find that the secondary constraints \( \chi \) and \( \chi_i j_i \) now lead to the tertiary constraints

\[
\left\{ \chi, \int d^{d-1} H_{S2} \right\} = -H_{ij}^i = -\tau
\]

(32a)

and

\[
\left\{ \chi_i, \int d^{d-1} H_{S2} \right\} = 2 \left( \frac{d-2}{d-1} \right) \chi_i + 2 \left( \Pi_{ij} - \Pi_{ij} \right)
\]

\[
= 2 \left( \frac{d-2}{d-1} \right) \chi_i + \tau = T_i.
\]

(32b)

No further constraints arise as

\[
\left\{ \tau, \int d^{d-1} H_{S2} \right\} = T_{ij}
\]

(33a)

\[
\left\{ \chi_i, \int d^{d-1} H_{S2} \right\} = \tau_i + 2(\chi_{ij} - \chi_{ij}) \chi_{ij}.
\]

(33b)

Furthermore, \( \left( \chi, \chi_i, \tau, \tau_i \right) \) all have vanishing DB amongst themselves. This shows that these constraints are all first class.
Using a technique outlined in [8], (the ‘HTZ’ approach) it is possible to find the gauge invariances present in an action from the first class constraints that are present. For the 1EH action, the first class constraints of equations (13), (14), (21), (22) have been shown to lead [1, 2] to the diffeomorphism gauge transformation

\[ \delta h^{\mu \nu} = h^{\mu \nu} \partial_0 \theta^\rho + h^{\rho \lambda} \partial_\lambda \theta^\mu - \partial_\lambda (h^{\mu \nu} \theta^\lambda) \]  

(34a)

\[ \delta G^\mu_{\nu} = -\partial_\mu \theta^\nu + \frac{1}{2} \left( \delta^\mu_\nu \partial_\alpha + \delta^\mu_\rho \partial_\rho \right) \partial_\alpha \theta - \partial_\nu \theta - \partial_\nu G^\rho_{\mu \rho} + G^\rho_{\mu \rho} \partial_\rho \theta^\nu - \left( G^\rho_{\mu \rho} \partial_\nu + G^\rho_{\nu \rho} \partial_\nu \right) \theta^\rho, \]  

(34b)

only provided the gauge parameter \( \theta^\mu \) takes the field-dependent form \( \theta = -hc, \theta^i = c^i - ch^i \) where \( (c, c^i) \) are arbitrary functions of \( x^\mu \), and the equation of motion for \( H^{ij} \) is satisfied.

For the 1S2 action, with the first class constraints of equations (30), (32), the form of the gauge generator is

\[ G = ap + a_p + b \chi + b_\chi + c \chi + c \chi. \]  

(35)

Here \((p, p_i)\) are the momenta associated with \((t, \vec{x})\) respectively; these are primary first class constraints. The HTZ formalism shows that in a system with canonical Hamiltonian \( H_c \), a set of first class constraints \( H_{c1} \) to the diffeomorphism gauge transformation action, the first class constraints of equations (13), (14), (21), (22) have been shown to lead invariances present in an action from the first class constraints that are present. For the 1S2 action, with the first class constraints of equations (30), (32), the form of the gauge generator is

\[ G = ap + a_p + b \chi + b_\chi + c \chi + c \chi. \]  

(36)

where \( Df/Dt \) denotes the total time derivative exclusive of time dependence through the canonical position and momentum variables. With the generator of equation (35), (36) leads to

\[ G = \left( -\dot{c} - \frac{d}{d - 1} \dot{c}_{i,i} \right) p + \left( \dot{c}_i - \dot{c}_i + c_i,j,j - c_{i,j,j} \right) p_i + \left( c_i - c_i \right) \chi \]

\[ + \left( -\dot{c}_i + c_i \right) \chi + c \chi + c \chi. \]  

(37)

From equation (37), we find that

\[ \delta^h = -\dot{c} + c_{i,i}, \quad \delta^h = -\dot{c} + c_i, \quad \delta H^{ij} = -c_i,j - c_{j,i} + 2\delta^{ij} c_{k,k}. \]  

(38a–c)

Using equation (26), we see that equation (38) is equivalent to

\[ \delta f^{\mu \nu} = \partial^\mu c^\nu + \partial^\nu c^\mu + \theta^{\mu \nu} \partial \cdot c \]  

(39)

where \( c^\mu = (c, c_i) \). Equation (39) is the usual spin two gauge transformation; it is the weak field limit of equation (34a).

We are now in a position to compare and contrast the canonical structure of the 1EH and 1S2 actions. The choice of variables made in equations (4)–(12) has been designed to facilitate this. In particular, if the weak field expansion of equation (2) is applied in equations (4)–(12) we end up with equations (25)–(27). Furthermore the weak field expansion reduces the secondary first class constraints of the 1EH action given by equations (13), (14) to the secondary first class constraints of the 1S2 action given by equations (30a), (30b).

However, beyond this point the canonical analysis of the 1EH and 1S2 actions diverge. We first note that after elimination of the second class constraints, the canonical Hamiltonian for the 1EH action is a linear combination of first class constraints (see equation (20)). This is not the case for the 1S2 action, as is apparent from equation (31).

The tertiary constraint \( \tau \) of equation (22) in the weak field limit (in which \( H^{ij} \approx \delta^{ij} \) reduces to \(-2\tau \)) of equation (32b). However, the weak field limit of \( \tau \) given by equation (21)
is not directly related to the constraint $\tau$ of equation (32a). Furthermore, the DB algebra of the first class constraints of the 1EH action given by equations (23), (24) does not reduce in the weak field limit to the DB algebra of the first class constraints of the 1S2 action. However, it is surprising that the weak field limit of the gauge transformation associated with the first class constraints following from the 1EH action is the gauge transformation for the 1S2 action following from its first class constraints.

It is apparent that if one were to make the expansion of equation (2) and substitute it into equation (1) without dropping terms in the action cubic to the fields, we would have in addition to the 1S2 action an interaction,

$$L_I = f_{\mu\nu} \left( \frac{1}{d-1} F_{\lambda\sigma}^\lambda F_{\sigma\nu}^\nu - F_{\sigma\lambda}^\lambda F_{\sigma\nu}^\nu \right).$$

Adding this interaction to $L_{S2}$ clearly does not restore the canonical structure of $L_{EH}$.

### 3. Discussion

In the preceding section we have demonstrated how expanding the metric about a flat background in the 1EH action alters the canonical structure of the theory. If the canonical structure of the 1EH is used in conjunction with the path integral to quantize gravity as in [2], then we are faced with an ambiguous situation when it comes to quantize fluctuations of the gravitational field about a flat background. One could either make use of the expansion of equation (2) in the path integral of [2], or insert the expansion of equation (2) into $L_{EH}$ (equation (1)) and then derive the path integral that follows from the canonical structure of $L_{S2} + L_I$ (equations (3), (40)). The two path integrals are not going to be equivalent because they are associated with two distinct canonical structures.

The usual background field method [9, 10] is used in conjunction with the Faddeev–Popov [11, 12] approach to defining the path integral. It has been shown that the use of the background field method is equivalent to what is obtained using canonical quantization when computing radiative corrections in Yang–Mills theory—but not in gravity. However, it has been used to compute loop corrections for the second-order EH (2EH) action [13, 14]. However, from the preceding section it is apparent that using the background field method in conjunction with the path integral as it is derived from the canonical structure of the 1EH action is not equivalent to what is obtained from the 1EH itself. The incompatibility of the path integral derived from the canonical structure of the 1EH action and that which follows from the Faddeev–Popov approach is discussed in [2]; this discrepancy is in part due to the presence of non-trivial ghosts that arise as a result of second class constraints that follow from the 1EH action. Quite likely this inequivalence is also a feature of the path integrals that follow from the 2EH action. (The canonical structure of the 2EH action is discussed in detail in [15–17]). In any case, one should recover the path integral based on the 2EH action from the path integral of the 1EH action by performing the path integral over the affine connection (provided this connection is not coupled to an external source), though the local measure for the path integral could be possibly altered. Problems associated with defining a path integral in systems with non-trivial second class constraints are also discussed in [18].

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References

[1] McKeon D G C 2010 Int. J. Mod. Phys. A 25 3453
[2] McKeon D G C and Chishtie F 2012 Class. Quantum Grav. 29 235016 (arXiv:1207.2302)
[3] Kiriushcheva N and Kuzmin S V 2011 Central Eur. J. Phys. 9 576
[4] Dirac P A M 2001 Lectures on Quantum Mechanics (Mineola, TX: Dover)
[5] Henneaux M and Teitelboim C 1992 Quantization of Gauge Systems (Princeton, NJ: Princeton University Press)
[6] Faddeev L D and Popov V N 1974 Sov. Phys.—Usp. 16 777
[7] Faddeev L D 1982 Sov. Phys.—Usp. 25 130
[8] Henneaux M, Teitelboim C and Zanelli J 1990 Nucl. Phys. B 332 169
[9] DeWitt B S 1967 Phys. Rev. 162 1195
[10] Abbott L 1981 Nucl. Phys. B 185 189
[11] Faddeev L D and Popov V N 1967 Phys. Lett. B 58 29
[12] Brandt F, Frenkel J and McKeon D G C 2007 Phys. Rev. D 76 105029
[13] 't Hooft G and Veltman M 1974 Ann. Inst. Henri Poincaré 20 69
[14] Göroff M H and Sagnotti A 1986 Nucl. Phys. B 266 709
[15] Kiriushcheva N, Kuzmin S, Racknor C and Valluri S R 2008 Phys. Lett. A 372 5101
[16] Kiriushcheva N and Kuzmin S 2011 Central Eur. J. Phys. 9 576
[17] Green K R, Kiriushcheva N and Kuzmin S 2011 Central Eur. J. Phys. C 71 1678
[18] Chishtie F and McKeon D G C 2012 Int. J. Mod. Phys. A 27 1250077