Error - disturbance uncertainty relations studied in neutron optics

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Abstract. Heisenberg’s uncertainty principle is probably the most famous statement of quantum physics and its essential aspects are well described by a formulations in terms of standard deviations. However, a naive Heisenberg-type error-disturbance relation is not valid. An alternative universally valid relation was derived by Ozawa in 2003. Though universally valid Ozawa’s relation is not optimal. Recently, Branciard has derived a tight error-disturbance uncertainty relation (EDUR), describing the optimal trade-off between error and disturbance. Here, we report a neutron-optical experiment that records the error of a spin-component measurement, as well as the disturbance caused on another spin-component to test EDURs. We demonstrate that Heisenberg’s original EDUR is violated, and the Ozawa’s and Branciard’s EDURs are valid in a wide range of experimental parameters, applying a new measurement procedure referred to as two-state method.

1. Introduction

Heisenberg’s uncertainty principle represents, without any doubt, one of the cornerstones of quantum mechanics. In his seminal paper, which was published in 1927, Heisenberg originally introduced a relation between the precision of a position measurement and the disturbance it induces on a subsequent momentum measurement of a particle. This is illustrated in the famous γ-ray microscope thought experiment, which is solely based on the Compton-effect: “At the instant when the position is determined - therefore, at the moment when the photon is scattered by the electron - the electron undergoes a discontinuous change in momentum. This change is the greater the smaller the wavelength of the light employed - that is, the more exact the determination of the position.” [1]. A relation is given by \( q_1 p_1 \sim \hbar \), for the product of the mean error \( q_1 \) of a position measurement (error) and the discontinuous change (disturbance) \( p_1 \) of the particle’s momentum. Heisenberg’s original formulation can be read in terms modern treatment of quantum mechanics as

\[
\epsilon(Q)\eta(P) \geq \frac{\hbar}{2},
\]

for error \( \epsilon(Q) \) of a measurement of the position observable \( Q \) and disturbance \( \eta(P) \) of the momentum observable \( P \). In the subsequent mathematical derivation of Eq. (1), based on the commutator relation \( [Q,P] = i\hbar \), Heisenberg showed that the product of the position and momentum standard deviations is given by

\[
\Delta Q \Delta P \geq \frac{\hbar}{2},
\]
for a class of Gaussian wavefunctions (with standard deviation \(\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} \)). Equation 2 was proven shortly afterwards in [2] for arbitrary wavefunctions by Kennard. In 1929 Robertson [3] generalized Eq. (2) to arbitrary pairs of observables \(A\) and \(B\) as

\[
\Delta Q \Delta P \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle |,
\]

for any state \(|\psi\rangle\), with \(\Delta Q \Delta P < \infty\) and commutator \([A, B] = AB - BA\). Equations 2 and 3 for standard deviations are uncontroversial and have been confirmed by many different experiments of various quantum systems [4, 5, 6].

However, Eqs. 2 and 3 describe the limitation of preparing quantum objects and has no direct relevance to the limitation of measurements on single systems, as suggested by Heisenberg’s γ-ray microscope thought experiment [7]. The generalized form of Heisenberg’s original error-disturbance relation, as in Eq. (1), would read

\[
\epsilon(A)\eta(B) \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle |.
\]

The validity of Eq. (4) is known to be limited to specific circumstances [8, 9, 10]. In 2003, Ozawa thus proposed in [11] a new error-disturbance uncertainty relation

\[
\epsilon(A)\eta(B) + \epsilon(A)\Delta(A) + \Delta(A)\eta(B) \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle |,
\]

and proved its universal validity in the general theory of quantum measurements. Here \(\epsilon(A)\) denotes the root-mean-square (r.m.s.) error of an arbitrary measurement for an observable \(A\), \(\eta(B)\) is the r.m.s. disturbance on another observable \(B\) induced by the measurement, and \(\Delta(A)\) and \(\Delta(B)\) are the standard deviations of \(A\) and \(B\) in the state \(|\psi\rangle\) before the measurement. Ozawa’s inequality Eq. (5) was tested experimentally in matter wave optics using neutrons [12, 13] and later with photons [14, 15].

Though universally valid Ozawa’s relations Eq. (5) is not optimal. Recently, Branciard [16] showed that one can improve on the sub-optimality of Ozawa’s proof and derived the following trade-off relation between error \(\epsilon(A)\) and disturbance \(\eta(B)\):

\[
\epsilon(A)\Delta(B)^2 + \Delta(A)^2\eta(B)^2 + 2\epsilon(A)\eta(B)\sqrt{\Delta(A)^2\Delta(B)^2 - C_{AB}^2} \geq C_{AB}^2,
\]

with \(C_{AB} = 1/2(|\langle \psi | [A, B] | \psi \rangle|)\). For the special case \(\langle \psi | A | \psi \rangle = \langle \psi | B | \psi \rangle = 0\), which implies \(\Delta(A) = \Delta(B) = 1\), and replacing \(\epsilon(A)\) and \(\eta(B)\) by \(\epsilon(A)\sqrt{1 - \frac{\epsilon(A)^2}{4}}\) and \(\eta(B)\sqrt{1 - \frac{\eta(B)^2}{4}}\), respectively, Eq. (6) can be strengthened yielding the tight relation

\[
\epsilon(A)^2 \left(1 - \frac{\epsilon(A)^2}{4}\right) + \eta(B)^2 \left(1 - \frac{\eta(B)^2}{4}\right) + 2\sqrt{1 - C_{AB}^2} \epsilon(A)\sqrt{1 - \frac{\epsilon(A)^2}{4}} \eta(B)\sqrt{1 - \frac{\eta(B)^2}{4}} \geq C_{AB}^2.
\]

which has been experimentally confirmed in [17, 18] with photons.

2. Measurement theory

For Eqs. (5-7) error \(\epsilon(A)\) and disturbance \(\eta(B)\) are defined via an indirect measurement model for an apparatus \(A\) measuring an observable \(A\) of an object system \(S\) as

\[
\epsilon(A) = \| (U^\dagger \otimes M) U - A \otimes I \| |\psi\rangle |\xi\rangle |,
\]

\[
\eta(B) = \| (U^\dagger \otimes I) U - B \otimes I \| |\psi\rangle |\xi\rangle |.
\]
Figure 1. (a) An indirect measurement model for apparatus A measuring a general object system S, defined on Hilbert space $H_{\text{obj}}$, is specified by a quadruple $(H_{\text{pro}}, |\xi\rangle, U, M)$. (b) A successive measurement scheme of qubit observables A and B exploited for demonstration of error-disturbance uncertainty relations. After state preparation (blue) apparatus $A_1$ carries out a projective measurement of $O_A$ instead of A (magenta), thereby disturbing observable B which is detected by apparatus $A_2$ (green). Between these two measurements a correction operation is performed, in order to minimize the disturbance $\eta(B)$. Error $\epsilon(A)$ and disturbance $\eta(B)$ are quantitatively determined by the four possible outcomes denoted as $(++)$, $(+ -)$, $(- +)$, and $(- -)$.

Here $|\psi\rangle$ is the initial state of system S, which is associated with Hilbert space $H_{\text{obj}}$. $|\xi\rangle$ is the state of the probe system P before the measurement, defined on Hilbert space $H_{\text{pro}}$, and an observable $M$, referred to as meter observable of P. The time evolution of the composite system $S + P$ during the measurement interaction is described by a unitary operator $U$ on $H_{\text{obj}} \otimes H_{\text{pro}}$. In Eq. (8) the Hilbert-Schmidt norm is used where the norm of a state vector in Hilbert space $X|\psi\rangle$ is given by the square root of its inner product: $\|X|\psi\rangle\| = \langle \psi | X^\dagger X | \psi \rangle^{1/2}$. A schematic illustration of a measurement apparatus A is given in Fig. 1(a).

A non-degenerate meter observable $M$ has a spectral decomposition $M = \sum_m m|\psi\rangle\langle m|$, where $m$ varies over eigenvalues of $M$, then the apparatus A has a family $\{M_m\}$ of operators, called the measurement operators, acting on $H_{\text{obj}}$ and defined as $M_m = \langle m | U | \xi \rangle$. Hence, the error is given by $\epsilon(A) = \sum_m \|M_m(m - A)|\psi\rangle\|^2$. If $M_m$ are mutually orthogonal projection operators sum and norm can be exchanged and the error can be written in compact form as

$$\epsilon(A) = \sum_m \| (O_A - A) |\psi\rangle\|^2;$$  

(9)

where $O_A$ is the output operator given as $O_A = \sum_m mM_m$. The disturbance can be written as

$$\eta(B) = \sum_m \| [M_m, B] |\psi\rangle\|^2.$$  

(10)

All these calculations are elaborated in detail in [10].

2.1. Threes-state versus two-state method

Due to occurrence of the term $O_A A + AO_A [X_B B + BX_B]$ when evaluating Eq. (9) [Eq. (10)] error $\epsilon(A)$ [disturbance $\eta(B)$] were claimed to be experimentally inaccessible [19, 20]. However, applying the operator identity $O_A A + AO_A = (A + I)O_A (A + I) - AO_A A - O_A$ measurement error and disturbance are given by

$$\epsilon(A) = 2 + \langle \psi | O_A | \psi \rangle + \langle \psi | AO_A A | \psi \rangle - \langle \psi | (A + I)O_A (A + I) | \psi \rangle,$$

$$\eta(B) = 2 + \langle \psi | X_B B | \psi \rangle + \langle \psi | B X_B B | \psi \rangle - \langle \psi | (B + I)X_B (B + I) | \psi \rangle,$$  

(11)
where \( X_B \) represents the modified output operator, given by 
\[ X_B = \sum_m M^\dagger_B m_B m. \]
Consequently error \( \epsilon(A) \) and disturbance \( \eta(B) \) can be expressed as a sum of expectation values in three input states \( |\psi\rangle, A|\psi\rangle, (A + \num{1})|\psi\rangle \) and \( |\psi\rangle, B|\psi\rangle, (B + \num{1})|\psi\rangle \), respectively, which was proposed in [21].
The three-state method was used in our previous experiments reported in [12, 13].

Here we introduce the so called two-state method using the operator identity \( O_A A + AO_A = (A + \num{1})O_A(A + \num{1}) - (A - \num{1})O_A(A - \num{1}) \) with error and disturbance denoted as
\[
\epsilon(A)^2 = 2 + \langle \psi| (A - \num{1})O_A(A - \num{1})|\psi\rangle - \langle \psi| (A + \num{1})O_A(A + \num{1})|\psi\rangle,
\eta(B)^2 = 2 + \langle \psi| (B - \num{1})X_B(B - \num{1})|\psi\rangle - \langle \psi| (B + \num{1})X_B(B + \num{1})|\psi\rangle,
\]
(12)
where \((A \pm \num{1})|\psi\rangle\) and \((B \pm \num{1})|\psi\rangle\) are the projections onto the eigenstates of \( A \) and \( B \) (multiplied by a factor 2). Note that in the two-state method the initial state \(|\psi\rangle\) is actually never sent onto the apparatus. Its effect is only enters via the auxiliary states \((A \pm \num{1})|\psi\rangle\) and \((B \pm \num{1})|\psi\rangle\).

2.2. Qubit measurements
In the case of qubit measurements the apparatus illustrated in Fig. 1 (a) has two output ports denoted as \((\num{+})\) and \((\num{-})\). The expectation values of \( O_A \ [X_B] \) from Eqs. (12), necessary for the determination of error \( \epsilon(A) \) [disturbance \( \eta(B) \)], are derived from the intensities at the four possible output ports, depicted the measurement scheme for qubit observables in Fig.1 (b), denoted as \( I_{++}, I_{+-}, I_{-+} \) and \( I_{--} \). The expectation value \( \langle \psi| O_A |\psi\rangle \) is obtained from the following combination of count rates
\[
\langle \psi| O_A |\psi\rangle = \frac{I_{++} + I_{+-} - I_{-+} - I_{--}}{I_{++} + I_{+-} + I_{-+} + I_{--}}.
\]
(13)

As already discussed due to the prior measurement of \( O_A \) the operator measured by apparatus \( A2 \) is modified from \( B \) to \( X_B \), with the corresponding expectation value expressed as
\[
\langle \psi| X_B |\psi\rangle = \frac{I_{++} + I_{+-} - I_{-+} - I_{--}}{I_{++} + I_{+-} + I_{-+} + I_{--}},
\]
(14)
required to determine the disturbance \( \eta(B) \). Consequently all expectation values necessary to determine error \( \epsilon(A) \) and disturbance \( \eta(B) \) can be derived from the intensities in the two input states \((A \pm \num{1})|\psi\rangle\) and \((B \pm \num{1})|\psi\rangle\), respectively.

3. Experiment
In our experiment the error-disturbance uncertainty relations, as defined in Eqs. (5) and (7), are tested via a successive measurement for qubit observables \( A \) and \( B \). The observables \( A \) and \( B \) are represented by the \( z \)- and \( y \)-components of the neutron \( \frac{1}{2} \) spin. A schematic illustration of the setup is given in Fig. 2. The error \( \epsilon(A) \) and the disturbance \( \eta(B) \) are defined for a joint measurement apparatus, where apparatus \( A1 \) measures the observable \( A = \sigma_z \) with error \( \epsilon(A) \) for an initial state \(|\psi_i\rangle = |+ \rangle \) and disturbs the observable \( B = \sigma_y \) thereby with disturbance \( \eta(B) \) during the measurement (here \( \sigma_z \) and \( \sigma_y \) denote the Pauli matrices; for simplicity, \( \hbar^2/4 \) is omitted in each spin component).
Finally apparatus \( A2 \) measures \( B = \sigma_y \). This particular combination of observables yields \( C_{AB} = 1 \) reducing Eq. 7 to
\[
\epsilon(A)^2 \left( 1 - \frac{\epsilon(A)^2}{4} \right) + \eta(B)^2 \left( 1 - \frac{\eta(B)^2}{4} \right) \geq 1.
\]
(15)
Since a precise measurement of \( A \) \( (\sigma_z) \) would disturb a posterior \( B \)-measurement maximally, an approximate measurement of \( A \), referred to as \( O_A \), is performed instead. Thus, apparatus \( A1 \) is designed to actually carry out not the maximally disturbing projective measurement \( A = \sigma_z \),
Figure 2. Neutron polarimetric setup for demonstration of the universally valid uncertainty relation for error and disturbance in neutron spin measurements. The setup is divided in four stages: state preparation (blue region), apparatus A1 carrying out the measurement of observable $O_A = \sigma_a(\theta) \cdot \sigma_B$ (magenta region), plus a correction operation (light green region) where the state is rotated by an angle $\theta_{AB} - \theta$, with $\theta_{AB}$ denoting the relative angle between the measurement directions of $A$ and $B$ (in our case that is $\theta_{AB} = \pi/2$). Apparatus A2 performs a subsequent measurement of observable $B = \sigma_y$ (dark green region). The required terms of Eqs. (5) and (7), i.e., error $\epsilon(A)$ and disturbance $\eta(B)$, are determined from the expectation values of the successive measurement of $O_A$ and $B$ (transparent coil DC$_{corr}$ is virtual, in practice the rotation is also carried out by DC-3).

but instead the projective measurement along a distinct axis $\sigma_a(\theta)$, in the $yz$ plane, denoted as $O_A = \tilde{\sigma}_a(\theta) \cdot \sigma_{y,z} = M_{+1} - M_{-1}$ (where $M_{\pm 1} = 1/2(\mathbb{1} \pm \tilde{\sigma}_a(\theta) \cdot \sigma_{y,z})$ and $\sigma_{y,z} = (\sigma_y, \sigma_z)^T$). Here $\theta$ denotes the angle of the measurement direction $\tilde{\sigma}_a$ and is our experimentally controlled detuning parameter, so that $\epsilon(A)$ and $\eta(B)$ are determined as a function of $\theta$. Next a unitary operation, as proposed in [16], given by $U_{corr} = e^{-i(\theta_{AB} - \theta)/2} \sigma_x$, is performed directly after the projective measurement of $O_A$, is supposed to reduce the influence of the prior $O_A$-measurement on the subsequent $B$-measurement optimally. For qubit-measurements this means that the state is rotated by an angle $\theta_{AB} - \theta$, where $\theta_{AB}$ denotes the relative angle between the measurement directions of $A$ and $B$. When there is no correction operation applied $U_{corr}$ is simply given by $U_{corr} = \mathbb{1}$. In order to detect the disturbance $\eta(B)$ on the observable $B$, induced by measuring $O_A$, apparatus A2 carries out the projective measurement of $B = \sigma_y$ in the state just after the first measurement.

The experiment was carried out at the tangential beam port of the 250 kW research reactor facility TRIGA Mark II of the TU-Wien, Austria. In the preparation section (Fig. 2, blue region)
Figure 3. (a) Experimental search of the optimal correction operation: After the projective measurement of $O_A$ (fixed at $\theta = 5\pi/18$) plus unitary rotations $U_{\text{corr}}$, the system leaves apparatus $A_1$ in the state $|\psi(\theta_{\text{corr}})\rangle$. A minimum of the disturbance $\eta(B)$ is found for $\theta_{\text{corr}} = \pi/2$, while the error $\epsilon(A)$ is not affected by the correction operation, as expected, and thus remains constant. (b) Results of error and disturbance plotted as $\epsilon(A)$ versus $\eta(B)$. Blue curve: the Branciard bound as defined in Eq. (6). Blue marker: Experimental results using the modified apparatus with optimal correction procedure, demonstrating the tightness of this relation. Orange curve: error and (sub-optimal) disturbance as in Eq. (5) lhs, without performing any correction operation. Green curve: bound imposed by the Heisenberg’s original error-disturbance relation $\epsilon(A)\eta(B) \geq |1/2\langle\psi| [A,B]|\psi\rangle|$, which is violated by our experimental results. Red curve: Ozawa’s relation Eq. (5) rhs, which is indeed satisfied, but is not saturated.

The auxiliary input states $|A\pm\mathbb{1}\rangle|\psi\rangle = |\pm z\rangle$ and $|B\pm\mathbb{1}\rangle|\psi\rangle = |\pm y\rangle$ are generated by spinor rotations within DC-1, due to Larmor precession about the static magnetic field that is produced inside the DC-1 coil, pointing in $+x$-direction. The projective measurement of $O_A$ (apparatus $A_1$ - magenta in Fig. 2) consist of two sequential operations: first the initially prepared state is projected onto the eigenstates of $O_A$ by DC-2, which rotates the respective spin component of $\vec{\sigma}_a$ belonging to $O_A$ in $+z$ direction. Then, in order to complete the projective measurement the spin, which is pointing in $+z$ after the analyzer, has to be prepared in an eigenstate of $O_A$. This is achieved by an appropriate adjusted magnetic field of DC-3 (thereby applying the same procedure as for DC-1 in the initial state preparation). The transparent coil DC$_{\text{corr}}$ (light green in Fig. 2) is only for illustrative purpose. In the actual experiment the spinor rotation of $\theta_{AB} - \theta$, ($\theta_{AB} = \pi/2$) about the $x$-axis is also carried out by DC-3. Finally the $B$-measurement is performed (apparatus $A_2$ - dark green in Fig. 2) utilizing DC-4 and the second analyzer. Unlike for the $O_A$-measurement subsequent preparation of the eigenstates of $B$ is not necessary because the count-rate detection done by a high-efficiency BF$_3$-detector is spin-insensitive.

3.1. Experimental results
No correction procedure - In the first experimental run no additional correction operation is applied, which means that $U_{\text{corr}} = \mathbb{1}$. The observables $O_A = \cos \theta \sigma_z + \sin \theta \sigma_y$ and $B = \sigma_y$ are successively measured, resulting in error $\epsilon(A)$ and disturbance $\eta(B)$ as defined for Ozawa’s universally valid error-disturbance uncertainty relations from Eq. (5). The obtained results, together with the theoretical prediction and the lower bound, are plotted in Fig. 3 (b), orange
curve. Though the theoretical prediction are reproduced evidently and the simple version of Heisenbergs’ error-disturbance relation $\epsilon(A)\eta(B) \geq \frac{1}{2} |\langle\psi||[A,B]||\psi\rangle|$ is violated for all values of our experimentally controlled parameter $\theta$, it can be seen that Eq. (5) is only tight for the trivial cases $\epsilon(A) = 0$ and $\eta(B) = \sqrt{2}$ or vice versa.

Optimal correction procedure - In order to strengthen Eq. (5) (apart from the two points $\eta(B) = 0, \sqrt{2}$) a correction operation can be applied to minimize the disturbance $\eta(B)$. The class of correction operations we study here are unitary transformations in principal realized by an additional spin turner device, denoted as DC$_{corr}$ (light green in Fig. 2). However, in the actual experiment state preparation after the $O_A$-measurement and correction operation $U_{corr}$ are both performed using DC-3 alone. In Fig. 3(a) experimental results are presented, where after the projective measurement of $O_A$ (with detuning fixed at $\theta = 5\pi /18$) the eigenstates of the $O_A$-measurement is rotated within the $zy$-plane onto states $|\psi_{corr}(\vartheta)\rangle = \cos \frac{\vartheta}{2}|+z\rangle + \sin \frac{\vartheta}{2}|-z\rangle$. A minimum of the disturbance $\eta(B)$ is found for $\vartheta = \pi/2$, while the error remains $\epsilon(A)$ remains constant because it is not affected by the correction operation. The angle $\vartheta = \pi/2$ is equivalent to a rotation of $\theta_{AB} - \theta$, with $\theta_{AB} = \pi/2$ after a $O_A$-measurement with detuning angle $\theta$. As seen in the blue curve of Fig. 3(b) and proposed in [16] this correction operation is indeed optimal, yielding a tight error-disturbance inequality.

4. Conclusions
We have demonstrated experimental validity of Ozawa’s universally valid error-disturbance uncertainty relation as well as its tightened version derived by Branciard in a neutron polarimetric experiment. The theoretical predictions are reproduced evidently at a high degree of accuracy.

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