Information transfer through a one-atom micromaser

Animesh Datta
Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131-1156, USA

Biplab Ghosh, A. S. Majumdar and N. Nayak
S. N. Bose National Centre for Basic Sciences, Block JD, Salt Lake, Kolkata 700 098, INDIA

(Dated: April 1, 2022)

We consider a realistic model for the one-atom micromaser consisting of a cavity maintained in a steady state by the streaming of two-level Rydberg atoms passing one at a time through it. We show that it is possible to monitor the robust entanglement generated between two successive experimental atoms passing through the cavity by the control decoherence parameters. We calculate the entanglement of formation of the joint two-atom state as a function of the micromaser pump parameter. We find that this is in direct correspondence with the difference of the Shannon entropy of the cavity photons before and after the passage of the atoms for a reasonable range of dissipation parameters. It is thus possible to demonstrate information transfer between the cavity and the atoms through this set-up.

PACS numbers: 42.50.Dv, 03.65.Ta

The generation of quantum entanglement in atomic systems is being vigorously pursued in recent years. The primary motivation for this upsurge of interest is to test the applicability of the ongoing conceptual developments in quantum information theory and through them the implementation of current quantum communication and computation protocols. Several schemes have been proposed recently to engineer the entanglement of two or more atoms. Many of these proposals are for generating entanglement in a probabilistic manner. Since a large number of these proposals rely on the trapping or slow passage of cold atoms through optical cavities, the efficient control of cavity leakage and atomic dissipation is a major concern. The value of the atom-cavity coupling $g$ is very close to the values of photonic and atomic decay rates $\kappa$ and $\Gamma$, respectively, in the parameter ranges operated by optical cavities. Thus decoherence effects are significant even in the time $O(1/g)$ needed for perceptible entanglement.

The micromaser, described below, is appreciated as a practical device for processing quantum information. The formation of atom-photon entanglement and the subsequent generation of correlations between spatially separated atoms has been shown using the micromaser. The nonlocal correlations developed in this fashion between two or more atoms can be used to test the violation of Bell-type inequalities. Since for Rydberg atoms tuned with microwave frequencies, $\kappa \ll g$, and $\Gamma$ is negligible, decoherence does not crucially affect the individual single atom dynamics. However, dissipation does build up over the passage of a number of atoms through the micromaser, and is revealed in the photonic statistics of the steady-state cavity field, as was discussed in Ref. 8. The entanglement between a pair of atoms pumped at the same time through a micromaser has been analysed in Ref. 8. It is rather difficult to practically realize such a set-up though. The genuine one-atom micromaser, on the other hand, can be operated over a reasonably large region of parameter space, and is thus a feasible device for generating entanglement between two or more atoms. Recently, Englert et al. have shown using a non-separability criterion, the generation of entanglement between two atoms that pass through a one-atom micromaser one after the other, in immediate succession.

In this Letter we propose a scheme to measure the entanglement generated between two successive atoms that stream through a real one-atom micromaser in such a manner that their flights through the cavity do not overlap. There is always a time gap between one atom leaving the cavity and the next atom entering the cavity. We show that successive atoms that emerge out of the cavity are entangled. This scheme does not require the spatial overlap between the two atoms at any stage. In the theory for the micromaser used by us, the interaction of the cavity with its reservoir is taken into account at all times. We indeed compute the effect of photon leakage on the entanglement measure. Such a model was earlier used by us to show the violation of a Bell-type inequality between two spatially non-overlapping atoms correlated via atom-photon interactions. This strongly suggests entanglement of the successive atoms that pass through the cavity. The generation of nonlocal correlations between the two atomic states emerging from the cavity can in general be understood using the Horodecki theorem.

Since the joint state of the two atoms emanating from the cavity is not a pure state, we quantify the entanglement using the well known measure appropriate for mixed states, i.e., the entanglement of formation. Information is transferred from the cavity to the atoms in order to build up entanglement. The amount of information transferred is expected to depend on the available information content of the cavity. The variance of the
photon number distribution of the cavity is an indicator of the information content of the cavity field, and we compute the variance over a range of atom-photon interaction times. The total information inside the cavity is however measured by its Shannon entropy \( S \) which has contributions from higher moments of the photon statistics as well. We therefore calculate the Shannon entropy of the photon distribution function in the cavity before and after the passage of the two atoms. The difference in the Shannon entropy is seen to be in remarkable correspondence with the entanglement of formation of the atoms up to a reasonably long atom-photon interaction time. We can thus claim that this scheme provides a concrete quantitative example of information transfer between the microwave cavity and the two atoms in a realistic set-up.

We begin with a description of the experimental scenario. A two-level atom initially in its upper excited state \(| e \rangle \) traverses a high-Q single mode cavity. The cavity is in a steady state built up by the passage of a large number of atoms, but only one at a time, with a fixed pump parameter and atom-photon interaction time, and is tuned to a single mode resonant with the transition \(| e \rangle \rightarrow | g \rangle \). The emerging single-atom wavefunction is a superposition of the upper \(| e \rangle \) and lower \(| g \rangle \) state and it leaves an imprint on the photonic wavefunction in the cavity. During this process, cavity leakage takes place, and is taken into account. Next, a second experimental atom, prepared also in state \(| g \rangle \), encounters the cavity photons. The difference in the parameters \( \theta \) is the phase difference between the two atoms and \( \alpha \) is the Rabi angle. After emanating from the cavity, each of the atoms encounters a \( \pi/2 \) pulse through an electromagnetic field whose phase may be varied for different atoms. The effect of the \( \pi/2 \) pulse is to transform the state \( \frac{1}{\sqrt{2}}(| e \rangle + | g \rangle) \) to \( | g \rangle \) and the other one to \( | e \rangle \). These resultant states may now be detected, and the corresponding outcomes used to signify the states emanating from the cavity.

The micromaser model has been described in details in Refs 7, 8. Here we merely outline its essential features. The cavity is pumped to its steady state by a Poissonian stream of atoms passing through it one at a time, with the time of flight through the cavity \( t \) being same for every atom. The dynamics of these individual flights are governed by the evolution equation with three kinds of interactions given by

\[
\dot{\rho} = \hat{\rho}|\text{atom-reservoir}\rangle + \hat{\rho}|\text{field-reservoir}\rangle + \hat{\rho}|\text{atom-field}\rangle \tag{1}
\]

where the strength of the three couplings are given by the parameters \( \Gamma \), (the atomic dissipation constant) \( \kappa \) (the cavity leakage constant) and \( g \) (the atom-field interaction constant, or \( gt \) the Rabi angle) and the individual expressions provided in Refs 7, 8. Obviously, \( \Gamma = 0 = g \) describes the dynamics of the cavity when there is no atom inside it. The finite temperature of the cavity is represented by the average thermal photons \( n_{\text{th}} \), obtained from B-E statistics. The density matrix of the steady state cavity field \( \rho_{ss} \) can be obtained by solving the above equation and tracing over the reservoir and atomic variables. The photon distribution function is then given by

\[
P_n^{(ss)} = \langle n| \rho_{ss}^{(ss)} | n \rangle \tag{2}
\]

in the photon number \( (n) \) representation, the expression for which in this model of the micromaser has been derived in Ref. 12.

The joint state of the two experimental atoms emanating successively from the cavity is not separable, and is represented by a mixed density operator \( \rho_a \) which is obtained by tracing over the field variables. \( \rho_a \) can be written as

\[
\rho_a = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_3 & \alpha_4 e^{i\theta} & 0 \\
0 & \alpha_4 e^{-i\theta} & \alpha_2 & 0 \\
0 & 0 & 0 & \alpha_5
\end{pmatrix}
\tag{3}
\]

where \( \theta \) is the phase difference between the two atoms introduced by the \( \pi/2 \) pulses, and the matrix elements are given by

\[
\alpha_1 = \text{Tr}_f \left( A A \rho_{ss}^{(ss)} A^\dagger A^\dagger \right), \tag{4}
\]

\[
\alpha_2 = \text{Tr}_f \left( A D \rho_{ss}^{(ss)} D^\dagger A^\dagger \right), \tag{5}
\]

\[
\alpha_3 = \text{Tr}_f \left( D A \rho_{ss}^{(ss)} A^\dagger D^\dagger \right), \tag{6}
\]

\[
\alpha_4 = \text{Tr}_f \left( D D \rho_{ss}^{(ss)} D^\dagger A^\dagger \right), \tag{7}
\]

\[
\alpha_5 = \text{Tr}_f \left( D D \rho_{ss}^{(ss)} D^\dagger D^\dagger \right) \tag{8}
\]

The field operators \( A \) and \( D \) are defined as

\[
A = \cos(gt \sqrt{a^\dagger a + 1}) \tag{9}
\]

\[
D = -i a^\dagger \sin(gt \sqrt{a^\dagger a + 1}) \sqrt{a^\dagger a + 1} \tag{10}
\]

where \( a(a^\dagger) \) is the usual photon annihilation (creation) operator.

The entanglement of formation \( E_F \) of the two-atom system can be expressed as

\[
E_F(\rho_a) = h \left( \frac{1}{2} \left[ 1 + \sqrt{1 - C(\rho_a)} \right] \right), \tag{11a}
\]

\[
h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x), \tag{11b}
\]
where $C(\rho_a)$, the concurrence of the state $\rho_a$, is defined as

$$C(\rho_a) \equiv \max\{0, \sqrt{\lambda_1 - \sqrt{\lambda_2 - \sqrt{\lambda_3 - \sqrt{\lambda_4}}}}\}, \quad (11)$$

in which $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of the matrix $\rho_a(\sigma_y \otimes \sigma_y)\rho_a^*(\sigma_y \otimes \sigma_y)$ in decreasing order and $\sigma_y$ the Pauli spin matrix. $E_F(\rho_a)$, $C(\rho_a)$, and the tang $\tau(\rho_a) \equiv C(\rho_a)^2$ are equivalent measures of entanglement inasmuch as they are monotonic functions of one another.

For the state $\rho_a$ in Eq. (8), the four eigenvalues of the matrix $\rho_a(\sigma_y \otimes \sigma_y)\rho_a^*(\sigma_y \otimes \sigma_y)$ are given by $(\alpha_4 + \sqrt{\alpha_2 \alpha_3}), (\alpha_4 - \sqrt{\alpha_2 \alpha_3}), \alpha_1 \alpha_5$, and $\alpha_1 \alpha_3$, respectively. We compute numerically the values of $E_F(\rho_a)$ for a range of micromaser parameters. In Fig.1 we plot the Entanglement of formation $E_F$ of the two-atom state versus the dimensionless cavity dissipation parameter $\kappa/g$. The value of the other parameters, i.e., $N$ (the micromaser pump rate denoting the number of atoms that pass through the cavity in a photon lifetime to maintain its steady state), $gt$ (the Rabi angle), $n_{th}$ (the number of thermal photons) are displayed in the figure caption. It is clearly seen that cavity dissipation leads to the loss of information transfer to the joint state of the two atom system. It may be further noted that the interaction times in the Curves II-IV correspond to the so-called “trapped states” conditions [10]. In such a situation, it is expected that the cavity field would evolve to a Fock state or a set of discrete states satisfying the condition $\sin(gt\sqrt{n_i} + 1) = 0$. Naturally, the correlation between different field states $|n_i\rangle >$ is not significant, and further deteriorates with increasing cavity leakage. This is reflected in Curves II-IV where the fall of entanglement is much faster than that in the situation where $gt$ does not satisfy “trapped state” conditions (Curve I). Note that the nonvanishing values of $E_F$ in Curves II-IV suggest that the field does not evolve to an exact Fock state of the photon field [12].

The steady-state photon statistics of the cavity field [8, 12] depending on the cavity parameters are reflected in the entanglement properties of the emerging atoms. As an illustration of this we compute the variance of the photon number distribution $\nu = \sqrt{[< n^2 > - < n >^2]} / < n >$ as function of the pump parameter $D = gt\sqrt{N}$. The entanglement of formation $E_F$ and the photon number variance $(\nu/10)$ are both plotted versus $D$ in Fig.2. The peaks in $\nu$ are related to the thresholds of micromaser operation (these characteristics of the micromaser are discussed in details in Refs. [11, 12]), and correspond to increased uncertainty or information content in the photon distribution function. It is hence not surprising that maximum information transfer to the atoms takes place around these parameter values as signified by the peaks of $E_F$. The functional relation of $E_F$ with $D$ is reminiscent of the behaviour of the Bell sum [8] in the same model. It can be shown using the Horodecki theorem [12] that a Bell-type inequality will be violated in this set-up if and only if $M(\rho_a) > 1$ where $M(\rho_a)$ is defined as the sum of the two largest eigenvalues of the $3 \times 3$ matrix $T^\dagger T$ with the elements $T_{ij} = \text{Tr}(\rho_a \sigma_i \otimes \sigma_j)$. Here the three eigenvalues of the matrix $T^\dagger T$ are given by $(2\alpha_4)^2$, $(2\alpha_4)^2$, and $(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_5)^2$ respectively. It is possible to confirm numerically that violation of a Bell-type inequality will occur for a range of parameter values.

The role of photon statistics on information transfer is further revealed by the computation of the Shannon entropy which for the steady-state cavity is defined as

$$S(\rho_f^{(ss)}) = - \sum_{n=1}^{\infty} P_n^{(ss)} \log P_n^{(ss)} \quad (12)$$

using the expression for the steady-state photon number distribution given in Refs. [8, 12], with the normalisation $\sum_n P_n = 1$. It is straightforward to derive the expression for the density operator of the cavity field after passage of the two experimental atoms [8]. We denote this cavity state as $\rho_f^{(2)}$, given by

$$\rho_f^{(2)} = AA\rho_f^{(ss)}A^\dagger A^\dagger + DD\rho_f^{(ss)}D^\dagger D^\dagger + AD\rho_f^{(ss)}A^\dagger D^\dagger + DA\rho_f^{(ss)}D^\dagger A^\dagger \quad (13)$$

and then compute its Shannon entropy

$$S(\rho_f^{(2)}) = - \sum_{n=1}^{\infty} P_n^{(2)} \log P_n^{(2)} \quad (14)$$

as well, through the corresponding photon number distribution $P_n^{(2)} = \langle n | \rho_f^{(2)} | n \rangle$. 

![Figure 1: The entanglement of formation $E_F$ plotted with respect to the dimensionless cavity dissipation parameter $\kappa/g$ with $N = 100$ and $n_{th} = 0.01$ for different values of the Rabi angle $gt$. It is seen that $E_F$ falls down with increasing cavity leakage $\kappa$ for all values of $gt$, including curve I ($gt = \pi/10$) which is also displayed in the inset for clarity. The other values of $gt$ in curves II ($gt = \pi/2$), III ($gt = 3\pi/2$) and IV ($gt = \pi$) each corresponds to trapped state conditions.](image-url)
We observe that the streaming atoms leave an imprint on the photon distribution function, as reflected in its reduced Shannon entropy. In Fig.2 we also plot the variation of $S(\rho_f^{(2)}) - S(\rho_f^{(ss)})$ (the difference of the cavity entropy before and after the passage of the atoms) versus the parameter $D$. The difference of the cavity entropy before and after the passage of the atoms is transported towards constructing the atomic entanglement. A striking feature of our results is the remarkable functional correspondence between the difference $S(\rho_f^{(2)}) - S(\rho_f^{(ss)})$ and the entanglement of formation $E_F(\rho_f)$ for the atoms up to a sufficiently long atom-photon interaction time $t$. Theory of the micromaser that we are using, takes into account dissipation during the atom-photon interaction time, as well as during the much longer time interval between the passage of the successive atoms. Dissipative builds up with the passage of a number of atoms through the cavity affecting the steady-state statistics. For short atom-photon interaction times before dissipative effects can creep in, the atomic entanglement exhibits peaks reflective of the characteristic micromaser thresholds. Corresponding peaks are observed in the difference of Shannon entropies characterizing the maximas of information loss from the cavity around these parameter values. For longer interaction times (corresponding to $D \geq 30$ in Fig.2) dissipative effects result in the saturation of the value of $v$ (around 1) and also of $E_F$. In contrast, the difference of the Shannon entropies continues to grow signifying greater loss of information from the cavity to the environment. The role of dissipative effects on hindering information transfer from the cavity to the atoms is clearly evident for these parameter values.

To summarize, we have considered a realistic model for the one-atom micromaser which is feasible to operate experimentally. In this device, correlations between two separate atoms build up via atom-photon interactions inside the cavity, even though no single atom interacts directly with another. The state-of-art of the device does not allow dissipative effects to reduce drastically the entanglement generated between a successive pair of atoms for an accessible range of parameters. Their effects nonetheless, are felt via the photon statistics of the cavity, on the transport of information from it, more of which is lost to the environment for longer interaction times. If our analysis were to be extended to include the quantification of the entanglement generated between three or more successive atoms, which is an interesting possibility within the present framework, then cavity photon loss is expected to directly bear upon such multipartite atomic entanglement. For Rydberg atoms undergoing microwave transitions, the atomic damping constant $\Gamma = 0$. However, in the one-atom laser operated at optical frequencies, a finite $\Gamma$ will further diminish the degree of entanglement. Thus, the generation of entangled bipartite atoms using their Rydberg energy levels and a microwave cavity seems to be more feasible compared to an optical cavity of the one atom laser.

We have seen that it is possible to demonstrate the transfer of information through the micromaser set-up. The initial joint state of two successive atoms that enter the cavity is entangled. Interactions mediated by the cavity photon field results in the final two-atom state being of a mixed entangled type. The information content of the final two-atom state characterised by its entanglement of formation emanates from the loss of information of the cavity field quantified by the reduction of Shannon entropy. Dissipation results in a part of the Shannon entropy, or information content of the cavity to be lost to the environment. Alternatively, the final entangled state of the two atoms could be viewed as arising through the interaction of two separate atoms with the common, but suitably tailored “environment”, the role of which is played by the cavity field. In this respect, the physics of entanglement generation in the micromaser is a special case of ‘environment induced entanglement’. Our approach enables the study of the relationship between a priori abstract information theoretic measures for systems in different Hilbert spaces describing the atomic states and cavity photons respectively. Further studies on dynamics of information transfer need to be undertaken in the contexts of different micromaser models. One particular model is that of the driven micromaser, which due to its solvable nature seems to be a promising candidate for investigation of details such as channel capacity of information transfer. Finally, it would be interesting to investigate the recently proposed fascinating conjecture of Bennett regarding the ‘monog-
amous' nature of entanglement through the micromaser device.

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Information* (Cambridge University Press, Cambridge, England, 2000).

[2] C. Cabrillo, J. I. Cirac, P. G-Fernandez and P. Zoller, Phys. Rev. A 59, 1025 (1999); S. Bose, P. L. Knight, M. B. Plenio and V. Vedral, Phys. Rev. Lett. 83, 5158 (1999); M. B. Plenio, S. F. Huelga, A. Beige and P. L. Knight, Phys. Rev. A 59, 2468 (1999); J. Hong and H.-W. Lee, Phys. Rev. Lett. 89, 237901, 2002; L.-M Duan, H. J. Kimble, Phys. Rev. Lett. 90, 253601, 2003.

[3] L. M. Duan, M. D. Lukin, J. I. Cirac and P. Zoller, Nature 414, 413 (2001); L. M. Duan, Phys. Rev. Lett. 88, 170402 (2002).

[4] E. Solano, G. S. Agarwal and H. Walther, Phys. Rev. Lett. 90, 027903 (2003); P. Lougovski, F. Casagrande, A. Lulii, B.-G. Englert, E. Solano and H. Walther, Phys. Rev. A 69, 023812 (2004).

[5] For a recent review, see, J. M. Raimond, M. Brune and S. Haroche, Rev. Mod. Phys. 73, 565 (2001).

[6] B. Tregenna, A. Beige and P. L. Knight, Phys. Rev. A 65, 032305 (2002); A. S. Sorensen and K. Mølmer, quant-ph/0202073; quant-ph/0304008; C. Marr, A. Beige and G. Rempe, Phys. Rev. A 68, 033817 (2003); S. Clark, A. Peng, M. Gu, S. Parkins, quant-ph/0307064.

[7] E. Hagley, X. Maitre, G. Nogues, C. Wunderlich, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 79, 1 (1997); M. Loffler, B. G. Englert and H. Walther, Appl. Phys. B63, 511 (1996); S. B. Zheng and G. C. Guo, Phys. Rev. Lett. 85, 2392 (2000).

[8] A. S. Majumdar and N. Nayak, Phys. Rev A 64, 013821 (2001).

[9] P. Maslak, Phys. Rev. A66, 023804 (2002).

[10] B. G. Englert, P. Lougovski, E. Solano, H. Walther, Laser Physics, 13, 355 (2003).

[11] G. Rempe, F. Schmidt-Kaler and H. Walther, Phys. Rev. Lett. 64, 2783 (1990).

[12] N. Nayak, Opt. Commun. 118, 114 (1995).

[13] R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A 200, 340 (1995).

[14] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022, (1997); W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[15] C. E. Shannon, Bell System Technical Journal, 27, 379 (1948).

[16] A. K. Rajagopal and F. W. Cummings, Phys. Rev. A39, 3414 (1989).

[17] D. Braun, Phys. Rev. Lett. 89, 277901 (2002).

[18] In preparation (2002).

[19] C. H. Bennett, Lecture course in the "School on Quantum Physics and Information Processing", TIFR, Mumbai (2002) [http://qpip-server.tcs.tifr.res.in/~qpip/HTML/Courses/Bennett/TIFR2.pdf], B. M. Terhal, quant-ph/0307120.