The infinitesimally bendable Euclidean hypersurfaces

M. Dajczer and Th. Vlachos

Abstract

The main purpose of this paper is to complete the work initiated by Sbrana in 1909 giving a complete local classification of the nonflat infinitesimally bendable hypersurfaces in Euclidean space.

In the final decades of the 19th century geometers were increasingly interested in the study of hypersurfaces in Euclidean space. Quite differently to what happens in the surface case, these submanifolds are not easily isometrically deformable. In fact, it was shown that hypersurfaces are isometrically rigid provided that they bend in enough directions. The first correct proof that hypersurfaces with at least three nonzero principal curvatures cannot be isometrically deformed was given in 1885 by Killing [9] after a claim made in 1876 by Beez [1].

The situation of hypersurfaces of rank two, that is, the ones with exactly two nonzero principal curvatures, remained to be understood. It turned out that even in this case hypersurfaces are “generically” rigid. After earlier work for the three-dimensional case by Bianchi [2], a parametric classification of all Euclidean hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, that admit non-trivial isometric deformations, was obtained in 1909 by Sbrana [11]. This was done in terms of the so called Gauss parametrization discussed in the next section. Cartan [3] in 1916 gave a more careful statement but now in the language of envelopes of hyperplanes. See Dajczer, Florit and Tojeiro [5] for a modern presentation and further results on the subject.

Perhaps the most interesting class in the classification discussed above is the one of isometrically bendable hypersurfaces, that is, when the hypersurface admits a smooth one-parameter variation by isometric hypersurfaces. These submanifolds can either be ruled, and then allow plenty of isometric bendings, or non-ruled in which case they just admit a single bending.

At around the same time, Sbrana [10] (who seems to have been a student of Bianchi) in an inspiring paper considered the problem of classifying Euclidean hypersurfaces that admit “infinitesimal deformations”, that is, they are infinitesimally bendable. Roughly speaking, this means that there is a smooth one-parameter variation by hypersurfaces that are isometric only to the “first order”. The precise definition of an infinitesimal bending is given in Section 3. Of course, any bendable hypersurface is infinitesimally...
bendable, but the latter class turns out to be much larger. In fact, what Sbrana did was to provide a complete description of one class of infinitesimally bendable hypersurfaces (in terms of the Gauss parametrization already used in [11]) but somehow ignored others.

It was very natural for Sbrana at that time to consider the infinitesimal version of the deformation problem. On one hand, because there was already a rich theory of infinitesimal bendings of surfaces; see Spivak [13]. On the other hand, it was known that any hypersurface that possesses at least three nonzero principal curvatures is infinitesimally rigid, that is, it is not infinitesimally bendable, a result that can be found in the book by Cesaro [4] from 1896. A modern proof of this fact follows from the more general result obtained by Dajczer and Rodríguez [8].

It is for us quite surprising that we were not able to find any reference to Sbrana’s contribution to the description of the hypersurfaces that admit infinitesimal bendings. In fact, the few places where his paper is referred to are quite old and do not discuss his result; see [12] and [14].

We should point out that all of the above results are of local nature, as is the case of this paper. By being local we mean that there is an open and dense subset of the manifold such that along any connected component the submanifold belongs to a class in the classification. In that respect, we observe that for isometric bendings it was already shown in [5] that hypersurfaces in different classes can be smoothly attached.

The main purpose of this paper is to give a complete local classification of the nonflat infinitesimally bendable hypersurfaces in modern terms. In order to give a description of all hypersurfaces $f: M^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, with two nonzero principal curvatures at any point that are infinitesimally bendable, we exclude from consideration the ones that are surface-like. Being surface-like means that $f$ is locally part of a cylinder either over a surface in $\mathbb{R}^3$ or the cone of a surface in $S^3 \subset \mathbb{R}^4$. The reason of exclusion is because in this case it can be shown that the infinitesimal bending of the hypersurface is given by an infinitesimal bending of the surface, and the surface case is not an object of this paper.

Among the infinitesimally bendable hypersurfaces there is the class of ruled hypersurfaces. A hypersurface $f: M^n \rightarrow \mathbb{R}^{n+1}$ is called ruled if $M^n$ admits a foliation by leaves of codimension one mapped by $f$ into affine subspaces of $\mathbb{R}^{n+1}$. In our context, this class is not very interesting because it turns out that any infinitesimal bending is determined by an isometric bending. And isometric bendings of ruled hypersurfaces are easily seen to be parametrized by the set of smooth functions on an interval.

Finally, there is the class of infinitesimally bendable hypersurfaces that admit a unique infinitesimal bending. These hypersurfaces are the really interesting ones since generically they are not bendable, as we argue at the end of this introduction. We next give a characterization of the hypersurfaces belonging to this class à la Cartan, that is, in terms of envelopes of hyperplanes. An equivalent statement in terms of the Gauss parametrization, the one used for the proof, is given later. The concepts of envelope of hyperplanes and the Gauss parametrization, as well as the relations between them, are
be discussed in the next section.

On an open subset $U \subset \mathbb{R}^2$ endowed with coordinates $(u, v)$ let $\{\varphi_j\}_{0 \leq j \leq n+1}$ be a set of solutions of the differential equation

$$\varphi_{z_1 z_2} + M \varphi = 0$$

where $(z_1, z_2)$ can be either $(u, v)$ or $(u + iv, u - iv)$ and $M \in C^\infty(U)$. Assume that the map $\varphi = (\varphi_1, \ldots, \varphi_{n+1}) : U \rightarrow \mathbb{R}^{n+1}$ is an immersion and consider the two-parameter family of affine hyperplanes

$$G(u, v) = \varphi_1 x_1 + \cdots + \varphi_{n+1} x_{n+1} - \varphi_0 = 0$$

where $(x_1, \ldots, x_{n+1})$ are canonical coordinates of $\mathbb{R}^{n+1}$.

Our main result says that any hypersurface $f : M^n \rightarrow \mathbb{R}^{n+1}$ in the last class is the envelope of a two-parameter family of hyperplanes as above which, in turn, means that $f$ is the solution of the system of equations $G = G_u = G_v = 0$.

**Theorem 1.** Let $f : M^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, be an infinitesimally bendable hypersurface of constant rank two that is neither surface-like nor ruled on any open subset of $M^n$. Then, there is an open and dense subset of $M^n$ such that along any connected component $f$ is the envelope of a two-parameter family of hyperplanes as above. Conversely, any hypersurface obtained as the envelope of a two-parameter family of hyperplanes as above admits locally a unique infinitesimal bending.

Parametrically, the hypersurface can be described by the Gauss parametrization and goes as follows: Let $g : U \rightarrow \mathbb{S}^n$ and $\gamma \in C^\infty(U)$ be given by

$$g = \frac{1}{\|\varphi\|} (\varphi_1, \ldots, \varphi_{n+1}) \quad \text{and} \quad \gamma = \frac{\varphi_0}{\|\varphi\|}.$$  

If $\Lambda$ denotes the normal bundle of $g$ and $h = i \circ g$ where $i : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ is the inclusion, then the map $\psi : \Lambda \rightarrow \mathbb{R}^{n+1}$ given by

$$\psi(x, w) = \gamma(x) h(x) + h \ast \text{grad} \gamma(x) + w$$

parametrizes the hypersurface.

We point out that for a hypersurface obtained as above, in order to be isometrically bendable the set of functions $\varphi_1, \ldots, \varphi_{n+1}$ must satisfy a strong additional condition, namely, the function $\phi = \|\varphi\|^2$ has to verify $\phi_{z_1 z_2} = 0$.

1 Parametrizations

In this section, we first recall how a Euclidean hypersurface of constant rank can be locally parametrized by the use of the Gauss parametrization. Then, we discuss a class of envelopes of hyperplanes depending on parameters as well how they can be described in terms of the Gauss parametrization.
1.1 The Gauss parametrization

Let $f : M^n \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of constant rank $k$ for $1 \leq k \leq n - 1$. By that we mean that its second fundamental form $A$ has constant rank $k$ or, equivalently, that the relative nullity subspaces, i.e., the kernels of its second form $\Delta(x) = \ker A(x)$, satisfy $\dim \Delta(x) = n - k$ at any $x \in M^n$. In this situation, it is a standard fact that the tangent distribution $x \mapsto \Delta(x)$ is integrable and that its totally geodesic leaves are mapped by $f$ into open subsets of affine subspaces of $\mathbb{R}^{n+1}$.

A hypersurface of constant rank can be locally parametrized in terms of the image of its Gauss map $N$ and its support function $\gamma = \langle f, N \rangle$. This parametrization is known as the Gauss parametrization and was described in [7] but it was already used by Sbrana in [10] and [11] long before.

Let $(g, \gamma)$ be a pair formed by an isometric immersion $g : L^k \rightarrow S^n$ into the unit sphere and a function $\gamma \in C^\infty(L)$. Denote by $\bar{\pi} : \Lambda \rightarrow L^{n-k}$ the normal bundle of $g$ and set $h = i \circ g$ where $i : S^n \rightarrow \mathbb{R}^{n+1}$ is the standard inclusion. It was shown in [7] that the map $\psi : \Lambda \rightarrow \mathbb{R}^{n+1}$ given by

$$\psi(x, w) = \gamma(x) h(x) + h_\ast \text{grad} \gamma(x) + i_\ast w$$

parametrizes (at regular points) a hypersurface of constant rank $k$ such that the fibers of $\Lambda$ are identified with the leaves of the relative nullity foliation of $\psi$.

Conversely, any hypersurface $f : M^n \rightarrow \mathbb{R}^{n+1}$ of constant rank $k$ can be locally parametrized as above. In fact, let $U \subset M^n$ be an open saturated subset of leaves of relative nullity and let $\pi : U \rightarrow L^{n-k}$ denote the projection onto the quotient space. The Gauss map $N$ of $f$ induces an immersion $g : L^{n-k} \rightarrow S^n$ given by $g \circ \pi = N$. Moreover, since the support function $\langle f, N \rangle$ is constant along the relative nullity leaves, hence it induces a function $\gamma \in C^\infty(L)$. Now the Gauss parametrization allows to recover $f$ by means of the pair $(g, \gamma)$.

The next statement presents some basic properties of the Gauss parametrization.

**Proposition 2.** The following assertions hold:

(i) The map $\psi : \Lambda \rightarrow \mathbb{R}^{n+1}$ is regular at $(x, w)$ if and only if the self adjoint operator

$$P_w(x) = \gamma(x) I + \text{Hess} \gamma(x) - A_w$$

on $T_x L$ is nonsingular. Here $A_w$ is the shape operator of $g$ with respect to $w$.

(ii) The map $\psi$ when restricted to the open subset $V$ of regular points is an immersed hypersurface having the map $N : \Lambda \rightarrow S^n$ given by $N(x, w) = g(x)$ as a Gauss map of rank $k$.
(iii) If \((x, w) \in V\) there is \(j(x, w): T_xL \to T_{(x, w)}\Lambda\) so that \(j: T_xL \to \Delta^+(x, w) \subset T_{(x, w)}\Lambda\) is an isometry such that

\[
h_* = \psi_* \circ j, \quad P_w^{-1} = \bar{\pi}_* \circ j \quad \text{and} \quad A \circ j = -j \circ P_w^{-1}
\]

where \(A\) is the shape operator of \(\psi\) at \((x, w)\) with respect to \(N\).

**Proof:** See [7].

### 1.2 Envelopes of hyperplanes

Let \(P_u: \mathbb{R}^n \to \mathbb{R}^{n+1}, n \geq 2\), denote a smooth \(k\)-parameter family of affine hyperplanes parametrized by \(u = (u_1, \ldots, u_k)\) on an open subset \(U \subset \mathbb{R}^k\) with \(1 \leq k \leq n - 1\).

We say that a hypersurface \(f: M^n \to \mathbb{R}^{n+1}\) is the **envelope of hyperplanes** of \((P_u)_{u \in U}\) if there exists a smooth totally geodesic foliation of \(M^n\) by leaves \((L_u)_{u \in U}\) of dimension \(n - k\) parametrized by an embedding \(h: U \to M^n\) transversal to the foliation and embeddings \(j_u: L_u \to P_u(\mathbb{R}^n)\) as open subset of \((n - k)\)-dimensional affine subspaces of \(\mathbb{R}^{n+1}\) such that \(j_u = f|_{L_u}\) and \(P_u(\mathbb{R}^n) = f_* T_{h(u)} M\) for any \(u \in U\).

Clearly, the leaves of \((L_u)_{u \in U}\) are contained in the relative nullity of \(f\). Also notice that any hypersurface \(f: M^n \to \mathbb{R}^{n+1}\) with constant index of relative nullity \(\nu = n - k\) is the envelope of the \(k\)-parameter family of tangent hyperplanes.

A \(k\)-parameter family of affine hyperplanes \((P_u)_{u \in U}\) can be given in terms of a smooth family of equations of the form

\[
G(u) = \varphi_1 x_1 + \cdots + \varphi_{n+1} x_{n+1} - \varphi_0 = 0
\]

where \(\varphi_j \in C^\infty(U), 0 \leq j \leq n + 1\), and \(x = (x_1, \ldots, x_{n+1})\) are coordinates in \(\mathbb{R}^{n+1}\) with respect to a canonical base.

Assume that the map \(\varphi = (\varphi_1, \ldots, \varphi_{n+1}): U \to \mathbb{R}^{n+1}\) is an immersion and without loss of generality that \(0 \notin \varphi(U)\). Let \(g: U \to S^n \subset \mathbb{R}^{n+1}\) and \(\gamma \in C^\infty(U)\) be given by

\[
g = \frac{1}{\phi}(\varphi_1, \ldots, \varphi_{n+1}) \quad \text{and} \quad \gamma = \frac{\varphi_0}{\phi}
\]

where \(\phi^2 = \sum_{j=1}^{n+1} \varphi_j^2\).

Being \(g\) is an immersion, the pair \((g, \gamma)\) gives a hypersurface \(f: M^n \to \mathbb{R}^{n+1}\) by means of the Gauss parametrization. Clearly \(f\) is the envelope of \((P_u)_{u \in U}\) and the leaves \((L_u)_{u \in U}\) of the envelope coincide with the relative nullity foliation of \(f\). Moreover, the envelope of \((P_u)_{u \in U}\) can be locally given as the solution of the system of equations

\[
(R) \begin{cases}
G(u) = 0 \\
G_{u_j}(u) = 0, \quad j = 1, \ldots, k.
\end{cases}
\]

We have shown the following fact.
Proposition 3. Any hypersurface $f: M^n \to \mathbb{R}^{n+1}$ of constant rank $k$ can be locally given as the envelope of a smooth family of affine hyperplanes

$$G(u) = \varphi_1 x_1 + \cdots + \varphi_{n+1} x_{n+1} - \varphi_0 = 0$$

where $(\varphi_1, \ldots, \varphi_{n+1}): U \to \mathbb{R}^{n+1}$ is an immersion of an open subset $U$ of $\mathbb{R}^k$. Then $f$ is locally the solution of the system of equations $(R)$.

2 A class of surfaces

A surface $g: L^2 \to S^n$ in the unit sphere is called hyperbolic (respectively, elliptic) if there exists a tensor $J$ on $L^2$ satisfying $J^2 = I$ and $J \neq I$ (respectively, $J^2 = -I$) and such that the second fundamental form $\alpha_g: TL \times TL \to N_g L$ of $g$ satisfies

$$\alpha_g(JX,Y) = \alpha_g(X,JY)$$

for all vector fields $X, Y \in \mathfrak{X}(L)$. Local coordinates $(u, v)$ on $L^2$ are called real-conjugate for $g$ if the condition

$$\alpha_g(\partial_u, \partial_v) = 0$$

holds where $\partial_u = \partial/\partial u$ and $\partial_v = \partial/\partial v$. They are called complex-conjugate if the condition $\alpha_g(\partial, \bar{\partial}) = 0$ holds where $\partial = \partial_z = (1/2)(\partial_u - i \partial_v)$, that is, if we have that

$$\alpha_g(\partial_u, \partial_u) + \alpha_g(\partial_v, \partial_v) = 0.$$

A simple argument (see [5]) gives the following result.

Proposition 4. Let $g: L^2 \to S^n$ be a hyperbolic (respectively, elliptic) surface. Then there exists locally a real-conjugate (respectively, complex-conjugate) system of coordinates on $L^2$ for $g$. Conversely, if there exists real-conjugate (respectively, complex-conjugate) coordinates on $L^2$, then $g$ is a hyperbolic (respectively, elliptic) surface.

Let $g: L^2 \to S^n$ be a simply-connected surface that carries a real-conjugate system of coordinates $(u, v)$. Equivalently, the isometric immersion $h = i \circ g: L^2 \to \mathbb{R}^{n+1}$ satisfies

$$h_{uv} - \Gamma^1_u h_v - \Gamma^2_v h_u + Fh = 0$$

where $\Gamma^1, \Gamma^2$ are the Christoffel symbols given by

$$\nabla_{\partial_u} \partial_v = \Gamma^1_u \partial_u + \Gamma^2_v \partial_v$$

and $F = \langle \partial_u, \partial_v \rangle$. 

6
We are interested in surfaces for which, in addition, the following system of differential equations admits solution:

\[ d\mu + 2\mu \omega = 0 \quad \text{where} \quad \omega = \Gamma^2 du + \Gamma^1 dv. \]  

(4)

This is the case if and only if the integrability condition

\[ \Gamma^1_u = \Gamma^2_v \]  

(5)

is satisfied.

**Proposition 5.** Let \( g: L^2 \to S^n \) be a hyperbolic surface with real conjugate coordinates \((u, v)\) such that the induced metric satisfies condition (5). Then there is a positive function \( \mu \in C^\infty(L) \) such that \( \varphi \) is a solution of (3) if and only if \( \psi = \sqrt{\mu} \varphi \) is a solution of

\[ \psi_{uv} + M\psi = 0 \]

where \( M \in C^\infty(L) \) is given by

\[ M = F - \frac{\mu_{uv}}{2\mu} + \frac{\mu_u\mu_v}{4\mu^2}. \]  

(6)

In particular, the immersion \( k = \sqrt{\mu} h: L^2 \to \mathbb{R}^{n+1} \) satisfies

\[ k_{uv} + Mk = 0. \]  

(7)

Conversely, let \( k: L^2 \to \mathbb{R}^{n+1} \) be an isometric immersion that for a system of coordinates \((u, v)\) satisfies (7) where \( M \in C^\infty(L) \). Then \((u, v)\) are real conjugate coordinates for the immersion \( g = (1/\|k\|) k: L^2 \to S^n \) and condition (5) is satisfied for the induced metric.

**Proof:** If \( g \) satisfies the integrability condition, then

\[ \Gamma^1 = -\frac{\mu_v}{2\mu}, \quad \Gamma^2 = -\frac{\mu_u}{2\mu} \]

where \( \mu = ce^{-2f_\omega} \) for any \( c \in \mathbb{R}_+ \). Hence (3) becomes

\[ h_{uv} + \frac{\mu_v h_u}{2\mu} + \frac{\mu_u h_v}{2\mu} + Fh = 0. \]  

(8)

It follows easily that \( k = \sqrt{\mu} h \) takes the form (7) where \( M \) is given by (6). The converse is a straightforward computation.

Let \( g: L^2 \to S^n \) be a simply-connected surface endowed with complex-conjugate coordinates \((z, \bar{z})\). Equivalently, the isometric immersion \( h = i \circ g: L^2 \to \mathbb{R}^{n+1} \) satisfies

\[ h_{zz} - \Gamma h_z - \bar{\Gamma} h_{\bar{z}} + Fh = 0 \]  

(9)
where the Christoffel symbols, obtained using the \( \mathbb{C} \)-linear extensions of the metric of \( L^2 \) and the corresponding connection, are given by
\[
\nabla_\partial \bar{\partial} = \Gamma \partial + \bar{\Gamma} \bar{\partial}
\]
and \( F = \langle \partial, \bar{\partial} \rangle \).

We are interested in surfaces for which, in addition, the following system of differential equations for \( \mu \) real admits solutions:
\[
\mu_z + 2\mu \Gamma = 0. \tag{10}
\]
This is the case if and only if the integrability condition
\[
\Gamma_z = \bar{\Gamma}_\bar{z}, \tag{11}
\]
that is, \( \Gamma_z \) is real, is satisfied.

**Proposition 6.** Let \( g: L^2 \to S^n \) be an elliptic surface with complex conjugate coordinates \((z, \bar{z})\) such that the induced metric satisfies condition \((11)\). Then, there is a positive solution \( \mu \in C^\infty (L) \) of \((10)\) such that \( \varphi \) is a solution of \((9)\) if and only if \( \psi = \sqrt{\mu} \varphi \) is a solution of
\[
\psi_{z\bar{z}} + M \psi = 0 \tag{12}
\]
where \( M \in C^\infty (L) \) is given by
\[
M = F - \frac{\mu_z \bar{z}}{2\mu} + \frac{\mu_z \mu_{\bar{z}}}{4\mu^2}. \tag{12}
\]
In particular, the immersion \( k = \sqrt{\mu} h: L^2 \to \mathbb{R}^{n+1} \) satisfies
\[
k_{z\bar{z}} + Mk = 0. \tag{13}
\]

Conversely, let \( k: L^2 \to \mathbb{R}^{n+1} \) be an isometric immersion that for a system of coordinates \((z, \bar{z})\) satisfies \((13)\) where \( M \in C^\infty (L) \). Then \((z, \bar{z})\) are complex conjugate coordinates for the immersion \( g = (1/\|k\|)k: L^2 \to S^n \) and condition \((11)\) is satisfied for the induced metric.

**Proof:** We have \( \mu = c e^{-2\int \omega} \) for any \( c \in \mathbb{R}_+ \) where \( \omega = \Gamma d\bar{z} \). Then \((9)\) takes the form
\[
h_{z\bar{z}} + \frac{\mu_z}{2\mu} h_z + \frac{\mu_{\bar{z}}}{2\mu} h_{\bar{z}} + F h = 0.
\]
It follows easily that \( k = \sqrt{\mu} h \) is as in \((13)\) where \( M \) is given by \((12)\). The converse is a straightforward computation. \( \blacksquare \)
3 The main result

After introducing the necessary terminology and definitions, we present the main result of the paper in terms of the Gauss parametrization, as is the case in the paper by Sbrana. The proof of the alternative version of the theorem in terms of env elopes of hyperplanes given in the introduction can easily be obtained from this version using results from the preceding sections.

By a variation $F$ of an isometric immersion $f: M^n \to \mathbb{R}^{n+1}$ we mean a smooth map $F: (-\epsilon, \epsilon) \times M^n \to \mathbb{R}^{n+1}$ such that $f_t = F(t, \cdot)$ is an immersion for each $t \in I = (-\epsilon, \epsilon)$ and $f = f_0$. The variational vector field of $F$ is the section $\mathcal{T} \in \Gamma(f^*(T\mathbb{R}^{n+1}))$ of the Riemannian vector bundle $f^*(T\mathbb{R}^{n+1})$ defined as

$$\mathcal{T}(x) = F_*\partial/\partial t|_{t=0}(x).$$

A variation $F$ of a given isometric immersion $f: M^n \to \mathbb{R}^{n+1}$ is called an isometric bending if $f_t$ is an isometric immersion for any $t \in I$. The variational vector field of an isometric bending satisfies

$$\langle \tilde{\nabla}_X \mathcal{T}, f_*Y \rangle + \langle f_*X, \tilde{\nabla}_Y \mathcal{T} \rangle = 0$$

for any $X, Y \in \mathfrak{X}(M)$. Equivalently, it satisfies that

$$\langle \tilde{\nabla}_X \mathcal{T}, f_*X \rangle = 0$$

for any $X \in \mathfrak{X}(M)$.

An isometric bending $F$ is called trivial if it is produced by a smooth one-parameter family of isometries of $\mathbb{R}^{n+1}$, that is, if there exist a smooth family $C: I \to O(n + 1)$ of orthogonal transformations of $\mathbb{R}^{n+1}$ and a smooth map $v: I \to \mathbb{R}^{n+1}$ such that

$$F(t, x) = C(t)f(x) + v(t).$$

For a trivial isometric bending the variational vector field is of the form

$$\mathcal{T}(x) = Df(x) + w$$

where $D = C'(0)$ is a skew-symmetric linear endomorphism of $\mathbb{R}^{n+1}$ and $w = v'(0)$ a vector in $\mathbb{R}^{n+1}$. Conversely, given a skew-symmetric linear endomorphism $D$ of $\mathbb{R}^{n+1}$ and a vector $w \in \mathbb{R}^{n+1}$, the map

$$F(t, x) = e^{tD}f(x) + tw$$

defines a trivial isometric bending that has $\mathcal{T} = Df + w$ as variational vector field.
By an infinitesimal bending $\mathcal{T}$ of an isometric immersion $f: M^n \to \mathbb{R}^{n+1}$ we mean an element of $\Gamma(f^*(T\mathbb{R}^{n+1}))$ that satisfies
\[ \langle \tilde{\nabla}_X \mathcal{T}, f_*Y \rangle + \langle f_*X, \tilde{\nabla}_Y \mathcal{T} \rangle = 0 \] (14)
for any $X,Y \in \mathfrak{X}(M)$. An infinitesimal bending is said to be trivial if
\[ \mathcal{T}(x) = \mathcal{D}f(x) + w \]
where $\mathcal{D}$ is a skew-symmetric linear endomorphism of $\mathbb{R}^{n+1}$ and $w \in \mathbb{R}^{n+1}$.

An isometric immersion $f: M^n \to \mathbb{R}^{n+1}$ is called infinitesimally bendable if it admits a nontrivial infinitesimal bending. Otherwise, it is said that $f$ is infinitesimally rigid.

Multiplying a given infinitesimal bending by a real constant and adding a trivial infinitesimal bending yields a new infinitesimal bending. In the sequel, we identify two infinitesimal bendings $\mathcal{T}_1$ and $\mathcal{T}_2$ if $\mathcal{T}_2 = \mathcal{T}_0 + c \mathcal{T}_1$ where $\mathcal{T}_0$ is a trivial infinitesimal bending and $0 \neq c \in \mathbb{R}$.

We have already observed that hypersurfaces of rank at least three at any point are infinitesimally rigid. Therefore, the interesting case to be considered is the one of constant rank two. We see next that even in this special case hypersurfaces are “generically” infinitesimally rigid.

We call the pair $(g, \gamma)$ a special hyperbolic pair (respectively, special elliptic pair) if $g: L^2 \to S^n$ is a hyperbolic (respectively, elliptic) surface so that system (4) (respectively, system (10)) has solution and $\gamma \in C^\infty(L)$ satisfies (3) (respectively, (9)).

**Theorem 7.** Let $f: M^n \to \mathbb{R}^{n+1}, n \geq 3$, be an infinitesimally bendable hypersurface of constant rank two that is neither surface-like nor ruled on any open subset of $M^n$. Then, there is an open and dense subset of $M^n$ such that along any connected component $f$ is parametrized in terms of the Gauss parametrization by a special hyperbolic or a special elliptic pair.

Conversely, any hypersurface parametrized in terms of the Gauss parametrization by a special hyperbolic or special elliptic pair admits locally a unique infinitesimal bending.

The case of ruled hypersurfaces that has been excluded from consideration in the above result is rather simple and will be treated separately in Section 5.

**4 Existence and uniqueness**

We study the system of differential equations of an infinitesimal bending of a Euclidean hypersurface and discuss its integrability conditions. This yields a kind of fundamental theorem for infinitesimal bendings that is, basically, contained in Sbrana’s paper [10].
In fact, the case of arbitrary codimension was later taken on by Schouten [12] but presented in a rather difficult terminology. We point out that in this section some long but straightforward computations are only indicated.

Given a hypersurface \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3 \), in the sequel we associate to any infinitesimal bending \( \mathcal{T} \) of \( f \) the variation \( F: \mathbb{R} \times M^n \to \mathbb{R}^{n+1} \) with variational vector field \( \mathcal{T} \) given by

\[
F(t, x) = f(x) + t\mathcal{T}(x).
\]

It is usually said that \( f_t = F(t, \cdot) \) is isometric to \( f \) up to first order for if

\[
\| f_t X \|^2 = \| f_* X \|^2 + t^2 \| \tilde{\nabla}_X \mathcal{T} \|^2
\]

for all \( X \in \mathfrak{X}(M) \).

Let \( g_t \) be the metric on \( M^n \) induce by \( f_t \). Then,

\[
\frac{\partial}{\partial t} \bigg|_{t=0} g_t(X, Y) = 0
\]

for all \( X, Y \in \mathfrak{X}(M) \). Consequently, we have that the associated one-parameter family of Levi-Civita connections and the corresponding family of curvature tensors satisfy

\[
\frac{\partial}{\partial t} \bigg|_{t=0} f_* \nabla^t X Y = 0
\]

and

\[
\frac{\partial}{\partial t} \bigg|_{t=0} \langle R^t(X, Y) Z, W \rangle = 0
\]

for all \( X, Y, Z, W \in \mathfrak{X}(M) \).

Let \( N(t) \) denote a Gauss map of \( f_t \) and \( A(t) \) the second fundamental form of \( f_t \) with respect to \( N(t) \) so that the map \( t \in \mathbb{R} \mapsto N(t) \) is smooth. Then \( N = N(0) \) is the Gauss map and \( A = A(0) \) is the second fundamental form of \( f \). Moreover, let us define \( L \in \Gamma(\text{End}(TM, f^*(T\mathbb{R}^{n+1})) \) by

\[
LX = \tilde{\nabla}_X \mathcal{T} = \mathcal{T}_X.
\]

Then (14) can be written as

\[
\langle LX, f_* Y \rangle + \langle f_* X, LY \rangle = 0
\]

for all \( X, Y \in \mathfrak{X}(M) \).

**Lemma 8.** We have that \( \mathcal{Y} = \frac{\partial}{\partial t} \bigg|_{t=0} N(t) \in \Gamma(f^*(T\mathbb{R}^{n+1})) \) satisfies

\[
\langle \mathcal{Y}, N \rangle = 0
\]

and

\[
\langle \mathcal{Y}, f_* X \rangle + \langle LX, N \rangle = 0
\]

for all \( X \in \mathfrak{X}(M) \).
Proof: The derivative with respect to $t$ at $t = 0$ of $\langle N(t), N(t) \rangle = 1$ gives \[18\] whereas of $\langle N(t), f_\ast X \rangle = 0$ yields \[19\].

Lemma 9. We have that $B = \partial/\partial t|_{t=0} A(t) \in \Gamma(\text{End}(TM))$ is symmetric and satisfies

\[ \nabla_X L(Y) = \langle BX, Y \rangle N + \langle AX, Y \rangle \mathcal{Y} \tag{20} \]

and

\[ \mathcal{Y}_\ast X = -f_\ast BX - LAX \tag{21} \]

for all $X, Y \in \mathfrak{X}(M)$.

Proof: The derivative with respect to $t$ at $t = 0$ of the Gauss formula

\[ \nabla_X f_\ast Y = f_\ast \nabla_X^t Y + g_t(A(t)X, Y)N(t) \]

easily gives \[20\]. As for the Weingarten formula

\[ \nabla_X N(t) = -f_\ast(A(t)X) \]

we have that its derivative at $t = 0$ yields \[21\].

If $T = \mathcal{D}f + w$ is a trivial infinitesimal bending then $L = \mathcal{D} \circ f_\ast$. It follows that $\mathcal{Y} = \mathcal{D}N$ and that $B = 0$ since

\[ \langle BX, Y \rangle = \langle (\nabla_X L)Y, N \rangle = \langle (\nabla_X \mathcal{D})Y, N \rangle = 0 \]

for all $X, Y \in \mathfrak{X}(M)$.

Proposition 10. The tensor $B$ is a symmetric Codazzi tensor, i.e.,

\[ (\nabla_X B)Y - (\nabla_Y B)X = 0 \tag{22} \]

such that

\[ BX \wedge AY - BY \wedge AX = 0 \tag{23} \]

for all $X, Y \in \mathfrak{X}(M)$.

Proof: The derivative at $t = 0$ of the Codazzi equation

\[ (\nabla_X^t A(t))Y = (\nabla_Y^t A(t))X \]

gives \[22\]. To obtain \[23\] we compute the derivative at $t = 0$ of the Gauss equation

\[ R^t(X, Y)Z = g(t)(A(t)Y, Z)A(t)X - g(t)(A(t)X, Z)A(t)Y \]

and use \[16\].

The next result is to be expected bearing in mind the nature of the Gauss and Codazzi equations as the integrability conditions for the system of differential equations associated to an isometric immersion as a hypersurface.
Lemma 11. Equations (22) and (23) are the integrability conditions of the system of differential equations (20) and (21) for \( L \) and \( Y \), that is,

\[
\begin{cases}
\mathcal{Y}_X Y = -LAX - f_s BX \\
(\tilde{\nabla}_X L)Y = \langle BX, Y \rangle N + \langle AX, Y \rangle \mathcal{Y}.
\end{cases}
\]

Proof: For the first equation, we have to show that

\[
\tilde{\nabla}_X \mathcal{Y}_Y Y - \tilde{\nabla}_Y \mathcal{Y}_X Y - \tilde{\nabla}_{[X,Y]} Y = 0
\]

for all \( X, Y \in \mathcal{X}(M) \). One has that

\[
\tilde{\nabla}_X \mathcal{Y}_Y Y = -(\tilde{\nabla}_X L)AY - L(\nabla_X A)Y - LA\nabla_X Y - f_s \nabla_X BY - \langle AX, BY \rangle N.
\]

Then (24) is equivalent to

\[
(\tilde{\nabla}_X L)AY - (\tilde{\nabla}_Y L)AX + f_s ((\nabla_X B)Y - (\nabla_Y B)X) + ((AX, BY) - (AY, BX)) N = 0.
\]

Replacing the first two terms by the use of the second equation in \((S)\) it is easily seen that (22) follows from (22).

It is easy to see that the integrability condition for the second equation is

\[
(\tilde{\nabla}_X \tilde{\nabla}_Y L - \tilde{\nabla}_Y \tilde{\nabla}_X L - \tilde{\nabla}_{[X,Y]} L)Z = -LR(X,Y) Z
\]

for all \( X, Y, Z, W \in \mathcal{X}(M) \). A straightforward computation using (20) gives

\[
(\tilde{\nabla}_X \tilde{\nabla}_Y L - \tilde{\nabla}_Y \tilde{\nabla}_X L - \tilde{\nabla}_{[X,Y]} L)Z = -(BY, Z) f_s AX - (AY, Z)(LA + f_s BX) + (AX, Z) f_s BX.
\]

That \( A \) is a Codazzi tensor together with (22) yields

\[
(\tilde{\nabla}_X \tilde{\nabla}_Y L - \tilde{\nabla}_Y \tilde{\nabla}_X L - \tilde{\nabla}_{[X,Y]} L)Z = -\langle BY, Z \rangle f_s AX - \langle AX, Z \rangle (LA + f_s BX)
\]

On the other hand, we have

\[
LR(X,Y) Z = \langle AY, Z \rangle LAX - \langle AX, Z \rangle LAY,
\]

and (25) follows using (23).

Next we consider the case of hypersurfaces of constant rank two.

Corollary 12. If \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3 \), is an infinitesimally bendable hypersurface of constant rank two, then \( \Delta \subset \ker \mathcal{B} \).
Proof: This follows easily from (23).

**Theorem 13.** Let $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a simply-connected hypersurface of constant rank two. Then, the set of all symmetric Codazzi tensors $B \in \Gamma(\text{End}(TM))$ such that $\Delta \subset \ker B$ and

$$BX \wedge AY - BY \wedge AX = 0$$

for all $X, Y \in \mathfrak{X}(M)$ is in one-to-one correspondence with the set of all infinitesimal bendings of $f$ so that $B = 0$ corresponds to the trivial one.

Proof: Given $B \in \Gamma(\text{End}(TM))$ as in the statement, we first prove that there exists a solution $Y$ and $L$ of system $(S)$ such that (17), (18) and (19) are satisfied. In particular, this gives the existence of an infinitesimal bending $\mathcal{T}$ such that $L = \mathcal{T}$. To see this, observe that by (20) the one-form $\omega = \langle L, v \rangle$ is closed for any $v \in \mathbb{R}^{n+1}$.

Given a solution $Y$ and $L$ of $(S)$, we define a smooth function by

$$\tau = \langle Y, N \rangle,$$

a smooth one-form by

$$\theta(X) = \langle Y, f_* X \rangle + \langle LX, N \rangle$$

and a smooth symmetric bilinear tensor by

$$\beta(X, Y) = \langle LX, f_* Y \rangle + \langle LY, f_* X \rangle.$$

A straightforward calculation gives that

$$d\tau = -\theta \circ A, \quad (26)$$

$$(\nabla_X \theta)Y = -\beta(AX, Y) + 2\tau \langle AX, Y \rangle \quad (27)$$

and

$$(\nabla_Z \beta)(X, Y) = \langle AX, Y \rangle \theta(Z) + \langle AX, Z \rangle \theta(Y) \quad (28)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

We claim that the system of differential equations formed by (26), (27) and (28) is completely integrable. The integrability condition for the first equation is easy to verify. For the second equation, we have to see that

$$(\nabla_X \nabla_Y \theta - \nabla_Y \nabla_X \theta - \nabla_{[X,Y]} \theta)Z = -\theta(R(X, Y)Z) \quad (29)$$

holds. Using (26) and (27) we obtain

$$(\nabla_X \nabla_Y \theta)Z = -(\nabla_X \beta)(AY, Z) - \beta(\nabla_X AY, Z) - 2\theta(AX) \langle AY, Z \rangle + 2\tau \langle \nabla_X AY, Z \rangle.$$
Hence
\[
(\nabla_X \nabla_Y \theta - \nabla_Y \nabla_X \theta - \nabla_{[X,Y]} \theta)Z = - (\nabla_X \beta)(AY, Z) + (\nabla_Y \beta)(AX, Z) - 2\theta(AX)(AY, Z) + 2\theta(AY)(AX, Z).
\]
Using (28) we obtain that
\[
(\nabla_X \nabla_Y \theta - \nabla_Y \nabla_X \theta - \nabla_{[X,Y]} \theta)Z = -\theta(AX)(AY, Z) + \theta(AY)(AX, Z).
\]
On the other hand, we have from the Gauss equation that
\[
\theta(R(X, Y)Z) = \langle AX, Z \rangle \theta(AX) - \langle AX, Z \rangle \theta(AY),
\]
and (29) follows.

Finally, the integrability condition for the last equation, namely, that
\[
(\nabla_X \nabla_Y \beta - \nabla_Y \nabla_X \beta - \nabla_{[X,Y]} \beta)(Z, W) = -\beta(R(X, Y)Z, W) - \beta(R(X, Y)W, Z) \quad (30)
\]
can be verified by a similar computation, and this proves the claim.

Start with a solution \( L^* \) and \( Y^* \) of system \( (S) \) with corresponding tensors \( \theta^*, \beta^* \) and function \( \tau^* \). Fix a point \( p_0 \in M^n \) and let \( L_0 \) and \( Y_0 \) be a solution of the integrable system
\[
(S_0) \begin{cases} 
Y_*, X = -LAX \\
(\nabla_X L)Y = \langle AX, Y \rangle \mathcal{Y}
\end{cases}
\]
with initial conditions \( \theta_0(p_0) = \theta^*(p_0), \beta_0(p_0) = \beta^*(p_0) \) and \( \tau_0(p_0) = \tau^*(p_0) \). Then \( L = L^* - L_0 \) and \( Y = Y^* - Y_0 \) are a solution of \( (S) \) such that \( \theta = \theta^* - \theta_0, \beta = \beta^* - \beta_0 \) and \( \tau = \tau^* - \tau_0 \). Clearly \( \theta(p_0) = \beta(p_0) = \tau(p_0) = 0 \). Since \( \theta, \beta \) and \( \tau \) solve the homogeneous integrable system (26), (27) and (28), hence \( \theta = \beta = \tau = 0 \).

Given any two pairs \( L_j, Y_j \), obtained as above, let \( T_j, 1 \leq j \leq 2 \), be the associated infinitesimal bendings. It remains to show that \( T = T_1 - T_2 \) is a trivial infinitesimal bending.

We have that the pair \( L = L_1 - L_2, Y = Y_1 - Y_2 \) satisfies \( (S_0) \) as well as (17), (18) and (19). Fix \( p_0 \in M^n \) and define a skew-symmetric linear endomorphism \( \mathcal{C} \) of \( \mathbb{R}^{n+1} \) by
\[
\mathcal{C} f_*(p_0)X = L(p_0)X \quad \text{and} \quad \mathcal{C} N(p_0) = Y(p_0)
\]
and a vector \( v \in \mathbb{R}^{n+1} \) by \( v = T(p_0) - \mathcal{C} f(p_0) \). Consider the trivial infinitesimal bending \( \tilde{T} = \mathcal{C} f + v \) and \( \tilde{Y} = \mathcal{C} N \). Then, the pair \( \tilde{L} \) and \( \tilde{Y} \) satisfies \( (S_0) \). Thus, also the pair \( L^* = L - \tilde{L}, Y^* = Y - \tilde{Y} \) solves system \( (S_0) \). Moreover, \( T^*(p_0) = 0, Y^*(p_0) = 0 \) and \( L^*(p_0) = L(p_0) - \tilde{L}(p_0) = 0 \). Thus \( T^* = 0 \) and hence \( T = \tilde{T} \). \( \blacksquare \)
5 The proof of Theorem \[7\]

In the sequel, let \( f : M^n \to \mathbb{R}^{n+1}, n \geq 3 \), be a hypersurface of constant rank two. Recall that the splitting tensor \( C : \Gamma(\Delta) \to \Gamma(\text{End}(\Delta^\perp)) \) is defined by

\[
C_T X = - (\nabla_X T)_{\Delta^\perp}
\]

for any \( T \in \Gamma(\Delta) \) and \( X \in \mathfrak{X}(M) \). From the Codazzi equation, it follows that

\[
\nabla_T A = AC_T = C_T^t A \tag{31}
\]

for any \( T \in \Gamma(\Delta) \).

**Proposition 14.** Assume that the splitting tensor at any point satisfies \( C_T \in \text{span}\{I\} \) for any \( T \in \Delta \), where \( I \) denotes the identity section of \( \text{End}(\Delta^\perp) \). Then \( f \) is surface-like.

**Proof:** See Lemma 6 in [5]. \( \blacksquare \)

Assume further that \( f \) is infinitesimally bendable. Locally and because of the rank assumption, there is an orthonormal tangent frame spanning \( \Delta^\perp \) such that

\[
A|_{\Delta^\perp} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}. \tag{32}
\]

**Lemma 15.** If \( B \neq 0 \) at any point of \( M^n \), then

\[
B|_{\Delta^\perp} = \begin{bmatrix}
\lambda \lambda_1 & b \\
b & -\lambda \lambda_2
\end{bmatrix}. \tag{33}
\]

**Proof:** By Corollary 12 we have that \( \Delta \subset \ker B \). Now (33) follows easily from (23). \( \blacksquare \)

**Lemma 16.** We have that \( D = (A|_{\Delta^\perp})^{-1} B|_{\Delta^\perp} \in \Gamma(\text{End}(\Delta^\perp)) \) satisfies:

(i) \( [D, C_T] = 0 \) for all \( T \in \Delta \),

(ii) \( \nabla_T D = 0 \) for all \( T \in \Delta \),

(iii) \( \text{tr} D = 0 \),

(iv) \( T(\text{det} D) = 0 \) for all \( T \in \Delta \).

**Proof:** We denote \( A = A|_{\Delta^\perp} \) and \( B = B|_{\Delta^\perp} \). From (31) we obtain \( \nabla_T B = BC_T \). Hence

\[
BC_T = C_T^t B.
\]
We have using (31) that
\[ ADC_T = BC_T = C_T^TB = C_T^TAD = AC_TD, \]
and (i) follows. We have
\[ A\nabla_T D = \nabla_T(AD) - (\nabla_T A)D = \nabla_TB - (\nabla_T A)D = BC_T - AC_TD \]
\[ = BC_T - C_T^TAD = BC_T - C_T^TB = 0, \]
and this yields (ii). We obtain from (32) and (33) that
\[ D = \begin{bmatrix} \lambda & b/\lambda_1 \\ b/\lambda_2 & -\lambda \end{bmatrix}, \]
which gives (iii). Now part (iv) follows from (ii) and (iii).

**Proposition 17.** Assume that \( f : M^n \to \mathbb{R}^{n+1} \), \( n \geq 3 \), is not surface-like on any open subset of \( M^n \). Then \( f \) is ruled along any open subset where \( D \neq 0 \) satisfies \( \det D = 0 \).

**Proof:** By Lemma 16 there is an orthogonal frame \( X,Y \) of \( \Delta^\perp \) with \( Y \) of unit length such that \( DY = 0 \) and \( DX = Y \). We claim that \( f \) is ruled by the integral leaves of the distribution \( \Delta \oplus \text{span}\{Y\} \). To see this, we have to show that
\[ (i) \langle AY,Y \rangle = 0, \quad (ii) \nabla_T Y = 0, \quad (iii) \langle \nabla_Y T, X \rangle = 0 \quad \text{and} \quad (iv) \langle \nabla_Y Y, X \rangle = 0 \]
for all \( T \in \Delta \). We have
\[ \langle AY,Y \rangle = \langle ADX,Y \rangle = \langle BX,Y \rangle = \langle BY,X \rangle = \langle ADY,X \rangle = 0. \]
Condition (ii) follows easily using \( \nabla_T D = 0 \). Since \( [D,C_T] = 0 \), we obtain
\[ \langle \nabla_Y T, X \rangle = -\langle C_T Y, X \rangle = -\langle C_T DX, X \rangle = -\langle DC_T X, X \rangle = 0. \]
We have that
\[ BY = ADY = 0 \quad \text{and} \quad BX = ADX = AY = \lambda X, \quad \lambda \neq 0, \]
and condition (iv) follows easily using (22).

In the sequel, we consider the case \( \det D \neq 0 \). By the above, this is always the case under the assumptions of Theorem 7.

By part (iii) of Lemma 16 the eigenvalues of \( D \) are the solutions of \( t^2 + \det D = 0 \). Therefore, on each connected component of an open subset of \( M^n \) either \( \det D < 0 \) and thus \( D \) has two smooth real eigenvalues \( \{\mu,-\mu\} \) or \( \det D > 0 \) and thus \( D \) has a pair of smooth complex eigenvalues \( \{i\mu,-i\mu\} \). Then \( J \in \Gamma(\text{End}(\Delta^\perp)) \) defined by
\[ D = \mu J \] (34)
satisfies \( J^2 = I \) in the first case and \( J^2 = -I \) in the second case.
Lemma 18. The eigenspaces of $D$ are parallel and the eigenvalues constant along the leaves of $\Delta$.

Proof: Follows from parts (ii) and (iv) of Lemma 16.

A hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank two is said to be hyperbolic (respectively, elliptic) if there exists $J \in \Gamma(\text{End}(\Delta^\perp))$ satisfying the following conditions:

(i) $J^2 = I$ and $J \neq I$ (respectively, $J^2 = -I$).

(ii) $\nabla_T J = 0$ for all $T \in \Gamma(\Delta)$.

(iii) $C_T \in \text{span}\{I, J\}$ for all $T \in \Gamma(\Delta)$.

Proposition 19. Assume that $f : M^n \to \mathbb{R}^{n+1}$ is neither surface-like nor ruled on any open subset $\tilde{M}^n$ of $M^n$. Then, there is an open and dense subset $\tilde{M}^n$ of $M^n$ such that the restriction of $f$ to any connected component of $\tilde{M}^n$ is either hyperbolic or elliptic.

Proof: Let $J \in \Gamma(\text{End}(\Delta^\perp))$ be defined by (34). The subspace $S$ of all elements in $\text{End}(\Delta^\perp)$ that commute with $D$, i.e., that commute with $J$, is $S = \text{span}\{I, J\}$. Thus condition (iii) in the above definition follows from part (i) of Lemma 16.

Given a submersion $\pi : M \to L$ between differentiable manifolds, then $X \in \mathfrak{X}(M)$ is said to be projectable if it is $\pi$-related to some $\bar{X} \in \mathfrak{X}(L)$, that is, if there exists $\bar{X} \in \mathfrak{X}(L)$ such that $\pi_*X = \bar{X} \circ \pi$.

In the sequel, we denote by $\pi : M^n \to L^2$ the submersion onto the (local) quotient space of leaves of $\Delta$, namely, onto $L^2 = M^n/\Delta$. A tensor $D \in \text{End}(\Delta^\perp)$ is said to be projectable with respect to $\pi$ if it is the horizontal lift of some tensor $\bar{D}$ on $L$. Clearly, $D$ is projectable with respect to $\pi$ if and only if for all $\bar{x} \in L$, $x, y \in \pi^{-1}(\bar{x})$, $v \in \Delta^\perp(x)$ and $w \in \Delta^\perp(y)$ with $\pi_*v = \pi_*w$, we have that $\pi_*Dv = \pi_* Dw$.

Lemma 20. Let $f : M^n \to \mathbb{R}^{n+1}$ be a hypersurface of rank two parametrized by a pair $(g, \gamma)$ in terms of the Gauss parametrization. If $f$ is hyperbolic (respectively, elliptic) with respect to $J \in \Gamma(\text{End}(\Delta^\perp))$ and $D = \mu J$ satisfies (i)–(iv) in Lemma 16, then $J$ and $D$ are the horizontal lifts of tensors $\bar{J}$ and $\bar{D} = \bar{\mu} \bar{J}$ on $L^2$ such that $\mu = \bar{\mu} \circ \pi$, $J^2 = \bar{J}$ (respectively, $J^2 = -\bar{J}$), the pair $(g, \gamma)$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$ and $\bar{D}$ satisfies:

(a) $\text{tr} \bar{D} = 0$,

(b) $\left(\nabla_X^\gamma \bar{D}\right) \bar{Y} - \left(\nabla_Y^\gamma \bar{D}\right) \bar{X} = 0$ for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$
where $\nabla'$ is the Levi-Civita connection of the metric induced by $g$.

Conversely, if the pair $(g, \gamma)$ is hyperbolic (respectively, elliptic) with respect to a tensor $\bar{J}$ on $L^2$ satisfying $\bar{J}^2 = \bar{I}$ (respectively, $\bar{J}^2 = -\bar{I}$), then the hypersurface $f$ is hyperbolic (respectively, elliptic) with respect to the horizontal lift $J$ of $\bar{J}$. In addition, the horizontal lift $D = \mu J$ of a tensor $\bar{D} = \bar{\mu} \bar{J}$, $\mu = \bar{\mu} \circ \pi$, satisfying $(a)$ and $(b)$ also fulfills the properties $(i)-(iv)$ in Lemma 16.

Proof: We have from parts $(i)$ and $(ii)$ of Lemma 16 and Corollary 13 in [6] that the tensor $D$ is projectable. Then part $(iii)$ of Lemma 16 gives $\text{tr} \bar{D} = \text{tr} D = 0$.

From part $(iv)$ of Lemma 16 we have that that $\text{det} D$ is projectable and from Lemma 18 that also $J$ is projectable. We have from the Gauss parametrization that

$$f_* AX = -N_* X = -h_* \pi_* X$$

where $h = i \circ g$. Hence,

$$f_* ADX = h_* \pi_* DX = -h_* \bar{D} \pi_* X$$

for any $X \in \mathfrak{X}(M)$. In particular,

$$f_* AD[X,Y] = -h_* \bar{D} \pi_* [X,Y] = -h_* \bar{D} [\pi_* X, \pi_* Y]$$

for any $X,Y \in \mathfrak{X}(M)$. Moreover,

$$f_* \nabla_X ADY = \tilde{\nabla}_X f_* ADY - \langle AX, ADY \rangle N$$

$$= -\nabla_{\pi_* X} h_* \bar{D} \pi_* Y - \langle h_* \pi_* X, h_* \bar{D} \pi_* Y \rangle h \circ \pi$$

$$= -h_* \nabla'_{\pi_* X} \bar{D} \pi_* Y - \alpha_g(\pi_* X, \bar{D} \pi_* Y) - \langle \pi_* X, \bar{D} \pi_* Y \rangle h \circ \pi$$

$$= -h_* \nabla'_{\pi_* X} \bar{D} \pi_* Y - \alpha_g(\pi_* X, \bar{D} \pi_* Y).$$

From (22) and the above, we have that

$$0 = f_*(\nabla_X B)Y - f_*(\nabla_Y B)X = f_* \nabla_X ADY - f_* \nabla_Y ADX - f_* AD[X,Y].$$

We conclude that part $(b)$ holds as well as

$$\alpha_g(\pi_* X, \bar{D} \pi_* Y) = \alpha_g(\bar{D} \pi_* X, \pi_* Y).$$

Since $\bar{D} \in \text{span}\{I, J\}$ but $\bar{D} \not\in \text{span}\{I\}$, the preceding equation is equivalent to

$$\alpha_g(\pi_* X, J \pi_* Y) = \alpha_g(J \pi_* X, \pi_* Y)$$

and thus $g$ is hyperbolic (respectively, elliptic) with respect to $\bar{J}$.

To deal with the function $\gamma$ we first show that condition (2) is equivalent to

$$(\text{Hess} h^v + h^v I)J = J^t (\text{Hess} h^v + h^v I)$$

19
where Hess $h^v$ is the endomorphism of $TL$ associated to the Hessian and $h^v = \langle h, v \rangle$ for any $v \in \mathbb{R}^{n+1}$. We have that the Hessian of $h^v$ satisfies

$$\text{Hess } h^v(X, Y) = \langle \alpha_h(X, Y), v \rangle = \langle i_*\alpha_g(X, Y) - \langle X, Y \rangle h, v \rangle$$

for all $X, Y \in \mathfrak{X}(L)$. Thus

$$\langle i_*\alpha_g(JX, Y) - i_*\alpha_g(X, JY), v \rangle = \langle ((\text{Hess } h^v + h^v I)J - J^t(\text{Hess } h^v + h^v I))X, Y \rangle$$

for all $X, Y \in \mathfrak{X}(L)$.

It remains to prove that

$$(\text{Hess } \gamma + \gamma I) \bar{J} = J^t(\text{Hess } \gamma + \gamma I). \tag{39}$$

By the Gauss parametrization, there exists a diffeomorphism $\theta : U \subset \Lambda \to M^n$ from an open neighborhood of the zero section of $\Lambda$ such that $\pi \circ \theta = \bar{\pi}$ and

$$f \circ \theta(x, w) = \gamma(x)h(x) + h^*\nabla \gamma(x) + w$$

for any $(x, w) \in \Lambda$. Let $j : T_xL \to T_{(x, w)}\Lambda$ be the linear isometry in Proposition 2. Then, for any $\bar{X}, \bar{Y} \in T_xL$ we obtain using (1) and (35) that

$$-\langle AD\theta_s j\bar{X}, \theta_s j\bar{Y} \rangle = -(f_*AD\theta_s j\bar{X}, f_*\theta_s j\bar{Y}) = \langle h_*\bar{D}\pi_s j\bar{X}, h_*\bar{Y} \rangle = \langle \bar{D}\pi_s j\bar{X}, \bar{Y} \rangle' = \langle \bar{D}P^{-1}_w \bar{X}, \bar{Y} \rangle'. \tag{40}$$

It follows that $\bar{D}P^{-1}_w = P^{-1}_w \bar{D}^t$, or equivalently, that $P_w \bar{D} = \bar{D}^t P_w$. And because $\bar{D} \in \text{span}\{I, \bar{J}\}$, this is equivalent to $P_w \bar{J} = \bar{J}^t P_w$. Moreover, using that $A_w \bar{J} = \bar{J}^t A_w$ as follows from (38), we conclude that (39) is satisfied. $\blacksquare$

**Lemma 21.** The following assertions on a surface $g : L^2 \to S^n$ are equivalent:

(i) The surface $g$ is hyperbolic (respectively, elliptic) with respect to a tensor $\bar{J}$ on $L^2$ satisfying $\bar{J}^2 = I$ (respectively, $\bar{J}^2 = -I$), and there is $\bar{D} = \bar{\mu}\bar{J}$, $\bar{\mu} > 0$, such that

(a) $\text{tr } \bar{D} = 0,$

(b) $\left(\nabla'_X \bar{D}\right) Y - \left(\nabla'_Y \bar{D}\right) X = 0$ for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$.

(ii) There exist real-conjugate (respectively, complex-conjugate) coordinates on $L^2$ such that system (4) (respectively, (10)) has solution.
Proof: We make use of Proposition 4. In the case of real coordinates \((u, v)\) and since we have \(\bar{D}\partial_u = \bar{\mu}\partial_u, \bar{D}\partial_v = -\bar{\mu}\partial_v\), we easily see that
\[
(\nabla'_{\partial_u} \bar{D}) \partial_v - (\nabla'_{\partial_v} \bar{D}) \partial_u = 0
\]
is equivalent to the system (4). The case of complex coordinates is similar. 

Proof of Theorem 7: By Proposition 19, on each connected component of an open and dense subset of \(M^n\) the hypersurface \(f\) is either hyperbolic or elliptic with respect to \(J \in \Gamma(\text{End}(\Delta^\perp))\). It follows from Lemma 16 that there exists \(D \in \Gamma(\text{End}(\Delta^\perp))\) satisfying the properties (i)–(iv).

Let \(f\) be parameterized by a pair \((g, \gamma)\) in terms of the Gauss parametrization. By Lemma 20 if \(f\) is hyperbolic (respectively, elliptic) with respect to \(J\), then \(J\) and \(D\) can be projected to tensors \(\bar{J}\) and \(\bar{D}\in \text{span}\{I, \bar{J}\}\) on \(L^2\), with \(\bar{J}^2 = I\) (respectively, \(\bar{J}^2 = -I\)). Moreover, the pair \((g, \gamma)\) is hyperbolic (respectively, elliptic) with respect to \(\bar{J}\) and \(\bar{D}\) satisfies (a) and (b) in Lemma 20. Now the proof of the direct statement follows from Lemma 21.

Conversely, let \(f: M^n \to \mathbb{R}^{n+1}\) be a simply-connected hypersurface parameterized in terms of the Gauss parametrization by a special hyperbolic or special elliptic pair \((g, \gamma)\). By Lemma 21 there exists \(D = \bar{\mu}\bar{J}\) satisfying equations (a) and (b). By Lemma 20 the hypersurface \(f\) is hyperbolic or elliptic with respect to the horizontal lift \(\bar{J}\) of \(J\), and the horizontal lift \(D = \mu\bar{J}\) of \(D\) satisfies (i)–(iv) in Lemma 16.

To conclude from Theorem 13 that \(f\) admits a unique nontrivial infinitesimal bending it remains to show that \(B \in \Gamma(\text{End}(TM))\) defined by \(B|_{\Delta^\perp} = A|_{\Delta^\perp} D\) and \(\Delta \subset \ker B\) is symmetric and satisfies equations (22) and (23). In fact, that \(B\) is symmetric follows easily from (40). Part (b) of Lemma 20 together with (36) and (37) imply that
\[
f_*(\nabla_X B Y - \nabla_Y B X) = f_* \nabla_X AD Y - f_* \nabla_Y AD X - f_* AD [X, Y] = 0
\]
for any \(X, Y \in \Gamma(\Delta^\perp)\). Since \(D\) is projectable, we can use Corollary 13 in [6] and deduce that
\[
\nabla_T D = [D, C_T]
\]
for any \(T \in \Delta\). Using (31) we obtain
\[
(\nabla_T B) X = (\nabla_T AD) X = (\nabla_T A) DX + A [D, C_T] X = ADC_T X
\]
for any \(T \in \Delta\) and \(X \in \Delta^\perp\). It follows that
\[
(\nabla_X B) T - (\nabla_T B) X = 0
\]
for any \(T \in \Delta\) and \(X \in \Delta^\perp\). Since
\[
(\nabla_S B) T - (\nabla_T B) S = B[S, T] = 0
\]
for any \(S, T \in \Delta\), we have shown that (36) holds. Since (37) is equivalent to \(\text{tr} \ D = 0\), the proof follows.
6 The ruled case

In this section, we discuss the infinitesimal bendings of ruled hypersurfaces that have not been considered yet.

Let \( f : M^n \rightarrow \mathbb{R}^{n+1}, n \geq 3 \), be a ruled hypersurface without flat points which is not surface-like on any open subset of \( M^n \). Then \( f \) has rank two and there exists locally an orthonormal frame \( X, Y \) of \( \Delta^\perp \) such that the second fundamental form is the form

\[
A |_{\Delta^\perp} = \begin{bmatrix} \lambda & \mu \\ \mu & 0 \end{bmatrix}.
\]

Note that if \( M^n \) is simply-connected then the set of all isometric immersions of \( M^n \) into \( \mathbb{R}^{n+1} \) consists of ruled immersions with the same rulings; see [5]. Moreover, this set can be parametrized by the set of all smooth functions in an interval. In fact, the second fundamental form of any other immersion must be of the form

\[
A |_{\Delta^\perp} = \begin{bmatrix} \lambda + \theta & \mu \\ \mu & 0 \end{bmatrix}
\]

where \( \theta \in C^\infty(M) \) is determined by choosing a smooth function along an integral curve of \( X \) and extending it to \( M^n \) by requiring that

\[
Y(\theta) = \langle \nabla_X X, Y \rangle \theta \quad \text{and} \quad T(\theta) = \langle \nabla_X X, T \rangle \theta
\]

for any \( T \in \Delta \).

**Proposition 22.** Let \( f : M^n \rightarrow \mathbb{R}^{n+1}, n \geq 3 \), be a simply-connected ruled hypersurface of constant rank two that is not surface-like on any open subset of \( M^n \). Then, any infinitesimal bending is the variational vector field of an isometric bending.

**Proof:** Since \( M^n \) is simply-connected, there is a global orthonormal frame \( \{X, Y\} \) of \( \Delta^\perp \) as above. By Lemma [16] and Proposition [17] the Codazzi tensor \( B \) on \( M^n \) is given by \( B |_{\Delta^\perp} = A |_{\Delta^\perp} D \) and \( \Delta \subset \ker B \), where \( D = \theta J \) and \( J \in \Gamma(End(\Delta^\perp)) \) is such that \( JX = Y \) and \( JY = 0 \). Moreover, \( \theta \in C^\infty(M) \) is arbitrarily prescribed along an integral curve of \( X \) and required to satisfy (41). Therefore, the one-parameter family of Codazzi tensors \( A(t) = A + tB, t \in \mathbb{R} \), gives rise to an isometric bending of \( f \) having the infinitesimal bending determined by \( B \) as its variational vector field.

**References**

[1] Beez, R., *Zur Theorie des Krümmungsmasses von Mannigfaltigkeiten höhere Ordnung*, Zeit. für Math. und Physik 21 (1876), 373–401.
[2] Bianchi, L., *Sulle varietà a tre dimensioni deformabili entro lo spazio euclideo a quattro dimensioni*, Memorie di Matematica e di Fisica della Società Italiana delle Scienze, serie III, t. XIII (1905), 261–323.

[3] Cartan, E., *La déformation des hypersurfaces dans l’espace euclidien réel a n dimensions*. Bull. Soc. Math. France 44 (1916), 65–99.

[4] Cesaro, E., “Lezioni di Geometria intrinseca”. Tipografia della R. Accademia delle scienze, Napoli, 1896.

[5] Dajczer, M., Florit, L. and Tojeiro, R., *On deformable hypersurfaces in space forms*. Ann. Mat. Pura Appl. 174 (1998), 361–390.

[6] Dajczer, M., Florit, L. and Tojeiro, R., *Euclidean hypersurfaces with genuine deformations in codimension two*. Manuscripta Math. 140 (2013), 621–643.

[7] Dajczer, M. and Gromoll, D., *Gauss parametrizations and rigidity aspects of submanifolds*. J. Differential Geom. 22 (1985), 1–12.

[8] Dajczer, M. and Rodríguez, L., *Infinitesimal rigidity of Euclidean submanifolds*. Ann. Inst. Fourier 40 (1990), 939–949.

[9] Killing, W., “Die nicht-Euklidischen Raumformen in Analytische Behandlung,” Teubner, Leipzig, 1885.

[10] Sbrana, U., *Sulla deformazione infinitesima delle ipersuperficie*. Ann. Mat. Pura Appl. 15 (1908), 329–348.

[11] Sbrana, U., *Sulla varietà ad n – 1 dimensioni deformabili nello spazio euclideo ad n dimensioni*. Rend. Circ. Mat. Palermo 27 (1909), 1–45.

[12] Schouten, J. A., *On infinitesimal deformations of V^m in V^n*. Proceedings Amsterdam 36 (1928), 1121–1131.

[13] Spivak, M., “A Comprehensive Introduction to Differential Geometry,” Publish or Perish Inc., Houston, 1979.

[14] Struik, D. J., “Grundzüge der Mehrdimensionalen Differentialgeometrie: In Direkter Darstellung”. Berlin, 1922.