ON ALMOST EVERYWHERE EXPONENTIAL
SUMMABILITY OF RECTANGULAR PARTIAL SUMS OF
DOUBLE TRIGONOMETRIC FOURIER SERIES

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Abstract. In this paper we study the a.e. exponential strong summability problem for the rectangular partial sums of double trigonometric Fourier series of the functions from $L \log L$.

1. Introduction

We denote the set of all non-negative integers by $\mathbb{N}$. Let $\mathbb{T} := [-\pi, \pi) = \mathbb{R}/2\pi$ and $\mathbb{R} := (-\infty, \infty)$. Denote by $L^1(\mathbb{T})$ the class of all measurable functions $f$ on $\mathbb{R}$ that are $2\pi$-periodic and satisfy

$$\|f\|_1 := \int_{\mathbb{T}} |f| < \infty.$$ 

The Fourier series of a function $f \in L^1(\mathbb{T})$ with respect to the trigonometric system is

$$\sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

are the Fourier coefficients of $f$. Denote by $S_n(x, f)$ the partial sums of the Fourier series of $f$ and let

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x, f)$$

be the $(C, 1)$ means of $(1)$. Fejér [1] proved that $\sigma_n(f)$ converges to $f$ uniformly for any $2\pi$-periodic continuous function. Lebesgue in [18] established almost everywhere convergence of $(C, 1)$ means if $f \in L^1(\mathbb{T})$. The strong

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summability problem, i.e. the convergence of the strong means

\[
\frac{1}{n} \sum_{k=0}^{n-1} |S_k(x, f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0,
\]

was first considered by Hardy and Littlewood in \cite{9}. They showed that for any \(f \in L^r(\mathbb{T})\) \((1 < r < \infty)\) the strong means tend to 0 a.e. as \(n \to \infty\). The trigonometric Fourier series of \(f \in L^1(\mathbb{T})\) is said to be \((H, p)\)-summable at \(x \in \mathbb{T}\) if the values \(2\) converge to 0 as \(n \to \infty\). The \((H, p)\)-summability problem in \(L^1(\mathbb{T})\) has been investigated by Marcinkiewicz \cite{19} for \(p = 2\), and later by Zygmund \cite{34} for the general case \(1 \leq p < \infty\).

Let \(\Phi : [0, \infty) \to [0, \infty)\), \(\Phi(0) = 0\), be a continuous increasing function. We say a series with the partial sums \(s_n\) strong \(\Phi\)-summable to a limit \(s\) if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|s_k - s|) = 0.
\]

In \cite{20} Oskolkov first considered the a.e strong \(\Phi\)-summability problem of Fourier series with exponentially growing \(\Phi\). Namely, he proved a.e strong \(\Phi\)-summability of Fourier series if \(\ln \Phi(t) = O(t/\ln \ln t)\) as \(t \to \infty\).

In \cite{21} Rodin proved

**Theorem R (Rodin).** If a continuous function \(\Phi : [0, \infty) \to [0, \infty)\), \(\Phi(0) = 0\), satisfies the condition

\[
\limsup_{t \to +\infty} \frac{\ln \Phi(t)}{t} < \infty,
\]

then for any \(f \in L^1(\mathbb{T})\) the relation

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|S_k(x, f) - f(x)|) = 0
\]

holds for a. e. \(x \in \mathbb{T}\).

Karagulyan \cite{11,12} proved that the exponential growth in Rodin’s theorem is optimal. Moreover, it was proved

**Theorem K (Karagulyan).** If a continuous increasing function \(\Phi : [0, \infty) \to [0, \infty)\), \(\Phi(0) = 0\), satisfies the condition

\[
\limsup_{t \to +\infty} \frac{\ln \Phi(t)}{t} = \infty,
\]

then there exists a function \(f \in L^1(\mathbb{T})\), for which the relation

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|S_k(x, f)|) = \infty
\]

holds everywhere on \(\mathbb{T}\).
In this paper we study the exponential summability problem for the rectangular partial sums of double Fourier series. Let $f \in L^1(\mathbb{T}^2)$ be a function with Fourier series

$$
\sum_{m,n=-\infty}^{\infty} c_{nm} e^{i(mx+ny)},
$$

where

$$
c_{nm} = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(x_1,x_2) e^{-i(mx_1+nx_2)} dx_1 dx_2
$$

are the Fourier coefficients of the function $f$. The rectangular partial sums of (4) are defined by

$$
S_{MN}(f) = S_{MN}(x_1,x_2,f) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} c_{nm} e^{i(mx_1+nx_2)}.
$$

We denote by $L \log L(\mathbb{T}^2)$ the class of measurable functions $f$, with

$$
\iint_{\mathbb{T}^2} |f| \log^+ |f| < \infty,
$$

where $\log^+ u := \mathbb{I}_{(1,\infty)} \log u, u > 0$. For the rectangular partial sums of two-dimensional trigonometric Fourier series Jessen, Marcinkiewicz and Zygmund [10] has proved for any $f \in L \log L(\mathbb{T}^2)$ that

$$
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (S_{ij}(x_1,x_2,f) - f(x_1,x_2)) = 0
$$

for a.e. $(x_1,x_2) \in \mathbb{T}^2$. They also showed that for every non-negative function $\omega : [0,\infty) \to [0,\infty)$ satisfying $\omega(t) \uparrow \infty$, $\omega(t) (\log^+ t)^{-1} \to 0$ as $t \to \infty$, there exists a function $f$ such that $|f| \omega(|f|) \in L^1(\mathbb{T}^2)$ and the $(C,1,1)$ means of double Fourier series of $f$ diverge a.e.. The two dimensional a.e. strong rectangular $(H,p)$-summability, i.e. the relation

$$
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{ij}(x_1,x_2,f) - f(x_1,x_2)|^p = 0 \text{ a.e.}
$$

was proved by Gogoladze [8] for $f \in L \log L(\mathbb{T}^2)$. These results show that in two dimensional case the optimal class of functions for $(C,1,1)$ summability and strong summability coincide. That is the class of functions $L \log L(\mathbb{T}^2)$.

We prove the following

**Theorem.** If a continuous increasing function $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, satisfies the condition

$$
\limsup_{t \to +\infty} \frac{\ln \Phi(t)}{\sqrt{t/\ln \ln t}} < \infty,
$$

(5)
then for any \( f \in L \log L (\mathbb{T}^2) \) the relation
\[
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi (|S_{ij} (x_1, x_2, f) - f(x_1, x_2)|) = 0
\]
holds for a. e. \((x_1, x_2) \in \mathbb{T}^2\).

As a corollary of this result we get the Gogoladze [8] theorem on a.e. \( H^p \)-summability of double Fourier series. From Jessen, Marcinkiewicz and Zygmund [10] theorem it follows that the class \( L \log L (\mathbb{T}^2) \) in our theorem is necessary in the context of strong summability question. That is, it is not possible to give a larger convergence space than \( L \log L (\mathbb{T}^2) \). Our method of proof do not allow to get (6) under the weaker condition
\[
\lim sup_{t \to +\infty} \frac{\ln \Phi (t)}{\sqrt{t}} < \infty.
\]
There is a conjecture that (7) is the optimal bound of \( \Phi \) ensuring a.e. rectangular strong summability (6) for every function \( f \in L \log L (\mathbb{T}^2) \).

The results on strong summability and approximation by trigonometric Fourier series have been extended for several other orthogonal systems, see Schipp [23, 24, 25], Leindler [14, 15, 16, 17], Totik [26, 27, 28, 29], Goginava, Gogoladze [5, 6], Goginava, Gogoladze, Karagulyan [7], Gat, Goginava, Karagulyan [3, 4], Weisz [30]-[33].

2. Auxiliary lemmas

The notation \( a \lesssim b \) will stand for \( a < c \cdot b \), where \( c > 0 \) is an absolute constant. We shall write \( a \sim b \) if the relations \( a \lesssim b \) and \( b \lesssim a \) hold at the same time. Everywhere below \( q > 1 \) will be used as the conjugate of \( p > 1 \), that is \( 1/p + 1/q = 1 \). \([a]\) denotes the integer part of \( a \in \mathbb{R} \).

The maximal function of a function \( f \in L^1 (\mathbb{T}) \) is defined by
\[
Mf (x) := \sup_{I: x \in I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f (y)| dy,
\]
where \( I \) is an open interval. The following one dimensional operators introduced by Gabisonia [2] are significant tools in the investigations of strong summability problems:
\[
G_p^{(n)} f (x) := \left( \sum_{k=1}^{[n\pi]} \left( \frac{n}{k} \int_{k \frac{n}{k}}^{k \frac{n}{k} + 1} |f (x + t)| + |f (x - t)| dt \right)^q \right)^{1/q},
\]
\[
G_p f (x) := \sup_{n \in \mathbb{N}} G_p^{(n)} f (x).
\]
Oskolkov’s following lemma plays key role in the proof of the basic lemma.
Lemma 1 (Oskolkov, [20]). For any family of pairwise disjoint intervals $\Delta_k \subset \mathbb{T}$ with centers $c_k$ it holds the inequality

$$\left\{ x \in \mathbb{T} : \sup_{p > 1} \frac{\sum_j \left( \frac{|\Delta_j|}{|x - c_j| + |\Delta_j|} \right)^q}{p \ln \ln(p + 2)} > \lambda \right\} \lesssim \exp(-c\lambda), \lambda > 0,$$

where $c > 0$ is an absolute constant.

One can easily check that

$$\sup_{p > 1} \frac{\left( \sum_j \left( \frac{|\Delta_j|}{|x - c_j| + |\Delta_j|} \right)^q \right)^{1/q}}{p \ln \ln(p + 2)} \lesssim \left\{ 1, \sup_{p > 1} \frac{\sum_j \left( \frac{|\Delta_j|}{|x - c_j| + |\Delta_j|} \right)^q}{p \ln \ln(p + 2)} \right\}.$$

Combining this with (8), we get

$$\int_{\mathbb{T}} \sup_{p > 1} \frac{\left( \sum_j \left( \frac{|\Delta_j|}{|x - c_j| + |\Delta_j|} \right)^q \right)^{1/q}}{p \ln \ln(p + 2)} \lesssim 1.$$

Lemma 2. If $f \in L^1(\mathbb{T})$, then

$$\left\{ x \in \mathbb{T} : \sup_{p > 1} \frac{G_p f(x)}{p \ln \ln(p + 2)} > \lambda \right\} \lesssim \left( \frac{1}{\lambda} \|f\|_1 \right)^{1/2}, \lambda > 0.$$

Proof. It is enough to prove the same estimate for the modified operators

$$G'_p f(x) := \sup_{n \in \mathbb{N}} \left( \sum_{k=1}^{[n\pi]} \left( \frac{n}{k} \int_{k-1/n}^{k/n} |f(x + t)| \, dt \right) \right)^{1/q}.$$

Using the Calderon-Zygmund lemma, for the maximal function we get the relation

$$R_\lambda := \left\{ x \in \mathbb{T} : Mf(x) > \sqrt{\lambda} \right\} = \bigcup_{k=0}^{\infty} \Delta_k, \lambda > 0,$$

where $\Delta_k \subset \mathbb{T}$ are disjoint open intervals such that

$$\sqrt{\lambda} \leq \frac{1}{|\Delta_k|} \int_{\Delta_k} |f(t)| \, dt \leq 2\sqrt{\lambda},$$

$$|R_\lambda| \leq \frac{1}{\sqrt{\lambda}} \|f\|_1.$$
Denote \( \delta_k^n := [(k - 1)/n, k/n] \) and \( \delta_k^n (x) := x + \delta_k^n \). Separating the terms in the sum (11) with \( k \) satisfying \( \delta_k^n (x) \subseteq R_\lambda \), we get

\[
G'_{p,f}(x) \leq \sup_{n \in \mathbb{N}} \left( \sum_{k: \delta_k^n (x) \subseteq R_\lambda} \left( \frac{n}{k} \int_{k-1/n}^{k} |f(x + t)| \, dt \right)^q \right)^{1/q} + \sup_{n \in \mathbb{N}} \left( \sum_{k: \delta_k^n (x) \not\subseteq R_\lambda} \left( \frac{n}{k} \int_{k-1/n}^{k} |f(x + t)| \, dt \right)^q \right)^{1/q} := I + II.
\]

From the definition of \( R_\lambda \) in the case \( \delta_k^n (x) \not\subseteq R_\lambda \) it follows that

\[
n \int_{k-1/n}^{k} |f(x + t)| \, dt \leq \sqrt{\lambda}.
\]

Thus we conclude

\[
II \leq \sqrt{\lambda} \left( \sum_{k=1}^{\infty} \frac{1}{k^q} \right)^{1/q} \lesssim \sqrt{\lambda} \left( \frac{1}{q - 1} \right)^{1/q} \lesssim p \sqrt{\lambda}.
\]

Given \( x \in \mathbb{T} \) set

\[
k_i(x) = \begin{cases} 
\min \{ k : \delta_k^n (x) \subseteq \Delta_i \} & \text{if } \{ k : \delta_k^n (x) \subseteq \Delta_i \} \neq \emptyset, \\
\infty & \text{if } \{ k : \delta_k^n (x) \subseteq \Delta_i \} = \emptyset.
\end{cases}
\]

Denote \( \tilde{R}_\lambda := \bigcup_{k=1}^{\infty} 3 \Delta_k \) and take an arbitrary point \( x \in \mathbb{T} \setminus \tilde{R}_\lambda \). One can easily check that if \( k_i(x) \neq \infty \), then

\[
\Delta_i \ni \frac{k_i(x)}{n} \sim |x - c_i|,
\]
where $c_i$ is the center of the interval $\Delta_i$. Thus for any $x \notin \tilde{R}_\lambda$ we obtain

\begin{equation}
I = \sup_{n \in \mathbb{N}} \left( \sum_{i=1}^{\infty} \sum_{k : \delta_k^n(x) \subset \Delta_i} \left( \frac{n}{k} \int_{\delta_k^n(x)} |f(t)| \, dt \right)^q \right)^{1/q}
\end{equation}

\begin{align*}
&\leq \sup_{n \in \mathbb{N}} \left( \sum_{i=1}^{\infty} \sum_{k : \delta_k^n(x) \subset \Delta_i} \frac{n}{k} \int_{\delta_k^n(x)} |f(t)| \, dt \right)^q \left( \frac{1}{\lambda} \int_{\Delta_i} |f(t)| \, dt \right)^q \left( \frac{1}{\Delta_i} \right)^q \\
&\leq \sqrt{\lambda} \sup_{n} \left( \sum_{i=1}^{\infty} \left( \frac{n}{k_i(x)} \right)^q \right)^{1/q} \\
&\leq \sqrt{\lambda} \left( \sum_{i=1}^{\infty} \left( \frac{1}{x-c_i+\Delta_i} \right)^q \right)^{1/q}, \quad x \notin \tilde{R}_\lambda.
\end{align*}

Using Chebyshev’s inequality, from (9), (16) and (17) it follows that

\begin{align*}
\left| \left\{ x \in \mathbb{T} \setminus \tilde{R}_\lambda : \sup_{p > 1} \frac{G_p'(f(x))}{p \ln p + 2} > \lambda \right\} \right| &\leq \left| \left\{ x \in \mathbb{T} \setminus \tilde{R}_\lambda : \sqrt{\lambda} \left( 1 + \sup_{p > 1} \left( \sum_{j} \frac{|\Delta_j|}{|x-c_j|+|\Delta_j|} \right)^q \right)^{1/q} \geq c \lambda \right\} \right| \\
&\leq \frac{1}{\sqrt{\lambda}} \int_{\mathbb{T}} \sup_{p > 1} \left( \sum_{j} \left( \frac{|\Delta_j|}{|x-c_j|+|\Delta_j|} \right)^q \right)^{1/q} \, dx \\
&\leq \frac{1}{\sqrt{\lambda}},
\end{align*}

for an appropriate absolute constant $c > 0$. Applying homogeneity, one can get

\begin{equation}
\left| \left\{ x \in \mathbb{T} \setminus \tilde{R}_\lambda : \sup_{p > 1} \frac{G_p'(f(x))}{p \ln p + 2} > \lambda \right\} \right| \lesssim \left( \frac{\|f\|_1}{\lambda} \right)^{1/2}, \quad \lambda > 0.
\end{equation}
Consequently, from (14)-(18) we get

\[
\left| \left\{ x \in T : \sup_{p>1} \frac{G^p_f(x)}{p \ln \ln(p + 2)} > \lambda \right\} \right| \\
\leq \left| \left\{ x \in T \setminus \tilde{R}_\lambda : \sup_{p>1} \frac{G^p_f(x)}{p \ln \ln(p + 2)} > \lambda \right\} \right| + |\tilde{R}_\lambda| \\
\lesssim \left( \frac{\|f\|_1}{\lambda} \right)^{1/2} + \frac{\|f\|_1}{\sqrt{\lambda}}.
\]

Again using homogeneity, we obtain (10). \(\square\)

We will need the following estimations.

**Lemma 3** (Gabisonia, [2]). If \( p > 1 \) and \( f \in L^1(\mathbb{T}^2) \), then

\[
\left( \frac{1}{n} \sum_{j=0}^{n-1} |S_j(x, f)|^p \right)^{1/p} \lesssim G^{(n)}_p f(x).
\]

**Lemma 4** (Schipp, [22]). If \( f \in L^1(\mathbb{T}^2) \), then

\[
\left( \frac{1}{n} \sum_{j=0}^{n-1} |S_j(x, f)|^p \right)^{1/p} \lesssim p G_2 f(x).
\]

Rodin [21] proved the weak \((1, 1)\)-type estimate for the operators \( G_p f(x) \) with a fixed \( p > 1 \). From this fact, applying a standard argument, one can derive

**Lemma 5** (Rodin, [21]). Let \( f \in L \log L(\mathbb{T}) \). Then

\[
\|G_2(f)\|_1 \lesssim 1 + \int_T |f| \log |f|.
\]

For any function \( f \in L^1(\mathbb{T}^2) \) define

\[
G_{p,1}(x_1, x_2; f) = G_p f_{x_2}(x_1), \quad G_{p,2}(x_1, x_2; f) = G_p f_{x_1}(x_2),
\]

\[
G^{(n)}_{p,1}(x_1, x_2; f) = G^{(n)}_p f_{x_2}(x_1), \quad G^{(n)}_{p,2}(x_1, x_2; f) = G^{(n)}_p f_{x_1}(x_2),
\]

where \( f_{x_2}(\cdot) = f(\cdot, x_2) \) and \( f_{x_1}(\cdot) = f(x_1, \cdot) \) are considered as functions on \( x_1 \) and \( x_2 \) respectively. Similarly one dimensional partial sums of \( f(x_1, x_2) \) with respect to each variables will be denoted by

\[
S_{n,1}(x_1, x_2, f) = S_n(x_1, f_{x_2}), \quad S_{n,2}(x_1, x_2, f) = S_n(x_2, f_{x_1}).
\]
Lemma 6. If \( f \in L \log L(\mathbb{T}^2) \), then

\[
\left\{ \left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(x_1, x_2, f)|^p \right)^{1/p} \right\}_{p>1, n,m \in \mathbb{N}} > \lambda
\]

\[
\sup_{n,m \in \mathbb{N}} \left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(x_1, x_2, f)|^p \right) > \lambda
\]

\[
\lesssim \left( \frac{1}{\lambda} \left( 1 + \int_{\mathbb{T}^2} |f| \log^+ |f| \right)^{1/2} \right)
\]

Proof. Using (19), (20) and generalized Minkowsi’s inequality, we get

\[
\frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(x_1, x_2, f)|^p = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,1}(x_1, x_2, S_j, f)|^p
\]

\[
\leq \frac{1}{m} \sum_{j=0}^{m-1} \left( G_{j,1}(x_1, x_2, |S_j, f|)^p \right)
\]

\[
\leq \left( G_{j,1}(x_1, x_2, \left( \frac{1}{m} \sum_{j=0}^{m-1} |S_j, f| \right)^{1/p}) \right)^p
\]

\[
\leq \left( G_{j,1}(x_1, x_2, \left( \frac{1}{m} \sum_{j=0}^{m-1} |S_j, f| \right)^{1/p}) \right)^p
\]

\[
\lesssim p^p \left( G_{j,1}(x_1, x_2, G_2, (f))^p \right).
\]

Hence we obtain

\[
\Omega = \left\{ (x_1, x_2) \in \mathbb{T}^2 : \sup_{p>1, n,m \in \mathbb{N}} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(x_1, x_2, f)|^p > \lambda \right\}
\]

\[
\subset \left\{ (x_1, x_2) \in \mathbb{T}^2 : \sup_{p>1} \frac{G_{j,1}(x_1, x_2, G_2, (f))}{p \ln \ln(p+2)} > \lambda \right\},
\]
then, applying Lemma 2 and 5 we conclude

\[ |\Omega| = \int_{\mathbb{T}^2} \mathbb{1}_{\Omega}(x_1, x_2) \, dx_1 \, dx_2 = \int_{\mathbb{T}} dx_2 \int_{\mathbb{T}} \mathbb{1}_{\Omega}(x_1, x_2) \, dx_1 \]

\[ \lesssim \int_{\mathbb{T}} \left( \frac{1}{\lambda} \int G_{2,2}(x_1, x_2, f) \, dx_1 \right)^{1/2} \, dx_2 \]

\[ \lesssim \int_{\mathbb{T}} \left[ \frac{1}{\lambda} \left( 1 + \int_{\mathbb{T}} |f(x_1, x_2)| \log^+ |f(x_1, x_2)| \, dx_1 \right) \right]^{1/2} \, dx_2 \]

\[ \lesssim \left[ \frac{1}{\lambda} \left( 1 + \int_{\mathbb{T}^2} |f(x_1, x_2)| \log^+ |f(x_1, x_2)| \, dx_1 dx_2 \right) \right]^{1/2}. \]

Lemma is proved. \(\square\)

3. PROOF OF THEOREM 1

Let \( L_M := L_M (\mathbb{T}^2) \) be Orlicz space of functions on \( \mathbb{T}^2 \) generated by the Young function \( M(t) = t \log^+ t \). It is known that \( L_M \) is a Banach space with respect to the Luxemburg norm

\[ \|f\|_M := \inf \left\{ \lambda : \lambda > 0, \int_{\mathbb{T}} M \left( \frac{|f|}{\lambda} \right) \leq 1 \right\} < \infty. \]

According to a theorem from ([13], Chap. 2, theorem 9.5) we have

\[ 0,5 \left( 1 + \int_{\mathbb{T}^2} M(|f|) \right) \leq \|f\|_M \leq 1 + \int_{\mathbb{T}^2} M(|f|) \]

provided \( \|f\|_M = 1 \). Hence from Lemma 6 we conclude

\[ \left( \sup_{p>1} \sup_{n,m \in \mathbb{N}} \left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_i,j(f)|^p \right)^{1/p} \right)^{1/2} > \lambda \]

\[ \approx \left( \frac{\|f\|_M^p}{\lambda} \right)^{1/2}. \]

Indeed, at first we deduce the case of \( \|f\|_M = 1 \), then using a homogeneity argument, we get the inequality in the general case.
Proof of Theorem. First we shall prove that for any \( f \in L \log L(T^2) \) the relation

\[
\lim_{n,m \to \infty} \sup_{p > 1} \left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f) - f|^p \right)^{1/p} = 0 \text{ a.e. .}
\]

(22)

Observe that (22) trivially holds for the double trigonometric polynomials. Indeed, let \( T \) be a trigonometric polynomial of degree \((s_1, s_2)\). Then we have

\[
S_{i,j}(T) - T = 0, \quad i \geq s_1, j \geq s_2,
\]

\[
S_{i,j}(T) - T = S_{s_1,j}(T) - T, \quad i \geq s_1, 0 \leq j < s_2,
\]

\[
S_{i,j}(T) - T = S_{i,s_2}(T) - T, \quad 0 \leq i < s_1, j \geq s_2.
\]

Thus for integers \( n > s_1 \) and \( m > s_2 \) we have

\[
\frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(T) - T|^p \leq \frac{1}{n} \sum_{i=0}^{s_1-1} |S_{i,s_2}(T) - T|^p + \frac{1}{m} \sum_{j=0}^{s_2-1} |S_{s_1,j}(T) - T|^p
\]

\[
+ \frac{1}{nm} \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} |S_{i,j}(T) - T|^p
\]

\[
\leq \frac{1}{n} \sum_{i=0}^{s_1-1} |S_{i,s_2}(T) - T|^p + \frac{1}{m} \sum_{j=0}^{s_2-1} |S_{s_1,j}(T) - T|^p
\]

\[
+ \frac{1}{nm} \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} |S_{i,j}(T) - T|^p
\]

\[
\leq c_1 + c_2,
\]

where \( c_1 \) and \( c_2 \) are constants depended on \( T \). Thus (22) holds if \( f = T \). To prove the general case it is enough to show that the set

\[
G_\lambda = \left\{ \lim_{n,m \to \infty} \sup_{p > 1} \left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f) - f|^p \right)^{1/p} > \lambda \right\}
\]

has measure zero for any \( \lambda > 0 \). Since \( M(t) \) satisfies the \( \Delta_2 \)-condition, the function \( f \) can be approximated by a trigonometric polynomial \( T \) (see [13]), that is

\[
\|f - T\|_M < \varepsilon, \quad \|f - T\|_{L^1} < \varepsilon
\]
Since \((22)\) holds for \(T\), applying \((21)\), one can obtain
\[
|G_\lambda| = \left\{ \limsup_{n,m \to \infty} \sup_{p > 1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f - T) - (f - T)|^p \right)^{1/p}}{p^2 \ln \ln(p + 2)} > \lambda \right\} \\
\leq \left\{ \sup_{n,m \in \mathbb{N}} \sup_{p > 1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f - T)|^p \right)^{1/p}}{p^2 \ln \ln(p + 2)} > \lambda/2 \right\} \\
+ \left\{ \sup_{p > 1} \frac{|f - T|}{p^2 \ln \ln(p + 2)} > \lambda/2 \right\} \\
\approx \left( \frac{\|f - T\|_M}{\lambda} \right)^{1/2} + \frac{\|f\|_{L^1}}{\lambda} \\
\leq \left( \frac{\varepsilon}{\lambda} \right)^{1/2} + \varepsilon/\lambda.
\]
Since \(\varepsilon > 0\) can be taken arbitrarily small, we conclude that \(|G_\lambda| = 0\) for any \(\lambda > 0\) and so \((22)\) holds. To prove \((6)\) observe that
\[
(23) \quad u(s) = \exp \left( \sqrt{\frac{s}{\ln \ln(s + 2)}} \right) \\
\leq v(s) = \sum_{k=1}^{\infty} \left( \frac{d}{k} \sqrt{\frac{s}{\ln \ln(k + 2)}} \right)^k, \quad s > 1,
\]
for some absolute constant \(d\). Indeed, if \(s \geq 1\), then one can check that
\[
1 < \sqrt{\frac{s}{\ln \ln(s + 2)}} < k(s) = \left[ \sqrt{\frac{s}{\ln \ln(s + 2)}} \right] + 1 < 2 \sqrt{\frac{s}{\ln \ln(s + 2)}},
\]
and therefore for enough bigger \(d\) we we will have
\[
v(s) \geq \left( \frac{d}{k(s)} \sqrt{\frac{s}{\ln \ln(k(s) + 2)}} \right)^{k(s)} \\
> \left( \frac{d}{2} \sqrt{\frac{\ln \ln(s + 2)}{\ln \ln(k(s) + 2)}} \right)^{k(s)} > e^{k(s)} \\
> u(s)
\]
and so \((23)\). If the function \(\Phi\) satisfies \((5)\), then one can check that
\[
\Phi(s) \leq \exp \left( \sqrt{\frac{A \cdot s}{\ln \ln(A \cdot s + 2)}} \right) = u(A s), \quad s > S,
\]
for some positive numbers $A > 1, S > 1$. Consider the functions

$$
\varphi_{i,j}(f) = S_{i,j}(f) - f,
$$

$$
\varphi^*_{i,j}(f) = \begin{cases} 
\varphi_{i,j}(f) & \text{if } |\varphi_{i,j}(f)| \leq S, \\
0 & \text{if } |\varphi_{i,j}(f)| > S,
\end{cases}
$$

$$
\varphi^{**}_{i,j}(f) = \varphi_{i,j}(f) - \varphi^*_{i,j}(f).
$$

From (23) and the definition of $\Phi$ it follows that

$$
\frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}(f)|) = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi^*_{i,j}(f)|) + \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi^{**}_{i,j}(f)|)
$$

$$
\leq \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi^*_{i,j}(f)|) + v(A|\varphi^{**}_{i,j}(f)|)
$$

$$
= \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi^*_{i,j}(f)|)
$$

$$
+ \sum_{k=1}^{\infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left( \frac{d \sqrt{A} \cdot |\varphi^{**}_{i,j}(f)|}{\ln \ln(k+2)} \right)^k
$$

$$
\leq \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi^*_{i,j}(f)|)
$$

$$
+ \sum_{k=1}^{\infty} (d \sqrt{A})^k \left( \sup_{p>1/2} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\varphi_{i,j}(f)|^p \right)^{1/p}}{4p^2 \ln \ln(2p+2)} \right)^{1/2}.
$$

The second term of the last expression tends to zero almost everywhere, since according to (22) we have

$$
\limsup_{n,m \to \infty} \sup_{p>1/2} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\varphi_{i,j}(f)|^p \right)^{1/p}}{4p^2 \ln \ln(2p+2)}
$$

$$
\leq \lim_{n,m \to \infty} \sup_{p>1} \frac{\left( \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |S_{i,j}(f) - f|^p \right)^{1/p}}{p^2 \ln \ln(p+2)} = 0 \text{ a.e.}
$$
Hence, to prove (6) it is enough to show the same for the first term. From (22) and Chebyshev’s inequality it follows that

\[ r_{n,m}(x_1, x_2) = \frac{\#\{i, j \in \mathbb{N} : 0 \leq i < n, 0 \leq j < m, \varphi_{i,j}(x_1, x_2) > \varepsilon\}}{nm} \leq \frac{1}{\varepsilon} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\varphi_{i,j}(x_1, x_2, f)| \to 0 \text{ a.e.}, \]

where \#C denotes the cardinality of a finite set C. Thus for a.e. \((x_1, x_2) \in \mathbb{T}^2\) we get

\[ \limsup_{n,m \to \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(x_1, x_2, f)|) \leq \limsup_{n,m \to \infty} (r_{n,m}(x_1, x_2)\Phi(S) + (1 - r_{n,m}(x_1, x_2))\Phi(\varepsilon)) = \Phi(\varepsilon) \text{ a.e.}. \]

Since \(\varepsilon > 0\) can be taken arbitrary small we get

\[ \lim_{n,m \to \infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi_{i,j}^*(x_1, x_2, f)|) = 0 \text{ a.e.} \]

and so (6).

\[ \Box \]

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