Convergence Rate of Euler-Maruyama Scheme for SDEs with Rough Coefficients

Jianhai Bao\textsuperscript{a)}, Xing Huang\textsuperscript{b)}, Chenggui Yuan\textsuperscript{c)}

\textsuperscript{a)}School of Mathematics and Statistics, Central South University, Changsha 410083, China
\textsuperscript{b)}School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
\textsuperscript{c)}Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK

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Abstract

In this paper, we are concerned with convergence rate of Euler-Maruyama scheme for stochastic differential equations with rough coefficients. The key contributions lie in (i), by means of regularity of non-degenerate Kolmogorov equation, we investigate convergence rate of Euler-Maruyama scheme for a class of stochastic differential equations, which allow the drifts to be Dini-continuous and unbounded; (ii) by the aid of regularization properties of degenerate Kolmogorov equation, we discuss convergence rate of Euler-Maruyama scheme for a range of degenerate stochastic differential equations, where the drift is locally Hölder-Dini continuous of order $\frac{2}{3}$ with respect to the first component, and is merely Dini-continuous concerning the second component.

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1 Introduction and Main Results

It is well-known that convergence rate of Euler-Maruyama (EM) for stochastic differential equations (SDEs) with regular coefficients is one-half, see, e.g., [10, 11]. With regard to convergence rate of EM scheme under various settings, we refer to, e.g., [2] for stochastic differential delay equations (SDDEs) with polynomial growth with respect to (w.r.t.) the delay variables, [8] for SDDEs under local Lipschitz and also under monotonicity condition, [6, 9, 15] for SDEs with discontinuous coefficients, and [24] for SDEs under log-Lipschitz condition.
Recently, convergence rate of EM scheme for SDEs with irregular coefficients has also gained much attention. For instance, by the Meyer–Tanaka formula, \[23\] revealed convergence rate in \(L^1\)-norm sense for a range of SDEs, where the drift term is Lipschitzian and the diffusion term is Hölder continuous w.r.t. spatial variable; Adopting the Yamada-Watanabe approximation approach, \[7\] extended \[23\] to discuss strong convergence rate in \(L^p\)-norm sense; Using the Yamada-Watanabe approximation trick and heat kernel estimate, \[14\] studied strong convergence rate in \(L^1\)-norm sense for a class of non-degenerate SDEs, where the bounded diffusion term satisfies a weak monotonicity and is of bounded variation w.r.t. a Gaussian measure and the diffusion term is Hölder continuous; Applying the Zvonkin transformation, \[17\] discussed strong convergence rate in \(L^p\)-norm sense for SDEs with additive noise, where the drift coefficient is bounded and Hölder continuous.

It is worth pointing out that \[14, 17\] focused on convergence rate of EM for SDEs with Hölder continuous and bounded drift, which, nevertheless, rules out some interesting examples. On the other hand, most of the existing literature on convergence rate of EM scheme is concerned with non-degenerate SDEs, the corresponding issue for degenerate SDEs is scarce. So, in this work, our goal is to discuss convergence rate of EM method for SDEs with rough coefficients, which may allow SDEs involved to be degenerate. For wellposedness of (path-dependent) SDEs with singular coefficients, we refer to, e.g., \[1, 19, 20, 21\] for more details.

Throughout the paper, the following notation will be used. Let \(\| \cdot \|\) and \(\| \cdot \|_{\text{HS}}\) stand for the usual operator norm and the Hilbert-Schmidt norm, respectively. Fix \(T > 0\) and set \(\|f\|_{T, \infty} := \sup_{t \in [0, T], x \in \mathbb{R}^m} \|f(t, x)\|\) for an operator-valued map \(f\) on \([0, T] \times \mathbb{R}^m\). Denote \(\mathcal{M}_n^{\text{non}}\) by the collection of all nonsingular \(n \times n\)-matrices and \(\mathcal{M}_n^{\text{cc}, \text{non}}\) by a closed and convex subset of \(\mathcal{M}_n^{\text{non}}\). Let \(\mathcal{D}_0\) be the collection of all slowly varying functions \(\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+\) at zero in Karamata’s sense (i.e., \(\lim_{t \to 0} t^\lambda \phi(t) = 1\) for any \(\lambda > 0\)), which are bounded from 0 and \(\infty\) on \([\varepsilon, \infty)\) for any \(\varepsilon > 0\). For more properties of slowly varying functions, we refer to, e.g., Bingham et al. \[3\]. Let \(\mathcal{D}_0\) be the family of Dini functions, i.e.,

\[\mathcal{D}_0 := \left\{ \phi \in \mathcal{D}_0 | \phi \text{ is increasing and } \int_0^1 \frac{\phi(s)}{s} \, ds < \infty \right\} .\]

A function \(f : \mathbb{R}^m \mapsto \mathbb{R}^n\) is called Dini-continuity if there exists \(\phi \in \mathcal{D}_0\) such that \(|f(x) - f(y)| \leq \phi(|x - y|)\) for any \(x, y \in \mathbb{R}^m\). Remark that every Dini-continuous function is continuous and every Lipschitz continuous function is Dini-continuous; Moreover, if \(f\) is Hölder continuous, then \(f\) is Dini-continuous. Nevertheless, there are numerous Dini-continuous functions, which are not Hölder continuous at all; see, e.g., \(\varphi(x) = (\log(c + x^{-1}))^{-(1 + \delta)}\), \(x > 0\), for some constants \(\delta > 0\) and \(c \geq e^{3 + 2\delta}\), and \(\varphi(x) = 0\) for \(x = 0\). For some sufficiently small \(\varepsilon \in (0, 1)\), set

\[\mathcal{D}_\varepsilon := \left\{ \phi \in \mathcal{D}_0 | \phi^{2(1 + \varepsilon)} \text{ is concave} \right\} .\]

Clearly, \(\varphi\) constructed above belongs to \(\mathcal{D}_0 \cap \mathcal{D}_\varepsilon\). A function \(f : \mathbb{R}^m \mapsto \mathbb{R}^n\) is called Hölder-Dini continuity of order \(\alpha \in [0, 1)\) if

\[|f(x) - f(y)| \leq |x - y|^\alpha \phi(|x - y|), \quad |x - y| \leq 1\]

for some sufficiently small \(\varepsilon \in (0, 1)\), set

\[\mathcal{D}_\varepsilon := \left\{ \phi \in \mathcal{D}_0 | \phi^{2(1 + \varepsilon)} \text{ is concave} \right\} .\]
For some $\phi \in \mathcal{D}_0$. For any measurable function $\psi : (0, 1] \mapsto \mathbb{R}_+$ and $f \in C(\mathbb{R}^n)$, let

$$[f]_\psi = \sup_{|x-y| \leq 1} \frac{|f(x) - f(y)|}{\psi(|x-y|)}, \quad \|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)| \quad \text{and} \quad \|f\|_\psi = [f]_\psi + \|f\|_\infty.$$ 

For notational simplicity, we shall write $\mathbb{R}^{2n}$ instead of $\mathbb{R}^n \times \mathbb{R}^n$. For any measurable functions $\psi_1, \psi_2 : (0, 1] \mapsto \mathbb{R}_+$ and $f \in C(\mathbb{R}^{2n})$, let

$$[f]_{\psi_1, \infty} = \sup_{x \in \mathbb{R}^n} [f(\cdot, x^{(2)})]_{\psi_1}, \quad [f]_{\infty, \psi_2} = \sup_{x \in \mathbb{R}^n} [f(x^{(1)}, \cdot)]_{\psi_2},$$

$$[f]_{\psi_1, \psi_2} = [f]_{\psi_1, \infty} + [f]_{\infty, \psi_2}, \quad \|f\|_{\psi_1, \psi_2} = [f]_{\psi_1, \psi_2} + \|f\|_\infty,$$

$$\|f\|_{\psi_1, \infty} = [f]_{\psi_1, \infty} + \|f\|_\infty, \quad \|f\|_{\infty, \psi_2} = [f]_{\infty, \psi_2} + \|f\|_\infty.$$ 

Write the gradient operator on $\mathbb{R}^{2n}$ as $\nabla = (\nabla^{(1)}, \nabla^{(2)})$, where $\nabla^{(1)}$ and $\nabla^{(2)}$ stand for the gradient operators for the first and the second components, respectively.

Before proceeding further, a few words about the notation are in order. Generic constants will be denoted by $c$; we use the shorthand notation $a \lesssim b$ to mean $a \leq cb$. If the constant $c$ depends on a parameter $p$, we shall also write $c_p$ and $a \lesssim_p b$.

### 1.1 Non-degenerate SDEs with Bounded Coefficients

In this subsection, we consider an SDE on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$

\begin{equation}
(1.1) \quad dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t, \quad t > 0, \quad X_0 = x,
\end{equation}

where $b : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n \otimes \mathbb{R}^n$, and $(W_t)_{t \geq 0}$ is an $n$-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

With regard to (1.1), we suppose that there exists $\phi \in \mathcal{D}$ such that for any $s, t \in [0, T]$ and $x, y \in \mathbb{R}^n$,

\begin{enumerate}
\item[(A1)] $\sigma_t \in C^2(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$, $\sigma_t \in \mathcal{M}_{\text{non}}$, and

\begin{equation}
(1.2) \quad \|b\|_{T, \infty} + \sum_{i=0}^2 \|\nabla^i \sigma\|_{T, \infty} + \|\nabla \sigma^{-1}\|_{T, \infty} + \|\sigma^{-1}\|_{T, \infty} < \infty,
\end{equation}

where $\nabla^i$ means the $i$-th order gradient operator;
\item[(A2)] (Regularity of $b$ w.r.t. spatial variables)

$$|b_t(x) - b_t(y)| \leq \phi(|x - y|);$$
\item[(A3)] (Regularity of $b$ and $\sigma$ w.r.t. time variables)

$$|b_s(x) - b_t(x)| + \|\sigma_s(x) - \sigma_t(x)\|_{\text{HS}} \leq \phi(|s - t|).$$
\end{enumerate}
Under (A1) and (A2), (1.1) admits a unique non-explosive strong solution \((X_t)_{t \in [0, T]}\); see, e.g., [19, Theorem 1.1].

Without loss of generality, we take an integer \(N > 0\) sufficiently large such that the stepsize \(\delta := T/N \in (0, 1)\). The continuous-time EM scheme corresponding to (1.1) is

\[
dY_t = b_t(Y_t)dt + \sigma_t(Y_t)dW_t, \quad t > 0, \quad Y_0 = X_0 = x.
\]

Herein, \(t_{\delta} := \lfloor t/\delta \rfloor \delta\) with \(\lfloor t/\delta \rfloor\) being the integer part of \(t/\delta\).

The first contribution in this paper is stated as follows.

**Theorem 1.1.** Under (A1)-(A3),

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right) \lesssim_T \phi(C_T \sqrt{\delta})^2
\]

for some constant \(C_T \geq 1\).

**Remark 1.2.** In Theorem 1.1, taking \(\phi(x) = x^\beta\) for \(x > 0\) and \(\beta \in (0, 1/2]\), we arrive at

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right) \lesssim_T \delta^\beta,
\]

which covers [17, Theorem 2.13] with \(\beta \in (0, 1/2]\) therein. On the other hand, by going carefully through the argument of Theorem 1.1, we can allow the Hölder exponent \(\beta \in (1/2, 1)\) whenever we consider the error bound of \(\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t - Y_t|^p \right)\) for any \(p \in [1, 1/\beta]\). Moreover, choosing \(\phi(x) = x, x \geq 0\), and inspecting closely the argument of Theorem 1.1, one has

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right) \lesssim_T \delta,
\]

which reduces to the classical result on strong convergence of EM scheme for SDEs with regular coefficients, see, e.g., [10].

### 1.2 Non-degenerate SDEs with Unbounded Coefficients

In Theorem 1.1, the coefficients are uniformly bounded, and that the drift term \(b\) satisfies the global Dini-continuous condition (see (A2) above), which seems to be a little bit stringent. Therefore, it is quite natural to replace uniform boundedness by local boundedness and global Dini continuity by local Dini continuity, respectively.

In lieu of (A1)-(A3), concerning (1.1) we assume that for any \(s, t \in [0, T]\) and \(k \geq 1\),

(A1') \(\sigma_t \in C^2(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n), \sigma_t \in \mathbb{M}^n_{\text{non}},\) and

\[
|b_t(x)| + \sum_{i=0}^2 \|\nabla^i \sigma_t(x)\|_{\text{HS}} + \|\nabla \sigma_t^{-1}(x)\|_{\text{HS}} + \|\sigma_t^{-1}(x)\|_{\text{HS}} \leq K_T(1 + |x|), \quad x \in \mathbb{R}^n
\]

for some constant \(K_T > 0\);
(A2') (Regularity of $b$ w.r.t. spatial variables) There exists $\phi_k \in \mathcal{D}$ such that
$$|b_t(x) - b_t(y)| \leq \phi_k(|x - y|), \quad |x| \vee |y| \leq k;$$

(A3') (Regularity of $b$ and $\sigma$ w.r.t. time variables) For $\phi_k \in \mathcal{D}$ such that (A2'),
$$|b_s(x) - b_t(x)| + \|\sigma_s(x) - \sigma_t(x)\|_{HS} \leq \phi_k(|s - t|), \quad |x| \leq k.$$

By the cut-off approach, Theorem 1.1 can be generalized to cover SDEs with local Dini-continuous coefficients, which is presented as below.

**Theorem 1.3.** Under (A1')-(A3'),

$$\lim_{\delta \to 0} \mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right) = 0. \quad (1.4)$$

Moreover, if $\phi_k(s) = e^{s^4} s^\alpha$, $s \geq 0$, for some $\alpha \in (0, 1/2]$ and $c_0 > 0$, then

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right) \lesssim \inf_{\varepsilon \in (0,1)} \left\{ (\log \log(\delta^{-\alpha}))^{-\frac{1}{2}} + \delta^{\alpha(1-\varepsilon)} \right\}. \quad (1.5)$$

**Remark 1.4.** For the case of bounded and Hölder continuous drift $b \in L^1(\mathbb{R})$ and bounded diffusion coefficient $\sigma$, [10] Theorem 2.6] studied convergence rate of EM scheme for a class of scalar non-degenerate SDEs. By a cut-off approach, [10, Theorem 2.7] extended [10, Theorem 2.6] to the case that the drift and diffusion terms are bounded. While, Theorem 1.3 reveals convergence rate of EM scheme for SDEs with rough coefficients, which allows the drift term to be unbounded and Hölder continuous.

### 1.3 Degenerate SDEs

So far, most of the existing literature on convergence of EM scheme for SDEs with irregular coefficients is concerned with non-degenerate SDEs; see, e.g., [11, 16, 17] for SDEs driven by Brownian motions, and [13, 17] for SDEs driven by jump processes. The issue for the setup of degenerate SDEs has not yet been considered to date to the best of our knowledge. Nevertheless, in this subsection, we make an attempt to discuss the topic for degenerate SDEs with rough coefficients.

Consider the following degenerate SDE on $\mathbb{R}^{2n}$

$$\begin{align*}
\text{d}X_t^{(1)} &= b_t^{(1)}(X_t^{(1)}, X_t^{(2)})\text{d}t, \\
\text{d}X_t^{(2)} &= b_t^{(2)}(X_t^{(1)}, X_t^{(2)})\text{d}t + \sigma_t(X_t^{(1)}, X_t^{(2)})\text{d}W_t,
\end{align*} \quad (1.6)$$

where $b_t^{(1)}, b_t^{(2)} : \mathbb{R}^{2n} \mapsto \mathbb{R}^n$, $\sigma_t : \mathbb{R}^{2n} \mapsto \mathbb{R}^n \otimes \mathbb{R}^n$, and $(W_t)_{t \geq 0}$ is an $n$-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. (1.6) is also called a stochastic Hamiltonian system, which has been investigated extensively in [5, 22, 25] on Bismut formulae, in [12] on ergodicity, in [18] on hypercontractivity, and in [14, 20, 21] on wellposedness, to name a few.

For any $x = (x^{(1)}, x^{(2)})$, $y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^{2n}$ and $t \in [0, T]$, assume that there exists $\phi \in \mathcal{D} \cap \mathcal{S}_0$ such that
In Sections 3, 4 and 5, we complete the proofs of Theorems 1.1, 1.3 and 1.5, respectively.

By applying the cut-off approach and refining the argument of [20, Theorem 2.3], the boundedness of coefficients can be removed. We herein do not go into details since the corresponding trick is quite similar to the proof of Theorem 1.3.

The outline of this paper is organized as follows: In Section 2, we elaborate regularity of nondegenerate Kolmogorov equation, which plays an important role in dealing with convergence rate of EM scheme for nondegenerate SDEs with rough and unbounded coefficients; In Sections 3, 4 and 5 we complete the proofs of Theorems 1.1, 1.3 and 1.5 respectively.

(C1) (Hypoellipticity) $\nabla^{(2)}b_{t}^{(1)} \in \mathbb{M}_{\non}^{n,cc}$, $\sigma_{t}(x) \in \mathbb{M}_{\non}^{n}$, and

$$
\|b^{(1)}\|_{T,\infty} + \|b^{(2)}\|_{T,\infty} + \|\nabla^{(2)}b^{(1)}\|_{T,\infty} + \sum_{i=0}^{2} \|\nabla^{i}\sigma\|_{T,\infty} + \|\sigma^{-1}\|_{T,\infty} < \infty;
$$

(C2) (Regularity of $b^{(1)}$ w.r.t. spatial variables)

$$
|b_{t}^{(1)}(x) - b_{t}^{(1)}(y)| \leq |x^{(1)} - y^{(1)}|^{\frac{3}{2}}\phi(|x^{(1)} - y^{(1)}|) \quad \text{if } x^{(2)} = y^{(2)},
$$

$$
\|\nabla^{(2)}b_{t}^{(1)}(x) - \nabla^{(2)}b_{t}^{(1)}(y)\|_{HS} \leq \phi(|x^{(2)} - y^{(2)}|)
$$

(C3) (Regularity of $b^{(2)}$ w.r.t. spatial variables)

$$
|b_{t}^{(2)}(x) - b_{t}^{(2)}(y)| \leq |x^{(1)} - y^{(1)}|^{\frac{1}{2}}\phi(|x^{(1)} - y^{(1)}|) + \phi\left(|x^{(2)} - y^{(2)}|\right);
$$

(C4) (Regularity of $b^{(1)}$, $b^{(2)}$ and $\sigma$ w.r.t. time variables)

$$
|b_{t}^{(1)}(x) - b_{s}^{(1)}(x)| + |b_{t}^{(2)}(x) - b_{s}^{(2)}(x)| + \|\sigma_{t}(x) - \sigma_{s}(x)\|_{HS} \leq \phi(|t - s|).
$$

Observe from (C2) and (C3) that $b^{(1)}(\cdot, x^{(2)})$ and $b^{(2)}(\cdot, x^{(2)})$ with fixed $x^{(2)}$ are locally Hölder-Dini continuous of order $\frac{2}{3}$, and $\nabla^{(2)}b^{(1)}(x^{(1)}, \cdot)$ and $b^{(2)}(x^{(1)}, \cdot)$ with fixed $x^{(1)}$ are merely Dini continuous. According to [20, Theorem 1.2], (1.6) admits a unique strong solution under the assumptions (C1)-(C3). In fact, (1.6) is wellposed under (C1)-(C3) with $\phi \in \mathcal{D}_{0} \cap \mathcal{J}_{0}$ in lieu of $\phi \in \mathcal{D} \cap \mathcal{J}_{0}$. Nevertheless, the requirement $\phi \in \mathcal{D} \cap \mathcal{J}_{0}$ is imposed in order to reveal the order of convergence for the EM scheme below.

The continuous-time EM scheme associated with (1.6) is as follows:

$$
\begin{align*}
\begin{cases}
    dY_{t}^{(1)} = b_{t}^{(1)}(Y_{t}^{(1)}, Y_{t}^{(2)})dt, \\
    dY_{t}^{(2)} = b_{t}^{(2)}(Y_{t}^{(1)}, Y_{t}^{(2)})dt + \sigma_{t}(Y_{t}^{(1)}, Y_{t}^{(2)})dW_{t},
\end{cases}
\end{align*}
$$

where $t_{\delta}$ is defined as in (1.3).

Another contribution in this paper reads as below.

**Theorem 1.5.** Under (C1)-(C4),

$$
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_{t} - Y_{t}|^{2}\right) \leq T \phi(C_{T}\sqrt{\delta})^{2}
$$

for some constant $C_{T} \geq 1$, in which

$X_{t} := \begin{pmatrix} X_{t}^{(1)} \\ X_{t}^{(2)} \end{pmatrix}$ and $Y_{t} := \begin{pmatrix} Y_{t}^{(1)} \\ Y_{t}^{(2)} \end{pmatrix}$.

**Remark 1.6.** By applying the cut-off approach and refining the argument of [20 Theorem 2.3], the boundedness of coefficients can be removed. We herein do not go into details since the corresponding trick is quite similar to the proof of Theorem 1.3.

The outline of this paper is organized as follows: In Section 2 we elaborate regularity of nondegenerate Kolmogorov equation, which plays an important role in dealing with convergence rate of EM scheme for nondegenerate SDEs with rough and unbounded coefficients; In Sections 3, 4 and 5 we complete the proofs of Theorems 1.1, 1.3 and 1.5 respectively.
2 Regularity of Non-degenerate Kolmogorov Equation

Let \((e_i)_{i \geq 1}\) be an orthogonal basis of \(\mathbb{R}^n\). For any \(\lambda > 0\), consider the following \(\mathbb{R}^n\)-valued parabolic equation:

\[
\partial_t u_\lambda^t + L_t u_\lambda^t + b_t + \nabla_b u_\lambda^t = \lambda u_\lambda^t, \quad u_\lambda^T = 0_n,
\]

where \(0_n\) is the zero vector in \(\mathbb{R}^n\) and

\[
L_t := \frac{1}{2} \sum_{i,j} \langle (\sigma_t \sigma_t^*) (\cdot) e_i, e_j \rangle \nabla e_i \nabla e_j
\]

with \(\sigma_t^*\) standing for the transpose of \(\sigma_t\). By solving the corresponding coupled forward-backward SDE, one has

\[
u_\lambda^s = \int_s^T e^{-\lambda(t-s)} P_{s,t}^0 \left\{ b_t + \nabla_b u_\lambda^t \right\} dt,
\]

where the semigroup \((P_{s,t}^0)_{t \leq s}\) is generated by \((Z_{t,s,x}^s)_{t \leq s}\) which solves the SDE below

\[
dZ_{t,s,x}^s = \sigma(Z_{t,s,x}^s) dW_t, \quad t > s, \quad Z_{s,s}^s = x.
\]

For notational simplicity, let

\[
\Lambda_{T,\sigma} = e^{\frac{T}{2} \|\nabla \sigma\|_{T,\infty}^2} \|\sigma^{-1}\|_{T,\infty}
\]

and

\[
\tilde{\Lambda}_{T,\sigma} = 48 e^{38 T^2 \|\nabla \sigma\|_{T,\infty}^2} \left\{ 6 \sqrt{2} e^{T \|\nabla \sigma\|_{T,\infty}^2} \|\sigma^{-1}\|_{T,\infty}^4 + T \|\nabla \sigma^{-1}\|_{T,\infty}^2 
\right. \\
\left. + 2 T^2 \|\nabla \sigma\|_{T,\infty}^2 \|\sigma^{-1}\|_{T,\infty}^2 e^{2 T \|\nabla \sigma\|_{T,\infty}^2} \right\}.
\]

Moreover, set

\[
\Upsilon_{T,\sigma} := \sqrt{\tilde{\Lambda}_{T,\sigma}} \left\{ 3 + 2 \|b\|_{T,\infty} + 28 \left( \Lambda_{T,\sigma} + \sqrt{\tilde{\Lambda}_{T,\sigma}} \right) \|b\|_{T,\infty}^2 \right\}.
\]

The lemma below plays a crucial role in investigating our numerical schemes.

**Lemma 2.1.** Under (A1) and (A2), for any \(\lambda \geq 9 \pi \Lambda_{T,\sigma}^2 \|b\|_{T,\infty}^2 + (\|b\|_{T,\infty} + \Lambda_{T,\sigma})^2\),

(i) \((2.1)\) (i.e., \((2.2)\)) enjoys a unique strong solution \(u_\lambda \in C([0, T]; C^1_b(\mathbb{R}^n; \mathbb{R}^n))\);

(ii) \(\|\nabla u_\lambda\|_{T,\infty} \leq \frac{1}{2}\); 

(iii) \(\|\nabla^2 u_\lambda\|_{T,\infty} \leq \Upsilon_{T,\sigma} \int_0^T \frac{e^{-\lambda t}}{t} \tilde{\phi}(\|\sigma\|_{T,\infty} \sqrt{t}) dt\), where \(\tilde{\phi}(s) := \sqrt{\phi^2(s) + s}, \quad s \geq 0\).
Proof. To show (i)-(iii), it boils down to refine the argument of [19] Lemma 2.1. (i) holds for any \( \lambda \geq 4(||b||_{T,\infty} + \Lambda_{T,\sigma})^2 \) via the Banach fixed-point theorem.

In what follows, we aim to show (ii) and (iii) one-by-one. It is easy to see from (2.3) that

\[
d\nabla \eta Z^{s,x}_t = (\nabla \nabla \eta Z^{s,x}_t \sigma_t)(Z^{s,x}_t) dW_t, \quad t \geq s, \quad \nabla \eta Z^{s,x}_s = \eta \in \mathbb{R}^n.
\]

Using Itô’s isometry and Gronwall’s inequality, one has

\[
E|\nabla \eta Z^{s,x}_t|^4 \leq |\eta|^4 e^{288 T^2 \|\nabla \sigma\|^4_{T,\infty}}.
\]

Recall from [19] (2.8) the Bismut formula below

\[
\nabla \eta P^{0}_{s,t} f(x) = \mathbb{E} \left[ \frac{f(Z^{s,x}_t)}{t-s} \int_s^t (\sigma^{-1}_r(Z^{s,x}_r) \nabla \eta Z^{s,x}_r, dW_r) \right], \quad f \in B_b(\mathbb{R}^n).
\]

By the Cauchy-Schwartz inequality, the Itô isometry and (2.8), we obtain that

\[
|\nabla \eta P^{0}_{s,t} f|^2(x) \leq \Lambda_{T,\sigma}^2 \frac{P^{0}_{s,t} f^2(x)}{t-s}, \quad f \in B_b(\mathbb{R}^n),
\]

where \( \Lambda_{T,\sigma} > 0 \) is defined in (2.4). So, one infers from (2.2) and (2.11) that

\[
||\nabla u^\lambda|| \leq \int_s^T e^{-\lambda(t-s)}||\nabla P^{0}_{s,t} \{b_t + \nabla b_t u^\lambda\}|| dt
\leq \Lambda_{T,\sigma}(1 + ||\nabla u^\lambda||_{T,\infty}) \int_0^T \frac{e^{-\lambda t}}{\sqrt{t}} dt
\leq \lambda^{-\frac{1}{2}} \sqrt{\pi} \Lambda_{T,\sigma} ||b||_{T,\infty}(1 + ||\nabla u^\lambda||_{T,\infty}).
\]

Thus, (ii) follows by taking \( \lambda \geq 9\pi \Lambda_{T,\sigma}^2 ||b||_{T,\infty}^2 \).

In the sequel, we intend to verify (iii). Set \( \gamma_{s,t} := \nabla \eta \nabla \eta' Z^{s,x}_t \) for any \( \eta, \eta' \in \mathbb{R}^n \). Notice from (2.7) that

\[
d\gamma_{s,t} = \{ \nabla \gamma_{s,t} \sigma_t(Z^{s,x}_t) + \nabla \nabla \gamma_{s,t} \cdot \nabla \eta \eta' Z^{s,x}_t \sigma_t(Z^{s,x}_t) \} dW_t, \quad t \geq s, \quad \gamma_{s,s} = 0_n.
\]
By the Doob submartingale inequality and the Itô isometry, besides the Gronwall inequality and (2.8), we get that

\[
\sup_{s \leq t \leq T} \mathbb{E}|\gamma_{s,t}|^2 \leq 16T\|\nabla^2 \sigma\|^2_{T,\infty} e^{288T^2\|\nabla \sigma\|^2_{T,\infty} + 2T\|\nabla \sigma\|^2_{T,\infty}} |\eta|^2 |\eta'|^2.
\]

From (2.10) and the Markov property, we have

\[
\nabla_\eta P_{s,t}^0 f(x) = \mathbb{E}\left( \frac{(P_{\frac{s+t}{2}, t}^0 f)(Z_{\frac{s+t}{2}})}{(t-s)/2} \int_s^{t+\frac{s+t}{2}} \langle \sigma_r^{-1}(Z_r^{s,x}) \nabla_\eta Z_r^{s,x}, dW_r \rangle \right).
\]

This further gives that

\[
\frac{1}{2} (\nabla_\eta \nabla_\eta P_{s,t}^0 f)(x) = \mathbb{E}\left( \frac{(P_{\frac{s+t}{2}, t}^0 f)(Z_{\frac{s+t}{2}})}{t-s} \int_s^{t+\frac{s+t}{2}} \langle \sigma_r^{-1}(Z_r^{s,x}) \nabla_\eta Z_r^{s,x}, dW_r \rangle \right)
\]

\[
+ \mathbb{E}\left( \frac{(P_{\frac{s+t}{2}, t}^0 f)(Z_{\frac{s+t}{2}})}{t-s} \int_s^{t+\frac{s+t}{2}} \langle (\nabla_\eta Z_r^{s,x}) \sigma_r^{-1}(Z_r^{s,x}) \nabla_\eta Z_r^{s,x}, dW_r \rangle \right)
\]

\[
+ \mathbb{E}\left( \frac{(P_{\frac{s+t}{2}, t}^0 f)(Z_{\frac{s+t}{2}})}{t-s} \int_s^{t+\frac{s+t}{2}} \langle \sigma_r^{-1}(Z_r^{s,x}) \nabla_\eta \nabla_\eta Z_r^{s,x}, dW_r \rangle \right).
\]

Thus, applying Cauchy-Schwartz’s inequality, [11, Theorem 7.1, p.39] and Itô’s isometry and taking (2.9), (2.11) and (2.12) into consideration, we derive that

\[
|\nabla_\eta \nabla_\eta P_{s,t}^0 f|^2(x)
\leq 12 \left\{ 6\|\sigma^{-1}\|^2_{T,\infty} \mathbb{E}|\nabla P_{\frac{s+t}{2}, t}^0 f|^2(Z_{\frac{s+t}{2}}) \frac{t-s}{(t-s)^{5/2}} \right.
\]

\[
\times \left( \mathbb{E}|\nabla_\eta Z_{\frac{s+t}{2}}^{s,x}|^4 \right)^{1/2} \left( \int_s^{t+\frac{s+t}{2}} \mathbb{E}|\nabla_\eta Z_r^{s,x}|^4 dr \right)^{1/2}
\]

\[
+ \frac{P_{s,t}^0 f^2(x)}{(t-s)^2} \left| \nabla \sigma^{-1}\|^2_{T,\infty} \int_s^{t+\frac{s+t}{2}} \left( \mathbb{E}|\nabla_\eta Z_r^{s,x}|^4 \right)^{1/2} \mathbb{E}|\nabla_\eta Z_r^{s,x}|^4 \right)^{1/2} dr
\]

\[
+ \frac{\tilde{\Lambda}_{T,\sigma} |\eta|^2 |\eta'|^2 P_{s,t}^0 f^2(x)}{(t-s)^2}
\]

\[
\leq \tilde{\Lambda}_{T,\sigma} |\eta|^2 |\eta'|^2 \frac{P_{s,t}^0 f^2(x)}{(t-s)^2},
\]

where \( \tilde{\Lambda}_{T,\sigma} > 0 \) is defined as in (2.5).

Set \( \tilde{f} := f - f(x) \) for \( f \in \mathcal{B}_b(\mathbb{R}^n) \) which verifies

\[
|f(x) - f(y)| \leq \phi(|x - y|), \quad x, y \in \mathbb{R}^n,
\]

(2.14)
where \( \phi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) is increasing and \( \phi^2 \) is concave. For \( f \in \mathcal{B}_b(\mathbb{R}^n) \) such that (2.14), (2.13) implies that

\[
|\nabla \eta \cdot \nabla \eta \cdot P_{s,t}^0 \cdot f|^2(x) = |\nabla \eta \cdot \nabla \eta \cdot P_{s,t}^0 \cdot \tilde{f}|^2(x) \leq \frac{\tilde{\Lambda}_{T,s} |\eta|^2 |\eta'|^2}{(t-s)^2} |E|f(Z_{s,x}^t) - f(x)|^2
\]

(2.15)

where in the second display we have used that

\[
Z_{s,x}^t - x = \int_s^t \sigma_r(Z_{s,x}^r) dW_r,
\]

and utilized Jensen’s inequality as well as Itô’s isometry.

Let \( f_t = b_t + \nabla b_t u_t^\lambda \). For any \( \lambda \geq 9 \pi \Lambda_{T,s}^2, \|b\|_{T,\infty}^2 + 4(\|b\|_{T,\infty} + \Lambda_{T,s})^2 \), note from (ii), (2.11) and (2.13) that

\[
|f_t(x) - f_t(y)| \leq (1 + \|u^\lambda\|_{T,\infty}) \phi(|x - y|) + \|b\|_{T,\infty} \|\nabla u_t^\lambda(x) - \nabla u_t(y)\|1_{\{|x - y| \geq 1\}}
\]

\[
+ \|b\|_{T,\infty} \|\nabla u_t^\lambda(x) - \nabla u_t(y)\|1_{\{|x - y| \leq 1\}}
\]

\[
\leq \frac{3}{2} \phi(|x - y|) + \|b\|_{T,\infty} \sqrt{|x - y|} 1_{\{|x - y| \geq 1\}}
\]

\[
+ 10(\Lambda_{T,s} + \sqrt{\tilde{\Lambda}_{T,s}}) \|b\|_{T,\infty}^2 \sqrt{|x - y|} \sqrt{|x - y|} \log \left(e + \frac{1}{|x - y|}\right) 1_{\{|x - y| \leq 1\}}
\]

\[
\leq \left\{ 3 + 2\|b\|_{T,\infty} + 28 \left( \Lambda_{T,s} + \sqrt{\tilde{\Lambda}_{T,s}} \right) \|b\|_{T,\infty}^2 \right\} \tilde{\phi}(|x - y|)
\]

with \( \tilde{\phi}(s) := \sqrt{\phi^2(s) + s}, s \geq 0 \), where in the second inequality we have used [19] Lemma 2.2 (1)], and that the function \([0, 1] \ni x \mapsto \sqrt{x} \log(e + \frac{1}{x})\) is non-decreasing. As a result, (iii) follows from (2.15).

\[
\square
\]

3 Proof of Theorem 1.1

With Lemma 2.4 in hand, we now in a position to complete the

Proof of Theorem 1.1. Throughout the whole proof, we assume \( \lambda \geq 9 \pi \Lambda_{T,s}^2, \|b\|_{T,\infty}^2 + 4(\|b\|_{T,\infty} + \Lambda_{T,s})^2 \) so that (i)-(iii) in Lemma 2.1 hold. For any \( t \in [0, T] \), applying Itô’s formula to \( x + u_t^\lambda(x), x \in \mathbb{R}^n \), we deduce from (2.1) that

\[
(3.1) \quad X_t + u_t^\lambda(X_t) = x + u_0(x) + \lambda \int_0^t u_s^\lambda(X_s)ds + \int_0^t \{I_{n \times n} + (\nabla u_s^\lambda(\cdot))(X_s)\sigma_s(X_s)\}dW_s,
\]
where $I_{n \times n}$ is an $n \times n$ identity matrix, and that
\begin{equation}
Y_t + u_t^\lambda(Y_t) = x + u_0^\lambda(x) + \lambda \int_0^t u_s^\lambda(Y_s)ds + \int_0^t \{I_{n \times n} + (\nabla u_s^\lambda)(\cdot)\}(Y_s)\sigma_{s\delta}(Y_{s\delta})dW_s
\end{equation}
\begin{equation}
+ \int_0^t \{I_{n \times n} + (\nabla u_s^\lambda)(\cdot)\}(Y_s)\{b_{s\delta}(Y_{s\delta}) - b_s(Y(s))\}ds
\end{equation}
\begin{equation}
+ \frac{1}{2} \int_0^t \sum_{k,j} \{(\sigma_{s\delta}\sigma_{s\delta}^*)(Y_{s\delta}) - (\sigma_s\sigma_s^*)(Y_s)\}e_k e_j (\nabla e_k \nabla e_j u_s^\lambda)(Y_s)ds.
\end{equation}

For notational simplicity, set
\begin{equation}
M_t^\lambda := X_t - Y_t + u_t^\lambda(X_t) - u_t^\lambda(Y_t).
\end{equation}

Using the elementary inequality: $(a + b)^2 \leq (1 + \varepsilon)(a^2 + \varepsilon^{-1}b^2)$ for $\varepsilon, a, b > 0$, we derive from (ii) that
\begin{equation}
|X_t - Y_t|^2 \leq (1 + \varepsilon)|M_t^\lambda|^2 + \varepsilon^{-1}|u_t^\lambda(X_t) - u_t^\lambda(Y_t)|^2
\end{equation}
\begin{equation}
\leq (1 + \varepsilon)\left(|M_t^\lambda|^2 + \frac{\varepsilon^{-1}}{4}|X_t - Y_t|^2\right).
\end{equation}

In particular, taking $\varepsilon = 1$ leads to
\begin{equation}
|X_t - Y_t|^2 \leq \frac{1}{2}|X_t - Y_t|^2 + 2|M_t^\lambda|^2.
\end{equation}

As a consequence,
\begin{equation}
\mathbb{E}\left(\sup_{0 \leq s \leq t} |X_s - Y_s|^2\right) \leq 4\mathbb{E}\left(\sup_{0 \leq s \leq t} |M_s^\lambda|^2\right).
\end{equation}

In what follows, our goal is to estimate the term on the right hand side of (3.4). Observe from the definition of the Hilbert-Schmidt norm that
\begin{equation}
\int_0^t \mathbb{E}\left|\sum_{k,j} \{(\sigma_{s\delta}\sigma_{s\delta}^*)(Y_{s\delta}) - (\sigma_s\sigma_s^*)(Y_s)\}e_k e_j (\nabla e_k \nabla e_j u_s^\lambda)(Y_s)\right|^2 ds
\end{equation}
\begin{equation}
\lesssim_T \|\nabla^2 u^\lambda\|_{T,\infty}^2 \int_0^t \mathbb{E}\|(\sigma_{s\delta}\sigma_{s\delta}^*)(Y_{s\delta}) - (\sigma_s\sigma_s^*)(Y_s)\|^2_{\text{HS}}ds.
\end{equation}

Thus, by Hölder’s inequality, Doob’s submartingale inequality and Itô’s isometry, it follows from (3.1), (3.2) and (3.5) that
\begin{equation}
\mathbb{E}\left(\sup_{0 \leq s \leq t} |M_s^\lambda|^2\right) \leq C_T \left\{\lambda^2 \int_0^t \mathbb{E}|u_s^\lambda(X_s) - u_s^\lambda(Y_s)|^2 ds + (1 + \|\nabla u\|_{T,\infty}^2) \int_0^t \mathbb{E}|b_{s\delta}(Y_s) - b_{s\delta}(Y_{s\delta})|^2 ds\right\}
\end{equation}
\[ + (1 + \|\nabla u\|^2_{T, \infty}) \int_0^t \mathbb{E}|b_s(Y_s) - b_{s,s}(Y_s)|^2 ds \]
\[ + \int_0^t \mathbb{E}\|\{(\nabla u_s^*)(X_s) - \nabla u_s^*(Y_s)\}\sigma_s(X_s)\|^2_{HS} ds \]
\[ + (1 + \|\nabla u\|^2_{T, \infty}) \int_0^t \mathbb{E}\|\sigma_s(X_s) - \sigma_{s,s}(Y_s)\|^2_{HS} ds \]
\[ + \|\nabla^2 u^\lambda\|^2_{T, \infty} \int_0^t \mathbb{E}\|\{\sigma_{s,s}(Y_s) - \sigma_{s,s}(Y_s)\}\sigma_{s,s}(Y_s)\|^2_{HS} ds \]
\[ + \|\nabla^2 u^\lambda\|^2_{T, \infty} \int_0^t \mathbb{E}\|\sigma_s(Y_s)\{\sigma_s(Y_s) - \sigma_{s,s}(Y_s)\}\|^2_{HS} ds \]
\[ + (1 + \|\nabla u\|^2_{T, \infty}) \int_0^t \mathbb{E}\|\sigma_s(X_s) - \sigma_{s,s}(X_s)\|^2_{HS} ds \]
\[ + \|\nabla^2 u^\lambda\|^2_{T, \infty} \int_0^t \mathbb{E}\|\sigma_s(Y_s)\{\sigma_s(Y_s) - \sigma_{s,s}(Y_s)\}\|^2_{HS} ds \]
\[ + \|\nabla^2 u^\lambda\|^2_{T, \infty} \int_0^t \mathbb{E}\|\{\sigma_s(Y_s) - \sigma_{s,s}(Y_s)\}\sigma_{s,s}(Y_s)\|^2_{HS} ds \]
\[ =: \sum_{i=1}^{10} I_i(t) \]

for some constant \( C_T > 0 \). Also, applying Hölder’s inequality and Itô’s isometry, we deduce from (A1) that

\[ (3.6) \quad \mathbb{E}|Y_t - Y_{t,i}|^2 \leq \beta_T \delta \]

for some constant \( \beta_T \geq 1 \). By Taylor’s expansion, it is readily to see that

\[ (3.7) \quad I_1(t) + I_2(t) \lesssim \{\lambda^2\|\nabla u^\lambda\|^2_{T, \infty} + \|\nabla^2 u^\lambda\|^2_{T, \infty}\|\sigma\|^2_{T, \infty}\} \int_0^t \mathbb{E}|X_s - Y_s|^2 ds. \]

From (A3), one has

\[ (3.8) \quad I_3(t) + \sum_{i=1}^{10} I_i(t) \lesssim_{T} \{1 + \|\nabla u^\lambda\|^2_{T, \infty} + \|\nabla^2 u^\lambda\|^2_{T, \infty}\|\sigma\|^2_{T, \infty}\} \phi(\sqrt{\delta})^2. \]

In view of (A2), we derive that

\[ I_2(t) + \sum_{i=5}^{7} I_i(t) \]
\[ \lesssim \{1 + \|\nabla u^\lambda\|^2_{T, \infty}\} \int_0^t \mathbb{E}\phi(|Y_s - Y_{s,s}|)^2 ds \]
\[ + \{1 + \|\nabla u^\lambda\|^2_{T, \infty}\}\|\nabla\sigma\|^2_{T, \infty} \int_0^t \mathbb{E}|X_s - Y_s|^2 ds \]
\[ + \{1 + \|\nabla u^\lambda\|^2_{T, \infty} + \|\nabla^2 u^\lambda\|^2_{T, \infty}\|\sigma\|^2_{T, \infty}\}\|\nabla\sigma\|^2_{T, \infty} \int_0^t \mathbb{E}|Y_s - Y_{s,s}|^2 ds. \]

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Thus, taking (3.6)-(3.9) into account and applying Jensen’s inequality gives that
\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |M_s^\lambda|^2 \right) \lesssim_T C_{T,\sigma,\lambda}\{\delta + \phi(\beta_T \sqrt{\delta})^2\} + C_{T,\sigma,\lambda} \int_0^t \mathbb{E}|X_s - Y_s|^2 ds,
\]
where
\[
(3.10) \quad C_{T,\sigma,\lambda} := \{1 + \|\nabla \sigma\|_{T,\infty}^2\}\left\{\frac{5}{4} + (1 + \lambda^2)\|\nabla^2 u\|_{T,\infty}^2\|\sigma\|_{T,\infty}^2\right\}.
\]
Owing to \( \phi \in \mathcal{D} \), we conclude that \( \phi(0) = 0 \), \( \phi' > 0 \) and \( \phi'' < 0 \) so that, for any \( c > 0 \) and \( \delta \in (0, 1) \),
\[
\phi(c\delta) = \phi(0) + \phi'(\xi)c\delta \geq \phi'(c)c\delta
\]
by recalling \( \delta \in (0, 1) \), where \( \xi \in (0, c\delta) \). This further implies that
\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |M_s^\lambda|^2 \right) \lesssim_T C_{T,\sigma,\lambda}\phi(\beta_T \sqrt{\delta})^2 + C_{T,\sigma,\lambda} \int_0^t \mathbb{E}|X_s - Y_s|^2 ds.
\]
Substituting this into (3.10) gives that
\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |X_s - Y_s|^2 \right) \lesssim_T C_{T,\sigma,\lambda}\phi(\beta_T \sqrt{\delta})^2 + C_{T,\sigma,\lambda} \int_0^t \mathbb{E}|X_s - Y_s|^2 ds.
\]
Thus, Gronwall’s inequality implies that there exists \( \tilde{C}_T > 0 \) such that
\[
(3.11) \quad \mathbb{E}\left( \sup_{0 \leq s \leq t} |X_s - Y_s|^2 \right) \leq \tilde{C}_T C_{T,\sigma,\lambda}e^{\tilde{C}_T C_{T,\sigma,\lambda}\phi(\beta_T \sqrt{\delta})^2}.
\]
So the desired assertion holds immediately.

4 Proof of Theorem 1.3

We shall adopt the cut-off approach to finish the proof.

Proof of Theorem 1.3. Take \( \psi \in C_b^\infty(\mathbb{R}^n) \) such that \( 0 \leq \psi \leq 1 \), \( \psi(r) = 1 \) for \( r \in [0, 1] \) and \( \psi(r) = 0 \) for \( r \geq 2 \). For any \( t \in [0, T] \) and \( k \geq 1 \), set define the cut-off functions
\[
b_t^{(k)}(x) = b_t(x)\psi(|x|/k) \quad \text{and} \quad \sigma_t^{(k)}(x) = \sigma_t(\psi(|x|/k)x), \quad x \in \mathbb{R}^n.
\]
It is easy to see that \( b^{(k)} \) and \( \sigma^{(k)} \) satisfy (A1). For fixed \( k \geq 1 \), consider the following SDE
\[
(4.1) \quad dX_t^{(k)} = b_t^{(k)}(X_t^{(k)})dt + \sigma_t^{(k)}(X_t^{(k)})dW_t, \quad t > 0, \quad X_0^{(k)} = X_0 = x.
\]
The corresponding continuous-time EM of (4.1) is defined by
\[
(4.2) \quad dY_t^{(k)} = b_t^{(k)}(Y_t^{(k)})dt + \sigma_t^{(k)}(Y_t^{(k)})dW_t, \quad t > 0, \quad Y_0^{(k)} = X_0 = x.
\]
Applying the BDG inequality, the Hölder inequality and the Gronwall inequality, we deduce from (A1') that

\begin{equation}
(4.3) \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t|^4\right) + \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t|^4\right) + \mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t^{(k)}|^4\right) + \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^{(k)}|^4\right) \leq C_T
\end{equation}

for some constant \(C_T > 0\). Note that

\begin{align*}
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - Y_t|^2\right) & \leq 2\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - X_t^{(k)}|^2\right) + 2\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t^{(k)} - Y_t^{(k)}|^2\right) \\
& \quad + 2\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t - Y_t^{(k)}|^2\right)
\end{align*}

\(=: I_1 + I_2 + I_3\).

For the terms \(I_1\) and \(I_3\), in terms of the Chebyshev inequality we find from (4.3) that

\begin{align*}
I_1 + I_3 & \leq \mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - X_t^{(k)}|^2 1_{\{\sup_{0 \leq t \leq T} |X_t| \geq k\}}\right) \\
& \quad + \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t - Y_t^{(k)}|^2 1_{\{\sup_{0 \leq t \leq T} |Y_t| \geq k\}}\right)
\end{align*}

\begin{align*}
\leq & \sqrt{\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t|^4\right) + \mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t^{(k)}|^4\right)} \frac{\sqrt{\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t|^2\right)}}{k} \\
& + \sqrt{\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t|^4\right) + \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^{(k)}|^4\right)} \frac{\sqrt{\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t|^2\right)}}{k} \\
\leq & T \frac{1}{k},
\end{align*}

where in the first display we have used the facts that \(\{X_t \neq X_t^{(k)}\} \subset \{\sup_{0 \leq s \leq t} |X_s| \geq k\}\) and \(\{Y_t \neq Y_t^{(k)}\} \subset \{\sup_{0 \leq s \leq t} |Y_s| \geq k\}\). Observe from (A1') that \(9\pi \Lambda_{T,\sigma^{(k)}}^2 \|b^{(k)}\|^2_T + 4(\|b^{(k)}\|_{T,\infty} + \Lambda_{T,\sigma^{(k)}})^2 \leq e^{ck^2}\) for some \(c > 0\). Next, according to (3.11), by taking \(\lambda = e^{ck^2}\) there exits \(C_T > 0\) such that

\[I_2 \leq e^{C_T C_{T,\sigma^{(k)},\lambda} \phi_k(\beta_T \sqrt{\delta})^2}.\]

Herein, \(C_{T,\sigma^{(k)},\lambda} > 0\) is defined as in (3.10) with \(\sigma\) and \(u^\lambda\) replaced by \(\sigma^{(k)}\) and \(u^{\lambda,k}\), respectively, where \(u^{\lambda,k}\) solves (2.2) by writing \(b^{(k)}\) instead of \(b\). Consequently, we conclude that

\begin{equation}
(4.4) \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - Y_t|^2\right) \leq \frac{\bar{c}_0}{k} + \bar{c}_0 e^{C_T C_{T,\sigma^{(k),\lambda} \phi_k(\beta_T \sqrt{\delta})^2}
\end{equation}

for some \(\bar{c}_0 > 0\). For any \(\varepsilon > 0\), taking \(k = \frac{2\bar{c}_0}{\varepsilon}\) and letting \(\delta\) go to zero implies that

\[\lim_{\delta \to 0} \mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - Y_t|^2\right) \leq \varepsilon.\]
Thus, (1.4) follows due to the arbitrariness of $\varepsilon$.

For $\phi_k(s) = e^{\varepsilon k t^4} s^\alpha$, $s \geq 0$, with $\alpha \in (0, 1/2]$, we deduce from Lemma 2.1 (iii) that

$$\|\nabla^2 u^\lambda_k\|_{T, \infty} \leq \frac{1}{2}$$

whenever

$$\lambda \geq \left\{2Y_{T, \sigma(k)} \left(e^{\varepsilon k t^4} \|\sigma(k)\|_{T, \infty}^\alpha \Gamma(\alpha/2) + \|\sigma(k)\|_{T, \infty}^{1/2} \Gamma(1/4)\right)\right\}^{2/\alpha} + 9\pi (\Lambda_{T, \sigma(k)})^2 \|b(k)\|^2_{T, \infty} + 4(\|b(k)\|_{T, \infty} + \Lambda_{T, \sigma(k)})^2.

Since the right hand side of (4.6) can be bounded by $e^{C_T k t^4}$ for some constant $C_T > 0$ due to (A1'), we can take $\lambda = e^{C_T k t^4}$ so that (1.3) holds. Thus, (1.4), together with (4.5) and (A1'), yields that

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t - Y_t|^2\right) \leq C_T \frac{k}{T} + C_T e^{C_T k t^4} \delta^\alpha$$

for some constants $C_T, \tilde{C}_T > 0$. Thus, (1.5) follows immediately by taking

$$k = (\tilde{C}_T \log \log \delta^{-\alpha})^{1/4}.

\square

5 Proof of Theorem 1.5

The proof of Theorem 1.5 relies on regularization properties of the following $\mathbb{R}^{2n}$-valued degenerate parabolic equation

$$\begin{align*}
\partial_t u_t^\lambda + \mathcal{L}_t^{b, \sigma} u_t^\lambda + b_t &= \lambda u_t^\lambda, \\
0 &= u_T^\lambda, \\
0 &= u_T, \\
0 &= 0_{2n}, \\
\end{align*}$$

where $0_{2n}$ is the zero vector in $\mathbb{R}^{2n}$,

$$b_t := \begin{pmatrix} b_t^{(1)} \\ b_t^{(2)} \end{pmatrix}$$

and

$$\mathcal{L}_t^{b, \sigma} u_t^\lambda := \frac{1}{2} \sum_{i,j=1}^n (\sigma^*_i \sigma^*_j) (\cdot) e_i \otimes e_j \nabla e_i \nabla e_j u_t^\lambda + \nabla b_t^{(1)} u_t^\lambda + \nabla b_t^{(2)} u_t^\lambda.$$

For any $\phi \in \mathcal{D}_0 \cap \mathcal{I}_0$, set

$$\mathcal{D}_\phi := \sup_{t \in [0, T]} \left\{ |b_t^{(1)}|_{\phi(2/3), \infty} + \|\nabla b_t^{(1)}\|_{\infty, \phi} + \|\sigma_t^{-1}\|_{\infty, \phi} + \|\sigma_t^{-1}\|_{\phi(2/3), \phi} + \|b_t^{(2)}\|_{\phi(2/3), \phi} \right\}$$

and

$$\mathcal{D}_\phi := \mathcal{D}_\phi + \sup_{t \in [0, T]} |b_t^{(2)}|_{\phi(2/3), \phi^{7/2}}, \quad \mathcal{D}'_\phi := \mathcal{D}_\phi + \sup_{t \in [0, T]} |\nabla b_t^{(2)}\|_{\phi(1/3), \infty},$$

where, for $\phi \in \mathcal{I}_0$,

$$\phi_{[a]}(t) := t^\alpha \phi(t) 1\{t \geq 1\} + 2c_\alpha t 1\{t > 1\} \quad \text{with} \quad c_\alpha := \sup_{s \in (0, 1]} (s^\alpha \phi(s)).$$

The following key lemma on regularity estimate of solution to (5.1) is cited from [20, Theorem 2.3] and is an essential ingredient in analyzing numerical approximation.
Lemma 5.1. Under (C1), (5.1) has a unique smooth solution such that for all \( t \in [0, T] \),
\[
\|\nabla u_t^\lambda\|_{1_{\{1/3, \infty\}}} + \|\nabla (\nabla^2) u_t^\lambda\|_{\infty} + \|\nabla (\nabla^2) u_t^\lambda\|_{\phi^{3/2}}
\leq C \int_0^t e^{-\lambda(t-s)} \frac{\phi((t-s)^{\frac{1}{2}})}{t-s} [b_s]_{\phi^{[3/2], \phi^{7/2}}} ds,
\]
(5.2)
and
\[
\|\nabla u_t^\lambda\|_{1_{\{1/3, \infty\}}} + \|\nabla (\nabla^2) u_t^\lambda\|_{\infty} \leq C' \int_0^t e^{-\lambda(t-s)} \frac{\phi((t-s)^{\frac{1}{2}})}{t-s} [b_s]_{\phi^{[3/2], \phi^{7/2}}} ds,
\]
(5.3)
where \( C = C(\phi, \mathcal{D}_\phi) \) and \( C' = C'(\phi, \mathcal{D}'_\phi) \) are increasing w.r.t. \( \mathcal{D}_\phi \) and \( \mathcal{D}'_\phi \), respectively.

From now on, we move forward to complete the

**Proof of Theorem 1.5.** For notational simplicity, set
\[
X_t := \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}, \quad Y_t := \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \end{pmatrix} \quad \text{and} \quad b_t(x) := \begin{pmatrix} b_t^{(1)}(x) \\ b_t^{(2)}(x) \end{pmatrix}, \quad x \in \mathbb{R}^{2n}.
\]
Then (1.6) and (1.7) can be reformulated respectively as
\[
dx_t = b_t(X_t)dt + \begin{pmatrix} 0_{n \times n} \\ \sigma_t \end{pmatrix} (X_t)dW_t, \quad t > 0, \quad X_0 = x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n},
\]
where \( 0_{n \times n} \) is an \( n \times n \) zero matrix, and
\[
dy_t = b_s(Y_t)dt + \begin{pmatrix} 0_{n \times n} \\ \sigma_{t_s} \end{pmatrix} (Y_t)dW_t, \quad t > 0, \quad Y_0 = x \in \mathbb{R}^{2n}.
\]
Note from (5.2) that there exists an \( \lambda_0 > 0 \) sufficiently large such that
\[
\|\nabla u^\lambda\|_{T, \infty} + \|\nabla (\nabla^2) u^\lambda\|_{T, \infty} + \|\nabla (\nabla^2) u^\lambda\|_{T, \infty} \leq \frac{1}{2}, \quad \lambda \geq \lambda_0.
\]
(5.4)
Applying Itô’s formula to \( x + u_t^\lambda(x) \) for any \( x \in \mathbb{R}^{2n} \), we deduce that
\[
X_t + u_t^\lambda(X_t) = x + u_0^\lambda(x) + \lambda \int_0^t u_s^\lambda(X_s)ds + \int_0^t \begin{pmatrix} 0_{n \times n} \\ \sigma_s \end{pmatrix} (X_s)dW_s + \int_0^t (\nabla (\nabla^2) u_s^\lambda)(X_s)ds,
\]
(5.5)
and that
\[
Y_t + u_t^\lambda(Y_t) = x + u_0^\lambda(x) + \lambda \int_0^t u_s^\lambda(Y_s)ds
\]
\[
+ \int_0^t \{I_{2n \times 2n} + (\nabla u_s)(\cdot)\} (Y_s) \{b_{s_s}(Y_{s_s}) - b_{s}(Y_s)\} ds
\]
\[
+ \int_0^t \begin{pmatrix} 0_{n \times n} \\ \sigma_{s_s} \end{pmatrix} (Y_s)dW_s + \int_0^t (\nabla (\nabla^2) u_s^\lambda)(Y_s)ds
\]
\[
+ \frac{1}{2} \sum_{k,j=1}^n \langle \{\sigma_{s_s} \sigma_{s_{s_s}}^*(Y_{s_s}) - (\sigma_s \sigma_s^*)(Y_s)\}e_k, e_j \rangle (\nabla (\nabla^2) u_s^\lambda)(Y_s)ds,
\]
(5.6)
where $\mathbf{I}_{2n \times 2n}$ is an $2n \times 2n$ identity matrix. Thus, using Hölder’s inequality, Doob’s submartingale inequality and Itô’s isometry and taking (3.5) into consideration gives that

\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |M^\lambda_s|^2 \right) \leq C_{0,T} \left\{ \int_0^t \mathbb{E} |u^\lambda_s(X_s) - u^\lambda_s(Y_s)|^2 \, ds \\
+ (1 + \|\nabla u^\lambda\|^2_{T,\infty}) \int_0^t \mathbb{E} |b_{s\delta}(Y_s) - b_{s\delta}(Y_s)|^2 \, ds \\
+ (1 + \|\nabla u^\lambda\|^2_{T,\infty}) \int_0^t \mathbb{E} |b_s(Y_s) - b_{s\delta}(Y_s)|^2 \, ds \\
+ \int_0^t \mathbb{E} \|\{(\nabla^2 u^\lambda_s)(X_s) - \nabla^2 u^\lambda_s(Y_s)\}\sigma_s(X_s)\|_{\text{HS}}^2 \, ds \\
+ (1 + \|\nabla^2 u^\lambda\|^2_{T,\infty}) \int_0^t \mathbb{E} \|\sigma_{s\delta}(X_s) - \sigma_{s\delta}(Y_s)\|_{\text{HS}}^2 \, ds \\
+ (1 + \|\nabla^2 u^\lambda\|^2_{T,\infty}) \int_0^t \mathbb{E} \|\sigma_s(X_s) - \sigma_{s\delta}(X_s)\|_{\text{HS}}^2 \, ds \\
+ ||\nabla^2 \nabla^2 u^\lambda|^2_{T,\infty} \int_0^t \mathbb{E} \|\{\sigma_s(Y_s) - \sigma_{s\delta}(Y_s)\}\sigma_{s\delta}^s(Y_s)\|_{\text{HS}}^2 \, ds \\
+ ||\nabla^2 \nabla^2 u^\lambda|^2_{T,\infty} \int_0^t \mathbb{E} \|\sigma_s(Y_s)\{\sigma_{s\delta}^s(Y_s) - \sigma_{s\delta}(Y_s)\}\|_{\text{HS}}^2 \, ds \\
+ ||\nabla^2 \nabla^2 u^\lambda|^2_{T,\infty} \int_0^t \mathbb{E} \|\sigma_{s\delta}(Y_s)\{\sigma_s^s(Y_s) - \sigma_{s\delta}(Y_s)\}\|_{\text{HS}}^2 \, ds \\
+ ||\nabla^2 \nabla^2 u^\lambda|^2_{T,\infty} \int_0^t \mathbb{E} \|\{\sigma_s(Y_s) - \sigma_{s\delta}(Y_s)\}\sigma_{s\delta}^s(Y_s)\|_{\text{HS}}^2 \, ds \right\}
\]

\[= \sum_{i=1}^{10} J_i(t) \]

for some constant $C_{0,T} > 0$, where $M^\lambda_t$ is defined as in (3.3). By using Hölder’s inequality and [11, Theorem 7.1, p.39], (C1) implies that

\[(5.7) \quad \mathbb{E}|Y_t - Y_{t\delta}|^p \lesssim \delta^{\frac{p}{2}}, \quad p \geq 1.\]

Utilizing Taylor’s expansion, one gets from (3.6) and (5.4) that

\[J_1(t) + J_4(t) + J_5(t) \lesssim \{1 + \|\nabla u^\lambda\|^2_{T,\infty} + \|\nabla^2 \nabla^2 u^\lambda\|^2_{T,\infty}\|\sigma\|^2_{T,\infty}\} \int_0^t \mathbb{E}|X_s - Y_s|^2 \, ds \]
\[+ \{1 + \|\nabla^2 u^\lambda\|^2_{T,\infty}\} \int_0^t \mathbb{E}|Y_s - Y_{s\delta}|^2 \, ds \]
\[\lesssim \delta + \int_0^t \mathbb{E}|X_s - Y_s|^2 \, ds.\]

Next, (C1), (C5) and (5.3) yield that

\[J_3(t) + J_6(t) + J_9(t) + J_{10}(t) \lesssim \phi(\sqrt{\delta})^2.\]
Additionally, by virtue of (C1), (C2), and (5.4), we infer from (C3) that

\[
J_2(t) + J_7(t) + J_8(t) \lesssim \delta + \int_0^t \mathbb{E}|b_{s_3}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}(Y_s^{(1)}, Y_s^{(2)})|^2 ds
\]
\[
+ \int_0^t \mathbb{E}|b_{s_3}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}(Y_s^{(1)}, Y_s^{(2)})|^2 ds
\]
\[
\leq C_{1,T} \left\{ \delta + \int_0^t \mathbb{E}|b_{s_3}^{(1)}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}^{(1)}(Y_s^{(1)}, Y_s^{(2)})|^2 ds
\]
\[
+ \int_0^t \mathbb{E}|b_{s_3}^{(2)}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}^{(2)}(Y_s^{(1)}, Y_s^{(2)})|^2 ds
\]
\[
+ \int_0^t \mathbb{E}|b_{s_3}^{(1)}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}^{(1)}(Y_s^{(1)}, Y_s^{(2)})|^2 ds
\]
\[
+ \int_0^t \mathbb{E}|b_{s_3}^{(2)}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}^{(2)}(Y_s^{(1)}, Y_s^{(2)})|^2 ds \right\}
\]
\[
= C_{1,T} \delta + \sum_{i=1}^4 \Lambda_i(t)
\]

for some constant \(C_{1,T} > 0\). From (C2), (C3), (5.7) and \(\phi \in \mathcal{D}^e\), we derive from Hölder’s inequality and Jensen’s inequality that

\[
\Lambda_1(t) + \Lambda_2(t) \lesssim \sum_{i=1}^2 \int_0^t \mathbb{E} \left( \left| \frac{b_{s_3}^{(i)}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}^{(i)}(Y_s^{(1)}, Y_s^{(2)})}{|Y_s^{(1)} - Y_s^{(1)}|^\frac{3}{2} \phi(|Y_s^{(1)} - Y_s^{(1)})|} \right|^2 ds
\]
\[
\lesssim \int_0^t \mathbb{E}(|Y_s^{(1)} - Y_s^{(1)}|^\frac{3}{2} \phi(|Y_s^{(1)} - Y_s^{(1)}|)|^2 ds
\]
\[
\lesssim \int_0^t \left( \mathbb{E} \phi(|Y_s^{(1)} - Y_s^{(1)})|^2 |Y_s^{(1)} - Y_s^{(1)}| \right)^\frac{1}{1+e} \left( \mathbb{E} |Y_s^{(1)} - Y_s^{(1)}|^{\frac{4(1+e)}{4+e}} \right)^\frac{1}{1+e} ds
\]
\[
\lesssim \delta^\frac{3}{2} \phi(C_{2,T} \sqrt{\delta})^2
\]

for some constant \(C_{2,T} > 0\). With regard to the term \(\Lambda_3(t)\), (C1) and (5.7) leads to

\[
\Lambda_3(t) \lesssim \|
abla^{(2)} b^{(1)} \|^2_{T,\infty} \int_0^t \mathbb{E}|Y_s^{(1)} - Y_s^{(1)}|^2 ds \lesssim \delta.
\]

Since \([b_t^{(2)}]_{\infty} < \infty\) due to (C3), observe from Jensen’s inequality and (5.7) that

\[
\Lambda_4(t) \lesssim \int_0^t \mathbb{E} \left( \frac{|b_{s_3}^{(2)}(Y_s^{(1)}, Y_s^{(2)}) - b_{s_3}^{(2)}(Y_s^{(1)}, Y_s^{(2)})|}{\phi(|Y_s^{(2)} - Y_s^{(2)}|)} \right)^2 ds
\]
\[
\lesssim \int_0^t \mathbb{E} \phi(|Y_s^{(2)} - Y_s^{(2)}|)^2 ds
\]
\[
\lesssim \phi(C_{3,T} \sqrt{\delta})^2
\]
for some constant $C_{3,T} > 0$. Consequently, we arrive at

$$
\mathbb{E}\left(\sup_{0 \leq s \leq t} |X_s - Y_s|^2\right) \lesssim_T \phi(C_{4,T} \sqrt{\delta})^2 + \int_0^t \mathbb{E}|X(s) - Y(s)|^2 ds
$$

for some constant $C_{4,T} \geq 1$. Thus, the desired assertion follows from the Gronwall inequality.

\begin{proof}
\end{proof}

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