UNIPOTENT AND PROUNIPOTENT GROUPS:
COHOMOLOGY AND PRESENTATIONS

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A pro-affine algebraic group $G$, over the field $k$ (which we always take to be algebraically closed of characteristic zero) is an inverse limit of affine algebraic groups [3]. If the algebraic groups in the inverse system are unipotent, we call $G$ prounipotent. Pro-affine algebraic groups arise naturally in the theory of finite-dimensional $k$-representations of discrete and analytic groups [3, 4, 9] and prounipotent groups arise naturally as the prounipotent radicals of pro-affine groups. Our interest in prounipotents is motivated by possible applications to finite-dimensional representation theory.

The extension of the category of unipotent groups to that of prounipotents makes possible "combinatorial group theory" (free groups and presentations):

If $X$ is a set, there is a prounipotent group $F(X)$ containing $X$ such that for every prounipotent group $H$ and function $f : X \rightarrow H$ with $\text{Card} \{X - f^{-1}(L)\}$ finite for every closed subgroup $L$ of finite codimension in $H$ there is a unique homomorphism $\tilde{f} : F(X) \rightarrow H$ extending $f$ [5, 2.1]. Every prounipotent group $G$ is a homomorphic image of a free prounipotent group $F$ so there is an exact sequence (*) $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$. We can choose (*) with $R \subseteq (F, F)$ and in this case we call (*) a proper presentation of $G$. If $F = F(X)$ in (*), we call $X$ generators for $G$ and we call generators of $R$, as a prounipotent normal subgroup of $F$, relations for $G$.

As for pro-$p$ groups [11], the numbers of generators and relations for $G$ have a cohomological interpretation. Cohomology here is in the category of polynomial representations as in [2]. There is a unique simple in this category (the one-dimensional trivial module $k$) so cohomological dimension is defined as $\text{cd}(G) = \inf \{i \mid H^n(G, k) = 0, n > i\}$.

**Theorem 1** [5, 2.8 and 2.9]. *The following are equivalent for prounipotent $G$:

(a) $G$ is free,
(b) $G$ is a projective group in the category of prounipotent groups,
(c) $\text{cd}(G) \leq 1$.

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PROPOSITION 2 [5, 1.14]. If $H$ is a pronipotent subgroup of the pronipotent group $G$ then $\text{cd}(H) \leq \text{cd}(G)$.

COROLLARY 3 [5, 2.10]. A closed subgroup of a free pronipotent group is free.

THEOREM 4 [5, 3.2 AND 3.11]. Let $1 \rightarrow R \rightarrow F(X) \rightarrow G \rightarrow 1$ be a proper presentation of the pronipotent group $G$. Then $d = \dim(H^1(G, k)) = \text{Card}(X)$ and $r = \dim(H^2(G, k))$ is the minimal number of normal generators of $R$ as a pronipotent subgroup of $F$. Thus, $d$ is the minimal number of generators and $r$ is the minimal number of relations for $G$.

The preceding results are proved similarly to the analogous results for pro-$p$ groups. (See [11].) Special properties of pronipotents establish

THEOREM 5 [5, 3.14]. If $G$ is pronipotent and $\dim(H^n(G, k)) = 1$ for some $n \geq 1$, then $\text{cd}(G) = n$.

If $G$ is one-relator, $\dim(H^2(G, k)) = 1$ by Theorem 4 so

COROLLARY 6 [5, 3.15]. A one-relator pronipotent group has cohomological dimension 2.

(Corollary 6 is the pronipotent analogue of [9, 11.2, p. 633].)

When $G$ is finite-dimensional, $\text{cd}(G) = \dim(G)$, so the only one-relator $G$ is $k \times k$. In general, there is a Golod-Shafarevich type inequality relating the numbers of generators and relations.

THEOREM 7 [7, 3.11]. Let $G$, $d$, and $r$ be as in Theorem 4 with $r \neq 0$ and $G$ finite-dimensional. Then $r \geq d^2/4$, with strict inequality unless $G = k \times k$, when $r = 1$ and $d = 2$.

The proof of Theorem 7 relies on the notion of a group algebra developed in [6 and 7]: The coordinate ring $k[G]$ of the pronipotent group $G$ is a $G$-bimodule so that the right translations define an embedding $\rho$ of $G$ in the units of the $G$-module endomorphism ring of $k[G]$ as a left $G$-module. We denote $\text{End}_G(k[G])$ by $k\langle G \rangle$.

When $G$ is finitely generated, $k\langle G \rangle$ is like a group algebra for $G$ (if $B$ is a finite-dimensional associative algebra, $U_1(B)$ is the group of units of $B$ congruent to 1 modulo the radical).

THEOREM 8 [7, 2.8]. If $G$ is a finitely generated pronipotent group and $B$ a finite-dimensional associative $k$-algebra any polynomial representation $G \rightarrow U_1(B)$ extends uniquely to an algebra homomorphism $k\langle G \rangle \rightarrow B$. Moreover, this property characterizes $k\langle G \rangle$. 

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Theorem 9 [7, 2.10]. Let $G$ be a prounipotent group with a proper presentation $1 \rightarrow R \rightarrow F\langle x_1, \ldots, x_d \rangle \rightarrow G \rightarrow 1$ where $\{s_1, \ldots, s_r\}$ is a minimal set of normal generators of $R$. Then $k\langle G \rangle$ is the formal (noncommutative) power series algebra $k\langle \rho(x_1) - 1, \ldots, \rho(x_d) - 1 \rangle$ modulo the ideal generated by $\{\rho(s_j) - 1\}$.

Theorem 9 is proved by first treating the case where $G$ is free on $\{x_1, \ldots, x_d\}$ [6, 1.5] (so $k\langle G \rangle$ is a formal power series algebra). Then the embedding $\rho: G \rightarrow k\langle G \rangle$ embeds $G$ in the ring of formal power series. This extends (in fact, reproves) the Magnus embedding [1, p. 151] of the free discrete group, and provides a concrete description of the free prounipotent group on $d$ generators as the Zariski closure of the subgroup generated by $\{1 + t_i\}$ in the group of units of constant term 1 in $k\langle t_1, \ldots, t_d \rangle$. Using this description, we obtain

Theorem 10 [6, 2.7]. The associated graded Lie algebra [1, p. 145] of the lower central series of a free prounipotent group on $d$ generators is a free $k$-Lie algebra on $d$ generators.

The proofs of the preceding theorems use a description of $k[G]$ as an ascending union of $G$-submodules $E_i(G)$ defined by $E_{-1}(G) = 0$ and $E_{i+1}(G)/E_i(G) = (k[G]/E_i(G))^G$. If $G$ is finitely generated then the numbers $c_i(G) = \dim(E_i(G))$ are all finite, and we have

Proposition 11 [6, 1.3 AND 7, 3.12]. Let $G$ be prounipotent,
(a) $G$ is free on $d$ generators if and only if $c_i(G) = 1 + d + d^2 + \cdots + d^i$ for $i \geq 0$.
(b) $G$ is finite-dimensional if and only if the series $\{c_i(G)\}$ has polynomial growth.

Finally, we record some applications to the finite-dimensional representation theory of a discrete group $\Gamma$. We let $A(\Gamma)$ be the pro-algebraic hull of $\Gamma$ [10, 2.2] and $R_u(\Gamma)$ the prounipotent radical of $A(\Gamma)$.

Theorem 12 [5, 4.3]. If $\Gamma$ contains a free subgroup of finite index, $R_u(\Gamma)$ is a free unipotent group.

If $\Gamma$ is torsion free nilpotent, then $R_u(\Gamma)$ is finite-dimensional, and there is an embedding $\Gamma \rightarrow R_u(\Gamma)$. (This is the Malcev embedding for which our methods provide a new proof [6, 5.12].) In this case we have $H^i(\Gamma, k) = H^i(R_u(\Gamma), k)$ [7, 3.8] so we can apply Theorem 7 to obtain an inequality relating the ranks of the first and second cohomology groups of $\Gamma$.

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