Indexed monoidal algebras are introduced as an equivalent structure for self-dual compact closed categories, and a coherence theorem is proved for the category of such algebras. Turing automata and Turing graph machines are defined by generalizing the classical Turing machine concept, so that the collection of such machines becomes an indexed monoidal algebra. On the analogy of the von Neumann data-flow computer architecture, Turing graph machines are proposed as potentially reversible low-level universal computational devices, and a truly reversible molecular size hardware model is presented as an example.

1 Introduction

The importance of reversibility in computation has been argued at several platforms in connection with the speed and efficiency of modern-day computers. As stated originally by Landauer [19] and re-emphasized by Abramsky [2]: “it is only the logically irreversible operations in a physical computer that necessarily dissipate energy by generating a corresponding amount of entropy for every bit of information that gets irreversibly erased”. Abramsky’s remedy for this situation in [2] is to translate high level functional programs in a syntax directed way into a simple kind of automata which are immediately seen to be reversible. The concept strong compact closed category [3] has been introduced and advocated as a theoretical foundation for this type of reversibility.

The problem of reversibility, however, does not manifest itself at the software level. Even if we manage to perform our programs in reverse, it is not guaranteed that information will not be lost during the concrete physical computation process. To the contrary, it may get lost twice, once in each direction. The solution must therefore be found at the lowest hardware level. Our model of Turing graph machines is being presented as a possible hardware solution for the problem of reversibility, but follows Abramsky’s structural approach. We even go one step further by showing how computations can be done in a virtually undirected fashion under the theoretical umbrella of self-dual compact closed categories. In practical terms we mean that, unlike in synchronous systems (e.g. sequential circuits), where the information is propagated through the interconnections (wires) between the functional elements (logical gates) always in the same direction, in a Turing graph machine the flow of information along these interconnections takes a direction that is determined dynamically by the current input and state of the machine. We are going to reconsider self-dual compact closed categories as indexed monoidal algebras and prove a coherence theorem to establish undirected graphs – constituting the basic underlying structure for Turing graph machines – as free indexed monoidal algebras generated by the ranked alphabet consisting of the star graphs.

Different parts of this paper need not be read in a strict sequential order. In-depth knowledge of algebra and category theory is only required in Section 2 and Section 3. The reader less familiar with

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*Work partially supported by Natural Science and Engineering Research Council of Canada, Discovery Grant #170493-03.
categories could still understand the concept of Turing automata and Turing graph machines in Section 5, and appreciate the main contribution of this work. The paper, being a short summary of rather complex theoretical results, admittedly elaborates only on those connections to these results that are directly related to their presentation. One may, however, recognize structures familiar from linear logic, game semantics, communicating concurrent processes, iteration theories, interaction nets, and the Geometry of Interaction program in general. These connections will be spelled out in a future extended version of the present summary.

2 Self-dual compact closed categories

In this section we shall assume familiarity with the concept of symmetric monoidal categories [21]. Even though our main concern is with strict monoidal categories, the algebraic constructions presented in Section 3 can easily be adjusted to cover the general case. From this point on, unless otherwise stated, by a monoidal category we shall always mean a strict symmetric one.

Let \( \mathcal{C} \) be a monoidal category with tensor \( \otimes \) and unit object \( I \). Recall from [18, 17] that \( \mathcal{C} \) is compact closed if every object \( A \) has a left adjoint \( A^* \) in the sense that there exist morphisms \( d_A : I \to A \otimes A^* \) (the unit map) and \( e_A : A^* \otimes A \to I \) (the counit map) for which the two composites below result in the identity morphisms \( 1_A \) and \( 1_{A^*} \), respectively.

\[
A = I \otimes A \to d_A \otimes 1_A (A \otimes A^*) \otimes A = A \otimes (A^* \otimes A) \to 1_A \otimes e_A A \otimes I = A,
\]

\[
A^* = A^* \otimes I \to 1_{A^*} \otimes d_A A^* \otimes (A \otimes A^*) = (A^* \otimes A) \otimes A^* \to e_A \otimes 1_{A^*}, I \otimes A^* = A^*.
\]

By virtue of the adjunctions \( A \dashv A^* \) there is a natural isomorphism between the hom-sets \( \mathcal{C}(B \otimes A, C) \) and \( \mathcal{C}(B, C \otimes A^*) \) for every objects \( B, C \), hence the name “compact closed” category. Category \( \mathcal{C} \) is self-dual compact closed (SDCC, for short) if \( A = A^* \) for each object \( A \). The category SDCC has as objects all locally small [21] SDCC categories, and as morphisms monoidal functors preserving the given self-adjunctions.

A well-known SDCC category (not strict, though) is the category \( (\text{Rel}, \times) \) of small sets and relations with tensor being the cartesian product \( \times \). We shall only use this category as an example to explain the idea of indexing on it. Recall from [13, 7] that an indexed family of sets is simply a functor \( I : \text{Ind} \to \text{Set} \), where \( \text{Ind} \) is the index category. In our example, \( \text{Ind} \) is the monoidal category \( (\text{Set}, \times) \) as a subcategory of \( (\text{Rel}, \times) \) and \( \mathcal{I} \) is the covariant powerset functor \( \mathcal{P} \), which is of course not monoidal. Relations \( A \to B \) are, however, still subsets of \( A \times B \), and as such they can be indexed by morphisms (functions) \( A \times B \to C \) in \( \text{Set} \). For any two objects (sets) one can then consider the binary operation \( \oplus_{A,B}(=\times) : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \times B) \), and the unary operation trace, \( \triangledown_{A,B} : \mathcal{P}(A \times B) \to \mathcal{P}(B) \) for which \( b \in \triangle_{A,B} R \iff \exists a \in A (a, b) \in R \). The concept indexed monoidal algebra arises from observing the equational algebraic laws satisfied by these operations and their relationship to indexing.

Regarding the index monoidal category \( \text{Ind} \), one would like to have it as narrow as possible. The best choice would be the collection of permutations in \( \text{Ind} \), which, unfortunately, fails to be a subcategory in general. To get around this problem we shall introduce so called permutation symbols as unary operations, which will be responsible for the task of indexing in a coherent way.

3 Indexed monoidal algebras

In this section we introduce the category IMA of indexed monoidal algebras along the lines of the pioneer work [7], and establish an equivalence between the categories IMA and SDCC.
Let $S$ be a set of abstract sorts, and consider the free monoid $S'$ generated by $S$. For a word (string) $w \in S'$, $|w|$ will denote the length of $w$ and () will stand for the empty string. By an $S$-permutation we mean a pair $(w, \pi)$, where $w = s_1 \ldots s_n$ is a string with $|w| = n \geq 0$, and $\pi$ is a permutation $n \to n$. We shall use the notation $w \pi$ for $(w, \pi)$, and say that $w \pi$ is an $S$-permutation $w \to \pi(w)$, where $\pi(w)$ is the string $s_{\pi(1)} \ldots s_{\pi(n)}$. If $v$ and $w$ are strings of length $n$ and $m$, respectively, then $c_{v,w}$ will denote the $S$-permutation $vw^\pi x_{n,m}$ in which $x_{n,m}$ is the block transposition $n+m \to m+n$.

The collection of $S$-permutations can naturally be equipped with the operations composition ($\bullet$) and tensor ($\otimes$), which structure, together with the identities $1_w = w^\pi id_w$ and symmetries $c_{v,w}$, defines a monoidal category $\Pi_S$ over the set of objects $S'$. See e.g. [5, Definition 1] covering the single-sorted case. The category $\Pi_S$ is $S$-initial in the sense that, for every monoidal category $\mathcal{C}$ and mapping $\chi$ from $S$ to the objects of $\mathcal{C}$, there exists a unique monoidal functor $\chi : \Pi_S \to \mathcal{C}$ extending $\chi$ on objects. See again [5, Corollary 1] for a proof in the single-sorted case.

Now let $(M, I, \otimes)$ be a cancellative monoid, fixed for the rest of the paper. Since the elements of $M$ are meant to be objects in an appropriate monoidal category, they will be denoted by capital letters. With a slight abuse of the notation, $M^*$ will no longer mean the free monoid generated by $M$, rather, its quotient by the equation $I = ()$. Accordingly, by $\Pi_M$ we mean the monoidal category of $M$-permutation symbols, rather than that of ordinary $M$-permutations. We do so in order to accommodate the assumption that our monoidal categories are strict. Permutations over $M$ in this new sense will then be called $M$-permutation symbols to restore unambiguity. Let $\varepsilon_M : M^* \to M$ be the unique homomorphism (counit map) determined by the identity function on $M$. Again, this time with a heavier abuse of the terminology and coherence, the $M$-permutation symbol $w \pi^M : w \to \pi(w)$ will also be called one with “domain” $A = \varepsilon_M(w)$ and “codomain” $B = \varepsilon_M(\pi(w))$. As an escape, however, we shall use the distinctive notation $w \pi^M : A \Rightarrow B$ and say that permutation symbols $\rho_1 : A \Rightarrow B$ and $\rho_2 : B \Rightarrow C$ are composable if they are such as proper morphisms in the category $\Pi_M$.

Let $\rho = w \pi^M$ be an $M^*$-permutation symbol with $w = u_1 \ldots u_n$, where $u_i = A_{i,1} \ldots A_{i,m_i}$. In the monoidal category $\Pi_M$, $\rho$ defines an $M$-permutation symbol $\varepsilon_M(\rho) : u_1 \ldots u_n \Rightarrow u_{\alpha(1)} \ldots u_{\alpha(n)}$. On the other hand, $\rho$ also gives rise naturally to the $M$-permutation symbol $\rho \varepsilon_M : B_1 \ldots B_n \Rightarrow B_{\alpha(1)} \ldots B_{\alpha(n)}$, where $B_i = \otimes_j A_{i,j}$. Clearly, $\varepsilon_M(\rho)$ and $\rho \varepsilon_M$ define the same permutation $\otimes_j B_i \Rightarrow \otimes_j B_{\alpha(i)}$ in every monoidal category having $(M, I, \otimes)$ as its object structure. Therefore we say that these two $M$-permutation symbols $\otimes_j B_i \Rightarrow \otimes_j B_{\alpha(i)}$ are equivalent and write $\varepsilon_M(\rho) \equiv \rho \varepsilon_M$. As a trivial, but representative example: $1_A \otimes 1_B \equiv 1_{AB} \equiv 1_A \otimes B$.

We shall be dealing with $M$-sorted algebras $\mathcal{M} \equiv \{\mathcal{M}_A \mid A \in M\}$ having the following operations and constants.

- For each $M$-permutation symbol $\rho : A \Rightarrow B$, a unary operation $\rho : \mathcal{M}_A \to \mathcal{M}_B$.
- For each $A, B \in M$, a binary operation sum, $\oplus : \mathcal{M}_A \times \mathcal{M}_B \to \mathcal{M}_{A \oplus B}$.
- For each $A \in M$, a constant $1_A \in \mathcal{M}_{A \oplus A}$.
- For each $A, B \in M$, a unary operation trace, $\downarrow_A : \mathcal{M}_{A \oplus A \oplus B} \to \mathcal{M}_B$.

To emphasize the categorical nature of such algebras we call the elements $f \in \mathcal{M}_A$ morphisms and write $f : A$. We also write $f : A \Rightarrow B$ as an alternative for $f : A \otimes B$. Note that cancellativity of $M$ is required in order to make the trace operation sound. Moreover, the accurate notation for trace would be $\downarrow_{A,B}$, but the intended object $B$ will always be clear from the context. Also notice the boldface notation $1_A : A \Rightarrow A$ as opposed to $1_w : w \Rightarrow w$. For better readability we shall write $f \cdot \rho$ for $fp$, that is, for indexing $f$ by permutation symbol $\rho$.

Composition ($\circ$) and tensor ($\otimes$) are introduced in $\mathcal{M}$ as derived operations in the following way.
– For \( f : A \rightarrow B \) and \( g : B \rightarrow C \), \( f \circ g = \downarrow_B ((f \oplus g) \cdot (c_{AB} \otimes 1_C)) \).
– For \( f : A \rightarrow B \) and \( g : C \rightarrow D \), \( f \circ g = (f \oplus g) \cdot (1_A \otimes c_{BC} \otimes 1_D) \).

See Fig. 1. Again, the accurate notation for \( \circ \) and \( \otimes \) would use the objects \( A, B, C, D \) as subscripts, but these objects will always be clear from the context. Observe that the above definition of composition and tensor is in line with the traced monoidal category axioms in \cite{17}. Regarding composition, see also \cite{5} Identity \( X_3 \). As we shall point out in Theorem 1 below, our trace operation models the so-called “canonical” trace concept (cf. \cite{17}) in SDCC categories.

![Diagram](image-url)

**Figure 1:** Composition (a) and tensor (b) in \( \mathcal{M} \)

**Definition 1** An indexed monoidal algebra over \( M \) is an \( M \)-sorted algebra \( \mathcal{M} = \{ \mathcal{M}_A | A \in M \} \) equipped with the operations and constants listed above, which satisfies the following equational axioms.

11. **Functoriality of indexing**
   \[
   f \cdot (\rho_1 \bullet \rho_2) = (f \cdot \rho_1) \cdot \rho_2 \quad \text{for } f : A \text{ and composable } \rho_1 : A \Rightarrow B, \rho_2 : B \Rightarrow C;
   \]
   \[
   f \cdot 1_A = f \quad \text{for } f : A.
   \]

12. **Naturality of indexing**
   \[
   (f \oplus g) \cdot (\rho_1 \otimes \rho_2) = f \cdot \rho_1 \oplus g \cdot \rho_2 \quad \text{for } f : A, g : B, \rho_1 : A \Rightarrow C, \rho_2 : B \Rightarrow D;
   \]
   \[
   (\uparrow_A f) \cdot \rho = \downarrow_A (f \cdot (1_A \otimes \rho)) \quad \text{for } f : A \otimes A \otimes B, \rho : B \Rightarrow C.
   \]

13. **Coherence**
   \[
   f \cdot \rho_1 = f \cdot \rho_2 \quad \text{for } f : A \Rightarrow B, \text{ whenever } \rho_1 \equiv \rho_2.
   \]

14. **Associativity and commutativity of sum**
   \[
   (f \oplus g) \oplus h = f \oplus (g \oplus h) \quad \text{for } f : A, g : B, h : C;
   \]
   \[
   f \oplus g = (g \oplus f) \cdot c_{AB} \quad \text{for } f : A, g : B.
   \]

15. **Right identity**
   \[
   f \circ 1_B = f \quad \text{and } f \otimes 1_I = f \quad \text{for } f : A \rightarrow B.
   \]

16. **Symmetry of identity**
   \[
   1_A \cdot c_{AA} = 1_A.
   \]

17. **Vanishing**
   \[
   \uparrow_A f = f \quad \text{for } f : A;
   \]
   \[
   \downarrow_{A \otimes B} f = \downarrow_B (\uparrow_A f \cdot (1_A \otimes c_{BA} \otimes 1_{BC})) \quad \text{for } f : A \otimes B \otimes A \otimes B \otimes C.
   \]

18. **Superposing**
   \[
   \downarrow_A (f \oplus g) = \downarrow_A f \oplus g \quad \text{for } f : A \otimes A \otimes B, g : C.
   \]

19. **Trace swapping**
   \[
   \downarrow_B (\uparrow_A f) = \downarrow_A (\downarrow_B (f \cdot (c_{AA, BB} \otimes 1_C))) \quad \text{for } f : A \otimes A \otimes B \otimes B \otimes C.
   \]

The algebra \( \mathcal{M} \) is called small if \( M \) is a small monoid in the sense of \cite{21} and the sets \( \mathcal{M}_A \) are also small for every \( A \in M \).

Let \( \mathcal{M} \) and \( \mathcal{M}' \) be indexed monoidal algebras. An indexed monoidal homomorphism \( F : \mathcal{M} \rightarrow \mathcal{M}' \) is a pair \( (h, \{ F_A | A \in M \}) \), where \( h \) is a monoid homomorphism \( M \rightarrow M' \) and \( F_A : \mathcal{M}_A \rightarrow \mathcal{M}'_{h(A)} \) are mappings that determine a homomorphism in the usual algebraic sense. With respect to indexing we mean that for every \( f : A \) and \( \rho : A \Rightarrow B, F_B (f \cdot \rho) = (F_A f) \cdot h^* \rho \), where \( h^* \) is the unique monoidal functor \( \Pi_M \rightarrow \Pi_{M'} \) determined by \( h \). The category \( \textbf{IMA} \) then consists of all small indexed monoidal algebras as objects and indexed monoidal homomorphisms as morphisms.
Theorem 1 The categories IMA and SDCC are equivalent.

Proof. Let $\mathcal{M}$ be a small indexed monoidal algebra over $M$, and define the monoidal category $\mathcal{C} = \mathcal{S} \mathcal{M}$ over the objects $M$ as follows. Morphisms $A \to B$ and identities in $\mathcal{C}$ are exactly those in $\mathcal{M}$, while composition and tensor are adopted from $\mathcal{M}$ as derived operations. Symmetries $c_{A,B} : A \otimes B \to B \otimes A$ in $\mathcal{C}$ are the morphisms $1_{A\otimes B} \cdot (1_{AB} \otimes c_{A,B})$. In general, every permutation symbol $\rho : \omega \to \omega'$ is represented in $\mathcal{S} \mathcal{M}$ as $1_{\varepsilon_M(\omega)} \cdot (1_{\varepsilon_M(\omega')} \otimes \rho) : \varepsilon_M(\omega) \to \varepsilon_M(\omega')$. For each self-adjunction $A \dashv A$, the unit map $d_A : I \to A \otimes A$ and the counit map $e_A : A \otimes A \to I$ are both the identity $1_A : A \otimes A$. It is essentially routine to check that $\mathcal{S} \mathcal{M}$ is a locally small SDCC category. Below we present the justification of some milestone equations, which can easily be developed into a complete rigorous proof.

1. Symmetry of trace, and canonical trace
   
   $\uparrow A f \uparrow A (f \cdot (c_{A,B} \otimes 1_B)) = 1_A \circ (A \otimes A) f$ for $f : A \otimes A \otimes B$.

   See Fig. 2.

   ![Figure 2: Symmetry of trace, and canonical trace (read from right to left, bottom-up)](image)

2. Left identity
   
   $1_A \circ f = f$ for $f : A \to B$

   See Fig. 3.

   Notice that the symmetry of trace and that of $1_A$ have both been used in the proof.

3. Tensor of identity
   
   $1_{A\otimes B} = 1_A \otimes 1_B$

   See Fig. 4.

   The definition of functor $\mathcal{S}$ on (homo-)morphisms is evident, and left to the reader.

   Conversely, let $\mathcal{C}$ be a locally small SDCC category over $M$ as objects, and define the indexed monoidal algebra $\mathcal{M} = \mathcal{S} \mathcal{C}$ as follows. For each $A \in M$, $\mathcal{M}_A = \mathcal{C}(I,A)$, the (small) set of morphisms $I \to A$ in $\mathcal{C}$. Since $\mathcal{C}$ is symmetric, every permutation symbol $\rho : A \Rightarrow B$ determines a permutation $\rho_C :$
A \rightarrow B$ in $\mathcal{C}$. Then, for $f : A$, define $f \cdot \rho = f \circ \rho_{\mathcal{C}}$. Notice that indexing indeed becomes the restriction of the covariant $\text{hom}$ functor to permutations, as intended. For $f : A$ and $g : B$, $f \otimes g = f \otimes_{\mathcal{C}} g : I \rightarrow A \otimes B$ and $1_A = d_A : I \rightarrow A \otimes A$. For $f : A \otimes A \otimes B$, $\uparrow^A f$ is defined as the canonical trace of the morphism $f_A : A \rightarrow A \otimes B$ in $\mathcal{C}$ that corresponds to $f$ according to compact closure. That is, $\uparrow^A f$ is the morphism $f \circ (\epsilon_A \otimes 1_B) : I \rightarrow B$ in $\mathcal{C}$.

In the light of this translation, each of the equations I1-I9 is either a standard monoidal category axiom or has been observed in [17, 18] for traced monoidal or compact closed categories. Thus, $\mathcal{M}$ is an indexed monoidal algebra. The specification of functor $\mathcal{I}$ on morphisms (monoidal functors) is again straightforward.

By definition, $\mathcal{I}(\mathcal{SM}) = \mathcal{M}$. On the other hand, the only difference between the monoidal categories $\mathcal{C}$ and $\mathcal{I}(\text{SCX})$ is that the hom-sets $A \rightarrow B$ in the latter are identified with the ones $I \rightarrow A \otimes B$ of the former, using the natural isomorphisms given by the self-adjunctions $A \dashv A$. In other words, morphisms $A \rightarrow B$ in $\mathcal{I}(\text{SCX})$ – as provided for by compact closure – are simply renamed as they appear in $\mathcal{C}(I,A \otimes B)$. Thus, there exists a natural isomorphism between the functors $1_{\text{SDCC}}$ and $\mathcal{I} \mathcal{F}$, so that the categories $\text{IMA}$ and $\text{SDCC}$ are equivalent as stated.

### 4 Coherence in indexed monoidal algebras

In general, a coherence result for some type $\tau$ of monoidal categories is about establishing a left-adjoint for a forgetful functor from the category $\text{T}$ of $\tau$-monoidal categories into an appropriate syntactical category, and providing a graphical characterization of the free monoidal $\tau$-categories so obtained. For some typical examples, see [21, 18, 5, 6]. In this section we present such a coherence result for SDCC categories, but phrase it in terms of indexed monoidal algebras. The graphical language arising from this result will justify our efforts in the previous section to reconsider SDCC categories in the given algebraic context.

For a set $S$ of sorts, an $S$-ranked alphabet (signature) is a set $\Sigma = \cup(\Sigma_w | w \in S^*)$, where $\Sigma_v \cap \Sigma_w = \emptyset$ if $v \neq w$. A morphism $\Omega : \Sigma \rightarrow \Delta$ between ranked alphabets of sort $S$ and $T$, respectively, is an alphabet mapping consisting of a function $\omega : S \rightarrow T$ and a family of mappings $\Omega_w : \Sigma_w \rightarrow \Delta_{\omega(w)}$. 

---

Figure 3: Left identity

\[
\begin{array}{c}
\begin{aligned}
A & \xrightarrow{f} A & \xrightarrow{\text{identity}} & A \\
\end{aligned}
\end{array}
\]

Figure 4: Tensor of identity, take $f = 1_{A \otimes B}$

\[
\begin{array}{c}
\begin{aligned}
A & \xrightarrow{f} A \\
\end{aligned}
\end{array}
\]
(The unique extension of ω to strings is denoted by ω as well.) Every indexed monoidal algebra \( \mathcal{M} = \{ \mathcal{M}_A | A \in M \} \) can be considered as an M-ranked alphabet \( \Sigma = \mathcal{A} \mathcal{M} \) in such a way that \( \Sigma_w = \{ f_w | f : \varepsilon_M(w) \in \mathcal{M} \} \) for every \( w \in M' \). (The identification \( l = ( \) is still in effect for \( M' \).) We use a subscript to distinguish between instances of \( f \) belonging to different ranks. If \( F = (h, \{ F_A | A \in M \}) : \mathcal{M} \rightarrow \mathcal{M}' \) is a homomorphism, then \( \mathcal{A} F : \mathcal{A} \mathcal{M} \rightarrow \mathcal{A} \mathcal{M}' \) is the alphabet mapping \( (h, \{ H_w | w \in M' \}) \) for which \( H_w(f_w) = F_{\varepsilon_M(w)}(f) \). Our aim is to provide a left adjoint for the functor \( \mathcal{A} \). In algebraic terms this amounts to constructing the indexed monoidal algebra freely generated by a given S-ranked alphabet \( \Sigma \).

Let \( \Sigma \) be an S-ranked alphabet. By a \( \Sigma \)-graph we mean a finite undirected and labeled multigraph \( G = (V, E, l) \) with vertices (nodes) \( V \), edges \( E \), and labeling \( l : V \to \Sigma \cup \{ \text{in}_A, \text{\textcircled{1}A} | A \in S \} \), where \( \text{in}_A \) and \( \text{\textcircled{1}A} \) are special symbols not in \( \Sigma \) with rank \( A \) and (), respectively. Vertices labeled by \( \text{in}_A \) (\( \text{\textcircled{1}A} \)) will be called interface (respectively, loop) vertices. All other vertices, as well as the edges connecting them will be called internal. It is required that the label of each node \( u \) be consistent with its degree \( d(u) \), so that if \( l(u) \in \Sigma_w \), then \( |w| = d(u) \). Each point at which an edge impinges on \( u \) is assigned a serial number \( 1 \leq i \leq n = d(u) \) and a sort \( A_i \) in such a way that \( w = A_1 \ldots A_n \). Adopting a terminology from [6] and [22], such points will be referred to as ports. Edges, too, must be consistent with the labeling in the sense that each edge connects two ports of the same sort, and each port is an endpoint of exactly one edge. The interface nodes themselves are assigned a serial number, so that one can speak of a \( \Sigma \)-graph \( G : w \) with \( w \) being the string of sorts assigned to (the unique ports of) the interface vertices in the given order. See Fig. 5a for an example \( \Sigma \)-graph \( G : AB \), where \( \Sigma = \Sigma_{BA} \uplus \Sigma_{ABA} \) with \( \Sigma_{BA} = \{ f \} \) and \( \Sigma_{ABA} = \{ g \} \). Symbols in \( \Sigma \) are represented as atomic \( \Sigma \)-graphs in the way depicted by Fig. 5b.

![Figure 5: \( \Sigma \)-graphs](image)

An isomorphism between \( \Sigma \)-graphs \( G, G' : w \) is a graph isomorphism that preserves the labeling information of the vertices. We shall not distinguish between isomorphic graphs. Let \( \mathcal{G}_w(\Sigma) \) denote the set of \( \Sigma \)-graphs of rank \( w \in S' \). The family \( \mathcal{G}(\Sigma) = \{ \mathcal{G}_w(\Sigma) | w \in S' \} \) is equipped with the indexed monoidal algebra operations (over the monoid \( S' \)) as follows.

- For a graph \( G : w \), each permutation symbol \( \rho : w \Rightarrow w' \) is interpreted as the relabeling of the interfaces according to the \( S \)-permutation (not symbol!) \( \varepsilon_{S'}(\rho) \), where \( \varepsilon_{S'} : (S')^* \rightarrow S' \) is the counit map.
- For graphs \( G_1 : w_1 \) and \( G_2 : w_2 \), \( G_1 \uplus G_2 \) is the disjoint union of \( G_1 \) and \( G_2 \) with the serial number of each interface vertex in \( G_2 \) incremented by \( |w_1| \).
- The identity graph \( 1_w : w \) for \( w = A_1 \ldots A_n \) is shown in Fig. 5c. The graph \( 1_{\emptyset} \) is empty.
- For a graph \( G : w w v \) with \( |w| = n \), the trace operation \( \varepsilon_v \) is defined by gluing together the pairs of edges incident with the interface vertices having serial numbers \( i \) and \( n + i \) for each \( 1 \leq i \leq n \), leaving out the interface vertices themselves. Whenever this procedure results in a loop of an even number of edges (but no internal vertices) glued together, a new loop vertex labeled by \( \text{\textcircled{1}A} \) is created and added to the graph, where \( A \) is the common sort of the interface ports involved in the loop.
See [7] for a more detailed description of $\dagger$ through examples. See also [16,11,5,6] for the corresponding standard definition of feedback/iteration in (directed) flowcharts. Interestingly, in all of these works, graphs (flowcharts) are equipped with a single loop vertex, so that loops do not multiply when taking the feedback. On the other hand, the loop vertex is present in the graph $I_1$ as well. Regarding the single-sorted case this amounts to imposing the additional axiom $\dagger 1 = 0$ (rather, its directed version, e.g. [5] Axiom S5: $\dagger 1 = 0$), which is not a standard traced monoidal category axiom. From the point of view of axiomatization this is a minor issue. Another issue, however, namely the assignment of an individual monoid to each object $A$ is extremely important and interesting. In terms of flowcharts, this allows one to erase begin vertices and join two incoming edges at any given port. See e.g. the constants $0_1: 0 \to 1$ and $\varepsilon: 2 \to 1$ in [5,6]. These constants (morphisms) were naturally incorporated in the axiomatization of schemes, both flowchart and synchronous. Concerning undirected graphs, the presence of such morphisms with a “circularly symmetric” interface allows for an upgrade of ordinary edges to hyperedges, exactly the way it is described in [22] for bigraphs. The axiomatization of undirected hypergraphs as SDCC categories will be presented in a forthcoming paper.

It is easy to check that the above interpretation of the indexed monoidal operations on $\mathcal{G}(\Sigma)$ satisfies the axioms I1-I9. Thus, $\mathcal{G}(\Sigma)$ is an indexed monoidal algebra over the monoid $S^*$. It is also clear that $\mathcal{G}(\Sigma)$ is generated by $\Sigma$, that is, by the collection of the atomic $\Sigma$-graphs. (See again Fig. 5b.) Indeed, every undirected graph can be reconstructed from its vertices as star graphs by adding internal edges one by one using the trace operation.

**Theorem 2** The algebra $\mathcal{G}(\Sigma)$ is freely generated by $\Sigma$.

**Proof.** Without essential loss of generality, we restrict our attention to the single-sorted case. One way to prove the statement is to copy the normal form construction for flowchart schemes as presented in [5]. Each step in this construction [8, Theorem 3.3] is completely analogous, except that one relies on undirected trace rather than directed feedback to create internal edges. Another idea that uses the corresponding result [5, Corollary 2] directly is the following. For each symbol $\sigma \in \Sigma_n$ ($n \geq 0$), consider the set of doubly ranked symbols $\sigma_{LR}^n$ of rank $(k,l)$ such that $L$ and $R$ are disjoint subsets of $[n] = \{1,\ldots,n\}$ with cardinality $k$ and $l$ respectively, and $k+l = n$. (Split the degrees into in-degrees and out-degrees in all possible ways.) Denote by $\overline{\Sigma}$ the doubly ranked alphabet consisting of these new symbols. Construct the free traced monoidal category $\text{Sch}(\overline{\Sigma})$ of $\overline{\Sigma}$-flowchart schemes [5], and consider the rank-preserving mapping $\Gamma: \sigma_{LR}^n \rightarrow \sigma$ from $\overline{\Sigma}$ into the SDCC category $\mathcal{G}(\Sigma)$. Now let $\mathcal{M}$ be an arbitrary indexed monoidal algebra over monoid $M$ and specify $A \in M$ arbitrarily together with a mapping $\Omega: \sigma \mapsto f_\sigma$, where $f_\sigma : nA = A \otimes \ldots \otimes A$ is an arbitrary morphism in $\mathcal{M}$. Since $\text{Sch}(\overline{\Sigma})$ is freely generated by $\overline{\Sigma}$, there are unique traced monoidal functors $\mathbb{F}_1$ and $\mathbb{F}_2$ from $\text{Sch}(\overline{\Sigma})$ into $\mathcal{G}(\Sigma)$ and $\mathcal{M}$ extending $\Gamma$ and $\Gamma \circ \Omega$, respectively. One can then easily prove by induction that the desired unique SDCC functor from $\mathcal{G}(\Sigma)$ into $\mathcal{M}$ extending $\Omega$ factors through an arbitrary inverse of $\mathbb{F}_1$ and $\mathbb{F}_2$. Hence, the statement of the theorem follows from Theorem 1.

A really elegant third proof, however, would use the $\text{Int}$ construction in [17] – alternatively, the $G$-construction in [11] – by duplicating each degree into an in-degree and a corresponding dual out-degree. The reader familiar with either of these constructions will instantly recognize the point in this argument. The statement of the theorem is, however, not an immediate consequence of applying the construction to an appropriate traced monoidal category, which is why we have chosen the above short and simple direct proof here. $\Box$

Let $\mathcal{M}$ be an arbitrary indexed monoidal algebra over $M$. An interpretation of $\Sigma$ in $\mathcal{M}$ is an alphabet mapping $\Omega: \Sigma \rightarrow \mathcal{M}$. By Theorem 2, every interpretation $\Omega$ can be extended in a unique way to a homomorphism $\overline{\Omega}: \mathcal{G}(\Sigma) \rightarrow \mathcal{M}$. Thus, $\mathcal{G}$ is indeed a left adjoint for the functor $\mathcal{A}$. 

**Turing Automata**
5 Turing automata and Turing graph machines

As an important example of indexed monoidal algebras, in this section we introduce the algebra of Turing automata and Turing graph machines. We shall use the monoidal category $(\text{Bset}, +)$ (small sets and bijections with disjoint union as tensor) as the index category. Unfortunately, this category is not strict, therefore the reader is asked to be lenient about the finer details. The only “shaky” ground will be the interpretation of set-permutation symbols as bijections in the category $(\text{Bset}, +)$, which is quite natural. For a set $A$, let $A_+ = A + \{\ast\}$, where $\ast$ is a fixed symbol, called the anchor.

**Definition 2** A Turing automaton $T : A$ is a triple $(A, Q, \delta)$, where $A$ is a set of interfaces, $Q$ is a nonempty set of states, and $\delta \subseteq (Q \times A_+)^2$ is the transition relation.

The role of the anchor as a distinguished interface will be explained later. The transition relation $\delta$ can either be considered as a function $Q \times A_+ \to \mathcal{P}(Q \times A_+)$ or as a function $Q \times (A_+ \times A_+) \to \mathcal{P}(Q)$, giving rise to a Mealy or Medvedev type automaton, respectively. We shall favor the latter interpretation, and define $T$ to be deterministic if $\delta$ is a partial function in this sense. Thus, an input to automaton $T$ is a pair $(a, b)$ of interfaces. Nevertheless, we still say that $T$ has a transition from $a$ to $b$ in state $q$, resulting in state $r$, if $((q, a), (r, b)) \in \delta$. By way of duality, one can also consider $T$ as an automaton with states $A_+$ and inputs $Q \times Q$. We shall reflect on this duality shortly. If $A$ is finite and $|A| = n$, then $T$ can be viewed as an $(n + 1) \times (n + 1)$ matrix, where each entry is a relation over $Q$.

**Example** The $n$-ary atomic switch is the Turing automaton $\mathcal{A}_n : [n] \to [n]$ having states $[n]$, so that

$$\delta = \{(i, j), (j, i) \mid 1 \leq i \neq j \leq n\} \cup \{(i, i), (j, j) \mid 1 \leq i \neq j \leq n\}$$

$$\cup \{(i, \ast), (j, \ast), (\ast, j), (\ast, i), (i, \ast), (j, \ast) \mid i, j \in [n]\}.$$  

For better readability, states, indicating a selected edge in an $n$-star graph, are written in boldface. In addition, if $n = 1$, then $((1, 1), (1, 1)) \in \delta$.

Heuristically, the $n$-ary atomic switch captures the behavior of an atom in a molecule having $n$ chemical bonds to neighboring atoms. Among these bonds exactly one is double, and referred to as the positive edge in the underlying star graph. The mechanism of switching is then clear by the definition above. The active ingredient (control) in this process is called the soliton, which is a form of energy traveling in small packets through chains of alternating single and double bonds within the molecule, causing the affected bonds to be flipped from single to double and vice versa. See [14, 9, 10] for the corresponding mathematical model. Note that, by our definition above, whenever the soliton enters an atom with a unique chemical bond (which must be double since $n = 1$), it bounces back immediately, producing no state change.

We now turn to defining the indexed monoidal algebra $\mathcal{T}$ of Turing automata. In this algebra, morphisms are Turing automata $T : A$. A permutation symbol $\rho : A \Rightarrow B$ is interpreted as a relabeling of the interfaces according to the unique bijection $A \to B$ determined by $\rho$ in $\text{Bset}$. (Elaboration of details regarding the transition relation is left to the reader.) The sum of $T : A$ and $T' : B$ having states $Q$ and $Q'$, respectively, is the automaton $T + T' = (A + B, Q \times Q', \delta \oplus \delta')$, where $\delta \oplus \delta' \subseteq ((Q \times Q') \times (A + B)_+)$. is defined by

$$((q, q'), x), ((r, r'), y)) \in \delta \oplus \delta'$$

iff either $q' = r'$ and $((q, x), (r, y)) \in \delta$, or $q = r$ and $((q', x), (r', y)) \in \delta'$.

(Notice the ambiguity in writing just $x, y$ rather than $\langle x, A \rangle, \langle y, A \rangle$ or $\langle x, B \rangle, \langle y, B \rangle$.) The definition, however, applies to the case $x = \ast$ and/or $y = \ast$, too, so that taking the sum of $T$ and $T'$ amounts to a selective performance of $\delta$ or $\delta'$ on $Q \times Q'$. The identity Turing automaton $1_A : A \Rightarrow A$ has a single state, in which there is a transition from $\langle a, 1 \rangle$ to $\langle a, 2 \rangle$ and back for every $a \in A$. 

The definition of $\downarrow_A T$ for a Turing automaton $T : A + A + B$ is complicated but natural, and it holds the key to understanding the Turing-machine-like behavior of this automaton. Intuitively, the definition models the behavior of loops in flowchart algorithms \cite{16} when implemented in an undirected environment. That is, control enters $\downarrow_A T$ at an interface $b \in B$, then, after alternating between corresponding interfaces in $A + A$ any number of times, it leaves at another (or the same) interface $b' \in B$. State changes are traced interactively during this process. For technical reasons we shall restrict the formal definition of trace to finitary Turing automata, whereby the number of interfaces is finite. This will allow us to set up an analogy with the well-known Kleene construction for converting a finite state automaton into a regular expression.

Let $T = (A + A + B, Q, \delta)$ be a Turing automaton and $C \subseteq A$ be arbitrary such that $A = C + C'$. Following the Kleene construction, concentrate on the relations $\lambda_{x,y}^C \subseteq Q^2 (x,y \in (C' + B)_+)$ as transitions of the automata $\downarrow_C T : C' + B$ from interface $x$ to interface $y$. Our goal is to satisfy the Kleene formula

$$\lambda_{x,y}^{C+\{z\}} = \lambda_{x,y}^C \cup \left( \lambda_{x,(z,1)}^C \lambda_{(z,2)}^C \right) \circ \left( \lambda_{(z,1),(z,1)}^C \lambda_{(z,2),(z,2)}^C \right) + \left( \lambda_{(z,1),(z,2)}^C \lambda_{(z,2),(z,2)}^C \right)$$

for every proper subset $C \subset A$ and $z \in A \setminus C$. In this formula, $\circ$ denotes alternating matrix product in the sense

$$U \circ \left( \begin{array}{c} V_1 \\ V_2 \end{array} \right) = U \cdot \left( \begin{array}{c} V_1 \\ V_2 \end{array} \right),$$

and $\ast$ is Kleene star of $2 \times 2$ matrices based on $\circ$, that is, $U^\ast = \cup_{n \geq 0} U^n$, where $U^0$ is the alternate identity matrix $I_2 \circ I_2$ and $U^{i+1} = U^i \circ U$. The underlying semiring $R$ is that of binary relations over $Q$ with union as sum, composition as product, $id_Q$ as unit and $\emptyset$ as zero. Notice the immediate relationship between our \ast and the star operation in star theories as defined in \cite{12}, e.g. in Conway matrix theories. See also the Example in \cite{17}, which originates from \cite{12}, too. As well, note the duality between states and input in comparison with the original Kleene formula. The alternating matrix product is another hidden allusion to the Int construction \cite{17} mentionned earlier, suggesting that the compact closed category resulting from that construction be restricted to its “self-dual” objects $(A,A)$.

We are using the above formula as a recursive definition for $\lambda_{x,y}^C$, starting from the basis step

$$\lambda_{x,y}^0 = \{ (q,q') \mid ((q,x),(q',y)) \in \delta \}.$$ 

The transition relation $\delta \subseteq (Q \times B)_+^2 of \downarrow_A T$ is then set in such a way that $((q,b),(q',b')) \in \hat{\delta}$ iff $(q,q') \in \lambda_{p,b'}^A$. In order to use this definition, one must prove that the specification of $\lambda_{x,y}^C$ does not depend on the order in which the elements $z \in C' = A \setminus C$ are left out. This statement is essentially equivalent to axiom I9 (trace swapping).

\textbf{Theorem 3} The algebra $\mathcal{T}$ of Turing automata is indexed monoidal.

\textbf{Proof.} At this point we can capitalize to a great extent on the simplicity of the indexed monoidal algebra axioms. Indeed, each of these axioms, except for vanishing (I7) and trace swapping (I9), holds naturally true in $\mathcal{T}$. The vanishing axiom expresses the fact that choosing the Kleene formula to define trace in $\mathcal{T}$ is right, and trace swapping ensures that the definition is correct. The proof of these two axioms is left to the reader as an exercise. \qed

Finally, we explain the role of the anchor $\ast$. We did not want all Turing automata of sort $I = \emptyset$ to have no transitions at all, like the automata $\downarrow_A = \downarrow_A 1_A$, which all coincide, having a unique state. The anchor is a fixed interface that is not supposed to be interconnected with any other, so that automata in $\mathcal{T}$ might still have transitions from $\ast$ to $\ast$. The index category itself, however, need not be that of pointed sets, because the anchor is not affected by any of the operations.
Let $D$ be a non-empty set of data. The indexed monoidal algebra $D\text{-}\text{dil}\mathcal{T}$ of $D$-flow Turing automata is defined in the following way.

- Morphisms of sort $A$ are Turing automata $T : D \times A$.
- Each permutation symbol $\rho : A \Rightarrow B$ is interpreted as a bijection (relabeling) $D \times A \rightarrow D \times B$, which is basically $\rho_{\mathcal{T}}$ performed on blocks of size $D$ in parallel.
- The operations sum and trace are adopted from $\mathcal{T}$ (assuming the identification of $D \times (A + B)$ with $D \times A + D \times B$), and the identities $1_A$ are the identities $1_{D \times A}$ in $\mathcal{T}$.

The notation $D\text{-}\text{dil}$ originates from [4], where the magmoid (single-sorted monoidal category) $k\text{-}\text{dil}\mathcal{M}$ was introduced for integer $k$ and magmoid $\mathcal{M}$ along these lines. Intuitively, a $D$-flow Turing automaton is a data-flow machine in which data in $D$ are passed along with each transition. Notice that the anchor does not emit or receive any data. As an immediate corollary to Theorem 3, the structure $D\text{-}\text{dil}\mathcal{T}$ of $D$-flow Turing automata is an indexed monoidal algebra.

![Figure 6: The von Neumann machine](image-url)

Consider, for example, the scheme $N$ of the classical von Neumann computer in Fig. 6 as a data-flow architecture. It consists of two interconnected single-sorted $D$-flow Turing automata: the processor $P : 2$, and the memory $M : 1$. The processor is a real finite state automaton, having state components like registers, the instruction counter, the PSW, etc. The transitions of $P$ are very complex. On the other hand, $M$ has (practically) infinite states, but its transitions are straightforward. The set $D$ consists of all pieces of information (data, control, and/or address) that can be transmitted along the bus line between $P$ and $M$ in either direction. The operation of $N$ need not be explained, and it is clearly that of a $D$-flow Turing automaton. It is a very important observation, however, that the machine can do as much as we want in one step, that is, from the time control enters port 1 of $P$ until it leaves at the same port. For example, it can execute one machine instruction stored in the memory, or even a whole program stored there. In other words, semantics is delay-free. In present-day digital computers this semantics is achieved by limiting the scope of what the machine can do in one step through introducing clock cycles and delay, which turn the computer into a synchronous system [6]. Theoretically speaking, undirected trace is turned into directed feedback with delay (or, using an everyday language, recursion is transformed into a loop), and computations become inevitably directed in a rigid way. According to the original scheme $N$, however, they need not be, yet they could be universal.

**Example (Continued)** The $n$-ary atomic alternating switch $\mathcal{A}_n^2$ augments the ordinary $n$-ary atomic switch by the passing of a digital information in the following way. Control from a negative interface (i.e., one not covered by the unique positive edge) can only take 0 for input and emits 1 for output. (Remember that in the meantime the positive edge is switched from the output side to the input side.) Conversely, control from a positive interface can only take 1 for input and emits 0 for output. Transitions from and to the anchor are as in the corresponding $2n$-ary switch.

For the rest of the paper, the alphabet $\Sigma$ will be single-sorted, that is, $\Sigma = \cup (\Sigma_n | n \geq 0)$.

**Definition 3** A $D$-flow Turing graph machine over $\Sigma$ is a triple $M = (G, D, \Omega)$, where $G$ is a $\Sigma$-graph and $\Omega$ is an interpretation of $\Sigma$ in $\mathcal{T}$ under which the single sort of $\Sigma$ is mapped into the finite set $D$. Equivalently, $\Omega$ is an interpretation in $D\text{-}\text{dil}\mathcal{T}$ that maps sort “1” to object $\{1\}$.

Intuitively, machine $M$ comes with an underlying graph $G$ that has a $D$-flow Turing automaton sitting in each of its internal vertices. The operation of $M$ as a complex Turing automaton is uniquely determined
by the given interpretation according to the homomorphism $\bar{\Omega}$. The classical Turing machine concept is recaptured by taking $\Sigma = \Sigma_2 = \{c\}$, where $c$ stands for “tape cell”. A Turing machine $TM$ is transformed into a $D$-flow Turing graph machine $M$ whose underlying graph is a linear array of cell vertices with the following interpretation $T_c : 2$ of $c$. The states of $T_c$ are the tape symbols of $TM$, and, by way of duality, elements of $D$ are the states of $TM$. The transition relation of $TM$ translates directly and naturally into that of $M$, using duality. The only shortcoming of this analogy is the finiteness of the underlying graph $G$, which can be “compensated” by making the set of states $Q$ infinite, e.g., taking the colimit of finite approximation automata in an appropriate extension of IMA to a 2-category, whereby the vertical structure is determined by homomorphisms of (Medvedev type) automata in the standard sense. Universality is then guaranteed either by the von Neumann machine $N$ or by the universal Turing machine, as special Turing automata.

Returning to our dilemma of reversible vs. irreversible computations, we define the “reverse” of a Turing automaton $T = (A, Q, \delta)$ simply as $T^R = (A, Q, \delta^{-1})$. Although this definition is quite natural, one cannot expect that, for every computation process represented by some Turing automaton $T$, both $T$ and $T^R$ be deterministic. Indeed, this restriction would directly undermine universality. Eventually, the point is not to actually perform the reverse of a given computation, rather, being able to carry it out on a device that is in principle reversible. Turing graph machines do have this capability by definition. Still, the effective construction of a universal Turing graph machine remains an enormous challenge. The soliton automaton model described below is an interesting try, but unfortunately it falls short of being universal even in terms of designing individual ad-hoc machines. To introduce this model as a Turing graph machine, let $\Sigma$ be the ranked alphabet consisting of a single symbol $c_n$ for each rank $n \geq 1$.

Example (Continued) A pre-soliton automaton is a Turing graph machine $S = (G, \{0, 1\}, \Omega)$, where $\Omega$ is the fixed interpretation that sends each symbol $c_n$ into the $n$-ary atomic alternating switch $\mathcal{A}^2_n$. Since the interpretation is fixed, we shall identify each pre-soliton automaton with its underlying graph. Moreover, since $\mathcal{A}^2_n$ is circularly symmetric, we do not need to order the ports (degrees) of the internal vertices. Thus, $G$ is an ordinary undirected graph (with its interface vertices still ordered).

Let $q$ be a state of graph $G$. By definition, each internal edge $e \in E$ is either consistent with respect to $q$, meaning that $e$ has the same sign (positive or negative) viewed from its two internal endpoints, or inconsistent if this is not the case. (Notice that a looping edge is always negative if consistent.) A soliton walk from interface $i$ to interface $j$ ($i, j \in [n] + \{\ast\}$) is a transition of $G$ from $i$ to $j$ in state $q$ according to the standard behavior of $G$ as a Turing automaton. The reader can now easily verify that this definition of soliton walks coincides with the original one given in [14], provided that $q$ is a perfect internal matching [20, 9] of $G$, that is, a state in which every edge of $G$ is consistent and the positive edges determine a matching by which the internal vertices are all covered. Indeed, the definition of $\mathcal{A}^2_n$ implies that the soliton can only traverse consistent edges in an alternating positive-negative fashion. A new feature of this model is a soliton walk from the anchor, which must return to the anchor if $q$ is a perfect internal matching, and in that case it defines a closed alternating walk (e.g. an alternating cycle).

At this point we stop elaborating on soliton automata, leaving them as a subject for future work. The key observation that allows one to restrict the states of pre-soliton automata to perfect internal matchings is the Gallai-Edmonds Structure Theorem [20], well-known in matching theory. On the basis of this theorem, the Gallai-Edmonds algebra of graphs having a perfect internal matching has been worked out in [8] as the homomorphic image of the indexed monoidal algebra of graphs. The algebra (SDCC category) of soliton automata then turns out to be the quotient of $\mathcal{G}(\Sigma)$ determined by the pushout of the Gallai-Edmonds algebra homomorphism and $\bar{\Omega}$ in the category IMA. This result will be presented in a forthcoming paper.
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