STOCHASTIC PARABOLIC ANDERSON MODEL WITH 
TIME-HOMOGENEOUS GENERALIZED POTENTIAL: MILD 
FORMULATION OF SOLUTION

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Abstract. A mild formulation for stochastic parabolic Anderson model with 
time-homogeneous Gaussian potential suggests a way of defining a solution to 
obtain its optimal regularity. Two different interpretations in the equation or in 
the mild formulation are possible with usual pathwise product and the Wick 
product: the usual pathwise interpretation is mainly discussed. We emphasize 
that a modified version of parabolic Schauder estimates is a key idea for the 
existence and uniqueness of a mild solution. In particular, the mild formulation 
is crucial to investigate a relation between the equations with usual pathwise 
product and the Wick product.

1. Introduction. We start with the stochastic parabolic Anderson model (PAM)
\[
\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x) W(x)
\] (1)
driven by multiplicative Gaussian white noise potential in d space dimensions, where \(\Delta\) is the Laplacian operator, and \(W(x)\) is time-homogeneous Gaussian white noise with mean zero and covariance \(E[W(x)W(y)] = \delta(x-y)\), where \(\delta\) is the Dirac-delta function. Note that the white noise is a generalized process, and we need to make 
sense of the multiplication \(uW\) in (1).

Since the notation \(W(x)\) stands for the formal derivative of a Brownian sheet \(W(x)\), the equation (1) may be written as
\[
\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x) \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W(x).
\] (2)

Depending on the multiplication between \(u\) and \(\frac{\partial^d}{\partial x_1 \cdots \partial x_d} W\), the equation (2) may be classified as follows:

- Stochastic PAM with usual pathwise product (Stratonovich interpretation):
\[
\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x) \cdot \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W(x).
\] (3)

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• Stochastic PAM with Wick product ◦ (Wick-Itô-Skorokhod interpretation):

\[
\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x) \circ \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W(x),
\]

(4)

As the idea of Stratonovich integral suggests, the equation (3) is equivalently interpreted as a pathwise limit of approximated equations

\[
\frac{\partial u^\varepsilon(t,x)}{\partial t} = \Delta u^\varepsilon(t,x) + u^\varepsilon(t,x) \cdot \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W^\varepsilon(x),
\]

where \( W^\varepsilon \) are smooth approximations of each sample path of \( W \) for \( \varepsilon > 0 \).

If \( d \geq 2 \), the usual pathwise product of \( u \cdot \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W \) is classically not well-defined, and constructing a solution of (3) presents a major challenge in the standard theory of stochastic partial differential equations. Indeed, it is known that \( d \)-dimensional Brownian sheet \( W \) has regularity \( 1/2 - \varepsilon \) in each parameter for any \( \varepsilon > 0 \). Therefore, \( \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W \) can be understood to have regularity \( -d/2 - \varepsilon \) in total. Then, the solution \( u \) is expected to have regularity \( 2 - d/2 - \varepsilon \) so that the sum of the regularity of \( u \) and \( \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W \) is strictly less than \( 0 \). Hence, unfortunately, classical integration theory cannot be applied to the product \( u \cdot \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W \), and advanced techniques are inevitably required.

When \( d = 2 \) and on the whole space \( \mathbb{R}^2 \), the paper [6] introduces a particular renormalization procedure and constructs a solution to (3) by subtracting a divergent constant from the equation. On a torus of \( \mathbb{R}^2 \), a solution of (3) is constructed independently using paracontrolled distributions in [3] and using the theory of regularity structures in [4]. When \( d = 3 \), using the theory of regularity structures, the paper [7] carries out the construction of (3) on the whole space \( \mathbb{R}^3 \). An alternative construction of solution to (3) on a torus of \( \mathbb{R}^3 \) is also established in [5].

It turns out that for the model (3) when \( d = 1 \), there are several ways to define a solution in the Stratonovich sense, and the regularity of solution is worth attention. In [9], the Feynman-Kac solution for (3) in the Stratonovich sense is introduced. The paper proves that the Feynman-Kac solution is almost Hölder \( 3/4 \) continuous in time and almost Hölder \( 1/2 \) continuous in space. Here, “almost” Hölder continuity of order \( \gamma \) means Hölder continuity of any order less than \( \gamma \). However, the standard parabolic theory implies that the spatial regularity can be improved. Indeed, consider the additive model

\[
\bar{u}_t(t,x) = u_{xx}(t,x) + \bar{W}(x)
\]

as a reference for optimal regularity. It is known that the explicit solution of (5) is almost Hölder \( 3/4 \) continuous in time and almost Hölder \( 3/2 \) continuous in space. In that sense, very recently, the paper [11] shows that a solution defined using change of variables is almost Hölder \( 3/4 \) continuous in time and almost Hölder \( 3/2 \) continuous in space as desired.

On the other hand, the Wick-Itô-Skorokhod interpretation (4) draws attention from several authors. For example, the paper [16] by Uemura when \( d = 1 \), and the paper [8] by Hu when \( d < 4 \), define the chaos solution using multiple Itô-Wiener integrals. Also, the paper [16] gives some regularity results: The chaos solution is almost Hölder \( 1/2 \) continuous in both time and space. Recently, the paper [10] pays attention to the Wick-Itô-Skorokhod interpretation of (4) in one space dimension and the optimal space-time regularity of solution. Using the chaos expansion (or
Fourier expansion), the paper proves that the chaos solution is almost Hölder 3/4 continuous in time and almost Hölder 3/2 continuous in space.

The objectives of this paper are:

- to introduce a new pathwise solution using the mild formulation (called a mild solution) of time-homogeneous parabolic Anderson model driven by Gaussian white noise on an interval with Dirichlet boundary condition

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) \cdot \frac{\partial}{\partial x} W(x), \quad t > 0, \quad 0 < x < \pi, \quad u(t,0) = u(t,\pi) = 0, \quad u(0,x) = u_0(x). \quad (6)$$

- to obtain the optimal space-time Hölder regularity of the mild solution. That is, the mild solution is almost Hölder 3/4 continuous in time and almost Hölder 3/2 continuous in space.

The main results are extended to any Hölder continuous function $W$ on $[0,\pi]$ of order $\gamma \in (0,1)$, which implies the derivative of $W$ is a generalized function. Note that our results can be applied to (6) on the whole line $\mathbb{R}$ as long as an appropriate norm of $W$ on $\mathbb{R}$ is bounded. We also show the importance of mild formulation: the mild formulation suggests a way of finding relations between Stratonovich interpretation and Wick-Itô-Skorokhod interpretation.

Section 2 discusses the classical Schauder estimate and a modified version of parabolic Schauder estimates. In Section 3, a mild solution of (6) is defined and the optimal space-time regularity of the mild solution is obtained. Section 4 gives conclusion and suggests further directions of research.

2. The Schauder estimates. In this section, we establish a modified version of the Schauder estimates for parabolic type on an interval in some spaces.

We start with the Hölder spaces on $\mathbb{R}^d$. Denote by $\frac{\partial}{\partial z_i}$ the differentiation operator with respect to $z_i$, and for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i \in \mathbb{N}_0$ and $|\alpha| = \sum_{i=1}^{d} \alpha_i < \infty$, denote

$$\partial^\alpha_z = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial z_d^{\alpha_d}}.$$

Let $G$ be a domain in $\mathbb{R}^d$. For $0 < \gamma < 1$, let

$$[u]_{\gamma} := \sup_{z \neq y \in G} \frac{|u(z) - u(y)|}{|z - y|^\gamma}.$$ 

We say that $u$ is Hölder continuous with Hölder exponent $\gamma$ (or Hölder $\gamma$ continuous) on $G$ if

$$\sup_{z \in G} |u(z)| + [u]_{\gamma} < \infty.$$ 

The collection of Hölder $\gamma$ continuous functions on $G$ is denoted by $C^\gamma(G)$ with the norm

$$\|u\|_{C^\gamma(G)} := \|u\|_{L^\infty(G)} + [u]_{\gamma},$$

where

$$\|u\|_{L^\infty(G)} := \sup_{z \in G} |u(z)|.$$

We say that $u$ is a $k$ times continuously differentiable function on $G$ if $\partial^\alpha_z u$ exists and is continuous for all $|\alpha| \leq k$. The collection of $k$ times continuously differentiable functions on $G$ such that

$$\partial^\alpha_z u \in C^\gamma(G), \quad |\alpha| = k$$
is denoted by $C^{k+\gamma}(G)$ with the norm
\[
\|u\|_{k+\gamma} := \sum_{|\alpha| \leq k} \sup_{z \in G} |\partial_2^{\alpha} u(z)| + \sum_{|\alpha| = k} [\partial_2^{\alpha} u], \gamma < \infty.
\]

In particular, due to the presence of time and space variables, we often write
\[
C^{k_1+\gamma_1,k_2+\gamma_2}((0, T) \times G)
\]
with the norm
\[
\|u\|_{k_1+\gamma_1,k_2+\gamma_2} := \sum_{n \leq k_1, |\alpha| \leq k_2} \sup_{t \in (0, T), x \in G} |\partial_2^n \partial_2^\alpha u(t, x)| + \sum_{|\alpha| = k_2} [\partial_2^\alpha u], \gamma_1, \gamma_2 < \infty.
\]

We restrict the space domain by $G = (0, \pi)$ and let us denote the Dirichlet heat kernel on $(0, \pi)$ by
\[
P_D(t, x, y) = \sum_{k=1}^{\infty} e^{-k^2 t} m_k(x)m_k(y), \ m_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \ k \geq 1.
\]

Define a convolution $\ast$ for a function $f$ by
\[
(P_D \ast f)(t, x) = \int_0^t \int_0^\pi P_D(t-s, x, y)f(s, y)dyds.
\]

Let $0 < \gamma \notin \mathbb{N}$ and $T > 0$ be given. The classical parabolic type of Schauder’s estimate in Hölder spaces (Theorem 5.2 of Chapter IV in [15]) says that the convolution mapping
\[
f \mapsto P_D \ast f
\]
is continuous from $C^{\gamma/2, \gamma}_{t, x}((0, T) \times (0, \pi))$ to $C^{1+\gamma/2, 2+\gamma}_{t, x}((0, T) \times (0, \pi))$. More specifically, there exists a Schauder constant $C_T > 0$ such that $C_T$ remains bounded as $T \to 0$ and
\[
\|P_D \ast f\|_{1+\gamma/2, 2+\gamma} \leq C_T \|f\|_{\gamma/2, \gamma}.
\]

Unfortunately, the classical Schauder constant $C_T$ does not give a good estimate for our purpose. We now give a relaxed version of Schauder’s estimate in fractional Sobolev spaces instead of in Hölder spaces. The modified results will be useful for the existence of mild solution in the next section.

Denote by $H^s_p$ the fractional Sobolev space as the collection of all functions $f$ such that
\[
\left\|(-\Delta)^{s/2} f\right\|_{L_p(0, \pi)} < \infty,
\]
where
\[
(-\Delta)^{s/2} f = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{t\Delta} - I)f \frac{dt}{t^{1+s/2}},
\]
\Gamma is the gamma function, and $e^{t\Delta}$ is the semigroup of Laplacian operator on $(0, \pi)$ with zero boundary conditions.

Define
\[
\Lambda := (-\Delta)^{1/2}.
\]

**Lemma 2.1.** Let $h \in L_p(0, \pi), p \geq 1$ and let $0 < \theta \leq 2$. For any $0 < t \leq T$, we have
\[
\left\|\Lambda^\theta \int_0^t P_D(t-s, y)h(y)dy\right\|_{L_p(0, \pi)} \leq C(T, \theta, p)t^{-\theta/2}\|h\|_{L_p(0, \pi)}
\]
for some $C(T, \theta, p) > 0$ depending on $T, \theta$ and $p$. 
Proof. From [12, Lemma 7.3], it is enough to show that
\[ \| \partial_t \int_0^\pi P^D(t, \cdot, y)h(y)dy \|_{L_p(0, \pi)} \leq \frac{C}{T} \| h \|_{L_p(0, \pi)} \] (7)
for some \( C > 0 \) depending only on \( p \). Indeed, (7) follows from the fact
\[ \| \partial_t e^{\Delta h} \|_{L_p(0, \pi)} = \| \partial_t \int_0^\pi P^D(t, \cdot, y)h(y)dy \|_{L_p(0, \pi)} = \left\| \int_0^\pi \partial_t P^D(t, \cdot, y)h(y)dy \right\|_{L_p(0, \pi)} \]
and [2, Lemma 2.5]. \( \Box \)

**Theorem 2.2.** Let \( 0 < \beta < \gamma < 1 \) and \( T > 0 \) be given. For \( p \geq 1 \), the convolution mapping \( f \mapsto P^D \ast f \) is continuous from \( L_\infty(0, T; H^\beta_p(0, \pi)) \) to \( L_\infty(0, T; H^{2+\beta}_p(0, \pi)) \).

In particular,
\[ \| P^D \ast f \|_{L_\infty(0,T;H^{2+\beta}_p(0,\pi))} \leq C T^{(\gamma-\beta)/2} \| f \|_{L_\infty(0,T;H^\beta_p(0,\pi))} \]
for some constant \( C > 0 \) depending only on \( \beta \) and \( \gamma \).

Proof. Let \( 0 < \beta < \gamma < 1 \) be given. By the triangle inequality, we have
\[ \| P^D \ast f(t, \cdot) \|_{H^{2+\beta}_p(0,\pi)} = \left\| \int_0^t \int_0^\pi P^D(t-s, \cdot, y)f(s, y)dy ds \right\|_{H^{2+\beta}_p(0,\pi)} \leq \int_0^t \int_0^\pi P^D(t-s, \cdot, y)f(s, y)dy ds \right\|_{H^{2+\beta}_p(0,\pi)} \leq C_0(t-s)^{-\theta/2} \| f(s, \cdot) \|_{L_p(0,\pi)} \]
by Lemma 2.1. Therefore, by the commutativity of \( \Lambda \),
\[ \| P^D \ast f(t, \cdot) \|_{H^{2+\beta}_p(0,\pi)} \leq C_0 \int_0^t (t-s)^{(2+\beta-\gamma)/2} \| f(s, \cdot) \|_{H^\beta_p(0,\pi)} ds \leq C_0 \| f \|_{L_\infty(0,T;H^\beta_p(0,\pi))} \int_0^t (t-s)^{(2+\beta-\gamma)/2} ds \leq C_1 \| f \|_{L_\infty(0,T;H^\beta_p(0,\pi))} T^{(\gamma-\beta)/2}, \]
completing the proof. \( \Box \)

Denote the Neumann heat kernel on \((0, \pi)\) by
\[ P^N(t, x, y) = \sum_{k=0}^\infty e^{-k^2 t} \tilde{n}_k(x) \tilde{n}_k(y), \ \tilde{n}_0(x) = \frac{1}{\sqrt{\pi}}, \ \tilde{n}_k(x) = \sqrt{\frac{2}{\pi}} \cos(kx), \ k \geq 1. \]

**Corollary 1.** Under the same assumptions of Theorem 2.2, the convolution map
\[ f \mapsto P^N \ast f \]
is continuous from \( L_\infty(0, T; H^\beta_p(0, \pi)) \) to \( L_\infty(0, T; H^{2+\beta}_p(0, \pi)) \). In other words,
\[ \| P^N \ast f \|_{L_\infty(0,T;H^{2+\beta}_p(0,\pi))} \leq C T^{(\gamma-\beta)/2} \| f \|_{L_\infty(0,T;H^\beta_p(0,\pi))} \]
for some constant \( C > 0 \) depending only on \( \beta \) and \( \gamma \).
3. The mild solution and its regularity. Let \( \{W(x)\}_{x \in [0, \pi]} \) be a standard Brownian motion on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Here, \( \mathcal{F} \) denotes the filtration generated by \( W \). It is known that Brownian motion is in \( C^\gamma(0, \pi) \) for any \( 0 < \gamma < 1/2 \).

Let \( W^\varepsilon \) be smooth approximations of \( W \) such that
\[
\|W^\varepsilon - W\|_{C^\gamma(0, \pi)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Consider the approximated equations of (6) for \( \varepsilon > 0 \):
\[
\begin{align*}
\frac{\partial u^\varepsilon(t, x)}{\partial t} &= \frac{\partial^2 u^\varepsilon(t, x)}{\partial x^2} + u^\varepsilon(t, x) \cdot \frac{\partial}{\partial x} W^\varepsilon(x), \quad t > 0, \quad 0 < x < \pi, \\
u^\varepsilon(t, 0) &= u^\varepsilon(t, \pi) = 0, \quad u^\varepsilon(0, x) = u_0(x).
\end{align*}
\]
(9)

Since \( W^\varepsilon \) is smooth, the equation (9) has the classical solution \( u^\varepsilon \). Denote
\[
P_0(t, x) = \begin{cases} 
  u_0(x) & \text{if } t = 0 \\
  \int_0^\pi P^D(t, x, y)u_0(y)dy & \text{if } t > 0.
\end{cases}
\]

Note that as \( t \to 0^+ \), we see that \( P_0(t, x) \to u_0(x) \) for each \( x \in [0, \pi] \). Then, the mild formulation for the equation (9) is given by
\[
u^\varepsilon(t, x) = P_0(t, x) + \int_0^t \int_0^\pi P^D(t - s, x, y)u^\varepsilon(s, y)\partial_y W^\varepsilon(y)dyds.
\]
(10)

By integration by parts with respect to \( y \) in the last term above, we rewrite (10) as
\[
u^\varepsilon(t, x) = P_0(t, x) - \int_0^t \int_0^\pi P^D(t - s, x, y)u^\varepsilon(s, y)W^\varepsilon(y)dyds
- \int_0^t \int_0^\pi P^D(t - s, x, y)u^\varepsilon_y(s, y)W^\varepsilon(y)dyds.
\]

Definition 3.1. We say that \( u \) is a mild solution of (6) if
\begin{itemize}
  \item for every \( 0 < t < T \) and \( 0 < x < \pi \), \( u \) is continuous in \( t \), continuously differentiable in \( x \), and it satisfies the equation
  \[
u(t, x) = P_0(t, x) - \int_0^t \int_0^\pi P^D(t - s, x, y)u(s, y)\partial_y W(y)dyds
  - \int_0^t \int_0^\pi P^D(t - s, x, y)u_y(s, y)W(y)dyds
  \]
  (11)
  \item for every \( 0 \leq t \leq T \), \( u(t, 0) = u(t, \pi) = 0 \);
  \item for every \( 0 \leq x \leq \pi \), \( \lim_{t \to 0^+} u(t, x) = u_0(x) \).
\end{itemize}

For the existence and the uniqueness of mild solution, we use a contraction mapping (or fixed point argument) on \( L_\infty \) \( (0, T; H_p^{1+\beta}(0, \pi)) \) with \( 0 < \beta < 1 \) and \( p \geq 1 \). Define a map
\[
\mathcal{M} : L_\infty \left(0, T; H_p^{1+\beta}(0, \pi)\right) \to L_\infty \left(0, T; H_p^{1+\beta}(0, \pi)\right)
\]
by
\[
(\mathcal{M}u)(t, x) = P_0(t, x) - \int_0^t \int_0^\pi P^D(t - s, x, y)u(s, y)\partial_y W(y)dyds
- \int_0^t \int_0^\pi P^D(t - s, x, y)u_y(s, y)W(y)dyds.
\]
(12)

We first prove the well-posedness of (12).
Lemma 3.2. Let $0 < \beta < \gamma < 1$. If $u_0 \in H^\beta_p(0, \pi)$ with $p \geq 1$, 
\[ \|P_0\|_{L^\infty(0,T;H^{1+\gamma}_p(0,\pi))} < \infty. \]

Proof. It is clear from Lemma 2.1. \qed

Theorem 3.3. Let $0 < \beta < \gamma < 1/2$. If $u_0 \in H^\beta_p(0, \pi)$ with $p \geq 1$, the mapping $\mathcal{M}$ on $L^\infty(0, T; H^{1+\beta}_p(0, \pi))$ is well-defined. Also, there exists a fixed point of $\mathcal{M}$. That is, the fixed point is the unique mild solution of (6).

Proof. Since $P^*_y(t, x, y) = -P^*_x(t, x, y)$ for each $t > 0$ and $x, y \in [0, \pi]$, we have
\[
\int_0^t \int_0^\pi P^*_y(t-s, x, y)u(s, y)W(y)dyds = -\int_0^t \int_0^\pi P^*_x(t-s, x, y)u(s, y)W(y)dyds.
\]
Then, we rewrite (12) by
\[
(\mathcal{M}u) (t, x) = P_0(t, x) + P^*_x *(uW)(t, x) - P^*_x *(u_x W)(t, x).
\]
We show the well-definedness term by term. By Lemma 3.2, 
\[
P_0 \in L^\infty(0, T; H^{1+\gamma}_p(0, \pi)).
\]
Fix $t \in (0, T)$. We have
\[
\|P^*_x *(uW(t, \cdot))\|_{H^{1+\gamma}_p(0, \pi)} = \|\partial_x (P^*_x * uW)(t, \cdot)\|_{H^{1+\gamma}_p(0, \pi)} \leq \|P^*_x *(uW)(t, \cdot)\|_{H^{1+\gamma}_p(0, \pi)}.
\]
Then, by Corollary 1 and [12, Lemma 5.2], 
\[
\|P^*_x * uW\|_{L^\infty(0, T; H^{1+\gamma}_p(0, \pi))} \leq \|P^*_x * uW\|_{L^\infty(0, T; H^{2+\beta}_p(0, \pi))}
\leq C_1 T^{(\gamma-\beta)/2} \|W\|_{C^\gamma(0, \pi)} \|u\|_{L^\infty(0, T; H^{\beta}_p(0, \pi))}
\leq C_2 T^{(\gamma-\beta)/2} \|W\|_{C^\gamma(0, \pi)} \|u\|_{L^\infty(0, T; H^{\beta}_p(0, \pi))}
\]
and similarly by (8), 
\[
\|P^*_x u_x W\|_{L^\infty(0, T; H^{1+\beta}_p(0, \pi))} \leq C_3 T^{1/2} \|W\|_{C^\gamma(0, \pi)} \|u\|_{L^\infty(0, T; H^{\beta}_p(0, \pi))}
\]
Let $u, v \in L^\infty(0, T; H^{1+\beta}_p(0, \pi))$. Then,
\[
\|\mathcal{M}u - \mathcal{M}v\|_{L^\infty(0, T; H^{1+\beta}_p(0, \pi))} \leq \|P^*_x *(u-v)W\|_{L^\infty(0, T; H^{1+\beta}_p(0, \pi))}
+ \|P^*_x *(u_x - v_x)W\|_{L^\infty(0, T; H^{1+\beta}_p(0, \pi))}
\leq C T^{1/2} \|W\|_{C^\gamma(0, \pi)} \|u-v\|_{L^\infty(0, T; H^{1+\beta}_p(0, \pi))}
\]
for some $C > 0$. Choose $\delta > 0$ such that $C \delta^{1/2} \|W\|_{C^\gamma(0, \pi)} < 1$. Then, clearly there exists a fixed point of $\mathcal{M}$ up to $\delta$. Consider a time partition 0 = $t_0 < \cdots < t_n = T$ such that $t_{i+1} - t_i \leq \delta$ for $i = 0, \cdots, n-1$. We now define $u$ recursively on $(t_i, t_{i+1}]$ with the initial condition $u(t_i, x)$ for $i = 0, \cdots, n-1$:
\[
(\mathcal{M}u)(t, x) = \int_0^t P^*_y (t_{i+1}, x, y) u(t_i, y) dy
- \int_0^{t_{i+1}} \int_0^\pi P^*_y (t_{i+1} - s, x, y) u(s, y) W(y) dy ds
- \int_0^{t_{i+1}} \int_0^\pi P^*_y (t_{i+1} - s, x, y) u_y (s, y) W(y) dy ds.
\]
The existence of the fixed point to (13) is guaranteed by the above arguments since $t_{i+1} - t_i \leq \delta$. Since $n$ is finite, we obtain the fixed point solution over the whole time interval $(0, T)$. We note that the fixed point solution satisfies the mild formulation (11). Since the fixed point solution is unique, the uniqueness of mild solution clearly holds.

**Remark 1.** From Theorem 3.3, we have that for all $t \in [0, T]$,

$$u(t, \cdot) \in H^{1+\beta}_p(0, \pi), \ p \geq 1.$$  

By the Sobolev embedding theorem, we have

$$H^{1+\beta}_p(0, \pi) \subset C^{1+\beta-1/p}(0, \pi)$$

for any $p \geq 1$. This shows that the mild solution $u$ is indeed almost Hölder $3/2$ continuous in space.

We show the mild solution of (6) is the limit of classical solutions $u^\varepsilon$ of (9) in $L^\infty(0, T; H^{1+\beta}_p(0, \pi)).$

**Theorem 3.4.** Let $0 < \beta < \gamma < 1/2$. If $u_0 \in H^{\beta}_p(0, \pi)$, then we have

$$\|u^\varepsilon - u\|_{L^\infty(0,T;H^{1+\beta}_p(0,\pi))} \to 0 \text{ as } \varepsilon \to 0.$$

**Proof.** For simplicity, we denote $\|\cdot\|_{C^\gamma} = \|\cdot\|_{C^{\gamma}(0,\pi)}$ and

$$\|\cdot\| = \|\cdot\|_{L^\infty(0,T;H^{1+\beta}_p(0,\pi))}.$$

Then, we can write $u^\varepsilon - u$ by

$$P^N_x \ast u^\varepsilon W^\varepsilon - P^N_x \ast uW - P^D \ast u^\varepsilon W^\varepsilon + P^D \ast u_x W.$$

By the triangle inequality,

$$\|u^\varepsilon - u\| \leq \left\|P^N_x \ast (u^\varepsilon - u)W^\varepsilon\right\| + \left\|P^N_x \ast u(W - W^\varepsilon)\right\|$$

$$+ \left\|P^D \ast (u^\varepsilon - u_x)W^\varepsilon\right\| + \left\|P^D \ast u_x (W - W^\varepsilon)\right\|$$

$$\leq C_1 \beta^{1/2} \|W^\varepsilon\|_{C^\gamma} \|u^\varepsilon - u\| + C_2 \beta^{1/2} \|u\| \|W - W^\varepsilon\|_{C^\gamma}$$

$$\leq C_1 \beta^{1/2} (\|W\|_{C^\gamma} + \|W - W^\varepsilon\|_{C^\gamma}) \|u^\varepsilon - u\| + C_2 \beta^{1/2} \|u\| \|W - W^\varepsilon\|_{C^\gamma}.$$

Note that the constants $C_1, C_2 > 0$ are independent of $\varepsilon$. Choose $\delta > 0$ such that

$$C_1 \beta^{1/2} (\|W\|_{C^\gamma} + \|W - W^\varepsilon\|_{C^\gamma}) < 1$$

for small $\varepsilon > 0$. Then, since $\|W - W^\varepsilon\|_{C^\gamma} \to 0$,

$$\|u^\varepsilon - u\| \to 0 \text{ as } \varepsilon \to 0.$$

We now consider a time partition $0 = t_0 < \cdots < t_n = T$ such that $t_{i+1} - t_i \leq \delta$ for $i = 0, \cdots, n - 1$. Finally, we iterate the above argument for $u^\varepsilon - u$ recursively on $(t_i, t_{i+1})$ to get

$$\|u^\varepsilon(t, \cdot) - u(t, \cdot)\| \to 0 \text{ as } \varepsilon \to 0$$

for $i = 1, \cdots, n - 1$. 

**Theorem 3.5.** Let $0 < \gamma < 1/2$. If $u_0 \in C^{1+\gamma}(0, \pi)$, then the mild solution of (6) is indeed in

$$C_1^{(1+\gamma)/2,1+\gamma}(0, T) \times (0, \pi))$$
Proof. Let $0 < \beta < \gamma < 1/2$. Recall, for each $\varepsilon > 0$, the approximated parabolic Anderson model
\[
\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + u^\varepsilon(t,x) \cdot \frac{\partial}{\partial x} W^\varepsilon(x), \quad t > 0, \ 0 < x < \pi,
\]
\[
u^\varepsilon(t,0) = u^\varepsilon(t,\pi) = 0, \ u^\varepsilon(0,x) = u_0(x).
\]
From the classical parabolic theory, there exists the unique classical solution $u^\varepsilon$. We note that by Theorem 3.4, $u^\varepsilon$ converges to the limit $u$ in $L_\infty(0,T; H^{1+\beta}_p(0,\pi))$

since $C^{1+\gamma}(0,\pi) \subset H^{\beta}_p(0,\pi)$ for any $p \geq 1$.

On the other hand, let $v^\varepsilon$ satisfy the equation
\[
\frac{\partial v^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 v^\varepsilon(t,x)}{\partial x^2} - 2W^\varepsilon \cdot \frac{\partial}{\partial x} v^\varepsilon(t,x) + (W^\varepsilon)^2 v^\varepsilon(t,x), \quad t > 0, \ 0 < x < \pi,
\]
\[
u^\varepsilon(t,0) = v^\varepsilon(t,\pi) = 0, \ v^\varepsilon(0,x) = u_0(x)e^{-\int_0^x W^\varepsilon(y)dy}.
\]
We observe that $u^\varepsilon(t,x) = v^\varepsilon(t,x)e^{\int_0^x W^\varepsilon(y)dy}$.

By [11, Theorem 2.3], $u^\varepsilon$ converges to a limit in $C^{(1+\gamma)/2,1+\gamma}_{t,x}(0,T \times (0,\pi))$.

and the limit of $u^\varepsilon$ is unique in $L_2((0,T); H^{1}_0(0,\pi)) \cap L_\infty((0,T); L_2(0,\pi))$ by [11, Theorem 3.5], where $H^{1}_0(0,\pi)$ is the closure of the set of smooth functions with compact support in $(0,\pi)$ with respect to the norm $\| \cdot \|_{H^{1}_0(0,\pi)}$.

Since, for $p \geq 2$,
\[
L_\infty(0,T; H^{1+\beta}_p(0,\pi)) \subset L_2((0,T); H^{1}_0(0,\pi)) \cap L_\infty((0,T); L_2(0,\pi)),
\]
the mild solution $u$ of (6) is indeed in $C^{(1+\gamma)/2,1+\gamma}_{t,x}(0,T \times (0,\pi)).$

\[\square\]

Remark 2. The reason why we set the upper bound of regularity $\gamma$ less than $1/2$ in Theorem 3.3, 3.4 and 3.5 is due to the regularity of Brownian motion. In fact, the Brownian motion $W$ can be replaced by any pathwisely Hölder $\gamma$ continuous process with $0 < \gamma < 1$ for Theorem 3.3, 3.4 and 3.5. For example, $W$ can be a standard fractional Brownian motion $W^H$ with the Hurst index $0 < H < 1$.

Remark 3. Consider the following equation on the whole line $\mathbb{R}$
\[
\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) \cdot \frac{\partial}{\partial x} W(x), \quad t > 0, \ x \in \mathbb{R},
\]
\[
u(0,x) = u_0(x), \ x \in \mathbb{R}.
\]

Theorem 3.3, 3.4 and 3.5 will also work on the whole line $\mathbb{R}$ if the Hölder $\gamma$ norm of $W$ on $\mathbb{R}$ is bounded with $0 < \gamma < 1$;

- The modified Schauder estimate result in Theorem 2.2 is sharper in Hölder spaces if $P^D$ is replaced by the Gaussian heat kernel $P(t,x-y)$ on $\mathbb{R}$: for any $f \in (0,T; C^\gamma(\mathbb{R}))$, we have
\[
\| P^* f \|_{L_\infty(0,T; C^{2+\gamma}(\mathbb{R}))} \leq CT^{\gamma/2} \| f \|_{L_\infty(0,T; C^\gamma(\mathbb{R}))},
\]
for some $C > 0$, which is independent of $T$. Note that a Brownian motion on $\mathbb{R}$ do not have a sample trajectory that has a bounded Hölder norm.

- Clearly, $P_y(t,x-y) = P_x(t,x-y)$ holds for $t > 0$.

4. Conclusion and further directions.

4.1. Spatial optimal regularity. The paper [9] gives the spatial (Hölder) regularity only less than $1/2$. However, Theorem 3.3 using the fixed point argument and [11, Theorem 2.3] show that the optimal spatial (Hölder) regularity of the solution $u$ is $3/2 - \varepsilon$ for any $\varepsilon > 0$ as long as $u_0 \in C^{3/2}(0,\pi)$. The result implies several important remarks:

- Achieve the spatial regularity higher than $1/2$. Since the standard Brownian motion $W$ is Hölder $1/2 - \varepsilon$ continuous almost surely, it is possible to apply Young’s integral: For each $s<t$, $\frac{\pi}{0} P(t_s,x,y)u(s,y) \frac{\partial}{\partial y} W(y) dy := \frac{\pi}{0} P(t_s,x,y)u(s,y) dW(y)$ appearing in the classical mild formulation of (6);

- The regularity $3/4 - \varepsilon$ in time and $3/2 - \varepsilon$ in space for $\varepsilon > 0$ are indeed in line with the standard parabolic partial differential equation theory.

Note that [11, Theorem 2.3] and Remark 1 give the same Hölder regularity in space, but by different approaches. More precisely, Theorem 3.3 implies the same spatial regularity with the one of [11, Theorem 2.3] by starting with the fractional Sobolev spaces first and by Sobolev embedding theorem.

4.2. Why mild formulation? In fact, the mild formulation is also applied to construct the Wick-Itô-Skorokhod solution of (4) on $[0,\pi]$ with Dirichlet boundary condition in [10]. It is known that for any initial function $\alpha_0 \in C^{3/2}(0,\pi)$, there exists the unique Wick-Itô-Skorokhod solution $u^\diamond$ satisfying the equation

$$ u^\diamond(t,x) = P_0(t,x) + \int_0^t \int_0^\pi P(t-s,x,y)u^\diamond(s,y) \diamond W(y) dy ds $$ (14)

almost surely in $C^{3/4-\varepsilon,3/2-\varepsilon}_{t,x} ((0,T) \times (0,\pi))$ for any $\varepsilon > 0$.

The natural further question is to find a meaningful relation between the usual solution of (6) and the Wick-Itô-Skorokhod solution of (4) with Dirichlet boundary condition in the mild formulation.

There are relations between the usual product and the Wick product (e.g. [13] and [14]). Let $\xi_k$ be i.i.d standard Gaussian random variables. Denote by $S$ the collection of all multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots)$ such that $\alpha_k \in \mathbb{N}_0$, $k = 1, 2, \cdots$, and $\sum_{k=1}^\infty \alpha_k < \infty$. For $\alpha, \beta \in S$, define

- $(0) = (0, 0, \cdots)$;
- $e(k)$ is the multi-index $\alpha$ such that $\alpha_k = 1$ and $\alpha_i = 0$ if $i \neq k$;
- $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots)$;
- $\alpha - \beta = (\max(\alpha_1 - \beta_1, 0), \max(\alpha_2 - \beta_2, 0), \cdots)$;
- $\alpha! = \prod_k \alpha_k!$.

We define the Hermite polynomial of order $n$ by

$$ H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$
and define, for each \( \alpha \in S \),
\[
\xi_{\alpha} = \prod_{k=1}^{\infty} \left( \frac{H_{\alpha k}(\xi_k)}{\sqrt{\alpha_k!}} \right).
\]

For any \( u, v \in L_2(\Omega) \), Cameron-Martin theorem [1] gives the following fact: For each \( t \in [0, T] \) and \( x \in [0, \pi] \), \( u \) and \( v \) may be written as
\[
u(t, x) = \sum_{\alpha \in S} u_{\alpha}(t, x) \xi_{\alpha} \text{ and } v(t, x) = \sum_{\alpha \in S} v_{\alpha}(t, x) \xi_{\alpha}.
\]

Also, we have the identity [13, Theorem 2.3]
\[
u \cdot v = u \diamond v + \sum_{\alpha \in S} \left( \sum_{\gamma \neq (0)} \sum_{0 \leq \beta \leq \alpha} \frac{\sqrt{\alpha!(\alpha - \beta + \gamma)!(\beta + \gamma)!}}{\beta!\gamma!(\alpha - \beta)!} u_{\alpha - \beta + \gamma} v_{\beta + \gamma} \right) \xi_{\alpha}.
\]

From the fact that standard Brownian motion on \([0, \pi]\) has an explicit formula
\[
W(x) = \sum_{k=1}^{\infty} \left( \frac{x}{\sqrt{\pi}} \right) m_k(x) dy \xi_{\epsilon(k)},
\]
we have a formal expression of Gaussian white noise on \([0, \pi]\) given by
\[
\dot{W}(x) = \sum_{k=1}^{\infty} m_k(x) \xi_{\epsilon(k)},
\]
where \( m_k \)'s are defined as before. Consider smooth approximations \( \dot{W}^\varepsilon \) of \( \dot{W}(x) \) as follows. Let \( \phi \) be a compactly supported smooth function such that
\[
0 \leq \phi \leq 1 \text{ and } \int_0^\pi \phi(x) dx = 1.
\]
Let \( \varepsilon > 0 \) be given. Denote
\[
\phi_{\varepsilon}(x) = \frac{1}{\varepsilon} \phi \left( \frac{x}{\varepsilon} \right).
\]
For each \( x \in [0, \pi] \) and \( 1 \leq p < \infty \),
\[
\dot{W}^\varepsilon(x) = \sum_{k=1}^{\infty} m_k^\varepsilon(x) \xi_{\epsilon(k)} \in L_p(\Omega),
\]
where \( m_k^\varepsilon(x) = m_k * \phi_{\varepsilon}(x) \). Here, * stands for the classical convolution operator.

It is also known [10] that for any \( u_0 \in L_p(0, \pi), 1 \leq p < \infty \), the Wick-Itô-Skorokhod solution of (14) has the basic regularity
\[
u^\varepsilon(t, x) \in L_p(\Omega), \quad t > 0, \ x \in [0, \pi].
\]

Also, the approximated Wick-Itô-Skorokhod solutions in the mild formulation
\[
(u^\varepsilon)^\diamond(t, x) = P_0(t, x) + \int_0^t \int_0^\pi P(t - s, x, y) (u^\varepsilon)^\diamond(s, y) \diamond \dot{W}^\varepsilon(y) dy ds.
\]
have the regularity
\[
(u^\varepsilon)^\diamond(t, x) \in L_p(\Omega), \quad t > 0, \ x \in [0, \pi].
\]

Then, we get the relation
\[
(u^\varepsilon)^\diamond(t, x) \cdot \dot{W}^\varepsilon(x) = (u^\varepsilon)^\diamond(t, x) \diamond \dot{W}^\varepsilon(x)
\]
Define the residual by
\[ \eta^\varepsilon(t,x) = \sum_{\alpha \in \mathcal{S}} \eta^\varepsilon_\alpha(t,x) \xi_\alpha := \sum_{\alpha \in \mathcal{S}} \sum_{k \geq 1} \left( \sqrt{\alpha_k + 1} (u^\varepsilon)^\alpha_{\alpha+\varepsilon(k)}(t,x) m_k^\varepsilon(x) \right) \xi_\alpha, \]
where
\[ \eta^\varepsilon_\alpha(t,x) := \sum_{k \geq 1} \sqrt{\alpha_k + 1} (u^\varepsilon)^\alpha_{\alpha+\varepsilon(k)}(t,x) m_k^\varepsilon(x). \]

Beyond the basic relations, let us find a further connection between usual solution and Wick-Itô-Skorokhod solution of (2). Consider the approximated mild solutions of (9)
\[ u^\varepsilon(t,x) = P_0(t,x) + \int_0^t \int_0^\pi P(t-s,x,y) u^\varepsilon(s,y) \cdot \dot{W}^\varepsilon(y) dy ds. \]

After we define \( Z^\varepsilon(t,x) \) by
\[ Z^\varepsilon(t,x) = u^\varepsilon(t,x) - (u^\varepsilon)^\varepsilon(t,x) \]
and using the relation (15), we have the equation
\[ Z^\varepsilon(t,x) = \int_0^t \int_0^\pi P(t-s,x,y) Z^\varepsilon(s,y) \cdot \dot{W}^\varepsilon(y) dy ds \]
\[ - \int_0^t \int_0^\pi P(t-s,x,y) \eta^\varepsilon(s,y) dy ds. \] (16)

Equivalently, the equation (16) is the mild formulation of
\[ \frac{\partial Z^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 Z^\varepsilon(t,x)}{\partial x^2} + Z^\varepsilon(t,x) \cdot \dot{W}^\varepsilon(x) - \eta^\varepsilon(t,x), \quad 0 < t < T, \quad 0 < x < \pi; \]
\[ Z^\varepsilon(0,x) = 0, \quad Z^\varepsilon(t,0) = Z^\varepsilon(t,\pi) = 0. \] (17)

Set
\[ Z(t,x) = u(t,x) - u^\varepsilon(t,x), \]
where \( u \) is the usual mild solution of (6) and \( u^\varepsilon \) is the Wick-Itô-Skorokhod solution of (4) on \([0, \pi]\) with Dirichlet boundary condition and initial condition in \( C_{3/2}^3(0, \pi) \). Then, for any \( 0 < \gamma_1 < 3/4 \) and \( 0 < \gamma_2 < 3/2 \),
\[ Z^\varepsilon \to Z \text{ in } C_{t,x}^{3-\gamma_1, \gamma_2}((0,T) \times (0,\pi)) \text{ as } \varepsilon \to 0. \]

We naturally expect that \( Z(t,x) \) satisfies the equation
\[ \frac{\partial Z(t,x)}{\partial t} = \frac{\partial^2 Z(t,x)}{\partial x^2} + Z(t,x) \cdot \dot{W}(x) - \eta(t,x), \quad 0 < t < T, \quad 0 < x < \pi; \]
\[ Z(0,x) = 0, \quad Z(t,0) = Z(t,\pi) = 0, \]
where
\[ \eta(t,x) = \sum_{\alpha \in \mathcal{S}} \sum_{k \geq 1} \left( \sqrt{\alpha_k + 1} u_{\alpha+\varepsilon(k)}(t,x) m_k(x) \right) \xi_\alpha. \]

Actually, we can show, for each \( t > 0 \), \( \eta^\varepsilon(t,\cdot) \) converges to \( \eta(t,\cdot) \) in \( L_2(\Omega; H_2^{-r}(0,\pi)) \) with \( r > 1/2 \). Here, \( H_2^{-r} \) is the dual space of the Sobolev space \( H_2^r \). Therefore, we can view the Wick-Itô-Skorokhod solution of (4) on \([0, \pi]\) with Dirichlet boundary condition as an approximation of usual mild solution of (6), and moreover, further investigation of the residual equation (17) is reasonable to give a rigorous relation between usual solution and Wick-Itô-Skorokhod solution.
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