ON THE COMPARISON OF NEARBY CYCLES VIA
\(b\)-FUNCTIONS

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Abstract. In this article, we give a simple proof of the comparison of nearby and vanishing cycles in the sense of Riemann-Hilbert correspondence following the idea of Beilinson and Bernstein, without using the Kashiwara-Malgrange \(V\)-filtrations.

1. Introduction

The idea of the nearby and vanishing cycles can be traced back to Grothendieck and they are first introduced by Deligne [Del73]. Nearby and vanishing cycles are widely studied from different perspectives, for instance by Beilinson [Bei87] algebraically and Kashiwara and Schapira [KS13] under the microlocal setting. They are also very useful, for instance Saito [Sai88] used nearby and vanishing cycles to give an inductive definition of pure Hodge modules.

Using the so-called Kashiwara-Malgrange filtration, as well as its refinement, the \(V\)-filtration, Kashiwara [Kas83] defined nearby and vanishing cycles for holonomic \(\mathcal{D}\)-modules and proved a comparison theorem in the sense of Riemann-Hilbert correspondence (see [Kas83, Theorem 2]).

Beilinson and Bernstein constructed the unipotent (or more precisely, nilpotent) nearby and vanishing cycles for holonomic \(\mathcal{D}\)-modules using \(b\)-functions under the algebraic setting in [BB93, §4.2] by using the complete ring \(\mathbb{C}[[t]]\); see also [BG12, §2.4] by using localization of \(\mathbb{C}[t]\) without completion. One then can “glue” the open part and the vanishing cycle of a holonomic \(\mathcal{D}\)-module along any regular functions in the sense of Beilinson [Bei87] (see also [Gin98, Theorem 4.6.28.1] and [Lic09]).

In this article, by using the theory of relative holonomic \(\mathcal{D}\)-modules by Maisonobe [Mai16] and its development in [WZ19, BVWZ19, BVWZ20], we give a slight refinement of the construction of nearby and vanishing cycles of Beilinson and Bernstein to other eigenvalues. Then we will give a simple proof of the comparison of nearby and vanishing cycles in the sense of Riemann-Hilbert correspondence without using \(V\)-filtrations (see Theorem 1.1), which we believe is new. However, essentially the proof has been hinted by Beilinson and Bernstein (see for instance [BB93, Remark 4.2.2(v)]). Another point of the proof of the comparison is that since the proof is purely algebraic, it can be transplanted on smooth varieties over fields of characteristic 0 without much modification.

Let \(X\) be a smooth algebraic variety over \(\mathbb{C}\) (or a complex manifold) and \(f\) a regular function on \(X\) (or a holomorphic function on \(X\)) and let \(\mathcal{M}\) be a holonomic \(\mathcal{D}_X\)-module. For \(\alpha \in \mathbb{C}\), we denote the \(\alpha\)-nearby cycle of \(\mathcal{M}\) along \(f\) by \(\Psi_{f,\alpha}\mathcal{M}\) and

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denote the vanishing cycle of $\mathcal{M}$ along $f$ by $\Phi_{f,\mathcal{M}}$ (see §2.2 for definitions). The sheaves $\Psi_{f,\mathcal{M}}$ and $\Phi_{f,\mathcal{M}}$ are the same as $\Psi_{\text{nil}}(\mathcal{M})$ and $\Phi_{\text{nil}}(\mathcal{M})$ in [BG12, §2.4].

From construction, $\Psi_{f,\mathcal{M}}$ and $\Phi_{f,\mathcal{M}}$ have the action by $s$, where $s$ is the independent variable introduced in defining $b$-functions (see §2.1) and $\Psi_{f,\mathcal{M}}$ only depends on $\mathcal{M}|_U$.

The $b$-function is also called the Bernstein-Sato polynomial. At least for $\mathcal{M} = \mathcal{O}_X$, there are algorithms to compute $b$-functions with the help of computer program (for instance Singular and Macaulay2). On the contrary, Kashiwara-Malgrange filtrations are more difficult to deal with from algorithmic perspectives as far as we know. Therefore, it seems easier to deal with nearby and vanishing cycles via $b$-functions.

Following Beilinson’s idea in [Bei87], we define the nearby and vanishing cycles for $\mathbb{C}$-perverse sheaves by using Jordan blocks. Let $K$ be a perverse sheaf of $\mathbb{C}$-coefficients on $X$. One can also work with perverse sheaves with arbitrary fields of coefficients, see for instance [Rei10]. When we talk about perverse sheaves on an algebraic variety over $\mathbb{C}$, we use the Euclidean topology by default. If one wants to work with algebraic varieties with other base fields of characteristic zero, then one can consider étale sheaves.

For $\lambda \in \mathbb{C}^*$ we define the $\lambda$-nearby cycle by
$$\psi_{f,\lambda}K := \lim_{m \to \infty} i^{-1}Rj_* (K|_U \otimes f_0^{-1}L_{m}^{1/\lambda}),$$
where $j : U = X \setminus D \hookrightarrow X$ and $D$ is the divisor defined by $f = 0$, $i : D \hookrightarrow X$ and $f_0 = f|_U$. Here $L_{m}^{\lambda}$ is isomorphic to the local system given by a $m \times m$ Jordan block with the eigenvalue $\lambda^{-1}$ on $\mathbb{C}^*$ (see §3 for the construction). For all $m \in \mathbb{Z}$, $L_{m}^{\lambda}$ naturally form a direct system with respect to the natural order on $\mathbb{Z}$. The vanishing cycle of $K^\bullet$ along $f$ is then defined by
$$\phi_f K := \text{Cone}(i^{-1}K \to \psi_{f,1}K).$$

We then have a canonical morphism
$$\text{can}: \psi_{f,1}K \to \phi_f K$$
fitting in the tautological triangle
$$i^{-1}K \to \psi_{f,1}K \xrightarrow{\text{can}} \phi_f K \xrightarrow{+1}.$$

The monodromy action on $L_{m}^{1/\lambda}$ naturally induces the monodromy action on both $\psi_{f,\lambda}K$ and $\phi_f K$, denoted by $T$. By construction, $T - \lambda$ acts on $\psi_{f,\lambda}K$ nilpotently. When $\lambda = 1$, log $T_u$ induces
$$\text{Var}: \phi_f K \to \psi_{f,1}K.$$

See [4] for details. The above definition of nearby and vanishing cycles coincides with Deligne’s construction (see [Bj93] Chapter VI. 6.4.6 for $\psi_{f,1}K$ and [Wu17, §3] and [Rei10] in general). The morphism Var corresponds to the “Var” morphism defined in [Kas83].

The rest of this paper is mainly about the proof of the following comparison theorem.

**Theorem 1.1.** Assume that $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module. Then we have
$$\text{DR}(\Psi_{f,\alpha}\mathcal{M}) \simeq i_*\psi_{f,\lambda}\text{DR}(\mathcal{M})[-1]$$
for every \( \alpha \in \mathbb{C} \) where \( \lambda = e^{2\pi \sqrt{-1} \alpha} \) and the monodromy action \( T \) corresponds to
\[
T \simeq \text{DR}(e^{-2\pi \sqrt{-1} \alpha}) = \text{DR}(\lambda \cdot e^{-2\pi \sqrt{-1}(s+\alpha)})
\]
under the isomorphisms (since \( s + \alpha \) acts on \( \Psi_{f,\alpha} \mathcal{M} \) nilpotently), and
\[
\text{DR}(\Phi_f \mathcal{M}) \simeq i_\ast \phi_f \text{DR}(\mathcal{M})[-1]
\]
where \( \text{DR} \) denotes the de Rham functor for \( \mathcal{D} \)-modules. Furthermore, we have natural morphisms of \( \mathcal{D} \)-modules
\[
v : \Phi_f \mathcal{M} \to \Psi_{f,0} \mathcal{M} \quad \text{and} \quad c : \Psi_{f,0} \mathcal{M} \to \Phi_f \mathcal{M}
\]
so that there exists an isomorphism of quivers
\[
\text{DR}(\Psi_{f,0} \mathcal{M}) \xrightarrow{v} \Phi_f \mathcal{M} \simeq i_\ast (\psi_{f,1}(\text{DR}(\mathcal{M}))) \xrightarrow{\text{can}} \phi_f (\text{DR}(\mathcal{M}))[\mathbb{Z}].
\]

Remark 1.2. In Theorem 1.1, we see that \( \Psi_{f,0+k} \mathcal{M} \) correspond to a unique nearby cycle of \( \text{DR}(\mathcal{M}) \) for all \( k \in \mathbb{Z} \). Namely, the Riemann-Hilbert correspondence for nearby cycles is \( \mathbb{Z} \)-to-1. However, \( \Psi_{f,0+k} \mathcal{M} \) is unique up to the \( \text{t} \)-action by Eq.4.

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2. Nearby and vanishing cycles for holonomic \( \mathcal{D} \)-modules

2.1. \( b \)-function and Localization. We recall the construction of \( b \)-functions. Let \( X \) be a smooth algebraic variety over \( \mathbb{C} \) of dimension \( n \) and let \( f \) be a regular function on \( X \). We denote by \( D \) the divisor defined by \( f = 0 \), by \( j : U = X \setminus D \hookrightarrow X \) the open embedding and by \( i : D \hookrightarrow X \) the closed embedding. We assume that \( \mathcal{M}_U \) is a (left) holonomic \( \mathcal{D}_U \)-module so that
\[
\mathcal{M}_U = \mathcal{D}_U \cdot \mathcal{M}_0|_U
\]
for some fixed coherent \( \mathcal{O}_X \)-submodule \( \mathcal{M}_0 \subseteq j_\ast (\mathcal{M}_U) \) throughout this section. We then introduce an independent variable \( s \) and consider the free \( \mathbb{C}[s] \)-module
\[
j_\ast (\mathcal{M}_U[s] \cdot f^s) = j_\ast (\mathcal{M}_U) \cdot f^s \otimes_{\mathbb{C}} \mathbb{C}[s].
\]
The module \( j_\ast (\mathcal{M}_U[s] \cdot f^s) \) has a natural \( \mathcal{D}_X[s] \)-module structure by requiring
\[
v(f^s) = sv(f)f^{s-1},
\]
for any vector field \( v \) on \( X \), where \( \mathcal{D}_X[s] := \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s] \). Notice that the module \( j_\ast (\mathcal{M}_U[s] \cdot f^s) \) is not necessarily coherent over \( \mathcal{D}_X[s] \). We then consider the coherent \( \mathcal{D}_X[s] \)-submodule generated by \( \mathcal{M}_0 \cdot f^{s+k} \)
\[
\mathcal{D}_X[s] \mathcal{M}_0 \cdot f^{s+k} \subseteq j_\ast (\mathcal{M}_U[s] \cdot f^s)
\]
for every \( k \in \mathbb{Z} \). It is obvious that we have inclusions
\[
\mathcal{D}_X[s] \mathcal{M}_0 \cdot f^{s+k_1} \subseteq \mathcal{D}_X[s] \mathcal{M}_0 \cdot f^{s+k_2}
\]
when \( k_1 \geq k_2 \).

Definition 2.1 (\( b \)-function). The \( b \)-function of \( \mathcal{M}_U \) along \( f \), also called the Bernstein-Sato polynomial, is the monic polynomial \( b(s) \in \mathbb{C}[s] \) of the least degree so that
\[
b(s) \text{ annihilates } \frac{\mathcal{D}_X[s] \mathcal{M}_0 \cdot f^s}{\mathcal{D}_X[s] \mathcal{M}_0 \cdot f^{s+1}}.
\]
In particular, if we pick $\mathcal{M}_U = \mathcal{O}_U$ and $\mathcal{M}_0 = \mathcal{O}_X$, then the above definition gives us the usual $b$-function for $f$ (see for instance [Kas77]). From definition, the roots of the $b$-function of $\mathcal{M}_U$ depends on the choice of $\mathcal{M}_0$. However, we will see that an arithmetic set generated by the roots is independent with the choice.

**Remark 2.2.** In the case that $X$ is a complex manifold and $f$ is a holomorphic function on $X$, for an analytic holonomic $\mathcal{D}_X$-module $\mathcal{M}$, one can use $\mathcal{M}(+D)$, the algebraic localization of $\mathcal{M}$ along $D$, to replace $j_*(\mathcal{M}_U)$ and define $b$-functions in the analytic setting in a similar way.

**Theorem 2.3** (Bernstein and Sato). The $b$-functions along $f$ exist for holonomic $\mathcal{D}_U$-modules.

The above theorem for $\mathcal{O}_U$ is due to Bernstein algebraically and Sato analytically. Björk extended it for arbitrary holonomic modules in the analytic setting (see [Bj93, Chapter VI]).

**Definition 2.4** (Localization). Assume that $\mathcal{N}$ is a (left) coherent $\mathcal{D}_X[s]$-module and $q$ is a prime ideal in $\mathbb{C}[s]$. Then we define the localization of $\mathcal{N}$ at $q$ by

$$\mathcal{N}_q = \mathcal{N} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]_q$$

where $\mathbb{C}[s]_q$ is the localization of $\mathbb{C}[s]$ at $q$. In particular, if $q$ is the ideal generated by $0 \in \mathbb{C}[s]$ (i.e. $q$ is the generic point of $\mathbb{C} = \text{Spec} \mathbb{C}[s]$), then $\mathcal{N}_q$ becomes a coherent $\mathcal{D}_X(s)$-module, where $X(s)$ is the variety defined over $\mathbb{C}(s)$ of $X$ after the base change $\mathbb{C} \to \mathbb{C}(s)$, where $\mathbb{C}(s)$ is the fractional field of $\mathbb{C}[s]$.

We write the localization of $\mathcal{D}_X[s]M_0 \cdot f^{*+k}$ and $j_*(\mathcal{M}_U[s] \cdot f^*)$ at $m$ by

$$\mathcal{D}_X[s]_m M_0 \cdot f^{*+k} \text{ and } j_*(\mathcal{M}_U[s]_m \cdot f^*)$$

respectively for a maximal ideal $m \subseteq \mathbb{C}[s]$ and by

$$\mathcal{D}_X(s)M_0 \cdot f^{*+k} \text{ and } j_*(\mathcal{M}_U(s) \cdot f^*)$$

the localization at the generic point.

**Definition 2.5** (Duality). Assume that $\mathcal{N}$ is a (left) coherent $\mathcal{D}_X[s]$-module and $\mathcal{N}_q$ is a coherent $\mathcal{D}_X[s]_q$-module for a prime ideal $q \subseteq \mathbb{C}[s]$. We then define the duality by

$$\mathcal{D}(\mathcal{N}) := \mathcal{R}\text{hom}_{\mathcal{D}_X[s]}(\mathcal{N}; \mathcal{D}_X[s]) \otimes_{\mathcal{D}_X[s]} \omega_X^{-1}[n],$$

and

$$\mathcal{D}(\mathcal{N}_q) := \mathcal{R}\text{hom}_{\mathcal{D}_X[s]_q}(\mathcal{N}_q; \mathcal{D}_X[s]_q) \otimes_{\mathcal{D}_X[s]_q} \omega_X^{-1}[n]$$

where $\omega_X$ is the dualizing sheaf of $X$. The twist by $\omega_X$ is to make the dual of $\mathcal{N}$ (resp. $\mathcal{N}_q$) a complex of left $\mathcal{D}_X[s]$-modules (resp. $\mathcal{D}_X[s]_q$-modules).

In the case that $\mathcal{D}(\mathcal{N})$ (resp. $\mathcal{D}(\mathcal{N}_q)$) has only the zero-th cohomological sheaf non-zero, we also use $\mathcal{D}(\mathcal{N})$ (resp. $\mathcal{D}(\mathcal{N}_q)$) to denote $\mathcal{H}^0(\mathcal{D}(\mathcal{N}))$ (resp. $\mathcal{H}^0(\mathcal{D}(\mathcal{N}_q))$).

Since the variable $s$ is in the center of $\mathcal{D}_X[s]$, one can easily check that duality and localization commute, i.e.

$$(1) \quad \mathcal{D}(\mathcal{N})_q \simeq \mathcal{D}(\mathcal{N}_q).$$

We can evaluate $\mathcal{N}$ at the residue field of a maximal ideal $m \subseteq \mathbb{C}[s]$: $\mathcal{N} \otimes_{\mathbb{C}[s]} \mathbb{C}_m$
where $\mathbb{C}_m \simeq \mathbb{C}$ is the residue field $\mathbb{C}[s]/m$ and the $\otimes^L_{\mathbb{C}[s]}$ denotes the derived tensor functor over $\mathbb{C}[s]$; it gives a complex of coherent $\mathcal{D}_X$-modules. Furthermore, since $\mathcal{D}_X[s]$ is free over $\mathbb{C}[s]$, one can check that evaluation and duality commute, i.e.

\[(2) \quad \mathcal{D}(N) \otimes^L_{\mathbb{C}[s]} \mathbb{C}_m \simeq \mathcal{D}(N \otimes^L_{\mathbb{C}[s]} \mathbb{C}_m),\]

where the second $\mathcal{D}$ denotes the duality functor for complexes of coherent $\mathcal{D}$-modules. Because of the evaluation functor and its commutativity with duality, we also call $\mathcal{D}_X[s]$-modules the relative $\mathcal{D}$-modules over $\mathbb{C}[s]$. See [WZ19 §5] for further discussions of relative $\mathcal{D}$-modules for the multi-variate $s$ and also [BVWZ19 §3] in general.

The following lemma is obvious to check; see also [BYWZ20 Lemma 5.3.1] for a multi-variate version.

**Lemma 2.6.** We have

$\mathcal{D}(\mathcal{M}_U[s] \cdot f^s) \simeq \mathcal{D}(\mathcal{M}_U)[s] \cdot f^{-s} \simeq \mathcal{D}(\mathcal{M}_U)[s] \cdot f^s$

where the last isomorphism is given by substituting $s$ by $-s$ (and hence it is not canonical).

**Lemma 2.7.** The $\mathcal{D}_X[s]$-module $\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ is $n$-Cohen-Macaulay for every $k \in \mathbb{Z}$, i.e. the complex $\mathcal{D}(\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s+k})$ only has one non-zero cohomology sheaf

$\mathcal{H}^0(\mathcal{D}(\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s+k})) \simeq \mathcal{E}xt^n(\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathcal{D}_X[s]) \otimes \omega_X^{-1}$.

**Proof.** By [Mai16 Proposition 14] (taking $p = 1$), we see that $\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ is $n$-pure (see for instance [BVWZ19 §4] for the definition of purity). Moreover, by [Mai16 Résultat 2], $\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ is relative holonomic (see [BYWZ19 Definition 3.2.3]). By [BVWZ19 Theorem 3.2.2], since we have a single $s$, $n$-purity is equivalent to $n$-Cohen-Macaulayness for relative holonomic modules over $\mathbb{C}[s]$. The proof is then done. □

By the above lemma and the isomorphism (1), we immediately have:

**Corollary 2.8.** For every prime ideal $q \subseteq \mathbb{C}[s]$, the $\mathcal{D}_X[s]_q$-module $\mathcal{D}_X[s]_q\mathcal{M}_0 \cdot f^{s+k}$ is $n$-Cohen-Macaulay for every $k \in \mathbb{Z}$, i.e. the complex $\mathcal{D}(\mathcal{D}_X[s]_q\mathcal{M}_0 \cdot f^{s+k})$ only has one non-zero cohomology sheaf

$\mathcal{H}^0(\mathcal{D}(\mathcal{D}_X[s]_q\mathcal{M}_0 \cdot f^{s+k})) \simeq \mathcal{E}xt^n(\mathcal{D}_X[s]_q\mathcal{M}_0 \cdot f^{s+k}, \mathcal{D}_X[s]_q) \otimes \omega_X^{-1}$.

The above corollary is the same as [BG12 Lemma 2(a) and Corollary 3]. But our proof (by using Lemma 2.7) is different from the approach in loc. cit. See also [WZ19 §5] for the multi-variate generalization.

For every $\alpha \in \mathbb{C}$, we denote by $m_\alpha$ the maximal ideal of $\alpha$ in $\mathbb{C}[s]$, that is, the ideal generated by $s - \alpha$, and $\mathbb{C}_\alpha$ its residue field.

**Lemma 2.9.** We have

$\mathcal{D}_X(s)\mathcal{M}_0 \cdot f^{s-k} = j_*(\mathcal{M}_U(s) \cdot f^s)$

for every $k \in \mathbb{Z}$. Moreover, for every $\alpha \in \mathbb{C}$, there exists $k_0 > 0$ so that

$\mathcal{D}_X[s]_{m_\alpha}\mathcal{M}_0 \cdot f^{s-k} = j_*(\mathcal{M}_U[s]_{m_\alpha} \cdot f^s)$

for all $k > k_0$. 

Example. For some $k$, the multiplication by $e$ the monodromy $T$ by assigning $t\partial \cdot f^s$ for every $\alpha \in \mathbb{C}$. Hence, the first statement follows. The second statement can be proved similarly.

We define $j_\alpha$-extensions
$$j_\alpha(M_U(s) \cdot f^s) := \mathbb{D} \circ j_\alpha \circ \mathbb{D}(M(s) \cdot f^s)$$
and
$$j_\alpha(M_U[s]_{m_\alpha} \cdot f^s) := \mathbb{D} \circ j_\alpha \circ \mathbb{D}(M[s]_{m_\alpha} \cdot f^s)$$
for every $\alpha \in \mathbb{C}$. By Lemma 2.10 Corollary 2.8 and Lemma 2.9 (both for $\mathbb{D}(M_U)$), they are both sheaves (instead of complexes).

For every $\alpha \in \mathbb{C}$, the multi-valued function $f^\alpha$ gives a local system on $U$. We then denote by $M_U \cdot f^\alpha$ the holonomic $\mathcal{D}_U$-module twisted by the local system given by $f^\alpha$. It is obvious by construction that $M_U \cdot f^\alpha$ is $\mathbb{Z}$-periodic, that is,
$$\mathcal{M}_U \cdot f^\alpha = \mathcal{M}_U \cdot f^{\alpha + k}$$
for every $k \in \mathbb{Z}$.

We write by $b(s)$ the $b$-function of $\mathcal{M}_U$. Since $b(s + k)$ is invertible in $\mathbb{C}(s)$, we have $\mathcal{D}_X(s)\mathcal{M}_0 \cdot f^s \cdot k = \mathcal{D}_X(s)\mathcal{M}_0 \cdot f^s$ for every $k \in \mathbb{Z}$. Hence, the first statement follows. The second statement can be proved similarly.

Theorem 2.10 (Beilinson and Bernstein). We have:

1. the natural morphism $j_\alpha(M_U(s) \cdot f^s) \to j_\alpha(M_U(s) \cdot f^s)$ is isomorphic and they are both equal to $\mathcal{D}_X(s)\mathcal{M}_0 \cdot f^{\alpha + k}$ for every $k \in \mathbb{Z}$;
2. the natural morphism $j_\alpha(M_U[s]_{m_\alpha} \cdot f^s) \to j_\alpha(M_U[s]_{m_\alpha} \cdot f^s)$ is injective for every $\alpha \in \mathbb{C}$;
3. for every $\alpha \in \mathbb{C}$, there exists $k_0 \in \mathbb{Z}_+$ so that for all $k > k_0$ we have
$$j_\alpha(M_U[s]_{m_\alpha} \cdot f^s) = \mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{-k},$$
$$j_\alpha(M_U[s]_{m_\alpha} \cdot f^s) = \mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{k},$$
$$j_\alpha(M_U \cdot f^\alpha) \cong \mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{-k} \otimes \mathcal{C}[s]_{m_\alpha} \mathcal{C}_\alpha$$

Proof. We write by $t\partial \cdot f^s$ the symbol of the multivalued function "$t^\alpha". Then the multi-valued flat section on $\mathbb{C}^*$, the punctured complex plane, is $e^{-\alpha \log t} \cdot t^\alpha$. Consequently, the monodromy $T$ of the underlying rank 1 local system (around the origin counterclockwise) is the multiplication by $e^{-2\pi \sqrt{-1} \alpha}$, by choosing different branches of log $t$.

By using the Deligne-Goresky-MacPherson extension (or the minimal extension), the following theorem is first proved by Ginsburg in [Gin86, §3.6 and 3.8], as well as in [BG12], which is essentially due to Beilinson and Bernstein. See also [WZ19, Theorem 5.3] for the multi-variate generalization.
and
\[ j_!(\mathcal{M}_U \cdot f^\alpha) \overset{\sim}{\longrightarrow} \mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{s+k} \otimes_{\mathbb{C}[s]} C_\alpha; \]
(4) for every \( \alpha \in \mathbb{C} \), if \( \alpha + k \) is not a root of the \( b \)-function of \( \mathcal{M}_U \) for every \( k \in \mathbb{Z} \), then we have
\[ j_!(\mathcal{M}_U[s]_{m_\alpha} \cdot f^s) = j_!(\mathcal{M}_U[s]_{m_\alpha} \cdot f^s) = \mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{s+k} \]
and
\[ j_!(\mathcal{M}_U \cdot f^\alpha) = j_!(\mathcal{M}_U \cdot f^\alpha) \overset{\sim}{\longrightarrow} \mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{s+k} \otimes_{\mathbb{C}[s]} C_\alpha \]
for every \( k \in \mathbb{Z} \), where \( q.i. \) stands for quasi-isomorphism.

2.2. Nearby and vanishing cycles. We now give constructions of nearby and vanishing cycles. We continue using the notations and setups in §2.1. We assume that \( \mathcal{M} \) is a holonomic \( \mathcal{D}_X \)-module so that \( \mathcal{M}_U \cong \mathcal{M}_U \).

**Definition 2.11.** For every \( \alpha \in \mathbb{C} \), the \( \alpha \)-nearby cycle of \( \mathcal{M} \) is
\[ \Psi_{f,\alpha} \mathcal{M} \cong \Psi_{f,\alpha} \mathcal{M}_U = \frac{j_!(\mathcal{M}_U[s]_{m_\alpha} \cdot f^s)}{j_!(\mathcal{M}_U[s]_{m_\alpha} \cdot f^s)}. \]

The above definition needs Theorem 2.10 (2) to get the quotient. From definition, the \( \alpha \)-nearby cycle of \( \mathcal{M} \) only depends on \( \mathcal{M}_U \).

Recall that \( \mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^s \) has a \( t \)-action given by
\[ t \cdot s = s + 1. \]
By definition, \( t \) acts on \( \Psi_{f,\alpha} \mathcal{M} \) and
\[ t \cdot \Psi_{f,\alpha} \mathcal{M} = \Psi_{f,\alpha+1} \mathcal{M}. \]
We define \( \Lambda \) by the discrete set: \( \mathbb{Z} \)-roots of the \( b \)-function of \( \mathcal{M}_U \). By Eq.(3), \( \Lambda \) is independent of choices of \( \mathcal{M}_0 \). By Theorem 2.10 (4), we see that
\[ \Psi_{f,\alpha} \mathcal{M}_U \neq 0 \text{ if } \alpha \in \Lambda. \]
When \( \alpha = 0 \), \( \Psi_{f,0} \mathcal{M}_U \) coincides with \( \Psi^{\text{all}}(\mathcal{M}) \) in [BGT12, §2.4].

**Proposition 2.12.** For every \( \alpha \in \mathbb{C} \), we have
1. \((s + \alpha)^N \) annihilates \( \Psi_{f,\alpha} \mathcal{M}_U \) for some \( N \gg 0 \).
2. \( \Psi_{f,\alpha} \mathcal{M}_U \) is holonomic \( \mathcal{D}_X \)-module supported on \( D \); moreover, if \( \mathcal{M}_U \) is regular holonomic, then so is \( \Psi_{f,\alpha} \mathcal{M}_U \);
3. \( \mathcal{D}(\Psi_{f,\alpha} \mathcal{M}_U) \cong \Psi_{f,-\alpha} \mathcal{D}(\mathcal{M}_U) \).

**Proof.** This proposition is essentially proved in [BGT97, §4.2]. We give a proof here for completeness.

By Theorem 2.10 (3), we have
\[ \Psi_{f,-\alpha} \mathcal{M}_U = \frac{\mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{s-k}}{\mathcal{D}_X[s]_{m_\alpha} \mathcal{M}_0 \cdot f^{s+k}}. \]
Therefore, \((s - \alpha)^N \) annihilates \( \Psi_{f,-\alpha} \mathcal{M}_U \) for some \( N \gg 0 \) by using the \( b \)-function of \( \mathcal{M}_U \). The first statement is thus proved.

For the second one, it is obvious that \( \Psi_{f,\alpha} \mathcal{M}_U \) is supported on \( D \). We then prove holonomicity. Using Theorem 2.10 (3) once more time, we obtain a short exact sequence
\[ 0 \rightarrow j_!(\mathcal{M}_U \cdot f^\alpha) \rightarrow \frac{j_!(\mathcal{M}_U[s]_{m_\alpha} \cdot f^s)}{(s - \alpha)^2 \cdot j_!(\mathcal{M}_U[s]_{m_\alpha} \cdot f^s)} \rightarrow j_!(\mathcal{M}_U \cdot f^\alpha) \rightarrow 0. \]
Hence, \( \frac{j_*(\mathcal{M}_U[s]M_{a} \cdot f^s)}{(s-\alpha)^2 \cdot j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)} \) is holonomic. By induction, we then have
\[
\frac{j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)}{(s-\alpha)^N \cdot j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)}
\]
is holonomic.

By Part (1), we have
\[
\frac{j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)}{(s-\alpha)^N \cdot j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)} = \frac{j_*(\mathcal{M}_U[s]M_{a} \cdot f^s)}{(s-\alpha)^N \cdot j_*(\mathcal{M}_U[s]M_{a} \cdot f^s)}.
\]
Therefore, \( \Psi \) is holonomic. Regularity can be proved similarly.

The third one follows from Lemma 2.6 and Eq. (1).

By construction, (5) and (6) are dual to each other. We now give an alternative description of \( \alpha \)-nearby cycle.

**Proposition 2.13.** For each \( \alpha \in \mathbb{C} \), \( \Psi_{f,\alpha}\mathcal{M}_U \) is canonically isomorphic to the generalized \( -\alpha \)-eigenspace of \( \mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k}/\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k} \) with respect to the s-action for \( k \gg 0 \).

**Proof.** Using the \( b \)-function, we first know that the s-action on \( \mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k}/\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k} \) admits a minimal polynomial for each \( k \geq 0 \). Hence, we have the generalized \( \alpha \)-eigenspace. Since as a \( \mathbb{C}[s] \)-module, \( \mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k}/\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k} \) is supported at a finite subset of Spec\( \mathbb{C}[s] \) (determined by the \( b \)-function). Therefore, its \( \alpha \)-eigenspace is naturally
\[
\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k}/\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k})_{m_{a}}.
\]
The proof is now done by Theorem 2.10(3).

We write by \( b_f(s) \) the \( b \)-function for \( \mathcal{O}_U \) along \( f \). Since \( \mathbb{D}(\mathcal{O}_U) \simeq \mathcal{O}_U \), we have the following well-known fact as an immediate corollary of Proposition 2.12(2):

**Corollary 2.14.** If \( \alpha \) is a root of \( b_f(s) \), then \( -\alpha + k \) is also a root of \( b_f(s) \) for some \( k \in \mathbb{Z} \).

**Definition 2.15 (Beilinson).** The maximal extension of \( \mathcal{M}_U \) is
\[
\Xi(\mathcal{M}_U) = \frac{j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)}{s \cdot j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)}.
\]

Using Theorem 2.10(3) with \( \alpha = 0 \), we then have the following two short exact sequences
\[
0 \rightarrow j!(\mathcal{M}_U) \xrightarrow{\alpha} \Xi(\mathcal{M}_U) \xrightarrow{\beta} \Psi_{f,0}\mathcal{M}_U \rightarrow 0
\]
and
\[
0 \rightarrow \Psi_{f,0}\mathcal{M}_U \xrightarrow{\beta} \Xi(\mathcal{M}_U) \xrightarrow{\alpha} j_+ (\mathcal{M}_U) \rightarrow 0,
\]
where \( \beta_+ \) is induced by the isomorphism
\[
\Psi_{f,0}\mathcal{M}_U \simeq \frac{s \cdot j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)}{s \cdot j_*(\mathcal{M}_U[s]m_{a} \cdot f^s)}.
\]
By construction, (5) and (6) are dual to each other.

Since \( \mathcal{M}_U \simeq \mathcal{M}_U \), we have natural morphisms
\[
\mathcal{M} \rightarrow j_*(\mathcal{M}_U) \text{ and } j!(\mathcal{M}_U) \rightarrow \mathcal{M}
\]
where the second one is obtained by taking duality of the first one. We then have the following commutative diagram

\[
\begin{array}{ccc}
\Xi(M_U) & \xrightarrow{\alpha^*} & j_*(M_U) \\
\downarrow \alpha^- & & \downarrow \\
j_!(M_U) & \rightarrow & M.
\end{array}
\]

**Definition 2.16 (Beilinson).** The vanishing cycle of $M$ is

\[
\Phi_f M := H^0([j_!(M_U) \rightarrow \Xi(M_U) \oplus M \rightarrow j_*(M_U)])
\]

where the complex is the total complex of the above double complex in degrees $-1$, $0$ and $1$.

Using the two short exact sequences (5) and (6), we have

\[
H^i([j_!(M_U) \rightarrow \Xi(M_U) \oplus M \rightarrow j_*(M_U)]) = 0 \text{ for } i \neq 0.
\]

We then have the morphisms of $D_X$-modules

\[
c : \Psi_{f,0} M \rightarrow \Phi_f M
\]
given by $c(\eta) = (\beta_+(\eta), 0)$, and

\[
v : \Phi_f M \rightarrow \Psi_{f,0} M
\]
given by $v(\xi, m) = \beta_-(\xi)$. Then

\[v \circ c = s \text{ and } c \circ v = (s, 0).
\]

The above construction of $\Xi(M_U)$ and $\Phi_f M$ exactly follows the recipe in [Bei87] and $\Phi_f M$ coincides with $\Phi^{nil}(M)$ in [BG12, §2.4].

The following corollary follows immediately from Proposition 2.12 and (8).

**Corollary 2.17.** We have:

1. $\Xi(M_U)$ and $\Phi_f M$ are both holonomic; moreover, if $M$ is regular holonomic, then so are $\Xi(M_U)$ and $\Phi_f M$;
2. $D(\Xi(M_U)) \simeq \Xi(D(M_U))$;
3. $D(\Phi_f M) \simeq \Phi_f(D(M))$.

### 3. Twisted $D_X[s]$-module by Jordan block

We discuss $D_X[s]$-modules twisted by local systems given by Jordan blocks. We first consider a key example: Local systems of Jordan blocks on $\mathbb{C}^*$.

For $\alpha \in \mathbb{C}$ and $m \geq 1$, we define a free $\mathcal{O}_{\mathbb{C}}[1/t]$-module

\[
K_m^\alpha = \bigoplus_{l=0}^{m-1} \mathcal{O}_{\mathbb{C}}[t^{-1}]e_l^\alpha
\]

with a naturally defined connection $\nabla$ by requiring

\[
\nabla e_l^\alpha = \frac{1}{t}(\alpha e_l^\alpha + e_{l-1}^\alpha),
\]

where $t$ is the coordinate of the complex plane $\mathbb{C}$. The generator $e_1^\alpha$ can be understood as the formal symbol of the multi-valued function $t^\alpha \frac{\log t}{t}$ and we conventionally set $e_{-1}^\alpha = 0$. 


We can identify $t \nabla$ with the action of $J_{\alpha,m}$, where $J_{\alpha,m}$ is the $m \times m$ Jordan block with the eigenvalue $\alpha$. The nilpotent part of $t \nabla$ is then $J_{0,m}$, or more explicitly
\[
(t \nabla)_{\text{nil}}(e_\alpha^t) = e_{\alpha t}^\alpha.
\]
It is then obvious that the multivalued $\nabla$-flat sections of $K_{\alpha,m}^\alpha$ (on $\mathbb{C}^*$) are the $\mathbb{C}$-span of
\[
\{e^{-J_{\alpha,m} \log t} \cdot e_k^\alpha \}_{k=0,\ldots,m-1}.
\]
We set $L_{\lambda,m}^\alpha$ the local system of the multivalued $\nabla$-flat sections of $K_{\alpha,m}^\alpha$ (on $\mathbb{C}^*$), or equivalently
\[
\text{DR}(K_{\alpha,m}^\alpha)|_{\mathbb{C}^*} \overset{\text{q.i.}}{\cong} L_{\lambda,m}^\alpha[1],
\]
where $\lambda = e^{2\pi \sqrt{-1} \alpha}$.

The monodromy action $T$ (around the origin of the complex plane counterclockwise) on $L_{\lambda,m}^\alpha$ is given by $e^{-2\pi \sqrt{-1} J_{\alpha,m}}$. In particular
\[
\log T_u = -2\pi \sqrt{-1} J_{0,m}
\]
where $T_u$ is the unipotent part of $T$ in the Jordan-Chevalley decomposition.

By construction, we have a direct system of $\mathcal{D}$-modules
\[
\cdots \to K_{\alpha,m}^\alpha \to K_{\alpha,m+1}^\alpha \to \cdots.
\]
Applying DR, we then obtain a direct system of local systems
\[
\cdots \to L_{\lambda,m}^\alpha \to L_{\lambda,m+1}^\alpha \to \cdots.
\]

We now define the $\mathcal{D}_X[s]$-module
\[
N_{\alpha,k}^\alpha_m := \bigoplus_{l=0}^{m-1} \mathcal{D}_X[s] \mathcal{M}_0 \cdot f^{s-k} \otimes e_l^\alpha,
\]
by assigning the $s$-action by
\[
s \cdot (\eta e_l^\alpha) = (s + \alpha)\eta e_l^\alpha - \eta e_{l-1}^\alpha
\]
for $\eta$ a section of $\mathcal{D}_X[s] \mathcal{M}_0 \cdot f^{s-k}$. We therefore have a direct symstem of $\mathcal{D}_X[s]$-modules
\[
\cdots \to N_{\alpha,k}^\alpha_m \to N_{\alpha,k}^\alpha_{m+1} \to \cdots.
\]

Let $\iota : X \hookrightarrow Y = X \times \mathbb{C}$ be the graph embedding of $f$, i.e.
\[
\iota(x) = (x, f(x)).
\]

By identifying $s$ with $-\partial_t$, we have a natural injection
\[
N_{\alpha,k}^\alpha_m \hookrightarrow \iota_*(j_*(\mathcal{M}_U)) \otimes_{\mathcal{O}_Y} p_1^* K_{m}^{-\alpha},
\]
where $\iota_+$ denotes the $\mathcal{D}$-module direct image functor and $p_1 : Y \to \mathbb{C}$ the projection (cf. [BMS06 §2.4]). The above injection is indeed a morphism of log $\mathcal{D}$-modules with the log structure along $X \times \{0\}$; see [WZ19 §2] for definitions.

**Lemma 3.1.** Assume that $b(s)$ is the $b$-function of $\mathcal{M}_U$ along $f$. Then
\[
b(s + \alpha)^m \text{ annihilates } N_{\alpha,0}^\alpha_m / N_{\alpha,-1}^\alpha_m.
\]
Proof. By construction, we have a short exact sequence of $\mathcal{D}_X[s]$-modules

$$0 \rightarrow \mathcal{N}^{\alpha,k}_{m-1} \rightarrow \mathcal{N}^{\alpha,k}_m \rightarrow \mathcal{Q} \rightarrow 0.$$ 

We also know that

$$\mathcal{N}^{\alpha,k}_1 \simeq \mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k+\alpha} \simeq \mathcal{Q}.$$ 

By substituting $s + \alpha$ for $s$, we know $b(s - k + \alpha)$ annihilates

$$\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k+\alpha}/\mathcal{D}_X[s]\mathcal{M}_0 \cdot f^{s-k+1+\alpha}.$$ 

Therefore, we obtain the required statement by induction.

\[\square\]

Proposition 3.2. For each $\alpha \in \mathbb{C}$, there exists $k_0 > 0$ so that for all $k \geq k_0$

$$\text{DR}(\mathcal{N}^{\alpha,k}_m \rightarrow \mathcal{N}^{\alpha,k}_m) \overset{q.i.}{\simeq} \iota_*\rho j_* (\text{DR}(\mathcal{M}_U) \otimes f_0^{-1}L_{m}^{1/\lambda})$$

and

$$\text{DR}(\mathcal{N}^{\alpha,-k}_m \rightarrow \mathcal{N}^{\alpha,-k}_m) \overset{q.i.}{\simeq} \iota_*\rho j_* (\text{DR}(\mathcal{M}_U) \otimes f_0^{-1}L_{m}^{1/\lambda})$$

for all $m$, where the complex $[\mathcal{N}^{\alpha,k}_m \rightarrow \mathcal{N}^{\alpha,k}_m]$ is in degrees $-1$ and $0$, and $\lambda = e^{2\pi\sqrt{-1}\alpha}$.

Proof. Under the inclusion (11), we have

$$\mathcal{N}^{\alpha,k}_m|_U \simeq \iota_+ j_+(\mathcal{M}_U) \otimes_{\mathcal{O}_Y} p_1^*K_m^{-\alpha}|_U \simeq \iota_+ j_+(\mathcal{M}_U)|_U \otimes_{\mathbb{C}} p_1^{-1}L_{m}^{1/\lambda}.$$ 

By using Theorem 2.10(3), we have that

$$\mathcal{N}^{\alpha,k}_m \otimes_{\mathbb{C}[s]} \mathbb{C}_0 \simeq (\mathcal{N}^{\alpha,k}_m)_{m_0} \otimes_{\mathbb{C}[s]_{m_0}} \mathbb{C}_0 \simeq j_*(\mathcal{N}^{\alpha,k}_m|_U) \otimes_{\mathbb{C}[s]} \mathbb{C}_0,$$

where $(\mathcal{N}^{\alpha,k}_m)_{m_0}$ is the localization of $\mathcal{N}^{\alpha,k}_m$ at $m_0$, the maximal ideal of $0 \in \mathbb{C}$. Moreover, since we identify $s$ with $-\partial_s t$, we further have

$$j_*(\mathcal{N}^{\alpha,k}_m|_U) \otimes_{\mathbb{C}[s]} \mathbb{C}_0 \simeq [\iota_+ j_+(\mathcal{M}_U) \otimes_{\mathcal{O}_Y} p_1^*K_m^{-\alpha} \overset{\partial_s}{\rightarrow} \iota_+ j_+(\mathcal{M}_U) \otimes_{\mathcal{O}_Y} p_1^*K_m^{-\alpha}].$$

By using the Koszul decompositions of de Rham complexes (see for instance [Wu17 §4.1]), we therefore have

$$\text{DR}(\mathcal{N}^{\alpha,k}_m \rightarrow \mathcal{N}^{\alpha,k}_m) \overset{q.i.}{\simeq} \text{DR}(\iota_+ j_+(\mathcal{M}_U) \otimes_{\mathcal{O}_Y} p_1^*K_m^{-\alpha}),$$

where the second DR is taken over the ambient space $Y$. Since

$$\iota_+(j_+(\mathcal{M}_U)) \otimes_{\mathcal{O}_Y} p_1^*K_m^{-\alpha}$$

is regular holonomic, we also naturally have

$$\text{DR}(\iota_+ j_+(\mathcal{M}_U) \otimes_{\mathcal{O}_Y} p_1^*K_m^{-\alpha}) \overset{q.i.}{\simeq} Rj_{Y*}(\iota_* \rho j_* (\text{DR}(\mathcal{M}_U) \otimes_{\mathbb{C}} p_1^{-1}L_{m}^{1/\lambda})).$$

where $j_Y$ and $\iota^o$ are as in the following diagram

$$\begin{array}{ccc}
U & \overset{\iota^o}{\rightarrow} & U_Y = Y \setminus X \times \{0\} \\
\downarrow j & & \downarrow j_Y \\
X & \overset{\iota}{\rightarrow} & Y;
\end{array}$$

see [Bj93 Chapter V.4]. By projection formula for local systems ([KST13 Proposition 2.5.11]), we further have

$$Rj_{Y*}(\iota_* \rho j_* (\text{DR}(\mathcal{M}_U) \otimes_{\mathbb{C}} p_1^{-1}L_{m}^{1/\lambda})) \overset{q.i.}{\simeq} \iota_* (Rj_* (\text{DR}(\mathcal{M}_U) \otimes_{\mathbb{C}} f_0^{-1}L_{m}^{1/\lambda})).$$
and the second quasi-isomorphism is obtained. The second quasi-isomorphism can be obtained similarly. The choice of \(k_0\) only depends on \(\alpha\) and the roots of the \(b\)-function annihilating \(N_0^\alpha / N_0^{\alpha-1}\). We therefore can choose a uniform \(k_0\) working for all \(m\) by Lemma 3.1. \(\Box\)

4. NEARBY CYCLES FOR PERVERSE SHEAVES VIA JORDAN BLOCKS

In this section, we define nearby and vanishing cycles via local systems given by Jordan blocks on \(\mathbb{C}^*\), the punctured complex plane. We keep the notations introduced in §2. Assume that \(K\) is a \(\mathbb{C}\)-perverse sheaf on \(X\).

**Definition 4.1.** For \(\lambda \in \mathbb{C}^*\), the \(\lambda\)-nearby cycle of \(K\) is
\[
\psi_{f,\lambda}(K) := \lim_{m \to -\infty} i^{-1} R j_*(j^{-1} K \otimes f^{-1}_0 L_1^{1/\lambda}).
\]

The vanishing cycle is
\[
\phi_f K := \text{Cone}(i^{-1} K \to \psi_{f,1}(K))
\]
where the morphism \(i^{-1} K \to \psi_{f,1}(K)\) is induced by the natural map \(K \to R j_*(j^{-1} K)\).

The monodromy action \(T\) of \(L_1^{1/\lambda}\) induces the monodromy action on \(\psi_{f,\lambda}(K)\) for each \(\lambda\), denoted also by \(T\). We then have the induced monodromy action \(T\) on \(\phi_f K\), by requiring \(T\) acting on \(i^{-1} K\) identically.

By construction, we have the tautological triangle
\[
i^{-1} K \to \psi_{f,1}(K) \to \phi_f K \to \psi_{f,1}(K)
\]
with the induced canonical map
\[
can : \psi_{f,1}(K) \to \phi_f K.
\]
We define
\[
\text{Var} : \phi_f K \to \psi_{f,1}(K)
\]
by
\[
\text{Var} := (0, -\log T_u / 2\pi \sqrt{-1}) = (0, J_{0,\infty})
\]
where \(J_{0,\infty} := \lim_{m \to -\infty} J_{0,m}\).

**Remark 4.2.** Using these definitions, one can prove the perversity of \(\psi_{f,\lambda} K\) and \(\phi_f K\) (with a shift of cohomological degrees) directly. Let us refer to [Rei10] for the proof of this point and other related results.

5. THE PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. Before we start, the following preliminary result about infinite Jordan blocks is needed.

**Lemma 5.1.** [Bj93, 6.4.5 Lemma] Let \(W\) be a \(\mathbb{C}\)-vector space, and let \(\varphi\) be a \(\mathbb{C}\)-linear operator on \(W\) admitting a minimal polynomial. Set \(W_\infty = \bigoplus_{k=0}^\infty W \otimes e_k\) and define
\[
\varphi_\infty(w \otimes e_k) = (\varphi - \alpha)w \otimes e_k - w \otimes e_{k-1}
\]
for \(w \in W\) (assume \(e_{-1} = 0\)). Then \(\varphi_\infty\) is surjective and \(\ker(\varphi_\infty) \cong W_\alpha\), where \(W_\alpha\) is the generalized \(\alpha\)-eigenspace.
Proof. Define a map $W_\alpha \to \ker(\varphi_\infty)$ by
$$w \mapsto \sum_{i \geq 0} (\varphi - \alpha)^i w \otimes e_i.$$ Clearly, this map is an isomorphism.

We then prove surjectivity. If $w \in W_\alpha$, then
$$\varphi_\infty(- \sum_{i \geq j} (\varphi - \alpha)^{i-j-1} w \otimes e_i) = w \otimes e_j.$$ If $w \in W_\alpha^\perp$, then then
$$\varphi_\infty(\sum_{i=1}^j (\varphi - \alpha)^{-i} w \otimes e_{j-i+1}) = w \otimes e_j.$$ Therefore, the surjectivity follows. □

Using the above lemma and Proposition 2.13, we immediately have:

**Corollary 5.2.** For a holonomic $D_U$-module $M_U$ and some $\alpha \in \mathbb{C}$, there exists $k \gg 0$ so that
$$\Psi_{f,\alpha} M_U[1] \cong \lim_{\to}^{-} \frac{N_{\alpha,k}^-}{N_{\alpha,k}^+} \quad \text{by Corollary 5.2 and Proposition 3.2.}$$

**Proof of Theorem 1.1.** We assume $M$ a regular holonomic $D_X$-module and write $M_{|U} = M_{|U}$. We first prove the comparison of nearby cycles.

Since $\text{DR}$ and the direct limit functor commute (since $\text{DR}$ is identified with $\omega_X \otimes L D^\bullet$, one can apply [Wei95, Corollary 2.6.17]), we have
$$\text{DR}(\Psi_{f,\alpha} M_U[1]) \cong \lim_{\to}^{-} \text{DR}(\psi_{f,\lambda} \text{DR}(M_U))$$
by Corollary 5.2 and Proposition 3.2.

One can check that the operator $J_{0,\infty}$ on $\lim_{\to}^{-} \Psi_{f,\omega} N_{m}^\alpha$ corresponds to the action $(s + \alpha)$ on $\Psi_{f,\alpha} M_U$ under the quasi-isomorphism in Corollary 5.2 by Lemma 5.1.

Taking $\text{DR}$, the operator $J_{0,\infty}$ becomes $- \frac{\log T_u}{2\pi\sqrt{-1}}$ on $\psi_{f,\lambda} \text{DR}(M_U)$ by the construction of the monodromy operator $T$. Or equivalently, the monodromy operator $T$ corresponds to
$$T \cong \text{DR}(e^{-2\pi\sqrt{-1} s}) = \text{DR}(\lambda \cdot e^{-2\pi\sqrt{-1}(s + \alpha)}).$$
We thus have proved the comparison for nearby cycles in Theorem 1.1.

We now prove the comparison for vanishing cycles. For simplicity, we write by $A^\bullet$ the complex
$$[\Xi(M_U) \to j_*(M_U)]$$
in degrees 0 and 1, by $B^\bullet$ the complex
$$[j_!(M_U) \to \Xi(M_U) \oplus M \to j_*(M_U)]$$
by $C^\bullet$ the complex
$$[j_!(M_U) \to M]$$
in degrees -1 and 0. Then we have a triangle
$$C^\bullet \to A^\bullet[1] \to B^\bullet[1] \xrightarrow{+1}.$$
Considering the two short exact sequences (5) and (6), we see that
\[ \Phi_f M \overset{\text{q.i.}}{\simeq} B^\bullet \]
and
\[ [\Psi_f, 0 M_{\cdot} \to \Phi_f M] \overset{\text{q.i.}}{\simeq} [A^\bullet \to B^\bullet]. \]
We hence have
\[ \Phi_f M[1] \overset{\text{q.i.}}{\simeq} \text{Cone}(C^\bullet \to [\Psi_f, 0 M_{\cdot} \to \Phi_f M[1])]. \]
Since
\[ \text{DR}(C^\bullet) \overset{\text{q.i.}}{\simeq} i_* i^{-1} \text{DR}(M), \]
we have
\[ \text{DR}(\Psi_f, 0 M_{\cdot} \to \Phi_f M[1]) \overset{\text{q.i.}}{\simeq} i_* (\psi, 0 \text{DR}(M) \overset{\text{can}}{\longrightarrow} \phi_f \text{DR}(M)). \]
Using the short exact sequence (6), we see that the \( \Psi_f, 0 M_{\cdot} \) in \[ \text{Cone}(C^\bullet \to [\Psi_f, 0 M_{\cdot} \to \Phi_f M[1]) \]
has a \( s \)-twist. Therefore, under the quasi-isomorphism (12)
\[ v : \text{Cone}(C^\bullet \to [\Psi_f, 0 M_{\cdot} \to \Phi_f M[1]) \to \Psi_f, 0 M_{\cdot} \]
is given by \((0, s)\). But the \( s \) action on \( \Psi_f, 0 M_{\cdot} \) is \( J_{0, \infty} \) after taking DR. Therefore, we obtain
\[ \text{DR}(\Phi_f M[1] \overset{\text{q.i.}}{\longrightarrow} \Psi_f, 0 M_{\cdot} \to \Phi_f M[1]) \simeq i_* (\phi_f \text{DR}(M) \overset{\text{Var}}{\longrightarrow} \psi, 0 \text{DR}(M)). \]

\[ \square \]

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