Hopf bifurcation of a computer virus propagation model with two delays and infectivity in latent period

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ABSTRACT
In this paper, a computer virus propagation model with two delays and infectivity in latent period is investigated. First, the conditions which guarantee the local stability of the positive equilibrium and the existence of the Hopf bifurcation are derived by choosing the different combination of the two delays as the bifurcation parameter. Moreover, some specific properties for determining the stability and direction of the Hopf bifurcation are obtained by employing the normal form theory and the centre manifold theorem. Finally, a numerical simulation is carried out to verify the correctness of our obtained theoretical analysis.

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1. Introduction
As the world becomes more electronically connected, computers connected to a network become more and more vulnerable to digital threats. Specially, computer viruses are one of the most digital threats in network which have posted a serious challenge for information security (Mishra and Pandey, 2012). To portray the important feature of computer viruses and control the propagation of the viruses, it is necessary and urgent to investigate the action of computer viruses throughout a network and to explore some effective defence measures.

Recently, some dynamical models for computer virus propagation have been proposed considering the high similarity between computer viruses and biological viruses. Ren, Yang, Zhu, Yang, and Zhang (2012), Piqueira and Araujo (2009) and Mishra and Pandey (2010) studied the stability of the SIR (susceptible-infectious-removed) computer virus model. However, in the real network, on adequate contact with an infectious computer, a susceptible computer may become exposed, that is, infected but not yet infectious. Based on this consideration, stability of the SEIR (susceptible-exposed-infectious-removed) computer virus model has been studied in Mishra and Pandey (2011), Peng, He, Huang, and Dong (2013) and Yuan and Chen (2008). Obviously, Mishra and Pandey (2011), Peng et al. (2013) and Yuan and Chen (2008) assumed that the recovered computers have a permanent immunization period and can no longer be infected. This is not consistent with the reality in the real network. Thus, SEIRS (susceptible-exposed-infectious-removed-susceptible) computer virus model is investigated in Guillen, Rey, and Encinas (2017) and Hosseini, Azgomi, and Rahmani (2016). Afterwards, considering the effect of quarantine strategy, SEIQRS (susceptible-exposed-infectious-quarantined-removed-susceptible) computer virus model has been proposed in Mishra and Jha (2010). There are also some other computer virus models have been prosed and studied, one can refer to Mishra and Keshri (2013), Gan, Yang, Zhu, Jin, and He (2013), Upadhyay, Kumari, and Misra (2017) and Yang (2015). Specially, Zhang and Yang (2013) studied an SEIRS computer virus model with two delays and analysed Hopf bifurcation of the model. However, they assume that the computers in latent period have no infectivity. This is not consistent with reality, since an infected computer which is in latency can also infect other computers through file copying or file downloading (Yang, Yang, Zhu, and Wen, 2013).

Based on the discussion above, and considering that it needs a period to clean the computer viruses in the latent period and the breaking computers for antivirus software, Zhang and Bi (2016) proposed an SLBS(susceptible-latent-breaking-susceptible) computer virus propagation model with delay and infectivity in latent period. They investigated the effect of the time delay on the proposed model. But they only consider the effect of the time delay due to the period that the antivirus software uses to clean the computer viruses in the latent computers and the
breaking computers. As we know, when the susceptible computers are infected by the computer viruses, there is usually a time delay before they develop into the breaking ones (Zhang and Bi, 2016). Namely, they neglect the latent characteristic of the computer viruses (Yang et al., 2013). Inspired by this idea, we incorporate the latent delay into system (1) and consider the computer virus propagation model with two delays and infectivity in latent period in the present paper. The main contributions of our research work are highlighted as follows: (1) A more general SLBS computer virus model is proposed; (2) The effects of the two delays on the stability of the model are investigated, including local stability and the existence of the Hopf bifurcation; (3) Properties of the Hopf bifurcation are discussed by regarding different combinations of the two delays as the bifurcating parameters; (4) Some policies have been suggested to control the occurrence of Hopf bifurcation so that the diffusion of computer viruses can be predicted and controlled.

The organization of this paper is as follows. Section 2 formulates the SLBS computer virus propagation model with two delays and infectivity in latent period. The local stability of the positive equilibrium and the existence of the Hopf bifurcation are discussed by regarding different combinations of the two delays as the bifurcating parameters in Section 3, and specific formulas for determining the direction and stability of the Hopf bifurcation are also derived. A numerical simulation is presented in order to validate the obtained theoretical results in Section 4. Concluding remarks are drawn in Section 5.

2. Model formulation

The proposed SLBS computer virus propagation model with delay and infectivity in latent period by Zhang and Bi (2016) is as follows

\[
\begin{align*}
\frac{dS(t)}{dt} &= \delta - \beta S(t)(L(t) + B(t)) + \gamma_1 L(t) - \tau_1 + \gamma_2 B(t) - \delta S(t), \\
\frac{dL(t)}{dt} &= \beta S(t)(L(t) + B(t)) - \gamma_1 L(t) - \alpha L(t) - \delta L(t), \\
\frac{dB(t)}{dt} &= \alpha L(t) - \gamma_2 B(t) - \delta B(t),
\end{align*}
\]

where \( S(t) \), \( L(t) \) and \( B(t) \) are the percentages of the susceptible computers, the latent computers and the breaking computers in the network at time \( t \), respectively. \( \delta \) is the rate at which the external computers connect to the network and it is also the rate at which the internal computers disconnect from the network. \( \beta \) is the infected rate of the susceptible computers due to the contact with the latent and the breaking-out computers; \( \alpha, \gamma_1 \) and \( \gamma_2 \) are the transition state rates of system (1); \( \tau \) is the time delay due to the period that the antivirus software uses to clean the computer viruses in the latent computers and the breaking computers.

It should be pointed out that system (1) assume that a computer is internal or external depending on whether it is currently connected to the Internet or not and only internal computers are concerned. Also, it assume that the total number of computers in the world is constant, considering that the total number of computers in the world would tend to saturation.

Considering that there is usually a time delay before they develop them into the breaking-out ones when the susceptible computers are infected by the computer viruses, we introduce the latent delay into system (1) and get the following computer virus propagation model with two delays:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \delta - \beta S(t)(L(t) + B(t)) + \gamma_1 L(t - \tau_1) + \gamma_2 B(t - \tau_1) - \delta S(t), \\
\frac{dL(t)}{dt} &= \beta S(t)(L(t) + B(t)) - \gamma_1 L(t) - \alpha L(t - \tau_2) - \delta L(t), \\
\frac{dB(t)}{dt} &= \alpha L(t - \tau_2) - \gamma_2 B(t) - \delta B(t),
\end{align*}
\]

where \( \tau_1 \) is the time delay due to the period that the antivirus software uses to clean the computer viruses in the latent computers and the breaking computers and \( \tau_2 \) is the time delay due to the latent period of the latent computers before they can infect other computers. The dynamical transfer is depicted in Figure 1.

To analyse dynamical behaviours of system (2) and get our main results, we need some assumptions which can guarantee the existence of Hopf bifurcation of system (2). They are listed in the following for clarity.

**Assumption 2.1:** \( A_{00} > 0, A_{02} > 0, A_{01}A_{02} > A_{00}, \) where

\[
\begin{align*}
A_{00} &= A_0 + B_0 + C_0 + D_0, & A_{01} &= A_1 + B_1 + C_1, \\
A_{02} &= A_2 + C_2.
\end{align*}
\]

![Figure 1. Schematic diagram for the flow of viruses in the network.](image-url)
and

\[ A_0 = a_{33}(a_{12}a_{21} - a_{11}a_{22}), \]
\[ A_1 = a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_{12}a_{21}, \]
\[ A_2 = -(a_{11} + a_{22} + a_{33}), \quad B_0 = a_{21}a_{33}b_{12}, \]
\[ B_1 = -a_{21}b_{12}, \]
\[ C_0 = a_{11}(a_{23}c_{32} - a_{33}c_{22}) - a_{21}a_{13}c_{32}, \]
\[ C_1 = c_{22}(a_{11} + a_{33}) - a_{23}c_{32}, \quad C_2 = -c_{22}, \]
\[ D_0 = -a_{21}b_{13}c_{32}, \]

with

\[ a_{11} = -(\beta L_s + \beta B_s + \delta), \quad a_{12} = -\beta S_s, \quad a_{13} = -\beta S_s, \]
\[ a_{21} = \beta (L_s + B_s), \quad a_{22} = \beta S_s - \gamma_1 - \delta, \quad a_{23} = \beta S_s, \]
\[ a_{33} = -(\gamma_2 + \delta), \quad b_{12} = \gamma_1, \quad b_{13} = \gamma_2, \quad c_{22} = -\alpha, \]
\[ c_{32} = \alpha, \]

and the meanings and definitions of \( L_s, B_s, S_s \) can be seen in Section 3.

**Assumption 2.2:** Equation (3) has at least one positive root \( \omega_{10} \).

\[ \omega_1^6 + g_{22}\omega_1^4 + g_{21}\omega_1^2 + g_{20} = 0, \tag{3} \]

where

\[ g_{20} = A_{10}^2 - B_{10}^2, \quad g_{21} = A_{11}^2 - B_{11}^2 - 2A_{10}A_{12}, \]
\[ g_{22} = A_{12}^2 - 2A_{11}, \]

and

\[ A_{10} = A_0 + C_0, \quad A_{11} = A_1 + C_1, \quad A_{12} = A_2 + C_2, \quad B_{10} = B_0 + D_0, \quad B_{11} = B_1. \]

**Assumption 2.3:** \( f_1(v_1^*) \neq 0, \) where

\[ f_1(v_1) = v_1^3 + g_{22}v_1^2 + g_{21}v_1 + g_{20}, \quad v_1^* = \omega_{10}^2. \]

**Assumption 2.4:** Equation (4) has at least one positive root \( \omega_{20} \).

\[ \omega_2^6 + g_{32}\omega_2^4 + g_{31}\omega_2^2 + g_{30} = 0, \tag{4} \]

where

\[ g_{30} = A_{20}^2 - C_{20}^2, \quad g_{31} = A_{21}^2 - 2A_{20}A_{22} - C_{21}^2 + 2C_{20}C_{22}, \]
\[ g_{32} = A_{22}^2 - 2A_{21} - C_{22}^2, \]

and

\[ A_{20} = A_0 + B_0, \quad A_{21} = A_1 + B_1, \quad A_{22} = A_2, \]
\[ C_{20} = C_0 + D_0, \quad C_{21} = C_1, \quad C_{22} = C_2. \]

**Assumption 2.5:** \( f_2(v_2^*) \neq 0, \) where

\[ f_2(v_2) = v_2^3 + g_{32}v_2^2 + g_{31}v_2 + g_{30}, \quad v_2^* = \omega_{20}^2. \]

**Assumption 2.6:** Equation (5) has at least one positive root \( \omega_0 \).

\[ g_{31}^2(\omega) + g_{32}^2(\omega) - g_{30}^2(\omega) = 1, \tag{5} \]

where

\[ g_{30}(\omega) = \omega^6 + (A_{32}^2 - 2A_{31})\omega^4 \]
\[ + (A_{31}^2 - 2A_{32}(A_{30} + D_{30}))\omega^2 + (A_{30} + D_{30})^2, \]
\[ g_{31}(\omega) = (B_{31} - A_{32}B_{32})\omega^4 + (B_{32}(A_{30} - D_{30}) - A_{31}B_{31} + A_{32}B_{30})\omega^2 - B_{30}(A_{30} - D_{30}), \]
\[ g_{32}(\omega) = B_{32}\omega^6 + (A_{32}B_{31} - A_{31}B_{32} - B_{30})\omega^4 \]
\[ + (A_{31}B_{30} - B_{31}(A_{30} + D_{30}))\omega^2, \]

and

\[ A_{30} = A_0, \quad A_{31} = A_1, \quad A_{32} = A_2, \quad B_{30} = B_0 + C_0, \]
\[ B_{31} = B_1 + C_1, \quad B_{32} = C_2, \quad D_{30} = D_0. \]

**Assumption 2.7:** \( P_{4R}Q_{4R} + P_{4L}Q_{4L} \neq 0, \) where

\[ P_{4R} = (A_{31} - 3\omega_0^2)\cos \tau_0\omega_0 - 2A_{32}\omega_0 \sin \tau_0\omega_0 + B_{31}, \]
\[ P_{4L} = (A_{31} - 3\omega_0^2)\sin \tau_0\omega_0 + 2A_{32}\omega_0 \cos \tau_0\omega_0, \]
\[ Q_{4R} = (A_{31}\omega_0^2 - \omega_0^4)\cos \tau_0\omega_0 \]
\[ - (A_{32}\omega_0^3 - A_{30}\omega_0 + D_{30}\omega_0) \sin \tau_0\omega_0, \]
\[ Q_{4L} = (A_{31}\omega_0^2 - \omega_0^4) \sin \tau_0\omega_0 \]
\[ + (A_{32}\omega_0^3 - A_{30}\omega_0 + D_{30}\omega_0) \cos \tau_0\omega_0. \]

**Assumption 2.8:** Equation (6) has at least one positive root \( \omega_{20} \).

\[ f_{50}(\omega_2^*) + f_{51}(\omega_2^*) \sin \tau_1\omega_2 + f_{52}(\omega_2^*) \cos \tau_1\omega_2 = 0, \tag{6} \]

where

\[ f_{50}(\omega_2^*) = (\omega_2^*)^6 + (A_{2}^2 - C_2^2 - 2A_1)(\omega_2^*)^4 + (A_1^2 - 2A_0A_2) \]
\[ + B_1^2 - C_1^2 + 2C_0C_2)(\omega_2^*)^2 + A_0^2 - C_0^2, \]
\[ f_{51}(\omega_2^*) = -2B_1(\omega_2^*)^4 + 2(A_1B_1 - A_2B_0 + C_2D_0)(\omega_2^*)^2 \]
\[ + 2(A_0B_0 - C_0D_0), \]
\[ f_{52}(\omega_2^*) = 2(B_0 - A_2B_1)(\omega_2^*)^3 + (A_0B_1 - A_1B_0 + C_1D_0)\omega_2. \]

**Assumption 2.9:** \( P_{SR}Q_{SR} + P_{SI}Q_{SI} \neq 0, \) where

\[ P_{SR} = (2C_2\omega_{20}^* + \tau_1\omega_{20}^* \sin \tau_1\omega_{20}) \sin \tau_2\omega_{20}^* \]
\[ + (C_1 - \tau_1D_0 \cos \tau_1\omega_{20}) \sin \tau_2\omega_{20}^* \]
\[ + (C_1 - \tau_1D_0 \cos \tau_1\omega_{20}) \sin \tau_2\omega_{20}^*. \]
For Theorem 3.1:

\[ P_{Sl} = (2C_2\omega_2^2 + \tau_1 D_0 \sin \tau_1 \omega_2) \cos \tau_2 \omega_2 - (C_1 - \tau_1 D_0 \cos \tau_1 \omega_2) \cos \tau_2 \omega_2 - \tau_1 B_1 \omega_2 \cos \tau_1 \omega_2 - (B_1 - \tau_1 B_0) \times \sin \tau_1 \omega_2 + 2A_2 \omega_2. \]

Remark 3.1: The unique positive equilibrium is

\[ \gamma^* = \frac{5 \lambda}{5 \tau + \frac{5}{\tau}} \]

system (2) undergoes a Hopf bifurcation at \( E_+ (S_*, L_*, B_*) \) when \( \tau_1 = \tau_{10} \) and a family of periodic solutions bifurcate from \( E_+ (S_*, L_*, B_*) \) and

\[ \tau_{10} = \frac{1}{\omega_{10}} \times \arccos \left( \frac{B_{11} \omega_{10}^4 + (A_{12} B_{10} - A_{11} B_{11})}{B_{11} \omega_{10}^2 + B_{10}^2} \frac{A_{11} \omega_{10}^2 + (A_{12} B_{10} - A_{11} B_{11})}{B_{11} \omega_{10}^2 - A_{10} B_{10}} \right). \]

Proof: When \( \tau_1 > 0 \) and \( \tau_2 = 0 \), Equation (8) reduces to

\[ \lambda^3 + A_{12} \lambda^2 + A_{11} \lambda + A_{10} + (B_{11} \lambda + B_{10}) e^{-\lambda \tau_1} = 0, \quad (9) \]

We assume that \( \lambda = i \omega_1 (\omega_1 > 0) \) is a root of Equation (9). Then,

\[ B_{11} \omega_1 \sin \tau_1 \omega_1 + B_{10} \cos \tau_1 \omega_1 = A_{12} \omega_1^2 - A_{10}, \]

Thus, we can obtain the expression of Equation (9). If all the values of the parameters in system (2) are given, then we can obtain all the roots of Equation (3) by means of Matlab software package. Thus, under Assumption 2.2, we know that Equation (9) has a pair of purely imaginary root \( \pm i \omega_1 \). For \( \omega_{10} \),

\[ \tau_{10} = \frac{1}{\omega_{10}} \times \arccos \left( \frac{B_{11} \omega_{10}^4 + (A_{12} B_{10} - A_{11} B_{11})}{B_{11} \omega_{10}^2 + B_{10}^2} \frac{A_{11} \omega_{10}^2 + (A_{12} B_{10} - A_{11} B_{11})}{B_{11} \omega_{10}^2 - A_{10} B_{10}} \right). \]

Differiating Equation (9) with respect to \( \tau_1 \), we have

\[ \left[ \frac{d \omega_{10}}{d \tau_1} \right]^{-1} = - \frac{3 \lambda^2 + 2A_{12} \lambda + A_{11}}{\lambda (\lambda^3 + A_{12} \lambda^2 + A_{11} \lambda + A_{10})} + \frac{B_{11}}{\lambda (B_{11} \lambda + B_{10})} \frac{\tau_1}{\lambda}. \]

Thus,

\[ \text{Re} \left[ \frac{d \omega_{10}}{d \tau_1} \right]_{\tau_1 = \tau_{10}} = \frac{f_{1i}(v_{1i})}{B_{11} \omega_{10}^2 + B_{10}^2}, \]

Therefore, we know that \( \text{Re} \left[ \frac{d \lambda}{d \tau_1} \right]_{\tau_1 = \tau_{10}} \neq 0 \) under Assumption 2.3. Based on the above discussion and according to the Hopf bifurcation theorem in Hassard, Kazarinoff, and Wan (1981), we have Theorem 3.1 and the proof is completed.

Theorem 3.1: For \( \tau_1 > 0, \tau_2 = 0 \), under Assumption 2.1–2.3,

(i) \( E_+ (S_*, L_*, B_*) \) is locally asymptotically stable for \( \tau_1 \in [0, \tau_{10}) \);
(ii) system (2) undergoes a Hopf bifurcation at $E_+(S_u, L_u, B_u)$ when $\tau_2 = \tau_0$ and a family of periodic solutions bifurcate from $E_+(S_u, L_u, B_u)$; and 

$$\tau_2 = \frac{1}{\omega_2} \times \arccos \left\{ \frac{(C_21 - A_22C_22)\omega_2^2 + (A_22C_20 + A_20C_22 - A_21C_21)\omega_2^2 - A_20C_20}{C_2^4\omega_2^2 + (C_21 - 2C_22C_22)\omega_2^2 + C_2^2} \right\}. $$

**Proof:** When $\tau_1 = 0$ and $\tau_2 > 0$, Equation (8) becomes

$$\lambda^3 + A_22\lambda^2 + A_21\lambda + A_20 + (C_22\lambda^2 + C_21\lambda + C_20) e^{-\lambda\tau_2} = 0.$$  \hspace{1cm} (10)

Let $\lambda = i\omega_2 (\omega_2 > 0)$ be a root of Equation (10). Then

$$C_21\omega_2 \sin \tau_2\omega_2 + (C_22 - C_22\omega_2^2) \cos \tau_2\omega_2 = A_22\omega_2^2 - A_20,$$

$$C_21\omega_2 \cos \tau_2\omega_2 - (C_22 - C_22\omega_2^2) \sin \tau_2\omega_2 = \omega_2^3 - A_21\omega_2,$$

from which one can obtain Equation (4). Under Assumption 2.4, we know that Equation (10) has a pair of purely imaginary roots $\pm i\omega_2$. For $\omega_2$,

$$\tau_2 = \frac{1}{\omega_2} \times \arccos \left\{ \frac{(C_21 - A_22C_22)\omega_2^2 + (A_22C_20 + A_20C_22 - A_21C_21)\omega_2^2 - A_20C_20}{C_2^4\omega_2^2 + (C_21 - 2C_22C_22)\omega_2^2 + C_2^2} \right\}. $$

In addition,

$$\left[ \frac{d\lambda}{d\tau_1} \right]^{-1} = - \frac{3\lambda^2 + 2A_22\lambda + A_21}{\lambda(\lambda^3 + A_22\lambda^2 + A_21\lambda + A_20)} + \frac{2C_22\lambda + C_21}{\lambda(C_22\lambda^2 + C_21\lambda + C_20) - \frac{\tau_2}{\lambda}}.$$ Further,

$$\text{Re} \left[ \frac{d\lambda}{d\tau_2} \right]^{-1} = \frac{f_2'(\upsilon_2^2)}{C_2^4\omega_2^2 + (C_21 - 2C_22C_22)\omega_2^2 + C_2^2}.$$ Therefore, we know that $\text{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2 = \tau_0} \neq 0$ under Assumption 2.5. Based on the above discussion and according to the Hopf bifurcation theorem in Hassard et al. (1981), we have Theorem 3.2 and the proof is completed.

**Theorem 3.3:** For $\tau_1 = \tau_2 = \tau > 0$, under Assumption 2.1, 2.6 and 2.7,

(i) $E_+(S_u, L_u, B_u)$ is locally asymptotically stable for $\tau \in [0, \tau_0]$;

(ii) system (2) undergoes a Hopf bifurcation at $E_+(S_u, L_u, B_u)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from $E_+(S_u, L_u, B_u)$, and

$$\tau_0 = \frac{1}{\omega_0} \times \arccos \left\{ \frac{g_31(\omega_0)}{g_30(\omega_0)} \right\}. $$

**Proof:** For $\tau_1 = \tau_2 = \tau > 0$, we have

$$\lambda^3 + A_32\lambda^2 + A_31\lambda + A_30 + (B_32\lambda^2 + B_31\lambda + B_30) e^{-\lambda\tau} + D_30 e^{-2\lambda\tau} = 0. $$  \hspace{1cm} (11)

Multiplying by $e^{\lambda\tau}$, Equation (11) becomes

$$B_32\lambda^2 + B_31\lambda + B_30 + (\lambda^3 + A_32\lambda^2 + A_31\lambda + A_30) e^{\lambda\tau} + D_30 e^{-\lambda\tau} = 0. $$ \hspace{1cm} (12)

Let $\lambda = i\omega (\omega > 0)$ be a root of Equation (12), then

$$(A_30 + D_30 - A_32\omega^2) \cos \tau\omega - (\omega^3 - A_31\omega) \sin \tau\omega = B_32\omega^2 - B_30,$$

$$(A_30 - D_30 - A_32\omega^2) \sin \tau\omega - (\omega^3 - A_31\omega) \cos \tau\omega = -B_31\omega,$$

from which one can obtain

$$\cos \tau\omega = \frac{g_31(\omega)}{g_30(\omega)}, \quad \sin \tau\omega = \frac{g_32(\omega)}{g_30(\omega)}.$$ Thus, we can get Equation (5). Under Assumption 2.6, we know that Equation (12) has a pair of purely imaginary roots $\pm i\omega_0$. For $\omega_0$, we have

$$\tau_0 = \frac{1}{\omega_0} \times \arccos \left\{ \frac{g_31(\omega_0)}{g_30(\omega_0)} \right\}. $$

Taking the derivative of $\lambda$ with respect to $\tau$, we obtain

$$\left[ \frac{d\lambda}{d\tau} \right]^{-1} = \frac{2B_32\lambda + B_31 + (3\lambda^2 + 2A_32\lambda + A_31) e^{\lambda\tau}}{D_30\lambda e^{-\lambda\tau} - (\lambda^4 + A_32\lambda^3 + A_31\lambda^2 + A_30\lambda) e^{\lambda\tau}} - \frac{\tau}{\lambda}. $$

Then, we can get that

$$\text{Re} \left[ \frac{d\lambda}{d\tau} \right]^{-1} = \frac{P_{4R}Q_{4R} + P_{4I}Q_{4I}}{Q_{4R}^2 + Q_{4I}^2}.$$ Therefore, under Assumption 2.7, $\text{Re}[d\lambda/d\tau]^{-1}_{\tau = \tau_0} \neq 0$. Therefore, based on the above discussion and the Hopf bifurcation theorem in Hassard et al. (1981), we obtain Theorem 3.3 and the proof is completed.
Theorem 3.4: For \( \tau_1 \in (0, \tau_{10}) \), \( \tau_2 > 0 \), under Assumption 2.8 and 2.9,

(i) \( E_+(S_+, L_+, B_+) \) is locally asymptotically stable for \( \tau_2 \in [0, \tau_{20}] \);
(ii) system (2) undergoes a Hopf bifurcation at \( E_+(S_+, L_+, B_+) \) when \( \tau_2 = \tau_{20} \) and a family of periodic solutions bifurcate from \( E_+(S_+, L_+, B_+) \), and

\[
\tau_{20} = \frac{1}{\omega_{20}} \times \arccos \left( \frac{g_{51}(\omega_{20}') \times g_{54}(\omega_{20}')} {g_{02}(\omega_{20})} + g_{52}(\omega_{20}') \times g_{53}(\omega_{20}) \right) .
\]

Proof: For \( \tau_2 > 0 \), \( \tau_1 \in (0, \tau_{10}) \). Let \( \lambda = i\omega_{20}' \) be the root of Equation (8). Then

\[
g_{51}(\omega_{20}') \sin \tau_2 \omega_{20}' + g_{52}(\omega_{20}') \cos \tau_2 \omega_{20}' = g_{53}(\omega_{20}'),
g_{51}(\omega_{20}') \cos \tau_2 \omega_{20}' - g_{52}(\omega_{20}') \sin \tau_2 \omega_{20}' = g_{54}(\omega_{20}'),
\]

where

\[
g_{51}(\omega_{20}') = C_1 \omega_{20}' - D_0 \sin \tau_1 \omega_{20}',
g_{52}(\omega_{20}') = C_0 - C_2 \omega_{20}'^2 + D_0 \cos \tau_1 \omega_{20}',
g_{53}(\omega_{20}') = A_2(\omega_{20}')^2 - A_0 - B_1 \omega_{20}' \sin \tau_1 \omega_{20}' - B_0 \cos \tau_1 \omega_{20}',
g_{54}(\omega_{20}') = (\omega_{20}')^3 - A_1 \omega_{20}' - B_1 \omega_{20}' \sin \tau_1 \omega_{20}' + B_0 \sin \tau_1 \omega_{20}.'
\]

Thus, we can obtain the equation with respect to \( \omega_{20}' \), Equation (6). Under Assumption 2.8, we know that Equation (8) has a pair of purely imaginary roots \( \pm i\omega_{20}' \). For \( \omega_{20}' \), we can obtain the expression of \( \tau_{20} \) as following

\[
\tau_{20} = \frac{1}{\omega_{20}} \times \arccos \left( \frac{g_{51}(\omega_{20}') \times g_{54}(\omega_{20}')} {g_{02}(\omega_{20})} + g_{52}(\omega_{20}') \times g_{53}(\omega_{20}) \right) .
\]

Differentiating Equation (8) with respect to \( \tau_2 \), one can get

\[
\left[ \frac{dx}{d\tau_2} \right]^{-1} = \frac{g_{55}(\lambda)}{g_{56}(\lambda)} - \frac{\tau_2}{\lambda},
\]

with

\[
g_{55}(\lambda) = 3\lambda^2 + 2A_2 \lambda + A_1 - (\tau_1 B_1 \lambda + \tau_1 B_0 - B_1) e^{-\lambda \tau_1} + (2C_2 \lambda + C_1) e^{-\lambda \tau_2} - \tau_1 D_0 e^{-\lambda (\tau_1 + \tau_2)},
g_{56}(\lambda) = (C_2 \lambda^3 + C_1 \lambda^2 + C_0 \lambda) e^{-\lambda \tau_2} + D_0 \lambda e^{-\lambda (\tau_1 + \tau_2)}.
\]

Thus,

\[
\text{Re} \left[ \frac{dx}{d\tau_2} \right]_{\tau_2 = \tau_{20}}^{-1} = \frac{P_{5R} Q_{5R} + P_{5S} Q_{5S}} {Q_{5R}^2 + Q_{5S}^2} .
\]

Under Assumption 2.9, we know that \( \text{Re}[dx/d\tau_2]_{\tau_2 = \tau_{20}}^{-1} \neq 0 \). Therefore, based on the above discussion and the Hopf bifurcation theorem in Hassard et al. (1981), we have Theorem 3.4 and the proof is completed.

Theorem 3.5: For system (2),

(i) if \( \mu_2 > 0 \), the Hopf bifurcation is supercritical; if \( \mu_2 < 0 \), the Hopf bifurcation is subcritical;
(ii) if \( \beta_2 < 0 \), the bifurcating periodic solutions are stable; if \( \beta_2 > 0 \), the bifurcating periodic solutions are unstable;
(iii) if \( T_2 > 0 \), the period of the bifurcating periodic solutions increases; if \( T_2 < 0 \), the period of the bifurcating periodic solutions decreases,

and the expressions of \( \mu_2, \beta_2 \) and \( T_2 \) are as follows:

\[
C_1(0) = \frac{i}{2 \omega_{20} \tau_{20}} \left( g_{11} g_{20} - 2 |g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2},
\]

\[
\mu_2 = - \frac{\text{Re} \{C_1(0)\}} {\text{Re} \{\lambda'(\tau_{20})\}},
\]

\[
\beta_2 = 2 \text{Re} \{C_1(0)\},
\]

\[
T_2 = - \frac{\text{Im} \{C_1(0)\} + \mu_2 \text{Im} \{\lambda'(\tau_{20})\}} {\omega_{20} \tau_{20}} .
\]

Proof: Suppose that \( \tau_{20}' < \tau_{20} \) where \( \tau_{10} \in (0, \tau_{10}) \). Let \( \tau_1 = \tau_{20}' + \mu, u_1 = S(\tau_{20}'), u_2 = L(\tau_2), u_3 = B(\tau_{20}), \mu \in R \). Then, \( \mu = 0 \) is the Hopf bifurcation value of system (2) and system (2) can be transformed into

\[
\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t),
\]

where \( u(t) = (u_1, u_2, u_3) \in C = C([-1, 0], R^3) \) and \( L_\mu: C \rightarrow R^3 \) and \( F: R \times C \rightarrow R^3 \) are given respectively by

\[
L_\mu \phi = (\tau_{20}' + \mu) (A_\mu \phi(0) + B_\mu \phi \left( \frac{-\tau_{10}} {\tau_{20}} \right) + C_\mu (-1))
\]

\[
F(\mu, \phi)(\tau_{20}' + \mu)(F_1, F_2, 0)^T,
\]

with

\[
A_\mu = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad B_\mu = \begin{bmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
C_\mu = \begin{bmatrix} 0 & c_{22} & 0 \\ 0 & c_{32} & 0 \end{bmatrix},
\]

and

\[
F_1 = - \beta (\phi(1)(\phi_2(0) + \phi(1)(\phi_3(0)),
\]

\[
F_2 = \beta (\phi(1)(\phi_2(0) + \phi(1)(\phi_3(0)),
\]

\[
\phi = (\phi_1, \phi_2, \phi_3)^T \in C([-1, 0], R^3).
\]

Thus, based on the Riesz representation theorem, we know that there exists a matrix \( \eta(\theta, \mu): [-1, 0] \rightarrow R^3 \) such
that

$$L_{μ}φ = \int_{−1}^{0} dη(θ, μ)φ(θ), \quad φ ∈ C([-1, 0], R^3).$$

In fact, choosing

$$η(θ, μ) = \begin{cases} (τ'_{20} + μ)(A_2 + B_2 + C_2), & θ = 0, \\ (τ'_{20} + μ)(B' + C_2), & θ ∈ \left[−\frac{τ'_{10}}{τ'_{20}}, 0\right), \\ (τ'_{20} + μ)C_2, & θ ∈ \left(−1, −\frac{τ'_{20}}{τ'_{10}}\right), \\ 0, & θ = -1. \end{cases}$$

For φ ∈ C([-1, 0], R^3), define

$$A(μ)φ = \begin{cases} \frac{dφ(θ)}{dθ}, & −1 ≤ θ < 0, \\ \int_{−1}^{0} dη(θ, μ)φ(θ), & θ = 0, \end{cases}$$

and

$$R(μ)φ = \begin{cases} 0, & −1 ≤ θ < 0, \\ F(μ), & θ = 0. \end{cases}$$

Then system (17) becomes

$$\dot{u}(t) = A(μ)u(t) + R(μ)u(t). \quad (18)$$

For φ ∈ C([-1, 0], R^3), ϕ ∈ C^1 ([0, 1], (R^3)*) it is defined

$$A^*(φ) = \begin{cases} \frac{dφ(s)}{ds}, & 0 < s ≤ 1, \\ \int_{−1}^{0} dη^T(s, 0)φ(-s), & s = 0, \end{cases}$$

and bilinear form

$$⟨φ, ϕ⟩ = \bar{φ}(0)φ(0) − \int_{−1}^{0} \int_{−1}^{θ} \bar{φ}(ξ − θ) dη(θ)φ(ξ) dξ,$$

where η(θ) = η(θ, 0), A = A(0) and A* are adjoint operators.

Let q(θ) = (1, q_2, q_3)^T e^{iω_0 τ'_{20}θ} be the eigenvector of A(0) corresponding to +iω_0 τ'_{20} and q*(θ) = K(1, q_2, q_3)^T e^{iω_0 τ'_{20}θ} be the eigenvector of A* corresponding to −iω_0 τ'_{20}. By direct computation, we obtain

$$q_2 = \frac{c_{32} e^{-iω_0 τ'_{20}}}{iω_0 τ'_{20} - a_{33}}, \quad q_2^* = -\frac{iω_0 τ'_{20} + a_{11}}{a_{21}},$$

$$q_3 = \frac{a_{13} + b_{13} e^{iω_0 τ'_{20}} + a_{23} q_2}{iω_0 τ'_{20} + a_{33}}.$$

From Equation (19), we get

$$\bar{R} = [1 + q_2 \bar{q}_2 + q_3 \bar{q}_3 + τ'_{10} e^{-iω_0 τ'_{10}} (b_{12} q_2 + b_{13} q_3) + τ'_{20} e^{-iω_0 τ'_{20}} q_2 (c_{22} \bar{q}_2 + c_{32} \bar{q}_3)]^{-1}.$$

such that (q*, q) = 1.

In what follows, we can obtain the coefficients by using the method introduced in Hassard et al. (1981) and the similar process as that in Jana, Agrawal, and Upadhyay (2014), Upadhyay and Agrawal (2016), Bianca, Ferrara, and Guerrini (2013), Meng, Huo, and Zhang (2011) and Xu (2017):

$$g_{20} = 2τ'_{20} \bar{R}(\bar{q}_2 - 1)β(q_2 + q_3),$$

$$g_{11} = τ'_{20} \bar{R}(\bar{q}_2 - 1)β(Re(q_2) + Re(q_3)),$$

$$g_{02} = 2τ'_{20} \bar{R}(\bar{q}_3 - 1)β(q_2 + \bar{q}_3),$$

$$g_{21} = 2τ'_{20} \bar{R}(\bar{q}_2 - 1)β(W_{11}^2(0)q_2$$

$$+ \frac{1}{2} W_{20}^2(0)q_2 + W_{21}^2(0) + \frac{1}{2} W_{20}^2(0))$$

$$+ W_{11}^2(0)q_3 + \frac{1}{2} W_{20}^2(0)q_3$$

$$+ W_{11}^3(0) + \frac{1}{2} W_{20}^3(0),$$

with

$$W_{20}(θ) = \frac{ig_{20} q(0)}{τ'_{20} α_{20}} e^{iτ'_{20} α_{20} θ} + \frac{ig_{02} \bar{q}(0)}{3τ'_{20} α_{20}} e^{-iτ'_{20} α_{20} θ}$$

$$+ E_1 e^{2iτ'_{20} α_{20} θ},$$

$$W_{11}(θ) = -\frac{ig_{11} q(0)}{τ'_{20} α_{20}} e^{iτ'_{20} α_{20} θ} + \frac{ig_{11} \bar{q}(0)}{τ'_{20} α_{20}} e^{-iτ'_{20} α_{20} θ} + E_2,$$

where

$$E_1 = 2 \begin{pmatrix} \frac{2iω_0 τ'_{20} - a_{11}}{a_{12} - b_{12}} & \frac{2iω_0 τ'_{20} - a_{12}}{e^{-2iω_0 τ'_{20}}} & \frac{-a_{13} - b_{13}}{e^{-2iω_0 τ'_{20}}} \\
-a_{21} & -c_{22} e^{-2iτ'_{20} α_{20}} & -a_{23} \\
0 & -c_{32} e^{-2iτ'_{20} α_{20}} & -a_{33} \end{pmatrix}^{-1}$$

$$× (-β(q_2 + q_3),$$

$$E_2 = -\begin{pmatrix} a_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\
a_{21} & a_{22} + c_{22} & a_{23} \\
0 & c_{32} & a_{33} \end{pmatrix}^{-1}$$

$$× (-β(Re(q_2) + Re(q_3)))$$

$$β(Re(q_2) + Re(q_3))).$$
Thus, we can obtain the expressions of $\mu_2, \beta_2$ and $T_2$. And based on the properties of the Hopf bifurcation stated in Hassard et al. (1981), we have Theorem 3.5 and the proof is competed.

### 4. Numerical simulations

In this section, we present some numerical simulation results of system (2) to illustrate our theoretical results. We choose the same values of the parameters as in Zhang and Bi (2016) and get the following specific case of system (2):

\[
\begin{align*}
\frac{dS(t)}{dt} &= 0.05 - 0.35S(t)(L(t) + B(t)) + 0.1L(t - \tau_1) \\
&\quad + 0.3B(t - \tau_1) - 0.05S(t), \\
\frac{dL(t)}{dt} &= 0.35S(t)(L(t) + B(t)) - 0.1L(t) \\
&\quad - 0.3L(t - \tau_2) - 0.05L(t), \\
\frac{dB(t)}{dt} &= 0.3L(t - \tau_2) - 0.3B(t) - 0.05B(t),
\end{align*}
\]

from which one can obtain $R_0 = 1.4444 > 1$ and $E_+(0.6924, 0.1657, 0.1420)$.

For $\tau_1 > 0$ and $\tau_2 = 0$. Based on the computation in Yang et al. (2013), we know that $\omega_{10} = 0.2766, \tau_{10} = 22.9208$. The numerical simulation can be shown as in Figures 2–3. As can be seen from Figure 2, $E_+$ is locally asymptotically stable when $\tau_1 = 21.85 < \tau_{10}$. In this case, the percentages of the susceptible computers, the latent computers and the breaking-out computers in system (20) will tend to $S^*, L^*, B^*$, respectively, and the computer viruses in system (20) can be predicted and controlled easily. When $\tau_1$ increases, $E_+$ loses its stability and a Hopf bifurcation occurs from $E_+$ and a family of periodic solutions bifurcate from $E_+$, which can be illustrated by Figure 3. From this we know that the computer viruses will be out of control. Similarly, we obtain $\omega_{20} = 3.0911$;
Figure 4. $E_+$ is locally asymptotically stable with $\tau_2 = 7.25 < \tau_{20}$.

Figure 5. $E_+$ is locally asymptotically stable with $\tau_2 = 8.35 > \tau_{20}$.

Figure 6. $E_+$ is locally asymptotically stable with $\tau_2 = 5.25 < \tau_0$.

and $\tau_{20} = 7.6025$. The corresponding waveform and the phase plots can be shown as in Figures 4–5.

For $\tau_1 = \tau_2 = \tau$. By some computation using Matlab software package, we obtain $\omega_0 = 1.3350$; and $\tau_0 = 5.3497$. We can conclude that $E_+$ is locally asymptotically stable when $\tau < \tau_0$ according to Theorem 3.3 and this property can be illustrated by Figure 6. Once the value of $\tau$ passes through the critical value $\tau_0$, $E_+$ will lose
Figure 7. $E_+$ is locally asymptotically stable with $\tau_2 = 5.65 > \tau_0$.

Figure 8. $E_+$ is locally asymptotically stable with $\tau_2 = 3.05 < \tau_2'$ and $\tau_1 = 15.65$.

Figure 9. $E_+$ is locally asymptotically stable with $\tau_2 = 3.95 > \tau_2'$ and $\tau_1 = 15.65$. 
its stability and a Hopf bifurcation occurs, which can be illustrated by Figure 7.

For $\tau_3 > 0$ and $\tau_1 = 15.65 \in (0, \tau_{10})$. By some complex computation, we have $\omega'_{20} = 0.9808$; and $\omega'_{20} = 3.2662$. The numerical simulation results are shown as in Figures 8–9. Figure 8 is illustrated by fixing $\tau_2 = 3.05 < \tau^*_{20}$, which is shown that $E_+$ is locally asymptotically stable. Figure 9 is illustrated by fixing $\tau_3 = 3.95$ and it is shown that a Hopf bifurcation occurs from $E_+$. In addition, in view of $\lambda'(\tau_{20}) = 0.3076 + 0.0081i$ and $C_1(0) = -0.9377 + 1.0008i$, we obtain $\mu_2 = 3.0484 > 0$, $\beta_2 = -1.8754 < 0$ and $T_2 = -0.1668 < 0$. Thus, we can find that the Hopf bifurcation is supercritical; the bifurcating periodic solutions are stable and decrease based on Theorem 3.5.

5. Conclusions

In this paper, a computer virus propagation model with two delays and infectivity in latent period was proposed by introducing the latent delay. Comparing with the model considered in Zhang and Bi (2016), we not only investigated the effect of the time delay due to the period that the antivirus software removes the computer viruses in the latent computers and the breaking computers, but also the effect of the time delay due to the latent period of the latent computers on the new model. Thus, the proposed computer virus propagation model is more general.

The main results are given in terms of the dynamics including the local stability and the local Hopf bifurcation of the computer virus propagation model. It was illustrated that the Hopf bifurcation occurs when the delay passes through the corresponding critical value under some certain conditions. According to the numerical simulation results, it can be concluded that the computer viruses can be predicted and controlled when the time delay is relatively small. Therefore, it is necessary to control the occurrence of the Hopf bifurcation by combining bifurcation control strategies and other relative features of computer virus prevalence. This will be our future work.

However, it should be pointed out that the main purpose of this paper is to investigate the effect of the two delays on system (2). And our study is restricted only to the theoretical and technical analysis of such phenomena in the Internet. It may be helpful for field investigation or experimental studies on the real situation. Namely, all the assumptions in this paper is only to guarantee the sufficient conditions for the occurrence of the Hopf bifurcation technically. But all the assumptions in this paper can be verified in the numerical simulations. In other words, the assumptions in this paper are not empty. In addition, we suppose that the rate at which the external computers connect to the network equals the rate at which the internal computers disconnect from the network, and that the period that the antivirus software uses to clean the computer viruses in the latent computers and the breaking computers is the same. This is may be too ideal in the real networks. Thus, it is definitely interesting to investigate the following more general model with multiple delays:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \delta - \beta S(t)(L(t) + B(t)) + \gamma_1 L(t - \tau_1) - \delta S(t) - \gamma_3 B(t) - \delta_3 B(t), \\
\frac{dL(t)}{dt} &= \beta S(t)(L(t) + B(t)) - \gamma_1 L(t) - \alpha L(t - \tau_3) - \delta_2 L(t), \\
\frac{dB(t)}{dt} &= \alpha L(t - \tau_3) - \delta_2 B(t) - \delta_3 B(t),
\end{align*}
\]

where $\delta$ is the rate at which the external computers connect to the network; $\delta_1$, $\delta_2$ and $\delta_3$ are the rates at which the susceptible computers, the latent computers and the breaking computers disconnect from the network, respectively; $\tau_1$ and $\tau_3$ are the time delay due to the period that the antivirus software uses to clean the computer viruses in the latent computers and the breaking computers, respectively; and $\tau_3$ is the time delay due to the latent period of the latent computers before they can infect other computers. We leave this as our future work.

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