Birack modules and their link invariants

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Abstract

We extend the rack algebra \( \mathbb{Z}[X] \) defined by Andruskiewitsch and Graña to the case of biracks, enabling a notion of birack modules. We use these birack modules to define an enhancement of the birack counting invariant generalizing the birack module counting invariant in [8]. We provide examples demonstrating that the enhanced invariant is not determined by the Jones or Alexander polynomials and is strictly stronger than the unenhanced birack counting invariant.

Keywords: Biracks, biquandles, Yang-Baxter equation, virtual knot invariants, enhancements of counting invariants

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1 Introduction

Biracks were first introduced in [7] as a generalization of racks, algebraic structures whose axioms encode blackboard-framed isotopy of oriented knots and links in \( \mathbb{R}^3 \). Initially called “wracks” alluding to the “wrack and ruin” resulting from keeping only conjugation in a group, racks were first considered by Conway and Wraith in the 1950s [6]. The current term “rack” (without the “w”) is due to Fenn and Rourke in [6], who dropped the “w” to denote dropping invariance under writhe-changing type I Reidemeister moves.

In [14], an integer-valued invariant of knots and links associated to a finite birack \( X \) known as the \textit{integral birack counting invariant}, \( \Phi^Z_X \) was introduced by the second author, generalizing the rack counting invariant from [13]. In [1] an associative algebra known as the \textit{rack algebra} \( \mathbb{Z}[X] \) was defined for every finite rack, and in [8] the rack counting invariant from [14] was enhanced with representations (known as \textit{rack modules}) of a modified form of \( \mathbb{Z}[X] \).

In this paper, for every finite birack \( X \) we define an associative algebra \( \mathbb{Z}_B[X] \) we call the \textit{birack algebra}. We use representations of \( \mathbb{Z}_B[X] \), known as birack modules or \( X \)-modules, to enhance the birack counting invariant from [14]. The new invariant is defined for classical and virtual knots and links.

The paper is organized as follows: in section 2 we review the basics of biracks and the birack counting invariant. In section 3 we define the birack algebra and give examples of birack modules. In section 4 we use birack modules to enhance the birack counting invariant and give examples to demonstrate that the birack module enhanced invariant is not determined by the Jones or Alexander polynomials and is strictly stronger than the unenhanced birack counting invariant. We conclude with a few questions for future research in section 5.

2 Biracks

We begin with a definition (see [5, 14]).

Definition 1 Let \( X \) be a set. A birack structure on \( X \) is an invertible map \( \text{B} : X \times X \rightarrow X \times X \) satisfying

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(i) $B$ is sideways invertible, that is, there exists a unique invertible map $S : X \times X \to X \times X$ satisfying for all $x, y \in X$ 
\[ S(B_1(x, y), x) = (B_2(x, y), y), \]
(ii) $B$ is diagonally invertible, that is, the compositions $S_1^{\pm 1} \circ \Delta$ and $S_2^{\pm 1} \circ \Delta$ of the diagonal map $\Delta : X \to X \times X$ given by $\Delta(x) = (x, x)$ with the components of the sideways map and its inverse are bijections, and
(iii) $B$ is a solution to the set-theoretic Yang-Baxter equation:
\[ (B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B). \]

We will often abbreviate $B_1(x, y) = y^x$ and $B_2(x, y) = x_y$.

**Example 1** Perhaps the simplest example of a birack structure is the class of constant action biracks: let $X$ be a set and let $\sigma, \tau : X \to X$ be bijections. Then the map $B(x, y) = (\sigma(y), \tau(x))$ defines a birack structure on $X$ iff $\sigma$ and $\tau$ commute; see [14].

**Example 2** Let $\hat{A} = \mathbb{Z}[t^{\pm 1}, s, r^{\pm 1}]/(s^2 - s(1 - tr))$. Then any $\hat{A}$-module $M$ is a birack, known as $(t, s, r)$-birack, under the map $B(x, y) = (sy + sx, rx)$. This structure generalizes the $(t, s)$-rack structure from [6]. The idea is to consider the case when the output components $B$ are linear combinations of the inputs, i.e., set $B(x, y) = (sx + ty, rx + py)$; then the invertibility conditions (i) and (ii) are satisfied if one of the four coefficients must be zero. Selecting $p = 0$, the Yang-Baxter equation is then satisfied provided we have $s^2 = (1 - tr)s$.

It is frequently useful to specify a birack structure on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ by listing the operation tables for the components of $B$ viewed as binary operations $(x, y) \mapsto B_1(x, y) = x^y$ and $(x, y) \mapsto B_2(x, y) = x_y$. Note the reversed order of the operands in $B_1$; this is for compatibility with previous work representing these operations as right actions. Then a birack operation $B$ on $X$ is specified by the $n \times 2n$ matrix $M_B = [U|L]$ with $U(i, j) = k$ and $L(i, j) = l$ where $x_k = B_1(x_j, x_i)$ and $x_l = B_2(x_i, x_j)$. Conversely, such a matrix specifies a birack structure iff the operation it defines via 
\[ B(x_i, x_j) = (x_U[i,j], x_L[i,j]) \]
satisfies the birack axioms.

**Example 3** The constant action birack on $X = \{x_1, x_2, x_3\}$ with bijections $\sigma = (123)$ and $\tau = (132)$ has birack matrix
\[
M_B = \begin{bmatrix}
2 & 2 & 2 & 3 & 3 & 3 \\
3 & 3 & 3 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 2 & 2 \\
\end{bmatrix}
\]
and the $(t, s, r)$-birack on $X = \mathbb{Z}_4$ with $t = r = 3$ and $s = 2$ has birack matrix
\[
M_B = \begin{bmatrix}
1 & 3 & 1 & 3 & 1 & 1 & 1 \\
4 & 2 & 4 & 2 & 4 & 4 & 4 \\
3 & 1 & 3 & 1 & 3 & 3 & 3 \\
2 & 4 & 2 & 4 & 2 & 2 & 2 \\
\end{bmatrix}
\]
where $0 = x_1, 1 = x_2, 2 = x_3$, and $3 = x_4$, using 4 as the representative of the class of 0 in $\mathbb{Z}_4$ since our row and column numbering starts with 1. Then for instance the entry in row 1 column 3 of the left matrix should be $B_1(x_3, x_1) = (tx_1 + sx_3) = 3(1) + 2(3) = 3 + 6 = 9 = 1$, while the entry in row 3 column 2 of the right matrix is $B_2(x_3, x_2) = r(x_3) = 2(2) = 4$.

As with other algebraic structures, we have the following standard notions:
Definition 2 Let $X$ and $X'$ be sets with birack operations $B : X \times X \rightarrow X \times X$ and $B' : X' \times X' \rightarrow X' \times X'$. Then:

- a map $f : X \rightarrow X'$ is a **birack homomorphism** if for all $x, y \in X$ we have $B'(f(x), f(y)) = (f(B_1(x, y)), f(B_2(x, y)))$;

- a subset $Y \subset X$ is a **subbirack** of $X$ if the restriction of $B$ to $Y \times Y$ defines a birack structure on $Y$; equivalently, $Y \subset X$ is a subbirack if $Y \times Y$ is closed under $B$.

The birack axioms encode the blackboard-framed Reidemeister moves with semiarcs labeled according to the rules below. At a positive kink, the invertibility properties of $B$ define bijections $\alpha : X \rightarrow X$ and $\pi : X \rightarrow X$ defined by $\alpha = (S_2^{-1} \circ \Delta)^{-1}$ and $\pi = S_1^{-1} \circ \Delta \circ \alpha$ which give us the labels on the other semiarcs in a blackboard framed type I move.

The exponent $N$ of the bijection $\pi$, that is, the smallest positive integer $N$ satisfying $\pi^N(x) = x$ for all $x \in X$, is known as the **birack rank** or **birack characteristic** of $X$. If $X$ is a finite set, then $N$ is guaranteed to exist, but infinite biracks may have infinite rank.

**Example 4** A birack with rank $N = 1$ is called a **strong biquandle** (see [5 11 15]); a birack satisfying $B_2(x, y) = x$ is a **rack** (see [6]). A strong biquandle which is also a rack is a **quandle**; see [9 12].

**Example 5** Every oriented blackboard framed link diagram $L$ has a **fundamental birack** $FB(L)$, defined as the set of equivalence classes of birack words in the semiarcs of $L$ modulo the equivalence relation determined by the birack axioms and the crossing relations in $L$. For example, the knot $5_1$ below has the fundamental birack presentation listed below:

$$FB(5_1) = \langle a, b, c, d, e, f, g, h, i, j \mid B(a, b) = (c, d), \ B(c, d) = (e, f),$$

$$B(e, f) = (g, h), \ B(g, h) = (i, h), \ B(i, j) = (a, b) \rangle.$$
A labeling of semi-arcs in a framed oriented link diagram with elements of a birack $X$ determines a homomorphism $f : FB(L) \to X$ if and only if the birack labeling condition is satisfied by the labels in $X$ at every crossing. Conversely, every homomorphism $f : FB(L) \to X$ determines a unique labeling of the semi-arcs of $L$. By construction, we have the following standard result:

**Theorem 1** If $L$ and $L'$ are oriented link diagrams which are related by blackboard framed Reidemeister moves and $X$ is a finite birack, then the sets of labelings of the semi-arcs of $L$ and $L'$ by elements of $X$ satisfying the birack labeling condition, $\text{Hom}(FB(L), X)$ and $\text{Hom}(FB(L'), X)$, are in bijective correspondence.

Thus, the cardinality of the set of birack labelings of an oriented link diagram by a finite birack $X$ is a positive integer valued invariant of framed isotopy. To obtain an invariant of unframed ambient isotopy, we observe as in [13] that the cardinalities of the sets of birack labelings are periodic in the birack rank $N(X)$ since birack labelings are preserved by the $N$-phone cord move:

$$
\pi^N(x) = x \quad \pi^2(x) \quad \pi(x) \quad x
$$

Thus, summing these cardinalities over a complete period of framings mod $N$ yields an invariant of ambient isotopy known as the integral birack counting invariant $\Phi^Z_X$. More formally, we have

**Definition 3** Let $X$ be a finite birack with birack rank $N$ and $L$ an oriented link with $c$ components. Let $W = (\mathbb{Z}_N)^c$ and let $(L, w)$ be a blackboard framed diagram of $L$ with writhe $w_k$ on component $k$ for each $k = 1, \ldots, c$. The integral birack counting invariant is the integer

$$
\Phi^Z_X(L) = \sum_{w \in W} |\text{Hom}(FB((L, w)), X)|.
$$

**Example 6** Let $X$ be the birack structure on $X = \{1, 2\}$ with birack matrix

$$
M_B = \begin{bmatrix}
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1
\end{bmatrix}.
$$

As a labeling rule, this birack requires switching labels from 1 to 2 and 2 to 1 on both semi-arcs at every crossing point. Thus, a component in a diagram has two valid labelings iff it has an even number of semi-arcs, but since a classical diagram with $n$ crossings has $2n$ semi-arcs, every diagram has 2 valid labelings. Since $N = 1$, we have $\Phi^Z_X(L) = 2^c$ for every classical link with $c$ components.
The integral birack counting invariant is also defined for virtual links via the usual technique of ignoring virtual crossings when dividing the link into semiarcs or, equivalently, regarding our virtual link diagrams as being drawn on a surface with sufficient genus to avoid virtual crossings.

**Example 7** While the birack in example 6 does not distinguish classical links with the same number of components via $\Phi^Z_X$, it can distinguish some virtual links from other virtual links with the same number of components. Recall that a virtual link diagram can include virtual crossings represented by circled self-intersections which we may regard as indicating genus in the surface on which the link diagram is drawn. For instance, the virtual Hopf link $v$Hopf below has no valid $X$-labelings, so $\Phi^Z_X(v$Hopf) $= 0 \neq 2 = \Phi^Z_X$(Unlink).

See [10] for more about virtual knots and links.

### 3 Birack algebras and modules

We would like to enhance the integral birack counting invariant by finding a way to distinguish between labelings rather than just counting how many total labeling we have. To this end, we will use a scheme analogous to the $(t, s, r)$-birack structure to define an associative algebra we call the birack algebra associated to a finite birack $X$. We will discuss the motivation for this definition later in this section.

**Definition 4** Let $X$ be a finite birack of birack rank $N$. The **birack algebra** associated to $X$ is the quotient of the polynomial algebra $\mathbb{Z}[t^\pm, s_{x,y}, r^\pm_{x,y}]$ generated by invertible elements $t_{x,y}, r_{x,y}$ and general elements $s_{x,y}$ for all $x, y \in X$, modulo the ideal $I$ defined by the relations

- $t_{x,y}r_{x,y} = r_{x,y}t_{x,y}$
- $t_{x,y}r_{x,y} = r_{x,y}t_{x,y}$
- $s_{x,y}t_{x,y} = t_{x,y}s_{x,y}$
- $t_{x,y}s_{x,y} = s_{x,y}t_{x,y}$
- $s_{x,y}t_{x,y} = t_{x,y}s_{x,y}$
- $s_{x,y}t_{x,y} = t_{x,y}s_{x,y}$
- $1 - \prod_{k=0}^{N-1} (t_{\pi^k(x),\alpha(\pi^k(x))}r_{\pi^k(y),\alpha(\pi^k(y))} + s_{\pi^k(x),\alpha(\pi^k(x))})$
We will denote the birack algebra of \( X \) by \( \mathbb{Z}_B[X] \). A representation of \( \mathbb{Z}_B[X] \), i.e. an abelian group \( G \) with automorphisms \( t_{x,y}, r_{x,y} : G \rightarrow G \) and endomorphisms \( s_{x,y} : G \rightarrow G \) such that the above listed maps are zero, will be called a \textit{birack module} or simply an \( X \)-module.

**Remark 1** This definition generalizes the quandle and rack module definitions from [1], which were used in [4, 8] to enhance the quandle and rack counting invariants respectively. In particular, if \( X \) is a rack then a rack module in the sense of [8] is a birack module over \( X \) in which every \( r_{x,y} = \text{Id}_G \). Note that in general, the set of rack modules over a rack \( X \) considered as a birack is a proper subset of the set of birack modules over \( X \).

The motivation behind the birack algebra definition is to define secondary labelings of \( X \)-labeled oriented blackboard framed link diagrams with a “bead” at every semiarc and to use the \((t, s, r)\)-birack operations on the beads at a crossing with each \( t, s \) and \( r \) coefficient indexed by the \( X \)-labels on the input strands at a positive crossing and the output strands at a negative crossing.

\[
\begin{align*}
c &= s_{x,y}a + t_{x,y}b \\
d &= r_{x,y}a
\end{align*}
\]

The birack algebra relations are chosen to preserve bead labelings under the blackboard-framed Reidemeister moves and the \( N \)-phone cord move. The choice of bead labeling rules guarantees that for every blackboard framed \( X \)-labeled diagram, the number of bead labelings is the same before and after type II moves and framed type I moves:

\[
\begin{align*}
b &= r_{x,\alpha(x)}a \\
c &= s_{x,\alpha(x)}a + t_{x,\alpha(x)}b \\
\Rightarrow c &= (s_{x,\alpha(x)} + t_{x,\alpha(x)}r_{x,\alpha(x)})a
\end{align*}
\]

The other type II and framed type I cases are similar.
Six of the seven birack algebra relations come from the type III move:

Comparing coefficients on the output beads $g, h$ and $i$ yields the first six relations.

The final birack algebra relation comes from the $N$-phone cord move. Pushing a bead on a strand labeled with birack element $x$ through a positive kink multiplies the bead by $\left( t_{x,\alpha(x)} r_{x,\alpha(x)} + s_{x,\alpha(x)} \right)$; thus, we need the product of these over a complete period of framings mod $N$ to be 1.

**Example 8** Let $X$ be a finite birack and $R$ a ring. We can give $R$ the structure of a $\mathbb{Z}_B[X]$-module by choosing invertible elements $t_{x,y}, r_{x,y} \in R$ and elements $s_{x,y}$ for all $x, y \in X$ such that the birack algebra relations are satisfied. We can represent such a birack module structure with a birack module matrix $M_R = \begin{bmatrix} T & S \\ R \end{bmatrix}$ where $T_{i,j} = t_{x_i,x_j},$ etc. For example, the birack from example 6 has birack modules on $\mathbb{Z}_3$ including $M_R = \begin{bmatrix} 2 & 2 & 0 & 2 & 2 & 1 \\ 2 & 2 & 2 & 0 & 1 & 2 \end{bmatrix}$.

**Example 9** Another important example of a $\mathbb{Z}_B[X]$-module is the fundamental $\mathbb{Z}_B[X]$-module of an $X$-labeled oriented blackboard framed link diagram $L$, denoted $Z_f[X]$, where $f : FB(L) \to X$ is the $X$-labeling of $L$. Starting with our $X$-labeled diagram, we put a bead on every semiarc and obtain a system of linear equations with coefficients in $\mathbb{Z}_B[X]$ determined at the crossings. For convenience, we will represent such a module with the coefficient matrix of the homogeneous system. For example, the link $L_{4a1}$ from example 6 with the labeling by the birack from the same example below has the listed fundamental $\mathbb{Z}_B[X]$-module.

\[ Z_f[X] = \begin{bmatrix} t_{2,1} & -1 & 0 & 0 & s_{2,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{2,1} & -1 & 0 & 0 \\ 0 & s_{2,1} & 0 & 0 & 0 & t_{2,1} & 0 & -1 \\ 0 & r_{2,1} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{2,1} & -1 & 0 & 0 & s_{2,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{2,1} & -1 & 0 \\ 0 & 0 & 0 & s_{2,1} & -1 & 0 & 0 & t_{2,1} \\ -1 & 0 & 0 & r_{2,1} & 0 & 0 & 0 & 0 \end{bmatrix} \]

The columns in this matrix represent the beads on the semiarc, while each row represents an equation in the beads determined at a crossing. For example, the upper left crossing determines the equations $t_{2,1}a + s_{2,1}c = b$ and $r_{2,1}e = f$, which give us the first two rows of the matrix.

### 4 Birack module enhancements of the counting invariant

We will now use an $X$-module $R$ to enhance the integral birack counting invariant by taking the set of $\mathbb{Z}_B[X]$-module homomorphisms $\text{Hom}_{\mathbb{Z}_B[X]}(Z_f[X], R)$ as a signature for each birack labeling $f \in \text{Hom}(FB(L), X)$. Since each homomorphism in $\text{Hom}_{\mathbb{Z}_B[X]}(Z_f[X], R)$ can be understood as a labeling of the beads by elements of $R$, we are effectively counting bead labelings of $X$-labelings of $L$, enhancing the birack counting invariant with a bead counting invariant of $X$-labeling of $L$. 

7
\textbf{Definition 5}  Let $X$ be a finite birack of rank $N$, $R$ a $\mathbb{Z}[X]$-module, and $L$ an oriented (classical or virtual) link of $c$ components. Let $W = (\mathbb{Z}_N)^c$ and for each $w \in W$, let $(L, w)$ be a diagram of $L$ with writhe $w_k$ on component $k$. Then the \textit{birack module enhanced multiset} is the multiset of $R$-modules

$$
\Phi^{B, M}_{X, R}(L) = \{ \text{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}_f[X], R) \mid f \in \text{Hom}(FB((L, w)), X), w \in W \}
$$

and the \textit{birack module enhanced polynomial} is

$$
\Phi^B_{X, R}(L) = \sum_{w \in W} \left( \sum_{f \in \text{Hom}(FB(L, w), X)} u^{\text{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}_f[X], R))} \right).
$$

If $R$ is not a finite set, we replace the cardinality $|\text{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}_f[X], R)|$ of the set of bead labelings with the rank of the $R$-module $\text{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}_f[X], R)$.

By construction, we have

\textbf{Proposition 2} If $L$ and $L'$ are ambient isotopic oriented classical or virtual links, then $\Phi^B_{X, R}(L) = \Phi^B_{X, R}(L')$ and $\Phi^{B, M}_{X, R}(L) = \Phi^{B, M}_{X, R}(L')$.

Note that the integral birack counting invariant $\Phi^Z_{X}(L)$ is recovered from the birack module enhanced invariant $\Phi^B_{X, R}(L)$ by evaluating at $u = 1$ and from $\Phi^{B, M}_{X, R}(L)$ by taking cardinality. In our next example we show that $\Phi^B_{X, R}(L)$ is stronger in general than $\Phi^Z_{X}(L)$.

\textbf{Example 10} Let $X = \mathbb{Z}_3$. We can give $X$ the structure of a $(t, s, r)$-birack by setting $t = 1$, $s = 2$ and $r = 2$ so we have $s^2 = 2^2 = 1 = (1 - 2(1))2 = (1 - tr)s$. Then $tr + s = 2 + 2 = 1$, so $X$ is a biquandle, and we have biquandle matrix

$$
M_X = \begin{bmatrix}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 2 & 1
\end{bmatrix}
$$

where $x_1 = 0, x_2 = 1$ and $x_3 = 2$. Our \textit{python} computations reveal 320 $\mathbb{Z}_B[X]$-module structures on $R = \mathbb{Z}_5$, including for instance

$$
M_R = \begin{bmatrix}
4 & 4 & 4 & 1 & 1 & 2 & 2 & 2 \\
1 & 2 & 2 & 1 & 4 & 3 & 4 & 4 & 4 \\
3 & 3 & 1 & 4 & 3 & 2 & 1 & 1 & 1
\end{bmatrix}
$$

The unknot has three $X$-labelings, one for each element of $X$, as is easily seen by considering the zero-crossing unknot diagram. It is also easy to see that for each $X$-labeling, the space of bead labelings by $R$ is one-dimensional, with a total of 5 bead labelings for each $X$-labeling. Thus, we have $\Phi^B_{X, R}(\text{Unknot}) = 3u^5$.

Now consider the virtual knot numbered 3.1 in the \textit{Knot Atlas} \cite{2}; it is the closure of the virtual braid diagram listed below. There are three classical crossings, all positive, and the bead and birack labels do not change at the virtual crossings. Thus, we have six semiarcs and six equations for each $X$-labeling. There are three $X$-labelings of 3.1, as listed in the table. Recall that in our matrix notation from Example 3 we have $x_1 = 0, x_2 = 1$ and $x_3 = 2$, so the first row gives the monochromatic $X$-labeling with all zeroes, while the two other $X$-labelings are nontrivial. For each $X$-labeling, we replace the coefficients in the matrix of
$\mathbb{Z}_f[X]$ with their values from $M_R$ and row-reduce to find the contribution to $\Phi_{X,R}^B(3.1)$.

The labeling of all semiarcs by $x_1 \in X$ has matrix

$$
M_{\mathbb{Z}_f[X]} = \begin{bmatrix}
0 & -1 & 0 & s_{u,v} & t_{u,v} & 0 \\
0 & 0 & -1 & r_{u,v} & 0 & 0 \\
-1 & 0 & s_{z,w} & 0 & 0 & t_{z,w} \\
0 & 0 & r_{z,w} & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & s_{v,w} & t_{v,w} \\
-1 & 0 & 0 & 0 & r_{v,w} & 0
\end{bmatrix}
$$

so there is only the zero-bead labeling and this $X$-labeling contributes $u^1$ to $\Phi_{X,R}^B(3.1)$. Similar computations reveal that the other $X$-labelings also contribute $u^1$, so we have $\Phi_{X,R}^B(3.1) = 3u$. Comparing this to the unknot, we have $\Phi_{X,R}^B(\text{Unknot}) = 3u^3 \neq 3u = \Phi_{X,R}^B(3.1)$, so this example shows that $\Phi_{X,R}^B$ detects the knottedness of $3.1$. In fact, this example shows much more: (1) since $\Phi_{X,R}^B(3.1) = 3 = \Phi_{X,R}^Z(\text{Unknot})$, this example shows that the birack module-enhanced invariant $\Phi_{X,R}^B$ is stronger than unenhanced counting invariant $\Phi_{X}^Z$, and (2) since $3.1$ and the unknot both have Jones polynomial equal to 1, this example shows that $\Phi_{X,R}^B$ is not determined by the Jones polynomial.

**Example 11** As we saw in example 6, the integral birack counting invariant with respect to the birack $X$ with matrix

$$
M_X = \begin{bmatrix}
2 & 2 & 2 & 2 \\
1 & 2 & 1 & 1
\end{bmatrix}
$$

has value $\Phi_{X}^Z(K) = 2$ for all classical knots $k$. However, the enhanced invariant $\Phi_{X,R}^B$ is effective at distinguishing knots and links. We randomly selected a $\mathbb{Z}[X]$-module structure on $\mathbb{Z}_3$ given by matrix

$$
M_R = \begin{bmatrix}
2 & 2 & 0 & 1 & 1 \\
1 & 2 & 0 & 2 & 2
\end{bmatrix}
$$

and used our python code to compute $\Phi_{X,R}^B$ for all prime knots with up to 8 crossings and all prime links.
with up to 7 crossings; the results are collected in the table below.

| \( \Phi^B_{X,R} \) | \( L \) |
|------------------|---------|
| \( 2u^4 \)       | Unknot, \( 4_1, 5_1, 5_2, 6_2, 6_3, 7_1, 7_2, 7_3, 7_4, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4, 8_6, 8_7, 8_8, 8_9, 8_{10}, 8_{11}, 8_{12}, 8_{13}, 8_{14}, 8_{16}, 8_{17} \) |
| \( 2u^9 \)       | \( 3_1, 6_1, 7_4, 7_7, 7_8, 8_10, 8_{11}, 8_{15}, 8_{19}, 8_{20}, 8_{21} \) |
| \( 2u^{27} \)    | \( 8_{18} \) |
| \( 2u^3 + 2u^9 \) | \( L_2a1, L_4a1, L_5a1, L_6a2, L_7a4, L_7a6 \) |
| \( 2u^3 + 2u^{27} \) | \( L_7a2, L_7a3, L_7n1, L_7n2 \) |
| \( 4u^9 \)       | \( L_6a1, L_6a3, L_7a1, L_7a5 \) |
| \( 2u^3 + 6u^{27} \) | \( L_6a4 \) |
| \( 2u^3 + 6u^9 \) | \( L_6n1, L_7a7 \) |
| \( 8u^9 \)       | \( L_6a5 \) |

**Example 12** For our final example, we note that our Python computations also show that for the same birack \( X \) from example 6 with the \( \mathbb{Z}[X] \)-module structure on \( \mathbb{Z}_3 \) given by the birack module matrix

\[
M_R = \begin{bmatrix}
1 & 2 & 0 & 2 & 1 & 1 \\
2 & 1 & 2 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\( \Phi^B_{X,R} \) distinguishes the knots \( 8_{18} \) and \( 9_{24} \) which both have Alexander polynomial

\[
\Delta(8_{18}) = -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3} = \Delta(9_{24})
\]

while \( \Phi^B_{X,R}(8_{18}) = 2u^{27} \neq 2u^9 = \Phi^B_{X,R}(9_{24}) \). In particular, \( \Phi^B_{X,R} \) is not determined by the Alexander polynomial. We note that this example also implies that \( \Phi^B_{X,R} \) is not determined by the generalized Alexander polynomial for virtual knots, since both \( 8_{18} \) and \( 9_{24} \) have generalized Alexander polynomial 0.

5 Questions

We end with a few open questions for future research.

Are there examples of biracks \( X \) and birack modules \( R \) whose \( \Phi^B_{X,R} \) invariants detect mutation or Homflypt-equivalent knots? Can \( \Phi^B_{X,R} \) distinguish knots and links with the same Khovanov or Knot Floer homology? What, if any, is the relationship between birack module invariants and Vassiliev invariants?

In [3], biquandle labelings are extended to define counting invariants of knotted surfaces in \( \mathbb{R}^4 \), and in particular it is shown that surface biquandles are just biquandles. Does the same hold for the surface biquandle algebra obtained by requiring bead-labeling invariance under Roseman moves?

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