Static vs Adaptive Strategies for Optimal Execution with Signals

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Abstract

We consider an optimal execution problem in which a trader is looking at a short-term price predictive signal while trading. In the case where the trader is creating an instantaneous market impact, we show that transactions costs resulting from the optimal adaptive strategy are substantially lower than the corresponding costs of the optimal static strategy. Later, we investigate the case where the trader is creating transient market impact. We show that strategies in which the trader is observing the signal a number of times during the trading period, can dramatically reduce the transaction costs and improve the performance of the optimal static strategy. These results answer a question which was raised by Brigo and Piat [6], by analyzing two cases where adaptive strategies can improve the performance of the execution.

1 Introduction

Market impact refers to the empirical fact that the execution of a large order affects the price of the underlying asset. Usually, this effect causes an unfavorable additional execution cost for the trader who is performing the exchange. As a result, a trader who wishes to minimize his trading costs has to split his order into a sequence of smaller orders which are executed over a finite time horizon. Academic efforts to reduce the transaction costs of large trades started with the seminal papers of Almgren and Chriss [2] and Bertsimas and Lo [5]. Both models deal with the trading process of one

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large market participant (for instance an asset manager or a bank) who would like to buy or sell a large amount of shares or contracts during a specified duration. The cost minimization problem takes into account market impact (see [11] and references therein) and therefore demands to trade slowly, or at least at a pace which takes into account the available liquidity. It is worth noticing that there are several types of market impact, including instantaneous, transient and permanent impact, and in this paper we will only consider instantaneous and transient impact. On the other hand, traders have an incentive to trade rapidly, because they do not want to carry the risk of an adverse price move far away from their decision price. The tradeoff between market impact and market risk is usually translated into a stochastic control problem where the trader’s strategy (i.e. the control) is the trading speed or the amount inventory liquidated at any time within the time horizon. Loosely speaking, the optimal strategy minimizes the risk-cost functional over a certain class of strategies.

Within the framework of optimal execution we usually distinct between two classes of trading strategies: static and adaptive. When seen from the initial time of the trade execution, static strategies are deterministic strategies that are completely decided at that time, based only on the information that is revealed to the trader at that initial time. Adaptive strategies are instead random when seen from the initial time, in that they will depend at each time point on the whole information that is available at that time. This models the fact that a trader will be able to react to new available information and adjust her strategy. Technically, adaptive strategies will be stochastic processes that are adapted to the relevant market information filtration in the given model. Clearly the class of static strategies is a subset of the class of adaptive strategies, therefore minimizing the cost functional over the class of adaptive strategies is expected to improve the results obtained when minimizing over the static class. In [6] this difference in the costs and in some cases risks was examined for two optimal trading frameworks: the discrete time Bertsimas and Lo model with an information signal and the continuous time Almgren and Chriss model that was studied by Gatheral and Schied in [9]. In both frameworks, the difference between the transaction costs resulting from the optimal adaptive strategies and the corresponding optimal static strategies were negligible, except in cases where one took unrealistic parameter values for either the asset dynamics or the market impact function. One of the main questions which was left open in [6], was whether there is any optimal trading framework in which the difference between the costs of adaptive vs static strategies will be considerable in a realistic setting. The main goal of this paper is to point out one such trading framework.

In [7], an optimal trading framework that incorporates signals (i.e. short term price predictors) into optimal trading problems was established. As we mentioned earlier, usually optimal execution problems focus on the tradeoff between market impact and market risk. In the simplest models we discussed above there is no continuous signal related to price predictors in the dynamics. However, in practice many traders and trading algorithms use short term price predictors. Most of such
documented predictors relate to orderbook dynamics \cite{10}. An example of such signal is the order book imbalance signal, measuring the imbalance of the current liquidity in the limit order book. This signal is computed by using the quantity of the best bid $Q_B$ and the best ask $Q_A$ of the order book,

$$\text{Imb}(\tau) = \frac{Q_B(\tau) - Q_A(\tau)}{Q_B(\tau) + Q_A(\tau)},$$

just before the occurrence of a transaction at time $\tau^+$. As argued in Section 4.2 of \cite{7}, the Ornstein–Uhlenbeck process is a good approximation for $\text{Imb}(\tau)$, hence we will assume such dynamics on the signal throughout this paper.

As in \cite{7} we will consider the following two types of market impact: instantaneous market impact and transient market impact with an exponential decay. In section 2 we compare the optimal static strategy to the optimal adaptive strategy in the case where the market impact is instantaneous. We first derive the static strategy in this setting. Then, we show that there is a significant improvement in the revenues when the agent trades with the optimal adaptive strategy, which was derived in Section 3 of \cite{7}.

In section 3 we consider the transient market impact case. The optimal static strategy in this case was derived Section 2.3 of \cite{7}, however, finding the optimal adaptive strategy remains an open problem. We propose a strategy which uses the value of the signal a few times during the trading window. This strategy, even though not necessarily optimal, increases the revenue of the agent significantly.

## 2 The instantaneous market impact case

In this section we define a model which incorporates a Markovian signal into the optimal trading framework with instantaneous market impact. Definitions and results from \cite{7} are used throughout this section.

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions, where $\mathcal{F}_0$ is trivial. Let $\tilde{W} = \{\tilde{W}_t\}_{t \geq 0}$ be a Brownian motion and $I = \{I_t\}_{t \geq 0}$ a homogeneous càdlàg Markov process satisfying,

$$E_{\epsilon}[|I_t|] \leq C(T)(1 + |\epsilon|), \quad \text{for all } \epsilon \in \mathbb{R}, \ 0 \leq t \leq T,$$

for some constant $C(T) > 0$, where $T$ is the final execution time. Here $E_{\epsilon}$ represents expectation conditioned on $I_0 = \epsilon$.

In our model $I$ represents a signal that is observed by the trader. We assume that the asset price process $P$, which is unaffected by trading transactions, is given by

$$P_t = P_0 + \int_0^t I_s ds + \sigma_P \tilde{W}_t,$$
hence the signal interacts with the price through the drift term, modeling the local trend of the price process. Here $\sigma_F$ is a positive constant modeling the price volatility.

Let $\mathcal{V}$ denote the class of progressively measurable control processes $r = \{r_t\}_{t \geq 0}$ for which $\int_0^T |r_t| \, dt < \infty$, $P$-a.s.

If $x \geq 0$ denotes the initial amount of inventory, we let

$$\dot{X}^r_t := x - \int_0^t r_s \, ds. \tag{2.3}$$

be the inventory trajectory with liquidation rate $r$; its marginal $X^r_t$ is the amount of inventory held by the trader at time $t$. We will often suppress the dependence of $X$ on $r$, to ease the notation. Note that $r_t = -\dot{X}_t$, namely the trader’s control is the trading speed. The price at which orders are executed is given by

$$S_t = P_t - \kappa r_t, \quad t \geq 0,$$

where $\kappa$ is a non-negative constant. This models the instantaneous linear market impact introduced in [3]. We observe that the affected price is impacted by the trading speed $r$, which is typically positive. Hence for positive $\kappa$ the impacted price $S$ will be smaller than the “mid” price $P$.

Finally, the investor’s cash $C_t$ is defined as follows

$$dC_t := -S_t dX^r_t = S_t r_t \, dt = (P_t - \kappa r_t) r_t \, dt,$$

with $C_0 = c$. Intuitively, $-S_t dX^r_t \approx S_{t+dt} (X^r_t - X^r_{t+dt})$ which is the revenue obtained from trading the inventory’s portion $X^r_t - X^r_{t+dt}$ at the affected price $S_{t+dt}$ in the time interval $[t, t + dt]$.

The purpose of the execution would be, ideally, to complete the order by time $T$ and have zero remaining inventory, $X_T = 0$. However, this is not always possible in practice. Therefore, as in Section 3 of [7], we add a penalty function $-\varrho X^2_T$ for the remaining inventory at time $T$ that has not been executed. Here $\varrho$ is a positive constant which is used to adjust the weight of penalty. Another ingredient in our optimal execution problem is the risk aversion term, which reflects the risk associated with holding a position $X_t$ at time $t$. A natural candidate is $\phi \int_0^T X^2_t \, dt$, where $\phi$ is a positive constant, see [11] [8] [13] and the discussion in Section 1.2 of [12]. When the value of $\phi$ is high, the trading speed tends to be higher at the beginning of the execution, i.e. the execution becomes more urgent. Finally, we add the term $P_T X_T$, which is the final value of the remaining inventory. The revenue-risk functional of the liquidation problem is

$$E_{\iota, c, x, p} \left[ C_T - \phi \int_0^T X^2_s ds + X_T (P_T - \varrho X_T) \right], \tag{2.4}$$

where $E_{\iota, c, x, p}$ represents expectation conditioned on $I_0 = \iota, C_0 = c, X_0 = x, P_0 = p$. 


We first formulate the optimal adapted solution relying on [7]. Introduce the following functions

\[ v_2(t) = \sqrt{\kappa \phi} \frac{1 + \zeta e^{2\beta(T-t)}}{1 - \zeta e^{2\beta(T-t)}}, \]

\[ v_1(t, \iota) = \int_t^T E[I_s|I_t = \iota] \exp \left( \frac{1}{\kappa} \int_t^s v_2(u) du \right) ds, \]

\[ v_0(t, \iota) = \frac{1}{4\kappa} \int_t^T E[v_1^2(s, I_s)|I_t = \iota] ds, \]

where the constants \( \zeta \) and \( \beta \) are given by

\[ \zeta = \frac{\varrho + \sqrt{\kappa \phi}}{\varrho - \sqrt{\kappa \phi}}, \quad \beta = \frac{\sqrt{\phi}}{\kappa}. \tag{2.5} \]

If \( \varrho \neq \sqrt{\kappa \phi} \), then the maximizer of the revenue functional in (2.4) exists, is unique and given by

\[ r^*_t = -\frac{1}{2\kappa} \left( 2v_2(t)X_t + \int_t^T e^{\frac{1}{\kappa} \int_t^s v_2(u) du} E[I_s|I_t] ds \right), \quad 0 \leq t \leq T, \]

where, for \( s > t \), \( E[I_s|I_t] \) is the expected value of \( I_s \) given \( I_t \). It is such reaction to the signal \( I_t \) that accounts for the adaptiveness of \( r^*_t \). The optimal revenue is given by \( c - xp + v_0(0, \iota) + xv_1(0, \iota) + x^2 v_2(0) \).

Assume further that the signal \( I \) follows an Ornstein-Uhlenbeck process,

\[ dI_t = -\gamma I_t dt + \sigma dW_t, \quad t \geq 0, \]

\[ I_0 = \iota, \tag{2.6} \]

where \( W \) is a standard Brownian motion independent of \( \tilde{W} \) and \( \gamma, \sigma > 0 \) are constants. This assumption is discussed and motivated in Section 4.2 of [7], where in particular the mean-reverting property of the order book imbalance is analysed. The parameter \( \gamma \), if positive, is the speed of mean reversion to zero for the signal starting at \( I_0 \). The parameter \( \sigma \) is the signal absolute volatility. Then, \( r^*_t \) has the form

\[ r^*_t = -\frac{1}{\kappa} v_2(t)X_t + \frac{1}{2\kappa} I_t \int_t^T \exp \left( -\gamma(s - t) + \frac{1}{\kappa} \int_t^s v_2(u) du \right) ds, \quad 0 \leq t \leq T. \]

**Remark 2.1.** One can impose a constraint on the admissible strategies to terminate without any inventory, that is to have \( X_T = 0 \). This constraint is often called a “fuel constraint” as the strategy is forced to terminate without any “fuel”. In our setting we could heuristically impose a fuel constraint on the strategy that maximizes (2.4) by
using the asymptotics of $r^*_t$ when $\varrho \to \infty$. In this case $\zeta \to 1$ and the limiting trading speed, which we denote by $r^f_t$, is

$$r^f_t = -\frac{1}{2\kappa} \left( 2\bar{v}_2(t)X_t + I_t \int_t^T e^{-\gamma(s-t)+\frac{1}{2} \int_t^s \bar{v}_2(u)du} ds \right), \quad 0 \leq t \leq T; \quad (2.7)$$

where

$$\bar{v}_2(t) = \sqrt{\kappa\phi \left( 1 + e^{2\beta(T-t)} \right) \left( 1 - e^{2\beta(T-t)} \right)}.$$

We now solve the static optimization under a fuel constraint. If $x$ denotes the quantity of asset to be liquidated, this means that the admissible strategies are those in the set

$$\mathcal{V}_S(x) = \left\{ r : \int_0^T |r_s| ds < \infty \text{ and } X^*_0 - X^*_T = \int_0^T r_s ds = x \right\}.$$ 

Notice that $\mathcal{V}_S$ is a subset of $\mathcal{V}$. As a consequence of such choice, the revenues functional will no longer have the penalisation on the inventory left after trading, and it will be defined as

$$E_{c,c,x,p} \left[ C_T - \phi \int_0^T X^2_s ds \right]. \quad (2.8)$$

In the following Theorem, we derive a necessary and sufficient condition to the maximiser of $(2.8)$ over the class of admissible strategies $\mathcal{V}_S(x)$.

**Theorem 2.2.** $r^*$ maximizes the revenue functional $(2.8)$ over $\mathcal{V}_S(x)$, if and only if there exists a constant $\lambda$ such that $r^*$ solves

$$2kr^*_t + 2\phi \int_0^t X^*_s ds - \int_0^t E[I_s] ds = \lambda, \quad \text{for all } 0 \leq t \leq T; \quad (2.9)$$

where $X^*_t = x - \int_0^t r^*_s ds$.

Recall that $\beta$ was defined in (2.5). From Theorem 2.2 we can easily deduce the following corollary.

**Corollary 2.3.** Assume that $I$ follows an OU-process as in (2.6). Then, the optimal static inventory $X^* := X^r$ is given by

$$X^*_t = x\psi(t) + \varphi(t), \quad (2.10)$$

where $\psi(t) = \frac{\sinh(\beta(T-t))}{\sinh(\beta T)}$ and

$$\varphi(t) = \frac{I_0}{2\kappa(\beta^2 - \gamma^2)} \left( 1 - \frac{e^{-\gamma(T-t)} \sinh(\beta t) + e^{\gamma t} \sinh(\beta(T-t))}{\sinh(\beta T)} \right). \quad (2.11)$$
In Figure 1 we present the optimal static inventory $X^*$ in (2.10) for the parameters: $\gamma = 0.1, \sigma = 0.1, T = 10, \kappa = 0.5, \phi = 0.1, X_0 = 10$. The influence of the initial value of the signal on the optimal strategy is demonstrated for $I_0 = 0.5, I_0 = 0$ and $I_0 = -0.5$. Since $I$ represents the local trend of the price $P$, we are assuming quite significant trends of 50% and −50%. Typical values of the signal which may initiate trading for static strategies appear in Fig. 4.2 and Fig 4.6 top left in [7] and 50% is in this range. In later examples we will adopt ±20%.

The reminder of this section is dedicated to a comparison between the signal adaptive strategy $r^f$ in (2.7) and the optimal static strategy $X^*$ from (2.10), and the comparison of their corresponding revenues. In Figure 2 (blue region) we simulate 1000 trajectories of the inventory $X^r$ resulting from $r^f$. In the black curve we present the optimal static inventory from (2.10). For the signal process $I$ parameters and the execution problem impact and boundary conditions we assume the following values:

$$\gamma = 0.1, \sigma = 0.1, I_0 = 0.2, T = 10, \kappa = 0.5, \phi = 0.1, X_0 = 10. \quad (2.12)$$

The parameters of the model are similar to the parameters of Figure 1 with the addition of $I_0 = 0.2$. We notice that even though the strategies start and end with the same innovatory values, the changes in the trading speed during $(0, T)$ can be substantial.

In Figure 3 (left) we compare the revenues resulting from the optimal static strategy (2.10) in blue, and the signal adaptive strategy (2.7) in orange. The revenues are plotted for different values of trading windows $T$ from 5 to 50. We observe that as the trading window increases, the difference in the expected revenues of the strategies increases drastically. In Figure 3 (right) we compare the revenues for different values of signal volatility $\sigma$. The model parameters (except form $\sigma$) are similar to the left plot. We observe that a signal with a large volatility will create a major difference between the revenues of the static and adaptive strategies.
Figure 1: Plot of the optimal static inventory $X^*$ in (2.10), for the parameters in (2.12) except for $I_0$. The optimal static strategy is presented for different initial values of the signal: $I_0 = 0.5$ (orange), $I_0 = 0$ (green) and $I_0 = -0.5$ (blue).

Figure 2: Simulation of the inventory $X^{r_f}$ resulting from the signal adapted trading speed $r_f$ in (2.7). The blue region is a plot of 1000 such trajectories of $X^{r_f}$. In the black curve we present the optimal static inventory (2.10). The parameters of the model are as in (2.12).
3 The transient market impact case

In this section consider the case where the market impact is exponentially decaying as in the Obizhaeva and Wang model \[11\]. The actual price process in this model is given by

\[ S_t = P_t + \kappa \rho \int_{\{s<t\}} e^{-\rho(t-s)} dX_s, \quad t \geq 0, \tag{3.1} \]

where \( P \) and \( I \) are given as in (2.2) and (2.6), receptively, and \( \kappa, \rho \) are positive constants. In this context we say that the inventory \( X \) is an admissible strategy, if it satisfies:

(i) \( t \rightarrow X_t \) is left–continuous and adapted.

(ii) \( t \rightarrow X_t \) has \( \mathbb{P}\)-a.s. bounded total variation.

(iii) \( X_0 = x \) and \( X_t = 0, \mathbb{P}\)-a.s. for all \( t > T \).

For the sake of readability we will assume that the risk-aversion constant \( \phi = 0 \). It was shown in Section 2.1 of \[7\] that the revenue functional which corresponds to an admissible strategy \( X \) is given by

\[ P_0x - E\left[ \int_0^t I_s \, ds \, dX_t + \frac{\kappa \rho}{2} \int \int \rho e^{-|t-s|} dX_s dX_t \right]. \tag{3.2} \]
The class of static strategies in this case is defined as follows,

\[ \Xi_S(x) = \{ X | \text{deterministic admissible strategy with } X_0 = x \text{ and support in } [0, T] \}. \]

In Corollary 2.7 of \cite{7} the unique static strategy \( X^* \) which maximises the revenue functional (3.2) was derived,

\[ X^*_t = (1 - b_0(t)) \cdot x + \frac{\lambda}{2K \rho^2 \gamma} \left\{ \frac{\rho^2 - \gamma^2}{\gamma} \cdot b_1(t) - (\rho + \gamma) \cdot b_2(t) - (\rho + \gamma) \cdot b_3(t) \right\}, \quad (3.3) \]

where

\[ b_0(t) = \frac{1_{\{t>0\}} + 1_{\{t>T\}} + \rho t}{2 + \rho T}, \]
\[ b_1(t) = 1 - e^{-\gamma t} - b_0(t)(1 - e^{-\gamma T}), \]
\[ b_2(t) = 1_{\{t>0\}} + \rho t - b_0(t)(1 + \rho T), \]
\[ b_3(t) = (b_0(t) - 1_{\{t>T\}})e^{-\gamma T}. \]

The optimal adaptive strategy for this model is an open problem (see Remark 2.9 in \cite{7}). Note that \( X^*_t \) has jumps at \( t = 0 \) and \( t = T \) and is continuous for \( 0 < t < T \). Moreover, \( X_t \) is a function of the initial signal value \( \iota \), initial inventory \( x \), initial time (which is set to 0 in (3.3)) and the terminal time \( T \). In what follows we will write \( X^*_t(I_s, x, s, T) \), for the optimal static strategy which starts at time \( 0 \leq s \leq T \) when the signal value is \( I_s \), the inventory held the trader at the initial time \( s \) is \( x \), and it terminates at time \( T \) (with \( X_T = 0 \)).

We will now propose a dynamic strategy which improves the results of the optimal static strategy \( X^* \). This new strategy \( \tilde{X}^{(n)}_t \), allows the agent to update the trading strategy at \( n - 1 \) intermediate times according to the new information available at these times. To formalise this we choose \( n \geq 1 \) and define a grid on \( [0, T] \) such that \( t_k = \frac{kT}{n}, \ \text{for } k = 0, \ldots, n \). We also define

\[ \tilde{X}^{(n)}_t = \begin{cases} X_0, & \text{if } t = 0, \\ X^*_t(I_{t_k-1}, \tilde{X}^{(n)}_{t_k-1}, t_{k-1}, T), & \text{if } t_{k-1} < t \leq t_k, \ \text{for } k = 1, \ldots, n. \end{cases} \quad (3.4) \]

Note that \( X^* = \tilde{X}^{(1)} \).

**Remark 3.1.** We remark at this point that it is not a-priori trivial that the revenue which is associated with \( \tilde{X}^{(n)}_t \) \( n \geq 2 \) is larger than the revenue of \( X^* \). Since the market impact is transient and does not vanish immediately, a trader who updates his strategy at time \( T/2 \) for example according to \( X^*_t(I_{T/2}, \tilde{X}^{(2)}_{T/2}, T/2, T) \), does not take into account the market impact which is caused by his strategy on the interval \([0, T/2]\), hence his strategy may be suboptimal (see Remark 2.9 in \cite{7} for detailed discussion).
Figure 4: Left: simulation of 50 trajectories of \( \tilde{X}^{(2)} \) from (3.4), where the update takes place at \( t = 5 \) (blue curves). The black curve presents the optimal static strategy \( X^* \) from (3.3). Right: Monte-Carlo simulations of the revenue functional (3.2) which corresponds to \( \tilde{X}_{t}^{(n)} \), for \( n = 1 \) (blue), \( n=2 \) (orange) and \( n = 3 \) (green). The parameters in both graphs are \( \gamma = 0.1, \sigma = 0.1, I_0 = 0.2, T = 10, \rho = 1, \kappa = 0.5, X_0 = 10 \) and \( P_0 = 10 \).

In Figure 3 we compare \( \tilde{X}^{(n)} \) with the optimal static strategy \( X^* \). On the left panel, in the blue curves, we plot 50 trajectories of \( \tilde{X}^{(2)} \) where the update takes place at \( t = 5 \). The black curve presents the optimal static strategy \( X^* \) from (3.3). One can observe that \( \tilde{X}^{(2)} \) has an additional jump at \( T/2 \) which is caused by the update of the strategy. On the right panel we show the results of Monte-Carlo simulations for the revenue functional (3.2) which corresponds to \( \tilde{X}_{t}^{(n)} \), for \( n = 1 \) (blue), \( n=2 \) (orange) and \( n = 3 \) (green). Note that the case where \( n = 1 \) is the static case. The graph shows the convergence of the expected revenue (y-axis) as a function of the number of trajectories \( N \) (x-axis) in the simulation. We observe that an increasing number of signals updates during the trading window improves the results of the execution, as the revenue functional increases. The parameters in both graphs are \( \gamma = 0.1, \sigma = 0.1, I_0 = 0.2, T = 10, \rho = 1, \kappa = 0.5, X_0 = 10 \) and \( P_0 = 10 \).

4 Conclusions and further research

In this work we considered a realistic model of execution with instantaneous and transient impact where there is a signal in the price process and where we are concerned with minimizing the cost and risk of the trade execution. The model had been proposed and studied in Lehalle and Neuman [7]. We studied the differences between optimal static strategies and optimal adaptive ones in this model, finding realistic cases where switching from static to adaptive improves considerably the revenues associated with the trade execution. This improves on the previous results of Brigo and
Piat [6], where a similar analysis had been run on the benchmark models of Bertsimas and Lo with information signal [5] and of Gatheral and Schied [9] after Almgren and Chriss [2]. Our conclusion is that there are settings where switching from static to adaptive strategies can pay off, even though the latter are much more complex to derive and study in general. In future research the static-adaptive comparison could be extended to broader classes of models.

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A Proofs

**Proof of Theorem 2.2** We will first prove the uniqueness of the optimal strategy. Let \( x > 0 \). For any \( r \in \mathcal{V}_S(x) \) define

\[
C(r) := C_1(r) + C_2(r) - K(r),
\]

where

\[
C_1(r) = \kappa \int_0^T r_s^2 ds, \quad C_2(r) = \phi \int_0^T X_t^2 dt, \quad K(r) = \int_0^T \int_0^t E[i_s] ds r_t dt.
\]

Note that \( C(x) \) is the revenue functional in (2.8) with a minus sign. From the fuel constraint and since \( x > 0 \) we have

\[
C_1(r) > 0, \quad C_2(r) > 0.
\]

Let \( r, v \in \mathcal{V}_S(x) \). We define the following cross functionals,

\[
C_1(r, v) = \kappa \int_0^T r_s v_s ds, \quad C_2(r, v) = \phi \int_0^T \int_0^t r_s v_s ds dt.
\]

Note that

\[
C_i(r, v) = C_i(v, r), \quad \text{for } i = 1, 2,
\]

and

\[
C_i(v - r) = C_i(v) + C_i(r) - 2C_i(v, r), \quad \text{for } i = 1, 2.
\]

We now can repeat the same steps as in the proof of Theorem 2.3 in [11] and argue that \( C(\cdot) \) is strictly convex to obtain existance of at most one minimizer to \( C(\cdot) \) in \( \mathcal{V}_S(x) \).

We now show that condition (2.9) is sufficient for optimality. Assuming that \( r^* \in \mathcal{V}_S(x) \) satisfy (2.9), we will show that \( r^* \) minimizes \( C(\cdot) \). Let \( r \) be any other
strategy in \( V_s(x) \). Define \( v = r - r^* \) and note that from the fuel constraint it follows that \( X_0^v = X_0^{r^*} = 0 \). We have
\[
C(r) = C(v + r^*)
\]
\[
= C_1(r^*) + C_1(v) + C_1(r^*) + C_1(v) + 2C_1(r^*, v)
\]
\[
+ C_2(r^*) + C_2(v) + 2C_2(r^*, v)
\]
\[
- K(r^*) - K(v)
\]
\[
= C(r^*) + C_1(v) + C_2(v) - K(v) + 2C_1(r^*, v) + 2C_2(r^*, v).
\]

Since \( C_i(\cdot) \geq 0, i = 1, 2 \), it follows that in order to prove the optimality of \( r^* \) we need to show that
\[
\ell(r^*, v) := 2C_1(r^*, v) + 2C_2(r^*, v) - K(v) \geq 0.
\]

Use (2.9) to get
\[
\ell(r^*, v) = 2k \int_0^T r^*_t v_t dt + 2\phi \int_0^T X_t^v X_t^{r^*} dt - \int_0^T \int_0^t E[I_s] ds v_t dt
\]
\[
= \lambda \int_0^T v_t dt - 2\phi \int_0^T \int_0^t X_s^{r^*} ds v_t dt + 2\phi \int_0^T X_t^v X_t^{r^*} dt.
\]

From integration by parts we have
\[
\int_0^T \int_0^t X_s^{r^*} ds v_t dt - \int_0^T X_t^v X_t^{r^*} dt = 0.
\]

From the fuel constraint it follows that \( \int_0^T v_t dt = 0 \), and therefore \( \ell(r^*, v) = 0 \).

**Proof of Corollary 2.3** In this case we have \( E[I] = \nu e^{-\lambda t} \). Assume a twice differentiable \( r_t \) and differentiate both sides of (2.9) to get
\[
- 2k\ddot{X}_t + 2\phi X_s - \nu e^{-\lambda t} = 0,
\]
for all \( 0 < t < T \), \( (A.5) \)

with the initial and terminal conditions \( X(0) = x \) and \( X(T) = 0 \). The solution to \( (A.5) \) is (2.10). By Theorem 2.2 this is the unique minimizer of (2.4).