Improving the bidirectional steerability between two accelerated partners via filtering process

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Abstract

The bidirectional steering between two accelerated partners sharing initially different classes of entangled states is discussed. Due to the decoherence, the steerability and its degree decrease either as the acceleration increases or the partners share initially a small amount of quantum correlations. The possibility of increasing the steerability is investigated by applying the filtering process. Our results show that by increasing the filtering strength, one can improve the upper bounds of the steerability and the range of acceleration at which the steerability is possible. Steering large coherent states is much better than steering less coherent ones.

Keywords: Steering; Accelerated system; Steerability; Decoherence.

1 Introduction

The concept of Einstein–Podolsky–Rosen (EPR) steering is suggested by Schrödinger in 1935 to discuss the EPR paradox\textsuperscript{1,2}. EPR steering is a significant quantum phenomenon in which a quantum state can be non-locally changed or steered another state remotely by performing some local measurements\textsuperscript{3}. Besides, the violation of EPR steering violation has been set as a measure of quantum correlations, which is laid out between the non-separability and Bell non-locality\textsuperscript{4,5}. Several experimental studies deal with interpreting the dynamics of quantum steering, such as; EPR steering game employing an all-versus-nothing criterion that depends on obtaining two different pure normalized conditional states has been explored\textsuperscript{6}. Measurement-device-independent steering protocols for optical polarization qubits and states that do not violate Bell inequality have been demonstrated\textsuperscript{7}. By polarizing the photons in a linear-optical setup, the temporal quantum steering has been applied for testing and securing the quantum communications\textsuperscript{8}. The convex steering monotone and measure of steerability depend on the observed statistics and the quantum inputs have been discovered\textsuperscript{9}. Quantum steering non-locality of high-dimensional quantum systems with isotropic noise fraction has been retrieved\textsuperscript{10}. Theoretically, the steering inequality and steerability have been reconstructed by using different relations as Heisenberg uncertainty principle\textsuperscript{11}, the standard geometric Bell inequalities\textsuperscript{12,13}, a version of the

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Clauser-Horne-Shimony-Holt inequality \cite{14,15}, fine-grained uncertainty relation \cite{16}, Abner Shimony inequalities \cite{17}, and steering witnesses \cite{18}. The violating of EPR inequalities indicates the possibility of steering process \cite{19,20}.

Due to the importance of steering in determining the quantum correlation, it has been used in many practical applications of quantum informatics. For example, it has been applied to investigate the security and feasibility of a one-sided device of the standard quantum key distribution \cite{21}. It also has been employed in quantum computation \cite{22} and securing quantum teleportation \cite{23}. Moreover, quantum steering has been discussed for some quantum systems such as, bipartite two-qubit X-state \cite{24,25}, Heisenberg chain models \cite{26}, two-level or three-level detectors \cite{27}. As quantum systems inevitably interact with the different environments, it is necessary to discuss the influence of these environments on the steering process. For instance, the effect of asymmetric dissipation relativistic motion \cite{28}, finite temperature \cite{29}, non-Markovian environment on the steering \cite{30}, and noise channel \cite{31} have been discussed. The steering entropic uncertainty of the qutrit system under amplitude damping decoherence has been examined \cite{32}.

In addition, the possibility of improving of quantum correlations via local filtering operation has been widely investigated, such as, improving the steering and nonlocality in Heisenberg XY mode \cite{33}, also protecting entanglement from the decoherence due to amplitude damping and acceleration \cite{34,35}. On the other hand, investigating the quantum correlation in an accelerated frame is one of the considerable areas in theoretical quantum processing, where the actual quantum systems are essentially non-inertial \cite{36,37}. The acceleration process may cause a dissipation of quantum correlations between an inertial observer and an accelerated one \cite{38}. Under the non-inertial frame, the entanglement of a general two-qubit system has been studied \cite{39}. The classicality and quantumness under different decoherence noisy channels in accelerated frame were investigated in \cite{40}. The general behaviour of EPR steering inequality and steerable between two users have been investigated, where the first in flat space-time and another in the non-inertial frame \cite{41}.

Our motivation in this work is to derive a generic version of the steering inequality for two quantum subsystems with two dimensions (two-qubit). We then apply the new inequality on a general form of a two-qubit system, as it is possible to generate some special classes of them \cite{42}. We assume that the actual quantum regimes are essentially in the non-inertial frame, where the relativistic framework is taken into account, whether we accelerate only one subsystem or accelerate the two subsystems collectively.

The paper is organized as follows: In Sec.\(2\), we derive a general form of steering inequality for a bipartite qubit system. In Sec.\(3\), we are defined the filtering process of the two-qubit state to improve the efficiency of shared entangled state. In Sec.\(4\) the EPR steering under the acceleration process of the generalized Werner state and the generic pure state has proposed. Additionally, the degree of steerable is enhanced via using the local filtering operation. Bidirectional steerable for the generic pure state from Alice to Bob is studied in Sec.\(5\), where Alice’s particle is accelerated and
Bob’s particle is at rest. Finally, we conclude our results in Sec.(6).

2 Entropic Steering:

In this section, we derive a generalized form of quantum steering for a bipartite system. According to the definition in [11], the entropic uncertainty steering inequality for the discrete observable in the even N-dimensional Hilbert space is defined as,

$$\sum_{i=1}^{N+1} H(R_i) \geq \frac{N}{2} \log_2 \left( \frac{N}{2} \right) + (1 + \frac{N}{2}) \log_2 (1 + \frac{N}{2}),$$  \hspace{1cm} (1)

where $H(R_i) = H(\rho_{ab}) - H(\rho_a)$ is the conditional Shannon entropy for the subsystems $a$ and $b$. By applying the Pauli spin operator ($\sigma_x, \sigma_y, \sigma_z$) as measurements, the EPR inequality of steering from $A$ to $B$ is given by:

$$\mathcal{I}_{ab} = H(\sigma_x^{(b)} | \sigma_x^{(a)}) + H(\sigma_y^{(b)} | \sigma_y^{(a)}) + H(\sigma_z^{(b)} | \sigma_z^{(a)}) \geq 2.$$  \hspace{1cm} (2)

In the computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, the general form of an arbitrary two-qubit system may be written as:

$$\hat{\rho}_{ab} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12}^* & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23}^* & \rho_{33} & \rho_{34} \\ \rho_{14}^* & \rho_{24}^* & \rho_{34}^* & \rho_{44} \end{pmatrix},$$  \hspace{1cm} (3)

By using Pauli matrices $\sigma_x, \sigma_y$, and $\sigma_z$ measurements, one can simplify the expression of EPR steering inequality in Eq.(2) reads,

$$\mathcal{I}_{ab} = \frac{1}{2} \sum_{i=1}^{4} \left\{ P_{x_i}^{ab} \log_2 [P_{x_i}^{ab}] + P_{y_i}^{ab} \log_2 [P_{y_i}^{ab}] + P_{z_i}^{ab} \log_2 [P_{z_i}^{ab}] \right\}$$

$$- \sum_{i=1}^{4} \left\{ P_{x_i}^{a} \log_2 [P_{x_i}^{a}] + P_{y_i}^{a} \log_2 [P_{y_i}^{a}] + P_{z_i}^{a} \log_2 [P_{z_i}^{a}] \right\},$$  \hspace{1cm} (4)

where $P_{x_i}^{ab}$, $P_{y_i}^{ab}$, and $P_{z_i}^{ab}$ represent the eigenvalues of the density operator $\hat{\rho}_{ab}$. Explicitly, they are given by

$$P_{x1}^{ab} = (1 + 2 \Re[\rho_{12} + \rho_{13} + \rho_{14} + \rho_{23} + \rho_{24} + \rho_{34}]),$$

$$P_{x2}^{ab} = (1 - 2 \Re[\rho_{12} + \rho_{13} + \rho_{14} - \rho_{23} + \rho_{24} + \rho_{34}]),$$

$$P_{x3}^{ab} = (1 - 2 \Re[\rho_{12} - \rho_{13} + \rho_{14} + \rho_{23} - \rho_{24} + \rho_{34}]),$$

$$P_{x4}^{ab} = (1 + 2 \Re[\rho_{12} - \rho_{13} + \rho_{14} - \rho_{23} - \rho_{24} + \rho_{34}]),$$

$$P_{y1,y2}^{ab} = (1 + 2 \Re[\rho_{23} - \rho_{14}] \pm 2 \Im[\rho_{12} + \rho_{13} + \rho_{24} + \rho_{34}]),$$

$$P_{y3,y4}^{ab} = (1 - 2 \Re[\rho_{23} - \rho_{14}] \pm 2 \Im[\rho_{12} - \rho_{13} - \rho_{24} + \rho_{34}]),$$

$$P_{z1}^{ab} = 4 \rho_{ii}.$$
Likewise, $P^a_{x_1}, P^a_{y_1},$ and $P^a_{z_1}$ are the eigenvalues of the reduced density operator $\hat{\rho}_A$, where,

$$
P^a_{x_1,x_2} = (1 \pm 2 \text{Re}(\rho_{13} + \rho_{24})), \quad P^a_{y_1,y_2} = (1 \pm 2 \text{Im}(\rho_{13} + \rho_{24})), \quad \text{and} \quad P^a_{z_1,z_2} = (1 \pm (\rho_{11} + \rho_{22} - \rho_{33} - \rho_{44})).
$$

(6)

The degree of steerability is quantified based on Alice’s measurements as follows [22],

$$
S^{A\rightarrow B} = \max\left\{0, \frac{I_{ab} - 2}{I_{\text{max}} - 2}\right\},
$$

(7)

where $I_{\text{max}}=6$ is calculated for a system initially prepared in Bell states. The factor $(I_{\text{max}} - 2)$ is introduced to ensure that the steering process is normalized. By exchanging the roles of A and B, the possibility of the steering by performing measurements on the subsystem B is given by,

$$
S^{B\rightarrow A} = \max\left\{0, \frac{I_{ba} - 2}{I_{\text{max}} - 2}\right\},
$$

(8)

where $I_{ba}$ quantifies the steering from Bob to Alice, it is given by, 

$$
I_{ba} = \frac{1}{2} \sum_{i=1}^{4} \left\{ P^b_{x_i} \log_2[P^b_{x_i}] + P^b_{y_i} \log_2[P^b_{y_i}] + P^b_{z_i} \log_2[P^b_{z_i}] \right\}
$$

$$
- \sum_{i=1}^{2} \left\{ P^b_{x_i} \log_2[P^b_{x_i}] + P^b_{y_i} \log_2[P^b_{y_i}] + P^b_{z_i} \log_2[P^b_{z_i}] \right\},
$$

(9)

with, 

$$
P^b_{x_1,x_2} = (1 \pm 2 \text{Re}(\rho_{12} + \rho_{34})), \quad P^b_{y_1,y_2} = (1 \pm 2 \text{Im}(\rho_{12} + \rho_{34})), \quad P^b_{z_1,z_2} = (1 \pm (\rho_{11} - \rho_{22} + \rho_{33} - \rho_{44})).
$$

(10)

Hereinafter, we shall discuss the Unruh effect on the steering process of the system $\hat{\rho}_{ab}$ and the possibility of improving this process via local filter operations.

3 Filter Process:

Filtering process is a method that may be used to improve the efficiency of a shared entangled state between two users Alice and Bob to perform some quantum information tasks. As we shall see in Sec.4, the acceleration process decreases the efficiency of steerability, therefore the users need to apply the filtering process on the accelerated shared state to improve its efficiency. The final output filtered state $\hat{\rho}^F_{ab}$ is given by [43],

$$
\hat{\rho}_{ab}^F = \frac{1}{\mathcal{N}} \mathcal{W}_a \mathcal{W}_b \cdot \hat{\rho}_{ab} \cdot (\mathcal{W}_a \mathcal{W}_b)^\dagger,
$$

(11)
The transformation (15) defines two regions in Rindler’s space: the first region \( I \) for \( \zeta < -\infty \) and the second region \( II \) for \( x < -|t| \).

A single mode \( k \) of fermions and anti-fermions in Minkowski space is described by the annihilation operators \( a_k \) and \( b_{-k} \) respectively, where \( a_k|0_k\rangle = 0 \) and \( b^\dagger_{-k}|0_k\rangle = 0 \).
Based on the approximation by Bruschi et al. [36,44], the relations between Minkowski and Rindler’s operators are given by Bogoliubov transformation,

\[ a_k = \cos r c_k^{(I)} - \exp(-i\phi) \sin r d_{-k}^{(II)}, \quad b_{-k}^\dagger = \exp(i\phi) \sin r c_k^{(I)} + \cos r d_{-k}^{(II)}, \]

where \( c_k^{(I)}, d_{-k}^{(II)} \) are the annihilation operators of Rindler’s space in the regions I (for fermions) and II (for anti-fermions) respectively, \( \tan\theta = e^{-\pi\omega/4} \), \( 0 \leq r \leq \pi/4 \), \( a \) is the acceleration such that \( 0 \leq a \leq \infty \), \( \omega \) is the frequency of the travelling qubits, \( c \) is the speed of light and \( \phi \) is an unimportant phase that can be absorbed into the definition of the operators [45,46]. The transformation (16) mixes a particle in first region I and an anti particle in the second region II [47]. In terms of Rindler’s modes, the Minkowski vacuum \( |0_k\rangle_M \) and the one particle state \( |1_k\rangle_M \) take the forms,

\[
|0_k\rangle_M = \cos r |0_k\rangle_I |0_{-k}\rangle_{II} + \sin r |1_k\rangle_I |1_{-k}\rangle_{II}, \\
|1_k\rangle_M = a_k^\dagger |0_k\rangle_M = |1_k\rangle_I |0_k\rangle_{II}. \tag{17}
\]

Then, by using these transformations and after tracing out the degree of freedom in the second region II, the initial state [3] may be written as,

\[
\hat{\rho}_a^{acc} = \begin{pmatrix}
C_a^2 C_b^2 \rho_{11} & C_a^2 C_b^{2} \rho_{12} & C_a C_b \rho_{13} & C_a C_b \rho_{14} \\
C_a^2 C_b \rho_{12} & C_a^2 (\rho_{22} + S_b^2 \rho_{11}) & C_a C_b \rho_{23} & C_a (\rho_{24} + S_b^2 \rho_{13}) \\
C_a C_b \rho_{13} & C_a C_b \rho_{23} & C_b^2 (\rho_{33} + S_a^2 \rho_{12}) & C_b (\rho_{34} + S_a^2 \rho_{14}) \\
C_a C_b \rho_{14} & C_a (\rho_{24} + S_b^2 \rho_{13}) & C_b (\rho_{34} + S_a^2 \rho_{12}) & C_b^2 (\rho_{22} + S_b^2 \rho_{11}) + S_b^2 \rho_{33} + \rho_{44}
\end{pmatrix},
\]

where \( C_i = \cos r_i, \) and \( S_i = \sin r_i, \) \( i = a, b, \)

Now, we are ready to investigate the steerability process between Alice and Bob who share the accelerated state. This idea will be clarified by using two different initial states namely; a generalized Werner state and a generic pure state.

4.1 Generalized Werner state:

This state is obtained by setting \( \vec{r} = \vec{0} = \vec{s}, \) and the cross dyadic \( C \) is defined by a diagonalize \( 3 \times 3 \) matrix, where \( \{c_{ij}\} = \text{diag}\{c_{11}, c_{22}, c_{33}\}. \) Then, Eq. (14) reduces to be;

\[
\hat{\rho}_{ab} = \frac{1}{4} \left( I_2^{(a)} \otimes I_2^{(b)} + \sum_{i,j} c_{ij} \sigma_i \otimes \tau_j \right), \quad i, j = x, y, z. \tag{19}
\]

Hence, the output accelerated system according to Eq.(18) is given by,

\[
\hat{\rho}_{a}^{acc} = C_a^2 C_b^2 A_{11} |00\rangle\langle 00| + C_a^2 (A_{22} + S_b^2 A_{11}) |01\rangle\langle 01| \\
+ C_b^2 (A_{22} + S_a^2 A_{11}) |10\rangle\langle 10| \\
+ (S_a^2 (A_{22} + S_b^2 A_{11}) + S_b^2 A_{22} + A_{11}) |11\rangle\langle 11| \\
+ C_a C_b \{A_{14} |00\rangle\langle 11| + A_{23} |01\rangle\langle 10| + h.c.\}, \tag{20}
\]
where,
\[
A_{11} = \frac{1 + c_{33}}{4}, \quad A_{22} = \frac{1 - c_{33}}{4}, \quad A_{14} = \frac{c_{11} + c_{22}}{4}, \quad A_{23} = \frac{c_{11} - c_{22}}{4},
\]
(21)

Figure 1: Steering from Alice to Bob with \(c_{11} = c_{22} = c_{33} = -1\) (a) \(\alpha = 0.1\), (b) \(\alpha = 0.4\), (c)\(\alpha = 0.7\). (d,e,f) the same as (a,b,c) respectively, but \(c_{11} = c_{22} = c_{33} = -0.8\).

In Fig. (1), the behavior of steerability is displayed, where we consider that the two qubit systems are initially prepared in the singlet state, namely we set \(c_{11} = c_{22} = c_{33} = -1\). Due to the symmetry, the behavior of \(I_{ab}\) is the same as \(I_{ba}\), thus we consider only \(I_{ab}\). In this discussion, it is assumed that, both qubits are accelerated. Fig. (1.(a)), shows the behavior of the steering inequality \(I_{ab}\) before applying the filtering process. The general behaviour shows that, the steerability is violated as the acceleration increases. However, it is well known that, the decoherence of the singlet state due to the acceleration is bounded, where the minimum entanglement is \(\sim 0.3\). Therefore the possibility that Alice steers Bob’s state is violated at \(r_a(r_b) > 0.4\). The effect of the filtering process on the steerability is displayed in Fig. (1(b), and Fig. (1(c)), where we set \(\alpha = 0.4\) and 0.7, respectively. The behavior of \(I_{ab}\) shows that, the steerability could be implemented at any value of the acceleration.

The effect of the initial state settings is displayed in Figs. (1(f)-(h)), where it is assumed that the users share Werner state, such that \(c_{11} = c_{22} = c_{33} = -0.8\). The behavior of the steerability is similar to that displayed for the singlet state, but it is violated at smaller values of the accelerations. As it is displayed from Fig. (1(f)), the inequality of steering is violated at \(r_a = r_b < 0.2\). By applying the filtering process, the steerability could be implemented at large values of the accelerations as it is displayed from Figs. (1(i)) and (1(j)), where the filtering strength is 0.4 and 0.7, respectively.
Figure 2: The same as Fig. (1), but for the degree of steerability $S_{A \rightarrow B}$.

The degree of steerability $S_{A \rightarrow B}$ that Alice can steer Bob’s qubit is displayed in Fig. (2), where it is assumed that, the steerer and the steered partner share a singlet state. The general behavior of $S_{A \rightarrow B}$ shows that, the degree of steerability decreases as the acceleration increases. In the absence of the filtering process, the degree of the steerability decreases fast, while when the filtering process is applied, the steerability decreases slowly as the acceleration of both qubits increases. As it is displayed from Fig. (2b) and Fig. (2c), where we set $\alpha = 0.4$, and 0.7, respectively, the degree of steerability that is depicted at $r_a = r_b = 0.6$ is given by $S_{A \rightarrow B} = 0.2$ and 0.7, respectively.

In Figs. (2d-2f) we quantify the degree of steerability $S_{A \rightarrow B}$, where the partners share a Werner state. It is clear that, its upper bounds are smaller than those displayed for the singlet state. The filtering process has the same effect, namely it increases as one increases the filtering strength $\alpha$.

4.2 Generic Pure state:

For this state, the coherence (Bloch) vectors of both qubits are equals, namely, $|\vec{r}| = p = |\vec{s}|$. The non-zero elements in the correlation matrix are defined by $c_{11} = -1$, and $c_{22} = -q = c_{33}$. In an explicit form, the state (14) may be written as:

$$
\hat{\rho}_{ab} = \frac{1}{4} \left( I_2^{(a)} \otimes I_2^{(b)} - \sigma_x \otimes \tau_x + p(\sigma_x \otimes I_2^{(b)} - I_2^{(a)} \otimes \tau_x) - q(\sigma_y \otimes \tau_y + \sigma_z \otimes \tau_z) \right),
$$

(22)
where, \( p = \sqrt{1 - q^2} \). After the acceleration process, the generic accelerated pure state is given by,

\[
\hat{\rho}_{\text{acc}}^{AB} = C_a^2 C_b^2 B_{11} |00\rangle \langle 00| + C_a^2 (B_{22} + S_b^2 B_{11}) |01\rangle \langle 01| + C_b^2 (B_{22} + S_a^2 B_{11}) |10\rangle \langle 10| +
(S_a^2 (B_{22} + S_b^2 B_{11}) + S_b^2 B_{22} + B_{11}) |11\rangle \langle 11| + C_a C_b \{B_{14} |00\rangle \langle 11| + B_{23} |01\rangle \langle 10| + B_{12} C_a (-|00\rangle \langle 01| + (1 + S_b^2) |01\rangle \langle 11|) + B_{12} C_b (|00\rangle \langle 10| - (1 + S_a^2) |10\rangle \langle 11|) + h.c.\},
\]

with,

\[
B_{11} = \frac{1 - q^2}{4} = -B_{14}, \quad B_{22} = \frac{1 + q^2}{4} = -B_{23}, \quad B_{12} = \frac{p^4}{4},
\]

(23)

Figure 3: quantum steering \( I_{ab} \) (green-solid curve), and \( I_{ba} \) (red-dot dash curve), (a) \( r_a = 0 = r_b \), (b) \( r_a = 0 \), \( r_b = 0 \), and (c) \( r_a = 0.5 \), \( r_b = 0.5 \).

The inequalities of steerability from Alice to Bob and vise versa, are displayed in Fig. (3), where they decrease gradually as \( p \) increases. Fig. (3a) shows the bidirectional steering’s inequalities when both the steerer and the steered are in the inertial frame, namely \( r_a = r_b = 0 \). It is clear that, the behavior of both inequalities coincides and decreases gradually as the parameter \( p \) increases. The steerability from both directions vanishes completely at \( p = 1 \), where the initial shared state turns into a separable state. As it is displayed from Fig. (3b), the steerability increases slightly if the steerer is accelerated and the steered particle is on the inertial frame, where \( I_{ab} \) satisfies the steerability criteria at large values of \( p \). The consistent behavior of both inequalities is displayed in Fig. (3c), where both users are accelerated with the same acceleration.

Fig. (4) shows the degree of steerability \( S_{A \rightarrow B} \), when the partners share a generic pure state. Different cases are considered, either both partners in the stationary frame, or only one partner or both partners qubits are accelerated. It is clear that, the steerability increases gradually to reach its maximum bound at \( p = 0 \), namely the shared state is a maximum entangled state. However, as \( p \) increases further, the steerability decreases gradually to vanish completely. As it is displayed in Fig. (4a), the largest upper bounds are reached when the partners’ qubits are in the stationary frame, while the smallest ones are depicted when both partners are accelerated. Figs. (4b) and (4c) display the effect of the filtering process on the degree of steerability. It is clear that, as one increases the filtering strength, the upper bounds of the steerability increase.
Figure 4: Steerability for generic pure state from Alice to Bob with zero acceleration (green curve), \( r_a = 0.3, r_b = 0 \) (red curve), \( r_a = 0.3, r_b = 0.3 \) (blue curve), under weak measurement, when we set \( q = \sqrt{1 - p^2} \). (a) \( \alpha = 0.1 \), (b) \( \alpha = 0.4 \), (c) \( \alpha = 0.8 \).

Figure 5: Steerability from Alice to Bob by using the generic pure state, when we set \( p = 0.5 \). (a) \( \alpha = 0.1 \), (b) \( \alpha = 0.4 \), (c) \( \alpha = 0.7 \).

The effect of the filtering process on an initial state of a generic pure state with \( p = 0.5 \) is displayed in Fig.(5), where it is assumed that, both qubits of the shared state are accelerated. In this situation, the decoherence arises only from the acceleration process, and consequently the degree of steerability decreases as the acceleration of both partners increases. However, as one switches on the filtering process, the possibility of achieving the steering process increases, namely the steering inequality is satisfied at large values of the acceleration. This phenomenon is displayed by comparing Figs.(5a), (5b), and (5c), where the steerable areas are enlarged as the filtering parameter \( \alpha \) increases. It is worth to mention that, the filtering process does not increase the degree of steerability, but increase the range of steerability.

To complete the discussion on the effect of the filtering process on the steerability and its degree, we consider several classes of initial state settings, where we plot the function \( S^{A \rightarrow B}(p, r_a, r_b = 0) \) in Fig.(6). In this case, there are two sources of decoherence: one due to the initial state settings, where the entanglement of the shared state decreases as \( p \) increases, and the second due to the acceleration process. Therefore, as it is expected, the degree of steerability decreases as one increases the decoherence parameter \( p \), and the acceleration \( r_a \). Figs.(6a-6c) show that, the filtering process has no effect on the upper bounds of \( S^{A \rightarrow B} \). The comparison of these figures shows that,
the range of steerability is enlarged as the filtering parameter increases. This means that, the steering inequality is satisfied at larger range of the acceleration.

5 Bidirectional steerability

In this section, we show that for the generic pure state the degree of steerability from Alice to Bob and vise versa, namely $S_{B\rightarrow A}$ depends on the type of steered information. Fig.7 shows the behavior of the bidirectional steerability when only Alice is accelerated and different initial state settings are considered. The general behavior shows that, the degree of steerability decreases as the acceleration increases. However, as it is displayed in Fig.7(a), if the partners share a maximum entangled state (singlet), both degrees of steerability coincide at small values of the acceleration, i.e. $r_a \in [0,0.4]$. However, at larger values of $r_a$, the degree of steerability from Bob to Alice, $S_{B\rightarrow A}$ is larger than that displayed from Alice to Bob, $S_{A\rightarrow B}$. On the other hand, since Alice particle is accelerated, then its decoherence increases and, consequently its local information decreases. Therefore, one can say that the possibility of steering states that coded coherent information is much larger than that coded mixed information.

Figure 7: Quantum steering for Alice is accelerated $r_a$, and Bob is fixed ($r_b = 0$) where $S_{A\rightarrow B}$ (green-solid curve), $S_{B\rightarrow A}$ (red-dot dash curve), $\Delta = |S_{A\rightarrow B} - S_{A\rightarrow B}|$ (blue-dash curve). (a) $p \rightarrow 0$, (b) $p = 0.6$, and (b) $p = 0.9$. 
In Fig.(7b), it is assumed that the partners share a partially entangled state with $p = 0.6$. A similar behavior is depicted in Fig.(7a), where the degree of steerability decreases as the acceleration $r_a$ increases. The displayed behavior shows that $S^{A\rightarrow B}$ and $S^{B\rightarrow A}$ have a slight difference, such that $S^{A\rightarrow B} > S^{B\rightarrow A}$. This means that, the decoherence due to the acceleration is smaller than that due to the decoherence parameter $p$. Consequently, the encoded local information on Bob’s qubit is more decoherent. Fig.(7c), shows the bidirectional steerability from both sides at larger value of the characteristic parameter $p$, namely the shared state has small degree of entanglement. The behavior of the steerability $S^{A\rightarrow B}$, is larger than that displayed for $S^{B\rightarrow A}$ and the difference between them $\Delta$, increases as $r_a$ increases. In the interval $r_a \in [0.1, 0.8]$, Bob’s information is more coherent, and consequently the degree of Alice steerability $S^{A\rightarrow B}$ is larger than that displayed for $S^{B\rightarrow A}$. From Fig.(7), it is clear that the bidirectional degree of steerability decreases as one increases the parameter $p$. The rate of decreasing due to the acceleration is much smaller than that displayed for the less entangled shared state between the steerer and the steered.

![Graphs showing decoherence for generic pure state](image)

Figure 8: decoherence for generic pure state where Alice is accelerated, decoherence $\rho_{AB}$ (Blue curve), decoherence $\rho_A$ (green curve), decoherence $\rho_B$ (red curve), (a)$p = 0.0$, (b)$p = 0.6$, (c)$p = 0.9$.

6 Conclusion

In this contribution, we investigated the bidirectional steerability between two partners who share different classes of initial states. It is assumed that, either the steerer or the steered partners are in inertial/ non-inertial frames. The effect of the initial state settings and the acceleration parameters on the steerability and its degree are discussed, where analytical forms are obtained for both quantities. However, the possibility of enhancing the degree of steerability is discussed by applying the filtering process.

The general behaviour of the steerability and its degree decreases, if both partners’s qubits are accelerated. The decay rate depends on whether one or both qubits are accelerated. It is shown that, the bidirectional steering is completely consistent if both partners are in the inertial frame or in the non-inertial frame with the same value of the acceleration. The bidirectional steerability and its degree are slightly different when one
of the partners’s qubit is at rest and the other is accelerated. The initial state settings play a significant role on maximizing/minimizing the steerability and its degree. It is shown that, the violation of the steering inequality appears as the shared state loses its quantum correlation. The largest upper bounds of the steerability and its degree are exhibited for the maximum entangled state. The two quantities are investigated for a class of a generic pure state. The violation of the steerability is displayed at larger values of this parameter, where the shared state turns into a separable state when one increases the decoherence parameter. However, due to the acceleration, the violation of steerability is depicted at small values of this decoherence parameter.

The possibility of improving the steerability and its degree is discussed by applying the filtering process. The results show that, as one increases the strength of the filtering process, the steerability increases. The filtering process restrains the decoherence that may be arisen from the accelerating process, and consequently, one enlarges the range of the acceleration in which the steerability could be achieved. At fixed acceleration, the filtering process increases the upper bounds of the degree of steerability.

The bidirectional steerability between the two users is discussed for the non-symmetric cases. It is clear that, if the users’ qubits are accelerated, then the degree of steerability between the two users coincide. It is shown that, for non-symmetric cases, the degree of steerability depends on the types of steerable information. However, due to the acceleration the coherence of the qubit decreases and the possibility of its steering it decreases. In this case, the difference between the steerability from both directions increases.

Finally, one may conclude that, the decay of the steerability and its degree is due to the decoherence from the initial state settings and the acceleration process. The filtering process improves the upper bounds of the degree of steerability and enlarges the range of acceleration in which the steerability could be achieved. The most information holds a great degree of steerability.

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