CHARACTERIZATIONS OF UNIMODULAR FINITE TENSOR CATEGORIES

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Abstract. Let $C$ be a finite tensor category, let $Z(C)$ denote its center, and let $L$ and $R$ be a left and a right adjoint functor of the forgetful functor $U : Z(C) \to C$. We show that the following assertions are equivalent: (i) $C$ is unimodular, (ii) $U$ is a Frobenius functor, (iii) $L$ preserves duality, (iv) $L(1)$ is self-dual, (v) $R$ preserves duality, and (vi) $R(1)$ is self-dual, where $1 \in C$ is the unit object. Some other equivalent assertions are also given. As an application, we generalize Ishii and Masuoka’s construction of an invariant of handlebody-links to unimodular finite tensor categories.

1. Introduction

A locally compact Hausdorff topological group is said to be unimodular if its left Haar measure is also a right Haar measure. Unimodularity of Hopf algebras [Mon93] is defined in an analogous way and is important in the theory of Hopf algebras: For example, a finite-dimensional Hopf algebra is symmetric if and only if it is unimodular and the square of its antipode is inner [Lor97], and a Verlinde-type formula is established for such a Hopf algebra [CW08]. It is also important for applications to topology: Given a finite-dimensional unimodular ribbon Hopf algebra, one can construct an invariant of closed 3-manifolds [Hen96, KR95]. Recently, Ishii and Masuoka [IM13] developed a method to construct an invariant of handlebody-links from finite-dimensional unimodular Hopf algebras.

A finite tensor category [EO04] is a class of monoidal categories including the representation category of a finite-dimensional Hopf algebra. To generalize the Radford $S^4$-formula for Hopf algebras [Rad76] to finite tensor categories, Etingof, Nikshych and Ostrik [ENO04] introduced the distinguished invertible object $D \in C$ of a finite tensor category $C$ over an algebraically closed field $k$. If $D$ is isomorphic to the unit object $1 \in C$, then $C$ is said to be unimodular. In this paper, in view of category-theoretical generalizations of the above-mentioned results for unimodular Hopf algebras, we give the following characterizations of unimodularity of finite tensor categories:

Theorem. Let $C$ be a finite tensor category over $k$, let $Z(C)$ denote its center, and let $L$ and $R$ be a left and a right adjoint functor of the forgetful functor $U : Z(C) \to C$, respectively. Then the following assertions are equivalent:

1. $C$ is unimodular.
2. $U$ is a Frobenius functor, i.e., $L \cong R$.
3. There exists a natural isomorphism $L(V^*) \cong L(V)^*$ for $V \in C$, where $(-)^*$ is the left duality functor.
4. $L(1)$ is self-dual, i.e., $L(1) \cong L(1)^*$.
5. $\text{Hom}_{Z(C)}(1, L(1)) \neq 0$. 

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(6) There exists a natural isomorphism $R(V^*) \cong R(V)^*$ for $V \in \mathcal{C}$.

(7) $R(1)$ is self-dual.

(8) $\text{Hom}_{Z(C)}(R(1), 1) \neq 0$.

Note that the equivalence between (1) and (2) has been obtained by Caenepeel, Militaru and Zhu in [CMZ02, §4, Theorem 53] in the case where $\mathcal{C}$ is the category of representations of a finite-dimensional Hopf algebra. We also give applications of our results to several constructions due to topology. In particular, we generalize Ishii and Masuoka’s construction of an invariant of handlebody-links [IM13] to unimodular finite tensor categories.

This paper is organized as follows: In Section 2, we recall basic notions in category theory. In Section 3, we first recall from [DS07, BV12] the fact that the center $Z(\mathcal{C})$ of a rigid monoidal category is isomorphic to the category of modules over a certain Hopf monad $Z$ on $\mathcal{C}$, called the central Hopf monad, provided that the following coend exists for all $V \in \mathcal{C}$.

\[(1.1) Z(V) = \int^{X \in \mathcal{C}} X^* \otimes V \otimes X \]

We show that a coend of certain type of functors, including (1.1), exists in a finite tensor category. As an application, we give an alternative proof of the fact that the center of a finite tensor category is again a finite tensor category [EO04].

Our main theorem is proved in Section 4. There is an algebra $A \in \mathcal{C} \otimes \mathcal{C}^{\text{rev}}$ which plays a crucial role in the definition of the distinguished invertible object of a finite tensor category $\mathcal{C}$. By using the results of Section 3, we express the algebra $A$ as a coend of a certain functor and relate it to the central Hopf monad on $\mathcal{C}$. Then it turns out that there exists equivalences $K$ and $\tilde{K}$ such that the diagram

\[
\begin{array}{ccc}
Z(\mathcal{C}) & \xrightarrow{\tilde{K}} & (\text{the category of } A\text{-bimodules}) \\
U & \downarrow & U_A \\
\mathcal{C} & \xrightarrow{K} & (\text{the category of right } A\text{-modules})
\end{array}
\]

commutes, where $U_A$ is the functor forgetting the left $A$-module structure. By using this commutative diagram, we obtain a natural isomorphism

\[(1.2) R(V) \cong L(D \otimes V) \quad (V \in \mathcal{C}),\]

where $D \in \mathcal{C}$ is the distinguished invertible object of $\mathcal{C}$ (Theorem 4.5). Once (1.2) is obtained, our main theorem (Theorem 4.9) follows without difficulty.

In Section 5 we give applications of our results to some constructions due to low-dimensional topology. The first application is a generalization of the construction of Ishii and Masuoka [IM13] to unimodular finite tensor categories. The second application concerns the object of integrals $\text{Int}(F)$ of a certain Hopf algebra $F$ in a braided finite tensor category, which is used to construct 3-dimensional topological quantum field theories in [KL01, §5.2]. We show that $\text{Int}(F)$ is precisely the dual of the distinguished invertible object.

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2. Preliminaries

2.1. Monoidal categories. For the basic theory of monoidal categories, we refer the reader to [BK01 Kas95 ML98]. We first fix some conventions for monoidal categories used throughout this paper. In view of Mac Lane’s coherence theorem, we may, and do, assume that all monoidal categories are strict. Given a monoidal category $C = (C, \otimes, 1)$ with tensor product $\otimes$ and unit object $1 \in C$, we set

$$C^{\text{op}} = (C^{\text{op}}, \otimes, 1) \quad \text{and} \quad C^{\text{rev}} = (C, \otimes^{\text{rev}}, 1),$$

where $\otimes^{\text{rev}}$ is the reversed tensor product given by $V \otimes^{\text{rev}} W = W \otimes V$.

Let $C$ and $D$ be monoidal categories. A monoidal functor from $C$ to $D$ is a functor $F : C \to D$ endowed with a morphism $F_0 : 1 \to F(1)$ and a natural transformation

$$F_2(V, W) : F(V) \otimes F(W) \to F(V \otimes W) \quad (V, W \in C)$$

satisfying certain axioms [ML98 XI.2]. If $F_0$ and $F_2$ are invertible, $F$ is said to be strong. A comonoidal functor is a monoidal functor from $C^{\text{op}}$ to $D^{\text{op}}$.

Following [Kas95], a left dual object of $V \in C$ is an object $V^* \in C$ endowed with morphisms $\text{ev}_V : V^* \otimes V \to 1$ and $\text{coev}_V : 1 \to V \otimes V^*$ in $C$ such that

$$(\text{coev}_V \otimes \text{id}_V)(\text{id}_V \otimes \text{ev}_V) = \text{id}_V \quad \text{and} \quad (\text{ev}_V \otimes \text{id}_{V^*})(\text{id}_{V^*} \otimes \text{coev}_V) = \text{id}_{V^*}.$$

One can extend $V \mapsto V^*$ to a strong monoidal functor $(-)^* : C^{\text{op}} \to C^{\text{rev}}$, called the left duality, provided that every object of $C$ has a left dual object. A right dual object $V^* \in C$ is a left dual object of $V$ in $C^{\text{rev}}$. Similarly to the above, one can extend $V \mapsto V^*$ to a strong monoidal functor $(\cdot)^* : C^{\text{op}} \to C^{\text{rev}}$ if every object of $C$ has a right dual object.

A monoidal category $C$ is said to be rigid (or autonomous) if every object of $C$ has both a left and a right dual object. If this is the case, the contravariant endofunctors $(-)^*$ and $(\cdot)^*$ on $C$ are mutually quasi-inverse. Moreover, by replacing $C$ with an equivalent one, we can choose dual objects so that

$$1^* = 1, \quad (V \otimes W)^* = W^* \otimes V^* \quad \text{and} \quad (V^*)^* = X = (\ast V)^*$$

hold for all $V, W \in C$ [Shi13].

2.2. Algebras in a monoidal category. Given an algebra $A (= \text{a monoid} [ML98 VII.3])$ in a monoidal category $C$, we denote by $A^C$ and $C_A$ the categories of left and right $A$-modules in $C$, respectively. If $M$ is a left $A$-module whose underlying object has a left dual object, then $M^*$ is a right $A$-module with action

$$M^* \otimes A \xrightarrow{id \otimes \text{id} \otimes \text{coev}_M} M^* \otimes A \otimes M \otimes M^* \xrightarrow{id \otimes \rho \otimes id} M^* \otimes M \otimes M^* \xrightarrow{\text{ev} \otimes id} M^*,$$

where $\rho : A \otimes M \to M$ is the action of $A$ on $M$. Similarly, a right dual object of a right $A$-module has a structure of a left $A$-module.

Given another algebra $B$ in $C$, we denote by $A^C_B$ the category of $A$-$B$-bimodules in $C$. The tensor product of $C$ induces a functor

$$A^C \times C_B \to A^C_B, \quad (X, Y) \mapsto X \otimes Y.$$

Lemma 2.1. Let $F_A : A^C_B \to C_B$ and $F_B : A^C_B \to A^C$ be the functors forgetting the action of $A$ and $B$, respectively. We denote by $A^A$ and $A^A_A$ the object $A$ viewed as a left and a right $A$-module by the multiplication of $A$, respectively. Then:
Then the following assertions are equivalent:

1. \( L_A = A \otimes (-) \) is left adjoint to \( F_A \)
2. \( R_A = \ast(A) \otimes (-) \) is right adjoint to \( F_A \) provided that \( \ast A \) exists.
3. \( L_B = (-) \otimes B_B \) is left adjoint to \( F_B \)
4. \( R_B = (-) \otimes (B_B)^* \) is right adjoint to \( F_B \) provided that \( B^* \) exists.

**Proof.** Given an \( A-B \)-bimodule \( M \) in \( C \), we denote by \( a_M : A \otimes M \to M \) the left action of \( A \) on \( M \). Define natural transformations \( \eta, \varepsilon, \overline{\eta} \) and \( \overline{\varepsilon} \) by

\[
\eta_V = u \otimes \text{id} : V \to F_A L_A(V), \quad \varepsilon_M = a_M : L_A F_A(M) \to M, \\
\overline{\eta}_M = \phi(a_M) : M \to R_A F_A(M), \quad \overline{\varepsilon}_V = \ast(u \otimes \text{id}_M) : F_A R_A(V) \to V
\]

for \( V \in C_B \) and \( M \in A C_B \), where \( \phi : \text{Hom}_C(A \otimes M, M) \to \text{Hom}_C(M, \ast A \otimes M) \) is the canonical isomorphism given by duality and \( u : 1 \to A \) is the unit of \( A \). One can check that \((L_A, F_A, \eta, \varepsilon)\) and \((F_A, R_A, \overline{\eta}, \overline{\varepsilon})\) are adjunctions between \( C_B \) and \( A C_B \). Hence (1) and (2) are proved. Replacing \( C \) with \( C^{\text{op}} \), we obtain (3) and (4). \( \square \)

Recall that a functor \( F \) is said to be Frobenius \([CMZ02]\) if it has a left adjoint functor which is also a right adjoint to \( F \). By Lemma 2.1, we have:

**Lemma 2.2.** Let \( A \) be an algebra in a monoidal category \( C \) such that \( A^* \) exists. Then the following assertions are equivalent:

1. The forgetful functor \( C_A \to C \) is Frobenius.
2. \( A_A \cong (A_A)^* \) as right \( A \)-modules.

A Frobenius algebra is an algebra \((A, m, u)\) endowed with a morphism \( \text{tr} : A \to 1 \), called the trace, such that \( A^* \) exists and the morphism

\[
(2.1) \quad A \xrightarrow{\text{id} \otimes \text{conv} A} A \otimes A \otimes A^* \xrightarrow{m \otimes \text{id}} A \otimes A^* \xrightarrow{\text{tr} \otimes \text{id}} A^*
\]

is an isomorphism in \( C \). If this is the case, then (2.1) is in fact an isomorphism of right \( A \)-modules and therefore the equivalent conditions of the above lemma are satisfied. Conversely, if we are given an isomorphism \( \phi : A_A \to (A A)^* \) of right \( A \)-modules, then \( A \) is a Frobenius algebra with trace

\[
\text{tr} : A \xrightarrow{\phi} A^* \xrightarrow{u^*} 1^* = 1.
\]

In view of this fact, we also say that \( A \) is Frobenius if it satisfies the equivalent conditions of Lemma 2.2.

### 2.3. Colax-lax adjunctions.

The category \( \text{Sets} \) of all sets is a monoidal category with respect to the Cartesian product. Let \( A, B \) and \( C \) be monoidal categories. If \( P : A \to C \) is a comonoidal functor and \( Q : B \to C \) is a monoidal functor, then

\[
H : A^{\text{op}} \times B \to \text{Sets, \quad } (V, W) \mapsto \text{Hom}_C(P(V), Q(W)) \quad (V \in A, W \in B)
\]

has a structure of a monoidal functor given by \( H_0(*) = G_0 \circ F_0 \) and

\[
H_2((V, W), (V', W')) : H(V, W) \times H(V', W') \to H(V \otimes V', W \otimes W'),
\]

\[
(f, f') \mapsto Q_2(W, W') \circ (f \otimes f') \circ P_2(V, V').
\]

Following Mac Lane \([ML98] \ IV\), we write

\[
(2.2) \quad (F, G, \eta, \varepsilon) : B \to C
\]

if \( F : B \to C \) is a functor, \( G \) is right adjoint to \( F \), and \( \eta \) and \( \varepsilon \) are the unit and the counit of the adjunction, respectively. Now suppose \( F \) is a comonoidal functor,
and $G$ is a monoidal functor. We say that \( \langle F, G, \eta, \varepsilon \rangle \) is a \textit{colax-lax adjunction} \cite{AM10} §3.9.1 if the natural isomorphism
\[
\text{Hom}_C(F(V), W) \cong \text{Hom}_C(V, G(W)) \quad (V \in \mathcal{B}, W \in \mathcal{C})
\]
of the adjunction is an isomorphism of monoidal functors. This notion is in fact an instance of doctrinal adjunctions \cite{Kel74} and therefore we have the following result (see \cite{AM10} §3.9.1 for details).

**Lemma 2.3.** Let \( \langle F, G, \eta, \varepsilon \rangle : \mathcal{B} \to \mathcal{C} \) be an adjunction between monoidal categories \( \mathcal{B} \) and \( \mathcal{C} \). If \( F \) is comonoidal (respectively, \( G \) is comonoidal), then there uniquely exists a monoidal structure of \( G \) (respectively, a comonoidal structure of \( F \)) such that \( \langle F, G, \eta, \varepsilon \rangle \) is a colax-lax adjunction.

An adjoint functor is often given only up to isomorphism. Thus we consider the case where two adjunctions \( \langle F, G, \eta, \varepsilon \rangle \) and \( \langle F', G', \eta', \varepsilon' \rangle \) are given. Then there are natural isomorphisms
\[
\text{(2.3)} \quad \text{Hom}_C(V, G(W)) \cong \text{Hom}_C(F(V), W) \cong \text{Hom}_C(V, G'(W)) \quad (V \in \mathcal{B}, W \in \mathcal{C})
\]
and therefore \( G \cong G' \) by the Yoneda lemma. We call the isomorphism \( G \cong G' \) obtained in this way the \textit{canonical isomorphism}. If \( F \) is comonoidal, then, by the above lemma, both \( G \) and \( G' \) have monoidal structures such that the isomorphisms in \((2.3)\) are monoidal. Again by the Yoneda lemma, we see that the canonical isomorphism \( G \cong G' \) is in fact an isomorphism of monoidal functors.

Similarly, if \( \langle F', G, \eta, \varepsilon \rangle \) and \( \langle F', G', \eta', \varepsilon' \rangle \) are adjunctions, then there is a canonical isomorphism \( F' \cong F' \). If \( G \) is monoidal, then \( F \) and \( F' \) are comonoidal and the canonical isomorphism is an isomorphism of comonoidal functors.

For a functor \( T \) between rigid monoidal categories, we set
\[
T^!(X) = ^*T(X^*).
\]
Let \( F : \mathcal{B} \to \mathcal{C} \) be a strong monoidal functor between rigid monoidal categories \( \mathcal{B} \) and \( \mathcal{C} \). There is an isomorphism \( F^! \cong F \) of monoidal functors \cite{NS07} Lemma 1.1. If \( L \) is left adjoint to \( F \), then \( L^! \) is right adjoint to \( F \) \cite{BV12} Lemma 3.5. Indeed, we have natural isomorphisms
\[
\text{Hom}_C(V, L^!(W)) \cong \text{Hom}_C(L(W^*), V^*)
\]
\[
\cong \text{Hom}_C(W^*, F(V^*))
\]
\[
\cong \text{Hom}_C(F^!(V), W) \cong \text{Hom}_C(F(V), W).
\]
Similarly, if \( R \) is right adjoint to \( F \), then \( R^! \) is left adjoint to \( F \).

Now suppose that \( F \) has a left adjoint \( L \). By Lemma \(\text{(2.3)}\) \( L \) is comonoidal. Hence the functor \( L^! \) is monoidal with monoidal structure \( ^*L_0 : 1 \to L^!(1) \) and
\[
L^!(X) \otimes L^!(Y) = (L(Y^*) \otimes L(X^*)) \xrightarrow{\text{\(L_2(X,Y)\)}} \text{^\(L_2(X,Y)\)}} ^*L(Y^* \otimes X^*) = L^!(X \otimes Y),
\]
where \( L_0 \) and \( L_2 \) are the comonoidal structure of \( L \). On the other hand, since \( L^! \) is right adjoint to \( F \), it has another monoidal structure by Lemma \(\text{(2.3)}\). The following lemma says that these two structures are the same.

**Lemma 2.4.** Let \( F : \mathcal{B} \to \mathcal{C} \) a strong monoidal functor between rigid monoidal categories. Suppose that \( F \) has a left adjoint \( L \) and a right adjoint \( R \). Then the canonical isomorphism \( L^! \cong R \) is an isomorphism of monoidal functors.
Applying this result to the functor $F^\text{rev} : \mathcal{B}^{\text{rev}} \to \mathcal{C}^{\text{rev}}$ induced by $F$, we also have an isomorphism $R \cong 1L$ of monoidal functors, where $1L = L(\ast)\ast$.

Since $R$ is monoidal, $A = R(1)$ is an algebra in $\mathcal{C}$ as the image of the trivial algebra $1 \in \mathcal{C}$. Similarly, since $L$ is comonoidal, $C = L(1)$ is a coalgebra in $\mathcal{C}$. The above lemma implies that $A \cong *C$ as algebras in $\mathcal{C}$.

**Proof of Lemma 2.4** The isomorphism $\text{Hom}_B(F(V), W) \cong \text{Hom}_C(V, L'(W))$ obtained in the above is in fact an isomorphism of monoidal functors. Hence

$\text{Hom}_C(V, R(W)) \cong \text{Hom}_B(F(V), W) \cong \text{Hom}_C(V, L'(W))$

as monoidal functors. Now the result follows from the Yoneda lemma. □

### 2.4. Ends and coends
Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and let $P$ and $Q$ be functors from $\mathcal{A} \times \mathcal{A}^{\text{op}}$ to $\mathcal{B}$. A dinatural transformation $\xi : P \Rightarrow Q$ is a family

$$\xi = \{\xi_X : P(X, X) \to Q(X, X)\}_{X \in \mathcal{A}}$$

of morphisms in $\mathcal{B}$ parametrized by the objects of $\mathcal{A}$ such that the diagram

$$
\begin{array}{ccc}
P(X, X) & \xleftarrow{P(X,f)} & P(X, Y) \\
\downarrow{\xi_X} & \ & \downarrow{\xi_Y} \\
Q(X, X) & \xrightarrow{Q(f, X)} & Q(Y, X)
\end{array}
$$

commutes for all morphism $f : X \to Y$ in $\mathcal{A}$.

We regard an object $X \in \mathcal{B}$ as the functor $\mathcal{A} \times \mathcal{A}^{\text{op}} \to \mathcal{B}$ sending all objects to $X$ and all morphisms to $\text{id}_X$. An end of a functor $Q : \mathcal{A} \times \mathcal{A}^{\text{op}} \to \mathcal{B}$ is a pair $(E, p)$ consisting of an object $E \in \mathcal{B}$ and a dinatural transformation $p : E \Rightarrow Q$ such that, for any such pair $(E', p')$, there uniquely exists a morphism $f : E' \to E$ in $\mathcal{B}$ such that $p'_{X} = f \circ p_{X}$ for all objects $X \in \mathcal{A}$. If it exists, an end $(E, p)$ of $Q$ is unique up to isomorphism. Following [ML98], we write the object $E$ as

$$E = \int_{X \in \mathcal{A}} Q(X, X).$$

A coend of $Q$ is a pair $(C, i)$ consisting of an object $C \in \mathcal{B}$ and a dinatural transformation $i : Q \Rightarrow C$ such that, for any such pair $(C', i')$, there uniquely exists a morphism $f : C \to C'$ such that $i_X = f \circ i'_{X}$ for all $X \in \mathcal{A}$. A coend $(C, i)$ of $Q$ is unique up to isomorphism if it exists and is written as

$$C = \int^{X \in \mathcal{A}} Q(X, X).$$

We refer the reader to [ML98] for general treatments of (co)ends. For reader’s convenience, we here collect some formulas for (co)ends. Suppose that $\mathcal{A}$ is essentially small. Given two functors $F_1, F_2 : \mathcal{A} \to \mathcal{B}$, we denote by $\text{Nat}(F_1, F_2)$ the set of natural transformations from $F_1$ to $F_2$. Then

$$p_X : \text{Nat}(F_1, F_2) \Rightarrow \text{Hom}_B(F_1(X), F_2(X)), \quad \alpha \mapsto \alpha_X \quad (X \in \mathcal{A})$$

is an end of $\text{Hom}_B(F_1(\ast), F_2(\ast))$. With integral notation, we have

$$\text{Nat}(F_1, F_2) = \int_{X \in \mathcal{A}} \text{Hom}_B(F_1(X), F_2(X)).$$
Suppose that a functor $F$ from $B$ is (co)continuous. If a (co)end $(E, p)$ of $Q$ exists, then $(F(E), F(p))$ is a (co)end of $F$. In particular, we have

$$\text{Hom}_B(V, \int_{X \in A} Q(X, X)) = \int_{X \in A} \text{Hom}_B(V, Q(X, X))$$

for all $V \in B$ provided that an end of $Q$ exists. Similarly, we have

$$\text{Hom}_B(\int_{X \in A} Q(X, X), V) = \int_{X \in A} \text{Hom}_B(Q(X, X), V)$$

for all $V \in B$ if a coend of $Q$ exists.

If $V$ is a complete category, then an end exists for any $Q : A \times A^{\text{op}} \to \mathcal{V}$. Since the category $\mathcal{Set}$ of all sets is complete, the ends of the right-hand side of (2.5) and (2.6) exist without the assumption that an end or a coend of $Q$ exists. By the parameter theorem for ends [ML98, IX.7], the right-hand side of (2.6) extends to a functor $Q^\triangledown : B \to \mathcal{Set}$, $V \mapsto \int_{X \in A} \text{Hom}_B(Q(X, X), V)$ ($V \in B$).

**Lemma 2.5.** The following assertions are equivalent:

1. A coend of $Q$ exists.
2. The functor $Q^\triangledown$ is representable.

Similarly, an end of $Q$ exists if and only if $Q_2 : B^{\text{op}} \to \mathcal{Set}$, $V \mapsto \int_{X \in A} \text{Hom}_B(V, Q(X, X))$ ($V \in B$).

**Proof.** It is obvious that (1) implies (2) by (2.6). Now we suppose (2). Let $C$ be an object representing the functor $Q^\triangledown$. By definition, there exists an isomorphism

$$\phi_V : \text{Hom}_B(C, V) \to \int_{X \in A} \text{Hom}_B(Q(X, X), V)$$

natural in $V \in B$. For each $X \in A$, we define $i_X : Q(X, X) \to C$ to be the image of the identity on $C$ under the following map:

$$\text{Hom}_B(C, C) \xrightarrow{\phi_C} \int_{X \in A} \text{Hom}_B(Q(X, X), C) \xrightarrow{p_X} \text{Hom}_B(Q(X, X), C).$$

One can check that $i = \{i_X\}$ is a dinatural transformation $i : Q \Rightarrow C$ and the pair $(C, i)$ is indeed a coend of $Q$. \qed

### 2.5. Hopf monads

Let $T = (T, \mu, \eta)$ be a monad [ML98, VI.1] on a category $\mathcal{C}$ with multiplication $\mu$ and unit $\eta$. By a $T$-module, we mean an object $M \in \mathcal{C}$ endowed with a morphism $\rho_M : T(M) \to M$ satisfying

$$\rho_M \circ \mu_M = \rho_M \circ T(\rho_M) \quad \text{and} \quad \rho_M \circ \eta_M = \text{id}_M.$$

This notion is also called a “$T$-algebra” in literature but we do not use this term in this paper. We denote by $T\mathcal{C}$ the category of $T$-modules (i.e., the Eilenberg-Moore category of $T$-algebras [ML98, VI.2]).

Now suppose that $\mathcal{C}$ is a monoidal category. A bimonad [BV07, BLV11] on $\mathcal{C}$ is a monad $T$ on $\mathcal{C}$ such that the functor $T$ is comonoidal and the natural transformations $\mu$ and $\eta$ are comonoidal natural transformations. Given a bimonad $T$
on \(C\), the category \(\tau C\) of \(T\)-modules is a monoidal category in such a way that the forgetful functor \(\tau C \rightarrow C\) is a strict monoidal functor.

A \textit{Hopf monad} on a monoidal category \(C\) is a bimonad such that certain natural transformations, called the \textit{fusion operators}, are invertible \cite{BLV11}. If \(C\) is rigid, then the notions of a left antipode and a right antipode for a bimonad on \(C\) are defined. A Hopf monad on a rigid monoidal category is characterized as a bimonad having a left and a right antipode \cite{BLV11} §3.4.

2.6. \textbf{Finite tensor categories.} Let \(k\) be a field. Given a \(k\)-algebra \(A\), we denote by \(A\text{-mod}\) and \(\text{mod} A\) the categories of finite-dimensional left and right \(A\)-modules, respectively. The following variant of the Eilenberg-Watts theorem \cite{Eil60,Wat60} will be used extensively:

\textbf{Lemma 2.6.} Let \(A\) and \(B\) be finite-dimensional \(k\)-algebras. For a \(k\)-linear functor \(F : \text{mod} A \rightarrow \text{mod} B\), the following three assertions are equivalent:

1. \(F\) is left exact.
2. \(F\) has a left adjoint.
3. \(F \cong \text{Hom}_A(M, -)\) for some finite-dimensional \(B\)-\(A\)-bimodule \(M\).

The following three assertions are also equivalent:

1.\' \(F\) is right exact.
2.\' \(F\) has a right adjoint.
3.\' \(F \cong (-) \otimes_A M\) for some finite-dimensional \(A\)-\(B\)-bimodule \(M\).

By a \textit{finite abelian category} over \(k\), we mean a \(k\)-linear abelian category equivalent to \(\text{mod} A\) for some finite-dimensional \(k\)-algebra \(A\). Following \cite{EO04}, a \textit{finite tensor category} over \(k\) is a monoidal category \(C\) such that

- \(C\) is a finite abelian category over \(k\),
- the tensor product \(\otimes : C \times C \rightarrow C\) is \(k\)-linear in each variable, and
- the unit object \(1 \in C\) is a simple object and \(\text{End}_C(1) \cong k\).

Let \(C\) be a finite tensor category over \(k\). The tensor product of \(C\) is exact in each variable, since there are adjunctions

\[ V^* \otimes (-) \vdash V \otimes (-) \dashv \ast V \otimes (-) \] \hspace{1cm} and \hspace{1cm} \[ (-) \otimes \ast V \vdash (-) \otimes V \dashv (-) \otimes V^* \]

for each \(V \in C\), where \(F \dashv G\) means that \(G\) is right adjoint to \(F\). Hence

\[ K_0(C) \times K_0(C) \rightarrow K_0(C), \quad [V] \cdot [W] = [V \otimes W] \quad (V, W \in C) \]

is a well-defined operation on the Grothendieck group \(K_0(C)\) of \(C\). With respect to this multiplication, \(K_0(C)\) is a ring.

The left multiplication of \(V \in C\) on \(K(C) := C \otimes_{\mathbb{Z}} K_0(C)\) can be represented by a matrix with non-negative entries. The largest positive eigenvalue of this matrix is called the \textit{Frobenius-Perron dimension} of \(V\) and denoted by \(\text{FPdim}(V)\) \cite{EO04} §2.4. It is known that the \(\mathbb{C}\)-linear map

\[ \text{FPdim} : K(C) \rightarrow \mathbb{C}, \quad [V] \mapsto \text{FPdim}(V) \quad (V \in C) \]

is a well-defined \(\mathbb{C}\)-algebra map. Note that the Frobenius-Perron dimension of an object \(X \in C\) is zero if and only if \(X = 0\). Hence, for \(V, W \in C\), we have

\[ V \otimes W = 0 \quad \implies \quad V = 0 \text{ or } W = 0. \]
2.7. Module categories. Let \( \mathcal{C} \) be a monoidal category. A \( \textit{left} \ \mathcal{C}-\text{module category} \) is a category \( \mathcal{M} \) endowed with a functor \( \otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M} \), called the \textit{action} of \( \mathcal{C} \), and natural isomorphisms

\[
1 \otimes M \cong M \quad \text{and} \quad (X \otimes Y) \otimes M \cong X \otimes (Y \otimes M) \quad (X, Y \in \mathcal{C}, M \in \mathcal{M})
\]
satisfying the axioms similar to those for monoidal categories. See [Ost03] for the precise definitions of a left \( \mathcal{C} \)-module category and related notions.

Now suppose that \( \mathcal{C} \) is a finite tensor category over a field \( k \). We say that a left \( \mathcal{C} \)-module category \( \mathcal{M} \) is \textit{finite} if its underlying category is a finite abelian category over \( k \) and the action \( \otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M} \) of \( \mathcal{C} \) is \( k \)-linear in each variable and right exact in the first variable. Note that the action \( \otimes \) is exact in the second variable since, for each \( V \in \mathcal{C} \), there are adjunctions

\[
V^* \dashv (\_)(V) \dashv \ast (V) \dashv V^* (\_)
\]

Now \( \mathcal{M} \) is a finite left \( \mathcal{C} \)-module category, then the functor

\[
\mathcal{C}^{\text{op}} \to \text{mod}_k, \quad V \mapsto \text{Hom}_\mathcal{M}(V \otimes M, N) \quad (V \in \mathcal{C})
\]
is representable for all \( N, M \in \mathcal{M} \) by Lemma 2.6. We denote by \( \text{Hom}(M, N) \) an object representing this functor. By definition, there is an isomorphism

\[
(2.8) \quad \text{Hom}_\mathcal{C}(V, \text{Hom}(M, N)) \cong \text{Hom}_\mathcal{M}(V \otimes M, N)
\]
natural in the variable \( V \). The assignment \( (M, N) \mapsto \text{Hom}(M, N) \) uniquely extends to a functor \( \text{Hom} : \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{C} \), called the \textit{internal Hom}, in such a way that (2.8) is natural also in the variables \( M \) and \( N \).

By the above arguments, there is an adjunction

\[
(2.9) \quad (-) \otimes M \dashv \text{Hom}(M, -)
\]
for each \( M \in \mathcal{M} \). The counit of this adjunction, denoted by

\[
\text{eval}_{M,N} : \text{Hom}(M, N) \otimes M \to N \quad (N \in \mathcal{M}),
\]
is called the \textit{evaluation}. For \( L, M, N \in \mathcal{M} \), the \textit{composition}

\[
(2.10) \quad \text{comp}_{L,M,N} : \text{Hom}(M, N) \otimes \text{Hom}(L, M) \to \text{Hom}(L, N)
\]
is defined to be the morphism corresponding to the morphism

\[
(\text{Hom}(M, N) \otimes \text{Hom}(L, M)) \otimes L \cong \text{Hom}(M, N) \otimes (\text{Hom}(L, M) \otimes L) \xrightarrow{id \otimes \text{eval}_{L,M}} \text{Hom}(M, N) \otimes M \xrightarrow{\text{eval}_{M,N}} N
\]
via natural isomorphism (2.8), and the \textit{identity}

\[
(2.11) \quad \text{id}_M : 1 \to \text{End}(M) \quad (= \text{Hom}(M, M))
\]
is the morphism corresponding to the canonical isomorphism \( 1 \otimes M \cong M \) via (2.8).

The composition and the identity behave like those in a usual category; in terms of category theory, \( \mathcal{M} \) has a structure of an enriched category over \( \mathcal{C} \).

\textbf{Example 2.7.} Set \( \mathcal{V} = \text{mod}_k \). Every finite abelian category \( \mathcal{M} \) over \( k \) has a natural structure of a finite left \( \mathcal{V} \)-module category with action \( "\cdot" \) determined by

\[
\text{Hom}_\mathcal{V}(V \cdot M, N) \cong \text{Hom}_k(V, \text{Hom}_\mathcal{M}(M, N)) \quad (V \in \mathcal{V}, M, N \in \mathcal{M}).
\]

By definition, \( \text{Hom}(M, N) = \text{Hom}_\mathcal{M}(M, N) \) for all \( M, N \in \mathcal{M} \). In this example, (2.10) and (2.11) coincide with the usual composition of maps and the usual identity map, respectively.
Example 2.8. Let \( \mathcal{B} \) and \( \mathcal{C} \) be a finite tensor categories, and let \( F : \mathcal{B} \to \mathcal{C} \) be a \( k \)-linear right exact strong monoidal functor. Then \( \mathcal{C} \) is a finite left \( \mathcal{B} \)-module category with action given by \( X \otimes V = F(X) \otimes V \) \( (X \in \mathcal{B}, V \in \mathcal{C}) \). By Lemma 2.4 \( F \) has a right adjoint functor \( R \). Since

\[
\text{Hom}_\mathcal{C}(X \otimes V, W) \cong \text{Hom}_\mathcal{C}(F(X), W \otimes V^*) \cong \text{Hom}_\mathcal{B}(X, R(W \otimes V^*)),
\]

the internal Hom is given by \( \text{Hom}(V, W) = R(W \otimes V^*) \). Note that \( R \) is a monoidal functor by Lemma 2.3. The composition is given by

\[
\text{Hom}(V, W) \otimes \text{Hom}(U, V) = R(W \otimes V^*) \otimes R(V \otimes U^*)
\]

\[
\xrightarrow{R} R(W \otimes V^* \otimes V \otimes U^*) \xrightarrow{R(\text{ev} \otimes U^*)} R(W \otimes U^*) = \text{Hom}(W, U),
\]

and the identity is given by

\[
1 \xrightarrow{R_0} R(1) \xrightarrow{R(\text{coev})} R(V \otimes V^*) = \text{End}(V).
\]

Example 2.9. Let \( A \) be an algebra in a finite tensor category \( \mathcal{C} \). The category \( \mathcal{C}_A \) of right \( A \)-modules in \( \mathcal{C} \) has a natural structure of a finite left \( \mathcal{C} \)-module category with action given by \( X \otimes M = X \otimes M \) for \( X \in \mathcal{C} \) and \( M \in \mathcal{C}_A \). We have

\[
\text{Hom}(M, N) = (M \otimes_A N)^* \quad (M, N \in \mathcal{C}_A),
\]

where \( \otimes_A \) is the tensor product over \( A \) [Ost03, Example 2.10.8].

We consider the comparison functor [ML98, VI.3] of adjunction (2.9). Fix an object \( M \in \mathcal{M} \). Note that \( A := \text{End}(M) \) is an algebra in \( \mathcal{C} \) with multiplication and unit given by (2.10) and (2.11), respectively. Following [Ost03], there is a natural isomorphism

\[
\text{Hom}(X \otimes M, Y \otimes N) \cong Y \otimes \text{Hom}(M, N) \otimes X^* \quad (M, N \in \mathcal{M}, X, Y \in \mathcal{C}).
\]

Hence the functor-part of the monad \( T \) associated to (2.9) is given by

\[
T = \text{Hom}(M, (-) \otimes M) \cong (-) \otimes \text{Hom}(M, M) = (-) \otimes A.
\]

With a bit more effort, we see that the category \( \mathcal{C}_T \) of \( T \)-modules can be identified with \( \mathcal{C}_A \). Thus the comparison functor for (2.9) is

\[
K_M : \mathcal{M} \to \mathcal{C}_A, \quad N \mapsto \text{Hom}(M, N) \quad (N \in \mathcal{M}),
\]

where the action of \( A \) on \( \text{Hom}(M, N) \) is given by (2.10) with \( L = M \). Note that \( \mathcal{C}_A \) is a finite left \( \mathcal{C} \)-module category (Example 2.9). By (2.12), the functor \( K_M \) is a functor of left \( \mathcal{C} \)-module categories.

Theorem 2.10 ([EGNO] Theorem 2.11.2 and Remark 2.11.3)). The functor \( K_M \) above is an equivalence of left \( \mathcal{C} \)-module categories if the following two conditions are satisfied:

(K1) The functor \( \text{Hom}(M, -) : \mathcal{M} \to \mathcal{C} \) is right exact.

(K2) Every object of \( \mathcal{M} \) is a quotient of \( V \otimes M \) for some \( V \in \mathcal{C} \).

Proof. Write \( G = \text{Hom}(M, -) : \mathcal{M} \to \mathcal{C} \). Since \( \mathcal{M} \) has all coequalizers, it follows from the Barr-Beck theorem that \( K_M \) is an equivalence if

(BB1) \( G \) has a left adjoint,

(BB2) \( G \) preserves all coequalizers, and

(BB3) \( G \) reflects isomorphisms
(see Exercises 3 and 7 of [ML98 VI.7]). (BB1) is trivial and (BB2) follows immediately from (K1). To show (3), let \( f : M_1 \rightarrow M_2 \) be a morphism in \( \mathcal{M} \) such that \( G(f) \) is an isomorphism. Since \( G \) is exact, we have
\[
G(\text{Ker}(f)) = \text{Ker}(G(f)) = 0 \quad \text{and} \quad G(\text{Coker}(f)) = \text{Coker}(G(f)) = 0.
\]
Now suppose that \( N \in \mathcal{M} \) is an object such that \( G(N) = 0 \). Then
\[
\text{Hom}_{\mathcal{M}}(V \otimes M, N) \cong \text{Hom}_{\mathcal{C}}(V, \text{Hom}(M, N)) = \text{Hom}_{\mathcal{C}}(V, G(N)) = 0
\]
for all \( V \in \mathcal{C} \). By the assumption (K2), we have \( N = 0 \). Applying this argument to (2.13), we have \( \text{Ker}(f) = 0 \) and \( \text{Coker}(f) = 0 \), i.e., \( f \) is an isomorphism. Hence (BB3) follows. The theorem is proved. \( \square \)

3. The central Hopf monad

3.1. The central Hopf monad. Let \( \mathcal{C} \) be a monoidal category. A half-braiding for \( V \in \mathcal{C} \) is a natural isomorphism \( \sigma_V : V \otimes (-) \rightarrow (-) \otimes V \) such that
\[
\sigma_V(X \otimes Y) = (id_X \otimes \sigma_Y(Y)) \circ (\sigma_V(X) \otimes id_Y)
\]
holds for all \( X, Y \in \mathcal{C} \). The center of \( \mathcal{C} \) is the category \( Z(\mathcal{C}) \) whose objects are the pairs \( (V, \sigma_V) \), where \( V \in \mathcal{C} \) and \( \sigma_V \) is a half-braiding for \( V \), and whose morphisms are the morphisms in \( \mathcal{C} \) compatible with the half-braidings. The category \( Z(\mathcal{C}) \) has a natural structure of a braided monoidal category; see, e.g., [Kas95 XIII.4].

Suppose that \( \mathcal{C} \) is a rigid monoidal category such that the coend
\[
Z(V) = \int^{X \in \mathcal{C}} X^* \otimes V \otimes X
\]
exists for all \( V \in \mathcal{C} \). By the parameter theorem for coends, \( V \mapsto Z(V) \) extends to an endofunctor \( Z \) on \( \mathcal{C} \). Day and Street [DS07] showed that the functor \( Z \) has a structure of a monad and \( Z \mathcal{C} \cong Z(\mathcal{C}) \) as categories. Following Bruguières and Virelizier [BV12], the monad \( Z \) has a structure of a quasitriangular Hopf monad and the isomorphism \( Z \mathcal{C} \cong Z(\mathcal{C}) \) is in fact an isomorphism of braided monoidal categories. We call the Hopf monad \( Z \) the central Hopf monad on \( \mathcal{C} \).

For later use, we recall from [DS07] and [BV12] the definition of the central Hopf monad and the construction of the isomorphism \( Z \mathcal{C} \cong Z(\mathcal{C}) \). For \( V, X \in \mathcal{C} \), we denote by \( i_V(X) : X^* \otimes V \otimes X \rightarrow Z(V) \) the component of the universal dinatural transformation. The comonoidal structure
\[
Z_0 : Z(1) \rightarrow 1 \quad \text{and} \quad Z_2(V, W) : Z(V \otimes W) \rightarrow Z(V) \otimes Z(W) \quad (V, W \in \mathcal{C})
\]
are defined to be the unique morphisms such that \( Z_0 \circ i_1(X) = ev_X \) and
\[
Z_2(V, W) \circ i_{V \otimes W}(X) = (i_V(X) \otimes i_W(X)) \circ (id_{X^*} \otimes id_V \otimes \text{coev}_X \otimes id_W \otimes id_X)
\]
for all \( X \in \mathcal{C} \), respectively. To define the multiplication of \( Z \), we note that
\[
i_V^{(2)}(X, Y) := i_{Z(V)}(Y) \circ (id_{X^*} \otimes i_V(X) \otimes id_Y) \quad (X, Y \in \mathcal{C})
\]
is a coend of \( (X_1, Y_1, X_2, Y_2) \mapsto X_2^* \otimes Y_2^* \otimes V \otimes X_1 \otimes Y_1 \) \( (X_1, X_2, Y_1, Y_2 \in \mathcal{C}) \) by the Fubini theorem for coends [ML98 IX.8]. Hence we can define \( \mu : Z^2 \rightarrow Z \) by
\[
\mu_V \circ i_V^{(2)}(X, Y) = i_V(X \otimes Y) \quad (V, X, Y \in \mathcal{C}).
\]
The unit is given by \( \eta_V = i_V(1) \) \( (V \in \mathcal{C}) \). We omit the descriptions of the left and right antipodes and the universal \( R \)-matrix of the Hopf monad \( Z \).
The isomorphism $\mathcal{Z}C \cong Z(C)$ is established as follows: By [2.74] and [2.76], there are isomorphisms

$$\text{Hom}_C(Z(V), V) \cong \int_{X \in C} \text{Hom}_C(X^* \otimes V \otimes X, V)$$

for each $V \in C$. One can check that a morphism $Z(V) \to V$ in $C$ makes $V$ into a $Z$-module if and only if the corresponding natural transformation $V \otimes (-) \to (-) \otimes V$ is a half-braiding for $V$. Therefore the objects of $\mathcal{Z}C$ and those of $Z(C)$ are in bijection. This bijection extends to an isomorphism $\mathcal{Z}C \cong Z(C)$ of monoidal categories. Note that the isomorphism so obtained commutes with the forgetful functors to $C$.

3.2. Existence of coends. To apply the above Hopf monadic description of the center to finite tensor categories, we show that a coend of certain type of functors, including [3.1], exists in a finite tensor category over a field $k$.

Given $k$-linear abelian categories $A_1, \ldots, A_n$ and $C$, we denote by

$$\text{LEX}_n(A_1, \ldots, A_n; C) \quad \text{(respectively, } \text{REX}_n(A_1, \ldots, A_n; C))$$

the category of functors from $A_1 \times \cdots \times A_n$ to $C$ being $k$-linear left exact (respectively, right exact) in each variable. For simplicity, we write

$$\text{LEX}(A, C) = \text{LEX}_1(A; C) \quad \text{and} \quad \text{REX}(A, C) = \text{REX}_1(A; C).$$

A tensor product [Del90, §5] of $k$-linear abelian categories $A_1, \ldots, A_n$ is a $k$-linear abelian category $\mathcal{T}$ endowed with $\boxtimes \in \text{REX}_n(A_1, \ldots, A_n; \mathcal{T})$ such that

$$\text{REX}(\mathcal{T}, C) \to \text{REX}_n(A_1, \ldots, A_n; C) \quad F \mapsto F \circ \boxtimes \quad (F \in \text{REX}(\mathcal{T}, C))$$

is an equivalence for any $k$-linear abelian category $C$. If it exists, it is unique up to equivalence and is denoted by $\mathcal{A}_1 \boxtimes \cdots \boxtimes \mathcal{A}_n$. Note that a tensor product of $k$-linear abelian categories does not always exist [Fra13]. A tensor product of finite abelian categories always exists and enjoys the following properties:

**Lemma 3.1** ([Del90] Proposition 5.13). Let $A$ and $B$ be finite abelian categories over a field $k$. Then the following statements hold:

1. A tensor product $A \boxtimes B$ exists and is a finite abelian category over $k$.
2. The functor $\boxtimes : A \times B \to A \boxtimes B$ is $k$-linear and exact in each variable.
3. The functor $\text{LEX}(A \boxtimes B, C) \to \text{LEX}_2(A, B; C)$ induced by $\boxtimes$ is an equivalence of categories for any $k$-linear abelian category $C$.
4. There is a natural isomorphism

$$\text{Hom}_{A \boxtimes B}(V \boxtimes W, X \boxtimes Y) \cong \text{Hom}_A(V, X) \otimes_k \text{Hom}_B(W, Y)$$

for $V, X \in A$ and $W, Y \in B$.

Suppose that $A = \text{mod}_A$ and $B = \text{mod}_B$ for some finite-dimensional $k$-algebras $A$ and $B$. Then $\text{mod}_{A \otimes_k B}$ is a tensor product of $A$ and $B$ with

$$\boxtimes : A \times B \to \text{mod}_{A \otimes_k B}, \quad (X, Y) \mapsto X \otimes_k Y \quad (X \in A, Y \in B)$$

[Del90] Proposition 5.3]. The above lemma is obtained by using this realization of a tensor product of finite abelian categories. We also have:
Lemma 3.2. For finite abelian categories $\mathcal{A}$ and $\mathcal{B}$ over $k$, the functor

\begin{equation}
(\mathcal{A} \boxtimes \mathcal{B}^{\text{op}} \rightarrow \text{Lex}(\mathcal{B}, \mathcal{A}), \quad V \boxtimes W \mapsto \text{Hom}_{\mathcal{B}}(W, -) \cdot V \quad (V \in \mathcal{A}, W \in \mathcal{B})
\end{equation}

is an equivalence.

Here, (3.3) means as follows: As shown in the below, $\mathcal{L} := \text{Lex}(\mathcal{A}, \mathcal{B})$ is a finite abelian category over $k$. Now we consider the functor

$$A \times \mathcal{B}^{\text{op}} \rightarrow \mathcal{L}, \quad (V, W) \mapsto \text{Hom}_{\mathcal{B}}(W, -) \cdot V \quad (V \in \mathcal{A}, W \in \mathcal{B}),$$

where "·" is the $\text{mod}_k$-action on $\mathcal{A}$ defined in Example 2.7. By Lemma 3.1 (3), this functor induces a left exact functor from $\mathcal{A} \boxtimes \mathcal{B}^{\text{op}}$ to $\mathcal{L}$. We express the functor obtained in such a way as in (3.3).

Proof of Lemma 3.2. We may assume that $\mathcal{A} = \text{mod}_A$ and $\mathcal{B} = \text{mod}_B$ for some finite-dimensional $k$-algebras $A$ and $B$. By Lemma 2.6 and the Yoneda lemma, we see that the following functor is an equivalence:

$$\text{Lex}(\mathcal{B}, \mathcal{A}) \rightarrow \mathcal{L}, \quad M \mapsto \text{Hom}_A(M, -) \quad (M \in \text{mod}_A),$$

where $\text{mod}_A$ is the category of finite-dimensional $A$-$B$-bimodules. Hence, in particular, $\mathcal{L}$ is a finite abelian category over $k$. In view of the above realization of a tensor product, we also have an equivalence

$$\mathcal{A} \boxtimes \mathcal{B}^{\text{op}} \rightarrow (\text{mod}_A \text{mod}_B)^{\text{op}}, \quad V \boxtimes W \mapsto V^* \otimes_k W \quad (V \in \mathcal{A}, W \in \mathcal{B}),$$

where $A$ acts on $V^* := \text{Hom}_k(V, k)$ by $a \cdot f = f(- \cdot a)$ ($a \in A, f \in V^*$). One can check that (3.3) is obtained by composing these equivalences. □

The following description of a quasi-inverse of (3.3) is important:

Lemma 3.3. Notations are the same as in Lemma 3.2. For all $F \in \text{Lex}(\mathcal{B}, \mathcal{A})$, a coend of the functor

\begin{equation}
(\mathcal{B} \times \mathcal{B}^{\text{op}} \rightarrow \mathcal{A} \boxtimes \mathcal{B}^{\text{op}}, \quad (X, Y) \mapsto F(X) \boxtimes Y \quad (X, Y \in \mathcal{B})
\end{equation}

exists. A quasi-inverse of (3.3) is given by

$$\text{Lex}(\mathcal{B}, \mathcal{A}) \rightarrow \mathcal{A} \boxtimes \mathcal{B}^{\text{op}}, \quad F \mapsto \int^X F(X) \boxtimes X \quad (F \in \text{Lex}(\mathcal{B}, \mathcal{A})).$$

Proof. For $F \in \text{Lex}(\mathcal{B}, \mathcal{A})$, there are isomorphisms

$$\text{Hom}_{\text{mod}_A \text{mod}_B}(F(X) \boxtimes Y, V \boxtimes W) \cong \text{Hom}_A(F(X), V) \otimes_k \text{Hom}_B(Y, W)$$

$$\cong \text{Hom}_A(F(X), \text{Hom}_B(W, Y) \cdot V)$$

natural in $V \in \mathcal{A}$ and $W, X, Y \in \mathcal{B}$ by Lemma 3.1 (4) and 2.7.2. Since both sides are $k$-linear and left exact in the variables $V$ and $W$, we obtain

$$\text{Hom}_{\text{mod}_A \text{mod}_B}(F(X) \boxtimes Y, M) \cong \text{Hom}_A(F(X), \Phi(M)(Y)) \quad (M \in \mathcal{A} \boxtimes \mathcal{B}^{\text{op}}),$$

where $\Phi$ is the equivalence given by (3.3). Taking ends, we get

$$\int_X \text{Hom}_{\text{mod}_A \text{mod}_B}(F(X) \boxtimes X, M) \cong \text{Nat}(F, \Phi(M)).$$

Let $\Phi$ be a quasi-inverse of $\Phi$. Since $\text{Nat}(F, \Phi(-))$ is represented by $\Phi(F)$, a coend of (3.3) exists and is isomorphic to $\Phi(F)$ by Lemma 2.5. □
Following Kerler and Lyubashenko \cite{KL01, §5.1.3}, a coend of $Q : A \times A^{\text{op}} \to \mathcal{B}$ exists if $Q$ is $k$-linear exact in each variable. Thus, in the case where $F$ is exact, the existence of a coend of (3.3) follows from their result. Theorem 3.4 below also follows from their result in such a case.

**Theorem 3.4.** Let $\mathcal{C}$ be a finite tensor category over a field $k$. Then coends
\[
\int_{X \in \mathcal{C}} F(X^*) \boxtimes X \quad \text{and} \quad \int_{X \in \mathcal{C}} F(X^*) \otimes X
\]
exist for all $F \in \text{LEX}(\mathcal{C}, \mathcal{C})$.

**Proof.** Note that $F(-^*) : \mathcal{C}^{\text{op}} \to \mathcal{C}$ is $k$-linear left exact if $F \in \text{LEX}(\mathcal{C}, \mathcal{C})$. Hence, applying the above lemma to $F(-^*)$, we see that the first coend exists. The second coend is obtained by applying the right exact functor
\[
\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}, \quad X \boxtimes Y \mapsto X \otimes Y \quad (X, Y \in \mathcal{C})
\]
to the first coend. \qed

**Remark 3.5.** For $F \in \text{LEX}(\mathcal{C}, \mathcal{C})$, there is an isomorphism
\[
\int_{X \in \mathcal{C}} F(X^*) \boxtimes X \cong \int_{X \in \mathcal{C}} F(X) \boxtimes *X.
\]
Indeed, for every object $C \in \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$, the map
\[
\text{Dinat}(F(-^*) \boxtimes *(-), C) \to \text{Dinat}(F(-^*) \boxtimes (-), C), \quad \{i_V\}_{V \in \mathcal{C}} \mapsto \{i_{V^*}\}_{V \in \mathcal{C}}
\]
is a bijection, where $\text{Dinat}(P, Q)$ means the set of dinatural transformations from $P$ to $Q$. Similarly, there is an isomorphism
\[
\int_{X \in \mathcal{C}} F(X^*) \otimes X \cong \int_{X \in \mathcal{C}} F(X) \otimes *X.
\]

3.3. **The center of finite tensor categories.** Applying Theorem 3.4 to $F = (-^*) \otimes V$, we see that the coend in the right-hand side of (3.1) always exists in a finite tensor category. As an application of this result, we prove:

**Theorem 3.6.** The center of a finite tensor category is a finite tensor category.

**Proof.** Let $\mathcal{C}$ be a finite tensor category over a field $k$. As we have remarked, the central Hopf monad $Z$ on $\mathcal{C}$ exists and therefore we can identify $Z(\mathcal{C})$ as the category $z\mathcal{C}$ of $Z$-modules. Set $Z^!(V) = *Z(V^*)$ for $V \in \mathcal{C}$. By Remark 3.5 we have
\[
\text{Hom}_\mathcal{C}(W, Z^!(V)) \cong \text{Hom}_\mathcal{C}(Z(V^*), W^*)
\]
for all $V, W \in \mathcal{C}$. This means that $Z^!$ is right adjoint to $Z$ (a special case of \cite[Corollary 3.12]{BV07}). Hence, by \cite[Proposition 5.3]{EM65}, $z\mathcal{C}$ is an abelian category and the forgetful functor $U : z\mathcal{C} \to \mathcal{C}$ preserves and reflects exact sequences.

We need to show that $\mathcal{C}$ is finite over $k$. Let $L$ be a left adjoint of $U$ (which exists since $U$ is monadic), and let $P$ be a projective generator of $\mathcal{C}$. Then $Q = L(P)$ is
projective, since \( \text{Hom}_{Z(C)}(L(P), -) \cong \text{Hom}_C(P, U(-)) \) is exact. Now let \( X \in ZC \).
Then there exists an epimorphism \( f : P^{\otimes m} \to U(X) \) in \( C \) for some \( m > 0 \). Note that \( L \) preserves epimorphisms as it is a left adjoint. Since \( U \) is faithful, the counit \( \varepsilon \) of the adjunction is epic [ML98 IV.3]. Hence the composition

\[
Q^{\otimes m} = L(P^{\otimes m}) \xrightarrow{L(f)} LU(X) \xrightarrow{\varepsilon} X
\]

is epic. Therefore \( Q \) is a projective generator. This implies the finiteness. \( \square \)

**Remark 3.7.** Let \( C \) and \( D \) be finite tensor categories over a field \( k \). Then \( C \boxtimes D \) is a \( k \)-linear monoidal category with tensor product

\[
(V \boxtimes W) \otimes (X \boxtimes Y) = (V \otimes X) \boxtimes (W \otimes Y) \quad (V, W, X, Y \in C)
\]

and unit \( 1 \boxtimes 1 \). Following Deligne [Del90 Proposition 5.17], \( C \boxtimes D \) is a finite tensor category provided that \( k \) is a perfect field. For general \( k \), a similar result does not seem to be proved.

Theorem 3.6 is proved in [EO04] under the assumption that the base field \( k \) is algebraically closed. Their proof does not apply to the case where \( k \) is not perfect, since it relies on the fact that \( C \boxtimes C^{\text{rev}} \) is a finite tensor category, which follows from the above-mentioned result of Deligne.

### 4. Characterizations of unimodularity

#### 4.1. The definition of unimodularity.

Let \( C \) be a finite tensor category over a field \( k \). Then \( C \boxtimes C^{\text{rev}} \) is a monoidal category with tensor product

\[
(V \boxtimes W) \otimes (X \boxtimes Y) = (V \otimes X) \boxtimes (W \otimes Y) \quad (V, W, X, Y \in C)
\]

and unit \( 1 \boxtimes 1 \). Throughout this section, we assume that (4.1)

\[
C^{\text{env}} := (C \boxtimes C^{\text{rev}}, \otimes, 1 \boxtimes 1)
\]

is rigid, which holds if \( k \) is perfect (see Remark 3.7). Under this assumption, \( C^{\text{env}} \) is a finite tensor category. We note that (4.1) is easily verified in some concrete cases such as the case where \( C = H^{\text{mod}} \) for some finite-dimensional Hopf algebra \( H \).

Following [ENO04], we recall the definition of the distinguished invertible object and the unimodularity of finite tensor categories. The category \( C \) has a structure of a finite \( C^{\text{env}} \)-module category determined by

\[
(V \boxtimes W) \otimes X = V \otimes X \boxtimes W \quad (V, W, X \in C).
\]

Now we set \( A = \text{End}(1, 1) \). The functor \( \text{Hom}(1, -) : C \to C^{\text{env}} \) is exact, since

\[
\text{Hom}(1, V) = \text{Hom}(1, (V \boxtimes 1) \boxtimes 1) \cong (V \boxtimes 1) \otimes A
\]

for all \( V \in C \) by (2.12). By Theorem 2.10, we see that the functor

\[
C \to (C^{\text{env}})_A, \quad V \mapsto (V \boxtimes 1) \otimes A
\]

(4.2)

is an equivalence of \( C^{\text{env}} \)-module categories. In view of this equivalence, there exists an object \( D \in C \), which is unique up to isomorphism, such that

\[
(D \boxtimes 1) \otimes A \cong (A \otimes D)^* \quad \text{(4.3)}
\]

By the theory of Frobenius-Perron dimensions [EO04], \( D \) is invertible, i.e.,

\[
D \otimes D^* \cong 1 \cong D^* \otimes D.
\]

**Definition 4.1** ([ENO04]). The object \( D \) is called the distinguished invertible object of \( C \), and the finite tensor category \( C \) is said to be unimodular if \( D \cong 1 \).
4.2. The algebra $A$ as a coend. The first step for the proof of our main theorem is to describe the algebra $A$ as a coend of a certain functor. Note that the left duality functor is an equivalence $(-)^* : C^{rev} \to C^{op}$ with quasi-inverse $^*(-)$. Hence, by Lemmas 3.2 and 3.3, the functor $j$ the dinatural transformation $V, W \in C \to X$ commutes for all $X$.

There is a natural isomorphism $\Psi : \text{LEX}(C) \to C^{env}$, $F \mapsto \int_{X \in C} F(X) \otimes *X$.

For $V, W \in C$, we set $H(V, W) = \Psi(W \otimes (-) \otimes V^*)$. The following lemma says that $H(V, W)$ is a realization of the internal Hom:

**Lemma 4.2.** There is a natural isomorphism

$$\text{Hom}_{C^{env}}(M, H(V, W)) \cong \text{Hom}_C(M \otimes V, W) \quad (V, W \in C, M \in C^{env}).$$

**Proof.** We may assume that $M \cong \Psi(F)$ for some $F \in \text{LEX}(C)$. Then:

$$\text{Hom}_{C^{env}}(M, H(V, W)) \cong \text{Nat}(F, W \otimes (-) \otimes V^*)$$

$$\cong \int_{X \in C} \text{Hom}_C(F(X), W \otimes X \otimes V^*)$$

$$\cong \int_{X \in C} \text{Hom}_C((F(X) \boxtimes *X) \otimes V, W)$$

$$\cong \text{Hom}_C(\Psi(F) \otimes V, W).$$  

Let $F \in \text{LEX}(C)$ and $V, W \in C$. We pay attention to the bijection $\text{Nat}(F, W \otimes (-) \otimes V^*) \cong \text{Hom}_C(\Psi(F) \otimes V, W)$ in the proof of Lemma 4.2. The morphism $f : \Psi(F) \otimes V \to W$ corresponding to a natural transformation $\alpha : F \to W \otimes (-) \otimes V^*$ via the above bijection is uniquely determined by the property that the diagram

$$\begin{array}{ccc}
(F(X) \boxtimes *X) \otimes V & \xrightarrow{j_F(X) \otimes V} & \Psi(F) \otimes V \\
\downarrow & & \downarrow f \\
F(X) \otimes V \otimes *X & \xrightarrow{\alpha_X \otimes V \otimes *X} & W \otimes X \otimes V^* \otimes V \otimes *X
\end{array}$$

commutes for all $X \in C$, where $j_F(X) : F(X) \boxtimes *X \to \Psi(F)$ is the component of the universal dinatural transformation. In particular, the evaluation $\text{eval}_{V,W}$ for $V, W \in C$ is the morphism making the diagram

$$\begin{array}{ccc}
((W \otimes X \otimes V^*) \boxtimes *X) \otimes V & \xrightarrow{j'_{V,W}(X) \otimes V} & H(V, W) \otimes V \\
\downarrow & & \downarrow \text{eval}_{V,W} \\
W \otimes X \otimes V^* \otimes V \otimes *X & \xrightarrow{W \otimes \text{ev}_{V \otimes *X}} & W
\end{array}$$

commutes for all $X \in C$, where $j'_{V,W} = j'_F$ with $F = W \otimes (-) \otimes V^*$.

Now we set $j = j'_{1,1}$. The algebra structure of $A = H(1, 1)$ is described by using the dinatural transformation $j$ as follows:
Lemma 4.3. With the above notation, the multiplication $m : A \otimes A \to A$ is a unique morphism such that the diagram

\[(4.5)\]

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{j(X \otimes j(Y))} & (X \boxtimes X) \otimes (Y \boxtimes Y) \\
m \downarrow & & \downarrow (j(X) \otimes j(Y)) \otimes 1 \\
A & \xrightarrow{j(X \otimes Y)} & (X \otimes Y) \boxtimes (*Y \otimes X)
\end{array}
\]

commutes for all $X, Y \in \mathcal{C}$. The unit $u : 1 \boxtimes 1 \to A$ is given by $u = j(1)$.

Proof. It is easy to see that the unit of $A$ is given as stated. For $X, Y \in \mathcal{C}$, we have a commutative diagram

\[
\begin{array}{ccc}
((X \boxtimes X) \otimes (Y \boxtimes Y)) \otimes 1 & \xrightarrow{(j(X) \otimes j(Y)) \otimes 1} & (A \otimes A) \otimes 1 \\
\cong & \cong & m \otimes 1 \\
(X \boxtimes X) \otimes ((Y \boxtimes Y) \otimes 1) & \xrightarrow{j(X) \otimes (j(Y) \otimes 1)} & A \otimes (A \otimes 1) \\
(X \boxtimes X) \otimes 1 & \xrightarrow{j(X) \otimes 1} & A \otimes 1 \\
& \xrightarrow{\text{eval}_{1,1}} & 1
\end{array}
\]

by (4.3) and the definition of $m$. Again by (4.4), the composition along the bottom row is $\text{ev} \cdot X$. Hence we obtain:

$$
\text{eval}_{1,1} \circ (m \otimes 1) \circ ((j(X) \otimes j(Y)) \otimes 1)
= \text{ev} \cdot X \circ (1 \boxtimes \text{id}_{X \boxtimes X} \otimes \text{ev} \cdot Y)
= \text{ev} \cdot X \circ (\text{id}_{X} \otimes \text{ev} \cdot Y \otimes \text{id}_{X})
= \text{eval}_{1,1} \circ (j(X \otimes Y) \otimes 1).
$$

Since the map $\text{Hom}_{\mathcal{C}^{env}}(M, A) \to \text{Hom}_{\mathcal{C}}(M \otimes 1, 1); f \mapsto \text{eval}_{1,1} \circ (f \otimes 1)$ is bijective, the commutativity of (4.5) follows. \qed

4.3. The algebra $A$ and the central Hopf monad. For $V, X \in \mathcal{C}$, we set

$$
Z(V) = A \otimes V \quad \text{and} \quad i_{V}(X) = j(X^{*}) \otimes V : X^{*} \otimes V \otimes X \to Z(V),
$$

where $A$ and $j$ are as before. Since $A$ is an algebra in $\mathcal{C}^{env}$, the functor $Z$ has a structure of a monad. More precisely, the multiplication of $Z$ is given by

$$
\mu_{V} : Z^{2}(V) \cong (A \otimes A) \otimes V \xrightarrow{m \otimes V} A \otimes V = Z(V) \quad (V \in \mathcal{C})
$$

and the unit of $Z$ is given by

$$
\eta_{V} : V \cong 1 \otimes V \xrightarrow{- \otimes V} A \otimes V = Z(V) \quad (V \in \mathcal{C}).
$$

Note that $\{i_{V}(X)\}_{X \in \mathcal{C}}$ is a coend, since $\otimes$ is right exact in the first variable. By Lemma 4.2 one can check that $\eta_{V} = i_{V}(1)$ for $V \in \mathcal{C}$ and $\mu$ is determined by the same formula as (4.2). In conclusion, the monad $Z$ under consideration is precisely the central Hopf monad on $\mathcal{C}$.

Let $K : \mathcal{C} \to (\mathcal{C}^{env})_{A}$ be the equivalence given by (4.2). Given a $Z$-module $M$ with action $\rho_{M}$, we can make the right $A$-module $K(M)$ into a $A$-$A$-bimodule by
Corollary 4.6.

\[ A \otimes K(M) \cong K(A \otimes M) = K(Z(M)) \xrightarrow{K(\rho_M)} K(M). \]

Since \( K \) is an equivalence of left \( \mathcal{C}^{\text{env}} \)-module categories, this construction extends to an equivalence

\[ \tilde{K} : z\mathcal{C} \xrightarrow{\cong} \mathcal{A}(\mathcal{C}^{\text{env}})_A, \quad M \mapsto K(M) \quad (M \in z\mathcal{C}) \]

of categories. Recall from [2.5] that \( z\mathcal{C} \) can be identified with \( Z(\mathcal{C}) \). By the definition of \( \tilde{K} \), it is obvious that the following diagram commutes:

\[
\begin{array}{ccc}
Z(\mathcal{C}) & \xrightarrow{\tilde{K}} & A(\mathcal{C}^{\text{env}})_A \\
\downarrow U & & \downarrow U_A \\
\mathcal{C} & \xrightarrow{K} & (\mathcal{C}^{\text{env}})_A,
\end{array}
\]

where \( U \) and \( U_A \) are the functors forgetting the half-braiding and the left \( A \)-module structure, respectively.

Remark 4.4. Etingof and Ostrik [EO04, Corollary 3.35] showed that \( A(\mathcal{C}^{\text{env}})_A \) is equivalent to \( Z(\mathcal{C}) \). However, since they did not give an equivalence in an explicit way, it is not clear that there exists a commutative diagram like (4.7). In this paper, we have given a somewhat explicit equivalence between \( A(\mathcal{C}^{\text{env}})_A \) and \( Z(\mathcal{C}) \) by investigating the relation between the algebra \( A \) and the monad \( Z \) on \( \mathcal{C} \). The commutativity of (4.7) is obvious from our point of view.

4.4. Characterizations of unimodularity. Recall our assumption that \( \mathcal{C} \) is a finite tensor category over a field \( k \) with property (4.1). Let \( L \) and \( R \) be a left and a right adjoint functor of the forgetful functor \( U : Z(\mathcal{C}) \to \mathcal{C} \). The difference of \( L \) and \( R \) are written by using the distinguished invertible object \( D \in \mathcal{C} \) as follows:

**Theorem 4.5.** There are natural isomorphisms

\[ R(V) \cong L(D \otimes V) \quad \text{and} \quad L(V) \cong R(D^* \otimes V) \quad (V \in \mathcal{C}). \]

**Proof.** Let \( \tilde{K} \) be a quasi-inverse of (4.6). By Lemma 2.1 (1), we have

\[ L(V) \cong \tilde{K}^{-1}(AA \otimes K(V)) \cong \tilde{K}^{-1}(AA \otimes (V \boxtimes 1) \otimes A_A). \]

By (4.3), \( ^*(A_A) \cong A_A \otimes (D \boxtimes 1) \) as left \( A \)-modules. By Lemma 2.1 (2),

\[
R(V) \cong \tilde{K}^{-1}(^*(A_A) \otimes (V \boxtimes 1) \otimes A_A)
\]

\[
\cong \tilde{K}^{-1}(A_A \otimes ((D \otimes V) \boxtimes 1) \otimes A_A) \cong L(D \otimes V).
\]

Hence the first natural isomorphism is obtained. Replacing \( V \) with \( D^* \otimes V \), we get the second one.

**Corollary 4.6.** There are natural isomorphisms

\[ R(D^* \otimes V^*) \cong R(V^*) \cong R(V^* \otimes D^*), \quad L(D \otimes V^*) \cong L(V^*) \cong L(V^* \otimes D). \]

**Proof.** By Lemma 2.1 and Theorem 4.3 we have

\[
R(V^*) \cong L(V^*), \quad R(D^* \otimes V^*), \quad R(V^*) \cong L(V \otimes D^*) \cong R(D^* \otimes V^*),
\]

\[
L(V)^* \cong R(V^*), \quad L(D \otimes V^*), \quad L(V)^* \cong R(D^* \otimes V^*) \cong L(V^* \otimes D). \]
Corollary 4.7. There is a chain of adjunctions

\[ \cdots \dashv L\gamma^n \dashv \gamma^{-n}U \dashv L\gamma^{n+1} \dashv \cdots \]

for all integers \( n \), where \( \gamma^n : C \to C \) is a functor defined by

\[ \gamma^p = D \otimes \cdots \otimes D \otimes (-) \]

\( \gamma^0(V) = \text{id}_C \)

\[ \gamma^{-p} = D^* \otimes \cdots \otimes D^* \otimes (-) \]

for \( p > 0 \).

Proof. Use Theorem 4.5 repeatedly (the theorem is the case for \( n = 0 \)). \( \square \)

Corollary 4.8. For a simple object \( V \in C \), we have

\[ \text{Hom}_{\mathbb{Z}(C)}(1, L(V)) \neq 0 \iff V \cong D, \]

\[ \text{Hom}_{\mathbb{Z}(C)}(R(V), 1) \neq 0 \iff V \cong D^*. \]

Proof. We only show the first equivalence, since the second one is obtained in a similar way. By Corollary 4.6, we have

\[ \text{Hom}_{\mathbb{Z}(C)}(1, L(V)) \cong \text{Hom}_{\mathbb{Z}(C)}(L(V)^*, 1) \]

\[ \cong \text{Hom}_{\mathbb{Z}(C)}(L(D \otimes V^*), 1) \cong \text{Hom}_C(D, V). \]

By Schur’s lemma, \( \text{Hom}_C(D, V) \) is non-zero if and only if \( D \cong V \). Hence the result follows. \( \square \)

Now we prove our main theorem:

Theorem 4.9. With the notation above, the following assertions are equivalent:

1. \( C \) is unimodular.
2. \( U \) is a Frobenius functor.
3. There exists a natural isomorphism \( L(V^*) \cong L(V)^* \) for \( V \in C \).
4. \( L(1) \cong L(1)^* \).
5. \( \text{Hom}_{\mathbb{Z}(C)}(1, L(1)) \neq 0 \).
6. There exists a natural isomorphism \( R(V^*) \cong R(V)^* \) for \( V \in C \).
7. \( R(1) \cong R(1)^* \).
8. \( \text{Hom}_{\mathbb{Z}(C)}(R(1), 1) \neq 0 \).

Proof. (1) \( \Rightarrow \) (2) follows from Theorem 4.5 and (2) \( \Rightarrow \) (3) from Corollary 4.6. It is obvious that (3) implies (4). If (4) holds, then we have

\[ \text{Hom}_{\mathbb{Z}(C)}(1, L(1)) \cong \text{Hom}_{\mathbb{Z}(C)}(L(1)^*, 1) \]

\[ \cong \text{Hom}_{\mathbb{Z}(C)}(L(1), 1) \cong \text{Hom}_C(1, 1) \neq 0, \]

which implies (5). (5) \( \Rightarrow \) (1) follows from Corollary 4.8. The proof is completed by showing (2) \( \Rightarrow \) (6) \( \Rightarrow \) (7) \( \Rightarrow \) (8) \( \Rightarrow \) (1) in a similar way. \( \square \)

Corollary 4.10. If \( C \) is unimodular, then we have

\[ \dim_k \text{Hom}_{\mathbb{Z}(C)}(1, L(1)) = \dim_k \text{Hom}_{\mathbb{Z}(C)}(1, R(1)) = 1. \]

Proof. This follows from (4.9) in the proof of Theorem 4.9. \( \square \)
4.5. Radford $S^4$-formula. One of main results of \cite{ENO04} is the following generalization of the Radford $S^4$-formula for finite-dimensional Hopf algebras to finite tensor categories: There exists an isomorphism
\begin{equation}
(\sim) \quad (-)^{****} \cong D \otimes (\sim) \otimes D^*
\end{equation}
of monoidal functors. We give comments on how this formula looks like through the equivalences $\Phi$ and $\Psi$, which are used to prove our main theorem.

For $F, G \in \text{LEX}(\mathcal{C})$, the \textit{Day convolution} is defined by
\[
F \star G = \int_{X,Y \in \mathcal{C}} \text{Hom}_C(X \otimes Y, \sim) \cdot (F(X) \otimes G(Y)).
\]
The coend exists and $\text{LEX}(\mathcal{C})$ is closed under $\star$ since
\[
\Phi(\Psi(F) \otimes \Psi(G)) = \int_{X,Y \in \mathcal{C}} \Phi(\Psi(F(X) \boxtimes X) \otimes \Psi(G(Y) \boxtimes Y))
\cong \int_{X,Y \in \mathcal{C}} \Phi((F(X) \otimes F(Y)) \boxtimes (Y \boxtimes X)) = F \star G.
\]
This operation is originally introduced by Day for the category $[\mathcal{A}, \mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{V}$, where $\mathcal{A}$ is a promonoidal category enriched over a symmetric closed monoidal category $\mathcal{V}$ \cite{Day70}. In the same way as $[\mathcal{A}, \mathcal{V}]$, $\text{LEX}(\mathcal{C})$ is a monoidal category with tensor product $\star$ and unit $\varepsilon = \text{Hom}_C(1, \sim) \cdot 1$. The above computation also shows that $\Psi$ is in fact a monoidal equivalence
\[
\Psi : (\mathcal{C}^\text{env}, \otimes, 1) \overset{\sim}{\longrightarrow} (\text{LEX}(\mathcal{C}), \star, \varepsilon).
\]
In view of this equivalence, the Radford $S^4$-formula is explained as follows: By the definition of $D$, there is an isomorphism
\begin{equation}
A^{**} \cong A^D \quad (:= (D \boxtimes 1) \otimes A \otimes (D^* \boxtimes 1))
\end{equation}
of algebras \cite[(3.4)]{ENO04}. Since $A^{**} \cong \int X X^{**} \boxtimes *** X$, we have
\[
\text{Hom}_C(\Psi(A^{**}))(V, W) \cong \int_{X \in \mathcal{C}} \text{Hom}_C(\text{Hom}_C(*** X^*, V) \cdot X^{**}, W)
\cong \int_{X \in \mathcal{C}} \text{Hom}_k(\text{Hom}_C(*** X^*, V), \text{Hom}_C(X^{**}, W))
\cong \int_{X \in \mathcal{C}} \text{Hom}_k(\text{Hom}_C(X, V^{**}), \text{Hom}_C(X, **W))
\cong \text{Nat}(\text{Hom}_C(\sim, V^{**}), \text{Hom}_C(\sim, **W))
\cong \text{Hom}_C(V^{**}, **W) \cong \text{Hom}_C(V^{****}, W)
\]
for $V, W \in \mathcal{C}$. Hence, $\Psi(A^{**}) \cong (\sim)^{****}$ by the Yoneda lemma. We also compute:
\[
\text{Hom}_C(\Psi(A^D))(V, W) \cong \int_{X \in \mathcal{C}} \text{Hom}_C(\text{Hom}_C(D^* \otimes X \otimes D, V) \cdot X, W)
\cong \int_{X \in \mathcal{C}} \text{Hom}_k(\text{Hom}_C(X, D \otimes V \otimes D^*), \text{Hom}_C(X, W))
\cong \text{Hom}_C(D \otimes V \otimes D^*, W)
\]
for \( V, W \in \mathcal{C} \), which implies that \( \Psi(A^D) \cong D \otimes (-) \otimes D^* \). Now (4.10) is obtained by applying \( \Psi \) to (4.11). The fact that (4.11) is an isomorphism of algebras translates into the fact that (4.10) is an isomorphism of algebras with respect to the Day convolution, i.e., monoidal functors [Day70, Example 3.2.2].

5. Applications

5.1. A result on Hopf modules. We give applications of our results to some constructions due to low-dimensional topology. As a preparation, we investigate a relation between Theorem 2.10 and the fundamental theorem of Hopf modules over a Hopf monad.

Let \( \mathcal{C} \) be a finite tensor category over a field \( k \) with property (4.1). As before, let \( U: Z(\mathcal{C}) \to \mathcal{C} \) be the forgetful functor, and let \( L \) and \( R \) be a left and a right adjoint functor of \( U \). We first prove the following lemma:

**Lemma 5.1.** \( L \) and \( R \) are faithful and reflect isomorphisms.

**Proof.** In view of Lemma 2.4, we only show that \( L \) is faithful and reflects isomorphisms. By (2.7) and (4.8) (in the proof of Theorem 4.5), we have

\[
L(X) = 0 \iff X = 0.
\]

Now let \( f: V \to W \) be a morphism in \( \mathcal{C} \) such that \( L(f) = 0 \). Since \( L \) is exact by Corollary 4.7, we have \( L(\text{Im}(f)) = \text{Im}(L(f)) = 0 \). Hence \( \text{Im}(f) = 0 \) by (5.1) and thus \( f = 0 \). This implies that \( L \) is faithful. That \( L \) reflects isomorphisms is proved in the same way as the proof of Theorem 2.10 by using (5.1). \( \square \)

\( Z(\mathcal{C}) \) acts on \( \mathcal{C} \) by \( X \otimes V = U(X) \otimes V \) (\( X \in Z(\mathcal{C}), V \in \mathcal{C} \)). Note that \( B := R(1) \) is an algebra in \( Z(\mathcal{C}) \) by Lemma 2.3. \( B \) acts on \( R(V) \) for \( V \in \mathcal{C} \) by

\[
R(V) \otimes B = R(V) \otimes R(1) \to R(V \otimes 1) = R(V).
\]

We denote by \( R(V)_B \) the \( B \)-module obtained in this way. By Example 2.8, \( K: \mathcal{C} \to Z(\mathcal{C}) \), \( V \mapsto R(V)_B \) (\( V \in \mathcal{C} \)) is the comparison functor for the adjunction \((-) \otimes 1 \dashv \text{Hom}(1, -) \).

**Theorem 5.2.** The functor \( K \) is an equivalence.

**Proof.** \( R = \text{Hom}(1, -) \) is exact by Corollary 4.7. Since \( R \) is faithful by the previous lemma, the counit \( \varepsilon_V: U R(V) \to V \) of the adjunction \( U \dashv R \) is an epimorphism for all \( V \in \mathcal{C} \). Hence every object \( V \in \mathcal{C} \) is a quotient of \( R(V) \otimes 1 = U R(V) \). Now the result is obtained by applying Theorem 2.10. \( \square \)

This theorem can be derived from the fundamental theorem of Hopf modules over a Hopf monad: By Lemma 2.3, \( C = L(1) \) is a coalgebra in \( Z(\mathcal{C}) \). The coalgebra \( C \) coacts on an object of the form \( L(V) \) by

\[
L(V) = L(1 \otimes V) \xrightarrow{L_2} L(1) \otimes L(V) = C \otimes L(V).
\]

We denote this \( C \)-comodule by \( C L(V) \). Now we recall that \( Z(\mathcal{C}) \) can be identified with the category \( \mathcal{Z} \) of \( C \)-modules over the central Hopf monad \( Z \) on \( \mathcal{C} \). Note that \( Z \) reflects isomorphisms by Lemma 5.1. By the fundamental theorem of Hopf modules [BV07, Theorem 4.6], the functor

\[
K': \mathcal{C} \to Z(\mathcal{C}) \quad (= \text{the category of left } C \text{-comodules}), \quad V \mapsto C L(V)
\]
is an equivalence. By Lemma 2.4, the diagram
\[
\begin{array}{ccc}
C & \xrightarrow{K} & Z(C)_B \\
\downarrow (-)^* & & \downarrow (-)^* \\
C & \xrightarrow{K'} & C Z(C)
\end{array}
\]
commutes up to isomorphism. Since the vertical arrows are anti-equivalences, that \(K\) is an equivalence is equivalent to that \(K'\) is an equivalence.

5.2. Handlebody TQFTs. We give applications of our results to handlebody topological quantum field theories (handlebody TQFT). A handlebody is a connected sum of solid tori, and a handlebody-link is a disjoint union of handlebodies embedded into the 3-dimensional Euclidean space. To construct an invariant of handlebody-links, Ishii and Masuoka [IM13] introduced the braided rigid monoidal category \(T\) of handlebody-tangles. The notion of handlebody TQFTs is formulated by using \(T\) as follows:

**Definition 5.3.** A handlebody TQFT is a braided monoidal functor \(T \to B\) from \(T\) to some braided monoidal category \(B\).

The equivalence classes of handlebody-links are in one-to-one correspondence between the set \(\text{End}_T(\emptyset)\), where \(\emptyset\) is the unit object of \(T\). Hence, given a handlebody TQFT \(F : T \to B\), we obtain an invariant of handlebody-links

\[F : \text{End}_T(\emptyset) \to \text{End}_B(1), \quad L \mapsto F(L).\]

As is well-known, given an object of a ribbon category \(R\), we can construct a braided monoidal functor from the category of framed tangles to \(R\) [Kas95]. In a similar manner, we can construct a handlebody TQFT \(T \to B\) if we are given the following type of object:

**Definition 5.4** (Ishii-Masuoka [IM13, Definition 4]). Let \(B\) be a braided monoidal category with braiding \(\sigma\). A quantum-commutative quantum-symmetric algebra (QCQSA) in \(B\) is a triple \((A, m, e)\) consisting of an object \(A \in B\) and morphisms

\[m : A \otimes A \to A \quad \text{and} \quad e : A \otimes A \to 1\]

satisfying the following conditions:

1. \((Q1)\) \(m \circ (m \otimes \text{id}_A) = m \circ (\text{id}_A \otimes m)\)
2. \((Q2)\) \(e \circ (m \otimes \text{id}_A) = e \circ (\text{id}_A \otimes m)\).
3. \((Q3)\) \(m\) is commutative, i.e., \(m \circ \sigma_{A,A} = m\).
4. \((Q4)\) \(e\) is symmetric, i.e., \(e \circ \sigma_{A,A} = e\).
5. \((Q5)\) There exists a morphism \(c : 1 \to A \otimes A\) such that the triple \((A, e, c)\) is a left dual object of \(A\).

One of main results of Ishii and Masuoka [IM13] is that the isomorphism classes of braided monoidal functors are in one-to-one correspondence between the isomorphism classes of QCQSAs.

Note that a QCQSA in a braided monoidal category \(B\) is defined as an “algebra without unit”. We say that a QCQSA \((A, m, e)\) is unital if there exists a morphism \(u : 1 \to A\) such that \((A, m, u)\) is an algebra. Unital QCQSAs are characterized as follows:
Proposition 5.5. Unital QCQSA in $\mathcal{B}$ are in one-to-one correspondence between commutative Frobenius algebras in $\mathcal{B}$.

Proof. Suppose that $(A, m, e)$ is a unital QCQSA with unit $u$. Then the algebra $(A, m, u)$ is a commutative Frobenius algebra with trace

$$\text{tr} : A \otimes u \otimes \text{id}_A \rightarrow A \otimes A \xrightarrow{e} 1.$$ 

Conversely, given a commutative Frobenius algebra $(A, m, u, \text{tr})$, we define $e : A \otimes A \xrightarrow{m} A \xrightarrow{\text{tr}} 1$.

Then $(A, m, e)$ is a unital QCQSA with unit $u$. It is easy to see that these constructions are mutually inverse. □

Hence, a commutative Frobenius algebra in a braided monoidal category yields a handlebody TQFT. We now give a construction of a commutative Frobenius algebra in the center of a unimodular finite tensor category:

Theorem 5.6. Let $\mathcal{C}$ be a finite tensor category over a field $k$ with property [4.1], let $D \in \mathcal{C}$ be the distinguished invertible object, and let $R$ be a right adjoint functor of the forgetful functor $U : Z(\mathcal{C}) \rightarrow \mathcal{C}$. Regarding the algebra $B = R(1)$, we have:

1. $B$ is commutative.
2. $(B B^*) \cong R(D^*)_B$ as right $B$-modules.
3. $B$ is Frobenius if and only if $\mathcal{C}$ is unimodular.

Proof. The part (1) follows from the proof of [DMNO13, Lemma 3.5]. To show (2) and (3), we note that the equivalence $K$ of Theorem 5.2 makes the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{K} & Z(\mathcal{C})_B \\
R & \downarrow & \downarrow U_B \\
Z(\mathcal{C}) & \xrightarrow{U} & Z(\mathcal{C})
\end{array}$$

commutes, where $U_B$ is the forgetful functor. By Corollary 4.7, $U \dashv R \dashv D^* \otimes U(-)$.

Hence there exists a natural isomorphism

$$R(D^* \otimes U(X))_B = K(D^* \otimes U(X)) \cong X \otimes (B_1)_B^* \quad (X \in Z(\mathcal{C})).$$

Now (2) is obtained by letting $X = 1$. To show (3), note that we have

$$B \text{ is Frobenius } \iff U_B \text{ is Frobenius } \iff R \text{ is Frobenius}$$

by the commutativity of (5.2). By (5.3), $R$ is Frobenius if and only if $D \cong 1$, i.e., $\mathcal{C}$ is unimodular. Hence (3) is proved. □

Remark 5.7. Let $\mathcal{C}$ and $B$ be as above, and suppose that $\mathcal{C}$ is unimodular. By the above theorem, there exists a morphism $t : B \rightarrow 1$ such that $(B, t)$ is Frobenius. It is easy to see that $t \neq 0$ and $(B, ct)$ is Frobenius for any $c \in k^*$. Since $\dim_k \text{Hom}_{Z(\mathcal{C})}(B, 1) = 1$ (Corollary 4.10), we have the following conclusion: Any non-zero morphism $\text{tr} : B \rightarrow 1$ is a trace of the algebra $B$. 

\[\text{UNIMODULAR FINITE TENSOR CATEGORIES 23}\]
Remark 5.8. Let $H$ be a finite-dimensional Hopf algebra over $k$ with comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$. A Yetter-Drinfeld module $[\text{Mon93}]$ over $H$ is a pair $(M, \rho)$ consisting of a left $H$-module $M$ and a left $H$-comodule structure of $F$ Hopf monad such that the Yetter-Drinfeld condition

$$\rho_M : M \to M \otimes_k H, \quad m \mapsto m(-1) \otimes m(0) \quad (m \in M)$$

such that the Yetter-Drinfeld condition

$$\rho_M(hm) = h(1)m(-1)S(h(3)) \otimes h(2)m(0)$$

holds for all $h \in H$ and $m \in M$, where $h(1) \otimes h(2) \otimes h(3) = (\Delta \otimes \text{id})\Delta(h)$. It is known that the category $HYD$ of finite-dimensional Yetter-Drinfeld modules can be identified with the center of $\mathcal{C} = \text{Hmod}$.

Given $V \in \mathcal{C}$, we make $R(V) = H \otimes_k V$ into a Yetter-Drinfeld module over $H$ by defining the action and the coaction of $H$ by

$$h \cdot (a \otimes v) = h(1)aS(h(3)) \otimes h(2)v \quad \text{and} \quad a \otimes v \mapsto a(1) \otimes (a(2) \otimes v)$$

for $a, h \in H$ and $v \in V$, respectively. $V \mapsto R(V)$ defines a functor $R : \mathcal{C} \to HYD$, which is right adjoint to the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$ under the identification $Z(\mathcal{C})$ with $HYD$ (see [Shi13, §7]; cf. [Rad03 §2]). The algebra $B = R(1)$ is precisely the one considered in [IM13] to construct invariants of handlebody-links.

5.3. The braided function algebra. Suppose that the coend

$$F = \int_{X \in \mathcal{C}} X \otimes^\ast X$$

exists in a braided rigid monoidal category $\mathcal{B}$. Then $F$ has a structure of a Hopf algebra in $\mathcal{B}$ defined as follows: Note that $F \cong Z(1)$ by Remark 5.5. The coalgebra structure of $F$ is defined in the same way as the comonoidal structure of the central Hopf monad $Z$. By using the universal dinatural transformation $i_X : X \otimes^\ast X \to F$, the multiplication $m$ is determined by

$$\begin{align*}
X \otimes^\ast X \otimes Y \otimes^\ast Y & \xrightarrow{i_X \otimes^\ast i_Y} F \otimes F \\
\downarrow \quad X \otimes \sigma_{X,Y} \otimes^\ast Y \otimes^\ast X & \quad \downarrow m \\
\quad X \otimes Y \otimes^\ast Y \otimes^\ast X & \xrightarrow{i_X \otimes^\ast i_Y} F \otimes F,
\end{align*}$$

and the unit $u$ is given by $u = i_1$. The Hopf algebra $F$ is called the braided function algebra; see [KL01 §5.2] for details.

As an application of our results, we determine the “object of integrals” of the braided function algebra $F$ in a braided finite tensor category. For the definition and basic properties of integrals, see [Tak99, BKLT00] and [KL01 §4.2.3]. Now we introduce the following terminology:

Definition 5.9. Let $(A, m, u)$ be an algebra in a rigid monoidal category $\mathcal{C}$, and let $I \in \mathcal{C}$ be an invertible object. An $I$-valued trace of $A$ is a morphism $t : A \to I$ such that the following composition is an isomorphism in $\mathcal{C}$:

$$A \xrightarrow{A \otimes \text{coev}} A \otimes A \otimes A^\ast \xrightarrow{m \otimes A^\ast} A \otimes A^\ast \xrightarrow{t \otimes A^\ast} I \otimes A^\ast.$$ 

Given a Hopf algebra in a braided rigid monoidal category, we denote by $\text{Int}(H)$ the object of integrals of $H$. By [KL01 Lemma 4.2.11], $\text{Int}(H)$ can be defined to be a unique (up to isomorphism) invertible object $I$ such that there exists an $I$-valued trace of $H$. 

Theorem 5.10. Let $\mathcal{C}$ be a braided finite tensor category with property (4.1), and let $D$ be the distinguished invertible object. Then the object of integrals of the braided function algebra $F$ in $\mathcal{C}$ is given by

$$\text{Int}(F) \cong D^*.$$  

Proof. Let $Z$ be the central Hopf monad on $\mathcal{C}$. Under the identification $Z(\mathcal{C}) \cong Z\mathcal{C}$, a left adjoint functor of the forgetful functor $U: Z(\mathcal{C}) \to \mathcal{C}$ is given by $L: \mathcal{C} \to Z(\mathcal{C})$, $V \mapsto (Z(V), \mu_V)$ ($V \in \mathcal{C}$).

Now let $R$ and $B$ be as in Theorem 5.2. By Lemma 2.4, we have $B_0 \cong U(R(1)) \cong U(*L(1)) = *U(L(1)) = *F$ as coalgebras. Hence we can make $B_0$ into a Hopf algebra in $\mathcal{C}$ by transporting the structure of $F$ via the above isomorphisms. By [KL01, Theorem 4.2.5], (5.4) $\text{Int}(F) \cong \text{Int}(B_0^*) \cong \text{Int}(B_0)^*$.

In what follows, we identify an object $V \in \mathcal{C}$ with $(V, \sigma_V, -) \in Z(\mathcal{C})$, where $\sigma$ is the braiding of $\mathcal{C}$. Since the equivalence $K$ of Theorem 5.2 is in fact an equivalence of left $Z(\mathcal{C})$-module categories, we have an isomorphism $R(V)_B = K(V \otimes 1) \cong V \otimes K(1) = V \otimes B_B$ of right $B$-modules for $V \in \mathcal{C}$. Hence, by Theorem 5.2

$$D \otimes (B_B)^* \cong D \otimes D^* \otimes B_B \cong B_B$$

Now let $f: B_B \to D \otimes (B_B)^*$ be the isomorphism so obtained and define $t$ by

$$t: B \xrightarrow{u \otimes B} B \otimes B \xrightarrow{f \otimes B} D \otimes B^* \otimes B \xrightarrow{D \otimes \text{ev}} D.$$  

One can check that the composition

$$B \xrightarrow{B \otimes \text{coev}} B \otimes B \otimes B^* \xrightarrow{m \otimes B^*} B \otimes B^* \xrightarrow{\otimes B^*} D \otimes B^*$$

is equal to $f$ and, in particular, is invertible. Hence $t$ is a $D$-valued trace of $B_0 = U(B)$. By [KL01, Lemma 4.2.11] mentioned above, $\text{Int}(B_0) \cong D$. Now the result follows from [5.4]. □

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