Renormalization Group Approach in Newtonian Cosmology

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We apply the renormalization group (RG) method to examine the observable scaling properties in Newtonian cosmology. The original scaling properties of the equations of motion in our model are modified for averaged observables on constant time slices. In the RG flow diagram, we find three robust fixed points: Einstein-de Sitter, Milne and Quiescent fixed points. Their stability (or instability) property does not change under the effect of fluctuations. Inspired by the inflationary scenario in the early Universe, we set the Einstein-de Sitter fixed point with small fluctuations, as the boundary condition at the horizon scale. Solving the RG equations under this boundary condition toward the smaller scales, we find generic behavior of observables such that the density parameter $\Omega$ decreases, while the Hubble parameter $H$ increases for smaller averaging volume. The quantitative scaling properties are analyzed by calculating the characteristic exponents around each fixed point. Finally we argue the possible fractal structure of the Universe beyond the horizon scale.

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1. INTRODUCTION

Several observational evidences imply scaling properties in the present Universe. For example, (a) G. de Vaucouleurs compiled data of density-size relation for galaxies and clusters of galaxies, and pointed out the scaling relation of \( \log \rho = -1.7 \log L \), where \( L \) is the linear scale of each astrophysical object and \( \rho \) is the mass density averaged at this scale \[1\]. This relation asserts the systematic decrease of the density at larger scale. (b) It is widely known that the observed two-point correlation function \( \xi(r) \) for galaxies or for clusters has the scaling property as \( \xi(r) \propto r^{-1.8} \) \[2\]. (c) Pietronero and his collaborators discuss a fractal structure of the Universe and claim that the fractal dimension is about two up to the scale of 1000 Mpc \[3\]. (d) The observations of mass-luminosity ratio for the various astronomical objects linearly increases toward the larger scale up to the scale about 10Mpc \[4\].

Motivated by the above observational evidences, we explore systematic underlying physics which may govern the scaling properties of observables in the present Universe. Usually the above scaling properties are understood as a result of an interplay of the scale invariant initial condition (Harrison-Zel’dovich spectrum) and the gravitational instability \[2\]. In this usual approach, a global uniform background is assumed. In our approach, we do not assume such uniform background a priori but consider general inhomogeneous distributions.

Let us consider the scaling properties carefully. We start our analysis with the equations of motion for fluid in the Newtonian Universe neglecting the pressure term for simplicity. This set of equations admits a naive scale invariance; a scaling of space-time coordinates as well as an appropriate scaling of observables leaves the set of equations unchanged. We define this naive scaling transformation as the operation-\( S \). Such a naive scaling property is, however, not necessarily reflected in the actual observations in a direct form. This is because in general the physically measurable quantities are strongly affected by the actual method of observations. Therefore the above naive scaling should be modified when we apply it to the actual observations. There are at least two important factors for the modification: (1) An observable region is limited by causality. (2) We may merely observe averaged quantities in a certain region in space.

We first consider the factor (1). Actually, we can only observe a light signal from a galaxy located on our past light-cone. In Newtonian cosmology, the observable quantities are located on a hyper-slice of constant time. Because in general the above naive scale transformation also requires a change of the time variable, we have to adjust it so that we can compare the observable quantities with different scales on the same time slice. This adjustment can be implemented by the time evolution of the observables by using the equations of motion for them. We define this adjustment as the operation-\( T \). In this context, we have to reconsider exactly what we are measuring. In this paper we consider scaling behavior of the cosmological parameters. For example, the density parameter \( \Omega \) is a dimensionless quantity and therefore it is invariant under the naive scale transformation. However in many cases, we can directly measure the matter density \( \rho \) and separately the cosmic expansion parameter \( H \) \[5\]. The combination of these observations induces the value of the density parameter \( \Omega \) which depends on the scale. In this case, the scale transformation and the necessary time adjustment are non-trivial. Thus we have to carefully distinguish the direct observables and the secondary induced observables.

Next we consider the factor (2). What we observe is necessarily an averaged quantity on some region of space-time because there are always spatial fluctuations and the resolution of our observation is limited. Therefore the observables are most generally dependent on the scale of averaging. We define this averaging as the operation-\( A \). However, it would be very difficult to consider the invariant averaging procedure in the full general relativity and to make connection with observables \[6\]. Therefore mainly from this reason we would better restrict our considerations to the Newtonian cosmology, in which an averaging procedure is properly defined and is explicitly calculated \[8\].

After the above adjustments (1) the operation-\( T \) and (2) the operation-\( A \) on the original naive scaling properties of the underlying equations of motion, we would obtain the relevant information for actually observable quantities. Thus in general, the difference of two observables at different scales would be related with each other by the naive scaling (operation-\( S \)) compensating time evolution (operation-\( T \)) and the inevitable averaging procedure (operation-\( A \)).

In order to unify the above operations, \( S \), \( T \), and \( A \), we need to introduce one more ingredient in our considerations.
The most sophisticated method so far to deal with such multiple operations would be the renormalization group (RG) method. This is a very general method to obtain the observable response of the system against the scale change and is widely used in various fields in physics including quantum field theory, statistical mechanics, fluid mechanics, etc. For example, in the quantum field theory, a naive scaling in the classical theory should be modified by the quantum fluctuations. In the statistical physics, the scaling property is used with the averaging operation to yield critical exponents.

In this paper, we try to apply this RG method to Newtonian Cosmology. In this model, the scaling invariance holds all the time. Thanks to the averaging procedure, we can define classical fluctuations of the observables. Then, naive scaling behaviors of the observables are modified by renormalization of the classical fluctuations. This is conceptually the same as RG methods in other fields of physics.

The RG equations we obtain give the effective scale change of averaged observables, and the effect of spatial fluctuations on observables. The flow diagram generated by the RG equations in the parameter space of observables represents how the observables actually change under the scale transformation. Each flow line represents a single physical system, in which the averaged variable at different scale corresponds to each point on the flow. Thus the RG flow gives a whole set of possible cosmological models. The fixed points, i.e. the stagnation points in the flow diagram, of the RG equations represent the possible “uniform” structure of observables, that is the averaged variables do not depend on the scale, and characterize diversities of the system.

There have been some applications of the RG method in cosmology so far. The RG method is an effective tool to examine the asymptotic behavior of a dynamical system. In fact, Koike, Hara and Adachi succeeded in explaining the critical behavior of a self-similar solution during the gravitational collapse of a radiation fluid using the RG method. Subsequent series of studies are in this line of study. In their cases, the aspects of (S) and (T) are reflected in their method. Thus their RG method does not necessarily contain an averaging procedure (A). On the other hand, Carfora and Marzuoli and Carfora and Piotrowska explicitly accounted for the spatial fluctuation of physical quantities in general relativity and derived the RG equation reflecting the aspects (S) and (A). Since they concentrated on the scaling in the spatial direction, the aspect (T) has not been manifestly reflected in their approach. Here in this paper, to examine the scale dependence of the observables on a constant hyper-slice, we apply the RG method from the full aspects of (S), (T), and (A) altogether, assuming some scaling property in averaging observables.

This paper is constructed as follows: In section II, we derive the RG equation for the averaged Newtonian fluid. The section III is the main part of our analysis. Here we examine the RG flow in the parameter space and show that there appear three robust fixed points including the Einstein-de Sitter universe. We examine the effects of shear, tidal force, and fluctuations, and derive the scale dependence of the physical variables such as $\Omega$ and $H$ around the Einstein-de Sitter universe. In section IV, we summarize our results. We also present the full RG equations including fluctuations up to the second order cumulants in the Appendix.

### II. DERIVATION OF RENORMALIZATION GROUP EQUATIONS

In Newtonian cosmology, the following equations describe the evolution of a self-gravitating fluid in the Eulerian coordinate system:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v), \quad \frac{\partial v}{\partial t} = -(v \cdot \nabla)v + g, \quad \nabla \cdot g = -4\pi G\rho,
\]

where $\rho$, $v$, and $g$ (≡ $-\nabla \phi$) respectively denote the mass density, the velocity of the fluid, and the gravitational acceleration. Now we transform these equations into the Lagrangian coordinate system. In order to eliminate the gravitational acceleration $g$, we introduce the spatial derivative of the velocity field and decompose it into a trace...
part (expansion $\theta = \nabla \cdot \mathbf{v}$), a symmetric traceless part (shear $\sigma^i_j$) and an anti-symmetric part (rotation $\omega^i_j$) \( (8) \). Then the above equations become

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho \theta, \\
\frac{d\theta}{dt} &= -\frac{1}{3} \theta^2 + 2 (\omega^2 - \sigma^2) - 4\pi G \rho, \\
\frac{d\sigma^i_j}{dt} &= -\frac{2}{3} \theta \sigma^i_j - \sigma^i_k \sigma^k_j - \omega^i_k \omega^k_j + \frac{2}{3} (\sigma^2 - \omega^2) \delta^i_j - E^i_j, \\
\frac{d\omega^i_j}{dt} &= -\frac{2}{3} \theta \omega^i_j - \sigma^i_k \omega^k_j - \omega^i_k \sigma^k_j,
\end{align*}
\]

where the Lagrange full time-derivative

\[
\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla
\]

is used on the left hand side. In these equations, $\sigma$ and $\omega$ denote the magnitudes of shear and rotation, respectively, defined by

\[
\sigma^2 = \frac{1}{2} \sigma^i_j \sigma^i_j \quad \text{and} \quad \omega^2 = \frac{1}{2} \omega^i_j \omega^i_j.
\]

The tensor $E^i_j = \nabla^i \nabla_j \phi - \delta^i_j \nabla^2 \phi / 3$ denotes the tidal force which comes from the spatial difference of the gravitational acceleration. In pure Newtonian dynamics, this is a slaving variable and there is no equation of motion for it. Fortunately, several excellent approximation schemes have been developed to deal with this highly non-local quantity. In this paper, we mainly follow the local tidal approximation (LTA) \[15\] \[16\], in which the equation of motion for the tensor $E^i_j$ is given by

\[
\frac{dE^i_j}{dt} = -\theta E^i_j - 4\pi G \rho \sigma^i_j.
\]

Hereafter we neglect the rotation for simplicity ($\omega^i_j = 0$). Actually the initial assumption of $\omega^i_j = 0$ is sufficient to guarantee this condition all the time as is seen from the above equation. Thus irrotational motions form a closed subclass of the full dynamics. Then, we can simultaneously diagonalize both the matrices $\sigma^i_j$ and $E^i_j$. We work in the frame where they are diagonalized as $\sigma^i_j$ and $E^i_j$ \( (i = 1, 2, 3) \), respectively. Since $\sigma^i_j$ and $E^i_j$ are traceless, only the two components of each are independent. Then it may be sometimes more convenient to introduce the following variables $\sigma_\pm$ and $E_\pm$:

\[
\begin{align*}
(\sigma_1, \sigma_2, \sigma_3) &= (\sigma_+ + \sqrt{3} \sigma_- , \sigma_+ - \sqrt{3} \sigma_- , -2 \sigma_+), \\
(E_1, E_2, E_3) &= (E_+ + \sqrt{3} E_- , E_+ - \sqrt{3} E_- , -2 E_+).
\end{align*}
\]

At a glance, we observe a scaling property in the above set of equations of motion \((2) - (8)\). They are invariant under the following scaling transformation:

\[
\begin{align*}
t &\to t' \equiv e^{s \Delta \tau} t, \\
x &\to x' \equiv e^{s \Delta \tau} x, \\
\theta &\to \theta' \equiv e^{-s \Delta \tau} \theta, \\
\sigma_\pm &\to \sigma_\pm' \equiv e^{-s \Delta \tau} \sigma_\pm, \\
E_\pm &\to E_\pm' \equiv e^{-s \Delta \tau} E_\pm, \\
\rho &\to \rho' \equiv e^{-2s \Delta \tau} \rho,
\end{align*}
\]

where $s$ is a constant free parameter and $\Delta \tau = \tau' - \tau$ measures a change of scale.
Now, based on these scaling properties, we derive renormalization group equations for spatially averaged observables on a constant time slice. To analyze the structure of the Universe, we adopt a scaling solution of our basic equations. Such a scaling could be realized as a result of a self-gravitating dynamical system with a scale invariant density fluctuation \[ \tilde{\delta} \], although here we do not specify any models to give such a scaling solution. If, however, we consider a scaling solution as (10) as it is, we find either a homogeneous Universe or inhomogeneous Universe with a "center", which is not consistent with the cosmological principle. We are interested in the inhomogeneous Universe model consistent with the cosmological principle such as a fractal universe. If we have a fractal universe, we expect some averaged values around an observer may show a scaling property. Such a scaling property will be always observed anywhere because of a fractal structure of the Universe, which is consistent with the cosmological principle. This is why we consider spatially averaged observables here. We assume a scaling property as (10) not only for averaged values of the variables in the above equations but also for those of the fluctuations such as a second order cumulant.

We define a spatial average \( \langle ... \rangle \) of an observable \( f(t, \mathbf{x}) \) at time \( t \) as

\[
\langle f \rangle_{\mathcal{D}(t)} = \frac{1}{V(t)} \int_{\mathcal{D}(t)} d^3x f(t, \mathbf{x}),
\]

where \( \mathcal{D}(t) \) is a spatial domain at time \( t \) and \( V(t) \) is its volume defined by \( V(t) = \int_{\mathcal{D}(t)} d^3x \). The observable \( f(t, \mathbf{x}) \) can be any function of \( \theta, \rho, \sigma_{\pm}, E_{\pm} \). We would like to derive the expression for the infinitesimal scale change of the averaged observable on the same time slice, i.e.

\[
\frac{\Delta \langle f \rangle_{\mathcal{D}(t)}}{\Delta \ell} = \frac{\langle f \rangle_{\mathcal{D}'(t)} - \langle f \rangle_{\mathcal{D}(t)}}{\Delta \ell},
\]

where the parameter \( \ell \) measures the true scale change on this time slice, which is defined by

\[
e^{\Delta \ell} = e^{\ell'-\ell} = \left( \frac{V'(t)}{V(t)} \right)^{1/3},
\]

and is different from the naive scaling parameter \( \tau \). \( \mathcal{D}'(t) \) and \( V'(t) \) denote the domain and volume transformed back to the original time \( t \)-slice by time evolution from \( \mathcal{D}'(t') \) and \( V'(t') \), which are obtained by a scale transformation (10). The above equation (12) can be decomposed into two parts:

(i) a change \( \Delta_S \langle f \rangle \) associated with a scale transformation from \( \mathcal{D}(t) \) to \( \mathcal{D}'(t') \),

(ii) a change \( \Delta_T \langle f \rangle \) associated with a time evolution from \( \mathcal{D}'(t') \) to \( \mathcal{D}'(t) \).

That is,

\[
\frac{\Delta \langle f \rangle_{\mathcal{D}(t)}}{\Delta \ell} = \frac{1}{\Delta \ell} \left[ \langle f \rangle_{\mathcal{D}'(t')} - \langle f \rangle_{\mathcal{D}(t)} + \langle f \rangle_{\mathcal{D}'(t)} - \langle f \rangle_{\mathcal{D}(t)} \right]
= \frac{1}{\Delta \ell} (\Delta_S \langle f \rangle + \Delta_T \langle f \rangle).
\]

As for part (i), we use the scaling law of the observable \( \langle f \rangle_{\mathcal{D}(t)} \), which is

\[
\langle f \rangle_{\mathcal{D}(t)} \rightarrow \langle f \rangle_{\mathcal{D}'(t')} = e^{\alpha_s \Delta \tau} \langle f \rangle_{\mathcal{D}(t)}.
\]

The second part (ii) is given by the Taylor expansion of \( f(t', \mathbf{x}') \) and \( \mathcal{D}'(t') \) with respect to time.

\[
\langle f \rangle_{\mathcal{D}'(t')} = \frac{1}{(1 + \theta \Delta t) V'(t)} \int_{\mathcal{D}'(t')} d^3x' \left[ f(t, \mathbf{x}') + \frac{df}{dt}(t, \mathbf{x}') \Delta t + O((\Delta t)^2) \right]
= \langle f \rangle_{\mathcal{D}'(t)} + \Delta t \left[ -\theta \langle f \rangle_{\mathcal{D}'(t')} + \theta \langle f \rangle_{\mathcal{D}'(t)} + \left( \frac{df}{dt} \right)_{\mathcal{D}'(t')} \right] + O((\Delta t)^2),
\]

where \( \Delta t = t' - t \).
The true scale change $\Delta \ell$ on a constant time slice is given as follows. In the scale transformation (i), the volume change is $V'(t')/V(t) = e^{3\Delta \tau} = 1 + 3\Delta \tau$, but it includes the expansion effect of the Universe because time is also transformed. To obtain the true scale change on the same time slice, we have to transform it back to the original time slice by the time evolution (ii), which gives $V'(t')/V(t) = 1 + \langle \theta \rangle \Delta t$. Then we have

$$\left( \frac{V'(t)}{V(t)} \right)^{1/3} = 1 + \Delta \tau - \frac{\langle \theta \rangle}{3} \Delta t$$

$$= 1 + \Delta \ell. \quad (17)$$

Thus we obtain

$$\Delta \ell = \Delta \tau - \frac{\langle \theta \rangle}{3} \Delta t. \quad (18)$$

The time interval $\Delta t = t' - t$ is given by the scale transformation as

$$\frac{\Delta t}{\Delta \tau} = st. \quad (19)$$

Hereafter we identify this fixed time $t$ as the present cosmic time $t_0$, when we observe cosmological quantities.

Thus, combining Eqs. (14) $\sim$ (19), we obtain the differential equation for the averaged observable $\langle f \rangle_{D(t)}$:

$$\frac{d(\langle f \rangle_{D(t)})}{d\ell} = S \left( \frac{\alpha}{t_0} (\langle f \rangle_{D(t)} + \langle \theta \rangle_{D(t)} \langle f \rangle_{D(t)} - \langle \theta \rangle_{D(t)} - \langle f \rangle_{D(t)}) \right), \quad (20)$$

where

$$S = \frac{s}{1 - s \langle \theta \rangle_{D(t)} t_0 / 3}. \quad (21)$$

The parameter $s$-dependence appears solely through this factor $S$.

The equation (20) is not yet sufficient to obtain a closed set of RG equations because of the non-linearity of the basic equations. Differential equations for the $n$-th order averaged quantities are not closed up to the $n$-th order. They necessarily contain $(n + 1)$-th order averaged quantities. This is the famous BBGKY hierarchy in statistical physics. The ordinary method to obtain the closed set of equations is to truncate this hierarchy at some order. In our case, we include the effect of fluctuations at the lowest non-trivial level, i.e. the second order cumulants. Then we will neglect all intrinsic fluctuations (cumulants) higher than the second order, finding the following truncation formula:

$$\langle fg \rangle \rightarrow \langle f \rangle \langle g \rangle + \langle fh \rangle + \langle gh \rangle - 2\langle f \rangle \langle g \rangle \langle h \rangle. \quad (22)$$

To write down the basic equations, we shall use the second order cumulant $\langle fg \rangle_c$ instead of the averaged quadratic quantity $\langle fg \rangle$, where $\langle fg \rangle_c \equiv \langle fg \rangle - \langle f \rangle \langle g \rangle$. Applying this reduction, we obtain a closed set of 27 differential equations for averaged variables and their second order cumulants. We will present the complete expression in the Appendix. Here we only show a reduced set of equations to the first order (the averaged variables) by setting $\langle fg \rangle_c = 0$:

$$\frac{d\langle \rho \rangle}{d\ell} = S \left( \langle \theta \rangle - 2 \right) \langle \rho \rangle, \quad (23)$$

$$\frac{d\langle \theta \rangle}{d\ell} = S \left[ \left( \frac{\langle \theta \rangle}{3} - 1 \right) \langle \theta \rangle + 4\pi G \langle \rho \rangle + 6(\langle \sigma_+ \rangle^2 + \langle \sigma_- \rangle^2) \right], \quad (24)$$

$$\frac{d\langle \sigma_+ \rangle}{d\ell} = S \left[ \left( \frac{2}{3} \langle \theta \rangle - 1 \right) \langle \sigma_+ \rangle - \langle \sigma_+ \rangle^2 + \langle \sigma_- \rangle^2 + \langle E_+ \rangle \right], \quad (25)$$

$$\frac{d\langle \sigma_- \rangle}{d\ell} = S \left[ \left( \frac{2}{3} \langle \theta \rangle - 1 \right) \langle \sigma_- \rangle + 2 \langle \sigma_+ \rangle \langle \sigma_- \rangle + \langle E_- \rangle \right], \quad (26)$$

$$\frac{d\langle E_+ \rangle}{d\ell} = S \left[ \langle \langle \theta \rangle - 2 \rangle \langle E_+ \rangle + 4\pi G \langle \rho \rangle \langle \sigma_+ \rangle \right], \quad (27)$$

$$\frac{d\langle E_- \rangle}{d\ell} = S \left[ \langle \langle \theta \rangle - 2 \rangle \langle E_- \rangle + 4\pi G \langle \rho \rangle \langle \sigma_- \rangle \right]. \quad (28)$$
These also form a closed set of equations at this order. We call these the first order RG equations, while those in the Appendix the second order RG equations.

Here and in Appendix, we have set $t_0 = 1$ and dropped the suffix of integration domain for averaged values just for simplicity. Hence, in this units, the expansion $\theta$ and the density $\rho$ for Einstein-de Sitter universe turn out to be $\theta_{E_{\text{dS}}} = 2$ and $\rho_{E_{\text{dS}}} = 1/6\pi G$

### III. ANALYSIS OF RG EQUATIONS : FIXED POINTS AND RG FLOW

We now examine the RG equations obtained in § 2. First, we analyze the first order RG equations (23) $\sim$ (28). Although this may not be realistic because the structure of the Universe could be highly inhomogeneous in our model, it shows a naive idea to understand the RG flow and give three important fixed points (including the Einstein-de Sitter spacetime), which will be also most relevant even in the case with fluctuations. We then examine the stability of these fixed points [18]. Next, introducing fluctuations, we study the second order RG equations (A1) $\sim$ (A27) and analyze the stability of those fixed points in a much wider parameter space. Finally in this section, we set the Einstein-de Sitter fixed point with small amount of fluctuations, as the boundary condition at the horizon scale in our present Universe and examine a scale dependence of averaged variables toward smaller scale.

#### A. Fixed points of the RG equations and Stability

First we analyze the structure of our RG equations. From Eqs. (A1) $\sim$ (A27), we find that those are written in a vector form as

$$
\frac{d\langle F \rangle}{d\ell} = S (A[\langle F \rangle] + C \cdot \langle FF \rangle_c)
$$

$$
\frac{d\langle FF \rangle_c}{d\ell} = S B[\langle F \rangle] \cdot \langle FF \rangle_c,
$$

where $\langle F \rangle$ and $\langle FF \rangle_c$ are 6- and 21-dimensional vectors defined as $\langle F \rangle \equiv \langle (f_i) \rangle$ and $\langle FF \rangle_c \equiv \langle (f_i f_j) \rangle$ with $f_i$’s $(i = 1 \sim 6)$ denoting $\rho, \theta, \sigma_+, \sigma_-, E_+$ and $E_-$. $A[\langle F \rangle], B[\langle F \rangle]$, and $C$ are 6-dimensional vector of quadratic functions of $\langle f_i \rangle$, 21$\times$21 matrix of linear functions of $\langle f_i \rangle$, and 6$\times$21 constant matrix, respectively. Setting $\langle FF \rangle_c = 0$, we have

$$
\frac{d\langle F \rangle}{d\ell} = SA[\langle F \rangle],
$$

which is exactly the first order RG equations (23) $\sim$ (28).

To analyze the structure of the RG equations, we first need to know the fixed point $\langle (F), \langle FF \rangle \rangle = \langle (F)^*, \langle FF \rangle_c^* \rangle$, which is defined by

$$
\left| \frac{d\langle F \rangle}{d\ell} \right|_{\langle F \rangle = \langle (F)^* \rangle, \langle FF \rangle_c = \langle (FF)_{c}^* \rangle} = \left| \frac{d\langle FF \rangle_c}{d\ell} \right|_{\langle F \rangle = \langle (F)^* \rangle, \langle FF \rangle_c = \langle (FF)_{c}^* \rangle} = 0.
$$

As for the first order RG equations, the fixed point $\langle F \rangle = \langle (F)^* \rangle$ is given by

$$
A[\langle F \rangle^*] = 0.
$$

The above equations (23), (33) guarantee that the fixed points $\langle F \rangle^*$ in the first order RG equations are always those for the second order RG equations with $\langle FF \rangle_{c}^* = 0$, although new additional fixed points appear in the second order.

The stability analysis is very important to know asymptotic behaviors near the fixed points. The stability is examined by linearizing the RG equations around the fixed point $\langle F \rangle^*, \langle FF \rangle_{c}^*$ as
where both $\mathcal{F}$ and $(\mathcal{F}_c)$ are small enough to make the linear perturbation treatment effective around $(F)^*$, $(FF)^*_c$.

Then, we find the perturbation equations in the form

$$\frac{d}{d\ell} \begin{pmatrix} \mathcal{F} \\ (\mathcal{F}_c) \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \mathcal{F} \\ (\mathcal{F}_c) \end{pmatrix},$$

where the perturbation matrix $W^{[2]} = (W_{AB})$ is given as

$$
\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial S_1(\langle F \rangle)}{\partial \langle F \rangle} & S^* C \\ \frac{\partial S_2(\langle F \rangle)}{\partial \langle F \rangle} & S^* B(\langle F \rangle) \end{pmatrix}
$$

with $S^* = S_{\langle F \rangle = \langle F \rangle^*}$. For the first order RG equations, it turns out to be

$$\frac{d\mathcal{F}}{d\ell} = W^{[1]}$$

where $W^{[1]}$ is given as

$$W^{[1]} = S^* \frac{\partial A(\langle F \rangle)}{\partial \langle F \rangle} |_{\langle F \rangle = \langle F \rangle^*}.$$  

This form guarantees that the eigenvalues of the first order matrix $W^{[1]}$ are always those of the second order matrix $W^{[2]}$. As we will show later, the other eigenvalues of the second order matrix are constructed by a sum of any two eigenvalues of the first order matrix (see Eqs. [18], [19]).

The positive (negative) eigenvalue of the matrix $W^{[2]}$ (or $W^{[1]}$) shows the instability (stability) of the fixed point $(\langle F \rangle^*, \langle FF \rangle^*_c)$ (or $(F)^*$) toward the larger scale, i.e. in the large $\ell$-direction. The stability and the instability are exchanged if we follow the RG flow in the opposite direction, i.e. toward the smaller scale.

The eigenvalues fully characterize the scale dependence of all physical variables in the vicinity of each fixed point. Although each observable is scaled according to its dimension, the effect of dynamics changes the scaling property from the original bare scaling law (10), so that the scaling of each observable is determined by the superposition of several eigen functions as follows:

$$\langle f_i \rangle = \langle f_i \rangle^* + \sum_{m=1}^{N} \epsilon_{i,m} e^{\lambda_m \ell}$$

$$\lambda_{\text{min}} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N = \lambda_{\text{max}},$$

where $\langle f_i \rangle^*$ is the value of $\langle f_i \rangle$ at a fixed point, $\epsilon_{i,m}$ is some small constant, and $\lambda_m$ ($m = 1 \sim N$) are eigenvalues. $N = n(n+3)/2$ for the second order RG equations, while $N = n$ for the first order RG equations, with $n$ being the number of observables. Since the maximum (minimum) eigenvalue, $\lambda_{\text{max}}(\lambda_{\text{min}})$ becomes the most relevant toward larger (smaller) scale, the scaling property of each observable around fixed point is determined by $\lambda_{\text{max}}(\lambda_{\text{min}})$.

The fixed points themselves are independent of the scaling parameter $s$, but the matrix $W^{[2]}$ (or $W^{[1]}$) and its eigenvalues depend on $s$. For the sake of explicit argument below, we restrict the range of the parameter $s$ as $0 < s < 1$ where the topology of the RG flow is not changed [19].
B. Fixed points and the flow of RG equations without fluctuations

Here we analyze the first order RG equations (23) \( \sim (28) \) in detail. First, by neglecting both shear and tidal force terms, we have simple RG equations (31) with

\[
A[\langle F \rangle] = \begin{pmatrix}
-\langle \theta \rangle + \frac{1}{3} \theta^2 + 4\pi G \langle \rho \rangle \\
\langle \theta \rangle - 2 \langle \rho \rangle
\end{pmatrix}
\]

(42)

for two averaged variables \( \langle F \rangle = (\langle \rho \rangle, \langle \theta \rangle) \). We easily find three fixed points \( \langle F \rangle^*_{(k)} \) \( (k = 1, 2, 3) \), which are listed in Table 1.

The stability at each fixed point \( \langle F \rangle^*_{(k)} \) is examined by linearizing the RG equations around the fixed point (Eq. (38)). All eigenvalues of \( W^{[1]} \) for three fixed points are also given in Table 1. For the first fixed point (1) where both \( \langle \theta \rangle^*_{(1)} \) and \( \langle \rho \rangle^*_{(1)} \) vanish, the dynamical term \( \Delta_T \langle F \rangle \) does not affect the scaling, so that the eigen vector points the direction of each observable. On the other hand, for other fixed points, the effect of dynamics changes a scaling property from the original bare scaling law (10).

As seen from Table 1, the fixed point (1) is stable toward larger scale. For increasing scale, the Universe approaches this fixed point, at which both the energy density and expansion vanish. Therefore we shall call it the fixed point Q as the Universe is quiescent. The second fixed point (2) is a saddle point. It represents the “Einstein-de Sitter” Universe because \( \langle \theta \rangle^*_{(2)} = \theta_{\text{EdS}} \) and \( \langle \rho \rangle^*_{(2)} = \rho_{\text{EdS}} \). We then call it the fixed point E. The third fixed point (3) is unstable in any direction. At first sight, it might look strange since the expansion of the Universe is finite in spite of vanishing energy density. This corresponds to the “Milne” Universe in which the expansion rate is apparently finite because of a specific choice of the time slice. We call it the fixed point M. This Universe is reduced to the “Minkowski” space by adjusting the time slice appropriately.

In Fig.1, to see more precisely those fixed points and the behaviors of global structure of the RG equations, we depict the RG flow on the \( H - \Omega \) plane, where the Hubble parameter \( H \) and the density parameter \( \Omega \) are defined by

\[
H \equiv \frac{\langle \theta \rangle}{3} \quad \text{and} \quad \Omega \equiv \frac{8\pi G \langle \rho \rangle}{3H^2}.
\]

(43)

(44)

The RG flow either approaches fixed point Q or escapes to infinity toward larger scale. These two kinds of flows are clearly separated by the divide which passes through two fixed points E and M (Fig.1). The flow escaping to infinity seems unphysical since both \( H \) and \( \Omega \) blow up to infinity at large scale.

Next we include shear and tidal force, which leads to Eqs. (23) \( \sim (28) \). The three fixed points Q, M and E found in the case without shear and tidal force survive, and their stabilities also remain unchanged (see Table 2). In this sense, the three fixed points Q, M and E are robust. Four new fixed points appear in addition to those three fixed points, which are listed in Table 2. Since all new fixed points are saddle points, any RG flow still approaches the fixed point Q or escapes to infinity toward larger scale, or it approaches the fixed point M toward smaller scale.

C. Fixed points and the flow of RG equations with fluctuations

Since our model of the Universe may be quite inhomogeneous and could be fractal in a microscopic point of view, an inclusion of fluctuations of the physical variables is inevitably important, when we compare our results with those for the real Universe. In fact, if we do not include fluctuations, it is mathematically equivalent to a self-similar solution on a fixed time slice [20], although the physical meaning is quite different. Then, we shall add fluctuations and see how the fluctuations affect the RG flow diagram. We just include second order cumulants to the RG equations and truncate this hierarchy there by ignoring all intrinsic fluctuations (cumulants) higher than the second order, although we may have another choice of higher-order cumulants [23].
There are 27 independent variables in this case and it becomes more difficult to find a full list of fixed points. Then, first we reduce the number of variables and start our study by the shear-free case. Since independent observables are just $\rho$ and $\theta$ in this case, the total number of variables including second order cumulants is just five: $\langle \rho \rangle$, $\langle \theta \rangle$, $\langle \rho^2 \rangle_c$, $\langle \theta^2 \rangle_c$, $\langle \rho \theta \rangle_c$. We list up all fixed points and show their stabilities in Table 3. The three important fixed points (1)~(3) in the first order RG equations still remain and their stabilities are also unchanged. They are quite robust. Since all of the new fixed points (8)~(10) are saddle points, global properties of RG flows toward both larger and smaller scales are solely determined by the fixed points $Q$ and $M$, respectively. The fixed point $E$ remains as a saddle point.

As we mentioned, the fixed points found in the first order equations are also those in the second order with zero cumulants. For the eigenvalues at those fixed points, we find eigenvalues in the first order are also those in the second order. To see more detail, we explicitly write down the perturbation matrix $W^{[2]}$ as

\[
W^{[1]} \equiv \begin{pmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{pmatrix} = S^* \begin{pmatrix}
\langle \theta \rangle^* - 2 & \langle \rho \rangle^* \\
4\pi G & \frac{2}{3} \langle \theta \rangle^* - 1
\end{pmatrix}, \quad S^* C = S^* \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{2}{3} & 0
\end{pmatrix}
\]

Then, the eigenvalue equation for the second order perturbation matrix $W^{[2]}$ is

\[
(\lambda^2 - \text{tr} W^{[1]} \cdot \lambda + \det W^{[1]})(\lambda - \text{tr} W^{[1]})(\lambda^2 - 2 \text{ tr} W^{[1]} \cdot \lambda + 4 \det W^{[1]})) = 0.
\]  

(46)

Since the eigenvalue equation for the first order matrix $W^{[1]}$ is

\[
\lambda^2 - \text{tr} W^{[1]} \cdot \lambda + \det W^{[1]} = 0,
\]  

(47)

supposing that $\lambda^{[1]}_1$ and $\lambda^{[1]}_2$ are its two solutions, we find the solutions of Eq. (43), $\lambda^{[2]}_i$ ($i = 1 \sim 5$), as

\[
\begin{align*}
\lambda^{[2]}_1 &= \lambda^{[1]}_1, \\
\lambda^{[2]}_2 &= \lambda^{[1]}_2, \\
\lambda^{[2]}_3 &= 2\lambda^{[1]}_1, \\
\lambda^{[2]}_4 &= \text{tr} W^{[1]} = \lambda^{[1]}_1 + \lambda^{[1]}_2, \\
\lambda^{[2]}_5 &= 2\lambda^{[1]}_2.
\end{align*}
\]

(48)

Therefore the minimum eigenvalues at fixed points in the first order, in particular the fixed points $Q$, $E$ and $M$, are given by

\[
\lambda^{[2]}_{\text{min}} = 2\lambda^{[1]}_{\text{min}},
\]  

(49)

if $\lambda^{[1]}_{\text{min}} < 0$ (this is the case for the fixed points $Q$ and $E$), otherwise

\[
\lambda^{[2]}_{\text{min}} = \lambda^{[1]}_{\text{min}},
\]  

(50)

(this is the case for the fixed point $M$).

Finally, we study a full model with 27 RG equations for 6 independent variables including shear and tidal force and their second order cumulants. Though it is very difficult to find all fixed points, all seven fixed points in the first order RG equations (listed in Table 2) remain as fixed points, as we mentioned. As for their eigenvalues, we also find the same result as that in the case without shear and tidal force, that is, the eigenvalue in the second order RG equations still remain and their stabilities are also unchanged. They are quite robust. Since all of the new fixed points (8)~(10) are saddle points, global properties of RG flows toward both larger and smaller scales are solely determined by the fixed points $Q$ and $M$, respectively. The fixed point $E$ remains as a saddle point.
where \( n \) is the number of observables. This result is understood from some special relation between the perturbation matrices \( W[1] \) and \( S^*B[\langle F\rangle^*] \) [24]:

\[
S^*B_{ij,kl}[\langle F\rangle^*] = \frac{1}{2} [\delta_{ik} w_{jl} + \delta_{il} w_{jk} + \delta_{jk} w_{il} + \delta_{jl} w_{ik}],
\]

(52)

where \( i, j, k, l \) denote the observables \( \rho, \theta, \sigma_\pm, E_\pm \).

In Table 4, we have only listed three important fixed points Q, E and M with their stabilities. These three fixed points still survive and their stabilities remain unchanged even after including fluctuations as well as shear and tidal force.

Let us just mention how our results change if we use another approximation scheme for the equation of motion for \( E_{ij} \). We mainly use LTA approximation in this paper. In the case of other local approximations like NMA [25], the RG equations become more complicated and it is more difficult to look for all fixed points and eigenvalues. However, it is still easy to show that the above three fixed points Q, M and E survive since the RG equations exactly reduce to the shear-free case when we set \( E_\pm = \sigma_\pm = 0 \). The stabilities of these fixed points can also be checked by examining the RG flow around them. We show that the stabilities of these fixed points remain unchanged even in the NMA. The minimum eigenvalue, which is the most relevant toward smaller scale, is also invariant irrespective of local approximation scheme of \( E_{ij} \), because the eigen-direction for \( \lambda_{\min}^{[2]} \) (or \( \lambda_{\min}^{[1]} \)) is orthogonal to both shear and tidal directions in 27-dimensional “phase” space of \( \langle F \rangle, \langle FF \rangle \) (or in 6-dimensional “phase” space of \( \langle F \rangle \)). Hence the effect of shear or tidal force seems irrelevant to the scaling property of observables around fixed point E or M toward smaller scale.

The three fixed points E, M, and Q are robust and change neither their stability nor the scaling property of observables around them toward smaller scale, regardless of shear, tidal force, and then the approximation scheme of the tidal force.

D. The flow of our Universe

The RG flow and the fixed points we argued in the previous subsections describe all the possible diversity of our system. Only a part of them, i.e. a particular integral line in the flow diagram which satisfies certain boundary condition, actually describes our real Universe. In order to elucidate such integral flow, we need a plausible boundary condition and examine the global/local scaling properties of observables.

One plausible boundary condition at the present horizon scale and beyond could be the fixed point E (“Einstein-de Sitter” Universe). This boundary condition would be inferred from the inflationary scenario in the early Universe, which predicts the flat FRW Universe with tiny fluctuations as our real Universe.

With this boundary condition, we solve the RG equations toward smaller scale and study the scale change of various observables. In solving the RG equations toward the smaller scale, we must remember that (i) the positive (negative) eigenvalue of the matrix \( W^{[2]} \) at a fixed point denotes stability (instability). (ii) The direction of the integration is opposite to that of flow indicated in Fig. 1. It is clear that the choice of the exact values of point E as a boundary condition is trivial and not interesting because it is the fixed point and the flow stays there forever. To find more interesting model of the Universe, we need to start from the close vicinity of the fixed point E.

Here we choose the boundary condition as follows. For the averaged variables, we use the exact value of the fixed point E:

\[
\langle \theta \rangle = \theta_{\text{E}dS} = 2, \quad \langle \rho \rangle = \rho_{\text{E}dS} = 1/6\pi G, \quad \langle \sigma_\pm \rangle = \langle E_\pm \rangle = 0,
\]

(53)

which gives \( \Omega = 1 \). Since we expect some fluctuations even in an inflationary model, we set small finite fluctuations at the fixed E point. We assume that the off-diagonal fluctuations have initially no intrinsic fluctuations [24], i.e.

\[
\langle f_i f_j \rangle = 0 \quad (i \neq j) \quad \text{at} \quad \ell = 0.
\]

(54)
Before going to show our numerical result, we shall discuss the asymptotic behaviors near three robust fixed points, Q, E, and M. We showed that the scaling property of observables around those fixed points toward smaller scale is determined by the minimum eigenvalue $\lambda^{[2]}_{\text{min}}$. This is always true for the second order cumulants $\langle f_i f_j \rangle_c$. The scaling property of the averaged variable $\langle f_i \rangle$ may, however, depend on the case, because the eigen vector with the eigenvalue $\lambda^{[1]}_{\text{min}}$ is perpendicular to the hypersurface of the cumulants in the “phase” space ($\langle F \rangle, \langle FF \rangle_c$). If we have fluctuations initially rather than a deviation from the fixed point, i.e. $\langle f_i^2 \rangle_c^* \neq 0$, then
\[
\langle f_i \rangle \sim (\langle f_i \rangle)^* + \epsilon_i e^{\lambda^{[2]}_{\text{min}} \ell},
\]
as $\ell$ decreases, unless $\langle f_i^2 \rangle_c^* \geq (\langle f_i \rangle)^2$. This is just because how to leave from the fixed point is determined by the fluctuations $\langle f_i^2 \rangle_c^*$, which scaling property is fixed by $\lambda^{[2]}_{\text{min}}$. On the other hand, if we have an initial deviation from the fixed point as $\langle f_i \rangle \neq (\langle f_i \rangle)^*$ without the initial cumulant $\langle f_i^2 \rangle_c^* = 0$, then $\langle f_i \rangle - (\langle f_i \rangle)^*$ behaves first as $\exp(\lambda^{[1]}_{\text{min}} \ell)$. Since our boundary condition is the former case, we expect the asymptotic behavior as Eq. (55).

It may be more comprehensive to show our results if we introduce the “dispersions” $\bar{\Delta}_{ij}$ instead of the cumulants $\langle f_i f_j \rangle_c$. For the diagonal cumulants $\langle f_i^2 \rangle_c$, we define them as
\[
\Delta_{\rho}^2 = \langle \rho^2 \rangle_c / \langle \rho \rangle^2, \quad \Delta_{\theta}^2 = \langle \theta^2 \rangle_c / \langle \theta \rangle^2, \quad \Delta_{\sigma^2}^2 = \langle \sigma^2 \rangle_c / \langle \theta \rangle^2, \quad \Delta_{E_{\pm}}^2 = \langle E_{\pm}^2 \rangle_c / \langle \theta \rangle^4,
\]
where for shear and tidal force, $\Delta_{\sigma^2}$ and $\Delta_{E_{\pm}}$, which are normalized by $\langle \theta \rangle$, have been used rather than the conventional dispersions because those averaged variables sometimes vanish even if their cumulants are finite, resulting in a divergence of the dispersions. The dispersion parameters $\Delta_{\rho}$, $\Delta_{\theta}$, $\Delta_{\sigma^2}$, $\Delta_{E_{\pm}}$ are assumed to be sufficiently small at the boundary $\ell = 0$. In what follows, just for simplicity, we also set $\Delta_{\rho} = \Delta_{\theta} \equiv \Delta_0$ at $\ell = 0$, which may be plausible because $\rho$ and $\theta$ may be closely connected with each other through the fluid equations and are expected to have the same order of fluctuations.

\section{The case without shear and tidal force}

First we consider the case without shear and tidal force. In Fig. 2, we depict the scale dependence of $\Omega, H, \Delta_{\rho}, \Delta_{\theta}$ in terms of $L \equiv e^\ell$, which is the ratio of the averaging scale to the horizon scale (or the scale at our boundary). Since we have chosen $\ell = 0$ as the horizon scale (or beyond), our observable Universe is described by negative $\ell$. As seen from Fig. 2, the integrated flow line always converges to the values at fixed point M, i.e. $\Omega_{\text{Milne}} = 0$ and $H_{\text{Milne}} = 1$. This means that the “Milne” Universe is the unique stable fixed point toward smaller scale. Once it comes close to fixed point M, the behavior of the RG flow at small scale is determined by the minimum eigenvalue at M, i.e. $\lambda^{[2]}_{\text{min}} = s/(1-s)$.

To see a role of the scaling parameter $s$, in Fig. 3, we plot the scale dependence of the density parameter $\Omega$, the expansion $H$, and the fluctuations $\Delta_{\rho}, \Delta_{\theta}$ for several values of $s$ with fixed initial fluctuations $\Delta_0 = 10^{-2}$. We find that the observables show the following universal behavior:

(i) $H$ increases toward smaller scale from $2/3$ to 1.
(ii) $\Omega$ reduces from 1 to 0 toward smaller scale. There is a plateau near the fixed point E, while a constant slope is found near the fixed point M.

The asymptotic values are given by those at the fixed points E and M. We can also explain a constant slope at the smaller scale and a plateau at larger scale in $\Omega$ diagram. Since $\langle \rho \rangle \sim O(\exp\{[s/(1-s)]\ell\})$ and $\langle \theta \rangle \sim 3$ near the fixed point M, $\Omega$ at smaller scale behaves as
\[
\Omega \sim O(\exp\{[s/(1-s)]\ell\}),
\]
where $s/(1-s)$ is the minimum eigenvalue evaluated at the fixed point M. Then the slope of the $\Omega$-curve near M turns out to be
\[
\frac{d \log \Omega}{d \log L} \sim \frac{s}{1-s}.
\]
On the other hand, the plateau appears near the fixed point E, because
\[ \langle \rho \rangle / \rho_{\text{EdS}} \sim 1 + \epsilon_\rho \exp\{-2s/(3-2s)\} \ell, \quad \langle \theta \rangle / \rho_{\text{EdS}} \sim 1 + \epsilon_\theta \exp\{-2s/(3-2s)\} \ell \]
and then
\[ \Omega \sim 1 + (\epsilon_\rho - 2\epsilon_\theta) \exp\{-2s/(3-2s)\} \ell \approx 1, \]
where we have used \( \lambda_{\min}^2 = -2s/(3-2s) \) evaluated at E. The turning point \( L_{cr} \) of the slope into the plateau is estimated easily. Since the deviations from the EdS values are due to the fluctuations \( \Delta_0, \epsilon_\rho, \epsilon_\theta \sim O(\Delta_0) \). Then, when \( \Delta_0 \exp\{-2s/(3-2s)\} \ell_{cr} \sim O(1) \), i.e.
\[ \ln L_{cr} = \ell_{cr} \sim [(3-2s)/2s] \ln \Delta_0, \]
we find a large deviation from the EdS values. Thus the minima of the eigen values at two fixed points E and M determine the asymptotic structure of our Universe.

Is there any observational data which may suggest a scale dependence of either \( \Omega \) or density? In discussion of dark matter, when we plot the mass-luminosity ratio in terms of the scale of the objects, we find a similar curve to Fig. 3(a). If we assume that the mass-luminosity ratio curve is what we are discussing here, we find some constraint of \( s \) as \( 0.3 \lesssim s \gtrsim 0.7 \).

2. The case with shear and tidal force

Next, we examine the effect of shear and tidal force. Just for simplicity, we assume that \( \sigma_- = E_- = 0 \), which is consistent with the RG equations. As we assumed, \( \langle \sigma_+ \rangle = \langle E_+ \rangle = 0 \). Then initial free parameters are \( \Delta_0, \Delta_{\sigma-0} \) and \( \Delta_{E-0} \), which are fluctuations at \( \ell = 0 \). Our numerical calculations, shown in Figs. 4, 5, reveal that they tend to prevent \( H \) and \( \Omega \) from converging to the fixed point M though “Milne” Universe is still a stable fixed point. More precisely, we observe three qualitatively different behaviors of the RG flow of our Universe:

(i) The RG flow monotonically converges to fixed point M. \( H \) and \( \Omega \) approach to the value determined by fixed point M toward smaller scale.

(ii) Initially the Universe comes near the fixed M point, resulting that \( H \) increases and \( \Omega \) decreases toward smaller scale. However it turns around before reaching the fixed point M and eventually diverges to infinity in the flow diagram as \( H \to -\infty \) and \( \Omega \to \infty \).

(iii) The RG flow monotonically diverges to infinity in the flow diagram as \( H \to -\infty \) and \( \Omega \to \infty \).

We can classify the parameter space \( (\Delta_0, \Delta_{\sigma+0}, \Delta_{E+0}) \) of fluctuations at the boundary (near the horizon) into three regions according to the above asymptotic behaviors of the RG flow. The class (i) is realized if the fluctuations of shear and tidal force are sufficiently smaller than those of the density and the expansion, i.e. \( \Delta_{\sigma+0}/\Delta_0 \sim \Delta_{E+0}/\Delta_0 < R_{cr} \). Otherwise, i.e. if \( \Delta_{\sigma+0}/\Delta_0 \sim \Delta_{E+0}/\Delta_0 > R_{cr} \), the class (iii) will happen. The critical value \( R_{cr} \) is almost independent of \( \Delta_0 \) and approximately \( R_{cr} \approx 0.15 \). The class (ii) is a narrow boundary between them. Hence, the class (i) (or (ii)) naturally occurs as long as the shear and tidal force are sufficiently smaller than the density fluctuations at the horizon scale. This situation is strongly suggested by recent observations \(^{28}\), i.e. the shear is strictly restricted around \( \sigma/H < 10^{-9} \) by CMB data. Hence we expect that our Universe belongs to the class (i) and its flow line converges into the fixed point M in smaller scale. The \( s \) dependence of \( \Omega \) or \( H \) is quite similar to the case without shear and tidal force (Fig. 6 (a), (b)). As for the shear and tidal force, their averaged values increase toward smaller scale, but eventually decrease if the flow will converge to the fixed point M (Fig. 6 (c), (d)). The scaling properties of the shear \( \langle \sigma_+ \rangle \) and the tidal force are...
\[ E \] are also determined by \( \lambda_{\text{min}}^2 \) at the fixed points \( E \) and \( M \). Both shear and tidal force are dumped off at smaller scale along the flow converging to the fixed point \( M \).

The behaviors of fluctuations \( \Delta \rho, \Delta \theta, \Delta_{\sigma+} \) and \( \Delta_{E+} \) are shown in Fig. 6 (e)∼(h). We find that the similar behaviors to the case without shear and tidal force for \( \Delta \rho \) and \( \Delta \theta \).

E. Beyond the horizon scale

In the previous subsections, we have integrated the RG equations from the horizon scale (\( \ell=0 \)) toward the smaller scale. It is also possible to solve them toward the larger scale with the same boundary conditions. The structure beyond the horizon scale is not observable at present but may give some relation with an inflationary scenario in the early Universe. In Fig. 7, we plot the scale dependence of \( \langle \rho \rangle, H \) and those fluctuations \( \Delta \rho \) and \( \Delta \theta \) in shear-free case with initial fluctuation \( \Delta_0 = 10^{-2.5} \). In the finite scale \( 10^{-5} < L < 10^{5} \), the flow stays in the vicinity of fixed point \( E \) and the power law behaviors of \( \Delta \rho \) and \( \Delta \theta \) toward larger scale are determined by \( \lambda_{\text{max}}^2 \) at fixed point \( E \), while in larger scale \( L > 10^{5} \), the flow converges to fixed point \( Q \) after \( H \) becomes negative and their power law behaviors are characterized by the eigenvalues of \( W^2 \) at the fixed point \( Q \). These eigenvalues are the same ones naively determined by the original scaling. This is consistent with the fact that the expansion ceases at the fixed point \( Q \) and the RG transformation reduces to a pure scale transformation. This property makes \( \Delta \rho \) constant around fixed point \( Q \) because of the cancellation between numerator and denominator in the definition of (56) (Fig. 7 (c)), while \( \Delta \theta \) diverges at the scale where \( H \) vanishes and turns negative from the definition (56) (Fig. 7 (d)). The declination of the slope in the \( \log \langle \rho \rangle - \log L \) diagram (Fig. 7 (a)) is \(-2s \) which is the minimum eigenvalue of \( W^2 \) associated with the fixed point \( Q \). If we assign a fractal dimension \( D \) for this Universe, \( D = 3 - 2s \) because \( \langle \rho \rangle \) reduces as \( e^{(D-3)t} \) from a fractal structure while \( \langle \rho \rangle \propto e^{-2st} \) by our scaling law. If we assign the value \( s = 0.5 \), the corresponding fractal dimension becomes \( D = 2 \). Because our model is a Newtonian cosmology, further analysis beyond the horizon scale would be irrelevant to the real Universe.

IV. CONCLUDING REMARKS

The renormalization group method has been applied to Newtonian cosmology in this paper and the scale dependence of the averaged observables on a fixed time slice including effects of fluctuations has been studied. The scaling assumption we proposed for the averaged observables is fulfilled in the case of either homogeneous distribution or non-analytic distribution such as a fractal, if we assume such a scaling at any position and at fixed time. Actual observation of the two point correlation function of galaxies may favor such a fractality at least in the finite scale range below 10 Mpc. Multi-fractality could be applied to explain the transition from finite fractal range at smaller scale to homogeneous distribution at larger scale [29]. Our scaling analysis of averaged observables here might give a hint to the behavior of observables in such a fractal matter distribution, if it exists.

To close our dynamical system and apply the RG method, here we have adopted an approximation of tidal force (LTA) and considered only fluctuations up to the second order. The higher order cumulants of averaged observables are ignored. We have found three robust fixed points of the RG equations.

1. The fixed point \( Q \) (“Quiescent” Universe) is stable in any directions toward larger scale.
2. The fixed point \( E \) (“Einstein-de Sitter” Universe) is a saddle point.
3. The fixed point \( M \) (“Milne” Universe) is unstable in any directions toward larger scale.

These fixed points do not change their stability even if we extend the “phase” space including fluctuations as well as shear and tidal force. Any other fixed points are saddle, and therefore the scaling property of the global RG flow toward larger (smaller) scale is determined by fixed point \( Q \) (M). We find that the Universe asymptotically approaches fixed point \( Q \) or diverges toward infinitely regardless of the detail at the smaller scale.

In order to find out the flow line which represents our real Universe, we have imposed the boundary condition fixed point \( E \) with tiny fluctuations at the horizon scale. The inflationary scenario in the early Universe would imply such
a boundary condition, i.e. the inflation predicts the Einstein-de Sitter Universe at the horizon scale and beyond. We have solved the RG equation toward smaller scale from this boundary. Then we find that, toward smaller scale, the expansion $H$ defined by $\langle \theta \rangle/3$ monotonically increases from $2/3$ to $1$, and the density parameter $\Omega = 8\pi G(\rho)/3H^2$ monotonically reduces from $1$ to $0$, provided the fluctuations of shear and tidal force are much smaller than the density fluctuations, which would be supported by the inflationary scenario [30] and observations of CMB.

We have two important free parameters, $s$ and $\Delta_0$. These values might be fixed by some of the following arguments:

(i) The scale dependence of the Mass-Luminosity ratio of astrophysical objects.
(ii) A systematic decrease of the Hubble parameter $H$, if any, toward the larger scale.
(iii) The scale dependence of the amplitude of the two point correlation functions.
(iv) Some of the inflationary models predict a fractal-like structure beyond the horizon scale [31].

Several comments related with our future work are in order.

(a) We definitely need the general relativistic generalization of our work. This is because the large scale structures at high redshift and beyond the horizon scale would be adequately described only by general relativity.
(b) In our analysis, we assume a scaling property of the averaged observables and their fluctuations. However we do not know whether such a solution is dynamically stable or not. To answer for this, we have to consider the dynamical stability of our system.
(c) So far we have considered fluctuations of variables up to the second order. Though this would be sufficient when we consider the vicinity of the horizon scale, we ultimately need to incorporate higher order fluctuations as well for the consistency of the theory.
(d) We have used the Local Tidal Approximation (LTA) for obtaining equations of motions for tidal force. By this method, we could avoid directly solving Poisson equation. This approximation would be justified upto mildly non-linear regime. However it may not be valid in the fully non-linear regime, especially in smaller scales, where the fluctuations grow $\Delta_\rho \gg 1$. In order to treat the smaller scales properly, we need to improve LTA in our future study. We would like to report the analysis on these issues in the near future.

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In this approximation, the third and higher order spatial derivatives are systematically neglected in the equation of motion for the tidal force. This is a generalization of the Zel’dovich approximation, and accuracies of these local approximations are examined for spheroidal configurations by Hui and Bertschinger [17]. It is also worth noticing that the choice of approximation scheme does not change our basic idea in this paper and many of them give similar results.

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[18] The RG equations are written in terms of a scale parameter \( \ell \). Then a “stability” here does not mean dynamical stability. It means what kind of asymptotic behavior of some averaged variables would be expected when a scale is changed.

[19] Other choices of \( s \) are mathematically viable, however they seem not to provide us any interesting model of the Universe.

[20] Such a similarity solution with \( s \neq 1 \) is related to “partial similarity” [21] or ”kinematic self-similarity” [22] in general relativity.

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[22] A.A. Coley, Class. Quantum Grav. 14, 87 (1997).

[23] If the structure of the Universe is described by Rayleigh-Lévy fractal, the higher-order cumulants are given as the three-point correlation function \( \zeta_{123} = \frac{1}{2}(\xi_{12}\xi_{23} + \xi_{23}\xi_{31} + \xi_{31}\xi_{12}) \) and the four-point correlation function \( \eta_{1234} = \frac{1}{4}(\xi_{12}\xi_{23}\xi_{34} + \cdots 12\text{terms}) \)

[24] This relation may be true only for the present case without higher-order cumulants.

[25] In this approximation, the magnetic part of the Weyl tensor is neglected.

[26] Here, “initial” means \( \ell = 0 \), which denotes the largest averaging scale of our Universe (~ horizon scale)

[27] M. Davis, and P.J.E. Peebles, Astrophys. J.267,465(1983); N.A. Bahcall and R.M. Soneira,Astrophys. J.270,20 (1983); M. Postman, J.P. Huchra,and M.J. Geller,Astrophys. J.384,404 (1992).

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[29] R. Balian and R. Schaeffer, Astron. Astrophys. 220, 1 (1989); 226, 374 (1989).

[30] J.D. Barrow and A.R. Liddle, preprint gr-qc/9705048 (1997) (to be published in General Relativity and Gravitation).

[31] M. Aryal and A. Vilenkin, Phys. Lett. B199, 351 (1984).
Table 1. The list of three fixed points and their eigenvalues in the case with neither shear and tidal force nor fluctuations.

|   | $6\pi G (\rho)^* (\theta)^*$ | $S^*$ | two eigenvalues |
|---|-------------------------------|--------|-----------------|
| 1 | Q                             | 0 0 s  | $\{-2S^*, -S^*\}$ |
| 2 | E                             | 1 2 $\frac{3s}{3-2s}$ | $\{-\frac{2}{3}S^*, S^*\}$ |
| 3 | M                             | 0 3 $\frac{s}{1-s}$ | $\{S^*, S^*\}$ |

Table 2. The list of 7 fixed points and their eigenvalues in the case without fluctuations in LTA. The seventh fixed point [7] is not an isolated point but a closed loop parameterized by one parameter $\eta$ as $\sigma^+_1 = \frac{1}{3} \cos \eta$, $\sigma^+_2 = \frac{1}{3} \sin \eta$, $E^+_1 = \frac{1}{3}(\cos 2\eta - 3\cos \eta)$, $E^+_2 = -\frac{1}{3}(\sin 2\eta + 3\sin \eta)$, where $0 \leq \eta < 2\pi$. $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ are three real solutions $(-1 \leq \bar{\lambda}_1 < \bar{\lambda}_2 \leq 0, \bar{\lambda}_3 \geq 1)$ of $\bar{\lambda}^3 - \bar{\lambda} - 4(1 - \cos 3\eta)/27 = 0$.

|   | $6\pi G (\rho)^* (\theta)^*$ | $\langle \sigma^+_1 \rangle^*$ | $\langle \sigma^- \rangle^*$ | $\langle E^+_1 \rangle^*$ | $\langle E^- \rangle^*$ | $S^*$ | six eigenvalues |
|---|-------------------------------|-------------------------------|-------------------------------|------------------------------|------------------------------|--------|-----------------|
| 1 | Q                             | 0 0 0 0 0 0 0                 | $s$                          | $\{-2S^*, -2S^*, -2S^*, -S^*, -S^*, -S^*, -S^*\}$ |
| 2 | E                             | 1 2 0 0 0 0                 | $\frac{3s}{3-2s}$           | $\{-\frac{2}{3}S^*, -\frac{2}{3}S^*, \frac{2}{3}S^*, S^*, S^*, S^*\}$ |
| 3 | M                             | 0 3 0 0 0 0                   | $\frac{s}{1-s}$           | $\{S^*, S^*, S^*, S^*, S^*, S^*\}$ |
| 4 |                               | 0 1 $-\frac{1}{3}$ 0 0 0       | $\frac{3s}{3-s}$           | $\{-S^*, -S^*, -S^*, -S^*, -S^*, -S^*, S^*\}$ |
| 5 |                               | 0 1 $\frac{1}{6}$ $\frac{1}{2\sqrt{3}}$ 0 0 | $\frac{3s}{3-s}$           | $\{-S^*, -S^*, -S^*, -S^*, -S^*, -S^*, S^*\}$ |
| 6 |                               | 0 1 $\frac{1}{6}$ $-\frac{1}{2\sqrt{3}}$ 0 0 | $\frac{3s}{3-s}$           | $\{-S^*, -S^*, -S^*, -S^*, -S^*, -S^*, S^*\}$ |
| 7 |                               | 0 2 $\sigma^+_1$ $\sigma^- \ E^+_1$ | $\frac{3s}{3-2s}$           | $\{\bar{\lambda}_1 S^*, \bar{\lambda}_2 S^*, 0, 0, S^*, \bar{\lambda}_3 S^*\}$ |
\[6\pi G(\rho)^* \langle \theta \rangle^* 36\pi^2 G^2(\rho^2)_{c}^* \langle \theta^2 \rangle_{c}^* 6\pi G(\rho^2)_{c}^* \] \[S^* \text{ five eigenvalues} \]

| (1) Q | 0 0 0 0 0 | \(s\) | \{-4S^*, -3S^*, -2S^*, -2S^*, -S^*\} |
| (2) E | 1 2 0 0 0 | \(\frac{3s}{3-2s}\) | \{-\frac{4}{3}S^*, -\frac{2}{3}S^*, \frac{1}{3}S^*, S^*, 2S^*\} |
| (3) M | 0 3 0 0 0 | \(\frac{s}{1-s}\) | \{S^*, S^*, 2S^*, 2S^*, 2S^*\} |
| (8) | 0 \(\frac{3}{2}\) 0 \(-\frac{9}{8}\) 0 | \(\frac{2s}{2-s}\) | \{-S^*, -S^*, -\frac{1}{2}S^*, -\frac{1}{2}S^*, S^*\} |
| (9) | 0 \(\frac{9}{5}\) 0 \(-\frac{27}{25}\) \(\frac{81}{250}\) | \(\frac{5s}{5-3s}\) | \{-\frac{1+\sqrt{13}}{5}S^*, -\frac{2}{5}S^*, -\frac{1}{5}S^*, -\frac{1+\sqrt{13}}{5}S^*, S^*\} |
| (10) | 0 2 \(-\frac{1}{4}\) \(-1\) \(\frac{1}{2}\) | \(\frac{3s}{3-2s}\) | \{-\frac{1+\sqrt{17}}{6}S^*, 0, -\frac{1+\sqrt{17}}{6}S^*, \frac{2}{3}S^*, S^*\} |

Table 3. The list of fixed points and their eigenvalues in the case without shear and tidal force including fluctuations. New fixed points (8) \sim (10) are unphysical because \(\langle \rho^2 \rangle_{c}^*\) and/or \(\langle \theta^2 \rangle_{c}^*\) are negative.

\[
\begin{align*}
\begin{array}{cccccc}
6\pi G(\rho)^* & \langle \theta \rangle^* & \langle \text{others} \rangle^* & S^* & 27 \text{ eigenvalues} \\
(1) Q & 0 & 0 & 0 & \(s\) & \{-4S^* \times 6, (-3S^*) \times 9, (-2S^*) \times 9, (-S^*) \times 3\} \\
(2) E & 1 & 2 & 0 & \(\frac{3s}{3-2s}\) & \left\{ \left(-\frac{4}{3}S^*\right) \times 6, \left(-\frac{2}{3}S^*\right) \times 3, \left(\frac{1}{3}S^*\right) \times 9, (S^*) \times 3, (2S^*) \times 6 \right\} \\
(3) M & 0 & 3 & 0 & \(\frac{s}{1-s}\) & \{S^* \times 6, (2S^*) \times 21\} \\
\end{array}
\end{align*}
\]

Table 4. Three important fixed points and their eigenvalues in the case with shear and tidal force in LTA as well as fluctuations.
APPENDIX A: THE BASIC EQUATIONS FOR THE CASE WITH FLUCTUATIONS

To be complete, here, we present a full set of our basic equations, which have been used in our renormalization analysis for Newtonian cosmology. Those include fluctuations up to second order cumulants as well as shear and tidal force.

(A) The RG equations for averaged variables

\[
\frac{d\langle \rho \rangle}{d\ell} = S \left[ (\langle \theta \rangle - 2) \langle \rho \rangle \right] \quad (A1)
\]

\[
\frac{d\langle \theta \rangle}{d\ell} = S \left[ \left( \frac{1}{3} \langle \theta \rangle - 1 \right) \langle \theta \rangle + 4\pi G \langle \rho \rangle + 6 \left( (\langle \sigma_+ \rangle)^2 + (\langle \sigma_- \rangle)^2 \right) - \frac{2}{3} \langle \theta^2 \rangle_c + 6 \left( (\langle \sigma_+^2 \rangle_c + (\langle \sigma_-^2 \rangle_c) \right) \right] \quad (A2)
\]

\[
\frac{d\langle \sigma_+ \rangle}{d\ell} = S \left[ \left( \frac{2}{3} \langle \theta \rangle - 1 \right) \langle \sigma_+ \rangle - \langle \sigma_+ \rangle^2 + \langle \sigma_- \rangle^2 + \langle E_+ \rangle - \frac{1}{3} \langle \theta \sigma_+ \rangle_c - \langle \sigma_+^2 \rangle_c + \langle \sigma_-^2 \rangle_c \right] \quad (A3)
\]

\[
\frac{d\langle \sigma_- \rangle}{d\ell} = S \left[ \left( \frac{2}{3} \langle \theta \rangle - 1 \right) \langle \sigma_- \rangle + 2 \langle \sigma_+ \rangle \langle \sigma_- \rangle + \langle E_- \rangle - \frac{1}{3} \langle \theta \sigma_- \rangle_c + 2 \langle \sigma_+ \sigma_- \rangle_c \right] \quad (A4)
\]

\[
\frac{d\langle E_+ \rangle}{d\ell} = S \left[ (\langle \theta \rangle - 2) \langle E_+ \rangle + 4\pi G \langle \rho \rangle \langle \sigma_+ \rangle + 4\pi G \langle \rho \sigma_+ \rangle_c \right] \quad (A5)
\]

\[
\frac{d\langle E_- \rangle}{d\ell} = S \left[ (\langle \theta \rangle - 2) \langle E_- \rangle + 4\pi G \langle \rho \rangle \langle \sigma_- \rangle + 4\pi G \langle \rho \sigma_- \rangle_c \right] \quad (A6)
\]

(B) The RG equations for second order cumulants

\[
\frac{d\langle \rho^2 \rangle_c}{d\ell} = 2S \left[ (\langle \theta \rangle - 2) \langle \rho^2 \rangle_c + \langle \rho \rangle \langle \rho \theta \rangle_c \right] \quad (A7)
\]

\[
\frac{d\langle \theta^2 \rangle_c}{d\ell} = 2S \left[ \left( \frac{2}{3} \langle \theta \rangle - 1 \right) \langle \theta^2 \rangle_c + 4\pi G \langle \rho \theta \rangle_c + 12 \left( \langle \sigma_+ \rangle \langle \theta \sigma_+ \rangle_c + \langle \sigma_- \rangle \langle \theta \sigma_- \rangle_c \right) \right] \quad (A8)
\]

\[
\frac{d\langle \rho \theta \rangle_c}{d\ell} = S \left[ \left( \frac{5}{3} \langle \theta \rangle - 3 \right) \langle \rho \theta \rangle_c + 4\pi G \langle \rho^2 \rangle_c + \langle \rho \rangle \langle \theta^2 \rangle_c + 12 \left( \langle \sigma_+ \rangle \langle \rho \sigma_+ \rangle_c + \langle \sigma_- \rangle \langle \rho \sigma_- \rangle_c \right) \right] \quad (A9)
\]

\[
\frac{d\langle \rho \sigma_+ \rangle_c}{d\ell} = S \left[ \left( \frac{5}{3} \langle \theta \rangle - 2 \langle \sigma_+ \rangle - 3 \right) \langle \rho \sigma_+ \rangle_c + 2 \langle \sigma_- \rangle \langle \rho \sigma_- \rangle_c + \frac{2}{3} \langle \sigma_+ \rangle \langle \rho \theta \rangle_c + \langle \rho \rangle \langle \theta \sigma_+ \rangle_c + \langle \rho \rangle \langle E_+ \rangle_c \right] \quad (A10)
\]

\[
\frac{d\langle \rho \sigma_- \rangle_c}{d\ell} = S \left[ \left( \frac{5}{3} \langle \theta \rangle + 2 \langle \sigma_+ \rangle - 3 \right) \langle \rho \sigma_- \rangle_c + 2 \langle \sigma_- \rangle \langle \rho \sigma_+ \rangle_c + \frac{2}{3} \langle \sigma_- \rangle \langle \rho \theta \rangle_c + \langle \rho \rangle \langle \theta \sigma_- \rangle_c + \langle \rho \rangle \langle E_- \rangle_c \right] \quad (A11)
\]

\[
\frac{d\langle \rho E_+ \rangle_c}{d\ell} = S \left[ (\langle \theta \rangle - 2) \langle \rho E_+ \rangle_c + \langle E_+ \rangle \langle \rho \theta \rangle_c + \langle \rho \rangle \langle \theta E_+ \rangle_c + 4\pi G \left( \langle \sigma_+ \rangle \langle \rho^2 \rangle_c + \langle \rho \rangle \langle \rho \sigma_+ \rangle_c \right) \right] \quad (A12)
\]

\[
\frac{d\langle \rho E_- \rangle_c}{d\ell} = S \left[ (\langle \theta \rangle - 2) \langle \rho E_- \rangle_c + \langle E_- \rangle \langle \rho \theta \rangle_c + \langle \rho \rangle \langle \theta E_- \rangle_c + 4\pi G \left( \langle \sigma_- \rangle \langle \rho^2 \rangle_c + \langle \rho \rangle \langle \rho \sigma_- \rangle_c \right) \right] \quad (A13)
\]
\[
\frac{d\langle \theta \sigma_{+} \rangle_c}{dt} = S \left[ 2 \left( \frac{2}{3} \langle \theta \rangle - \langle \sigma_{+} \rangle - 1 \right) \langle \theta \sigma_{+} \rangle_c + 2 \langle \sigma_{-} \rangle \langle \theta \sigma_{-} \rangle_c + \frac{2}{3} \langle \sigma_{+} \rangle \langle \theta^2 \rangle_c + \langle \theta E_{+} \rangle_c + 4 \pi G \langle \rho \sigma_{+} \rangle_c 
\right.
\]
\[
+ 12 \left( \langle \sigma_{+} \rangle \langle \sigma_{+}^2 \rangle_c + \langle \sigma_{-} \rangle \langle \sigma_{+} \sigma_{-} \rangle_c \right) \tag{A14}
\]
\[
\frac{d\langle \theta \sigma_{-} \rangle_c}{dt} = S \left[ 2 \left( \frac{2}{3} \langle \theta \rangle + \langle \sigma_{+} \rangle - 1 \right) \langle \theta \sigma_{-} \rangle_c + 2 \langle \sigma_{-} \rangle \langle \theta \sigma_{+} \rangle_c + \frac{2}{3} \langle \sigma_{-} \rangle \langle \theta^2 \rangle_c + \langle \theta E_{-} \rangle_c + 4 \pi G \langle \rho \sigma_{-} \rangle_c 
\right.
\]
\[
+ 12 \left( \langle \sigma_{-} \rangle \langle \sigma_{-}^2 \rangle_c + \langle \sigma_{+} \rangle \langle \sigma_{+} \sigma_{-} \rangle_c \right) \tag{A15}
\]
\[
\frac{d\langle \theta E_{+} \rangle_c}{dt} = S \left[ \left( \frac{5}{3} \langle \theta \rangle - 3 \right) \langle \theta E_{+} \rangle_c + \langle E_{+} \rangle \langle \theta^2 \rangle_c + 4 \pi G \left( \langle \rho \rangle \langle \theta \sigma_{+} \rangle_c + \langle \sigma_{+} \rangle \langle \rho \sigma_{+} \rangle_c + \langle \rho E_{+} \rangle \right) 
\right.
\]
\[
+ 12 \left( \langle \sigma_{+} \rangle \langle \sigma_{+} E_{+} \rangle_c + \langle \sigma_{-} \rangle \langle \sigma_{-} E_{+} \rangle_c \right) \tag{A16}
\]
\[
\frac{d\langle \theta E_{-} \rangle_c}{dt} = S \left[ \left( \frac{5}{3} \langle \theta \rangle - 3 \right) \langle \theta E_{-} \rangle_c + \langle E_{-} \rangle \langle \theta^2 \rangle_c + 4 \pi G \left( \langle \rho \rangle \langle \theta \sigma_{-} \rangle_c + \langle \sigma_{-} \rangle \langle \rho \sigma_{-} \rangle_c + \langle \rho E_{-} \rangle \right) 
\right.
\]
\[
+ 12 \left( \langle \sigma_{+} \rangle \langle \sigma_{+} E_{-} \rangle_c + \langle \sigma_{-} \rangle \langle \sigma_{-} E_{-} \rangle_c \right) \tag{A17}
\]
\[
\frac{d\langle \sigma_{+}^2 \rangle_c}{dt} = S \left[ 2 \left( \frac{2}{3} \langle \theta \rangle - 2 \langle \sigma_{+} \rangle - 1 \right) \langle \sigma_{+}^2 \rangle_c + \frac{4}{3} \langle \sigma_{+} \rangle \langle \theta \sigma_{+} \rangle_c + 4 \langle \sigma_{-} \rangle \langle \sigma_{+} \sigma_{-} \rangle_c + 2 \langle \sigma_{+} \rangle \langle \sigma_{+} E_{+} \rangle_c \right] \tag{A18}
\]
\[
\frac{d\langle \sigma_{-}^2 \rangle_c}{dt} = S \left[ 2 \left( \frac{2}{3} \langle \theta \rangle + 2 \langle \sigma_{+} \rangle - 1 \right) \langle \sigma_{-}^2 \rangle_c + \frac{4}{3} \langle \sigma_{-} \rangle \langle \theta \sigma_{-} \rangle_c + 4 \langle \sigma_{-} \rangle \langle \sigma_{+} \sigma_{-} \rangle_c + 2 \langle \sigma_{-} \rangle \langle \sigma_{-} E_{-} \rangle_c \right] \tag{A19}
\]
\[
\frac{d\langle \sigma_{+} \sigma_{-} \rangle_c}{dt} = S \left[ 2 \left( \frac{2}{3} \langle \theta \rangle - 1 \right) \langle \sigma_{+} \sigma_{-} \rangle_c + \frac{2}{3} \left( \langle \sigma_{+} \rangle \langle \theta \sigma_{-} \rangle_c + \langle \sigma_{-} \rangle \langle \theta \sigma_{+} \rangle_c \right) + 2 \langle \sigma_{-} \rangle \left( \langle \sigma_{+}^2 \rangle_c + \langle \sigma_{-}^2 \rangle_c \right) 
\right.
\]
\[
+ \langle \sigma_{+} \rangle \langle \sigma_{-} \rangle \langle \sigma_{+} \sigma_{-} \rangle_c \tag{A20}
\]
\[
\frac{d\langle E_{+}^2 \rangle_c}{dt} = 2S \left[ \left( \langle \theta \rangle - 2 \right) \langle E_{+}^2 \rangle_c + \langle E_{+} \rangle \langle \theta E_{+} \rangle_c + 4 \pi G \langle \rho \rangle \langle \sigma_{+} \rangle \langle \sigma_{+} E_{+} \rangle_c + \langle \sigma_{+} \rangle \langle \rho E_{+} \rangle \right) \right] \tag{A21}
\]
\[
\frac{d\langle E_{-}^2 \rangle_c}{dt} = 2S \left[ \left( \langle \theta \rangle - 2 \right) \langle E_{-}^2 \rangle_c + \langle E_{-} \rangle \langle \theta E_{-} \rangle_c + 4 \pi G \langle \rho \rangle \langle \sigma_{-} \rangle \langle \sigma_{-} E_{-} \rangle_c + \langle \sigma_{-} \rangle \langle \rho E_{-} \rangle \right) \right] \tag{A22}
\]
\[
\frac{d\langle E_{+} E_{-} \rangle_c}{dt} = S \left[ 2 \left( \langle \theta \rangle - 2 \right) \langle E_{+} E_{-} \rangle_c + \langle E_{+} \rangle \langle \theta E_{-} \rangle_c + \langle E_{+} \rangle \langle \theta E_{-} \rangle_c + 4 \pi G \langle \rho \rangle \left( \langle \sigma_{+} \rangle \langle \sigma_{-} \rangle \langle \sigma_{+} \sigma_{-} \rangle_c + \langle \sigma_{-} \rangle \langle \rho E_{-} \rangle \right) 
\right.
\]
\[
+ 4 \pi G \langle \rho \rangle \langle \sigma_{+} \rangle \langle \sigma_{-} \rangle \langle \rho E_{-} \rangle \rangle_c \tag{A23}
\]
\[
\frac{d\langle \sigma_{+} E_{-} \rangle_c}{dt} = S \left[ \left( \frac{5}{3} \langle \theta \rangle - 2 \langle \sigma_{+} \rangle - 3 \right) \langle \sigma_{+} E_{-} \rangle_c + 2 \langle \sigma_{-} \rangle \langle \sigma_{-} E_{+} \rangle_c + \langle \sigma_{+} \rangle \langle \sigma_{+} E_{+} \rangle_c + E_{+} \langle \theta \sigma_{+} \rangle_c + 2 \langle \sigma_{+} \rangle \langle \sigma_{+} E_{+} \rangle_c 
\right.
\]
\[
+ \frac{2}{3} \langle \sigma_{+} \rangle \langle \theta E_{-} \rangle_c + 4 \pi G \langle \rho \rangle \langle \sigma_{+}^2 \rangle_c + \langle \sigma_{+} \rangle \langle \rho E_{-} \rangle \rangle_c \tag{A24}
\]
\[
\frac{d\langle \sigma_+ E_- \rangle_c}{d\ell} = S \left[ \left( \frac{5}{3} \langle \theta \rangle - 2 \langle \sigma_+ \rangle - 3 \right) \langle \sigma_+ E_- \rangle_c + 2 \langle \sigma_- \rangle \langle \sigma_- E_- \rangle_c + \langle \theta \rangle \langle \sigma_+ \rangle_c + \langle E_- \rangle \langle \sigma_+ \rangle_c \\
+ \frac{2}{3} \langle \sigma_+ \rangle \langle \theta E_- \rangle_c + 4\pi G (\langle \rho \rangle \langle \sigma_+ \sigma_- \rangle_c + \langle \sigma_- \rangle \langle \rho \sigma_+ \rangle_c) \right] \tag{A25}
\]

\[
\frac{d\langle \sigma_- E_+ \rangle_c}{d\ell} = S \left[ \left( \frac{5}{3} \langle \theta \rangle + 2 \langle \sigma_+ \rangle - 3 \right) \langle \sigma_- E_+ \rangle_c + 2 \langle \sigma_- \rangle \langle \sigma_- E_+ \rangle_c + \langle \theta \rangle \langle \sigma_- \rangle_c + \langle E_+ \rangle \langle \sigma_- \rangle_c \\
+ \frac{2}{3} \langle \sigma_- \rangle \langle \theta E_+ \rangle_c + 4\pi G (\langle \rho \rangle \langle \sigma_+ \sigma_- \rangle_c + \langle \sigma_+ \rangle \langle \rho \sigma_- \rangle_c) \right] \tag{A26}
\]

\[
\frac{d\langle \sigma_- E_- \rangle_c}{d\ell} = S \left[ \left( \frac{5}{3} \langle \theta \rangle + 2 \langle \sigma_+ \rangle - 3 \right) \langle \sigma_- E_- \rangle_c + 2 \langle \sigma_- \rangle \langle \sigma_- E_- \rangle_c + \langle \theta \rangle \langle \sigma_- \rangle_c + \langle E_- \rangle \langle \sigma_- \rangle_c \\
+ \frac{2}{3} \langle \sigma_- \rangle \langle \theta E_- \rangle_c + 4\pi G (\langle \rho \rangle \langle \sigma_-^2 \rangle_c + \langle \sigma_- \rangle \langle \rho \sigma_- \rangle_c) \right] \tag{A27}
\]
FIG. 1. The RG flow toward larger scale on the $H$-$\Omega$ plane in the case with neither shear nor fluctuations. Here we define the Hubble parameter $H$ and the density parameter $\Omega$ as $H \equiv (\theta)/3$ and $\Omega \equiv 8\pi G(\rho)/3H^2$, respectively. The three important fixed points exist: (1) fixed point $Q$ (Quiescent Universe), (2) fixed point $E$ ("Einstein-de Sitter" Universe) (3) fixed point $M$ ("Milne" Universe). The RG flow converges to the fixed point $Q$ or escapes to infinity. The fixed point seems to be a line in the $H$-$\Omega$ plane, but it is a point that all physical variables vanish in the "phase" space of $(F) = (\rho, \theta)$. $\Omega$ turns out to be some finite value, which depends on the limit of $\langle \rho \rangle \to 0$ and $\theta \to 0$

FIG. 2. The scale dependence of (a) $\log \Omega$, (b) $H$, (c) $\Delta_\rho$, and (d) $\Delta_\sigma$ in shear-free case with $s = 0.5$ and $\Delta_0 = (1) 10^{-1}$, (2) $10^{-1.5}$, (3) $10^{-2}$, (4) $10^{-2.5}$. We solve the RG flows from the fixed point $E$ toward smaller scale. $L \equiv \exp \xi$ represents the ratio of the scale of averaged region to the horizon scale. Every flow monotonically converges to $M$ as $H$ becomes larger ($H \nearrow$) and $\Omega$ gets smaller ($\Omega \searrow$) toward smaller scale. The critical scale where $\Omega$ and $H$ deviate from the EdS values depends on the magnitude of $\Delta_0$.

FIG. 3. The $s$ dependence of the RG flows of (a) $\log \Omega$, (b) $H$, (c) $\Delta_\rho$, and (d) $\Delta_\sigma$ from the fixed point $E$ at horizon scale in shear-free case. Here we depict the cases of $\Delta_0 = 10^{-2}$ and $s = (1) 0.3$, (2) 0.5, (3) 0.7, (4) 0.9. $\Omega$ begins to decline beyond a critical scale, a few decades smaller than horizon scale, which depends on the magnitude of $\Delta_0$. The slope at smaller scale is determined by $\lambda_{\text{min}}^2 = s/(1-s)$ at the fixed point $M$. The slope for the case (2) is equal to 1. $H$ also changes from the value at $E$ to the value at $M$ beyond the above critical scale. As the value of $s$ increases, the critical scale shifts toward smaller scale.

FIG. 4. The RG flow toward smaller scale on the $H$-$\Omega$ plane in the case that $s = 0.5$ and $\bar{D}_{s+,0} = \bar{D}_{E+,0} = 10^{-2}$. As for the density perturbations, we choose $\Delta_0 = (1) 10^{-0.75}$, (2) $10^{-1}$, (3) $10^{-1.25}$, (4) $10^{-1.5}$. If both $\bar{D}_{s+,0}/\Delta_0$ and $\bar{D}_{E+,0}/\Delta_0$ are less than $R_{cr} \approx 0.1495$ ((1), (2)), every flow converges to $M$ as $H$ increases ($H \nearrow$) to 1 and $\Omega$ decreases ($\Omega \searrow$) to 0 toward smaller scales. On the other hand, if $\bar{D}_{s+,0}$ or $\bar{D}_{E+,0}$ are larger than the above critical value $R_{cr}$ ((3), (4)), the flow heading for $M$ comes to turn around and go away to infinity.

FIG. 5. The RG flow of (a) $\log \Omega$, (b) $H$, (c) $\log(\langle \sigma_\pm \rangle/H)$ and (d) $\log(\langle E_\pm \rangle/H^2)$ from the fixed point $E$ toward smaller scale in the case that $s = 0.5$, $\bar{D}_{s+,0} = \bar{D}_{E+,0} = 10^{-2}$ and $\Delta_0 = (1) 10^{-0.75}$, (2) $10^{-1}$, (3) $10^{-1.25}$, (4) $10^{-1.5}$. In the case of (1) or (2) ($\bar{D}_{s+,0}/\Delta_0, \bar{D}_{E+,0}/\Delta_0 \leq R_{cr} \approx 0.1495$), the RG flow converges to $M$ toward smaller scale without shear and tidal force (see Fig. 2). In the case of (3) or (4) ($\bar{D}_{s+,0}/\Delta_0, \bar{D}_{E+,0}/\Delta_0 > R_{cr}$), however, the RG flow does not converge to $M$ but rather diverges to infinity. We also depicted the flow of fluctuations; (e) $\log \Delta_\rho$, (f) $\log \Delta_\sigma$, (g) $\log \bar{D}_{s+}$, and (h) $\log \bar{D}_{E+}$ for the same initial values as the above. In the case of (1) or (2), all fluctuations but $\Delta_\rho$ damp off as the RG flow converges to $M$, while $\Delta_\rho$ approaches the finite value. In the case of (3) or (4), all of them diverges to infinity.

FIG. 6. The $s$ dependence of the RG flows from fixed point $E$ of (a) $\log \Omega$, (b) $H$, (c) $\log(\langle \sigma_\pm \rangle/H)$ and (d) $\log(\langle E_\pm \rangle/H^2)$. Here we depict the cases of $s = (1) 0.3$, (2) 0.5, (3) 0.7, (4) 0.9 with the boundary conditions, $\Delta_0 = 10^{-3.5}$ and $\bar{D}_{s+,0} = \bar{D}_{E+,0} = 10^{-2.5}$ at horizon scale. The results for $\log \Omega$ and $H$ are quite similar to Fig. 3, if the flow converges to the fixed point $M$. The slopes of $\langle \sigma_\pm \rangle$ and $\langle E_\pm \rangle$ at larger and smaller scale are determined by $\lambda_{\text{min}}^2$ at fixed points $E$ and $M$, i.e. $-4s/(3-2s)$ and $2s/(1-s)$, respectively. As for the $s$ increases, the turning point of the slope shifts towards smaller scale and the slope becomes less steep. We also depicted the flow of fluctuations; (e) $\log \Delta_\rho$, (f) $\log \Delta_\sigma$, (g) $\log \bar{D}_{s+}$, and (h) $\log \bar{D}_{E+}$ for the same range of $s$. $\Delta_0 = 10^{-1.5}$ and $\bar{D}_{s+,0} = \bar{D}_{E+,0} = 10^{-2.5}$ at horizon scale in the case of LTA. We find that $\lambda_{\text{min}}^2 = \text{constant}$ near the fixed point $M$, while it decreases linearly near the fixed point $E$, which behavior is determined by the eigenvalue $-4s/(3-2s)$.

FIG. 7. The scale dependence of (a) $\log(\rho)$, (b) $H$, (c) $\log \Delta_\rho$, and (d) $\log \Delta_\sigma$ beyond horizon scale for $s = 0.5$ with the boundary conditions, $\Delta_0 = 10^{-2.5}$ at horizon scale in shear-free case. In the flow departing from the fixed point $E$, $\langle \rho \rangle$ approaches zero toward both smaller and larger scales, for which the slopes at larger and smaller scale are determined by $\lambda_{\text{min}}^2$ at fixed points, $Q$ and $M$, i.e. $-2s$ and $2s/(1-2s)$, respectively. $H$ also vanishes at larger scale, after it becomes negative around $L = 10^5$, which causes the divergence of $\Delta_\rho$ around the scale. The slope of both $\Delta_\rho$ and $\Delta_\sigma$ around the fixed point $E$ are determined by $\lambda_{\text{max}}^2 = -4s/(3-2s)$ toward smaller scale and $\lambda_{\text{max}}^2 = 6s/(3-2s)$ toward larger scale at the point.
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