Circuit theory for crossed Andreev reflection and nonlocal conductance

Jan Petter Morten1,3, Arne Brataas1,3, Wolfgang Belzig2,3

1 Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway
2 University of Konstanz, Department of Physics, D-78457 Konstanz, Germany
3 Centre for Advanced Study, Drammensveien 78, N-0271 Oslo, Norway

Received: 31 January 2007 / Revised version: 12 April 2007

Abstract Nonlocal currents, in devices where two normal-metal terminals are contacted to a superconductor, are determined using the circuit theory of mesoscopic superconductivity. We calculate the conductance associated with crossed Andreev reflection and electron transfer between the two normal-metal terminals, in addition to the conductance from direct Andreev reflection and quasiparticle tunneling. Dephasing and proximity effect are taken into account.

1 Introduction

Transport between a normal-metal and a superconductor at subgap energy is possible through Andreev reflections, where an incident electron from the normal-metal is retro-reflected as a hole and a Cooper pair is transferred into the superconductor [1]. However, since Andreev reflection is a nonlocal process on the scale of the coherence length, the retro-reflected hole can end up in another normal-metal contacted to the superconductor. This process is known as crossed Andreev reflection (CA) [2, 3] and contributes to a nonlocal conductance. We define the nonlocal conductance in a three-terminal device (see Fig. 1) as the current response in one normal-metal terminal (N1) to a voltage bias between another normal-metal terminal (N2) and a superconducting terminal (S).

In our recent paper Ref. [4], we used the circuit theory of mesoscopic superconductivity [16] to calculate the conductances in a three terminal device where two normal-metal terminals and one superconducting terminal are connected to a region where chaotic scattering takes place. We assumed that the energy of the injected particles was much smaller than the gap of the superconducting terminal. In this case, the only transport process involving only one of the normal-metal terminals and the superconductor is direct Andreev reflection (DA), where the hole is backreflected into the same normal-metal as the incident electron. We now extend this approach to take into account situations where the bias voltage is comparable to the gap of the superconducting terminal. In this case, the only transport process involving only one of the normal-metal terminals and the superconductor is direct Andreev reflection (DA), where the hole is backreflected into the same normal-metal as the incident electron. We now extend this approach to take into account situations where the bias voltage is comparable to the gap of the superconductor so that incident electrons from a normal-metal may be transferred into the superconductor as quasiparticles (QP). Taking these processes into account, the current

In our previous paper Ref. [4], this transport process was referred to as electron cotunneling, but as this phrase applies to the tunneling limit we will here use the more general term electron transfer.
at energy $E$ out of $N_1$ can be written
\[
I_1(E) = \frac{G_{CA}(E)}{e} \left[ 1 - f_1(E) - f_2(-E) \right] + \frac{G_{ET}(E)}{e} \left[ f_2(E) - f_1(E) \right] + 2 \frac{G_{DA}(E)}{e} \left[ 1 - f_1(E) - f_1(-E) \right] + \frac{G_{GP}(E)}{e} \left[ f_S(E) - f_1(E) \right].
\]
(1)

Here, the functions $f_n(\pm E)$ denote the Fermi-Dirac distribution functions in terminals $n = 1, 2, 3$ at energy $\pm E$ and we have defined energy dependent conductances $G(E)$ for the transport processes discussed above. Total charge current is given by $I_{\text{charge}, 1} = \int dE I_1(E)$. The nonlocal differential conductance is obtained from Eq. (1),
\[
\frac{\partial I_{\text{charge}, 1}}{\partial V_2} = -\int dE \left[ G_{ET}(E) - G_{CA}(E) \right] \frac{\partial f(E - eV_2)}{\partial E}.
\]
(2)

This shows how the nonlocal conductance is determined by competing contributions from crossed Andreev reflections and electron transfer.

2 Circuit theory

In the circuit theory of mesoscopic superconductivity [10], a system can be modeled as a network of terminals, connectors and nodes in a similar manner as electrical circuits on the macroscale are modeled using classical circuit theory. The theory is formulated in terms of matrix Green’s functions and matrix currents in Nambu-Keldysh space and $e.g.$ describes quantum effects and the flow of charge, energy and particle-hole correlations.

To describe the three-terminal devices at hand we introduce equilibrium Green’s functions depending on temperature and local chemical potential for the two normal-metal terminals and the superconducting terminal. The terminals are coupled to a region where scattering takes place, which we will refer to as a cavity. Our circuit theory model is shown in Fig. 1. If we assume that the distance between the contacts is very small on the scale of the coherence length, the properties of the cavity are spatially homogeneous so that it can be described by one nonequilibrium Green’s function $\tilde{G}_c$. If spatial variation in the scattering region is important, it can be modeled as a network of cavities with different Green’s functions. We assume that $\tilde{G}_c$ is isotropic due to chaotic or diffusive scattering.

Connectors between terminals and the cavity are described by their sets of transmission probabilities $\{T^{(n)}\}$, with flow of matrix currents $\tilde{I}_n$ depending on the Green’s functions of adjacent elements,
\[
\tilde{I}_n = -\frac{2e^2}{\pi \hbar} \sum_k T^{(n)}_k \frac{\tilde{G}_n \tilde{G}_c}{4 + T^{(n)}_k (\tilde{G}_n \tilde{G}_c - 2)}.
\]
(3)

The matrix current describes not only the flow of charge, spin, and energy currents, but $e.g.$ also the flow of correlations [10].

Finally, the theory is completed by a generalized “Kirchhoff’s rule”: The sum of matrix currents flowing into a cavity should vanish. This determines the nonequilibrium Green’s function of the cavity in our present circuit. The spectral charge current through connector $n$ can be determined $I_{\text{T}, n} = \text{Tr} \left\{ \tilde{\sigma}_3 \tilde{I}^{(n)} \right\} / 8e$ once the Green’s functions $\tilde{G}_n, \tilde{G}_c$ have been determined. Similarly, the spectral energy current becomes $I_{\text{L}, n} = \text{Tr} \left\{ \tilde{E}^{(n)} \right\} / 8e$. The K superscript denotes the Keldysh matrix block of the current.

Using the generic circuit theory model above, we can describe crossed Andreev reflection in a wide range of different experimental systems. For example, consider a system where two adjacent normal-metal electrodes are deposited on and connected by metallic contacts to a mesoscopic superconductor. The terminal Green’s functions correspond to properties of the normal-metals and the superconducting sample away from the contact region. The cavity corresponds to the small region of the superconductor between the normal-metal contacts. Metallic contacts are modeled by putting $T^{(n)}_k = 1$ for the conducting modes of contacts $n = 1, 2$. The contact between the cavity and the superconducting terminal can be modeled as a diffusive connector introducing a bimodal distribution of transmission probabilities [17]. Other systems for experimental study of crossed Andreev reflection may be fabricated from superconductors coupled to semiconductors where the geometry is defined by deposition of gates. For example, the semiconductor can consist of a ballistic cavity with two point contacts to normal reservoirs. In this case, the cavity Green’s function describes the nonequilibrium state of the ballistic cavity. The point contacts are modeled by putting $T^{(n)}_k = 1$ for the open channels and zero otherwise for $n = 1, 2$. The transparency and number of conducting modes of the contact to the superconductor can also be determined experimentally.

Superconducting pairing and dephasing in the cavity are described in circuit theory by introducing a “leakage current” $\tilde{I}_{\text{leakage}} = -ie^2v_0 V_e \tilde{G}_c$ in the matrix current conservation on the cavity [10]. Here $v_0$ is the density of states, $V_e$ the volume, $\tilde{H}_c = E\tilde{\sigma}_3 + i\bar{\sigma}_1 \text{Re} \{\Delta_c\} + i\bar{\sigma}_2 \text{Im} \{\Delta_c\}$ the Hamiltonian, and $\Delta_c$ the gap in the cavity. The energy-dependent term in $\tilde{H}_c$ describes dephasing between electrons and holes, and sets an energy scale for the proximity effect which we will refer to as the effective Thouless energy of the cavity. We now disregard
Josephson effects, the phase of $\Delta_c$ can be chosen arbitrarily e.g. purely imaginary and inspection of the retarded part of the matrix current conservation reveals that pairing inside the cavity appears with the same matrix structure as the coupling to the superconducting terminal. Therefore, the effect of pairing inside the cavity can be described by a renormalization of the coupling between the cavity and the superconducting terminal. Thus the difference between a normal and a superconducting cavity is equivalent to rescaling this conductance, i.e. only quantitative modifications.

3 Results

We will now discuss the structure of the Green’s functions and the solution of the matrix equations. The normal terminals have Green’s functions $\hat{G}_{1(2)} = \sigma_3 \hat{\tau}_3 + (\sigma_3 h_{1(2)} + \hat{1} h_{\tau(2)})(\hat{\tau}_1 + \hat{i}\tau_2)$, where we have introduced the charge- and energy-distribution functions $f_T(E)$, $h_L(E)$ that can be written in terms of the particle distribution function $f(E)$ as $h_T = 1 - f(E) - f(-E)$ and $h_L = -f(E) + f(-E)$, see Ref. [1]. The retarded (advanced) part of the Green’s function of the superconducting reservoir is $\hat{G}_S^{R(A)} = ((E+\text{i}\delta)\hat{\sigma}_3 \pm \Delta \hat{\sigma}_2)/\Omega$, where $\Omega = \sqrt{(E+\text{i}\delta)^2 - \Delta^2}$, $\Delta$ is the gap. The Keldysh part is obtained from $\hat{G}_S^K = \hat{G}_S^R \hat{n}_S - \hat{n}_S \hat{G}_S^A$ where $\hat{n}_S = \hat{\sigma}_3 h_{\text{LS}} + \hat{1} h_{\tau S}$. We parametrize the Green’s function of the cavity as $\hat{G}_c^K = \hat{\sigma}_3 \cosh(\theta) + \hat{i}\sigma_2 \sinh(\theta)$ and $\hat{G}_c^A = -\hat{\sigma}_3 \hat{G}_c^K \hat{\sigma}_3$, the Keldysh part is given by $\hat{G}_c^K = \hat{G}_c^R \hat{n}_c - \hat{n}_c \hat{G}_c^A$, where $\hat{n}_c = \hat{1} h_{\text{L.c}} + \hat{\sigma}_3 h_{\tau c}$.

With the Green’s functions specified as above, we impose matrix current conservation in the cavity, $\sum_n \hat{I}_n + \hat{I}_{\text{leakage}} = 0$ to obtain equations that determine $\theta$, $h_{\tau c}$, and $h_{\text{L.c}}$. The equations for the distribution functions are conservation of charge (energy) at each energy,

$$\sum_n G_{T(L),n}(h_{T(L),n} - h_{T(L),c}) = 0, \quad (4)$$

for $n = 1, 2, S$, where we have defined effective, energy dependent conductances for charge (energy) $G_{T(L),n}[\theta(E)]$ between reservoir $n$ and the cavity. Using the solution for $\hat{G}_c$ that we have obtained, we calculate the current out of terminal $N_1$ and compare the result to Eq. (1). This allows us to determine the conductances for the transport processes:

$$G_{QP}(E) = \frac{G_{L_1} G_{L_S}}{G_{L_1} + G_{L_2} + G_{L_S}}, \quad (5a)$$

$$G_{DA}(E) = \frac{1}{4} \left( \frac{G_{T_1} (G_{T_2} + G_{T_S})}{G_{T_1} + G_{T_2} + G_{T_S}} \right)$$

$$G_{ET}(E) = \frac{1}{2} \left( \frac{G_{L_1} G_{L_2}}{G_{L_1} + G_{L_2} + G_{L_S}} \pm \frac{G_{T_1} G_{T_2}}{G_{T_1} + G_{T_2} + G_{T_S}} \right), \quad (5b)$$

The conductance for quasiparticle transport into the superconducting terminal, $G_{QP}$, is proportional to $G_{L_S}$ which is the energy conductance for transport from the cavity into the superconductor. This quantity vanishes at subgap energies where Andreev reflection of particles from opposite sides of the Fermi surface is the only possible transport process. The symmetry between $G_{ET}$ and $G_{CA}$ in Eq. (5c) was discussed in our previous paper Ref. [4], and shows that the differential nonlocal conductance, given by $G_{ET} - G_{CA}$ see Eq. (2), is always positive. In that paper, we also discussed the limit that there is no resistance between the superconducting reservoir and the cavity. We see from Eq. (5c) that in this case $G_{ET(CA)}$ vanishes because of the large terms $G_{T,S} G_{L_S}$.

The equations that determine $\theta$ must generally be solved numerically. We have performed such calculations, and show in Fig. 2 the result for a system where connectors to $N_1$ and $N_2$ are point contacts and the connector to $S$ has transparency $T_k^{(S)} = 0.5$ for the conducting modes. Defining $g_n = e^2 \sum_k \gamma_k^{(n)}/(\pi \hbar)$ we choose parameters $g_2/g_3 = 0.1$ for $j = 1, 2$. There are now two energy scales in the problem, $\Delta$ and $E_{Th}$, and we have chosen $\Delta/E_{Th} = 6$ in Fig. 2. The nonlocal conductance is largest for energy smaller than the effective Thouless energy, defined as $E_{Th} = \hbar g_3/(2e^2\hbar V')$, and decreases in two steps at $E_{Th}$ and $\Delta$ with increasing energy. The nonlocal conductance above the gap corresponds to the normal state result $\partial I_1/\partial V_2 = g_1 g_2/(g_1 + g_2 + g_3)$ for quasiparticle transport in a three terminal network. In this energy range we also have a contribution from quasiparticle transfer into $S$.

When the transmission probabilities of the interface to $S$ are in the tunneling limit, i.e., all $T_k^{(S)} \ll 1$ the CA conductance will be suppressed in comparison to the case shown in Fig. 2. The ET conductance, on the other hand, is enhanced by the reduced transmission of the contact to $S$. In Fig. 3 we show the conductances for a device where $N_1$ and $N_2$ are connected by point contacts, and $S$ by a tunneling barrier. The conductance of the tunneling barrier is the same as the conductance of the contact to $S$ in Fig. 2. For energies between $E_{Th}$ and $\Delta$, we see that $G_{CA}$ is strongly suppressed. The total nonlocal conductance, $G_{ET} - G_{CA}$, has a minimum in the subgap
regime at energy corresponding to $E_{\text{Th}}$, and is largest for an energy close to $\Delta$. This is qualitatively different from the device where $S$ is connected by an interface of intermediate transparency (Fig. 2), where the maximum nonlocal conductance in the subgap regime is found at very small energy, and then decreases with increasing energy due to increasing CA conductance.

Fig. 3 Conductances when $N_1$ and $N_2$ are connected by point contacts, and $S$ by a tunnel barrier. In contrast to the the device where $S$ was connected by an interface of intermediate transparency (Fig. 2), the CA conductance below the gap is strongly suppressed in this case.

4 Conclusion

In conclusion, we have studied nonlocal transport in a three-terminal device with two normal-metal terminals and one superconducting terminal. To this end we have applied the circuit theory of mesoscopic transport. The connectors between the circuit elements are represented by general expressions, relevant for a wide range of contacts. Dephasing is taken into account, and gives rise to an effective Thouless energy. We calculate the conductance for crossed Andreev reflection, electron transfer between the normal-metal terminals, and direct Andreev reflection and quasiparticle transport between one normal-metal terminal and the superconducting terminal. The nonlocal conductance is generally dominated by electron transfer in our model, similar predictions were made in Refs. [12, 4]. We showed in Ref. [4] that for this model, in the limit that there is no resistance between the device and the superconducting terminal, our results agree with Ref. [5] and the total nonlocal conductance vanishes. We numerically calculate the conductances for experimentally relevant combinations of contacts to demonstrate the appearance of two energy scales in the conductances: The effective Thouless energy and the gap of the superconducting terminal. The conductance for crossed Andreev reflection depends strongly on the transparency of the interface to superconducting terminal as demonstrated in our numerical calculations.

Acknowledgments

This work was supported in part by The Research Council of Norway through Grants No. 167498/V30, 162742/V00, 1534581/432, 1585181/143, 1585471/431, the DFG through SFB 513, the Landesstiftung Baden-Württemberg, the National Science Foundation under Grant No. PHY99-07949, and EU via project NMP2-CT-2003-505587 ‘SFINx’.

References

1. A. F. Andreev, Sov. Phys. JETP 19, 1228 (1964).
2. J. M. Byers and M. E. Flatte, Phys. Rev. Lett. 74, 306 (1995).
3. G. Deutsher and D. Feinberg, Appl. Phys. Lett. 76, 487 (2000).
4. J. P. Morten, A. Brataas, and W. Belzig, Phys. Rev. B 74, 214510 (2006).
5. D. Beckmann, H. B. Weber, and H. v. Lohneysen, Phys. Rev. Lett. 93, 197003 (2004).
6. S. Russo, M. Kroug, T. M. Klapwijk, and A. F. Morpurgo, Phys. Rev. Lett. 95, 027002 (2005).
7. P. Cadden-Zimansky and V. Chandrasekhar, Phys. Rev. Lett. 97, 237003 (2006).
8. G. Falcì, D. Feinberg, and F. W. J. Hekking, Europhys. Lett. 54, 255 (2001).
9. T. Yamashita, S. Takahashi, and S. Maekawa, Phys. Rev. B 68, 174504 (2003).
10. D. Sanchez, R. Lopez, P. Samuelsson, and M. Buttiker, Phys.Rev. B 68, 214501 (2003).
11. N. M. Chitcholkatchev, JETP Lett. 78, 230 (2003).
12. R. Mélin and D. Feinberg, Phys. Rev. B 70, 174509 (2004).
13. M. S. Kalenkov and A. D. Zaikin, Phys. Rev. B 75, 172503 (2007).
14. A. Brinkman and A. A. Golubov, Phys. Rev. B 74, 214512 (2006).
15. A. Levy Yeyati, F. S. Bergeret, A. Martin-Rodero, and T. M. Klapwijk, (2006), cond-mat/0612027.
16. Y. V. Nazarov, Superlatt. Microstruct. 25, 1221 (1999).
17. W. Belzig, A. Brataas, Y. V. Nazarov, and G. E. W. Bauer, Phys. Rev. B 62, 9726 (2000).
18. W. Belzig, F. K. Wilhelm, C. Bruder, G. Schön, and A. D. Zaikin, Superlatt. Microstruct. 25, 1251 (1999).