Small Field Polynomial Inflation:
Reheating, Radiative Stability and Lower Bound

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Abstract: We revisit the renormalizable polynomial inflection point model of inflation, focusing on the small field scenario which can be treated fully analytically. In particular, the running of the spectral index is predicted to be \( \alpha = -1.43 \times 10^{-3} + 5.56 \times 10^{-5} (N_{\text{CMB}} - 65) \), which might be tested in future. We also analyze reheating through perturbative inflaton decays to either fermionic or bosonic final states via a trilinear coupling. The lower bound on the reheating temperature from successful Big Bang nucleosynthesis gives lower bounds for these couplings; on the other hand radiative stability of the inflaton potential leads to upper bounds. In combination this leads to a lower bound on the location \( \phi_0 \) of the near inflection point, \( \phi_0 > 3 \times 10^{-5} \) in Planckian units. The Hubble parameter during inflation can be as low as \( H_{\text{inf}} \sim 1 \text{ MeV} \), or as high as \( \sim 10^{10} \text{ GeV} \). Similarly, the reheating temperature can lie between its lower bound of \( \sim 4 \text{ MeV} \) and about \( 4 \times 10^8 (10^{11}) \text{ GeV} \) for fermionic (bosonic) inflaton decays. We finally speculate on the “prehistory” of the universe in this scenario, which might have included an epoch of eternal inflation.
1 Introduction and Motivations

Inflation neatly solves the horizon, flatness and monopole problems of (old) standard cosmology [1–4]. The simplest inflationary model uses a single elementary scalar “inflaton” field $\phi$ to drive slow-roll (SR) inflation, with a monomial $\lambda \phi^p$ potential; at sufficiently large field values this even allows eternal inflation (where the inflaton field undergoes random walk) [5, 6]. However recent Planck 2018 measurements [7] have disfavored those models with $p \geq 1$: these potentials are too steep and therefore predict too large a tensor–to–scalar ratio $r$. Agreement with these observations can be obtained for smaller values of $p$, which however are not easy to realize in complete particle physics models. We refer to ref. [8] for a review for inflationary modes.

In this paper, we instead consider the most general renormalizable single–field model, where the potential is a polynomial of degree four [9–13]. We will assume that the density perturbations observed in the CMB and other cosmological probes were produced when the inflaton field had values not larger than the Planck scale, so that the energy scale during inflation is far below the Planck scale; hence insisting on renormalizability seems reasonable. Since the linear term can be removed via a shift of the inflaton field and the constant term is at most of the order of today’s cosmological constant, which is essentially zero relative to the energy scales during inflation, the potential only contains three terms. It turns out that all three terms are needed in order to reproduce the measurements by the Planck collaboration. In particular, the potential is sufficiently flat only if it has a
(near) inflection point where both the first and the second derivative of the potential are very small. Such an inflection point might arise from radiative corrections [14–23], but here we generate it already at the tree level. Inflation near an inflection point of the potential has been discussed previously in a supersymmetric context, often using non–renormalizable potentials or just analyzing the motion of the field around the inflection point [24–30]. It should be noted that this model does allow for eternal inflation, at much larger (trans–Planckian) field values but still sub–Planckian energy densities. Assuming an early phase of “eternal” inflation alleviates the initial condition problem; in fact, it is not clear whether one can meaningfully speak of “initial conditions” in such a case [6]. Eternal inflation also offers the only known physical mechanism that might allow to sample a “landscape”, i.e. a (complicated) potential with a very large number of minima [31].

The goal of this paper is to study the non–supersymmetric small field polynomial inflation model. We wish to explore the entire allowed parameter space in a complete model, which also includes a coupling that allows the inflaton to decay; this is required so that the universe can reheat at the end of inflation. To this end, we first analytically calculate the number of e–folds and inflationary predictions (power spectrum, tensor–to–scalar ratio, spectral index and its running). Once the overall size of the density perturbations and the spectral index have been fixed, essentially only the location $\phi_0$ of the near–inflection point remains as free parameter. It is bounded from below by the requirement that the reheating temperature is sufficiently high [34, 35], with inflaton couplings that are sufficiently small not to disturb the flatness of the potential through radiative corrections. We find that $\phi_0$ has to be larger than $3 \cdot 10^{-5}$ (in Planckian units). The resulting tensor–to–scalar ratio is much too small to be detectable. On the other hand, the running of the spectral index, which turns out to be independent of $\phi_0$, just might be detectable in future precision measurements. Within the allowed parameter space, the inflationary scale can be as low as $H_{\text{inf}} \sim 1$ MeV; such a low inflationary energy scale might help to embed the QCD axion as dark matter with a wider cosmologically allowed window, i.e. larger decay constant $f_a$ than is usually considered [36, 37], and would greatly alleviate the cosmological moduli problem [38]. On the other hand, for larger (still sub–Planckian) values of $\phi_0$ the reheat temperature might exceed $10^{10}$ GeV, which would allow standard thermal leptogenesis [39, 40]; however, this requires an inflationary Hubble parameter of order $10^{9}$ GeV.

The remainder of this paper is organized as follows. In Sec. 2 we give a complete analytical description of the small field polynomial inflection point model. In Sec. 3 the model parameters and predictions of cosmological observables are investigated. In Sec. 4 we calculate the reheating temperature and discuss the corresponding constrains from BBN; we analyze two scenarios, where the inflaton dominantly decays into two fermions or two bosons, respectively. In Sec. 5, the radiative stability of the potential under one–loop corrections is investigated and the resulting lower bound on $\phi_0$ is derived. In Sec. 6, we investigate the inflationary scale and reheating temperature within the parameter space we have obtained. In Sec. 7, we briefly describe a possible “prehistory” of our model, starting from a phase of eternal inflation. Finally, in Sec. 8 we sum up our findings and end with

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1For reviews on eternal inflation, see e.g. [32, 33].
some prospects to embed our inflation model into some well motivated BSM scenarios.

2 The Setup

In this Section we introduce our potential. We show that inflation can occur at small field values, \( \phi \leq 1 \) in Planckian units, only in the presence of a very flat region, which requires that the potential almost possesses a saddle point. If this is the case, the problem can be treated fully analytically to excellent approximation.

2.1 The Potential

A general renormalizable potential of a single real scalar inflaton \( \phi \) has terms \( \propto \phi^n \) with \( n \in \{0, 1, 2, 3, 4\} \). However, the linear term can be eliminated by shifting the field, such that the origin is an extremum of the potential. We also neglect the constant term, which could produce the cosmological constant, which is tiny compared to the energy scales of interest here. This leaves us with the potential

\[
V(\phi) = b\phi^2 + c\phi^3 + d\phi^4. \tag{2.1}
\]

In order to guarantee the potential to be bounded from below we require \( d > 0 \). The origin is the absolute minimum of the potential if \( b > 0 \). Since the potential is invariant under the simultaneous transformation \( c \rightarrow -c, \phi \rightarrow -\phi \), we can take \( c \leq 0 \) without loss of generality. We work in Planckian units, where the reduced Planck mass \( M_P \simeq 2.4 \cdot 10^{18} \text{ GeV} \equiv 1 \).

The derivatives of the potential are:

\[
V'(\phi) = 2b\phi + 3c\phi^2 + 4d\phi^3;
V''(\phi) = 2b + 6c\phi + 12d\phi^2. \tag{2.2}
\]

At a true saddle point, \( V'(\phi_0) = V''(\phi_0) = 0 \). This happens at

\[
\phi_0 = -\frac{3c}{8d}, \tag{2.3}
\]

if the parameters of the potential satisfy the relation

\[
b = \frac{9c^2}{32d}. \tag{2.4}
\]

In general the parameters will not obey eq.(2.4). Allowing the cubic term to deviate from this relation by a factor \( 1 - \beta \), we can rewrite the potential, still in full generality, as

\[
V(\phi) = d \left[ \phi^4 + \frac{c}{d} (1 - \beta) \phi^3 + \frac{9}{32} \left( \frac{c}{d} \right)^2 \phi^2 \right]
= d \left[ \phi^4 + A (1 - \beta) \phi^3 + \frac{9}{32} A^2 \phi^2 \right], \tag{2.5}
\]

where we have introduced the quantity

\[
A = -\frac{8}{3} \phi_0, \tag{2.6}
\]
which controls the location of the (would-be) inflection point.

As noted in the Introduction, for small field values ($\phi \leq 1$) inflation can occur only if the potential indeed “almost” has a saddle point, i.e. $\beta$ has to be small. This can be seen as follows. As well known, SR inflation requires the parameters $\epsilon_V = 0.5(V'/V)^2$ and $\eta_V = V''/V$ to be small [41]. For our potential, we find

$$
\epsilon_V = \frac{8}{\phi^2} f(\phi);
\eta_V = \frac{12}{\phi^2} g(\phi).
$$

Here the functions $f$ and $g$ approach 1 for $\phi \gg |A|$; in the opposite limit, $\phi \ll |A|$, we have $f(\phi) \to 1/4$, $g(\phi) \to 1/6$. “Generically” these functions will therefore be of order unity, or slightly below. Clearly SR inflation would then require $\phi \gg 1$, i.e. large field values. Here we are interested in small-field inflation, $\phi \lesssim 1$. Since $f \propto (V')^2$, $g \propto V''$, $\epsilon_V$ and $\eta_V$ can evidently only be simultaneously small if for some range of field values both the first and the second derivative of $V$ are small; which requires the existence of a near saddle point, i.e. we need $|\beta| \ll 1$.

This parameter controls the flatness of the potential for $\phi \sim \phi_0$, i.e. the larger $\beta$ is, the more the potential around $\phi_0$ deviates from a flat plateau. Note that $\beta < 0$ would lead to a negative slope at $\phi_0$, and hence to a second minimum at some $\phi > \phi_0$. This would require some finetuning of initial conditions, since the universe could easily get “stuck” in this second minimum if $\phi$ was initially large. We therefore require $\beta \geq 0$.

As already noted, the model parameter $A$ determines the position of the saddle point (or flat region of the potential). Finally, the model parameter $d$ determines the amplitude of the potential, which can be constrained by the power spectrum near the plateau.

Although the inflaton potential (2.5) only contains the three parameters $d$, $A$ and $\beta$, the predictions for cosmological observables also depend on the value of the inflaton field $\phi$ at the time when observable density perturbations were produced. As we will show now, this four-dimensional parameter space can be explored fully analytically in the region of interest.

### 2.2 Analytical Analysis

In this paper, we consider $\phi_0 \leq 1$. In this case $\phi_{\text{CMB}}$ (the field value when the “pivot” scale $k_\ast = 0.05$ Mpc$^{-1}$ crossed out of the horizon) is very close to $\phi_0$ (see Fig. 1). We therefore introduce the field parameter $\delta$:

$$
\phi = \phi_0(1 - \delta),
$$

so decreasing $\phi$ corresponds to increasing $\delta$. Since both $\delta$ and $\beta$ are rather small (as we will see, $\beta \ll \delta \ll 1$), we keep terms up to linear $\beta$ and up to quadratic in $\delta$ in our analysis, and also drop terms $O(\beta \delta)$.

The following definitions for SR parameters, number of e-folds and inflationary predictions are based on standard literature, see e.g. Ref.[41]. For our model the SR parameters
Figure 1: Schematic plot for inflaton potential with an (near) inflection-point at $\phi_0$.

are given by:

$$
\epsilon_V = \frac{1}{2} \left( \frac{V'}{V} \right)^2 \simeq \frac{72 (-2\beta (\delta - 1) + \delta^2)^2}{\phi_0^2} \simeq \frac{72 (2\beta + \delta^2)^2}{\phi_0^2}; \\
\eta_V = \frac{V''}{V} \simeq \frac{12 (-4\beta (\delta - 1) + \delta (3\delta - 2))}{\phi_0^2} \simeq \frac{24 (2\beta - \delta)}{\phi_0^2}; \\
\xi_V^2 = \frac{V''V'''}{V^2} \simeq \frac{288 (4\beta^2 + \beta (2 - 10\delta) + \delta^2)}{\phi_0^4} \simeq \frac{288 (2\beta + \delta^2)}{\phi_0^4}.
$$

As already stated, SR requires $\epsilon_V, |\eta_V| < 1$. The first two eqs. (2.9) show that $\epsilon_V \ll |\eta_V|$ in our case, i.e. the beginning and the end of inflation is determined by $|\eta_V| = 1$, with

$$
\delta_{\text{end}} \simeq \frac{\phi_0^2}{24}. \tag{2.10}
$$

The third slow-roll parameter turns out to always be small if $|\eta_V| < 1$; it affects the running of the spectral index, as we will see shortly.

The number $N_{\text{CMB}}$ of $e$–folds of inflation after the pivot scale $k_* = 0.05 \text{ Mpc}^{-1}$ crossed out of the horizon is given by:

$$
N_{\text{CMB}} = \int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{1}{\sqrt{2\epsilon_V}} d\phi \\
= -\frac{\phi_0^2}{12} \int_{\delta_{\text{end}}}^{\delta_{\text{CMB}}} \frac{d\delta}{(2\beta + \delta^2)} \\
= -\frac{\phi_0^2}{12\sqrt{2\beta}} \left[ \arctan \left( \frac{\delta_{\text{CMB}}}{\sqrt{2\beta}} \right) - \arctan \left( \frac{\delta_{\text{end}}}{\sqrt{2\beta}} \right) \right] \simeq \frac{\phi_0^2}{12\sqrt{2\beta}} \left[ \frac{\pi}{2} - \arctan \left( \frac{\delta_{\text{CMB}}}{\sqrt{2\beta}} \right) \right], \tag{2.11}
$$

where $\delta_{\text{CMB}}$ can be obtained from eq.(2.8):

$$
\delta_{\text{CMB}} = 1 - \frac{\dot{\phi}_{\text{CMB}}}{\dot{\phi}_0}.
$$
In order to resolve the flatness and horizon problems at least 50 e−folds of inflation are needed; in this paper we will take as typical value $N_{\text{CMB}} = 65$. Eq.(2.11) then implies $\sqrt{2\beta} \ll \phi_0^2/12$, i.e. $\delta_{\text{end}}$ of eq.(2.10) is much larger than $\sqrt{2\beta}$ so that $\arctan(\delta_{\text{end}}/\sqrt{2\beta}) \simeq \pi/2$.

Eq.(2.11) also shows that $\delta_{\text{CMB}}$ cannot be much larger than $\sqrt{2\beta}$, but it does not exclude the possibility $\delta_{\text{CMB}} \ll \sqrt{2\beta}$. In order to decide this, we look at the spectral index of the density perturbations:

$$n_s = 1 - 6\epsilon_V + 2\eta_V \simeq 1 - \frac{48(\delta - 2\beta)}{\phi_0^2}.$$  \hfill (2.12)

Observations imply $n_s < 1$, i.e. we need $\delta_{\text{CMB}} > 0$. The second term in the last line of eq.(2.11) therefore reduces the number of e−folds of inflation. Ignoring this term and requiring $N_{\text{CMB}} > 50$ thus implies $\beta < 3.4 \cdot 10^{-6}\phi_0^4$, which in turn shows that the term $\propto \beta$ in eq.(2.12) can be neglected:

$$\delta_{\text{CMB}} \simeq (1 - n_s) \phi_0^2/48.$$  \hfill (2.13)

Eq.(2.11) then requires $\sqrt{2\beta}$ to be of order $\delta_{\text{CMB}}$, so that $\beta \sim O(\delta_{\text{CMB}}^2) \ll \delta_{\text{CMB}}$, as claimed at the beginning of this Subsection.

During SR inflation, the power spectrum of curvature perturbation can be approximated by:

$$P_\zeta = \frac{V}{24\pi^2\epsilon_V} \simeq \frac{d\phi_0^6}{5184\pi^2(\delta^2 + 2\beta)^2}.$$  \hfill (2.14)

This is the only observable of interest that depends on the strength of the quartic coupling $d$.

There are two additional observables, whose values are currently not so well known but where significant progress is expected in the coming years. One is the running of the spectral index, which is given by:

$$\alpha = 16\epsilon_V\eta_V - 24\epsilon_V^2 - 2\xi_V^2 \simeq -\frac{576(2\beta + \delta^2)}{\phi_0^4}.$$  \hfill (2.15)

Due to the smallness of $\epsilon_V$, $\alpha$ is dominated by the contribution $\propto \xi_V^2$, and is negative in our model. The second observable is the power in gravitational fields produced during inflation. It is usually described by the tensor–to–scalar ratio $r$, which is given by:

$$r = 16\epsilon_V \simeq \frac{1152(2\beta + \delta^2)^2}{\phi_0^6}.$$  \hfill (2.16)

3 Model Parameters and Inflationary Predictions

Of course, any potentially realistic model of inflation has to reproduce known facts. Of particular interest are the Planck 2018 measurements [7] at the pivot scale $k_\star = 0.05$ Mpc$^{-1}$:

$$P_\zeta = (2.1 \pm 0.1) \times 10^{-9}; \; n_s = 0.9649 \pm 0.0042; \; \alpha = -0.0045 \pm 0.0067; \; r < 0.061.$$  \hfill (3.1)
We see that two quantities, $P_\zeta$ and $n_s$, are already known quite accurately. In addition, we have to satisfy eq.(2.11) with $N_{CMB} \simeq 65$. Altogether we can thus essentially fix three of the four free parameters of our model.

We chose to keep $\phi_0$ as a free parameter. The model parameter $\delta_{CMB}$ is fixed by the spectral index using eq.(2.13). Choosing a value of $N_{CMB}$ then fixes $\beta$ via eq.(2.11). Finally, we use eq.(2.14) to fix the quartic coupling $d$.

For the central values of $n_s$ and $P_\zeta$ and our standard choice $N_{CMB} = 65$ we find in this way:

\begin{align}
\delta_{CMB} &= 7.31 \times 10^{-4} \phi_0^2 ; \\
\beta &= 9.73 \times 10^{-7} \phi_0^4 ; \\
d &= 6.61 \times 10^{-16} \phi_0^2 .
\end{align}

The scaling with powers of $\phi_0$ can be traced back to eq.(2.13); the numerical factor in (3.2) corresponds to the result with $n_s = 0.9649$ and $\phi_0 = 1$. Since $\delta_{CMB} \propto \phi_0^2$, we see from (2.11) that $\beta$ should be $\propto \phi_0^4$ in order to yield a fixed $N_{CMB}$. The numerical pre–factor in (3.3) comes from the numerical factor in (3.2) and $N_{CMB} = 65$. Finally, $d \propto \phi_0^4$ (from eq.(2.14)) is required to have a fixed power $P_\zeta = 2.1 \times 10^{-9}$.

With eqs.(3.2), (3.3), (3.4) and (2.16), one obtains the prediction

\begin{align}
r &= 7.09 \times 10^{-9} \phi_0^6 .
\end{align}

For $\phi_0 \leq 1$ this is well below the sensitivity of any currently conceivable observation. Varying $N_{CMB}$ and $n_s$ over their allowed ranges does not change this conclusion. On the other hand, eq.(2.15) predicts for the running of spectral index

\begin{align}
\alpha &= -1.43 \times 10^{-3} .
\end{align}

This might be within the sensitivity of a combination of future CMB measurements with greatly improved investigations of structures at smaller scale, in particular the so–called Lyman–$\alpha$ forest [42]. We note that $\alpha$ is independent of $\phi_0$, i.e. this is a clear prediction of our model.

Recently there has been quite a bit of interest in production mechanisms of primordial black holes (PBHs). In principle they can be produced by the gravitational collapse of domains that have a high over–density after inflation. However, this requires [43] a power of $O(10^{-2})$. From eq.(2.14) we see that in our model the highest power occurs at $\delta = 0$; however, eqs.(3.3) and (3.4) show that this maximal power only amounts to $O(10^{-8})$, independent of $\phi_0$. Moreover, eq.(2.14) implies that the power decreases monotonically as $\delta$ increases, i.e. with decreasing scale, as also indicated by $n_s < 1$ and $\alpha < 0$. Hence the current model does not lead to PBH formation from primordial density fluctuations.

Eqs.(3.2) to (3.6) hold for the central value of $n_s$ and $N_{CMB} = 65$. Deviations from these values are explored in Fig. 2. We see that $\beta$ is of order $10^{-6} \phi_0^4$ for the entire allowed parameter space. The results shown in this figure can again be understood analytically. To that end we first expand (around the central values)

\begin{align}
1 - n_s &= 0.0351(1 + \epsilon_n) 
\end{align}
Figure 2: $N_{\text{CMB}}$ as function of $n_s$ and $\beta$ for $\phi_0 = 1$; for other values of $\phi_0$, $\beta$ has to be rescaled by $\phi_0^4$. The vertical black line denotes the current central value of $n_s$, which crosses the contour line with $N_{\text{CMB}} = 65$ for $\beta = 9.73 \times 10^{-7}$.

and

$$\sqrt{\frac{\beta}{\phi_0^4}} = 9.86 \times 10^{-4} (1 + \epsilon_b). \quad (3.8)$$

Taylor expanding the arctan function in eq.(2.11) around the central value then yields:

$$\epsilon_b = \frac{65 - N_{\text{CMB}}}{40.4} - 0.61 \epsilon_n. \quad (3.9)$$

Eq.(3.9) enables us to obtain an analytical expression for $\beta$ as function of $N_{\text{CMB}}$ and $n_s$:

$$\sqrt{\frac{\beta}{\phi_0^4}} = 9.86 \times 10^{-4} \left\{ 1 + \left[ \frac{65 - N_{\text{CMB}}}{40.4} - 0.61 \left( \frac{1 - n_s}{0.0351} - 1 \right) \right] \right\}, \quad (3.10)$$

which agrees very well with the numerical results shown in Fig. 2.

As already noted, $r$ remains tiny, of order $10^{-8} \phi_0^6$, over the entire allowed parameter space. The dependence of the running of the spectral index $\alpha$ on $n_s$ and $N_{\text{CMB}}$ is given by

$$\alpha = -\frac{576(2\beta + \delta^2)}{\phi_0^4}$$

$$= -1.43 \times 10^{-3} - 5.56 \times 10^{-5} \left[ 65 - N_{\text{CMB}} \right] + 0.02149 \left[ 0.9649 - n_s \right] - 0.25 \left[ 0.9649 - n_s \right]^2,$$

(3.11)
which still does not depend on $\phi_0$; the result is shown in Fig. 3.

**Figure 3:** Prediction for the running of the spectral index $-\alpha/10^{-3}$ as function of $n_s$ and $N_{\text{CMB}}$. Our model predicts $\alpha \sim -10^{-3}$ when $n_s$ lies in the vicinity of its current central value (vertical black dashed line) and $50 < N_{\text{CMB}} < 80$, which might be testable in future.

Evidently our model requires a very small but positive value of $\beta$, see eq.(3.3). Eq.(3.10) shows that $\beta$ varies approximately linearly when $n_s$ and/or $N_{\text{CMB}}$ are varied over their allowed ranges. In that sense $\beta$, while undoubtedly very small, is not very finely tuned. On the other hand, setting $\beta = 0$ does not enhance the symmetry of the potential. This means that radiative corrections to $\beta$ – or, more accurately, to the first and second derivative of the potential at $\phi = \phi_0$ – need not be proportional to $\beta$. In order to compute these corrections, we first have to expand the scope of our model to include reheating. After inflation the inflaton field has to decay away to produce relativistic Standard Model particles, i.e. radiation; otherwise no hot Big Bang will result. This requires some coupling(s) of the inflaton to lighter particles. These couplings will contribute to the radiative corrections to the inflaton potential. Before computing these corrections, we therefore need to discuss reheating.
4 Reheating

After inflation ends, the inflaton field oscillates around the minimum of its potential and transfers energy to other degree of freedoms. This process is usually called reheating. In general it consists of a non-perturbative “preheating” stage followed by the perturbative decay of the remaining inflaton particles [47]. Finally, the decay products have to thermalize.

In this paper we focus on the simplest possibility, perturbative decays of the inflaton through trilinear couplings of the inflaton to lighter particles. Such a coupling is always required, since otherwise at least some inflaton particles would remain at the end of the reheating period. At the end of this Section we will argue that non-perturbative effects cannot significantly deplete the inflaton number in our model; and in the next Section we will see that all inflaton couplings have to be so small that inflaton annihilation reactions are completely negligible.

We compute the reheating temperature in the instantaneous decay approximation. By setting the energy density in inflaton matter, \( \rho = m_\phi n_\phi \), equal to the radiation density \( \rho_R = \pi^2 g_* T_{re}^4 / 30 \) at time \( t = 2 / (3H) = 1 / \Gamma_\phi \), with \( H^2 = \rho / 3 \) as usual in FRW cosmology, we find (still using Planckian units)

\[
T_{re} \simeq 1.41 g_*^{-1/4} \Gamma_\phi^{1/2}.
\]

Here \( g_* \) is the number of light degrees of freedom forming the thermal plasma, and \( \Gamma_\phi \) is the perturbative inflaton decay width. For \( T_{re} > 1 \text{ GeV} \), \( g_* \) is of order 100.

For completeness we allow the inflaton to decay into a Dirac fermion \( \chi \) and/or a scalar \( \phi' \). Since \( \phi \) is a singlet under the SM gauge group, \( \chi \) would have to be in a vector-like representation of that group, i.e. it cannot be an SM fermion. On the other hand, \( \phi' \) might be the Higgs field of the SM. We will treat this as our standard case, i.e. we will assume that \( \phi' \) contains four degrees of freedom, just like the Dirac fermion \( \chi \). The relevant parts of the Lagrangian are given by

\[
\mathcal{L} = i \bar{\chi} \gamma^\mu \partial_\mu \chi + (\partial_\mu \phi')^\dagger \partial^\mu \phi' - m_\chi \bar{\chi} \chi - y_\chi \phi \bar{\chi} \chi - m_{\phi'}^2 |\phi'|^2 - g_\phi |\phi'|^2 + \lambda_\phi |\phi'|^4.
\]

The total decay width of the inflaton is then given by

\[
\Gamma_\phi = \frac{y_\chi^2 m_\phi}{8\pi} \left( 1 - \frac{4m_{\phi'}^2}{m_\phi^2} \right)^{3/2} + \frac{g_\phi^2}{8\pi m_\phi} \sqrt{1 - \frac{4m_{\phi'}^2}{m_\phi^2}} \simeq \frac{y_\chi^2 m_\phi}{8\pi} + \frac{g_\phi^2}{8\pi m_\phi},
\]

where the inflaton mass is\(^3\)

\[
m_\phi^2 = \frac{\partial^2 V}{\partial \phi^2} \bigg|_{\phi = 0} \simeq \frac{9}{16} dA_0^2 = 4dA_0^2.
\]

\(^2\)For reviews on (p)reheating, see e.g. Refs. [44–46].

\(^3\)This is the mass of inflaton particles after inflation, and therefore not directly related to the SR parameter \( \eta_V \), which is also computed from the second derivative of the inflaton potential.
In the numerical analysis we will assume that one of the two terms in eq.(4.3) dominates; the other one may then even vanish. Moreover, we assume that the mass of $\chi$ or $\phi'$ is much smaller than $m_\phi$; this minimizes the upper bound on the couplings $y$ and $g$ which we wish to derive.

If fermionic decays dominate, the reheat temperature is given by

$$T_{\chi}^{\text{re}} \simeq 1.41 g_*^{-1/4} \left( 2\phi_0 \frac{y^2}{8\pi} \sqrt{d} \right)^{1/2},$$  \hspace{1cm} (4.5)$$

while for the bosonic decay mode,

$$T_{\phi'}^{\text{re}} \simeq 1.41 g_*^{-1/4} \left( \frac{g^2}{8\pi 2\phi_0 \sqrt{d}} \right)^{1/2}.$$  \hspace{1cm} (4.6)$$

Successful BBN requires $T_{\text{re}} \geq 4$ MeV, i.e. $T_{\text{re}} \geq 1.67 \cdot 10^{-21}$ in Planckian units [34, 35]. Taking $g_* = 10.75$, as appropriate for a temperature of 4 MeV, and $d = 6.61 \times 10^{-16} \phi_0^2$ from Eq.(3.4), we finally obtain lower bounds on the inflaton couplings:

$$y\phi_0 \geq 4.7 \times 10^{-17},$$  \hspace{1cm} (4.7)$$

if fermionic decays dominate, and

$$\frac{g}{\phi_0} \geq 2.4 \times 10^{-24},$$  \hspace{1cm} (4.8)$$

for bosonic inflaton decays. The scaling with $\phi_0$ can be understood from the observation that $m_\phi \propto \sqrt{d}\phi_0 \propto \phi_0^2$, from eqs.(4.4) and (3.4). Eq.(4.1) shows that a constant reheat temperature corresponds to a constant decay width $\Gamma_\phi$. From eq.(4.3) this requires constant $y^2 m_\phi$, i.e. constant $y^2 \phi_0^2$, if fermionic decays dominate, but constant $g^2/m_\phi$, i.e. constant $g^2/\phi_0^2$ for bosonic inflaton decays; note that $g$ has dimension of mass in natural units, whereas $y$ is dimensionless.

Before concluding this Section we come back to the issue of non–perturbative inflaton depletion mechanisms. In principle the couplings in the Lagrangian (4.2) allow both fermionic and bosonic preheating. However, due to Pauli blocking, fermionic preheating is usually very inefficient [48, 49], i.e. it reduces the initial inflaton energy by less than 1\%. For bosonic inflaton decays, our trilinear $\phi\phi'|^2$ coupling leads to a tachyonic instability if $|g\phi| > m_{\phi'}^2$, where the squared mass of the daughter particle is negative for part of each oscillation of the inflaton field. As shown in [50, 51] this can build up a sizable $\phi'$ number density after a very small number of $\phi$ oscillations. However, even with very small $\phi'$ self–interactions of $\mathcal{O}(10^{-6})$ this only allows to transfer less than 10\% of the initial inflaton energy [50]. In our default scenario, where $\phi'$ is the SM Higgs field, $\phi'$ does have a sizable self interaction of the form $\lambda_{\phi'}\phi'^4$. This gives an extra positive effective squared mass $\sim \lambda_{\phi'} \langle \phi'^2 \rangle$ (with $\langle \phi'^2 \rangle$ denoting the variance) once the daughter $\phi'$ particles are copiously produced. This back–reaction counteracts the negative tachyonic mass and quickly terminates preheating, making it even less efficient [50]; in a lattice simulation [52]
we found that less than 1% of the initial inflaton energy is depleted through preheating also for these bosonic decay products. Preheating can thus indeed be neglected in our model.\footnote{In the literature on preheating some scenarios have been suggested which could deplete inflaton energy more efficiently, see e.g. instant preheating \cite{53} or combined reheating \cite{54–56}. Here one introduces additional perturbative decay channels for the produced daughter particles in order to get rid of the back reaction problem. However, the decay width of the SM Higgs, our default bosonic decay product, is just 4 MeV, which is 5 orders of magnitude smaller than the minimal allowed value of $m_\phi$ in our model (see below). The preheating time scale, which is $\mathcal{O}(1/m_\phi)$, is thus very much shorter than the $\phi'$ lifetime, in which case $\phi'$ decays cannot affect the preheating dynamics.}

5 Radiative Corrections and Stability

The lower bounds (4.7) and (4.8) on the inflaton couplings imply lower bounds on the radiative corrections to the inflaton potential caused by these couplings. The self–couplings of the inflaton, described by the potential (2.5), also contribute to the radiative corrections. In this Section we investigate the impact of these corrections in 1–loop order. This will lead to upper bounds on the couplings; together with the lower bounds derived in the previous Section this will finally yield a lower bound on the remaining free parameter $\phi_0$.

The starting point of this analysis is the expression for the 1–loop effective potential, in the formalism of Coleman and Weinberg (CW) \cite{57}:

$$\Delta V(\phi) = \frac{1}{64\pi^2} \sum_\psi (-1)^{2s_\psi} g_\psi \tilde{m}_\psi(\phi)^4 \left( \ln \left( \frac{\tilde{m}_\psi(\phi)^2}{Q_0^2} \right) - \frac{3}{2} \right).$$

(5.1)

The sum runs over all fields $\psi$ that couple to the inflaton field $\phi$. $s_\psi$ is the spin of $\psi$; the factor $(-1)^{2s_\psi}$ therefore implies that bosons (fermions) contribute with positive (negative) sign to $\Delta V$. $g_\psi$ is the number of degrees of freedom of the field $\psi$; it includes a spin multiplicity factor $2s_\psi + 1$. Finally, $\tilde{m}_\psi(\phi)$ is the $\phi$–dependent mass of $\psi$ (not to be confused with the physical mass), and $Q_0$ is a renormalization scale.

In our case, up to three fields couple to the inflaton: the inflaton itself, as well as the fermionic and bosonic decay products $\chi$ and $\phi'$ introduced in the previous Section. Their field–dependent masses are given by:

$$
\begin{align*}
\tilde{m}_\phi^2(\phi) &= 12d\phi^2 + 6dA(1 - \beta)\phi + \frac{9}{16}dA^2; \\
\tilde{m}_\chi^2(\phi) &= (m_\chi + y\phi)^2; \\
\tilde{m}_{\phi'}^2(\phi) &= m_{\phi'}^2 + g\phi.
\end{align*}
$$

(5.2)

We want to make sure that the predictions derived in Sec. 3 are stable under radiative corrections. To this end we need to investigate the potential around the point $\phi_0$, where inflation happens. In fact, the tree–level potential $V_0$ itself is not particularly suppressed at $\phi = \phi_0$: $V_0(\phi_0) \rightarrow d\phi_0^4/3$ as $\beta \rightarrow 0$. On the other hand, it is essential that the first and second derivatives of the potential are suppressed at $\phi_0$; this is why $\phi_0$ is a near inflection point. Recall also that $V'_0$ and $V''_0$ directly determine $N_{\text{CMB}}$ and $n_s$, respectively.


eq. (2.5) with \( A = -8\phi_0/3 \) we have

\[
\begin{align*}
V_0'(\phi_0) &= 8d\beta \phi_0^3; \\
V_0''(\phi_0) &= 16d\beta \phi_0^2. \\
\end{align*}
\] (5.3)

On the other hand, from eq. (5.1) the derivatives of the CW correction to the potential can be written as

\[
\Delta V' = \frac{1}{32\pi^2} \sum_{\psi} (-1)^{2s\psi} g_{\psi} \tilde{m}_{\psi}^2 \tilde{m}_{\psi}^2 \left( \ln \left( \frac{\tilde{m}_{\psi}^2}{Q_0^2} \right) - 1 \right);
\]

\[
\Delta V'' = \frac{1}{32\pi^2} \sum_{\psi} (-1)^{2s\psi} g_{\psi} \left\{ \left[ (\tilde{m}_{\psi}^2)^2 + \tilde{m}_{\psi}^2 \tilde{m}_{\psi}^{2n} \right] \ln \left( \frac{\tilde{m}_{\psi}^2}{Q_0^2} \right) - \tilde{m}_{\psi}^2 \tilde{m}_{\psi}^{2n} \right\}. \] (5.4)

Here \( \tilde{m}_{\psi}^{2n} \) and \( \tilde{m}_{\psi}^{2n} \) are the first and second derivatives of \( \tilde{m}_{\psi}^2 \) with respect to \( \phi \).

The loop corrections are minimized if the bare masses \( m_\chi \) and \( m_{\phi'} \) vanish. Recall also that these masses must be below half the physical inflaton mass; using eqs. (4.4) and (3.4) this implies \( m_\chi, m_{\phi'} < \sqrt{d}\phi_0 = 2.6 \cdot 10^{-8}\phi_0 \), which is already quite small. In the subsequent analysis we will therefore assume \( m_\chi \ll y\phi_0 \) and \( m_{\phi'} \ll g\phi_0 \), so that the bare mass terms can be neglected. Moreover, we set \( Q_0 = \phi_0 \), since this is the field value we are interested in; this means that the Lagrangian parameters \( y \) and \( g \) should be interpreted as running couplings, taken at scale \( Q_0 \). The derivatives of the correction to the potential at \( \phi = \phi_0 \) are then given by:

\[
\Delta V'(\phi_0) = \frac{\phi_0^3}{4\pi^2} \left[ y^4 - 16d^2\beta - y^4 \ln(y^2) + 16d^2\beta \ln(16d\beta) \right] + \frac{g^2\phi_0}{8\pi^2} \ln \left( \frac{g}{\phi_0} \right) - 1; \]

\[
\Delta V''(\phi_0) = \frac{\phi_0^3}{4\pi^2} \left[ y^4 - 3y^4 \ln(y^2) + 8d^2 \ln(16d\beta) \right] + \frac{g^2}{8\pi^2} \ln \left( \frac{g}{\phi_0} \right). \] (5.5)

In the first eq. (5.5) we have ignored terms of order \( d^2\beta^2 \). We see that all corrections from the inflaton self–coupling \( d \) are proportional to \( \beta \), which means that these terms are automatically smaller than the tree–level result given in the first eq. (5.3). In the second eq. (5.5) we neglected also terms linear in \( \beta \). We see that nevertheless a finite one–loop correction \( \propto d^2 \) remains.

In order to ensure stability of our inflationary model against radiative corrections, we will require that the terms \( \propto d^2, \propto y^4 \) and \( \propto g^2 \) are separately smaller than the tree–level results of eqs. (5.3). We just saw that in case of \( d^2 \) only the correction to the second derivative of the potential can be dangerous. Demanding that it is smaller in magnitude than the tree–level result leads to the constraint

\[
\left| \frac{d^2\ln(16d\beta)}{\pi^2} \right| < 8d\beta. \] (5.6)

Using the numerical values from eqs. (3.3) and (3.4) this implies

\[
|\ln(10^{-20}\phi_0^6)| < 1.16 \cdot 10^{11}\phi_0^2, \]
which in turn implies 
\[ \phi_0 > 3 \cdot 10^{-5}. \]  

(5.7)

The strongest upper bound on the Yukawa coupling also comes from the second derivative of the potential:
\[ \left| \frac{y^4 - 3y^4 \ln(y^2)}{4\pi^2} \right| < 16d\beta. \] 

(5.8)

In order to turn this into a lower bound on \( \phi_0 \), we again use eqs. (3.3) and (3.4) for the right-hand side, and insert the lower limit (4.7) from reheating for \( y \); this gives
\[ \phi_0 > 3.4 \cdot 10^{-5}, \] 

(5.9)

which is slightly stronger than the bound (5.7).

On the other hand, the strongest bound on the coupling \( g \) originates from the first derivative of the potential; it reads
\[ \left| \frac{g^2}{8\pi^2} \ln\left( \frac{g}{\phi_0} \right) - 1 \right| < 8d\beta \phi_0^2. \] 

(5.10)

Replacing \( g \) by its lower bound (4.8) then implies
\[ \phi_0 > 3.1 \cdot 10^{-5}, \] 

(5.11)

very close to the bound (5.7) which is independent of reheating.

**Figure 4**: The light blue region is the allowed parameter space, yielding \( T_{\text{re}} \geq 4 \text{ MeV} \) while keeping the inflaton potential stable against radiative corrections. The left (right) frame is for fermionic (bosonic) inflaton decays.

The constraints on the parameter space spanned by \( \phi_0 \) and the coupling that is responsible for reheating are shown in Fig. 4; the left and right frames are for fermionic and bosonic inflaton decays, respectively. The allowed parameter space, shown in blue, ends
at the values of $\phi_0$ given by the bounds (5.9) and (5.11). Evidently the allowed range of couplings opens up when $\phi_0$ increases; for the maximal value we consider, $\phi_0 = 1$, it ranges over 11 orders of magnitude for $y$, and 13 orders of magnitude for $g$. Nevertheless, even for $\phi_0 = 1$ the maximal allowed value of the Yukawa coupling is about $10^{-5}$, which is only slightly larger than the Yukawa coupling of the electron in the SM.

Recall that we assumed that four (bosonic or fermionic) degrees of freedom couple to the inflaton, i.e. $g_{\phi'} = g_\chi = 4$. In case of bosonic decays, both the lower bound on $g^2$ from reheating and the upper bound from radiative stability scale like $1/g_{\phi'}$, i.e. the resulting lower bound (5.11) does not depend on $g_{\phi'}$. On the other hand, for fermionic decays the lower bound on $y^2$ scales like $1/g_\chi$ while the upper bound scales like $1/\sqrt{g_\chi}$; the bound (5.9) therefore roughly scales like $g^{-1/10}$ in the SM. However, it is in any case already quite close to the bound (5.7) which is independent of reheating.

We ignored a possible quartic coupling $\lambda \phi^2 |\phi'|^2$ in our discussion of (p)reheating. Such a coupling would also contribute to the CW corrections to the potential. Demanding that this contribution to the first derivative of the inflaton potential at $\phi_0$ does not exceed the tree–level value gives the quite stringent upper bound $\lambda \sqrt{|\ln(\lambda)|} < 4.5 \cdot 10^{-10} \phi_0^3$. The largest quartic coupling $\lambda$ allowed by this bound is of $\mathcal{O}(10^{-10})$ even for $\phi_0 = 1$. Preheating with such small coupling is not efficient [47], i.e. reheating has to proceed via perturbative inflaton decay as we analyzed in Sec. 4.

We finally note that the upper bounds on the inflaton couplings we derived in this Section imply that the rate for inflaton annihilation reactions, $\phi\phi \rightarrow \chi\bar{\chi}$ or $\phi\phi \rightarrow \phi'\bar{\phi}'$, is always much smaller than the Hubble rate $H$. The annihilation rate is given by $\langle \sigma v \rangle n_\phi$, where $\sigma$ is the relevant annihilation cross section, $\langle ... \rangle$ denotes averaging over the ensemble of inflaton particles, and $n_\phi$ is the inflaton density. Right after inflation one can estimate $n_\phi \sim V_{\text{inf}}/m_\phi$ and $H \sim H_{\text{inf}} \sim \sqrt{V_{\text{inf}}}$, but even at this high inflaton density the annihilation rate is many orders of magnitude smaller than the Hubble rate. The ratio becomes even smaller at later times, since $H \propto 1/t$ while $n_\phi \propto 1/t^2$ during matter domination. Therefore inflaton annihilation plays no role in the dynamics of reheating.

6 The Scales of Inflation

Having derived a lower bound on $\phi_0$ we can discuss the range of energy scales during and just after inflation that can be realized in our model. With this we mean both the vacuum energy during inflation (or, equivalently, the Hubble parameter), and the range of reheating temperatures after inflation.

Since $\phi_{\text{CMB}}$ is very close to $\phi_0$, the inflationary scale $H_{\text{inf}}$ is essentially equal to that at the inflection-point $\phi_0$. From eqs. (2.5) and (3.4) we have

$$V(\phi_0) = \frac{1}{3} d\phi_0^4 \simeq 2.2 \cdot 10^{-16} \phi_0^6,$$

where we have neglected $\beta$ and used $A = -8\phi_0/3$. This corresponds to a Hubble parameter

$$H_{\text{inf}} = \sqrt{\frac{V(\phi_0)}{3}} \simeq 8.6 \cdot 10^{-9} \phi_0^3.$$
In the previous section we saw that $\phi_0 \gtrsim 3 \times 10^{-5}$; the lower bound on the bound on the Hubble parameter during inflation is thus

$$H_{\text{inf}} \gtrsim 2.3 \times 10^{-22} \simeq 0.6 \text{ MeV}.$$  \hfill (6.3)

In such a low scale inflationary scenario, the cosmological moduli problem can be relaxed [38]. Besides the isocurvature bound of QCD axion can be easily satisfied, making our model a good candidate to embed QCD axion as dark matter, which can even allow a wider cosmological window with larger decay constant $f_a$ [36, 37]. On the other hand, for $\phi_0 \simeq 1$, $H_{\text{inf}} \sim 10^{10}$ GeV is possible, which allows for the non–thermal production of various particles, and hence non–standard post–inflationary cosmologies.

It is instructive to compare the inflationary Hubble parameter (6.2) with the change of the inflaton field during one Hubble time due to the slow–roll of the field. The latter is given by

$$\Delta \phi = \frac{|\dot{\phi}|}{H} = \frac{|V'|}{3H^2} = \frac{|V'|}{V} = \frac{24\beta}{\phi_0} = 2.3 \cdot 10^{-5} \phi_0^3,$$  \hfill (6.4)

which is much larger than $H_{\text{inf}}/(2\pi)$. This means that even near the inflection point the dynamics of the inflaton field is entirely dominated by the classical (SR) equation of motion. The other energy scale of interest in inflationary model building is the reheating temperature. As long as we don’t fix the relevant coupling $y$ or $g$, we cannot make a firm prediction; however, the upper bounds on these couplings that we derived in the previous Section allow to derive an upper bound on $T_{\text{re}}$ for given $\phi_0$. This is shown in Fig. 5, where we have again used the instantaneous reheating approximation. We see that for fermionic (bosonic) inflaton decay, the reheating temperatures as high as $4 \cdot 10^8$ GeV ($10^{11}$ GeV) are possible. This allows for standard thermal leptogenesis [40]. Of course, the fermionic decay product $\chi$ might itself be right–handed neutrinos (which contribute $g_\chi = 2$ for each generation), allowing for non–thermal leptogenesis if the coupling $y$ is (well) below its upper bound.

The slopes of the curves can be understood as follows. For fermionic decays, $\Gamma_\phi \propto y^2 m_\phi$, with $m_\phi \propto \phi_0^2$ from eqs. (4.4) and (3.4) while $y^2 \propto \phi_0^5$ (up to logarithmic corrections) from the constraint (5.8), hence $T_{\text{re max}} \propto 1/2 \phi_{\text{max}} \propto \phi_0^{5/2}$. For bosonic decays, $\Gamma_\phi \propto g^2 / m_\phi$ and $g_\text{max}^2 \propto \phi_0^3$ (again up to logarithmic corrections) from (5.10), hence $T_{\text{re max}} \propto \phi_0^3$. In these simple estimates we have ignored the dependence of $g_\chi$ on $T_{\text{re}}$, which has been included in Fig. 5. When the temperature is around 0.1 GeV, the QCD deconfinement transition happens, leading to a rapid change of $g_\chi$ [58]; this is the reason for the features in the red and green curves at $T_{\text{re max}} \sim 0.1$ GeV.

Recall from eq. (4.4) that $m_\phi/2 = \sqrt{Q_0} = 6.2 \cdot 10^{10}$ GeV × $(\phi_0 / M_{\text{Pl}})^2$, which is somewhat above the maximal reheat temperature for fermionic inflaton decays as shown in Fig. 5. For fermionic decays a scenario with $T_{\text{re}} > m_\phi/2$ is difficult to realize; instead, Pauli blocking would delay inflaton decays such that $T_{\text{re}} \lesssim m_\phi/2$. For bosonic decays $T_{\text{re}} > m_\phi/2$ is possible, since several relative soft bosons can combine into a smaller number of more energetic bosons.

We also remind the reader that the highest temperature of the thermal background can be considerably higher than $T_{\text{re}}$ [59]; parametrically, in Planckian units $T_{\text{max}} \sim \sqrt{T_{\text{re}} H_{\text{inf}}}^{1/4} \sim$
Figure 5: Allowed range of the post–inflationary reheat temperature as a function of $\phi_0$ (in the range $[3 \cdot 10^{-5}, 1]$), for bosonic (red) and fermionic (green) inflaton decays. The blue line shows the lower bound of 4 MeV from BBN considerations, and the purple line denotes half of inflaton mass within the parameter space.

$\phi_0^{3/4}/\sqrt{T_{re}}$. In our case this is indeed always several orders of magnitude above $T_{re}$, with $T_{\text{max}}/T_{\text{re,max}}$ scaling like $\phi_0^{-1/2}(\phi_0^{-3/4})$ for fermionic (bosonic) inflaton decays. However, for fermionic inflaton decays one also has to require $T_{\text{max}} \leq m_\phi/2$, as we argued above.

7 Prehistory

So far our analysis has only been concerned with field values $\phi \lesssim \phi_0$, which we limited to be not larger than 1 (in Planckian units). In that sense our model is a “small field” model of inflation.

In this Section we nevertheless wish to briefly describe the dynamics at much larger field values. After all, except for possible quantum gravity effects our model can be UV complete, i.e. it might describe the dynamics also at much larger field values.\(^5\)

\(^5\)It has been conjectured that complete models of quantum gravity “always” contain many relatively light
For field values $\phi \gg \phi_0$ the potential (2.5) is dominated by the quartic term $d\phi^4$. The dynamics in this range is therefore that of quartic chaotic inflation [63]. In particular, the deterministic change of $\phi$ during one Hubble time, $|\dot{\phi}|/H$, will be smaller than the random variation $H/(2\pi)$ if
\begin{equation}
|\dot{\phi}|/H < \frac{1}{\sqrt{(2\pi)}} H_0^{1/3},
\end{equation}
where we have again used eq.(3.4) for the strength of the quartic coupling. If $\phi$ ever satisfied this bound, a period of “eternal” inflation started; in fact, in this case it should continue even now in “most” of space. This epoch of eternal inflation might allow to sample a “landscape” of minima of the (total) effective potential, which seems to be a feature of superstring theory [31].

Of course, in our patch of the universe eternal inflation must have ended at some point. It would have been followed by a long period of deterministic inflation, since for $\phi \lesssim \phi_{\text{ch, min}}$ the SR parameters are still very small. This first phase of deterministic SR inflation ended at
\begin{equation}
\phi = \phi_{e,1} \simeq \sqrt{12} + \frac{2}{3} \phi_0,
\end{equation}
where we have neglected terms of order $\phi_0^2$. This first phase of deterministic inflation, where $\phi_{\text{ch, min}} > \phi > \phi_{e,1}$, lasted for
\begin{equation}
N_{\text{det,1}} \simeq 1.8 \cdot 10^5 \phi_0^{-2/3}
\end{equation}
e−folds. It should be noted that any initial field value $\phi_i > \phi_{e,1}$ would lead to large–field SR inflation; large field inflation is much less sensitive to initial conditions than small–field inflation [64]. Of course, if our universe indeed underwent a period of eternal inflation, the question of initial conditions might be moot [6].

For $\phi < \phi_{e,1}$ the field underwent fast roll (or overshooting), until it reached the vicinity of the near–inflection point $\phi_0$ [65]. Here we can use an expansion as in eq.(2.8) again, but now $\delta$ is negative, at least initially. SR inflation then starts again once $|\eta_V| < 1$, which is true for
\begin{equation}
\phi < \phi_b \simeq \phi_0 \left(1 + \frac{\phi_0^2}{24} - \frac{\phi_0^4}{384} + O(\phi_0^6)\right).
\end{equation}
Here we have neglected terms of order $\beta$. Eventually $\phi$ reached the value $\phi_{\text{CMB}} = \phi_0(1 - \delta_{\text{CMB}})$, with $\delta_{\text{CMB}}$ given by eq.(3.2). SR inflation with $\phi_b > \phi > \phi_{\text{CMB}}$ gave rise to another
\begin{equation}
N_{\text{pre–CMB}} \simeq 120
\end{equation}
e−folds of inflation, with Hubble parameter given by eq.(6.2).

This second deterministic stage of SR inflation would have been sufficient to completely dilute any relics from possible earlier large–field inflationary phases, even before density perturbations on CMB scales were generated. Therefore the “pre–history” sketched in this Section most likely does not have any direct observational consequences.

degrees of freedom if the inflaton field moves over trans–Planckian field ranges, which means that one might lose control over the theory [60, 61]. However, explicit counter–examples in the framework of string theory seem to exist [62]. We also note that additional light fields need not affect the dynamics of the inflaton, even if they “generically” do.
8 Summary and Conclusions

In this paper, we have revisited the renormalizable small field polynomial inflation model. This model can reproduce cosmological data only if the potential possesses an “almost” inflection point \( \phi_0 \), such that \( \phi \simeq \phi_0 \) during inflation. Expanding in \( \phi_0 - \phi \) allowed us to derive accurate analytical expressions for all relevant quantities. This includes the number of e−folds of inflation after the pivot scale crossed out of the horizon, \( N_{\text{CMB}} \), given in eq.(2.11), as well as the power spectrum, spectral index, its running, and the tensor–to–scalar ratio \( r \), as shown in eqs.(2.12-2.16).

As usual for small–field models of inflation, \( r \) is too small to be detectable by currently conceivable experiments, i.e. a convincing detection of gravitational waves of inflationary origin would exclude our model. A second prediction is a negative running of the spectral index, given by
\[
\alpha = -1.43 \times 10^{-3} + 5.56 \times 10^{-5} (N_{\text{CMB}} - 65),
\]
which might be detectable in future [42]. Note that this is independent of \( \phi_0 \), which is the only free parameter of our model once we have fixed the overall power of the density perturbations, their spectral index, and \( N_{\text{CMB}} \).

A complete model also has to provide for a mechanism to reheat the universe after inflation ends. Here we considered inflaton decays into either fermions or bosons via trilinear interactions. For given \( \phi_0 \) the corresponding coupling strengths can be bounded from below by demanding that the reheating temperature is sufficiently high for successful BBN. On the other hand, we showed that the radiative stability of the inflaton potential near the inflection point leads to upper bounds on these couplings, which again depend on \( \phi_0 \). These constraints on the parameter space are summarized in Fig. 4. In particular, radiative stability requires \( \phi_0 > 3 \cdot 10^{-5} \) in Planckian units. Within the allowed parameter space the Hubble parameter during inflation (\( H_{\text{inf}} \)) can be as low as \( \sim 1 \) MeV, which makes our model a good candidate to embed QCD axion as dark matter allowing wider cosmological window [36, 37]. On the other hand, \( H_{\text{inf}} \) can also be as high as \( 10^{10} \) GeV if \( \phi_0 \simeq 1 \). In this case the reheat temperature might be as high as \( 4 \cdot 10^8 \) \( (10^{11}) \) GeV for fermionic (bosonic) inflaton decays as shown in Fig. 5. We finally showed that our potential also allows for a phase of “eternal” inflation if the field ever was large enough. While this does not directly affect any observables, it can address conceptual issues involving the “landscape” of superstring theory and the initial conditions for inflation.

Of course, if large field values are admissible, one can also consider scenarios where the near inflection point lies at \( \phi_0 > 1 \). Since \( \delta_{\text{CMB}} \propto \phi_0^2 \) the expansion we used in this paper will no longer work when \( \phi_0 \) becomes large. Qualitatively new features will then become possible, including a sizable value of \( r \) and two distinct epochs of eternal inflation. We will investigate the large field version of this model in a future publication.

The least attractive feature of this model is that one has to engineer \( \phi_0 \) to “almost” be an inflection point; specifically, the parameter \( \beta \), which controls the flatness of the potential around \( \phi_0 \), has to be of order \( 10^{-6} \phi_0^4 \), see eq.(3.3). Actually, when written in the form of eq.(2.5) the finetuning is not obvious; after all, \( \beta \), while small, is not terribly finely tuned. On the other hand, the coefficient of the cubic term is tuned. This conclusion can be avoided only if \( A \) and \( \beta \) in eq.(2.5) can be considered to be independent parameters. This is another
example where conclusions about finetuning depend strongly on what are considered to be independent parameters. At any rate, our upper bounds on the relevant couplings imply that the model is at least technically natural, in the sense that radiative corrections are under control.

On the other hand, the model we consider is renormalizable, and can thus serve as the inflationary sector of some well motivated extensions of the standard model of particle physics; examples are the $\nu$MSM [66, 67], or the new minimal standard model (NMSM) [68] which can explain cosmological dark matter, neutrino masses and the baryon asymmetry. This offers avenues for future research. See Ref. [69] for a recent study along this direction.

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