Invariant differential operators for non-compact Lie groups: the SO∗(12) case

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Abstract. In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact algebra so∗(12). We give the main multiplets of indecomposable elementary representations. Due to the recently established parabolic relations the multiplet classification results are valid also for the algebra so(6, 6) with suitably chosen maximal parabolic subalgebra.

1. Introduction
Invariant differential operators play very important role in the description of physical symmetries. In a recent paper [1] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on the algebra so∗(12). The algebras so∗(4n) are form a subclass of the class of algebras, which we call 'conformal Lie algebras' in [2], which have very similar properties to the canonical conformal algebras of Minkowski space-time. In our subclass we have the algebras: so∗(4), so∗(8), so∗(12), ... However the first two cases are reduced to well known conformal algebras due to the coincidences: so∗(4) ≃ so(3) ⊕ so(2, 1), so∗(8) ≃ so(6, 2). Thus, the algebra so∗(12) is the lowest nontrivial member of our subclass.

This paper is a sequel of [1] and [3] and due to the lack of space we refer to these papers for motivations and extensive list of literature on the subject.

2. Preliminaries
Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition G = KA0N0, where A0 is abelian simply connected vector subgroup of G, N0 is a nilpotent simply connected subgroup of G preserved by the action of A0. Further, let M0 be the centralizer of A0 in K. Then the subgroup P0 = M0A0N0 is a minimal parabolic subgroup of G. A parabolic subgroup P = MAN is any subgroup of G which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [4–6].

Let ν be a (non-unitary) character of A, ν ∈ A∗, let μ fix an irreducible representation Dμ of M on a vector space Vμ.
We call the induced representation $\chi = \text{Ind}_G^H(\mu \otimes \nu \otimes 1)$ an elementary representation of $G$ [7]. Their spaces of functions are:

$$C_\chi = \{ \mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(g) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) \mathcal{F}(g) \}$$

(1)

where $a = \exp(H) \in A$, $H \in A$, $m \in M$, $n \in N$. The representation action is the left regular action:

$$(T^\lambda(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g') \ , \ g, g' \in G \ .$$

(2)

For our purposes we need to restrict to maximal parabolic subgroups $P$, so that rank $A = 1$. Thus, for our representations the character $\nu$ is parameterized by a real number $d$, called the conformal weight or energy.

An important ingredient in our considerations are the highest/lowest weight representations of $G$. These can be realized as (factor-modules of) Verma modules $V^\Lambda$ over $G^C$, where $\Lambda \in (H^C)^*$, $H^C$ is a Cartan subalgebra of $G^C$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from $\chi$ [8,9].

Actually, since our ERs will be induced from finite-dimensional representations of $M$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use generalized Verma modules $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector $v_0$ is taken by the space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight $d$. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called multiplets [9,10]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair $(\beta, m)$, where $\beta$ is a (non-compact) positive root of $G^C$, $m \in N$, such that the BGG [11] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m \ , \ \beta^\vee \equiv 2\beta/(\beta, \beta) \ .$$

(3)

When (3) holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and $\beta$ non-compact) is embedded in the Verma module $V^\Lambda$ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector $v_\beta$ determined by a polynomial $P_{m, \beta}(G^-)$ in the universal enveloping algebra $U(G^-)$ $v_0$, $G^-$ is the subalgebra of $G^C$ generated by the negative root generators [12]. More explicitly, [9], $v_{m, \beta} = P_{m, \beta} v_0$ (or $v_{m, \beta} = P_{m, \beta} v_\mu v_0$ for GVMs). Then there exists [9] an intertwining differential operator

$$D^m_\beta : C_{\chi(\Lambda)} \to C_{\chi(\Lambda - m\beta)}$$

(4)

given explicitly by:

$$D^m_\beta = P^m_\beta (\tilde{G}^-)$$

(5)

where $\tilde{G}^-$ denotes the right action on the functions $\mathcal{F}$, cf. (1).
3. The non-compact Lie algebras $so^*(12)$
The group $G = SO^*(2n)$ consists of all matrices in $SO(2n, \mathbb{C})$ which commute with a real skew-symmetric matrix times the complex conjugation operator $C$:

$$SO^*(2n) \triangleq \{ g \in SO(2n, \mathbb{C}) \mid J_n C g = g J_n C \}$$  \hspace{1cm} (6)

The Lie algebra $\mathcal{G} = so^*(2n)$ is given by:

$$so^*(2n) \triangleq \{ X \in so(2n, \mathbb{C}) \mid J_n C X = X J_n C \} = \{ X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in gl(n, \mathbb{C}), \quad ^t a = -a, \quad ^t b = b \} .$$  \hspace{1cm} (7)

$\dim_R \mathcal{G} = n(2n - 1)$, rank $\mathcal{G} = n$.

The Cartan involution is given by: $\Theta X = -X^\dagger$. Thus, $\mathcal{K} \cong u(n)$:

$$\mathcal{K} = \{ X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in gl(n, \mathbb{C}), \quad ^t a = -a = -\bar{a}, \quad ^t b = b = \bar{b} \} .$$  \hspace{1cm} (8)

Thus, $\mathcal{G} = so^*(2n)$ has discrete series representations and highest/lowest weight representations. The complimentary space $\mathcal{P}$ is given by:

$$\mathcal{P} = \{ X = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a, b \in gl(n, \mathbb{C}), \quad ^t a = -a = -\bar{a}, \quad b^\dagger = b = \bar{b} \} .$$  \hspace{1cm} (9)

$\dim_R \mathcal{P} = n(n - 1)$. The split rank is $r \equiv [n/2]$. The subalgebras $\mathcal{N}_0^{\pm}$ which form the root spaces of the root system $(\mathcal{G}, \mathcal{A}_0)$ are of real dimension $n(n - 1) - [n/2]$.

The maximal parabolic subalgebras have $\mathcal{M}$-factors as follows [1]:

$$\mathcal{M}_j^{\text{max}} = so^*(2n - 4j) \oplus su^*(2j) , \quad j = 1, \ldots, r .$$  \hspace{1cm} (10)

The $\mathcal{N}^\pm$ factors in the maximal parabolic subalgebras have dimensions: $\dim (\mathcal{N}_j^{\pm})^{\text{max}} = j(4n - 6j - 1)$.

For even $n = 2r$ we choose a maximal parabolic $\mathcal{P} = \mathcal{MAV}$ such that $\mathcal{A} \cong so(1,1)$, $\mathcal{M} = \mathcal{M}_r^{\text{max}} = su^*(n)$. We note also that

$$\mathcal{K}^C \cong u(1)^C \oplus sl(n, \mathbb{C}) \cong \mathcal{A}^C \oplus \mathcal{M}^C .$$  \hspace{1cm} (11)

Thus, the factor $\mathcal{M}$ has the same finite-dimensional (nonunitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra of $\mathcal{K}$. The property (11) distinguishes the class we called 'conformal Lie algebras' [2], to which class the algebras $so^*(4r)$ belong.

Further we restrict to our case of study $\mathcal{G} = so^*(12)$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1, n_2, n_3, n_4, n_5; c \} , \quad n_j \in \mathbb{Z}_+, \quad c = d - \frac{15}{2}$$  \hspace{1cm} (12)

where the last entry of $\chi$ labels the characters of $\mathcal{A}$, and the first five entries are labels of the finite-dimensional (nonunitary) irreps of $\mathcal{M} = su^*(6)$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$.  

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

\[(n_1, n_2, n_3, n_4, n_5)^* = (n_5, n_4, n_3, n_2, n_1)\]  \hspace{1cm} (13)

The ERs in the multiplet are related also by intertwining integral operators introduced in [13]. These operators are defined for any ER, the general action being:

\[G_{KS} : C_\chi \rightarrow C_\chi' , \quad \chi = \{n_1, \ldots, n_5; c\} , \quad \chi' = \{(n_1, \ldots, n_5)^*; -c\}.\]  \hspace{1cm} (14)

Further, we need the root system of the complexification \(G^C = so(12, \mathbb{C})\). The positive roots are given standardly as:

\[\alpha_{ij} = \epsilon_i - \epsilon_j , \quad 1 \leq i < j \leq 6 , \quad \beta_{ij} = \epsilon_i + \epsilon_j , \quad 1 \leq i < j \leq 6\]  \hspace{1cm} (15)

where \(\epsilon_i\) are standard orthonormal basis: \(\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}\). The compact roots are \(\alpha_{ij}\) - they form (by restriction) the root system of the semisimple part of \(K^C \cong su^*(6)^C\), while the roots \(\beta_{ij}\) are noncompact.

Further, we give the correspondence between the signatures \(\chi\) and the highest weight \(\Lambda\). The connection is through the Dynkin labels:

\[m_i \equiv (\Lambda + \rho, \alpha_i^\vee) = (\Lambda + \rho, \alpha_i) , \quad i = 1, \ldots, 5,\]  \hspace{1cm} (16)

where \(\Lambda = \Lambda(\chi), \rho\) is half the sum of the positive roots of \(G^C\). The explicit connection is:

\[n_i = m_i , \quad c = -\frac{1}{2}(m_1 + 2m_2 + 3m_3 + 4m_4 + 2m_5 + 3m_6)\]  \hspace{1cm} (17)

Finally, we remind that according to [3] the above considerations are applicable also for the algebra \(so(6, 6)\) with parabolic \(M\)-factor \(sl(6, \mathbb{R})\).

4. Main multiplets of SO*(12)

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of \(so^*(12)\), i.e., they are labelled by the six positive Dynkin labels \(m_i \in \mathbb{N}\).

The number of ERs/GVMs in the corresponding multiplets is (cf. [3]):

\[|W(G^C, H^C)| / |W(K^C, H^C)| = |W(so(12, \mathbb{C}))| / |W(sl(6, \mathbb{C}))| = 32\]  \hspace{1cm} (18)

where \(H\) is a Cartan subalgebra of both \(G\) and \(K\).

The signatures of the 32 ERs/GVMs in a main multiplet are given in the following pair-wise
manner:

\[
\chi^+ = \{ (m_1, m_2, m_3 m_4, m_5)^\pm; \pm \frac{1}{2} (m_1 + 2 m_2 + 3 m_3 + 4 m_4 + 2 m_5 + 3 m_6) \}
\]

\[
\chi^a = \{ (m_1, m_2, m_3, m_4, m_5)^\pm; \pm \frac{1}{2} (m_1 + 2 m_2 + 3 m_3 + 4 m_4 + 2 m_5 + m_6) \}
\]

\[
\chi^b = \{ (m_1, m_2, m_3, m_4, m_5)^\pm; \pm \frac{1}{2} (m_1 + 2 m_2 + 3 m_3 + 2 m_4 + 2 m_5 + m_6) \}
\]

\[
\chi^c = \{ (m_1, m_2, m_3, m_4, m_5)^\pm; \pm \frac{1}{2} (m_1 + 2 m_2 + 3 m_3 + 2 m_4 + m_6) \}
\]

\[
\chi^d = \{ (m_1, m_2, m_3, m_4, m_5)^\pm; \pm \frac{1}{2} (m_1 + 2 m_2 + m_3 + 2 m_4 + m_6) \}
\]

\[
\chi^e = \{ (m_1, m_2, m_3, m_4, m_5)^\pm; \pm \frac{1}{2} (m_1 + 2 m_2 + m_3 + m_6) \}
\]

\[
\chi^f = \{ (m_1, m_2, m_3, m_4, m_5)^\pm; \pm \frac{1}{2} (m_1 + m_2 + m_3 + 2 m_4 + m_6) \}
\]

\[
\chi^c' = \{ (m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm \frac{1}{2} (m_1 + m_2 + m_3 + m_6) \}
\]

\[
\chi^e' = \{ (m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm \frac{1}{2} (m_1 + m_2 + m_3 + m_6) \}
\]

\[
\chi^f' = \{ (m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm \frac{1}{2} (m_1 + m_2 + m_3 + m_6) \}
\]

\[
\chi^g = \{ (m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm \frac{1}{2} (m_1 + m_2 + m_3 + -m_6) \}
\]

\[
\chi^g' = \{ (m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm \frac{1}{2} (m_1 + m_2 + m_3 + m_6) \}
\]

\[
\chi^g'' = \{ (m_1, m_2, m_3, m_4, m_5, m_6)^\pm; \pm \frac{1}{2} (m_1 + m_2 + m_3 + m_6) \}
\]

where \((k_1, k_2, k_3, k_4, k_5)^- = (k_1, k_2, k_3, k_4, k_5), (k_1, k_2, k_3, k_4, k_5)^+ = (k_1, k_2, k_3, k_4, k_5)^\ast\). They are given explicitly in Fig. 1. The pairs \(\Lambda^\pm\) are symmetric w.r.t. to the bullet in the middle of the figure - this represents the Weyl symmetry realized by the Knapp-Stein operators (14): \(G_{KS} : C_{\chi^+} \leftrightarrow C_{\chi^\pm}\).

Matters are arranged so that in every multiplet only the ER with signature \(\chi^\pm_0\) contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace \(E\). The latter corresponds to the finite-dimensional irrep of \(so(12)\) with signature \(\{m_1, \ldots, m_6\}\). The subspace \(E\) is annihilated by the operator \(G^+\), and is the image of the operator \(G^-\). The subspace \(E\) is annihilated also by the intertwining differential operator acting from \(\chi^\pm_0\) to \(\chi^\pm_a\). When all \(m_i = 1\) then \(\dim E = 1\), and in that case \(E\) is also the trivial one-dimensional UIR of the whole algebra \(G\). Furthermore in that case the conformal weight is zero: \(d = \frac{15}{2} + c = \frac{10}{2} - \frac{1}{2} (m_1 + 2m_2 + 3m_3 + 4m_4 + 2m_5 + 3m_6)_{|m_i=1} = 0\).

In the conjugate ER \(\chi^\pm_0\) there is a unitary discrete series subrepresentation in an infinite-dimensional subspace \(D\). It is annihilated by the operator \(G^-\), and is the image of the operator \(G^+\).

All the above is valid also for the algebra \(so(6,6)\), cf. [3]. However, the latter algebra does not have highest/lowest weight representations while the algebra \(so^*(12)\) has highest/lowest weight series representations.

Thus, for \(so^*(12)\) the ER with signature \(\chi^\pm_0\) contains both a holomorphic discrete series representation and a conjugate anti-holomorphic discrete series representation. The direct sum of the holomorphic and the antiholomorphic representations spaces form the invariant subspace \(D\) mentioned above. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

In Fig. 1 we use the notation: \(\Lambda^\pm = \Lambda(\chi^\pm)\). Each intertwining differential operator is represented by an arrow accompanied by a symbol \(i_{jk}\) encoding the root \(\alpha_{jk}\) and the number
$m_{\alpha_{jk}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data $\beta, m_\beta$, which is involved in the embedding $V^\Lambda \hookrightarrow V^{\Lambda-m_\beta,\beta}$ turns out to involve only the $m_i$ corresponding to simple roots, i.e., for each $\beta, m_\beta$ there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 5\}$, such that $m_\beta = m_i$. Hence in Fig. 1. the data $\alpha_{jk}$, $m_{\alpha_{jk}}$ is represented by $i_{jk}$ on the arrows.

![Diagram](image-url)

**Fig. 1.** $SO^*(12)$ main multiplets
Acknowledgments
The author would like to thank the Organizers for the kind hospitality and invitation to present a talk at the XXX International Colloquium on Group Theoretical Methods in Physics (Ghent, July 2014).

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