Pseudoconvex domains with smooth boundary in projective spaces

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Abstract

Given a pseudoconvex domain $U$ with $C^1$-boundary in $\mathbb{P}^n$, $n \geq 3$, we show that if $H^{2n-2}_{dR}(U) = 0$, then there is a strictly psh function in a neighborhood of $\partial U$. We also solve the $\bar{\partial}$-equation in $X = \mathbb{P}^n \setminus U$, for data in $\mathcal{C}_{0,(1)}(X)$. We discuss Levi-flat domains in surfaces. If $Z$ is a real algebraic hypersurface in $\mathbb{P}^2$, (resp a real-analytic hypersurface with a point of strict pseudoconvexity), then there is a strictly psh function in a neighborhood of $Z$.

Keywords Levi-flat · $\bar{\partial}$-Equation · Pseudo-concave sets · Strictly plurisubharmonic functions

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1 Introduction

In this paper we discuss the pluri-potential theory on a smooth hypersurface $Z$ in $\mathbb{P}^n$. This includes the question of existence of positive closed (resp. $dd^c$-closed) currents supported on $Z$ and also the question of the existence of a strictly plurisubharmonic function (psh) in a neighborhood of $Z$. We will sometimes need a pseudo-convexity hypothesis on a component of the complement. We also give some results in the case where $Z$ is a closed set satisfying some geometric assumptions.

Recall that a complex manifold of dimension $n$ is strongly $q$-complete if it admits a smooth exhaustion function $\rho$ whose Levi form at each point has at least $(n - q + 1)$ strictly positive eigenvalues. The main result in that theory is the following Theorem.

Theorem (Andreotti–Grauert [1]) Let $U$ be a strongly $q$-complete manifold. Then for every coherent analytic sheaf $\mathcal{S}$ over $U$, $H^k(U, \mathcal{S}) = 0$, for $k \geq q$. 

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In particular, if $k \geq q$, $H^{n,k}(U, \mathbb{C}) = 0$

Indeed, for a holomorphic bundle $E$, $H^{n,k}(U, E) = H^k(U, \Lambda^{n,0}U \otimes E)$.

Our main result will use the above theorem.

**Theorem 1.1** Let $U$ be a domain in $\mathbb{P}^n$, $n \geq 3$, with $\mathscr{C}^1$ boundary. Assume $U$ is strongly $(n-2)$-complete (i.e., it admits a smooth exhaustion function whose Levi form has at least $3$ strictly positive eigenvalues at each point). Assume also that $H^{2n-2}_d(U, \mathbb{C}) \neq 0$. Then there is a strictly psh function near the boundary of $U$.

As a consequence of Theorem 1.1 and of Proposition 2.1 below, we get the following result.

**Corollary 2.2** In $\mathbb{P}^n$, $n \geq 3$, there is no $\mathscr{C}^1$ hypersurface $Z$ such that the two components $U^\pm$ of $\mathbb{P}^n \setminus Z$ are both strongly $(n-2)$-complete and one them, say $U^-$, satisfies $H^{2}_{dR}(U^-) = 0$.

Observe that in $\mathbb{P}^n$ the hypothesis $H^{2}_{dR}(U^-) = 0$ for a domain with $\mathscr{C}^1$ boundary implies that $H^{2n-2}_{dR}(U^+, \mathbb{C}) \neq 0$. See the proof of Proposition 2.1 below.

Y. T. Siu has proved the following result. [19].

**Theorem** (Siu) In $\mathbb{P}^n$, $n \geq 3$, there is no Levi-flat hypersurface $Z$, i.e., both sides are $1$-complete (exhaustion with $n$ strictly positive eigenvalues).

The meaning of Levi flat is that the Levi form of a defining function $r$ for $Z$, is identically zero on the complex tangent space. Since the Levi problem has a positive solution in $\mathbb{P}^n$, this implies that both components $U^\pm$ are Stein and hence strongly $1$-complete. In particular they are strongly $(n-2)$-complete for $n \geq 3$.

We also study the solvability of the $\overline{\partial}$-equation (resp. $\partial \overline{\partial}$-equation) on a pseudo-concave set $X$ in the $\mathscr{C}^\infty$ category, i.e., we assume that $\mathbb{P}^n \setminus X$ is Stein. We use the Hörmander duality method (see [2] for example).

The existence of $dd^c$-closed currents on $Z$, gives an obstruction to the resolution of the $\overline{\partial}$- equation in the smooth category, see Theorem 6.1.

In the last section we discuss the same problem in $\mathbb{P}^2$. So far it is not known if there are smooth Levi-flat hypersurfaces in $\mathbb{P}^2$.

**2 Real hypersurfaces in $\mathbb{P}^n$**

Let $Z$ be a real hypersurface in $\mathbb{P}^n$, $n \geq 2$, of class $\mathscr{C}^1$. Since $\mathbb{P}^n$ is simply connected, $\mathbb{P}^n \setminus Z$ has two components $U^\pm$. Denote by $\omega$ a Kähler form of mass one on $\mathbb{P}^n$.

**Proposition 2.1** If $n = 2$, the form $\omega$ is $d$-exact either on a neighborhood of $\overline{U^-}$ or on a neighborhood of $\overline{U^+}$.

If $n \geq 3$, then: $H^{2n-2}_{dR}(U^+, \mathbb{C}) \neq 0$, iff $H^2_{dR}(\overline{U^-}) = 0$. In particular the form $\omega$ is $d$-exact, in a neighborhood of $\overline{U^-}$.

**Proof** Assume first $n = 2$. If $\omega$ is not $d$-exact in a neighborhood of $\overline{U^+}$, there is a $2$-cycle $\sigma^+$ in $\overline{U^+}$ which is non-trivial. Using that $Z$ is $\mathscr{C}^1$, we can retract $\sigma^+$ as a cycle in $U^+$. Similarly, we would get a nontrivial cycle $\sigma^-$ in $U^-$. But by Poincaré duality, $\sigma^+ \sim c_+ \omega$ and $\sigma^- \sim c_- \omega$, with $c_+, c_-$ non-zero. Since $\sigma^+$ and $\sigma^-$ are disjoint, we get $\sigma^+ \sim \sigma^- = 0$, and hence $c_+ c_- = 0$, a contradiction.

If $n \geq 3$ and $H^{2n-2}_{dR}(U^+, \mathbb{C}) \neq 0$. Then there is a $(2n-2)$-cycle $\sigma^+$ in $U^+$, with $\sigma^+ \sim c_+ \omega^{2n-2}$. If $H^2_{dR}(\overline{U^-}) \neq 0$, we would construct as above a non-trivial $2$-cycle in $U^-$ and get a contradiction as above. □
Observe that we cannot have $\omega^{n-1}$, $d$-exact near $\overline{U}^+$ and $\omega$, $d$-exact near $\overline{U}^-$. Otherwise, assume that $\omega^{n-1} = d(\phi_+)$ near $\overline{U}^+$ and $\omega = d(\phi_-)$ near $\overline{U}^-$. Then, $\omega^{n-1} - d(\chi_+\phi_+)$ and $\omega - d(\chi_-\phi_-)$ would have disjoint support for appropriate cut-off functions $\chi_\pm$, contradicting that the cup-product should be non-zero.

For a compact $X$ in $\mathbb{P}^n$, we will write $\mathcal{H}^2(X) = 0$, if there is an open neighborhood $V \supset X$, such that the de Rham cohomology group $H^2_{\text{dR}}(V, \mathbb{C}) = 0$.

**Proposition 2.2** Assume that $\omega$ is $d$-exact in a neighborhood of $\overline{U}^-$.

1. There is no closed current $A$ of order zero and dimension $2p$ supported on $\overline{U}^-$, such that $\{A\} \neq 0$. Here $\{\}$ denotes the de Rham cohomology class. In particular, there is no non-zero positive closed current supported on $\overline{U}^-.$

2. If $T$ is a real $\bar{\partial}\bar{\partial}$-closed current (non-closed) of bidegree $(1, 1)$ supported on $\overline{U}^-$ and $\{T\} \neq 0$. Then there is a $\bar{\partial}$-closed $(2, 0)$ holomorphic form non-identically zero in $U^+.$

**Proof** Since $\omega = d\phi$, near $\overline{U}^-$, we have

$$(A, \omega^p) = (A \wedge \omega^{p-1}, d\phi) = 0.$$ 

Hence $\{A\} = 0$. If $A$ is positive closed non-zero, necessarily $\{A\} \neq 0$.

It is also possible to define the cohomology class of a $\bar{\partial}\bar{\partial}$-closed current in a compact Kähler manifold. It suffices to use the $\bar{\partial}\bar{\partial}$-lemma, and Poincaré’s duality.

When $T$ is a real $\bar{\partial}\bar{\partial}$-closed current of bi-degree $(1, 1)$, it follows from basic Hodge theory, see [9] that when $\{T\} = \{\omega\}$, then

$$T = \omega + \partial\sigma + \bar{\partial}\bar{\sigma},$$

where $\sigma$ is a $(0, 1)$-current. Define

$$T_c := \omega + \partial\sigma + \bar{\partial}\bar{\sigma} + \bar{\partial}\sigma + \partial\bar{\sigma}.$$ 

It is easy to check that $dT_c = 0$. If $\bar{\partial}\sigma = 0$ on $\overline{U}^+$, then $T_c$ is closed and supported on $\overline{U}^-$. Since $\{T_c\} = \{\omega\}$ we get a contradiction. It follows that $\partial\sigma$ is not identically zero in $\overline{U}^+$. But $\bar{\partial}\bar{\sigma} = -\bar{\partial}T$ is supported on $\overline{U}^-$. Hence, $\bar{\partial}(\partial\sigma) = 0$ on $U^+$. Therefore, $\partial\sigma$ is a holomorphic $(2, 0)$-form. \hfill $\square$

**Corollary 2.3** Let $Z$ be a $\mathcal{C}^1$ hypersurface in $\mathbb{P}^n$. There is no positive closed current of bidegree $(1, 1)$ supported on $Z$.

**Proof** If $n = 2$, this follows from Proposition 2.2, since we can consider that $Z$ bounds $\overline{U}^-$, and that $H^2_{\text{dR}}(\overline{U}^-) = 0$.

Assume $n \geq 3$. Let $T$ a positive closed current of bidegree $(1, 1)$ supported on $Z$. Fix a point $p \notin Z$ and consider subspaces $L_p$ of co-dimension $(n - 2)$ through $p$. For almost all $L_p$, $L_p \cap Z$ is of class $\mathcal{C}^1$, and the slice of $T$ at $L_p$ is a positive closed current. Hence using the case $n = 2$, almost all slices vanish. This is true for all $p$ out of $Z$. It follows then, from slicing theory, that $T = 0$. \hfill $\square$

**Remark 2.4** If $T$ is a closed and flat current, supported on $Z$ then slicing theory is valid. We obtain that $\{T\} = 0$. Indeed the class of a slice is the slice of the class.

**Corollary 2.5** Let $\overline{U}^-$, be a domain in $\mathbb{P}^2$ with $\mathcal{C}^1$ boundary. Assume, $H^2_{\text{dR}}(U^-) = 0$. Then, there is a neighborhood of $\overline{U}^-$, which is Kobayashi hyperbolic.

**Proof** Otherwise, in an arbitrary neighborhood of $\overline{U}^-$, we will have a non-constant holomorphic image of $\mathbb{C}$. This will permit to construct an Ahlfors current. In particular, we will have a positive closed current of mass one on $\overline{U}^-$. Hence it’s cohomology class is non-zero, contradicting Proposition 2.2. \hfill $\square$
3 Strictly psh functions near a compact $X$ and currents.

Let $(M, \omega)$ be a complex Hermitian manifold of dimension $n$. Let $X \Subset M$ be a compact set. We are interested in some general facts about the existence of strictly psh functions near $X$. Strictly psh functions are the starting point in order to use Hörmander’s $L^2$ estimates, see for example J. J. Kohn [14]. In particular, they permit to prove regularity at the boundary, for the $\bar{\partial}$-equation.

**Proposition 3.1** Let $X \Subset M$ be a compact set. There is a positive $\partial \bar{\partial}$-closed current $T$ of bi-dimension $(1,1)$ supported on $X$ iff there is no smooth strictly psh function $u$ in a neighborhood of $X$. Moreover, for any $C^2$ function $r$, vanishing on $X$, any such current $T$, satisfies the following equations:

$$T \wedge \partial r = 0, \quad T \wedge \partial \overline{\partial} r = 0.$$  \hspace{1cm} (3.1)

**Proof** The proof is essentially the same as in [18] Proposition 2.1. There, $X = \partial U$ and $U$ is smooth and pseudoconvex. Indeed, pseudoconvexity is not needed. The result is used for arbitrary $X$, in [18] Theorem 4.3.

If $u$ is a $C^2$ psh function in a neighborhood of $X$ and $T$ is a positive current supported on $X$, then

$$\langle T, i \partial \overline{\partial} u \rangle = \langle i \partial \overline{\partial} T, u \rangle.$$  

So if $i \partial \overline{\partial} T = 0$, we get that $T = 0$, near every point where $u$ is strictly psh. Hence if there is a strictly psh function near $X$, then $T = 0$.

We just show that any positive $\partial \bar{\partial}$-closed current $T$ of mass one supported on $X$ satisfies the above relations. The proof of the other assertions is identical to the one in [18], mainly Hahn-Banach Theorem.

Since $T$ is $\partial \bar{\partial}$-closed then $\langle T, i \partial \overline{\partial} r^2 \rangle = 0$. Expanding and using that $T$ is positive and is supported on $\{r = 0\}$, we get: $T \wedge \partial r \wedge \bar{\partial} r = 0$. Therefore, $T \wedge \partial r = 0$.

Let $\chi$ be a smooth non-negative function with compact support. Using that $T \wedge \partial r = 0$, we get that:

$$0 = \langle T, i \partial \overline{\partial} (\chi r) \rangle = \langle T, \chi i \partial \overline{\partial} r \rangle.$$  

Since $\chi$ is arbitrary, the measure, $T \wedge i \partial \overline{\partial} r = 0$. \hfill $\square$

**Remark 3.2** If $T$ is positive and $\partial \bar{\partial}$-closed on $M$, then the calculus can be extended to continuous psh functions near $\text{Support}(T)$ [5]. In fact if a function $u$ is continuous on $\text{Support}(T)$ and is locally approximable, on $\text{Support}(T)$, by continuous psh function, then $T \wedge i \partial \overline{\partial} u = 0$.

In particular, let $U$ be a pseudo-convex domain with boundary of class $C^2$ in $\mathbb{P}^n$. According to [16], $U$ admits a bounded, strictly psh continuous exhaustion function $u$. Since $\partial U$ is of class $C^2$. For $p \in \partial U$, $u$ is approximable by psh functions in a fixed neighborhood of $p$. Indeed, it suffices to push functions in the normal direction at $p$. It follows that for $T$ positive of bi-dimension $(1,1)$, $\partial \bar{\partial}$-closed and supported on $\overline{U}$, we have: $T \wedge i \partial \overline{\partial} u = 0$. Hence $T$ is supported on $\partial U$.

**Corollary 3.3** Let $X \Subset M$ be a compact subset. Assume there is no strictly psh function in a neighborhood of $X$. Then there is a compact $X_\infty \subset X$ with $X_\infty = \bigcup_\alpha X_\alpha$, each $X_\alpha$, is compact connected and every continuous psh function, in a neighborhood of $X_\alpha$, is constant.
on $X_\alpha$. Moreover, there is a positive $dd^c$-closed current $T$ of bidimension $(1,1)$ such that $X_\infty = \text{Support}(T)$.

For any compact $K \subset X$ with $K \cap X_\infty = \emptyset$, there are strictly psh functions near $K$.

**Proof** Assume there is no strictly psh function near $X$. Then there is a positive $\partial \bar{\partial}$-closed current $T$ of bi-dimension $(1,1)$ supported on $X$, of mass $1$. We can assume $T$ is extremal and define $X' = \text{Support}(T)$. Then according to Proposition 4.2 in [18], every continuous psh function near $X'$ is constant on $X'$.

Let $C_{1,1}$ denote the convex compact set of positive $dd^c$-closed currents supported on $X$, of bi-dimension $(1,1)$ and of mass $1$. Let $(T_\alpha)$ be the family of extremal elements in $C_{1,1}$. Let $X_\alpha := \text{Support}(T_\alpha)$. Define $X_\infty = \overline{\bigcup_{\alpha} X_\alpha}$. Since $(T_\alpha)$ is extremal, it’s support $X_\alpha$ is connected.

Let $(T_n)$, $n \geq 1$, be a dense sequence in $C_{1,1}$. Define $T_\infty = \sum_n 2^{-n} T_n$, then $T_\infty \in C_{1,1}$ and $X_\infty = \text{Support}(T_\infty)$.

As we have seen every $X_\alpha$ has the property that continuous psh function in a neighborhood of $X_\alpha$, is constant on $X_\alpha$. If $K \subset X$ and $K \cap X_\infty = \emptyset$, then there are strictly psh functions near $K$.

Indeed by Krein-Milman Theorem $C_{1,1}$, is the closed convex hull of its extremal elements. Hence there is no positive $\partial \bar{\partial}$-closed current $T$ of bi-dimension $(1,1)$ supported on $K$. Proposition 3.1 implies that there is a strictly psh function near $K$. □

**Remark 3.4** (1) Following [21], one should call $X_\infty$ the Poincaré set of $X$ and $T_\infty$, a Poincaré current for $X$.

There are many examples of the above decomposition in holomorphic dynamics. It could happen that $X \setminus X_\infty$, contains a biholomorphic image of $\mathbb{C}^2$, this is the case in the dynamics of Hénon maps, if we take $X = \overline{K^+}$, see [6].

If $X$ is the the closure of the Torus in Grauert’s example for the Levi problem, as described in [18], then there are uncountably many $X_\alpha$, each one being a real torus and also the closure of an image of $\mathbb{C}$.

In fact for a current $T_\alpha$ as above, a continuous $T_\alpha$ subharmonic functions (in the sense of [17]) is necessarily constant. These are the function which are decreasing limits of $C^2$ functions $u$, satisfying $dd^c u \wedge T_\alpha \geq 0$. This is a more intrinsic property, since it depends on the “complex directions” of $T_\alpha$ i.e. the infinitesimal complex structure of $X$, independently of any smoothness assumption.

(2) A similar decomposition is given in [18] Theorem 4.3, for a domain $U$ admitting a continuous psh exhaustion function $\varphi$. The obstruction to Steiness, is the existence of positive $\partial \bar{\partial}$-closed currents supported on the level sets $\varphi = c$.

The “Poincaré” decomposition of an arbitrary domain $V \subset M$, could be introduced following [18], Definition 4.1. The Poincaré set $V_\infty$ is the union of support of Liouville currents, i.e. positive current of bi-dimension $(1,1)$, such that $i \partial \bar{\partial} T = 0$ and $T \wedge i \partial \bar{\partial} \varphi = 0$, for every bounded continuous psh function in $V$.

One can prove that $V_\infty$ is the support of a Liouville current $T_\infty$ and that $V_\infty$ is 1-pseudo-convex.

(3) According to [11] Corollary 2.6, if $T$ a positive bi-dimension $(1,1)$, $\partial \bar{\partial}$-closed current, then $M \setminus \text{Support}(T)$ is 1-pseudo-convex or with another terminology, $\text{Support}(T)$ is...
1-pseudo-concave. So $X_\infty$ is 1-pseudo-concave. Hence the existence of a positive bi-
dimension $(1, 1)$, $\partial \overline{\partial}$-closed current, always implies the existence of a 1-pseudo-concave
set. In fact there is a Poincaré decomposition $(X_\alpha)$ of $\text{Support}(T)$ and each $X_\alpha$ is 1-
pseudo-concave.

In dimension 2 and if $M$ is a surface where the Levi-problem has a positive solution, there is a smooth strictly psh exhaustion function, on $M \backslash X_\infty$.

(4) Consider a compact set $X \subset M$, and a closed pluripolar set $E \subset X$. Assume $X \setminus E$, satisfies the local maximum principle for continuous psh functions. More precisely if $p \in X \setminus E$, and $V_p$ is a neighborhood of $p$, disjoint from $E$. Then for any continuous psh function $u$ near $\overline{V}_p$, we have

$$ u(p) \leq \max_{z \in X \setminus \partial V_p} u(z). $$

Then there is no strictly psh function on $X$ and hence there are extremal positive $dd^c$-
closed currents supported on $X$.

The proof is basically the same as in [2]. Suppose there is a strictly psh function $u$ near $X$. Assume it reaches it’s maximum on $X$ at the point $p$. We can assume $p = 0$ in a local chart, with local coordinates $z$. Let $v$ be a psh near $p$, such that $v = -\infty$ on $E$. For $\epsilon$ small enough, the function $u + \epsilon v(z)$, will have a maximum at a point $q \notin E$ near $p$. So we can assume $0 = p \notin E$. Then for an appropriate cut-off function $\chi$, equal to 1 near 0, the function $u - \epsilon \chi(z).\|z\|^2$, will have a strict maximum at $p$, contradicting the local maximum principle.

In particular if $X = E$ is pluripolar, either there is a strictly psh function near $E$ or it admits a Poincaré decomposition.

(5) Slodkowski [20] has shown, that a closed set $Z \subset M$ satisfies the local maximum principle iff it is 1-pseudo-concave.

(6) It follows from Corollary 3.3 that smooth psh functions in a neighborhood of $X$ separate points in $X$, iff there is a strictly psh function near $X$. Indeed, if they separate points, there is no $T_\alpha$, hence there is a strictly psh function. For the converse, one can observe, that if $u$ is a strictly psh function near $X$, then smooth function on a level set of $u$ can be extended to a strictly psh function.

**Corollary 3.5** Let $U \subset M$ be a domain with $\partial^2$ boundary. Let $r$ be a defining function for
$\partial U$. Let $W \subset \partial U$ denote the set of points where the Levi-form is not positive or negative
definite. Assume every component of $W$ is of 2-Hausdorff measure zero. Then, there is a
smooth strictly psh function $u$ in a neighborhood of $\partial U$.

**Proof** We can assume that $U := \{z \in U_1 : r(z) < 0\}$ where $U_1$ is a neighborhood of $\overline{U}$. We have that $\partial r$ does not vanish on $\partial U$.

Assume there is no strictly psh function near $\partial U$. Let $T$ be an extremal positive $\partial \overline{\partial}$-closed
current, of mass 1, supported on $\partial U$.

Recall that the Levi-form is defined on the complex tangent space of the boundary. If $\langle \partial \overline{r}(z), t \rangle = 0$, then the Levi-form at the point $z$ for the direction $t$, is given by: $\langle i \partial \overline{\partial} r(z), it \wedge \overline{t} \rangle$.

At points of the boundary where $i \partial \overline{\partial} r > 0$ or $i \partial \overline{\partial} r < 0$, on the complex tangent space, it follows from the equation $T \wedge i \partial \overline{\partial} r = 0$, that the current $T$ has no mass there, hence it is supported on $W$. Since it is extremal it is supported on a component of $W$. But as observed in
positive $\bar{\partial}\bar{\partial}$-closed currents give no mass to sets of 2-Hausdorff measure zero. So $T = 0$ and the assertion follows.

**Remark 3.6** Let $X$ be a compact set in $M$. Let $E$ be a closed subset of $X$, of 2-Hausdorff measure zero. Assume that for every point $p \in X \setminus E$ there is a neighborhood $V_p$ of $p$ and a continuous psh function $u_p$ in $V_p$ peaking at $p$ on $X \cap V_p$. Then there is a strictly psh function in a neighborhood of $X$.

Indeed, one can construct a continuous psh function, $v_p$ in a neighborhood of $X$, strictly psh at $p$ and peaking at $p$. Using Remark 3.2, one shows that a $dd^c$-closed current $T$ supported on $X$ has no mass near $p$. As above, it follows that $T = 0$.

A similar argument gives the following. Let $X$ be a real compact sub-manifold in $M$. If the set $E$ of points in $X$ where there is a complex tangent is of 2-Hausdorff measure zero, then there is a smooth strictly psh function $u$ in a neighborhood of $X$. Indeed any positive $\bar{\partial}\bar{\partial}$-closed current of bi-dimension $(1, 1)$, has to be supported on $E$.

**Corollary 3.7** Let $C$ be a compact connected real surface in $M$. There is a strictly psh function near $C$ iff $C$ is not a complex curve.

**Proof** It is clear that if $C$ is a complex curve, there is no strictly psh function near $C$ (by maximum principle).

Recall that the support of a positive $\bar{\partial}\bar{\partial}$-closed currents, satisfies the local maximum principle for local psh functions, [17] Theorem 3.2.

Assume $C$ is not a complex curve. Let $T$ be a positive $\bar{\partial}\bar{\partial}$-closed current of bi-dimension $(1, 1)$ supported in $C$. Since $C$ is a manifold, equations (3.1) permit to consider $T$ as a current on $C$. Let $E_c$ denote the set of points in $C$ where the tangent space is complex. Since $C$ is not a complex curve, then $E_c$ admits boundary points in $C$. The current $T$ is of bidimension $(1, 1)$ and is supported on $E \subset E_c$. Let $p$ be a boundary point of $E$ in $C$. There are psh functions in a fixed neighborhood $V$ of $p$ with a unique peak point on $V$ near $p$. This contradicts the above local maximum principle. So there is no such $T$.

**4 Constructing strictly psh functions**

We give a stronger version of Theorem 1.1. We do not assume, that $X$ is a domain with $C^1$ boundary. When $X = U^-$, is a domain with $C^1$ boundary, it is equivalent to assume that $H^2_{dR}(U^-) = 0$ or that $\mathcal{H}^2(X) = 0$, as explained in Proposition 2.1.

**Theorem 4.1** Let $X$ be a compact set in $\mathbb{P}^n$, $n \geq 3$, such that $\mathcal{H}^2(X) = 0$. Assume that the open set $U^+ := \mathbb{P}^n \setminus X$ is strongly $(n - 2)$-complete. Then there is a strictly psh function in a neighborhood of $X$.

**Proof** Let $\mathcal{W}_{(0,1)}^\infty(X)$ denote the space of smooth forms of bidegree $(0, 1)$ on $X$. Here the smoothness is in the Whitney sense with the usual $\mathcal{W}_{\infty}$-Fréchet topology. Smooth functions on $X$, in the Whitney sense, do extend as smooth functions in a neighborhood of $X$. They admit also an intrinsic characterization using only the jet on $X$ i.e. the collection of derivatives. The jet extends and it is the jet of a smooth function. This permit to give the space a Fréchet topology, “uniform convergence on derivatives” [15].

The dual space of $\mathcal{W}_{(0,1)}^\infty(X)$ is the space of currents $R$ of bidegree $(n, n - 1)$ on $\mathbb{P}^n$, supported on $X$. 

[2]
Let \( \mathcal{E} \) denote the closure in \( C^\infty_{(0,1)}(X) \) of \( \{ \overline{\partial}u \}, u \in C^\infty(X) \). We want to use the Hahn-Banach Theorem, to show that \( \varphi^{0,1} \) is in \( \mathcal{E} \). Let \( R \) be a current, supported on \( X \), vanishing on the subspace \( \mathcal{E} \). We need to show that \( (R, \varphi^{0,1}) = 0 \). Since the current \( R \) is supported on \( X \), and vanishes on the subspace \( \mathcal{E} \), then \( \overline{\partial}R = 0 \) on \( \mathbb{P}^n \). It follows from, \( H^{n,n-1}(\mathbb{P}^n) = 0 \), that there is \( S \) of bidegree \((n,n-2)\) such that \( R = \overline{\partial}S \). Moreover, \( S \) is smooth on \( U^+ \). Indeed, \( S \) is constructed using canonical solutions of the Hodge Laplacean, which satisfy the same regularity as the right hand side. Here we are using the local regularity in Hodge theory, [3].

The Andreotti–Grauert Theorem implies that on \( U^+ \), since \( \overline{\partial}S = 0 \), there is a form \( B \) such that \( S = \overline{\partial}B \), on \( U^+ \). Let \( V \supset X \) be an open neighborhood of \( X \) such that on \( V \),

\[
\omega = d\varphi = \partial\varphi^{0,1} + \overline{\partial}\varphi^{1,0}, \quad \overline{\partial}\varphi^{0,1} = 0.
\]

Let \( \chi \) be a cutoff function with \( \chi = 1 \) in a neighborhood of \( U^+ \setminus V \) and vanishing near \( X \). Then \( R = \overline{\partial}S = \overline{\partial}[S - \overline{\partial}(\chi B)] \). Observe that \( S_1 := S - \overline{\partial}(\chi B) \) is supported on \( V \), where \( \overline{\partial}\varphi^{0,1} = 0 \). Hence,

\[
(R, \varphi^{0,1}) = (\overline{\partial}S_1, \varphi^{0,1}) = -(S_1, \overline{\partial}\varphi^{0,1}) = 0.
\]

It follows, by Hahn-Banach theorem, that \( \varphi^{0,1} \in \mathcal{E} \). Hence, there is a family \((u_\varepsilon)\) of smooth functions such that \( \overline{\partial}u_\varepsilon \rightarrow \varphi^{0,1} \) in \( C^\infty(X) \). Then \( \omega = \lim_{\varepsilon \to 0} i\partial\overline{\partial}(u_\varepsilon - \overline{\partial}u_\varepsilon) \).

As a consequence, for \( \varepsilon > 0 \) small enough and \( v_\varepsilon := \frac{u_\varepsilon - \overline{\partial}u_\varepsilon}{\varepsilon}, i\partial\overline{\partial} v_\varepsilon \geq \frac{\varepsilon}{2} \omega \) on \( X \). Hence, \( v_\varepsilon \) is strictly psh near \( X \).

To get Theorem 1.1, we should take \( X = \overline{U}^- \).

In order to prove Corollary 1.2, we will use the following version of the maximum principle, implicit in [17].

**Lemma 4.2** Let \( \rho \) be a function of class \( \mathcal{C}^2 \) in a neighborhood of a closed ball \( \overline{B} \) in \( \mathbb{C}^k \). Assume that for every point \( z \in B \), there is a direction \( t_z \) such that \((i\partial\overline{\partial}\rho(z), it_z \wedge \overline{t}_z) > 0 \). Then there is no local maximum of \( \rho \) in \( B \).

**Proof** Assume by contradiction, that \( \rho \) has a local maximum at a point \( p \in B \). Consider a complex disc \( D_p \), at \( p \) in the direction \( t_p \). The restriction of \( \rho \) to \( D_p \), near \( p \). It cannot have a local maximum at \( p \). \( \square \)

We now prove Corollary 1.2.

**Proof** Assume to get a contradiction that the component \( U^- \), satisfies \( H^2(\overline{U}^-) = 0 \). According to Theorem 4.1, there is a strictly psh function \( v \) near \( Z \). There is a point \( z_0 \in Z \) where \( v(z_0) = \max Z v \). Using that \( v \) is strictly psh, we can assume the maximum at \( z_0 \) is strict and \( v(z_0) = 0 \). Indeed, it suffices to add a negative small perturbation vanishing to second order at \( z_0 \). So, \( \{ v < 0 \} \), is a strictly pseudoconvex domain near \( z_0 \). Hence there is a germ of complex hypersurface \( W \) tangent to \( U^- \) at \( z_0 \) and such that \( W \setminus \{ z_0 \} \subset U^+ \). Then \( W \setminus \{ z_0 \} \) is strongly \((n-2)\)-complete, with an exhaustion function \( \rho \), with \( 2 \) strictly positive eigenvalues at each point, going to \( +\infty \) at \( z_0 \). Recall that the restriction of a strongly \( q \)-complete function to a submanifold is still \( q \)-complete.

Without loss of generality, we can assume that \( W \setminus \{ z_0 \} \) is a pointed ball \( B^* \) of dimension \((n-1)\). We can find a sequence of balls \( (B_j) \) of dimension \((n-2)\) whose centers \((z_j)\) converge to \( z_0 \). On \( B_j \), the function \( \rho \) has a Levi-form, with one strictly positive eigenvalue. It satisfies the maximum principle, given in Lemma 4.2. Since \( \rho \), is uniformly bounded on \( \bigcup (\partial B_j) \), it cannot converge to \( +\infty \) at \( 0 \). This finishes the proof. The last part shows that the pointed ball \( B^* \) of dimension \((n-1)\), is not strongly \((n-2)\)-complete. \( \square \)
5 \overline{\partial} Equation on pseudo-concave sets

**Theorem 5.1** Let $X$ be a compact set in $\mathbb{P}^n$, $n \geq 3$. Assume $U^+ := \mathbb{P}^n \setminus X$ is pseudoconvex (hence Stein). Then, the following properties hold.

(i) Let $\beta$ be a smooth $(0, 1)$-form on $X$ such that $\overline{\partial} \beta = 0$ in $C^\infty_{(0, 1)}(X)$. Then for each integer $k$, there is a function $v \in C^k(X)$ such that $\overline{\partial} v = \beta$.

(ii) If $H^2(X) = 0$, then there is a strictly psh function near $X$.

Recall that on a pseudoconvex domain $U$ in $\mathbb{P}^n$, Takeuchi [22] and Elencwajg [8], proved that if $\delta$ denotes the distance to the boundary of $U$ (with respect to the Fubini-Study metric $\omega$), there is a constant $C$ such that near the boundary of $U$,

$$i \partial \overline{\partial}(- \log \delta) \geq C \omega.$$ 

In particular, pseudoconvex domains are Stein. Indeed, the result is valid when $U$ is pseudoconvex in a compact Kähler manifold $M$, with positive holomorphic bisectional curvature. Moreover the constant $C$ depends only on the curvature. See [8], in particular, inequality (39), and Greene-Wu [12].

We will use the following result which is a consequence of Serre’s duality and Hörmander’s estimates. One solves the $\overline{\partial}$-equation with the weight $e^\varphi$, with $\varphi$ psh instead of the classical $e^{-\varphi}$, one need a vanishing of the forms on the boundary, see for example [2].

**Theorem 5.2** Let $U$ be a pseudoconvex domain in $\mathbb{P}^n$. Let $\alpha \in L^2_{p,q}(U, \text{loc})$, $q < n$, be a $\overline{\partial}$-closed form such that for a given $s > 0$

$$\int |\alpha|^2 \frac{1}{\delta^{s+2}} d\lambda < \infty.$$ 

Then there exists $u \in L^2_{p,q-1}(U, \text{loc})$ such that

$$\overline{\partial} u = \alpha$$

and

$$\int |u|^2 \frac{1}{\delta^{s+2}} d\lambda \leq \frac{1}{C} \int |\alpha|^2 \frac{1}{\delta^{s+2}} d\lambda.$$ 

Here $d\lambda$ denotes the volume form associated to $\omega$.

**Proof** We can consider that $\beta$ is extended as a smooth $(0, 1)$-form in $\mathbb{P}^n$ and that $\overline{\partial} \beta$ vanishes to infinite order on $X$. Indeed, the jet of $\beta$ on $X$ satisfies $\overline{\partial} \beta = 0$ as a jet. So the extension of $\beta$ will satisfy the asserted property.

Let $\delta$ denote the Fubini-Study distance to $X$ on $U^+$. We know that $\varphi = - \log \delta$ is psh. We use the above Hörmander’s type result. For $s > 0$, there is an $l \geq 2$ and a form $\psi$, such that $\overline{\partial} \psi = \overline{\partial} \beta$ on $U^+$, with the following estimate

$$\int |\psi|^2 \frac{1}{\delta^{s+l}} d\lambda \leq C \int |\overline{\partial} \beta|^2 \frac{1}{\delta^{s+l}} d\lambda < \infty.$$ 

We can choose $s = 2n + l$. Hence for $z \in U^+$, and $k > 0$ fixed,

$$\frac{|\psi(z)|^2}{\delta^k(z)} \leq \frac{1}{\delta^{2n+k}(z)} \int_{B(z, \delta(z))} |\psi|^2 + \frac{1}{\delta^k(z)} \sup_{B(z, \delta(z))} |\overline{\partial} \psi|$$

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\[
\leq \int \frac{|\phi(z)|^2}{\delta^{2n+k}(z)} + o(\delta)
\]

Hence, \(|\psi(z)| = O(\delta^k)\). Consequently, \(\psi\) vanishes on \(X\) to any given fixed order. If extended by zero on \(X\), it is in \(\mathcal{C}^k(\mathbb{D}^n)\). We can now solve in \(\mathbb{D}^n\) the equation

\[
\overline{\partial} u = \beta - \psi.
\]

The restriction of \(u\) to \(X\) satisfies \(\overline{\partial} u = \beta\) and is in \(\mathcal{C}^k(X)\).

If \(\mathcal{H}^2(X) = 0\), then \(\omega = \overline{\partial} \psi_{0,1} + \overline{\partial} \phi_{0,1}\) near \(X\), with \(\overline{\partial} \phi_{0,1} = 0\) near \(X\). We solve \(\overline{\partial} u = \phi_{0,1}\) on \(X\). Then \(\omega = \overline{\partial} \overline{\partial} u + \overline{\partial} \overline{d} \overline{u} = i \overline{\partial} \overline{d} \left( \frac{u - \overline{u}}{t} \right)\). The function \(v := \frac{u - \overline{u}}{t}\) is strictly psh on \(X\) and hence in a neighborhood of \(X\). \(\square\)

**Remark 5.3**

1. A similar result can be obtained for \((p, q)\)-forms with \(q + 1 < n\).
2. Suppose \(U^+\) is a Stein domain with smooth boundary and that \(H^{2n-2}(U^+) \neq 0\). As we have seen, there is a strictly psh function on a neighborhood of \(\partial U^+\). Then a theorem of J. J. Kohn [14] asserts that one can solve the \(\overline{\partial}\)-equation in the Sobolev spaces \(H^s(U^+)\) for \(s\) large enough. One can solve it also in \(\mathcal{C}^\infty\).
3. The results are valid if we replace \(\mathbb{D}^n\), by a compact simply connected Kähler manifold \(M\), of dimension \(n \geq 3\), with positive holomorphic bisectional curvature, such that \(H^{0,1}(M) = 0\), and \(H^{1,1}(M)\) is one-dimensional.
4. To get a strictly psh function on a neighborhood of \(X\), it is enough to assume the existence of a 1-form \(\psi\) near \(X\) with \(\partial \psi_{0,1} + \overline{\partial} \psi_{1,0} > 0\) on \(X\) and \(\overline{\partial} \psi_{0,1} = 0\) on \(X\). This is satisfied when \(\mathcal{H}^2(X) = 0\).

Observe however that the existence of such a 1-form \(\psi\) implies the following geometric condition on \(X\). There is no closed current \(T\) of dimension 2, with \(\{T\} \neq 0\), such that the component of bi-dimension \((1, 1), T_{1,1}\) is positive. Indeed, since \(T\) is closed and on \(X\), it follows that \(\partial \overline{\partial} (T_{1,1}) = 0\). We then have, since \(\overline{\partial} \psi_{0,1} = 0\),

\[
0 = \langle T, d \psi \rangle = \langle T_{1,1}, \partial \psi_{0,1} + \overline{\partial} \psi_{1,0} \rangle.
\]

This implies that \(T_{1,1} = 0\), and hence \(\{T\} = 0\).

### 6 The case of surfaces

Let \(U^-\) be a pseudoconvex domain in \(\mathbb{P}^2\) with \(\mathcal{C}^2\)-boundary. Let \(r\) be a defining function for the boundary \(Z = \partial U^-\). If for every \(z \in Z\),

\[
\langle i \partial \overline{\partial} r(z), i t \wedge \overline{t} \rangle = 0 \quad \text{if} \quad \langle \partial r(z), t \rangle = 0
\]

then we say that the boundary is Levi flat.

It is not known if such domains exist, even if we assume that the boundary is real analytic. However for arbitrary Kähler surfaces \(M\) and for a Levi flat surface, there is a positive current, \(\partial \overline{\partial}\)-closed of mass 1 directed by the foliation on the boundary \(Z\). It is shown in [9] for \(\mathbb{P}^2\) and in [4] in general, that when there is no positive closed current of mass 1 directed by the foliation, then \(T\) is unique. As we have seen in Proposition 2.2, for \(\mathbb{P}^2\) or more generally for simply connected Kähler surfaces, there is no positive closed current on \(Z\). Indeed, we can assume that \(\overline{U}^-\) satisfies \(H^2_{\overline{\partial}}(\overline{U}^-) = 0\).
Theorem 6.1 Let $X$ be a compact set in a compact Kähler surface $M$, such that $\mathcal{H}^2(X) = 0$. Assume $X$ supports a positive $\overline{\partial S}$-closed current $T$ of mass 1. Then there is a real analytic $(0,1)$-form $\varphi^{0,1}$, $\overline{\partial S}$-closed in a neighborhood of $X$ and such that there is no function $u$ in the Sobolev space $W^2(M)$, with $\overline{\partial} u = \varphi^{0,1}$ on $X$.

In particular if $X$ is a real hypersurface, there is no solution $u \in W^1_2(X)$, for the equation $\overline{\partial} u = \varphi^{0,1}$.

Proof For arbitrary $X$ the meaning of $\overline{\partial} u = \varphi^{0,1}$ on $X$, is that for every current $R \in W^{-1}(M)$, supported on $X$, $(R, \varphi^{0,1}) = (R, \overline{\partial} u)$. If $X$ is a $C^1$, hypersurface, a function in $u \in W^{3,2}(X)$, extends as a function in $W^2(M)$, and a function in $W^2(M)$, restricts to $W^{3,2}(X)$, so $\overline{\partial} u = \varphi^{0,1}$ on $X$, makes sense and the two notions coincide.

Since $\mathcal{H}^2(X) = 0$, the current $T$ is not closed and hence $\partial T$ is non-zero. It is shown in [9] that if $T \geq 0$, $\overline{\partial} T = 0$ and $\int T \wedge \Omega = 1$ then $T = \Omega + \partial S + \overline{\partial} S + i \overline{\partial} v$, with $S$, $\partial S$, $\overline{\partial} S \in L^2$ and $v \in L^p$ for all $p < 2$. Here $\Omega$ is smooth and represents the class of $T$.

It follows that
\[
\partial T = -\overline{\partial} \partial S
\]
is in the Sobolev space $W^{-1}(M)$. If $u \in W^2(M)$, since $\overline{\partial} \partial T = 0$,
\[
(\partial T, \overline{\partial} u) = 0.
\]
On the other hand in a neighborhood of $X$, $\omega = \partial \varphi^{0,1} + \overline{\partial} \varphi^{1,0}$, $\overline{\partial} \varphi^{0,1} = 0$. Hence
\[
1 = (T, \omega) = -2 \text{Re}(\partial T, \varphi^{0,1}).
\]
So $(\partial T, \varphi^{0,1}) \neq 0$. Hence we cannot have $\varphi^{0,1} = \overline{\partial} u$ on $X$, with $u$ having an extension in $W^2(M)$.

Remark 6.2 If $X = \partial U^-$ is Levi-flat with $H^2_{\overline{\partial} R}(\overline{U^-}) = 0$, then $\partial T$ is of order zero [9], and the same proof shows there is no continuous function $u$ on $\overline{U^-}$ such that $\overline{\partial} u = \varphi^{0,1}$.

Question Suppose $U$ is a smooth pseudoconvex domain in $\mathbb{P}^n$, $n \geq 2$. Assume $H^2_{\overline{\partial} R}(U) = 0$. Is there a strictly psh function near the boundary?

There is an example of a compact Kähler surface $M$, with a Stein domain $U \subset M$ with real analytic boundary, but all bounded psh functions in $U$ are constant [16].

Theorem 6.3 Let $M$ be a compact complex manifold of dimension $n$. Let $Z$ be an irreducible real-analytic set of real dimension $m$. Suppose that there is in $Z$ a germ of a complex analytic set of dimension $p \geq 1$. Define $A_p$ as the set of points $z \in Z$, with a germ of complex analytic set of dimension $\geq p$ through $z$. Then $A_p$ is closed. If there is a strictly psh function near $Z$ then $A_1$ is empty.

Proof It is possible to cover $Z$ with finitely many open sets $(V_i)$ such that on $V_i$, the equation of $Z$, is $\rho_i(z, \overline{z}) = 0$. Here $\rho_i(z, \overline{z})$ a real-valued analytic function in $z, \overline{z}$. We can assume that the functions $\rho_i(z, w)$ are holomorphic in $V_i \times V_i'$, where $V_i'$ is the image of $V_i$ by conjugation.

Consider a germ of a complex analytic set $L$ parametrized by a holomorphic map $u : \mathbb{D}^p \to L$. Then, the function $\rho_i(u(t), \overline{u}(s))$ on $\mathbb{D}^{2p}$ vanishes when $s = \overline{t}$, where $\overline{u}(s) := u(\overline{s})$. Since this function is holomorphic, its zero set is a complex analytic set. Hence, it vanishes everywhere in $\mathbb{D}^{2p}$. Fixing an arbitrary $s$, we deduce that $\rho_i(z, u(s)) = 0$ for $z \in L$. 

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Define \( L' := \{ \bar{z} \in V_i, \cap_{j} \rho_i(z, u(s)) = 0 \} \). Then \( L' \) is an extension of the germ \( L \) to an analytic set in \( V_i \), contained in \( Z \). The size of the \( V_i \)'s is fixed. It follows easily that \( A_p \) is closed. This is precisely the Segre argument, see [7] and [10, Example 7].

If \( A_1 \) is non-empty, then it satisfies the local maximum principle for psh, functions in a neighborhood. If \( u \) is strictly psh in a neighborhood of \( Z \), it reaches its maximum on \( A_1 \) at a point \( p \). As in Remark 3.4 (4), we can arrange that the maximum is strict. A contradiction.

\[
\square
\]

**Theorem 6.4**  Let \( M \) be a compact complex surface. Let \( U \) be a smooth domain with connected real analytic boundary \( Z \). Assume that \( Z \) admits a point of strict pseudoconvexity. Then either there is a compact complex curve on the boundary, or there is a strictly psh function near the boundary. In particular if \( Z \subset \mathbb{P}^2 \), there is a strictly psh function in a neighborhood of \( Z \).

**Proof**  Suppose there is no compact complex curve on the boundary. Let \( S \) be the union of strictly pseudoconvex points and strictly pseudoconcave points on \( Z \). Let \( W = Z \setminus S \). Since the defining function \( r \) is real analytic, then \( W \) is a real analytic set of dimension \( \leq 2 \). It admits a stratification by smooth manifolds. If there is a curve of real dimension 2 with complex tangents, then it is a complex curve. By the above theorem it has no boundary and it is necessarily of finite area, since this is the case for \( W \). Hence it is a compact complex curve. Since this is not possible, then the set \( C \) with complex tangents is at most of finite one dimensional Hausdorff measure.

On the other hand, if there is no strictly psh function near the boundary, there is a positive \( \partial \bar{\partial} \)-closed current \( T \) of bi-dimension \( (1, 1) \) of mass 1 supported on \( \partial U \). Such a current has no mass on the set of strictly pseudoconvex points, nor on the set of strictly pseudoconcave points, as follows from equations (3.1). Hence it is supported on \( W \). Since \( T \) is of bi-dimension \( (1, 1) \) it is supported on \( C \). But such currents don’t give mass to sets of 2-Hausdorff dimension zero. See [18] for more details on the geometry of such currents. It follows that there is a strictly psh function near \( \partial U \). \( \square \)

**Remark**  It is shown in [7] and [10, Example 7] that no Levi-flat hypersurface \( Z \) exists in \( \mathbb{P}^2 \) if we assume it is smooth and real algebraic. Hence, there is a point of strict pseudoconvexity. It follows that for smooth real algebraic surfaces in \( \mathbb{P}^2 \) there is a strictly psh function in a neighborhood of \( Z \).

**Theorem 6.5**  Let \( X \) be a compact set in a compact Kähler surface \( M \).

1. If \( X \) is (locally) pluripolar, then either there is a positive closed current \( T \) of bi-degree \( (1, 1) \), and of mass one supported on \( X \), or there is a strictly psh function in a neighborhood of \( X \).

2. If \( X \) is of Lebesgue measure zero, either there is a strictly psh function in a neighborhood of \( X \) or a positive closed current \( T \) of mass one supported on \( X \), or there is a \((2, 0)\)-form \( A \) in \( L^2(\Omega^{1,0}(M)) \) of norm 1 and holomorphic in \( M \setminus X \).

**Proof**  Assume \( X \) is pluripolar. If there is no strictly psh function near \( X \), then there is a positive \( \partial \bar{\partial} \)-closed current \( T \) of mass 1, supported on \( X \).

Moreover, as we have seen that, \( T = \Omega + \partial S + \bar{\partial} S + i \partial \bar{\partial} u \), with \( S, \partial S, \bar{\partial} S \in L^2 \) and \( u \in L^p \) for all \( p < 2 \). Hence \( \partial T = -\bar{\partial} S \). So \( \bar{\partial} S \) is holomorphic out of \( X \), which is pluripolar. Since holomorphic functions in \( L^2 \), extend holomorphically through pluripolar sets, it follows that \( \partial \bar{\partial} S = 0 \) in \( M \). Hence \( T \) is closed.

If we assume only that \( X \) is of Lebesgue measure zero, and there is no positive closed current supported on \( X \), then \( A = \partial \bar{\partial} S \) is a non-identically zero, holomorphic form in \( M \setminus X \), which is in \( L^2 \). \( \square \)
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