Exact Complexity of Exact-Four-Colorability

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Abstract

Let \( M_k \subseteq \mathbb{N} \) be a given set that consists of \( k \) noncontiguous integers. Define Exact-\( M_k \)-Colorability to be the problem of determining whether \( \chi(G) \), the chromatic number of a given graph \( G \), equals one of the \( k \) elements of the set \( M_k \) exactly. In 1987, Wagner [Wag87] proved that Exact-\( M_k \)-Colorability is \( \text{BH}_{2k}(\text{NP}) \)-complete, where \( M_k = \{6k+1,6k+3,\ldots,8k-1\} \) and \( \text{BH}_{2k}(\text{NP}) \) is the 2\( k \)th level of the boolean hierarchy over \( \text{NP} \). In particular, for \( k = 1 \), it is DP-complete to determine whether \( \chi(G) = 7 \), where \( \text{DP} = \text{BH}_2(\text{NP}) \). Wagner raised the question of how small the numbers in a \( k \)-element set \( M_k \) can be chosen such that Exact-\( M_k \)-Colorability still is \( \text{BH}_{2k}(\text{NP}) \)-complete. In particular, for \( k = 1 \), he asked if it is DP-complete to determine whether \( \chi(G) = 4 \). In this note, we solve this question of Wagner and determine the precise threshold \( t \in \{4,5,6,7\} \) for which the problem Exact-\{\( t \)\}-Colorability jumps from \( \text{NP} \) to DP-completeness: It is DP-complete to determine whether \( \chi(G) = 4 \), yet Exact-\{3\}-Colorability is in \( \text{NP} \). More generally, for each \( k \geq 1 \), we show that Exact-\( M_k \)-Colorability is \( \text{BH}_{2k}(\text{NP}) \)-complete for \( M_k = \{3k+1,3k+3,\ldots,5k-1\} \).

1 Exact-\( M_k \)-Colorability and the Boolean Hierarchy over \( \text{NP} \)

To classify the complexity of problems known to be \( \text{NP} \)-hard or \( \text{coNP} \)-hard, but seemingly not contained in \( \text{NP} \cup \text{coNP} \), Papadimitriou and Yannakakis [PY84] introduced DP, the class of differences of two \( \text{NP} \) problems. They showed that DP contains various interesting types of problems, including uniqueness problems, critical graph problems, and exact optimization problems. For example, Cai and Meyer [CM87] proved the DP-completeness of Minimal-3-Uncolorability, a critical graph problem that asks whether a given graph is not 3-colorable, but deleting any of its vertices makes it 3-colorable. A graph is said to be \( k \)-colorable if its vertices can be colored using no more than \( k \) colors such that no two adjacent vertices receive

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the same color. The chromatic number of a graph $G$, denoted $\chi(G)$, is defined to be the smallest $k$ such that $G$ is $k$-colorable.

Generalizing DP, Cai et al. [CGH+88,CGH+89] defined and studied the boolean hierarchy over NP. Their papers initiated an intensive work and many papers on the boolean hierarchy; e.g., [Wag87,KSW87,Kad88,Wag90,Bei91,Cha91,HR97] to name just a few. To define the boolean hierarchy, we use the symbols $\land$ and $\lor$, respectively, to denote the complex intersection and the complex union of set classes. That is, for classes $C$ and $D$ of sets, define

\[ C \land D = \{ A \cap B | A \in C \text{ and } B \in D \}; \]
\[ C \lor D = \{ A \cup B | A \in C \text{ and } B \in D \}. \]

**Definition 1** ([CGH+88]) The boolean hierarchy over NP is inductively defined as follows:

\[ BH_1(NP) = NP, \]
\[ BH_2(NP) = NP \land \text{coNP}, \]
\[ BH_k(NP) = BH_{k-2}(NP) \lor BH_2(NP) \text{ for } k \geq 3, \text{ and} \]
\[ BH(NP) = \bigcup_{k \geq 1} BH_k(NP). \]

Equivalent definitions in terms of different boolean hierarchy normal forms can be found in the papers [CGH+88,Wag87,KSW87]; for the boolean hierarchy over arbitrary set rings, we refer to the early work by Hausdorff [Hau14]. Note that DP $= BH_2(NP)$.

In his seminal paper [Wag87], Wagner provided sufficient conditions to prove problems complete for the levels of the boolean hierarchy. In particular, he established the following lemma for $BH_{2k}(NP)$.

**Lemma 2** [Wag87, Thm. 5.1(3)] Let $A$ be some NP-complete problem, let $B$ be an arbitrary problem, and let $k \geq 1$ be fixed.

If there exists a polynomial-time computable function $f$ such that, for all strings $x_1, x_2, \ldots, x_{2k} \in \Sigma^*$ satisfying $(\forall j : 1 \leq j < 2k)[x_{j+1} \in A \implies x_j \in A]$, it holds that

\[ \|\{i | x_i \in A\}\| \text{ is odd } \iff f(x_1, x_2, \ldots, x_{2k}) \in B, \quad (1.1) \]

then $B$ is $BH_{2k}(NP)$-hard.

For fixed $k \geq 1$, let $M_k = \{ 6k + 1, 6k + 3, \ldots, 8k - 1 \}$, and define the problem Exact-$M_k$-Colorability $= \{ G | \chi(G) \in M_k \}$. In particular, Wagner applied Lemma 2 to prove that, for each $k \geq 1$, Exact-$M_k$-Colorability is $BH_{2k}(NP)$-complete. For the special case of $k = 1$, it follows that Exact-$\{7\}$-Colorability is DP-complete.

Wagner [Wag87, p. 70] raised the question of how small the numbers in a $k$-element set $M_k$ can be chosen such that Exact-$M_k$-Colorability still is $BH_{2k}(NP)$-complete. Consider the special case of $k = 1$. It is easy to see that Exact-$\{3\}$-Colorability is in NP and, thus, cannot be DP-complete unless the boolean hierarchy collapses; see Proposition 3 below. Consequently, for
$k = 1$, Wagner’s result leaves a gap in determining the precise threshold $t \in \{4, 5, 6, 7\}$ for which the problem Exact-$\{t\}$-Colorability jumps from NP to DP-completeness. Closing this gap, we show that it is DP-complete to determine whether $\chi(G) = 4$. More generally, answering Wagner’s question for each $k \geq 1$, we show that Exact-$M_k$-Colorability is $\text{BH}_{2k}(\text{NP})$-complete for $M_k = \{3k + 1, 3k + 3, \ldots, 5k - 1\}$.

2 Solving Wagner’s Question

**Proposition 3** Fix any $k \geq 1$, and let $M_k$ be any set that contains $k$ noncontiguous positive integers including 3. Then, Exact-$M_k$-Colorability is in $\text{BH}_{2k-1}(\text{NP})$; in particular, for $k = 1$, Exact-$\{3\}$-Colorability is in NP.

Hence, Exact-$M_k$-Colorability is not $\text{BH}_{2k}(\text{NP})$-complete unless the boolean hierarchy, and consequently the polynomial hierarchy, collapses.

**Proof.** Fix any $k \geq 1$, and let $M_k$ be given as above. Note that

$$\text{Exact-$M_k$-Colorability} = \bigcup_{i \in M_k} \text{Exact-$\{i\}$-Colorability}.$$ 

Since for each $i \in M_k$, Exact-$\{i\}$-Colorability $= \{G \mid \chi(G) \leq i\} \cap \{G \mid \chi(G) > i - 1\}$ and since the set $\{G \mid \chi(G) \leq i\}$ is in NP and the set $\{G \mid \chi(G) > i - 1\}$ is in coNP, each of the $k - 1$ sets Exact-$\{i\}$-Colorability with $i \in M_k - \{3\}$ is in DP. However, Exact-$\{3\}$-Colorability is even contained in NP, since it can be checked in polynomial time whether a given graph is 2-colorable, so $\{G \mid \chi(G) > 2\}$ is in P. It follows that Exact-$M_k$-Colorability is in $\text{BH}_{2k-1}(\text{NP})$.

**Theorem 4** For fixed $k \geq 1$, let $M_k = \{3k + 1, 3k + 3, \ldots, 5k - 1\}$. Then, Exact-$M_k$-Colorability is $\text{BH}_{2k}(\text{NP})$-complete. In particular, for $k = 1$, it follows that Exact-$\{4\}$-Colorability is DP-complete.

**Proof.** We apply Lemma 3 with $A$ being the NP-complete problem 3-SAT and $B$ being the problem Exact-$M_k$-Colorability, where $M_k = \{3k + 1, 3k + 3, \ldots, 5k - 1\}$ for fixed $k$. The standard reduction $\sigma$ from 3-SAT to 3-Colorability has the following property [GJ79]:

$$\phi \in \text{3-SAT} \implies \chi(\sigma(\phi)) = 3 \quad \text{and} \quad \phi \notin \text{3-SAT} \implies \chi(\sigma(\phi)) = 4. \quad (2.2)$$

Using the PCP theorem, Khanna, Linial, and Safra [KLS00] showed that it is NP-hard to color a 3-colorable graph with only four colors. Guruswami and Khanna [GK00] gave a novel proof of the same result that does not rely on the PCP theorem. We use their direct transformation, call it $\rho$, that consists of two subsequent reductions—first from 3-SAT to the independent set problem, and then from the independent set problem to 3-Colorability—such that $\phi \in \text{3-SAT}$ implies $\chi(\rho(\phi)) = 3$, and $\phi \notin \text{3-SAT}$ implies $\chi(\rho(\phi)) \geq 5$. Guruswami and Khanna [GK00] note that the graph $H = \rho(\phi)$ they construct always is 6-colorable. In fact, their construction even gives that $H$ always is 5-colorable; hence, we have:

$$\phi \in \text{3-SAT} \implies \chi(\rho(\phi)) = 3 \quad \text{and} \quad \phi \notin \text{3-SAT} \implies \chi(\rho(\phi)) = 5. \quad (2.3)$$
To see why, look at the reduction in [GK00]. The graph $H$ consists of tree-like structures whose vertices are replaced by $3 \times 3$ grids, which always can be colored with three colors, say 1, 2, and 3. In addition, some leaves of the tree-like structures are connected by leaf-level gadgets of two types, the “same row kind” and the “different row kind.” The latter gadgets consist of two vertices connected to some grids, and thus can always be colored with two additional colors. The leaf-level gadgets of the “same row kind” consist of a triangle whose vertices are adjacent to two grid vertices each. Hence, regardless of which 3-coloring is used for the grids, one can always color one triangle vertex, say $t_1$, with a color $c \in \{1, 2, 3\}$ such that $c$ is different from the colors of the two grid vertices adjacent to $t_1$. Using two additional colors for the other two triangle vertices implies $\chi(H) \leq 5$, which proves Equation (2.3).

The join operation $\oplus$ on graphs is defined as follows: Given two disjoint graphs $A = (V_A, E_A)$ and $B = (V_B, E_B)$, their join $A \oplus B$ is the graph with vertex set $V_{A\oplus B} = V_A \cup V_B$ and edge set $E_{A\oplus B} = E_A \cup E_B \cup \{(a, b) \mid a \in V_A$ and $b \in V_B\}$. Note that $\oplus$ is an associative operation on graphs and $\chi(A \oplus B) = \chi(A) + \chi(B)$.

Let $\phi_1, \phi_2, \ldots, \phi_{2k}$ be $2k$ given boolean formulas satisfying $\phi_{j+1} \in \text{3-SAT} \implies \phi_j \in \text{3-SAT}$ for each $j$ with $1 \leq j < 2k$. Define $2k$ graphs $H_1, H_2, \ldots, H_{2k}$ as follows. For each $i$ with $1 \leq i \leq k$, define $H_{2i-1} = \rho(\phi_{2i-1})$ and $H_{2i} = \sigma(\phi_{2i})$. By Equations (2.2) and (2.3),

$$\chi(H_j) = \begin{cases} 3 & \text{if } 1 \leq j \leq 2k \text{ and } \phi_j \in \text{3-SAT} \\ 4 & \text{if } j = 2i \text{ for some } i \in \{1, 2, \ldots, k\} \text{ and } \phi_j \notin \text{3-SAT} \\ 5 & \text{if } j = 2i - 1 \text{ for some } i \in \{1, 2, \ldots, k\} \text{ and } \phi_j \notin \text{3-SAT}. \end{cases} \quad (2.4)$$

For each $i$ with $1 \leq i \leq k$, define the graph $G_i$ to be the disjoint union of the graphs $H_{2i-1}$ and $H_{2i}$. Thus, $\chi(G_i) = \max\{\chi(H_{2i-1}), \chi(H_{2i})\}$, for each $i$ with $1 \leq i \leq k$. The construction of our reduction $f$ is completed by defining $f(\phi_1, \phi_2, \ldots, \phi_{2k}) = G$, where the graph $G = \bigoplus_{i=1}^{k} G_i$ is the join of the graphs $G_1, G_2, \ldots, G_k$. Thus,

$$\chi(G) = \sum_{i=1}^{k} \chi(G_i) = \sum_{i=1}^{k} \max\{\chi(H_{2i-1}), \chi(H_{2i})\}. \quad (2.5)$$

It follows from our construction that

$$\|\{i \mid \phi_i \in \text{3-SAT}\}\| \text{ is odd} \iff (\exists i : 1 \leq i \leq k) [\phi_1, \ldots, \phi_{2i-1} \in \text{3-SAT} \text{ and } \phi_{2i}, \ldots, \phi_{2k} \notin \text{3-SAT}].$$

Thus, $\chi(G) \in M_k = \{3k + 1, 3k + 3, \ldots, 5k - 1\}$

Hence, Equation (1.1) is satisfied. By Lemma 3, Exact-$M_k$-Colorability is BH$_{2k}$($\text{NP}$)-complete.

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