CLASSIFICATION OF ALMOST YAMABE SOLITONS IN EUCLIDEAN SPACES

TATSUYA SEKO AND SHUN MAETA

Abstract. In this paper, we completely classify almost Yamabe solitons on hypersurfaces in Euclidean spaces arisen from the position vector field. Some results of almost Yamabe solitons with a concurrent vector field and almost Yamabe solitons on submanifolds in Riemannian manifolds equipped with a concurrent vector field are also presented. Moreover, we classify complete Ricci solitons on minimal submanifolds in non-positively curved space forms. For almost Yamabe solitons, all of results in this paper can be applied to Yamabe solitons.

1. Introduction

\((M, g, v, \rho)\) is called a Yamabe soliton if it satisfies

\[
(R - \rho)g = \frac{1}{2} \mathcal{L}_v g,
\]

where \(\mathcal{L}_v g\) is the Lie-derivative, \(R\) is the scalar curvature of \(M\) and \(\rho\) is a constant. If \(\rho > 0\), \(\rho = 0\), \(\rho < 0\), then a Yamabe soliton \((M, g, v, \rho)\) is called a shrinking, a steady or an expanding Yamabe soliton, respectively.

A Yamabe soliton \((M, g, v, \rho)\) is called a gradient Yamabe soliton if \(v\) is the gradient of some function \(f\) on \(M\). We denote a gradient Yamabe soliton by \((M, g, f, \rho)\).

Yamabe solitons are special solutions to the Yamabe flow. Under some conditions, Yamabe solitons have been studied (cf. \([6]\), \([7]\) and \([8]\)). In particular, it is shown that any compact Yamabe soliton has constant scalar curvature (cf. \([6]\) and \([7]\)).

E. Barbosa and E. Ribeiro introduced a generalization of Yamabe solitons in \([1]\) as follows.

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Definition 1.1 ([1]). A Riemannian manifold \((M, g, v, \rho)\) is an almost Yamabe soliton if there exist a complete vector field \(v\) and a smooth soliton function \(\rho\) on \(M\) satisfying

\[(R - \rho)g = \frac{1}{2} \mathcal{L}_v g.\]

An almost Yamabe soliton \((M, g, v, \rho)\) is called a gradient almost Yamabe soliton if \(v\) is the gradient of some function \(f\) on \(M\). We denote a gradient almost Yamabe soliton by \((M, g, f, \rho)\).

Remark 1.2. From the definition, if \(\rho\) is constant, almost Yamabe solitons are Yamabe solitons.

A vector field \(v\) on \(M\) is called a concurrent vector field if it satisfies

\[\nabla_X v = X,\]

for any vector field \(X\) on \(M\), where \(\nabla\) is the Levi-Civita connection on \(M\). One of the most important example of Riemannian manifolds with a concurrent vector field is Euclidean spaces. Because, the position vector field on Euclidean spaces satisfies (3). Riemannian manifolds endowed with concurrent vector fields have been studied (cf. [4], [9] and [10]).

In this paper, we completely classify almost Yamabe solitons on hypersurfaces in Euclidean spaces arisen from the position vector field \(v\). We denote the tangential and the normal components of \(v\) by \(v^T\) and \(v^\perp\), respectively.

Theorem 1.3 (Theorem 5.1). Any almost Yamabe soliton \((M, g, v^T, \rho)\) on a hypersurface in a Euclidean space \(\mathbb{E}^{n+1}\) is contained in either a hyperplane or a sphere.

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. In section 3 we show that any Yamabe soliton \((M, g, v, \rho)\) with a concurrent vector field \(v\) is a gradient expanding Yamabe soliton with \(\rho = -1\) and the scalar curvature is zero. In section 4 we consider almost Yamabe solitons on submanifolds in Riemannian manifolds endowed with a concurrent vector field. Section 5 is devoted to the proof of Theorem 1.3. Finally in Appendix, we completely classify complete gradient Ricci solitons on minimal submanifolds in non-positively curved space forms.
2. Preliminaries

Let \((N, \tilde{g})\) be an \(m\)-dimensional Riemannian manifold and \((M, g)\) be an \(n\)-dimensional submanifold in \((N, \tilde{g})\). We denote Levi-Civita connections on \((M, g)\) and \((N, \tilde{g})\) by \(\nabla\) and \(\tilde{\nabla}\), respectively.

For any vector fields \(X, Y\) tangent to \(M\) and \(\eta\) normal to \(M\), the formula of Gauss is given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),
\]

where \(\nabla_X Y\) and \(h(X, Y)\) are the tangential and the normal components of \(\tilde{\nabla}_X Y\). The formula of Weingarten is given by

\[
\tilde{\nabla}_X \eta = -A_\eta(X) + D_X \eta,
\]

where \(-A_\eta(X)\) and \(D_X \eta\) are the tangential and the normal components of \(\tilde{\nabla}_X \eta\). \(A_\eta(X)\) and \(h(X, Y)\) are related by

\[
g(A_\eta(X), Y) = \tilde{g}(h(X, Y), \eta).
\]

The mean curvature vector \(H\) of \(M\) in \(N\) is given by

\[
H = \frac{1}{n} \text{trace } h.
\]

For any vector fields \(X, Y, Z, W\) tangent to \(M\), the equation of Gauss is given by

\[
\tilde{g}(\tilde{R}m(X, Y)Z, W) = g(Rm(X, Y)Z, W)
+ \tilde{g}(h(X, Z), h(Y, W))
- \tilde{g}(h(X, W), h(Y, Z)),
\]

where \(Rm\) and \(\tilde{R}m\) are Riemannian curvature tensors of \(M\) and \(N\), respectively. The equation of Codazzi is given by

\[
(\tilde{R}m(X, Y)Z) \perp = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z),
\]

where \((\tilde{R}m(X, Y)Z) \perp\) is the normal component of \(\tilde{R}m(X, Y)Z\) and \(\tilde{\nabla}_X h\) is defined by

\[
(\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\]

If \(N\) is a space of constant curvature, then the equation of Codazzi reduces to

\[
0 = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z).
\]
3. Almost Yamabe solitons with a concurrent vector field

Firstly, we show a formula of almost Yamabe solitons which is useful for study of almost Yamabe solitons.

Lemma 3.1.

\( (n - 1)\Delta(R - \rho) + \frac{1}{2} g(\nabla R, \nabla f) + R(R - \rho) = 0 \).

Proof. Since
\[
\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla_j f,
\]
\[
\Delta \nabla_i f = \nabla_k \nabla_k \nabla_i f = \nabla_k ((R - \rho)g_{ki}) = \nabla_i (R - \rho),
\]
and
\[
\nabla_i \Delta f = \nabla_i (n(R - \rho)) = n \nabla_i (R - \rho),
\]
we have
\( (n - 1)\nabla_i (R - \rho) + R_{ij} \nabla_j f = 0 \),
where \( R_{ij} \) is the Ricci curvature of \( M \). By applying \( \nabla_l \) to the both side of (5), we obtain
\( (n - 1)\nabla_l \nabla_i (R - \rho) + \nabla_l R_{ij} \cdot \nabla_j f + R_{ij} \nabla_l \nabla_j f = 0 \).
Taking the trace, we obtain (4). \( \square \)

Proposition 3.2. If an almost Yamabe soliton \( (M, g, v, \rho) \) has a concurrent vector field \( v \), then \( M \) is a gradient almost Yamabe soliton with \( R = \rho + 1 \).

Proof. Firstly, we show that an almost Yamabe soliton with a concurrent vector field is a gradient almost Yamabe soliton. Set
\[
f = \frac{1}{2} g(v, v).
\]
Then we have
\[
g(\nabla f, X) = X(f) = g(v, \nabla_X v) = g(v, X),
\]
for any vector field \( X \) on \( M \).
Secondly, we show that \( R = \rho + 1 \). Since \( v \) is a concurrent vector field, we have
\( \mathcal{L}_v g = 2g \).
Combining \( \mathcal{L}_v g = 2g \) with \( (2) \), we obtain
\( R = \rho + 1 \). \( \square \)
Corollary 3.3. Any compact almost Yamabe soliton with a concurrent vector field is a gradient expanding Yamabe soliton with zero scalar curvature and \( \rho = -1 \).

Proof. By Proposition 3.2 and (2), we have \( \Delta f = n \). By applying maximum principle, we get \( f \) is constant. From this and (4), we have \( R = 0 \) and \( \rho = -1 \). \( \square \)

By applying Proposition 3.2 to Yamabe solitons, we can get the following.

Corollary 3.4. Any Yamabe soliton with a concurrent vector field is a gradient expanding Yamabe soliton with zero scalar curvature and \( \rho = -1 \).

Proof. Since \( \rho \) is constant, by Proposition 3.2 and (4), we have \( R = 0 \) and \( \rho = -1 \). \( \square \)

4. Almost Yamabe solitons on submanifolds

In this section, we assume that \( (N, \tilde{g}) \) is a Riemannian manifold endowed with a concurrent vector field \( v \) and \( (M, g) \) is a submanifold in \( (N, \tilde{g}) \). We denote the tangential and the normal components of \( v \) by \( v^T \) and \( v^\perp \), respectively.

To classify almost Yamabe solitons on a submanifold, we show the following Lemma which will be used in the proof of Proposition 4.3 and Theorem 5.1.

Lemma 4.1. Any almost Yamabe soliton \( (M, g, v^T, \rho) \) on a submanifold \( M \) in \( N \) satisfies

\[
(R - \rho - 1) g(X, Y) = g(A_{v^\perp}(X), Y),
\]

for any vector fields \( X, Y \) on \( M \).

Proof. Since \( v \) is a concurrent vector field and by using formulas of Gauss and Weingarten, we have

\[
X = \tilde{\nabla}_X v = \tilde{\nabla}_X (v^T + v^\perp)
\]

\[
= (\nabla_X v^T + h(X, v^T)) + (-A_{v^\perp}(X) + D_X v^\perp),
\]

for any vector field \( X \) on \( M \). By comparing the tangential and the normal components of (10), we obtain

\[
\nabla_X v^T = X + A_{v^\perp}(X), \quad h(X, v^T) = -D_X v^\perp.
\]

From the definition of Lie-derivative and (11), we have

\[
(\mathcal{L}_v^T g)(X, Y) = 2g(X, Y) + 2g(A_{v^\perp}(X), Y),
\]
for any vector fields $X, Y$ on $M$. Combining (12) with (2), we obtain □.

**Proposition 4.2.** Any almost Yamabe soliton $(M, g, v^T, \rho)$ on a submanifold $M$ in $N$ is a gradient almost Yamabe soliton.

*Proof.* Set

$$f = \frac{1}{2} \tilde{g}(v, v).$$

Then we have

$$g(\nabla f, X) = X(f) = \tilde{g}(v, \tilde{\nabla} X v) = \tilde{g}(v, X) = g(v^T, X),$$

for any vector field $X$ on $M$. □

**Proposition 4.3.** If an almost Yamabe soliton $(M, g, v^T, \rho)$ on a submanifold $M$ in $N$ is minimal, then $R = \rho + 1$.

*Proof.* From Proposition 4.2, we know that any almost Yamabe soliton on a submanifold is a gradient almost Yamabe soliton. Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame on $M$. From Lemma 4.1 we have

$$(R - \rho - 1)g_{ij} = g(A_{v^\perp}(e_i), e_j).$$

Since $M$ is minimal and taking the trace, we obtain

$$n(R - \rho - 1) = n\tilde{g}(H, v^\perp) = 0.$$

Therefore we conclude that

$$R = \rho + 1.$$

□

**Corollary 4.4.** Any compact almost Yamabe soliton on a minimal submanifold in $N$ is a gradient expanding Yamabe soliton with zero scalar curvature and $\rho = -1$.

*Proof.* By Proposition 4.3 and (2), we have $\Delta f = n$. By applying maximum principle, we get $f$ is constant. From this and (4), we have $R = 0$ and $\rho = -1$. □

**Corollary 4.5.** Any Yamabe soliton on a minimal submanifold in $N$ is a gradient expanding Yamabe soliton with zero scalar curvature and $\rho = -1$.

*Proof.* Since $\rho$ is constant, by Proposition 4.3 and (4), we have $R = 0$ and $\rho = -1$. □
5. Classification of almost Yamabe solitons in Euclidean spaces

In this section, we give the proof of Theorem 1.3, namely, we completely classify almost Yamabe solitons on hypersurfaces in Euclidean spaces arisen from the position vector field. Let $v$ be the position vector field on Euclidean spaces.

**Theorem 5.1.** Any almost Yamabe soliton $(M, g, v^T, \rho)$ on a hypersurface in a Euclidean space $\mathbb{E}^{n+1}$ is contained in either a hyperplane or a sphere.

**Proof.** Let $\alpha$ be a mean curvature and $\lambda$ be a support function of $M$, i.e. $H = \alpha N$ and $\lambda = \tilde{g}(N, v)$ with a unit normal vector field $N$. From Lemma 4.1,

$$ (R - \rho - 1)g_{ij} = \tilde{g}(h(e_i, e_j), v^\perp) = \tilde{g}(\kappa_i g_{ij} N, v) = \kappa_i g_{ij} \lambda, $$
where $A_N(e_i) = \kappa_i e_i, \ i = 1, \cdots, n$. So we have

$$ R - \rho - 1 = \lambda \kappa_i. \quad (13) $$

Taking the summation, we obtain

$$ R - \rho - 1 = \lambda \alpha. \quad (14) $$

Comparing (13) and (14), we have

$$ \kappa_i = \alpha. $$

Therefore $M$ is a totally umbilical submanifold with $A_N(e_i) = \alpha e_i$ and $h$ satisfies $h(X, Y) = \alpha g(X, Y) N$. Now we have

$$ 0 = \nabla_X(\tilde{g}(N, N)) = 2\tilde{g}(\nabla_X N, N) = 2\tilde{g}(D_X N, N). $$

Therefore $D_X N = 0$. So we obtain

$$ (\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = X(\alpha)g(Y, Z) N, $$

for any vector fields $X, Y, Z$ on $M$. From the equation of Codazzi, we have

$$ X(\alpha)Y = Y(\alpha)X. $$

Taking $X$ and $Y$ linearly independent, we conclude that $\alpha$ is a constant.

**Case 1:** $\alpha = 0$. From $\nabla_X N = 0$, $N$, restricted to $M$, is a constant in $\mathbb{E}^{n+1}$ and we have

$$ \nabla_X(\tilde{g}(v, N)) = \tilde{g}(\nabla_X v, N) + \tilde{g}(v, \nabla_X N) = \tilde{g}(X, N) = 0. $$
This shows that $\tilde{g}(v, N)$ is constant when $v$ and $N$ is restricted to $M$. Therefore $M$ is contained in the hyperplane normal to $N$.

\textbf{Case 2: $\alpha \neq 0$.} We have

$$\tilde{\nabla}_X(v + \alpha^{-1}N) = X + \alpha^{-1}\tilde{\nabla}_X N = X + \alpha^{-1}(-AN(X)) = 0.$$ 

This shows that the vector field $v + \alpha^{-1}N$, restricted to $M$, is a constant in $\mathbb{E}^{n+1}$. Therefore $M$ is contained in the sphere. \hfill $\square$

6. Appendix

In this appendix, we completely classify complete gradient Ricci solitons on minimal submanifolds in Euclidean spaces or hyperbolic spaces. $(M, g, f, \rho)$ is called a gradient Ricci soliton if it satisfies (see for example [5]),

$$\text{(15)} \quad \text{Ric} + \nabla \nabla f + \rho^2 g = 0.$$ 

Some recent progress on the subject can be found in [2].

\textbf{Theorem 6.1.} (i) Any complete gradient Ricci soliton on a minimal submanifold $M$ in a Euclidean space is an affine subspace.

(ii) There is no complete gradient Ricci soliton on a minimal submanifold $M$ in a hyperbolic space.

\textbf{Proof.} Let $c$ be a sectional curvature of a Euclidean space and a hyperbolic spaces, namely, $c = 0$ or $-1$. From the equation of Gauss,

$$\text{Ric}(X, Y) = c(n - 1)g(X, Y) - \tilde{g}(h(X, e_i), h(e_i, Y)) + \tilde{g}(h(X, Y), nH),$$ 

for any vector fields $X, Y$ on $M$. Since $M$ is a minimal, we have

$$\text{Ric}(X, Y) = c(n - 1)g(X, Y) - \tilde{g}(h(X, e_i), h(e_i, Y)).$$ 

Therefore,

$$\text{(16)} \quad R = cn(n - 1) - |h|^2.$$ 

In [3], B. L. Chen showed that any complete gradient Ricci soliton has non-negative scalar curvature $R \geq 0$.

\underline{Case 1: $c = 0$.} By (16), $0 \leq R = -|h|^2$. Therefore, $M$ is a totally geodesic submanifold and it is an affine subspace in $\mathbb{E}^{n+1}$.

\underline{Case 2: $c = -1$.} By the same argument, $|h|^2 \leq -n(n - 1)$, which can not happen. \hfill $\square$

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