Dynamical Stationarity as a Result of Sustained Random Growth

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In sustained growth with random dynamics stationary distributions can exist without detailed balance. This suggests thermodynamical behavior in fast growing complex systems. In order to model such phenomena we apply both a discrete and a continuous master equation. The derivation of elementary rates from known stationary distributions is a generalization of the fluctuation-dissipation theorem. Entropic distance evolution is given for such systems. We reconstruct distributions obtained for growing networks, particle production, scientific citations and income distribution.

I. INTRODUCTION

Statistical physics methods are applied to problems related to complex system evolution in an increasing manner. While these are powerful enough to describe essential properties of statistical data and their distributions, the meaning of parameters behind such distributions can be understood deeper if derived from dynamical models. Following Occam’s razor principle (among competing hypotheses, the one with the fewest assumptions should be selected), simple rules for the dynamics are welcome.

The dynamics of many complex systems can be studied by using a simple master-equation approach [1–4]. beside physics, such studies are also popular in network science [5–8], biology [9], economics [10], chemistry [11], epidemics [12, 13], scientometrics [14, 15] and sociology [16]. Generally such dynamical processes tend to a stationary state with an invariant limiting distribution [17].

In the master equation approach to the evolution of general probability distributions, we know several statements for systems satisfying the detailed balance condition in their stationary state [18], but much less is known for fast growing complex systems without detailed balance. In particular, if the microprocesses are not reversible, the entropy growth and the global stability of stationary solutions are not guaranteed even for generalized entropies. Such cases occur in open systems.

In this paper we investigate a promising subset of unbalanced master equations leading to stationary distributions. Such an approach can be applied to understand several complex phenomena. In this work we refer to application examples for emerging particle distributions in high-energy accelerator experiments, to income distributions following from redistribution and taxation strategies, to scientific citation dynamics and to evolution of growing complex networks.

In this framework, the stationary distribution, $Q_n$, is determined by two microscopic rates: $\mu_n$ describes the transition rate from a state with $n$ quanta to $n+1$ inside a chain of states, while $\gamma_n$ describes a loss rate for the state $n$ towards an unspecified environment. We assume that there is no $n$ to $n-1$ process, so the transition dynamics is unidirectional. Without having a state dependent loss rate, $\gamma_n = 0$, the only possibility for a stationary distribution would be $Q_n \sim 1/\mu_n$, a trivial case. Already for constant and linearly $n$-dependent rates a rich structure of possible solutions emerges.

Since there is no reverse process inside the chain of states, a detailed balance condition cannot be fulfilled. We illustrate the difference between the classical scheme allowing detailed balance and the presently discussed one-sided growth picture with the flow diagrams on Fig. 1.

State-dependent loss rates, $\gamma_n \neq 0$, open the door to nontrivial stationary distributions. Usually models are constructed with assumed transition rates $\mu_n$ and $\gamma_n$ and the stationary (limiting) distribution, $Q_n$, is derived. However, the reverse problem is also interesting: by observing a distribution, $Q_n$, and knowing the interaction rate with the environment, $\gamma_n$, one wishes to reconstruct the internal dynamics of the system governed by the rates, $\mu_n$. We interpret the quantity $n$ in high-energy experiments as number of hadrons produced in energetic collisions. In studying income distributions $n$ is the amount of money received by consumer units. In studying the impact of scientific papers $n$ is the number

Figure 1. Schematic view of master equations for balanced (top) and sustained random growth (bottom) processes.
of received citations. For complex random networks \( n \) means the number of connections starting from a given node and \( Q_n \) is therefore the degree distribution.

We present both a master equation approach over discrete states labelled by \( n \), and its continuous limit. Finally the stability of stationary distributions obtained from given transition rates is investigated in terms of a generalized entropic distance.

II. MASTER AND FLOW EQUATION FRAMEWORK

Now we turn to the definition of the underlying mathematical formalism. We consider linear and first order time evolution equations for the distribution, \( P_n(t) \) and its continuous version, \( P(x,t) \). The corresponding stationary distributions, \( Q_n \) and \( Q(x) \), respectively, shall be determined by the same equations with vanishing time derivative. Beyond finding out what stationary distributions, i.e. results of the long term evolution, belong to given rates \( \mu_n, \gamma_n \) (or \( \mu(x), \gamma(x) \)) one is interested in the whole process starting from arbitrary initial distributions as well as in the stability and basin of attraction for the final distribution.

A. Discrete state space master equation

The sustained growth master equation hereafter is given as depicted in the lower part of Fig. 1

\[
\dot{P}_n = \mu_{n-1} P_{n-1} - (\mu_n + \gamma_n) P_n
\]

for \( n \geq 1 \). The corresponding equation for the \( n = 0 \) term can be obtained from the normalization condition \( \sum_{n=0}^{\infty} P_n(t) = 1 \):

\[
\dot{P}_0 = \langle \gamma \rangle P_0 - (\mu_0 + \gamma_0) P_0.
\]

Here we used the abbreviation \( \langle \gamma \rangle = \sum_{n=0}^{\infty} \gamma_n P_n \). This system allows for stationary solutions satisfying:

\[
\mu_{n-1} Q_{n-1} = (\mu_n + \gamma_n) Q_n
\]

for \( n \geq 1 \) and \( Q_0 = \langle \gamma \rangle Q_0/ (\mu_0 + \gamma_0) \). Eqs. (1) and (2) constitute a specific realization of a general, continuous-time Markov process:

\[
\dot{P}_n = \sum_m (w_{n\to m} P_m - w_{m\to n} P_n)
\]

with

\[
w_{n\to m} = \mu_m \delta_{m,n-1} + \gamma_m \delta_{n,0}.
\]

The inflow and outflow in each patch in Fig. 1 balance each other in the stationary state. This also offers a strategy to reconstruct the link connection probability rate to a link with already \( m \) connections or to increase a conveniently discretized income from \( m \) to \( m+1 \), \( \mu_m \), by observing the stationary distribution, \( Q_n \), and the loss rate \( \gamma_n \). We simply sum up eq. (2) from \( n = m+1 \) to infinity and obtain

\[
\mu_m = \frac{1}{Q_m} \sum_{n=m+1}^{\infty} \gamma_n Q_n.
\]

This relation reminds to the fluctuation-dissipation theorem, in particular when the stationary distribution is exponential, \( Q_n = e^{-\beta n}/Z = (1-q)q^n \), and the loss rate due to environmental effects is constant \( \gamma_n = \gamma \). In this case eq. (6) delivers a constant inner-chain rate reminding to the quantum Kubo formula:

\[
\mu_m^{\text{exp}} = \gamma \frac{q}{1-q} = \gamma \frac{1}{e^{\beta\gamma} - 1}.
\]

The general solution of the recursion represented by eq. (6) is given as a ratio of \( n \)-fold products,

\[
Q_n = Q_0 \prod_{i=0}^{n-1} \frac{\mu_i}{\mu_i + \gamma_i}.
\]

\( Q_0 \) can either be obtained from the normalization condition \( \sum_{n=0}^{\infty} Q_n = 1 \) or applying eq. (2) with \( Q_0 = 0 \). It is not trivial that these are equivalent procedures: the product form (8) and the definition of the expectation value delivers

\[
Q_0 = \frac{\langle \gamma \rangle Q_0}{\mu_0 + \gamma_0} = Q_0 \sum_{n=0}^{\infty} \frac{\gamma_n}{\mu_0 + \gamma_0} \prod_{i=0}^{n-1} \frac{\mu_i}{\mu_i + \gamma_i}.
\]

Consistency can easily be re-formulated in terms of the basic ratios, \( r_i = \gamma_i/\mu_i \), after dividing both sides with \( Q_0 \neq 0 \):

\[
1 = \sum_{n=0}^{\infty} r_n \prod_{i=0}^{n-1} \frac{1}{1 + r_i}.
\]

It is at the first glance surprising, but true, that this identity is fulfilled for any infinite series of \( r_i \neq -1 \) ratios. A short mathematical proof is given in the Appendix.

B. Continuum approach

It is instructive to obtain the above equations in a continuous version. We set up the following Markovian
framework:

$$\frac{\partial}{\partial t} P(x,t) = \int [w(x,y)P(y) - w(y,x)P(x)] \, dy$$

(11)

with

$$w(x,y) = \frac{1}{\Delta x} \mu(y) \delta(y - x + \Delta x) + \gamma(y) \delta(x).$$

(12)

Next we take the $\Delta x \to 0$ limit leading to

$$\frac{\partial}{\partial t} P(x,t) = - \frac{\partial}{\partial x} (\mu(x)P(x,t)) - \gamma(x)P(x,t) + \langle \gamma \rangle_P \delta(x).$$

(13)

Please note that this is an integro-differential equation containing

$$\langle \gamma \rangle_P = \int \gamma(y)P(y,t) \, dy.$$  

(14)

Equation (13) describes a flow with general velocity field, $\mu(x)$, a loss rate $\gamma(x)$ and a feeding at $x = 0$. From now on we restrict the discussion to $x > 0$, and all effects stemming from the singular term $\langle \gamma \rangle_P \delta(x)$ are treated by enforcing the normalization condition.

The stationary distribution in the continuous model satisfies

$$\frac{d}{dx} (\mu(x)Q(x)) = -\gamma(x) Q(x),$$

(15)

revealing the solution

$$Q(x) = \frac{K}{\mu(x)} e^{-\int \frac{\gamma(x)}{\mu(x)} \, dx}.$$  

(16)

The constant, $K$, is specified by the normalization $\int_{0}^{\infty} Q(x) \, dx = 1$.

We note that this form can also directly be obtained from the discrete solution, eq. (8), when it is written in the alternative form

$$Q_n = Q_0 \frac{\mu_0}{\mu_n} e^{\sum_{j=1}^{n} \ln \left(1 + \frac{\gamma_j}{\sigma} \right)}.$$  

(17)

The approximation, $\gamma_j/\mu_j = \gamma(x) \Delta x/\mu(x) \ll 1$, defines the continuous limit and one arrives at

$$Q_n \approx Q_0 \frac{\mu_0}{\mu_n} e^{-\sum_{j=1}^{n} \frac{\gamma_j}{\mu_j} \Delta x},$$

(18)

to an obvious analog of eq. (16) with $K = \mu(0)Q(0)\Delta x$.

The inner-chain growth rate, $\mu(x)$, can be reconstructed from the known stationary distribution, $Q(x)$, and loss rate, $\gamma(x)$:

$$\mu(x) = \frac{1}{Q(x)} \int_{x}^{\infty} \gamma(u)Q(u) \, du.$$  

(19)

The validity of this formula is tested by applying a derivation with respect to $x$ and eq. (15). For the exponential distribution, $Q(x) \sim e^{-x/T}$, and constant $\gamma(x) = \gamma$ we obtain

$$\mu^{\text{exp}}(x) = \gamma T.$$  

(20)

The temperature-like parameter in the exponential distribution, $T$, becomes a factor between two elementary rates $\gamma$ and $\mu$. In a physical picture $\gamma$ describes dissipation, $\mu(x)$ random advances towards larger $x$ values.

### III. PARTICULAR RATES AND DISTRIBUTIONS

In the followings we discuss the simplest choices for the involved rates. First we keep the loss rate a positive constant, $\gamma_n = \gamma > 0$, and vary the growth rate, $\mu_n$. This is relevant for a wide class of distributions considered in statistics. For a constant $\mu_j = \sigma$ we obtain the geometrical distribution,

$$Q_n = \frac{\gamma}{\sigma} (1 + \gamma/\sigma)^{-1-n},$$  

(21)

shortly $Q_n = (1-q)q^n$, with $q = \sigma/(\sigma + \gamma)$. This is also called the exponential, or Boltzmann–Gibbs distribution in the form $Q_n = e^{-\beta n}/Z$ with $Z = 1 + \sigma/\gamma$ and $\beta = \ln(1 + \gamma/\sigma) > 0$.

For fast growing systems, like networks, citations or energetic hadronization, the most prevalent is the next simplest case, $\mu_j = \sigma(j + b)$, describing a growth rate with thresholded linear preference. Often $b = 1$ is taken when investigating the evolution of degree distribution in networks [19]. Eq. (8) delivers

$$Q_n = \frac{\gamma}{\sigma b + \gamma} \frac{(b)_n}{(b + 1 + \gamma/\sigma)_n}$$  

(22)

with the Pochhammer symbol:

$$(b)_n = b \cdot (b + 1) \cdot \ldots \cdot (b + n - 1) = \Gamma(b + n)/\Gamma(b).$$  

(23)

The Waring distribution [20–22] in eq. (22) has a power-law tail for large $n$,

$$\lim_{n \to \infty} Q_n \propto n^{-1-\gamma/\sigma}.$$  

(24)

This behavior is based on the leading order behavior of
Gamma functions for large arguments:

\[
\lim_{n \to \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1. \tag{25}
\]

Our result eq.\(^\text{22}\) coincides with eq.\(^7\) in Ref.\(^\text{19}\) at \(b = 1\). The asymptotic power is steeper than minus one for positive rate factors \(\gamma\) and \(\sigma\), but it can be anything, depending on the ratio of the universal driving rate \(\gamma\) and the preference scale of the individual growth rates, \(\sigma = \mu_n - \mu_{n-1}\). For \(\gamma \to 0^+\) the stationary distribution tail \(^\text{24}\) leads to Zipf’s law \(^\text{23}\).

Now we analyze these particular growth rates by a constant loss rate, \(\gamma(x) = \gamma\) in the continuous model. We expect the same asymptotic behavior for the tail of the distribution. For a constant growth rate, \(\mu(x) = \sigma\), the stationary PDF becomes again the exponential distribution,

\[
Q(x) = \frac{\gamma}{\sigma} e^{-\gamma/\sigma} x. \tag{26}
\]

This is the \(\gamma \ll \sigma\) limit of the result in the discrete case, eq.\(^\text{21}\). For a linearly preferential rate, \(\mu(x) = \sigma \cdot (x+b)\), we obtain a cut power-law in the form of the Tsallis–Pareto distribution \(^\text{23–27}\)

\[
Q(x) = \frac{\gamma}{b \sigma} \left(1 + \frac{x}{b}\right)^{-1-\gamma/\sigma}. \tag{27}
\]

Beyond these reassuring results a further question arises: what is the stationary distribution for weaker or stronger than linear preferences in the attachment probability rate \(^\text{28}\) \(^\text{29}\)? By assuming a general power, \(\mu(x) = \sigma (x+b)^a\), one obtains the stretched exponential distribution,

\[
Q(x) = \frac{\gamma}{\sigma (x+b)^a} e^{-\sigma (x+b)^{1-a}} \tag{28}
\]

for \(a < 1\) with \(\alpha = \gamma/[(\sigma - a)]\). For \(a > 1\) it delivers \(Q(x) \sim \gamma/\mu(x)\) tail. Eq.\(^\text{28}\) represents also a 3-parameter Weibull distribution, with \(\alpha = 1 - k, b = -\theta\) and \(\gamma/\sigma = k \lambda^{-k}\) \(^\text{30}\). On the other hand for an exponential preference rate, \(\mu(x) = \sigma e^{\alpha x}\), one obtains the Gompertz distribution \(^\text{31}\):

\[
Q(x) = \frac{\gamma}{1-e^{-\gamma/\sigma}} e^{-\alpha x} e^{-\gamma/\sigma} (1-e^{-\alpha x}). \tag{29}
\]

We note by passing that the particular form for the stationary distribution, \(Q(x)\) in eq.\(^\text{16}\) with constant \(\gamma\), is term by term compatible with the usual notions used in survival analysis in demography, finance and insurance statistics\(^\text{32}\); the PDF (probability density function) has in this case the form

\[
Q(x) = h(x) e^{-H(x)} \tag{30}
\]

with \(h(x)\) being the hazard rate and \(H(x) = \int_0^x h(t) dt\) the cumulative hazard. The factor \(R(x) = e^{-H(x)}\) is called survival rate. The growth rate inside the chain is simply related to the hazard rate: \(\mu(x) = \gamma/h(x)\). This is again a special case for the fluctuation–dissipation correspondence summarized in eq.\(^\text{19}\). The same relation has been called truncated expectation value theorem in Refs.\(^\text{33}\) \(^\text{34}\). A similar result has been derived by generalizing the thermodynamical fluctuation–dissipation relation between the diffusion and damping coefficients for a general Fokker–Planck equation stemming from a particularly colored, i.e. energy dependent, multiplicative noise Langevin equation \(^\text{35–36}\) and eq.(5.46) in \(^\text{37}\).

Finally we mention two important examples, frequently encountered in complex system applications, which do not fit in the above scheme. We consider loss rates, \(\gamma_n\), which can be negative for some low \(n\). Such a mechanism has been suggested a.o. in Ref.\(^\text{38}\) for describing the multiplicity distribution of hadrons in high-energy collision events. The linear rates

\[
\gamma_n = \sigma (n-kf), \quad \mu_n = \sigma f(n+k) \tag{31}
\]

will lead to a negative binomial stationary distribution:

\[
Q_n = \frac{(f \sigma)^n \prod_{i=0}^{n-1} (j+k)}{(\sigma (1+f))^{n+1} \prod_{i=1}^{n}} = \frac{(n+k-1)}{n} f^n (1+f)^{-n-k}. \tag{32}
\]

We note that in this case \(\langle \gamma \rangle_Q = 0\). In order to achieve a normalized stationary distribution obviously \(\gamma_n + \mu_n > 0\) for all \(n\).

A similar arrangement of the rates in the continuous model,

\[
\gamma(x) = \sigma (ax - c), \quad \mu(x) = \sigma x, \tag{33}
\]

leads to the two-parameter gamma distribution,

\[
Q(x) = \frac{K}{\sigma x} e^{-f(x-c)/\sigma} dx = \frac{a^c}{\Gamma(c)} x^{c-1} e^{-ax}. \tag{34}
\]

This stationary distribution emerges as a result of a pure (unthresholded) linear preference in the growth rate and a linear, but not overall positive loss rate to the environment. The negative values of \(\gamma(x)\) actually mean a feeding from the environment (Fig.2).

Such a gamma distribution fits income data very well \(^\text{39}\). We risk the conclusion that in the background of such processes, beyond the linear preference rate, \(\mu(x) = \sigma x\) (often cited as the Matthias principle in market economies), a taxation and a social welfare redistribution system acts.

### IV. EVOLUTION OF ENTROPIC DISTANCE

The entropy – probability connection is also interpreted as a measure of a distance to the minimal informa-
It is very common to deal with entropic distances defined via a function of the ratio of the respective probabilities only, \( s(\xi_n) \) with \( \xi_n = P_n/Q_n \). Then, from the property of zero distance from itself one concludes that \( s(1) = 0 \), and it is different from zero if there is an index \( n \) such that \( \xi_n \neq 1 \).

The change of the entropic distance is governed by its definition and the evolution equation for the distribution. The entropic distance of an actual, time-dependent distribution, \( P_n(t) \), to the stationary distribution, \( Q_n \) has the trace form \( 40 \):

\[
\rho = \sum_n s(\xi_n) Q_n. \tag{37}
\]

For all concave \( s(\xi) \) functions the following Jensen inequality applies:

\[
\rho \geq s \left( \sum_n \xi_n Q_n \right) = s \left( \sum_n P_n \right) = s(1) = 0. \tag{38}
\]

The time derivative of the entropic distance is given by

\[
\dot{\rho} = \sum_n s'(\xi_n) \dot{\xi}_n Q_n = \sum_n s'(\xi_n) \dot{P}_n. \tag{39}
\]

Now we utilize eq.(4) and obtain

\[
\dot{\rho} = \sum_{n,m} s'(\xi_n) \left[ w_{n\rightarrow m} Q_m \xi_m - w_{n\rightarrow m} Q_n \xi_n \right]. \tag{40}
\]

As a first step we write \( \xi_m = \xi_n + (\xi_m - \xi_n) \) and use the property

\[
0 = \sum_m \left[ w_{n\rightarrow m} Q_m - w_{m\rightarrow n} Q_n \right] \tag{41}
\]

for the stationary state. The above formula transforms to

\[
\dot{\rho} = \sum_{n,m} s'(\xi_n) (\xi_m - \xi_n) w_{n\rightarrow m} Q_m. \tag{42}
\]

In the second step we use the remainder theorem for Taylor series in its Lagrange form:

\[
s(\xi_m) = s(\xi_n) + (\xi_m - \xi_n)s'(\xi_n) + \frac{1}{2}(\xi_m - \xi_n)^2s''(c_{nm}) \tag{43}
\]

with the internal point \( c_{nm} \) lying between \( \xi_n \) and \( \xi_m \). Expressing the first order term in eq.(43), eq.(42) becomes

\[
\dot{\rho} + \sum_{m,n} \left[ s(\xi_m) - s(\xi_n) \right] w_{m\rightarrow n} Q_m - \frac{1}{2} \sum_{n,m} (\xi_m - \xi_n)^2 s''(c_{nm}) w_{n\rightarrow m} Q_m. \tag{44}
\]

Here the first sum on the right hand side vanishes due to the stationarity \( 41 \). This can be seen by exchanging the summation indices \( m \) and \( n \) in the first part, leading to

\[
\sum_n s(\xi_n) \sum_m \left[ w_{m\rightarrow n} Q_n - w_{n\rightarrow m} Q_m \right] = 0. \tag{45}
\]
For positive transition rates, \( w_{n\rightarrow m} > 0 \), the remainder term is always negative for concave, \( s''(\xi) > 0 \) functions.

In the special case of the avalanche dynamics with loss, \( w_{n\rightarrow m} = \mu_m \delta_{m,n+1} + \gamma_m \delta_{m,0} \), we obtain

\[
\dot{\rho} = -\frac{1}{2} \sum_{n} (\xi_n - \xi_{n-1})^2 s''(c_{n,n-1}) \mu_{n-1} Q_{n-1} - \frac{1}{2} \sum_{n} (\xi_n - \xi_{0})^2 s''(c_{n,0}) \gamma_n Q_n. \tag{46}
\]

For positive rates \( \gamma_n \) and \( \mu_n \) therefore \( \dot{\rho} < 0 \) unless the equilibrium state is achieved where \( \xi_n = 1 \) for all \( n \).

Finally let us briefly discuss cases when some \( \gamma_n \) can be negative. We encountered this for processes leading to negative binomial or gamma distributions. The remainder result (46) in such a case does not guarantee a steady approach towards the stationary distribution in terms of a general entropic distance. Henceforth further investigations are necessary.

V. CONCLUSION

In the present work we have proposed a unified mathematical framework based on a master equation approach to complex systems governed by random dynamics. In particular we have focused on transition rates which do not lead to a detailed balance. A wide variety of stationary distributions known from complex network research, particle physics, scientometrics, econophysics, biology and demography are successfully reproduced.

This view is able to clarify why only the linear preference rate leads to a power-law tailed degree distribution in random networks as well as to a Pareto-type distribution of wealth when the preference expressed by ”the rich gets richer” principle is linear. Similarly, the distribution of scientific citations, known to be power-law tailed, has been explained earlier on the basis of such an evolution equation [12]. The exponential (geometrical) distribution is obtained for constant rates and the power-law tailed Waring (in the continuum limit Tsallis–Pareto) distribution for a linear preference growth rate. The method outlined in this paper is able to deliver further well-known and frequently used distributions, such as the Weibull or the Gompertz distribution or the streched exponential.

Beyond the above mentioned practical application possibilities we have established connections to the fundamental fluctuation–dissipation relation central in statistical physics. In the simplified version with a constant loss rate, \( \gamma_n = \gamma \), the stationary PDF, \( Q(x) \), are proved to be related to quantities familiar from general statistics: the necessary growth rate is reciprocal to the hazard rate, \( \mu(x) = \gamma/h(x) \). The correspondence between this hazard rate and the cumulated hazard was generalized to a ”fluctuation–dissipation” type relation between the growth rate, \( \mu(x) \), and the loss rate \( \gamma(x) \) in eq.[19]. A similar general relation was derived for the discrete version in eq. [6]. The specific case \( m = 0 \) gives the key to reconstruct the first attachment rate \( \mu_0 \) from observing \( Q_n \) and measuring \( \langle \gamma \rangle_Q \equiv \mu_0 = \langle \gamma \rangle_Q / Q_0 - \gamma_0 \).

Finally, while seeking answer to the question which entropy formula could be the optimal one for such unbalanced growth processes in random systems, we proved that any entropic distance based on a general concave function of the probability ratio, \( s(\xi) \), will decrease to zero for \( \gamma(x) > 0 \).

Generalizing further the dynamics for \( \gamma_n \) containing both positive and negative elements we have discussed two models. First with \( \gamma_n = \sigma(n - fk) \) and \( \mu_n = \sigma f(n + k) \) a model for high energy hadron production, leading to a negative binomial stationary distribution was evoked. Second, with \( \gamma(x) = \sigma(ax - b) \) and \( \mu(x) = \sigma x \) a continuous model for the income distribution was recited. This model assumes a constant percentage taxation and social welfare amendments, leading to a gamma distribution.

The unified mathematical treatment outlined in this paper should be a primary tool in understanding intriguing universality classes reported in complex systems. Important questions are left open for further research: what are the precise conditions for entropy growth in cases involving partially negative \( \gamma(x) \) rates (while \( \gamma(x) + \mu(x) > 0 \) is still satisfied); what are the minimal conditions for gaining a stationary distribution in unbalanced random processes; or how the transient dynamics towards the stationary state is displayed with time.

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Here we prove eq. (10). We define the summed expression of product chain as

$$S_0 = \sum_{n=0}^{\infty} r^n \prod_{i=0}^{n} \frac{1}{1 + r_i}.$$  \hspace{1cm} (47)

The first few terms are:

$$S_0 = \frac{r_0}{1 + r_0} + \frac{1}{1 + r_0} \frac{r_1}{1 + r_1} + \frac{1}{1 + r_0} \frac{1}{1 + r_1} \frac{r_2}{1 + r_2} + \ldots \hspace{1cm} (48)$$

By rearranging the sum starting at the second term,

$$S_0 = \frac{r_0}{1 + r_0} + \frac{1}{1 + r_0} \frac{r_1}{1 + r_1} + \frac{1}{1 + r_0} \frac{1}{1 + r_1} \frac{r_2}{1 + r_2} + \ldots,$$

we realize that

$$S_0 = \frac{r_0}{1 + r_0} + \frac{1}{1 + r_0} S_1$$ \hspace{1cm} (50)

with an obvious notation, $S_1$, for the same infinite sum starting with terms containing $r_1$. After a linear rearrangement it is convincing that this relation,

$$(S_0 - S_1) = r_0 (1 - S_0),$$ \hspace{1cm} (51)

holds for an arbitrary $r_0$ if and only if $S_1 = S_0 = 1$. The same proof is valid for starting at any $m$-th element. $S_0 = 1$ proves the original statement.

**APPENDIX**

The same proof is valid for starting at any $r$. By rearranging the sum starting at the second term,

$$S_0 = \frac{r_0}{1 + r_0} + \frac{1}{1 + r_0} \frac{r_1}{1 + r_1} + \frac{1}{1 + r_0} \frac{1}{1 + r_1} \frac{r_2}{1 + r_2} + \ldots$$

By rearranging the sum starting at the second term,

$$S_0 = \frac{r_0}{1 + r_0} + \frac{1}{1 + r_0} \frac{r_1}{1 + r_1} + \frac{1}{1 + r_0} \frac{1}{1 + r_1} \frac{r_2}{1 + r_2} + \ldots,$$

we realize that

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with an obvious notation, $S_1$, for the same infinite sum starting with terms containing $r_1$. After a linear rearrangement it is convincing that this relation,

$$(S_0 - S_1) = r_0 (1 - S_0),$$ \hspace{1cm} (51)

holds for an arbitrary $r_0$ if and only if $S_1 = S_0 = 1$. The same proof is valid for starting at any $m$-th element. $S_0 = 1$ proves the original statement.