The Lost Proof of Fermat’s Last Theorem

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PREMISE

This work contains two papers: the first entitled "Some Euler double equations equivalent to Fermat’s last theorem" presents a marvellous proof through the so-called discordant forms of appropriate Euler’s double equations, which could have entered in a not very narrow margin, i.e. in only a few pages (less than 14).

The second instead, entitled "Some Diophantus-Fermat double equations equivalent to Frey’s elliptic curve" provides the possible proof, which Fermat has not published in detail, but which uses the characteristic of all right-angled triangles with sides equal to whole numbers, or the famous Pythagorean identity (the pages are less than 8).

After these two works, a session is provided which clarifies the direct and extraordinary connection of the two elementary proofs and it is necessary if you want to understand how two different proofs of Fermat’s Last Theorem are possible.

Regarding the first paper, a method is used that drastically simplifies Wiles’ theory, a theory that has received much honors from the entire mathematical community.

More precisely, through the aid of a Diophantine equation of second degree solved at first not directly, but as a consequence of the resolution of the double Euler equations that originated it and finally in a direct way, I was able to obtain the following result: the intersection of the infinite solutions of Euler’s double equations gives rise to an empty set and this only by exploiting a well-known Legendre Theorem, which concerns the properties of all the Diophantine equations of the second degree, homogeneous and ternary.

I report that the Journal of Analysis and Number Theory has made in part (5 pages); this paper available online at

http://www.naturalspublishing.com/ContIss.asp?IssID=1779

The author is looking for editors of magazines specialized in Number Theory and indexed by Scopus (or rather a journal that is reviewed in MathSciNet or Zentralblatt), able to review and accept one of the two papers indicated at the beginning or even both, despite colleagues and experienced referees are indeed reluctant to volunteer their time and skills for tracing out what such papers really contribute.

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DIGRESSION

Fermat wrote that his proof would not fit into the margin of his copy of Arithmetica, and Wiles's 100 pages of dense mathematics certainly fulfils this criterion, but surely the Frenchman did not invent modular forms, the Taniyama-Shimura conjecture, Galois Groups and the Kolyvagin-Flach method centuries before anyone else.

If Fermat did not have Wiles's proof then what did he have?

Mathematicians are divided into two camps.

The hard-headed sceptics believe that Fermat’s Last Theorem was the result of a rare moment of weakness by the 17th-century genius.

They claim that although Fermat wrote, "I have discovered a truly marvellous proof", he had in fact found only a flawed proof.

Other mathematicians, the romantic optimists, believe that Fermat may have had a genuine proof.

Whatever this proof might have been, it would have been based on 17th-century techniques, and would have involved an argument so cunning that it has eluded everybody.

Indeed, there are plenty of mathematicians who believe that they can still achieve fame and glory by discovering Fermat’s original proof.

In my case it is pure passion for the Mathematics and the desire to do justice to Fermat and his genius !!
I) Some Euler double equations equivalent to Fermat’s last theorem.

abstract

In this work we illustrate that a possible proof of Fermat’s Last Theorem derives from an appropriate use of the concordant forms of Euler and from an equivalent ternary quadratic homogeneous Diophantine equation able to accommodate a solution of Fermat’s extraordinary equation. Following a similar and almost identical approach to that of A. Wiles, I tried to translate the link between Euler’s double equations (concordant / discordant forms) and Fermat’s Last Theorem into a possible proof of the Fermat Theorem. More precisely, through the aid of a Diophantine equation of second degree, homogeneous and ternary, solved not directly, but as a consequence of the resolution of the double Euler equations that originated it, I was able to obtain the following result: the intersection of the infinite solutions of Euler’s double equations gives rise to an empty set and this only by exploiting a well-known Legendre Theorem, which concerns the properties of all the Diophantine equations of the second degree, homogeneous and ternary. The impossibility of solving the second degree Diophantine equation thus obtained is certainly possible also through methods known and discovered by Fermat.

1. Introduction

Fermat’s last theorem affirms: If \( n \) is an integer, greater than 2, there are not any positive integers \( X, Y, Z \), so that it can be valid:

\[
X^n + Y^n = Z^n.
\]

Fermat himself proved it for \( n=4 \) ([7], pp. 108-112),([6], II, Chap. XIII, § 202–209); it is consequent its validity also for \( n \) as a multiple of 4, because, if \( n \) is equal \( 4p \), for some positive integer \( p \),

\[
X^n + Y^n = Z^n \Rightarrow (X^p)^4 + (Y^p)^4 = (Z^p)^4
\]

and this is impossible.

In the same way if we succeed in proving the theorem for a certain \( k \)--exponent, then it is valid for all the multiples of \( k \).

As every positive integer greater than 2 is divisible either by a prime odd number (that is different from 2), or by 4, it will be then sufficient to prove the theorem for all those cases in which the exponent is a prime odd number ([9], pp. 203-207).

In this proof we will discuss all those cases in which the exponent \( n \) is an odd number > 1 and, from now onwards, we will indicate the Fermat Last Theorem with the acronym F.L.T..

2. Indeterminate Analysis of Second Degree

Our goal is to take care of the resolution, into integers, of quadratic equation with integer coefficients, depending on \( n \) unknowns ([1], Cap. I, pp. 60-69).

We will develop our considerations on the equation in three unknowns:

\[
F(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0
\]

warning that, all what we will say, extends immediately to the case of \( n \) unknowns.

Since the (1) is an equation homogeneous, if \((A, B, C)\) are the solutions also \((mA, mB, mC)\) are solutions.
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Therefore we deem identical two solutions such as \((A, B, C)\) and \((mA, mB, mC)\).

Such assumption, will narrow the search to the only primitive solutions of Eq.(1), that is, to those in which \(X, Y\) and \(Z\) are pairwise relatively prime.

Let \((x, y, z)\) be a solution in integers of the Eq.(1) and then \(F(x, y, z) = 0\) and we put:

\[
X = \rho \cdot x + \xi, Y = \rho \cdot y + \eta, Z = \rho \cdot z + \zeta
\]

(2)

where \(\xi, \eta, \zeta\) are arbitrary integer constants and \(\rho\) an unknown to be determined, so that Eqs.(2) provide an integer solution for Eq.(1).

It must be:

\[
F(X, Y, Z) = \rho^2 \left[ ax^2 + by^2 + cz^2 + dxy + exz + fyz \right]
\]

\[
+ \rho \left[ 2a\xi \cdot x + 2b\eta \cdot y + 2c\zeta \cdot z + d(\xi \cdot y + \eta \cdot x) + e(\xi \cdot z + \zeta \cdot x) + f(\eta \cdot z + \zeta \cdot y) \right]
\]

\[
+ \left[ a\xi\xi + b\eta\eta + c\zeta\zeta + d\xi\eta + e\xi\zeta + f\eta\zeta \right] = 0.
\]

But the coefficient of \(\rho^2\), equal to \(F(x, y, z)\) , is null and the known term is \(F(\xi, \eta, \zeta)\) , so, set equal to \(M\) (with \(M \neq 0\) due to the arbitrary of \(\xi, \eta, \zeta\)), the coefficient \(\rho\) of the above equation is equal to \(\rho = -\frac{F(\xi, \eta, \zeta)}{M}\).

Consequently, if it is known an integer solution of Eq.(1), we have infinite other, by putting in Eqs.(2), in place of \(\rho\), the value now found; then, without the divisor \(M\), we have:

\[
X = \xi \cdot M - xF(\xi, \eta, \zeta) ; Y = \eta \cdot M - yF(\xi, \eta, \zeta) ;
\]

(3)

\[
Z = \zeta \cdot M - zF(\xi, \eta, \zeta).
\]

These are the general solutions of Eq.(1).

To prove it, we will show, by appropriately selecting \(\xi, \eta, \zeta\), the previous solutions provide a solution of Eq.(1), given arbitrarily.

Let this \((A, B, C)\),it is meanwhile \(F(A, B, C) = 0\); if now, in Eqs.(3) we write \(\xi = A, \eta = B, \zeta = C\),we have the solution: \(X = AM; Y = BM; Z = CM\), that, without the factor \(M\), it is identified with the one already provided.

In conclusion:

**Theorem 2.1:** Let \((x, y, z)\) be an integer solution of Eq.(1). All its integer solutions are given by Eqs.(3), without the integer divisor \(M\).

Now we solve the equation \(F(X, Y, Z) = X^2 + aY^2 - Z^2 = 0\) in integer numbers.

Keeping in mind that this equation is homogeneous we know that we can consider identical the two solutions, as \((1,0,1)\) and \((m,0, m)\).
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Let's consider, at this point, the trivial solution \((1,0,1)\) and we will have: \(M = 2(\xi - \zeta)\); 
\[ F(\xi, \eta, \zeta) = \xi^2 + a\eta^2 - \zeta^2 \]
for which all the solutions, keeping in mind the Eqs.(3), are given by the relations:

\[
X = 2\xi (\xi - \zeta) - \xi^2 + a\eta^2 + \zeta^2 = (\xi - \zeta)^2 - a\eta^2 \quad ; \quad Y = 2\eta (\xi - \zeta) \\
Z = 2\zeta (\xi - \zeta) - \xi^2 - a\eta^2 + \zeta^2 = - (\xi - \zeta)^2 - a\eta^2.
\]

Therefore assumed \((\xi - \zeta) = \theta\) and observed that from a solution \((x, y, z)\) we get others changing sign to one, or two, or all \((x, y, z)\), we have:

\[
X = \theta^2 - a\eta^2 \quad ; \quad Y = 2\theta \eta \quad ; \quad Z = \theta^2 + a\eta^2
\]

which provide us with all the primitive integer solutions of quadratic equation, without an appropriate integer divisor \(M\).

In general we have that all integer solutions for the equation \(X^2 + aY^2 = Z^2\) are:

\[
X = k (\theta^2 - a\eta^2) \quad ; \quad Y = k (2\theta \eta) \quad ; \quad Z = k (\theta^2 + a\eta^2).
\]

where \(\theta, \eta\) are natural numbers and \(k\) a rational proportionality factor (see also [3], kap. V, §29, pp. 39-44).

3. On Homogeneous Ternary Quadratic Diophantine Equations \(aX^2 + bY^2 - cZ^2 = 0\)

**Theorem 3.1**: Let \(x^n + y^n = z^n\), with \((x, y) = 1\) and \(n \geq 3\) has a solution, then there exists an equation \(ax^2 + by^2 = cz^2\), where \(a, b, c\) are relatively prime and reduced to the minimum terms, whose a solution could be reduced to a solution of Fermat’s equation.

**Proof.**

Let \(X_1, Y_1, Z_1\) be three whole numbers pairwise relatively prime such as to satisfy the Fermat equation \(x^n + y^n = z^n\).

Then the following homogeneous ternary quadratic Diophantine equation, with \((V, T, P) = 1\) exists:

\[
X_1^n V^2 + Y_1^n T^2 = Z_1^n P^2.
\]

We observe that with the following particular nontrivial solutions:

\[ V = 1, T = 1 \text{ and } P = 1 \text{ or } V = T = P \text{ in Eq.(5) we obtain the fundamental Hypothesis (Reductio ad Absurdum) of the F.L.T.} : \]

\[ X_1^n + Y_1^n = Z_1^n. \]

Now by the evident solutions, indicated above, we can derive an infinite number of solutions of Eq.(5).

Let's remember that for Legendre’s Theorem if a ternary quadratic homogeneous Diophantine equation (assuming \(a, b\) and \(c\) are fixed) has an integral solution, then the number of possible solutions is infinite.
Having said this, it is possible to transform the previous Diophantine equation (5) into the following equivalent Diophantine equation, with \((V', T', P') = 1\):

\[
X_1 V'^2 + Y_1 T'^2 = Z_1 P'^2. \tag{6}
\]

It is sufficient to assume \(V' = X_1^k V, T' = Y_1^k T, P' = Z_1^k P\) where \(k = \frac{n-1}{2}\) and \(n > 1\) odd number.

Using the "fundamental theorem of Arithmetic" we can represent ([13], Theorem 19, p. 31):

\[
X_1 = X_0 U^2_1, \quad Y_1 = Y_0 U^2_2, \quad Z_1 = Z_0 U^2_3.
\]

In this case is possible to transform the previous Diophantine equation (6) into the following equivalent Diophantine equation with the relative coefficients reduced to the minimum terms:

\[
X_0 V''^2 + Y_0 T''^2 = Z_0 P''^2.
\]

In fact just assume \(V'' = U_1 V', T'' = U_2 T', P'' = U_3 P'\)

We observe that \(X_0, Y_0, Z_0\) are pairwise relatively prime and square-free numbers.

The proof ends here by properly verifying also the nature of exponent \(n\).

4. From the Concordant Forms of Euler to Fermat’s Last Theorem

Let \(m, n \in \mathbb{Z} / \{0\}\) be integers with \(m \neq n\). Following Euler (see [5]), the quadratic forms \(X^2 + mY^2\) and \(X^2 + nY^2\) (or the numbers \(m\) and \(n\) themselves) are called **concordant** if there are integers \((X, Y, Z, T)\) with \(Y \neq 0\) such that:

\[
X^2 + mY^2 = Z^2, \quad X^2 + nY^2 = T^2. \tag{7}
\]

In 1780 Euler seeks criteria for the treatment of the double equations (7) and his interest and our own turns to proofs of impossibility for the cases \(m=1, n=3\) or 4 and others equivalent to these two ([15], Chap. III, §XVI, pp. 253-254).

In practice, Euler called \(X^2 + mY^2\) and \(X^2 + nY^2\) **concordant** forms if they can both be made squares by choice of integers \(X, Y\) each not zero; otherwise, **discordant** forms. At this stage, let us introduce the following Euler double equations:

\[
P^2 + Y_1^2 Q^2 = V^2, \quad P^2 - X_1^2 Q^2 = T^2 \tag{8}
\]

with \(X_1^n + Y_1^n = Z_1^n\) and \(n > 1\) odd number.

By multiplying the first two equations (8) together, and multiplying by \(\frac{P^2}{Q^2}\), with \(P \neq 0\) and \(Q \neq 0\), we get([8]):

\[
\frac{p^2 V^2 T^2}{Q^6} = \frac{P^6}{Q^6} + (Y_1^n - X_1^n) \frac{P^4}{Q^4} - \frac{P^2}{Q^2} Y_1^n. \tag{9}
\]

If we then replace \(\frac{p^2}{Q^2}\) by \(X\) and also \(\frac{pV}{Q}\) by \(Y\) we find that

\[
Y^2 = X (X - X_1^n) (X + Y_1^n).
\]

This is known as Frey Elliptic curve ([4], pp. 154–156).
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In Mathematics, a Frey curve or Frey–Hellegouarch curve is the elliptic curve:

\[ Y^2 = X (X - X_1^n) (X + Y_1^n) \]  

(10)

or, equivalently:

\[ Y^2 = X \left[ X^2 + X (Y_1^n - X_1^n) - X_1^n Y_1^n \right] \]  

(11)

associated with a (hypothetical) solution of Fermat’s equation: \( X_1^n + Y_1^n = Z_1^n \).

In fact, the discriminant

\[ \Delta = \sqrt{(Y_1^n - X_1^n)^2 + 4X_1^n Y_1^n} = X_1^n + Y_1^n = Z_1^n, \]

that determines the existence of the polynomial

\[ (X - X_1^n) (X + Y_1^n) = X^2 + X (Y_1^n - X_1^n) - Y_1^n X_1^n \]

is a perfect power of order \( n \).

Frey suggested, in 1985, that the existence of a non-trivial solution to \( X^n + Y^n = Z^n \) would imply the existence of a non-modular elliptic curve, viz. \( Y^2 = X(X - X^n)(X + Y^n) \).

This suggestion was proved by Ribet in 1986.

This curve is semi-stable and in 1993 Wiles announced a proof (subsequently found to need another key ingredient, furnished by Wiles and Taylor) that every semi-stable elliptic curve is modular, the semi-stable case of the Taniyama-Shimura-Weil conjecture ([16] and [14]).

Hence no non-trivial \( X^n + Y^n = Z^n \) can exist.

Basically thanks to the spectacular work of A. Wiles, today we know that Frey’s elliptic curve not exist and from this derives indirectly, as an absurd, the F.L.T..

Now, multiplying the first two equations (8) respectively by \( X_1^n \) and by \( Y_1^n \) and at end adding together we get the following homogeneous ternary quadratic equation (see Section 3):

\[ X_1^n V^2 + Y_1^n T^2 = Z_1^n P^2 \]  

(12)

with the identity \( X_1^n + Y_1^n = Z_1^n \) and \( n > 1 \) odd number.

So, we can also enunciate the following theorem:

**Theorem 4.1:** Fermat’s Last Theorem is true only if the homogeneous ternary quadratic Diophantine equation (12) does not exist.

Nobody prevents us from assuming the evident solution \( V = T = P = 1 \) or \( V = T = 0 \) in the equation (12) and with this we obtain the solution of Fermat equation: \( X_1^n + Y_1^n = Z_1^n \).

Now from the Euler double equations (8) by subtracting, we have:

\[ V^2 - T^2 = Z_1^n Q^2. \]

This equation together with equation (12) gives rise to a system perfectly equivalent to Euler’s double equations (8) (see section 6).

We have also with \( V = T = 1 \) or \( V = T = 0 \):

\[ V^2 - T^2 = Z_1^n Q^2 = 0. \]

By definition, in Euler’s concordant forms, \( Q \) is absolutely non-zero integer.
Further verification of these conclusions is also possible in this way. Let us introduce the following Euler double equations:

\[ P'^2 + Y_1^n Q^2 = V^2, \quad P''^2 - X_1^n Q^2 = T''^2 \]  

(13)

with \( X_1^n + Y_1^n = Z_1^n \) and \( n > 1 \) odd number or

\[ P'^2 + Y_1^n Q^2 = V^2, \quad P''^2 - X_1^n Q^2 = T''^2 \]  

(14)

with \( X_1^n + Y_1^n = Z_1^n \) and \( n > 1 \) odd number.

From Eqs.(4) we have the following solutions of first Euler equation of Eqs.(13):

\[ P' = k \left( \theta^2 - Y_1^n \eta^2 \right), \quad Q = k \left( 2 \theta \eta \right), \quad V = k \left( \theta^2 + Y_1^n \eta^2 \right) \]  

(15)

and the following solutions of second Euler equation of Eqs.(13):

\[ P'' = k \left( \theta^2 + X_1^n \eta^2 \right), \quad Q = k \left( 2 \theta \eta \right), \quad T'' = k \left( \theta^2 - X_1^n \eta^2 \right) \]  

(16)

or the following solutions of second Euler equation of Eqs.(14):

\[ P''' = k' \left( \theta'^2 - X_1^n \eta'^2 \right), \quad Q' = k' \left( 2 \theta' \eta' \right), \quad T''' = k' \left( \theta'^2 + X_1^n \eta'^2 \right) \]  

(17)

Now assuming \( V = T = P \) in the equations (8) with \( Q \) non-zero integer or in the equivalent equation (12) we have the following result due to Eqs.(15) and Eqs.(16):

\[ P = P' = P'' \Rightarrow -Y_1^n = X_1^n \Rightarrow Z_1^n = 0 \quad \text{and} \quad V = T'' \Rightarrow Y_1^n = -X_1^n \Rightarrow Z_1^n = 0. \]

While, with Eqs.(15) and Eqs.(17), we have:

\[ P = P' = V \Rightarrow -Y_1^n = Y_1^n \Rightarrow Y_1^n = 0 \quad \text{and} \quad V = T''' \Rightarrow X_1^n = -X_1^n \Rightarrow X_1^n = 0 \]

and therefore still \( Z_1^n = 0 \).

In conclusion what has been described so far in relation to Theorem 4.1 obviously does not have a demonstrative value, but allows us to state the following equivalent theorem:

**Fundamental Theorem:** *Fermat’s Last Theorem is true if and only if is not possible a solution in integers of Eqs.(8) with \( Q \) non-zero integer, that is these are discordant forms.*

In practice, this means that if the system of quadratic Eqs.(8) admits only the trivial solutions \((m,0,\pm m, \pm m)\), that include also \((1,0,1,1)\), then the quadratic forms \( P^2 + Y_1^n Q^2 \) and \( P^2 - X_1^n Q^2 \) are a fortiori called *discordant*.

A complete and direct proof of this Theorem is formed in section 6.
5. The Nature of Euler’s Double Equations Through the Algebraic Geometry

In this section we will concentrate on the following Euler’s concordant/discordant forms Eqs(8):

\[ P^2 + Y^n Q^2 = V^2, \quad P^2 - X^n Q^2 = T^2 \]

with \( X^n_1 + Y^n_1 = Z^n_1 \) and \( n \geq 3 \).

In determining the nature of the Euler double equations and of an appropriate equivalent Diophantine system, we will make use of the description given by A. Weil ([15], Chap. II, App. IV, pp. 140–149) in order to provide some theoretical background to Fermat’s and Euler’s method of descent employed in the treatment of elliptic curves.

For simplicity we consider the case where the roots of a cubic \( \Gamma \) are rational integers \( \alpha, \beta \) and \( \gamma \). The cubic \( \Gamma \) is then given by

\[ y^2 = f(x) = (x - \alpha) (x - \beta) (x - \gamma). \quad (18) \]

Weil consider an oblique quartic \( \Omega(A, B, C) \) in the space \((u, v, w)\)

\[ Au^2 + \alpha = Bv^2 + \beta = Cw^2 + \gamma \quad (19) \]

with \( u, v, w \in \mathbb{Q} \) and the following mapping of \( \Omega \) in \( \Gamma \)

\[ x = Au^2 + \alpha, \quad y = \sqrt{ABCuvw} \quad (20) \]

where \( A \cdot B \cdot C \) has to be a square.

In practice Weil states that the determination of rational points of the curve \( \Gamma \) can be reduced to that of finding rational points of one or more appropriate quartics, such as (19), given a set of integers \( A, B, C \) (positive or negative), considered squarefree, that is, not divisible by any square greater than 1, and such that the product \( A \cdot B \cdot C \) is a square.

In homogeneous coordinates, \( \Omega(A, B, C) \) may be regarded as defined by the equation

\[ AU^2 + \alpha T^2 = BV^2 + \beta T^2 = CW^2 + \gamma T^2, \quad (21) \]

with integers \( U, V, W, T \) without a common divisor.

Subsequently, after affirming that Eq.(21) admits at least one solution, instead of defining \( \Omega = \Omega(A, B, C) \) through (19), Weil writes it through the equation of two quadrics in \( \mathbb{P}^3 \), that is:

\[ \Phi = \sum_{i,j=1}^{4} a_{ij} X_i Y_j \quad \text{and} \quad \Psi = \sum_{i,j=1}^{4} b_{ij} X_i Y_j, \] with the condition \( \Phi = \Psi = 0 \).

In detail, one has:

\[ \Phi(U, V, W, T) = \alpha (\beta - \gamma) \left( AU^2 + \alpha T^2 \right) + \beta (\gamma - \alpha) \left( BV^2 + \beta T^2 \right) + \gamma (\alpha - \beta) \left( CW^2 + \gamma T^2 \right) \]

\[ \Psi(U, V, W, T) = (\beta - \gamma) AU^2 + (\gamma - \alpha) BV^2 + (\alpha - \beta) CW^2 \]

where one has put \( \delta = (\beta - \gamma) (\gamma - \alpha) (\alpha - \beta) \).
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With this in mind, we consider the following assumptions

\[ A = 1, \quad \alpha = 0, \quad B = 1, \quad \beta = X^n_1, \quad C = 1, \quad \gamma = -Y^n_1. \]  

(22)

In this case Eq.(18) would be reduced to the Frey elliptic curve:

\[ Y^2 = f(X) = (X)(X - X^n_1)(X + Y^n_1). \]  

(23)

and the Euler double equations (8) with the following assumptions, in order: \( P = U, \quad T = W, \quad Q = T \) would be reduced to the oblique quartic \( \Omega (A, B, C) = \Omega (1, 1, 1) \):

\[ U^2 = V^2 - Y^n_1T^2 = W^2 + X^n_1T^2. \]  

(24)

The product \( ABC \) is, as required, a perfect square, and therefore it is certainly possible the application (19) of the quartic \( \Omega \) on cubic \( \Gamma \).

The expressions of the two quadrics in \( P^3 \) become

\[ \Phi (U, V, W, T) = -Y^n_1X^n_1V^2 + X^n_1Y^n_1W^2 + Z^n_1X^n_1Y^n_1T^2 \]

and

\[ \Psi (U, V, W, T) = -(Y^n_1 + X^n_1)U^2 + X^n_1V^2 + Y^n_1W^2 = -(Z^n_1)U^2 + X^n_1V^2 + Y^n_1W^2. \]

Finally, by \( \Phi = \Psi = 0 \), they are translated into

\[ (V^2 - W^2) = (Z^n_1)T^2 \]  

(25)

and

\[ X^n_1V^2 + Y^n_1W^2 = Z^n_1U^2. \]  

(26)

Now Eq.(25) and Eq.(26) with the following replacements:

\[ T \Rightarrow W, \quad Q \Rightarrow T, \quad P \Rightarrow U \]

are none other than the equations what we have described in the section 4, that is:

\[ (V^2 - T^2) = (Z^n_1)Q^2 \]

and

\[ X^n_1V^2 + Y^n_1T^2 = Z^n_1P^2. \]

This alternative procedure confirms the validity of the our conclusions: more precisely, I am referring to the fact that Euler’s double equations, as representatives of an evident oblique quartic of genus 1, can also be defined by means of a pair of equations of two quadrics in \( P^3 \), which establish uniquely that the following Diophantine systems are perfectly equivalent:

\[ \begin{align*}
  P^2 + Y^n_1Q^2 &= V^2 \\
  P^2 - X^n_1Q^2 &= T^2
\end{align*} \quad \begin{align*}
  X^n_1V^2 + Y^n_1T^2 &= Z^n_1P^2 \\
  Z^n_1Q^2 &= V^2 - T^2.
\end{align*} \]  

(27)
6. The determination of the parameter $Q$ in Euler’s double equations

Let us consider the first Diophantine equation of the second system (27):

$$X^n_1 V^2 + Y^n_1 T^2 = Z^n_1 P^2$$

and we apply Theorem 2.1.

Now we solve the equation $F(X, Y, Z) = aX^2 + bY^2 - cZ^2 = 0$.

Keeping in mind that this equation is homogeneous we known that we can consider identical the two solutions, as $(1, 1, 1)$ and $(m, m, m)$.

Let’s consider, at this point, the solutions $(1, 1, 1)$ and we will have:

$$M = 2(aξ + bη - cζ) ; \quad F(ξ, η, ζ) = aξ^2 + bη^2 - cζ^2$$

for which all the solutions, without the integer divisor $M$, keeping in mind Eq.(3), are given by the relations:

$$X = aξ^2 - bη^2 + 2bξη + cζ (ζ - 2ξ) ; \quad Y = -aξ^2 + bη^2 + 2aξη + cζ (ζ - 2η)$$

$$Z = -aξ^2 - bη^2 - ζ [cζ - 2(aξ + bη)]$$.

Without loss of generality, we assume that $ζ = 0$, therefore we reduce the intervention of the three integers $ξ, η$ and $ζ$ and to only two of them.

In practice we use the following equations instead of Eqs.(2):

$$X = ρ · x + ξ, \quad Y = ρ · y + η, \quad Z = ρ · z$$

and eliminates the parameter $ρ$ to obtain the following parametric solutions of Eq.(28):

$$V = λ \left( X^n_1 ξ^2 - Y^n_1 η^2 + 2Y^n_1 ξη \right) ; \quad T = λ \left( -X^n_1 ξ^2 + Y^n_1 η^2 + 2X^n_1 ξη \right) ;$$

$$P = λ \left( X^n_1 ξ^2 + Y^n_1 η^2 \right) .$$

Where $ξ$ and $η$ are coprime integers and $λ$ is rational proportionality factor.

Moreover $ξ$, $η$ and $λ$ are uniquely determinated, up to a simultaneous change of sign of $ξ$, and $η$.

One standard method of obtaining the above parametrization can be found also in ([2], §6.3.2, pp. 343-346).

Now from the second equation of the second system (27) with the Eqs.(29) and $(V, T) = 1$, we have with $λ = 1/M$:

$$Z^n_1 Q^2 = V^2 - T^2 = \frac{1}{M^2} \left[ 4ξη (ξ - η) (X^n_1 ξ + Y^n_1 η) (X^n_1 + Y^n_1) \right] \Rightarrow$$

$$Q^2 = \frac{1}{M^2} 4ξη (ξ - η) (X^n_1 ξ + Y^n_1 η) .$$

For the last factor $(X^n_1 ξ + Y^n_1 η)$ we can consider the following linear equation:

$$(X^n_1 ξ + Y^n_1 η) = hZ^n_1$$

which certainly, admitting the obvious solution $ξ = η = h$, provides us all solutions also with $ξ \neq η$, that is:

$$ξ = h + Y^n_1 θ; \quad η = h - X^n_1 θ.$$
Besides we have:

\[(\xi - \eta) = Z_1^n \theta. \quad (33)\]

Therefore bearing in mind that \((X_1, Y_1, Z_1) = 1, (V, T, P) = 1\) and \((\xi, \eta) = 1\), we have also that \((h, \theta) = 1\).

Now, Eq.(30) with Eq.(31), Eq.(33) and in addition with \(M = 2(a\xi + b\eta) = 2(X_1^n\xi + Y_1^n\eta)\) provides:

\[
Q^2 = \frac{1}{4(X_1^n\xi + Y_1^n\eta)^2} \cdot 4\xi\eta (\xi - \eta)(X_1^n\xi + Y_1^n\eta) = \frac{\xi\eta\theta Z_1^n}{h Z_1^n} = \xi\eta \frac{\theta}{h}. \quad (34)
\]

Now we will resort to the Corollary 6.3.8 ([2], p. 346).

In the case of \((V, T, P) = 1\) we have that the rational proportionality parameter in the Eqs.(29) is \(\lambda = \frac{1}{r}\) with \(r|2Y_1^nZ_1^n\).

Now \(\lambda = \frac{1}{M} \Rightarrow h = \frac{V_1^n}{m}\) with \(m|Y_1^n\).

Without loss of generality, we can verify only the following extreme case \(m = 1\) and \(m = Y_1^n\) [see Section 8: Appendix].

In fact, thanks to the solutions (32), a single and appropriate value of \(h\) is sufficient for these equations to constitute the general solution of the linear equation (31).

It follows that for \(\theta = 0, \pm 1, \pm 2, \ldots\) formulas (32) give all the integral solutions of equation (31).

The necessary condition is that \(h\) is an exact divisor of \(Y_1^n\) and consequently \(h = Y_1^n\) or \(h = 1\) both satisfy this condition.

In the first case with \(h = Y_1^n\) we have from Eq.(34): \(Q^2 = (1 + \theta)(\theta)(Y_1^n - X_1^n\theta)\) with the three positive factors in brackets that are pairwise relatively prime.

By the uniqueness of the prime decomposition we have \((1 + \theta)\) and \(\theta\) should be equal to squares and this is absurd.

In the second case with \(h = 1, \theta > 0\) and \(X_1^n < 0\) we have from Eq.(34): \(Q^2 = (1 + Y_1^n\theta)(\theta)(1 - X_1^n\theta)\) with the three positive factors in brackets that are pairwise relatively prime.

By the uniqueness of the prime decomposition we have that:

\[
\xi = (1 + Y_1^n\theta) = V_1^n; \eta = (1 - X_1^n\theta) = T_1^n; P_1^n = 1; \theta = Q_1^n.
\]

In conclusion we have the further double Euler equations:

\[
P_1^n + Y_1^n Q_1^n = V_1^n \quad ; \quad P_1^n - X_1^n Q_1^n = T_1^n
\]

with \(Q > Q_1\), if compared with the double Euler equations of the first Diophantine system (27).

Repeating the argument indefinitely would the give a sequence of positive integer \(Q > Q_1 > Q_2 > Q_3 > \ldots\) which decreased indefinitely.

This is impossible, because imply an "infinite descent" for parameter \(Q\).

The determination of the parameter \(Q\), as rational integer not equal to zero, ends here, but we must remember that the Eq.(34) was determined only thanks by assuming the obvious solution \(\xi = \eta = h\) of the linear equation (31).

In this case due to Eq.(33), assuming \(Z_1^n > 0\), we have \(\theta = 0\) and this results in the zeroing of the parameter \(Q\).
The double equations of Euler are discordant forms and so the F.L.T. turns out to be true, just as honestly announced by Fermat himself.

7. Conclusions

In this paper we have try to prove F.L.T. making use of elementary techniques, certainly known to P. Fermat.

We show that making use of the concordant forms of Euler and a ternary quadratic homogeneous Diophantine equation, it is possible to derive a proof of the F.L.T. without recurring to modern techniques, but exploiting the important criterion of Legendre for determining the solutions of ternary quadratic homogeneous equation.

The proof, here presented, is valid in the case of all odd exponents greater than one (see the proof of the Theorem 3.1).

We observe however that also in the case of exponent \( n = 4 \) the double equations of Euler are discordant: in this case, in the double equations of Euler, defined by the expressions (7) is just assume that \( m = -n = 1 \).

More precisely we have the following system of equations:

\[
\begin{align*}
X^2 + Y^2 &= Z^2 \\
X^2 - Y^2 &= T^2
\end{align*}
\]

that has no solutions in the natural numbers.

This theorem of a "congruent number" was anticipated by Fibonacci in his book "The Book of squares" ([12], Chap. III, § VI-2, pp. 310–311), but with a demonstration does not complete (then he thought of this Fermat with the equivalent Theorem: No Pythagorean triangle has square area) ([13], Chap. II, pp. 50–56).

In this work we have not used the proof of non-existence of the Frey elliptic curve, but we have limited ourselves to proof of non-existence of the single homogeneous ternary quadratic equation Eq.(5), defined in the proof of the Theorem 3.1, but whose origin [see Eq.(12)] is implicit in the nature of Euler’s double equations.

The double equations of Euler gave rise in different ways to the elliptic curve of Frey and to a particular homogeneous ternary quadratic equation: both characterized by the presence of \( X_1^n \), \( Y_1^n \) and \( Z_1^n \) in their coefficients.

For this it was possible to use a similar strategy to build a proof of the F.L.T..

Additional Remarks

Remark 1. This work is a reworking of the paper "Euler’s double equations equivalent to Fermat’s Last Theorem" [11] with the aim of making more accessible a Theorem of which Fermat claimed to have a proof and which generations of mathematicians have tried in vain to try to rediscover it.

Remark 2. In 1753 Euler calls the Fermat Last Theorem \( \ll \) a very beautiful theorem \( \gg , \) adding that he could only prove it for \( n = 3 \) and \( n = 4 \) and in no other case ([15], Chap. III, § 
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5-d, p. 181).

In 1770, He gave a proof with exponent $p = 3$, in his Algebra ([6], II, Chap. XV, § 243), but his proof by infinite descent contained a major gap.

However, since Euler himself had proved the lemma necessary to complete the proof in other work, he is generally credited with the first proof.

The author of this paper has done nothing but complete a work begun and masterly conducted by Euler himself.

For this reason, he considers himself as a co-author of this proof, but hopes, as shown elsewhere ([10]), that this way of working can become a normal habit.

8. Appendix

Let us consider the following homogeneous linear equation $ax + by + cz = 0$.

All integer solutions are given by formulas:

$$x = \frac{k}{\delta} (b\alpha), \quad y = \frac{k}{\delta} (c\beta - a\alpha), \quad z = -\frac{k}{\delta} (b\beta)$$

where $k$, $\alpha$, $\beta$ are integers, $(\alpha, \beta) = 1$ and $\delta = (b\alpha, c\beta - a\alpha, b\beta)$.

Having said this, let us consider the equation $X^1 \xi + Y^1 \eta - Z^1 h = 0$.

We will have the following integer solutions:

$$\xi = \frac{k}{\delta} (Y^1 \alpha), \quad \eta = \frac{k}{\delta} (Z^1 \beta - X^1 \alpha), \quad h = \frac{k}{\delta} (Y^1 \beta)$$

(35)

where $(\alpha, \beta) = 1$ and $\delta = (Y^1 \alpha, Z^1 \beta - X^1 \alpha, Y^1 \beta)$.

Alongside these we also consider Eqs.(32), that is:

$$\xi = h + Y^1 \theta, \quad \eta = h - X^1 \theta.$$

(36)

Resulting in any case $h \mid Y^1$ and $(\xi, \eta) = 1$ we have $k = 1$ and $(h, \theta) = 1$.

Furthermore, in order to determine values for the parameter $h$, we consider the following equation [see Eq.(34)]:

$$Q^2 = \frac{1}{h} \xi \eta \theta$$

(37)

From Eqs.(35) we have: $\frac{\xi}{h} = \frac{\delta}{\beta} \Rightarrow \beta = 1$ and

$$\frac{\xi}{h} = \alpha.$$  

(38)

Furthermore, again from Eqs.(35)

$$\xi - \eta = \frac{Y^1 \alpha}{\delta} - \frac{1}{\delta} (Z^1 - X^1 \alpha) = \frac{1}{\delta} Z^1 (\alpha - 1).$$

(39)
The Lost Proof of Fermat’s Last Theorem

From Eqs.(36) we have:

\[ \xi - \eta = Z^n \theta. \] (40)

The Eq.(39) and Eq.(40) \[ \Rightarrow \]

\[ \theta \delta = \alpha - 1. \] (41)

Now resulting:

\[ h \delta = Y^n_1 \] (42)

we also have: \[ \frac{h}{\delta} = \frac{Y^n_1}{\alpha - 1} \Rightarrow \]

\[ h = \frac{\theta}{\alpha - 1} Y^n_1 \text{ or } Y^n_1 = \frac{\alpha - 1}{\theta} h. \] (43)

From Eqs.(36) with Eqs.(43) we obtain

\[ \xi = h + Y^n_1 \theta = Y^n_1 \theta \frac{\alpha}{\alpha - 1} ; \quad \eta = h - X^n_1 \theta = \theta \left( \frac{Z^n - X^n_1 \alpha}{\alpha - 1} \right). \] (44)

From Eq.(37) with Eq.(38) and Eqs.(44) we have:

\[ Q^2 = \alpha \theta^2 \left( \frac{Z^n - X^n_1 \alpha}{\alpha - 1} \right) = \frac{\theta \alpha}{\alpha - 1} (Z^n - \alpha X^n_1) \theta. \]

At end with Eqs.(43) we obtain the following equivalent equations:

\[ Q^2 = \left( \frac{\xi}{Y^n_1} \right) (Z^n - \alpha X^n_1) \theta \]

or

\[ Q^2 = \xi \left( \frac{Z^n - \alpha X^n_1}{Y^n_1} \right) \theta. \]

The determination of the parameter \( Q \), as rational integer not equal to zero, ends here.

The former equation \[ \Rightarrow h = Y^n_1 \text{ and } \delta = 1 \] [see Eq.(37) and Eq.(42)] and the latter equation \[ \Rightarrow \delta = Y^n_1 \text{ and } h = 1 \] [see Eq.(37) and second formula of Eqs.(35)].

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II) Some Diophantus-Fermat double equations equivalent to Frey’s elliptic curve.

abstract

In this work I demonstrate that a possible origin of the Frey elliptic curve derives from an appropriate use of the double equations of Diophantus-Fermat and from an isomorphism: a birational application between the double equations and an elliptic curve.

From this origin I deduce a Fundamental Theorem which allows an exact reformulation of Fermat’s Last Theorem.

A complete proof of this Theorem, consisting of a system of homogeneous ternary quadratic Diophantine equations, is certainly possible also through methods known and discovered by Fermat, in order to solve his extraordinary equation.

9. The «double equations» and the «Frey elliptic curve»

A careful reading of the existing documentation about the Diophantine problems, reveals that Fermat, and especially Euler, often used the so-called "double equations" of Diophantus, that is $a x^2 + b x + c = z^2$; $a' x^2 + b' x + c' = t^2$ with the conditions that $a$ and $a'$, or $c$ and $c'$ are squares. These conditions ensure the existence of rational solutions of the double equations.

These equations can be written in a more general form as:

$$ax^2 + 2bxy + cy^2 = z^2 \quad a'x^2 + 2b'xy + c'y^2 = t^2$$  \hspace{1cm} (1)

and usually both Fermat and Euler considered only the curves of those forms which have, in the projective space, at least one "visible" rational point.

Fermat and Euler derive from few evident solutions an infinite number of solutions.

Under this last hypothesis ([5], Chap. II, Appendix III, pp. 135–139) the curve determined by the equations (1) results isomorphic to the one given by

$$Y^2 = X \left( (b'X - b)^2 - (a'X - a) (c'X - c) \right)$$  \hspace{1cm} (2)

i.e. an elliptic curve (see also Appendix A).

In fact, an elliptic curve, which has at least one rational point, can be written as a cubic $y^2 = f(x)$, where $f$ is a polynomial of degree 3.

Given this, we consider the following system, consisting of a pair of «double equations»

$$\begin{cases} (3)_1 & X_1^nV^2 + Y_1^nT^2 = U'^2 \quad V^2 - T^2 = W^2 \\ (3)_2 & X_1^nW^2 + Z_1^nT^2 = U'^2 \quad W^2 + T^2 = V^2 \end{cases}$$  \hspace{1cm} (3)

where $X_1, Y_1, Z_1$ are integer numbers (positive or negative), pairwise relatively primes, $n > 2$ is a natural number and $U', V, W, T$ are integer variables.

Applying the isomorphism described by Eq. (2) we obtain, from the first two equations of the system (3), i.e. the $(3)_1$, the elliptic curve

$$Y^2 = X (X - X_1^n) (X + Y_1^n),$$  \hspace{1cm} (4)

and from the other two equations, the $(3)_2$, the further elliptic curve

$$Y^2 = -X (X - X_1^n) (X - Z_1^n).$$  \hspace{1cm} (5)
Combining Eq. (4) and Eq. (5) and using the relation \( X = X_1^n / 2 \) one obtains the following identity

\[ X_1^n + Y_1^n = Z_1^n. \]  

(6)

Now the elliptic curve (4), together with the identity (6), is nothing but the Frey elliptic curve ([1], pp.154–156).

In Mathematics, a Frey curve, or Frey–Hellegouarch curve, is the elliptic curve:

\[ Y^2 = X (X - X_1^n) (X + Y_1^n) \]  

(7)

or, equivalently :

\[ Y^2 = X \left[ X^2 + X (Y_1^n - X_1^n) - X_1^n Y_1^n \right] \]  

(8)

associated with a (hypothetical) solution of Fermat’s equation : \( X_1^n + Y_1^n = Z_1^n \).

In fact, the discriminant

\[ \Delta = \sqrt{(Y_1^n - X_1^n)^2 + 4X_1^n Y_1^n} = X_1^n + Y_1^n = Z_1^n, \]

that determines the existence of the polynomial \( (X - X_1^n) (X + Y_1^n) = X^2 + X (Y_1^n - X_1^n) - X_1^n Y_1^n \) is a perfect power of order \( n \).

Frey suggested, in 1985, that the existence of a non-trivial solution to \( X^n + Y^n = Z^n \) would imply the existence of a non-modular elliptic curve, viz. \( Y^2 = X(X - X^n)(X + Y^n) \).

This suggestion was proved by Ribet in 1986.

This curve is semi-stable and in 1993 Wiles announced a proof (subsequently found to need another key ingredient, furnished by Wiles and Taylor) that every semi-stable elliptic curve is modular, the semi-stable case of the Taniyama-Shimura-Weil conjecture ([6] and [4]).

Hence no non-trivial \( X^n + Y^n = Z^n \) can exist.

Moreover, as Euler found out, treating similar problems, regarding algebraic curves of genus 1, the two problems, connected to curves (4) and (5), are completely equivalent.

In our case it is simple to verify that the elliptic curve (5) can be reduced to (4) by the transformation \( X \Rightarrow -X + X_1^n \) and the identity (6).

### 10. The Diophantine System

One can reduce the system (3) to the following Diophantine system

\[
\begin{align*}
X_1^n V^2 + Y_1^n T^2 &= U^2 \\
X_1^n W^2 + Z_1^n T^2 &= U'^2 \\
W^2 + T^2 &= V^2.
\end{align*}
\]  

(9)

Our proof of Fermat’s Last Theorem consists in the demonstration that it is not possible a resolution in whole numbers, all different from zero, of a system derived from system (9), but analogous, [see section 11 and system (19)], with integer coefficients and using integer variables \( U, W', T', V' \).

From the first two equations of the system (9) one obtains

\[ X_1^n V^2 + Y_1^n T^2 = X_1^n W^2 + Z_1^n T^2. \]  

(10)

Now from Eq.(10) is

\[ X_1^n \left( V^2 - W^2 \right) = (Z_1^n - Y_1^n) T^2. \]  

(11)
Eq. (11) results in identity (6) if the third equation in the systems (9), \( W^2 + T^2 = V^2 \), is satisfied. In fact, since this equation is the Pythagorean triangle, in general, it accepts the following integer solutions, where \( p, q \) are natural numbers and \( k \) a proportionality factor (the values of \( W \) and \( T \) are interchangeable if necessary):

\[
W = k \left( 2pq \right); \quad T = k \left( p^2 - q^2 \right); \quad V = k \left( p^2 + q^2 \right).
\]

We can therefore consider also the primitive integer solutions with \( p, q \in \mathbb{N} \)

\[
W = 2pq; \quad T = p^2 - q^2; \quad V = p^2 + q^2.
\]  (12)

Thus Eq.(11), with \( p \) and \( q \) relatively prime, of opposite parity and \( p > q > 0 \) now is reduced to the identity (6).

11. On Homogeneous Ternary Quadratic Diophantine Equations \( aX^2 + bY^2 - cZ^2 = 0 \)

**Theorem 3.1:** Let \( x^n + y^n = z^n \), with \( (x, y) = 1 \) and \( n \geq 3 \) has a solution, then there exists an equation \( ax^2 + by^2 = cz^2 \), where \( a, b, c \) are relatively prime and reduced to the minimum terms, whose a solution could be reduced to a solution of Fermat’s equation.

**Proof.**

Let \( X_1, Y_1, Z_1 \) be three whole numbers pairwise relatively prime such as to satisfy the Fermat equation \( x^n + y^n = z^n \).

Then the following homogeneous ternary quadratic Diophantine equation, with \( (V, T, P) = 1 \) exists:

\[
X_1^nV^2 + Y_1^nT^2 = Z_1^nP^2. \quad (13)
\]

We observe that with the following particular nontrivial solutions:

\( V = 1, T = 1 \) and \( P = 1 \) or \( V = T = P \) in Eq.(13) we obtain the fundamental Hypothesis (Reductio ad Absurdum) of the F.L.T.:

\[
X_1^n + Y_1^n = Z_1^n.
\]

Now by the evident solutions, indicated above, we can derive an infinite number of solutions of Eq.(13).

Let’s remember that for Legendre’s Theorem if a ternary quadratic homogeneous Diophantine equation (assuming \( a, b \) and \( c \) are fixed) has an integral solution, then the number of possible solutions is infinite.

Having said this, it is possible to transform the previous Diophantine equation (13) into the following equivalent Diophantine equation, with \( (V', T', P') = 1 \) :

\[
X_1V'^2 + Y_1T'^2 = Z_1P'^2. \quad (14)
\]

It is sufficient to assume \( V' = X_1^kV, T' = Y_1^kT, P' = Z_1^kP \) where \( k = \frac{n-1}{2} \) and \( n > 1 \) odd number.

Using the "fundamental theorem of Arithmetic" we can represent ([3], Theorem 19, p. 31):

\[
X_1 = X_2U_1^2, \quad Y_1 = Y_2U_2^2, \quad Z_1 = Z_2U_3^2.
\]
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In this case is possible to transform the previous Diophantine equation (14) into the following equivalent Diophantine equation with the relative coefficients reduced to the minimum terms:

$$X_2 V'^2 + Y_2 T'^2 = Z_2 P'^2.$$ 

In fact just assume $V'' = U_1 V', T'' = U_2 T', P'' = U_3 P'$

We observe that $X_2, Y_2, Z_2$ are pairwise relatively prime and square-free numbers.

The proof ends here.

**Theorem 3.2**

Let's suppose that $x^n + y^n = z^n$, with $n \geq 3$ has a solution, in this case we will have an equation $ax^2 + by^2 = cz^2$, where $c$ is a square, whose solution could be reduced to a solution of Fermat's equation.

**Proof.**

From Theorem 3.1 [see Eq.(13)] we have the following equation $X^n V^2 + Y^n T^2 = Z^n U^2$ that could be reduced to a solution of Fermat’s equation with $n \geq 3$ odd integer.

Now multiplying the coefficient $X^n, Y^n, Z^n$ by factor $U^n$ we have

$$X^n Z^n V^2 + Y^n Z^n T^2 = Z^n Z^n U^2$$

and with $X_0 = X_1 Z_1$, $Y_0 = Y_1 Z_1$, $Z_0 = Z^n$ we get

$$X_0^n V^2 + Y_0^n T^2 = Z_0^n U^2$$

(15)

and $Z_0^n = (Z^n)^2$, that is a square.

In this case we have also, with g.c.d.($X_0, Y_0$) = $Z_1$, the equation of Fermat

$$X_0^n + Y_0^n = Z_0^n.$$ (16)

The proof ends here.

**12. The Lost Proof**

At this point, multiplying the Eq.(15) by factoring quadratic $Z_0^n$ we have

$$X_0^n Z_0^n V^2 + Y_0^n Z_0^n T^2 = Z_0^n Z_0^n U^2$$

and finally with $V' = Z_0^2 V$, $T' = Z_0^2 T$ we obtain

$$X_0^n V'^2 + Y_0^n T'^2 = (Z_0^n)^2 U^2.$$ (17)

That said, let's consider the following double equations of Diophantus-Fermat, necessary to give rise to the following, well known, Frey’s elliptic curve $Y^2 = X (X - X_0^n) (X + Y_0^n)$:

$$X_0^n V'^2 + Y_0^n T'^2 = (Z_0^n)^2 U^2 ; \quad V'^2 - T'^2 = W'^2$$

that together with the identity (16) can be rewritten in

$$Z_0^n U^2 = X_0^n V'^2 + Y_0^n T'^2 = X_0^n W'^2 + Z_0^n T'^2.$$ (18)

In practice we have rewritten the system (9) in the following Diophantine system:

$$\begin{cases}
X_0^n V'^2 + Y_0^n T'^2 = Z_0^n U^2 \\
X_0^n W'^2 + Z_0^n T'^2 = Z_0^n U^2 \\
W'^2 + T'^2 = V'^2.
\end{cases}$$ (19)
Eqs.(18) give us:

$$U^2 [Z_0^n]^2 - V'^2 [Z_0^n] + W'^2 Y_0^n = 0$$

or equivalently

$$U^2 [Z_0^n]^2 - T'^2 [Z_0^n] - W'^2 X_0^n = 0.$$  \hfill (21)

$Z_0^n$ is a square, so the product of the two roots in Eq.(20) is

$$[Z_0^n]_1 \cdot [Z_0^n]_2 = \frac{W'^2 Y_0^n}{U^2} \Rightarrow Y_0^n,$$

which is a square,

and in Eq.(21) is

$$[Z_0^n]_1 \cdot [Z_0^n]_2 = -\frac{W'^2 X_0^n}{U^2} \Rightarrow -X_0^n,$$

which is a square.

From Theorem 3.1 we have that $X_1, Y_1, Z_1$ are pairwise relatively prime and with $Y_0^n = [\Box = Y_1^n Z_1^n]$ and $X_0^n = [\Box = X_1^n Z_1^n]$ we obtain:

$$Z_1^n = \Box; Y_1^n = \Box; X_1^n = -\Box.$$  

With this last result, obtained also thanks to the use of a Pythagorean equation [see Eqs.(17)], one finds also:

$$[Z_0^n]_1 \cdot [Z_0^n]_2 = \frac{W'^2 Y_0^n}{U^2} = -\frac{W'^2 X_0^n}{U^2}. $$

This gives finally the special solution:

$$Y_1^n = -X_1^n \Rightarrow Z_1^n = 0.$$  

Consequently the Diophantine system (19) does not admit integer solutions.

### 13. Analytical digressions

There is no doubt that the system (19), inspired by system (9), represents a true "lockpick" of the Fermat Last Theorem.

Through the former system, keeping in mind always the possibility of exchanging the role of $X_0$ and $Y_0$ into identity (16), we are able to establish the following Fundamental Theorem:

*The Fermat Last Theorem is true if and only if a solution in integers, all different from zero, of the following Diophantine system, made of three homogeneous equations of second degree, with integer coefficients $X_0^n, Y_0^n, Z_0^n$, where $n$ is a natural number $> 2$ and with $U, T', V', W'$ integer indeterminates is not possible.*

\[
\begin{align*}
X_0^n V'^2 + Y_0^n T'^2 &= Z_0^n U^2 \\
X_0^n W'^2 + Z_0^n T'^2 &= Z_0^n U^2 \\
W'^2 + T'^2 &= V'^2.
\end{align*}
\hfill (22)
\]

The presence of a Pythagorean equation in this system has been proved to be essential, not only to connect the most general Fermat’s equation to the supposed Frey’s elliptic curve, but to demonstrate the above indicated Fundamental Theorem (see Section 11) and at the end to provide also a proof of Fermat’s Last Theorem, using a method of Reductio ad Absurdum.

\[^1\text{The symbol } \Box \text{ represents an indeterminate square.}\]
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14. Conclusions

In this paper I demonstrate that a possible origin of Frey’s elliptic curve derives from an appropriate use of the so-called "double equations" of Diophantus-Fermat and from an isomorphism: a birational application between the double equations and an elliptic curve.

This Frey elliptic curve does not exist ([1], pp. 154–156) and from this derives indirectly, as an absurd, the Fermat Last Theorem.

In this work we wanted to emphasize that a proof of the Fermat Last Theorem can not be separated by the strong links with the supposed Frey elliptic curve, although this does not mean that Fermat, in another way, was unable to produce our own proof.

Appendix A. Elliptic Curves from Frey to Diophantus

In Mathematics, a Frey curve or Frey–Hellegouarch curve is the elliptic curve:

\[ Y^2 = X(X - X_1^n) (X + Y_1^n) \]  \hfill (23)

or, equivalently:

\[ Y^2 = X \left[ X^2 + X(Y_1^n - X_1^n) - X^n Y_1^n \right] \]  \hfill (24)

associated with a (hypothetical) solution of Fermat’s equation: \( X_1^n + Y_1^n = Z_1^n \). In the language of Diophantus and of Fermat, we consider the following "double equation":

\[ ax^2 + 2bxy + cy^2 = z^2 \quad a'x^2 + 2b'y + c'y^2 = t^2. \]  \hfill (25)

In Weil’s Appendix III ([5], Ch. II, pp.135-139) he established (modulo the existence of a rational point) an isomorphism between the curve defined by the equations (25) and a certain elliptic curve defined by:

\[ Y^2 = X \left[ (b'X - b)^2 - (a'X - a) (c'X - c) \right] = X \left[ (b'^2 - a'c') X^2 + (ca' + ac' - 2bb') X - ac + b^2 \right]. \]  \hfill (26)

Let’s suppose that the first double equation is \( ax^2 + Y_1^n y^2 = z^2 \).

In this case we have considered the following assumptions in Eq.(26):

\[ b = 0 \quad \text{and} \quad c = Y_1^n. \]

Now the coefficient of \( X^2 \) in Eq.(24) is equal to coefficient of \( X^2 \) in Eq.(26): \( b'^2 - a'c' = 1 \) and the coefficient of \( X \) and the known term in Eq.(24) are equal to the ones in Eq.(26):

\[ (ca' + ac' - 2bb') = Y_1^n - X_1^n; \quad -ac + b^2 = -X_1^n Y_1^n, \]  \hfill (27)

but with \( b = 0 \) and \( c = Y_1^n \) we have

\[ -ac = -X_1^n Y_1^n \Rightarrow -aY_1^n = -X_1^n Y_1^n \Rightarrow a = X_1^n. \]

From the first of Eq.(27) we have \( Y_1^n a' + X_1^n c' = Y_1^n - X_1^n \Rightarrow a' = 1, \)

\[ c' = -1 \Rightarrow b' = 0. \]

With these results we have the following double equation of Diophantus:

\[ X_1^n x^2 + Y_1^n y^2 = z^2 \quad \text{and} \quad x^2 - y^2 = t^2 \]  equivalently to \( X_1^n V^2 + Y_1^n T^2 = U^2 \)

and \( V^2 - T^2 = W^2 \)  [see the equations of the system (3), i.e. the (3)_1].
Additional Remarks

REMARK 1. Fermat’s idea, in my opinion, to prove his Last Theorem, could take place through the following logical steps:

1- Define a quadratic and homogeneous ternary equation, in the normal form of Lagrange, able to accommodate a solution, with $n$ greater than or equal to 3, of its extraordinary equation (6) [see Theorem 3.2].

2- Connect this appropriate Diophantine equation of 2nd degree to the classic Pythagorean equation [see Eqs.(17)] to build a complete Diophantine system capable of determining its possible whole solution [see system (19)].

3- Establish that this Diophantine system does not admit congruent integer solutions and therefore as a consequence of this, there are no three integers that satisfy Fermat’s equation (6).

REMARK 2. The truth is that the impossibility to solve single equations can be proved as deduction from the impossibility of solving a system of equations.

The Fundamental Theorem is a reformulation of the Fermat Last Theorem: his following statements are equivalent:

(A) Fermat’s Last Theorem is true $\iff$ (A') The Diophantine System does not allow integer solutions different from zero.

Let $n > 2$; there is a bijection between the following sets:

(S) the set of solutions $(x, y, z)$ of Fermat’s Equation, where $x, y, z$ are nonzero natural numbers; and

(S') the set of solutions $(u, t', v', w')$ of the Diophantine System, where $u, t', v', w'$ are nonzero natural numbers.

The set of solutions of (S) and (S') are the same, that gives rise to an empty set, as shown in the Fundamental Theorem.

In the literature there are other Diophantine equations, that were compared to Fermat’s equation, i.e. a first result, due to Lebesgue in 1840, is the following Theorem:

If Fermat’s Last Theorem is true for the exponent $n \geq 3$ then the equation $X^{2n} + Y^{2n} = Z^2$ has only trivial solutions.

The proof of this theorem is extremely simple and is found in [2].

In this case, however, it cannot be said that Lebesgue’s theorem is equivalent to Fermat’s Last Theorem, while on the contrary, the Fundamental Theorem is just equivalent to Fermat’s last theorem.
REMARK 3. I conclude this work with the following observation by A. Weil ([5], Chap. IV, § VI, pp. 335–336): "Infinite descent a' la Fermat depends ordinarily upon no more than the following simple observation: if the product $\alpha \cdot \beta$ of two ordinary integers (resp. two integers in an algebraic number-field) is equal to an $m$-th power, and if the g.c.d. of $\alpha$ and $\beta$ can take its values only in a given finite set of integers (resp. of ideals), then both $\alpha$ and $\beta$ are $m$-th powers, up to factors which can take their values only in some assignable finite set." (See the end of section 11: The Lost Proof.)

References

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5. A. Weil, *Number Theory: an Approach Through History from Hammurapi to Legendre*, reprint of 1984 Edition, Birkhäuser, Boston, 2007.
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A direct link between the two elementary Proofs of the Fermat Last theorem.

Given that the two elementary proofs present respectively in the works:

(I) "Some Euler double equations equivalent to Fermat’s last theorem"

and

(II) "Some Diophantus-Fermat double equations equivalent to Frey’s Elliptic Curve" are based on the proof of non-existence of an appropriate Diophantine equation, ternary and homogeneous of the second degree, capable of accommodating a possible integer solution of the Fermat equation, here we will illustrate some useful elements to clarify the direct and extraordinary connection between the two proofs.

In the second (II) (the one attributable to Fermat) we have deduced the following equations with $Z_0^n = (Z_1^n)^2$ [see Eq.(20) e Eq.(21)]:

$$U^2 [Z_0^n]^2 - V'^2 [Z_0^n] + W'^2 Y_0^n = 0$$  \hspace{1cm} (1)

or equivalently

$$U^2 [Z_0^n]^2 - T'^2 [Z_0^n] - W'^2 X_0^n = 0.$$  \hspace{1cm} (2)

and with these we have established that $X_0^n = X_1^n Z_1^n = -\Box^2$, $Y_0^n = Y_1^n Z_1^n = \Box$ and also being $(X_1^n, Y_1^n, Z_1^n) = 1$ we have that:

$$X_1^n = -\Box, \ Y_1^n = \Box e \ Z_1^n = \Box.$$

Now keeping in mind that Eq. (1) and Eq. (2) arose from the rewriting of the original System (9) in System (19), we must consider the various replacements that we have subsequently applied and in particular from $V' = Z_0^n V, T' = Z_0^n T$ and $W' = Z_0^n W$ (because of the Pythagorean identity $V^2 = T^2 + W^2$) we can rewrite Eq. (1) and Eq. (2) in the following way:

$$[Z_1^n]^3 \left[ Z_1^n \left( U^2 - V^2 \right) + W^2 Y_1^n \right] = 0$$  \hspace{1cm} (3)

or equivalently

$$[Z_1^n]^3 \left[ Z_1^n \left( U^2 - T^2 \right) - W^2 X_1^n \right] = 0.$$  \hspace{1cm} (4)

At this point, canceling the second factors of the two products (3) and (4) we have:

$$\left[ Z_1^n \left( V^2 - U^2 \right) \right] = W^2 Y_1^n$$

and

$$\left[ Z_1^n \left( U^2 - T^2 \right) \right] = W^2 X_1^n.$$

\[2\] The symbol $\Box$ represents an indeterminate square.
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By dividing them among them and simplifying them we get back to the original Diophantine equation:

\[ X_1^n V^2 + Y_1^n T^2 = Z_1^n U^2. \]

In work (I), see chapter 5, we have proved the perfect equivalence of the following two Diophantine systems:

\[
\begin{align*}
P^2 + Y_1^n Q^2 &= V^2 \\
Q^2 - X_1^n Q^2 &= T^2
\end{align*}
\]

\[
\begin{align*}
X_1^n V^2 + Y_1^n T^2 &= Z_1^n P^2 \\
Z_1^n Q^2 &= V^2 - T^2.
\end{align*}
\]

For this reason, since the Pythagorean identity is valid in work (II) \( V^2 - T^2 = W^2 \), we can also write in place of \( Z_1^n Q^2 = V^2 - T^2 \):

\[ Z_1^n Q^2 = W^2. \] (5)

The result (5) is perfectly consistent with the fact that in the work (II) we established that \( Z_1^n = \square \).

With Eq. (5), Eq. (3) and Eq. (4) are:

\[ [Z_1^n]^4 \left[ U^2 - V^2 + Q^2 Y_1^n \right] = 0 \] (6)

or equivalently

\[ [Z_1^n]^4 \left[ U^2 - T^2 - Q^2 X_1^n \right] = 0. \] (7)

Such products [see Eq. (6) and Eq. (7)] are null a condition of having \( Z_1^n = 0 \) or that the following system is valid

\[
\begin{align*}
U^2 + Y_1^n Q^2 &= V^2 \\
U^2 - X_1^n Q^2 &= T^2.
\end{align*}
\]

(8)

It is evident that we are in the presence of double equations of Euler [note that System (8) is precisely the solved System in work (I) in section 6].

Given this, we can determine from Eq. (1) and Eq. (2) also that:

\[ [Z_0^n]_1 + [Z_0^n]_2 = \frac{V'^2}{U^2} \]

and

\[ [Z_0^n]_1 + [Z_0^n]_2 = \frac{T'^2}{U^2}. \]

The two sums must be equal, therefore from \( V'^2 = Z_0^n V^2 \) and \( T'^2 = Z_0^n T^2 \) we get that

\[ V'^2 = T'^2 \Rightarrow V^2 = T^2 \Rightarrow W^2 = 0. \] (9)

The result (9) implies for Eq. (5): \( Z_1^n Q^2 = 0 \) or \( Z_1^n = 0 \) or \( Q^2 = 0 \).

In both cases Fermat’s Last Theorem results proved, in fact \( Z_1^n = 0 \) is the final result determined in section 12 of work (II), while \( Q^2 = 0 \) it is the result of section 6 of work (I), which establishes that Euler’s double equations are discordant.
In conclusion, from what has been illustrated it is fundamental the role of the Pythagorean identity in the resolution of Fermat’s last Theorem.

In work (II), or the proof attributable to Fermat, this role is evident, in the work (I), that is the attributable proof to Euler, it is implicit in the use and determination of the parameter $Q$.

The annulment of this parameter undoubtedly resulted from the relationship $Z^n Q^2 = V^2 - T^2 = W^2$ with $W^2 = 0$.

Finally I attach the page of the work on the F.L.T. by U. Bini, in Italian language, who inspired me in many years of study.

see [U. Bini, *La risoluzione delle equazioni $x^n + y^n = M$ e l’ultimo teorema di Fermat*, “Archimede”, anno IV n° 2, Le Monnier ed., Firenze, 1952.]
LA RISOLUZIONE DELLE EQUAZIONI $x^n \pm y^n = M$
E L’ULTIMO TEOREMA DI FERMAT

In questo scritto risolvo in numeri interi e positivi le equazioni

$$x^n \pm y^n = M,$$

qualunque sia il numero intero positivo $n$; e siccome il procedimento risolutivo di ogni equazione implica eventualmente l'accertamento dell'impossibilità sua, così ho provato la impossibilità dei tipi

$$x^n \pm y^n = M^n,$$

sotto condizioni molto ampie per $M$. Questi risultati sono ovviamente aspetti dell'ultimo teorema di Fermat, che mi sembrano non privi d'interesse, sopratutto perché, volendomi storicamente porre al tempo (1601-1665) del più grande dilettante di matematica di tutti i tempi, sono stati ottenuti con mezzi del tutto elementari. Posso dire infatti di essermi appoggiato soltanto allo sviluppo del binomio ed ai suoi immediati derivati, come il teorema di Fermat-Eulero.

Com'è ben noto Fermat leggendo i Commentaria in Diophantum di C. G. Bachet de Mézières, aveva preso l'abitudine di annotarli a margine.

A proposito dell'ottavo problema diofantico, che richiede e dà la risoluzione in numeri razionali dell'equazione $x^2 + y^2 = a^2$, Fermat postillò «Al contrario è impossibile dividere un cubo nella somma di due cubi, una quarta potenza in due quarte potenze, e, in generale, una potenza qualunque di grado superiore a due, in due potenze dello stesso grado. Ho scoperto una mirabile dimostrazione di tale teorema generale che questo margine è troppo piccolo per contenere».

Riflettendo al modo serio ed onesto col quale il magistrato al Parlamento di Tolosa presentava al mondo matematico le sue scoperte aritmetiche, tutte, prima o poi, risultate veritiere, non ci sono buone ragioni per credere che il teorema in discorso sia frutto di una sua avventata affermazione o di una smania. In una sola occasione, che io sappia, disse cosa risultata poi non esatta a proposito della serie

$$2^n + 1,$$

i cui termini reputò tutti primi, mentre tali erano i primi cinque.

Ed allora se è da credersi che Fermat abbia veramente trovato la dimostrazione generale dell'impossibilità in numeri interi non nulli primi tra loro della $x^n + y^n + z^n = 0$, non si capisce perché coloro che oggi si occupano della questione ritengano, per vincerne le difficoltà, di far ricorso a nuovi concetti, a nuove teorie nei campi dell'analisi superiore moderna. Mi pare che si voglia ricorrere alla bomba all'idrogeno, mentre basta forse un abile colpo di archibugio; uno di quei colpi di cui era maestro il grande Eulero.
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