COMPACT OBJECTS IN GENERAL RELATIVITY: FROM BUCHDAHL STARS TO QUASIBLACK HOLES

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A Buchdahl star is a highly compact star for which the boundary radius $R$ obeys $R = \frac{9}{8} r_+$, where $r_+$ is the gravitational radius of the star itself. A quasiblack hole is a highly compact object, or more generically a highly compact object, for which the boundary radius $R$ obeys $R = r_+$. Quasiblack holes are objects on the verge of becoming black holes. Continued gravitational collapse ends in black holes and has to be handled with the Oppenheimer-Snyder formalism. Quasistatic contraction ends in a quasiblack hole and should be treated with appropriate techniques. Quasiblack holes, not black holes, are the real descendants of Mitchell and Laplace dark stars. Quasiblack holes have many interesting properties. We develop the concept of a quasiblack hole, give several examples of such an object, define what it is, draw its Carter-Penrose diagram, study its pressure properties, obtain its mass formula, derive the entropy of a nonextremal quasiblack hole, and through an extremal quasiblack hole give a solution to the puzzling entropy of extremal black holes.

Keywords: black holes; quasiblack holes; Carter-Penrose diagrams; mass formula; entropy

1. Introduction

In general relativity, a compact object is a body whose radius $R$ is not much larger than its own gravitational radius $r_+$. Compact objects are realized in compact stars. The concept of a compact object within general relativity achieved full form with the work of Buchdahl where it was proved on quite general grounds that for any nonsingular static and spherically symmetric perfect fluid body configuration of radius $R$ with a Schwarzschild exterior, the radius $R$ of the configuration is bounded by $R \geq \frac{9}{8} r_+$, with $r_+ = 2m$ in this case, $m$ being the spacetime mass, and we use units in which the constant of gravitation and the velocity of light are set equal to one. Objects with $R = \frac{9}{8} r_+$ are called Buchdahl stars, and are highly compact stars. A Schwarzschild star, i.e., what is called the Schwarzschild interior solution with energy density $\rho$ equal to a constant, is a realization of this bound.
Schwarzschild stars can have any relatively large radius $R$ compared to their gravitational radius $r_+$, but when the star has radius $R = \frac{3}{2} r_+$, i.e., it is a Buchdahl star, the inner pressure goes to infinity and the solution becomes singular at the center, solutions with smaller radii $R$ being even more singular. From here one can infer that when the star becomes a Buchdahl star, i.e., its radius $R$, by a quasistatic process say, achieves $R = \frac{3}{2} r_+$, it surely collapses. A neutron star, of radius of the order $R = 3 r_+$, although above the Buchdahl limit, is certainly a compact star, and its unequivocal existence in nature to Oppenheimer and others, led Oppenheimer himself and Snyder to deduce that complete gravitational collapse should ensue. By putting some interior matter to collapse, matched to a Schwarzschild exterior, it was found by them that the radius of the star crosses its own gravitational radius and an event horizon forms with radius $r_+$, thus discovering Schwarzschild black holes in particular and the black hole concept in general. Note that when there is a star $r_+$ is the gravitational radius of the star, whereas in vacuum $r_+$ is the horizon radius of the spacetime, so that when the star collapses, the gravitational radius of the star gives place to the horizon radius of the spacetime. In its full vacuum form, the Schwarzschild solution represents a wormhole, with its two phases, the expanding white hole and the collapsing black hole phase, connecting two asymptotically flat universes, see\textsuperscript{5}. There are other black holes in general relativity, belonging to the Kerr-Newman family, having as particular cases, the Reissner-Nordström solution with mass and electric charge, and the Kerr solution with mass and angular momentum, see\textsuperscript{6}. Classically, black holes are well understood from the outside. For their inside, however, it is under debate whether they harbor spacetime singularities or have a regular core. Clearly, the understanding of the black hole inside is an outstanding problem in gravitational theory. Quantically, black holes still pose problems related to the Hawking radiation and entropy. Both are low energy quantum gravity phenomena, whereas the singularity itself, if it exists, is a full quantum gravity problem. Black holes form quite naturally from collapsing matter, and the uniqueness theorems are quite powerful, but a time immemorial question is: Can there be matter objects with radius $R$ obeying $R = r_+$?

I.e., are there black hole mimickers? Unquestionably, it is of great interest to conjecture on the existence of compact objects that might obey $R = r_+$. Speculations include gravastars, highly compact boson stars, wormholes, and quasiblack holes. Here we advocate the quasiblack hole. It has two payoffs. First, it shows the behavior of highly compact objects and second, it allows a different point of view to better understand a black hole, both the outside and the inside stories. To bypass the Buchdahl bound and go up to the stronger limit $R \geq r_+$, that excludes trapped surfaces within matter, one has to put some form of charge. Then a new world of objects and states opens up, which have $R = r_+$. The charge can be electrical, or angular momentum, or other charge. Indeed, by putting electric charge into the gravitational system, Andréasson\textsuperscript{7} generalized the Buchdahl bound and found that for those systems the bound is $R \geq r_+$. Thus, systems with $R = r_+$ are indeed possible, see\textsuperscript{8} for a realization of this bound. Systems with $R = r_+$ are called quasiblack
holes.

To see how it works, let us start with static electrically charged massive particles in Newtonian gravitation, i.e., let us work in Newton-Coulomb theory. Two massive electrically charged particles with electric charges of the same sign, each of mass \( m \) and charge \( q \) and separated by a distance \( r \), attract each other with a gravitational force given by \( F_g = \frac{m^2}{r^2} \) and repel each other with an electric force given by \( F_e = \frac{q^2}{r^2} \), see Figure 1. When \( m = q \), clearly the forces of attraction and of repulsion on each particle are the same and one has \( F_g = F_e \). In this set up, a particle at rest remains at rest in equipoise with the other particle. Now, we can put another particle into the system with the same mass \( m \) and charge \( q \), and indeed any number of these particles, even a continuous distribution, with any symmetry, in any configuration, and the result holds, the system stays in equipoise. Incidentally, Mitchell and Laplace dark stars of Newtonian gravitation, which were never implemented as concrete systems, can easily be built from this type of matter, by making from it an actual ball with radius \( R = r + \), such that the escape velocity from its surface is equal to the speed of light.

General relativity plus electric charged matter and electromagnetism yields the Einstein-Maxwell system of equations. For a static situation, one can write the line element as \( ds^2 = -W^2(x^i) dt^2 + g_{ij}(x^k) dx^i dx^j \), with \((t, x^i)\) as the time and spatial coordinates, respectively, and \( i \) a spatial index running from 1 to 3, \( W(x^i) \) the metric potential, and \( g_{ij}(x^k) \) the metric form for the 3-space. In the case where \( W^2(x^i) \) depends strictly on the electric potential \( \phi(x^i) \), i.e., \( W^2 = W^2(\phi) \), then in electrovacuum the following relation holds, \( W^2 = (\phi + b)^2 + c \), where \( b \) and \( c \) are constants. Moreover, in the particular case that \( c = 0 \), and so \( W^2(\phi) = (\phi + b)^2 \), one can show that the solution corresponds to two general relativistic particles, i.e., two extremal black holes, with mass equal to charge \( m = q \) in equilibrium. The solution also holds for a number of extremal black holes scattered around in equipoise, and generically such type of solutions is called Majumdar-Papapetrou. If one includes matter, maintaining the condition \( W^2(\phi) = (\phi + b)^2 \), then the matter density \( \rho \) and the electric density \( \rho_e \) are related by \( \rho = \rho_e \). This type of matter is called extremal matter or Majumdar-Papapetrou matter, and the corresponding Majumdar-Papapetrou solutions are the general relativistic versions of the simple Newtonian gravitation solutions.

One can make a star out of Majumdar-Papapetrou, i.e., extremal matter. One
puts a boundary at some radius $R$ on the matter, with the interior being of extremal matter and the exterior being an extremal Reissner-Nordström spacetime. The global solution, interior plus exterior, is a Bonnor star.[9,10] Examples of Bonnor stars can be given. One example is a star made of a continuous fluid distribution of small clouds, such that each cloud has $10^{18}$ neutrons and 1 proton, see Figure 2. The star has, at its boundary, $m = q$ and the outside is extremal Reissner-Nordström. Another example is a spherical star made of any continuous distribution of supersymmetric, or otherwise, stable particles, each particle with mass equal to charge. The star has then, of course, total mass $m$ and total charge $q$ obeying $m = q$.

It is clear that for any star radius $R$ the star is in equilibrium. In a series of quasistatic steps, one can bring the radius $R$ of the star into its own gravitational radius $r_+$, i.e., one can achieve a configuration for which $R = r_+$, as near as one likes. Since $r_+$ is the gravitational radius of the configuration, which is on the verge of becoming a horizon, something special has to happen. Indeed, at $R = r_+$ a quasiblack hole forms. Thus, a quasiblack hole is an object, a star for instance, with its boundary at its own gravitational radius. In this sense quasiblack holes are the real general relativistic successors of the Mitchell and Laplace stars. Moreover, since they can be realized as stars, and these stars are frozen at their own gravitational radius, quasiblack holes epitomize naturally the concept of a frozen star, a name that Zel’dovich and Novikov gave instead rather to mean a black hole and that was in turn superseded by it, see[5].
2. Examples of quasiblack holes

Stars that can be brought to the quasiblack hole state do not need to be Bonnor stars, these are only one example that yields quasiblack holes. Under certain conditions, several type of stars that shrink to $R = r_+$, do form quasiblack holes, see Figure 3. For instance, it is possible to have stars with a nonextremal interior, that nonetheless the condition $m = q$ at the boundary is obeyed. These stars have as exterior the extremal Reissner-Nordström solution, and the quasiblack hole with $R = r_+$ is a solution. Furthermore, one can generalize the concept of a quasiblack hole to simply a solution in which one finds $R = r_+$.

There are many examples of quasiblack holes. They are: (i) Majumdar-Papapetrou quasiblack holes asymptotic to the extremal Reissner-Nordström solution.\(^{11}\) (ii) Bonnor quasiblack holes with a sharp boundary.\(^{12}\) see also\(^{9,10}\) (iii) Spheroidal quasiblack holes made of extremal charged matter.\(^{13}\) (iv) Quasiblack holes with pressure: Relativistic charged spheres as the frozen stars.\(^{11,17}\) see also\(^{18}\) (v) Yang-Mills-Higgs magnetic monopole quasiblack holes.\(^{19,20}\) (vi) Rotating matter at the extremal limit.\(^{21,22}\) (vii) Quasiblack holes made of fundamental fields.\(^{23}\) (viii) Matter with spin in Einstein-Cartan theory at the quasiblack hole state, an example that can be worked out. (ix) Quasiblack hole shells of matter, i.e., a thin shell at its own gravitational radius, with zero pressure in the extremal case and unbound pressure in the nonextremal case, as an exercise for a student in general relativity and gravitation.\(^{24}\) For a review of these examples see also.\(^{25}\)

Since there are ubiquitous solutions one should consider the core properties of those solutions, the most independently as possible from the matter they are made, in much the same way as one does for black holes.

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![Fig. 3. Sequence of star configurations to a quasiblack hole where $R = r_+$.](image-url)
3. Definition of quasiblack holes and generic properties

Consider a static spherically symmetric line element written in $(t, r, \theta, \phi)$ spherical coordinates as

$$ds^2 = -B(r) \ dt^2 + A(r) \ dr^2 + r^2 \ (d\theta^2 + \sin^2 \theta \ d\phi^2) \ ,$$

(1)

where $A$ and $B$ are metric potentials depending on the radial coordinate $r$, $A = A(r)$ and $B = B(r)$. Assume the line element represents an interior and an exterior region, and so is valid and nonsingular for $0 \leq r < \infty$. At infinity the metric is asymptotically flat.

Consider that the solution for the metric potentials $B(r)$ and $\frac{1}{A(r)}$ has the following properties: (a) the function $\frac{1}{A(r)}$ attains a minimum at some $r^* \neq 0$, such that $\frac{1}{A(r^*)} = \epsilon$, with $\epsilon \ll 1$. If one prefers an invariant definition one can replace $\frac{1}{A(r)}$ by $(\nabla r)^2$, where $\nabla$ is the gradient. (b) For such a small but nonzero $\epsilon$ the configuration is regular everywhere with a nonvanishing metric function $B(r)$; (c) In the limit $\epsilon \to 0$ the metric coefficient $B(r) \to 0$ for all $r \leq r^*$. In this limit $r^*$ is the horizon radius $r^+, r^* = r^+$. These three features define a quasiblack hole. For further details, see $^{26, 27}$

It is relevant to compare the form of the metric potentials of a quasiblack hole with the form of the metric potentials of an extremal electrically charged black hole. For an extremal electrically charged black hole, i.e., an extremal Reissner-Nordström black hole with mass equal to charge, $m = q$, one has for the metric potentials the relations $B(r) = \frac{1}{A(r)} = \left(1 - \frac{r^+}{r}\right)^2$, with also $r^+ = m$. In Figure 4 typical plots of $B(r)$ and $\frac{1}{A(r)}$ as functions of $r$ are displayed for quasiblack and black holes showing clearly the differences between the two cases.

![Graphs showing metric potentials B(r) and 1/A(r) for quasiblack and black holes.](image)

Fig. 4. The metric potentials $B(r)$ and $\frac{1}{A(r)}$ as functions of $r$ for an extremal quasiblack hole and for an extremal black hole. In the region $r < r_+$ the functions are totally different while in the exterior they are the same.
There are several important and generic properties of quasiblack holes that should be mentioned, see also. 1. A quasiblack hole is on the verge of forming an event horizon, instead, a quasihorizon appears. 2. Quasiblack holes with finite stresses must be extremal to the outside. 3. The curvature invariants of extremal and nonextremal quasiblack hole spacetime remain regular everywhere. 4. A free-falling observer in a quasiblack hole spacetime finds in its frame infinite tidal forces at the quasihorizon. This shows some form of degeneracy, i.e., a combination of features typical of regular and singular systems, at the quasihorizon. 5. Both in an extremal and in a nonextremal quasiblack hole spacetime, outer and inner regions become mutually impenetrable and disjoint. An interesting example is the Lemos-Weinberg solution, where the interior is a Bertotti-Robinson spacetime, the quasihorizon region is extremal Bertotti-Robinson, and the exterior is extremal Reissner-Nordström. 6. There are infinite redshift whole 3-regions. 7. For far away observers a quasiblack hole spacetime is indistinguishable from that of a black hole. Quasiblack holes are black hole mimickers.

4. Carter-Penrose diagram for quasiblack holes

Carter-Penrose diagrams are a useful tool to understand the conformal and causal structure of a spacetime. For a spherically symmetric star composed of vacuum and a thin shell, the corresponding Carter-Penrose diagram is composed of the timelike origin, the Minkowski interior bounded by the timelike radius \( R \) of the thin shell star, and the exterior with the past and future null infinities, together with the past and future timelike infinities and the spatial infinity. For the Carter-Penrose diagram for a quasiblack hole, made of a Minkowski spacetime inside, a thin shell made of some matter at the boundary \( R = r_+ \), and an exterior Reissner-Nordström spacetime outside see where a comparison with the thin shell star is also made.

The Carter-Penrose diagram for a generic static spherically symmetric quasiblack hole spacetime, extremal or nonextremal, is shown in Figure 5. There are two separated causal regions which are mutually impenetrable and disjoint. This is connected to the fact that tidal forces are infinite at the boundary which acts as a barrier. Infinite energy particles could, in principle, penetrate this barrier, but those would destroy the spacetime. The interior region is composed of some form of matter that extends up to the timelike boundary \( R = r_+ \). In the interior, outgoing light rays are reflected at the timelike boundary, to become ingoing light rays, and so forth. The outer region is indistinguishable from a Reissner-Nordström black hole outer region, it has the past and future null horizons \( r_+ \), the past and future null infinities, together with the past and future timelike infinities and the spatial infinity. The null horizons \( r_+ \) are naked horizons as they show some form of singular behavior. One can see that quasiblack holes have special causal properties, and their Carter-Penrose diagrams show that they are a natural blend of stars, regular black holes, and null naked horizons.
Fig. 5. Carter-Penrose diagram for a generic quasiblack hole, extremal and nonextremal. The interior is a spacetime with matter, has the timelike \( r = 0 \) origin, has its past and future timelike infinity, \( i^+ \) and \( i^- \), respectively, and the infinity \( \text{scri} I \) which is an impenetrable barrier bounded by a timelike line at radius \( R = r_+ \). The exterior region is identical to the outer Reissner-Nordström black hole region, with the past and future horizons at \( r_+ \), which are null lines, the past and future timelike infinity, \( i^- \) and \( i^+ \), respectively, the spatial infinity \( i^0 \), and the past and future null infinity, \( \text{scri minus} I^- \) and \( \text{scri plus} I^+ \), respectively. Interior and exterior are mutually impenetrable and disjoint with \( r_+ \) playing the role of a kind of singular boundary. The symbol \( \# \) means the connected sum of the two disjoint spacetimes.

5. Pressure properties of quasiblack holes

A quasiblack hole has matter in its interior and so it is important to study its pressure properties, particularly at the boundary \( R = r_+ \). We restrict to the quasiblack holes that are extremal on the exterior, the quasiblack holes that are nonextremal have unbound pressure at \( R = r_+ \).

Let us analyze the case in which the matter is nonextremal in the interior region, to start. Imposing finite Riemann tensor components \( R_{abcd} \) in an orthonormal frame, and denoting radial pressure by \( p_r \), we find

\[
p_{r}^{\text{in}}(r_+) = -\frac{1}{8\pi r_+^3}.
\]

For an extremal exterior region one also has

\[
p_{r}^{\text{out}}(r_+) = -\frac{1}{8\pi r_+^3}.
\]

So, we find

\[
p_{r}^{\text{in}}(r_+) = p_{r}^{\text{out}}(r_+),
\]

i.e., this type of quasiblack holes have continuous pressure at the boundary, surely a neat result. If matter in the interior region is not electrical then

\[
p_{r}^{\text{in}}(r_+) = p_{r}^{\text{in\,\,matter}}(r_+),
\]

and the quasiblack holes in question are supported by tension. If matter in the interior region is electrical then through the whole interior the pres-
sure \( p^\text{in}_r(r) \) is composed of a matter part \( p^\text{in}_r \text{matter}(r) \) and of an electromagnetic part \( p^\text{in}_r \text{em}(r) \), such that \( p^\text{in}_r(r) = p^\text{in}_r \text{matter}(r) + p^\text{in}_r \text{em}(r) \), but at \( r_+ \) one finds obligatory that \( p^\text{in}_r \text{matter}(r_+) = 0 \), and so \( p^\text{in}_r(r_+) = p^\text{in}_r \text{em}(r_+) \). This means that in this case all pressure support at the boundary comes from the electric part and there is no pressure support from matter at \( r_+ \). This case, in contrast to the previous one, can arise from a quasistatic shrinking star. Indeed, a star with radius \( R \) has to have zero pressure at the boundary, \( p^\text{in}_r \text{matter}(R) = 0 \), so that the boundary is static. One can then contract the star in quasistatic way to \( r_+ \) to yield a quasihole with \( p^\text{in}_r \text{matter}(r_+) = 0 \). Note also, that since the matter inside is not extremal, by assumption, there is a jump in the density at \( r_+ \), but such jumps in density pose no problems.

Let us now analyze the quasihole case in which the matter is extremal in the interior region. Imposing finite Riemann tensor components \( R_{abcd} \) in an orthonormal frame we find

\[
p_r(r+) = -\rho(r+).
\]

This is the same condition as found for dirty black holes. One can prove further, in this case of matter being extremal in the interior region, the following: (i) One cannot build an interior extremal quasihole entirely from phantom matter, i.e., one cannot build such a quasihole with the matter violating the null energy condition, namely, \( p_+ + \rho < 0 \), everywhere inside. (ii) In case there is phantom matter, it cannot border the quasihorizon, it has to be in the inner region. At least in a vicinity of the quasihorizon the null energy condition, \( p_+ + \rho \geq 0 \), is satisfied.

### 6. Mass formula for quasiholes

The quasihole, being an object on the verge of becoming a black hole shares several properties with black holes themselves. We now work out the mass formula for quasiholes. One needs two steps. In the first one makes sure that at the quasihorizon the Kretschmann scalar is finite. In the second one uses the Tolman mass definition to find a mass formula for quasiholes.

For the first step it is useful to rewrite the metric given in Eq. (1) and put it in Gaussian coordinates valid near the quasihorizon. Redefining the metric potentials \( B \) and \( A \) in Eq. (1) such that \( N = B \) and the new radial coordinate \( l = \frac{dr}{\sqrt{A}} \), the metric takes the form

\[
ds^2 = -N^2 dt^2 + dl^2 + g_{ab} dx^a dx^b,
\]

with \( a, b = 1, 2 \). The Kretschmann scalar \( K_r \) for this metric is given by

\[
K_r = P_{ijkl} P_{ijkl} + 4 C_{ij} C_{ij},
\]

where \( i, j = 1, 2, 3 \) are spatial indices, \( P_{ijkl} \) is the curvature tensor for a \( t = \text{const} \) hypersurface, and

\[
C_{ij} = \frac{N_{ij}}{N},
\]
with \( \mathcal{D} \) denoting a covariant derivative. As the metric of the 3-space is positive definite, all terms enter the expression with a positive sign, so if we impose finiteness for \( \kappa r \) this means each term should be finite separately. The scalar \( P_{ijkl}P_{ijkl} \) for the metric (4) is clearly finite. So we have to deal with the term \( 4C_{ij}C^{ij} \). By definition, quasiblack hole implies that the metric potential \( N \) in Eq. (4) satisfies \( N = N(x^a) \to 0 \). Choose \( l = 0 \) on the surface of the object, without loss of generality. Putting \( \frac{\partial}{\partial r} \) and \( \kappa \alpha \) the covariant derivative for the two-metric \( g_{ab} \) in Eq. (4), we find from Eq. (6)

\[
\lim_{l \to 0} C_{ll} = \lim_{l \to 0} \frac{N''}{N_0}, \quad \lim_{l \to 0} C_{a\ell} = \lim_{l \to 0} \frac{N'_a}{N_0}.
\] (7)

Finiteness of \( \kappa r \) implies \( \lim_{r \to 0} \lim_{l \to 0} N'' = 0 \) and \( \lim_{r \to 0} \lim_{l \to 0} N'_a = 0 \). Near the quasihorizon, we expand the metric function \( N \) as \( N = N_0 + \kappa_1(x^a, \epsilon)l + \kappa_2(x^a, \epsilon)r^2 + \kappa_3(x^a, \epsilon)r^3 + O(l^4) \), where \( N_0 \) is some constant, \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) are functions that have to be determined. The first part of Eq. (7) implies \( \lim_{r \to 0} \kappa_2 = 0 \). The second part of Eq. (7) implies \( \lim_{r \to 0} \kappa_1(x^a, \epsilon) = \kappa \), where \( \kappa \) is a constant, which is identified with the surface gravity of the surface. So at the quasihorizon one has

\[
N = N_0 + \kappa l + \kappa_3(x^a)\frac{r^3}{3!} + O(l^4).
\] (8)

For the second step we use the fact that when there is matter there is mass and that mass is given by the Tolman formula. The Tolman formula for the mass \( m \) of an object is

\[
m = \int (-T^0_0 + T^i_i)\sqrt{-g} \, d^3x,
\] (9)

where \( T^0_0 \) and \( T^i_i \) are the components of the energy-momentum tensor, \( g \) is the determinant of the metric, and the integral is performed over the region of interest. Here, it is convenient to split the mass \( m \) into three parts, namely,

\[
m = M_{\text{in}} + M_{\text{surf}} + M_{\text{out}},
\] (10)

where \( M_{\text{in}} \) is the interior mass, \( M_{\text{surf}} \) is the surface mass, and \( M_{\text{out}} \) is the outer mass. Let us analyze each one in turn. The interior mass is \( M_{\text{in}} = \int_{m} (-T^0_0 + T^i_i) N \sqrt{g} \, d^3x \), so that \( M_{\text{in}} \leq N_B (M_0 + M_k) \), with \( N_B \) being the value of the metric potential at the boundary and \( M_0 \) and \( M_k \) being the components of the mass in an obvious notation. Since for a quasiblack hole \( N_B \to 0 \) one has

\[
M_{\text{in}} = 0.
\] (11)

For the surface, from Dirac-\( \delta \) contributions, we define \( S^\mu_\nu \) as \( S^\mu_\nu = \int T_{\mu\nu} \, dl \). So that one gets \( M_{\text{surf}} = \int (-S^0_0 + S^\alpha_\alpha) N \, d\sigma \). Now, one has \( 8\pi S^\mu_\nu = ([K^\mu_\nu - \delta^\mu_\nu K]) \), where \( K^\mu_\nu \) is the extrinsic curvature tensor, \([\ldots] = ([\ldots]_+ - ([\ldots]_-), + and - refer to the outer and inner sides. After calculating \( K^\nu_\nu \) for the metric (4) one finds \( M_{\text{surf}} = \frac{1}{4\pi} \int_{\text{surf}} \left[ \left( \frac{\partial N}{\partial r} \right)_+ - \left( \frac{\partial N}{\partial r} \right)_- \right] \, d\sigma \), where \( d\sigma \) is the two-surface element. Now
\((\partial N/\partial l)_- \to 0\), and since from Eq. (8) one has \(N = N_0 + \kappa l + \kappa_3 (x^a)^3 + O(l^4)\), one finds \((\partial N/\partial l)_+ = \kappa\). So finally,

\[ M_{\text{surf}} = \frac{\kappa A_+}{4\pi}, \]  

(12)

\(A_+\) being the quasihorizon area. For the outer mass \(M_{\text{out}}\) one has \(M_{\text{out}} = \int_{\text{out}} (-T_{00} + T_k^k) N \sqrt{g} d^3x\), so when one includes an electromagnetic field one finds

\[ M_{\text{out}} = \varphi_+ q + M_{\text{matter}}^{\text{out}}, \]  

(13)

where \(\varphi_+\) is the electric potential at the quasihorizon \(r_+\), \(q\) is the outer electric charge, and \(M_{\text{matter}}^{\text{out}}\) is the matter that lingers outside \(r_+\), e.g., matter in an accretion disk.

With steps one and two in hand, one can now put all the masses together to obtain the total mass \(m\) of a static system containing a quasiblack hole, namely,

\[ m = \frac{\kappa A_+}{4\pi} + \varphi_+ q + M_{\text{out}}^{\text{matter}}, \]  

(14)

In vacuum, \(M_{\text{out}} = 0\), and the mass formula becomes \(m = \frac{\kappa A_+}{4\pi} + \varphi_+ q\). For the extremal case \(\kappa A_+ \) goes to zero, as \(\kappa = 0\), and since one can set \(\varphi_+ = 1\) one gets \(m = q\) as the mass formula for a pure extremal quasiblack hole. Note that in the nonextremal case one has \(M_{\text{surf}} \neq 0\) and the surface of the quasiblack hole makes no contribution to the mass. When there is rotation the quasiblack hole has angular velocity \(\omega_+\) and so there is also angular momentum \(J\). In this case the mass formula for quasiblack holes is

\[ m = \frac{\kappa A_+}{4\pi} + 2\omega_+ J + \varphi_+ q + M_{\text{out}}^{\text{matter}}, \]  

(15)

In vacuum, \(M_{\text{out}} = 0\), and the mass formula becomes \(m = \frac{\kappa A_+}{4\pi} + 2\omega_+ J + \varphi_+ q\).

Two comments are in order. First, the mass formulas that appear in Eqs. (14) and (15) have the same form as the mass formulas for pure black holes, i.e., the Smarr formula\(^{32}\) and for black holes and surroundings\(^{33}\) but were obtained from totally different means. For black holes, since these are vacuum solutions, the appropriate mass definition is the Komar mass, which is a totally different definition from the Tolman mass that we used for quasiblack holes. Strikingly, either mass definition yields the same mass formula. Second, the surface gravity \(\kappa\) that appears in the mass formulas of Eqs. (14) and (15) have for black holes the interpretation of being the acceleration a test particle experiences at \(r_+\) redshifted to infinity. For quasiblack holes one can have an alternative interpretation, \(\kappa\) being the surface density of the matter at the surface of the quasiblack hole. This interpretation was unveiled when studying the mass formula for black holes from the membrane paradigm perspective.\(^{35}\)
7. Entropy of quasiblack holes

7.1. Rationale

A fundamental test for the theory of quasiblack holes is to find their entropy $S$. Imagine a collapsing body. Its matter has some entropy that might grow or not during the collapse, but when the surface is at the horizon $r_+$ there is no apparent reason for the entropy to turn into $S = \frac{1}{4} A_+$, it appears as a jump. We will see that by working with a quasistatic contraction and a quasiblack hole approach one sheds some light on the origin of this $S = \frac{1}{4} A_+$ entropy.

We use the first law of thermodynamics for the matter and gravitational fields. In general, to find the entropy of a system one needs an equation of state. One then integrates the first law on a path along the energy first, say, and then along a volume, or a length, or some other useful quantity. With the quasiblack hole approach one can do differently, dispensing altogether an equation of state for the matter. For a quasiblack hole one picks a different path, i.e., one chooses a sequence of configurations such that all members remain on the threshold of horizon formation and then one integrates over this subset of members. The answer has to be independent of the model and independent of the equation of state for the matter. The approach explores the fact that the boundary of the matter tends to a quasihorizon in the quasiblack hole limit.

7.2. Entropy of spherical quasiblack holes

7.2.1. Generics

Let us suppose that there is a spherically symmetric star spacetime composed of some interior spacetime with a fluid with energy density $\rho$, tangential pressure, radial pressure $p_r$, and a boundary surface at radius $R$, plus an outside Schwarzschild spacetime which is characterized by the spacetime mass $m$ or equivalently by the gravitational radius $r_+ = 2m$. We want to take the radius of the star to the quasiblack hole limit, $R = r_+$, analyze the first law of thermodynamics for the system and find its entropy $S$.

For this purpose we rewrite the metric of Eq. (1), now putting $B(r) = -N^2(r)e^{2\psi(r)}$ and $A(r) = \frac{1}{N^2(r)}$, so that the star’s static spherically symmetric metric is written as

$$ds^2 = -N^2(r)e^{2\psi(r)} dt^2 + \frac{dr^2}{N^2(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  

(Eq. 16)

Einstein field equations yield for this metric

$$N^2(r) = 1 - \frac{8\pi}{r} \int_0^r d\bar{r} \bar{r}^2 \rho,$$  

(Eq. 17)

$$\psi(r) = 4\pi \int_0^r d\bar{r} \frac{(\rho + p_r) \bar{r}}{N^2(\bar{r})}.$$  

(Eq. 18)
There is another equation involving the tangential pressure that we do not write here. The matter is constrained to the region \( r \leq R \), so that for \( r \geq R \) one has \( \psi(r) = 0 \) and \( N^2 = 1 - \frac{r_+}{r} \), i.e., the Schwarzschild solution.

For a gravitational system the first law of thermodynamics is given in terms of boundary values where the boundary can be put at any radius \( r \). Since we are working with a spherically symmetric spacetime the first law of thermodynamics can be written as

\[
TdS = dE + pdA,
\]

where \( T \) is the local temperature, \( S \) the entropy, \( E \) the energy, \( p \) the tangential pressure, and \( A \) the area, with all these quantities being locally defined quantities at a radius \( r \). In particular the quantities are well defined at the interface \( R \), see Figure 6.

![A spherical star approaching its quasiblack hole state and its thermodynamic quantities at the quasihorizon \( R = r_+ \).](image)

Fig. 6. A spherical star approaching its quasiblack hole state and its thermodynamic quantities at the quasihorizon \( R = r_+ \).

We want to find the entropy \( S \) when the radius of the star is at its own gravitational radius, \( R = r_+ \), i.e., at the quasiblack hole limit. We use the quasiblack hole condition that \( N \to 0 \). Then, we change simultaneously the radius \( R \) and the thermodynamic energy \( E \), keeping the interface \( R \) near the gravitational radius \( r_+ \), with the condition \( N \to 0 \), for all configurations of interest in this process. Mathematically we write \( R = r_+ (1 + \delta) \) for some \( \delta \) and send \( \delta \to 0 \) ensuring the star is kept near the quasihorizon. Then we integrate the first law along such a sequence of quasihorizon configurations, counting different members of the same family of states and obtain \( S \) at a given \( r_+ \).
7.2.2. Example: Spherically symmetric quasiblack holes with vacuum
Minkowski interior and a thin shell at the boundary

An interesting system to work with, which allows not only a thermodynamic solution through the quasiblack hole approach but also an exact thermodynamic solution, is a thin shell system. The whole system is composed of Minkowski spacetime in the interior, a thin shell of radius \( R \) at the junction, and Schwarzschild spacetime in the exterior region. The exterior spacetimes mass is \( m \) and its gravitational radius is \( r_+ = 2m \), so that \( m \) and \( r_+ \) can be interchanged. The metric potential \( N \) of Eq. (16) for the exterior is given by

\[
N^2 = 1 - \frac{r_+}{R},
\]

and the other metric potential is \( \psi = 0 \). The thin shell at radius \( R \) has proper mass \( M \) which is also the thermodynamic energy \( E \), i.e., \( M = E \). The relation between \( m, E \) and \( R \) is

\[
m = E - \frac{E^2}{2R},
\]

and since \( r_+ = 2m \), one has \( r_+ = 2E - \frac{E^2}{R} \), or solving for \( E \) gives the proper mass or thermodynamic energy \( E \) as

\[
E = R \left( 1 - N \right).
\]

The tangential pressure \( p \) that supports the thin shell is taken from the junction condition and is given by

\[
p = \frac{(1 - N)^2}{16\pi RN}.
\]

The shell has a temperature \( T \) and area \( A = 4\pi R^2 \). The first law is given in Eq. (19).

Now we use the quasiblack hole approach. In general to integrate the first law (19) one needs equations of state for \( p \) and \( T \). But not here. Here we want to integrate the first law when the system is near \( r_+ \). Then, in the process of integrating the first law, all three quantities \( R, r_+, \) and \( E \), change, but we impose that they change in such a way that \( R = r_+ (1 + \delta) \), and so \( N^2 = \delta \), with \( \delta \) small. In other words, we change simultaneously and proportionally \( R \) and \( r_+ \) when passing from one equilibrium configuration to another. Then the pressure term \( p \) in Eq. (22),

\[
p = \frac{1}{16\pi RN} \approx \frac{1}{16\pi r_+ N},
\]

is huge since \( N = \sqrt{\delta} \) is small. Also one has \( dA = 8\pi r_+ dr_+ \). The term \( dE \) given from Eq. (21) is \( dE \approx dR \approx dr_+ \), and so is negligible. The local temperature \( T \) at the shell and the temperature at infinity \( T_0 \), say, are related by the Tolman formula, namely, \( T = \frac{T_0}{N} \), where in the quasiblack hole state \( N = \sqrt{\delta} \). One should now note that near the quasihorizon \( R = r_+ \) the backreaction of quantum fields is divergent unless \( T_0 \) is the Hawking temperature \( T_H \), i.e., \( T_0 = T_H = \frac{\kappa}{4\pi} = \frac{1}{4\pi r_+} \). Then, putting altogether in the first law (19), we find \( dS = 2\pi dr_+ \), i.e.,

\[
S = \frac{1}{4} A_+,
\]

where \( A_+ = 4\pi r_+^2 \) is the area of the quasihorizon, and we have put the constant of integration to zero. This is the Bekenstein-Hawking entropy for black holes. For details see. For a related treatment involving the membrane paradigm see.
Now we use the thin shell approach, as the thin shell spacetime offers an alternative route to perform the calculations different from the quasiblack hole approach, and permits an exact solution of the thermodynamic problem. Indeed, the solution for the thin shell thermodynamic problem is exact and valid for all $R$, not only $r_+$. Take the first law as given in Eq. (19). Again, the junction condition gives the tangential pressure for any radius $R$ as \[ p = \frac{1}{16\pi} \frac{(1-N)^2}{RN^2}, \] with $N^2 = 1 - \frac{r_+}{R}$. Now, take into account the integrability condition that comes out of the first law (19), and change variables from $(E,R)$ to $(r_+, R)$. This integrability condition gives that the local temperature $T$ has to have the form \[ T = \frac{T_0(r_+)}{r_+}, \] necessarily, where $T_0(r_+)$ has the usual meaning of the temperature measured by an observer at infinity. Clearly this equation for $T$ is the other equation of state needed to integrate the first law (19). Inserting the two equations of state, namely for $p$ and for $T$, into the first law we find \[ dS = \frac{dr_+}{2T_0(r_+)}. \] Hence the entropy can be found by direct integration once the equation of state $T_0(r_+)$ is known. In particular, we can choose then $T_0$ as the Hawking temperature $T_H$, i.e., \[ T_0(r_+) = T_H = \frac{1}{4\pi r_+}. \] In this case we find $S = \frac{1}{4} A_+$. For a thin shell with the Hawking temperature, the formula is valid for any shell radius $R$, including the quasiblack hole radius $R = r_+$. Moreover, for a quasiblack hole $T_0(r_+) = T_H = \frac{1}{4\pi r_+}$ is the only equation of state possible, to avoid infinite back reaction effects. So $S = \frac{1}{4} A_+$ is the only solution for the entropy in this case, in conformity with the quasiblack hole approach of the last paragraph. Note also that the entropy of a thin shell does not depend on $R$. This is a consequence of the fact that there is no matter inside, one has $\frac{dS}{dR} = 0$ everywhere. For the thermodynamics of a Schwarzschild thin shell see the thermodynamics of the thin shell at its gravitational radius limit and quasiblack hole limit were taken in this paragraph.

7.2.3. Example: Spherically symmetric quasiblack holes with a continuous distribution of matter in the inside

We now find the entropy of a spherically symmetric quasiblack hole with a continuous distribution of matter in the inside, rather than a thin shell. For spherical symmetric general configurations with a continuous distribution one has that $\frac{dS}{dR}$ is nonzero. In general, an exact thermodynamic solution for a star with a continuous distribution of matter cannot be found. For such a star, the temperature depends on $r_+$ and $R$ in some complicated fashion, so that the entropy $S$ of the star will also depend on $r_+$ and $R$. The integrability conditions are of no use and it is virtually impossible to make progress. However, if the star is at the quasiblack hole limit, it is possible to implement the quasiblack hole approach and find the entropy from the first law of thermodynamics.

Consider then a general metric for a spherically symmetric distribution of matter. The metric is the one given in Eq. (16). For spherically symmetric systems, the first law of thermodynamics given in Eq. (19) can be written alternatively and usefully in terms of the temperature at infinity $T_0$, $r_+$, $R$, and the radial pressure.
of the matter as
\[ T_0 \, dS = \frac{1}{2} \exp \psi(R) \, (dr_+ + 8\pi p_+ R^2 dR). \] (24)

Although, without knowing the system details, one cannot find \( S(r_+, R) \) in general, for spherical quasiblack holes one can bypass this lack of knowledge and deduce the entropy of the system through a series of steps. The steps are: (i) Since we want \( R \to r_+ \) we also have to put \( T_0 \) as the Hawking temperature in order that the backreaction of quantum fields remains finite, \( T_0 \to T_H \). Now, for the metric (16), \( T_H \) is given by
\[ T_H = e^{\psi(r_+)} \frac{4}{4\pi} \sqrt{g} \, dN \, (r_+), \] i.e., at the quasihorizon \( R = r_+ \) one finds, \( T_0 = T_H = e^{\psi(r_+)} \left( 1 - 8\pi \rho(r_+) r_+^2 \right) \). Thus, substituting into the first law of Eq. (24) we obtain
\[ dS = \exp(\psi(R) - \psi(r_+)) \frac{2\pi r_+}{1 - 8\pi \rho(r_+)} dr_+ + 8\pi p_+ r_+^2 dR. \] This equation gives the change of the entropy in terms of the changes of \( r_+ \) and \( R \). (ii) We are interested in the quasihorizon limit, we want to move along the line \( R \approx r_+ \) in the space of parameters, so that \( dR \approx dr_+ \). Then for \( R \to r_+ \), the factor \( \exp(\psi(R) - \psi(r_+)) \) in \( dS \) drops out, so
\[ dS = 2\pi r_+ \frac{dr_+ + 8\pi p_+ R^2 dR}{1 - 8\pi \rho(r_+)} \] Also, from the regularity conditions on the pressure at the quasihorizon one has \( p_+(r_+) = -\rho(r_+) \) and the terms in the numerator and denominator cancel. Thus, putting this into the first law yields
\[ dS = 2\pi r_+ dr_+. \] (iii) Then upon integration on finds \( S = \frac{1}{4} A_+ \), i.e., one gets back the Bekenstein-Hawking entropy.

### 7.3. Entropy of nonspherical quasiblack holes

One can also study the thermodynamics and the entropy of nonspherical quasiblack holes, see Figure 7. In this generic nonspherical case it is appropriate to use again Gaussian coordinates for which the line element can be written as in Eq. (4). Suppose that the boundary of the compact body is at \( l = 0 \) without loss of generality.

The local Tolman temperature at the surface \( l = 0 \) is denoted by \( T \), whereas \( T_0 \) is the temperature at asymptotically flat infinity. The relation between the two temperatures is the Tolman relation \( T = \frac{T_0}{N} \). Since now there is no spherical symmetry we cannot write the first law, Eq. (19) in terms of the quantities \( S, E, \) and \( A \), which were defined over the whole sphere. Instead we have to resort to densities, i.e., quantities per unit area. They are \( s, \varepsilon, \) and \( a \), the entropy density, energy density, and unit area, respectively. The first law of thermodynamics in terms of these densities at the boundary can then be written as
\[ T d(\sqrt{g} \, s) = d(\sqrt{g} \, \varepsilon) + \frac{\Theta_{ab}}{2} \sqrt{g} \, dg_{ab}. \] (25)

Here \( g \) is the determinant of the two-metric \( g_{ab} \) defined in Eq. (4), the energy density on the layer is defined as \( \varepsilon = \frac{K_{ab}}{8\pi} \) with \( K = K_{ab} g^{ab} = -\frac{1}{\sqrt{g}} \sqrt{g} \, \varepsilon \), a prime meaning derivative with respect to \( l \), \( K_{ab} \) being the extrinsic curvature of the two-surface, and the spatial energy-momentum tensor on the layer \( \Theta_{ab} \) is \( 8\pi \Theta_{ab} = K_{ab} + \left( \frac{N}{N'} - K \right) g_{ab} \). These quantities include matter as well as gravitational fields.
We want to integrate the first law of thermodynamics as given in Eq. (25) to obtain the entropy $S$ of the system in a quasiblack hole state. The need for equations of state are avoided because we use the quasiblack hole approach. Again, we choose a sequence of configurations such that all members remain on the threshold of horizon formation and then integrate over this very subset, the answer must be once again model independent. Several conditions must be met. The quasiblack hole limit means that $N \to 0$. To have a regular horizon to an outside observer one has $K_{ab} = k_{ab} l + O(l^2)$, where $k_{ab}$ is some constant tensor. Then the energy density $\varepsilon$ remains finite, indeed $\varepsilon = \frac{K}{8\pi}$. The spatial stresses, $\Theta_{ab}$ given by $\Theta_{ab} = \frac{1}{8\pi} \left( K_{ab} + \left( \frac{\dot{N}}{N} - K \right) g_{ab} \right)$, diverge due to the term $\frac{\dot{N}}{N}$, and $N = 0$ in the quasiblack hole state. In the outer region we have seen that $\dot{N} = \kappa$, where $\kappa$ is the surface gravity. So the dominant contribution to the first law given in Eq. (25) comes from the term $\frac{\Theta^{ab}}{2} \sqrt{g} dg_{ab}$, which then becomes $d(\sqrt{g} s) = \frac{\kappa}{16\pi T_0} \sqrt{g} g^{ab} dg_{ab}$. Take again into account that near the quasihorizon the backreaction of quantum fields becomes divergent unless one makes the choice that $T_0$ is the Hawking temperature $T_H$, i.e., $T_0 = T_H = \frac{\kappa}{4\pi}$. Thus the first law is now $d(\sqrt{g} s) = \frac{1}{2} d\sqrt{g}$, and so the entropy density at the quasihorizon of a quasiblack hole is $s\sqrt{g} = \frac{1}{4} \sqrt{g}$, up to a constant which we put to zero. Upon integration over the surface, i.e., $\int d^2 x$, we obtain $S = \frac{1}{4} A_+$, where again $A_+$ is the area of the horizon of the quasiblack hole, for details see. This is the Bekenstein-Hawking entropy for a black hole.

7.4. Entropy of electrically charged quasiblack holes

The entropy of electrically charged spherically symmetric quasiblack holes can also be dealt with using the quasiblack hole approach and the calculation follows the same lines as above, see also. In particular one can calculate the entropy of a
electrically charged thin shell with a Minkowski interior and a Reissner-Nordström exterior. The electric thin shell also has, through the integrability conditions of the first law of thermodynamics, an exact thermodynamic solution, which yields \( S = \frac{A}{4} \), when the temperature of the shell is the Hawking temperature. When the radius of the shell is put to its own gravitational radius, then the entropy of this quasiblack hole is the Bekenstein-Hawking entropy, for details see.\(^{40}\)

7.5. **Entropy of other quasiblack holes: rotating black holes in three dimensions, and black holes in \( d \) dimensions**

The quasiblack hole approach can be applied to a number of other situations. The thin shell approach, through its integrability conditions, it is also of great interest in these cases. It has been applied for nonrotating and rotating shells in three-dimensional anti-de Sitter spacetimes and, upon taking the quasiblack hole limit, one find the entropy of the corresponding BTZ black hole, for details see.\(^{41,42}\) The entropy of static shells in \( d \)-dimensions and the quasiblack hole limit has also been analyzed, see.\(^{43}\)

8. **Entropy of extremal quasiblack holes and implications to the entropy of extremal black holes**

The ultimate test of the quasiblack hole approach and formalism is the study of the entropy of quasiblack holes in the extremal case. This is because the entropy of extremal black holes is a particularly intriguing and interesting problem. Arguments based on the periodicity of the Euclidean section of the black hole lead one to assign zero entropy in the extremal case, \( S = 0 \). This value \( S = 0 \) is obtained because the Euclidean time, and so the temperature, is not fixed in a classical calculation of the action for extremal black holes, indeed \( T \) or its inverse \( \beta \) can take any value, and this forces a zero entropy value in the path-integral action approach.\(^{44,45}\) However, extremal black hole solutions in string theory typically have the conventional value given by the area formula \( S = \frac{A}{4} \), a value that is obtained from counting string states of a black hole within string theory.\(^{46}\) Up to now the issue has not been settled. Neither it has been showed that \( S \) should be nonzero for extremal black holes nor it has been shown that it should indeed be \( S = \frac{A}{4} \). The situation remains to be clarified.

The quasiblack hole approach yields a method of facing this problem and proposes a solution on the basis of pure thermodynamics. We use the quasiblack hole approach and we also use the thermodynamic exact solution of a thin shell to find a thermodynamic solution for the entropy.

Let us apply the quasiblack hole approach for extremal systems. The metric is given in Eq. (16), and for the outside one has \( \psi = 0 \) and \( N^2 = \left(1 - \frac{r_-}{r_+}\right)^2 \), where \( r_+ = m = q \), since we are dealing with the extremal case. The electric potential is written generically as \( \phi(r) = \frac{r_-}{r} \) up to a constant. The first law given in Eq. (19)
in terms of boundary values is now generalized to include electric charge

\[ TdS = dE + pdA - \varphi dq, \]  

(26)

where \( \varphi \) is the thermodynamic electric potential at the boundary and \( q \) its electric charge, and the other quantities were defined previously. Note that \( p \) is a thermodynamic tangential pressure defined in \( TdS \).

We go through the terms \( dE, pdA, -\varphi dq \) carefully in order to understand the term \( TdS \) in the end. First, we deal with \( dE \). Clearly, in the extremal case \( E = r_+ \), so \( dE = dr_+ \). Second, we deal with \( pdA \). One can show that

\[ 8\pi pR = \frac{4\pi p_{\text{matter}} R^2}{1 - \frac{r_+}{R}}. \]

It is clear that \( p \) is a two-dimensional pressure and \( p_{\text{matter}} \) is a three-dimensional pressure, and also that \( p \) is a blue shifted pressure to \( R \). Now, to make progress we have to understand the system at the threshold of being a quasiblack hole. We have to take into account that on the quasihorizon \( p_{\text{matter}}(r_+) = 0 \) according to our general results on pressure. When matter is absent in the inner region, as in a thin shell, this condition is exact. When there is matter, one can write quite generally \( p_{\text{matter}}(r) = \frac{b(r_+, R)}{4\pi R^2} (1 - \frac{r_+}{R}) \), valid near \( R = r_+ \) and with the function \( b(r_+, R) \) being model-dependent. The point here is that when the body is sufficiently compressed it follows that \( p_{\text{matter}}(r_+) = 0 \). Thus, \( p = \frac{1}{4\pi} \frac{b(r_+, R)}{R} \).

Now, the area \( A \) is defined as \( A = 4\pi R^2 \), so that \( dA = 8\pi R dR \), and for \( R = r_+ \) we have \( dA = 8\pi r_+ dr_+ \). Third, we deal with \( -\varphi dq \). In the first law, the thermodynamic electric potential \( \varphi \) is the difference between the electric potential \( \phi_0 \) at a reference point and the electric potential \( \phi(R) = \frac{\varphi}{R} \) at the boundary \( R \), blue-shifted from infinity to \( R \) through the factor \( \frac{1 - \frac{r_+}{R}}{1} \), i.e., \( \varphi = \frac{(\phi_0 - \frac{\varphi}{R})}{(1 - \frac{r_+}{R})} \). Now for \( R \) near \( r_+ \) one has \( \phi_0 \to 1 \) so we can put \( \varphi = f(r_+, R) \). For \( dq \) in the first law, since at the quasihorizon limit \( q = r_+ \), one has \( dq = dr_+ \). We are now ready to analyze the entropy of extremal quasiblack holes.

We assume that the integrability conditions for the system are valid, otherwise there is no thermodynamic system. Then, since \( S \) is a total differential one can integrate along any path. Choose the path \( R = r_+(1 + \delta) \) with \( \delta \) constant and small, so that from what we found above we get the first law in the form

\[ TdS = (1 + b_+ - f_+) dr_+. \]

Clearly, the integrability condition yields that the local temperature \( T \) is of the form \( T = T(r_+) \). Then, \( dS = \frac{(1 + b_+ - f_+)}{T(r_+)} dr_+ \) where, \( b_+ = b(r_+, R = r_+) \) and \( f_+ = f(R = r_+) \). One can now integrate this equation to obtain \( S \), already with the limit taken \( R \to r_+ \), to obtain

\[ S = S(r_+) = \int_0^{r_+} d\bar{r}_+ \frac{D(\bar{r}_+)}{T(\bar{r}_+)}, \]  

(27)

where, \( D(\bar{r}_+) = 1 + b_+ - f_+ \). In general, one should require only that \( 1 + b_+ - f_+ > 0 \) to ensure the positivity of the entropy. Note that if the density of matter inside vanishes for \( r < R = r_+ \), we have a thin shell situation, \( b_+ \to 0 \) and \( f_+ \to 0 \), and so \( D(\bar{r}_+) = 1 \).
Now, the local temperature $T$ at the boundary $R$ is in general a function of $r_+$ and $R$, $T = T(r_+, R)$, and it is related to the temperature $T_0$ at infinity by the Tolman formula, $T = \frac{T_0}{N(r_+, R)}$, i.e., $T = \frac{T_0}{1 - r_+/R}$. But we have just deduced above that in the extremal case at the quasiblack hole limit $T$ is a function of $r_+$ solely, and not of $R$, $T = T(r_+)$. So, from the Tolman formula, the temperature at infinity $T_0$ has thus the form,

$$T_0 = T(r_+) \left(1 - \frac{r_+}{R}\right),$$

(28)

and therefore $T_0 = T_0(r_+, R)$. With Eqs. (27) and (28) in hand we can now draw several conclusions relatively to the entropy of extremal quasiblack holes.

One possible case, from Eq. (27), is the one that has finite generic local temperature $T(r_+)$, the entropy $S(r_+)$ of quasiblack holes is positive, $S(r_+) > 0$, and can be any well behaved function of $r_+$. Moreover, in this case, the temperature at infinity $T_0$, see Eq. (28), goes to zero in this limit, $T_0 \to 0$.

Another possible case in this approach is when the local temperature $T(r_+)$ behaves as $T(r_+) \to \infty$ when one assumes $T_0$ finite, i.e., $T_0 \neq 0$, see Eq. (28). This means that the local temperature $T(r_+)$ is infinite for every $r_+$ in the integration process of Eq. (27) and so one obtains $S = 0$. This case of extremal quasiblack hole behavior is equivalent to the path integral prescription given in 44, 45 for extremal pure black holes. Somehow, this case avoids the problem of the quantum stress-energy tensor that is given by $T^\text{quant}_{\nu\mu} = T^4 f_{\nu\mu} + h_{\nu\mu}$, where $f_{\nu\mu}$ and $h_{\nu\mu}$ are some finite tensorial functions, and that blows up when the local temperature $T$ goes to infinity.

Thus, on taking into account the two cases, i.e., $T_0 = 0$, and $T_0 > 0$, altogether, one can say that the entropy $S$ is a function of $r_+$, although an undetermined function, $S = S(r_+)$. In the extremal case the stresses are finite, and so one can deduce that not all possible modes are excited when the quasiblack hole state is approached. Since for nonextremal quasiblack holes $S = \frac{1}{4} A_+$ and all the possible modes due to the infinite stresses are excited here, one concludes that the entropy of extremal quasiblack holes should be $S \leq \frac{1}{4} A_+$. Changing the variable $r_+$ to $A_+$, we obtain that the entropy of an extremal quasiblack hole is $S = S(A_+)$ with $S(A_+)$ arbitrary, bounded from below by 0 and from above by $\frac{1}{4} A_+$, i.e.,

$$0 \leq S(A_+) \leq \frac{1}{4} A_+.$$

(29)

In brief, we showed consistently that the thermodynamic treatment does not give an unambiguous universal result for $S(A_+)$. The entropy depends on the properties of the working material and, moreover, on the manner the temperature approaches the zero value. In particular, $S = \frac{A_+}{4}$ is not singled out beforehand for the extremal black hole entropy. This holds for extremal quasiblack holes and by continuity for extremal black holes. So, our approach points to the conclusion that the extremal entropy depends on the manner the quasiblack hole, and thus the black hole, has formed, for details see 47, see also 48.
There is another important approach for the thermodynamics and entropy of extremal quasiblack holes that yields exact results. It is the thin shell approach, which is an exact solution as a general relativistic system and has an exact solution as a thermodynamic system. For an extremal shell one finds that there are three distinct cases. First, one starts with a nonextremal shell, studies its thermodynamics, puts it at the gravitational radius, and turns it extremal there. Not surprisingly, the entropy is \( S = \frac{1}{4} A_+ \), in this case only the pressure term contributes and all the modes have been excited. Second, one turns the shell extremal concomitantly with its approaching of its own gravitational radius. Surprisingly, one obtains \( S = \frac{1}{4} A_+ \), and in this case all forms of energy in the first law of thermodynamics contribute to give this value. Third, one turns the shell extremal and only afterwards one approaches the gravitational radius. One finds here that \( S = S(A_+) \), the entropy is any well behaved function of \( A_+ \). So can conclude again, now from the thin shell solution approach and its three distinct cases, that \( 0 \leq S \leq \frac{1}{4} A_+ \), i.e., Eq. (29) holds. This result shows that the quasiblack hole and the thin shell approaches are consistent. For details see. 49, 50 For extremal three-dimensional rotating black holes in anti-de Sitter spacetimes see, 51, 52 where the same type of results and conclusions are drawn.

9. Conclusions

Quasiblack hole solutions are matter solutions up to a boundary \( R \) which is a quasihorizon, i.e., \( R = r_+ \). A quasiblack hole is a regular solution in the sense that the Kretschmann scalar is finite everywhere, although there is some form of degeneracy at the horizon, namely, for external observers the horizon is a naked singular horizon. This degeneracy, that combines features typical of regular and singular systems, is made clear in the Carter-Penrose diagram of a quasiblack hole, where the interior and exterior regions are disjoint. The pressure properties of a quasiblack hole at the quasihorizon can be calculated and regularity imposes very strict conditions on them. A mass formula for quasiblack holes can be obtained, which is identical to the mass formula for black holes, although derived from totally different techniques. In studying the entropy of a nonextremal quasiblack hole we recover the Bekenstein-Hawking entropy. The result and the way it is established suggest that the degrees of freedom are on the horizon, since it is when a horizon is formed and the system has to settle to the Hawking temperature that the entropy takes the value \( S = \frac{A_+}{4} \). The results also suggests that the degrees of freedom are gravitational modes, since when the nonextremal quasiblack hole state is approached the tangential pressure goes to infinite or, more appropriately, to the Planck pressure. Then modes, presumably quantum gravitational modes, are induced. The difficult issue of the entropy of extremal black holes is a high point of the quasiblack hole approach. One finds that the entropy is a generic function of \( A_+ \), \( S = S(A_+) \), and the precise function depends on the manner the quasiblack hole has been formed. Moreover, the entropy of extremal quasiblack holes, and thus extremal black holes,
should be bounded by the Bekenstein-Hawking value \( \frac{A_+}{4} \), so that \( 0 \leq S \leq \frac{A_+}{4} \).

Several final remarks can be drawn. First, quasiblack holes are stars at the quasiblack hole limit and as such can be considered the genuine frozen stars, now that the frozen star name endorsed in the past for black holes has not this specific use anymore. Second, quasiblack holes, not black holes, are the real descendants of Mitchell and Laplace stars. Third, quasiblack holes are Schwarzschild and Buchdahl stars pushed, by use of some added repulsive charge, to their maximum compactification, i.e., the trapped surface limit. Fourth, in this sense, quasiblack holes are objects on the verge of becoming black holes, and as such can be envisaged as a metastate of spacetime and matter. Continued gravitational collapse ends in black holes, whereas quasistatic contraction passes through a quasiblack hole phase. Fifth, quasiblack holes have special properties, and their Carter-Penrose diagrams manifestly incorporate features of normal stars, regular black holes, and null naked horizons. Sixth, the quasiblack hole approach, in some of its aspects, is a cousin of the membrane paradigm. By taking a timelike matter surface into a null horizon we are recovering the membrane paradigm. One difference is that the quasiblack hole membrane is not fictitious like in the membrane paradigm, it is made of real matter. Finally, studies to understand the entropy of systems with other types of horizon can be taken with the quasiblack hole approach.

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