The Warm-starting Sequential Selection Problem and its Multi-round Extension

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Abstract

In the Sequential Selection Problem (SSP), immediate and irrevocable decisions need to be made as candidates randomly arrive for a job interview. Standard SSP variants, such as the well-known secretary problem, begin with an empty selection set (cold-start) and perform the selection process once over a single candidate set (single-round). In this paper we address these two limitations. First, we introduce the novel Warm-starting SSP (WSSP) setting which considers at hand a reference set, a set of previously selected items of a given quality, and tries to update optimally that set by (re-)assigning each job at most once. We adopt a cutoff-based approach to optimize a rank-based objective function over the final assignment of the jobs. In our technical contribution, we provide analytical results regarding the proposed WSSP setting, we introduce the algorithm Cutoff-based Cost Minimization (CCM) (and the low failures-CCM, which is more robust to high rate of resignations) that adapts to changes in the quality of the reference set thanks to the translation method we propose. Finally, we implement and test CCM in a multi-round setting that is particularly interesting for real-world application scenarios.

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1 Introduction

Since its introduction in the early 60’s, the secretary problem \cite{14, 16, 26} has been perhaps the most famous optimal stopping problem: \( n \) randomly incoming candidate secretaries are interviewed one after the other for a job position. In each interview, the decision maker (DM) acquires information about a candidate’s competence which allows her to rank him among the so far examined candidates. She can decide when to terminate the process by selecting the last candidate interviewed. The DM has no knowledge of who will come later on, yet her decisions should be immediate and irrevocable after each interview. This describes a Sequential Selection Problem (SSP)\(^1\). The class of SSP problems is attractive for theoretical analysis and for practical use, due to its generality and evident relevance to online selection under realistic constraints. Same as in this work, SSPs are usually presented in the intuitive recruitment context.

The goal of the original problem is to select none but the best among the sequence of \( n \) candidates, while in each interview the DM only realizes the relative quality of the examined candidate, that is his relative rank. The standard algorithm, first proposed in \cite{26}, is a cutoff-based approach which comprises two phases: the learning phase where a number (referred to as cutoff) of candidates are automatically rejected, and the selection phase where the first candidate ranked above the best recorded during the first phase is hired (or the last one, by default). In essence, the former phase learns a threshold that is subsequently used in the latter to spot the first candidate to beat it. For instance, the optimal cutoff for maximizing the probability to find the best candidate is \( c^* = \lfloor n/e \rfloor \) asymptotically. Note that the multi-choice problem is a natural extension of the above (see Sec. 2).

Motivation and contribution. Our motivation derives from real-world recruitment processes that take place in large organizations or companies whose aim is to dynamically adapt in their operating environments. This setting goes beyond the existing SSP models in literature that have one important limitation, namely they consider a cold-start initialization where there is no assignment of jobs at the beginning of the selection process.

To address this issue, we introduce a new online initialized problem that we call Warm-starting SSP (WSSP): at the beginning of the selection, the DM has at hand a reference set of referents for whom she knows the status of availability (referents are allowed to quit their jobs just before the beginning of the interviews), and eventually the average quality w.r.t. the new candidates. The selection strategy operates as in the standard cutoff-based fashion, however having a reference set of a given size, the question whether the learning phase of the DM should be longer or shorter is not obvious. We thereby propose an algorithm for the Warm-starting SSP, called Cutoff-based Cost Minimization (CCM), that gives the optimal learning time (i.e. the optimal cutoff value) according to the main parameters of the problem, while trying to minimize a regret defined as the average sum of the ranks of the selected items.

As for the technical contributions, we analyze the Warm-starting SSP and derive analytical formulas for: i) the initialization, specifically the expected rank of the referents (available or not) and the minimal regret of an offline strategy, and ii) the expectation of the main parameters of the process when using CCM, i.e. the acceptance threshold for each candidate, the number of new hires, and the regret. From the latter, we infer the optimal cutoff \( c^*(b,r,n) \), given the number of jobs \( b \), the number of candidates \( n \), and the number of resignations \( r \) starting with the case where the quality of the reference set is average; thereafter we propose a translation method that permits to derive \( c^* \) for every value of the quality \( q \) and highlights some interesting results. We then propose the low failures-CCM (lf-CCM) variation that is more robust to high resignation rates and hence prevents from accepting the very last candidates by default.

The rest of the paper is organized as follows. Sec. 2 presents the background of our work including related research; in Sec. 3.2 we present a new formalism for a broad range of SSPs called

\[^1\]Depending on the context, the last letter of the abbreviations SSP and the herein presented MSSP may refer to the respective selection ‘Problems’ or the associated selection ‘Processes’.
Generalized SSP (GSSP), and introduce one of its specific instance, the Warm-starting SSP.

Sec. 4 details the proposed CCM algorithm, tries to answer the question of the optimal learning
time, describes the translation method and the lf-CCM. Then, Sec. 7 gives an implementation of
the CCM algorithm in a multi-round fashion and, finally, our conclusions and future work are
presented in Sec. 8.

2 Related work

Various extensions of the basic secretary problem have been investigated; for non-exhaustive
surveys see [14, 16]. Importantly, a change in the setting or in the objective function, changes
also the optimal cutoff. In some scenarios, the DM can not only compute the relative rank of an
interviewed candidate among those examined earlier, but also assess candidate’s true quality
score. This score can be thought of as a random variable associated with each candidate. In [5],
candidates are drawn from a uniform distribution on [0,1] but the DM can only rank candidates
relatively to those she has seen before, and the objective is to maximize the expectation of the
score of the selected candidate. They have shown that in this case, the optimal cutoff becomes
c∗ = √n − 1. On the other end, Robbin’s problem [9] seeks to minimize the expectation of the
rank of the selected candidate (note: low ranks are better). However, the analytical solution to
this problem remains unknown, even when the score distribution of the candidates is known.

Notable variants are those related to multiple stopping, or simply b-choice, where the DM
has to select b candidates [3, 4, 6, 7, 17, 21, 25, 27]. In that case, the objective set function can
be modular (i.e. equivalent to adding up the independent application of the function to the set
of elements), submodular [4, 11], or subject to matroid constraints [2, 12, 13]. Non-modularity
introduces interesting set evaluation aspects, such as the complementarity or mutual-enhancement
among the selected candidates, which are however out of the scope of this work. Regarding
modular objective functions, [3] studies the b-choice problem with the objective to maximize the
sum of scores of the selected candidates, that arrive in a random order, without assuming prior
knowledge of the score distribution. An interesting finding is that the optimal cutoff for that
setting does not depend on b: c∗ = ⌊n/e⌋.

Very few papers study the algorithmic notions related to repeated selections [29], as well
as the human capacity to learn the right cutoff after reviewing multiple independent candidate
sets [6, 18]. However, [29] develops a non cutoff-based strategy which is implemented regarding
two distinct aims: to maximize the probability of selecting the best, or to maximize the
expected score of the selected candidate. That work concludes by stating that learning the
score distribution does contribute to the efficiency of the selection only w.r.t. the second aim.
An experimental comparison of simpler and intuitive non cutoff-based heuristics is provided in
[28]. More sophisticated adaptive strategies worth to be mentioned are the Bruss’ odds theorem
[8] and the work in [27]. A rather different scenario concerns a startup company (or a new
ambitious business unit) which is initially funded by a handful of people but is about to grow
larger. The so-called hiring problem [7] refers to the SSP that aims at driving the optimal growth
of personnel using an adaptive selection threshold based on the already employed items. Among
heuristics, such as hiring above the worst or the best current referents, hiring above the mean
referent score shown to be the best performing strategy. Similar settings where a set of selected
candidates increases through time are considered in [15, 19, 23, 24], while [19] makes a thorough
analysis of hiring above the m-th best strategies. In [15] the temp secretary problem is introduced
where contracts are of a fixed duration, thus temporary. The improved algorithm presented in
[20] generalizes towards general packing constraints and arbitrary hiring durations.
3 A general class of Sequential Selection Processes and the novel Warm-starting setting

Notations. A bold symbol denotes a vector, for instance, \( \mathbf{A} = (A_1, \ldots, A_k) \in \mathbb{R}^k \), \( \forall k \in \mathbb{N}^* \), in which with little abuse we omit the symbol of the transpose. The concatenation of matrices is denoted by \( (\mathbf{A}, \mathbf{B}) \). Moreover, \( \mathbb{1}\{\cdot\} \) is the indicator function, which is 1 if the input condition is true, and otherwise 0; also, \( \mathbb{1}_{[l]} \) is the unit vector of length \( l \).

3.1 Generalized Sequential Selection Process

In a standard Sequential Selection Process (SSP), candidates for a job position arrive sequentially in random order. The qualitative skills of each candidate can be assessed independently on his arrival by the decision maker (DM), allowing the relative ranking of the examined candidates against each other. According to this evaluation, the DM chooses who to hire in order to optimize a given objective function.

Definition 1. Generalized SSP (GSSP): Online selection process described by the following elements organized in several categories:

1. Background \( B \): collection of information known upfront by the DM, including the set \( \mathcal{A} \) of all possible actions the DM can take (e.g. hire, fire, add in queue, put on standby, etc.).
2. Sequential Arrivals
   - \( S = (S_j)_{j \geq 1} \): sequence of candidate scores s.t. \( S_j \in \mathcal{S} \subset \mathbb{R} \), drawn from distribution \( f_j \), \( \forall j \).
3. Decision Process
   - \( \pi = (\pi_j)_{j \leq 1} \): policy, i.e. sequence of mappings where \( \pi_j : \mathcal{S}^j \times \mathcal{A}^{j-1} \rightarrow \mathcal{A} \);
   - \( \mathcal{A} = (A_j)_{j \geq 1} \): sequence of decisions regarding the candidates, according to the policy, i.e. \( A_j = \pi_j(S_1, \ldots, S_j, A_1, \ldots, A_{j-1}) \in \mathcal{A}, \forall j \).
4. Evaluation
   - \( \ell : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}_+ \): loss function s.t. \( \ell(s, a) \) is the loss for taking decision \( a \) after observing \( s \);
   - \( L(S, \mathcal{A}) = \sum_{j \leq 1} \ell(S_j, A_j) \): cumulative loss;
   - Let \( P \) be the distribution of \( (S, \mathcal{A}) \). The evaluation criterion, called regret, is evaluated at the end of the process and defined as \( \Phi(\pi) = \mathbb{E}_P[\phi_B(\mathcal{A} | S)] \), where:
     \[
     \phi_B(\mathcal{A} | S) = |L(S, \mathcal{A}) - \phi^*| \in \mathbb{R}_+,
     \]  \hspace{1cm} (1)

With the high-level formalization of the GSSP class, we can summarize several well-known processes, such as the indicative ones mentioned below.

Examples of well-known GSSPs:

- Standard secretary problem [10]: A GSSP setting where \( B = (b, n, \mathcal{A} = \{0, 1\}) \), where \( b = 1 \) is the number of job position, \( n \) is the finite number of candidates, and a candidate is either selected (hired, \( A_j = 1 \)) or rejected (\( A_j = 0 \)). It is assumed that decisions are immediate and irrevocable, that candidates arrive in a random order, and that their scores are not independent (each candidate’s score depends on those examined before) nor identically distributed. This is equivalent to having relative ranks as observations, i.e. a triangular array \( (X_{i,j}) \) where \( X_{i,j} \in \{1, \ldots, j\} \) is the relative rank of the \( i \)-th incoming candidate after having examined \( j \geq i \) of them. The vector of absolute ranks, evaluated at the end, is given by \( X = (X_{1,n}, \ldots, X_{n,n}) \in \mathcal{P}_n \), where \( \mathcal{P}_l \) is the set of all permutations of the elements of \( \{1, \ldots, n\} \). The evaluation criterion to maximize is the probability to select the best candidate (the one with absolute rank 1 at the
end of the selection), which can be expressed by \( \ell(X_{j,j}, A_j) = I\{X_{j,n} A_j = 1\}, \forall j \leq n \), therefore \( \phi^* = 0 \) and \( \Phi = P(X^1 A = 1) \).

- **Hiring problem** [7]: A multi-choice GSSP setting where, by respecting the trade-off between the rate of hires and the quality of the hired candidates, the objective is to grow the company as much as possible while keeping maximal the average score of the employees. The recruitment process has infinite horizon. Therefore we have \( \mathcal{B} = (b, n, \mathcal{A} = \{0, 1\}) \), where \( b \to \infty \) and \( n \to \infty \). It is assumed that decisions are immediate and irrevocable, and that observations are i.i.d. scores drawn from a uniform distribution, i.e. \( S_j \sim \mathcal{U}(0,1) \).

**Remark 1.** The number of job positions \( b \) and of candidates \( n \) are usually included in the background \( \mathcal{B} \); however variants of the standard secretary problem [1, 22] may involve a random number of candidates. Note that \( b > 1 \) usually refers to a multi-choice or multi-stopping problem.

**Remark 2.** In most GSSP settings, the loss suffered at each decision is the score of an accepted item, i.e. \( \ell(S_j, A_j) = \pm S_j A_j \), with a positive (resp. negative) sign if the goal is to minimize (resp. maximize) the sum of scores. The evaluation criterion is further detailed into two cases: 1) the ‘no regret’ case, where the DM merely tries to optimize its selection i.e. for \( \phi^* = 0 \), and 2) the ‘with regret’ case, where the online selection is to be compared to the best associated offline selection \( \pi_{\text{off}} \), where the DM knows the entire sequence of candidates beforehand, in this case \( \phi^* = \min_{O \in \pi_{\text{off}}} \phi(S, O) \) (or \( \phi^* = \max_{O \in \pi_{\text{off}}} \phi(S, O) \) when the goal is to maximize the sum of the scores).

### 3.2 The Warm-starting Sequential Selection Process

**Description and rules of the game.** The Warm-starting SSP (WSSP) is a particular GSSP instance that overcomes the limitations of standard cold-starting SSP frameworks. Its characteristics is to start with a set of items at hand, called reference set and composed of referents, each of them having also a status of availability. The total number of job positions determines the size of the reference set. Items can therefore be of two types, candidate or referent. The value of each item is observed through a fixed real-valued relative score, i.e. each item’s score depends on the scores of those already seen. Although the referent’s availability status can be broad (e.g. on vacation, sick leave, resigned, etc.), we only allow resignations, i.e. a referent is unavailable if he resigned (leaving his position empty) and available otherwise (in other words, he is preselected). In this paper, we work under the simple assumption that resignations are independent. The DM therefore seeks highly-skilled candidates to 1) fill up empty positions and 2) replace non-competitive available referents; by respecting the following specific constraints.

**Assumption 1.** On the sequence of arriving candidates:
   1.A) Candidates arrive in a random order.
   1.B) Scores are not observed, the DM can only make pairwise comparisons between items.

**Assumption 2.** On the decision policy:
   2.A) The availability status is known upfront, and fixed throughout the process.
   2.B) Decisions are immediate and irrevocable.
   2.C) Every position must be filled at the end of the process.

**Formal definition.** We add a dot on top of a variable to refer explicitly to the reference set, e.g. \( \hat{S} = (\hat{S}_{(1)}, \hat{S}_{(2)}, \ldots, \hat{S}_{(b)}) \in \mathbb{R}^b \) gives the value represented by the variable \( S \) (here, scores) of the referents in descending order: the best, the second best, etc. Let the ranking function \( R_N : \mathbb{R} \times \mathbb{N} \to \{1, \ldots, N\} \), be the function that gives to each element of a collection of values its rank from 1 to \( N \) when compared to the other values, s.t. \( R_N(s, \Sigma) = \sum_{i=1}^{N} I\{\Sigma_i \leq s\}, \forall s \in \Sigma \), where \( \Sigma \) is a finite number set. 5
Definition 2. Warm-starting SSP (WSSP): A particular GSSP with the following characteristics:

1. Background
   \( \mathcal{B} = (n, b, \mathbf{A}, \mathbf{A}_0) \), where the included elements are:
   - \( n \in \mathbb{N}^* \): finite number of candidates to appear;
   - \( b \in \mathbb{N}^* \): number of job positions s.t. \( b \leq n \);
   - \( \mathcal{A} = \{0, 1\} \): the set of possible actions the DM can take, respectively reject or hire;
   - \( \mathbf{A}_0 = (\mathbf{A}_{(1),0}, \ldots, \mathbf{A}_{(b),0}) \in \{0, 1\}^b \): availability status of the reference set s.t. \( \mathbf{A}_{(i),0} = 1 \) if the
     \( i \)-th best referent is available.

2. Sequential Arrivals and 3. Decision Process as in Definition \[ \]

4. Rank-based evaluation
   The following simplified notation for the absolute ranks is written \( R(s) = R_{b+n}(s, (\mathbf{S}, \mathbf{S})) \),
   where \( \mathbf{S} = (\mathbf{S}_{(1)}, \ldots, \mathbf{S}_{(b)}) \in \mathbb{R}^b \) gives the referents scores (sorted in descending value order for
   convenience).
   - \( \mathbf{X} = (\hat{X}_{(1)} = R(\mathbf{S}_{(1)}), \ldots, \hat{X}_{(b)} = R(\mathbf{S}_{(b)})) \in \mathbb{R}^b \): referents’ absolute ranks,
   - \( \mathbf{X} = (X_1 = R(S_1), \ldots, X_n = R(S_n)) \in \mathbb{R}^n \): candidates’ absolute ranks,
   - Let \( P \) be the distribution of \( (\mathbf{X}, \mathbf{A}) \). The evaluation criterion, called regret, is evaluated at
     the end of the process and defined as \( \Phi(\pi) = \mathbb{E}_P[\phi_B(\mathbf{A} \mid \mathbf{X})] \), where:

   \[
   \phi_B(\mathbf{A} \mid \mathbf{X}) = \left( \mathbf{X}^T \mathbf{A} + \mathbf{X}^T \mathbf{A} \right) - \min_{(\mathbf{O}_n, \mathbf{O}) \in \pi_{\text{off}, B}} \left( \mathbf{X}^T \mathbf{O}_n + \mathbf{X}^T \mathbf{O} \right) \in \mathbb{R}_+ ,
   \]

   where \( \pi_{\text{off}, B} = \left\{ (\mathbf{O}_n, \mathbf{O}) \in \{0, 1\}^{n+b} : (\mathbf{O}_n, \mathbf{O})^T \mathbf{1}_{[n+b]} = b \right\} \) and \( \mathbf{A}_n \in \{0, 1\}^b \) is the hiring de-
   cisions of the referents after \( n \) interviews of candidates.

The first term in Eq. [2] is the sum of the ranks of the items to which jobs have been assigned
at the end of the selection. The second term is the minimal regret achievable by an offline oracle
strategy that, knowing the ranks, would select the best out of the available referents (i.e. for
some \( i \): \( \mathbf{A}_{(i),0} = 1 \)) and the candidates.

Remark 3. In this work we make no assumptions at all about the source and nature of the
scores. This is why we adopt a rank-based criterion to assess the selection strategy, which is a
standard approach in nonparametric statistics.

4 The proposed Cutoff-based Cost Minimization policy

In this section we present our novel algorithm for the WSSP, called Cutoff-based Cost Min-
imization (CCM). It takes as input a cutoff value \( c \in \mathbb{N} \) representing the size of the learning
phase, i.e. the number of candidates to be rejected by default from which the DM learns valuable
information about the overall sample. In the next section, we will analyze its optimality.

4.1 Cutoff-based strategies

Inspired by the secretary problem, we develop the Cutoff-based Cost Minimization (CCM) policy,
see Alg. \[ \]. We consider a cutoff-based strategy for the following reasons: i) the DM should
somehow define a value above which a candidate might be accepted, value that needs to be
consistent with the current candidate sample (and not necessarily with the reference set) hence
the need to explore before making any decision, ii) in a finite-horizon settings with limited
and constrained budget, the DM should not rush into hiring since decisions are irrevocable,
iii) exploring the sample before making any decisions helps to estimate the quality of the reference
set when we do not make the assumption that it is given to the DM, and iv) the intriguing behavior of the learning phase when the reference set has a given quality raised our curiosity.

Two other points concerning the cutoff-based CCM strategy. First, in practical situations where the quality of the reference set is good enough, that leads to an optimal cutoff value of \( c^* = 0 \), i.e. it degenerates to a non cutoff-based strategy. The second point is that despite its name, the cutoff value \( c \) is not the only parameter involved (the quality threshold, \( \tau_j \), is another one, see Definition 3). However, we found that it is the most critical parameter in driving the performance of the proposed strategy. The cutoff value being one of the key parameter of CCM algorithm, the policy is written \( \pi(c) \) and therefore the regret becomes \( \phi(c) \).

### 4.2 Acceptance threshold

Derived from the learning phase, the CCM policy dictates a set of threshold values specific to each job position (i.e. specific to each referent that filled them) that candidates need to exceed to be accepted. Since it depends on the available referents, we first need to define the number of resignations by \( r \in \{1, \ldots, b\} \), and therefore \( r := b - \hat{A}_{0}^{T}1_{[y]} \). The available referents’ scores are then denoted by \( \hat{S}^{+} = (\hat{S}_{(i)})_{i \in I} \), where \( I = \{i : \hat{A}_{(i),0} = 1\} \), and is thus of size \( b - r \).

In practice, during the selection phase, the acceptance threshold for each candidate is set to be the score of the \( b \)-th best up to the end of the learning phase. This set, called updated reference set, is defined as \( Y = (Y_{(i)})_{i \leq b} \) where each term belongs to the concatenation of both the referents and the rejected candidates, i.e. the \( c \) first candidates, hence \( Y_{(i),c} \in (\hat{S}, S_{1}, \ldots, S_{c}) \) s.t. \( Y_{(1),c} > \ldots > Y_{(b),c} \). The threshold is a fixed value, which might not be optimal when every empty job positions have been filled. In fact, in the latter case, the threshold should be adapted to the scores of the available referents, so that no position gets filled by a worse item. Note that, during the learning phase candidates are rejected by default, hence the acceptance threshold is defined only during the selection phase. Under these conditions, the acceptance threshold is defined as follows.

**Definition 3.** Step-specific acceptance threshold \( (\tau_j) \): Score value to beat at step \( j > c \) of the WSSP when the CCM policy is applied with cutoff value \( c \):

\[
\tau_j(c) := \begin{cases} 
Y_{(b),c} & l < r + \sum_{j=1}^{c} \mathbb{1}\{S_j \geq Y_{(b),c}\}; \\
\hat{S}^{+}_{(b-l)} & \text{otherwise},
\end{cases} 
\]

where \( l = \sum_{i=c+1}^{j-1} A_i \). The second term in the condition is the number of candidates from the learning phase that have been added in the updated reference set.

Following the definition of the acceptance threshold, the decision variable is therefore given by:

\[
A_j = \mathbb{1}\{j > c\} \mathbb{1}\left\{ \sum_{i=c+1}^{j-1} A_i < b \right\} \mathbb{1}\{S_j \geq \tau_j\},
\]

where the second indicator function ensures that no more than \( b \) items can be selected. In the rest of the paper, the number of candidates accepted up to step \( j \) (included) is denoted by \( \tilde{A}_j = \sum_{i=1}^{j} A_i \). The CCM algorithm is fully described in Alg. 1.

**Remark 4.** Due to the finite horizon, the DM might select candidates by necessity, regardless their quality. This may occur in order to prevent having vacant positions in the output when the very end of the sequence is reached.
Algorithm 1: The proposed Cutoff-based Cost Minimization policy for WSSP

**Input:** the number of \( b \) jobs, the number of candidates \( n \), the number of resignations \( r \), the reference set scores from best to worst \( \hat{S} = (\hat{S}_1, \ldots, \hat{S}_n) \), the initial vector of reference set availability \( \hat{A}_0 = (\hat{A}_{(1),0}, \ldots, \hat{A}_{(n),0}) \), and the cutoff value \( c \).

**Output:** the set of final job assignment \( (\hat{A}_n, \hat{A}) \)

- **Learning phase**
  1: \( A_1, \ldots, c \leftarrow 0 \) // reject by default all \( c \) first candidates
  2: \( Y \leftarrow \text{top}_{0} \text{rank}(b, (\hat{S}, S_1, \ldots, S_c)) \) // \( b \)-best from \( \hat{S} \) and \( (S_1, \ldots, S_c) \), in descending value order
  3: \( n_{\text{rej}} \leftarrow \sum_{j=1}^{c} 1 \{ S_j > Y(b), c \} \) // the number of candidates among the \( c \) first that beat...
  // ...the threshold, i.e. here, the last rating of the updated reference set
  4: \( \hat{S}^+ \leftarrow (\hat{S}_{(i)})_{i \leq I} \) where \( I = \{ i : \hat{A}_{(i),j} = 1 \} \) // initialize the selection with the available reference set
  5: \( l \leftarrow 0 \) // the number of jobs assigned so far in the selection

- **Selection phase**
  6: for \( j = c+1 \) to \( n \) do
    7: if \( l < n_{\text{rej}} + r \) then // set the threshold that the \( j \)-th candidate should beat (see Definition 3)
    8: \( \tau_j = Y(b) \)
    9: else \( \tau_j = \hat{S}_{l-j}^+ \)
  10: end if
  11: if \( l < b \) and \( (S_j > \tau_j \text{ or } j-l=n-r+1) \) then
    12: \( A_j \leftarrow 1 \)
    13: if \( l \geq r \) then
      14: \( \hat{A}_{(b-l),j} \leftarrow 0 \) // remove job from reference set
    15: end if
  16: \( l \leftarrow l+1 \)
  17: else \( A_j \leftarrow 0 \)
  18: end if
  19: end for

5 Optimal Cutoff-based Cost Minimization

We now propose an in-depth study of the properties of the cutoff strategies that takes advantage of the rank-based perspective used in the evaluation setup.

5.1 Defining the quality

A natural question that arises from the existence of the reference set concerns the ‘value’ (or quality) of the referents compared to the candidates next to come. How ‘good’ is our initial set with respect to the arriving candidates? Besides, a notion of ‘good’ should also be defined. We address the latter interrogation by introducing the ‘goodness’ of \( \hat{X} \) for \( X \), which we call quality of the reference set and denote as \( q \) (see Definition 4). This parameter quantifies how the reference set ranks on average compared to the candidates.

**Definition 4.** True rank-based relative quality of reference set \((q)\): For a WSSP, \( q \) is the average normalized rank of the \( b \) items of the reference set compared to the \( n \) candidates:

\[
q := 1 - \frac{\frac{1}{b} \hat{X}^T 1_{[b]} - 1}{n + b - 1},
\]

where \( \hat{X} = (\hat{X}_1 = R(\hat{S}_1), \ldots, \hat{X}_b = R(\hat{S}_b)) \) are the referents absolute ranks, \( q \in [0, 1] \), with \( q \to 1 \) as the reference set gets better skilled and \( q = 1/2 \) corresponds to the medium quality s.t. \( \frac{1}{b} \hat{X}^T 1_{[b]} = \frac{1}{2} (n + b + 1) \).
5.2 Offline analysis

**Initialization.** This analysis concerns the initialization of the process, i.e. before the arrival of candidates, and is independent on the chosen strategy. The DM has information about the average quality of the referents, but we are particularly interested in the available ones, i.e. those with ranks $\tilde{X}^+ = (\tilde{X}_{(i)})_{i \in I}$, where $I = \{i : \tilde{A}_{(i),0} = 1\}$. These preselected referents might end up, if competitive enough, in the final selection.

**Proposition 1.** Let a given WSSP starting with $r \leq b$ resignations. The expectation of the rank of the $l$-th item from the available reference set $\tilde{X}^+$ is given by:

$$E[\tilde{X}^+_{(l)}] = \frac{\gamma_0(b+1)}{b(b-r+1)},$$

where $\gamma_0 := E[\tilde{X}_{(b)}] = (1-q)\frac{2b(n+b-1)}{b+1} + \frac{2b}{b+1}$.

s.t. $\gamma_0$ is the expectation of the $b$-th item from the reference set $\tilde{X}$, and a function of the relative quality $q$ of the reference set.

**Offline selection.** It is desirable for any online algorithm to perform as close as possible to the optimal offline case where the DM knows the $b$-best items and can directly select them. Hence, we want our strategy to converge towards the offline case and have $\phi$ as small as possible. The offline output $\phi^* \in \mathbb{R}_+$ is given by Definition 2 as:

$$\phi^* = \min_{(\tilde{O}_n, O) \in \pi_{\text{off},B}} \left( \tilde{X}^T \tilde{O}_n + X^T O \right),$$ (8)

where $\pi_{\text{off},B} = \{ (\tilde{O}_n, O) \in \{0,1\}^{n+b} : (\tilde{O}_n, O)^T 1_{[n+b]} = b \}$.

**Proposition 2.** In the WSSP context, the expected minimal regret an offline algorithm can achieve, by selecting the $b$-best out of the $n+b-r$ candidates and available referents, is:

$$E[\phi^*] = \frac{b(b+1)}{2} + \frac{rb^2(\gamma_0 + r)}{2\gamma_0^2},$$ (9)

where $\gamma_0$ is given in Proposition 1.

The first term of Eq. (9) accounts for the standard average offline regret, i.e. the sum of the $b$-best ranks, while the second term represents the increase due to potentially unavailable items from the $b$-best.

5.3 Optimal cutoff and WSSP main parameters for $q = 1/2$

Let us first consider that, on average, referents have a medium quality i.e. $q = 1/2$. Indeed, the analytical computation of the main variables of the problem is more challenging when $q \neq 1/2$, therefore we provide what we call a translation method to ‘translate’ any setting of arbitrary $q$ to the situation where $q = 1/2$ for which we have analytical results (see Sec. 5.4).

**Lemma 1.** Let a WSSP with $n$ candidates, and a reference set of size $b$. Using Eq. (4), a candidate is accepted if his rank beats the rank-based threshold, $\gamma_j = R(\tau_j)$, and less than $b$ candidates have been accepted. The probability for the number of accepted candidates at step $j$ to be smaller than $b$ is given by:

$$g_j(b) := P(\tilde{A}_{j-1} < b) = \begin{cases} 1, & b > j-c-1; \\ e^{-\lambda_j} \sum_{i=0}^{b-1} \frac{\lambda_j^i}{i!} + o(\sigma_j^2) & b \leq j-c-1, \end{cases}$$ (10)
where \( \lambda_{j-1} = \sum_{i=c+1}^{j-1} \frac{\gamma_{i-1}}{n+b} \) and \( \sigma_{j-1}^2 = \sum_{i=c+1}^{j-1} \left( \frac{\gamma_{i-1}}{n+b} \right)^2 \).

**Theorem 1.** Applying the CCM algorithm with parameter \( c \) as cutoff value, given that \( r \) referents resigned, and using Lemma 1, the WSSP exhibits the following features:

- **Expected rank-based acceptance threshold for candidate** \( j \) is given by \( \gamma_j := \mathbb{E}[R_j] \) s.t.:
  \[
  \gamma_j = \gamma g_j(\Delta) + \frac{\gamma_0(b+1)}{b(b+r-1)} \left( b - \sum_{i=1}^{j-1} \frac{\gamma_{i-1}}{n+b} g_i(b) \right) (1 - g_j(\Delta)),
  \]
  \[
  \quad \text{where} \quad \gamma := \mathbb{E}[R(Y_{b,j})] = \frac{b(b+n)}{b+r} \Delta = r + c \frac{\gamma - 1}{n+b} \text{ and } \gamma_0 \text{ is given in Proposition 1.}
  \]

- **Expected number of new hires at the end of the selection** \( \hat{A}_n \leq b \):
  \[
  \mathbb{E}[\hat{A}_n] = \sum_{j=1}^{n} \frac{\gamma_j - \gamma_0}{n+b} g_j(b)
  \]

- **Expected regret function to minimize**, i.e. expected average rank of the selected items:
  \[
  \mathbb{E}[\phi(c)] = \frac{1}{(n+b)} \sum_{j=c+1}^{n} g_j(b) \left( \frac{\gamma_j(\gamma_j - 1)}{2} + \frac{\gamma_0(b+1)}{2b(b+r-1)} (b - \mathbb{E}[\hat{A}_n])(b+1 - \mathbb{E}[\hat{A}_n]) - \mathbb{E}[\phi^*] \right),
  \]
  \[
  \text{where } \mathbb{E}[\phi^*] \text{ is the expected minimal offline loss defined in Proposition 2.}
  \]

Eq. 13 holds a good approximation of the expected regret of WSSP when \( q = 1/2 \). Recall that we want to find the optimal cutoff value \( c^* = \arg\min_c \mathbb{E}[\phi(c)] \) which is equivalent to finding \( c^* \) s.t. \( \frac{\partial}{\partial c} \mathbb{E}[\phi(c)] \big|_{c=c^*} = 0 \). Unfortunately, this equation is analytically intractable unless approximations or restrictive assumptions are made, however we can easily spot \( c^* \) numerically by tracking the lowest point of the curve \( \mathbb{E}[\phi(c)] \) using Eq. 13 \( \forall b, \forall n \), and store the results in \( c^*(b,r,n) \).

**Remark 5.** Note that in practice \( \hat{A}_n \) is actually equal to \( \max(\hat{A}_n, r) \) to avoid empty positions at the end of the selection. An approximation of \( \mathbb{E}[\max(\hat{A}_n, r)] \) can be found in the Appendix, as well as an empirical verification.

**Example.** Imagine a WSSP instance where \( n = 100 \) candidates are going to be sequentially interviewed. The DM handles \( b = 5 \) job positions, each of them already filled by available referents (i.e. \( r = 0 \)) of a given quality \( q = 0.75 \) w.r.t. to the candidates next to come. Using Theorem 1 the length of the learning phase is \( c^* = [38.36] = 38 \), the expected average rank of the selected items is \( \mathbb{E}[\phi(c^*)]/5 = 2.60 \), and the expected number of accepted candidates is \( \mathbb{E}[\hat{A}_n] = 0.997 \). The latter is low since the initial quality is quite good, hence the DM expect to fire only his worse referent (available referent). Now, with the same setup of WSSP parameters except with full resignations, i.e. \( r = b \), we get \( c^* = [28.27] = 28 \), \( \mathbb{E}[\phi(c^*)]/5 = 3.40 \) and \( \mathbb{E}[\hat{A}_n] = 5 \), which is coherent with the fact that all positions are initially empty. The length of the learning phase is reduced compared that of the previous example, implying a less competitive acceptance threshold. Justifiably, the DM is less demanding on the quality of the accepted items, to avoid the risk of having to select last incoming candidates by default, called a failure (see Sec. 6).

**Simulations.**

In order to guarantee the accuracy of our analytical approximation \( \mathbb{E}[\phi(c)] \) in Eq. 13 we simulate each WSSP scenario 1000 times: for a fixed number of candidates \( n = 100 \) and a fixed reference set quality \( q = 1/2 \). The top row of Fig 1 displays a heatmap of the average empirical
regret (simulated) w.r.t. the number of jobs \( b \) (x-axis) and the value of the cutoff \( c \) (y-axis). The white plain line in each heatmap follows the path of the lowest simulated value of the heatmap, referred to as \( c^*_{\text{sim}}(b) = c^*_{\text{sim}} \). These plots should be put in comparison with those in the bottom row which show the heatmaps of the expected regret according to our analysis. The white dashed line follows again the path of the lowest heatmap value, which we denote as \( c^*(b) = c^* \). From Fig. 1, it becomes clear that the law of large number complies with the lemmas and propositions of Sec. 5.3 which are consistent in these experiments.

5.4 Optimal cutoff for arbitrary \( q \)

5.4.1 The translation method

In Sec. 5.3, we derived an analytical expression for \( E[\phi(c)] \) given a relative quality of the reference set \( q = 1/2 \). However, when \( q \neq 1/2 \), the analytical computation of the WSSP’s main variables is highly complex. We introduce a rather simple trick to efficiently overcome this difficulty. More specifically, we provide a way translate any setting of arbitrary \( q \) to a \( \gamma_0 \)-similar setting where the quality of the reference set is set to be \( q = 1/2 \) and for which we can use the results presented in Sec. 5.3. We introduce a notion of similarity between two different settings’ reference set \( \gamma_0 \) (see Definition 5) and come up with what we call as the translation method described below (see Proposition 3).

Definition 5. \( \gamma_0 \)-similarity: Suppose each WSSP instance, denoted by WSSP\(_x\), starts with \( b_x \) jobs positions filled with the available referents \( \hat{X}_x^+ \), and thereafter interviews \( n_x \) candidates using CCM (see Alg. 1) with the optimal cutoff value \( c_x^* \). Then, the settings of two instances, WSSP\(_x\) and WSSP\(_y\), are said to be \( \gamma_0 \)-similar if their reference sets (even those unavailable) have the

![Figure 1: Comparative heatmaps for the average empirical regret (top line) and the expected regret (bottom line) derived from Theorem 1 over different resignation numbers \( r = \{0, 0.1b, 0.5b, b\} \), and both for reference set quality \( q = 1/2 \). In each case, the heatmap of the regret is presented over the parametrization of the cutoff value \( c \) and the number of jobs \( b \) (budget).](image)
Algorithm 2 Translation method between two WSSP settings

Input: the WSSP setting of interest, WSSP\textsubscript{t} (subscript \textit{t} for ‘target’ and, below, \textit{s} for ‘source’), and its main parameters: \(n_t\) candidates, \(b_t\) jobs, a reference set of a relative rank-based quality \(q_t\) from which \(r_t\) resigned, and the vector \(c^*(b, n)\) with the optimal cutoffs for any sequence length \(n\), for \(q = 1/2\), as described in Theorem 3

Output: the optimal cutoff \(c^*_t\) for the WSSP\textsubscript{t} setting.

- Find a \(\gamma_0\)-similar setting to WSSP\textsubscript{t}, let that be WSSP\textsubscript{s}
  
  1. Require \(b_s = b_t = b\), as in Definition 5
  2. Impose \(q_s = 1/2\)
  3. Compute \(n_s = [(n_t + b - 1) \frac{1-q_t}{1-q_s} - b + 1]\), as suggested by Proposition 3

- Translate the setting WSSP\textsubscript{s} to WSSP\textsubscript{t}
  
  4. Find the best cutoff value \(c^*_s = c^*(b, n_s)\) from the input vector
  5. Compute \(c^*_t = \lfloor c^*_s \frac{n_t + b}{n_t + b} \rfloor\), according to Proposition 3

---

| Algorithm 2 | Translation method between two WSSP settings |
|-------------|---------------------------------------------|
| **Input:**  | the WSSP setting of interest, WSSP\textsubscript{t} (subscript \textit{t} for ‘target’ and, below, \textit{s} for ‘source’), and its main parameters: \(n_t\) candidates, \(b_t\) jobs, a reference set of a relative rank-based quality \(q_t\) from which \(r_t\) resigned, and the vector \(c^*(b, n)\) with the optimal cutoffs for any sequence length \(n\), for \(q = 1/2\), as described in Theorem 3 |
| **Output:** | the optimal cutoff \(c^*_t\) for the WSSP\textsubscript{t} setting. |

- Find a \(\gamma_0\)-similar setting to WSSP\textsubscript{t}, let that be WSSP\textsubscript{s}
  
  1. Require \(b_s = b_t = b\), as in Definition 5
  2. Impose \(q_s = 1/2\)
  3. Compute \(n_s = [(n_t + b - 1) \frac{1-q_t}{1-q_s} - b + 1]\), as suggested by Proposition 3

- Translate the setting WSSP\textsubscript{s} to WSSP\textsubscript{t}
  
  4. Find the best cutoff value \(c^*_s = c^*(b, n_s)\) from the input vector
  5. Compute \(c^*_t = \lfloor c^*_s \frac{n_t + b}{n_t + b} \rfloor\), according to Proposition 3

---

Figure 2: The optimal cutoff w.r.t. the number of jobs \(b\) (x-axis), according to the simulations in plain lines and to our analytical approximation (see Eq. 2) in dashed lines for different values of the relative quality of the reference set \(q = \{\frac{1}{2}, \frac{3}{2}; \frac{3}{4}, \frac{5}{4}\}\), for \(n = 100\) candidates.

**Examples.** Let us illustrate the translation method with one example. Imagine the DM deals with WSSP\textsubscript{t}, where no referent resigned, with \(n_t = 100\), \(b_t = 15\), \(q_t = 0.8\), and she is interested in knowing \(c^*_t\). One possible \(\gamma_0\)-similar setting, WSSP\textsubscript{s}, has the following features \(q_s = 1/2\) and \(b_s = b_t = b = 15\). Using Proposition 3, we get \(n_s = [(n_t + b - 1) \frac{1-q_t}{1-q_s} - b + 1]\) = \(\lfloor 114/0.2 - 14 \rfloor = 31\); then using Theorem 1 we compute \(c^*_s\) numerically for \(n_s = 31\) (which is feasible as long as \(q_s = 1/2\)) and get \(c^*_s = 9\). Finally we obtain \(c^*_t = \lfloor c^*_s \frac{n_t + b}{n_t + b} \rfloor = 22\); the DM rejects the first \(\frac{22}{100}\) candidates, that is 22% of the total sample, before starting to select.

**Simulations.** For a fixed quality \(q\), it is worth pointing out that \(c^*(b)\) is not a monotonic function but rather has two distinct regimes indicated by the sign(\(\partial c^*/\partial b\)). This can be better...
explained as the following trade-off. Suppose fixed n and r (see Fig. 2) and that we start with
b = 1: increasing b would mean more jobs to assign, hence, the DM should very quickly (w.r.t.
budget increase) increase the length of the rejection phase to make sure that she learns sufficiently
before taking the many decisions (regime $\frac{\partial c}{\partial r} \geq 0$). From a point and further, though, increasing
b would also mean a) to have a less competitive threshold (which depends on the quality of
the worst current referents), b) that the whole process becomes less selective as less and less
candidates need to be rejected, c) to have a higher expected number of resignations (if $r > 0$),
which makes the exploration for the DM less safe. Hence, the DM should start shortening
her learning phase (regime $\frac{\partial c}{\partial r} < 0$). The optimal cutoff values get lower as the number of
resignations r increases (see the curves across the plots of Fig. 2), as well as with the decrease of
reference set quality q (see the compared curves in each plot of Fig. 2).

6 Adjusted policy: low failures-CCM

In real-life scenarios the proportion of referents that resign compared to those who stay is often
relatively small; therefore, in the presented recruitment context, the more relevant results of this
work concern situations where the number of resignations is small (e.g. $2r \leq b$). However when
the latter is quite high (i.e. most job positions are empty), the DM might have to accept last
arriving candidate (s) in order to fill vacant positions, this event is called a failure (described in
Definition 5) and is similar to hiring random candidate (s) which ends up increasing the regret.

**Definition 6.** Failure and failure rate ($f_j$): A failure at step $j$ is the event of accepting a last
incoming candidate by default (to fill empty job positions) whose score did not beat its associated
sum of the number of failures divided by the number of tests.

Simulations show that in some settings the failure rate is indeed significant, for instance it reaches
$\rho_f = 0.58$ for $b = 20$, $r = b$, and $q = 0.81$. This phenomenon appears due to the high quality of the
updated reference set, i.e. the threshold becomes too competitive and hence difficult to beat for
most candidates. Our idea to mitigate this effect is to estimate the expected number of accepted
candidates at step $j$, denoted by $\hat{\mu}_j(c,r)$, given that the total number of accepted candidates
(i.e. at the end of the selection) is greater or equal to the number of resignations $r$; formally that is:
$\hat{\mu}_j(c,r) := E[\hat{A}_j \mid \hat{A}_n \geq r], \forall j$, i.e. there is no failure (see Proposition 4).

**Proposition 4.** The expectation of the number of candidates accepted at step $j$ given that there
is no failure is given by:

$$\hat{\mu}_j(c,r) := E[\hat{A}_j \mid \hat{A}_n \geq r] = \lambda_j g_{j+1}(b-1) + \frac{b(1-g_{j+1}(b))}{1-g_{n+1}(r)},$$

(16)

where $\lambda_j = \sum_{i=1}^{j} \frac{n-i}{n+1}$ and $g_j(x) := P(\hat{A}_{j-1} < x)$ (see Lemma 1).

The proof is detailed in the Appendix. We use Proposition 4 to compute $\hat{\mu}_j(c^*,r)$, $\forall j \in \{1,\ldots, n\}$, and compare it to the current number of accepted candidates at step $j$ (included),
denoted by $\hat{A}_j$. From this comparison we introduce the notion of zone in Definition 7 that we
use to adjust the threshold. In Fig. 3 that zone is enclosed by dashed lines and shaded in gray.

**Definition 7.** Zone (Z): Area around the expectation of the number of accepted candidates
inside which the threshold $\gamma_j$ is identical to that of CCM. It is defined between the two curves
$\hat{\mu}_j + w_j$ and $\hat{\mu}_j - w_j$ where $w_j = w(j)$ is the function that defines the zone’s thickness at step $j$.
$Z = \int_0^{\hat{A}_j} dx 2w(x)$. 

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When it is below (resp., above) the zone, then the threshold for the next candidate is reduced.

The threshold of this adjusted algorithm low failures-CCM (lf-CCM) is defined as:

$$\tilde{y}^j(c) := \begin{cases} 
Y_{(b),c} & \text{in the zone } Z, \\
Y_{(m+D^+_{j-1}),j} & \text{below the zone } Z, \\
Y_{(m-D^-_{j-1}),j} & \text{above the zone } Z,
\end{cases}$$

where $Y_{(i),j} \in (S_i,S_{i+1},...,S_j)$ is the score of the $i$-th best seen out of the reference set and up to the $j$-th candidate s.t. $Y_{(1),j} > \ldots > Y_{(b),j}$, and $m \leq b$ is s.t. $Y_{(m),j} = Y_{(b),c}$. The $D^+_{j-1}$ and $D^-_{j-1}$ functions define how the threshold will change provided that a point $(j, A_j)$ is outside the zone $Z$. More precisely, when that point lies in the zone, the threshold is constant and equal to $Y_{(b),c}$. When it is below (resp., above) the zone, then the threshold for the next candidate is reduced (resp., increased) by $\lfloor D^+_{j} \rfloor$ (resp., $\lceil D^-_{j} \rceil$) positions from the former, as many times as needed until the point is inside the zone again, and the threshold goes back to the original one (i.e. $Y_{(b),c}$) for the next candidate. Finally, $d_j = D^+_{j} - D^-_{j-1} = D^-_{j} - D^-_{j-1}$ is the increment in the position each time a point has been above (resp. below) the zone in a row. For simplicity, we assume that an optimal cutoff value $c^*$ for CCM is also an optimal cutoff value for lf-CCM. The tuning of parameters $w_j$ and $d_j$ is done empirically, and the relevant ones are used in the simulations (see Sec. 7).

7 A multi-round extension

7.1 General setting and assumptions

In this section we build upon the WSSP that was described thoroughly in previous sections, and introduce the Multi-round Sequential Selection Problem (MSSP). The process takes place in multiple successive rounds such that the output of a given round constitutes the input of the following one. The environment of the problem is set to be on a large population $\mathcal{C}$ of job-seekers, i.e. candidates. Essentially, each round constitutes a separate WSSP (see Sec. 3.2) on a sample of candidates.

**Assumptions.** The MSSP requires further assumptions: i) the environment is considered to be fixed during each WSSP round, however changes may occur between any two rounds regarding the referents availability since any referent can resign, and ii) sample are obtained by a random picking of candidates in the population. The process may have an arbitrary number of WSSP rounds. Therefore, the challenge for the DM is to improve, or at least adapt, the personnel.
in the course of the multi-round process: at the end of any round that is to have selected the $b$-best items she could have chosen under the above assumptions and while respecting all the management constraints described for a single round in Sec. 3.2. We use the notations introduced in Sec. 3.2 and add a subscript $k$ at each variable to refer to a precise round $k$, for instance $n_k$ is the number of candidates at round $k$.

### 7.2 Implementing CCM in a MSSP

In the previous section we created two algorithms (Cutoff-based Cost Minimization, lf-CCM) that aim at selecting good candidates in a single-round horizon. In this section, we intend to plug these algorithms in the multi-round setting (MSSP) in order to iteratively improve the DM’s selection. For the simulations of this section we use the following parametrization. Firstly, each multi-round simulation considers a population of $|C| = 1000$ items and for all rounds we set the number of candidates to $n = 100$. Secondly, the resignation probability $P_r \leq 0 \leq 1$ is considered to be known in advance by the DM, and is kept constant for every round $k$ and equal for all referents.

**Cutoff-choice and resignations.** Fig. 4 displays the average regret $\phi_k$ w.r.t. the round number $k$ for different resignation probabilities. We first observe that, regardless the resignation probability, our proposed cutoff $c^*$ (red curves) outperforms other alternatives originating from the general SSP literature, or heuristics such as the case $c = 0$. As presented, MSSP allows for referents to resign their job at the beginning of a round, with probability $P_r$. Notice that, the cutoff $c = n/e$ is a decent alternative to $c = c^*$ when $P_r = 0$ (see Fig.4(a)), although failing at reducing the regret when $P_r = 1$ (see Fig.4(c)). Large number of resignations can occur when the environment changes abruptly (e.g. company’s future, changes in the job market, etc.), or when the time-interval between two subsequent rounds is very long and more referents may happen to resign.

Another observation on this scenario is that CCM seems to struggle to make the regret converge towards zero, and as stated in Sec. 6 this effect is a consequence of being forced to select the last candidate (s) in order to assign all vacant jobs (i.e. failure), hence the low failures-CCM. A comparison of CCM and lf-CCM can be found in Fig. 5 and illustrates the fact that lf-CCM is more efficient at improving the selection through rounds than CCM, although it requires more adaptation from the DM.
Figure 5: Average regret \( \phi_k \) w.r.t. the round number \( k \) in the stationary case, for \( n = 100 \) candidates. The red curve uses CCM when the orange yellow green and blue ones use If-CCM. The parameters for the blue curve are s.t. \( v_1 = 82 \) and \( v_2 = 14 \). The dashed lines are not CCM strategies: MEAN accepts a candidate if its score is above the mean of the current referents and RAND accepts a candidate if its score is above a randomly computed threshold. Top line \( b = 5 \) and bottom line \( b = 50 \).

8 Conclusion

In this paper we introduced the Warm-starting SSP (WSSP), where a DM has at hand a set of referents, some of which still available, randomly incoming one-by-one. Following the well-known secretary problem, we developed a cutoff-based strategy, the Cutoff-based Cost Minimization (and a tuned version low failures-CCM), composed of a learning phase and a selection phase. The optimal length of the former according to the number of initially empty jobs is an intriguing question for which we brought interesting and not always straight to see results. The rank-based regret function that we used enables our algorithm to be efficient for arbitrary candidate scores. We approximate analytically this objective function by deriving main parameters’ expectations in closed-form (e.g. the acceptance threshold, the number of accepted candidates, the regret, etc.).

In the second part of the paper we implemented our algorithm CCM in a multi-round framework (MSSP). That process is motivated by the needs of real-world recruitment processes that are constantly trying to improve the personnel of an organization or a company.

The conducted simulations are consistent with our analytical work, and demonstrated that CCM is efficient in reducing the regret at the course of the multi-round process while being robust to scores, resignations or number of jobs changes. Moreover, our experiments showed that our proposed optimal cutoff \( c^* \) compares favorably against various cutoff values presented in literature for other sequential selection settings.

In our future work, we plan on adopting and testing CCM for MSSP in various applications. In addition, the multi-round setting creates plenty of room for developing statistical learning methods aiming to learn efficiently candidates scores, were they to come from a given distribution.
Appendix - Technical proofs

Proof. Proof of Proposition \[\text{Eq.}\] Derives from the definition of the quality in Definition \[\text{Eq.}\] and uses the fact that \(\mathbb{E}[X_{(i)}] = \mathbb{E}[X_{(b)}]/i = \frac{\gamma_0}{m}, \forall i \leq b.\)

The best available referent, i.e. with rank \(\tilde{X}^+_1\), is therefore expected to have a rank at best \(\gamma_0/b\) and at worst \(\gamma_0(r+1)/b\). He has an expected rank of \(\gamma_0/b\) iff the available item(s) are any of the \(b-1\) below him in the ranking, i.e. with a probability \(\left(\frac{r-1}{r}\right)^i\). Then, he has an expected rank of \(2\gamma_0/b\) iff the best referent resigned and the \(r-1\) other unavailable referents are any of the \(b-2\) below him in the ranking, i.e. with probability \(\left(\frac{r-2}{r}\right)\). Finally:

\[
\mathbb{E}[\tilde{X}^+_1] = \sum_{i=1}^{r+1} \mathbb{P}(\tilde{X}^+_1 = \mathbb{E}[\tilde{X}^+] | \mathbb{E}[\tilde{X}^+] = \sum_{i=1}^{r+1} \left(\frac{r-1}{r}\right)^i \frac{\gamma_0}{b} = \frac{\gamma_0}{b} \sum_{i=1}^{r+1} \left(\frac{b-1}{r+1-i}\right) i, \quad (18)
\]

from the multiset relation \(\sum_{i=0}^{n} \binom{m+i-1}{i} = \binom{n+m}{n}\) we obtain:

\[
\mathbb{E}[\tilde{X}^+_1] = \frac{\gamma_0}{b} \left(\frac{r+1}{b+1} \left(\frac{b+1}{r+1}\right) - (b-r) \left(\frac{b+1}{r}\right)\right) = \frac{\gamma_0(b+1)}{b(b-r+1)}. \quad (19)
\]

Using \(\mathbb{E}[\tilde{X}^+_1] = \mathbb{E}[\tilde{X}^+] | \forall l \in \{1, \ldots, b-r\}\), we obtain \(\mathbb{E}[\tilde{X}^+_1] = \frac{\gamma_0(b+1)}{b(b-r+1)}. \quad \square\)

Proof. Proof of Proposition \[\text{Eq.}\] We begin by deriving the variable \(\eta\) that gives the expected number of referents that belong to the \(b\)-best, i.e. \(\eta = \mathbb{E}[\sum_{i=1}^{b} \mathbbm{1}\{\tilde{X}^+_1 \leq b\}]\):

\[
\eta = \frac{r}{b} \mathbb{E}\left[\sum_{l=1}^{b} \mathbbm{1}\{\tilde{X}(l) \leq b\}\right] = \frac{r}{b} \sum_{l=1}^{b} \mathbb{P}(\tilde{X}(l) \leq b) = \frac{r}{b} \sum_{l=1}^{b} \left(\frac{\gamma_0}{b}\right) \mathbbm{1}\{l \leq b\} = \frac{r}{b} \sum_{l=1}^{b} 1 \quad \Leftrightarrow \quad \eta = \frac{rb}{\gamma_0}. \quad (20)
\]

In \((\mathbf{X}, \mathbf{X}) \in \mathcal{P}_{n+b}\), candidates and reference set are ranked jointly, regardless if the referents resigned or not. The optimal regret is defined as the average sum of the \(b\)-best available ranks. If one of the unavailable referents is among the \(b\)-best, his rank is replaced by the next best available rank (same for multiple unavailable referents), which increases the expected offline regret. Formally:

\[
\mathbb{E}[\phi^*] = \frac{b-\eta}{b} \sum_{m=1}^{b} m + \sum_{m=b+1}^{b+\eta} m = (b-\eta) \frac{b+1}{2} + \eta b + \frac{\eta(\eta+1)}{2} \quad \Leftrightarrow \quad (21)
\]

\[
\mathbb{E}[\phi^*] = \frac{b(b+1+\eta+\eta^2/b)}{2} \quad \Leftrightarrow \quad \mathbb{E}[\phi^*] = \frac{b(b+1)}{2} + \frac{rb^2/\gamma_0 + r}{2\gamma_0^2}. \quad (22)
\]

\square

Proof. Proof of Lemma \[\text{Eq.}\] Set \(p_j = \mathbb{P}(X_j < \gamma_j), \ Z_j \sim \text{Bernoulli}(j-c, p_j), \ \text{and} \ \tilde{Z}_j = \sum_{i=c+1}^{j} Z_i. \)

Thus:

\[
\mathbb{P}(\tilde{A}_{j-1} < b) = \mathbb{P}\left(\min\left(\sum_{i=c+1}^{j-1} Z_i, b\right) < b\right) = \mathbb{P}\left(\sum_{i=c+1}^{j-1} Z_i < b\right) = \mathbb{P}(\tilde{Z}_{j-1} < b). \quad (23)
\]

We have \(q = 1/2\), hence \((\mathbf{X}, \mathbf{X})\) is uniformly distributed in \(\{1, \ldots, n+b\}\). Therefore, \(p_j = \frac{\gamma_{j-1}}{n+b}\) and since \(j \leq n, \ j \rightarrow \infty \Rightarrow n \rightarrow \infty \) thus, \(\lim_{j \rightarrow \infty} \sum_{i=c+1}^{j} \left(\frac{\gamma_{j-1}}{n+b}\right)^j = 0. \) Therefore \(\lim_{j \rightarrow \infty} \sum_{i=c+1}^{j} p_i^2 = 0, \) in other words, the more candidates there are, the smaller the probability for each of them to be accepted. Set \(\sigma_j^2 = \sum_{i=c+1}^{j} p_i^2 \) and \(\lambda_j = p_{c+1} + \ldots + p_j. \) From Le Cam’s theorem we
have \( \sum_{m=0}^{\infty} P(\tilde{Z}_j = m) - \frac{\lambda_j e^{-\lambda_j}}{m!} < 2\sigma_j^2 \), and since \( \lim_{j \to \infty} \sigma_j^2 = 0 \), \( \tilde{Z}_j \) tends to a Poisson distribution with parameter \( \lambda_j \). Its cumulative distribution is therefore, \( P(\tilde{Z}_j \leq x) = P(\tilde{A}_j \leq x) = e^{-\lambda_j - 1} \sum_{i=0}^{x} \frac{\lambda_j^i}{i!} + o(\sigma_j^2) \).

**Proof.** Proof of Theorem 1. We handle each bullet point separately:

- First, we investigate the rank-based expected threshold to beat for the first candidate incoming just after the learning phase, \( \gamma := E[Y_{(b,c)}] \). The proof is done by backward induction. We first consider the case where the number of rejected candidates \( c \) is s.t. \( c = n \); the updated reference set is composed of the \( b \)-best items of \( (X, X) \) since every candidate has been rejected and their scores are stored in the updated reference set. Thus \( \gamma = b \). Let us go one step ahead and consider the case where \( c = n - 1 \), which implies that \( \gamma = b \) if the candidate that has not been examined is not among the \( b \)-best items, and \( b + 1 \) if he is. Hence, \( \gamma(c = n - 1) = b \frac{c}{b + c} + (b + 1) \frac{b}{b + c} \).

  By recursion, we get:

  \[
  \gamma(c) = \sum_{m=0}^{n-c} \binom{n-c}{m} \frac{b}{b+c}^{n-c-m} \frac{c}{b+c}^m b + m = \frac{1}{(b+c)^{n-c}} \sum_{m=0}^{n-c} \binom{n-c}{m} c^{n-c-m} b^m (b + m)
  \]

  \[
  \Leftrightarrow \gamma(c) = \frac{b(n+b)}{b+c}.
  \]

  When \( j > c \), after multiple repetitions of the selection, each acceptance threshold is replaced by its expectation, in particular \( Y_{(b,c)} \) tends towards its expectation \( \gamma := E[Y_{(b,c)}] \). Hence \( \delta = r + \sum_{j=1}^{c} \mathbb{I}\{X_j < Y_{(b,c)}\} \) tends to \( E[r + \sum_{j=1}^{c} \mathbb{I}\{X_j < \gamma\}] \), i.e. \( \Delta := E[\delta] = r + \sum_{j=1}^{c} E[X_j < \gamma] = r + c \frac{2 - 1}{n + b} \). Then, the evolving threshold becomes \( \gamma_j = \gamma E[\mathbb{I}\{\tilde{A}_j - 1 < \Delta\}] + E[\tilde{X}_{(b-\tilde{A}_{j-1})}^+] \mathbb{I}\{\tilde{A}_{j-1} \geq \Delta\} \).

  In order to use the fact that \( E[\tilde{X}_{(b)}^+] = E[\tilde{X}_{(1)}^+], \forall \in \{1, ..., b\} \), in the proof we approximate \( \tilde{X}_{(b-\tilde{A}_{j-1})}^+ \) by \( \tilde{X}_{(b-E[\tilde{A}_{j-1}])}^+ \) by considering that \( \tilde{A}_j \) has a small variance, which is given by \( \sigma_{j-1}^2 \).

  Therefore:

  \[
  \gamma_j = \gamma E[\mathbb{I}\{\tilde{A}_j - 1 < \Delta\}] + E[\tilde{X}_{(b-\tilde{A}_{j-1})}^+] E[\mathbb{I}\{\tilde{A}_{j-1} \geq \Delta\}] + o(\sigma_{j-1}^2)
  \]

  \[
  = \gamma P(\tilde{A}_{j-1} < \Delta) + \frac{\gamma_0 (b+1)}{b(b-r+1)} (b - E[\tilde{A}_{j-1}]) P(\tilde{A}_{j-1} \geq \delta) + o(\sigma_{j-1}^2)
  \]

  We have \( E[\tilde{A}_j] = \sum_{i=c+1}^{j} E[A_i] = \sum_{i=c+1}^{j} P(A_i = 1) = \sum_{i=c+1}^{j} P(X_i < \gamma_i) P(\tilde{A}_{i-1} < b) \); hence:

  \[
  \gamma_j = \gamma g_j(\Delta) + \frac{\gamma_0 (b+1)}{b(b-r+1)} \left( b - \sum_{i=1}^{j} \frac{\gamma_i - 1}{n + b} g_i(b) \right) (1 - g_j(\Delta)) + o(\sigma_{j-1}^2).
  \]

  where \( g_j(x) := P(\tilde{A}_{j-1} < x) \) is computed using Lemma 1

- Since \( \tau_j \) tends to \( \gamma_j \), we get \( A_j = \mathbb{I}\{j > c\} \mathbb{I}\{\tilde{A}_{j-1} < b\} \mathbb{I}\{X_j < \gamma_j\} \), hence:

  \[
  E[\tilde{A}_n] := \sum_{j=1}^{n} E[A_j] = \sum_{j=1}^{n} P(A_j = 1) = \sum_{j=1}^{n} P(X_j < \gamma_j) P(\tilde{A}_{j-1} < b) = \sum_{j=1}^{n} \frac{\gamma_j - 1}{n + b} g_j(b).
  \]

Recall the definition of the regret \( \phi := X^T \tilde{A}_n + X^T A - \phi^* \). Set \( \phi_1 = X^T \tilde{A}_n \) and \( \phi_2 = X^T A \) that give respectively the reference set and the candidates contribution to the regret. We start with the candidates, \( \phi_2 = \sum_{j=1}^{n} X_j A_j \). Its expectation is given by \( E[\phi_2] = E[\sum_{j=1}^{n} X_j A_j] \). We
Proof. Proposition 4. Set $\tilde{E}_j$ as a candidate with rank lower than the threshold $\gamma_j$ will never be accepted, hence:

$$\mathbb{E}[\phi_2] = \sum_{j=c+1}^{n} \sum_{m=1}^{n+b} \sum_{a\in\{0,1\}} P(X_j = m, A_j = a) am = \sum_{j=c+1}^{n} \sum_{m=1}^{n+b} P(A_j = 1 | X_j = m) P(X_j = m)m$$

(30)

A candidate with rank lower than the threshold $\gamma_j$ will never be accepted, hence:

$$\mathbb{E}[\phi_2] = \sum_{j=c+1}^{n} \sum_{m=1}^{n+b} P(A_j = 1 | X_j = m) P(X_j = m)m.$$

(31)

A candidate with rank lower than the threshold is accepted if there were less than $b$ candidates accepted before him. Moreover, we use the fact that $P(X_j = m) = \frac{1}{n+b}$ to write:

$$\mathbb{E}[\phi_2] = \sum_{j=c+1}^{n} \sum_{m=1}^{n+b} P(\tilde{A}_{j-1} < b) \frac{m}{n+b} = \sum_{j=c+1}^{n} \sum_{m=1}^{n} g_j(b) \frac{m}{n+b} = \frac{1}{n+b} \sum_{j=c+1}^{n} g_j(b) \frac{\gamma_j(\gamma_j-1)}{2}. \quad (32)$$

Following up with the reference set contribution: $\phi_1 = \sum_{l=1}^{b} \tilde{X}^{-1}(l) \{ l \leq b - \tilde{A}_n \} = \sum_{l=1}^{b-\tilde{A}_n} \tilde{X}^{-1}(l)$ is the regret associated with the available referents that were not fired at the end of the selection. Its expectation is given by $\mathbb{E}[\phi_1] = \mathbb{E}[\sum_{l=1}^{b-\tilde{A}_n} \tilde{X}^{-1}(l)]$. We suppose that variables $\tilde{A}_n$ and $\tilde{X}(l)$ are independent $\forall l$, which is a reasonable assumption since we consider a reference set with medium quality, i.e. medium average rank, and we use $\mathbb{E}[\tilde{X}^{-1}(l)] = \frac{\gamma_0(b+1)}{b(b-r+1)}$ (see Proposition 1):

$$\mathbb{E}[\phi_1] = \sum_{l=1}^{b-\tilde{A}_n} \frac{\gamma_0(b+1)}{b(b-r+1)} = \frac{\gamma_0(b+1) l}{b(b-r+1)} \sum_{l=1}^{b-\tilde{A}_n} l = \frac{\gamma_0(b+1)}{2b(b-r+1)} (b - \mathbb{E}[\tilde{A}_n])(b+1 - \mathbb{E}[\tilde{A}_n]). \quad (33)$$

Proof. Proof Proposition 4. Set $\tilde{Z}_j = \sum_{i=c+1}^{j} Z_i$ where $Z_j \sim \text{Bernoulli}(j-c,p_j)$ and set $\lambda_j = \sum_{i=c+1}^{j} p_i$, $\forall j$. We have $\mathbb{E}[A_j | \tilde{A}_n \geq r] = \sum_{k=0}^{b} k P(\tilde{A}_j = k | \tilde{A}_n \geq r)$. Hence:

$$\mathbb{E}[\tilde{A}_j | \tilde{A}_n \geq r] = \sum_{k=0}^{b} \frac{k P(\text{min}(\tilde{Z}_j,b) = k, \text{min}(\tilde{Z}_n,b) \geq r))}{P(\tilde{A}_n \geq r)}$$

$$\mathbb{E}[\tilde{A}_j | \tilde{A}_n \geq r] = \frac{1}{P(\tilde{A}_n \geq r)} \sum_{k=0}^{b-1} k P(\{\tilde{Z}_j = k\} \cap \{\tilde{Z}_n \geq r\}) + \frac{b}{P(\tilde{A}_n \geq r)} P(\{\tilde{Z}_j \geq b\} \cap \{\tilde{Z}_n \geq r\})$$

$$\mathbb{E}[\tilde{A}_j | \tilde{A}_n \geq r] = \frac{P(\tilde{Z}_n \geq r)}{P(\tilde{A}_n \geq r)} \sum_{k=0}^{b-1} k P(\tilde{Z}_j = k) + \frac{b P(\tilde{Z}_j \geq b)}{P(\tilde{A}_n \geq r)}$$

$$\mathbb{E}[\tilde{A}_j | \tilde{A}_n \geq r] = \sum_{k=0}^{b-1} k P(\tilde{Z}_j = k) + \frac{b(1-g_{b+1}(b))}{1-g_{b+1}(r)}.$$
From Le Cam’s theorem we have \( \sum_{m=0}^{\infty} |P(\tilde{Z}_j = m) - \frac{\lambda_j^m e^{-\lambda_j}}{m!}| < 2\sigma_j^2 \); and since \( \lim_{j \to \infty} \sigma_j^2 = 0 \), \( \tilde{Z}_j \) tends a Poisson distribution with parameter \( \lambda_j \), see proof of Lemma 1, hence:

\[
E[\tilde{A}_j | \tilde{A}_n \geq r] = \lambda_j \sum_{k=0}^{b-2} \frac{\lambda_j^k e^{-\lambda_j}}{k!} + \frac{b(1-g_{j+1}(b))}{1-g_{n+1}(r)} + o(\sigma_j^2)
\]

\[
E[\tilde{A}_j | \tilde{A}_n \geq r] = \lambda_j \sum_{k=0}^{b-1} \frac{\lambda_j^k e^{-\lambda_j}}{(k-1)!} + \frac{b(1-g_{j+1}(b))}{1-g_{n+1}(r)} + o(\sigma_j^2)
\]

\[
E[\tilde{A}_j | \tilde{A}_n \geq r] = \lambda_j g_{j+1}(b-1) + \frac{b(1-g_{j+1}(b))}{1-g_{n+1}(r)}.
\]

\( \square \)

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