Log concavity of \((1 + x)^m(1 + x^k)\)

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Let \(m\) and \(k\) be positive integers. We determine necessary and sufficient conditions for the polynomial \(P = (1 + x)^m(1 + x^k)\) to be strongly unimodal (frequently known as log concave; see [S] for an extensive discussion of the combinatorial significance). Among the surprises is that the criterion, \(m \geq k^2 - 3\), is also the criterion for \(P\) to be merely unimodal (that is, for \(P\) of this form, unimodality implies strong unimodality).

Let \(p = \sum_{i=0}^{n} a_i x^i\) be a polynomial with nonnegative coefficients. Then \(p\) is unimodal if the distribution of coefficients, \((a_0, a_1, \ldots, a_n)\) is unimodal, that is, there exists \(k\) with \(0 \leq k \leq n\) such that \(a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n\) (in this definition, if the index \(j\) is less than zero or exceeds \(n\), then \(a_j = 0\)). The polynomial \(p\) is strongly unimodal if the function \(i \mapsto a_i\) is log concave (which amounts to \(a_i^2 \geq a_{i+1}a_{i-1}\) and \(a_ia_{i+2} \neq 0\) implies \(a_{i+1} \neq 0\).

In the combinatorics literature, strongly unimodal polynomials are referred to as log concave (this can cause confusion when the polynomials are treated as functions, in which case log concave has a different meaning). The term strongly unimodal dates back to an undeservedly-neglected 1956 paper of Ibragimov [I], wherein he proves (among other things) that a product of a strongly unimodal polynomial with a unimodal one is unimodal, and this property characterizes strong unimodularity.

A reparameterization [BH] (known as a tilting in \([P]\)) of the polynomial \(p\) is the new polynomial \(p_\lambda(x) := p(\lambda x) = \sum a_i \lambda^i x^i\) for some positive real \(\lambda\). It is an easy observation from [BH] that \(p\) is strongly unimodal if and only if for every reparameterization, \(p_\lambda\) is unimodal.

It is known [H] that if \(p\) is any real polynomial with no positive real roots, then there exists \(N\) such that \((1 + x)^N p\) is strongly unimodal. So it seemed of interest to determine the smallest choice \(N\) when \(p\) has particularly spread out roots (if all roots of \(p\) lie in the segment \(\arg z - \pi| \leq \pi/3\), then \(p\) is already strongly unimodal; on the other hand, if some roots have small argument, then the \(N\) will likely have to be large).

We use inner product notation to denote coefficients; thus if \(p = \sum a_i x^i\), then \(a_i = (p, x^i)\).

**THEOREM** Let \(m\) and \(k\) be positive integers, and set \(P = (1 + x)^m(1 + x^k)\). The following are equivalent.

(a) \(P\) is strongly unimodal

(b) \(P\) is unimodal

(c) \[
\begin{cases}
(P, x^{m+k-3/2}) \leq (P, x^{m+k-1/2}) & \text{if } m + k \text{ is odd} \\
(P, x^{m+k-2/2}) \leq (P, x^{m+k}) & \text{if } m + k \text{ is even}
\end{cases}
\]

(d) \(m \geq k^2 - 3\).

Obviously (a) implies (b). Since \(P\) is symmetric about \((m + k)/2\), (b) implies (c). The implication (c) \(\implies\) (d) is not difficult. The bulk of the work involves showing (d) implies (a).

For the proof of (c) implies (d) and a small portion of the proof of (d) implies (a), we compute \((P, x^j)\) for several values of \(j\) near the centre of the distribution. Here \(P = (1 + x)^m(1 + x^k)\), and \(j = (m + k - 1)/2, (m + k - 3)/2, (m + k - 5)/2, (m + k - 7)/2\) if \(m + k\) is odd, and \(j = (m + k)/2, (m + k - 2)/2\) if \(m + k\) is even. Write \(j = (m + k - \rho)/2\), where \(0 \leq \rho \leq 5\) and is an odd integer.

First, suppose that \(m + k\) is odd. Set \(Q(\rho) = (P, x^j)((m + k - 5)/2)!((m - k - 5)/2)!/m!\).
Then

\[ Q(1) = \frac{32}{(m+k-3)(m+k-1)(m-k-3)(m-k-1)(m-k+1)} + \frac{32}{(m+k-3)(m+k-1)(m-k-3)(m-k-1)(m+k+1)} \]
\[ = \frac{32}{(m+k-3)(m+k-1)(m-k-3)(m-k-1)} \left( \frac{1}{m-k+1} + \frac{1}{m+k+1} \right) \]
\[ = \frac{64(m+1)}{((m-3)^2 - k^2)((m-1)^2 - k^2)((m+1)^2 - k^2)}; \]

\[ Q(3) = \frac{32}{(m+k-3)(m-k-3)(m-k-1)(m+k+1)(m-k+3)} + \]
\[ = \frac{32}{(m-3)^2 - k^2} \left( \frac{1}{(m-k-1)(m+k+1)(m-k+3)} + \frac{1}{(m+k-1)(m+k+1)(m+k+3)} \right) \]
\[ = \frac{64(m+1)(m^2 + 2m + 3(k^2 - 1))}{((m-3)^2 - k^2)((m-1)^2 - k^2)((m+1)^2 - k^2)((m+3)^2 - k^2)} \]

\[ Q(5) = \frac{32}{\prod_{l=0}^{4}(m-k+2l-3)} + \frac{32}{\prod_{l=0}^{4}(m+k+2l-3)} \]
\[ = \frac{32}{\prod_{l=0}^{4}(m+k+2l-3)} \left( \prod_{l=0}^{4}(m-k+2l-3) + \prod_{l=0}^{4}(m+k+2l-3) \right) \]
\[ = \frac{64((m+1)(m^4 + 4m^3 + (10k^2 - 14)m^2 + (20k^2 - 36)m + 5k^4 - 50k^2 + 45))}{\prod_{l=0}^{4}((m+2l-3)^2 - k^2)} \]

\[ Q(7) = \frac{32(m+k-5)}{\prod_{l=0}^{5}(m-k+2l-3)} + \frac{32(m-k-5)}{\prod_{l=0}^{5}(m+k+2l-3)} \]
\[ = \frac{64(m+1)(m^6 + 6m^5 + (21k^2 - 41)m^4 + (84k^2 - 204)m^3 + (35k^4 - 434k^2 + 463)m^2 + (70k^4 - 1036k^2 + 1350)m - 245k^4 + 7k^6 + 1813k^2 - 1575)}{\prod_{l=0}^{5}(m+k+2l-3)} \]

The bottom lines for \(Q(5)\) and \(Q(7)\) were found with Maple. In the latter case, the second factor in the numerator had to be broken over two lines (fortunately, it simplifies drastically when we substitute \(m = k^2 - 3\)). We don’t require those two until much later in the paper, but this is a convenient time to deal with them.

We first examine \(Q(3)/Q(1)\); by symmetry of the distribution of coefficients of \(P\), if the latter is unimodal, then \(Q(3)/Q(1) \leq 1\). We show that this is the case if and only if \(m \geq k^2 - 3\) (still under the assumption that \(m + k\) is odd), and moreover, if \(m = k^2 - 3\) (so that \(m + k\) is odd), then \(Q(3) = Q(1)\).
\[
\frac{Q(3)}{Q(1)} = \frac{(m+1)(m^2 + 2m + 3(k^2 - 1))}{(m+1)((m+3)^2 - k^2)}
= \frac{m^2 + 2m + 3(k^2 - 1)}{(m+3)^2 - k^2}
\]

Expanding this (and assuming, as usual, that \(m, k \geq 2\)), we deduce that \(P, x^{\frac{m+k-3}{2}} \leq (P, x^{\frac{m+k-1}{2}})\) if and only if \(m \geq k^2 - 3\), with equality if and only if \(m = k^2 - 3\).

For \(m+k\) even, computing \(P, x^{\frac{m+k-2}{2}}\)/\(P, x^{\frac{m+k}{2}}\) (the ratio of one term away from the centre to the central term) is very easy, and we obtain
\[
\frac{P, x^{\frac{m+k-2}{2}}}{P, x^{\frac{m+k}{2}}} = \frac{1}{2} \left( \frac{m+k}{m-k+2} + \frac{m-k}{m+k+2} \right)
= \frac{m^2 + k^2 + 2m}{m^2 + 4m + 4 - k^2}.
\]

It follows immediately that when \(m+k\) is even, \(P, x^{\frac{m+k-2}{2}} \leq (P, x^{\frac{m+k}{2}})\) if and only if \(m \geq k^2 - 2\) with equality only when \(m = k^2 - 2\).

The two cases (depending on the parity of \(m+k\)) yield (c) implies (d). The first case also shows that when \(m = k^2 - 3\) (which entails that \(m+k\) is odd), the four consecutive middle coefficients (that is, positions \(m+k\pm 3)/2, (m+k\pm 1)/2\) are equal.

Next we show that when \(m = k^2 - 3\), \(P, x^{\frac{m+k-5}{2}} < (P, x^{\frac{m+k-3}{2}})\), equivalently, \(Q(5) < Q(3)\); more importantly, the ratio is determined exactly. Note that \(m+1\) divides the numerator of \(Q(5)\); this allows a simplification. We note that
\[
\frac{Q(5)}{Q(3)} = \frac{m^4 + 4m^3 + (10k^2 - 14)m^2 + (20k^2 - 36)m + 5k^4 - 50k^2 + 45}{(m^2 + 2m + 3(k^2 - 1)((m + 5)^2 - k^2))}
= \frac{m^4 + 4m^3 + (10k^2 - 14)m^2 + (20k^2 - 36)m + 5k^4 - 50k^2 + 45}{m^4 + 12m^3 + (42 + 2k^2)m^2 + (20 + 28k^2)m - 3k^4 + 78k^2 - 75}.
\]

Now assume \(m = k^2 - 3\); the numerator factors (courtesy of Maple) as \(k^2(k-1)(k-2)(k+2)(k+1)(k^2 + 7)\) and the denominator as \(k^2(k-1)(k+1)(k^2 + k + 2)(k^2 - k + 2)\), so
\[
\frac{Q(5)}{Q(3)} \bigg|_{m = k^2 - 3} = \frac{(k-2)(k+2)(k^2 + 7)}{(k^2 + k + 2)(k^2 - k + 2)}
= \frac{k^4 + 3k^2 - 28}{k^4 + 3k^2 + 4} = 1 - \frac{32}{k^4 + 3k^2 + 4}.
\]

In particular, when \(m = k^2 - 3\), \(P, x^{\frac{m+k-5}{2}} < (P, x^{\frac{m+k-3}{2}})\).

Both \(Q(5)\) and \(Q(7)\) have common factors of \(64(m+1)\); when we factor these out and set \(m = k^2 - 3\), the result for \(Q(7)\) factors as simply \(k^2(k^2 - 1)(k^2 - 4)(k^6 + 14k^4 - 63k^2 - 272)\). Since
\[(m + 7)^2 - k^2\] evaluates to \((k^2 + 4)^2 - k^2\), we deduce
\[
\frac{Q(7)}{Q(5)} \bigg|_{m = k^2 - 3} = \frac{k^6 + 14k^4 - 63k^2 - 272}{((k^2 + 4)^2 - k^2)(k^2 + 7)}
\]
\[
= \frac{k^6 + 14k^4 - 63k^2 - 272}{k^6 + 14k^4 + 65k^2 + 112} = 1 - \frac{128(k^2 - 3)}{k^6 + 14k^4 + 65k^2 + 112}.
\]

Now we prove (d) implies (a). It suffices to do this only in the case that \(m = k^2 - 3\) because multiplication by \(1 + x\) (or any other strongly unimodal polynomial) preserves strong unimodality. Maple has verified strong unimodality of \(P\) for \(2 \leq k \leq 41\). For each positive \(\lambda\), define \(P \equiv P_\lambda\) via
\[
P(x) = (1 + \lambda x)^m(1 + (\lambda x)^k).
\]
Assume \(m = k^2 - 3\), where \(k \geq 42\) is an integer. We show that for all \(\lambda > 0\), \(P_\lambda\) is unimodal, hence \(P_1\) (our original \(P\)) is strongly unimodal. Symmetry of \(P_1\) implies that it is enough to show unimodality of \(P_\lambda\) for all \(0 < \lambda \leq 1\). The first observation is that the sequence \(\left(\lambda \cdot P, x^j\right)\) is increasing for \(0 < j \leq \lambda m / (1 + \lambda)\) and decreasing for \(j \geq \lambda m / (1 + \lambda) + k\). Hence we need only consider the integers \(j\) in the interval \((\lambda m / (1 + \lambda), \lambda m / (1 + \lambda) + k)\). It is trivial to verify that \(P_\lambda\) is unimodal for \(0 < \lambda \leq k(1 + \lambda) / m\), so we may assume \(\lambda > k(1 + \lambda) / m\).

Define (for \(\lambda\) fixed),
\[
p(j) := (P, x^{j+1} - x^j); \text{ then,}
\]
\[
A(j) := \frac{p(j)(m - j)}{\lambda^j \binom{m}{j+1}} = \lambda(m - j) - (j + 1) + \frac{\binom{m}{j+1-k}}{\binom{m}{j+1}} \left( \frac{\lambda(m - j)(m - j + k) - (m - j)(j - k + 1)}{m - j + k} \right)
\]
\[
= \lambda m - (1 + \lambda)j - 1 + \frac{\binom{m}{j+1-k}}{\binom{m}{j+1}} \frac{m - j}{m - j + k} \left( \lambda m - (1 + \lambda)j + (1 + \lambda)k - 1 \right).
\]

Next we observe that
\[
\frac{\binom{m}{j+1-k}}{\binom{m}{j+1}} = \prod_{s=1}^{k} \frac{j - k + s + 1}{m - j - 1 + s}
\]
\[
= \prod_{s=1}^{k} \frac{j - s + 2}{m - j - 1 + s}, \text{ and moreover,}
\]
\[
\prod_{s=1}^{k} \frac{m - j + k - s}{m - j + s - 1} = \frac{1}{\prod_{s=2}^{k+1} (m - j - 1 + s)} = \frac{1}{\prod_{s=1}^{k} (m - j + s)}.
\]

Thus
\[
A(j) \prod_{1}^{k} (m - j + s) = (\lambda m - (1 + \lambda)j - 1) \prod_{1}^{k} (m - j + s) + (\lambda m - (1 + \lambda)j + (1 + \lambda)k - 1) \prod_{1}^{k} (j - s + 2);
\]

now setting \(r = j - \lambda m / (1 + \lambda)\) with \(0 < r < k\),
\[
= (-1 - (1 + \lambda)r) \prod_{1}^{k} \left( \frac{m}{1 + \lambda} - r + s \right) + ((1 + \lambda)(k - r) - 1) \prod_{1}^{k} \left( \frac{\lambda m}{1 + \lambda} + r - s + 2 \right)
\]
\[
= \lambda^k ((1 + \lambda)(k - r) - 1) \prod_{1}^{k} \left( \frac{m}{1 + \lambda} + \frac{r - s + 2}{\lambda} \right) - ((1 + \lambda)r + 1) \prod_{1}^{k} \left( \frac{m}{1 + \lambda} - r + s \right).
\]
In particular, \( p(j) \geq 0 \) (that is, the coefficients of \( P \) are increasing at \( j \)) if and only if

\[
\frac{(1 + \lambda)(k - r) - 1}{(1 + \lambda)r + 1} \geq \prod_{i=1}^{k} \frac{m - r + s}{m + \lambda} + r - s + 2
\]

\[
= \lambda^{-k} \prod_{i=1}^{k} \frac{m - r + s}{m + \lambda} + r - s + 2
\]

\[
= \lambda^{-k} \prod_{i=1}^{k} \left( 1 + \frac{(s - r)(1 + 1/\lambda) - 2/\lambda}{m + \lambda - s - r/\lambda} \right)
\]

(We use the forms of both lines 1 and 3 of (1) in what follows.)

Set \( M = m/(1 + \lambda) \). The left side of (1) is less than \( k \). The right side of the expression (first line) is larger than \(((M - k)/(M \lambda + k))^{k}\); since this is monotone decreasing in \( \lambda \), the minimum the expression achieves on the interval \((k/M, \alpha)\) (where \( \alpha \) is some number less than \( 1 \)) occurs when \( \lambda = \alpha \).

We evaluate it, assuming \( k \geq 42 \) and \( 1 > \alpha > 1/2 \).

\[
\left( \frac{M - k}{M \alpha + k} \right)^k = \alpha^{-k} \left( 1 - \frac{k(1 + 1/\alpha)}{M + k/\alpha} \right)^k
\]

\[
= \alpha^{-k} \exp -k \left( \frac{k(1 + 1/\alpha)}{M + k/\alpha} + \frac{1}{2} \left( 1 - \frac{k(1 + 1/\alpha)}{M + k/\alpha} \right)^2 \right) + \ldots
\]

\[
\geq \alpha^{-k} \left( 1 - \frac{(1 + 1/\alpha)^2}{2M^2/k^3} \right) \exp \left( -\frac{k^2(1 + 1/\alpha)}{M + k/\alpha} \right)
\]

\[
\geq \frac{1}{\alpha^k} e^{-9/2} (1 - 10/k) \geq \frac{1}{\alpha^k} .75e^{-9/2}.
\]

If we choose any \( \alpha > 1/2 \) with \( \alpha \leq 1 - (5 + \ln k)/k \), then it is easy to see that \( .75e^{-9/2}/\alpha^k \geq k \).

Hence \( p(j) \) is negative for \( 0 < r < k \), and thus for any \( \lambda \leq 1 - (5 + \ln k)/k \), \( P_\lambda \) is decreasing on the interval \( Z \cap (\lambda M, \lambda M + k) \); in particular, this implies that \( P_\lambda \) is unimodal. So we are reduce to considering the case that \( 1 \geq \lambda > 1 - (5 + \ln k)/k \). The next, highly intricate, stage is to show unimodality for \( \lambda < 1 - 7/k^2 \), and finally there will remain only a few special cases to consider.

Let \( G(r) \) be the logarithm of the left side of (1) and \( H(r) \) the logarithm of the right side [first line]. From \( \ln g = \ln((1 + \lambda)(k - r) - 1) - \ln((1 + \lambda)(k - r) + 1) \), we have

\[
G'(r) = -\frac{1 + \lambda}{(1 + \lambda)(k - r) - 1} + \frac{1 + \lambda}{(1 + \lambda)r + 1}
\]

\[
= -\frac{k}{(k - r - 1/\lambda)(r + 1/\lambda)}.
\]

Set \( M = m/(1 + \lambda) \). From \( H = \ln h = -k \ln \lambda + \sum_{1}^{k} (\ln(M + s - r) - \ln(M - (s - r - 2)/\lambda)) \), we have

\[
H'(r) = -\sum_{s=1}^{k} \left( \frac{1}{M + s - r} + \frac{1}{M - (s - r - 2)/\lambda} \right).
\]

Now suppose that \( g(r_0) < h(r_0) \), so that \( G(r_0) < H(r_0) \). If \( G - H \) is decreasing on \([r_0, r]\), then \( G - H \) is negative on this interval, so that \( g < h \) on the interval, hence \( p(j) < 0 \), and if we
can arrange that this hold for \( r \) sufficiently close to \( k/2 \), then \( (P, x^j) \) is unimodal. We consider when the following inequality holds,

\[
\frac{k}{(k - r - \frac{1}{1 + \lambda})(r + \frac{1}{1 + \lambda})} > \sum_{s=1}^{k} \left( \frac{1}{M + s - r} + \frac{1}{M - (s - r - 2)/\lambda} \right).
\]

When this holds, \((G - H)' < 0\), so that \( G - H \) is decreasing. We estimate the right side using the first few terms of Euler-Maclaurin summation,

\[
\sum_{i=0}^{n} f(i) = \int_{0}^{n} f + \frac{1}{2} (f(n) + f(0)) + \frac{1}{12} (f''(n) - f''(0)) + R_{0},
\]

where \(|R_{0}| \leq \frac{2}{(2\pi)^{n/2}} f^{(n)}(0)|f^{(3)}|\), and if \( f^{(3)} \) has constant sign on \([0, n]\), then sign \( R = \text{sign} f^{(3)} \).

Set \( M_0 = M - r + 1 \), so that \( \sum_{s=1}^{k} 1/(M + s - r) = \sum_{s=1}^{k-1} 1/(M_0 + s) \), and set \( n = k - 1 \) and \( f(x) = 1/(M_0 + x) \). Then \( f' = -1/(M_0 + x)^2 \), \( f'' = 2/(M_0 + x)^3 \), \( f''' = -6/(M_0 + x)^4 \) so that the remainder term, \( R_0 \), is negative. Thus

\[
\sum_{s=1}^{k} \frac{1}{M + s - r} \leq \int_{0}^{k-1} \frac{dx}{M_0 + x} + \frac{1}{2} \left( \frac{1}{M_0 + k - 1} + \frac{1}{M_0} \right) + \frac{1}{12} \left( \frac{1}{(M_0 + k - 1)^2} + \frac{1}{M_0^2} \right)
\]

\[
= \ln \frac{M_0 + k - 1}{M_0} + \frac{1}{2} \left( \frac{1}{M_0 + k - 1} + \frac{1}{M_0} \right) + \frac{1}{12} \left( \frac{1}{(M_0 + k - 1)^2} + \frac{1}{M_0^2} \right)
\]

and \(|R_{0}| \leq 4/(2\pi)^3 (1/M_0^3 - 1/(M_0 + k - 1)^3) < k/20M_0^4\).

The second sum is slightly different: \( \sum_{s=1}^{k} 1/(M - (s - r - 2)/\lambda) = \sum_{s=1}^{k-1} M - (s - r - 1)/\lambda \); set \( M_1 = M + (r + 1)/\lambda \) and \( f = 1/(M_1 - x/\lambda) \). Then all derivatives of \( f \) are nonnegative, so the remainder term, \( R_1 \), is positive. Thus

\[
\sum_{s=1}^{k} \frac{1}{M - s - r - 2/\lambda} = \int_{0}^{k-1} \frac{dx}{M_1 - \frac{x}{\lambda}} + \frac{1}{2} \left( \frac{1}{M_1 - \frac{k-1}{\lambda}} + \frac{1}{M_1} \right) + \frac{1}{12\lambda} \left( \frac{1}{(M_1 - \frac{k-1}{\lambda})^2} - \frac{1}{M_1^2} \right) + R_{1}
\]

\[
= \lambda \ln \frac{M_1}{M_1 - \frac{k-1}{\lambda}} + \frac{1}{2} \left( \frac{1}{M_1 - \frac{k-1}{\lambda}} + \frac{1}{M_1} \right) + \frac{1}{12\lambda} \left( \frac{1}{(M_1 - \frac{k-1}{\lambda})^2} - \frac{1}{M_1^2} \right) + R_{1}
\]

and \( 0 \leq R_1 \leq k/20M_1^4 \).

Now we estimate the various terms and add them up.

\[
\ln \frac{M_0 + k - 1}{M_0} = \ln \left( 1 + \frac{k - 1}{M_0} \right) = \frac{k - 1}{M_0} - \frac{1}{2} \left( \frac{k - 1}{M_0} \right)^2 + \frac{1}{3} \left( \frac{k - 1}{M_0} \right)^3 - \frac{1}{4} \left( \frac{k - 1}{M_0} \right)^4 + \ldots
\]

\[
\leq \frac{k - 1}{M_0} - \frac{1}{2} \left( \frac{k - 1}{M_0} \right)^2 + \frac{1}{3} \left( \frac{k - 1}{M_0} \right)^3
\]

\[
\lambda \ln \frac{M_1}{M_1 - \frac{k-1}{\lambda}} = \lambda \left( -\ln \frac{M_1 - \frac{k-1}{\lambda}}{M_1} \right)
\]

\[
= \lambda \left( \frac{k - 1}{\lambda M_1} + \frac{1}{2} \left( \frac{k - 1}{\lambda M_1} \right)^2 + \frac{1}{3} \left( \frac{k - 1}{\lambda M_1} \right)^3 + \ldots \right)
\]

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A simple computation reveals that $1/M_0^2 - 1/(M_0 + k - 1)^2 < 3(k - 1)/M_0^3$ when $k > 11$ and similarly $1/(M_1 - (k - 1)/\lambda)^2 - 1/M_1^2 \leq 3(k - 1)/\lambda M_1^3$ for $k$ at least 11. We expand $1/(M_0 + k - 1)$ and $1/(M_1 - (k - 1)/\lambda)$ by their obvious geometric series expansions, e.g.,

$$\frac{1}{M_0 + k - 1} = \frac{1}{M_0} \cdot \frac{1}{1 + \frac{k - 1}{M_0}} = \frac{1}{M_0} \left(1 - \frac{k - 1}{M_0} + \left(\frac{k - 1}{M_0}\right)^2 - \left(\frac{k - 1}{M_0}\right)^3 + \ldots\right)$$

$$\frac{1}{M_1 - \frac{k - 1}{\lambda}} = \frac{1}{M_1} \cdot \frac{1}{1 - \frac{k - 1}{\lambda M_1}} = \frac{1}{M_1} \left(1 + \frac{k - 1}{\lambda M_1} + \left(\frac{k - 1}{\lambda M_1}\right)^2 + \left(\frac{k - 1}{\lambda M_1}\right)^3 + \ldots\right).$$

A simple computation also yields that $1/\lambda M_1^2 - 1/M_0^2 \leq 2(1 - \lambda)/M^2$ for $k \geq 11$ (the left side may be negative; this does no harm).

Next, we just add everything to obtain an upper bound for $-H'(r)$. We also expand $1/M_0 = (1/M)(1 - (r - 1)/M) = M^{-1} \sum((r - 1)/M)^j$ and similarly for $1/M_1$. The sum of the cubic and higher terms in the expansions of the logarithm is bounded above by $k^4/2M^4$, and that of the fourth order and higher terms in all the geometric expansions is bounded by $(k - 1)/M^4$.

$$-H'(r) \leq k \left(\frac{1}{M_0} + \frac{1}{M_1}\right) + \frac{k^2 - k}{2} \left(\frac{1}{\lambda M_1^2} - \frac{1}{M_0^2}\right) + (\frac{k - 1}{4} + \frac{(k - 1)^2}{2} + \frac{(k - 1)^3}{3}) \left(\frac{1}{M_0^3} + \frac{1}{\lambda^2 M_1^2}\right) + \frac{1}{M^4} \left(\frac{k^4}{2} + (k - 1)\right) + R_1$$

$$\leq \frac{2}{M} \frac{r(1 - 1/\lambda) - (1 + 1/\lambda)}{M^2} + \frac{r^2}{M^3} (1 + 1/\lambda) + \ldots + \frac{1 - \lambda}{2\lambda k^3} + \frac{C'}{k^4}$$

$$\leq \frac{1}{M} \left(2 + \frac{(1 - \lambda)M}{2\lambda k^3} + \frac{C'M}{k^4}\right).$$

where $0 \leq C' \leq ??$

Set $R = r + 1/(1 + \lambda)$ (so $1/(1 + \lambda) < R < k + 1/(1 + \lambda)$). The inequality (2) thus holds if

$$R(k - R) \left(2 + \frac{(1 - \lambda)M}{2\lambda k^3} + \frac{C'M}{k^4}\right) \leq M = \frac{k^2 - 3}{1 + \lambda}.$$

We view this as a quadratic in $R$. Expanding

$$\frac{M}{2 + \frac{(1 - \lambda)M}{2\lambda k^3} + \frac{C'M}{k^4}} = \frac{k^2 - 3}{2(1 + \lambda)} \left(1 - \frac{(1 - \lambda)M}{4\lambda k^3} - \frac{C'M}{2k^2} + \ldots\right)$$

$$\geq \frac{k^2 - 3}{2(1 + \lambda)} \left(1 - \frac{(1 - \lambda)M}{4\lambda k^3}\right)$$

Hence we need only examine the (now normalized) quadratic inequality

$$R^2 - kR + \frac{k^2 - 3}{2(1 + \lambda)} \left(1 - \frac{(1 - \lambda)M}{4\lambda k^3}\right) > 0; \text{ that is,}$$

$$R^2 - kR + \frac{k^2 - 3}{2(1 + \lambda)} \left(1 - \frac{(1 - \lambda)M}{4\lambda k^3}\right) > 0.$$

(3) $(R - k/2)^2 > -k^2 \left(\frac{1}{4} - \frac{1}{2(1 + \lambda)} + \frac{(1 - \lambda)(k^2 - 3)}{4\lambda(1 + \lambda)^2 k^3}\right) + \frac{3}{2(1 + \lambda)} - \frac{3(1 - \lambda)(k^2 - 3)}{4\lambda(1 + \lambda)^2 k^3}$

$$= -k^2 \left(\frac{1 - \lambda}{4(1 + \lambda)} \left(1 - \frac{1-k}{k^2}\right) \frac{1}{\lambda(1 + \lambda)}\right) + \frac{3}{2(1 + \lambda)}$$
We first observe that if $\lambda > 1 - (\ln k + 5)/k$ (as we may assume by earlier argument), then sufficient for the right side (the discriminant) to be negative is that $1 - \lambda > 6/(k^2(1 - \alpha/k)$ for any $\alpha < 1$. Hence if $\lambda < 1 - (6 + 1/k)/k^2$, the inequality holds for all values of $R$ in the relevant interval, and thus $P_\lambda$ is unimodal. If however, $\lambda > 1 - (6 + 1/k)/k^2$, we reduce the problem to calculating a small number of coefficients.

Now assume that $\lambda = 1 - \gamma/k^2$ where $0 \leq \gamma < 7$ and the inequality in (3) does not hold. The right side of (3) becomes

$$\frac{3}{4 - 2/\gamma^2} - \left(\frac{\gamma}{8 - 4\gamma/k^2} \left(1 - \frac{1}{k} \left(1 - \frac{2}{k^2}\right)\right) \frac{1}{\lambda(2 - \gamma/k^2)}\right) \leq \frac{3 - \gamma/2}{4} + \frac{1}{k}.$$

If $\gamma > 6 + 4/k$, this is negative, so the inequality holds for all relevant values of $R$. For $\gamma \leq 6 + 4/k$, the inequality is valid providing

$$|R - \frac{k}{2}| \leq \frac{\sqrt{3 - \gamma/2 + 4/k}}{2}.$$

So now we investigate what happens when this last inequality fails. We recall that $\lambda m/(1 + \lambda) + r$ must be an integer. Now

$$\frac{\lambda m}{1 + \lambda} + r = \frac{(1 - \gamma/k^2)(k^2 - 3)}{2 - \gamma/k^2} + R - \frac{1}{2 - \gamma/k^2}$$

$$= R + \frac{1}{2} \frac{1}{1 - \gamma/2k^2}(k^2 - 4 - \gamma + 3\gamma/k^2)$$

$$= R + \frac{1}{2} (k^2 - 4 - \gamma + 3\gamma/k^2) \sum_{j=0}^{\infty} (\gamma/2k^2)^j$$

$$= R + \frac{k^2 - 4 - \gamma/2}{2} + \frac{\gamma^2 + 2\gamma}{2k^2} + e,$$

where $-3\gamma/8k^4 < e < \gamma/10k^4$. If $k$ is odd, this entails that the fractional part of $R$ be the fractional part of $1/2 + \gamma/4 - f$ where $f \sim (-\gamma^2/2 + 2\gamma)/4k^2$ (to within $\gamma/10k^4$). If $k$ is even, the fractional part of $R$ is the fractional part of $\gamma/4 - f$.

In the former case, write $R = c + 1/2 + \gamma/4 - f$, and $k = 2t + 1$ (where $c$ and $t$ are integers). Then $R - k/2 = c - t + \gamma/4 - f$, and the absolute value of this is supposed to be less than $\sqrt{3 - \gamma/2 + 4/k}/2$. If $k$ is instead even, then write $R = c + \gamma/4 - f$ and $k = 2t$; then $R - k/2$ is exactly the same as in the odd case.

Set $C = c - t - f$, so that $C$ is within a tiny fraction of an integer. The inequality $(R - k/2)^2 < 3/4 - \gamma/8 + 1/k$ then translates to $(C + \gamma/4)^2 < 3/4 - \gamma/8 + 1/k$, that is

$$\gamma^2 + (8C + 2)\gamma + 16C^2 - 12 - 16/k < 0.$$

If $C > |f|$, then it is at least as large as a number close to 1, and we see quickly that when $k > 10$, the inequality is impossible. If $C$ is close to zero, then $\gamma < 2.7$ (assuming $k \geq 42$). If $C$ is close to $-1$, then $\gamma < 3.85$, and finally if $C$ is close to $-2$ or any smaller integer, there are no solutions. The fact that $C$ must be close to either 0 or $-1$ means we really only have to calculate a few coefficients of the original polynomial.
If $C$ is close to 0, then $c = t$, and we can evaluate $\lambda m/(1 + \lambda) + r$ (an integer):

(i) If $k$ is odd. The nearest integer to $R + (1/2)(k^2 - 4 - \gamma/2 - 2f + 2e)$, where $R = k/2 + \gamma/4 - f + e$ and $\gamma < 2.7$; this is $(k^2 + k - 4)/2$ (second term to the left of the centre).

(ii) If $k$ is even. The nearest integer to the same expression, where $R = c + \gamma/4 - f + e$ and $k = 2t$; this is the nearest integer to $k/2 + (k^2 - 4)/2$, exactly as in case (i).

If $C$ is close to $-1$, then $c = t - 1$; when $k = 2t + 1$ is odd, $R = t - 1/2 + \gamma/4 - f + e$, so $\lambda m/(1 + \lambda) + r$ is $t - 1/2 + \gamma/4 - f + (k^2 - 4/2 + 2f - 2e)/2$, which is just $(k^2 + k - 6)/2$. The result for even $k$ is the same.

Thus far, we have shown that if $\lambda < 1 - 3.85/k^2$, then $P_\lambda$ is unimodal; and if $1 > \lambda \geq 1 - 3.85/k^2$, the difference of consecutive coefficients in $P_\lambda$ is decreasing up to $j = (k^2 + k - 8)/2$ and for $j$ at least as large as $(k^2 + k - 2)/2$.

It remains to show that $P_\lambda$ is unimodal for $1 > \lambda > 1 - 3.85/k^2$ (notice that if $P_\lambda$ is unimodal for all $\lambda < 1$, then it is unimodal for $\lambda = 1$). We have that $(P_\lambda, x^{j + 1} - x^j)$ is decreasing for all $j$ in the intervals $[0, (k^2 + k - 8)/2]$ and $[(k^2 + k + 2)/2, k^2 + k - 3]$. Because of the overlap, it is sufficient to show $((P_\lambda, x^j))$ is unimodal for $\lambda m/(1 + \lambda) < j < k + \lambda m/(1 + \lambda)$.

Since $(P_\lambda, x^{j + 1} - x^j)$ is decreasing on the interval $[0, (k^2 + k - 8)/2]$, it follows that the sequence $(P_\lambda, x^j)_{j=0}^{(k^2+k-8)/2}$ is unimodal. Denote $(k^2 + k - 8)/2 = (m + k - 5)/2$ by $s$. By the $Q(5)/Q(3)$ formula,

$$(P_1, x^s) < (P_1, x^{s+1}) = (P_1, x^{s+2}) = (P_1, x^{s+3}) = (P_1, x^{s+4}) > (P_1, x^{s+5}).$$

Hence the sequence $((P_1, x^{x+i}))_{i=0}^5$ is strongly unimodal, and therefore the sequence $((P_\lambda, x^{x+i}))_{i=0}^5 = (\lambda x^i(P_1, x^{x+i}))_{i=0}^5$ is unimodal (it is of course strongly unimodal, but this is unnecessary here). Moreover, if $\lambda < 1$, the sequence $((P_\lambda, x^{x+i}))_{i=1}^5$ (beginning at $i = 1$ rather than $i = 0$) is monotone decreasing. As $(P_\lambda, x^{j+1} - x^j)$ is decreasing for $j \geq s - 5$, it follows that the entire tail, $((P_\lambda, x^{x+i}))_{i=1}^5$ is monotone decreasing.

Let $\nu$ denote the rightmost mode (integer where the maximum is attained). If $\nu = s$, then $(P_\lambda, x^j)_{j=0}^s$ is increasing and thus the entire sequence is unimodal (with mode at $s$). Suppose that $\nu < s$. Then unimodality of the entire sequence will result if $(P_\lambda, x^s) \geq (P_\lambda, x^{s+1})$. Since $(P_\lambda, x^s)/(P_\lambda, x^{s+1}) = \lambda^{-1}(P_1, x^s)/(P_1, x^{s+1})$, and the second factor is just $Q(5)/Q(3)$ evaluated at $m = k^2 - 3$, we see that if $\lambda \leq 1 - 32/(k^4 + 3k^2 + 4)$, then $P_\lambda$ is unimodal.

This leaves the possibility that $\lambda > 1 - 32/(k^4 + 3k^2 + 4)$ but $\nu < s$. We show this is impossible. Since $\nu < s$ and $\nu$ is the largest mode, $(P_\lambda, x^{s-1}) > (P_\lambda, x^s)$. Then the ratio $(P_\lambda, x^{s-1})/(P_\lambda, x^s) = \lambda^{-1}(P_1, x^{s-1})/(P_1, x^s)$, and again the right factor is $Q(7)/Q(5)$ evaluated at $m = k^2 - 3$. This yields

$$\frac{Q(7)}{Q(5)}\bigg|_{m=k^2-3} > \lambda > 1 - \frac{32}{k^4 + 3k^2 + 4},$$

so

$$1 - \frac{128(k^2 - 3)}{k^6 + 14k^4 + 65k^2 + 112} > 1 - \frac{32}{k^4 + 3k^2 + 4};$$

equivalently

$$\frac{1}{k^4 + 3k^2 + 4} > \frac{4(k^2 - 3)}{k^6 + 14k^4 + 65k^2 + 112}.$$

A quick check shows that this is false merely if $k > 3$. This completes the proof of the theorem.

**Addendum**

Let $f$ be a complex-valued function defined on a neighbourhood of the unit circle, that is at least $C^2$, and such that $f(1) = 1$. The variance of $f$ is defined as $V(f) = f''(1) + f'(1) - f'(1)^2$. Define, as in [H2], the class of entire functions,

$$E = \left\{ \text{entire} f : C \to C, f(1) = 1 \ \text{if} \ |z| = 1, \text{then} \ |f(z)|^2 \leq e^{-\Re V(f)}|1-z|^2 \right\}.$$
This class is closed with respect to products and uniform convergence on compact sets. It includes all real polynomials all of whose roots lie in the segment $|\arg z - \pi| \leq \pi/3$. It is known (and easy to prove) [H2] that if $p$ is a real polynomial with $p(1) = 1$, then there exists $N$ such that $((1+x)/2)^N p$ belongs to $\mathcal{E}$. All known examples of real polynomials in $\mathcal{E}$ with no negative coefficients are strongly unimodal, so it will be of interest to check what the optimal value of $N$ is when $p = (1 + x^k) / 2$.

Another surprise: $N$ must be of order $k^4$.

Set $f = (1 + x)^m (1 + x^k) / 2^{m+1}$. Then $\mathcal{V}(f) = m \mathcal{V}((1 + x) / 2) + \mathcal{V}(1 + x^k) / 2 = (m + k^2) / 4$. From $|1 - e^{i\theta}|^2 = 4 \sin^2 \theta / 2$, we have that for $e = e^{i\theta}$ (it is enough to deal with $0 \leq \theta \leq \pi$, since $f$ has only real coefficients) to

$$|f(z)|^2 = \cos^{2m} \theta / 2 \cos^2(k\theta / 2)$$

$$e^{-(\mathcal{V}(f)^{1-\alpha})^2} = \exp(-(m + k^2) \sin^2 \theta / 2)$$

Thus the inequality $|f(z)|^2 \leq e^{-(\mathcal{V}(f)^{1-\alpha})^2}$ is equivalent (on multiplying by $e^{m+k^2}$ and then taking $m$th roots, $e^{1+k^2/m} (\cos^2 \theta / 2) \cdot |\cos(k\theta / 2)|^{2/m} \leq \exp((1 + k^2/m) \cos^2 \theta / 2)$).

Consider the expression obtained by replacing the obnoxious term by $1$ and the $\cos^2 \theta / 2$ by $\alpha$ (with $0 \leq \alpha \leq 1$, and for convenience, replace $1 + k^2/m$ by $s$ (so $1 \leq s$, and typically, $s$ will be much less than 2),

$$F(\alpha) := e^{\alpha} - \alpha e^{\alpha}.$$ 

On the interval $[0,1]$, $F$ has a unique minimum, when $e^s = se^{\alpha}$, that is, at $\alpha_0 = 1 - \ln s / s$, and $F(\alpha_0) = -(s - 1 - \ln s)e^s / s < 0$. In fact, if $k^2/m < 1$ (as it will be), then $s - 1 - \ln s = k^2/m - \ln(1 + k^2/m) = (k^2/m)^2 (1/k^2 - \ldots) > 0$. If $k^2/m$ is substantially less than 1, $F(\alpha_0) = k^4 / 2m^2 + e$, where $e < k^6 / 3m^3$ is usually good enough in what follows.

Since $F(0) = 1$, $F$ has a unique zero on $[0,1)$, $\alpha_1$, and $F(\alpha) > 0$ when $\alpha < \alpha_1$; in particular, a necessary condition for $f$ to belong to $\mathcal{E}$ is that $\cos^2 \pi / k \leq \alpha_1$ (since if $\theta = 2\pi / k$, the obnoxious term evaluates to 1). We can approximate (for $k \geq 42$) $\alpha_1$ by $1 - 2t + 8t^2 / 3 + O(t^3)$ where $t = k^2 / m$ will be surprisingly small.

Now consider $p(x) = (1 + x^k) / 2$; then $\mathcal{V}(p) = k / 4$, and we consider $e^{-(\mathcal{V}(p)^{1-\alpha})^2} - |p(z)|^2$, that is, $g(\theta) = e^{-k \sin^2 \theta / 2 - \cos^2(k\theta / 2)}$ where $z = e^{i\theta}$. It is a triviality (from the additivity of $\mathcal{V}$, that is, $\mathcal{V}(ab) = \mathcal{V}(a) + \mathcal{V}(b)$, and that $(1 + x) / 2$ belongs to $\mathcal{E}$) to see that if $\theta$ satisfies $g(\theta) \geq 0$, then $|f(z)|^2 \leq e^{-(\mathcal{V}(f)^{1-\alpha})^2}$. It is routine to verify that $g$ is positive on an interval of the form $(0, \beta \equiv \beta(k)$, and a quick check shows that if $k = 42$, then $\beta = (3.45101 \pm 10^{-5})\pi / 2k$. Since increasing $k$ decreases $\cos^2 \beta(k)$ (easy), it follows that for all $k \geq 42$, $g$ is positive on $(0, 3.45\pi / 2k)$. Set $\alpha_2 = \cos^2 3.45\pi / 4k$. In particular, a sufficient condition for $f$ to belong to $\mathcal{E}$ is that $\alpha_2 \leq \alpha_1$. It is relatively easy to verify that when we permit larger $k$, the coefficient $3.45$ can be replaced by anything less than 4 (for sufficiently large $k$).

Doing the numerics, the necessary condition includes $m > k^4 / 3\pi^2 \approx 0.34k^4$ and the sufficient condition is $m > 0.448k^4 / \pi^2 \approx 0.045k^4$. (As $k$ increases, the difference between the constants becomes arbitrarily small.) In particular, the $N = m$ in the formulation of this problem must be of order $k^4$.

Why is it so much more difficult for $(1 + x)^m (1 + x^k) / 2^{m+1}$ to belong to $\mathcal{E}$ than for it to be strongly unimodal? I don’t know.

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