Green’s Function and Pointwise Behaviors of the Vlasov-Poisson-Fokker-Planck System

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Abstract
The pointwise space-time behaviors of the Green’s function and the global solution to the Vlasov-Poisson-Fokker-Planck (VPFP) system in spatial three dimension are studied in this paper. It is shown that the Green’s function consists of the diffusion waves decaying exponentially in time but algebraically in space, and the singular kinetic waves which become smooth for all \((t, x, v)\) when \(t > 0\). Furthermore, we establish the pointwise space-time behaviors of the global solution to the nonlinear VPFP system when the initial data is not necessarily smooth in terms of the Green’s function.

Key words. Vlasov-Poisson-Fokker-Planck system, Green’s function, pointwise behavior, spectrum structures.

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1 Introduction
The Vlasov-Poisson-Fokker-Planck (VPFP) system can be used to model the time evolution of dilute charged particles governed by the electrostatic force coming from their (self-consistent) Coulomb interaction [21]. The collision term in the kinetic equation is the Fokker-Planck operator that describes the Brownian force. In general, the scaled Vlasov-Poisson-Fokker-Planck (VPFP) system for one species reads

\[
\begin{align*}
\partial_t F + v \cdot \nabla_x F + \nabla_x \Phi \cdot \nabla_v F &= \nabla_v \cdot (\nabla_v F + v F), \\
\Delta_x \Phi &= \int_{\mathbb{R}^3} F dv - \bar{\rho}, \\
F(0, x, v) &= F_0(x, v),
\end{align*}
\]
Vlasov-Poisson-Fokker-Planck System

where $F = F(t, x, v)$ is the distribution function of charged particles with $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}_3^3 \times \mathbb{R}^3$, and $\Phi(t, x)$ denotes the electrostatic potential. The background density $\bar{\rho}$ is assumed to be constant 1 in this paper.

There are many important progress made on the well-posedness and asymptotic behaviors of solutions to the initial value problem or the initial boundary value problem of the VPFP system. We refer to [2, 24, 29, 11] for results on the existence of classical solutions and to [3, 5, 6, 28] for the existence of weak solutions and their regularity. Concerning the long-time behavior of the VPFP system, we refer to [1, 4, 7, 11, 13]. The diffusion limit of the weak solution to the VPFP system had been studied extensively in the literature (cf. [8, 9, 10, 22, 23]). The spectrum structure and the optimal decay rate of the classical solution to the VPFP system were investigated in [13].

We also mention some works related to the study in this paper. The Green’s function and pointwise space-time behavior of the Boltzmann equation were first studied in the pioneering works [16, 17, 18, 19]. For the linear Fokker-Planck equation, the structure of the Green’s function was investigated in [12, 15, 25, 26]. Recently, Li, Yang and Zhong established the Green’s function and pointwise space-time behavior of the unipolar Vlasov-Poisson-Boltzmann system in [14].

In this paper, we study the pointwise space-time behaviors of the Green’s function and the global solution to the VPFP system (1.1)–(1.2) based on the spectral analysis [13]. Note that the VPFP system (1.1)–(1.2) has an equilibrium state $(F^*, \Phi^*) = (M(v), 0)$ with the normalized Maxwellian $M(v)$ given by

$$M(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2 / 2}, \quad v \in \mathbb{R}^3.$$
we introduce the projections $P_m, P_2$ in the invariant subspace of $L$ by
\[
P_m f = \sum_{k=1}^{3} (f, v_k \sqrt{M}) v_k \sqrt{M}, \quad P_2 = P_0 + P_m, \quad P_3 = I - P_2. \tag{1.11}
\]

Corresponding to the linearized operator $L$, we define the following dissipation norm:
\[
\|f\|_2^2 = \|\nabla_v f\|^2 + \|\langle v \rangle f\|^2, \quad \langle v \rangle = \sqrt{1 + |v|^2}. \tag{1.12}
\]

From [20], the linearized operator $L$ is non-positive and locally coercive in the sense that there is a constant $0 < \mu < 1$ such that
\[
(Lf, f) \leq -\|P_1 f\|^2, \quad (Lf, f) \leq -\mu \|P_1 f\|_0^2. \tag{1.13}
\]

Since we only consider the pointwise behavior with respect to the space-time variable $(t, x)$, it’s convenient to regard the Green’s function $G(t, x)$ as an operator on $L^2(\mathbb{R}^3)$ defined by
\[
\left\lfloor \begin{array}{l}
\partial_t G = B G, \quad t > 0, \\
G(0, x) = \delta(x)I_v
\end{array} \right., \tag{1.14}
\]
where $I_v$ is the identity in $L^2(\mathbb{R}^3)$ and the operator $B$ is given by
\[
Bf = Lf - v \cdot \nabla_x f + v \cdot \nabla_x (\Delta_x)^{-1} P_0 f. \tag{1.15}
\]

Then, the solution for the initial value problem of the linearized VPFP equation
\[
\left\lfloor \begin{array}{l}
\partial_t f = Bf, \\
f(0, x, v) = f_0(x, v)
\end{array} \right., \tag{1.16}
\]
can be represented by
\[
f(t, x) = G(t) * f_0 = \int_{\mathbb{R}^3} G(t, x - y)f_0(y)dy, \tag{1.17}
\]
where $f_0(y) = f_0(y, v)$.

For any $(t, x)$ and $f \in L^2(\mathbb{R}^3)$, we define the $L^2$ norm of $G(t, x)$ by
\[
\|G(t, x)\| = \sup_{\|f\|_{L^2} = 1} \|G(t, x)f\|_{L^2}, \tag{1.18}
\]
and define the norms of operators $T_1: L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ and $T_2: L^2(\mathbb{R}^3) \rightarrow C$ by
\[
\|T_1\| = \sup_{\|f\|_{L^2} = 1} \|T_1f\|_{L^2}, \quad |T_2| = \sup_{\|f\|_{L^2} = 1} |T_2f| \tag{1.19}.
\]

**Notations:** Before state the main results in this paper, we list some notations. For any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$, denote
\[
\partial^\alpha_x = \partial^\alpha_{x_1} \partial^\alpha_{x_2} \partial^\alpha_{x_3}, \quad \partial^\beta_v = \partial^\beta_{v_1} \partial^\beta_{v_2} \partial^\beta_{v_3}.
\]
The Fourier transform of $f = f(x, v)$ is denoted by
\[
\hat{f}(\xi, v) = \mathcal{F}f(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x, v) dx,
\]
where and throughout this paper we denote $i = \sqrt{-1}$.

For $\gamma \geq 0$, define
\[
\|f\|_{L^\infty_{x,v}} = \sup_{v \in \mathbb{R}^3} |f(x, v)|(1 + |v|)^\gamma.
\]
Define a pseudo-differential operator $\chi_R(D)$:
\[
\chi_R(D)f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \chi_R(\xi) \hat{f}(\xi) d\xi,
\]
where $\chi_R(\xi)$ is a smooth cut-off function satisfying
\[
\chi_R(\xi) = 0, \ |\xi| \leq R; \quad \chi_R(\xi) = 1, \ |\xi| \geq 2R.
\]

First, we have the pointwise space-time behaviors of the Green’s function for the linearized VPFP system.

**Theorem 1.1.** Let $G(t, x)$ be the Green’s function for the VPFP equation defined by (1.14). Then, there exists a small constant $R > 0$ such that the Green’s function $G(t, x)$ can be decomposed into
\[
G(t, x) = G_L(t, x) + G_H(t, x),
\]
where $G_L(t, x) = [I - \chi_R(D)]G(t, x)$ is the low frequency part and $G_H(t, x) = \chi_R(D)G(t, x)$ is the high frequency part with $\chi_R(D)$ defined by (1.20). Moreover, the following estimates hold for $G_L(t, x)$ and $G_H(t, x)$:

1. For any multi-index $\alpha \in \mathbb{N}^3$, the low frequency part $G_L(t, x)$ satisfies
   \[
   \|\partial^\alpha_x P_0 G_L(t, x)\| \leq Ce^{-\frac{4}{3}t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}},
   \]
   \[
   \|\partial^\alpha_x P_m G_L(t, x)\| \leq Ce^{-\frac{4}{3}t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}},
   \]
   \[
   \|\partial^\alpha_x P_3 G_L(t, x)\| \leq Ce^{-\frac{4}{3}t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}},
   \]
   and in particular
   \[
   \|\partial^\alpha_x P_0 G_L(t, x)P_1\| \leq Ce^{-\frac{4}{3}t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}},
   \]
   \[
   \|\partial^\alpha_x P_m G_L(t, x)P_1\| \leq Ce^{-\frac{4}{3}t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}},
   \]
   \[
   \|\partial^\alpha_x P_3 G_L(t, x)P_1\| \leq Ce^{-\frac{4}{3}t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}},
   \]
   where $C > 0$ is a constant dependent of $\alpha$, and the operators $P_0, P_1, P_m$ and $P_3$ are defined by (1.10) and (1.11).

2. For any integer $n > 0$ and multi-index $\alpha \in \mathbb{N}^3$, there exist constants $0 < \eta_0 < 1/4$ such that the high frequency part $G_H(t, x)$ satisfies
   \[
   \|\partial^\alpha_x (G_H(t, x) - W_\alpha(t, x))\| \leq Ce^{-\eta_0 t}(1 + |x|^2)^{-n},
   \]
   where $C > 0$ is a constant dependent of $\eta_0$ and $\alpha$, and $W_\alpha(t, x)$ is the high frequency singular kinetic wave constructed by
   \[
   W_\alpha(t, x) = \sum_{k=0}^{\gamma+|\alpha|/2} \chi_R(D)J_k(t, x),
   \]
   where
   \[
   J_0(t, x) = e^{tA}\delta(x) I_v,
   \]
   \[
   J_k(t, x) = \int_0^t e^{(t-s)A}(2 + \nabla_x \Delta_x^{-1} P_0)J_{k-1} ds, \quad k \geq 1.
   \]
   Here $A = L - 2 - (v \cdot \nabla_x)$, $I_v$ is an identity operator in $L_v^2$ and the operator $e^{tA}$ is defined by
   \[
   e^{tA}h(x, v) = G_1(t) \ast h(x, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G_1(t, x, v; y, u)h(y, u)dydu,
   \]
   where $G_1(t, x, v; y, u)$ is the Green’s function of the operator $A$ defined by (3.65).
Then, we have the pointwise space-time behavior of the global solution to the nonlinear VPFP system \((1.4)-(1.6)\) as follows.

**Theorem 1.2.** There exists a small constant \(\delta_0 > 0\) such that if the initial data \(f_0\) satisfies
\[
\|f_0(x)\|_{L_{\infty}^\gamma} \leq C\delta_0(1 + |x|^2)^{-n}, \quad n \geq 2, \tag{1.29}
\]
then there exists a unique global solution \((f, \Phi)\) to the VPFP system \((1.4)-(1.6)\) satisfying
\[
\|P_0 f(t, x)\|_{L_2^\gamma} \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-2}, \tag{1.30}
\]
\[
\|P_m f(t, x)\|_{L_2^\gamma} + |\nabla_x \Phi(t, x)| \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-1}, \tag{1.31}
\]
\[
\|P_0 f(t, x)\|_{L_2^\gamma} \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-2}, \tag{1.32}
\]
\[
\|f(t, x)\|_{L_{\infty}^\gamma} + h(t)\|\nabla_v f(t, x)\|_{L_{\infty}^\gamma} \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-1}, \tag{1.33}
\]
where \(h(t) = \sqrt{t}/(1 + \sqrt{t})\). Furthermore, if the initial data satisfies \((f_0, \sqrt{M}) = 0\) and \((1.29)\) for \(n \geq 5/2\), then
\[
\|P_0 f(t, x)\|_{L_2^\gamma} \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-\frac{2}{5}}, \tag{1.34}
\]
\[
\|P_m f(t, x)\|_{L_2^\gamma} + |\nabla_x \Phi(t, x)| \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-\frac{2}{5}}, \tag{1.35}
\]
\[
\|P_0 f(t, x)\|_{L_2^\gamma} \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-\frac{2}{5}}, \tag{1.36}
\]
\[
\|f(t, x)\|_{L_{\infty}^\gamma} + h(t)\|\nabla_v f(t, x)\|_{L_{\infty}^\gamma} \leq C\delta_0 e^{-\rho t}(1 + |x|^2)^{-\frac{2}{5}}. \tag{1.37}
\]

**Remark 1.3.** The Green’s function \(G_1(t, x; y, u)\) becomes smooth for all \((t, x, v)\) when \(t > 0\) immediately. In particular,
\[
G_1(t, x; y, u) = e^{-2t} \frac{1}{2\pi} e^{-\frac{|x-y|^2}{4t}} \frac{1}{|u-y|^2}, \quad t \to 0;
\]
\[
G_1(t, x; y, u) = e^{-2t} \frac{1}{2\pi} e^{-k|x-y|^2} \frac{1}{|u-y|^2}, \quad t \to \infty.
\]
That is, \(e^{2t}G_1(t, x; y, u)\) behaves like the Kolmogorov function at short time and like the heat kernel at large time. Thus, the singular waves \(J_k(t)g_0\) with \(g_0 \in L^2(\mathbb{R}^3)\) are also smooth for all \((t, x, v)\) when \(t > 0\).

**Remark 1.4.** Note that we can represent \(G\) in terms of \(G_1\) as
\[
G(t) = G_1(t) + \int_0^t G_1(t-s) * (2G + v \cdot \nabla \Delta^{-1}P_0 G) ds.
\]
Thus, the Green’s function \(G(t, x)g_0\) with \(g_0 \in L^2(\mathbb{R}^3)\) is smooth for all \((t, x, v)\) when \(t > 0\). Moreover, by the above relation, Theorem [1.1] and Lemmas [4.2], we can obtain that for any \(g_0(x, v)\) satisfying
\[
\|g_0(x)\|_{L_{\infty}^\gamma} \leq C(1 + |x|^2)^{-n}, \quad n \geq 2, \gamma \geq 3,
\]
\(G\) and its velocity directrice \(\nabla_v G\) satisfy the following estimate
\[
\|(G(t) * g_0)(x)\|_{L_{\infty}^\gamma} \leq C e^{-\rho t(1 + |x|^2)}^{-1},
\]
\[
\|\nabla_v(G(t) * g_0)(x)\|_{L_{\infty}^\gamma} \leq C(1 + t^{-\frac{1}{2}}) e^{-\rho t(1 + |x|^2)}^{-1}.
\]

We now outline the main idea and make some comments on the proof of the above theorems. The results in Theorem [1.1] on the pointwise behavior of the Green’s function to the VPFP system is proved based on the spectral analysis [13] and the ideas inspired by [14] [15]. Indeed, we first decompose the Green function \(G\) into the lower frequency part \(G_L\) and the high frequency part \(G_H\). By the virtue of the spectrum analysis of the VPFP system, the lower frequency part \(G_L\) decays at \(e^{-Ct}\), and in particular decays at \(e^{-Ct/2}(1 + |x|^2)^{-k}\) in the region \(|x| \leq e^{Ct/(2k)}\). Furthermore, we apply the Picard’s iteration as the VPB system \([1.4]\) to estimate \(G_L\).
outside the region $|x| \leq e^{Ct/(2k)}$. To be more precisely, we apply the macro-micro decomposition to construct the approximate sequence $(\tilde{I}_k, \tilde{J}_k)$ of $(P_{2k}G_L, P_{3k}G_L)$ such that $\tilde{I}_k(t, \xi)$ and $\tilde{J}_k(t, \xi)$ are the solutions to the Navier-Stokes-Poisson system with damping and the microscopic VPFP system respectively, and $I_k(t, \xi)$ and $J_k(t, \xi)$ are smooth and compact supported functions in $\xi$ and satisfy for any $g_0 \in L^2(\mathbb{R}^3_\xi)$ (refer to Lemma 3.2)

$$\|\partial_\xi^2 \tilde{I}_k(t, \xi)g_0\|_\xi \leq Ce^{-Ct|\xi|^{2k-|\alpha|-1}}\|g_0\|, \quad \|\partial_\xi^2 \tilde{J}_k(t, \xi)g_0\| \leq Ce^{-Ct|\xi|^{2k-|\alpha|}}\|g_0\|.$$  

Moreover, we establish the energy estimates on the remaining part $\tilde{V}_k(t, \xi)$ and their frequency derivatives to show that for any $g_0 \in L^2(\mathbb{R}^3_\xi)$ and any $\delta \in (0, \delta_0)$ (refer to Lemma 3.3)

$$\|\partial_\xi^2 \tilde{V}_k(t, \xi)g_0\|_\xi \leq C\delta^{-2|\alpha|}e^{2\delta t}|\xi|^{2k-|\alpha|}\|g_0\|.$$  

This implies that the remaining part $V_k(t, x)$ decays at $e^{2\delta t}(1 + |\delta x|^2)^{-k}$. Combining the above decompositions and estimates together, we can obtain the pointwise space-time behaviors of the low frequency part $G_L(t, x)$ as listed in (1.22) and (1.23).

What left is to deal with the high frequency part $G_H(t, x)$. Since the Fourier transform of $G_H$ is not $L^1$ integrable in frequency space, $G_H$ can be decomposed into the singular part and the remaining smooth part. We apply the refined Picard’s iteration as Fokker-Planck equation [15] to construct the singular kinetic waves $J_k$ of $G_H$ where $J_k(t, x)$ are the solutions to the Fokker-Planck equations with damping given by (1.20)–(1.22). In particular, $J_k(t, x)g_0$ with $g_0 \in L^2(\mathbb{R}^3_\xi)$ are smooth for all $(t, x, v)$ when $t > 0$, and $\tilde{J}_k(t, \xi)$ satisfy (refer to Lemma 3.4)

$$\|\partial_\xi^2 \tilde{J}_k(t, \xi)g_0\|_\xi \leq C(1 + t^{-3/2})e^{-t(1 + |\xi|)^{-3}}\|g_0\|, \quad \forall j \geq 0.$$  

By the energy estimate in frequency space, we can show that the remaining part $\tilde{R}_k(t, \xi)$ is smooth and satisfies for any $l \geq 0$ and $\delta \in (0, \delta_0)$ (refer to Lemma 3.7)

$$\|\partial_\xi^l \tilde{R}_k(t, \xi)g_0\|_\xi \leq C\delta^{-1-2|\alpha|}e^{2\delta t}(1 + |\xi|)^{-4-l}\|g_0\|,$$  

where $k \geq 6 + 3l/2$. This implies that the remaining part $R_k(t, x)$ decays at $e^{-n_0t}(1 + |x|^2)^{-n}$ for any $n \geq 1$ (refer to Lemma 3.8). Combining the above decompositions and estimates together, we can obtain the pointwise estimates of the high frequency part $G_H(t, x)$ as listed in (1.24) and (1.25).

Finally, making use of the estimates of the Green’s function, one can establish the pointwise space-time behaviors of global solution to the nonlinear VPFP system as in Theorem 1.2. Compare to the results of the VPB system in [13], we can obtain the pointwise estimates of the solution when the initial data is not smooth. The main difficulty is that the nonlinear term $H(f)$ contain the high order derivative term $\nabla_v f$. To overcome this difficulty, we introduce the smooth function $G_1(t, x, v; y, u)$ to represent the solution $f$ as

$$f(t) = G_1(t) * f_0 + \int_0^t G_1(t-s) * (2f + v \cdot \nabla_x \Phi \sqrt{M} + H(f))ds.$$  

By using the explicit expression of $G_1(t, x, v; y, u)$ in (3.63), we can obtain the key estimates of $G_1(t) * g_0$ and $\nabla_v(G_1(t) * g_0)$ as in Lemmas 4.1 and 4.2 for any non smooth function $g_0(x, v)$. Combining the above relation, Lemmas 4.1, 4.2 and Theorem 1.1, we can obtain the pointwise space-time estimates of the global solution as in Theorem 1.2 after a straightforward computation.

The rest parts of this paper are organized as follows. In Section 2 we present the results about the spectrum analysis of the linear operator related to the linearized VPFP system. In Section 3 we establish the pointwise space-time estimates of the Green’s functions to the linearized VPFP system. In Section 4 we prove the pointwise space-time estimates of the global solutions to the original nonlinear VPFP system by making use of the estimates of the Green’s function.
2 Spectral analysis

In this section, we review the spectrum structure of the linearized VPFP operator $B(\xi)$ defined by (2.2) in order to study the pointwise estimate of the Green’s function.

Taking the Fourier transform in (1.14) with respect to $x$, we obtain

$$\begin{aligned}
\frac{\partial_t \hat{G}}{\hat{G}} &= B(\xi) \hat{G}, \quad t > 0, \\
\hat{G}(0, \xi) &= 1(\xi)I_v,
\end{aligned}$$

where the operator $B(\xi)$ is defined for $\xi \neq 0$ by

$$B(\xi) = L - i(v \cdot \xi) - i\frac{v \cdot \xi}{|\xi|^2}P_0.$$ 

(2.2)

Introduce the weighted Hilbert space $L^2_\xi(\mathbb{R}^3_v)$ for $\xi \neq 0$ as

$$L^2_\xi(\mathbb{R}^3_v) = \{ f \in L^2(\mathbb{R}^3_v) | \| f \|_\xi = \sqrt{(f, f)_\xi} < \infty \},$$

equipped with the inner product

$$(f, g)_\xi = (f, g) + \frac{1}{|\xi|^2}(P_0 f, P_0 g).$$

We have the following results about the spectrum structure and semigroup of the operator $B(\xi)$.

**Lemma 2.1** ([13]). The operator $B(\xi)$ generates a strongly continuous contraction semigroup on $L^2_\xi(\mathbb{R}^3_v)$, which satisfies

$$\| e^{tB(\xi)} f \|_\xi \leq \| f \|_\xi, \quad \forall \ t > 0, \ f \in L^2_\xi(\mathbb{R}^3_v).$$

(2.3)

**Lemma 2.2** ([13]). Let $\sigma(B(\xi))$ denote the spectrum set of the operator $B(\xi)$. We have

1. There exists a constant $r_0 > 0$ such that for $|\xi| \leq r_0$,

$$\sigma(B(\xi)) \subset \{ \lambda \in \mathbb{C} | \text{Re} \lambda < -1/2 \},$$

(2.4)

and there exists a constant $\beta_0 = \beta_0(r_0) > 0$ such that for $|\xi| > r_0$,

$$\sigma(B(\xi)) \subset \{ \lambda \in \mathbb{C} | \text{Re} \lambda < -\beta_0 \}. $$

(2.5)

2. The semigroup $e^{tB(\xi)}$ satisfies

$$\| e^{tB(\xi)} f \|_\xi \leq C e^{-\beta_1 t} \| f \|_\xi, \quad \forall f \in L^2_\xi(\mathbb{R}^3_v),$$

(2.6)

where $\beta_1 = \min\{\beta_0, 1/2\}$ is a constant.

3 Green’s function

In this section, we establish the pointwise space-time estimates of the Green’s function defined by (1.14). First, based on the spectral analysis given in section 2, we divide the Green’s function into the low frequency part and the high frequency part and show that the low frequency part decaying exponential in time. Then, we introduce the Picard’s iteration as VPB system [14] to construct the approximate sequences for the low frequency part, and estimate the remaining part by the energy method in frequency space. Finally, we refine the Picard’s iteration in Fokker-Planck equation [15] to construct the approximate sequences (singular kinetic waves) for the high frequency part which are smooth for $(t, x, v)$ when $t > 0$, and establish the pointwise behavior of the remaining part by the energy method in frequency space.
3.1 Low frequency part

In this subsection, we establish the pointwise estimate of the low frequency part of the Green’s function based on the spectral analysis given in section 2. To begin with, we decompose the operator \( G(t, x) \) into low-frequency part and high-frequency part:

\[
\begin{cases}
  G(t, x) = G_L(t, x) + G_H(t, x), \\
  G_L(t, x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi + tB(\xi)} \chi_1(\xi) d\xi, \\
  G_H(t, x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi + tB(\xi)} \chi_2(\xi) d\xi,
\end{cases}
\]

(3.1)

where

\[
\chi_1(\xi) = 1 - \chi_R(\xi), \quad \chi_2(\xi) = \chi_R(\xi),
\]

and \( R = r_0/2 \) with \( r_0 > 0 \) given by Lemma 2.2.

From [13], we have the following estimates of each part of \( \hat{G}(t, \xi) \) defined by (2.1).

**Lemma 3.1.** For any \( g_0 \in L^2_\xi(\mathbb{R}^3_\xi) \), there exists a positive constant \( C \) such that

\[
\|\hat{G}(t, \xi)g_0\|_\xi, \|\hat{G}_H(t, \xi)g_0\|_\xi \leq Ce^{-\beta_1 t}\|g_0\|_\xi,
\]

(3.2)

\[
\|\hat{G}_L(t, \xi)g_0\|_\xi \leq Ce^{-\frac{\beta_1}{2}t}\|g_0\|_\xi,
\]

(3.3)

where \( \hat{G}_L(t, \xi) \) and \( \hat{G}_H(t, \xi) \) are the Fourier transforms of \( G_L(t, x) \) and \( G_H(t, x) \) defined by (3.1), and \( \beta_1 > 0 \) is given by Lemma 2.2.

In the following, we introduce the Picard’s iteration as [14] and make use of the energy estimates to establish the pointwise space-time behaviors of the low frequency part \( G_L(t, x) \). To this end, we apply the macro-micro decomposition to construct the approximate sequences to \( \hat{G}_L(t, \xi) \) with increasing regularity at \( \xi = 0 \), and deal with the remaining term by the weighted energy estimate. Note that \( \hat{G}_L(t, \xi) \) satisfies

\[
\partial_t \hat{G}_L + i(v \cdot \xi)\hat{G}_L - L\hat{G}_L + i(v \cdot \xi)|\xi|^{-2}P_0\hat{G}_L = 0,
\]

(3.4)

\[
G_L(0, \xi) = \chi_1(\xi)|v|.
\]

Since

\[
L\sqrt{M} = 0, \quad L(v_j\sqrt{M}) = -v_j\sqrt{M}, \quad j = 1, 2, 3,
\]

it follows that \( N_1 = \text{span}\{\sqrt{M}, v_j\sqrt{M}, j = 1, 2, 3\} \) is an invariant subspace of \( L \) and hence

\[
P_2LP_3 = P_3LP_2 = 0, \quad P_2LP_2 = LP_2, \quad P_3LP_3 = LP_3,
\]

where \( P_2, P_3 \) are defined by (1.11).

We apply the projections \( P_j, j = 2, 3 \) to (3.4) to get

\[
\partial_t P_2\hat{G}_L + iP_2(v \cdot \xi P_2\hat{G}_L) - LP_2\hat{G}_L + i(v \cdot \xi)|\xi|^{-2}P_0\hat{G}_L = -iP_2(v \cdot \xi P_3\hat{G}_L),
\]

\[
\partial_t P_3\hat{G}_L + iP_3(v \cdot \xi P_3\hat{G}_L) - LP_3\hat{G}_L = -iP_3(v \cdot \xi P_2\hat{G}_L).
\]

Based on the above decomposition, we can construct an approximate solution \( (\hat{I}_k, \hat{J}_k) \in (N_1, N_1^\perp) \) for \( (P_2\hat{G}_L, P_3\hat{G}_L) \) as follows

\[
\partial_t \hat{I}_0 + iP_2(v \cdot \xi \hat{I}_0) - L\hat{I}_0 - i(v \cdot \xi)|\xi|^{-2}P_0\hat{I}_0 = 0,
\]

(3.5)

\[
\partial_t \hat{J}_0 + iP_3(v \cdot \xi \hat{J}_0) - L\hat{J}_0 = -iP_3(v \cdot \xi)\hat{I}_0,
\]

(3.6)

\[
\hat{I}_0(0, \xi) = \chi_1(\xi)P_2, \quad \hat{J}_0(0, \xi) = \chi_1(\xi)P_3.
\]

(3.7)
and
\[
\begin{align*}
\partial_t \hat{I}_k + i P_2(v \cdot \xi \hat{I}_k) - L \hat{I}_k &= -i v \cdot \xi|\xi|^{-2} P_0 \hat{I}_k = -i P_2(v \cdot \xi) \hat{J}_{k-1}, \\
\partial_t \hat{J}_k + i P_3(v \cdot \xi \hat{J}_k) - L \hat{J}_k &= -i P_3(v \cdot \xi) \hat{I}_k, \\
\hat{I}_k(0, \xi) &= 0, \quad \hat{J}_k(0, \xi) = 0,
\end{align*}
\]
(3.8)

(3.9)

(3.10)

for \( k \geq 1 \). Denote
\[
B_1(\xi) = -i P_2(v \cdot \xi) P_2 - i(v \cdot \xi)|\xi|^{-2} P_0 + LP_2,
\]
\[
B_2(\xi) = LP_3 - i P_3(v \cdot \xi) P_3.
\]
Since the operators \( B_1(\xi) \) and \( B_2(\xi) \) are dissipate on \( N_1 \) and \( N_1^\perp \), it follows that \( B_1(\xi) \) and \( B_2(\xi) \) generate contraction semigroups on \( N_1 \) and \( N_1^\perp \), respectively. By (3.3)–(3.10), we have
\[
\begin{align*}
\hat{I}_0(t, \xi) &= e^{tB_1(\xi)} \chi_1(\xi) P_2, \\
\hat{J}_0(t, \xi) &= e^{tB_2(\xi)} \chi_1(\xi) P_3 - \int_0^t e^{(t-s)B_2(\xi)} i P_3(v \cdot \xi) \hat{I}_0 ds,
\end{align*}
\]
(3.11)

(3.12)

and
\[
\begin{align*}
\hat{I}_k(t, \xi) &= -\int_0^t e^{(t-s)B_1(\xi)} i P_2(v \cdot \xi) \hat{J}_{k-1} ds, \\
\hat{J}_k(t, \xi) &= -\int_0^t e^{(t-s)B_2(\xi)} i P_3(v \cdot \xi) \hat{I}_k ds, \quad k \geq 1.
\end{align*}
\]
(3.13)

(3.14)

Define the approximate solution and the remaining part as
\[
\begin{align*}
U_k(t, x) &= \sum_{n=0}^k (I_n + J_n)(t, x), \quad Y_k(t, x) = \Delta_x^{-1}(U_k, \sqrt{M}), \\
V_k(t, x) &= G_k(t, x) - U_k(t, x), \quad Z_k(t, x) = \Delta_x^{-1}(V_k, \sqrt{M}).
\end{align*}
\]
(3.15)

(3.16)

Then, it follows that \( \hat{U}_k(t, \xi) \) and \( \hat{Y}_k(t, \xi) \) satisfy
\[
\begin{align*}
\partial_t \hat{U}_k + i v \cdot \xi \hat{U}_k - L \hat{U}_k &= -|\xi|^2 \hat{Y}_k = (\hat{U}_k, \sqrt{M}), \\
-|\xi|^2 \hat{Y}_k &= (\hat{U}_k, \sqrt{M}), \\
\hat{U}_k(0, \xi) &= \chi_1(\xi) I_v,
\end{align*}
\]
and \( \hat{V}_k(t, \xi) \) and \( \hat{Z}_k(t, \xi) \) satisfy
\[
\begin{align*}
\partial_t \hat{V}_k + i v \cdot \xi \hat{V}_k - L \hat{V}_k &= -|\xi|^2 \hat{Z}_k = (\hat{V}_k, \sqrt{M}), \\
-|\xi|^2 \hat{Z}_k &= (\hat{V}_k, \sqrt{M}), \\
\hat{V}_k(0, \xi) &= 0.
\end{align*}
\]
(3.17)

For any \( \xi \neq 0 \), we define a weighted norm of the operator \( T : L^2(\mathbb{R}_+^3) \to L^2(\mathbb{R}_+^3) \) as
\[
|T|_\xi = \sup_{\|f\|_\xi = 1} \|Tf\|_\xi.
\]
Then, we have the following estimates for the approximate sequence \( (I_k, J_k) \).

**Lemma 3.2.** For any \( k \geq 0 \) and \( \alpha \in \mathbb{N}^3 \), we have
\[
\|\partial_x^\alpha P_0 I_k(t, x)\| \leq Ce^{-\frac{t}{4}}(1 + |x|^2)^{-\frac{\alpha_1 + \alpha_2}{2}},
\]
(3.18)
\[
\| \partial_x^2 P_m I_k(t, x) \| \leq C e^{-\frac{k}{4} t} (1 + |x|^2)^{-\frac{5 + |\alpha|}{2}}, \\
\| \partial_x^2 J_k(t, x) \| \leq C e^{-\frac{k}{4} t} (1 + |x|^2)^{-\frac{5 + |\alpha|}{2}},
\]

and
\[
\| \partial_x^2 P_0 I_k(t, x) P_1 \| \leq C e^{-\frac{k}{4} t} (1 + |x|^2)^{-\frac{5 + |\alpha|}{2}}, \\
\| \partial_x^2 P_m I_k(t, x) P_1 \| \leq C e^{-\frac{k}{4} t} (1 + |x|^2)^{-\frac{5 + |\alpha|}{2}}, \\
\| \partial_x^2 J_k(t, x) P_1 \| \leq C e^{-\frac{k}{4} t} (1 + |x|^2)^{-\frac{5 + |\alpha|}{2}}.
\]

where \( C > 0 \) is a constant dependent on \( \alpha \). In particular, \( \hat{I}_k(t, \xi) \) and \( \hat{J}_k(t, \xi) \) are smooth and supported in \( \{ \xi \in \mathbb{R}^3 | |\xi| \leq 2R \} \) satisfying
\[
\| \partial^2 \hat{I}_k(t, \xi) \|_{\xi} \leq C \sum_{n=0}^{\infty} t^{k+n} |\xi|^{2k-|\alpha|-1} e^{-\frac{k}{4} t}, \\
\| \partial^2 \hat{J}_k(t, \xi) \| \leq C \sum_{n=0}^{\infty} t^{k+n} |\xi|^{2k-|\alpha|} e^{-\frac{k}{4} t},
\]

for any \( |\alpha| \geq 0 \).

**Proof.** Define \( H_1(t, x) \) and \( H_2(t, x) \) by their fourier transforms:
\[
\hat{H}_1(t, \xi) = e^{iB_1(\xi)} \chi_1(\xi) P_2, \quad \hat{H}_2(t, \xi) = e^{iB_2(\xi)} \chi_1(\xi) P_3.
\]

To estimate \( \hat{H}_1(t, \xi) \), we consider the eigenvalue problem of \( B_1(\xi) \) as follows
\[
\begin{pmatrix}
-\hat{I}_2(v \cdot \xi) P_2 - i \frac{v \cdot \xi}{|\xi|^2} P_0 + LP_2
\end{pmatrix} = \lambda \psi, \quad \xi \in \mathbb{R}^3.
\]

By taking inner product between \( 3.26 \) and \( \{ \sqrt{M}, v\sqrt{M} \} \), we obtain
\[
\lambda \hat{n} = -i (\hat{n} \cdot \xi), \\
\lambda \hat{m} = -i \left( \xi + \frac{\xi}{|\xi|^2} \right) \hat{n} - \hat{m},
\]

where \( \hat{n} = (\psi, \sqrt{M}) \) and \( \hat{m} = (\psi, v\sqrt{M}) \).

It is easy to verify that \( \lambda_j(|\xi|) \) and \( \psi_j(\xi) \), \( j = 0, 1, 2, 3 \) are the eigenvalues and eigenfunctions of \( B_1(\xi) \) for \( \xi \in \mathbb{R}^3 \) satisfying
\[
\begin{align*}
\lambda_0(|\xi|) &= -\frac{1}{2} + \frac{i}{2} \sqrt{4|\xi|^2 + 3}, \\
\lambda_1(|\xi|) &= -\frac{1}{2} + \frac{i}{2} \sqrt{4|\xi|^2 + 3}, \\
\lambda_j(|\xi|) &= -1, \quad j = 2, 3, \\
\psi_k(\xi) &= |\xi|a_k(|\xi|)\sqrt{M} + b_k(|\xi|)|\xi|^{-1} (v \cdot \xi)^{\sqrt{M}}, \quad k = 0, 1, \\
\psi_j(\xi) &= (v \cdot Y^j)\sqrt{M}, \quad j = 2, 3,
\end{align*}
\]

where \( a_k(s), b_k(s) \) are analytic functions of \( s \) satisfying \( a_k = -ib_k/\lambda_k \) and \( b_k^2 = \lambda_k^2/(\lambda_k^2 - s^2 - 1) \) for \( k = 0, 1 \), and \( Y^j \) are orthonormal vectors satisfying \( Y^j \cdot \xi = 0, j = 2, 3 \).

By \( 3.26 \) and \( 3.27 \), we obtain
\[
\hat{H}_1(t, \xi) = \sum_{j=0}^{3} e^{-\lambda_j(|\xi|) t} \psi_j(\xi) \odot \left( \langle \psi_j(\xi) + \frac{1}{|\xi|^2} (P_0 \psi_j(\xi)) \rangle \right) \chi_1(\xi).
\]
Here for any $f, g \in L^2(\mathbb{R}^3)$, the operator $f \otimes \langle g \rangle$ on $L^2(\mathbb{R}^3)$ is defined by

$$f \otimes \langle g \rangle h = (h, \overline{g})f, \quad h \in L^2(\mathbb{R}^3).$$

It follows from (3.28) and (3.27) that

$$P_0 \hat{H}_1 = \sum_{j=0,1} e^{\lambda_j(|\xi|)} \chi_1(\xi) a_j(|\xi|) \sqrt{M} \otimes (|\xi|^2 + 1) a_j(|\xi|) \sqrt{M} + b_j(|\xi|)(v \cdot \xi) \sqrt{M},$$

$$P_m \hat{H}_1 = \sum_{j=0,1} e^{\lambda_j(|\xi|)} \chi_1(\xi) b_j(|\xi|) \frac{v \cdot \xi}{|\xi|^2} \sqrt{M} \otimes (|\xi|^2 + 1) a_j(|\xi|) \sqrt{M} + b_j(|\xi|)(v \cdot \xi) \sqrt{M}$$

$$+ \chi_1(\xi)e^{-t} \sum_{k=1}^3 v_k \sqrt{M} \otimes \left\langle v_k \sqrt{M} - \frac{(v \cdot \xi) \xi_k}{|\xi|^2} \right\rangle,$$

which lead to

$$\begin{align*}
\|\partial_\xi^2 P_0 H_1(t, x)\| &\leq Ce^{-\frac{4}{3}t(1 + |x|^2)} - k, \\
\|\partial_\xi^2 P_m H_1(t, x)\| &\leq Ce^{-\frac{4}{3}t(1 + |x|^2)} - \frac{m}{2}, \quad (3.29)
\end{align*}$$

where $k \geq 0$ is any integer, and $C > 0$ is a constant dependent of $k$ and $\alpha$.

By (1.12), the operator

$$B_2(\xi_0 + i\xi_0) = LP_3 - iP_3(v \cdot \xi)P_3 + P_3(v \cdot \xi_0)P_3, \quad (\xi_1, \xi_0) \in \mathbb{R}^3 \times \mathbb{R}^3$$

is dissipative for $\xi_1 \in \mathbb{R}^3$ and $|\xi_0| \leq 2\mu$, namely,

$$-Re(B_2(\xi_1 + i\xi_0) f, f) \geq (\mu - \frac{1}{2}|\xi_0|)\|f\|^2, \quad \forall f \in N^+_1.$$

It follows that $B_2(\xi)$ with $\xi = \xi_1 + i\xi_0 \in \mathbb{C}^3$ generates a contraction semigroup $e^{tB_2(\xi)}$ in $N^+_1$ for $\xi_1 \in \mathbb{R}^3$ and $|\xi_0| \leq 2\mu$, which satisfies

$$\|e^{tB_2(\xi)} f\| \leq e^{-(1 - \frac{|\xi_0|}{2\mu})t}\|f\|, \quad \forall t > 0, \quad f \in N^+_1,$$

where we have used

$$-(L f, f) \geq (1 - \frac{|\xi_0|}{2\mu})\|f\|^2 + \frac{|\xi_0|}{2}\|f\|^2, \quad \forall f \in N^+_1.$$

Moreover, the semigroup $e^{tB_2(\xi)}$ is analytic in $\{\xi \in \mathbb{C}^3 \mid |\text{Im}\xi| \leq 2\mu\}$. Rewrite

$$e^{tB_2(\xi)} \chi_1(\xi) = e^{tB_2(\xi)} \frac{1}{(1 + |\xi|^2)^2(1 + |\xi|^2)^2} \chi_1(\xi).$$

By Cauchy theorem, it holds for any $g_0 \in N^+_1$ and $|\xi_0| \leq |\mu$ that

$$\begin{align*}
\int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{tB_2(\xi)} g_0 \frac{1}{(1 + |\xi|^2)^2} d\xi \\
= \int_{\mathbb{R}^3} e^{ix \cdot (\xi + i\xi_0)} e^{-(1 - P_3(v \cdot \xi)P_3 + P_3(v \cdot \xi_0)P_3)t} g_0 \frac{1}{(1 + |\xi + i\xi_0|^2)^2} d\xi \\
\leq e^{-|\xi_0| \cdot |x|} e^{-\frac{4}{3}t\|g_0\|}.
\end{align*}$$

(3.32)

Since $(1 + |\xi|^2)^2 \chi_1(\xi)$ is a smooth and compact supported function, we can prove that for any $k > 0$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} e^{ix \cdot \xi} (1 + |\xi|^2)^2 \chi_1(\xi) d\xi \leq C(1 + |x|^2)^{-k}. \quad (3.33)$$
Thus, it follows from (3.31)–(3.33) that
\[
\|\partial_x^k H_2(t, x)\| \leq C \int_{\mathbb{R}^3} e^{-\frac{t}{4}} e^{-\frac{1}{2^k} |x|^2} (1 + |y|^2)^{-k} dy \\
\leq C e^{-\frac{t}{2^k}} (1 + |x|^2)^{-k}, \quad \forall k \in \mathbb{N}.
\] (3.34)

Combining (3.11)–(3.14), (3.29) and (3.34), we can prove (3.18)–(3.23).

Next, we prove (3.21) and (3.24) as follows. For any $g_0 \in L^2(\mathbb{R}^3)$, define
\[
\hat{I}_k(t, \xi) = \hat{I}_k(t, \xi)g_0, \quad \hat{J}_k(t, \xi) = \hat{J}_k(t, \xi)g_0.
\]
By (3.27), (3.28) and
\[
\text{Re}(B_2(\xi, f, f)) \leq -(f, f), \quad \forall f \in N^1_0, \quad \xi \in \mathbb{R}^3,
\]
we have
\[
\|e^{tB_1(\xi)f_0}\| \leq Ce^{-\frac{t}{2}}\|f_0\|, \quad \forall f_0 \in N_1, \quad (3.35)
\]
\[
\|e^{tB_2(\xi)f_1}\| \leq e^{-\frac{t}{2}}\|f_1\|, \quad \forall f_1 \in N^1_1. \quad (3.36)
\]

Thus, it follows from (3.35), (3.36) and (3.11)–(3.14) that
\[
\|\hat{I}_0\| \leq Ce^{-\frac{t}{2}}\|P_2g_0\| \leq C|\xi|^{-1}e^{-\frac{t}{2}}\|g_0\|,
\]
\[
\|\hat{J}_0\| \leq e^{-t}\|P_3g_0\| + \int_0^t e^{-\frac{(t-s)}{2}}|\xi||\hat{I}_0||ds \leq Ce^{-\frac{t}{2}}\|g_0\|,
\]
and
\[
\|\hat{I}_k\| \leq C\int_0^t e^{-\frac{(t-s)}{2}}|\xi||\hat{J}_{k-1}||ds \leq Ct^k|\xi|^{2k-1}e^{-\frac{t}{2}}\|g_0\|,
\]
\[
\|\hat{J}_k\| \leq C\int_0^t e^{-\frac{(t-s)}{2}}|\xi||\hat{I}_k||ds \leq Ct^k|\xi|^{2k}e^{-\frac{t}{2}}\|g_0\|, \quad k \geq 1.
\]

Taking the derivative $\partial_x^\alpha$ to (3.33) with $|\alpha| \geq 1$, we have
\[
\partial_x^\alpha \hat{I}_0 + B_1(\xi)\partial_x^\alpha \hat{I}_0 = G_{0,\alpha}, \quad (3.37)
\]
where
\[
G_{0,\alpha} = \sum_{\beta \leq \alpha, |\beta| \geq 1} C_{\alpha}^\beta \left( -iP_2\partial_\xi^\alpha (v \cdot \xi)\partial_\xi^{\alpha-\beta} \hat{I}_0 + i\partial_\xi^{\beta} \left( \frac{v \cdot \xi}{|\xi|^2} \right) P_0\partial_\xi^{\alpha-\beta} \hat{I}_0 \right).
\]

By (3.37) and (3.38), we have
\[
\|\partial_x^\alpha \hat{I}_0\| \leq Ce^{-\frac{t}{2}}\|\partial_x^\alpha \chi_1(\xi)\||P_2g_0||\xi
\]
\[
+ C\int_0^t e^{-\frac{t}{2}} \left( \frac{1}{|\xi|} \|\partial_\xi^{\alpha-1} \hat{I}_0\| + \sum_{n=1}^{|\alpha|} \frac{1}{|\xi|^{n+1}} \|\partial_\xi^{-n} (\hat{I}_0, \sqrt{M}) \|\right) ds
\]
\[
\leq C_{\alpha} \sum_{n=0}^{|\alpha|} t^n |\xi|^{-|\alpha|}e^{-\frac{t}{2}}\|g_0\|.
\]

Taking the derivative $\partial_x^\alpha$ to (3.36) with $|\alpha| \geq 1$, we have
\[
\partial_\xi \partial_x^\alpha \hat{J}_0 + iP_0\partial_\xi (v \cdot \xi)\hat{J}_0 - L\partial_\xi \hat{J}_0 = iP_0\partial_\xi (v \cdot \xi)\hat{I}_0.
\] (3.38)

Taking the inner product between (3.38) and $\partial_x^\alpha \hat{J}_0$ and using Cauchy inequality, we obtain
\[
\frac{d}{dt}\|\partial_x^\alpha \hat{J}_0\|^2 + \frac{3}{2} \|\partial_x^\alpha \hat{J}_0\|^2 + \frac{\mu}{4} \|\partial_x^\alpha \hat{I}_0\|^2.
\]
which leads to
\[
\|\partial_\xi^\alpha \hat{J}_0\|^2 \leq C e^{-\frac{2}{t}} \|\partial_\xi^\alpha \chi_1(\xi)\|^2 \|P_0 g_0\|^2 \\
+ C \int_0^t e^{-\frac{2}{s}(t-s)} \left( |\alpha|^2 \|\partial_\xi^{-1} \hat{J}_0\|^2 + |\xi|^2 \|\partial_\xi^{\alpha-1} \hat{I}_0\|^2 \right) ds \\
\leq C_\alpha \sum_{n=0}^{|\alpha|} t^{2n+2|\alpha|-1} e^{-\frac{2}{t}} \|g_0\|^2.
\]

We take the derivative \( \partial_\xi^\alpha \) to (3.39) with \( |\alpha| \geq 1 \) to get
\[
\partial_\xi^\alpha \hat{I}_k + B_1(\xi) \partial_\xi^\alpha \hat{I}_k = G_{k,\alpha},
\]
where
\[
G_{k,\alpha} = \sum_{\beta \leq \alpha, |\beta| \geq 1} C_{\alpha}^\beta \left( iP_2 \partial_\xi^\beta (v \cdot \xi) \partial_\xi^{\alpha-\beta} \hat{I}_k + i\partial_\xi^\beta \left( \frac{v \cdot \xi}{|\xi|^2} \right) P_0 \partial_\xi^{\alpha-\beta} \hat{I}_k \right) \\
+ \sum_{\beta \leq \alpha} C_{\alpha}^\beta iP_2 \partial_\xi^\beta (v \cdot \xi) \partial_\xi^{\alpha-\beta} \hat{J}_{k-1}.
\]

It follows from (3.39) and (3.35) that
\[
\|\partial_\xi^\alpha \hat{I}_k\|_{\xi} \leq C \int_0^t e^{-\frac{2}{s}(t-s)} \left( |\xi|^2 \|\partial_\xi^\alpha \hat{J}_{k-1}\| + \|\partial_\xi^{\alpha-1} \hat{I}_{k-1}\| \\
+ \frac{1}{|\xi|^2} \|\partial_\xi^{\alpha-1} \hat{I}_k\| + \sum_{n=1}^{|\alpha|} \frac{1}{|\xi|^2} \|\partial_\xi^{\alpha-n} (\hat{I}_k, \sqrt{M})\| \right) ds \\
\leq C_\alpha \sum_{n=0}^{|\alpha|} t^{2n+2|\alpha|-1} e^{-\frac{2}{t}} \|g_0\|.
\]

By taking the derivative \( \partial_\xi^\alpha \) to (3.39) with \( |\alpha| \geq 1 \), we have
\[
\partial_\xi^\alpha \hat{J}_k + iP_2 \partial_\xi^\alpha (v \cdot \xi) \hat{J}_k - L \partial_\xi^\alpha \hat{J}_k = i\partial_\xi^\alpha P_0 (v \cdot \xi \hat{I}_k).
\]

Taking the inner product between (3.40) and \( \partial_\xi^\alpha \hat{J}_k \) and using Cauchy inequality, we obtain
\[
\frac{d}{dt} \|\partial_\xi^\alpha \hat{J}_k\|^2 + \frac{3}{2} \|\partial_\xi^{\alpha-1} \hat{I}_k\|^2 + \frac{1}{4} \|\partial_\xi^\alpha \hat{J}_k\|^2 \\
\leq C |\xi|^2 \|\partial_\xi^\alpha \hat{I}_k\|^2 + C |\alpha|^2 \|\partial_\xi^{\alpha-1} \hat{I}_k\|^2 + C |\alpha|^2 \|\partial_\xi^{\alpha-1} \hat{J}_k\|^2,
\]
which leads to
\[
\|\partial_\xi^\alpha \hat{J}_k\|^2 \leq C \int_0^t e^{-\frac{2}{s}(t-s)} \left( |\alpha|^2 \|\partial_\xi^{-1} \hat{J}_k\|^2 + |\xi|^2 \|\partial_\xi^{\alpha} \hat{I}_k\|^2 + |\alpha|^2 \|\partial_\xi^{\alpha-1} \hat{I}_k\|^2 \right) ds \\
\leq C_\alpha \sum_{n=0}^{|\alpha|} t^{2(k+n)+2|\alpha|-1} e^{-\frac{2}{t}} \|g_0\|^2.
\]

The proof is completed.

With the help of Lemma 3.2, we have the following the pointwise behaviors for the remaining terms \( V_k(t, x) \) and \( \nabla_x Z_k(t, x) \) defined by (3.11).
Lemma 3.3. For any $k \geq 1$ and $\alpha \in \mathbb{N}^3$, there exists a small constant $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$,

$$\|\partial_\xi^\alpha \hat{V}_k(t, x)\| + |\partial_\xi^\alpha \nabla_x Z_k(t, x)| \leq C e^{2\delta t} (1 + |\delta x|^2)^{-(k + 3\alpha)|\alpha|/2},$$  

(3.41)

where $C > 0$ is a constant dependent of $k$ and $\alpha$.

Proof. For any $g_0 \in L^2(\mathbb{R}^3_x)$, define

$$\hat{V}_k(t, \xi) = \hat{V}_k(t, \xi)g_0, \quad \hat{Z}_k(t, \xi) = \hat{Z}_k(t, \xi)g_0.$$

We claim that for any $k \geq 1$ and $\alpha \in \mathbb{N}^3$,

$$\|\partial_\xi^\alpha \hat{V}_k(t, \xi)\| \leq C \delta^{-2|\alpha|} e^{2\delta t} |\xi|^{2k-|\alpha|} \|g_0\|.$$

(3.42)

We prove (3.42) by induction. Taking the inner product $(\cdot, \cdot)_\xi$ between $\hat{V}_k$ and (3.17) and choosing the real part, we have

$$\frac{1}{2} \frac{d}{dt} \|\hat{V}_k\|^2 + \frac{\mu}{2} \|P_1 \hat{V}_k\|^2 \leq C |\xi|^2 \|\hat{J}_k\|^2,$$

(3.43)

which leads to

$$\|\hat{V}_k\|^2 \leq C |\xi|^{4k+2} \|g_0\|^2 \int_0^t s^{2k} e^{-\delta s} ds \leq C |\xi|^{4k+2} \|g_0\|^2.$$

(3.44)

Suppose that (3.42) holds for $|\alpha| \leq j - 1$. Taking the derivative $\partial_\xi^\alpha$ to (3.17) with $|\alpha| = j$, we have

$$\partial_\xi^\alpha \hat{V}_k + iv \cdot \xi \partial_\xi^\alpha \hat{V}_k + L \partial_\xi^\alpha \hat{V}_k + i \frac{v}{|\xi|^2} P_0 \partial_\xi^\alpha \hat{V}_k = G_{k, \alpha},$$

(3.45)

where

$$G_{k, \alpha} = \sum_{\beta \leq \alpha, |\beta| \geq 1} C_\alpha^\beta \left( i \partial_\xi^\beta (v \cdot \xi) \partial_\xi^{\alpha-\beta} \hat{V}_k + i \partial_\xi^\beta \left( \frac{v}{|\xi|^2} \right) P_0 \partial_\xi^{\alpha-\beta} \hat{V}_k \right) + i P_2 \partial_\xi^\alpha (v \cdot \xi \hat{J}_k).$$

Taking the inner product $(\cdot, \cdot)_\xi$ between (3.45) and $\partial_\xi^\alpha \hat{V}_k$ and using Cauchy inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\partial_\xi^\alpha \hat{V}_k\|^2 + \frac{\mu}{2} \|P_1 \partial_\xi^\alpha \hat{V}_k\|^2 \leq \frac{C}{\delta} \left( |\xi|^2 \|\partial_\xi^\alpha \hat{J}_k\|^2 + \|\partial_\xi^{\alpha-1} \hat{J}_k\|^2 \right) + \frac{C}{\delta} |\alpha|^2 \|P_1 \partial_\xi^{\alpha-1} \hat{V}_k\|^2$$

$$+ \frac{C}{\delta} \left( \frac{1}{|\xi|^2} \|\partial_\xi^{\alpha-1} \tilde{n}_k\|^2 + \sum_{l=1}^{\alpha} \frac{1}{|\xi|^{2l+2}} \|\partial_\xi^{\alpha-l} \tilde{n}_k\|^2 \right) + \delta \frac{1}{|\xi|^2} \|\partial_\xi^\alpha \tilde{n}_k\|^2,$$

(3.46)

where $\delta \in (0, \delta_0)$ with $\delta_0 > 0$ sufficiently small, and

$$\tilde{n}_k = (\hat{V}_k, \sqrt{M}), \quad \tilde{n}_k = (\hat{V}_k, v \sqrt{M}).$$

Apply Gronwall’s inequality to (3.46) and using (3.42), we have

$$\|\partial_\xi^\alpha \hat{V}_k\| \leq \frac{C}{\delta} |\xi|^{4k+2-2|\alpha|} \|g_0\|^2 \|\tilde{n}_k\|^2 \int_0^t e^{2\delta (t-s)} e^{4\delta s} ds$$

$$+ \frac{C}{\delta} |\xi|^{4k+2-2|\alpha|} \|g_0\|^2 \|\tilde{n}_k\|^2 \int_0^t e^{2\delta (t-s)} s^{2k+2n} e^{-\delta s} ds$$

$$\leq C \delta^{-2|\alpha|} |\xi|^{4k+2-2|\alpha|} e^{4\delta t} \|g_0\|^2.$$  

(3.47)
This proves (3.32) for $|\alpha| = j$.

Therefore, it holds for any $\gamma \in \mathbb{N}^3$,

$$\|\partial_\xi^\alpha (\xi^\gamma \hat{V}_k)\| + |\partial_\xi^\alpha (\xi^\gamma \hat{Z}_k)|$$

$$\leq \sum_{\beta \leq \alpha} C_\alpha^\beta \left( \|\partial_\xi^\beta (\xi^\gamma) \partial_\xi^{\alpha-\beta} \hat{V}_k\| + |\partial_\xi^\beta (\xi^\gamma) \hat{n}_k| \right)$$

$$\leq C \sum_{|\beta| = 0} |\xi|^{\gamma - |\beta|} \|\partial_\xi^{\alpha-\beta} \hat{V}_k\|_\xi \leq C \delta^{-|\alpha|} |\xi|^{2k - |\alpha| + |\gamma| + 1} e^{2M} \|g_0\|,$$

which leads to

$$\left( \|\partial_x^\alpha V_k(t,x)\| + |\partial_x^\alpha \nabla_x Z_k(t,x)| \right) x^{2\alpha}$$

$$\leq C \int_{|\xi| \leq 2R} \left( \|\partial_\xi^{2\alpha} (\xi^\gamma \hat{V}_k)\| + |\partial_\xi^{2\alpha} (\xi^\gamma \hat{Z}_k)| \right) d\xi$$

$$\leq C e^{2M \delta^{-2|\alpha|} \|g_0\|} \int_{|\xi| \leq 2R} |\xi|^{2k - 2|\alpha| + |\gamma| + 1} d\xi$$

$$\leq C e^{2M \delta^{-2|\alpha|} \|g_0\|},$$

for any $|\alpha| \leq k + |\gamma|/2 + 3/2$. This proves the lemma.

With the help of Lemmas 3.2 and 3.3, we can obtain the following pointwise space-time estimates of $G_L(t,x)$ defined by (3.11).

**Theorem 3.4.** For any $\alpha \in \mathbb{N}^3$, the low frequency part $G_L(t,x)$ satisfies

$$\|\partial_\xi^\alpha P_0 G_L(t,x)\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}}} \quad (3.48)$$

$$\|\partial_\xi^\alpha P_m G_L(t,x)\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{2+|\alpha|}{2}}} \quad (3.49)$$

$$\|\partial_\xi^\alpha P_3 G_L(t,x)\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}}} \quad (3.50)$$

where $C > 0$ is a constant dependent of $\alpha$. Moreover,

$$\|\partial_\xi^\alpha P_0 G_L(t,x) P_1\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}}} \quad (3.51)$$

$$\|\partial_\xi^\alpha P_m G_L(t,x) P_1\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}}} \quad (3.52)$$

$$\|\partial_\xi^\alpha P_3 G_L(t,x) P_1\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{4+|\alpha|}{2}}} \quad (3.53)$$

**Proof.** By (3.10), we can decompose $G_L(t,x)$ into

$$G_L(t,x) = U_k(t,x) + V_k(t,x).$$

By Lemma 3.2, we have

$$\left\{ \begin{array}{l} \|\partial_\xi^\alpha P_0 U_k(t,x)\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}}} \quad (3.55) \\
\|\partial_\xi^\alpha P_m U_k(t,x)\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}}} \quad (3.55) \\
\|\partial_\xi^\alpha P_3 U_k(t,x)\| \leq C e^{-\frac{4\pi}{t}(1 + |x|^2)^{-\frac{3+|\alpha|}{2}}} \end{array} \right.$$
By (3.54), Lemmas 3.1 and 3.2, we obtain
\[ \| \hat{V}_k(t, \xi) \|_\xi \leq \| (\hat{G}_L - \hat{U}_k)(t, \xi) \| \leq C|\xi|^{-1}e^{-\beta_2 t}, \quad |\xi| \leq 2R, \]
where \( \beta_2 \in (0, 1/2) \) is a constant, which leads to
\[ \| \partial_x^n V_k(t, x) \| + |\partial_x^n \nabla_x Z_k(t, x)| \leq Ce^{-\beta_2 t}. \]
This together with (3.41) imply that
\[ \| \partial_x^n V_k(t, x) \| + |\partial_x^n \nabla_x Z_k(t, x)| \leq Ce^{-\frac{\beta_2}{2} t + \frac{\beta_2}{2} t} (1 + |x|^2)^{-\frac{3+\alpha}{2}}, \]
for \( k \geq 6 + |\alpha| \). Combining (3.54)–(3.57), we prove the theorem. \( \square \)

### 3.2 High frequency part

In this subsection, we extract the singular part for the high frequency part \( G_H \) and establish the pointwise estimate of the remaining part. Since \( G_H \) does not belong to \( L^1(\mathbb{R}^3_+) \), \( G_H \) can be decomposed into the singular part and the remaining smooth part. Indeed, we apply a refined Picard’s iteration as [15] to construct the approximate sequences of \( \hat{G}_H(t, \xi) \) and estimate the smooth remaining term by the energy estimate. Note that \( \hat{G}_H(t, \xi) \) satisfies
\[ \partial_t \hat{G}_H + i(v \cdot \xi)\hat{G}_H - L\hat{G}_H + i(v \cdot \xi)|\xi|^{-2} P_0 \hat{G}_H = 0, \]
\[ \hat{G}_H(0, \xi) = \chi_2(\xi) I_v. \]

Set
\[ A(\xi) = L - 2 - i(v \cdot \xi). \]  

**Lemma 3.5.** For any \( k \geq 0 \) and \( g_0 \in L^2(\mathbb{R}^3_+) \), we have
\[ \| \nabla_x e^{tA(\xi)} g_0 \|_2^2 \leq C(1 + t^{-1})e^{-4t}\| g_0 \|_2^2, \] \[ \| \xi|e^{tA(\xi)} g_0 \|_2^2 \leq C(1 + t^{-3k})e^{-4t}\| g_0 \|_2^2, \]
for \( C > 0 \) a positive constant.

**Proof.** Let \( h(t, x, v) \) be the solution of the following linear Fokker-Planck equation:
\[ \partial_t h + v \cdot \nabla_x h = \nabla_v \cdot (\nabla_v h + vh), \] \[ h(0, x, v) = h_0(x, v). \]

Then \( h \) can be represented by
\[ h(t, x, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G_0(t, x, v; y, u) h_0(y, u) dy du, \]
where the Green’s function \( G_0 \) is given by [25]
\[ G_0(t, x, v; y, u) = \frac{1}{2\pi^6} \left( \frac{\pi}{\sqrt{D(t)}} \right)^3 \exp \left\{ -\frac{1}{4D(t)} \left[ \frac{1}{2} (1 - e^{-2t})|\hat{x}|^2 \right. \right. \]
\[ - (2(1 - e^{-t}) - (1 - e^{-2t})) \hat{x} \cdot \hat{v} \]
Thus it is easy to verify that
\begin{align*}
D(t) &= \frac{1}{2} \left( t(1 - e^{-2t}) - 2(1 - e^{-t})^2 \right), \\
\dot{x} &= x - (y + u(1 - e^{-t})), \quad \dot{v} = v - ue^{-t}.
\end{align*}

By a direct computation, we obtain
\begin{equation}
G_0(t, x, v; y, u) = \frac{1}{(2\pi)^6} \left( \frac{\pi}{\sqrt{D(t)}} \right)^3 \exp \left\{ -\frac{1}{8D(t)} \left| x - y - (v + u) \frac{1 - e^{-t}}{1 + e^{-t}} \right|^2 \right\} - \frac{1}{2(1 - e^{-2t})} |v - u e^{-t}|^2 ,
\tag{3.63}
\end{equation}

Set
\[ h = e^{2t} \sqrt{M} g. \]

It follows from (3.61)–(3.62) that \( g \) satisfies
\begin{align*}
\partial_t g + v \cdot \nabla_x g - L g + 2g &= 0, \\
g(0, x, v) &= g_0(x, v) = M^{-1/2} h_0(x, v).
\end{align*}

Thus
\begin{equation}
g(t, x, v) = e^{tA} g_0 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G_1(t, x, v; y, u) g_0(y, u) dy du,
\tag{3.64}
\end{equation}

where
\begin{align*}
G_1(t, x, v; y, u) &= e^{-2t} \frac{1}{\sqrt{M(v)}} G_0(t, x, v; y, u) \sqrt{M(u)} \\
&= \frac{1}{(2\pi)^6} \left( \frac{\pi}{\sqrt{D(t)}} \right)^3 \exp \left\{ -\frac{1}{8D(t)} \left| x - y - (v + u) \frac{1 - e^{-t}}{1 + e^{-t}} \right|^2 \right\} - \frac{1}{4(1 - e^{-2t})} \left| v \frac{2e^{-t}}{1 + e^{-2t}} - u \right|^2 \frac{1 - e^{-2t}}{4(1 + e^{-2t})} |v|^2 - 2t \right\}.
\tag{3.65}
\end{align*}

Taking the Fourier transform to (3.64) with respect to \( x \), we obtain
\begin{equation}
g(t, \xi, v) = e^{tA(\xi)} \hat{g}_0 = \int_{\mathbb{R}^3} \hat{G}_1(t, \xi, v; u) \hat{g}_0(\xi, u) du,
\tag{3.66}
\end{equation}

where
\begin{align*}
\hat{G}_1(t, \xi, v; u) &= \frac{1}{(2\pi)^3} \left( \frac{4}{1 - e^{-2t}} \right)^{3/2} \exp \left\{ -i \xi \cdot (v + u) \frac{1 - e^{-t}}{1 + e^{-t}} \right. - 2D(t) \frac{1 - e^{-2t}}{1 - e^{-2t}} |\xi|^2 \\
&\quad - \left. \frac{1}{4(1 - e^{-2t})} \left| v \frac{2e^{-t}}{1 + e^{-2t}} - u \right|^2 - \frac{1}{4(1 + e^{-2t})} |v|^2 - 2t \right\}.
\tag{3.67}
\end{align*}

It is easy to verify that
\begin{equation}
\hat{G}_1(t, \xi, v; u) = \hat{G}_1(t, \xi, u; v),
\tag{3.68}
\end{equation}

\begin{equation}
\int_{\mathbb{R}^3} \left| \hat{G}_1(t, \xi, v; u) \right| du \leq C e^{-2t} e^{- \frac{2|\xi|^2}{1 - e^{-2t}}}.
\tag{3.69}
\end{equation}

Since \( D(t) > 0, \ t > 0 \) is strictly increasing and satisfies
\begin{equation}
D(t) = \frac{1}{12} t^4 + O(t^5), \ t \to 0; \quad D(t) = \frac{1}{2} t + O(1), \ t \to \infty,
\tag{3.70}
\end{equation}

\begin{align*}
\int_{\mathbb{R}^3} \left| \hat{G}_1(t, \xi, v; u) \right| du &\leq C e^{-2t} e^{- \frac{2|\xi|^2}{1 - e^{-2t}}} \\
&\leq C e^{-2t} e^{- \frac{2|\xi|^2}{1 - e^{-2t}}}.
\end{align*}
it follows from (3.66), (3.68), (3.70) that for any \( g_0 \in L^2(\mathbb{R}_v^3) \),
\[
\left\| |\xi|^k e^{iA(\xi)} g_0 \right\|^2 \leq \left\| |\xi|^k \int_{\mathbb{R}^3} \hat{G}_1(t, \xi, \nu; u) g_0(u) du \right\|^2 dv
\]
\[
\leq |\xi|^{2k} \sup_v \int_{\mathbb{R}^3} |\hat{G}_1(t, \xi, \nu; u)| du \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |\hat{G}_1(t, \xi, \nu; u)| dv \right) g_0^2(u) du
\]
\[
\leq C \left( 1 + \frac{1}{t^{5k}} \right) e^{-\frac{2D(t)}{1+e^{-\frac{v}{2}}} |\xi|^2} \left\| g_0 \right\|^2.
\]

By a direct computation, one has
\[
\left| \nabla_v \hat{G}_1(t, \xi, \nu; u) \right| = \left| -i \xi \frac{1-e^{-t}}{1+e^{-t}} - \frac{e^{-t}}{1+e^{-2t}} \left( \nu \frac{2e^{-t}}{1+e^{-2t}} - u \right) - \frac{1-e^{-2t}}{2(1+e^{-2t})} \right| \hat{G}_1(t, \xi, \nu; u)
\]
\[
\leq C \left( \frac{(1-e^{-t})^3}{D(t)} + \frac{e^{-t}}{1+e^{4t}} + \frac{1-e^{-2t}}{1+e^{-2t}} e^{-2t} \left( \frac{4}{1-e^{-2t}} \right)^{3/2} \right)
\exp \left\{ - \frac{D(t)}{8(1-e^{-2t})} |\xi|^2 - \frac{1+e^{-2t}}{8(1-e^{-2t})} \left| \left( \frac{2e^{-t}}{1+e^{-2t}} - u \right) - \frac{1-e^{-2t}}{8(1+e^{-2t})} |v|^2 \right. \right\}.
\]

Thus, it follows from (3.66), (3.70) and (3.71) that for any \( g_0 \in L^2(\mathbb{R}_v^3) \),
\[
\left\| \nabla_v e^{iA(\xi)} g_0 \right\|^2 \leq \sup_v \int_{\mathbb{R}^3} |\nabla_v \hat{G}_1(t, \xi, \nu; u)| du \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |\nabla_v \hat{G}_1(t, \xi, \nu; u)| dv \right) g_0^2(u) du
\]
\[
\leq C \left( 1 + \frac{1}{t} \right) e^{-\frac{4D(t)}{1+e^{-\frac{v}{2}}} |\xi|^2} \left\| g_0 \right\|^2.
\]

This completes the proof. \( \square \)

We define the approximate sequence \( \hat{I}_k \) for the high frequency part \( \hat{G}_H \) as follow
\[
\begin{align*}
\partial_t \hat{I}_0 + iv \cdot \hat{I}_0 - (L - 2) \hat{I}_0 &= 0, \\
|\xi|^2 \hat{E}_0 &= - (\hat{I}_0, \sqrt{M}), \\
\hat{I}_0(0, \xi) &= \chi_2(\xi) I_v,
\end{align*}
\] (3.72)

and
\[
\begin{align*}
\partial_t \hat{I}_k + iv \cdot \hat{I}_k - (L - 2) \hat{I}_k &= 2 \hat{I}_{k-1} + iv \cdot \xi \sqrt{M} \hat{E}_{k-1}, \\
|\xi|^2 \hat{E}_k &= - (\hat{I}_k, \sqrt{M}), \\
\hat{I}_k(0, \xi) &= 0,
\end{align*}
\] (3.73)

for \( k \geq 1 \). Define the singular waves and the remaining part as
\[
W_k(t, x) = \sum_{j=0}^{k} I_j(t, x), \quad \psi_k(t, x) = \sum_{j=0}^{k} E_j(t, x),
\] (3.74)

\[
R_k(t, x) = G_H(t, x) - W_k(t, x), \quad \phi_k(t, x) = \Phi_H(t, x) - \psi_k(t, x),
\] (3.75)

with \( \Phi_H(t, x) = \Delta_x^{-1} (G_H(t, x), \sqrt{M}) \).

It follows from (3.72) and (3.73) that \( \hat{W}_k(t, \xi) \) and \( \hat{\psi}_k(t, \xi) \) satisfy
\[
\begin{align*}
\partial_t \hat{W}_k + iv \cdot \xi \hat{W}_k - LW_k - iv \cdot \xi \sqrt{M} \hat{\psi}_k &= -2 \hat{I}_k - iv \cdot \xi \sqrt{M} \hat{E}_k, \\
|\xi|^2 \hat{\psi}_k &= (W_k, \sqrt{M}), \\
\hat{W}_k(0, \xi) &= \chi_2(\xi) I_v,
\end{align*}
\] (3.76)
and $\dot{R}_k(t, \xi)$ and $\dot{\phi}_k(t, \xi)$ satisfy

$$\begin{cases}
\partial_t \dot{R}_k + iv \cdot \xi \dot{R}_k - LR_k - iv \cdot \xi \sqrt{M} \dot{\phi}_k = 2\dot{I}_k + iv \cdot \xi \sqrt{M} \dot{E}_k, \\
|\xi|^2 \dot{\phi}_k = (\dot{R}_k, \sqrt{M}), \\
\dot{R}_k(0, \xi) = 0.
\end{cases}$$

(3.77)

With the help of Lemma 3.5, we can show the spatial derivative estimate of the approximate sequence $I_k(t, x)$ and $E_k(t, x)$ as follow.

**Lemma 3.6.** For each $k, j \geq 0$ and $\alpha \in \mathbb{N}^3$, $\dot{I}_j(t, \xi)$ and $\dot{E}_j(t, \xi)$ are smooth and supported in $\{\xi \in \mathbb{R}^3 \mid |\xi| \geq R\}$, and they satisfy the following estimates.

1. For $0 < t \leq 1$,

$$\|\xi|^k \partial^\alpha \dot{I}_j(t, \xi)\| + \|\xi|^k \partial^\alpha \dot{E}_j(t, \xi)\| \leq Ct^{-3k/2},$$

(3.78)

2. For $t > 1$,

$$\|\xi|^k \partial^\alpha \dot{I}_j(t, \xi)\| + \|\xi|^k \partial^\alpha \dot{E}_j(t, \xi)\| \leq Ct^{-2}e^{-t}.$$  

(3.79)

**Proof.** First, we want to show that for any $0 < t \leq 1$,

$$\|\xi|^k \dot{I}_j(t)\| \leq t^{-3k/2+j}, \quad \forall j, k \geq 0.$$  

(3.80)

The estimate of $\dot{I}_0$ is immediately from Lemma 3.5. For $j \geq 1$, it follows from (3.73) that

$$I_j(t) = \int_0^t e^{(t-s)A(\xi)}(2I_{j-1} + i(v \cdot \xi)|\xi|^{-2}P_0 I_{j-1})ds.$$  

By induction and Lemma 3.5, we obtain

$$|\xi|^k \|\dot{I}_j(t)\| \leq \int_0^{t/2} \|\xi|^k e^{(t-s)A(\xi)}\| (\|\dot{I}_{j-1}\| + |\xi|^{-1}\|\dot{I}_{j-1}\|)ds + \int_{t/2}^t \|e^{(t-s)A(\xi)}\| (|\xi|^k \|\dot{I}_{j-1}\| + |\xi|^{-1}\|\dot{I}_{j-1}\|)ds$$

$$\leq C \int_0^{t/2} (t-s)^{-3k/2}s^{-1}ds + C \int_{t/2}^t s^{-3k/2+j-1}ds$$

$$\leq Ct^{-3k/2+j}, \quad \forall j \geq 1.$$  

This proves (3.80).

Next, we claim that for $0 < t \leq 1$ and $|\alpha| \geq 1$,

$$|\xi|^k \|\partial^\alpha \dot{I}_j(t)\| \leq Ct^{-3k/2+j}, \quad \forall j, k \geq 0.$$  

(3.81)

Taking $\partial^\alpha \xi$ to the right of (3.72), we obtain

$$\partial_\xi \partial^\alpha \dot{I}_0 + i(v \cdot \xi) \partial^\alpha \dot{I}_0 - (L - 2) \partial^\alpha \dot{I}_0 = - \sum_{\beta \leq \alpha, |\beta| = 1} i v^\beta \partial^{\alpha-\beta} \dot{I}_0.$$  

Then we can represent $\partial^\alpha \dot{I}_0$ as

$$\partial^\alpha \dot{I}_0 = e^{tA(\xi)} \partial^\alpha \chi_2(\xi) - \sum_{\beta \leq \alpha, |\beta| = 1} \int_0^t e^{(t-s)A(\xi)} i v^\beta \partial^{\alpha-\beta} \dot{I}_0 ds.$$  

Since

$$\int_{\mathbb{R}^3} |\hat{G}_1(t, \xi, v; u)||u|^2 du$$
it follows from (3.68)–(3.69) that for any $g_0 \in L^2(\mathbb{R}^3)$,

$$
|\xi|^{2k}\|e^{t\phi_1(\xi)}v|g_0\|^2
\leq |\xi|^{2k}\sup_v \int_{\mathbb{R}^3} |G_1(t, \xi, v; u)|u^2 du \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |G_1(t, \xi, v; u)|dv \right) g_0^2(u)du
\leq \left( 1 + \frac{\alpha}{\beta + 1} \right) e^{-\alpha t} \epsilon^{1/2} \|g_0\|^2.
$$

(3.82)

By induction, (3.80) and (3.81), we obtain that for $0 < t \leq 1$ and $|\alpha| \geq 1$,

$$
|\xi|^k \|\partial_\xi^\alpha \hat{I}_0\| \leq C|\xi|^k\|e^{t\phi_1(\xi)}v|\|\|\partial_\xi^\alpha - 1 \hat{I}_0\|ds
+ \int_0^t \|e^{(t-s)\phi_1(\xi)}v|\|\|\partial_\xi^\alpha - 1 \hat{I}_0\|ds
\leq e^{-t\alpha} \int_0^t (t-s)^{-\alpha/2} ds + C \int_0^t (t-s)^{-\alpha/2} ds
\leq Ct^{-\alpha/2}, \ \forall k \geq 0.
$$

(3.83)

Taking $\partial_\xi^\alpha$ to (3.83) with $|\alpha| \geq 1$, we obtain

$$
\partial_\xi^\alpha \hat{I}_j + i(v \cdot \xi) \partial_\xi^\alpha \hat{I}_j - (L - 2) \partial_\xi^\alpha \hat{I}_j
= \partial_\xi^\alpha \left( 2\hat{I}_{j-1} + i(v \cdot \xi)|\xi|^{-2}P_0 \hat{I}_{j-1} \right) - \sum_{\beta \leq \alpha, |\beta| = 1} i\beta \partial_\xi^\beta \hat{I}_j.
$$

Then we can represent $\partial_\xi^\alpha \hat{I}_j$ for $j \geq 1$ as

$$
\partial_\xi^\alpha \hat{I}_j = \int_0^t e^{(t-s)\phi_1(\xi)} \partial_\xi^\alpha \left( 2\hat{I}_{j-1} + i(v \cdot \xi)|\xi|^{-2}P_0 \hat{I}_{j-1} \right) ds
- \sum_{\beta \leq \alpha, |\beta| = 1} \int_0^t e^{(t-s)\phi_1(\xi)} i\beta \partial_\xi^\beta \hat{I}_j ds.
$$

Thus, by induction, (3.80) and (3.82) we can obtain that for $|\alpha| \geq 1$ and $j \geq 1$,

$$
|\xi|^k \|\partial_\xi^\alpha \hat{I}_j\| \leq C \int_0^t \|\xi|^k e^{(t-s)\phi_1(\xi)}\| \left( \|\partial_\xi^\alpha \hat{I}_{j-1}\| + \sum_{\beta \leq \alpha} \frac{1}{|\xi|^{1+|\beta|}} \|\partial_\xi^\beta \hat{I}_{j-1}\| \right) ds
+ C \int_0^t \|\xi|^k e^{(t-s)\phi_1(\xi)}\| \left( \|\partial_\xi^\alpha \hat{I}_j\| + \sum_{\beta \leq \alpha} \frac{1}{|\xi|^{1+|\beta|}} \|\partial_\xi^\beta \hat{I}_j\| \right) ds
+ \int_0^t \|\xi|^k e^{(t-s)\phi_1(\xi)}\| \|\partial_\xi^1 \hat{I}_j\| ds
+ \int_0^t \|\xi|^k e^{(t-s)\phi_1(\xi)}\| \|\partial_\xi^0 \hat{I}_j\| ds
$$

(3.84)
we have it follows from (3.80) and Lemma 3.5 that

\[ |\xi|^k (\partial_{\xi}^\alpha (|\xi|^k \hat{I}_j)) \leq C \sum_{\beta \leq \alpha} \partial_{\xi}^\beta \left( \frac{\xi}{|\xi|^2} \right) |\xi|^k (|\partial_{\xi}^{\alpha-\beta} \hat{I}_j, \sqrt{M}) \leq Ct^{-3k/2+j}, \]  

(3.85)

for \( 0 < t \leq 1 \). By combining 3.83–3.85, we prove (3.81).

Next, we want to show that for \( t > 1 \),

\[ |\xi|^k \| \hat{I}_j(t) \| \leq Ct^j e^{-2t}, \quad \forall j, k \geq 0. \]

Since

\[ \hat{I}_0(t) = e^{(t-1)A(\xi)} \hat{I}_0(1), \quad t > 1, \]

it follows from 3.80 and Lemma 3.3 that

\[ |\xi|^k \| \hat{I}_0(t) \| \leq Ce^{-2(t-1)\| \hat{I}_0(1) \|} \leq Ce^{-2(t-1)}. \]  

(3.86)

Noting that

\[ \hat{I}_j(t) = e^{(t-1)A(\xi)} \hat{I}_j(1) + \int_1^t e^{(t-s)A(\xi)} (2\hat{I}_j - 1 + i(v : \xi)|\xi|^{-2}P_0 \hat{I}_j - 1) ds, \]

we have

\begin{align*}
|\xi|^k \| \hat{I}_j(t) \| & \leq Ce^{-2(t-1)\| \hat{I}_j(1) \|} + \int_1^t e^{-2(t-s)\| \hat{I}_j \|} ds
\leq Ce^{-2(t-1)} + \int_1^t e^{-2(t-s)} s^{j-1} e^{-2s} ds
\leq Ct^j e^{-2t}, \quad t > 1.
\end{align*}

(3.87)

For \( t > 1 \), \( \partial_{\xi}^\alpha \hat{I}_0 \) can be written as

\[ \partial_{\xi}^\alpha \hat{I}_0(t) = e^{(t-1)A(\xi)} \partial_{\xi}^\alpha \hat{I}_0(1) - \sum_{\beta \leq \alpha, |\beta| = 1} \int_1^t e^{(t-s)A(\xi)} i\epsilon \partial_{\xi}^{\alpha-\beta} \hat{I}_0 ds. \]

By induction, 3.80 and 3.82, we obtain that for \( t > 1 \) and \( |\alpha| \geq 1 \),

\begin{align*}
|\xi|^k \| \partial_{\xi}^\alpha \hat{I}_0(t) \| & \leq Ce^{-2(t-1)\| \hat{I}_0(1) \|} + \int_1^t \| e^{(t-s)A(\xi)} (|\xi|^k v |\partial_{\xi}^{\alpha-1} \hat{I}_0|) \| ds
\leq Ce^{-2(t-1)} + C \int_1^t \left( 1 + (t-s)^{-\frac{k}{2}} \right) e^{-2(s)} s^{j-1} e^{-2s} ds
\leq Ct^j e^{-2t}, \quad \forall k \geq 0.
\end{align*}

(3.88)

By the similar arguments as 3.80 and 3.81, we can prove

\[ |\xi|^k \| \partial_{\xi}^\alpha \hat{I}_j(t) \| + |\xi|^k \| \partial_{\xi}^\alpha \hat{E}_j(t) \| \leq Ct^j e^{-2t}, \quad j \geq 0, \quad t > 1. \]

The proof of the lemma is completed. \( \square \)

With the help of Lemma 3.3, we can show the pointwise estimates of the remaining terms \( R_k(t,x) \) and \( \phi_k(t,x) \) defined by 3.77 as follows.

**Lemma 3.7.** For any \( n \geq 1 \) and \( \alpha \in \mathbb{N}^3 \), there exists a small constant \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \),

\[ \| \partial_{\xi}^\alpha R_k(t,x) \| + \| \partial_{\xi}^\alpha \nabla_x \phi_k(t,x) \| \leq C\delta^{n/2} e^{2\delta t} (1 + |\delta x|^2)^{-n}, \]

(3.89)

where \( k \geq 6 + 3|\alpha|/2 \) and \( C > 0 \) is a constant dependent of \( n \) and \( \alpha \).
Proof. For any \( g_0 \in L^2(\mathbb{R}^3_v) \), we define
\[
\hat{R}_k(t, \xi) = \hat{R}_k(t, \xi)g_0, \quad \hat{\Omega}_k(t, \xi) = \hat{\phi}_k(t, \xi)g_0.
\]
We claim that for any \( l \ge 0 \) and \( \alpha \in \mathbb{N}^3 \),
\[
\| \partial^\alpha \hat{R}_k(t, \xi) \|_\xi \le C \delta^{-1-2|\alpha|}e^{2lt}(1 + |\xi|)^{-4-l} \|g_0\|,
\]
where \( k \ge 6 + 3l/2 \) and \( \delta \in (0, \delta_0) \).

We prove (3.90) by induction. From Lemma 3.6, it holds for any \( k \ge 6 + 3l/2 \) and \( \alpha \in \mathbb{N}^3 \),
\[
\| \partial^\alpha \hat{I}_k \| + |\partial^\alpha (\xi \hat{E}_k)| \le C e^{-t}(1 + |\xi|)^{-4-l}.
\]
Taking the inner product between (3.77) and \( \hat{R}_k \) and using Cauchy inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \| \hat{R}_k \|_\xi^2 + \frac{\mu}{2} \| P_1 \hat{R}_k \|_\sigma^2 \le -C \| \hat{R}_k \|_\xi^2 + |\xi^\alpha | \| \hat{E}_k \|_\xi^2 + \delta \| P_0 \hat{R}_k \|_\xi^2,
\]
where \( \delta \in (0, \delta_0) \) with \( \delta_0 > 0 \) small enough, and
\[
\hat{I}_k(t, \xi) = \hat{I}_k(t, \xi)g_0, \quad \hat{E}_k(t, \xi) = \hat{E}_k(t, \xi)g_0.
\]
Applying Gronwall’s inequality to (3.92) and using (3.91), we obtain
\[
\| \hat{R}_k \|_\xi^2 \le \frac{C}{\delta} \|g_0\|^2 \int_0^t e^{2s(t-s)}(1 + |\xi|)^{-8-2l}e^{-2s}ds
\]
\[
\le C \delta^{-2}(1 + |\xi|)^{-8-2l}e^{2t} \|g_0\|^2.
\]
Suppose that (3.90) holds for \(|\alpha| \le j - 1\). Taking the derivative \( \partial^\alpha \) to (3.17) with \(|\alpha| = j\) to get
\[
\partial_t \partial^\alpha \hat{R}_k + i v \cdot \xi \partial^\alpha \hat{I}_k - L \partial^\alpha \hat{E}_k - i v \cdot \xi \partial^\alpha \hat{R}_k = G_{k,\alpha},
\]
where
\[
G_{k,\alpha} = \sum_{\beta \le \alpha, |\beta| \ge 1} C_{\alpha} \left( \partial^\beta (v \cdot \xi) \partial^{\alpha-\beta} \hat{R}_k + i \partial^\beta \left( \frac{v \cdot \xi}{|\xi|^2} \right) \right) P_0 \partial^\beta \hat{R}_k
\]
\[
= i \partial^\alpha (v \cdot \xi \hat{E}_k) \sqrt{M} + 2 \partial^\alpha \hat{I}_k.
\]
Taking the inner product between (3.93) and \( \partial^\alpha \hat{R}_k + |\xi|^{-2} P_0 \partial^\alpha \hat{R}_k \) and using Cauchy inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha \hat{R}_k \|_\xi^2 + \frac{\mu}{2} \| P_1 \hat{R}_k \|_\sigma^2
\]
\[
\le \frac{C}{\delta} \left( \| \partial^\alpha \hat{I}_k \|_\xi^2 + |\partial^\alpha (\xi \hat{E}_k)| \right) + \frac{C}{\delta} \left( |\partial^\alpha \hat{R}_k| \right)^2 \sum_{|\beta| = 1} |\partial^\beta \hat{n}_k|^2
\]
\[
+ \frac{C}{\delta} \| P_1 \hat{R}_k \|_\xi^2 + \delta \| P_0 \hat{R}_k \|_\xi^2,
\]
where
\[
\hat{n}_k = (\hat{R}_k, \sqrt{M}), \quad \hat{m}_k = (\hat{R}_k, v \sqrt{M}).
\]
Thus, it follows from (3.93) and (3.95) that
\[
\| \partial^\alpha \hat{R}_k \|_\xi^2 \le \frac{C}{\delta} \|g_0\|^2 (1 + |\xi|)^{-8-2l} \left( \int_0^t e^{2s(t-s)}e^{-2s}ds + \delta^{-2}|\alpha| \int_0^t e^{2s(t-s)}e^{4s}ds \right)
\]
\[
\le C \delta^{-2-2|\alpha|}(1 + |\xi|)^{-8-2l}e^{4t} \|g_0\|^2.
\]
This proves (3.90) for $\alpha = j$.

Therefore, it holds for any $\gamma \in \mathbb{N}^3$,

$$\|\partial_\xi^\gamma (\xi^j \hat{R}_k)\| + |\partial_\xi^\gamma (\xi^j \xi \hat{R}_k)|$$

$$\leq \sum_{\beta \leq \alpha} C_\beta^\alpha \left( \|\partial_\xi^\beta (\xi^j) \partial_\xi^{\alpha-\beta} \hat{R}_k\| + |\partial_\xi^\beta (\xi^j \xi) \partial_\xi^{\alpha-\beta} \hat{R}_k| \right)$$

$$\leq C \sum_{|\beta|=0} |\xi|^{\gamma - |\beta|} \|\partial_\xi^{\alpha - \beta} \hat{R}_k\| \leq C \delta^{-1 - |\alpha|} (1 + |\xi|)^{-4} e^{2\delta t} \|g_0\|,$$

where $k \geq 6 + 3|\gamma|/2$, which leads to

$$(\|\partial_\xi^\alpha [G_H(t, x) - W_k(t, x)]\| + |\partial_\xi^\alpha \nabla_x \Omega_k(t, x)|) x^{2\alpha} \leq C \int_{|\xi| \geq R} (\|\partial_\xi^{2\alpha} (\xi^j \hat{R}_k)\| + |\partial_\xi^{2\alpha} (\xi^j \xi \hat{R}_k)|) d\xi$$

$$\leq C \delta^{-1 - 2|\alpha|} e^{2\delta t} \|g_0\|.$$

This proves the lemma. □

**Theorem 3.8.** Let $G_H(t, x)$ be the high frequency part defined by (3.51). For any $n \geq 0$ and $\alpha \in \mathbb{N}^3$, there exists a constant $0 < \eta_0 < 1/4$ such that

$$\|\partial_\xi^\alpha [G_H(t, x) - W_k(t, x)]\| \leq C e^{-\eta_0 t} (1 + |x|^2)^{-n},$$

(3.96)

for $k \geq 6 + 3|\alpha|/2$ and $C > 0$ a constant dependent of $n$ and $\alpha$, where $W_k(t, x)$ is the singular kinetic wave defined by (3.74).

**Proof.** By (3.75), we have

$$G_H(t, x) = W_k(t, x) + R_k(t, x).$$

(3.97)

From (3.77), (3.81) and Lemma 3.1 we can obtain that for $k \geq 6 + 3|\alpha|/2$,

$$\|\hat{R}_k(t, \xi)\| \leq \int_0^t \left\| G_H(t-s) (2\hat{I}_k + i(v \cdot \xi) \sqrt{M} \hat{E}_k)(s) \right\| \xi ds$$

$$\leq C \int_0^t e^{-\beta_1(t-s)} (\|\hat{I}_k(s)\| \xi + |\xi| \|\hat{E}_k(s)|) ds$$

$$\leq C e^{-\beta_1 t} (1 + |\xi|)^{-4|\alpha|},$$

which gives

$$\|\partial_\xi^\alpha R_k(t, x)\| + |\partial_\xi^\alpha \nabla_x \phi_k(t, x)| \leq C e^{-\beta_1 t}.$$  \hspace{1cm} (3.98)

It follows from (3.97), (3.98) and (3.99) that

$$\|\partial_\xi^\alpha [G_H(t, x) - W_k(t, x)]\| \leq C e^{-\eta_0 t} (1 + |x|^2)^{-n},$$

where $\eta_0 = \beta_1/2 \in (0, 1/4)$, which lead to (3.96). □

**Theorem 1.1** directly follows from Theorem 3.8 and Theorem 3.4.

## 4 The nonlinear system

In this section, we prove Theorem 1.2 on the pointwise behaviors of the global solution to the nonlinear VPFP system with the help of the estimates of the Green’s function given in Sections 3.
Lemma 4.1. Let $\gamma, n \geq 0$ and $\alpha \in \mathbb{N}^3$. If the function $g_0(x, v)$ satisfies

$$|g_0(x, v)| \leq C(1 + |x|^2)^{-n}(1 + |v|)^{-\gamma},$$

then we have

$$|G_1(t) * g_0(x, v)| \leq C e^{-t}(1 + |x|^2)^{-n}(1 + |v|)^{-\gamma},$$

$$|\nabla_v G_1(t) * g_0(x, v)| \leq C t^{-\frac{1}{2}} e^{-t}(1 + |x|^2)^{-n}(1 + |v|)^{-\gamma},$$

where $G_1$ is defined by (3.65). In addition,

$$|W_\alpha(t) * g_0(x, v)| \leq C e^{-t}(1 + |x|^2)^{-n}(1 + |v|)^{-\gamma},$$

where $W_\alpha$ is defined by (1.25).

Proof. First, we deal with (1.11) and (1.2). Since it follows from (3.70) that

$$\frac{1}{D(t)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{1}{2} e^{-t}|x - y - (v + u)|^2} (1 + |y|^2)^{-n} dy$$

$$= \frac{1}{D(t)^{3/2}} \left( \int_{|y| \leq y_0} + \int_{|y| \geq y_0} \right) e^{-\frac{1}{2} e^{-t} |x - y - (v + u)|^2} (1 + |y|^2)^{-n} dy$$

$$\leq C \frac{1}{(1 - e^{-2t})^{3/2}} \left( e^{-\frac{1}{2} e^{-t} |x - (v + u)|^2} + (1 + |x - (v + u)|^2) e^{-t} \right)^{-n}$$

$$\leq C \frac{1}{(1 - e^{-2t})^{3/2}} (1 + n) \left( 1 + |x - (v + u)|^2 \right)^{-n},$$

where $y_0 = \frac{1}{2} |x - (v + u)| \frac{1}{1 + e^{-t}},$ it follows that

$$\left| \int_{\mathbb{R}^6} G_1(t, x, v; y, u) g_0(y, u) dy du \right|$$

$$\leq C e^{-t} \frac{1}{D(t)^{3/2}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\frac{1}{2} e^{-t} |x - y - (v + u)|^2} (1 + |y|^2)^{-n} dy$$

$$e^{-\frac{1}{2} e^{-t} |v|^2} e^{-\frac{1}{2} e^{-t} |x - y - (v + u)|^2} (1 + |y|^2)^{-n} du$$

$$\leq C (1 + n) \frac{e^{-2t}}{(1 - e^{-2t})^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{1}{2} e^{-t} |x - (v + u)|^2} (1 + |x - (v + u)|^2)^{-n} du.$$
Thus, it follows from (4.4)–(4.6) that
\[ I \]
where we have used (3.70).

Let
\[ I_2 = \left( 1 + \left| x - v \right| e^{-2t} \right)^{-n} \left( 1 + \left| \frac{e^{-t}}{1 + e^{-2t}} \right| e^{-\frac{t}{4(1 + e^{-2t})}} \right)^2. \]

It holds that $I_1 \leq C(1 + |v|)^{-\gamma}$ for $t < 1$ and $I_1 \leq e^{-|v|^2}$ for $t \geq 1$, and $I_2 \leq C(1 + |x|^2)^{-n}$ for $|x| \geq 2|v|$, namely,
\[ I_1 \leq C(1 + |v|)^{-\gamma}, \quad I_2 \leq C(1 + |x|^2)^{-n}. \]

Thus, it follows from (4.7)–(4.8) that
\[ |G_1(t) \ast g_0(x, v)| \leq C(1 + t)^n e^{-2t} (1 + |x|)^{-n} (1 + |v|)^{-\gamma} e^{-\frac{t}{4(1 + e^{-2t})}} |v|^2. \]
where

\[ J_0(t, x, v) = G_1(t) * g_0 = \int_{\mathbb{R}^3} G_1(x - y, v, t; u) g_0(y, u) du, \]

\[ J_k(t, x, v) = \int_0^t G_1(t - s) * (2 + v \cdot \nabla_x \Delta_x^{-1} P_{01}) J_{k-1}(s) ds, \quad k \geq 1. \]

To estimate \( \chi_R(D) \), we decompose

\[ \chi_R(D) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} d\xi + \int_{\mathbb{R}^3} e^{ix \cdot \xi} \chi_1(\xi) d\xi = \delta(x) + F(x). \]  

(4.12)

Since \( \chi_1(\xi) \) is a smooth and compact supported function, it follows that for any \( k \geq 0 \),

\[ |F(x)| \leq C(1 + |x|^2)^{-k}. \]

Combining (4.9) and (4.12), we obtain

\[ |\chi_R(D)J_0(t, x, v)| \leq |J_0(t, x, v)| + |F(x) * J_0(t, x, v)| \leq C(1 + t)^n e^{-2t} (1 + |v|)^{-2} e^{-\frac{1 + e^{-2t}}{4/k+1/4} |v|^2} \]

\[ \left( (1 + |x|^2)^{-n} + \int_{\mathbb{R}^3} (1 + |x - y|^2)^{-n-2} (1 + |y|^2)^{-n} dy \right) \leq C e^{-t} (1 + |x|^2)^{-n+2} (1 + |v|^2)^{-n}. \]

(4.13)

Since

\[ \left| \int_{\mathbb{R}^3} e^{ix \cdot \xi} \frac{\xi_j}{|\xi|^2} \chi_2(\xi) d\xi \right| \leq C \frac{1}{|x|^2} (1 + |x|^2)^{-k}, \quad j = 1, 2, 3, \]

for any \( k \geq 0 \), we have

\[ |\chi_R(D)\nabla_x \Delta_x^{-1}(J_0(t, x, \sqrt{M}))| \leq C(1 + t)^n e^{-2t} \left( \int_{|x-y| \leq \frac{1}{2}} + \int_{|x-y| > \frac{1}{2}} \right) \frac{1}{|x-y|^2} (1 + |x-y|^2)^{-n-2} (1 + |y|^2)^{-n} dy =: C(1 + t)^n e^{-2t} (I_3 + I_4). \]

(4.14)

Note that

\[ I_3 \leq C(1 + |x|^2)^{-n} \int_{|x-y| \leq \frac{1}{2}} \frac{1}{|x-y|^2} (1 + |x-y|^2)^{-n-2} dy \]

\[ \leq C(1 + |x|^2)^{-n}, \]  

(4.15)

\[ I_4 \leq C \int_{|x-y| > \frac{1}{2}} (1 + |x-y|^2)^{-n-2} (1 + |y|^2)^{-n} dy \]

\[ \leq C(1 + |x|^2)^{-n}. \]  

(4.16)

Combining (4.14), (4.15) and (4.16), we have

\[ |\chi_R(D)\nabla_x \Delta_x^{-1}(J_0(t, x, \sqrt{M}))| \leq C(1 + t)^n e^{-2t} (1 + |x|^2)^{-n}. \]

(4.17)

Thus, it follows from (4.15), (4.17) and (4.10) that

\[ |\chi_R(D)J_1(t, x, v)| = |\chi_R(D) \int_0^t G_1(s) * (2J_0 + v \cdot \nabla_x \Delta_x^{-1} P_0 J_0)(t - s) ds| \]
\[ Ce^{-t} \int_0^t \int_{\mathbb{R}^n} e^{s} G_1(s, x, v; y, u)(1 + |y|)^{-n}(1 + |u|)^{-\gamma} dydu ds \]
\[ \leq Ce^{-t} \int_0^t e^{-s} (1 + s)^n (1 + |x|)^{-n} (1 + |v|)^{-\gamma} ds \]
\[ \leq Ce^{-t} (1 + |x|)^{-n} (1 + |v|)^{-\gamma}. \]

By a similar argument as above, we obtain
\[ |\chi_{R(D)} J_k(t, x, v)| \leq Ce^{-t} (1 + |x|)^{-n} (1 + |v|)^{-\gamma}, \quad \forall k \geq 2, \]
which proves (4.3). The proof of the lemma is completed. □

Lemma 4.2. Let \( \gamma, n \geq 0 \). If the functions \( F_i(t, x, v), i = 0, 1, 2 \) satisfy
\[ |F_k(t, x, v)| \leq Ct^{-\frac{k}{2} \beta} e^{-bt} (1 + |x|^2)^{-n} (1 + |v|)^{-\gamma}, \quad k = 0, 1, \]
\[ \| F_2(t, x) \|_{L^2} \leq Ce^{-bt} (1 + |x|^2)^{-n}, \]
with \( 0 < b < 2 \), then we have
\[ \left| \int_0^t G_1(t - s) * F_k(s, x, v) ds \right| \leq Ce^{-bt} (1 + |x|^2)^{-n} (1 + |v|)^{-\gamma-1}, \quad (4.18) \]
\[ \left| \int_0^t \nabla_v G_1(t - s) * F_k(s, x, v) ds \right| \leq Ce^{-bt} (1 + |x|^2)^{-n} (1 + |v|)^{-\gamma}, \quad (4.19) \]
\[ \left| \int_0^t G_1(t - s) * F_2(s, x, v) ds \right| \leq Ce^{-bt} (1 + |x|^2)^{-n}, \quad (4.20) \]
\[ \left| \int_0^t W_k(t - s) * F_k(s, x, v) ds \right| \leq Ce^{-bt} (1 + |x|^2)^{-n} (1 + |v|)^{-\gamma-1}, \quad (4.21) \]
where \( k = 0, 1 \) and \( C > 0 \) is a constant.

Proof. By (4.19), we have
\[ \left| \int_0^t G_1(t - s) * F_k(s, x, v) ds \right| \]
\[ \leq Ce^{-bt} \int_0^t e^{-2bs}(1 + s)^n (1 + |x|)^{-n} (1 + |v|)^{-\gamma} e^{-\frac{b(1 + |x|^2)}{2} (t - s)^{-\frac{1}{2}}} ds \]
\[ \leq Ce^{-bt} \int_0^t e^{-2bs}(1 + s)^n (1 + |x|)^{-n} (1 + |v|)^{-\gamma-1} \frac{1}{\sqrt{1 - e^{-2bs}(t - s)^{-\frac{1}{2}}}} ds \]
\[ \leq Ce^{-bt} (1 + |x|)^{-n} (1 + |v|)^{-\gamma-1}, \quad k = 0, 1. \]

By (4.11), we have
\[ \left| \int_0^t \nabla_v G_1(t - s) * F_k(s, x, v) ds \right| \]
\[ \leq Ce^{-bt} \int_0^t e^{-2bs}(1 + s)^n (1 + |x|)^{-n} (1 + |v|)^{-\gamma} (t - s)^{-\frac{1}{2}} ds \]
\[ \leq Ce^{-bt} (1 + |x|)^{-n} (1 + |v|)^{-\gamma}, \quad k = 0, 1. \]

Thus, we obtain (4.18) and (4.19).

Finally, we want to show (4.20). By changing variable \( v = \frac{2e^{-\frac{b}{2}}}{1 + e^{-\frac{b}{2}}} - u \to z \), we obtain
\[ \int_{\mathbb{R}^n} |G_1(t, x, v; y, u)|^2 du \]
Finally, by (4.18) and a similar argument as Lemma 4.1, we can prove (4.21).

By (4.24)–(4.26), Theorem 1.1 and Lemmas 4.1–4.2, and noting that $\frac{H}{2t(1+e^{-2t})}$, we obtain

\[
\begin{align*}
&\leq C e^{-4t} \frac{D(t)}{t^3} \left\{ \frac{1}{2} e^{-2t} \left| x-y - \frac{1}{1+e^{-2t}} \right|^2 + \frac{1}{2} e^{-2t} \left| |x|^2 - \frac{1}{1+e^{-2t}} \right|^2 \right\} \\
&\quad + C e^{-2t} \left| \frac{1}{2} e^{-2t} \left( \frac{D(t)}{1-e^{-2t}} \right)^{3/2} \left( \frac{1+e^{-1}}{1-e^{-1}} \right)^{3/2} \right\}.
\end{align*}
\]

(4.22)

By (3.70), (4.22) and a similar argument as (4.9), we obtain

\[
\left| \int_0^t G_1(t-s) * F_2(s, x, v) ds \right|
\leq \int_0^t \| G_1(s, x, v; y) \|_{L^2_x} \| F_2(t-s, y) \|_{L^2_y} dy ds
\leq C e^{-bt} (1 + |x|^2)^{-n} \int_0^t e^{-2s} s \left\{ \frac{(1+s)^n}{(1-e^{-2s})^2} + \left[ \frac{(1-e^{-s})^3}{D(s)} \right] \right\} ds
\leq C e^{-bt} (1 + |x|^2)^{-n}.
\]

Finally, by (4.12) and a similar argument as Lemma 4.1, we can prove (4.21).

With the help of Theorem 1.1 and Lemmas 4.1–4.2, we are able to prove Theorem 1.2 as follows.

**Proof of Theorem 1.2** First, we deal with (1.30)–(1.33). Let $f$ be a solution to the IVP problem (1.4)–(1.6) for $t > 0$. We can represent this solution as

\[
f(t, x, v) = G(t) * f_0 + \int_0^t G(t-s) * H(f)(s) ds,
\]

(4.23)

where $H(f)$ is the nonlinear term defined by

\[
H(f) = \frac{1}{2} (v \cdot \nabla_x \Phi)f - \nabla_x \Phi \cdot \nabla_v f.
\]

Define

\[
Q(t) = \sup_{0 \leq s \leq t} \left\{ \left( \| P_0 f \|_{L^2_x} (1 + |x|^2)^{1/2} + \| P_3 f \|_{L^2_x} \right) e^{\eta_0 s} (1 + |x|^2)^{\frac{1}{2}}
\right.
\]

\[
\left. + (\| P_{n} f \|_{L^2_x} + \| \nabla_x \Phi \|) e^{\eta_0 s} (1 + |x|^2)
\right.
\]

\[
\left. + \left( \| f \|_{L^2_{x,v}} + \| \nabla_v f \|_{L^2_{x,v}} \sqrt{1 + \sqrt{s}} \right) e^{\eta_0 s} (1 + |x|^2) \right\}.
\]

It holds that for $0 \leq s \leq t$,

\[
|H(s, x, v)| \leq \frac{1}{2} |\nabla_x \Phi| |vf| + |\nabla_v \Phi| |\nabla_v f|
\leq C Q^2(t) (1 + s^{-\frac{1}{2}}) e^{-2\eta_0 s} (1 + |x|^2)^{-2} (1 + |v|)^{-2}.
\]

(4.24)

By Theorem 1.1, we decompose

\[
G(t) * f_0 = G_L(t) * f_0 + W_0(t) * f_0 + (G_H - W_0)(t) * f_0,
\]

(4.25)

\[
\int_0^t G(t-s) * H(s) ds = \int_0^t G_L(t-s) * H(s) ds + \int_0^t W_0(t-s) * H(s) ds
\]

\[
+ \int_0^t (G_H - W_0)(t-s) * H(s) ds.
\]

(4.26)

By (4.24)–(4.26), Theorem 1.1 and Lemmas 4.1–4.2, and noting that $(H(t, x), \sqrt{M}) = 0$, we obtain

\[
\| P_0 f(t, x) \|_{L^2_x} \leq C \delta_0 e^{-\eta_0 t} (1 + |x|^2)^{-2} + C e^{-\eta_0 t} (1 + |x|^2)^{-2} Q^2(t),
\]

(4.27)
\[ \|P_m f(t,x)\|_{L^2} \leq C\delta_0 e^{-\gamma_0 t} (1 + |x|^2)^{-1} + Ce^{-\gamma_0 t} (1 + |x|^2)^{-1} Q^2(t), \] (4.28)

\[ \|P_1 f(t,x)\|_{L^2} \leq C\delta_0 e^{-\gamma_0 t} (1 + |x|^2)^{-\frac{1}{2}} + Ce^{-\gamma_0 t} (1 + |x|^2)^{-\frac{5}{2}} Q^2(t). \] (4.29)

In addition, it holds that
\[ |\nabla_x \Phi(t,x)| = \left| \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} (f(t,y), \sqrt{M}) dy \right| \leq Ce^{-\gamma_0 t} (1 + |x|^2)^{-1} (\delta_0 + Q^2(t)). \] (4.30)

By (1.4), we have
\[ \frac{\partial_t f - Af}{2} = 2f + v \cdot \nabla_x \Phi \sqrt{M} + H, \]
where \( A = L - 2 - (v \cdot \nabla_x) \). Thus, we can represent \( f \) as
\[ f(t,x,v) = G_1(t) * f_0 + \int_0^t G_1(t-s) * (2f + v \cdot \nabla_x \Phi \sqrt{M} + H) ds. \] (4.31)

By Lemma 4.1 it holds that
\[ |G_1(t) * f_0(x,v)| \leq C\delta_0 e^{-\gamma_0 t} (1 + |x|^2)^{-2} (1 + |v|)^{-3}. \] (4.32)

By (4.24) - (4.29), we have
\[ \|f(t,x)\|_{L^2} \leq C(\delta_0 + Q(t)^2) e^{-\gamma_0 t} (1 + |x|^2)^{-1}, \]
which together with (4.20), (4.24) and (4.30), we have
\[ \left| \int_0^t G_1(t-s) * (2f + v \cdot \nabla_x \Phi \sqrt{M} + H) ds \right| \leq C(\delta_0 + Q(t)^2) e^{-\gamma_0 t} (1 + |x|^2)^{-1}. \] (4.33)

Thus, it follows from (4.31) - (4.33) that
\[ |f(t,x,v)| \leq C(\delta_0 + Q(t)^2) e^{-\gamma_0 t} (1 + |x|^2)^{-1}. \]

By induction and (4.18), we have
\[ |f(t,x,v)| \leq C(\delta_0 + Q(t)^2) e^{-\gamma_0 t} (1 + |x|^2)^{-1} (1 + |v|)^{-3}. \] (4.34)

Next, we estimate \( \nabla_v f \) as follows. By (4.31), we have
\[ \nabla_v f(t,x) = \nabla_v G_1(t) * f_0 + \int_0^t \nabla_v G_1(t-s) * (2f + v \cdot \nabla_x \Phi \sqrt{M} + H) ds. \] (4.35)

By Lemma 4.1 it holds that
\[ |\nabla_v G_1(t) * f_0(x,v)| \leq C\delta_0 t^{-\frac{1}{2}} e^{-\gamma_0 t} (1 + |x|^2)^{-2} (1 + |v|)^{-3}. \] (4.36)

By (4.31), (4.30), (4.24) and (4.19), we have
\[ \left| \int_0^t \nabla_v G_1(t-s) * (2f + v \cdot \nabla_x \Phi \sqrt{M} + H) ds \right| \leq C(\delta_0 + Q(t)^2) e^{-\gamma_0 t} (1 + |x|^2)^{-1} (1 + |v|)^{-2}. \] (4.37)

Thus, it follows from (4.35) - (4.37) that
\[ |\nabla_v f(t,x,v)| \leq C(\delta_0 + Q(t)^2) (1 + t^{-\frac{1}{2}}) e^{-\gamma_0 t} (1 + |x|^2)^{-1} (1 + |v|)^{-2}. \] (4.38)
Combining (4.37), (4.39), (4.41) and (4.35), we have
\[ Q(t) \leq C\delta_0 + CQ(t)^2, \]
from which (1.30)–(1.33) can be verified so long as \( \delta_0 > 0 \) is small enough. Similarly, we can prove (1.34)–(1.37) for the case of \( P_0f_0 = 0 \), the details are omitted.

Finally, we prove the existence of the solution \( f \) by the following iteration: \( f^0 \equiv 0 \),
\[ f^n(t, x, v) = G(t) * f_0 + \int_0^t G(t - s) * H(f^{n-1})ds, \quad n \geq 1. \] (4.39)
We want to show that \( \{f^n\}_n \) is a Cauchy sequence in \( L^\infty(\mathbb{R}_+ \times \mathbb{R}_3^3 \times \mathbb{R}_3^3) \). By the above estimates and using induction argument, we can obtain that when \( \delta_0 > 0 \) sufficiently small,
\[ \|f^n(t, x)\|_{L_{x,v}^\infty} + h(t)\|\nabla_v f^n(t, x)\|_{L_{x,v}^2} + \|\nabla_x \Phi^n(t, x)\| \leq C\delta_0 e^{-\eta_0 t}(1 + |x|^2)^{-1}, \] (4.40)
where \( h(t) = \sqrt{t}/(1 + \sqrt{t}), \) \( \Phi^n(t, x) = \Delta_x^{-1}(f^n(t, x), \sqrt{\mathcal{M}}) \), and \( C > 0 \) is a constant independent of \( n \). Let
\[ g^n = f^{n+1} - f^n, \quad \phi^n = \Phi^{n+1} - \Phi^n, \quad n \geq 0. \]
Then, \( g^n \) satisfies
\[ g^n(t, x, v) = \int_0^t G(t - s) * (H^n_1 + H^n_2)ds, \quad n \geq 1, \] (4.41)
where
\[ \begin{cases} H^n_1 = \left( \frac{1}{2}v^n f^n + \nabla_v f^n \right) \cdot \nabla_x \phi^{n-1}, \\ H^n_2 = \left( \frac{1}{2}v^n g^{n-1} + \nabla_v g^{n-1} \right) \cdot \nabla_x \Phi^{n-1}. \end{cases} \]
By (4.40), (4.41) and using induction argument, we have
\[ \|g^n(t, x)\|_{L_{x,v}^\infty} + h(t)\|\nabla_v g^n(t, x)\|_{L_{x,v}^2} + \|\nabla_x \phi^n(t, x)\| \leq (C\delta_0)^{n+1} e^{-\eta_0 t}(1 + |x|^2)^{-1}. \]
Thus \( \{f^n\}_n \) is a Cauchy sequence in \( L^\infty(\mathbb{R}_+ \times \mathbb{R}_3^3 \times \mathbb{R}_3^3) \), there exists a unique function \( f(t, x, v) \) such that
\[ \|f^n(t, x) - f(t, x)\|_{L_{x,v}^\infty} \leq (C\delta_0)^{n+1} e^{-\eta_0 t}(1 + |x|^2)^{-1} \to 0, \quad n \to \infty. \] (4.42)
This together with (4.39) and (4.42) imply that \( f(t, x, v) \) satisfies the following equation:
\[ f(t, x, v) = G(t) * f_0 + \int_0^t G(t - s) * H(f)ds. \]
Thus \( f(t, x, v) \) is the unique solution to the VPFP system (1.3)–(1.6). This completes the proof. \( \square \)

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