Enhanced radiative ion cooling

E.G. Bessonov, Lebedev Physical Institute RAS, Moscow, Russia

(Dated: March 31, 2022)

PACS numbers: 29.20.Dh, 07.85.Fv, 29.27.Eg

I. INTRODUCTION

In the method of ordinary three-dimensional radiative cooling of ion beams a laser beam overlaps an ion beam, its transverse position is motionless, all ions interact with the laser beam independent of their energy and amplitude of betatron oscillations. The difference in rates of momentum loss of ions having maximum and minimum energies in the beam is small. That is why the cooling time of the ion beam is high [1] - [3].

When a cooling is produced in a dispersion-free straight section and the laser beam intensity is constant inside the area of the laser beam occupied by a being cooled ion beam, then the damping times of the horizontal vertical and phase oscillations are:

\[ \tau_x = \tau_y = \frac{\tau_e}{1+D} = \frac{2\varepsilon}{P}, \]

where \( P \) is the average power of scattered radiation; \( \varepsilon \), ion energy; \( D \), saturation parameter.

The physics of radiative ion cooling is similar to synchrotron radiation damping. Friction causes the appearance of a reaction force on the emitting particle. Liouville’s theorem does not validate for such non-conservative system. At the same time in the case of non-selective interaction the Robinson’s damping criterion is valid: the sum of damping decrements \( \tau_{x,y,s}^{-1} \) is a constant. In a particular choice of lattice and laser or material target, damping rates can be shifted between different degrees of freedom. However the decrements are limited by the criterion by the value \( \tau_{x,y,s} \geq 2(1+D)\varepsilon/(3+D)P \) when all decrements are greater then or equal to zero [4, 5].

II. ENHANCED LASER COOLING

The method of enhanced resonance laser cooling of ion beams in the longitudinal plane is based on the Rayleigh scattering of ”monochromatic” laser photons by not fully stripped ion beams or by complicated nuclei when the radio-frequency (RF) system of the storage ring is switched off [6] - [10]. In this method the laser beam overlaps the ion beam and has a chirp of frequency. Ions interact with the laser beam at resonance energy, decrease their energy in the process of the laser frequency scanning until all of them reach the minimum energy of ions in the beam. At this frequency the laser beam is switched off. The higher energy of ions the earlier they begin interaction with the laser beam, the longer the time of interaction. Ions of minimum energy do not interact with the laser beam at all. In such a way, in this method, the selective interaction is realized. The damping time of the ion beam in the longitudinal plane is determined by the dispersion \( \sigma_z \) of its energy spread

\[ \tau_e = \frac{2\sigma_z}{P}. \]

The damping time (2) is \( \varepsilon/\sigma_z \sim 10^3 \) times less than (1). At the same time the transverse decrements are zero. From this it follows that the selectivity of interaction leads to the violation of the Robinson’s damping criterion and open the possibility of the enhanced cooling.

Below we will consider the non-resonance methods of enhanced cooling of ion beams in the longitudinal and transverse planes. These methods can be used for cooling of another particles as well. A universal kind of selective interaction of particles with moving in the radial direction laser or media targets will be used. By analogy with the resonance laser cooling the interaction region in this case change its radial position in the being cooled beam for the cooling process when the dispersion function of the storage ring is not equal to zero at the interaction region. Robinson’s damping criterion does not work in this case as well (see Appendix).

A. New enhanced cooling methods

For the sake of simplicity we will neglect the emission of the synchrotron radiation by ions in the bending magnets of a storage ring, supposing that the RF system of the ring is switched off and the broadband laser beams (or material targets) are homogeneous and have sharp edges in the radial directions. We suppose that the jump of instantaneous orbits of ions caused by the energy loss is less than the amplitude of their betatron oscillations.

In a smooth approximation, the motion of an ion relative to its instantaneous orbit is described by the equation \( x_{\beta} = A \cos(\Omega t + \varphi) \), where \( x_{\beta} = x - x_0 \) is the ion deviation from the instantaneous orbit \( x_0 ; \ x \), its radial coordinate; \( A \) and \( \Omega \), the amplitude and the frequency of betatron oscillations. If the coordinate \( x_{\beta 0} \) and transverse radial velocity of the ion \( \dot{x}_{\beta 0} = -A\Omega \sin(\Omega t_0 + \varphi) \) correspond to the moment \( t_0 \) of change of the ion energy in a laser beam then the amplitude of betatron oscillations of the ion before an interaction is \( A_0 = \)
\[ \sqrt{x_0^2 + \dot{x}_0^2 / \Omega^2} \]. After the interaction, the position of the ion instantaneous orbit will be changed by a value \( \delta x \), the deviation of the ion relative to the new orbit will be \( x_0 - \delta x \), and the direction of the electron velocity will not be changed. The new amplitude will be

\[ A_1 = \sqrt{(x_0 - \delta x)^2 + x_0^2 / \Omega^2} \]

and the change of the square of the amplitude

\[ \delta(A)^2 = A_1^2 - A_0^2 = -2x_0\delta x + (\delta x)^2. \quad (3) \]

When \( |\delta x| < |x_0| < A_0 \) then in the first approximation the value \( \delta A = -x_0 \delta x \). From this it follows that to produce the enhanced cooling of an ion beam in the transverse plane we must create such conditions when ions interact with a laser beam under deviations from the instantaneous orbit \( x_0 \) of one sign. In this case the value \( \delta A \) has one sign and the rate of change of amplitudes of betatron oscillations of ions is maximum. A selective interaction of ions with the laser beam is necessary to realize this case.

\[ \sqrt{x_0^2 + \dot{x}_0^2 / \Omega^2} \]. After the interaction, the position of the ion instantaneous orbit will be changed by a value \( \delta x \), the deviation of the ion relative to the new orbit will be \( x_0 - \delta x \), and the direction of the electron velocity will not be changed. The new amplitude will be

\[ A_1 = \sqrt{(x_0 - \delta x)^2 + x_0^2 / \Omega^2} \]

and the change of the square of the amplitude

\[ \delta(A)^2 = A_1^2 - A_0^2 = -2x_0\delta x + (\delta x)^2. \quad (3) \]

When \( |\delta x| < |x_0| < A_0 \) then in the first approximation the value \( \delta A = -x_0 \delta x \). From this it follows that to produce the enhanced cooling of an ion beam in the transverse plane we must create such conditions when ions interact with a laser beam under deviations from the instantaneous orbit \( x_0 \) of one sign. In this case the value \( \delta A \) has one sign and the rate of change of amplitudes of betatron oscillations of ions is maximum. A selective interaction of ions with the laser beam is necessary to realize this case.

The degree of overlapping is changed by moving uniformly the laser beam position from inside in the direction of the being cooled ion beam with some velocity \( v_{r1} \). When the laser beam reaches the instantaneous orbit corresponding to ions of maximum energies then the laser beam must be switched off and returned to a previous position. All ions of the beam will have small amplitudes of betatron oscillations and increased energy spread. Ions with high amplitudes of betatron oscillations will start to interact with a laser beam first, their duration of interaction and absolute decrease of amplitudes of betatron oscillations will be higher.

To realize the enhanced cooling of an ion beam in the longitudinal plane we can use a broadband laser beam \( T_2 \) located in the region of a storage ring with non zero dispersion function. The radial laser beam position is moving uniformly from outside in the direction of the being cooled ion beam with a velocity \( v_{r2} \) higher than maximum velocity \( \dot{x}_{1,m} \) of the ion instantaneous orbit deepened in the laser beam. At the initial moment, the laser beam overlaps only a small part of the ion beam. The degree of overlapping is changed in such a way that ions of maximum energy, first and then ions of lesser energy, come into interaction. When the laser beam reaches the orbit of ions of minimum energy then it must be switched off and returned to the previous position. In this case, the rate of the energy loss of ions in the beam will not be increased, but the difference in duration of interaction and hence in the energy losses of ions having maximum and minimum energies will be increased essentially. As a result all ions will be gathered at the minimum energy in a short time.

---

1 Instantaneous orbits can be moved in the direction of the laser beam, instead of moving of a laser beam. A kick, decreasing of the value of the magnetic field in bending magnets of the storage ring, a phase displacement or eddy electric fields can be used for this purpose.
B. Interaction of ion beams with transversely moving laser beams

In the methods of enhanced laser cooling of ion beams the internal and external laser beam positions are displaced in the transverse directions. Below the evolution of amplitudes of betatron oscillations and positions of instantaneous orbits in the process of the energy loss of ions in laser beams will be analyzed.

The velocity of an ion instantaneous orbit \( \dot{x}_n \) depends on the distance \( x_{T_2} - x_n \) between the edge of the laser beam and the instantaneous orbit, and on the amplitude of betatron oscillations. When the orbit enters the laser beam at the depth higher than the amplitude of betatron oscillations then ions interact with the laser beam every turn and their velocity reaches the maximum value \( \dot{x}_{n_{in}} \) which is given by the intensity and the length of the interaction region of the ion and laser beams. In the general case, the velocity \( \dot{x}_n \) can be presented in the form \( \dot{x}_n = W \cdot \dot{x}_{n_{in}} \), where \( W \) is the probability of an ion crossing the laser beam. \( W \) is the ratio to a period of a part of the period of betatron oscillations of the ion determined by the condition \( |x_{T_2} - x_n| \leq |x_0| \leq A \) when the deviation of the ion from the instantaneous orbit is directed to the laser beam and is greater than the distance between the orbit and the laser beam. Probability can be presented in the form \( W = \varphi_{1.2}/\pi \), where \( \varphi_1 = \pi - \arccos \xi_1, \varphi_2 = \arccos \xi_2, \xi_1 = (x_{T_2} - x_n)/A, \) indices 1,2 correspond to laser beams.

The behavior of the amplitudes of betatron oscillations of ions, according to (3), is determined by the equation \( \partial A/\partial x_n = -<x_{30}>/A \), where \( <x_{30}> > 0 \) is the ion deviation from the instantaneous orbit averaged through the range of phases 2\( \varphi_{1.2} \) of betatron oscillations where ions cross the laser beam. The value \( <x_{30}> := \pm \text{sinc} \varphi_{1.2}, \) where \( \text{sinc} \varphi_{1.2} = \sin \varphi_{1.2}/\varphi_{1.2}, \) signs + and − are related to the first and second laser beams. Thus the cooling processes are determined by the system of equations

\[
\frac{\partial A}{\partial x_n} = \pm \text{sinc} \varphi_{1.2}, \quad \frac{\partial x_n}{\partial t} = \frac{\dot{x}_{n_{in}}}{\pi} \varphi_{1.2}.
\]  

(4)

From equations (4) and the expression \( \partial A/\partial x_n = [\partial A/\partial t]/[\partial x_n/\partial t] \) it follows:

\[
\frac{\partial A}{\partial t} = \frac{\dot{x}_{n_{in}}}{\pi} \sin \varphi_{1.2} = \frac{\dot{x}_{n_{in}}}{\pi} \sqrt{1 - \xi_1^2}.
\]  

(5)

Let the initial instantaneous ion orbits be distributed in a region \( \pm \sigma_{x,0} \) relative to the location of the middle instantaneous orbit \( x_n \), and the initial amplitudes of ion radial betatron oscillations \( A_0 \) be distributed in a region \( \sigma_{x,0} \) relative to their instantaneous orbits, where \( \sigma_{x,0} \) and \( \sigma_{x,0} \) are dispersions. The dispersion \( \sigma_{x,0} \) is determined by the initial energy spread \( \sigma_{x,0} \).

Suppose that the initial spread of amplitudes of betatron oscillations \( \sigma_{x,0} \) is identical for all instantaneous orbits of the beam. The velocities of the instantaneous orbits in a laser beam \( \dot{x}_{n_{in}} < 0 \), the transverse velocities of the laser beams \( v_{T_1} > 0, v_{T_2} < 0 \). Below we will use the relative radial velocities of the laser beam displacement \( k_{1,2} = v_{T_1}/\dot{x}_{n_{in}} \), where \( v_{T_1} = dx_{T_1}/dt \). In our case \( k_1 < 0, k_2 > 0 \).

From the definition of \( \xi_1 \) we have a relation \( x_n = x_{T_1} - \xi_1 A(\xi_1) \). The time derivative is \( \partial x_n/\partial t = v_{T_1} - [A + \xi_1(\partial A/\partial \xi_1)]\xi_1/\partial t \). Equating this value to the second term in (4) we will receive the time derivative

\[
\frac{\partial \xi_1}{\partial t} = \frac{\dot{x}_{n_{in}}}{\pi} \frac{\pi k_{1,2} - \varphi_{1.2}}{\xi_1 A(\xi_1) + \xi_1(\partial A/\partial \xi_1)}.
\]  

(6)

Using this equation we can transform the first value in (4) to the form \( \pm \text{sinc} \varphi_{1.2}(\xi_1) = (\partial A/\partial \xi_1)(\partial \xi_1/\partial t)/(\partial x_n/\partial t) = (\pi k_{1,2} - \varphi_{1.2})(\partial A/\partial \xi_1)/[A + \xi_1(\partial A/\partial \xi_1)] \), \( \varphi_{1.2} \) which can be transformed to \( \partial \ln A/\partial \xi_1 = \pm \sin \varphi_{1.2}/\pi k_{1,2} - (\varphi_{1.2} \pm \xi_1 \sin \varphi_{1.2}) \). The solution of this equation is

\[
A = A_0 \exp \int_{\xi_1}^{\xi_1} \frac{\pm \sin \varphi_{1.2} d\xi_1}{\pi k_{1,2} - (\varphi_{1.2} \pm \xi_1 \sin \varphi_{1.2})},
\]  

(7)

where the index 0 correspond to the initial time. Substituting the values \( A \) and \( \partial A/\partial \xi_1 \) determined by (7) in (6) we find the relation between time of observation and parameter \( \xi_1 \)

\[
\frac{t - t_0}{\pi A_0} \psi(k_{1.2}, \xi_1) = \int_{\xi_1}^{\xi_1} A_0/\pi k_{1,2} - (\varphi_{1.2} \pm \xi_1 \sin \varphi_{1.2}) d\xi_1.
\]  

(8)

The equations (8) determine the time dependence of the functions \( \xi_1(t - t_0) \). The dependence of the amplitudes \( A(\xi_1(t - t_0)) \) is determined by the equation (7) through the functions \( \xi_1(t - t_0) \) in a parametric form. The dependence of the position of the instantaneous orbit follows from the definition of \( \xi_1 \)

\[
x_n(t - t_0) = x_{T_1} + v_{T_1}(t - t_0) - A(\xi_1(t - t_0)) \cdot \xi_1(t - t_0).
\]  

(9)

The function \( \psi(k_2, \xi_2) \) for the case \( k_2 > 0 \) according to (8) can be presented in the form

\[
\psi(k_2, \xi_2) = \int_{\xi_2}^{1} dx \exp \int_{x_1}^{x_2} \frac{\sqrt{1 - t^2}/(\pi k_2 - \arccos t + t \sqrt{1 - t^2} / \pi k_2 - \arccos x + x \sqrt{1 - x^2})}{dt}.
\]  

(10)

The instantaneous orbits of ions having initial amplitudes of betatron oscillations \( A_0 \) will be deepened into
the laser beam to the depth greater than their final amplitudes of betatron oscillations \( A_f \) at a moment \( t_f \). According to (8), \( t_f = t_0 + \pi A_0 \psi(k_2, \xi_2, f)/|x_{fin}| \), where \( \xi_{2,f} = \xi(t_f) = 1 \). During the interval \( t_f - t_0 \) the laser beam \( T_2 \) will pass a way \( l_f = |\psi(T_2)| (t_f - t_0) = \pi k_2 \psi(k_2, \xi_2, f) A_0 \). The dependence \( \psi(k_2, \xi_2) \) determined by (10) is presented in Table 1.

### Table 1

| \( k_2 \) | 1.0 | 1.02 | 1.03 | 1.05 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.7 | 2.0 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \psi \) | 13.8 | 9.90 | 6.52 | 3.71 | 2.10 | 1.51 | 1.18 | 0.98 | 0.753 | 0.538 |

Numerical calculations of the dependence \( \psi(k_2, \xi_2) \) on \( \xi_2 \) for the cases \( k_2 = 1.0, k_2 = 1.1 \) and \( k_2 = 1.5 \) are presented in Tables 2, 3, and 4, respectively. It can be presented in the next approximate form

\[
\psi(k_2, \xi_2) \simeq C_3(k_2) \psi\left(\frac{1 - \xi_2}{k_2 + \xi_2}\right),
\]

where \( C_3(k_2) \simeq 0.492 - 0.680(k_2 - 1)^2 + 0.484(k_2 - 1)^3 + \ldots \), \( \psi((1 - \xi_2)/(k_2 + \xi_2))_{k_2=1} \simeq (1 - \xi_2)/(1 + \xi_2) \).

#### Table 2 (\( k_2 = 1.0 \))

| \( \xi_2 \) | 0.1 | 0.5 | 0.2 | 0 | -0.2 | -0.5 | -0.8 | -0.9 | -1.0 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \psi \) | 0.162 | 0.341 | 0.492 | 0.716 | 1.393 | 4.388 | 10.187 | -∞ |

#### Table 3 (\( k_2 = 1.1 \))

| \( \xi_2 \) | 0.1 | 0.5 | 0.2 | 0 | -0.2 | -0.5 | -0.8 | -0.9 | -1.0 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \psi \) | 0.163 | 0.360 | 0.423 | 0.599 | 1.033 | 2.076 | 2.759 | 3.740 |

#### Table 4 (\( k_2 = 1.5 \))

| \( \xi_2 \) | 0.1 | 0.5 | 0.2 | 0 | -0.2 | -0.5 | -0.8 | -0.9 | -1.0 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \psi \) | 0.116 | 0.202 | 0.273 | 0.353 | 0.466 | 0.602 | 0.772 | 0.980 |

The time of the laser beam cooling and the final total radial dimension of the beam are equal to

\[
\tau_{x,1} \simeq \frac{\sigma_{x,0}}{v_{T_1} \sigma_{x,f} |k_1|}, \quad \sigma_{x,f} |k_1| \ll 1 \simeq \frac{\sigma_{x,0}}{|k_1|} + \sigma_{x,\tau,0},
\]

where \( \sigma_{x,0} = \sigma_{x,b,0} + \sigma_{x,x,0} \) is the total initial radial dimension of the ion beam. For the time \( \tau_{x,1} \) the instantaneous orbits of ions of a beam having minimum energy and maximum amplitudes of betatron oscillations at \( |k_1| \ll 1 \) pass the distance \( |\dot{x}_{fin}| \tau_{x,1} \).

According to (13) and (14) the enhanced transverse laser cooling can lead to an appreciable degree of cooling of ion beams in the transverse plane and a much greater degree of heating in the longitudinal one.

### D. The enhanced longitudinal laser cooling

In the method of the enhanced longitudinal laser cooling of ion beams a laser beam \( T_2 \) is located in the region \((x_{T_2}, x_{T_2} + a)\), where \( a \) is the laser beam width. The degree of the transverse cooling of the ion beam is determined by (7). The final amplitude in this case can be presented in the form

\[
A_f = A_0 \exp \left( \int_{\xi_{1,0}}^{\xi_{1,f}} \frac{\sqrt{1 - \xi_1^2} d\xi_1}{\pi k_1 - \pi + \arccos \xi_1 - \xi_1 \sqrt{1 - \xi_1^2}} \right).
\]

The numerical calculations of the dependence of the ratio \( A_f/A_0 \) on the relative radial velocity \( k_1 \) of the laser beam displacement are presented in Fig.2 and Table 5 for the case \( \xi_{1,0} = -1, \xi_{1,f} = 1 \). This dependence can be presented by the approximate expression

\[
A_f \simeq A_0 \sqrt{\left| \frac{k_1}{|k_1| + 1} \right|}.
\]
the working region of the storage ring in the direction of a being cooled ion beam. The instantaneous orbits of ions will go in the same direction with a velocity \(|\dot{x}_{in}| \leq |\dot{x}_{in}|\) beginning from the moment of their first interaction with the laser beam. When the laser beam reaches the instantaneous orbit of ions having minimum initial energies it must be removed to the initial position.

The law of change of the amplitudes of ion betatron oscillations is determined by (7), which can be presented in the form

\[
A = A_0\exp\int_{\xi_2,0}^{\xi_2} \frac{-\sqrt{1-\xi_2^2}d\xi_2}{\pi k_2 - \arccos\xi_2 + \xi_2\sqrt{1 - \xi_2^2}}. \tag{15}
\]

![Graph](image)

**FIG. 3: The dependence of the ratio \(A_f/A_0\) on \(k_2\).**

| \(k_2\)  | 1.0001  | 1.001  | 1.01   | 1.1    | 1.5    | 2.0    |
|----------|---------|--------|--------|--------|--------|--------|
| \(A_f/A_0\) | 100.0005 | 31.64  | 10.04  | 3.32  | 1.73  | 1.414  |

The dependence of the ratio of a final amplitude of ion betatron oscillations \(A_f = A(\xi_2 = 1)\) to the initial one on the relative velocity \(k_2\) of the second laser beam is presented in Fig.3 and Table 6. This ratio can be presented by the next approximate expression

\[
A_f \simeq A_0\sqrt{\frac{k_2}{k_2-1}}. \tag{16}
\]

The evolution of instantaneous orbits of ions interacting with the laser beam depends on the initial amplitudes of betatron oscillations of these ions. First of all the laser beam \(T_2\) interacts with ions having the largest initial amplitudes of betatron oscillations \(A_0 = \sigma_{x,b,0}\) and the highest energies. The instantaneous orbit of these ions, according to (7) - (9), is changed by the law \(x_{n_2} = x_{T_2,0} + \nu_{T_2}(t - t_{0_1}) - \xi_2\sigma_{x,b}(\xi_2)\) up to the time \(t = t_f\), where \(t_{0_1}\) is the initial time of interaction of ions with the laser beam. At the same time instantaneous orbits \(x_{n_2}\) of ions having the same maximum energy but zero amplitudes of betatron oscillations are at rest up to the moment \(t_{0_2} = t_{0_1} + \sigma_{x,b,0}/|v_{T_2}|\). The orbit \(x_{n_1}\) is displaced relative to the orbit \(x_{n_2}\) by the distance \(\Delta x_{n_1-2} = (x_{n_1} - x_{n_2})\). At the moment \(t_{0_2}\), when \(x_{n_2} = x_{T_2}\), this distance reaches the minimum

\[
\Delta x_{n_1-2}(t_{0_2}) = -\xi_2(t_{0_2}) \cdot \sigma_{x,b}(t_{0_2}) < 0, \tag{17}
\]

where the parameter \(\xi_2(t_{0_2})\), according to (8) and the condition \(|v_{T_2}|(t_{0_2} - t_{0_1}) = \sigma_{x,b,0}\), will be determined by the equation \(\psi[k_2, \xi_2(t_{0_2})] = 1/\pi k_2\). The value \(\psi[k_2, \xi_2(t_{0_2})]|_{t_{0_2} \geq 1} \approx 1/\pi, \xi_2(k_2, t_{0_2})|_{t_{0_2} \geq 1} \approx 0.22\) (see Tables 2-4), \(\sigma_{x,b}(t_{0_2}) = 1.26\sigma_{x,b,0}\) and the distance \(\Delta x_{n_1-2}(t_{0_2}) \approx 0.28\sigma_{x,b,0}\). This distance is decreased with increasing \(k_2\).

The instantaneous orbit of particles \(x_{n_2}\) inside the interval \(t_{0_2} < t \leq t_f\) is changed by the law \(x_{n_2} = x_{T_2,0} - \sigma_{x,b,0} + \dot{x}_{n_1}(t - t_{0_1} + \sigma_{x,b,0}/|v_{T_2}|)\) and the distance

\[
\Delta x_{n_1-2} = \frac{(k_2 - 1)}{k_2}[\sigma_{x,b,0} + v_{T_2}(t - t_{0_1})] - \xi_2\sigma_{x,b}(\xi_2) =
\]

\[
- \left[\frac{k_2 - 1}{k_2}\left(\frac{v_{T_2}}{\sigma_{x,b,0}} - 1\right) + \xi_2\sigma_{x,b,0}\right] \phi(\xi_2, \xi_2) \tag{18}
\]

where \(D_2 = D_2(k_2, \xi_2) = \sigma_{x,b}/\sigma_{x,b,0} = A/A_0\), \(t_{T_2} = x_{T_2} - x_{T_2,0} = \pi k_2|v_{T_2}|\sigma_{x,b,0} \leq l_f\) is the displacement of the laser beam. The typical dependence \(D_2\) defined by (15) is presented in Fig.4.

![Graph](image)

**FIG. 4: The dependence on \(\xi_2\) of the ratio of a current amplitude of betatron oscillations to an initial amplitude \(D_2 = A/A_0 = \sigma_{x,b}/\sigma_{x,b,0}\).**

When \(t > t_f\) then the value \(\xi_2 = \xi_{2,f} = -1, l_{T_2} = l_f, D_2 = \sqrt{k_2/(k_2-1)}\) and (18) have the maximum
The instantaneous orbit \( x_{n_2} \) will be at a distance \( x_{n_2-3} = ((k_2 - 1)/k_2)\sigma_{x,\eta,0} \) from the motionless instantaneous orbit \( x_{n_3} \) of ions having minimum energy and zero amplitudes of betatron oscillations when the laser beam is stopped at the position \( x_{n_3} \).

If we take into account that the instantaneous orbits of ions having maximum amplitudes of betatron oscillations and minimum energy, by the value 0.28\( \sigma_{x,b,0} \), at the moment of the laser beam stopping then the total radial dispersion of the instantaneous orbits of the beam can be presented in the form

\[
\sigma_{x,\varepsilon,f} \leq \frac{k_2 - 1}{k_2} \sigma_{x,0} + \left[ \sqrt{\frac{k_2}{k_2 - 1}} - \pi(k_2 - 1)\psi(k_2, \xi_2,f) \right]
\]

\[
+ 0.28\sigma_{x,b,0}, \quad A_{T_2} > I_f, \sigma_{x,0}. \tag{20}
\]

The damping time of the ion beam in the longitudinal plane \( \tau_c = 2\sigma_{x,0}(1 + \sigma_{x,b,0}/\sigma_{x,\varepsilon,0})/P \).

According to (20) the efficiency of the enhanced longitudinal laser cooling is the higher the less the ratio of the spread of the initial amplitudes of betatron oscillations to the spread of the instantaneous orbits of the being cooled ion beam.

According to (16) and (20) the enhanced longitudinal laser cooling can lead to a high degree of cooling of ion beams in the longitudinal plane and a much lesser degree of heating in the transverse one.

\[\Delta x_{n_2-1}|_{t>t_f} = \left(\frac{k_2 - 1}{k_2}\right) + \sqrt{\frac{k_2}{k_2 - 1}} \]

\[\pi(k_2 - 1)\psi(k_2, \xi_2,f)\sigma_{x,b,0}. \tag{19}\]

The instantaneous orbit \( x_{n_2} \) can be presented in the form

\[
\Delta x_{n_2-1}|_{t>t_f} = \left(\frac{k_2 - 1}{k_2}\right) + \sqrt{\frac{k_2}{k_2 - 1}} \]

\[\pi(k_2 - 1)\psi(k_2, \xi_2,f)\sigma_{x,b,0}. \tag{19}\]

The instantaneous orbit \( x_{n_2} \) can be presented in the form

\[
\Delta x_{n_2-1}|_{t>t_f} = \left(\frac{k_2 - 1}{k_2}\right) + \sqrt{\frac{k_2}{k_2 - 1}} \]

\[\pi(k_2 - 1)\psi(k_2, \xi_2,f)\sigma_{x,b,0}. \tag{19}\]

To cool an ion beam both in the transverse and longitudinal planes we must look for combinations of enhanced nonresonance methods of cooling with other methods.

In the nonresonance method of enhanced longitudinal laser cooling, contrary to the transverse one, the degree of longitudinal cooling is much greater than the degree of heating in the transverse plane. That is why we can use the emittance exchange between longitudinal and transverse planes when the RF system is switched on and a synchro-betatron resonance \([3] - [10]\) or dispersion coupling by additional motionless wedge-shaped targets \([1, 2, 3]\) are used together with the moving there and back target \( T_2 \). In such a way the enhanced two-dimensional cooling of the ion beam based on the longitudinal laser cooling can be realized\(^2\). In this case the energy losses of ions in the laser beam per turn must be higher than the maximum energy gain in the RF system and the motion of the target \( T_2 \) must be limited by the region \( x_{T_2} > 0 \). Cooling of particle beams in the RF buckets is another problem to be considered elsewhere.

The enhanced nonresonance transverse method of laser cooling together with the enhanced resonance longitudinal one can be used for cooling of ion beams when the RF system is switched off. A heating of the ion beam in the longitudinal plane in the process of the enhanced transverse cooling will be compensated completely by its following cooling in the longitudinal plane.

\[\text{III. DISCUSSION}\]

In the nonresonance method of the enhanced transverse laser cooling of ion beams, according to (13) and (14), the degree of decrease of betatron oscillations \( C_1 = \sigma_{x,b,0}/\sigma_{x,b,f} = A_0/A = \sqrt{(1 + |k_1|)/|k_1|} \) and the degree of increase of the spread of the instantaneous orbits of the beam is much greater: \( D_1 \approx C_1^2 \). In the nonresonance method of the enhanced longitudinal laser cooling, according to (16) and (20), there is a significant decrease in the spread of instantaneous orbits of ions \( C_2 = \sigma_{x,e,0}/\sigma_{x,e,f} \), and a much lesser value of increase in the amplitudes of betatron oscillations: \( D_2 \approx \sqrt{C_2} \). From this it follows that cooling of ion beams both in the transverse and longitudinal planes, in turn, does not lead to their total cooling in these planes. We can cool ion beams either in the transverse or longitudinal planes.

\[\text{IV. APPENDIX}\]

We start by reviewing very briefly the derivation of Robinson’s damping criterion \([3]\).

The general method of describing the motion of a particle in a circular accelerator is to determine an equilibrium orbit, and then analyze small deviations from this orbit as a linear combinations of normal modes of oscillation. The characteristics of the modes are determined by solving for the principal values of the matrix. If the particle motion is stable such that the particle oscillate about the equilibrium position, and since the transfer matrix is real, the principal values will be three pairs of complex conjugate numbers, which determine the frequencies and damping rates of the oscillation modes.

To find damping rates K.W.Robinson considered an element of the accelerator of infinitesimal length and calculated the six order transfer matrix for this element. This matrix has infinitesimal nondiagonal terms which are first order in the length of the element, and the diagonal terms differ from unity by a quantity which is

\(^2\) For the wedge-shaped target the fast cooling effect in one direction is equal to near the same heating effect in the other one. That is why the combination of the fast emittance exchange with the enhanced longitudinal cooling can lead to the two-dimensional enhanced cooling effect. Cooling of muon beams by material targets can be done similar way.
proportional to the infinitesimal length of the element. In order to determine damping, the determinant of the transfer matrix of the infinitesimal element was evaluated. The only terms in the determinant which are first order in the length of the element are due to the diagonal terms of the matrix. The determinant of the transfer matrix is given by \(1 + \Sigma \delta_{nn}\). The diagonal terms for \(x, y, z\) are zero as changes in \(x, y\) are only related to \(x', y'\) and changes in \(z\) related to \(x\). In this case \(x, x', y, y'\) represent the variation of displacement and angular deviation in the radial and vertical planes, \(\Delta \varepsilon\) and \(z\) represent the variation in energy and azimuthal position from the values of an equilibrium particle, as measured at the time the particle transverses the infinitesimal element.

The diagonal term for \(\Delta \varepsilon\) was determined from the characteristics of the radiation loss \(P_s \sim E^2 B^2\).

\[
P_s = P_{s, \gamma}(1 + 2\Delta B/B + 2\Delta \varepsilon/\varepsilon), \quad (21)
\]

where \(B\) is only function of position. The diagonal term for \(\Delta \varepsilon\) due to radiation loss is \(1 - 2\delta \varepsilon_{loss}/\varepsilon_s\) with \(\delta \varepsilon_{loss}\) the radiation loss for an ideal particle in the infinitesimal element. The energy gain from the RF system is not dependent on \(\Delta \varepsilon\) and contributes no change in the \(\Delta \varepsilon\) diagonal term.

The difference from unity of \(x'\) and \(y'\) diagonal terms is determined from the energy gain from the RF system and is unaffected by radiation loss. The energy increase due to the RF system add a momentum change parallel to the equilibrium orbit and will reduce the angular variation for the value \(\delta x' = -(\delta \varepsilon_{RF}/\varepsilon_s)x\) and the diagonal term for \(x'\) is \(1 - \delta \varepsilon_{RF}/\varepsilon_s\), with \(\delta \varepsilon_{RF}\) the energy gain from the RF system for an equilibrium particle. Similarly the diagonal element for \(y'\) is \(1 - \delta \varepsilon_{RF}/\varepsilon_s\). Then the determinant for the infinitesimal element is

\[
D^{inf} = 1 + \Sigma \delta_{nn} = 1 - 2\delta \varepsilon_{loss}/\varepsilon_s - 2\delta \varepsilon_{RF}/\varepsilon_s. \quad (22)
\]

The determinant of the transfer matrix for one complete period is the product of the transfer matrices of the infinitesimal elements of that period

\[
D = 1 + \Sigma \delta_{nn} = 1 - 2\varepsilon_{loss}/\varepsilon_s - 2\varepsilon_{RF}/\varepsilon_s, \quad (23)
\]

where \(\varepsilon_{loss}\) and \(\varepsilon_{RF}\) are the radiation loss and energy gain from the RF system in one period.

The characteristics of the principal modes of oscillation are determined by solving for the principal values of the transfer matrix for one complete period. If all modes are oscillatory the principal values will be of the form \(\exp \gamma\). Then \(D = 1 - 2\varepsilon_{loss}/\varepsilon_s - 2\varepsilon_{RF}/\varepsilon_s\) or \(\exp \Sigma 2\alpha = 1 - 2\varepsilon_{loss}/\varepsilon_s - 2\varepsilon_{RF}/\varepsilon_s\), where \(\gamma_i = \alpha_i \pm i\nu_i\), \(\alpha_i\) is the fractional damping of a mode in one period of the accelerator.

For equilibrium conditions the radiation loss is equal to the energy gain from the RF system for one complete period, \(\varepsilon_{loss} = \varepsilon_{RF}\). Then for \(\Sigma |\alpha|\alpha < 1\) \(= -2\varepsilon_{loss}/\varepsilon_s\).

From this it follows the Robinson’s damping criterion for the sum of the damping rates of the three modes of oscillation

\[
\Sigma \beta_i = -2P_{s, \gamma}/\varepsilon_s, \quad (24)
\]

where \(P_{s, \gamma}\) is the average rate of radiation loss, and the amplitude of an oscillation varies as \(\exp(\beta_i t)\).

When the vertical and radial oscillations are not coupled then damping decrements for vertical betatron and synchrotron oscillations can be derived lightly and the decrement for the radial betatron oscillations can be derived from the Robinson’s damping criterion. The decrement for the radial betatron oscillations can be derived directly as well.

Below we would like to pay attention on the limits of applicability of this criterion.

As a starting point for his proof K.W. Robinson took some assumptions. He wrote:

1. "For small deviations from the principal orbit, a transfer matrix for a complete period may be written relating initial to final deviations. This is usually done for radial and vertical displacements and velocities, and may be extended to sixth order transfer matrix relating initial and final vertical displacements and velocities, and also energy variation, and longitudinal displacement, from the values of a particle on the principal orbit. For this general transfer matrix, the complete periods of the accelerator are defined so as to be identical in both magnet structure and RF accelerating system."

2. "The diagonal term for \(\Delta \varepsilon\) may be determined from the characteristics of the radiation loss for small variations in \(E\) and \(B\) from the values for an ideal particle. \(B\) is only a function of position, ... " (see eq. (21)).

3. "If all modes are oscillatory the principal values will be of the form \(\exp \gamma_i\) with six values of \(\gamma_i\) being three pairs of complex conjugates."

4. "... This is a general result for any type of electron accelerator if the average electron energy is constant." (He has in view (24) for the result).

We can see that the Robinson’s damping criterion was received for the stationary conditions when the guiding magnetic field and the amplitude of the RF voltage does not depend on time, particle energy loss is a linear function of both the energy deviation of the particle \(\Delta \varepsilon\) and the magnetic field deviation \(\Delta B\) in the region occupied by the beam, all modes are oscillatory.

Linear dependence on \(\Delta \varepsilon\), \(\Delta B\) and stationary conditions are important for the exponential decay and the concept of decrement. The equations of motion in our case are the linear homogeneous Hill’s equations for the transverse degrees of freedom and the pendulum equation for the longitudinal one when we neglect coupling and losses. The losses lead to additional terms at the first order derivatives in the corresponding equations.

The violation of the Robinson’s assumptions can lead to the violation of his damping criterion. Simple examples can justify this statement.
Example 1. Laser cooling of ion beams in the longitudinal plane. The RF system of the storage ring is switched off. The homogeneous laser beam overlaps the ion beam. The chirp of the laser frequency is used.

In this example there is no oscillatory motion of ions in the longitudinal plane. The instantaneous orbits of ions which are at resonance with the laser beam are moving with a constant velocity to the motionless non-resonance orbit of ions having minimal energy. When the orbit of resonance ions reach the orbit of ions having minimal energy the laser beam is switched off. We have non-exponential low of bringing closer of resonance and non-resonance ions. The damping time is determined by a value (2), which is out of the Robinson’s damping criterion. The degree of cooling for this time can be very high and is determined by the natural line width of the laser beam and the maximal energy of scattered photons. Heating of the ion beam in the transverse plane will not appear when the homogeneous laser beam is used.

This is the simplest example of the violation of the Robinson’s damping criteria. Experiments confirm this conclusion [6] - [10].

We can use the broadband laser beam without chirp and with sharp frequency edges. Ions in this case will be gathered at the orbit corresponding to the greatest frequency of the laser beam.

The violation of the Robinson’s damping criterion in this example is the consequence of the non-linear dependence of the power of the emitted radiation on the deviation of the energy of interacting and non-interacting ions $\Delta \varepsilon$: ions of the minimal energy and less does not interact with the laser beam and keep their position. At the same time ions interacting with the laser beam have equal velocities. The rate of bringing closer of the energy of interacting and non-interacting ions is maximum. At the same time the heating in the transverse direction is absent.

Example 2. Laser cooling of ion beams in the longitudinal plane. The RF system of the storage ring is switched on. The homogeneous laser beam overlap the ion beam. The broadband laser beam without chirp and with sharp frequency edges is used. The lowest frequency of the laser beam corresponds to the equilibrium energy of ions. The power of radiation scattered by ions of the energy greater then equilibrium $\mathcal{P}_{\varepsilon>\varepsilon_s}$ is much higher than the maximum power which the ion can extract from the RF system then all ions of the energy higher then equilibrium one will be gathered at the equilibrium energy for a short time (much less then the period of phase oscillation). Then we can switch off the laser beam, wait a quarter of the period of phase oscillation, switch on laser beam again and wait for damping of the next part of the beam which appeared in the region of the energy $\varepsilon > \varepsilon_s$. Such beam manipulations we can repeat three times. The beam will be cooled in the bucket in the longitudinal plane. Heating of the ion beam in the transverse plane will not appear in this case.

In this paper we have considered another, more complicated schemes of cooling of particle beams based on non-linear interaction of moving target with the being cooled beam.

[1] E.G. Bessonov, Proc. Internat. Linear Accel. Conf. LINAC94, Tsukuba, KEK, 1994, Vol.2, pp.786; Journal of Russian Laser Research, 15, No 5, (1994), p.403.
[2] E.G. Bessonov and Kwang-Je Kim, Preprint LBL-37458 UC-414, 1995; Phys. Rev. Lett., 1996, vol.76, p.431.
[3] E.G. Bessonov, K.-J. Kim, Proc. 5th European Particle Accelerator Conference, Sitges, Barcelona, 10-14 June 1996, v.2, p. 1196.
[4] H. Wiedemann, Particle Accelerator Physics I and II (Springer-Verlag, New York, 1993).
[5] E.G. Bessonov, Physics/0202040.
[6] P.J. Channel, J. Appl. Physics, v. 52(6), p.3791 (1981).
[7] P.J. Channel, L.D. Selvo, R. Bonifacio, W. Barletta, Optics Communications, v.116, (1995), p.374.
[8] S. Shröder, R. Klein, N. Boos, et al., Phys. Rev. Lett., v. 64, No 24, p.2901 (1990).
[9] J.S. Hangst, M. Kristensen, J.S. Nielsen, O. Poulsen, J.P. Schiffer, P. Shi, Phys. Rev. Lett., v. 67, 1238 (1991).
[10] J.S. Hangst, K. Berg-Sorensen, P. S. Jessen, et al., Proc. IEEE Part. Accel. Conf., San Francisco, May 6-9, NY, 1991, v.3, p.1764.
[11] O’Neil G., Phys. Rev., 102, 1418 (1956);
[12] A. Shoch, Nucl. Instr. Meth, v. 11, p.40 (1961).
[13] K.W. Robinson, Phys. Rev., 1958, v.111, No 2, p.373.
[14] A. Hoffman, R. Little, J. Peterson, Proc. VI Int. Conf. High Energy Accel. Cambridge (Mass.), 1967, p.123.
[15] H. Okamoto, A.M. Sessler, and D. Möhl, Phys. Rev. Lett. 72, 3977 (1994).
[16] T. Kihara, H. Okamoto, Y. Iwashita, et al., Phys. Rev. E, v.59, No 3, p.3594, (1999).
[17] D.V. Neuffer, Nucl. Instr. Methods, 1994, v.A350, p.24.