Abstract. Consider a set of jobs connected to a directed acyclic task graph with a fixed source and sink. The edges of this graph model precedence constraints and the jobs have to be scheduled with respect to those. We introduce the Server Cloud Scheduling problem, in which the jobs have to be processed either on a single local machine or on one of many cloud machines. Both the source and the sink have to be scheduled on the local machine. For each job, processing times both on the server and in the cloud are given. Furthermore, for each edge in the task graph, a communication delay is included in the input and has to be taken into account if one of the two jobs is scheduled on the server, the other in the cloud. The server can process jobs sequentially, whereas the cloud can serve as many as needed in parallel, but incurs costs. We consider both makespan and cost minimization. The main results are an FPTAS with respect for the makespan objective for a fairly general case and strong hardness for the case with unit processing times and delays.

Keywords: Scheduling · Cloud · Precedence Constraints · Communication Delays · Approximation · NP-hardness.

1 Introduction

Scheduling with precedence constraints with the goal of makespan minimization is widely considered a fundamental problem. It has already been studied in the 1960s by Graham [6] and receives a lot of research attention up to this day (see e.g. [3, 4, 7]). One problem variant that has received particular attention recently, is the variant with communication delays (e.g. [1, 10, 12]). Another, more contemporary topic concerns scheduling using external resources like, for instance, machines from the cloud and several models in this context have been considered of late (e.g. [1, 10, 12]). In this paper, we introduce and study a model closely connected to both settings, where jobs with precedence constraints may...
either be processed on a single server machine or on one of many cloud machines. Here, communication delays may occur only if the computational setting is changed, the server and cloud machines may behave heterogeneously, i.e., jobs may have different processing times on the server and in the cloud, and scheduling in the cloud incurs costs proportional to the computational load performed in this context. Both makespan and cost minimization is considered. We believe that the present model provides a useful link between scheduling with precedence constraints and communication delays on the one hand and cloud scheduling on the other.

**Problem** We consider a scheduling problem $SCS$ in which a task graph $G = (\mathcal{J}, E)$ has to be scheduled on a combination of a local machine (server) and a limitless number of remote machines (cloud). The task graph is a directed, acyclic graph with exactly one source $S \in \mathcal{J}$ and exactly one sink $T \in \mathcal{J}$. Each job $j \in \mathcal{J}$ has a processing time on the server $p_s$ and on the cloud $p_c$. We consider $p_s(S) = p_s(T) = 0$ and $p_c(S) = p_c(T) = \infty$. For every other job the values of $p_s$ and $p_c$ can be arbitrary in $\mathbb{N}$, meaning that the server and the cloud are unrelated machines in our default model. An edge $e = (i,j)$ denotes precedence, job $i$ has to fully processed before job $j$ can start. Furthermore an edge $e = (i,j)$ has a communication delay of $c(i,j) \in \mathbb{N}$, which means, that after job $i$ finished, job $j$ has to wait an additional $c(i,j)$ time steps before it can start, if $i$ and $j$ are not both scheduled on the same type of machine (server or cloud).

A schedule $\pi$ is given as the tuple $(\mathcal{J}^*, \mathcal{J}^c, C)$. $\mathcal{J}^*$ and $\mathcal{J}^c$ are a proper partition of $\mathcal{J}$: $\mathcal{J}^* \cap \mathcal{J}^c = \emptyset$ and $\mathcal{J}^s \cup \mathcal{J}^c = \mathcal{J}$. $\mathcal{J}^s$ denotes jobs that are processed on the server in $\pi$, $\mathcal{J}^c$ respectively for the cloud. $C : \mathcal{J} \rightarrow \mathbb{N}$ maps jobs to their completion time.

We define a bit of notations before we formally define the validity of a schedule. Let $p^*(j)$ be equal to $p_s(j)$ iff $j \in \mathcal{J}^s$, and $p_c(j)$ iff $j \in \mathcal{J}^c$. $p^*(j)$ denotes the actual processing time of job $j$ in $\pi$. Let $E^* := \{(i,j) \in E | i \in \mathcal{J}^s \land j \in \mathcal{J}^c \lor (i \in \mathcal{J}^c \land j \in \mathcal{J}^s)\}$ be the set of edges between jobs on different machines. Intuitively, for all the edges in $E^*$ we have to take the communication delays into consideration, for all edges in $E \setminus E^*$ we only care about the precedence. We call a schedule $\pi$ valid if and only if the following conditions are met:

a) There is always at most one job processing on the server:

\[ \forall i \in \mathcal{J}^s, \forall j \in \mathcal{J}^c \setminus \{i\} : (C(i) \leq C(j) - p^*(j)) \lor (C(i) - p^*(i) \geq C(j)) \]

b) Tasks are not started before the previous tasks has been finished/ the required communication is done:

\[ \forall (i,j) \in E \setminus E^* : (C(i) \leq C(j) - p^*(j)) \]

\[ \forall (i,j) \in E^* : (C(i) + c(i,j) \leq C(j) - p^*(j)) \]

The makespan of a schedule is given by the completion time of the sink $T$: $C(T)$. The cost of a schedule is given by the time it spends processing tasks on the cloud: $\sum_{i \in \mathcal{J}^c} p^*(i)$. Note here, that by requiring $p_s(S) = p_s(T) = 0$ and $p_c(S) = p_c(T) = \infty$, we assume every job to start and end on the server. This is done only for convenience as it defines a clear start and end state for each schedule.
Naturally two different optimization problems arise from the definition. First, given a deadline $d$, find a schedule with lowest cost and $C(T) \leq d$. Second, given a cost budget $b$, find a schedule with smallest makespan and $\sum_{i \in J} p^c(i) \leq b$. In both instances the $d$, respectively the $b$, is strict. The natural decision variant is: given both $d$ and $b$ find a schedule that adheres to both, if one exists. Note here, that the decision variant is naturally easier (or at least as easy) as the optimization problems.

Remark 1. Instances of SCS might contain schedules with a makespan (and therefore cost) of 0. We can check for those in polynomial time: First, remove all edges with communication delay 0, we get a set of connected components $K$. Iff $\forall k \in K (\forall j \in k \ p_s(j) = 0) \lor (\forall j \in k \ p_c(j) = 0)$, then there is a schedule with makespan of 0. For the rest of the paper we will assume that our algorithms check that beforehand and are only interested in schedules with makespan $> 0$.

Results We start by establishing (weak) NP-hardness already for the case without communication delays and very simple task graphs. More precisely, for the case in which the task graph forms one chain starting with the source and ending with the sink and the case in which the graph is fully parallel, i.e., each job $j \in J \setminus \{S, T\}$ is only preceded by the source and succeeded by the sink. On the other hand, we establish FPTAS results for both the chain and fully parallel case with arbitrary communication delays and with respect to both objective functions. Furthermore, we present a 2-approximation for the case without delays and identical server and cloud machines ($p_c = p_s$) but arbitrary task graph and the makespan objective and show that the respective algorithm can also be used to solve the problem optimally with respect to both objectives in the case of unit processing times. These results are all relatively simple and are discussed in Section 2. In Section 3 we aim to generalize the FPTAS result for chain graphs regarding the makespan as much as possible. We are able to show that an FPTAS can be achieved as long as the maximum cardinality source and sink dividing cut $\psi$ is constant. Intuitively, this parameter upper bounds the number of edges that have to be considered together in a dynamic program and in many relevant problem variants it can be bounded or replaced by the longest anti-chain length. We provide a formal definition in Section 3. Next, we turn our attention to strong NP-hardness results in Section 4. We are able to show, that a classical reduction due to Lenstra and Rinnooy Kan can be adapted to prove NP-hardness already for the variant of SCS without communication delays and processing times equal to one or two. Now, in the case of unit processing times we know that problem can be solved in polynomial time, and hence we are interested in the case with unit processing times and communication delays. We design an intricate reduction to show that this very basic case is NP-hard as well. Note that in this setting the server and cloud machines are implicitly identical. Furthermore, we are able to show that a slight variation of this reduction implies that no constant approximation with respect to the cost objective can be achieved regarding the general problem. Lastly, in Section 5, we consider approximation algorithms for the case with unit processing times and delays. We
show that a relatively simple approach yields a \( \frac{1+\varepsilon}{2\varepsilon} \)-approximation for \( \varepsilon \in (0, 1] \) regarding the cost objective if we allow a makespan of \((1+\varepsilon)d\).

Table 1: An overview of the results of this paper.

| Algorithmic Results | Hardness Results |
|---------------------|------------------|
| fully parallel or chain task graph | fully parallel or chain task graph, \( c = 0 \) |
| task graph with constant \( \psi \) | \( \forall j \in J : c(j) = 0, p_c(j), p_s(j) \in \{1, 2\} \) |
| \( c = 0, p_c = p_s \) (no delays, identical machines) | \( c = p_c = p_s = 1 \) (unit delays, unit sizes) |
| \( 2 \)-approximation w.r.t. makespan | no constant approximation w.r.t. cost |
| \( c = 0, p_c = p_s = 1 \) | (weakly) NP-hard |
| \( \frac{1+\varepsilon}{2\varepsilon} \)-approximation w.r.t. cost with makespan at most \((1+\varepsilon)d\) | \( (\text{strongly}) \) NP-hard |
| polynomial w.r.t. makespan and cost | (strongly) NP-hard |
| FPTAS w.r.t. cost and makespan | general problem |

Related Work Probably the closest related model to the one considered in this paper was studied by Aba et al. \[1\]. In this paper the input is very similar, however, in both computational settings an unbounded number of machines may be used and the goal is makespan minimization. The authors show NP-hardness on the one hand, and identify cases that can be solved in polynomial time on the other. In the conclusion of this paper a model very similar to the one studied in this work is mentioned as an interesting research direction. For a detailed discussion of related models, we refer to the preprint version of the above work \[2\].

The present model is closely related to the classical problem of makespan minimization on parallel machines with precedence constraints, where a set of jobs with processing times, a precedence relation on the jobs (or a task graph), and a set of \( m \) machines are given. The goal is to assign the jobs to starting times on machines such that the precedence constraints are met and the last job finishes as soon as possible. In the 1960’s, Graham \[6\] introduced the list scheduling heuristic for this problem and proved it to be a \( (2 - \frac{1}{m}) \)-approximation. Interestingly, to date, this is essentially the best result for the general problem. On the other hand, Lenstra and Rinnooy Kan \[8\] showed that no better than \( \frac{4}{3} \) approximation can be achieved for the problem with unit processing times, unless \( P=NP \). In more recent days, there has been a series of exciting new results for this problem starting with a paper by Svensson \[13\] who showed that no better than \( 2 \)-approximation can be hoped for assuming a variant of the unique games conjecture. Furthermore, Levey and Rothvoss \[9\] presented an approximation scheme with nearly quasi-polynomial running time for the variant with unit
processing times and a constant number of machines, and Garg \cite{5} improved the running time to quasi-polynomial shortly thereafter. These results utilized so called LP-hierarchies to strengthen linear programming relaxations of the problems. This basic approach has been further explored in a series of subsequent works, e.g. \cite{3, 4, 7}, which in particular also investigate the problem variant where a communication delay is incurred for pairs of precedence-constrained jobs running on different machines. The latter problem variant is closely related to our setting as well.

Lastly, there is at least a conceptual relationship to problems where jobs are to be executed in the cloud. For example, a problem was considered by Saha \cite{12} in which cloud machines have to be rented in fixed time blocks in order to schedule a set of jobs with release dates and deadlines minimizing the costs which are proportional to the rented time blocks. Another example is a work by Mäcker et al. \cite{10} in machines of different types can be rented from the cloud and machine dependent setup times have to be payed before they can be used. Jobs arrive in an online fashion and the goal is again cost minimization. Both papers reference further work in this context.

2 Preliminary Results

In this section we collect some results that can be considered low hanging fruits and give a first overview concerning the complexity and approximability of our problem. In particular, we show weak NP-hardness already for cases with very simple task graphs and without communication delays. Furthermore, we discuss complementing FPTAS results and a 2-approximation for the case with identical cloud and server machines and without communication delays.

2.1 Hardness

We show that SCS is NP-hard even for two very simple types of taskgraphs and in a case where every communication time is 0. For both of these reductions we use the decision variant of the problem: given both \(d\) and \(b\), find a schedule that fulfills both. Naturally this will show the hardness of both the cost minimization as well as the makespan minimization problem. We start by reducing the decision version of knapsack to SCS with a chain graph as its task graph. The knapsack problem is given as a capacity \(C\), a value threshold \(V\) and a set of items \{1, \ldots, n\} with weights \(w_i\) and value \(v_i\).

The question is, if is there a subset of items \(S\) such that \(\sum_{i \in S} w_i \leq C\) and \(\sum_{i \in S} v_i \geq V\). We create the respective SCS problem as follows. For every item \(i \in \{1, \ldots, n\}\) create a task with \(p_s(i) = w_i + v_i\) and \(p_e(i) = v_i\). Consider a task graph with those tasks as a chain (in an arbitrary order) and each resulting edge \((i, j)\) has \(c(i, j) = 0\). We set the deadline to \(d = \sum_{1 \leq i \leq n} v_i + C\) and the budget to \(b = \sum_{1 \leq i \leq n} v_i - V\). It is left to show, that there is a solution to the knapsack problem with value \(\geq V\) if and only if there is a schedule to our transformed problem with cost \(\leq b\). Basically we show that there is a one to one relation...
between our schedules and knapsack solutions. Assume some solution (subset of items $S$) for the knapsack problem with value $V'$. For each $i \in S$ we put the respective task in $J^s$ and the rest in $J^c$. Since the task graph is a chain we can compute a minimal completion time from this partition: $\sum_{1 \leq i \leq n} v_i + \sum_{i \in S} w_i$ which is smaller or equal to $d$ if and only if $\sum_{i \in S} w_i \leq C$. The cost for the schedule is equal to $\sum_{1 \leq i \leq n} v_i - V'$. Therefore, the cost for the schedule is smaller or equal to $b$ exactly when $V' \geq V$. It is easy to see that we can construct a knapsack solution from a schedule in a similar vein, therefore we conclude:

**Theorem 1.** The SCS problem is weakly NP-hard for chain graphs and without communication delays.

Secondly we look at problems with fully parallel task graphs, which means that every job $j$ besides $S$ and $T$ has exactly two edges: $(S, j)$ and $(j, T)$. Here we do a simple partition reduction. Given a set $S$ of natural numbers, the question is, if there is a partition into sets $S_1$ and $S_2$ such that $\sum_{i \in S_1} i = \sum_{i \in S_2} i$? For every element $i$ in $S$ we create a task with $p_s = p_c = i$, set $d = b = \frac{1}{2} \sum_{i \in S_1} i$. We arrange the tasks into a fully parallel task graph where each edge $(i, j)$ has $c(i, j) = 0$. Imagine a solution $S_1, S_2$ for the partition problem. We schedule every task related to an integer in $S_1$ on the server and every other task on the cloud. Since everything is fully parallel and there are no communication delays we can conclude a makespan of $\max\{\sum_{i \in S_1} i, \max_{i \in S_2} i\}$ and costs of $\sum_{i \in S_1} i$. This is a correct solution for the scheduling problem if and only if $\sum_{i \in S_1} i = \sum_{i \in S_2} i$. Again it is easy to see that an equivalent argument can be made for the other direction.

**Theorem 2.** The SCS problem is weakly NP-hard for fully parallel graphs and without communication delays.

### 2.2 Algorithms

In the following, we present complementing FPTAS results for the variants of SCS with fully parallel and chain task graphs. Furthermore, in both of the above reductions we did have no communication delays and in one of them the jobs had the same processing time on the server and the cloud. Hence, we take a closer look at this case as well and present a simple 2-approximation even for arbitrary task graphs and with respect to the makespan objective.

**Fully Parallel Case** We show that the variant of SCS with fully parallel task graph can be dealt with using straight-forward applications of well-known results and techniques. In particular, we can design two simple dynamic programs for the search version of the problem that consider for each job the two possibilities of scheduling them on the cloud or on the server and compute for each possible budget or deadline the lowest makespan or cost, respectively, that can be achieved with the jobs considered so far. These dynamic programs can then be combined with suitable rounding procedures that reduce the number of considered states and search procedures for approximate values for the optimal cost or makespan, respectively, yielding:
Theorem 3. There is an FPTAS for SCS with fully parallel task graph with respect to both the cost and the makespan objective.

Proof. We start by designing the dynamic programs for the search version of the problem with budget \( b \) and deadline \( d \). Without loss of generality, we assume \( \mathcal{J} = [n] \cup \{\mathcal{S}, \mathcal{T}\} \) and set \( c(j) = c(\mathcal{S}, j) + c(j, T) \).

For each deadline \( d' \in \{0, 1, \ldots, d\} \) and \( j \in \{0, 1, \ldots, n\} \), we want to compute the smallest cost \( C[j, d'] \) of all the schedules of the jobs \( 1, \ldots, j \) adhering to the deadline \( d' \) on the server. We may do so by setting \( C[0, d'] = 0 \) for each \( d' \) and considering the two possibilities of scheduling job \( j \) on the cloud or server. In particular, let \( C_1[j, d'] = C[j-1, d'] + p_c(j) \) if \( p_c(j) + c(j) \leq d \) and \( C_1[j, d'] = \infty \) otherwise, and, furthermore, \( C_2[j, d'] = C[j-1, d' - p_c(j)] \) if \( p_c(j) \leq d' \) and \( C_2[j, d'] = \infty \) otherwise. Then, we may set \( C[j, d'] = \min\{C_1[j, d'], C_2[j, d']\} \). Now, if \( C[n, d] > b \), we know that there is no feasible solution for the search version, and otherwise we can use backtracking starting from \( C[n, d] \) to find one. The time and space complexity is polynomial in \( d \) and \( n \).

In the second dynamic program, we compute the smallest makespan \( M[j, b'] \) of all the schedules of the jobs \( 1, \ldots, j \) adhering to the budget \( b' \), for each budget \( b' \in \{0, 1, \ldots, b\} \) and \( j \in \{0, 1, \ldots, n\} \). Again, we set \( M[0, b'] = 0 \) for each \( b' \) and consider the two possibilities of scheduling job \( j \) on the cloud or server. To that end, let \( M_1[j, b'] = M[j-1, b' - p_c(j)] \) if \( p_c(j) + c(j) \leq d \) and \( M_1[j, b'] = \infty \) otherwise, and, furthermore, \( M_2[j, b'] = M[j-1, b' - p_s(j)] \). Then, we may set \( M[j, b'] = \min\{M_1[j, b'], M_2[j, b']\} \). Again, if \( M[n, b] > d \), we know that there is no feasible solution for the search version, and otherwise we can use backtracking starting from \( M[n, b] \) to find one. The time and space complexity is polynomial in \( b \) and \( n \).

For both programs, we can use rounding and scaling approaches to trade the complexity dependence in \( d \) or \( b \) with a dependence in \( \text{poly}(n, \frac{1}{\varepsilon}) \) incurring a loss of a factor \( (1 + O(\varepsilon)) \) in the makespan or cost, respectively, if a solution is found. This can then be combined with a suitable search procedure for approximate values of the optimal makespan or cost. For details, we refer to [Section 3](#), where such techniques are used and described in more detail. In addition to the techniques mentioned there, the possibility of a cost zero solution has to be considered which can easily be done in this case. \( \square \)

Chain Graph Case We present FPTAS results for the variant of SCS with chain task graph. The basic approach is very similar to the fully parallel case.

Theorem 4. There is an FPTAS for SCS with chain task graph with respect to both the cost and the makespan objective.

Proof. We again start by designing dynamic programs for the search version of the problem with budget \( b \) and deadline \( d \). Without loss of generality, we assume \( \mathcal{J} = \{0, 1, \ldots, n + 1\} \) with \( \mathcal{S} = 0, \mathcal{T} = n + 1 \), and \( j \in [n] \) being the \( j \)-th job in the chain.

For each deadline \( d' \in \{0, 1, \ldots, d\} \), job \( j \in \{0, 1, \ldots, n + 1\} \), and location \( \ell \in \{s, c\} \) (referring to the server and cloud) we want to compute the smallest
cost \( C[d', j, \ell] \) of all the schedules of the jobs \( 1, \ldots, j \) adhering to the deadline \( d' \) and with the job \( j \) being scheduled on \( \ell, p_c \). To that end, we set \( C[d', 0, s] = 0, \) \( C[d', 0, c] = \infty \), and with slight abuse of notation use the convention \( C[z, j, \ell] = \infty \) for \( z < 0 \). Further values can be computed via the following recurrence relations:

\[
C[d', j, s] = \min \{C[d' - p_s(j) - c(j - 1, j), c], C[d' - p_s(j), s]\}
\]

\[
C[d', j, c] = \min \{C[d' - p_c(j), c] + p_c(j), C[d' - p_c(j) - c(j - 1, j), s] + p_c(j)\}
\]

If \( C[d, n + 1, s] > b \), we know that there is no feasible solution for the search version, and otherwise we can use backtracking starting from \( C[d, n + 1, s] \) to find one. The time and space complexity is polynomial in \( d \) and \( n \).

In the second dynamic program, we compute the smallest makespan \( M[j, b', \ell] \) of all the schedules of the jobs \( 0, \ldots, j \) adhering to the budget \( b' \) and with job \( j \) placed on location \( \ell \), for each \( b' \in \{0, 1, \ldots, b\}, j \in \{0, 1, \ldots, n + 1\} \) and \( \ell \in \{s, c\} \). We set \( M[b', 0, s] = 0, M[b', 0, c] = \infty \), use the convention \( M[z, j, \ell] = \infty \) for \( z < 0 \), and the recurrence relations:

\[
M[b', j, s] = \min \{M[b', c] + p_s(j) + c(j - 1, j), M[b', s] + p_s(j)\}
\]

\[
M[b', j, c] = \min \{M[b' - p_c(j), c] + p_c(j), M[b' - p_c(j), s] + p_c(j) + c(j - 1, j)\}
\]

If \( M[b, n + 1, s] > d \), we know that there is no feasible solution for the search version, and otherwise we can use backtracking starting from \( M[b, n + 1, s] \) to find one. The time and space complexity is polynomial in \( b \) and \( n \).

Like in the fully parallel case, we can use rounding and scaling approaches to trade the complexity dependence in \( d \) or \( b \) with a dependence in \( \text{poly}(n, \frac{1}{\varepsilon}) \) incurring a loss of a factor \((1 + O(\varepsilon))\) in the makespan or cost, respectively, if a solution is found. This can then be combined with a suitable search procedure for approximate values of the optimal makespan or cost. For details, we refer to Section 3, where such techniques are used and described in more detail. In addition to the techniques mentioned there, the possibility of a cost zero solution has to be considered which can easily be done in this case as well.

\[ \square \]

**No Delays and Identical Machines** We design a simple heuristic for the case in which the server and the cloud machines behave the same, that is, \( p_c(j) = \) \( p_s(j) \) for each job \( j \) (except for the source and sink), and the communication delays all equal zero. In this case, we may define the length of a chain in the task graph as the sum of the processing times of the jobs in the chain. The first step in the algorithm is to identify a longest chain in the task graph, which can be done in polynomial time. The jobs of the longest chain are scheduled on the server and the remaining jobs on the cloud each as early as possible. Now, the makespan of the resulting schedule is the length of a longest chain, which is optimal (or better) and there are no idle times on the server. However, the schedule may not be feasible since the budget may be exceeded. Hence, we repeatedly do the following: If the budget is still exceeded, we pick a job scheduled on the cloud with maximal starting time and move it on to the server right before its first
successor (which may be the sink). Some jobs on the server may be delayed by this but we can do so without causing idle times. If all the processing times are equal this procedure produces an optimal solution and otherwise there may be an additive error of up to the maximal job size. Hence, we have:

**Theorem 5.** There is a 2-approximation for SCS without communication delays and identical server and cloud machines.

It is easy to see, that the analysis is tight considering an instance with three jobs: One with size \( b \), one with size \( b + \varepsilon \), and one with size \( 2\varepsilon \). The first jobs precedes the last one. Our algorithm will place everything on the server, while the first job is placed on the cloud in the optimal solution.

Note that we can take a similar approach to find a solution with respect to the cost objective by placing more and more jobs on the server as long as the deadline is still adhered to. However, an error of one job can result in an unbounded multiplicative error in the objective in this case. On the other hand, it is easy to see that in the case with unit processing times, there will be no error at all in both procedures yielding:

**Corollary 1.** The variant of SCS without communication delays and unit processing times can be solved in polynomial time with respect to both the makespan and the cost objective.

### 3 Constant Cardinality Source and Sink Dividing Cut

We introduce the concept of a maximum cardinality source and sink dividing cut. For \( G = (\mathcal{J}, E) \), let \( \mathcal{J}_S \) be a subset of jobs, such that \( \mathcal{J}_S \) includes \( S \) and there are no edges \((j, k)\) with \( j \in \mathcal{J} \setminus \mathcal{J}_S \) and \( k \in \mathcal{J}_S \). In other words, in a running schedule \( \mathcal{J}_S \) and \( \mathcal{J} \setminus \mathcal{J}_S \), could represent already processed jobs and still to be processed jobs respectively. Denote by \( \mathcal{J}_G^S \) the set of all such sets \( \mathcal{J}_S \). We define

\[
\psi := \max_{\mathcal{J}_S \in \mathcal{J}_G^S} |\{(j, k) \in E | j \in \mathcal{J}_S \land k \in \mathcal{J} \setminus \mathcal{J}_S\}|,
\]

the maximum number of edges between any set \( \mathcal{J}_S \) and \( \mathcal{J} \setminus \mathcal{J}_S \) in \( G \). In a series-parallel task graph \( \psi \) is equal to the maximum anti-chain size of the graph.

In this chapter we discuss how to solve or approximate SCS problems with a constant size \( \psi \), but otherwise arbitrary task graphs. We first consider the deadline confined cost minimization, in [Theorem 7] we show how to adapt this to the budget confined makespan minimization. We give a dynamic program to optimally solve instances of SCS with arbitrary task graphs. At first we will not confine the algorithm to polynomial time. Consider a given problem instance with \( G = (\mathcal{J}, E) \), its source \( S \) and sink \( T \), processing times \( p_s(j) \) and \( p_c(j) \) for each \( j \in \mathcal{J} \), communication delays \( c(i, j) \) for each \( (i, j) \in E \) and a deadline \( d \).

We define intermediate states of a (running) schedule, as the states of our dynamic program. Such a state contains two types of variables. First we have two global variables, the timestamp \( t \) and the number of time steps the server
has been unused $f_s$. In other words, the server has not finished processing a job since $t - f_s$. The second type is defined per open edge. An open edge is a $e = (j,k)$ where $j$ has already been processed, but $k$ has not. For each such edge add the variables $e = (j,k)$ (the edge itself), $loc_j \in \{s,c\}$ denoting if $j$ was processed on the server ($s$) or the cloud ($c$) and $f_j$ denoting the number of time steps that have passed since $j$ finished processing. If a job $j$ is contained in multiple open edges, $loc_j$ and $f_j$ are still only included once. Write the state as $[t,f_s,e_1,\ldots,e_m,loc_1,\ldots,loc_m,f_1,\ldots,f_m]$, where $e_1,\ldots,e_m$ denote all open edges. Note here, that there are informations that we purposefully drop from a state: the completion time and location of every processed job without open edges, as those are not important for future decisions anymore. There might be multiple ways to reach a specific state, but we only care about the minimum possible cost to achieve that state, which is the value of the state.

We iteratively calculate the value of every reachable state with $t = 0, 1, 2, \ldots$. We start with the trivial state $[t = 0, f_s = 0, e_1,\ldots,e_m,loc_S = s, f_S = 0] = 0$. This state forms the beginning of our (sorted) state list. We keep this list sorted in an ascending order of state values (costs) at all times. We exhaustively calculate every state that is reachable during a specific time step, given the set of states reachable during the previous time step. Intuitively, we try every possible way to “fill up” the still undefined time windows $f_s$ and $f_j$.

Finally, we give the actual dynamic program in Algorithm 1. After the dynamic program finished, we iterate through the state list one last time and take the first state $[t = d, f_d]$. The value of that state is the minimum cost possible to schedule $G$ in time $d$. One can easily adapt this procedure to also yield such a schedule, by keeping a list of all processed jobs per state containing their location and completion time.

**Lemma 1.** DPrGG’s runtime is bounded in $O(d^{2^\psi+3} \cdot n^{2^\psi+1})$.

**Proof.** At any point there are a maximum of $O(d \cdot (d \cdot n)^\psi)$ states in the state list. For every $t$ we look at every state. Since we never insert a state in front of the state we are currently inspecting (costs can only increase), this traverses the list exactly once. For each of those states we calculate every possible successor, of which there are $O(\psi)$ and traverse the state list an additional time to correctly insert or update the state. We iterate from $t = 0$ to $d$ and therefore get a runtime of:

$$O(d \cdot ((d \cdot (d \cdot n)^\psi) \cdot \psi \cdot (d \cdot (d \cdot n)^\psi))) = O(d^3 \cdot n \cdot (d \cdot n)^{2\psi}) \leq O(d^{2^\psi+3} \cdot n^{2^\psi+1}).$$

$\square$
Algorithm 1 DPfGG: Dynamic Program for General Graphs

1: initialize state list \( SL \) with start state (as defined above)
2: for all \( state \in SL \) do
3:   let \( J_{state} \) be the set of all jobs that are endpoints in open edges from \( state \)
4:   for all \( j \in J_{state} \) do
5:     if \( \forall (k,j) \in E : (k,j) \) also open edge in \( state \) then
6:       if \( f_s \geq p_s(j) \) then
7:         \( jFits \leftarrow TRUE \)
8:       for all \( (k,j) \in E \) do
9:         if \( loc_k = s \land f_k < p_s(j) \) or \( loc_k = c \land f_k < p_s(j) + c(k,j) \) then
10:            \( jFits \leftarrow FALSE \)
11:       if \( jFits = TRUE \) then
12:         calculate resulting state \( state' \), value equal to \( state \)
13:     for all \( (k,j) \in E \) do
14:       remove \( (k,j) \) from \( state' \)
15:       if \( j \) is last open successor of \( k \) then
16:         remove \( f_k \) and \( loc_k \) from \( state' \)
17:       add \( f_j = 0, loc_j = s \) and all new open edges to \( state' \)
18:     if \( state' \in SL \) then
19:       update value of \( state' \) in \( SL \) if new value lower
20:       else add \( state' \) to correct position in \( SL \) (always after \( state \))
21:     can \( j \) be processed on the server? analogously to the previous case, cost value of \( state' \) increased by \( p_c(j) \)
22:     check end condition
23:   if a state \( [t = d, f_s] \in SL \) then
24:     return lowest value of such states
25:   if \( t < d \) then
26:     move from \( t \) to \( t+1 \)
27:     for all each \( state \in SL \) do
28:       increase \( t, f_s \) and each \( f_j \) in \( state \) by \( 1 \)
29: Back to step 2

3.1 Rounding the Dynamic Program

We use a rounding approach on DPfGG to get a program that is polynomial in \( n = |J| \), given that \( \psi \) is constant. We scale \( d, c, p_c, \) and \( p_s \) by a factor \( \varsigma := \frac{\epsilon \cdot d^2}{2n} \). Denote by \( \hat{d} := \lceil \frac{\hat{d}}{\varsigma} \rceil \leq \frac{d^2}{\varsigma^2} + 1, \hat{p}_s(j) := \lceil \frac{p_s(j)}{\varsigma} \rceil, \hat{p}_c(j) := \lceil \frac{p_c(j)}{\varsigma} \rceil \) and \( \hat{c}(x) := \lceil \frac{c(x)}{\varsigma} \rceil \). Note here, that we round up \( d \) but everything else down. We run the dynamic program with the rounded values, but still calculate the cost of a state with the original unscaled values.

We transform the output \( \pi' \) to the unscaled instance, by trying to start every job \( j \) at the same (scaled back up) point in time as in the scaled schedule. Since we rounded down, there might now be points in the schedule where a
job \( j \) cannot start at the time it is supposed to. This might be due to the server not being free, a parent node of \( j \) that has not been fully processed or an unfinished communication delay. We look at the first time this happens and call the mandatory delay on \( j \Delta \) and increase the start time of every remaining job by \( \Delta \). Repeat this process until all jobs are scheduled. We introduce no new conflicts with this procedure, since we always move everything together as a block. Call this new schedule \( \pi \).

**Theorem 6.** Assuming a constant number \( \psi \) DPfGG combined with the scaling technique finds a schedule \( \pi \) with at most optimal cost and a makespan \( \leq (1+\varepsilon)d \) in time \( \text{poly}(n, \frac{1}{\varepsilon}) \), for any \( \varepsilon > 0 \).

**Proof.** We start by proving the runtime of our algorithm. We can scale the instance in polynomial time, this holds for both scaling down and scaling back up. The dynamic program now takes time in \( O(\hat{d}^{2\psi+3} \cdot n^{2\psi+1}) \), where \( \hat{d} \leq \frac{2n}{\varepsilon}+1 \). Since \( \psi \) is constant this results in an dynamic program runtime in \( \text{poly}(n, \frac{1}{\varepsilon}) \). In the end we transform the schedule as described above, for that we go trough the schedule once and delay every job no more than \( n \) times. Trivially, this can be done in polynomial time as well.

Secondly we show that the makespan of \( \pi \) is at most \( (1+\varepsilon)d \). Every valid schedule for the unscaled problem is also valid in the scaled problem, meaning that there is no possible schedule we overlook due to the scaling. In the other direction this might not hold. First, while scaling everything down we rounded the deadline up. This means, that scaled back we might actually work with a deadline of up to \( d + \varsigma \). Secondly, we had to delay the start of jobs to make sure that we only start jobs when it is actually possible. In the worst case we delay the sink \( T \) a total of \( n - 2 \) times, once for every job other than \( S \) and \( T \). Each time we delay all remaining jobs we can bound the respective \( \Delta \) by \( \varsigma \). This is due to the fact that each of the delaying options can not delay by more than \( \varsigma \) (as that is the maximum timespan not regarded in the scaled problem) and only a direct predecessor job and the communication from it needing longer can coincide to a non-parallel delay. Taking both of these into account, a valid schedule for the scaled problem might use time up to

\[
d + \varsigma + (n - 2) \cdot (2\varsigma) \leq d + 2n\varsigma = (1+\varepsilon) \cdot d
\]

in the unscaled instance.

Lastly, we take a look at the cost of \( \pi \). While rounding, we did not change the calculation of a states value, and with every valid schedule of the unscaled instance being still valid in the scaled instance we can conclude that the cost of \( \pi \) is smaller or equal to an optimal solution of the original problem.

**Theorem 7.** DPfGG combined with the scaling technique and a binary search over the deadline yields an FPTAS for the cost budget makespan problem, for graphs with a constant number \( \psi \).

**Proof.** Theorem 6 can be adapted to solve this, assuming that we know a reasonable makespan estimate of an optimal solution to use in our scaling factor.
During the algorithm discard any state with costs bigger than the budget and terminate when the first state \([t, f_t]\) is reached. The \(t\) gives us the makespan.

Using a makespan estimate that is too big will lead to a rounding error that is not bounded by \(\varepsilon \cdot \text{mspan}_{\text{OPT}}\). A too small estimate might not find a solution. To solve this, we start with an estimate that is purposefully large. Let \(d_{\text{max}} = \sum_{j \in J} p_s(j)\) be the sum over all processing times on the server. There is always a schedule with 0 costs and makespan \(d_{\text{max}}\). We run our algorithm with the scaling factor \(\varsigma^0 := \frac{\varepsilon \cdot d_{\text{max}}}{4n}\). Iteratively repeat this process with scaling factor \(\varsigma^i = \frac{1}{2^{i-1}} \varsigma^0\) for increasing \(i\) starting with 1. At the same time, halve the original deadline estimate in each step, which leads to \(d\), and therefore the runtime, to stay the same in each iteration. End the process when the algorithm does not find a solution for the current \(i\) and deadline estimation. This infers that there is no schedule with the wanted cost budget and a makespan smaller or equal to \(\frac{1}{2^i}d_{\text{max}}\) (in the unscaled instance), therefore \(\frac{1}{2^i}d_{\text{max}} < \text{mspan}_{\text{OPT}}\). We look at the result of the previous run \(i - 1\): The scaled result was optimal, therefore the unscaled version has a makespan of at most

\[
\text{mspan}_{\text{ALG}} \leq \text{mspan}_{\text{OPT}} + 2n \cdot \varsigma^{i-1} \\
= \text{mspan}_{\text{OPT}} + 2n \cdot \frac{1}{2^{i-1}} \cdot \frac{\varepsilon \cdot d_{\text{max}}}{4n} \\
= \text{mspan}_{\text{OPT}} + \varepsilon \cdot \frac{1}{2^i}d_{\text{max}} \leq (1 + \varepsilon)\text{mspan}_{\text{OPT}}.
\]

It should be easy to infer from Lemma 1 that each iteration of this process has polynomial runtime. Combined with the fact that we iterate at most \(\log d_{\text{max}}\) times we get a runtime that is in \(\text{poly}(n, \frac{1}{\varepsilon})\).

Remark 2. The results of this chapter work, as written, for a constant \(\psi\). Note here, that for series parallel digraphs, this is equivalent to a constant anti-chain size. The algorithms can also be adapted to work on any graph with constant anti-chain size, if the communication delays are \(\text{locally small}\). Delays are locally small, if for every \((j, k) \in E\), \(e(j, k)\) is smaller or equal than every \(p_c(k'), p_s(j'), p_c(j')\) and \(p_s(j')\), where \(k'\) is every direct successor of \(j\) and \(j'\) every direct predecessor of \(k\) \(\text{[11]}\).

4 Strong NP-Hardness

In this section, we consider more involved reductions than in Section 2 in order to gain a better understanding for the complexity of the problem. First, we show that a classical result due to Lenstra and Rinnooy Kan \(\text{[8]}\) can be adapted to prove that already the variant of SCS without communication delays and processing times equal to one or two is NP-hard. This already implies strong NP-hardness. Remember that we did show in Section 2 that SCS without communication delays and with unit processing times can be solved in polynomial time. Hence, it seems natural to consider the problem variant with unit processing times and communication delays. We prove this problem to be NP-hard.
as well as via an intricate reduction from 3SAT that can be considered the main result of this section. Lastly, we show that the latter reduction can be easily modified to get a strong inapproximability result regarding the general variant of SCS and the cost objective.

No Delays and Two Sizes We show strong hardness for the case without communication delays and \( p_c(j), p_s(j) \in \{1, 2\} \) for each job \( j \). The reduction is based on a classical result due to Lenstra and Rinnooy Kan [8].

Let \( G = (V, E) \), \( k \) be a clique instance with \(|E| > \binom{k}{2}\), and let \( n = |V| \) and \( m = |E| \). We construct an instance of the cloud server problem in which the communication delays all equal zero and both the deadline and the cost bound is \( 2n + 3m \). There is one vertex job \( J(v) \) for each node \( v \in V \) and one edge job \( J(e) \) for each edge \( e \in E \) and \( J(\{u, v\}) \) is preceded by \( J(u) \) and \( J(v) \). The vertex jobs have size 1 and the edge jobs size 2 both on the server and on the cloud.

Furthermore there is a dummy structure. First, there is a chain of \( 2n + 3m \) many jobs called the anchor chain. The \( i \)-th job of the anchor chain is denoted \( A(i) \) for each \( i \in \{0, \ldots, 2n + 3m - 1\} \) and has size 1 on the cloud and size 2 on the server. Furthermore, there are gap jobs each of which has size 1 both on the server and the cloud. Let \( k^* = \binom{k}{2} \) and \( v \prec w \) indicate that an edge from \( v \) to \( w \) is included in the task graph. There are four types of gap jobs, namely \( G(1, i) \) for \( i \in \{0, \ldots, k - 1\} \) with edges \( A(2i) \prec G(1, i) \prec A(2i + 1) \), \( G(2, i) \) for \( i \in \{0, \ldots, k^* - 1\} \) with \( A(2k + 3i + 1) \prec G(2, i) \prec A(2k + 3(i + 1)) \), \( G(3, i) \) for \( i \in \{0, \ldots, (n - k) - 1\} \) with \( A(2k + 3k^* + 2i) \prec G(3, i) \prec A(2k + 3k^* + 2(i + 1)) \), and \( G(4, i) \) for \( i \in \{0, \ldots, (m - k^*) - 1\} \) with \( A(2n + 3k^* + 3i + 1) \prec G(4, i) \prec A(2n + 3k^* + 3(i + 1)) \) for \( i < (m - k^*) - 1 \) and \( A(2n + 3m - 2) \prec G(4, (m - k^*) - 1) \) for \( i = (m - k^*) - 1 \). Lastly, there are the source and the sink which precedes or succeeds all of the above jobs, respectively.

Lemma 2. There is a \( k \)-clique, if and only if there is a schedule with length and cost at most \( 2n + 3m \).

Proof. First note that in a schedule with deadline \( 2n + 3m + 1 \) the anchor chain has to be scheduled completely on the cloud. If the schedule additionally satisfies the cost bound, all the other jobs have to be scheduled on the server. Furthermore, for the gap and anchor chain jobs there is only one possible time slot due to the deadline. In particular, \( A(i) \) starts at time \( i \), \( G(1, i) \) at time \( 2i + 1 \), \( G(2, i) \) at time \( 2k + 3i + 2 \), \( G(3, i) \) at time \( 2k + 3k^* + 2i + 1 \), and \( G(4, i) \) at time \( 2n + 3k^* + 3i + 2 \). Hence, there are \( k \) length 1 slots positioned directly before the \( G(1, i) \) jobs left on the server, as well as, \( k^* \) length 2 slots directly before the \( G(2, i) \) jobs, \( n - k \) length 1 slots directly before the \( G(3, i) \) jobs, and \( m - k^* \) length 2 slots directly before the \( G(2, i) \) jobs (see also Figure 2). The \( m \) edge jobs have to be scheduled in the length 2 slots, and hence the vertex jobs have to be scheduled in the length 1 slots.

\( \Rightarrow \) : Given a \( k \)-clique, we can position the \( k \) clique vertices in the first \( k \) length 1 slots, the corresponding \( k^* \) edges in the first length 2 slots, the remaining
vertex jobs in the remaining length 1 slots, and the remaining edge jobs in the
remaining length 2 slots.
\[\iff \text{ Given a feasible schedule, the vertices corresponding to the first length}
1 \text{ slots have to form a clique. This is the case, because there have to be } k^* \text{ edge}
jobs in the first length 2 slots and all of their predecessors are positioned in
the first length 1 slots. This is only possible if these edges are the edges of a}
k\text{-clique.} \]

Fig. 2: The dummy structure for the reduction from the clique problem to a
special case of \textit{SCS}. Time flows from left to right, the anchor chain jobs are
positioned on the cloud, and the gap jobs on the server.

Hence, we have:

\textbf{Theorem 8.} The \textit{SCS} problem with job sizes 1 and 2 and without communi-
caction delays is strongly NP-hard.

In the above reduction the server and the cloud machines are unrelated relative
to each other due to different sizes of the anchor chain jobs. However, it is easy
to see that the reduction can be modified to a uniform setting where the cloud
machines have speed 2 and the server speed 1. If we allow communication delays,
even identical machines can be achieved.

\textbf{Unit Size and Unit Delay} We consider a unit time variant of our model in
which all \(p_c = p_s = 1\) and all \(c = 1\). Note here, that this also implies that the
server and the cloud are identical machines (the cloud still produces costs, while
the server does not). As usual for reductions we look at the decision variant of
the problem: Is there a schedule with cost smaller or equal to \(b\) while adhering
to the deadline \(d\).

\textbf{Theorem 9.} The \textit{SCS}\textsuperscript{1} problem is strongly NP-hard.

We give a reduction 3\textit{SAT} \(\leq_p\) \textit{SCS}\textsuperscript{1}. Let \(\phi\) be any boolean formula in
3-CNF, denote the variables in \(\phi\) by \(X = \{x_1, x_2, \ldots, x_m\}\) an the clauses by
\(C = \{C_1^\phi, C_2^\phi, \ldots, C_n^\phi\}\). Before we define the reduction formula we want to give
an intuition and a few core ideas used in the reduction.

The main idea is that we ensure that nearly everything has to be processed
on the cloud, there are only a few select jobs that can be handled by the server.
For each variable there will be two jobs, of which one can be processed on the server, the selection will represent an assignment. For each clause there will be a job per literal in that clause, only one of which can be processed on the server, and only if the respective variable job is ‘true’. Only if for each variable and for each clause one job is handled by the server the schedule will adhere to both the cost and the time limits.

A core technique of the reduction is the usage of an anchor chain. An anchor chain of length \( l \) consists of two chains of the same length \( l := d - 2 \), where we interlock the chains by inserting \((a_i, b_{i+1})\) and \((b_i, a_{i+1})\) for two parallel edges \((a_i, a_{i+1})\) and \((b_i, b_{i+1})\). The source \( S \) is connected to the two start nodes of the anchor chain, the two nodes at the end of the chain are connected to \( T \).

![Anchor Chain Diagram](image)

**Fig. 3:** schematic representation of an anchor chain

**Lemma 3.** If the task graph of a \( SCS^1 \) problem contains an anchor chain, every valid schedule has to schedule all but one of \( a_1, b_1 \) and one of \( a_l, b_l \) on the cloud. For every job \( a_i, b_i \) \( 1 < i < l \) the time step in which it will finish processing on the cloud in every valid schedule is \( i + 1 \).

Finally we give the reduction function \( f(\phi) = G, d, b \), where \( G = (J, E) \). Set \( d = 12 + m + n \) and \( b = |J| - (2 + m + n) \). We define \( G \) by constructively giving which jobs and edges are created by \( f \). Create an anchor chain of length \( d - 2 \), this will be used to limit parts of a schedule to certain time frames. Note that by **Lemma 3** we know that every valid schedule of \( G = (J, E), d, k \) has every node pair of the anchor chain (besides the first and last) on the cloud at a specific fixed timestamp. More specifically, the completion time of \( a_i \) and \( a_{i+j} \) differ by exactly \( j \) time units. For each variable \( x_i \in X \) create two jobs \( j_{x_i} \) and \( j_{\neg x_i} \) and edges \((a_{1+i}, j_{x_i}), (a_{1+i}, j_{\neg x_i})\) and \((j_{x_i}, a_{5+i}), (j_{\neg x_i}, a_{5+i})\). For each clause \( C_\phi \) create a clause job \( j_{C_\phi} \) and edges \((a_{7+m+p}, j_{C_\phi}), (j_{C_\phi}, a_{9+m+p})\). Let \( L^{1}_p, L^{2}_p, L^{3}_p \) be the literals in \( C_\phi \). Create jobs \( j_{L_1^p}, j_{L_2^p}, j_{L_3^p} \) and edges \((j_{L_1^p}, C_\phi), (j_{L_2^p}, C_\phi), (j_{L_3^p}, C_\phi)\) for these literals. For every literal job \( j_{L_i^p} \) connect it to the corresponding variable job \( j_{x_i}, j_{\neg x_i} \) by a chain of length \( 1 + (m - i) + p \). Also create an edge from \( a_{3+i} \) to the start of the created chain and an edge from the end of the chain to \( a_{6+m+p} \).

It remains to show that there is a schedule of length at most \( d \) with costs at most \( b \) in \( f(\phi) = G, d, b \) if and only if there is a satisfying assignment for \( \phi \).
Lemma 4. In a deadline adhering schedule for $f(\phi) = G, d, b$ every job in the anchor chain (except on at the front and one at the end), every job in the variable and clause literal connecting chains and every clause job has to be scheduled on the cloud.

Proof. By Lemma 3 we already know that every node in the anchor chain except one of $v_1, w_1$ and one of $v_l, w_l$ has to be scheduled on the cloud. We also know, that the jobs in the anchor chain have fixed time steps in which they have to be processed. We look at some chain and its connection to the anchor chain. The start of the chain of length $1 + (m - i) + p$ is connected to $a_{i+1}$, the end to $a_{6 + m + p}$. Between the end of $a_{i+1}$ and the start of $a_{6 + m + p}$ are $6 + m + p - 1 - (3 + i) = 2 + m + p - i$ time steps. So with the processing time required to schedule all $1 + (m - i) + p$ jobs of the chain, there is only one free time step, but we would need at least 2 free time steps to cover the communication cost to and from the server. (Recall here that both $a_{i+1}$ and $a_{6 + m + p}$ have to be processed on the cloud). The same simple argument fixes each clause job to a specific time step on the server.

Lemma 5. In a deadline adhering schedule for $f(\phi) = G, d, b$ only one of $j_x$, and $j_{\overline{x}}$ can be processed on the server for every variable $x \in X$. The same is true for $j_{L_1^p}, j_{L_2^p}, j_{L_3^p}$ of clause $C_\phi^p$.

Proof. $j_x$ and $j_{\overline{x}}$ are both fixed to the same time interval via the edges $(a_{i+1}, j_x)$, $(a_{i+1}, j_{\overline{x}})$ and $(j_x, a_{6+m+p})$. Since $a_{i+1}$ and $a_{6+m+p}$ will be processed on the cloud and keeping communication delays in mind, only the middle of the three time steps in between can be used to schedule $j_x$ or $j_{\overline{x}}$ on the server. Since the server is only a single machine only one of them can be processed on the server. Note here that the other job can be scheduled a time step earlier which we will later use. The argument for $j_{L_1^p}, j_{L_2^p}, j_{L_3^p}$ works analogously to the statement above.

Lemma 6. There is a deadline adhering schedule for $f(\phi) = G, d, b$ with costs of $|J| - (2 + m + n)$ if and only if there is a satisfying assignment for $\phi$. The variable jobs processed on the cloud represent this satisfying assignment.
Proof. From Lemma 3, Lemma 4 and Lemma 5 we can infer that a schedule with costs of $|J| - (2 + m + n)$ has two jobs of the anchor chain, one job for each pair of variable jobs and one job per clause on the server. Two jobs of the anchor chain can always be placed on the server, the choice of variable jobs is also free. It remains to show, that we can only schedule a literal job per clause on the server if and only if the respective clause is fulfilled by the assignment inferred by the variable jobs.

The clause job $j_{C_p}^i$ of $C_p$ has to be processed in time step $9 + m + p$ (between $a_{7+m+p}$ and $a_{9+m+p}$). Therefore, $j_{L_p}^i$ has to be processed no later than $8 + m + p$ or $7 + m + p$ if it is processed on the cloud or server respectively. Let $j_{x_i}$ be the variable job connected to $j_{L_p}^i$ via a connection chain.

If $j_{x_i}$ is true (scheduled on the cloud), it can finish processing at time step $3 + i$, which does not delay the start of the connection chain (which is connected to $a_{3+i}$, finishing in time step $4 + i$). This means that the chain can finish in time step $4 + i + 1 + (m - i) + p = 5 + m + p$, the time step $6 + m + p$ can be used for communication, allowing $j_{L_p}^i$ to be processed by the server in $7 + m + p$.

If $j_{x_i}$ is false (scheduled on the server), it finishes processing at time step $4 + i$, which, combined with the induced communication delay, delays the start of the chain by 1. Therefore, the chain only finishes in time step $6 + m + p$, and $j_{L_p}^i$ has to be processed on the cloud, since there is not enough time for the communication back and forth.

Trivially, the same argument holds true for $j_{L_2}^i$ and $j_{L_3}^i$.

It should be easy to see that the reduction function $f$ is computable in polynomial time. Combined with Lemma 6 this concludes the proof of our reduction $3SAT \leq_p SCS^1$. The correctness of Theorem 9 trivially follows from that.

The General Case Adapting the previous reduction we can show an even stronger result for the general case of SCS. Basically we are able to degenerate the reduction output in a way, that a satisfying assignment results in a schedule with cost 0, while every other assignment (schedule) has costs of at least 1. It should be obvious, that this also means that there is no approximation algorithm for this problem with a fixed multiplicative performance guarantee, if $P \neq NP$.

In this version we will use processing times and communication delays of 0, $\infty$ and values in between. Note here that $\infty$ can be replaced by $d + 1$, and it is also possible to realize this construction without 0, for readability and space we will only do the version using 0. To keep the following part readable we will substitute ”an edge $(j, j')$ with cost $c(j, j') = k$” simply by ”an edge $c(j, j') = k$”

We follow the same general structure (an anchor chain, variable-, clause- and connection gadgets). The anchor chain now looks as follows: For every step create two jobs $a_i$ and $a_i'$ with $p_s(a_i) = 0$, $p_c(a_i) = \infty$, $p_s(a_i') = \infty$, $p_c(a_i') = 0$ and an edge $c(a_i, a_i') = 0$. These chain links are than connected by an edge $c(a_i', a_{i+1}) = 1$. Finally we create $c(S, a_1) = 1$ and $c(a_d, T) = 0$. It should be easy to see, that every schedule will process $a_i$ and $a_i'$ in time step $i$ on the server and the cloud respectively. This gives us anchors to the server and to the cloud.
for every time step, without inducing congestion or costs. Since the anchor jobs themselves have processing time of 0, the "usable" time interval between some \( a_i \) and \( a_{i+1} \) is one full time step.

For each variable \( x_i \in X \) create two jobs \( j_{x_i}, j_{\bar{x}_i} \) with \( p_s(j_{x_i}) = p_s(j_{\bar{x}_i}) = 1 \) and \( p_c(j_{x_i}) = p_c(j_{\bar{x}_i}) = 0 \). Create edges \( c(a_i, j_{x_i}) = 1 \) and \( c(a_i, j_{\bar{x}_i}) = 1 \) and \( c(j_{x_i}, a_{i+1}) = 0 \), \( c(j_{\bar{x}_i}, a_{i+1}) = 0 \). In short, only one of them can be processed on the server, the other on the cloud. Both will finish in time step \( i + 1 \), the one processed on the server is "true", therefore processing both on the cloud is possible, but not helpful.

For each clause \( C^p \) create a clause job \( j_{C^p} \) with \( p_s(j_{C^p}) = \infty \), \( p_c(j_{C^p}) = 0 \) and edges \( c(a'_5 + m + 3p, j_{C^p}) = \infty \) and \( c(j_{C^p}, a'_6 + m + 3p) = \infty \). This means, that \( j_{C^p} \) has to finish processing by time step \( 6 + m + 3p \). Let \( L^p_1, L^p_2, L^p_3 \) be the literals in \( C^p \). Create jobs \( j_{L^p_1}, j_{L^p_2}, j_{L^p_3} \) each with \( p_c = p_s = 1 \) and edges \( c(j_{L^p_1}, C^p) = 0 \), \( c(j_{L^p_2}, C^p) = 0 \), \( c(j_{L^p_3}, C^p) = 0 \) for these literals. Create edges \( c(a_3 + m + 3p, j_{L^p_1}) = 0 \), \( c(a_3 + m + 3p, j_{L^p_2}) = 0 \) and \( c(a_3 + m + 3p, j_{L^p_3}) = 0 \), so that, in theory, all three of the literal jobs can be processed on the server, finishing in time steps \( 4 + m + 3p \), \( 5 + m + 3p \) and \( 6 + m + 3p \) respectively.

Lastly, for every literal job \( j_{L^p_i} \) connect it to the corresponding variable job \( j_{x_i} \) (or \( j_{\bar{x}_i} \)) by an edge with communication delay of \( m - i + 3p + 3 \). Since \( j_{x_i} \) (or \( j_{\bar{x}_i} \)) finish processing in time step \( i + 1 \), this means that \( j_{L^p_i} \) can start no earlier than \( m + 3p + 4 \) (and therefore finish processing in \( 5 + m + 3p \)), if \( j_{x_i} \) (or \( j_{\bar{x}_i} \)) were processed on the cloud.

Recall here, that a variable job being scheduled on the server denotes that it is "true". So only a literal job that evaluates to true, can be scheduled so that it finishes processing in time step \( 4 + m + 3p \) on the cloud.

It follows directly, that a schedule for this construction will have costs of 0 if and only if the assignment derived from the placement of the variable jobs fulfills every clause.

**Theorem 10.** There is no approximation algorithm for SCS that has a fixed performance guarantee, assuming that \( P \neq NP \).

## 5 Unit Size and Unit Delay

As the last step of this paper we explore simple algorithms on unit size instances with arbitrary task graphs. Recall that we proved these to be strongly NP-hard. We use resource augmentation and ask: given a SCS\(^1\) problem instance with deadline \( d \), find a schedule in poly. time that has a makespan of at most \((1 + \varepsilon) \cdot d\) that approximates the optimal cost in regards to the actual deadline \( d \).

If there is a chain of length \( d \) or \( d - 1 \), that chain has to be scheduled on the server, since there is no time for the communication delay. For instances with a chain of size \( d \) that is trivially optimal, for those with \( d - 1 \) we can check in polynomial time if any other job also fits on the server, again, finding an optimal solution. From now we assume that there is no chain of length more than \( d - 2 \).
First, construct a schedule which places every job on the cloud, as fast as possible. The resulting schedule from time step (ts) 1 to $(1+\varepsilon)\cdot d$ looks as follows: one ts of communication, at most $d - 2$ ts of processing on the server, another ts for communication followed by at least $\varepsilon d$ empty ts. Now pull (one of) the last job(s) that is processed on the cloud to the last empty ts and process it on the server instead. Repeat this process until the last job can not be moved to the server anymore. Do the whole procedure again, but this time starting with the cloud schedule in the end of the schedule, and each time pulling the first job to the beginning. Keep the result with lower costs. Note that one can always fill the ts being used solely for communicating from the server to the cloud with processing one job on the server, that otherwise would be one of the first jobs being processed on the cloud (the same holds for the other direction).

**Theorem 11.** The described algorithm yields a schedule with approximation factor of $\frac{1+\varepsilon}{2\varepsilon}$ while having a makespan of at most $(1 + \varepsilon)\cdot d$.

**Proof.** Case $n \leq (1 + \varepsilon)d$: The algorithm places all jobs on the server, the cost is 0 and therefore optimal.

Case $(1 + \varepsilon)d < n < (1 + 2\varepsilon)d$: Assume that the preliminary cloud-only schedule needs $d - 2$ ts on the cloud, if that is not the case, we stretch the schedule to that length. There are $n$ jobs distributed onto $d - 2$ ts. Therefore, either from the front or from the end, there is an interval of length $\frac{d}{2} - 1$ with at least $\frac{d}{2} - 1$ and at most $\frac{d}{2} < (1 + 2\varepsilon)d = \frac{d}{2} + \varepsilon d$ many jobs. It should be easy to see, that the algorithm will schedule those at most $\frac{d}{2} + \varepsilon d - 1$ jobs to the $\frac{d}{2} - 1$ plus the free $\varepsilon d$ many time slots. If the interval included less than $\frac{d}{2} + \varepsilon d - 1$ jobs, it will simply continue until the $\frac{d}{2} - 1 + \varepsilon d$ are filled with jobs being processed on the server. With the one job we can process on the server during the communication ts we process $\frac{d}{2} + \varepsilon d$ jobs on the server and have costs of $n - (\frac{d}{2} + \varepsilon d)$. An optimal solution has costs of at least $n - d$. For $\varepsilon \geq 0.5$ it holds that: $\text{cost}_{\text{ALG}} = n - (\frac{d}{2} + \varepsilon d) \leq n - d \leq \text{cost}_{\text{OPT}}$, otherwise:

$$\frac{\text{cost}_{\text{ALG}}}{\text{cost}_{\text{OPT}}} \leq \frac{n - (\frac{d}{2} + \varepsilon d)}{n - d} \leq \frac{(1 + \varepsilon)d - (\frac{d}{2} + \varepsilon d)}{(1 + \varepsilon)d - d} \leq \frac{0.5d}{\varepsilon d} = \frac{1}{2\varepsilon}$$

Case $(1 + 2\varepsilon)d \leq n$: In this case we simply observe that our algorithm places at least $\varepsilon d$ many jobs on the server. For $\varepsilon \geq 1$ it holds that: $\text{cost}_{\text{ALG}} = n - \varepsilon d \leq n - d \leq \text{cost}_{\text{OPT}}$, otherwise:

$$\frac{\text{cost}_{\text{ALG}}}{\text{cost}_{\text{OPT}}} \leq \frac{n - \varepsilon d}{n - d} \leq \frac{(1 + 2\varepsilon)d - \varepsilon d}{(1 + 2\varepsilon)d - d} = \frac{d + \varepsilon d}{2\varepsilon d} = \frac{1 + \varepsilon}{2\varepsilon}$$

$\Box$

### 6 Conclusion

We give a small overview over the future research directions that emerge from our work. SCS, Section 4 gives a strong inapproximability result for the general
case with regards to the cost function. For two easy cases (chain and fully parallel graphs) we could establish FPTAS results, for graphs with a constant $\psi$ we have an algorithm that finds optimal solutions with a $(1+\varepsilon)$ deadline augmentation. Here one could explore if there are FPTAS results for different assumptions, are there approximation algorithms without resource augmentation for constant $\psi$ instances, and lastly are there approximation algorithms with resource augmentation for the general case. For the makespan function we already have a FPTAS for graphs with a constant $\psi$. It remains to explore approximation algorithms or inapproximability results for the general case of this problem. SCS$^1$: We show strong NP-hardness even for this simplified problem. Since this is a special case of the general problem all constructive results still hold, additionally we were able to give a first simple algorithm for cost optimization in general graphs. Here it would be interesting to look into more involved approximation algorithms that give better performance guarantees, maybe without resource augmentation.

Finally one could think about expanding the model. Natural first steps might be multiple (identical or different) server machines and/or different cloud environments. But keep in mind here, that additional simplifying assumptions might be necessary to keep constructive results possible.
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