INFINITELY MANY SUBHARMONIC SOLUTIONS FOR NONLINEAR EQUATIONS WITH SINGULAR $\phi$-LAPLACIAN

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Abstract. In this paper we prove the existence and multiplicity of subharmonic solutions for nonlinear equations involving the singular $\phi$-Laplacian. Such equations are in particular motivated by the one-dimensional mean curvature problems and by the acceleration of a relativistic particle of mass one at rest moving on a straight line. Our approach is based on phase-plane analysis and an application of the Poincaré-Birkhoff twist theorem.

1. Introduction. In this paper we deal with the existence and multiplicity of subharmonic solutions of nonlinear second order equation

$$(\phi(x'))' + g(t, x) = 0$$

(1)

involving the singular $\phi$-Laplacian, where $\phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, and $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and periodic function with period $2\pi$ with respect to time $t$. Also, we assume that, there are constants $\varepsilon_0 > 0$ and $d_0 > 0$, such that $g(t, x)$ satisfies the following sign condition:

$$(G_0) \quad \text{sign}(x) g(t, x) \geq \varepsilon_0, \quad \forall t \in [0, 2\pi] \text{ and } |x| \geq d_0.$$

In case $\phi$ is the identity operator, Eq. (1) is the classical second order forced Duffing equation

$$x'' + g(t, x) = 0.$$  

(2)

The existence and multiplicity of subharmonic solutions have been studied by many researchers under various growth conditions on the function $g$. For example, see [29, 10, 11, 12, 15]. Also, some extensions can be found in [16, 22, 23].

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In case $\phi$ is an increasing homeomorphism from $(-a,a)$ to $\mathbb{R}$, the existence of solutions of various boundary value problems have been studied by Bereanu and Mawhin based on the method of upper and lower solutions and Leray-Schauder degree, see [2, 3, 4, 5]. We also note the work of Bereanu and Torres on the multiple existence of periodic solutions of relativistic forced pendulum by using critical point theory [6].

However, there are few results on the multiplicity of subharmonic solutions of Eq. (1). Recently, Boscaggin, Garrione and Feltrin, Fonda and Toader, Donde and Zanolin applied the Poincaré-Birkhoff twist theorem to discuss related problems, see [8, 7, 19, 14]. This paper is another result in this direction. Using the careful phase-plane analysis based on the sign condition \((G_0)\), we apply the Poincaré-Birkhoff twist theorem to prove the following multiplicity result.

**Theorem 1.1.** Assume that $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, and $g(t,x)$ is a continuous function and $2\pi$-periodic in $t$, such that solutions of Eq. (1) are unique with respect to initial value. If \((G_0)\) holds, then Eq. (1) has at least one $2\pi$-periodic solution. Moreover, for any prime number $n$, there is $m_0(n) > 0$ such that, for any positive integer $m \geq m_0(n)$, Eq. (1) has at least two subharmonic solutions $x_{n,m,k}(t)$, $k = 1, 2$, with minimal period $2m\pi$.

Furthermore,

$$\lim_{m \to +\infty} \min_{t \in [0,2m\pi]} (|x_{n,m,k}(t)| + |\phi(x'_{n,m,k}(t))|) = +\infty, \quad k = 1, 2.$$ 

Some parts of our arguments are motivated by Ding and Zanolin [12]. In [12], they proved the multiplicity of subharmonic solutions for sublinear Duffing equation (2) via the Poincaré-Birkhoff twist theorem. To apply the Poincaré-Birkhoff twist theorem, one needs to construct an annulus such that the solution $z(t; z_0) = (x(t; x_0, y_0), y(t; x_0, y_0))$ satisfies that

$$\arg(z(T; z_0)) - \arg(z_0) + 2n\pi < 0 \quad (0 < 0), \quad z_0 \in \Gamma^-,$$

$$\arg(z(T; z_0)) - \arg(z_0) + 2n\pi > 0 \quad (0 < 0), \quad z_0 \in \Gamma^+,$$

where $\Gamma^-$ and $\Gamma^+$ are the inner and outer boundary of the annulus, respectively. The difficulty of constructing such annulus for sublinear Duffing equation (2) is how to find the inner boundary $\Gamma^-$. Sometimes, the solutions may pass through the origin. In this case we can not compute and estimate the argument because the argument function $\arg(z)$ is not well-defined at $z = 0$. A good idea from Ding and Zanolin can be used to overcome this difficulty. They introduced a transformation $y(t) = x(t) - x_0(t)$ with $x_0(t)$ a $2\pi$-periodic solution of Eq. (2) (obtained by a simple application of topological method since there is a natural priori bound for sublinear equations). Then Eq. (2) is equivalent to

$$y'' + g(t, y + x_0) - g(t, x_0) = 0.$$  

Each nonzero solution of Eq. (3) does not pass through the origin by uniqueness of solutions. Thus one can compute and estimate the argument $\arg(z(t))$ for all $t \in \mathbb{R}$. But in our case, $\phi$ is not linear. The method introduced above is not easy to use. Fortunately, we can prove that the solutions of (1) have spiral properties, that is, there are two spiral curves guiding the solutions of (1) in the phase plane, and forcing them to rotate around the origin as they increase in norm. Then each solution of (1) with fixed number of zeros starting from outside of a large disk in the phase plane does not pass through the origin. This property and the angular feature of
the fixed points obtained by Pincaré-Birkhoff twist theorem inspire us to consider a modified equation of (1) instead. We mention here that in recent, Fonda and Sfecci [17] developed the so-called admissible spiral method, a tool introduced by the same authors in [18], to prove the existence of infinitely many periodic solutions for weakly coupled superlinear second order systems. The argument for the spiral property is also used to obtain the existence of infinitely many periodic solutions for superlinear second order equation (2) without the sign condition \((G_0)\) [32].

The rest of paper is organized as follows. For special \(\phi(x) = \frac{x}{\sqrt{1-x^2}}\), in Section 2, we give an application of Theorem 1.1 in nonlinear systems under relativistic effect. In Section 3, we will prove some basic geometric properties of the solutions of the modified system. In particular, we will prove that the solutions of the modified system have spiral properties, that is, there are two spiral curves guiding the solutions in the phase plane, and forcing them to rotate around the origin as they increase in norm. Finally, in Section 4, we will give the proof of Theorem 1.1 using the Poincaré-Birkhoff twist theorem.

2. Subharmonic mechanical vibrations of nonlinear systems under relativistic effect. Relativistic oscillator models for particle motions are a subject of rather broad interest nowadays, which have been widely used in different branches of theoretical physics such as quantum mechanics, statistical mechanics, superconductivity theory, nuclear physics, and so forth [21, 27, 1].

Consider the equation describing the relativistic motion of molecule that interacts with another molecule

\[
\frac{d}{dt} \left( m_0 x' \sqrt{1 - \frac{x'^2}{c^2}} \right) + g(x) = p(t),
\]

(4)

where the restoring term \(g(x)\) derives from one of the one-dimensional potentials and the forced term \(p(t)\) is \(2\pi\)-periodic and continuous. Here, \(m_0\) is the rest unit mass of the particle and \(c\) is the light speed. We normalize the equation by restricting the light speed \(c = 1\). Some recent papers [24, 25, 26], in this field, have taken into account chaos, the relativistic dependence between the oscillator energy and the order of the formed resonances with various special potentials.

In case there is no forced term, Eq. (4) is the autonomous system

\[
x' = \frac{y}{\sqrt{1+y^2}}, \quad y' = -g(x),
\]

(5)

Define the energy function by \(H(x, y) = \sqrt{1+y^2} - 1 + G(x)\), where \(G(x) = \int_0^x g(s)ds\). It is easy to check that, if the sign condition

\[(G_1) \quad \liminf_{|x| \to \infty} \text{sign}(x) (g(x) - \|p\|_{\infty}) > 0\]

holds, then for large enough \(h\), all the solutions with the orbit \(\Gamma_h = \{(x,y) : H(x,y) = h\}\) \((c_-,0)\) are periodic with the fundamental period

\[
T_h = 2 \int_{c_-}^{c_+} \frac{h + 1 - G(x)}{\sqrt{(h + 1 - G(x))^2 - 1}} dx + 2 \int_0^{c_+} \frac{h + 1 - G(x)}{\sqrt{(h + 1 - G(x))^2 - 1}} dx,
\]

where \(c_- < 0 < c_+\) and \(G(c_+) = h\).

For various potentials, numerical results on the relations between the fundamental period \(T_h\) and “energy” \(h\) (see Fig. 1) have shown that \(T_h\) can be arbitrarily
Figure 1. The relations between the fundamental period $T_h$ and "energy" $h$ with various potentials: (a) Toda potential $G(x) = k(x + e^{-x})$ with $k = 1$; (b) Sublinear potential $G(x) = \frac{4}{5}|x|^{5/4}$; (c) Harmonic potential $G(x) = \frac{1}{2}x^2$; (d) Superlinear potential $G(x) = \frac{25}{2}|x|^{5/2}$.

large, if $h$ is large enough, that is, Eq. (5) has a natural property of second order sublinear system. Thus, if

$$\lim_{h \to +\infty} T_h = +\infty,$$

then it follows that there is $n^* > 0$ such that, for any integer $n \geq n^*$, Eq. (5) has at least one subharmonic solution with minimal period $2n\pi$. As a direct application of Theorem 1.1, we have the same conclusion as the autonomous system for the periodically perturbed equation (4).

**Theorem 2.1.** Assume that $g(x)$ is a continuous function such that solutions of Eq. (4) are unique with respect to initial value, and $p(t)$ is $2\pi$-periodic and continuous. If $(G_1)$ holds, then Eq. (4) has at least one $2\pi$-periodic solution and, there is $N^* > 0$ such that, for any positive integer $N > N^*$, Eq. (1) has at least one subharmonic solution $x_N(t)$ with minimal period $2N\pi$. Moreover,

$$\lim_{N \to +\infty} \min_{t \in [0, 2N\pi]} \left( |x_N(t)| + |\phi(x_N'(t))| \right) = +\infty.$$

3. **Geometric properties of the solutions of a modified system.** Consider the equivalent system of Eq. (1) of the form

$$x' = \phi^{-1}(y), \quad y' = -g(t, x),$$

We assume in what follows that $g(t, x)$ is continuous and $(G_0)$ holds. Moreover, we assume, without loss of generality, that every solution of system (6) is unique with respect to initial value.

**Remark 3.1.** If the solution of system (6) is not unique with respect to initial value, we can get same result by using an approximation approach as in [11].
For any given constants \( R_0, R_1 \) with \( R_1 > R_0 > 0 \), let \( \mathcal{K}(\cdot) \in C^\infty \) be a function such that
\[
\mathcal{K}(x^2 + y^2) = \begin{cases} 
0, & 0 \leq x^2 + y^2 \leq R_0^2; \\
1, & x^2 + y^2 \geq R_1^2.
\end{cases}
\]
We consider the following modified system
\[
\begin{align*}
x' &= \frac{\partial H}{\partial y} = \phi^{-1}(y) + 2yK'(x^2 + y^2) \left[ G(t, x) - \frac{x^2}{2} \right], \\
y' &= -\frac{\partial H}{\partial x} = -2xK'(x^2 + y^2) \left[ G(t, x) - \frac{x^2}{2} \right] + K(x^2 + y^2) [g(t, x) - x] - x
\end{align*}
\]
with the Hamiltonian
\[
H(x, y) = \Phi^{-1}(y) + \mathcal{K}(x^2 + y^2)G(t, x) + \left[ 1 - \mathcal{K}(x^2 + y^2) \right] \frac{x^2}{2},
\]
where
\[
G(t, x) = \int_0^x g(t, s)ds, \quad \Phi^{-1}(y) = \int_0^y \phi^{-1}(s)ds.
\]

In this section we will prove some basic geometric properties of the solutions of (7). In particular, we will prove that solutions of (7) have spiral properties, that is there are two spiral curves guiding the solutions of (7) in the phase plane and forcing them to rotate around the origin as they increase in norm.

**Lemma 3.1.** Every solution of the Cauchy problem associated with system (7) is defined on the whole t-axis.

**Proof.** Consider the first equality of (7). When \( x^2 + y^2 \leq R_0^2 \) or \( x^2 + y^2 \geq R_1^2 \), we have \( x' = \phi^{-1}(y) \). Thus from the bounded range of the homeomorphism \( \phi^{-1} \), it implies that \( |x'(t)| \leq a \). When \( R_0^2 \leq x^2 + y^2 \leq R_1^2 \), since functions \( K'(x^2 + y^2) \) and \( G(t, x) \) are continuous, the right terms of the first equality of (7) are bounded, that is, we have \( M_1 > a \), such that
\[
|x'(t)| < M_1.
\]
Let \((x(t; x_0, y_0), y(t; x_0, y_0))\) be the solution of system (7) with the initial value condition \( x(0; x_0, y_0) = x_0, y(0; x_0, y_0) = y_0 \). For any given finite time \( T \), we have
\[
|x(t; x_0, y_0)| \leq |x_0| + M_1T, \quad \text{for } |t| \leq T.
\]
For \( |t| \leq T \), when \( x^2 + y^2 \leq R_0^2 \), we have \( y' = -x \), then \( |y'(t)| \leq R_0 \);
when \( x^2 + y^2 \geq R_1^2 \), we have \( y' = -g(t, x) \), then
\[
|y'(t)| \leq h = \max\{|g(t, x)| : t \in [0, 2\pi], |x| \leq |x_0| + M_1T\};
\]
when \( R_0^2 \leq x^2 + y^2 \leq R_1^2 \), since functions \( K(x^2 + y^2), K'(x^2 + y^2) \) and \( G(t, x) \) are continuous, we can get that there exist \( M_2 > \max\{R_1, h\}, \) such that
\[
|y'(t)| \leq M_2, \quad \text{for } |t| \leq T.
\]
Moreover,
\[
|y(t; x_0, y_0)| \leq |y_0| + M_2T, \quad \text{for } |t| \leq T.
\]
Thus the solution \((x(t; x_0, y_0), y(t; x_0, y_0))\) can not go to infinity for \( |t| \leq T \). Therefore the global existence of solutions is proved.\(\)
Note that the uniqueness of the solution with respect to initial value implies the continuity of the solution with respect to initial value combined with the global existence of solutions we have the following lemma

**Lemma 3.2.** The solutions of (7) have elastic property, that is, for any given positive constants \( T > 0 \) and \( b_0 > 0 \), there is \( r_{b_0} > 0 \) such that the inequality \(|(x_0, y_0)| \geq r_{b_0} \) implies \(|(x(t; x_0, y_0), y(t; x_0, y_0))| \geq b_0, \) for \(|t| \leq T.\)

Moreover, \((0,0)\) is the unique solution of (7) passing through the origin \( O. \) Thus if \((x_0, y_0) \neq (0,0), (x(t; x_0, y_0), y(t; x_0, y_0)) \neq (0,0)\) for every \( t \in \mathbb{R}. \) Then, polar coordinates

\[
x(t; x_0, y_0) = r(t; \theta_0, r_0) \cos \theta(t; \theta_0, r_0), \quad y(t; x_0, y_0) = r(t; \theta_0, r_0) \sin \theta(t; \theta_0, r_0)
\]

are well defined. Moreover, when \( x(t; x_0, y_0) = 0 \) we have

\[
x'(t; x_0, y_0) = \phi^{-1}(y(t; x_0, y_0)) \neq 0 \text{ if } y(t; x_0, y_0) \neq 0,
\]

which implies that non-trivial solutions can never perform clockwise rotations at \( y\)-axis. More precisely, we have for any \( t_2 > t_1 > 0 \)

\[
\theta(t_2; \theta_0, r_0) - \theta(t_1; \theta_0, r_0) < \pi.
\]

Moreover, for any \( k \in \mathbb{Z}, \)

\[
\text{if } \theta(t_1; \theta_0, r_0) \leq k\pi + \pi/2, \text{ then } \theta(t_2; \theta_0, r_0) < k\pi + \pi/2.
\]

Next we will describe some geometric properties of the solutions of (7) using polar coordinates. To simplify the notation, we denote \((\theta(t; r_0, \theta_0), r(t; r_0, \theta_0))\) by \((\theta(t), r(t)).\)

**Lemma 3.3.** Let \((\theta(t), r(t))\) be the solution of system (7). Then there exists a positive constant \( R_2 > \max\{d_0, R_1\} \) such that

\[
\theta'(t) < 0, \quad \text{for } r(t) \geq R_2.
\]

**Proof.** For \( r(t) \geq R_1, \) we have

\[
\theta'(t) = \frac{\frac{x(t, x)}{r^2}}{y\phi^{-1}(y)}.
\]

In case \(|x(t)| \geq d_0\), using condition \((G_0)\) and \(y\phi^{-1}(y) > 0\), inequality (9) holds. In case \(|x(t)| \leq d_0\), we have

\[
|y(t)| \geq \sqrt{r^2(t) - d_0^2}.
\]

It follows that

\[
\theta'(t) \leq \frac{d_0M_3 - y\phi^{-1}(y)}{r^2} \leq \frac{d_0M_3}{r^2} \sqrt{r^2(t) - d_0^2} \phi^{-1}(\sqrt{r^2(t) - d_0^2}),
\]

where \(M_3 = \max_{t \in [0, 2\pi]} |g(t, x)|.\)

Note that \(y\phi^{-1}(y) \to +\infty, \) as \( y \to +\infty. \) Hence, taking sufficient large number \(R_2 > 0,\) we obtain the inequality (9). The proof is thus complete.

Furthermore, we prove that the solutions of system (7) have slow oscillatory properties.
Lemma 3.4. For any given $R > R_2 > \max\{R_1, \sqrt{2}d_0\}$, if the solution $(\theta(t), r(t))$ of system (7) satisfies
\[
\lim_{R \to +\infty} (t_1 - t_0) = +\infty.
\]
then we have
\[
r(t) \geq R, \text{ for } t \in (t_0, t_1), \quad \text{and} \quad \theta(t_0) - \theta(t_1) = 2\pi,
\]
Proof. Note that if $R$ is large enough, then $\theta'(t) < 0$. The expression of $(\theta(t), r(t))$ in rectangular coordinates system $(x(t), y(t))$ satisfies
\[
\begin{cases}
x' = \phi^{-1}(y), \\
y' = -g(t, x).
\end{cases}
\]
Moreover, if $\theta(t_0) - \theta(t_1) = 2\pi$, then there exist $\alpha_+ > 0$ and $t_{\alpha+}, t'_{\alpha+} \in (t_0, t_1)$ such that $x(t_{\alpha+}) = \alpha + 2$, $x(t'_{\alpha+}) = \alpha + 2$ and $y(t_{\alpha+}) = 0$. From the first equation of (10), we have
\[
t_{\alpha+} - t_0 \geq \int_{\alpha_{\alpha+}/2}^{\alpha_+} \frac{1}{\phi^{-1}(y)} \, dx \geq \frac{\alpha_+ - a}{2} \to +\infty,
\]
as $R \to +\infty$. The proof is thus complete. \qed

Finally, we will prove that solutions of (7) have spiral properties, which is crucial in our approach.

Lemma 3.5. Let $(\theta(t), r(t))$ be the solution of system (7). For every positive integer
\[
N > 0,\text{ there is a constant } R_3^{(N)} > R_2 \text{ and there are two strictly monotone increasing functions } \zeta_N, \zeta_N : [R_3^{(N)}, +\infty) \to \mathbb{R}^+, \text{ with } \zeta_N(s) = r(s) \to +\infty \text{ as } s \to +\infty. \quad \text{Moreover, for every solution } (\theta(t), r(t)) \text{ of system (7) and every } r^* > N, \text{ if } r(t_0) = r^* \text{ and } r(t_0) > \zeta_N(r^*) \text{ (or } r(t_0) < \zeta_N(r^*)), \text{ then we have } t_N \in (t_0, t_N), \text{ such that }
\]
\[
\theta(t_N) - \theta(t_0) = 2N\pi,
\]
and
\[
\zeta_N(r^*) \leq r(t) \leq \zeta_N(r^*), \quad \text{for } t \in [t_0, t_N].
\]
Proof. At first, we assume $N = 1$. We will prove that for sufficient large $r_0$, we have $\zeta_{\pm 1}(r_0)$ such that the solution $z(t) = (x(t), y(t)) = (\theta(t), r(t))$ with the initial value $|z(t_0)| = r_0$ will complete at least one clockwise turn before it leaves annular $\zeta_{-1}(r_0) \leq r \leq \zeta_{+1}(r_0)$. To prove this we divide the region $\bar{D}$
\[
\bar{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq R_2\}
\]
into six parts
\[
\bar{D}_1 = \{(x, y) \in \bar{D} : -R_2 \leq x \leq R_2, y \geq 0\};
\]
\[
\bar{D}_2 = \{(x, y) \in \bar{D} : x \geq R_2, y \geq 0\};
\]
\[
\bar{D}_3 = \{(x, y) \in \bar{D} : x \geq R_2, y \leq 0\};
\]
\[
\bar{D}_4 = \{(x, y) \in \bar{D} : -R_2 \leq x \leq R_2, y \leq 0\};
\]
\[
\bar{D}_5 = \{(x, y) \in \bar{D} : x \leq -R_2, y \leq 0\};
\]
\[
\bar{D}_6 = \{(x, y) \in \bar{D} : x \leq -R_2, y \geq 0\}.
\]
We will describe the behaviour of the solution in the following steps.

**Step 1.** Without loss of generality, let \( z(t) \) start from the line \( x = -R_2 \) and choose \( y_0 \geq R_2 \). For \( |x(t)| \leq R_2 \) and \( y(t) \geq R_2 \) we have
\[
\frac{dy}{dx} = \frac{g(t, x)}{\phi^{-1}(y)} \leq \max_{t \in [0, 2\pi]} |g(t, x)|/\phi^{-1}(R_2).
\]

Using \( x'(t) = \phi^{-1}(y(t)) > 0 \), we get
\[
|y(t) - y_0| = \left| \int_{t_0}^{t} y'(t)dt \right| = \left| \int_{x_0}^{x(t)} \frac{dy}{dx} dx \right| \leq c_1 := 2R_2 \max_{t \in [0, 2\pi]} |g(t, x)|/\phi^{-1}(R_2).
\]

Then for \( r_0 \geq c_1 + 3R_2 \) we have
\[
y(t) \geq y_0 - c_1 \geq r_0 - R_2 - c_1 \geq 2R_2
\]
which implies that \( x'(t) \geq \phi^{-1}(2R_2) \) for \( |x(t)| \leq R_2 \). Hence, for sufficient large \( r_0 \) there is \( s_1 > t_0 \), such that \( |x(t)| \leq R_2 \) for \( t \in [t_0, s_1] \) and \( x(s_1) = R_2 \). Moreover, from \( |x(t) - x_0| \leq 2R_2 \) and \( |y(t) - y_0| \leq c_1 \), we get
\[
|r(t) - r_0| = |z(t) - z(t_0)| \leq |z(t) - z(t_0)| \leq \sqrt{(x(t) - x(t_0))^2 + (y(t) - y(t_0))^2} \leq \sqrt{4R_2^2 + c_1^2}.
\]

Denote by \( \xi_{\pm 1}(r_0) = r_0 \pm \sqrt{4R_2^2 + c_1^2} \), then
\[
\xi_{-1}(r_0) \leq r(t) \leq \xi_{+1}(r_0), \quad \forall t \in [t_0, s_1].
\]

**Step 2.** Let \( z(t) \in \overline{D}_2 \) for \( t \in [s_1, s_2] \). To give the estimation of \( z(t) \in \overline{D}_2 \), we define the functions
\[
\overline{\mathcal{H}}(x, y) = G(x) + \Phi^{-1}(y) \quad \text{and} \quad \underline{\mathcal{H}}(x, y) = G(x) + \Phi^{-1}(y),
\]
where \( \overline{G}(x) = \int_0^x \max_{t \in [0, 2\pi]} g(t, s)ds \) and \( \underline{G}(x) = \int_0^x \min_{t \in [0, 2\pi]} g(t, s)ds \), respectively. By the fact that \( \phi \) is an increasing homeomorphism onto \( \mathbb{R} \) and the condition \( (G_0) \), we have
\[
\overline{\mathcal{H}}(x, y) \to +\infty \iff \sqrt{x^2 + y^2} \to +\infty \iff \underline{\mathcal{H}}(x, y) \to +\infty. \tag{11}
\]

Moreover,
\[
\Gamma_{-2}(r(s_1)) : \overline{\mathcal{H}}(x, y) = \overline{G}(x(s_1)) + \Phi^{-1}(y(s_1)), \quad (x, y) \in \overline{D}_2,
\]
\[
\Gamma_{+2}(r(s_1)) : \underline{\mathcal{H}}(x, y) = \underline{G}(x(s_1)) + \Phi^{-1}(y(s_1)), \quad (x, y) \in \overline{D}_2
\]
are two convex curves connecting the line \( x = R_2 \) and the \( x \)-axis in \( \overline{D}_2 \).

Let \( \overline{h}(t) = \overline{\mathcal{H}}(x(t), y(t)) \) and \( \underline{h}(t) = \underline{\mathcal{H}}(x(t), y(t)) \). It is easy to verify that
\[
\overline{h}(t) \geq 0 \quad \text{and} \quad \underline{h}(t) \leq 0, \quad \text{for} \quad r(t) \geq R_2 \quad \text{and} \quad z(t) \in \overline{D}_2.
\]

Thus, we have
\[
\overline{h}(t) \geq \overline{h}(s_1) \quad \text{and} \quad \underline{h}(t) \leq \underline{h}(s_1), \quad \text{for} \quad t \in [s_1, s_2]. \tag{12}
\]

Therefore, from (12), we have
\[
\xi_{-2}(r(s_1)) \leq r(t) \leq \xi_{+2}(r(s_1)), \quad \forall t \in [s_1, s_2]. \tag{13}
\]
where
\[
\xi^{-2}(r(s_1)) = \min \left\{ \sqrt{x^2 + y^2} : (x, y) \in \Gamma^{-2}(r(s_1)) \right\},
\]
\[
\xi^{+2}(r(s_1)) = \max \left\{ \sqrt{x^2 + y^2} : (x, y) \in \Gamma^{+2}(r(s_1)) \right\}
\]
with
\[
\xi^{\pm 2}(r(s_1)) \to +\infty \iff r(s_1) \to +\infty.
\]

On the other hand, from Lemma 3.3, \(\theta'(t) < 0\) in
\[
E_2 = \overline{D}_2 \cap \{ \xi^{-2}(r(s_1)) \leq r \leq \xi^{+2}(r(s_1)) \}.
\]
Thus, there is \(c_2 > 0\) such that \(\theta'(t) < -c_2\) for \(z(t) \in E_2\) which implies \(z(t)\) will escapes from \(E_2\) at some time \(t'_2\). Hence, we can assume that \(t'_2 = s_2\), that is \(y(s_2) = 0\) and \(x(s_2) = r(s_2)\).

**Step 3.** The arguments for \(\overline{D}_3, \overline{D}_5\) and \(\overline{D}_6\) are similar to that for \(\overline{D}_2\), and the argument for \(\overline{D}_4\) is similar to that for \(\overline{D}_1\). Therefore, we have times \(s_i\) and functions \(\xi^{\pm i}(r(s_i))\), \(i = 3, 4, 5, 6\), such that
\[
\xi^{-i}(r(s_{i-1})) \leq r(t) \leq \xi^{+i}(r(s_{i-1})), \quad \text{and} \quad z(t) \in \overline{D}_i,
\]
for \(t \in [s_{i-1}, s_i]\), \(i = 3, 4, 5, 6\), respectively. Moreover,
\[
\xi^{\pm i}(r(s_i)) \to +\infty \iff r(s_i) \to +\infty, \quad i = 3, 4, 5, 6.
\]

Obviously, we can choose \(\xi^{\pm i}(r(s_i))\) to be strictly monotonic increasing functions with respect to \(r(s_i)\), \(i = 1, \ldots, 6\). We define
\[
\zeta_1(r) = \xi^{+6} \circ \cdots \circ \xi^{+1}(r), \quad \zeta_{-1}(r) = \xi^{-6} \circ \cdots \circ \xi^{-1}(r),
\]
respectively. Let \(t_1 = s_6\), then we have
\[
\theta(t_1) - \theta(t_0) = -2\pi,
\]
and
\[
\zeta_{-1}(r^*) \leq r(t) \leq \zeta_1(r^*), \quad \text{for} \quad t \in [t_0, t_1].
\]

In the same manner we can prove above conclusion when the initial value point \((\theta_0, r_0)\) is in any subregions of \(\overline{D}\). Moreover, there exists a constant \(K_1 > \zeta_{-1}(R_2)\) such that, for every \(r^* > K_1\), if \(r(t_0) = r^*\) and \(r(t'_0) > \zeta_1(r^*)\) \(t'_0 > t_0\) or \(r(t'_0) < \zeta_{-1}(r^*)\) \(t'_0 > t_0\), then we have \(t_1 \in (t_0, t'_0)\), such that (15) and (16) hold.

Finally, for given positive integer \(N\), define
\[
\zeta_N(r) = \underbrace{\zeta_1 \circ \cdots \circ \zeta_1}_{N}(r), \quad \zeta_{-N}(r) = \underbrace{\zeta_{-1} \circ \cdots \circ \zeta_{-1}}_{N}(r).
\]
Take \(R_3^{(N)} = \zeta_{-N}(R_2)\), then \(\zeta_N(r), \zeta_{-N}(r)\) are the functions such that the conclusion of the lemma holds. The proof is thus completed. \(\square\)
4. Existence of infinitely many subharmonic solutions. In this section we will give the proof of Theorem 1.1 by using the Poincaré-Birkhoff twist theorem.

For any prime number \( n \), we will prove that there is \( m_0(n) > 0 \), such that for any positive integer \( m \geq m_0(n) \), there exists an annulus \( A_{n,m} \) such that the Poincaré-Birkhoff twist theorem can be used on \( A_{n,m} \). Firstly, we find the inner boundary of \( A_{n,m} \). According to Lemma 3.5, for given \( R_3^{(n)} > R_2 \), there exist

\[
R_4^{(n)} = \zeta_n^{-1}(R_3^{(n)}), \quad R_5^{(n)} = \zeta_n(R_3^{(n)}).
\]

Denote the annulus by

\[
A(R_3^{(n)}, R_5^{(n)}) = \{(\theta, r) \in \mathbb{R}^2 : R_3^{(n)} \leq r < R_5^{(n)} \}.
\]

By Lemma 3.3, \( \theta'(t) < 0 \) holds on the compact annulus \( A(R_3^{(n)}, R_5^{(n)}) \). Note that from the polar form of the system (7), \( \theta'(t) = \Phi(t, \theta(t), r(t)) \), where \( \Phi(t, \theta, r) \) is a continuous function, \( 2\pi \)-periodic with respect to \( t \) and \( \theta \). So \( \Phi(t, \theta, r) \) has negative lower and upper bounds on compact annulus \( A(R_3^{(n)}, R_5^{(n)}) \). That is, we can choose positive constant \( b_1, b_2 > 0 \), such that

\[-b_2 < \theta'(t) < -b_1, \quad \text{for all } (\theta, r) \in A(R_3^{(n)}, R_5^{(n)}).
\]

Denote by \( \Gamma_n^- = \{ r = R_4^{(n)} \} \). For any positive integer \( m \geq m_0 = \left\lceil \frac{n}{b_1} \right\rceil + 1 \), where \([x]\) denotes the integer part of \( x \), we claim that, every solution of Eq. (7) starting from \( \Gamma_n^- \) goes at least \( n \) clock-wise turns around the origin for \( t \in [0, 2m\pi] \).

In fact, if \( (r, \theta) \in A(R_3^{(n)}, R_5^{(n)}) \), for all \( t \in [0, 2m\pi] \), then we have

\[
\theta(2m\pi) - \theta_0 = \int_0^{2m\pi} \theta'(t) \, dt < -2mb_1\pi
\]

\[
< -b_12\left( \left\lceil \frac{n}{b_1} \right\rceil + 1 \right) \pi < -2n\pi.
\]

If there exists \( t_1 \in (0, 2m\pi) \) such that \( r(t_1) < R_3^{(n)} \) or \( r(t_1) > R_5^{(n)} \), according to Lemma 3.5, we know that

\[
\theta(t_1) - \theta(0) < -2(n + 1)\pi.
\]

And using (8), we obtain

\[
\theta(2m\pi) - \theta(t_1) < \pi.
\]

It follows that

\[
\theta(2m\pi) - \theta(0) < \theta(2m\pi) - \theta(t_1) + \theta(t_1) - \theta(0) < -2n\pi.
\]

Next we find the outer boundary \( \Gamma_n^+ \). By Lemma 3.4, there exists \( R_6^{(m)} > R_5^{(n)} \) large enough such that, for every solution with \( r(t) > R_6^{(m)} \), \( t \in [0, 2m\pi] \), we have

\[
\theta(t) - \theta(0) > -2n\pi.
\]

Also, Lemma 3.2 implies that there exist \( R_7^{(m)} \) such that every solution starting from \( r = R_7^{(m)} \) satisfies

\[
r(t) > R_6^{(m)}, \quad \text{for all } t \in [0, L].
\]

Let \( \Gamma_m^+ = \{ r = R_7^{(m)} \} \). Then the solution starting from \( \Gamma_m^+ \) satisfies

\[
\theta(2m\pi) - \theta(0) > -2n\pi.
\]
Denote the annulus, using the polar lifting, by
\[ A_{n,m} = \{ (\theta, r) \in \mathbb{R}^2 : R_4^{(n)} \leq r \leq R_7^{(m)} \} \]
and the \( m \) order iteration of Poincaré map by
\[ \tilde{P}_m : (\theta_0, r_0) \mapsto (\theta(2m\pi; \theta_0, r_0), r(2m\pi; \theta_0, r_0)) \].

With the arguments discussed above, \( \tilde{P}_m \) is an area-preserving homeomorphism with boundary twisting on the annulus \( A_{n,m} \).

By the generalized Poincaré–Birkhoff twist theorem (see [33, 13, 20, 31], and [30] for its various versions) we have that, there are at least two fixed points of \( \tilde{P}_m \)
\[ \eta_{n,m,k} = (\theta_{n,m,k}, r_{n,m,k}) \in A_m, \quad k = 1, 2, \]
which correspond two geometrically distinct fixed points
\[ z_{n,m,k} = (x_{n,m,k}, y_{n,m,k}), \quad k = 1, 2, \]
of the \( m \) order iteration of Poincaré map \( P_m \) for Eq. (7). Thus we have two \( 2m\pi \)-periodic solutions
\[
\begin{align*}
z_{n,m,1}(t) &= (x(t; x_{n,m,1}, y_{n,m,1}), y(t; x_{n,m,1}, y_{n,m,1})), \\
z_{n,m,2}(t) &= (x(t; x_{n,m,2}, y_{n,m,2}), y(t; x_{n,m,2}, y_{n,m,2}))
\end{align*}
\]
of Eq. (7), such that \( z_{n,m,1}(t) \) and \( z_{n,m,2}(t) \) complete exactly \( n \) clock-wise turns around the origin \( O \) in \( [0, 2m\pi] \), respectively.

Let \( z_{n,m,1}(t) = (r_{n,m,1}(t) \cos \theta_{n,m,1}(t), r_{n,m,1}(t) \sin \theta_{n,m,1}(t)) \). Note that
\[ r_{n,m,1}(0) \geq R_4^{(n)} = \zeta_{-1}(R_3^{(n)}). \]
If there exists \( t_1 \in (0, 2m\pi) \) such that \( r_{n,m,1}(t_1) < R_3^{(n)} \), according to Lemma 3.5, we know that
\[ \theta_{n,m,1}(t_1) - \theta_{n,m,1}(0) < -2(n + 1)\pi. \]
Using (8), we obtain
\[ \theta_{n,m,1}(2m\pi) - \theta_{n,m,1}(t_1) < \pi. \]
It follows that
\[ \theta_{n,m,1}(2m\pi) - \theta_{n,m,1}(0) \leq \theta_{n,m,1}(2m\pi) - \theta_{n,m,1}(t_1) + \theta_{n,m,1}(t_1) - \theta_{n,m,1}(0) < -(2n + 1)\pi. \]
That is, \( z_{n,m,1}(t) \) completes \( n \) and half more clock-wise turns around the origin \( O \) in \( [0, 2m\pi] \). This is a contradiction. Thus, \( z_{n,m,1}(t) \) is the solution of Eq. (6). The same argument can be used to prove that \( z_{n,m,2}(t) \) is also the solution of Eq. (6). Therefore, \( x(t; x_{n,m,k}, y_{n,m,k}), \quad k = 1, 2, \) are the \( 2m\pi \)-periodic solutions of Eq. (1) which have exactly \( 2n \) zeros in \( [0, 2m\pi] \). Since \( g(t, x) \) is \( 2\pi \)-least period with respect to \( t \), \( x(t; x_{n,m,k}, y_{n,m,k}), \quad k = 1, 2, \) have exactly \( 2n \) zeros in \( [0, 2m\pi] \) and \( m \) is prime with \( n \), so that \( x(t; x_{n,m,k}, y_{n,m,k}), \quad k = 1, 2, \) have minimal period \( 2m\pi \).

Besides, since the \( 2\pi \)-periodic system (6) admits a positively bounded solution and it’s Poincaré map is defined on \( \mathbb{R}^2 \), we can use the Massera theorem (see Massera [28] or Theorem 4.8 and Corollary 4.3 in [9]) to prove that Eq. (1) has at least one \( 2\pi \)-periodic solution.

Moreover, for fixed \( n \) and \( k \), if there are a sequence \( \{m_l\} \) and a bounded annulus
\[ A_\ast = \{ (r, \theta) \in \mathbb{R}^2 : R_3^{(n)} \leq r \leq R_\ast \}, \]
with \( R_* < +\infty \), such that
\[
z_{n,m,k}(t) \in A_*, \quad \text{for} \quad t \in [0, 2m\pi].
\]
Reasoning as to find \( \Gamma_-^* \), we have \( b_* > 0 \) such that the polar angle component \( \theta_{n,m,k}(t) \) satisfies
\[
\theta_{n,m,k}'(t) < -b_*, \quad \text{for} \quad t \in [0, 2m\pi].
\]
Thus,
\[
\theta(2m\pi) - \theta(0) < -2mb_* \to -\infty, \quad \text{as} \quad l \to \infty,
\]
which contradicts
\[
\theta(2m\pi) - \theta(0) = -2n\pi.
\]
Therefore, we have
\[
\lim_{m \to +\infty} \max_{t \in [0, 2m\pi]} \left( |x_{n,m,k}(t)| + |\phi(x_{n,m,k}'(t))| \right) = +\infty.
\]
Furthermore, if there is a sequence \( \{m_p\} \) and a bounded annulus
\[
A_{**} = \{(\theta, r) \in \mathbb{R}^2 : R_3^{(n)} \leq r \leq R_{**}\}
\]
with \( R_{**} > 0 \) fixed, such that
\[
\max_{t \in [0, 2m_p\pi]} \left( |x_{n,m_p,k}(t)| + |\phi(x_{n,m_p,k}'(t))| \right) \in A_{**},
\]
then there exist \( t_p < t_p' \), such that \( 0 < t_p' - t_p < 2m_p\pi \), \( z_{n,m_p,k}(t_p) \in A_{**} \), and
\[
|z_{n,m_p,k}(t_p')| \to +\infty, \quad \text{as} \quad p \to \infty.
\]
Using Lemma 3.5, we have
\[
\theta_{n,m_p,k}(t_p') - \theta_{n,m_p,k}(t_p) \to -\infty, \quad \text{as} \quad p \to \infty.
\]
Combining (8), we have
\[
\theta_{n,m_p,k}(2m_p\pi) - \theta_{n,m_p,k}(0) \to -\infty, \quad \text{as} \quad p \to \infty.
\]
This is a contradiction.
Therefore, we have
\[
\lim_{m \to +\infty} \min_{t \in [0, 2m\pi]} \left( |x_{n,m,k}(t)| + |\phi(x_{n,m,k}'(t))| \right) = +\infty.
\]

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REFERENCES
[1] N. Atakishiev and R. Mir-Kasimov, Generalized coherent states for relativistic model of a linear oscillator, *Theor. Math. Phys.*, 67 (1986), 362–367.
[2] C. Bereanu and J. Mawhin, Nonlinear Neumann boundary value problems with \( \phi \)-Laplacian operators, *An. Stiint. Univ. Ovidius Constanta Ser. Mat.*, 12 (2004), 73–82.
[3] C. Bereanu and J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular-Laplacian, *NoDEA: Nonlinear Differ. Equ. Appl.*, 15 (2008), 159–168.
[4] C. Bereanu and J. Mawhin, Multiple periodic solutions of ordinary differential equations with bounded nonlinearities and \( \phi \)-Laplacian, *J. Differential Equations*, 243 (2007), 536–557.
[5] C. Bereanu and J. Mawhin, Boundary value problems for some nonlinear systems with singular \( \phi \)-Laplacian, *J. Fixed Point Theory Appl.*, 4 (2008), 57–75.
NONLINEAR SYSTEMS WITH SINGULAR φ-LAPLACIAN

[6] C. Bereanu and P. J. Torres, Existence of at least two periodic solutions of the forced relativistic pendulum, Proc. Amer. Math. Soc., 140 (2012), 2713–2720.

[7] A. Boscaggin and G. Feltrin, Positive periodic solutions to an indefinite Minkowski-curvature equation, arXiv:1905.06659.

[8] A. Boscaggin and M. Garrione, Sign-changing subharmonic solutions to unforced equations with singular φ-Laplacian, Differential and Difference Equations with Applications, Springer Proceedings in Mathematics and Statistics, 47, 321–329.

[9] T. Ding, Approaches to the Qualitative Theory of Ordinary Differential Equations: Dynamical Systems and Nonlinear Oscillations, Peking University Series in Mathematics, World Scientific Publishing Co. Pte. Ltd., Singapore, 2007.

[10] T. Ding, R. Iannacci and F. Zanolin, On periodic solutions of sublinear Duffing equations, J. Math. Anal. Appl., 158 (1991), 316–332.

[11] T. Ding and F. Zanolin, Periodic solutions of Duffing’s equations with superquadratic potential, J. Differential Equations, 97 (1992), 328–378.

[12] T. Ding and F. Zanolin, Subharmonic solutions of second order nonlinear equations: a time-map approach, Nonlinear Anal., 20 (1993), 509–532.

[13] W. Ding, A generalization of the Poincaré-Birkhoff theorem, Proc. Amer. Math. Soc., 88 (1983) 341–346.

[14] T. Donde and F. Zanolin, Multiple periodic solutions for one-sided sublinear systems: A refinement of the Poincaré-Birkhoff approach, arXiv:1901.09406.

[15] A. Fonda, R. Manásevich and F. Zanolin, Subharmonic solutions for some second-order differential equations with singularities, SIAM J. Math. Anal., 24 (1993), 1294–1294.

[16] A. Fonda and M. Ramos, Large-amplitude subharmonic oscillations for scalar second-order differential equations with asymmetric nonlinearities, J. Differential Equations, 109 (1994), 354–372.

[17] A. Fonda and A. Sfecci, Periodic solutions of weakly coupled superlinear systems, J. Differential Equations, 260 (2016), 2150–2162.

[18] A. Fonda and A. Sfecci, A general method for the existence of periodic solutions of differential equations in the plane, J. Differential Equations, 252 (2012), 1369–1391.

[19] A. Fonda and R. Toader, Subharmonic solutions of Hamiltonian systems displaying some kind of sublinear growth, Adv. Nonlinear Anal., 8 (2019), 583–602.

[20] J. Franks, Generalizations of the Poincaré-Birkhoff theorem, Ann. Math., 128 (1988), 139–151.

[21] J. Ginocchio, Relativistic symmetries in nuclei and hadrons, Phys. Rep., 414 (2005), 155–261.

[22] Z. Guo and J. Yu, The existence of periodic and subharmonic solutions of subquadratic second order difference equations, J. London Math. Soc., 68 (2003), 419–430.

[23] Q. Jiang and C. Tang, Periodic and subharmonic solutions of a class of subquadratic second-order Hamiltonian systems, J. Math. Anal. Appl., 328 (2007), 380–389.

[24] J. Kim and H. Lee, Nonlinear resonance and chaos in the relativistic phase space for driven nonlinear systems, Phys. Rev. E, 52 (1995), 473–480.

[25] J. Kim and H. Lee, Relativistic chaos in the driven harmonic oscillator, Phys. Rev. E, 51 (1995), 1579–1581.

[26] A. Kolovsky, Relativistic chaos for an electron in a standing microwave field, EPL-Europhysics Lett., 41 (1998), 257.

[27] D. Kulikov and R. Tutik, Oscillator model for the relativistic fermion-boson system, Phys. Lett. A, 372 (2008), 7105–7108.

[28] J. Massera, The existence of periodic solutions of systems of differential equations, Duke Math. J., 17 (1950), 475–475.

[29] Z. Opial, Sur les solutions périodiques de l’équation différentielle $x'' + g(x) = p(t)$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 8 (1960), 151–156.

[30] D. Qian, Infinity of Subharmonics for Asymmetric Duffing Equations with the Lazer-Leach-Dancer Condition, J. Differential Equations, 171 (2001), 233–250.

[31] D. Qian and P. J. Torres, Periodic motions of linear impact oscillators via the successor map, SIAM J. Math. Anal., 36 (2005), 1707–1725.

[32] D. Qian, P. J. Torres and P. Wang, Periodic solutions of second order equations via rotation numbers, J. Differential Equations, 266 (2019), 4746–4768.

[33] C. Rebelo, A note on the Poincaré-Birkhoff fixed point theorem and periodic solutions of planar systems, Nonlinear Anal., 29 (1997), 291–311.
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