ONTO INTERPOLATING SEQUENCES FOR THE DIRICHLET SPACE

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ABSTRACT. We describe two new classes of onto interpolating sequences for the Dirichlet space, in particular resolving a question of Bishop. We also give a complete description of the analogous sequences for a discrete model of the Dirichlet space.

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1. Overview

We begin with an informal overview; precise definitions and statements will be given later.

1.1. Interpolating Sequences for the Hardy Space. The study of interpolating sequences for the Dirichlet space evolved from the study of interpolating sequences for the Hardy space and we begin by recalling that work.

The Hardy space, $H^2$, is a Hilbert space of holomorphic functions on the unit disk. It is normed by, for $f = \sum_{n \leq 0} a_n z^n$, $\|f\|_{H^2}^2 = \sum |a_n|^2$. It has reproducing kernels $k_z^H = (1 - \bar{z}w)^{-1}$. The kernel has norm $\|k_z^H\|_{H^2} = (1 - |z|^2)^{-1/2}$ and we denote the normalized kernels by $\hat{k}_z^H$; $\hat{k}_z^H = (1 - |z|^2)^{1/2}k_z^H$.

Given a sequence $Z = \{z_i\}$ contained in the disk we define a weighted restriction map $R_{H^2} = R_{H^2,Z}$ mapping functions on the disk to functions on $Z$ by $(R_{H^2}f)(z_i) = \|k_{z_i}^H\|_{H^2}^{-1}f(z_i)$. Straightforward Hilbert space considerations insure that $R_{H^2}$ maps $H^2$ into $\ell^\infty(Z)$. If, in fact, $R_{H^2}$ maps $H^2$ into and onto $\ell^2(Z)$ then we say that $Z$ is an interpolating sequence for the Hardy space.

If $R_{H^2}$ maps onto $\ell^2(Z)$ then norm control is possible; there is a $C > 0$ so that for each $\gamma \in \ell^2(Z)$ there is a $\Gamma \in H^2$ with $R_{H^2}\Gamma = \gamma$ and $\|\Gamma\|_{H^2} \leq C \|\gamma\|$. To see this note that, letting $V_Z$ be the closed subspace of functions in $H^2$ that vanish on $Z$, there is a well defined linear map $\Lambda$ of $\ell^2(Z)$ into $V_Z^\perp$ such that $R_{H^2}\Lambda$ is the identity on $\ell^2(Z)$. The closed graph theorem insures that $\Lambda$ is continuous and thus, given $\gamma \in \ell^2(Z)$, the choice $\Gamma = \Lambda\gamma$ meets the requirements. In particular, if $R_{H^2}$ is onto then there is a $C > 0$ so that for any $i, j, i \neq j$ we can find $f_{ij} \in H^2$ with

$$R_{H^2}f_{ij}(z_i) = 1, \quad R_{H^2}f_{ij}(z_j) = 0$$

and $\|f_{ij}\|_{H^2} \leq C$. On the other hand, given $z_i, z_j \in \mathbb{D}$, any $f_{ij} \in H^2$ which satisfies (1.1) then it must satisfy

$$\|f_{ij}\| \geq \left(1 - \left|\langle \hat{k}_{z_i}^H, \hat{k}_{z_j}^H \rangle \right|^2\right)^{-1/2}.$$

If $Z$ is an interpolating sequence then we can combine this with the previous estimate for $\|f_{ij}\|$ and find that the points of $Z$ satisfy a separation condition: $\exists \varepsilon > 0, \forall i, j, i \neq j$

(Sep(Hardy))

$$\left|\langle \hat{k}_{z_i}^H, \hat{k}_{z_j}^H \rangle \right| \leq 1 - \varepsilon.$$

An equivalent condition, written using $\beta$, the hyperbolic metric on the disk, is $\exists \varepsilon' > 0, \forall i, j, i \neq j$

(Sep'(Hardy))

$$\beta(z_i, z_j) \geq \varepsilon'.$$

The interpolating sequence $Z$ must satisfy (Sep(Hardy)) because $R_{H^2}$ is onto. The requirement that $R_{H^2}$ be into, that is, that $R_{H^2}$ be bounded, gives a different requirement. Associate to the sequence $Z$ the measure $\lambda_Z$ given by

$$\lambda_Z = \sum_{j=1}^{\infty} \|\hat{k}_z\|_{H^2}^{-2} \delta_{z_j}.$$
The condition that $R_{H^2}$ be bounded, that there is a $C > 0$ so that for all $f \in H^2$ \[ \|R_{H^2}f\| \leq \sqrt{C} \|f\|_{H^2}, \] is equivalent to the condition that $\lambda_Z$ be a Carleson measure (for the Hardy space). That is, there is a $C > 0$ so that for $\lambda = \lambda_Z$ and all $f \in H^2$

\[(\text{Car}(\text{Hardy})) \quad \int |f|^2 \, d\lambda \leq C \|f\|^2_{H^2}.\]

If this condition holds for some $\lambda$ then a fortiori it holds when $f$ is a reproducing kernel. This implies the simple condition: $\exists C > 0, \forall z \in \mathbb{D}$

\[(\text{CarSimp}(\text{Hardy})) \quad \lambda(T(z)) \leq C |I_z|.\]

Here $I_z$ is the boundary interval with center $z/|z|$ and length $2\pi(1 - |z|)$, $|I_z|$ is its length, and $T(z)$ is the tent over $I_z$, the convex hull of $z$ and $I_z$.

In fact, these conditions characterize interpolating sequences. Combining the results of Carleson in 1958 [C] and Shapiro and Shields in 1961 [SS] we have

**Theorem 1.** Given $Z$, the following conditions are equivalent:

1. $R_{H^2,Z}$ maps $H^2$ into and onto $\ell^2(Z)$, that is, $Z$ is an interpolating sequence for the Hardy space.
2. $Z$ satisfies $\text{(Sep}(\text{Hardy}))$ and $\lambda_Z$ satisfies $\text{(Car}(\text{Hardy}))$.
3. $Z$ satisfies $\text{(Sep}(\text{Hardy}))$ and $\lambda_Z$ satisfies $\text{(CarSimp}(\text{Hardy}))$.
4. $R_{H^2,Z}$ maps $H^2$ onto $\ell^2(\lambda_Z)$.

The equivalence of the first two statements is the traditional description of interpolating sequences for the Hardy space. The second and third are equivalent because for any positive measure $\lambda$ the conditions $\text{(Car}(\text{Hardy}))$ and $\text{(CarSimp}(\text{Hardy})$ are equivalent. The first condition certainly implies the last; the converse of that implication is a subtle consequence of the details of the analysis.

1.2. Interpolating Sequences for the Dirichlet Space. The Dirichlet space, $B_2$, is a Hilbert space of holomorphic functions on the unit disk. It is normed by, for $f = \sum_{n \leq 0} a_n z^n$, $\|f\|^2 = |a_0|^2 + \sum_{n > 0} n |a_n|^2$. It has reproducing kernels $k_z = -\bar{z}w \log(1 - |z|^2)$. Note that $\|k_z\|^2 = -|z|^2 \log(1 - |z|^2)$ and that for $z$ near the boundary, the only case of interest for us, $\|k_z\|^2 \sim -\log(1 - |z|^2)$.

We denote the normalized kernels by $\hat{k}_z$; $\hat{k}_z = \|k_z\|^{-1} k_z$.

Given a sequence $Z = \{z_i\} \subset \mathbb{D}$ we now define a weighted restriction map $R = R_z$ by $(Rf)(z_i) = \|k_{z_i}\|^{-1} f(z_i)$. As before it is automatic that $R$ maps $B_2$ into $\ell^\infty(Z)$. If, in fact, $R$ maps $B_2$ into and onto $\ell^2(Z)$ then we say that $Z$ is an interpolating sequence for the Dirichlet space.

As in the Hardy space case, there are two natural necessary conditions for $R$ to be an interpolating sequence. The fact that $R$ maps onto $\ell^2(Z)$ insures that the points of $Z$ satisfy a separation condition: $\exists \varepsilon > 0, \forall i, j, i \neq j$

\[(\text{Sep}) \quad \left|\langle \hat{k}_{z_i}, \hat{k}_{z_j} \rangle \right| \leq 1 - \varepsilon.\]

This condition can also be given an equivalent reformulation using $\beta : \exists C > 0, \forall i, j, i \neq j$

\[(\text{Sep}') \quad \beta(z_i, z_j) \geq C(1 + \beta(0, z_j)).\]
The map $R$ is bounded if and only if the measure $\mu_Z$ defined by

\[
\mu_Z = \sum_{j=1}^{\infty} \|k_z\|^{-2} \delta_{z_j}.
\]

is a Carleson measure, but now a Carleson measure for the Dirichlet space. That is, there is a $C > 0$ so that for $\mu = \mu_Z$ and all $f \in B_2$

\[(\text{Car}) \quad \int |f|^2 \, d\mu \leq C \|f\|^2.\]

With the choice $f = k_z$ this estimate implies the Dirichlet space version of the simple condition: $\exists C > 0, \forall z \in \mathbb{D}, |z| \sim 1$

\[(\text{CarSimp}) \quad \mu(T(z)) \leq C(-\log(1 - |z|^2))^{-1}.\]

In unpublished work in the early 1990’s Bishop [Bi] and, independently, Marshall-Sundberg [MS] characterized the interpolating sequences for the Dirichlet space. The first published proof was given by Böe [Bo] in 2002 using different techniques.

**Theorem 2.** Given $Z$, the following conditions are equivalent:

1. $R_Z$ maps $B_2$ into and onto $\ell^2(Z)$, that is, $Z$ is an interpolating sequence for the Dirichlet space.

2. $Z$ satisfies (Sep) and $\mu_Z$ satisfies (Car).

This is a very satisfying analogy with the equivalence of the first two statements in Theorem 1. The first statements of the theorems are similar by design, (Sep’(Hardy)) and (Sep’) differ in detail but are similar in spirit, and (Car(Hardy)) and (Car) are formally the same. However there are also fundamental differences between the two theorems. The subtle condition (Car(Hardy)) is equivalent to the more straightforward geometric condition (CarSimp(Hardy)). On the other hand the simple condition (CarSimp), while necessary for (Car) is not equivalent to it.

The fact that condition (5) in Theorem 1 is equivalent to the others is one of the deeper parts of that theorem and the analogous statement fails for the Dirichlet space. Bishop had noted that there are sequences for which the restriction map is onto, i.e. $\ell^2(\mu_Z) \subset R(B_2)$, but the restriction map is not bounded; and hence those sequences are not interpolating sequences. We call sequences $Z$ for which the restriction map is onto but not necessarily bounded onto interpolating sequences (for the Dirichlet space). The study of such sequences is the theme of this paper.

1.3. **Onto Interpolating Sequences for the Dirichlet Space.** If $Z$ is an onto interpolating sequence for the Dirichlet space then the arguments that led to (Sep) and (Sep’) still apply and $Z$ must be satisfy those conditions. On the other hand, condition (Car) was a reformulation of the requirement that $R_Z$ be bounded. That requirement is not imposed in this case and (Car) is not necessary; in fact it plays no role in what follows.

However there are three conditions on $\mu_Z$ that we will be working with. The first condition is that $\mu_Z$ be finite, $\|\mu_Z\| < \infty$. The second condition is that $\mu_Z$ satisfy the simple condition (CarSimp).

The third condition is a weaker variation on the simple condition. To describe it we introduce Bergman trees which will be a basic tool in our analysis. We describe them now informally, the detailed description is in [ArRoSa], [ArRoSa2], or [Sa]. A Bergman tree is a set $T = \{\alpha_i\} \subset \mathbb{D}$ for which there is a positive lower bound on
the hyperbolic distances between distinct points and so that for some constant $C$
the union of hyperbolic balls, $\bigcup B(\alpha_i, C)$, cover $\mathbb{D}$. We regard $\{\alpha_i\}$ as the vertices
of a rooted tree with $o$, the vertex nearest the origin, as root. Each point $\alpha_i$ of $T$,
except $o$, is connected by an edge to its predecessor $\alpha_i^-$, a nearby point closer
to the origin. Being a rooted tree, $T$ has a partial ordering; $\alpha \leq \beta$ if $\alpha$
is on the geodesic connecting $\beta$ to $o$. For $\alpha \in T$ let $d(\alpha)$ be the number of vertices on the
tree geodesic connecting $\alpha$ to $o$; in particular, for any $\alpha$, $d(\alpha) \geq 1$.
We will assume that each $\alpha \in T$ is the predecessor of exactly two other points of $T$,
the successors, $\alpha_\pm$. This assumption is a notational convenience; our trees automatica lly have an
upper bound on their branching number and all our discussions extend to that case
by just adding notation.

If $Z$ is separated then there is a positive lower bound on the hyperbolic distance
between distinct points of $Z$ and, given this, it is easy to see that we can construct
a Bergman tree $T$ for the disk which contains $Z$ among its nodes. So, without loss
of generality, we may assume that $Z \subset T$. When the points of $Z$ are regarded as
elements of $T$ we will often denote them with lower case Greek letters.

For any $\alpha \in T$ we denote the shadow of $\alpha$ by $S(\alpha)$:

$$S(\alpha) = \{ \beta \in T : \beta \geq \alpha \}.$$ 

It is the tree analog of the tent $T'(\alpha)$. If $\mu_Z$ satisfies (CarSimp) then, regarded as a
measure on $T$, the measure satisfies $\exists C > 0 \forall \alpha \in T$

$$\mu(S(\alpha)) \leq Cd(\alpha)^{-1}.$$ 

In this case will say that the measure or the sequence satisfy the simple condition.
We say they satisfy the weak simple condition if:

$$\exists C > 0, \forall \alpha \in T \sum_{\beta \in Z, \beta \geq \alpha} \mu(\gamma) = 0 \text{ for } \alpha < \gamma < \beta \mu(\beta) \leq Cd(\alpha)^{-1}.$$ 

That is, in this case, for each $\alpha$ we now only consider points of $Z \cap S(\alpha)$ which have
an unobstructed view of $\alpha$.

The following was shown by Bishop [Bi] and, as was noted in [Bo], it is also a
consequence of the proof in [Bo].

**Theorem 3.** If the sequence $Z \subset \mathbb{D}$ is separated, the measure $\mu_Z$ is finite, and the
measure satisfies the simple condition; then $Z$ is an onto interpolating sequence.

We prove two theorems which extend this result.

**Theorem A.** If the sequence $Z \subset \mathbb{D}$ is separated, the measure $\mu_Z$ is finite, and the
measure satisfies the weak simple condition; then $Z$ is an onto interpolating sequence.

Our next theorem removes the hypothesis that the measure is finite thus, in
particular, answering a question of Bishop who had asked if an onto interpolating
sequence had to be associated with a finite measure.

Our proof requires additional geometric structure for $Z$, that it be tree-like. We
say that a sequence $Z$ which is separated and satisfies the weak simple condition
is tree-like if whenever $\alpha, \beta \in Z$ and $\alpha$ is in a certain expanded version of $S(\beta)$
then, in fact $\alpha \in S(\beta)$. Specifically, with $C$ the constant from (Sep) there is a
$\beta \in (1 - C/2, 1)$ so that

(Tree-like) if $z_j, z_k \in \mathbb{Z}$, $|z_j| \geq |z_k|$, and $|z_j - |z_k|^{-1} z_k| \leq \left(1 - |z_k|^2\right)^{\beta}$

then $z_j \in T(z_k)$.

**Theorem B.** If the sequence $Z \subset \mathbb{D}$ is separated, the measure $\mu_Z$ satisfies the weak simple condition; and $Z$ is tree-like, then $Z$ is an onto interpolating sequence.

The proofs of these theorems uses an elaboration of the constructive techniques of \[Bo\] and \[ArRoSa2\].

1.4. The Böe space. The building blocks for our constructions are a type of function introduced by Böe (Lemma 2 below), and the interpolating functions we construct all lie in a closed subspace of $B_2$ spanned by those functions. We call that span the Böe space. We will, in fact, prove refinements of Theorems A and B which also include the result that if onto interpolation is possible for a separated sequence $Z$ using only functions from the Böe space then $Z$ must be separated and satisfy the weak simple condition.

1.5. Tree Interpolation and Tree Capacities. Recall that the Dirichlet space $B_2(\mathcal{T})$ of the tree $\mathcal{T}$ consists of all complex-valued functions $f$ on $\mathcal{T}$ for which the norm

\[
\|f\|_{B_2(\mathcal{T})} = \left\{ \left| f(\alpha) \right|^2 + \sum_{\beta \in \mathcal{T}} \left| \Delta f(\beta) \right|^2 \right\}^{1/2} < \infty,
\]

Here $\Delta f(\beta) = f(\beta) - f(\beta^-)$. The analog of Theorem 2 for $B_2(\mathcal{T})$ is Theorem 26 in \[ArRoSa\]. Here we complement that and characterize the onto interpolating sequences for $B_2(\mathcal{T})$, that is, sequences for which the restriction map taking functions on $\mathcal{T}$ to functions on $Z$ takes $B_2(\mathcal{T})$ onto $\ell^2(\mu)$.

These sequences are analogs of onto interpolation sequences for $B_2$. Furthermore, it was shown in \[ArRoSa2\] Sec. 7.2 that the restriction map $f \to \{f(\alpha)\}_{\alpha \in \mathcal{T}}$ is bounded from $B_2$ to $B_2(\mathcal{T})$. As a consequence

(Restriction) Every onto interpolating sequence for $B_2$

is an onto interpolating sequence for $B_2(\mathcal{T})$.

We have a characterization of the onto interpolating sequence for $B_2(\mathcal{T})$. We say a subset $Z$ of $\mathcal{T}$ satisfies the tree capacity condition if $\exists C > 0, \forall \alpha \in Z$

(TreeCap) $\inf \left\{ \sum_{\beta \in \mathcal{T}} |\Delta f(\beta)|^2 : f(\alpha) = 1, f(\gamma) = 0 \forall \gamma \in \mathbb{Z} \setminus \{\alpha\} \right\} \leq \frac{C}{d(\alpha)}$

**Theorem C.** The sequence $Z \subset \mathcal{T}$ is an onto interpolating sequence for $B_2(\mathcal{T})$ if and only if it satisfies the tree capacity condition.

This result is analogous to the capacitary characterization of onto interpolating sequences for the Dirichlet space $B_2$ given by Bishop in \[Bi\], namely that $Z$ is onto interpolating for the Dirichlet space $B_2$ if and only if for any for $z, w \in \mathbb{Z}$, there is an $F_w \in B_2$ with $F_w(z) = \delta_{z,w}$ and $\|F_w\|_{B_2}^2 \leq C(- \log(1 - |w|^2))^{-1}$. This condition is called weak interpolation by Schuster and Seip \[ScSe\]. Thus, for both $B_2(\mathcal{T})$ and $B_2$ a weak interpolating sequence is an onto interpolating sequence.
1.6. Relations Between the Conditions. We construct a sequence covered by Theorem A but not by Theorem 3 and one covered by Theorem B but not Theorem A. That second example answers Bishop’s question by giving an example of an onto interpolating $Z$ with $\|\mu Z\| = \infty$.

In analysis on the tree we will show that the tree separation condition (Sep) and the weak simple condition (WeakSimp) imply the tree capacity condition (TreeCap). On the other hand, using a recursive scheme for computing tree capacities which is developed while proving Theorem C, we construct an example of an onto interpolating sequence for $B_2(T)$ that fails not only the weak simple condition, but also fails to be contained in any separated sequence satisfying the weak simple condition.

Taking into account (Restriction) we see the tree capacity condition is necessary for $Z$ to be onto interpolating for the classical Dirichlet space $B_2$. This gives a geometric condition stronger the separation condition (Sep) which any onto interpolating sequence must satisfy. Bishop [B] gave a similar necessary condition involving logarithmic capacity on the circle.

We also construct an example of a separated sequence with $\|\mu Z\| < \infty$ for which $Z$ fails the tree capacity condition (TreeCap) and hence, by Theorem C and (Restriction), is not an onto interpolating for $B_2(T)$ or for $B_2$. Thus separation and finite measure alone are not enough for onto interpolation.

1.7. Contents. Theorems A, and B are proved in Sections 2 and 3 but the proof of the necessity of the conditions under additional hypotheses is postponed to Section 7. Theorem C is proved in Sections 5. Sections 4 and 6 contain examples and further discussion of the relationship between the various conditions.

2. Theorem A

In fact, the hypotheses (Sep), (WeakSimp) and $\|\mu\| < \infty$ yield more than Theorem A. The more general theorem includes a converse, that under certain extra conditions one can conclude that $Z$ satisfies both (Sep) and (WeakSimp). In this section we state the general theorem and prove the half that implies Theorem A. The other half is proved in Section 7.

Suppose $Z$ is given and fixed. For $w \in Z$ we denote by $\varphi_w$ the function introduced by Böe in [B] in his work on interpolation. By construction $\varphi_w$ is a function in $B_2$ which is essentially 1 on the tent $T(w)$ and small away from that region. The details of the construction and properties are recalled in Lemma 2 below. Actually there are various choices in Lemmas 1 and 2 below. We assume that allowable choices have been made once and for all. Also, we further require that the chosen parameters satisfy

\[ \beta < \alpha < \frac{2\beta \eta}{(\eta + 1)} \]

\[ s > \frac{(\alpha - \rho)}{(\rho - \beta)} \]

as we need that in our proof of the necessity of (WeakSimp) for a certain type of interpolation, Proposition 7 below.

We define the Böe space, $B_{2,Z}$, to be the closed linear span in $B_2$ of the functions $\{\varphi_w\}_{w \in Z}$. It follows from (2.45) in Lemma 4 and Proposition 7 below that for an
appropriate cofinite subset \( \{ \zeta_j \}_{j=1}^\infty \) of \( Z \) we have

\[
B_{2,Z} = \left\{ \varphi = \sum_{j=1}^\infty a_j \varphi_{\zeta_j} : \sum_{j=1}^\infty |a_j|^2 \mu(\zeta_j) < \infty \right\},
\]

\[
\|\varphi\|_{B_{2,Z}} \approx \left\| \{a_j\}_{j=1}^\infty \right\|_{\ell^2(\mu)}.
\]

We will say \( Z \) is an onto interpolating sequence for \( B_{2,Z} \) if it is an onto interpolating sequence for the Dirichlet space \( B_2 \) and, if further, the interpolating functions can all be selected from \( B_{2,Z} \). In the case that \( \mu_Z \) is finite, the next theorem completely characterizes onto interpolation for \( B_{2,Z} \) in terms of separation and the weak simple condition. Theorem A in an immediate corollary.

**Theorem 4.** Let \( Z \subset \mathbb{D} \) and suppose \( \|\mu_Z\| < \infty \). Then \( Z \) is an onto interpolating sequence for the \( \mathcal{B}_\infty \) space \( B_{2,Z} \) if and only if both the separation condition \( \text{(Sep)} \) and the weak simple condition \( \text{(WeakSimp)} \) hold.

We will need the following lemma from [MS], (see also [ArRoSa2]). Let \( C \) be the constant in \( \text{(Sep)} \). For \( w \in \mathbb{D} \) and \( 1 - C/2 < \beta < 1 \), define

\[
V_w = V^\beta_w = \left\{ z \in \mathbb{D} : |z - |w|^{-1} w| < (1 - |w|^2)^\beta \right\}.
\]

**Lemma 1.** Suppose the separation condition in \( \text{(Sep)} \) holds. Then for every \( \beta \) satisfying \( 1 - C/2 < \beta < 1 \) there is \( \eta > \beta \eta > 1 \) such that if \( V^\beta_{z_i} \cap V^\beta_{z_j} \neq \emptyset \) and \( |z_i| \geq |z_j| \), then \( z_i \notin V^\beta_{z_j} \) and

\[
(1 - |z_j|) \leq (1 - |z_i|)^\eta.
\]

We have the following useful consequence of Lemma 1. If \( \sigma > 0 \) and \( \mu \) satisfies \( \text{(WeakSimp)} \), then

\[
\sum_{|z_j| \geq z_k} (1 - |z_j|)^{\sigma} \leq C_\sigma (1 - |z_k|)^{\sigma}.
\]

Indeed, if \( \mathcal{G}_1(z_k) = \{ \alpha_m^k \} \) consists of the minimal elements in \( \{S(z_k) \setminus \{z_k\} \} \cap Z \), \( \mathcal{G}_2(z_k) = \cup_m \mathcal{G}_1(\alpha_m^k) \), etc., we have using \( \text{(WeakSimp)} \),

\[
\sum_{z_j \geq z_k} (1 - |z_j|)^{\sigma} = (1 - |z_k|)^{\sigma} + \sum_{\ell=1}^\infty \sum_{\beta \in \mathcal{G}_{\ell-1}(z_k)} \sum_{\alpha \in \mathcal{G}_{\ell}(\beta)} (1 - |\alpha|)^{\sigma}
\]

\[
\leq (1 - |z_k|)^{\sigma} + C_\delta \sum_{\ell=1}^\infty \sum_{\beta \in \mathcal{G}_{\ell-1}(z_k)} \sum_{\alpha \in \mathcal{G}_{\ell}(\beta)} (1 - |\alpha|)^{\sigma - \delta} \left( \log \frac{1}{1 - |\alpha|} \right)^{-1}
\]

\[
\leq (1 - |z_k|)^{\sigma} + C_\delta \sum_{\ell=1}^\infty \sum_{\beta \in \mathcal{G}_{\ell-1}(z_k)} (1 - |\beta|)^{\sigma - \delta} \eta C \left( \log \frac{1}{1 - |\beta|} \right)^{-1}
\]

\[
\leq (1 - |z_k|)^{\sigma} + C_\delta \sum_{z_j \geq z_k} (1 - |z_j|)^{(\sigma - \delta) \eta}.
\]

Now we can choose \( \delta > 0 \) so small that \( (\sigma - \delta) \eta - \sigma > 0 \), and \( R \) such that \( C_\delta (1 - R)^{\delta} = \frac{1}{2} \), so that for \( |z_k| \geq R \) we have

\[
C_\delta \sum_{z_j \geq z_k} (1 - |z_j|)^{(\sigma - \delta) \eta} \leq \left\{ C_\delta \sup_{j \geq 1} (1 - |z_j|)^{\theta} \right\} \sum_{z_j \geq z_k} (1 - |z_j|)^{\sigma} \leq \frac{1}{2} \sum_{z_j \geq z_k} (1 - |z_j|)^{\sigma}.
\]
Thus \( \sum_{j \geq k} (1 - |z_j|)^{\gamma} \leq 2 (1 - |z_k|)^{\gamma} \), proving (2.4) for \( |z_k| \geq R \). Now the number of points \( z_k \) in the ball \( B(0, R) \) depends only on \( R \) and the separation constant \( C \) in (Sep), and it is now easy to obtain (2.4) in general.

We will also use a lemma from [Bo] which constructs a holomorphic function \( \varphi_w = \Gamma_s g_w \), where \( \Gamma_s \) is the projection operator below, that is close to 1 on the Carleson region associated to a point \( w \in \mathbb{D} \), and decays appropriately away from the Carleson region. Again let \( 1 - C/2 < \beta < 1 \) where \( C \) is as in (Sep). Given \( \beta < \rho < \alpha < 1 \), we will use the cutoff function \( c_{\rho, \alpha} \) defined by

\[
(2.5) \quad c_{\rho, \alpha} (\gamma) = \begin{cases} 
0 & \text{for } \gamma < \rho \\
\frac{\gamma - \rho}{\alpha - \rho} & \text{for } \rho \leq \gamma \leq \alpha \\
1 & \text{for } \alpha < \gamma
\end{cases}
\]

**Lemma 2.** (Lemma 4.1 in [Bo]) Suppose \( s > -1 \), \( C \) is as in (Sep), and \( 1 - C/2 < \beta < 1 \). There are \( \beta_1, \rho \) and \( \alpha \) satisfying \( \beta < \beta_1 < \rho < \alpha < 1 \) such that for every \( w \in \mathbb{D} \), we can find a function \( g_w \) so that

\[
\varphi_w (z) = \Gamma_s g_w (z) = \int_{\mathbb{D}} \frac{g_w (\zeta) (1 - |\zeta|^2)^s}{(1 - \zeta z)^{1+s}} d\zeta
\]

satisfies, with \( c_{\rho, \alpha} \) as is in (2.4), and \( \gamma_w (z) \) is defined by \( |z - |w|^{-1} w| = \left( 1 - |w|^2 \right)^{\gamma_w (z)} \),

\[
(2.6) \quad \begin{cases} 
\varphi_w (w) = 1 \\
\varphi_w (z) = c_{\rho, \alpha} (\gamma_w (z)) + O \left( \left( \frac{\log \frac{1}{1 - |w|^2}}{1 - |w|^2} \right)^{-1} \right), & z \in V_1^w \\
|\varphi_w (z)| \leq C \left( \frac{1}{1 - |w|^2} \right)^{-1} \left( 1 - |w|^2 \right)^{\rho - \beta_1 (1+s)}, & z \notin V_1^w.
\end{cases}
\]

Furthermore we have the estimate

\[
(2.7) \quad \int_{\mathbb{D}} |g_w (\zeta)|^2 d\zeta \leq C \left( \frac{1}{1 - |w|^2} \right)^{-1}.
\]

**2.0.1. The Sufficiency Proof.** We now prove that the hypotheses of Theorem 3 are sufficient for interpolation.

Order the points \( \{ z_j \}_{j=1}^{\infty} \) so that \( 1 - |z_{j+1}| \leq 1 - |z_j| \) for \( j \geq 1 \). We now define a “forest structure” on the index set \( \mathbb{N} \) by declaring that \( j \) is a child of \( i \) (or that \( i \) is a parent of \( j \)) provided that

\[
(2.8) \quad i < j, \quad V_{z_j} \subset V_{z_i}, \quad V_{z_j} \nsubseteq V_{z_k} \text{ for } i < k < j.
\]

Note if we have competing indices \( i \) and \( i' \) with \( V_{z_j} \subset V_{z_i} \cap V_{z_{i'}} \), then the child \( j \) chooses the “nearest” parent \( i \). We define a partial order associated with this parent-child relationship by declaring that \( j \) is a successor of \( i \) (or that \( i \) is a predecessor of \( j \)) if there is a “chain” of indices \( \{ i = k_1, k_2, \ldots, k_m = j \} \subset \mathbb{N} \) such that \( k_{r+1} \) is a child of \( k_r \) for \( 1 \leq r < m \). Under this partial ordering, \( \mathbb{N} \) decomposes into a disjoint union of trees. Thus associated to each index \( \ell \in \mathbb{N} \), there is a unique tree containing \( \ell \) and, unless \( \ell \) is the root of the tree, a unique parent \( P (\ell) \) of \( \ell \) in that tree. Denote by \( G_\ell \) the unique geodesic joining the root of the tree to \( \ell \). We
will usually identify \( \ell \) with \( z_\ell \) and thereby transfer the forest structure \( F \) to \( Z \) as well.

If \( f(z) \in B_2 \) and \( f(z_0) = 0 \) for some \( z_0 \in \mathbb{D} \) then \( f(z)/(z - z_0) \in B_2 \). Using this it is easy to show that \( Z \) is an onto interpolating sequence if and only if some cofinite subsequence if \( Z \) is. With this observation, and recalling the hypothesis that \( ||\mu|| < \infty \), we see that it suffices to do the proof under the additional assumption that

\[
\|\mu\| = \sum_{j=1}^{\infty} \mu(z_j) = \sum_{j=1}^{\infty} \left( \log \frac{1}{1 - |z_j|^2} \right)^{-1} < \varepsilon.
\]

Where \( \varepsilon \) is a small quantity to be specified later. With this done we now further suppose that the sequence \( \{z_j\}_{j=1}^{J} \) is finite, and obtain an appropriate estimate independent of \( J \geq 1 \). Fix \( \alpha, s > -1 \) and a sequence of complex numbers \( \{\xi_j\}_{j=1}^{J} \) in \( \ell^2(\mu) \) where

\[
\left\| \{\xi_j\}_{j=1}^{J} \right\|_{\ell^2(\mu)} = \left\| \left\{ \frac{\xi_j}{\|\xi_j\|_{B_2}} \right\}_{j=1}^{J} \right\|_{\ell^2(\mu)}.
\]

We will define a function \( \varphi = S\xi \) on the disk \( \mathbb{D} \) by

\[
\varphi(z) = S\xi(z) = \sum_{j=1}^{J} a_j \varphi_{z_j}(z), \quad z \in \mathbb{D},
\]

that will be our candidate for the interpolating function of \( \xi \). We follow the inductive scheme of B"oe that addresses the main difficulty in interpolating holomorphic functions, namely that on the sequence \( Z \) the building blocks \( \varphi_{z_j} \) take a large set of values (rather than just 0 and 1 as in the tree analogue).

Recall that \( P_{z_j} \) denotes the parent of \( z_j \) in the forest structure \( F \) and that \( G_\ell \) is the geodesic from the root to \( z_\ell \) in the tree containing \( z_\ell \). In order to define the coefficients \( a_j \) we will use the doubly indexed sequence \( \{\beta_{i,j}\} \) of numbers given by

\[
\beta_{i,j} = \varphi_{P_{z_j}}(z_i).
\]

We consider separately the indices in each tree of the forest \( \{1, 2, \ldots, J\} \), and define the coefficients inductively according to the natural ordering of the integers. So let \( \mathcal{Y} \) be a tree in the forest \( \{1, 2, \ldots, J\} \) with root \( k_0 \). Define \( a_{k_0} = \xi_{k_0} \). Suppose that \( k \in \mathcal{Y} \setminus \{k_0\} \) and that the coefficients \( a_j \) have been defined for \( j \in \mathcal{Y} \) and \( j < k \). Let

\[
\mathcal{G}_k = [k_0, k] = \{k_0, k_1, \ldots, k_{m-1}, k_m = k\}
\]

be the geodesic \( \mathcal{G}_k \) in \( \mathcal{Y} \) joining \( k_0 \) to \( k \), and note that \( \mathcal{G}_k = \mathcal{G}_{k_{m-1}} \cup \{k\} \). Define

\[
f_k(z) = f_{k_m}(z) = \sum_{i=1}^{m} a_k \varphi_{z_{k_i}}(z) = f_{k_{m-1}}(z) + a_k \varphi_{z_k}(z)
\]

and

\[
\omega_k = f_{k_{m-1}}(z_k) = \sum_{i=1}^{m} a_{k_{i-1}} \varphi_{z_{k_{i-1}}}(z_k) = \sum_{i=1}^{m} \beta_{k,i} a_{k_{i-1}}, \quad k \geq 1.
\]

Then define the coefficient \( a_k \) by

\[
a_k = \xi_k - \omega_k, \quad k \geq 1.
\]
This completes the inductive definition of the sequence \( \{a_k\}_{k \in \mathcal{Y}} \), and hence defines the entire sequence \( \{a_i\}_{i=1}^J \).

We first prove the following \( \ell^2 (d\mu) \) estimate for the sequence \( \{a_j^m\}_{j=1}^J \) given in terms of the data \( \{\xi_j^m\}_{j=1}^J \) by the scheme just introduced. This is the difficult step in the proof of sufficiency.

**Lemma 3.** The sequence \( \{a_i\}_{i=1}^J \) constructed in (2.12) above satisfies

\[
\|\{a_j\}_{j=1}^J\|_{\ell^2(d\mu)} \leq C \|\{\xi_j\}_{j=1}^J\|_{\ell^2(d\mu)}.
\]

**Proof:** Without loss of generality, we may assume for the purposes of this proof that the forest of indices \( \{j\}_{j=1}^J \) is actually a single tree \( \mathcal{Y} \). Now fix \( \ell \). At this point it will be convenient notation to momentarily relabel the points \( \{z_j\}_{j \in \mathcal{G}_\ell} = \{z_{k_1}, z_{k_2}, \ldots, z_{k_m}\} \) as \( \{z_0, z_1, \ldots, z_m\} \), with similar relabeling of the and similarly relabel the \( \{\alpha_j\}, \{\zeta_j\}, \) and \( \{\beta_j\} \) so that

\[
a_k = \zeta_k - \sum_{i=1}^k \beta_{k,i}a_{i-1}, \quad 0 \leq k \leq \ell.
\]

In other words, we are restricting attention to the geodesic \( \mathcal{G}_\ell \) and relabeling sequences so as to conform to the ordering in the geodesic. We also rewrite \( f_k(z) \) and \( \omega_k \) as

\[
f_k(z) = \sum_{i=1}^k a_i \varphi_{z_i}(z) = f_{k-1}(z) + a_k \varphi_{z_k}(z)
\]

and

\[
\omega_k = f_{k-1}(z) = \sum_{i=1}^k a_{i-1} \varphi_{z_{i-1}}(z) = \sum_{i=1}^k \beta_{k,i}a_{i-1}, \quad k \geq 1.
\]

so that the coefficients \( a_k \) are given by

\[
a_0 = \xi_0, \quad a_k = \zeta_k - \omega_k, \quad k \geq 1.
\]

We now claim that

\[
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_k
\end{pmatrix}
= \begin{bmatrix}
b_{1,1} & 0 & \cdots & 0 \\
b_{2,1} & b_{2,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
b_{k,1} & b_{k,2} & \cdots & b_{k,k}
\end{bmatrix}
\begin{pmatrix}
\xi_0 \\
\xi_1 \\
\vdots \\
\xi_{k-1}
\end{pmatrix}, \quad 1 \leq k \leq \ell,
\]

where

\[
b_{i,j} = 0, \quad i < j, \\
b_{i,i} = \beta_{i,i}, \\
b_{i,j} = b_{i-1,j} - b_{i-1,j}\beta_{i,i}, \quad i > j,
\]

and the \( b_{i,j} \) are defined in the following calculations. We also claim that the \( b_{i,j} \) are bounded:

\[
|b_{i,j}| \leq C.
\]
Indeed, from (2.14), (2.15) and the equality 
\[ \xi = \omega, \]
we readily obtain

\[ z \]
which proves (2.16) for \( k = 1 \).

For this we will use the estimate (see Lemma 7 below)

\[ |\varphi'_w(z)| \leq \left(1 - |w|^2\right)^{-\alpha}, \quad z \in \mathbb{D}. \]

Note first that

\[ b_{1,1} = \beta_{1,1} = \varphi_{z_0}(z_1) \]
since then (2.14) and (2.15) yield

\[ b_{1,1}z_0 = \omega_1, \]

which is (2.16) for \( k = 1 \). We also have (2.18) for \( 1 \leq j \leq i = 1 \) since (2.6) yields

\[ |b_{1,1}| \leq 1 + \lambda(z_0), \]

where we have introduced the convenient notation

\[ \lambda(z_j) = \left(\log \frac{1}{1 - |z_j|^2}\right)^{-1}. \]

We now define a function \( b_{1,1}(z) \) by

\[ b_{1,1}(z) = \varphi_{z_0}(z), \]

i.e. we replace \( z_1 \) by \( z \) throughout the formula for \( b_{1,1} \). If we then define

\[ b_{1,1}^* = b_{1,1}(z_2) = \varphi_{z_0}(z_2), \]

we readily obtain

\[ b_{2,1} = b_{1,1}^* - b_{1,1}\beta_{2,2} = \varphi_{z_0}(z_2) - \varphi_{z_0}(z_1)\varphi_{z_1}(z_2), \]
\[ b_{2,2} = \beta_{2,2} = \varphi_{z_1}(z_2). \]

Indeed, from (2.14), (2.15) and the equality \( \xi_1 = a_1 + \omega_1 = a_1 + \varphi_{z_0}(z_1) a_0 \), we have

\[ b_{2,1}z_0 + b_{2,2}z_1 = [\varphi_{z_0}(z_2) - \varphi_{z_0}(z_1)\varphi_{z_1}(z_2)]a_0 + \varphi_{z_1}(z_2)[a_1 + \varphi_{z_0}(z_1) a_0] \]
\[ = \varphi_{z_0}(z_2)a_0 + \varphi_{z_1}(z_2)a_1 \]
\[ = \omega_2, \]

which proves (2.16) for \( k = 2 \). We also have (2.18) for \( 1 \leq j \leq i = 2 \) since the bound

\[ |b_{2,2}| \leq 1 + \lambda(z_1) \]
is obvious from (2.6), and the bound for \( b_{2,1} \) follows from (2.6), (2.19) and Lemma 1.

\[ |b_{2,1}| \leq |\varphi_{z_0}(z_2) - \varphi_{z_0}(z_1)| + |\varphi_{z_0}(z_1)||1 - \varphi_{z_1}(z_2)| \]
\[ \leq |\varphi_{z_0}'(z_0)||z_2 - z_1| + (1 + \lambda(z_0))(1 + \lambda(z_1)), \]

and since

\[ |\varphi_{z_0}'(z_0)||z_2 - z_1| \leq \left(1 - |z_0|^2\right)^{-\alpha}|z_2 - z_1| \]
\[ \leq \left(1 - |z_0|^2\right)^{-\alpha}\left(1 - |z_1|^2\right)^{\beta} \]
\[ \leq \left(1 - |z_0|^2\right)_{\beta\alpha - \alpha}, \]
we obtain
\begin{equation}
|b_{2,1}| \leq \left(1 - |z_0|^2\right)^{\beta \eta - \alpha} + e^{\lambda(z_0) + \lambda(z_1)}.
\end{equation}

We now define functions $b_{2,1}(z)$ and $b_{2,2}(z)$ by
\begin{align*}
b_{2,1}(z) &= \varphi_{z_0}(z) - b_{1,1}\varphi_{z_1}(z), \\
b_{2,2}(z) &= \varphi_{z_1}(z),
\end{align*}
i.e. we replace $z_2$ by $z$ throughout the formulas for $b_{2,1}$ and $b_{2,2}$. If we then set
\begin{align*}
b_{2,1}^* &= b_{2,1}(z_3), \\
b_{2,2}^* &= b_{2,2}(z_3),
\end{align*}
we obtain as above that
\begin{align*}
b_{3,1} &= b_{2,1}^* - b_{2,1} \beta_{3,3} = [\varphi_{z_0}(z_3) - b_{1,1}\varphi_{z_1}(z_3)] - [\varphi_{z_0}(z_2) - b_{1,1}\varphi_{z_1}(z_2)] \varphi_{z_2}(z_3), \\
b_{3,2} &= b_{2,2}^* - b_{2,2} \beta_{3,3} = \varphi_{z_1}(z_3) - \varphi_{z_1}(z_2) \varphi_{z_2}(z_3), \\
b_{3,3} &= \beta_{3,3} = \varphi_{z_2}(z_3),
\end{align*}
which proves \((2.16)\) for $k = 3$. Moreover, we again have \((2.18)\) for $1 \leq j \leq i = 3$. Indeed,
\[|b_{3,3}| \leq 1 + \lambda(z_2),\]
and the arguments used above to obtain \((2.20)\) show that
\[|b_{3,2}| \leq \left(1 - |z_1|^2\right)^{\beta \eta - \alpha} + e^{\lambda(z_1) + \lambda(z_2)}.
\]
Finally,
\begin{align*}
|b_{3,1}| &\leq |b_{2,1}^* - b_{2,1}| + |b_{2,1}| |1 - \beta_{3,3}| \\
&\leq \left\{|\varphi'_{z_0}(z_0)| + |b_{1,1}| |\varphi'_{z_1}(z_1)|\right\} |z_2 - z_3| \\
&\quad + \left(1 - |z_0|^2\right)^{\beta \eta - \alpha} + e^{\lambda(z_0) + \lambda(z_1) + \lambda(z_2)},
\end{align*}
and since
\begin{align*}
\left\{|\varphi'_{z_0}(z_0)| + |b_{1,1}| |\varphi'_{z_1}(z_1)|\right\} &|z_2 - z_3| \\
&\leq \left\{(1 - |z_0|^2)^{-\alpha} + (1 + \lambda(z_0)) \left(1 - |z_1|^2\right)^{-\alpha}\right\} |z_2 - z_3| \\
&\leq \left\{(1 - |z_1|^2)^{-\alpha} + (1 + \lambda(z_0)) \left(1 - |z_1|^2\right)^{-\alpha}\right\} \left(1 - |z_2|^2\right)^{\beta} \\
&\leq \left\{(1 + A\lambda(z_0)) \left(1 - |z_1|^2\right)^{-\alpha}\right\} \left(1 - |z_1|^2\right)^{\beta \eta} \\
&\quad \leq (1 + A\lambda(z_0)) \left(1 - |z_1|^2\right)^{\beta \eta - \alpha},
\end{align*}
for a large constant $A$, we have
\[|b_{3,1}| \leq e^{A\lambda(z_0)} \left(1 - |z_1|^2\right)^{\beta \eta - \alpha} + e^{\lambda(z_0) + \lambda(z_1) + \lambda(z_2)}.
\]
Continuing in this way with
\begin{equation}
(b_{i,j}(z) = b_{i-1,j}(z) - b_{i-1,j}(z_i),
\end{equation}
\begin{equation}
(b_{i,j}^* = b_{i,j}(z_{i+1}),
\end{equation}
we can prove (2.16) and (2.18) by induction on $k$ and $i$ (see below). The bound $C$ in (2.18) will use the fact that
\begin{equation}
\lambda(z_0) + \lambda(z_1) + \lambda(z_2) + \cdots \leq C\lambda(z_0).
\end{equation}
To see (2.22) we use Lemma 1.

Now if $G_\ell = [k_0, k_1, \ldots, k_{m-1}, k_m]$, then by applying (2.3) repeatedly, we obtain
\begin{equation}
(1 - |z_{k_i}|^2) \leq \left(1 - |z_{k_0}|^2\right)^{\eta},
\end{equation}
and so combining these estimates we have
\begin{equation}
\lambda(z_0) + \lambda(z_1) + \lambda(z_2) + \cdots \leq C \sum_{i \in G_\ell \setminus \{k_0\}} \left(\log \frac{1}{1 - |z_{P(i)}|}\right)^{-1}
\end{equation}
\begin{equation}
\leq C \left(\sum_{j=0}^{m-1} \eta^{-j} \left(\log \frac{1}{1 - |z_{k_0}|}\right)^{-1}\right)
\leq C\eta \left(\log \frac{1}{1 - |z_{k_0}|^2}\right)^{-1} = C\eta \lambda(z_0)
\end{equation}
since $\eta > 1$, which yields (2.22).

We now give the induction details for proving (2.16) and (2.18). The proof of (2.16) is straightforward by induction on $k$, so we concentrate on proving (2.18) by induction on $i$. If we denote the $i^{th}$ row
\begin{equation}
\begin{array}{ccccccc}
 b_{i,1} & b_{i,2} & \cdots & b_{i,i} & 0 & \cdots & 0 \\
\end{array}
\end{equation}
of the matrix in (2.16) by $B_i$, the corresponding row of starred components
\begin{equation}
\begin{array}{ccccccc}
 b_{i,1}^* & b_{i,2}^* & \cdots & b_{i,i}^* & 0 & \cdots & 0 \\
\end{array}
\end{equation}
by $B_i^*$, and the row having all zeroes except a one in the $i^{th}$ place by $E_i$, then we have the recursion formula
\begin{equation}
B_i = B_{i-1}^* - B_{i-1} + (1 - \beta_{i,i}) B_{i-1} + \beta_{i,i} E_i
\end{equation}
\begin{equation}
= \{(1 - \beta_{i,i}) B_{i-1} + \beta_{i,i} E_i\} - (B_{i-1} - B_{i-1}^*)
\end{equation}
which expresses $B_i$ as a “convex combination” of the previous row and the unit row $E_i$, minus the difference of the previous row and its starred counterpart. In terms of the components of the rows, we have
\begin{equation}
b_{i,j} = [b_{i-1,j}^* - b_{i-1,j}] + (1 - \beta_{i,i}) b_{i-1,j} + \beta_{i,i} \delta_{i,j}.
\end{equation}

For a large constant $A$ that will be chosen later so that the induction step works, we prove the following estimate by induction on $i$:
\begin{equation}
|b_{i,j}| \leq e^{A(\lambda(z_{j-1}) + \cdots + \lambda(z_{i-1}))}, \quad i \geq j.
\end{equation}
The initial case $i = j$ follows from

$$|b_{j,j}| = |\varphi_{z_{j-1}}(z_j)| \leq 1 + \lambda(z_{j-1}) \leq e^{\lambda(z_{j-1})}.$$  

Now (2.21) yields

$$b'_{i,j}(z) = b'_{i-1,j}(z) - b_{i-1,j} \varphi'_{z_{i-1}}(z),$$

and so by the induction assumption for indices smaller than $i$, we have from (2.19) that

(2.27)

$$\|b_{i,j}\|_\infty \leq \|b_{i-1,j}\| \left(1 - |z_{i-1}|^2\right)^{-\alpha} + \|b'_{i-1,j}\|_\infty$$

$$\leq \|b_{i-1,j}\| \left(1 - |z_{i-1}|^2\right)^{-\alpha} + |b_{i-1,j}| \left(1 - |z_{i-2}|^2\right)^{-\alpha} + \|b'_{i-2,j}\|_\infty$$

$$\vdots$$

$$\leq \left\{ \sup_{j \leq k \leq i-1} |b_{k,j}| \right\} \left[ \left(1 - |z_{i-1}|^2\right)^{-\alpha} + \left(1 - |z_{i-2}|^2\right)^{-\alpha} \right]$$

$$\leq e^{A\lambda(z_{i-1}) + \ldots + \lambda(z_{i-2})} \left[ \left(1 - |z_{i-1}|^2\right)^{-\alpha} + \left(1 - |z_{i-2}|^2\right)^{-\alpha} \right]$$

$$\leq e^{A\lambda(z_{i-1}) + \ldots + \lambda(z_{i-2})} \left[ \left(1 - |z_{i-1}|^2\right)^{-\alpha} + \left(1 - |z_{i-2}|^2\right)^{-\alpha} \right]$$

where the last line uses (2.29). Thus we have from (2.27), (2.25) and (2.21),

(2.28)

$$|b_{i,j}| \leq |b'_{i-1,j} - b_{i-1,j}| + |(1 - \beta_{i,1}) b_{i-1,j}|$$

$$\leq |b_{i-1,j} (z_{i+1}) - b_{i-1,j} (z_i)| + |1 - \beta_{i,1}| |b_{i-1,j}|$$

$$\leq \|b'_{i,j}\|_\infty |z_{i+1} - z_i| + (1 + \lambda(z_{i-1})) |b_{i-1,j}|$$

$$\leq e^{A\lambda(z_{i-1}) + \ldots + \lambda(z_{i-2})} \times$$

$$\left\{ 1 + \lambda(z_{i-1}) + \left[ \left(1 - |z_{i-1}|^2\right)^{-\alpha} + \ldots + \left(1 - |z_{i-1}|^2\right)^{-\alpha} \right] \left(1 - |z_{i-1}|^2\right)^{\beta\eta} \right\},$$

upon using the inequality $|z_{i+1} - z_i| \leq \left(1 - |z_i|^2\right)^{\beta} \leq \left(1 - |z_{i-1}|^2\right)^{\beta\eta}$, which follows from Lemma [1].

Finally we use the inequality (see below for a proof)

(2.29)

$$\left(1 - |z_{i-1}|^2\right)^{-\alpha} + \ldots + \left(1 - |z_{i-1}|^2\right)^{-\alpha} \leq C_\eta \left(1 - |z_{i-1}|^2\right)^{-\alpha} + 2(i - j + 1)$$
to obtain that

\[
\left( 1 - |z_{i-1}|^2 \right)^{-\alpha} + \ldots + \left( 1 - |z_{i-1}|^2 \right)^{-\frac{\beta \eta}{\beta - \gamma}} \left( 1 - |z_{i-1}|^2 \right)^{\beta \eta} \leq 2 \left\{ C_\eta \left( 1 - |z_{i-1}|^2 \right)^{\beta \eta - \alpha} + 2i \left( 1 - |z_{i-1}|^2 \right)^{\beta \eta} \right\} \leq (A - 1) \log \left( \frac{1}{1 - |z_{i-1}|^2} \right) = (A - 1) \lambda (z_{i-1}),
\]

for all \( i \) if \( A \) is chosen large enough. With such a choice of \( A \), (2.28) yields

\[
|b_{i,j}| \leq e^{A \{ z_{i-1}, \ldots, z_{j-1} \} \{ 1 + \lambda (z_{i-1}) + (A - 1) \lambda (z_{i-1}) \}} \leq e^{A \{ z_{i-1}, \ldots, z_{j-1} \}},
\]

which proves (2.26), and hence (2.18) by (2.22). To see (2.29), we rewrite it as

\[
\sum_{\ell=0}^{N} R^{\eta - \ell} \leq C \eta R + 2N + 2,
\]

and to prove this, note that for \( R^{\eta - \ell} > 2 \) the ratio of the consecutive terms \( R^{\eta - (\ell + 1)} \) and \( R^{\eta - \ell} \) is \( \left( R^{\eta - \ell} \right)^{1 - \eta} < 2^{1 - \eta} \), i.e. this portion of the series is supergeometric. Thus we have

\[
\sum_{\ell=0}^{N} R^{\eta - \ell} \leq \sum_{\ell=0 : R^{\eta - \ell} > 2} R^{\eta - \ell} + \sum_{\ell \leq N : R^{\eta - \ell} \leq 2} R^{\eta - \ell} \leq R \sum_{j=0}^{\infty} (2^{1 - \eta})^j + 2(N + 1).
\]

We now claim the following crucial property. Recall that \( P(m) = m - 1 \). If \( \sigma > 0 \) and \( \gamma_{m-1} (z_m) > \alpha + \sigma \), i.e. \( z_m \in V^{\alpha + \sigma}_{m-1} \), then

\[
|b_{i,j}| \leq C \left( 1 - |z_{m-1}|^2 \right)^{\sigma} \text{ for all } j < m \leq i.
\]

We first note that from (2.19), we have for \( z_m \in V^{\alpha + \sigma}_{m-1} \),

\[
\beta_{m,m} = \varphi_{z_{m-1}} (z_m) = \varphi_{z_{m-1}} (z_m) + \left[ \varphi_{z_{m-1}} (z_m) - \varphi_{z_{m-1}} (z_{m-1}) \right] = 1 + O \left( \left( 1 - |z_{m-1}|^2 \right)^{-\alpha} \left| z_m - z_{m-1} \right| \right) = 1 + O \left( \left( 1 - |z_{m-1}|^2 \right)^{-\alpha} \left( 1 - |z_{m-1}|^2 \right)^{\alpha + \sigma} \right) = 1 + O \left( \left( 1 - |z_{m-1}|^2 \right)^{\sigma} \right).
\]

From (2.24) we then obtain

\[
\|B_m - \beta_{m,m} E_m\|_\infty \leq \|B_{m-1} - B_{m-1}\|_\infty + O \left( \left( 1 - |z_{m-1}|^2 \right)^{\sigma} \right) \|B_{m-1}\|_\infty.
\]

Next, the estimate

\[
\|B_{m-1} - B_{m-1}\|_\infty \leq C \left( 1 - |z_{m-1}|^2 \right)^{\sigma},
\]
follows from (2.21), (2.27) and (2.30) with $\beta \eta$ replaced with $\alpha + \sigma$:

$$b_{m-1,j}^* - b_{m-1,j} = |b_{m-1,j} (z_m) - b_{m-1,j} (z_{m-1})|$$

$$\leq \|b_{m-1,j}^*\|_{L^\infty} |z_m - z_{m-1}|$$

$$\leq \sum_{k=1}^{m} \left\{ C_\eta \left(1 - |z_{k-1}|^2\right)^{-\alpha} + 2k \right\} \left(1 - |z_{m-1}|^2\right)^{\alpha+\sigma}$$

$$\leq C_{\sigma} \left(1 - |z_{m-1}|^2\right)^{\sigma}.$$

Thus altogether we have proved that the top row of the rectangle $R_m = [b_{i,j}]_{j<m \leq i}$ satisfies (2.31), i.e. $b_{m,j} \leq C \left(1 - |z_{m-1}|^2\right)^{\sigma}$ for $j < m$. The proof for the remaining rows is similar using (2.25).

For convenience in notation we now define

$$\Gamma = \left\{ m : z_m \in V_{z_m-1}^{\alpha+\sigma} \right\}.$$

If we take $0 < \sigma \leq (\eta - 1) \alpha$ and iterate the proof of (2.31) and use (2.18), we obtain the improved estimate

$$|b_{i,j}| \leq C \prod_{m \in \Gamma ; j < m \leq i} \left(1 - |z_{m-1}|^2\right)^{\sigma}, \quad i > j.$$

To see this we first look at the simplest case when $2, 3 \in \Gamma$ and establish the corresponding inequality

$$|b_{3,1}| \leq C \left(1 - |z_1|^2\right)^{\sigma} \left(1 - |z_2|^2\right)^{\sigma}.$$

We have from (2.25) that

$$b_{3,1} = [b_{2,1}^* - b_{2,1}] + (1 - \beta_{3,3}) b_{2,1}.$$

From (2.32) and (2.31) we have

$$|b_{2,1}| |1 - \beta_{3,3}| \leq C \left(1 - |z_1|^2\right)^{\sigma} \left(1 - |z_2|^2\right)^{\sigma}.$$

From (2.21) we have

$$|b_{2,1} - b_{2,1}^*| = |b_{2,1} (z_2) - b_{2,1} (z_3)|$$

$$= |[b_{1,1} (z_2) - b_{1,1} \varphi_{z_1} (z_2)] - [b_{1,1} (z_3) - b_{1,1} \varphi_{z_1} (z_3)]|$$

$$\leq |b_{1,1} (z_2) - b_{1,1} (z_3)| + |b_{1,1} | \varphi_{z_1} (z_2) - \varphi_{z_1} (z_3)|$$

$$\leq \|b_{1,1}^*\|_{\infty} |z_2 - z_3| + |b_{1,1} | \varphi_{z_1}^* \|_{\infty} |z_2 - z_3|$$

where

$$\|b_{1,1}^*\|_{\infty} |z_2 - z_3| \leq C \left(1 - |z_0|^2\right)^{-\alpha} \left(1 - |z_2|^2\right)^{\alpha+\sigma}$$

$$\leq C \left(1 - |z_1|^2\right)^{-\alpha\eta} \left(1 - |z_1|^2\right)^{\eta \alpha} \left(1 - |z_2|^2\right)^{\sigma}$$

and

$$|b_{1,1} | \varphi_{z_1}^* \|_{\infty} |z_2 - z_3| \leq C \left(1 - |z_1|^2\right)^{-\alpha} \left(1 - |z_2|^2\right)^{\alpha+\sigma}$$

$$\leq C \left(1 - |z_1|^2\right)^{(\eta - 1) \alpha} \left(1 - |z_2|^2\right)^{\sigma}.$$
are both dominated by $C \left( 1 - |z_1|^2 \right)^{\sigma} \left( 1 - |z_2|^2 \right)^{\sigma}$ if $0 < \sigma \leq (\eta - 1) \alpha$. Altogether we have proved (2.34).

Now we suppose that $4 \in \Gamma$ as well and prove the estimate

\begin{equation}
|b_{4,1}| \leq C \left( 1 - |z_1|^2 \right)^{\sigma} \left( 1 - |z_2|^2 \right)^{\sigma} \left( 1 - |z_3|^2 \right)^{\sigma}.
\end{equation}

Again we have from (2.25) that

$$b_{4,1} = [b_{4,1}^* - b_{3,1}] + (1 - \beta_{4,4}) b_{3,1},$$

and from (2.32) and (2.34) we have

$$|b_{3,1}| |1 - \beta_{4,4}| \leq C \left( 1 - |z_1|^2 \right)^{\sigma} \left( 1 - |z_2|^2 \right)^{\sigma} \left( 1 - |z_3|^2 \right)^{\sigma}.$$ 

From (2.21) we have

$$|b_{3,1} - b_{3,1}^*| = |b_{3,1} (z_3) - b_{3,1}^* (z_4)|$$

$$= |(b_{2,1} (z_3) - b_{2,1} (\varphi_{z_2} (z_3))) - (b_{2,1} (z_4) - b_{2,1} (\varphi_{z_2} (z_4)))|$$

$$\leq |b_{2,1} (z_3) - b_{2,1} (z_4)| + |b_{2,1} (\varphi_{z_2} (z_3) - \varphi_{z_2} (z_4))|$$

$$\leq \|b_{2,1}\|_{\infty} |z_3 - z_4| + |b_{2,1}\|_{\varphi_{z_2}} |z_3 - z_4|,$$

where

$$\|b_{2,1}\|_{\infty} |z_3 - z_4| \leq C \left( 1 - |z_1|^2 \right)^{-\alpha} \left( 1 - |z_2|^2 \right)^{\alpha + \sigma}$$

$$\leq C \left( 1 - |z_1|^2 \right)^{-\alpha} \left( 1 - |z_2|^2 \right)^{\eta \alpha} \left( 1 - |z_3|^2 \right)^{\sigma}$$

$$\leq C \left( 1 - |z_1|^2 \right)^{(\eta - 1) \alpha} \left( 1 - |z_2|^2 \right)^{(\eta - 1) \alpha} \left( 1 - |z_3|^2 \right)^{\sigma}$$

and

$$|b_{2,1}|_{\varphi_{z_2}} |z_3 - z_4| \leq C \left( 1 - |z_1|^2 \right)^{\sigma} \left( 1 - |z_2|^2 \right)^{-\alpha} \left( 1 - |z_3|^2 \right)^{\alpha + \sigma}$$

$$\leq C \left( 1 - |z_1|^2 \right)^{\sigma} \left( 1 - |z_2|^2 \right)^{(\eta - 1) \alpha} \left( 1 - |z_3|^2 \right)^{\sigma}$$

are both dominated by $C \left( 1 - |z_1|^2 \right)^{\sigma} \left( 1 - |z_2|^2 \right)^{\sigma} \left( 1 - |z_3|^2 \right)^{\sigma}$ if $0 < \sigma \leq (\eta - 1) \alpha$. This completes the proof of (2.35), and the general case is similar.

The consequence we need from (2.43) is that if $m_1 < m_2 < \ldots < m_N \leq k$ is an enumeration of the $m \in \Gamma$ such that $m \leq k$, then

\begin{equation}
|a_k| \leq |\xi_k - \omega_k| \leq |\xi_k| + |\omega_k|
\end{equation}

$$\leq |\xi_k| + C \sum_{i=1}^{N} \left\{ \prod_{i \leq \ell \leq N} \left( 1 - |z_{m_i-1}|^2 \right)^{\sigma} \right\} \sum_{m_{j-1} \leq j \leq m_i} |\xi_j| - |\omega_k|$$

for $0 \leq k \leq \ell$.

We now return our attention to the tree $Y$. For each $\alpha \in Y$, with corresponding index $j \in \{j\}_{j=1}^{J}$, there are values $a(\alpha) = a_j$, $\xi(\alpha) = \xi_j$ and $m(\alpha) = z_{m(j)} \in Y$.

Define functions $f(\alpha) = |a(\alpha)|$ and $g(\alpha) = |\xi(\alpha)|$ on the tree $Y$. Note that we are
are comparable in $T$ in the tree $T$ uniformly in $k$.

The following crucial property of the sequence $Z$ is

$$J_k g (\alpha) = \sum_{m_{k-1}(\alpha) \leq \beta \leq m_k(\alpha)} g (A\beta),$$

$$J_\infty g (\alpha) = g (\alpha) + \sum_{m_{N(\alpha)}(\alpha) \leq \beta \leq \alpha} g (A\beta)$$

on the tree $Y$, then inequality (2.36) implies in particular that

$$f (\alpha) \leq C \left( J_\infty g (\alpha) + \sum_{k=1}^{N(\alpha)} \left\{ \prod_{k \leq \ell \leq N(\alpha)} \left( 1 - |z_{m_{\ell-1}}|^2 \right)^\sigma \right\} J_k g (\alpha) \right), \quad \alpha \in Y.$$

Recall that we are assuming that the measure $d\mu = \sum_{\alpha \in Y} \left( \log \frac{1}{1 - |z_\alpha|^2} \right)^{-1} d\alpha$, where $z_\alpha = z_j \in \mathbb{D}$ if $\alpha$ corresponds to $j$, satisfies the weak simple condition,

$$\beta (0, t) \sum_{j:z_j \in S(t) \text{ is minimal}} \mu (j) \leq C, \quad t \in T.$$

Note that this last inequality refers to the tree $T$ rather than to $Y$. Using the fact that $\beta (0, \alpha) \approx \log \frac{1}{1 - |z_\alpha|^2}$, we obtain from this weak simple condition that if $S(t) \approx V_{z_k}$, i.e. $t = \left[ 1 - \left( 1 - |z_k|^2 \right) \right] z_k$, then

$$\sum_{j:z_j \in S(t) \text{ is minimal}} \mu (j) \leq C \beta (0, t)^{-1} \approx C \left( \log \frac{1}{1 - |z_k|^2} \right)^{-1} \approx C \mu (z_k).$$

by the definition of the region $V_{z_k}$. To utilize this inequality on the tree $Y$ we need the following crucial property of the sequence $Z$: if $[\alpha, \beta]$ is a geodesic in $Y$ such that $\gamma \notin \Gamma$ for all $\alpha < \gamma \leq \beta$, then the geodesic $[\alpha, \beta]$, considered as a set of points in the tree $T$, is scattered in $T$ in the sense that no two distinct points $\gamma, \gamma' \in [\alpha, \beta]$ are comparable in $T$, i.e. neither $\gamma \leq \gamma'$ nor $\gamma' \leq \gamma$ in $T$. With this observation we obtain that on the tree $Y$, the adjoint $J_k^* \mu$ of $J_k$ satisfies

$$J_k^* \mu (\alpha) \leq C \mu (\alpha), \quad \alpha \in Y.$$

Now (2.13) will follow from (2.37) together with the inequality

$$\sum_{\alpha \in Y} J_k g (\alpha)^2 \mu (\alpha) \leq C \sum_{\alpha \in Y} g (\alpha)^2 \mu (\alpha), \quad g \geq 0,$$

uniformly in $k$, and thus it suffices to show the equivalence of (2.40) and (2.39).

To see this we first claim that the inequality

$$\sum_{\alpha \in Y} I g (\alpha)^2 \mu (\alpha) \leq C \sum_{\alpha \in Y} g (\alpha)^2 \mu (\alpha),$$

is equivalent to

$$I^* \mu (\alpha) \leq C \mu (\alpha), \quad \alpha \in Y.$$
Indeed, \( (2.42) \) is obviously necessary for \( (2.41) \). To see the converse, we use our more general tree theorem, Theorem 3 of [ArRoSa], for the tree \( Y \):

\[
\sum_{\alpha \in Y} I g(\alpha)^2 w(\alpha) \leq C \sum_{\alpha \in Y} g(\alpha)^2 v(\alpha), \quad g \geq 0,
\]

if and only if

\[
(2.43) \quad \sum_{\beta \geq \alpha} I^* w(\beta)^2 v(\beta)^{-1} \leq C I^* w(\alpha) < \infty, \quad \alpha \in Y.
\]

With \( w = v = \mu \), \( (2.42) \) yields condition \( (2.43) \) as follows:

\[
\sum_{\beta \geq \alpha} I^* \mu(\beta)^2 \mu(\beta)^{-1} \leq C \sum_{\beta \geq \alpha} \mu(\beta)^2 \mu(\beta)^{-1} = C \sum_{\beta \geq \alpha} \mu(\beta) = C I^* \mu(\alpha),
\]

and this completes the proof of the claim.

In general, condition \( (2.41) \), a consequence of the weak simple condition, does not imply the simple condition \( (2.42) \). However, we can again exploit the crucial property of the sequence \( Z \) mentioned above - namely that if \( [\alpha, \beta] \) is a geodesic in \( Y \) with \( (\alpha, \beta) \cap \Gamma = \emptyset \), then \( [\alpha, \beta] \), considered as a set of points in the tree \( T \), is scattered in \( T \). Now decompose the tree \( Y \) into a family of pairwise disjoint forests \( Y_\ell \) as follows. Let \( Y_1 \) consist of the root \( o \) of \( Y \) together with all points \( \beta > o \) having \( \gamma \notin \Gamma \) for \( o < \gamma \leq \beta \). Then let \( Y_2 \) consist of each minimal point \( \alpha \) in \( Y \setminus Y_1 \) together with all points \( \beta > \alpha \) having \( \gamma \notin \Gamma \) for \( \alpha < \gamma \leq \beta \), then let \( Y_3 \) consist of each minimal point \( \alpha \) in \( Y \setminus (Y_1 \cup Y_2) \) together with all points \( \beta > \alpha \) having \( \gamma \notin \Gamma \) for \( \alpha < \gamma \leq \beta \), etc.

A key property of this decomposition is that on \( Y_\ell \) the operator \( J_k \) sees only the values of \( g \) on \( Y_\ell \) itself. A second key property is that since the geodesics in \( Y_\ell \) are scattered, we see that the restriction \( \mu_\ell \) of \( \mu \) to the forest \( Y_\ell \) satisfies the simple condition, rather than just the weak simple condition. As a consequence, upon decomposing each forest \( Y_\ell \) into trees and applying the above claim with \( \mu_\ell \) in place of \( \mu \), i.e. \( (2.41) \) holds if and only if \( (2.42) \) holds, we conclude that

\[
\sum_{\alpha \in Y_\ell} J_k g(\alpha)^2 \mu(\alpha) \leq C \sum_{\alpha \in Y_\ell} g(\alpha)^2 \mu(\alpha), \quad \alpha \geq 0,
\]

uniformly in \( k \) for each \( \ell \geq 1 \). Summing in \( \ell \) and using the finite overlap, we obtain the sufficiency of \( (2.39) \) for \( (2.40) \).

Finally, to see that \( (2.13) \) now follows from \( (2.37) \), we use that

\[
\sum_{i=1}^{N} \prod_{i \leq \ell \leq N} \left( 1 - |z_{m_i - 1}|^2 \right)^{\sigma_i} \leq C
\]

in \( (2.37) \) to obtain \( (2.13) \). This completes the proof of Lemma 3.

Now we prove that the function \( \varphi = \sum_{j=1}^{J} a_i \varphi_i \) constructed above comes close to interpolating the data \( \{\xi_j\}_{j=1}^{J} \) provided we choose \( \varepsilon > 0 \) sufficiently small in \( (2.49) \).

Lemma 4. Suppose \( s > -1 \), that \( \{\xi_j\}_{j=1}^{J} \) is a sequence of complex numbers, and let \( 0 < \delta < 1 \). Let \( \varphi_j, g_j \) and \( \gamma_j \) correspond to \( z_j \) as in Lemma 2 and with the same \( s \).
Then for \( \varepsilon > 0 \) sufficiently small in (2.44), there is \( \{a_i\}_{i=1}^J \) such that \( \varphi = \sum_{i=1}^J a_i \varphi_i \) satisfies
\[
\left\| \{ \xi_j - \varphi (z_j) \}_{j=1}^J \right\|_{\ell^2(\mu)} < \delta \left\| \{ \xi_j \}_{j=1}^J \right\|_{\ell^2(\mu)}
\]
and
\[
\| \varphi \|_{B_2} \leq C \left\| \{ a_j \}_{j=1}^J \right\|_{\ell^2(\mu)}.
\]

**Remark 1.** The construction in the proof below shows that both the sequence \( \{a_i\}_{i=1}^J \) and the function \( \varphi \) depend linearly on the data \( \{ \xi_j \}_{j=1}^J \).

**Proof:** We now show that both (2.44) and (2.45) hold for the function \( \varphi = \sum_{i=1}^J a_i \varphi_i \) constructed above. Fix an index \( \ell \in \mathbb{N} \), and with notation as above, let \( \mathcal{F}_\ell = \mathbb{N} \setminus \mathcal{G}_\ell \) and write using (2.12),
\[
\sum_{j=1}^J \frac{1}{2} \varphi (z_j^i) \leq B_\ell,
\]
\[
\sum_{i=1}^J \mu (z_i).
\]
We first note that if \( z_\ell \notin V_{z_1} \), then
\[
\left| \varphi (z_\ell) \right| \leq C \left( 1 - \left| z_\ell \right|^2 \right) ^\sigma, \quad \sigma > 0,
\]
by the third line in (2.6). On the other hand, if \( z_\ell \in V_{z_1} \), then \( |z_1| < |z_\ell| \), and if \( \mathcal{G}_\ell = [k_0, k_1, \ldots, k_{m-1}, k_m] \), then either \( |z_1| < |z_{k_0}| \) or there is \( j \) such that \( |z_{k_{j-1}}| < |z_{k_j}| \). Note however that equality cannot hold here by Lemma 1 and so we actually have \( \left| z_{k_{j-1}} \right| < |z_1| < \left| z_{k_j} \right| \). From (2.8) we obtain that no index \( m \in (k_{j-1}, k_{j}) \) satisfies \( V_{z_{k_{j}}} \subset V_{z_{m}} \). Since \( i \notin \mathcal{G}_\ell \), we have \( i \in (k_{j-1}, k_j) \) and thus we have both
\[
V_{z_{k_{j}}} \subset V_{z_i} \quad \text{and} \quad |z_{k_j}| > |z_i|.
\]
Now using Lemma 1 and \( \beta \eta > 1 \), we obtain
\[
\left( 1 - \left| z_{k_j} \right|^2 \right)^\beta \leq \left( 1 - \left| z_i \right|^2 \right)^{\beta \eta} \leq \left( 1 - \left| z_i \right|^2 \right)^\beta.
\]
If we choose \( w \in V_{z_{k_{j}}} \setminus V_{z_i} \), then \( w, z_\ell \in V_{z_{k_{j}}} \) implies \( |z_\ell - w| \leq C \left( 1 - \left| z_{k_j} \right|^2 \right)^\beta \) by definition, and \( w \notin V_{z_i} \) implies \( |1 - w \cdot P_{z_i}| \geq C \left( 1 - \left| z_i \right|^2 \right)^\beta \). Together with the
reverse triangle inequality we thus have
\[ |1 - \overline{T} \cdot Pz_i| \geq |1 - \overline{w} \cdot Pz_i| - |\overline{T} \cdot Pz_i - \overline{w} \cdot Pz_i| \]
\[ \geq C \left( 1 - |z_i|^2 \right)^\beta - C \left( 1 - |z_i|^2 \right)^\beta \eta \]
\[ \geq (1 - |z_i|)^{\beta_1}, \]
for some $\beta_1 \in (\beta, \rho)$ (again provided the $|z_i|$ are large enough). Thus in the case $z_i \in V_{\ell}$, estimate \((2.48)\) again follows from the third line in \((2.7)\). Finally, the estimate \((1 - |z_i|^2) \sigma \leq C \mu(z_i)\) is trivial and this yields \((2.47)\).

Combining \((2.47)\) and \((2.9)\) we then have for the sequence \(\{\xi_j - \varphi(z_j)\}_{j=1}^J\),
\[ \|\{\xi_j - \varphi(z_j)\}_{j=1}^J\|_{L^2(d\mu)} \leq C \left\{ \begin{array}{l} \|B_j\|_{L^2(d\mu)} \\ \sum_{j=1}^J |a_j| \mu(z) \end{array} \right\}^{1/2} \]
\[ \leq C \left\{ \sum_{j=1}^J |a_j|^2 \mu(z) \right\}^{1/2} \left\{ \sum_{j=1}^J \mu(z) \right\}^{1/2} \]
\[ = C \|\mu\| \left\{ \sum_{j=1}^J |a_j| \right\}_{L^2(d\mu)} \]
\[ < C \varepsilon \left\{ \sum_{j=1}^J |a_j| \right\}_{L^2(d\mu)}. \]
This completes the proof of \((2.41)\).

We now prove the estimate $\|\varphi\|_{B_2} \leq C$ in \((2.45)\) whenever $\|\{a_j\}_{j=1}^J\|_{L^2(d\mu)} = 1$, independent of $J \geq 1$. Thus we must show that
\[ \int_D |\nabla \varphi(z)|^2 dz \leq C, \]
independent of $J \geq 1$. Now
\[ \varphi = \sum_{i=1}^J a_i \varphi_i = \sum_{i=1}^J a_i \Gamma_s g_i = \Gamma_s g \]
where $g = \sum_{i=1}^J a_i g_i$ with $\|\{a_i\}_{i=1}^J\|_{L^2(d\mu)} = 1$. Moreover,
\[ |\nabla \Gamma_s g(z)| \leq C_s \widehat{T}_s |g(z)| \]
where the operator $\widehat{T}_s$ is given by
\[ \widehat{T}_s f(z) = C_s \int_D \frac{f(w) \left( 1 - |w|^2 \right)^s}{|1 - \overline{w} \cdot z|^{1+s}} dw. \]
Thus we must estimate $\int_D \left| \widehat{T}_s |g(z)| \right|^2 dz$. Now by Theorem 2.10 in [Zhu], $\widehat{T}_s$ is bounded on $L^2$ if and only if $s > -\frac{1}{2}$ so, for any such $s$
\[ \int_D |\nabla \varphi(z)|^2 dz \leq C \int_D |g(z)|^2 dz \]
Since the supports of the \( g_i \) are pairwise disjoint by the separation condition, we obtain from (2.7) that \( g = \sum_{i=1}^{J} a_i g_i \) satisfies
\[
\int_D |g(z)|^2 \, dz = \sum_{i=1}^{J} |a_i|^2 \int_D |g_i(z)|^2 \, dz \\
\leq C \sum_{j=1}^{J} |a_i|^2 \left( \log \frac{1}{1 - |z_j|^2} \right)^{-1} \\
= C \left\{ \sum_{j=1}^{J} a_i \right\} J \| a_j \|_{\ell^2(\mu)} = C.
\]
This completes the proof of Lemma 4.

Now we finish proving the sufficiency portion of Theorem 4. Fix \( s > -1, 0 < \delta < 1 \) and \( \{\xi_j\}^J_{j=1} \) with \( \|\{\xi_j\}^J_{j=1}\|_{\ell^2(\mu)} = 1 \). Then by Lemma 4 there is \( f_1 = \sum_{i=1}^{J} a_i^1 \varphi_i \in B_2 \) such that \( \|\{\xi_j - f_1(z_j)\}^J_{j=1}\|_{\ell^2(\mu)} < \delta \) and using Lemma 3 as well, \( \|a_i^1\|^J_{\ell^2(\mu)}, \|f_1\|_{B_2} \leq C \) where \( C \) is the product of the constants in (2.45) and (2.13). Now apply Lemma 3 to the sequence \( \{\xi_j - f_1(z_j)\}^J_{j=1} \) to obtain the existence of \( f_2 = \sum_{i=1}^{\infty} a_i^1 \varphi_i \in B_2 \) such that \( \|\{\xi_j - f_1(z_j) - f_2(z_j)\}^J_{j=1}\|_{\ell^2(\mu)} < \delta^2 \) and again using Lemma 3 as well, \( \|a_i^2\|^J_{\ell^2(\mu)}, \|f_2\|_{B_2} \leq C\delta \) where \( C \) is again the product of the constants in (2.45) and (2.13). Continuing inductively, we obtain \( f_m = \sum_{i=1}^{J} a_i^m \varphi_i \in B_2 \) such that
\[
\left\{ \xi_j - \sum_{i=1}^{m} f_i(z_j) \right\}^J_{j=1} < \delta^m, \\
\left\| \sum_{i=1}^{m} a_i^m \varphi_i \right\|_{\ell^2(\mu)}, \|f_m\|_{B_2} \leq C\delta^{m-1}.
\]
If we now take
\[
\varphi = \sum_{m=1}^{\infty} f_m = \sum_{m=1}^{\infty} \left\{ \sum_{i=1}^{J} a_i^m \varphi_i \right\} = \sum_{i=1}^{J} a_i \varphi_i,
\]
we have
\[
\xi_j = \varphi(z_j), \quad 1 \leq j \leq J,
\]
\[
\|a_i\|^J_{\ell^2(\mu)} \leq C, \\
\|\varphi\|_{B_2} \leq C,
\]
if \( \varepsilon > 0 \) is chosen small enough in (2.9). A limiting argument using \( J \to \infty \) now completes the sufficiency proof of Theorem 4.

3. Theorem B

We follow the pattern of the previous section. We will state a result more general than Theorem B and prove the half of that result that contains Theorem B. We return to the other half of the proof in Section [7]
In this section we drop the requirement that \( \| \mu_Z \| < \infty \). The key is to let \( Z \) be a subtree of \( T \), so that if \( z \in Z \) is a Böe child of \( w \in Z \), then \( z \) actually lies in the Bergman successor set \( S \) of \( w \), and hence the value of \( c_{\rho, \alpha} (\gamma_w (z)) \) in Lemma 2 is 1, which is exploited in (3.4) below. The advantage when assuming (Tree-like) is that we may dispense with the complicated inductive definition of the coefficients \( a_k \) in (2.12) for the holomorphic function \( \xi \) in (2.10) approximating \( \xi \) on \( Z \), and instead use the elementary construction in (3.2) below of a holomorphic function \( M \xi \) approximating the integrated sequence \( I \xi \) of \( \xi \) on \( Z \). This permits us to interpolate the \( \alpha \)-approximation \( \xi \) in (3.4) below of a holomorphic function \( M \xi \) approximating the integrated sequence \( I \xi \) using the operator \( \Delta M \), whose kernel is better localized. Of course in the absence of (Tree-like), the values \( c_{\rho, \alpha} (\gamma_w (z)) \) may lie in \([0,1)\) and then \( M \xi \) will not be a good approximation to \( I \xi \) on \( Z \).

**Theorem 5.** Suppose \( Z = \{ z_j \}_{j=1}^\infty \subset \mathbb{D} \) is a subtree of \( T \) that satisfies the separation condition \([\text{Sep}]\), and if \( C \) is the constant in \([\text{Sep}]\), that there is \( \beta \in (1 - C/2, 1) \) satisfying (Tree-like). Then \( Z \) is onto interpolating for the Böe space \( B_{2, \mathbb{C}} \) if and only if the weak simple condition \([\text{WeakSimp}]\) holds.

**Proof** (of the sufficiency of the conditions in the Theorem): Fix \( \{ \xi_j \}_{j=1}^\infty \) with

\[
\left\| \left\{ \frac{\xi_j}{\| k_{z_j} \|_{B_2} } \right\}_{j=1}^\infty \right\|_{\ell^2} = 1.
\]

Recall that \( \| k_{z_j} \|_{B_2} \approx \left( \log \frac{1}{1 - |z_j|^2} \right)^{1/2} \) and that we may suppose \( Z \subset T \). We note that (2.4) holds here - in fact the proof is simpler using the separation condition \([\text{Sep}]\) and the assumption that \( Z \) is a subtree of \( T \) (and hence has branching number at most 2). We can adjoin the origin to \( Z \) in which case (2.4) yields that \( \sum_{j=1}^\infty (1 - |z_j|)^\sigma < \infty \). Thus, as at the start of the previous proof, given any \( \sigma > 0 \), we can discard all points from \( Z \) that lie in some ball \( B(0, R) \), \( R < 1 \), and reorder the remaining points so that

\[
(3.1) \quad \left( \log \frac{1}{1 - R^2} \right)^{1/2}, 1 - R^2, \sum_{j=1}^\infty (1 - |z_j|)^\sigma < \epsilon.
\]

We next, in addition, suppose that the sequence \( Z = \{ z_j \}_{j=1}^J \) is finite, and obtain an appropriate estimate independent of \( J \geq 1 \). Given a sequence of complex numbers \( \xi = \{ \xi_j \}_{j=1}^J \) we define a holomorphic function \( M \xi \) on the ball by

\[
(3.2) \quad M \xi (z) = \sum_{j=1}^J \xi_j \varphi_{z_j} (z), \quad z \in \mathbb{D},
\]

where \( \varphi_{z_j} (z) \) is as in Lemma 2. View \( \mu \) as the measure assigning mass \( \left( \log \frac{1}{1 - |z_j|^2} \right)^{-1} \) to the point \( j \in \{ 0, 1, 2, ..., J \} \). We have

\[
\left\| (\xi_j)_{j=1}^J \right\|_{\ell^2(\text{d} \mu)} \approx \left\| \left\{ \frac{\xi_j}{\| k_{z_j} \|_{B_2} } \right\}_{j=1}^J \right\|_{\ell^2},
\]
for any complex sequence \( \{ \xi_j \}_{j=1}^J \). We will use another useful consequence of Lemma 1.

\[
(3.3) \quad 1 - |A z_j|^2 \leq \left(1 - |z_j|^2 \right)^\eta, \quad \text{for } z_j \in \mathcal{V}_{z} \setminus \mathcal{C} \left( z \right).
\]

Indeed, if \( z_j \in \mathcal{V}_{z} \setminus \mathcal{C} \left( z \right) \), then \( A z_j \neq z \) and \( |A z_j| \geq |z| \) by the construction in (2.8). Then \( \mathcal{V}_{z} \cap V_{A z_j} \) contains \( z_j \) and is thus nonempty, and Lemma 1 now shows that \( 1 - |A z_j|^2 \leq \left(1 - |z_j|^2 \right)^\eta \).

Now define a linear map \( T \) from \( \ell^2 (d\mu) \) to \( \ell^2 (d\mu) \) by

\[
T \xi = \triangle \left( M \xi \right) |z = \{ M \xi (z_k) - M \xi (A z_k) \}_{j=1}^J = \left\{ \sum_{j=1}^J \xi_j \left[ \varphi_{z_j} (z_k) - \varphi_{z_j} (A z_k) \right] \right\}_{j=1}^J,
\]

where \( A z_j \) denotes the predecessor of \( z_j \) in the forest structure on \( Z \) defined in (2.8) above (we identify \( z_k \) with \( k \) here). Let \( \mathcal{R} \) denote the set of all roots of maximal trees in the forest. In the event that \( z_k \notin \mathcal{R} \), then \( A z_k \) isn’t defined and our convention is to define \( \varphi_{z_j} (A z_k) = 0 \). We claim that \( T \) is a bounded invertible map on \( \ell^2 (d\mu) \) with norms independent of \( J \geq 1 \). To see this it is enough to prove that \( I - T \) has small norm on \( \ell^2 (d\mu) \) where \( I \) denotes the identity operator. We have

\[
(1 - T) \xi = \left\{ \xi_k - \sum_{j=1}^J \xi_j \left[ \varphi_{z_j} (z_k) - \varphi_{z_j} (A z_k) \right] \right\}_{k=1}^J
\]

\[
= \left\{ \xi_k \varphi_{z_k} (A z_k) \right\}_{k=1}^J - \left\{ \sum_{j \neq k} \xi_j \left[ \varphi_{z_j} (z_k) - \varphi_{z_j} (A z_k) \right] \right\}_{k=1}^J.
\]

since \( \varphi_{z_k} (z_k) = 1 \).

Now we estimate the kernel \( K (k, j) \) of the operator \( I - T \). We have on the diagonal,

\[
|K (k, k)| = \left\{ \left| \varphi_{z_k} (A z_k) \right| \leq \left(1 - |z_k|^2 \right)^{\rho - \beta_1 (1 + s)} \right. \quad \text{if } z_k \notin \mathcal{R},
\]

\[
0 \quad \text{if } z_k \in \mathcal{R},
\]

by the third estimate in (2.6).

Suppose now that \( z_k \notin \mathcal{R} \) and \( j \neq k \). Lemma 11 shows that \( \left| \varphi'_{z_j} (\zeta_k) \right| \leq \left(1 - |z_j|^2 \right)^{-\alpha} \) and the definition of \( \mathcal{V}_{z_j} \) shows that \( |z_k - A z_k| \leq \left(1 - |A z_k|^2 \right)^{\beta} \).

Thus if \( 1 - |A z_k|^2 \leq \left(1 - |z_j|^2 \right)^{\eta} \), then

\[
|K (k, j)| = |\varphi_{z_j} (z_k) - \varphi_{z_j} (A z_k)| \leq \left| \varphi'_{z_j} (\zeta_k) \right| |z_k - A z_k|
\]

\[
\leq C \left(1 - |z_j|^2 \right)^{-\alpha} \left(1 - |A z_k|^2 \right)^{\beta}
\]

\[
\leq C \left(1 - |z_j|^2 \right)^{\eta (\beta - \delta) - \alpha} \left(1 - |A z_k|^2 \right)^{\delta},
\]

where the exponent \( \eta (\beta - \delta) - \alpha \) is positive if we choose \( \delta \) small enough, since \( \alpha < 1 < \beta \eta \) by Lemma 1.
Suppose instead that \(1 - |Az_k|^2 > \left(1 - |z_j|^2\right)^\eta\). Then \(Az_k \notin V_{z_j}\) by Lemma 1. If \(z_k \notin \mathcal{C}(z_j)\), then \(z_k \notin V_{z_j}\) by (3.3), and this time we use the third estimate in (2.6) to obtain

\[
|K(k,j)| = |\varphi_{z_j}(z_k) - \varphi_{z_j}(Az_k)| \leq |\varphi_{z_j}(Az_k)| + |\varphi_{z_j}(z_k)| \\
\leq C \left(1 - |z_j|^2\right)^{(\rho - \beta_1)(1+s)} \\
\leq C \left(1 - |z_j|^2\right)^{(\rho - \beta_1)(1+s)} \left(1 - |Az_k|^2\right)^\delta.
\]

On the other hand, if \(z_k \in \mathcal{C}(z_j)\), then \(|z_k| \geq |z_j|\) and our hypothesis (Tree-like) implies that \(z_k \in S(z_j)\). Then we have

\[
(3.4) \quad |K(k,j)| = |\varphi_{z_j}(z_k)| \leq C \left(1 - |z_j|^2\right)^{(\rho - \beta_1)(1+s)},
\]

by the first two estimates in (2.6) since \(c_{p,\alpha}(\gamma_{z_j}(z_k)) = 1\) in Lemma 2 if \(z_k \in S(z_j)\).

Finally, we consider the case when \(z_k \in \mathcal{R}\) and \(j \neq k\). The third estimate in (2.6) shows that

\[
|K(k,j)| = |\varphi_{z_j}(z_k)| \leq C \left(1 - |z_j|^2\right)^{\delta},
\]

where the exponent \((\rho - \beta_1)(1+s)\) can be made as large as we wish by taking \(s\) sufficiently large. Combining all cases we have in particular the following estimate for some \(\sigma_1, \sigma_2 > 0:\)

\[
|K(k,j)| \leq C \begin{cases} 
\left(1 - |z_j|^2\right)^{\sigma_1} \left(1 - |Az_k|^2\right)^{\sigma_2}, & \text{if } z_k \notin \mathcal{R} \text{ and } z_k \notin \mathcal{C}(z_j) \\
\left(\log \frac{1}{1 - |z_j|^2}\right)^{-1}, & \text{if } z_k \notin \mathcal{R} \text{ and } z_k \in \mathcal{C}(z_j) \\
\left(1 - |z_j|^2\right)^{\delta}, & \text{if } z_k \in \mathcal{R}
\end{cases}
\]

Now we obtain the boundedness of \(I - T\) on \(\ell^2(d\mu)\) with small norm by Schur’s test. It is here that we use the assumption that \(\mu\) satisfies the weak simple condition (WeakSimp). With \(\xi \in \ell^2(d\mu)\) and \(\eta \in \ell^2(d\mu)\), we have

\[
\left|\langle (I - T) \xi, \eta \rangle_{\mu}\right| = \left|\sum_k \left(\sum_j K(k,j) \xi_j\right) \eta_k \mu(k)\right| \\
\leq C \sum_j \sum_{k \notin \mathcal{R}, z_k \notin \mathcal{C}(z_j)} \left(1 - |z_j|^2\right)^{\sigma_1} \left(1 - |Az_k|^2\right)^{\sigma_2} |\xi_j| |\eta_k| \mu(k) \\
+ C \sum_j \sum_{k \notin \mathcal{R}, z_k \notin \mathcal{C}(z_j)} \left(\log \frac{1}{1 - |z_j|^2}\right)^{-1} |\xi_j| |\eta_k| \mu(k) \\
+ C \sum_j \sum_{k \in \mathcal{R}} \left(1 - |z_j|^2\right)^{\delta} |\xi_j| |\eta_k| \mu(k),
\]

where \(\sigma_1, \sigma_2 > 0\).
and since \( \mu(j) \leq \left(1 - |z_j|^2 \right)^{\varepsilon} \), we have with \( \sigma'_1 = \sigma_1 - \varepsilon \),

\[
|\langle (\mathbb{I} - T) \xi, \eta \rangle_{\mu}| \leq C \sum_j \sum_{k \notin \mathbb{R}, z_k \notin \mathbb{C}(z_j)} \left(1 - |z_j|^2 \right)^{\sigma'_1} \left(1 - |Az_k|^2 \right)^{\sigma_2} |\xi_j| \mu(j) |\eta_k| \mu(k) \\
+ C \sum_j \sum_{k \notin \mathbb{R}, z_k \in \mathbb{C}(z_j)} |\xi_j| \mu(j) |\eta_k| \mu(k) \\
+ C \sum_j \sum_{k \in \mathbb{R}} \left(1 - |z_j|^2 \right)^{2} |\xi_j| \mu(j) |\eta_k| \mu(k).
\]

By Schur’s test it suffices to show

\[
\mu(Ak) + \sum_{j=1}^{J} \left(1 - |z_j|^2 \right)^{\sigma'_1} \mu(j) < C\varepsilon < 1, \\
\sum_{k \in \mathbb{R}, z_k \in \mathbb{C}(z_j)} \mu(k) + \sum_{k \notin \mathbb{R}} \left(1 - |Az_k|^2 \right)^{\sigma_2} \mu(k) < C\varepsilon < 1, \\
\sum_{k \in \mathbb{R}} \left(1 - |z_j|^2 \right)^{3} \mu(k) < C\varepsilon < 1.
\]

Now (3.1) yields

\[
\sum_{j=1}^{J} \left(1 - |z_j|^2 \right)^{\sigma'_1} \mu(j) \leq C \sum_j \left(1 - |z_j|^2 \right)^{\sigma''_1} < C\varepsilon,
\]

and combined with the weak simple condition \( \text{WeakSimp} \), we have

\[
\sum_{k=1}^{J} \left(1 - |Az_k|^2 \right)^{\sigma_2} \mu(k) = \sum_{\ell} \left(1 - |z_\ell|^2 \right)^{\sigma_2} \left( \sum_{z_k \in \mathbb{C}(z_\ell)} \mu(k) \right) \\
\leq C \sum_{\ell} \left(1 - |z_\ell|^2 \right)^{\sigma_2} \mu(\ell) \\
\leq C \sum_{\ell} \left(1 - |z_\ell|^2 \right)^{\sigma'_1} < C\varepsilon.
\]

Finally we write the annulus \( B(0, 1) \setminus B(0, R) \) as a pairwise disjoint union \( \cup_{i=1}^{N} B_i \) of Carleson boxes of “size” \( R \) where \( N \approx \left(1 - R^2\right)^{-1} \). Then

\[
\sum_{z_k \in B_i, k \in \mathbb{R}} \mu(k) \leq C \left(1 + \log \frac{1}{1 - R^2}\right)^{-1} \leq C
\]

by the weak simple condition \( \text{WeakSimp} \), and thus the left side of the final estimate in (3.5) satisfies

\[
\sum_{k \in \mathbb{R}} \left(1 - |z_j|^2 \right)^{2} \mu(k) \leq (1 - R^2)^2 \sum_{i=1}^{N} \sum_{z_k \in B_i, k \in \mathbb{R}} \mu(k) \\
\leq C (1 - R^2)^2 N \\
\leq C (1 - R^2) < C\varepsilon,
\]

by (3.1) as required.
Thus $T^{-1}$ exists uniformly in $J$. Now we take $\xi \in \ell^2 (d\mu)$ and set $\eta = \Delta \xi$. Here we use the convention that $\xi (A\alpha) = 0$ if $\alpha$ is a root of a tree in the forest $Z$. By the weak simple condition we have the estimate

\begin{equation}
\|\eta\|^2_{\ell^2(d\mu)} = \sum_j |\eta_j|^2 \mu (j) = \sum_j |\xi_j - \xi_{A_j}|^2 \mu (j)
\end{equation}

\begin{align*}
&\leq C \sum_j |\xi_j|^2 \mu (j) + C \sum_{\ell} |\xi_{\ell}|^2 \left( \sum_{z_j \in C (z_{\ell})} \mu (j) \right) \\
&\leq C \sum_j |\xi_j|^2 \mu (j) + C \sum_{\ell} |\xi_{\ell}|^2 \mu (\ell) \\
&\leq C \|\xi\|^2_{\ell^2(d\mu)} .
\end{align*}

Then let $h = M (T^{-1} \eta)$ so that

$$
\Delta h \big|_Z = \Delta (MT^{-1} \eta) \big|_Z = TT^{-1} \eta = \eta = \Delta \xi.
$$

Thus the holomorphic function $h$ satisfies

$$
h \big|_Z = \xi.\n$$

Finally, from \((2.7)\) and then \((3.6)\) we have the Besov space estimate \((ArRoSa2)\),

$$
\|h\|_{B_2}^2 \leq C \sum_{j=1}^J \left| (T^{-1} \eta)_j \right|^2 \int_\mathbb{D} \left( 1 - |\zeta|^2 \right)^2 \left| g_w (\zeta) \right|^2 \mu (\zeta) d\zeta
\end{equation}

\begin{align*}
&\leq C \sum_{j=1}^J \left| (T^{-1} \eta)_j \right|^2 \left( \log \frac{1}{1 - |w|^2} \right)^{-1} \\
&\leq C \|T^{-1} \eta\|^2_{\ell^2(d\mu)} \leq C \|\eta\|^2_{\ell^2(d\mu)} \leq C \|\xi\|^2_{\ell^2(d\mu)} .
\end{align*}

Since all of this is uniform in $J$ we may let $J \to \infty$ and use a standard normal families argument to complete the proof of the sufficiency of the hypotheses in Theorem 5.

4. Relations Between Conditions

4.1. An Example Covered by Theorem B but not Theorem A. Let $a, b > 1$ satisfy $[a^{k+1}b] \geq [a^k] + 1$ for all $k \geq 0$ (in particular this will hold if $(a-1)b \geq 2$), and define a Cantor-like sequence $Z$

$$
Z = Z_{a,b} = \cup_{k=0}^\infty \{ z_j^k \}_{j=1}^{2k} \subset T
$$

as follows. Pick a point $z_0^0$ of $T$ satisfying $d (z_0^0) = [b]$. Then choose $2^1$ points $\{ z_1^1, z_1^1 \} \subset T$ that are successors to distinct children of $z_0^0$ and having $d (z_1^1) = [ab]$, $1 \leq j \leq 2^1$, and, recalling that $\beta$ is the hyperbolic distance on the disk, $\beta (z_1^1, z_1^1) \geq [ab]$ for $i \neq j$. Then choose $2^2$ points $\{ z_1^2, z_2^2, z_3^2, z_4^2 \} \subset T$ that are successors to distinct children of the points in $\{ z_1^1, z_2^1 \}$ and having $d (z_1^2) = [a^2b]$, $1 \leq j \leq 2^2$, and $\beta (z_1^2, z_2^2) \geq [a^2b]$ for $i \neq j$. Having constructed $2^k$ points $\{ z_j^k \}_{j=1}^{2^k} \subset T$ in this way, we then choose $2^{k+1}$ points $\{ z_j^{k+1} \}_{j=1}^{2^{k+1}} \subset T$ that are successors to distinct children of the points in $\{ z_j^k \}_{j=1}^{2^k}$ and having $d (z_j^{k+1}) = [a^{k+1}b]$, $1 \leq j \leq 2^{k+1}$, and
\( \beta(z_i^{k+1}, z_j^{k+1}) \geq [a^{k+1}b] \) for \( i \neq j \). Note that the condition \([a^{k+1}b] \geq [a^kb] + 1\) allows for the existence of such points. Then \( Z = \cup_{k=0}^{\infty} \{z_j^k\}_{j=1}^{2^k} \) satisfies the separation condition \( \text{Sep} \) with constant roughly \( a - 1 \) and condition \( \text{Tree-like} \) with \( \beta \) close to 1, and the associated measure \( \mu_Z \) satisfies the weak simple condition \( \text{WeakSimp} \) with constant 2. Thus Theorem 5 applies to show that \( Z \) is onto interpolating for \( B_2 \). Yet the total mass of the measure \( \mu_Z \) satisfies

\[
\|\mu_Z\| = \sum_{k=0}^{\infty} 2^k [a^kb]^{-1} \approx \frac{1}{b} \sum_{k=0}^{\infty} \left( \frac{2}{a} \right)^k = \infty
\]

if \( a \leq 2 \).

4.2. An Example Covered by Theorem A but not Theorem 3. We now use a similar construction to give a separated sequence \( W \) in the disk with \( \text{finite} \) measure \( \mu = \mu_W \) satisfying the weak simple condition but not the simple condition. This yields an example of a sequence which fails the simple condition, but to which Theorem A applies.

We continue the notation of the previous example. We choose \( a = 2 \) for convenience, let \( b, N \) be large integers, and replace the sequence \( Z_{2,b} \) above with the truncated sequence \( Z_{2,b,N} = \cup_{k=0}^{N} \{z_j^k\}_{j=1}^{2^k} \). Then \( Z_{2,b,N} \) satisfies the separation condition \( \text{Sep} \) with constant roughly 1, the associated measure \( \mu_{Z_{2,b,N}} \) satisfies the weak simple condition \( \text{WeakSimp} \) with constant 2, and the total mass of \( \mu_{Z_{2,b,N}} \) is about \( \frac{N}{b} \):

\[
\|\mu_{Z_{2,b,N}}\| = \sum_{k=0}^{N} 2^k [2^kb]^{-1} \approx \frac{1}{b} \sum_{k=0}^{N} \left( \frac{2}{2} \right)^k \approx \frac{N}{b}.
\]

On the other hand the constant \( C(\mu_{Z_{2,b,N}}) \) in the simple condition \( \text{(2.38)} \) for \( \mu_{Z_{2,b,N}} \) satisfies

\[
C(\mu_{Z_{2,b,N}}) \geq N,
\]

since

\[
\frac{N}{b} \approx \|\mu_{Z_{2,b,N}}\| = \sum_{\alpha \geq 0} \mu_{Z_{2,b,N}}(\alpha) \leq C \frac{1}{d(z_n)} = \frac{C}{b}.
\]

It is now an easy exercise to choose sequences of parameters \( \{b(n)\}_{n=1}^{\infty} \) and \( \{N(n)\}_{n=1}^{\infty} \), and initial points \( \{z^0_1(n)\}_{n=1}^{\infty} \) so that the corresponding sequences \( Z_{2,b(n),N(n)} = \cup_{k=0}^{N(n)} \{z^k_j(n)\}_{j=1}^{2^k} \) satisfy

\[
\|\mu_{Z_{2,b(n),N(n)}}\| \approx \frac{N(n)}{b(n)} \leq 2^{-n}
\]

and

\[
\lim_{n \to \infty} N(n) = \infty,
\]

along with the nested property

\[
z^0_1(n + 1) \geq z^b(n) \geq 1 , \quad n \geq 1.
\]

Then the union \( W = \cup_{n=1}^{\infty} Z_{2,b(n),N(n)} \) satisfies the separation condition and the associated measure \( \mu_W \) is finite by \( \text{(4.2)} \), satisfies the weak simple condition by \( \text{(4.4)} \), yet fails the simple condition by \( \text{(4.1)} \) and \( \text{(4.3)} \).
5. Tree Interpolation and Theorem C

5.1. Reduction to a Basic Construction. In this section we prove Theorem C.

Previously our tree $T$ was constructed to contain our given sequence $Z$. In this section we regard $T$ as a given, fixed, Bergman tree and we will be interested in subsets of $T$. Our tree has a bounded branching number but to keep the notation simple we suppose it is a dyadic tree. We want to characterize the subsequences $Z = \{\alpha_j\}_{j=1}^\infty$ of $T$ which are onto interpolating sequence for $B_2(T)$. We had defined that class of sequences using the weighted restricting operator taking note of the estimate $d(\alpha_j) \sim \|k_z\|_{B_2}^2$, we can alternatively characterize the sequences by

\begin{equation}
\text{(5.1)} \quad \text{for every sequence } \{\xi_j\}_{j=1}^\infty \text{ with } \left\|\left\{d(\alpha_j)^{-1/2}\xi_j\right\}_{j=1}^\infty\right\|_2 = 1,
\end{equation}

there is $f \in B_2(T)$ with $\|f\|_{B_2(T)} \leq C$ and $f(\alpha_j) = \xi_j$, $j \geq 1$.

Suppose $Z \subset T$ is given and (TreeCap) holds. One immediate consequence is the tree separation condition,

\begin{equation}
\text{(5.2)} \quad \forall z, w \in Z, \quad d(z, w) \geq d(\alpha, z)
\end{equation}

This holds because the left hand side of (TreeCap) a majorant for

$$
\inf \left\{ \sum_{\xi \in T} |\Delta f(\xi)|^2 : f(z) = 1, f(w) = 0 \right\} = \frac{1}{d(z, w) - 1}.
$$

This separation is certainly similar to (Sep'), however, because of "edge effects" this condition is weaker.

We will prove lemma about the existence of certain almost extremal functions in $B_2(T)$. They only take the values 0, 1 on $Z$, their discrete derivatives have disjoint support, and they have controlled norms. Using the lemma the proof of Theorem C is immediate.

Let $E, F$ be disjoint subsets of $T$. The capacity $Cap(E, F)$ of the condenser $(E, F)$ is defined as

$$
Cap(E, F) = \inf \{ \|h\|_{L^2} : Ih|_E = 1, Ih|_F = 0 \}.
$$

For $Z$ a sequence in $T$ set $\gamma(z, Z) = Cap(z, Z \setminus \{z\})$. With this notation our tree capacity condition (TreeCap) can be written as

\begin{equation}
\text{(5.3)} \quad \gamma(z, Z) \leq Cd(z)^{-1}.
\end{equation}

We say that $S \subset T$ is a stopping region if every pair of distinct points in $S$ are incomparable in $T$.

**Lemma 5.** Given a subset $Z$ of $T$ that satisfies the tree separation condition (TreeCap), there are functions $H_w = Ih_w$ on $T$ and a constant $C$ depending only on $C$ in (5.2) satisfying

1. $H_w(z) = \delta_{w,z}$ for $w, z \in Z$
2. $\text{supp}(h_w) \cap \text{supp}(h_z) = \emptyset$ for $z, w \in Z, z \neq w$,
3. $\|H_w\|_{B_2(T)} \leq 2C\gamma(w, Z)$ for $w \in Z$,
4. If $S$ is a stopping region in $T$, then $\sum_{\alpha \in S} |h_w(\alpha)| \leq 2\gamma(w, Z)$ for $w \in Z$.

**Proof of Theorem C given the Lemma:** We’ve already mentioned that (5.1) implies (TreeCap). Conversely, if (TreeCap) holds, then so does (5.2) and hence also properties 1, 2 and 3 of Lemma 5. Let $Z = \{z_j\}_{j=1}^\infty$. If $\xi = \{\xi_j\}_{j=1}^\infty \in l^2(\mu)$,
then \( f = \sum_{j=1}^{\infty} \xi_j H_{z_j} \in B_2(T) \) with \( B_2(T) \) norm at most \( C\|\xi\|_{L^2(\mu)} \) by properties 2 and 3; and \( f(z_j) = \xi_j \) for all \( j \geq 1 \) by property 1.

5.2. Extremal Functions. We now prove a string of results on capacity that will culminate in the proof of Lemma \( \Box \). More precisely, properties 1, 2 and 3 of Lemma 5 will follow from Proposition 4 below, and property 4 will follow from Proposition 5.

We use the following notation. If \( x \) is an element of the tree \( T \), \( x^{-1} \) denotes its immediate predecessor in \( T \). If \( z \) is an element of the sequence \( Z \subset T \), \( Pz \) denotes its predecessor in \( Z \): \( Pz \in Z \) is the maximum element of \( Z \cap [o,z] \) Let \( \Omega \subseteq T \). A point \( x \in T \) is in the interior of \( \Omega \) if \( x,x^{-1},x_+ ,x_- \in \Omega \). A function \( H \) is harmonic in \( \Omega \) if

\[
H(x) = \frac{1}{3}(H(x^{-1}) + H(x_+) + H(x_-))
\]

for every point \( x \) which is interior in \( \Omega \). If \( H = Ih \) is harmonic in \( \Omega \), then we have that

\[
h(x) = h(x_+) + h(x_-)
\]

whenever \( x \) is in the interior of \( \Omega \).

5.2.1. Basic Properties.

**Proposition 1.** Let \( T \) be a dyadic tree.

1. If \( E \) and \( F \) are finite, there is an extremal function \( H = Ih \) such that \( \text{Cap}(E,F) = \|h\|^2_{L^2} \).
2. The function \( H \) is harmonic on \( T \setminus (E \cup F) \).
3. Let \( E = \{z\} \), \( F = Z - \{z\} \). Then the support of \( h \) consists of (at most) three connected components. The upper support consists of the segment \( (Pz,z) \) and of all segments \( [\zeta(w),w] \), where \( w \in Z \) and \( \zeta(w),w \) has some intersection with the component of \( T \setminus Z \) lying above \( z \) (i.e., the component containing \( z^{-1} \)). The lower support consists of all segments \( [\zeta(w),w] \), where \( w \in Z \) and \( \zeta(w),w \) has some intersection with one of the (at most) two components of \( T \setminus Z \) lying below \( z \) (i.e., the components containing, respectively, \( z_+ \) and \( z_- \)).
4. The function \( h \) is positive on \( (Pz,z) \), negative on the segments \( [\zeta(w),w] \) and vanishes everywhere else.

In classical potential theory capacities can be recovered from the derivative of the Green potential. An analogous result holds here for the capacity of \( \gamma^- \) with \( h \) playing the role of that derivative.

**Proposition 2.** Let \( \gamma_+, \gamma_p \) be the \( L^2 \)-sum of \( h \) over the lower and upper components of its support, respectively. Then,

\[
\gamma_+ = -h(z_+), \quad \gamma_p = h(z^-).
\]

**Proof of both propositions:** We consider first the case of \( \gamma_+ \). Let \( T^+_z \) be the component of \( T \setminus Z \) containing \( x_+ \) and let \( \overline{T}^+_z \) be its forward closure, which is obtained by adding to \( T^+_z \) all points \( w \in Z \), \( w \neq z \), such that \( d(w,T^+_z) = 1 \). For each \( x \in T^+_z \), \( x_+ \in T^+_z \). We proceed by induction on the cardinality of \( Z \cap \overline{T}^+_z \).

If \( |Z \cap \overline{T}^+_z| = 1 \), \( Z \cap \overline{T}^+_z = \{w\} \), then \( \text{supp}(h) = \{z,w\} \) and \( h = -d(z,w)^{-1} \) over its support, so that \( \gamma_+ = d(z,w) \cdot d(z,w)^{-2} = d(z,w)^{-1} = -h(z_+) \).
Suppose we know that $N \geq 1$ and that the property holds when $\sharp(Z \cap T^+_z) \leq N$ and suppose now that $\sharp(Z \cap T^+_z) = N+1$. Consider $W_z = \{w \wedge w' : w, w' \in Z \cap T^+_z\}$, the subtree of $T^+_z$ generated by $Z \cap T^+_z$. Then, $W_z$ has a minimal element $\zeta \in T^+_z$ ($\zeta \not\in T$ because $N + 1 \geq 2$). Let $U_+ = S(\zeta_+) \cap Z \cap T^+_z$ and $U_- = S(\zeta_-) \cap Z \cap T^+_z$. Then, the function $H$ goes from 1 to $H(\zeta) > 0$ on $[z, \zeta]$ and from $H(\zeta)$ to 0 as $x$ moves from $\zeta$ to $U_\pm = S(\zeta_\pm) \cap Z \cap T^+_z$ in $\{\zeta\} \cup X_\pm = \{\zeta\} \cup S(\zeta_\pm) \cap T^+_z$. Note that

$$H(\zeta) = 1 - d(z, \zeta)h(z_+) = 1 - d(z, \zeta)h(\zeta).$$

The function $h|_{X_\pm}$ has minimal $\ell^2$ norm with the property that $\sum_{z \approx \zeta_\pm} = -h(\zeta)$ whenever $w \in U_+$, otherwise we could modify it to obtain a global $H$ with better $\ell^2$ norm, contradicting the hypothesis that $H$ was optimal. Also, observe that $\sharp(U_+) \leq N$.

Let $H_+ = Ih_+$ be the function such that $H_+(\zeta) = 1$, $H_+|_{U_+} = 0$ and which has minimal $\ell^2$ norm with these properties. Similarly define $H_- = Ih_-$ and $H' = Ih'$, the latter with the conditions $H'(z) = 1$ and $H'(\zeta) = 0$. By the induction hypothesis,

$$\|h_+\|_{\ell^2} = h_+(\zeta_+), \quad \|h_-\|_{\ell^2} = h_-(\zeta_-), \quad \|h'\|_{\ell^2} = h'(z_+) = h'(\zeta).$$

By uniqueness of the extremal function $H$ and by homogeneity of the minimization problem, we have that, with constants $a_+ = H(\zeta) = a_-, a' = 1 - H(\zeta)$,

$$h = a_\pm \cdot h_\pm, \ h = a' \cdot h'.$$

The norm of $h$ is then

$$\|h\|_{\ell^2}^2 = a_+^2 \|h_+\|_{\ell^2}^2 + a_-^2 \|h_-\|_{\ell^2}^2 + a'^2 \|h'\|_{\ell^2}^2
= H(\zeta)^2 \cdot (h_+(\zeta_+) + (h_-(\zeta_-)) + (1 - H(\zeta))^2 \cdot h'(\zeta)
= H(\zeta) \cdot [h(\zeta_+) + h(\zeta_-)] + (1 - H(\zeta)) \cdot h(\zeta)
= h(\zeta) = h(z_+),$$

as wished. In the fourth equality, we used the harmonicity of $H = Ih$.

Exactly the same argument works for $\gamma_-$ and a variation thereof gives the desired formula for $\gamma_p$. 

**Remark 2.** The proof gives a useful formula for computing capacities. Given $z, \zeta \in T$, $z < \zeta$ and given $U_\pm \subset S(\zeta_\pm)$ we have

$$\begin{equation}
\text{Cap}(z, U_+ \cup U_-) = \frac{\text{Cap}(\zeta, U_+) + \text{Cap}(\zeta, U_-)}{1 + d(z, \zeta)\text{Cap}(\zeta, U_+) + \text{Cap}(\zeta, U_-)).}
\end{equation}
$$

To see this note that,

$$\text{Cap}(z, U_+ \cup U_-) = h(z_+) = h(\zeta) = h(\zeta_+) + h(\zeta_-)
= H(\zeta)[h_+(\zeta_+) + h_-(\zeta_-)]
= (1 - d(z, \zeta)h(\zeta_+))\text{Cap}(\zeta, U_+) + \text{Cap}(\zeta, U_-)]
= (1 - d(z, \zeta) \cdot \text{Cap}(z, U_+) \cup U_-)\text{Cap}(\zeta, U_+) + \text{Cap}(\zeta, U_-)].$$

Later in this section we will use this formula to develop a simple, computable and geometric algorithm for calculating capacities.
5.2.2. Disjoint supports. From now on, we consider a finite sequence $Z$ satisfying (5.2). We want to show that the functions $h = h_z$ can be replaced by near extremal functions $k_z$, with the extra property that $\text{supp}(k_z) \cap \text{supp}(k_w) = \emptyset$ if $z \neq w$.

We will assume that $Z = \{z_j : j \geq 0\}$ is ordered in such a way that $d(z_n) \leq d(z_{n+1})$. We will also assume that $z_0 = o = 0$ belongs to $Z$. We define

$$Z_n = \{z_0, \ldots, z_n\}$$

and

$$\mathcal{T}_n = \bigcup_{j=1}^n [o, z_j]$$

We also need $\mathcal{T}_{\infty} = \bigcup_{j=1}^\infty [o, z_j]$, the minimum subset of $\mathcal{T}$ containing $Z$ which is geodesically connected. The landing point of $z = z_{n+1}$ is, by definition,

$$\xi(z) = \max ([o, z] \cap \mathcal{T}_n)$$

By construction, $Z \cap (\xi(z), z) = \emptyset$ and, if $z \neq w$, $[\xi(z), z]$ and $[\xi(w), w]$ are either disjoint, or they intersect in $\xi(z)$, or in $\xi(w)$. If $i_j \geq i_k$, then $\xi(z_{i_k}) \in [\xi(z_{i_k}), z_{i_j}]$ if and only if $\xi(z_{i_j}) = \xi(z_{i_k})$.

**Lemma 6.** ([ArRoSa]: Lemma 27) Let $Z$ be a sequence satisfying (5.2). Then, for some positive constant $\eta$,

$$d(\xi(z), z) \geq \eta d(z)$$

for all $z$ in $Z$.

As a consequence, by removing a finite number of points from $Z$ we can assume that $d(\xi(z), z) \geq 3$ for all $z \neq o$.

We record some further properties of the functions $H = H_z$.

**Proposition 3.** The function $H$ is increasing on $[P_z, z]$ and decreasing on all segments of the form $[\zeta(w), w]$. Moreover, $H$ is convex on all intervals of the form $[\zeta(w), w]$ and on $[P_z, z]$.

For $z$ in $Z$, let

$$\mathcal{N}(z) = \bigcup_{w \geq P(z)} [w, P(z)] \cap \mathcal{T}_\infty \\cap (Z \setminus \{z\})$$

The set $\mathcal{N}(z)$ is the “downward closure” of the connected component $\mathcal{N}_0(z)$ of $\mathcal{T}_\infty \\setminus (Z \setminus \{z\})$ containing $z$. We have

$$\mathcal{N}_0(z) = \mathcal{N}(z) \setminus (Z \setminus \{z\})$$

All of the interesting action takes place inside $\mathcal{N}(z)$.

The main tool we need is the following. We write $DK = k$ if $K = Ik$.

**Proposition 4.** If (5.3) holds, then to each $z$ in $Z$ we can associate a function $K_z$ in $B_2$ such that $K_z(w) = \delta_z(w)$ for $w \in Z$, $K_z$ is harmonic on $\text{supp}(DK_z) \setminus Z$, $\|K_z\|_{B_2} \leq C\gamma(z, Z)$ and such that $\text{supp}(DK_z) \cap \text{supp}(DK_w) = \emptyset$ if $z \neq w$. In fact the $K_z$ are pairwise disjoint as well: $\text{supp}(K_z) \cap \text{supp}(K_w) = \emptyset$ if $z \neq w$.

**Remark 3.** Because the $K_z$ have disjoint support we have an immediate solution, $\sum_{j=1}^\infty \xi_j K_{z_j}$, to the problem of interpolating $\xi = \{\xi_j\}_{j=1}^\infty$ on the tree $T$. 

Proof of the proposition: Fix $z \in Z$. By Proposition there is a function $H$ such that $\|H\|_{B_2}^2 = \gamma(z, Z)$ and $H(z) = 1$, $H|_{Z \setminus \{z\}} \equiv 0$, $H \geq 0$, supp$(DH) \subseteq N(z)$ and $H$ is convex on intervals of the form $[w, \xi(w)]$. Let $Q_z$ be the first point on $[P(z), z]$ such that $d(x, z) \leq \frac{1}{3}d(P(z), z)$. For each $w \in N(z) \cap (Z \setminus \{z\}) \cap S(\xi(z))$, let $Q_w$ be the first point $y$ in $[\xi(w), w]$ such that $d(y, \xi(w)) \geq \frac{1}{3}d(\xi(w), w)$. As in the proof of Proposition, $N(z) \cap (Z \setminus \{z\}) \cap S(\xi(z)) = \{z_i : j \geq 0\}$, where $z = z_{i_0}$, $d(z_{i+1}) \geq d(z_i)$ and $\xi(z_{i+1}) \in T_i$. The function $K_z$ is constructed inductively, in such a way that

$$\text{supp}(k_z) \subseteq N_r(z) \equiv \bigcup_{j}(Q_{z_{i_0}}, Q_{z_i}], \quad k_z = DK_z.$$ 

It is clear that if $z \neq w$ are points in $Z$, then $N_r(z) \cap N_r(w) = \phi$.

Step $j = 0$. Construct a new function $K_0$ as follows. On $[P(z), Q_z]$, set $K_0 = 0$. For $x \in [Q_z, z]$, set

$$K_0(x) = \frac{H(x) - H(Q_z)}{H(z) - H(Q_z)}$$

If $y$ is such that $y \wedge z \in [P(z), z]$, set

$$K_0(y) = \frac{K_0(y \wedge z)}{H(y \wedge z)}H(y)$$

Set $K_0(y) = H(y)$ otherwise. The function $K_0$ has the following properties (with $\eta$ from and with the parameter $\ell = 0$),

1. $K_0$ is admissible for $z$, $K_0 \geq 0$, $K_0$ is harmonic on $N_r(z)$, $K_0$ is convex on $[\xi(z_i), z_i]$ for $j \geq 1$;
2. $K_0 \leq H$, pointwise;
3. if $x \notin [Q_z, z]$, then $|DK_0(x)| \leq |DH(x)|$;
4. if $x \in [Q_z, z]$, then $|DK_0(x)| \leq C|DH(x)|$, where $C = 3/\eta$;
5. $\text{supp}(k_0) \subset [Q_z, z] \cup (\cup_{j \geq 1}[z_{i_j}, \xi(z_{i_j})])$.

Properties 1, 2, 3 and 5 immediately follow from the construction. We show 4.

By convexity,

$$\frac{1}{d(z) - d(Q_z)} = \frac{H(z) - H(Q_z)}{d(z) - d(Q_z)} \geq \frac{H(z) - H(o)}{d(z) - d(o)} = \frac{1}{d(z) - 1},$$

hence,

$$H(z) - H(Q_z) \geq \frac{d(z) - d(Q_z)}{d(z) - 1} \geq \frac{\eta}{3}$$

by Lemma and by the definition of $Q_z$. Thus, if $x \in [Q_z, z]$, then $|DK_0(x)| = |DH(x)|/(H(z) - H(Q_z)) \leq C|DH(x)|$.

Induction. Pick now $z_{i_1}$. If $\xi(z_{i_1}) \in P(z), Q_z$, let $K_1 = K_0$. If $\xi(z_{i_1}) \in (z, Q_z)$, construct $K_1$ as we did above, changing first the values of $K_0$ on $[\xi(z_{i_1}), z_{i_1}]$ in such a way $K_1$ vanishes on $(Q_{z_{i_1}}, z_{i_1})$, then adjusting the values elsewhere to make $K_1$ admissible for $z$. Observe that $\{x : K_1 \neq K_0\} \cap [P(z), z] = \phi$. The function $K_1$ has ($\ell = 1$) properties 1, 2 (that is, $K_0 \leq K_{\ell-1}$), 3 (that is, if $x \notin [Q_{z_{i}}, z_{i}]$, then $|DK_1(x)| \leq |DK_{\ell-1}(x)|$), 4 (that is, if $x \in [Q_{z_{i}}, z_{i}]$, then $|DK_1(x)| \leq C|DK_{\ell-1}(x)| \leq C|DH(x)|$, and this time $C = 3$ suffices). Property 5 becomes

$$\text{supp}(K_1) \subset [Q_z, z] \cup (\cup_{1 \leq j \leq \ell}[Q_{z_{i_j}}, z_{i_j}]) \cup (\cup_{j \geq \ell+1}[z_{i_j}, \xi(z_{i_j})])$$
Moreover, we have that
\[ DK_\ell(x) = DK_{\ell-1}(x) \text{ on the set } B_{\ell-1} = \bigcup_{1 \leq j \leq \ell-1} [z_i, z_i]. \]

Passing to the limit, we find a function \( K_z \) with all the desired properties, since the estimates for \(|DK_\ell|\) add up nicely.

5.2.3. Stopping estimates. Let \( S \) be a stopping region in the tree \( T \). By this, we mean that there are no \( x, y \in S \) such that \( x > y \); equivalently \( S(x) \cap S(y) = \emptyset \) if \( x \neq y \).

**Proposition 5.** Let \( \hat{K}_z = I\hat{k}_z \) be the function in Proposition 4 and let \( S \) be a stopping region. Then,
\[ \sum_{x \in S} |\hat{k}_z(x)| \leq 2\gamma(z, Z). \]

**Proof:** Let \( k = \hat{k}_z \). We know that \( k \geq 0 \) on \([Q_z, z]\) and that \( k \leq 0 \) elsewhere. Let \( S \) be a stopping time in \( T \). Then \( S \cap [Q_z, z] \) consists of at most one element \( x \) and
\[ 0 \leq k(x) \leq k(z) = \gamma_P, \]
since \( k \) is convex on \([Q_z, z]\).

Let \( x_0 = Q_z, x_1, \ldots, x_N = z \) be an ordered enumeration of the points in \([Q_z, z^{-1}]\).

Let \( S_j = S \cap (S(x_j) - S(x_j+1)) \), \( j = 0, \ldots, N - 1 \). Without loss of generality, we can assume that \( x_{j+1} = x_j^+ \), so that \( S(x_j) - S(x_{j+1}) = S(x_j^-) \). By harmonicity of \( k \), (5.5) and easy induction, we have that
\[ \sum_{x \in S_j} |k(x)| = - \sum_{x \in S_j} k(x) = -k(x_j^-) = -[k(x_j) - k(x_{j+1})]. \]

Summing over \( j \), we have
\[ \sum_j \sum_{x \in S_j} |k(x)| = k(z) - k(Q_z) \leq k(z) = \gamma_P. \]

Let \( S_\pm = S \cap S(z_\pm) \). Induction and harmonicity show that
\[ \sum_j \sum_{x \in S_\pm} |k(x)| = |k(z_\pm)| = \gamma_\pm. \]

All points in the support of \( k \) fall in \( S(Q_z) \cup \cdots \cup S(z) \), hence
\[ \sum_{x \in S} |k(x)| \leq 2\gamma_P + \gamma_+ + \gamma_- \leq 2\gamma(z, Z). \]

6. More Relations Between Conditions

6.1. The Weak Simple Condition is not Necessary. Fix a point \( z_0 \in T \) with \( d(z_0) \) large. Let \( N \) and \( b \) be positive integers. Fix a geodesic segment \([z_0 = w_0, w_1, w_2, \ldots, w_N]\) in \( T \) and choose for each \( 1 \leq n \leq N \) a point \( z_n \) satisfying
\[ d(w_n, z_n) = b, \]
\[ w_{n+1} \notin [w_n, z_n], \quad 1 \leq n < N. \]
Thus \( z_n \) and \( w_{n+1} \) lie on different branches below \( w_n \) and \( z_n \wedge w_{n+1} = w_n \). Set \( Z_N = \{ z_n \}_{n=1}^{N} \) and \( \beta = \frac{1}{\beta} \), so that \( \text{Cap}(w_n, \{ z_n \}) = \beta \) for each \( n \). Then the formula in (5.6) above shows that

\[
\text{Cap}(z_0, Z_N) = \frac{1}{1 + \text{Cap}_{w_n}(\{ z_1 \}) + \ldots + \frac{1}{1 + \text{Cap}_{w_n}(\{ z_N \})}} = \frac{1}{1 + \frac{1}{\beta + \frac{1}{\beta + \ldots + \frac{1}{\beta}}} = \gamma_N.}
\]

The function \( \varphi_\beta(x) = \frac{1}{1 + \frac{1}{\beta x}} \) is strictly increasing so if we take \( \gamma_0 = 0 \) and note that \( \gamma_{N+1} = \varphi_\beta(\gamma_N) \) we see that \( \gamma_N < \gamma_{N+1} \) for all \( N \geq 0 \). If \( \gamma_\infty = \lim_{N \to \infty} \gamma_N \) denotes the corresponding infinite continued fraction,

\[
\gamma_\infty = \frac{1}{1 + \frac{1}{\beta + \frac{1}{\beta + \ldots}}}
\]

then \( \text{Cap}_{z_0}(Z_N) = \gamma_N < \gamma_\infty \). Since \( \gamma_\infty = \varphi_\beta(\gamma_\infty) = \frac{1}{1 + \frac{1}{\beta + \gamma_\infty}} \) we compute that

\[
\gamma_\infty = \sqrt{\beta^2 + 4\beta - \beta} < \frac{\beta}{2}
\]

since \( 0 < \beta < 1 \). Altogether then,

\[
\text{Cap}(z_0, Z_N) < \sqrt{\beta}, \quad N \geq 1.
\]

Now we fix \( N = b = d(z_0)^2 \) so that

\[
\text{Cap}(z_0, Z_N) < \sqrt{\frac{1}{d(z_0)^2}} = \frac{1}{d(z_0)}.
\]

From this we obtain that

\[
Z = \{ z_0 \} \cup Z_N = \{ z_n \}_{n=0}^{N}
\]

satisfies the tree capacity condition \( \text{TreeCap} \) with constant \( C \) (for \( n \geq 1 \) the capacity \( \text{Cap}(z_n, \{ z_m \}_{m \neq n}) \) is easily seen to be bounded by \( C d(z_0)^{-2} \) since the distance from \( z_n \) to \( w_n \) is \( d(z_0)^2 \), and the geodesics from \( z_n \) to another point of \( Z \) must pass through \( w_n \)). The separation condition \( \text{TreeCap} \) holds with constant close to \( \frac{1}{2} \) since \( d(z_i, z_j) \geq d(z_0)^2 \) and \( d(z_n) \leq d(z_0) + 2d(z_0)^2 \).

**Remark 4.** It is possible to choose the points \( z_j \) above so that they are separated with the same constant in the Bergman metric in the disk \( \mathbb{D} \).

On the other hand, the weak simple condition constant for \( Z \) is quite large since \( \{ z_n \}_{n=1}^{N} \) are the children of \( z_0 \) and

\[
\sum_{n=1}^{N} \mu(z_n) = \sum_{n=1}^{N} \frac{1}{d(z_n)} = \sum_{n=1}^{N} \frac{d(z_0)^2}{d(z_0) + n + d(z_0)^2} \\
\approx \log \frac{d(z_0) + 2d(z_0)^2}{d(z_0) + 1 + d(z_0)^2} \approx \log 2,
\]

where the final step uses the fact that \( d(z_0)^2 \) is much larger than \( d(z_0) \).
which is much larger than \(\mu(z_0) = \frac{1}{d(z_0)}\) if \(d(z_0)\) is large.

Finally, even the constant in the *enveloping weak simple condition* (i.e. there is \(Z' \supset Z\) such that \([\text{WeakSimp}]\) holds for \(Z'\)) must remain quite large if the separation condition constant is to remain under control, i.e. not go to zero. Indeed, suppose we can add points \(\{v_k\}_{k=1}^K\) to \(Z\) so that the weak simple condition for \(Z' = \{v_k\}_{k=1}^K \cup Z\) holds with constant \(C\). Without loss of generality we may assume that the points \(\{v_k\}_{k=1}^K\) lie along the geodesic segment \([w_1, w_2, ..., w_{d(z_0)^2}]\).

If we consider the weak simple condition at \(w_m\) where \(m = d(z_0)^2 - RC\) (and where \(R\) is a sufficiently large positive integer: \(R = 10\) surely works), then we must have a point \(v_K\) lying below \(w_m\) since otherwise

\[
\sum_{n=m+1}^{N} \mu(z_n) = \sum_{n=d(z_0)^2 - RC + 1}^{d(z_0)^2} \frac{1}{d(z_0)^2 + n + d(z_0)^2} \approx \log \frac{d(z_0)^2 + 2d(z_0)^2}{d(z_0)^2 + 2d(z_0)^2 - RC + 1} \approx \frac{RC}{d(z_0)^2},
\]

which is not bounded by

\[
C \mu(w_m) = \frac{C}{d(w_m)} = \frac{C}{d(z_0)^2 + d(z_0)^2 - RC} \approx \frac{C}{d(z_0)^2},
\]

provided \(RC\) is much smaller than \(d(z_0)^2\) and \(R\) is sufficiently large. The same argument shows that there must be a point \(v_{K-1}\) lying in the segment \([w_p, w_m]\) where \(p = m - RC = d(z_0)^2 - 2RC\). But then \(d(v_{K-1}, v_K) \leq RC\) while

\[
\min \{d(v_{K-1}), d(v_K)\} = d(z_0) + d(z_0)^2 - 2RC,
\]

which shows that the separation constant is at most

\[
\frac{d(v_{K-1}, v_K)}{\min \{d(v_{K-1}), d(v_K)\}} = \frac{RC}{d(z_0)^2 + d(z_0)^2 - 2RC} \approx \frac{RC}{d(z_0)^2},
\]

a spectacularly small number if \(d(z_0)\) is large.

### 6.2. Separation Plus Weak Simple Implies Tree Capacity.

**Proposition 6.** If \(Z\) satisfies both \([5,2]\) and \([\text{WeakSimp}]\), then \(Z\) satisfies the tree capacity condition \((\text{TreeCap})\).

**Proof:** Fix \(z_0 \in Z\) and let \(Z' = \{z_j\}_{j=1}^\infty\) be those points in \(Z\) whose geodesic to \(z_0\) contains no other points of \(Z\). Consider for the moment the case where \(Z' \subset S(z_0)\). Arrange the sequence \(Z'\) so that \(|z_{j+1}| \geq |z_j|\) for all \(j \geq 1\). Define \(\tau_k\) to be the smallest connected subset of \(T\) containing \(\{z_j\}_{j=0}^k\). We then define the landing point \(\xi_k\) of \(z_k\) on \(\tau_{k-1}\) as the maximal point on the geodesic segment \([0, z_k] \cap \tau_{k-1}\). We now claim that

\[
d(z_k, \xi_k) \geq \frac{C}{2} d(z_k), \quad k \geq 1,
\]

where \(C\) is a separation constant.
where \( C \) is the constant in (5.2). Indeed, there is \( z_j \in Z \cap \tau_{k-1} \) with \( j < k \) such that \( z_j \geq \xi_k \), and it follows that

\[
Cd(z_k) \leq d(z_j,\xi_k) \leq d(z_j,\xi_k) + d(\xi_k, z_k) = d(z_j) - d(\xi_k) + d(z_k) - d(\xi_k) \leq 2d(z_k) - 2d(\xi_k) = 2d(z_k,\xi_k).
\]

Now define \( h_k = \frac{1}{d(z_k,\xi_k)} \chi(\xi_k, z_k) \) for \( k \geq 1 \) and \( h_0 = \frac{1}{\lambda(z_0,\xi_0)} \). If we set \( h = h_0 - \sum_{k=1}^{\infty} h_k \) and \( f = Ih \), then we have \( f(o) = 0 \), \( f(\xi_0) = 1 \), and \( f(z_j) = 0 \) for \( j \geq 1 \). Since \( Z \subset S(z_0) \) by our momentary assumption, we actually have \( f(z) = 0 \) for all \( z \in Z \setminus \{z_0\} \). We also have the norm estimate

\[
\|f\|^2_{B_2(T)} = \sum_{k=0}^{\infty} \frac{1}{d(z_0)} + \sum_{k=1}^{\infty} \frac{1}{d(z_k,\xi_k)} \leq \frac{1}{d(z_0)} + 2 \sum_{k=1}^{\infty} \frac{1}{d(z_k)} \leq C_1 \frac{1}{d(z_0)}
\]

by (6.4) and the weak simple condition (WeakSimp). Thus we obtain \( \gamma(z_0, Z') \leq C_\mu(z_0) \), and so (TreeCap) holds for \( \alpha = z_0 \).

The general case, where not all points in \( Z' \) lie in \( S(z_0) \), is handled as follows.

Let \( w \) be the point on the geodesic \( (o,z_0) \) satisfying \( d(w,z_0) = \left[ Cd(z_0) \right] \). Note that by (5.2) there are no points of \( Z \) in the segment \( (w,z_0) \). Now redefine \( h_0 = \frac{1}{d(w,z_0)} \chi(\xi_0, z_0) \). For each point \( \xi \) in the segment \( (w,z_0) \), let \( \xi' \) be the child of \( \xi \) on the geodesic \( (w,z_0) \), and consider the subset

\[ Z'_\xi = Z' \cap (S(\xi) \setminus S(\xi')) \]

Just as we did for \( Z' \cap S(z_0) \) above, we construct a function \( h_\xi = \sum_{z_k \in Z'_\xi} h_k \) such that \( Ih_\xi(\xi) = 0 \) and \( Ih_\xi(z_k) = 1 \) for \( z_k \in Z'_\xi \). Now define

\[ h = h_0 - \sum_{z_k \in S(z_0)} h_k - \sum_{\xi \in (w,z_0)} \sum_{z_k \in Z'_\xi} h_0(\xi) h_k. \]

Then \( f = Ih \) satisfies \( f(o) = 0 \), \( f(z_0) = 1 \), and \( f(z) = 0 \) for all \( z \in Z \setminus \{z_0\} \). Moreover we have the norm estimate

\[
\|f\|^2_{B_2(T)} = \|h_0\|^2_{\ell^2} + \sum_{z_k \in S(z_0)} \|h_k\|^2_{\ell^2} + \sum_{\xi \in (w,z_0)} |h_0(\xi)|^2 \sum_{z_k \in Z'_\xi} \|h_k\|^2_{\ell^2} \leq \frac{1}{d(z_0)} + \frac{2}{C} \sum_{z_k \in S(z_0)} \frac{1}{d(z_k)} + \frac{2}{C} \sum_{\xi \in (w,z_0)} |h_0(\xi)|^2 \sum_{z_k \in Z'_\xi} \frac{1}{d(z_k)} \leq \frac{1}{d(z_0)} + \frac{2}{C} \sum_{z_k \in S(z_0)} \frac{1}{d(z_k)} + \frac{2}{C} \sum_{\xi \in (w,z_0)} \frac{1}{d(z_k)} \leq \frac{1}{d(z_0)} + \frac{1}{d(z_0)} + \frac{1}{d(w)} \leq C \frac{1}{d(z_0)}
\]
Thus again $\gamma(z_0, Z') \leq C(\mu (z_0))$ and (TreeCap) holds for $\alpha = z_0$.

6.3. Separation Plus Finite Measure Doesn’t Imply Interpolation. We use the sequence $Z_N$ in constructed in Subsection 6.1 to obtain a separated sequence $Z$ with $\|\mu_Z\| < \infty$ that fails the tree capacity condition (TreeCap), and hence by Theorem C and (Restriction) fails to be onto interpolating sequence for the tree or the disk.

Let $N = b$ in the construction example $Z_N$ above. Recall from that construction that

$$\text{Cap}(z_0, Z_N) = \gamma_N < \gamma_\infty < \sqrt{\beta} = \frac{1}{\sqrt{N}}$$

We claim that for $0 < \beta < \frac{1}{36}$, the affine function

$$\psi_\beta (x) = \frac{\beta}{2} + (1 - 3\sqrt{\beta}) x$$

satisfies

$$\psi_\beta (x) < \varphi_\beta (x) < \gamma_\infty, \quad 0 \leq x < \gamma_\infty.$$  

Indeed, $\varphi_\beta - \psi_\beta$ is positive at 0 and has positive derivative in the interval $(0, \sqrt{\beta})$ provided $0 < \beta < \frac{1}{36}$. If we now let $\delta_n = \psi_\beta (\delta_{n-1})$, $n \geq 1$, and $\delta_0 = 0$, then we have by induction

$$\delta_n < \gamma_n < \gamma_\infty, \quad n \geq 1.$$  

Indeed, $\delta_n = \psi_\beta (\delta_{n-1}) < \varphi_\beta (\delta_{n-1}) < \varphi_\beta (\gamma_{n-1}) = \gamma_n < \gamma_\infty$.

Now we compute

$$\delta_n - \delta_{n-1} = \psi_\beta (\delta_{n-1}) - \psi_\beta (\delta_{n-2}) = (1 - 3\sqrt{\beta}) (\delta_{n-1} - \delta_{n-2})$$

$$= (1 - 3\sqrt{\beta})^{n-1} (\delta_1 - \delta_0) = \frac{\beta}{2} (1 - 3\sqrt{\beta})^{n-1},$$

and so

$$\delta_N = \sum_{n=1}^{N} (\delta_n - \delta_{n-1}) = \frac{\beta}{2} \sum_{n=1}^{N} (1 - 3\sqrt{\beta})^{n-1}$$

$$= \frac{\beta}{2} \frac{1 - (1 - 3\sqrt{\beta})^N}{1 - (1 - 3\sqrt{\beta})} = \frac{\sqrt{\beta}}{6} \left\{ 1 - \left( 1 - 3\sqrt{\beta} \right)^N \right\}$$

$$= \frac{1}{6\sqrt{N}} \left\{ 1 - \left( 1 - \frac{3}{\sqrt{N}} \right)^N \right\} > \frac{1}{12\sqrt{N}},$$

for $N$ large since $\left( 1 - \frac{3}{\sqrt{N}} \right)^N \to 0$ as $N \to \infty$ by l’Hospital’s rule. Altogether we have

$$\text{Cap}_{z_0} (Z_N) = \gamma_N > \delta_N > \frac{1}{12\sqrt{N}}$$

for large $N$. If we now take $N = d (z_0)^\theta$, $d (z_0)$ large and $1 \leq \theta < 2$, we obtain that the separation constant $C$ of $Z_N$ in (6.2) is at least 1, and that

$$\|\mu_{Z_N}\| = \frac{1}{d(z_0)} + \sum_{n=1}^{N} \frac{1}{d(w_n)} = \frac{1}{N^\theta} + \sum_{n=1}^{N} \frac{1}{d(z_0) + n + N} \leq 2,$$
yet
\[ \text{Cap}_{ \mathcal{Z}_N } (Z_N) > \frac{1}{12} \left( \frac{1}{d(z_0)} \right)^{\frac{2}{3}} \gg \frac{1}{d(z_0)}. \]

6.4. The Simple Condition and Interpolation in the Böe Space. Suppose that \( Z \subset \mathbb{D} \) satisfies the separation condition \([\text{Sep}]\) and that the associated measure \( \mu \) is finite. Here we show that if the simple condition \([2.38]\) holds then \( R(B_{2,Z}) \subset \ell^2(\mu) \), and in the other direction, if \( R(B_{2,Z}) \subset \ell^2(\mu) \) then a weaker version \([6.4]\) of condition \([2.38]\) holds. To see this we fix \( f = \sum_{i=1}^{\infty} a_i \varphi_{z_i} \in B_{2,Z}, \ z_i \in Z \), and, as Subsection \( 2.0.1 \) we let \( \mathcal{Y} \) be the Böe tree containing \( j \) and
\[ \mathcal{G}_j = [j_0,j] = \{ j_0, j_1, \ldots, j_{m-1}, j_m = j \} \]
be the geodesic \( \mathcal{G}_j \) in \( \mathcal{Y} \) joining \( j_0 \) to \( j \). Then we have
\[ f(z_j) = \sum_{k=0}^{m} a_{jk} \varphi_{z_{jk}} (z_j) + \sum_{i \not\in \{j_0, j_1, \ldots, j_m\}} a_i \varphi_{z_i} (z_j). \]

From Hölder’s inequality, \([2.2]\) (which follows from Proposition \( 7 \)) and the third estimate in \([2.6]\) we obtain
\[
\left| \sum_{i \not\in \{j_0, j_1, \ldots, j_m\}} a_i \varphi_{z_i} (z_j) \right|^2 \leq C \left\{ \sum_i |a_i|^2 \mu (z_i) \right\} \left\{ \sum_{i \not\in \{j_0, j_1, \ldots, j_i\}} |\varphi_{z_i} (z_j)|^2 \mu (z_i)^{-1} \right\}
\leq C \| f \|_{B_{2,Z}}^2 \left\{ \sum_{i \not\in j_0} d(z_i)^{-1} \left( 1 - |z_i|^2 \right)^{\sigma} d(z_i) \right\}
\leq C \| f \|_{B_{2,Z}}^2.
\]

We also have
\[
\left| \sum_{k=0}^{m} a_{jk} \varphi_{z_{jk}} (z_j) \right| \leq C \sum_{k=0}^{m} |a_{jk}| = CI |a| (z_j)
\]
where \( I \) denotes the summation operator on the Böe tree \( \mathcal{Y} \). Thus we have
\[
\| R f \|_{\ell^2(\mu)} \leq C \| I |a| \|_{\ell^2(\mu)} + C \| f \|_{B_{2,Z}} \| \mu \|^\frac{1}{2}.
\]

By Theorem 3 in \([\text{ArRoSa}]\) \( I \) is bounded on \( \ell^2(\mu) \) if and only if
\[
\sum_{\beta \geq \alpha} \frac{I^* \mu (\beta)}{\mu (\beta)}^2 \leq C I^* \mu (\alpha), \quad \alpha \in \mathcal{Y}.
\]

Now if \( \mu \) satisfies the simple condition \([2.38]\) then \( I^* \mu (\beta) \leq C \mu (\beta) \) for \( \beta \in \mathcal{Y} \), and we see that \([6.3]\) holds. Thus \( \| I |a| \|_{\ell^2(\mu)} \leq C \| a \|_{\ell^2(\mu)} \approx \| f \|_{B_{2,Z}} \) and this combined with \([6.2]\) completes the proof that \( R \) maps \( B_{2,Z} \) boundedly into \( \ell^2(\mu) \).

Conversely, if \( R \) is bounded from \( B_{2,Z} \) to \( \ell^2(\mu) \), then we have
\[
\sum_{z_k \in V_{\mathcal{Z}_j}^z} \mu (z_k) \leq \| R \varphi_{z_j} \|_{\ell^2(\mu)}^2 \leq C \| \varphi_{z_j} \|_{B_{2,Z}}^2 = C \mu (z_j)
\]
for all \( z_j \in Z \), a weaker version of the simple condition \([2.38]\).
7. Converse Results for Böe Space Interpolating Sequences

7.0.1. Riesz bases of Böe functions. The proof that the weak simple condition \text{(WeakSimp)} is necessary for onto interpolation for the Böe space \(B^2\) requires additional tools, including the fact that the Böe functions \(\{\varphi_{z_j}\}_{j=1}^\infty\) corresponding to a separated sequence \(Z = \{z_j\}_{j=1}^\infty\) in the disk \(\mathbb{D}\) form a Riesz basis for the Böe space \(B^2\), at least in the presence of a mild summability condition on \(Z\). It is interesting to note that for a separated sequence \(Z\) in \(\mathbb{D}\), the set of Dirichlet reproducing kernels \(\{k_{z_j}\}_{j=1}^\infty\) form a Riesz basis if and only if \(\mu_Z\) is \(B_2\)-Carleson \((\text{Bö})\), a condition much stronger than the mild summability used for the Böe functions. This points to an essential advantage of the set of Böe functions \(\{\varphi_{z_j}\}_{j=1}^\infty\) over the set of corresponding normalized reproducing kernels \(\{k_{z_j}(z_j)^{-1}k_{z_j}\}_{j=1}^\infty\). The feature of Böe functions responsible for this advantage is the fact that the supports of the functions \(g_{z_j}\) are pairwise disjoint.

**Proposition 7.** Let \(Z = \{z_j\}_{j=1}^\infty\subset \mathbb{D}\) satisfy the separation condition \(\text{Sep}\) and the mild summability condition \(\sum_{j=1}^\infty (1-|z_j|^2)^\sigma < \infty\) for all \(\sigma > 0\). Then there is a finite subset \(S\) of \(Z\) such that \(\{\varphi_{z_j}\}_{j\in Z\setminus S}\) is a Riesz basis for the closed linear span \(B^2\) of \(\{\varphi_{z_j}\}_{j=1}^\infty\) in the Dirichlet space \(B^2\).

**Proof:** A sequence of Böe functions \(\{\varphi_{z_j}\}_{j=1}^\infty\) is a Riesz basis if

\[
C^{-1}\left\|(a_j)_{j=1}^\infty\right\|_{\ell^2(\mu)}^2 \leq \left\|\sum_{j=1}^\infty a_j\varphi_{z_j}\right\|_{B^2}^2 \leq C\left\|(a_j)_{j=1}^\infty\right\|_{\ell^2(\mu)}^2
\]

holds for all sequences \(\{a_j\}_{j=1}^\infty\) with a positive constant \(C\) independent of \(\{a_j\}_{j=1}^\infty\). Here \(\mu = \sum_{j=1}^\infty \|\varphi_{z_j}\|_{B^2}^2 \delta_{z_j}\) and \(\mu(z_j) = \|\varphi_{z_j}\|_{B^2}^{-2} \approx d(z_j)^{-1}\). The inequality on the right follows from \((7.1)\) and the disjoint supports of the \(g_{z_j}\) - see the argument use to prove \((2.2)\) above - so we concentrate on proving the leftmost inequality in \((7.1)\) for an appropriate set of Böe functions. We begin with

\[
\left\|\sum_{j=1}^\infty a_j\varphi_{z_j}\right\|_{B^2}^2 = \int_{\mathbb{D}} \left\|\sum_{j=1}^\infty a_j\varphi'_{z_j}(z)\right\|^2 dz = \sum_{j,k=1}^\infty a_j a_k \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz = \sum_{j=1}^\infty \left|a_j\right|^2 \mu(z_j) + \sum_{j\neq k} a_j \overline{a_k} \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz.
\]

We now claim that by discarding finitely many points of \(Z\), we have

\[
\left|\sum_{j\neq k} a_j \overline{a_k} \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz\right| < \frac{1}{2} \sum_{j=1}^\infty \left|a_j\right|^2 \mu(z_j).
\]

Indeed, we will estimate \((7.2)\) using the following derivative estimates for Böe functions in the unit disk.
Lemma 7. Let $\varphi_w(z)$ be as in Lemma 3. Then we have

$$
\begin{align*}
|\varphi'_w (z)| &\leq C \left( 1 - |w|^2 \right)^{-\alpha}, \\
|\varphi'_w (z)| &\leq C \left| z - w |w|^{-1} \right|^{-1} \leq \left( 1 - |w|^2 \right)^{-\alpha}, \\
|\varphi'_w (z)| &\leq C \left| z - w |w|^{-1} \right|^{1+s} \leq \left( 1 - |w|^2 \right)^{-\sigma},
\end{align*}
$$

where $V^\beta_w = \{ z \in \mathbb{D} : \gamma_w (z) \geq \beta \}$ and $\gamma_w (z)$ is given by

$$
\left| z - w |w|^{-1} \right| = \left( 1 - |w|^2 \right)^{\gamma_w(z)}.
$$

Proof. This follows readily from the formula

$$
\varphi_w(z) = \Gamma_s g_w (z) = \int_{\mathbb{D}} \frac{g_w (\zeta) \left( 1 - |\zeta|^2 \right)^s}{(1 - \overline{\zeta} z)^{1+s}} \, d\zeta,
$$

together with the estimate in [ArRoSa2, (5.45)],

$$
|g_w (\zeta)| \leq C \left( \log \frac{1}{1 - |w|^2} \right)^{-1} \left| \zeta - w |w|^{-1} \right|^{-1}, \quad \zeta \in \mathbb{D},
$$

and the fact that the support of $g_w$ lives in the annular sector $S$ centred at $w |w|^{-1}$ given as the intersection of the annulus

$$
\mathcal{A} = \mathcal{A}_w = \left\{ \zeta \in \mathbb{D} : \left( 1 - |w|^2 \right)^{\alpha} \leq \left| \zeta - w |w|^{-1} \right| \leq \left( 1 - |w|^2 \right)^{\gamma} \right\}
$$

and the 45° angle cone $C_w$ with vertex at $w |w|^{-1}$. Note that the cone $C_w$ corresponds to the geodesic in the Bergman tree $T$ joining the root to the “boundary point” $w |w|^{-1}$.

The estimate we will prove is, for $j \neq k$,

$$
|\langle \varphi_{z_j}, \varphi_{z_k} \rangle| = \left| \int_{\mathbb{D}} \overline{\varphi'_{z_j} (z)} \varphi'_{z_k} (z) \, dz \right| 
\leq C \left( 1 - |z_j|^2 \right)^{\sigma} \left( 1 - |z_k|^2 \right)^{\sigma} \mu (z_j) \mu (z_k),
$$

for some $\sigma > 0$. We may assume that $1 - |z_j|^2 \leq 1 - |z_k|^2$ and write

$$
\left| \int_{\mathbb{D}} \overline{\varphi'_{z_j} (z)} \varphi'_{z_k} (z) \, dz \right| \leq \int_{\mathbb{D}} \left| \varphi'_{z_j} (z) \right| \left| \varphi'_{z_k} (z) \right| \, dz
= \left\{ \int_{V_{z_j}} + \int_{\mathbb{D} \setminus V_{z_j}} \right\} \left| \varphi'_{z_j} (z) \right| \left| \varphi'_{z_k} (z) \right| \, dz
= I + II.
$$
To estimate $II$ we use Lemma $[7]$ to obtain

$$II \leq C \int_{D \setminus V_j} \frac{(1 - |z_j|^2)^{\rho(1 + s)}}{|1 - \overline{z_j} z_k|^{2 + s}} \left(1 - |z_k|^2\right)^{-\alpha} \, dz$$

$$= C \left(1 - |z_k|^2\right)^{-\alpha} \left(1 - |z_j|^2\right)^{\rho(1 + s)} \int_{D \setminus V_j} \frac{dz}{|1 - \overline{z_j} z_k|^{2 + s}} \leq C \left(1 - |z_k|^2\right)^{-\alpha} \left(1 - |z_j|^2\right)^{\rho(1 + s) - \beta s - \alpha}. $$

Using (2.1) we see that the exponent $\rho (1 + s) - \beta s - \alpha$ is positive, and using $1 - |z_j|^2 \leq 1 - |z_k|^2$ we easily obtain (7.3).

To estimate $I$ we consider two cases. In the case that $V_{z_j} \cap V_{z_k} \neq \phi$, we have from Lemma $[7]$ and the estimate $|V_{z_j}| \leq C \left(1 - |z_j|^2\right)^{2\beta}$ that

$$I \leq C \sup_D |\varphi'_{z_j}| \sup_{V_j} |\varphi'_{z_k}| |V_{z_j}|$$

$$\leq C \left(1 - |z_j|^2\right)^{-\alpha} \left(1 - |z_k|^2\right)^{-\alpha} \left(1 - |z_j|^2\right)^{2\beta} \leq C \left(1 - |z_j|^2\right)^{2\beta - \alpha} \left(1 - |z_k|^2\right)^{-\alpha},$$

and now Lemma $[1]$ yields

$$I \leq C \left(1 - |z_j|^2\right)^{2\beta - \alpha} \left(1 - |z_j|^2\right)^{-\frac{\alpha}{\beta}} = C \left(1 - |z_j|^2\right)^{2\beta - \alpha - \frac{\alpha}{\beta}}.$$

Now using (2.1) we see that the exponent $2\beta - \alpha - \frac{\alpha}{\beta}$ is positive and we again obtain (7.3). On the other hand, if $V_{z_j} \cap V_{z_k} = \phi$, then we use

$$I \leq C \sup_{V_j} |\varphi'_{z_j}| \sup_{V_j} |\varphi'_{z_k}| |V_{z_j}|$$

$$\leq C \left(1 - |z_j|^2\right)^{-\alpha} \left(1 - |z_k|^2\right)^{\rho(1 + s)} \left(1 - |z_j|^2\right)^{2\beta} \leq C \left(1 - |z_j|^2\right)^{2\beta - \alpha} \left(1 - |z_k|^2\right)^{\rho(1 + s) - \beta (2 + s)} \leq C \left(1 - |z_j|^2\right)^{2\beta - \alpha - \varepsilon} \left(1 - |z_k|^2\right)^{\rho(1 + s) - \beta (2 + s) + 2\beta - \alpha - \varepsilon},$$

upon using $\left(1 - |z_j|^2\right)^{2\beta - \alpha - \varepsilon} \leq \left(1 - |z_k|^2\right)^{2\beta - \alpha - \varepsilon}$ in the last line. Now choosing

$$s > \frac{\alpha - \rho + \varepsilon}{\rho - \beta},$$

the exponent $\rho (1 + s) - \beta (2 + s) + 2\beta - \alpha - \varepsilon$ is positive, and once more we obtain (7.3).
Now we can estimate the left side of (7.2) by (7.3) and (2.2) to obtain

\[
\left| \sum_{j \neq k} a_j \sum_{i \in F} \varphi_{z_j}(z) \varphi_{z_k}(z) dz \right| \leq C \sum_{j \neq k} |a_j a_k| \left( 1 - |z_j|^2 \right)^{\sigma} \left( 1 - |z_k|^2 \right)^{\sigma} \mu(z_j) \mu(z_k)
\]

\[
\leq C \left( \sum_{k} \left( 1 - |z_k|^2 \right)^{\sigma} \mu(z_k) \right) \sum_{j=1}^{\infty} |a_j|^2 \mu(z_j)
\]

\[
< \frac{1}{2} \sum_{j=1}^{\infty} |a_j|^2 \mu(z_j)
\]

if \( \sum_{k} \left( 1 - |z_k|^2 \right)^{\sigma} \mu(z_k) \) is sufficiently small, which can be achieved by discarding a sufficiently large finite subset \( F \) from \( Z \). This shows that \( \{ \varphi_{z_j} \}_{z_j \in Z \setminus F} \) is a Riesz basis. However, if \( w \in F \) is not in the closed linear span of the Riesz basis \( \{ \varphi_{z_j} \}_{z_j \in Z \setminus F} \cup \{ \varphi_w \} \), then it is immediate that \( \{ \varphi_{z_j} \}_{z_j \in Z \setminus F} \cup \{ \varphi_w \} \) is also a Riesz basis.

We can continue adding Boë functions \( \varphi_w \) with \( w \in G \subset F \) so that \( \{ \varphi_{z_j} \}_{z_j \in Z \setminus F} \cup \{ \varphi_w \}_{w \in G} \) is a Riesz basis, and such that all of the remaining Boë functions \( \varphi_w \) with \( w \in F \setminus G \) lie in the closed linear span of the Riesz basis \( \{ \varphi_{z_j} \}_{z_j \in Z \setminus F} \cup \{ \varphi_w \}_{w \in G} \).

This completes the proof of Proposition 7 with \( S = F \setminus G \).

7.0.2. Completion of the Proofs of Theorems 4 and 5. Now we consider the necessity of the two conditions (Sep), or equivalently (Sep'), and (WeakSimp) in Theorem 4. We noted when we introduced (Sep') that it is necessary. To see that (WeakSimp) is necessary, we note that by Proposition 7 above (the summability hypothesis there is a consequence of \( \| \mu \| < \infty \)), we can remove a finite subset \( S \) from \( Z \) so that \( B_{2, Z \setminus S} (\mathbb{D}) = B_{2, Z} \) and \( \{ \varphi_{z_j} \}_{z_j \in Z \setminus S} \) is a Riesz basis. We can obviously add finitely many points to a sequence satisfying the weak simple condition and obtain a new sequence satisfying the weak simple condition. Thus we may assume that (2.2) holds for \( Z \).

Now let \( e_j \) be the function on \( Z \) that is 1 at \( z_j \) and vanishes on the rest of \( Z \). Denote the collection of all children of \( z_j \) in the forest structure \( \mathcal{F} \) by \( \mathcal{C}(z_j) \), and let \( \mu = \mu_Z \). We now claim that for \( j \) sufficiently large,

\[
S e_j = \varphi_{z_j} - \sum_{z_i \in \mathcal{C}(z_j)} \varphi_{z_j}(z_i) \varphi_{z_i} + f_j,
\]

where \( f_j \in B_{2, Z} \) has the form

\[
f_j = \sum_{i=1}^{\infty} a_i \varphi_{z_i}
\]

with \( \{ a_i \}_{i=1}^{\infty} \in \ell^2(\mu) \) and

\[
|a_j| < \frac{1}{2} \quad |a_i| < \frac{1}{2} \quad \text{if } z_i \in \mathcal{C}(z_j).
\]

Indeed, by (2.2) we have

\[
S e_j = \sum_{i=1}^{\infty} b_i \varphi_{z_i},
\]

with \( \{ b_i \}_{i=1}^{\infty} \in \ell^2(\mu) \) and \( \| \{ b_i \}_{i=1}^{\infty} \|_{\ell^2(\mu)} \approx \mu(z_j) \).
Now let \( \mathcal{Y} \) be the Böe tree containing \( j \) and 
\[
\mathcal{G}_j = [j_0, j] = \{j_0, j_1, \ldots, j_m \mid j_m = j\}
\]
be the geodesic \( \mathcal{G}_j \) in \( \mathcal{Y} \) joining \( j_0 \) to \( j \). If we evaluate both sides of (7.6) at \( z_{j\ell} \) where \( 0 \leq \ell < m \), we have
\[
0 = S e_j (z_{j\ell}) = \sum_{k=0}^{\ell} b_{jk} \varphi_{z_{jk}} (z_{j\ell}) + \sum_{i \notin \{j_0, j_1, \ldots, j_{\ell}\}} b_i \varphi_{z_i} (z_{j\ell}) .
\]
Subtracting the cases \( \ell \) and \( \ell + 1 \) in (7.6) we obtain
\[
0 = S e_j (z_{j\ell+1}) - S e_j (z_{j\ell})
= \sum_{k=0}^{\ell-1} b_{jk} \left[ \varphi_{z_{jk}} (z_{j\ell+1}) - \varphi_{z_{jk}} (z_{j\ell}) \right] + b_{j\ell+1} \left( \varphi_{z_{j\ell}} (z_{j\ell+1}) - 1 \right)
+ b_{j\ell+1} + \sum_{i \notin \{j_0, j_1, \ldots, j_{\ell+1}\}} b_i \varphi_{z_i} (z_{j\ell+1}) - \sum_{i \notin \{j_0, j_1, \ldots, j_{\ell}\}} b_i \varphi_{z_i} (z_{j\ell}) .
\]
From Hölder’s inequality and the third estimate in (2.6) we obtain
\[
(7.7) \quad \left| \sum_{i \notin \{j_0, j_1, \ldots, j_{\ell}\}} b_i \varphi_{z_i} (z_{j\ell}) \right| \leq \left\{ \sum_i \left| b_i \right|^2 \mu (z_i) \right\}^{\frac{1}{2}} \left\{ \sum_i \left| \varphi_{z_i} (z_{j0}) \right|^2 \mu (z_i)^{-1} \right\}^{\frac{1}{2}}
\leq C \mu (z_{j})^{\frac{1}{2}} \left\{ \sum_{i \notin j_0} |d (z_i)|^{-1} \left( 1 + |z_i|^2 \right)^{\sigma} |d (z_i)| \right\}^{\frac{1}{2}}
\leq C \mu (z_{j})^{\frac{1}{2}},
\]
where the final term in braces is bounded by hypothesis. We also have from (2.6)
\[
\left| \varphi_{z_{j\ell}} (z_{j\ell+1}) - 1 \right| \leq (1 + C \mu (z_{j\ell}))
\]
and
\[
\left| \sum_{k=0}^{\ell-1} b_{jk} \left[ \varphi_{z_{jk}} (z_{j\ell+1}) - \varphi_{z_{jk}} (z_{j\ell}) \right] \right|
\leq C \sum_{k=0}^{\ell-1} |b_{jk}| \mu (z_{jk}) \leq C \| \mu \|^{\frac{1}{2}} \left\{ \sum_{k=0}^{\ell-1} |b_{jk}|^2 \mu (z_{jk}) \right\}^{\frac{1}{2}}.
\]
Altogether then we have
\[
|b_{j\ell+1}| \leq |b_{j\ell}| \left( 1 + C \mu (z_{j\ell}) \right) + C \| \mu \|^{\frac{1}{2}} \left\{ \sum_{k=0}^{\ell-1} |b_{jk}|^2 \mu (z_{jk}) \right\}^{\frac{1}{2}} + 2 C \mu (z_{j})^{\frac{1}{2}}
\leq |b_{j\ell}| \left( 1 + C \mu (z_{j\ell}) \right) + C_1 \mu (z_{j})^{\frac{1}{2}} .
\]
Now the case \( \ell = 0 \) of (7.6) together with (7.7) yields
\[
|b_{j0}| \leq \sum_{i \notin \{j_0, j_1, \ldots, j_{\ell}\}} b_i \varphi_{z_i} (z_{j0}) \leq C_0 \mu (z_{j})^{\frac{1}{2}},
\]
and now by induction on $\ell$ we obtain that for $0 \leq \ell \leq m - 1$, $|b_{j_\ell}|$ is dominated by
\[
C_0 \mu (z_{j_\ell}) \cdot \left\{ \prod_{k=0}^{\ell-1} \left( 1 + C d(z_{j_k})^{-1} \right) + \prod_{k=1}^{\ell-1} \left( 1 + C d(z_{j_k})^{-1} \right) + \cdots + \left( 1 + C d(z_{j_{\ell-1}})^{-1} \right) \right\}.
\]
In particular,
\[
|b_{j_\ell}| \leq C_0 \mu (z_{j_\ell}) \frac{\ell}{\ell} \exp \left( C \sum_{k=0}^{\ell-1} d(z_{j_k})^{-1} \right)
\]
for $0 \leq \ell \leq m - 1$.

Now evaluate both sides of (7.3) at $z_j = z_{j_m}$ to obtain
\[
1 = b_j + \sum_{k=0}^{m-1} b_{j_k} \varphi_{j_k}(z_j) + \sum_{i \not\in \{j_0,j_1,\ldots,j_m\}} b_i \varphi_{z_i}(z_j),
\]
which by the argument above yields
\[
|b_{j} - 1| \leq C_0 \mu (z_{j_\ell}) \frac{m}{m} \exp \left( C \sum_{k=0}^{m-1} d(z_{j_k})^{-1} \right).
\]
Similarly, for $z_i \in C (z_j)$ we obtain
\[
|b_i - b_{j} \varphi_{z_i}(z_j)| \leq C_0 \mu (z_{j_\ell}) \frac{m + 1}{m + 1} \exp \left( C d(z_i)^{-1} + C \sum_{k=0}^{m} d(z_{j_k})^{-1} \right).
\]
Now the separation condition $\text{Sep}'$ yields $d(z_{j_k}) \geq (1 + C) d(z_{j_{k-1}})$ for $1 \leq k \leq m$ and it follows that
\[
\sum_{k=0}^{m} d(z_{j_k})^{-1} \leq C
\]
independent of $j$. Thus we see that
\[
|b_j - b_{j} \varphi_{z_i}(z_j)| \leq C (m + 1) \mu (z_{j_\ell}) \frac{m}{m}, \quad z_i \in C (z_j),
\]
with a constant $C$ independent of $j$. If we take $j_0$ large enough, then since $d(z_j) = d(z_{j_m}) \geq (1 + C)^m d(z_{j_0})$, we have
\[
|b_{j} - 1| \leq C m \mu (z_{j_\ell}) \frac{m}{m + 1} \exp \left( C d(z_j)^{-1} + C \sum_{k=0}^{m} d(z_{j_k})^{-1} \right) \leq \frac{m}{(1 + C)^m} d(z_{j_0})^{-\frac{m}{2}} < \frac{1}{2}.
\]
It follows that
\[
|b_i - \varphi_{z_i}(z_j)| \leq |b_i - b_{j} \varphi_{z_i}(z_j)| + |b_{j} - 1| |\varphi_{z_i}(z_j)| < \frac{1}{2}, \quad z_i \in C (z_j),
\]
which proves (7.4).

By (7.2) we then have using (7.4) and the fact that $\varphi_{z_i}(z_j) = 1$ for $z_i \in C (z_j) \cap V'_{z_i}$,
\[
\|S e_j\|_{B_{2,2}} \approx \left\{ \sum_i |b_i|^2 \mu (z_i) \right\} \frac{1}{\ell} \geq \frac{1}{2} \left\{ \sum_{z_i \in C (z_j) \cap V'_{z_j}} \mu (z_i) \right\} \frac{1}{\ell}.\]

It follows that
\[
\mu (z_j) = \|e_j\|^2_{L^2(\mu)} \geq C^2 \|S e_j\|_{B_{2,2}}^2 \geq C' \sum_{z_i \in C (z_j) \cap V'_{z_j}} \mu (z_i),
\]
which yields (WeakSimp) for $\alpha = z_j \in Z$ with $j$ large, and hence for all $j$ with a worse constant.
Now we suppose that \( \alpha \in T \setminus Z \). We claim that with either \( z_0 = \alpha \) or \( z_0 = A^M \alpha \), where \( M = \left[ \frac{\mu}{\ell} d (\alpha) \right] \) and \( C \) is as in \([\text{Sep}]\), the set \( Z' = Z \cup \{ z_0 \} \) is separated with separation constant in \([\text{Sep}]\) at least \( C/100 \). Indeed, if \( Z \cup \{ \alpha \} \) fails to satisfy \([\text{Sep}]\) with separation constant \( C/100 \), then there is some \( w \) in \( Z \) such that

\[
\beta (\alpha, w) < \frac{C}{50} (1 + \beta (\alpha, w)).
\]

From this we obtain that

\[
\beta (A^M \alpha, w) \geq \beta (A^M \alpha, \alpha) - \beta (\alpha, w) > \frac{C}{10} (1 + \beta (\alpha, w)) - \frac{C}{50} (1 + \beta (\alpha, w)) > \frac{C}{20} (1 + \beta (\alpha, w)),
\]

and then for any \( z \in Z \setminus \{ w \} \),

\[
\beta (A^M \alpha, z) \geq \beta (w, z) - \beta (A^M \alpha, w) > \beta (w, z) - \left\{ \beta (A^M \alpha, \alpha) + \beta (\alpha, w) \right\} > C (1 + \beta (\alpha, w)) - \left\{ \frac{C}{10} (1 + \beta (\alpha, w)) + \frac{C}{50} (1 + \beta (\alpha, w)) \right\}
\]

\[
> \frac{C}{2} (1 + \beta (\alpha, w)),
\]

which shows that \( Z \cup \{ A^M \alpha \} \) satisfies \([\text{Sep}]\) with separation constant \( C/2 \). Now we associate a Böe function \( \varphi_{z_0} \) with \( z_0 \), but take the parameters \( \beta, \beta_1, \rho, \alpha \) so close to 1 for this additional function \( \varphi_{z_0} \) that the extended set of Böe functions \( \{ \varphi_z \}_{z \in Z'} = \{ \varphi_z \}_{z \in Z} \cup \{ \varphi_{z_0} \} \) satisfy the property that the supports of the associated functions \( g_z \) are pairwise disjoint for \( z \in Z' \).

Now we define a bounded linear operator \( S' \) from \( \ell^2 (\mu_{Z'}) \) into \( B_{2, Z'} (T) \) by

\[
S' [\xi' ] = S \xi + (\xi_0 - S \xi (z_0)) \{ \varphi_{z_0} - S [\varphi_{z_0} \mid z] \},
\]

where \( \xi' = (\xi_0, \xi) = (\xi_0, \xi, \xi, \ldots) \). For \( j \geq 1 \) we have

\[
S' [\xi' ] (z_j) = \xi_j + (\xi_0 - S \xi (z_0)) \{ \varphi_{z_0} (z_j) - S [\varphi_{z_0} \mid z] (z_j) \}
\]

\[
= \xi_j + (\xi_0 - S \xi (z_0)) (0) = \xi_j,
\]

and for \( j = 0 \),

\[
S' [\xi' ] (z_0) = S \xi (z_0) + (\xi_0 - S \xi (z_0)) \{ 1 - S [\varphi_{z_0} \mid z] (z_0) \}
\]

\[
= \xi_0 - S [\varphi_{z_0} \mid z] (z_0) (\xi_0 - S \xi (z_0)).
\]

Now \( S [\varphi_{z_0} \mid z] (z_0) \) is small by the argument used to prove \([7.4]\) above, and in fact \([7.8]\) and \([7.9]\) of that argument yield

\[
| S [\varphi_{z_0} \mid z] (z_0) | \leq C \mu (z_0)^{\frac{1}{2}}.
\]

At this point we may assume that \( C \mu (z_0)^{\frac{1}{2}} < \varepsilon \) since there are only finitely many (depending on \( \varepsilon > 0 \)) points \( \alpha \) in the tree \( T \) having such a point \( z_0 \) that fails this condition. Thus \( S' \) is an approximate bounded right inverse to the restriction map \( \mathcal{U} \), and in fact,

\[
US' \xi' - \xi' = S [\varphi_{z_0} \mid z] (z_0) (\xi_0 - S \xi (z_0)) e_{z_0},
\]
so that
\[ \|US' \xi' - \xi'\|_{\ell^2(\mu)} \leq \varepsilon C \|\xi'\|_{\ell^2(\mu)} < \frac{\varepsilon}{2} \|\xi'\|_{\ell^2(\mu)} \]
if \( \varepsilon > 0 \) is small enough. Then \( US' \) is invertible on \( \ell^2(\mu) \), and so the operator \( S'' = S'(US')^{-1} \) is an exact bounded right inverse to the restriction map \( U \) since \( US'' = US'(US')^{-1} = I_{\ell^2(\mu)} \). Then the result proved in the previous paragraph shows that the weak simple condition \(^{\text{(WeakSimp)}}\) holds at \( z_0 \) with a controlled constant, and thus also at \( \alpha \) with a controlled constant. This completes the proof of Theorem 4.

It remains to show the necessity of \(^{\text{(WeakSimp)}}\) in the context of Theorem 5. For that situation, when \( Z \) is onto interpolating for the Böe space \( B_{2,Z} \), we note that a subtree of a dyadic tree has branching number at most 2, and it follows easily from the separation condition that
\[ \sum_{j=1}^{\infty} \left( 1 - |z_j|^2 \right)^\sigma < \infty \]
for all \( \sigma > 0 \). Thus Proposition 7 can be applied together with the argument used above to prove necessity of \(^{\text{(WeakSimp)}}\) in the case \( \|\mu_Z\| < \infty \).

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