On some notions of rank for matrices over tracts

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Abstract. Given a tract $F$ in the sense of Baker and Bowler and a matrix $A$ with entries in $F$, we define several notions of rank for $A$. In this way, we are able to unify and find conceptually satisfying proofs for various results about ranks of matrices that one finds scattered throughout the literature.

1. Introduction

In [4], the first author and Nathan Bowler introduced a new class of algebraic objects called tracts which generalize not only fields but also partial fields and hyperfields. Given a tract $F$, Baker and Bowler also define a notion of $F$-matroid.\footnote{In fact, one finds two notions (weak and strong) of $F$-matroids in [4]; we will work exclusively in this paper with strong $F$-matroids. Over many tracts of interest, the two notions coincide.} If $F = K$ is a field, then a $K$-matroid of rank $r$ on the finite set $E = \{1, \ldots, n\}$ is just an $r$-dimensional subspace of $K^n$, and matroids over the Krasner hyperfield $\mathbb{K}$ are just matroids in the usual sense.

In this paper, we use the theory of $F$-matroids to define a new notion of rank for matrices with entries in a tract $F$. When $F = K$ is a field, this gives the usual notion of rank, and when $F = \mathbb{K}$ is the Krasner hyperfield we recover an intriguing notion of rank for matrices of zero-nonzero patterns recently introduced by Deaett [13]. We also introduce a relative notion of rank for matrices over $F$ which depends on the choice of a tract homomorphism $\varphi : F' \to F$; this is, arguably, the more important notion. We compare these new notions of rank to more familiar notions like row rank, column rank, and determinental rank (all of which have straightforward generalizations to matroids over tracts), providing a number of general inequalities as well as some inequalities which only hold under additional hypotheses.

Our motivation for studying these concepts comes from a desire to unify, and find conceptually satisfying proofs for, various results about ranks of matrices that one finds scattered throughout the literature. For example, we will give a unified proof and conceptual generalization of the following results:

**Theorem 1.1.** (1) (Berman et. al. [8, Proposition 2.5] Let $A = (a_{ij})$ be a zero-nonzero pattern, i.e., an $m \times n$ matrix whose entries are each 0 or *, and suppose that each row of $A$ has at least $k$ *s. Then for any infinite field $K$, there exists an
\(m \times n\) matrix \(A' = (a'_{ij})\) over \(K\) with zero-nonzero pattern \(A^2\) and having rank at most \(m - k + 1\).

(2) (Alon-Spencer)[1, Lemma 13.3.3] Let \(A = (a_{ij})\) be a full sign pattern, i.e., an \(m \times n\) matrix whose entries are each + or −, and suppose that there are at most \(k\) sign changes in each row of \(A\). Then there exists an \(m \times n\) matrix \(A' = (a'_{ij})\) over \(\mathbb{R}\) with sign pattern \(A^3\) and having rank at most \(k + 1\).

Our proof naturally yields an extension of (2) to the non-full case, see Theorem 4.22 below.

Our results also give a unified way of viewing (and proving) results like the following. For the statement of (2), we define a sequence \(z_1, \ldots, z_n\) of complex numbers to be \textbf{colopsided} if 0 is not in their convex hull (when viewed as elements of \(\mathbb{R}^2 \cong \mathbb{C}\)).

**Theorem 1.2.**

(1) (Camion-Hoffman) [12, Theorem 3], [16, Theorem 4.6] Let \(A = (a_{ij})\) be an \(n \times n\) matrix with non-negative real entries. Then every complex matrix \(A' = (a'_{ij})\) with \(|a'_{ij}| = a_{ij}\) for all \(i, j\) is non-singular iff there exists an \(n \times n\) permutation matrix \(P\) and an \(n \times n\) diagonal matrix \(D\) with non-negative real entries such that \(PAD\) is strictly diagonally dominant.

(2) (McDonald et. al.) [17, Lemma 3.2], [19, Lemma 3.2] Let \(A = (a_{ij})\) be an \(n \times n\) matrix with entries in \(S^1 \cup \{0\}\), where \(S^1\) is the complex unit circle. Then every complex matrix \(A' = (a'_{ij})\) with phase\((a'_{ij}) = a_{ij}\) for all \(i, j\) is non-singular iff there does not exist a scaling of its rows by elements of \(S^1 \cup \{0\}\), not all zero, such that no column is colopsided.

Theorem 1.1(1) (resp. (2)) is obtained by applying Theorem 4.9 below to the natural homomorphism \(K \to \mathbb{K}\) (resp. the natural homomorphism \(\text{sign} : \mathbb{R} \to \mathbb{S}\), where \(\mathbb{S}\) is the sign hyperfield).

Theorem 1.2(1) (resp. (2)) is obtained by applying Theorem 4.15 below to the natural homomorphism \(\mathbb{C} \to \mathbb{V}\), where \(\mathbb{V}\) is Viro’s triangle hyperfield (resp. the natural homomorphism \(\mathbb{C} \to \mathbb{P}\), where \(\mathbb{P}\) is the phase hyperfield).

We conclude the paper with some open questions for future study.

Philosophically, the approach in the present paper shares much in common with the paper [6] by the first author and Oliver Lorscheid. In that paper, Baker and Lorscheid define a notion of \textbf{multiplicity} for roots of a polynomial over a hyperfield \(F\), and given a homomorphism \(\varphi : F' \to F\) of hyperfields, they prove a general inequality relating the multiplicities of roots of a polynomial \(p \in F'[x]\) and its image \(\varphi(p)\) in \(F[x]\) (along with some sufficient conditions for equality to hold). They also show how both Descartes’ Rule of Signs and Newton’s Polygon Rule are special cases of this general inequality.

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2 This means that \(A'_{ij} = 0\) (resp. \(A'_{ij} \neq 0\)) iff \(A_{ij} = 0\) (resp. \(A_{ij} = \ast\)).

3 This means that \(A'\) has non-zero entries and \(A'_{ij} = +\) (resp. \(A'_{ij} = -\)) iff \(A_{ij} > 0\) (resp. \(A_{ij} < 0\)).
2. Review of tracts, hyperfields, and matroids over tracts

For more details concerning the definitions and concepts in this section, as well as numerous examples, see [4].

2.1. Tracts. Given an abelian group $G$, let $\mathbb{N}[G]$ denote the group semiring associated to $G$.

**Definition 2.1.** A tract is a multiplicatively written commutative monoid $F$ with an absorbing element 0 such that $F^\times := F \setminus \{0\}$ is a group, together with a subset $N_F$ of $\mathbb{N}[F^\times]$ satisfying:

(T1) The zero element of $\mathbb{N}[F^\times]$ belongs to $N_F$.
(T2) There is a unique element $\epsilon \neq 0$ of $F^\times$ with $1 + \epsilon \in N_F$.
(T3) $N_F$ is closed under the natural action of $F^\times$ on $\mathbb{N}[F^\times]$.

We call $N_F$ the null set of $F$, and write $-1$ instead of $\epsilon$.

**Definition 2.2.** A homomorphism of tracts is a map $\varphi : F' \to F$ such that $\varphi(0) = 0$, $\varphi$ induces a group homomorphism from $(F')^\times$ to $F^\times$, and $\varphi(N_{F'}) \subseteq N_F$.

**Definition 2.3.** Given a tract $F$, a natural number $n$, and vectors $X = (X_i), Y = (Y_i) \in F^n$, we say that $X$ is orthogonal to $Y$, denoted $X \perp Y$, if $\sum_i X_i Y_i \in N_F$.

**Definition 2.4.** Given a tract $F$, we say that vectors $X_1, \ldots, X_k \in F^n$ are linearly dependent over $F$ if there exist $c_1, \ldots, c_k \in F$, not all zero, such that $\sum c_i X_i \in (N_F)^n$, and linearly dependent otherwise.

2.2. Hyperfields and partial fields. Roughly speaking, a hyperfield is an algebraic structure which behaves like a field except that addition is allowed to be multivalued. We focus here on a particular class of hyperfields called quotient hyperfields (for the general definition, see [4]). Every hyperfield can be viewed in a natural way as a tract.

**Definition 2.5.** Let $K$ be a field and let $H \leq K^\times$ be a multiplicative subgroup. Then the quotient monoid $F = K/H = (K^\times/H) \cup \{0\}$ is naturally a tract: the null set $N_F$ consists of all expressions $\sum_{i=1}^k x_i$ such that there exist $c_i \in H$ with $\sum_{i=1}^k c_i x_i = 0$ in $K$. We call a tract of this form a quotient hyperfield. Note that the natural map $\varphi : K \to F$ is a homomorphism of tracts.

**Example 2.6.**

(1) The Krasner hyperfield $\mathbb{K}$ can be defined as the quotient $K/K^\times$ for any field $K$ with $|K| > 2$.
(2) The sign hyperfield $\mathbb{S}$ is equal to $\mathbb{R}/\mathbb{R}_{>0}$.
(3) The triangle hyperfield $\mathbb{V}$ is equal to $\mathbb{C}/\mathbb{S}^1$, where $\mathbb{S}^1$ is the complex unit circle.
(4) The phase hyperfield $\mathbb{P}$ is equal to $\mathbb{C}/\mathbb{R}_{>0}$.
(5) The tropical hyperfield $\mathbb{T}$ is equal to $K/\ker(v)$, where $K$ is any valued field with value group $\mathbb{R}$ and $v : K^\times \to \mathbb{R}$ is the valuation map.

Partial fields are another class of algebraic objects which can naturally be viewed as tracts.
Definition 2.7. Let $R$ be a commutative ring with 1 and let $H \trianglelefteq R^\times$ be a multiplicative subgroup containing $-1$. Then the multiplicative monoid $P = H \cup \{0\}$ is naturally a tract: the null set $N_P$ consists of all expressions $\sum_{i=1}^k x_i$ such that $\sum_{i=1}^k x_i = 0$ in $R$. We call a tract of this form a partial field.

Example 2.8. If we take $R = \mathbb{Z}$ and $H = \{\pm 1\}$ in Definition 2.7, we obtain a tract called the regular partial field.

2.3. Matroids over tracts. Let $F$ be a tract and let $E$ be a finite set. For simplicity of notation we identify $E$ with the set $[n] := \{1, \ldots, n\}$. For $V \in F^n$, the support of $V$ is defined to be the set of $i \in [n]$ such that $V_i \neq 0$.

For our purposes, it is most convenient to define (strong) $F$-matroids on $E$ as follows. (We assume the reader is familiar with the basic concepts of matroid theory.)

Definition 2.9. An $F$-matroid $M$ of rank $r$ on $E$ is a matroid $M$ of rank $r$ on $E$, together with subsets $\mathcal{C}(M) \subseteq F^n$ and $\mathcal{C}^*(M) \subseteq F^n$ (called the $F$-circuits and $F$-cocircuits of $M$, respectively), such that:

1. $\mathcal{C}(M)$ and $\mathcal{C}^*(M)$ are both closed under multiplication by elements of $F^\times$.
2. For any $C \in \mathcal{C}(M)$, the support of $C$ is a circuit of $M$, and for any $C^* \in \mathcal{C}^*(M)$, the support of $C^*$ is a cocircuit of $M$.
3. For any circuit $C$ of $M$, there is a projectively unique (meaning unique up to multiplication by some element of $F^\times$) $F$-circuit $C'$ whose support is $C$, and for any cocircuit $C^*$ of $M$, there is a projectively unique $F$-cocircuit $C^*$ whose support is $C^*$.
4. For any $F$-circuit $C \in \mathcal{C}(M)$ and any $F$-cocircuit $C^* \in \mathcal{C}^*(M)$, we have $C \perp C^*$.

Definition 2.10. If $M$ is an $F$-matroid, the dual $F$-matroid $M^*$ is the $F$-matroid obtained by replacing $M$ with its dual matroid $M^*$ and interchanging $F$-circuits and $F$-cocircuits.

Definition 2.11. If $M$ is an $F$-matroid, the set $\text{Vec}(M)$ of $F$-vectors of $M$ is the set of all $X \in F^n$ such that $X \perp C^*$ for every $F$-cocircuit $C^*$ of $M$. Similarly, the set $\text{Cov}(M)$ of $F$-covectors of $M$ is the set of all $X \in F^n$ such that $X \perp C$ for every $F$-circuit $C$ of $M$.

Example 2.12. (1) [4, Example 3.30] If $F = \mathbb{K}$ is a field, a $\mathbb{K}$-matroid of rank $r$ on $[n]$ is the same thing as an $r$-dimensional $\mathbb{K}$-linear subspace of $\mathbb{K}^n$.

(2) [4, Example 3.31] If $F = \mathbb{K}$ is the Krasner hyperfield, a $\mathbb{K}$-matroid is the same thing as a matroid in the usual sense.

(3) [4, Example 3.32] If $F = \mathbb{S}$ is the sign hyperfield, an $\mathbb{S}$-matroid is the same thing as an oriented matroid.

(4) [4, Example 3.33] If $F = \mathbb{T}$ is the tropical hyperfield, a $\mathbb{T}$-matroid is the same thing as a valuated matroid in the sense of Dress and Wenzel.

\footnote{Note that partial fields are defined differently in \cite{7}; there the null set is by definition generated by expressions of length at most 3. For our purposes it is simpler to use the present definition.}
Definition 2.13. If $\varphi : F' \to F$ is a homomorphism of tracts and $M'$ is an $F'$-matroid, the push-forward $\varphi_*(M')$ is the $F$-matroid whose set of $F$-circuits (resp. $F$-cocircuits) is given by all subsets of $F^n$ of the form $cf(X)$, where $X$ is an $F'$-circuit (resp. $F'$-cocircuit) of $M'$ and $c \in F^\times$.

Example 2.14. If $\varphi : K \to K$ is the canonical map from a field $K$ to the Krasner hyperfield and $W \subset K^n$ is a linear subspace, identified with the corresponding $K$-matroid, the push-forward $\varphi^*(W)$ coincides with the linear matroid associated to any matrix $A$ whose rows form a basis for $W$.

3. Several notions of rank for matrices over tracts

Let $F$ be a tract. In this section we define several different notions of rank for an $m \times n$ matrix $A$ with entries in $F$, and establish some inequalities between them.

Definition 3.1. The column rank of $A$, denoted $r_{\text{col}}(A)$, is the maximum number of linearly independent columns of $A$.

Definition 3.2. The matroidal rank of $A$, denoted $r_{\text{mat}}(A)$, is the minimal rank of an $F$-matroid $M$ on $[n]$ such that every row of $A$ is a covector of $M$.

Note that $\text{Cov}(U_{n,n}) = F^n$, so $r_{\text{mat}}(A) \leq n$ and in particular the matroidal rank of $A$ is well-defined.

Remark 3.3. One can also define the row rank of $A$, denoted $r_{\text{row}}(A)$, as the column rank of $A^T$, i.e., $r_{\text{row}}(A) = r_{\text{col}}(A^T)$. Similarly, one can define the transpose matroidal rank $r_{\text{tmat}}(A) = r_{\text{mat}}(A^T)$. In general, the row and column ranks of a matrix are not equal, nor are the matroidal and transpose matroidal ranks, cf. Remark 3.7 and Remark 3.9.

When $F$ is a field, these various notions of rank all agree and coincide with the “usual” notion of rank:

Proposition 3.4. If $F = K$ is a field and $A$ is a matrix with entries in $K$, then

\[ r_{\text{row}}(A) = r_{\text{col}}(A) = r_{\text{mat}}(A) = r_{\text{tmat}}(A). \]

Before giving the proof, we need the following result.

Proposition 3.5. $r_{\text{col}}(A) \leq r_{\text{mat}}(A)$, i.e., the column rank of $A$ is at most the matroidal rank of $A$.

Proof. It suffices to show that any $r + 1$ columns of $A$ are linearly dependent. Let $I$ be any subset of the set $E = [n]$ of columns of $A$ of size $r + 1$, and consider the matroid $M|I$. We have $r(M|I) \leq r < r + 1 = |I|$, so there must be at least one nonzero $F$-circuit $C \in C(M|I)$. If $A_i$ denotes the column of $A$ corresponding to $i \in I$, one verifies easily that $\sum_{i \in [n]} C_i A_i \in N_F$. □

It is possible to have strict inequality in Proposition 3.5:
Example 3.6. Let $A$ be the following $3 \times 4$ matrix with coefficients in the sign hyperfield $\mathbb{S}$, which has $r_{\text{col}}(A) = 2$:

\[
\begin{pmatrix}
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{pmatrix}
\]

Suppose there exists a rank 2 oriented matroid $M$ such that the rows of $A$ are covectors of $M$. Then the underlying matroid $M$ of $M$ must be $U_{2,4}$, since otherwise there would be a circuit of size at most 2, but any 2 columns of $A$ are linearly independent over $\mathbb{S}$.

Since there is a unique $\mathbb{S}$-linear combination of any three columns giving a linear dependence, we compute that $\mathcal{C}(M) = \{(-1, 1, 1, 0), (0, 1, 1, 1), (-1, 0, 1, 1), (-1, 1, 0, 1)\}$. By [5, Theorem 2.16], any 3 oriented circuits in $\mathcal{C}(M)$ must be $\mathbb{S}$-linearly dependent. However, it is easy to check that $\{0, 1, 1, 1\}, (-1, 0, 1, 1), (-1, 1, 0, 1)\}$ are not linearly dependent. Hence, $r_{\text{mat}}(A) \geq 3$. (In fact, one checks easily that $r_{\text{mat}}(A) = 3$.)

Remark 3.7. Example 3.6 also shows that in general $r_{\text{col}}(A) \neq r_{\text{row}}(A)$. Indeed, the rows of $A$ are $\mathbb{S}$-linearly independent, so $r_{\text{col}}(A^T) = r_{\text{mat}}(A^T) = 3$.

For another example, let $P$ be the regular partial field (cf. Example 2.8) and let $A$ be the following matrix with entries in $P$:

\[
\begin{pmatrix}
1 & -1 & -1 & -1 \\
1 & 0 & 1 & -1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

Then one checks easily that $r_{\text{col}}(A) = r_{\text{mat}}(A) = 4$ and $r_{\text{row}}(A) = r_{\text{mat}}(A^T) = 3$.

The following example shows that in general we do not have $r_{\text{row}}(A) \leq r_{\text{mat}}(A)$.

Example 3.8. Let $A$ be the following $3 \times 4$ matrix with entries in the phase hyperfield $\mathbb{P}$:

\[
\begin{pmatrix}
1 & 1+i & 1 & 0 \\
2+i & 1+4i & 1 & 1 \\
2+i & 1+5i & 1 & 1
\end{pmatrix}
\]

Laura Anderson shows in [2, Example 4.7] that $r_{\text{mat}}(A) = 2$. Conversely, since the rows of $A$ are linearly independent, $r_{\text{mat}}(A^T) = r_{\text{row}}(A) = 3$.

Remark 3.9. Examples 3.7 and 3.8 also show that in general $r_{\text{mat}}(A) \neq r_{\text{tmat}}(A)$.

We now give the promised proof of Proposition 3.4.

Proof of Proposition 3.4. It is well-known that $r_{\text{row}}(A) = r_{\text{col}}(A)$, and $r_{\text{col}}(A) \leq r_{\text{mat}}(A)$ due to Proposition 3.5. Hence, it is sufficient to show $r_{\text{mat}}(A) \leq r_{\text{row}}(A)$. For this, let $L$ be the row space of $A$. We obtain the cocircuits of a rank $r_{\text{row}}(A)$ matroid $M$ by taking the vectors with minimal nonzero support in $L$. The set of covectors of $M$ is precisely $L$, and hence $r_{\text{mat}}(A) \leq r_{\text{row}}(A)$. \qed
Remark 3.10. When $F = \mathbb{K}$ is the Krasner hyperfield, the matroidal rank of a matrix $A$ over $\mathbb{K}$ coincides with a notion of rank introduced by Deaett [13]. Indeed, given an $m \times n$ zero-nonzero matrix pattern $A$, Deaett defines $R(A)$ to be the collection of all matroids $M$ on ground set $\{1, \ldots, n\}$ such that for each row $R$ of $A$, the set of zero positions of $R$ is a flat of $M$. Deaett then defines $\text{mrR}(A)$ to be the minimum rank of a matroid in $R(A)$. A flat is an intersection of hyperplanes, a hyperplane is the complement of a cocircuit, and a covector is a union of cocircuits, so the complement of a flat of $M$ is the same thing as a covector of $M$. From this, it follows easily that $r_{\text{mat}}(A) = \text{mrR}(A)$.

Remark 3.11. We have already seen in Remark 3.9 that in general one does not have $r_{\text{mat}}(A) = r_{\text{mat}}(A^T)$. Example 25 from [13], when combined with Remark 3.10, provides yet another example. Indeed, according to [13, Example 25], the $8 \times 7$ matrix

$$X = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}$$

over $\mathbb{K}$ has $4 = r_{\text{mat}}(X^T) < r_{\text{mat}}(X)$.

Given a zero-nonzero pattern $A$, Deaett defines $r_{\text{tri}}(A)$ to be the maximum $r$ such that we can permute the rows and columns of $A$ in order to obtain a matrix with an $r \times r$ upper-triangular submatrix. In particular, $r_{\text{tri}}(A) = r_{\text{tri}}(A^T)$ and $r_{\text{tri}}(A) \leq \min\{r_{\text{col}}(A), r_{\text{row}}(A)\}$. For the matrix above, Deaett shows that $r_{\text{tri}}(X) = 4$, so both $r_{\text{col}}(X)$ and $r_{\text{row}}(X)$ are at least 4. Moreover, $r_{\text{mat}}(X^T) = 4$, so $r_{\text{row}}(X) = 4$ and hence $r_{\text{col}}(X) = 4$ as well. This provides another example where we have strict inequality in Proposition 3.5.

4. Relative notions of rank

Suppose $\varphi : F' \to F$ is a homomorphism of tracts. Given a matrix $A$ over $F$, we will define some additional notions of rank which depend on the map $\varphi$ and not just on $F$.

Definition 4.1. The $\varphi$-matroidal rank of $A$, denoted $r_{\varphi-\text{mat}}(A)$, is the minimal rank of an $F'$-matroid $M'$ such that every row of $A$ is a covector of $\varphi_*(M')$.

We say that a matrix $A'$ over $F'$ is a lift of $A$ relative to $\varphi$ if $\varphi(A') = A$. If $X$ is a set and $f : X \to \mathbb{N}$ is a function, we extend $f$ to a function on subsets $A$ of $X$ by setting $f(A) = \min_{a \in A} f(a)$.

Definition 4.2. We define $r_{\text{mat}}(\varphi^{-1}(A))$ to be the minimum matroidal rank of a lift of $A$ (or $+\infty$ if $A$ does not lift).
Remark 4.3. By replacing \( r_{\text{mat}} \) with a different notion of rank for matrices over \( F' \) (e.g. \( r_{\text{col}} \)), we get other relative rank functions. If \( F' \) is a field, we sometimes write \( r(\varphi^{-1}(A)) \) instead of \( r_{\text{mat}}(\varphi^{-1}(A)) \) since all of the basic rank functions agree.

The basic inequality which makes these notions of interest is:

**Proposition 4.4.** \( r_{\text{mat}}(\varphi^{-1}(A)) \geq r_{\varphi-\text{mat}}(A) \geq r_{\text{mat}}(A) \).

**Proof.** For the first inequality, let \( A' \) be any lift of \( A \) achieving the minimum in Definition 4.2 (if no lift exists, there is nothing to prove). By definition, there exists an \( F'-\text{matroid} \ M' \) of rank \( r_{\text{mat}}(\varphi^{-1}(A)) \) such that every row of \( A' \) is a covector of \( M' \). By definition 2.13, every row of \( A \) is a covector of \( \varphi_*(M') \). We therefore have \( r_{\text{mat}}(\varphi^{-1}(A)) = \text{rank}(M') \geq r_{\varphi-\text{mat}}(A) \).

For the second inequality, by definition there exists an \( F'-\text{matroid} \ M' \) such that \( \text{rank}(M') = r_{\varphi-\text{mat}}(A) \) and every row of \( A \) is a covector of \( \varphi_*(M') \). The push-forward \( \varphi_*(M') \) is an \( F-\text{matroid} \) of rank \( r_{\varphi-\text{mat}}(A) \) such that every row of \( A \) is a covector of \( \varphi_*(M') \). Hence, \( r_{\text{mat}}(A) \leq \text{rank}(\varphi_*(M')) = r_{\varphi-\text{mat}}(A) \). \( \square \)

In particular, \( r_{\varphi-\text{mat}}(A) \) is a lower bound for the rank of any lift \( A' \) of \( A \), and this bound is better than the one we would obtain by just using the absolute rank \( r_{\text{mat}}(A) \) (or, say, the column rank \( r_{\text{col}}(A) \) of \( A \), which would give an even worse bound).

In general, both inequalities in Proposition 4.4 can be strict, as the next two examples show.

**Example 4.5.** Let \( \varphi : \mathbb{C}\{\{T\}\} \to \mathbb{T} \) be the natural homomorphism. Let \( A \) be the following matrix:

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\]

Consider the Fano matroid, viewed in the tautological way as a valuated matroid \( M \). One can explicitly verify that every row of \( A \) is a covector of \( M \) and that the first 3 columns of \( A \) are linearly independent. Using Proposition 3.5, it follows that \( r_{\varphi-\text{mat}}(A) > r_{\text{mat}}(A) = 3 \).

**Remark 4.6.** For the natural homomorphism \( \varphi : \mathbb{C}\{\{T\}\} \to \mathbb{T} \), we always have the equality \( r(\varphi^{-1}(A)) = r_{\varphi-\text{mat}}(A) \), cf. Proposition 4.13 below. In the tropical algebra literature, this quantity is called the **Kapranov rank** of \( A \) relative to the ground field \( \mathbb{C} \). More precisely, the Kapranov rank of \( A \) is defined to be \( r_{\varphi-\text{mat}}(A^T) \) in [14, Definition 1.2], and [14, Theorem 3.3] yields \( r(\varphi^{-1}(A)) = r_{\varphi-\text{mat}}(A^T) = r_{\varphi-\text{mat}}(A) \).

As in [14, Definition 3.9], one can change the ground field to obtain a different notion of Kapranov rank. In [20, Definition 5.3.2], one finds a notion of Kapranov rank that...
does not depend on the choice of a valued field $K$; it is defined as the minimal Kapranov rank over all such $K$. By [20, Theorem 5.3.21], this Kapranov rank is not necessarily equal to $r_{\text{mat}}(A)$. For example, one can choose $A = \mathcal{C}(M)$, where $M$ is the non-Pappus matroid (which is not representable over any field).

For a matrix $A$ over $\mathbb{T}$, there are (at least) two other notions of rank in the literature, namely the tropical rank [20, Definition 5.3.1] and the Barvinok rank [20, Definition 5.3.3]. The tropical rank coincides with the determinantal rank defined in Definition 5.1 below. There is a well-known inequality [20, Theorem 5.3.4] which says that $\text{tropical rank}(A) \leq \text{Kapranov rank}(A) \leq \text{Barvinok rank}(A)$.

**Example 4.7** (cf. [13, Example 30]). Consider the natural homomorphism $\varphi : \mathbb{F}_2 \to \mathbb{K}$ and let $A$ be the following matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
$$

Then $r_{\text{mat}}(\varphi^{-1}(A)) = 4$, because $\varphi^{-1}(A)$ is a singleton. However, $r_{\varphi,\text{-mat}}(A) = 3$, because one can take $M'$ to be the rank 3 matroid represented over $\mathbb{F}_2$ by the following matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}
$$

Note that $[0, 1, 1, 1]$ is not a covector of $M'$, but it is a covector of $\varphi_*(M')$.

What goes wrong in the previous example is that the natural map $\text{Cov}(M') \to \text{Cov}(\varphi_*(M'))$ is not surjective. We now show that when the map on covectors is surjective, the situation is nicer. For this, it is convenient to introduce the following definition:

**Definition 4.8.** A homomorphism $\varphi : F' \to F$ of tracts is epic if the natural map $\text{Vec}(M') \to \text{Vec}(\varphi_*(M'))$ is surjective for every $F'$-matroid $M'$. (By duality, this holds iff the natural map $\text{Cov}(M') \to \text{Cov}(\varphi_*(M'))$ is surjective for every $F'$-matroid $M'$.)

Our proof of the following result is inspired by [13, Theorem 28].

**Theorem 4.9.** Let $K$ be a field. If $\varphi : K \to F$ is epic then $r_{\text{mat}}(\varphi^{-1}(A)) = r_{\varphi,\text{-mat}}(A)$.

**Proof.** It suffices to show that $r_{\text{mat}}(\varphi^{-1}(A)) \leq r_{\varphi,\text{-mat}}(A)$. Suppose there exists a $K$-matroid $M$ of rank $r$ such that every row of $A$ is a covector of $\varphi_*(M)$. We want to prove that there exists a lift $A'$ of $A$ with $r(A') \leq r$.

We know that there exists a matrix $B$ with $n$ columns such that $B$ represents the matroid $M$ over $K$, but we do not necessarily have $\varphi(B) = A$. (In fact, $B$ might not even have the correct number of rows!) However, matroid duality now comes to the rescue.

---

5Not to be confused with the category-theoretic usage of the word “epic”, which simply means an epimorphism.
Since $M$ is representable over $K$, $M^*$ is also representable over $K$, so let $B'$ be a $K$-matrix such that $\varphi_*(M') = M^*$, where $M' = M[B']$. Let $v$ be a row of $A$, so that $v \in \text{Cov}(M) = \text{Vec}(M^*)$. Since $\varphi$ is epic, there exists a vector $v'$ of $M'$ such that $\varphi(v') = v$. Since $K$ is a field and $B'$ represents $M'$, we have $\text{Vec}(M') = \ker(B')$. Replacing each row of $A$ by a corresponding vector of $M'$ mapping to it under $\varphi$, we get a $K$-matrix $A'$ with $\varphi(A') = A$ and such that the row space of $A'$ is contained in the kernel of $B'$. Thus,

$$\text{rank}(A') \leq \ker(B') = n - \text{rank}(B') = n - \text{rank}(M^*) = \text{rank}(M) = r.$$ 

$\square$

The question of whether a given homomorphism $\varphi : K \to F$ is epic or not seems subtle. For example, the natural map $\varphi : \mathbb{C} \to \mathbb{P}$ is not epic, as the following example shows:

**Example 4.10** (cf. [2, Example 4.7]). Let $M$ be the matroid over $\mathbb{C}$ corresponding the the row space of

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 4i & 0 & 1 \end{pmatrix}$$

and let $\varphi : \mathbb{C} \to \mathbb{P}$ be the natural map. Then the induced map from $\text{Cov}(M)$ to $\text{Cov}(\varphi_*(M))$ is not surjective, because the image does not contain $\varphi([2+i, 1+4i, 1, 1])$.

On the positive side, the proof of [13, Theorem 28] immediately gives:

**Lemma 4.11.** If $K$ is an infinite field, the canonical map $\varphi : K \to \mathbb{K}$ is epic.

The main idea of the proof is that if $W$ is a $K$-matroid with underlying matroid $M$, the canonical map from $K$-circuits of $W$ to circuits of $M$ is surjective by definition (cf. Definition 2.13). Since a vector of a matroid is just a union of circuits and a vector of a $K$-matroid is just a $K$-linear combination of $K$-circuits, it suffices to take a suitably general linear combination (which exists since $K$ is infinite) of some set of circuits mapping surjectively onto $\mathcal{C}(M)$.

As another example of a positive result, we have:

**Proposition 4.12.** The natural map $\text{sign} : \mathbb{R} \to \mathbb{S}$ is epic.

**Proof.** Let $M$ be an $\mathbb{R}$-matroid on $[n] = \{1, \ldots, n\}$ and let $\text{sign}_*(M)$ be the associated oriented matroid. By definition, the circuits of $\text{sign}_*(M)$ are the push-forward of the circuits of $M$, i.e. $C' \in \mathcal{C}(\text{sign}_*(M))$ iff $C' = \text{sign}(C)$ for some $C \in \mathcal{C}(M)$.

On the other hand, recall that by the vector axioms for oriented matroids [9, Definition 3.7.1], the vectors of an oriented matroid are precisely the conformal compositions of circuits. Hence, it suffices to show that the image of the natural map $\text{Cov}(M) \to \text{Cov}(\text{sign}_*(M))$ is closed under conformal composition.
Suppose $X'$ and $Y'$ are two covectors in $\text{Cov}(\text{sign}_+(M))$ and that there exist $X$ and $Y$ in $\text{Cov}(M)$ satisfying $X'_i = \text{sign}(X_i)$ and $Y'_i = \text{sign}(Y_i)$ for $i \in [n]$. Let $\epsilon$ be a positive real number that is smaller than $\frac{X_i}{Y_i}$ for all $i \in [n]$. Then $X + \epsilon Y$ is a covector of $M$ and $\text{sign}(X + \epsilon Y) = X' \circ Y'$.

A similar argument can be used to show:

**Proposition 4.13.** Let $K$ be an infinite field. The natural valuation map $\nu : K\{\{T\}\} \to \mathbb{T}$ is epic.

**Proof.** As in the proof of Proposition 4.12, there is a binary composition operation for vectors of valued matroids, defined by $(X \circ Y)_i = \max\{X_i, Y_i\}$, such that vectors are precisely the compositions of circuits, cf. [10, Lemma 22]. It suffices to show that the image of the natural map $\text{Cov}(M) \to \text{Cov}(\nu_+(M))$ is closed under composition.

Suppose $X'$ and $Y'$ are two covectors in $\text{Cov}(\nu_+(M))$ and that there exist $X$ and $Y$ in $\text{Cov}(M)$ satisfying $X'_i = \nu(X_i)$ and $Y'_i = \nu(Y_i)$ for all $i \in [n]$. Let $c_i$ and $d_i$ be the coefficients of the initial terms of $X_i$ and $Y_i$. Since $K$ is an infinite field, there exists some $a \in K$ such that $a \cdot d_i \neq c_i$ for all $i$. Then, it is easy to verify that $\nu(X + a \cdot Y) = X \circ Y$. \hfill \Box

**Example 4.14.** The natural maps $\mathbb{S} \to \mathbb{K}$ and $\mathbb{T} \to \mathbb{K}$ are both epic. This follows from the fact that a vector of a matroid is the same thing as a union of circuits, together with the observation that the composition operations in Proposition 4.12 and Proposition 4.13 satisfy $\text{supp}(V_1 \circ V_2) = \text{supp}(V_1) \cup \text{supp}(V_2)$.

In general, it seems hard to say precisely when equality holds for the various inequalities we’ve touched upon so far in this paper. However, there is at least one case where things are relatively nice.

**Theorem 4.15.** Let $K$ be a field, let $H$ be a subgroup of $K^\times$, and let $\varphi : K \to F$ be the canonical quotient map to the hyperfield $F = K^\times/H$. Let $A$ be an $m \times n$ matrix over $F$ with $m \geq n$. Then the following are equivalent:

1. $r_{\text{mat}}(\varphi^{-1}(A)) = n$.
2. $r_{\varphi-\text{mat}}(A) = n$.
3. $r_{\text{col}}(A) = n$.
4. $r_{\text{mat}}(A) = n$.

**Proof.** In view of $r_{\text{mat}}(\varphi^{-1}(A)) \geq r_{\varphi-\text{mat}}(A) \geq r_{\text{mat}}(A) \geq r_{\text{col}}(A)$, it suffices to show $r_{\text{mat}}(\varphi^{-1}(A)) = n$ implies $r_{\text{col}}(A) = n$. We show the inequality that $r_{\text{col}}(A) < n$ implies $r_{\text{mat}}(\varphi^{-1}(A)) < n$.

Suppose $\sum a_{ij}x_j \in N_F$ for some $x_j \in F$ which are not all zero. If we let $\tilde{a}_{ij}$ be any lift of $a_{ij}$ and let $\tilde{x}_j$ be any lift of $x_j$, then by definition of $F$ there exist $c_{ij} \in H^\times$ s.t. $\sum_j c_{ij} \tilde{a}_{ij} \tilde{x}_j = 0$ for all $i$. If we take the lift $A'$ of $A$ to be given by $(c_{ij} \tilde{a}_{ij})$, then $\text{rank}(A') < n$. \hfill \Box

**Remark 4.16.** Theorem 4.15 does not extend to the case where there are more columns than rows. Indeed, we already saw in Example 3.6 that there exists a $3 \times 4$ matrix $A$ over $\mathbb{S}$ with $r_{\text{mat}}(A) = 3$ but $r_{\text{row}}(A) = 2$. 
Corollary 4.17. With notation as in Theorem 4.15, the following are equivalent for an 
\( n \times n \) matrix \( A \) over \( F \):

1. Every matrix \( A' \) over \( K \) with \( \varphi(A') = A \) is nonsingular.
2. The columns of \( A \) are \( F \)-linearly independent.
3. \( r_{\text{mat}}(A) = n \).
4. \( r_{\text{mat}}(A^T) = n \).

Applying Corollary 4.17 to various specific examples recovers several known results 
from the literature in a unified manner. For example, applying the corollary to the 
natural map \( C \rightarrow V \) gives (with a small amount of extra effort) the Camion-Hoffman 
theorem Theorem 1.2(1). And for the natural map \( C \rightarrow P \), we recover Theorem 1.2(2).

Applied to sign : \( R \rightarrow S \), Corollary 4.17 recovers the following result from [3]:

Corollary 4.18. [3, Corollary 3.4] Let \( A \) be an \( m \times n \) sign pattern and let \( \text{row}(A) \) denote the set of rows of \( A \). Then \( r(\text{sign}^{-1}(A)) = n \) if and only if for every nonzero sign vector \( x \in \{+,-,0\}^n \), \( \text{row}(A) \not\subseteq x^\perp \).

Here is an application of Theorem 4.9 inspired by the proof of [13, Theorem 30].

Theorem 4.19. Let \( \chi \) be an \( m \times n \) matrix over \( K \) i.e. a zero-nonzero pattern. Let \( K \) be an infinite field, and \( \varphi : K \rightarrow K \) be the natural map. If \( \chi \) has at least \( t \) nonzero entries in each row, then \( r(\varphi^{-1}(\chi)) \leq n - t + 1 \), i.e., there exists matrix \( A \) over \( K \) with zero-nonzero pattern \( \chi \) and \( \text{rank}(A) \leq n - t + 1 \).

Proof. Any \( v \in K^n \) with at least \( t \) nonzero elements is a covector of \( U_{n-t+1,n} \), since the circuits are all subsets of size \( n - t + 2 \). Furthermore, the matroid \( U_{n-t+1,n} \) is repre- sentable over any infinite field (for example, it can be represented by the \((n-t+1) \times n \) Vandermonde matrix \( V_{ij} = x_i^{j-1} \) for any distinct elements \( x_1, \ldots, x_{n-t+1} \)). Hence, \( r(\varphi^{-1}(\chi)) = r_{\varphi^{-1}}(A) \leq \text{rank}(U_{n-t+1,n}) = n - t + 1 \). 

Proposition 4.12, together with the idea behind the proof of Theorem 4.19, allows us 
to obtain a similar theorem involving sign patterns. Before stating the theorem, we will 
need the following definition.

Definition 4.20. Given \( V \in \{0,1,-1\}^n \), let \( V^+ = \{i \mid V_i = 1\} \) and \( V^- = \{i \mid V_i = -1\} \). 
Define the \textbf{generalized number of sign changes} \( \sigma(V) \) to be the maximal number of 
sign changes of a sign vector \( X \in \{-1,1\}^n \) such that \( V^+ \subseteq X^+ \) and \( V^- \subseteq X^- \). (In 
other words, we allow a zero entry of \( V \) to count as either 1 or \( -1 \) and then count the 
maximum possible number of sign changes in such a vector.)

Our proof of Theorem 4.22 will make use of the alternating oriented matroid \( C^{n,r} \) of 
rank \( r \) on \( [n] \), see [9, Section 9.4] for a definition.

Lemma 4.21. \( V \in \mathbb{S}^n \) is a covector of \( C^{n,r} \) if \( \sigma(V) < r \).
Proof. Suppose $V$ is not perpendicular to some circuit $C$. Without loss of generality, we may assume that $V_i \cdot C_i \in \{0, 1\}$ for all $i \in [n]$. Define

$$V'_i = \begin{cases} V_i, & \text{if } V_i \neq 0 \\ C_i, & \text{if } V_i = 0 \text{ and } C_i \neq 0 \\ 1, & \text{if } V_i = C_i = 0. \end{cases}$$

Then $V'$ is a vector in $\{X \in \{1, -1\}^n \mid V^+ \subseteq X^+, V^- \subseteq X^-\}$, so $\sigma(V') < r$. On the other hand, $V'_i \cdot C_i = 1$ for all $i \in \text{supp}(C)$. Hence, the restriction of $V'$ to the support of $C$ has $r$ sign changes, which yields $\sigma(V') \geq r$, a contradiction. □

**Theorem 4.22.** Let $\chi$ be an $m \times n$ matrix over $\mathbb{S}$, i.e., a sign pattern. Let $\text{sign} : \mathbb{R} \to \mathbb{S}$ be the natural map. If the generalized number of sign changes for each row $\chi$ is less than $k$, then $\text{rank}(\text{sign}^{-1}(\chi)) \leq k$, i.e., there exists matrix $A$ over $\mathbb{R}$ with sign pattern $\chi$ such that $\text{rank}(A) \leq k$.

**Proof.** By Lemma 4.21, every row of $A$ is a covector of $C^{n,k}$. Moreover, $C^{n,k}$ is realizable by [9, Proposition 9.4.1]. We have $r(\text{sign}^{-1}(\chi)) = r_{\text{sign-mat}}(A) \leq \text{rank}(C^{n,r}) = k$. □

## 5. Some open questions

Let $F$ be a tract and let $A$ be an $m \times n$ matrix with entries in $F$.

### 5.1. Determinantal rank.

**Definition 5.1.** The **determinant** of an $n \times n$ matrix $A = \{a_{i,j}\}$ over a tract $F$ is the following formal sum, thought of as an element of $\mathbb{N}[F^\times]$:

$$\sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

The **determinantal rank** of $A$, denoted $r_{\text{det}}(A)$, is the maximal $r$ such that $A$ has a $r \times r$ submatrix $A'$ with $\text{det}(A') \notin N_F$.

One has the following inequality, which follows from a non-trivial theorem of Dress and Wenzel.

**Theorem 5.2** ([15, Theorem 4.9]). *If the tract $F$ is perfect* $^6$, *then $r_{\text{mat}}(A)\geq r_{\text{det}}(A)$.*

**Remark 5.3.** The inequality $r_{\text{mat}}(A)\geq r_{\text{det}}(A)$ over a perfect tract $F$ can be strict. A counterexample is provided by the matrix $\mathcal{X}$ from Remark 3.11, which can also be viewed as a matrix over $\mathbb{T}$ and has $r_{\text{det}}(\mathcal{X}) = r_{\text{det}}(\mathcal{X}^T) \leq r_{\text{mat}}(\mathcal{X}^T) < r_{\text{mat}}(\mathcal{X})$.

**Question 5.4.** What can one say about the relationship between $r_{\text{det}}(A)$ and $r_{\text{col}}(A)$ over a perfect tract $F$?

$^6$This means that for every $F$-matroid $M$, every $F$-vector of $M$ is orthogonal to every $F$-covector of $M$. Examples of perfect tracts include fields and the hyperfields $\mathbb{K}, \mathbb{S}, \mathbb{T}$, see [4].
In this direction, here are a couple of known results for square matrices when $F = \mathbb{S}$ or $\mathbb{T}$:

**Theorem 5.5.**  
(1) [18, Theorem 3.6] Let $A$ be an $n \times n$ square matrix over $\mathbb{T}$. Then $r_{\det}(A) = n$ if and only if $r_{\col}(A) = n$.

(2) [11, Theorem 1.2.5] Let $A$ be a $n \times n$ square matrix over $\mathbb{S}$. Then, $r_{\det}(A) = n$ if and only if $r_{\col}(A) = n$.

Using Theorem 5.5 (1) as a building block, Izhakian and Rowen prove the following more general result:

**Theorem 5.6.** Let $A$ be an $m \times n$ rectangular matrix over $\mathbb{T}$. Then $r_{\det}(A) = r_{\col}(A) = r_{\row}(A)$.

**Remark 5.7.** It is **not** true that $r_{\mat}(A)$ coincides with these other three notions of rank for a rectangular matrix $A$ over $\mathbb{T}$. This follows immediately from Remark 5.3.

**Remark 5.8.** The analogue of Theorem 5.6 does **not** hold over $\mathbb{S}$, as one sees from Remark 3.7.

### 5.2. Rank of $A$ versus rank of $A^T$.

We have seen in Remark 3.7, in which $F = \mathbb{S}$, that the column rank of a matrix $A$ over a tract $F$ is not always equal to the row rank. Of course, the two ranks are equal when $F$ is a field.

**Question 5.9.** Can we characterize the tracts for which $r_{\col}(A) = r_{\row}(A)$ for all matrices $A$ over $F$, or at least give a nontrivial sufficient condition for this to hold?

Similarly, we have seen in Remark 3.9 that the matroidal ranks of $A$ and $A^T$ are not always equal. But we have not yet found either a proof or refutation of the following:

**Question 5.10.** If $A$ is a matrix over the sign hyperfield $\mathbb{S}$, is $r_{\mat}(A) = r_{\mat}(A^T)$?

More generally:

**Question 5.11.** Can we characterize the tracts $F$ for which $r_{\mat}(A) = r_{\mat}(A^T)$ for all matrices $A$ over $F$, or at least give a nontrivial sufficient condition for this to hold?

### 5.3. Matrices versus systems of linear equations.

In linear algebra over a field $K$, solving a system of homogeneous linear equations is equivalent to studying the null space of the matrix $A$ of coefficients, and the rank-nullity theorem applied to $A$ shows that if there are more unknowns than equations then there is a nonzero solution. For a system of homogeneous linear “equations” over a tract $F$, each of the form $\sum_j a_{ij}x_j \in N_F$, we can still view the solution set as the null space of a matrix, but the rank-nullity theorem no longer holds in general, and a nonzero solution does not always exist.

**Example 5.12.** Consider the following $3 \times 4$ matrix $A = (a_{ij})$ over the regular partial field $\mathbb{F}_1^\pm$:

\[
\begin{pmatrix}
1 & -1 & -1 & -1 \\
1 & 0 & 1 & -1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]
For the corresponding system of homogeneous linear “equations”, there are more unknowns than equations but \((0,0,0,0)\) is the only solution.

**Question 5.13.** If \(F\) is a hyperfield, does a system of homogeneous linear equations with more unknowns than equations always have a nonzero solution?

If \(F = K/H^K\) is a quotient hyperfield then the answer to Question 5.13 is yes, since we can deduce the result directly from the corresponding result over \(K\). However, we do not know what happens for non-quotient hyperfields.

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