1 Appendix A: Wavelet transform

Continuous wavelet transform (CWT) decomposes a time series in time-frequency domain by successively convolving the time series with a mother wavelet function $\psi_0$ which is stretched in time by varying its scale ($s$) and normalized to have unit energy (Torrence and Compo 1998). In this work we used the Morelet wavelet ($\omega_0 = 6$) defined as:

$$\psi_0(\eta) = \pi^{-\frac{1}{4}} e^{i\omega_0 \eta} e^{-\frac{\eta^2}{2}}$$

The continuous wavelet transform of a time series $X$ of length $N$ with values $x_n$ ($n = 1, ..., N$) sampled from a continuous signal at a time step of $\Delta t$ is defined as:

$$W_n^X(s) = \left(\frac{\Delta t}{s}\right)^\frac{1}{2} \sum_{n'=1}^{N} x_{n'}^* \psi_0^* \left( (n - n') \frac{\Delta t}{s} \right)$$

where $s$ is the stretch parameter used to change the scale, $n$ is the translation parameter used to slide the wavelet function in time, and $*$ indicates the complex conjugate (Grinsted et al. 2004). Large scales correlate with the low-frequency components of the signal, while small scales are associated with the high-frequency components. In analogy to Fourier analysis, a wavelet power spectrum (Tian et al. 2016) of a time series $X$ with values $x_n$ can be defined as follows:

$$W_n^{XX}(s) = W_n^X(s) W_n^{X*}(s) = |W_n^X(s)|^2$$

Given two time series $X$ and $Y$ with values $x_n$ and $y_n$ and wavelet transforms $W_n^X(s)$ and $W_n^Y(s)$, the cross wavelet transform (XWT) of $X$ and $Y$ is defined as:

$$W_n^{XY}(s) = W_n^X(s) W_n^{Y*}(s)$$

Where $*$ denotes the complex conjugate.

The cross-wavelet power between $X$ and $Y$ is defined as $|W_n^{XY}(s)|$ and reveals areas with high common power, while the complex argument of $W_n^{XY}(s)$ represents the relative phase between $X$ and $Y$ (Grinsted et al. 2004).

The gain between two time series $X$ and $Y$ can be expressed as follows:

$$G_n^{XY}(s) = \frac{|W_n^{XY}(s)|}{|W_n^{XX}(s)|}$$
The squared cross-wavelet coherence $R_n^2(s)$ measures the localized correlation coefficient between two time series $X$ and $Y$ in the time-frequency domain and ranges between 0 and 1. The squared cross-wavelet coherence wavelet coherence is defined as follow.

$$R_n^2(s) = \frac{|(s^{-1}W_{n}^{XY}(s))|^2}{(s^{-1}|W_{n}^{X}(s)|^2)(s^{-1}|W_{n}^{Y}(s)|^2)}$$

where $\langle . \rangle$ is a smoothing operator in both time and scale dimensions. Smoothing is required to remove the singularities in wavelet power spectra, and enhance regions of significant power, which can be accomplished using a weighted running average in both the time and scale directions, as described by (Torrence and Compo 1998).

The statistical significance threshold of $R_n^2(s)$ can be estimated using a Monte Carlo simulation with a large ensemble of surrogate data set pairs having the same coefficients as the real input data pair based on the first-order autoregressive (AR1) model (Grinsted et al. 2004).
2 Appendix B: Convergent cross mapping

Convergent cross mapping is a technique used to calculate the bidirectional causal relationship between two time series $X (x_t, t = 1, ..., L)$ and $Y (y_t, t = 1, ..., L)$ where $L$ is the length of the time series. CCM relies on state-space reconstruction to infer causality by measuring the extent to which historical values of $X$ can be used to accurately estimate the states of $Y$ (cross-mapping) (Sugihara and May 1990; Sugihara et al. 2012). To do so, the lagged coordinates of variables $X$ and $Y$, are used to construct the shadow manifold of $X(M_X)$ and $Y(M_Y)$ respectively. The lagged coordinates of $X$ ($\tilde{x}_t$) and $Y$ ($\tilde{y}_t$) are formed (Tsonis et al. 2018; Barraquand et al. 2021) as follows:

$$\tilde{x}_t = (x_{t-\tau}, x_{t-2\tau}, ..., x_{t-(E-1)\tau})$$  \hspace{1cm} (7)

$$\tilde{y}_t = (y_{t-\tau}, y_{t-2\tau}, ..., y_{t-(E-1)\tau})$$  \hspace{1cm} (8)

Where $t = 1 + (E - 1)\tau, ..., L$, $E$ is the embedding dimension, and $\tau$ is the time lag. Each of the vectors $\tilde{x}_t$ and $\tilde{y}_t$ represents a point in the $E - 1$ dimensional space. The set of vectors $\{\tilde{x}_t\}$ and $\{\tilde{y}_t\}$ constitute the reconstructed $M_X$ and $M_Y$ manifolds, respectively. The next step is to find the minimum $E + 1$ nearest neighbors of each $\tilde{x}_t$ in $M_X$. Let’s note the time indices (from closest to farthest) of the $E + 1$ nearest neighbors of $\tilde{x}_t$ by $t_1, t_2, ..., t_{E+1}$. The nearest neighbors of $\tilde{x}_t$ in $M_X$ are used to estimate $Y$ as follows:

$$\hat{Y}|M_X = \sum_{i=1}^{E+1} w_i y_{t_i}$$  \hspace{1cm} (9)

With $w_i = u_i / \sum_{j=1}^{E+1} u_j$, $u_j = \exp \left[-d(\tilde{x}_{t_i}, \tilde{x}_{t_j})/d(\tilde{x}_{t_i}, \tilde{x}_{t_E})\right]$, and $d(\tilde{x}_t, \tilde{x}_s)$ is the Euclidean distance between the two vectors $\tilde{x}_t$, and $\tilde{x}_s$. Predicting $Y$ by $M_X$ is equivalent to $Y$ causing $X$, and the strength of causality flowing from $Y$ to $X$ is quantified by calculating the Pearson correlation coefficient between the original time series $Y$ and the estimated $\hat{Y}|M_X$. Similarly, to know if $X$ is causing $Y$ (cross mapping of $X$ by using $M_Y$: $\hat{X}|M_Y$), we can calculate the Pearson correlation coefficient between $X$ and $\hat{X}|M_Y$. 
3 References

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