Stress tensor for massive fields on flat spaces of spatial topology $\mathbb{R}^2 \times S^1$

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Abstract
We calculate the expectation values of the energy–momentum tensor, $T_{\mu\nu}$, for massive scalar and spinor fields, in the Minkowski-like vacuum states on the two flat spaces which are quotients of Minkowski space under the discrete isometries $(t, x, y, z) \mapsto (t, x, y, z + 2a)$ and $(t, x, y, z) \mapsto (t, -x, -y, z + a)$.

The results on the first space confirm the literature. The results on the second space are new. We note some qualitative differences between the massless and massive fields in the limits of large $a$ and large $x^2 + y^2$.

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1. Introduction

We present the expectation values of the energy–momentum tensor, in the Minkowski-like vacuum, for massive scalar and spinor fields in the flat but topologically non-trivial spacetimes $M_0$ and $M_-$. These are quotients of Minkowski space under the groups generated by the discrete isometries $J_0 : (t, x, y, z) \mapsto (t, x, y, z + 2a)$ and $J_- : (t, x, y, z) \mapsto (t, -x, -y, z + a)$, respectively. For details of these spacetimes and for the calculations in the massless case, see [1, 2].

The expectation values for massive scalar and spinor fields on $M_0$ are in the literature [3, 4]. The results for the massive fields on $M_-$ are new. It is seen that in all cases the values decay exponentially in the limit of large mass, and the leading order corrections for small mass are $O(m^2)$. Further we note some qualitative differences between the behaviour of the massless and massive field values in the limits of large $a$ and large $x^2 + y^2$. Interest in this problem arises from the role of $M_0$ and $M_-$ in modelling, via accelerated observers on flat spacetimes, the Hawking(-Unruh) effect on respectively the Kruskal manifold and the $\mathbb{R}P^3$ geon [1]. As an aside we also present the expectation values

1 The expectation values for the massless scalar field on $M_0$ appear in a number of places in the literature [1, 11, 12]. Those for two-component spinors are also found in [11, 12].
of the stress tensor, in the Minkowski-like vacuum, for a massive scalar field on Minkowski space with an infinite straight plane boundary.

We work throughout in natural units $\hbar = c = 1$ and with metric signature $(+,−,−,−)$.

2. The massive scalar field

The case of massive scalar fields in multiply connected flat spacetimes was considered by Tanaka and Hiscock in [3]. In particular they consider $(0|T_{\mu\nu}|0)$ on flat spacetimes with topology $\mathbb{R}^3 \times S^1$ (which is denoted by $M_0$ here), $\mathbb{R}^2 \times T^2$ and $\mathbb{R}^1 \times T^3$. It is seen in [3] that the magnitude of the energy density decreases with an increasing field mass. Here we will reproduce the result on $M_0$ and present the expectation values on $M_\infty$.

The energy–momentum tensor for the massive scalar field in a general four-dimensional curved spacetime in our conventions is [5]

$$T_{\mu\nu} = (1 - 2\xi)g_{\mu\nu}\phi, + (2\xi - \frac{1}{2})g_{\mu\nu}g^{\rho\sigma}\phi,_{\rho}\phi,_{\sigma} - 2\xi\phi,_{\mu}\phi,_{\nu} + \frac{1}{2}\xi g_{\mu\nu}\phi,^2 - \xi$$

$$\times \left[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{2}\xi Rg_{\mu\nu} \right] \phi^2 + \frac{1}{3}(1 - 3\xi)m^2g_{\mu\nu}\phi^2,$$

(1)

which, with the help of the field equation $[\square + m^2 + \xi R]\phi = 0$, may be written on a flat spacetime as

$$T_{\mu\nu} = (1 - 2\xi)g_{\mu\nu}\phi, + (2\xi - \frac{1}{2})g_{\mu\nu}g^{\rho\sigma}\phi,_{\rho}\phi,_{\sigma} - 2\xi\phi,_{\mu}\phi,_{\nu} + 2\xi g_{\mu\nu}\phi,^2 + \frac{1}{2}m^2g_{\mu\nu}\phi^2,$$

(2)

where $\xi$ gives the coupling to the gravitational field ($\xi = 0$ for minimal coupling and $\xi = 1/6$ for conformal coupling). By the point splitting technique [6], where we split the points in the above quadratic expressions and take the coincidence limit at the end, this may be written as

$$T_{\mu\nu} = \frac{1}{2}\lim_{\delta \to 0} \left[ (1 - 2\xi)\nabla_\mu \nabla_\nu + (2\xi - \frac{1}{2})g_{\mu\nu}\nabla^a \nabla^a - 2\xi \nabla_\mu \nabla_\nu + \frac{1}{2}m^2g_{\mu\nu}\right] G(1)(x, x'),$$

(3)

where $[a, b]$ is an anticommutator.

The spacetimes we consider here are $M_0$ and $M_\infty$. These are built as quotients of Minkowski space under certain discrete isometry groups. Both spacetimes admit a global timelike Killing vector $\partial_t$ and a vacuum state built from it (which we denote by $|0\rangle$ in all cases, the particular vacuum being considered being that relevant for the spacetime under consideration). This vacuum state is that induced by the usual vacuum on Minkowski space.

The expectation value of $T_{\mu\nu}$ in this vacuum state is then

$$\langle 0|T_{\mu\nu}|0 \rangle = \frac{1}{2}\lim_{\delta \to 0} \left[ (1 - 2\xi)\nabla_\mu \nabla_\nu + (2\xi - \frac{1}{2})g_{\mu\nu}\nabla^a \nabla^a - 2\xi \nabla_\mu \nabla_\nu + \frac{1}{2}m^2g_{\mu\nu} \right] G(1)(x, x'),$$

(4)

where $G(1)(x, x') = \langle 0|\phi(x)\phi(x')|0 \rangle = G^s(x, x') + G^-(x, x')$ is the scalar Hadamard function which on Minkowski space is given by (see Appendix C of [7])

$$G(1)(x, x') = \frac{m}{4\pi (t-t')^2 - R^2} Y_1(m\sqrt{(t-t')^2 - R^2})\Theta((t-t')^2 - R^2)$$

$$+ \frac{m}{2\pi^2(R^2 - (t-t')^2)^{3/2}} K_1(m\sqrt{R^2 - (t-t')^2})\Theta(R^2 - (t-t')^2),$$

(5)

where $Y, K$ are Bessel functions and $R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$. The Hadamard function on the quotient spaces $M_0$ and $M_\infty$ may be found by the method of images [10, 11],

2 Note the sign difference in the second term from that in Fulling’s book [9] p 85 which is the reference used by Tanaka and Hiscock [3]. Further, note the typographical error in [3], where the $Y$ Bessel has been replaced by the $I$ Bessel function. This does not affect the results in [3].
as

\[ G_{M_0}^{(1)}(x, x') = \sum_{n \in \mathbb{Z}} \eta^n G_{M_0}^{(1)}(x, J^0_n x') \]  
\[ G_{M_1}^{(1)}(x, x') = \sum_{n \in \mathbb{Z}} \eta^n G_{M_1}^{(1)}(x, J^0_n x'), \]

where \( J_0 : (t, x, y, z) \mapsto (t, x, y, z + 2a) \), \( J_- : (t, x, y, z) \mapsto (t, -x, -y, z + a) \) and \( \eta = 1, (-1) \) labels standard (twisted) fields, respectively. The calculation of \( \langle 0 | T_{\mu\nu} | 0 \rangle \) on these spaces is now reduced to that of finding derivatives of these Hadamard functions and applying (4) to (6) and (7). Renormalization is achieved as usual on these flat spaces [5] by dropping the divergent Minkowski contribution coming from the \( n = 0 \) terms in the above sums. Next we present the results.

2.1. \( M_0 \)

For \( M_0 \), in the coincidence limit \( G_{M_0}^{(1)}(x, x') \) becomes a function only of \((2na)^2\) which is positive. This means that only the \( K \) Bessel term in (5) contributes. Further we note that \( \lim_{x \to y} \partial_{x^\mu} G_{M_0}^{(1)}(x, x') = -\lim_{x \to y} \partial_{x^\mu} G_{M_0}^{(1)}(x, x') \). These observations simplify the calculations somewhat. The result is

\[ \langle 0 | T_{\mu\nu} | 0 \rangle_{M_0} = -\langle 0 | T_{zx} | 0 \rangle_{M_0} = -\langle 0 | T_{yz} | 0 \rangle_{M_0} = -\sum_{n=1}^{\infty} \eta^n \frac{m^2}{2\pi^2(2na)^2} K_2(2mna) \]

\[ \langle 0 | T_{zz} | 0 \rangle_{M_0} = \sum_{n=1}^{\infty} \eta^n \left[ \frac{m^2}{2\pi^2(2na)^2} K_2(2mna) - \frac{m^3}{2\pi^2(2na)} K_3(2mna) \right] \]

with all other components vanishing. The results agree with [3]. The massless limit may be easily checked by noting that near \( z = 0 \)

\[ K_\nu(z) = \frac{\Gamma(\nu)}{2} \left( \frac{z}{2} \right)^{-\nu} \]

when the real part of \( \nu \) is positive [8]. These agree with [1, 12] and also with [11], the extra factor of 4 in [11] arising because the authors consider a multiplet of two complex massless scalar fields. The leading corrections for small mass are of the order \( O(m^2) \). \( \langle 0 | T_{\mu\nu} | 0 \rangle \to 0 \) exponentially in the large mass and large \( a \) limits, in contrast to the massless case where \( \langle 0 | T_{\mu\nu} | 0 \rangle \) vanishes as \( O(a^{-2}) \).

2.2. \( M_- \)

For \( M_- \) in the coincidence limit \( G_{M_-}^{(1)}(x, x') \) becomes a function only of \((x + (1)^n x)^2 + (y + (1)^n y)^2 + (na)^2 \) which is positive. Again only the \( K \) Bessel term in (5) contributes. Further we note the following:

\[ \lim_{x \to y} \partial_{x^\mu} G_{M_-}^{(1)}(x, x') = -\lim_{x \to y} \partial_{x^\mu} G_{M_-}^{(1)}(x, x') \quad \text{for} \quad \mu \in \{t, z\} \]

\[ \lim_{x' \to x} \partial_{x'^\mu} G_{M_-}^{(1)}(x, x') = -(+ \lim_{x' \to x} \partial_{x'^\mu} G_{M_-}^{(1)}(x, x') \quad \text{when} \quad n \text{ even (odd)} \quad \text{and} \quad \mu \in \{x, y\}. \]

As \( G_{M}^{(1)}(x, x') = \sum_{n \in \mathbb{Z}} \rho^n G^{(1)}(x, J^0_n x') \), where \( \rho = +1, (-1) \) labels untwisted (twisted) fields, the result here may be split into two parts where the part coming from the even terms in the above sum leads to the same expectation values as on \( M_0 \) for untwisted fields. Therefore
we write \( \langle 0|T_{\mu\nu}|0\rangle_{M_0} = \langle 0|T_{\mu\nu}|0\rangle_{M_0(n=1)} + \rho \langle 0|T_{\mu\nu}|0\rangle_{\text{odd}} \), where we find

\[
\langle 0|T_{\mu\nu}|0\rangle_{\text{odd}} = \sum_{n \in \mathbb{Z}} \left[ -(4\xi - 1) \frac{m^3}{4\pi^2\sigma_n} K_3(m\sigma_n) \left[ 1 - \frac{(2na + a)^2}{\sigma_n^2} \right] \right. \\
+ \left. \left( 2\xi - \frac{3}{4} \right) \frac{m^2}{\pi^2\sigma_n^3} K_2(m\sigma_n) \right]
\]

\[
\langle 0|T_{\rho\sigma}|0\rangle_{\text{odd}} = \sum_{n \in \mathbb{Z}} \left[ (4\xi - 1) \frac{m^3}{4\pi^2\sigma_n} K_3(m\sigma_n) - \left( 2\xi - \frac{1}{2} \right) \frac{m^2}{2\pi^2\sigma_n} K_2(m\sigma_n) \right]
\]

\[
\langle 0|T_{\rho\sigma}|0\rangle_{\text{odd}} = \sum_{n \in \mathbb{Z}} \left[ (4\xi - 1) \frac{m^3}{4\pi^2\sigma_n} K_3(m\sigma_n) - \left( 2\xi - \frac{1}{2} \right) \frac{m^2}{2\pi^2\sigma_n^3} K_2(m\sigma_n) \right]
\]

\[
\langle 0|T_{\rho\sigma}|0\rangle_{\text{odd}} = \sum_{n \in \mathbb{Z}} \left[ (4\xi - 1) \frac{m^3}{4\pi^2\sigma_n} K_3(m\sigma_n) - \left( 2\xi - \frac{3}{4} \right) \frac{m^2}{\pi^2\sigma_n^3} K_2(m\sigma_n) \right]
\]

where \( \sigma_n = ((2x)^2 + (2y)^2 + (2na + a)^2)^{1/2} \) and the sum is over all \( n \) including \( n = 0 \). Other components vanish. Again it is a simple matter to check the massless limit. The results agree with those of \([1, 11]\) in this limit. The leading correction for small mass is \( O(m^2) \), and \( \langle 0|T_{\mu\nu}|0\rangle \) vanishes exponentially in the large mass and, for non-zero mass, the large \( a \) limits. The difference between \( \langle 0|T_{\mu\nu}|0\rangle \) on \( M_0 \) and \( M_0 \) vanishes exponentially as \( r^2 := x^2 + y^2 \to \infty \). This behaviour is qualitatively different to the massless case where the difference vanishes as \( O(r^{-3}) \). The result in the massive case is, to our knowledge, new.

### 3. The massive Dirac field

In this section we repeat the above calculations for the massive Dirac field. The result on \( M_0 \) was given recently in \([4]\) where it is shown that, as for the scalar field, the magnitude of energy density decreases with increasing field mass. We shall present the results for \( M_0 \) and \( M_{-\infty} \) and comment on various limits.

The energy–momentum tensor for the massive Dirac field is \([5]\)

\[
T_{\mu\nu} = \frac{i}{4} \text{Tr}(\bar{\psi} \gamma_{[\mu} \nabla_{\nu]} \psi - \nabla_{[\mu} \bar{\psi} \gamma_{\nu]} \psi) \tag{12}
\]

where \( A_{(\mu} B_{\nu)} = 1/2(A_{\mu} B_{\nu} + A_{\nu} B_{\mu}) \). This may be written as

\[
T_{\mu\nu} = \frac{i}{4} \text{Tr}(\gamma_{[\mu} [\nabla_{\nu]} \bar{\psi}, \bar{\psi} - \gamma_{[\mu} [\bar{\psi}, \nabla_{\nu]} \bar{\psi}]). \tag{13}
\]

Further we use the point splitting technique \([6]\) to write this as

\[
T_{\mu\nu} = \frac{i}{8} \lim_{x' \to x} \text{Tr}(\gamma_{[\mu} [\nabla_{\nu]} \bar{\psi}(x'), \bar{\psi}(x)] + \gamma_{[\mu} [\nabla_{\nu]} \bar{\psi}(x'), \bar{\psi}(x')] \\\n- \gamma_{[\mu} [\bar{\psi}(x), \nabla_{\nu]} \bar{\psi}(x')] - \gamma_{[\mu} [\bar{\psi}(x'), \nabla_{\nu]} \bar{\psi}(x)) \tag{14}
\]
The expectation value of $T_{\mu\nu}$ in the vacuum state $|0\rangle$ may now be expressed in terms of the spinor Hadamard function $S_{\alpha\beta}^{\text{Minkowski}}(x, x') = \langle 0|\gamma_\mu(x)\gamma_\nu(x')|0\rangle$:

$$\langle 0|T_{\mu\nu}|0\rangle = \frac{1}{8}\lim_{x \to x'} \text{Tr}(\gamma_\mu \nabla_\nu S^{\text{Minkowski}}(x', x) + \gamma_\mu \nabla_\nu S^{\text{Minkowski}}(x, x') - \gamma_\mu \nabla_\nu S^{\text{Minkowski}}(x', x)).$$

(15)

Further the spinor Hadamard function may be expressed in terms of the scalar one $S^{\text{scalar}}(x, x') = -(i\gamma_\rho \nabla_\rho + m)G^{\text{scalar}}(x, x')$, where $G^{\text{scalar}}(x, x')$ is given by (5) in Minkowski space, and we note that

$$\lim_{x \to x'} \nabla_\mu S^{(1)}(x', x) = \lim_{x \to x'} \nabla_\mu S^{(1)}(x, x')$$

(16)

and so

$$\langle 0|T_{\mu\nu}|0\rangle = \frac{1}{4}\lim_{x \to x'} \text{Tr}[\gamma_\mu (\nabla_\nu S^{(1)}(x, x') - \nabla_\nu S^{(1)}(x, x'))]$$

$$= \frac{1}{8}\lim_{x \to x'} \text{Tr}[(\gamma_\mu \nabla_\nu + \gamma_\nu \nabla_\mu - (\gamma_\mu \nabla_\nu + \gamma_\nu \nabla_\mu))\gamma^\rho \nabla_\rho G^{\text{scalar}}(x, x').$$

(17)

Note that this expression is twice that of (10) of [4] where the Majorana spinors are considered.

We are concerned with these expectation values on quotient spaces of Minkowski space. As with the scalar field the Hadamard function may be found here by the method of images, but extra care must be taken with the local Lorentz frames with respect to which the spinors are expressed. We shall work throughout with a vierbein aligned along the usual Minkowski coordinate axes. This has the advantage of making covariant and partial derivatives coincide.

As this vierbein is invariant under $J_0$ the calculation on $M_0$ is then reduced to a straightforward calculation of these derivatives of the Hadamard function and applying equation (17). The Minkowski vierbein, however, is not invariant under $J_+$ and more care must be taken on $M_-$.

### 3.1. $M_0$

Recall that on $M_0$

$$G^{\text{scalar}}_{M_0}(x, x') = \sum_{n \in \mathbb{Z}} \eta^n G^{\text{scalar}}_n(x, J_0^n x'),$$

(18)

where renormalization is performed again by simply dropping the Minkowski $n = 0$ term in the sum. Here again we note that in the coincidence limit $G^{\text{scalar}}_{M_0}(x, x')$ becomes a function only of $(2na)^2$ which is positive and so only the $K$ Bessel term in (5) contributes. Also

$$\lim_{x \to x'} \partial_\nu G^{\text{scalar}}_{M_0}(x, x') = -\lim_{x \to x'} \partial_\mu G^{\text{scalar}}_{M_0}(x, x').$$

With these observations (17) reduces to

$$\langle 0|T_{\mu\nu}|0\rangle = \lim_{x \to x'} \nabla_\mu \nabla_\nu G^{\text{scalar}}_{M_0}(x, x'),$$

(19)

and we find for the non-zero expectation values

$$\langle 0|T_{\mu\nu}|0\rangle = \begin{cases} -\langle 0|T_{xx}|0\rangle & \mu = 0 \\ \langle 0|T_{yy}|0\rangle & \mu = \nu = 1 \\ \langle 0|T_{zz}|0\rangle & \mu = \nu = 2 \\ \langle 0|T_{xy}|0\rangle & \mu = \nu = 3 \\ \langle 0|T_{yz}|0\rangle & \mu = \nu = 1 \\ \langle 0|T_{xz}|0\rangle & \mu = \nu = 2 \\ \langle 0|T_{yz}|0\rangle & \mu = \nu = 3 \\ \frac{2m^2}{\pi(2na)^2} K_2(2mna) & \mu = \nu = 1 \\ \frac{2m^3}{\pi(2na)^3} K_3(2mna) & \mu = \nu = 2 \\ \frac{2m^2}{\pi(2na)^2} K_2(2mna) & \mu = \nu = 3 \\ \frac{2m^3}{\pi(2na)^3} K_3(2mna) & \mu = \nu = 4 \end{cases}$$

(20)

where $\eta = +1$, $(-1)$ labels periodic (twisted) spinors with respect to the standard Minkowski vierbein (that is, $\eta$ labels the two possible spin structures on $M_0$ [2]). The $\eta = -1$ spin structure is energetically preferred. The results are $-4t$ times those of the massive scalar field (8). The factor of $-1$ is due to different statistics while the factor of $4$ is due to degrees
of freedom. The massless limit agrees with twice the massless two-component expectation values \[2, 11, 12\] as expected.

3.2. \(M_\pm\)

The standard Minkowski vierbein is not invariant under \(J_\pm\). In an invariant vierbein the spinor Hadamard function on \(M_\pm\) would be given directly by the method of images, that is

\[
S^{(1)}_{M_\pm}(x, x') = \sum_{n \in \mathbb{Z}} \rho^n S^{(1)}(x, J^nx'),
\]

with \(\rho = +1, (-1)\), however as we choose to work in the Minkowski vierbein the image expression is different. One vierbein which is invariant under \(J_\pm\) is one which rotates by \(\pi\) in the \(x - y\)-plane as \(z \to z + a\). The transformation from this vierbein to the standard Minkowski one is clearly the corresponding rotation by \(-\pi\). The associated transformation of the spinor Hadamard function is

\[
S^{(1)}_{SM}(x, x') = e^{-\frac{|x'|^2}{2}} S^{(1)}_{R(M)}(x, x') e^{-\frac{|x|^2}{2}},
\]

the \(R(S)\) subscript denotes the rotating (standard) vierbein, respectively. Therefore on \(M_\pm\) in the standard vierbein

\[
S^{(1)}_{SM}(x, x') = \sum_{n \in \mathbb{Z}} \rho^n S^{(1)}_{SM}(x, J^nx') e^{-\frac{|x|^2}{2}}.
\]

In terms of the scalar Hadamard function this translates to using the following expression in \((17)\)

\[
O^{(1)}_M(x, x') = \sum_{n \in \mathbb{Z}} \rho^n G^{(1)}(x, J^n x') e^{\frac{m|y|^2}{4}},
\]

where the \(n = 0\) term is dropped. Here \(\rho = +1, (-1)\) labels periodic (antiperiodic) spinors with respect to the vierbein which rotates by \(\pi\) as \(z \to z + a\). Thus, \(\rho\) labels the two inequivalent spin structures on \(M_\pm\) \[2\]. In the coincidence limit \(G^{(1)}_M(x, x') \to K_2(m\sigma_n)\) which is positive so that again only the \(K\) Bessel term in \((5)\) contributes. Further we note again the relations \((10)\). We now therefore just apply \((17)\) to \((24)\). The calculation is made easier by splitting the sum in \((24)\) into odd and even terms. The even terms lead to the expectation values on \(M_\pm\) in the twisted spin structure there and \(\langle 0| T_{\mu\nu} | 0 \rangle_{M_\pm} = \langle 0| T_{\mu\nu} | 0 \rangle_{M_\pm, (\rho = -1)} + \rho \langle 0| T_{\mu\nu} | 0 \rangle_{odd} \) with

\[
\langle 0| T_{\mu\nu} | 0 \rangle_{odd} = \sum_{n \in \mathbb{Z}} (-1)^n \frac{m^3 y (2na + a)}{\pi^2 \sigma_n} K_2(m\sigma_n)
\]

\[
\langle 0| T_{\mu\nu} | 0 \rangle_{odd} = \sum_{n \in \mathbb{Z}} (-1)^n \frac{m^3 x (2na + a)}{\pi^2 \sigma_n} K_2(m\sigma_n),
\]

where \(\sigma_n = ((2x)^2 + (2y)^2 + (2na + a)^2)^{1/2}\).

We see that the spinor expectation values on \(M_\pm\) are not \(-4\) times those of the scalar field. In particular it is interesting to note that for the scalar field the only non-zero cross term is \(\langle 0| T_{\mu\nu} | 0 \rangle\) while for the spinor field this term is 0 and \(\langle 0| T_{\mu\nu} | 0 \rangle\) and \(\langle 0| T_{\mu\nu} | 0 \rangle\) are non-zero. Also note that these two terms for the spinor field change sign under a change of spin structure. In the massless limit the results agree with twice those of the massless two-component spinor results found in \[2\] where it is shown that the expectation values for right- and left-handed two-component spinors are the same (see also \[11\]). The leading order corrections for small mass are \(O(m^2)\) and \(\langle 0| T_{\mu\nu} | 0 \rangle\) vanishes exponentially in the limits of large mass and
large $a$. The difference between $\langle 0 | T_{\mu\nu} | 0 \rangle$ on $M_-$ and on $M_0$ vanishes at $r^2 := x^2 + y^2 = 0$ and vanishes exponentially as $r \to \infty$.

4. Minkowski space with infinite plane boundary

In this section we consider briefly a massive scalar field in four-dimensional Minkowski space with an infinite plane boundary at $x = 0$. While the stress–energy for a massless field is well known (see e.g. [5, 13]), the massive results to our knowledge are new. There is some similarity with $M_-$ as both spaces may be considered as quotients of Minkowski space with the quotient group including a reflection in $x$ about $x = 0$.

Again here the scalar Hadamard function is given by the method of images,

$$G^+_B(x, x') = G^+_M(x, x') + \eta G^+_M(x, J_B x'),$$

(26)

where $J_B : (t, x, y, z) \mapsto (t, -x, y, z)$ and $\eta = +1, (-1)$ labels Neumann (Dirichlet) boundary conditions on the plate. The first term leads to the expectation values $\langle 0 | T_{\mu\nu} | 0 \rangle$ on Minkowski space which are divergent and are dropped by the usual renormalization procedure. Therefore from the second term we get

$$\langle 0 | T_{tt} | 0 \rangle = -\langle 0 | T_{yy} | 0 \rangle = -\langle 0 | T_{zz} | 0 \rangle = \eta \left[ \frac{(1 - 4\xi)m^3}{4\pi^2 |2x|} K_3(m |2x|) \right. + \left. \frac{(2\xi - 1)m^2}{8\pi^2 |2x|^2} K_2(m |2x|) \right],$$

(27)

$$\langle 0 | T_{xx} | 0 \rangle = 0.$$

In the massless limit these agree with the literature [5]. For the massive field $\langle 0 | T_{\mu\nu} | 0 \rangle$ is non-zero and has non-vanishing trace for both conformal and minimal coupling. The mass breaks the conformal invariance of the field. $\langle 0 | T_{\mu\nu} | 0 \rangle$ vanishes exponentially as the mass and as $x$ goes to infinity, in contrast to the massless case which behaves as $O(x^{-4})$ for the minimally coupled field and is identically 0 for conformal coupling. It is interesting to note that for conformal coupling the leading order correction for small mass is $O(m^2)$, while for minimal coupling it is $O(m^4)$.

5. Discussion

We have calculated the expectation values $\langle 0 | T_{\mu\nu} | 0 \rangle$ for the massive scalar and Dirac fields on the flat spacetimes $M_0$ and $M_-$. For the scalar field our results on $M_0$ agree with those in [3]. For the spinor field on $M_0$ our results are twice those of [4] as expected. On $M_-$ the results for the massive fields are new. Further in the massless limit our expectation values agree with the previous literature [1, 2, 11, 12]. In all cases the values fall off exponentially in the large $m$ limit, and the leading order correction for small mass is $O(m^2)$. Further it is noted that for the scalar field in the large $a$ limit on $M_-$ there is an exponential decay in the massive case while for the massless field the behaviour is $a^{-4}$. For the massive field the difference between $\langle 0 | T_{\mu\nu} | 0 \rangle$ on $M_-$ and the corresponding values on $M_0$ vanishes exponentially in the limit of large $x^2 + y^2$, while for the massless field it behaves as $O(r^{-3})$. As an aside we also presented the expectation values $\langle 0 | T_{\mu\nu} | 0 \rangle$ for a massive scalar field on Minkowski space with an infinite straight plane boundary.

While we have focussed the present paper on the stress–energy in its own right, our underlying interest in this problem arises from the role of $M_0$ and $M_-$ in modelling, in the context of accelerated observers on flat spacetimes, the Hawking(-Unruh) effect on
respectively the Kruskal manifold and the $\mathbb{RP}^3$ geon [1]. Certain aspects of the thermal and non-thermal effects for scalar and spinor fields on $M_0$ and $M_-$ are at present understood from the viewpoint of Bogoliubov transformations and particle detector analyses [1, 2], but the connections between (non-)thermality and stress-energy remain less clear. We view our results, in conjunction with those in [1, 2], as data points to which we anticipate future work on this question to provide a deeper understanding.

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