CLOSED GEODESICS ON COMPACT SYMMETRIC SPACES OF HIGHER RANK

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ABSTRACT. In this article, we consider a compact symmetric space $M$ of higher rank. Let $P(t)$ be the set of free-homotopy classes containing a closed geodesic on $M$ with length at most $t$, and $\#P(t)$ its cardinality. We obtain the following asymptotic estimates:

$$\#P(t) = \frac{e^{ht}}{ht}(1 + O(e^{-ut}))$$

for some $u > 0$, where $h$ is the topological entropy of the geodesic flow.

1. Introduction

Let $X$ be a globally symmetric space of noncompact type, with a reference point $p \in X$. Then $X \cong G/K$ where $G = \text{Isom}^0(X)$ is the connected component containing the identity of the isometry group of $X$ and $K$ is the isotropy group of $p$ in $G$. $G$ is a semisimple Lie group with trivial center and no compact factors, and $K$ is a maximal compact subgroup in $G$. $X$ admits a unique $G$-invariant Riemannian metric with nonpositive curvature. In fact by the higher rank rigidity theorem [2, 3, 4, 5, 6], any simply connected irreducible Riemannian manifold of nonpositive curvature and of geometric rank greater than one is isometric to a globally symmetric space of noncompact type.

Let $\Gamma < G$ be a cocompact lattice in $G$ so that $M = \Gamma \backslash X = \Gamma \backslash G/K$ is a compact symmetric space of noncompact type. The geodesic flow is defined on the unit tangent bundle $SM$. There is correspondence between closed geodesics on $M$ and periodic orbits under the geodesic flow, and they are important from both geometric and dynamical points of view. Let $P(t)$ be the set of free-homotopy classes containing a closed geodesic with length at most $t$. In the case $G = \text{PSL}(2, \mathbb{R})$, $X = \mathbb{H} = G/K$ and $M = \Gamma \backslash \mathbb{H}$ being a hyperbolic surface, Huber [17] in 1959 showed that

$$\#P(t) \sim e^t/t$$

where $f \sim g$ means $\lim_{t \to \infty} \frac{f(t)}{g(t)} = 1$.

However in the higher rank symmetric spaces, the asymptotic behavior of $\#P(t)$ is much less known. In his 1983 thesis [37], Spatzier studied
the topological entropy of the geodesic flow on higher rank symmetric spaces. He obtained a closing lemma and showed that

$$\lim_{t \to \infty} \frac{1}{t} \log \# P_\epsilon(t) = h$$

where $h$ is the topological entropy of the geodesic flow and $P_\epsilon(t)$ is the maximal set of $\epsilon$-separated closed geodesics. He also obtained

$$\lim_{t \to \infty} \frac{1}{t} \log \sum_F \text{vol}(F) = h$$

where the sum is over all compact flats $F$ with regular systol no more than $t$, and $\text{vol}(F)$ denotes the volume of $F$.

Knieper [21] showed there exists a unique measure of maximal entropy for the geodesic flow which is the Haar measure on an ergodic component. He also showed that a maximal set of $\epsilon$-separated closed geodesics is uniformly distributed with respect to the measure of maximal entropy. He asked the following question:

**Question 1.1.** ([16, Question 14.5], [21, Remark p. 175]) Do we have

$$\lim_{t \to \infty} \frac{1}{t} \log \# P(t) = h?$$

Link [25] proved that if $\Gamma < G$ is a Schottky group (then $M$ has infinite volume),

$$\frac{1}{r} e^{\delta(\Gamma)t} \leq \# P(t) \leq \frac{1}{t} e^{\delta(\Gamma)t}$$

where $r$ is the rank of $X$ and $\delta(\Gamma)$ is the critical exponent of $\Gamma$. Recently, Dang-Li [11, Theorem 1.2] obtained the following asymptotic formula:

$$\sum_F |\Lambda(F) \cap B_\sigma^+(0,t)| \text{vol}(F) = CT^{\frac{r-1}{2}} e^{ht}(1 + O(e^{-ut}))$$

for some $u > 0$, where the sum is taken over all compact flat tori, $\Lambda(F)$ is the set of periods of $F$, and $B_\sigma^+(0,t)$ is the set of vectors in a Weyl chamber with length less than $t$.

However, as far as we know the asymptotic estimates of $\# P(t)$ is still unknown. The main result of this paper is the following asymptotic estimates of $\# P(t)$ which answers Question 1.1.

**Theorem A.** Let $M$ be a compact symmetric space of higher rank, then

$$\# P(t) = \frac{e^{ht}}{ht} (1 + O(e^{-ut}))$$

for some $u > 0$, where $h$ denotes the topological entropy of the geodesic flow on $SM$. 
We also obtain the following result of equidistribution of periodic orbits with respect to the MME.

**Theorem B.** Suppose that $M$ is a compact symmetric space of higher rank, and $\epsilon \in (0, \text{inj}(M)/2)$ is fixed where $\text{inj}(M)$ is the injectivity radius of $M$. For $t > 0$, let $C(t)$ be any maximal set of pairwise non-free-homotopic closed geodesics with lengths in $(t - \epsilon, t]$, and define the measure

$$\nu_t := \frac{1}{C(t)} \sum_{c \in C(t)} \frac{\text{Leb}_c}{t}$$

where $\text{Leb}_c$ is the Lebesgue measure along the curve $\dot{c}$ in the unit tangent bundle $SM$.

Then the measures $\nu_t$ converge in the weak* topology to the unique measure of maximal entropy as $t \to \infty$.

We explore Margulis’s approach to symmetric spaces of higher rank. Margulis [27] constructed a measure of maximal entropy for the geodesic flow on compact manifolds of negative curvature. This measure is mixing, and the conditional measures on stable/unstable manifolds contract/expand with a uniform rate under the flow. By counting the intersections of flow boxes, he obtained an asymptotic formula:

$$\#P(t) \sim \frac{e^{ht}}{ht}.$$  

An alternative approach for Axiom A flows is given by Parry-Pollicott [29] and Pollicott-Sharp[32], based on zeta functions and symbolic dynamics. See the survey by Sharp in [27] for further progress that has been made since Margulis’s thesis.

For compact rank one manifolds of nonpositive curvature, Knieper [18, 19, 20] proved the uniqueness of measure of maximal entropy and obtained the following asymptotic estimates: there exists $C > 0$ such that for $t$ large enough

$$\frac{1}{C} \frac{e^{ht}}{t} \leq \#P(t) \leq Ce^{ht}.$$  

Recently, a breakthrough was made by Ricks [34], who proved the Margulis-type asymptotic estimates for compact rank one manifolds of nonpositive curvature. Then a remarkable progress was made by Climenhaga, Knieper and War [8] for a class of manifolds (including all surfaces of genus $\geq 2$) without conjugate points. The asymptotic formula was also generalized to compact rank one manifolds without focal points [42].
2. Preliminaries

In this section, we present some geometric and ergodic results on compact symmetric space of higher rank, which will be used in the following discussions.

2.1. Geometric properties. Let $X$ be a globally symmetric space of noncompact type, $G = \text{Isom}^0(X)$ the connected component containing the identity of the isometry group of $X$. Fix a reference point $p \in X$, and $K$ the isotropy group of $p$ in $G$. Then it is well known that $G$ is a semisimple Lie group with trivial center and no compact factors, and $K$ is a maximal compact subgroup in $G$. Since $G$ acts transitively on $X$, we have $X \cong G/K$.

Let $g$ and $k$ denote the Lie algebra of $G$ and $K$ respectively. The geodesic symmetry in $p$ induces a Cartan involution $\theta$ on $g$, namely $g = k \oplus p$. We can identify $p \cong T_pX$, and the Killing form $B$ on $g$ induces a scalar product $<X,Y> := -B(X,\theta Y)$, $X,Y \in g$.

The scalar product restricted to $p$ induces the unique $G$-invariant Riemannian metric $d_X$ on $X$. $X$ has nonpositive sectional curvature with respect to $d_X$. Let $a \subset p$ be a maximal abelian subspace. The dimension $r$ of $a$ is called the rank of $X$.

Choose an open positive Weyl chamber $a^+ \subset a$. Denote $a^+_+ := \{H \in a^+ : \|H\| = 1\}$. Then it induces a Cartan decomposition $G = K e^{a^+} K$. Let $\Sigma$ be the set of roots of the pair $(g,a)$, and $\Sigma^+ \subset \Sigma$ the set of positive roots determined by the Weyl chamber $a^+$. We denote $g_\alpha$ the root space of $\alpha \in \Sigma, n^+ := \sum_{\alpha \in \Sigma^+} g_\alpha$, and $N^+ = e^{n^+}$, the Lie group exponential of the nilpotent Lie algebra $n^+$. Then $G = KAN^+$, which is called the Iwasawa decomposition associated to the Cartan decomposition $G = KA^+K$.

Consider a cocompact lattice $\Gamma$ in $G$ and then $M = \Gamma \backslash G/K = \Gamma \backslash X$ is a compact locally symmetric space. Let $pr : X \to M$ the universal cover of $M$. Then $\Gamma \simeq \pi_1(M)$ is the group of deck transformations on $X$, so each $\gamma \in \Gamma$ acts isometrically on $X$. Since $M = X/\Gamma$ is compact, each $\gamma \in \Gamma$ is axial (cf. [9] Lemma 2.1), that is, there exists a geodesic $c$ and $t_0 > 0$ such that $\gamma(c(t)) = c(t + t_0)$ for every $t \in \mathbb{R}$. Correspondingly $c$ is called an axis of $\gamma$ and we denote $|\gamma| := t_0$ where $t_0$ is minimal with the above property.

We still denote by $pr : SX \to SM$ and $\gamma : SX \to SX$ the map on unit tangent bundles induced by $pr$ and $\gamma \in \Gamma$. From now on, we use an underline to denote objects in $M$ and $SM$, e.g. for a geodesic $c$ in
X and \( v \in SX, \varphi := pr c, \varphi := pr v \) denote their projections to \( M \) and \( SM \) respectively.

2.1.1. Geometric boundary. Suppose that \( c_1 \) and \( c_2 \) are both geodesics in \( X \). We call \( c_1 \) and \( c_2 \) are positively asymptotic or just asymptotic if there is a positive number \( C > 0 \) such that

\[
(1) \quad d_X(c_1(t), c_2(t)) \leq C, \quad \forall t \geq 0.
\]

We say \( c_1 \) and \( c_2 \) are negatively asymptotic if (1) holds for all \( t \leq 0 \). \( c_1 \) and \( c_2 \) are said to be bi-asymptotic or parallel if they are both positively and negatively asymptotic. The relation of positive/negative asymptoticity is an equivalence relation between geodesics on \( X \). The class of geodesics that are positively/negatively asymptotic to a given geodesic \( c_0 \) is denoted by \( c_0(+\infty)/c_0(-\infty) \) or \( v^+/v^- \) respectively. We call them points at infinity. Obviously, \( c_0(-\infty) = c_0(+\infty) \). We use \( \partial X \) to denote the set of all points at infinity, and call it the geometric boundary. If \( \eta = v^+ \in \partial X \), we say \( v \) points at \( \eta \).

The following are the fundamental properties of the geometry of \( X \).

**Lemma 2.1.** (Cf. [3, 4])

1. \( d(c_1(t), c_2) \) and \( d(c_1(t), c_2(t)) \) are convex in \( t \in \mathbb{R} \) for any two geodesics \( c_1 \) and \( c_2 \).

2. Let \( c_1 \) and \( c_2 \) be distinct geodesics with \( c_1(0) = c_2(0) \). Then for \( t > 0 \), both \( d(c_1(t), c_2) \) and \( d(c_1(t), c_2(t)) \) are strictly increasing and tend to infinity as \( t \to \infty \).

3. Let \( c_1 \) and \( c_2 \) be asymptotic geodesics. Then both \( d(c_1(t), c_2) \) and \( d(c_1(t), c_2(t)) \) are nonincreasing functions of \( t \in \mathbb{R} \).

4. For any geodesic \( c \) and each \( p \in X \), there exists a unique geodesic through \( p \) and asymptotic to \( c \).

5. (Flat Strip Lemma) If \( c_1 \) and \( c_2 \) are bi-asymptotic, then they bound a flat strip, i.e., there is an isometric embedding \( \phi : [0, a] \times \mathbb{R} \to X \) such that \( \phi(0, t) = c_1(t) \) and \( \phi(a, t) = c_2(t) \) up to parametrization.

We can define the visual topology on \( \partial X \) following [12] and [13]. For each \( p \), by Lemma 2.1, there is a bijection \( f_p : S_pX \to \partial X \) defined by \( f_p(v) = v^+ \), \( v \in S_pX \). So for each \( p \in M \), \( f_p \) induces a topology on \( \partial X \) from the usual topology on \( S_pX \). Given \( p, q \in X \), let \( \psi : S_pX \to S_qX \) be the map such that \( \psi(v) \) is the unique vector in \( S_qX \) asymptotic to \( v \in S_pX \).

**Lemma 2.2.** ([41, Lemma 2.6]) The map \( \psi : S_pX \to S_qX \) above is continuous.
It follows that $\psi$ is in fact a homoeomorphism. Hence the topology on $\partial X$ induced by $f_p$ is independent of $p \in X$, and is called the visual topology on $\partial X$.

The above topology on $\partial X$ and the manifold topology on $X$ can be extended to $\overline{X} := X \cup \partial X$ naturally by requiring the map $\varphi$ defined as follows is a homeomorphism. Fix $p \in X$. For each $v \in T_pX$ with $\|v\| \leq 1$, define

$$\varphi(v) := \begin{cases} \exp\left(\frac{v}{1-\|v\|}\right) & \text{if } \|v\| < 1; \\ f_p(v) & \text{if } \|v\| = 1. \end{cases}$$

This topology is usually called the cone topology. Under this topology, $\overline{X}$ is homeomorphic to the closed unit ball in $\mathbb{R}^{\dim(X)}$, and $\partial X$ is homeomorphic to the unit sphere $S^{\dim(X)-1}$.

The following continuity property is useful.

**Lemma 2.3.** ([41, Lemma 6.4]) Let $p,p_n \in X$ with $p_n \to p$, and $x_n, \zeta \in \overline{X}$ with $x_n \to \zeta$. Then $c_{p_n,x_n}(0) \to c_p(\zeta)$, where $c_{q\xi}$ is the unique geodesic from $q \in X$ and pointing at $\xi \in \partial X$.

2.1.2. Continuity of Busemann function. For each pair of points $(p, q) \in X \times X$ and each point at infinity $\xi \in \partial X$, the Busemann function based at $\xi$ and normalized by $p$ is

$$b_{\xi}(q, p) := \lim_{t \to +\infty} \left(d(q, c_{p, \xi}(t)) - t\right),$$

where $c_{p, \xi}$ is the unique geodesic from $p$ and pointing at $\xi$. The Busemann function $b_{\xi}(q, p)$ is well-defined since the function $t \mapsto d(q, c_{p, \xi}(t)) - t$ is bounded from above by $d(p, q)$, and decreasing in $t$ (this can be checked by using the triangle inequality). If $v \in SX$ points at $\xi \in \partial X$, we also write $b_v(q) := b_{\xi}(q, p)$.

The level sets of the Busemann function $b_{\xi}(q, p)$ are called the horospheres centered at $\xi$. The horosphere through $p$ based at $\xi \in \partial X$, is denoted by $H_p(\xi)$. For more details of the Busemann functions and horospheres, please see [10, 35, 36]. Here we are concerned with the continuity property of the horospheres and Busemann functions.

We say that the manifold $M$ satisfies the axiom of asymptoticity if for every $v \in SX$ the following statement holds: For any choice of $x_n, x \in X, v_n \in SX, x_n \to x, v_n \to v$ and a sequence of numbers $t_n \to +\infty$, the sequence $c_{v_n}(t_n)$ of geodesic segments joining $x_n$ to $c_{v_n}(t_n)$ converges to an asymptote of $c_v$.

**Proposition 2.4.** (Cf. [31, Theorem 6.1] [35, Lemma 1.2]) Let $M$ be a closed Riemannian manifold without conjugate points that satisfies the axiom of asymptoticity. Then for every $p \in X$, the map $\xi \mapsto H_p(\xi)$
is continuous in the following sense: if $\xi_n \to \xi \in \partial X$ and $K \subset X$ is compact, then $H_{\xi_n}(p) \cap K \to H_\xi(p) \cap K$ uniformly in the Hausdorff topology.

**Proposition 2.5.** (Cf. [31, Theorem 5.2]) If the manifold $M$ has no focal points, then it satisfies the axiom of asymptoticity.

By Proposition 2.5, Proposition 2.4 applies to manifolds without focal points and hence manifolds of nonpositive curvature. So we have the following corollary (see also [26, Proposition 9]):

**Corollary 2.6.** The functions $(v,q) \mapsto b_v(q)$ and $(\xi,p,q) \mapsto b_\xi(p,q)$ are continuous on $SX \times X$ and $\partial X \times X \times X$ respectively.

We would like to obtain certain equicontinuity of Busemann function $v \mapsto b_v(q)$. The idea is to extend the function to the compact space $X \cup \partial X$. Given $x, p, q \in X$, define $b_x(q,p) := d(q,x) - d(p,x)$.

**Lemma 2.7.** ([26, Proposition 10]) For each pair of points $p, q \in X$, if there is a sequence $\{x_n\} \subset X$ with $\lim_{n \to +\infty} x_n = \xi \in \partial X$, then $\lim_{n \to +\infty} b_{x_n}(q,p) = b_\xi(q,p)$.

**Lemma 2.8.** Let $p \in X$, $A \subset S_pX$ be closed, and $B \subset X$ be such that $A^+ := \{v^+ : v \in A\}$ and $B^- := \{\lim_{n \to +\infty} q_n \in \partial X : q_n \in B\}$ are disjoint subsets of $\partial X$. Then the family of functions $A \to \mathbb{R}$ indexed by $B$ and given by $v \mapsto b_v(q)$ are equicontinuous: for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\angle_p(v,w) < \delta$, then $|b_v(q) - b_w(q)| < \epsilon$ for every $q \in B$.

**Proof.** The proof is a repetition of that of [8, Lemma A.1] and thus omitted. Notice that in our setting, Lemma 2.7 should be used instead of [7, Corollary 2.18]. \qed

2.1.3. Furstenberg boundary. We denote $A^+ = e^{a^+}$ and $A = e^a$. Then $A^+p \subset X$ is a closed geometric Weyl chamber based at $p \in X$. A geometric Weyl chamber is a subset of $X$ of the form $gA^+p$ where $g \in G$. The space of geometric chambers can be identified

$$G.\overline{A^+p} \cong G/M$$

where $M := Z_K(A)$ is the centralizer of $A$ in $K$.

We introduce the following asymptotic equivalence relation between geometric Weyl chambers:

$$g_1\overline{A^+p} \sim g_2\overline{A^+p} \iff \sup_{a \in A^+} d_X(g_1ap, g_2ap) < \infty.$$ 

Denote by $\eta_0$ the asymptotic class of $\overline{A^+p}$, and $\zeta_0$ of $\overline{A^{-1}p}$. 

Remark 2.9. From the above definition, if two geometric Weyl chambers \( g_1 A^+ p \) and \( g_2 A^+ p \) are asymptotic, then the isometry \( g_2 g_1^{-1} \) maps a geodesic ray in \( g_1 A^+ p \) to a unique asymptotic geodesic ray in \( g_2 A^+ p \).

The set of asymptotic geometric Weyl chambers is called Furstenberg boundary \( \mathcal{F} \), which can be identified
\[
\mathcal{F} \cong ( (G.A^+ p) / \sim ) \cong G/P \cong K/M \cong K. \eta_0
\]
where \( P := MAN \). Moreover, the following \( G \)-equivariant map is a diffeomorphism:
\[
G/M \to X \times F
\]
\[
gM \mapsto (gp, g\eta_0).
\]

For \( k \in K \) and \( H \in a_1^+ \), we denote by \( (k, H) \) the unique class in \( \partial X \) which contains the geodesic ray \( c(t) = ke^{Ht}p, t > 0 \). We call \( k \) the angular projection, and \( H \) the Cartan projection of \( (k, H) \). The Cartan projection \( H \) of a point \( \xi \in \partial X \) is unique, whereas its angular projection \( k \) is only determined up to right multiplication by an element in the centraliser of \( H \) in \( K \).

If \( r = \text{rank}(X) > 1 \), we define the regular boundary \( \partial X^{\text{reg}} \) as the set of classes with Cartan projection \( H \in a_1^+ \). We have a natural projection \( \pi^B : \partial X^{\text{reg}} \to K/M \) such that \( \pi^B(k, H) = kM \). The following lemma relates the cone topology to the topology of \( K/M \).

Lemma 2.10. (\cite[Lemma 2.3]{24}) A sequence \( (\xi_n) \in \partial X^{\text{reg}} \) converges to \( \xi = (k, H) \in \partial X^{\text{reg}} \) in the cone topology if and only if \( \pi^B(\xi_n) \) converges to \( kM \) in \( K/M \) and the Cartan projections of \( \xi_n \) converge to \( H \) in \( a_1^+ \).

The Iwasawa decomposition induces a natural projection \( \pi^I : G \to K/M \) with \( g = kan \mapsto kM \). The action of \( G \) on geometric and Furstenberg boundaries can be described as follows.

Lemma 2.11. (\cite[Lemma 2.4]{24}) Let \( g \in G \) and \( \xi = (k, H) \in \partial X \) with \( k \in K \) and \( H \in a_1^+ \). If \( \pi^I(gk) = k' M \) for some \( k' \in K \), then \( g\xi = (k', H) \). In particular, if \( \xi = (k, H) \in \partial X^{\text{reg}} \), then \( g\pi^B(\xi) = \pi^B(g\xi) = k'M \).

2.1.4. Busemann and Iwasawa cocycle. Recall the Iwasawa decomposition \( G = KAN \). For every \( \xi \in \mathcal{F} \) and \( g \in G \), there exists a unique element \( \sigma(g, \xi) \in a \) such that if \( k_\xi \in K \) satisfies \( k_\xi \eta_0 = \xi \), then
\[
gk_\xi \in Ke^{\sigma(g, \xi)}N.
\]
\( \sigma(g, \xi) \) is called the Iwasawa cocycle, as it satisfies the cocycle relation:
\[
\sigma(g_1 g_2, \xi) = \sigma(g_2, \xi) + \sigma(g_1, g_2 \xi).
\]
For every $x, y \in X$ and $\xi \in F$, we define the Busemann cocycle by

$$\beta_\xi(x, y) := \sigma(h_{x}^{-1}h_{y}, h_{y}^{-1}\xi)$$

independently of the choice of $h_{x}, h_{y} \in G$ such that $h_{x}o = x$ and $h_{y}o = y$. It is easy to check that

$$\beta_\xi(x, y) = \beta_{g\xi}(gx, gy)$$

$$\beta_\xi(x, z) = \beta_\xi(x, y) + \beta_\xi(y, z).$$

Let $X \in a_{+}^{1}$. For $\eta = k\eta_{0} \in F$, we introduce the geodesic ray on $G/K$ given by $m(t) := k e^{tx}p$. The geometric interpretation of the Iwasawa cocycle comes from the equality

$$<X, \sigma(g, \eta)> = \lim_{t \to \infty} d(g^{-1}p, m(t)) - d(p, m(t)).$$

The righthand side above is $b_{(k, X)}(g^{-1}p, p)$. By definition, $\sigma(g, \eta) = \beta_{\eta}(g^{-1}p, p)$. Thus we have

$$<X, \beta_{\eta}(g^{-1}p, p)> = b_{(k, X)}(g^{-1}p, p).$$

### 2.1.5. Hopf coordinates

Let $v_{0} \in SX$ be a regular vector such that $p := \pi(v_{0})$, which will be the reference point in the following discussions.

Let $H \in a_{+}^{1}$ be such that $v_{0}^{+} = (id, H)$. Then $Gv_{0} \cong G/(Z(H) \cap K)$ where $Z(H)$ is the centralizer of $H$ in $K$. Since $v_{0}$ is regular, $Z(H) = M$ and thus $Gv_{0} \cong G/M$.

Define $F^{(2)} := \{(g\zeta_{0}, g\eta_{0}) : g \in G\}$ to be the set of ordered opposite pairs in $F \times F$. The Hopf map $H : Gv_{0} \to F^{(2)} \times a$ for $p \in X$ is defined as

$$H(gv_{0}) := (g\zeta_{0}, g\eta_{0}, s(gv_{0}))$$

where $s(gv_{0}) := -\sigma(g, \zeta_{0})$.

Suppose that $v^{+} = (k_{v}, H_{v})$ where $k_{v} \in K$ and $H_{v} \in a_{1}^{+}$. From definition, we see

$$s(\phi^{t}v) = s(v) + tH_{v} = s(v) + t(0, \ldots, 0, 1)$$

for any $v \in Gv_{0}$ and $t \in \mathbb{R}$. $s$ is continuous by Corollary 2.6.

Using $Gv_{0} \cong G/M \cong X \times F$, we have

$$H(x, \xi) = (\xi_{x}^{\perp}, \xi, \beta_{\xi_{x}^{\perp}}(x, p))$$

where $\xi_{x}^{\perp}$ is the unique point in $F$ opposite to $\xi$ such that the flat $F_{\xi\xi_{x}^{\perp}}$ connecting $\xi$ and $\xi_{x}^{\perp}$ passes through $x$. 

2.2. Entropy and Patterson-Sullivan measure. Let $H \in \mathfrak{a}^+$ and $v \in X$ with $v^+ = (id, H) \in \partial X$. Then $Gv$ is an ergodic component of the geodesic flow with respect to the Haar measure on $S.X$. Indeed, the disjoint union $\partial X = \bigcup_{H \in \mathfrak{a}^+} G.(id, H)$ corresponds to all ergodic components of the geodesic flow.

Spatzier [37] showed that there exists a unique $H \in \mathfrak{a}^+$ such that the Haar measure on $G.v$ with $v^+ = (id, H)$ has metric entropy equal to the topological entropy of the geodesic flow. In other words, the geodesic flow has a unique measure of maximal entropy (MME for short), which is the (normalized) Haar measure on $G.v \cong G/M$. Moreover, the topological entropy is equal to $2\rho(H)$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$ is the half sum of the positive roots with multiplicities.

Patterson-Sullivan measures [30, 38] are generalized to symmetric spaces of higher rank by [1, 33, 22, 23, 39, 11]. We use the one described in [11].

For $x \in X$, let $K_x$ be the stabilizer group of $x$ in $G$. Let $\mu_x$ be the unique $K_x$ invariant probability measure on the Furstenberg boundary $\mathcal{F} \cong K/M$. Then we have for $g \in G$ and $x \in X$, $g_\ast \mu_x = \mu_{gx}$. This relation holds because the stabilizer of $gx$ is given by $gK_xg^{-1} = K_{gx}$.

By [33, Lemma 6.3],

$$\frac{dg_\ast \mu_p}{d\mu_p}(\xi) = e^{-2\rho(g^{-1}, \xi)},$$

and hence

$$\frac{d\mu_x}{d\mu_p}(\xi) = e^{-2\rho(h_x^{-1}, \xi)},$$

where $h_x p = x$.

Then we define measures $\nu_x$ on $\mathcal{F}^{(2)}$ by

$$d\nu_x(\xi, \eta) := e^{2\rho(\xi|\eta)} d\mu_p(\xi) d\mu_p(\eta),$$

where $(\xi|\eta)_x$ is the Gromov product at point $x$ (see [11, Definition 2.12]) which satisfies

$$(g\xi|g\eta)_p - (\xi|\eta)_p = \iota \sigma(g, \xi) + \sigma(g, \eta)$$

where $\iota$ is the inverse involution on $\mathfrak{a}$.

By [11, Propositions 2.13, 2.15] and [28], we have

**Proposition 2.12.**

1. For all $x \in X$, $\nu_x$ is $G$-invariant and $\nu_x = \nu_p$. We denote it by $\nu$.

2. $\nu \otimes \text{Leb}$ on $\mathcal{F}^{(2)} \times \mathfrak{a}$ is a disintegration in Hopf coordinates of a Haar measure $m$ on $Gv_0$ or $G/M$. 

(3) $m$ is exponentially mixing. In particular, for any two sets $A, B$ with $m(A) > 0, m(B) > 0$, there exists $u > 0$ and $D = C(A, B) > 0$ such that for every $t > 0$.

$$|m(A \cap \phi^{-t}(B)) - m(A)m(B)| \leq De^{-ut}.$$ 

2.3. Local Product flow boxes. From now on we fix $v_0 \in SX$ such that $p = \pi(v_0)$ and $\rho(Hv_0) = \|\rho\| = \max_{H \in a^+} \rho(H)$. For simplicity, we will suppress $v_0$ and $p$ from the notation. Recall that $\eta_0 = \pi^B(v_0^+) \in F$ and $\zeta_0 = \pi^B(v_0^-) \in F$. We also fix a scale $e \in (0, \min\{\frac{1}{8}, \frac{\text{inj}(M)}{4}\})$.

There exists a distance $d$ on the Furstenberg boundary $F$ arising from representations of semisimple groups. Given $\xi, \eta \in F$

$$d(\xi, \eta) := \sup_{\alpha \in \Pi} d(x^\alpha(\xi), x^\alpha(\eta))$$

where $\Pi \subset \Sigma^+$ is the set of simple roots, $x^\alpha$ is defined through an irreducible and proximal representation $(\rho^\alpha, V^\alpha)$ and $\bar{d}$ is a distance in the projective space $\mathbb{P}(V^\alpha)$. See Section 2.1 in [11] for details. Denote the ball with respect to this distance $d$ on $F$ as

$$B(\xi, \epsilon) := \{\eta \in F : d(\xi, \eta) < \epsilon\}.$$ 

There also exists a function $\delta$ on $F$ defined as

$$\delta(\xi, \eta) := \inf_{\alpha \in \Pi} d(x^\alpha(\xi), x^\alpha(\eta))$$

where $\bar{d}$ is a function in the projective space $\mathbb{P}(V^\alpha)$. Define the balls for $\delta$ by

$$\mathcal{V}_\epsilon(\xi) := \{\eta \in F : \delta(\xi, \eta) < \epsilon\}.$$ 

There exists a distance $d_2$ on $G/M$ via Hopf coordinates:

$$d_2((\xi^-, \xi^+, v), (\eta^-, \eta^+, w)) := \max\{d(\xi^-, \eta^-), d(\xi^+, \eta^+), \|v - w\|\}.$$ 

Given $\alpha, \beta > 0$, denote

$$R_{\alpha, \beta} := [-\beta, \beta]^{r-1} \times [0, \alpha].$$ 

Following [8], we define for each $\theta > 0$ and $0 < \alpha < \frac{3}{2} \epsilon, 0 < \beta < \frac{\epsilon}{8r},$

$$P = P_\theta := B(\eta_0, \theta),$$

$$F = F_\theta := B(\zeta_0, \theta),$$

$$B = B_\theta^{\alpha, \beta} := H^{-1}(P \times F \times R_{\alpha, \beta}),$$

$$S = S_\theta := B_\theta^{\epsilon^2, \epsilon^2} = H^{-1}(P \times F \times R_{\epsilon^2, \epsilon^2}).$$

$B = B_\theta^{\alpha, \beta}$ is called a flow box with depth $\alpha$ and $\beta$, and $S = S_\theta$ is a slice with depth $\epsilon^2$. We will consider $\theta > 0$ small enough, which will be specified in the following.
Given $s > 0$ and $x \in X$, denote
\[
\mathcal{F}^{(2)}(x, s) := \{ (\xi, \eta) \in \mathcal{F}^{(2)} : d_X(x, F_{\xi \eta}) < s \}
\]
where $F_{\xi \eta}$ denote the flat connecting $\xi$ and $\eta$.

**Lemma 2.13.** Let $v_0, p, \epsilon$ be as above. Given any $s > 0$, there exists $\theta_1$ sufficiently small such that
\[
P \times F \subset \mathcal{F}^{(2)}(p, s).
\]

**Proof.** Let $\xi_n = k_n \eta_0 \rightarrow \eta_0$ and $\eta_n = k'_n \eta_0 \rightarrow \zeta_0$ with $k_n, k'_n \in K$. Then $(k_n, H) \rightarrow (id, H)$ and $(k'_n, H) \rightarrow (id, -H)$ in the cone topology of $\partial X$ by Lemma 2.10. Let $c_n$ be the geodesic connecting $c_{p,(k_n,H)}(\infty)$ and $c_{p,(k'_n,H)}(\infty)$. Then $d(p, c_n) \rightarrow 0$. Indeed, otherwise assume without generality that $c'_n(0) \rightarrow w$. Then by Lemma 2.1, $c_{v_0}$ and $c_w$ bound a flat strip. But $c_w$ is outside of the flat $F_{v_0}$, which implies that $v_0$ is a singular vector, a contradiction. So $d(p, c_n) \rightarrow 0$. It follows that $c'_n(0)$ is regular and thus lies in a unique flat $F_n$. Obviously, the flat $F_n$ connects $\xi_n$ and $\eta_n$ and for $n$ large enough, $d(p, F_n) < s$. \[\square\]

**Lemma 2.14.** Let $v_0, p, \epsilon$ be as above and $\theta_1$ be given. Then for any $0 < \theta < \theta_1, 0 < \alpha < \frac{3\epsilon}{2}, 0 < \beta < \frac{\epsilon}{8r}$,

1. $\text{diam } \pi H^{-1}(P \times F \times [-\beta, \beta]^{-r-1} \times \{0\}) < \frac{\epsilon}{2}$;
2. $H^{-1}(P \times F \times [-\beta, \beta]^{-r-1} \times \{0\}) \subset SX$ is compact;
3. $\text{diam } \pi B^\alpha_{\theta} < 4\epsilon$ for any $0 < \alpha, \beta \leq \frac{\epsilon}{2}$.

**Proof.** The proof essentially follows from the one of [11, Lemma 3.8]. Let $z = (\xi, \eta, 0) \in P \times F \times [-\beta, \beta]^{-r-1} \times \{0\}$. Then $d_2(z, (\zeta_0, \eta_0, 0)) < d(\eta, \eta_0) + d(\xi, \zeta_0) < 2\theta$. As in the proof of [11, Lemma 3.8], we have
\[
d_X(\pi z, p) < \frac{\epsilon}{4}
\]
provided $\theta_1$ is small. Then the diameter of $\pi H^{-1}(P \times F \times [-\beta, \beta]^{-r-1} \times \{0\})$ is less than $\frac{\epsilon}{4} + 2\sqrt{r - 1}\beta < \frac{\epsilon}{2}$. By continuity of $H$, $H^{-1}(P \times F \times [-\beta, \beta]^{-r-1} \times \{0\}) \subset SX$ is closed. So it is compact since its projection under $\pi$ is bounded.

By triangle inequality, $\text{diam } \pi B^\alpha_{\theta} < \epsilon/2 + 2\alpha < 4\epsilon$. \[\square\]

At last, we also assume that the choice of $\theta \in (0, \theta_1)$ satisfies the following properties. Since $\theta \mapsto \bar{\mu}(P_\theta \times F_\theta)$ is nondecreasing and hence has at most countably many discontinuities, we always choose $\theta$ to be the continuity point of this function, i.e.,
\[
\lim_{\rho \rightarrow \theta} \nu(P_\rho \times F_\rho) = \nu(P_\theta \times F_\theta).
\]
Furthermore, $\theta$ is chosen so that $\nu(\partial P_\theta \times \partial F_\theta) = 0$. By the product structure of $B$ and $S$ and the definition of $m$, we have for any $\alpha \in (0, \frac{2}{\epsilon})$,

$$\lim_{\rho \to \theta} m(S_\rho) = m(S_\theta), \quad \lim_{\rho \to \theta} m(B^\alpha_\rho) = m(B^\alpha_\theta), \quad m(\partial B^\alpha_\theta) = 0. \tag{3}$$

The following is a direct corollary of Lemma 2.8.

**Corollary 2.15.** Given $v_0, p, \epsilon > 0$ as above, there exists $\theta_2 > 0$ such that for any $0 < \theta < \theta_2$, if $\xi, \eta \in P_\theta$ and any $q$ lying within $4\epsilon$ of $\pi H^{-1}(P_\theta \times F_\theta \times [0, \infty))$, we have $|\beta_\xi(q, p) - \beta_\eta(q, p)| < \epsilon^2$. Similar result holds if the roles of $P_\theta$ and $F_\theta$ are reversed.

**Proof.** Note that $\beta_\xi(q, p) = \sigma(h_q^{-1}, \xi)$. Take $X \in \mathfrak{a}^+$. By (2), we have $<X, \beta_\xi(q, p)> = b(k, X)(q, p)$.

By Lemma 2.10, we can choose $\theta > 0$ small enough, so that

$$\angle_p((k_1, X), (k_2, X)) < \delta$$

for any $k_1 \eta_0, k_2 \eta_0 \in P_\theta$, where $\delta$ is given in Lemma 2.8. By Lemma 2.8, $|<X, \beta_\xi(q, p)> - <X, \beta_\eta(q, p)>| < \epsilon^2$ for any $\xi_1, \xi_2 \in P_\theta$ and any $X \in \mathfrak{a}^+$. Thus $|\beta_\xi(q, p) - \beta_\eta(q, p)| < \epsilon^2$. $\square$

Let $\theta_0 := \min\{\theta_1, \theta_2\}$, where $\theta_1$ is given in Lemma 2.14, $\theta_2$ is given in Corollary 2.15. In the following, we always suppose that $0 < \theta < \theta_0$ and (3) is satisfied.

### 3. Closing lemma

Recall that by Cartan decomposition, for every element $g \in G$ there exist $k, l \in K$ and a unique element $g_0(\xi) \in \mathfrak{a}^+$ such that $g = ke^{\xi}(y)^{-1}$. For every $x, y \in X$, denote the $d_\mathfrak{a}(x, y) = a(h_x^{-1}h_y)$ where $h_xp = x$ and $h_yp = y$.

**Definition 3.1.** For all $x \in X$ and $g \in G$, we define $d_\mathfrak{a}(g) = d_\mathfrak{a}(x, gx)$. We say that $g$ is $x$-Cartan regular if $d_\mathfrak{a}(g) \in \mathfrak{a}^+$.

Let $g$ be an $x$-Cartan regular element. Consider $h, h' \in G$ such that $hp = h'p = x$ with $he^{d_\mathfrak{a}(g)}p = gx$ and $h'e^{d_\mathfrak{a}(g^{-1})}p = g^{-1}x$. We set $g_x^+ := h\eta_0$ and $g_x^- := h'n_0$. In particular, when $x = p$, we can take $h = k$ and $h' = lk$, where $k, n_0 = \zeta_0$.

Note that $(x, g_x^+) \in X \times \mathcal{F}$ (resp. $(x, g_x^-)$) is the Weyl chamber based on $x$ containing $gx$ (resp. $g^{-1}x$).

The aim of this section is to establish a type of closing lemma 3.3. The lemma below follows from [11, Proposition 3.1] and its proof. We need recall some constants first.
For all $x \in X$, we define $C_x = 8C_2C_1e^{C_0d_X(p,x)}$. $r_0$ is defined as the unique zero in $(0,1)$ of the function
\[ r \mapsto -\log r - \max\{C_3,2\}r. \]
For every $\varepsilon > 0$ and $x \in X$ we define a function $t_0(x,\varepsilon)$ such that
\[ 2\log C_x - 2\log \varepsilon = O(t_0(x,\varepsilon)). \]
For the specific values of constants $C_0, C_1, C_2, C_3$, please refer to [11].

**Theorem 3.2.** ([11, Proposition 3.1]) Let $x \in X$, $r \in (0,r_0)$ and $\varepsilon \in (0,\min\{C_x^{-1}r,\varepsilon_0\})$. Suppose that $\gamma \in G$ satisfies:

1. $a_{\gamma}(\gamma) \in a^+$ and $d(a_{\gamma}(\gamma),\partial a^+) \geq t_0(x,\varepsilon)$,
2. $d(\gamma_x^+,\gamma_x^-) \in \mathcal{F}(2)$ and $d(x,F_{c_\varepsilon+c_\varepsilon}) < r$.

Then $\gamma(\gamma_x^+) \in B(\gamma_x^+,\varepsilon)$, $\gamma^{-1}\mathcal{V}(\gamma^-_x) \in B(\gamma^-_x,\varepsilon)$, and the attracting and repelling point of $\gamma$ satisfy $\gamma^\pm \in B(\gamma^\pm_x,\varepsilon)$.

In the next section, we will count the number of elements in certain subsets of $\Gamma$. Let us collect the definitions here for convenience.

\[
\Gamma(t,\alpha,\beta) = \Gamma_0(t,\alpha,\beta) := \{\gamma \in \Gamma : S_\theta \cap \phi^{-t}\gamma B_{\delta}^{\alpha,\beta} \neq \emptyset\},
\]
\[
\Gamma^* = \Gamma^*_0 := \{\gamma \in \Gamma : \gamma F_{\theta} \subset F_{\theta} \text{ and } \gamma^{-1}P_{\theta} \subset P_{\theta}\},
\]
\[
\Gamma^*(t,\alpha,\beta) := \Gamma^* \cap \Gamma(t,\alpha,\beta),
\]
\[
\Gamma'(t,\alpha,\beta) := \{\gamma \in \Gamma^*(t,\alpha,\beta) : \gamma \neq \delta^\alpha \text{ for any } \delta \in \Gamma, n \geq 2\}.
\]

**Lemma 3.3** (Closing lemma). Let $M$ be a compact symmetric space of higher rank, $v_0, p, \varepsilon$ be fixed as in Section 2.3, and $\theta_1$ be small enough. Then for every $0 < \rho_1 < \theta < \theta_1$, there exists some $t_0 > 0$ such that for all $t \geq t_0$, we have $\Gamma^*_\rho(t,\alpha,\beta) \subset \Gamma^*_0$.

**Proof.** Take $\rho < \rho_1 < \theta$ and $0 < \varepsilon < \frac{\theta-\rho_1}{2}$. We claim that there exists some $t_0 > 0$ such that for all $t \geq t_0$ and $\gamma \in \Gamma$, if $S_\rho \cap \phi^{-t}\gamma B_{\rho}^{\alpha} \neq \emptyset$, then $\gamma(F_{\theta}) \subset F_{\theta}$.

Let us prove the claim. Assume not. Then for each $i$, there exist $t_i \rightarrow \infty$ and $\gamma_i \in \Gamma$ such that $v_i \in S_\rho \cap \phi^{-t_i}\gamma_i B_{\rho}^{\alpha}$, but $\gamma_i(F_{\theta}) \not\subset F_{\theta}$. Clearly, for any $x \in X$, $\gamma_i x$ goes to infinity. By passing to a subsequence, let us assume that $\gamma_i x \rightarrow \xi \in \partial X$.

Let $v_i \in S_\rho \cap \phi^{-t_i}\gamma_i B_{\rho}^{\alpha}$. By Lemma 2.14, $S_\rho$ and $B_{\rho}^{\alpha}$ are both compact. By passing to a subsequence, we may assume that $v_i \rightarrow v \in S_\rho$ and $\gamma_i^{-1}\phi^{t_i}v_i \rightarrow w \in B_{\rho}^{\alpha}$. Note that $\gamma_i \pi w \rightarrow \xi \in \partial X$. Since $d(\gamma_i w, \phi^{t_i}v_i) \rightarrow 0$, we have $\xi = \lim_i \pi \phi^{t_i}v_i$ and thus $\pi B(\xi) \in F_{\rho_i}$. Denote $u_i = \dot{\pi}w, \gamma_i \pi(w)(0)$. As $\lim_i u_i(+\infty) \rightarrow \xi$, we see that $(\gamma_i)_{\pi w} \in F_{\rho_1}$ for sufficiently large $i$ by Lemma 2.10.
We may assume that $\gamma_i^{-1} \pi v \to \eta \in \partial X$. Let $w_i = \gamma_i^{-1} \phi^t v_i \in \mathcal{P}_\rho$. Then $d(\gamma_i^{-1} v, \phi^{-t_i} w_i) = d(\gamma_i^{-1} v, \gamma_i^{-1} v_i) \to 0$, and thus $d(\gamma_i^{-1} v, \pi \phi^{-t_i} w_i) \to 0$. We then see that $\eta = \lim \pi \phi^{-t_i} w_i$ and $\pi^B(\eta) \in \mathcal{P}_\rho$. Denote $u_{-i} = \hat{c}_{\pi w, \gamma_i^{-1} \pi(w)}(0)$. As $\lim u_{-i}(+\infty) \to \eta$, we see that $(\gamma_i)_{-w} \in \mathcal{P}_{\rho_1}$ for sufficiently large $i$.

Let us verify the conditions (1) and (2) in Theorem 3.2 for $\gamma_i$.

As $u_i(\infty) \to \xi = v(\infty)$ and $v \in S_\rho$, by Remark 2.9, there exist isometries $g_i \in G$ such that $g_i u_i \to v$. So $u_i \in \mathfrak{a}^+$ and thus $\mathfrak{a}_x(\gamma_i) \in \mathfrak{a}^+$. By enlarging $i$ if necessary, we have $d(\mathfrak{a}_x(\gamma_i), \partial \mathfrak{a}^+) \geq t_0(\pi w, \varepsilon)$.

Since $((\gamma_i)_{\pi w}^+ (\gamma_i)_{\pi w}^-) \in \mathcal{P}_\rho_1 \times \mathcal{F}_\rho_1$, we see that $d_X(\pi w, F_{\gamma_i}((\gamma_i)_{\pi w}^- (\gamma_i)_{\pi w}^+)) < r$ by Lemma 2.13.

Applying Theorem 3.2, we have $\gamma_i^+ \in B((\gamma_i)_{\pi w}^+, \varepsilon)$ and $\gamma_i^- \in B((\gamma_i)_{\pi w}^-, \varepsilon)$. We claim that $\mathcal{F}_\theta \subset \mathcal{V}_\varepsilon ((\gamma_i)_{\pi w}^-) \subset B((\gamma_i)_{\pi w}^+, \varepsilon)$. The proof is an application of Corollary 3.11 that $\delta((\gamma_i)_{\pi w}^+, (\gamma_i)_{\pi w}^-) \geq C_{\pi w}^{-1}e^{-C_{3r}} \geq (8C_2C_1)^{-1}e^{-4C_{3r}}$. By Lemma 2.13, we choose $\theta > 0$ small enough so that $r$ is small. Moreover, $\varepsilon$ is fixed small, so by shrinking $\varepsilon$ at the beginning, we can get $\delta((\gamma_i)_{\pi w}^+, (\gamma_i)_{\pi w}^-) > 2\varepsilon$. Then $B((\gamma_i)_{\pi w}^+, 2\theta) \subset \mathcal{V}_\varepsilon ((\gamma_i)_{\pi w}^-)$. Then

$$\mathcal{F}_\theta \subset B((\gamma_i)_{\pi w}^+, 2\theta) \subset \mathcal{V}_\varepsilon ((\gamma_i)_{\pi w}^-).$$

Similarly, we can show $\mathcal{P}_\theta \subset \mathcal{V}_\varepsilon ((\gamma_i)_{\pi w}^+).$ This proves the claim.

So $\gamma_i \mathcal{F}_\theta \subset \mathcal{V}_\varepsilon ((\gamma_i)_{\pi w}^+ \subset B((\gamma_i)_{\pi w}^+, \varepsilon) \subset \mathcal{F}_\theta$. Similarly, and $\gamma_i^- \mathcal{P}_\theta \subset \mathcal{P}_\theta$. Thus $\gamma_i \in \Gamma_\theta^*$.\hfill $\square$

4. Using scaling and mixing

In this section, we use the scaling and mixing properties of Knieper measure $m$, to give an asymptotic estimates of $\# \Gamma^*(t, \alpha, \beta)$ and $\# \Gamma(t, \alpha, \beta)$.

4.1. Depth of intersection. To start, we want to show the relation between $t$ and $|\gamma|$ when $\gamma \in \Gamma^*(t, \alpha, \beta)$.

Given $\xi \in \partial X$ and $\gamma \in \Gamma$, define $b^\gamma_\xi := \beta_\xi(\gamma p, p)$.

**Lemma 4.1.** Let $\xi, \eta \in \mathcal{P}$ and $c \in \Gamma(t, \alpha)$ with $t > 0$. Then $|b^\gamma_\xi - b^\gamma_\eta|^2 < \varepsilon^2$.

**Proof.** The proof is an application of Corollary 2.15. The computation is completely parallel to that in [8, Lemma 4.11], and hence omitted here.\hfill $\square$

**Lemma 4.2.** Let $c$ be an axis of $\gamma \in \Gamma$ and $\xi = c(-\infty)$. Then $|b^\gamma_\xi| = |\gamma|$.
Proof. Take $q$ to be any point on $c$. Then $|\gamma| = d(\gamma q, q) = |b_{\xi}(\gamma q, q)|$. Since $\gamma \xi = \xi$, we have
\begin{align*}
\beta_{\xi}(\gamma q, q) - \beta_{\xi}(\gamma p, p) = & (\beta_{\xi}(\gamma q, \gamma p) + \beta_{\xi}(\gamma p, q)) - (\beta_{\xi}(\gamma p, q) + \beta_{\xi}(q, p)) \\
= & \beta_{\xi}(\gamma q, \gamma p) - \beta_{\xi}(\gamma q, \gamma p) = 0.
\end{align*}
So $|\gamma| = |\beta_{\xi}(\gamma q, q)| = |\beta_{\xi}(\gamma p, p)| = |b_{\xi}|$. \qed

Lemma 4.3. Given any $\gamma \in \Gamma$ and any $t \in \mathbb{R}$, we have
\[ S \cap \phi^{-t}_{\gamma} B^{\alpha, \beta} = \{ w \in E^{-1}(P \times \gamma F) : s(w) \in R_{\xi_{w}, \xi_{0}} \cap (b_{w}^{-} - tH_{w} + R_{\alpha, \beta}) \}. \]

Proof. Take any $\omega \in S \cap \phi^{-t}_{\gamma} B^{\alpha, \beta}$. Then $w \in S$ implies that $\pi^{B}(\omega^{-}) \in P$. Similarly, $w \in \phi^{-t}_{\gamma} B^{\alpha, \beta}$ implies that $\pi^{B}(\omega^{+}) \in \gamma F$. Thus $S \cap \phi^{-t}_{\gamma} B^{\alpha, \beta} \subseteq E^{-1}(P \times \gamma F)$.
\[ \text{Given } \omega \in E^{-1}(P \times \gamma F), \omega \in S \text{ if and only if } s(\omega) \in R_{\xi_{w}, \xi_{0}} \text{ by definition. It remains to show that if } \omega \in S, \text{ then } \omega \in \phi^{-t}_{\gamma} B^{\alpha, \beta} \iff s(\omega) \in b_{w}^{-} - tH_{w} + R_{\alpha, \beta}. \]
Since $\omega \in S$, $\beta_{\omega}(\pi \phi^{t}_{\omega}, \pi \omega) = tH_{w}$.
\[ \omega \in \phi^{-t}_{\gamma} B^{\alpha, \beta} \] \[ \iff \gamma^{-1}_{\phi^{t}_{\omega}} \omega \in B^{\alpha, \beta} \] \[ \iff \omega \in E^{-1}(\gamma P \times \gamma F) \text{ and } \beta_{\omega}(\pi \gamma^{-1}_{\phi^{t}_{\omega}} \omega, \pi \omega) \in R_{\alpha, \beta} \] \[ \iff \beta_{\omega}(\pi \phi^{t}_{\omega}, \gamma p) \in R_{\alpha, \beta} \] \[ \iff \beta_{\omega}(\pi \phi^{t}_{\omega}, p) - b_{w}^{-} \in R_{\alpha, \beta} \] \[ \iff \beta_{\omega}(\pi \omega, p) + tH_{w} - b_{w}^{-} \in R_{\alpha, \beta} \] \[ \iff s(\omega) \in b_{w}^{-} - tH_{w} + R_{\alpha, \beta}. \]
\[ \Box \]

Lemma 4.4. If $\gamma \in \Gamma^{*}(t, \alpha, \beta)$, then $|\gamma| \in [t - \alpha - e^{2}, t + 2e^{2} + \sqrt{r - 1}(\beta + e^{2})]$. 

Proof. Since $S \cap \phi^{-t}_{\gamma} B^{\alpha, \beta}$ is nonempty, by Lemma 4.3, there exist $\zeta \in P$ and $\tau \in R_{\xi_{w}, \xi_{0}}$ such that $\tau \in b_{\xi}^{\gamma} - tH_{w} + R_{\alpha, \beta}$. It follows that $b_{\xi}^{\gamma} \in tH + \tau - R_{\alpha, \beta}$. So
\[ |b_{\xi}^{\gamma}| \in [t - \alpha, t + e^{2} + \sqrt{r - 1}(\beta + e^{2})]. \]
By Lemmas 4.1 and 4.2, $|\gamma| \in [t - \alpha - e^{2}, t + 2e^{2} + \sqrt{r - 1}(\beta + e^{2})]$. \[ \Box \]

The following lemma implies that the intersections also have product structure.
Lemma 4.5. If $\gamma \in \Gamma^*(t, \alpha, \beta)$, then

$$S \cap \phi^{-(t+2\epsilon)} \gamma B^{\alpha+4\epsilon^2, \beta+4\epsilon^2} \supset H^{-(t+4\epsilon^2, \beta+4\epsilon^2)} := S'$$

Proof. Let $\gamma \in \Gamma^*(t, \alpha, \beta)$, then $S \cap \phi^{-t} \gamma B^{\alpha+\beta} \neq \emptyset$. By Lemma 4.3, there exists $\eta \in \mathcal{P}$ such that $(H = (0, \cdots, 0, 1))$

$$R_{\epsilon^2, \epsilon^2} \cap (b_{\eta} - tH + R_{\alpha, \beta}) \neq \emptyset.$$ 

It follows that $R_{\epsilon^2, \epsilon^2} \subset (b_{\eta} - tH - \epsilon^2H + R_{\beta+2\epsilon^2, \alpha+2\epsilon^2})$. Then by Lemma 4.1, for any $\xi \in \mathcal{P}$ we have

$$R_{\epsilon^2, \epsilon^2} \cap (b_{\xi} - tH - \epsilon^2H + R_{\beta+2\epsilon^2, \alpha+2\epsilon^2}) \neq \emptyset,$$

which in turn implies that

$$R_{\epsilon^2, \epsilon^2} \subset (b_{\xi} - tH - 2\epsilon^2H + R_{\beta+4\epsilon^2, \alpha+4\epsilon^2}).$$

We are done by Lemma 4.3. □

4.2. Scaling and mixing calculation. We use the following notations in the asymptotic estimates.

$$f(t) = e^{\pm C} g(t) \Leftrightarrow e^{-C} g(t) \leq f(t) \leq e^{C} g(t) \text{ for all } t;$$

$$f(t) \lesssim g(t) \Leftrightarrow \limsup_{t \to \infty} \frac{f(t)}{g(t)} \leq 1;$$

$$f(t) \gtrsim g(t) \Leftrightarrow \liminf_{t \to \infty} \frac{f(t)}{g(t)} \geq 1;$$

$$f(t) \sim g(t) \Leftrightarrow \lim_{t \to \infty} \frac{f(t)}{g(t)} = 1;$$

$$f(t) \sim e^{\pm C} g(t) \Leftrightarrow e^{-C} g(t) \lesssim f(t) \lesssim e^{C} g(t).$$

Lemma 4.6. There exists $C > 0$ such that if $\gamma \in \Gamma^*$, then

$$m(S^\gamma) = e^{\pm Ch\epsilon} e^{-h|\gamma|} m(S).$$

Proof. The main work is to estimate $\beta_p(\xi, \eta)$ and $b_\eta(\gamma^{-1} p, p)$ given $\xi \in \mathcal{P}, \eta \in \mathcal{F}$.

Firstly, consider the flat $F_{\xi\eta}$ connecting $\xi$ and $\eta$. Then

$$|\beta_p(\xi, \eta)| \leq C_3 d(p, F_{\xi\eta}) < \frac{C_3 \epsilon}{2}$$

where we used Lemma 2.14 in the last inequality.

Secondly, since $\gamma \in \Gamma^*$, we know $\gamma^{-1}$ has a regular axis $c$. By Lemma 4.2, $|\beta_{c(-\infty)}(\gamma^{-1} p, p)| = |\gamma^{-1}| = |\gamma|$. Then by Corollary 2.15, we have $||\beta_\eta(\gamma^{-1} p, p)| - \beta_{c(-\infty)}(\gamma^{-1} p, p)| < \epsilon^2$ for any $\eta \in \mathcal{F}$. 
Notice that since $\gamma F \subset F$, we have $\gamma \eta \in F$ if $\eta \in F$. Thus we have

$$\frac{m(S^\gamma)}{m(S)} = \frac{\bar{\mu}(P \times \gamma F)}{\bar{\mu}(P \times F)} = \frac{e^{+C_3 h e/2} \mu_\rho(P) \mu_\rho(\gamma F)}{e^{+C_3 h e/2} \mu_\rho(P) \mu_\rho(F)} = e^{+C_3 h e} \frac{\mu_{\gamma^{-1}p}(F)}{\mu_\rho(F)} = e^{+C_3 h e} \int_F e^{-2p_\rho(\gamma^{-1}p, p)} d\mu_\rho(\eta) = e^{+C_3 h e} e^{-h |\gamma|}$$

for some $C > \max\{C_3, 1\}$.

Combining Lemma 4.4 and Lemma 4.6, we have

**Corollary 4.7** (Scaling). Given $0 < \alpha \leq \frac{3\epsilon}{2}, 0 < \beta \leq \frac{3\epsilon}{2\epsilon}$ and $\gamma \in \Gamma^*(t, \alpha, \beta)$, we have $|t - |\gamma|| \leq 2\epsilon$, and thus

$$m(S^\gamma) = e^{+4C_3 h e} e^{-ht} m(S).$$

**Remark 4.8.** It is clear that the conclusions in Lemma 4.6 and Corollary 4.7 hold if $m, S, S^\gamma$ are replaced by $\bar{m}, \bar{S}, \bar{S}^\gamma$ respectively.

Finally, we combine scaling and mixing properties of Knieper measure to obtain the following asymptotic estimates. The proof is a repetition of [8, Section 5.2]. Since it is a key step, we provide a detailed proof here.

**Proposition 4.9.** We have

$$e^{-4C_3 h e} - De^{-ut} \lesssim \frac{\# \Gamma_\rho^*(t, \alpha, \beta)}{e^{ht} m(B_0^{\alpha, \beta})} \lesssim e^{4C_3 h e} (1 + \frac{4\epsilon^2}{\beta}) r^{-1} (1 + \frac{4\epsilon^2}{\alpha}) + De^{-ut},$$

$$e^{-4C_3 h e} - De^{-ut} \lesssim \frac{\# \Gamma_\rho(t, \alpha, \beta)}{e^{ht} m(B_0^{\alpha, \beta})} \lesssim e^{4C_3 h e} (1 + \frac{4\epsilon^2}{\beta}) r^{-1} (1 + \frac{4\epsilon^2}{\alpha}) + De^{-ut}.$$

**Proof.** Recall that $\alpha \in (0, \frac{3\epsilon}{2}], \beta \in (0, \frac{3\epsilon}{2\epsilon}]$. By Lemmas 3.3 and 4.5, for any $0 < \rho < \theta$ and $t$ large enough, we have

$$S_\rho \cap \phi^{-t} B_\theta^{\alpha, \beta} \subset \bigcup_{\gamma \in \Gamma^*_\rho(t, \alpha, \beta)} S_\rho^\gamma \cap \phi^{-(t+2\epsilon^2)} B_\theta^{\alpha+4\epsilon^2, \beta+4\epsilon^2}.$$

By Corollary 4.7, $m(S_\rho^\gamma) = e^{4C_3 h e} e^{-ht} m(S_\rho)$. Thus we have

$$e^{-4C_3 h e} m(S_\rho \cap \phi^{-t} B_\theta^{\alpha, \beta}) \leq \# \Gamma_\rho^*(t, \alpha, \beta) e^{-ht} m(S_\rho) \leq e^{4C_3 h e} m(S_\rho \cap \phi^{-(t+2\epsilon^2)} B_\theta^{\alpha+4\epsilon^2, \beta+4\epsilon^2}).$$

Dividing by $m(S_\rho)m(B_\theta^{\alpha, \beta})$ and using mixing of $\bar{m}$ (cf. Lemma 2.12(3)), we get

$$e^{-4C_3 h e} \frac{m(S_\rho)m(B_\theta^{\alpha, \beta})}{m(S_\rho)m(B_\theta^{\alpha, \beta})} - De^{-ut} \lesssim \frac{\# \Gamma_\rho^*(t, \alpha, \beta)}{e^{ht} m(B_\theta^{\alpha, \beta})} \lesssim e^{4C_3 h e} \frac{m(B_\theta^{\alpha+4\epsilon^2, \beta+4\epsilon^2})}{m(B_\theta^{\alpha, \beta})} + De^{-ut}$$
for some $u > 0$ and $D > 0$. By (3), letting $\rho \searrow \theta$, we obtain the first equation in the proposition.

To prove the second equation, we consider $\theta \prec \rho \prec \theta_0$. Then by Lemma 3.3, $\Gamma^*_\theta(t, \alpha, \beta) \subset \Gamma_\theta(t, \alpha, \beta) \subset \Gamma^*_\rho(t, \alpha, \beta)$. By (4),

$$e^{-4\chi_{\rho}m(S_\rho)m(B^\alpha_\rho, \beta)} - D e^{-ut} \lesssim \frac{\# \Gamma^*_\theta(t, \alpha, \beta)}{e^{ht}m(B^\alpha_\theta, \beta)} \lesssim \frac{\# \Gamma_\theta(t, \alpha, \beta)}{e^{ht}m(B^\alpha_\theta, \beta)} \lesssim \frac{e^{4\chi_{\rho}m(B^{\alpha+4\epsilon^2, \beta+4\epsilon^2}_\rho)} - D e^{-ut}}{m(B^{\alpha, \beta}_\rho)}.$$

Letting $\rho \searrow \theta$ and by (3), we get the second equation in the proposition. □

**Remark 4.10.** The constant $D$ depends on the flow boxes. But we can choose a uniform $D$ for all $\rho \searrow \theta$ and $\rho \prec \theta$ by approximating characteristic functions by smooth functions.

## 5. Measuring along periodic orbits

Recall that $C(t)$ is any maximal set of pairwise non-free-homotopic closed geodesics with length $(t - \epsilon, t]$ in $M$. We obtain in this section an upper bound and a lower bound respectively for $\#C(t)$.

### 5.1. Upper bound for $\#C(t)$

By definition of $\nu_t$, we have

$$\#C(t) = \frac{\sum_{\varepsilon \subset C(t)} \text{Leb}_\varepsilon(B^\alpha_\rho)}{t \nu_t(B^\alpha_\rho)}.$$

Define

$$\Pi(t) := \{ \dot{c}(s) \in \text{pr} H^{-1}(P \times F \times \{0\}^r) : c \in C(t), s \in \mathbb{R} \}.$$

Then we have

$$(5) \quad \#C(t) = \frac{\alpha \# \Pi(t)}{t \nu_t(B^\alpha_\rho)}.$$  

Now we define a map $\Theta: \Pi(t) \to \Gamma(t, \epsilon, \epsilon)$ as follows. Given $\bar{v} \in \Pi(t)$, let $\ell = \ell(\bar{v}) \in (t - \epsilon, t]$ be such that $\phi^\ell \bar{v} = \bar{v}$. Let $v$ be the unique lift of $\bar{v}$ such that $v \in H^{-1}(P \times F \times \{0\}^r) \subset B^\alpha_\rho$. Define $\Theta(v)$ to be the unique axial isometry of $X$ such that $\phi^\ell v = \Theta(v)v$. Then $|\Theta(v)| = \ell$. If $\gamma = \Theta(v)$, then $\phi^\ell v = \phi^{\ell-\gamma} \gamma v \in \gamma B^\epsilon_\rho$. So $v \in S_\theta \cap \phi^{-t} \gamma B^\epsilon_\rho$, and we get

$$(6) \quad \Theta(\Pi(t)) \subset \Gamma(t, \epsilon, \epsilon).$$

To estimate $\#\Pi(t)$, we first show that $\Theta$ is injective.

**Lemma 5.1.** $\Theta$ is injective.
Proof. Suppose that \( v, w \in \Pi(t) \) are such that \( \Theta(v) = \Theta(w) =: \gamma \). Let \( v, w \in B_{0,3}^\alpha \) be the lifts of \( v, w \) respectively. Then by definition, both \( c_v \) and \( c_w \) are axes of \( \gamma \).

We claim that \( v^+ = w^+ \) and \( v^- = w^- \). Indeed, first notice that \( \gamma^n c_v(0) \rightarrow v^+ \) and \( \gamma^n c_w(0) \rightarrow w^+ \). On the other hand,

\[
d(\gamma^n c_v(0), \gamma^n c_w(0)) = d(c_v(0), c_w(0))
\]

for all \( n \geq 0 \). So \( \gamma^n c_v(0) \) and \( \gamma^n c_w(0) \) converge to the same point at infinity. Thus we get \( v^+ = w^+ \). Similarly, \( v^- = w^- \).

It follows that \( c_v \) and \( c_w \) are bi-asymptotic. If \( c_v \) and \( c_w \) are geometrically distinct, then they bound a flat strip by Lemma 2.1(4). Since \( v \) and \( w \) are both regular vectors, they lie in a common flat. So there exist \( \xi, \eta \in F \) such that \( v, w \in H^{-1}(\{\xi\} \times \{\eta\} \times \{0\}^r) \). Hence \( v = w \) and \( v = w \). So \( \Theta \) is injective. \( \square \)

Proposition 5.2. We have

\[
\#C(t) \leq \frac{\epsilon \#\Gamma(t, \epsilon, \epsilon)}{t \nu(B_{\epsilon, \epsilon}^r)}.
\]

Proof. The proposition follows from (5), (6) and Lemma 5.1. \( \square \)

5.2. Lower bound for \( \#C(t) \). First we deal with the multiplicity of \( \gamma \in \Gamma \). Given \( \gamma \in \Gamma \), let \( d = d(\gamma) \in \mathbb{N} \) be maximal such that \( \gamma = \delta^d \) for some \( \delta \in \Gamma \). \( \gamma \in \Gamma \) is called primitive if \( d(\gamma) = 1 \), i.e., \( \gamma \neq \delta^d \) for any \( \delta \in \Gamma \) and any \( d \geq 2 \).

Define \( \Gamma_2(P, F, t) \) to be the set of all \( \gamma \in \Gamma \) such that

1. \( \gamma \) has an axis \( c \) with \( \pi^B c(-\infty) \in P, \pi^B c(\infty) \in F \);
2. \( |\gamma| \in (t - \epsilon, t) \);
3. \( d(\gamma) \geq 2 \).

Lemma 5.3. There exists \( K > 0 \) such that for any \( t > 0 \) we have

\[
\sum_{\gamma \in \Gamma_2(P, F, t)} d(\gamma) \leq Ke^{\frac{2}{3}ht}.
\]

Proof. Since \( X \) is a simply connected manifold of nonpositive curvature, we have by [15]

\[
h = \lim_{r \rightarrow \infty} \frac{1}{r} \log \text{Vol}(B(q, r))
\]

where \( B(q, r) \) is the ball of radius \( r \) around \( q \) in \( X \). So the proof is almost analogous to that in [8, Lemma 4.5] with only minor modifications as follows.

If \( \gamma = \delta(\gamma)^d(\gamma) \) for some \( \delta(\gamma) \in \Gamma \), we must argue that \( \delta(\gamma) \) has an axis with endpoints in \( P \) and \( F \), so that we can choose \( v \in H^{-1}(P \times F \times \{0\}^r) \) tangent to such an axis with \( \phi^{d(\gamma)} v = \delta(\gamma) v \). We cannot use [8, Lemma
2.10] in our setting. Nevertheless, since $\gamma$ has a regular axis $c$ with $\hat{c}(0) \in H^{-1}(\mathbf{P} \times \mathbf{F} \times \{0\})$, we know from the proof of Lemma 5.1 that all axes of $\gamma$ are parallel to $c$ and they are tangent to a common flat. Every axis of $\delta(\gamma)$ is an axis of $c$, so $c$ is an axis of $\delta(\gamma)$, as we want.

Repeat the remaining part of the proof of [8, Lemma 4.5] and we are done. □

Recall that $\Gamma'(t, \alpha, \beta) := \{\gamma \in \Gamma^*(t, \alpha, \beta) : \gamma \neq \delta^n \text{ for any } \delta \in \Gamma, n \geq 2\}.

**Lemma 5.4.** Let $a = 2 + 2\sqrt{r - 1}$ and $\alpha = \epsilon - ae^{3/2} - 2e^2$. Then $\Theta(\Pi(t)) \supset \Gamma'(t - ae^{3/2}, \alpha, e^{3/2})$ and

$$\#C(t) \geq \frac{\alpha \#\Gamma'(t - ae^{3/2}, \alpha, e^{3/2})}{tv_1(\mathcal{B}^{\alpha, e^{3/2}})}.$$

**Proof.** Let $\gamma \in \Gamma'(t - ae^{3/2}, \alpha, e^{3/2})$. Then there exists $v \in H^{-1}(\mathbf{P} \times \mathbf{F} \times \{0\})$ such that $\phi^{(1)}v = \gamma v$. By Lemma 4.4, we have

$$|\gamma| \geq (t - ae^{3/2}) - \alpha - e^2 > t - \epsilon,$$

$$|\gamma| \leq (t - ae^{3/2}) + 2e^2 + 2\sqrt{r - 1}e^{3/2} < t.$$

It follows that $\mathcal{C}_v$ is a closed geodesic with length $|\gamma| \in (t - \epsilon, t)$. Note that if $\mathcal{C}$ is another closed geodesic in the free-homotopic class of $\mathcal{C}_v$, then we can lift $\mathcal{C}$ to a geodesic $c$ such that $c$ and $c_v$ are bi-asymptotic. So $c$ and $c_v$ bound a flat strip by Lemma 2.1(4). Since $v$ is regular, we see that $c$ and $c_v$ are in a common flat. As a consequence, $v \in \Pi(t)$ and $\gamma = \Theta(v)$. So $\Theta(\Pi(t)) \supset \Gamma'(t - ae^{3/2}, \alpha, e^{3/2})$ and thus by (5),

$$\#C(t) \geq \frac{\alpha \#\Gamma'(t - ae^{3/2}, \alpha, e^{3/2})}{tv_1(\mathcal{B}^{\alpha, e^{3/2}})}.$$ □

**Proposition 5.5.** Consider $\alpha = \epsilon - ae^{3/2} - 2e^2$. We have

$$\#C(t) \geq \frac{\alpha}{tv_1(\mathcal{B}^{\alpha, e^{3/2}})} \cdot (\#\Gamma^*(t - ae^{3/2}, \alpha, e^{3/2}) - Ke^{2ht}).$$

**Proof.** From the proof of Lemma 5.4, we also see that $|\gamma| \in (t - \epsilon, t]$ if $\gamma \in \Gamma^*(t - ae^{3/2}, \alpha, e^{3/2})$. Thus

$$\Gamma^*(t - ae^{3/2}, \alpha, e^{3/2}) \setminus \Gamma'(t - ae^{3/2}, \alpha, e^{3/2}) \subset \Gamma_2(\mathbf{P}, \mathbf{F}, t).$$

Then by Lemma 5.3,

$$\#\Gamma^*(t - ae^{3/2}, \alpha, e^{3/2}) - \#\Gamma'(t - ae^{3/2}, \alpha, e^{3/2}) \leq Ke^{2ht}.$$ This together with Lemma 5.4 proves the proposition. □
6. Equidistribution and completion of the proof

The following result is standard in ergodic theory, which is a corollary of the classical proof of variational principle [40, Theorem 9.10].

**Lemma 6.1.** [14, Proposition 4.3.12] Let \( Y \) be a compact metric space and \( \phi \) a continuous flow on \( Y \). Fix \( \epsilon > 0 \) and suppose that \( E_t \subset Y \) is a \((t, \epsilon)\)-separated set for all sufficiently large \( t \). Define the measures \( \mu_t \) by

\[
\mu_t(A) := \frac{1}{\#E_t} \sum_{v \in E_t} \frac{1}{t} \int_0^t \chi_A(\phi^s v) \, ds.
\]

If \( t_k \to \infty \) and the weak* limit \( \mu = \lim_{k \to \infty} \mu_{t_k} \) exists, then

\[
h_\mu(\phi^1) = \limsup_{k \to \infty} \frac{1}{t_k} \log \#E_{t_k}.
\]

*Proof of Theorem B.* We claim that the set \( \{ \hat{c}_1(0) : c \in C(t) \} \) is \((t, \epsilon)\)-separated for any \( 0 < \epsilon < \inj(M)/2 \).

Indeed, if it were not, then \( C(t) \) would contain two closed geodesics \( \hat{c}_1, \hat{c}_2 \) in distinct free-homotopic classes such that \( d(\hat{c}_1(s), \hat{c}_2(s)) \leq \epsilon < \inj(M)/2 \) for all \( s \in [0, t] \). Define \( v = \hat{c}_1(0) \) and \( w = \hat{c}_2(0) \). We can lift \( v, w \) to \( v, w \in SX \), and \( \hat{c}_1, \hat{c}_2 \) to \( c_1, c_2 \) respectively, such that \( d(c_1(s), c_2(s)) \leq \epsilon \) for all \( s \in [0, t] \). Moreover, there exist \( \gamma_1, \gamma_2 \in \Gamma \) and \( t_1, t_2 \in (t - \epsilon, t] \) such that \( \gamma_1 v = g^{t_1} v \) and \( \gamma_2 w = g^{t_2} w \). Then

\[
d(\gamma_2^{-1}\gamma_1 c_1(0), c_2(0)) = d(\gamma_2^{-1} c_1(t_1), \gamma_2^{-1} c_2(t_2)) = d(c_1(t_1), c_2(t_2)) \leq d(c_1(t_1), c_2(t_1)) + |t_2 - t_1|.
\]

Hence \( d(\gamma_2^{-1}\gamma_1 c_1(0), c_2(0)) \leq 3\epsilon < 2\inj(M) \), which is possible only if \( \gamma_1 = \gamma_2 \). Then \( c_1 \) and \( c_2 \) are both axes for a common \( \gamma = \gamma_1 = \gamma_2 \). By the proof of Lemma 5.1, we see that \( c_1 \) and \( c_2 \) must be bi-asymptotic and consequently bound a flat strip. It follows that \( \hat{c}_1 \) and \( \hat{c}_2 \) are free-homotopic. A contradiction, so the claim holds.

Now by Propositions 5.5 and 4.9, we know

\[
\#C(t) \geq \frac{\alpha}{tv_t(B^{\alpha, \epsilon^{3/2}})} \cdot \left( \#\Gamma^*(t - a\epsilon^{3/2}, \alpha, \epsilon^{3/2}) - Ke^{\frac{2}{3}ht} \right)
\]

\[
\geq \frac{\alpha}{tv_t(B^{\alpha, \epsilon^{3/2}})} \cdot \left( (e^{-4C'h\epsilon} - De^{-ut})e^{ht}m(B^{\alpha, \epsilon^{3/2}}) - Ke^{\frac{2}{3}ht} \right)
\]

for some \( C' > C \). So \( \liminf_{t \to \infty} \frac{1}{t} \log \#C(t) \geq h \). Applying Lemma 6.1, we know any limit measure of \( \nu_t \) has entropy equal to \( h \), and thus it must be \( m \), the unique MME. This proves Theorem B. \( \square \)

**Proposition 6.2.** We have

\[
\#C(t) \sim (e^{Qe^{1/2}} + O(e^{-ut})) \frac{e^{ht}}{t^e}
\]
where $Q > 0$ is a universal constant depending only on $h$.

**Proof.** By Theorem B and (3), we have $\nu_t(B^\alpha,\beta) \to m(B^\alpha,\beta) = m(B^0,\beta)$ for $\alpha = \epsilon$ and $\alpha = \epsilon - a\epsilon^{3/2} - 2\epsilon^2$.

Consider $\alpha = \epsilon$. By Propositions 5.2 and 4.9, we have

$$\#C(t) \lesssim \frac{\epsilon \# \Gamma(t,\epsilon,\epsilon)}{tm(B^{\epsilon^3/4})} \lesssim (e^{4Ch\epsilon}(1 + 4\epsilon)^r + De^{-ut}) \frac{\epsilon}{t} e^{ht}.$$ 

Now consider $\alpha = \epsilon - a\epsilon^{3/2} - 2\epsilon^2$. By Propositions 5.5 and 4.9,

$$\#C(t) \gtrsim \frac{\alpha}{tm(B^{\epsilon^3/2})} \cdot (\# \Gamma^*(t - a\epsilon^{3/2}, \alpha, \epsilon^{3/2}) - Ke^{\frac{3}{4}ht}) \gtrsim (1 - 2\epsilon - a\epsilon^{1/2})(e^{-4C''\epsilon h} - D e^{-ut}) \frac{\epsilon}{t} e^{ht}.$$

If $\epsilon > 0$ is small enough, $1 - 2\epsilon - a\epsilon^{1/2} > e^{-C''\epsilon^{1/2}}$ for some $C'' > 0$. So there exists a universal $Q > 0$ depending only on $h$ such that $\#C(t) \sim (e^{Q\epsilon^{1/2}} + O(e^{-ut})) \frac{\epsilon}{h} e^{ht}$. \hfill $\square$

**Proof of Theorem A.** The last step of the proof of Theorem A is to estimate $\#P(t)$ via $\#C(t)$ and a Riemannian sum argument. Indeed, a verbatim repetition of the proof in [8, Section 6.2] gives

$$\#P(t) \sim (e^{\pm 2(Q\epsilon^{1/2} + h\epsilon)} + O(e^{-ut})) \frac{e^{ht}}{ht}.$$ 

Since $\epsilon > 0$ can be arbitrarily small, we get $\#P(t) = \frac{e^{ht}}{ht}(1 + O(e^{-ut}))$ which completes the proof of Theorem A. \hfill $\square$

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