Phase transition and thermodynamical geometry for Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime

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Abstract

We study thermodynamics and thermodynamic geometry of a five-dimensional Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime by treating the cosmological constant as the number of colors in the boundary gauge theory and its conjugate quantity as the associated chemical potential. It is found that the chemical potential is always negative in the stable branch of black hole thermodynamics and it has a chance to be positive, but appears in the unstable branch. We calculate scalar curvatures of the thermodynamical Weinhold metric, Ruppeiner metric and Quevedo metric, respectively and we find that the divergence of scalar curvature is related to the divergence of specific heat with fixed chemical potential in the Weinhold metric and Ruppeiner metric, while in the Quevedo metric the divergence of scalar curvature is related to the divergence of specific heat with fixed number of colors and the vanishing of the specific heat with fixed chemical potential.

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I. INTRODUCTION

The thermodynamical properties of black holes in anti-de Sitter (AdS) space are quite different from those of black holes in asymptotically flat or de Sitter space. The main reason is that the AdS space acts as a confined cavity so that black holes in AdS space can be thermodynamically stable. In particular, there exists a minimal Hawking temperature for a Schwarzschild black hole in AdS space, below which there does not exist any black hole solution, instead a stable thermal gas solution exists. For a given temperature above the minimal one, there exist two black hole solutions. The black hole with smaller horizon is thermodynamically unstable with a negative heat capacity, while the black hole with larger horizon is thermodynamically stable with a positive heat capacity. And Hawking and Page find that a phase transition, named Hawking-Page phase transition, will happen between the stable large black hole and thermal gas in AdS space [1]. According to AdS/CFT correspondence, which says that there is an equivalence between a weakly coupled gravitational theory in $d$-dimensional AdS spacetime and a strongly coupled conformal field theory (CFT) in a $(d-1)$-dimensional boundary of the AdS space [2–5] (for a review, see [6]), thermodynamical properties of black holes in AdS space can be identified with those of dual strongly coupled CFT. The Hawking-Phase phase transition for black holes in AdS space is interpreted as the confinement/deconfinement phase transition in gauge theory [5]. Thus it becomes quite interesting to study thermodynamics and phase structure of black holes in AdS space. Indeed, in the past few years there have been a lot of works on thermodynamics and phase transition for black holes in AdS space.

In ordinary thermodynamic systems, a divergence of heat capacity is usually associated with a second order phase transition. For Kerr-Newmann black holes in Einstein-Maxwell theory, some heat capacities diverge at some black hole parameters. Based on this, Davies argued that some second order phase transitions will happen in Kerr-Newmann black holes [7]. For a Reissner-Nordström AdS (RN-AdS) black hole, such a phase transition was studied in some details in [8]. In a canonical ensemble with a fixed charge, it was found that there exists a phase transition between small and large black holes. This phase transition be-
haves very like the gas/liquid phase transition in a Van der Waals system [8, 9]. However, an identification between RN-AdS black hole and Van der Waals system was recently realized in [11], where the negative cosmological constant plays the role as pressure, while its conjugate acts as thermodynamic volume of the black hole in the so-called extended phase space [12, 13] (For a recent review, see [14]). Recently, thermodynamics and phase transition in the extended phase space for black holes in AdS space have been extensively studied in the literature [15].

In the framework of AdS/CFT correspondence, the negative cosmological constant is related to the degrees of freedom of dual CFT. Thus it is an interesting question as to whether the interpretation of the cosmological constant as pressure is applicable to the boundary CFT. Very recently, it was argued that it is more suitable to view the cosmological constant as the number of colors in gauge field and its conjugate as associated chemical potential [16–18]. This interpretation was examined in the case of $AdS_5 \times S^5$, for $\mathcal{N}=$4 supersymmetric Yang-Mills theory at large $N$ in [16]. The chemical potential conjugate to the number of colors, is calculated. It is found that the chemical potential in the high temperature phase of the Yang-Mills theory is negative and decreases as temperature increases. For spherical black holes in the bulk the chemical potential approaches zero as the temperature is lowered below the Hawking-Page temperature and changes its sign at a temperature near the temperature at which the heat capacity diverges.

On the other hand, applying the geometrical ideas to ordinary thermodynamical systems gives us an alternative way to study phase transition in those systems. Weinhold [19] first introduced a sort of metric defined as the second derivatives of internal energy with respect to entropy and other extensive quantities of a thermodynamic system. Soon later, based on the fluctuation theory of equilibrium thermodynamics, Ruppeiner [20] introduced another metric which is defined as the minus second derivatives of entropy with respect to the internal energy and other extensive quantities of a thermodynamic system. It was argued that the scalar curvature of the Ruppeiner metric can reveal the micro interaction and its divergence is related to some phase transition in the thermodynamical system [21]. In addition, it was shown that the Weinhold metric is conformal to the Ruppeiner metric [22]. However, both of
the Weinhold metric and Ruppeiner metric are not invariant under Legendre transformation and sometimes contradictory results will be produced [23, 24]. In order to solve this puzzle, Quevedo et al. [25–28] proposed a method to obtain a new formulism of Geometrothermodynamics whose metric is Legendre invariant in the space of equilibrium states. To the best of our knowledge, applying the thermodynamical geometry to black hole thermodynamics was initiated in [29], there it was found that the Weinhold metric is proportional to the metric on the moduli space for supersymmetric extremal black holes, whose Hawking temperature is zero, and the Ruppeiner metric governing fluctuations naively diverges, which is consistent with the argument that near the extremal limit, the thermodynamical description breaks down. Applying the thermodynamical geometry approach to phase transition of black holes was followed in [30, 31], and for more relevant references see the recent review [32] and references therein.

In this paper, we will study thermodynamics and thermodynamical geometry for a five-dimensional Schwarzschild AdS black hole in $AdS_5 \times S^5$ by viewing the number of colors as a thermodynamical variable from the view of point of dual CFT. For this, we will calculate energy density and entropy density for the dual CFT and then obtain the chemical potential associated with the number of colors. The main difference from that in [16] is that we remove the effect of the change of space volume where the dual CFT lives when one varies the number of colors. For details see the next section. In Sec. III we will calculate the thermodynamical curvatures of the Weinhold metric, Ruppeiner metric and Quevedo metric, respectively, for the thermodynamical system, in order to see the relations between the thermodynamical curvature and phase transition. Note that such calculations can not be done if one views the cosmological constant as a true constant, or even in the case of the extended phase space in the sense [12, 13], because in the latter case, the heat capacity $C_V$ always vanishes. We end the paper with conclusions in Sec. IV.
II. THERMODYNAMICS OF SCHWARZSCHILD ADS BLACK HOLE IN $AdS_5 \times S^5$

In $AdS_5 \times S^5$ spacetime, the line element for a five-dimensional Schwarzschild AdS black hole reads \[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij}dx^i dx^j + L^2 d\Omega_5^2, \] (1)

where $d\Omega_5^2$ is the metric of a five-dimensional sphere with unit radius, $h_{ij}dx^i dx^j$ is the line element of a three-dimensional Einstein space $\Sigma_3$ with constant curvature $6k$, and the metric function $f(r)$ is given by

\[ f(r) = k - \frac{m}{r^2} + \frac{r^2}{L^2}. \] (2)

where $L$ is the $AdS$ radius and $m$ is an integration constant. The cosmological constant is $\Lambda = -6/L^2$. Without loss of generality, one can take the scalar curvature parameter $k$ of the three-dimensional space $\Sigma_3$ as $k = 1, 0, \text{or} -1$, respectively. The ten-dimensional spacetime (1) can be viewed as the near horizon geometry of $N$ coincident $D3$-branes in type IIB supergravity. In that case, the AdS radius $L$ has a relation to the number $N$ of $D3$-branes \[ L^4 = \sqrt{2} N \ell_p^4 \equiv \alpha^2 N, \] (3)

where $\ell_p$ is the ten-dimensional Planck length. According to AdS/CFT correspondence, the spacetime (1) can be regarded as the gravity dual to $N=4$ supersymmetric Yang-Mills theory. Then $N$ is nothing, but the rank of the gauge group of the supersymmetric $SU(N)$ Yang-Mills Theory. In the large $N$ limit, the number of degrees of freedom of the $N=4$ supersymmetric Yang-Mills theory is proportional to $N^2$ (in fact, it is that of $8N^2$ massless bosons and fermions in the weak coupling limit [33]).

The event horizon $r_h$ of the black hole is determined by the equation $f(r) = 0$. Then according to Eq. (2), the mass of black hole can be expressed as

\[ M = \frac{3\omega_3}{16\pi G_5} m = \frac{3\omega_3 r_h^2}{16\pi G_5 L^2} (kL^2 + r_h^2), \] (4)

where $\omega_3$ is the volume of $\Sigma_3$. Using the Bekenstein-Hawking entropy formula of black hole,
we have \[ S = \frac{A}{4G_5} = \frac{\omega_3 r_h^3}{4G_5}. \] (5)

Note that \( G_5 = G_{10}/(\pi^3 L^5) \) and \( G_{10} = \frac{r^8}{\ell_p} \). Furthermore, let us notice that the dual CFT to the Schwarzschild AdS black hole lives in the AdS boundary with a metric (up to a conformal factor)

\[ ds^2 = -dt^2 + L^2 h_{ij} dx^i dx^j. \] (6)

Namely the CFT lives in a space with volume \( V_3 = \omega_3 L^3 \). We see that the volume depends on the number of colors \( N^2 \). In order to remove the effect of volume change when one varies the number of colors, let us consider the energy density and entropy density of the dual CFT as

\[ \rho = \frac{M}{V_3} = \frac{3\pi^2 L^2}{16G_{10}} r_h^2(k + \frac{r_h^2}{L^2}), \]

\[ s = \frac{S}{V_3} = \frac{\pi^3}{4G_{10}} L^2 r_h^3. \] (7)

Note that the Hawking temperature of the black hole is given by

\[ T = \frac{1}{2\pi r_h} \left( k + 2\frac{r_h^2}{L^2} \right). \] (9)

And then we see that when \( k = 0 \), or say, in the large horizon radius limit, one has

\[ \rho \sim N^2 T^4, \quad s \sim N^2 T^3. \] (10)

This is expected from the dual CFT side. Note that in [16], the mass and entropy of the black hole do not have the scaling relation as that in Eq. (10), in the case of \( k = 0 \). This is the main deference from that in [16]. The energy density can be expressed in terms of entropy density and the number of colors as

\[ \rho = \frac{3\pi^2}{16G_{10}} \left( \frac{4G_{10}}{\pi^3 \alpha} \right)^{2/3} \left[ \alpha k N^{1/6} s^{2/3} + N^{-2/3} s^{4/3} \left( \frac{4G_{10}}{\pi^3 \alpha} \right)^{2/3} \right]. \] (11)

The temperature of the CFT can be calculated as

\[ T = \left( \frac{\partial \rho}{\partial s} \right)_{N^2} = \frac{\pi^2}{8G_{10}} \left( \frac{4G_{10}}{\pi^3 \alpha} \right)^{2/3} \left[ \alpha k N^{1/6} s^{-1/3} + 2N^{-2/3} s^{1/3} \left( \frac{4G_{10}}{\pi^3 \alpha} \right)^{2/3} \right]. \] (12)
It is easy to check that this temperature is just the Hawking temperature Eq. (9) of the black hole. The chemical potential $\mu$ conjugate to the number of colors is

$$\mu = \left. \left( \frac{\partial \rho}{\partial N^2} \right) \right|_s = \frac{\pi^2}{16 G_{10}} \left( \frac{4G_{10}}{\pi^3 \alpha} \right)^{2/3} \left[ \frac{1}{4} \alpha k N^{-11/6} s^{2/3} - N^{-8/3} s^{4/3} \left( \frac{4G_{10}}{\pi^3 \alpha} \right)^{2/3} \right],$$

which is the measure of the energy cost to the system of increasing the number of colors. Note that here we simply take $N^2$ as the number of colors following [16]. Thus we have the first law of thermodynamics

$$d\rho = T ds + \mu dN^2. \quad (14)$$

The Gibbs free energy density can be calculated as

$$\mathcal{G} = \rho - Ts = \frac{\alpha \pi^2}{16 G_{10}} \left( \frac{4G_{10}}{\pi^3 \alpha} \right)^{2/3} N^{1/6} s^{2/3} \left[ k - \left( \frac{1}{\alpha^2 N} \right)^{5/6} \left( \frac{4G_{10} s}{\pi^3} \right)^{2/3} \right]. \quad (15)$$

For convenience in the later discussions, we introduce the notation $D = \left[4G_{10}/(\pi^3 \alpha)\right]^{2/3}$ in the following.

For the cases of $k = 0$ and $k = -1$, it is easy to see from Eq. (12) for a fixed $N^2$ that the Hawking temperature increases monotonically with the entropy density $s$. However, in the case of $k = 1$, the Hawking temperature is not a monotonic function but has a minimum at

$$s_1 = \alpha^{3/2} N^{5/4} / (2 \sqrt{2} D^{3/2}),$$

equivalently, at $r_h = L/\sqrt{2}$. We plot the behavior of temperature with respective to entropy density in Fig. 1. The corresponding minimal temperature is $T_{\text{min}} = \sqrt{2}/(\pi L)$. Namely under the minimal temperature there is no black hole solution. Above the minimal temperature, there exist two branches, as we will see shortly, the branch with small entropy (horizon radius) is thermodynamically unstable, while the branch with large entropy (horizon radius) is thermodynamically stable.

One can see easily from the Gibbs free energy density Eq. (15) that when $k = 1$, $\mathcal{G} < 0$ if $r_h > L$, while it is positive as $r_h < L$. The Hawking-Page phase transition happens at $r_h = L$ with the phase temperature $T_{\text{HP}} = 3/(2\pi L)$, which is larger than $T_{\text{min}}$. And the corresponding entropy density for the Hawking-Page transition is

$$s_2 = \alpha^{3/2} N^{5/4} / D^{3/2}.$$
FIG. 1: The temperature with respect to entropy density. Here we take $\ell_p = 1$, $k = 1$ and $N = 3$. The temperature arrives at the minimal value when $s = s_1 = N^{5/4}\alpha^{3/2}/(2\sqrt{2}D^{3/2}) \approx 0.9539$.

We can see that $s_2 > s_1$. On the other hand, when $k = 0$ or $k = -1$, the Gibbs free energy density is always negative, no phase transition happens. In Fig. 2, we show the Gibbs free energy density with respect to the Hawking temperature $T$ for some fixed $N$. For a fixed $N$, there exist two branches. The upper branch corresponding to the small entropy black holes is thermodynamically unstable, while the down one is thermodynamically stable with large black hole entropy.

Now we study the chemical potential conjugate to the number of colors. We see from Eq. (13) that when $k = 0$ or $k = -1$, the chemical potential is always negative. When $k = 1$, however, the situation is changed. In Fig. 3 we show the chemical potential as a function of entropy density $s$ for a fixed $N$. We see that the chemical potential is positive when $s$ is small, while it changes to be negative when $s$ is large. The chemical changes its sign at

$$s_3 = \alpha^{3/2}N^{5/4}/(8D^{3/2}).$$

We see that

$$s_3 < s_1 < s_2.$$  

In the large black hole horizon limit, we have from Eq. (13) that $\mu \sim -T^4$, which is independent of $N$, while in the same limit the chemical potential depends on $N$ in [16]. Indeed for a classical gas, the chemical potential is always negative, and will become more negative as temperature increases. When quantum effect comes into play, the chemical potential may
FIG. 2: The Gibbs free energy density as a function of the temperature for various numbers of colors. Here we take \( \ell_p = 1 \) and \( k = 1 \). The down branch Gibbs free energy density for a fixed \( N \) changes its sign at the point \( s = s_2 = N^{5/4} \alpha^{3/2} / D^{3/2} \), which corresponds to the Hawking-Page transition point.

change its sign and become positive [34]. It was argued that the vanishing of the chemical potential would be a signal of Bose-Einstein condensation for a bosonic system or playing a role of the exclusions principle for a fermionic system [16]. But we see that the vanishing of the chemical potential appears in the unstable branch. This implies that the vanishing of the chemical potential does not make any sense from the point of view of dual supersymmetric Yang-Mills theory. In Fig. 4 we plot the chemical potential as a function of temperature \( T \) for a fixed \( N \), while in Fig. 5 the chemical potential is plotted as a function of \( N \) in the case with a fixed entropy density \( s \). We can see that when

\[
N < N_1 = (4D/\alpha)^{6/5} s^{4/5},
\]

the chemical potential is negative, otherwise it is positive.

In the following section, we will study thermodynamical geometry of the Schwarzschild AdS black hole in the extended phase space by viewing the cosmological constant as the number of colors. We pay attention to the case of \( k = 1 \), since the cases of \( k = 0 \) and \( k = -1 \) are trivial.
III. THERMODYNAMICAL GEOMETRY OF THE SCHWARZSCHILD ADS BLACK HOLE

When the corresponding number of colors $N^2$ is kept fixed, this corresponds to the case in a canonical ensemble. In this case, the specific heat for a fixed $N^2$ can be obtained as

$$C_{N^2} = T \left( \frac{\partial s}{\partial T} \right)_{N^2} = \frac{6D s^{5/3} + 3N^{5/6} s \alpha}{2Ds^{2/3} - N^{5/6} \alpha}.$$

(21)

The specific heat diverges at the point of $s_1 = N^{5/4} \alpha^{3/2}/(2\sqrt{2}D^{3/2})$ (i.e., $r_h = L/\sqrt{2}$) which just coincides with the point corresponding to the minimal Hawking temperature for a fixed $N^2$. When $s < s_1$, the specific heat is negative, indicating the thermodynamical instability, while it is positive as $s > s_1$. We show the behavior of $C_{N^2}$ as a function of $s$ in Fig. 6.

In the grand canonical ensemble with fixed chemical potential $\mu$, corresponding specific heat can be obtained as

$$C_{\mu} = T \left( \frac{\partial s}{\partial T} \right)_{\mu} = \frac{128D^2 s^{7/3} + 42D s^{5/3} \alpha N^{5/6} - 11 N^{5/3} s \alpha^2}{3N^{5/3} \alpha^2 - 18Ds^{2/3}N^{5/6} \alpha}.$$

(22)

The specific heat is plotted in Fig. 7. We see that the specific heat diverges at

$$s = s_4 = \frac{N^{5/4} \alpha^{3/2}}{6\sqrt{6}D^{3/2}},$$

(23)

which corresponds to the horizon radius $r_h = L/\sqrt{6}$. Clearly $s_4 < s_1$, namely the divergence happens in the small black hole branch. There exists only a very limited region with a
FIG. 4: The chemical potential as a function of temperature $T$ for a fixed $N = 3$. Here we take $\ell_p = 1$ and $k = 1$. Note that to show the figure clearly, we multiply the chemical potential by a factor of 5 in the left plot, while in the right plot, we show the positive part of chemical potential by multiplying a factor of $10^3$.

positive specific heat between $s_4 < s < s_5$, where

$$s_5 = \frac{11\sqrt{11}N^{5/4}a^{3/2}}{512D^{3/2}},$$

namely, $r_h = L\sqrt{11}/8$, which has a vanishing specific heat. Note that $s_5$ is also less than $s_1$. This is quite different from the classical gas with negative chemical potential. When the chemical potential approaches zero and becomes positive, quantum effects should come into playing some role [16].

Now we turn to the thermodynamical geometry of the black hole and want to see whether
FIG. 5: The chemical potential as a function of $N$ for a fixed $s = 0.1$. Here we take $\ell_p = 1$ and $k = 1$. The maximum of the chemical potential corresponds to the point with $s = s_5 = 11\sqrt{11}N^{5/4}\alpha^{3/2}/(512D^{3/2})$, namely, $N = 1.7783$.

FIG. 6: The specific heat in the case with a fixed $N = 3$ as a function of entropy density $s$. Here we take $k = 1$ and $\ell_p = 1$. The divergence corresponds to the point $s = s_1 = N^{5/4}\alpha^{3/2}/(2\sqrt{2}D^{3/2}) \approx 0.9539$.

the thermodynamical curvature can reveal the singularity of these two specific heats. The Weinhold metric [19] is defined as the second derivatives of internal energy with respect to entropy and other extensive quantities in the energy representation, while the Ruppeiner metric [20] is related to the Weinhold metric by a conformal factor of temperature [22]

$$ds^2_R = \frac{1}{T}ds^2_W.$$  \hspace{1cm} (25)

The Weinhold metric and Ruppeiner metric, which are dependent of the choice of thermo-
FIG. 7: The specific heat for a fixed $\mu$ vs entropy density $s$ for $N = 3, k = 1$ and $\ell_p = 1$. The divergence corresponds to the point $s = s_4 = N^{5/4} \alpha^{3/2} / (6\sqrt{6} D^{3/2}) \approx 0.1836$. The vanishing specific heat point is at $s = s_5 = 11\sqrt{11} N^{5/4} \alpha^{3/2}/(512 D^{3/2}) \approx 0.1923$. Note that there is a trivial zero specific heat point at $s = 0$, which will not be considered here.

dynamic potentials, are not Legendre invariant.

Quevedo et al. [25–28] proposed a method to obtain a thermodynamical metric from a Legendre invariant thermodynamic potential. This method allows one to obtain a new formulism of Geometrothermodynamics whose metric is Legendre invariant in the space of equilibrium states. In what follows, we will first briefly review the formulism of Geometrothermodynamics. Define an $(2n+1)$-dimensional thermodynamic phase space $\mathcal{T}$ which can be described by the coordinates of $\{\phi, E^a, I^a\}$, $a = 1, \ldots, n$, where $\phi$ denotes the thermodynamic potential, $E^a$ and $I^a$ respectively represent the set of extensive variables and the set of intensive variables. Then the fundamental Gibbs 1-form can be defined on the space $\mathcal{T}$ as $\Theta = d\phi - \delta_{ab} I^a dE^b$ with $\delta_{ab} = \text{diag}(1, 1, \ldots, 1)$. Under the assumption that $\mathcal{T}$ is differentiable and $\Theta$ satisfies the condition of $\Theta \wedge (d\Theta)^n \neq 0$, the pair $(\mathcal{T}, \Theta)$ defines a contact manifold. Considering $G$ as a non-degenerate Riemannian metric on the space $\mathcal{T}$, especially, the geometric properties of metric $G$ do not depend on the choice of thermodynamic potential in its construction because of Legendre invariance, then the set $(\mathcal{T}, \Theta, G)$ can define a Riemannian contact manifold or the phase manifold. As a result, an $n$-dimensional Riemannian submanifold $\varepsilon \subset \mathcal{T}$ can be defined as the space of thermodynamic equilibrium states.
(equilibrium manifold) by a smooth map \( \varphi : \varepsilon \to \mathcal{T} \), i.e., \( \varphi : (E^a) \mapsto (\phi, E^a, I^a) \) where the pullback of the map should satisfy the condition \( \varphi^*(\Theta) = 0 \). Furthermore, Quevedo metric \( g \) can be induced on the equilibrium manifold \( \varepsilon \) by using \( \varphi^*(G) \). The non-degenerate Riemannian metric \( G \) can be chosen as

\[
G = (d\phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\eta_{cd}dE^c dI^d), \quad \eta_{cd} = \text{diag}(-1, 1, \ldots, 1).
\]  

(26)

Then Quevedo metric reads

\[
g = \varphi^*(G) = \left( E^c \frac{\partial \phi}{\partial E^c} \right) \left( \eta_{ab} \delta^{bc} \frac{\partial^2 \phi}{\partial E^c \partial E^d} dE^a dE^d \right).
\]  

(27)

Now we calculate the thermodynamical curvature for the Schwarzschild AdS black hole. The Weinhold metric is given by

\[
g^W = \begin{pmatrix}
\rho_{ss} & \rho_{sN^2} \\
\rho_{N^2s} & \rho_{N^2N^2}
\end{pmatrix},
\]  

(28)

where \( \rho_{ij} \) stands for \( \partial^2 \rho / \partial x^i \partial x^j \), and \( x^1 = s, x^2 = N^2 \). The scalar curvature of this metric can be calculated directly. Substituting Eq. (11) and Eq. (12) into Eq. (28), the scalar curvature yields

\[
R^W = \frac{16G_{10}(5N^{5/6}\alpha - 24Ds^{2/3})}{3N^{1/6}\pi^2\alpha(N^{5/6}\alpha - 6Ds^{2/3})^2}.
\]  

(29)

On the other hand, considering Eq. (25), the Ruppeiner metric can be written as

\[
g^R = \frac{1}{T} \begin{pmatrix}
\rho_{ss} & \rho_{sN^2} \\
\rho_{N^2s} & \rho_{N^2N^2}
\end{pmatrix},
\]  

(30)

and the corresponding curvature of this metric is

\[
R^R = -\frac{4D\left(48D^3s^2 + 20D^2N^{5/6}s^{4/3}\alpha + 20DN^{5/3}s^{2/3}\alpha^2 - 5N^{5/2}\alpha^3\right)}{3N^{5/6}s^{1/3}\alpha(6Ds^{2/3} - N^{5/6}\alpha)^2}\frac{2Ds^{2/3} + N^{5/6}\alpha}{(2Ds^{2/3} + N^{5/6}\alpha)^2}.
\]  

(31)

From Eq. (29) and Eq. (31), we can conclude that both the scalar curvatures of the Weinhold metric and Ruppeiner metric possess the same singularity at \( s = s_4 = \frac{N^{5/4}\alpha^{3/2}}{6\sqrt{6}D^{3/2}} \), i.e., \( r_h^2 = L^2/6 \) (see Fig. 8). This singularity just coincides with the divergence of the specific heat \( C_{\mu} \) for fixed chemical potential (comparing Fig. 7 with Fig. 8). Therefore, we may conclude
FIG. 8: Scalar curvature vs entropy density for the Weinhold metric (a) and Ruppeiner metric (b) with $N = 3$, $k = 1$ and $\ell_p = 1$. Both scalar curvatures diverge at $s = s_4 = N^{5/4} \alpha^{3/2}/(6\sqrt{6}D^{3/2}) \approx 0.1836$.

that both the Weinhold metric and Ruppeiner metric can reveal the phase transition of the Schwarzschild AdS black hole in $AdS_5 \times S^5$ in grand canonical ensemble.

The Quevedo metric reads

$$g^Q = (sT + N^2 \mu) \begin{pmatrix} -\rho_{ss} & 0 \\ 0 & \rho_{N^2N^2} \end{pmatrix}. \tag{32}$$

Calculating its scalar curvature gives

$$R^Q = A_1/B_1, \tag{33}$$

where $A_1$ and $B_1$ are given by

$$A_1 = 2048N^{13/6}\alpha G^2_{10}(12160 D^3 s^2 + 46540 D^2 N^{5/6} s^{4/3} \alpha + 3003 D N^{5/3} s^{2/3} \alpha^2 - 2424 N^{5/2} \alpha^3),$$

$$B_1 = 3D \pi^4 s^{2/3} (4D s^{2/3} + 3N^{5/6} \alpha)^3 \left(128D^2 s^{4/3} - 86D N^{5/6} s^{2/3} \alpha + 11N^{5/3} \alpha^2\right)^2.$$

The scalar curvature is plotted in Fig. 9, we see that there exist two divergent points at

$$s_1 = \frac{N^{5/4} \alpha^{3/2}}{2\sqrt{2}D^{3/2}}, \quad \text{and} \quad s_5 = \frac{11\sqrt{11}N^{5/4} \alpha^{3/2}}{512D^{3/2}}, \tag{34}$$
FIG. 9: Scalar curvature vs entropy density $s$ for the Quevedo metric with $N = 3, \ell_p = 1$ and $k = 1$. There exist two divergences at $s = s_5 = 11\sqrt{11}N^{5/4}\alpha^{3/2}/(512D^{3/2}) \approx 0.1923$ and $s = s_1 = N^{5/4}\alpha^{3/2}/(2\sqrt{2}D^{3/2}) \approx 0.9539$, respectively. The first one just coincides with the divergent point of $C_{N^2}$, while the second one corresponds to $C_\mu = 0$. This result is consistent with the recent study in [35, 36] that the divergences of scalar curvature for the Quevedo metric correspond to divergence or zero for specific heat. These results are meaningful to further understand the relation between phase transition and thermodynamical curvature.

IV. CONCLUSIONS

In this paper, we have studied thermodynamics of a Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime in the extended phase space where the cosmological constant is viewed as the number of colors in the dual supersymmetric Yang-Mills theory. We calculated and discussed the chemical potential associated with the number of colors, and found that the chemical potential is always negative in the stable branch of black hole thermodynamics. The chemical potential has a chance to be positive, but it appears in the unstable branch.

The specific heats with fixed number of colors $C_{N^2}$ and with fixed chemical potential $C_\mu$ have been calculated, respectively. It is found that $C_{N^2}$ diverges at the minimal temperature
of the black hole, while $C_\mu$ diverges at a smaller horizon radius.

In the extended phase space, we have a chance to study the thermodynamical geometry associated with the Schwarzschild AdS black hole. By calculating scalar curvatures of the Weinhold metric, Ruppeiner metric and Quevedo metric, we see that in the Weinhold metric and Ruppeiner metric both the scalar curvatures diverge at the same divergent point of $C_\mu$, while in the Quevedo metric, the scalar curvature diverges at the divergence of $C_{N^2}$, besides at the point of $C_\mu = 0$. These results indicate that the divergence of thermodynamical curvature indeed is related to some divergence of specific heats, but the divergence of thermodynamical curvature may be also related to the vanishing points of the thermodynamic potential, temperature and specific heat, etc [35, 36]. This is helpful to further understand the relation between phase transition and divergence of thermodynamical curvature. For a further study of this relation, it should be of great interest to discuss thermodynamics and thermodynamical curvature for other black holes in AdS space in the extended phase space.

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