How Quantum Entanglement Helps to Coordinate Non-Communicating Players

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Abstract

We consider a coalitional game with the same payoff for all players. To maximize the payoff, the players need to use one collective strategy, if all players are in certain states, and the other strategy otherwise. The current state of each player changes according to external conditions and is not known to the other players. In one example of such a game, quantum entanglement between players results in the optimal payoff thrice the maximal payoff for unentangled players.

1 Introduction

Suppose that several players with the same payoff in the game need to coordinate their moves, even though they cannot exchange information with each other during the game. The players can agree in advance on their collective strategy. However, the optimal strategy for each player may depend on what the other players "see" during the game, the knowledge the players cannot communicate to each other.

We concentrate on a specific example how to coordinate non-communicating players. In our example, the "quantum" solution, in which players share entangled qubits pairs, is clearly superior to the classical one.

2 Discussion

Consider a two-player game where players somehow need to switch between the two opposite strategies. If both players are in a certain state, they score by making the same simultaneous moves. Otherwise, i.e. when none or only one player is in this state, the players score my making different moves at the same time. Without loss of generality, let each player have a binary statespace \{0, 1\} and make binary moves \{A, B\}. Let \(q_{ij}\) be the state-dependent probabilities to make different simultaneous moves when players one and two are in states \(i\) and \(j\).
The arbiter cannot observe the *exact* probabilities. Instead, we could use empirical probabilities and let the number of moves go to infinity, but such mathematical strictness hardly adds any physical insight. The payoff $P$ is defined as:

$$ P = \frac{q_{00}}{\max\{q_{01}, q_{10}, q_{11}\}}. $$

Let us compare the optimal strategies and payoffs in classical and quantum case.

### 2.1 Classical strategy

We can start with an arbitrary binary sequence $X_0$ of length $N \gg 1$. For easier comparison with quantum case, let $X_0$ be a coin-flipping sequence:

$X_0 \sim \text{Bern}(1/2)$. Let us choose nonzero probability $q < 1/3$. Sequence $X_1$ is generated from $X_0$ by "flipping" $qN$ bits of $X_0$. Sequence $X_2$ is generated from $X_1$ by flipping $qN$ "other" bits of $X_1$. Finally, sequence $X_3$ is generated from $X_2$ by flipping $qN$ bits of $X_2$, which have not changed when generating $X_1$ and $X_2$. The Hamming distances $d$ between the sequences are then given by:

$$ d(X_\alpha, X_\beta) = qN|\alpha - \beta|; \ \alpha, \beta = 0, ..., 3. $$

Player one transmits sequences $X_0$ and $X_2$ in states "0" and "1", respectively. Player two sends sequences $X_3$ and $X_1$ in states "0" and "1", respectively. Thus the state-dependent probabilities to make different moves $q_{ij}$ are given by:

$$ q_{01} = q_{10} = q_{11} = q, \ q_{00} = 3q. $$

The optimal classical payoff is then $P = 3$. (The optimality check is a simple exercise.) The same payoff occurs in the limit $q \to 0$, if each subsequent sequence $X_{\alpha+1}$ is obtained by passing its preceding sequence $X_\alpha$ through a binary symmetric channel with crossover probability $q$.

### 2.2 Quantum strategy

The simultaneous moves depend on the spin measurements of qubit singlets shared between the players. The players agree on the four measurement directions, say in $XY$ plane, defined by angles

$$ \phi_\alpha = \alpha \delta; \ \alpha = 0, ..., 3; \ \delta \ll 1 $$

Player one measures the component of qubit spin along $\phi_0$ and $\phi_2$ in states "0" and "1", respectively. Player two measures the component of qubit spin along $\pi + \phi_3$ and $\pi + \phi_1$ in states "0" and "1", respectively. Then the angle between the measurement directions for the players is $(\pi - 3\delta)$, if both players are in state "0", and $(\pi \pm \delta)$ otherwise. Each player makes "A" ("B") move when the spin projection on the measurement direction is positive (negative), respectively.
The state-dependent probabilities to make different moves $q_{ij}$ are given by \[1\]:

\[ q_{00} = \frac{1 - \cos 3\delta}{2} = \frac{9}{4}\delta^2 + O(\delta^4), \] 

(5)

\[ q_{01} = q_{10} = q_{11} = \frac{1 - \cos \delta}{2} = \frac{1}{4}\delta^2 + O(\delta^4). \] 

(6)

The optimal classical payoff is then $P \to 9$, achieved in the limit $\delta \to 0$.

(The proof of optimality is to be verified and reported elsewhere).

2.3 Generalization

It is straightforward to calculate the optimal strategies and payoffs under more general conditions, by means of inequalities like

\[ q_{00} \leq (\sqrt{q_{01}} + \sqrt{q_{10}} + \sqrt{q_{11}})^2 \]

for quantum case and

\[ q_{00} \leq q_{01} + q_{10} + q_{11} \]

for classical case. Other generalizations will be reported elsewhere.

3 Conclusions

We considered a game-theoretical example, in which the payoff depends on the correlation between simultaneous moves of non-communicating players. Player coordination by quantum entanglement triples the optimal payoff.

Similar techniques apply to other games with restrictions on interplayer communication.

References

[1] John S. Bell. Speakable and Unspeakable in Quantum Mechanics. Cambridge University Press, Cambridge, 1987.