GROUP C*-ALGEBRAS, METRICS AND AN OPERATOR THEORETIC INEQUALITY.

CRISTINA ANTONESCU AND ERIK CHRISTENSEN

Abstract. On a discrete group $G$ a length function may implement a spectral triple on the reduced group C*-algebra. Following A. Connes, the Dirac operator of the triple then can induce a metric on the state space of reduced group C*-algebra. Recent studies by M. Rieffel raise several questions with respect to such a metric on the state space. Here it is proven that for a free non Abelian group, the metric on the state space is bounded. Further we propose a relaxation in the way a length function is used in the construction of a metric, and we show that for groups of rapid decay there are many metrics related to a length function which all have all the expected properties. The boundedness result for free groups is based on an estimate of the completely bounded norm of a certain Schur multiplier and on some techniques concerning free groups due to U. Haagerup. At the end we have included a noncommutative version of the Arzelá-Ascoli Theorem.

1. Introduction

In the article [Co1] Connes demonstrates that the geodesic distance on a compact, spin, Riemannian manifold $M$ can be expressed in terms of an unbounded Fredholm module over the C*-algebra $C(M)$. The distance between points $p, q$ in $M$ is obtained via the Dirac operator $D$ by the formula

$$d(p, q) = \sup \{|a(p) - a(q)| \mid a \in C(M), \| [D, a] \| \leq 1\}.$$ 

Inspired by the compact manifold $\mathbb{T}$, i.e. the unit circle, and the well known identity $C^*(\mathbb{Z}) = C(\mathbb{T})$, it is natural to consider discrete groups with length functions and for such a pair $(G, \ell)$ to define a Dirac operator $D$ on $l^2(G)$ by $(D\xi)(g) = \ell(g)\xi(g)$. This set up is discussed in [Co1] and Connes proves that $(l^2(G), D)$ is an unbounded Fredholm module for $C^*_r(G)$ if $\ell^{-1}([0, a])$ is a finite set for each $a \in \mathbb{R}_+$. Still following [Co1], one can then try to define a metric on the state space $S(C^*_r(G))$ of $C^*_r(G)$ by

$$d_\ell(\varphi, \psi) = \sup \{|\varphi(a) - \psi(a)| \mid a \in C^*_r(G), \| [D, a] \| \leq 1\}.$$ 

In the first place it is not clear that $d_\ell(\varphi, \psi) < \infty$ for all pairs, but if that is the case then one could ask if $S(C^*_r(G))$ is a bounded set with respect to this metric, and if so does $d_\ell$ generate the $w^*$-topology on $S(C^*_r(G))$.

Especially Rieffel has studied these questions and some more general questions concerning the set of all metrics on $S(C^*_r(G))$, where $G$ is a group with reasonable properties. We will concentrate on the three questions mentioned above. We will prove that for a free (non Abelian) group $\mathbb{F}_n$, on finitely many generators $n$, and $\ell$ the usual length function on $\mathbb{F}_n$, $d_\ell$ is bounded. We think that $d_\ell$ induces the $w^*$-topology too, but the combinatorics involved in proving this seems to be rather difficult. On the other hand we have realized that if one allows a variation in the definition of $d_\ell$ then one can quite easily get all the...
desired properties of \(d_\ell\). This change works for the groups of rapid decay \([J]\), a fundamental concept which will be defined properly in the text. For a discrete group which is of rapid decay with respect to some length function \(\ell\) there exists a \(k \in \mathbb{N}\) such that the metric \(d_{k,\ell}\) on the state space \(S\) of \(C^*_r(G)\) defined by
\[
d_{k,\ell}(\varphi, \psi) = \sup\{||\varphi(a) - \psi(a)|| \mid a \in C^*_r(G), \|[[D, D, \ldots, [D, a], \ldots]]\| \leq 1\},
\]
has all the wanted properties. The proof of this result is on the other hand very easy, and this may be seen as an indication in favor of this alternative construction of a metric from a length function.

The boundedness result for \(d_\ell\) in the case of free groups is based on an estimate of the completely bounded norm of a certain Schur multiplier. This is a purely operator theoretic result which is not far from the known results which are collected in the book by Pisier \([P2]\). This variant is apparently unknown and a proof of this result fills section 3.

At the end of the paper we have included some thoughts on what a noncommutative version of the Arzelá-Ascoli Theorem could be. Using some classical functional analysis techniques it turns out that at least one way to look at the Arzelá-Ascoli Theorem is that if you know a norm compact convex balanced subset, say \(K\), of a unital \(C^*\)-algebra, say \(A\), such that \(K\) separates the states of \(A\), then you can tell whether any other subset, say \(H\) of \(A\) is norm precompact. In order to explain the result we will use the terminology that \(A_\varepsilon\) denotes the closed ball in \(A\) of radius \(\varepsilon\). Having this, a subset \(H\) of \(A\) is norm precompact if and only if
\[
\mathcal{H} \text{ is bounded and } \forall \varepsilon > 0 \exists N \in \mathbb{R}_+ \quad \mathcal{H} \subseteq A_\varepsilon + NK + CI.
\]
The result is not deep at all, but it emphasizes that the concept of a metric on the state space \(S(A)\) generating the \(w^*\)-topology is closely related to compact convex subsets of \(A\) and once you know one such set then the other ones are not too far away. This close connection between metrics on \(S(A)\) and precompact subsets of \(A\) is studied in details in the papers \([R2], [Pav]\); from where we have learned it.

2. Notation and preliminaries

Most of our investigations deal with properties of the \(C^*\)-algebra generated by the left regular representation \(\lambda\) of a discrete group \(G\) on \(l^2(G)\). We refer to chapter 6 of \([KR]\) for the basic properties of this \(C^*\)-algebra, but we will use a slightly different notation which is inspired by Connes’ presentations in \([Co1]\) and \([Co2]\). This means that for an \(x\) in the group algebra \(C[G]\) we will write \(\lambda(x) = \sum_g x(g)\lambda_g\) for the convolution operator on \(l^2(G)\), and for \(g \in G\), \(\delta_g\) denotes the natural basis element in \(l^2(G)\). The \(C^*\)-algebra generated by \(\lambda(C[G])\) in \(B(l^2(G))\) is called the reduced group \(C^*\)-algebra and denoted \(C^*_r(G)\). Any element \(x\) in this algebra has a unique representation in \(l^2(G)\) by \(x \rightarrow x\delta_e\) so in a natural way we have
\[
l^1(G) \subseteq C^*_r(G) \subseteq l^2(G)
\]
and for \(x \in l^1(G)\)
\[
\|x\|_2 \leq \|\lambda(x)\| \leq \|x\|_1.
\]

For a discrete group of rapid decay (RD) one has a kind of inverse to the first inequality and such a type of inequality is very powerful as we shall see. In order to explain the concept of rapid decay we remind the reader that a length function \(\ell\) on a group \(G\) is a mapping \(\ell : G \rightarrow \mathbb{R}_+ \cup \{0\}\) such that
expression, generalization of this mapping to operators on an abstract Hilbert space.

Let $G$ be a discrete group with a length function $\ell$ by $\sum_{n \in \mathbb{Z}} |x_n| \leq \left( \sum_{n \in \mathbb{Z}} (1 + |n|)^2 |x_n|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}} (1 + |n|)^{-2} \right)^{1/2}.

The article [Jo] by Jolissaint contains a lot of results on groups of rapid decay and among his results he proves that a discrete group is of rapid decay, if it is of polynomial growth with respect to some set of generators and the corresponding length function. In Connes’ book [Co2] he presents in Theorem 5, p. 241 a proof of the fact that the word hyperbolic groups of Gromov all are of rapid decay.

As mentioned before Connes defines in [Co1] a metric on a non-commutative C*-algebra via an unbounded Fredholm module. For a discrete group $G$ with a length function $\ell$ he also obtains an unbounded Fredholm module if $\ell$ has the sort of finiteness property defined in the next definition.

**Definition 2.2.** Let $G$ be a discrete group and $\ell : G \to \mathbb{R}_+ \cup \{0\}$ a length function. If, for each $c \in \mathbb{R}_+$, $\ell^{-1}([0, c])$ is a finite set then we say that $\ell$ is proper.

The Fredholm module for $C^*_r(G)$ is the Hilbert space $l^2(G)$ and the Dirac operator $D$ on $l^2(G)$ is the selfadjoint unbounded multiplication operator which multiplies $\xi \in l^2(G)$ by $\ell$ pointwise.

**Definition 2.3.** Let $G$ be a discrete group with a length function $\ell$ and let $S$ denote the state space of $C^*_r(G)$ then $d_\ell : S \times S \to [0, \infty]$ is defined by

$$d_\ell(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| \mid a \in C^*_r(G), \|[D, a]\| \leq 1 \}.$$ 

A computation involving the properties of a length function shows that for any pair $g \in G$ and $\xi \in l^2(G)$ we have

$$\left([D, \lambda_g]\xi\right)(k) = \left(\ell(g^{-1}k) - \ell(k)\right)\xi(g^{-1}k).$$

So $\|[D, \lambda_g]\| \leq \ell(g)$ and we see that $d_\ell$ must separate the points in $S$. It is not clear if $d_\ell(\varphi, \psi) < \infty$ always, but except for that, $d_\ell$ behaves exactly as a metric on $S$ so we will call $d_\ell$ a possibly infinite metric on $S$.

3. THE COMPLETELY BOUNDED NORM OF CERTAIN SCHUR MULTIPLIERS ON $B(H)$

We will let $M_n(\mathbb{C})$ denote the $n \times n$ complex matrices. A matrix $A \in M_n(\mathbb{C})$ induces a mapping $A_s : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by the Schur multiplication which is given by the expression, $M_n(\mathbb{C}) \ni X = (x_{ij}) \to (a_{ij}x_{ij}) = A_s(X) \in M_n(\mathbb{C})$. Here we will consider a generalization of this mapping to operators on an abstract Hilbert space $H$, which for some
discrete group $G$ is decomposed into a sum of orthogonal subspaces $H_g, g \in G, H = \bigoplus_g H_g$. It should be remarked that for any $g \in G$ we do permit that $H_g = 0$ and we want to emphasize that the results in this section are designed to work for the well known Abelian group $\mathbb{Z}$, whereas the work in the in rest of the paper aims at general discrete groups of rapid decay. Despite the fact that this section really is devoted to a result on the group $\mathbb{Z}$ we present the first result in terms of a general discrete group $G$, a Hilbert space $H$ and a decomposition of $H$ indexed by $G$. In this setting where we have a decomposition of $H = \bigoplus_g H_g$ we get a matrix decomposition of the operators in $B(H)$ such that for any $x$ in $B(H)$ we can write $x = (x_{s,t}), s, t \in G$ where each $x_{s,t}$ is in $B(H_t, H_s)$. It is not possible to generalize the Schur multiplication directly to this setting since $B(H_t, H_s)$ is not an algebra unless $s = t$. On the other hand it is possible to multiply any operator $x_{s,t} \in B(H_t, H_s)$ with a complex scalar, so for an infinite scalar matrix $\Lambda = (\lambda_{s,t})$ it is possible to perform a formal Schur multiplication $B(\bigoplus H_g) \ni x = (x_{s,t}) \rightarrow (\lambda_{s,t} x_{s,t}) = \Lambda(x)$. The latter matrix may not correspond to a bounded operator, but the product is well defined as an infinite matrix and we will call it a formal Schur product. The theorem just below, provides a criterion on $\Lambda$ for the boundedness of the Schur product $\Lambda_s(x)$ for any $x$ in $B(H)$. We do a little more since we do compute the so called completely bounded norm of $\Lambda_s : B(H) \rightarrow B(H)$. The theory connected to completely bounded operators is described in Paulsen’s book [Pau]. In order to explain this concept shortly we consider for $n \in \mathbb{N}$ the mapping $(\Lambda_s)_n : M_n(\mathbb{C}) \otimes B(H) \rightarrow M_n(\mathbb{C}) \otimes B(H)$ which is given as $(\Lambda_s)_n = \text{id}_{M_n(\mathbb{C})} \otimes \Lambda_s$. If the sequence of norms defined by $\|\Lambda_s\|_n := \|(\Lambda_s)_n\|$ is bounded, $\Lambda_s$ is said to be completely bounded and the completely bounded norm, $\|\Lambda_s\|_{cb}$, is given as the sup$_n \|\Lambda_s\|_n$. Our criterion for $\|\Lambda_s\|_{cb}$ to be finite is based upon a generalization of a theorem by M. Božejko and G. Fendler [BF]. In Pisier’s book [Pi], he presents this result in Theorem 6.4. This theorem deals with the situation where each of the summands $H_g$ above is one-dimensional, i.e. $H_g = \mathbb{C}$.

**Theorem 3.1.** Let $G$ be a discrete group, $H$ be a Hilbert space which is decomposed into a sum of orthogonal subspaces $H_g, g \in G$ and let $\varphi$ be a complex function on $G$. If the linear operator $T_\varphi : \lambda(\mathbb{C}G) \rightarrow \lambda(\mathbb{C}G)$ which is defined by $T_\varphi(\lambda_g) = \varphi(g)\lambda_g$ extends to a completely bounded operator on $C^*_r(G)$ then for the matrix $\Lambda$ given by $\Lambda = (\lambda_{s,t})$ $s, t \in G, \lambda_{s,t} = \varphi(st^{-1})$ the mapping $\Lambda_s$ is completely bounded, $\|\Lambda_s\|_{cb} \leq \|T_\varphi\|_{cb}$ and $\Lambda_s$ is an ultraweakly continuous, or normal, operator on $B(H)$.

**Proof.** Suppose that $T_\varphi$ is completely bounded, then for the case where $H_g = \mathbb{C}$ for every $g \in G$ the result follows from [P]. The proof of in that book of the scalar case, as presented in the proof of the implication (i) $\Rightarrow$ (iii) of Theorem 6.4 of [P], shows that there exist a Hilbert space $K$ and two functions say $\xi$ and $\eta$ on $G$ with values in $K$ such that

$$\forall s, t \in G \quad \varphi(st^{-1}) = (\xi(t), \eta(s)) \quad \text{and} \quad \forall g \in G \quad \|\xi(g)\| \leq \sqrt{\|T_\varphi\|_{cb}}, \quad \|\eta(g)\| \leq \sqrt{\|T_\varphi\|_{cb}}.$$ 

We will now turn to the operator $\Lambda_s$ and show that it is completely bounded. The proof of this can be obtained as a modification of a part of the proof of Theorem 5.1 of [P]. In fact the representation of $\varphi$, we have obtained above, makes it possible to construct operators say $x$ and $y$ in $B(H, H \otimes K)$ such that $\Lambda_s$ can be expressed as a completely bounded operator in terms of these operators. We recall that $H = \bigoplus H_g$ and define the operators $x$ and $y$ on a vector $\alpha = (\alpha_g)_{g \in G}$ by $x\alpha = (\alpha_g \otimes \xi(g))_{g \in G}$ and similarly $y\alpha = (\alpha_g \otimes \eta(g))_{g \in G}$ and we find that both operators are of norm at most $\sqrt{\|T_\varphi\|_{cb}}$. Let $\pi$
denote the representation of $B(H)$ on $H \otimes K$ which is simply the amplification $a \to a \otimes I_K$, then an easy computation shows that for any pair of vectors $\alpha = (\alpha_g), \beta = (\beta_g)$ from $H = \oplus H_g$ and any $a$ in $B(H)$ we have

\begin{align*}
(\pi(a)x, y) &= \sum_{s,t \in G} (a_{st}, \beta_s)(\xi(t), \eta(s)) \\
(\pi(a)x)_{s,t} &= \sum_{s,t \in G} (\varphi(st^{-1})a_{st}, \beta_s) \\
(\Lambda_s(a)\alpha, \beta) &= (\Lambda_s(a)\alpha, \beta).
\end{align*}

Consequently $\Lambda_s(a) = y^*\pi(a)x$ and the cb-norm of $\Lambda_s$ is at most $\|T_\varphi\|_{cb}$. The concrete description $\Lambda_s(.) = y^*\pi(.)x$ where $\pi$ is just an amplification shows that $\Lambda_s$ is ultraweakly continuous.

**Corollary 3.2.** If $G$ is an Abelian discrete group and $T_\varphi$ extends to a bounded operator on $C_0(G)$ then the mapping $\Lambda_s$ is completely bounded, $\|\Lambda_s\|_{cb} \leq \|T_\varphi\|$ and $\Lambda_s$ is an ultraweakly continuous, or normal, operator on $B(H)$

**Proof.** We have to prove that $T_\varphi$ extends to a completely bounded mapping if it extends to a bounded mapping and that the two norms on $T_\varphi$ agree. In order to do so we remark that for the compact Abelian dual group $\hat{G}$ and any natural number $k$ we have $C_0(G) \otimes M_k(\mathbb{C}) = C(\hat{G}, M_k(\mathbb{C}))$, the continuous $M_k(\mathbb{C})$ valued functions on $\hat{G}$. Then for any finite sum $x = \sum_{g} \lambda_g \otimes m_g$ in $C(\hat{G}) \otimes M_k(\mathbb{C})$ we have

\[ \|x\| = \max\{\|\sum_{g} \chi(g)m_g\|_{M_k(\mathbb{C})} \mid \chi \in \hat{G}\} .\]

The norm in $M_k(\mathbb{C})$ is determined by the functionals of norm one on this algebra, so let $M_k(\mathbb{C})^*_1$ denote this unit ball and we get

\[ \|x\| = \max\{\|\sum_{g} \chi(g)\psi(m_g)\| \mid \chi \in \hat{G} \text{ and } \psi \in M_k(\mathbb{C})^*_1}\} .\]

Let us now suppose that $T_\varphi$ extends to a bounded operator on the group algebra of norm at most 1. For $x$ as above of norm at most 1 in $C_0(G) \otimes M_k(\mathbb{C})$ we get for any pair $\chi, \psi$ as above that $\|\sum_{g} \chi(g)\psi(m_g)\| \leq 1$, hence for this fixed $\psi$ we get in $C_0(G)$ the estimate $\|\sum_{g} \varphi(m_g)\lambda_g\| \leq 1$. Since $\|T_\varphi\| \leq 1$ we also get $\|\sum_{g} \varphi(g)\psi(m_g)\lambda_g\| \leq 1$, but this holds for any $\psi$ so we can go back and note that $T_\varphi$ is completely bounded of norm at most 1.

The following corollary shows that a square summable function $\varphi$ on a commutative discrete group $G$ induces a completely bounded operator $\Lambda_s$. Besides these functions and the positive definite functions on $G$ we do not know of any other general results which can guarantee the complete boundedness of $\Lambda_s$.

**Corollary 3.3.** Let $G$ be an Abelian discrete group, $\varphi \in \ell^2(G)$ and let $\Lambda : G \times G \to \mathbb{C}$ be given by $\Lambda(s, t) = \varphi(st^{-1})$. Then $\Lambda_s$ is completely bounded and $\|\Lambda_s\|_{cb} \leq \|\varphi\|_2$.

**Proof.** The operator $T_\varphi$ on $C_0^*(G)$ can, when we look at the latter algebra as $C(\hat{G})$, be expressed as the convolution operator implemented by the Fourier transform, $\hat{\varphi} \in L^2(\hat{G})$. Here we have chosen the probability Haar measure on the compact group $\hat{G}$ such that the Fourier transform is an isometric operator between the 2 Hilbert spaces. It follows
directly from The Cauchy-Schwarz inequality that the norm of the convolution operator is dominated by the norm $\|\hat{\varphi}\|_2$, and the corollary follows.

The purpose of the previous corollary is actually to compute the norm of the partial inverse of certain derivations on $B(H)$. Let $D$ be a possibly unbounded self adjoint operator on $B(H)$ with spectrum contained in the set of integers $\mathbb{Z}$, then the Hilbert space $H$ decomposes as a direct sum of the eigenspaces $H_m$ of $D$. Many of these spaces may vanish, but anyway we can write $H = \bigoplus_{m \in \mathbb{Z}} H_m$ and we will be able to use the results from above concerning the norm of certain Schur multipliers. The question we are going to deal with, is to give a description of a bounded operator $a \in B(H)$ which has the property that the commutator $[D, a]$ is bounded and of norm at most 1. Clearly all bounded operators which commute with $D$ must play a special role in this set up. This set is a von Neumann algebra and consists of the operators in the main diagonal of $B(H)$, when the latter algebra is viewed as infinite matrices with respect to the decomposition $H = \bigoplus H_m$. Consequently we will let $\mathfrak{D}_a$ denote the commutant of $D$ and for $k \in \mathbb{Z}$ we will define the $k$'th diagonal of $B(H)$ by

$$\mathfrak{D}_k = \{(x_{ij}) \in B(H) \mid i - j \neq k \Rightarrow x_{ij} = 0\}.$$ 

For $k \in \mathbb{Z}$ there is a natural projection of $B(H)$ onto $\mathfrak{D}_k$, say $\rho_k$ given by the expressions

$$\rho_k((x_{ij}))_{m,n} = \begin{cases} x_{m,n} & \text{if } m - n = k, \\ 0 & \text{if } m - n \neq k. \end{cases}$$

If one computes $\rho_k(x)^*\rho_k(x)$ it is easy to realize that $\rho_k$ is a projection onto the $k$'th diagonal and of norm at most one. The problem we are facing is analogous to well known problems concerning convergence of Fourier series; it is not easy to give norm estimates of the norm of a general finite sum $\sum_{k \in \mathbb{Z}} \rho_k(x)$. If we disregard convergence questions for some time, it follows from elementary algebraic manipulations that for an operator $a = (a_{m,n}) \in B(H)$, the commutator $[D, a]$ must have the formal infinite matrix $c = (c_{m,n})$ given by $c_{m,n} = (m - n)a_{m,n}$. So at least formally we can write

$$[D, a] = \sum_{k \in \mathbb{Z}} k\rho_k(a),$$

and we see that this operator on $B(H)$ in fact is an unbounded Schur multiplier. Further it follows - so far formally - that

$$a - \rho_0(a) = \sum_{k \in \mathbb{Z} \text{ and } k \neq 0} \frac{1}{k}\rho_k([D, a]),$$

Hence the partial inverse to the derivation $B(H) \ni a \rightarrow [D, a]$ is a Schur multiplier which according to the results above will turn out to be completely bounded. With this notation in mind we can offer norm estimates for such sums in the next theorem. The theorem is a generalization of the well known fact that the Fourier series for a differentiable $2\pi$ periodic function on $\mathbb{R}$ is uniformly convergent.

**Theorem 3.4.** Let $D$ be a self adjoint operator on a Hilbert space $H$ such that the spectrum of $H$ is contained in $\mathbb{Z}$ and let $K = \{a \in B(H) \mid \|[D, a]\| \leq 1 \text{ and } \rho_0(a) = 0\}$, then every element in $K$ is of norm at most $\frac{\pi}{\sqrt{3}}$. For $a \in B(H)$ such that $\|[D, a]\| \leq 1$ the sum $\sum_{m \in \mathbb{Z}} \rho_m(a)$ is norm convergent and $\forall k \in \mathbb{N}$, $\|\sum_{|m| > k} \rho_k(a)\| \leq \frac{\sqrt{2}}{k}$. 
Proof. Suppose \( a \in B(H) \) satisfies \( \| [D, a] \| \leq 1 \) and \( \rho_0(a) = 0 \), then the first statement in the theorem is, as we shall see, just a special case of the second corresponding to \( k = 0 \), although the estimates are slightly different. Let then \( k \in \mathbb{N}_0 \) be given and consider the set of Hilbert spaces \( H_m, m \in \mathbb{Z} \) where the space \( H_m \) is defined as above, i.e. the eigenspace for \( D \) corresponding to the eigenvalue \( m \). We can then apply Corollary 3.3 for the group \( \mathbb{Z} \) and the function \( \varphi_k : \mathbb{Z} \rightarrow \mathbb{R} \) given by

\[
\varphi_k(m) = \begin{cases} 
  m^{-1}, & \text{if } |m| > k, \\
  0, & \text{if } |m| \leq k.
\end{cases}
\]

Hence we see that

\[
\| a - \sum_{i=-k}^{k} \rho_i(a) \| \leq \|[D, a]\| \|\varphi_k\|_2 \leq (2 \sum_{j>k} j^{-2})^{1/2}.
\]

After recalling the well known sum \( \sum_{j\in\mathbb{N}} j^{-2} = \frac{\pi^2}{6} \) for \( k = 0 \) and the integral estimate \( \sum_{j>k} j^{-2} < \frac{1}{k} \) for \( k > 0 \), the theorem follows.

\[\square\]

Remark 3.5. The proof of Theorem 3.4 depends on the fact that the spectrum of \( D \) is contained in the Abelian discrete group \( \mathbb{Z} \). This is not likely to be a relevant condition for a result of this type and we would think that this theorem must have a more general version which is valid for an unbounded self adjoint operator whose spectrum consists of points \( (s_k)_{k \in \mathbb{Z}} \) such that \( |s_k| \to \infty \) for \( |k| \to \infty \) and \( \inf\{|s_m - s_n| \mid m, n \in \mathbb{Z}\} > 0 \). We are aware of, and have already mentioned the fact that an estimate similar to the one above does exist for ordinary differentiation on \( C(\mathbb{T}) \), but we have not found a general operator theoretic treatment of this problem.

4. On unital C*-algebras as noncommutative compact metric spaces.

This section is mainly devoted to the study of some metrics on the state space of a C*-algebra which is generated by the left regular representation of a discrete group of rapid decay. We will start by recalling Connes’ construction [Co1] which defines a metric on the state space of a discrete group C*-algebras in terms of an unbounded Fredholm module. Let then \( G \) be a discrete group with a length function \( \ell : G \to \mathbb{N}_0 \) such that \( \ell \) is proper and \( G \) is of rapid decay with constants \( C, k \) with respect to \( \ell \). As described in [Co2] p. 241 such a length function \( \ell \) on a discrete group \( G \) with values in \( \mathbb{N}_0 \) induces a decomposition of \( \ell^2(G) \) into an orthogonal sum of subspaces \( H_m \), each one being the closed linear span of the basis vectors \( \delta_g \) for which \( \ell(g) = m \). The Dirac operator on \( \ell^2(G) \) is the self adjoint unbounded operator \( D \) which is the closure of the operator \( D_0 \) defined on the linear span of the basis vectors \( \delta_g, g \in G \) and acts by \( D_0\delta_g = \ell(g)\delta_g \). The operator \( D \) clearly has it’s spectrum contained in \( \mathbb{Z} \) and the eigenspaces all vanish for \( m < 0 \) and equals \( H_m \) for \( m \geq 0 \). The Theorem 3.4 is designed to deal with this situation, but it does not work as well as expected. The theorem provides a norm estimate of \( \| a - \rho_0(a) \| \) in terms of \( \|[D, a]\| \) for an \( a \) in \( C^*_r(G) \), and we believed that from this it would be easy to get an estimate of \( \|\rho_0(a)\| \), because a group is such a rigid object. Unfortunately this is not so and we can only get an estimate which also takes care of the main diagonal part \( \rho_0 \) if the group is a one of the free non Abelian groups \( \mathbb{F}_n \).
Theorem 4.1. Let \( G \) be a free non Abelian group on finitely many generators and \( \ell \) the natural length function on \( G \), then the metric \( d_\ell \) on the state space \( S(C_\ell^*(G)) \) is bounded, and the diameter of the state space is at most 5.

Proof. We will prove the boundedness of the metric by studying the convex and balanced subset \( K \) of \( C_\ell^*(G) \) given by \( K = \{ a \in C_\ell^*(G) \mid ||[D, a]|| \leq 1 \} \). By Theorem 3.4 we know that for an \( a \in K \) we will have \( ||a - \rho_0(a)|| \leq \frac{\pi}{\sqrt{3}} \), so we only have to get an estimate of the norm \( \rho_0(a) \). In order to control \( ||\rho_0(a)|| \) we do first restrict to the case where \( a \) is of finite support in \( G \) and secondly we make the extra assumption that \( (a\delta_e, \delta_e) = 0 \). If the latter is not the case we simply subtract the corresponding multiple of the unit from \( a \). This operation has of course no effect on the commutator \([D, a]\). Since we know by assumption that this commutator is of norm at most 1, we get the first estimate

\[
1 \geq ||[D, a]\delta_e||^2 = \sum_{g \in G} \ell(g)^2|a(g)|^2.
\]

Let us now pick a unit vector \( \xi \in H_m \) and let \( p_m \) denote the orthogonal projection from \( H \) onto \( H_m \). We can now try to estimate \( ||\rho_0(a)|| \) by estimating \( p_m a \xi \). So let \( g \in G \) be of length \( m \) then \( g \) can be expressed uniquely in terms of generators as \( g = g_1 g_2 \ldots g_m \), and when we have to compute the value of the convolution \( a \ast \xi(g) \) we have to remember that \( \xi \) is supported on words of length \( m \) so the sum will be an expression of the type

\[
a \ast \xi(g) = \sum_{k=1}^{m} \sum_{\{s_1, \ldots, s_k \mid s_k \neq g_k \text{ and } s_k \neq g_{k+1}^{-1}\}} a(g_1, \ldots, g_k s_k^{-1}, \ldots s_1^{-1})\xi(s_1, \ldots, s_k g_{k+1}, \ldots g_m),
\]

We can now imitate a trick from the proof of \cite{Ha} Lemma 1.3. In order to do so we define for a fixed \( k, 1 \leq k \leq m \) (remember \( a(e) = 0 \) so \( k > 0 \) ) a function \( b_k \) supported on words of length \( k \) and a vector \( \eta_{m-k} \) supported on words of length \( m - k \) by

\[
(4.1) \quad b_k(g_1, \ldots, g_k) = \left( \sum_{\{s_1, \ldots, s_k \mid s_k \neq g_k\}} |a(g_1, \ldots, g_k s_k^{-1}, \ldots s_1^{-1})|^2 \right)^{\frac{1}{2}},
\]

\[
(4.2) \quad b_m(g_1, \ldots, g_m) = |a(g_1, \ldots, g_m)|
\]

\[
(4.3) \quad \eta_{m-k}(g_{k+1}, \ldots, g_m) = \left( \sum_{\{s_1, \ldots, s_k \mid s_k \neq g_{k+1}^{-1}\}} |\xi(s_1, \ldots, s_k g_{k+1}, \ldots g_m)|^2 \right)^{\frac{1}{2}}
\]

\[
(4.4) \quad \eta_0(e) = \|\xi\|_2 = 1.
\]

Having this we get

\[
|a \ast \xi(g)| \leq \sum_{k=1}^{m} b_k(g_1, \ldots, g_k) \eta_{m-k}(g_{k+1}, \ldots, g_m) = (\sum_{k=1}^{m} b_k \ast \eta_{m-k})(g).
\]

As in \cite{Ha} we will let \( \chi_m \) denote the characteristic function on the words of length \( m \) and we will further use the statement contained in Lemma 1.3 of \cite{Ha} which says that for functions like \( b_k \) which is supported on words of length \( k \) and \( \eta_{m-k} \) which is supported on words of length \( m - k \) one has

\[
\|b_k \ast \eta_{m-k} \chi_m\|_2 \leq \|b_k\|_2 \|\eta_{m-k}\|_2.
\]

A combination of the inequalities above then yields
\[ \|p_m a \xi \|_2 = \| a \ast \xi \chi_m \|_2 \]

\[ \leq \| \sum_{k=1}^{m} b_k \ast \eta_{m-k} \chi_m \|_2 \]

\[ \leq \sum_{k=1}^{m} \| b_k \ast \eta_{m-k} \chi_m \|_2 \]

\[ \leq \sum_{k=1}^{m} \| b_k \|_2 \| \eta_{m-k} \|_2 \]

\[ \leq \sum_{k=1}^{m} \| b_k \|_2, \text{ since } \| \xi \|_2 = 1 \]

\[ = \sum_{k=1}^{m} k \| b_k \|_2 (1/k) \]

\[ \leq \left( \sum_{k=1}^{m} k^2 \| b_k \|_2^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{m} k^{-2} \right)^{\frac{1}{2}} \]

\[ \leq \frac{\pi \sqrt{6}}{\sqrt{3}} \left( \sum_{g, \ell(g) = 2k} |a(g)|^2 \right)^{\frac{1}{2}} \]

\[ \leq \frac{\pi \sqrt{6}}{4} \sum_{g \in G} (\ell(g)^2 |a(g)|^2)^{\frac{1}{2}} \]

\[ \leq \frac{\pi}{2\sqrt{6}}. \]

The computations just above shows that for an \( a \in K \) we have \( \| \rho_0(a) - (a \delta_e, \delta_e) I \| \leq \frac{\pi}{2\sqrt{6}} \) and from Theorem 3.4 we know that \( \| a - \rho_0(a) \| \leq \frac{\pi}{\sqrt{3}} \), so we obtain \( \| a - (a \delta_e, \delta_e) I \| < 2.5 \), and the diameter of the state space is at most 5.

In [Ri2], Rieffel introduces the concept of a lower semicontinuous Lipschitz seminorm \( L \) on a \( C^* \)-algebra \( A \). The term Lipschitz means that the kernel of the seminorm consists of the scalars and the terms lower semicontinuous means that the set \( \{ a \in A \mid L(a) \leq 1 \} \) is norm closed. In our context the operator \( D \) induces several Lipschitz seminorms whose domains of definition always contain the dense subalgebra \( \lambda(CG) \) of \( C^*_r(G) \). In order to define these seminorms we fix the setting as above. Let \( G \) be a discrete group with a length function \( \ell : G \rightarrow \mathbb{N}_0 \) such that \( \ell^{-1}(0) = \{ e \} \). Let \( D \) denote the corresponding Dirac operator and \( \delta \) the unbounded derivation on \( C^*_r(G) \) given by \( \delta(a) = \text{closure}([D, a]) \), if the commutator \( [D, a] \) is bounded and densely defined. For any natural number \( k \) we define a seminorm \( L_D^k \) by

\[ \text{domain}(L_D^k) = \text{domain}(\delta^k) \text{ and } L_D^k(a) = \| \delta^k(a) \|. \]

Having this notation we can state a theorem of quite general validity.

**Theorem 4.2.** Let \( G \) be a discrete group with a length function \( \ell : G \rightarrow \mathbb{N}_0 \) such that \( \ell^{-1}(0) = \{ e \} \). For any natural number \( k \) the seminorm \( L_D^k \) on \( C^*_r(G) \) is a lower semicontinuous Lipschitz seminorm.
Proof. The condition \( \ell^{-1}(0) = \{e\} \) implies that the only operators in \( C^*_r(G) \) which commute with \( D \) are the multiples of the unit in \( C^*_r(G) \). Let us define \( \varphi : \mathbb{Z} \to \mathbb{R} \) by

\[
\varphi(m) = \begin{cases} 
    m^{-1}, & \text{if } m \neq 0, \\
    0, & \text{if } m = 0.
\end{cases}
\]

and let \( \Lambda_s \) denote the Schur multiplier on \( B(\oplus H_m) \) implemented by the function \( \lambda(m, n) = \varphi(m-n) \). Then by Theorem 3.4 we know that \( \Lambda_s \) is a completely bounded and ultraweakly continuous operator on \( B(H) \). Let now \( B(H)_1 \) denote the unit ball in \( B(H) \) then for any \( k \in \mathbb{N} \) we have \( \Lambda^k_s(B(H)_1) \) is ultraweakly closed. We can now control most of the set

\[
\{ a \in C^*_r(G) \mid L^k_D(a) \leq 1 \}
\]

the only part missing is the main diagonal \( \rho_0(C^*_r(G)) \). For \( B(H) \) we have \( \rho_0(B(H)) = D_0 \) i. e. the main diagonal which is clearly ultraweakly closed. The sum \( \Lambda^k_s(B(H)_1) + \rho_0(B(H)) \) is then ultraweakly closed and consequently also norm closed and the intersection below is norm closed too.

\[
\{ a \in C^*_r(G) \mid L^k_D(a) \leq 1 \} = [\Lambda^k_s(B(H)_1) + \rho_0(B(H))] \cap C^*_r(G).
\]

□

It is rather easy to check that the metric \( d_\ell \) introduced in Definition 2.3 induces a topology which is finer than the \( w^* \)-topology. In fact the norm dense group algebra \( \lambda(C(G)) \) is obviously contained in the domain of definition for the derivation \( \delta \) and the metric clearly induces a topology on the state space which is finer than pointwise convergence on the operators \( \lambda_g \). The question is whether the 2 topologies agree. In the first place we thought that once the boundedness question was settled this question ought to be easy to settle because Theorem 3.4 controls problems involving norms of the diagonals with large indices. A closer analysis shows that the problems involved seems to be much more complex and probably are of a difficult combinatorial nature. The short formulation of the problem is that for a group element - say \( g \) - such that \( \ell(g) \) is “large” the unitary operator \( \lambda_g \) may have a lot of non vanishing diagonals with “small” indices. At least in principle this makes it possible for the algebra \( \mathbb{C}G \) to have the property that for any natural number \( N \), there exists an operator \( x = \sum x(g)\lambda_g \) such that \( x(g) \neq 0 \Rightarrow \ell(g) > N \), and \( \| \sum_{|k| \leq N} \rho_k(x) \| \) is big whereas \( \| \sum_{|k| > N} \rho_k(x) \| \) is small. Since we have not been able to solve these problems we have looked for alternative constructions of metrics which will induce the \( w^* \)-topology on the state space of \( C^*_r(G) \). The most obvious thing to do, seemed to be to restrict the attention to the analysis of discrete groups of rapid decay [262, 10].

Before we state and prove our result we want to mention that it follows from Rieffel’s works [21], [22] and the work of Pavlović [Pav] that a lower semicontinuous Lipschitz seminorm \( L \) on a unital C*-algebra \( A \) is bounded and induces the \( w^* \)-topology on the state space of \( A \) if and only if the set

\[
\{ a \in A : L(a) \leq 1 \}
\]

has a compact image in the quotient space \( A/CI \), equipped with the quotient norm.

Theorem 4.3. Let \( G \) be a discrete group with a length function \( \ell : G \to \mathbb{N}_0 \) such that \( \ell^{-1}(e) = 0 \), \( \ell \) is proper and \( G \) is of rapid decay with respect to \( \ell \). Then there exists a \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \), \( L^k_D \) is lower semicontinuous, the metric generated by \( L^k_D \) on \( S(C^*_r(G)) \) is bounded and the topology generated by the metric equals the \( w^* \)-topology.
The number $k_0$ is then defined by $k_0 = \lfloor s \rfloor + 1$, and given this we will fix a $k \in \mathbb{N}$ such that $k \geq k_0$. According to the statement just in front of this theorem we have to prove that the set, say $\tilde{K}_k$ defined by

$$\tilde{K}_k = \{ a \in C^*_r(G) \mid L_D^k(a) \leq 1 \}$$

has precompact image in $C^*_r(G)/CI$. The way we obtain this is by choosing the element from each equivalence class in $\tilde{K}_k$ which is of trace 0. This set is denoted $K_k$ and is clearly precompact if and only if $\tilde{K}_k$ has precompact image in the quotient space $C^*_r(G)/CI$. Consequently $K_k$ is given by

$$K_k = \{ a \in C^*_r(G) \mid L_D^k(a) \leq 1 \text{ and } (a \delta_e, \delta_e) = 0 \}$$

The first observation we need has already been used before, namely that any element $a \in C^*_r(G)$ can be expressed as an $l^2$ convergent infinite sum $\sum a(g)\delta_g$ and that $\|a\|_2 = \|a \delta_e\|$. Having this, and the fact that $D\delta_e = 0$ we get for an $a \in K_k$ that

$$1 \geq L_D^k(a) = \|\delta^k(a)\| \geq \|\delta^k(a) \delta_e\| = \|\sum \ell(g)^k a(g) \delta_g\|.$$ 

In particular we get for an $a \in K_k$ that

$$\sum \ell(g)^{2k}|a(g)|^2 \leq 1.$$ 

The properness condition on $\ell(g)$ implies that there are only finitely many group elements of length less than any natural number $n$. Hence in order to prove that $K_k$ is precompact it is sufficient to show that for any positive real $\varepsilon$ there exists a natural number $n$ such that for any $a \in K_k$

$$\|\sum_{\ell(g) \geq n} a(g)\lambda_g\|_{C^*_r(G)} \leq \varepsilon$$

but this is on the other hand easily obtainable from the inequality at the top of the proof. In fact let $n \in \mathbb{N}$ then for $g \in G$ with $\ell(g) \geq n \geq 1$ we get

$$(1 + \ell(g))^{2s} \leq 2^{2s}\ell(g)^{2s} \leq 2^{2s}n^{(2s-2k)}\ell(g)^{2k}.$$ 

Since $2s - 2k < 0$ there exists an $n \in \mathbb{N}$ such that $2^{2s}n^{(2s-2k)} \leq \frac{\varepsilon^2}{C^2}$. For this $n$ we then obtain

$$\|\sum_{\ell(g) \geq n} a(g)\lambda_g\|_{C^*_r(G)}^2 \leq C^2 \sum_{\ell(g) \geq n} (1 + \ell(g))^{2s}|a(g)|^2 \leq C^2 \frac{\varepsilon^2}{C^2} L_D^k(a)^2 \leq \varepsilon^2$$

and the theorem follows. 

Given the common use of language which says: a noncommutative compact topological space is a unital $C^*$-algebra, it seems natural to propose a definition of a metric on this noncommutative space in terms of an object which is related directly to the algebra and not to its state space. If there is an obvious smooth structure in terms of a spectral triple, this object should be preferred since it contains much more information, but for a general noncommutative, unital and separable $C^*$-algebra $A$ without any particularities it seems that any precompact balanced and convex subset of $A$ which separates the states on $A$ contains all the information needed. The reason why we propose to look at such a set as a
noncommutative metric is partly due to our reading of the works by Rieffel and Pavlović, but also partly due to the result in our next section.

**Definition 4.4.** Let $\mathcal{A}$ be a unital $C^*$-algebra. A subset $\mathcal{K}$ of $\mathcal{A}$ is called a metric set if it is norm compact, balanced, convex and separates the states on $\mathcal{A}$.

The term balanced will be defined properly in the next section. With this definition at hand one can easily construct metric sets for separable unital $C^*$-algebras and for instance for a countable group $G = \{g_n \mid n \in \mathbb{N}\}$ a metric set in $C^*_r(G)$ could be given by the following expression where $\text{conv}$ means the closed convex hull.

$$
\mathcal{K} := \text{conv}\left( \cup_{n=1}^{\infty} \{\alpha g_n + \beta \lambda g_n^* \mid \alpha, \beta \in \mathbb{C} \text{ and } |\alpha| + |\beta| \leq 1/n\} \right).
$$

The next section will contain proofs which hopefully will justify this introduction of yet another concept.

### 5. A noncommutative Arzelá-Ascoli Theorem

The classical Arzelá-Ascoli Theorem gives a characterization of precompact subsets of $C(X)$ for a compact topological space $X$. If $X$ is a equipped with a metric $\rho$ generating the topology on $X$ one can construct a convex subset $\tilde{\mathcal{K}}$ of $C(X)$ by

$$
\tilde{\mathcal{K}} = \{f \in C(X) \mid \forall x, y \in X \ |f(x) - f(y)| \leq \rho(x, y) \}
$$

This set is unbounded since any constant function belongs to $\tilde{\mathcal{K}}$. If one normalizes the set by considering the subset consisting of those elements which all vanish at a certain point $x_0$ then the classical Arzelá-Ascoli Theorem shows that the set, say $\mathcal{K}$, given by

$$
\mathcal{K} = \{f \in C(X) \mid \forall x, y \in X \ |f(x) - f(y)| \leq \rho(x, y) \text{ and } f(x_0) = 0 \}
$$

will be a compact balanced convex subset of $C(X)$ which separates the points in $X$. The Arzelá-Ascoli-Theorem measures any other subset of $C(X)$ against this set in order to see whether this subset is precompact or not. What we do in the following lines is just to transfer this measuring process to the noncommutative case. The methods we use are elementary functional analytic duality results. So we have wondered if this sort of result is valid in a much wider generality like operator spaces $\mathbb{K}$, $\mathbb{ES}$. It seems that the validity of a generalization of Lemma 5.1 to this new setting is the crucial thing. Before we start we want to introduce some more notation. We will be considering the self adjoint part of a unital $C^*$-algebra which is denoted $\mathcal{A}_h$ and we want to think of the elements in $\mathcal{A}$ as affine complex $w^*$-continuous functions on the state space $\mathcal{S}$ of $\mathcal{A}$, so we will let $\mathcal{A}(\mathcal{S})$ denote the space of $w^*$-continuous affine complex functions on $\mathcal{S}$ and for an element $a \in \mathcal{A}$, $\hat{a}$ will denote the corresponding affine function in $\mathcal{A}(\mathcal{S})$. This presentation of $\mathcal{A}$ is called Kadison’s functional representation of $\mathcal{A}$. It is well known that the functional representation is isometric on $\mathcal{A}_h$, but for a general element $a \in \mathcal{A}$ we only have the estimates

$$
\|a\| \geq \sup |\hat{a}(\varphi)| = \|\hat{a}\| \geq \frac{1}{2} \|a\|.
$$

In particular this shows that a subset $\mathcal{H}$ of $\mathcal{A}$ is bounded if and only if the subset $\hat{\mathcal{H}}$ of $\mathcal{A}(\mathcal{S})$ is bounded.

The term balanced is used in the sense that a subset $\mathcal{H}$ of $\mathcal{A}$ is balanced if for any complex number $\mu$ such that $|\mu| \leq 1$ we have $\mu \mathcal{H} \subseteq \mathcal{H}$. We remind the reader that a subscript attached to a Banach space like $Y_\mu$, means that we consider the closed ball of
radius $\mu$ in $Y$ and an asterix attached to $Y$ like $Y^*$ means the dual space. For a pair of Banach spaces like $A$ and $A^*$ we will use the duality result known under the name of the bipolar theorem. Here the polar of a set $H \subseteq A$ is denoted $H^\circ$ and defined by

$$H^\circ = \{ \gamma \in A^* \mid \forall h \in H \mid \gamma(h) \mid \leq 1 \}$$

The bipolar theorem with respect to this polar, then states that the bipolar $H^{\circ\circ}$, which now is a subset of $A$, is the smallest balanced, convex and norm closed set in $A$ which contains $H$.

We can now start the presentation of the generalization of the Arzelá-Ascoli Theorem and our first lemma is closely connected to the very fundamental structure in $C^*$-algebra theory, that a continuous real linear functional can be decomposed in a unique way into a difference of two positive functionals, such that a certain norm identity holds.

**Lemma 5.1.** Let $A$ be a unital $C^*$-algebra and $S$ the state space of $A$, then

$$S - S = (A^*_h)_2 \cap \{ CI \}^\perp.$$ 

**Proof.** The inclusion ”$\subseteq$” is obvious. To prove the remaining inclusion ”$\supseteq$” let us take an arbitrary element $f$ in $(A^*_h)_2 \cap \{ CI \}^\perp$. It is well known that for $f$ in $(A^*_h)_2$ we can find two positive linear functionals $f^+$ and $f^-$ such that $f = f^+ - f^-$ and $\|f\| = \|f^+\| + \|f^-\|$. If $f = 0$ we can write $f$ as a difference $g - g$ where $g$ is any state on the unital algebra $A$. If $f \neq 0$ the condition $f(I) = 0$ implies that $0 \neq \|f^+\| = \|f^-\| = \frac{1}{2}\|f\| \leq 1$. Based on $f^+$ we can then define a positive functional $g$ of norm $\|g\| = 1 - \|f^+\|$ by

$$g = \frac{(1 - \|f^+\|)}{\|f^+\|} f^+.$$ 

By construction it follows that $f^+ + g$ and $f^- + g$ are both states and from the equality

$$f = (f^+ + g) - (f^- + g)$$

we can conclude that $f \in S - S$, and the lemma follows. $\square$

Following the ideas of Rieffel [R11] we deduce the following result

**Lemma 5.2.** Let $A$ be a unital $C^*$-algebra, $S$ the state space of $A$ and $K$ a norm compact subset of $A$ which separates the points in the state space. Then for states $\varphi, \psi$ on $A$ the formula

$$d_K(\varphi, \psi) := \sup_{k \in K} |(\varphi - \psi)(k)|$$

defines a metric on the state space $S$ which generates the $w^*$-topology.

**Proof.** The separation property and the compactness assumption show that $d_K$ is a bounded metric on $S$. The norm compactness of $K$ and the boundedness of $S$ further implies that the topology induced by $d_K$ is a Hausdorff topology weaker than the compact $w^*$-topology on $S$. A well known theorem from topology then tells that the two topologies do agree, and the lemma follows. $\square$

We can now state and prove the result of this section.

**Theorem 5.3.** Let $A$ be a unital $C^*$-algebra and $K$ a metric subset of $A$. For any subset $H$ of $A$ the following conditions are equivalent

(i) The set $H$ is norm precompact.

(ii) The set of affine functions $\{ h \in A(S) \mid h \in H \}$ is bounded and equicontinuous with respect to the $w^*$-topology on $S$. 


(iii) The set \( H \) is bounded and for every \( \varepsilon > 0 \) there exists a real \( N > 0 \) such that

\[ H \subseteq A_{\varepsilon} + NK + CI. \]

Proof. The equivalence between (i) and (ii) follows from the classical Arzelá-Ascoli Theorem and the fact mentioned above that \( \hat{H} \) is bounded if and only if \( \hat{H} \) is bounded.

To prove the equivalence between (ii) and (iii) we start with (iii) \( \Rightarrow \) (ii). From the boundedness of \( H \) it follows that the set \( \{ \hat{h} \mid h \in H \} \) is bounded. To prove the equicontinuity of this set let us fix an \( \varepsilon > 0 \) and find a positive real \( N \) which fulfills the condition (iii) with respect to \( \varepsilon^4 \). Moreover let \( \varphi \) and \( \psi \) be two states such that

\[ d_K(\varphi, \psi) < \frac{\varepsilon}{2N}. \]

then we will show that for any \( h \in H \), \( |\hat{h}(\varphi) - \hat{h}(\psi)| \leq \varepsilon \). Let now \( h \) be an arbitrary element in \( H \), then by (iii) we can find an element \( a \in A_1 \), an element \( k \in K \) and a complex number \( \mu \) such that

\[ h = \frac{\varepsilon}{4}a + Nk + \mu 1. \]

then we obtain

\begin{align*}
(5.1) & \quad |\hat{h}(\varphi) - \hat{h}(\psi)| = |(\varphi - \psi)(h)| \\
(5.2) & \quad \leq |(\varphi - \psi)(\frac{\varepsilon}{4}a)| + |(\varphi - \psi)(Nk)| \\
(5.3) & \quad \leq \frac{\varepsilon}{2} + Nd_K(\varphi, \psi) \\
(5.4) & \quad < \varepsilon.
\end{align*}

and the equicontinuity of \( \hat{H} \) has been established.

To prove the last implication (ii) \( \Rightarrow \) (iii) we again first mention that \( H \) is bounded. Let then \( \varepsilon > 0 \) be given and find, via the equicontinuity assumption on \( \hat{H} \), a \( \delta > 0 \) such that

\[ \forall h \in H \forall \varphi, \psi \in S : \quad d_K(\varphi, \psi) \leq \delta \Rightarrow |(\varphi - \psi)(h)| = |\hat{h}(\varphi) - \hat{h}(\psi)| \leq \varepsilon. \]

We will now use the bipolar theorem and remark that the expression \( d_K(\varphi, \psi) \leq \delta \) exactly means that \( \varphi - \psi \in \delta(K^\circ) \). It is clear that \( \varphi - \psi \in S - S \) and an application of Lemma 5.1 then shows that the implication above can just as well be expressed as

\[ \forall h \in H \forall \gamma \in (A_{\varepsilon}^*)_2 \cap \{CI\}^\perp \cap \delta(K^\circ) : \quad |\gamma(h)| \leq \varepsilon. \]

This statement is not sufficient for our computations because it involves the space \( A_{\varepsilon}^* \) rather that just \( A^* \). Since a functional on \( A \) vanishes on the identity \( I \) if and only both it’s hermitian and it’s skew hermitian part vanish on \( I \), we can change from \( A_{\varepsilon}^* \) to \( A^* \) at the compensation of a factor of 2, so we have

\[ \forall h \in H \forall \gamma \in A_2^* \cap \{CI\}^\perp \cap \delta(K^\circ) : \quad |\gamma(h)| \leq 2\varepsilon. \]

Since all the sets involved now are convex and balanced the bipolar theorem can be applied very easily. Moreover \( K \) is norm compact so any set of the form \( A_{\varepsilon} + CI + NK \) is norm closed balanced and convex. The relation we just established gives immediately the first inclusion below and the rest follows by some well known “polar techniques” and an application of the bipolar theorem.
(5.5) $\mathcal{H} \subseteq 2\varepsilon \left( A_2^* \cap \{ CI \}^\perp \cap \delta(\mathcal{K}^\circ) \right)^\circ$

(5.6) $= 2\varepsilon \left( A_2^* \cup CI \cup \frac{1}{\delta} \mathcal{K} \right)^\infty$

(5.7) $= 2\varepsilon \operatorname{conv} \left( A_2^* \cup CI \cup \frac{1}{\delta} \mathcal{K} \right)$

(5.8) $\subseteq 2\varepsilon \left( A_2^* + CI + \frac{1}{\delta} \mathcal{K} \right).$

In conclusion for the given $\varepsilon$ we found the number $N = \frac{2\varepsilon}{\delta}$ such that

$\mathcal{H} \subseteq A_\varepsilon + CI + NK$

which proves the desired implication. $\square$

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Institut for Matematiske Fag University of Copenhagen, Universitetsparken 5, DK 2100 Copenhagen Ø, Denmark

E-mail address: chris@math.ku.dk, echris@math.ku.dk