Trajectory structures and transport

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The special problem of transport in 2-dimensional divergence-free stochastic velocity fields is studied by developing a statistical approach, the nested subensemble method. The nonlinear process of trapping determined by such fields generates trajectory structures whose statistical characteristics are determined. These structures strongly influence the transport.

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I. INTRODUCTION

Test particle motion in stochastic velocity fields is a generic problem in various topics of fluid and plasma turbulence or solid state physics [1]-[4]. This problem was very much studied and several analytical approaches were developed. Most of them are based on Corrsin [5], [1] and direct-interaction [6], [7] approximations.

In this context, particle motion in 2-dimensional divergence-free velocity fields represents a special case. Kraichnan has shown for the first time, in a study based on numerical simulations [8], that the existing analytical methods are not adequate for this type of problems. The cause of this anomaly is the trapping of the particles, which appears in such velocity fields when they have slow time variation. The trapping consists in trajectory winding on almost closed paths. A typical trajectory has a complicated shape with such localized trapping events separated by long jumps. Consequently, the probability distribution function is non-Gaussian. Thus Corrsin and direct-interaction approximation which are based on this hypothesis are not adequate for this specific case. A more recent analysis of the effects of trapping is presented in Ref. [9] where a non-Gaussian peaked distribution of the displacements and a long negative tail in the Lagrangian velocity correlations are evidenced for numerically calculated trajectories.

This special problem of diffusion in 2-dimensional divergence-free velocity fields describes for instance the transport in turbulent magnetized plasmas or in incompressible fluids. It was studied especially by means of direct numerical simulations ([10] and the reference there in) or on the basis of simplified models [11], [12]. There is also a qualitative theoretical estimation of the scale law for the asymptotic diffusion coefficient [13] based on an analogy with the percolation process in stochastic landscapes. The case of collisional particle motion in such static velocity fields was analyzed by means of the renormalization group techniques ([2] and the references there in) and the asymptotic time behavior of the mean square displacement was determined. The evolution of the diffusion process is determined only in [14] where a statistical approach, the decorrelation trajectory method, is developed. It yields analytical expressions for the time dependent diffusion coefficient $D(t)$ and for the correlation of the Lagrangian velocity $L(t)$, that are qualitatively valid for the whole range of the Kubo number (see next Section for the definitions). The basic idea consists of determining the Lagrangian velocity correlation by means of a set of average Lagrangian velocities estimated in subensembles of realizations of the stochastic field. This method could be extended to more complicated physical systems which contain particle collisions [15], average velocities [16] or a supplementary component of the motion perpendicular to the 2-dimensional plane [17], [18]. It was shown that the presence of trapping strongly influences the diffusion coefficients and their scaling laws determining a rich class of anomalous diffusion regimes. These studies have shown that the decorrelation trajectory method provides a qualitatively good description of the trapping process. However, due to the rather strong approximation introduced in this method (see Section III), there are several qualitative aspects that are not well described [19], [20]. They are related to trajectory fluctuations and their correlation with the stochastic velocity.

The above results concern the effect of trapping on the individual trajectories. The trapping has also collective effects. It determines coherence in the stochastic motion in the sense that bundles of neighboring trajectories form localized structures similar with fluid vortices. The formation of these structures strongly influences the transport. The aim of this paper is to study the statistical characteristics of these trajectory structures determined by the intrinsic trapping appearing in 2-dimensional divergence-free stochastic velocity fields. The average, the dispersion and the probability distribution function for the trajectories in such structures are determined. The evolution of these statistical quantities saturates showing that they do not contribute to the asymptotic transport. The statistical evolution of the distance between two neighboring trajectories is also studied and its mean, dispersion and probability distri-
bution are determined. We show that for the trapped trajectories in a static velocity field, the average decays to zero and the dispersion remains very small for long time and eventually saturates. A very strong clump effect is found for the trapped trajectories. This demonstrates the existence of trajectory structures and their effect of strongly reducing the relative transport. The later is produced only by a small part of the stochastic trajectories which are not contained in these vortical trajectory structures.

The method developed for this study is a semi-analytical statistical approach which extends and improves the decorrelation trajectory method [14] by introducing the fluctuations of the trajectories in the subensembles.

II. THE PROBLEM

Particle motion in a 2-dimensional stochastic velocity field is described by the nonlinear Langevin equation:

$$\frac{dx(t)}{dt} = \nabla \cdot \mathbf{v}(x(t),t), \quad x(0) = 0$$  \hspace{1cm} (1)

where \(x(t)\) represents the trajectory in Cartesian coordinates \(x \equiv (x_1, x_2)\). The stochastic velocity field \(\mathbf{v}(x, t)\) is divergence-free: \(\nabla \cdot \mathbf{v}(x, t) = 0\) and thus its two components \(v_1\) an \(v_2\) can be determined from a stochastic scalar field \(\phi(x, t)\), as:

$$v_i(x, t) = \varepsilon_{ij} \frac{\partial \phi(x, t)}{\partial x_j}$$  \hspace{1cm} (2)

where \(\varepsilon_{ij}\) is the antisymmetric tensor \((\varepsilon_{12} = 1, \varepsilon_{21} = -1, \varepsilon_{11} = \varepsilon_{22} = 0)\). In the studies of turbulence of magnetized plasmas, \(\phi(x, t)\) is essentially the potential \(\phi = -\phi^e / B\) where \(\phi^e / x, t)\) is the electrostatic potential and \(B\) is the magnetic field strength) and in fluid turbulence \(\phi(x, t)e_3\) is the stream function \((e_3\) is the unitary vector along the axis perpendicular on the plane \((x_1, x_2)\)).

The potential \(\phi(x, t)\) is considered to be a stationary and homogeneous Gaussian stochastic field, with zero average and given two-point Eulerian correlation function (EC)

$$E(x, t) \equiv \langle \phi(x_1, t_1) \phi(x_1 + x, t_1 + t) \rangle$$  \hspace{1cm} (3)

where \(\langle \ldots \rangle\) denotes the statistical average over the realizations of \(\phi(x, t)\). The statistical properties of the space derivatives of the potential are completely determined by those of the potential. They are stationary and homogeneous Gaussian stochastic fields like \(\phi(x, t)\). The two-point Eulerian correlations of the derivatives of \(\phi(x, t)\) are obtained as derivatives of the potential EC, \(E(x, t)\). We introduce the notation

$$E_{i..k..}(x, t) \equiv \langle \phi_{i..}(x_1, t_1) \phi_k..(x_1 + x, t_1 + t) \rangle$$

where \(\phi_i(x, t) \equiv \frac{\partial \phi(x, t)}{\partial x_i}\) and the subscript of \(E\) contains the indices of the derivatives of the potential in \(x_1, t_1\) \(t_1\) (left factor) separated by semicolon from the indices of the derivatives of the potential in \(x_1 + x, t_1 + t\) (right factor).

The absence of indices correspond to a factor \(\phi\) inside the average (see Eqs. (5) for examples). One obtains

$$E_{i..k..}(x, t) = (-1)^n \frac{\partial}{\partial x_i} \ldots \frac{\partial}{\partial x_k} E(x, t)$$  \hspace{1cm} (4)

with \(n\) equal to the number of derivatives of the first factor \(\phi_{\ldots}(x_1, t_1)\) inside the above average. In particular, the velocity \(\mathbf{v}(x, t)\) is such a stationary and homogeneous Gaussian stochastic field. The correlation of the velocity components and the potential-velocity correlations are obtained using the definition (2) of the velocity and Eq. (4) as

$$\langle v_1(0, 0) v_1(x, t) \rangle = E_{2;2}(x, t) = -\frac{\partial^2}{\partial x_2^2} E(x, t),$$  \hspace{1cm} (5)

$$\langle v_2(0, 0) v_2(x, t) \rangle = E_{1;1}(x, t) = \frac{\partial^2}{\partial x_1^2} E(x, t),$$

$$\langle v_1(0, 0) v_2(x, t) \rangle = -E_{2;1}(x, t) = -\frac{\partial^2}{\partial x_1 \partial x_2} E(x, t),$$

$$\langle \phi(0, 0) v_1(x, t) \rangle = \varepsilon_{ij} E_{i;1}(x, t) = -\varepsilon_{ij} \frac{\partial}{\partial x_j} E(x, t),$$

$$\langle \phi(0, 0) v_2(x, t) \rangle = \varepsilon_{ij} E_{j;1}(x, t) = \varepsilon_{ij} \frac{\partial}{\partial x_j} E(x, t).$$

These correlations (4)-(5) will be used in the following calculations.

The potential is a continuous function of \(x\) and \(t\) in each realization and it determines an unique trajectory as the solution Eq. (1). Starting from the above statistical description of the stochastic potential and from an explicit EC, \(E(x, t)\), one has to determine the statistical properties of the trajectories. The later can be obtained from the Lagrangian velocity correlation (LVC), defined by:

$$L_{ij}(t) \equiv \langle v_i [x(0, 0)] v_j [x(t, t)] \rangle$$  \hspace{1cm} (6)

for a stationary process. The mean square displacement \(\langle x_i^2(t) \rangle\) and its derivative, the running diffusion coefficient \(D_i(t)\), are determined by this function [21] as:

$$\langle x_i^2(t) \rangle = 2 \int_0^t d\tau L_{ii} (\tau) (t - \tau),$$  \hspace{1cm} (7)

$$D_i(t) = \int_0^t d\tau L_{ii} (\tau).$$  \hspace{1cm} (8)

The probability distribution function of the trajectories can be obtained from the equation for particle density, once the LVC is known [22]. Thus, the solution of the
above Langevin problem consists essentially in determining the LVC (6) corresponding to a given EC (3) of the stochastic potential.

This kind of Langevin problem, sometimes named in the literature diffusion by continuous movements, is non-linear due to the space dependence of the potential, which leads to \( x \)-dependence of the EC (3). The importance of the nonlinearity is characterized by the Kubo number defined by

\[
K = \frac{V \tau_c}{\lambda_c}
\]

where \( V \) is the amplitude of the stochastic velocity, \( \tau_c \) is the correlation time and \( \lambda_c \) is the correlation length. These parameters appear in the EC of the velocity as the maximum value in the origin \( [V^2 = E_{ul}(0,0)] \) and the characteristic decay time and length of this functions.

The Kubo number is thus the ratio of \( \tau_c \) to the average time of flight of the particles over the correlation length, \( \tau_{fl} = \lambda_c/V \). It measures the particle’s capacity of exploring the space structure of the stochastic velocity field before it changes.

For small Kubo numbers the time variation of the velocity field is fast and the particles cannot "see" the space structure of the velocity field. The condition \( K \ll 1 \) \((\tau_c \ll \tau_{fl})\) defines the quasilinear regime (or the weak turbulence case) for which the results are well established: the diffusion coefficient is \( D_{nl} = (\lambda_c^2/\tau_c)K^2 \) and the trajectories have Gaussian distribution.

For \( K > 1 \) \((\tau_c > \tau_{fl})\) the time variation of the stochastic potential is slow and the trajectories approximately follow the contour lines of \( \phi(x,t) \). This produces a trapping effect: the trajectories are confined for long periods in small regions. A typical trajectory shows an alternation of large displacements and trapping events. The latter appear when the particles are close to the maxima or minima of the potential and consist of trajectory winding on almost closed small size paths. The large displacements are produced when the trajectories are at small absolute values of the potential. Thus there is a strong influence of trapping on individual trajectories. Trajectory trapping appears for \( K > 1 \) and becomes stronger as \( K \) increases up to the limit of static field \((K, \tau_c = \infty)\) where the trapping is permanent. It determines the decrease of the diffusion coefficient and the change of its dependence on the Kubo number from the Bohm scaling \([23, 24], D_B \sim (\lambda_c^2/\tau_c)K\), to a trapping scaling, \( D_{tr} \sim (\lambda_c^2/\tau_c)K^\gamma \) with \( \gamma < 1 \). This process was statistically described in [14], [15] and the exponent \( \gamma \) was evaluated for given EC of the potential. In the limit of static potential field \((K, \tau_c = \infty)\) the transport is sub-diffusive with \( D(t) \rightarrow 0 \) and \( \langle x^2(t) \rangle \sim t^\alpha \) with \( \alpha < 1 \). It can be shown (see Section V) that \( \alpha = \gamma \) and thus the same power law describes the large time behavior of the mean square displacement in a static potential and the dependence on \( K \) of the asymptotic diffusion coefficient in a time dependent potential.

Besides this influence on individual trajectories, the trapping has collective effects and it generates trajectory structures similar with fluid vortices. The trapping appears coherently for bundles of neighboring trajectories leading to eddying regions. We analyze here the statistical characteristics of these trajectory structures. We show that the dispersion of the trapped trajectories saturates and that the mean square of the distance between trajectories evolve slowly and eventually saturates. The motion of trapped particles is almost coherent and leads to structures which do not contribute to the transport (trajectory dispersion) nor to the relative transport.

### III. THE NESTED SUBENSEMBLE METHOD

We start from the main idea of the decorrelation trajectory method [14]. It consists in studying the Langevin equation (1) in subensembles (S) of realizations of the stochastic field, which are determined by given values of the potential and of the velocity in the starting point of the trajectories:

\[
(S) : \quad \phi(0,0) = \phi^0, \quad v(0,0) = v^0.
\]

We note that similar subensemble averages of the Eulerian stochastic velocity field were studied in Ref. [25] with the aim of showing that eddies and structures exist even in isotropic turbulence. Subensemble Lagrangian averages are estimated in [26] on the basis of a rather strong assumption and in [12] for a model of rotating fluid layers. Our approach is different.

The statistical properties of the stochastic potential and velocity, reduced in the subensemble (S) defined by condition (10) are derived in [14]. They are Gaussian fields but non-stationary and non-homogeneous, with space and time dependent averages and dispersions. The averages depend on the parameters of the subensemble and are defined by

\[
\Phi^E(x,t; S) \equiv \langle \phi(x,t) \rangle_S, \quad V^E(x,t; S) \equiv \langle v(x,t) \rangle_S
\]

where \( \langle \ldots \rangle_S \) denotes the average taken on the realizations in (S) and the superscript \( E \) is used to underline the Eulerian nature of these quantities. They are equal to the corresponding imposed condition (10) in \( x = 0 \) and \( t = 0 \) and decay to zero at large distance and/or time. The mean squares of the potential and velocity fluctuations are zero in \( x = 0, t = 0 \) and increase up to the values corresponding to the whole set of realizations at large distance and/or time. The existence of an average Eulerian velocity in the subensemble determines an average motion (decorrelation trajectory). This decorrelation trajectory is estimated in each subensemble and an approximation for the LVC is derived in terms of the set of these trajectories.
More precisely, the LVC (6) for the whole set of realizations is obtained by summing up the contributions of each subensemble. The latter can be written as \( \langle v_i \langle x(0), 0 \rangle \rangle \langle v_j \langle x(t), t \rangle \rangle_S = \langle v_i \langle x(t), t \rangle \rangle_S \). Thus, the problem of evaluating the LVC reduces to the determination of the average Lagrangian velocity in each subensemble (S), \( \mathbf{V}^L(t; S) \equiv \langle \mathbf{v} \rangle_{\langle x(t), t \rangle} \). This is one of the advantages brought by the subensemble analysis: the two-point LVC (6) can be expressed as a function of one-point averages \( \langle \mathbf{V}^L(t; S) \rangle \) in subensembles which correspond to given initial velocity:

\[
L_{ij}(t) = \int \int d\phi^0 d\nu^0 P_i(S) v_i^0 V_j^L(t; S) \tag{12}
\]

where \( P_i(S) \) is the one-point Gaussian probability density for the values of the potential and velocity at any moment perpendicular to the gradient of the potential in the instantaneous position \( x(t) \). It represents the probability that a realization belongs to the subensemble (S) and is obtained as:

\[
P_i(S) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{E(0,0)E_{11}(0,0)E_{22}(0,0)}} \times \exp \left( -\frac{(\phi^0)^2}{2E(0,0)} - \frac{(v_i^0)^2}{2E_{11}(0,0)} - \frac{(v_j^0)^2}{2E_{22}(0,0)} \right) \tag{13}
\]

because the potential and the velocity components are not correlated in the same point \( \langle \phi(0,0)v_j(0,0) \rangle = 0 \) when \( E(x) \) has a maximum in \( x = 0 \), as happens in most cases. Eq. (12) is an exact equation.

The approximation introduced in the decorrelation trajectory method is in the estimation of \( \mathbf{V}^L(t; S) \) and essentially consists in neglecting the fluctuations of the trajectories around the average trajectory in (S). Thus the average Lagrangian velocity in (S) is approximated with the average Eulerian velocity calculated along the average trajectory:

\[
\langle \mathbf{v} \rangle_{\langle x(t), t \rangle} \approx \langle \mathbf{v} \rangle_{\langle x(t), t \rangle} \quad \text{for} \quad \phi(0,0) = 0, \quad \text{when} \quad E(x) \text{ has a maximum in} \ x = 0.
\]

i.e.

\[
\mathbf{V}^L(t; S) \approx \mathbf{V}^E \langle \mathbf{x}(t, S), t; S \rangle \tag{15}
\]

where \( \mathbf{X}(t, S) \) is the average trajectory in (S), \( \mathbf{x}(t, S) \approx \langle \mathbf{x}(t), t \rangle_S \). Thus, this average trajectory in (S) (decorrelation trajectory) is determined from the equation

\[
\frac{d\mathbf{X}(t, S)}{dt} = \mathbf{V}^E \langle \mathbf{x}(t, S), t; S \rangle \tag{16}
\]

so that an explicit expression for the average Lagrangian velocity is obtained. It was shown that Eq. (16) is of Hamiltonian type with the Hamiltonian function equal to the average Lagrangian potential \( \Phi^E [\mathbf{X}(t, S), t; S] \). Thus, in the static case the average trajectories obtained from the approximation (14) are periodic and evolve on closed paths. They provide a representation of the trapping and lead to a subdiffusive transport for the static potential and to trapping scaling for the asymptotic diffusion coefficient, \( D_{tr} \sim K^\gamma \) with \( \gamma < 1 \), in time dependent stochastic potentials with slow variation (\( K > 1 \)).

The approximation (14) seems to be rather rough but, because it is performed in the subensemble, there are two aspects which contribute to improving its accuracy. One is due to the fact that the fluctuations of the velocity in (S), which are the source of the trajectory fluctuations, are zero in the starting point of the trajectories and become important only if the trajectory reaches large enough distances. The second is related to the fact that the trajectories in the subensemble are superdetermined. Besides the necessary and sufficient initial condition \( x(0) = 0 \), they have supplementary initial conditions determined by the definition (10) of the subensembles. This reduces the differences between the trajectories in (S) and thus the fluctuations. The first description of the trapping process in qualitative agreement with the numerical simulations was obtained using this approximations [14]-[18]. However, there are important qualitative aspects that are not obtained from this approximation. The most evident concerns the average trajectory in the subensemble. In the static case the trajectory in each realization (solution of (1)) is periodic but the average of such trajectories cannot be periodic (as obtained from (16)) since they have different periods (distributed around some average value which depends on (S)).

Another aspect is discussed in [19], [20] and concerns the average Lagrangian velocity in a biased stochastic potential. It is thus necessary to improve the decorrelation trajectory method by taking into account the fluctuations of the trajectories in the subensembles. This development is also required by the aim of this paper. The decorrelation trajectory method as presented in [14] is not able to describe the statistics of the trajectories in (S) nor the two-particle statistical quantities.

The analysis of the decorrelation trajectory method leads to the conclusion that this method succeeds in describing the trajectory trapping due to the fact that it maintains the invariance of the average Lagrangian potential in (S). The motion described by Eqs. (1), (2) has the velocity at any moment perpendicular to the gradient of the potential in the instantaneous position \( x(t) \) and the time variation of the Lagrangian potential is produced only by the explicit time dependence of \( \phi(x, t) \)

\[
\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial \phi}{\partial t} \frac{dt}{dt} = \frac{\partial \phi}{\partial t}.
\]\n
Thus, for the static case where \( \partial \phi/\partial t = 0 \) (\( \tau_s, K \to \infty \)), the potential is an invariant of the motion. The trajectories are on the contour lines of the potential and the
motion is periodic. For slowly varying or large amplitude potentials such that \( \tau_e > \tau_{fl} \) (i.e., \( K > 1 \)), \( \partial \phi / \partial t \) is small and there is an approximate invariance of the potential along the trajectory in each realization. Trajectory trapping is essentially related to this invariance of the Lagrangian potential.

The average Lagrangian potential in (S) is invariant in the frame of the decorrelation trajectory method. Indeed, one obtains, by neglecting trajectory fluctuations as in (14), the average Lagrangian potential in (S) as

\[
\Phi^L(t; S) \equiv \langle \phi \{ x(t), t \} \rangle \equiv \Phi^E \{ X(t), t; S \} .
\]  

(18)

Since the Eulerian quantities \( V^E(x, t; S) \) and \( \Phi^E(x, t; S) \) are related by an equation similar to (2)

\[
V^E_i(x, t; S) = \varepsilon_{ij} \frac{\partial \Phi^E(x, t; S)}{\partial x_j},
\]  

(19)

the time derivative of (18) is zero for the static case and, due to the definition (10) of the subensembles (S), \( \Phi^L(t; S) = \Phi^E(0; S) = \phi^0 \) at any time.

The aim of this paper is to consider the fluctuations of the trajectories in the subensemble (S) and to study their effect. The approximations (14), (14) that neglect trajectory fluctuations have to be replaced. As suggested by the above discussions, this development must be performed having in mind the idea of maintaining the condition of invariance of \( \Phi^L(t; S) \). Moreover, the invariance of the Lagrangian potential applies to each trajectory, i.e., in each realization. Thus, it determines, besides the invariance of \( \Phi^L(t; S) \), other statistical constraints. Actually, in the static case the probability distribution function for the Lagrangian potential in a subensemble (S) is

\[
P^S(\phi, t) \equiv \langle \delta \{ \phi - \phi \{ x(t) \} \} \rangle_S = \delta(\phi - \phi^0)
\]  

(20)

because \( \phi \{ x(t) \} = \phi^0 \) for all trajectories in (S) and for any time moment. We note that the knowledge of \( P^S(\phi, t) \) is not very helpful in solving the problem of determining the statistical properties of the trajectories but it rather imposes strong conditions on the possible approximations. For example, the hypothesis that trajectory fluctuations in (S) are Gaussian around the average trajectory is excluded by Eq. (20). It can be shown that such distribution is not compatible with the \( \delta \)-distribution of the Lagrangian potential.

Trajectory fluctuations in the subensembles (S) are considered here in agreement with the condition (20) by separating the realizations in (S) in subensembles (S2) corresponding to fixed values of the second derivatives of the potential in \( x = 0, t = 0 \)

\[
(S2) : \quad \phi_{ij}(0, 0) \equiv \frac{\partial^2 \phi(0, 0)}{\partial x_i \partial x_j} |_{x=0, t=0} = \phi^0_{ij}
\]  

(21)

where \( ij = 11, 12, 22 \). The Langevin equation (1) is studied in these subensembles (S2). The average trajectory is approximately determined by neglecting trajectory fluctuations in (S2). Pushed to the subensembles (S2), this approximation is much more accurate than taken in (S) because the trajectories in (S2) are much more superdetermined than those in (S); three supplementary initial conditions (21) are added to the initial conditions (10). Moreover, the source of trajectory fluctuations, the velocity fluctuations, has smaller amplitude in (S2) than in (S). Thus the accuracy of this method is much increased compared to the accuracy of the decorrelation trajectory method. But the main advantage of performing this development consists in the fact that it provides for each subensemble (S) a collection of trajectories instead of one decorrelation trajectory. It is thus possible to determine the statistical properties of the trajectories in (S) by performing averages over the subensembles (S2) contained in (S). We determine using this nested subensemble procedure the dispersion of the trajectories in (S) and their probability distribution function, besides the average trajectory. The statistical evolution of the distance between two trajectories is also studied by this method and the average, the dispersion and the probability distribution function are determined as functions of time.

We note that this nested subensemble approach can be further developed by introducing subensembles defined by higher order derivatives of the potential, (S3), (S4), .... This systematic expansion fulfills at each order higher than 1 all the conditions required by the invariance of the Lagrangian potential. It is however expected that the main statistical properties of the stochastic trajectories in (S) are already obtained at the second order and that the higher orders contribute with corrections to these results. The nested subensembles (S), (S2) are considered in this study.

The explicit calculations consist of the following steps. First, the statistical properties of the stochastic potential and velocity, reduced in the subensemble (S2) defined by conditions (21) and (10) are derived. Namely, the probability that a realization belongs to a subensemble (S2) and the subensemble average Eulerian velocity and potential are determined. These average quantities lead to the equation for the average trajectory in (S2) by neglecting the fluctuations of the trajectories in (S2). Then the statistical properties of the trajectories and of the distance between two trajectories in the subensemble (S) are expressed as functions of the average trajectories in all subensembles (S2) contained in (S). The calculations are for the static stochastic potential corresponding to the strongest trapping. The transport in time-dependent potential is discussed in Section VI B.

A. Eulerian statistics in the subensemble (S2)

The (one-point) probability that a realization of the stochastic potential belongs to the subensemble (S2) con-
as the derivative at a parameter \( a \) with an additional term the exponential of the above linear combination performing the inverse Fourier transforms 

\[
\phi(x; S2) = \frac{E(x)}{E(0)} \left[ \phi^0 + \frac{a_1}{c} (E_{22}(0)c_{12} - E_{11}(0)c_2) + \frac{a_2}{c} (E_{11}(0)c_{12} - E_{22}(0)c_1) \right] + \frac{E_2(x)}{E_{22}(0)} \frac{\phi^0}{E_{11}(0)} \phi^0 + \frac{E_{12}(x)}{E_{12}(0)} \phi^0 \left[ (a_1 c_2 - a_2 c_{12}) + E_{22}(0)(a_2 c_1 - a_1 c_{12}) \right].
\]

The subensemble (S2) average potential (28) equals \( \phi^0 \) for all \( x = 0, t = 0 \) and it decays to zero at large \( x \), as the average Eulerian potential in the upper subensemble (S). Its expression is more complicated and depends on the second order derivatives \( \phi^0_{ij} \) that label (S2). The potential considered only in the realizations in the subensemble (S2) is a non-stationary and non-homogeneous Gaussian field having a space-time dependent average. As in the whole ensemble and as in (S), the statistical properties of the velocity field in the subensemble (S2) are deduced from those of the potential in (S2). The velocity in the subensemble (S2) (21) is a non-stationary and non-homogeneous Gaussian stochastic field having a space-time dependent average. This average Eulerian velocity is calculated by the same procedure used for the potential (28). The relation (2) between velocity and potential in each realization holds between the respective average quantities calculated in the nested subensembles. It was obtained in the subensemble (S), Eq. (19), and it can be shown that:

\[
V^E_i(x; S2) = \varepsilon_{ij} \frac{\partial \phi^E(x; S2)}{\partial x_j}.
\]

Thus the average velocity in the subensemble (S2) is divergence-free: \( \nabla \cdot V^E_i(x; S2) = 0 \).

It is interesting to note that the potential and the velocity in the subensembles (S) and (S2) are deterministic quantities in \( x = 0 \) \( \phi(0) = \phi^0, v(0) = v^0 \) for all realizations in (S), thus also in (S2). As \( x \) grows, the average values decay to zero and the fluctuations build up progressively and eventually become the same as in the global statistical ensemble. The amplitude of the fluctuations in a point \( x \) is smaller in the subensemble (S2) than in the subensemble (S).

This nested subensemble procedure evidences, in the zero-average stochastic velocity field, a set of average velocities (corresponding to each subensemble). They depend on the statistical characteristics of the velocity field (the correlation and the constraint imposed in the problem, i.e., the zero-divergence condition). In the nested subensembles introduced here, the following relation holds between the (S2) average velocities and the (S) average velocity:

\[
\Phi^E(x; S2) = \frac{E(x)}{E(0)} \left[ \phi^0 + \frac{a_1}{c} (E_{22}(0)c_{12} - E_{11}(0)c_2) + \frac{a_2}{c} (E_{11}(0)c_{12} - E_{22}(0)c_1) \right] + \frac{E_2(x)}{E_{22}(0)} \frac{\phi^0}{E_{11}(0)} \phi^0 + \frac{E_{12}(x)}{E_{12}(0)} \phi^0 \left[ (a_1 c_2 - a_2 c_{12}) + E_{22}(0)(a_2 c_1 - a_1 c_{12}) \right].
\]
\[ V^E(x; S) = \int d\phi_0^0 d\phi_{12}^0 d\phi_{22}^0 P_1(S2) V^E(x; S2). \] (30)

Similar equations can be written for all statistical quantities defined in the nested subensembles.

**B. Average Lagrangian velocity in the subensemble (S2)**

The average Eulerian velocity (29) determines an average displacement in the subensemble (S2), \( \mathbf{X}(t; S2) \). It is the time integral of the average Lagrangian velocity in (S2), \( V^L(t; S2) \). The latter is evaluated using an approximation similar to (14) as

\[ V^L(t; S2) \equiv V^E [\mathbf{X}(t; S2); S2]. \] (31)

by neglecting the fluctuations of the trajectories in the subensemble (S2). A nonlinear equation for \( \mathbf{X}(t; S2) \) is so obtained

\[ \frac{d\mathbf{X}(t, S2)}{dt} = V^E [\mathbf{X}(t; S2); S2]. \] (32)

With this approximation the average Lagrangian potential in (S2) is \( \langle \phi [x(t)] \rangle_{S2} \equiv \Phi^E [\mathbf{X}(t; S2); S2] \) and due to Eq. (29)

\[ V^E [\mathbf{X}(t; S2); S2] = \varepsilon_{ij} \frac{\partial}{\partial X_j} \Phi^E [\mathbf{X}(t; S2); S2]. \]

which shows that Eq. (32) has a Hamiltonian structure. The solution of this equation with the initial condition \( \mathbf{X}(0; S2) = 0 \) ensures the invariance of its time-independent Hamiltonian function \( \Phi^E [\mathbf{X}(t; S2); S] \).

Since the Eulerian average potential (28) has the value \( \phi^0 \) in \( x = 0 \), the average Lagrangian potential equals \( \phi^0 \) at any time moment.

**C. Lagrangian statistics in the subensemble (S)**

The nested subensemble method provides for a subensemble (S) an ensemble of trajectories, the average trajectories \( \mathbf{X}(t; S2) \), one for each subensemble (S2) contained in (S). The statistical properties of these trajectories can be determined performing averages over the subensembles (S2).

The average velocity in (S), \( V^L(t; S) \), is determined by averaging \( V^L(t; S2) \) obtained from the solution of Eq. (32) over all subensembles (S2) contained in (S)

\[ V^L(t; S) = \int d\phi_0^0 d\phi_{12}^0 d\phi_{22}^0 P_1(S2) V^E [\mathbf{X}(t; S2); S2]. \] (33)

Similar equations hold for all the statistical quantities corresponding to the subensemble (S). The average trajectory in the subensemble (S), \( \mathbf{X}(t; S) \equiv \langle \mathbf{x}(t) \rangle_S \), is determined by averaging \( \mathbf{X}(t; S2) \), the solution of Eq. (32)

\[ \mathbf{X}(t; S) = \int d\phi_0^0 d\phi_{12}^0 d\phi_{22}^0 P_1(S2) \mathbf{X}(t; S2). \] (34)

The dispersion of the trajectories, 

\[ d_i(t; S) \equiv \langle (x_i(t) - \mathbf{X}(t; S))^2 \rangle_S \]

is

\[ d_i(t; S) = \int d\phi_{12}^0 d\phi_{22}^0 P_1(S2) X_i^2(t; S2) - X_i^2(t; S). \] (35)

The probability distribution function (pdf) for the trajectories in (S) is determined by integrating the pdf in the subensemble (S2) which in this approximation is \( \delta [x - \mathbf{X}(t; S2)] \)

\[ P^S(x, t) = \int d\phi_0^0 d\phi_{12}^0 d\phi_{22}^0 P_1(S2) \delta [x - \mathbf{X}(t; S2)]. \] (36)

The pdf for the Lagrangian potential \( \phi [x(t)] \) in the subensemble (S) obtained by this method equals \( \delta (\phi - \phi^0) \) since the potential is equal to \( \phi^0 \) on all trajectories considered in the average. This shows that the approximation (31) introduced in this method ensures entirely the statistical properties of the Lagrangian potential.

The statistical properties in the subensemble (S) of the distance between two trajectories \( \delta (t) \equiv x'(t) - x(t) \) starting from \( x'(0) = \delta_0 \) and \( x(0) = 0 \) respectively can also be determined using the average over the subensembles (S2). The average trajectory in (S2), \( \langle x(t) \rangle_{S2} \), is the solution of Eq. (32) with the initial condition \( x'(0; S2) = \delta_0 \). The average, the dispersion and the pdf of \( \delta (t) \) in the subensemble (S) are determined by equations similar to (34)-(36) where \( \mathbf{X}(t; S2) \) is replaced by \( \mathbf{X}'(t; S2) - \mathbf{X}(t; S2) \).

**D. Running diffusion coefficient**

The correlation of the Lagrangian velocity in the whole set of realizations is determined using Eq. (12) where \( V^L(t; S) \) is the time derivative of the average trajectory in (S), given by Eq. (34). The running diffusion coefficient (8) is obtained from (12) and (34) as

\[ D_i(t) = \int \int d\phi^0 d\nu^0 P_1(S) \nu_i^0 X_i(t; S). \] (37)

In the case of an isotropic stochastic field, the integral over the orientation of the velocity \( \mathbf{v}^0 \) can be performed analytically [14] and one obtains for the static case

\[ D(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{E(0,0)E_{1:1}(0,0)}} \times \] (38)
\[ \int_0^\infty d\phi^0 \int_0^\infty du \: u^2 \exp \left( - \frac{(\phi^0)^2}{2E(0,0)} - \frac{u^2}{2E(1;0,0)} \right) X_1(t;S), \]
and
\[ L(t) = D'(t), \quad (39) \]

where \( X_1(t;S) \) is the component of the average trajectory along \( \mathbf{v}^0 \), determined from Eq.(34), and \( D'(t) \) is the derivative of the function \( D(t) \). We note that the same analytical expression for \( D(t) \) in terms of the average trajectory \( X_1(t;S) \) is obtained in [14] by means of the decorrelation trajectory method. But the average trajectory was determined there as solution of a Hamiltonian equation while here it is the average (34) of the average trajectories in (S2).

IV. EXPLICIT CALCULATIONS

The nested subensemble method actually is based on the classification of the stochastic trajectories in groups (subensembles) according to some resemblance between them. The most important criterion in this classification is the value of the potential in the starting point of the trajectories. All trajectories contained in such a group evolve on contour lines with the same value of the potential. Consequently their paths and periods are statistically similar in the sense that they have an average size and period. Thus this condition determines a global resemblance of the trajectories (extended at long time). This condition is imposed beginning with the first level of classification (in the subensembles (S)). Other criteria of the classification are the velocity and the derivatives of the velocity in the origin. These are not conserved quantities and they influence the shape of the trajectory only at small time for time intervals that grow with the number of imposed conditions. The value of the initial velocity is fixed in the subensemble (S), then each subensemble (S) is divided in smaller subensembles (S2) according to the value of the derivatives of the velocity (second derivatives of the potential). This classification can continue in principle and at each step the resemblance of the trajectories contained in a group is increased and the number of groups grows. The approximation consists in neglecting the differences between the trajectories in a group. With this condition it is possible to determine an average trajectory for each subensemble. Thus the nested subensemble method determines a set of trajectories \( \mathbf{X}(t;S2) \) and a weighting factor for each one. Then, the statistical properties of the stochastic trajectory are obtained by performing averages over these trajectories.

Except for some special case, the trajectories \( \mathbf{X}(t;S2) \) have to be numerically calculated by solving the Hamiltonian system (32). This procedure appears to be very similar with a direct numerical study of the simulated trajectories. There are however essential differences. The average trajectories are obtained from a rather smooth and simple time-independent Hamiltonian. They are periodic functions and thus are calculated only for a period. The number of trajectories is much smaller than in the numerical study due to the weighting factor determined analytically. This reduced very much the calculation time, such that it can be performed on PC. Moreover such a calculation performed for a static stochastic potential with a given EC determines the solution for the time dependent potential with arbitrary time factor in the EC (see Section VI B).

We have developed an algorithm for calculating the statistical characteristics of the trajectories in subensembles (S) and the running diffusion coefficient (38) for given EC of the potential. The trajectories are calculated for a period using a variable integration step determined monitored by the precision obtained for the potential. Values of this precision of \( 10^{-3} - 10^{-4} \) ensure the stability of the calculated \( D(t) \). The order of performing the integrals in Eq.(38) appears to be important. The integral over \( u \) is first calculated. This parameter is factorized in the expression of the average Eulerian potential in (S2) (28) such that it appears as a factor in the right hand side of Eq.(32). This factor is introduced in the time variable and so only the trajectories with \( u = 1 \) need to be calculated. The values of the function \( X_1(nt;S2) \) are determined by interpolation and using the periodicity, the range of \( u \) is determined from the range of the exponential factor and the step \( du \) is determined at each integration such that a large enough number of points (30-50) exists on each period. When there are more than about 50 periods on the range of \( u \) the integration is not performed because its value is negligible. The next integrations are over \( \phi^0_{12}, \phi^0_{11} \) and \( \phi^0_{12} \). The integrand for each of these integrals is over the range determined by the exponential factor and we have calculated it on a mesh with constant step with 31-61 points. The last is performed the integral over \( \phi^0 \). Due to trajectory trapping, the range of this integral is continuously reduced as time increases (\( \phi^0_{\text{max}} \rightarrow 0 \) when \( t \rightarrow \infty \)). The range of this integral is calculated as a function of time for the interval of calculation of \( D(t) \). The calculations start with a large value of \( \phi^0_{\text{max}} \), as obtained from the exponential factor. At the time when the integrand becomes approximately zero on half of this range \( \phi^0_{\text{max}} \) is reduced and the integration of the trajectories is taken again from \( t = 0 \) for the new values of \( \phi^0 \). The mesh for \( \phi^0 \) has variable steps that increase toward large values of \( \phi^0 \) because the function has strong variations at small \( \phi^0 \). The tests performed with this code have shown that the numerical calculations are rather fast and accurate and they can be advanced up to large values of time. For instance, using the decorrelation trajectory method the duration of the calculation of \( D(t) \) up to time of the order \( 10^2 \) is of the order of 10 seconds on an usual PC. Using the nested subensemble method the calculation time is of the order
of one day because the number of calculated trajectories increases with a factor $10^4$.

The nested subensemble method determines the LVC for test particles moving in a stochastic potential with given EC. The main condition for using this method is that the transport is stationary, which usually corresponds to stationary and homogeneous stochastic potentials. The potential field is Gaussian. The time-dependent diffusion coefficient (38) corresponds to an isotropic potential but this is not a restriction for this method. The above calculation are for a static potential but they can be extended to time dependent case (see Section VI B and [14]).

V. TRAJECTORY STRUCTURES

We present in this Section typical results obtained for the statistical characteristics of the trajectories in a subensemble (S2). We need to specify the EC of the stochastic potential, which we choose as

$$E(x) = \frac{1}{1 + (x_1^2 + x_2^2)/2}. \tag{40}$$

This is the EC of a normalized stochastic potential with amplitude $E(0) = 1$, $\lambda_c = 1$ and the time with $\tau_{fl} = 1$. The velocity $v^0$ that defines the subensembles (S) is taken along the $x_1$ axis.

The average trajectory in the subensemble (S2), solution of Eq. (32), is a periodic function of time and evolves on a closed path for most of the subensembles (S2). There are also some open paths for subensembles with $\phi^0 = 0$ and trajectories with very large periods for small values of $|\phi^0|$. Some examples of paths of the average trajectories in (S2) are presented in Fig. 1. There is a clear difference between the trajectories corresponding to small $|\phi^0|$ (Fig. 1a for $\phi^0 = 0$) and large $|\phi^0|$ (Fig. 1b for $\phi^0 = 1$). In the first case there are open trajectories, large displacements and large periods for the closed paths. In the second case the trajectories have small size and their periods are much smaller. The size of the path and the period of the trajectory depend on the six parameters that define the nested subensembles (S), (S2).

The average trajectory in the upper subensemble (S) is obtained from (34). Typical average trajectories in subensemble (S) are presented in Fig. 2. The time dependence of $X(t; S)$ is presented in Fig. 3. These trajectories are not periodic. They evolve on spiral shaped paths, except for the subensemble with $\phi^0 = 0$ which yields a continuous displacement along $v^0$. The size of the paths depends on the parameters of (S), $\phi^0$ and $u \equiv |v^0|$: it is large for small $|\phi^0|$ and large $u$ and it decreases as $|\phi^0|$ increases. The displacement along the initial velocity $v^0$ decays to zero in a characteristic time $\tau_s$, while the displacement perpendicular to $v^0$ saturates at a finite value whose sign is the same as the sign of $\phi^0$. The saturation time $\tau_s$ depends on the parameters of the subensemble (S). It increases when $|\phi^0|$ decreases (as the size of the paths) and when $\phi^0 \to 0$ it becomes infinite. Thus the average trajectories in (S) obtained here are completely different from those obtained by means of the decorrelation trajectory method [14]. The later are periodic functions of time and their paths are closed (see Fig. 2 for comparison). This means that the fluctuations of the trajectories have a strong influence on the average trajectory in (S). They determine the time-saturation of the average trajectory in (S) by the mixing of the closed periodic trajectories.

The dispersion of the trajectories in the subensemble (S) obtained from Eq. (35) is presented in Fig. 3 as a function of time for $\phi^0 = 0$ (Fig. 3a) and $\phi^0 = 1$ (Fig. 3b). One can see that in the first case the dispersion continuously increases (Fig. 3a) while in the second case it saturates after a more complicated evolution (Fig. 3b). The saturation time is the same as for the average trajectory. The amplitude of the trajectory fluctuations is comparable with the average displacement.

The trajectories in the subensemble (S) are Gaussian at small time $t \ll \tau_{fl}$ but their distribution is strongly modified as time increases. The pdf obtained from Eq. (36) is represented in Fig. 4. Important differences can be observed between the subensembles with $\phi^0 \cong 0$ (Fig. 4a) and those with large $|\phi^0|$ (Fig. 4b). In the first case the pdf is symmetric around $v^0$, it develops a narrow maximum in $x = 0$ and an annulus that expands continuously in the direction of $v^0$ as time increases (Fig. 4a). The velocity of this part of trajectories is larger than the average velocity. The path of the average displacement is also represented in Fig. 4a. The end point of this curve is the average position at the moment corresponding to the representation of the pdf ($t = 100\tau_{fl}$). It is located between the two maxima of the pdf in a region where the later is practically zero. The pdf for subensembles with large $|\phi^0|$ is completely different. It saturates in a time $\tau_s$ at a function that has a narrow maximum in $x = 0$ and extends only on a small region (with $x_2 > 0$ for $\phi^0 > 0$).

Thus the statistical characteristics of the trajectories in a subensemble (S) with a large values of $|\phi^0|$ are completely different of those corresponding to subensembles with $\phi^0 \cong 0$. The average, the dispersion and the pdf of these trajectories saturate. This shows that there is a quasi-coherent motion in such subensembles consisting in trajectory rotation on closed paths, with localized pdf and small saturated dispersion. The trajectories form structures similar with fluid vortices. These structures or eddying regions are permanent in static stochastic potentials. The saturation time $\tau_s$ represents the average time necessary for the formation of the structure. The average size of the structure is represented by the asymptotic average displacement $|X(t; S)|$ at $t \gg \tau_s$. The dispersion of the trajectories in the structure is given by the
asymptotic value of \(d_i(t)\). We have found that these characteristic parameters of the trajectory structures depend on the parameters of the subensemble (S). The size, the dispersion and the build up time of the structures increase when \(|\phi^0|\) decreases and go to infinity at \(\phi^0 = 0\).

The existence of the trajectory structures is confirmed by the statistical properties of the distance between two neighboring trajectories, \(\delta(t) \equiv x'(t) - x(t)\). Typical results obtained for the second moment of the relative displacement \(\delta(t), \langle \delta_i^2(t) \rangle_S\), are presented in Fig. 5 (continuous lines) compared with the second moments of the absolute displacements, \(\langle x_i^2(t) \rangle_S\) (dashed lines). In the subensembles with large \(|\phi^0|\) (Fig. 5b), the evolution of \(\langle \delta_i^2(t) \rangle_S\) shows that it maintains long time the initial value \(\delta_0^2\) and that it reaches values comparable with the absolute displacement \(\langle x_i^2(t) \rangle_S\) only after a very long time (of the order of 100\(\tau_{fl}\)). Thus the relative motion is strongly hindered and a very strong clump effect appears in the subensembles with large \(|\phi^0|\). There is a very strong degree of coherence of the relative motion for these trajectories showing that they form structures. On the contrary, the clump effect is very weak (practically absent) for the trajectories which are not in the structures (those in the subensembles with \(\phi^0 \cong 0\)). As seen in Fig. 5a, \(\langle \delta_i^2(t) \rangle_S\) has values comparable to those of \(\langle x_i^2(t) \rangle_S\) since the first stage of the evolution, at time much smaller \(\tau_{fl}\). The pdf of \(\delta(t)\) in a subensemble (S) shows that the relative motion is not Gaussian. In the case of structures (large \(|\phi^0|\)), the pdf remains very localized around zero and saturates (Fig. 6b). In the case of free trajectories (\(\phi^0 \cong 0\)), the pdf has a more complicated shape and extends continuously. At large times it is similar with the pdf of the trajectories (Fig. 6a).

Thus, the trapping of the trajectories has a strong influence on the statistical characteristics of the relative motion. It produces an anomalous clump effect. In the absence of trapping the clump effect appears as an exponential time dependence of the average square distance between two trajectories, of the type \(\langle \delta_i^2(t) \rangle = \delta_0^2 \exp(\tau/\tau_{cl})\), where the clump characteristic time \(\tau_{cl}\) is of the order of the diffusion time which is the flight time \(\tau_{fl}\) at large K (see [28] and the references therein or the review paper [4]). The distance between two neighboring trajectories remains close to the initial value during a time \(\tau_{cl}\) and then \(\langle \delta_i^2(t) \rangle\) grows rapidly and reaches a diffusive behavior with the diffusion coefficient equal to 2\(D\). The process of trajectory trapping determines a complete change of the clump effect. It appears only for a part of the trajectories, those contained in subensembles (S) with large \(|\phi^0|\), and is very strong. The life time of the clump is much larger than \(\tau_{fl}\) and than the saturation time \(\tau_s\). The time evolution of the relative square displacement is much slower. Neighboring particles have thus a coherent motion for a long time. For the other part of the trajectories, those that move along contour lines of the potential with \(\phi^0 \cong 0\), the clump effect is absent and the relative motion become rapidly incoherent, after a time interval smaller than \(\tau_{fl}\).

VI. TRANSPORT

A. Static stochastic potential

The Lagrangian velocity correlation and the time dependent diffusion coefficient for the whole ensemble of trajectories are determined from Eq. (38). The integral over the parameters of the subensembles (S) of the average displacement in (S) has to be calculated. Since \(X_1(t; S)\) decays to zero in a time \(\tau_s(S)\), the trajectory structures have only a transient contribution to the running diffusion coefficient. At times larger that \(\tau_s(S)\) the contribution of the subensemble (S) vanishes. As time increases the diffusion coefficient \(D(t)\) is determined by a smaller and smaller number of trajectories, those contained in large structures with large saturation time. The results obtained for \(D(t)\) are presented in Fig. 7 where the dimensionless function \(F(t) \equiv D(t)/D_B\) is plotted (continuous line). The transport is subdiffusive in such static stochastic potential. One can observe the change that appears at \(t \gtrsim \tau_{fl}\). The running diffusion coefficient begins to decrease and eventually goes to zero. A power law decay was obtained at \(t > \tau_{fl}\) as \(D(t) = \nu c(t/\tau_{fl})^{-0.43}\). The LVC becomes negative at this time and after a minimum it has a long negative algebraic tail that decays to zero. The positive and the negative parts of \(L(t)\) have equal time integral such that \(\int_0^t L(t) dt = D(t) \rightarrow 0\). The mean square displacement is \(\langle x^2(t) \rangle \sim t^{0.57}\) and thus the process is subdiffusive.

The result obtained for \(D(t)\) with the decorrelation trajectory method is also plotted in Fig. 7 (dashed line). It is surprisingly close to the result of the nested subensemble method although the two methods yield completely different average trajectories in the subensembles (S) (Fig. 2). This shows that by introducing the subensembles (S2) in the nested subensemble method a strong qualitative improvement of the statistical results in the next upper subensemble (S) is obtained and only a small correction at the level of the whole set of realizations. It is thus expected that the development of the method by introducing higher order derivatives and the corresponding nested subensembles (S3), (S4), ...would yield only small corrections for the physically interesting results that concern the diffusion coefficient \(D(t)\) and the statistical properties of the trajectory structures. This nested subensemble method appears to be fast convergent. This is a consequence of the fact that the mixing of periodic trajectories, which characterizes this nonlinear stochastic process, is directly described at each order of our approach. The results obtained in first order (the
decorrelation trajectory method) for $D(t)$ are thus validated by the present second order calculations.

**B. Time-dependent stochastic potential**

For a time-dependent potential $\phi(x,t)$ (finite $\tau_c$ and $K$), it is also possible to apply the nested subensemble method following the same procedure as above. A very simple analytical solution is obtained when the stochastic potential has independent time and space variations such that its EC is $E(x)h(t)$.

In this case, the average Eulerian potential in the subensemble (S) is given by Eq.(28) multiplied with the factor $h(t)$. This factor is transmitted to the average Eulerian velocity in (S) (29) and it appears in the equation (32) for the average trajectory in (S2). A change of variable from $t$ to

$$\theta(t) = \int_0^t h(\tau)d\tau$$

(41)

can be performed in Eq.(32) and thus the average trajectory in (S2) for a time dependent potential can be written in terms of the average trajectory for the static case as $X(\theta(t); S2)$. The argument $\theta(t)$ determined by the time-dependence of the potential is $\theta(t) \approx t$ at small $t$ and saturates at a constant which is the decorrelation time $\theta(t) \rightarrow \tau_c$. The same expression (38) is eventually obtained for the time dependent diffusion coefficient but with $X_1(t; S)$ replaced by $X_1(\theta(t); S)$ and thus the diffusion coefficient is

$$D_{td}(t) = D(\theta(t)).$$

(42)

The limit for $t \rightarrow \infty$ is finite which shows that the transport is diffusive in a time dependent stochastic potential and the diffusion coefficient is

$$D_{td} = D(\tau_c) = D_B F(\tau_c).$$

(43)

This equation shows that the asymptotic diffusion coefficient is determined by the time dependent diffusion coefficient $D(t)$ corresponding to the static potential ($F(t)$ is the function plotted in Fig.7 and represents the normalized diffusion coefficient in the static potential and $D_B = (\lambda^2_c/\tau_c)K = V\lambda_c$ is the Bohm diffusion coefficient obtained when trajectory trapping is neglected). In the limit of small $K$, the quasilinear result is recovered from Eq.(43) and, at large $K$, $D_{td}$ is reduced compared to the Bohm diffusion coefficient by a factor $F(K) < 1$ which accounts for trajectory trapping. For the above EC of the potential, Eq.(43) gives the large $K$ scaling law $D_{td} \approx (\lambda^2_c/\tau_c)K^\gamma$ with $\gamma = 0.57$. The exponent $\gamma$ depends on the EC of the potential, namely on its space dependence at large distances. It is not a fixed value as in the estimation based on percolation theory [13]. A detailed study of the effect of the EC of the potential on the scaling of the diffusion coefficient is in progress.

Thus, the study of the static case permits to determine the asymptotic diffusion coefficient in a time dependent stochastic potential. This property appears in the results of the nested subensemble method (and in the decorrelation trajectory method) but it is possible to demonstrate in general that the time dependence of the diffusion coefficient in the subdiffusive static case determines the Kubo number dependence of the asymptotic diffusion coefficient in a time dependent potential. The subdiffusive transport corresponds to Lagrangian correlations $L(t)$ which have the property:

$$D(t) = \int_0^t L(t)dt \rightarrow 0.$$  \hspace{1cm} (44)

This shows that such a correlation has negative parts that compensate the small time correlation which is always positive. We suppose that $D(t)$ decays to zero as $D(t) \approx (t/\tau_{fl})^{-\alpha}$ and consequently the LVC behaves as $L(t) \approx (t/\tau_{fl})^{-\alpha-1}$. In the time-dependent case, the variation of the stochastic field produces the decorrelation of the Lagrangian velocity and consequently the Lagrangian correlation decays to zero at $t \gtrapprox \tau_c$. The asymptotic diffusion coefficient can be evaluated as

$$D = \int_0^\infty L(t)dt \approx \int_0^{\tau_c} L(t)dt$$

and using Eq. (44) one can write

$$D \approx -\int_0^{\tau_c} L(t)dt \approx (\tau_c/\tau_{fl})^{-\alpha} = K^{-\alpha}.$$  \hspace{1cm} (45)

Thus the exponent $\alpha$ of the time decay of the subdiffusive transport coefficient in the static case determines the exponent of the $K$ dependence of the asymptotic diffusion coefficient in the time-dependent case. This means that Eq.(43) holds even if the evolution of $D_{td}(t)$ is not given by Eq.(42) as happens for example when the EC of the potential is not factorized.

The time variation of the potential determines a decorrelation effect. After a time of the order $\tau_c$ the configuration of the stochastic potential changes. A competition appears between the intrinsic tendency of the trajectories to form structures and the destruction of these structures produced by the time variation of the potential field. Structures with $\tau_s(S) \gtrapprox \tau_c$ cannot exist and the corresponding trajectories produce a diffusive transport. Small structures that build up rapidly (with $\tau_s(S) \ll \tau_c$) still exist if the correlation time of the field is longer than the flight time ($\tau_c > \tau_{fl}$, $K > 1$). These vortical structures do not contribute to the large time values of the diffusion coefficient and the transport is reduced.

**VII. CONCLUSIONS**

We have studied the special problem of test particle transport in 2-dimensional divergence-free stochas-
tic velocity fields, which is characterized by the intrinsic trapping of the trajectories on the contour line of the stochastic potential. We have developed a semi-analytical statistical approach, the nested subensemble method. The time dependent diffusion coefficient is determined by means of a set of deterministic trajectories, the average trajectories in subensembles with given values of the potential and of the velocity in their starting point. These trajectories are obtained by dividing each subensemble in a class of subensembles defined by the values of the second derivatives of the potential. Thus, the nested subensemble approach reduces the problem of determining the statistical behavior of the stochastic trajectories to the calculation of weighted averages of some smooth, deterministic trajectories determined from the EC of the stochastic potential.

The statistical characteristics of subensembles of trajectories are obtained with this method. We have shown that the statistical behavior of the trapped trajectories is completely different from that of the free trajectories. The trapped trajectories have a quasi-coherent behavior. Their average displacement, dispersion and probability distribution function saturate. A very strong anomalous clump effect characterizes neighboring trapped trajectories. Their clump life time is very large compared to the time of flight. This shows that these trajectories form structures similar with fluid vortices. The statistical parameters of these structures (size, build-up time, dispersion) are determined. The trajectories contained in such structures do not contribute to the large time diffusion coefficient. The later is determined by the free trajectories which have a continuously growing average displacement and dispersion. The probability distribution function for both types of trajectories are non-Gaussian.

The time dependent diffusion coefficient is determined as a functional of the Eulerian correlation of the stochastic potential.

The general conclusion of this work is that the existence of an invariant in the evolution equation (the potential) determines long-time correlations (memory effects) and coherence (trajectory structures) in the stochastic motion.

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Figure 1
Exemples of paths of average trajectories in subensembles (S2) obtained from Eq. (32) for several values of $\phi_0^i$ and for the values of $\phi^0$ and $u$ mentioned on the figure.
Figure 2
Examples of paths of average trajectories in subensembles (S) obtained from Eq. (34) (continuous lines). The results obtained with the decorrelation trajectory method are also plotted for comparison (dashed lines).
Figure 3
Time evolution of the average trajectory and of the dispersion [Eq. (35)] in subensembles (S) with $\phi^0 = 0$ (a) and for $\phi^0 = 1$ (b).
Contour plot of the pdf [Eq. (36)] of the trajectories in a subensemble (S) for (a) $\phi^0 = 0$, $u = 1$, $t = \tau_{fl}$ and (b) $\phi^0 = 1$, $u = 1$, $t = \tau_{fl}$ (at saturation). The path of the average trajectory is also represented (red line) on the interval $[0, t]$.

Figure 4
The second moments of the relative displacement $\langle \delta_1^2(t) \rangle_S$ (continuous blue line) and $\langle \delta_2^2(t) \rangle_S$ (continuous black line) compared to $\langle x_1^2(t) \rangle_S$ (dashed blue line) and $\langle x_2^2(t) \rangle_S$ (dashed black line) for two subensembles (S): one with $\phi^0 = 0$, $u = 1$ (Fig. 5a) and the other with $\phi^0 = 1$, $u = 1$ (Fig. 5b).
Figure 6
Contour plot of the pdf of the relative motion in a subensemble (S) for (a) $\phi^0 = 0$, $u = 1$, $t = \tau_{fl}$ and (b) $\phi^0 = 1$, $u = 1$, $t = \tau_{fl}$ (at saturation).
Figure 7
The time dependent diffusion coefficient $D(t)$ obtained from Eq. (38) with the nested subensemble method (continuous line) compared with the result of the decorrelation trajectory method (dashed line).