DEFORMING A HYPERSURFACE BY PRINCIPAL RADI OF CURVATURE AND SUPPORT FUNCTION

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ABSTRACT. We study the motion of smooth, closed, strictly convex hypersurfaces in \( \mathbb{R}^{n+1} \) expanding in the direction of their normal vector field with speed depending on the \( k \)-th elementary symmetric polynomial of the principal radii of curvature \( \sigma_k \) and support function \( h \). A homothetic self-similar solution to the flow that we will consider in this paper, if it exists, is a solution of the well-known \( L^p \)-Christoffel-Minkowski problem \( \varphi h^{1-p} \sigma_k = c \). Here \( \varphi \) is a preassigned positive smooth function defined on the unit sphere, and \( c \) is a positive constant. For \( 1 \leq k \leq n-1 \), \( p \geq k+1 \), assuming the spherical hessian of \( \varphi^{\frac{1}{p}} \) is positive definite, we prove the \( C^\infty \) convergence of the normalized flow to a homothetic self-similar solution. One of the highlights of our arguments is that we do not need the constant rank theorem/deformation lemma of [20] and thus we give a partial answer to a question raised in [21]. Moreover, for \( k = n \), \( p \geq n+1 \), we prove the \( C^\infty \) convergence of the normalized flow to a homothetic self-similar solution without imposing any further condition on \( \varphi \). In the final section of the paper, for \( 1 \leq k < n \), we will give an example that spherical hessian of \( \varphi^{\frac{1}{p}} \) is negative definite at some point and the solution to the flow loses its smoothness.

1. An expanding flow

Suppose \( F_0 : M^n \to \mathbb{R}^{n+1} \) is a smooth parametrization of a closed, strictly convex hypersurface \( M_0 \) and suppose the origin of \( \mathbb{R}^{n+1} \) is in the interior of the region enclosed by \( M_0 \). In this paper, we study the long-time behavior of a family of hypersurfaces \( M_t \) given by the smooth map \( F : M^n \times [0, T) \to \mathbb{R}^{n+1} \) satisfying the initial value problem

\[
\begin{align*}
\frac{\partial}{\partial t} F(x, t) &= \varphi(\nu(x, t)) \langle F(x, t), \nu(x, t) \rangle^{2-p} \frac{E_n}{E_n}(x, t) \nu(x, t); \\
F(\cdot, 0) &= F_0(\cdot),
\end{align*}
\]

where \( p \in \mathbb{R} \), \( E_i \) is the \( i \)-th elementary symmetric polynomial of principal curvatures for \( 0 \leq i \leq n \) normalized so that \( E_i(1, \ldots, 1) = 1 \) (and \( E_0 \equiv 1 \)), \( \nu(\cdot, t) \) is the outer unit normal vector of \( M_t := F(M^n, t) \) and \( \varphi \) is a positive smooth function defined on the unit sphere \( S^n \).

Assuming \( M_t \) is strictly convex, its support function as a function on the unit sphere is given by

\[ h_{M_t}(x) = h(x, t) := \langle F(\nu^{-1}(x, t), t), x \rangle. \]

Write \( \bar{g} \) and \( \nabla \) for the standard round metric and the Levi-Civita connection of \( S^n \). Recall that the principal radii of curvature are the eigenvalues of the matrix

\[ r_{ij} := \nabla_i \nabla_j h + g_{ij} h \]

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with respect to $\tilde{g}$. Also write $\sigma_k$ for the $k$th elementary symmetric polynomial of the principal radii of curvature, normalized so that $\sigma_k(1, \ldots, 1) = 1$.

We also define

$$M_t := \left\{ \left( \frac{f_{\varphi} h dx}{\int_{\mathbb{S}^n} \varphi dx} \right)^{\frac{1}{p}} M_t, \begin{array}{ll} \text{if} \ p \neq 0; \\
\exp \left( -\frac{1}{\int_{\mathbb{S}^n} \log h(x,t) dx} \int_{\mathbb{S}^n} \log h(x,t) dx \right) M_t, \end{array} \right. \quad \text{if} \ p = 0. \tag{1.2}$$

Here $dx$ is the Lebesgue measure on $\mathbb{S}^n$ and $\omega_n = \int_{\mathbb{S}^n} dx$.

By direct calculation, we find $h : \mathbb{S}^n \times [0, T) \to \mathbb{R}$ satisfies

$$\begin{cases}
\partial_t h = \varphi h^{2-p} \sigma_k; \\
h(x, 0) = \langle F_0(v^{-1})(x), x \rangle.
\end{cases} \tag{1.3}$$

We consider a normalization of the flow $(1.3)$ given by

$$\partial_{\tau} h = \varphi h^{2-p} \sigma_k - \frac{\int_{\mathbb{S}^n} h \sigma_k dx}{\int_{\mathbb{S}^n} \varphi dx}, \tag{1.4}$$

Caveat. We always distinguish between the solutions to $(1.3)$ and $(1.4)$ respectively through the parameters $t, \tau$.

Note that for $p \neq 0$,

$$\frac{d}{d\tau} \int_{\mathbb{S}^n} \frac{h^p(x, \tau)}{\varphi(x)} dx = p \int_{\mathbb{S}^n} h \sigma_k dx - p \int_{\mathbb{S}^n} \frac{h}{\varphi} dx \int_{\mathbb{S}^n} \frac{h \sigma_k dx}{\varphi dx}$$

$$= p \int_{\mathbb{S}^n} h \sigma_k dx \left( \int_{\mathbb{S}^n} \frac{1}{\varphi} dx - \int_{\mathbb{S}^n} \frac{h}{\varphi} dx \right).$$

If the solution to $(1.4)$ at time $\tau = 0$ satisfies $\int_{\mathbb{S}^n} \frac{h}{\varphi} dx = \int_{\mathbb{S}^n} \frac{1}{\varphi} dx$, then at any later time this identity still holds. If $p = 0$, we always have $\frac{d}{d\tau} \int_{\mathbb{S}^n} \log h(x, \tau) dx = 0$. Note the support functions of $M_t$ after a suitable time re-parametrization solve $(1.4)$.

Our motivation to study the flow $(1.3)$ is due to the significance of its solitons in convex geometry. A positive homothetic self-similar solution of $(1.3)$, when exists, is a solution to

$$\varphi h^{1-p} \sigma_k = c. \tag{1.5}$$

for some $c > 0$. One would like to find necessary and sufficient conditions on a function $\varphi$ such that a positive strictly convex solution exists. Here the strict convexity of a solution, $h$, is understood as the strict convexity of the associated closed hypersurface. The pairs $(p = 1, k = 1)$, $(p = 1, k = n)$, $(p \neq 1, k = n)$ of this equation are known in order as the Christoffel problem, the Minkowski problem and the $L_p$-Minkowski problem. In general, this equation is known as the $L_p$-Christoffel-Minkowski problem. This equation is of considerable interest in convex geometry, and it is related to the problem of existence of a convex body (a compact convex set with non-empty interior) whose $L_p$ surface area of order $n - k$ is prescribed.

Let us briefly explain how $(1.5)$ arises naturally in the $L_p$ Brunn-Minkowski theory. Good references for this material are [25, 33]. Let $p \geq 1$ and $\zeta, \eta \geq 0$ and let $K, L$ be two convex bodies with the origin of $\mathbb{R}^{n+1}$ in their interiors. In the following, $\zeta \cdot K := \zeta^\sharp K$ and $\eta \cdot L := \eta^\flat L$. Define the $L_p$-linear combination $\zeta \cdot K + \eta \cdot L$ as the convex body whose support function is given by $(\zeta h_K^p + \eta h_L^p)^{\frac{1}{p}}$. The mixed
with respect to $W_p,0(K,L), \ldots , W_{p,n}(K,L)$ are defined as the first variation of the usual Quermassintegrals\footnote{For a convex body $K, W_0(K), \ldots , W_{n+1}(K)$ notate the Quermassintegrals of $K$. In particular, $W_0(K)$ is volume of $K, nW_1(K)$ is the surface area of $K$ and $W_{n+1}(K)$ is the volume of the unit ball.} with respect to $L_p$-sum:

\[
\frac{n+1-k}{p} W_{p,k}(K,L) = \lim_{\varepsilon \to 0^+} \frac{W_k(K+p \varepsilon \cdot L) - W_k(K)}{\varepsilon}.
\]

Moreover, the unit sphere is said to be rotationally symmetric if $\phi$ is continuous and has finite limits as $\phi$ tends to $\pm \frac{\pi}{2}$:

\begin{itemize}
  \item[i:] \frac{1}{\phi} is continuous and has finite limits as $\theta$ tends to $\pm \frac{\pi}{2}$,
  \item[ii:] $\int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(\alpha) \sin(\alpha)}{\phi(\alpha)} d\alpha > 0$ and zero for $\theta = -\frac{\pi}{2}$,
  \item[iii:] $\frac{1}{\phi(\theta)} > \frac{n-k}{\cos^{n}(\theta)} \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(\alpha) \sin(\alpha)}{\phi(\alpha)} d\alpha.$
\end{itemize}

It is necessary and sufficient that in some coordinates on $S^n$, $S_k(K,\cdot)$, and over $\int_{S^n} \frac{x}{\phi(x)} dx = 0$.

Miraculously this condition suffices for the Minkowski problem; see, for example, [7]. The $L_p$-Minkowski problem is also well-understood (except the case $p \le -n-1$) and we refer the reader to the essential papers [5, 11, 25–27] for motivation and the most comprehensive list of results, see also [33, Chapter 9.2, Notes for Section 9.2]. An application of the existence of solutions to the $L_p$-Minkowski problem appears in Lutwak, Yang, Zhang [28]. If $p = 1, k < n$, much less is known and in addition to (1.6) further restrictions need to be imposed on $\phi$. For example, let us consider the case when $\phi$ is rotationally symmetric. A function $\phi$ defined on the unit sphere is said to be rotationally symmetric if $\phi(\theta) = \phi(x_1, \ldots , x_{n+1})$ with $x_{n+1} = \sin(\theta)$ where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Note that $\theta$ is the angle that the vector from the origin to $(x_1, \ldots , x_{n+1})$ makes with $x_{n+1} = 0$. In [16], Firey has found that in order for a continuous function $1/\phi$ to be the $k$th elementary symmetric function of the principal radii of a $C^2$ smooth, closed strictly convex hypersurface of revolution, it is necessary and sufficient that in some coordinates on $S^n$, $\phi$ is a function of the latitude $\theta$ alone, and over $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$:

\begin{itemize}
  \item[i:] $\frac{1}{\phi}$ is continuous and has finite limits as $\theta$ tends to $\pm \frac{\pi}{2}$,
  \item[ii:] $\int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(\alpha) \sin(\alpha)}{\phi(\alpha)} d\alpha > 0$ and zero for $\theta = -\frac{\pi}{2}$,
  \item[iii:] $\frac{1}{\phi(\theta)} > \frac{n-k}{\cos^{n}(\theta)} \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(\alpha) \sin(\alpha)}{\phi(\alpha)} d\alpha.$
\end{itemize}
Due to symmetry, the assumption for $\theta = -\frac{\pi}{2}$ in item (ii) is the same as the closure equation (1.6). The main consequence of item (iii) is that the principal radii of curvature are positive:

$$\frac{1}{\varphi(\theta)} - \frac{n - k}{\cos^n(\theta)} \int_0^{\frac{\pi}{2}} \cos^{n-1}(\alpha) \sin(\alpha) \varphi(\alpha) d\alpha = \left(\frac{n - 1}{k - 1}\right) \left(\frac{h''(\theta)}{k(\theta) + h(\theta)}(h(\theta) - h'(\theta) \tan \theta)^{k-1}\right).$$

In [30], Pogorelov proved if $\varphi^{-1} - (\varphi^{-1})_{ss} > 0$ on every great circle parameterized by arc-length $s$, then $1/\varphi$ is the sum of the principal radii of curvature in Euclidean 3-space (this is not a necessary condition). The case $p = 1, k = 1$ without any dimensional restriction was eventually solved by Firey [14, 15] where he gave a necessary and sufficient condition, settling a hundred year old problem posed by Christoffel.\textsuperscript{2} An application of Firey’s [14] existence result to the study of surfaces of constant width appears in Fillmore [17]. The solution to Christoffel’s problem was independently discovered by Berg [4]. See also [33, Chapter 8.3.2] for the explicit construction of the solution to the Christoffel problem and [19, Theorem 6.1] for the corresponding regularity properties and [33, Notes for Section 8.4]. In [20], Guan-Ma proved a deformation lemma which allowed them to establish if a function $\varphi \in C^2(\mathbb{S}^n)$ is $k$-convex, e.g., $\nabla_i \nabla_j \varphi^\perp + \tilde{g}_{ij} \varphi^\perp$ is non-negative definite, then the equation (1.5) for $p = 1, k < n$ has a strictly convex solution. Note that Guan-Ma’s condition for $p = 1, k = 1$ is weaker than Pogorelov’s condition. Later in [24], using the deformation lemma, Hu-Ma-Shen proved that if $p \geq k + 1, k < n$ and $\varphi \in C^2(\mathbb{S}^n)$ is $(p + k - 1)$-convex, then (1.5) admits a positive strictly convex solution.\textsuperscript{3} Recently, for $1 < p < k + 1$ and for even prescribed data, under the $(p + k - 1)$-convexity of $\varphi$, an existence result was proved by Guan and Xia in [21] using a refined gradient estimate and the constant rank theorem.

Before we state our main theorems, we draw attention to an interesting feature of the flow (1.3); however, this property is not used in this paper. Suppose that for a positive, smooth rotationally symmetric $\varphi$ and a smooth, rotationally symmetric, strictly convex hypersurface $M_0$ with the support function $h_0$ we have

\begin{itemize}
  \item[iii:] \[ h_{-1}(\theta) \varphi(\theta) > \frac{n - k}{\cos^n(\theta)} \int_0^{\frac{\pi}{2}} \cos^{n-1}(\alpha) \sin(\alpha) h_{-1}^{n-1}(\alpha) \varphi(\alpha) d\alpha. \]
\end{itemize}

If we start the flow (1.3) from $M_0$, then for all $t > 0$, $M_t$ satisfies the previous two properties provided $p > 1$. To see this for the item (iii), note that

\[ \frac{d}{dt} \left( \frac{h^{p-1}(\theta, t)}{\varphi(\theta)} \right) = \left( p - 1 \right) \left( \frac{\sigma_k(\theta, t)}{\cos^n(\theta)} \right) \left( \frac{n - k}{\cos^n(\theta)} \right) \left( \int_0^{\frac{\pi}{2}} \cos^{n-1}(\alpha) \sin(\alpha) \sigma_k(\alpha, t) d\alpha \right) > 0. \]

\textsuperscript{2} Firey also explains in [14, page 11] how Pogorelov’s condition connects to his.

\textsuperscript{3} The statement of Hu-Ma-Shen’s theorem is erroneous and in item (i) it should be read “if $f$ is spherical convex” and in item (ii) it should be read “if $f$ is spherical convex”.

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\[ \frac{\partial f}{\partial t} = \left( f_{p-1} \right)_\theta. \]
One can see similarly that item (ii) is preserved along the flow. For the case 
\( \varphi \equiv 1, k = n, p = -n - 1 \), preserving a property similar to (ii) played a role in the
proofs of [13].

In this paper, we prove the following theorems about the asymptotic behavior of
the flow.

**Theorem 1.1.** Suppose \( p \geq k + 1, k < n \) and \( \varphi \in C^\infty(S^n) \) is a positive function
such that \( \nabla_i \nabla_j \varphi \frac{1}{k+1} + \tilde{g}_{ij} \varphi \frac{1}{n+1} \) is positive definite. Then there exists a unique
smooth, closed strictly convex solution \( \{ \tilde{M}_t \} \) to (1.1) such that \( \{ \tilde{M}_t \} \) converges in
\( C^\infty \) to a smooth, closed strictly convex solution hypersurface whose support function is positive and solves (1.5).

Our proof of the convergence to solitons does not employ the deformation lemma
(constant rank theorem) and thus provides a partial answer to the following question
raised in [21]: Is there a direct effective way to derive an estimate for \( r_{ij} \)
from under the same convexity conditions without using the constant rank
theorem? Here our parabolic approach to the problem (1.5) allows us to obtain a
uniform positive lower bound on \( r_{ij} \) along the normalized flow by using a very simple
auxiliary function (see Lemma 2.7 below) and hence we can avoid the constant
rank theorem when the assumptions of Theorem 1.1 are satisfied.

In the last section, for \( 1 \leq k < n, p + k - 1 > 0 \) we show the existence of
a rotationally symmetric \( \varphi \) with

\[
\left( (\varphi \frac{1}{n+1})_{\theta \theta} + \varphi \frac{1}{n+1} \right)_{(n+1)} < 0 \quad \text{and a smooth,}
\]
closed, strictly convex initial hypersurface for which the solution to the flow (1.1)
with \( k < n \) will lose smoothness. Therefore \( (p + k - 1) \)-convexity of \( \varphi \) is essential
to ensure the smoothness of the solution is preserved.

For \( k = n, p \geq n + 1 \), we can improve [3, Theorem 1] by dropping the evenness assumption and allowing general \( \varphi \).

**Theorem 1.2.** Suppose \( k = n, p \geq n + 1 \) and \( \varphi \in C^\infty(S^n) \) is a positive function.
Then there exists a unique smooth, closed strictly convex solution \( \{ M_t \} \) to (1.1) such
that \( \{ M_t \} \) converges in \( C^\infty \) to a smooth, closed strictly convex solution hypersurface
whose support function is positive and solves (1.5).

We should point out the only new ingredient required to prove this last theorem
is the gradient estimate established in Lemma 2.5; this allows us to obtain uniform
lower and upper bounds on the support function of the normalized solution even
if the initial hypersurface is not origin-symmetric. In particular, the curvature
estimate of [3, Lemma 8] is crucial.

Finally in view of Chow-Gulliver’s gradient estimate [9] for the case \( p > 2, \varphi \equiv 1 \),
we have the following result.

**Theorem 1.3.** Suppose \( p > 2 \) and \( \varphi \equiv 1 \). Then there exists a unique smooth,
closed strictly convex solution \( \{ M_t \} \) to (1.1) such that \( \{ M_t \} \) converges in \( C^\infty \) to
the unit sphere.

To conclude this section, we draw attention to some earlier works on the flow
(1.3). For \( p = 2, \varphi \equiv 1 \), the \( C^1 \) convergence was established by Chow-Tsai [10] and
recently the \( C^\infty \) convergence was proved by Gerhardt in [18]. For \( p > 2, \varphi \equiv 1 \), the \( C^1 \) convergence follows from the work of Chow-Gulliver [9] (up-to showing convexity
is preserved). For \( p = -n - 1, k = n, \varphi \equiv 1 \), the flow was studied in [12, 13] and for
\( p > -n - 1, k = n, \varphi \not\equiv 1 \) in [3].
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2. Regularity estimates

For convenience we put
\[ \eta := \frac{\int_{\mathcal{S}^{n}} h \sigma_{k} dx}{\int_{\mathcal{S}^{n}} \frac{1}{p} dx}, \quad \Theta := \varphi h^{2-p}, \quad \mathcal{L} := \Theta \sigma_{k}^{ab} \nabla_{a} \nabla_{b}, \quad \rho := \sqrt{h^{2} + |\nabla h|^{2}}. \]

If \( h \in C^{\infty}(\mathcal{S}^{n}) \) determines a smooth, closed strictly convex hypersurface, we write \([h] \) for the associated hypersurface. For such a hypersurface define
\[ A_{\varphi}^{k,p}[h] := \begin{cases} \int_{\mathcal{S}^{n}} h \sigma_{k} dx \left( \int_{\mathcal{S}^{n}} \frac{h^{p}}{\varphi} dx \right)^{\frac{k+1}{p}}, & \text{if } p \neq 0; \\ \exp \left( -\frac{k+1}{f_{\mathcal{S}^{n}} \log h} \right) \int_{\mathcal{S}^{n}} h \sigma_{k} dx, & \text{if } p = 0. \end{cases} \]

The functionals \( A_{\varphi}^{k,p} \) are well-known and have appeared for example in [1].

Lemma 2.1. \( A_{\varphi}^{k,p}[h(\cdot, \tau)] \) is non-decreasing.

Proof. We only consider the case \( p \neq 0 \). Using the divergence theorem, we calculate
\[ \frac{d}{d\tau} A_{\varphi}^{k,p}[h(\cdot, \tau)] = \int_{\mathcal{S}^{n}} \varphi h^{2-p} \sigma_{k}^{2} dx \int_{\mathcal{S}^{n}} \frac{b^{p}}{\varphi} dx - \left( \int_{\mathcal{S}^{n}} h \sigma_{k} dx \right)^{2}. \]

Therefore by the Hölder inequality \( A_{\varphi}^{k,p}[h(\cdot, \tau)] \) is non-decreasing along the flow. \( \Box \)

Lemma 2.2. Suppose \( p \geq 2 \). \( \eta(\tau) \) is uniformly bounded above and below.

Proof. The uniform lower bound on \( \eta(\tau) \) follows from Lemma 2.1. To prove that \( \eta \) is uniformly bounded above we proceed as follows. Along the flow \( \partial_{\tau} h = \sigma_{k} \), \( A_{k,2}^{1}[h(\cdot, t)] \) is monotone. Thus by the result of [18], for any smooth, closed strictly convex hypersurface with support function \( h \) we have \( A_{k,2}^{1}[h] \leq A_{k,2}^{1}[1] \). Now observe that for \( p \geq 2 \), the inequality
\[ \left( \int_{\mathcal{S}^{n}} h^{2} dx \right)^{\frac{k+1}{p}} \geq \left( \int_{\mathcal{S}^{n}} h^{p} dx \right)^{\frac{2}{p}} \left( \frac{\omega_{n}^{-1} - \frac{2}{p}}{p} \right)^{\frac{k+1}{p}} \]
gives
\[ \int_{\mathcal{S}^{n}} h \sigma_{k} dx \left( \int_{\mathcal{S}^{n}} h^{p} dx \right)^{\frac{2}{p}} \left( \frac{\omega_{n}^{-1} - \frac{2}{p}}{p} \right)^{\frac{k+1}{p}} \leq A_{k,2}^{1}[1]. \]

So \( A_{k,2}^{1} \) is bounded above. The proof of the lemma is done.\( \Box \)

In the next lemma \( r^{ij} \) signifies the entries of the inverse matrix of \([r_{ij}]\).

\( ^{4} \)In fact, only knowing the asymptotic behavior of the flow \( \partial_{\tau} h = \sigma_{1} \) is sufficient here; using quermassintegral inequalities we can control \( \int_{\mathcal{S}^{n}} h \sigma_{k} dx \) from above by \( (\int_{\mathcal{S}^{n}} h \sigma_{1} dx)^{\frac{k+1}{p}} \).
Lemma 2.3. The following evolution equation holds along the flow (1.4).

\[(\partial_\tau - L)_{ij} = \Theta \sigma_k^{ab,mn} \nabla_i \eta^{ab} \nabla_j r_{mn} + (k+1) \Theta \sigma_k ^{ab} \eta_{ij} \]

- \[\Theta \sigma_k^{ab} \rho_{ab} r_{ij} + \Theta (\sigma_k^{ai} r_{aj} - \sigma_k^{aj} r_{ai}) + \nabla_i \Theta \nabla_j \sigma_k + \sigma_k \nabla_i \nabla_j \Theta - \eta r_{ij}, \]

\[(\partial_\tau - L) r^{ij} = -(k+1) \Theta \sigma_k r^{jp} r^{pj} + \Theta \sigma_k^{ab} \rho_{ab} r^{ij} \]

- \[\Theta r^{il} r^{js} (2\sigma_k^{am} r_{mb} + \sigma_k^{bm,mn}) \nabla_l r_{ab} \nabla_s r_{mn} \]

- \[\Theta r^{jp} r^{ij} (\sigma_k^{ap} r_{aq} - \sigma_k^{aq} r_{ap}) \]

- \[\eta r^{ia} r^{jb} (\nabla_a \Theta \nabla_b \sigma_k + \nabla_k \Theta \nabla_a \sigma_k + \sigma_k \nabla_a \nabla_b \Theta) + \eta r^{ij}, \]

\[(\partial_\tau - L) h = (1-k) \Theta \sigma_k + \Theta h \sigma_k^{ij} \rho_{ij} - \eta h, \]

\[(\partial_\tau - L) (\varphi^{2-p} \sigma_k) = (2-p) \varphi^{2} h^{3-2p} \sigma_k^{ij} + \varphi^{2} h^{4-2p} \sigma_k^{ij} \eta_{ij} - (2-p+k) \eta \varphi^{2-p} \sigma_k, \]

\[(\partial_\tau - L) \frac{\rho^2}{2} = (k+1) \text{h} \Theta \sigma_k - \eta \rho^2 + \sigma_k \eta_{ij} \nabla_i \text{h} \nabla_j \Theta - \Theta \sigma_k^{ab,mn} r_{mb}. \]

Proof. For the computation of the evolution equations of \( r_{ij} \) and \( r^{ij} \) see \([10,31]\). Deriving the evolution equations of \( h, \varphi^{2-p} \sigma_k \) is straightforward. For \( \rho^2/2 \) we calculate

\[(\partial_\tau - L) \frac{\rho^2}{2} = h \partial_\tau h + \eta \rho^2 \nabla_i h \nabla_j \partial_\tau h \]

- \[\Theta \sigma_k^{ab} \left( h \nabla_a \nabla_k h + \nabla_a h \nabla_k h + \nabla_m h \nabla_a \nabla_k \nabla_m h + \nabla_a \nabla_m h \nabla_k \nabla_m h \right) \]

= \[h \partial_\tau h - \eta |\nabla h|^2 + \sigma_k \eta \nabla_i \nabla_j \Theta + \Theta \rho^2 \nabla_i \nabla_j \sigma_k \]

- \[\Theta \sigma_k^{ab} \left( \nabla_a h \nabla_k h + \nabla_m h \nabla_a (r_{bm} - \eta \rho_{bm} h) \right) \]

- \[\Theta \sigma_k^{ab} (r_{am} - \eta \rho_{am} h)(r_{bm} - \eta \rho_{bm} h) \]

- \[\Theta \sigma_k^{ab} h(r_{ab} - \eta \rho_{ab}) \]

= \[h \partial_\tau h - \eta |\nabla h|^2 + \sigma_k \rho^2 \nabla_i \nabla_j \Theta + k \text{h} \Theta \sigma_k - \Theta \sigma_k^{ab,mn} r_{mb} \]

= \[(k+1) h \text{h} \Theta \sigma_k - \eta \rho^2 + \sigma_k \rho^2 \nabla_i \nabla_j \Theta - \Theta \sigma_k^{ab,mn} r_{mb}. \]

\]

\[\square \]

Lemma 2.4. Suppose \( p \geq k+1 \). Then \( \varphi^{1-p} \sigma_k \) remains uniformly bounded above and below.

Proof. By Lemma 2.3 we have

\[\partial_\tau (\varphi^{1-p} \sigma_k) = \varphi^{2-p} \sigma_k^{ij} \nabla_i \nabla_j (\varphi^{1-p} \sigma_k) + 2 \varphi^{1-p} \sigma_k^{ij} \nabla_i (\varphi^{1-p} \sigma_k) \nabla_j h \]

+ \[(1-k) \varphi^{1-p} \sigma_k(2) + (p-k-1) \eta \varphi^{1-p} \sigma_k. \]

Since we have control over \( \eta \), the claim follows from the maximum principle. \[\square \]

In the next lemma, using the concavity of \( \varphi^{k_{+1}} \) we obtain a gradient estimate for \( \log h(\cdot, \tau) \) provided \( p \geq k+1 \). In general, due to the examples in \([21]\), such an estimate does not exist for \( 1 < p < k+1 \).

Lemma 2.5. Suppose \( p \geq k+1 \). There exists a positive constant \( \gamma \) depending only on the initial hypersurface and \( \varphi \) such that \(|\nabla \log h(\cdot, \tau)| \leq \gamma \).
Proof. The proof is a parabolic version of the estimates in [24, Lemma 2]. Using Lemma 2.3 we deduce

$$(\partial_\tau - \mathcal{L})(\rho^2 - Ah^2) = 2(k + 1)h\Theta_k - 2\eta p^2 + 2\sigma_k \bar{g}^{ij} \bar{\nabla}_i h \bar{\nabla}_j \Theta - 2\Theta \sigma_k^{ab} r_a r_m m_{mb}$$

$$- 2Ah((1 - k)\Theta_k + \Theta h a_i g_{ij} - \eta h) + 2A \Theta \sigma_k^{ab} \bar{\nabla}_a h \bar{\nabla}_b h.$$ 

Let us put $v := \log h$. Pick $A > 1$ such that

$$(\rho^2 - Ah^2)(\cdot, 0) < 0.$$ 

We will show that this inequality will be preserved, perhaps for a larger value of $A$ to be determined later. If it were otherwise, there would be a point $(u_\tau, \tau)$ with $\tau > 0$ that for the first time $(\rho^2 - Ah^2)(u_\tau, \tau) = 0$. At this point $\bar{\nabla}|\bar{\nabla}v|^2 = 0$ and we may choose an orthonormal frame $\{e_i\}$ such that

$$\bar{\nabla}v = (\bar{\nabla}_e v)e_1.$$ 

For the rest of the proof it is more convenient to put

$$v_1 := \bar{\nabla}_e v, \quad v_{ij} := \bar{\nabla}_i \bar{\nabla}_j v.$$ 

Since $\bar{\nabla}|\bar{\nabla}v|^2 = 0$ at $u_\tau$, a rotation of $\{e_i\}_{i \geq 2}$ diagonalizes $(v_{ij})$ at $u_\tau$,

$$a_{ij} := v_{ij} + v_i v_j + \delta_{ij} = \text{diag}(1 + v_1^2, 1 + v_{22}, 1 + \ldots, 1 + v_{nn}).$$ 

Also, note that

$$\sigma_k (h_{ij} + \delta_{ij} h) = h^k \sigma_k (a_{ij}), \quad \sigma_k^{cd} (h_{ij} + \delta_{ij} h) = h^{k-1} \sigma_k^{cd} (a_{ij}).$$

Therefore at $(u_\tau, \tau)$ we have

$$0 \leq (k + 1) + (2 - p) v_1^2 + v_1 (\log \varphi)_1 - \frac{\sigma_k^{ii} (a_{ij})}{\sigma_k (a_{ij})} (a_{ij})^2$$

$$+ A(k - 1) - A \sum_l \frac{\sigma_k^{il} (a_{ij})}{\sigma_k (a_{ij})} + A \frac{\sigma_k^{11} (a_{ij})}{\sigma_k (a_{ij})} v_1^2.$$ 

Since $A = 1 + v_1^2 = a_{11}$, we may rewrite the previous estimate as

$$0 \leq 2k + (1 + k - p)(A - 1) + v_1 (\log \varphi)_1 - \frac{\sigma_k^{11} (a_{ij})}{\sigma_k (a_{ij})} A^2$$

$$- A \sum_l \frac{\sigma_k^{il} (a_{ij})}{\sigma_k (a_{ij})} + A(A - 1) \frac{\sigma_k^{11} (a_{ij})}{\sigma_k (a_{ij})}.$$ 

Thus for $p > k + 1$ we arrive at

$$(p - k - 1)(A - 1) \leq 2k + |(\log \varphi)_1| \sqrt{A - 1}.$$ 

Choosing $A$ large enough ensures that $\rho^2 - Ah^2$ always remains negative.

If $p = k + 1$, by Lemma 2.4, $\sigma_k (a_{ij})$ is uniformly bounded above. Also, since $\sigma_k^1$ is concave, we have $\sum l \sigma_k^{il} (a_{ij}) \geq k \sigma_k^{p-1} (a_{ij})$; see, for instance, [2]. Thus for a positive constant $c_1$ depending only on the initial hypersurface and $\varphi$ we have

$$c_1 A \leq A \sum l \frac{\sigma_k^{il} (a_{ij})}{\sigma_k (a_{ij})} \leq 2k + |(\log \varphi)_1| \sqrt{A - 1}.$$ 

Choosing $A$ large enough proves the claim.

Lemma 2.6. Suppose $p \geq k + 1$. There exist positive constants $a, b, c, d$ depending only on the initial hypersurface and $\varphi$ such that $a \leq h(\cdot, \tau) \leq b$ and $c \leq \sigma_k (\cdot, \tau) \leq d$. 

\]
Proof. Along the normalized flow \( \int_{S^n} \frac{h^p(x, \tau)}{\varphi(x)} \, dx \) is constant and
\[
\frac{h^p_{\min}(\tau)}{\varphi_{\max}} \leq \frac{1}{\omega_n} \int_{S^n} h^p(x, \tau) \, dx \leq \frac{h^p_{\max}(\tau)}{\varphi_{\min}}.
\]
Therefore Lemma 2.5 gives uniform lower and upper bounds on \( h(\cdot, \tau) \). Now the lower and upper bounds on \( \sigma_k(\cdot, \tau) \) follow from Lemma 2.4.

Next we will obtain a lower bound on the principal radii of curvature under an additional assumption on \( \varphi \). This in turns implies that the normalized hypersurfaces are uniformly convex. It is only in the following lemma that we require \( \nabla_i \nabla_j \varphi \frac{r^{ij}}{\bar{\varphi}} + g_{ij} \varphi \frac{r^{ij}}{\bar{\varphi}} \) to be positive definite. Our example in the final section shows that one cannot hope for a positive lower for \( r_{ij} \) if \( \nabla_i \nabla_j \varphi \frac{r^{ij}}{\bar{\varphi}} + g_{ij} \varphi \frac{r^{ij}}{\bar{\varphi}} \) is negative definite at some point.

Lemma 2.7. Let \( p \geq k + 1 \). Suppose \( \nabla_i \nabla_j \varphi \frac{r^{ij}}{\bar{\varphi}} + g_{ij} \varphi \frac{r^{ij}}{\bar{\varphi}} \) is positive definite. Then the principal curvatures satisfy \( l \leq \kappa_i(\cdot, \tau) \leq L \) for some positive constants \( l, L \) depending only on the initial hypersurface and \( \varphi \).

Proof. We provide two proofs.

First proof. We apply the maximum principle to \( \frac{r^{ij}}{\bar{\varphi}} \); see also [32, Lemma 3.3]. By a rotation of frame we may assume the maximum eigenvalue of \( \bar{r}^{ij} \) over \( S^n \) at time \( \tau \) is attained at a point \( u_\tau \) in the direction of unit tangent vector \( e_1 \in T_{u_\tau} S^n \). In particular, \( r^{ij} = 0 \) for \( i \neq j \). Using Lemma 2.3 we calculate
\[
\partial_{\tau} \frac{r^{11}}{h} = \Theta \sigma_k^{1b} \nabla_a \nabla_b \frac{r^{11}}{h} - (k + 1) \frac{\Theta}{h} \sigma_k(r^{11})^2 + \frac{\Theta}{h}(\sigma_k^{ij} g_{ij}) r^{11} + \frac{(r^{11})^2}{h}(2 \nabla_1 \Theta \nabla_1 \sigma_k + \sigma_k \nabla_1 \nabla_1 \Theta) - \frac{\Theta r^{11} \sigma_k}{h^2} + \frac{2 \Theta}{h} \sigma_k^{ab} \nabla_a \nabla_b \frac{r^{11}}{h} + \frac{\Theta r^{11}}{h^2} \sigma_k^{ab} (r_{ab} - g_{ab} h) + 2 \frac{\eta}{h} r^{11}.
\]
Balancing the terms gives
\[
(\partial_{\tau} - \mathcal{L} - 2 \eta) \frac{r^{11}}{h} = 2 \frac{\Theta}{h} \sigma_k^{ab} \nabla_a \nabla_b \frac{r^{11}}{h} - (k + 1) \frac{\Theta}{h} \sigma_k(r^{11})^2 - \frac{\Theta}{h}(r^{11})^2 (2 \sigma_k^{am} r^{nb} + \sigma_k^{ab, mn}) \nabla_1 r_{ab} \nabla_1 r_{mn} - \frac{(r^{11})^2}{h}(2 \nabla_1 \Theta \nabla_1 \sigma_k + \sigma_k \nabla_1 \nabla_1 \Theta) + (k - 1) \frac{\Theta \sigma_k r^{11}}{h^2}.
\]
To suitably group the terms on the right-hand side, note that
\[
(1) \ f := \sigma_k^{\downarrow} \text{ is inverse concave; therefore, by } [31, (3.49)] \text{ we get}
\]
\[
(2.2) \quad (2 f^{am} r^{bn} + f^{ab, mn}) \nabla_1 r_{ab} \nabla_1 r_{mn} \geq 2 \left( \frac{\nabla_1 f}{f} \right)^2.
\]
This in turn implies that
\[
(2 \sigma_k^{am} r^{bn} + \sigma_k^{ab, mn}) \nabla_1 r_{ab} \nabla_1 r_{mn} \geq (k^2 + k) f^{k-2} \left( \frac{\nabla_1 f}{f} \right)^2 = \frac{k + 1}{k} \frac{(\nabla_1 f)^2}{f^k}.
\]
(2) By the Schwartz inequality,

$$2|\nabla_1 \Theta \nabla_1 f^k| \leq \frac{k+1}{k} \frac{\Theta (\nabla_1 f^k)^2}{f^k} + \frac{k}{k+1} \frac{f^k(\nabla_1 \Theta)^2}{\Theta}.$$ 

Due to the preceding estimates, at \((u_\tau, \tau)\) there holds

$$\partial_\tau \frac{r^{11}}{h} \leq -\left(\frac{r^{11}}{h}\right)^2 \sigma_k \left((k+1)\Theta + \nabla_1 \nabla_1 \Theta - \frac{k(\nabla_1 \Theta)^2}{(k+1)\Theta} + (1-k)\frac{\Theta r^{11}}{h}\right) + \frac{2\eta r^{11}}{h}.$$ 

Let \(s\) be the arc-length of the great circle passing through \(u_\tau\) with unit tangent vector \(e_1\). The sum of the first three terms in the bracket may be expressed as

$$(k+1)\Theta \frac{s}{\pi} \left(\Theta \frac{s}{\pi} + (\Theta \frac{s}{\pi})_{ss}\right).$$

Expand this last expression for \(p+k-1 > 0\),

$$\frac{\Theta \frac{s}{\pi} + (\Theta \frac{s}{\pi})_{ss}}{\Theta \frac{s}{\pi}} = 1 + 2 \frac{2-p}{(k+1)^2} h s \varphi s + \frac{(p-2)(p+k-1)}{(k+1)^2} h^2 \frac{\varphi s}{h^2} \frac{2-p h s}{k+1} \frac{\varphi s}{\varphi} + \frac{1}{k+1} \frac{\varphi s}{\varphi} = 2 \frac{2-p h + h s}{k+1} \frac{h}{h} + \frac{2-p}{(k+1)^2} h s \left(\frac{h}{h} \frac{(p+k-1) \varphi}{h} + \varphi s \left(\frac{h}{h} \frac{(p+k-1) \varphi}{h}\right)^2 \right) + \frac{1}{\varphi(k+1)} \left\{ (p+k-1) \varphi - \frac{1}{(k+1)^2} \frac{p-2}{p+k-1} \frac{k}{(k+1)^2} \frac{\varphi s}{\varphi} + \frac{\varphi s}{\varphi} \right\}$$

$$= 2 \frac{2-p h + h s}{k+1} \frac{h}{h} + \frac{1}{k+1} \left\{ (p+k-1) - \frac{p+k-2}{p+k-1} \left(\frac{\varphi}{\varphi} \right)^2 + \frac{\varphi_s}{\varphi} \right\}$$

$$+ \frac{p-2}{(k+1)^2} h s \left(\frac{h}{h} \frac{(p+k-1) \varphi}{h} + \varphi s \left(\frac{h}{h} \frac{(p+k-1) \varphi}{h}\right)^2 \right).$$

Let \(p \geq 2\) and assume either of the following equivalent conditions hold

\[(2.3) \quad \{ (\varphi \frac{p+k-1}{k})_{ss} + \varphi \frac{p+k-1}{k} > 0 \} \text{ on every great circle,} \]

$$\tilde{\nabla}_i \tilde{\nabla}_j \varphi \frac{p+k-1}{k} + \tilde{g}_{ij} \varphi \frac{p+k-1}{k} \text{ is positive definite.}$$

Under this condition we conclude that

$$\partial_\tau \frac{r^{11}}{h} \leq -\Theta \sigma_k \left(\frac{r^{11}}{h}\right)^2 \left(c_0 h + (3-k-p) r_{11}\right) + 2\eta \frac{r^{11}}{h} \leq -c_1 \left(\frac{r^{11}}{h}\right)^2 + c_2 \frac{r^{11}}{h},$$

where \(c_0 > 0\) depends on the smallest eigenvalue of \(\tilde{\nabla}_i \tilde{\nabla}_j \varphi \frac{p+k-1}{k} + \tilde{g}_{ij} \varphi \frac{p+k-1}{k}\) with respect to \(\tilde{g}\) and we used the lower and upper bounds on \(h, \sigma_k, \eta\) from Lemmas 2.2, 2.6. By the maximum principle \(r^{11} \leq L\) for some \(L\) depending on \(M_0, \varphi\). Thus the principal radii of curvature satisfy \(\frac{1}{L} \leq \lambda_i\). To finish the proof, note that \(\frac{1}{\max \lambda_i} \leq \sigma_k \leq d\).

Second proof. Arrange the principal radii of curvature as \(\lambda_1 \leq \cdots \leq \lambda_n\). We show that \(h \lambda_1\) satisfies a suitable differential inequality in a viscosity sense.
Let us fix a point \((u, \tau)\) with \(\tau > 0\) and suppose at this point the multiplicity of \(\lambda_1 \) is \(\mu\); that is, \(\lambda_1 = \cdots = \lambda_{\mu} < \lambda_{\mu+1} < \cdots \leq \lambda_n\). Choose an orthonormal frame for \(T_u S^n\) such that \(r_{ij} = \lambda_i \delta_{ij} \) and \(g_{ij} = \delta_{ij}\).

Let \(\xi\) be an arbitrary \(C^2\) lower support of \(h\lambda_1\) at \((u, \tau)\). That is, for some \(\varepsilon > 0\) and an open neighborhood \(\mathcal{O}_{u, \tau}\) of \((u, \tau)\), we have \(\xi(u, t) \leq (h\lambda_1)(u, t)\) for all \((u, t) \in \mathcal{O}_{u, \tau} \times (\tau - \varepsilon, \tau]\) and \(\xi(u, \tau) = (h\lambda_1)(u, \tau)\). With similar calculations as in [6, Lemma 5] at \((u, \tau)\) we obtain

\[
\begin{align*}
\nabla_i \nabla_j \xi &\leq \nabla_i \nabla_j hr_{11} - 2h \sum_{b > \mu} \frac{\langle \nabla a r_{1b} \rangle^2}{\lambda_b - \lambda_1}, \\
r_{kl} \nabla_i h + h \nabla_i r_{kl} &\geq \delta_{kl} \nabla_i \xi \quad \text{for all} \ 1 \leq k, l \leq \mu.
\end{align*}
\]

In addition, note that

\[
\forall (u, t) \in \mathcal{O}_{u, \tau} \times (\tau - \varepsilon, \tau]: \quad \frac{h \frac{r_{1k} g^{kl} r_{1l}}{r_{11}}}{r_{11}} \geq h \lambda_1 \geq \xi.
\]

In fact, assume the hypersurface is given as an embedding of \(S^n\) via the inverse Gauss map. Then the second fundamental form is \(r_{ij}\). By [8, Proposition 4.1], in any chart we have \(\frac{\partial r_{ij}}{\partial y^i} \leq \frac{1}{\lambda_i}\) where \(g_{ij}\) is the metric of the hypersurface. Due to the identity \(g_{ii} = r_{ik} g^{kl} r_{li}\), we obtain \(\frac{h \frac{r_{1k} g^{kl} r_{1l}}{r_{11}}}{r_{11}} \geq h \lambda_1 \geq \xi\). Moreover, this last inequality becomes an equality at \((u, \tau)\), we get

\[
\partial_t (hr_{11}) \big|_{(u, \tau)} = \partial_t \left( \frac{h \frac{r_{1k} g^{kl} r_{1l}}{r_{11}}}{r_{11}} \big|_{(u, \tau)} \right) = \partial_t \xi \big|_{(u, \tau)},
\]

Putting all these facts together yields

\[
(\partial_\tau - \mathcal{L})\xi \geq (\partial_t - \mathcal{L}) r_{11} + \xi (\partial_\tau - \mathcal{L}) h + 2h \Theta a a \sum_{b > \mu} \frac{(\nabla a r_{1b})^2}{\lambda_b - \lambda_1}
+ 2 \Theta a a \frac{(\nabla a \xi - \lambda_1 \nabla a h)^2}{\xi} - 2 \Theta a b (\nabla a \xi \nabla b \xi \log \frac{\xi}{\bar{h}}).
\]

Then owing to Lemma 2.3 and (2.2), we arrive at the estimate

\[
(\partial_\tau - \mathcal{L})\xi + 2 \Theta a b (\nabla a \xi \nabla b \xi \log \frac{\xi}{\bar{h}}) + ((k - 1) \varphi h^{1-p} \sigma_k + 2\eta)\xi
\geq
\]

\[
2h \Theta \left( a a \sum_{b > \mu} \frac{(\nabla a r_{1b})^2}{\lambda_b - \lambda_1} - a a b \frac{\langle \nabla_1 r_{ab} \rangle^2}{\lambda_1 h^2} + a a \frac{(\nabla a \xi - \lambda_1 \nabla a h)^2}{\lambda_1 h^2} \right)
+ (k + 1) h \sigma_k + 2h \nabla_1 \nabla_1 \xi + h \sigma_k \nabla_1 \nabla_1 \xi + \frac{k + 1}{h} \Theta (\nabla_1 \sigma_k)^2.
\]

By Schwartz's inequality we obtain

\[
(\partial_\tau - \mathcal{L})\xi + 2 \Theta a b (\nabla a \xi \nabla b \xi \log \frac{\xi}{\bar{h}}) + ((k - 1) \varphi h^{1-p} \sigma_k + 2\eta)\xi
\geq
\]

\[
2h \Theta \left( a a \sum_{b > \mu} \frac{(\nabla a r_{1b})^2}{\lambda_b - \lambda_1} - a a b \frac{\langle \nabla_1 r_{ab} \rangle^2}{\lambda_1 h^2} + a a \frac{(\nabla a \xi - \lambda_1 \nabla a h)^2}{\lambda_1 h^2} \right)
+ h \sigma_k \left( (k + 1) \Theta + \nabla_1 \nabla_1 \Theta - \frac{k}{k + 1} \frac{(\nabla_1 \Theta)^2}{\Theta} \right).
\]
We show that
\[ R := \sigma^a_k \sum_{b > \mu} \frac{(\nabla_a r_{1b})^2}{\lambda_b - \lambda_1} - \sigma^a_k r^{bb} (\nabla_a r_{1b})^2 + \sigma^a_k (\nabla_a \xi - \lambda_1 \nabla_a h)^2 \lambda_1 h^2 \]
is non-negative.

To see this, note that we can estimate the second term in \( R \) as follows
\[ r^{bb} (\nabla_a r_{1b})^2 = \sum_{b \leq \mu} r^{bb} (\nabla_a r_{1b})^2 + \sum_{b > \mu} r^{bb} (\nabla_a r_{1b})^2 = \frac{(\nabla_a \xi - \lambda_1 \nabla_a h)^2}{\lambda_1 h^2} + \sum_{b > \mu} \frac{1}{\lambda_b} (\nabla_a r_{1b})^2. \]

We may continue as in the final part of the first proof to deduce that at \((u_\tau, \tau)\),
\[ (2.4) \quad \partial_\tau \xi \geq L\xi - 2\Theta \sigma^a_k \nabla_a \xi \nabla_b \log \frac{i}{h} - ((p + k - 3)\varphi h^{1-p}\sigma_k + 2\eta)\xi + h^3-p\sigma_k c_\varphi \]
for a positive constant depending on the \( C^2 \) norm of \( \varphi \). Now suppose for the sake of contradiction that \( h\lambda_1 \) for the first time \( \tau > 0 \) at \( u_\tau \) equals to
\[ \xi := \frac{1}{2} \min \left\{ \frac{\min(h^{3-p}\sigma_k) c_\varphi}{\max((p + k - 3)\varphi h^{1-p}\sigma_k + 2\eta)}, \min(h\lambda_1(\cdot, 0)) \right\}. \]
So \( \xi \) serves as a lower support for \( h\lambda_1 \) on \( M^n \times [0, \tau] \). But (2.4) yields a contradiction. \( \square \)

**Remark 2.8.**

(1) Since mixed volumes are monotonic increasing in each argument [33, page 282],
\[ h^{k+1} \leq \int_{S^n} h \sigma_k dx \leq h_{\max}^{k+1}. \]

If we have lower and upper estimates for \( h(\cdot, \tau) \), then we have lower and upper bounds on \( \eta(\tau) \).

(2) From the proof of the previous lemma and the first remark it is clear that if \( p \geq 2 \) and one can establish lower and upper bounds for \( h, \sigma_k \) along the normalized flow, then under the assumption (2.3) there are lower and upper bounds on the principal curvatures.

(3) For \( p \geq k + 1 \) we could avoid using the result of [18] to prove Lemma 2.2. In fact, since \( \nabla_u \nabla_j h \) is non-negative at the maximum point of \( h \), by the monotonicity of the entropy we have
\[ \partial_\tau h_{\max} \leq h_{\max}(\varphi_{\max} h_{\max}^{1+k-p} - \eta(\tau)) \leq h_{\max}(\varphi_{\max} h_{\max}^{1+k-p} - \eta(0)) \]
in the sense of the \( l \) sup of forward difference quotients; see, [22]. Since \( p > k + 1 \), if \( h_{\max} \) becomes too large, then the right-hand side will be negative. So \( h \) remains bounded above by some constant \( b \). To get a lower bound on \( h \), note that \( \eta(\tau) \leq b^{k+1} \); therefore,
\[ \partial_\tau h_{\min} \geq h_{\min} \left( \varphi_{\min} h_{\min}^{1+k-p} - \eta(\tau) \right) \geq h_{\min} \left( \varphi_{\min} h_{\min}^{1+k-p} - b^{k+1} \right). \]

If \( h_{\min} \) becomes strictly less than a critical value, then the right-hand side will be positive; therefore, \( h(\cdot, \tau) \geq a \) for some positive constant \( a \). So we have shown for \( p > k + 1 \), \( a^{k+1} \leq \eta \leq b^{k+1} \). For \( p = k + 1 \), the proof of Lemma 2.5 does not need any control on \( \eta \).

(4) If \( k = 1 \), in the proof of the previous lemma we could apply the maximum principle directly to \( r_{11} \) instead of \( hr_{11} \).
To obtain lower and upper bounds on $\sigma_k$ without the limitation set by homogeneity degree of the speed, we will need to use auxiliary functions that are not homogeneous. The next lemma gives a lower bound on $\sigma_k$ in a large generality. The proof does not use any particular structure of $\sigma_k$.

**Lemma 2.9.** Let $h(\cdot, \tau)$ be a solution to the flow \((1.4)\) such that $a \leq h(\cdot, \tau) \leq b$ for some positive constants $a, b$. Then $\sigma_k(\cdot, \tau) \geq c$ for a positive constant $c$.

**Proof.** Let $A > 0$ be a constant to be determined later. We will apply the maximum principle to the following auxiliary function considered in [29],

$$\chi := \log(\Theta \sigma_k) - A\rho^2 \over 2.$$  

Owing to Lemma 2.3 we have

$$(\partial_{\tau} - \mathcal{L}) \log(\phi h^{2-p} \sigma_k) = \Theta \sigma_k^{ab} \nabla_a \log(\phi h^{2-p} \sigma_k) (\nabla_b \log(\phi h^{2-p} \sigma_k) ) + (2-p) \phi h^{1-p} \sigma_k + \phi h^{2-p} \sigma_k^{ij} \bar{g}_{ij} - (2 + k-p) \eta.$$  

Consequently using Lemma 2.3 the evolution equation of $\chi$ reads as

$$(\partial_{\tau} - \mathcal{L}) \chi = \Theta \sigma_k^{ab} \nabla_a \log(\phi h^{2-p} \sigma_k) (\nabla_b \log(\phi h^{2-p} \sigma_k) ) + (2-p) \phi h^{1-p} \sigma_k + \phi h^{2-p} \sigma_k^{ij} \bar{g}_{ij} - (2 + k-p) \eta$$  

$$- A(k+1) \Theta \sigma_k + A\rho^2 \rho \sigma_k \bar{g}^{ij} \nabla_i \nabla_j \Theta + A\Theta \sigma_k r_a^m r_{mb}.$$  

 Dropping some positive terms and rearranging terms yield

$$(\partial_{\tau} - \mathcal{L}) \chi \geq \left( \frac{A}{2} \eta \rho^2 + (2 - p) \frac{\rho^2}{h} - (2 + k-p) \right) \frac{\rho^2}{h}$$

$$A \Theta \sigma_k \left( \frac{\rho^2}{2e^{\chi + A\rho^2}} - \bar{g}^{ij} \nabla_i \nabla_j \Theta - (k+1) \right).$$

Choose $A > \max \frac{\rho^2}{h} (2 + k-p)$. Thus if $\chi$ becomes very negative then the right-hand side becomes positive; this is due to the uniform upper bound on $|\nabla_a h(\cdot, \tau)|$ and lower bound on $\eta$; see Remark 2.8.  

The next lemma gives an upper bound on $\sigma_k(\cdot, \tau)$ for every $p, k$. The proof employs the following property of $\sigma_k$ (due to its inverse concavity; see, e.g., [2]):

(2.5)  

$$\sigma_k^{ab} r_a^m r_{mb} \geq k \sigma_k^{1+\frac{1}{2}}.$$  

**Lemma 2.10.** Let $h(\cdot, \tau)$ be a solution to the flow \((1.4)\) such that $\varepsilon \leq \rho^2(\cdot, \tau) \leq \frac{1}{\varepsilon}$ for an $0 < \varepsilon < 1$. Then $\sigma_k(\cdot, \tau) \leq d$ for some positive constant $d$.

**Proof.** We will apply the maximum principle to the auxiliary function

$$\chi(\cdot, \tau) := \frac{\phi h^{1-p} \sigma_k(\cdot, \tau)}{1 - \varepsilon \rho^2}.$$  

Meanwhile note that for two positive smooth functions $f, g : S^n \times [0, T) \to \mathbb{R}$,

$$(\partial_{\tau} - \mathcal{L}) \frac{f}{g} = \frac{1}{g} (\partial_{\tau} - \mathcal{L}) f - \frac{f}{g^2} (\partial_{\tau} - \mathcal{L}) g + 2 \Theta \sigma_k^{ij} \nabla_i \log g \nabla_j \frac{f}{g}.$$
Therefore in view of the evolution equation (2.1) and Lemma 2.3, at the point where $\chi$ attains its maximum we have

$$
\partial_{\tau} \chi \leq \frac{1}{1 - \varepsilon \rho^2} \left(2 \varphi h^{1-p} \sigma_k \bar{\nabla}_i (\varphi h^{1-p} \sigma_k) \bar{\nabla}_j h + (1 + k - p) (\varphi h^{1-p} \sigma_k)^2 + (p - k - 1) \eta \rho \varphi h^{1-p} \sigma_k \right)
$$

and

$$
\bar{\nabla}_i (\varphi h^{1-p} \sigma_k) = -\varepsilon \rho \frac{\varphi h^{1-p} \sigma_k \bar{\nabla}_i \rho}{1 - \varepsilon \rho^2} = -\varepsilon \frac{\varphi h^{1-p} \sigma_k \bar{\nabla}_i \rho}{1 - \varepsilon \rho^2}.
$$

By Remark 2.8, $\eta$ is bounded above. Putting (2.7) into (2.6), using (2.5) and the lower and upper bounds on $\rho, h, \Theta$ and $|\bar{\nabla} \Theta|, |\bar{\nabla} h|$, we find that there exist positive constants $c_1, c_2, c_3$ such that

$$
\partial_{\tau} \chi \leq c_1 \chi + c_2 \chi^2 - c_3 \chi^2 + k.
$$

The maximum principle completes the proof. \qed

**Remark 2.11.** The auxiliary function we considered in the previous lemma is very robust. Consider a curvature flow $\partial_t h = \varphi h^{\alpha} f^p$, where $p > 0$ and $f$ is a 1-homogeneous function of principal radii of curvature satisfying $f^{ij} r_i^k r_j^k \geq c f^2$ for some $c > 0$ and $\varphi$ is a positive smooth function defined on the unit sphere. In particular, if $f$ is inverse concave or we are privileged with a pinching estimate the inequality holds. Then in the presence of lower and upper bounds on the support function we can apply the maximum principle to $\varphi h^{1-p} \sigma_k$ to obtain a uniform upper bound on the speed.

### 3. CONVERGENCE OF THE NORMALIZED SOLUTION

In this section we complete the proofs of our main theorems.

- **Theorem 1.1:** uniform $C^2$ regularity estimates were obtained in previous section.

- **Theorem 1.2:** uniform lower and upper bounds on the support function and $\sigma_n(\cdot, \tau)$ follow from Lemma 2.6. Moreover, uniform $C^2$ regularity estimates were proved in [3, Lemma 8].

- **Theorem 1.3:** the lower and upper bounds on $h(\cdot, \tau)$ follow from the strong gradient estimates of Chow-Gulliver [9]. Therefore, by Remark 2.8-(1), $\eta(\tau)$ is controlled from above and below. Now the lower and upper bounds on $\sigma_k(\cdot, \tau)$ follow from Lemmas 2.9 and 2.10. Then by Remark 2.8-(2), we can deduce uniform lower and upper bounds on the principal curvatures $\kappa_i(\cdot, \tau)$.

Now in all cases higher order regularity estimates follow from Krylov and Safonov [23] and Schauder theory. The convergence of the normalized solution for a subsequence of times to a soliton then follows from monotonicity of $A_{k,p}^\tau [h(\cdot, \tau)]$ established in Lemma 2.1. The convergence for the full sequence of the normalized
solution follows from the uniqueness result of [25]; see also [24, page 149] for another proof of uniqueness.\footnote{For $p = k + 1$, the uniqueness of a strictly convex solution to (1.5) is up to dilations. Since here we are dealing with normalized solutions of the flow such that $\int_{\mathbb{S}^n} \frac{h^{k+1}(x,x)}{\varphi(x)} dx$ is constant, the limit is unique.}

4. Loss of smoothness

**Example 4.1.** Suppose $p + k - 1 > 0$ and $k < n$. There exist a rotationally symmetric positive function $\varphi \in C^2(\mathbb{S}^n)$ satisfying $((\varphi \cdot \nabla)|_{\theta} + \varphi |_{\theta})_{|\theta = 0} < 0$ and a smooth, closed, strictly convex initial hypersurface $M_0$ such that the solution to the flow (1.1) will lose smoothness.

**Proof.** We follow the same approach as in [2, Corollary 1]. Let $0 < h(\cdot, 0) \in C^{\infty}(\mathbb{S}^n)$ be a rotationally symmetric support function (e.g., non-negative spherical hessian) such that

$$h_{\theta\theta} + h \geq 0, \quad \sigma_k(\nabla_i \nabla_j h(\cdot, 0) + \tilde{g}_{ij} h(\cdot, 0)) > 0.$$ 

Since $\sigma_k(\cdot, 0) > 0$, a rotationally symmetric solution to the following equation exists for a short time,

$$(4.1) \quad h : \mathbb{S}^n \times [0, T) \to \mathbb{R}$$

$$\partial_t h(\cdot, t) = \varphi h^{2-p} \sigma_k(\nabla_i \nabla_j h + \tilde{g}_{ij} h).$$

The eigenvalues of $\nabla_i \nabla_j h + \tilde{g}_{ij} h$ with respect to $\tilde{g}$ are given by

$$\zeta_1 = h_{\theta\theta} + h, \quad \zeta_2 = \cdots = \zeta_n = h - \tan(\theta) h_{\theta\theta}.$$ 

Also, note that

$$(\zeta_2)_{\theta} = \tan(\theta)(\zeta_2 - \zeta_1), \quad (\zeta_2)_{\theta\theta} = (1 + 2\tan^2(\theta))(\zeta_2 - \zeta_1) - \tan(\theta)(\zeta_1)_{\theta}.$$ 

From the definition of $\zeta_1$, we obtain

$$\partial_t \zeta_1 = \varphi h^{2-p} \frac{\partial \sigma_k}{\partial \zeta_1}(\zeta_{1})_{\theta\theta} + \varphi h^{2-p} \frac{\partial^2 \sigma_k}{\partial \zeta_1 \partial \zeta_j}(\zeta_{1})_{\theta j} + \varphi h^{2-p} \sigma_k + 2(\varphi h^{2-p})_{\theta}(\sigma_k)_{\theta} + \sigma_k(\varphi h^{2-p})_{\theta\theta}.$$ 

Thus for the particular choice $\theta = 0$, we have

$$\left(\partial_t \zeta_1\right)_{|(0,t)} = \varphi h^{2-p} \frac{\partial \sigma_k}{\partial \zeta_1}(\zeta_{1})_{\theta\theta} + (n-1)\varphi h^{2-p} \frac{\partial \sigma_k}{\partial \zeta_2}(\zeta_{2} - \zeta_1) + \varphi h^{2-p} \sigma_k + 2\frac{\partial \sigma_k}{\partial \zeta_1}(\varphi h^{2-p})_{\theta}(\zeta_{1})_{\theta} + \sigma_k(\varphi h^{2-p})_{\theta\theta}$$

$$= \varphi h^{2-p} \frac{\partial \sigma_k}{\partial \zeta_1}(\zeta_{1})_{\theta\theta} - \varphi h^{2-p}((n-1)\frac{\partial \sigma_k}{\partial \zeta_2} + \frac{\partial \sigma_k}{\partial \zeta_1})\zeta_1$$

$$+ 2\frac{\partial \sigma_k}{\partial \zeta_1}(\varphi h^{2-p})_{\theta}(\zeta_{1})_{\theta} + \sigma_k(\varphi h^{2-p})_{\theta\theta} + (k+1)\varphi h^{2-p} \sigma_k.$$ 

Let $r$ be a $\pi$-periodic, $C^\infty$ function such that $r(\theta) = r(-\theta)$, it is zero on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and positive elsewhere in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Now define

$$h(\theta, 0) := \sin(\theta) \int_0^\theta r(\alpha) \cos(\alpha) d\alpha + \cos(\theta) \int_\theta^{\frac{\pi}{2}} r(\alpha) \sin(\alpha) d\alpha.$$
Note that \( (h(\cdot, 0))_\theta |_{\theta=0} = 0 \) and for all \( \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \) we have
\[
\zeta_1(0, 0) = (\zeta_1)_\theta(0, 0) = (\zeta_1)_{\theta\theta}(0, 0) = 0.
\]
Hence we obtain
\[
(\partial_t \zeta_1) |_{(0, 0)} = (\sigma_k h^2 - p(\varphi_{\theta\theta} + (p + k - 1)\varphi)) |_{(0, 0)}.
\]
Pick any positive function such that \( \varphi(\theta) = 0 \), \( (\varphi_{\theta\theta} + (p + k - 1)\varphi) |_{\theta=0} < 0 \).

For example, \( \varphi(\theta) = (\cos^2(\theta) + \frac{1}{2})^{p+k-1} \). So \( \zeta_1(0, t) \) becomes negative for \( t > 0 \) sufficiently small.

Since the solution to (4.1) depends continuously on the initial data, the nearby smooth, closed strictly convex hypersurfaces will lose smoothness. Since \( \sigma_k(0, 0) = 0 \), the argument fails if \( k = n \), as expected in view of the results in [3] and also Theorem 1.2 here.

\[ \square \]

Remark 4.2. It would be interesting to improve Theorem 1.1 by allowing
\[
\bar{\nabla}_i \bar{\nabla}_j \varphi^{n+1-n} + \bar{g}_{ij} \varphi^{n+1-n}
\]
to be non-negative definite. We conclude the paper with the following questions.

Question 1. Is it possible to obtain a gradient bound for the support function of the normalized flow for \( k < n, 1 < p < k+1 \) in the class of origin-symmetric solutions?

Question 2. If \( k < n, p < 2, \varphi \equiv 1 \), what can be said about the asymptotic behavior of the flow?

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