The prime spectrum of the universal enveloping algebra of
the 1-spatial ageing algebra and of $U(\mathfrak{gl}_2)$

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Abstract

For the algebras in the title, their prime, primitive and maximal spectra are explicitly
described. For each prime ideal an explicit set of generators is given. An explicit description
of all the containments between primes is obtained.

Key Words: Prime ideal, maximal ideal, a universal enveloping algebra, prime spectrum.

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1 Introduction

Let $K$ be a field of characteristic zero, $K^* := K \setminus \{0\}$, $N = \{0, 1, 2, \ldots\}$ and $N_+ = \{1, 2, \ldots\}$.

The goal of the paper. Let $\mathcal{H}$ be the 3-dimensional Heisenberg Lie algebra and $b$ be the
Borel subalgebra of $\mathfrak{sl}_2$. The semidirect products of Lie algebras $\mathfrak{s} = \mathfrak{sl}_2 \rtimes \mathcal{H}$ and $\mathfrak{a} := b \ltimes \mathcal{H}$
are called the Schrödinger algebra and the 1-spatial ageing algebra, respectively. Clearly, $\mathfrak{a} \subseteq \mathfrak{s}$. Let $U(\mathfrak{s})$ and $\mathcal{A} = U(\mathfrak{a})$ be the universal enveloping algebras of the Lie algebras $\mathfrak{s}$ and $\mathfrak{a}$. Then
$\mathcal{A} \subseteq U(\mathfrak{s})$.

Let $\mathfrak{gl}_2$ be the Lie algebra of $2 \times 2$ matrices over $K$ and $U(\mathfrak{gl}_2)$ be its universal enveloping
algebra. The aim of the paper is for the algebras $\mathcal{A}$ and $U(\mathfrak{gl}_2)$ to classify their prime, primitive
and maximal ideals (Theorem 1.1, Corollary 2.6, Corollary 2.8, Theorem 1.2, Corollary 3.2 and
Corollary 3.1.(1)). For each prime ideal an explicit (finite) set of generators is given. All the
containments between primes are explicitly described, i.e., the Zariski–Jacobson topology on the
spectra are described. In [6], the prime, completely prime, maximal and primitive ideals of the
algebra $U(\mathfrak{s})$ are classified. Generating sets are given for all prime ideals of $U(\mathfrak{s})$ apart from an explicit set $\{I_n^s \mid n \in N_+\}$ (see [6, Theorem 3.3]). Primitive ideals of $U(\mathfrak{s})$ with nonzero-central charge
were described by Dubsky, Mazorchuk and Zhao [13] in the following way: Each such ideal is
the annihilator of a simple highest weight $U(\mathfrak{s})$-module with nonzero central charge. The prime
ideals of the quantum spatial ageing algebra are classified in [3]. Modules over the Schrödinger
algebra are studied in [9, 12, 13, 14, 17, 19, 20, 21]. In [4], simple weight $\mathfrak{s}$-modules are classified.

The prime spectrum of the algebra $\mathcal{A}$. Recall that $\mathfrak{sl}_2 = K F \oplus K H \oplus K E$ where $[H, E] = 2E, [H, F] = -2F$ and $[E, F] = H$ and $b := K H \oplus K E$ is its Borel subalgebra. The 3-dimensional
Heisenberg Lie algebra $\mathcal{H} = K X \oplus K Y \oplus K Z$ where $[X, Y] = Z$ and $Z$ is a central element of $\mathcal{H}$. As an abstract algebra, the algebra $\mathcal{A}$ is generated by the elements $H, E, X, Y$ and $Z$ subject to the defining relations:

$$[H, E] = 2E, \quad [X, Y] = Z, \quad [H, X] = X, \quad [H, Y] = -Y, \quad [E, X] = 0, \quad [E, Y] = X,$$

and $Z$ is a central element of $\mathcal{A}$. Recall that the algebra $\mathcal{A}$ is the subalgebra of $U(\mathfrak{s})$ generated
by the elements $H, E, X, Y$ and $Z$ and the algebra $\mathcal{A}$ is isomorphic to the enveloping algebra $U(\mathfrak{a})$
of the solvable Lie subalgebra \(a\). By [18, Theorem 8.3.36], all prime ideals of the algebra \(A\) are completely prime.

Let \(H' := H + Z^{-1}XY - \frac{1}{2}\) and \(E' := E - \frac{1}{2}Z^{-1}X^2\). Then \([H', E'] = 2E'\). It was proved in [6] that the algebra \(A_Z\) (the localization of \(A\) at the powers of the central element \(Z\)) is a tensor product of three algebras

\[
A_Z = \mathbb{K}[Z^{\pm 1}] \otimes \mathbb{K}[H'][E'; \sigma] \otimes A_1
\]

where \(\mathbb{K}[H'][E'; \sigma]\) is a skew polynomial algebra where \(\sigma(H') = H' - 2\) and \(A_1 := \mathbb{K}[X, Y]\) is the Weyl algebra, \([X, Y] = 1\), where \(X := Z^{-1}X\). The algebras \(A\) and \(A_Z\) are Noetherian algebras of Gelfand-Kirillov dimension 5.

Let \(E := E'Z = EZ - \frac{1}{2}X^2\) and \(\mathcal{H} := H'Z + \frac{1}{2}Z = HZ + XY\). Then \(E\) is a normal element of the algebra \(A\), i.e., \(E.A = AE\). In fact, \(AE = (H - 2)E, AE = E\mathcal{E}, \mathcal{E}Y = Y\mathcal{E}, \mathcal{E}X = X\mathcal{E}\) and \(EZ = Z\mathcal{E}\).

Let \(P := \{p \in \text{Spec}(\mathbb{K}[Z, \mathcal{E}]) \mid ht(p) = 1\}\). Recall that every height 1 prime ideal \(p\) is generated by a simple irreducible polynomial which is unique up to multiplicative non-zero constant. Let \(\bar{A} := A/(\mathcal{E})\) and \(\mathcal{A} := A/(\mathcal{E}, Z)\). For an element \(a \in A\), let \(\bar{a} := a + (\mathcal{E}) \in \bar{A}\) and \(\mathcal{a} := a + (\mathcal{E}, Z) \in \mathcal{A}\).

In the diagram below, \(\mathcal{P}\) means \(p \subseteq q\), and \(\mathcal{G}\) means obvious inclusions between the sets of primes in sets \(G\) and \(H\).

\[\begin{array}{c}
\{ (Y, E, p) \mid p \in \text{Max}(\mathbb{K}[H]) \} \\
\downarrow (Y, E) \\
\{ (X, q') \mid q' \in Q' \} \\
\downarrow (Y) \\
\downarrow (E) \\
\{ (\mathcal{E}, m) \mid m \in M \} \\
\{ A \cap (\mathcal{E}, p_0) \mid p_0 \in P_0 \} \\
\{ (\mathcal{E}, p_1) \mid p_1 \in P_1 \} \\
\{ (\mathcal{E}, q) \} \\
\{ 0 \}
\end{array}\]

where \(q' \in Q' := \text{Max}(\mathbb{K}[Z']) \setminus \{(Z')\}\) and \(Z' := EY^2\), \(q \in Q := \text{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}\), \(p_0 \in P_0 := \{ p_0' \in P \mid p_0' \neq (Z), p_0' \subseteq (Z, \mathcal{E}) \}, p_1 \in P_1 := P \setminus (P_0 \cup \{ q' \mid q' \in \text{Max}(\mathbb{K}[Z]) \})\) and \(M := \{ m \in \text{Max}(\mathbb{K}[Z, \mathcal{E}]) \mid Z \notin m \}\). Furthermore, let \(p_0 \in P_0\), then \(p_0 = (f)\) for some irreducible polynomial \(f = a_0(Z) + a_1(Z)\mathcal{E} + \cdots + a_n(Z)\mathcal{E}^n \in \mathbb{K}[Z, \mathcal{E}] \setminus (\mathbb{K} \cup \mathbb{K}Z)\) with \(a_0(0) = 0\) and (see Theorem 2.5.2), then

\[
A \cap (\mathcal{E}, f)Z = \begin{cases} (\mathcal{E}, \tilde{f}_{\text{red}}), & \text{if } \tilde{f}_{\text{red}} \notin (X), \\ (\mathcal{E}, \tilde{f}_{\text{red}}, Z^{-1}(f_{\text{red}})X), & \text{if } \tilde{f}_{\text{red}} \in (X), \end{cases}
\]

see (9) for the definition of \(f_{\text{red}}\).
The prime spectrum of $U(\mathfrak{gl}_2)$. The Lie algebra $\mathfrak{gl}_2$ is the direct product of the Lie algebra $\mathfrak{sl}_2$ and a 1-dimensional Lie algebra $\mathbb{K}Z$. So, $Z$ is a central element of the algebra $\mathcal{U} := U(\mathfrak{gl}_2)$. Clearly, $\mathcal{U} = U(\mathfrak{sl}_2) \otimes \mathbb{K}[Z]$ is a tensor product of algebras. Hence, $Z(\mathcal{U}) = Z(U(\mathfrak{sl}_2)) \otimes \mathbb{K}[Z] = \mathbb{K}[\Delta, Z]$ where $\Delta = 4FE + H^2 + 2H$ is the Casimir element of $U(\mathfrak{sl}_2)$.

**Theorem 1.2.** Let $\mathbb{K}$ be a field of characteristic zero. Then the set of prime ideals of the algebra $\mathcal{U} = U(\mathfrak{gl}_2)$ and all the containments between primes are given in the following diagram:

\[
\begin{array}{c}
(I_n, q) \\
(\Delta - \lambda_n, q) \\
(q) \quad \{ (m) | m \in \mathbb{H}_2 \} \\
0 \quad \{ (p) | p \in \mathbb{H}_1 \}
\end{array}
\]

where $n \in \mathbb{N}_+$, $q \in \text{Max}(\mathbb{K}[Z])$, $\mathbb{H}_1 := \{ p \in \text{Spec}(\mathbb{K}[Z, \Delta]) | \text{ht}(p) = 1, p \cap \mathbb{K}[Z] = 0, p \neq (\Delta - \lambda_n) \text{ for all } n \in \mathbb{N}_+ \}$, $\mathbb{H}_2 := \text{Max}(\mathbb{K}[Z, \Delta]) \setminus \{ (\Delta - \lambda_n, q) | n \in \mathbb{N}_+, q \in \text{Max}(\mathbb{K}[Z]) \}.$

2 Prime ideals of the algebra $\mathcal{A}$

In this section, classifications of prime, primitive and maximal ideals of the algebra $\mathcal{A}$ are given (Theorem 1.1, Corollary 2.8 and Corollary 2.6). Moreover, for each of the prime ideals of $\mathcal{A}$ an explicit set of generators is given.

For an algebra $R$, let $\text{Spec}(R)$ be the set of its prime ideals. The set $(\text{Spec}(R), \subseteq)$ is a partially ordered set (poset) with respect to inclusion of prime ideals. Each element $r \in R$ determines two maps from $R$ to $R$, $r : x \mapsto rx$ and $r : x \mapsto xr$ where $x \in R$. An element $a \in R$ is called a normal element if $Ra = aR$.

**Proposition 2.1.** ([3]) Let $R$ be a Noetherian ring and $s$ be an element of $R$ such that $S_s := \{ s^i | i \in \mathbb{N} \}$ is a left denominator set of the ring $R$ and $(s^i) = (s)^i$ for all $i \geq 1$ (e.g., $s$ is a normal element). Then $\text{Spec}(R) = \text{Spec}(R, s) \cup \text{Spec}_s(R)$ where $\text{Spec}(R, s) := \{ p \in \text{Spec}(R) | s \not\in p \}$, $\text{Spec}_s(R) = \{ q \in \text{Spec}(R) | s \not\in q \}$ and

(a) the map $\text{Spec}(R, s) \to \text{Spec}(R/(s))$, $p \mapsto p/(s)$, is a bijection with the inverse $\pi^{-1}(q)$ where $\pi : R \to R/(s), r \mapsto r + (s),$

(b) the map $\text{Spec}_s(R) \to \text{Spec}(R_s)$, $p \mapsto S_s^{-1}p$, is a bijection with the inverse $\sigma^{-1}(q)$ where $\sigma : R \to R_s := S_s^{-1}R, r \mapsto \frac{r}{s}.$

(c) For all $p \in \text{Spec}(R, s)$ and $q \in \text{Spec}_s(R), p \not\subseteq q.$

Since $Z$ is a central element of $\mathcal{A}$, by Proposition 2.1, we have the disjoint union

$$\text{Spec}(\mathcal{A}) = \text{Spec}(\mathcal{A}/(Z)) \cup \text{Spec}(\mathcal{A}_Z).$$

We identify the sets of prime ideals in (5) via the bijections given in the statements (a) and (b) of Proposition 2.1. The factor algebra $\mathcal{A}/(Z)$ is studied in [7] where a classification of prime ideals of $\mathcal{A}/(Z)$ is given, see [7, Theorem 2.5]. By [7, Theorem 2.5], the set of prime ideals of $\mathcal{A}$ containing the element $Z$ contains precisely the ideals in (2) over $(Z)$. So, it remains to describe the set $\text{Spec}(\mathcal{A}_Z)$, i.e., the set of prime ideals of $\mathcal{A}$ that do not contain the central element $Z$.  

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For an element \( a \in \mathcal{A}_Z \), we denote by \((a)_Z\) the ideal of \( \mathcal{A}_Z \) generated by the element \( a \). Clearly, \((\mathcal{E})_Z = (E')_Z\). Let \( \mathcal{A}_{Z,\mathcal{E}} \) be the localization of the algebra \( \mathcal{A}_Z \) at the powers of the element \( \mathcal{E} \). Notice that \( E' \) and \( \mathcal{E} \) are normal elements of \( \mathcal{A}_Z \). Clearly, \( \mathcal{A}_{Z,E'} = \mathcal{A}_{Z,\mathcal{E}} \). By Proposition 2.1,

\[
\text{Spec} \left( \mathcal{A}_Z \right) = \text{Spec} \left( \mathcal{A}_Z / (\mathcal{E})_Z \right) \sqcup \text{Spec} \left( \mathcal{A}_{Z,\mathcal{E}} \right).
\]

Now, by (1), the factor algebra

\[
\mathcal{A}_Z / (\mathcal{E})_Z \cong \mathbb{K}[Z^\pm 1, H'] \otimes A_1 = \mathbb{K}[Z^\pm 1] \otimes A_1
\]

is a tensor product of algebras. By (1), the localized algebra

\[
\mathcal{A}_{Z,\mathcal{E}} = \mathbb{K}[Z^\pm 1] \otimes \mathbb{K}[H'][E^\pm 1, \sigma] \otimes A_1
\]

is a tensor product of algebras where the algebras \( \mathbb{K}[H'] [E^\pm 1, \sigma] \) and \( A_1 \) are central simple algebras.

The algebra \( \mathcal{A} \) is a Noetherian domain. Let \( \text{Frac}(\mathcal{A}) \) be its skew field of fractions. The next lemma describes the centres of the algebras \( \mathcal{A} \) and \( \text{Frac}(\mathcal{A}) \).

**Lemma 2.2.** \( Z(\mathcal{A}) = \mathbb{K}[Z] \) and \( Z(\text{Frac}(\mathcal{A})) = \mathbb{K}(Z) \).

**Proof.** By (8), \( Z(\mathcal{A}_{Z,\mathcal{E}}) = \mathbb{K}[Z^\pm 1] \). Since \( \mathbb{K}[Z] \subseteq Z(\mathcal{A}) \subseteq Z(\mathcal{A}_{Z,\mathcal{E}}) \cap \mathcal{A} = \mathbb{K}[Z^\pm 1] \cap \mathcal{A} = \mathbb{K}[Z] \), we have \( Z(\mathcal{A}) = \mathbb{K}[Z] \). The second equality in statement 1 follows directly from (8). \( \square \)

Next, we represent the factor algebra \( \mathcal{A} / (\mathcal{E}) \) as an iterated Ore extension. It follows from this fact that \((\mathcal{E})_Z \cap \mathcal{A} = (\mathcal{E})\).

**Lemma 2.3.** \( \mathcal{A} / (\mathcal{E}) \cong \Lambda[Y; \delta_1][H; \delta_2] \) is an iterated Ore extension over the commutative domain \( \Lambda := \mathbb{K}[E, X, Z] / (\mathcal{E}) \) where the derivations \( \delta_1 \) and \( \delta_2 \) are defined in the proof. In particular, \( (\mathcal{E}) \) is a completely prime ideal of \( \mathcal{A} \) and \( (\mathcal{E})_Z \cap \mathcal{A} = (\mathcal{E}) \).

**Proof.** Notice that \( \mathcal{A} = \bigoplus_{i,j \in \mathbb{N}} \mathbb{K}[E, X, Z] Y^i X^j \). Since \( \mathcal{E} \) is a normal element of \( \mathcal{A} \), we have \((\mathcal{E}) = \mathcal{E} \mathcal{A} = \bigoplus_{i,j \in \mathbb{N}} \mathcal{E} \mathbb{K}[E, X, Z] Y^i X^j \). Hence, \( \mathcal{A} / (\mathcal{E}) \cong \Lambda[Y; \delta_1][H; \delta_2] \) where \( \delta_1 \) is the derivation of \( \Lambda \) defined by the rule

\[
\delta_1(E) = -X, \delta_1(X) = -Z, \delta_1(Z) = 0
\]

and \( \delta_2 \) is the derivation of \( \Lambda[Y; \delta_1] \) defined by the rule \( \delta_2(E) = 2E, \delta_2(X) = X, \delta_2(Z) = 0 \) and \( \delta_2(Y) = -Y \).

Since \( \mathcal{E} = EZ - \frac{1}{2} X^2 \) is an irreducible polynomial in \( \mathbb{K}[E, X, Z] \), the factor algebra \( \Lambda = \mathbb{K}[E, X, Z] / (\mathcal{E}) \) is a domain. Then the iterated Ore extension \( \mathcal{A} / (\mathcal{E}) \) is a domain, i.e., \( (\mathcal{E}) \) is a completely prime ideal of \( \mathcal{A} \). Now, let \( u \in (\mathcal{E})_Z \cap \mathcal{A} \). Then \( Z^i u \in (\mathcal{E}) \) for some \( i \in \mathbb{N} \). Since \( (\mathcal{E}) \) is a completely prime ideal of \( \mathcal{A} \) and \( Z \notin (\mathcal{E}) \), we have \( u \in (\mathcal{E}) \). Hence, \((\mathcal{E})_Z \cap \mathcal{A} = (\mathcal{E})\). \( \square \)

Let \( \Gamma := \mathbb{K}[E, Z] \). Then \( \Lambda = \Lambda \oplus \Gamma X \) is a free \( \Gamma \)-module of rank 2. Clearly, \( X \Lambda = \Lambda EZ \oplus \Gamma X \) (since \( \mathcal{E} = EZ - \frac{1}{2} X^2 \)) and \( Z^i \Lambda = Z^i \Gamma \oplus Z^i \Gamma X \) for all \( i \in \mathbb{N} \). So, in the algebra (see Lemma 2.3)

\[
\mathcal{A} := \Lambda / (\mathcal{E}) = (\Lambda \oplus \Gamma X) [Y; \delta_1][H; \delta_2]
\]

it is very easy to decide whether an element belongs to the left ideal \( X \mathcal{A} \) or the left ideal \( Z^i \mathcal{A} \). Clearly, \( \bigcap_{i \geq 0} Z^i \mathcal{A} = 0 \). So, for every nonzero element \( a \in \mathcal{A} \), there is a unique natural number \( d = v_Z(a) \) such that \( a \in Z^d \mathcal{A} \setminus Z^{d+1} \mathcal{A} \). The map \( Z : : \mathcal{A} \to \mathcal{A}, b \mapsto Zb \) is an injection. Therefore, \( Z^{-v_Z(a)}a \in \mathcal{A} \) and \( v_Z(Z^{-v_Z(a)}a) = 0 \). The element

\[
a_{\text{red}} := Z^{-v_Z(a)}a
\]

is called the reduced form of \( a \). Since \( \mathcal{E} = EZ - \frac{1}{2} X^2 \), we have the equality of ideals \( (\mathcal{E}, Z) = (Z, X^2) \) in \( \mathcal{A} \). By Lemma 2.3,

\[
\mathcal{A} := \mathcal{A} / (\mathcal{E}, Z) = \mathcal{A} / (Z, X^2) \cong \frac{\mathbb{K}[E, X]}{X^2 \mathbb{K}[E, X]} [Y; \delta_1][H; \delta_2].
\]
The element $X \in \tilde{A}$ is a normal element: $XY = YX$ and $XH = (H-1)X$ in $\tilde{A}$. Clearly, $X^2 = 0$, $(X)^2 = (X^2) = 0$ in $\tilde{A}$ and

$$\tilde{A}/(X) \simeq \mathcal{A}/(X) \simeq \mathbb{K}[E][Y; \delta_1][H; \delta_2]$$

is a domain. Therefore, the set of left zero divisors of $\tilde{A}$ is equal to the set of right zero divisors of $\tilde{A}$ and is equal to the ideal $(X)$ of $\tilde{A}$ since the left/right $\tilde{A}$-module $(X)$ is isomorphic to $\tilde{A}$. The set $C_\tilde{A}$ of regular elements (i.e., nonzero divisors) of $\tilde{A}$ is equal to $\tilde{A} \setminus (X)$. The algebra $\tilde{A}$ is an epimorphic image of $\mathcal{A}$.

The next lemma is used in the proof of Theorem 2.5.

**Lemma 2.4.** $\mathcal{H}^2 \equiv \Phi Z \text{ mod } (\mathcal{E})$ where $\Phi := H^2Z + 2HXY + 2EY^2 - XY$.

**Proof.** Using the defining relations of $\mathcal{A}$,

$$\mathcal{H}^2 = (HZ + XY)^2 = H^2Z^2 + 2HXYZ + (XY)^2 = H^2Z^2 + 2HXYZ + X^2Y^2 - XYZ$$

$$=(H^2Z^2 + 2HXYZ + 2EY^2Z - XYZ) - 2EY^2Z + X^2Y^2 = \Phi Z - 2\mathcal{E}Y^2,$$

and the result follows. \qed

The next theorem is a key result in finding explicit generators for prime ideals of the algebra $\mathcal{A}$. In general, as a rule, it is not possible to find explicit generators for a prime ideal of the form $\mathcal{A} \cap P$ where $P$ is a prime ideal of a localization $S^{-1} \mathcal{A}$ of $\mathcal{A}$.

**Theorem 2.5.** Let $f = a_0(Z) + a_1(Z)\mathcal{H} + a_2(Z)\mathcal{H}^2 + \cdots + a_n(Z)\mathcal{H}^n \in \mathbb{K}[Z, \mathcal{H}] \setminus (\mathcal{K} \cup \mathbb{K}Z)$ where all $a_i(Z) \in \mathbb{K}[Z]$ and $a_0(Z) \neq 0$. Then

1. $\mathcal{A} \cap (\mathcal{E}, f) = (\mathcal{E}, f)$ iff $a_0(0) \neq 0$ iff $f \notin (Z, \mathcal{H})$ where $(Z, \mathcal{H})$ is the maximal ideal of the polynomial algebra $\mathbb{K}[Z, \mathcal{H}]$ generated by the elements $Z$ and $\mathcal{H}$.

2. Suppose, in addition, that $a_0(0) = 0$. Let $d = v_Z(f)$, $\tilde{f}_{\text{red}} = Z^{-d}\tilde{f}$ and $\tilde{f}_{\text{red}}$ is the image of the element $\tilde{f}_{\text{red}}$ under the epimorphism $\mathcal{A} \to \tilde{A}$.

   (a) If $\tilde{f}_{\text{red}} \notin (X)$ then $\mathcal{A} \cap (\mathcal{E}, f) = (\mathcal{E}, \tilde{f}_{\text{red}})$.

   (b) If $\tilde{f}_{\text{red}} \in (X)$ then $\mathcal{A} \cap (\mathcal{E}, f) = (\mathcal{E}, \tilde{f}_{\text{red}}, Z^{-1}\tilde{f}_{\text{red}}X) = (\mathcal{E}, \tilde{f}_{\text{red}}) + Z^{-1}\tilde{f}_{\text{red}}X \mathcal{A}$.

**Proof.** 1. The equality $\mathcal{A} \cap (\mathcal{E}, f) = (\mathcal{E}, f)$ holds iff the image $\tilde{f} \in \tilde{A}$ of the element $f$ under the epimorphism $\mathcal{A} \to \tilde{A} \to \tilde{A}$ is a nonzero divisor, i.e., $f \notin (X)$. By Lemma 2.4, $f = a_0(0) + a_1(0)XY$ (since $\mathcal{H} = HZ + XY$). So, $\tilde{f} \notin (X)$ iff $a_0(0) \neq 0$.

2. Clearly, $(\mathcal{E}, f) = (\mathcal{E}, \tilde{f}_{\text{red}})$ in $\tilde{A}$ since $v_Z(\tilde{f}) = 0$.

   (a) If $\tilde{f}_{\text{red}} \notin (X)$ then $\mathcal{A} \cap (\mathcal{E}, f) = \mathcal{A} \cap (\mathcal{E}, \tilde{f}_{\text{red}}) = (\mathcal{E}, \tilde{f}_{\text{red}})$, by statement 1.

   (b) If $\tilde{f}_{\text{red}} \in (X)$ then $\tilde{f}_{\text{red}} = \alpha X$ for some nonzero element $\alpha \in \mathcal{A}/(X)$ since

   $$\tilde{A}X = \mathcal{A}/(X) \cdot X$$

   (as $(X)^2 = 0$ in $\tilde{A}$). The element $\alpha$ is unique since the left $\mathcal{A}/(X)$-module $\mathcal{A}/(X) \cdot X$ is free of rank 1. If $\alpha \in \mathcal{A} \cap (\mathcal{E}, \tilde{f}_{\text{red}}) \setminus (\mathcal{E}, \tilde{f}_{\text{red}})$ then the element $\tilde{a} = a + (\mathcal{E}) \in \tilde{A}$ is a product $\tilde{a} = Z^{-i}\tilde{f}_{\text{red}}b$ for some $b \in \mathcal{A}$ with $v_Z(b) = 0$ where $i = v_Z(\tilde{f}_{\text{red}}b) \geq 1$. We may assume that $v_Z(\tilde{a}) = 0$. The image $\tilde{b}$ of the element $b$ in $\tilde{A}$ is a nonzero one since $v_Z(b) = 0$. Since $\tilde{f}_{\text{red}}\tilde{b} = 0$ in $\tilde{A}$ and $\tilde{f}_{\text{red}} \neq 0$, the element $\tilde{b} \in \tilde{A}$ is a zero divisor, i.e., $\tilde{b} \in X\tilde{A}$. Therefore, $\tilde{b} = X\beta$ for a nonzero element $\beta \in \mathcal{A}/(X)$ since $X \cdot \tilde{A} = X \cdot \mathcal{A}/(X)$ (as $(X)^2 = 0$ in $\tilde{A}$). Now,

   $$Z\tilde{a} = \tilde{f}_{\text{red}}\tilde{b} \equiv \tilde{f}_{\text{red}}\tilde{b} \equiv \alpha X \cdot X \beta \equiv \alpha 2EZ\beta \equiv 0 \mod (Z\tilde{A}).$$

Since the image of the element $\alpha 2EZ\beta$ in the domain $\mathcal{A}/(X)$ is a nonzero one, we must have $i = 1$ (since $v_Z(\tilde{a}) = 0$ and $v_Z(Z\tilde{a}) = i = v_Z(\tilde{a}) = i$). Therefore,

$$\tilde{a} = Z^{-1}\tilde{f}_{\text{red}}b \in Z^{-1}\tilde{f}_{\text{red}}(X\tilde{A} + Z\tilde{A}) \subseteq Z^{-1}\tilde{f}_{\text{red}}X\tilde{A} + \tilde{f}_{\text{red}}\tilde{A}.$$

This means that $\mathcal{A} \cap (\mathcal{E}, f) = \mathcal{A} \cap (\mathcal{E}, \tilde{f}_{\text{red}}) = (\mathcal{E}, \tilde{f}_{\text{red}}) + Z^{-1}\tilde{f}_{\text{red}}X\tilde{A} = (\mathcal{E}, \tilde{f}_{\text{red}}, Z^{-1}\tilde{f}_{\text{red}}X)$. \qed
Example. Let $f = \mathcal{H}$. Then $p_0 = (f) \in \mathcal{P}_0$ and $\mathcal{A} \cap (\mathcal{E}, p_0)_Z = (\mathcal{E}, \mathcal{H}, HX + 2EY)$: Clearly, $p_0 \in \mathcal{P}_0$. Then $\tilde{f}_{\text{red}} = \hat{f} = \mathcal{H} = HZ + XY$ and $Z^{-1}\tilde{f}_{\text{red}}X = Z^{-1}(HZX + X^2Y) = HX + Z^{-1}2EYZ = HX + 2EY$. Hence, $\tilde{f}_{\text{red}} = 1 + \mathcal{H} + 2EY$. Now, the result follows from Theorem 1.1.

Example. Let $f = Z + Z\mathcal{H} + \mathcal{H}^2$. Then $p_0 = (f) \in \mathcal{P}_0$ and $\mathcal{A} \cap (\mathcal{E}, p_0)_Z = (\mathcal{E}, 1 + \mathcal{H} + 2EY^2)$: Clearly, $p_0 \in \mathcal{P}_0$ and $f = Z + Z\mathcal{H} + X^2Y^2 = Z(1 + \mathcal{H} + 2EY^2)$. Hence, $\tilde{f}_{\text{red}} = 1 + \mathcal{H} + 2EY^2$. Now, the result follows from Theorem 1.1.

Example. Let $f = Z^2 + Z\mathcal{H} + \mathcal{H}^2$. Then $p_0 = (f) \in \mathcal{P}_0$ and $\mathcal{A} \cap (\mathcal{E}, p_0)_Z = (\mathcal{E}, Z + 2EYX, 1 + 4E^2Y^2)$: Clearly, $p_0 \in \mathcal{P}_0$ and $f = Z^2 + Z\mathcal{H} + X^2Y^2 = Z(1 + \mathcal{H} + 2EY^2)$. Hence, $\tilde{f}_{\text{red}} = Z + \mathcal{H} + 2EY^2$. Since $\tilde{f}_{\text{red}} = XY + 2EY^2 \notin (X)$, we have the result, by Theorem 1.1.

Example. Let $f = Z^2 + Z^3$. Then $p_0 = (f) \in \mathcal{P}_0$ and $\mathcal{A} \cap (\mathcal{E}, p_0)_Z = (\mathcal{E}, Z + 2EYX, 1 + 4E^2Y^2)$: Clearly, $p_0 \in \mathcal{P}_0$ and $f = Z^2 + Z^3 = Z^2 + 2ZEYX$. Then $\tilde{f}_{\text{red}} = Z + 2EY$. Since $\tilde{f}_{\text{red}} = 2EYX \in (X)$, $Z^{-1}\tilde{f}_{\text{red}}X = 1 + Z^{-1}2E^2Y^2 = 1 + Z^{-1}2E \cdot 2ZEY = 1 + 4E^2Y$. Now, the result follows from Theorem 1.1.

Proof of Theorem 1.1 By (5) and (6), the set Spec($\mathcal{A}$) is a disjoint union of the sets Spec($\mathcal{A}/(\mathcal{Z})$), Spec($\mathcal{A}_Z/(\mathcal{E})_Z$) and Spec($\mathcal{A}_{Z,E}$). Recall that we view these sets as subsets of Spec($\mathcal{A}$). We have seen above that Spec($\mathcal{A}/(\mathcal{Z})$) contains precisely the prime ideals over ($\mathcal{Z}$) in the diagram (2).

(i) The set (as a subset of Spec($\mathcal{A}$)) Spec($\mathcal{A}_{Z,E}$) is equal to $\{0\} \cup \{(q) | q \in \mathcal{Q}\}$ where $\mathcal{Q} = \text{Max}(\mathcal{K}(Z)) \setminus \{(Z)\}$: By (8), each nonzero element of Spec($\mathcal{A}_{Z,E}$) is equal to $\mathcal{A} \cap (q)_{Z,E}$ for some $q \in \mathcal{Q}$. We have to show that $\mathcal{A} \cap (q)_{Z,E} = q\mathcal{A}$. The ideal $q = q\mathcal{A}$ is a completely prime ideal of $\mathcal{A}$ since, by (1),

$$\mathcal{A}/q \simeq \mathcal{A}_Z/q \simeq \mathcal{K}[Z^{\pm 1}]/(q)_Z \otimes \mathcal{K}[H^\prime][\mathcal{E}; \sigma] \otimes \mathcal{A}_1,$$

a domain. Hence, $(q) = \mathcal{A} \cap (q)_{Z}$. By (10), $\mathcal{A} \cap (q)_{Z,E} = \mathcal{A} \cap (q)_{Z}$, and so $(q) = \mathcal{A} \cap (q)_{Z,E}$, as required.

Let us describe the set Spec($\mathcal{A}_Z/(\mathcal{E})_Z$). By (7), the set Spec($\mathcal{A}_Z/(\mathcal{E})_Z$) (as a subset of Spec($\mathcal{A}$)) consists of the elements: $(\mathcal{E}) = \mathcal{A} \cap (\mathcal{E})_{Z}$ (by Lemma 2.3), $\mathcal{A} \cap (\mathcal{E}, m)$ where $m \in \mathcal{M}$, and $\mathcal{A} \cap (\mathcal{E}, p)$ where $p$ is a prime, height 1 ideal of $\mathcal{K}[Z, \mathcal{E}]$ such that $p \neq (Z)$. The last set of prime ideals is equal to the set $\{\mathcal{A} \cap (\mathcal{E}, p_0)_Z, \mathcal{A} \cap (\mathcal{E}, p_1)_Z | p_0 \in \mathcal{P}_0, p_1 \in \mathcal{P}_1\}.

(ii) For all $m \in \mathcal{M}$, $(\mathcal{E}, m) = \mathcal{A} \cap (\mathcal{E}, m)_Z$ is a maximal (completely prime) ideal of $\mathcal{A}$: The element $Z$ is a unit of the field $F_m := \mathcal{K}[Z, \mathcal{E}]/(\mathcal{M})$. Therefore, by (1),

$$\mathcal{A}/(\mathcal{E}, m) \simeq \mathcal{A}_Z/(\mathcal{E}, m)_Z \simeq F_m \otimes \mathcal{A}_1,$$

a simple domain. Hence, the ideal $(\mathcal{E}, m) = \mathcal{A} \cap (\mathcal{E}, m)_Z$ is a maximal (completely prime) ideal of $\mathcal{A}$.

(iii) For every $q \in \mathcal{Q}$, the ideal $(\mathcal{E}, q) = \mathcal{A} \cap (\mathcal{E}, q)_Z$ of $\mathcal{A}$ is completely prime: The element $Z$ is a unit of the field $L_q := \mathcal{K}[Z]/q$. By (1),

$$\mathcal{A}/(\mathcal{E}, q) \simeq \mathcal{A}_Z/(\mathcal{E}, q)_Z \simeq L_q \otimes \mathcal{K}[H^\prime] \otimes \mathcal{A}_1,$$

a domain. Hence, the ideal $(\mathcal{E}, q) = \mathcal{A} \cap (\mathcal{E}, q)_Z$ is a completely prime ideal.

(iv) For all $p_1 \in \mathcal{P}_1$, $\mathcal{A} \cap (\mathcal{E}, p_1)_Z = (\mathcal{E}, p_1)$: This follows from Theorem 2.5, (1).

(v) Let $p_0 \in \mathcal{P}_0$, then $(p_0) = (f)$ for some irreducible polynomial $f \in \mathcal{K}[Z, \mathcal{E}] \setminus (\mathcal{K} \cup \mathcal{K}Z)$ as in Theorem 2.5 with $a_0(0) = 0$. Then (3) holds: This follow from Theorem 2.5, (2).

So, we have proved that the diagram (2) contains precisely all the prime ideals of the algebra $\mathcal{A}$.

(vi) All the containments between primes are given in (2): Repeating twice Proposition 2.1.(c), we see that possible inclusions between prime ideals $P_1, P_2$ and $P_3$ of the three sets Spec($\mathcal{A}/(\mathcal{Z})$), Spec($\mathcal{A}_Z/(\mathcal{E})_Z$) and Spec($\mathcal{A}_{Z,E}$), respectively, can only possibly be $P_1 \supset P_2$, $P_1 \supset P_3$ or $P_2 \supset P_3$. It is easy to see that we have the inclusions between prime ideals as in the diagram (2) (see also below for details). Clearly, the only possible inclusions $P_1 \supset P_3$ and $P_2 \supset P_3$ are as in the diagram (2).
It remains to sort out inclusions of the type \( P_1 \supset P_2 \). By the statement (ii), \( P_2 \) is not equal to the maximal ideal \((\mathcal{E}, \mathfrak{m})\) where \( \mathfrak{m} \in \mathcal{M} \). By the very definition the elements \( Z, \mathcal{E} \) and \( \mathcal{K} \) belong to the ideal \((X)\). Hence, \((X) \supset (\mathcal{E})\). It remains to consider the case where \( P_2 \) is of type \( A \cap (\mathcal{E}, \mathfrak{p}) \) where \( \mathfrak{p} \in P_0 \cup P_1 \). Clearly, \((X) \supset A \cap (\mathcal{E}, \mathfrak{p}) \) for all \( \mathfrak{p} \in P_0 \) (since \( Z, \mathcal{E}, \mathcal{K} \in (X) \)) and \( P_1 \supset A \cap (\mathcal{E}, \mathfrak{p}_1) \) for all \( P_1 = (Y, E, \mathfrak{p}) \) where \( \mathfrak{p} \in \text{Max}(\mathcal{K}[H]) \) and all \( \mathfrak{p}_1 \in P_1 \) since \( P_1 \supset (X) \) and \( Z, \mathcal{E} \in (X) \) (and so the maximal ideal of \( \mathcal{K}[Z, \mathcal{E}] \) generated by the elements \( Z \) and \( \mathcal{E} \) is contained in \((X)\) and, therefore, in all \( P_1 = (Y, E, \mathfrak{p}) \); clearly, the ideal of \( \mathcal{K}[Z, \mathcal{E}] \) generated by \( Z, \mathcal{E} \) and \( P_1 \) is (1)). So, all inclusions between primes are as in (2).

As a corollary of Theorem 1.1 we obtain a classification of maximal ideals of \( \mathcal{A} \).

**Corollary 2.6.** The set of maximal ideals of \( \mathcal{A} \) is equal to \( \{(X, q') \mid q' \in Q'\} \cup \{(Y, E, \mathfrak{p}) \mid \mathfrak{p} \in \text{Max}(\mathcal{K}[H])\} \cup \{(\mathcal{E}, \mathfrak{m}) \mid \mathfrak{m} \in \mathcal{M}\} \) where \( Q' \) and \( \mathcal{M} \) are defined in Theorem 1.1.

**Proof.** The corollary follows from Theorem 1.1.

**Corollary 2.7.** Not every nonzero prime ideal of the algebra \( \mathcal{A} \) meets the centre.

**Proof.** The statement follows from (2).

**The set \( \text{Prim}(\mathcal{A}) \) of primitive ideals of the algebra \( \mathcal{A} \).** An ideal of a ring \( R \) is called a **primitive ideal** if it is the annihilator of a simple left \( R \)-module. The set of primitive ideals of \( R \) is denoted by \( \text{Prim}(R) \). A prime ideal \( P \) of a ring \( R \) is said to be **locally closed** if the set \( \{P\} \) is locally closed in the topological space \( \text{Spec}(R) \) where \( \text{Spec}(R) \) is equipped with Zariski-Jacobson topology ([8, II.1.1]). A prime ideal \( P \) of a Noetherian \( \mathcal{K} \)-algebra \( R \) is said to be **rational** if the field \( \text{Frac}(R/P) \) is algebraic over \( \mathcal{K} \) where \( \text{Frac}(R/P) \) is the left (right) quotient ring of the Noetherian prime algebra \( R/P \). We say that the **Dixmier-Moeglin equivalence** holds for a Noetherian \( \mathcal{K} \)-algebra \( A \) if for each prime ideal \( P \) of \( A \) we have the following equivalences:

\[ P \text{ is locally closed} \iff P \text{ is primitive} \iff P \text{ is rational.} \]

The next corollary describes the set of primitive ideals \( \text{Prim}(\mathcal{A}) \) of the algebra \( \mathcal{A} \).

**Corollary 2.8.** \( \text{Prim}(\mathcal{A}) = \text{Max}(\mathcal{A}) \cup \{(Y), (E), (Z)\} \cup \{(q) \mid q \in Q\} \).

**Proof.** Since \( \mathcal{A} \) is a universal enveloping algebra of a finite dimensional Lie algebra it satisfies the Dixmier-Moeglin equivalence. By [8, Lemma II.7.7], a prime ideal \( P \) in a ring \( R \) is locally closed iff the intersection of all prime ideals properly containing \( P \) is also an ideal properly containing \( P \). Clearly, all the maximal ideals are primitive ideals. By (2), the set of locally closed prime ideals is \( \text{Max}(\mathcal{A}) \cup \{(Y), (E), (Z)\} \cup \{(q) \mid q \in Q\} \). Then the corollary follows from the Dixmier-Moeglin equivalence for \( \mathcal{A} \).

**The semicentre of \( \mathcal{A} \).** Recall that \( \mathcal{A} = U(\mathfrak{a}) \) is the enveloping algebra of the solvable Lie algebra \( \mathfrak{a} \). For each \( \lambda \in \text{Hom}(\mathfrak{a}, \mathcal{K}) \), let \( \mathcal{A}_\lambda := \{a \in \mathcal{A} \mid \text{ad}_\lambda(a) = \lambda(a)x, \forall x \in \mathfrak{a}\} \). Any nonzero element \( a \in \mathcal{A}_\lambda \) is called a **semi-invariant** with weight \( \lambda \). The sum of the \( \mathcal{A}_\lambda \) is direct and is denoted by \( Sz(\mathcal{A}) \) which contains the centre of \( \mathcal{A} \). \( Sz(\mathcal{A}) \) is called the **semicentre** of \( \mathcal{A} \), which is a commutative domain, (see [15, Corollary 4.6]).

**Proposition 2.9.** \( Sz(\mathcal{A}) = \mathcal{K}[Z, \mathcal{E}] \).

**Proof.** It is clear that \( Z \) and \( \mathcal{E} \) are semi-invariants of the algebra \( \mathcal{A} \). Since \( Z(\text{Frac}(\mathcal{A})) = \mathcal{K}(Z) \), by [10, Proposition 16.(c)], \( Sz(\mathcal{A}) \) is a polynomial algebra over \( \mathcal{K} \). Since \( \text{ind}(\mathfrak{a}) = 1 \) where \( \text{ind}(\mathfrak{a}) \) is the index of the Lie algebra \( \mathfrak{a} \) and \( \text{deg}(Z) + \text{deg}(\mathcal{E}) = 3 = \frac{1}{2}(\dim \mathfrak{a} + \text{ind}(\mathfrak{a})) \), by [10, Lemma 17] and [16, Theorem 1.1], we have \( Sz(\mathcal{A}) = \mathcal{K}[Z, \mathcal{E}] \), as required.

**The factor algebra \( \mathcal{A}/(Z - \lambda) \).** For \( \lambda \in \mathcal{K}^* \), let \( \mathcal{A}(\lambda) := \mathcal{A}/(Z - \lambda) \).
Lemma 2.10. \( \mathcal{A}(\lambda) \simeq \mathbb{K}[H_{\lambda}] | E_{\lambda}; \sigma] \otimes A_{1} \) where \( H_{\lambda} = H + \lambda^{-1}XY - \frac{1}{2} \), \( E_{\lambda} = E - \frac{1}{2} \lambda^{-1}X^{2} \), \( \sigma \) is the automorphism of the algebra \( \mathbb{K}[H_{\lambda}] \) defined by \( \sigma(H_{\lambda}) = H_{\lambda} - 2 \) and \( A_{1} = \mathbb{K}(\lambda^{-1}X,Y) \) is the first Weyl algebra.

Proof. Notice that the element \( Z + (Z - \lambda) = \lambda + (Z - \lambda) \in \mathcal{A}(\lambda) \) is an invertible element, so \( \mathcal{A}(\lambda) \simeq \mathcal{A}(Z/(Z - \lambda)Z) \). Then the lemma follows from (1). \( \square \)

Whittaker \( \mathcal{A}(\lambda) \)-modules. Let \( U^{+} = \mathbb{K}[H] | E; \sigma] \) where \( \sigma(H) = H - 2 \). Then \( U^{+} \) is the ‘positive’ part of the enveloping algebra \( U = U(sl_{2}) \).

Lemma 2.11. For \( \alpha \in \mathbb{K} \), let \( M = U^{+}/U^{+}(E - \alpha) \) be the left \( U^{+} \)-module. Then \( M \) is a simple module iff \( \alpha \neq 0 \).

For \( \mu, \delta \in \mathbb{K} \), let \( W(\mu, \delta) := \mathcal{A}(\lambda)/\mathcal{A}(\lambda)(E - \mu, X = \delta) \), a left \( \mathcal{A}(\lambda) \)-module. We call \( W(\mu, \delta) \) the Whittaker \( \mathcal{A}(\lambda) \)-module of type \( (\mu, \delta) \). The following proposition is a simplicity criterion of the module \( W(\mu, \delta) \).

Proposition 2.12. \( W(\mu, \delta) \) is a simple \( \mathcal{A}(\lambda) \)-module iff \( \delta^{2} - 2\lambda \mu \neq 0 \).

Proof. Let \( U^{+} = \mathbb{K}[H_{\lambda}] | E_{\lambda}; \sigma] \) where \( \sigma(H_{\lambda}) = H_{\lambda} - 2 \). By Lemma 2.10, \( \mathcal{A}(\lambda) = U^{+} \otimes A_{1} \). Then

\[
W(\mu, \delta) = \mathcal{A}(\lambda)/\mathcal{A}(\lambda)(E_{\lambda} + \frac{1}{2} \lambda^{-1}X^{2} - \mu, X - \delta) = \mathcal{A}(\lambda)/\mathcal{A}(\lambda)(E_{\lambda} + \frac{1}{2} \lambda^{-1}\delta^{2} - \mu, X - \delta) \\
\simeq U^{+}/U^{+}(E_{\lambda} + \frac{1}{2} \lambda^{-1}\delta^{2} - \mu) \otimes A_{1}/A_{1}(X - \delta).
\]

Let \( W' := U^{+}/U^{+}(E_{\lambda} + \frac{1}{2} \lambda^{-1}\delta^{2} - \mu) \). Notice that \( A_{1}/A_{1}(X - \delta) \) is a simple \( A_{1} \)-module with \( \text{End}(A_{1}/A_{1}(X - \delta)) = \mathbb{K} \). Then \( W(\mu, \delta) \) is a simple \( \mathcal{A}(\lambda) \)-module iff \( W' \) is simple \( U^{+} \)-module iff \( \frac{1}{2} \lambda^{-1}\delta^{2} - \mu \neq 0 \), i.e. \( \delta^{2} - 2\lambda \mu \neq 0 \), by Lemma 2.11. \( \square \)

3 Prime ideals of the universal enveloping algebra \( U(sl_{2}) \)

In this section, we describe the prime, maximal, completely prime and primitive ideals of the enveloping algebra \( U(sl_{2}) \) (Theorem 1.2, Corollary 3.1 and Corollary 3.2). Recall that the Lie algebra \( sl_{2} \) is the direct product of the Lie algebra \( sl_{2} \) and a 1-dimensional Lie algebra \( \mathbb{K}Z \). So, \( Z \) is a central element of the algebra \( \mathcal{U} := U(sl_{2}) \). Clearly, \( \mathcal{U} = U(sl_{2}) \otimes \mathbb{K}[Z] \) is a tensor product of algebras. Hence, \( Z(\mathcal{U}) = Z(U(sl_{2})) \otimes \mathbb{K}[Z] = \mathbb{K}[\Delta, Z] \) where \( \Delta = 4FE + H^{2} + 2H \) is the Casimir element of \( U(sl_{2}) \). The algebra

\[
\mathcal{U} = \mathbb{K}[Z, \Delta, H][E, F; \sigma, a = \frac{1}{4}(\Delta - H^{2} - 2H)]
\]

is a generalized Weyl algebra (GWA) where \( \sigma(Z) = Z, \sigma(\Delta) = \Delta \) and \( \sigma(H) = H - 2 \), see [1, 2] for the definition of GWAs and classification of ideals of GWAs. Clearly, \( \mathcal{U} \) is a free module over its centre \( \mathbb{K}[Z, \Delta] \).

Proof of Theorem 1.2. Let \( Z = \mathbb{K}[Z, \Delta] \).

(i) For each prime ideal \( p \) of \( Z \), the ideal \( (p) = p\mathcal{U} \) of \( \mathcal{U} \) is a completely prime ideal: The statement (i) follows from the fact that the factor algebra

\[
\mathcal{U}/p\mathcal{U} \simeq Z/p[H][E, F; \sigma, a]
\]

is a GWA which is a domain.

(ii) For all non-zero prime ideals \( P \) of \( \mathcal{U} \), \( P \cap Z \neq 0 \): The localization of the algebra \( \mathcal{U} \) at the set \( Z \setminus \{0\} \) is the simple GWA \( \mathbb{K}(Z, \Delta)[H][E, F; \sigma, a] \), since the element \( a \) is irreducible in
the Dedekind domain $\mathbb{K}(Z, \Delta)[H]$ and the group $\langle \sigma \rangle$ acts freely on $\text{Max}(\mathbb{K}(Z, \Delta)[H])$, see the description of all the ideals in [2]. Now, the statement (ii) is obvious.

Till the end of the proof of the theorem $P$ is a nonzero prime ideal of $\mathcal{U}$. Then $P' := Z \cap P \in \text{Spec}(Z)$. Let $T := \mathbb{K}[Z] \setminus \{0\}$. Then

$$T^{-1}\mathcal{U} = \mathbb{K}(Z) \otimes \mathcal{U} = \mathbb{K}(Z)[\Delta, H][E, F; \sigma, a]$$

is the universal enveloping algebra of $\mathfrak{sl}_2$ over the field $\mathbb{K}(Z)$. Then

$$\text{Spec}(\mathcal{U}) = \bigsqcup_{q \in \text{Max}(\mathbb{K}(Z))} \text{Spec}(\mathcal{U}/q\mathcal{U}) \sqcup \text{Spec}(T^{-1}\mathcal{U}).$$

Let $L_q := \mathbb{K}[Z]/q$, a finite field extension of $K$. For every $q \in \text{Max}(\mathbb{K}[Z])$, the algebra

$$\mathcal{U}/q\mathcal{U} = L_q \otimes \mathcal{U} = L_q[\Delta, H][E, F; \sigma, a]$$

is a GWA. Using the classification of the ideals of the algebra $U$ in [2],

(a) the set $\text{Spec}(\mathcal{U}/q\mathcal{U})$, as a part of $\text{Spec}(\mathcal{U})$, is identified with the set

$$\{(q), (\Delta - \lambda_n, q), (I_n, q), (m) \mid n \in \mathbb{N}_+, m \in \text{Max}(Z) \text{ and } q \subseteq m\},$$

and

(b) the set $\text{Spec}(T^{-1}\mathcal{U})$, as a part of $\text{Spec}(\mathcal{U})$, is identified with the set

$$\{0, (\Delta - \lambda_n), (I_n), (p), (m) \mid n \in \mathbb{N}_+, p \in \mathbb{H}_1 \text{ and } m \in \mathbb{H}_2\}.$$

Hence, the set of prime ideals is as in (4). The inclusions in (4) are obvious. \hfill $\square$

The next corollary classifies all the maximal and completely prime ideals of $\mathcal{U}$.

**Corollary 3.1.**

1. Max(\mathcal{U}) = $\{(I_n, q) \mid n \in \mathbb{N}_+, q \in \text{Max}(\mathbb{K}[Z])\} \cup \{(m) \mid m \in \mathbb{H}_2\}$ where $\mathbb{H}_2 = \text{Max}(\mathbb{K}[Z, \Delta]) \setminus \{(\Delta - \lambda_n, q) \mid n \in \mathbb{N}_+, q \in \text{Max}(\mathbb{K}[Z])\}$.

2. The set of completely prime ideals of $\mathcal{U}$ is equal to $\text{Spec}(\mathcal{U}) \setminus \{(I_n), (I_n, q) \mid n \geq 2; q \in \text{Max}(\mathbb{K}[Z])\}$.

**Proof.** 1. Statement 1 follows from (4).

2. For all $n \in \mathbb{N}_+$ and $q \in \text{Max}(\mathbb{K}[Z])$, $\mathcal{U}/(I_n) = \mathbb{K}[Z] \otimes U/I_n \simeq \mathbb{K}[Z] \otimes M_n(\mathbb{K})$ and

$$\mathcal{U}/(I_n, q) \simeq \mathbb{K}[Z]/q \otimes U/I_n \simeq \mathbb{K}[Z]/q \otimes M_n(\mathbb{K}).$$

For all $P \in \text{Spec}(\mathcal{U}) \setminus \{(I_n), (I_n, q) \mid n \geq 2, q \in \text{Max}(\mathbb{K}[Z])\}$, $\mathcal{U}/P \simeq Z/P'\{H\}[E, F; \sigma, a]$ is a GWA which is a domain. Now statement 2 is obvious. \hfill $\square$

The next corollary describes the set of primitive ideals of $\mathcal{U}$.

**Corollary 3.2.** The set of primitive ideals of $\mathcal{U}$ is equal to $\text{Max}(\mathcal{U}) \cup \{(\Delta - \lambda_n, q) \mid n \in \mathbb{N}_+, q \in \text{Max}(\mathbb{K}[Z])\}$.

**Proof.** The maximal ideals are prime. For all $n \in \mathbb{N}_+$ and $q \in \text{Max}(\mathbb{K}[Z])$, the factor algebra $\mathcal{U}/(\Delta - \lambda_n, q) \simeq \mathbb{K}[Z]/q[H][E, F; \sigma, a]$ is a GWA for which zero ideal is primitive. Let $P$ be a prime ideal of $\mathcal{U}$ which does not belong to the union in the corollary, then the centre of the algebra $\mathcal{U}/P$ contains a transcendental element over $\mathbb{K}$. Hence, the ideal $P$ cannot be primitive. \hfill $\square$

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