From Euler elements and 3-gradings to non-compactly causal symmetric spaces

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Dedicated to Karl Heinrich Hofmann on the occasion of his 90th birthday

Abstract. In this article we discuss the interplay between causal structures of symmetric spaces and geometric aspects of Algebraic Quantum Field Theory (AQFT). The central focus is the set of Euler elements in a Lie algebra, i.e., elements whose adjoint action defines a 3-grading. In the first half of this article we survey the classification of reductive causal symmetric spaces from the perspective of Euler elements. This point of view is motivated by recent applications in AQFT. In the second half we obtain several results that prepare the exploration of the deeper connection between the structure of causal symmetric spaces and AQFT. In particular, we explore the technique of strongly orthogonal roots and corresponding systems of $\mathfrak{sl}_2$-subalgebras. Furthermore, we exhibit real Matsuki crowns in the adjoint orbits of Euler elements and we describe the group of connected components of the stabilizer group of Euler elements.

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1. Introduction

From the group theoretic perspective, symmetric spaces are quotients $M = G/H$, where $G$ is a Lie group, $\tau$ is an involutive automorphism of $G$ and $H \subseteq G^\tau$ is an open subgroup. Symmetric spaces subsume quadrics on which pseudo-orthogonal groups act and Lie groups $G$ on which the product group $G \times G$ acts by left and right translations. For an axiomatic approach to symmetric spaces we refer to O. Loos’ monograph [Lo69].

Causal symmetric spaces $G/H$ carry a $G$-invariant field of pointed generating closed convex cones $C_m \subseteq T_m(M)$ in their tangent spaces. They subsume time-orientable Lorentzian symmetric spaces, but it is not required that the cones come from an invariant Lorentzian metric. They permit to study causality aspects of spacetimes in a highly symmetric environment. On some of these spaces the causal curves define a global order structure with compact intervals (they are called globally hyperbolic) and in this context one can also prove the existence of a global “time function” with group theoretic methods (see [Ne91]). We refer to the monograph [HÓ97] for more details and a complete exposition of the classification of irreducible symmetric spaces.

Recent interest in causal symmetric spaces in relation with representation theory arose from their role as analogs of spacetime manifolds in the context of Algebraic Quantum Field Theory (AQFT) in the sense of Haag–Kastler, where one considers nets of von Neumann algebras $\mathcal{M}(\mathcal{O})$ of operators on a fixed Hilbert space $\mathcal{H}$, associated to regions $\mathcal{O}$ in some spacetime manifold $M$ (Ha96). The hermitian elements of the algebra $\mathcal{M}(\mathcal{O})$ represent observables that can be measured in the “laboratory” $\mathcal{O}$. One typically requires the following properties:

(1) Isotony: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$

(L) Locality: $\mathcal{O}_1 \subseteq \mathcal{O}_2'$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$, where $\mathcal{O}'$ is the “causal complement” of $\mathcal{O}$, i.e., the maximal open subset that cannot be connected to $\mathcal{O}$ by causal curves.

(RS) Reeh–Schlieder property: There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for $\mathcal{M}(\mathcal{O})$ if $\mathcal{O} \neq \emptyset$.

(Cov) Covariance: There is a Lie group $G$ acting on $M$ and a unitary representation $U: G \to U(\mathcal{H})$ such that $U_g \mathcal{M}(\mathcal{O}) U_g^{-1} = \mathcal{M}(g\mathcal{O})$ for $g \in G$.

(BW) Bisognano–Wichmann property: $\Omega$ is separating for some “wedge region” $W \subseteq M$ and there exists an element $h \in g$ with $\Delta^{-it/2\pi} = U(\exp th)$ for $t \in \mathbb{R}$, where $\Delta$ is the modular operator corresponding to $(\mathcal{M}(W), \Omega)$ in the sense of the Tomita–Takesaki Theorem [BR87, Thm. 2.5.14].

(Vac) Invariance of vacuum: $U(g)\Omega = \Omega$ for every $g \in G$.

The (BW) property gives a geometrical meaning to the dynamics provided by the modular group $\Delta^{it}$ of von Neumann algebras associated to wedge regions with respect to the vacuum state. This also holds under the weaker hypothesis of the modular covariance property (MC). Under one of these assumptions, the relations among

\[ \Delta^{it}_{\mathcal{M}(W), \Omega} \mathcal{M}(\mathcal{O}) \Delta^{it}_{\mathcal{M}(W), \Omega} = \mathcal{M}(\exp(2\pi \theta) \mathcal{O}), \ O \subset M. \]
the modular groups of the wedge algebras can be used to reconstruct a positive energy representation of the Poincaré group, acting covariantly on the net of von Neumann algebra on Minkowski spacetime \((\text{GL95 Bo98})\). In particular, one can start with a finite configuration of von Neumann algebras with a cyclic and separating vector in some specific relative position to determine a large group of symmetries generated by their modular groups whose action on the family of von Neumann algebras is generating an AQFT on the spacetime manifold. For instance, in the chiral theories, such configurations are represented by half-sided modular inclusions \([\text{Wie93 AZ05}]:\) An inclusion \(\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}(\mathcal{H})\) of von Neumann algebras with a common cyclic and separating vector \(\Omega \in \mathcal{H}\), such that \(\Delta_{\mathcal{H}_\Omega}^{it} \mathcal{A} \subset \mathcal{A}\) for all \(t \geq 0\). On Minkowski spacetime these structures have been discussed by \([\text{Wie98}]\). The homogeneous spacetimes occurring naturally in AQFT are causal symmetric spaces associated to their symmetry groups (Minkowski spacetime for the Poincaré group, de Sitter space for the Lorentz group and anti-de Sitter space for \(\text{SO}_{2,d}(\mathbb{R})\)) and the localization in wedge regions is ruled by the acting group.

In our abstract context a natural question is, given a symmetry group \(G\), to which extent such nets of von Neumann algebras exist on causal symmetric spaces. For representations \((U, \mathcal{H})\) of \(G\) for which the positive cone

\[ C_U := \{ x \in \mathfrak{g} : -i \cdot \partial U(x) \geq 0 \} \tag{1} \]

spans \(\mathfrak{g}\) such nets can be constructed via Second Quantization from nets of so-called standard subspaces. We refer to \([\text{NÖ21}]\) for left invariant nets on reductive Lie groups, to \([\text{Oeh21}]\) for left invariant nets on non-reductive Lie groups, and to \([\text{NÖ22a}]\) for invariant nets on compactly causal symmetric spaces. In all these constructions the non-triviality of the cone \(C_U\) is a crucial assumption, but this restricts the class of representations considerably. These papers construct so-called one-particle nets on symmetric spaces from which nets of von Neumann algebras can be obtained by second quantization functors. An abstract description of wedge spaces is introduced in \([\text{MN21}]\). Here the spectral condition \(C_U \neq \{0\}\) is only needed to encode non-trivial inclusions among the wedge regions. For instance, \(C_U = \{0\}\) for non-trivial representations of the Lorentz groups \(\text{SO}_{1,d}(\mathbb{R})_e\) and in de Sitter space there are no proper inclusions for wedge regions. On the other hand, \([\text{BM96}]\) shows that covariant nets for the Lorentz group exist for de Sitter space. Moreover, according to \([\text{MN22}]\) the potential generators \(h \in \mathfrak{g}\) of the modular groups in \((\text{BW})\) are Euler elements, i.e., \(\text{ad } h\) defines a 3-graded

\[ \mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h), \quad \text{where} \quad \mathfrak{g}_\lambda(h) = \ker(\text{ad } h - \lambda 1). \]

This leads to the question how the existence and the choice of the Euler element affects the geometry of symmetric spaces.

The goal of this article is twofold. First, we present an approach to reductive causal symmetric spaces and their classification from the perspective of Euler elements that should be accessible to a large readership beyond the Lie group community (Sections 2-4). Second, we intend to lay the foundation for the exploration of the deeper connection between the structure of causal symmetric spaces and AQFT (Sections 5-7). In particular, a better understanding of the locality condition \((\text{L})\) and “causal complements” is under development; see in particular \([\text{MNO22a}]\) and
We recall some basic terminology concerning symmetric spaces and symmetric Lie algebras:

- A **symmetric Lie algebra** is a pair \((\mathfrak{g}, \tau)\), where \(\mathfrak{g}\) is a finite-dimensional real Lie algebra and \(\tau\) is an involutive automorphism of \(\mathfrak{g}\). We write
  \[
  \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{h} = \mathfrak{g}^\tau = \text{ker}(\tau - 1) \quad \text{and} \quad \mathfrak{q} = \mathfrak{g}^{-\tau} = \text{ker}(\tau + 1).
  \]

- A **causal symmetric Lie algebra** is a triple \((\mathfrak{g}, \tau, C)\), where \((\mathfrak{g}, \tau)\) is a symmetric Lie algebra and \(C \subseteq \mathfrak{q}\) is a pointed generating closed convex cone invariant under the group \(\text{Inn}_\mathfrak{g}(\mathfrak{h}) := \langle \text{ad}\mathfrak{h} \rangle \subseteq \text{Aut}(\mathfrak{g})\). We call \((\mathfrak{g}, \tau, C)\)
  
  - **compactly causal (cc)** if \(C\) is elliptic in the sense that, for \(x \in C^\circ\) (the interior of \(C\)), the operator \(\text{ad}\, x\) is semisimple with purely imaginary spectrum.
  
  - **non-compactly causal (ncc)** if \(C\) is hyperbolic in the sense that, for \(x \in C^\circ\), the operator \(\text{ad}\, x\) is diagonalizable.

Let \((\mathfrak{g}, \tau, C)\) be an ncc symmetric Lie algebra, \(G\) a connected Lie group with Lie algebra \(\mathfrak{g}\), \(\tau_G\) an involution on \(G\) integrating the involution \(\tau\) on \(\mathfrak{g}\) and \(H \subseteq G^{cc}\) be an open subgroup with \(\text{Ad}(H)C = C\). Then we obtain the structure of a **causal symmetric space** on \(M = G/H\), specified by the \(G\)-invariant field of open convex cones \(g.C^\circ \subseteq T_{gH}(M), g \in G\).

If \(G\) is semisimple with finite center, then, for compactly causal spaces, there are closed causal curves, so that no global causal order exists on \(M\), but non-compactly causal spaces carry a global order which is **globally hyperbolic** in the sense that all order intervals are compact ([HÖ97, Thm. 5.3.5]). In general the semi-Riemannian metric on semisimple causal symmetric spaces is not Lorentzian and the cone \(C\) may not have a smooth boundary away from 0.

The two main examples of non-flat Lorentzian symmetric spaces are de Sitter space and anti de Sitter space. **De Sitter space**

\[
dS^d := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{1,d}: x_0^2 - x_1^2 - \cdots - x_d^2 = -1\}
\]  

(2)

is a non-compactly causal irreducible symmetric space with \(G = \text{SO}_{1,d}(\mathbb{R})e\), \(H = G_{e_1} = \text{SO}_{1,d-1}(\mathbb{R})e\), and \(C \subseteq T_{e_1}(dS^d) \cong \mathfrak{e}_1^\perp\) given by

\[
C = \{(x_0, 0, x_2, \ldots, x_d): x_1 = 0, x_0 \geq 0, x_2^2 \geq x_2^2 + \cdots + x_d^2\}.
\]

Likewise **anti-de Sitter space** is the compactly causal irreducible symmetric space

\[
\text{AdS}^d := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{2,d-1}: x_0^2 + x_1^2 - x_2^2 - \cdots - x_d^2 = 1\}
\]  

(3)

with \(G = \text{SO}_{2,d-1}(\mathbb{R})e\), \(H = G_{e_1} \cong \text{SO}_{1,d-1}(\mathbb{R})e\), and \(C \subseteq T_{e_1}(\text{AdS}^d) \cong \mathfrak{e}_1^\perp\) given by

\[
C = \{(x_0, 0, x_2, \ldots, x_d): x_0 \geq 0, x_0^2 \geq x_2^2 + \cdots + x_d^2\}.
\]

In Appendix C we recall the well-known result that any irreducible Lorentzian \(d\)-dimensional causal symmetric space is either locally isomorphic to anti-de Sitter
space (if compactly causal) or to de Sitter space (if non-compactly causal). However, there are many reducible Lorentzian symmetric spaces (see Appendix C).

The contents of this paper is as follows: We start in Section 2 by introducing Euler elements and their classification, as presented in [MN21]. Since Euler elements correspond to 3-gradings of Lie algebras, their classification is also contained implicitly in the work of S. Kaneyuki (cf. [Kan98, p. 600] or [Kan00]).

In Section 3 we explore the close relation between Euler elements, H-elements ([Sa80]) and invariant cones. Recall that an element $z$ of a reductive Lie algebra $g$ is called an H-element if $\ker(\text{ad} z)$ is maximal compactly embedded in $g$ and $iz \in g_C$ is an Euler element. In Section 3.1 we recall the classification of simple hermitian Lie algebras in terms of Euler elements of their complexification. Using maximal sets of strongly orthogonal roots, we explore in Section 3.2 how the symmetry of an Euler element is related to the projection of the root system to the subspace spanned by the strongly orthogonal roots: These projections are either of type $C_r$ or $BC_r$, and the first case occurs if and only if $h$ is symmetric, i.e., to $-h \in \text{Inn}(g)h$. This implies in particular that hermitian Lie algebras of tube type correspond to symmetric Euler elements in complex simple Lie algebras. This is closely related to their characterization in terms of the existence of a “morphism of hermitian Lie algebras” $sl_2(\mathbb{R}) \to g$ whose range contains $h$ ([Sa80, Cor. III.1.6]). We recall in Section 3.3 how H-elements are related to Cartan involutions and in Section 3.4 we discuss the duality between H-elements and Euler elements. We conclude Section 3 with a review concerning invariant cones in hermitian Lie algebras and irreducible symmetric spaces (Section 3.5).

In Section 4 we present a classification of irreducible non-compactly causal symmetric spaces based on Euler elements. Here a key concept is that of a causal Euler element that is explored in Section 4.1. If $(g, \tau)$ is a symmetric Lie algebra, an Euler element $h \in q$ is said to be causal if it is contained in the interior of an $\text{Inn}_q(h)$-invariant pointed convex cone in $q$. Theorem 4.2 asserts that, for any pair $(\theta, h)$ of a Cartan involution $\theta$ and an Euler element satisfying $\theta(h) = -h$, the involution $\tau = \tau_h \theta$ with $\tau_h = e^{\pi i \text{ad} h}$ makes $h$ causal for $(g, \tau)$. In Section 4.3 we use this construction to classify irreducible non-compactly causal symmetric Lie algebras in terms of $\text{Inn}(g)$-orbits of Euler elements (Theorem 4.21). From the dual perspective, focusing on H-elements, this classification goes back to [Ol91], which ties in naturally with [HNØ94, Prop. II.3], where it is shown that the maximal generating invariant cones in $g$ (and by duality the minimal ones) are parametrized by adjoint orbits of H-elements.

In general there are many locally isomorphic causal symmetric spaces $G/H$ corresponding to the same triple $(g, \tau, C)$. The maximal one is the universal covering space of $\text{Inn}(g)/\text{Inn}(g)^\tau$, but the “minimal model” is not as obvious. In Section 4.2 we show that the minimal symmetric space $\text{Inn}(g)/\text{Inn}(g)^\tau$ for the irreducible symmetric Lie algebra $(g, \tau)$ is this minimal model for $(g, \tau, C)$ if and only if the corresponding causal Euler element is not symmetric; otherwise we have to pass to a two-fold covering. This can be understood in terms of wedge regions in non-compactly causal symmetric spaces as described in [NO09, Prop. II.3], where it is shown that the maximal generating invariant cones in $g$ (and by duality the minimal ones) are parametrized by adjoint orbits of H-elements.

In general there are many locally isomorphic causal symmetric spaces $G/H$ corresponding to the same triple $(g, \tau, C)$. The maximal one is the universal covering space of $\text{Inn}(g)/\text{Inn}(g)^\tau$, but the “minimal model” is not as obvious. In Section 4.2 we show that the minimal symmetric space $\text{Inn}(g)/\text{Inn}(g)^\tau$ for the irreducible symmetric Lie algebra $(g, \tau)$ is this minimal model for $(g, \tau, C)$ if and only if the corresponding causal Euler element is not symmetric; otherwise we have to pass to a two-fold covering. This can be understood in terms of wedge regions in non-compactly causal symmetric spaces as described in [NO09, Prop. II.3], where it is shown that the maximal generating invariant cones in $g$ (and by duality the minimal ones) are parametrized by adjoint orbits of H-elements.
then, on the minimal symmetric space $M = \text{Inn}(g)/\text{Inn}(g)^\tau$ we cannot distinguish between $C$ and $-C$. Hence the causality of the manifold $M$ is lost. In order to preserve causality one has to consider the minimal causal symmetric space $\text{Inn}(g)/H_C$, where $H_C \subseteq \text{Inn}(g)^\tau$ is the subgroup of those elements $g$ satisfying $gC = C$. In this case causally complementary wedge regions correspond to opposite Euler elements: if $h$ determines the wedge region $W_h \subset M$, then $-h$ determine $W'_h \subset M$, where the prime refers to the locality condition (L). A prominent example is de Sitter spacetime which is discussed in Remark 4.20.

In Section 4.3 we present a structured table with the classification of the irreducible non-compactly causal symmetric Lie algebras from the perspective of causal Euler elements. Together with Theorem 4.21 it provides a complete classification of the local structure of non-compactly causal symmetric spaces given by non-compactly causal simple symmetric Lie algebras.

In Section 6 we extend some of the results obtained in \cite{NO22b} for the special class of modular ncc spaces to general semisimple non-compactly causal symmetric spaces. Our main result is that, if the cone $C$ is maximal and $(\mathfrak{g}, \tau)$ is semisimple without Riemannian ideals, then the connected component of $h$ in the intersection of the adjoint orbit $O_h = \text{Inn}(g)h$ with the real tube domain $\mathcal{T}_C = h + C^\circ \subseteq g$ is the Matsuki crown of the Riemannian symmetric space $\mathcal{O}^q_h := \text{Inn}_g(h)h \cong e^{\text{ad}h^\tau}h$, i.e.,

$$(\mathcal{T}_C \cap O_h)_h = \text{Inn}_q(h) e^{\text{ad}q^\tau}h \subseteq O_h \quad \text{for} \quad \Omega_{q^\tau} = \left\{ x \in q^\tau : \rho(\text{ad} x) < \frac{\pi}{2} \right\}.$$  

Here $\rho(\text{ad} x)$ is the spectral radius of $\text{ad} x$ (Theorem 6.6) and $q^\tau = q \cap \mathfrak{e}$ for a Cartan decomposition $g = \mathfrak{e} \oplus \mathfrak{p}$ with $h \in \mathfrak{p}$. In \cite{MNO22a} we actually show that $E(\mathfrak{g}) \cap \mathcal{T}_C$ is connected.

Section 7 is devoted to an analysis of the group $\pi_0(G^h)$ of connected components of the centralizer $G^h$ of an Euler element $h$ in a simple real Lie algebra. By the polar decomposition $G^h = K^h \exp(\mathfrak{h}_p)$, this group equals $\pi_0(K^h)$. As $K/K^h \cong \mathcal{O}^K_h := \text{Ad}(K)h$ is a compact symmetric space, we discuss this problem in Section 7.1 in the context of compact symmetric spaces, where $\pi_0(K^h)$ appears as a quotient of
\[ \pi_1(K/K^h) \text{ in the long exact homotopy sequence} \]
\[ \pi_1(K) \to \pi_1(K/K^h) \to \pi_0(K^h) \to \pi_0(K) = 1. \]

In Section 7.2 we explore this situation further, using that \( O^K_h \) actually is a symmetric R-space (cf. [Lo85]). Here the strongly orthogonal roots come in handy and permit us to show that \( \text{Ad}(G)^h \) is connected if \((g, \tau)\) is either of complex type or non-split type (cf. Section 4.4), and if it is of split type or Cayley type, then it either is trivial or \( \mathbb{Z}_2 \) (see Theorem 7.8 for details). In particular \( \text{Ad}(G)^h \) has at most two connected components. In Section 7.3 we finally collect some consequences of this result such as the identity
\[ \text{Inn}_{\mathfrak{g}^c}(\mathfrak{g}) \cap \text{Inn}_{\mathfrak{g}^c}(\mathfrak{g}) = \text{Inn}_{\mathfrak{g}^c}(\mathfrak{k}) \heh \]
f for the c-dual Lie algebra \( \mathfrak{g}^c := \mathfrak{k} + i\mathfrak{q} \).

We conclude this paper with three short appendices containing some calculations in \( \mathfrak{sl}_2(\mathbb{R}) \) (Appendix A), some general facts on invariant cones and their extensions (Appendix B) and on Lorentzian symmetric spaces (Appendix C).

**Notation:**
- We write \( e \in G \) for the identity element in the Lie group \( G \) and \( G_e \) for its identity component.
- For \( x \in \mathfrak{g} \), we write \( G^x := \{ g \in G : \text{Ad}(g)x = x \} \) for the stabilizer of \( x \) in the adjoint representation and \( G^x_e = (G^x)_e \) for its identity component.
- For \( h \in \mathfrak{g} \) and \( \lambda \in \mathbb{R} \), we write \( \mathfrak{g}_\lambda(h) := \ker(\text{ad}h - \lambda \mathbf{1}) \) for the corresponding eigenspace in the adjoint representation.
- If \( \mathfrak{g} \) is a Lie algebra, we write \( \mathcal{E}(\mathfrak{g}) \) for the set of *Euler elements* \( h \in \mathfrak{g} \), i.e., \( \text{ad}h \) is non-zero and diagonalizable with \( \operatorname{Spec}(\text{ad}h) \subseteq \{-1, 0, 1\} \). We write \( O_h = \text{Inn}(\mathfrak{g})h \) for the adjoint orbit of \( h \) and call it *symmetric* if \( -h \in O_h \).
- For a Lie subalgebra \( \mathfrak{s} \subseteq \mathfrak{g} \), we write \( \text{Inn}_\mathfrak{g}(\mathfrak{s}) = \langle e^{\text{ad}z} \rangle \subseteq \text{Aut}(\mathfrak{g}) \) for the subgroup generated by \( e^{\text{ad}z} \). We call \( \mathfrak{s} \) *compactly embedded* if the group \( \text{Inn}_\mathfrak{g}(\mathfrak{s}) \) has compact closure.
- For a convex cone \( C \) in a vector space \( V \), we write \( C^\circ := \text{int}_{C-C}(C) \) for the relative interior of \( C \) in its span.
- For a symmetric space \( M = G/H \) we write \( \text{Exp} : T(M) \to M \) for the exponential function which, in the base point \( eH \) takes the form \( \text{Exp}_{eH}(x) = \exp xH \) if we identify \( \mathfrak{q} \) with \( T_{eH}(M) \).

### 2. The classification of Euler elements

In this section we introduce Euler elements and their classification, as presented in [MN21]. Since Euler elements correspond to 3-gradings of Lie algebras, their classification is also contained implicitly in the work of S. Kaneyuki (cf. [Kan98 p. 600] or [Kan00]).
Definition 2.1. We call an element \( h \) of the finite dimensional real Lie algebra \( \mathfrak{g} \) an Euler element if \( \text{ad} \, h \) is non-zero and diagonalizable with \( \text{Spec}(\text{ad} \, h) \subseteq \{-1, 0, 1\} \). Then the eigenspace decomposition of \( \mathfrak{g} \) with respect to \( \text{ad} \, h \) defines a 3-grading of \( \mathfrak{g} \):

\[
\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h), \quad \text{where} \quad \mathfrak{g}_\nu(h) = \ker(\text{ad} \, h - \nu \text{id}_\mathfrak{g})
\]

Then \( \tau_h(y_j) = (-1)^j y_j \) for \( y_j \in \mathfrak{g}_j(h) \) defines an involutive automorphism of \( \mathfrak{g} \) that can also be written as \( e^{\pi i \text{ad} \, h} \) such that

\[
\mathfrak{g}^{\tau_h} = \mathfrak{g}_0(h) \quad \text{and} \quad \mathfrak{g}^{-\tau_h} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_{-1}(h).
\]

We write \( \mathcal{E}(\mathfrak{g}) \) for the set of Euler elements in \( \mathfrak{g} \). The orbit of an Euler element \( h \) under the group \( \text{Inn}(\mathfrak{g}) = \langle e^{\text{ad} \, g} \rangle \) of inner automorphisms is denoted with \( \mathcal{O}_h = \text{Inn}(\mathfrak{g}) \subseteq \mathfrak{g} \). We say that \( h \) is symmetric if \( -h \in \mathcal{O}_h \).

Definition 2.2. Let \( \theta \) be a Cartan involution of the semisimple Lie algebra \( \mathfrak{g} \) and \( \mathfrak{a} \subseteq \mathfrak{p} \) maximal abelian, so that we obtain the restricted root system \( \Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) \). Then the Cartan–Killing form \( \kappa(x, y) = \text{tr}(\text{ad} \, x \, \text{ad} \, y) \) restricts to a scalar product on \( \mathfrak{a} \). For \( \alpha \in \Sigma \), we define the coroot \( \alpha^\vee \in \mathfrak{a} \) as the unique element which is orthogonal to \( \ker \alpha \) and satisfies

\[
\alpha(\alpha^\vee) = 2.
\]

Then the corresponding reflection is given on \( \mathfrak{a} \) by

\[
s_\alpha : \mathfrak{a} \to \mathfrak{a}, \quad s_\alpha(x) = x - \alpha(x)\alpha^\vee,
\]

and on \( \mathfrak{a}^\ast \) by

\[
s_\alpha : \mathfrak{a}^\ast \to \mathfrak{a}^\ast, \quad s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha.
\]

The Weyl group \( W := W(\mathfrak{g}, \mathfrak{a}) \) is the finite subgroup of \( \text{GL}(\mathfrak{a}) \), generated by these reflections.

If \( \mathfrak{g} \) is simple, we call a root long if its length is maximal ([Hu90, §2.9]).

Remark 2.3. An element \( h \in \mathfrak{a} \) is an Euler element if and only if it represents a so-called minuscule weight of the root system \( \Sigma^\vee := \{\alpha^\vee : \alpha \in \Sigma\} \) dual to \( \Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \). This follows from the remark at the end of §7, no. 3 in [Bo90b, Ch. 8], that also provides an interesting relation with the corresponding affine root system (see also Remark 4.15).

If \( h \neq 0 \) is hyperbolic, then there exists a maximal abelian hyperbolic subspace \( \mathfrak{a} \) containing \( h \). Then \( h \) is an Euler element if and only if

\[
\Sigma(\mathfrak{g}, \mathfrak{a})(h) \subseteq \{-1, 0, 1\}
\]

where \( \Sigma(\mathfrak{g}, \mathfrak{a}) \) is the system of restricted roots of \( \mathfrak{a} \) in \( \mathfrak{g} \). Therefore Euler elements are easy to detect in terms of the structure of \( \Sigma(\mathfrak{g}, \mathfrak{a}) \). For a given set \( \{\alpha_1, \ldots, \alpha_r\} \) of simple roots contained in the half-space \( \{\lambda \in \mathfrak{a}^\ast : \lambda(h) \geq 0\} \) and with the highest root \( \alpha = \sum_{j=1}^r n_j \alpha_j \), the different 3-gradings correspond to those indices \( j \) with \( n_j = 1 \).

The following theorem describes the conjugacy classes of Euler elements in simple real Lie algebras.
Theorem 2.4. ([MN21 Thm. 3.10]) Suppose that \( g \) is a non-compact simple real Lie algebra and that \( \mathfrak{a} \subseteq g \) is maximal ad-diagonalizable with restricted root system \( \Sigma = \Sigma(g, \mathfrak{a}) \subseteq \mathfrak{a}^* \) of type \( X_n \). We follow the conventions of the tables in [Bo90a] for the classification of irreducible root systems and the enumeration of the simple roots \( \alpha_1, \ldots, \alpha_n \). For each \( j \in \{1, \ldots, n\} \), we consider the uniquely determined element \( h_j \in \mathfrak{a} \) satisfying \( \alpha_k(h_j) = \delta_{jk} \). Then every Euler element in \( g \) is conjugate under inner automorphism to exactly one \( h_j \). For every irreducible root system, the Euler elements among the \( h_j \) are the following:

\[
\begin{align*}
A_n &: h_1, \ldots, h_n, \\
B_n &: h_1, \\
C_n &: h_n, \\
D_n &: h_1, h_{n-1}, h_n, \\
E_6 &: h_1, h_6, \\
E_7 &: h_7.
\end{align*}
\]

For the root systems \( BC_n, E_8, F_4 \) and \( G_2 \) no Euler element exists (they have no 3-grading). The symmetric Euler elements are

\[
\begin{align*}
A_{2n-1} &: h_n, \\
B_n &: h_1, \\
C_n &: h_n, \\
D_n &: h_1, \\
D_{2n} &: h_{2n-1}, h_{2n}, \\
E_7 &: h_7.
\end{align*}
\]

3. Euler elements, H-elements and invariant cones

In this section we turn to the close relation between Euler elements, H-elements ([HNØ94, Sa80]) and invariant cones. In Section 3.1 we recall the classification of simple hermitian Lie algebras in terms of Euler elements. Using arguments involving maximal sets of strongly orthogonal roots, we explore in Section 3.2 how the symmetry of an Euler element is related to the projection of the root system to the subspace spanned by the strongly orthogonal roots: These projections are either of type \( C_r \) or \( BC_r \), and the first case is equivalent to \( h \) being symmetric. This implies in particular that hermitian Lie algebras of tube type correspond to symmetric Euler elements in complex simple Lie algebras. We recall in Section 3.3 how H-elements are related to Cartan involutions and in Section 3.4 the duality between H-elements and Euler elements. We conclude this section with a review on invariant cones in hermitian Lie algebras and irreducible symmetric Lie algebras in Section 3.5.

3.1. Hermitian real forms of complex simple Lie algebras

Definition 3.1. Let \( \mathfrak{g} \) be a reductive Lie algebra and let \( z \in \mathfrak{g} \) be an Element for which \( iz \) is an Euler element of \( \mathfrak{g}_C \). Then \( \theta_z := e^{\pi \text{ad} z} \) is an involution of \( \mathfrak{g} \) whose fixed point set is \( \ker(\text{ad} z) \). We call \( z \) an H-element if \( \ker(\text{ad} z) \) is maximal compactly embedded, i.e., if \( \theta_z \) restricts to a Cartan involution on the commutator algebra \( [\mathfrak{g}, \mathfrak{g}] \) of \( \mathfrak{g} \).

We call a pair \( (\mathfrak{g}, z) \), where \( \mathfrak{g} \) is a reductive Lie algebra and \( z \in \mathfrak{g} \) an H-element a reductive Lie algebra of hermitian type ([Sa80]). A reductive Lie algebra containing an H-element is called quasihermitian.

Remark 3.2. If \( \mathfrak{g} \) is a simple non-compact real Lie algebra, then an H-element exists if and only if \( \mathfrak{g} \) is hermitian, i.e., for any Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \), the center \( \mathfrak{z}(\mathfrak{k}) \) of \( \mathfrak{k} \) is non-zero. Then \( \mathfrak{z}(\mathfrak{k}) = \mathbb{R} z \) for an H-element \( z \). In particular \( iz \) is an Euler element in \( \mathfrak{g}_C \) and, for the conjugation \( \sigma \) with respect to the real form \( \mathfrak{g} \), the involution \( \theta = \sigma \tau_{iz} \) is a Cartan involution of \( \mathfrak{g}_C \) with \( \theta(iz) = -iz \).
If, conversely, $g$ is a complex simple Lie algebra and $h \in g$ an Euler element, then there exists a Cartan involution $\theta$ of $g$ (automatically antilinear) with $\theta(h) = -h$, and then $\sigma := \theta_{\gamma_h}$ defines the real form $h := g^\sigma$ with $h_C \cong g$. Then $\sigma(h) = -h$ implies that $z := ih \in h$ is an H-element, and thus $h$ is hermitian. By this construction, the Euler elements in the root systems of simple complex Lie algebras, listed in [14], specify the simple hermitian Lie algebras as the associated real forms:

| $h$ (hermitian) | $\Sigma(h, a_h)$ | $g = h_C$ | $\Sigma(g, a)$ | Euler elt. |
|-----------------|------------------|----------|----------------|------------|
| $\mathfrak{so}_{p,q}(\mathbb{C})$, $1 \leq p \neq q$ | $BC_p$ ($p < q$) | $\mathfrak{sl}_{p+q}(\mathbb{C})$ | $A_{p+q-1}$ | $h_p$ |
| $\mathfrak{sp}_{2n}(\mathbb{R})$ | $C_p$ ($p = q$) | $\mathfrak{sp}_{2n}(\mathbb{C})$ | $C_n$ | $h_n$ |
| $\mathfrak{so}_{2,n}(\mathbb{R})$, $n > 2$ | $C_2$ | $\mathfrak{so}_{2n}(\mathbb{C})$ | $B_m$ ($n = 2m - 1$) | $h_n$ |
| $\mathfrak{so}^*(2n)$ | $BC_m$ ($n = 2m + 1$) | $\mathfrak{so}_{2n}(\mathbb{C})$ | $D_m$ ($n = 2m - 2$) | $h_n$ |
| $\mathfrak{so}_{6(-14)}$ | $BC_2$ | $\mathfrak{e}_6$ | $E_6$ | $h_{11}, h_6$ |
| $\mathfrak{so}_{7(-25)}$ | $C_3$ | $\mathfrak{e}_7$ | $E_7$ | $h_7$ |

Table 1: Simple hermitian Lie algebras $h$ and corresponding Euler elements.

Note that $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}(\mathbb{R}) \cong \mathfrak{su}_{1,1}(\mathbb{C})$. More exceptional isomorphisms are discussed in [HN12, §17]. We recall that the hermitian real form $\mathfrak{so}^*(2n)$ of $\mathfrak{so}_{2n}(\mathbb{C})$ is given by

$$\mathfrak{so}^*(2n) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathbb{C}) : a^* = -a, b^* = b \right\}.$$

For hermitian Lie algebras $h$, the restricted root system $\Sigma = \Sigma(h, a_h)$ is either of type $C_r$ or $BC_r$ (cf. [HC56, Lemmas 13-16], [Mo64, Thm. 2], [Ne00, Thm. XII.1.14] or Proposition 3.3 below). We say that $g$ is of tube type if the restricted root system is of type $C_r$ (cf. [KW65]). A comparison of Table 1 with the list [5] reveals that simple hermitian Lie algebras of tube type $h$ correspond to symmetric Euler elements in $h_C$. Since the Euler elements $h_{n-1}$ and $h_n$ for the root system of type $D_n$ are conjugate under a diagram automorphism, they correspond to isomorphic hermitian real forms. These Euler elements are symmetric if and only if $n$ is even (Theorem 2.4).

| $h$ (hermitian) | $\Sigma(h, a_h)$ | $g = h_C$ | $\Sigma(g, a)$ | symm. Euler element |
|-----------------|------------------|----------|----------------|---------------------|
| $\mathfrak{su}_{n,n}(\mathbb{C})$ | $C_n$ | $\mathfrak{sl}_{2n}(\mathbb{C})$ | $A_{2n-1}$ | $h_n$ |
| $\mathfrak{sp}_{2n}(\mathbb{R})$ | $C_n$ | $\mathfrak{sp}_{2n}(\mathbb{C})$ | $C_n$ | $h_n$ |
| $\mathfrak{so}_{2,n}(\mathbb{R})$, $n > 2$ | $C_2$ | $\mathfrak{so}_{2n}(\mathbb{C})$ | $B_m$ ($n = 2m - 1$) | $h_1$ |
| $\mathfrak{so}^*(4n)$ | $C_n$ | $\mathfrak{so}_{4n}(\mathbb{C})$ | $D_m$ ($n = 2m - 2$) | $h_{2n-1}, h_{2n}$ |
| $\mathfrak{so}_{7(-25)}$ | $C_3$ | $\mathfrak{e}_7$ | $E_7$ | $h_7$ |

Table 2: Simple hermitian Lie algebras $h$ of tube type

### 3.2. Strongly orthogonal roots and symmetric Euler elements

In this subsection we consider maximal systems of long strongly orthogonal roots in
\(\Sigma_1\), where \(\Sigma\) is a 3-graded irreducible root system. As we shall see in Section 5.1 strongly orthogonal roots provide a powerful tool to reduce problems on groups and symmetric spaces to the Lie algebra \(\mathfrak{sl}_2(\mathbb{R})\).

**Proposition 3.3.** (Harish–Chandra) Suppose that \(g\) is simple with restricted root system \(\Sigma = \Sigma(g, a)\). Let \(h \in a\) be an Euler element and consider the corresponding 3-grading of the restricted root system \(\Sigma = \Sigma(g, a)\), defined by

\[
\Sigma_j := \{\alpha \in \Sigma : \alpha(h) = j\}.
\]

Let

\[
\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Sigma_1
\]

be a maximal set of long strongly orthogonal roots (the sums and differences are no roots) and \(\Sigma^+ \subseteq \Sigma\) a positive system containing \(\Sigma_1\). The orthogonal projection \(\text{pr}_\Gamma\) from \(a\) to the span of \(\{\gamma_1, \ldots, \gamma_r\}\) is given by

\[
\text{pr}_\Gamma(\alpha) := \sum_{j=1}^r \frac{\alpha(\gamma_j)}{2} \gamma_j, \quad \alpha \in a^*.
\]

We put \(C_0 := \Sigma_0 \cap \text{pr}_\Gamma^{-1}(0)\) and consider the subsets

\[
C_j := \Sigma_0 \cap \text{pr}_\Gamma^{-1}\left(\frac{\gamma_j}{2}\right), \quad P_j := \Sigma_1 \cap \text{pr}_\Gamma^{-1}\left(\frac{\gamma_j}{2}\right), \quad \text{for} \quad j = 1, \ldots, r,
\]

and

\[
C_{jk} := \Sigma_0 \cap \text{pr}_\Gamma^{-1}\left(\frac{\gamma_j - \gamma_k}{2}\right), \quad P_{jk} := \Sigma_1 \cap \text{pr}_\Gamma^{-1}\left(\frac{\gamma_j + \gamma_k}{2}\right) \quad \text{for} \quad j < k.
\]

Then the following assertions hold:

\[
\Sigma_1 = \Gamma \cup \bigcup_j P_j \cup \bigcup_{j < k} P_{jk} \quad \text{and} \quad \Sigma^+_0 \subseteq C_0 \cup \bigcup_j C_j \cup \bigcup_{j < k} C_{jk}
\]

where all the unions are disjoint. Further, \(\text{pr}_\Gamma(\Sigma) \setminus \{0\}\) either is a root system of type \(BC_r\) or of type \(C_r\).

**Proof.** Since every 3-graded root system arises in the complexification of a hermitian simple Lie algebra \(g\) (cf. Theorem 2.4) can be derived from [HC56, Lemmas 13-16] or [Mo64, §2].

However, we have to show that our choice of \(\Gamma\) is equivalent to the construction of the set \(\Gamma'\) of strongly orthogonal roots in [HC56], which proceeds as follows: Let \(\gamma'_1\) be the maximal root (with respect to \(\Sigma^+\)), and choose \(\gamma'_j\) maximal in \(\Sigma_1\) orthogonal to \(\gamma'_1, \ldots, \gamma'_{j-1}\).

The set \(\Sigma_1\) is the weight set of the irreducible representation of the Lie subalgebra \(g_{C,0}(h)\) on \(g_{C,1}(h)\). The long roots in \(\Sigma_1\) are the extremal weights of this representation. Therefore the Weyl group \(W_0\) of the subsystem \(\Sigma_0\) acts transitively on this set. Harish–Chandra’s results ([HC56, Lemmas 13-16]) show in particular that, for \(r > 1\), the root system \((\gamma'_1)^\perp\) is irreducible. Here the main point is that, for any \(\alpha \in C_0\), there exists a \(\beta \in \Sigma_1\) with \(\alpha + \beta \in \Sigma_1\). This implies the existence of \(\beta' \in \Sigma_1 \setminus \alpha^\perp\).
Now we show that Harish–Chandra’s set $\Gamma'$ of strongly orthogonal roots is conjugate under $W_0$ to $\Gamma$. As all long roots in $\Sigma_1$ are conjugate under the Weyl group $W_0$, we may assume that $\gamma_1 = \gamma'_1 \in \Gamma$. Now one proceeds inductively to see that in the irreducible root system $\gamma_1' \cap \Sigma$, the roots $\gamma_2$ and $\gamma'_2$ are conjugate under the Weyl group $W_0$. We conclude that $\Gamma$ is conjugate under the Weyl group $W_0$ to $\Gamma'$.

The set $\text{pr}_\Gamma(\Sigma) \setminus \{0\}$ is a root system of type $BC_r$ if all sets $C_j, P_j, C_{jk}, P_{jk}$ are non-empty, and of type $C_r$ if the sets $C_j$ and $P_j$ are empty (Morinelli, Neeb, ´Olafsson §2).

**Example 3.4.** (a) For $g = \mathfrak{sl}_3(\mathbb{R})$, the restricted root system is of type $A_2$:

$$\Sigma = \{\pm (\varepsilon_1 - \varepsilon_2), \pm (\varepsilon_2 - \varepsilon_3), \pm (\varepsilon_1 - \varepsilon_3)\}.$$  

For the 3-grading defined by the Euler element $h = \frac{1}{3}(2, -1, -1)$, we have

$$\Sigma_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3\}, \quad \Sigma_0 = \{\pm (\varepsilon_2 - \varepsilon_3)\}, \quad W_0 \cong S_2,$$

and $\Gamma$ is $\{\varepsilon_1 - \varepsilon_2\}$ or $\{\varepsilon_1 - \varepsilon_3\}$. The projection $\text{pr}_\Gamma(\Sigma) \setminus \{0\}$ is a root system of type $BC_1$.

(b) For $g = \mathfrak{sp}_4(\mathbb{R})$, the restricted root system is of type $C_2$:

$$\Sigma = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm \varepsilon_1 \pm \varepsilon_2\}.$$  

For the 3-grading defined by $h := \frac{1}{3}(1, 1)$, we then have

$$\Sigma_1 = \{2\varepsilon_1, 2\varepsilon_2, \varepsilon_1 + \varepsilon_2\} \quad \text{and} \quad \Sigma_0 = \{\pm (\varepsilon_1 - \varepsilon_2)\}.$$

Here $\Gamma = \{2\varepsilon_1, 2\varepsilon_2\}$ is a maximal subset of long strongly orthogonal roots. The subset $\{\varepsilon_1 + \varepsilon_2\}$ is also maximal strongly orthogonal in $\Sigma_1$, but it consists of a short root.

**Corollary 3.5.** In the notation of Proposition 3.3 consider $h_s := \frac{1}{2} \sum_{j=1}^r \gamma_j^\vee$. Then the following are equivalent:

(a) The Euler element $h$ is symmetric.

(b) $h = h_s$.

(c) The sets $P_j$ and $C_j$ are empty.

(d) $\text{pr}_\Gamma(\Sigma) \setminus \{0\}$ is a root system of type $C_r$.

**Proof.** Let $s_\alpha(x) = x - \alpha(x)\alpha^\vee$ be the reflection corresponding to the root $\alpha$. We consider the Euler element

$$h' := s_{\gamma_1} \cdots s_{\gamma_r}(h) = h - \sum_{j=1}^r \gamma_j(h)\gamma_j^\vee = h - \sum_{j=1}^r \gamma_j^\vee.$$

Since the Weyl group acts by automorphisms of the root system (Hu90 §1.2), $h'$ also is an Euler element. Next we observe that $\gamma_j(h') = -1$ for each $j$, $\alpha(h') = 1 - 1 = 0$ for $\alpha \in P_j$ and $\alpha(h') = 1 - 2 = -1$ for $\alpha \in P_{jk}$. Moreover, $\alpha(h') = 0$ for $\alpha \in C_0$, $\alpha(h') = -1$ for $\alpha \in C_j$ and $\alpha(h') = 0$ for $\alpha \in C_{jk}$. This shows that $\alpha(h') \leq 0$ for all
The following proposition translates the information on symmetry of the Euler element to the Lie algebra context.

**Proposition 3.6.** Let \( g \) be a simple real Lie algebra and \( h \in g \) be an Euler element. Let \( a \ni h \) be a maximal abelian hyperbolic subspace, \( \Sigma := \Sigma(g,a) \) the corresponding set of restricted roots, \( \Gamma = \{ \gamma_1, \ldots, \gamma_r \} \subseteq \Sigma \) a maximal set of strongly orthogonal roots and \( g(\gamma_j) \cong \mathfrak{sl}_2(\mathbb{R}) \) corresponding \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebras. Then the following are equivalent:

(a) \( h \) is symmetric.

(b) \( h \in \sum_{j=1}^r g(\gamma_j) \).

**Proof.** (a) \( \Rightarrow \) (b): From Corollary 3.5 we know that \( h \) is symmetric if and only if \( h = h_s \), and \( h_s \) is contained in \( \sum_{j=1}^r g(\gamma_j) \) because \( \gamma_j \in g(\gamma_j) \).

(b) \( \Rightarrow \) (a) follows from the fact that all Euler elements in \( \mathfrak{sl}_2(\mathbb{R}) \) are symmetric, and this is inherited by \( \mathfrak{sl}_2(\mathbb{R})' \).  

---

### 3.3. Cartan involutions and H-elements

In this subsection we briefly discuss the relation between Cartan involutions and H-elements in quasihermitian semisimple Lie algebras.

**Lemma 3.7.** If \( g \) is semisimple without compact ideals, then the following assertions hold:

(a) If \( \sigma \) is an involutive automorphism for which \( g^\sigma \) is compactly embedded. Then \( \sigma \) is a Cartan involution.\(^2\)

(b) If \( C_g^o \subseteq g \) is a pointed generating invariant cone and \( z \in C_g^o \) such that \( h := iz \) is an Euler element in \( g_C \), then \( z \) is an H-element.

**Proof.** (a) Let \( \mathfrak{k} \supseteq g^\sigma \) be a maximal compactly embedded subalgebra and \( \theta \) the corresponding Cartan involution with \( \mathfrak{k} = g^\theta \). Then

\[ p := g^{-\theta} = \mathfrak{k}^\perp \subseteq (g^\sigma)^\perp = g^{-\sigma}, \]

\(^2\)Every non-trivial involution on a compact simple Lie algebra shows that this conclusion is false if \( g \) contains compact ideals.
where \( \perp \) refers to the Cartan–Killing form. As \( \mathfrak{g} \) contains no compact ideals, we have
\[
\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]
\]
because the right hand side is an ideal containing all non-compact simple ideals. Since \( \theta \) and \( \sigma \) coincide on \( \mathfrak{p} \), they coincide, i.e., \( \sigma \) is a Cartan involution.

(b) First we observe that \( \text{Spec}(\text{ad} z) = \{0, \pm i\} \) and that \( \ker(\text{ad} z) \) is compactly embedded because \( z \in C^\circ_\theta \) \([\text{Ne00, Prop. V.5.11}]\). Hence \( \sigma := e^{\pi \text{ad} z} \in \text{Aut}(\mathfrak{g}) \) is an involution for which \( \mathfrak{g}^\sigma = \ker(\text{ad} z) \) is compactly embedded. Now (a) implies that \( \theta_z \) is a Cartan involution, i.e., \( z \) is an H-element.

Leon 3.8. Let \( \mathfrak{g} \) be a simple Lie algebra and \( \theta_1 \) and \( \theta_2 \) two Cartan involutions. Then \( \theta_1 \) and \( \theta_2 \) commute if and only if \( \theta_1 = \theta_2 \).

Proof. If \( \theta_1 = \theta_2 \) then they clearly commute. For the other direction write \( \mathfrak{g}_j = \mathfrak{k}_j \oplus \mathfrak{p}_j \), \( j = 1, 2 \), for the Cartan decomposition corresponding the \( \theta_j \). Then \( \mathfrak{k}_2 \) is \( \theta_1 \)-stable and hence
\[
\mathfrak{k}_2 = (\mathfrak{k}_2 \cap \mathfrak{k}_1) \oplus (\mathfrak{k}_2 \cap \mathfrak{p}_1).
\]
As \( \mathfrak{k}_2 \) is compactly embedded, we have \( \mathfrak{k}_2 \cap \mathfrak{p}_1 = \{0\} \), hence \( \mathfrak{k}_2 \subseteq \mathfrak{k}_1 \). We likewise obtain \( \mathfrak{k}_1 \subseteq \mathfrak{k}_2 \), hence equality. Then \( \mathfrak{p}_1 = \mathfrak{p}_2 \) is the orthogonal space with respect to the Cartan–Killing form and thus \( \theta_1 = \theta_2 \).

Lemma 3.9. If \( \mathfrak{g} \) is a simple hermitian Lie algebra with Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) and \( z \in j(\mathfrak{k}) \) is an H-element, then \( \pm z \) are the only H-elements in \( \mathfrak{k} \).

Proof. Clearly, \( \pm z \) are both H-elements in \( \mathfrak{k} \), defining the same Cartan involution \( \theta = \theta_z = e^{\pi \text{ad} z} \). If \( z' \in \mathfrak{k} \) is another H-element, then \( [z, z'] = 0 \), so that \( \theta_z \) and \( \theta_{z'} \) are two commuting Cartan involutions and hence \( \theta_z = \theta_{z'} \) by Lemma 3.8 so that \( z, z' \in j(\mathfrak{k}) \). But \( \dim j(\mathfrak{k}) = 1 \) and hence \( z' = \pm z \).

3.4. The duality between Euler elements and H-elements

The following lemma relates Euler elements in \( \mathfrak{p} = \mathfrak{g}^{-\theta} \) to symmetric Lie algebras \((\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}, \tau^c, z)\) of hermitian type.

Lemma 3.10. Let
\[
\mathcal{A} := \{(\mathfrak{g}, \theta, h) : \mathfrak{g} \text{ semisimple, } h \in \mathcal{E}(\mathfrak{g}), \theta \text{ Cartan involution with } \theta(h) = -h\}
\]
and
\[
\mathcal{B} := \{(\mathfrak{g}, \tau, z) : \mathfrak{g} \text{ semisimple, } z \text{ H-element, } \tau \text{ involution with } \tau(z) = -z\}.
\]
Then we have a bijection
\[
\Phi : \mathcal{A} \to \mathcal{B}, \quad \Phi(\mathfrak{g}, \theta, h) := (\mathfrak{g}^c, \tau^c, ih),
\]
where
\[
\tau := \tau_h \theta, \quad \mathfrak{g}^c = \mathfrak{g}^\tau + i\mathfrak{g}^{-\tau}, \quad \tau^c(x + iy) = x - iy.
\]
Proof. Let \((g, \theta, h) \in \mathcal{A}\). As \(\theta\) commutes with \(\tau_h\), the product \(\tau := \tau_h \theta\) is an involution on \(g\). It satisfies

\[
h := g^\tau = \mathfrak{k}^{\tau} \oplus \mathfrak{p}^{^{-\tau}} \quad \text{and} \quad q := g^{-\tau} = \mathfrak{k}^{-\tau} \oplus \mathfrak{p}^{\tau},
\]

so that

\[
g^{\tau} = \ker(\text{ad} h) = h_\mathfrak{t} \oplus q_\mathfrak{p} \quad \text{and} \quad h \in q_\mathfrak{p}. \tag{7}
\]

We associate to \((g, \tau)\) the dual symmetric Lie algebra \((g^c, \tau^c)\). It contains \(z := ih\) with centralizer \(\ker(\text{ad} z) = h_\mathfrak{t} + iq_\mathfrak{p} = \mathfrak{k}^c\) maximal compactly embedded in \(g^c\) (where \(\mathfrak{h} = g^\tau, q = g^{-\tau}\)). Therefore \(z \in iq_\mathfrak{p}\) is an H-element and the involution \(\tau^c\) on \(g^c\) satisfies \(\tau^c(z) = -z\). This shows that \((g^c, \tau^c, ih) \in B\).

If, conversely, \((g, \tau, z) \in B\), then the dual symmetric Lie algebra \((g^c, \tau^c)\) contains the Euler element \(h := -iz\) and \(\theta := \tau^c \tau_h\) is a Cartan involution because its fixed point set

\[
\mathfrak{k} := h^{\tau} + iq^{\tau} = h_\mathfrak{t} + iq_\mathfrak{p}
\]

is maximal compactly embedded in \(g\). This shows that \((g^c, \theta, h) \in \mathcal{A}\). As \(\tau^c = \theta \tau_h\), we have \(\Phi(g^c, \theta, h) = (g, \tau, z)\). Therefore \(\Phi\) is bijective. ■

Remark 3.11. Since the Lie algebras \(g\) and \(g^c\) are real forms of the same complex Lie algebra \(g_\mathbb{C}\), one can also represent the triples \((g, \theta, h)\) by triples \((\sigma, \theta, h)\), where \(\sigma\) is an antilinear involution of \(g_\mathbb{C}\), \(\theta\) is a linear involution of \(g_\mathbb{C}\) commuting with \(\sigma\) such that \(\sigma\theta\) is a Cartan involution of \(g_\mathbb{C}\), and \(h \in \mathcal{E}(g_\mathbb{C})\) satisfies \(\sigma(h) = h\) and \(\theta(h) = -h\).

Likewise elements of \(B\) can be represented by triples \((\sigma, \tau, z)\), where \(\sigma\) is an antilinear involution of \(g_\mathbb{C}\), \(\tau\) is a linear involution of \(g_\mathbb{C}\) commuting with \(\sigma\) and \(iz \in \mathcal{E}(g_\mathbb{C})\) satisfies \(\sigma(z) = z\) and \(\tau(z) = -z\). In these terms \(\Phi\) maps \((\sigma, \theta, h)\) to \((\sigma \tau \theta, \tau, ih)\).

3.5. Basic facts on invariant cones

3.5.1. Invariant cones in hermitian Lie algebras

Let \(g\) be a simple hermitian Lie algebra and \(g = \mathfrak{k} \oplus \mathfrak{p}\) be a Cartan decomposition. Then \(\mathfrak{z}(\mathfrak{k}) = \mathbb{R}z\) is one-dimensional and generated by an H-element \(z\) satisfying \(\theta = e^{\pi \text{ad} z}\) (Remark 3.2). Now every closed convex \(\text{Inn}(g)\)-invariant cone \(C_g \subseteq C\) satisfies

\[
C_{\mathfrak{z}(\mathfrak{k})} := p_{\mathfrak{z}(\mathfrak{k})}(C_g) = C_g \cap \mathfrak{z}(\mathfrak{k}) \quad \text{and} \quad C_{\mathfrak{z}(\mathfrak{k})}^o = C_g^o \cap \mathfrak{z}(\mathfrak{k}) \tag{8}
\]

where the projection \(p_{\mathfrak{z}(\mathfrak{k})} : g \to \mathfrak{z}(\mathfrak{k})\) is the composition of the fixed point projection \(g \to \mathfrak{k}\) for the compact group \(\text{Inn}_g(\mathfrak{z}(\mathfrak{k})) \cong \mathbb{T}\) and the fixed point projection for the compact group \(\text{Inn}_g(\mathfrak{z})\) (Proposition 3.4). Therefore every non-trivial invariant cone \(C_g\) either contains \(z\) or \(-z\).

Conversely, it is easy to see with the Iwasawa decomposition that

\[
C_g^{\min}(z) := \mathbb{R}_+ \text{conv}(\text{Inn}_g(z))
\]

is a pointed invariant cone ([HN93, Thm. 7.25]). For any invariant cone \(C_g \subseteq g\) containing \(z\) we then have

\[
C_g^{\min}(z) \subseteq C_g. \tag{9}
\]
The Cartan–Killing form \( \kappa(x, y) = \text{tr}(\text{ad} \, x \, \text{ad} \, y) \) is negative definite on \( \mathfrak{f} \) and positive definite on \( \mathfrak{p} \), so that

\[
C_{\mathfrak{g}}^{\text{max}}(z) := \{ x \in \mathfrak{g} : (\forall y \in C_{\mathfrak{g}}^{\text{min}}(z)) \, \kappa(x, y) \leq 0 \}
\]

is a pointed generating invariant cone containing \( z \), hence also \( C_{\mathfrak{g}}^{\text{min}}(z) \). Dualizing (9) implies that every invariant cone \( C_{\mathfrak{g}} \) containing \( z \) is contained in \( C_{\mathfrak{g}}^{\text{max}}(z) \):

\[
C_{\mathfrak{g}}^{\text{min}}(z) \subseteq C_{\mathfrak{g}} \subseteq C_{\mathfrak{g}}^{\text{max}}(z). \tag{10}
\]

If \( \mathfrak{g} \) is a semisimple Lie algebra and \( z \in \mathfrak{g} \) an H-element, then \( \mathfrak{g} = \mathfrak{g}_k \oplus \mathfrak{g}_n \), where \( \mathfrak{g}_k \) is the sum of all simple ideals commuting with \( z \) (these are compact) and \( \mathfrak{g}_n \) is a sum of hermitian simple ideals. Applying the preceding discussion to all simple ideals in \( \mathfrak{g}_n \), we obtain pointed generating invariant cones \( C_{\mathfrak{g}_n}^{\text{min}}(z) \subseteq C_{\mathfrak{g}_n}^{\text{max}}(z) \).

We then put

\[
C_{\mathfrak{g}}^{\text{min}}(z) := C_{\mathfrak{g}_n}^{\text{min}}(z) \quad \text{and} \quad C_{\mathfrak{g}}^{\text{max}}(z) := \mathfrak{g}_k \oplus C_{\mathfrak{g}_n}^{\text{max}}(z). \tag{11}
\]

In this context it is still true that, for every pointed generating invariant cone \( C_{\mathfrak{g}} \subseteq \mathfrak{g} \), there exists an H-element \( z \) with

\[
C_{\mathfrak{g}}^{\text{min}}(z) \subseteq C_{\mathfrak{g}} \subseteq C_{\mathfrak{g}}^{\text{max}}(z). \tag{12}
\]

(cf. [HNO94, Prop. II.3]).

### 3.5.2. Invariant cones in irreducible symmetric Lie algebras

Let \( \mathfrak{g} \) be a simple Lie algebra with Cartan involution \( \theta \) and \( h \in \mathfrak{g}^{-\theta} \) an Euler element. For the involution \( \tau = \tau_h \theta \) we then have for \( \mathfrak{h} = \mathfrak{g}^\tau \) and \( \mathfrak{q} = \mathfrak{g}^{-\tau} \)

\[
\mathfrak{h} = \mathfrak{h}_t \oplus \mathfrak{h}_p, \quad \mathfrak{q} = \mathfrak{q}_t \oplus \mathfrak{q}_p \quad \text{with} \quad \mathfrak{g}^\theta = \ker(\text{ad} \, h) = \mathfrak{h}_t \oplus \mathfrak{q}_p.
\]

It follows that \( z := ih \in \mathfrak{g}^c \) is an H-element and that its centralizer \( \mathfrak{t}^c = \mathfrak{h}_t + i\mathfrak{q}_p \) is a maximal compact subalgebra of \( \mathfrak{g}^c \) (Lemma 3.11). The closed convex \( \text{Inn}_{\mathfrak{g}}(\mathfrak{h}) \)-invariant cone \( C_{\mathfrak{q}}^{\text{min}}(h) \subseteq \mathfrak{q} \) generated by \( h \) is pointed because it is contained in the pointed cone \( -iC_{\mathfrak{g}}^{\text{min}}(ih) \). By (8) \( h \) is contained in its interior (cf. also [HO97, Prop. 3.1.3]).

**Lemma 3.12.** (cf. [HO97, Prop. 3.1.11]) If \( (\mathfrak{g}, \tau, C) \) is irreducible ncc, then the following assertions hold:

(a) Every element in \( C_{\mathfrak{q}}^{\circ} \) fixed by \( \text{Inn}_{\mathfrak{g}}(\mathfrak{h}_t) \) is contained in \( \mathfrak{q}_p \).

(b) \( \mathfrak{z}_{\mathfrak{q}_p}(\mathfrak{h}_t) = \mathbb{R} h \) for an Euler element \( h \).

(c) \( C_{\mathfrak{q}}^{\circ} \cap \mathfrak{q}_p \) contains an Euler element \( h \).
Proof. The group $H := \text{Inn}_q(h)$ is the identity component of the group $\text{Aut}(g)^+$, hence closed. It has the polar decomposition $H = HKe^{\text{ad}h_p}$, where $H_K = \text{Inn}_q(h_t)$ ([HN12 Prop. 13.1.5]). As the simplicity of $g$ implies that $h = [q, q]$, the representation $\text{Ad}_q$ of $H$ on $q$ is faithful and its image in $\text{GL}(q)$ is closed with maximal compact subgroup $\text{Ad}(H_K)$.

(a) Let $x \in C^c \cap q_p$ be fixed by the compact group $H_K$. As the stabilizer of $x$ in $\text{Ad}_q(H)$ is compact ([Ne00 Prop. V.5.11]), $H^x = H_K$. Since $x$ is hyperbolic, there exists an element $g \in H$ with $y := \text{Ad}(g)x \in q_p$ ([KN96 Cor. II.9]). Then $\theta(y) = -y$ implies that the stabilizer $H^y$ is $\theta$-invariant. So $g_1 = h_1e^{adz} \in H^y$ with $h_1 \in H_K$ and $z \in h_p$ implies $\theta(q_1)g_1 = e^{2adz} \in H^q$. The compactness of $H^y = gH^yg^{-1} = gH_Kg^{-1}$ now entails $z = 0$. Hence $H^y \subseteq H_K$ and thus $H^y = H_K$ as $H^y$ is maximally compact. We conclude that $gH_Kg^{-1} = H_K$ which further implies $g \in H_K$, the stabilizer group of the base point in the Riemannian symmetric space $H/H_K$. This shows that $y = x \in q_p$.

(b) Let $x \in q_p$ be fixed by $H_K$, i.e., $x$ commutes with $h_t$. Then $[x, q_p] \subseteq h_t$, and

$$\kappa([x, q_p], h_t) = \kappa(q_p, [x, h_t]) = \{0\}$$

implies $[x, q_p] = \{0\}$, so that $x$ is central in the maximal compactly embedded subalgebra $\mathfrak{t}^c := \mathfrak{h}_t + iq_p$ of $\mathfrak{g}^c = \mathfrak{h} + iq$. As $g$ is simple, either $\mathfrak{g}^c$ is simple as well (this happens if $(g, \tau)$ is not of complex type), or if $(g, \tau)$ is of complex type and then $(\mathfrak{g}^c, \tau) \cong (\mathfrak{h} \oplus \mathfrak{h}, \tau_{\mathfrak{h}\mathfrak{h}})$. In the first case $\mathfrak{g}^c$ is simple hermitian, so that $\mathfrak{h}(\mathfrak{t}^c) = \mathbb{R}izc$ contains an $H$-element and thus $x$ is a multiple of an Euler element. In the second case $\mathfrak{h}$ is also simple hermitian and $\mathfrak{t}^c = \mathfrak{h}_t \oplus \mathfrak{h}_s$ has 2-dimensional center with $\mathfrak{h}(\mathfrak{t}^c) \cap iq_p = \mathbb{R}(z, -z)$ for an $H$-element $z \in h_t$. Again, some positive multiple of $x$ is an Euler element and $\mathfrak{h}(\mathfrak{t}^c) = \mathbb{R}x$.

(c) As the fixed point projection for the compact group $H_K$ leaves the interior of $C_q$ invariant (Proposition [13.4], $C^q_0$ contains an $H_K$-fixed point $x$. By (a) it is contained in $q_p$ and by (b) it is a multiple of some Euler element $h$. Hence $C^q_0$ contains $h$ or $-h$.

It follows from Lemma [3.12(c)] that, for every $\text{Inn}_q(h)$-invariant pointed generating closed convex cone $C_q \subseteq q$, the interior $C^q_0$ contains $h$ or $-h$. Further, $h \in C^q_0$ implies

$$C^q_{\text{min}}(h) \subseteq C_q.$$

Similar arguments as above, using the Cartan–Killing form on $q$, now lead to a maximal $\text{Inn}_q(h)$-invariant cone $C^q_{\text{max}}(h)$ with

$$C^q_{\text{min}}(h) \subseteq C_q \subseteq C^q_{\text{max}}(h) = C^q_{\text{min}}(h)^* = \{x \in q; (\forall y \in C^q_{\text{min}}(h)) \kappa(x, y) \geq 0\}.$$

If $g$ is only semisimple, we decompose it as $g = g_r \oplus g_s$, where $g_s$ is the sum of all simple ideals not commuting with $h$. We then obtain a pointed $\text{Inn}_q(h)$-invariant cone $C^q_{\text{min}}(h) \subseteq q_s := q \cap g_s$ whose dual cone $C^q_{\text{max}}(h)$ with respect to the Cartan–Killing form satisfies

$$C^q_{\text{min}}(h) \subseteq C^q_{\text{max}}(h).$$

Both cones are adapted to the decomposition of $(g, \tau)$ into irreducible summands. Further, each pointed generating $\text{Inn}_q(h)$-invariant cone $C_q$ containing $h$ satisfies

$$C^q_{\text{min}}(h) \subseteq C_q \subseteq C^q_{\text{max}}(h).$$
Here the first inclusion is obvious, and the second one follows from the fact that $h$ is also contained in the dual cone

$$C_q^* = \{ y \in q : (\forall x \in C_q) \kappa(x, y) \geq 0 \}.$$  

This leads to $C_q^{\min}(h) \subseteq C_q^*$, and thus to $C_q \subseteq C_q^{\min}(h)^* = C_q^{\max}(h)$.

4. Euler elements and non-compactly causal symmetric spaces

In this section we present a classification of irreducible non-compactly causal symmetric spaces based on Euler elements. Here a key concept is that of a causal Euler element that is explored in Section 4.1. If $(g, \tau)$ is a symmetric Lie algebra and $h \in q$ an Euler element, then we call it causal if it is contained in the interior of an $\text{Inn}_q(h)$-invariant pointed convex cone in $q$. Theorem 4.2 asserts that, for any pair $(\theta, h)$ of a Cartan involution $\theta$ and an Euler element $h$ satisfying $\theta(h) = -h$, the involution $\tau = \tau_h \theta$ makes $h$ causal for $(g, \tau)$. In Section 4.3 we use this construction to classify irreducible non-compactly causal symmetric Lie algebras in terms of $\text{Inn}(g)$-orbits of Euler elements (Theorem 4.21), which essentially goes back to [OL91].

In general there are many locally isomorphic causal symmetric spaces $G/H$ corresponding to a causal symmetric Lie algebra $(g, \tau, C)$. It is therefore natural to ask for a “minimal model”. In Section 4.4 we present a structured table with the classification of the irreducible non-compactly causal symmetric Lie algebras, based on causal Euler elements.

Throughout this section the Lie algebra $g$ is assumed to be semisimple if not stated otherwise.

4.1. Causal Euler elements

**Definition 4.1.** Let $(g, \tau)$ be a symmetric Lie algebra and $h \in E(g) \cap q$ an Euler element. We say that $h$ is causal if there exists a pointed generating $\text{Inn}_q(h)$-invariant cone $C \subseteq q$ with $h \in C^\circ$. We write $E_c(q) \subseteq E(g) \cap q$ for the set of causal Euler elements in $q$.

We start with a reductive Lie algebra $g$. Let $h \in E(g)$ be an Euler element. As $h$ is hyperbolic, there exists a Cartan involution $\theta$ with $\theta(h) = -h$, where we assume that $z(g) \subseteq g^{-\theta}$. To make $h$ causal, one has to find an appropriate involution $\tau$ on $g$. It turns out that $\tau := \tau_h \theta$ is the natural choice making $h$ a causal Euler element (cf. Lemma 3.10).

**Theorem 4.2.** (Euler elements vs. causal symmetric spaces) Let $h \in E(g)$ be an Euler element in the reductive Lie algebra $g$ and $\theta$ a Cartan involution with $\theta(h) = -h$ and $z(g) \subseteq g^{-\theta}$. Then the following assertions hold:

(a) $\tau := \tau_h \theta$ defines an involution on $g$.

(b) The symmetric Lie algebra $(g, \tau)$ is non-compactly causal and there exists a pointed generating $\text{Inn}_q(h)$-invariant cone $C \subseteq q$ with $h \in C^\circ$. 
All ideals of \( g \) contained in \( g^\tau \) are compact.

This theorem implies in particular that every Euler element is causal with respect to a suitably chosen involution \( \tau \) on \( g \). In [HÖ07] causal Euler elements in \( q_p \) are called “cone generating elements” because they generate pointed generating invariant cones (see also [O91] and Section 3.5.2).

**Proof.** Let \( g = z(g) \oplus g_1 \oplus \cdots \oplus g_n \) denote the composition of \( g \) into the center and simple ideals. Recall that \( \theta \) preserves all these ideals because all Cartan involutions are conjugate under \( \text{Inn}(g) \) ([HN12 Cor. 13.2.13]) and all simple ideals possess Cartan involutions ([HN12 Thm. 13.2.10]). Now \( h = h_z + h_1 + \cdots + h_n \), where either \( h_j = 0 \) or \( h_j \in g_j \) is an Euler element. As \( \theta \) and \( \tau_h \) preserve \( g_j \), the involution \( \tau := \tau_h \theta \) defines on \( g_j \) an involution \( \tau_j \) which is Cartan if \( h_j = 0 \), and if this is not the case, then \( h_j \in q_j := g_j^{-\tau_j} \). Note that \( \text{ad} h \neq 0 \) implies that some \( h_j \) is non-zero. As all ideals of \( g \) contained in \( g^\tau \) commute with \( h \), they are contained in \( g^\theta \), hence are compact Lie algebras.

From the discussion in Section 3.5.2 we know that \( h_j \) is contained in the interior of a pointed generating \( \text{Inn}_q(h_j) \)-invariant cone \( C_j := C^\min_{q_j}(h_j) \subseteq q_j \). As \( h_j \) is hyperbolic and contained in \( C^\circ_j \), the causal symmetric Lie algebra \((g_j, \tau_j, C_j)\) is ncc. The Lie algebra

\[
\mathfrak{h}_r := \sum_{h_j \neq 0} h_j
\]

is compactly embedded and

\[
C_s := \sum_{h_j \neq 0} C_j = C^\min_{q_s}(h) \subseteq q_s := \sum_{h_j \neq 0} q_j
\]

is pointed and generating. Further

\[
\mathfrak{h}_s := \sum_{h_j \neq 0} h_j
\]

satisfies \( \text{Inn}_q(\mathfrak{h}_s)|_q \subseteq \text{SL}(q) \). Hence Lemma [B.2] implies the existence of a pointed generating \( \text{Inn}_q(h) \)-invariant cone \( C \subseteq q = z(g) \oplus q_r \oplus q_s \) with \( C \cap q_s = C_s \) and \( h \in C^\circ \subseteq C^\circ \).

In the preceding proof, we have seen that it is convenient to decompose the Lie algebra \( g \) as

\[
g = g_k \oplus g_r \oplus g_s, \tag{15}
\]

where \( g_s \) is the sum of all simple ideals not commuting with \( h \) (the strictly ncc part), \( g_r \) is the sum of all non-compact simple ideals commuting with \( h \) on which \( \tau = \theta \) (the non-compact Riemannian part), and \( g_k \) is the sum of all compact ideals (they commute with \( h \)). All these ideals are invariant under \( \theta \) and \( \tau = \tau_h \theta \), so that we obtain decompositions

\[
g_s = \mathfrak{h}_s \oplus q_s, \quad g_r = \mathfrak{h}_r \oplus q_r \quad \text{and} \quad g_k = \mathfrak{h}_k, \tag{16}
\]

where \( \mathfrak{h}_r \oplus \mathfrak{h}_k \) is a compact ideal of \( \mathfrak{h} \) and \( \mathfrak{h}_r \oplus q_r \) is a Cartan decomposition of \( g_r \).
Remark 4.3. Let \( q = q_r \oplus q_s \) be the decomposition of \( q \) into the Riemannian part \( q_r \) and its orthogonal complement which is ncc. We write \( p_s : q \to q_s \) for the projection with kernel \( q_r \), which is the fixed point projection for the compact group \( \text{Inn}_q(h_k \oplus h_r) \). Then every \( \text{Inn}_q(h) \)-invariant closed convex cone \( C \subseteq q \) satisfies
\[
p_s(C) = C \cap q_s =: C_s \quad \text{and} \quad C_s^0 = C^0 \cap q_s
\]
(Proposition B.4).

Corollary 4.4. Let \( (\mathfrak{g}, \tau) \) be a reductive symmetric Lie algebra and write it as \( \mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_r \oplus \mathfrak{g}_s \), where \( \mathfrak{g}_c \) is the sum of all compact ideals of \( \mathfrak{g} \) contained in \( \mathfrak{h} = \mathfrak{g}^\tau \), \( \mathfrak{g}_r \) is the sum of all Riemannian summands on which \( \tau \) is a Cartan involution, and \( \mathfrak{g}_s \) is the sum of all non-Riemannian irreducible summands. Then \( \mathfrak{g} \) is non-compactly causal if and only if \( \mathfrak{g}_s \neq \{0\} \) and \( \mathfrak{g}_s \) is a sum of irreducible non-compactly causal symmetric Lie algebras.

Proof. If \( \mathfrak{g}_s \) is non-zero and non-compactly causal, then the proof of Theorem 4.2 implies that \( (\mathfrak{g}, \tau) \) is non-compactly causal. If, conversely, this is the case, then Remark 4.3 shows that \( \mathfrak{g}_s \neq \{0\} \) and that \( (\mathfrak{g}_s, \tau) \) is non-compactly causal.

Theorem 4.5. (Uniqueness of the causal Euler elements) Let \( (\mathfrak{g}, \tau, C) \) be a semisimple ncc symmetric Lie algebra for which all ideals of \( \mathfrak{g} \) contained in \( \mathfrak{h} \) are compact, and \( \theta \) a Cartan involution commuting with \( \tau \). Then the following assertions hold:

(a) \( C_s^0 \cap q_p \) contains a unique Euler element \( h \) and \( \tau = \tau_h \).
(b) \( \text{Inn}_q(h) \) acts transitively on \( C_s^0 \cap \mathcal{E}(\mathfrak{g}) \).
(c) For every Euler element \( h \in C_s^0 \), the involution \( \tau\tau_h \) is Cartan.

Proof. (a) We write \( (\mathfrak{g}, \tau) \) as a direct sum of irreducible symmetric Lie algebras \( (\mathfrak{g}_j, \tau_j) \), \( j = 1, \ldots, n \) and a compact algebra \( \mathfrak{g}_k \) of \( \mathfrak{g} \) contained in \( \mathfrak{h} \) (cf. [NÖ22b, Prop. 2.14]). Let \( x \in C \) be such that all components \( x_j \in \mathfrak{q}_j = \mathfrak{q} \cap \mathfrak{g}_j \) are non-zero. Then
\[
\mathcal{O}_x := \text{Inn}_q(h)x = \mathcal{O}_{x_1} + \cdots + \mathcal{O}_{x_n} \subseteq C
\]
implies that all the orbits \( \mathcal{O}_{x_j} \) are contained in \( \text{Inn}_q(h_j) \)-invariant closed convex subsets not containing affine lines. If \( \mathcal{O}_{x_j} \) is bounded, then \( (\mathfrak{g}_j, \tau_j) \) is Riemannian, and if it is unbounded, \( (\mathfrak{g}_j, \tau_j) \) is causal. Moreover, the fact that the projection of \( C^0 \) to \( \mathfrak{q}_j \) consists of hyperbolic elements implies that \( (\mathfrak{g}_j, \tau_j) \) is non-compactly causal. We enumerate the simple ideals in such a way that \( (\mathfrak{g}_j, \tau_j) \) is Riemannian for \( j \leq m \) and causal for \( j > m \).

The action of the compact group \( \text{Inn}_q(h_j) \) on \( C^0 \) has a fixed point \( x = \sum_{j=m+1}^n x_j \in C^0 \) (Proposition B.3) with \( x_j \) contained in \( \mathfrak{q}_{p,j} \) (Lemma 3.12(a)). By Lemma 3.12 there exist \( \lambda_j > 0 \) for \( j = m + 1, \ldots, n \), such that \( h_j := \lambda x_j \) is an Euler element in \( \mathfrak{g}_j \). Then
\[
h := \sum_{j=m+1}^n h_j \in \sum_{j=m+1}^n C^0_{\min}(h_j) \subseteq C_s^0
\]
(see [14] and [17] for the last inclusion) is a causal Euler element in \( g \). Further

\[
\mathfrak{z}_g(h) = \sum_{j=1}^{m} \mathfrak{g}_j + \sum_{j=m+1}^{n} \mathfrak{z}_g(x_j) = \mathfrak{h}_t \oplus q_p
\]

(18)

implies \( \tau = \tau_0 \theta \).

To verify the uniqueness of \( h \in q_p \cap C_s^0 \), let \( h_1 \in q_p \cap C_s^0 \) be an Euler element. We pick \( a \subseteq q_p \) maximal abelian containing \( h_1 \) and observe that \( h \in a \) follows from \( [h, q_p] = \{0\} \). We choose a positive system \( \Sigma^+ \subseteq \Sigma(g, a) \) such that

\[
C \cap a \subseteq C_a^\text{max} := C_q^\text{max} \cap a = \{x \in a: (\forall \alpha \in \Sigma_1) \alpha(x) \geq 0\}
\]

([KN96, Thm. VI.6]). As all positive non-compact roots are positive on the interior of \( C_a^\text{max} \), we must have \( \alpha(h_1) = 1 \) for every \( \alpha \in \Sigma_1 \). Next we observe that \( \Sigma_1 \) spans \( (a \cap q_p)^\circ \) ([KN96, §V]), so that we obtain \( h = h_1 \). Therefore \( h \) is the only Euler element in \( q_p \cap C_s^0 \).

(b) If \( h_1 \in C_s^0 \) is an Euler element, then it is in particular hyperbolic, hence conjugate under \( \text{Im}_g(h) \) to an element \( h_2 \in q_p \) ([KN96, Cor. II.9]). Then \( h_2 \in q_p \cap C_s^0 \) is an Euler element, hence equal to the Euler element \( h \) from (a). This implies that \( h_1 \in \text{Im}_g(h)h_1 \), and thus \( \text{Im}_g(h) \) acts transitively on \( C_s^0 \cap E(g) \).

(c) The assertion holds for the unique Euler element \( h \in C_s^0 \cap q_p \) by (a). If \( h_1 \in C_s^0 \) is another Euler element, then (b) implies the existence of \( \varphi \in \text{Im}_g(h) \) with \( h_1 = \varphi(h) \). Then

\[
\tau h_1 = \tau \varphi \tau h \varphi^{-1} = \varphi \tau h \varphi^{-1} = \varphi \theta \varphi^{-1}
\]

is a Cartan involution.

Assertion (b) in Theorem 4.5 has important consequences for the possible choices of an open subgroup \( H \subseteq \text{Inn}(g)^\tau \) preserving the cone \( C \).

**Corollary 4.6.** If \((g, \tau, C)\) is a semisimple ncc symmetric Lie algebra for which \( h \) contains no non-compact ideal of \( g \), \( h \in C_s^0 \) is a causal Euler element, and \( H \subseteq \text{Inn}(g)^\tau \) is an open subgroup preserving \( C \), then the following assertions hold:

(a) \( H = H_h H^h \), i.e., every connected component of \( H \) meets \( H^h \), which is equivalent to \( H_h \) being connected.

(b) \( H \) is closed and \( H^h \) is a maximal compact subgroup of \( H \).

(c) \( H^h = H^h_\tau \) and \( \tau_h \) induces a Cartan involution on \( H \).

(d) \( \tau \) induces a Cartan involution on \( H^h \) for which \( (H^h_\tau)^\tau = e^{\text{ad}h_\tau} \) is connected.

**Proof.** (a) From Theorem 4.5(b) and \( H_e = \text{Inn}_g(h) \), we derive that \( H_h \subseteq C_s^0 \cap E(g) = H_e h \), so that \( H = H_h H^h \).

(b) Since \( \text{Aut}(g)^\tau \) is an algebraic group, it is closed and has only finitely many connected components. It contains \( H \) as an open subgroup, so that \( H \) is also closed. Further, \( H^h \) fixes the element \( h \) in the interior of \( C \), so that \( H^h \) acts on \( q \) as a relatively compact group. Next we use that \( [q, q] + q \leq g \) is an ideal of \( g \) and since \( h \) contains no non-compact ideal, it follows that \( h = [q, q] \oplus h_k \), where \( h_k \subseteq h \cap k \) is the
sum of all compact ideals of $\mathfrak{g}$ contained in $\mathfrak{h}$. Therefore the closed subgroup $H^h$ of $\text{Aut}(\mathfrak{g})$ is compact. Its maximality in $H$ now follows from the polar decomposition $H = H^h \exp(\mathfrak{h}_p)$ and (a). Note that Riemannian components correspond to compact ideals of $\mathfrak{h}$, and the corresponding subgroups of $\text{Inn}(\mathfrak{g})$ are compact, commute with $\tau$ and fix $h$.

(c) From $H^{\tau_h} : \mathfrak{h} \subseteq C_s^2 \cap \mathfrak{q}_p$ and Theorem 4.5(a), it follows that $H^{\tau_h}$ fixes $h$, so that $H^{\tau_h} \subseteq H^h$. Conversely, every $h \in H^h \subseteq \text{Aut}(\mathfrak{g})$ commutes with $\text{ad} \ h$, hence also with $\tau_h$, and thus $H^h = H^{\tau_h}$. In view of (b), $\tau_h$ induces a Cartan involution on $H$.

(d) On $H^h = H^{\tau_h}$ the involution $\tau = \tau_h \theta$ acts like $\theta$. Restricting to the identity component $H^h_e$, we obtain (d) because the group of fixed points of a Cartan involution is connected.

\[\text{Corollary 4.7.} \quad \text{(Characterization of causal Euler elements) Let } (\mathfrak{g}, \tau) \text{ be a semisimple symmetric Lie algebra for which all ideals of } \mathfrak{g} \text{ contained in } \mathfrak{h} \text{ are compact and } h \in \mathcal{E}(\mathfrak{g}) \text{ (cf. 1(b)). Consider the following assertions:} \]

\[(a) \ h \text{ is causal, i.e., there exists a pointed generating } \text{Inn}_\mathfrak{g}(\mathfrak{h})\text{-invariant closed convex cone } C \subseteq \mathfrak{q} \text{ with } h \in C^\circ.\]

\[(b) \ \tau \tau_h \text{ is a Cartan involution.}\]

\[(c) \ ih \text{ is an } H\text{-element in } \mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}.\]

Then (a) $\iff$ (b) $\iff$ (c) and if $h \in \mathfrak{q}_s$, then (a),(b),(c) are equivalent.

\[\text{Proof.} \quad (a) \Rightarrow (b): \text{If } h \in \mathfrak{q}_s, \text{ then this implication follows from Theorem 4.5(c).} \]

(b) $\Rightarrow$ (a) follows from Theorem 4.2 and the fact that $\tau \tau_h(h) = \tau(h) = -h$.

(b) $\Rightarrow$ (c): Suppose that $\theta := \tau \tau_h$ is a Cartan involution of $\mathfrak{g}$. As $\theta(h) = -h$, Lemma 3.10 implies that $ih$ is an $H$-element of $\mathfrak{g}^c$.

(c) $\Rightarrow$ (b) follows immediately from Lemma 3.10 because $(\mathfrak{g}^c, \tau^c, ih) \in \mathcal{B}$.

\[\text{Remark 4.8.} \quad \text{In Theorem 4.5(b) it is crucial to restrict to the cone } C_s = C \cap \mathfrak{q}_s \text{ in } \mathfrak{q}_s.\]

(a) First we observe that, for a Riemannian symmetric Lie algebra $(\mathfrak{g}, \theta)$, where $\theta$ is a Cartan involution, we have $\mathfrak{p} = \mathfrak{q}_p$. If $h \in \mathfrak{p}$ is an Euler element then $\tau_h \theta$ is never a Cartan involution. The hyperbolic space $\mathbb{H}^d \cong \text{SO}_{1,d}(\mathbb{R})/\text{SO}_d(\mathbb{R})$ is an example showing that this situation does in fact occur. Moreover, $\text{Inn}_\mathfrak{g}(\mathfrak{t})$ need not act transitively on $\mathcal{E}(\mathfrak{g}) \cap \mathfrak{p}$. If the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ ($\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian) is of type $A_n, n \geq 2, D_n, n \geq 4$ or $E_6$, then there exists more than one conjugacy class of Euler elements in $\mathfrak{p}$ (cf. Theorem 2.4).

(b) (cf. also Example 4.22 below) We consider the example

$$\mathfrak{g} = (\mathfrak{sl}_2(\mathbb{R}), \theta) \oplus (\mathfrak{sl}_2(\mathbb{R}), \tau_h),$$

where $h = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

which is ncc. For $G := \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ and $H := \text{SO}_2(\mathbb{R}) \times \text{SO}_{1,1}(\mathbb{R})$, the symmetric space

$$M = G/H \cong \mathbb{H}^2 \times dS^2$$

is an example for $h \in (C^\lambda)^g$ with $\lambda > 1$. 

\[\footnote{\text{If } h \text{ is not contained in } \mathfrak{q}_s, \text{ then (b) does not follow from (a). Remark 4.8(b) provides an example for } h \in (C^\lambda)^g \text{ with } \lambda > 1.}\]
Lemma 4.11. If \( \Gamma(\theta) \) is a Cartan involution with \( \theta(h) = -h \) and \( \tau = \tau_h \theta \). Then

\[
\mathfrak{g}^{\tau_h} = \mathfrak{g}^{\tau^G} = \mathfrak{h}_t \oplus \mathfrak{q}_p.
\]

Let \( G \) be a connected Lie group with semisimple Lie algebra \( \mathfrak{g} \) on which \( \tau^G \), hence also \( \tau^G \), exist (cf. the discussion in [NÖ22a, Rem. 2.12]). From the polar decomposition \( G = K \exp(\mathfrak{p}) \) we derive for the centralizer of \( h \) the decomposition

\[
G^h = K^h \exp(\mathfrak{q}_p). \tag{19}
\]

We also obtain

\[
G^{\tau^G} = K^{\tau^G} \exp(\mathfrak{h}_p) = K^{\tau^G} \exp(\mathfrak{h}_p). \tag{20}
\]

In general \( K^h \neq K^{\tau^G} \), as the simply connected covering group \( G = \widetilde{\text{SL}}_2(\mathbb{R}) \) of \( \text{SL}_2(\mathbb{R}) \) shows (see [NÖ22a, Rem. 5.3]). If \( Z(G) = \{e\} \), i.e., \( G \cong \text{Ad}(G) = \text{Inn}(\mathfrak{g}) \), then \( G^h \subseteq G^{\tau^G} \) implies \( K^h \subseteq K^{\tau^G} = K^{\tau^G} \).

Remark 4.10. For a semisimple symmetric Lie algebra \( (\mathfrak{g}, \tau) \) for which all ideals of \( \mathfrak{g} \) contained in \( \mathfrak{h} \) are compact, Corollary 4.7 provides a map\n
\[
\Gamma: \mathcal{E}_c(\mathfrak{q}_a) \to \text{Cart}(\mathfrak{g})^\tau, \quad h \mapsto \tau \tau_h
\]

to the set \( \text{Cart}(\mathfrak{g})^\tau \) of Cartan involutions on \( \mathfrak{g} \) commuting with \( \tau \). For a given Cartan involution \( \theta \) commuting with \( \tau \), and \( K := \text{Inn}(\mathfrak{g})^\theta \), the transitivity of the action of \( G := \text{Inn}(\mathfrak{g}) \) on \( \text{Cart}(\mathfrak{g}) \) ([HN12, Cor. 13.2.13]) implies that

\[
\text{Cart}(\mathfrak{g}) \cong G/K \quad \text{and} \quad \text{Cart}(\mathfrak{g})^\tau \cong \text{Exp}_{eK}(\mathfrak{h}_p) = \text{Inn}_g(\mathfrak{h}) \theta.
\]

The equivariance of \( \Gamma \) with respect to \( \text{Inn}_g(\mathfrak{h}) \) thus shows that \( \Gamma \) maps any \( \text{Inn}_g(\mathfrak{h}) \)-orbit in \( \mathcal{E}_c(\mathfrak{q}_a) \) surjectively onto \( \text{Cart}(\mathfrak{g})^\tau \).

If \( \Gamma(h) = \Gamma(h') = \theta \) for \( h, h' \in \mathcal{E}_c(\mathfrak{q}) \), then \( h, h' \in \mathfrak{q}_a \). If \( \mathfrak{g} \) is simple, then the following lemma implies that \( h' \in \{\pm h\} \).

Lemma 4.11. Let \( (\mathfrak{g}, \tau) \) be a simple symmetric Lie algebra, \( h \in \mathcal{E}_c(\mathfrak{q}) \) and \( \theta = \tau h \). Then the following assertions hold:
(a) $E_c(q) \cap q_p = \{ \pm h \}$.

(b) $\text{Inn}_g(h)$ has two orbits in the set $E_c(q)$ of causal Euler elements in $q$. They are represented by $\pm h$ for any $h \in E_c(q)$.

**Proof.** (a) Let $g^c := h + iq$ be the dual symmetric Lie algebra. Then $E^c = h + iq$ is maximally compact in $g^c$ and $z := ih \in \mathfrak{z}(E^c)$ is an H-element with $\ker(\text{ad} z) = E^c$. We distinguish two cases according to $(g, \tau)$ being of complex type or not.

**The complex case:** If $g$ is complex and $\tau$ antilinear, then $g \cong h_c$, where $h$ is simple hermitian. Then $q = ih$, and $E_c(q) \cap q_p = E_c(q) \cap q_h$ consists of all elements of the form $h = iz$, where $z \in E^c = iq_p$ is an H-element. Hence the assertion follows from the fact that $\pm z$ are the only H-elements of $g^c$ contained in $E^c$ (Lemma 3.9).

**The real case:** If $g$ is not complex, then $g^c$ is a simple hermitian Lie algebra with $E^c = h + iq$ maximal compactly embedded. If $h, h' \in q_p$ are causal Euler elements, then $ih, ih' \in E^c$ are H-elements, and Lemma 3.9 implies that $h' \in \{ \pm h \}$.

(b) Let $h \in q$ be a causal Euler element and consider the corresponding Cartan involution $\theta = \tau rh$. Then $h \in q_p$ and any other causal Euler element $h_1 \in q$ is hyperbolic, hence conjugate under $\text{Inn}_g(h)$ to an element of $q_p$ ([KN96 Cor. II.9]), and therefore by (a) to $h$ or $-h$. \hfill \blacksquare

**Remark 4.12.** For every irreducible ncc symmetric Lie algebra $(g, \tau, C)$ and $h \in E(g)$, the intersection $O_h \cap C^\circ$ is either empty or a single orbit of $\text{Inn}_g(h)$ (Theorem 4.12(b)). Therefore at most one $\text{Inn}(g)$-orbit in $E(g)$ intersects $C^\circ$. In general $\text{Inn}(g)$ does not act transitively on $E(g)$. A typical example is $g = sl_n(\mathbb{R})$ with $n - 1$ orbits of $\text{Inn}_g(h)$ in $O_h \cap q$ (cf. Example 4.22 below; see also [MNO22a Rem. 4.15]).

**Proposition 4.13.** (Classification of $\text{Inn}_g(h)$-orbits in $O_h \cap q$) Let $(g, \tau)$ be a reductive symmetric Lie algebra with $\mathfrak{z}(g) \subseteq q$. Further, let $h \in q_p$ be a causal Euler element with $\theta = \tau rh$ and $a \subseteq q_p$ maximal abelian. Let $\Sigma = \Sigma(g, a)$ denote the corresponding set of restricted roots, $\Sigma_0 := \{ \alpha \in \Sigma : \alpha(h) = 0 \}$, and $W_0 \subseteq W$ the corresponding Weyl group. Then the map

$$W_0 \backslash W/W_0 \to (O_h \cap q)/\text{Inn}_g(h), \quad W_0wW_0 \mapsto \text{Inn}_g(h).wh$$

is a bijection from the set of $W_0$-double cosets in $W$ to the set of orbits of the group $\text{Inn}_g(h)$ in $O_h \cap q$.

**Proof.** As $\mathfrak{z}(g) \cap p = q_p$, the subspace $a$ is also maximal abelian in $p$. Clearly, $a$ contains $h$. Then $O_h \cap a = W_h$ by [KN96 Thm. III.10] and elementary Coxeter theory implies that the stabilizer of $h$ in $W$ is $W^h = W_0$ ([Ne00 Prop. V.2.7]). Every $\text{Inn}_g(h)$-orbit in $O_h \cap q$ intersects $a$ ([KN96 Cor. II.9], [Ne00 Prop. VII.2.10]) and for $x \in a$ we have

$$\text{Inn}_g(h)x \cap a = \text{Inn}_g(h)x \cap a = W_0x.$$ Therefore

$$\text{Inn}_g(h) \backslash (O_h \cap q) \cong W_0 \backslash Wh \cong W_0 \backslash W/W_0.$$ \hfill \blacksquare

**Remark 4.14.** Let $H := \text{Inn}_g(h) \supseteq H_K = \text{Inn}_g(h_t)$ and $K := \text{Inn}_g(\mathfrak{t})$. We show
that, if $H_K$ fixes the causal Euler element $h$, then

$$H_K.x \cap a = \mathcal{W}_0 x \quad \text{for} \quad x \in a.$$ 

So let $x \in a \subseteq q_p$ and $g \in H_K \subseteq K^h$ with $x' := g.x \in a$. Then

$$a' := g.a \subseteq q_p \cap \mathcal{h}(x')$$

is maximal abelian in $q_p$, hence in particular in $q_p \cap \mathcal{h}(x')$, which also contains $a$. Therefore [KN96, Thm. III.3] implies the existence of an $g_1 \in \text{Inn}_q(h_t \cap \mathcal{h}(x'))$ with $g_1.a' = a$. Then $g_2 := g_1g \in H_K$ satisfies $x' = g_1.x' = g_2.x$ and $g_2.a = a$. Therefore $g_2 \in N_{H_K}(a) \subseteq N_K(a)$ acts on $a$ as a Weyl group element $w \in W$ ([KN96, Def. III.9]). As $H_K$ fixes $h$ by assumption, we have $w \in W^h = \mathcal{W}_0$ ([Ne00, Prop. V.2.7]).

**Remark 4.15.** (Supplement to Proposition 4.13) Let $g$ be a simple real Lie algebra. Euler elements in $q$ are classified by their representatives in a closed Weyl chamber $a_+ \subseteq a$, where $g = \mathfrak{t} \oplus \mathfrak{p}$ is a Cartan involution, and $a \subseteq \mathfrak{p}$ is maximal abelian. Theorem 2.4 provides a concrete list for all types of root systems $\Sigma(g, a)$ for which Euler elements exist. For $r := \dim a$, the representatives $h_j$ are labeled by a subset of $\{1, \ldots, r\}$. Any such Euler element defines an involution $\tau_j := \theta h_j$, for which $h_j \in a \subseteq q_j = g^{-\tau_j}$. The stabilizer group $W^h_j \subseteq W = W(q, a)$ is generated by the reflections fixing $h_j$ ([Ne00, Prop. V.2.7]). Now Proposition 4.13 applies to this situation and identifies the set of $\text{Inn}_q(h_j)$-orbits in $O_{h_j} \cap q_j$ with the $W^h_j$-double cosets in $W$.

**Example 4.16.** For $g = \mathfrak{sl}_n(\mathbb{R})$, $n = p + q$, $p \leq q$, the Euler element

$$h_p := \frac{1}{p+q} \begin{pmatrix} q1_p & 0 \\ 0 & -p1_q \end{pmatrix},$$

and the Cartan involution $\theta(x) = -x^\top$, we obtain $\tau(x) = -\tau_h(x)^\top$, so that we may take

$$a = \left\{ \text{diag}(x_1, \ldots, x_n) : \sum_j x_j = 0 \right\}.$$

Now

$$W \cong S_n \subseteq W_0 \cong S_p \times S_q,$$

and the orbit space $W/W_0$ corresponds to the set of $p$-element subsets $F_p \subseteq \{1, \ldots, n\}$. The orbits of $W_0$ on this set are represented by $F_p \cap \{1, \ldots, p\}$. Parametrized by the cardinality of this intersection, we have $p + 1$ orbits, and these correspond to $p + 1$ orbits of $\text{Inn}_q(h)$ in $O_h \cap q$. They are obtained by permuting the entries of the diagonal matrix $h_p$.

### 4.2. On the causality of $G/G^\tau$

In this section we show that the minimal irreducible symmetric space $\text{Inn}(g)/\text{Inn}(g)^\tau$ is a causal symmetric space for the triple $(g, \tau, C)$ if and only if the corresponding causal Euler element is not symmetric; otherwise we have to pass to a two-fold covering to obtain a causal space.
Definition 4.17. Suppose that \((g, \tau)\) is a semisimple symmetric Lie algebra. 
(a) We call the corresponding symmetric space 
\[
\mathcal{M}(g, \tau) := \text{Inn}(g) / \text{Inn}(g)^\tau
\]
a minimal symmetric space associated to the symmetric Lie algebra \((g, \tau)\). All other connected symmetric spaces \(M = G/H\) corresponding to \((g, \tau)\) are equivariant coverings of \(\mathcal{M}(g, \tau)\) by the map 
\[
gh \mapsto \text{Ad}(gh)\tau \text{Ad}(g)^{-1} \in \text{Ad}(G).\tau \cong \mathcal{M}(g, \tau).
\]
(b) If \((g, \tau, C)\) is a causal semisimple symmetric Lie algebra, 
\[G = \text{Inn}(g), \quad H_C := \{g \in G^\tau : \text{Ad}(g)C = C\},\]
then we call \(G/H_C\) the minimal causal symmetric space associated to \((g, \tau, C)\).

Minimal symmetric spaces corresponding to causal symmetric Lie algebras \((g, \tau, C)\) are not always causal because \(G^\tau\) does not always leave a pointed generating cone \(C \subseteq q\) invariant, which is equivalent to \(\text{cone}(G^\tau.h)\) being pointed for a causal Euler element \(h \in \mathcal{E}_c(q)\). The following proposition and its corollary make this requirement easy to check.

Proposition 4.18. Suppose that \((g, \tau)\) is simple ncc and \(h \in \mathcal{E}_c(q)\) a causal Euler element, \(G\) a connected Lie group with Lie algebra \(g\) to which \(\tau\) integrates, and \(H \subseteq G^\tau\) an open subgroup. Then two mutually exclusive cases occur:

(a) \(\text{Ad}(H)h = \text{Ad}(H_e)h\) and \(G/H\) is causal.

(b) \(-h \in \text{Ad}(H)h\) and \(G/H\) is not causal.

Proof. Clearly, \(\text{Ad}(H)h \subseteq \mathcal{E}_c(g) \cap q\). In view of Lemma 4.11(b), we either have \(\text{Ad}(H)h = \text{Ad}(H_e)h\) or \(-h \in \text{Ad}(H)h\). In the first case \(G/H\) is causal and in the second case it is not.

The following corollary identifies the minimal causal symmetric space in terms of the symmetry property of the corresponding Euler element.

Theorem 4.19. Let \((g, \tau)\) be simple ncc with causal Euler element \(h \in \mathcal{E}_c(q)\) and \(G = \text{Inn}(g)\). Then \(\mathcal{M}(g, \tau) \cong G/G^\tau\) is causal if and only if \(h\) is not symmetric, i.e., \(-h \not\in \mathcal{O}_h\).

Proof. Let \(H := \text{Inn}(g)^\tau\), so that \(G/H \cong \mathcal{M}(g, \tau)\) and consider the pointed generating closed convex cone \(C \subseteq q\) generated by \(H_e.h\). If \(h\) is symmetric, then \(-h = gh\) for some \(g \in G\). Proposition 4.13 implies the existence of a Weyl group element \(w \in \mathcal{W}\) with \(w.h = -h\). As every \(w \in \mathcal{W}\) is a restriction of \(\text{Ad}(k)\) to \(a\) for some \(k \in K = G^\theta\) (cf. [KN96, Def. III.9]), we may assume that \(g \in G^\theta\). Further \(g.h = -h\) implies that \(g\) commutes with \(\tau_h\), hence also with \(\tau\), i.e., \(g \in G^\tau = H\). Therefore Proposition 4.18 implies that \(\mathcal{M}(g, \tau)\) is not causal.

If, conversely, \(\mathcal{M}(g, \tau)\) is not causal, then Proposition 4.18 implies that \(-h \in H.h \subseteq \mathcal{O}_h\), so that \(h\) is symmetric.
Remark 4.20. (a) Suppose that \((g, \tau)\) is an irreducible ncc symmetric Lie algebra and \(h\) an Euler element with \(\theta(h) = -h\) and \(\tau = \tau_h\theta\). Let \(G\) be a connected Lie group with Lie algebra \(g\) on which \(\tau\) integrates to an involution \(\tau^G\) and let \(H \subseteq G^{\tau_G}\) be an open \(\theta\)-invariant subgroup, so that
\[
H = H_K \exp(h_p) \quad \text{for} \quad H_K = H^\theta = H \cap K,
\]
and Lemma 4.11 implies that
\[
H_K,h \subseteq \mathcal{E}(q) \cap q_p = \{\pm h\}.
\]
Therefore the causality of \(G/H\) is equivalent to \(H_K \subseteq G^h\) (cf. Proposition 4.18). (b) If \(G = \text{Inn}(g)\), then \(K^h \subseteq K^\tau h = K^\tau \subseteq G^\tau\), so that
\[
H_{C^{\max}} := K^h e^{ad h_p} \subseteq G^\tau
\]
is the maximal open subgroup \(H \subseteq G^\tau\) for which \(G/H_{C^{\max}}\) is causal, i.e., \(G/H_{C^{\max}}\) is a minimal causal symmetric space associated to \((g, \tau, C^{\max})\). In view of Theorem 4.19, \(H_{C^{\max}} = G^\tau\) if and only if \(h\) is not symmetric. We shall see in Theorem 5.4 below that, if \(h\) contains Euler elements, then \(h\) is symmetric. Therefore \(H_{C^{\max}}\) is an index 2-subgroup of \(G^\tau = \text{Inn}(g)^\tau\).
(c) (Symmetric Euler element and de Sitter space) The symmetry group of de Sitter spacetime is \(G = \text{SO}_{1,d}(\mathbb{R})_e\). On its Lie algebra \(\theta(x) = -x^\top\) is a Cartan involution. Let \(h = h_{W_1} \in g\) be the Euler element such that
\[
e^{th_{W_1}} p = (\cosh(t)p_0 + \sinh(t)p_1, \sinh(t)p_0 + \cosh(t)p_1, p_2, \ldots, p_d)
\]
and the corresponding wedge reflection is implemented by the linear map
\[
\tau_h = \text{diag}(-1, -1, 1, 1, \ldots, 1) \in \text{SO}_{1,d}(\mathbb{R}).
\]
Then
\[
G^\tau \simeq G_{e_1} \rtimes \{1, r_{12}\} \simeq \text{SO}_{1,d-1}(\mathbb{R})_e \rtimes \mathbb{Z}_2,
\]
where \(r_{12} = \text{diag}(1, -1, -1, 1, \ldots, 1)\) is the point reflection in the \(x_1-x_2\)-plane. The orbit map of \(e_1 = (0, 1, 0, \ldots, 0)\) induces a diffeomorphism \(G/G_{e_1} \to dS^d\). Note that \(G_{e_1}\) is the identity component \(G^e_\tau\) in \(G^\tau\). The involution \(r_{12}\) is not contained in \(G^e_\tau\). It normalizes \(G_{e_1}\), so that it acts by right multiplication on \(G/G_{e_1}\). We thus obtain a \(G\)-equivariant involution \(\varphi(gG_{e_1}) := gr_{12}G_{e_1} \in G/G_{e_1}\). The corresponding involution \(\psi\) on \(dS^d \cong G/G_{e_1}\) is \(G\)-equivariant and maps \(e_1\) to \(r_{12}(e_1) = -e_1\), hence satisfies \(\psi(x) = -x\) for all \(x \in dS^d\). As a consequence, the symmetric space \(G/G^\tau\) is the projective de Sitter space \(dS^d/\{\pm 1\}\). This is not a causal space because multiplication by \(-1\) reverses the causal structure on \(dS^d\).

4.3. Classifying non-compactly causal structures by Euler elements
Let \(G\) be a connected Lie group with simple Lie algebra \(g\). We now describe a bijection between the set \(E(g)/G\) of \(G\)-orbits in \(E(g)\) and the isomorphism classes of non-compactly causal symmetric Lie algebras \((g, \tau, C)\), where \(C = C^{\min}_q \subseteq q\) is a minimal \(\text{Inn}_g(\mathfrak{h})\)-invariant pointed closed convex cone. We write
\[
\mathcal{NCC}^{\min}(g)
\]
for the \(\text{Inn}(g)\)-orbits in the set of all pairs \((\tau, C)\) for which \((g, \tau, C)\) is an ncc symmetric Lie algebra and \(C\) is minimal.
Theorem 4.21. (Classification of ncc structures by Euler elements) Let \( g \) be simple. To \( h \in \mathcal{E}(g) \), we associate the ncc symmetric Lie algebra \((g, \tau, C^\min(h, \theta)) \in \mathcal{NCC}^\min(g)\), where \( \tau := \tau_h \theta \) for a Cartan involution \( \theta \) satisfying \( \theta(h) = -h \) and

\[
C^\min(h, \theta) = \overline{\text{cone}(\text{Inn}_g(g^\tau)h)}.
\]

This assignment induces a bijection \( \Gamma \) from the set \( \mathcal{E}(g)/\text{Inn}(g) \) of adjoint orbits of Euler elements in \( g \) onto \( \mathcal{NCC}^\min(g) \).

With the notation from Remark 4.15, we see in particular that, if \( a_+ \subseteq a \) is a closed Weyl chamber (a fundamental domain for the \( \mathcal{W} \)-action), then

\[
\mathcal{E}(g)/\text{Inn}(g) \cong (\mathcal{E}(g) \cap a)/\mathcal{W} \cong \mathcal{E}(g) \cap a_+
\]

is a finite set.

Proof. Theorem 4.12 implies that \((g, \tau, C^\min(h, \theta)) \) is ncc with \( h \in C^\min(h, \theta)^\circ \). If \( \theta_1 \) is another Cartan involution with \( \theta_1(h) = -h \), then \( \theta_1 \theta(h) = h \). Write \( \theta_1 = e^{ad x} \theta e^{-ad x} \) with \( \theta(x) = -x \) (cf. [HN12, Thm. 13.1.7]). Then

\[
h = \theta_1 \theta(h) = e^{ad x} \theta e^{-ad x} \theta(h) = e^{2ad x} \theta \theta(h) = e^{2ad x} h
\]

implies \( [x, h] = 0 \) because \( ad x \) is diagonalizable. Therefore

\[
\tau_1 := \tau_h \theta_1 \tau_h = \tau_h e^{ad x} \theta e^{-ad x} = e^{ad x} \tau \theta e^{-ad x},
\]

and

\[
e^{ad x} : (g, \tau, C^\min(h, \theta)) \to (g, \tau_1, C(h, \theta_1))
\]

is an isomorphism of ncc symmetric Lie algebras fixing \( h \). Therefore the isomorphism class of \((g, \tau, C^\min(h, \theta)) \) does not depend on the choice of \( \theta \).

If \( h_1 = g \cdot h \), then \( \theta_1 := g \theta g^{-1} \) is a Cartan involution of \( g \) with \( \theta_1(h_1) = -h_1 \), so that

\[
\tau_1 := \tau_h \theta_1 = g \tau g^{-1}
\]

leads to an ncc symmetric Lie algebra \((g, \tau_1, C(h_1, \theta_1)) \) with \( C(h_1, \theta_1) = g \cdot C^\min(h, \theta) \).

Clearly,

\[
g : (g, \tau, C^\min(h, \theta)) \to (g, \tau_1, C(h_1, \theta_1))
\]

is an isomorphism of causal symmetric Lie algebras. This shows that the isomorphism class of \((g, \tau, C^\min(h, \theta)) \) only depends on the orbit \( C^\circ \subseteq \mathcal{E}(g) \).

If, conversely, \((g, \tau, C) \) is an ncc symmetric Lie algebra with \( C \) minimal, and \( \theta \) a Cartan involution commuting with \( \tau \), then \( C^\circ \cap \mathfrak{q}_p \) contains a causal Euler element \( h \) by Theorem 4.5, and minimality implies \( C = C^\min(h, \theta) \). Therefore \( \Gamma \) is surjective.

To see that it is also injective, we recall that any two Euler elements in \( C^\circ \) are conjugate under \( \text{Inn}_g(h) \) (Theorem 4.4(b)), so that any two Euler elements defining isomorphic triples \((g, \tau, C) \) lie in the same \( \text{Inn}(g) \)-orbit.

Example 4.22. We consider the simple real Lie algebra \( g = \mathfrak{sl}_n(\mathbb{R}) \). For \( 1 \leq p < n \) and \( q := n - p \), we obtain an Euler element

\[
h_p := \frac{1}{p+q} \begin{pmatrix} q & 0 \\ 0 & -p & q \end{pmatrix}
\]
(cf. Example 4.16). Then
\[
\tau_{h_p} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}
\]
and \( \theta(x) = -x^\top \) is a Cartan involution with \( \theta(h_p) = -h_p \). For \( \tau = \tau_{h_p} \theta \) we then have
\[
\tau(x) = -\tau_{h}(x^\top), \quad \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^\top & c^\top \\ b^\top & -d^\top \end{pmatrix}.
\]
Therefore \( h = \mathfrak{so}_{p,q}(\mathbb{R}) \) and
\[
q = \left\{ \begin{pmatrix} a & b \\ -b^\top & d \end{pmatrix} : a^\top = a, d^\top = d, \text{tr}(a) + \text{tr}(d) = 0 \right\}.
\]
A typical invariant cone in \( q \) is
\[
C = \left\{ \begin{pmatrix} a & b \\ -b^\top & d \end{pmatrix} \in q : \begin{pmatrix} a & b \\ b^\top & -d \end{pmatrix} \geq 0 \right\}.
\]
It contains \( h_p \) in its interior.

Note that the subspace \( \mathfrak{a} \) of diagonal matrices in \( q \) is also maximal abelian in \( \mathfrak{g} \), so that all \( G \)-orbits in \( \mathcal{E}(\mathfrak{g}) \) intersect \( \mathfrak{a} \), hence also \( q \) (cf. Example 4.16).

**Remark 4.23.** (a) Let \((\mathfrak{g}, \tau)\) be a symmetric Lie algebra of complex type, where \( \mathfrak{g} = \mathfrak{h}_\mathbb{C} \) and \( \mathfrak{h} \) is simple hermitian. If \( \mathfrak{t} \subseteq \mathfrak{h} \) is a compactly embedded Cartan subalgebra, then \( \mathfrak{a} := i\mathfrak{t} \subseteq \mathfrak{q} = i\mathfrak{h} \) is maximal hyperbolic abelian. As \( \Sigma(\mathfrak{g}, \mathfrak{a}) \) may have several 3-gradings, there are many Euler elements in \( \mathfrak{a} \) which are not contained in an \( \text{Inn}\mathfrak{g}(\mathfrak{h}) \)-invariant cone in \( i\mathfrak{h} \). This happens for \( \mathfrak{h} = \mathfrak{su}_{p,q}(\mathbb{C}) \), where we have \( p+q-1 \) \( \text{Inn}\mathfrak{h} \)-orbits of Euler elements in \( i\mathfrak{h} \), represented by \( h_1, \ldots, h_{p+q-1} \) (see Proposition 4.13 and Theorem 2.4). The only Euler elements contained in the interior of an invariant cone are those of the form \( h = iz \), where \( z \in \mathfrak{z}(\mathfrak{h}_\mathbb{R}) \) for a Cartan decomposition \( \mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus \mathfrak{h}_\mathbb{P} \). This is the case for \( h = h_p \).

(b) The example in (a) shows that, for a simple symmetric Lie algebra \((\mathfrak{g}, \tau)\), there may be Euler elements in \( q \) which are not contained in any hyperbolic \( \text{Inn}\mathfrak{h}(\mathfrak{h}) \)-invariant cone \( C \subseteq q \), i.e., not every Euler element in \( q \) is causal for \((\mathfrak{g}, \tau)\). However, the picture changes if we are free to choose \( \tau \). Then Theorem 4.21 implies that every Euler element \( h \in \mathcal{E}(\mathfrak{g}) \) is causal for a suitable choice of \( \tau \).

### 4.4. The classification of irreducible ncc symmetric Lie algebras

We have seen above that irreducible non-compactly causal symmetric Lie algebras \((\mathfrak{g}, \tau)\) are classified by Euler element \( h \) in real simple Lie algebras \( \mathfrak{g} \) and Cartan involutions \( \theta \) on \( \mathfrak{g} \) satisfying \( \theta(h) = -h \) (Section 4.3). In this case \((\mathfrak{g}, \tau)\) with \( \tau = \theta \tau_h \) is the corresponding causal symmetric Lie algebra. Along these lines, one obtains a complete classification of these structures, which is described in Table 3 below. It lists all irreducible non-compactly causal symmetric Lie algebras \((\mathfrak{g}, \tau)\) according to the subdivision into the following 4 types:

- **Complex type:** \( \mathfrak{g} = \mathfrak{h}_\mathbb{C} \) and \( \tau \) is complex conjugation with respect to \( \mathfrak{h} \). In this case \( \mathfrak{g}^c \cong \mathfrak{h}^{\mathbb{R}^2} \), so that \( \text{rk}_\mathbb{R}(\mathfrak{g}^c) = 2 \text{rk}_\mathbb{R}(\mathfrak{h}) \).
• Cayley type (CT): $\tau = \tau_{h_1}$ for an Euler element $h_1 \in \mathfrak{h}$. Then $\text{rk}_\mathbb{R}(\mathfrak{g}^c) = \text{rk}_\mathbb{R}(\mathfrak{g}) = \text{rk}_\mathbb{R}(\mathfrak{h})$.

• Split type (ST): $\text{rk}_\mathbb{R}(\mathfrak{h}) = \text{rk}_\mathbb{R}(\mathfrak{g}^c)$ and $(\mathfrak{g}, \tau)$ is not of Cayley type.

• Non-split type (NST): $\text{rk}_\mathbb{R}(\mathfrak{g}^c) = 2 \text{rk}_\mathbb{R}(\mathfrak{h})$ and $(\mathfrak{g}, \tau)$ is not of complex type.

Remark 4.24. (a) If $(\mathfrak{g}, \tau)$ is a simple symmetric Lie algebra, then either the dual symmetric Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ is simple ($(\mathfrak{g}, \tau)$ is not of complex type) and if $(\mathfrak{g}, \tau)$ is of complex type, then $(\mathfrak{g}^c, \tau) \cong (\mathfrak{h} \oplus \mathfrak{h}, \tau_{\text{flip}})$.

(b) For an irreducible ncc symmetric Lie algebra $(\mathfrak{g}, \tau)$, the Lie algebra $\mathfrak{g}$ is simple: Since $\tau = \theta \tau_h$ for a causal Euler element $h$, this follows from the fact that $\theta$ and $\tau_h$, hence also $\tau$, preserve all simple ideals.

(c) For an irreducible ncc symmetric Lie algebra $(\mathfrak{g}, \tau)$, either $\mathfrak{g}^c$ is simple hermitian or isomorphic to $(\mathfrak{h} \oplus \mathfrak{h}, \tau_{\text{flip}})$ (if $\mathfrak{g}$ is complex).

In the table below we write $r = \text{rk}_\mathbb{R}(\mathfrak{g}^c)$ and $s = \text{rk}_\mathbb{R}(\mathfrak{h})$. Further $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal abelian of dimension $r$. For root systems $\Sigma(\mathfrak{g}, \mathfrak{a})$ of type $A_{n-1}$, there are $n-1$ Euler elements $h_1, \ldots, h_{n-1}$, but for the other root systems there are less; see Theorem 2.4 for the concrete list. For $1 \leq j < n$ we write $j' := \min(j, n-j)$. 
| Complex type | $\mathfrak{g}$ | $\mathfrak{g}' = \mathfrak{h} + iq$ | $r$ | $\mathfrak{h} = \mathfrak{g}^{\mathfrak{h},\mathfrak{h}}$ | $s$ | $\Sigma(\mathfrak{g}, \mathfrak{a})$ | $h$ | $\mathfrak{g}_1(h)$ |
|--------------|---------------|----------------|-----|---------------------|-----|-----------------|-----|---------------|
| $\mathfrak{sl}_n(\mathbb{C})$ | $\mathfrak{su}_{j,n-j}(\mathbb{C}) \oplus \mathfrak{su}_{j,n-j}(\mathbb{C})$ | $2j'$ | $\mathfrak{su}_{j,n-j}(\mathbb{C})$ | $j'$ | $A_{n-1}$ | $h_j$ | $M_{j,n-j}(\mathbb{C})$ |
| $\mathfrak{sp}_{2n}(\mathbb{C})$ | $\mathfrak{sp}_{2n}(\mathbb{R})$ | $2n$ | $\mathfrak{sp}_{2n}(\mathbb{R})$ | $n$ | $C_n$ | $h_n$ | Sym$_n(\mathbb{C})$ |
| $\mathfrak{so}_n(\mathbb{C})$, $n > 4$ | $\mathfrak{so}_{2,n-2}(\mathbb{R})$ | $4$ | $\mathfrak{so}_{2,n-2}(\mathbb{R})$ | $2$ | $D_{2n}^1$, $B_{2n}^1$ | $h_1$ | $\mathbb{C}^{n-2}$ |
| $\mathfrak{so}_{2n}(\mathbb{C})$ | $\mathfrak{so}^*(2n)$ | $2\frac{n}{2}$ | $\mathfrak{so}^*(2n)$ | $\frac{3}{2}$ | $D_n$ | $h_{n-1}, h_n$ | Skew$_n(\mathbb{C})$ |
| $\mathfrak{e}_6(\mathbb{C})$ | $\mathfrak{e}_6(-14)$ | $4$ | $\mathfrak{e}_6(-14)$ | $2$ | $E_6$ | $h_1, h_6$ | $M_{1,2}(\mathbb{O})_C$ |
| $\mathfrak{e}_7(\mathbb{C})$ | $\mathfrak{e}_7(-25)$ | $6$ | $\mathfrak{e}_7(-25)$ | $3$ | $E_7$ | $h_7$ | $\text{Herm}_3(\mathbb{O})_C$ |

| Cayley type | $\mathfrak{su}_{r,r}(\mathbb{C})$ | $\mathfrak{su}_{r,r}(\mathbb{C})$ | $r$ | $\mathbb{R} \oplus \mathfrak{su}_r(\mathbb{C})$ | $r$ | $C_r$ | $h_r$ | $\text{Herm}_r(\mathbb{C})$ |
| $\mathfrak{sp}_2(\mathbb{R})$, $d > 2$ | $\mathfrak{sp}_2(\mathbb{R})$ | $2$ | $\mathbb{R} \oplus \mathfrak{su}_r(\mathbb{C})$ | $r$ | $C_r$ | $h_r$ | Sym$_r(\mathbb{R})$ |
| $\mathfrak{e}_6(-14)$ | $\mathfrak{e}_6(-14)$ | $3$ | $\mathbb{R} \oplus \mathfrak{su}_r(\mathbb{C})$ | $r$ | $C_r$ | $h_r$ | $\text{Herm}_r(\mathbb{C})$ |

| Split type | $\mathfrak{sl}_n(\mathbb{R})$ | $\mathfrak{su}_{j,n-j}(\mathbb{C})$ | $j'$ | $\mathfrak{so}_{j,n-j}(\mathbb{R})$ | $j'$ | $A_{n-1}$ | $h_j$ | $M_{j,n-j}(\mathbb{R})$ |
| $\mathfrak{so}_{n,n}(\mathbb{R})$ | $\mathfrak{so}^*(2n)$ | $2\frac{n}{2}$ | $\mathfrak{so}_{1,n}(\mathbb{R}) \oplus \mathfrak{so}_{1,q}(\mathbb{R})$ | $\frac{3}{2}$ | $D_n$ | $h_{n-1}, h_n$ | Skew$_n(\mathbb{R})$ |
| $\mathfrak{so}_{p+1,q+1}(\mathbb{R})$, $p > 1$ | $\mathfrak{so}_{p+1,q+1}(\mathbb{R})$ | $2$ | $\mathfrak{so}_{p+1,q+1}(\mathbb{R})$ | $\frac{3}{2}$ | $D_{p+1}$ | $h_{n-1}, h_n$ | Skew$_n(\mathbb{R})$ |
| $\mathfrak{e}_6(\mathbb{R})$ | $\mathfrak{e}_6(-14)$ | $3$ | $\mathfrak{e}_6(-14)$ | $\mathfrak{u}_{2,2}(\mathbb{H})$ | $2$ | $E_6$ | $h_1, h_6$ | $M_{1,2}(\mathbb{O}_\text{split})$ |
| $\mathfrak{e}_7(\mathbb{R})$ | $\mathfrak{e}_7(-25)$ | $3$ | $\mathfrak{e}_7(-25)$ | $\mathfrak{u}_{2,2}(\mathbb{H}) = \mathfrak{u}^*(8)$ | $3$ | $E_7$ | $h_7$ | $\text{Herm}_3(\mathbb{O}_\text{split})$ |

| Non-split type | $\mathfrak{su}_{2j,2n-2j}(\mathbb{C})$ | $2j'$ | $\mathfrak{u}_{j,n-j}(\mathbb{H})$ | $j'$ | $A_{n-1}$ | $h_j$ | $M_{j,n-j}(\mathbb{H})$ |
| $\mathfrak{sp}_{2n}(\mathbb{C})$ | $\mathfrak{sp}_{2n}(\mathbb{C})$ | $2n$ | $\mathfrak{sp}_{2n}(\mathbb{C})$ | $n$ | $C_n$ | $h_n$ | Aherm$_n(\mathbb{H})$ |
| $\mathfrak{so}_{1,d+1}(\mathbb{R})$ | $\mathfrak{so}_{1,d+1}(\mathbb{R})$ | $2$ | $\mathfrak{so}_{1,d+1}(\mathbb{R})$ | $1$ | $A_1$ | $h_1$ | $\mathbb{R}^d$ |
| $\mathfrak{e}_6(-26)$ | $\mathfrak{e}_6(-26)$ | $2$ | $\mathfrak{e}_6(-26)$ | $1$ | $A_2$ | $h_1, h_2$ | $M_{1,2}(\mathbb{O})$ |

Table 3: Irreducible ncc symmetric Lie algebras with corresponding causal Euler elements $h \in \mathfrak{a}$

5. Strongly orthogonal roots

In this section we introduce a key technical tool in the structure theory of causal symmetric spaces: maximal $\tau$-invariant sets of long strongly orthogonal roots. In Section 5.1 we recall the construction of such sets from [10] and discuss its basic properties. In particular we connect Euler elements with strongly orthogonal roots and the corresponding $\mathfrak{sl}_2$-subalgebras (Proposition 5.2). An important application of this technique is Theorem 5.3 that characterizes irreducible non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau, C)$ for which $\mathfrak{h}$ contains Euler elements as those, for which the causal Euler element is symmetric.

5.1. A $\tau$-invariant set of strongly orthogonal roots

Let $(\mathfrak{g}, \tau)$ be an irreducible non-compactly causal symmetric Lie algebra and recall that this implies that $\mathfrak{g}$ is simple ([10] Prop. 2.13, [11] Rem. 3.1.9)). We fix a causal Euler element $h \in \mathfrak{q}$ and the corresponding Cartan involution $\theta = \tau \tau_h$
The set $\Gamma$ of strongly orthogonal roots specifies a subalgebra of $g$ if $(\Gamma)$ if of Cayley type then $s$ contains $h$ and is also maximal abelian in $q$ and $p$, so that
\[
\dim a = \text{rk}_R(g) = \text{rk}_R(g^h).
\]

Let $c \subseteq g$ be a Cartan subalgebra containing $a$. Then $c_t := c \cap \mathfrak{t} \subseteq h_t$ and $c$ is invariant under $\tau$ and $\theta$, which coincide on $c$.

For the root decomposition of $g$ with respect to $c_C$, we then have
\[
\Sigma = \Sigma(g_C, c_C) \subseteq ic_t^* \oplus a^*, \quad \text{with} \quad -\tau\alpha = \overline{\alpha} \quad \text{for} \quad \alpha \in \Sigma.
\]

Further, $h$ induces a 3-grading of the root system
\[
\Sigma = \Sigma_{-1} \cup \Sigma_0 \cup \Sigma_1 \quad \text{with} \quad \Sigma_j = \{\alpha \in \Sigma : \alpha(h) = j\}.
\]

The $\tau$-invariance of $c$ implies that $\tau$ acts on $\Sigma$, and since $\tau(h) = -h$, we have
\[
\tau\Sigma_0 = \Sigma_0 \quad \text{and} \quad \tau\Sigma_1 = \Sigma_{-1}.
\]

Note that $\Sigma_0 = \Sigma(c, c)$ is the root system of the subalgebra $g^h$.

According to [Ol91] Thm. 3.4, there exists a maximal subset $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Sigma_1$ of strongly orthogonal long roots, i.e., $\gamma_j \pm \gamma_k \notin \Sigma$ for $j \neq k$, such that $-\tau(\Gamma) = \Gamma$ (cf. Proposition 3.3). Then $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 := \{\gamma \in \Gamma : -\tau\gamma = \gamma\}$ is the subset of real-valued roots in $\Gamma$, and $-\tau$ acts by complex conjugation without fixed points on $\Gamma_1$, so that this set has an even number of elements. For $r_0 := |\Gamma_0|$ and $r_1 := |\Gamma_1|/2$, we obviously have
\[
r = r_0 + 2r_1 \quad \text{and put} \quad s := r_0 + r_1.
\]

By [Ol91] Lemma 4.3,
\[
\text{rk}_R(h) = s. \quad (22)
\]

If $(g, \tau)$ if of Cayley type then $h = g^{h}$ for an Euler element $h_1 \in h$ (see Section 4.4), hence $h$ is of real rank $s$ and $g \cong g^{c}$ is of real rank $r$. We therefore have $r = s$, which is equivalent to $r_1 = 0$.

The set $\Gamma$ of strongly orthogonal roots specifies a subalgebra
\[
\mathfrak{s}_C := \sum_{\gamma \in \Gamma} \mathfrak{g}_C^\gamma + \mathfrak{g}_C^{-\gamma} + \mathbb{C}\gamma^\vee \cong sl_2(\mathbb{C})^r \quad (23)
\]
of $g_C$ invariant under the $\mathbb{C}$-linear extension of $\tau$. The involution $\tau$ leaves all ideals in $\mathfrak{s}_C$ corresponding to roots in $\Gamma_0$ invariant and induces flip involutions on the ideals corresponding to $(-\tau)$-orbits in $\Gamma_1$. We thus obtain
\[
\mathfrak{s} \cong sl_2(\mathbb{R})^{r_0} \oplus sl_2(\mathbb{C})^{r_1} \quad \text{and} \quad \mathfrak{s}^\tau \cong so_{1,1}(\mathbb{R})^{r_0} \oplus su_{1,1}(\mathbb{C})^{r_1}. \quad (24)
\]

For any Euler element $h_s \in \mathfrak{s} \cap \mathfrak{q}_p$ whose kernel contains no non-zero ideal, the Lie subalgebra generated by $h_s$ and $t_q$ is isomorphic to $sl_2(\mathbb{R})^r$. This is easily verified by considering the two cases $sl_2(\mathbb{R})$ and $sl_2(\mathbb{C})$ separately.
For the \( \mathfrak{s}_2(R) \)-ideals of \( \mathfrak{s} \) we have \( \mathfrak{s}_2(R)^c = \mathfrak{su}_{1,1}(\mathbb{C}) \) and for the \( \mathfrak{s}_2(C) \) ideals, we have \( \mathfrak{s}_2(C)^c \cong \mathfrak{su}_{1,1}(\mathbb{C})^{\otimes 2} \), so that

\[
\mathfrak{s}^c = \mathfrak{s}_C \cap \mathfrak{g}^c \cong \mathfrak{su}_{1,1}(\mathbb{C})^*.
\]

In particular \( \mathfrak{s}^c \subseteq \mathfrak{g}^c \) is a subalgebra of full real rank \( r = \text{rk}_R(\mathfrak{g}^c) \).

The Cayley transform \( \kappa_h := e^{\frac{1}{2} \text{ad } h} \) induces a complex structure on \( \mathfrak{h}_p + i\mathfrak{q}_e \) for which \( \tau \) acts as an antilinear involution. Therefore \( \kappa_h \) maps \( \mathfrak{h}_p \) bijectively to \( i\mathfrak{q}_e \), and thus \([22]\) implies that the maximal abelian subspaces of \( \mathfrak{q}_e \) are also of dimension \( s \). Note also that \( \text{ad } h \) defines a bijection \( \mathfrak{h}_p \to i\mathfrak{q}_e \), commuting with \( \text{Inn}_h(\mathfrak{h}_e) \).

In view of \([22]\), \( \mathfrak{s} \cap \mathfrak{q}_e \) contains a maximal abelian subspace \( \mathfrak{t}_q \) of \( \mathfrak{q}_e \). With respect to the decomposition of \( \mathfrak{s} \) in \([21]\), we may choose

\[
\mathfrak{t}_q = \mathfrak{so}_2(R)^{r_0 + r_1} = \mathfrak{so}_2(R)^s.
\]

**Lemma 5.1.** For \( x \in \mathfrak{t}_q \), we have \( \rho(\text{ad } x) = \rho(\text{ad } x|_s) \), where \( \rho \) denotes the spectral radius. With the basis

\[
z^j = \left(0, \ldots, 0, \frac{1}{2}, 0, -1, 0, 0, \ldots, 0\right), \quad j = 1, \ldots, s,
\]

in \( \mathfrak{so}_2(R)^s \) we have for \( x = \sum_{j=1}^s x_j z^j \)

\[
\rho(\text{ad } x) = \max\{|x_j|: j = 1, \ldots, s\}.
\]

**Proof.** That \( \rho(\text{ad } x|_s) \) equals \( \max\{|x_j|: j = 1, \ldots, s\} \) follows by a matrix calculation in \( \mathfrak{s}_2(R) \). As complexification does not change the spectral radius, \( \rho(\text{ad } x|_s) = \rho(\text{ad } x|_s) \). It therefore remains to observe that for semisimple elements

\[
y = \sum_{j=1}^r y_j z^j \in \mathfrak{s}_C \cong \mathfrak{s}_2(C)^*,
\]

we have

\[
\rho(\text{ad } y) = \rho(\text{ad } y|_{s_C}).
\]

Clearly, \( \rho(\text{ad } y) \geq \rho(\text{ad } y|_{s_C}) \). From \([30]\) in Proposition 3.3 it follows that, for the simple ideals in \( \mathfrak{s}_C \), the Lie algebra \( \mathfrak{g}_C \) contains only simple modules of dimension 1, 2 and 3. This implies that \( \rho(\text{ad } y) \leq \rho(\text{ad } y|_{s_C}) \) for any semisimple element \( y \in \mathfrak{s}_C \). ■

**Proposition 5.2.** Let \((\mathfrak{g}, \tau, C)\) be a simple ncc symmetric Lie algebra. Pick a causal Euler element \( h \in C^0 \), a Cartan involution \( \theta \) with \( \theta(h) = -h \), so that \( \tau = \tau_h \theta \), and \( \mathfrak{t}_q \subseteq \mathfrak{q}_e \) maximal abelian. Then \( \dim \mathfrak{t}_q = s = r_0 + r_1 \), and the following assertions hold:

(a) The Lie algebra \( \mathfrak{l} \) generated by \( h \) and \( \mathfrak{t}_q \) is reductive.

(b) The commutator algebra \([\mathfrak{l}, \mathfrak{l}]\) is isomorphic to \( \mathfrak{s}_2(R)^s \).

(c) \( \mathfrak{g}(\mathfrak{l}) = \mathbb{R}h_0 \) for some hyperbolic element \( h_0 \) satisfying \( \tau(h_0) = -h_0 = \theta(h_0) \) which is zero if and only if \( \mathfrak{g}^c \) is of type tube or a sum of two ideals of type tube.
(d) The Lie algebra $\mathfrak{l}$ is $\tau$-invariant with $\mathfrak{l}' \cong \mathfrak{so}_{1,1}(\mathbb{R})^\tau$. It is also $\theta$-invariant with $\mathfrak{l}' = t_q = \mathfrak{so}_2(\mathbb{R})^\theta$.

Note that (c) implies that $\mathfrak{l}$ is semisimple, i.e., $h \in [\mathfrak{l}, \mathfrak{l}]$, if and only if $\mathfrak{g}^c$ is of tube type.

**Proof.** The complex case: If $\mathfrak{g}$ is complex, then $\mathfrak{g} \cong \mathfrak{h}_C$, where $\tau$ is complex conjugation, $\mathfrak{h}$ is simple hermitian and $\mathfrak{q} = i\mathfrak{h}$. For a Cartan decomposition $\mathfrak{h} = \mathfrak{h}_t \oplus \mathfrak{h}_p$, we then have $\mathfrak{t} = \mathfrak{h}_t + i\mathfrak{h}_p$ and $\mathfrak{p} = \mathfrak{h}_p + i\mathfrak{h}_t$. Now $t_q = ia$, where $a \subseteq \mathfrak{h}_p$ is maximal abelian and $s = rk_\mathbb{R}(h)$ and $r = rk_\mathbb{R}(\mathfrak{g}^c) = rk_\mathbb{R}(\mathfrak{h} \oplus \mathfrak{h}) = 2s$, so that $r_0 = 0$. A maximal system $\{\gamma_1, \ldots, \gamma_s\}$ of long strongly orthogonal roots in $\Sigma(\mathfrak{h}, a)$ now leads to a complex subalgebra

$$\mathfrak{s} := \sum_{j=1}^s \mathfrak{g}(\gamma_j) \cong \mathfrak{sl}_2(\mathbb{C})^s \subseteq \mathfrak{g}, \quad \text{where} \quad \mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}],$$

which is invariant under $\mathfrak{ad} h$. Next we recall that $\mathfrak{h}$ is of tube type if and only if the corresponding system of restricted roots is of type $C_r$. By Corollary 3.3, this is equivalent to $h$ being symmetric and by Proposition 3.6, this is equivalent to $h \in \mathfrak{s}$. It is easy to see that $h$ generates with $t_q \cong \mathfrak{so}_2(\mathbb{R})^\tau$ a subalgebra $\mathfrak{l} = \mathbb{R} h + \mathfrak{l}'$ with $\mathfrak{l}' \cong \mathfrak{sl}_2(\mathbb{R})^\tau$. The assertion thus follows with $r_0 = 0$ and $r_1 = s$. We write $h_0$ for the component of $h$ in the center of $\mathfrak{l}$. Then $\tau(h_0) = -h_0$ follows from $\tau(h) = -h$. Likewise $\theta(h) = -h$ implies $\theta(h_0) = -h_0$, so that $h_0$ is hyperbolic.

The real case: If $\mathfrak{g}$ is not complex, then the $c$-dual Lie algebra $\mathfrak{g}^c := \mathfrak{h} + iq$ is simple hermitian. The subspace $it_q \subseteq i\mathfrak{q}_\mathfrak{c} = \mathfrak{q}_\mathfrak{c}^c$ is maximal abelian. Since all maximal abelian subspaces in $\mathfrak{q}_\mathfrak{c}$ are conjugate under the group $\text{Inn}_\mathfrak{c}(\mathfrak{h}_t)$, we may w.l.o.g. assume that $it_q \subseteq \mathfrak{q}_\mathfrak{c}$, where $\mathfrak{q}_\mathfrak{c} \cong \mathfrak{sl}_2(\mathbb{C})^r$ is constructed from a $-\tau$-stable maximal set of strongly orthogonal roots as in (23) above. The action of $\tau$ and complex conjugation with respect to $\mathfrak{g}$ on $\mathfrak{q}_\mathfrak{c}$ now show with (23) above that

$$\mathfrak{s} := \mathfrak{g} \cap \mathfrak{q}_\mathfrak{c} \cong \mathfrak{sl}_2(\mathbb{R})^\tau \oplus \mathfrak{sl}_2(\mathbb{C})^r \quad \text{and} \quad \mathfrak{s}^\tau \cong \mathfrak{so}_{1,1}(\mathbb{R})^\tau \oplus \mathfrak{su}_{1,1}(\mathbb{C})^r.$$  

As $\mathfrak{s}^\tau \cap \mathfrak{t} \cong \mathfrak{so}_2(\mathbb{R})^{\tau + r_1}$ is contained in a maximal abelian subspace $t_q \subseteq \mathfrak{q}_\mathfrak{c}$, it follows that

$$t_q = \mathfrak{s}^\tau \cap \mathfrak{t} = \mathfrak{q}_\mathfrak{c} \cap \mathfrak{s}.$$  

As in the complex case, we see that $\mathfrak{g}^c$ is of tube type if and only if $h \in \mathfrak{q}_\mathfrak{c}$ (Corollary 3.3, Proposition 3.6). Then $h$ is an Euler element in $\mathfrak{s}$, contained in $\mathfrak{s}^\tau$. Inspecting the configurations we obtain for the components in the simple ideals of $\mathfrak{s}$, it now follows easily that the Lie algebra $\mathfrak{l}$ generated by $h$ and $t_q$ has the asserted form. Here the main point is to see that $\mathfrak{so}_2(\mathbb{R})$ generates with an Euler element in $i\mathfrak{sl}_2(\mathbb{R}) \subseteq \mathfrak{sl}_2(\mathbb{C})$, a 3-dimensional Lie subalgebra isomorphic to $\mathfrak{sl}_3(\mathbb{R})$.  

**5.2. Characterization of modular ncc symmetric Lie algebras**

In this subsection we characterize non-compactly causal Lie algebras $(\mathfrak{g}, \tau, C)$ for which $\mathfrak{h}$ contains an Euler element by the symmetry of the corresponding causal Euler element. This condition plays an important role in [NO22b], where we study the flows of Euler elements in $\mathfrak{h}$ on the symmetric space.
Definition 5.3. A modular causal symmetric Lie algebra is a quadruple \((\mathfrak{g}, \tau, C, h')\), where \((\mathfrak{g}, \tau, C)\) is a causal symmetric Lie algebra and \(h' \in \mathfrak{h} = \mathfrak{g}^\tau\) is an Euler element. We sometimes call \((\mathfrak{g}, \tau, C)\) modular if such an Euler element \(h'\) exists.

Theorem 5.4. For a simple ncc symmetric Lie algebra \((\mathfrak{g}, \tau, C)\) with causal Euler element \(h \in \mathfrak{q}_\mathfrak{p}\) satisfying \(\tau = \tau_\mathfrak{g} \theta\), the following are equivalent:

(a) \(\mathfrak{h} \cap \mathcal{E}(\mathfrak{g}) \neq \emptyset\).

(b) \(\mathfrak{g}^\tau\) is either simple of tube type or a direct sum of two such ideals.

(c) \(h\) is a symmetric Euler element, i.e., \(-h \in \mathcal{O}_h = \text{Inn}(\mathfrak{g})h\).

(d) There exists a second Euler element \(h_1\) such that \(\tau_{h_1}(h) = -h\).

(e) There exists an \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebra containing \(h\) that is invariant under \(\tau\) and \(\theta\).

(f) \(\mathfrak{h} \cap \mathcal{O}_h \neq \emptyset\).

Proof. (a) \(\Rightarrow\) (b): Suppose that \(h_1 \in \mathfrak{h}\) is an Euler element. Then \(h_1\) also is an Euler element in the dual Lie algebra \(\mathfrak{g}^\tau\). Either \(\mathfrak{g}^\tau\) is simple (if \(\mathfrak{g}\) is not of complex type) or \((\mathfrak{g}^\tau, \tau^\tau) \cong (\mathfrak{h} \oplus \mathfrak{h}, \tau_{\mathfrak{h}_{\mathfrak{p}}}^\tau)\) (if \(\mathfrak{g}\) is of complex type). Now [MN21, Prop. 3.11(b)] implies that, in the first case, \(\mathfrak{g}^\tau\) is of tube type, and, in the second case, the same argument shows that \(\mathfrak{h}\) is of tube type (cf. also Corollary 3.5).

(b) \(\Rightarrow\) (c): From Proposition 5.2(c) we infer that the Lie subalgebra \(l\) generated by \(h\) and \(t_q\) is isomorphic to \(\mathfrak{sl}_2(\mathbb{R})\). Therefore (c) follows from the fact that all Euler elements in \(\mathfrak{sl}_2(\mathbb{R})\) are symmetric.

(c) \(\Leftrightarrow\) (d): The equivalence of (c) and (d) follows from [MN21, Thm. 3.13(b)].

(d) \(\Rightarrow\) (e): [MN21, Thm. 3.13] implies that we also have \(\tau_{h_1}(h_1) = -h_1\), i.e., \(h_1 \in \mathfrak{g}^{-\tau_h}\). As \(h_1\) is hyperbolic, it is conjugate under \(\text{Inn}_h(\mathcal{O}_h) = \text{Inn}_h(\mathfrak{z}_h)\) to an element of \(\mathfrak{g}^{-\tau_h} \cap \mathfrak{p}\) ([KN96, Cor. II.9]). We may therefore assume that \(\theta(h_1) = -h_1\). Then \(\tau = \tau_\mathfrak{g} \theta\) satisfies \(\tau(h_1) = h_1\), i.e., \(h_1 \in \mathfrak{h}\). [MN21, Thm. 3.13] further implies that \(\mathfrak{b} := \text{span}\{h, h_1, [h, h_1]\} \cong \mathfrak{sl}_2(\mathbb{R})\). This subalgebra is clearly invariant under \(\theta\) and \(\tau\).

(e) \(\Rightarrow\) (f): If \(\mathfrak{b} \subseteq \mathfrak{g}\) is isomorphic to \(\mathfrak{sl}_2(\mathbb{R})\), invariant under \(\tau\) and \(\theta\) and \(h \in \mathfrak{b}\), then \(\mathfrak{b}^\tau\) contains an Euler element \(h_1\) of \(\mathfrak{b}\), but then \(h_1 \in \text{Inn}(\mathfrak{b})h\) implies that \(h_1 \in \mathcal{O}_h \cap \mathfrak{h}\).

(f) \(\Rightarrow\) (a) is trivial.

Corollary 5.5. Let \((\mathfrak{g}, \tau) = (\mathfrak{h}_\mathfrak{C}, \tau)\) be an irreducible ncc symmetric Lie algebra of complex type and \(h \in \mathcal{E}_c(\mathfrak{q})\) a causal Euler element. Then the following are equivalent:

(a) The real form \(\mathfrak{h}\) is of tube type.

(b) \(h\) is symmetric, i.e., \(-h \in \mathcal{O}_h\).

Proof. As \(\mathfrak{g}^\tau \cong \mathfrak{h} \oplus \mathfrak{h}\), this follows from the equivalence of (b) and (c) in Theorem 5.4.
Remark 5.6. Any pair of Euler elements \((h, h_1)\) which are orthogonal in the sense that \(\tau_h(h_1) = -h_1\) and \(\tau_{h_1}(h) = -h\) (cf. \[MN21\] Thm. 3.13) leads to an embedding of symmetric Lie algebras \((l, \tau_l) \hookrightarrow (g, \tau_g)\) with \(h, h_1 \in l \cong \mathfrak{sl}_2(\mathbb{R})\). Here we may w.l.o.g. assume that \(h_1 \in h_p\), so that \(l_q = [h_1, l] = \mathbb{R}h + \mathbb{R}[h_1, h]\). The centralizer of \(h_1\) in \(L := \text{SL}_2(\mathbb{R})\) is \(L_{h_1} \cong \text{SO}_{1,1}(\mathbb{R})\). The fact that \(l\) contains an Euler element of \(g\) implies that the adjoint action of \(L\) on \(g\) descends to a homomorphism \(\text{Ad}(L) \cong \text{PSL}_2(\mathbb{R}) \hookrightarrow \text{Inn}(g)\), which leads to an embedding \(dS^2 \cong L/L_{h_1} \hookrightarrow \text{Inn}(g) / \text{Inn}(g)_{\text{h}	ext{p}}\) and also to an embedding of projective 2-dimensional de Sitter space \(PdS^2 := dS^2 / \{\pm 1\} \hookrightarrow \text{Inn}(g) / \text{Inn}(g)_{\text{h}\text{p}}\) (cf. Remark 4.20(c)).

6. Real crown domains and real tubes

Let \((G, \tau_G)\) be a connected symmetric semisimple Lie group with non-compactly causal symmetric Lie algebra \((g, \tau, C)\), \(H \subseteq G\) an open subgroup satisfying \(\text{Ad}(H)C = C\) and \(M = G/H\) the associated non-compactly causal symmetric space. In this section we extend some of the results obtained in [N ´O22b] for the special class of modular ncc spaces (Definition 5.3), where \(h\) contains an Euler element, to general semisimple non-compactly causal symmetric Lie algebras.

Our main result is that, if the cone \(C\) is maximal and \((g, \tau)\) is semisimple without Riemannian ideals, then the connected component of \(h\) in the intersection of the adjoint orbit \(O_h\) with the real tube domain \(T_C = h + C^\circ \subseteq g\) of the Riemannian symmetric space \(O_h^q = \text{Inn}_g(h)h = e^{\text{ad}h_p}h\), i.e.,

\[ \text{Inn}_g(h)e^{\text{ad}\Omega_{q_t}}h \quad \text{for} \quad \Omega_{q_t} = \left\{ x \in q_t : \rho(\text{ad} x) < \frac{\pi}{2} \right\}, \]

where \(\rho(\text{ad} x)\) is the spectral radius of \(\text{ad} x\) (Theorem 6.6) and \(q_t = q \cap \mathfrak{k}\) for a Cartan decomposition \(g = \mathfrak{k} \oplus \mathfrak{p}\) with \(h \in \mathfrak{p}\).

Definition 6.1. (a) Let \(h \in q_p \cap C^\circ\) be a causal Euler element, then \(\mathfrak{z}_h(h) = \mathfrak{h}_t\) implies that

\[ O_h^q := \text{Inn}_g(h)h = e^{\text{ad}h_p}h \]

is the non-compact Riemannian symmetric space associated to the symmetric Lie algebra \((\mathfrak{h}, \theta)\).

(b) For a convex cone \(C \subseteq q\), we write

\[ T_C := h + C^\circ \]

for the corresponding open real tube domain in \(g\).

(c) We define the Matsuki crown of the Riemannian symmetric space \(O_h^q\) by

\[ C(O_h^q) := \text{Inn}_g(h)e^{\text{ad}\Omega_{q_t}}h = e^{\text{ad}h_p}e^{\text{ad}\Omega_{q_t}}h, \quad \text{where} \quad \Omega_{q_t} = \left\{ x \in q_t : \rho(\text{ad} x) < \frac{\pi}{2} \right\}. \]
Here we have used the polar decomposition

\[ \text{Inn}_g(h) = \text{Inn}_g(h_t)e^{ad h_p} \quad \text{and} \quad \text{Inn}_g(h_t)\Omega_{q_t} = \Omega_{q_t} \]

so see that \( \text{Inn}_g(h_t)e^{ad \Omega_{q_t} h} = e^{ad \Omega_{q_t} h} \).

**Remark 6.2.** (The connection with Matsuki’s domains) To see that the inverse image of our domain \( C(O^q_{\mathfrak{g}}) \) in \( G \) is one of the domains considered by Matsuki in [Ma03, §1.2], let us recall his setting: On \( G \) we consider a Cartan involution \( \theta \) with \( \theta(h) = -h \) and the involution \( \tau = \tau_h \theta \) and consider the the connected groups

\[ H := G^\tau_e \quad \text{and} \quad H' := G^{	au_{r\theta}}_e = G^{\tau_{h}}_e. \]

Their Lie algebras are

\[ \mathfrak{h} = \mathfrak{h}_t \oplus \mathfrak{h}_p \quad \text{and} \quad \mathfrak{h}' = \mathfrak{h}_t \oplus \mathfrak{q}_p = \ker(\text{ad} \mathfrak{h}). \]

We choose a maximal abelian subspace \( t_q \subseteq \mathfrak{q}_t \). As \( \theta \) fixes \( t_q \), it leaves the corresponding weight spaces \( \mathfrak{g}_C^q \) invariant, so that \( \mathfrak{g}_C^q = \mathfrak{t}^q \oplus \mathfrak{p}_C^q \), and Matsuki considers the subset

\[ \Sigma(p_C, t) := \{ \alpha \in \Sigma(p_C, t) : p_C^\alpha \neq \{0\} \} \]

and the domain

\[ t^+ := \left\{ y \in t : (\forall \alpha \in \Sigma(p_C, t)) |\alpha(y)| < \frac{\pi}{2} \right\} \supseteq \left\{ y \in t : \rho(\text{ad} y) < \frac{\pi}{2} \right\} \quad (26) \]

which in turn determines the domain

\[ \mathcal{M}(H, H') := H \exp(t^+)H' \subseteq G. \]

The discussion of the following two examples, combined with Lemma [5.1] now implies that we have equality in (26).

(a) For \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \) and \( h = \frac{1}{2} \text{diag}(1, -1) \) we have \( \theta(x) = -x^\top \),

\[ \tau_h \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix} \quad \text{and} \quad \tau \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = -\begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}^\top = \begin{pmatrix} -a & c \\ b & a \end{pmatrix}. \]

So \( \mathfrak{t} = \mathfrak{so}_2(\mathbb{R}) = \mathfrak{q}_t \subseteq \mathfrak{q} \) and \( \mathfrak{t} = \mathfrak{q}_t = \mathfrak{so}_2(\mathbb{R}) \). We conclude that \( \Sigma(\mathfrak{g}_C, t) = \Sigma(p_C, t) \).

(b) For \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \supseteq \mathfrak{h} = \mathfrak{su}_{1,1}(\mathbb{C}) \), we have \( \theta(x) = -x^* \),

\[ \mathfrak{h} = \left\{ \begin{pmatrix} ia & b \\ b & -ia \end{pmatrix} : a \in \mathbb{R}, b \in \mathbb{C} \right\}, \quad \mathfrak{q} = i \mathfrak{su}_{1,1}(\mathbb{C}) \]

and

\[ \mathfrak{t} = \mathfrak{su}_2(\mathbb{C}), \quad \mathfrak{q}_t = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbb{C} \right\} \supseteq \mathfrak{t} = \mathfrak{so}_2(\mathbb{R}). \]

In this case \( \Sigma(\mathfrak{g}_C, t) = \{ \pm \alpha \} \) with \( \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2i, \quad \dim_C(\mathfrak{g}_C^q) = 2 \), and both roots occur in \( p_C \) for \( p = i \mathfrak{su}_2(\mathbb{C}) \).
Example 6.3. We consider the hermitian simple Lie algebra \( g = \mathfrak{su}_{p,q}(\mathbb{C}) \). It is of tube type if and only if \( p = q \) (cf. Section 3.1). We assume that \( p \leq q \), so that \( r := \text{rk}_g(g) = p \). Then

\[
\begin{pmatrix}
q_{1p} & 0 \\
0 & -p_{1q}
\end{pmatrix}
\]

is an H-element, the subspace of diagonal matrices \( t \) is a compactly embedded Cartan subalgebra with

\[
\Sigma(g,t) = \{ \varepsilon_j - \varepsilon_k: j \neq k, j, k = 1, \ldots, p + q \} \cong A_{p+q-1}.
\]

We consider the adapted positive system

\[
\Sigma^+ = \{ \varepsilon_j - \varepsilon_k: 1 \leq j < k \leq p + q \}
\]

with

\[
\Sigma_1 = \{ \varepsilon_j - \varepsilon_k: j \leq p < k \} = \{ \alpha \in \Sigma: -i\alpha(z) > 0 \}.
\]

A set of strongly orthogonal roots in \( \Sigma_1 \) is

\[
\Gamma = \{ \gamma_j: j = 1, \ldots, p \}, \quad \gamma_j := \varepsilon_j - \varepsilon_{p+j}.
\]

In \( g(\gamma) \cong \mathfrak{su}_{1,1}(\mathbb{C}) \), we have the H-element

\[
z^j = \frac{i}{2}(E_{jj} - E_{j+p,j+p}).
\]

We then obtain

\[
z = z^0 + \sum_{j=1}^p z^j \quad \text{with} \quad z^0 = \frac{i}{p+q} \begin{pmatrix}
\frac{q-p}{2} & 0 \\
0 & -p_{1q-p}
\end{pmatrix}.
\]

Note that \( z^0 = 0 \) if and only if \( p = q \), i.e., if \( g \) is of tube type.

With the notation \( x = \text{diag}(x_1, \ldots, x_{p+q}) \), we then have

\[
C^\max_t = \{-ix \in t: (\forall \alpha \in \Sigma_1) i\alpha(-ix) = \alpha(x) \geq 0 \}
\]

\[
= \{-ix: x_j > x_k \text{ for } j \leq p, k > p \}
\]

and

\[
C^\min_t = \text{cone}\{i\alpha^\vee: \alpha \in \Sigma_1 \} = \text{cone}\{-i(E_{jj} - E_{kk}): j \leq p < k \}.
\]

Considering extreme points of a basis of this cone, we see that

\[
C^\min_t = \text{cone}\{ -ix: x_j \geq 0, x_k \leq 0, j \leq p < k; \sum_j x_j = 0 \}.
\]

If \( p < q \), then the entry in position \( p + 1 \) of \( z^0 \) is negative, so that \( z^0 \notin C^\min_t \).

The preceding example shows in particular that the following lemma does not hold for the minimal cone \( C^\min_q \); see also Example 6.8 below.

Lemma 6.4. Let \((g,\tau)\) be irreducible ncc, \( h \in C^\max_q \) a causal Euler element and write \( h_0 \) for its central component in the subalgebra \( l \) generated by \( h \) and \( t_q \) (Proposition 5.2). Then \( h_0 \in C^\max_q \).
Proof. (a) We start with a discussion concerning hermitian Lie algebras, which corresponds to the “complex case”. Let \( g \) be simple hermitian and isomorphic to the reductive subalgebra generated by an H-element \( z \in \mathfrak{z}(\mathfrak{t}) \) and the subalgebras \( g(\gamma_j) \) associated to a set \( \{ \gamma_1, \ldots, \gamma_r \} \) of strongly orthogonal positive non-compact roots in \( \Delta(g_C, t_C) \), where \( t \subseteq \mathfrak{t} \) is a Cartan subalgebra and

\[
\Delta^+_p = \{ \alpha \in \Delta : i \alpha(z) = 1 \}.
\]

Then

\[
z = z^0 + \sum_{j=1}^r z^j, \quad z^j = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})
\]

and

\[
t_t = \text{span}\{z^j : j = 0, \ldots, r\} \subseteq t \cap I
\]

is a compactly embedded Cartan subalgebra of \( I \). We then have

\[
z^j \in C^\min_g \quad \text{for} \quad j > 0,
\]

(see Section 3.5.1 for \( C^\min_g \)) but in general \( z^0 \notin C^\min_g \) (Example 6.3). We claim that

\[
z^0 \in C^\max_{t_I} := C^\max_g \cap t = \{ x \in t : (\forall \alpha \in \Delta^+_p) \ i \alpha(x) \geq 0 \}.
\]

From [3] in Proposition 3.3 it follows that, for \( \alpha \in \Delta^+_p \), we have

\[
i \alpha(z^j) \in \left\{ \frac{1}{2}, 1 \right\} \quad \text{and} \quad \sum_{j=1}^r i \alpha(z^j) \in \left\{ \frac{1}{2}, 1 \right\}.
\]

From \( i \alpha(z) = 1 \) it now follows that

\[
i \alpha(z^0) \in \left\{ 0, \frac{1}{2} \right\}, \quad \text{so that} \quad z^0 \in C^\max_{t_I}.
\]

(b) (The real situation) Now we prove the lemma. Let \( (g, \tau) \) be simple ncc not of complex type; the complex type is covered by (a). Then \( g^c \) is simple hermitian and

\[
g = h \oplus q \subseteq g_C = g^c \oplus ig^c.
\]

We then have maximal invariant cones

\[
C^\max_q \subseteq q \quad \text{and} \quad C^\max_{ig^c} \subseteq ig^c,
\]

and [Ol91, Lemma 7.10] (see also [KN96, Thm. VIII.1]) implies that

\[
C^\max_q = q \cap C^\max_{ig^c}.
\]

As the Euler element \( h \in C^\max_q \cap q_p \) also is an Euler element in \( it^c \subseteq ig^c \). The central component \( h_0 \in \mathfrak{z}(I) \) coincides with the central component of \( h \), considered as an Euler element in \( ig^c \) (as in (a)). Therefore (a) implies that

\[
h_0 \in q \cap C^\max_{ig^c} = C^\max_q.
\]
We shall need the following lemma ([N ´O22b, Lemma 4.8]).

**Lemma 6.5.** Let \((G, \tau^G)\) be a connected reductive Lie group. If \(D \subseteq T C_q = h + C_q^0 \subseteq q\) is compact and \(H \subseteq G^*\) an open subgroup with \(\text{Ad}(H)C_q = C_q\), then \(\text{Ad}(H)D\) is closed in \(g\). Moreover, there exists a smooth \(\text{Ad}(H)\)-invariant function \(\psi : T C_q \to (0, \infty)\) such that \(z_n \to z_0 \in \partial T C_q\) for \(z_n \in T C_q\) implies \(\psi(z_n) \to \infty\).

The following theorem has a twin in [N ´O22b, Thm. 4.10], where we consider only the modular case, where \(h\) contains an Euler element, and this stronger assumption permits us to work with a general cone \(C_q\). Here we work with general ncc spaces, where \(h\) may not contain an Euler element, but in this case we have to assume that \(C_q\) is maximal.

**Theorem 6.6.** (Crown Theorem for causal Euler elements) Let \((g, \tau)\) be a semisimple ncc symmetric Lie algebra without \(\tau\)-invariant Riemannian ideals and \(h \in q\) a causal Euler element. Let

\[ H = \text{Inn}_q(h) \subseteq G = \text{Inn}(g) \quad \text{and} \quad C := C_q^{\text{max}} \subseteq q \]

be the maximal pointed generating \(\text{Ad}(H)\)-invariant cone containing \(h\). Then the connected component of \(h\) in the open subset \(O_h \cap T C\) of \(O_h\) is the Matsuki crown \(C(O_h^h) = \text{Ad}(H)e^{\text{ad} \Omega h}\).

In [MNO22a, §7] we show that \(O_h \cap T C_q^{\text{max}}\) is actually connected.

**Proof.** Our assumptions on \((g, \tau)\) imply that all simple \(\tau\)-invariant ideals are irreducible ncc and that \(C\) is the product of the maximal \(\text{Ad}(H)\)-invariant cones in the irreducible factors. Therefore both sets whose equality is to be shown are adapted to the decomposition into irreducible factors and we may therefore assume that \((g, \tau)\) is irreducible and that \(G = \text{Inn}(g)\) has trivial center (cf. [N ´O22b, Prop. 2.14]). Then \(K = G^h\) is compact and the polar decomposition of \(H\) implies that \(H = H_K e^{\text{ad} h}\) with \(H_K = H^h\) (Corollary 1.16).

We will derive the theorem from the following three claims:

(a) \(C(O_h^h)\) is connected and open in \(O_h\).

(b) \(C(O_h^h) \subseteq T C\).

(c) \(C(O_h^h)\) is relatively closed in \(O_h \cap T C\).

As \(C(O_h^h)\) is connected by definition (a), (b) and (c) imply that it is the connected component of \(h\) in the open subset \(O_h \cap T C\).

(a) To see that \(C(O_h^h)\) is connected, we use the polar decomposition \(H = H_K \exp(h_p)\). Then \(\text{Ad}(H_K)\Omega h_t = \Omega h_t\) and \(\text{Ad}(H_K)h = \{h\}\) (cf. Lemma 4.11). This implies that

\[ C(O_h^h) = \text{Ad}(H)e^{\text{ad} \Omega h} = e^{\text{ad} h} e^{\text{ad} \Omega h} \text{Ad}(H_K)h = e^{\text{ad} h} e^{\text{ad} \Omega h} h, \]

which is obviously connected.

Next we use [N ´O22b, Lemma C.3(a)] to see that the exponential map

\[ \text{Exp} : q_t \to O_h, \quad x \mapsto e^{\text{ad} x h} \]
is regular in $x \in \Omega_{q_t}$ because $\rho(\text{ad} \, x) < \pi/2 < \pi$. Now [NÔ22b, Lemma C.3(b)] implies that the map

$$\Phi: H \times \Omega_{q_t} \to \mathcal{O}_h, \quad (g, x) \mapsto \text{Ad}(g)e^{\text{ad}x}h$$

is regular in $(g, x)$ because $\text{Spec}(\text{ad} \, x) \subseteq (-\pi/2, \pi/2)i$ does not intersect $(\pi/2 + Z\pi)i$. This implies that the differential of $\Phi$ is surjective in each point of $H \times \Omega_{q_t}$; hence its image is open.

(b) We observe that both sides of (b) are $\text{Ad}(H)$-invariant, and

$$C(\mathcal{O}_h^d) = \text{Ad}(H)e^{\text{ad}\Omega_t}h = \text{Ad}(H)e^{\text{ad}\Omega_t}h$$

for a maximal abelian subspace $t_q \subseteq q_t$ and $\Omega_t := \Omega_{q_t} \cap t_q$. Here we use that $H_K$ fixes $h$ and $\text{Ad}(H_K)t_q = q_t$. Therefore it suffices to show $e^{\text{ad} \, x}h \in T_C$ for

$$x \in \Omega_{t_q} = \{ y \in t_q : \rho(\text{ad} \, y) < \frac{\pi}{2} \}.$$ 

From Proposition 5.2 we infer that the causal Euler element $h$ and $t_q$ generate a $\tau$-invariant reductive subalgebra $l$ with

$$[l, l] \cong \mathfrak{sl}_2(\mathbb{R})^s \quad \text{and} \quad l^\tau \cong \mathfrak{so}_{1,1}(\mathbb{R})^s$$

in which $h$ is a causal Euler element, contained in the interior of the pointed generating cone $C_l := C \cap l \subseteq l^\tau$ which is invariant under $\text{Inn}(\Gamma)$. Let $s := r_0 + r_1$.

For elements of $\mathfrak{sl}_2(\mathbb{R})$, we recall the notation from (38) and (39) above:

$$h^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (27)$$

For a suitable isomorphism $[l, l] \to \mathfrak{sl}_2(\mathbb{R})^{r_0} \oplus \mathfrak{sl}_2(\mathbb{C})^{r_1}$, we obtain

$$h = h_0 + \sum_{j=1}^s h_j \quad \text{and} \quad t_q = \mathfrak{so}_2(\mathbb{R})^s,$$

where we write $h_j$ for the Euler elements $h^0$ in the $j$th summand. For

$$x = \sum_{j=1}^s x_j \frac{e_j - f_j}{2} \in t_q,$$

we then obtain with (11)

$$p_q(e^{\text{ad} \, x}h) = \cosh(\text{ad} \, x)(h) = h_0 + \sum_{j=1}^s \cos(x_j)h_j. \quad (28)$$

Since $\rho(\text{ad} \, x) < \frac{\pi}{2}$ for $x \in \Omega_{q_t}$ and $ix_j \in \text{Spec}(\text{ad} \, x)$, we have $|x_j| < \frac{\pi}{2}$ for $j = 1, \ldots, s$, hence $\cos(x_j) \in (0, 1]$. As $h_0 \in C_l$ (Lemma 5.3) and $h_j \in C_l$ for $j = 1, \ldots, s$, we have

$$p_q(e^{\text{ad} \, x}h) \in C^\circ_{a_s} \quad \text{for} \quad C_{a_s} := C \cap a_s, \quad a_s := \text{span}\{h_j : j = 0, \ldots, s\}.$$
Now \( h \in C^0 \cap a_s \) implies \( C^0_{\alpha_s} = C^0 \cap a_s \subseteq C_1 \) (Lemma B.1), so that
\[
p_h(e^{\text{ad}x}h) \in C^0.
\]
This proves (b).

(c) We have to show that \( C(O^q_h) \) is relatively closed in \( O_h \cap T_C \). If \( D \subseteq \Omega_q \) is compact, then \( \text{Ad}(H)e^{\text{ad}D}h \) is closed in \( g \) by Lemma 6.5, and by (a) it is contained in \( T_C \).

Now suppose that the sequence \( \text{Ad}(h_n)e^{\text{ad}x_nh} \), \( h_n \in H, \ x_n \in \Omega_q \), converges to some element in \( T_C \) which is not contained in \( C(O^q_h) \). As we may assume that the bounded sequence \( x_n \in \Omega_q \) converges in \( t_q \), it converges by the preceding paragraph to a boundary point \( y \in \partial \Omega_q \). Writing
\[
y = \sum_{j=1}^{s} y_j e_j - f_j/2,
\]
we claim that there exists a \( j \) with \( |y_j| = \pi/2 \). As \( \rho(\text{ad}y||i) = \max\{|y_j| : j = 1, \ldots, s\} \), this follows from \( \rho(\text{ad}y||i) = \rho(\text{ad}y) \) (Lemma 5.1). Now (28) and
\[
C \cap \text{span}\{h_0, h_1, \ldots, h_s\} \subseteq \mathbb{R}h_0 + \sum_{j=1}^{s} [0, \infty)h_j.
\]
implies that
\[
e^{\text{ad}y}h \in \partial T_{C^{\text{max}}_q}.
\]
For the \( H \)-invariant function \( \psi \) from Lemma 6.5 this leads to
\[
\psi(\text{Ad}(h_n)e^{\text{ad}x_nh}) = \psi(e^{\text{ad}x_nh}) \to \infty,
\]
contradicting the convergences of the sequence \( \text{Ad}(h_n)e^{\text{ad}x_nh} \) in \( T_{C^{\text{max}}_q} \).

Remark 6.7. In the special case where \((g, \tau)\) is Riemannian, \( q = q_p \) and \( h = h_k \), the real crown domain \( C(O^q_h) \) reduces to a point. Hence there is no interesting analog of the preceding theorem in the Riemannian case, and we therefore assume that \((g, \tau)\) contains no Riemannian summands.

Example 6.8. We consider the Lie algebra \( g = \mathfrak{su}_{2,1}(\mathbb{C}) \) which is hermitian, but not of tube type. In particular \((g_C, \tau)\), \( \tau(z) = \overline{z} \), is ncc of complex type, but \( g \) contains no Euler element. Concretely, we have
\[
\mathfrak{su}_{2,1}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ b^* & -\text{tr}a \end{pmatrix} : a \in \mathfrak{u}_2(\mathbb{C}), b \in \mathbb{C}^2 \right\}.
\]
For the Cartan involution \( \theta(x) = -x^* \), we obtain
\[
\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -\text{tr}a \end{pmatrix} : a \in \mathfrak{u}_2(\mathbb{C}) \right\} \cong \mathfrak{u}_2(\mathbb{C})
\]
with
\[
\mathfrak{z}(\mathfrak{k}) = \mathbb{R}ih_c, \quad h_c = \frac{1}{3}\text{diag}(1, 1, -2).
\]
The subspace $t \subseteq \mathfrak{t}$ of diagonal matrices is a compactly embedded Cartan subalgebra of $\mathfrak{g}$. In $\Delta = \Delta(\mathfrak{g}_C, t_C)$ we have

$$\Delta_k = \{ \pm(\varepsilon_1 - \varepsilon_2) \}, \quad \Delta^+_p = \{ \alpha \in \Delta_p : \alpha(h_c) = 1 \} = \{ \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3 \}$$

and $\Gamma := \{ \varepsilon_1 - \varepsilon_3 \}$ is a maximal system of strongly orthogonal roots. In $\mathfrak{p}$ the set $\Gamma$ leads to the maximal abelian subspace

$$a = \mathbb{R}h \quad \text{with} \quad h := \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which generates with $a$ the Lie algebra

$$\mathfrak{s} = \mathfrak{t} + [\mathfrak{s}, \mathfrak{s}] \cong \mathfrak{gl}_3(\mathbb{R}) \quad \text{with} \quad [\mathfrak{s}, \mathfrak{s}] = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ b & 0 & -a \end{pmatrix} : a \in i\mathbb{R}, b \in \mathbb{C} \right\} \cong \mathfrak{su}_{1,1}(\mathbb{C})$$

In $\mathfrak{s}$ the element $ih_c$ decomposes into $ih^0_c + ih^1_c$, where $ih^0_c$ is central and $ih^1_c \in [\mathfrak{s}, \mathfrak{s}]$. Concretely, we have

$$h^0_c = \frac{1}{6} \text{diag}(-1, 2, -1), \quad h^1_c = \frac{1}{3} \text{diag}(1, 0, -1).$$

In $i\mathfrak{t}$ we have the cone

$$C^{\text{max}} := \{ \text{diag}(x_1, x_2, -x_1 - x_2) : 2x_1 + x_2 \geq 0, 2x_2 + x_1 \geq 0 \} \supseteq C^{\text{min}} = \text{cone}((\text{diag}(1, 0, -1), \text{diag}(0, 1, -1)) \right.$$

We have

$$p_0(e^{\frac{\pi}{2} \text{ad} h_c}) = \cos \left( \frac{\pi}{2} \text{ad} h \right) h_c = h^0_c \in \partial C^{\text{max}} \setminus C^{\text{min}}.$$  

Therefore the real crown domain in $\text{Ad}(G_C)h_c$ is not contained in the tube domain $\mathfrak{g} + iC^{\text{min}} \subseteq \mathfrak{g}_C$.

7. Components of the stabilizer group $G^h$

This section is devoted to an analysis of the group $\pi_0(G^h)$ of connected components of the centralizer $G^h$ of an Euler element $h$ in a simple real Lie algebra. With a polar decomposition $G^h = K^h \exp(\mathfrak{h}_p)$, this group equals $\pi_0(K^h)$. As $K/K^h \cong \mathcal{O}^K_h := \text{Ad}(K)h$ is a compact symmetric space, we discuss this problem in Section 7.1 in the context of compact symmetric spaces, where $\pi_0(K^h)$ appears as a quotient of $\pi_1(K/K^h)$ in the long exact homotopy sequence

$$\pi_1(K) \to \pi_1(K/K^h) \to \pi_0(K^h) \to \pi_0(K) = 1.$$  

In Section 7.2 we explore this situation further, using that $\mathcal{O}^K_h$ actually is a symmetric $R$-space. Here the strongly orthogonal roots come in handy and permit us in Theorem 7.8 to show that $G^h$ is connected if $(\mathfrak{g}, \tau)$ is either of complex type or non-split type (cf. Section 4.3), and if it is of split type or Cayley type, then it either
is trivial or \( \mathbb{Z}_2 \). In Section 7.3 we finally collect some consequences of this result such as the identity
\[
\text{Inn}_{\mathfrak{g}_c}(\mathfrak{g}^c) \cap \text{Inn}_{\mathfrak{g}_c}(\mathfrak{g}) = \text{Inn}_{\mathfrak{g}_c}(\mathfrak{t})^h e^{ad hp}.
\]

7.1. The fundamental group of a compact symmetric space

Let \( G \) be a connected symmetric Lie group with compact Lie algebra \( \mathfrak{g} \), \( \tau \) an involutive automorphism of \( G \), \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \) the corresponding eigenspace decomposition, \( H \subseteq G^\tau \) an open subgroup, \( X = G/H \) the corresponding compact symmetric space and \( q_X : G \to X \) the quotient map with the base point \( e_X := q(e) \). We pick a maximal abelian subspace \( \mathfrak{t}_q \subseteq \mathfrak{q} \) and enlarge it to a Cartan subalgebra \( \mathfrak{t} \subseteq \mathfrak{g} \), so that \( \mathfrak{t} \) is \( \tau \)-invariant and \( \mathfrak{t} = \mathfrak{t}_q \oplus \mathfrak{t}_q^\perp \).

We write \( q_G : \tilde{G} \to G \) for the simply connected covering. For a subgroup \( B \subseteq G \), we write \( B^* := q_G^{-1}(B) \) for its inverse image in \( \tilde{G} \). Then \( X = G/H \cong \tilde{G}/H^* \), which leads to an isomorphism \( \tilde{X} \cong \tilde{G}/H^*_e \) and thus to \( \pi_1(X) \cong \pi_0(H^*) \) ([HN12 Cor. 11.1.14]).

In \( G \) we have the tori
\[
T_Q := \exp(\mathfrak{t}_q) \subseteq T := \exp(\mathfrak{t}).
\]
Now \( T_X := \text{Exp}(\mathfrak{t}_q) = q_X(T_Q) \subseteq X \) is a maximal flat torus whose fundamental group is the lattice
\[
\Gamma_X := \{ x \in \mathfrak{t}_q : \text{Exp}(x) = e_X \}.
\]

We likewise have \( T_{\tilde{X}} \subseteq \tilde{X} \) and a discrete subgroup \( \Gamma_{\tilde{X}} \subseteq \Gamma_X \). On the group level we likewise define
\[
\Gamma_G := \{ x \in \mathfrak{t} : \exp_G x = e \} \cong \pi_1(T) \quad \text{and} \quad \Gamma_{\tilde{G}} := \{ x \in \mathfrak{t} : \exp_{\tilde{G}} x = e \} \cong \pi_1(T^*).
\]

In the following lemma note that \( Z(G) = \{ e \} \) if and only if \( G = \text{Inn}(\mathfrak{g}) \).

**Lemma 7.1.** The following assertions hold:

(a) The inclusion \( T \hookrightarrow G \) induces an isomorphism \( \Gamma_G/\Gamma_{\tilde{G}} \to \pi_1(G) \).

(b) If \( Z(G) = \{ e \} \), then \( \pi_1(G) \cong Z(\tilde{G}) \) and \( \Gamma_G = \{ x \in \mathfrak{t} : e^{ad x} = 1 \} \).

(c) \( G = \exp_{\tilde{G}} \mathfrak{h} \exp_{\tilde{G}} \mathfrak{q} \).

(d) \( H^* = H^*_e \exp(\{ x \in \mathfrak{t}_q : e^{ad x} = 1 \}) \subseteq H^*_e (H^* \cap T_Q) \).

(e) \( H = H_e (H \cap T_Q) \).

**Proof.**

(a) follows from [HN12 Cor. 14.2.10] (see also [He78 Thm. VII.6.7]).

(b) If \( Z(G) = \{ e \} \), then \( G \) is semisimple and \( Z(\tilde{G}) = \ker q_G \cong \pi_1(G) \).

(c) First we assume that \( G \) is semisimple, hence compact. As a closed subgroup of \( G \), the group \( H \) has only finitely many connected components and \( \tilde{X} := G/H_e \) is still compact. The surjectivity of the Riemannian exponential function of \( \tilde{X} \) ([He78 Thm. I.10.3.4/4]) implies \( \tilde{X} = \text{Exp}(\mathfrak{q}) \), and therefore \( G = \exp(\mathfrak{q})H_e = \exp \mathfrak{q} \exp \mathfrak{h} \).

Here we use that the exponential function of the compact group \( H_e \) is surjective.
Now we turn to the general case. As the commutator group $G'$ with the semisimple Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is compact, the preceding argument yields

$$G' = \exp(\mathfrak{q} \cap \mathfrak{g}') \exp(\mathfrak{h} \cap \mathfrak{g}').$$

For the, possibly non-compact, group $G$, we now obtain

$$G = Z(G) e_{G'} = \exp(\mathfrak{h} \cap \mathfrak{j}(\mathfrak{g})) \exp(\mathfrak{q} \cap \mathfrak{j}(\mathfrak{g})) \exp(\mathfrak{q} \cap \mathfrak{g}') \exp(\mathfrak{h} \cap \mathfrak{g}') = \exp(\mathfrak{q}) \exp(\mathfrak{h}).$$

(d) (cf. [He78, Thm. VII.9.1]) If $h \in H^*$, then $q_G(h) \in G^0$ implies that $z := h\theta(h)^{-1} \in \ker(q_G) \subseteq Z(\tilde{G})$. In view of (c), there exists an $x \in \mathfrak{q}$ and $g \in H^*_e = \overset{\sim}{G}^*$ with $h = \exp(x)g$, so that

$$z = h\theta(h)^{-1} = \exp(2x) \in Z(\tilde{G}).$$

Now we pick $h_1 \in \overset{\sim}{G}^*$ with $x_1 := \Ad(h_1)x \in \mathfrak{t}_q$ ([KN96, Thm. II.8]) and find

$$z = h_1zh^{-1} = \exp(2\Ad(h_1)x) = \exp(2x_1).$$

Then $1 = \Ad(z) = e^{2\ad x_1}$, and (d) follows.

(e) follows immediately from (d) by applying $q_G$. ⊓⊔

**Proposition 7.2.** The homomorphism $\iota : \pi_1(T_X) \cong \Gamma_X \to \pi_1(X)$ induced by the inclusion $T_X \hookrightarrow X$ is surjective and induces an isomorphism

$$\Gamma_X / \Gamma_{\overset{\sim}{X}} \to \pi_1(X).$$

**Proof.** First we show that $\iota$ is surjective. We shall use the exact sequence

$$\pi_1(G) \to \pi_1(X) \to \pi_0(H),$$

which is part of the long exact homotopy sequence of the principal $H$-bundle $G \to X$. Every element in $\pi_0(H)$ can be represented by an element $g = \exp x$ in $T_Q \cap H$ (Lemma 7.1(e)), and then $x \in \Gamma_X$. This shows that $\delta \circ \iota$ is surjective.

It remains to see that $\ker \delta \subseteq \ker \iota$. So let $[\gamma] \in \ker \delta$. By (30), $[\gamma]$ comes from a loop in $G$, hence from a loop in $T$ (Lemma 7.1(a)). We may therefore assume that $\gamma(t) = \exp(tx)H$ with $x \in \mathfrak{t}$ satisfying $\exp(x) = e$. Writing $x = x_h + x_q$ with $x_h \in \mathfrak{t}_h$ and $x_q \in \mathfrak{t}_q$, we then have

$$\gamma(t) = \exp(tx_q)H = \Exp(tx_q) \quad \text{with} \quad \Exp(x_q) = \exp(x_q)H = e_X.$$}

This implies $x_q \in \Gamma_X$ and shows that $\iota$ is surjective.

That $\ker \iota$ is $\Gamma_{\overset{\sim}{X}}$ follows immediately from the fact that, for $x \in \Gamma_X$, the curve $\gamma_x(t) := \Exp_X(tx)$ lifts to a loop in $\overset{\sim}{X}$ if and only if $\Exp_{\overset{\sim}{X}}(x) = e_{\overset{\sim}{X}}$, which means that $x \in \Gamma_{\overset{\sim}{X}}$. ⊓⊔

**Remark 7.3.** [He78, Thm. VII.8.5] also provides a description of the discrete subgroup $\Gamma_{\overset{\sim}{X}} \subseteq \mathfrak{t}_q$ in terms of root data. We have

$$\Gamma_{\overset{\sim}{X}} = \frac{1}{2} \Gamma_{T_Q} \quad \text{for} \quad \Gamma_{T_Q} := \{x \in \mathfrak{t}_q : \exp_G(x) = e\}. $$

(31)
Let $G$ be the quotient group. Then defined by an Euler element in $h$.

Remark 7.5. (a) If $h \in p$ is an Euler element and $\tau_h$ is the corresponding involution, then $p$ decomposes into eigenspaces $p^h$ and $p^{-h}$ and the $K$-orbit $O^K_h = \text{Ad}(K)h \subseteq p$ is $\tau_h$-invariant. Moreover, $\tau_h$ induces on this orbit an involution turning it into a so-called extrinsic symmetric space, for the definition see [Fe80].

(b) The compact symmetric spaces of the form $O^K_h$ are precisely the symmetric R-spaces, when considered as homogeneous spaces of $K$ ([Lo85]). We refer to [MNO22a] for their interpretation as real flag manifolds $G/P^-$ on which $K$ acts transitively with point stabilizer $K \cap P^- = K^h$.

Let $G = \text{Inn}(g), K = \text{Inn}_q(\mathfrak{k})$, and consider the symmetric R-space

$$X := \text{Ad}(K)h \subseteq p$$

defined by an Euler element in $h \in p$. We want to determine the group $\pi_0(G^h) \cong \pi_0(K^h)$ of connected components of $G^h$, resp., $K^h$.

We consider the torus $T_X := e^{ad x}h \subseteq X$ with the fundamental group

$$\Gamma_X = \{x \in t_q : e^{ad x}h = h\}.$$}

We shall need the following piece of the long exact homotopy sequence of the $K^h$-principal bundle $K \to X$:

$$\pi_2(X) \to \pi_1(K^h) \to \pi_1(K) \to \pi_1(X) \to \pi_0(K^h) \to 1,$$
where \( q : K \to X \cong K/K^h \) is the orbit map. Then the surjectivity of the homomorphism \( \pi_1(X) \to \pi_0(K^h) \) implies that

\[
\pi_0(K^h) \cong \coker(\pi_1(q)).
\]

With Proposition 7.2 we see that \( \pi_1(X) \cong \Gamma_X/\Gamma_X^q \), so that we obtain a surjective map

\[
\delta : \Gamma_X \to \pi_0(K^h), \quad x \mapsto [e^{ad x}].
\]

We shall now determine the range of this map with the aid of Proposition 5.2, where we have seen that \( h \) and \( t_q \) generate a reductive Lie algebra

\[
l = \mathbb{R}h_0 \oplus \mathfrak{sl}_2(\mathbb{R})^s \supseteq t_q = \mathfrak{so}_2(\mathbb{R})^s, \quad s = r_0 + r_1.
\]

**Lemma 7.6.** The following assertions hold:

(a) The torus \( T_Q := \exp(t_q) \) satisfies \( T_Q^h = \exp(\Gamma_X) = Z(L') \), where \( L' \subseteq L \) denotes the commutator subgroup.

(b) Write \( L = Z(L_1)eL_1 \cdots L_s \), where \( L_j \) is the integral subgroup corresponding to the \( j \)-th simple ideal \( I_j \) in \( I \). Then \( Z(L) = Z(L_1)eZ(L_1) \cdots Z(L_s) \), where the group \( Z(L_j) \) is either trivial or contains two elements. In particular we have

\[
T_Q^h = Z(L_1) \cdots Z(L_s).
\]

(c) The subgroup \( Z(L_1) \cdots Z(L_{r_0}) \subseteq T_Q^h \) maps surjectively onto \( \pi_0(K^h) \).

**Proof.** (a) In \( \mathfrak{sl}_2(\mathbb{R}) \) and \( \mathfrak{sl}_2(\mathbb{C}) \) any element \( x \in \mathfrak{so}_2(\mathbb{R}) \) satisfying \( e^{ad x}h = h \) for an Euler element \( h \) also satisfies \( e^{ad x} = 1 \). Therefore \( T_Q^h \subseteq Z(L') \). The converse inclusion follows from \( Z(L') \subseteq \exp(t_q) = T_Q ^{c} \), which implies that \( Z(L') \subseteq T_Q^h \).

(b) As the center \( Z(L_j) \) commutes for each \( j \) with all of \( L \), we have \( Z(L) = Z(L_1)eZ(L_1) \cdots Z(L_s) \). The subgroup \( Z(L_j) \) is the image of the center of a group of the type \( SL_2(\mathbb{R}) \), hence contains at most two elements. Now \( Z(L_1) \cdots Z(L_s) = Z(L') \), combined with (a), implies (b).

(c) For \( j > r_0 \), the ideal \( I_j \) is contained as \( \mathfrak{sl}_2(\mathbb{R}) \) in a complex \( \tau \)-invariant Lie algebra \( \mathfrak{g}_j \cong \mathfrak{sl}_2(\mathbb{C}) \). If \( S_j := \{ \exp s_j \} \), then \( K_j := K \cap S_j \cong SU_2(\mathbb{C}) \) or \( PSU_2(\mathbb{C}) \) and \( \text{Ad}(K_j)h \) is a 2-sphere, hence simply connected. Therefore \( K_j^h \) is connected and therefore \( Z(L_j) \) does not contribute to \( \pi_0(K^h) \). This implies (c).

**Remark 7.7.** (a) The fact that \( h \in I \) is an Euler element of \( g \) restricts the possibilities for irreducible \( I \)-submodules of \( g \) significantly. Writing \( V_j^k \) for the \( k \)-dimensional simple complex module of the \( j \)-th ideal \( I_j \), we see that the only non-trivial simple \( I \)-submodules of \( g \) are the adjoint modules \( V_1^1 \), tensor products \( V_{j_1}^{j_1} \otimes V_{j_2}^{j_2} \) for \( j_1 \neq j_2 \), and, if \( h_0 \neq 0 \), also 2-dimensional modules \( V_2^j \) on which \( h_0 \) acts by \( \pm \frac{1}{2}I \). On the other two types the central element \( h_0 \) acts trivially. Let \( z_j \in Z(L_j) \) be generators. Then \( z_j \) acts trivially on \( V_{3j_1}^{j_1} \) and \( V_{j_1}^{j_1} \otimes V_{j_2}^{j_2} \), but non-trivially on \( V_{3j_1}^{j_1} \otimes V_{3j_2}^{j_2} \) for \( j_3 \neq j_1, j_2 \).

(b) Write \( h \in I \) as \( h = h_0 + h_1 \) with \( 0 \neq h_0 \in \mathfrak{z}(I) \) and \( h_1 \in [I, I] \). Then the structure of the restricted root system implies that \( h_1 \) defines a 5-grading of \( g \):

\[
g = g_{-1}(h_1) \oplus g_{-1/2}(h_1) \oplus g_0(h_1) \oplus g_{1/2}(h_1) \oplus g_1(h_1),
\]
where $h_0$ commutes with $\mathfrak{g}_{\pm 1}(h_1)$. Therefore

$$\mathfrak{g}_{\text{ut}} := \mathfrak{z}_g(h_0) = \mathfrak{g}_{-1}(h_1) \oplus (\mathfrak{g}_0(h_0) \cap \mathfrak{g}_0(h_1)) \oplus \mathfrak{g}_1(h_1) \subseteq \mathfrak{g}$$

is a 3-graded $\tau$-invariant subalgebra in which $h_1$ is an Euler element. Let $r'_0, r'_1$ and $s' := r'_0 + r'_1$ be the corresponding numbers. We claim that

$$r_0 = r'_0, \quad r_1 = r'_1 \quad \text{and} \quad s' = s. \quad (33)$$

In fact, as $h_0$ is hyperbolic $r' = \text{rk}_\mathbb{R}(\mathfrak{g}_{\text{ut}}) = \text{rk}_\mathbb{R}(\mathfrak{g}^c) = r$. Further $t_q \subseteq \mathfrak{t} \subseteq \mathfrak{g}_{\text{ut}}$ shows that $s' = \text{rk}_\mathbb{R}(\mathfrak{h}_{\text{ut}}) = s$. This implies that $r'_1 = r' - s' = r - s = r_1$ and $r'_0 = s' - r'_1 = s - r_1 = r_0$.

**Theorem 7.8.** (The group $\pi_0(G^h)$) Let $G = \text{Inn}(\mathfrak{g})$. For the group $\pi_0(K^h) \cong \pi_0(G^h)$, the following assertions hold:

(a) If $(\mathfrak{g}, \tau)$ is of **complex type** or **non-split type**, then $G^h$ is connected.

(b) If $(\mathfrak{g}, \tau)$ is of **Cayley type** or **split type**, then $G^h$ is connected if $r$ is odd.

If $r$ is even, then it is not connected and $\pi_0(G^h) \cong \mathbb{Z}_2$ in the following cases of Cayley type:

- $r = 2$ and $\mathfrak{g} = \mathfrak{so}_{2,n}(\mathbb{R})$ with $n \geq 3$ odd.
- $r \geq 4$ even and $\mathfrak{g} = \mathfrak{sp}_{2r}(\mathbb{R})$,

and the following cases of split type:

- $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{R})$ with $h = \frac{1}{2}(1_n, -1_n)$ and $n$ even.
- $\mathfrak{g} = \mathfrak{so}_{p,q}(\mathbb{R})$ with $p, q > 2$ and $p + q$ odd.
- $\mathfrak{g} = \mathfrak{so}_{2n,2n}(\mathbb{R})$.

**Proof.** (a) If $(\mathfrak{g}, \tau)$ is of **complex type**, then $\mathfrak{g} = \mathfrak{h}_C$ and $\tau$ is antilinear. Here $\mathfrak{c} = \mathfrak{a}_C$, where $\mathfrak{a}_C \subseteq \mathfrak{h}_C$ is a compactly embedded Cartan subalgebra of $\mathfrak{h}$. For every root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{c})$ we have $(-\tau)\alpha = \tau$. As the roots are complex linear on $\mathfrak{a}_C$, no root is fixed by $-\alpha$, and thus $r_0 = 0$ and $s = r_1$. Hence $K^h$ is connected by Lemma 7.6(c).

If $(\mathfrak{g}, \tau)$ is of **non-split type**, then $s = \text{rk}_\mathbb{R}(\mathfrak{h})$ and $r = \text{rk}_\mathbb{R}(\mathfrak{g}^c) = 2s$, so that $r_0 = 0$ and $r_1 = s$. Therefore Lemma 7.6(c) implies that $K^h$ is connected.

(b) We first discuss **Cayley type** Lie algebras. Let

$$r := \text{rk}_\mathbb{R}(\mathfrak{g}) \quad \text{and} \quad s = \text{rk}_\mathbb{R}(\mathfrak{h}) = \text{rk}_\mathbb{R}(\mathfrak{g}^c).$$

Then $s \leq r$ and there exists a subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})^r$ containing $\mathfrak{h}$, so that we must have $s = r$, and therefore $r_1 = 0$.

If $r = 1$, then $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $K^h = Z(G)$ is trivial.

If $r > 1$, then the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type $C_r$ (\mathfrak{g} is hermitian of tube type), which implies that the simple $L$-submodules of $\mathfrak{g}$ are the adjoint modules $I_j$ and the tensor products $V_{j_1,j_2} = V_{2}^{j_1} \otimes V_{2}^{j_2}$. Note that Cayley type algebras are hermitian of tube type, so that $h_0 = 0$ follows from Proposition 5.2(c). For any
product \( z := z_{j_1} \cdots z_{j_k} \in Z(L) \) with \( j_1 < \ldots < j_k \) and \( k < r \), we then find a tensor product \( V_{j_1,j_2} \) on which \( z \) acts non-trivially, and \( z_1 \cdots z_r = 1 \). This shows that \( Z(L) \cong \mathbb{Z}_2^{r-1} \).

The restricted root system \( \Sigma(\mathfrak{g}, \mathfrak{a}) \) is of type \( BC_r \) with the subsystem \( \Sigma(\mathfrak{h}, \mathfrak{a}) \) of type \( A_{r-1} \). The corresponding Weyl group is the symmetric group \( S_r \) which acts by permutations of the central involutions \( z_1, \ldots, z_r \) of \( L \). Since the Weyl group elements are induced by the normalizer of \( a \) in the connected group \( \text{Inn}_\mathfrak{g}(R) \), it follows that \( z_1, \ldots, z_r \) all lie in the same connected component of \( K^h \). As these elements are involutions, all even products \( z_1 \cdots z_k \) are contained in \( K_e^h \). It follows that \( |\pi_0(K^h)| \leq 2 \) and, \( z_1 \cdots z_r = 1 \) further implies that \( \pi_0(K^h) \) is trivial if \( r \) is odd. So the only cases that have to be inspected in detail arise for \( r \) even.

**Case 1:** \( r = 2 \). Then \( \mathfrak{g} = \mathfrak{so}_{2,n}(\mathbb{R}) \) for some \( n \geq 3 \). In this case \( G \cong \text{PSO}_{2,n}(\mathbb{R})_e \). Let \( G^* := \text{SO}_{2,n}(\mathbb{R})_e \) with maximal compact subgroup \( K^* = \text{SO}_2(\mathbb{R}) \times \text{SO}_n(\mathbb{R}) \). Here \( h \in \mathfrak{so}_{2,n}(\mathbb{R}) \) is a diagonalizable rank-2 element whose \( \pm 1 \)-eigenspaces are isotropic, f.i., \( e_2 \pm e_3 \) (up to conjugacy). Hence its centralizer leaves the plane \( F := \mathbb{R}e_2 + \mathbb{R}e_3 \) invariant and we thus obtain a homomorphism \((K^*)^h \to O_{1,1}(\mathbb{R}) \). The centralizer of the Lorentz boost in \( \mathfrak{so}_{1,1}(\mathbb{R}) \) is \( O_{1,1}(\mathbb{R}) \) which is not connected and contains \(-\text{id}_{\mathbb{R}^2} \). As \( \text{diag}(-1, -1, -1, -1, 1, \ldots) \in (K^*)^h \) maps to \(-\text{id}_{\mathbb{R}^2} \), it follows that \((K^*)^h \) has at least 2-connected components.

The invariance of \( F \) under \((K^*)^h \) also shows that it leaves \( \mathbb{R}e_2 \) and \( \mathbb{R}e_3 \) invariant, so that

\[
(K^*)^h \subset \text{SO}(O_{1,1}(\mathbb{R})) \times \text{SO}(O_{n-1}(\mathbb{R})) \cong \{ \pm 1_{\mathbb{R}^2} \} \times \text{SO}(O_1(\mathbb{R}) \times O_{n-1}(\mathbb{R})),
\]

where the group on the right has 4 connected components. As the group on the right maps surjectively onto \( O_{1,1}(\mathbb{R}) \) and \((K^*)^h \) onto \( O_{1,1}(\mathbb{R}) \), we see that \((K^*)^h \) has exactly 2 components.

If \( n \) is odd, then \(-1 \notin G^* \), so that \( G \cong G^* \) and \( K \cong K^* \). It follows that \( G^h \) has two connected components.

If \( n \) is even, then \(-1 \in \text{SO}_{2,n}(\mathbb{R})_e \) and \( G \cong \text{SO}_{2,n}(\mathbb{R})_e / \{ \pm 1 \} \). Then \(-1 \notin (K^*)^h_e \) (consider the restriction to \( F \)) and the fact that \((K^*)^h \) has 2 connected components imply that \( K^h \) is connected.

**Case 2:** \( r \geq 3 \). Let \( E = \mathfrak{g}_1(h) \) be the euclidean Jordan algebra for which \( \mathfrak{g} \) is the conformal Lie algebra.

- For \( E = \text{Sym}_r(\mathbb{R}) \), we consider \( G^* := \text{Sp}_2(\mathbb{R}) \) with \((G^*)^h \cong \text{GL}_r(\mathbb{R}) \), which has 2 connected components and contains \( Z(G^*) = \{ \pm 1 \} \). Therefore \( G^h \cong (G^*)^h / \{ \pm 1 \} \) is connected if and only if \(-1 \notin \text{GL}_r(\mathbb{R})_e \), which is equivalent to \( \det(-1) = (-1)^r = -1 \). This corresponds to \( r \) odd. If \( r \) is even, then \( G^h \) is not connected.

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\(^4\) The group \( G = \text{PSO}_{2,n}(\mathbb{R})_e \) acts by causal automorphisms on the causal compactification \( \hat{M} \cong (S^1 \times S^{n-1}) / \{ \pm 1 \} \) of \( n \)-dimensional Minkowski space \( M = \mathbb{R}^{1,n-1} \). If \( \hat{M} \) is orientable, then the connected group preserves the orientation. As the subgroup \( G^h \cong \mathbb{R}_+^r \text{SO}_{1,n-1}(\mathbb{R})_e \) fixes 0 and acts by linear maps, this implies that \( G^h \cong \mathbb{R}_+^r \text{SO}_{1,n-1}(\mathbb{R})_e \), and since the latter group is connected, it follows that \( G^h \cong \text{SO}_{1,n-1}(\mathbb{R})_e \) is connected. The manifold \( \hat{M} \) is orientable if and only if the antipodal map is orientation preserving on \( S^{n-1} \), i.e., if \( n \) is even. If this is not the case, then \( \hat{M} \) is not orientable and \( G^h \cong \mathbb{R}_+^r \text{SO}_{1,n-1}(\mathbb{R})_e \) is not connected.
For $E = \text{Herm}_r(\mathbb{C})$ we consider $G^* := \text{SU}_{r,r}(\mathbb{C})$ with
\[(G^*)^h \cong \{ g \in \text{GL}_r(\mathbb{C}) : \det(g) \in \mathbb{R} \}.
\]
This group has two connected components, corresponding to the sign of the real-valued determinant. It contains
\[Z(G^*) = \{ \zeta 1_{2r} : \zeta \in \mathbb{C}^\times, \zeta^{2r} = 1 \} \cong \{ \zeta 1_r : \zeta \in \mathbb{C}^\times, \det(\zeta 1_r) = \zeta^r \in \{ \pm 1 \} \} \cong C_{2r}.
\]
If $\zeta \in \mathbb{C}^\times$ is an $r$th root of $-1$, then $\zeta 1_r \in Z(G^*) \cap (G^*)^h$ with $\det(\zeta 1_r) < 0$ shows that both connected components of $(G^*)^h$ intersects $Z(G^*)$. Therefore $G^h = \text{Ad}((G^*)^h) \cong (G^*)^h/Z(G^*)$ is connected.

For $E = \text{Herm}_r(\mathbb{H})$ we consider
\[G^* := \text{SO}^*(4r) := \{ g \in \text{SU}_{2r,2r}(\mathbb{C}) : g \top Ag = A \} \quad \text{for} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Then
\[K^* := G^* \cap U_{4r}(\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-\top} \end{pmatrix} : a \in U_{2r}(\mathbb{C}) \right\} \cong U_{2r}(\mathbb{C}).
\]
In the Lie algebra
\[\mathfrak{so}^*(4r) = \left\{ \begin{pmatrix} a & b \\ b^* & -a^\top \end{pmatrix} : a \in \text{u}_{2r}(\mathbb{C}), b \in \text{Skew}_{2r}(\mathbb{C}) \right\}
\]
an Euler element is given by
\[h = \frac{1}{2} \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \quad \text{for} \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{Skew}_{2r}(\mathbb{C}).
\]
An element $k = \text{diag}(a,a^{-\top})$, $a \in K^*$, commutes with $h$ if and only if $J = aJa^\top = aJa^{-1}$, i.e., $aJ = Ja$. One readily checks that this is equivalent to $a$ being of the form $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, i.e., to $a \in \text{U}_r(\mathbb{H}) \subseteq \text{U}_{2r}(\mathbb{C})$. We conclude that $(K^*)^h \cong \text{U}_r(\mathbb{H})$ is connected. This implies that $G^h = \text{Ad}((G^*)^h)$ is connected.

For $E = \text{Herm}_3(\mathbb{O})$ and $\mathfrak{g} = \mathfrak{e}_7(-25)$, the rank $r = 3$ is odd, so that $G^h$ is connected by the discussion above.

Now we turn to split type Lie algebras. Then
\[s = \text{rk}_\mathbb{R}(h) = \text{rk}_\mathbb{R}(g^c) = r, \quad \text{so that} \quad r_0 = r = s \quad \text{and} \quad r_1 = 0
\]
(see Table 3). As in (c), we see that $G^h$ is connected if $r$ is odd and that it has at most 2 connected components.

For $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{R})$ and $G^* = \text{SL}_{2n}(\mathbb{R})$, we have
\[(G^*)^h = \text{S}(\text{GL}_n(\mathbb{R}) \times \text{GL}_n(\mathbb{R})),
\]
which has two connected components. As $-1 \in (G^*)^h$ is contained in the identity component if and only if $n$ is even, it follows that $G^h$ is connected if and only if $n$ is odd.
7.3. The maximal compact subgroup of $H$

In this subsection we collect some consequences of our discussion of the preceding subsection. In particular, we show that, for the c-dual Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$, we consider $G^* = \text{SO}_{p,q}(\mathbb{R})_e$, with maximal compact subgroup $K^* = \text{SO}_p(\mathbb{R}) \times \text{SO}_q(\mathbb{R})$. Here $h \in \mathfrak{so}_{p,q}(\mathbb{R})$ is a diagonalizable rank-2 element whose $\pm 1$-eigenspaces are one-dimensional isotropic, i.e., $e_p \pm e_{p+1}$. Hence its centralizer $(G^*)^h$ leaves the plane $F := \mathbb{R}e_p + \mathbb{R}e_{p+1}$ invariant. This shows that

$$(K^*)^h \subseteq \text{SO}(1) \times \text{O}_{p-1}(\mathbb{R})) \times \text{SO}(1) \times \text{O}_{q-1}(\mathbb{R})) \cong \text{O}_{p-1}(\mathbb{R}) \times \text{O}_{q-1}(\mathbb{R}).$$

The fact that $(K^*)^h$ preserves the 1-dimensional subspaces generated by $e_p \pm e_{p+1}$ shows that $(K^*)^h$ has 2-connected components and that its restriction to the Minkowski plane $F$ contains $-1$.

If $p + q$ is odd, then $-1 \not\in G^*$, so that $G \cong G^*$ and $G^h$ has two connected components.

If $p + q$ is even, then $-1 \in G^*$ and $G \cong G^*/\{\pm 1\}$. Then $-1 \not\in (K^*)^h_c$ (consider the restriction to $F$) implies that $K^h$ is connected.

- We realize the split real form $\mathfrak{so}_{2n,2n}(\mathbb{R}) \subseteq \mathfrak{so}_{4n}(\mathbb{C})$ as

$$\mathfrak{so}_{2n,2n}(\mathbb{R}) = \left\{ X \in \mathfrak{gl}_{2n}(\mathbb{R}) : XI + IX^\top = 0 \right\} = \left\{ \begin{pmatrix} a & b \\ c & -a^\top \end{pmatrix} : b, c \in \text{Skew}_{2n}(\mathbb{R}), a \in \mathfrak{gl}_{2n}(\mathbb{R}) \right\}$$

for

$$I = \begin{pmatrix} 0 & 1_{2n} \\ 1_{2n} & 0 \end{pmatrix}.$$ 

This exhibits the Euler element

$$h := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad \mathfrak{g}_1(h) \cong \text{Skew}_{2n}(\mathbb{R})$$

and we obtain for $G^* := \text{SO}_{2n,2n}(\mathbb{R})_e$ the subgroup

$$(G^*)^h = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-\top} \end{pmatrix} : a \in \text{GL}_{2n}(\mathbb{R}) \right\} \cong \text{GL}_{2n}(\mathbb{R})$$

with 2 connected components. Further, $Z(G^*) = \{\pm 1\} \subseteq (G^*)_c^h$, so that $G^h = \text{Ad}((G^*)^h)$ also has 2 connected components.

- The split real form $\mathfrak{g} = \mathfrak{e}_7(\mathbb{R})$ of type $E_7$: Then $\mathfrak{g}^c$ is the hermitian real form of $\mathfrak{e}_7(\mathbb{C})$, hence of real rank $r = 3$. Therefore $r_0 = r = 3$ is odd, so that $G^h$ is connected by the discussion above.

7.3. The maximal compact subgroup of $H = G \cap G^c$

In this subsection we collect some consequences of our discussion of $\pi_0(G^h)$ in the preceding subsection. In particular, we show that, for the c-dual Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$, we have

$$\text{Inn}_{\mathfrak{g}^c}(\mathfrak{g}^c) \cap \text{Inn}_{\mathfrak{g}^c}(\mathfrak{g}) = \text{Inn}_{\mathfrak{g}^c}(\mathfrak{h})^h e^{ad_b}.$$
Proposition 7.9. Let \((g, \tau, C)\) be a semisimple non-compactly causal symmetric Lie algebra, where \(\tau = \tau_0\theta\), \(h \in g_\mathfrak{s} \cap C^0\) is a causal Euler element,
\[
G := \text{Inn}_{g_\mathfrak{c}}(g) \cong \text{Inn}(g), \quad K := G^\theta = \text{Inn}_g(\mathfrak{t}), \quad \text{and} \quad G^c := \text{Inn}_{g_\mathfrak{c}}(g^c).
\]
Then \(H := G \cap G^c\) is \(\tau\)-invariant and satisfies
\[
H = K^h \exp(\mathfrak{h}_p) \quad \text{and} \quad H \cap K = K^h.
\]

Proof. First we write \(g\) as a direct sum \(g_\mathfrak{s} \oplus g_\mathfrak{r} \oplus g_\mathfrak{c}\) as in (15), where \(g_\mathfrak{s}\) is the sum of all simple ideals not commuting with \(h\) (the strictly ncc part), \(g_\mathfrak{r}\) is the sum of all non-compact simple ideals commuting with \(h\) on which \(\tau = \theta\) (the non-compact Riemannian part), and \(g_\mathfrak{c}\) is the sum of all compact ideals (they commute with \(h\)). As \(h \in g_\mathfrak{s}\), the corresponding subgroups \(G_\mathfrak{k}\) and \(G_\mathfrak{r}\) are contained in \(G^h\) and \(\mathfrak{h}_p \cap \mathfrak{g}_k = \{0\} = \mathfrak{h}_p \cap \mathfrak{g}_r\). Therefore we may assume that \(g = g_\mathfrak{s}\) and, by decomposition into simple ideals, even that \((g, \tau)\) is irreducible non-compactly causal.

We start with the polar decomposition \(G^h = K^h \exp(p^{-\tau_0})\) which implies that
\[
H_K := H \cap K = G^c \cap K = G \cap K^c = K \cap K^c.
\]

Using the strongly orthogonal roots and the subgroup \(L \subseteq G\) (cf. Lemma 7.6), we find with Lemma 7.1(d), applied with the compact group \(K\) (in the lemma) and the subgroup \(K^h\) (\(H\) in the lemma), that
\[
K^h = K_\mathfrak{c}^h(K^h \cap Z(L)) \quad \text{with} \quad K^h \cap Z(L) \subseteq T_Q = \exp(\mathfrak{t}_q). \tag{35}
\]

We claim that all generators \(z_j \in Z(L_j)\) are contained in \(G \cap G^c\). In fact, we have \(I_j \cong \mathfrak{sl}_2(\mathbb{R})\), \(I_j^c \cong \mathfrak{su}_{1,1}(\mathbb{C})\) and \(I_j^c \cong \mathfrak{sl}_2(\mathbb{C})\). Clearly \(-1 \in \text{SL}_2(\mathbb{R}) \cap \text{SU}_{1,1}(\mathbb{C})\) holds in \(\text{SL}_2(\mathbb{C})\), so that \(z_j \in L_j \cap L_j^c\). This implies that \(T_Q \subseteq Z(L) \subseteq L \cap L^c\) is contained in \(H = G \cap G^c\), and hence by (35) that \(K^h \subseteq H\). Since we also have \(H_K \subseteq K^h\) (a consequence of \(\text{Ad}(H_K)\) fixing \(h \in C\)) (Lemma 1.11), we obtain \(H_K = K^h\). ■

Lemma 7.10. If \(H' \subseteq G^\tau\) is another open subgroup satisfying
\[
\text{Ad}(H')C^\tau_{\mathfrak{q}} = C^\tau_{\mathfrak{q}}, \tag{36}
\]
then \(H' \subseteq H = K^h \exp(\mathfrak{h}_p)\) and we obtain an equivariant covering of ordered symmetric spaces
\[
M' := G/H' \to M = G/H, \quad gH' \mapsto gH. \tag{37}
\]

Proof. In the polar decomposition \(H' = H_K' \exp(\mathfrak{h}_p)\) we have \(H_K' \subseteq K^\tau = K^{\tau_0}\). Since every element \(g \in G^{\tau_0}\) normalizes \(g_0(h)\) and preserves \(g_0(h) + g_{-1}(h)\), which are inequivalent representations of the Lie algebra \(g_0(h)\), we either have \(\text{Ad}(g)g_{\pm 1}(h) = g_{\pm 1}(h)\) or \(g_{\pm 1}(h)\). In the first case \(\text{Ad}(g)h = h\), and in the second \(\text{Ad}(g)h = -h \in -C^\tau_{\mathfrak{q}}\). Therefore (36) implies \(H_K' \subseteq K^h\), so that \(H' \subseteq H = G_D\). We therefore have an equivariant covering of homogeneous spaces as in (37). ■
As an important application of the preceding discussion, we obtain the following result on the fundamental group of $M$, connecting it to connected components of the stabilizer of $h$, and not to connected components of $H$, as usual.

**Proposition 7.11.** Let $q_G: \tilde{G} \to G$ denote the simply connected covering group. Then $q_G$ induces the universal covering

$$q_M: \tilde{M} := \tilde{G}/\tilde{G}^\tau \to M = G/H$$

for $H = K^h \exp(h_p)$ and

$$\pi_1(M) \cong \pi_0(\tilde{G}^h).$$

**Proof.** The subgroup $\tilde{G}^\tau$ is connected because $\tilde{G}$ is simply connected ([Lo69, Thm. IV.3.4]). Thus $\tilde{M}$ is the simply connected covering of $M = G/H \cong \tilde{G}/q_G^{-1}(H)$. This further implies that

$$\pi_1(M) \cong \pi_0(q_G^{-1}(H))$$

([HN12, Cor. 11.1.14]). As $H = K^h \exp(h_p)$, the inclusion $\ker(q_G) \subseteq \tilde{K}^h$ leads to the polar decompositions

$$q_G^{-1}(H) = q_G^{-1}(H_K) \exp(h_p) = q_G^{-1}(K^h) \exp(h_p) = \tilde{K}^h \exp(h_p)$$

and $\tilde{G}^h = \tilde{K}^h \exp(q_p)$, which imply that $\pi_0(\tilde{G}^h) \cong \pi_0(\tilde{K}^h) \cong \pi_0(q_G^{-1}(H)) \cong \pi_1(M)$. 

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**A. Some calculations in $\mathfrak{sl}_2(\mathbb{R})$**

In this subsection we collect some formulas concerning the 3-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ that we shall use below. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, we fix the Cartan involution $\theta(x) = -x^\top$, so that

$$\mathfrak{t} = \mathfrak{so}_2(\mathbb{R}) \quad \text{and} \quad \mathfrak{p} = \{x \in \mathfrak{sl}_2(\mathbb{R}): x^\top = x\}.$$ 

The basis elements

$$h^0 := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$h^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(e^0 + f^0), \quad e^1 = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad f^1 = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

satisfy

$$[h^j, e^j] = e^j, \quad [h^j, f^j] = -f^j, \quad [e^j, f^j] = 2h^j \quad \text{and} \quad \theta(e^j) = -f^j \quad \text{for} \quad j = 1, 2.$$ 

For the involution

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

we have $\mathfrak{h} = \mathbb{R} h^0$ and $\mathfrak{q} = \mathfrak{g} - \mathfrak{h} = \mathbb{R} h^1 + \mathbb{R}(e^0 - f^0)$.
and 

\[ C = [0, \infty) e^0 + [0, \infty) f^0 \]

is a hyperbolic \( \text{Inn}(h) \)-invariant cone in \( q \), containing \( h^1 \) as a causal Euler element. The subspace \( t_q := \mathbb{R}(e^0 - f^0) = \mathfrak{so}_2(\mathbb{R}) \) of \( q \) is maximal elliptic. For

\[ x_0 := \frac{\pi}{4}(e^0 - f^0) = \frac{\pi}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in t_q \]

and

\[ g_0 := \exp(x_0) = \begin{pmatrix} \cos \left( \frac{\pi}{4} \right) & \sin \left( \frac{\pi}{4} \right) \\ -\sin \left( \frac{\pi}{4} \right) & \cos \left( \frac{\pi}{4} \right) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \]

we then have

\[ \text{Ad}(g_0)h^1 = h^0 \quad \text{and} \quad \text{Ad}(g_0)h^0 = -h^1, \quad (40) \]

More generally, we have for \( t \in \mathbb{R} \)

\[ e^{\frac{t}{2}\text{ad}(e^0 - f^0)}h^0 = \cos(t)h^0 - \sin(t)h^1 \quad (41) \]

because

\[ \left[ \frac{e^0 - f^0}{2}, h^0 \right] = -h^1, \quad \left[ \frac{e^0 - f^0}{2}, h^1 \right] = h^0. \]

B. Some general facts on invariant cones

Lemma B.1. Let \( E \) be a finite dimensional real vector space, \( C \subseteq E \) a closed convex cone and \( E_1 \subseteq E \) a linear subspace. If the interior \( C^o \) of \( C \) intersects \( E_1 \), then \( C^o \cap E_1 \) coincides with the relative interior \( C^o_1 \) of the cone \( C_1 := C \cap E_1 \) in \( E_1 \).

Proof. Clearly, \( C^o \cap E_1 \subseteq C_1^o \). Pick \( x_0 \in C^o \cap E_1 \). If \( x \in C_1 = C \cap E_1 \) and \( t \in [0, 1) \), then \( x_t := x_0 + t(x - x_0) = (1 - t)x_0 + tx \in C^o \) shows that \( C^o \cap E_1 \) is dense in \( C_1 \). If, in addition, \( x \in C_1^o \), then there exists an \( s > 1 \) with \( x_s \in C_1 \). Then the argument from above shows that

\[ x = x_1 = x_0 + \frac{1}{s}(x_s - x_0) \in C^o. \]

Lemma B.2. (Cone Extension Lemma) Let \( E = E_1 \oplus E_2 \) be a finite dimensional real vector space and \( p_1 : E \to E_1 \) the projection along \( E_2 \). Further, let \( K \subseteq \text{GL}(E_2) \subseteq \text{GL}(E) \) be a compact subgroup and \( H \subseteq \text{SL}(E_1) \subseteq \text{SL}(E) \) a subgroup. If \( C_1 \subseteq E_1 \) is pointed generating \( H \)-invariant closed convex cone and \( h \in E \) with \( p_1(h) \in C_1^o \), then there exists an \( H \times K \)-invariant pointed generating closed convex cone \( C \subseteq E \) such that

(a) \( p_1(C) = C \cap E_1 = C_1 \).

(b) \( C^o \cap E_1 = C_1^o \).

(c) \( h \in C^o \).
Proof. As $H \subseteq \text{SL}(E_1)$, the characteristic function

$$\varphi : C_1^o \to (0, \infty), \quad \varphi(x) = \int_{C_1^o} e^{-\alpha(x)} \, d\alpha$$

is smooth and $H$-invariant with the property that $x_n \to x_0$ with $x_n \in C_1^o$ and $x_0 \in \partial C_1$ implies $\varphi(x_n) \to \infty$ ([Ne00, Thm. V.5.4]). Let

$$B := \{ x \in C_1 : \varphi(x) \leq 2 \varphi(p_1(h)) \}.$$ 

This is a closed convex $H$-invariant subset with $C_1 = \text{cone}(B) = \mathbb{R}_+ B$. Further, let $D \subseteq E_2$ be a $K$-invariant compact 0-neighborhood such that $h - p_1(h) \in \text{int}_{E_2}(D)$. Then $B + D$ is a $H \times K$-invariant closed convex subset not containing 0, and thus $C := \text{cone}(B + D)$ is a $H \times K$-invariant pointed generating invariant cone which satisfies

$$C \cap E_1 \subseteq p_1(C) \subseteq C_1 \subseteq C \cap E_1.$$ 

Moreover, $h \in \text{int}_{E_1}(B) + \text{int}_{E_2}(D) \subseteq C^o$ and

$$C^o \cap E_1 \subseteq C_1^o = (0, \infty) B \subseteq C^o. \quad \blacksquare$$

Example B.3. The condition $H \subseteq \text{SL}(E_1)$ is crucial, as the following example shows. We consider

$$E = \mathbb{R}^2, \quad K = \{1\} \quad \text{and} \quad H = \{\text{diag}(t, 1) : t > 0\}.$$ 

Then every open neighborhood $U \subseteq E$ of a point $(t, 0) \in E_1$ has the property that $\{0\} \times \mathbb{R}$ is contained in the closed convex cone generated by $H.U$. Therefore the cone $C = [0, \infty) \times \{0\}$ does not extend to a pointed generating $H$-invariant cone $\tilde{C}$ containing $(0, \infty) \times \{0\}$ in its interior. However, there are $H$-invariant pointed generating invariant cones $\tilde{C}$ with $\tilde{C} \cap E_1 = C$, but they contain $C$ in their boundary.

Proposition B.4. ([Ne10 Prop. 2.11]) Let $K$ be a compact group acting continuously on the finite-dimensional real vector space $E$ by the representation $\pi : K \to \text{GL}(E)$ and $p(v) := \int_K \pi(k)v \, d\mu_K(k)$ the corresponding fixed point projection, where $\mu_K$ is a normalized Haar measure on $K$. If $\Omega \subseteq E$ is an open or closed $K$-invariant convex subset, then

$$p(\Omega) = \Omega \cap E^K.$$ 

The preceding proposition implies in particular for any convex invariant subset with non-empty interior the relation

$$p(\Omega^o) = \Omega^o \cap E^K \subseteq (\Omega \cap E^K)^o.$$ 

That we actually have equality follows from Lemma [B.1].
C. Lorentzian symmetric spaces

Time-oriented Lorentzian symmetric spaces are in particular causal symmetric spaces. Not all such spaces are reductive, as the biinvariant Lorentzian structures on the 4-dimensional (solvable) oscillator group shows ([HN93]). If, however, \((g, \tau, \beta)\) is reductive and Lorentzian (\(\beta\) denoting the Lorentzian form on \(q\)), then we may assume that \(z(g) \subseteq g^{-\tau}\). Accordingly

\[
g = z(g) \oplus \bigoplus_{j=1}^{n} (g_j, \tau_j),
\]

where each \((g_j, \tau_j)\) is irreducible and the corresponding direct sum decomposition

\[
(q, \beta) = (z(g), \beta_z) \oplus \bigoplus_{j=1}^{n} (q_j, \beta_j)
\]

is orthogonal. Therefore at most one summand contains timelike vectors and the other summands are space-like, hence correspond to Riemannian symmetric spaces. So two types occur:

(CL) The central type, where \(\beta_z\) is Lorentzian and all forms \(\beta_j\) are negative definite.

(SL) The simple type, where some \(\beta_{j_0}\) is Lorentzian and all other summands are negative definite.

For the classification of Riemannian symmetric spaces, we refer to Helgason’s monograph [He78]. These determine the spaces of central type and to understand the other type, one needs to know the irreducible Lorentzian spaces, whose classification we recall below.

**Theorem C.1.** If \((g, \tau)\) is an irreducible semisimple symmetric Lie algebra and the corresponding \(d\)-dimensional symmetric space \(M\) is Lorentzian, then it is locally isomorphic to de Sitter space \(dS^d\) or anti-de Sitter space \(AdS^d\). Accordingly, \((g, h)\) is isomorphic to

\[
(s0_{1,d}(\mathbb{R}), s0_{1,d-1}(\mathbb{R})) \quad \text{or} \quad (s0_{2,d-1}(\mathbb{R}), s0_{1,d-1}(\mathbb{R})).
\]

**Proof.** Let \((g, \tau)\) be an irreducible semisimple symmetric Lie algebra. We are interested in a description of all Lorentzian symmetric spaces \(G/H\). To see if \((g, \tau)\) is Lorentzian, we choose a Cartan involution \(\theta\) of \(g\) commuting with \(\tau\). We then have

\[
q = q_\tau \oplus q_\rho,
\]

where the Cartan–Killing form of \(g\) is negative definite on \(q_\tau\) and positive definite on \(q_\rho\). That a corresponding symmetric space \(G/H\) is Lorentzian is equivalent to one of the two subspaces \(q_\tau\) or \(q_\rho\) to be one-dimensional, because this implies that a suitable multiple of the Cartan–Killing form on \(q\) is a Lorentzian form. As \((g, \tau)\) is Lorentzian if and only if the dual pair \((g^e = h \oplus iq, \tau^e)\) is Lorentzian, and \((iq)_\tau = iq_\rho\) and \((iq)_\rho = iq_\tau\), we may assume that \(q_\rho\) is one-dimensional. Then \(\theta\) defines a dissecting involution on the associated 1-connected symmetric space \(G/H\), where \(G\)
is 1-connected and $H = G^\tau$. Then the classification of irreducible symmetric spaces with dissecting involutions in \[\text{NO20}\] implies that $G/H$ is locally isomorphic to a quadric. In particular $\mathfrak{g} \cong \mathfrak{so}_{p,q}(\mathbb{R})$ and $\mathfrak{h} \cong \mathfrak{so}_{p-1,q}(\mathbb{R})$. The Lorentzian property now implies that $q = 1$ or $p = 2$, which corresponds to de Sitter space and Anti-de Sitter space.

**Remark C.2.** In addition to irreducible Lorentzian symmetric spaces, there is also the class of indecomposable solvable symmetric spaces; classified by Cahen and Wallach in \[\text{CW70}\]. According to \[\text{CW70, Thm. 3}\], all indecomposable Lorentzian symmetric Lie algebras are either semisimple or solvable. A detailed exposition of the classification can be found in \[\text{KO08, §3.3}\].

The corresponding symmetric Lorentzian Lie algebras $(\mathfrak{g}, \tau, \beta)$, where $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is an invariant non-degenerate symmetric bilinear form, have the following structure. We start with a quadratic space $(\mathfrak{a}, \kappa_\mathfrak{a})$ of signature $(p + 2q, p)$ and orthogonal basis $\{e_1, \ldots, e_{2q}, e'_1, \ldots, e'_{2p}\} \subset \mathfrak{a}$ with $\kappa_\mathfrak{a}(e_j, e_j) = 1$, $\kappa_\mathfrak{a}(e'_j, e'_j) = \begin{cases} 1 & \text{for } j \leq p \\ -1 & \text{for } j > p. \end{cases}$

We also have a skew-symmetric endomorphism $D \in \mathfrak{so}(\mathfrak{a}, \kappa_\mathfrak{a})$ acting by

\[
D e_j = \lambda_j e_{j+q}, \quad D e_{j+q} = -\lambda_j e_j, \quad D e'_j = \mu_j e'_{j+p}, \quad D e'_{j+p} = \mu_j e'_j
\]

with $0 < \lambda_1 \leq \cdots \leq \lambda_q$, $0 < \mu_1 \leq \cdots \leq \mu_p$.

Now $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{a} \oplus \mathbb{R}$ with the Lie bracket

\[
[(z, a, t), (z', a', t')] := (\kappa_\mathfrak{a}(Da, a'), tDa' - t'Da, 0)
\]

and the invariant symmetric bilinear form

$$\beta((z, a, t), (z', a', t')) = zt' + z't + \kappa_\mathfrak{a}(a, a').$$

The involution $\tau$ is given by

$$\tau(z, a, t) = (-z, \tau_\mathfrak{a}(a), -t),$$

and

$$\mathfrak{a}^{-\tau} = \text{span}\{e_1, \ldots, e_q, e'_1, \ldots, e'_{q+p}\}, \quad \mathfrak{a}^{\tau} = \text{span}\{e_{q+1}, \ldots, e_{2q}, e'_{p+1}, \ldots, e'_{2p}\}.$$

For $p = 0$ these Lie algebras are called oscillator algebras and the form $\beta$ on $\mathfrak{g}$ is Lorentzian because $\kappa_\mathfrak{a}$ is positive definite. In general the form $\beta$ is positive definite on $\mathfrak{a}^{-\tau} = \mathfrak{a} \cap \mathfrak{q}$, so that the restriction of $\beta$ to $\mathfrak{q} = \mathfrak{g}^{-\tau}$ is Lorentzian, but the form $\beta$ on $\mathfrak{g}$ has signature $(p + 1 + 2q, p + 1)$. 

In addition to Lorentzian symmetric spaces, also natural generalizations of conformal spacetime-geometries have been discussed from the physics perspective in [MdR07], where the conformal compactifications of simple euclidean Jordan algebras are studied.

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