BOUNDARY VALUES OF HOLOMORPHIC SEMIGROUPS AND FRACTIONAL INTEGRATION

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Abstract. The concept of boundary values of holomorphic semigroups in a general Banach space is studied. As an application, we consider the Riemann-Liouville semigroup of integration operator in the little Hölder spaces $\text{lip}_\alpha^0[0,1], 0 < \alpha < 1$ and prove that it admits a strongly continuous boundary group, which is the group of fractional integration of purely imaginary order. The corresponding result for the $L^p$-spaces ($1 < p < \infty$) has been known for some time, the case $p = 2$ dating back to the monograph by Hille and Phillips. In the context of $L^p$ spaces, we establish the existence of the boundary group of the Hadamard fractional integration operators using semigroup methods. In the general framework, using a suitable spectral decomposition, we give a partial treatment of the inverse problem, namely: Which $C_0$-groups are boundary values of some holomorphic semigroup of angle $\pi/2$?

1. Introduction

The concept of boundary values of holomorphic semigroups that we use in the present work originates in the treatise [22]. Specifically, in [22, Chapter 17], the authors give a necessary and sufficient condition for a holomorphic semigroup of angle $\pi/2$ to admit a boundary value group. The converse question of which groups are boundary values of holomorphic semigroups is answered in [22, Theorem 17.10.1]. Using this concept, Arendt, El-Mennaoui and Hieber [3] gave an elementary proof to the classical result of L. Hörmander [3, Theorem 3.9.4 and Chapter 8] to the effect that the Schrödinger operators $\pm i\Delta$ generate $C_0$-semigroups on $L^p(\mathbb{R}^n)$ if and only if $p = 2$. Later the same approach was also used in [18] to prove a similar result in $L^p(\Pi^n)$ with Dirichlet, Neumann or periodic boundary conditions (where $\Pi^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the $n$-dimensional torus).

The following proposition [22, Theorem 17.9.1 and Theorem 17.9.2] answers the question of which holomorphic semigroups in the right half-plane admit boundary value groups.

Proposition 1.1. Let $A$ be the generator of a holomorphic $C_0-$semigroup $T$ of angle $\pi/2$ on a Banach space $E$. Then $iA$ (resp. $-iA$) generates a $C_0-$semigroup (which is the boundary value of $T$) if and only if $T$ is bounded on $D_+ := \{z \in \mathbb{C}; \text{Re}(z) > 0, \text{Im}(z) > 0, |z| \leq 1\}$ (resp. on $D_- := \{z \in \mathbb{C}; \text{Re}(z) > 0, \text{Im}(z) < 0, |z| \leq 1\}$).

This result was extended by El-Mennaoui to cover holomorphic semigroups which admit as boundary values exponentially bounded integrated semigroups. The results are presented in [11, Section 3.14] and applied to the Schrödinger equation in $L^p$ spaces.

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Our focus in the present paper is on the Riemann-Liouville semigroup which describes the fractional integration. It is also the basis for the definition of the most commonly used concepts of fractional derivatives, namely the Riemann-Liouville and the Caputo fractional derivatives. The explicit representation of this semigroup \((J(z))_{Re(z)>0}\) is given by

\[
J(z)f(t) = \frac{1}{\Gamma(z)} \int_0^t (t-s)^{z-1} f(s)ds \quad (f \in L^1[0,1], t \in (0,1)).
\]

We will show that the semigroup \(J\) is bounded on \(D := D_+ \cup D_-\) with respect to the norm of the space \(\text{lip}_\alpha[0,1]\) of all \(\alpha\)-Lipschitz continuous functions on \([0,1]\). In view of Proposition 1.1 we conclude the existence of a boundary value group on \(\text{lip}_\alpha[0,1]\) which is the largest subspace of \(\text{lip}_\alpha[0,1]\) on which \(J\) is strongly continuous.

The Riemann-Liouville semigroup was studied extensively in \([22]\) Section 23.16] where it is proved that it is strongly continuous and holomorphic in \(L^p(0,1), 1 \leq p < \infty\). In particular, a description of its infinitesimal generator is given \([22]\) Section 23.16.1. The importance of this semigroup in spectral and ergodic theory is stressed and it is proved that when \(p = 2\), a boundary group exists. The proof in that monograph relies on a lemma by Kober \([22]\) Lemma 23.16.2. An interesting result is the explicit computation of the norm of boundary group \(\|J(it)\|_{L^2} = e^{\frac{1}{2}|t|}, t \in \mathbb{R}\). This means in particular that the boundary group is not uniformly bounded.

Let \(-B\) be the generator of the translation semigroup on \(L^p[0,1]\). In section 3 below we recall the well known fact that \(J(z)\) may be identified with \(B^{-z} (\text{Re}(z) > 0)\). It was proved in \([18]\) (see also \([1]\) Section 3.14]), using the transference principle due to R. Coifman and G. Weiss \([12]\) and/or \([13]\) that \(J\) is bounded on \(D_\pm\) as a holomorphic semigroup of angle \(\pi/2\) acting on \(L^p[0,1]\) if and only if \(1 < p < +\infty\). Here the notation \(D_\pm\) will stand henceforth for \(D_+\) or \(D_-\) and will mean either the first object or the second one. A similar notation prevails for all mathematic objects along this paper like \(E_\pm\) and \(\pm i\Delta\) in section 2. It should be noted that a similar result had appeared earlier (see the two papers \([17], [24]\)) using different methods for the proofs. While Kalish relies on Kober’s results on integral operators on \(L^p\)-spaces, Fisher uses the Mikhlin multiplier theorem. The method of Kalish is therefore close to the one used by Hille-Phillips \([22]\) when \(p = 2\). The Riemann-Liouville semigroup \(J(z)\) may be viewed as the semigroup of fractional powers of the Volterra operator (for details, see e.g. \([20]\) Section 8.5], in particular Theorem 8.5.8).

Let \(\mu \in \mathbb{R}\). The Hadamard type fractional integration operators of order \(\alpha > 0\) are given by

\[
(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (\frac{t}{x})^\mu (\ln(\frac{t}{x}))^{\alpha-1} f(t) \frac{dt}{t} \quad (x > 0),
\]

and

\[
(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (\frac{x}{u})^\mu (\ln(\frac{x}{u}))^{\alpha-1} f(u) \frac{du}{u} \quad (x > 0).
\]

The fractional integral \((1.2)\) (in the case \(\mu = 0\)) appears first in Hadamard’s paper \([21]\) dedicated to the study of fine properties of functions which are representable by power series. It is obtained as a modification of the Riemann-Liouville fractional integral. Our approach to the Hadamard fractional derivative seems to be new in that it involves...
the abstract theory of fractional powers of semigroup generators while the classical approach consists in modifying the Riemann-Liouville fractional integral (see for example \[35, \text{Section 18.3}\]). In the case of the Riemann-Liouville semigroup, the connection with fractional powers of the generator of the translation semigroup is well documented (see e.g. \[1, 20, 35\]).

We prove that in the spaces \(X_c\) (see Section 3 below), and for \(1 < p < \infty\), these semigroups with parameter \(\alpha\) are holomorphic of angle \(\pi\) and for appropriate values of \(c, \mu\) they admit boundary values on the imaginary axis. We derive the above representations in a way similar to the Riemann-Liouville case described earlier. From this, the semigroup property which was proved in the papers \[7, 8, 26\] without appeal to semigroup theory (relying instead on direct computation and the use of the Mellin transform), is obtained as a consequence. More specifically, our proofs are based on the following two operator semigroups

\[
(T_1(t)f)(x) = f(xe^t), \quad x > 0, \quad t > 0,
\]

\[
(T_2(t)f)(x) = f(xe^{-t}), \quad 0 < x < a, \quad t > 0,
\]

acting on appropriate weighted \(L^p\) spaces, where \(1 \leq p < \infty\) and \(a \in (0, \infty]\). More precisely, we consider the semigroups \((e^{-\mu t}T_j(t))\), \(j = 1, 2\). Such semigroups have been studied extensively in connection with spectral theory and asymptotic behavior the Black-Scholes equation \(u_t = x^2 u_{xx} + xu_x\) of financial mathematics (see \[2\]). This is a degenerate parabolic equation. The semigroup \((T_1)\) is also important in spectral theory where it is used to provide counterexamples in various contexts (see e.g. \[1, \text{Chapter 5}\], \[2\]). We are able to recover some results of Boyd \[5\] on the powers of the Cesàro operator.

In recent years, fractional calculus has gained increasing interest due to its suitability in modeling several phenomena (deterministic or stochastic) in science and engineering, most notably phenomena with memory effects such as anomalous diffusion, fractional Brownian motion and problems in material science, to name a few. Some information as well references on these topics can be found in \[28, 32, 33\] and \[35\].

The right nilpotent translation semigroup \(S\) is of contraction operators on \(C[0, 1]\) whose maximal subspace of strong continuity is \(C_0[0, 1] := \{f \in C[0, 1], f(0) = 0\}\). The Lipschitz space

\[
\text{Lip}_0^\alpha[0, 1] = C_0[0, 1] \cap \text{Lip}^\alpha[0, 1] = \left\{ f \in C_0[0, 1], \sup_{t \neq s} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty \right\}
\]

where \(0 < \alpha < 1\) is a subspace of \(C_0[0, 1]\) invariant under the semigroup \(S\) but strong continuity fails as well. The maximal subspace of \(\text{Lip}_0^\alpha[0, 1]\) on which strong continuity holds is the little Hölder space \(\text{lip}_0^\alpha[0, 1]\). A more concrete description of the little Hölder space \(\text{lip}_0^\alpha[0, 1]\) is the following:

\[
(1.4) \quad \text{lip}_0^\alpha[0, 1] = \left\{ f \in C_0[0, 1], \lim_{\delta \to 0} \sup_{0 < |t-s| < \delta} \frac{|f(t) - f(s)|}{|t - s|^\alpha} = 0 \right\}.
\]

The lack of strong continuity in the space \(\text{Lip}_0^\alpha[0, 1]\) is no accident. Indeed, by a theorem of Ciesielski \[9\] (see also \[9, \text{Section 2.7}\]), for each \(\alpha \in (0, 1)\), the space \(\text{Lip}_0^\alpha[0, 1]\) is isomorphic to the space \(l^\infty(\mathbb{N})\) of bounded sequences. In spaces of this type (namely
Grothendieck spaces with the Dunford-Pettis property), all generators of strongly continuous semigroups are bounded operators (see [1, Theorem 4.3.18 and Corollary 4.3.19]). The typical spaces in this class are the spaces $L^\infty(\Omega, \mu)$ where $(\Omega, \mu)$ is a measure space, in particular $l^\infty$, the Hardy space $H^\infty(D)$ of bounded holomorphic functions on the open unit disc, as well as the spaces $C(K)$ of continuous function on a compact Hausdorff space when $K$ is extremally disconnected. On the other hand, another result of Ciesielski from [9] states that $lip_0^\alpha[0, 1]$ is isomorphic to the space $c_0$ of complex sequences converging to 0.

In the present paper, we shall prove that a boundary group exists in the Hölder spaces $lip_0^\alpha[0, 1], 0 < \alpha < 1$. The paper is organized as follows. In Section 2, we study boundary values of holomorphic semigroups by analysing existence of boundary values for individual trajectories $T(z)x$. The semigroups we consider are holomorphic in some sector $\Sigma_\theta := \{ z \in \mathbb{C} \setminus \{0\}, \ |\arg(z)| < \phi \}$ where $\phi \in (0, \pi/2]$ is given. The Coifman-Weiss transference principle is applied to the Hadamard type semigroups in Section 3 to prove the existence of boundary value groups on the imaginary axis. On the way to this, we are able to recover some results of Boyd [3] on the powers of the Cesàro operator. We prove existence of the boundary for the Riemann-Liouville semigroup as our main result in Section 4 by direct estimates allowing us to apply Proposition 1.1. In the final Section 5, we use a spectral approach to study the question of which groups are boundary value groups. In particular, a generalization of the spectral decomposition in [15] will be established in the more general spectral situation. Precisely, we obtain a direct suitable space decomposition when the spectrum of the generator is assumed to be union of two connected components which belong disjointly to $\mathbb{C}_+$ and $\mathbb{C}_-$.

2. Boundary values of a holomorphic semigroup

We first make precise what we mean by boundary values of holomorphic semigroups. Let $A$ be a linear operator in a complex Banach space $E$. For $\theta \in (0, \pi]$ let $\Sigma_\theta$ be the sector in the complex plane:

$$\Sigma_\theta := \{ z \in \mathbb{C}; \ z \neq 0, \ |\arg(z)| < \theta \}.$$

We recall that $A$ generates a bounded holomorphic semigroup of angle $\phi \in (0, \pi/2]$ if $A$ generates a $C_0$-semigroup $T$ which has a bounded extension to each subsector $\Sigma_{\phi'}$ (where $0 < \phi' < \phi$) of the sector $\Sigma_\phi$. The bound will in general depend on $\phi'$. Then by analytic continuation, the holomorphic extension, still denoted by $T$, satisfies the semigroup property: $T(z + z') = T(z)T(z')$ for $z, z' \in \Sigma_{\phi'}$ and $\lim_{z \to 0, z \in \Sigma_{\phi'}} T(z) = I$ in the strong operator topology. The operator $A$ generates a holomorphic semigroup $T$ of angle $\phi$ if for all $\theta \in (0, \phi)$ there exists $\omega \geq 0$ such that $A - \omega I$ generates a bounded holomorphic semigroup $T_\theta$ of angle $\theta$ (and then $T(z) = e^{\omega z}T_\theta(z)$, $z \in \Sigma_\theta$). Then for $\theta \in (-\phi, \phi)$, $(T(te^{i\theta}))_{t \geq 0}$ is a $C_0$-semigroup and its generator is $e^{i\theta}A$. Conversely, suppose $A$ generates a $C_0$-semigroup. Then, if for each $\theta \in (-\phi, \phi)$ where $\phi \in (0, \pi/2]$, the operator $e^{i\theta}A$ is the generator of a $C_0$-semigroup of contractions, then and only then, $A$ generates a holomorphic semigroup of contractions of angle $\phi$ (see [25, Theorem 1.54] where this and similar results are presented).

Due to their importance in the area of partial differential equations (more precisely, those of parabolic type) and in the theory of stochastic processes, holomorphic semigroups have been extensively studied from the early days of the theory of strongly continuous
semigroups. We refer to \cite{1 6 19 22 31} for more information on the subject. Throughout this paper we denote respectively by $D(A)$, $\sigma(A)$ and $\rho(A)$ the domain, the spectrum and the resolvent set of $A$. In this first section we are interested in the boundary values of holomorphic semigroups. In this context, we reformulate Proposition 1.1 as:

**Proposition 2.1.** Let $A$ be the generator of a holomorphic $C_0$–semigroup $T$ of angle $\phi \in (0, \pi/2]$. Then $T(te^{\pm i\theta}) := \lim_{\epsilon \to 0} T(\epsilon + te^{\pm i\theta})x$ exists uniformly in $t \in [0,1]$ for all $x \in E$ if and only if

$$\sup_{z \in \Sigma_\phi \cap D_\pm} \|T(z)\| < \infty.$$ 

Proposition 2.1 is an obvious combination of \cite[Proposition 1.1, Proposition 1.2]{3}. In fact, in this case, the operators $(T(te^{\pm i\phi}))_{0 \leq t \leq 1}$ can then be canonically extended to the half line $[0, \infty)$ as $C_0$–semigroups. We call these semigroups $(T(te^{\pm i\phi}))_{t \geq 0}$ the boundary values of $T$.

When we deal with parabolic partial differential equations in $L^p$ spaces, for instance, the holomorphic semigroup $T$ may not be locally bounded on $\Sigma_\phi$ but the trajectories for smooth initial data may be locally bounded. In order to assign also boundary values, to these semigroups, we consider for every holomorphic semigroup $T$ of angle $\phi$ the $T$–invariant subspaces shortly denoted $E_T^T$ and $E_T$:

$$E_T := \{x \in E; \lim_{\epsilon \to 0} T(\epsilon + te^{i\phi})x \text{ converges uniformly for } t \text{ in compact subsets of } [0, +\infty)\},$$

$$E_T := \{x \in E; \lim_{\epsilon \to 0} T(\epsilon + te^{-i\phi})x \text{ converges uniformly for } t \text{ in compact subsets of } [0, +\infty)\},$$

where the convergence is to be understood in $E$. As a consequence of the above uniform convergence we also have $x \in E_T$ (respectively $x \in E_T$) if and only if $T(\epsilon + z)x$ converges uniformly for $z$ in the compact subsets of $\Sigma_\phi := \{z \in \Sigma_\phi; \Im(z) \geq 0\}$ (respectively $\Sigma_\phi := \{z \in \Sigma_\phi; \Im(z) \leq 0\}$). We then set $T(te^{\pm i\phi}) := \lim_{\epsilon \to 0} T(\epsilon + te^{\pm i\phi})x$ for all $x \in E_T$.

**Proposition 2.2.** Let $A$ be the generator of a holomorphic $C_0$–semigroup $T$ of angle $\phi \in (0, \pi/2]$ and $E_T, E_T$ be the subspaces defined above. Then:

i) $E_T \cap E_T$ is dense in $E$.

ii) For all $x \in E_T$, $z, z' \in \Sigma_\phi$ (respectively $x \in E_T$, $z, z' \in \Sigma_\phi$) we have $T(z')T(z)x = T(z + z')x$ and $\Sigma_\phi \ni z \mapsto T(z)x$ (respectively $\Sigma_\phi \ni z \mapsto T(z)x$) is continuous with values in $E$.

**Proof.** (i) Let $x \in E$. For all $t_0 > 0$ since $T$ is strongly uniformly continuous on the compact subsets of the sub-sectors $\Sigma_\theta$, $0 < \theta < \phi$ we have $T(t_0)x \in E_T \cap E_T$ and $T(t_0)x \to x$ as $t_0 \to 0$. The claim follows.

(ii) Let $z \in \Sigma_\phi$, $z' \in \Sigma_\phi$ and $x \in E_T$. We have $T(z)x = \lim_{\epsilon \to 0} T(\epsilon + z)x$ uniformly for $z$ in the compact subsets of $\Sigma_\phi$. Then

$$T(z')T(z)x = \lim_{\epsilon \to 0} T(z')T(\epsilon + z)x$$

$$= \lim_{\epsilon \to 0} T(\epsilon + z + z')x = T(z + z')x.$$
since \(z + z' \in \Sigma_\phi\). The continuity of \(z \mapsto T(z)x\) on \(\overline{\Sigma_\phi}\) follows from the uniform convergence. The same argument prevails for \(x \in E_T\) and \(z \in \Sigma_\phi\). \(\square\)

The above proof uses the fact that \(\bigcup_{t>0} T(t)(E)\) is dense in \(E\), which follows from the \(C_0\)- property. We mention that for general holomorphic semigroups, more is true: \(\bigcap_{t>0} T(t)(E)\) is dense in \(E\) (see e.g. \[10\]).

**Example 2.3.** Let \(E = l^2(\mathbb{N})\) and \((A, D(A))\) be given by

\[
A(x_n) := ((-n + in + i\ln(n + 1))x_n)
\]

\[
D(A) := \{ (x_n) \in l^2(\mathbb{N}) \mid (nx_n) \in l^2(\mathbb{N}) \}.
\]

Then \(A\) with domain \(D(A)\) generates a holomorphic \(C_0\)-semigroup \(T\) of angle \(\pi/4\) given by

\[
T(z)(x_n) := \left( \exp((-n + in + i\ln(n + 1))z)x_n \right)
\]

for all \((x_n) \in l^2(\mathbb{N})\) and \(z \in \Sigma_{\pi/4}\). Moreover, we have \(E_T^+ = E\) and \(E_T^- = \{(x_n) \in l^2(\mathbb{N}), (nx_n) \in l^2(\mathbb{N})\} \) for all \(\alpha > 0\). In fact, for all \(z \in \Sigma_{\pi/4}\) and all integer \(n\) we have

\[
\|T(z)(x_n)\| = \|\exp((-n + in + i\ln(n + 1))z)x_n\|
\]

and for the upper boundary values \(z = te^{i\pi/2}, t \geq 0\), we obtain

\[
\|T(z)(x_n)\| = \|\exp((-n + in + i\ln(n + 1))te^{i\pi/2})x_n\|
\]

\[
= \|\exp((-n\sqrt{2}e^{-i\pi/4} + i\ln(n + 1))te^{i\pi/2})x_n\|
\]

\[
= \|\exp(-n\sqrt{2}t - \frac{\ln(n + 1)}{\sqrt{2}}t)x_n\|.
\]

From this it follows that \(T(te^{i\pi/2})\) is well defined for all \(t \geq 0\) and all \((x_n) \in E\), which means that \(E_T^+ = E\).

Let us now examine the behavior of the semigroup for lower boundary values \(z = te^{-i\pi/2}, t \geq 0\). For such values \(z\) we obtain

\[
\|T(z)(x_n)\| = \|\exp((-n + in + i\ln(n + 1))te^{-i\pi/2})x_n\|
\]

\[
= \|\exp(i\frac{\ln(n + 1)}{\sqrt{2}}t)x_n\|
\]

\[
= \left( \sum_{n \geq 0} (n + 1)^{2t/\sqrt{2}}|x_n|^2 \right)^{1/2}.
\]

In order to ensure the existence of boundary values for semigroup \(T(z)_{Re(z) \geq 0}\), one must have \((n^\alpha x_n) \in \ell^2(\mathbb{N})\) for all \(\alpha \geq 0\) and all \((x_n) \in \ell^2(\mathbb{N})\). This is obviously not the case for an arbitrary \(x\) in \(\ell^2(\mathbb{N})\). However, we conclude that

\[
E_T = \{(x_n) \in \ell^2(\mathbb{N}) \mid (n^\alpha x_n) \in \ell^2(\mathbb{N}), \forall \alpha > 0\}.
\]

This is a dense subspace of \(\ell^2(\mathbb{N})\) as it contains the finitely supported sequences.
This example shows that the semigroup lives until a maturity which depends on the regularity of the initial data. So, for more regular positions (e.g. \( x \in D^\infty = \cap_{n\geq0} D(A^n) \)) such as \( x = (e^{-n})_{n\geq0} \), the limit \( \lim_{t\to0} T(\epsilon + te^{-i\theta}) \) exists uniformly for all \( t \geq 0 \) in compact subsets of \( \mathbb{R}^+ \).

Now, \( t \mapsto T(te^{i\phi}) \) and \( t \mapsto T(te^{-i\phi}) \) seen as functions on \([0, +\infty)\) with values respectively in the spaces \( \mathcal{L}(E^T, E) \) and \( \mathcal{L}(E_T, E) \), where \( E^T \) and \( E_T \) are equipped with an adequate topology, are called the boundary values of \( T \). The cases when \( E^T = E \) (resp. \( E_T = E \)) or \( D(A^n) \subset E^T \) (resp. \( D(A^n) \subset E_T \)) for some integer \( n \geq 1 \) are of particular interest. The first case is characterized in Proposition 1.1. The second one, which includes many partial differential equations, is the subject of the following proposition.

**Proposition 2.4.** Let \( A \) be the generator of a holomorphic \( C_0 \)-semigroup \( T \) of angle \( \phi \in (0, \pi/2] \) and \( n \geq 1 \) be an integer. Then \( D(A^n) \subset E^T \) (resp. \( D(A^n) \subset E_T \)) if and only if \( \|T(z)\|_{\mathcal{L}(D(A^n), E)} \) is locally bounded on \( \Sigma^+_{\phi} \) (resp. \( \Sigma^-_{\phi} \)).

**Proof.** We prove the claim for \( E^T \) and \( \Sigma^+_{\phi} \). Analogously, the same argument remains true for \( E_T \) and \( \Sigma^-_{\phi} \). Assume that \( n \geq 1 \) and \( D(A^n) \subset E^T \) and let \( \lambda_0 \in \rho(A) \). Then \( \lim_{t\to0} T(\epsilon + z)(\lambda_0 - A)^{-n}x \) converges for all \( x \in E \) uniformly for \( z \) in compact subsets of \( \Sigma^+_{\phi} \). It follows, by the uniform bounded principle \( \|T(\epsilon + z)(\lambda_0 - A)^{-n}\| \) is locally bounded in \( \Sigma^+_{\phi} \).

Conversely, if \( T(z)(\lambda_0 - A)^{-n} \) is locally bounded in \( \Sigma_{\phi} \cap \{z; \pm \text{Im } (z) \geq 0\} \), since \( E^T_\pm \) is dense it follows that \( \lim_{t\to0}(\lambda_0 - A)^{-n}T(\epsilon + t \exp(+i\phi))x = (\lambda_0 - A)^{-n} \lim_{t\to0} T(\epsilon + t \exp(+i\phi))x \) exists for all \( x \in E \) uniformly for \( t \) in compact subsets of \([0, +\infty)\). 

In practice the estimate on \( \|T(z)\|_{\mathcal{L}(D(A^n), E)} \) is not easy to establish and one prefers to estimate \( \|T(z)\|_{\mathcal{L}(E)} \). The following proposition gives a sufficient condition on \( \|T(z)\|_{\mathcal{L}(E)} \) to guarantee \( D(A^n) \subset E^T \cap E_T \).

**Proposition 2.5.** Let \( A \) be the generator of a holomorphic \( C_0 \)-semigroup \( T \) of angle \( \phi \in (0, \pi/2] \) and \( n \geq 1 \) be an integer. Assume that there exists some \( \gamma \in \{0, n\} \) such that the function \( z \mapsto (|z|d(z, \partial D_{\phi}))^\gamma \|T(z)\|_{\mathcal{L}(E)} \) is locally bounded on \( \Sigma_{\phi} \). Then \( D(A^n) \subset E^T \cap E_T \).

Observe first that for all \( z \) in the sector \( \Sigma_{\phi} \), one has \( d(z, \partial \Sigma_{\phi}) = |\sin(\phi - |\arg(z)|)| \). This identity will be used in the following proof.

**Proof.** Without loss of generality we can assume that \( 0 \in \rho(A) \). Let \( n \geq 1 \), \( x \in E \) and \( z \in \Sigma^+_{\phi} \). By the Taylor formula we have

\[
T(z)A^{-n}x = A^{-n}x + zA^{-n+1}x + \ldots + \frac{z^{n-1}A^{-1}x}{(n-1)!} + \frac{1}{(n-1)!} \int_0^z (z - \zeta)^{n-1}T(\zeta)xd\zeta.
\]

It suffices then to show that \( z \mapsto \int_0^z \frac{(z - \zeta)^{n-1}}{(n-1)!}T(\zeta)xd\zeta \) is locally bounded on \( \Sigma^+_{\phi} \). Let us denote \( (|z|, z) \) the circular path \( |z|e^{i\theta}, 0 \leq \theta \leq \arg(z) \). Writing

\[
\int_0^z (z - \zeta)^{n-1}T(\zeta)xd\zeta = \int_0^{|z|} (z - \zeta)^{n-1}T(\zeta)xd\zeta + \int_{(|z|, z)} (z - \zeta)^{n-1}T(\zeta)xd\zeta,
\]

we get
and since \((T(t))_{t \geq 0}\) is strongly continuous, we need only to show the local boundedness of the second integral. Let \(K \subset \Sigma^+_{\phi}\) be a compact and \(c_K > 0\) such that
\[
(|z|\sin(\phi - |\arg(z)|))^{\gamma} \|T(z)\|_{\mathcal{L}(E)} \leq c_K
\]
for all \(z \in K\). A direct calculation gives
\[
\left\| \int_{\{|z|,z\}} \frac{(z-\zeta)^{n-1}}{(n-1)!} T(\zeta) x d\zeta \right\| \leq \frac{1}{(n-1)!} \int_0^{\phi} |\sin(\phi - |\arg(\zeta)|)|^{\gamma} \int_0^{\phi} \sin^{n-1-\gamma}(\theta) d\theta.
\]
When \(z \in \Sigma^-_{\phi}\), one may consider mutatis mutandis the circular path \(|z|e^{i\theta}\), \(\arg(z) \leq \theta \leq 0\) and do the same calculation. \(\square\)

**Remark 2.6.** Assuming in Proposition 2.5 only
\[
\sup_{z \in \Sigma_{\phi} \cap D} (|z|d(z, \partial \Sigma))^\gamma \|T(z)\| < \infty
\]
then we obtain analogously for \(x \in D(A^{np})\) that \(\lim_{\epsilon \to 0} T(\epsilon + z)x\) converges uniformly for \(z \in \Sigma^+_{\phi}\), \(|z| \leq p\) for all integer \(p \geq 1\). In other words, to guarantee the existence of the boundary value \(T(te^{i\pm\phi})x\) for more time \((0 \leq t \leq p)\) we need more regularity on the initial data (compare with [4]).

### 3. Hadamard-type fractional integrals

In the articles [7, 8], [26], the Hadamard fractional integral was considered along with a generalization and the semigroup property was established. The Hadamard fractional integral first appeared in the paper [21] on the study of functions presented by power series. This fractional integral has been studied extensively in the monograph [35] along with the associated concept of fractional derivative. Here, we prove the semigroup property and obtain that the semigroups involved are holomorphic with angle \(\pi/2\) on a class of weighted \(L^p\) spaces. Moreover, we show that these semigroups admit a boundary value group when \(1 < p \leq \infty\). We shall consider two of the operator families studied in [26, 7]. The other families can be treated with the methods of the present section. Our approach uses abstract semigroup theory in contrast with [7, 8, 26] where the authors proceed with direct computation. Moreover, the above cited papers do not consider complex parameters in the semigroups. We shall obtain as a consequence the representation of the semigroup powers of the Cesàro averaging operator.

The (generalized) Hadamard fractional integral of a function \(f\) is defined as follows (see [26, 7, 8])
\[
(J^\alpha_\mu f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{t^{\mu}(\ln(\frac{x}{t}))^{\alpha-1} f(t)}{t^{\mu}} dt \quad (x > 0),
\]
where \(\alpha > 0\) and \(\mu \in \mathbb{R}\). The original definition corresponds to \(\mu = 0\) and is discussed in [35, Chapter 4, Section 18.3]. As in [26, 7, 8], we consider the Banach spaces
\[
X^p_{c} = \{ f : (0, 1) \to \mathbb{C}, f \text{ measurable and } \|f\|_{c,p} = \left( \int_0^1 |x^c f(x)|^p \frac{dx}{x} \right)^{1/p} < \infty \}.
\]
The space $X^p_0$ is a Banach space and if $c = \frac{1}{p}$, it coincides with $L^p(0, 1)$. We note that if $\alpha = \mu = 1$ then $(J_1^1 f)(x) = \frac{1}{x} \int_0^x f(t)dt$, $x > 0$, which is the Cesàro operator, so that $L^p$ boundedness of $J_1^1$ is to be compared to Hardy’s inequality. It follows that the semigroup $(J_1^\alpha)$ represents the fractional powers of the Cesàro operator.

We shall prove the following result on the boundary values of the Hadamard -type fractional integral.

**Theorem 3.1.** Let $1 \leq p < \infty$ and $\mu, c \in \mathbb{R}$ with $\mu > c$. Then the family of operators $(J_\mu^\alpha)_{\alpha>0}$ acting on the space $X^p_0$ forms a strongly continuous semigroup which has an analytic extension to the right half-plane $\mathbb{C}_+ = \{\alpha \in \mathbb{C} \mid \text{Re}(\alpha) > 0\}$. Moreover, the semigroup $(J_\mu^\alpha)_{\alpha>0}$ has a boundary $C_0$-group. More precisely, $(J_\mu^{is})_{s \in \mathbb{R}}$, given by

$$
(J_\mu^{is} f) = \lim_{\sigma \to i\infty} (J_\mu^{\sigma+is} f), \quad \forall f \in X^p_0
$$

and $(J_\mu^{is})_{\mu \in \mathbb{R}}$ forms a $C_0$-group, provided $1 < p < \infty$.

**Proof.** We consider the operator family

$$
(T(t)f)(x) = e^{-\mu t} f(e^{-t}x), \quad x \in (0, 1), \quad t > 0.
$$

Then it is readily verified that $T = (T(t))_{t \geq 0}$ is a strongly continuous group on the space $X^p_0$ defined above. The infinitesimal generator of $T$ is the operator $A = x \frac{d}{dx} - \mu I$. If $\mu > c$ then $T$ is exponentially stable. In fact,

$$
\|T(t)f\|_{X^p_0}^p = \int_0^1 e^{-p(x^c f(e^{-t}x))} \frac{dx}{x} = e^{-p(\mu-c)t} \int_0^1 e^{-u} \|u^{c} f(u)\|_{X^p_0} \frac{du}{u} \\
\leq e^{-p(\mu-c)t} \|f\|_{X^p_0}^p.
$$

The fractional powers $A^{-\alpha}$ for $\alpha > 0$ are given by the well-known formula (see e.g. [16, Formula (3.56)], [27, Proposition 11.1]) or [20] Proposition 3.3.5 and Corollary 3.3.6).

$$
((-A)^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (T(t)f)(x) dt, \quad f \in X.
$$

By a change of variable in the integral, we have for $f \in X$ :

$$
((-A)^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (T(t)f)(x) dt \\
= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\mu t} f(e^{-t}x) dt \\
= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left( \frac{t}{x} \right)^\mu (\ln(x/t))^{\alpha-1} f(t) \frac{dt}{t}
$$

which is (3.3). From this representation, the semigroup property for the family $(J_\mu^\alpha)_{\alpha>0}$ follows by the general theory of fractional powers of operators (see e.g. [20] Proposition 3.2.3]). Analyticity also follows from the general theory.
In order to obtain the last assertion, we note that \((T(t))\) is a strongly continuous semigroup of positive contraction operators on the space \(X = L^p(0, 1), 1 < p < \infty\). The conclusion is obtained by application of the Coifman-Weiss transference principle (see [1] Theorem 3.9.5, [12]).

Let us consider the particular case where \(\mu = 1\).

**Corollary 3.2.** Assume that \(1 < p < \infty\). Then the family
\[
T(z)f(x) = ((-A)^{-z}f)(x) = \frac{1}{x\Gamma(z)} \int_0^x (\ln(\frac{x}{t}))^{z-1} f(t) dt, \quad x > 0, \quad \text{Re}(z) > 0
\]
forms a holomorphic semigroup of angle \(\frac{\pi}{2}\) in the space \(L^p(0, 1)\). This semigroup admits a boundary value group on the imaginary axis.

Another consequence of the above representation is the explicit description of the powers of the averaging operator of the Cesàro operator \(C\),
\[
(Cf)(x) := \frac{1}{x} \int_0^x f(s) ds, \quad f \in L^p(0, 1).
\]
Clearly \(C = T(1)\). The strong continuity of \(C\) in \(L^p(0, 1), 1 < p < \infty\) yields the Hardy’s inequality. Since Hardy’s inequality does not hold for \(p = 1\), we see that the condition \(\mu > c\) in 3.1 is sharp.

**Corollary 3.3.** For each \(n \in \mathbb{N}\) and \(f \in L^p(0, 1)\) we have
\[
(C^nf)(x) = \frac{1}{x(n-1)!} \int_0^x (\ln(\frac{x}{t}))^{n-1} f(t) dt.
\]
This is of course a direct consequence of the semigroup property: \(C^n = (T(1))^n = T(n)\).

We observe that this formula was obtained by D. W. Boyd [5, Lemma 2] who used mathematical induction. He used this result to study the spectral radius of averaging operators. The spectral theory of the Cesàro operator (including the discrete version) has been studied in several papers, (see for example [2, Section 2] where the Boyd indices are used in the description of the spectrum in various Banach function spaces).

Boyd obtains the following formula (we consider the case \(a = 0\) in his formula)
\[
(C^nf)(x) = \frac{1}{(n-1)!} \int_0^1 (\ln(\frac{1}{s}))^{n-1} f(sx) ds, \quad f \in L^p(0, 1)
\]
which is readily obtained from (3.5) by a change of variable. In fact, the semigroup \((T(t))\) in Corollary 3.2 can be written as follows
\[
(T(z)f)(x) = \frac{1}{\Gamma(z)} \int_0^1 (\ln(\frac{1}{\sigma}))^{z-1} f(\sigma x) d\sigma, \quad f \in L^p(0, 1), \quad \text{Re}(z) > 0.
\]
We observe that the above theorem and its corollaries remain valid if we replace \(L^p(0, 1)\) with \(L^p(0, a)\) where \(a \in (0, \infty]\). We state this below for the case \(a = \infty\). For that we introduce the space
\[
X^p_c := \{ f : (0, \infty) \longrightarrow \mathbb{C}, \ f \ \text{measurable} \text{ and } \|f\|_{c,p} = \left( \int_0^\infty |x^c f(x)|^p \frac{dx}{x} \right)^{1/p} < \infty \}.
\]
Theorem 3.4. Let \( 1 \leq p < \infty, \mu, c \in \mathbb{R}, \) with \( \mu > c \in \mathbb{R} \) then the family of operators \((J_\mu^\alpha)_{\alpha>0}\) acting on the space \(X^p_c\) forms a strongly continuous semigroup which has an analytic extension to the right half-plane \(\mathbb{C}_+\). Moreover, the semigroup \((J_\mu^\alpha)_{\alpha>0}\) has a boundary \(C_0\)-group on \(X^p_c\) denoted \((J_\mu^s)^{\alpha} f(x) = \lim_{\sigma \to 0^+} (J_\mu^{\sigma + i\sigma} f)(x)\) provided \(1 < p < \infty\).

For the remainder of this section, we consider a second form of the Hadamard fractional integral operator to which the above construction applies (see again [26, 7, 8] and [35, Chapter 4, Section 18.3] for the case \(\mu = 0\)). Here we set
\[
(I_\mu^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{u^\mu (\ln \frac{u}{x})^{\alpha-1}}{u} f(u) \frac{du}{u}, \quad x > 0.
\]
We obtain the following counterpart of Theorem 3.1.

Theorem 3.5. Let \(1 \leq p < \infty\) and let \(\mu \in \mathbb{R}\) such that \(c + \mu > 0\). Then the family of operators \((I_\mu^\alpha)_{\alpha>0}\) acting on \(X^p_c\) forms a strongly continuous semigroup which has an analytic extension to the right half-plane \(\mathbb{C}_+\). Moreover, the semigroup has a boundary \(C_0\)-group on \(X^p_c\).

Proof. In order to obtain the result, we consider the semigroup:
\[
(T(t)f)(x) = e^{-\mu t} f(e^t x), \quad x > 0, \quad t > 0.
\]
acting on the space \(X^p_c\). Let denote by \(A\) its infinitesimal generator. Then
\[
\|T(t)f\|_{X^p_c} = \int_0^\infty |e^{-\mu t} f(e^t x)|^p \frac{dx}{x} = e^{-\mu pt} \int_0^\infty e^{-c pt} |u^c f(u)|^p \frac{du}{u} = e^{-p(c+\mu)t} \|f\|_{X^p_c}
\]
for every \(f \in X^p_c\) and \(t > 0\). Since \(c + \mu > 0\), we deduce that \(T\) is exponentially stable. Next, for \(\alpha > 0\) we have
\[
((-A)^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (T(t)f)(x) dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\mu t} f(e^t x) dt, \quad f \in X.
\]
Again, by a change of variable in the integral, we have for \(f \in X\) :
\[
((-A)^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \left(\frac{u}{x}\right)^\mu (\ln \frac{u}{x})^{\alpha-1} f(u) \frac{du}{u}, \quad x > 0,
\]
\[
= (I_\mu^\alpha f)(x).
\]
Now the proof is completed in the same way as in the proof of Theorem 3.1. \(\square\)

We remark that we can consider in the above results \(\mu \in \mathbb{C}\), in which case the conditions in Theorem 3.1 and Theorem 3.5 become \(\text{Re}(\mu) > c\) and \(\text{Re}(\mu) > -c\) respectively.
We observe that the theorem holds for $X = L^p(0, \infty)$ provided $\mu > -\frac{1}{p}$ because $X_c = L^p(0, \infty)$ if $c = \frac{1}{p}$. We observe that a consequence of the theorem is the second Hardy inequality which gives the $L^p$–continuity of $\mathcal{I}_0^1$ where $(\mathcal{I}_0^1 f)(x) = \int_x^\infty f(u) \frac{du}{u}$, $f \in L^p(0, \infty)$ for $1 \leq p < \infty$.

We now check that the above semigroups may be studied in the Lipschitz and Hölder spaces as the Riemann-Liouville semigroup considered in the next section. We will do this for the semigroup \([5,3]\) which we further simplify by taking $\mu = 0$. For $h > 0, t > 0$ and $f \in C[0, 1]$, we have by the Mean Value theorem:

$$
|t^{-\alpha}(T(t)f - f)(x)| = t^{-\alpha}|f(e^{-t}x) - f(x)| = t^{-\alpha}|f(x - te^{-\eta t}x) - f(x)| = \sup_{h > 0}(h^{-\alpha}|f(x - h) - f(x)|)
$$

It follows that Lip$^\alpha[0, 1] \subset F_\alpha$ for $0 < \alpha < 1$ as well as lip$^\alpha[0, 1] \subset X_\alpha$. Here, $F_\alpha$ and $X_\alpha$ are denote respectively the Favard space and the abstract Hölder space of order $\alpha$ (see [19, Chapter II]). On the other hand, if $0 < \alpha \leq 1$ and $s > 0$, from

$$
|t^{-\alpha}(T(t)T(s)f - T(s)f)| = t^{-\alpha}T(s)(T(t)f - f)
$$

that Lip$^\alpha[0, 1]$ is $T(t)$–invariant.

More information about mapping properties of operators with power logarithmic kernels such as the Hadamard fractional integrals can be found in [35, Paragraph 21] where $L^p$ spaces and spaces of Hölder continuous functions are considered. Our aim in this paper is to study the purely imaginary powers from the viewpoint of holomorphic semigroups.

4. THE RIEMANN-LIOUVILLE SEMIGROUP

We discuss the Riemann-Liouville semigroup in connection with the right translation semigroup $S$ acting on Lip$^\alpha[0, 1]$. Later, we shall study the Riemann-Liouville semigroup in $C_0[0, 1]$ and Lip$^\alpha_0[0, 1]$.

The spaces, which can be defined in connection with any strongly continuous semigroup, are called Favard classes (for Lip$^\alpha_0$, which are denoted by $F_\alpha$ in [19]) and Hölder spaces (for lip$^\alpha_0$, denoted by $X_\alpha$ in [19]) and are studied systematically in [8] in connection with approximation theory. In particular, one can find equivalent descriptions of these spaces in terms of the resolvent of the infinitesimal generator of the semigroup under consideration. They are also related to the continuous interpolation spaces and have been studied in relation to maximal regularity (for a reference, see [19, page 155]).

Analogously the Riemann-Liouville semigroup $J := (J(z))_{\Re(z) > 0}$ on $C[0, 1]$ given by \([1.1]\) is not strongly continuous since $J(t)1(s) = \frac{s^t}{\Gamma(t+1)}$ does not converge to 1 as $t \to 0^+$ (one may consider for example the point evaluations at $s = \frac{1}{2m}$ and $t = \frac{1}{n}$, $n \in \mathbb{N}$). But, its part in $C_0[0, 1]$ defines a $C_0$–holomorphic semigroup of angle $\pi/2$. Denoting the generator of the nilpotent translation semigroup $S$ (acting in $C_0[0, 1]$) by $B$ we have $\rho(B) = \mathbb{C}$ and

$$
J(z)f = \frac{1}{\Gamma(z)} \int_0^{+\infty} s^{z-1} S(s)f \, ds = (-B)^{-z}f,
$$
for all \( f \in C_0[0,1] \) where \((-B)^{-z}\) are the complex powers of \( B \) as defined for instance in 31 11 19 27. Recall that, for all \( 0 < \alpha < 1 \) the domain of the fractional power \((-B)^\alpha\), or equivalently the range of \((-B)^{-\alpha}\), equipped with the norm \( \|f\|(-B)^\alpha := \|(-B)^{-\alpha}f\|_\infty \) is a Banach space.

**Theorem 4.1.** The family of operators \( J = (J(z))_{Re(z) > 0} \) defines a strongly continuous holomorphic semigroup of angle \( \pi /2 \) in the space \( C_0[0,1] \). Moreover, there exist \( C_1, C_2 > 0 \) such that

\[
(4.1) \quad C_1 \frac{|z|}{Re(z)} \leq \|J(z)\|_\infty \leq C_2 \frac{|z|}{Re(z)} \quad (Re(z) > 0, |z| \leq 1).
\]

Furthermore, as operators acting on the space \( Lip_0^0[0, 1] \), \( 0 < \alpha < 1 \), we have

\[
(4.2) \quad \|J(z)\|_\alpha \leq C \frac{|z|}{Re(z)} \quad (Re(z) > 0)
\]

for some constant \( C > 0 \).

**Proof.** The fact that \( J \) is a holomorphic semigroup follows from the general discussion preceding the statement of the theorem. It remains to prove the estimates (4.1) and (4.2).

Set \( \mathcal{D}_+ := \{z \in \mathbb{C} \mid Re(z) > 0 \text{ and } |z| \leq 1\} \).

**Step 1:** Let \( \epsilon > 0, z \in \mathbb{C}^+ \). Consider the two functions \( f_\epsilon \in C[0,1] \) and \( g_\epsilon \in C_0[0,1] \) respectively defined by \( f_\epsilon(t) = (\epsilon + 1 - t)^{-i \text{Im}(z)} \) and \( g_\epsilon(t) := f_\epsilon(t) - f_\epsilon(0) \). For \( \epsilon \) small enough, one may find \( t_0 \in [0,1] \) such that \( |g_\epsilon| = 2 \) and then one has \( \|g_\epsilon\|_{Lip} = 2 \).

\[
\|J(z)\|_\infty \geq \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{2} \left| (J(z)g_\epsilon)(1) \right|
\]

\[
= \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\Gamma(z)} \int_0^1 y^{z-1} g_\epsilon(1 - y) dy
\]

\[
\geq \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\Gamma(z)} \int_0^1 e^{i \text{Im}(z) + Re(z) - 1} \ln(y) e^{-i \text{Im}(z) \ln(e+y)} dy
\]

\[
- \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\Gamma(z)} \int_0^1 y^{z-1} (\epsilon + 1)^{-i \text{Im}(z)} dy
\]

\[
\geq \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\Gamma(z)} \int_0^1 e^{(Re(z)-1) \ln(y)} \ln(e+y) dy - \frac{1}{2 \Gamma(z)} |z|.
\]

To obtain the first part in inequality in (4.1), we distinguish two cases: \( |\arg(z)| \geq \frac{\pi}{4} \) and \( |\arg(z)| < \frac{\pi}{4} \). In the latter case, To justify the result on \( C_0[0,1] \), it suffices to consider unity approximation sequence \( g_n(t) = \frac{1}{n} \left| 0, \frac{1}{n} \right| (t) + 1_{\left[ \frac{1}{n}, 1 \right]} (t) \). So
\[
\|J(z)\|_{L^\infty(C_0)} \geq \|J(z)g_n\|_{\infty} \\
\geq |\int_0^t g_n(t-s)s^{z-1}ds| \\
\geq |\liminf \int_0^t g_n(t-s)s^{z-1}ds| \\
\geq |\int_0^t \liminf g_n(t-s)s^{z-1}ds| \\
\geq \frac{1}{|z\Gamma(z)|} \\
\geq \frac{1}{2\Gamma(z+1)\Re(z)}
\]

In this case the desired estimate is obvious. In the second case (i.e \(|\arg(z)| \geq \frac{\pi}{4}\)), using Lebesgue’s dominated convergence theorem in the last estimate above we deduce that

\[
\|J(z)\|_{\infty} \geq \frac{1}{2|\Gamma(z)|\Re(z)} \left(1 - \frac{\Re(z)}{|z|}\right) \\
\geq \frac{1}{2|\Gamma(z)|\Re(z)} \left(1 - \frac{\sqrt{2}}{2}\right) \\
\geq \frac{1}{4|\Gamma(z+1)|\Re(z)}.
\]

This last estimate establishes the first part in inequality (4.1) with \(C_1 = \frac{1}{4|\Gamma(z+1)|}\).

**Step 2:** For each \(f \in C_0[0,1]\) and \(z \in D_+\) we have

\[
|\Gamma(z)J(z)f(t)| \leq \int_0^t |f(t-s)|s^{\Re(z)-1}ds \leq \frac{\|f\|_{\infty}}{\Re(z)}.
\]

It follows that the second inequality in (4.1) holds with \(C_2 := \sup_{z \in D_+} \frac{1}{|\Gamma(z+1)|}\).

**Step 3:**
Let \(f \in \text{lip}_0^\alpha[0,1]\) and \(z \in \mathbb{C}\) such that \(\Re(z) > 0\). By a direct computation we obtain that

\[
\|J(z)f\|_{\alpha} = \sup_{t \neq t'} \left| \frac{(t-t')^{-\alpha}}{\Gamma(z)} \left( \int_0^t \frac{f(t-s) - f(t'-s)}{s^{1-z}}ds + \int_t^{t'} \frac{f(t'-s)}{s^{1-z}}ds \right) \right|.
\]

It follows
Proposition 4.2. Let \( \beta < \alpha < 1 \) and \( J \) be the Riemann-Liouville semigroup in \( C_0[0, 1] \). Then:

(i) \( J(\alpha)C_0[0, 1] \subset \text{Lip}_0^\alpha[0, 1] \) and \( \|J(\alpha)f\|_\alpha \leq \frac{2}{|\text{Re}(\alpha+1)|} \|f\|_\infty \).

(ii) \( \text{Lip}_0^\alpha[0, 1] \hookrightarrow D((-B)^\beta) \).

In the next theorem, which is our main result, we show that the Riemann-Liouville semigroup \( J \) admits a boundary group on \( \text{Lip}_0^\alpha[0, 1] \). This group defines therefore the operation of fractional integration of imaginary order as a bounded strongly continuous group in the Hölder spaces. It will be important to note that in Proposition 2.1, we can replace \( D \) by \( D_R := \{ z \in \mathbb{C}, \text{Re}(z) > 0 \} | z | \leq R \} \) where \( R \) is any fixed positive real number. This simple fact is seen by observing that if \( R > 0 \) is a given number, and \( (T(z)) \) is a holomorphic semigroup of angle \( \frac{\pi}{2} \) whose infinitesimal generator is \( A \), then

\[
\sup_{z \in D_R} \|T(z)\|_{L(E)} = \sup_{z \in D} \|T(Rz)\|_{L(E)}.
\]

It suffices then to note that \( (T(Rt))_{t \geq 0} \) is a holomorphic semigroup and its generator is \( RA \).

Theorem 4.3. Let \( 0 < \alpha < 1 \). Consider the Riemann-Liouville semigroup \( J \) in the space \( \text{Lip}_0^\alpha[0, 1] \). Then
The first item (i) is an immediate consequence of Theorem 4.1. We shall prove the

\begin{proof}
In addition, \( \sup_{z \in D_R} \|J(z)\|_{L(E)} < \infty \).

Consequently the semigroup \( J \) admits a boundary group on the imaginary axis.

\begin{proof}
The first item (i) is an immediate consequence of Theorem 4.1. We shall prove the estimate (ii) with \( R = \frac{1}{2\pi} \). Let \( h = t - t' > 0 \) where \( (t, t') \in [0, 1]^2 \) and \( z \in D_R \). Then we have

\[
\begin{align*}
\Gamma(z)(J(z)f(t) - J(z)f(t')) & = \int_0^t f(t - s)s^{z-1}ds - \int_0^{t'} f(t' - s)s^{z-1}ds \\
& = \int_{-h}^{t'} f(t' - s)(s + h)^{z-1}ds + \int_0^t f(t' - s)s^{z-1}ds \\
& = \int_{-h}^0 f(t' - s)(s + h)^{z-1}ds + \int_0^{t'} f(t' - s)((s + h)^{z-1} - s^{z-1})ds \\
& = \int_{-h}^0 f(t' - s)(s + h)^{z-1}ds + \int_0^{t'} (f(t') - f(t'))((s + h)^{z-1} - s^{z-1})ds + \\
& + \int_0^{t'} f(t')((s + h)^{z-1} - s^{z-1})ds \\
& =: I_1 + I_2 + I_3.
\end{align*}
\end{proof}

First, we estimate the sum of the two integrals \( I_1 \) and \( I_3 \). We have by a direct computation

\[
\begin{align*}
I_1 + I_3 & = \int_0^h f(t' - s + h)s^{z-1}ds + f(t') \int_0^{t'} [(s + h)^{z-1} - s^{z-1}]ds \\
& = \int_0^h f(t - s)s^{z-1}ds + f(t') \int_0^{t'} [(s + h)^{z-1} - s^{z-1}]ds \\
& = \int_0^h \frac{f(t - s) - f(t)}{s^\alpha}s^{\alpha+z-1}ds + \int_0^h f(t)s^{z-1}ds + f(t') \int_0^{t'} [(s + h)^{z-1} - s^{z-1}]ds \\
& = \int_0^h \frac{f(t - s) - f(t)}{s^\alpha}s^{\alpha+z-1}ds + \int_0^h \frac{f(t)}{z}h^z + \frac{f(t)}{z}(t^z - h^z - t'^z) \\
& + \int_0^h \frac{f(t)}{z}h^z + \frac{f(t)}{z}(t^z - t'^z). \\
& =: K_1 + K_2 + K_3
\end{align*}
\]

It is easy to check that for \( \Re(z) > 0 \),

\[
|K_1| \leq \int_0^h \|f\|_\alpha s^{\Re(z)+\alpha-1}ds \leq \|f\|_\alpha \frac{h^{\alpha+\Re(z)}}{\alpha + \Re(z)} \leq \|f\|_\alpha h^{\alpha}/\alpha.
\]
and

\[(4.4) \quad |K_2| \leq \frac{h^{\alpha + \text{Re}(z)} \|f\|_\alpha}{|z|} \leq \frac{h^{\alpha} \|f\|_\alpha}{|z|}.
\]

To estimate the term \(K_3\) we discuss two following two cases

**Case 1:** \(t' \leq 2h\). Then

\[
\left| \frac{1}{h^\alpha} \frac{f(t')}{z} (t^z - t'^z) \right| \leq \frac{t'^\alpha}{h^\alpha} \left| t^z - t'^z \right| \frac{\|f\|_\alpha}{|z|}
\]

\[
\leq \frac{t'^\alpha}{h^\alpha} (t^{\text{Re}(z)} + t'^{\text{Re}(z)}) \frac{\|f\|_\alpha}{|z|}
\]

\[
\leq 2^{1+\alpha} \frac{|z|}{h^\alpha} \|f\|_\alpha
\]

where we have used that \(|f(t')| \leq t'^{\alpha} \|f\|_\alpha\).

**Case 2:** \(t' > 2h\). Then we have

\[
\left| \frac{1}{h^\alpha} \frac{f(t')}{z} (t^z - t'^z) \right| \leq \frac{t'^\alpha}{h^\alpha} \frac{\|f\|_\alpha}{|z|} |t^z - t'^z|
\]

\[
\leq \frac{1}{|z|} \frac{t'^\alpha}{h^\alpha} \frac{\|f\|_\alpha}{|z|} \left| \int_{t'}^t z^u u^{\alpha - 1} du \right|
\]

\[
\leq \|f\|_\alpha h^{1-\alpha} t'^\alpha \sup_{t' < u < t} u^{\text{Re}(z) - 1}
\]

\[
\leq \|f\|_\alpha h^{1-\alpha} t'^\alpha + \text{Re}(z) - 1
\]

\[
\leq \|f\|_\alpha 2^{\alpha + \text{Re}(z) - 1} h^{\alpha + \text{Re}(z) - 1}
\]

\[
\leq \|f\|_\alpha
\]

Thus we conclude that

\[(4.5) \quad K_3 \leq \frac{2^{1+\alpha}}{|z|} h^\alpha \|f\|_\alpha.
\]

Combining (4.3), (4.4) and (4.5) we deduce that

\[(4.6) \quad I_1 + I_3 \leq h^\alpha \|f\|_\alpha \left( \frac{1 + 2^{1+\alpha}}{|z|} + \alpha^{-1} \right).
\]
Now we come back to $I_2$

$$I_2 = \int_0^t (f(t' - s) - f(t'))[v(s + h)z^{-1} - s^{-1}]ds$$

| ≤ \|f\|_\alpha \int_0^t s^\alpha |v(s + h)z^{-1} - s^{-1}|ds |
| ≤ \|f\|_\alpha \int_0^t s^\alpha Re(z)^{-1}[(1 + \frac{s}{h})z^{-1} - (\frac{s}{h})z^{-1}]ds |
| ≤ \|f\|_\alpha \int_0^t \frac{h}{Re(z)} + \alpha s^\alpha |v(1 + s)z^{-1} - s^{-1}|ds |
| ≤ \|f\|_\alpha \frac{h}{Re(z)} + \alpha \int_0^{\infty} s^{\alpha + Re(z) - 1}[(1 + \frac{1}{s})z^{-1} - 1]ds |
| = \|f\|_\alpha \frac{h}{Re(z)} + \alpha (J_1 + J_2). |

As above, we will treat separately $J_1$ and $J_2$. For $J_1$, one can verify that if $Re(z) < 1$, we have

$$J_1 = \int_0^2 s^{\alpha + Re(z) - 1}[(1 + \frac{1}{s})z^{-1} - 1]ds$$

| ≤ \int_0^2 s^{\alpha + Re(z) - 1}[(1 + \frac{1}{s})Re(z)* - 1]ds |
| ≤ \int_0^2 s^{\alpha + Re(z) - 1}(2Re(z)* - 1)ds |
| ≤ (2Re(z)* - 1)(\frac{2Re(z) + \alpha}{Re(z) + \alpha}) |
| ≤ (\frac{z}{Re(z) + \alpha}) |

We now estimate $J_2$. For that we note that $|(1 + \frac{1}{s})z^{-1} - 1| \leq \frac{1}{s}|z - 1|$ for every $s > 0$ and $Re(z) < 1$. Indeed, if $g(u) = (1 + u)z^{-1}$, $u \geq 0$ then $g'(u) = (z - 1)(1 + u)z^{-2}$. By the mean value inequality, $|g(u) - g(0)| = |(1 + u)z^{-1} - 1| \leq u \sup_{c > 0} |(z - 1)(1 + c)z^{-2}|$. But $|(z - 1)(1 + v)z^{-2}| = |(z - 1)|(1 + v)Re(z) - 2| \leq |z - 1|$ for $Re(z) < 1$. From this, it follows that

$$J_2 = \int_2^{+\infty} s^{\alpha + Re(z) - 1}[(1 + \frac{1}{s})z^{-1} - 1]ds \leq |z - 1| \int_2^{+\infty} s^{\alpha + Re(z) - 2}ds$$

| ≤ \frac{|z - 1|}{1 - \alpha - Re(z)} |
for all $z \in \mathbb{C}^+$ such that $1 - (\text{Re}(z) + \alpha) > 0$. Therefore, we have
\[
I_2 \leq \|f\|_\alpha h^{\alpha + \text{Re}(z)} \left[ \frac{2^{1+\alpha + \text{Re}(z)}}{\text{Re}(z) + \alpha} + \frac{|z - 1|}{1 - \alpha - \text{Re}(z)} \right].
\]
From this, we conclude that for $0 < |z| \leq \frac{1 - \alpha}{2}$ we have
\[
I_2 \leq h^{\alpha} \|f\|_\alpha \left( \frac{4}{\alpha} + \frac{2|z - 1|}{1 - \alpha} \right).
\]
Finally, combining (4.6) and (4.7) we obtain
\[
\|J(z)\|_\alpha \leq \frac{1}{\Gamma(z)} \left( \frac{5}{\alpha} + \frac{2|z - 1|}{1 - \alpha} + \frac{1 + 2^{1+\alpha}}{\Gamma(z + 1)} \right)
\]
for $0 < |z| \leq \frac{1 - \alpha}{2}$. This completes the proof of the theorem.

Recall that $S$ is a strongly continuous semigroup on $C_0$. Since $C_{00} \subset D((-B)^\alpha) = J(\alpha)C_0 \hookrightarrow \text{Lip}_0^\alpha[0,1]$ we obtain that $D((-B)^\alpha)$ is dense in $\text{Lip}_0^\alpha[0,1]$. Seen as a linear application on $C_0[0,1]$ with values in $D((-B)^\alpha)$, $J(\alpha)$ is an isomorphism. We deduce from above that $J$ induced also a holomorphic semigroup in $D((-B)^\alpha)$ of angle $\pi/2$ which is not locally bounded. The following diagrams illustrate the foregoing description.

On one hand, the behavior of $J(z)$ on $C_0[0,1]$ and on $D((-B)^\alpha)$ is the same since it embeds the first space in the second so the following diagram is commutative
\[
\begin{array}{ccc}
C_0[0,1] & \xrightarrow{J(z)} & C_0[0,1] \\
\downarrow J(\alpha) & & \uparrow J(\alpha)^{-1} \\
D((-B)^\alpha) & \xrightarrow{J(z)} & D((-B)^\alpha)
\end{array}
\]
On the other hand, according to theorem 4.3, $J(z)_{z}$ is locally bounded on $\text{Lip}_0^\alpha[0,1]$ but not on $D((-B)^\alpha)$. It yields that the diagram
\[
\begin{array}{ccc}
D((-B)^\alpha) & \xrightarrow{J(z)} & D((-B)^\alpha) \\
\Downarrow & & \Downarrow \\
\text{Lip}_0^\alpha[0,1] & \xrightarrow{J(z)} & \text{Lip}_0^\alpha[0,1]
\end{array}
\]
cannot commute.

It is important to recall that the norms considered here are the natural ones induced by the considered embeddings. For example, $D((-B)^\beta)$ is endowed with the norm $\|y\|_{D((-B)^\beta)} = \|J(\beta)x\|_\infty$ where $x$ is the unique element such as $J(\beta)x = y$. As an immediate consequence we remark that since $J(z)$ is locally bounded in $\text{Lip}_0^\alpha[0,1]$ but it is not in $D((-B)^\alpha)$, then in truth $D((-B)^\alpha) \subset \text{Lip}_0^\alpha[0,1]$. 
5. FROM THE BOUNDARY GROUP TO THE SEMIGROUP

This section is devoted to the inverse problem on the half plane. Let \((T(it))_{t \in \mathbb{R}}\) be a \(C_0\)-group with generator \(iA\). Is it the boundary value of some holomorphic semigroup of angle \(\pi/2\)?

A great work was done before in this direction. This problem was studied in [16] and [18] and affirmatively solved under spectral conditions \((s(A) := \sup \{\Re(z); \ z \in \sigma(A)\} < +\infty\) on the generator and a growth condition (non quasi analyticity) on the group. The problem was also studied by Zsidó and Cioranescu [11] in terms of analytic generators.

Let \(iA\) be the generator of the \(C_0\)-group \((T(it))_{t \in \mathbb{R}}\) in the \(E\) and \(w \geq 0\) such that \(\sup_{t \in \mathbb{R}} e^{-w|t|}\|T(it)\| < \infty\). Then \(\sigma(iA) \subset \{z \in \mathbb{C}; \ -w \leq \Re(z) \leq w\}\) which means \(\sigma(A) = \sigma \subset \{z \in \mathbb{C}; \ -w \leq \Im(z) \leq w\}\). In this section, we assume that:

\[
\sigma = \sigma_1 \cup \sigma_2 \text{ such that } \sigma_1 \subset \{z \in \mathbb{C}, \Re(z) < -\delta\} \text{ and } \sigma_2 \subset \{z \in \mathbb{C}, \Re(z) > +\delta\}
\]

for some \(\delta > 0\). This situation generalizes that one treated in [15] where the spectrum \(\sigma\) is assumed belonging entirely in a half plane with a single connected component \((\sigma \subset \mathbb{C}_{-})\).

A similar and more general result is given by the following:

**Proposition 5.1.** Let \((T(it))_{t \in \mathbb{R}}\) be a \(C_0\)-group in \(E\) with generator \(iA\). Assume furthermore that \(\sigma(A) = \sigma_1 \cup \sigma_2\) with \(\sigma_1 \subset \{z \in \mathbb{C}, \Re(z) < -\delta\}\) and \(\sigma_2 \subset \{z \in \mathbb{C}, \Re(z) > +\delta\}\). Then there exist two \(T(it)\)-invariant and closed subspaces \(E_1\) and \(E_2\) of \(E\) such that:

(i) \(D(A) \subset E_1 \oplus E_2\),
(ii) \(A_1 := A|_{E_1}\) generates a holomorphic semigroup \(T_1\) of angle \(\pi/2\) in \(E_1\) and \(\sigma(A_1) = \sigma_1\),
(iii) \(-A_2 := -A|_{E_2}\) generates a holomorphic semigroup \(T_2\) of angle \(\pi/2\) and \(\sigma(A_2) = \sigma_2\).

Proposition 5.1 generalizes the splitting theorem established in [15]. Indeed, it suffices to choose \(\sigma_1 = \emptyset\) in order to recover the main theorem 1 therein.

The proof gives an explicit construction of these spaces and semigroups based on a \(A\)-bounded spectral projection.

Let \((T(it))_{t \in \mathbb{R}}\) be a \(C_0\)-group in \(E\) with generator \(iA\). Then there exists \(w \geq 0\) such that \(M := \sup_{t \in \mathbb{R}} e^{-w|t|}\|T(it)\| < \infty\). In particular \(\sigma_1 \subset \{z \in \mathbb{C}; \ -w \leq \Im(z) \leq w, \Re(z) \geq \delta\}\) and \(\sigma_2 \subset \{z \in \mathbb{C}; \ -w \leq \Im(z) \leq \omega, \Re(z) \leq -\delta\}\). Moreover, the resolvent of \(\pm iA\) is given by the Laplace transform of \((\pm it)\) and consequently

\[
|\Re(\lambda) - w||R(\lambda, \pm iA)| \leq M \quad \text{for } \Re(\lambda) \geq w.
\]

The latter estimate guarantees in particular the absolute convergence of all integrals involved below.

Let \(0 < r < \delta\) and \(\beta \in (\pi/4, \pi)\) such that \(\sigma_1\) lies in the left side of the path \(\Gamma = (-\infty e^{-i\beta}, re^{-i\beta}] \cup [re^{-i\beta}; -\beta \leq \theta \leq \beta] \cup [re^{i\beta}, +\infty e^{i\beta})\). For all \(t > 0\) define

\[
T_1(t)x := \frac{1}{2i\pi} \int_{\Gamma} e^{z\lambda}R(\lambda, A)x d\lambda, \quad T_2(t)x := \frac{1}{2i\pi} \int_{\Gamma} e^{z\lambda}R(\lambda, -A)x d\lambda,
\]

then \(T_k(t) \in L(E)\) is well defined and does not depend on the choice of \(\beta, r\) for \(k = 1, 2\). Furthermore, we have the following lemma.

**Lemma 5.2.** Let \(k \in \{1, 2\}\). The family \((T_k(t))_{t \geq 0}\) satisfies the semigroup property .
Proof. Let \( k = 1 \) and \( t, s > 0 \). Let \( \Gamma' = \Gamma'(r', \beta') \) with \( \beta' > \beta \) and \( r' < r \). We have by the resolvent equation and the Cauchy formula:

\[
T_1(t)T_1(s)x = T_1(t) \frac{1}{2\pi i} \int_{\Gamma'} e^{s\lambda} R(\lambda, A)x d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} R(\lambda, A) (\frac{1}{2\pi i} \int_{\Gamma'} e^{s\mu} R(\mu, A) d\mu) d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} (\frac{1}{2\pi i} \int_{\Gamma'} e^{s\mu} R(\mu, A) R(u, A) x d\mu) d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} (\frac{1}{2\pi i} \int_{\Gamma'} e^{su} \frac{R(\lambda, A) + R(\mu, A)}{\mu - \lambda} x d\mu) d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} R(\lambda, A) x d\lambda
\]

\[
= T_1(t + s)x \quad \text{for all} \quad x \in E.
\]

Replacing \( A \) by \(-A\) we obtain also by this calculation \( T_2(t)T_2(s) = T_2(t + s) \). \qed

Observe that for all \( s > 0 \) and all \( x \in D(A) \) we have

\[
T_1(s)x - x = \frac{1}{2\pi i} \int_{\Gamma} e^{s\lambda} R(\lambda, A)x d\lambda - x
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} e^{s\lambda} [R(\lambda, A)x - \frac{x}{\lambda}] d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} e^{s\lambda} \frac{R(\lambda, A)Ax}{\lambda} x d\lambda.
\]

The right hand side in the last equality does not always coincide with \( x \) when \( s \to 0 \), but thanks to the Lebesgue theorem, one sees that

\[
\lim_{s \to 0} T_1(s)x = \frac{1}{2\pi i} \int_{\Gamma} \frac{R(\lambda, A)Ax}{\lambda} d\lambda + x,
\]

for all \( x \in D(A) \). It is so allowed to define an unbounded projection \( P \quad x \mapsto Px = \frac{1}{2\pi i} \int_{\Gamma} \frac{R(\lambda, A)Ax}{\lambda} d\lambda + x \).

The relation (5.2) says that

\[
\lim_{s \to 0} T_1(s)x = Px.
\]

The semigroup property and identity (5.3) ensure that \( P \) is an unbounded projector, which means \( P^2x = Px \) for all \( x \in D(A^2) \). Indeed one may write \( \lim_{t+s \to 0} T_1(t + s)x = Px \) where \( t \) and \( s \) are assumed to be positive and so converge both to zero. But \( \lim_{t \to 0} T_1(t) \lim_{s \to 0} T_1(s)x = \lim_{t \to 0} T(t)Px = P^2x \). (One may simplify by taking \( t = s \).)

We consider the spaces:

\[
E_1 = R(PA^{-1}) = \{ x \in X, \ PA^{-1}x = A^{-1}x \}
\]
and

\[ E_2 = \text{Ker}(PA^{-1}) = \{ x \in X; \ PA^{-1}x = 0 \}. \]

Denoting again by \( T_k \) the restriction of \( T_k \) to \( E_k \) \((k = 1, 2)\), we have:

**Lemma 5.3.** The operators \( A_1 \) and \( A_2 \) in Proposition 5.7 satisfy the following:
i) \( A_1 \) generates the holomorphic semigroup \( T_1 \) on \( E_1 \).
ii) \( -A_2 \) generates a holomorphic semigroup \( T_2 \) on \( E_2 \). \( \sigma(A_1) = \sigma_1 \) and \( \sigma(A_2) = \sigma_2 \).

**Proof.** i) For all \( x \in E_1 \) and \( \Re(z) > \delta \) we have

\[
\int_0^\infty \exp(-zt)T_1(t)A_1^{-1} x dt = \frac{1}{2\pi i} \int_\Gamma \exp(-zt) \exp(\lambda t) \frac{R(\lambda, A)x + A^{-1}x}{-\lambda} d\lambda dt
\]

\[
= -\frac{1}{2\pi i} \int_\Gamma \int_0^\infty \exp((-z + \lambda)t) \frac{R(\lambda, A)x}{\lambda} dt d\lambda - \frac{1}{2\pi i} \int_0^\infty \exp((-z + \lambda)t) \frac{A^{-1}x}{\lambda} dt d\lambda
\]

\[
= \frac{1}{2\pi i} \int_\Gamma \frac{R(\lambda, A)x}{\lambda} \, d\lambda + \frac{1}{2\pi i} \int_\Gamma \frac{A^{-1}x}{\lambda(-z + \lambda)} \, d\lambda = \int_\Gamma \frac{R(\lambda, A)x}{\lambda} \, d\lambda + \frac{A^{-1}x}{z}.
\]

From which we deduce that

\[
(z - A_1) \int_0^\infty \exp(-zt)T_1(t)A_1^{-1} x dt = PA^{-1}x = A^{-1}x,
\]

which means that \( R(z, A_1)A_1^{-1}x = \int_0^\infty \exp(-zt)T_1(t)A_1^{-1} x dt \) if \( z \in \rho(A_1) \). It follows from the Phragmen-Lindelöf Theorem that

\[
\sup_{z \in \rho(A_1), \Re(z) > 0} ||zR(z, A_1)|| < \infty.
\]

Then \( \{\Re(z) > 0\} \subset \rho(A_1) \) and \( \sup_{\Re(z) > 0} ||zR(z, A_1)|| < \infty \). The claim follows by the well known Phragmén-Lindelöf Theorem (see \[36\] or \[37\]).

ii) Analogously, considering \( -A \) instead of \( A \) and using a the change of variable \( \lambda \rightarrow -\lambda \) we obtain \( -A_2 \) is the generator of an holomorphic semigroup with angle \( \pi/2 \) in \( E_2 \).

iii) From (i) we conclude \( \sigma(A_1) \subset \sigma_1 \). A similar purpose prove that \( \sigma(A_2) \subset \sigma_2 \).

It suffices now to prove that \( \sigma(A_1) \supseteq \sigma_1 \). If it is not the case, there exists \( \lambda_1 \in \mathbb{C} \) such that \( \lambda_1 \in \sigma_1 \setminus \sigma(A_1) \). This implies that \( \lambda_1 \in \rho(A_1) \). To obtain a contradiction, we prove that \( \lambda_1 \in \rho(A) \). Let us establish that it is injective and surjective:

\( \lambda_1 - A \) is injective: for \( x \neq 0 \) verifying \( Ax = \lambda_1x \), we have for all \( \lambda \in \Gamma \) the identity

\[
R(\lambda, A)x = \frac{x}{\lambda - \lambda_1},
\]

\[
PA^{-1}x = \frac{1}{2\pi i} \int_\Gamma \frac{R(\lambda, A)x}{\lambda} d\lambda + A^{-1}x
\]

\[
= \frac{1}{2\pi i} \int_\Gamma \frac{x}{\lambda(\lambda_1 - \lambda)} d\lambda + A^{-1}x
\]

\[
= \frac{x}{\lambda_1} + A^{-1}x
\]

\[
= 0.
\]
But $PA^{-1}x = 0$ implies that $x \in E_1$ so $Ax = A_1x = \lambda_1 x$. Since $\lambda_1 \in \rho(A_1)$ then $x = 0$.

The operator $\lambda_1 - A$ is surjective: To prove that it is surjective, we will construct a preimage $x$ under $\lambda_1 - A$ for an arbitrarily chosen $y \in E$. It suffices to consider

$$x = -A^{-1}y + \lambda_1 R(\lambda_1, A_1)(A^{-1}y - PA^{-1}y) + \lambda_1 R(\lambda_1, A_2)(PA^{-1}y).$$

The identity (5.4) is well defined because $A^{-1}y - PA^{-1}y \in E_1$ and $PA^{-1}y \in E_2$. It is easy to verify that $(\lambda_1 - A)x = y$.

In general, the sum $E_1 \oplus E_2$ in Proposition (5.1) is not always closed as the examples below prove. The example below (i.e Example 5.4) is based on an interesting result of harmonic analysis due to D. J. Newman [30].

Example 5.4. Let $E = L^1_{2\pi}$ and $A$ be given by $Af := -if'$ with domain $D(A) := \{f \in L^1_{2\pi}: f' \in L^1_{2\pi}\}$. Then $iA$ generates the $C_0$-group $T$ given by $T(it)f(x) := f(x + t)$. We have: $E_1 = \{f \in L^1_{2\pi}: \int_0^{2\pi} e^{-inx}f(x)dx = 0 \text{ for } n \leq 0\} = H^1$ and $E_2 = \{f \in L^1_{2\pi}: \int_0^{2\pi} e^{-inx}f(x)dx = 0 \text{ for } n > 0\}$. Moreover, by a result due to D. J. Newman [30], there is no bounded projection of $L^1_{2\pi}$ onto the Hardy space $H^1$. It follows that $E_1 \oplus E_2 \neq E$. For more information about this important result on can see also [34] or more recently [23].

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