COMBINATORICS OF \((q, y)-LAGUERRE POLYNOMIALS AND THEIR MOMENTS\)

QIONGQIONG PAN AND JIANG ZENG

To Christian Krattenthaler on the occasion of his 60th birthday

Abstract. We consider a \((q, y)\)-analogue of Laguerre polynomials \(L_n^{(\alpha)}(x; y | q)\) for integral \(\alpha \geq -1\), which turns out to be a rescaled version of Al-Salam–Chihara polynomials. A combinatorial interpretation for the \((q, y)\)-Laguerre polynomials is given using a colored version of Foata and Strehl’s Laguerre configurations with suitable statistics. When \(\alpha \geq 0\), the corresponding moments are described using certain classical statistics on permutations, and the linearization coefficients are proved to be a polynomial in \(y\) and \(q\) with nonnegative integral coefficients.

1. Introduction

The monic Laguerre polynomials \(L_n^{(\alpha)}(x)\) are defined by the generating function

\[
(1 + t)^{-\alpha - 1} \exp \left( \frac{xt}{t + 1} \right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \frac{t^n}{n!}.
\]

They are the multiple of the usual (general) Laguerre polynomials [16, pp. 241–242] by \((-1)^n n!\). We have the explicit formula

\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k! (n-k)!} \left( \frac{n+\alpha}{n-k} \right) x^k
\]

and the three-term recurrence relation

\[
L_{n+1}^{(\alpha)}(x) = (x - (2n + \alpha + 1))L_n^{(\alpha)}(x) - n(n + \alpha)L_{n-1}^{(\alpha)}(x).
\]

The Laguerre polynomials \(L_n^{(\alpha)}(x)\) are orthogonal with respect to the moments \(\mathcal{L}(x^n) = (\alpha+1)_n\), where \((x)_n = x(x+1)\cdots(x+n-1)\) \((n \geq 1)\) is the shifted factorial with \((x)_0 = 1\), and \(\mathcal{L}\) is the linear functional defined by

\[
\mathcal{L}(f) = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} f(x)x^{\alpha}e^{-x}dx.
\]
The linearization formula \([23]\) reads as follows:
\[
\mathcal{L}(L_{n_1}^{(\alpha)}(x)L_{n_2}^{(\alpha)}(x)L_{n_3}^{(\alpha)}(x)) = \sum_{s \geq 0} \frac{n_1!n_2!n_3!2^{n_1+n_2+n_3-2s}(\alpha+1)_s}{(s-n_1)!(s-n_2)!(s-n_3)!(n_1+n_2+n_3-2s)!}.
\] (1.5)

A combinatorial model for Laguerre polynomials with parameter \(\alpha\) was first given by Foata and Strehl \([7]\). Recall that a Laguerre configuration on \([n] := \{1, \ldots, n\}\) is a pair \((A, f)\), where \(A \subset [n]\) and \(f\) is an injection from \(A\) to \([n]\). A Laguerre configuration can be depicted by a digraph on \([n]\) by drawing an edge \(i \to j\) if and only if \(f(i) = j\). Clearly, such a graph has two types of connected components called cycles and paths, see Figure 1. Let \(\mathcal{LC}_{n,k}\) be the set of Laguerre configurations \((A, f)\) on \([n]\) with \(|A| = n-k\). Then Foata and Strehl’s interpretation \([7]\) reads
\[
\sum_{(A,f) \in \mathcal{LC}_{n,k}} (\alpha+1)^{\text{cyc}(f)} = \frac{n!}{k!} \binom{n+\alpha}{n-k},
\] (1.6)
where \(\text{cyc}(f)\) is the number of cycles of \(f\).

Note that one can derive (1.6) from any of the three formulas (1.1)–(1.3), see \([1, 7]\). The aim of this paper is to study combinatorial aspects of more general \((q,y)\)-Laguerre polynomials \(L_n^{(\alpha)}(x; y \mid q) (n \geq 0)\) defined by the three term-recurrence relation
\[
L_{n+1}^{(\alpha)}(x; y \mid q) = (x - (y[n+\alpha+1]_q + [n]_q)) L_n^{(\alpha)}(x; y \mid q)
- y[n]_q[n+\alpha]_q L_{n-1}^{(\alpha)}(x; y \mid q), \quad \alpha \geq -1, \ n \geq 1,
\] (1.7)
with \(L_0^{(\alpha)}(x; y \mid q) = 1, \ L_{-1}^{(\alpha)}(x; y \mid q) = 0\). Here and throughout this paper, we use the standard \(q\)-notations: \([n]_q = \frac{1-q^n}{1-q}\) for \(n \geq 0\), the \(q\)-analogue of \(n\)-factorial \(n!_q = \prod_{i=1}^n[i]_q\), and the \(q\)-binomial coefficient
\[
\binom{n}{k}_q = \frac{n!_q}{k!_q(n-k)!_q} \quad \text{for} \quad 0 \leq k \leq n.
\]
Clearly we have \(L_n^{(0)}(x; 1 \mid 1) = L_n^{(0)}(x)\). Kasraoui et al. \([17]\) gave a combinatorial interpretation for the linearization coefficients of the polynomials \(L_n^{(0)}(x; y \mid q)\) and pointed out
that a combinatorial model for \( L_n^{(0)}(x; y \mid q) \) can be derived from Simion and Stanton’s model for octabasic \( q \)-Laguerre polynomials in [22]. For \( k \in \mathbb{Z} \), let

\[
\mathbb{N}_k := \{ n \in \mathbb{Z} : n \geq k \}
\]

and \( \mathbb{N} := \mathbb{N}_1 \). Recently, using the theory of \( q \)-Riordan matrices, Cheon, Jung and Kim [3] derived a combinatorial model for the \( q \)-Laguerre polynomials \( L_n^{(\alpha)}(x; q \mid q) \) when \( \alpha \in \mathbb{N}_0 \). It is then natural to search for a combinatorial structure unifying the above two special cases, as was alluded to at the end of [3]. Our first goal is to give such a combinatorial model for \( L_n^{(\alpha)}(x; y \mid q) \) with variable \( y \) and integer \( \alpha \in \mathbb{N}_1 \) by using a \( q \)-analogue of Foata and Strehl’s Laguerre configurations. Moreover, for \( \alpha \in \mathbb{N}_0 \), the \((q, y)\)-Laguerre polynomials \( L_n^{(\alpha)}(x; y \mid q) \) are orthogonal polynomials. It is our second goal to give a combinatorial interpretation for the moments of \((q, y)\)-Laguerre polynomials and prove that the linearization coefficients are polynomials in \( y \) and \( q \) with nonnegative integral coefficients. We achieve this by making use of the combinatorial theory of continued fractions.

By (1.7), the first few values of \( L_n^{(\alpha)}(x; y \mid q) \) are

\[
L_1^{(\alpha)}(x; y \mid q) = x - y[\alpha + 1]_q,
\]
\[
L_2^{(\alpha)}(x; y \mid q) = x^2 - (y[\alpha + 1]_q + y[\alpha + 2]_q + 1) x + [\alpha + 1]_q[\alpha + 2]_q y^2,
\]
\[
L_3^{(\alpha)}(x; y \mid q) = x^3 - (y([\alpha + 1]_q + [\alpha + 2]_q + [\alpha + 3]_q) + 2 + q)x^2
\]
\[
+ (y^2([\alpha + 1]_q[\alpha + 2]_q + [\alpha + 2]_q[\alpha + 3]_q + [\alpha + 1]_q[\alpha + 3]_q)
\]
\[
+ y([\alpha + 3]_q + [2]_q[\alpha + 1]_q) + [2]_q)x - y^3[\alpha + 1]_q[\alpha + 2]_q[\alpha + 3]_q.
\]

For convenience, we introduce the signless \((q, y)\)-Laguerre polynomials

\[
L_n^{(\alpha)}(x; y \mid q) := (-1)^n L_n^{(\alpha)}(-x; y \mid q) = \sum_{k=0}^{n} \ell_{n,k}^{(\alpha)}(y; q)x^k. \tag{1.8}
\]

For \( \alpha \in \mathbb{N}_1 \), we observe that \( \ell_{n,k}^{(\alpha)}(y; q) \) is a polynomial in \( y, q \) with nonnegative integral coefficients, which is far from obvious from the explicit Formula (2.8). For \( \alpha \in \mathbb{N}_1 \), Formula (1.6) implies that \( \ell_{n,k}^{(\alpha)}(1; 1) \) is equal to the number of Laguerre configurations in \( \mathcal{LC}_{n,k} \) such that each cycle carries a color \( \in [1 + \alpha] \). In particular, the number of Laguerre configurations in \( \mathcal{LC}_{n,k} \) without cycles (i.e., consisting of only \( k \) paths) is equal to the \( \text{Lah numbers} \) [18]:

\[
\ell_{n,k}^{(-1)}(1; 1) = \frac{n!}{k!} \binom{n - 1}{k - 1}.
\]

**Remark 1.** Two different \( q \)-analogues of Lah numbers were defined and studied by Garsia and Remmel [9] and Lindsay et al. [19], respectively. Moreover an elliptic analogue of Garsia and Remmel’s \( q \)-Lah numbers was constructed by Schlosser and Yoo [21].
The organization of this paper is as follows. In Section 2 we identify the \((q, y)\)-Laguerre polynomials as a rescaled version of Al-Salam–Chihara polynomials and derive several expansion formulas for \((q, y)\)-Laguerre polynomials. In Section 3 we present a combinatorial interpretation for the \((q, y)\)-Laguerre polynomials in terms of \(\alpha\)-Laguerre configurations, which are in essence the product structure of “cycles” and “paths”. In Section 4 we give a combinatorial interpretation for the moments of \((q, y)\)-Laguerre polynomials and prove that the linearization coefficients are polynomials in \(y\) and \(q\) with nonnegative integral coefficients. As the Laguerre polynomials play an important role in the theory of rook polynomials, we translate our \(\alpha\)-Laguerre configurations in terms of rook placements in Section 5 and set up the connection between our \(\alpha\)-Laguerre configurations and the matching model of complete bipartite graphs \(K_{n,n+\alpha}\) (see Godsil and Gutman [12]).

2. A detour to Al-Salam–Chihara polynomials

The \(q\)-Pochhammer symbol or \(q\)-shifted factorial \((a; q)_n\) is defined by

\[
(a; q)_n = \begin{cases} 
\prod_{i=0}^{n-1} (1 - aq^i) & \text{for } n \in \mathbb{Z}^+ \cup \{\infty\}, \\
1 & \text{for } n = 0.
\end{cases}
\]

The Al-Salam–Chihara polynomials \(Q_n(x) := Q_n(x; a, b \mid q)\) are defined by the generating function (see [16, Chapter 14])

\[
\sum_{n=0}^{\infty} Q_n(x; a, b \mid q) \frac{t^n}{(q; q)_{\infty}} = \frac{(at, bt; q)_{\infty}}{(te^{i\theta}, te^{-i\theta}; q)_{\infty}},
\]

(2.1)

with \((a, b; q)_{\infty} = (a; q)_{\infty}(b; q)_{\infty}\), and they satisfy the recurrence relation (op. cit.)

\[
\begin{cases} 
Q_{-1}(x) = 0, & Q_0(x) = 1, \\
Q_{n+1}(x) = (2x - (a + b)q^n)Q_n(x) - (1 - q^n)(1 - abq^{n-1})Q_{n-1}(x), & n \geq 0.
\end{cases}
\]

(2.2)

We have the explicit formula

\[
Q_n(x; a, b \mid q) = \frac{(ab; q)_n}{a^n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (au; q)_k (au^{-1}; q)_k}{(ab; q)_k(q; q)_k} q^k,
\]

(2.3)

where \(x = \frac{u+u^{-1}}{2}\) or \(x = \cos \theta\) if \(u = e^{i\theta}\).

Comparing (1.7) with (2.2) and using (1.8), we see that our polynomials \(L_n^{(\alpha)}(x; y \mid q)\) are a rescaled version of the Al-Salam–Chihara polynomials:

\[
L_n^{(\alpha)}(x; y \mid q) = \left(\frac{\sqrt{y}}{1-q}\right)^n Q_n \left(\frac{(1-q)x + y + 1}{2\sqrt{y}}, \frac{1}{\sqrt{y}}, \sqrt{y}q^{\alpha+1} \mid q\right).
\]

(2.4)
The Al-Salam–Chihara polynomials (see [16, pp. 455–456] and [14]) are orthogonal with respect to the linear functional \( \mathcal{L}_q \) defined by

\[
\mathcal{L}_q(f) = \frac{(q, ab; q)_\infty}{2\pi} \int_{-1}^{+1} \frac{f(x)dx}{\sqrt{1-x^2}} \prod_{k=0}^{\infty} \frac{1 - 2(2x^2 - 1)q^k + q^{2k}}{[1 - 2xaq^k + a^2q^{2k}][1 - 2xbq^k + b^2q^{2k}]} .
\]

(2.5)

Hence, for \( \alpha \in \mathbb{N}_0 \), the polynomials \( L_n^{(\alpha)}(x; y \mid q) \) are orthogonal with respect to the linear functional \( \mathcal{L}_q \) given by

\[
\mathcal{L}_q(f) = \frac{(q, q^{\alpha+1}; q)_\infty}{2\sqrt{y}} \int_{B_+}^{B_-} \frac{f(x)dx}{\sqrt{1-v(x)^2}} \prod_{k=0}^{\infty} \frac{[1 - 2(2v(x)^2 - 1)q^k + q^{2k}]}{[1 - 2v(x)q^k/\sqrt{y} + q^{2k}/y][1 - 2v(x)q^{k+\alpha+1}/\sqrt{y} + q^{2k+2\alpha+2}]} ,
\]

(2.6)

where \( B_\pm = [(1 \pm \sqrt{y})^2, 1 - q] \) and

\[
v(x) = \frac{1}{2\sqrt{y}}((q - 1)x + (y + 1)).
\]

(2.7)

Now, by (2.4), we may derive an explicit formula from (2.3), namely

\[
L_n^{(\alpha)}(x; y \mid q) = \sum_{n,k=0}^{\infty} \frac{n!}{k!q^k} \left[ n + \alpha \right]_q q^{nk} y^{n-k} \prod_{j=0}^{k-1} (x + (1 - yq^{-j})[j]_q) ,
\]

(2.8)

and, from (2.1), the generating function

\[
\mathcal{L}^{(\alpha)}(x; y; t \mid q) := \sum_{n \geq 0} L_n^{(\alpha)}(x; y \mid q) \frac{t^n}{n!q}
\]

\[
= \prod_{k=0}^{\infty} \frac{(t; q)_\infty (ytq^{\alpha+1}; q)_\infty}{[1 - ((1 - q)x + y + 1)qt + yt^2q^{2k}]} .
\]

(2.9)

which can be written as

\[
\mathcal{L}^{(\alpha)}(x; y; t \mid q) = \mathcal{L}^{(\alpha)}(0; y; t \mid q) \cdot \mathcal{L}^{(-1)}(x; y; t \mid q).
\]

(2.10)

Define the “vertical generating function”

\[
\mathcal{L}_k^{(\alpha)}(y; t \mid q) := [x^k] \mathcal{L}^{(\alpha)}(x; y; t \mid q) = \sum_{n \geq k} \ell_n^{(\alpha)}(y, q) \frac{t^n}{n!q} ,
\]

(2.11)

and the \( q \)-derivative operator \( D_q \) for \( f(t) \in \mathbb{R}[[t]] \) by

\[
D_q(f(t)) = \frac{f(t) - f(qt)}{(1-q)t} ,
\]

where \( \mathbb{R} = \mathbb{C}[[x, y, q, \ldots]] \). Thus \( D_q(1) = 0 \) and \( D_q(t^n) = [n]_q t^{n-1} \) for \( n > 0 \).
It follows from (2.9) that
\[
D_q \mathcal{L}^{(-1)}(x; y \mid t) = \frac{x}{(1-t)(1-yt)} \mathcal{L}^{(-1)}(x; y \mid t),
\] (2.12)
which in particular gives
\[
D_q \mathcal{L}^{(-1)}_1(y; t \mid q) = \frac{1}{(1-t)(1-yt)} \sum_{n \geq 0} n! [n+1]_y t^n.
\] (2.13)
So we can rewrite (2.12) as
\[
D_q \mathcal{L}^{(-1)}(x; y \mid t) = x \cdot D_q \mathcal{L}^{(-1)}_1(y; t \mid q) \cdot \mathcal{L}^{(-1)}(x; y \mid t),
\] (2.14)
which is equivalent to the following result.

**Proposition 2.** For \( n \in \mathbb{N} \), we have
\[
L^{(-1)}_{n+1}(x; y \mid q) = x \sum_{k=0}^{n} \binom{n}{k} [k+1]_y L^{(-1)}_{n-k}(x; y \mid q).
\] (2.15)

Now, applying the \( q \)-binomial formula (see [10, Chapter 1])
\[
\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}
\]
with \( a = q^{\alpha+1} \) and \( z = yt \), we have
\[
\mathcal{L}^{(\alpha)}(0; y \mid t) = \frac{(ytq^{\alpha+1}; q)_\infty}{(yt; q)_\infty} = \sum_{n \geq 0} \left( \prod_{k=1}^{n} [\alpha + k]_q \right) \frac{(yt)^n}{n!_q}.
\] (2.16)
Substitution of the latter into (2.10) gives the following result.

**Proposition 3.** For \( n \in \mathbb{N} \), we have
\[
L^{(\alpha)}_n(x; y \mid q) = \sum_{k=0}^{n} \binom{n}{k} \left( \prod_{j=1}^{k} [\alpha + j]_q \right) y^k L^{(-1)}_{n-k}(x; y \mid q).
\] (2.17)

**Remark 4.** (1) More generally we can prove the following connection formula for \( \alpha \geq \beta \geq -1 \):
\[
L^{(\alpha)}_n(x; y \mid q) = \sum_{k=0}^{n} \binom{n}{k} \left( \prod_{j=0}^{k-1} [\alpha - \beta + j]_q \right) (yq^{\beta+1})^k L^{(\beta)}_{n-k}(x; y \mid q).
\] (2.18)
(2) For \( q \to 1 \), Identity (2.9) reduces to

\[
\sum_{n \geq 0} L_n^{(\alpha)}(x; y \mid 1) \frac{t^n}{n!} = (1 - yt)^{-(\alpha+1)} \left( 1 - \frac{(1 - y)t}{1 - yt} \right)^{-x/(1-y)}.
\]

Comparing with the generating function of the Meixner polynomials (see [16, Equation (1.9.11)])

\[
\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x; \beta, c)t^n = (1 - t)^{-x-\beta} (1 - t/c)^x,
\]

we derive

\[
L_n^{(\alpha)}(x; y \mid 1) = y^n(\alpha + 1)_n M_n \left( \frac{-x}{1-y}; \alpha + 1, y \right).
\]

Hence the \((q, y)\)-Laguerre polynomials \( L_n^{(\alpha)}(x; y \mid q) \) are a \( q \)-analogue of rescaled Meixner polynomials.

3. Combinatorial interpretation of \((q, y)\)-Laguerre polynomials

The reader is referred to [1, 6, 13] for the general combinatorial theory of exponential generating functions for labeled structures. For our purpose we need only a \( q \)-version of this theory for special labeled structures. A labeled structure on a (finite) set \( A \subset \mathbb{N} \) is a graph with vertex set \( A \). Consider a family of labeled \( \mathcal{F} \)-structures \( \mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n \), where \( \mathcal{F}_n \) consists of the \( \mathcal{F} \)-structures on \([n]\). If \( A = \{a_1, \ldots, a_n\} \subset \mathbb{N} \), where \( a_1 < \cdots < a_n \), an \( \mathcal{F} \)-structure on \( A \) is obtained by replacing \( i \) by \( a_i \) for \( i = 1, \ldots, n \) in the elements of \( \mathcal{F}_n \).

Let \( \mathcal{F}[A] \) denote the set of \( \mathcal{F} \)-structures on \( A \) and associate a weight \( u(f) \) to each object \( f \in \mathcal{F} \). For the set of weighted \( \mathcal{F} \)-structures \( \mathcal{F}_u \) (where the valuation \( u \) may involve the parameter \( q \)), the \( q \)-generating function is defined as

\[
\mathcal{F}_u(t) = \sum_{f \in \mathcal{F}_u} u(f) \frac{t^{\mid f \mid}}{\mid f \mid_q!},
\]

where \( \mid f \mid = n \) if \( f \in \mathcal{F}[n] \). If \( \mathcal{F}_u \) and \( \mathcal{G}_v \) are two weighted structures, we denote by \((\mathcal{F} \cdot \mathcal{G})_w[n]\) the set of pairs \((f, g) \in \mathcal{F}[S] \times \mathcal{G}[T]\) with weight

\[
w(f, g) = u(f) \cdot v(g) \cdot q^{\text{inv}(S,T)},
\]

where \((S, T)\) is an ordered bipartition of \([n]\) and \( \text{inv}(S, T) \) is the number of pairs \((i, j) \in S \times T\) such that \( i > j \). Recall (see [13, p. 98]) that

\[
\sum_{(S, T)} q^{\text{inv}(S,T)} = \begin{bmatrix} n \end{bmatrix}_q,
\]
where the sum is over all ordered bipartitions \((S, T)\) of \([n]\) with \(|S| = k\). It is folklore and immediately checked that

\[
(F \cdot G)_w(t) = F_u(t) \cdot G_v(t). \tag{3.1}
\]

We need some further definitions.

(a) For a permutation \(\sigma\) of a set \(A \subset \mathbb{N}\), let the word \(\hat{\sigma}\) denote its linear representation in the usual sense, i.e., \(\hat{\sigma} = \sigma(i_1) \ldots \sigma(i_n)\) if \(A = \{i_1, \ldots, i_n\}\) with \(i_1 < \cdots < i_n\).

(b) A list of (nonnegative) integers, taken as a word over \(\mathbb{N}\), is strict if no element occurs more than once. For a strict list \(\rho\) let \(r(\rho)\) be the number of elements that come after the maximum element.

(c) For a set \(\lambda\) of \(k\) non-empty and disjoint strict lists of integers, order these lists according to their minimum element (increasing). This gives a list of \(k\) words \((\lambda_1, \ldots, \lambda_k)\), which will be identified with \(\lambda\). Then \(\underline{\lambda} = \lambda_1 \ldots \lambda_k\) denotes the concatenation of these lists.

Two particular structures will be used to interpret the \((q, y)\)-Laguerre polynomials.

(d) The structures \(S^{(\alpha)}\) consist of permutations \(\sigma\), where each cycle carries a color \(\in\{0, 1, 2, \ldots, \alpha\}\). Write \(\sigma\) as a product of unicolored permutations, \(\sigma = \sigma_0 \cdot \sigma_1 \cdots \sigma_\alpha\), where \(\sigma_i\) is the product of cycles with color \(i\). Now consider the concatenation

\[
\underline{\sigma} = \hat{\sigma}_0 \cdot \hat{\sigma}_1 \cdots \hat{\sigma}_\alpha
\]

and the word with letters from \(\{0, 1\}\) given by

\[
\underline{\sigma} = 0^{|\hat{\sigma}_0|} 1 10^{|\hat{\sigma}_1|} 1 \cdots 10^{|\hat{\sigma}_\alpha|}.
\]

Define the valuation \(u\) on \(S^{(\alpha)}\) by

\[
u(\sigma) = q^{\gamma_{\text{inv}}(\sigma) + \gamma_{\text{inv}}(\underline{\sigma})}.
\]

(e) The structures \(Lin^{(k)}\) consist of sets \(\lambda = (\lambda_1, \ldots, \lambda_k)\) of \(k\) nonempty and disjoint strict lists (cf. (c)). Define the valuation \(v\) on \(Lin^{(k)}\) by

\[
v(\lambda) = q^{r(\lambda)} q^{\gamma_{\text{inv}}(\underline{\lambda}) - r(\lambda)}
\]

where \(r(\lambda) = \sum_{i=1}^k r(\lambda_i)\).

Let \(LC_{n,k}^{(\alpha)} := S^{(\alpha)} \cdot Lin^{(k)}[n]\). For any \(\alpha\)-Laguerre configuration \((\sigma, \lambda) \in S^{(\alpha)}[A] \times Lin^{(k)}[B]\) with \(A \cap B = \emptyset\) and \(A \cup B = [n]\), in order to invoke the folklore statement (3.1), one should use as valuation

\[
w(\sigma, \lambda) = u(\lambda) \cdot v(\lambda) \cdot q^{\gamma_{\text{inv}}(A, B)}
\]

where

\[
\begin{align*}
&= y^{\gamma(\sigma)} q^{\gamma_{\text{inv}}(\sigma) + \gamma_{\text{inv}}(\underline{\sigma})} \cdot y^{r(\lambda)} q^{\gamma_{\text{inv}}(\underline{\lambda}) - r(\lambda)} q^{\gamma_{\text{inv}}(A, B)} \\
&= y^{\gamma(\sigma) + r(\lambda)} q^{\gamma_{\text{inv}}(\sigma) + \gamma_{\text{inv}}(\underline{\sigma}) + \gamma_{\text{inv}}(\underline{\lambda}) - r(\lambda)} q^{\gamma_{\text{inv}}(A, B)} \\
&= y^{\gamma(\sigma) + r(\lambda)} q^{\gamma_{\text{inv}}(\underline{\lambda}) - r(\lambda) + \gamma_{\text{inv}}(\underline{\sigma})}.
\end{align*}
\]

\[
\tag{3.2}
\]
The essential point is $\text{inv}(\sigma) + \text{inv}(\lambda) + \text{inv}(A, B) = \text{inv}(\sigma, \lambda)$. This describes the weighted configurations $(\mathcal{LC}_{n,k})_w$. An element of $(\mathcal{LC}_{n,k})_w$ is called an $\alpha$-Laguerre configuration on $[n]$, see Figure 2.

Lemma 5. For $\alpha \in \mathbb{N}$, we have

$$S^{(\alpha)}_u(t) = \mathcal{L}^{(\alpha)}(0; y; t | q).$$

Proof. Let $P(n, \alpha)$ be the set of words of length $n + \alpha$ with $n$ 0’s and $\alpha$ 1’s, i.e., lattice paths from $(0, 0)$ to $(n, \alpha)$. For $\sigma \in S^{(\alpha)}[n]$, the word $\sigma$ can be seen as the linear representation of an (ordinary) permutation $\tilde{\sigma} \in S^{(0)}[n]$, whereas $\underline{\sigma} \in P(n, \alpha)$. The mapping

$$S^{(\alpha)}[n] \rightarrow S^{(0)}[n] \times P(n, \alpha)
\sigma \mapsto (\tilde{\sigma}, \underline{\sigma})$$

is a bijection, and from summing both contributions separately, one obtains

$$\sum_{\sigma \in S^{(\alpha)}[n]} q^{\text{inv}(\sigma)} = \sum_{\sigma \in S^{(0)}[n]} q^{\text{inv}(\sigma)} \sum_{\underline{\sigma} \in P(n, \alpha)} q^{\text{inv}(\underline{\sigma})} = n!_q \left\lfloor \frac{n + \alpha}{\alpha} \right\rfloor_q,$$

which is $\prod_{i=1}^{\alpha}[\alpha + i]_q$. So we get

$$S^{(\alpha)}_u(t) = \sum_{n \geq 0} \left( \prod_{i=1}^{\alpha}[\alpha + i]_q \right)(yt)^n.$$

The result then follows from (2.16). □

Lemma 6. For integers $k \geq 1$, we have

$$\mathcal{L}^{-\alpha}_u(k)(t) = \mathcal{L}^{(\alpha)}_{k+1}(y; t | q).$$

Proof. We proceed by induction on $k \geq 1$. 
• The case $k = 1$. For a single list $\lambda = \lambda \in \mathcal{L}^{(1)}[n+1]$, let $j_{\lambda}$ be the position of the maximum element, let $\lambda' = \lambda' \in \mathcal{L}^{(1)}[n]$ be the list obtained by deleting this maximum element. Then

$$\mathcal{L}^{(1)}[n+1] \rightarrow \mathcal{L}^{(1)}[n] \times [n+1]$$

$$\lambda \mapsto (\lambda', j_{\lambda})$$

is a bijection such that $\text{inv}(\lambda) = \text{inv}(\lambda') + \text{rl}(\lambda)$. Furthermore, we have

$$\sum_{\lambda \in \mathcal{L}^{(1)}[n+1]} q^{\text{rl}(\lambda)} q^{\text{inv}(\lambda') - n(\lambda)} = \sum_{\lambda' \in \mathcal{L}^{(1)}[n]} q^{\text{inv}(\lambda')} \sum_{j \in [n+1]} y^{n+1-j},$$

and thus

$$\sum_{\lambda \in \mathcal{L}^{(1)}[n+1]} v(\lambda) = n! q [n+1],$$

which, in view of (2.13), gives

$$\mathcal{D}_q \mathcal{L}^{(1)}_1(t) = \mathcal{D}_q \mathcal{L}^{(-1)}_1(y; t \mid q),$$

and by $q$-integration

$$\mathcal{L}^{(1)}_1(t) = \mathcal{L}^{(-1)}_1(y; t \mid q)$$

because the series on both sides have a zero constant term.

• The case $k > 1$. Assuming that $\mathcal{L}^{(k)} v(t) = \mathcal{L}^{(-1)}_k(y; t \mid q)$ has already been proved for $k \geq 1$, the goal is to show

$$\mathcal{L}^{(k+1)} v(t) = \mathcal{L}^{(-1)}_{k+1}(y; t \mid q).$$

Comparing the coefficients of $x^{k+1}$ on both sides of Equation (2.14), we obtain

$$\mathcal{D}_q \mathcal{L}^{(-1)}_{k+1}(y; t \mid q) = \mathcal{D}_q \mathcal{L}^{(-1)}_1(y; t \mid q) \cdot \mathcal{L}^{(-1)}_k(y; t \mid q).$$

If we can show that similarly

$$\mathcal{D}_q \mathcal{L}^{(k+1)} v(t) = \mathcal{D}_q \mathcal{L}^{(1)} v(t) \cdot \mathcal{L}^{(k)} v(t),$$

(3.3)

then we would be done. Again, the final integration step poses no problem because in both $\mathcal{L}^{(k+1)} v(t)$ and $\mathcal{L}^{(-1)}_{k+1}(y; t \mid q)$ the first $k + 1$ coefficients vanish. Recall that a configuration $\lambda \in \mathcal{L}^{(k+1)}[n]$ consists of a list of $k+1$ disjoint strict lists, written as a list $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k)$, with $\lambda_i \in \mathcal{L}^{(1)}[A_i]$, where

$$\bigcup_{i=0}^{k} A_i = [n] \quad \text{and} \quad \min A_{i-1} < \min A_i, \quad 1 \leq i \leq k.$$
where $\lambda' = (\lambda_1, \ldots, \lambda_k)$ and $A' = \bigcup_{i=1}^{k} A_i$, which also satisfies the requirement for applying the folklore statement (3.1):

$$v(\lambda) = v(\lambda_0) \cdot v(\lambda') \cdot q^{\text{inv}(A_0, A')}.$$  

All this holds only if for the bipartition $A = A_0 \cup A'$ it is guaranteed that $\min A_0 < \min A'$. This is where the derivative $D_q$ comes into play. Differentiation for a collection of structures means that the minimum element of the underlying set of a structure is tagged and no longer counted in the $w$-valuation of the base set. In the present situation, this implies that only structures are considered where tagging the minimum element of $\lambda$ means the same as tagging the minimum element of $\lambda_0$. This shows that (3.3) holds. □

**Theorem 7.** For integers $\alpha \geq -1$, we have

$$\ell_{n,k}^{(\alpha)}(y; q) = \sum_{(\sigma; \lambda) \in \mathcal{LC}^{(\alpha)}_{n,k}} y^{|\sigma| + r(\lambda)} q^{\text{inv}(\sigma, \lambda) - n(\lambda) + \text{inv}(\sigma)}.$$  

**Proof.** From (2.10) we infer

$$\mathcal{L}_k^{(\alpha)}(y; t | q) = \mathcal{L}^{(\alpha)}(0; y; t | q) \mathcal{L}_k^{(-1)}(y; t | q),$$

and the result follows from Lemmas 1 and 2. □

Here we give an example to illustrate the $\alpha$-Laguerre configurations.

**Example 8.** Consider the 1-Laguerre configuration $(\sigma; \lambda) \in \mathcal{LC}_{15,4}^{(1)}$ in Figure 2. We have

$\sigma = \sigma_0 \cdot \sigma_1$ with $\sigma_0 = (15)(7 4)$, $\sigma_1 = (14)(13 5 2)$;

$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $\lambda_1 = 13$, $\lambda_2 = 12611$, $\lambda_3 = 108$, $\lambda_4 = 9$.

Thus,

$$\sigma = \hat{\sigma}_0 \cdot \hat{\sigma}_1 = 7415 \cdot 132514,$$

$$\sigma = 0^3 10^4;$$

$$\lambda = 13 \cdot 12611 \cdot 108 \cdot 9.$$  

We have $|\sigma| = 7$, $r(\lambda) = 3$, $\text{inv}(\sigma) = 4$, and $\text{inv}(\sigma \cdot \lambda) = 52$.

**Remark 9.** Our model of $\alpha$-Laguerre configurations is simpler than the model in [3]. Actually, the $\alpha$-Laguerre configurations are essentially the Laguerre configurations of which each cycle has a color in $\{0, \ldots, \alpha\}$. A linear order of paths and colored cycles is needed only for the valuation $w$ in (3.2).
4. Moments of \((q, y)\)-Laguerre polynomials

For \(\alpha \in \mathbb{N}_0\), by (2.6) the moments of the \((q, y)\)-Laguerre polynomials are defined by

\[
\mu_n^{(\alpha)}(q, y) := L_q(x^n). \tag{4.1}
\]

According to the theory of orthogonal polynomials (see [4]) and the three-term recurrence relation (1.7), we have the orthogonality relation

\[
L_q(L_n^{(\alpha)}(x; y|q)L_m^{(\alpha)}(x; y|q)) = y^n n! q \left( \prod_{j=1}^{n} (\alpha + j) q \right) \delta_{nm}. \tag{4.2}
\]

Moreover, we have the following continued fraction expansion:

\[
\sum_{n \geq 0} \mu_n^{(\alpha)}(q, y) t^n = \frac{1}{1 - b_0 t - \lambda_1 t^2}, \tag{4.3}
\]

where \(b_n = y[n + \alpha + 1]_q + [n]_q\) and \(\lambda_n = y[n]_q [n + \alpha]_q\).

Let \(S_n\) be the set of permutations of \(\{1, 2, \ldots, n\}\). For \(\sigma \in S_n\), we define three statistics, namely:

- the number of weak excedances, \(\text{wex}(\sigma)\), given by
  \[\text{wex}(\sigma) = |\{i \in [n] : \sigma(i) \geq i\}|;\]

- the number of records (or left-to-right maxima), \(\text{rec}(\sigma)\), given by
  \[\text{rec}(\sigma) = |\{i \in [n] : \sigma(i) > \sigma(j) \text{ for all } j < i\}|;\]

- the number of crossings, \(\text{cros}(\sigma)\), given by
  \[\text{cros}(\sigma) = |\{(i, j) \in [n] \times [n] : i < j \leq \sigma(i) < \sigma(j) \text{ or } \sigma(j) < \sigma(i) < j < i\}|.\]

**Theorem 10.** Let \(\beta = [\alpha + 1]_q\). Then

\[
\mu_n^{(\alpha)}(y, q) = \sum_{\sigma \in S_n} \beta^{\text{rec}(\sigma)} y^{\text{wex}(\sigma)} q^{\text{cros}(\sigma)}. \tag{4.4}
\]

The first values of the moments are as follows:

\[
\begin{align*}
\mu_1^{(\alpha)}(y, q) &= y\beta, \\
\mu_2^{(\alpha)}(y, q) &= y\beta + y^2\beta^2, \\
\mu_3^{(\alpha)}(y, q) &= y\beta + \beta(1 + (2 + q)\beta)y^2 + y^3\beta^3.
\end{align*}
\]
Due to the contraction formula [13, p. 292], we can rewrite (4.3) as

$$
\sum_{n \geq 0} \mu_n^{(\alpha)}(q, y)t^n = \frac{1}{1 - \frac{\gamma_1 t}{1 - \gamma_2 t}}.
$$

(4.5)

where \(\gamma_{2n} = [n]_q\) and \(\gamma_{2n+1} = y[n + \alpha]_q = y([n]_q + [\alpha + 1]_q q^n)\) for \(n \geq 0\).

Recall that a *Dyck path* of length \(2n\) is a sequence of points \((\omega_0, \ldots, \omega_{2n})\) in \(\mathbb{N}_0 \times \mathbb{N}_0\) satisfying \(\omega_0 = (0, 0), \omega_{2n} = (2n, 0)\) and \(\omega_{i+1} - \omega_i = (1, 1)\) or \((1, -1)\) for \(i = 0, \ldots, 2n - 1\). Clearly we can also identify a Dyck path with its sequence of steps (or *Dyck word*) \(s = s_1 \ldots s_{2n}\) on the alphabet \(\{u, d\}\), and we use \(|s|_u\) and \(|s|_d\) to denote the number of \(u\)'s and \(d\)'s, respectively, in \(s\). So, for a Dyck word \(s\), we have \(|s|_u = |s|_d = n\) and \(|s_1 \ldots s_k|_u \geq |s_1 \ldots s_k|_d\) for \(k \in [2n]\). The height \(h_k\) of step \(s_k\) is defined to be \(h_1 = 0\) and

\[h_k = |s_1 \ldots s_{k-1}|_u - |s_1 \ldots s_{k-1}|_d \quad \text{for} \quad k = 2, \ldots, 2n.\]

A *Laguerre history* of length \(2n\) is a pair \((s, \xi)\), where \(s\) is a Dyck word of length \(2n\) and \(\xi = (\xi_1, \ldots, \xi_{2n})\) is a sequence of integers such that \(\xi_i = 1\) if \(s_i = u\) and \(1 \leq \xi_i \leq |h_i/2|\) if \(s_i = d\). Let \(\mathcal{LH}_n\) be the set of Laguerre histories of length \(2n\). We essentially use Biane's bijection [2] to construct a bijection \(\Phi\) from \(\mathcal{S}_n\) to \(\mathcal{LH}_n\).

**Proof of Theorem 10.** We identify a permutation \(\sigma \in \mathcal{S}_n\) with the bipartite graph \(\mathcal{G}\) on \([1, \ldots, n; 1', \ldots, n']\) with an edge \((i, j')\) if and only if \(\sigma(i) = j\). We display the vertices on two rows called top row and bottom row as follows:

\[
\begin{pmatrix}
1 & 2 & \cdots & n \\
1' & 2' & \cdots & n'
\end{pmatrix},
\]

and we read the graph column by column from left to right and from top to bottom. In other words, the order of vertices is \(v_1 = 1, v_2 = 1', \ldots, v_{2n-1} = n, v_{2n} = n'\).

For \(k = 1, \ldots, 2n\), the \(k\)-th restriction of \(\mathcal{G}\) is the graph \(\mathcal{G}_k\) on \([v_1, v_2, \ldots, v_k]\) with edge \((v_i, v_j)\) in \(\mathcal{G}_k\) if and only if \(i, j \in [k]\), so isolated vertices may exist in \(\mathcal{G}_k\).

For \(i = 1, \ldots, n\), the Dyck path \(s = s_1 \ldots s_{2n}\) is defined as follows:

- if \(\sigma^{-1}(i) > i < \sigma(i)\) (i.e., \(i\) is a cycle valley), then \(s_{2i-1}s_{2i} = uu\);
- if \(\sigma^{-1}(i) < i < \sigma(i)\) (i.e., \(i\) is a cycle double ascent), then \(s_{2i-1}s_{2i} = ud\);
- if \(\sigma^{-1}(i) > i > \sigma(i)\) (i.e., \(i\) is a cycle double descent), then \(s_{2i-1}s_{2i} = du\);
- if \(\sigma^{-1}(i) < i > \sigma(i)\) (i.e., \(i\) is a cycle peak), then \(s_{2i-1}s_{2i} = dd\);
- if \(\sigma^{-1}(i) = i = \sigma(i)\) (i.e., \(i\) is a fixed point), then \(s_{2i-1}s_{2i} = ud\).

It is easy to see that

- \(s\) is a Dyck path;
- the height \(h_i\) is the number of isolated vertices in \(\mathcal{G}_{i-1}\) for \(i \in [2n]\) with \(\mathcal{G}_{i-1} = \emptyset\); thus \(h_{2i-1}\) (respectively \(h_{2i}\)) is even (respectively odd) for \(i = 1, \ldots, n\) and there are \([h_{2i}/2]\) isolated vertices in the top row.

Next, the sequence \(\xi = (\xi_1, \ldots, \xi_{2n})\) is defined as follows:
\[ s_i = u \] then \( \xi_i = 1; \)
\[ s_i = d, \] then
- if \( \sigma(i) < i \) (i.e., \( i \) is a cycle double descent or cycle peak), then \( h_{2i-1} > 0; \) let \( \xi_i = m \) if \( \sigma(2i) \) is the \( m \)-th isolated vertex in the bottom row of \( G_{2i-2} \) from right-to-left \( (1 \leq m \leq \lfloor h_{2i}/2 \rfloor) \); clearly the value \( i \) will contribute \( m - 1 \) crossings \( l < k < i < j \) such that \( l = \sigma(i), k = \sigma(j); \)
- if \( \sigma^{(-1)}(i) \geq i \) (i.e., \( i \) is a cycle double ascent, cycle peak or fixed point), then \( h_{2i} > 0; \) let \( \xi_i = m \) if \( \sigma^{(-1)}(i) \) is the \( m \)-th isolated vertex in the top row of \( G_{2i-2} \) from right-to-left, so \( 1 \leq m \leq \lceil h_{2i}/2 \rceil \); clearly the value \( i \) will contribute \( m - 1 \) crossings \( l < k < i < j \) such that \( l = \sigma^{(-1)}(i), k = \sigma(j), \) and \( i \) is a record if and only if \( m = \lceil h_{2i}/2 \rceil. \)

Let \( \Phi(\sigma) = (s, \xi). \) Then
\[
\begin{align*}
\text{wex}(\sigma) &= |\{i \in [n] : s_{2i} = d\}|, \\
\text{rec}(\sigma) &= |\{i \in [n] : s_{2i} = d, \xi_{2i} = \lceil h_{2i}/2 \rceil\}|, \\
\text{cros}(\sigma) &= \sum_{i : s_i = d} (\xi_i - 1).
\end{align*}
\]

Therefore,
\[
\sum_{\sigma \in \Theta_n} \beta^{\text{rec}(\sigma)} y^{\text{wex}(\sigma)} q^{\text{cros}(\sigma)} = \sum_{(s, \xi) \in LH_n \ i : s_i = d} \prod_{i : s_{2i} = d} q^{\xi_i - 1} \prod_{i : s_i = d} y^\chi(\xi_{2i} = \lceil h_{2i}/2 \rceil)
\]
\[
= \sum_{s \in \text{Dyck}_n \ i : s_i = d} \prod_{i : s_i = d} w(s_i), \tag{4.6}
\]
where \( \text{Dyck}_n \) denotes the set of Dyck paths of semilength \( n, \) and the weight of each down step \( s_i = d \) is defined by
\[
w(s_i) = \begin{cases} 1 + q + \cdots + q^{k-1}, & \text{if } h_i = 2k, \\ y(1 + q + \cdots + q^{k-1} + \beta q^k), & \text{if } h_i = 2k + 1. \end{cases}
\]
A folklore theorem [5] implies that the generating function of (4.6) has the continued fraction expansion (4.5), and we are done. \( \square \)

**Example 11.** If \( \sigma = 412796583 \in \mathcal{S}_9, \) then the Laguerre history \( \Phi(\sigma) = (s, \xi) \) is given by
\[
\begin{pmatrix} s \\ \xi \end{pmatrix} = \begin{pmatrix} uu & du & du & ud & uu & ud & dd & ud & dd \\ 11 & 11 & 11 & 12 & 11 & 11 & 12 & 11 & 11 \end{pmatrix}.
\]

**Theorem 12.** Let \( \alpha \in \mathbb{N}_0. \) For nonnegative integers \( n_1, \ldots, n_k, \) the linearization coefficient
\[
\mathcal{L}_q \left( \prod_{k=1}^m L_n^{(\alpha)}(x; y \mid q) \right) \tag{4.7}
\]
is a polynomial in \( \mathbb{N}[y, q] \).

Proof. In view of the orthogonality (4.2), it suffices to prove the \( m = 3 \) case. Indeed, we can derive the following explicit formula from [17, Theorem 1]:

\[
\mathcal{L}_q(L_{n_1}^{(\alpha)}(x; y | q)L_{n_2}^{(\alpha)}(x; y | q)L_{n_3}^{(\alpha)}(x; y | q)) = n_1! q^{n_2} n_3! q^{n_2} q^{n_3} \sum_{s \geq \max(n_1, n_2, n_3)} y^s \binom{n_1 + n_2 + n_3 - 2s, s - n_3, s - n_2, s - n_1}{q} \times \left[ \alpha + s \right] \sum_{k \geq 0} \binom{n_1 + n_2 + n_3 - 2s}{k} q^{(n_1+n_2+n_3-2s-k)+ka},
\]

where the \( q \)-multinomial coefficients

\[
\binom{a + b + c + d}{a, b, c, d}_q = \frac{(a + b + c + d)!}{a!_q b!_q c!_q d!_q}
\]

are known to be polynomials in \( \mathbb{N}[q] \) for integral \( a, b, c, d \geq 0 \), see [13]. Hence, the right-hand side of (4.8) is a polynomial in \( \mathbb{N}[y, q] \), and we are done. \( \square \)

For arbitrary \( \alpha \), a combinatorial interpretation of (4.7) was given by Foata and Zeilberger [8] with \( y = q = 1 \), and generalized by the second author [23] to \( q = 1 \) (see also [24]), while for \( \alpha = 0 \) a combinatorial interpretation of (4.7) was given by Kasraoui et al. [17]. Thus, the following problem suggests itself.

Problem. What is the combinatorial interpretation of (4.7) for \( \alpha \in \mathbb{N}_0 \) unifying the two special cases with \( \alpha = 0 \) or \( q = 1 \)?

5. Connection with rook polynomials and matching polynomials

In this section we show how the model of \( \alpha \)-Laguerre configurations is connected with the models of non-attacking rook placements and matchings of complete bipartite graphs.

5.1. Interpretation in rook polynomials. An \( m \) by \( n \) board \( B \) is a subset of an \( m \times n \) grid of cells (or squares). A rook is a chessboard piece which takes on rows and columns. If \( r_k \) is the number of ways of putting \( k \) non-attacking rooks on this board, then the ordinary rook polynomial is defined by

\[
R_{m,n}(x) = \sum_k r_k x^k.
\]

Thus, the Laguerre polynomials (1.2) can be written as

\[
L_n^{(\alpha)}(x) = (-1)^n n! R_{n,n+\alpha}(-x^{-1}).
\]

A \( k \)-rook placement on a board \( B \) is a subset \( C \subset B \) of \( k \) cells such that no two cells are in the same row or column of \( B \). We refer the reader to Riordan’s classical book [20, Chapters 7 and 8] for many problems formulated in terms of configurations of non-attacking rooks on “chessboards” of various shapes.
Definition 13. For integers $n, k \geq 0$ and $\alpha \geq -1$, let $m = (m_0, \ldots, m_\alpha)$ and $n = (n_1, \ldots, n_k)$ be nonnegative integer sequences such that $m_0 + m_1 + \cdots + m_\alpha + n_1 + \cdots + n_k = n$ with $m_i \geq 0$ and $n_j \geq 1$. We define $\mathcal{B}_{n,k}^{(\alpha)}(m; n)$ as the set of $n \times n$ squares of color shape $B := (B^{(1)}; B^{(2)})$ with

\begin{align}
B^{(1)} := (n^{m_0}, \ldots, n^{m_\alpha}), \\
B^{(2)} := (n^{n_1}, \ldots, n^{n_k}).
\end{align}

By convention, if $\alpha = -1$ (respectively $k = 0$), then $B^{(1)} = \emptyset$ (respectively $B^{(2)} = \emptyset$). Let

$$\mathrm{cw}(B) = \sum_{i=0}^{\alpha} m_i$$
and
$$\mathrm{cd}(B) = \sum_{i=0}^{\alpha} i \cdot m_i.$$ 

Let $\mathcal{B}_{n,k}^{\alpha}(m; n)$ denote the set of all ordered pairs $R = (B, C)$, where $B \in \mathcal{B}_{n,k}^{(\alpha)}(m; n)$ and $C$ is an $n$-rook placement on $B$ such that

$$\min(C \cap B_1^{(2)}) < \min(C \cap B_2^{(2)}) < \cdots < \min(C \cap B_k^{(2)}),$$

where $\min(C \cap B_1^{(2)})$ is the minimum row index of cells in $C \cap B_1^{(2)}$. For each block $B_i^{(2)} = (n^{n_i})$, we define $\mathrm{ind}(C \cap B_i^{(2)})$ as the number of rooks in $C \cap B_i^{(2)}$ whose column indices are greater than the column index of the rook which has the maximum row index in $B_i^{(2)}$, and let $\mathrm{ind}(R) = \sum_{i=1}^{k} \mathrm{ind}(C \cap B_i^{(2)})$. Let

$$\mathcal{B}_{n,k}^{(\alpha)} = \bigcup_{m,n} \mathcal{B}_{n,k}^{\alpha}(m; n)$$
with

$$\sum_{i=0}^{\alpha} m_i + \sum_{j=1}^{k} n_j = n.$$
An element \( \mathcal{R} = (B, C) \in BC^{(\alpha)}_{n,k} \) is called a colored rook configuration.

Remark 14. One can imagine that each column of a board in \( BC^{(\alpha)}_{n,k}(m; n) \) is colored with colors in \( \{0, 1, \ldots, \alpha + k\} \) from left to right as follows: the first \( m_0 \) columns get color 0, the next \( m_1 \) columns get color 1, \ldots, the last \( n_k \) columns get color \( \alpha + k \).

Theorem 15. The coefficient \( \ell^{(\alpha)}_{n,k}(y; q) \) in (1.8) is the following generating polynomial of colored rook configurations in \( BC^{(\alpha)}_{n,k} \):

\[
\ell^{(\alpha)}_{n,k}(y; q) = \sum_{\mathcal{R} = (B, C) \in BC^{(\alpha)}_{n,k}} y^{cw(B) + \text{ind}(\mathcal{R})} q^{\text{inv}(C) + \text{cd}(B) - \text{ind}(\mathcal{R})}.
\]

Proof. Let \( \mathcal{L}^{(\alpha)}_{n,k}(m; n) \) be the set of \( \rho := (\sigma_0, \ldots, \sigma_{\alpha}; \lambda_1, \ldots, \lambda_k) \in \mathcal{L}^{(\alpha)}_{n,k} \) such that \( |\rho| = (|\sigma_0|, \ldots, |\sigma_{\alpha}|; |\lambda_1|, \ldots, |\lambda_k|) = (m; n) \). We define the map \( \phi : \mathcal{L}^{(\alpha)}_{n,k}(m; n) \to BC^{(\alpha)}_{n,k}(m; n) \) by \( \phi(\rho) = (B, C) \) for \( \rho = (\sigma_0, \ldots, \sigma_{\alpha}; \lambda_1, \ldots, \lambda_k) \in \mathcal{L}^{(\alpha)}_{n,k}(m; n) \) as follows:

(i) The colored board \( B = (B^{(1)}, B^{(2)}) \) is given by

\[
B^{(1)} = (n^{|\sigma_0|}, \ldots, n^{|\sigma_{\alpha}|}) \quad \text{and} \quad B^{(2)} = (n^{|\lambda_1|}, n^{|\lambda_2|}, \ldots, n^{|\lambda_k|}).
\]

(ii) If \( w := \sigma_0 \sigma_1 \cdots \sigma_{\alpha} \lambda_1 \lambda_2 \cdots \lambda_k = w_1 \ldots w_n \), which is a permutation of \([n]\), let \( C = \{(j, w_j) : j \in [n]\} \).
It is clear that $\phi(\rho) \in \mathcal{BC}_{n,k}^{(\alpha)}$, and the procedure is reversible. Hence $\phi$ is a bijection. It is easy to verify that $\text{inv}(C) = \text{inv}(\rho)$, $\text{ind}(B_i^{(2)}) = \text{rl}(\lambda_i)$, and $\text{cd}(B) = \sum_{i=0}^{\alpha} i|\sigma_i|$, which implies that
\[
|\sigma| + \text{rl}(\lambda) = \text{cw}(B) + \text{ind}(\mathcal{R}); \\
\text{inv}(\sigma, \lambda) - \text{rl}(\lambda) + \text{inv}(\sigma) = \text{inv}(C) + \text{cd}(B) - \text{ind}(\mathcal{R}).
\]
The result then follows from Theorem 7.

Example 16. Let $\rho = ((74)(15), (1325)(14); 13, 12, 6, 11, 10, 8, 9) \in \mathcal{L}_1^{(1)}$. Then $\phi$ maps $\rho$ to the placement of 15 rooks on the board $B = (15^7; 15^2, 10^3, 8^2, 7)$ shown in Figure 4. We find $\text{cw}(B) = 1$, $\text{cd}(B) = 4$; $\text{inv}(C) = 52$ and $\text{ind}(\mathcal{R}) = 3$.

5.2. Interpretation in matching polynomials. Recall that a matching of a graph $G$ is a set of edges without common vertices. For any graph $G$ with $n$ vertices, the matching polynomial of $G$ is defined by
\[
m(G, x) = \sum_{k=0}^{[n/2]} (-1)^k m_k x^{n-2k},
\]
where $m_k$ is the number of $k$-edge matchings of $G$. Let $K_{n,m}$ denote the set of complete bipartite graphs on the two disjoint sets $A = [n]$ and $B = \{1', \ldots, m\}$, that is, there is an edge $(a, b)$ if and only if $a \in A$ and $b \in B$. From the explicit formula (1.2) it is quite easy to derive the connection formula
\[
m(K_{n,n+\alpha}, x) = x^\alpha L_n^{(\alpha)}(x^2), \quad \alpha \geq -1. \tag{5.4}
\]
Godsil and Gutman [12] proved (5.4) by showing that the matching polynomials satisfy the same three-term recurrence relation (1.3). Here we give a simple bijection between our $\alpha$-Laguerre configuration model and the above matching model of complete bipartite graphs. Let $\mathcal{M}_{n,m}^{n-k}$ be the set of matchings of $K_{n,m}$ with $n - k$ edges.

Proposition 17. For integers $n, k \geq 1$ and $\alpha \geq -1$, there exists an explicit bijection $\phi : \mathcal{L}_n^{(\alpha)} \rightarrow \mathcal{M}_{n,n+\alpha}^{n-k}$.

Proof. We construct such a bijection $\phi$. Let $\rho = (\sigma_0, \sigma_1, \ldots, \sigma_\alpha; \lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{L}_n^{(\alpha)}$ be an $\alpha$-Laguerre configuration. We define a matching $\gamma$ of $K_{n,n+\alpha}$ such that $(a, b') \in A \times B$ is an edge in $\gamma$ if and only if $(a, b)$ satisfies one of the following three conditions:

1. $\sigma_0(a) = b$, i.e., the image of $a$ is $b$ through the action of permutation $\sigma_0$;
2. $a$ and $b$ are consecutive letters in the word $\hat{\sigma}_1(n+1)\hat{\sigma}_2(n+2)\ldots\hat{\sigma}_\alpha(n+\alpha)$;
3. $a$ and $b$ are consecutive letters in the word $\lambda_j$ for some $j \in [k]$.

By convention, if $\alpha = -1$ (respectively $\alpha = 0$) there are no words of types (1) and (2) (respectively type (2)). Since $\sigma_0\sigma_1\ldots\sigma_\alpha \lambda_1 \ldots \lambda_k$ is a permutation of $[n]$, it is clear that there are $n - k$ such edges $(a, b')$. The above procedure is obviously reversible. \qed
Example 18. For the 1-Laguerre configuration
\[ \rho = ((7\ 4)(15)\ (13\ 2\ 5)(14); 1\ 3\ 12\ 6\ 11, 10\ 8, 9) \in \mathcal{L}^{(1)}_{15,4} \]
in Example 8, the corresponding matching \( \gamma \) of \( K_{15,16}^{11} \) is shown in Figure 5.

Remark 19. We leave it to the interested reader to find the \((q, y)\)-version of the above matching polynomials for \((q, y)\)-Laguerre polynomials \( L_n^{(\alpha)}(x; y | q) \).

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Univ Lyon, Université Lyon1, UMR 5208 du CNRS, Institut Camille Jordan, F-69622, Villeurbanne Cedex, France
Email address: qpan@math.univ-lyon1.fr

Univ Lyon, Université Lyon1, UMR 5208 du CNRS, Institut Camille Jordan, F-69622, Villeurbanne Cedex, France
Email address: zeng@math.univ-lyon1.fr