BISTABLE BEHAVIOUR OF A JUMP-DIFFUSION DRIVEN BY A PERIODIC STABLE-LIKE ADDITIVE PROCESS

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Dedicated to Björn Schmalfuß on the occasion of his 60th birthday

ABSTRACT. We study a bistable gradient system perturbed by a stable-like additive process with a periodically varying stability index. Among a continuum of intrinsic time scales determined by the values of the stability index we single out the characteristic time scale on which the system exhibits the metastable behaviour, namely it behaves like a time discrete two-state Markov chain.

1. Introduction. In this work we study the small noise behaviour of a jump diffusion in a double-well potential driven by a stable-like additive process with independent increments and periodic stability index.

The motivation for this work stems from the observation made by P. Ditlevsen that the abrupt climate transitions between glacial and interstadial states during the last glacial period (the Dansgaard–Oeschger events) can be well understood as a realization of a bistable dynamical system perturbed by noise which contains a heavy-tail $\alpha$-stable Lévy component, see [4, 5]. The proxy signal for the global Earth's temperature is modelled there as a solution of a stochastic differential equation (SDE)

$$X^\varepsilon_t = x - \int_0^t V'(X^\varepsilon_s) \, ds + \varepsilon L_t. \quad (1.1)$$

The function $x \mapsto V(x)$ is a double-well potential with two stable minima $m_- < 0 < m_+$ identified with the cold and warm climate states and separated by the unstable local maximum $m_0 = 0$, e.g. a standard quartic potential $V(x) = x^4 - x^2$. The process $L$ is a symmetric $\alpha$-stable Lévy process with the jump measure $\nu(dx) = |x|^{-1-\alpha}dx$, $\alpha \in (0, 2)$. It was noted that big (heavy) jumps of the driving process $L$ induce fast transitions of $X^\varepsilon$ between the potential wells.

The use of an $\alpha$-stable noise in (1.1) is rather ad hoc and the values of $\alpha$ can be derived not from the physical laws but rather from the data. Using a histogram analysis, Ditlevsen estimated the stability index as $\alpha \approx 1.75$. Hein et al. in [13] studied the empirical $p$-variation of the time series and obtained $\alpha \approx 0.75$. On the other hand, Gairing et al. [10] studied the heavy tailed noises without an assumption
of α-stability and discovered that for large |x| the tails of the Lévy measure should decay as \( \nu(dx) \approx |x|^{-1-\alpha}dx \), with \( \alpha \approx 3.5 \).

In the present paper we will continue to work with the technically simpler α-stable distributions. One can expect however that our results will hold for noises with polynomial regularly varying tails, see [21].

The metastable behaviour of \( X^\varepsilon \) in the small noise regime, namely the laws of transitions of \( X^\varepsilon \) between the wells, was studied in [20, 21, 18]. In particular it was shown (see, e.g. Theorem 1.1 in [21]) that the finite dimensional laws of the jump diffusion \( (X^\varepsilon_{\alpha\varepsilon^{-1}t})_{t \geq 0} \) on a new fast time scale converge to those of a continuous time Markov chain on the state space \( \{m_-, m_+ \} \) with the generator

\[
Q = \begin{pmatrix} -|m_-|^{-\alpha} & |m_-|^{-\alpha} \\ -|m_+|^{-\alpha} & -|m_+|^{-\alpha} \end{pmatrix}.
\]

This means that in the small noise limit, the transition times normalized by the factor \( \alpha^{-1}\varepsilon^\alpha \) are exponentially distributed with means \( |m_\pm|^{-\alpha} \).

A more thorough analysis (see, e.g. Schulz [31]) of the palaeoclimatic data from the Greenland ice cores which motivated Ditlevsen’s conjectures detected a periodic signal with the estimated period of 1470 years. The existence of such a periodic component may be interpreted as a response of the climate system to an external periodic forcing and can be explained within the paradigm of the stochastic resonance, see [1, 11, 6] for more information and a discussion. In the SDE setting, such dynamics can be explained as follows. Consider a non-autonomous SDE

\[
X^\varepsilon_{t} = x - \int_0^t U'(X^\varepsilon_{s}, \frac{s}{2T}) \, ds + \varepsilon L_t,
\]

with a time-periodic double-well potential \( U, U(\cdot, t) = U(\cdot, t + 1) \), and \( U'(x, t) = \frac{1}{\pi^2}U(x, t) \). The dependence of \( U \) on time can be interpreted as a perturbation of a fixed double-well potential by a weak period signal, e.g. \( U(x, t) = V(x) + ax \cos(2\pi t), \ a > 0 \) being small. Then, for noise amplitudes \( \varepsilon \) within a certain range (the optimal tuning amplitudes), the exit times of \( X^\varepsilon_{t} \) from the potential wells get synchronized with the weak periodic signal. In climate modelling, the weak periodic forcing is often related to Earth’s orbital changes or deep ocean processes.

Stochastic resonance in diffusions driven by small Gaussian noise (\( L = B \) being a standard Brownian motion) was firstly studied in [9] in the framework of large deviations theory, and also in [19, 2, 14, 15, 17, 16]. The crucial role in the description of the stochastic resonance is played by the exponentially large exit times from the wells which obey the so-called Kramers law. In a time homogeneous case, \( U(x, t) = V(x) \) they are of the order \( \exp(2\Delta V_\pm/\varepsilon^2) \), where \( \Delta V_\pm \) are the heights of the potential barriers separating the wells. In the presence of a weak periodic perturbation, the barriers \( \Delta U_\pm(t/2T) \) are time dependent. However it is intuitively clear that in the so-called adiabatic limit when \( T \) is very large, the stochastic resonance can only be observed if the noise amplitude \( \varepsilon \) and the period \( 2T \) are related in such a way that a Brownian particle has enough time to exit the shallow well during the half period (see, e.g. Theorem 1 in [9] and Chapter 3.1 in [16]).

Stochastic resonance in a jump-diffusion (1.3) driven by Lévy processes with heavy tails was studied recently in [25, 26, 27]. The characteristic feature of the jump-diffusion (1.3) driven by an α-stable Lévy process \( L \) is the unique (up to a constant factor) time scale of the order \( \varepsilon^{-\alpha} \) on which the transitions between the wells occur. It was shown that a non-trivial behaviour of \( X^\varepsilon_{t} \) can be observed only for the periods of the order \( 2T(\varepsilon) = \text{Const} \cdot \varepsilon^{-\alpha} \). In [25, 26, 27], the authors
determined the “optimal tuning” pre-factor which maximizes the probability to exit
the well at time instants when the distance \( |m_+(t) - m_0(t)| \) between the current
well’s minimum and the saddle point is minimal.

The obvious drawback of the models \((1.1)\) and \((1.3)\) is the unique time scale
\(\mathcal{O}(\varepsilon^{-\alpha})\) on which the non-trivial metastable behaviour can be observed. In the
present paper we are going to modify the equation \((1.1)\) and consider a bistable
jump-diffusion driven by a heavy-tail noise which has a continuum of intrinsic time
scales of different orders w.r.t. \(\varepsilon\). This can be achieved by making the stability
index \(\alpha\) of the driving process depending on time and/or the current location of the
jump-diffusion. Such perturbations are called stable-like.

Elements of the theory of stable-like jump diffusions with a spatially dependent
stability index can be found in [23, Chapter 5] and [24, Chapter 7]. A scaling
limit behaviour of stable-like processes with a spatially periodic stability index was
performed in [7, 8]. Large deviations type theorems for general Markov processes
with heavy tails were obtained in [12].

In this work we assume that the stability index \(\alpha\) depends only on time, namely
the function \(t \mapsto \alpha(t)\) is 1-periodic. The corresponding stochastic process \(L = A\) is
an additive process, i.e. a process with independent but in general not stationary
increments. Such processes resemble very much Lévy processes and we can use many
of the methods developed for the Lévy setting. However the dependence of \(\alpha\) on time
implies the existence of a continuum of intrinsic time scales of the order \(\varepsilon^{-\mu}, \mu \in
[\min \alpha(t), \max \alpha(t)]\) and the behaviour of the time non-autonomous jump-diffusion
diffs strongly from its Lévy driven counterpart. In view of our previous results
on metastability of Lévy driven diffusions it is not surprising that the transition
behaviour is determined by the lowest value \(\alpha_*\), i.e. the by the heaviest component
of the driving process. However it is rather unexpected that the characteristic time
scale turns out to be slightly longer than the time scale \(\varepsilon^{-\alpha_*}\) discovered in [21],
namely it is of the order \(\varepsilon^{-\alpha_*} \sqrt{\ln \varepsilon}\).

The paper is organized as follows. In the next Section 2 we formulate the setting
and the main result of the paper. Section 3 is devoted to a rather elementary
analysis of a simple two-state Markov chain with periodic transition probabilities
which mimics the behaviour of the jump diffusion. We present the proof of the main
result in Sections 4 and 5. In this paper, the complement of a set \(A\) is denoted by
\(A^c\), and \(B_r(x)\) stands for an open ball of radius \(r\) centred at \(x\).

2. The setting and the main result. Let \(V \in C^2(\mathbb{R}^d, \mathbb{R}_+)\) be a double-well
potential. We assume that \(\nabla V(y) = 0\) only for \(y = m_-\) and \(m_+\) and a saddle \(m_0\).
The Hesse matrix of \(V\) at \(m_-\) and \(m_+\) is positive definite while the Hessian at \(m_0\)
is indefinite and has non-zero eigenvalues. We also assume that \(\lim_{\|y\| \to \infty} V(y) = +\infty\)
and the derivatives up to order two of \(y \mapsto \ln(1 + V(y))\) are bounded.

Let \((y_t(y_0))_{t \geq 0}\) be the solution of the ordinary differential equation \(\dot{y}_t = -\nabla V(y_t)\)
with the initial value \(y_0 \in \mathbb{R}^d\). Define the domains of attraction of the stable points
\(m_\pm\) by
\[
\Omega_\pm := \{ y_0 \in \mathbb{R}^d : \lim_{t \to \infty} y_t(y_0) = m_\pm \}.
\]

Let \(\Gamma := \mathbb{R}^d \setminus (\Omega_+ \cup \Omega_-)\) be the separatrix between \(\Omega_-\) and \(\Omega_+\).

On a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), let \(N^T(dx, ds), T > 0\), be a family of Poisson
random measures with intensity measures \(\nu_{1/2T}(dx)dt\) where
We assume that \( \alpha \in C^2(\mathbb{R}_+, \mathbb{R}) \) and \( c \in C(\mathbb{R}_+, \mathbb{R}_+) \) are 1-periodic functions, \( 0 < \alpha_* := \min_{t \in [0,1]} \alpha(t) < \max_{t \in [0,1]} \alpha(t) =: \alpha^* < 2 \), \( t \mapsto \alpha(t) \) has no intervals of constancy, and the minimum of \( t \mapsto \alpha(t) \) on \( t \in [0,1] \) is attained at a unique point \( a \in (0,1) \), \( \alpha(a) = \alpha_* \). Moreover we assume that \( \alpha''(a) > 0 \), and \( \min_{c \in [0,1]} c(t) > 0 \).

Let \( \gamma = \gamma(t) \) be a bounded \( d \)-dimensional function which has bounded variation on finite time intervals, and \( \sigma \) be a bounded continuous \( d \times d \)-matrix-valued function. Let \( B \) be a \( d \)-dimensional Brownian motion independent of the Poisson random measures \( N^T \).

Let \( A^T = (A^T_t)_{t \geq 0} \) be a \( d \)-dimensional additive process with the Lévy–Itô representation

\[
A^T_t = \int_0^t \gamma(s) \, ds + \int_0^t \sigma(s) \, dB_s + \int_0^t \int_{\|x\| < 1} x(N^T(dx, ds) - \nu_1/2T(dx)ds) + \int_0^t \int_{\|x\| \geq 1} x N^T(dx, ds),
\]

i.e. a stochastically continuous process with independent increments, càdlàg paths and \( A_0 = 0 \) a.s. (see, e.g. Chapter 14 in [3] or §9 in [30]).

Let \( Y^\varepsilon,T = (Y^\varepsilon,T_t)_{t \geq 0} \) be a unique global solution of the SDE

\[
Y^\varepsilon,T_t = y - \int_0^t \nabla V(Y^\varepsilon,T_s) \, ds + \varepsilon A^T_s, \quad y \in \mathbb{R}^d, \ t \geq 0,
\]

which exists under the conditions on \( V \) and \( A^T \) formulated above (see [25] for details). In our notation, we emphasize the dependence of \( Y^\varepsilon,T \) on the noise intensity \( \varepsilon \) and the period \( 2T \) of the stability index \( \alpha(\cdot) \).

Denote the exit times of \( Y^\varepsilon,T \) from the domains \( \Omega_\pm \) by

\[
\tau_\pm = \inf\{t \geq 0 : Y^\varepsilon,T_t \notin \Omega_\pm\}.
\]

Our first theorem describes the behaviour of the jump diffusion \( Y^\varepsilon,T \) on the algebraic time scales of the order \( \varepsilon^{-\mu}, \mu > 0 \).

**Theorem 2.1.** Let \( y \in \Omega_\pm, \mu > 0 \) and \( 2T = 2T(\varepsilon) = \varepsilon^{-\mu}, \)

(i) If \( \mu \leq \alpha_* \) then for all \( t > 0 \) we have \( \lim_{\varepsilon \to 0} \mathbb{P}_y(\tau_{\pm}(\varepsilon)^{\varepsilon^{-\mu}} \geq t) = 1 \).

(ii) If \( \mu \geq \alpha^* \) then for all \( t > 0 \) we have \( \lim_{\varepsilon \to 0} \mathbb{P}_y(\tau_{\pm}(\varepsilon)^{\varepsilon^{-\mu}} \geq t) = 1 \).

(iii) For \( \mu \in (\alpha_*, \alpha^*) \) we define \( u_\mu := \inf\{s \geq 0 : \alpha(s) < \mu\} \). Then for any \( \delta > 0 \) we have \( \lim_{\varepsilon \to 0} \mathbb{P}_y(\tau_{\pm}(\varepsilon)^{\varepsilon^{-\mu}} \in [u_\mu - \delta, u_\mu + \delta]) = 1 \).

The result of this Theorem can be interpreted as follows. If \( 2T(\varepsilon) = \varepsilon^{-\mu} \) with \( \mu \leq \alpha_* \), then the time horizon of the order \( \varepsilon^{-\mu} \) is too short to observe any jump, so that the jump diffusion stays in a small neighbourhood of one of the wells (see Fig. 1 (left)). If the time scale is of the order \( \varepsilon^{-\mu} \) with \( \mu \geq \alpha^* \), the jump-diffusion jumps chaotically (see Fig. 1 (right)), i.e. the waiting times between transitions converge to zero and the current position of the jump-diffusion cannot be predicted. Finally in the intermediate regime \( \mu \in (\alpha_*, \alpha^*) \) the intervals of chaotic and deterministic behaviour repeat themselves periodically, namely on the time intervals where \( \alpha(t) < \mu \), the jump-diffusion is chaotic, and if \( \alpha(t) > \mu \) then it is almost constant (see Fig. 1 (middle)). Similar intervals of chaotic behaviour were discovered by Herrmann and Imkeller in [14, 15] in the context of stochastic resonance in diffusions.
The stability index $t \mapsto \alpha(t)$, the scaling rate $\mu$ and typical paths of the jump-diffusion $t \mapsto Y_{2T(\varepsilon)}(t)$ with $2T(\varepsilon) = \varepsilon^{-\mu}$ for different values of $\mu$: $\mu \leq \alpha_*$ (left), $\mu \in (\alpha_*, \alpha^*)$ (middle), and $\mu \geq \alpha^*$ (right).

The main result of this paper formulated in the following theorem recovers another sub-algebraic time scale at which the chaotic behaviour degenerates so that the limiting transition behaviour of the jump-diffusion can be described probabilistically.

**Theorem 2.2.** Let $Y_{\varepsilon,T}$ be a solution of the SDE (2.4) and let the half-period $T = T(\varepsilon)$ and the rate $\lambda = \lambda(\varepsilon)$ be defined as

$$2T(\varepsilon) = \frac{1}{\lambda(\varepsilon)} = \sqrt{\frac{|\ln \varepsilon|}{\varepsilon^\alpha}}. \quad (2.6)$$

Then for any $y \in \Omega_\pm$ and $t \geq 0$ the following limit holds true:

$$\lim_{\varepsilon \to 0} \mathbb{P}_y(\tau_{\varepsilon,T(\varepsilon)} \leq t) = 1 - e^{-c_{\pm} g(t)}, \quad (2.7)$$

where

$$g(t) = \begin{cases} 0, & t < a, \\ \frac{1}{2} + k, & t = a + k, \, k \geq 0, \\ k + 1, & t \in (a + k, a + k + 1), \, k \geq 0, \end{cases} \quad (2.8)$$

and

$$c_{\pm} = \sqrt{\frac{2\pi}{\alpha''(a)}} \nu_\alpha(\Omega_\pm^c - m_\pm). \quad (2.9)$$

In particular, for any $\delta > 0$, any $k \geq 0$ and any $y \in \Omega_\pm$,

$$\lim_{\varepsilon \to 0} \mathbb{P}_y(\tau_{\varepsilon,T(\varepsilon)} \lambda(\varepsilon) \in [a + k - \delta, a + k + \delta]) = (1 - e^{-c_{\pm}}). \quad (2.10)$$

In other words, for small $\varepsilon$ and a properly chosen period $2T(\varepsilon) = \varepsilon^{-\alpha_*} \sqrt{|\ln \varepsilon|}$, the first exit time of the process $(Y_{2T(\varepsilon)}(t))_{t \geq 0}$ with $2T(\varepsilon) = \varepsilon^{-\alpha_*} \sqrt{|\ln \varepsilon|}$ from a potential well $\Omega_\pm$ occurs most likely during short time windows around the time instants $t_k := a + k, \, k \geq 0$, when the stability index $t \mapsto \alpha(t)$ attains its minimal value. The probability law of $\tau_{\varepsilon,T(\varepsilon)} \lambda(\varepsilon)$ reminds of a geometric probability distribution with the parameter $p = 1 - e^{-c_{\pm}} \in (0, 1)$ and support on the set $t_k := a + k, \, k \geq 0$, see Fig. 2.
Figure 2. The 1-periodic stability index $\alpha = \alpha(t)$ with global minima at $t_k = \frac{1}{2} + k, k \geq 0$ (above), and a sample path of a stable-like jump diffusion $Y^{\varepsilon,T(\varepsilon)}$ in a double-well potential with minima at $\pm 1$ for $T(\varepsilon) = \varepsilon^{-\alpha_*} \sqrt{\ln \varepsilon}$ (below). Transitions of $Y^{\varepsilon,T(\varepsilon)}$ between the potential wells occur during short time windows around the time instants $t_k$ with probabilities given by the r.h.s. of (2.10).

In the next section we perform an elementary analysis of a two-state continuous time Markov chain which dynamics reminds of those of the jump discussion $Y^{\varepsilon,T}$ in a symmetric potential $V$.

3. Approximation by a two-state Markov chain. In accordance with the results known for the Lévy driven jump diffusion (1.1), for $T > 0$ and $\varepsilon > 0$ we define a non-autonomous Markov chain $Z^{\varepsilon,T} = (Z^{\varepsilon,T}_t)_{t \geq 0}$ on the state space $\{-1, 1\}$ with the matrix of the infinitesimal probabilities

$$Q^{\varepsilon,T}(t) = \begin{pmatrix} -\varphi^{\varepsilon,T}(t) & \varphi^{\varepsilon,T}(t) \\ \varphi^{\varepsilon,T}(t) & -\varphi^{\varepsilon,T}(t) \end{pmatrix}, \quad t \geq 0,$$

where

$$\varphi^{\varepsilon,T}(t) = p \left( \frac{t}{2T} \right) e^{\alpha(t/2T)}.$$

The function $t \mapsto p(t)$ is continuous, 1-periodic, $p(t) = p(t + 1)$, $t \geq 0$, and $\min_{t \in [0,1]} p(t) > 0$. We assume that $\alpha \in C^2([0,1], \mathbb{R})$, is a 1-periodic function, $0 < \alpha_* := \min_{t \in [0,1]} \alpha(t) < \max_{t \in [0,1]} \alpha(t) =: \alpha^* < 2$ and the minimum of $\alpha(\cdot)$ is attained at a unique point $a \in (0, 1)$, $\alpha(a) = \alpha_*$. Moreover we assume that $\alpha''(a) > 0$, and $t \mapsto \alpha(t)$ has no intervals of constancy.

Let $T^{\varepsilon,T}_n$ denote the $n$-th jump time of $Z^{\varepsilon,T}$ and define the normalized times $\tau_n = \frac{T^{\varepsilon,T}_n}{2T}$.
The following asymptotics is obtained as a result of a Laplace type expansion of an asymptotic integral, see Chapter 3.7 in [28].

**Lemma 3.1.** Let $\alpha \in C^2(\mathbb{R}, \mathbb{R}_+)$ be 1-periodic with $\min_{t \in [0,1]} \alpha(t) = a_0 > 0$ for the unique $a \in (0,1)$. Let $f \in C(\mathbb{R}, \mathbb{R})$ be also 1-periodic with $f(a) > 0$. Then for any $t \geq 0$

$$\lim_{\varepsilon \to 0} \sqrt{\ln \varepsilon} \int_0^t \varepsilon^{\alpha(s) - a} f(s) \, ds = \frac{\sqrt{2\pi} f(a)}{\sqrt{\alpha''(a)}} g(t), \quad (3.1)$$

where the function $g$ is defined in (2.8).

**Lemma 3.2.** Let $\mu > 0$ and $2T = 2T(\varepsilon) = \varepsilon^{-\mu}$. Then three cases of limiting behaviour of $\tau_{n,T}^\varepsilon$ can be distinguished.

(i) If $\mu \leq a_0$ then for any $t \geq s$ we have $\lim_{\varepsilon \to 0} \mathbb{P}(\tau_{n,T}^\varepsilon \geq t | \tau_{n-1,T}^\varepsilon = s) = 1$.

(ii) If $\mu \geq a_0$ then for any $t > s$ we have $\lim_{\varepsilon \to 0} \mathbb{P}(\tau_{n,T}^\varepsilon \leq t | \tau_{n-1,T}^\varepsilon = s) = 1$, i.e. the conditional law of $\tau_n$ converges weakly to the Dirac measure in $s$.

(iii) For $\mu \in (a_0, a_\ast)$ we define $u_{s,\mu} = \inf\{t > s : \alpha(t) < \mu\} \in [s, s+1)$. Then the weak limit of the conditional laws $\mathbb{P}(\tau_{n,T}^\varepsilon | \tau_{n-1,T}^\varepsilon = s)$ is the Dirac measure in $u_{s,\mu}$.

**Proof.** The statement of the lemma is obtained by a straightforward analysis of the conditional density of $\tau_{n,T}^\varepsilon$ given $\tau_{n-1,T}^\varepsilon = s$ which equals to

$$f_{s,T}^\varepsilon(t) = \mathbb{1}_{[s,\infty)}(t) 2T p(t) \varepsilon^{\alpha(t)} \exp\left(-2T \int_s^t p(r) \varepsilon^{\alpha(r)} \, dr\right), \quad t \geq 0. \quad (3.2)$$

In particular for any $t \geq s$

$$\mathbb{P}\left(\tau_{n,T}^\varepsilon \in [s, t] | \tau_{n-1,T}^\varepsilon = s\right) = \int_s^t f_{s,T}^\varepsilon(r) \, dr = 1 - \exp\left(-2T \int_s^t p(r) \varepsilon^{\alpha(r)} \, dr\right). \quad (3.3)$$

Note that $2T(\varepsilon)^{\varepsilon^{\alpha(t)}} = \varepsilon^{\alpha(t) - \mu}, t \geq 0$.

(i) Let $\mu \leq a_0$ and $t \geq s$. Then $\varepsilon^{\alpha(r) - \mu} \to 0$ almost everywhere on $[s,t]$. Consequently the dominated convergence theorem implies that

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\tau_{n,T}^\varepsilon \in [s, t] | \tau_{n-1,T}^\varepsilon = s\right) = 0. \quad (3.4)$$

(ii) and (iii) Due to the continuity of $t \mapsto \alpha(t)$, there is $\delta^* > 0$ such that $\max_{t \in (u_{s,\mu}, u_{s,\mu} + \delta)} (\alpha(t) - \mu) < 0$. Consequently, for any $\delta \in (0, \delta^*)$ we get

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\tau_{n,T}^\varepsilon \in [u_{s,\mu}, u_{s,\mu} + \delta] | \tau_{n-1,T}^\varepsilon = s\right) = 1. \quad (3.5)$$

Analogously for all $\delta$ small enough

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\tau_{n,T}^\varepsilon \in [u_{s,\mu} - \delta, u_{s,\mu}] | \tau_{n-1,T}^\varepsilon = s\right) = 0. \quad (3.6)$$

As we see, for different scalings $2T(\varepsilon) = \varepsilon^{-\mu}$, the behaviour of the Markov chain $Z_{s,T}^\varepsilon$ can be almost deterministic, chaotic or repeatedly deterministic or chaotic. In the next proposition we determine such an sub-algebraic optimal period $2T(\varepsilon)$ such that the intervals of chaotic behaviour degenerate to points $t_k = a + k, k \geq 0$.

Motivated by a question of the best synchronization of the transitions of $Z_{s,T}^\varepsilon$ with the periodic signal $\alpha(t/2T)$ let us determine the optimal half-period $T = T(\varepsilon)$
at which the probability for $Z^{\varepsilon,T}$ to change a state is maximal. The proof of the next proposition is straightforward.

**Proposition 3.3.** For any $\delta > 0$, $\varepsilon > 0$, and $k \geq 0$ the map $T \mapsto \mathbb{P}_{-1}(\tau^{\varepsilon,T}_1 \in [k + a - \delta, k + a + \delta])$ attains its largest value at the optimal half-period length $T_{k,\delta}(\varepsilon) =$

$$
T_{k,\delta}(\varepsilon) = \frac{1}{2} \ln \int_0^{k+a+\delta} p(r) \varepsilon^{\alpha(r)} \ dr - \frac{1}{2} \ln \int_0^{k+a-\delta} p(r) \varepsilon^{\alpha(r)} \ dr.
$$

As $\delta \to 0$, the optimal half-period $T_{k,\delta}(\varepsilon)$ tends to the value

$$
T_k(\varepsilon) = \left( 2 \int_0^{k+a} p(r) \varepsilon^{\alpha(r)} \ dr \right)^{-1}.
$$

Moreover, the following asymptotics holds as $\varepsilon \to 0$ (Laplace’s method):

$$
2T_k(\varepsilon) \approx \sqrt{\ln \varepsilon \over \varepsilon^{\alpha_*}} \sqrt{\alpha''(a)} / 2\pi (a + k)p(a).
$$

In other words, the most synchronized behaviour of $Z^{\varepsilon,T}$ is attained on a time scale of the order $T = T(\varepsilon) = \varepsilon^{-\alpha_*} \sqrt{\ln \varepsilon}$. The next two sections will be devoted to the proof of the Theorems 2.1 and 2.2.

4. **The basic properties of the jump-diffusion $Y^{\varepsilon,T}$.** For the additive process $A^T$ defined in (2.3), let us consider another Lévy–Itô decomposition into a big and small jump components. For $\rho \in (0,1)$ and $\varepsilon \in (0,1]$, denote

$$
A^\varepsilon_t = \tilde{A}^\varepsilon_t + Q^\varepsilon_t,
$$

$$
Q^\varepsilon_t = \int_0^t \int_{\|x\| \geq \varepsilon^{-\rho}} x N^T(dx,ds),
$$

$$
\tilde{A}^\varepsilon_t = \int_0^t \gamma(s) \ ds + \int_0^t \sigma(s) \ dB_s + \int_0^t \int_{\|x\| \leq \varepsilon^{-\rho}} x (N^T(dx,ds) - \nu_s/2T(dx) \ ds).
$$

Note that since the jump measure $\nu_t(dx)$ is symmetric, the drift components of the processes $\tilde{A}^\varepsilon$ and $A^T$ coincide.

Denote the jump times of the rescaled non-autonomous compound Poisson process $\varepsilon Q^\varepsilon_T$ by $(\tau_{j}^{\varepsilon,T})_{j \geq 0}$ with $\tau_0^{\varepsilon,T} \equiv 0$ and the corresponding jump sizes by $(W^{\varepsilon,T}_j)_{j \geq 1}$. Let

$$
\beta^\varepsilon(t) = \nu_t(B^{\varepsilon,0}_{\varepsilon^{-\rho}}(0)) = \varepsilon^{\alpha^\varepsilon(t)} \nu_t(B^0(0))
$$

denote the instant jump intensity. Note that $\beta^\varepsilon(t)$ can be estimated as a function of $\varepsilon$ uniformly over $t$ as

$$
\varepsilon^{-\rho} \min_{t \in [0,1]} \nu_t(B^0(0)) \leq \beta^\varepsilon(t) \leq \varepsilon^{\alpha^\varepsilon(t)} \max_{t \in [0,1]} \nu_t(B^0(0)).
$$

Since $\varepsilon Q^\varepsilon_T$ is an additive process its inter-jump times $\tau^{\varepsilon,T}_{j+1} - \tau^{\varepsilon,T}_j$, $j \geq 1$, are independent and have the following conditional probability density:

$$
f^{\varepsilon,T}(t|\tau^{\varepsilon,T}_j = r) = \beta^\varepsilon \left( r + t \over 2T \right) \exp \left( - \int_r^{r+t} \beta^\varepsilon \left( s \over 2T \right) \ ds \right), \quad t \geq 0.
$$
The conditional probability law of the jump sizes $W^{\epsilon,T}_\cdot$ of the process $\epsilon Q^{\epsilon,T}$ is given by

$$\mathbb{P}\left(W^{\epsilon,T}_j \in B \mid \tau^{\epsilon,T}_j = t\right) = \frac{1}{\beta(t/2T)} \nu_{(t/2T)} \left((\epsilon^{-1}B) \cap B_{\epsilon^{-1}}(0)\right), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

and the set $\epsilon^{-1}B := \{\epsilon^{-1}x \in \mathbb{R}^d : x \in B\}$.

We impose no condition of the increase rate of the potential $V$ at infinity and thus localize the dynamics of $Y^{\epsilon,T}$ in a large domain which contains the local minima of the potential $V$. Since $V(y) \to +\infty$ as $\|y\| \to \infty$, for any ball $B_R = \{y : \|y\| < R\}$ there is $L > 0$ big enough such that the level set $O_L := \{y: V(y) < L\}$ is a connected bounded domain which contains the extrema $m_{\pm}$ and $m_0$ and $B_R \subset O_L$. The domain $O_L$ is positively invariant w.r.t. the dynamical system $\dot{y} = -\nabla V(y)$.

The bounded domains of attraction are defined as $\Omega_{\pm,L} := \Omega_{\pm} \cap O_L$.

We are not able to control the behaviour of $Y^{\epsilon,T}$ near the separatrix $\Gamma$ and thus also exclude a small neighbourhood of $\Gamma$ from $O_{\pm,L}$. Namely, for any $\delta > 0$ we define the reduced domains of attraction

$$\Omega_{\pm,L}(\delta) = \{y \in \mathbb{R}^d : B_{\delta}(y) \subseteq \Omega_{\pm,L}, \text{ for all } t \geq 0\}.$$  

Let $Y^{\epsilon,T}$ be the unique strong solution of the SDE (2.4), and let $\mathbb{F} = \mathbb{F}^{\epsilon,T}$ be its augmented own filtration. Between the two successive jumps of the compound Poisson process $Q^{\epsilon,T}$ the dynamics of $Y^{\epsilon,T}$ is described by the SDE

$$\tilde{Y}^{\epsilon,T}_{s,t}(y) = y - \int_{s}^{t} \nabla V(\tilde{Y}^{\epsilon,T}_{s,r}) \; dr + \epsilon(\tilde{A}^{\epsilon,T}_{s+t} - \tilde{A}^{\epsilon,T}_s),$$

with $\tau^{\epsilon,T}_j = s, t \in [0, \tau^{\epsilon,T}_{j+1} - \tau^{\epsilon,T}_j)$ and $y = Y^{\epsilon,T}_{\tau^{\epsilon,T}_j}$. The driving process $\epsilon(\tilde{A}^{\epsilon,T}_{s+t} - \tilde{A}^{\epsilon,T}_s)$ consists of a continuous component of the order $\epsilon$ and a pure jump martingale with jumps which do not exceed the threshold $\epsilon^{-1-p}$. Thus the jump diffusion $\tilde{Y}^{\epsilon,T}_{s,t}(y)$ can be seen as a small noise perturbation of the deterministic dynamical system $\dot{y} = -\nabla V(y)$. The next Lemma shows that $\tilde{Y}^{\epsilon,T}_{s,t}(y)$ indeed follows the deterministic solution with high probability.

**Lemma 4.1.** For $s \geq 0$, let $\tau^{\epsilon,T}_s$ be a random variable independent of $(\tilde{A}^{\epsilon,T}_{s+t})_{t \geq 0}$ with the probability density (4.4). Then for any $L > 0$ big enough and $\delta > 0$ small enough there are $\varepsilon_0, p_0, \gamma_0 > 0$ such that the following inequality holds true for all $\varepsilon < \varepsilon_0, p < p_0, \gamma < \gamma_0$:

$$\sup_{s \geq 0, T > 0} \sup_{y \in \Omega_{\pm,L}(\delta)} \mathbb{P}\left(\sup_{t \in [0, \tau^{\epsilon,T}_s]} \left\|\tilde{Y}^{\epsilon,T}_{s,t}(y) - y_t(y)\right\| > \varepsilon^\gamma\right) \leq e^{-\varepsilon^{-p}}.$$  

An analogous statement was proven for SDE driven by a heavy tail Lévy process in [29]. The extension of the proof to the case of the additive noise can be found in [25]. The proof is based on the combination of an exponential inequality for martingales (see, e.g. Theorem 23.17 in [22]), Gronwall’s lemma, and estimates with the help of Lyapunov functions in the neighbourhoods of the local minima of $V$. We emphasize that the estimate is uniform in $T$ and $s$ due to the following uniform estimate for the tail of the random variable $\tau^{\epsilon,T}_s$

$$\mathbb{P}(\tau^{\epsilon,T}_s \geq \varepsilon^{-\kappa}) \leq \exp\left(-\varepsilon^\rho a^{-\kappa} \min_{t \in [0,1]} \nu(B^\epsilon_t(0))\right), \quad \kappa > 0,$$

and a uniform estimate for the martingale component of $\tilde{A}^{\epsilon,T}_s$ over $t \in [0, \varepsilon^{-\kappa}]$. 

For the argument in the next section, for \( L > 0, \delta > 0, \varepsilon, \gamma > 0 \) from Lemma 4.1, we introduce the relaxation time \( R_{\gamma}(\varepsilon) = R_{\gamma}(\varepsilon; L, \delta) \) such that the trajectory \( y_t(y) \) satisfies

\[
\sup_{y \in \Omega_{\pm,L}(\delta)} \sup_{t \geq R_{\gamma}(\varepsilon)} \| y_t(y) - m_{\pm} \| \leq \varepsilon^\gamma.
\]

(4.10)

The non-degeneracy of the potential \( V \) at the local minima \( m_{\pm} \) imply that \( R_{\gamma}(\varepsilon) \leq C| \ln \varepsilon | \) for small \( \varepsilon > 0 \).

5. Exit from a reduced domain of attraction. For a fixed big \( L > O \) and small \( \delta_1 > 0 \), define the exit times \( \sigma_{\pm,T} \) of \( Y_{\varepsilon,T} \) from reduced domains \( \Omega_{\pm,L}(\delta_1) \) by

\[
\sigma_{\pm,T} = \sigma_{\pm,T}(L, \delta_1) = \inf \{ t \geq 0 : Y_{\varepsilon,T}^t(y) \notin \Omega_{\pm,L}(\delta_1) \}.
\]

(5.1)

For the next proposition we have to localize the initial values of the jump diffusion \( Y_{\varepsilon,T} \) in a smaller subdomain of \( \Omega_{\pm,L}(\delta_1) \). Namely, for \( 0 < \delta_2 < \delta_1 \) we set

\[
\Omega_{\pm,L}(\delta_1, \delta_2) = \{ y \in \mathbb{R}^d : B_{\delta_2}(y) \subseteq \Omega_{\pm,L}(\delta_1) \text{ for all } t \geq 0 \}.
\]

(5.2)

**Proposition 5.1.** Let \( T = T(\varepsilon) \) and \( \lambda = \lambda(\varepsilon) \) be defined according to (2.6), and \( t > 0 \). Then for any \( \eta > 0 \) there are \( L > 0 \) and \( 0 < \delta_2 < \delta_1 \) such that the estimate

\[
\lim sup_{\varepsilon \to 0} \| \mathbb{P}_y(\sigma_{\pm,T}(\varepsilon) \geq t) - e^{-g(t)c_{\pm}} \| \leq \eta
\]

holds true uniformly over \( y \in \Omega_{\pm,L}(\delta_1, \delta_2) \).

**Proof.** From now on, we will consider only the “left” well \( \Omega_- \) and prove for the exit time \( \sigma_{-T}(\varepsilon) \) that

\[
\lim inf_{\varepsilon \to 0} \mathbb{P}_y(\sigma_{-T}(\varepsilon) \geq t) \geq 1 - e^{-g(t)c_-} - \eta.
\]

(5.4)

The estimate from above can be obtained analogously, but is a bit more involved technically. The proof can be found in [25].

Choose \( \rho \in (\frac{2}{3}, 1) \) and define \( k_\varepsilon = [\varepsilon^{-\kappa}] \) for some \( \kappa \in (\alpha_s - 1, \frac{1}{2} \alpha_s \rho) \). For \( y \in \Omega_{-L}(\delta_1, \delta_2) \) and \( W \in \mathbb{R}^d \) define the events

\[
B_{s,r}(y,W) = \{ \omega : \tilde{Y}_{s,t}^\varepsilon(y) \in \Omega_{-L}(\delta_1), r \in [0, t-s), \tilde{Y}_{s,t-s}^\varepsilon(y) + W \in \Omega_{-L}(\delta_1, \delta_2) \},
\]

\[
C_{s,r}(y,W) = \{ \omega : \tilde{Y}_{s,t}^\varepsilon(y) \in \Omega_{-L}(\delta_1), r \in [0, t-s), \tilde{Y}_{s,t-s}^\varepsilon(y) + W \notin \Omega_{-L}(\delta_1) \},
\]

\[
E_{s,t}(y) = \{ \omega : \sup_{r \in [0, t-s]} \| \tilde{Y}_{s,t}^\varepsilon(y) - y_r(y) \| \leq \varepsilon^\gamma \},
\]

(5.5)

where \( \gamma \) is chosen as in Lemma 4.1. Then

\[
\mathbb{P}_y(\sigma_{-T}(\varepsilon) \geq t) \geq \sum_{k=1}^{k_\varepsilon} \mathbb{P}_y(\sigma_{-T} = \sigma_{-T}(\varepsilon \leq t)) \geq \sum_{k=1}^{k_\varepsilon} \mathbb{E}_y \left[ \prod_{j=0}^{k-2} \left[ B_{s_{j+1}, s_j}(Y_{s_j}^\varepsilon, W_{s_j}) \right] \times \left[ C_{s_{k-1}, s_k}(Y_{s_k}^\varepsilon, W_{s_k}) \right] \right]
\]

(5.6)
We exploit the fact that on the event $E_{\tau_j,\tau_{j+1}}(y)$ the small jump process $\tilde{Y}^{\varepsilon,T}(y)$ follows the deterministic trajectory $y_{(y)}$. If the inter-jump time $\tau_{j+1}^{\varepsilon,T} - \tau_j^{\varepsilon,T}$ exceeds the relaxation time $R_\gamma(\varepsilon)$, we obtain that $Y^{\varepsilon,T}_{\tau_j,\tau_{j+1}}(y)$ belongs to a small neighbourhood of $m_-$, and thus we can determine the location of $Y^{\varepsilon,T}_{\tau_j,\tau_{j+1}}(y) \approx m_+ + W^{\varepsilon,T}_{j+1}$. To make the last approximate relation precise we introduce another smaller reduced domain $\Omega_{-L}(\delta_1, \delta_2)$ defined analogously to (5.2) as

$$\Omega_{-L}(\delta_1, \delta_2) = \{y \in \mathbb{R}^d : B_{\delta_2}(y) \subseteq \Omega_{-L}(\delta_1, \delta_2) \text{ for all } t \geq 0\}.$$ 

Then we obtain

$$\Phi_k^{\varepsilon,T}(y) \geq \mathbb{E}_y \left[ \left( \tau_k^{\varepsilon,T} \lambda(\varepsilon) \leq t \right) \prod_{j=0}^{k-1} \left( \tau_j^{\varepsilon,T} - \tau_j^{\varepsilon,T} \geq R_\gamma(\varepsilon) \right) \right] \Bigg[ \prod_{j=0}^{k-2} \left( E_{\tau_j^{\varepsilon,T},\tau_{j+1}^{\varepsilon,T}}(Y^{\varepsilon,T}_{\tau_j^{\varepsilon,T}}) \right) \left( B_{\tau_j^{\varepsilon,T},\tau_{j+1}^{\varepsilon,T}}(Y^{\varepsilon,T}_{\tau_j^{\varepsilon,T}}, W^{\varepsilon,T}_{j+1}) \right) \right] \Bigg[ \left( E_{\tau_k^{\varepsilon,T},\tau_{k+1}^{\varepsilon,T}}(Y^{\varepsilon,T}_{\tau_k^{\varepsilon,T}}) \right) \left( B_{\tau_k^{\varepsilon,T},\tau_{k+1}^{\varepsilon,T}}(Y^{\varepsilon,T}_{\tau_k^{\varepsilon,T}}, W^{\varepsilon,T}_{k+1}) \right) \right] \prod_{j=0}^{k-1} \left( m_- + W^{\varepsilon,T}_{\tau_j^{\varepsilon,T}} \in \Omega_L \right) \prod_{j=0}^{k-2} \left( m_- + W^{\varepsilon,T}_{\tau_j^{\varepsilon,T}} \in \Omega_L(\delta_1, \delta_2) \right) \Bigg] \Bigg\{ m_--W^{\varepsilon,T}_{\tau_k^{\varepsilon,T}} \in \Omega_L \Bigg\} \leq \sum_{k=0}^{k-1} \sup_{y \in \Omega_L(\delta_1, \delta_2)} \mathbb{P} \left( E_{\tau_j^{\varepsilon,T},\tau_{j+1}^{\varepsilon,T}}(y) \right).$$

The last error term is estimated with the help of Lemma 4.1 as $ke^{-1/\varepsilon^p}$.

Using the density (4.4) of the inter-jump times $\tau_{j+1}^{\varepsilon,T} - \tau_j^{\varepsilon,T}$, $j = 0, \ldots, k-1$, and recalling the explicit formula (4.5) for the law of $W^{\varepsilon,T}_{j}$ we disintegrate

$$\Phi_k^{\varepsilon,T}(y) \geq \int_0^t \int_0^{s_k} \prod_{j=1}^k \beta^\varepsilon(s_j) \exp \left( - \int_0^{s_k} \beta^\varepsilon(u) \, du \right) \times \frac{1}{\beta^\varepsilon(s_k)} \nu_k \left( \frac{\Omega_L - m_-}{\varepsilon} \right) \prod_{j=1}^{k-1} \frac{1}{\beta^\varepsilon(s_j)} \nu(s_j) \left( B_{\varepsilon-\rho}(0) \cap \frac{\Omega_L(\delta_1, \delta_2, \delta_2) - m_-}{\varepsilon} \right) \times \prod_{j=0}^{k-1} \left( 2T(s_{j+1} - s_j) \geq R_\gamma(\varepsilon) \right) \, ds_1 \ldots ds_k \leq k^{-1/\varepsilon^p}.$$ 

We estimate the term which involves the logarithmic return times and extend the finite sum to an infinite series to obtain

$$\mathbb{P}_y(\sigma^{\varepsilon,T}_- \lambda(\varepsilon) \leq t) \geq \sum_{k=1}^{\infty} \int_0^t \int_0^{s_k} \prod_{j=1}^k 2T^\varepsilon(s_j) \exp \left( - \int_0^{s_k} 2T \beta^\varepsilon(u) \, du \right)$$

We denote $\mathbb{P}_y(\sigma^{\varepsilon,T}_- \lambda(\varepsilon) \leq t)$ as $\mathbb{P}_y^{n}(\sigma^{\varepsilon,T}_- \lambda(\varepsilon) \leq t)$.
First we estimate the error terms in (5.9).

The second sum is an error term of order \(\varepsilon^{\alpha - p - 2\kappa} \ln \varepsilon\). Indeed, taking into account that for any \(s \geq 0\), \(\int_0^s \beta^\varepsilon(s + t) \exp(-\int_s^{s+t} \beta^\varepsilon(u)\,du)\,dt = 1\) we get for \(0 \leq l \leq k - 1\) and \(R_{\gamma}(\varepsilon) \leq C \ln \varepsilon\)

\[
\int_0^\infty \int_0^{s_2} \cdots \int_0^{s_k} \prod_{j=1}^k 2\beta^\varepsilon(s_j) \exp\left(-\int_0^{s_k} 2\beta^\varepsilon(u)\,du\right) \\
\times \mathbb{I}\left(2T(s_{l+1} - s_l) < R_{\gamma}(\varepsilon)\right)\,ds_1 \cdots ds_k \\
= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^k 2\beta^\varepsilon(t_1 + \cdots + t_j) e^{-\int_{t_0+\cdots+t_{j-1}}^{t_1+\cdots+t_j} 2\beta^\varepsilon(u)\,du} \\
\times \mathbb{I}\left(2Tt_{l+1} < R_{\gamma}(\varepsilon)\right)\,dt_k \cdots dt_1 \\
= \int_0^\infty \cdots \int_0^\infty \left[ \prod_{j=1}^l 2\beta^\varepsilon(t_1 + \cdots + t_j) e^{-\int_{t_0+\cdots+t_{j-1}}^{t_1+\cdots+t_j} 2\beta^\varepsilon(u)\,du} \\
\times \left(1 - \exp\left(-\int_{t_0+\cdots+t_l}^{t_1+\cdots+t_l+R_{\gamma}(\varepsilon)/2T} 2\beta^\varepsilon(u)\,du\right)\right)\right] \,dt_l \cdots dt_1 \\
\leq C \max_{t \in [a, b]} \nu_t(B_t^\varepsilon(0)) e^{\alpha \varepsilon - \kappa} \ln \varepsilon.
\]
Recalling that $\beta^\varepsilon(t) \leq \text{Const} \cdot \varepsilon^{\alpha \rho}$ and the definition of $T = T(\varepsilon)$ from (2.6), it is easy to check with the help of Stirling’s formula that for $\kappa > \alpha_*(1 - \rho)$ the r.h.s. of (5.11) converges to zero.

Finally, the fourth term is easily estimated by $k_2^2 \varepsilon^{-1/\varepsilon^n}$ and converges to 0 as $\varepsilon \to 0$.

The first series in (5.9) contributes to the main part of the probability we look for. To treat this term, we recall the identity

$$
\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^n f(t_j) \, dt_1 \cdots dt_n = \frac{1}{n!} \left( \int_0^c f(s) \, ds \right)^n.
$$

which holds for any $c > 0$, $n \geq 1$, and any non-negative integrable function $f$. Since the jump measure is self-similar, i.e. $\nu_t(\frac{1}{2}B) = \varepsilon^{\alpha(t)} \nu_t(B)$ for any $B \in \mathcal{B}(\mathbb{R}^d)$, we get that the first term in (5.9) equals to

$$
\int_0^t 2T \varepsilon^{\alpha(t)} \nu_s(\Omega_-^c - m_-)
\times \exp \left( - \int_0^s 2T \varepsilon^{\alpha(r)} \nu_r(\Omega_-^c - \delta_1, \delta_2) - m_- \right) dr \right) \, ds.
$$

Choose $L$ large and $\delta_1, \delta_2$ sufficiently small so that $\nu_r(\Omega_-^c - \delta_1, \delta_2) - m_- \leq \nu_r(\Omega_-^c - m_-) + \eta$ for $r \in [0,1]$. Eventually, we recall again that $2T(\varepsilon) = \varepsilon^{-\alpha_*} \sqrt{\ln \varepsilon}$ to get for small $\varepsilon$ that

$$
P_y(\sigma_-^\varepsilon T \lambda(\varepsilon) \leq t) \geq (1 - \eta) \int_0^\varepsilon \sqrt{\ln \varepsilon} e^{\alpha(s) - \alpha_*} \nu_s(\Omega_-^c - m_-)
\times \exp \left( - \int_0^s \sqrt{\ln \varepsilon} e^{\alpha(r) - \alpha_*} (\nu_r(\Omega_-^c - m_-) + \eta) \right) dr \right) \, ds - \eta.
$$

We evaluate the integral in the exponent in the limit $\varepsilon \to 0$ with the help of the Laplace method (Lemma 3.1) and find that for some $C > 0$ and $\varepsilon$ small enough

$$
P_y(\sigma_-^\varepsilon T \lambda(\varepsilon) \leq t) \geq 1 - e^{-\varepsilon - g(t)} - C \eta,
$$

what finishes the proof.

6. Proof of Theorem 2.1 and 2.2.

Proof. (Theorem 2.1) For definiteness we consider the “left” well $\Omega_-$. For any $t > 0$ and $\eta > 0$, we perform the localization and consider the reduced domains of attraction and the corresponding exit times $\sigma_-^\varepsilon T$ and $\tau_+^\varepsilon T$ (see (2.5) and (5.1)).

(i) Since $\sigma_-^\varepsilon T(\varepsilon) \leq \tau_+^\varepsilon T(\varepsilon)$ a.s., the inequality $\mathbb{P}_y(\tau_+^\varepsilon T(\varepsilon) \varepsilon^\mu \leq t) \leq \mathbb{P}_y(\sigma_-^\varepsilon T(\varepsilon) \varepsilon^\mu \leq t)$ is immediate. We can repeat the arguments of the proof of Proposition 5.1 and verify that for $\varepsilon \to 0$

$$
\left| \mathbb{P}_y(\sigma_-^\varepsilon T(\varepsilon) \varepsilon^\mu \leq t) - 1 + \exp \left( - \int_0^t e^{\alpha(s) - \mu} \nu_s(\Omega_-^c - m_-) \, ds \right) \right| \leq \eta.
$$

For any $t \geq 0$ and $\mu \leq \alpha_*$, the integral in the exponent is of order $\varepsilon^{\alpha_* - \mu} |\ln \varepsilon|^{-1/2}$ and converges to zero.

(ii) If $\mu > \alpha_*$ we use the estimate

$$
\mathbb{P}_y(\sigma_-^\varepsilon T(\varepsilon) \varepsilon^\mu \leq t) \geq \mathbb{P}_y(\tau_-^\varepsilon T(\varepsilon) \varepsilon^\mu \leq t, \tau_-^\varepsilon T(\varepsilon) = \sigma_-^\varepsilon T(\varepsilon))
= \mathbb{P}_y(\sigma_-^\varepsilon T(\varepsilon) \varepsilon^\mu \leq t, Y_\sigma_-^\varepsilon T(\varepsilon) \notin \Omega_-).
$$

(6.2)
Inspecting the arguments of the Proposition 5.1 we can derive
\[
\mathbb{P}_y(\sigma^{-,T(\varepsilon)} \leq t, Y_{\sigma^{-,T(\varepsilon)}} \notin \Omega_-) \geq -\eta + 1 - \exp\left(-\int_0^t \varepsilon^{\alpha(s)-\mu} \nu_s(\Omega^- - m_-) \, ds\right).
\] (6.3)

For \(\mu > \alpha^*\) and any \(t \geq 0\) the integral in the exponent is of order \(\varepsilon^{\alpha^*-\mu} |\ln \varepsilon|^{-1/2}\) and diverges to \(+\infty\), so the assertion (ii) follows.

(iii) Assume that \(\mu \in (\alpha_+, \alpha^*], \alpha(0) > \mu\) and \(t < u\). Apply the inequality (6.1) to get
\[
\mathbb{P}_y(\sigma^{-,T(\varepsilon)} \leq t) \leq -\eta + 1 - \exp\left(-t\varepsilon^\min_{s \in [0,1]} \alpha(s)-\mu \max_{s \in [0,1]} \nu_s(\Omega_- - m_-)\right).
\]

Since \(\min_{s \in [0,1]} \alpha(s) - \mu > 0\) we have proven the first assertion.

To prove the second assertion we set \(v := \inf\{s > u: \alpha(s) > \mu\} \land t\). For some \(\delta' > 0\) small enough we note that \(C := \max_{r \in [u+\delta', u-\delta']} \alpha(s) - \mu < 0\). Then, the inequalities (6.2) and (6.3) imply
\[
\mathbb{P}_y(\tau_{-}^{\varepsilon, T(\varepsilon)} \leq t) \geq -\eta + 1 - \exp\left(-t\varepsilon^\min_{s \in [0,1]} \alpha(s)-\mu \max_{s \in [0,1]} \nu_s(\Omega_- - m_-) dr\right) \geq 1 - 2\eta.
\] (6.4)

For \(\mu > \alpha(0)\) define analogously for \(\delta' > 0\) small enough the constant \(C := \max_{r \in [u+\delta', u-\delta']} \alpha(r) - \mu < 0\) and obtain an estimate
\[
\mathbb{P}_y(\tau_{-}^{\varepsilon, T(\varepsilon)} \leq t) \geq -\eta + 1 - \exp\left(-\int_0^{t\land(u-\delta')} \varepsilon^C \min_{s \in [0,1]} \nu_s(\Omega_- - m_-) \, dr\right) \geq 1 - 2\eta.
\] (6.5)

in the limit \(\varepsilon \to 0\).

Proof. (Theorem 2.2) For any \(t > 0\) and \(\eta > 0\), we perform the localization and consider the reduced domains of attraction and the corresponding exit times \(\sigma_+^{\varepsilon, T}\) and \(\tau_+^{\varepsilon, T}\) (see (2.5) and (5.1)). The inequality \(\sigma_+^{\varepsilon, T} \leq \tau_+^{\varepsilon, T}\) and Proposition 5.1 yield
\[
\mathbb{P}_y(\tau_+^{\varepsilon, T(\varepsilon)} \leq t) \leq \mathbb{P}_y(\sigma_+^{\varepsilon, T(\varepsilon)} \leq t) \leq 1 - e^{-g(t)c_+^\pm} - \eta.
\] (6.6)

for small \(\varepsilon\). The following lower bound for \(\mathbb{P}_y(\tau_+^{\varepsilon, T(\varepsilon)} \leq t)\) can be proven with similar methods as in the proof of Proposition 5.1 and Lemma 2.1:
\[
\mathbb{P}_y(\tau_+^{\varepsilon, T(\varepsilon)} \leq t) \geq \mathbb{P}_y\left(\sigma_+^{\varepsilon, T(\varepsilon)} \leq t, Y_{\sigma_+^{\varepsilon, T(\varepsilon)}} \notin \Omega_\pm\right) \geq 1 - e^{-c_\pm g(t)} - \eta.
\] (6.7)

This finishes the proof of the Theorem.

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