QCD EQUATIONS FOR
GENERATING FUNCTIONALS
AND MULTIPARTICLE
CORRELATIONS
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Abstract
QCD equations for generating functionals are solved at coinciding
momenta of particles. As a result, the relations for $q$-particle corre-
lation functions at equal momenta are obtained. They are directly
connected with the previously derived results about the factorial and
cumulant moments of multiplicity distributions. It is predicted that
the correlations at coinciding points decrease, first, as a function of
the rank $q$, and then start oscillating when represented in a form of the
ratio of cumulant to factorial correlations. In particular, the cumulant
function of the fifth rank should become negative at the origin. Some
experimental data are briefly discussed.

1 INTRODUCTION
Multiparticle correlations in hadroproduction at high energies are very com-
licated both for experimental studies and for theoretical description. Beside
rather sophisticated dynamics, the problems due to the huge number of avail-
able variables enter the game. The choice of any particular variable leads to
some specific correlations being emphasized (see, e.g., Ref.[1]).

The multiparticle correlation functions integrated over momenta give rise
to the moments of multiplicity distributions. Some special features of the mo-
ments have been revealed from the solution of QCD equations for generating
functions (for the review, see Ref.[2]).

Surely, the dependence of the correlation functions on particle momenta
provides more detailed information. Unfortunately, their study in the $3q$-
dimensional phase space is not an easy task. That is why up to now we have
got experimental data on 2-particle (and, sometimes, on 3-particle) correlation functions only plotted as functions of a particular variable (usually, it is the relative (pseudo)rapidity or 4-momentum squared).

Theoretical approach asks for the equations for generating functionals. In QCD they were written rather long time ago [3, 4]. However, no general solution of them has been found. It is common to rewrite those equations in the form of cascade equations \textit{a la} Altarelli-Parisi, and to find their solution for inclusive distributions or for some specific correlations [3, 4, 5, 6].

We shall show how to get some information about correlations directly from the equations for generating functionals using the experience we got from the solution of corresponding equations for the generating functions [2]. We demonstrate it with the illustrative example of the relations between the correlation functions of various ranks at coinciding momenta. In particular, some predictions for the correlations of higher order (yet unavailable in experiment) are obtained. The more sophisticated case of unequal momenta will be considered in a separate publication.

2 DEFINITIONS AND NOTATIONS

The main ingredient of any correlation analysis is the \( q \)-particle inclusive density

\[
\rho_q^{(f)}(k_1, \ldots, k_q) \equiv \frac{1}{\sigma} \frac{d^q\sigma^{(f)}}{d\omega_1, \ldots, d\omega_q},
\]

where the invariant momentum phase space is given by

\[
d\omega_q = \frac{d^3k_q}{(2\pi)^32E_q},
\]

and the set of indices \( f_i \) denotes the internal quantum numbers of all \( q \) particles. We shall omit it in what follows.

The generating functional of inclusive distributions is defined by the formula

\[
G[z(k)] = \sum_{q=0}^{\infty} \frac{1}{q!} \int d\omega_1 \ldots \int d\omega_q \rho_q z(k_1) \ldots z(k_q),
\]

where the integration is over available \( 3q \)-dimensional phase space.
From (3) one gets

\[ \rho_q(k_1, \ldots, k_q) = \frac{\delta^q G[z]}{\delta z(k_1) \ldots \delta z(k_q)} \bigg|_{z=0}. \] (4)

The normalized factorial correlation function is defined by

\[ F_q(k_1, \ldots, k_q) = \frac{\rho_q(k_1, \ldots, k_q)}{\rho_1(k_1) \ldots \rho_1(k_q)}. \] (5)

The logarithm of the functional (3) is the generating functional of the so-called cumulant correlation functions

\[ \kappa_q(k_1, \ldots, k_q) = \frac{\delta^q \ln G[z]}{\delta z(k_1) \ldots \delta z(k_q)} \bigg|_{z=0}. \] (6)

The corresponding normalized function is

\[ K_q(k_1, \ldots, k_q) = \frac{\kappa_q(k_1, \ldots, k_q)}{\rho_1(k_1) \ldots \rho_1(k_q)}. \] (7)

In quantum field theory, the functions \( \rho_q \) are determined by the whole set of Feynman graphs while the functions \( \kappa_q \) are given by the connected graphs only. In correlation analysis, correspondingly, \( \rho_q \) take into account all correlations, and \( \kappa_q \) correspond to the “genuine” \( q \)-particle correlations.

Integration of these functions over momenta gives rise to factorial and cumulant moments of multiplicity distributions. Their generating function is described by the same formula as (3) at constant values of the functions \( z(k) \equiv z = \text{const.} \)

### 3 Equations for generating functionals and their solution

The QCD Lagrangian defines the dynamics of hadroproduction, and, consequently, the correlations of particles produced. It determines the relationship of generating functionals in the form of non-linear integro-differential equations which were proposed a long time ago [3, 4]. To simplify the equations, we shall consider here the gluodynamics where quark degrees of freedom are
neglected. It is reasonable for qualitative conclusions we are aimed at as our previous experience \[2\] of solution of equations for generating functions shows.

Thus the equation for the generating functional of gluodynamics is

$$\frac{\partial G(z(k), Y)}{\partial Y} = \int_0^1 dx K(x) \gamma_0^2 [G(z, Y + \ln x)G(z, Y + \ln(1-x)) - G(z, Y)]. \quad (8)$$

Here \(Y = \ln(p\theta/p_0)\), \(p\) is the initial momentum, \(\theta\) is the angular width of the gluon jet considered, \(p_0 = \text{const}\),

$$\gamma_0^2 = \frac{6\alpha_s}{\pi}, \quad (9)$$

\(\alpha_s\) is the coupling constant, and the kernel of the equation is

$$K(x) = \frac{1}{x} - (1 - x)[2 - x(1 - x)]. \quad (10)$$

It is the non-linear integro-differential equation with shifted arguments in the non-linear part which take into account the conservation laws, and with the initial condition

$$G(z, Y = 0) = 1 + z(k = p), \quad (11)$$

and the normalization

$$G(z = 0, Y) = 1. \quad (12)$$

In particular, at constant \(z\) one gets the generating function

$$G(z, Y) = \sum_{n=0}^{\infty} P_n(Y)(1 + z)^n, \quad (13)$$

where \(P_n\) is the probability of \(n\)-particle events.

The condition (13) normalizes the total probability to 1, and the condition (11) declares that there is a single particle at the very initial stage. The corresponding equation for the generating function (13) looks quite similar and is obtained from eq. (8) at \(z = \text{const}\). Its solution was considered in \[7\].

In a similar way, one can get the solution of (8) at the special point where all the momenta of the \(q\) registered particles (actually, gluons in our treatment) are equal i.e. \(k_1 = k_2 = \ldots = k_q\). Therefore, the momenta differences are
chosen equal to zero. It means that we are considering the correlations at coinciding points where $\Delta k = 0$. Since the correlation functions depend on the differences of momenta and on the overall momentum (i.e. on $Y$) the only dependence left in our case is the dependence on $Y$.

After Taylor series expansion and differentiation in eq. (8) similar to the procedure in [7], one gets the same differential equation

$$\ln G(Y) = \gamma_0^2 [G(Y) - 1 - 2h_1 G(Y) + h_2 G^2(Y)],$$

(14)

where $h_1 = 11/24; h_2 = (67 - 6\pi^2)/36 \approx 0.216$.

In general, one can define the anomalous dimension $\gamma(\omega_i, \alpha_S(Y))$ by the formula which generalizes ones commonly used for average multiplicity or single inclusive distributions (see [3, 4])

$$\rho_q(\omega_i, Y) = \rho_q(Y_0) \exp[q \int_{Y_0}^Y dy \gamma(\omega_i, \alpha_S(y))].$$

(15)

Substituting (15) in (14) one gets after differentiation for terms with equal powers of $z(\omega_i)$

$$H_q(\omega_i) = \frac{K_q(\omega_i)}{F_q(\omega_i)} = \frac{\kappa_q(\omega_i)}{\rho_q(\omega_i)} =
\frac{\gamma_0^2 [1 - 2h_1 q\gamma(\omega_i) + h_2 (q^2 \gamma^2(\omega_i) + q\gamma'(\omega_i))]}{q^2 \gamma^2(\omega_i) + q\gamma'(\omega_i)}.$$  

(16)

Even though we write down the explicit dependence on $\omega_i$, it was assumed in (14) when differentiating over $Y$ that correlation functions are constant in $\omega_i$ i.e. the correlation lengths are much larger than the interval of momenta considered. That is why we omit $\omega_i$ in what follows considering it equal to zero. Apart from it, the condition $qk \ll E$, where $E$ is the total energy, should be fulfilled to ensure no additional dependence on $Y$. Then all calculations are just the same as for the moments of multiplicity distributions in Ref.[7], and from the normalization condition $H_1(0) = 1$ one gets the same relation between $\gamma(0)$ and $\gamma_0$

$$\gamma(0) \approx \gamma_0 - \frac{1}{2} h_1 \gamma_0^2 + \frac{1}{8} (4h_2 - h_1^2) \gamma_0^3 + O(\gamma_0^4).$$

(17)

The value of $\gamma_0$ is measured in experiment (it is equal 0.48 at the mass of $Z^0$-boson). Therefore it is possible to estimate the values of correlation functions at coinciding points according to (14) and (17).
The main conclusion is that the ratio of cumulant to factorial correlation functions at the origin should be approximately equal to the ratio of their integrals

\[ H_q(0) = \frac{K_q(0)}{F_q(0)} \approx \frac{K_q}{F_q} = H_q. \]  

(18)

According to results of [7, 2], it predicts that this ratio should decrease fastly at \( q = 2, 3, 4 \) and reach the negative value at \( q = 5 \). At ever higher values of \( q \) it should oscillate (for the review see [2]).

4 Theoretical results and experimental data

Our main theoretical statement relates two sets of experimentally accessible quantities, namely, moments of multiplicity distributions and the values of correlation functions at the origin. It could be verified without any theoretical formulae. However, one should clearly recognize possible shortcomings which could be of various origin. First, the prediction is done for gluons while in experiment we deal with pions mostly. However, the success in predicting the corresponding qualitative features of multiplicity distributions (see [2]) is encouraging. More important problems could be related to hadronization effects just at the origin because of resonances, Bose-Einstein correlations etc.

Moreover, we have now the experimental data only about two- and three-particle correlation functions (with rather low precision) while the data about moments of multiplicity distributions are much more precise and admit good predictions. It appeals to experimentalists.

At the moment I was able to get the necessary ratios from the data about densities \( \rho_q \) provided by NA22 Collaboration using the formulae

\[ H_2(0) = \frac{C_2(0)}{\rho_2(0)} = 1 - \frac{\rho_1^2(0)}{\rho_2(0)}, \]  

(19)

\[ H_3(0) = \frac{C_3(0)}{\rho_3(0)} = \frac{\rho_3(0) - 3\rho_1(0)\rho_2(0) + 2\rho_1^3(0)}{\rho_3(0)}. \]  

(20)

Using the data for non-singlediffractive \( \pi^+p \) events at 250 GeV/c, one gets

\[ H_2^{cc}(0) = 0.34; \ H_5^{ccc}(0) = 0.13; \]  

(21)

\[ H_2^{--}(0) = 0.28; \ H_3^{---}(0) = 0.1; \]  

(22)
for charged ($c$), negatives ($-$) and positives ($+$). These values agree qualitatively with those shown in [2] for moments of multiplicity distributions in $p\bar{p}$ data (e.g., UA5 Collaboration data for charged-charged correlations are $H_2 = 0.26; H_3 = 0.07$) and show the same tendency as the ratios of moments. Hardly, more elaborated tests can be done at present stage but one could try to check it in Monte Carlo models as well. The most impressive prediction of the negative value of the fifth order correlation function at the origin can be also verified in Monte Carlo models.

5 Conclusions

The solution of the equations for generating functionals in gluodynamics gives a hint to the equality of the ratios of integral cumulant and factorial moments to the corresponding ratios of the values of cumulant and factorial correlation functions at the origin. In particular, it follows that latest ratio should also decrease from $q = 1$ to $q = 5$ becoming negative, and oscillating at ever higher values of ranks $q$. It would be desirable to verify the prediction using the data of same experiments. The comparison to Monte Carlo models is welcome too. From the theoretical side it would indicate once again the importance of a parameter $q\gamma$ in QCD (at the inclusive level, it is closely related to the exclusive factor $n!\alpha_S^n$ widely discussed in vector boson production at threshold and in connection with Parke-Taylor amplitudes). Besides, the negative values of the ratio $H_5$ at asymptotically high energies would show the decline from the negative binomial distribution which predicts the positive values of $H_q$ at any $q$.

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