VALUES OF TWISTS BY DIRICHLET CHARACTERS OF ARTIN L-FUNCTIONS

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Dedicated to the memory of Paul J. Sally, Jr.

ABSTRACT. This note gives a simple proof that certain values of Artin's L-function, for a representation \( \rho \) with character \( \chi_\rho \), are stable under twisting by an even Dirichlet character \( \chi \), up to an element generated over \( \mathbb{Q} \) by the values of \( \chi \) and \( \chi_\rho \), and a product with a power of the Gauss sum \( \tau(\chi) \) equal to the dimension of \( \rho \). This extends a result due to J. Coates and S. Lichtenbaum.

1. INTRODUCTION

We let \( K|F \) denote a finite Galois extension of algebraic number fields, and \( \chi \) will denote the character associated with a representation of \( \text{Gal}(K|F) \). If the given representation is specified as \( \rho \), then the associated character will simply be written as \( \chi_\rho \). Via Shintani's [6] unit theorem, C.L. Siegel and H. Klingen [3, 7] showed that, if \( \chi \) denotes a Dirichlet character and \( K \) is a totally real number field, then the Dirichlet L-function \( L(\chi, s) \) lies in \( \mathbb{Q}(\chi) \) when \( s \) takes negative integer values. For general representations \( \rho \) when \( K \) is a finite Galois extension of \( \mathbb{Q} \), J. Coates and S. Lichtenbaum [1] decomposed the factors at infinity in the functional equation [4, VII.12.6]

\[
\Lambda(K|F, \chi_\rho, s) = W(\chi_\rho)\Lambda(K|F, \chi_\rho, 1-s)
\]

(1)

\( (W(\chi_\rho)) = 1 \)

to show that, at a negative integer \( s = m \) that is a critical point for the Artin L-function \( L(K|\mathbb{Q}, \chi_\rho, s) \), either

(I) \( m \) is odd, and the fixed field \( K_\rho \) of the kernel of \( \rho \) is totally real; or

(II) \( K_\rho \) is totally imaginary, conjugation is central in \( \text{Gal}(K_\rho|\mathbb{Q}) \), and \( \chi_\rho(\sigma) = -\dim(\rho) \).

It follows by Brauer's theorem on induced characters and functorial properties of Artin’s L-function that \( L(K|\mathbb{Q}, \chi_\rho, m) \in \mathbb{Q}(\chi_\rho) \) when \( m \) is a negative integer critical point of \( L(K|\mathbb{Q}, \chi_\rho, s) \). Of course, for all other negative integers \( m \), this L-function takes the value zero (as can easily be seen, for example, by the functional equation (1)).

In this paper, we again consider a finite Galois extension \( K|\mathbb{Q} \), and a representation \( \chi_\rho \) of \( \text{Gal}(K|\mathbb{Q}) \). We also let \( \chi \) denote an even Dirichlet character, i.e., a one-dimensional character of \( \text{Gal}(K|\mathbb{Q}) \) which acts trivially on the conjugation automorphism, and we let \( \mathbb{Q}(\chi_\rho, \chi) \) denote the field generated over \( \mathbb{Q} \) by the values of \( \chi_\rho \) and \( \chi \). The character \( \chi \otimes \chi_\rho = \chi \cdot \chi_\rho \) is associated with the representation
Theorem 1. We will say that the Gauss sum of \( \chi \) via the Artin symbol \( \left( \frac{K_{\chi} | \mathbb{Q}, \cdot}{J_{\mathbb{Q}} / P_{\mathbb{Q}}^i} \right) : J_{\mathbb{Q}}^i / P_{\mathbb{Q}}^i \to \text{Gal}(K_{\chi} | \mathbb{Q}) \).

This allows us to define the Gauss sum of \( \chi \) in the expected way.

**Definition.** [4, VII.6.3] For an algebraic number field \( F \) with ring of integers \( \mathfrak{o}_F \), a Dirichlet character \( \chi \) of \( F \) with finite part \( \chi_f \) as a Größencharakter and conductor \( f \), \( \mathfrak{d}_{F, \mathbb{Q}} \) the different of \( F | \mathbb{Q} \), \( y \in f^{-1} \mathfrak{d}_{F, \mathbb{Q}}^{-1} \) and \( Tr_{F | \mathbb{Q}}(\cdot) \) the trace from \( F \) to \( \mathbb{Q} \), the Gauss sum of \( \chi \) at \( y \) is defined as

\[
\tau(\chi, y) = \sum_{x \bmod f} \chi_f(x)e^{2\pi i Tr_{\mathbb{Q}(xy)}},
\]

where the sum is taken over a set of representatives of \( (\mathfrak{o}_F / f)^* \).

Also, we define \( \chi \) to be zero for any ideals not relatively prime to \( f \), and we set \( \tau(\chi) = \tau(\chi, 1) \). Finally, for a field \( K \), a subfield \( F \subset K \), and two elements \( a, b \in K \), we will say that \( a \sim_F b \) if \( a = kb \) for some \( k \in F \). We are now prepared to state the main result.

**Theorem 1.** If \( s = m \) is a negative integer and critical point of \( L(K | \mathbb{Q}, \chi_{\rho}, s) \), then

\[
L(K | \mathbb{Q}, \chi \otimes \chi_{\rho}, 1 - m) \sim_{Q(\chi_{\rho})} \tau(\chi)^{\dim(\rho)} L(K | \mathbb{Q}, \chi_{\rho}, 1 - m).
\]

Viewing \( \chi_{\rho} \) as a character of \( \text{Gal}(K_{\rho} | \mathbb{Q}) \), we may write, by Brauer’s theorem on induced characters [2, 5.2],

\[
(2) \quad \chi_{\rho} = \sum_{i=1}^{l} n_i \text{Ind}_{H_i}^{G}(\chi_i),
\]

where \( G = \text{Gal}(K_{\rho} | \mathbb{Q}) \), and for each \( i = 1, \ldots, l \), \( \chi_i \) is a character of degree one of a subgroup \( H_i \subset G \) and \( n_i \in \mathbb{Z} \). By Galois theory, for each \( i = 1, \ldots, l \), let \( F_i \) be the subfield of \( K_{\rho} \) so that \( H_i = \text{Gal}(K_{\rho} | F_i) \). We also let \( G_i = \text{Gal}(K | F_i) \) for each \( i = 1, \ldots, l \). The notation in the decomposition (2) will be used henceforth. Our first lemma relates the Gauss sums of restrictions of \( \chi \) to \( L \)-function values.

**Lemma 1.** If \( s = m \) is a negative integer and critical point of \( L(K | \mathbb{Q}, \chi_{\rho}, s) \), then

\[
\prod_{i=1}^{l} \left( \tau(\chi_{|G_i}) \right)^{n_i} \sim_{Q(\chi_{\rho})} \frac{L(K | \mathbb{Q}, \chi_{\rho}, 1 - m)}{L(K | \mathbb{Q}, \chi \otimes \chi_{\rho}, 1 - m)}. \]

This lemma is proven using the functional equation and Galois theory for Gauss sums. We must then relate this product of restricted Gauss sums to the value of the unrestricted Gauss sum, which is the content of our second lemma.

**Lemma 2.** \( \tau(\chi)^{\dim(\rho)} \sim_{Q(\chi)} \prod_{i=1}^{l} \left( \tau(\chi_{|G_i}) \right)^{n_i} \).
This is proven using ramification and Galois theory. It is no surprise that ramification groups have a role in this, as we must measure the action of characters on primes. We also use the "fundamental identity"

$$\sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}|\mathfrak{p}} f_{\mathfrak{P}|\mathfrak{p}} = [K:F],$$

where $e_{\mathfrak{P}|\mathfrak{p}}$ denotes ramification index and $f_{\mathfrak{P}|\mathfrak{p}}$ inertia degree of $\mathfrak{P}|\mathfrak{p}$, for primes $\mathfrak{P}$ and $\mathfrak{p}$ of $K$ and $F$, respectively [4, I.8.2]. Finally, we relate the Gauss sum of a character to that of its conjugate, which we include as a lemma.

**Lemma 3.** $\tau(\overline{\chi}) \sim_{\mathbb{Q}(\chi)} \overline{\tau(\chi)}$.

The importance of requiring that $\chi$ be an even Dirichlet character is clear: this ensures that the critical points of $L(K|\mathbb{Q}, \chi \otimes \chi_p, s)$ and $L(K|\mathbb{Q}, \chi_p, s)$ are the same, by conditions (I) and (II).

## 2. Proof of the Main Result

For Lemma 1, we apply automorphisms to Gauss sums and representations. We take advantage of the interaction of Brauer’s theorem with the Artin $L$-function, precisely as was done in the original proof of meromorphicity of the Artin $L$-function.

**Proof of Lemma 1.** Let us fix $i \in \{1, \ldots, l\}$. We may choose ideal numbers $d_i, f_i$ of $F_i$ so that the different $\mathfrak{d}_{F_i|Q}$ of $F_i|\mathbb{Q}$ satisfies $\mathfrak{d}_{F_i|Q} = (d_i)$ and the conductor $f_i = f(i)$ of $\chi_i$ satisfies $f_i = (f_i)$ [4, VII.7]. We let $p_i = (p_i, \tau)_{\tau \in \text{Hom}(F_i, \mathbb{C})}$ be the exponent of the infinite part of the decomposition of $\chi_i$ as a Größencharakter,

$$\chi_i((x)) = \chi_{i,f}(x)\chi_{i,\infty}(x) = \chi_{i,f}(x)N_{F_i|\mathbb{Q}}\left(\frac{x}{|x|}\right)^{p_i},$$

where $N_{F_i|\mathbb{Q}}(\cdot)$ denotes the norm from $F_i$ to $\mathbb{Q}$. As $\chi_i$ is a character of the group of ideals of $F_i$ via the Artin map, it may be viewed as a Dirichlet character, and thus has $p_i, \tau = 0$ unless $\tau \in \text{Hom}(F_i, \mathbb{R})$. The root number $W(\chi_i)$ from the functional equation (1) satisfies

$$W(\chi_i) = \left[\omega_{4}^{tr_{F_i|\mathbb{Q}}(\mathfrak{P})}N_{F_i|\mathbb{Q}}\left(\left(\frac{f_{id_i}}{|f_{id_i}|}\right)^{p_i}\right)\right]^{-1} \frac{\tau(\chi_i)}{\mathcal{N}_{F_{i|\mathbb{Q}}}(\mathfrak{f})},$$

where $tr_{F_i|\mathbb{Q}}(\cdot)$ denotes the trace from $F_i$ to $\mathbb{Q}$, $\mathcal{N}_{F_{i|\mathbb{Q}}}(\cdot)$ the ideal norm from $F_i$ to $\mathbb{Q}$, and $\omega_{4}$ the complex number $i$, to avoid confusion with indices. We let $d_{F_i}$ denote the discriminant of $F_i$, and we let

$$N_{i,+} = \frac{[F_i: \mathbb{Q}]}{2} + \frac{1}{2} \sum_{\mathfrak{p}} \chi_i(\phi_{\mathfrak{p}}), \quad N_{i,-} = \frac{[F_i: \mathbb{Q}]}{2} - \frac{1}{2} \sum_{\mathfrak{p}} \chi_i(\phi_{\mathfrak{p}}),$$
where for each prime \( p \) appearing in these sums, \( \Psi \) may be any choice of prime of \( K \) so that \( \Psi \mid p \) and \( \phi_p \) is a generator of the decomposition group \( G_\Psi(K|F) \). This yields the following expression for the functional equation:

\[
L(K|F, \chi_i, 1 - s) = \pm \left[ \omega_{K/F}(\Psi) \right]^{-1} \frac{\tau(\chi_i)}{\sqrt{\Delta_{F_i}(\xi)}} \frac{1}{|d_{F_i}|} \left( \cos \left( \frac{\pi s}{2} \right) \right)^{N_+} \times \left( \sin \left( \frac{\pi s}{2} \right) \right)^{N_-} (2\pi)^{-s} \Gamma(s) |L(K|F, \xi, s)|
\]

All of this holds equally well with \( \chi | G_i \otimes \chi_i \) in place of \( \chi_i \). For an automorphism \( \sigma \in \text{Gal}({\overline{Q}}(\chi_\rho, \chi)) \), induction by characters (2) and functoriality of the Artin L-function give

\[
\frac{L(K|Q, \chi \otimes \chi_\rho, m)}{L(K|Q, \chi_\rho, m)} = \frac{L(K|Q, \sigma(\chi \otimes \chi_\rho), m)}{L(K|Q, \sigma(\chi_\rho), m)} = \frac{\prod_{i=1}^l L(K|F_i, \sigma(\chi | G_i \otimes \chi_i), m)^{n_i}}{\prod_{i=1}^l L(K|F_i, \sigma(\chi_i), m)^{n_i}} = \frac{\prod_{i=1}^l L(K|F_i, \chi | G_i \otimes \chi_i, m)^{n_i}}{\prod_{i=1}^l L(K|F_i, \chi_i, m)^{n_i}} = \alpha \left[ \prod_{i=1}^l \left( \frac{\tau(\chi | G_i \otimes \chi_i)}{\tau(\chi_i)} \right)^{n_i} \right] \frac{L(K|Q, \chi \otimes \chi_\rho, 1 - m)^{n_i}}{L(K|Q, \chi_\rho, 1 - m)^{n_i}}
\]

where \( \alpha \in \mathbb{Q} \). The automorphism \( \sigma \) holds the following fixed:

1. (a) the (rational-valued) infinite part of either Größencharakter \( \chi | G_i \otimes \chi_i \) or \( \chi_i \);
2. (b) the conductors of \( \chi | G_i \otimes \chi_i \) or \( \chi_i \); and
3. (c) the action of \( \chi | G_i \otimes \chi_i \) or \( \chi_i \) on the generators of decomposition groups for infinite primes.

By the functional equation (3) without the use of \( \sigma \), it follows from (1.a)-(1.c) that

\[
\frac{L(K|Q, \chi \otimes \chi_\rho, m)}{L(K|Q, \chi_\rho, m)} = \alpha \left[ \prod_{i=1}^l \left( \frac{\tau(\chi | G_i \otimes \chi_i)}{\tau(\chi_i)} \right)^{n_i} \right] \frac{L(K|Q, \chi \otimes \chi_\rho, 1 - m)}{L(K|Q, \chi_\rho, 1 - m)}
\]

so that (4) and (5) give

\[
\prod_{i=1}^l \left( \frac{\tau(\chi | G_i \otimes \chi_i)}{\tau(\chi_i)} \right)^{n_i} = \prod_{i=1}^l \left( \frac{\tau(\chi | G_i \otimes \chi_i)}{\tau(\chi_i)} \right)^{n_i}
\]
Furthermore, as in Theorem 1.2 of [1], we have
\[
\frac{L(K\mathcal{O}, \chi \otimes \chi_p, m)}{L(K\mathcal{O}, \chi_p, m)} \in \mathbb{Q}(\chi_p, \chi),
\]
by none other than the fact that the value of each \( L \)-function in the numerator and denominator of (7) lies in \( \mathbb{Q}(\chi_p, \chi) \).

We let \( \zeta \) be a root of unity, chosen so that \( \mathbb{Q}(\zeta) \) contains all of the values of the characters \( \chi \) and \( \chi_i \), as well as any roots of unity appearing in the Gauss sums \( \tau(\chi_i), \tau(\chi_i \otimes \chi_i), \) and \( \tau(\chi_i | \mathcal{I}) \), for all \( i = 1, \ldots, l \). Let \( \sigma \in \text{Gal}(\mathbb{Q}(\zeta) | \mathbb{Q}(\chi_p, \chi)) \), so that \( \sigma(\zeta) = \zeta^d \) for some integer \( d \). We apply \( \sigma \) to the expression
\[
\prod_{i=1}^{l} \left( \frac{\tau(\chi_i | \mathcal{I})}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i}.
\]

We note that, for any Dirichlet character \( \chi \) of an algebraic number field, it follows from the definition of the Gauss sum that \( \tau(\chi, \mathcal{I}) = \overline{\tau}(\chi) \tau(\chi) \) if \( (a, \mathcal{I}) = 1 \), but also that \( \tau(\chi, \mathcal{I}) = 0 = \overline{\tau}(\chi) \tau(\chi) \) if \( (a, \mathcal{I}) \neq 1 \), as we have defined \( \chi \) to be zero for ideals not relatively prime to \( \mathcal{I} \). Thus, for each \( i = 1, \ldots, l \),
\[
\text{(2.a) } \sigma(\tau(\chi_i | \mathcal{I})) = \tau(\chi_i | \mathcal{I}) \tau(\chi_i),
\]
\[
\text{(2.b) } \sigma(\tau(\chi_i \otimes \chi_i)) = \tau(\chi_i \otimes \chi_i),
\]
\[
\text{(2.c) } \sigma(\tau(\chi_i)) = \tau(\chi_i).
\]
In particular, notice this requires that \( d \) be relatively prime to the conductors of each of these characters (or else Gauss sums are mapped to zero). By (6) and (2.a)-(2.c), we obtain
\[
\sigma \left( \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i} \right)
\]
\[
= \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i} \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i}
\]
\[
= \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i} \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i}
\]
\[
= \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i} \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i}
\]
\[
= \prod_{i=1}^{l} \left( \frac{\tau(\chi_i \otimes \chi_i)}{\tau(\chi_i \otimes \chi_i)} \right)^{n_i}. \]
By Galois theory, it follows that

\[
\prod_{i=1}^{l} \left( \frac{\tau(\chi|\mathbb{Q}) \otimes \chi_i}{\tau(\chi|\mathbb{Q}) \otimes \tau(\chi_i)} \right)^{n_i} \in \mathbb{Q}(\chi_\rho, \chi).
\]

By (4), (7), and (9), the result follows. \qed

Lemma 2 exploits the arithmetic of Dedekind domains; for the proof, it is also necessary to examine Frobenius elements in residue fields, so we may interpret the action of the character \(\chi\).

**Proof of Lemma 2.** As in the proof of Lemma 1, we select an appropriate choice of root of unity \(\zeta\), which contains the values of the characters \(\chi|_{G_i}\) and \(\chi\), as well as all roots of unity appearing in \(\tau(\chi|_{G_i})\) and \(\tau(\chi)\), and we consider some \(\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}(\chi))\), where \(\sigma(\zeta) = \zeta^d\) for some integer \(d\). As before, \(d\) must be relatively prime to the conductors of \(\chi\) and \(\chi|_{G_i}\), for all \(i = 1, \ldots, l\). Similarly to previous arguments, we obtain

\[
\sigma \left( \prod_{i=1}^{l} (\tau(\chi|_{G_i}))^{n_i} \right) = \frac{\prod_{i=1}^{l} (\chi|_{G_i})_{\sigma}^{f_i}(d)^{n_i}}{\chi^{\sigma f_i}\left( (d)^{\dim(\rho)} \right)} \left( \prod_{i=1}^{l} (\tau(\chi|_{G_i}))^{n_i} \right).
\]

Note that \(\sigma\) makes no appearance in this final expression, as \(\sigma\) fixes the values of \(\chi\) and \(\chi|_{G_i}\), for all \(i = 1, \ldots, l\). Let \(K_\chi\) and \(K_{\chi|_{G_i}}\) denote the fixed fields of the kernels of \(\chi\) and \(\chi|_{G_i}\), for each \(i = 1, \ldots, l\), respectively. We have for each \(i = 1, \ldots, l\) that \((\chi|_{G_i})_{\sigma}(d) = \chi|_{G_i}(d))\), where \((d)\) is viewed as an ideal of \(K_{\chi|_{G_i}}\), and likewise that \(\chi_f(d) = \chi((d))\), where \((d)\) is viewed as an ideal of \(K_\chi\). We write \(d = \prod_k p_k^{\alpha_k}\) as the prime factorization of \(d\) in \(\mathbb{Z}\). As \(d\) is relatively prime to the conductor of \(\chi\), each prime \(p_k\) is unramified in \(K_\chi\). This implies, for a prime \(\mathfrak{p}|p_k\) of \(K\) with decomposition group \(G_{\mathfrak{p}}\) and inertia group \(I_{\mathfrak{p}}\) of \(\mathfrak{p}|p_k\), that the character \(\chi\) acts on \(G_{\mathfrak{p}}/I_{\mathfrak{p}}\).

Let us fix some \(i \in \{1, \ldots, l\}\). For each \(k\), we write \(p_k = \prod_j \mathfrak{p}_{k,j}^{e_{\mathfrak{p}_{k,j}}}\) in \(K\) and \(p_k = \prod_m \mathfrak{p}_{k,m}^{e_{\mathfrak{p}_{k,m}}}\) in \(F_i\). For each \(j\), we write \(\phi_{\mathfrak{p}_{k,j}}\) for a representative element of \(G_{\mathfrak{p}_{k,j}}\) which maps to the Frobenius in the residue field of \(\mathfrak{p}_{k,j}|p_k\), and we write \(\phi_{\mathfrak{p}_{k,j}}'\) for the analogous representative in \(\text{Gal}(K|F_i)\) \([4, \text{VII.10}]\). Let us also fix a choice of \(k\) and \(j\). Let \(\mathfrak{p}_{k,m}^j\) denote the prime of \(F_i\) with \(\mathfrak{p}_{k,m}^j|\mathfrak{p}_{k,m}|p_k\), and let \(f_{\mathfrak{p}_{k,m}^j} = [\sigma_{F_i}/\mathfrak{p}_{k,m}^j : \mathbb{Z}/p_k]\mathbb{Z}\). By definition of \(\phi_{\mathfrak{p}_{k,j}}\) and \(\phi_{\mathfrak{p}_{k,j}}'\), the image

\[
\phi_{\mathfrak{p}_{k,m}^j}(\alpha \mod \mathfrak{p}_{k,j}) := \left( \phi_{\mathfrak{p}_{k,m}^j}(\alpha) \mod \mathfrak{p}_{k,j} \right)
\]

in the residue field of \(\mathfrak{p}_{k,j}|p_k\) is equal to

\[
\phi_{\mathfrak{p}_{k,j}}'(\alpha \mod \mathfrak{p}_{k,j}) := \left( \phi_{\mathfrak{p}_{k,j}}'(\alpha) \mod \mathfrak{p}_{k,j} \right).
\]
Thus \( \phi_{p_k,m} \) and \( \phi'_{p_k,m} \) belong to the same coset of \( H_{p_k,j} \) in \( G_{p_k,j} \). As \( \chi \) is trivial on \( H_{p_k,j} \), this implies that

\[
(11) \quad \chi\left(\phi_{p_k,j}\right) = \chi\left(\phi'_{p_k,j}\right).
\]

By \( (11) \) and the definition of the Artin symbol, we obtain

\[
(12) \quad \chi((p_k))^{f_{p_k,m}} = \chi\left(\phi_{p_k,j}\right)^{f_{p_k,m}} = \chi\left(\phi'_{p_k,j}\right)^{f_{p_k,m}} = \chi\left(\phi'_{p_k,j}\right).
\]

By \( (12) \) and the identity \( \sum_m e_{p_k,m} f_{p_k,m} = [F_i : \mathbb{Q}] \), we obtain

\[
(13) \quad \chi((d))^{[F_i : \mathbb{Q}]} = \prod_k \left(\chi((p_k))^{[F_i : \mathbb{Q}]}\right)^{a_k} = \prod_k \left(\chi((p_k))^{\sum_m e_{p_k,m} f_{p_k,m}}\right)^{a_k}
\]

\[
= \prod_k \left(\chi|_{G_i}((p_k))^{e_{p_k,m} f_{p_k,m}}\right)^{a_k} = \prod_k \left(\chi|_{G_i}\left(\prod_m \chi_{p_k,m}\right)\right)^{a_k}
\]

\[
= \prod_k \left(\chi|_{G_i}((d))\right)^{a_k} = \chi|_{G_i}((d)).
\]

Returning to Brauer’s theorem on induced characters \( \phi \), we have

\[
(14) \quad \dim(\rho) = \chi_{\rho}(1) = \sum_{i=1}^l n_i \text{Ind}_{H_i}^{G_i} (\chi_i)(1) = \sum_{i=1}^l n_i [G : H_i]
\]

\[
= \sum_{i=1}^l n_i [K_{p_i} : \mathbb{Q}] = \sum_{i=1}^l n_i [F_i : \mathbb{Q}].
\]

We thus obtain by \( (13) \) and \( (14) \) that

\[
(15) \quad \frac{\prod_{i=1}^l \left(\chi|_{G_i}((d))\right)^{n_i}}{\chi((d))^\dim(\rho)} = \frac{\prod_{i=1}^l \left(\chi|_{G_i}((d))\right)^{n_i}}{\chi((d))^\dim(\rho)} = \frac{\prod_{i=1}^l \left(\chi((d))^{[F_i : \mathbb{Q}]}\right)^{n_i}}{\chi((d))^\dim(\rho)}
\]

\[
= \frac{\chi((d))^{\sum_i n_i [F_i : \mathbb{Q}]}}{\chi((d))^\dim(\rho)} = 1.
\]

By \( (10) \) and \( (15) \), it follows that \( \sigma \) fixes

\[
\frac{\prod_{i=1}^l \left(\tau(\chi|_{G_i})\right)^{n_i}}{\tau(\chi)^\dim(\rho)},
\]

which by Galois theory must therefore lie in \( \mathbb{Q}(\chi) \).

The proof of Lemma 3 uses arguments we have seen before; we give the short proof for completeness.

\textit{Proof of Lemma 3.} Once again, let \( \zeta \) be a root of unity so that \( \mathbb{Q}(\zeta) \) contains the values of \( \chi, \chi' \), and all roots of unity contained in \( \tau(\chi) \) and \( \tau(\chi') \). Let \( \sigma \in \text{Gal}(\mathbb{Q}(\chi) / \mathbb{Q}) \) be an automorphism. Then \( \sigma(\chi) \) is also a root of unity, and \( \sigma(\chi) = \zeta^r \chi \) for some integer \( r \). Since \( \chi \) is trivial on \( H_{p_k,j} \), we have

\[
\sigma(\chi|_{H_{p_k,j}}) = \zeta^r \chi|_{H_{p_k,j}}.
\]

By the definition of the Artin symbol, we obtain

\[
\sigma(\chi|_{G_i}) = \zeta^r \chi|_{G_i}.
\]
so that $\sigma(\zeta) = \zeta^d$ for some integer $d$. Of course, as the values of $\chi$, and thus $\mathcal{X}$, are roots of unity, $\sigma$ must also fix the values of $\mathcal{X}$. We have

$$
\sigma(\tau(\chi) \tau(\mathcal{X})) = \sigma(\tau(\chi)) \sigma(\tau(\mathcal{X})) = \tau(\sigma(\chi), d) \tau(\sigma(\mathcal{X}), d)
$$

$$
= \tau(\chi, d) \tau(\mathcal{X}, d) = \mathcal{X}(d) \tau(\chi) \tau(\mathcal{X})
$$

$$
= \mathcal{X}(d) \chi(d) \tau(\chi) \tau(\mathcal{X}) = \tau(\chi) \tau(\mathcal{X}).
$$

By (16) and Galois theory, it follows that $\tau(\chi) \tau(\mathcal{X}) \in \mathbb{Q}(\chi)$.

\[ \Box \]

**Proof of Theorem 1.** This is an almost immediate application of the previous lemmas. We note that throughout this analysis, there is no difficulty with exchanging the roles of characters and their conjugates: as $\mathcal{X}(g) = \chi(\rho^{-1})$, equivalences remain valid up to elements of $\mathbb{Q}(\chi, \chi)$. Applying Lemmas 1, 2, and 3 (in that order), we obtain

$$
\tau(\chi) \dim(\rho) L(K|\mathbb{Q}, \chi, 1-m)
$$

$$
\sim_{\mathbb{Q}(\chi, \chi)} \tau(\chi) \dim(\rho) \prod_{i=1}^{l} (\tau(\mathcal{X}|G_i))^{\nu_i} L(K|\mathbb{Q}, \chi \otimes \chi, 1-m)
$$

$$
\sim_{\mathbb{Q}(\chi, \chi)} \tau(\chi) \dim(\rho) \tau(\mathcal{X}) \dim(\rho) L(K|\mathbb{Q}, \chi \otimes \chi, 1-m)
$$

$$
\sim_{\mathbb{Q}(\chi, \chi)} L(K|\mathbb{Q}, \chi \otimes \chi, 1-m).
$$

\[ \Box \]

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