Quantum fluctuations: Enhancement or suppression of chaos?

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Abstract

We have studied the effects of quantum fluctuations on dynamical behavior by using squeezed state approach. Our numerical results of the kicked harmonic oscillator demonstrate qualitatively and quantitatively that quantum fluctuations can not only enhance chaos but also suppress classical diffusion. In addition, the squeezed state approach also gives a simple picture of dynamical localization.

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Unlike classical chaos, which is characterized by sensitive dependence on initial conditions, "quantum chaos" is not a well-defined concept. Nevertheless, much effort has been made to study the dynamical behavior of quantum systems. Most of the works in this field have been devoted to two aspects. On the one hand, people have been trying to seek a generic behavior of the quantum spectra of a classical system, which exhibits either full chaos or a mixture of regular and chaotic behavior (see e.g. [1]). On the other hand, a large amount of contributions have been made in looking for manifestations of classical chaos in the structure of individual eigenstates, and trying to invoke the periodic orbit theory to understand many extraordinary phenomena such as the scar [2]. In addition, there has been another approach, the so-called squeezed state approach (also being called semiquantal approach) [3–6], to quantum chaos. The main purpose of this approach is to study how quantum fluctuations manifest themselves in classical trajectories. It starts directly from the quantum systems without refering to the classical limit. In fact, it has been shown [3] that the squeezed state dynamics exists even for systems without a well-defined classical dynamics.

The squeezed state approach starts from the time-dependent variational principle (TDVP) formulation, wherein the action is,

\[ S = \int dt \langle \Psi, t | i\hbar \frac{\partial}{\partial t} - \hat{H} | \Psi, t \rangle. \] (1)

The time evolution of the quantum system is then determined by \( \delta S = 0 \) against independent variations of \( \langle \Psi, t | \) and \( | \Psi, t \rangle \), which yields the Schrödinger equation and its complex conjugate. The true solution may be approximated by restricting the choice of states to a subspace of the full Hilbert space and finding the path along which \( \delta S = 0 \) within this subspace. Since in the presentation of coherent state the width of the wavepacket is fixed, which is not suitable for studying dynamics, we shall take the squeezed state as the trial wave function. In this case, as we shall see later, in addition to the dynamics of the centroid of the wave packet, we will also have equations for the motion of the fluctuations (the spread of wave packet).

Until now, very few works have been done in this direction, and there is still no general conclusion as to how the quantum fluctuations affect classical chaos in a quantum system. Zhang and his coworkers have observed suppression of chaos in the kicked spin and kicked rotator models [3,4]. In the study of a 1-D problem with a Duffing potential without any external perturbation, Pattanayak and Schieve [6] have shown that the semiquantal behavior is chaotic. They therefore concluded that quantum fluctuations induce chaos.

In this letter, we would like to demonstrate that the quantum fluctuations may not only enhance chaos but suppress chaos as well. To this end, we shall make use of the quantum kicked harmonic oscillator (KHO) [7]. The authors in Ref. [7] have studied the quantum behavior of KHO. In the special case of a rational frequency ratio \( 1/4 \), KHO can be cast into the kicked Harper model which attracts a great attention [8]. Here we use the quantum KHO model just to illustrate how the quantum fluctuations affect its dynamical behavior. As is well known, if the kick is absent, the quantum fluctuations will be decoupled from the classical motion and the time evolution of the wave packet will behave exactly like a classical particle. This was what Schrödinger concluded in 1926 [9].

The Hamiltonian of KHO is [10],
\[ \hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2} \hat{q}^2 - K \sin \hat{q} \delta_T, \]  

(2)

where \( \delta_T = \sum_{n=-\infty}^{\infty} \delta(t - nT) \). Using the squeezed state as the trial wave function of the Hamiltonian (2), one can readily obtain (3),

\[ \hat{H} \equiv \langle \Psi, t | \hat{H} | \Psi, t \rangle = \frac{\hat{p}^2}{2} + \frac{\Delta p^2}{2} + \frac{\omega^2 q^2}{2} + \frac{\omega^2 \Delta q^2}{2} - K e^{-\frac{hG}{2}} \sin q \delta_T \]

(3)

where, \( p \equiv \langle \Psi, t | \hat{p} | \Psi, t \rangle \) and \( q \equiv \langle \Psi, t | \hat{q} | \Psi, t \rangle \) are expectation values of the momentum and coordinate operators, respectively. \( \Delta q^2 \equiv \langle \Psi, t | (\hat{q} - q)^2 | \Psi, t \rangle = hG \) is fluctuation of coordinate, and \( \Delta p^2 \equiv \langle \Psi, t | (\hat{p} - p)^2 | \Psi, t \rangle = h \left( \frac{1}{\delta_T} + 4\Pi_n^2 G \right) \) fluctuation of momentum. From the TDVP, we have equations for the centroid of wave packet and the fluctuations. In time interval, \( nT < t < (n + 1)T \) the harmonic oscillator undergoes free motion governed by \( \dot{q}_n = p_n, \quad \dot{p}_n = -\omega q_n \); and \( \dot{G}_n = 4\Pi_n G_n, \quad \dot{\Pi}_n = -2\Pi_n^2 - \frac{\omega^2}{2} q_n^2 \). At the time \( t = (n + 1)T \), the harmonic oscillator is kicked by the external potential, thus at this point the momentum and its fluctuation undergo a jump,

\[
\begin{align*}
  p_{n+1}(T^+) &= p_n(T^-) + K e^{-\frac{hG_n(T^-)}{2}} \cos q_n(T^-), \\
  \Pi_{n+1}(T^+) &= \Pi_n(T^-) - \frac{K}{2} e^{-\frac{hG_n(T^-)}{2}} \sin q_n(T^-). 
\end{align*}
\]

(4)

It is obvious that unlike classical motion, the squeezed state dynamics is described by four coupled differential equations. Thus we expect much more complicated dynamical behavior. In fact, before we discuss any detailed numerical calculation, we can give some qualitative analysis of the dynamical properties of this system. Firstly, since the quantum fluctuations are always positive, (they are equal to zero only at limit \( \hbar = 0 \)), the effective potential strength \( K_{eff} = K \exp \left( \frac{-hG_n}{2} \right) \) is always less than \( K \). This reduction would suppress classical diffusion if the classical motion is chaotically diffusive. Secondly, the quantum fluctuations in coordinate and momentum are in fact add two dimensions to the classical problem. We thus expect that invariant curves in the classical phase space would not be able to prevent the trajectories from penetrating or crossing them in the squeezed state dynamics. These two mechanisms coexist. They compete with each other and determine the dynamical behavior of the underlying system. Therefore, we conjecture that the quantum fluctuations may enhance chaos in a certain case, and suppress chaos as well.

In Fig. 1(a) we plot the classical phase space \((q_{cl}, p_{cl})\) for a trajectory starting from \((0,0)\) and evolving \(10^4\) kicks. Here \( K = 1 \) and \( \sigma = \pi \), where \( \sigma = \omega_T / \omega \), is the ratio between the angular frequency of the kick \( \omega_T \) \( (\omega_T = 2\pi / T, T \) is the period of kicks) and angular frequency \( \omega \). In our calculations, we put \( \omega = 1 \). Fig. 1(b) shows the semiquantal phase space, where a wave packet starts from \((q_0(0), p_0(0), G_0(0), \Pi_0(0)) = (0, 0, 0.5, 0), \) with \( \hbar = 0.1 \). It is worth pointing out that we have tested numerically and found that our results given in this letter do not depend on initial conditions. However, the selection of the initial conditions must be physically meaningful, i.e. \((G_0, \Pi_0)\) are required by the criterion of minimum uncertainty, while the \((q_0, p_0)\) to be the expectation values of eigenstates of harmonic oscillator, i.e. \((0, 0)\). As was shown that if there was no kicks, the wave packet starts from this point will
evolve exactly along the particle’s classical trajectories, the quantum fluctuations both in momentum and coordinate are constant and independent of time. In this case the squeezed state approach describes exactly the classical behavior. Now if we switch on the kicks, the situation changes dramatically. As is shown in Fig. 1(a), the initial point just lies in the stochastic sea, thus it is evident that in the classical case this trajectory will never enter stable islands due to the existence of invariant curves. However, as we predicted, the invariant curves could not prevent the trajectory from crossing it via other dimensions semiquantally. This is demonstrated in Fig. 1(b), where all the stable islands in the classical phase space are "visited" by the semiquantal trajectory. Moreover, the trajectory diffuses into a wider stochastic region than in the classical case. As a quantitative verification, we have numerically calculated the Lyapunov exponents $\lambda_n = \lim_{n \to \infty} \lambda_n$, for the trajectories in both cases. The time behavior of $\lambda_n$ is shown in Fig. 1(c). It is clearly seen that after a certain time the semiquantal $\lambda_n$ becomes larger than its classical counterpart, which means that the enhancement mechanism becomes dominant, and as a consequence leads to the enhancement of chaos. This fact provides a quantitative evidence for the enhancement of chaos by the quantum fluctuations.

What happens if we increase the external potential strength $K$? Classically, when $K$ increases, the classical motion becomes more and more chaotic. At $K = 6$ and $\sigma = \pi$, the phase space is completely chaotic as shown in Fig. 2(a). Like Fig. 1(a), Fig. 2(a) is for the trajectory that starts from the origin and undergoes $10^4$ kicks. The classical chaotic and diffusive process is easily seen from the evolution of this phase plot. To demonstrate the suppression of chaos, we start a wave packet from $(0, 0, 0, \frac{\pi}{2})$ in the 4-D semiquantal phase space. The evolution is shown in Fig. 2(b). Comparing Fig. 2(a) and Fig. 2(b), it is obvious that in the classical case, the phase space is chaotic and diffusive, whereas in the semiquantal case the diffusive process is largely slowed down and suppressed. Furthermore, an invariant-curve-like structure appears in the semiquantal phase space, which seems to form a barrier for the diffusion and thus suppress chaos. Again, the suppression of chaos is quantitatively illustrated by a large decrement of $\lambda_n$ as shown in Fig. 2(c), where the suppression mechanism is most important.

As a further investigation of the suppression, it is convenient to calculate energy diffusion with time $n$ (in unit of kicks). The diffusion is defined by $\langle E_n \rangle - \langle E_0 \rangle$, where $\langle E_0 \rangle$ is the initial averaging energy. For the classical case, $E_n = \frac{1}{2}(p_n^2 + q_n^2)_{cl}$ In our calculations we have taken an ensemble averaging over $10^4$ initial points which are uniformly distributed inside a disk centred at the origin with a radius of $\pi$. As for the semiquantal dynamics, $\langle E_n \rangle$ is defined by,

$$\langle E_n \rangle = \frac{1}{2} \langle \Psi | \hat{p}_n^2 + \hat{q}_n^2 | \Psi \rangle \rangle = \frac{1}{2} \left(p_n^2 + \hbar \left( \frac{1}{4G_n} + 4\Pi_n^2 G_n \right) \right) + \frac{1}{2} \left( q_n^2 + \hbar G_n \right).$$

(5)

In Fig. 3 we show the energy diffusion at $K = 6$ and $\sigma = \pi$ in the classical and semiquantal cases. The suppression of the classical diffusion is very obvious. All the results shown in this letter have been tested to be true for the external potential (kick) in Eq.(2) having even parity.
Our results demonstrate that in the small $K$ regime, the quantum fluctuations enhance chaos, whereas in the large $K$ regime, they suppress chaos. In the limiting case of $\omega = 0$, the KHO model is reduced to the famous kicked rotator model, in which chaotic diffusion would be completely suppressed by the quantum fluctuations and results in dynamical localization, a well established fact observed by Casati et al.\cite{12} numerically and confirmed recently by experiments.\cite{13}. The squeezed state approach captures this phenomenon in a direct and simple way (see also \cite{4}). In the big $K$ case, the suppression factor determines the dynamical behavior of the system. In the opposite case, i.e. at small $K$, the harmonic oscillator potential is dominant. In this case, there are many stable islands among the stochastical layer in classical phase space, the enhancement mechanism becomes significant. We thus expect the existence of a threshold value $K_c$ distinguishing the suppression and enhancement. We have carried out extensive numerical investigations over a wide range of $K$. Our results have confirmed this fact. However, the $K_c$ is $\hbar$ dependent.

Finally, we would like to discuss the validity of the squeezed state approach. It is obvious from Eq. (1) that the power of the TDVP depends on the selection of the trial wave function. In our approach, the squeezed state is chosen. This approach has been proved to be far better than the semiclassical approach, both for integrable and for nonintegrable systems. As for the integrable system such as the harmonic oscillator, we have shown very recently that the squeezed state approach can produce not only the exact eigenvalues but also eigenfunctions. As for the non-integrable system, Pattanayak and Schieve\cite{14} have successfully used the squeezed state approach to calculate the low-lying eigenenergies of a classically chaotic system, for which the WKB method completely fails. The calculated eigenenergies agree with exact quantum (numerical) results within a few percent. Furthermore, we have applied the squeezed state approach to a 1-D many-body system, the Frenkel-Kontorova model, the result agrees with that of the quantum Monte Carlo method excellently\cite{15}.

In summary, using the squeezed state as a trial wavefunction for the quantum KHO, and from the TDVP, we obtained four coupled dynamical equations for the expectation values and the quantum fluctuations, with which we have shown qualitatively and quantitatively that the quantum fluctuations can not only enhance chaos, but also suppress chaos. The squeezed state approach gives simple picture of dynamical localization.

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REFERENCES

[1] O. Bohigas in "Chaos and Quantum Systems (Les Houches Session LII)" eds. M. J. Giannoni, A. Voros and J. Zinn-Justin, Amsterdam, Elsevier 1991; F. Haake, Quantum Signatures of Chaos, Springer-Verlag 1992, Berlin, Heidelberg
[2] M. Gutzwiller, "Chaos in Classical and Quantum Mechanics", Springer-Verlag, 1990.
[3] W. M. Zhang, J. M. Yuan, D. H. Feng, R. Qi and J. Tjon, Phys. Rev. A, 42, 3615(1990); W. M. Zhang and D. H. Feng, J. Mod. Phys. A, 8, 1417(1993)
[4] W. M. Zhang and M. T. Lee, Preprint IP-ASTP-26-94.
[5] W. M. Zhang and D. H. Feng, Phys. Rep., 252, 1(1995).
[6] A. K. Pattanayak and W. C. Schieve, Phys. Rev. Lett., 72, 2855(1994)
[7] G. P. Berman, V. Yu. Rubaev and G. M. Zaslavsky, Nonlinearity, 4, 543 (1991), D. Shepelyansky and C. Sire. Europhys. Lett., 20, 95 (1992); I. Dana, Phys. Rev. Lett., 73, 1609 (1994) and Phys. Lett. A 197, 413 (1995); F. Borgonovi and L. Rebuzzini, Phys. Rev. E. 52, 2302 (1995)
[8] P. Leboeuf, J. Jurchan, M. Feingold, and D. P. Arovas, Phys. Rev. Lett. 65, 3076 (1990); R. Lima and D. Shepelyansky, ibid 67, 1377 (1991); T. Geisel, R. Ketzmerick, and G. Petschel, ibid 68, 3826 (1992); R. Ketzmerick G. Petschel, and T. Geisel, ibid 69, 695 (1992); R. Artuso, F. Borgonovi, I. Guarneri, L. Rebuzzini, and G. Casati ibid 68, 3302 (1992); R. Artuso, F. Borgonovi, G. Casati, I. Guarneri, and L. Rebuzzini, Int. J. Mod. Phys. B 8, 207 (1994).
[9] E. Schrödinger, Z. für Naturwiss., 14, 644(1926)
[10] G.M. Zaslavsky, R.Z. Sagdeev, D.A.Usikov and A.A.Chernikov, "Weak Chaos and Quasi-regular Patterns", Cambridge University Press, 1992.
[11] Y. Tsui and Y. Fujiwara, Prog. Theor. Phys., 86, 443, 469 (1991); Y. Tsui ibid 88, 911 (1992)
[12] G. Casati, B. Chirikov, J. Ford and F. M. Izrailev, in Stochastic behavior in Classical and Quantum Hamiltonian Systems, Lecture Notes in Physics, edited by G. Casati and J. Ford, (Springer-Verlag, Berlin 1979), Vol. 93, P.334.
[13] F.L. Moore, J.C. Robinson, C.F. Bharucha, B. Sundaram and M.G. Raizen, Phys. Rev. Lett., 75, 4598(1995).
[14] A. K. Pattanayak and W. C. Schieve. Phys. Rev. E, 56 278 (1997).
[15] B. Hu, B. Li, and W.M. Zhang, Preprint HKBU-CNS-9702.
FIGURES

FIG. 1. Comparison between the classical phase space (a) and the semiquantal one ($\hbar = 0.1$) (b) at $K = 1$ with $\sigma = \pi$. (c) the time (in unit of the kick) behavior of $\lambda_n$ for the trajectory shown in (a).

FIG. 2. The same as Fig.1 but $K = 6$. (a) the classical phase space; (b) the semiquantal ($\hbar = 1$) $(q,p)$; (c) the time behavior of $\lambda_n$ for the trajectory shown in (a).

FIG. 3. The diffusion at $K = 6$ with $\sigma = \pi$, for classical and semiquantal ($\hbar = 1$) cases. The ensemble averaging is taken over $10^4$ initial points.
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The graph depicts the relationship between \( n \) and \( \frac{\Lambda}{V} \). The dashed line represents the classical case, and the solid line represents the semiquantum case. The y-axis ranges from 0 to 1500, and the x-axis ranges from 0 to 1000.