Symmetries of Nonrelativistic Phase Space and the Structure of Quark-Lepton Generation *

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Abstract

According to the Hamiltonian formalism, nonrelativistic phase space may be considered as an arena of physics, with momentum and position treated as independent variables. Invariance of $x^2 + p^2$ constitutes then a natural generalization of ordinary rotational invariance. We consider Dirac-like linearization of this form, with position and momentum satisfying standard commutation relations. This leads to the identification of a quantum-level structure from which some phase space properties might emerge.

Genuine rotations and reflections in phase space are tied to the existence of new quantum numbers, unrelated to ordinary 3D space. Their properties allow their identification with the internal quantum numbers characterising the structure of a single quark-lepton generation in the Standard Model. In particular, the algebraic structure of the Harari-Shupe preon model of fundamental particles is reproduced exactly and without invoking any sub-particles.

Analysis of the Clifford algebra of nonrelativistic phase space singles out an element which might be associated with the concept of lepton mass. This element is transformed into a corresponding element for a single coloured quark, leading to a generalization of the concept of mass and a different starting point for the discussion of quark unobservability.

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“I do not believe that a real understanding of the nature of elementary particles can ever be achieved without a simultaneous deeper understanding of the nature of spacetime”

Roger Penrose - [1]

1 Elementary particles and space

In our search for the underlying components of matter we have identified several species of fundamental elementary fermions. According to the Standard Model, there are three generations of such particles, each generation composed of eight objects: two leptons and two triplets of quarks. Interactions of these particles proceed through an exchange of various gauge bosons. The particles themselves (and systems composed thereof) differ in their properties, with the differences corresponding to different eigenvalues of various quantum numbers.

Two groups of such quantum numbers may be identified. The first one is composed of the so-called “spatial” quantum numbers, the other one comprises “internal” quantum numbers.

The spatial quantum numbers of a particle are those standardly written in the form “$J^PC$”. Here $J$ denotes the spin of the particle, $P$ denotes its parity, and $C$ is the so-called charge-conjugation parity. They are all connected with the properties of spacetime: spin is related to ordinary rotations, parity - to reflections in our 3D space, while $C$-parity - being related to complex conjugation - is clearly connected with time reflection. It is important to notice that all these spatial quantum numbers are nonrelativistic in origin. In particular, the emergence of antiparticles is not a relativistic phenomenon as the Dirac equation might suggest: they appear also when the strictly nonrelativistic Schrödinger equation is linearized [2].

As far as internal quantum numbers are concerned, many physicists believe that these quantum numbers should also be connected with some “classical arena” which would constitute some kind of an extension of spacetime. In view of the nonrelativistic nature of all spatial quantum numbers it seems then natural to expect that the minimal extension of the concept of space needed for an understanding of internal quantum numbers should be nonrelativistic as well.

2 Space, time and quantum

It might be argued that the reasoning of the preceding section, based on an expected analogy between spatial and internal quantum numbers, is not sufficient to justify the adoption of a nonrelativistic approach well enough. After all, the standard form of the theory of special relativity involves transformations which mix time with space, thereby undermining the very concept of absolute simultaneity.
In fact, however, the connection between space and time is more subtle than the standard form of special relativity would suggest: the latter form emerges only when the Einstein radiolocation prescription for the synchronization of distant clocks is adopted. Yet, distant clocks may be synchronised in various ways, reflecting the presence of gauge freedom related to the impossibility of measuring the one-way speed of light. Absolute simultaneity may then be achieved with the help of a suitable gauge [3].

A different argument in favour of a nonrelativistic approach may also be given. Namely, although relativistic field theory does unite special relativity and quantum physics, this marriage of quantum and relativistic ideas is in the opinion of many physicists somewhat uneasy [4]. The actual wording takes various forms, such as e.g. “The construction of a fully objective theory of state-vector reduction which is consistent with the spirit of relativity is a profound challenge, since ‘simultaneity’ is a concept (...) foreign to relativity”[5]. Considerations of this type lead to a widely advocated idea that space and time are emergent phenomena, absent at the underlying quantum level.

We conclude therefore that it is well justified to adopt an approach in which one does not start from mixing time with space. Yet, if the internal quantum numbers are to be connected with the properties of the macroscopic “space”, the ordinary 3D space clearly has to be somehow extended into a broader “arena”.

3 The arena

3.1 Max Born: a hint

The issue of a possible relation between some particle properties, such as mass, and the surrounding “emergent” space was of significant concern already to Max Born, over half a century ago. In his 1949 paper [6] he writes: “I think that the assumption of the observability of the 4-dimensional distance of two events inside atomic dimensions (no clocks or measuring rods) is an extrapolation...” He then continues with the discussion of a difference between the position and momentum spaces for elementary particles. First, he notes that the concept of mass appears in the relation $p^2 = m^2$, and that different observed elementary particles correspond to different discrete values of $m^2$, thus rendering $p^2$ observable. Then, he stresses that $x^2$, the corresponding invariant in coordinate space (with $x^2$ of atomic dimensions), seems to be “no observable at all”.

At the same time, he points out that the laws of nature such as

$$\dot{x}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial x_k},$$

$$[x_k, p_l] = i\hbar \delta_{kl},$$

$$L_{kl} = x_k p_l - x_l p_k$$

(1)
are invariant under “reciprocity” transformations:

\[ x_k \rightarrow p_k, \quad p_k \rightarrow -x_k. \]  \hspace{1cm} (2)

Noting that the reciprocity symmetry somehow does not apply to elementary particles, he concludes: “This lack of symmetry seems to me very strange and rather improbable”.

### 3.2 Phase space as the arena

In view of the preceding arguments and with quantum mechanics “living in phase space”, it seems natural to start from mixing the 3D space of positions with the 3D space of momenta, and to look for any additional quantum numbers that might possibly emerge in such a broader scheme. In other words, instead of implicitly identifying the ordinary 3D space of positions with the nonrelativistic arena on which physical processes take place, I propose that it is the nonrelativistic phase space that might and should be viewed as such an arena.

It should be stressed that the procedure of mixing the spaces of positions and momenta leads to such a generalization of the concept of ordinary space, which could be termed “minimal”. In particular no additional dimensions - “hidden” from our sight - are introduced in this way. We just have to recognize that this macroscopic arena is around us, fully visible.

In the following, I will infer the existence of some internal quantum numbers from the properties of this generalized arena. In a more fundamental approach, however, one has to reverse the arrow of implications and - starting from the underlying quantum structure, as manifested in quantum numbers of elementary particles - actually build the “Emergent Phase Space” (EPS).

### 3.3 Phase space and the Standard Model

The simplest phase-space generalization of the 3D concepts of reflection and rotation requires a fully symmetric treatment of the two O(3) invariants, \( x^2 \) and \( p^2 \), which is achieved by adding them together:

\[ x^2 + p^2 \]  \hspace{1cm} (3)

This generalization leads to \( O(6) \), which goes beyond both Born’s reciprocity and the familiar symmetries of 3D space. Treating \( x^2 \) and \( p^2 \) as operators we now require their commutators to be form invariant. As is well known from the case of the 3D harmonic oscillator the original \( O(6) \) symmetry is then reduced to \( U(1) \otimes SU(3) \). The \( U(1) \) factor describes (in particular) Born’s reciprocity transformations and their squares: 3D reflections. The \( SU(3) \) factor takes care of standard rotations (among other transformations).
The appearance of the $U(1) \otimes SU(3)$ group from the first principles and the presence of the same group in the Standard Model (SM) raises the possibility that the SM internal symmetry group is actually related to phase space symmetries. A confirmation of this suggestion seems to require the construction of the SM gauge prescription from and/or upon the underlying quantum structure. The gauge structure would have to appear alongside the emerging (phase) space. Consequently, I think it lies beyond our reach at the moment. In the following, I shall show, however, that the structure of quantum numbers obtained at the quantum level of the phase-space-related approach exactly parallels that observed in the real world.

4 Linearization of $x^2 + p^2$

In order to reach the quantum level, we linearize $x^2 + p^2$ à la Dirac. Using anticommuting $A_k$ and $B_k$ ($k = 1, 2, 3$), with explicit representation

$$
A_k = \sigma_k \otimes \sigma_0 \otimes \sigma_1 \\
B_k = \sigma_0 \otimes \sigma_k \otimes \sigma_2 \\
B_7 = \sigma_0 \otimes \sigma_0 \otimes \sigma_3
$$

($B_7$ is the seventh anticommuting element of the relevant Clifford algebra), one finds

$$(A \cdot p + B \cdot x)(A \cdot p + B \cdot x) = (p^2 + x^2) + \sum_1^3 \sigma_k \otimes \sigma_k \otimes \sigma_3. \quad (5)$$

The first term on the r.h.s., denoted below by $R$, appears here because all six elements $A_k$ and $B_l$ anticommute among themselves. The second term, denoted by $R^\sigma$, is due to the fact that $x_k$ and $p_k$ do not commute. These two terms sum up to a total $R^{tot} = R + R^\sigma$.

The $SU(4)/SO(6)$ generators are constructed as antisymmetric bilinears of $A_k, B_l$. In particular, the generator of standard rotation has the explicit form

$$S_k = \frac{1}{2} (\sigma_k \otimes \sigma_0 + \sigma_0 \otimes \sigma_k) \otimes \sigma_0 \quad (6)$$

and corresponds to simultaneous (same size and sense) rotations in momentum and position subspaces.

5 Eigenvalues of $R^{tot}$

5.1 Gell-Mann-Nishijima relation

We now introduce operator $Y$, to be identified shortly with the (weak) hypercharge:

$$Y \equiv \frac{1}{3} R^a B_7 = \frac{1}{3} \sum_1^3 \sigma_k \otimes \sigma_k \otimes \sigma_0 \equiv \sum_1^3 Y_k. \quad (7)$$
Table 1: Decomposition of eigenvalue of $Y$ into eigenvalues of its components

| colour | 0   | 1   | 2   | 3   |
|--------|-----|-----|-----|-----|
| $Y$    | $-1$| $+\frac{1}{3}$| $+\frac{1}{3}$| $+\frac{1}{3}$|
| $Y_1$  | $-\frac{1}{3}$| $-\frac{1}{3}$| $+\frac{1}{3}$| $+\frac{1}{3}$|
| $Y_2$  | $-\frac{1}{3}$| $+\frac{1}{3}$| $-\frac{1}{3}$| $+\frac{1}{3}$|
| $Y_3$  | $-\frac{1}{3}$| $+\frac{1}{3}$| $+\frac{1}{3}$| $-\frac{1}{3}$|

Since the “partial hypercharges” $Y_k$ commute among themselves, they may be simultaneously diagonalized. One then gets the pattern shown in Table 1 (as the matrices are $8 \times 8$, this pattern is obtained twice).

In [7] a conjecture was put forward that the electric charge $Q$ is just an appropriately normalized operator $R^{\text{tot}} B_7$, evaluated for the lowest level of $R$, i.e.:

$$ Q = \frac{1}{6} (R_{\text{lowest}} + R') B_7 = I_3 + \frac{Y}{2} $$  \hspace{1cm} (8)$$

where, with $R_{\text{lowest}} = (p^2 + x^2)_{\text{lowest}} = 3$, the eigenvalues of $I_3 = B_7/2$ are $\pm 1/2$. The above equation, derived here in a phase-space-related approach, is known as the Gell-Mann-Nishijima relation (with $I_3$ known as weak isospin), and constitutes a law of nature. With the help of Table 1, it yields the charges of all eight leptons and quarks from a single SM generation.

5.2 Harari-Shupe rishons

With the growing number of fundamental fermions, the problem of understanding why they group into generations composed of eight particles was addressed by many physicists. The most widely cited proposal (over 320 citations) is due to Haim Harari and Michael Shupe [8]. The Harari-Shupe model describes the structure of a single SM generation with the help of a composite model. It builds all eight fermions of a single generation from only two spin-1/2 “preons” $V$ and $T$ (or “rishons” as Harari dubbed them), of charges 0 and $+1/3$ respectively. This is shown in Table 2, where total charges and hypercharges of particles are also listed. Note that rishons obey strange statistics, i.e. it is the states with ordered rishons (e.g. $VTT$, $VT T$, and $TT V$ ) which are to correspond to the three colours of $SU(3)$ and which, consequently, are deemed different.

The rishon model, though algebraically very economical, has several drawbacks, however. These include, among others: the issue of preon confinement at extremely small distance scales (when confronted with the uncertainty principle), the apparent
absence of spin-3/2 fundamental particles, and the lack of explanation as to why the ordering of three rishons is important and leads to $SU(3)$.

### 5.3 Preonless resolution of problems

A comparison of Tables 1, 2 (using Eq. (8)) shows that the phase-space approach reproduces the main structure of the Harari-Shupe model exactly. In fact, it does not only that: it also solves the three problems of the rishon model.

Namely, the phase-space approach explains the structure of charge eigenvalues without assuming any subparticle components of quarks and leptons. Therefore, there is no problem of “where are spin-3/2 fundamental fermions”, and there is no problem with preon confinement. Furthermore, the strange statistics of rishons and its connection with $SU(3)$ are naturally explained.

One can readily understand the meaning of the “ordered rishon structure” (such as $VTT$) in phase-space terms. Thus, the position of rishon corresponds to one of three directions in our macroscopic 3D space: $VTT$ corresponds therefore to the partial hypercharge eigenvalue of $-1/3$ in direction $(x, p_x)$ and to the same eigenvalue of $+1/3$ in both remaining directions, $(y, p_y)$ and $(z, p_z)$. Now, any discussion of rotations requires three directions, not just one. Hence the concept of spin simply cannot be applied to a single rishon. This is in line with Heisenberg’s opinion [9] concerning the idea of dividing matter again and again: “…the antinomy of the smallest dimensions is solved in particle physics in a very subtle manner, of which neither Kant nor the ancient philosophers could have thought: The word ‘dividing’ loses its meaning”.

### 6 Transformations in phase space

#### 6.1 Genuine $SO(6)$ transformations

In order to see the relation between quarks and leptons consider [10, 11] a transformation generated by $F^\sigma_{\pm 2}$, one of six “genuine” $SU(4)/SO(6)$ generators $F^\sigma_{\pm n}$.
\( n = 1, 2, 3 \):

\[
F_{-n}^\sigma = \frac{1}{2} (\sigma_0 \otimes \sigma_n - \sigma_n \otimes \sigma_0) \sigma_3
\]

(9)

\[
F_{+n}^\sigma = \frac{1}{2} \epsilon_{nkl} \sigma_k \otimes \sigma_l \otimes \sigma_3.
\]

(10)

Under \( F_{-2}^\sigma \)-generated transformations one obtains:

\[
A_k' = A_1 \cos \phi - A_3 \sin \phi \quad B_1' = B_1 \cos \phi + B_3 \sin \phi
\]

\[
A_2' = A_2 \quad B_2' = B_2
\]

\[
A_3' = A_3 \cos \phi + A_1 \sin \phi \quad B_3' = B_3 \cos \phi - B_1 \sin \phi
\]

i.e. \( A \) and \( B \) rotate in opposite senses. For \( \phi = \pm \pi/2 \) one then gets:

\[
Y = Y_1 + Y_2 + Y_3 \rightarrow Y' = -Y_3 + Y_2 - Y_1
\]

(12)

and consultation of Table 1 shows that lepton and quark \# 2 are exchanged, while the remaining two quarks are left untouched. The same result is obtained when the analogous transformation generated by \( F_{+2}^\sigma \) is considered.

### 6.2 Rotations in phase space

When the corresponding transformations in phase space are considered, one gets (for \( F_{-2}^\sigma \)-generated rotations):

\[
[x'_k, x'_l] = [p'_k, p'_l] = 0
\]

(13)

\[
[x'_k, p'_l] = i \Delta_{kl}
\]

(14)

with

\[
\Delta = \begin{bmatrix}
\cos 2\phi & 0 & \sin 2\phi \\
0 & 1 & 0 \\
-\sin 2\phi & 0 & \cos 2\phi
\end{bmatrix}.
\]

(15)

For the case of lepton-quark \# 2 interchange (\( \phi = \pm \pi/2 \)) one then obtains (for both \( F_{-2}^\sigma \) - and \( F_{+2}^\sigma \)-generated rotations):

\[
\text{(quark) } \Delta = \begin{bmatrix}
-1 & 0 & 0 \\
0 & +1 & 0 \\
0 & 0 & -1
\end{bmatrix} \quad \leftrightarrow \quad \Delta = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad \text{(lepton)}
\]

(16)

It may be said therefore that quark is a lepton rotated in phase space.

Taking into account the remaining two types of genuine \( SO(6) \) transformations we get the following four sets of generalized commutation relations:

| lepton | quark 1 | quark 2 | quark 3 |
|--------|--------|--------|--------|
| \([x_1, p_1] = i\) | \([x_1, p_1] = i\) | \([p_1, x_1] = i\) | \([p_1, x_1] = i\) |
| \([x_2, p_2] = i\) | \([p_2, x_2] = i\) | \([x_2, p_2] = i\) | \([p_2, x_2] = i\) |
| \([x_3, p_3] = i\) | \([p_3, x_3] = i\) | \([p_3, x_3] = i\) | \([x_3, p_3] = i\). |
Table 3: Classification of Clifford algebra elements according to their $U(1) \otimes SU(3)$ properties

| $U(1)$ | $SU(3)$ | elem $Y_l$ | $Y_r$ | $U(1)$ | $SU(3)$ | elem $Y_l$ | $Y_r$ |
|--------|---------|------------|-------|--------|---------|------------|-------|
| $-2$   | $3^*$   | $H_{m0}^+$ | $-1$  | $+\frac{1}{3}$ | $+1$   | $3^*$      | $U_k^\dagger$ | $+\frac{1}{3}$ | $+\frac{1}{3}$ |
| $+2$   | $3$     | $H_{0m}^+$ | $+\frac{1}{3}$ | $-1$  | $-\frac{1}{3}$ | $-1$ | $3$ | $V_k$ | $+\frac{1}{3}$ | $-1$ |
| $0$    | $8$     | $F_a^+$   | $+\frac{1}{3}$ | $+\frac{1}{3}$ | $-1$  | $3$ | $W_k$ | $-1$ | $-\frac{1}{3}$ |
| $0$    | $1$     | $Y_{+1}^-$ | $-1$  | $-1$  | $1$   | $6$ | $G_{\{kl\}}$ | $+\frac{1}{3}$ | $+\frac{1}{3}$ |
| $0$    | $1$     | $Y_{+\frac{1}{3}}^+$ | $+\frac{1}{3}$ | $+\frac{1}{3}$ | $-3$ | $1$ | $G_0$ | $-1$ | $-1$ |

6.3 Reflections in phase space

When a reflection in phase space (e.g., $p'_k = p_k$, $x'_k = -x_k$, $i \rightarrow i$) is performed, the number of sets is doubled from four to eight, with $i$ in Eqs (17) changed to $-i$ (this corresponds to doublets of weak isospin and is different from charge conjugation). Thus, we obtain 8 disjoint sectors, corresponding to 8 particles of a single generation of the Standard Model.

The fact that the three additional sets of commutation relations in Eq. (17) are not rotationally invariant is in my opinion an asset of the approach. Namely, the only condition that quarks must really fulfill is that it is the systems composed thereof (i.e., mesons, baryons) that must be covariant under rotations. Thus, my conjecture is that quarks must conspire (see example in the next section).

7 Clifford algebra and mass

The $U(1) \otimes SU(3)$ structure of (a half of) the Clifford algebra of nonrelativistic phase space is shown in Table 3. Here the even elements (linear combinations of $SU(4)/SO(6)$ generators and the unit element) with $I_3 = +1/2$ (hence superscript ‘+’) are given on the left, while the odd elements (linear combinations of products of an odd number of $A_m$ and $B_n$) with left and right eigenvalues of $I_{3l} = +1/2$ and $I_{3r} = -1/2$ are shown on the right. In the columns marked $Y_l$ and $Y_r$, the left and right eigenvalues of $Y$ are given. A more detailed explanation of entries in this table may be found in [12].

The algebraic counterpart of lepton mass should be odd (just like the odd $A_m$ is associated with $p_m$) and is identified with the only odd scalar element in Table 3, i.e., with $G_0$. The $F_{\pm2^2}$-generated transformation from the lepton to the quark sector changes $G_0$ into $G_{\{22\}}$, which is a member of the $SU(3)$ sextet, and is not rotationally invariant. However, the sum of the three quark mass terms, i.e., $G_{\{kk\}}$, is rotationally invariant, just as the idea of quark conspiracy suggests.
Figure 1: Four of the eight “DICE1979” corresponding to the Harari-Shupe model. All dice are identical when their rotations are admitted. Each corner corresponds to one particle of a single SM generation. Each face corresponds to a rishon. The three dice to the right (coloured quarks) show the leftmost die (lepton) rotated around axes 1, 2, 3 by $\pi$, so that different corners (as marked) present themselves to the reader.

8 Summary

The phase-space approach provides a possible theoretical explanation of the structure of a single generation of the Standard Model. The symmetry obtained involves $U(1) \otimes SU(3)$ and resembles the full $U(1) \otimes SU(3) \otimes SU(2)_L$ symmetry group of the Standard Model quite closely (in fact the $SU(2)$ partners of $I_3 = B_7/2$, i.e. $I_{1,2} = \sigma_0 \otimes \sigma_0 \otimes \sigma_{1,2}$, automatically do not commute with the 3D reflections, thus suggesting parity violation). We have derived the Gell-Mann-Nishijima relation and reproduced the structure of the Harari-Shupe model (visualized in Fig. 1), while evading its main problems related to the introduction of “confined preons”. The existence of eight particles in a single SM generation has been related to the $2^3 = 8$ possible sets of $[x_k, p_k] = \pm i$ commutation relations, with the $\pm$ sign adopted independently for each direction in our 3D space.

The proposed approach obviously raises many questions. The phase-space-related modification of the way in which the imaginary unit enters into our theories brings in the question of whether it is possible to extend the concept of the arena further, so that parity violation in weak interactions (together with the emergence of three quark-lepton generations and the related appearance of the $CP$-violating $i$) could be described in a more realistic way. Then, there are other important questions such as the issue of the emergence of points, the construction of composite systems, etc. I hope to be able to address some of them in the future.

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