ON THE MISSING BRANCHES OF THE BRUHAT-TITS TREE.

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Abstract. Let $k$ be a local field and let $\mathfrak{A}$ be the two-by-two matrix algebra over $k$. In our previous work we developed a theory that allows the computation of the set of maximal orders in $\mathfrak{A}$ containing a given suborder. This set is given as a subtree of the Bruhat-Tits tree that is called the branch of the order. Branches have been used to study the global selectivity problem and also to compute local embedding numbers. They can usually be described in terms of two invariants. To compute these invariants explicitly, the strategy in our past work has been visualizing branches through the explicit representation of the Bruhat-Tits tree in terms of balls in $k$. This is easier for orders spanning a split commutative subalgebra, i.e., an algebra isomorphic to $k \times k$. In the present work, we develop a theory of branches over field extension that can be used to extend our previous computations to orders spanning a field. We use the same idea to compute branches for orders generated by arbitrary pairs of non-nilpotent pure quaternions. In fact, the hypotheses on the generators are not essential.

1. Introduction

Let $\Omega$ be an order in the local matrix algebra $\mathfrak{A} = \mathbb{M}_2(k)$, where $k$ is a local field. The set $S(\Omega)$ of maximal orders in $\mathfrak{A}$ containing $\Omega$ plays a significant role in the study of several interesting arithmetical phenomena, like the selectivity problem, determining whether a global order embeds into all or just into some of the maximal orders in a quaternion algebra over a global field $F$ [2], [4], or describing the normalizers in $\text{PSL}_2(F)$ of Eichler orders and congruence subgroups [5]. The study of this set plays also a significant role in determining quotient graphs for some arithmetically important subgroups of the general linear group [3].

Usually, we describe this set of orders as the vertex set of a subgraph $\mathcal{s}(\Omega)$ of the Bruhat-Tits tree $t(k)$, a tree whose vertices are the maximal orders in $\mathfrak{A}$, while two of them $\mathcal{D}$ and $\mathcal{D}'$ are neighbors if in some basis they have the form

$$\mathcal{D} = \left( \begin{array}{cc} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{array} \right) \quad \text{and} \quad \mathcal{D}' = \left( \begin{array}{cc} \mathcal{O} & \pi^{-1} \mathcal{O} \\ \pi \mathcal{O} & \mathcal{O} \end{array} \right),$$

where $\mathcal{O}$ is the ring of integers, and $\pi$ is a uniformizing parameter of $k$. The graph $\mathcal{s}(\Omega)$, which we call the branch of $\Omega$, is usually a tubular neighborhood of a path $\mathcal{p}$, i.e., a thick path with stem $\mathcal{p}$ as defined in §2, except for a couple of very simple orders (c.f. [2] Prop 5.3 and [2] Prop 5.4):

1. $\Omega = \mathcal{O}$, identified with the ring of scalar matrices with integral entries.
2. $\Omega = \mathcal{O}[u]$ where $u \in \mathfrak{A}\setminus\{0\}$ is nilpotent. This is called an idempotent order.

In the latter case, the branch of $\Omega$ is a graph called an infinite leaf [1], and can be seen as the limit of a sequence of thick paths, with a common vertex of valency one, whose stems get infinitely far away. This type of set is called a horoball in some previous literature on diophantine approximation [9].
The graph $t(k)$ can alternatively be defined as a graph whose vertices are the balls in $k$, while two balls are neighbors if one is a maximal proper sub-ball of the other. This observation becomes a powerful tool to compute branches, while embeddings can be easily interpreted graphically in this context, provided that the order $\Omega$ is the intersection of a family of maximal orders. This property allowed us to compute local embedding numbers for orders spanning algebras whose maximal semisimple quotient is the split commutative algebra $k \times k$ [6]. Unfortunately, this is not the case for many interesting orders, like those contained in maximal subfields.

The purpose of the present work is to develop a technique that can be applied to the latter orders. With this in mind, we construct an embedding of the graph $t(k)$, or more precisely an appropriate subdivision of it, into the graph $t(L)$ for any finite field extension $L/k$. In order to show the scope of this new tool, we extend two of our previous results. The first one extends the explicit formulas obtained in [4] to compute the invariant describing the branch $s(\Omega)$, when $\Omega = \mathcal{O}[i, j]$ is the order generated by two orthogonal pure quaternions, i.e., two matrices satisfying the relations

\begin{equation}
    i^2 = a, \quad j^2 = b, \quad ij + ji = 0,
\end{equation}

which are the standard generators of a quaternion algebra, and therefore play a central role in the theory. However, orders generated by more general pairs of pure quaternions do appear naturally in practical problems, making desirable to extend this computation in a more general setting (c.f. [7]). More precisely, in this work we no longer require the orthogonality condition.

Our second result extends the embedding number computations in [6] to the case of orders contained in fields, the only orders of non-maximal rank that failed to be considered in our previous work.

**Conventions on graphs and walks.** In all that follows, a graph $\mathcal{g}$ is a set of vertices $V_{\mathcal{g}}$ together with a symmetric relation called the neighborhood relation in $\mathcal{g}$. A subgraph of $\mathcal{g}$ is any graph $\mathcal{h}$, satisfying $V_{\mathcal{h}} \subseteq V_{\mathcal{g}}$ and whose neighborhood relation implies the induced relation. If the neighborhood relation in $\mathcal{h}$ is the induced relation, we call it a full subgraph. We are not concerned with non-full subgraphs in this work. The intersection of a family of full subgraphs is also a full subgraph with the natural conventions. The valency of a vertex, in a given graph $\mathcal{g}$, is the number of its neighbors. Vertices of valency 1 are called optimal, since, when $\mathcal{g} = s(\Omega)$ as before, they correspond to maximal orders in which $\Omega$ is optimal [6]. A finite walk in $\mathcal{g}$ is a sequence of vertices $v_0 v_1 \ldots v_r$ satisfying the following conditions:

1. Each pair of consecutive vertices are neighbors.
2. There is no backtracking, i.e., $v_i \neq v_{i+2}$ for every $i = 0, \ldots, r-2$.

We usually emphasize the initial and last vertex in the walk by saying a walk from $v_0$ to $v_r$. A graph $\mathcal{g}$ is connected if there is a walk from every vertex $v_0 \in V_{\mathcal{g}}$ to every vertex $v_r \in V_{\mathcal{g}}$. A cycle is a walk $v_0 v_1 \ldots v_r$ satisfying $v_r = v_0$. A tree is a connected graph with no cycles. Equivalently, a graph is a tree if there is a unique walk from $v_0$ to $v_r$ for any pair $(v_0, v_r) \in V_{\mathcal{g}} \times V_{\mathcal{g}}$. A walk in a tree has no repeated vertices. All graph considered here are trees. We call $r$ the length of the walk $v_0 \ldots v_r$, and we admit the walk $v_0$ of length 0. The tree distance in $\mathcal{g}$ is the metric $\delta$ in $V_{\mathcal{g}}$ defined by $\delta(v, w) = r$ if the walk from $v$ to $w$ has length $r$. We also consider two types of infinite walks:
(1) An infinite walk is a sequence of the form \( v_0v_1 \ldots \) with one vertex for each natural number.

(2) A double infinite walk is a sequence of the form \( \ldots v_{-1}v_0v_1 \ldots \) with one vertex for each integer.

We identify the double infinite walks \( \ldots v_{-1}v_0v_1 \ldots \) and \( \ldots v'_{-1}v'_0v'_1 \ldots \) provided \( v'_t = v_{t+m} \) for a fixed integer \( m \) and every integer \( t \). We also define an equivalence relation between infinite walks, where \( v_0v_1 \ldots \) and \( v'_0v'_1 \ldots \) are related whenever \( v'_t = v_{t+m} \) for a fixed integer \( m \) and every big enough integer \( t \). Equivalence classes in the latter sense are called ends. We usually represent an end graphically by a dot beyond the walk. Furthermore, for any subgraph \( h \) of \( g \), there is a natural embedding from the set of ends of \( h \) to the set of ends of \( g \). We identify the ends of \( h \) with the corresponding ends of \( g \), and usually write expressions like the end \( a \) is in \( h \), or belongs to \( h \), in this sense, and even write \( a \in h \).

2. Main results

In all that follows, \( k, O, A \) and \( \Omega \) are as before. Let \( \pi \) be a uniformizing parameter in \( k \), and let \( \nu : k^* \to \mathbb{Z} \) be the usual valuation, normalized in a way that \( \nu(\pi) = 1 \). We let \( N \) be the number of quadratic classes of ramified units in \( O^* \), i.e., units whose square roots generate ramified extensions, so that the set of square classes in \( k^* \) is

\[
\frac{k^*}{k^{*2}} = \{1, \Delta, \bar{u}_1, \ldots, \bar{u}_N, \bar{\pi}_1, \ldots, \bar{\pi}_{N+2}\},
\]

where \( \Delta \) is a unit of minimal quadratic defect \([8]\), \( u_1, \ldots, u_N \) are ramified units, and \( \pi_1, \ldots, \pi_{N+2} \) are uniformizing parameters. Our results are usually stated in terms of the quadratic defect \( \delta \). For any element \( a \in k^* \), the quadratic defect is the smallest fractional ideal in \( k \) spanned by an element of the form \( a - b^2 \) (c.f. \([8]\)). For the square class representatives shown above, this is computed as follows:

\[
\delta(1) = \{0\}, \quad \delta(\Delta) = (4), \quad \delta(\bar{\pi}_a) = (\pi), \quad \delta(u_n) = (\pi^{2s+1})
\]

for some integer \( s = s(u_n) \) satisfying \( 0 \leq s < \nu(2) \).

Let \( t = t(k) \) be the Bruhat-Tits tree, with the tree-distance \( \delta \) defined in \( \S1 \). For every vertex \( v \), we define the ball of radius \( p \) around \( v \), as the full subgraph \( b = b_v[p] \) whose vertex set satisfies \( V_b = \{w \in V|\delta(v, w) \leq p\} \). For every walk \( w = \ldots v_{i-1}v_iv_{i+1} \ldots \) in \( t \), finite or not, we call the interval \( I(w) = \{\ldots, i-1, i, i+1, \ldots\} \subseteq \mathbb{Z} \) its index interval. A thick path is a full subgraph of the form \( s = \bigcup_{n \in I(w)} b_{v_n}[p] \), where \( w = \ldots v_{i-1}v_iv_{i+1} \ldots \) is a walk. The integer \( p \leq 0 \) is called the depth of \( s \), while \( m = \{v_i|i \in I(w)\} \) is called the stem. Note that \( s = m \) if \( p = 0 \). As mentioned in \( \S1 \), the branch \( s(\Omega) \) is a thick path for most orders \( \Omega \). Many combinatorial properties of the branch can be described in terms of two invariants. The stem length \( l \), i.e., the length of \( w \), and the depth \( p \). The computation of local embedding numbers \([6]\) or representation fields for global orders \([2] \), \([3]\), reduces to determining these invariants. This was done explicitly in \([1]\), for orders generated by a pair of orthogonal pure quaternions as in \([1]\). In fact, we already gave in \([1] \) Prop. 2.3 and \([4] \) Prop. 2.4 a method to do this in full generality, provided that the relative position between the branches is known. The latter piece of data was collected, for orders generated by orthogonal pure quaternions, by thickening the stem of the thick path corresponding to either generator, until we found a minimal setting where both branches do intersect, or equivalently, there exists a maximal order containing each of the corresponding orders. This kind of computations could be extended with enough work, but the
method shown here is far simpler. It consist in giving a precise location for the stem of an order in terms of the simmetric product $ij + ji \in K$. One this is known, the invariants can be computed by the results in [1].

**Theorem 2.1.** Let $i, j \in \mathbb{M}_2(k)$ be pure quaternion satisfying $i^2 = \alpha$, $j^2 = \beta$ and $ij + ji = 2\lambda$. Assume $\alpha$ and $\beta$ belong to a set of representatives of the form

$$Q = \{1, \Delta, u_1, \ldots, u_N, \pi_1, \ldots, \pi_{N+2}\},$$

of all square classes, where $\Delta$ is a unit of minimal quadratic defect, $\{u_1, \ldots, u_N\}$ is a set of representatives of all ramified units, while $\{\pi_1, \ldots, \pi_{N+2}\}$ is a set of representatives of all uniformizing parameters. Let $d_f$ be the function defined case by case as follows:

1. If $\alpha, \beta \in \{1, \Delta\}$, then $d_f = -\frac{1}{2}\nu\left(\frac{\lambda^2 - \alpha\beta}{4}\right)$.
2. If $\alpha \in \{1, \Delta\}$, while $\beta \notin \{1, \Delta\}$ and $\partial(\beta) = (\pi^{2t+1})$, then $d_f = t - \frac{1}{2}\nu(\lambda^2 - \alpha\beta)$.
3. If $\alpha \notin \{1, \Delta\}$ and $\partial(\alpha) = (\pi^{2s+1})$, while $\beta \in \{1, \Delta\}$, then $d_f = s + t - \frac{1}{2}\nu(\lambda^2 - \alpha\beta)$.
4. If $\{\alpha, \beta\} \cap \{1, \Delta\} = \emptyset$, while $\partial(\alpha) = (\pi^{2s+1})$ and $\partial(\beta) = (\pi^{2t+1})$, then $d_f = \min\{-2d_f, l(i), l(j)\}$.

Then if $d_f > 0$, it equals the distance between the stems, otherwise the length of the intersection is

$$\min\{-2d_f, l(i), l(j)\},$$

where $l(q)$ is the stem length of $q$, which is 0, 1 or $\infty$ [2].

In what follows, the possibly negative function $d_f$ is referred to as the fake distance. It is a distance only when it is non-negative.

Recall that an embedding $\phi : \Omega \to \mathfrak{D}$ is called optimal if $\hat{\phi}^{-1}(\mathfrak{D}) = \Omega$, where $\hat{\phi} : k \otimes _O \Omega \to \mathbb{M}_2(k)$ is the natural extension. A suborder $\Omega \subset \mathfrak{D}$ is optimal when the inclusion is an optimal embedding. For any quadratic extension $L/k$, any order $\Omega = \mathcal{O}_L^{(t)}$ of maximal rank in $L$, where $\mathcal{O}_L$ is the ring of integers in $L$, and any Eichler order $\mathfrak{E} \subset \mathbb{M}_2(k)$ of level $\rho$, we let $X$ be the set of optimal embeddings $\phi : \Omega \to \mathfrak{E}$ and let $Y$ be the set of optimal suborders of $\mathfrak{E}$ that are isomorphic to $\Omega$. We also let $\Gamma_1 = k^* \mathfrak{E}^*$, where $A^*$ denotes the group of units of a ring $A$, and let $\Gamma_2$ be the normalizer of $\mathfrak{E}$ in $\mathrm{GL}_2(k)$. The embedding numbers $\epsilon_i = \epsilon_i(\mathfrak{E} \mid \Omega)$ are the following quantities:

$$\epsilon_1 = \#(X/\Gamma_1), \quad \epsilon_2 = \#(X/\Gamma_2), \quad \epsilon_3 = \#(Y/\Gamma_1), \quad \epsilon_4 = \#(Y/\Gamma_2).$$

We call them, respectively, first, second, third, and fourth embedding number. We use the embedding vector $\overrightarrow{e} = (e_1, e_2, e_3, e_4)$ to simplify the statements below. When $\alpha$ is a real number, we denote its integral part by $\lfloor \alpha \rfloor = \max\{n \in \mathbb{Z} | n \leq \alpha\}$. For any triple $(r, u, t) \in (\mathbb{Z}_{\geq 0})^2$ satisfying $v \leq u \leq [r/2]$, for $v = \max\{0, r - t\}$, we consider the cardinality

$$(2) \quad \chi(r, u, t) = \# \left\{ \bar{a} \in \left( \frac{\mathcal{O}}{\nu(t-r+2u)\mathcal{O}} \right)^* \Bigg| \bar{a}^2 = 1 \quad \text{and} \quad \nu(a-1) = \nu(\pi^{t-r+u}) \quad \text{for any lifting} \ a \in \mathcal{O} \text{ of} \ \bar{a} \right\},$$

which we set as 1 for $u = 0$. Note that $\nu(a-1)$ depends only on $\bar{a}$ if $\bar{a} \neq \bar{1}$. Let $\chi_3 = \chi_3(r, t) = \sum_{u=0}^{h} \chi(r, u, t)$, and set it as 0 if the sum is empty.
Table 1. The invariants \( m \) and \( \chi_2 \) for the order \( \mathcal{O}_L^{[t]} \).

| \( L/k \) | \( r \) | \( m \) | \( \chi_2 \) |
|---|---|---|---|
| Unramified | \( r = 2h + 1 < 2t \) | 0 | 0 |
| Unramified | \( r = 2h < 2t \) | \( (q - 1)q^{h-1} \) | \( \chi(r, h, t) \) |
| Unramified | \( r = 2t \) | \( q^t \) | \( \chi(r, t, t) \) |
| Ramified | \( r = 2h + 1 < 2t \) | 0 | 0 |
| Ramified | \( r = 2h < 2t \) | \( (q - 1)q^{h-1} \) | \( \chi(r, h, t) \) |
| Ramified | \( r = 2t \) | \( (q - 1)q^{t-1} \) | \( \chi(r, t, t) \) |
| Ramified | \( r = 2t + 1 \) | \( q^t \) | \( \chi(r, t, t) \) |

Theorem 2.2. Let \( L/k \) be a quadratic extension. Let \( \mathcal{E} \) be an Eichler order of level \( r > 0 \) and let \( \Omega \subseteq \mathcal{E} \) be an order isomorphic to \( \mathcal{O}_L^{[t]} \). Then there exists optimal embeddings of \( \Omega \) into \( \mathcal{E} \) if and only if one of the following conditions hold:

- \( r \leq 2t \) and \( L/k \) is unramified, or
- \( r \leq 2t + 1 \) and \( L/k \) is ramified.

Furthermore, when optimal embeddings do exist, the values for the embedding numbers are given by the formula

\[
\frac{\delta}{e} = \frac{p^{[r/2]}}{2} (4, 2, 2, 1) - \frac{m}{2} (2, 1, 1, 0) + \frac{\chi_2}{4} (0, 2, 0, 1) + \frac{\chi_3}{4} (0, 0, 2, 1),
\]

where \( \chi_3 \) is as above, while \( m \) and \( \chi_2 \) are as in Table 1. If \( r = 0 \), then \( \frac{\delta}{e} = (1, 1, 1, 1) \).

3. Trees and Ghost branches

The ends of the Bruhat-Tits tree \( t(k) \), as defined in §1, are in correspondence with the elements of the projective line \( \mathbb{P}^1(k) \) [4, §4]. This can be seen by associating, to each ball \( B_a^{[r]} = B_a[[x^{[r]}]] \), the endomorphism ring of the lattice \( \left\langle \left( \begin{array}{c} a \\ 1 \end{array} \right), \left( \begin{array}{c} \pi^r \\ 0 \end{array} \right) \right\rangle \), which is a maximal order. This defines a correspondence between balls and maximal orders that can be used to define a tree whose vertices are balls, while two balls are neighbors if one is a maximal sub-ball of the other.

The largest subgraph whose vertices contain an order \( \Omega \), as before, is denoted \( s(\Omega) \), or \( s_k(\Omega) \) is we need to emphasize the field, as it is the case in all that follows. If \( \Omega = \mathcal{O}[a_1, \ldots, a_n] \), we write \( s_k(a_1, \ldots, a_n) \). When \( s_k(\Omega) \) is a thick path, which is usually the case, as described in §1, we denote its stem by \( m_k(\Omega) \). The notation \( m_k(a_1, \ldots, a_n) \) is defined analogously. Note that \( s_k(a_1, \ldots, a_n) = \bigcup_{i=1}^n s_k(a_i) \), but the corresponding property for stems is usually false.

With the preceding definitions, the vertices of Bruhat-Tits tree \( t(k) \), for a finite field extension \( L \) of \( k \), can be identified with a subset of the vertices in \( t(L) \), via \( \mathcal{D} \mapsto \mathcal{O}_L \otimes_{\mathcal{O}_k} \mathcal{D} \). However, the map \( t(k) \mapsto t(L) \) is not a morphism of graphs subgraph, unless \( L/k \) is an unramified extension. To fix this, we normalize, in all that follows, for any finite extension \( L/k \), both the valuation in \( L \) and the tree distance in \( t(L) \), in a way that both are extensions of the corresponding functions defined on \( k \). In particular, for any uniformizing parameter \( \pi_L \in L \) we set \( \nu(\pi_L) = \frac{1}{e} \), where \( e = e(L/k) \) is the ramification index, and let \( |a| = c^{\nu(a)} \), for a suitable positive constant \( c < 1 \), denote the absolute value. Similarly, \( \delta(\nu, \nu') = \frac{1}{e} \) for neighboring
vertices in \(\mathfrak{t}(L)\). For any pair of ends \(a, b \in \mathbb{P}^1(L)\) we denote by \(p(a, b)\) the smallest tree containing these two ends. This is a maximal path, i.e., a graph whose vertices are precisely the vertices in a double infinite walk. For any subgraph \(b\), and any vertex \(v\), we denote by \(p_a(v; b)\) the radius of the largest ball in \(\mathfrak{t}(k)\), with center \(v\), contained in \(b\). The depth \(p_k(b)\) of the subgraph \(b\) is the maximal depth of its vertices. Ball radii and depth are normalized according to our general pattern, e.g., \(p_a(v; b)\) is a multiple of \(\frac{1}{2}\), while \(B_0^{[1/e]}\) denotes the ball of radius \(|\pi_L|\) centered at 0. With this conventions, the vertices in \(\mathfrak{s}_k(a_1, \ldots, a_n)\) can be identified with the corresponding vertices in \(\mathfrak{s}_L(a_1, \ldots, a_n)\). It is not always the case, however, that the vertices in \(\mathfrak{m}_k(a_1, \ldots, a_n)\) are vertices of \(\mathfrak{m}_L(a_1, \ldots, a_n)\) (c.f. §5).

When \(L/k\) is a Galois extension, the galois group \(\text{Gal}(L/k)\) acts on both \(\mathbb{P}^1(L)\) and \(\mathfrak{t}(L)\). It is not hard to see, using the explicit correspondence between balls and maximal orders mentioned at the begining of this section, that these two actions are compatible, in the sense that an element \(\sigma \in \text{Gal}(L/k)\) maps the maximal path \(p(a, b)\) defined above onto the maximal path \(p\left(\sigma(a), \sigma(b)\right)\).

On the other hand, the action of \(\text{Gal}(L/k)\) on \(V_t(L)\) leaves invariant many vertices that fail to belong to \(V_t(k)\). In fact, a ball \(B = B_a[|u|]\) is invariant if and only if \(|\sigma(a) - a| \leq |u|\) for every element \(\sigma \in \text{Gal}(L/k)\). This condition is satisfied by every ball of the form \(B_a[|u|]\) with \(a \in k\) even if \(|u| \notin \{k^n\}^*\). However, there are many invariant balls without an invariant center. An example is the ball \(B_2[w]\) in Figure 3C (c.f. §5).

There is also a natural action of the group of Moebius transformations on the set of balls that correspond to the \(\text{PSL}_2(k)\)-action on maximal orders by conjugation. It can be define by associating, to each ball \(B\), a partition \(\mathbb{P}^1(k) = B^c \cup B_1 \cup \cdots \cup B_q\) of the projective line into balls and complements of balls. The balls \(B_1, \ldots, B_q\) are the neighboring sub-balls of \(B\). Moebius transformations act naturally on those partitions [6]. This action is compatible with the action of the Galois group.

In what follows, we refer to vertices in \(\mathfrak{t}(L)\) that are not in \(\mathfrak{t}(k)\) as ghost vertices. Similarly, any maximal path \(p(a, b) \subset \mathfrak{t}(L)\) with \(a, b \in k\) is identified with the corresponding path in \(\mathfrak{t}(k)\). Maximal paths that are not of this form are called ghost paths. This is particularly important in the sequel, as the branch of the order, in \(\mathbb{M}_2(L)\), generated by an integral domain \(\Omega \subseteq \mathbb{M}_2(k)\), have a doubly infinite path as a stem for an appropriate quadratic extension \(L\). We refer to this path as the ghost stem of \(\Omega\), and usually fails to contain the stem \(\mathfrak{m}_k(\Omega)\) which has either one or two vertices [2]. A similar convention applies to finite paths.

## 4. THE PATH OF AN IDEMPOTENT

For every pair \((a, b)\) of different elements in \(k\) we define \(\tau_{a,b} \in \mathbb{M}_2(k)\) as the only idempotent satisfying the following conditions:

\[
\ker(\tau_{a,b}) = \left\langle \begin{pmatrix} a \\ 1 \end{pmatrix} \right\rangle, \quad \text{Im}(\tau_{a,b}) = \left\langle \begin{pmatrix} b \\ 1 \end{pmatrix} \right\rangle.
\]

If \(a\) or \(b\) is \(\infty\), we replace the corresponding generator by \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\). Thus \(\tau_{a,b}\) is defined for every pair \((a, b)\) of different elements in \(\mathbb{P}^1(k)\). It is easy to see that \(\tau_{b,a} = 1 - \tau_{a,b}\).

Recall that the group \(\text{PGL}_2(k)\), which is isomorphic to the group \(\mathcal{M}(k)\) of M"obius transformations, acts transitively on both, nontrivial idempotents by conjugation,
and ordered pairs of distinct elements in $\mathbb{P}^1(k)$ as Möbius transformations. It is immediate from the definitions that both actions are compatible, namely, if $\mu$ is the Möbius transformation corresponding to the matrix $A$, then $A\tau_{a,b}A^{-1} = \tau_{\mu(a),\mu(b)}$. Since any non-trivial idempotent in $\mathbb{M}_2(k)$ has a one-dimensional image and a one-dimensional kernel, every idempotent equals $\tau_{a,b}$ for some ordered pair $(a, b) \in \mathbb{P}^1(k)^2$ satisfying $a \neq b$. Furthermore, by applying a Möbius transformation, we can assume $a = 0$ and $b = \infty$, so $\tau_{a,b} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is contained in exactly the maximal orders of the form $\mathcal{D}_n = \left( \begin{array}{cc} \mathcal{O} & \mathcal{O} \\ \pi^n\mathcal{O} & \pi^{-n}\mathcal{O} \end{array} \right)$, which are the ones corresponding to balls in the path joining 0 and $\infty$. We conclude that the same holds for every pair $(a, b) \in \mathbb{P}^1(k)^2$ where $a \neq b$. Therefore, we can identify idempotents with directed maximal paths, or more precisely, doubly infinite walks, on the tree. The walk from the end $a$ to the end $b$ contains exactly the maximal orders containing $\tau_{a,b}$. We use this identification in all that follows without further ado. We can actually give an explicit formula for these idempotents, namely

$$\tau_{a,b} = \frac{1}{b-a} \begin{pmatrix} b & -ab \\ 1 & -a \end{pmatrix}.$$ 

Many result in this section and §5 can be proved by extensive computations using the above formula. We have chosen, however, indirect or geometrical proofs whenever possible for the sake of brevity.

For any non-trivial idempotent $\tau$, the element $i = 1 - 2\tau$ is a non-trivial solution of $x^2 = 1$, and conversely, every non-trivial solution of the equation is of this form. Replacing $\tau$ by $1 - \tau$ has the effect of replacing $i$ by $-i$. Recall that the matrix algebra $\mathfrak{A} = \mathbb{M}_2(k)$, together with the map sending each matrix $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ to its adjugate matrix $\bar{A} = \begin{pmatrix} y & z \\ -w & x \end{pmatrix}$, is isomorphic, as an algebra with involution, to the split quaternion algebra $(\mathbb{H}_{\mathbb{k}})$. We identify these two algebras in the remainder of this section. A matrix satisfying $\bar{A} = -A$ is called a pure quaternion. By a simple computation, we have $ij + ji = ij + ji \in k$ for every pair of pure quaternions $i$ and $j$.

**Lemma 4.1.** Let $\lambda \in k$ be any element, and let $\mathcal{A}(\lambda)$ be the algebra defined, in terms of generators and relations, by

$$\mathcal{A}(\lambda) = k\left[i, j \mid i^2 = j^2 = 1, ij + ji = 2\lambda\right].$$

Then there is, up to conjugation, a unique representation $\phi : \mathcal{A}(\lambda) \to \mathbb{M}_2(k)$, for which $\phi(i)$ and $\phi(j)$ are linearly independent. Furthermore, $\phi$ is an isomorphism unless $\lambda = \pm 1$.

**Proof.** It is immediate from the definition that $\dim_k \left(\mathcal{A}(\lambda)\right) \leq 4$. If $\lambda \neq \pm 1$, we observe that $i$ and $j' = \lambda i - j$ satisfy the standard relations among the generators of a quaternion algebra, since

$$ij' = -j'i, \quad (j')^2 = (\lambda i - j)^2 = 1 - \lambda^2 \neq 0.$$ 

It follows that $\mathcal{A}(\lambda)$ is a quotient of a quaternion algebra and therefore it is a quaternion algebra since quaternion algebras are simple. In particular $\dim_k \left(\mathcal{A}(\lambda)\right) = 4,$
so that $i$ and $1$ are linearly independent and $A(\lambda)$ cannot be a division algebra since the commutative subalgebra $k[i]$ has too many squares roots of $1$. We conclude that $A(\lambda) \cong \mathbb{M}_2(k)$, so the result follows from Skolem-Noether’s Theorem.

Assume now that $\lambda = 1$ or $\lambda = -1$. Replacing $j$ by $-j$ if needed, we can assume that $\lambda = 1$. Let $\phi : A(1) \to \mathbb{M}_2(k)$ be a representation satisfying the hypotheses. Let $\omega = \frac{\phi(i)+1}{2}$ and $\eta = \frac{\phi(i)+1}{2}$. Note that $\omega$ and $\eta$ are idempotents, whence we can write $\omega = \tau_{a,b}$ and $\eta = \tau_{c,d}$, and the branches $S(i)$ and $S(j)$ are tubular neighborhoods of the corresponding paths. Now consider the nilpotent element $u_n = \pi^{-n}\phi(i-j)$, satisfying $u_n\phi(i)+\phi(i)u_n = u_n\phi(j)+\phi(j)u_n = 0$. It is not hard to check that $\phi(j)$ and $u_n$ span an order containing also $\phi(i)$. On the other hand, the sequence $\{u_n\}$ leaves every compact subset of $\mathbb{M}_2(k)$ and therefore $i$ and $j$ are contained simultaneously in infinite many maximal orders. This is only possible if the paths $p(a,b)$ and $p(c,d)$ have a common end. They cannot coincide, since this would imply $\eta = 1-\omega$ and therefore $i = -j$, or else $\eta = \omega$ and therefore $i = j$. We can assume therefore that $\omega = \tau_{a,b}$ and $\eta = \tau_{c,d}$. Since the group of M"obius transformations act transitively on triples in $\mathbb{P}^1(k)^3$, we can assume $(a,b,d) = (\infty, 0, 1)$, whence $\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The result follows.

Now consider a pair $(i,j)$ of pure quaternions satisfying $i^2 = j^2 = 1$, and let $\lambda = \frac{1}{2}(ij + ji)$. By the previous lemma, the conjugacy class of the pair $(i,j)$ is completely determined by $\lambda$. On the other hand, if $\tau_{a,b}$ and $\tau_{c,d}$ are the corresponding idempotents, the orbit of the quartet $(a,b,c,d)$ under M"obius transformations is completely determined by the cross-ratio $t = [a:b;c,d]$. In fact, applying a M"obius transformation if needed, we can assume $(a,b,c,d) = (\infty, 0, 1, t)$. In this case we say that the pair of paths is in the first standard form (c.f. Figure 1). It follows that also $t$ is a complete invariant of the quartet.

Lemma 4.2. Let $\lambda \in k$ be any element, and let $i$ and $j$ be linearly independent pure quaternions satisfying $i^2 = j^2 = 1$ and $ij + ji = 2\lambda$. Assume $j = 2\tau_{a,b} - 1$ and $i = 2\tau_{c,d} - 1$. Then

$$
\lambda = \frac{t+1}{t-1}, \quad t = [a:b;c,d].
$$

Proof. Since $\lambda$ is certainly a rational function of $(a,b,c,d)$, while both $t$ and $\lambda$ are complete invariants of the quartet, it must be of the form $\lambda = \mu(t)$ where $\mu$ is a M"obius transformation. Since transposing $a$ and $b$ has the effect of changing the sign of $i$, and therefore also the sign of $\lambda$, we have $\mu(t^{-1}) = -\mu(t)$. We conclude that either $\mu(t) = \xi \cdot \frac{t+1}{t-1}$, or $\mu(t) = \xi \cdot \frac{t+1}{t-1}$. The first possibility is discarded out since $t = -1$ should correspond to a finite value of $\lambda$. Now $\xi = 1$ follows by choosing

![Figure 1](image-url)
any particular example. For instance, the value \( t = \infty \), corresponding to the quartet \((\infty, 0, 1, \infty)\), gives us \( \tau_{\infty, 0} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \tau_{\infty, 1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \), whence

\[
(2\tau_{\infty, 0} - 1)(2\tau_{1,\infty} - 1) + (2\tau_{1,\infty} - 1)(2\tau_{\infty, 0} - 1) = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

\( \square \)

Applying the transformation \( \mu \) in the preceding proof to the coordinates of the quartet \((\infty, 0, 1, t)\), we get \((-1, 1, \infty, \lambda)\), so that both paths are in one of the configurations shown in Figure 2, which we call the second invariant form. In the pictures, \(|\pi^u|\) is the radius of the smallest ball containing \( \lambda \) and either element in \( \{1, -1\} \). Note that \( \nu(\lambda + 1) = \varepsilon =: \nu(2) \) in Figure 2A, while \( w = \nu(\lambda - 1) = \nu(\lambda + 1) \) in Figure 2C. Next result is now apparent:

**Lemma 4.3.** Let \( \lambda \in K \) be any element, and let \( i \) and \( j \) be linearly independent pure quaternions satisfying \( i^2 = j^2 = 1 \) and \( ij + ji = 2\lambda \). Let \( d_f = -\frac{1}{2} \nu \begin{pmatrix} \lambda^2 - 1 \\ 4 \end{pmatrix} \), then the following holds:

1. If \( \lambda \neq \pm 1 \) and \( d_f \leq 0 \), then the stems \( m_k(i) \) and \( m_k(j) \) intersect non-trivially, and the intersection is a path of length \(-2d_f\) (see Figures 2A-B).
2. If \( \lambda \neq \pm 1 \) and \( d_f > 0 \), then \( d_f \) is the distance between the stems \( m_k(i) \) and \( m_k(j) \) (see Figure 2C).
3. If \( \lambda = \pm 1 \), the intersection between the two stems is a ray, and \( d_f = \infty \).

5. Galois action on ghost branches

In this section we let \( \tau = \tau_{z,u} \) be an idempotent in the algebra \( \mathbb{M}_2(L) \) generating an algebra \( L[\tau] = L[i] \), where \( i \in \mathbb{M}_2(k) \subseteq \mathbb{M}_2(L) \) is a pure quaternion satisfying \( k[i] \cong L \). Certainly \( L/k \) is a quadratic extension. Since the algebra \( k[i] \) has no non-trivial idempotents, necessarily \( \tau_{z,u} \notin \mathbb{M}_2(k) \), and therefore \( z \) and \( u \) cannot be both in \( k \). On the other hand, since \( L[i] \) is obtained from \( k[i] \) by extension of scalars, and \( \tau_{z,u} \) and \( \tau_{u,z} \) are the only two idempotents in this algebra, the non-trivial element \( \sigma \) in the Galois group \( \text{Gal}(L/k) = \langle \sigma \rangle \) necessarily permutes these two idempotents, and therefore also \( z \) and \( u = \sigma(z) \). If \( i^2 = \alpha \), we can write \( L = k[\sqrt{\alpha}] \), while \( z = a + b\sqrt{\alpha} \) and \( u = \bar{z} = a - b\sqrt{\alpha} \). This can be used to easily recover the description for branches of pure quaternions given in [1, Lemma 3.4].

Let \( \xi \in k \) be chosen so that its distance to \( z \) is minimal, as in Figure 3A. Then \( \xi \) must be equidistant from the extremes \( z \) and \( \bar{z} \). In fact, \( z \) and \( \bar{z} \) are equidistant from every element in \( k \), so minimizing \(|z - \xi|\) is equivalent to minimizing \(|(z - \xi)(\bar{z} - \xi)| = |(a - \xi)^2 - \alpha b^2|\). This is achieved by the element \( \xi = a + b\delta \), where \( \delta \in k \) is the
conclude that the vertex in the
Note that using the normalized distance make no difference in this for mula. We
has depth \( \alpha \) from the path
\( s \mid \mbox{element minimizing} \ | \delta^2 - \alpha| \). In particular, the ideal \((\delta^2 - \alpha)\) is the quadratic defect of \( \alpha \). The path joining \( \xi \) and \( \infty \) is called a \( k \)-vine for \( \tau \). The normalized tree-distance from the path \( s_L(\tau) \) to the \( k \)-vine is

\[
\nu(\bar{z} - z) - \nu(\xi - z) = \nu(2b\sqrt{\alpha}) - \nu(b(\delta - \sqrt{\alpha}))
\]

\[
= -\nu \left( \frac{\delta - \sqrt{\alpha}}{2\sqrt{\alpha}} \right) = -\frac{1}{2} \nu \left( \frac{\delta^2 - \alpha}{4\alpha} \right).
\]

Recall that the depth of the branch \( s_L(i) \subseteq t(L) \) is \( p = \nu(2\sqrt{\alpha}) \) [4, Lemma 3.4]. Note that using the normalized distance make no difference in this formula. We conclude that the vertex in the \( k \)-vine that is closest to the stem of the branch \( s_L(i) \) has depth \( p = \frac{1}{2} \nu(\delta^2 - \alpha) \) in that branch. Replacing \( i \) by \( \pi^{-n}i \) if needed, we can always assume that \( \alpha \in Q \), as defined in Theorem 2.1.

- If the extension \( L/k \) is unramified, we have that \( \alpha = \Delta \in Q \) is a unit of minimal quadratic defect. The vertex \( v_0 \) in the \( k \)-vine that is closest to the stem \( m_L(i) \) has depth \( p = -\nu(2) \). We conclude that \( v_0 \) in Figure 3B is indeed a vertex of \( t(k) \), and therefore it is the stem of \( s_k(i) \). The depth of this branch, in this case, equals \( p \).

- If the extension \( L/k \) is ramified, the vertex \( v_0 \) in the \( k \)-vine that is closest to the stem \( m_L(i) \) has depth \( p = -\nu(\delta^2 - \alpha) \) in that branch, which is an odd number, as it is the valuation of the quadratic defect [8, 63.2]. We conclude that \( v_0 \) is a ghost vertex, and therefore the midpoint of the stem \( m_k(i) \) (see Figure 3C). As the normalized distance in \( t(L) \) of neighboring vertices is \( 1/2 \), the depth of this branch is, therefore, \( \frac{1}{2} \).

Now assume we have two pure quaternions \( i \) and \( j \) en \( \mathbb{H}_2(k) \) satisfying the relations

\[
i^2 = \alpha, \quad j^2 = \beta, \quad ij + ji = 2\lambda.
\]

Then, the elements \( i_0 = \frac{i}{\sqrt{\alpha}} \) and \( j_0 = \frac{j}{\sqrt{\beta}} \) in \( \mathbb{H}_2(L) \), where \( L = K(\sqrt{\alpha}, \sqrt{\beta}) \), satisfy

the relations

\[
i_0^2 = j_0^2 = 1, \quad i_0j_0 + j_0i_0 = 2\lambda = \frac{2\lambda}{\sqrt{\alpha\beta}}.
\]
so that the results in last section apply to them. In this general setting, we have a classification of pairs of pure quaternions satisfying the relations in (3) that is entirely analog to the one in §4, namely:

**Lemma 5.1.** Let $\alpha, \beta, \lambda \in k$, with $\alpha \beta \neq 0$, and let $A = A(\alpha, \beta, \lambda)$ be the algebra defined, in terms of generators and relations, by

$$A = k[i, j]i^2 = \alpha, j^2 = \beta, ij + ji = 2\lambda.$$

Then there is, up to conjugation, at most one representation $\phi : A \to M_2(k)$, for which $\phi(i)$ and $\phi(j)$ are linearly independent. Furthermore $\phi$, if it exists, is an isomorphism unless $\lambda^2 = \pm \alpha \beta$. In the latter case $\alpha$ and $\beta$ are squares.

**Proof.** If $\lambda \neq \pm \sqrt{\alpha \beta}$, it follows from Lemma 4.1 that $L \otimes_k A$, for $L$ as above, is a quaternion algebra. We conclude that also $A$ is a quaternion algebra and the result follows from Skolem-Noether Theorem as before. We assume thus $\lambda^2 = \alpha \beta$, so $\beta/\alpha = (\lambda/\alpha)^2$. Replacing $j$ by $\frac{j}{\sqrt{\lambda}}$ if needed, we can assume $\beta = \alpha = \pm \lambda$. If this common value is a square, we are in the case of Lemma 4.1 so we assume this is not the case. Then any 2-dimensional representation of $A$ restrict to a two dimensional representation of $k[i]$ which is unique up to conjugation, so we may assume $\phi(i) = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$. Now we claim that the only matrices $X$ satisfying $X\phi(i) + \phi(i)X = \pm 2\alpha$ and $X^2 = \alpha$ are $X = \pm \phi(i)$. In fact, the first condition implies $X = \begin{pmatrix} x & \alpha(\pm 2 - z) \\ z & -x \end{pmatrix}$, while the second gives $x^2 = \alpha(-z \pm 1)^2$, which implies $x = -z \pm 1 = 0$, since $\alpha$ is not a square. The result follows. \(\Box\)

Now we characterize exactly when, given non-zero elements $\alpha, \beta \in k$, there exists two linearly independent pure quaternions in $M_2(k)$ satisfying the relations in (3). Note that we can assume $\lambda^2 \neq \alpha \beta$ by Lemma 5.1 since otherwise, replacing $i$ and $j$ by suitable multiples if needed, we can assume that $\alpha = \beta = 1$, and in this case the representation exists by Lemma 4.1. In case $\lambda^2 \neq \alpha \beta$, quaternions satisfying (3) exist precisely when $A(\alpha, \beta, \lambda) \cong M_2(k)$. When either $\alpha$ or $\beta$ is a square, the algebra $A(\alpha, \beta, \lambda)$ contains a non-trivial idempotent, either $\frac{i + \sqrt{\alpha}}{2\sqrt{\alpha}}$ or $\frac{i + \sqrt{\beta}}{2\sqrt{\beta}}$, and therefore it is a matrix algebra. In the remaining case we apply next result:

**Lemma 5.2.** Let $\alpha, \beta, \lambda \in k$, with $\lambda^2 \neq \alpha \beta \neq 0$. Assume neither $\alpha$ nor $\beta$ is a square. Then $A(\alpha, \beta, \lambda)$, defined as above, is a matrix algebra, if and only if there exists elements $a, b, c, d \in k$ with $bd \neq 0$, satisfying the relation

$$\lambda = \frac{b^2 \alpha + d^2 \beta - (a - c)^2}{2bd}.$$

**Proof.** We can assume that $i = \sqrt{\alpha}(2z_1, z_2 - 1)$ and $j = \sqrt{\beta}(2z_3, z_4 - 1)$, where

$$(z_1, z_2, z_3, z_4) = (a + b\sqrt{\alpha}, a - b\sqrt{\alpha}, c + d\sqrt{\beta}, c - d\sqrt{\beta}).$$

Then, the formulas in the preceding section give

$$\frac{2\lambda}{\sqrt{\alpha \beta}} = \frac{i}{\sqrt{\alpha \sqrt{\beta}}} + \frac{j}{\sqrt{\beta \sqrt{\alpha}}} = \frac{t + 1}{t - 1},$$

where $t = [z_1, z_2; z_3, z_4]$, which under a little algebraic manipulation becomes

$$\frac{\lambda}{\sqrt{\alpha \beta}} = \frac{(z_1 - z_4)(z_2 - z_3) + (z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3) - (z_1 - z_3)(z_2 - z_4)} =$$
\[-\frac{2(z_1z_2 + z_3z_4) - (z_1 + z_2)(z_3 + z_4)}{(z_1 - z_2)(z_3 - z_4)} = -\frac{2(a^2 - b^2\alpha) + 2(c^2 - d^2\beta) - 4ac}{4bd\sqrt{\alpha\beta}}.
\]

Conversely, if elements \(a, b, c, d\) satisfying (1) exist, the above formulas define two paths that are invariant under the Galois group, so they actually correspond to matrices \(i\) and \(j\) with coefficients in \(k\).

Note that, when \(\lambda = 0\), this reduces to the well known criterion stating that the quaternion algebra \((\alpha, \beta)\) splits if and only if the quadratic form in the numerator is isotropic.

**Lemma 5.3.** Let \(L/k\) be a multiple quadratic extension of non-archimedean local fields with Galois group \(G = \text{Gal}(L/k)\), and let \((x_1, x_2, x_3), (x_1', x_2', x_3') \in L^3\) be two triples satisfying the following conditions

1. Each set \(\{x_1, x_2, x_3\}\) and \(\{x_1', x_2', x_3'\}\) is \(G\) invariant.
2. For any \(\sigma \in G\) and any \(i, j \in \{1, 2, 3\}\) we have \(\sigma(x_i) = x_j\) if and only if \(\sigma(x_i') = x_j'\).

Then the Möbius transformation \(\tau\) satisfying \(\tau(x_i) = x_i'\) has coefficients in \(k\).

**Proof.** It follows from the hypotheses, and the uniqueness of the Möbius transformation taking one ordered triple onto another, that for any matrix \(A \in \text{GL}_2(L)\) defining \(\tau\), and for any element \(\sigma \in G\), we have \(\sigma(A) = \lambda_\sigma A\), where the map \(\sigma \mapsto \lambda_\sigma\) is a cocycle with values in \(L^*\), so the result follows from Hilbert’s Theorem 90.

For instance, if \(a, b, c, d \in k\), the Möbius transformation sending the triple \((c + d\sqrt{\alpha}, c - d\sqrt{\alpha}, a)\) onto \((c + d\sqrt{\alpha}, c - d\sqrt{\alpha}, b)\) is defined over \(k\).

6. **Proof of Theorem 2.1**

If \(\alpha = \beta = 1\), then the result is a direct application of Lemma 4.3. If one of them, say \(\alpha\) equals \(\Delta\), the stem \(m_k(i)\) of the branch \(s_k(i)\) contains exactly one point, namely the highest point \(v_0\) of the ghost path \(m_L(i)\). It is clear from Figure 4A, since \(v_0\) is the only point in the ghost path that is defined over \(k\), that the distance from any vertex \(u \in V(L)\) to \(m_L(i)\) equals the distance from \(u\) to \(v_0\), and therefore the same formulas hold in this case, after a substitution as in (5). Note that the fake distance \(d_f\) is nonnegative here. If \(\alpha = \beta = \Delta\), the same argument holds (Figure 4B), unless both ghost paths do intersect nontrivially, which means that the fake distance is negative. In that case the only vertex defined over \(k\) on that intersection is, necessarily, the highest point on each ghost path (Figure 4C). The intersection consists, therefore, of a single vertex, as required in this case, since \(l(i) = l(j) = 0\).

Assume now that \(\alpha = 1\), while \(\beta\) is either a ramified unit of a uniformizing parameter. Then Lemma 4.3 and a substitution as in (5), prove that the normalized distance, between the stems \(m_L(i)\) and \(m_L(j)\) is \(-\frac{1}{2} \nu \left(\frac{\lambda^2 - \beta}{4\beta}\right)\), which is always positive by the properties of the quadratic defect. However, the stem \(m_k(j)\) lies at a distance \(\nu(2\sqrt{\beta} - \nu(0 - \sqrt{\beta}) = -\frac{1}{2} \nu \left(\frac{\eta^2 - \beta}{4\beta}\right)\) from the stem \(m_L(j)\), assuming that the base \(\xi\) of the \(k\)-vine in Figure 3C satisfies \(\xi = a + b\eta\), where \(\bar{\sigma}(\beta) = (\eta^2 - \beta)\). It follows that, if the stem \(m_L(i)\), which is identified as a path with the stem \(m_k(i)\), fails to contain the stem \(m_k(j)\), then the distance between these stems is

\[-\frac{1}{2} \nu \left(\frac{\lambda^2 - \beta}{4\beta}\right) + \frac{1}{2} \nu \left(\frac{\eta^2 - \beta}{4\beta}\right) - \frac{1}{2} = -\frac{1}{2} \nu \left(\frac{\lambda^2 - \beta}{4\beta}\right) - \frac{1}{2} = t - \frac{1}{2} \nu(\lambda^2 - \beta) = d_f,
\]
Figure 4. Relative position for $\alpha = \Delta$ and $\beta = 1$ (A) and $\beta = \Delta$ (B,C). The long dashes in (C) denote the ghost intersection $m_L(i) \cap m_L(j)$ of the two ghost stems.

Figure 5. Relative position for $\alpha = 1$ and $\beta$ a ramified unit or a uniformizing parameter. The stem $m_k(j)$ can be contained in the stem $m_k(i)$ (A) or not (B).

Figure 6. Relative position for $\alpha = \Delta$ and $\beta$ a ramified unit or a uniformizing parameter.

where the summand $1/2$ is the normalized distance between $v_0$ and an endpoint of the stem $m_k(j)$, as in Figure 5B. Note that $m_k(i)$ contains $m_k(j)$ precisely when the fake distance is $-1/2$, as this is the case if and only if the $k$-vine can be chosen in a way that $\eta = \lambda$ as in Figure 5A. Furthermore, $d_f = -1/2$ is indeed the minimum possible value since the valuation $\nu\left(\frac{2\eta - \beta}{2}\right)$ above cannot be positive by definition of $\eta$. The case where $\alpha = \Delta$, and $\beta$ is either a ramified unit of a uniformizing parameter, is handled similarly, see Figure 6. Cases B and C can easily be reduced to case A, by applying a Moebius transformation taking $\infty$ to $\eta$, while preserving the ends of $m_L(i)$ (c.f. Lemma 5.3). In this case $d_f = 0$ is possible, when the vertex in $m_k(j)$, the highest point of $m_L(j)$, coincide with the lower endpoint of $m_k(i)$ in Figure 6A. However, $d_f = -\frac{1}{2}$ is not possible, as the vertex in $m_k(j)$ is defined over $k$, and cannot, therefore, be the midpoint of $m_k(i)$. Note that the stems $m_L(i)$ and $m_L(j)$ are defined over different quadratic extensions of $k$ in this case.
Figure 7. Possible locations of the stems, when each element in \( \{\alpha, \beta\} \) is either a ramified unit or a uniformizing parameter. In (D) the arrows might coincide.

In all remaining cases, either stem, \( m_k(i) \) or \( m_k(j) \), is an edge located in the \( k \)-vine minimizing the distance to the corresponding stem, either \( m_L(i) \) or \( m_L(j) \), as in Figure 3C. If these two \( k \)-vines are the maximal paths joining \( \infty \) with \( \eta_1 \) and \( \eta_2 \), respectively, then the stems of the \( k \)-branches are located as shown in one of the Figures 3A-C, unless they coincide, and in the latter case, this common stem is located as in Figure 3D. The ghost stems \( m_L(i) \) and \( m_L(j) \) are located in the direction of either arrow in these Figures. The arrows could coincide in Figure 3D, but this does not affect the proof. In the first 3 cases, the unique path in the tree \( t(L) \) from \( m_L(i) \) to \( m_L(j) \) passes through one of the endpoints of each, \( m_k(i) \) and \( m_k(j) \), so that the distance between them, by a similar argument as before, is

\[
-\frac{1}{2}\nu \left( \frac{\lambda^2 - \alpha \beta}{4\alpha \beta} \right) + \frac{1}{2}\nu \left( \frac{\eta_1^2 - \alpha}{4\alpha} \right) + \frac{1}{2}\nu \left( \frac{\eta_2^2 - \beta}{4\beta} \right) - 1 =
\]

\[
-\frac{1}{2}\nu \left( \frac{4(\lambda^2 - \alpha \beta)}{(\eta_1^2 - \alpha)(\eta_2^2 - \beta)} \right) - 1 = t + s - \frac{1}{2}\nu \left( 4(\lambda^2 - \alpha \beta) \right) = d_f,
\]

where, as before, there is a final 1 to take care of the distance 1/2 from one endpoint of each stem to its center.

In the last case, i.e., when the \( k \)-stems coincide, we claim that

\[
t + s - \frac{1}{2}\nu \left( 4(\lambda^2 - \alpha \beta) \right) = d_f < 0.
\]

Note that the expression on the left equals

\[
-\frac{1}{2}\nu \left( \frac{\lambda^2 - \alpha \beta}{4\alpha \beta} \right) + \frac{1}{2}\nu \left( \frac{\eta_1^2 - \alpha}{4\alpha} \right) + \frac{1}{2}\nu \left( \frac{\eta_2^2 - \beta}{4\beta} \right) - 1,
\]

where the first term is the distance between the \( L \)-stems, while the additive inverses of the second and third terms are the distance from each of them to the common \( k \)-stem. Since, in this case, the path from one \( L \)-stem to the other cannot pass through either endpoint of the \( k \)-stem, the result follows. \( \square \)

7. Computing embedding numbers via gost branches

In this section we prove Theorem 2.2. For this we make extensive use of Lemma 5.3.

Let us begin by recalling the method employed in [6] to compute embedding numbers for an order \( \Omega \) spanning an algebra isomorphic to \( k \times k \). Essentially, it depends on the following observations:
To compute the embedding numbers, we first compute the number of walks, the walk cannot exceed \( v \) which can be chosen among \( q \) at depth \( p \). Since every vertex, in a thick path whose stem has \( t \) that is not in \( E \), we have \( p_b(v) \leq \rho(b) = t \), whence \( i \geq r - t \). The smallest value of \( i \) satisfying these inequalities is called the returning point \( r_0 \). In fact, \( r_0 = r - \lfloor r/2 \rfloor \) in each case in our setting, since \( r \leq 2t + 1 \). Since every vertex, in a thick path whose stem has length 0 or 1, has at most one neighbor whose depth is not smaller, the vertices \( v_1 \) through \( v_m \) are completely determined by \( v_0 \). Each subsequent vertex \( v_{m+1}, \ldots, v_r \) can be chosen among \( q \) different choices, as every path of length \( r - r_0 \) from a vertex at depth \( p_b(v_{r_0}) = r_0 \leq r - r_0 \) is completely contained in the branch \( b \). The claim follows.

Similarly, the invariant \( m \) denotes, in each case, the number of these walks for which \( v_r \) is also optimal. If \( r \) is odd, then \( m = 0 \), unless we can have a path of length
$r$ with two optimal endpoints, which necessarily contains a stem edge. This is only possible in the last case in the table. If $r$ is the diameter of $s_k(\phi(\Omega))$, then $m = p^{[r/2]}$. Otherwise, $v_r$ is optimal and only if $p_b(v_{r_0+1}) < p_b(v_{r_0})$, and the proportion of paths satisfying this inequality is precisely $\frac{2^{r_1}}{2^r}$, since there is exactly one choice of $v_{r_0+1}$ that fails to satisfy it.

Every embedding $\phi : \Omega \rightarrow M_2(k)$ corresponds to a unique walk $w$ in $t(L)$ whose corresponding path is the ghost stem $m_L(\phi(\Omega))$. The ends of this path must be a Gal($L/k$)-orbit, for $\phi$ to be defined over $k$. Every Eichler order corresponds to a unique finite path in $t(k)$, or equivalently, two walks. We conclude that every pair $(\phi, \mathcal{E})$, where $\phi : \Omega \rightarrow M_2(k)$ is an embedding, and $\mathcal{E}$ is an Eichler order optimally containing $\phi(\Omega)$, corresponds to two pairs of walks $(w, u_1)$ and $(w, u_2)$ where $u_1$ and $u_2$ define the same path. Furthermore, the initial vertex of either $u_1$ or $u_2$ is optimal in the corresponding branch $s_k(\phi(\Omega))$. Recall that $\Gamma_1$ is the stabilizer of the walk $u_1$, while $\Gamma_2$ is the stabilizer of the pair \{u_1, u_2\}. The group of Moebius transformations acts transitively on triplets of elements in $k$, whence the orbit of a triplet $(v, a, b)$, where $v$ is a vertex, while $a$ and $b$ are ends of the BT-tree, is completely determined by the distance from $v$ to the path joining $a$ and $b$. A vertex $v_0$ is optimal in the branch $s_k(\phi(\Omega))$, if and only if its distance to the maximal path corresponding to $\phi$ equals $t$. This information is given by the invariant of the pair of walks. In fact, if $u_1$ is the path from $C$ to $D$ in Figure 8, the distance $l$ from $V$ to $U$ in given by the formula $l = \nu(t - 1)$, where $t = [a, b; c, d]$. This is immediate, since we can assume $(a, b, c, d) = (\infty, 0, 1, t)$.

To compute $e_1$, we fix a walk $u_1$ corresponding to the Eichler order $\mathcal{E}$. Assume $u_1$ is the walk from $A$ to $B$ in Figure 8. Observe that $\Gamma_1$ is the stabilizer of this path. Two optimal embeddings of $\Omega$ into $\mathcal{E}$, corresponding to the doubly infinite walks $w$ and $w'$ from $c$ to $d$, and from $c'$ to $d'$, respectively, are conjugate by an element stabilizing the path $u_1$ if and only if the invariant is the same, i.e.,

$$[a, b; c, d] \equiv [a, b; c', d'] \mod \pi_L^{e(t)}(s; y),$$

as in this case $u = x = \infty$. Note that both $(c, d)$ and $(c', d')$ are Gal($L/k$)-orbits, while $a$ and $b$ can be chosen in $k$. Assume $s \leq y$. Set $b' = \mu(b)$, while $\mu$ is the Moebius transformation satisfying $\mu(a, c, d) = (a, c', d')$, which is defined over $k$ by Lemma 5.3. Then by [6 Cor. 5.1], $b'$ is an end beyond $B$ in Figure 8. Now $\mu$ leaves $u_1$ invariant, while sends $w$ to $w'$, and is defined over $k$, again by Lemma 5.3. We conclude that, to compute the number of orbits, we just need to compute the number of possible invariants. Note however that we need to consider embeddings whose corresponding walk has an initial optimal vertex, and also embeddings whose corresponding walk has an final optimal vertex. This leads us to the formula $e_1 = 2n - m = 2p^{[r/2]} - m$.

To compute $e_2$, we replace the stabilizer $\Gamma_1$ of the walk $u_1$ by the stabilizer $\Gamma_2$ of the corresponding path. Note that $\Gamma_1$ is normal in $\Gamma_2$, and the corresponding quotient acts on the set of $\Gamma_1$-orbits, so we have $e_2 = \frac{1}{2}(e_1 + \chi_2)$ as soon as we prove that $\chi_2$ is the number of invariant orbits. The orbit of an embedding $\phi$ is invariant if there exists two elements $\sigma \in \Gamma_1$ and $\lambda \in \Gamma_2/\Gamma_1$ satisfying $\sigma \phi(\omega) \sigma^{-1} = \lambda \phi(\omega) \lambda^{-1}$ for every $\omega \in \Omega$. This is equivalent to the existence of a Moebius transformation flipping the ends of the path $s_k(\mathcal{E})$, while leaving the ends of the branch $m_L(\phi(\Omega))$ invariant. This is only possible if the following conditions are satisfied:
(1) Both endpoints of the path \( s_k(E) \) are equidistant to the ghost stem \( \mathfrak{m}_L(\phi(\Omega)) \).

(2) The invariant \([a, b; c, d]\) of the quartet is its own inverse in the quotient ring in equation (2) (c.f. §2).

The computation of \( e_3 = \frac{1}{2}(e_1 + \chi_3) \) is similar, but we no longer require condition (1) above, as we just need to flip the ends of the infinite path.

To compute \( e_4 \) we need to compute the number of orbits of pairs of walks, as before, under an action of the Klein group \( C_2 \times C_2 \) that reverses either walk. This can be done via Burnside’s Counting Lemma [10, §26.10]. We already know the number of invariants that remain invariant when we reverse either walk, they are \( \chi_2 \) and \( \chi_3 \) respectively. If we reverse both walks simultaneously, every walk with two optimal endpoints in invariant since the cross ratio has the symmetry \([a, b; c, d] = [b, a; d, c]\). We conclude that

\[
e_4 = \frac{1}{4}\left((2n - m) + \chi_2 + \chi_3 + m\right) = \frac{n}{2} + \frac{1}{4}(\chi_2 + \chi_3).
\]

If \( r = 0 \) then \( \Gamma_1 = \Gamma_2 \), which implies \( e_1 = e_2 \) and \( e_3 = e_4 \). Furthermore \( e_1 \geq e_3 \) as conjugate embeddings have conjugate images. It suffices therefore to see that \( e_1 = 1 \).

By another application of Lemma 5.3, the group of Möbius transformations acts transitively on triples \((v, a, b)\), where \( a \) and \( b \) are ends of the BT-tree \( t(L) \) in the same Galois orbit, while \( v \in V_{t(k)} \) is a vertex at a fixed distance, as above. The result follows.

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