Nonlinear dynamics of intense EM pulses in plasma

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Abstract. The evolution of laser beam in underdense/overdense plasma medium which is key to understanding of several nonlinear processes and underlying physics is governed by nonlinear parabolic equation. The nonlinearity considered here is of relativistic as well as of ponderomotive type. We have set Lagrangian for the problem and reduced Lagrangian problem is solved using appropriate trial function. Equation for the beam width and phase are derived. Further, these equations are used to solve eigen value problem for the stability of laser beam evolution and Hurwitz condition is satisfied.

1. Introduction
The study of evolution of high intensity lasers as they propagate through the underdense plasmas in an active area of research due to its importance in potential applications such as plasma based accelerators \([1]\), inertial confinement fusion \([2,3]\) and new radiation sources \([4,5,6,7,8]\). For these applications, a long propagation distance of intense lasers in the medium is desirable. As several effects emerge in such transit of high power laser beam through plasma medium, many instabilities and nonlinear phenomenon, such as self-modulation, the filamentation instability, group velocity dispersion (GVD), finite pulse effects, relativistic self-focusing effects, plasma waves etc., become important. Therefore, it is important to study analytically and numerically these effects. Among these, the self-focusing is genuinely nonlinear basic phenomenon which plays important role in the propagation. Self-focusing is due to increase of the on-axis index of refraction relative to edge of the laser beam resulting from averaged quiver motion of the electrons with their subsequent expulsion from the region of high intensity ponderomotive self-focusing. Also the effect of quiver motion is to reduce the local plasma frequency which leads to relativistic self-focusing of the laser beam. Experimental observations of relativistic self-focusing and ponderomotive self-channeling have been reported in Refs. \([9,10,11,12,13,14]\). Most general form of the field envelope in the absence of dispersion and under slowly varying envelope approximation is:

\[
2ik \frac{\partial \psi}{\partial z} + \nabla_\perp^2 \psi + Q(\vert \psi \vert^2)\psi = 0
\] (1)

This governing equation of field envelope determines the beam evolution, is nonlinear, so conventional separate techniques using fourier expansions are not appropriate. These higher dimensional nonlinear Schrodinger equation (NLSE) are not integrable so that they do not have special solution such as soliton solutions (spatial or temporal). However, they possess stationary
solution which are unstable on propagation. For deeper insight into the physical process, we solve equation (1) by using some approximate models. Thus, the resulting expressions provide qualitative rather than quantitative estimates. There are several approximate analytical approaches to analyse the effect of self-focusing namely a systematic perturbation theory [15,16], ray equation approximation [17], an approach based on Fermat’s principle [18], variational approach [19], a paraxial ray approximation (PRA) [20,21,22,23], moment theory approach [19,24,25] and source-dependent expansion method [26]. Most commonly used theory being PRA, suffers from a drawback of being local, i.e. it overemphasizes the field closest to beam axis and lacks global pulse dynamics. Further, it predicts unphysical phase relationship [27]. With some partial remedies to PRA, moment theory of self-focusing gives results closer to computer simulations. This theory was not actively pursued as it lacks generalization and phase description. Another global approach which is in vogue these days is the variational method, which find diverse applications in a number of fields for the study of nonlinear propagation. Lagrangian for the problem \( L \), chosen to make the corresponding variational function stationary i.e.

\[
\delta \int L dsdz = 0
\]  

where \( \psi \) solves the evolution equation. Since ponderomotive and relativistic channeling occur together, we investigate their combined effects on the evolution of intense laser beam in plasma. The organisation of the paper is as follows. In section 2, a model is set up in a weakly relativistic limit starting from Maxwell’s equations and hydrodynamic equations. These equations under approximate conditions leads to an evolution equation. Lagrangian for the problem is set up and variational approach is used.

2. Basic Formulation

The present model is set up in a weakly relativistic limit starting from Maxwell’s equations and hydrodynamic equations. Two coupled equations for density perturbation and laser beam vector potential in a preformed plasma channel are given as follows:

\[
(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\psi = \frac{\omega_p^2}{c^2} (1 + \frac{r^2}{r_{ch}^2} + \frac{\delta n}{n_0} - \frac{\left|\psi\right|^2}{2})\psi
\]  

(3)

\[
\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)n_0\frac{\delta n}{c^2 \nabla^2} = c^2 \nabla^2 \frac{\left|\psi\right|^2}{2}
\]  

(4)

Assuming parabolic density profile and long pulse limit approximation and in slowly varying envelope approximation, we obtain the following evolution equation governing the electric field envelope in collisionless plasmas as follows:

\[
(2\kappa k \frac{\partial}{\partial z} + \nabla_\perp^2 - k_p^2 \frac{r_{ch}^2}{r_p^2} - \nabla_\perp^2 \frac{\left|\psi\right|^2}{2} + k_p^2 \frac{\left|\psi\right|^2}{2})\psi(r, z) = 0
\]  

(5)

Equation (5) which is a special case of equation (1), is nonlinear parabolic partial differential equation in which second term has its origin in diffractional divergence, third and fourth corresponds to channel and ponderomotive self-focusing while the last one has its origin to relativistic self-focusing respectively. This can be solved by a number of approximate techniques. It may be mentioned that another techniques invoke these days, is source-dependent technique. This has frequently been used in free electron laser theory, where nonlinear term is expanded in
terms of associated Laguerre polynomials. However, it amounts to solve nonlinear problems in terms of linear theory as nonlinear source term is expanded by superposition principle i.e. in terms of Laguerre polynomial lacks/needs justification. Variational approach which have rigorous basis, as applied in other fields, is used here to nonlinear wave propagation. We can reformulate equation (5) into a variational problem corresponding to a Lagrangian \( L \), so as to make \( \frac{\partial L}{\partial \psi} = 0 \), is equivalent to equation (5). Thus Lagrangian \( L \), is given by:

\[
L = ik \left( \psi^* \frac{\partial \psi}{\partial z} - \psi \frac{\partial \psi^*}{\partial z} \right) - \left| \frac{\partial \psi}{\partial x} \right|^2 - \left| \frac{\partial \psi}{\partial y} \right|^2 + \frac{1}{4} \left( \frac{\partial |\psi|^2}{\partial x} \right)^2 + \left( \frac{\partial |\psi|^2}{\partial y} \right)^2 - \frac{x^2}{r_{ch}^2} k_p |\psi|^2 - \frac{y^2}{r_{ch}^2} k_p |\psi|^2
\]

\[+ k_p^2 \left| \psi \right|^4 \frac{4}{4}
\]

Thus, the solution to the variational problem

\[
\delta \int \int \int L dx dy dz = 0
\]

also solves the nonlinear Schrödinger equation (5). Using the trial function as elliptic Gaussian beam of the form as follows:

\[
\psi(r, z) = \psi_o(z) \exp \left[ -\frac{x^2}{2X^2(z)} - \frac{y^2}{2Y^2(z)} + i(x^2 b_x(z) + y^2 b_y(z) + \phi(z)) \right]
\]

where \( X, Y \) are the width parameters of the beam in the \( x \) and \( y \) direction respectively. It may be further noted that \( X \) and \( Y \) are real functions of \( z \). As most of the laser systems usually generate a beam which is more nearly elliptical than circular in cross-section. This makes our choice of trial function of above form as more realistic one. Using the ansatz, with expression for \( \psi \) as trial function and substituted in \( L \), we can perform the integration to write:

\[
< L > = k \Delta Y \psi_o \frac{\partial \psi_o}{\partial z} - \psi_o \frac{\partial \psi_o}{\partial z} - \left| \psi_o \right|^2 \pi X^4 Y^2 \frac{4}{4} \left[ \frac{db_x}{dz} + 4b_x^2 + \frac{1}{X^4} - \frac{1}{Y^4} \right] \left| \psi_o \right|^2 \pi X^4 Y^2 \frac{4}{4} \left[ \frac{db_y}{dz} + 4b_y^2 + \frac{1}{X^4} - \frac{1}{Y^4} \right] \left| \psi_o \right|^2
\]

\[
2k \Delta Y \left| \psi_o \right|^2 \frac{d \phi}{dz} - k_p \left| \psi_o \right|^2 \pi X^4 Y - k_p \left| \psi_o \right|^2 \pi X^4 Y + \frac{\left| \psi_o \right|^4}{16X} \pi X + \frac{\left| \psi_o \right|^4}{16Y} \pi Y + k_p \left| \psi_o \right|^2 \frac{8}{8} \pi X Y
\]

Variation with respect to \( \psi, \psi^*, X, Y \) etc. and using the procedure of [28], we arrive at the following equations for \( X, Y, \phi \):

\[
\frac{d^2 X}{dz^2} = \frac{2}{k^2 X^3} - \frac{3}{4} \frac{\left| \psi_o \right|^2}{k^2 X^2} - \frac{\left| \psi_o \right|^2}{4 k^2 X Y^2} - k_p \frac{\left| \psi_o \right|^2}{2 k^2 X} - \frac{2 k_p^2}{k^2 r_{ch}^2} X
\]

\[
\frac{d^2 Y}{dz^2} = \frac{2}{k^2 Y^3} - \frac{3}{4} \frac{\left| \psi_o \right|^2}{k^2 Y^2} - \frac{\left| \psi_o \right|^2}{4 k^2 X Y^2} - k_p \frac{\left| \psi_o \right|^2}{2 k^2 Y} - \frac{2 k_p^2}{k^2 r_{ch}^2} Y
\]

\[
k \frac{d \phi}{dz} = -\frac{1}{8 X^2} - \frac{1}{8 Y^2} + \frac{\left| \psi_o \right|^2}{16 X^2 Y} + \frac{\left| \psi_o \right|^2}{16 Y^2 X} + \frac{3}{32} \frac{k_p^2}{k^2} \left| \psi_o \right|^2
\]

After normalization, we get the following equations:
\[
\frac{d^2 X}{d\eta^2} = \frac{2}{k^2 X^3} - \frac{3}{4 k^4 X^3} \frac{|\psi_0|^2}{4 k^4 X Y^2} - k_p^2 \frac{|\psi_0|^2}{2 k^2 X} - \frac{2k_p^2}{k^4 r_{ch}^2} X \tag{13}
\]

\[
\frac{d^2 Y}{d\eta^2} = \frac{2}{k^2 Y^3} - \frac{3}{4 k^4 Y^3} \frac{|\psi_0|^2}{4 k^4 X^2 Y} - k_p^2 \frac{|\psi_0|^2}{2 k^2 Y} - \frac{2k_p^2}{k^4 r_{ch}^2} Y \tag{14}
\]

\[
\frac{d\phi}{d\eta} = -\frac{1}{8k^4 X^2} - \frac{1}{8k^4 Y^2} + \frac{|\psi_0|^2}{16k^4 X^2} + \frac{|\psi_0|^2}{16k^4 Y^2} + 3 \frac{k_p^2}{32 k^2} |\psi_0|^2 \tag{15}
\]

where \(|\psi_0|^2_{XY} = A_0^2 X_0 Y_0\) is constant. We write the above equations in dimensionless form using \(\eta = \frac{z}{z_r}\). Equations can be manipulated algebraically in the following form:

\[
\frac{1}{2} \frac{d^2 \rho^2}{d\eta^2} = \left(\frac{X^2}{Y^2} - 1\right) \left(\frac{2}{k^2} - \frac{|\psi_0|^2}{k^2}\right) + \frac{k_p^2}{k^4} \left(\frac{|\psi_0|^2}{k^4}\right) + \frac{2k_p^2}{k^4 r_{ch}^2} \left(X^2 + Y^2\right) + \left(\frac{dX}{d\eta}\right)^2 + \left(\frac{dY}{d\eta}\right)^2 \tag{16}
\]

where \(\rho^2 = X^2 + Y^2\)

For an initially parallel beam, initial conditions are:

\[
dX_0 \left/ dz\right. = dY_0 \left/ dz\right. = 0 \tag{18}
\]

implies

\[
\left. \frac{\partial \rho^2}{\partial \eta}\right|_{\eta=0} = 0 \tag{19}
\]

3. Discussion

Equations (13) and (14) describes the beam dynamics in plasma with relativistic self-focusing, ponderomotive self-channeling (PSF) and channel focusing. The equations (13) and (14) are nonlinearly coupled equations governs the beam width parameters \(X\) and \(Y\) in the \(x\) and \(y\) directions. Further equation (15) describes the longitudinal phase. It is observed that R.H.S of these equations contain various terms, each representing some physical mechanism responsible for focusing / defocusing of laser beam. e.g. the first term on R.H.S of (13) is responsible for diffractional divergence of the laser beam. It has its origin in the Laplacian appearing in the evolution equation (5). The other terms represent the combined effects of relativistic self-focusing, PSF and self-channeling. In the absence of these terms, beam diverges due to diffraction. However, as mentioned in introduction, long distance of several Rayleigh lengths are required for novel applications of lasers. Earlier research work e.g. relativistic and ponderomotive self-focusing using different approaches have been reported in various context. However, results of variational approach is more appropriate and we here study the effects of various physical mechanisms in the evolution of beam. We have chosen the following set of parameters for numerical computation:

\[X_0 = 0.002cm, Y_0 = 0.0016cm, k = 1.25 \times 10^3 cm^{-1}, |\psi_0|^2 = 0.1, k_p \frac{\omega_p}{c}, r_{ch} = 0.0017cm.\]
In the absence of plasma channel and for the above mentioned parameters, the diffraction is the dominant mechanism leading to monotonically increase in normalized beam width parameters $X_n$ and $Y_n$. This is in contrast to earlier observations where nonlinearity due to ponderomotive and relativistic mechanism are sufficient to result in self-focusing phenomenon. The results are investigated and succinctly shown in figure 1 where $X_n$, $Y_n$ and $\rho_n^2$ have been plotted. However, situation changes drastically as finite channel is introduced. The nonlinear ordinary differential equations start the occurrence of individual self-focusing of $X_n$ and $Y_n$ as well as phenomenon of cross-focusing is observed. In this process, focusing of $X_n$ leads to defocusing of $Y_n$ and vice versa. Figure 2(a) and 2(b) exhibits the oscillatory self-focusing of $X_n$ and $Y_n$ with the distance of propagation. Figure 3 exhibits the graphs of $X_n$, $Y_n$ and $\rho_n^2$ for all other parameters mentioned alongwith channel. Further, the phenomenon of waveguide is not observed as $X_n$ and $Y_n$ keeps on oscillatory self-focusing. From variational approach, it is observed that propagation extends to several Rayleigh lengths. This confirms the experimental observations of [29]. In figure 4, we have shown longitudinal phase, which is negative with distance of propagation.

4. Stability Criterion

To study the stability properties of the system [30], the following Jacobi determinant is constructed from derivatives with respect to amplitude, width and curvature in terms of $S$, $F$ and $G$, where

\[
S = \frac{d\psi_0}{dz} = \frac{2b_x |\psi_0|}{k},
\]

\[
F = \frac{dX}{dz} = \frac{4Xb_x}{k},
\]

\[
G = \frac{db_x}{dz} = \frac{-2b_x^2}{k} + \frac{1}{2kX} - \frac{3}{4} \frac{A_0^2 X_0 Y_0}{kX^3Y} - \frac{A_0^2 X_0 Y_0}{16kX^3Y^3} - \frac{k^2 A_0^2 X_0 Y_0}{8kX^3Y} - \frac{k^2}{2kr_{ch}^2}.
\]

\[
\det[J - \lambda I] = \begin{vmatrix}
\frac{\partial S}{\partial \psi_0} - \lambda & \frac{\partial S}{\partial X} & \frac{\partial S}{\partial b_x} \\
\frac{\partial F}{\partial \psi_0} & \frac{\partial F}{\partial X} & \frac{\partial F}{\partial b_x} \\
\frac{\partial G}{\partial \psi_0} & \frac{\partial G}{\partial X} & \frac{\partial G}{\partial b_x} - \lambda
\end{vmatrix} = 0
\]

This leads to the following characteristic equation cubic in $\lambda$:

\[
\lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0
\]

where

\[
\alpha_1 = -\frac{2b_x}{k}
\]

\[
\alpha_2 = \frac{8}{kX^4} - 15 \frac{|\psi_0|^2}{k^2 X^4} - \frac{3}{4} \frac{|\psi_0|^2}{k^2 X^2 Y^2} - \frac{16b_x^2}{k^2} - \frac{3}{2} \frac{k^2 |\psi_0|^2}{k^2 X^2}
\]
\[ \alpha_3 = \frac{32b^3}{k^3} - \frac{16b_x}{k^3 X^4} + \frac{30|\psi_0|^2 b_x}{k^3 X^4} + \frac{3}{2} \frac{|\psi_0|^2 b_x}{k^3 X^3} + \frac{3k^2_b |\psi_0|^2}{k^3 X^2} \]  

Equation (27)

In order to have Lyapunov’s stability, Hurwitz conditions must be fulfilled, i.e. \( \alpha_1 \alpha_2 - \alpha_3 \) must be positive. According to Routh-Hurwitz criterion, a necessary and sufficient condition for the stationary solution to be stable is:

\[ \alpha_1 \alpha_2 - \alpha_3 > 0 \]  

Equation (28)

Equation (24) has a pair of imaginary roots at a critical point:

\[ \lambda = \pm i \nu, \nu > 0 \]  

We may substitute (29) into (24) and we get:

\[ \nu^2 - \alpha_2 = 0 \]  

and

\[ \alpha_1 \nu^2 - \alpha_3 = 0 \]  

The critical condition for Hopf bifurcation is:

\[ f = \alpha_1 \alpha_2 - \alpha_3 = 0 \]  

\( f > 0 \) is a necessary condition for the stationary solution to be stable, \( f < 0 \) is a necessary condition for the Hopf bifurcation to emerge. It is observed that the condition \( f > 0 \) is satisfied for chosen set of parameters and therefore Hopf bifurcation, resulting from the unstable fixed point, does not come into play, leading to overall stability of the beam dynamics.

5. References
[1] Sarkisov, G S et al. 1999 Phys. Rev. E 59 7042
[2] Tabak M, Hamer J, Glinsky M E, Kruer W L, Wilks S C, Woodworth J, Campbell E M, Perry M D, Mason R J 1994 Phys. Plasmas 1 1626
[3] Regan S P, Bradley D K, Chirokikh A V, Craxton R S, Meyerhofer D D, Seka W, Short R W, Simon A, Town R P J, Yaakobi B 1999 Phys. Plasmas 6 2072
[4] Suckewer S, Skinner C H 1990 Science 247 1553 ; Suckewer S, Skinner C H 1995 Comments At. Mol. Phys. 30 331
[5] Benware B R, Macchietto C D, Moreno C H, Rocca J J 1998 Phys. Rev. Lett. 81 5804
[6] Yu W, Yu M Y, Zhang J, Xu Z 1998 Phys. Rev. E 57 2531
[7] Foldes I B, Bakos J S, Bakonyi Z, Nagy T, Szatmari S 1999 Phys. Lett. A 258 312
[8] Fedotov A B, Naumov A N, Silin V P, Uryupin S A, Zheiltikov A M, Tarasevitch A P, Von der Linde D 2000 Phys. Lett. A 271 407
[9] Borisov A B, Borovskiy A V, Korobkin V V, Prokhorov A M, Shiryaev O B, Shi X M, Luk T S, McPherson A, Solem J C, Boyer K and Rhodes C K 1992 Phys. Rev. Lett. 68 2309
[10] Monot P, Auguste T, Gibbon P, Jakober F, Mainfray G, Dulieu A, Louis-Jacquet M, Malka G and Miquel J L 1995 Phys. Rev. Lett. 74 2953
[11] Krushelnick K, Ting A, Moore C I, Burris H R, Esarey E, Sprangle P and Baine M 1997 Phys. Rev. Lett. 78 4047
[12] Wagner R, Chen S Y, Maksimchuk A and Umstadter D 1997 Phys. Rev. Lett. 78 3125
[13] Chen S Y, Sarkisov G S, Maksimchuk A, Wagner R and Umstadter D 1998 Phys. Rev. Lett. 80 2610
[14] Borisov A B, Longworth J W, Boyer K and Rhodes C K 1998 *Proc. Natl. Acad. Sci. USA* **95** 7854
[15] Fibich G 2000 *Opt. Lett.* **25** 335
[16] Fibich G and Papanicolan G 1999 SIAM *J. Appl. Math* **60** 183
[17] Marburger J H 1975 *Prog. Quantum. Elect.* **4** 35
[18] Boyd R W 2003 *Nonlinear optics* Academic Press, Sandiego, USA
[19] Wirth W J 1977 *Opt. Commun.* **20** 226
[20] Akhmanov S A, Sukhorukov A P and Khokhlov R V 1996 *Sov. Phys. JETP* **23** 1025
[21] Akhmanov S A, Sukhorukov A P and Khokhlov R V 1968 *Sov. Phys. Usp* **10** 609
[22] Sodha M S, Ghatak A K and Tripathi V K 1974 *Self-focusing of laser beam in dielectric plasma and semi-conductors* Tata McGra-Hill, New York
[23] Sodha M S, Ghatak A K and Tripathi V K 1976 *Prog. Optics* **13**
[24] Lam J F, Lippmann B and Tappert F 1975 *Optics Commun.* **15** 419
[25] Lam J F, Lippmann B and Tappert F 1977 *Phys. Fluids* **20** 1176
[26] Sprangle P, Hafizi B and Penano JR 2000 *Phys. Rev. E* **61** 4381
[27] Karlsson M, Anderson D and Lisak M 1991 *Opt. Lett.* **16** 1374
[28] Anderson D, Bommedal M and Lisak M 1979 *Self-trapped cylindrical laser beams* *Phys. Fluids* **22** 1838-40
[29] Fauser C and Langhoff H 2000 *Appl. Phy. B* **71** 607-09
[30] Lakshman M and Rajasekar S 2003 *Nonlinear dynamics* Springer Verlag
Figure 1. Variation of beam widths $X_n$, $Y_n$, $\rho_n^2$ with normalized distance of propagation $\eta$ for the following set of parameters: $X_0 = 0.002$ cm, $Y_0 = 0.0016$ cm, $k = 1.25 \times 10^3$ cm$^{-1}$. Channel term is assumed to be zero. $|\psi_0|^2 = 0.1$. 
Figure 2(a). Variation of beam width $X_n$ with $\eta$ for the same set of parameters as in figure 1 except channel radius, $r_{ch} = 0.0017$ cm.
**Figure 2 (b).** Variation of beam width $Y_n$ with $\eta$ for the same parameters as in figure 1 except $r_{ch} = 0.0017$ cm.
Figure 3. Variation of beam widths $X_n$, $Y_n$, $\rho_n^2$ with normalized distance $\eta$ for the same set of parameters as mentioned above except $r_{ch} = 0.0017$ cm.
Figure 4. Plot of longitudinal phase ($\phi$) versus $\eta$ for different values of $|\psi_0|^2$ and other parameters are same as mentioned in figure 1 with $r_{ch} = 0.0017$ cm. Curve 1 corresponds to $|\psi_0|^2 = 0.1$. Curve 2 corresponds to $|\psi_0|^2 = 0.13$. Curve 3 corresponds to $|\psi_0|^2 = 0.15$. 