Research Article

Faber Polynomial Coefficient Bounds for m-Fold Symmetric Analytic and Bi-univalent Functions Involving q-Calculus

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In our present investigation, by applying q-calculus operator theory, we define some new subclasses of m-fold symmetric analytic and bi-univalent functions in the open unit disk \( U = \{ z : |z| < 1 \} \) and use the Faber polynomial expansion to find upper bounds of \( |a_{m+1}| \) and initial coefficient bounds for \( |a_{n+1}| \) and \( |a_{2n+1}| \) as well as Fekete-Szego inequalities for the functions belonging to newly defined subclasses. Also, we highlight some new and known corollaries of our main results.

1. Introduction, Definitions, and Motivation

Let \( \mathcal{A} \) denote the class of all analytic functions \( f(z) \) in the open unit disk \( U = \{ z : |z| < 1 \} \) and have the series expansion of the form

\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

By \( \mathcal{S} \), we mean the subclass of \( \mathcal{A} \) consisting of univalent functions. The inverse \( f^{-1} \) of univalent function \( f \) can be defined as

\[
    f^{-1}(f(z)) = z, \quad z \in U,
\]

\[
    f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4},
\]

where

\[
    a_1(w) = f^{-1}(w) = w - a_1 w^2 + (2a_1^2 - a_1) w^3 - (5a_1^3 - 5a_1a_2 + 4a_3) w^4 + \cdots
\]

According to the Koebe one-quarter theorem [1], an analytic function \( f \) is called bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). Let \( \Sigma \) denote the class all bi-univalent functions in \( U \). For \( f \in \Sigma \), Lewin [2] showed that \( |a_2| < 1/51 \) and Brannan and Clunie [3] proved that \( |a_2| \leq \sqrt{2} \). Netanyahu [4] showed that \( \max |a_2| = 4/3 \). Brannan and Taha [5] introduced a certain subclass of bi-univalent functions for class \( \Sigma \). In recent years, Srivastava et al. [6], Frasin and Aouf [7], Altinkaya and Yalcin [8, 9], and Hayami and Owa [10] studied the various subclasses of analytic and bi-univalent function. For a brief history, see [11].

In [12], Faber introduced Faber polynomials, and after that, Gong [13] studied Faber polynomials in geometric function theory. In their published works, some contributions have been made to finding the general coefficient bounds \( |a_n| \) by applying Faber polynomial expansions. By using Faber polynomial expansions, very little work has been done for the coefficient bounds \( |a_n| \) for \( n \geq 4 \) of Maclaurin’s series. For more studies, see [14–17].

A domain \( U \) is said to be \( m \)-fold symmetric if

\[
    f\left(e^{(2\pi im)/m}z\right) = f^{(2\pi im)}(z), \quad z \in U, f \in \mathcal{A}, m \in \mathbb{N}.
\]
The univalent function $h(z)$ maps the unit disk $\mathcal{U}$ into a region with $m$-fold symmetry and can be defined as

$$h(z) = \sqrt[m]{f(z^m)}, \quad f \in \mathcal{S}.$$  

A function $f$ is said to be $m$-fold symmetric [18] if it has the series expansion of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}.$$  

The class of all $m$-fold symmetric univalent functions is denoted by $\mathcal{S}^m$, and for $m = 1$, then $\mathcal{S}^m = \mathcal{S}$.

In [19], Srivastava et al. proved the inverse $f_m^{-1}$ series expansion for $f \in \Sigma_m$, which is given as follows:

$$g(w) = f_m^{-1}(w) = w - a_{m+m}w^{m+1} + \left( (m+1)a_{m+m} - a_{2m+m} \right) w^{2m+1} - \frac{1}{2} \left( (m+1)(m+2)a_{m+m} - (3m+2)a_{2m+m}a_{2m+1} + a_{3m+1} \right) w^{3m+1} + \ldots$$  

(7)

Here, we will denote $m$-fold symmetric bi-univalent functions by $\Sigma_m^m$. For $m = 1$, equation (7) coincides with equation (3) of the class $\Sigma$. The coefficient problem for $f \in \Sigma_m$ is one of the favorite subjects of geometric function theory in these days (see [20-23]).

The quantum (or $q$-) calculus has great importance because of its applications in several fields of mathematics, physics, and some related areas. The importance of $q$-derivative operator ($D_q$) is pretty recognizable by its applications in the study of numerous subclasses of analytic functions. Initially, in 1908, Jackson [24] introduced a $q$-derivative operator and studied its applications. Further, in [25], Ismail et al. defined a class of $q$-starlike functions; after that, Srivastava [26] studied $q$-calculus in the context of univalent function theory; also, numerous mathematicians studied $q$-calculus in the context of univalent function theory. Further, the $q$-analogue of the Ruscheweyh differential operator was defined by Kanas and Raducanu [27] and Arif et al. [28] discussed some of its applications for multivalent functions while Zhang et al. in [29] studied $q$-starlike functions related with the generalized conic domain. Srivastava et al. published the articles (see [30, 31]) in which they studied the class of $q$-starlike functions. For some more recent investigations about $q$-calculus, we may refer to [32-34].

For a better understanding of the article, we recall some concept details and definitions of the $q$-difference calculus. Throughout the article, we presume that

$$0 < q < 1.$$  

Definition 1. The $q$-factorial $[n]_q!$ is defined as

$$[n]_q! = \prod_{k=1}^{n} [k]_q \quad (n \in \mathbb{N}),$$  

and the $q$-generalized Pochhammer symbol $[t]_{n,q}$, $t \in \mathbb{C}$, is defined as

$$[t]_{n,q} = [t]_q [t+1]_q [t+2]_q \cdots [t+n-1]_q \quad (n \in \mathbb{N}).$$  

Remark 2. For $n = 0$, then $[n]_q! = 1$, and $[t]_{n,q} = 1$.

Definition 3. The $q$-number $[t]_q$ for $q \in (0, 1)$ is defined as

$$[t]_q = \begin{cases} 
1 - q^t & (t \in \mathbb{C}), \\
\sum_{k=0}^{n-1} q^k & (t = n \in \mathbb{N}).
\end{cases}$$  

(11)

Definition 4 (see [24]). The $q$-derivative (or $q$-difference) operator $D_q$ of a function $f$ is defined, in a given subset of $\mathbb{C}$, by

$$D_q f(z) = \begin{cases} 
\frac{f(z) - f(qz)}{1-qz} & \text{for } z \neq 0, \\
f'(0) & \text{for } z = 0,
\end{cases}$$  

(12)

provided that $f'(0)$ exists.

From Definition 4, we can observe that

$$\lim_{q \rightarrow 1^{-}} (D_q f(z)) = \lim_{q \rightarrow 1^{-}} \frac{f(z) - f(qz)}{1-qz} = f'(z),$$  

(13)

for a differentiable function $f$ in a given subset of $\mathbb{C}$. It is also known from (1) and (12) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$  

(14)

Here, in this paper, we use the $q$-difference operator to define new subclasses of $m$-fold symmetric analytic and bi-univalent functions and then apply the Faber polynomial expansion technique to determine the general coefficient bounds $|a_{mk+1}|$ and initial coefficient bounds $|a_{m+1}|$ as well as Fekete-Szegö inequalities.

Definition 5. A function $f \in \Sigma_m$ is said to be in the class $\mathcal{R}_b(\varphi, m, q)$ if and only if

$$1 + \frac{1}{b} (D_q f(z) - 1) < \varphi(z),$$  

(15)

$$1 + \frac{1}{b} (D_q g(w) - 1) < \varphi(w),$$

where $\varphi \in \mathcal{P}$, $b \in \mathbb{C} \setminus \{0\}$, and $f, w \in \mathcal{U}$, and $g(w) = f_m^{-1}(w)$ is defined by (7).
Remark 6. For \( q \to 1^- \) and \( m = 1 \), then the class \( \mathcal{R}_b(\varphi, m, q) \) reduces into the class \( \mathcal{R}_b(\varphi) \) introduced by Hamidi and Jahangiri in [35].

Definition 7. A function \( f \in \Sigma_m \) is said to be in the class \( \mathcal{S}^+_{\Sigma_m}(\varphi, q) \) if and only if

\[
\frac{zDq f(z)}{f(z)} < \varphi(z),
\]

\[
\frac{wDq g(w)}{g(w)} < \varphi(w),
\]

where \( \varphi \in \mathcal{P} \), \( b \in \mathbb{C} \setminus \{0\} \), and \( \{x, w\} \in \mathcal{H} \), and \( g(w) = f^{-1}(w) \) is defined by (7).

Remark 8. For \( q \to 1^- \), \( m = 1 \), and \( \varphi(z) = (1 + Az)/(1 + Bz) \), then the class \( \mathcal{S}^+_{\Sigma_m}(\varphi, q) \) reduces into the class \( \mathcal{S}(A, B) \), introduced by Hamidi and Jahangiri in [36].

2. Main Results

Using the Faber polynomial expansion of functions \( f \in \mathcal{A} \) of the form (1), the coefficients of its inverse map \( g = f^{-1} \) may be expressed as [15] given by

\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n!} K_{n-1}^m(a_2, a_3, \ldots) w^n,
\]

for an expansion of \( K_{n-1}^m \) (see [37]). In particular, the first three terms of \( K_{n-1}^m \) are

\[
\begin{align*}
\frac{1}{2} K_1^2 &= -a_2, \\
\frac{1}{3} K_2^3 &= 2a_2^2 - a_3, \\
\frac{1}{4} K_3^4 &= -(5a_2^2 - 5a_2a_3 + a_4).
\end{align*}
\]

In general, for any \( p \in \mathbb{N} \) and \( n \geq 2 \), an expansion of \( K_{n-1}^p \) is as (see [15])

\[
K_{n-1}^p = pa_n + \frac{p(p - 1)}{2} E_{n-1}^2 + \frac{p!}{(p - 3)!} E_{n-1}^3 + \cdots + \frac{p!}{(p - n + 1)!} E_{n-1}^n,
\]

where \( E_{n-1}^p = E_{n-1}^p(a_2, a_3, \ldots) \), and by [37],

\[
E_{m-1}^m(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1! \cdots \mu_n!}, \quad \text{for } m \leq n,
\]

while \( a_1 = 1 \), and the sum is taken over all nonnegative integers \( \mu_1, \ldots, \mu_n \) satisfying

\[
\mu_1 + \mu_2 + \cdots + \mu_n = m, \quad \mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n - 1.
\]

Evidently, \( E_{m-1}^m(a_2, \ldots, a_n) = a_2^{m-1} \) (see [14]), or equivalently,

\[
E_{m-1}^m(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1! \cdots \mu_n!}, \quad \text{for } m \leq n,
\]

while \( a_1 = 1 \), and the sum is taken over all nonnegative integers \( \mu_1, \ldots, \mu_n \) satisfying

\[
\mu_1 + \mu_2 + \cdots + \mu_n = m, \quad \mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n - 1.
\]

Similarly, using the Faber polynomial expansion of functions \( f \in \mathcal{A} \) of the form (6), that is,

\[
f(z) = z + \sum_{k=1}^{\infty} K_k^m(a_2, a_3, \ldots, a_{k+1}) z^{mk+1}.
\]

The coefficients of its inverse map \( g = f^{-1} \) may be expressed as

\[
g(z) = f^{-1}(z) = w + \sum_{k=1}^{\infty} \frac{1}{mk + 1} K_k^{mk+1} (a_{mk+1}, a_{mk+2}, \ldots, a_{mk+k}) w^{mk+1}.
\]

Theorem 9. For \( b \in \mathbb{C} \setminus \{0\} \), let \( f \in \mathcal{R}_b(\varphi, m, q) \) be given by (6), and if \( a_{mk+j} = 0 \), \( 1 \leq j \leq k-1 \), then

\[
|a_{mk+1}| \leq \frac{|b|}{1 + mk}, \quad \text{for } k \geq 2.
\]

Proof. By definition, for the function \( f \in \mathcal{R}_b(\varphi, m, q) \) of the form (6), we have

\[
1 + \frac{1}{b} D_q f(z) - 1 = 1 + \sum_{k=1}^{\infty} \frac{(1 + mk)q}{b} a_{mk+1} z^{mk},
\]

and for its inverse map \( g = f^{-1} \), we have

\[
1 + \frac{1}{b} D_q g(w) - 1 = 1 + \sum_{k=1}^{\infty} \frac{(1 + mk)q}{b} A_{mk+1} w^{mk},
\]

where

\[
A_{mk+1} = \frac{1}{mk + 1} K_{mk+1}^{-(mk+1)} (a_{mk+1}, a_{mk+2}, \ldots, a_{mk+k}), \quad k \geq 1.
\]
On the other hand, since \( f \in \mathcal{B}_b(\phi, m, q) \) and \( g = f^m \in \mathcal{B}_b(\phi, m, q) \) by definition, we have

\[
p(z) = c_1z^m + c_2z^{2m} + \cdots = \sum_{k=1}^{\infty} c_k z^{mk},
\]

\[
q(w) = d_1w^m + d_2w^{2m} + \cdots = \sum_{k=1}^{\infty} d_k w^{mk},
\]

where

\[
\phi(p(z)) = 1 + \sum_{k=1}^{a_{mk+1}} q_k \phi_1(c_1, c_2, \ldots, c_k) z^{mk},
\]

\[
\phi(q(w)) = 1 + \sum_{k=1}^{a_{mk+1}} q_k \phi_1(d_1, d_2, \ldots, d_k) w^{mk}.
\]

Comparing the coefficients of (27) and (31), we have

\[
\frac{1}{b^1}[1 + mk] q_{a_{mk+1}} = \sum_{l=1}^{k-1} q_k \phi_1(c_1, c_2, \ldots, c_k).
\]

Similarly, comparing coefficients of (28) and (32), we have

\[
\frac{1}{b^1}[1 + mk] q_{a_{mk+1}} = \sum_{l=1}^{k-1} q_k \phi_1(d_1, d_2, \ldots, d_k).
\]

Note that for \( a_{mk+1} = 0, 1 \leq j \leq k - 1 \), we have

\[
A_{mk+1} = -a_{mk+1},
\]

and so

\[
\frac{1}{b^1}[1 + mk] q_{a_{mk+1}} = \phi_1 c_k,
\]

\[
-\frac{1}{b^1}[1 + mk] q_{a_{mk+1}} = \phi_1 d_k.
\]

Now taking the absolute of (36) and (37) and using the fact that \( |\phi_1| \leq 2 \), \( |c_k| \leq 1 \), and \( |d_k| \leq 1 \), we have

\[
|a_{mk+1}| \leq \frac{|b|}{|1 + mk| q_{a_{mk+1}}} |\phi_1 c_k| = \frac{|b|}{|1 + mk| q_{a_{mk+1}}} |\phi_1 d_k|,
\]

\[
|a_{mk+1}| \leq \frac{2|b|}{1 + mk| q_{a_{mk+1}}},
\]

which completes the proof of Theorem 9.

Corollary 10. For \( b \in \mathbb{C} \setminus \{0\} \), let \( f \in \mathcal{B}_b(\phi, q) \), and if \( a_{j+1} = 0 \), \( 1 \leq j \leq n \), then

\[
|a_{n+1}| \leq \frac{2|b|}{|n+1| q_{a_{n+1}}}, \quad \text{for } n \geq 3.
\]

For \( q \rightarrow 1^- \), \( m = 1 \), and \( k = n - 1 \), in Theorem 9, we obtain the following known corollary.

Corollary 11 (see [35]). For \( b \in \mathbb{C} \setminus \{0\} \), let \( f \in \mathcal{B}_b(\phi, q) \), and if \( a_{j+1} = 0 \), \( 1 \leq j \leq n \), then

\[
|a_{n+1}| \leq \frac{2|b|}{n^{a_{n+1}}}, \quad \text{for } n \geq 3.
\]

Theorem 12. For \( b \in \mathbb{C} \setminus \{0\} \), let \( f \in \mathcal{B}_b(\phi, m, q) \) be given by (6), and then

\[
|a_{m+1}| \leq \begin{cases} 
\frac{2|b|}{|m+1| q_{a_{m+1}}} & \text{if } |b| < \frac{8}{(m+1)|m+1| q_{a_{m+1}}} \\
\frac{8|b|}{(m+1)|m+1| q_{a_{m+1}}} & \text{if } |b| \geq \frac{8}{(m+1)|m+1| q_{a_{m+1}}} 
\end{cases}
\]

\[
|a_{2m+1}| \leq \begin{cases} 
\frac{2|b|}{|m+1| q_{a_{2m+1}}} + \frac{2(m+1)|b|^2}{(m+1)|m+1| q_{a_{2m+1}}} & \text{if } |b| < \frac{2}{|m+1| q_{a_{2m+1}}} \\
\frac{4|b|}{|m+1| q_{a_{2m+1}}} & \text{if } |b| \geq \frac{2}{|m+1| q_{a_{2m+1}}}
\end{cases}
\]

\[
|a_{2m+1} - (m+1)a_{m+1}| \leq \frac{4|b|}{|m+1| q_{a_{m+1}}},
\]

\[
|a_{2m+1} - (m+1)a_{m+1}| \leq \frac{2|b|}{|m+1| q_{a_{m+1}}},
\]

Proof. Replacing \( k \) by 1 and 2 in (33) and (34), respectively, we have

\[
\frac{1}{b^1}[m+1] q_{a_{m+1}} = \phi_1 c_1,
\]

\[
\frac{1}{b^1}[m+1] q_{a_{m+1}} = \phi_1 c_2 + \phi_2 c_1,
\]

\[
\frac{1}{b^1}[m+1] q_{a_{m+1}} = \phi_1 c_1 + \phi_2 c_1 + \phi_2 c_1,
\]

\[
\frac{1}{b^1}[m+1] q_{a_{m+1}} = \phi_1 d_1 + \phi_2 d_1,
\]

\[
\frac{1}{b^1}[m+1] q_{a_{m+1}} = \phi_1 d_2 + \phi_2 d_2,
\]

From (42) and (44), we have

\[
|a_{m+1}| \leq \frac{|b|}{|m+1| q_{a_{m+1}}} |\phi_1 c_1| = \frac{|b|}{|m+1| q_{a_{m+1}}} |\phi_1 d_1| \leq \frac{2|b|}{|m+1| q_{a_{m+1}}}. \quad \text{(46)}
\]
Adding (43) and (45), we have
\[
a_{m+1}^2 = \frac{b \{ \varphi_1 (c_2 + d_2) + \varphi_2 (c_2^2 + d_2^2) \}}{(m+1)[2m+1]_q}.
\]
(47)

Taking the absolute value (47), we have
\[
|a_{m+1}| \leq \sqrt{\frac{8|b|}{(m+1)[2m+1]_q}}.
\]
(48)

Now, the bounds given for $|a_{m+1}|$ can be justified since
\[
|b| < \sqrt{\frac{8|b|}{(m+1)[2m+1]_q}}, \quad \text{for } |b| < \frac{8}{(m+1)[2m+1]_q}.
\]
(49)

From (43), we have
\[
|a_{2m+1}| = \frac{|b| \{ \varphi_1 c_2 + \varphi_2 c_2^2 \}}{2[2m+1]_q} \leq \frac{4|b|}{2[2m+1]_q}.
\]
(50)

Next, we subtract (45) from (43), and we have
\[
\frac{2[2m+1]_q}{b} \left\{ a_{2m+1} - \frac{(m+1)}{2} a_{m+1} \right\} = \varphi_1 (c_2 - d_2) + \varphi_2 (c_2^2 - d_2^2) = \varphi_1 (c_2 - d_2),
\]
(51)

or
\[
a_{2m+1} = \frac{(m+1)}{2} a_{m+1} + \frac{\varphi_1 b (c_2 - d_2)}{2[2m+1]_q}.
\]
(52)

After some simple calculation and by taking the absolute, we have
\[
|a_{2m+1}| \leq \frac{|\varphi_1||b||c_2 - d_2|}{2(2m+1)} + \frac{(m+1)}{2} |a_{m+1}|.
\]
(53)

Using the assertion (46), we have
\[
|a_{2m+1}| \leq \frac{2|b|}{2[2m+1]_q} + \frac{2(m+1)|b|^2}{(m+1)[2m+1]_q}.
\]
(54)

From (50) and (54), we note that
\[
\frac{2|b|}{2[2m+1]_q} + \frac{2(m+1)|b|^2}{(m+1)[2m+1]_q} \leq 4|b|, \quad \text{if } |b| < \frac{2}{2[2m+1]_q}.
\]
(55)

Now, we rewrite (45) as
\[
\frac{1}{b} [2m+1]_q \left\{ (m+1) a_{m+1}^2 - a_{2m+1} \right\} = \varphi_1 d_2 + \varphi_2 d_1.
\]
(56)

Taking the absolute value, we have
\[
|a_{2m+1} - (m+1) a_{m+1}^2| \leq \frac{4|b|}{[2m+1]_q}.
\]
(57)

Finally, from (51), we have
\[
\frac{2[2m+1]_q}{b} \left\{ a_{2m+1} - \frac{(m+1)}{2} a_{m+1}^2 \right\} = \varphi_1 (c_2 - d_2).
\]
(58)

Taking the absolute value, we have
\[
|a_{2m+1} - \frac{(m+1)}{2} a_{m+1}^2| \leq \frac{2|b|}{[2m+1]_q}.
\]
(59)

\[\square\]

For $m = 1$ and $k = n - 1$, in Theorem 12, we obtain the following corollary.

**Corollary 13.** For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathcal{B}_b(\varphi, q)$ be given by (1), and then
\[
|a_2| \leq \begin{cases} 
\frac{2|b|}{[2]_q}, & \text{if } |b| < \frac{4}{[3]_q}, \\
\frac{4|b|}{[3]_q}, & \text{if } |b| \geq \frac{4}{[3]_q},
\end{cases}
\]
(60)

\[
|a_j| \leq \begin{cases} 
\frac{2|b|}{[3]_q} + \frac{4|b|^2}{(2)_q^2}, & \text{if } |b| < \frac{2}{[3]_q}, \\
\frac{4|b|}{[3]_q}, & \text{if } |b| \geq \frac{2}{[3]_q},
\end{cases}
\]
(60)

\[
|a_3 - 2a_{m+1}^2| \leq \frac{4|b|}{[3]_q},
\]

\[
|a_{2m+1} - a_{m+1}^2| \leq \frac{2|b|}{[3]_q}.
\]

For $q \to 1^-$, $m = 1$, and $k = n - 1$, in Theorem 12, we obtain the following corollary.
Corollary 14 (see [35]). For $b \in \mathbb{C} \setminus \{0\}$, let $f \in \mathcal{B}_k(\phi)$ be given by (1), and then

$$
|a_2| \leq \begin{cases} 
|b|, & \text{if } |b| < \frac{4}{3}, \\
\frac{4|b|}{3}, & \text{if } |b| \geq \frac{4}{3},
\end{cases}
$$

$$
|a_3| \leq \begin{cases} 
\frac{2|b|}{3} + |b|^2, & \text{if } |b| < \frac{2}{3}, \\
\frac{4|b|}{3}, & \text{if } |b| \geq \frac{2}{3},
\end{cases}
$$

For the inverse map $g = f_m^{-1} \in \mathcal{S}_{\varphi}^+(\mathbb{C}, q)$, we obtain

$$
\frac{zD_g(w)}{g(w)} = 1 - \sum_{k=1}^{\infty} F_k(b_{m+1}, b_{2m+1}, \ldots, b_{mk+1})w^{mk},
$$

where

$$
A_{mk+1} = \frac{1}{mk + 1} K^{(mk+1)}_k(a_{m+1}, a_{2m+1}, \ldots, a_{mk+1}), \quad k \geq 1.
$$

On the other hand, since $f \in \mathcal{S}_{\varphi}^+(\mathbb{C}, q)$ and $g = f_m^{-1} \in \mathcal{S}_{\varphi}^+(\mathbb{C}, q)$ by definition, we have

$$
p(z) = \epsilon_1 z^m + \epsilon_2 z^{2m} + \cdots = \sum_{k=1}^{\infty} \epsilon_k z^{mk},
$$

$$
q(w) = d_1 w^m + d_2 w^{2m} + \cdots = \sum_{k=1}^{\infty} d_k w^{mk},
$$

where

$$
\varphi(p(z)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_k K^l_k(\epsilon_1, \epsilon_2, \ldots, \epsilon_k) z^{mk},
$$

$$
\varphi(q(w)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_k K^l_k(d_1, d_2, \ldots, d_k) w^{mk}.
$$

Comparing the coefficients of (63) and (70), we have

$$
-[mk] q_{amk+1} = \sum_{j=1}^{k-1} \varphi_k K^j_k(\epsilon_1, \epsilon_2, \ldots, \epsilon_k).
$$

Similarly, comparing the coefficients of (67) and (71), we have

$$
-[mk] q_{bmk+1} = \sum_{j=1}^{k-1} \varphi_k K^j_k(d_1, d_2, \ldots, d_k).
$$

Note that for $a_{mj+1} = 0$, $1 \leq j \leq k - 1$, we have

$$
A_{mk+1} = -a_{mk+1},
$$

and so

$$
-[mk] q_{amk+1} = \varphi_1 \epsilon_1,
$$

$$
[mk] q_{bmk+1} = \varphi_1 d_1.
$$

Taking the absolute values of (75) and (76) and using the fact that $|\varphi_1| \leq 2$, $|\epsilon_1| \leq 1$, and $|d_1| \leq 1$, we have
Theorem 17. Let \( f \in \mathcal{S}^\ast (\varphi, q) \) be given by (6), and then

\[
|a_{m+1}| \leq \frac{2}{|m|_q},
\]

\[
|a_{2m+1}| \leq \frac{4(m+1)}{m|2m|_q} + \frac{2}{|2m|_q},
\]

\[
|a_{2m+1} - \frac{|m|_q(2m+1)}{|2m|_q} a_{m+1}^2| \leq \frac{4}{|2m|_q},
\]

\[
|a_{2m+1} - \frac{|m|_q(m+1)}{|2m|_q} a_{m+1}^2| \leq \frac{2}{|2m|_q}.
\]

Proof. Replacing \( k \) by 1 and 2 in (72) and (73), respectively, we have

\[
|m|_q a_{m+1} = \varphi_1 c_1,
\]

\[
|2m|_q a_{2m+1} - |m|_q a_{m+1}^2 = \varphi_1 c_2 + \varphi_2 c_1^2,
\]

\[
-|m|_q a_{m+1} = \varphi_1 d_1,
\]

\[
|m|_q(2m+1) a_{m+1}^2 - |2m|_q a_{2m+1} = \varphi_1 d_2 + \varphi_2 d_1^2.
\]

Adding (80) and (82), we have

\[
|a_{m+1}| \leq \frac{1}{|m|_q} |\varphi_1 c_1| = \frac{1}{|m|_q} |\varphi_1 d_1| \leq \frac{2}{|m|_q}.
\]

Adding (81) and (83), we have

\[
a_{m+1}^2 = \frac{\varphi_1 (c_2 + d_2) + \varphi_2 (c_1^2 + d_1^2)}{2|m|_q}.
\]

Taking the absolute value (85), we have

\[
|a_{m+1}| \leq \frac{2}{\sqrt{|m|_q}}.
\]

Next, we subtract (83) from (81), and we have

\[
\left\{ 2|m|_q a_{2m+1} - 2|m|_q (m+1) a_{m+1}^2 \right\}
\]

\[
= \varphi_1 (c_2 - d_2) + \varphi_2 (c_1^2 - d_1^2),
\]

or

\[
a_{2m+1} = \frac{|m|_q(m+1)}{\frac{2m}{|2m|_q}} a_{m+1}^3 + \frac{|\varphi_1 (c_2 - d_2)|}{2|m|_q}.
\]

After some simple calculation of (88) and by taking the absolute, we have

\[
|a_{2m+1}| \leq \frac{|\varphi_1||c_2 - d_2|}{|2m|_q} + \frac{|m|_q(m+1)}{|2m|_q} |a_{m+1}^2|.
\]

Using the assertion (86), we have

\[
|a_{2m+1}| \leq \frac{4(m+1)}{m|2m|_q} + \frac{2}{|2m|_q}.
\]

For the third part, we rewrite (83) as

\[
|m|_q(2m+1) a_{m+1}^2 - |2m|_q a_{2m+1} = |\varphi_1 d_2 + \varphi_2 d_1^2|.
\]

Taking the absolute value, we have

\[
|a_{2m+1} - \frac{|m|_q(2m+1)}{|2m|_q} a_{m+1}^2| \leq \frac{4}{|2m|_q}.
\]

Finally, from (87), we have

\[
2|m|_q a_{2m+1} - |m|_q(m+1) a_{m+1}^2 = |\varphi_1(c_2 - d_2)|.
\]

Taking the absolute value, we have

\[
|a_{2m+1} - \frac{|m|_q(m+1)}{|2m|_q} a_{m+1}^2| \leq \frac{2}{|2m|_q}.
\]

For \( q \rightarrow 1^+ \), \( m = 1 \), and \( k = n - 1 \), in Theorem 17, we get the following corollary.
Corollary 18. Let \( f \in \mathcal{S}^*(\varphi) \) be given by (1), and then
\[
|a_2| \leq 2,
\]
\[
|a_3| \leq 5,
\]
\[
|a_3 - \frac{3}{2}a_1^2| \leq 2,
\]
\[
|a_3 - a_2^2| \leq 1.
\]

3. Conclusion
In this paper, we have applied \( q \)-calculus operator theory to define some new subclasses of \( m \)-fold symmetric analytic and bi-univalent functions in open unit disk \( \mathbb{U} \) and used the Faber polynomial expansion to find upper bounds \( |a_{m+1}| \) and initial coefficient bounds \( |a_{m+1}| \) and \( |a_{2m+1}| \) as well as Fekete-Szego inequalities for the functions belonging to newly defined subclasses of \( m \)-fold symmetric analytic and bi-univalent function. Also, we highlighted some new and known consequences of our main results.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no competing interests.

Authors’ Contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

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References
[1] P. L. Duren, Univalent Functions, Springer-Verlag, New York, NY, USA, 1983.
[2] M. Lewin, “On a coefficient problem for bi-univalent functions,” Proceedings of the American Mathematical Society, vol. 18, no. 1, pp. 63–68, 1967.
[3] D. A. Brannan and J. Clunie, “Aspects of contemporary complex analysis,” Proceedings of the NATO Advanced Study Institute Held at University of Durham, Academic Press, New York, NY, USA, 1979.
[4] E. Netanyahu, “The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in \( |z| < 1 \),” Archive for Rational Mechanics and Analysis, vol. 32, pp. 100–112, 1967.
[5] D. A. Brannan and T. S. Taha, “On some classes of bi-univalent function,” Studia Universitatis Babeş-Bolyai Mathematica, vol. 31, no. 2, pp. 70–77, 1986.
[6] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, “Certain subclasses of analytic and bi-univalent functions,” Applied Mathematics Letters, vol. 23, no. 10, pp. 1188–1192, 2010.
[7] B. A. Frasin and M. K. Aouf, “New subclasses of bi-univalent functions,” Applied Mathematics Letters, vol. 24, no. 9, pp. 1569–1573, 2011.
[8] S. Altinkaya and S. Yalçın, “Coefficient bounds for a subclass of bi-univalent functions,” Journal of Pure and Applied Mathematics, vol. 6, no. 2, pp. 180–185, 2015.
[9] S. Altinkaya and S. Yalçın, “Borne des coefficients des développements en polynômes de Faber d’une sous-classe de fonctions bi-univalentes,” Comptes Rendus Mathematique, vol. 353, no. 12, pp. 1075–1080, 2015.
[10] T. Hayami and S. Owa, “Coefficient bounds for bi-univalent functions,” Panamerican Mathematical Journal, vol. 22, no. 4, pp. 15–26, 2012.
[11] S. Hussain, S. Khan, M. A. Zaighum, M. Darus, and Z. Shareef, “Coefficient bounds for certain subclass of bi-univalent functions associated with Ruscheweyh \( q \)-differential operator,” Journal of Complex Analysis, vol. 2017, Article ID 2826514, 9 pages, 2017.
[12] G. Faber, „Über polynomische Entwicklung,” Mathematische Annalen, vol. 57, no. 3, pp. 389–408, 1903.
[13] S. Gong, “The Bieberbach conjecture, translated from the 1989 Chinese original and revised by the author,” in AMS/IP Studies in Advanced Mathematics, Journal of the American Mathematical Society, Providence, RI, USA, 1999.
[14] H. Airault, “Remarks on Faber polynomials,” International Mathematical Forum, vol. 3, no. 9, pp. 449–456, 2008.
[15] H. Airault and H. Bouali, “Differential calculus on the Faber polynomials,” Bulletin des Sciences Mathematiques, vol. 130, no. 3, pp. 179–222, 2006.
[16] S. Bulut, “Estimations des coefficients polynomes de Faber pour une sous-classe complete de fonctions analytiques bi-univalentes,” Comptes Rendus Mathematique, vol. 352, no. 6, pp. 479–484, 2014.
[17] S. Yalçın, S. Khan, and S. Hussain, “Faber polynomial coefficients estimates of bi-univalent functions associated with generalized Salagean \( q \)-differential operator,” Konuralp Journal of Mathematics, vol. 7, no. 1, pp. 25–32, 2019.
[18] W. Koepf, “Coefficients of symmetric functions of bounded boundary rotation,” Proceedings of the American Mathematical Society, vol. 105, no. 2, pp. 324–329, 1989.
[19] H. M. Srivastava, S. Sivasubramanian, and R. Sivakumar, “Initial coefficient bounds for a subclass of \( m \)-fold symmetric bi-univalent functions,” Tbilisi Mathematical Journal, vol. 7, no. 2, pp. 1–10, 2014.
[20] A. Akgul, “A new general subclass of \( m \)-fold symmetric bi-univalent functions given by subordination,” Turkish Journal of Mathematics, vol. 43, no. 3, pp. 1688–1698, 2019.
[21] A. Akgul, “On the coefficient estimates of analytic and bi-univalent \( m \)-fold symmetric functions,” Mathematical Aeterna, vol. 7, no. 3, pp. 253–260, 2017.
[22] S. G. Hamidi and J. M. Jahangiri, “Unpredictability of the coefficients of \( m \)-fold symmetric bi-starlike functions,” International Journal of Mathematics, vol. 25, no. 7, pp. 1450064–1450068, 2014.
[23] S. Sümer Eker, “Coefficient bounds for subclasses of \( m \)-fold symmetric bi-univalent functions,” Turkish Journal of Mathematics, vol. 40, no. 3, pp. 641–646, 2016.
[24] F. H. Jackson, “On $q$-functions and a certain difference operator,” Transactions of the Royal Society of Edinburgh, vol. 46, pp. 253–281, 1909.

[25] M. E. H. Ismail, E. Merkes, and D. Styer, “A generalization of starlike functions,” Complex Variables, Theory and Application: An International Journal, vol. 14, no. 1-4, pp. 77–84, 1990.

[26] H. M. Srivastava, “Operators of basic (or $q$-) calculus and fractional $q$-calculus and their applications in geometric function theory of complex analysis,” Iranian Journal of Science and Technology, Transactions A: Science, vol. 44, pp. 327–344, 2020.

[27] S. Kanas and D. Raducanu, “Some class of analytic functions related to conic domains,” Mathematica Slovaca, vol. 64, no. 5, pp. 1183–1196, 2014.

[28] M. Arif, M. Raza, K. I. Noor, and S. N. Malik, “On strongly Bazilevic functions associated with generalized Robertson functions,” Mathematical and Computer Modelling, vol. 54, no. 5-6, pp. 1608–1612, 2011.

[29] X. Zhang, S. Khan, S. Hussain, H. Tang, and Z. Shareef, “New subclass of $q$-starlike functions associated with generalized conic domain,” AIMS Mathematics, vol. 5, no. 5, pp. 4830–4848, 2020.

[30] H. M. Srivastava, B. Khan, N. Khan, and Q. A. Ahmad, “Coefficient inequalities for $q$-starlike functions associated with the Janowski functions,” Hokkaido Mathematical Journal, vol. 48, no. 2, pp. 407–425, 2019.

[31] H. Tang, S. Khan, S. Hussain, and N. Khan, "Some general classes of $q$-starlike functions associated with the Janowski functions," Symmetry, vol. 11, no. 2, p. 292, 2019.

[32] A. Akgul and F. M. Sakar, "A certain subclass of bi-univalent analytic functions introduced by means of the $q$-analogue of Noor integral operator and Horadam polynomials," Turkish Journal of Mathematics, vol. 43, no. 5, pp. 2275–2286, 2019.

[33] H. Tang, S. Khan, S. Hussain, and N. Khan, "Hankel and Toeplitz determinant for a subclass of multivalent $q$-starlike functions of order $\alpha$," AIMS Mathematics, vol. 6, no. 6, pp. 5421–5439, 2021.

[34] Z.-G. Wang, S. Hussain, M. Naeem, T. Mahmood, and S. Khan, "A subclass of univalent functions associated with $q$-analogue of Choi-Saigo-Srivastava operator," Hacettepe Journal of Mathematics and Statistics, vol. 49, no. 4, pp. 1471–1479, 2020.

[35] S. G. Hamidi and J. M. Jahangiri, "Faber polynomial coefficient estimates for bi-univalent functions defined by subordinations," Bulletin of the Iranian Mathematical Society, vol. 41, no. 5, pp. 1103–1119, 2014.

[36] S. G. Hamidi and J. M. Jahangiri, "Polynomes de Faber et coefficients des fonctions bi-subordonnees," Comptes Rendus Mathematique, vol. 354, no. 4, pp. 365–370, 2016.

[37] H. Airault, "Symmetric sums associated to the factorizations of Grunsky coefficients," in Conference, Groups and Symmetries, pp. 3–16, Montreal, Canada, April 2007.