Optimal construction of a layer-ordered heap∗

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Abstract
The layer-ordered heap (LOH) is a simple, recently proposed data structure used in optimal selection on $X + Y$, the algorithm with the best known runtime for selection on $X_1 + X_2 + \cdots + X_m$, and the fastest method in practice for computing the most abundant isotope peaks in a chemical compound. Here, we introduce a few algorithms for constructing LOHs, analyze their complexity, and demonstrate that one algorithm is optimal for building a LOH of any rank $\alpha$. These results are shown to correspond with empirical experiments of runtimes when applying the LOH construction algorithms to a common task in machine learning.

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1 Introduction

Layer-ordered heaps (LOHs) are used in algorithms that perform optimal selection on $A + B$ [8], algorithms with the best known runtime for selection on $X_1 + X_2 + \cdots + X_m$ [6], and the fastest known method for computing the most abundant isotopes of a compound [7].

A LOH of rank $\alpha$ consists of $\ell$ layers of values $L_0 \leq L_1 \leq L_2 \leq \cdots \leq L_{\ell-1}$, where each $L_i$ is an unordered array and the ratio of the sizes, $|L_{i+1}|/|L_i|$, tends to $\alpha$ as the index, $i$, tends to infinity. One possible way to achieve this is to have the exact size of each layer, $|L_i|$, be $p_i - p_{i-1}$ where $p_i$, the $i^{th}$ pivot, is calculated as $p_i = \lceil \sum_{j=0}^{i} \alpha^j \rceil$. The size of the last layer is the difference between the size of the array and the last pivot. Figure 1 depicts a LOH of rank $\alpha = 2$.

Soft heaps [5] are qualitatively similar in that they achieve partial ordering in theoretically efficient time; however, the disorder from a soft heap occurs in a less regular manner, requiring client algorithms to cope with a bounded number of “corrupt” items. Furthermore, they are less efficient in practice because of the discontinuous data structures and the greater complexity of implementation.

Throughout this paper, the process of constructing a LOH of rank $\alpha$ from an array of length $n$ will be denoted “LOHification.” While LOHify with $\alpha = 1$ is equivalent to comparison sort and $\alpha \gg 1$ can be performed in $O(n)$ [6], the optimal runtime for an arbitrary $\alpha$ is unknown. Likewise, no optimal LOHify algorithm is known for arbitrary $\alpha$.

Here we derive a lower bound runtime for LOHification, describe a few algorithms for LOHification, prove their runtimes, and demonstrate optimality of one method for any $\alpha$. We then demonstrate the practical performance of these methods on a non-parametric stats test.

| 1 | 3 | 2 | 7 | 6 | 4 | 5 | 9 | 10 | 8 | 11 | 13 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $L_0$ | $L_1$ | $L_2$ | $L_3$ |

Figure 1: A LOH of rank 2 Pivot indices are shaded in gray. Notice that the last layer is not full.

2 Methods

2.1 A lower bound on LOHification In this section we will prove an asymptotic lower bound on the complexity of constructing a LOH in terms of $n$ and $\alpha$ by first proving bounds on variables and then using those to bound the process as a whole.

2.1.1 Bounds on variables

**Lemma 2.1. (Upper bound on number of layers)** An upper bound on the number of layers, $\ell$, in a LOH of $n$ elements is $\log_\alpha(n \cdot (\alpha - 1) + 1)$.

Proof. Because the final pivot can be no more than $n$, the size of our array, we have the following inequality:

\[
\left[ \sum_{i=0}^{\ell-2} \alpha^i \right] \leq n
\]

\[
\left[ \sum_{i=0}^{\ell-2} \alpha^i \right] \leq n
\]

\[
\frac{\alpha^{\ell-1} - 1}{\alpha - 1} \leq n
\]

\[
\alpha^{\ell-1} - 1 \leq n \cdot (\alpha - 1)
\]

\[
\alpha^{\ell-1} - 1 \leq n \cdot (\alpha - 1) + 1
\]

\[
\ell - 1 \leq \log_\alpha(n \cdot (\alpha - 1) + 1)
\]

\[
\ell \leq \log_\alpha(n \cdot (\alpha - 1) + 1) + 1
\]

**Lemma 2.2. (Lower bound on number of layers)** A lower bound on the number of layers, $\ell$, in a LOH of $n$ elements is $\log_\alpha(n \cdot (\alpha - 1) + 1)$.

Proof. Because an additional pivot (after the final pivot) must be more than $n$, the size of our array, we have the following inequality:

\[
\left[ \sum_{i=0}^{\ell-1} \alpha^i \right] > n
\]

\[
\sum_{i=0}^{\ell-1} \alpha^i \geq n, \text{ because } n \text{ is discrete;}
\]

\[
\frac{\alpha^{\ell} - 1}{\alpha - 1} \geq n
\]

\[
\alpha^{\ell} - 1 \geq n \cdot (\alpha - 1)
\]

\[
\alpha^{\ell} \geq n \cdot (\alpha - 1) + 1
\]

\[
\ell \geq \log_\alpha(n \cdot (\alpha - 1) + 1)
\]

\[
\ell > \log_\alpha(n \cdot (\alpha - 1))
\]

**Lemma 2.3. (Asymptotic number of layers)** The number of layers as $n$ grows is asymptotic to $\log_\alpha(n \cdot (\alpha - 1) + 1)$.

Proof. For $\alpha = 1$, the number of layers is $n$.

\[
\lim_{\alpha \to 1} \log_\alpha(n \cdot (\alpha - 1) + 1)
\]
\[
\lim_{\alpha \to 1} \frac{\log(n \cdot (\alpha - 1) + 1)}{\log(\alpha)} = n
\]

For \( \alpha > 1 \) we know \( \log_\alpha (n \cdot (\alpha - 1) + 1) \leq \ell \leq \log_\alpha (n \cdot (\alpha - 1) + 1) + 1 \).

\[
\lim_{n \to \infty} \frac{\log(n \cdot (\alpha - 1) + 1) + 1}{\log(n \cdot (\alpha - 1) + 1)} = \lim_{n \to \infty} \frac{\log(n \cdot (\alpha - 1) + 1) + 1}{\log(n \cdot (\alpha - 1) + 1)} = 1
\]

\[\square\]

**Lemma 2.4. (Upper bound on size of layers)** An upper bound on the size of layer \( i \) is \( |L_i| \leq \lfloor \alpha^i \rfloor \).

**Proof.** \( |L_i| \), as defined above, can be calculated by:

\[
|L_i| = p_i - p_{i+1} = \left\lfloor \sum_{j=0}^{i} \alpha^j \right\rfloor - \left\lfloor \sum_{j=0}^{i-1} \alpha^j \right\rfloor \leq \lfloor \alpha^i \rfloor + \sum_{j=0}^{i-1} \alpha^j - \sum_{j=0}^{i-1} \alpha^j \leq \lfloor \alpha^i \rfloor
\]

\[\square\]

### 2.1.2 Lower bound of LOHification

Here we will show that, for any \( \alpha > 1 \), LOHification is in \( \Omega\left(n \log\left(\frac{1}{\alpha-1}\right) + \frac{n \cdot \log(\alpha)}{\alpha-1}\right) \).

**Theorem 2.1.** \( \forall \alpha > 1 \), \( \text{LOHification} \in \Omega\left(n \log\left(\frac{1}{\alpha-1}\right) + \frac{n \cdot \log(\alpha)}{\alpha-1}\right) \)

**Proof.** If \( \alpha = 1 \), we are sorting, which is known to be in \( \Omega(n \log(n)) \). Hence, for the following derivation, we shall assume that \( \alpha > 1 \). From \( n! \) possible unsorted arrays, LOHification produces one of \( |L_0| \cdot |L_1| \cdots |L_{\ell-1}| \) possible valid results; hence, using an optimal decision tree, \( r(n) \) is in \( \Omega\left(\log_2 \left(\frac{n!}{|L_0| \cdot |L_1| \cdots |L_{\ell-1}|}\right)\right) \); hence,

\[
r(n) \in \Omega\left(\log\left(\frac{n!}{\prod_{i=0}^{\ell-1} |L_i|!}\right)\right)
\]

\[
= \Omega\left(n \log(n) - \sum_{i=0}^{\ell-1} \log(|L_i|!\right)\)
\]

\[
= \Omega\left(n \log(n) - \sum_{i=0}^{\ell-1} \log(|\alpha^i|!\right)\)
\]

\[
= \Omega\left(n \log(n) - \sum_{i=0}^{\ell-1} |\alpha^i| \cdot \log(|\alpha^i|\right)\)
\]

\[
= \Omega\left(n \log(n) - \sum_{i=0}^{\ell-1} |\alpha^i| \cdot \log(|\alpha^i|\right)\) \text{ by Lemma 2.3}
\]

\[
= \Omega\left(n \log(n) - \log(\alpha) \cdot \left(\alpha^{\ell+1} \cdot (\ell-1) \alpha + \alpha^{-\ell} \cdot \ell\right)\)
\]

\[
= \Omega\left(n \log(n) - \log(\alpha) \cdot \left(\frac{\alpha^{\ell+1} \cdot (\ell-1) + \alpha - \alpha^{-\ell} \cdot \ell}{(\alpha-1)^2}\right)\)
\]

\[
= \Omega\left(n \log(n) - \log(\alpha) \cdot \left(\frac{\alpha^{\ell+1} \cdot (\ell-1) + \alpha - \alpha^{-\ell} \cdot \ell}{(\alpha-1)^2}\right)\) \text{ by Lemma 2.3}
\]

\[
= \Omega\left(n \log(n) - \log(\alpha) \cdot \left(\frac{\alpha^{\ell+1} \cdot (\ell-1) + \alpha - \alpha^{-\ell} \cdot \ell}{(\alpha-1)^2}\right)\)
\]

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\[= \frac{(n \cdot (\alpha - 1) + 1) \cdot \alpha \log(\alpha)}{(\alpha - 1)^2} + \frac{\alpha \log(\alpha)}{(\alpha - 1)^2} \]

\[= \Omega\left(n \log(n) - \frac{(n \cdot (\alpha - 1) + 1) \cdot \log(n \cdot (\alpha - 1) + 1) \cdot (\alpha - 1)}{(\alpha - 1)^2} + \frac{\alpha \log(\alpha)}{(\alpha - 1)^2}\right) \]

\[= \Omega\left(n \log(n) - \frac{(n \cdot (\alpha - 1) + 1) \cdot \log(n \cdot (\alpha - 1) + 1) \cdot (\alpha - 1)}{(\alpha - 1)^2} + \frac{\alpha \log(\alpha)}{(\alpha - 1)^2}\right) \]

\[= \Omega\left(n \log(n) - \frac{(n \cdot (\alpha - 1) + 1) \cdot \log(n \cdot (\alpha - 1) + 1)}{\alpha - 1} + \frac{\alpha \log(\alpha)}{\alpha - 1}\right) \]

\[= \Omega\left(n \log\left(\frac{n \cdot (\alpha - 1) + 1}{\alpha - 1}\right) + \frac{\alpha \log(\alpha)}{\alpha - 1}\right) \]

\[= \Omega\left(n \log(n) + \lim_{\alpha \to 1} \frac{n \cdot \log(\alpha) + 1}{1}\right) \]

by L'Hôpital's rule

\[= \Omega(n \log(n) + n) \]

\[= \Omega(n \log(n)) \]

\[\square \]

We have now established LOHification to be in
\[\Omega(n \log(n)) \] trivially LOHifies an array (sorting can be done using any LOHification method by setting \(\alpha = 1\)); note that this also guarantees the LOH property for any \(\alpha \geq 1\), because any partitioning of layers in a sorted array will have \(L_i \leq L_{i+1}\). Hence, LOHification is in \(O(n \log(n))\).

### 2.2 LOHification via sorting

Sorting in \(\Theta(n \log(n))\) trivially LOHifies an array (sorting can be done using any LOHification method by setting \(\alpha = 1\)); note that this also guarantees the LOH property for any \(\alpha \geq 1\), because any partitioning of layers in a sorted array will have \(L_i \leq L_{i+1}\). Hence, LOHification is in \(O(n \log(n))\).

#### 2.2.1 When sorting is optimal \(\alpha = 1\) indicates each layer has \(|L_i| = 1\), meaning an ordering over all elements; this means that sorting must be performed. Thus, for \(\alpha = 1\), sorting is optimal. Furthermore, we can find an \(\alpha^*\) where sorting is optimal for all \(\alpha \leq \alpha^*\). Doing this, we find that, for any constant, \(C > 0\), sorting is optimal for \(\alpha^* \leq 1 + \frac{C}{n}\).

**Theorem 2.3.** For any constant, \(C > 0\), sorting is optimal for \(\alpha \leq 1 + \frac{C}{n}\).

**Proof.** Because decreasing \(\alpha\) can only increase the number of layers (and therefore the work), it suffices to show that sorting is optimal at \(\alpha^* = (1 + \frac{C}{n})\).

\[r(n) \leq \Omega\left(n \log\left(\frac{1}{\alpha^* - 1}\right) + \frac{n \cdot \alpha^* \log(\alpha^*)}{\alpha^* - 1}\right) \]

\[= \Omega\left(n \log\left(\frac{1}{1 + \frac{C}{n}}\right) + \frac{n \cdot (1 + \frac{C}{n}) \log((1 + \frac{C}{n}))}{1 + \frac{C}{n} - 1}\right) \]

\[= \Omega\left(n \log\left(\frac{n \log\left(\frac{C}{n}\right)}{\frac{C}{n}}\right) + \frac{(n^2 + C \cdot n) \log((1 + \frac{C}{n}))}{C}\right) \]

\[= \Omega\left(n \log(n) + \frac{(n^2 + n) \cdot \log\left(1 + \frac{C}{n}\right)}{C}\right) \]

\[\square \]
\[ = \Omega(n \cdot \log(n) + o(n \cdot \log(n))) \text{ Lemma A.4} \]
\[ \subseteq \Omega(n \cdot \log(n)) \]

Therefore,

\[ \text{LOH} \subseteq \Theta(n \cdot \log(n)) \quad \forall \alpha \leq \left(1 + \frac{C}{n}\right) \]

Because sorting is optimal for these values of \(\alpha\), we know that, for all \(\alpha\) at most \(\alpha^* = (1 + \frac{C}{n})\), LOHification is in \(\Theta(n \log(\frac{n}{\alpha \cdot (\alpha - 1)}) + \frac{n \cdot \alpha \cdot \log(n)}{\alpha - 1})\). Next we will look at LOHification methods that are based on selection.

### 2.3 LOHification via iterative selection

LOHs can be constructed using one-dimensional selection (one-dimensional selection can be done in linear time via median-of-medians \([4]\)). In this section, we will describe a LOHification algorithm that selects away layers from the end of the array, prove its complexity, and find for which values of \(\alpha\) it is optimal.

#### 2.3.1 Selecting away the layer with the greatest index

This algorithm repeatedly performs a linear-time one-dimensional selection on the value at the first index (were the array in sorted order) in \(L_{\ell-1}\), then the LOH is partitioned about this value. This is repeated for \(L_{\ell-2}, L_{\ell-3}\), and so on until the LOH has been partitioned about the minimum value in each layer. We will prove that this algorithm is in \(\Theta\left(\frac{\alpha \cdot n}{\alpha - 1}\right)\).

**Lemma 2.5.** Selecting away the layer with the greatest index is in \(\Omega\left(\frac{\alpha \cdot n}{\alpha - 1}\right)\)

**Proof.** By using a linear time one-dimensional selection, we can see that the runtime for selecting away the layer with the greatest index is:

\[ r(n) = \Theta\left(\sum_{i=0}^{\ell-1} \left(n - \sum_{j=\ell-i}^{\ell-1} |L_j|\right)\right) \]

\[ \subseteq \Omega\left(\sum_{i=0}^{\ell-1} \left(n - \sum_{j=\ell-i}^{\ell-1} \alpha^j\right)\right) \]

\[ \subseteq \Omega\left(\sum_{i=0}^{\ell-1} \left(n - \sum_{j=\ell-i}^{\ell-1} (\alpha^j + 1)\right)\right) \]

\[ = \Omega\left(n \cdot \ell - \frac{\ell^2 - \ell}{2} - \frac{1}{\alpha - 1} \cdot \left(\ell \sum_{i=0}^{\ell-1} (\alpha^\ell - \alpha^{\ell-i})\right)\right) \]

\[ = \Omega\left(n \cdot \ell - \frac{\ell^2 - \ell}{2} - \frac{1}{\alpha - 1} \cdot \left((\ell - 1)\alpha^{\ell+1} - \alpha \cdot (\alpha^\ell + \alpha)\right)\right) \]

\[ = \Omega\left(n \cdot \ell - \frac{\ell^2 - \ell}{2} - \frac{1}{\alpha - 1} \cdot \left((\alpha - 1) \cdot \ell \cdot \alpha^\ell - \alpha \cdot (\alpha^\ell - 1)\right)\right) \]

\[ \subseteq \Omega\left(n \cdot \ell - \frac{\ell^2 - \ell}{2} - \frac{1}{\alpha - 1} \cdot \left((\alpha - 1) \cdot \ell \cdot \alpha^\ell - \alpha \cdot (\alpha^\ell - 1)\right)\right) \]

**Theorem 2.4.** Selecting away the layer with the greatest index is in \(\Theta\left(\frac{\alpha \cdot n}{\alpha - 1}\right)\)

**Proof.** Using a linear time one-dimensional selection, we can see that the runtime for selecting away the layer with the greatest index is:

\[ r(n) \in \Theta\left(\sum_{i=0}^{\ell-1} \left(n - \sum_{j=\ell-i}^{\ell-1} |L_j|\right)\right) \]

\[ \subseteq O\left(\sum_{i=0}^{\ell-1} \left(n - \sum_{j=\ell-i}^{\ell-1} \alpha^j\right)\right) \]

\[ = O\left(n \cdot \ell - \frac{1}{\alpha - 1} \cdot \left(\ell \sum_{i=0}^{\ell-1} (\alpha^\ell - \alpha^{\ell-i})\right)\right) \]

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Lemma 2.6. Iterative selection is optimal for all \( \alpha > 1 \) as sorting is optimal for \( \alpha = 1 \). We will prove that this method is optimal for all values of \( \alpha \) less than two.

Theorem 2.5. Iterative selection is optimal for all \( \alpha \geq 2 \)

Proof. LOHification is trivially done in \( \Omega(n) \), as that is the cost to load the data. As \( \alpha \) increases, the number of layers (hence the work) can only decrease, thus it suffices to show iterative selection is optimal at \( \alpha = 2 \).

\[
 r(n) \in O\left(\frac{2 \cdot n}{2 - 1}\right) \quad \text{Theorem 2.4}
\]

\[
 r(n) \in O(n)
\]

therefore;

\[\text{LOH} \in \Theta(n) \quad \forall \quad \alpha \geq 2\]

Lemma 2.6. Iterative selection is sub-optimal for \( \alpha = \alpha^* = 1 + \frac{C}{n} \) where \( C \) is any constant > 0.

Proof.

\[
r(n) \in \Theta\left(\frac{\alpha^* \cdot n}{\alpha^* - 1}\right) \quad \text{Theorem 2.4}
\]

\[
= \Theta\left(\frac{(1 + \frac{C}{n}) \cdot n}{(1 + \frac{C}{n}) - 1}\right)
\]

\[
= \Theta\left(\frac{n + C}{\frac{C}{n}}\right)
\]

\[
= \Theta\left(\frac{n^2 + C \cdot n}{C}\right)
\]

\[
\subseteq \Theta(n^2)
\]

\[
\subseteq \omega(n \cdot \log(n))
\]

\[\square\]

Theorem 2.6. Iterative selection is sub-optimal for \( 1 < \alpha < 2 \)

Proof. For this derivation, we shall look at the runtime of iterative selection as a function of \( \alpha \) defined by \( f(\alpha) = \frac{\alpha \cdot n}{\alpha - 1} \). We can see that \( f'(\alpha) = \frac{-n}{(\alpha - 1)^2} \) is negative for all \( \alpha > 1 \), thus it is decreasing on the interval \( \alpha \) in \((1, \infty)\). Because decreasing \( \alpha \) can only increase the number of layers (hence the runtime), we know the runtime is sub-optimal for \( \alpha \leq \alpha^* \) by Lemma 2.6. Because \( f(\alpha) = \frac{\alpha \cdot n}{\alpha - 1} \) is continuous and decreasing on the interval \( \alpha \) in \((1, \infty)\) and sub-optimal at \( \alpha = \alpha^* \); it is sub-optimal for \( \alpha^* \leq \alpha < \alpha' \) where \( \alpha' \) is the first value of \( \alpha \), greater than 1, for which \( f(\alpha) = \frac{\alpha \cdot n}{\alpha - 1} \) is optimal. We can find \( \alpha' \) by solving:

\[
\frac{\alpha' \cdot n}{\alpha' - 1} = n \log_2 \left(\frac{1}{\alpha' - 1}\right) + \frac{n \cdot \alpha' \cdot \log_2(\alpha')}{\alpha' - 1}
\]

Which can be simplified to:

\[
\frac{\alpha'}{\alpha' - 1} = \log_2 \left(\frac{1}{\alpha' - 1}\right) + \frac{\alpha' \cdot \log_2(\alpha')}{\alpha' - 1}
\]

We see that \( \alpha' = 2 \) is our solution. Therefore, iterative selection is sub-optimal for \( 1 < \alpha < 2 \). \[\square\]

2.4 Selecting to divide remaining pivot indices in half For this algorithm, we first calculate the pivot indices in \( O(n) \). Then, we perform a linear-time one-dimensional selection on the layers up to the median pivot. We then recurse on the sub-problems until the array is LOHified.

2.4.1 Runtime Because one-dimensional selection is in \( \Theta(n) \), the cost of every layer in the recursion is in \( \Theta(n) \). Because splitting at the median pivot creates a balanced-binary recursion tree, the cost of the algorithm

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is in $\Theta(n \cdot d)$ where $d$ is the depth of the recursion tree. Because the number of pivots in each recursive call is one less than half of the number of pivots in the parent call, we have $d = \log_2(\ell)$. Hence:

$$
\begin{align*}
    r(n) &
    \in \Theta(n \cdot \log(\ell)) \\
    &= \Theta(n \cdot \log_\alpha(n \cdot (\alpha - 1) + 1)) \\
    &= \Theta\left(n \cdot \log\left(\frac{\log(n \cdot (\alpha - 1) + 1)}{\log(\alpha)}\right)\right)
\end{align*}
$$

2.4.2 When selecting to divide remaining pivot indices in half is optimal

Here we will show that this method is optimal for the values of $\alpha$ where sorting is optimal, i.e. $1 \leq \alpha \leq \alpha^* = 1 + \frac{C}{\omega(n)}$ for any constant, $C > 0$. Then, however, we will show that it is not optimal for some interval between $\alpha^*$ and two.

**Lemma 2.7.** Selecting to divide remaining pivot indices in half is optimal for $\alpha = \alpha^* = 1 + \frac{C}{\omega(n)}$ for any constant, $C > 0$.

**Proof.**

$$
\begin{align*}
    r(n) &
    \in \Theta\left(n \cdot \log\left(\frac{\log(n \cdot (\alpha^* - 1) + 1)}{\log(\alpha^*)}\right)\right) \\
    &= \Theta\left(n \cdot \log\left(\frac{\log(n \cdot ((1 + \frac{C}{\omega(n)}) - 1) + 1)}{\log(1 + \frac{C}{\omega(n)})}\right)\right) \\
    &= \Theta\left(n \cdot \log\left(\frac{\log(C)}{\log(1 + \frac{C}{\omega(n)})}\right)\right) \\
    &= \Theta(n \cdot \log(n)) \text{ Lemma 4.7}
\end{align*}
$$

**Lemma 2.8.** Selecting to divide remaining pivot indices in half is sub-optimal for $\alpha = 2$

**Proof.**

$$
\begin{align*}
    r(n) &
    \in \Theta\left(n \cdot \log\left(\frac{\log(n \cdot (2 - 1) + 1)}{\log(2)}\right)\right) \\
    &= \Theta(n \cdot \log(n + 1)) \\
    \subseteq \Theta(n \cdot \log(n)) \\
    \subseteq \omega(n)
\end{align*}
$$

**Theorem 2.7.** Selecting to divide remaining pivot indices in half is sub-optimal for some interval in $\alpha^* < \alpha \leq 2$

**Proof.** For this derivation, we shall look at the runtime of dividing the remaining pivot indices in half as a function of $\alpha$ defined by $f(\alpha) = n \cdot \log\left(\frac{\log(n \cdot (\alpha - 1) + 1)}{\log(\alpha)}\right)$. By Lemma 2.7 and Lemma 2.8 $f(\alpha)$ is optimal at $\alpha^*$ and sub-optimal at 2. Because

$$
\begin{align*}
f'(\alpha) &= \frac{n \cdot \log(\alpha) \cdot \left(n \cdot \log(\alpha) - (n \cdot (\alpha - 1) + 1) \cdot \log(n \cdot (\alpha - 1) + 1)\right)}{(n \cdot (\alpha - 1) + 1) \cdot \log^2(\alpha) \cdot \alpha} \\
\end{align*}
$$

is negative for large $n$ and $\alpha > 1$, the algorithm performs better as $\alpha$ increases. Because it is sub-optimal at $\alpha = 2$ there must be an interval in $(\alpha^*, 2]$ where $f(\alpha)$ is sub-optimal.

2.5 Partitioning on the pivot closest to the center of the array

For this implementation of the algorithm, we start by computing the pivots and then performing a linear-time selection algorithm on the pivot closest to the true median of the array to partition the array into two parts. We then recurse on the parts until all layers are generated. In this section, we will describe the runtime recurrence in detail, and then prove that this method has optimal performance at any $\alpha$.

2.5.1 The runtime recurrence

Let $n_s$ and $n_e$ be the starting and ending indices (respectively) of our (sub)array. Let $m(n_s, n_e)$ be the number of pivots between $n_s$ and $n_e$ (exclusive). Let $x(n_s, n_e)$ be the index of the pivot closest to the middle of the (sub)array starting at $n_s$ and ending at $n_e$. Then the runtime of our algorithm is $r(0, n)$ where

$$
\begin{align*}
r(n_s, n_e) &= \begin{cases} 
0 & n_s \geq n_e \\
0 & m(n_s, n_e) = 0 \\
n_e - n_s + r(n_s, x(n_s, n_e) - 1) + r(x(n_s, n_e) + 1, n_e) & \text{else}
\end{cases}
\end{align*}
$$

The recurrence for this algorithm is solved by neither the master theorem, nor the more general Akra-Bazzi method. Instead, we will bound the runtime by bounding how far right we go in the recursion tree, $t_{\text{max}}$, and using this to find the deepest layer, $d^*$ for which all branches have work. Because performing two selections is in $O(n)$, we will bound the size of the recursions by half of the parent by selecting on the pivots on both sides of the true median (if the true median is a pivot we just pay for it twice). From there, the bound on the runtime can be computed as $O(d^* \cdot n) + O\left(\sum_{d=d^*}^{\infty} \sum_{t=1}^{t_{\text{max}}} \frac{n}{2^t}\right)$. This scheme is depicted in Figure 2.
Figure 2: The recursion tree for partitioning on the pivot closest to the center of the array. The work at “Top” is in $O(n \cdot d^r)$ and the work done at “Bottom” is in $O\left(\sum_{d=d^*}^{\log(n)} \sum_{t=1}^{t_{max}} \frac{n}{n^\alpha}\right)$.

2.5.2 Bounds on variables

For the derivation of the following bounds, we shall assume that $\alpha > 1$.

**Lemma 2.9.** The number of pivots between any two points is $m(n_s, n_e) \leq \log_{\alpha} \left(\frac{n_e \cdot (\alpha - 1) + 1}{n_s \cdot (\alpha - 1) + 1}\right)$.

**Proof.** By our definition, the $i^{th}$ pivot, $p_i$, occurs at $p_i = \left[\sum_{j=0}^{i} \alpha^j\right] = \left[\frac{\alpha^{i+1} - 1}{\alpha - 1}\right]$. Let $n_s$ be the start of our (sub)array and $n_e$ be the end of our (sub)array. Then the number of pivots, $p_e$, occurring before $n_e$ is bound by the inequality:

$$n_e \geq \left[\frac{\alpha^{p_e+1} - 1}{\alpha - 1}\right]$$
$$n_e \geq \frac{\alpha^{p_e+1} - 1}{\alpha - 1}$$
$$n_e \cdot (\alpha - 1) \geq \alpha^{p_e+1} - 1$$
$$n_e \cdot (\alpha - 1) + 1 \geq \alpha^{p_e+1}$$
$$\log_{\alpha}(n_e \cdot (\alpha - 1) + 1) \geq p_e + 1$$
$$\log_{\alpha}(n_e \cdot (\alpha - 1) + 1) - 1 \geq p_e$$

Similarly, the number of pivots, $p_s$, occurring before $n_s$ is bound by the inequality:

$$n_s \leq \left[\frac{\alpha^{p_s+1} - 1}{\alpha - 1}\right]$$
$$n_s \leq \frac{\alpha^{p_s+1} - 1}{\alpha - 1} + 1$$
$$n_s - 1 \leq \alpha^{p_s+1} - 1\]$$
$$(n_s - 1) \cdot (\alpha - 1) \leq \alpha^{p_s+1} - 1$$

By combining these two inequalities, we can find an upper bound on the number of pivots in the (sub)array, $m(n_s, n_e)$:

$$m(n_s, n_e) \leq \log_{\alpha}(n_e \cdot (\alpha - 1) + 1) - (\log_{\alpha}(n_s - 1) \cdot (\alpha - 1) + 1)$$

$$m(n_s, n_e) \leq \log_{\alpha}(n_e \cdot (\alpha - 1) + 1) - \log_{\alpha}(n_s - 1) \cdot (\alpha - 1) + 1)$$

$$m(n_s, n_e) \leq \log_{\alpha}\left(\frac{n_e \cdot (\alpha - 1) + 1}{n_s - 1} \cdot (\alpha - 1) + 1\right)$$

**2.5.3 A bound on the runtime recurrence**

For the following bounds, we will assume that $\alpha > 1$. Let $d$ be the depth of our current recursion (indexed at 0) and $t$ be how far right in the tree we are at our current recursion (indexed at 1). To get an upper bound on the recurrence, we will compute the cost of selecting for both the first index before the true middle and the first index after the true middle. We will then treat the true middle as $x(n_s, n_e)$ for our recursive calls. Under these restrictions, $n_s = \frac{n \cdot (t - 1)}{2^r}$ and $n_e = \frac{n \cdot t}{2^r}$ for a given $t$ and $d$. Knowing this, we can calculate bounds for $m(n_s, n_e)$ in terms of $t$ and $d$.

$$m(n_s, n_e) \leq \log_{\alpha}\left(\frac{n_e \cdot (\alpha - 1) + 1}{n_s - 1} \cdot (\alpha - 1) + 1\right)$$

by **Lemma 2.9**

$$m(n_s, n_e) \leq \log_{\alpha}\left(\frac{n_e \cdot (\alpha - 1) + 1}{(n \cdot t - 1) \cdot (\alpha - 1) + 1}\right)$$

$$m(n_s, n_e) \leq \log_{\alpha}\left(\frac{n \cdot t \cdot (\alpha - 1) + 2^d}{(n \cdot t - 1) - 2^d} \cdot (\alpha - 1) + 2^d\right)$$

We can then use this to calculate $t$, in terms of $\alpha, n$ and $d$ for which $m(n_s, n_e) < 1$. This will give us a bound on how far right we go in the recursion tree.

$$\log_{\alpha}\left(\frac{n \cdot t \cdot (\alpha - 1) + 2^d}{(n \cdot t - 1) - 2^d} \cdot (\alpha - 1) + 2^d\right) < 1$$

$$n \cdot t \cdot (\alpha - 1) + 2^d < \alpha \cdot (n \cdot (t - 1) - 2^d) \cdot (\alpha - 1) + \alpha \cdot 2^d$$

$$n \cdot t \cdot (\alpha - 1) < \alpha \cdot (n \cdot (t - 1) - 2^d) \cdot (\alpha - 1) + \alpha \cdot 2^d$$
closest to the center of the array is in Theorem 2.8. For
2.5.4 The runtime of partitioning on the pivot
Let $r$ at layer $d$ have work. Because
on our runtime recurrence where $r = (\alpha - 1) \cdot \frac{2^d}{n}$
Because $2^d \leq n$ at any layer of the recursion, $t_{max} = \frac{\alpha}{\alpha-1} + 1$. Using this, we can define $r^*$, an upper bound
on our runtime recurrence where $r(0, n) \leq r^*(1, 0)$ and
$r^*(t, d) = \begin{cases} 0 & \text{if } t > t_{max} \\ 0 & \text{if } 2^d > n \\ \frac{n}{2^d} + r(2 \cdot t - 1, d+1) + r(2 \cdot t, d+1) & \text{else.} \end{cases}$

2.5.4 The runtime of partitioning on the pivot closest to the center of the array

Theorem 2.8. For $\alpha > 1$, partitioning on the pivot closest to the center of the array is in $O \left( n \log \left( \frac{\alpha}{\alpha-1} \right) \right)$

Proof. Let $d^*$ be the largest $d$ for which all branches at layer $d$ have work. Because $t_{max} = \frac{\alpha}{\alpha-1} + 1$, $d^* = \log_2(\frac{\alpha}{\alpha-1} + 1)$. This yields:

$$ r(n) \leq r^*(n) \leq O \left( n \cdot d^* \right) $$

$$ \leq O \left( \log(n) \cdot \sum_{d=d^*}^{\frac{\alpha}{\alpha-1}} \frac{n}{2^d} \right) + O(n \cdot d^*) $$

$$ \leq O \left( \log(n) \cdot \frac{\alpha}{\alpha-1} \right) + O \left( n \cdot \log \left( \frac{\alpha}{\alpha-1} + 1 \right) \right) $$

$$ \leq O \left( \frac{n \cdot \alpha}{\alpha-1} \cdot \sum_{d=d^*}^{\frac{\alpha}{\alpha-1}} \frac{1}{2^d} \right) + O \left( n \cdot \log \left( \frac{\alpha}{\alpha-1} \right) \right) $$

$$ \leq O \left( \frac{n \cdot \alpha}{\alpha-1} \cdot (2^{1-d^*} - 2^{-\log(n)}) \right) + O \left( n \cdot \log \left( \frac{\alpha}{\alpha-1} \right) \right) $$

$$ \leq O \left( \alpha \cdot \left( \frac{\alpha-1}{\alpha} \right) \right) + O \left( n \cdot \log \left( \frac{\alpha}{\alpha-1} \right) \right) $$

$$ \leq O \left( \alpha \cdot \left( \frac{\alpha-1}{\alpha} \right) \right) + O \left( n \cdot \log \left( \frac{\alpha}{\alpha-1} \right) \right) $$

$$ \leq O \left( n \cdot \log \left( \frac{\alpha}{\alpha-1} \right) \right) $$

Theorem 2.9. For $\alpha = 1$, partitioning on the pivot closest to the center of the array is optimal.

Proof. Because we are sorting in this case, it suffices to show that this method is in $O(n \log(n))$. Let $d^*$ be the largest $d$ for which all branches at that layer have work. Because $\alpha = 1$, all branches have work. Thus
$d^* = \log_2(n)$. This yields:

$$ r(n) \leq r^*(n) \leq O(n \cdot d^*) $$

$$ \leq O(n \log(n)) $$

Lemma 2.10. partitioning on the pivot closest to the center of the array is optimal for $1 \leq \alpha \leq \alpha^* = 1 + \frac{C}{n}$ for any constant, $C > 0$.

Proof. By Theorem 2.9 this method sorts an array in $O(n \log(n))$. Because a sorted array is also a LOH of any order and LOHification of order $\alpha^* = 1 + \frac{C}{n}$ for any constant, $C > 0$, is in $\Omega(n \log(n))$ by Theorem 2.3 this method is optimal for $1 \leq \alpha \leq \alpha^*$.

Lemma 2.11. partitioning on the pivot closest to the center of the array is optimal for $\alpha \geq 2$

Proof.

$$ r(n) \leq O \left( n \cdot \log \left( \frac{\alpha}{\alpha-1} \right) \right) $$

$$ = O(n) $$

$$ \leq \Theta(n) $$

Theorem 2.10. partitioning on the pivot closest to the center of the array is optimal for all $\alpha \geq 1$

Proof. By Lemma 2.10 and Lemma 2.11 it suffices to show that partitioning on the pivot closest to the center of the array is optimal for $\alpha^* < \alpha < 2$. Suppose $\alpha^* < \alpha < 2$. Then:

$$ r(n) \leq \Omega \left( n \log \left( \frac{1}{\alpha-1} \right) + n \cdot \frac{\alpha}{\alpha-1} \log(\alpha) \right) $$

by Theorem 2.1.
array and assume without loss of generality that \( H_{\text{ify}} \) is in \( \Theta(n) \).

The expected runtime for Quick LOHify can be thought of as a Quick-Selection algorithm, we partition on a random element, record the index of this element in an auxiliary array and then recurse on the left side until the best element is selected. While this method is probabilistic with a worst case construction in \( O(n^2) \), it performs well in practice and has a linear expected construction time.

2.7 Quick LOHify

For this implementation of the algorithm, we partition on a random element, record the index of this element in an auxiliary array and then recurse on the left side until the best element is selected. While this method is probabilistic with a worst case construction in \( O(n^2) \), it performs well in practice and has a linear expected construction time.

2.7.1 Expected Runtime of Quick LOHify

Quick LOHify can be thought of as a Quick-Selection with \( k = 1 \) and a constant number of operations per recursion for the auxiliary array. By this, we know the expected runtime to be in \( \Theta(n) \). A direct proof is also provided.

**Theorem 2.11.** The expected runtime for Quick LOHify is in \( \Theta(n) \)

**Proof.** The runtime is proportional to the number of comparisons. Suppose \( x_i \) is the \( i \)th element in the sorted array and assume without loss of generality that \( i < j \). We compare \( x_i \) and \( x_j \) only when one of these values is the pivot element. This makes the greatest possible probability that these elements are compared \( \frac{2}{j} \) as \( j \) is the minimum range that contains these elements. The expected number of comparisons can be found by summing this probability over all pairs of elements. This yields:

\[
E = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \frac{2}{j}
\]

\[
= 2 \cdot \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \left( \frac{1}{j} \right)
\]

\[
= 2 \cdot \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right)
\]

\[
= 2 \cdot (n-1) - 2 \cdot n - 2 \in \Theta(n)
\]

2.7.2 Expected \( \alpha \) of Quick LOHify

Unlike other constructions of a LOH, an \( \alpha \) is not specified when performing Quick LOHify nor is it guaranteed to be the same across different runs. We can, however, determine that the expected value of \( \alpha \) to be in \( \Theta(\log(n)) \).

**Theorem 2.12.** The expected \( \alpha \) for Quick LOHify is in \( \Theta(\log(n)) \)

**Proof.** The average \( \alpha \), \( \alpha' \), can be computed as the average ratio of the last two layers. This can be found by dividing the sum of all ratios by the number of ways to choose the pivots. This yields:

\[
\alpha' = \frac{1}{\binom{n}{2}} \cdot \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \frac{n-j}{j-i}
\]

\[
= \frac{2}{n^2 - n} \cdot \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \frac{n-j}{j-i}
\]

\[
= \frac{2}{n^2 - n} \cdot \sum_{k=1}^{n-i-1} \frac{n-i-k}{k}
\]

\[
= \frac{2}{n^2 - n} \cdot \sum_{i=0}^{n-2} \left( \sum_{k=1}^{n-i-1} \frac{n-i-k}{k} - 1 \right)
\]

\[
= \frac{2}{n^2 - n} \cdot \sum_{i=0}^{n-2} \left( \sum_{k=1}^{n-i-1} \frac{n-i}{k} - (n-i-2) \right)
\]

\[
= \frac{2}{n^2 - n} \cdot \sum_{i=0}^{n-2} \left( (n-i) \cdot \left( \sum_{k=1}^{n-i-1} \frac{1}{k} \right) - n + i + 2 \right)
\]

\[
= \frac{2}{n^2 - n} \cdot \sum_{i=0}^{n-2} \left( (n-i) \cdot H_{n-i-1} - n + i + 2 \right)
\]
\[
\frac{2}{n^2 - n} \sum_{i=0}^{n-2} (n \cdot H_{n-i-1} - i \cdot H_{n-i-1} - n + i + 2)
= \frac{2}{n^2 - n} \cdot \left(\frac{n \cdot \sum_{i=0}^{n-2} (H_{n-i})}{n - \sum_{i=0}^{n-2} (i \cdot H_{n-i})} - \frac{n - \sum_{i=0}^{n-2} (n - k - 1) \cdot H_k}{n - \sum_{i=0}^{n-2} (n - k - 1) \cdot H_k} \right)
= \frac{2}{n^2 - n} \cdot \left(\frac{n \cdot \sum_{k=1}^{n-1} H_k - n - \sum_{k=1}^{n-1} (n - k - 1) \cdot H_k}{n - \sum_{k=1}^{n-1} H_k} \right)
= \frac{2}{n^2 - n} \cdot \left(\frac{2 \cdot n^2}{2} \cdot H_n - \frac{n^2}{4} - \frac{3 \cdot n}{4} + \frac{n^2 + 5 \cdot n - 6}{2} \right)
\in \Theta(\log(n))
\]

† Simplified with Wolfram Mathematica \(\text{Sum[Sum[(k + 1)/i, i, 1, k], k, 1, -1 + n]}\)

3 Results
We compare the runtimes of various LOHify algorithms to compute the most permissive score threshold at which a given false discovery rate (FDR) \(\tau\) occurs. This is traditionally accomplished by sorting the scored hypotheses (which are labeled as TP or FP) best first and then advancing one element at a time, updating the FDR to the current \(FDR = \frac{\#FP}{\#FP + \#TP}\) at the threshold, finding the worst score at which \(FDR \leq \tau\) occurs.

The LOH method behaves similarly, but they compute optimistic bounds on the FDRs in each layer (if all TPs in the layer come first) and pessimistic bounds (if all FPs in the layer come first). When these bounds include \(\tau\), the layer is recursed on, until the size of the list is in \(O(1)\).

Table 1 demonstrates the performance benefit and the influence of \(\alpha\) on practical performance.

| \(\alpha\) | SLWGI | SDRPIH | PPCCA | QUICK |
|---|---|---|---|---|
| 2\(^{28}\) | 27.1712 | 3.60989 | 6.64702 | 3.55166 | 1.20981 |
| 2\(^{27}\) | 13.4568 | 2.67432 | 2.98285 | 2.74130 | 0.840145 |
| 2\(^{26}\) | 6.4227 | 1.03872 | 1.86184 | 1.05260 | 0.596104 |
| 2\(^{25}\) | 3.06724 | 0.58973 | 0.890603 | 0.58956 | 0.266691 |

| \(\alpha\) | SLWGI | SDRPIH | PPCCA |
|---|---|---|---|
| 1.5 | 8.92916 | 11.0468 | 8.12265 |
| 2.0 | 8.24106 | 9.24010 | 8.21261 |
| 3.0 | 5.73344 | 7.94349 | 5.60023 |
| 4.0 | 3.83187 | 6.35753 | 3.84014 |
| 6.0 | 3.60989 | 6.47042 | 3.55166 |
| 8.0 | 4.62627 | 6.90307 | 4.5759 |

Table 1: Runtimes (seconds) of different LOHification methods with various \(\alpha\). Reported runtimes are averages over 10 iterations. SORT is sorting, SLWGI is selecting the layer with the greatest index, SDRPIH is selecting to divide the remaining pivot indices in half, PPCCA is partitioning on the pivot closest to the center of the array, and QUICK is Quick-LOHify. Quick-LOHify generates its own partition indices, which are not determined by \(\alpha\).

4 Discussion
Due to the \(\Omega(n \log(n))\) bound on comparison-based sorting, ordering values using only pairwise comparison is generally considered to be an area for little practical performance benefit; however, LOHs have been used to replace sorting in applications where sorting is a limiting factor. Optimal LOHify for any \(\alpha\) and the practically fast Quick-LOHify variant are useful to replace sorting in applications such as finding the most abundant isotopes of a compound \[7\] (fast in practice with \(1 < \alpha \ll 2\)) and finding the score at which a desired FDR threshold occurs (fast in practice with \(\alpha \gg 2\)).

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Appendix

A Lemmas used in methods

**LEMMA A.1.** \( \forall \alpha > 1, [\alpha^i] \cdot \log([\alpha^i]) \sim \alpha^i \cdot \log(\alpha^i) \)

**Proof.** \( \alpha^i \cdot \log(\alpha^i) \leq [\alpha^i] \cdot \log([\alpha^i]) \leq (\alpha^i+1) \cdot \log(\alpha^i+1) \)

\[
\lim_{i \to \infty} \frac{(\alpha^i + 1) \cdot \log(\alpha^i + 1)}{\alpha^i \cdot \log(\alpha^i)} = \lim_{i \to \infty} \frac{\alpha^i \log(\alpha^i) + \log(\alpha^i + 1)}{\alpha^i \log(\alpha^i)} = \lim_{i \to \infty} \frac{\log(\alpha^i + 1) + \log(\alpha^i + 1)}{\alpha^i \log(\alpha^i)}
\]

which, by L'Hôpital's rule,

\[
= \lim_{i \to \infty} \frac{\frac{\alpha^i \log(\alpha^i)}{\alpha^i} + \frac{\alpha^i \cdot \log(\alpha) \cdot (\log(\alpha^i) + 1)}{\alpha^i \log(\alpha^i)}}{\alpha^i + 1 + (\alpha^i + 1) \cdot (\log(\alpha^i) + 1)}
\]

\[
= \lim_{i \to \infty} \frac{\alpha^i}{\alpha^i + 1} \cdot \frac{1}{\alpha^i + 1} = \lim_{i \to \infty} 1 - \frac{1}{\alpha^i + 1} = 1
\]

**LEMMA A.2.** \( \forall \alpha > 1, \frac{\log(n \cdot (\alpha - 1) + 1)}{\alpha - 1} \in o(n) \)

**Proof.**

\[
\lim_{n \to \infty} \frac{\log(n \cdot (\alpha - 1) + 1)}{n \cdot (\alpha - 1)} = \lim_{n \to \infty} \frac{\log(n \cdot (\alpha - 1) + 1)}{n} \cdot \frac{1}{(\alpha - 1)}
\]

by L'Hôpital's rule

\[
= \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n}{(\alpha - 1) \cdot (n \cdot (\alpha - 1) + 1)} = \lim_{n \to \infty} \frac{1}{n \cdot (\alpha - 1) + 1} = 0
\]

**LEMMA A.3.** \( \forall \alpha > 1, n \log\left(\frac{n}{n \cdot (\alpha - 1) + 1}\right) \sim n \log\left(\frac{1}{\alpha - 1}\right) \)

**Proof.**

\[
\lim_{n \to \infty} \frac{n \log\left(\frac{n}{n \cdot (\alpha - 1) + 1}\right)}{n \log\left(\frac{1}{\alpha - 1}\right)} = \lim_{n \to \infty} \frac{\log\left(\frac{n}{n \cdot (\alpha - 1) + 1}\right)}{\log\left(\frac{1}{\alpha - 1}\right)} = \frac{1}{\log\left(\frac{1}{\alpha - 1}\right)} \cdot \left(\lim_{n \to \infty} \log\left(\frac{n}{n \cdot (\alpha - 1) + 1}\right)\right)
\]

\[
= \frac{1}{\log\left(\frac{1}{\alpha - 1}\right)} \cdot \left(\log\left(\lim_{n \to \infty} \frac{n}{n \cdot (\alpha - 1) + 1}\right)\right)
\]

\[
= \frac{1}{\log\left(\frac{1}{\alpha - 1}\right)} \cdot \left(\log\left(\frac{1}{\alpha - 1}\right) \right)
\]

by L'Hôpital's rule

\[
= 1
\]

**LEMMA A.4.** For any constant, \( C > 0, (n^2 + n) \cdot \log(1 + \frac{C}{n}) \in o(n \cdot \log(n)) \)

**Proof.**

\[
\lim_{n \to \infty} \frac{(n^2 + n) \cdot \log(1 + \frac{C}{n})}{n \cdot \log(n)} = \lim_{n \to \infty} \frac{(n + 1) \cdot \log(1 + \frac{C}{n})}{n \cdot \log(n)}
\]

\[
= \lim_{n \to \infty} \frac{n \cdot \log(1 + \frac{C}{n}) + \log(1 + \frac{1}{n})}{n \cdot \log(n)} = \lim_{n \to \infty} \frac{n \cdot \log(1 + \frac{C}{n})}{n \cdot \log(n)}
\]

\[
= \lim_{n \to \infty} \log(1 + \frac{C}{n}) - \frac{C}{n + C} \text{ by L'Hôpital's rule}
\]

\[
= \lim_{n \to \infty} \log(1 + \frac{C}{n}) - \frac{n}{n + C}
\]

\[
= \lim_{n \to \infty} -\frac{C}{n^2 + C \cdot n} - 1 \text{ by L'Hôpital's rule}
\]

\[
= \lim_{n \to \infty} \frac{-C}{n^2 + C \cdot n}
\]

\[
= \lim_{n \to \infty} \frac{n}{n + C} - 1 \text{ by L'Hôpital's rule}
\]

\[
= 1 - 1 \text{ by L'Hôpital's rule}
\]

\[
= 0
\]

**LEMMA A.5.** \( \forall \alpha > 1, \left(\frac{\log_\alpha(n \cdot (\alpha - 1) + 1)^2 - \log_\alpha(n \cdot (\alpha - 1))}{2}\right) \in O\left(\frac{\alpha \cdot n}{\alpha - 1}\right) \)

**Proof.**

\[
\lim_{n \to \infty} \frac{\left(\log_\alpha(n \cdot (\alpha - 1) + 1)^2 - \log_\alpha(n \cdot (\alpha - 1))\right)}{\left(\frac{\alpha \cdot n}{\alpha - 1}\right)} = \lim_{n \to \infty} \frac{\log_\alpha(n \cdot (\alpha - 1) + 1)^2 - \log_\alpha(n \cdot (\alpha - 1))}{\left(\frac{\alpha \cdot n}{\alpha - 1}\right)}
\]

\[
= \lim_{n \to \infty} \frac{\alpha - 1}{\log(\alpha)^2} \cdot \left(\log(n \cdot (\alpha - 1) + 1)^2 - \log(n \cdot (\alpha - 1) + 1)\right)
\]

\[
= \lim_{n \to \infty} \frac{\alpha - 1}{\log(\alpha)^2} \cdot \left(2 \cdot \alpha \cdot n \cdot \log(n \cdot (\alpha - 1) + 1) - \frac{\alpha - 1}{2} \cdot \alpha \cdot n \cdot (\alpha - 1) + 1\right)
\]

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\[ \lim_{n \to \infty} \frac{2(\alpha-1)^2}{(\log(n))^{2}} \cdot \frac{\log(n \cdot (\alpha-1)+1)}{n \cdot (\alpha-1)+1} = 2 \cdot \alpha \]

by L'Hôpital's rule

\[ = \lim_{n \to \infty} \frac{(\alpha-1)^2}{\alpha \cdot (\log(\alpha))^2} \cdot \frac{\log(n \cdot (\alpha-1)+1)}{n \cdot (\alpha-1)+1} \]

\[ = \frac{(\alpha-1)^2}{\alpha \cdot (\log(\alpha))^2} \cdot \left( \lim_{n \to \infty} \frac{\log(n \cdot (\alpha-1)+1)}{n \cdot (\alpha-1)+1} \right) \]

\[ = \frac{(\alpha-1)^2}{\alpha \cdot (\log(\alpha))^2} \cdot \frac{1}{\lim_{n \to \infty} n \cdot (\alpha-1)+1} \]

by L'Hôpital's rule

\[ = 0 \]

\[ \square \]

**Lemma A.6.** \( \forall \alpha > 1, \left( \frac{\log(n \cdot (\alpha-1))}{\alpha-1} \right) \in o \left( \frac{\alpha \cdot n}{\alpha-1} \right) \)

**Proof.**

\[ \lim_{n \to \infty} \left( \frac{\log(n \cdot (\alpha-1))}{\alpha-1} \right) \]

\[ = \lim_{n \to \infty} \frac{\log(n \cdot (\alpha-1))}{\alpha \cdot n} \]

\[ = \lim_{n \to \infty} \frac{1}{\alpha \cdot \log(\alpha)} \cdot \frac{\log(n \cdot (\alpha-1))}{n} \]

\[ = \frac{1}{\alpha \cdot \log(\alpha)} \cdot \left( \lim_{n \to \infty} \frac{\log(n \cdot (\alpha-1))}{n} \right) \]

\[ = \frac{1}{\alpha \cdot \log(\alpha)} \cdot \left( \lim_{n \to \infty} \frac{\alpha}{n \cdot (\alpha-1)} \right) \]

by L'Hôpital's rule

\[ = 0 \]

\[ \square \]

**Lemma A.7.** \( n \cdot \log \left( \frac{\log(C)}{\log(1+\frac{C}{n})} \right) \in \Theta(n \cdot \log(n)) \)

**Proof.**

\[ \lim_{n \to \infty} n \cdot \log \left( \frac{\log(C)}{\log(1+\frac{C}{n})} \right) \]

\[ = \lim_{n \to \infty} \log \left( \frac{\log(C)}{\log(1+\frac{C}{n})} \right) \]

\[ = \lim_{n \to \infty} \frac{C}{n \cdot (n+C) \cdot \log(1+\frac{C}{n})} \]

by L'Hôpital's rule

\[ = \lim_{n \to \infty} \frac{1}{(n+C) \cdot \log(1+\frac{C}{n})} \]