Phenomenological Consequences of sub-leading Terms in See-Saw Formulas

Hans Hettmansperger*, Manfred Lindner†, Werner Rodejohann‡

Max–Planck–Institut für Kernphysik,
Postfach 103980, D–69029 Heidelberg, Germany

Abstract

Several aspects of next-to-leading (NLO) order corrections to see-saw formulas are discussed and phenomenologically relevant situations are identified. We generalize the formalism to calculate the NLO terms developed for the type I see-saw to variants like the inverse, double or linear see-saw, i.e., to cases in which more than two mass scales are present. In the standard type I case with very heavy fermion singlets the sub-leading terms are negligible. However, effects in the percent regime are possible when sub-matrices of the complete neutral fermion mass matrix obey a moderate hierarchy, e.g. weak scale and TeV scale. Examples are cancellations of large terms leading to small neutrino masses, or inverse see-saw scenarios. We furthermore identify situations in which no NLO corrections to certain observables arise, namely for $\mu-\tau$ symmetry and cases with a vanishing neutrino mass. Finally, we emphasize that the unavoidable unitarity violation in see-saw scenarios with extra fermions can be calculated with the formalism in a straightforward manner.

*email: hhettman@googlemail.com
†email: manfred.lindner@mpi-hd.mpg.de
‡email: werner.rodejohann@mpi-hd.mpg.de
1 Introduction

Neutrino masses are small. This simple fact is usually attributed to the presence of a seesaw mechanism. In its simplest and most often studied manifestation, the type I seesaw, the low energy Majorana neutrino mass matrix is

\[ m_\nu = -m_D^T M_R^{-1} m_D, \]

where \( m_D \) is a Dirac and \( M_R \) a Majorana mass matrix. The above relation is formally obtained in the limit of \( "M_R \gg m_D" \), which means that the eigenvalues of \( M_R \) are much larger than the entries of \( m_D \). While one usually gives the expression for \( m_\nu \) with an equality sign \( "=" \), it should strictly speaking be an approximative equal sign \( "\simeq" \), as there are higher order, or next-to-leading order (NLO), terms which correct it. In the standard type I seesaw with heavy fermion singlets the corrections are usually negligible. In the present paper we will study the NLO corrections in detail and identify phenomenologically relevant applications. A formalism to give those terms at arbitrary order has been developed by Grimus and Lavoura in Ref. [2]. We generalize that formalism to several of the seesaw variants which are discussed in the literature, such as the double [4], inverse [5], linear [6] or singular [7] seesaw. We note that in scenarios in which the seesaw scale is lowered to TeV scale, NLO corrections in the percent regime are possible, which in the light of upcoming neutrino precision experiments is surely not negligible. This occurs for instance in scenarios in which cancellations of large terms lead to small neutrino mass, or in inverse seesaw frameworks. These cases have in common that sub-matrices of the complete neutral fermion mass matrix obey a moderate hierarchy.

One common aspect of all seesaw mechanisms with additional fermions is the violation of unitarity of the 3 \( \times \) 3 mixing matrix which describes the mixing of the three active neutrinos. The formalism to calculate NLO seesaw corrections is applicable to calculate the magnitude and structure of those. This allows to obtain formulae for unitarity violation in a simple and straightforward manner for the seesaw variants.

We furthermore identify situations in which no corrections to certain parameters arise. We will show that in case of \( \mu-\tau \) symmetry there are no NLO corrections to \( U_{e3} = 0 \) and \( \theta_{23} = \pi/4 \). Another finding is that if one neutrino is massless, then the mixing matrix elements associated with this massless state receive no corrections. The massless neutrino does not mix with the heavy ones.

The paper is build up as follows: in Section 2 the formalism to calculate higher order corrections to the type I seesaw term is reviewed. The connection to the inherent unitarity violation is noted. In Section 3 we discuss examples on the application of the NLO terms, and identify cases which are stable. In Section 4 we show how the formalism can be applied to seesaw variants, before we conclude in Section 5.

\(^*\)A similar Ansatz for the unitary matrix diagonalizing the full neutral fermion mass matrix has been proposed in Ref. [3]. The same results could be obtained with this approach.

\(^{†}\)We will discuss here only the corrections to the low energy mass matrix \( m_\nu \) and leave consequences for the heavy singlets for further study.
2 NLO Terms to the Type I See-Saw Mechanism

In what follows we will review the derivation of the NLO terms to the type I see-saw formula. The reader familiar with it can continue in Section 2.2, where some of its applications are studied.

2.1 Derivation of NLO Terms

The conventional type I see-saw mechanism [1] contains after electroweak symmetry breaking two mass terms in the Lagrangian:

\[ \mathcal{L} = \overline{N_R} m_D \nu_L + \frac{1}{2} \overline{N_R} M_R N_R^c + h.c. \]  

(2.1)

Here \( \nu_L \) are left-handed neutrinos, \( N_R \) are right-handed singlets, and \( m_D (M_R) \) is the Dirac (Majorana) mass matrix. The effective mass matrix at low energy is conventionally obtained by integrating out the heavy states \( N_R \), and the result is

\[ m_\nu \simeq -m_D^T M_R^{-1} m_D. \]  

(2.2)

The approximative nature of this expression is noteworthy. Formally, the effective mass matrix is obtained by diagonalizing the total mass matrix in the basis \( (\nu_L^c, N_R) \):

\[ M = \begin{pmatrix} 0 & m_D^T M_R & \\ m_D & M_R \end{pmatrix}. \]  

(2.3)

We note here for later use that the inverse of a matrix of this texture is

\[ M^{-1} = \begin{pmatrix} -m_D^{-1} M_R (m_D^T)^{-1} & m_D^{-1} \\ (m_D^T)^{-1} & 0 \end{pmatrix}. \]  

(2.4)

We will assume in what follows that the involved matrices are invertible square matrices, unless otherwise noted. However, in cases with non-invertible \( m_D \) (Sections 3.2 and 3.3) we will be able to exactly solve the problem without the need of an expansion. If the eigenvalues of \( M_R \) are all much heavier than the entries of \( m_D \), then this situation will be described as “\( M_R \gg m_D \)”. Block diagonalization of \( M \) is now possible, and the block-diagonal matrices are approximately given by

\[ -m_D^T M_R^{-1} m_D \quad \text{and} \quad M_R. \]  

(2.5)

In Ref. [2] a formalism to evaluate the corrections to these expressions to arbitrary order has been developed. Let us shortly summarize the derivation of that result. A unitary transformation diagonalizes \( M \) according to

\[ U^T M U = \begin{pmatrix} \tilde{m}_\nu & 0 \\ 0 & \tilde{M}_R \end{pmatrix} \]  

(2.6)
and transforms the states \((\nu_L, N^c_R)\) to the mass states \((\nu_l, \nu_h)\), where the subscript “l” denotes light and “h” denotes heavy:

\[
\mathcal{U}^\dagger \left( \begin{array}{c} \nu_L \\ N^c_R \end{array} \right) = \left( \begin{array}{c} \nu_l \\ \nu_h \end{array} \right)_L.
\]

(2.7)

The matrix \(\mathcal{U}\) can be written as [2]

\[
\mathcal{U} = \left( \begin{array}{cc} \sqrt{1 - BB^\dagger} & B \\ -B^\dagger & \sqrt{1 - B^\dagger B} \end{array} \right), \quad \mathcal{U}^\dagger = \left( \begin{array}{cc} \sqrt{1 - BB^\dagger} & -B \\ B^\dagger & \sqrt{1 - B^\dagger B} \end{array} \right),
\]

(2.8)

where \(B\) is a complex \(3 \times 3\) matrix (in general it has the dimension of \(m_D\)), and the square root is to be understood as

\[
\sqrt{1 - BB^\dagger} = 1 - \frac{1}{2} BB^\dagger - \frac{1}{8} BB^\dagger BB^\dagger - \cdots - \frac{\Gamma(-\frac{1}{2} + n)}{n! \Gamma(-\frac{1}{2})} (BB^\dagger)^n - \cdots
\]

(2.9)

With this Ansatz the matrix \(\mathcal{U}\) is unitary order by order in \(BB^\dagger\). The analogy of the form of \(\mathcal{U}\) with a real two-by-two mixing matrix is obvious. We can insert \(\mathcal{U}\) in Eq. (2.6) and the result for the three independent entries of the r.h.s. is

\[
\sqrt{1 - B^*B^T} m_D^T \sqrt{1 - B^\dagger B} - B^* m_D B - B^* M_R \sqrt{1 - B^\dagger B} = 0,
\]

\[
-B^* m_D \sqrt{1 - B^\dagger B} - \sqrt{1 - B^*B^T} m_D^T B^\dagger + B^* M_R B^\dagger = \tilde{m}_\nu,
\]

(2.10)

Now the see-saw approximation enters the game, by assuming that \(B\) can be written as a power series in terms of \(1/M_R\), i.e., in terms of the eigenvalues of \(M_R\), which are assumed to be much heavier than the entries of \(m_D\). Hence, \(B = B_1 + B_2 + \ldots\), where \(B_i\) is of order \((1/M_R)^i\). The square root then reads

\[
\sqrt{1 - B^\dagger B} \approx 1 - \frac{1}{2} B^\dagger B_1 - \frac{1}{2} \left( B^\dagger_1 B_2 + B^\dagger_2 B_1 \right) - \cdots
\]

(2.11)

A recursive solution of Eq. (2.10) is now possible. At leading order the solution of Eq. (2.10) is given by

\[
B^*_1 = m^T_D M^{-1}_R.
\]

(2.12)

The next order term \(B_2\) in the expansion is, in the limit of a vanishing triplet contribution, zero. This is true for all \(B_i\), where \(i\) is even [2]. We obtain for the 3rd and 5th order terms

\[
B^*_3 M_R = -\frac{1}{2} B^\dagger_1 B^T_1 m_D^T - \frac{1}{2} m_D^T B^\dagger_1 B_1 - B^* m_D B_1 + \frac{1}{2} B^*_1 M_R B^\dagger_1 B_1
\]

\[
= -\frac{1}{2} m_D^T M^{-1}_R (M^*_R)^{-1} m_D^T m_D^T M^{-1}_R M^*_R M^{-1}_R m_D^T (M^*_R)^{-1},
\]

(2.13)

\[
B^*_5 M_R = -\frac{1}{2} \left( B^*_1 B^T_3 + B^*_3 B^T_1 + \frac{1}{4} B^*_1 B^T_1 B^*_1 B^T_1 \right) m_D^T - B^*_1 m_D B_3
\]

\[
- B^*_3 m_D B_1 + \frac{1}{2} B^*_3 M_R B^\dagger_1 B_1.
\]
Inserting $B_1$ and $B_3$ in the 11- and 22-entries of Eq. (2.6) yields

$$\tilde{m}_\nu = -m_D^T M_R^{-1} m_D + \frac{1}{2} m_D^T M_R^{-1} \left[ m_D^T (M_R^*)^{-1} + (M_R^*)^{-1} m_D^T \right] M_R^{-1} m_D ,$$

$$\tilde{M}_R = M_R + \frac{1}{2} \left[ m_D^T (M_R^*)^{-1} + (M_R^*)^{-1} m_D^T \right].$$

One is lead to define the symmetric matrix

$$X \equiv A + A^T , \quad \text{where} \quad A \equiv m_D^T (M_R^*)^{-1} . \quad (2.14)$$

The order of magnitude of $X$ is $m_D^2/M_R$ and one can simplify the relations to

$$\tilde{m}_\nu = -m_D^T M_R^{-1} m_D + \frac{1}{2} m_D^T M_R^{-1} X M_R^{-1} m_D , \quad (2.15a)$$

$$\tilde{M}_R = M_R + \frac{1}{2} X . \quad (2.15b)$$

For completeness, we also give the NNLO terms to $m_\nu$ and $M_R$, which are

$$\tilde{m}_\nu^{\text{NNLO}} = \frac{1}{2} m_D^T M_R^{-1} \left[ \frac{1}{4} A M_R^{-1} A + \frac{1}{4} A^T M_R^{-1} A^T + \frac{1}{2} A^T M_R^{-1} A + \frac{1}{2} (M_R^*)^{-1} A^* A^T + \frac{1}{2} A A^* (M_R^*)^{-1} + A A^* (M_R^*)^{-1} + (M_R^*)^{-1} A^* A^T \right] M_R^{-1} m_D ,$$

$$\tilde{M}_R^{\text{NNLO}} = -\frac{1}{2} \left[ A A^* (M_R^*)^{-1} + (M_R^*)^{-1} A^* A^T + \frac{1}{4} A M_R^{-1} A + \frac{1}{4} A^T M_R^{-1} A^T \right].$$

The zeroth order terms of the light and heavy mass matrices are $m_D^2/M_R$ and $M_R$, respectively. The relative NLO corrections are of order $X/M_R = m_D^2/M_R^2$ for both. The absolute correction is of order $m_D^4/M_R^3$ for the light neutrinos and $m_D^2/M_R^2$ for the heavy neutrinos. Note that this NLO correction vanishes when $A$ is antisymmetric. The absolute order of magnitude of the NNLO terms is $m_D^6/M_R^5$ for the light neutrinos and $m_D^4/M_R^3$ for the heavy ones. In general, the $N^{n+1}$LO term of the heavy neutrinos has the same absolute order of magnitude than the NLO term of the light ones.

A comment to be made here is that the same expressions for the corrections are obtained in type III see-saw scenarios \[11\], for which $M_R$ is the mass term of the neutral component of a weak fermion triplet.

In this work we will mostly ignore the possibility of the presence of a Higgs triplet, which would fill the upper left entry of $\mathcal{M}$ in Eq. (2.3) with a term $m_L$ \[10\]. In this case, the first order correction to $m_\nu$ is \[2\]

$$\tilde{m}_\nu = m_L - m_D^T M_R^{-1} m_D + \frac{1}{2} m_D^T M_R^{-1} X M_R^{-1} m_D - \frac{1}{2} (C + C^T) , \quad (2.16)$$

where $C = m_D^T M_R^{-1} (M_R^*)^{-1} m_D^T m_L$. 

5
2.2 Some possible Applications

We will continue with a few examples on the possible consequences of the NLO terms. The typical order of magnitude of the terms is \( m_D \simeq 10^2 \text{ GeV} \) and \( M_R \simeq 10^{14} \text{ GeV} \), for which \( m_\nu \simeq 0.1 \text{ eV} \). In this case, of course, the NLO terms are negligible (the same is true for the unitarity violation, see below). An exception is when the Majorana singlets are put to TeV scale, which is largely motivated by current collider opportunities. The mixing with the singlets is naively of order \( m_D/M_R \), and hence the requirement of \( m_\nu = 0.1 \text{ eV} \) would lead to small \( m_D \) and thus small mixing. However, it is possible that small neutrino masses are an effect of cancellation of large terms, with \( M_R \simeq \text{TeV} \) and \( m_D \simeq \nu \), in which case the ratio of leading order and NLO terms is \( m_D^2/M_R^2 \sim 10^{-2} \), a percent effect! In the light of future precision experiments, this is a correction one surely should take into account.

Let us give an illustrative example on this. We enter now the basis in which \( M_R \) is real and diagonal. For a two neutrino case, and a diagonal right-handed neutrino mass matrix \( M_R \), and if the Dirac mass matrix is written in its most general form

\[
m_D = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

(2.17)

the neutrino mass matrix is at leading order:

\[
m_\nu = \frac{1}{M_1} \begin{pmatrix} a^2 & ab \\ c^2 & cd \end{pmatrix} + \frac{1}{M_2} \begin{pmatrix} a^2 & c \cdot d^2 \\ c \cdot d^2 & d^2 \end{pmatrix}.
\]

(2.18)

One might imagine that \( M_{1,2} \) lie around \( \text{TeV} \), \( m_D \) lies around \( \nu \), and that at leading order the neutrino mass matrix vanishes, \( m_\nu = 0 \). In order for \( m_\nu \) to vanish\(^\text{\ref{footnote-1}}\), the requirements

\[
m_D = \begin{pmatrix} x & ax \\ y & ay \end{pmatrix} \quad \text{and} \quad \frac{x^2}{y^2} = -\frac{M_1}{M_2}
\]

(2.19)

must hold simultaneously\(^\text{\ref{footnote-1}}\). A small correction in one or both of those two conditions will generate small but non-zero neutrino mass. For instance, violating the second condition as \( x = iy \sqrt{\frac{M_1}{M_2}} (1 + \epsilon) \), one finds

\[
m_\nu^0 = -m_D^T M_R^{-1} m_D = \frac{y^2}{M_2} \epsilon (2 + \epsilon) \begin{pmatrix} 1 & a \\ a^2 \end{pmatrix},
\]

(2.20)

which has eigenvalues \( m_1 = 0 \) and \( m_2 = (1 + a^2) y^2/M_2 (2 + \epsilon) \epsilon \), obtained by diagonalizing \( m_\nu^0 \) with the mixing matrix

\[
U^0 = \frac{1}{\sqrt{1 + a^2}} \begin{pmatrix} -a & 1 \\ 1 & a \end{pmatrix}.
\]

\(^\text{\footnote{This will hold true to all orders, see below.}}\)
Note that large mixing would imply \( a = \mathcal{O}(1) \). For, \( a = 1 \), \( y \simeq 100 \) GeV and \( M_2 \simeq \text{TeV} \), it follows that \( \epsilon \simeq 10^{-12} \) to give \( m_2 = 0.05 \) eV. This illustrates the enormous tuning which has to present in order for this mechanism to work.

What are the corrections to \( m^0_\nu \)? We have shown above that the NLO term to \( m^0_\nu \) is \( \frac{1}{2} m^T_D M^{-1}_R X M^{-1}_R m_D \), where \( X = A + A^T \) and \( A = m_D m^T_D (M^{-1}_R)^* \). Evaluating this with our example gives

\[
m^1_\nu = -\frac{(2 + \epsilon) \epsilon}{M_1 M_2^3} (M_1 + (1 + \epsilon)^2 M_2) y^4 (1 + a^2) \left( \begin{array}{c} 1 \\ a \\ a^2 \end{array} \right).
\]

Comparing the zeroth and first order term, we have \( m^0_\nu = \mathcal{O}(\epsilon y^2 / M) \) and \( m^1_\nu = \mathcal{O}(\epsilon y^4 / M^3) \), which for \( y \simeq 100 \) GeV and \( M \simeq \text{TeV} \) results in a NLO term being suppressed only at the percent level. The mixing matrix stays the same in this example, and the non-zero eigenvalue of \( m^0_\nu + m^1_\nu \) is different from the non-zero eigenvalue of \( m^0_\nu \) by order \( \epsilon y^4 / M^3 \).

It should be clear to realize that the moderate hierarchy between the submatrices \( m_D \) and \( M_R \) of \( M \) leads here to sizable effects.

One may wonder what happens when \( m_\nu = 0 \) at leading order, for instance if

\[
m_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad \text{and} \quad M_R = \begin{pmatrix} 0 & 0 & M_1 \\ \cdot & M_2 & 0 \\ \cdot & \cdot & 0 \end{pmatrix}.
\]

However, one can show [8, 2] that the vanishing of \( m_\nu \) will remain true to all orders§ (the rank of \( M \) is three). We will discuss the realistic case of one vanishing eigenvalue of \( m_\nu \) in Section 3.3.

Another aspect is that zeros entries could be filled by NLO terms. Consider the case

\[
m_D = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & b_3 & 0 \end{pmatrix} \quad \text{and} \quad M_R = \begin{pmatrix} M_1 & 0 & 0 \\ \cdot & 0 & M_2 \\ \cdot & \cdot & 0 \end{pmatrix}.
\]

The resulting low energy mass matrix at zeroth order is

\[
m^T_D M^{-1}_R m_D = \begin{pmatrix} 0 & \frac{a_3 b_2}{M_2} & \frac{a_3 c_2}{M_2} \\ \cdot & 2 b_2 b_3 & \frac{b_3 c_2}{M_2} \\ \cdot & \cdot & \frac{c_3}{M_3} \end{pmatrix}.
\]

and forbids neutrino-less double beta decay because the 11-entry of \( m_\nu \) vanishes. The NLO term fills this entry with a contribution

\[
(m_\nu)_{11} = \frac{a_3^2 b_2 b_3}{M_2^3}.
\]

§A proof for the case when a triplet is present can be found in [9].
However, this term is suppressed with respect to the non-zero entries in $m_D^T M_R^{-1} m_D$ by many orders of magnitude, and can safely be neglected. It is typically even smaller than terms of order $U_{ei}^2 m_i^3 / q^2$ ($q^2 \approx 0.1$ GeV$^2$, the typical momentum exchange in double beta decay), which are in general non-zero when $U_{ei} m_i = 0$. An exception could be when such zero textures are generated in scenarios in which small $m_\nu$ is generated by cancellations of large terms, see above.

### 2.3 Connection to Unitarity Violation

It is well-known that an intrinsic unitarity violation is present in those see-saw scenarios which contain extra fermions (i.e., not in a pure type II see-saw). This can be shown easily by diagonalizing Eq. (2.6) in a slightly different way, namely via

$$ U = \begin{pmatrix} N & S \\ T & V \end{pmatrix}, \quad (2.26) $$

where $N, S, T, V$ are $3 \times 3$ mixing matrices which are in general non-unitary. By evaluating

$$ U^T M_\nu U = \begin{pmatrix} m_\nu^{\text{diag}} & 0 \\ 0 & M_R^{\text{diag}} \end{pmatrix}, \quad (2.27) $$

and by assuming that $S, T$ are of order $m_D/M_R$, one obtains from the 12-entry of $U^T M_\nu U$ that at leading order $T^T \simeq -N^T m_D^T M_R^{-1}$. Inserting this in the 11-entry gives

$$ m_\nu^{\text{diag}} \simeq -N^T m_D^T M_R^{-1} m_D N \quad (2.28) $$

and $V^T M_R V \simeq M_R^{\text{diag}}$. The PMNS matrix $N$ is therefore non-unitary, because $NN^\dagger = 1 - SS^\dagger$ and $N^\dagger N = 1 - T^\dagger T$. Phenomenologically, the non-unitarity can be described by writing

$$ N = (1 + \eta) U_0, \quad (2.29) $$

where $U_0$ is unitary and $\eta$ hermitian. The latter matrix contains three phases, but current constraints exist only for its absolute values $|\eta|$:

$$ |\eta| < \begin{pmatrix} 4.0 \times 10^{-3} & 1.2 \times 10^{-5} & 3.2 \times 10^{-3} \\ \cdot & 1.6 \times 10^{-3} & 2.1 \times 10^{-3} \\ \cdot & \cdot & 5.3 \times 10^{-3} \end{pmatrix}. \quad (2.30) $$

By comparing Eq. (2.29) with (2.26) we can identify $\eta \simeq -\frac{1}{2} SS^\dagger$. By inserting in $N^\dagger N = 1 - T^\dagger T$ the above relation for $T$ and the phenomenological description for $N$ from Eq. (2.29), one finds

$$ \eta \simeq -\frac{1}{2} m_D^\dagger (M_R^{-1})^* M_R^{-1} m_D = -\frac{1}{2} B_1 B_1^\dagger. \quad (2.31) $$
Hence, we can read off the amount of unitarity violation from the first order expression of $B$, which is given as $B_1 = m_D^\dagger (M_R^{-1})$ in Eq. (2.12). If a triplet term $m_L$ is present, the same calculation can be performed and the result from Eq. (2.31) stays the same as long as $m_L \ll M_R$. A triplet term does therefore not induce unitarity violation. As we will see in Section 4, we can calculate the NLO terms for see-saw variants like the inverse or double see-saw in the same way as we have done above for the type I see-saw. This makes it possible to simply write down the magnitude and structure of unitarity violation for those scenarios.

3 Special Cases in Type I See-Saw

There are certain cases in which the specific (flavor) structure of the mass matrices leaves imprints on the higher order see-saw corrections and the unitarity violating parameters. We will discuss some examples, starting first with $\mu-\tau$ symmetry and then move on to scaling ($m_3 = \theta_{13} = 0$ in the inverted hierarchy), which we will generalize to scenarios containing a vanishing neutrino mass with arbitrary mixing and mass ordering.

3.1 $\mu-\tau$ Symmetric See-Saw

Consider $\mu-\tau$ symmetric $m_D$ and $M_R$, i.e., \[ M_R = \begin{pmatrix} X & Y & Y \\ W & Z & \cdot \\ \cdot & \cdot & W \end{pmatrix} \text{ and } m_D = \begin{pmatrix} a & b & b \\ d & e & f \\ d & f & e \end{pmatrix}. \] (3.1)

As a result of such a structure the leading term of the low energy mass matrix is

\[ m_\nu^0 = -m_D^T M_R^{-1} m_D = \begin{pmatrix} A & B & B \\ \cdot & D & E \\ \cdot & \cdot & D \end{pmatrix}, \] (3.2)

where $A, B, D, E$ are functions of the parameters in $m_D$ and $M_R$. The above matrix $m_\nu^0$ predicts to the eigenvalue $D - E$ the eigenvector $(0, -1, 1)^T$. Hence, if $D - E$ corresponds to the largest (smallest) mass, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ in the normal (inverted) mass ordering.

\[ \] Actually, this is a generalized form of the usually considered $\mu-\tau$ symmetry, which denotes the invariance under exchange of flavor indices $\mu$ and $\tau$ in $m_\nu$. A more correct name would be 2–3 symmetry, but we stick to the name $\mu-\tau$ symmetry.
is predicted. Interestingly, the unitarity violating parameter $\eta$ is also $\mu-\tau$ symmetric,

$$\eta = -\frac{1}{2} m_D^\dagger (M_R^{-1})^* M_R^{-1} m_D = \begin{pmatrix} x & y & y^* & |z| \\ y & |z| & w^* & \sqrt{M^2 + 4 m^2} \\ y^* & w & |z| & \sqrt{M^2 + 4 m^2} \\ |x| & y & y^* & |z| \end{pmatrix}, \quad (3.3)$$

where the new parameters $x, y, z, w$ are functions of the entries in $m_D$ and $M_R$ in Eq. (3.1). This implies in particular that $\eta_{\mu\tau}$ is predicted to be extremely small and below values which can be probed in future neutrino oscillation facilities [15].

The eigenvalue $D - E = (e - f)^2/(w - z)$ of the zeroth order matrix is in its exact form (that is, by diagonalizing $M$ instead of $m_0^0$) given as

$$m_3 = \frac{1}{2} \left( (z - w) - \sqrt{(z - w)^2 + 4 (e - f)^2} \right) \equiv M - \sqrt{M^2 + 4 m^2}. \quad (3.4)$$

The exact eigenvector to this eigenvalue can be written as

$$\begin{pmatrix} U_{e3} \\ U_{\mu3} \\ U_{\tau3} \\ U_{N_3} \\ U_{N_3} \\ U_{N_3} \end{pmatrix} = N \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -\frac{1}{2m} (M - \sqrt{M^2 + 4 m^2}) \\ \frac{1}{2m} (M - \sqrt{M^2 + 4 m^2}) \end{pmatrix}, \quad (3.5)$$

where we included its normalization in the factor $N$. We therefore showed that $U_{e3} = 0$ and maximal mixing in the sense $|U_{\mu3}| = |U_{\tau3}|$ is not modified by higher order corrections in $\mu-\tau$ symmetric see-saw scenarios. Interestingly, the state with mass $m_3$ does not mix with one heavy neutrino and mixes in equal amounts with the other two.

### 3.2 Scaling

The next example deals with “scaling” [16] (see also [17]), for which the Dirac mass matrix has the following texture:

$$m_D = \begin{pmatrix} a_1 & b_1 & b_1/c \\ a_2 & b_2 & b_2/c \\ a_3 & b_3 & b_3/c \end{pmatrix}. \quad (3.6)$$

The third column is proportional to the second one, the relevant factor being the “scaling constant” $c$. Interestingly, independent on the form of $M_R$ (other than being non-singular) the low energy mass matrix has the form [18]

$$m_D^T M_R^{-1} m_D = \begin{pmatrix} A & B & B/c \\ \cdot & D & D/c \\ \cdot & \cdot & D/c^2 \end{pmatrix}. \quad (3.7)$$

Such a low energy mass matrix has been derived for instance in explicit flavor symmetry models based on $D_4$ in Ref. [16]. The prediction of this particular texture is that the
eigenvector to the zero eigenvalue (note that the rank of $m_D^T M_R^{-1} m_D$ is 2) is $(0, -1/c, 1)^T$, and hence scaling predicts an inverted hierarchy, with $U_{e3} = 0$ and $\tan^2 \theta_{23} = 1/|c|^2$ [10]. It is easy to see that with $m_D$ given in Eq. (3.6) the full $6 \times 6$ mass matrix $M$ has rank 5 and the eigenvector corresponding to the zero eigenvalue is

$$
\begin{pmatrix}
U_{e3} \\
U_{\mu 3} \\
U_{\tau 3} \\
U_{N 13} \\
U_{N 23} \\
U_{N 33}
\end{pmatrix}
= \begin{pmatrix}
0 \\
-1/c \\
1 \\
0 \\
0 \\
0
\end{pmatrix}
\frac{1}{\sqrt{1 + |1/c|^2}}.
$$

(3.8)

Therefore, there are no corrections to the predictions $U_{e3} = 0$ and $\tan^2 \theta_{23} = 1/|c|^2$ in see-saw scenarios obeying scaling. The massless neutrino does not mix with the heavy ones. The unitarity violation obeys the relations

$$
\begin{align*}
|\eta_{e\mu}| &= |\eta_{\mu\mu}| = |\eta_{e\tau}| = |\eta_{\tau\tau}| = |c| = \cot \theta_{23},
\end{align*}
$$

(3.9)

and again the implied value of $\eta_{e\tau}$ is very small.

### 3.3 Vanishing Eigenvalue

The specific example discussed in the last subsection had a vanishing neutrino mass in $m_\nu^0 = -m_D^T M_R^{-1} m_D$, and higher order corrections did not induce a non-zero mass (this is actually trivial, since the rank of $M$ is five), nor did they modify the mixing matrix elements of the vanishing eigenvalue. Is this true in general? In what follows we will show that this is indeed the case.

If $m_\nu^0$ is to have rank 2, it follows that there is an eigenvector $|\psi\rangle$ to it such that $m_\nu^0 |\psi\rangle = 0$. If $M_R$ is non-singular (we treat this case of $\det M_R = 0$ later) then this means that $m_D$ has rank 2. Hence, there is an eigenvector $|\Phi\rangle$ to $m_\nu$ with the property $m_D |\Phi\rangle = 0$. With the definition of $m_\nu^0$ it follows that $m_\nu^0 |\Phi\rangle = 0$. Since $m_\nu^0$ can not have two zero eigenvalues, $|\Phi\rangle$ must be proportional to $|\psi\rangle$, and hence $m_D |\psi\rangle = 0$, or to be more specific:

$$
m_D \begin{pmatrix}
U_{e1} \\
U_{\mu 1} \\
U_{\tau 1}
\end{pmatrix} = 0 \text{ or } m_D \begin{pmatrix}
U_{e3} \\
U_{\mu 3} \\
U_{\tau 3}
\end{pmatrix} = 0,
$$

(3.10)

depending on whether the normal or inverted ordering is present. Note again that this is independent on the form of $M_R$. Taking the normal ordering as an example, Eq. (3.10) implies that the Dirac mass matrix takes the form

$$
m_D = \begin{pmatrix}
a_1 & b_1 & -\frac{U_{e1} a_1 + U_{\mu 1} b_1}{U_{\tau 1}} \\
a_2 & b_2 & -\frac{U_{e2} a_2 + U_{\mu 2} b_2}{U_{\tau 1}} \\
a_3 & b_3 & -\frac{U_{e3} a_3 + U_{\mu 3} b_3}{U_{\tau 1}}
\end{pmatrix},
$$

(3.11)
or similar relations in the first or second column of $m_D$. In case of an inverted hierarchy, $U_{a1}$ has to be replaced with $U_{a3}$, and if in this case $U_{e3} = 0$, we have the scaling scenario described above in Section 3.2. Inserting Eq. \(3.11\) in the $6 \times 6$ matrix $\mathcal{M}$ reveals that it has rank 5, and the exact eigenvector to the zero mass state is simply

\[
\begin{pmatrix}
U_{ei} \\
U_{\mu i} \\
U_{\tau i} \\
0 \\
0 \\
0
\end{pmatrix},
\]

(3.12)

with $i = 1 \ (3)$ for the normal (inverted) ordering. Thus, if there is a vanishing eigenvalue $m_i$ of $m_0^\nu$, then there are no corrections arising to its mixing parameters $U_{a_i}$, where $\alpha = e, \mu, \tau$. In addition, this neutrino does not mix with heavy ones.

Is this conclusion valid in both ways? Let us assume that the $6 \times 6$ mass matrix $\mathcal{M}$ has a vanishing eigenvalue, i.e.,

\[
\mathcal{M} \vec{a} = 0,
\]

(3.13)

where $\vec{a} = (a_1, a_2, a_3, a_4, a_5, a_6)^T$ is the eigenvector of the zero eigenvalue. Solving this equation for, say, the third column of $m_D$ and $a_{4,5,6}$ gives nothing but Eqs. \(3.11\) and \(3.12\). Hence, we have shown that the eigenvector to a vanishing neutrino mass receives no corrections from higher order see-saw terms.

### 3.4 Almost vanishing $m_\nu$

Let us return to the scenarios in which small $m_\nu$ is generated by a small perturbation to $m_\nu = 0$. In a 3 family framework the condition for vanishing $m_\nu$ is for diagonal $M_R$ that \[12\]

\[
m_D = \begin{pmatrix}
x & ax & bx \\
y & ay & by \\
z & az & bz
\end{pmatrix}
\quad \text{and} \quad \frac{x^2}{M_1} + \frac{y^2}{M_2} + \frac{z^2}{M_3} = 0.
\]

(3.14)

The generalization to non-diagonal $M_R$ has recently been discussed in Ref. \[20\], and the possibility of percent effects of the NLO terms has been discussed in Section 2.2. The structure of the implied unitarity violation has been analyzed recently in Ref. \[21\]. With Eq. \(3.14\) and the definition of $\eta$ in terms of $B_1$ we have

\[
\eta = -\frac{1}{2} \left( \frac{x^2}{M_1} + \frac{y^2}{M_2} + \frac{z^2}{M_3} \right) \begin{pmatrix}
1 & a & b \\
\frac{a^*}{|a|^2} & a^* b & |b|^2
\end{pmatrix}
\]

(3.15)

As $|\eta_{e\mu}|$ is known to be very small, $a$ can essentially be set to zero and the flavor structure of $\eta$ becomes very simple \[21\].

12
4 NLO Terms to See-Saw Variants

There are popular variants of the type I see-saw, in which additional singlets $S$ are added to the theory, and the $(\nu_L^c, N_R)$ basis is extended to a $(\nu_L^c, N_R, S)$ basis:

$$M = \begin{pmatrix} m_L & m_D^T & m_{DS}^T \\ m_D & M_R & m_{RS}^T \\ m_{DS} & m_{RS} & M_S \end{pmatrix}. \quad (4.1)$$

The diagonal entries are complex symmetric, while the off-diagonal elements are arbitrary complex matrices. As mentioned above, we will not consider the presence of a triplet term $m_L$ here. The frequently discussed variants of the type I see-saw are obtained from this equation by setting some terms to zero and assuming a hierarchy in the eigenvalues of the surviving terms. We will discuss in the following these variants and apply the formalism discussed in Section 2 to analyze the NLO terms and the order of magnitude of the unitarity violation. The results of this Section are summarized in Table 1.

4.1 Double See-Saw

In the double see-saw scenario we have [1]:

$$M = \begin{pmatrix} 0 & m_D^T & 0 \\ m_D & 0 & m_{RS}^T \\ 0 & m_{RS} & M_S \end{pmatrix}. \quad (4.2)$$

with the conditions $m_D, m_{RS} \ll M_S$ and $m_D \ll m_{RS}^2/M_S$. To block-diagonalize $M$, we define

$$\mathbb{M}_D := \begin{pmatrix} m_D \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbb{M}_R := \begin{pmatrix} 0 & m_{RS}^T \\ m_{RS} & M_S \end{pmatrix}, \quad (4.3)$$

and write

$$M = \begin{pmatrix} 0 & \mathbb{M}_D^T \\ \mathbb{M}_D & \mathbb{M}_R \end{pmatrix}. \quad (4.4)$$

The eigenvalues of the symmetric block $\mathbb{M}_R$ are of order $M_S$ and $m_{RS}^2/M_S$ which are, because of the above mentioned conditions, much bigger than the entries in $\mathbb{M}_D$. Thus, if we compare Eq. (4.4) with Eq. (2.3), we recognize that the double see-saw formulas and their corrections simply follow from the type I equations which we presented in Section 2. We only have to perform the substitutions

$$m_D \rightarrow \mathbb{M}_D, \quad M_R \rightarrow \mathbb{M}_R. \quad (4.5)$$

Note that $\mathbb{M}_R$ has the same structure as the type I see-saw matrix (2.3) whose inverse is given in Eq. (2.4), and the inverse of $\mathbb{M}_R$ can therefore simply be read off from that expression. For illustration, let us determine the usual double see-saw formula and its first
order correction. The relevant relations are given in Eq. (2.15). Applying (4.3), equation (2.14) translates into

\[ X = A + A^T, \quad \text{where} \quad A = M_D M_D^T (M_R^*)^{-1}, \]

and we obtain for the mass of the lightest neutrinos the expression

\[ \tilde{m}_\nu = m_D^T m_{RS}^{-1} M_S (m_{RS}^T)^{-1} m_D + \mathcal{O} \left( M_S m_D^4 \frac{M_R^2}{m_{RS}^2} \left( 1 + \frac{M_S^2}{m_{RS}^2} \right) \right), \]

which follows from Eq. (2.13) after inserting Eq. (4.5). The first term on the right-hand side represents the well known double see-saw formula. By setting in the suggestive values \( m_D \approx 100 \, \text{GeV}, M_S \approx M_{\text{Pl}} \) and \( m_{RS} \approx M_{\text{GUT}} \approx 10^{16} \, \text{GeV} \), we can generate the correct order of magnitude for neutrino masses. The second term in Eq. (4.7) gives the order of magnitude of the very lengthy NLO corrections, which however can be obtained in a straightforward manner by inserting Eq. (4.6) in Eq. (2.15). For the sake of completeness, let us quote the result:

\[
-2 m_\nu^1 = m_D^T m_{RS}^{-1} M_S (m_{RS}^T)^{-1} \left( m_D m_D^T (m_{RS}^*)^{-1} M_S^* (m_{RS}^*)^{-1} + (\text{last term})^T \right) m_{RS}^{-1} M_S (m_{RS}^T)^{-1} m_D \\
+ m_D^T m_{RS}^{-1} \left( M_S (m_{RS}^T)^{-1} m_D m_D^T (m_{RS}^*)^{-1} + (\text{last term})^T \right) (m_{RS}^T)^{-1} m_D. \]

The first term of order \( M_S^2 m_D^4 m_{RS}^{-6} \) is the leading one for the double see-saw. By setting in the suggestive values given above, we realize that the latter gives a contribution of the same order, if not larger, than the correction \( m_D^4 / M_R^5 \) for the type I see-saw formula. It is however, with the indicated values of \( m_D, M_S \) and \( m_{RS} \) a negligibly small contribution, which may change in other realizations.

Let us now determine the unitarity violation. From Section 2.3 we know that we can read off its amount from the first order expression of \( B \) (cf. Eq. (2.31)). Thus,

\[ \eta \simeq - \frac{1}{2} M_D^T (M_R^{-1})^* M_R^{-1} M_D, \]

which has the following explicit form:

\[
\eta \simeq - \frac{1}{2} \left( m_D^T (m_{RS}^*)^{-1} M_S^* (m_{RS}^*)^{-1} m_{RS}^{-1} M_S (m_{RS}^T)^{-1} m_D + m_D^T (m_{RS}^*)^{-1} (m_{RS}^T)^{-1} m_D \right) \\
= \mathcal{O} \left( m_D^4 \frac{M_S^2}{m_{RS}^2} \left( 1 + \frac{M_S^2}{m_{RS}^2} \right) \right). \]

Again, for the double see-saw the term \( M_S^2 m_D^4 m_{RS}^{-6} / m_{RS}^4 \) in the second row, giving the order of magnitude of \( \eta \), is expected to be the dominating one.

Finally, we mention the possibility of “screening” \cite{22}, in which case \( m_D = \epsilon m_{RS}^T \). It follows that the zeroth plus first order terms are given as

\[ \tilde{m}_\nu = \epsilon^2 m_S - \frac{1}{2} \epsilon^4 m_S \left( M_S^* (m_{RS}^*)^{-1} + (m_{RS}^{-1} M_S^*) \right) M_S - \epsilon^4 M_S. \]

\[ (4.11) \]
One sees that the leading order term has its flavor structure determined by the high (possibly Planck) scale physics, while the corrections include additional flavor terms. The unitarity violation simplifies to
\[
\eta = -\frac{1}{2} \epsilon^2 \left( \mathbb{1} + M_S^* (m_{RS}^*)^{-1} m_{RS}^{-1} M_S \right). \tag{4.12}
\]

### 4.2 Inverse See-Saw

The inverse see-saw \cite{6} is a variant of the double see-saw. The texture of the neutral fermion mass matrix is the same as in Eq. (4.2), but now obeys the condition \( M_S \ll m_D \ll m_{RS} \).

In the limit of \( M_S \to 0 \) lepton number is conserved and the scenario is natural in the 't Hooft sense \cite{23}. It is the preferred scenario to arrange for sizable unitarity violation. A recent discussion can be found in Ref. \cite{24}, and the results for \( \eta \) in this paper agree.

The calculation of the NLO terms proceeds in the same way as for the double see-saw, because we can perform the same replacement as in Eq. (4.3): the eigenvalues of \( M_R \) (which form a Pseudo-Dirac pair) are much larger than the entries in \( M_D \). The effective light mass matrix \( \tilde{m}_\nu \) and the unitarity violating parameter \( \eta \) look exactly as in Eq. (4.7) and Eq. (4.10), respectively. However, the term of order \( M_S^2 m_D^2 / m_{RS}^4 \) is not anymore the dominating one, but can be neglected instead. This means that \( \eta \) does basically not depend on \( M_S \) and is given by
\[
\eta \simeq -\frac{1}{2} m_D^T (m_{RS}^*)^{-1} (m_{RS}^*)^{-1} m_D. \tag{4.13}
\]

With the suggestive values \( m_D = 100 \text{ GeV}, m_{RS} = 1 \text{ TeV} \) and \( M_S = 0.1 \text{ keV} \) it follows that \( \eta \) is of order \( 10^{-2} \). The leading term of the NLO correction to the mass matrix is
\[
m_\nu^1 = -\frac{1}{2} m_D^T m_{RS}^{-1} \left( M_S (m_{RS}^*)^{-1} m_D m_D^T (m_{RS}^*)^{-1} + (\text{last term})^T \right) (m_{RS}^*)^{-1} m_D. \tag{4.14}
\]

It is of order \( m_D^4 m_{RS}^{-4} M_S \) and for \( m_D = 100 \text{ GeV}, m_{RS} = 1 \text{ TeV} \) and \( M_S = 0.1 \text{ keV} \) of order \( 10^{-2} \) eV. In analogy to the case treated in Section 2.2, the sizable NLO term has its origin in the moderate hierarchy of two sub-matrices in the total neutral fermion mass matrix.

### 4.3 Linear See-Saw

The linear see-saw mechanism \cite{6} arises when the neutral fermion mass matrix has the following form:
\[
\mathcal{M} = \begin{pmatrix}
0 & m_D^T & m_D^T M_S \\
 m_D & 0 & m_D^T M_S \\
 m_D M_S & m_D M_S & M_S
\end{pmatrix}.
\tag{4.15}
\]

Here the non-zero 31 entry \( m_{DS} \) is assumed to be of weak scale, i.e. of order \( m_D \). In the following we will assume that \( m_{RS} \) is much larger than \( m_D \) and \( m_{DS} \). Its flavor structure may or may not be related to the flavor structure of \( m_{DS} \). The relations between the other
block matrices in $\mathcal{M}$ are that of the double or the inverse see-saw and are given above. We can introduce the notation
\[
\mathbb{M}_D := \begin{pmatrix} m_D & \end{pmatrix} \quad \text{and} \quad \mathbb{M}_R := \begin{pmatrix} 0 & m_{RS}^T \end{pmatrix}.
\]
(4.16)
and write
\[
\mathcal{M} = \begin{pmatrix} 0 & \mathbb{M}_D^T \\ \mathbb{M}_D & \mathbb{M}_R \end{pmatrix}.
\]
(4.17)
The eigenvalues of $\mathbb{M}_R$ are much bigger than the entries in $\mathbb{M}_D$ which allows us again to apply the method of Section 2 on matrix (4.17). The uncorrected low energy mass matrix is easily obtained as
\[
m^0_{\nu} = m_D^T m_{RS}^{-1} M_S (m_{RS}^T)^{-1} m_D - [m_D^T m_{RS}^{-1} m_{DS} + m_{DS}^T (m_{RS}^T)^{-1} m_D].
\]
(4.18)
Note that if the first term was negligible and $m_{DS} \propto m_{RS}$, the flavor structure of $m_\nu$ is determined by the flavor structure of $m_D$. The unitarity violation $\eta$ is again determined by Eq. (2.31). There are in total 5 different terms, two of which are the known ones from Eq. (4.10), and the remaining three are
\[
\eta \sim \eta^{\text{double}} - \frac{1}{2} \left( m_{DS}^\dagger (m_{RS}^\dagger)^{-1} m_{DS}^{-1} m_{DS} - m_D^\dagger (m_{RS}^*)^{-1} M_S^* (m_{RS}^*)^{-1} m_{DS}^{-1} m_D \right)
\]
\[
= \mathcal{O} \left( \frac{m_{DS}^2}{m_{RS}^2} \left( 1 + \frac{m_D^2}{m_{DS}^2} + M_S \frac{m_D}{m_{DS} m_{RS}} + M_S^2 \frac{m_D^2}{m_{RS}^2 m_{DS}^2} \right) \right).
\]
(4.19)
As mentioned above, quite often it holds in explicit realizations that $m_{DS} = \epsilon m_{RS}$, in which case the first new term in Eq. (4.19) is proportional to $\epsilon^2 \mathbb{1}$, and the contribution to the mass matrix is $-\epsilon (m_D + m_D^T)$. If we assume that the terms containing $M_S$ are absent or sufficiently suppressed, then unitarity violation is determined by terms of order $(m_{DS}^2 + m_D^2)/m_{RS}^2$. Sizable violation of unitarity could be achieved if $m_D$ or $m_{DS}$ are sizable and not much smaller than $m_{RS}$.

4.4 Singular See-Saw
Cases with a vanishing determinant of $M_R$ are called singular see-saw [7], and have recently received some new attention in the framework of sterile neutrino hints in LSND or MiniBooNE data [25]. We will shortly apply our approach to this case now.
In a three generation framework, the mass matrix is
\[
\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & a_1 & b_1 & c_1 \\ 0 & 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & a_3 & b_3 & c_3 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & M_1 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & M_2 \end{pmatrix}.
\]
(4.20)
There are two heavy mass states of order $M_{1,2}$, two light states of order $m_D^2/M_R$ and two intermediate states of order $m_D$, which form a Pseudo-Dirac pair. Realistic cases with 3 light neutrinos would require that $m_D$ and $M_R$ are $4 \times 4$ matrices, the latter having rank 3.

We can remove first the heavy states from the discussion by identifying

$$M_L = \begin{pmatrix}
0 & 0 & 0 & a_1 \\
\cdot & 0 & 0 & a_2 \\
\cdot & \cdot & 0 & a_3 \\
\cdot & \cdot & \cdot & 0
\end{pmatrix},
M_D = \begin{pmatrix}
b_1 & b_2 & b_3 & 0 \\
c_1 & c_2 & c_3 & 0
\end{pmatrix},
M_R = \begin{pmatrix}
M_1 & 0 \\
0 & M_2
\end{pmatrix}. \quad (4.21)$$

The low mass states (i.e., the small masses and the Pseudo-Dirac pair) are obtained from diagonalizing

$$M_L - M_D^T M_R^{-1} M_D = -\begin{pmatrix}
\frac{b_1^2}{M_1} + \frac{c_1^2}{M_2} + \frac{b_1 b_3}{M_1} + \frac{c_1 c_3}{M_2} & -a_1 \\
\cdot & \frac{b_2^2}{M_1} + \frac{c_2^2}{M_2} + \frac{b_2 b_3}{M_1} + \frac{c_2 c_3}{M_2} & -a_2 \\
\cdot & \cdot & \frac{b_3^2}{M_1} + \frac{c_3^2}{M_2} & -a_3 \\
\cdot & \cdot & \cdot & 0
\end{pmatrix}. \quad (4.22)$$

The corrections to this matrix can be evaluated using the expression for the NLO term in case a triplet is present, see Eq. (2.16). There are two terms, one stemming from $M_R$, the other from $M_L$. Their structure is different, the contribution from $M_R$ looks like

$$\begin{pmatrix}
* & * & * & 0 \\
* & * & * & 0 \\
* & * & * & 0 \\
* & * & * & 0
\end{pmatrix}, \quad (4.23)$$

where the non-zero entries are of order $m_D^4/M_R^3$. The contribution from $M_L$ has the structure

$$\begin{pmatrix}
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & *
\end{pmatrix}, \quad (4.24)$$

where the non-zero entries are of order $m_D^3/M_R^2$. The relative correction to all entries is therefore the same, namely of order $m_D^2/M_R$. As mentioned in Section 2.3 a triplet term does not contribute to unitarity violation (as long as $m_L \ll M_R$), and $\eta$ is determined solely by $M_D$ and $M_R$. The result is

$$\eta = -\frac{1}{2} \begin{pmatrix}
\frac{|b_1|^2}{M_1^2} + \frac{|c_1|^2}{M_2^2} & \frac{b_1 b_3}{M_1^2} + \frac{c_1 c_3}{M_2^2} & \frac{b_1 b_3}{M_1^2} + \frac{c_1 c_3}{M_2^2} & 0 \\
\cdot & \frac{|b_2|^2}{M_1^2} + \frac{|c_2|^2}{M_2^2} & \frac{b_2 b_3}{M_1^2} + \frac{c_2 c_3}{M_2^2} & 0 \\
\cdot & \cdot & \frac{|b_3|^2}{M_1^2} + \frac{|c_3|^2}{M_2^2} & 0 \\
\cdot & \cdot & \cdot & 0
\end{pmatrix}. \quad (4.25)$$
Though its entries are arbitrary, $\eta$ is effectively only a $3 \times 3$ matrix, having no effect for the fourth state, which is one of the Pseudo-Dirac states.

If for instance in the double see-saw of Section 4.1 the matrix $m_{RS}$ was singular, we could now apply similar steps. After suitable diagonalization of $M_R$ in Eq. (4.3) it would (recall that $M_S \gg m_{RS}$) take a form corresponding to $\text{diag}(0, m^2_{RS}/M_S, m^2_{RS}/M_S, M_S, M_S, M_S)$. Here the entries are understood as being of order $m^2_{RS}/M_S$ and $M_S$, respectively. Because of $m^2_{RS}/M_S \gg m_D$ we can redefine $M_R$ as being a diagonal $5 \times 5$ matrix of the form $\text{diag}(m^2_{RS}/M_S, m^2_{RS}/M_S, M_S, M_S, M_S)$ and follow the procedure of this subsection, finding a Pseudo-Dirac pair in the general case etc. Interestingly, if $m_{RS}$ was rank 2, we could write it as

$$m_{RS} = \begin{pmatrix} x_1 & x_2 & x_2/c \\ y_1 & y_2 & y_2/c \\ z_1 & z_2 & z_2/c \end{pmatrix}. \tag{4.26}$$

The eigenvector of the vanishing eigenvalue of $M_R$ (which has rank 5, if $M_S$ is non-singular) is proportional to $(0, -1/c, 1, 0, 0)^T$, i.e. a similar situation as for scaling treated in Section 3.2. Note that with $m_{RS}$ having rank 2, and $M_R$ being rank 5, the full $9 \times 9$ mass matrix has rank 9; there is no vanishing eigenvalue. We will not discuss the cases of “singular double see-saw” or “singular inverse see-saw” any further.

## 5 Conclusions and Summary

With increasing precision in the experimental determination of neutrino mass and lepton mixing parameters, care has to be taken in giving theoretical predictions. In the present paper we have revisited higher order corrections to the see-saw mechanism. The conventional type I see-saw, as well as several popular variants were considered, and a strategy to determine the next-to-leading order (NLO) terms was developed, based on the well-known formalism for the type I see-saw. This can be applied to determine both the NLO terms, as well as to obtain the structure of the unitarity violation connected to see-saw mechanisms with additional neutral fermions. Table 1 summarizes the structure of the zeroth and next-to-leading order terms, as well as of the parameter describing the unitarity violation. We have identified situations in which no corrections arise to certain observables, namely vanishing neutrino masses or $\mu-\tau$ symmetry.

While the standard type I see-saw implies insignificant NLO terms, there are cases with phenomenologically interesting effects. This occurs when sub-matrices of the complete neutral fermion mass matrix obey a moderate hierarchy (say, TeV and weak scale). Examples are scenarios which explain the smallness of neutrino masses through cancellations of large terms, or inverse see-saw frameworks. NLO corrections in the percent regime can arise in those cases.
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|                | $m_\nu^0$                                      | $m_\nu^1$                                      | $\eta$                                      |
|----------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| type I         | $(\frac{m_D}{10^2 \text{GeV}})^2 (\frac{10^{13} \text{GeV}}{M_R}) \text{ eV}$ | $10^{-22} (\frac{m_D}{10^2 \text{GeV}})^4 (\frac{10^{13} \text{GeV}}{M_R})^{3} \text{ eV}$ | $10^{-22} (\frac{m_D}{10^2 \text{GeV}})^2 (\frac{10^{13} \text{GeV}}{M_R})^{2} \text{ eV}$ |
| double         | $(\frac{m_D}{10^2 \text{GeV}})^2 (\frac{10^{16} \text{GeV}}{m_{RS}})^2 (\frac{M_S}{10^{19} \text{GeV}}) \text{ eV}$ | $10^{-22} (\frac{m_D}{10^2 \text{GeV}})^4 (\frac{10^{16} \text{GeV}}{m_{RS}})^6 (\frac{M_S}{10^{19} \text{GeV}})^{3} \text{ eV}$ | $10^{-22} (\frac{m_D}{10^2 \text{GeV}})^2 (\frac{10^{16} \text{GeV}}{m_{RS}})^4 (\frac{M_S}{10^{19} \text{GeV}})^2 \text{ eV}$ |
| inverse        | $(\frac{m_D}{10^2 \text{GeV}})^2 (\frac{\text{TeV}}{m_{RS}})^2 (\frac{M_S}{0.1 \text{keV}}) \text{ eV}$ | $10^{-2} (\frac{m_D}{10^2 \text{GeV}})^4 (\frac{\text{TeV}}{m_{RS}})^4 (\frac{M_S}{0.1 \text{keV}}) \text{ eV}$ | $10^{-2} (\frac{m_D}{10^2 \text{GeV}})^2 (\frac{\text{TeV}}{m_{RS}})^2 \text{ eV}$ |
| linear         | $(\frac{m_D}{10^2 \text{GeV}}) (\frac{m_{DS}}{10^2 \text{GeV}}) (\frac{10^{13} \text{GeV}}{m_{RS}}) \text{ eV}$ | $10^{-22} \left( (\frac{m_D}{10^2 \text{GeV}})^3 \text{ or } 1 (\frac{m_{DS}}{10^2 \text{GeV}})^1 \text{ or } 3 \right) (\frac{10^{13} \text{GeV}}{m_{RS}})^{3} \text{ eV}$ | $10^{-22} \left( (\frac{m_D}{10^2 \text{GeV}})^2 + (\frac{m_{DS}}{10^2 \text{GeV}})^2 \right) (\frac{10^{13} \text{GeV}}{m_{RS}})^{2} \text{ eV}$ |

Table 1: See-saw variants and their “typical” orders of magnitude for the zeroth order mass matrix $m_\nu^0$, the NLO term $m_\nu^1$ and the unitarity violating parameter $\eta$. 
