Invariant manifolds in control problems

Alois Steindl

Institut für Mechanik und Mechatronik, TU Wien, Getreidemarkt 9, 1060 Wien

Invariant manifolds are useful tools for the investigation of nearly all nonlinear systems. Especially for the determination of stabilizing controls the center-stable manifold characterizes the proper feedback controls. The method is demonstrated for the stabilization of a tethered satellite in the local vertical position by applying tension control. While in-plane perturbations can be extinguished in finite time, the tension control acts as parametric excitation for out-of-plane perturbations and is only able to cause a slow algebraic decay for both kinds of perturbations. An analytical or numerical power series expansion of the center-stable manifold at the target state provides the proper feedback controls.

1 Introduction

In this article it will be explained, how invariant manifolds can be used to determine feedback laws for control problems, such that the state variables approach a steady solution. The method is well established for equilibria, at which the linearization has stable and unstable eigenvalues. In this case it is usually sufficient to calculate the eigenvectors of the stable eigenvalues, but for higher precision also nonlinear expansions of the stable manifold might be required. We will focus on an application, where the eigenvalues of the linearized system lie on the imaginary axis and higher expansions are necessary to enforce the desired behaviour.

2 Stabilization of a tethered satellite

As a demonstration example we consider a tethered satellite, which is connected by a massless tether to a space station rotating around the earth along a Keplerian circle with constant angular speed (Fig. 1). By applying a tension force on the tether we try to steer the satellite from a nearby configuration to the steady local vertical position. The scaled nonlinear dynamics of the tether is given by (1)

\[ \ddot{p} + \{(\ddot{\vartheta} - 1)^2 + 3 \cos^2 \vartheta\}/2\sin 2\vartheta\ell + 2\dot{\ell}\dot{\vartheta} = 0, \]  
\[ (\ddot{\vartheta} + 3/2\sin 2\vartheta) \cos \vartheta - 2(\ddot{\vartheta} - 1)\dot{\vartheta} \sin \vartheta\ell + 2\dot{\ell}(\ddot{\vartheta} - 1) \cos \vartheta = 0, \]  
\[ \dot{\ell} + [1 - \{(\ddot{\vartheta} - 1)^2 + 3 \cos^2 \vartheta\}] \cos^2 \vartheta \dot{\vartheta} - \dot{\vartheta}\dot{\vartheta} \ell = -u. \]  

The variables \( \vartheta \) and \( \psi \) denote the in-plane and out-of-plane angles and \( \ell \) is the scaled tether length; the control variable \( u \) denotes the tension force with \( u = 3 \) in the steady vertical configuration. In order to find a smooth control, which minimizes the cost

\[ C = \int_0^\infty F(q, u)dt \quad \text{with } F(q, u) = \frac{(u - 3)^2 + \vartheta^2 + \psi^2 + (\ell - 1)^2 + \dot{\vartheta}^2 + \dot{\psi}^2 + \dot{\ell}^2}{2} \]  

we apply Optimal Control theory (2): We state the Hamiltonian

\[ H(q, p, u) = -F + p \cdot f(q, u), \]  

where \( q = (\vartheta, \psi, \ell, \dot{\vartheta}, \dot{\psi}, \dot{\ell})^T \) and \( f(q, u) \) denotes the right hand side of the first order system corresponding to (1) and \( p \) are the adjoint variables. The optimal control \( u^* \) is obtained from the Maximum principle

\[ u^* = \arg \max_u H(q, p, u) = 3 - pu. \]  

The adjoint variables satisfy the differential equations

\[ \dot{p}_i = -\frac{\partial H}{\partial q_i}. \]  

At the start all values of the state variables \( q_i \) are given: \( q_i(0) = q_i^0 \). The controlled system should converge to the steady configuration

\[ \lim_{t \to \infty} q(t) = q_e = (0, 0, 1, 0, 0, 0)^T. \]
The eigenvalues of the Jacobian of the Hamiltonian system at the equilibrium point $z_e(q_e, p_e)$ with $p_e = 0$ are displayed in Fig. 2. There are two pairs of real eigenvalues (black diamonds), a quadruple of complex eigenvalues (blue circles) and a double pair of imaginary eigenvalues $\pm 2i$, displayed by red squares. This purely imaginary pair occurs, because the out-of-plane oscillation $\psi(t)$ is influenced by the parametric excitation term $2\ell \dot{\psi}$.

### 2.1 Pure out-of-plane oscillation

To leading order the dynamics of the out-of-plane oscillation is given by the parametrically excited oscillation equation

$$\ddot{\psi} = -4\psi - \frac{2\ell}{\ell} \dot{\psi}. \quad (7)$$

Regarding $v = \dot{\ell}/\ell$ as control variable, we state the optimal control problem

$$\min_v \int_0^\infty \frac{\psi_1^2 + \psi_2^2 + v^2}{2} dt \quad (8)$$

subject to the first order system

$$\dot{\psi}_1 = 2\psi_2, \quad \dot{\psi}_2 = -2\psi_1 - v\psi_2. \quad (9)$$

Introducing the Hamiltonian

$$H(\psi, p, v) = -\frac{\psi_1^2 + \psi_2^2 + v^2}{2} + 2p_1\psi_2 + p_2(-2\psi_1 - v\psi_2) \quad (10)$$

we find the optimal control

$$v^* = \arg \max_v H = -p_2\psi_2 \quad (11)$$

and the corresponding Hamiltonian

$$H^*(\psi, p) = H(\psi, p, v^*) = -\frac{\psi_1^2 + \psi_2^2}{2} + 2(p_1\psi_2 - p_2\psi_1) - \frac{(p_2\psi_2)^2}{2}. \quad (12)$$

The Jacobian of the canonical equations ($\dot{\psi}_i = \partial H^*/\partial p_i, \quad \dot{p}_i = -\partial H^*/\partial q_i$) becomes

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{pmatrix}. \quad (13)$$
It has a non-semisimple pair of purely imaginary eigenvalues $\pm 2i$ and is already in real Jordan Normal Form. In [3] the Hamiltonian Hopf bifurcation, which occurs precisely with this critical matrix, was investigated and a new set of variables

\[
X = (\psi_1^2 + \psi_2^2)/2, \\
S = p_1\psi_2 - p_2\psi_1, \\
Y = (p_1^2 + p_2^2)/2, \\
Z = p_1\psi_1 + p_2\psi_2,
\]

which are the invariant functions for the flow generated by the purely rotatory semi-simple part of $A$, was introduced. These functions satisfy the relation

\[
Z^2 + S^2 = 4XY. \tag{14}
\]

Using Normal Form theory ([3,4]) it is possible to simplify the Hamiltonian (12) to

\[
\tilde{H}(X,Y,S,Z) = -X + 2S + Z^2/8. \tag{15}
\]

\textbf{Remark 2.1} In [3] it is shown, that by Normal Form simplification also the term $Z^2/8$ could be eliminated. In this case one would need higher order terms to obtain the proper behaviour of the system, so we keep this term in the reduced Hamiltonian.

Since $S$ is a first integral and we are looking for solutions converging to $\psi = 0$, we restrict our attention to the leaf $S = 0$. By (14) the reduced dynamics therefore takes place on the cone $Z^2 = 4XY$, and the trajectories of the system are the level curves of $\tilde{H}$ on this manifold, as displayed in Fig. 3. Along the dot-dashed line $(X,Z) = (0,0)$ the reduced dynamics vanishes, in the original dynamics the co-state variables $p_i$ perform a harmonic oscillation with constant amplitude $\sqrt{2Y}$. At $(X,Y,Z) = (0,2,0)$ there exists a saddle point, from which the parabolic invariant curve $Y = 2, 8X = Z^2$ emerges. Along the lower branch of this parabola the solution tends to the saddle, along the upper branch it diverges from the saddle point. Along the stable manifold the costate variables $p_i$ oscillate with constant amplitude and angular frequency $\omega = 2$, while the state variables $\psi_i$ oscillate with slowly decaying amplitude. The control function $v^*$ oscillates with frequency $2\omega$; from the conditions $S = 0$ and $Z < 0$ it follows, that $\psi$ and $p$ oscillate in paraphase. In this example the dichotomy of the Mathieu equation in primary resonance is used to extinguish the out of plane oscillations and the stable manifold in the reduced system helped us to find the proper initial values for $p_i$.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig3}
\caption{Invariant cone $Z^2 = 4XY$ and level curves of the Hamiltonian $\tilde{H}$. The thick line corresponds to the stable (lower branch) and unstable (upper branch) manifold.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig4}
\caption{Out of plane oscillation $\psi_2(t)$ and optimal control $v(t)$ for the simplified control model (8) and (9).}
\end{figure}

### 3 Treatment of the 3-dimensional system

Since the control $v = \dot{\ell}/\ell$ of the simple problem is always positive, the tether length would increase indefinitely; furthermore the single control simultaneously influences the in-plane and out-of-plane oscillations. Since we found a two-dimensional center-stable manifold for the pure out-of-plane system, we expect to find a 3-dimensional stable manifold for the full system (1) to (5). As the formulas quickly become quite involved, the calculations are carried out using a symbolic algebra system ([5]).
In order to determine the dynamics of the out-of-plane subsystem and the center manifold of the system, we apply Normal Form calculations for the Hamiltonian function using the recursive algorithm explained in detail in [4]: Let

\[ H(z, \varepsilon) = H_0^0(z) + \varepsilon H_1^0(z) + \frac{\varepsilon^2}{2} H_2^0(z) \]

be the initial power series expansion of the Hamiltonian, where \( \varepsilon \) is a scaling parameter; \( H_0^0 \) contains the quadratic parts of \( H \), \( H_1^0 \) and \( H_2^0 \) the cubic and quartic ones, respectively. Let

\[ G(z, \varepsilon) = H_0^0(z) + \varepsilon H_1^0(z) + \frac{\varepsilon^2}{2} H_2^0(z) \]

and \( W(z, \varepsilon) = W_1(z) + \varepsilon W_2(z) \) be the transformed Hamiltonian and an auxiliary function for the coordinate transform, respectively. The functions \( H_j^i \) satisfy the recursive relations

\[ H_j^i = H_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} \{ H_{j-k}^{i-1}, W_{k+1} \}, \quad (16) \]

where \( \{ H, W \} \) denotes the Poisson bracket. At lowest order we obtain the relations

\[ H_0^0 = H_1^0 + \{ H_0^0, W_1 \}, \quad H_1^2 = H_2^0 + \{ H_1^0, W_1 \} + \{ H_0^0, W_2 \}. \]

By a proper choice of \( W_1 \) all cubic terms can be eliminated; the coefficient of \( Z^2 \) in \( H_0^2 \) is given by \( a = 1248/18725 \); the amplitude \( \sqrt{2Y} \) of the periodic orbit for the costate variables is therefore given by \( \sqrt{4/a} \). The lowest order approximation of the center manifold is given by

\[ q_i = -\partial W_1/\partial p_i, \quad p_i = \partial W_1/\partial q_i, \quad \text{evaluated at } \vartheta = \dot{\vartheta} = \dot{\ell} = 0, \quad \ell = 1. \]

Numerical solutions corresponding to the initial values \( \psi(0) = 0.5, \dot{\psi}(0) = \vartheta(0) = \dot{\vartheta}(0) = \dot{\ell}(0) = 0, \ell(0) = 1 \) are displayed in Figs. 5 and 6. The variable \( \dot{\ell} \) oscillates with twice the frequency of \( \psi \). Fig. 6 demonstrates, that the numerical solution (solid curve) converges quickly to the approximate center manifold, depicted by the dashed line.

**Fig. 5:** Evolution of the variables \( \psi(t) \) and \( \ell(t) \)

**Fig. 6:** Phase portrait in the \((\psi, \vartheta)\)-plane. The dashed curve shows the center manifold approximation.

### References

[1] D. P. Jin and H. Y. Hu, Nonlinear Dynamics 46, 161–178 (2006).

[2] G. Leitman, The Calculus of Variations and Optimal Control – An Introduction (Plenum Press, 1981).

[3] J. C. V. der Meer, The Hamiltonian Hopf Bifurcation, Lecture Notes in Mathematics (Springer-Verlag, 1985).

[4] K. R. Meyer and G. R. Hall, Introduction to Hamiltonian Dynamical Systems and the \( N \)-Body Problem, No. 90 in Applied Mathematical Sciences (Springer-Verlag, 1992).

[5] I. Wolfram Research, Mathematica (Wolfram Research, Inc., Champaign, Illinois, 2010).