GREEDY BASES FOR BESOV SPACES

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Abstract. We prove that the Banach space \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}\), which is isomorphic to certain Besov spaces, has a greedy basis whenever \(1 \leq p \leq \infty\) and \(1 < q < \infty\). Furthermore, the Banach spaces \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}\), with \(1 < p \leq \infty\), and \((\oplus_{n=1}^{\infty} c_0^n)_{\ell_p}\), with \(1 \leq p < \infty\) do not have a greedy bases. We prove as well that the space \((\oplus_{n=1}^{\infty} \ell_p^n)_{\ell_1}\) has a 1-greedy basis if and only if \(1 \leq p = q \leq \infty\).

1. Introduction

Let \(X\) be a Banach space and let \((x_i)\) be a Schauder basis for \(X\) with biorthogonal sequence \((x_i^*)\). For \(x \in X\) and \(n \geq 1\), the error in the best \(n\)-term approximation to \(x\) (using \((x_i)\)) is given by

\[
\sigma_n(x) := \inf \left\{ \left\| x - \sum_{i \in A} a_i x_i \right\| : (a_i) \subset \mathbb{R}, |A| \leq n \right\}.
\]

Let \(A_n(x) \subset \mathbb{N}\) be the indices corresponding to a choice of \(n\) largest coefficients of \(x\) in absolute value, i.e. \(A_n(x)\) satisfies

\[
\min \left\{ |e_i^*(x)| : i \in A_n(x) \right\} \geq \max \left\{ |e_i^*(x)| : i \in \mathbb{N} \setminus A_n(x) \right\}.
\]

Then \(G_n(x) := \sum_{i \in A_n(x)} x_i^* x_i\) is called an \(n^{th}\) greedy approximant to \(x\). We say that \((x_i)\) is greedy with constant \(C\) if

\[
\| x - G_n(x) \| \leq C \sigma_n(x) \quad (x \in X, n \geq 1).
\]

If \(C = 1\) then \((x_i)\) is said to be 1-greedy. Temlyakov [11] proved that the Haar system for \(L_p[0,1]^d\) \((1 < p < \infty, d \geq 1)\) is greedy, which provides an important theoretical justification for the thresholding procedure used in data compression. Subsequently, Konyagin and Temlyakov [5] gave a very useful abstract characterization of greedy bases. To state their result, we recall that \((x_i)\) is unconditional with constant \(K\) if, for all choices of signs, we have

\[
\left\| \sum_{i=1}^{\infty} \pm x_i^* x_i \right\| \leq K \| x \| \quad (x \in X).
\]
We say that \((x_i)\) is democratic with constant \(\Delta\) if, for all finite \(A, B \subset \mathbb{N}\) with \(|A| = |B|\), we have
\[
\left\| \sum_{i \in A} x_i \right\| \leq \Delta \left\| \sum_{i \in B} x_i \right\|.
\]

**Theorem A.** [KT] Suppose that \((x_i)\) is unconditional with constant \(K\) and democratic with constant \(\Delta\). Then \((x_i)\) is greedy with constant \(K + K^3 \Delta\). Conversely, if \((x_i)\) is greedy with constant \(C\) then \((x_i)\) is unconditional with constant \(C\) and democratic with constant \(C^2\).

Theorem A was used in [W1, W2] to prove that \(L_p[0,1] (p \neq 2)\) has a greedy basis that is not equivalent to a subsequence of the Haar basis, and in [DHK] to prove that \(\ell_p\) and \(L_p[0,1] (p \neq 2)\) have a continuum of mutually non-equivalent greedy bases. It was also used in [DKKT] to study duality for greedy bases, and a similar theorem was proved in [DKKT] to characterize the larger class of almost greedy bases (see also [DKK]).

Some examples of greedy bases are given in [W2]. In most cases these bases are greedy simply because they are symmetric (e.g. Riesz bases for a Hilbert space, which are equivalent to the unit vector basis of \(\ell_2\), or good wavelet bases for the Besov spaces \(B^p_{r,p}(\mathbb{R})\), which are equivalent to the unit vector basis of \(\ell_p\), or because they are equivalent to the Haar basis (e.g. good wavelet bases for \(L_p(\mathbb{R}^d)\)) or to a subsequence of the Haar basis (e.g. generalized Haar systems [K]). In [GH] certain wavelet bases in the Triebel-Lizorkin spaces \(f^s_{p,r}\) are shown to be greedy. In [AW] it is proved that 1-symmetric bases (e.g. the unit vector bases of Orlicz and Lorentz sequence spaces) are in fact 1-greedy. On the other hand, there are examples of spaces with an unconditional basis but no democratic unconditional basis, and hence no greedy basis, e.g. certain spaces with a unique unconditional basis up to permutation [BCLT], the spaces \(\ell_p \oplus \ell_q\) and \(\ell_p \oplus c_0\) for \(1 \leq p < q < \infty\) [AW], and the original Tsirelson space \(T^*\) [T2]. Wojtaszczyk [W3] proved that the \(L_p\) spaces \((1 < p < \infty)\) are the only rearrangement-invariant function spaces on \([0,1]\) for which the Haar system is greedy.

Using Theorem A we prove that for every \(1 \leq p \leq \infty\) and \(1 < q < \infty\) the Banach space \((\oplus_{n=1}^\infty \ell_p^n)\ell_q\) has a greedy basis. Furthermore, we show that the Banach space \((\oplus_{n=1}^\infty \ell_p^n)\ell_1\) does not have a greedy basis whenever \(1 < p \leq \infty\). This answers a question posed by P. Wojtaszczyk, who asked when such spaces have a greedy basis. The problem of finding a greedy basis for Banach spaces of the form \((\oplus_{n=1}^\infty \ell_p^n)\ell_q\) is particularly pertinent from the approximation theoretical standpoint as such spaces are isomorphic to certain Besov spaces on the circle [R]. As shown in [R] Theorem 2 (see also [Pi] page 255), the Besov space \(B^p_{\alpha,q} [0,1]\), where \(1 \leq p \leq \infty, 1 \leq q < \infty, m \in \{-1,0,1,2\ldots\}\), and \(\alpha \in (1, m + 1 + 1/p)\), is isomorphic to \(\ell_1^{m+2} \oplus (\oplus_{n=0}^\infty \ell_p^n)\ell_q\) which is easily seen to be isomorphic to \((\oplus_{n=1}^\infty \ell_p^n)\ell_q\).

The greedy bases which we construct in Theorem 1 differ from the examples discussed above in that they are neither subsymmetric (see [LT] p. 114 for this notion) nor equivalent to a subsequence of the Haar basis.

The following result completely characterizes for which pairs \((p,q)\) the space \((\oplus_{n=1}^\infty \ell_p^n)\ell_q\) has a greedy basis.

**Theorem 1.** Let \(1 \leq p, q \leq \infty\).

a) If \(1 < q < \infty\) then the Banach space \((\oplus_{n=1}^\infty \ell_p^n)\ell_q\) has a greedy basis.
b) The spaces \((\oplus_{n=1}^\infty e_n^p)_{\ell_q}\), with \(1 < p \leq \infty\), and \((\oplus_{n=1}^\infty e_n^p)_{c_0}\), with \(1 \leq p < \infty\), do not have greedy bases.

The following result yields that only in the trivial case that \(p = q\) does \((\oplus_{n=1}^\infty e_n^p)_{\ell_q}\) have a 1-greedy basis.

**Theorem 2.** Let \(1 \leq p \leq \infty\) and let \((E_n)_{n=1}^\infty\) be a sequence of finite dimensional Banach spaces. If \((x_i)_{i=1}^\infty\) is a normalized 1-greedy basis for the space \((\oplus_{n=1}^\infty E_n)_{\ell_p}\) then \((x_i)_{i=1}^\infty\) is 1-equivalent to the standard unit vector basis for \(\ell_p\) (as usual, if \(p = \infty\) we consider the \(c_0\)-sum).

As the cases \(\ell_p \oplus \ell_q\) and \((\oplus_{n=1}^\infty e_n^p)_{\ell_q}\) are settled the following spaces might be interesting to consider.

**Problem 3.** Assume \(1 < p \neq q < \infty\). Does \(\ell_q(\ell_p) = (\oplus \ell_p)_{\ell_q}\) have a greedy basis?

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## 2. Proof of Theorems 1 and 2

Part (a) of Theorem 1 will follow easily from the following Lemma, whose proof will require some work.

**Lemma 4.** Let \(1 \leq p \leq \infty\) and \(1 < q < \infty\) and let \(\varepsilon > 0\). There is a constant \(1 \leq K < \infty\) such that for all \(N \in \mathbb{N}\) there exist \(M = M_N\) and a finite normalized sequence \((x_i)_{i=1}^M \subset \ell_q(e_n^N)\) such that

a) \((x_i)_{i=1}^M\) is 1-unconditional,

b) \((1 - \varepsilon)|A| \leq \left\| \sum_{i \in A} x_i \right\|^q \leq (1 + \varepsilon)|A|\) for all \(A \subset \{1, \ldots, M\}\),

c) the span of \((x_i)_{i=1}^M\) is \(K\)-complemented in \(\ell_q(e_n^N)\), and

d) \(e_n^N\) is isometric to a \(K\)-complemented subspace of the span of \((x_i)_{i=1}^M\).

Using the lemma, we give a quick proof of the first part of Theorem 1.

**Proof of Theorem 1 (a).** Let \(1 \leq p \leq \infty\) and \(1 < q < \infty\). It will be more convenient for us to work with the space \(X := (\oplus_{N=1}^\infty \ell_q(e_n^N))_{\ell_q}\) instead of \((\oplus_{n=1}^\infty e_n^p)_{\ell_q}\). That these spaces are isomorphic follows from Pelczynski’s Decomposition Method [Pe], which says that if two Banach spaces are complementably embedded in each other, and one of them is isomorphic to the (countably infinite) \(\ell_r\)-sum, \(1 \leq r < \infty\), or \(c_0\)-sum of itself, then they are isomorphic. It is easy to observe that \(X\) and \((\oplus_{n=1}^\infty e_n^N)_{\ell_q}\) are 1-complemented in each other, and that \(X\) is isometric to its \(\ell_q\) sum.

We let \(\varepsilon > 0\) and choose, for each \(N \in \mathbb{N}\), a sequence \((x_i^{(N)})_{i=1}^{M_N}\) in the \(N\)th coordinate of \(X = (\oplus_{N=1}^\infty \ell_q(e_n^N))_{\ell_q}\) which satisfies Lemma 4. From the conditions (a) and (b) in Lemma 4 it follows that \((x_i^{(N)})_{N \in \mathbb{N}, 1 \leq i \leq M_N}\) is a 1-unconditional and \(\frac{\varepsilon}{2}\)-democratic sequence in \(X\). As the span of \((x_i^{(N)})_{i=1}^{M_N}\) is \(K\)-complemented in \(\ell_q(e_n^N)\), for each \(N \in \mathbb{N}\), it follows that the closed span of \((x_i^{(N)})_{N \in \mathbb{N}, 1 \leq i \leq M_N}\) is \(K\)-complemented in \(X\). Furthermore, as \(e_n^N\) is \(K\)-complemented in the span of \((x_i^{(N)})_{i=1}^{M_N}\) for each \(N \in \mathbb{N}\), it follows that \((\oplus_{N=1}^\infty e_n^N)_{\ell_q}\) is \(K\)-complemented in

the closed span of $\{x_i^{(N)}\}_{N \in \mathbb{N}, 1 \leq i \leq M_N}$. Thus by the Pelczyński decomposition theorem, $X$ is isomorphic to the closed span of $\{x_i^{(N)}\}_{N \in \mathbb{N}, 1 \leq i \leq M_N}$. Hence $X$, and thus $(\oplus_{n=1}^\infty \ell_p^n)_{\ell_q}$, has a greedy basis.

**Proof of Lemma 4.** For $\varepsilon > 0$, we choose numbers $\varepsilon_i \searrow 0$ such that $\prod_{i=1}^\infty (1 + \varepsilon_i) < 1 + \varepsilon$ and $\prod_{i=1}^\infty (1 - \varepsilon_i) > 1 - \varepsilon$. For each $n \in \mathbb{N}$, we denote by $(e(i,n))_{i=1}^N$ the unit vector basis for the $n$th coordinate of $\ell_q(\ell_p^n)$, and we denote by $(e^*_i)_{i=1}^N$ their biorthogonal functionals. Thus the norm on $\ell_q(\ell_p^n)$ is calculated by

$$
\left\| \sum_{i=1}^N a(i,n) e(i,n) \right\| = \left( \sum_{n=1}^\infty \left( \sum_{i=1}^N |a(i,n)|^p \right)^{q/p} \right)^{1/q}
$$

for $(a(i,n))_{1 \leq i \leq N, n \in \mathbb{N}} \subset \mathbb{R}$.

If $x \in \ell_q(\ell_p^n)$, then we denote the support of $x$ by supp$(x) = \{(i,n) | e^*_i(x) \neq 0\}$.

Before proceeding we fix two sequences of integers $(m_i), (k_i) \in \mathbb{N}^\omega$ with $m_1 = k_1 = 1$ and which satisfy the following inequalities for all $i > 1$.

1. $1/\varepsilon_i < m_i$,
2. $m_i^{1/q} + 1 < (1 + \varepsilon_i)^{1/q} m_i^{1/q}$,
3. $(1 + (m_i/k_i)^{1/q})^q < 1 + \varepsilon_i$, and
4. $1 - \varepsilon_i < (1 - (m_i/k_i)^{1/q})^q$.

The above inequalities can be easily guaranteed by first choosing $m_i$ large enough to satisfy i) and ii), and then choosing $k_i$ large enough to satisfy iii) and iv). For the sake of convenience we define $n_j = \prod_{i=1}^j k_i$ for all $j \in \mathbb{N}$. We define the finite family $(x(i,j))_{1 \leq i \leq N, 1 \leq j \leq n_N/n_i}$ by

$$
x(i,j) = \frac{1}{n_i^{1/q}} \sum_{s=1}^{n_i} e(i,s+(j-1)n_i)
$$

for all $1 \leq i \leq N$ and $1 \leq j \leq n_N/n_i$.

It is clear that $(x(i,j))$ is a normalized and 1-unconditional basic sequence as the sequence has pairwise disjoint support. Also, $\bigcup_{i \leq N, j \leq n_N/n_i} \text{supp}(x(i,j)) = \{1, 2, \ldots, N\} \times \{1, 2, \ldots, n_N\}$. For each integer $1 \leq \ell \leq N$ and subset

$$
A \subset \{ (i,j) \in \mathbb{N}^2 | 1 \leq i \leq \ell \text{ and } 1 \leq j \leq n_N/n_i \},
$$

we will prove by induction on $\ell$ that

$$
|A| \prod_{i=2}^\ell (1 - \varepsilon_i) \leq \left\| \sum_{(i,j) \in A} x(i,j) \right\|^q \leq |A| \prod_{i=2}^\ell (1 + \varepsilon_i).
$$

First note that if $\ell = 1$ and $A \subset \{ (1,j) \in \mathbb{N}^2 | 1 \leq j \leq n_N \}$ then

$$
\left\| \sum_{(1,j) \in A} x(1,j) \right\|^q = \left\| \sum_{(1,j) \in A} e(1,j) \right\|^q = |A|.
$$

Thus (1) is trivially satisfied. We now assume that equation (1) is satisfied for a given $1 \leq \ell < N$ and we will prove it for $\ell + 1$. We first partition the set $\Omega = \{ (i,j) \in \mathbb{N}^2 : 1 \leq i \leq \ell + 1 \text{ and } 1 \leq j \leq n_N/n_i \}$, into sets $\Omega_1, \Omega_2, \ldots, \Omega_{n_N/n_{\ell+1}}$, defined for each $1 \leq r \leq n_N/n_{\ell+1}$ by

$$
\Omega_r := \{ (i,j) \in \mathbb{N}^2 | 1 \leq i \leq \ell + 1, (r-1)n_{\ell+1}/n_i + 1 \leq j \leq rn_{\ell+1}/n_i \}.
$$
Observe that for $1 \leq r \leq n_N/n_{\ell+1}$, $1 \leq i \leq \ell + 1$, and $1 \leq j \leq n_N/n_i$

$$(i,j) \in \Omega_n \iff (r-1)\frac{n_{\ell+1}}{n_i} + 1 \leq j \leq (r-1)\frac{n_{\ell+1}}{n_i}$$

$$\iff \text{supp}(x_{(i,j)}) = \{i\} \times [(j-1)n_i+1, jn_i] \subset \{i\} \times [(r-1)n_{\ell+1}+1, rn_{\ell+1}].$$

Since $\text{supp}(x_{(\ell+1,r)}) = \{\ell + 1\} \times [(r-1)n_{\ell+1}+1, rn_{\ell+1}]$, it follows that

$$(i,j) \in \Omega_r \iff \{(\ell + 1, s) : (i,s) \in \text{supp}(x_{(i,j)})\} \subset \text{supp}(x_{(\ell+1,r)}).$$

Given the set $A \subset \Omega$, we partition $A$ into $A_1, A_2, \ldots, A_{n_N/n_{\ell+1}}$, by defining $A_r = A \cap \Omega_r$, for all $1 \leq r \leq n_N/n_{\ell+1}$. We note that

$$\left\| \sum_{(i,j) \in A} x_{(i,j)} \right\|^q = \sum_{r=1}^{n_N/n_{\ell+1}} \left\| \sum_{(i,j) \in A_r} x_{(i,j)} \right\|^q$$

and that $(x_{(i,j)})_{(i,j) \in \Omega_1}$ is $1$-equivalent to $(x_{(i,j)})_{(i,j) \in \Omega_r}$, for all $1 \leq r \leq n_N/n_{\ell+1}$. Thus, to prove the inequality (1), we just need to consider the case $A = A_1$. We first note that if $(\ell + 1, 1) \notin A_1$, then the inequality (1) is immediately true by the induction hypothesis. Thus we now assume that $(\ell + 1, 1) \in A_1$ and $A_1 \setminus \{(\ell + 1, 1)\} \neq \emptyset$.

Roughly speaking, we will argue that either $|A_1|$ is large enough so that $\sum_{(i,j) \in A_1} x_{(i,j)}$ can be replaced by $\sum_{(i,j) \in A_1 \setminus \{\ell+1,1\}} x_{(i,j)}$, or $|A_1|$ is so small that a large part of the support of $x_{(\ell+1,1)}$ is disjoint from

$$B_1 = \{(\ell + 1, n) : (i,n) \in \text{supp}(x_{(i,j)}) \text{ for some } j \in \{1,2,\ldots,n_N/n_i\}\},$$

and we can approximate $x_{(\ell+1,1)}$ by its projection onto $\text{span}(e_{(1,\ell+1)} : n \in \{1,2,\ldots,N\} \setminus B_1)$.

The first case we consider is that $|A_1| \geq m_{\ell+1}$. This assumption, together with the inequality $m_{\ell+1} > 1/\varepsilon_{\ell+1}$, yields

$$\frac{|A_1|}{|A_1| - 1} \leq \frac{m_{\ell+1}}{m_{\ell+1} - 1} < \frac{1}{1 - \varepsilon_{\ell+1}}.$$

This allows us to obtain the desired lower estimate. Indeed,

$$\left\| \sum_{(i,j) \in A_1} x_{(i,j)} \right\| \geq \left\| \sum_{(i,j) \in A_1 \setminus \{\ell+1,1\}} x_{(i,j)} \right\| \geq \left| A_1 \right| - 1 \prod_{i=2}^{\ell} (1 - \varepsilon_i) \geq |A_1| \prod_{i=2}^{\ell+1} (1 - \varepsilon_i).$$

To prove the upper estimate in (1), we use that $|A_1| \geq m_{\ell+1}$ together with ii) to get

$$\left( \frac{|A_1| - 1}{|A_1|} \right)^{1/q} + \frac{m_{\ell+1}^{1/q}}{m_{\ell+1}^{1/q}} < (1 + \varepsilon_{\ell+1})^{1/q}.$$

Thus,

$$\left\| \sum_{(i,j) \in A_1} x_{(i,j)} \right\| \leq \left\| \sum_{(i,j) \in A_1 \setminus \{\ell+1,1\}} x_{(i,j)} \right\| + 1$$

$$\leq \left( |A_1| - 1 \right)^{1/q} \left( \prod_{i=2}^{\ell} (1 + \varepsilon_i) \right)^{1/q} + \left( \prod_{i=2}^{\ell} (1 + \varepsilon_i) \right)^{1/q} < |A_1|^{1/q} \left( \prod_{i=2}^{\ell+1} (1 + \varepsilon_i) \right)^{1/q}. $$
This completes the proof of (1) for \( \ell + 1 \) in the case that \(|A_1| \geq m_{\ell+1}\). We now assume that \(|A_1| < m_{\ell+1}\). The size of the support of each \( x_{i,j} \) is given by \(|\text{supp}(x_{i,j})| = n_i\) for all \(1 \leq i \leq \ell + 1\) and \(1 \leq j \leq n_N/n_i\). We thus have a simple estimate for the size of the union of the supports,

\[
|\bigcup_{(i,j) \in A_1 \setminus \{(\ell+1,1)\}} \text{supp}(x_{i,j})| = \sum_{(i,j) \in A_1 \setminus \{(\ell+1,1)\}} |\text{supp}(x_{i,j})| < |A_1| n_\ell.
\]

We define sets

\[
B_1 := \{ (\ell + 1, n) \in \mathbb{N}^2 : (m, n) \in \bigcup_{(i,j) \in A_1 \setminus \{(\ell+1,1)\}} \text{supp}(x_{i,j}) \text{ for some } 1 \leq m \leq \ell \} \text{ and } B_2 := \text{supp}(x_{(\ell+1,1)}) \setminus B_1.
\]

The inequality (2) gives that \(|B_1| < |A_1| n_\ell\). For \(i = 1, 2\), we define \(P_{B_i}x_{(\ell+1,1)}\) by

\[
P_{B_i}x_{(\ell+1,1)} = \frac{1}{n_{\ell+1}} \sum_{(\ell+1,1) \in B_i} x_{(\ell+1,1)}.
\]

We may estimate the value \(\|P_{B_1}x_{(\ell+1,1)}\|\) by

\[
\|P_{B_1}x_{(\ell+1,1)}\| = \frac{1}{n_{\ell+1}} |B_1|^{1/q} < \left(\frac{|A_1| n_\ell}{n_{\ell+1}}\right)^{1/q} \left(\frac{m_{\ell+1} n_\ell}{n_{\ell+1}}\right)^{1/q} = \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}.
\]

We use this to obtain the following estimate.

\[
\left\| \sum_{(i,j) \in A_1} x_{(i,j)} \right\|^q = \left\| P_{B_2}x_{(\ell+1,1)} + P_{B_1}x_{(\ell+1,1)} + \sum_{(i,j) \in A_1 \setminus \{(\ell+1,1)\}} x_{(i,j)} \right\|^q
\]

\[
\leq 1 + \left[ \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \left(\left|A_1\right| - 1 \prod_{i=1}^{\ell} (1 + \varepsilon_i)\right)^{1/q} \right]^q
\]

\[
= 1 + \left[ \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \left(\left|A_1\right| - 1 \prod_{i=1}^{\ell} (1 + \varepsilon_i)\right)^{-1/q} \right]^q \left(\left|A_1\right| - 1 \prod_{i=1}^{\ell} (1 + \varepsilon_i)\right)^{1/q}
\]

\[
\leq \left[ 1 + \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \left(\prod_{i=1}^{\ell} (1 + \varepsilon_i)\right)^{-1/q} \right]^q \left|A_1\right| \prod_{i=1}^{\ell} (1 + \varepsilon_i)
\]

\[
\leq \left[ 1 + \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \left(\prod_{i=1}^{\ell} (1 + \varepsilon_i)\right)^{-1/q} \right]^q \left|A_1\right| \prod_{i=1}^{\ell} (1 + \varepsilon_i)
\]

\[
\leq |A_1| \prod_{i=1}^{\ell+1} (1 + \varepsilon_i) \quad \text{by iii}).
\]
For proving the remaining lower inequality in (1), we will use the following estimate for \( \|P_{B_2}x_{(\ell+1,1)}\| \) which follows from (3).

\[
\|P_{B_2}x_{(\ell+1,1)}\| \geq 1 - \|P_{B_1}x_{(\ell+1,1)}\| > 1 - \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q}.
\]

This yields

\[
\left\| \sum_{(i,j) \in A_1} x_{(i,j)} \right\|^q = \|P_{B_2}x_{(\ell+1,1)}\|^q + \left(1 - \|P_{B_1}x_{(\ell+1,1)}\|\right) \left(\sum_{(i,j) \in A_1 \setminus \{(\ell+1,1)\}} x_{(i,j)}\right)^q \\
\geq \|P_{B_2}x_{(\ell+1,1)}\|^q + \left(1 - \|P_{B_1}x_{(\ell+1,1)}\|\right) \left(\sum_{(i,j) \in A_1 \setminus \{(\ell+1,1)\}} x_{(i,j)}\right)^q \\
= \|P_{B_2}x_{(\ell+1,1)}\|^q \\
+ \left(1 - \|P_{B_1}x_{(\ell+1,1)}\|\right) \left(\sum_{(i,j) \in A_1 \setminus \{(\ell+1,1)\}} x_{(i,j)}\right)^q \\
\geq \left(1 - \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \right) + \left(1 - \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \right) (|A_1| - 1) \prod_{i=1}^{\ell+1} (1 - \varepsilon_i) \\
\geq \left(1 - \left(\frac{m_{\ell+1}}{k_{\ell+1}}\right)^{1/q} \right) |A_1| \prod_{i=1}^{\ell+1} (1 - \varepsilon_i) \\
> \prod_{i=1}^{\ell+1} (1 - \varepsilon_i)|A_1| \quad \text{by iv}.
\]

Thus we have proven the inequalities (1) in all cases. It remains to prove that there exists a constant \( 1 \leq K < \infty \), independent of \( N \in \mathbb{N} \), such that \( X := \text{span}(x_{(i,j)}) \) is \( K \)-complemented in \( \ell_q(\ell^N_p) \) and \( \ell^N_p \) is isometric to a \( K \)-complemented subspace of \( X \). For each \( 1 \leq i \leq N \), we define the vector \( y_i \) as

\[
y_i := \frac{1}{n_i^{1/q}} \sum_{j=1}^{n_i} x_{(i,j)} = \frac{1}{n_i^{1/q}} \sum_{j=1}^{n_i} e_{(i,j)}.
\]

It should be clear that \( (y_i)_{i=1}^N \) is 1-equivalent to the unit vector basis for \( \ell^N_p \). Indeed, if \( (a_i) \in \ell^N_p \) then

\[
\left\| \sum_{i=1}^{N} a_i y_i \right\| = \left( \sum_{j=1}^{n_i} \left( \sum_{i=1}^{N} \frac{|a_i|^p}{n_i^{p/q}} \right)^{q/p} \right)^{1/q} = \left( \frac{1}{n_i} \sum_{j=1}^{n_i} \left( \sum_{i=1}^{N} |a_i|^p \right)^{q/p} \right)^{1/q} = \left( \sum_{i=1}^{N} |a_i|^p \right)^{1/p}.
\]

We let \( Y = \text{span}(y_i) \) and define projections \( T_X : \ell_q(\ell^N_p) \to X \) and \( T_Y : \ell_q(\ell^N_p) \to Y \) by

\[
T_X \left( \sum a_{(i,j)} e_{(i,j)} \right) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} \sum_{s=1}^{n_i} \left( \frac{1}{n_i} \sum_{k=1}^{n_i} a_{(ik+(j-1)n_i)} \right) e_{(i,s+(j-1)n_i)}
\]
It is simple to check that $T_X$ and $T_Y$ are projections of $\ell_q(\ell_p^N)$ onto $X$ and $Y$ respectively. As $Y$ is a subspace of $X$, we have that $T_Y$ restricted to $X$ is a projection of $X$ onto $Y$. Thus we just need to prove that there exists a uniform constant $K$ such that $\|T_X\|, \|T_Y\| \leq K$. We note that $T_X = T_Y$ if, considering a more general case, $n_i = n_N$ for all $1 \leq i \leq N$. Thus proving there exists a constant $K$ independent of $(n_i)_{i=1}^N$ and $N$ such that $\|T_X\| \leq K$ will prove the inequality for $\|T_Y\|$ as well. We first consider the case that $p = q$. In this case, the basis $(e_{(i,j)})$ for the space $\ell_q(\ell_p^N)$ is symmetric. The operator $T_X$ is then an averaging operator on a space with a symmetric basis, and hence has norm one.

Now if $q < p < \infty$, then $\ell_q(\ell_p^N)$ is an interpolation space for the spaces $\ell_q(\ell_p^N)$ and $\ell_q(\ell_p^N)$. Then by the vector valued Riesz-Thorin interpolation theorem [BP] (see also [G]), if $\theta = \frac{q}{p}$ then

$$
\|T_X\|_{\ell_q(\ell_p^N)} \leq \|T_X\|_{\ell_q(\ell_p^N)}^{1-\theta} \leq \|T_X\|_{\ell_q(\ell_p^N)}.
$$

Thus if we prove that there exists a uniform constant $K$ such that $\|T_X\|_{\ell_q(\ell_p^N)} \leq K$ then the result will follow as well for all $1 < q < p < \infty$. On the other hand, if $1 \leq p < q < \infty$ then $1 < q' < p' \leq \infty$, with $q' = q/(q-1)$ and $p' = p/(p-1)$. It is simple to check that our operator $T_X : \ell_q(\ell_p^N) \to \ell_q(\ell_p^N)$ has adjoint $T_X = T_X : \ell_{q'}(\ell_{p'}^N) \to \ell_{q'}(\ell_{p'}^N)$. We thus have that $\|T_X\|_{\ell_q(\ell_p^N)} = \|T_X\|_{\ell_{q'}(\ell_{p'}^N)} \leq K$.

All that remains is to prove that there exists a uniform constant $K$ such that $\|T_X\|_{\ell_{q'}(\ell_{p'}^N)} \leq K$. This constant $K$ will come from a discretization of the classical Hardy-Littlewood maximal operator [HL], which is defined as

$$
n_u(g)(x) = \sup_{y < x \leq z} \frac{1}{z - y} \int_y^z |g(t)| \, dt \quad \text{for } x \in \mathbb{R} \text{ and } g \in L_{\text{loc}}(\mathbb{R}).
$$

It is known that the operator $n_u$ is of strong type $(q,q)$ for $1 < q \leq \infty$ and for a proof of this see [G] Theorem 8.9.1 and Corollary 8.9.1. In other words, there exists a constant $1 \leq K < \infty$ such that

$$
\left( \int |n_u(g)(x)|^q \, dx \right)^{1/q} \leq K \left( \int |g(x)|^q \, dx \right)^{1/q} \quad \text{for all } g \in L_q(\mathbb{R}).
$$

By applying this to step functions whose discontinuities are contained in $\mathbb{N}$, we get the following inequality for $\ell_q$.

$$
\left( \sum_{j \in \mathbb{N}} \left( \sup_{n \leq j \leq n+1} \frac{1}{n-m+1} \sum_{k=m}^{n} |a_k| \right)^q \right)^{1/q} \leq K \left( \sum_{j \in \mathbb{N}} |a_j|^q \right)^{1/q} \quad \text{for all } (a_j) \in \ell_q.
$$

We now prove that $\|T_X\|_{\ell_q(\ell_p^N)} \leq K$. As $\ell_q(\ell_p^N)$ is reflexive, the operator $T_X$ attains it’s norm at an extreme point of the ball $B_{\ell_q(\ell_p^N)}$. The set of extreme points of $B_{\ell_q(\ell_p^N)}$ is given
Thus there exists constants \( \varepsilon_{(i,j)} = \pm 1 \) and a sequence \( (a_j) \in \mathcal{S}_{\ell_1} \) such that

\[
\|T_X\|_{\ell_1(\ell_1^\infty)} = \left\| T_X \left( \sum_{j=1}^\infty a_j \sum_{i=1}^N \varepsilon_{(i,j)} e_{(i,j)} \right) \right\| \\
= \left\| \sum_{i=1}^N \sum_{j=1}^{n_i} \frac{1}{n_i} \sum_{s=1}^{n_j} \varepsilon_{(i,k+(j-1)n_i)} a_{k+(j-1)n_i} e_{(i,s+(j-1)n_i)} \right\| \\
\leq \left\| \sum_{i=1}^N \sum_{j=1}^{n_i} \sum_{s=1}^{n_j} \left( \frac{1}{n_j} \sum_{k=1}^{n_i} |a_{k+(j-1)n_i}| \right) e_{(i,s+(j-1)n_i)} \right\| \\
\leq \left( \sum_{j=1}^{n_N} \left( \sup_{m \leq j \leq n} \left( \frac{1}{n + 1 - m} \sum_{k=m}^{n} |a_k| \right) q \right)^{1/q} \right) \leq K \left( \sum_{j=1}^{n_N} |a_k|^q \right)^{1/q} \text{ by (5).}
\]

\[\square\]

**Theorem 5.** The Banach space \( (\oplus_{n=1}^{\infty} \ell_1^p)_{\ell_1} \) does not have a greedy basis whenever \( 1 < p \leq \infty \).

We recall that Bourgain, Casazza, Lindenstrauss, and Tzafriri proved that the spaces \( (\oplus_{n=1}^{\infty} \ell_1^p)_{\ell_1} \) and \( (\oplus_{n=1}^{\infty} \ell_\infty)_{\ell_1} \) each have unconditional bases which are unique up to permutation [BCLT]. In particular, these spaces cannot have a greedy basis, as they each have an unconditional basis which is not greedy. Thus we need only to prove Theorem 5 for the case \( 1 < p < \infty \). This is important for us as \( \ell_p \) has non-trivial type and cotype when \( 1 < p < \infty \). We rely on the following proposition, which was used in [BCLT] to prove, among other uniqueness results, that \( (\oplus_{n=1}^{\infty} \ell_1^p)_{\ell_1} \) has a unique unconditional basis up to permutation.

**Proposition B.** [BCLT] Proposition 2.1 Let \( V \) be a Banach lattice of type \( p \) and cotype \( q \), for some \( 1 \leq p \leq q < \infty \) and let \( C_v, C_u \geq 1 \) be constants. There exists a uniform constant \( K \geq 1 \) which satisfies the following statement. If \( Z \) is a \( C_v \)-complemented subspace of the direct sum \( X = (\oplus_{n=1}^{\infty} V)_{\ell_1} \) and \( Z \) has a normalized basis \( (z_n)_{n=1}^k \) with unconditional constant \( C_u \), then there exists a partition of the integers \( \{1, 2, \ldots, k\} \) into mutually disjoint...
subsets \( \{ \tau_s \}_{s=1}^r \) so that, for any choice of scalars \( \{ \alpha_n \}_{n=1}^k \), we have

\[
K^{-1} \max_{1 \leq s \leq r} \left( \sum_{n \in \tau_s} |\alpha_n|^q \right)^{1/q} \leq \left\| \sum_{n=1}^{k} \alpha_n z_n \right\| \leq K \max_{1 \leq s \leq r} \left( \sum_{n \in \tau_s} |\alpha_n|^p \right)^{1/p}.
\]

Using Proposition B, we are now prepared to give a proof of Theorem 5.

**Proof of Theorem 5** Let \( 1 < p < \infty \). To reach a contradiction, we assume that \( X = (\oplus_{n=1}^\infty \ell_p^n)_{\ell_1} \) has a normalized basis \( (x_i)_{i=1}^\infty \) which is \( C_d \)-democratic and \( C_u \)-unconditional for some constants \( C_d, C_u \geq 1 \). Let \( (x_i^*) \subset (\oplus_n \ell_p^n)_{\ell_1} \) be the biorthogonal functionals. We let \( (e_{(i,n)})_{1 \leq i \leq n < \infty} \) be the unit vector basis for \( X \) with biorthogonal functionals \( (e_{(i,n)}^*)_{1 \leq i \leq n < \infty} \). By a standard perturbation argument we may assume that \( \text{supp}(x_j) = \{(i, n) : n \in \mathbb{N}, i \leq n, e_{(i,n)}^*(x_j) \neq 0\} \) is finite for each \( j \in \mathbb{N} \).

We now fix \( N \in \mathbb{N} \). There exists \( M_N \in \mathbb{N} \) such that \( (x_j)_{j=1}^N \subset (\oplus_{n=1}^{M_N} \ell_p^n)_{\ell_1} \) for all \( n > M_n \) and \( i \leq n \). Define \( (z_{(N,j)})_{j=1}^N \subset (\oplus_{n=1}^{M_N} \ell_p^n)_{\ell_1} \) by \( z_{(N,j)} = x_j^* |_{(\oplus_{n=1}^{M_N} \ell_p^n)_{\ell_1}} \) for all \( 1 \leq j \leq N \). As \( (x_j)_{j=1}^\infty \) has basis constant at most \( C_u \), it is easy to show both that \( (x_j^*)_{j=1}^N \) is \( C_u \)-equivalent to \( (z_{(N,j)})_{j=1}^N \), and that the span of \( (z_{(N,j)})_{j=1}^N \) is \( C_u \)-complemented in \( (\oplus_{n=1}^{M_N} \ell_p^n)_{\ell_1} \). Indeed, let \( J : \text{span}(x_i)_{i=1}^N \to X \) be the inclusion map. Then \( J^* : X^* \to \text{span}(x_i^*)_{i=1}^N \) is a quotient map of norm 1. Let \( H : \text{span}(x_i^*)_{i=1}^N \to \text{span}(z_{(N,j)})_{j=1}^N \) be the isomorphism defined by \( H(x_i^*) = z_{(N,j)} \), which has norm at most \( C_u \). Then \( H \circ J^* : X^* \to \text{span}(z_{(N,j)})_{j=1}^N \) is a quotient map of norm at most \( C_u \). We just need to check that \( H \circ J^*(z_{(N,j)}) = z_{(N,j)} \). Indeed, for \( x \in \text{span}(x_i) : j \leq N \), \( J^*(z_{(N,j)})(x) = z_{(N,j)}(x) = x_i^*(x) \). Hence, \( J^*(z_{(N,j)}) = x_i^* \) and \( H \circ J^*(z_{(N,j)}) = z_{(N,j)} \). Thus \( H \circ J^* \) is a projection of norm at most \( C_u \).

We will be applying Proposition B for the space \( V = \ell_q \) and consider the spaces \( \text{span}(z_{(N,i)}) : i \leq N \), \( N \in \mathbb{N} \), (in the natural way) to be \( C_u \)-complemented subspaces of \( c_0(\ell_q) \). For the sake of convenience, we denote \( q = \min(q, 2) \) and \( \check{q} = \max(q, 2) \). We have thus defined \( q \) and \( \check{q} \) exactly so that \( \ell_q \) has type \( q \) and cotype \( \check{q} \). By Proposition B, there exists a constant \( K \) independent of \( N \in \mathbb{N} \), and there exists for all \( N \in \mathbb{N} \) a partition of the integers \( \{1, 2, \ldots, N\} \) into mutually disjoint subsets \( \{\tau_s^N\}_{s=1}^r \) so that, for any choice of scalars \( \{\alpha_n^N\}_{n=1}^N \), we have

\[
K^{-1} \max_{1 \leq s \leq r} \left( \sum_{n \in \tau_s^N} |\alpha_n|^q \right)^{1/q} \leq \left\| \sum_{n=1}^{N} \alpha_n z_{(N,n)} \right\| \leq K \max_{1 \leq s \leq r} \left( \sum_{n \in \tau_s^N} |\alpha_n|^p \right)^{1/p}.
\]

We first consider the case that \( \sup_{N \in \mathbb{N}} \max_{1 \leq s \leq r_N} |\tau_s^N| < \infty \). In this case we have that if \( N \in \mathbb{N} \) and \( \{\alpha_n^N\}_{n=1}^N \subset \mathbb{R} \) then

\[
\left\| \sum_{i=1}^{N} \alpha_i x_i^* \right\| \leq C_u \left\| \sum_{i=1}^{N} \alpha_i z_{(N,i)}^* \right\| \leq C_u K \left( \sup_{M \in \mathbb{N}} \max_{1 \leq s \leq M} |\tau_s^M| \right) \max_{1 \leq i \leq N} |\alpha_i|.
\]
Hence \((x_i^*)\) is equivalent to the unit vector basis of \(c_0\), which implies that \((x_i)\) is equivalent to the unit vector basis of \(\ell_1\). This is a contradiction as \((x_i)\) is a basis for \(X = (\oplus_{n=1}^\infty \ell_p^n)_{\ell_1}\), and \(X\) is not isomorphic to \(\ell_1\) as \(1 < p < \infty\). We now consider the remaining case that \(\sup_{N \in \mathbb{N}} \max_{1 \leq s \leq r_N} |\tau_s^N| = \infty\).

First note that there exists a subsequence of \((x_i)_{i=1}^\infty\) which is equivalent to the unit vector basis of \(\ell_1\). Indeed, there is a subsequence \((x'_j)\) which converges in the \(w^*\)-topology (considering \(X\) as the dual of \((\oplus_{n=1}^\infty \ell_p^n)_{c_0}\)) to some \(x\). If \(x \neq 0\), then it follows from the assumed unconditionality that \((x'_j)\) is equivalent to the unit vector basis of \(\ell_1\), and if \(x = 0\), then we can find a perturbation \((z'_j)\) of a further subsequence, so that for each \(n \in \mathbb{N}\), there is at most one \(j\) so that the \(\ell_p^n\) component of \(z'_j\) is not 0. But this also implies that there is a subsequence equivalent to the unit vector basis of \(\ell_1\).

Thus as \((x_i)\) is democratic, there is a \(C \geq 1\) so that \(C \left\| \sum_{i \in A} x_i \right\| > |A|\) for all finite sets \(A \subset \mathbb{N}\). We choose \(N \in \mathbb{N}\) and \(1 \leq s \leq r_N\) such that \(|\tau_s^N|^{1/q} > 2KCC_u^2\). We have the following estimates.

\[
\left\| \sum_{i \in \tau_s^N} x_i \right\| \leq 2C_u \left( \sum_{i \in \tau_s^N} |b_i x_i^*| \right) \left( \sum_{i \in \tau_s^N} x_i \right) \text{ for some } (b_i) \in c_0, \text{ with } \left\| \sum_{i \in \tau_s^N} b_i x_i^* \right\| = 1
\]

\[
\leq 2C_u^2 \left\| \sum_{i \in \tau_s^N} |b_i| \right\| \left( \sum_{i \in \tau_s^N} \|b_i z_{(N,i)}^*\| \right) \\
\leq 2KC_u^2 \sum_{i \in \tau_s^N} |b_i| \left( \sum_{i \in \tau_s^N} \|b_i\| |q|^{1/q} \right) \\
\leq 2KC_u^2 |\tau_s^N|^{1-1/q} \text{ by Hölder's inequality.}
\]

Combining this result with the inequality \(|\tau_s^N|^{1/q} > 2KCC_u^2\) gives the following contradiction.

\[
2KC_u^2 |\tau_s^N|^{1-1/q} < |\tau_s^N| < C \left\| \sum_{i \in \tau_s^N} x_i \right\| \leq 2KC_u^2 |\tau_s^N|^{1-1/q}
\]

As both possible cases result in a contradiction, we see that \((\oplus_{n=1}^\infty \ell_p^n)_{\ell_1}\) cannot have a greedy basis when \(1 < p < \infty\).

Finally the case \(q = \infty\) and \(1 \leq p < \infty\) in Theorem 1 is easy to handle.

**Proposition 6.** For \(1 \leq p < \infty\) the space \((\oplus \ell_p^n)_{c_0}\) does not have a greedy basis.

**Proof.** Assume that \((x_n)\) is a greedy basis of \((\oplus \ell_p^n)_{c_0}\). Since \((x_n)\) is democratic and must contain a subsequence which is equivalent to the unit vector basis of \(c_0\) it follows that for some constant \(C \geq 1\) that,

\[
\left\| \sum_{n \in A} x_n \right\| \leq C, \text{ for all finite } A \subset \mathbb{N}.
\]

This together with the unconditionality of \((x_n)\) implies that \((x_n)\) is equivalent to the unit basis of \(c_0\), which is a contradiction since \((\oplus \ell_p^n)_{c_0}\) is not isomorphic to \(c_0\). \qed
We have now finished the proof of Theorem 1 and determined which spaces of the form \((\bigoplus_{n=1}^{\infty} l_p^N)_{\ell_p}\) have a greedy basis. We now turn to the proof of Theorem 2.

We rely on the concept of greedy permutations developed by Albiac and Wojtaszczyk [AW], which we recall here. Let \(M(x)\) denote the subset of the support of \(x\) consisting of the largest coordinates of \(x\) in absolute value. We will say that a one-to-one map \(\pi: \text{supp}(x) \rightarrow \mathbb{N}\) is a greedy permutation of \(x\) if \(\pi(j) = j\) for all \(j \in \text{supp}(x) \setminus M(x)\) and if \(j \in M(x)\) then, either \(\pi(j) = j\) or \(\pi(j) \notin \text{supp}(x)\).

**Definition.** A basic sequence \((e_n)\) is defined to have property \((A)\) if for any \(x \in \text{span}(e_i)\) we have

\[
\left\| \sum_{n \in \text{supp}(x)} e_n^*(x)e_n \right\| = \left\| \sum_{n \in \text{supp}(x)} \theta_{\pi(n)} e_n^*(x)e_{\pi(n)} \right\|
\]

for all greedy permutations \(\pi\) of \(x\) and all sequences of signs \((\theta_k)\) with \(\theta_{\pi(n)} = 1\) if \(\pi(n) = n\).

We recall that \((e_n)\) is called \(C\)-suppression unconditional, for some \(C \geq 1\), if for any \((a_i) \subset c_{00}\) and any \(A \subset \mathbb{N}\),

\[
\left\| \sum_{i \in A} a_i e_i \right\| \leq C \left\| \sum_{i \in \mathbb{N}} a_i e_i \right\|.
\]

**Theorem C.** [AW] Theorem 3.4 | A basic sequence \((e_n)\) is 1-greedy if and only if \((e_n)\) is \(1\)-suppression unconditional and satisfies property \((A)\).

**Proof of Theorem 2.** We first consider the case that \(1 < p < \infty\). We show that if \(A \subset \mathbb{N}\) is any finite set, then \(\left\| \sum_{i \in A} a_i x_i \right\| = (\sum_{i \in A} |a_i|^p)^{1/p}\) for all \((a_i) \in c_{00}\). This is trivial if \(|A| = 1\) as \((x_i)\) is normalized. We now assume that the equality holds for \(|A| \leq k\) for some \(k \geq 1\). Let \((a_i) \in c_{00}\) and \(A \subset \mathbb{N}\) such that \(|A| = k + 1\). Choose \(N \in A\) such that \(|a_N| = \max_{i \in A} |a_i|\). We define \(\pi_j : A \rightarrow \mathbb{N}\) by \(\pi_j(N) = j\) and \(\pi_j(n) = n\) for all \(n \neq N\). The map \(\pi_j\) is a greedy permutation whenever \(j \notin A\), and hence by Theorem C we have the following equalities.

\[
\left\| \sum_{i \in A} a_i x_i \right\| = \left\| \sum_{i \in A, i \neq N} a_i x_i + a_N x_j \right\| \quad \text{for all} \ j \notin A
\]

\[
= \lim_{j \rightarrow \infty} \left\| \sum_{i \in A, i \neq N} a_i x_i + a_N x_j \right\|
\]

\[
= \left( \left\| \sum_{i \in A, i \neq N} a_i x_i \right\|^{p} + |a_N|^p \right)^{1/p}, \quad \text{as} \ (x_j) \text{ is normalized and weakly null,}
\]

\[
= (\sum_{i \in A} |a_i|^p)^{1/p}, \quad \text{by the induction hypothesis.}
\]

This finishes the proof of the induction step, and, thus, the proof of our claim.

The case \(p = \infty\), in which case we consider the \(c_0\)-sum of the \(E_n\), works similarly, as every normalized unconditional sequence in \((\sum E_n)_{\ell_0}\) must be weakly null.

We now consider the \(\ell_1\) case. Let \((x_i)\) be a \(1\)-greedy basis for \((\sum E_n)_{\ell_1}\). If \((x_i)\) is \(w^*\)-null with respect to the \(w^*\) topology given by \((\sum E_n^*)_{\ell_\infty}\), then the proof that \((x_i)\) is 1-equivalent...
to the unit vector basis for $\ell_1$ is the same as the previous case $1 < p < \infty$. If $(x_i)$ is not $w^*$-null, then $(x_i)$ has a subsequence $(x_{k_i})$ which converges $w^*$ to some non-zero $x \in (\sum E_n)\ell_1$. Hence $(x_{k_i} - x)$ is $w^*$-null. This implies that $\lim_{i \to \infty} \| y + x_{k_i} - x \| = \| y \| + \lim_{i \to \infty} \| x_{k_i} - x \|$ for all $y \in (\sum E_n)\ell_1$. We use this to achieve the following equalities.

$$\lim_{n \to \infty} \lim_{i \to \infty} \| x_{k_i} - x_{k_i} \| = \lim_{n \to \infty} \lim_{i \to \infty} \| (x_{k_i} - x) - (x_{k_i} - x) \|$$

$$= \lim_{n \to \infty} \| x_{k_i} - x \| + \lim_{i \to \infty} \| x_{k_i} - x \| = 2 \lim_{i \to \infty} \| x_{k_i} - x \|.$$

Furthermore,

$$\lim_{n \to \infty} \lim_{i \to \infty} \| x_{k_i} + x_k \| = \lim_{n \to \infty} \lim_{i \to \infty} \| (x_{k_i} - x) + (x_{k_i} - x) + 2x \|$$

$$= \lim_{n \to \infty} \| x_{k_i} - x + 2x \| + \lim_{i \to \infty} \| x_{k_i} - x \|$$

$$= 2\| x \| + \lim_{n \to \infty} \| x_{k_i} - x \| + \lim_{i \to \infty} \| x_{k_i} - x \| = 2\| x \| + 2 \lim_{i \to \infty} \| x_{k_i} - x \|.$$

As $(x_i)$ is 1-greedy, we must have, by Theorem C, that $\lim_{n \to \infty} \lim_{i \to \infty} \| x_{k_i} - x_{k_i} \| = \lim_{n \to \infty} \lim_{i \to \infty} \| x_{k_i} + x_{k_i} \|$. This however implies that $\| x \| = 0$, which is a contradiction with our assumption that $x$ is non-zero. \hfill \Box

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