Two-component abelian sandpile models

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In one-component abelian sandpile models, the toppling probabilities are independent quantities. This is not the case in multi-component models. The condition of associativity of the underlying abelian algebras impose nonlinear relations among the toppling probabilities. These relations are derived for the case of two-component quadratic abelian algebras. We show that abelian sandpile models with two conservation laws have only trivial avalanches.

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Keywords: Sandpile models are important toy models to understand self-organized criticality [1]. In an abelian sandpile model, the toppling rules can be encoded in an abelian algebra. This was pointed out by Dhar [2, 3, 4]. The structure of these algebras [5] (see also [6]) is very simple, their physical relevance will be shown later in the text. They are defined by taking graphs (see Fig. 1) and attaching to each vertex a generator \( a_i \) of the algebra. All the generators commute with each other. Two vertices are connected by at most two links oriented in opposite directions. To each vertex ”\( i \)” we attribute a polynomial relation which expresses a power of \( a_i \) (say \( n \)) as a polynomial in the generators attached to the sites reached by the outgoing arrows starting at ”\( i \)” as well as \( a_i \). The degree of the polynomial is at most equal to \( n \). This implies for example that for the vertex ”\( 0 \)” in Fig. 1 we have

\[
a_0^n = P(a_0, a_1, a_2, a_3, a_4). \tag{1}
\]

In the corresponding sandpile model, \( a_0^k \) is interpreted as having \( k \) grains of sand on the vertex 0. The coefficients in the polynomial \( P(a_0, a_1, a_2, a_3, a_4) \) are nonnegative and their sum is equal to 1, as we are going to see, they are going to be interpreted as probabilities.

A simple and very relevant example [1] is the case of a two-dimensional square lattice (coordinates \( (i, j) \)) the abelian algebra being given by the relations:

\[
a_{i,j}^4 = a_{i+1,j} a_{i,j+1} a_{i-1,j} a_{i,j-1}, \quad [a_{i,j}, a_{i',j'}] = 0. \tag{2}
\]

We didn’t specify the boundary conditions.

The sandpile model is defined by a stochastic process which gives the stationary state and the rules how the sand grains act.

In continuous time and a lattice with \( N \) vertices (sites) the time evolution of the system is given by a Hamiltonian \( H \):

\[
H = \sum_{i=1}^{N} w_i (1 - a_i), \tag{3}
\]

where the nonnegative coefficients \( w_i \) are transition rates and we chose the unit of time by taking:

\[
\sum_{i=1}^{N} w_i = 1. \tag{4}
\]

The Hamiltonian \( H \) acts in an \( M \)-dimensional vector space given by all the independent \( M \) monomials in the generators \( a_i \). There is a correspondence between each monomial and a configuration in the sandpile model in which to each generator appearing in the monomial corresponds sand grains on the respective sites. Their number equals the power of which the generator appears in the monomial.

The action of \( H \) on the configuration space can be understood in the following way: take a configuration, in a unit of time, with a probability \( w_i \) a grain of sand is...
One adds with a probability \( u \) at the site "i". As a result this configuration goes to others, with probabilities given by the toppling rules (1).

The unnormalized probabilities \( P_m(t) \) to find the system in a configuration "m" at the time \( t \) can be obtained from the master equation:

\[
\frac{dP(t)}{dt} = -HP(t),
\]

where

\[
P(t) = \sum_{m=1}^{M} P_m(t)W(m),
\]

\( W(m) \) is the monomial corresponding to the configuration "m".

The stationary PDF will be denoted by \( |0 \rangle \) (\( H|0 \rangle = 0 \)).

Except for positivity there are no supplementary constraints in the relations (11). This implies that one can choose at will the toppling probabilities, however as we are going to see this is not the case if one considers two-component sandpiles models.

### II. Sandpile Models with Two Kinds of Sand

In two-component abelian sandpile models, one assumes that one has two types of sand (say "a" and "b"). One adds with a probability \( u_i \) (\( v_i \)) a grain of sand "a" (respectively "b") to the site "i". Grains topple if there are more than one, of any kind, on a given site. In the toppling process, the two types of sand mix and eventually transform into each other. We are going to consider the simplest model of this kind.

Consider a one-dimensional directed lattice (Fig.2) with \( N \) sites \( \mathbb{R} \). To each site "i" we attach two generators \( a_i \) and \( b_i \), all of them being mutually commuting

\[
[a_i, a_{i'}] = [a_i, b_{i'}] = [b_i, b_{i'}] = 0 \quad \forall \ i, i' = 1, \ldots, N.
\]

For simplicity we consider quadratic algebras only. The most general quadratic relations involving nearest-neighbor interactions only are

\[
a_i^2 = \alpha_1 a_i a_{i+1} + \alpha_2 a_i b_{i+1} + \beta_1 b_i a_{i+1} + \beta_2 b_i b_{i+1} + \xi_1 a_i^2 + \xi_2 b_i^2 + \xi_3 a_i b_{i+1},
\]

\[
b_i^2 = \gamma_1 a_i a_{i+1} + \gamma_2 a_i b_{i+1} + \delta_1 b_i a_{i+1} + \delta_2 b_i b_{i+1} + \eta_1 a_i^2 + \eta_2 b_i^2 + \eta_3 a_i b_{i+1},
\]

\[
a_i b_i = \mu_1 a_i a_{i+1} + \mu_2 a_i b_{i+1} + \nu_1 b_i a_{i+1} + \nu_2 b_i b_{i+1} + \zeta_1 a_i^2 + \zeta_2 b_i^2 + \zeta_3 a_i b_{i+1},
\]

for \( i = 1, \ldots, N \), in which we take:

\[
a_{N+1} = b_{N+1} = 1.
\]

This implies that the two types of sand may leave the system on the site \( N \). The constants in (8) are positive, so that

\[
\alpha + \beta + \xi = \gamma + \delta + \eta = \mu + \nu + \zeta = 1,
\]

where we used the notation \( \alpha = \sum_{i=1}^{2} \alpha_i \), \( \beta = \sum_{i=1}^{2} \beta_i \), \( \xi = \sum_{i=1}^{3} \xi_i \), etc.

The relations (7)–(9) don’t define yet an algebra since one has still to impose associativity (diamond conditions):

\[
(a_i^2)b_i = a_i(a_i b_i) = (a_i b_i) a_i.
\]

Introducing (8) in (11) we find the following 12 relations:

\[
\xi_i = \beta_i (\mu_i - \delta_i) + \nu_i (\nu_i - \alpha_i), \quad i = 1, 2,
\]

\[
\xi_3 = \beta_1 (\mu_2 - \delta_2) + \beta_2 (\mu_1 - \delta_1) + \nu_1 (\nu_2 - \alpha_2) + \nu_2 (\nu_1 - \alpha_1),
\]

\[
\eta_i = \gamma_i (\nu_i - \alpha_i) + \mu_i (\mu_i - \delta_i), \quad i = 1, 2,
\]

\[
\eta_3 = \gamma_1 (\nu_2 - \alpha_2) + \gamma_2 (\nu_1 - \alpha_1) + \mu_1 (\mu_2 - \delta_2) + \mu_2 (\mu_1 - \delta_1),
\]

\[
\zeta_i = \beta_i \gamma_i - \mu_i \nu_i, \quad i = 1, 2,
\]

\[
\zeta_3 = \beta_1 \gamma_2 + \beta_2 \gamma_1 - \mu_1 \nu_2 - \mu_2 \nu_1,
\]

and

\[
\beta(1 + \mu - \delta) = (1 - \nu)(1 + \nu - \alpha),
\]

\[
\gamma(1 + \nu - \alpha) = (1 - \mu)(1 + \mu - \delta),
\]

\[
\beta \gamma = (1 - \mu)(1 - \nu).
\]

Out of the 3 relations (13)–(15) only two are independent since multiplying (13) with (14) one obtains (15).

As one can see from (12)–(15), in the two-component case the various toppling probabilities are constrained by nonlinear relations.

The vector space in which the Hamiltonian

\[
H = \frac{1}{N} \sum_{i=1}^{N} (1 - u_i a_i - v_i b_i), \quad u_i + v_i = 1,
\]

acts is made out of monomials in which either \( a_i \), \( b_i \) or 1 appears for a given site "i". The physical interpretation...
of these monomials is obvious. If in the monomial $a_i \, (b_j)$ appears, on the site \( \ell \) one has a grain of sand of type \( \ell \) of sand of type \( \ell \). If neither \( a_i \) or \( b_j \) appear in the monomial, there is a vacancy on the site \( \ell \).

It is easy to show the the stationary state PDF is of product form:

\[
|0\rangle = \prod_{i=1}^N \left[ \frac{(1-\alpha)(1-\beta)-\mu\nu+(1+\mu-\beta)a_i+(1+\nu-\alpha)b_i}{(2-\alpha)(2-\beta)-(1-\alpha)(1-\beta)} \right].
\]

(17)

An obvious question is if there is a solution of \([12]-[15]\) in which each type of sand is conserved separately. This would imply perhaps a new universality class of sandpile described in which each type of sand is conserved separately. This is a surprising result. We have no proof of a similar result for general quadratic algebras.

We consider now a more general graph where the vertex \( \ell \) is linked by outgoing arrows to a number of sites which we label by index \( \ell \). If one asks for two conservation laws and imposes the associativity condition \([11]\), the generalization of the solution \([18]-[20]\) reads:

\[
a_i^2 = \sum_x \alpha_x a_x a_{i+1} + \sum_{x,y} \xi_{x,y} a_x a_y,
\]

(22)

\[
b_i^2 = \sum_x \delta_x b_x b_{i+1} ,
\]

(23)

\[
a_i b_i = \sum_x \nu_x b_x a_{i+1},
\]

(24)

where the toppling probabilities satisfy the relations

\[
\sum_x \delta_x = \sum_x \nu_x = 1, \quad \alpha_x \leq \nu_x ,
\]

and

\[
\xi_{x,y} = \frac{1}{2} \{ \nu_x (\nu_y - \alpha_y) + \nu_y (\nu_x - \alpha_x) \}.
\]

(25)

Notice that, similar to the one-dimensional case, only the grains of sand of type \(a\) topple. This has as consequence that for an arbitrary graph the stationary state of the system is an absorbing state, like \([21]\), in which only grains of sand of type \(b\) are present. The proof is straightforward. Only trivial avalanches can be obtained.

We have shown that using other representations of the algebra \([15]-[20]\) which may define other sandpile models (see \([5]\), one obtains the same result: two conservation laws are compatible with trivial avalanches only.

We have checked, only for quadratic algebras, that for a vertex connected by arrows to several vertices (there was only one in example \([5]\)), the associativity conditions \([11]\) are incompatible with the existence of non-trivial avalanches in the sandpiles with two conservation laws. This is a surprising result. We have no proof of a similar statement for more general algebras (cubic, quartic, ...). Simple examples like the generalization of \([22]\), give the same result.

If we don’t insist on two conservation laws, there are various solutions of the constraints \([12]-[15]\) and therefore two-component sandpile models. They all share the property that during the toppling process, grains of sand of type \(a\) and \(b\) mutate (we have not identified a physical process which leads to such a phenomenon). We present a simple example in which we consider the abelian algebra:

\[
a_i^2 = (1-\nu) a_i a_{i+1} + \frac{\nu}{1+\mu} b_i a_{i+1} + \frac{\nu \phi}{1+\mu} a_{i+1} b_{i+1},
\]

(26)

\[
b_i^2 = (1-\mu) b_i b_{i+1} + \phi (1+\mu - 2\phi) a_i a_{i+1} + \phi^2 a_i a_{i+1} + (1-\phi)(\mu - \phi) b_{i+1}^2,
\]

(27)

\[
a_i b_i = \phi a_i a_{i+1} + (1-\phi) a_{i+1} b_{i+1}.
\]

(28)

and the Hamiltonian \([16]\) with \( u_1 = v_i = 1/2 \). This implies that we take the stationary state

\[
|0\rangle = \prod_{i=1}^N \frac{\mu \nu + (1+\mu) a_i + \mu b_i}{(1+\mu)(1+\nu)}
\]

(29)

and add with equal probability a grain of sand of type \(a\) and \(b\) on the first site. Notice that \( |0\rangle \) is independent on \( \phi \).

We are interested to know what is the probability \( P_a(T) \) to have an avalanche ending with a grain \(a\) and having a duration \(T\). The duration of the avalanche, in this one dimensional case, is the number of sites where the topplings occur. \( P_a(T) \) has a similar meaning. In \([2]\) it was shown that in the case of the one-component model which is obtained by considering only \( [22] \) in which one takes \( \phi = 0 \), at large values of \( T \), one has

\[
P_a(T) \sim T^{-3/2}.
\]

(30)

This means that for the one-component case the avalanches are in the random walk universality class \([6]\). In Fig.3 we present the results of Monte-Carlo sim-
FIG. 3: (Color online) Probability distribution functions $P_a(T)$ and $P_b(T)$ to have avalanches with duration $T$ ending with grains of sand of type "a", respectively, "b" obtained in the model given by (26)–(28). The values $\mu = 1$ and $\phi = 1/2$ were used for the parameters of the model. $4 \times 10^9$ avalanches were observed in the Monte Carlo simulations. In the figure $P_a(T)$ and $P_b(T)$ are multiplied by $T^{3/2}$ in order to show their large $T$ behavior.

We have also shown, in an example of a one-dimensional directed model, that once we allow the two components to mutate in each other during the toppling process, the one-component and the two-component models belong to the same universality class.

We believe that our conclusions apply to any multi-component models.

We would like to mention a possible extension of the quadratic algebra (8). If we omit the last of the three equations (8), two grains of sand, one of type "a" and the other one of type "b", don’t topple. As a result the stationary state is a linear combination of $4^N$ instead of $3^N$ states. If we are not interested in mutations ($a \leftrightarrow b$), one has a direct product of two algebras. One contains only $a_i$ generators (like (18)) and the other one contains only $b_i$ generators. One gets avalanches in which grains of type "a" and "b" don’t mix. If one consider the possibility of mutations the situation is different. Take for example the algebra defined by (26) and (27) (we have omitted (28)!), the total number of "grains" is not conserved since $a_{i+1}b_{i+1}$ has to be looked upon as a new kind grain. As a result, the PDFs of the duration of avalanches ending in a grain of sand of type "a", "b" or "ab" have an exponential falloff. We have also checked that if we change the algebra by allowing mutations, while conserving the number of sand grains, the avalanches belong to the random walker universality class [7].

**III. CONCLUSIONS**

The main message of this paper is that if one is interested in two-component abelian sandpile models, the toppling probabilities are not arbitrary. They have to satisfy nonlinear relations coming from the condition that the algebra is associative. These constraints don’t exist in the one-component models. An unexpected result is that, at least for the case of quadratic algebras, the coexistence of two conservation laws in the bulk (one for each component), and of nontrivial avalanches, is impossible.

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