The radial propagation of turbulence in gyro-kinetic toroidal systems

P Migliano, R Buchholz, S R Grosshauser, W A Hornsby and A G Peeters

Physics department, University of Bayreuth, Universitätstraße 30, 95447 Bayreuth, Germany

E-mail: pierluigimigliano@gmail.com

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Abstract
In this paper, a conservation equation is derived for the radially dependent entropy in toroidal geometry using the local approximation of the gyro-kinetic equation. This naturally leads to an operative definition for the turbulence intensity. It is shown that the conservation equation can be split into two contributions, one describing the dynamics of the zonal modes and one for the non-zonal modes. In essence the paper provides an operative tool for both analytic as well as numeric studies of the radial propagation of turbulence in tokamak plasmas.

Keywords: radial propagation, turbulence, gyro-kinetic

(Some figures may appear in colour only in the online journal)

1. Introduction
A detailed understanding of turbulent transport in magnetically confined plasmas is essential for the development of nuclear fusion devices. One of the key questions regards the relation between local and global model descriptions of plasma turbulence. A fundamental issue of the latter research area is understanding the role of the transport of turbulence intensity (turbulence spreading) that occurs in the global model, but is lacking in any local description. Several authors have considered this problem in the past. In [1] a fluid model is used to show that mode coupling provides an efficient mechanism for the radial propagation of turbulence in tokamaks. Furthermore, a conservation equation, for the evolution of the local intensity $I$ of the turbulence, is given in [2] in the form of a Fisher–Kolmogorov equation [3, 4] with an inhomogeneous diffusion coefficient. In the case of weak turbulence (see [5]) it takes the form

$$\frac{\partial I}{\partial t} - \frac{\partial}{\partial \psi} \left[ D(I) \frac{\partial I}{\partial \psi} \right] = \gamma I - k_\perp^2 I^2.$$  

Equation (1) has been largely used by these and other authors (see for example [8 and 9]) to tackle the problem of turbulence spreading. It provides a very useful model for the discussion of turbulence spreading, but it is affected by some deficiencies. Although physically motivated, it is not derived from first principles. Indeed, the evolution of the local turbulence intensity defined as the squared modulus of the electrostatic potential can be shown not to satisfy a conservation equation of the form given. Furthermore, the scaling of the diffusion coefficient $D = D_0 I$ itself cannot be derived. Therefore, numerical calculations can not be directly interpreted in terms of the dynamics described by this equation. Another point of concern is that there is no clear separation between turbulent and zonal intensity. With the latter, we refer to the potential perturbation connected with the zonal ($n = 0$ toroidal) mode. It is not obvious if the turbulent intensity should contain (or not contain) the zonal contribution. The points raised above...
provide a motivation to investigate the possibility of deriving analytically a conservation equation of the form given by equation (1). In this paper, we undertake this task starting from the gyro-kinetic framework. The goal is to give a solid foundation to the discussion of turbulence spreading and to derive analytic expressions for the form of the turbulent flux of turbulence intensity.

Our starting point is the choice of a quantity describing the intensity of the turbulence. A reasonable candidate is the entropy of the system, since the entropy is a measure of the departure from equilibrium and the entropy satisfies a proper conservation equation. The idea of using the entropy to define the intensity of the turbulence has already been suggested in the literature, for instance in [10] where a balance equation for the entropy density is given starting from the drift kinetic equation in cylindrical geometry. Entropy conservation in a gyro-kinetic toroidal system has already been extensively treated in the literature (see for instance [11–17]). In particular, in this paper we perform a calculation close to the one given in [16, 17] (which was performed for the case of the local limit approximation [18], considering the total entropy of the system, i.e. integrated over the entire computational domain), but we exclude the integral over the radial coordinate in order to explicitly keep track of the radial dependence of the perturbations. This procedure leads to an equation for the evolution of the radially dependent entropy of the system considered. The form of the conservation equation for the entropy leads naturally to an operative definition for the intensity of the turbulence and its conservation equation. Further analysis allows this equation to be split into two separate equations, one describing the dynamics of the turbulence intensity in the zonal ($n = 0$ toroidal) mode and the other the turbulence intensity in the perturbations (non-zonal $n ≠ 0$ toroidal modes). The symmetry and simplicity of the resulting system of equations give a genuine insight into the connection between the dynamics of zonal and non-zonal modes.

2. Turbulence intensity balance in gyro-kinetic theory

In this section, we derive the conservation equation for the radially dependent turbulence intensity of a collisionless plasma with no rotation, in the electrostatic case, for general toroidal geometry. The calculation is performed in the local limit approximation [18]. In particular, we consider the case in which background quantities do not vary across the perpendicular (to the magnetic field) extent of the domain, applying periodic boundary conditions in the binormal direction, excluding the integral over the radial coordinate. This choice, although it does not describe the most general case, allows us to study the behavior of radial inhomogeneities in the perturbations of the system.

We need an operative definition for the intensity of the turbulence, i.e. we look for a quantity which is radially dependent and satisfies a conservation equation in the form of equation (1), so that numerical results from gyro-kinetic simulations can then be properly interpreted in terms of the dynamics described by this equation. As already pointed out in the introduction, a natural candidate is the entropy of the system. We define the radially dependent entropy of the particles of the sp-species as

$$e_{sp} = -\int dx \, dv \, f_{sp}^{tot} \ln \frac{f_{sp}^{tot}}{f_{M}^{sp}}, \tag{2}$$

the radially dependent entropy of all particles is obviously obtained taking the sum over all species. In equation (2) we have $dx \, dv = 2\pi(B/m_{sp})J dx \, d\psi \, dv_{\|} \, d\mu$ with $J = \sqrt{g}$ the Jacobian of the transformation ($g$ being the determinant of the metric tensor, $m_{sp}$ the mass of the sp-species and $B$ the background magnetic field strength) and $f_{sp}^{tot} = f_{M} + f_{sp}$ is the total distribution of the sp-species written as a sum of $f_{M}$, the equilibrium Maxwell distribution as given in equation (66) of [19] and a small perturbation $f_{sp}$ of order $\rho_{s}$ to the equilibrium (where $\rho_{s} = \rho/\rho_{0}$ is the normalized reference Larmor radius, with $R$ the tokamak reference major radius and $\rho = \sqrt{2T/m} / \omega_{i}$ where $m$ is the reference mass, $\omega_{i}$ is the reference cyclotron frequency, $T$ is the reference temperature). We use gyro-center field-aligned Hamada coordinates ($X, \psi, v_{\|}, \mu$), $\psi = (\psi, s, \zeta)$ being the gyro-center position (with $\psi, s$ and $\zeta$ respectively radial, field line and binormal coordinates), $v_{\|}$ the parallel (to the magnetic field) velocity and $\mu$ the magnetic moment $\mu = m_{sp}v_{\|}^{2}/(2B)$ where $v_{s}$ is the velocity component perpendicular to the equilibrium magnetic field. We choose the Maxwell distribution $f_{M}$ as the reference distribution in the definition of the entropy to make the maximum entropy state correspond to the physical equilibrium distribution ($e_{sp}$ has a maximum when $f_{sp} = 0$). It is important to stress again that in contrast to [17] here the integral is performed over the phase space excluding the radial coordinate $\psi$ in order to explicitly keep track of the radial dependence of the perturbations.

We make a Taylor expansion of equation (2) to the second order in $\rho_{s}$ then the following approximation holds

$$e_{sp} \approx -\int dx \, dv \left( f_{sp} + \frac{f_{sp}^{2}}{2f_{M}} \right). \tag{3}$$

Note that the first term does not vanish in this case since the integral is not performed over the entire phase space. We build the equation which describes the time evolution of $e_{sp}$ using the gyro-kinetic equation given in equation (69) of [19], considering the case of a plasma as described at the beginning of this section. The time derivative of the first term in equation (3) simply gives

$$\frac{\partial n_{sp}}{\partial t} + \frac{\partial \Gamma_{n_{sp}}}{\partial \psi} = 0, \tag{4}$$

e.i. the continuity equation for the mass density, here $\psi$ is the radial coordinate, $t$ is the time, $n_{sp} = \int dx \, dv \, f_{sp}$ is the radially dependent perturbed particle density of the sp-species and $\Gamma_{n_{sp}}$ its relative radial flux given by

$$\Gamma_{n_{sp}} = \int dx \, dv \left[ \left( f_{sp} + \frac{Z_{sp}}{f_{sp}^{tot}} F_{sp} \right) v_{\|}^{2} + f_{sp} \right], \tag{5}$$

Note that the first term does not vanish in this case since the integral is not performed over the entire phase space. We build the equation which describes the time evolution of $e_{sp}$ using the gyro-kinetic equation given in equation (69) of [19], considering the case of a plasma as described at the beginning of this section. The time derivative of the first term in equation (3) simply gives
where \( Z_{sp} \) and \( T_{sp} \) are respectively the electric charge and the temperature of the sp-species, \( \chi = \mathcal{G}(\phi) \) is the gyro-averaged perturbed electrostatic potential (\( G \) is the gyro-average operator and \( \phi \) the perturbed electrostatic potential), \( v^{\perp}_{sp} \) is the radial component of the perturbed gyro-averaged \( \mathbf{E} \times \mathbf{B} \) velocity and \( v^{\parallel}_{sp} \) is the radial component of the drift velocity. The time derivative of the second term in equation (3) can be rewritten in the form

\[
\int dx \, dv \left( \frac{f^{2}_{sp}}{2F_{M}} \right) = \int dx \, dv \left( \frac{f^{2}_{sp}}{F_{M}} \frac{\partial f^{\parallel}_{sp}}{\partial t} \right),
\]

therefore we find

\[
\int dx \, dv \left[ \frac{f^{2}_{sp}}{2F_{M}} + \frac{Z_{sp} \frac{\partial f^{\parallel}_{sp}}{\partial t}}{T_{sp}} \right] = - \int dx \, dv \left( \frac{\partial }{\partial \psi} \left( \frac{f^{2}_{sp}}{2F_{M}} \right) \right) + \int dx \left( \frac{1}{L_n} - \frac{3}{2L_T} \right) J_{sp} + \frac{1}{L_T} K_{sp} \right],
\]

where \( 1/L_n \) and \( 1/L_T \) are the inverse density and temperature background gradient lengths, \( J_{sp} \) and \( K_{sp} \) are given by

\[
J_{sp} = \int dv \left[ h_{sp} (v^{\perp}_{D} + v^{\parallel}_{E}) \right],
\]

\[
K_{sp} = \int dv \left[ \frac{m_{sp} v^{2}}{2} h_{sp} (v^{\perp}_{D} + v^{\parallel}_{E}) \right],
\]

where \( h_{sp} = f_{sp} + (Z_{sp} T_{sp}) \chi F_{M} \) is the sp-species non-adiabatic gyro-center response, \( m_{sp} \) the mass of the sp-species and \( v^{2} = v^{2}_{\parallel} + v^{2}_{\perp} \) with \( v^{\parallel} \) and \( v^{\perp} \) velocity space coordinates as defined at the beginning of this section.

Equation (7) does not quite show the features of a proper conservation equation in the form of equation (1). The problem is clearly the second term in the first line which requires particular attention. In the following, we discuss how to deal with it. When integrating over the entire phase space a proper scalar product between functions of the gyro-center coordinates can be defined. Therefore, the following relation holds exactly

\[
\int d\psi \, dx \, dv \left[ G(s) t \right] = \int d\psi \, dx \, dv \left[ sG(t) \right], \tag{9}
\]

where \( s \) and \( t \) are any functions of the gyro-center coordinates and the hermiticity of the gyro-average operator \( G = \mathcal{G}^{\dagger} \) (with \( \mathcal{G}^{\dagger} \) adjoint gyro-average operator) has been used because of the local limit approximation (this identity is analogous to equation (28) in [17]). In our case, equation (9) can not be directly applied since the integration over the radial coordinate is not performed, but using periodic boundary conditions we can write

\[
\int dx \, dv \left[ G(s) t \right] = \int dx \, dv \left[ sG(t) \right] + \frac{\partial }{\partial \psi} (\Gamma_{GA}), \tag{10}
\]

with \( \Gamma_{GA} \) a periodic function of the radial coordinate only; its physical meaning will soon be clarified.

We now manipulate the second term in the first line of equation (7) according to equation (10), then we use the quasi-neutrality condition written in the form

\[
\sum_{sp} \int dv \left[ Z_{sp} G(f_{sp}) + \frac{Z_{sp} F_{M}}{T_{sp}} (G(\chi) - \phi) \right] = 0, \tag{11}
\]

and combining equations (5) and (7) we can write

\[
\frac{\partial }{\partial t} (\epsilon + w) + \frac{\partial }{\partial \psi} (\Gamma + \Gamma_{GA}) + C = 0, \tag{12}
\]

where we have defined

\[
\epsilon = \sum_{sp} \epsilon_{sp},
\]

\[
w = \sum_{sp} w_{sp} = \sum_{sp} \int dx \, dv \left[ \frac{Z_{sp} F_{M}}{2T_{sp}^{2}} (\phi^{2} - \phi^{2}) \right],
\]

\[
\Gamma = - \sum_{sp} \int dx \, dv \left[ \left( \frac{f^{2}_{sp}}{2T_{sp}^{2}} + \frac{Z_{sp} \chi F_{M}}{2T_{sp}^{2}} \right) v^{\parallel}_{D} + \left( \frac{f^{2}_{sp}}{2T_{sp}^{2}} + \frac{Z_{sp} \chi F_{M}^{2}}{2T_{sp}^{2}} \right) v^{\perp}_{D} \right],
\]

\[
C = \sum_{sp} \int dx \left( \frac{1}{L_n} - \frac{3}{2L_T} \right) J_{sp} + \frac{1}{L_T} K_{sp},
\]

while \( \Gamma_{GA} \) is the term arising from leaving out the integration over \( \psi \) when performing the operation in equation (10) with the gyro-average operator. It is interesting to note that because of equation (11) it is not possible to write a conservation equation for the entropy of one species (\( \epsilon_{sp} + w_{sp} \)). The conserved quantity is the entropy of the whole system.

Since equation (12) appears in the proper form of a conservation equation we can read out of it the physical meaning of each single term: \( \epsilon + w \) is the radially dependent entropy of the system, with \( \epsilon \) entropy in the particles and \( w \) the contribution of the electrostatic field to the entropy of the system; \( \Gamma + \Gamma_{GA} \) is the radial flux of entropy, which means that the physical effect of equation (10) is giving rise to an additional contribution to the radial flux; the last term \( C \) represents sources and sinks as fluxes in the background gradients.

The contribution of \( \Gamma_{GA} \) can be shown to be of a higher order in the Larmor radius compared to \( \Gamma \) as follows: by approximating the gyro-average operator as

\[
G \approx 1 - \frac{1}{4} \mathcal{P}^{2} \Delta,
\]

where \( \Delta \) is the normalized Laplacian operator and applying for each species the gyro-kinetic ordering

\[
\frac{f_{sp}}{F_{M}} \approx \frac{Z_{sp} \phi}{T_{sp}} \approx \rho_{s},
\]

it is straightforward to show that

\[
\Gamma_{GA} \approx \rho_{s} \Gamma.
\]
We can therefore neglect the contribution of $\Gamma_{GA}$ to the total flux of entropy. Furthermore, our purpose is to find a balance equation whose form can be directly related to equation (1) in the context of gyro-kinetic theory. Thus, we quantitatively miss a small part of the radial flux but it does not qualitatively destroy the form of the balance equation.

From now on, for simplicity in the notation, we omit the sum over the species and we get rid of the sp-index. However, each quantity in the equations has to be understood as related to a particular species and the physical equations are obtained performing the sum over all species in the system.

We consider equation (12) and subtract from it the continuity equation for the mass density (5), neglecting the contributions of $\Gamma_{GA}$ we are left with a conservation equation of the form

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial \psi} = C,$$

for the quantity

$$I = \int dx \ d\psi \left[ \frac{f^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} (\phi^2 - \chi^2) \right],$$

where $I$ is given by

$$I = \int dx \ d\psi \left[ \left( \frac{f^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) v_E^2 + \left( \frac{f^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) v_B^2 \right],$$

and $C$ given in equation (13). It is clear that equation (17) has the same form as equation (1), i.e. term by term starting from the left we have the time derivative of $I$, the radial derivative of its radial flux and the source terms. Furthermore, the quantity in equation (18) is quadratic in the perturbation. For these reasons we choose $I$ as the definition for the intensity of the turbulence equation for the mass density (5), neglecting the contributions of $\Gamma_{GA}$.

The turbulence intensity flux $\Gamma_I$ can be written in the spectral representation for $\zeta$ using the Parseval’s theorem together with the convolution theorem. Thus we obtain

$$\Gamma_I = \int d\sigma \sum_{m,n} \left( \frac{f_M}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) \alpha_n^* \alpha_m \right) v_B^2 \right],$$

where we have renamed the Fourier transform in the binormal direction of the $E \times B$ velocity as $\alpha_n = (v_B^2)_n$ in order to lighten the notation. Unrolling now the sums over $n$ and $m$ in equation (23) and using the identities

$$\alpha_0 = 0 \int d\sigma \sum_n \alpha_n \alpha_n^* = 0,$$

which hold due to the fact that $\alpha_n \propto ik_n \chi_0$, one can see that the flux $\Gamma_I$ can be split similarly to the turbulence intensity in the form $\Gamma_I = \Gamma_{I_M} + \Gamma_{I_P}$ where

$$\Gamma_{I_M} = \int d\sigma \left[ \sum_n \left( \frac{f_M}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) \alpha_n^2 \right] v_B^2 \right],$$

$$\Gamma_{I_P} = \int d\sigma \left[ \sum_{m,n} \left( \frac{f_M}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) \alpha_n^* \alpha_m \right) v_B^2 \right],$$

where the sum runs over all integers $n \in (-\infty, +\infty)$.

We can therefore neglect the contribution of $\Gamma_{GA}$ to the total flux of entropy. Furthermore, our purpose is to find a balance equation whose form can be directly related to equation (1) in the context of gyro-kinetic theory. Thus, we quantitatively miss a small part of the radial flux but it does not qualitatively destroy the form of the balance equation.

From now on, for simplicity in the notation, we omit the sum over the species and we get rid of the sp-index. However, each quantity in the equations has to be understood as related to a particular species and the physical equations are obtained performing the sum over all species in the system.

We consider equation (12) and subtract from it the continuity equation for the mass density (5), neglecting the contributions of $\Gamma_{GA}$ we are left with a conservation equation of the form

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial \psi} = C,$$

for the quantity

$$I = \int dx \ d\psi \left[ \frac{f^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} (\phi^2 - \chi^2) \right],$$

where $I$ is given by

$$I = \int dx \ d\psi \left[ \left( \frac{f^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) v_E^2 + \left( \frac{f^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) v_B^2 \right],$$

and $C$ given in equation (13). It is clear that equation (17) has the same form as equation (1), i.e. term by term starting from the left we have the time derivative of $I$, the radial derivative of its radial flux and the source terms. Furthermore, the quantity in equation (18) is quadratic in the perturbation. For these reasons we choose $I$ as the definition for the intensity of the turbulence.

Although the intensity $I$ satisfies a conservation equation, it still contains both zonal ($n = 0$) and non-zonal contributions. The intensity $I$ does not provide a meaningful definition of the turbulence intensity simply because it does not depend on turbulent fluctuations only. The entropy depends on the temperature and the intensity $I$ is, therefore, affected by the evolution of the temperature profile, preventing the separation of turbulence intensity and temperature profile evolution. In order to describe the turbulence intensity in terms of fluctuating quantities only, the possibility of separating the evolution of the zonal and the non-zonal part of $I$ is investigated. Surprisingly, it turns out that it is possible to split the evolution equation of $I$ into two separate conservation equations, one describing the dynamics of the zonal part and one describing the non-zonal part. Below, this splitting is discussed in detail.

The binormal coordinate $\zeta$ is an ignorable coordinate. Therefore, it can be treated spectrally. Using the Parseval’s theorem the integral over $\zeta$ can be replaced by a sum over the toroidal modes ($n$), then the definition of the turbulence intensity given in equation (18) is

$$I = \int d\sigma \sum_n \left[ \frac{|f|^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} (\phi^2_n - |\zeta|^2) \right].$$

The turbulence intensity flux $\Gamma_I$ is given by

$$\Gamma_I = \sum_{m,n} \left( \frac{f_M}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) \alpha_n^* \alpha_m \right) v_B^2 \right].$$

where $\alpha_n = (v_B^2)_n$ in order to lighten the notation. Unrolling now the sums over $n$ and $m$ in equation (23) and using the identities

$$\alpha_0 = 0 \int d\sigma \sum_n \alpha_n \alpha_n^* = 0,$$

which hold due to the fact that $\alpha_n \propto ik_n \chi_0$, one can see that the flux $\Gamma_I$ can be split similarly to the turbulence intensity in the form $\Gamma_I = \Gamma_{I_M} + \Gamma_{I_P}$ where

$$\Gamma_{I_M} = \int d\sigma \left[ \sum_n \left( \frac{h_0}{F_M} f_n \alpha_n^* \right) + \left( \frac{|f|^2}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) \alpha_n \right) v_B^2 \right],$$

$$\Gamma_{I_P} = \int d\sigma \left[ \sum_{m,n} \left( \frac{f_M}{2 F_M} + \frac{Z^2 F_M}{2 T^2} \right) \alpha_n^* \alpha_m \right) v_B^2 \right],$$

where $h_0 = f_0 + (ZT)\chi_0 F_M$ is the zonal modes component of the non-adiabatic gyro-center response.

Equation (25) can be considered the main result of this work. In fact, it is shown for the first time that the turbulence
intensity flux can be split into two terms, one of which (the second one) does not contain any contribution from the zonal modes.

The physical meaning of this property can be understood as follow: considering the relation
\[
\int dx \, dv \left( \frac{f_0}{F_M} \frac{\partial f}{\partial t} \right) = \int d\sigma \, \frac{\partial}{\partial t} \left( \frac{f_0}{2F_M} \right),
\]
(26)
it is clear that multiplying the gyro-kinetic equation by \( f_0/F_M \) and integrating over the entire phase space apart from the radial direction, one can find a separate conservation equation for the intensity in the zonal modes \( I_{ZM} \). In fact, applying this procedure we obtain
\[
\frac{\partial I_{ZM}}{\partial t} + \frac{\partial I_{ZM}}{\partial \psi} = C_{ZM},
\]
(27)
where \( C_{ZM} \) is a source term written as
\[
C_{ZM} = \int d\sigma \left( \frac{1}{F_M} \frac{\partial f_0}{\partial \psi} \sum_n (f_n \alpha_n^2) \right)
+ \left( \frac{1}{L_n} - \frac{3}{2} \frac{1}{L_T} + \frac{mv^2}{2} \frac{1}{L_T} \right) \hbar \nu \psi^2 \right],
\]
(28)
i.e. it is given by the flux in the gradient of the zonal perturbation plus a part arising from the drift term. It provides the source for the zonal turbulence intensity.

It is remarkable that the first term in equation (25) is exactly the turbulence intensity flux connected with the zonal modes, i.e. equation (27) shows that the zonal modes give a specific separate contribution to the turbulence intensity flux. Therefore, we can now subtract equation (27) from equation (17) and obtain a conservation equation for the turbulence intensity in the perturbations \( I_P \), i.e.
\[
\frac{\partial I_P}{\partial t} + \frac{\partial I_P}{\partial \psi} = C - C_{ZM}.
\]
(29)
This equation shows that the zonal modes enter the equation for \( I_P \) only as a modification of the source term: \( C_{ZM} \) can be interpreted as a correction to the source term of equation (17) which together with the flux in the background gradient provides the total source for the non-zonal turbulence intensity.

3. Conclusions

We have shown that starting from the conservation equation for the entropy, it is possible to write two separate conservation equations: equation (27) describes the evolution of the turbulence intensity in the zonal modes \( I_{ZM} \) and equation (29) the turbulence intensity in all the other perturbations \( I_P \). The turbulence flux connected to \( I_P \), as shown in equation (25), does not receive any contribution from the zonal modes. Equation (29) shows that the zonal modes contribute to the conservation equation for \( I_P \) as a correction to the source term given by the flux in the gradient of the zonal perturbation.

The turbulence flux non-linearity due to the \( \mathbf{E} \times \mathbf{B} \) velocity is found to be cubic in the perturbation amplitude while the one in the drift velocity is quadratic. Many works in the literature are based on the assumption that the turbulence flux can be expressed as a quadratic non-linearity in turbulence intensity (therefore 4th order, or quartic, in field amplitude), leading to equation (1), or similar formulations. The reason is that many authors use a closure scheme. In fact, this leads to an approximate expression for the turbulence flux, whereas the conservation equation derived above is exact.

This treatment gives an operative tool to actually measure the flux of turbulence in gyro-kinetic numerical calculations and can therefore be used to quantitatively study the problem of the radial propagation of turbulence in tokamak plasmas.

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