Vanishing of the first reduced cohomology with values in an $L^p$-representation.

Romain Tessera

September 11, 2018

Abstract

We prove that the first reduced cohomology with values in a mixing $L^p$-representation, $1 < p < \infty$, vanishes for a class of amenable groups including connected amenable Lie groups. In particular this solves for this class of amenable groups a conjecture of Gromov saying that every finitely generated amenable group has no first reduced $\ell^p$-cohomology. As a byproduct, we prove a conjecture by Pansu. Namely, the first reduced $L^p$-cohomology on homogeneous, closed at infinity, Riemannian manifolds vanishes. We also prove that a Gromov hyperbolic geodesic metric measure space with bounded geometry admitting a bi-Lipschitz embedded 3-regular tree has non-trivial first reduced $L^p$-cohomology for large enough $p$. Combining our results with those of Pansu, we characterize Gromov hyperbolic homogeneous manifolds: these are the ones having non-zero first reduced $L^p$-cohomology for some $1 < p < \infty$.

1 Introduction

1.1 A weak generalization of a result of Delorme.

In [Del], Delorme proved the following deep result: every connected solvable Lie groups has the property that every weakly mixing\footnote{A unitary representation is called weakly mixing if it contains no finite dimensional sub-representation.} unitary representation $\pi$ has trivial first reduced cohomology, i.e. $\overline{H}^1(G, \pi) \neq 0$. This was recently extended to connected amenable Lie groups, see [Ma, Theorem 3.3], and to a large class of amenable groups including polycyclic groups by Shalom [Sh]. Shalom also proves that this property, that he calls Property $H_{FD}$, is invariant under quasi-isometry between amenable discrete groups. Property $H_{FD}$ has nice implications in various contexts. For instance, Shalom shows that an amenable finitely generated group
with Property $H_{FD}$ has a finite index subgroup with infinite abelianization [Sh, Theorem 4.3.1]. In [CTV1], we prove [CTV1, Theorem 4.3] that an amenable finitely generated group with Property $H_{FD}$ cannot quasi-isometrically embed into a Hilbert space unless it is virtually abelian.

It is interesting and natural to extend the definition of Property $H_{FD}$ to isometric representations of groups on certain classes of Banach spaces.

In this paper, we prove that a weak version of Property $H_{FD}$, also invariant under quasi-isometry, holds for isometric $L^p$-representations of a large class of amenable groups including connected amenable Lie groups and polycyclic groups: for $1 < p < \infty$, every strongly mixing isometric $L^p$-representation $\pi$ has trivial first reduced cohomology (see Section 2 for a precise statement).

1.2 $L^p$-cohomology.

The $L^p$-cohomology (for $p$ non necessarily equal to 2) of a Riemannian manifold has been introduced by Gol’dshtein, Kuz’minov, and Shvedov in [GKS]. It has been intensively studied by Pansu [Pa2, Pa3, Pa6] in the context of homogeneous Riemannian manifolds and by Gromov [Gro2] for discrete metric spaces and groups. The $L^p$-cohomology is invariant under quasi-isometry in degree one [HS]. But in higher degree, the quasi-isometry invariance requires some additional properties, like for instance the uniform contractibility of the space [Gro2] (see also [BP, Pa6]). Most authors focus on the first reduced $L^p$-cohomology since it is easier to compute and already gives a fine quasi-isometry invariant (used for instance in [B, BP]). The $\ell^2$-Betti numbers of a finitely generated group, corresponding to its reduced $\ell^2$-cohomology [CG], have been extensively studied in all degrees by authors like Gromov, Cheeger, Gaboriau and many others. In particular, Cheeger and Gromov proved in [CG] that the reduced $\ell^2$-cohomology of a finitely generated amenable group vanishes in all degrees. In [Gro2], Gromov conjectures that this should also be true for the reduced $\ell^p$-cohomology. For a large class of finitely generated groups with infinite center, it is known [Gro2, K] that the reduced $\ell^p$-cohomology vanishes in all degrees, for $1 < p < \infty$. The first reduced $\ell^p$-cohomology for $1 < p < \infty$ is known to vanish [BMV, MV] for certain non-amenable finitely generated groups with “a lot of commutativity” (e.g. groups having a non-amenable finitely generated normal subgroup with infinite centralizer).

A consequence of our main result is to prove that the first reduced $\ell^p$-cohomology, $1 < p < \infty$, vanishes for large class of finitely generated amenable groups, in-

\footnote{We write $\ell^p$ when the space is discrete.}
cluding for instance polycyclic groups.

On the other hand, it is well known \cite{Gro2} that the first reduced $\ell^p$-cohomology of a Gromov hyperbolic finitely generated group is non-zero for $p$ large enough. Although the converse is false\footnote{In \cite{CTV2} for instance, we prove that any non-amenable discrete subgroup of a semi-simple Lie group of rank one has non-trivial reduced $L^p$-cohomology for $p$ large enough. On the other hand, non-cocompact lattices in $\mathrm{SO}(3,1)$ are not hyperbolic. See also \cite{BMV} for other examples.} for finitely generated groups, we will see that it is true in the context of connected Lie groups. Namely, a connected Lie group has non-zero reduced first $L^p$-cohomology for some $1 < p < \infty$ if and only if it is Gromov hyperbolic.

Acknowledgments. I would like to thank Pierre Pansu, Marc Bourdon and Hervé Pajot for valuable discussions about $L^p$-cohomology. Namely, Marc explained to me how one can extend a Lipschitz function defined on the boundary $\partial_\infty X$ of a Gromov hyperbolic space $X$ to the space itself, providing a non-trivial element in $H^1_p(X)$ for $p$ large enough (see the proof of Theorem 9.2 in Section 9). According to him, this idea is originally due to Gabor Elek. I would like to thank Yaroslav Kopylov for pointing out to me the reference \cite{GKS} where the $L^p$-cohomology was first introduced. I am also grateful to Yves de Cornulier, Pierre Pansu, Gilles Pisier, and Michael Puls for their useful remarks and corrections.

2 Main results

(The definitions of first $L^p$-cohomology, $p$-harmonic functions and of first cohomology with values in a representation are postponed to Section 4.)

Let $G$ be a locally compact group acting by measure-preserving bijections on a measure space $(X, m)$. We say that the action is strongly mixing (or mixing) if for every measurable subset of finite measure $A \subset X$, $m(gA \cap A) \to 0$ when $g$ leaves every compact subset of $G$. Let $\pi$ be the corresponding continuous representation of $G$ in $L^p(X,m)$, where $1 < p < \infty$. In this paper, we will call such a representation a mixing $L^p$-representation of $G$.

Definition 2.1. \cite{T1} Let $G$ be a locally compact, compactly generated group and let $S$ be a compact generating subset of $G$. We say that $G$ has Property (CF) (Controlled Følner) if there exists a sequence of compact subsets of positive measure $(F_n)$ satisfying the following properties.

- $F_n \subset S^n$ for every $n$;
• there is a constant $C < \infty$ such that for every $n$ and every $s \in S$,

$$\frac{\mu(sF_n \triangle F_n)}{\mu(F_n)} \leq C/n.$$ 

Such a sequence $F_n$ is called a controlled Følner\textsuperscript{4} sequence.

In [T1], we proved that following family\textsuperscript{5} of groups are (CF).

1. Polycyclic groups and connected amenable Lie groups;

2. semidirect products $\mathbb{Z}[\frac{1}{mn}] \rtimes \mathbb{Z}$, with $m, n$ co-prime integers with $|mn| \geq 2$ (if $n = 1$ this is the Baumslag-Solitar group $BS(1,m)$); semidirect products $(\bigoplus_{i \in I} \mathbb{Q}_{p_i}) \rtimes \mathbb{Z}$ with $m, n$ co-prime integers, and $(p_i)_{i \in I}$ a finite family of primes (including $\infty$: $\mathbb{Q}_\infty = \mathbb{R}$) dividing $mn$;

3. wreath products $F \wr \mathbb{Z}$ for $F$ a finite group.

Our main result is the following theorem.

**Theorem 1.** Let $G$ be a group with Property (CF) and let $\pi$ be a mixing $L^p$-representation of $G$. Then the first reduced cohomology of $G$ with values in $\pi$ vanishes, i.e. $\overline{H}^1(G, \pi) = 0$.

**Invariance under quasi-isometry.** The proof of [Sh, Theorem 4.3.3] that Property $H_{FD}$ is invariant under quasi-isometry can be used identically in the context of $L^p$-representations and replacing the hypothesis “weak mixing” by “mixing” since the induced representation of a mixing $L^p$-representation is also a mixing $L^p$-representation. As a result, we obtain that the property that $\overline{H}^1(G, \pi) = 0$ for every mixing $L^p$-representation is invariant under quasi-isometry between discrete amenable groups. It is also stable by passing to (and inherited by) co-compact lattices in amenable locally compact groups.

It is well known [Pu] that for finitely generated groups $G$, the first reduced cohomology with values in the left regular representation in $\ell^p(G)$ is isomorphic to the space $HD_p(G)$ of $p$-harmonic functions with gradient in $\ell^p$ modulo the constants. We therefore obtain the following corollary.

**Corollary 2.** Let $G$ be a discrete group with Property (CF). Then every $p$-harmonic function on $G$ with gradient in $\ell^p$ is constant.

\textsuperscript{4}A controlled Følner sequence is in particular a Følner sequence, so that Property (CF) implies amenability.

\textsuperscript{5}This family of groups also appears in [CTV1].
Using Von Neumann algebra technics, Cheeger and Gromov \cite{CG} proved that every finitely generated amenable group $G$ has no nonconstant harmonic function with gradient in $\ell^2$, the generalization to every $1 < p < \infty$ being conjectured by Gromov.

To obtain a version of Corollary \ref{cor:cg} for Lie groups, we prove the following result (see Theorem \ref{thm:cg}).

**Theorem 2.2.** Let $G$ be a connected Lie group. Then for $1 \leq p < \infty$, the first $L^p$-cohomology of $G$ is topologically (canonically) isomorphic to the first cohomology with values in the right regular representation in $L^p(G)$, i.e.

$$H^1_{p}(G) \simeq H^1(G, \rho_{G,p}).$$

Now, since this isomorphism induces a natural bijection

$$HD_p(G) \simeq \overline{H^1}(G, \rho_{G,p}),$$

we can state the following result that was conjectured by Pansu in \cite{Pa3}. Recall that a Riemannian manifold is called closed at infinity if there exists a sequence of compact subsets $A_n$ with regular boundary $\partial A_n$ such that $\mu_{d-1}(\partial A_n)/\mu_d(A_n) \to 0$, where $\mu_k$ denotes the Riemannian measure on submanifolds of dimension $k$ of $M$.

**Corollary 3.** Let $M$ be a homogeneous Riemannian manifold. If it is closed at infinity, then for every $p > 1$, every $p$-harmonic function on $M$ with gradient in $L^p(TM)$ is constant. In other words, $HD_p(M) = 0$.

Together with Pansu’s results \cite[Théorème 1]{Pa4}, we obtain the following dichotomy.

**Theorem 4.** Let $M$ be a homogeneous Riemannian manifold. Then the following dichotomy holds.

- Either $M$ is quasi-isometric to a homogeneous Riemannian manifold with strictly negative curvature, and then there exists $p_0 \geq 1$ such that $HD_p(M) \neq 0$ if and only if $p > p_0$;
- or $HD_p(M) = 0$ for every $p > 1$.

We also prove

**Theorem 5.** (see Corollary \ref{cor:cg2}) A homogeneous Riemannian manifold $M$ has non-zero first reduced $L^p$-cohomology for some $1 < p < \infty$ if and only if it is non-elementary\footnote{By non-elementary, we mean not quasi-isometric to $\mathbb{R}$.} Gromov hyperbolic.
To prove this corollary, we need to prove that a Gromov hyperbolic Lie group has non-trivial first reduced $L^p$-cohomology for $p$ large enough. This is done in Section 9. Namely, we prove a more general result.

**Theorem 6.** (see Theorem 9.2) Let $G$ be a Gromov hyperbolic metric measure space with bounded geometry having a bi-Lipschitz embedded 3-regular tree, then for $p$ large enough, it has non-trivial first reduced $L^p$-cohomology.

Corollary 8 and Pansu’s contribution to Theorem 4 yield the following corollary.

**Corollary 7.** A non-elementary Gromov hyperbolic homogeneous Riemannian manifold is quasi-isometric to a homogeneous Riemannian manifold with strictly negative curvature.

(See [He] for an algebraic description of homogeneous manifolds with strictly negative curvature).

## 3 Organization of the paper.

In the following section, we recall three definitions of first cohomology:

- a coarse definition of the first $L^p$-cohomology on a general metric measure space which is due to Pansu;
- the usual definition of first $L^p$-cohomology on a Riemannian manifold;
- the first cohomology with values in a representation, which is defined for a locally compact group.

In Section 5, we construct a natural topological isomorphism between the $L^p$-cohomology of a connected Lie group $G$ and the cohomology with values in the right regular representation of $G$ in $L^p(G)$. We use this isomorphism to deduce Corollary 3 from Theorem 1.

The proof of Theorem 1 splits into two steps. First (see Theorem 6.1), we prove that for any locally compact compactly generated group $G$ and any mixing $L^p$-representation $\pi$ of $G$, every 1-cocycle $b \in Z^1(G, \pi)$ is sublinear, which means that for every compact symmetric generating subset $S$ of $G$, we have

$$\|b(g)\| = o(|g|_S)$$

when $|g|_S \to \infty$, $|g|_S$ being the word length of $g$ with respect to $S$. Then, we adapt to this context a remark that we made with Cornulier and Valette (see [CTV1]...
Proposition 3.6): for a group with Property (CF), a 1-cocycle belongs to $E^1(G, \pi)$ if and only if it is sublinear. The part “only if” is an easy exercise and does not require Property (CF). To prove the other implication, we consider the affine action $\sigma$ of $G$ on $E$ associated to the 1-cocycle $b$ and use Property (CF) to construct a sequence of almost fixed points for $\sigma$.

In Section 8, we propose a more direct approach to prove Corollary 3. The interest is to provide an explicit approximation of an element of $D_p(G)$ by a sequence of functions in $W^{1,p}(G)$ using a convolution-type argument.

Finally, in Section 9, we prove that a Gromov hyperbolic homogeneous manifold has non-trivial $L^p$-cohomology for $p$ large enough. This section can be read independently.

4 Preliminaries

4.1 A coarse notion of first $L^p$-cohomology on a metric measure space

The following coarse notion of (first) $L^p$-cohomology is essentially due to [Pa6] (see also the chapter about $L^p$-cohomology in [Gro2]).

Let $X = (X, d, \mu)$ be a metric measure space, and let $p \geq 1$. For all $s > 0$, we write $\Delta_s = \{(x, y) \in X^2, d(x, y) \leq s\}$.

First, let us introduce the $p$-Dirichlet space $D_p(X)$.

- The space $D_p(X)$ is the set of measurable functions $f$ on $X$ such that
  \[ \int_{\Delta_s} |f(x) - f(y)|^p d\mu(x) d\mu(y) < \infty \]
  for every $s > 0$.

- Let $D_p(X)$ be the Banach space $D_p(X)/\mathbb{C}$ equipped with the norm
  \[ \|f\|_{D_p} = \left( \int_{\Delta_1} |f(x) - f(y)|^p \mu(x) d\mu(y) \right)^{1/p}. \]

- By a slight abuse of notation, we identify $L^p$ with its image in $D_p$.

**Definition 4.1.** The first $L^p$-cohomology of $X$ is the space

\[ H^1_p(X) = D_p(X)/L^p(X), \]

7However, the ingredients are the same: sublinearity of cocycles, and existence of a controlled Følner sequence.
and the first reduced $L^p$-cohomology of $X$ is the space
\[ \overline{H}_p^1(X) = D_p(X)/L_p^p(D_p(X)). \]

**Definition 4.2. (1-geodesic spaces)** We say that a metric space $X = (X, d)$ is 1-geodesic if for every two points $x, y \in X$, there exists a sequence of points $x = x_1, \ldots, x_m = y$, satisfying
- $d(x, y) = d(x_1, x_2) + \ldots + d(x_{m-1}, x_m)$,
- for all $1 \leq i \leq m - 1$, $d(x_i, x_{i+1}) \leq 1$.

**Remark 4.3.** Let $X$ and $Y$ be two 1-geodesic metric measure spaces with bounded geometry in the sense of [Pa6]. Then it follows from [Pa6] that if $X$ and $Y$ are quasi-isometric, then $H^1_p(X) \simeq H^1_p(Y)$ and $\overline{H}_p(X) \simeq \overline{H}_p(Y)$.

**Example 4.4.** Let $G$ be a locally compact compactly generated group, and let $S$ be a symmetric compact generating set. Then the word metric on $G$ associated to $S$,
\[ d_S(g, h) = \{ n \in \mathbb{N}, g^{-1}h \in S^n \}, \]
defines a 1-geodesic left-invariant metric on $G$. Moreover, one checks easily two such metrics (associated to different $S$) are bilipschitz equivalent. Hence, by Pansu’s result, the first $L^p$-cohomology of $(G, \mu, d_S)$ does not depend on the choice of $S$.

**Definition 4.5. (coarse notion of $p$-harmonic functions)** Let $f \in D_p(X)$ and assume that $p > 1$. The $p$-Laplacian of $f$ is
\[ \Delta_p f(x) = \frac{1}{V(x, 1)} \int_{d(x,y)\leq 1} |f(x) - f(y)|^{p-2}(f(x) - f(y))d\mu(y), \]
where $V(x, 1)$ is the volume of the closed ball $B(x, 1)$. A function $f \in D_p(X)$ is called $p$-harmonic if $\Delta_p f = 0$. Equivalently, the $p$-harmonic functions are the minimizers of the variational integral
\[ \int_{\Delta_1} |f(x) - f(y)|^p d\mu(x)d\mu(y). \]

**Definition 4.6.** We say that $X$ satisfies a Liouville $D_p$-Property if every $p$-harmonic function on $X$ is constant.

---

*Here we define a coarse $p$-Laplacian at scale 1: see [T2, Section 2.2] for a more general definition.*
As $D_p(X)$ is a strictly convex, reflexive Banach space, every $f \in D_p(X)$ admits a unique projection $\tilde{f}$ on the closed subspace $L^p(X)$ such that $d(f, \tilde{f}) = d(f, L^p(X))$. One can easily check that $f - \tilde{f}$ is $p$-harmonic. In conclusion, the reduced cohomology class of $f \in D_p(X)$ admits a unique $p$-harmonic representant modulo the constants. We therefore obtain

**Proposition 4.7.** A metric measure space $X$ has Liouville $D_p$-Property if and only if $H^1_p(X) = 0$.

### 4.2 First $L^p$-cohomology on a Riemannian manifold

Let $M$ be Riemannian manifold, equipped with its Riemannian measure $m$. Let $1 \leq p < \infty$.

Let us first define, in this differentiable context, the $p$-Dirichlet space $D_p$.

- Let $D_p$ be the vector space of continuous functions whose gradient is (in the sense of distributions) in $L^p(TM)$.

- Equip $D_p(M)$ with a pseudo-norm $\|f\|_{D_p} = \|\nabla f\|_p$, which induces a norm on $D_p(M)$ modulo the constants. Denote by $D_p(M)$ the completion of this normed vector space.

- Write $W^{1,p}(M) = L^p(M) \cap D_p(M)$. By a slight abuse of notation, we identify $W^{1,p}(M)$ with its image in $D_p(M)$.

**Definition 4.8.** The first $L^p$-cohomology of $M$ is the quotient space

$$H^1_p(M) = D_p(M)/W^{1,p}(M),$$

and the first reduced $L^p$-cohomology of $M$ is the quotient

$$\overline{H}^1_p(M) = D_p(M)/\overline{W^{1,p}(M)},$$

where $\overline{W^{1,p}(M)}$ is the closure of $W^{1,p}(M)$ in the Banach space $D_p(M)$.

**Definition 4.9.** ($p$-harmonic functions) A function $f \in D_p(M)$ is called $p$-harmonic if it is a weak solution of

$$\text{div}(|\nabla f|^{p-2}\nabla f) = 0,$$

that is,

$$\int_M \langle |\nabla f|^{p-2}\nabla f, \nabla \varphi \rangle dm = 0,$$
for every $\varphi \in C_0^\infty$. Equivalently, $p$-harmonic functions are the minimizers of the variational integral

$$\int_M |\nabla f|^p dm.$$ 

**Definition 4.10.** We say that $M$ satisfies a Liouville $D_p$-Property if every $p$-harmonic function on $M$ is constant.

As $D_p(M)$ is a strictly convex, reflexive Banach space, every $f \in D_p(M)$ admits a unique projection $\tilde{f}$ on the closed subspace $W^{1,p}(M)$ such that $d(f, \tilde{f}) = d(f, W^{1,p}(M))$. One can easily check that $f - \tilde{f}$ is $p$-harmonic. In conclusion, the reduced cohomology class of $f \in D_p(M)$ admits a unique $p$-harmonic representant modulo the constants. Hence, we get the following well-known fact.

**Proposition 4.11.** A Riemannian manifold $M$ has Liouville $D_p$-Property if and only if $H^{1,p}(M) = 0$.

**Remark 4.12.** In [Pa6], Pansu proves (in particular) that if a Riemannian manifold has bounded geometry (which is satisfied by a homogeneous manifold), then the first $L^p$-cohomology defined as above is topologically isomorphic to its coarse version defined at the previous section. In particular, the Liouville $D_p$-Property is invariant under quasi-isometry between Riemannian manifolds with bounded geometry.

### 4.3 First cohomology with values in a representation

Let $G$ be a locally compact group, and $\pi$ a continuous linear representation on a Banach space $E = E_\pi$. The space $Z^1(G, \pi)$ is defined as the set of continuous functions $b : G \to E$ satisfying, for all $g, h$ in $G$, the 1-cocycle condition $b(gh) = \pi(g)b(h) + b(g)$. Observe that, given a continuous function $b : G \to E$, the condition $b \in Z^1(G, \pi)$ is equivalent to saying that $G$ acts by affine transformations on $E$ by $\alpha(g)v = \pi(g)v + b(g)$. The space $Z^1(G, \pi)$ is endowed with the topology of uniform convergence on compact subsets.

The subspace of coboundaries $B^1(G, \pi)$ is the subspace (not necessarily closed) of $Z^1(G, \pi)$ consisting of functions of the form $g \mapsto v - \pi(g)v$ for some $v \in E$. In terms of affine actions, $B^1(G, \pi)$ is the subspace of affine actions fixing a point.

The first cohomology space of $\pi$ is defined as the quotient space

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi).$$

The first reduced cohomology space of $\pi$ is defined as the quotient space

$$\overline{H}^1(G, \pi) = Z^1(G, \pi)/\overline{B}^1(G, \pi),$$
where $\overline{B}^1(G, \pi)$ is the closure of $B^1(G, \pi)$ in $Z^1(G, \pi)$ for the topology of uniform convergence on compact subsets. In terms of affine actions, $\overline{B}^1(G, \pi)$ is the space of actions $\sigma$ having almost fixed points, i.e. for every $\varepsilon > 0$ and every compact subset $K$ of $G$, there exists a vector $v \in E$ such that for every $g \in K$,

$$\|\sigma(g)v - v\| \leq \varepsilon.$$ 

If $G$ is compactly generated and if $S$ is a compact generating set, then this is equivalent to the existence of a sequence of almost fixed points, i.e. a sequence $v_n$ of vectors satisfying

$$\lim_{n \to \infty} \sup_{s \in S} \|\sigma(s)v_n - v_n\| = 0.$$ 

### 5 $L^p$-cohomology and affine actions on $L^p(G)$. 

Let $G$ be a locally compact group equipped with a left-invariant Haar measure. Let $G$ act on $L^p(G)$ by right translations, which defines a representation $\rho_{G,p}$ defined by

$$\rho_{G,p}(g)f(x) = f(xg) \quad \forall f \in L^p(G).$$ 

Note that this representation is isometric if and only if $G$ is unimodular, in which case $\rho_{G,p}$ is isomorphic to the left regular representation $\lambda_{G,p}$. In particular, in this case, we can identify the first reduced cohomologies.

Now suppose that the group $G$ is also compactly generated and equipped with a word metric $d_S$ associated to a compact symmetric generating subset $S$. In this section, we prove that the first cohomology with values in the regular $L^p$-representation $\rho_{G,p}$ is topologically isomorphic to the first $L^p$-cohomology $H^1_p(G)$ (here, we mean the coarse version, see Section 4.1). By the result of Pansu mentioned in Remark 4.12, if $G$ is a connected Lie group equipped with left-invariant Riemannian metric $m$, we can also identify $H^1(G, \rho_{G,p})$ with the first $L^p$-cohomology on $(G, m)$ (see Section 4.2). We also obtain a direct proof of this fact.

We consider here the two following contexts: where $G$ is a compactly generated locally compact group equipped with a length function $d_S$; or $G$ is a connected Lie group, equipped with a left-invariant Riemannian metric.

Consider the linear map $J : D_p(G) \to Z^1(G, \rho_{G,p})$ defined by

$$J(f)(g) = b(g) = f - \rho_{G,p}(g)f.$$ 

$J$ is clearly well defined and induces a linear map $HJ : H^1_p(G) \to H^1(G, \rho_{G,p})$. 

11
Theorem 5.1. For $1 \leq p < \infty$, the canonical map $HJ : H^1_p(G) \to H^1(G, \rho_{G,p})$ is an isomorphism of topological vector spaces.

Let us start with a lemma.

Lemma 5.2. Let $1 \leq p < \infty$ and $b \in Z^1(G, \rho_{G,p})$. Then there exists a 1-cocycle $c$ in the cohomology class of $b$ such that

1. the map $(g, x) \mapsto c(g)(x)$ is continuous;
2. the continuous map $f(x) = c(x^{-1})(x)$ satisfies $c(g) = f - \rho_{G,p}(g)f$;
3. moreover if $G$ is a Riemannian connected Lie group, then $c$ can be chosen such that $f$ lies in $D_p(G)$ (and in $C^\infty(G)$).

Proof of the lemma. Let $\psi$ be a continuous, compactly supported probability density on $G$. We define $c \in Z^1(G, \rho_{G,p})$ by

$$c(g) = \int_G b(gh)\psi(h)dh - \int_G b(h)\psi(h)dh = \int_G b(h)(\psi(g^{-1}h) - \psi(h))dh.$$ 

Clearly it satisfies $c(1) = 0$ and we have

$$c(gg') = \int_G b(gg'h)\psi(h)dh - \int_G b(h)\psi(h)dh = \rho_{G,p}(g)\int_G b(g'h)\psi(h)dh + \int_G b(g)\psi(h)dh - \int_G b(h)\psi(h)dh.$$

But note that

$$\int_G b(g)\psi(h)dh = \int_G b(ghh^{-1})\psi(h)dh = \rho_{G,p}(g)\int_G b(h^{-1})\psi(h)dh + \int_G b(gh)\psi(h)dh = -\rho_{G,p}(g)\int_G b(h)\psi(h)dh + \int_G b(gh)\psi(h)dh.$$

So we obtain

$$c(gg') = \rho_{G,p}(g)\left(\int_G b(g'h)\psi(h)dh - \int_G b(h)\psi(h)dh\right) + \int_G b(gh)\psi(h)dh - \int_G b(h)\psi(h)dh = \rho_{G,p}(g)c(g') + c(g).$$

So $c$ is a cocycle.
Let us check that $c$ belongs to the cohomology class of $b$. Using the cocycle relation, we have

\[ c(g) = \int_G (\rho_{G,p}(g)b(h) - b(g)\psi(h))dh - \int_G b(h)\psi(h)dh \]

\[ = b(g) + \int_G (\rho_{G,p}(g)b(h) - b(h)\psi(h))dh \]

\[ = b(g) + \rho_{G,p}(g)\int_G b(h)\psi(h)dh - \int_G b(h)\psi(h)dh. \]

But since \( \int_G b(h)\psi(h)dh \in L^p(G) \), we deduce that $c$ belongs to the cohomology class of $b$.

Now, let us prove that \((g,x) \mapsto c(g)(x)\) is continuous. It is easy to see from the definition of $c$ that $g \mapsto c(g)(x)$ is defined and continuous for almost every $x$: fix such a point $x_0$. We conclude remarking that the cocycle relation implies

\[ c(g)(x) = c(xg)(x_0) - c(g)(x_0). \]

Now we can define $f(x) = c(x^{-1})(x)$ and again the cocycle relation for $b$ implies that $c(g) = f - \rho_{G,p}(g)f$.

Finally, assume that $G$ is a Lie group and choose a smooth $\hat{\psi}$. The function $\hat{\psi}$ defined by

\[ \hat{\psi}(g) = \psi(g^{-1}) \]

is also smooth and compactly supported. We have

\[ c(g)(x) = f(x) - f(xg) = \int_G b(h)(x)(\hat{\psi}(h^{-1}g) - \hat{\psi}(h^{-1}))dh. \]

Hence, $f$ is differentiable and

\[ \nabla f(x) = \int_G b(h)(x)(\nabla \hat{\psi})(h^{-1})dh, \]

and so $\nabla f \in L^p(TG)$. ■

Proof of Theorem 5.1. The last statement of the lemma implies that $HJ$ is surjective. The injectivity follows immediately from the fact that $f$ is determined up to a constant by its associated cocycle $b = I(f)$.

We now have to prove that the isomorphism $HJ$ is a topological isomorphism. This is immediate in the context of the coarse $L^p$-cohomology. Let us prove it for a Riemannian connected Lie groups. Let $S$ be a compact generating subset of $G$ and define a norm on $Z^1(G,\rho_{G,p})$ by

\[ \|b\| = \sup_{s \in S} \|b(s)\|_p. \]
Let \( \psi \) be a regular, compactly generated probability density on \( G \) as in the proof of Lemma 5.2. Denote 
\[
    f^{} \ast \psi(x) = \int_G f(xh) \psi(h).
\]
We have

**Lemma 5.3.** There exists a constant \( C < \infty \) such that for every \( f \in D_p(G) \),
\[
    C^{-1} \| f^{} \ast \psi \|_{D_p} \leq \| J(f) \| \leq C \| f \|_{D_p}.
\]

**Proof of the lemma.** First, one checks easily that if \( b \) is the cocycle associated to \( f \), then the regularized cocycle \( c \) constructed in the proof of Lemma 5.2 is associated to \( f^{} \ast \psi \).

We have
\[
    \nabla (f^{} \ast \psi)(x) = \int (f(xh) - f(x)) \nabla_x (\psi(xh)) dh \\
    = \int (f(xh) - f(x)) \nabla_x (\hat{\psi}(h^{-1}x)) dh
\]
So
\[
    \| \nabla (f^{} \ast \psi) \|_p \leq \sup_{h \in \text{Supp}(\hat{\psi})} \int |f(xh) - f(x)|^p \| \nabla \hat{\psi} \|_\infty dx
    = \sup_{h \in \text{Supp}(\hat{\psi})} \| b(h) \|^p \| \nabla \hat{\psi} \|_\infty dx,
\]
which proves the left-hand inequality of Lemma 5.3. Let \( g \in G \) and \( \gamma : [0, d(1, g)] \rightarrow G \) be a geodesic between 1 and \( g \). For any \( f \in D_p(G) \) and \( x \in G \), we have
\[
    (f - \rho_{G,p}(g)f)(x) = f(x) - f(xg) = \int_0^{d(1, g)} \nabla f(x) \cdot \gamma'(t) dt.
\]
So we deduce that
\[
    \| f - \rho_{G,p}(g)f \|_p \leq d(1, g) \| \nabla f \|_p,
\]
which proves the right-hand inequality of Lemma 5.3.

Continuity of \( HJ \) follows from continuity of \( J \) which is an immediate consequence of Lemma 5.3.

Let us prove that the inverse of \( HJ \) is continuous. Let \( b_n \) be a sequence in \( Z^1(G, \rho_{G,p}) \), converging to 0 modulo \( B^1(G, \rho_{G,p}) \). This means that there exists a sequence \( a_n \) in \( B^1(G, \rho_{G,p}) \) such that \( \| b_n + a_n \| \rightarrow 0 \). By Lemma 5.2, we can assume that \( b_n(g) = f_n - \rho_{G,p}(g)f_n \) with \( f \in D_p(G) \). On the other hand,
\[ a_n = h - \rho_{G,p}(g)h \] with \( h \in L^p(G) \). As compactly supported, regular functions on \( G \) are dense in \( L^p(G) \), we can assume that \( h \) is regular. So finally, replacing \( f_n \) by \( f_n + h_n \), which is in \( D_p(G) \), we can assume that \( J(f_n) \rightarrow 0 \). Then, by Lemma 5.3 \( \|f_n * \psi\|_{D_p} \rightarrow 0 \). But by the proof of Lemma 5.2 \( f_n * \psi \) is in the class of \( L^p \)-cohomology of \( f_n \). This finishes the proof of Theorem 5.1. \[ \square \]

6 Sublinearity of cocycles

**Theorem 6.1.** Let \( G \) be a locally compact compactly generated group and let \( S \) be a compact symmetric generating subset. Let \( \pi \) be a mixing \( L^p \)-representation of \( G \). Then, every 1-cocycle \( b \in Z^1(G, \pi) \) is sublinear, i.e.

\[ \|b(g)\| = o(|g|_S) \]

when \( |g|_S \rightarrow \infty \), \( |g|_S \) being the word length of \( g \) with respect to \( S \).

Let \( L^p(X, m) \) the \( L^p \)-space on which \( G \) acts. We will need the following lemma.

**Lemma 6.2.** Let us keep the assumptions of the theorem. There exists a constant \( C < \infty \) such that for any fixed \( j \in \mathbb{N} \),

\[ \|\pi(g_1)v_1 + \ldots + \pi(g_j)v_j\|_p \rightarrow \|v_1\|_p + \ldots + \|v_j\|_p \]

when \( d_S(g_k, g_l) \rightarrow \infty \) whenever \( k \neq l \), uniformly with respect to \( (v_1, \ldots, v_j) \) on every compact subset of \( (L^p(X, m))^j \).

**Proof of Lemma 6.2.** First, let us prove that if the lemma holds pointwise with respect to \( \sigma = (v_1, \ldots, v_j) \), then it holds uniformly on every compact subset \( K \) of \( (L^p(X, m))^j \). Let us fix some \( \varepsilon > 0 \). Equip \( (L^p(X, m))^j \) with the norm

\[ \|\sigma\| = \max_i \|v_i\|_p, \]

and take a finite covering of \( K \) by balls of radius \( \varepsilon \): \( B(\sigma, \varepsilon) \), \( \sigma \in W \), where \( W \) is a finite subset of \( K \). Take \( \min_{1 \leq k \neq l \leq j} d_S(g_k, g_l) \) large enough so that for any \( \sigma \in W \), \( \|\pi(g_1)v_1 + \ldots + \pi(g_j)v_j\|_p \) is closed to \( \|v_1\|_p + \ldots + \|v_j\|_p \) up to \( \varepsilon \). As \( \pi(g) \) preserves the \( L^p \)-norm for every \( g \in G \), we immediately see that for any \( \sigma \) in \( K \), \( \|\pi(g_1)v_1 + \ldots + \pi(g_j)v_j\|_p \) is closed to \( \|v_1\|_p + \ldots + \|v_j\|_p \) up to some \( \varepsilon' \) only depending on \( K \), \( p \) and \( \varepsilon \), and such that \( \varepsilon' \rightarrow 0 \) when \( \varepsilon \rightarrow 0 \).

So now, we just have to prove the lemma for \( v_1, \ldots, v_j \) belonging to a dense subset of \( L^p(X, m) \). Thus, assume that for every \( 1 \leq k \leq j \), \( v_k \) is bounded and compactly supported. Let us denote by \( A_k \) the support of \( v_k \). For every finite sequence \( \overline{\sigma} = g_1, \ldots, g_j \) of elements in \( G \), we write, for every \( 1 \leq i \leq j \),

\[ \text{Regular here, means either continuous, or } C^\infty \text{ if } G \text{ is a Lie group.} \]
\[ U_{i,j} = \left( \bigcup_{l \neq i} g_l A_l \right) \cap g_i A_i; \]
\[ A_{i,j} = g_i A_i \setminus U_{i,j}. \]

The key point of the proof is the following observation

**Claim 6.3.** For every \( 1 \leq i \leq j \),
\[ m(U_{i,j}) \to 0, \]
when the relative distance between the \( g_k \) goes to infinity.

**Proof of the claim.** For \( u,v \in L^2(G,m) \), write \( \langle u,v \rangle = \int_X u(x)v(x)dm(x) \). For every \( 1 \leq i \leq j \),
\[ m\left( \left( \bigcup_{l \neq i} g_l A_l \right) \cap g_i A_i \right) = \left\langle \sum_{l \neq i} \pi(g_l)1_{A_l}, \pi(g_i)1_{A_i} \right\rangle \]
\[ = \sum_{l \neq i} \langle \pi(g_l)1_{A_l}, \pi(g_i)1_{A_i} \rangle \]
\[ = \sum_{l \neq i} \langle \pi(g_l^{-1} g_i)1_{A_l}, 1_{A_i} \rangle \]
\[ = \sum_{l \neq i} m(g_l^{-1} g_i \cap A_l) \to 0 \]
by mixing property of the action. \( \Box \)

**Proof of the lemma.** First, observe that by the claim,
\[ \| \pi(g_i)v_i 1_{U_{i,j}} \|^p \leq \| v_i \|^p \cdot m(U_{i,j}) \to 0, \]
when the relative distance between the \( g_k \) goes to infinity. In other words, as \( \pi(g_i)v_i = \pi(g_i)v_i 1_{A_{i,j}} + \pi(g_i)v_i 1_{U_{i,j}} \),
\[ \| \pi(g_i)v_i 1_{A_{i,j}} - \pi(g_i)v_i \|^p \to 0. \]
In particular,
\[ \| \pi(g_i)v_i 1_{A_{i,j}} \|^p \to \| v_i \|^p. \]
On the other hand, the \( A_{i,j} \) are piecewise disjoint. So finally, we have
\[ \lim_{d_S(g_l,g_k) \to \infty} \| \pi(g_1)v_1 + \ldots + \pi(g_j)v_j \|^p = \lim_{d_S(g_l,g_k) \to \infty} \| \pi(g_1)v_1 1_{A_{1,j}} + \ldots + \pi(g_j)v_j 1_{A_{j,j}} \|^p \]
\[ = \lim_{d_S(g_l,g_k) \to \infty} \| \pi(g_1)v_1 1_{A_{1,j}} \|^p + \ldots + \| \pi(g_j)v_j 1_{A_{j,j}} \|^p \]
\[ = \| v_1 \|^p + \ldots + \| v_j \|^p. \]
which proves the lemma. ■

**Proof of Theorem 6.1.** Fix some $\varepsilon > 0$. Let $g = s_1 \ldots s_n$ be a minimal decomposition of $g$ into a product of elements of $S$. Let $m \leq n$, $q$ and $r < m$ be positive integers such that $n = qm + r$. To simplify notation, we assume $r = 1$. For $1 \leq i < j \leq n$, denote by $g_j$ the prefix $s_1 \ldots s_j$ of $g$ and by $g_{i,j}$ the subword $s_{i+1} \ldots s_j$ of $g$. Developing $b(g)$ with respect to the cocycle relation, we obtain

$$b(g) = b(s_1) + \pi(g_1)b(s_2) + \ldots + \pi(g_{n-1})b(s_n).$$

Let us put together the terms in the following way

$$b(g) = \left[b(s_1) + \pi(g_m)b(s_{m+1}) + \ldots + \pi(g_{q-1}m)b(s_{q-1}m+1)\right]$$

$$+ \left[\pi(g_1)b(s_2) + \pi(g_{m+1})b(s_{m+2}) + \ldots + \pi(g_{q-1}m+1)b(s_{q-1}m+2)\right]$$

$$+ \ldots + \left[\pi(g_{m-1})b(s_m) + \pi(g_{2m-1})b(s_{2m}) + \ldots + \pi(g_{qm})b(s_{qm+1})\right]$$

In the above decomposition of $b(g)$, consider each term between $[\cdot]$, e.g. of the form

$$\pi(g_k)b(s_{k+1}) + \ldots + \pi(g_{(q-1)m+k})b(s_{(q-1)m+k+1})$$

(6.1)

for $0 \leq k \leq m - 1$ (we decide that $s_0 = 1$). Note that since $S$ is compact and $\pi$ is continuous, there exists a compact subset $K$ of $E$ containing $b(s)$ for every $s \in S$. Clearly since $g = s_1 \ldots s_n$ is a minimal decomposition of $g$, the length of $g_{i,j}$ with respect to $S$ is equal to $j - i - 1$. For $0 \leq i < j \leq q - 1$ we have

$$d_S(g_{im+k}, g_{jm+k}) = |g_{im+k,jm+k}|s = (j - i)m \geq m.$$

So by Lemma 6.2 for $m = m(q)$ large enough, the $p$-power of the norm of (6.1) is less than

$$\|b(s_{k+1})\|_p^p + \|b(s_{m+k+1})\|_p^p + \ldots + \|b(s_{(q-1)m+k+1})\|_p^p + 1.$$

The above term is therefore less than $2q$. Hence, we have

$$\|b(g)\|_p \leq 2mq^{1/p}.$$

So for $q \geq q_0 = (2/\varepsilon)^{p/(p-1)}$, we have

$$\|b(g)\|_p/n \leq 2q^{1-1/p} \leq \varepsilon.$$

Now, let $n$ be larger than $m(q_0)q_0$. We have $\|b(g)\|_p/|g| \leq \varepsilon.$ ■
7 Proof of Theorem 1

Theorem 1 results from Theorem 6.1 and the following result, which is an immediate generalization of [CTV1, Proposition 3.6]. For the convenience of the reader, we give its short proof.

Proposition 7.1. Let $G$ be a group with property (CF) and let $\pi$ be a continuous isometric action of $G$ on a Banach space $E$. Let $b$ a 1-cocycle in $Z^1(G, \pi)$. Then $b$ belongs to $B^1(G, \pi)$ if and only if $b$ is sublinear.

**Proof**: Assume that $b$ is sublinear.

Let $(F_n)$ be a controlled Følner sequence in $G$. Define a sequence $(v_n) \in E^N$ by

$$v_n = \frac{1}{\mu(F_n)} \int_{F_n} b(g) dg.$$

We claim that $(v_n)$ defines a sequence of almost fixed points for the affine action $\sigma$ defined by $\sigma(g)v = \pi(g)v + b(g)$. Indeed, we have

$$\|\sigma(s)v_n - v_n\| = \left\| \frac{1}{\mu(F_n)} \int_{F_n} \sigma(s)b(g) dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g) dg \right\|$$

$$= \left\| \frac{1}{\mu(F_n)} \int_{F_n} b(sg) dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g) dg \right\|$$

$$= \left\| \frac{1}{\mu(F_n)} \int_{s^{-1}F_n} b(g) dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g) dg \right\|$$

$$\leq \frac{1}{\mu(F_n)} \int_{s^{-1}F_n \triangle F_n} \|b(g)\| dg.$$

Since $F_n \subset S^n$, we obtain that

$$\|\sigma(s)v_n - v_n\| \leq C \frac{1}{n} \sup_{|g| \leq n+1} \|b(g)\|$$

which converges to 0. This proves the non-trivial implication of Proposition 7.1. ■

8 Liouville $D_p$-Properties: a direct approach.

In this section, we propose a direct proof of Corollary 3. Instead of using Theorem 1 and Theorem 5.1, we reformulate the proof, only using Theorem 6.1 and [TI, Theorem 11]. The interest is to provide an explicit approximation of an element of $D_p(G)$ by a sequence of functions in $W^{1,p}(G)$ using a convolution-type argument. Since Liouville $D_p$-Property is equivalent to the vanishing of $\overline{H}^1_p(G)$,
we have to show that for every $p$-Dirichlet function on $G$, there exists a sequence of functions $(f_n)$ in $W^{1,p}(G)$ such that the sequence $(\|\nabla (f - f_n)\|_p)$ converges to zero. Let $(F_n)$ be a right controlled Følner sequence. By a standard regularization argument, we can construct for every $n$, a smooth 1-Lipschitz function $\varphi_n$ such that

- $0 \leq \varphi_n \leq 1$;
- for every $x \in F_n$, $\varphi_n(x) = 1$;
- for every $y$ at distance larger than 2 from $F_n$, $\varphi_n(y) = 0$.

Denote by $F'_n = \{x \in G : d(x, F_n) \leq 2\}$. As $F_n$ is a controlled Følner sequence, there exists a constant $C < \infty$ such that

$$\mu(F'_n \setminus F_n) \leq C\mu(F'_n)/n$$

and

$$F'_n \subset B(1, Cn).$$

Define

$$p_n = \frac{\varphi_n}{\int_G \varphi_n d\mu}.$$ 

Note that $p_n$ is a probability density satisfying for every $x \in X$,

$$|\nabla p_n(x)| \leq \frac{1}{\mu(F_n)}.$$ 

For every $f \in D_p(G)$, write $P_n f(x) = \int_X f(y)p_n(y^{-1}x) d\mu(y)$. As $G$ is unimodular,

$$P_n f(x) = \int_X f(yx^{-1})p_n(y^{-1}) d\mu(y).$$

We claim that $P_n f - f$ is in $W^{1,p}$. For every $g \in G$ and every $f \in D_p$, we have

$$\|f - \rho(g)f\|_p \leq d(1, g)\|\nabla f\|_p.$$ 

Recall that the support of $p_n$ is included in $F'_n$ which itself is included in $B(1, Cn)$. Thus, integrating the above inequality, we get

$$\|f - P_n f\|_p \leq Cn\|\nabla f\|_p,$$ 

so $f - P_n f \in L^p(G)$.

It remains to show that the sequence $(\|\nabla P_n f\|_p)$ converges to zero. We have

$$\nabla P_n f(x) = \int_G f(y)\nabla p_n(y^{-1}x) d\mu(y).$$
Since $\int_G \nabla p d\mu = 0$, we get

$$\nabla P_n f(x) = \int_G (f(y) - f(x^{-1})) \nabla p_n (y^{-1}x) d\mu(y)$$

$$= \int_G (f(yx^{-1}) - f(x^{-1})) \nabla p_n (y^{-1}) d\mu(y).$$

Hence,

$$\|\nabla P_n f\|_p \leq \int_G \|\lambda(y)f - f\|_p |\nabla p_n(y^{-1})| d\mu(y)$$

$$\leq \frac{1}{\mu(F_n)} \int_{F_n \setminus F_{n-1}} \|\lambda(y)f - f\|_p d\mu(y)$$

$$\leq \frac{\mu(F_{n+1} \setminus F_n)}{\mu(F_n)} \sup_{|g| \leq Cn} \|b(g)\|_p$$

$$\leq \frac{C}{n} \sup_{|g| \leq Cn} \|b(g)\|_p$$

where $b(g) = \lambda(g)f - f$. Note that $b \in Z^1(G, \lambda_{G,p})$. Thus, by Theorem 6.1,

$$\|\nabla P_n f\|_p \to 0.$$

This completes the proof of Corollary 3. 

\[\blacksquare\]

9 Non-vanishing of the first reduced $L^p$-cohomology on a non-elementary Gromov hyperbolic space.

Let us start with a remark about first $L^p$-cohomology on a metric measure space.

Remark 9.1. (Coupling between 1-cycles and 1-cocycles) A 1-chains on $(X,d,\mu)$ is a functions supported on $\Delta_r = \{(x,y) \in X^2, d(x,y) \leq r\}$ for some $r > 0$. The $L^p$-norm of a (measurable) 1-chain $s$ is the norm

$$\left(\int_{X^2} |s(x,y)|^p d\mu(x) d\mu(y)\right)^{1/p}.$$

A 1-chain $s$ is called a 1-cycle if $s(x,y) = s(y,x)$.

Given $f \in D_p$, we define a 1-cocycle associated to $f$ by $c(x,y) = f(x) - f(y)$, for every $(x,y) \in X^2$. Let $s$ be a 1-cycle in $L^q$, with $1/p + 1/q = 1$. We can form a coupling between $c$ and $s$

$$\langle c, s \rangle = \int_{X^2} c(x,y)s(x,y) d\mu(x) d\mu(y) = \int_{X^2} (f(x) - f(y))s(x,y) d\mu(x) d\mu(y).$$
Clearly, if \( f \in L^p \), then as \( s \) is a cycle, we have \( \langle c, s \rangle = 0 \). This is again true for \( f \) in the closure of \( L^p(X) \) for the norm of \( D_p(X) \). Hence, to prove that a 1-cocycle \( c \) is non-trivial in \( \overline{H}_p(X) \), it is enough to find a 1-cycle in \( L^q \) whose coupling with \( c \) is non-zero.

The main result of this section is the following theorem.

**Theorem 9.2.** Let \( X \) be a Gromov hyperbolic 1-geodesic metric measure space with bounded geometry having a bi-Lipschitz embedded 3-regular tree, then for \( p \) large enough, it has non-trivial first reduced \( L^p \)-cohomology.

From this theorem, we will deduce

**Corollary 8.** A homogeneous Riemannian manifold \( M \) has non-zero first reduced \( L^p \)-cohomology for some \( 1 < p < \infty \) if and only if it is non-elementary Gromov hyperbolic.

**Proof of Corollary 8.** By Theorem 4, if \( M \) has non-zero \( \overline{H}_p(M) \) for some \( 1 < p < \infty \), then being quasi-isometric to a negatively curved homogeneous manifold, it is non-elementary Gromov hyperbolic.

Conversely, let \( M \) be a Gromov hyperbolic homogeneous manifold. As \( M \) is quasi-isometric to its isometry group \( G \), which is a Lie group with finitely many components, we can replace \( M \) by \( G \), and assume that \( G \) is connected. If \( G \) has exponential growth, then [CT, Corollary 1.3] it has a bi-Lipschitz embedded 3-regular tree \( T \), and hence Theorem 9.2 applies. Otherwise \( G \) has polynomial growth, and we conclude thanks to the following classical fact.

**Proposition 9.3.** A non-elementary Gromov-hyperbolic connected Lie group has exponential growth.

**Proof:** Let \( G \) be a connected Lie group with polynomial growth. By [Gui], \( G \) is quasi-isometric to a simply connected nilpotent group \( G \), whose asymptotic cone [Pa1] is homeomorphic to another (graded) simply connected nilpotent Lie group with same dimension. Hence, unless \( G \) is quasi-isometric to \( \mathbb{R} \), the asymptotic cone of \( G \) has dimension larger or equal than 2. But [Gro2, page 37] the asymptotic cone of a Gromov hyperbolic space is an \( \mathbb{R} \)-tree, and therefore has topological dimension 1. ■

**Proof of Theorem 9.2.** The proof contains ideas that we found in [Gro2, page 258]. Roughly speaking, we start by considering a non-trivial cycle defined on a bi-Lipschitz embedded 3-regular subtree \( T \) of \( X \). To construct a 1-cocycle which has non-trivial reduced cohomology, we take a Lipschitz function \( F \) defined
on the boundary of $X$, such that $F$ is non-constant in restriction to the boundary of the subtree $T$. We then extend $F$ to a function defined $f$ on $X$ which defines a 1-cocycle in $D_p(X)$. Coupling this cocycle with our cycle on $T$ proves its non-triviality in $\overline{H}_1^p(X)$.

**Boundary at infinity of a hyperbolic space.** To denote the distance between to points in $X$ or in its boundary, we will use indifferently the notation $d(x, y)$, or the notation of Gromov $|x - y|$. Let us fix a point $o \in X$. We will denote $|x| = |x - o| = d(x, o)$.

Consider the Gromov boundary (see [Gro1, Chapter 1.8] or [GH]) of $X$, i.e. the set of geodesic rays issued from $o$ up to Hausdorff equivalence.

For $\varepsilon$ small enough, there exists [GH] a distance $| \cdot |_\varepsilon$ on $\partial_\infty X$, and $C < \infty$ such that

$$|u - v|_\varepsilon \leq \limsup_{t \to \infty} e^{-\varepsilon(v(t)|w(t))} \leq C|u - v|_\varepsilon.$$

for all $v, w \in \partial_\infty X$, where $(\cdot | \cdot)$ denotes the Gromov product, i.e.

$$(x|y) = \frac{1}{2}(|x| + |y| - |x - y|).$$

**Reduction to graphs.** A 1-geodesic metric measure space with bounded geometry is trivially quasi-isometric to a connected graph with bounded degree (take a maximal 1-separated net, and join its points which are at distance 1 by an edge). Hence, we can assume that $X$ is the set of vertices of a graph with bounded degree.

**A Lipschitz function on the boundary.** By [Gro2, page 221], $T$ has a cycle which has a non-zero pairing with every non-zero 1-cochain $c$ on $T$ supported on a single edge. Hence, to prove that $\overline{H}_1^p(X) \neq 0$, it is enough to find an element $\overline{c}$ in $D_p(X)$ whose restriction to $T$ is zero everywhere but on $e$.

The inclusion of $T$ into $X$ being bi-Lipschitz, it induces a homeomorphic inclusion of the boundary of $T$, which is a Cantor set, into the boundary of $X$. We therefore identify $\partial_\infty T$ with its image in $\partial_\infty X$. Now, consider a partition of $\partial_\infty T$ into two clopen non-empty subsets $O_1$ and $O_2$. As $O_1$ and $O_2$ are disjoint compact subsets of $\partial_\infty X$, they are at positive distance from one another. Hence, for $\delta > 0$ small enough, the $\delta$-neighborhoods $V_1$ and $V_2$ of respectively $O_1$ and $O_2$ in $\partial_\infty X$ are disjoint. Denote by $e$ the edge of $T$ joining the two complementary subtrees $T_1$ and $T_2$ whose boundaries are respectively $O_1$ and $O_2$.

Now, take a Lipschitz function $F$ on $\partial_\infty X$ which equals 0 on $V_1$ and 1 on $V_2$. 

22
Extension of $F$ to all of $X$.

Let us first assume that every point in $X$ is at bounded distance from a geodesic ray issued from $o$.

Let us define a function $f$ on $X$: for every $x$ in $X \setminus \{o\}$, we denote element by $u_x$ a geodesic ray issued from $o$ and passing at distance at most $C$ from $x$ ($C$ is a constant). Define

$$f(x) = F(u_x) \quad \forall x \in X \setminus \{o\}.$$ 

Let us prove that for $p$ large enough, $f \in D_p(X)$. Take two elements $x$ and $y$ in $X$ such that $|x - y| \leq 1$, we have

$$|u_x(t) - u_y(t)| \leq |u_x(t) - x| + |x - u_y(t)| \leq |u_x(t) - x| + |y - u_y(t)| + |x - y| \leq |u_x(t) - x| + |y - u_y(t)| + 1.$$

So for large $t$,

$$(u_x(t)u_y(t)) = |u_x(t)| + |u_y(t)| - |u_x(t) - u_y(t)| \geq |u_x(t)| + |u_y(t)| - |u_x(t) - x| - |y - u_y(t)| - 1 \geq |x| + |y| - 2C - 1 \geq 2|x| - 2(C + 1)$$

Hence, since $F$ is $K$-Lipschitz,

$$|f(x) - f(y)|^p = |F(u_x) - F(u_y)|^p \leq K^p|u_x - u_y|^p \leq K^p \lim_{t \to \infty} e^{-p\varepsilon(u_x(t)u_y(t))} \leq K^p e^{-2p\varepsilon|x|+2(C+1)p}$$

On the other hand, as $\mu(B(o, |x|)) \leq C e^{\lambda|x|}$ for some $\lambda$, if $2p\varepsilon > \lambda$, then $f$ is in $D_p(X)$.

Now, let us consider the values of $f$ along $T$. To fix the ideas, let us assume that $o$ is a vertex of $T$. For $i = 1, 2$, take $x_i$ a vertex of $T_i$. Let $e_{x_i}$ be the edge whose one extremity is $x_i$ and that separates $o$ and $x_i$. Let $T_{x_i}$ be the connected component of $T \setminus \{e_{x_i}\}$ contained in $T_i$. Let $y$ be a vertex of $T_{x_i}$ and let $z$ be an element of $\partial_\infty T_{x_i}$. One the one hand $z$ can be interpreted as a geodesic $t$ issued from $o$ and passing through $y$ in $T$, and on the other hand as a geodesic ray $v$ issued from $o$ in $X$. As $T$ is bi-Lipschitz embedded in $X$, $v$ is a quasi-geodesic ray.
in $X$. Hence it stays at bounded distance—say less than $C$—from $v$. In particular, $v$ passes at distance less than $C$ from $y$. So for $d(o,y)$ large enough, $|u_y - v| \leq \delta$. Hence, choosing $x_i$ far enough from $o$ in $T_i$, we have that all $u_y$ where $y \in T_{x_i}$ belong to $V_i$. Now, up to modify $T$, we can suppose that $x_1$ and $x_2$ are the end points of the edge $e$. Then, along $T$, $c$ takes only two values: 0 on $T_1 = T_{x_1}$ and 1 on $T_2 = T_{x_2}$. Hence its coupling with the cycle of [Gro2 page 221] is non-zero, which implies that $\overline{H}_p^1(X) \neq 0$.

**Reduction to the case when every point in $X$ is at bounded distance from a geodesic ray issued from $o$.**

Consider the graph $\tilde{X}$ obtained by gluing a copy of $N$ to every vertex of $X$. Clearly $\tilde{X}$ satisfies that every point in $X$ is at bounded distance from a geodesic ray issued from $o$. Applying the above to $\tilde{X}$, we construct an element $\tilde{f}$ in $D_p(\tilde{X})$ that has a non-trivial coupling with the cycle that we considered on $T$. Clearly the restriction $f$ of $\tilde{f}$ to $X$ also satisfies these properties and hence defines a non-trivial cocycle in $\overline{H}_p^1(X)$.

**References**

[B] M. Bourdon. *Cohomologie $L^p$ et produits amalgamés.* Geom. Ded., Vol. 107 (1), 85-98(14), 2004.

[BMV] M. Bourdon, F. Martin and A. Valette. *Vanishing and non-vanishing of the first $L^p$-cohomology of groups.* Comment. math. Helv. 80, 377-389, 2005.

[BP] M. Bourdon and H. Pajot. *Cohomologie $L^p$ et Espaces de Besov.* Journal fur die Reine und Angewandte Mathematik 558, 85-108, 2003.

[CG] A. Cheeger, M. Gromov. *$L^2$-cohomology and group cohomology.* Topology 25, 189-215, 1986.

[CT] Y. de Cornulier and R. Tessera. *Quasi-isometrically embedded free sub-semigroups.* Preprint, 2006.

[CTV1] Y. de Cornulier, R. Tessera, A. Valette. *Isometric group actions on Hilbert spaces: growth of cocycles.* Preprint, 2005.

[CTV2] Y. de Cornulier, R. Tessera, A. Valette. *Isometric group actions on Banach spaces and representations vanishing at infinity.* math.GR/0509527, 2006.
[Del] P. Delorme. 1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations. Bull. Soc. Math. France 105, 281-336, 1977.

[GH] E. Ghys and P. de la Harpe. Sur les groupes hyperboliques d’après Mikhael Gromov. Progress in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1990.

[GKS] V. M. Gol’dshteĭn, V. I. Kuz’minov, I. A. Shvedov. $L_p$-cohomology of Riemannian manifolds. (Russian) Trudy Inst. Mat. (Novosibirsk) 7, Issled. Geom. Mat. Anal. 199, 101–116, 1987.

[Gro1] M. Gromov. Hyperbolic groups. Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[Gro2] M. Gromov. Asymptotic invariants of groups. Cambridge University Press 182, 1993.

[Gui] Y. Guivarc’h. (1973). Croissance polynômiale et périodes des fonctions harmoniques. Bull. Sc. Math. France 101, 333-379.

[He] E. Heintze. On Homogeneous Manifolds with Negative Curvature. Math. Ann. 211, 23-34, 1974.

[Ho] N. Holopainen. Rough isometries and $p$-harmonic functions with finite Dirichlet integral. Rev. Mat. Iberoamericana 10, 143-176, 1994.

[HS] N. Holopainen, G. Soardi. A strong Liouville theorem for $p$-harmonic functions on graphs. Ann. Acad. Sci. Fenn. Math. 22 (1), 205-226, 1997.

[K] E. Kappos. $\ell^p$-cohomology for groups of type $FP_n$. math.FA/0511002, 2006.

[Ma] F. Martin. Reduced 1-cohomology of connected locally compact groups and applications. J. Lie Theory, 16, 311-328, 2006.

[MV] F. Martin, A. Valette. On the first $L^p$-cohomology of discrete groups. Preprint, 2006.

[Pa1] P. Pansu. Métriques de Carnot-Caratheodory et quasi-isométries des espaces symétriques de rang un. Ann. Math. 14, 177-212, 1989.
[Pa2] P. Pansu. Cohomologie $L^p$ des variétés à courbure négative, cas du degré 1. Rend. Semin. Mat., Torino Fasc. Spec., 95-120, 1989.

[Pa3] P. Pansu. Cohomologie $L^p$, espaces homogènes et pincement. Unpublished manuscript, 1999.

[Pa4] P. Pansu. Cohomologie $L^p$ en degré 1 des espaces homogènes, Preprint, 2006.

[Pa5] P. Pansu. Cohomologie $L^p$ et pincement. To appear in Comment. Math. Helvetici, 2006.

[Pa6] P. Pansu. Cohomologie $L^p$: invariance sous quasiisométries. Preprint, 1995.

[Pu] M. J. Puls. The first $L^p$-cohomology of some finitely generated groups and $p$-harmonic functions. J. Funct. Ana. 237, 391-401, 2006.

[Sh] Y. Shalom. Harmonic analysis, cohomology, and the large scale geometry of amenable groups. Acta Math. 193, 119-185, 2004.

[T1] R. Tessera. Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces. math.GR/0603138, 2006.

[T2] R. Tessera. Large scale Sobolev inequalities on metric measure spaces and applications. math.MG/0702751.

Romain Tessera
Department of mathematics, Vanderbilt University, Stevenson Center, Nashville, TN 37240 United,
E-mail: tessera@clipper.ens.fr