On Minimum Uncertainty States

Pankaj Sharan
Physics Department, Jamia Millia Islamia, New Delhi, 110 025, India

Abstract

Necessary and sufficient condition for the existence of a minimum uncertainty state for an arbitrary pair of observables is given.

Let the states of a physical system be represented by normalized vectors in a Hilbert space $H$. For two vectors $\phi$ and $\psi$ in $H$, denote the inner product by $(\psi, \phi)$ and define the norm $\|\phi\|$ of $\phi$ by $\|\phi\|^2 = (\phi, \phi)$. Let $A$ and $B$ be two observables; that is, self-adjoint operators. Let the observable $C$ be defined by the commutator $[A, B] = iC$. The expectation value $(\psi, A\psi)$ of $A$ is denoted by $a$. Similarly, expectation values of $B$ and $C$ in the state $\psi$ are denoted $b$ and $c$ respectively.

The statement of the uncertainty inequality is

$$\Delta A \Delta B \geq \frac{1}{2} |c|,$$

(1)

where the variance (or uncertainty) of $A$ in the state $\psi$ is defined as $\Delta A = \| (A - a) \psi \|$ and a similar formula for $\Delta B$. We say that $\psi$ is a minimum uncertainty state (MUS) for the pair $A, B$ if the equality is achieved in (1) above, that is, if

$$\Delta A \Delta B = \frac{1}{2} |c|.$$

(2)

The proof of the uncertainty inequality is a direct application of the Schwarz inequality which states that

$$|(\psi, \phi)| \leq \|\psi\| \|\phi\|$$

(3)

for any two vectors $\phi$ and $\psi$ in $H$. We assume that one of the vectors (say $\phi$) is non-zero to avoid triviality. The Schwarz inequality becomes an equality if and only if $\psi$ can be written as the other (non-zero) vector $\phi$ multiplied by a complex number $z$

$$\psi = z\phi.$$

(4)
The proof of the uncertainty inequality is as follows. Denote by $\text{Im} \ z$ the imaginary part of a complex number $z$. The Schwarz inequality implies

$$\Delta A \Delta B = \| (A - a) \psi \| \| (B - b) \psi \|$$

$$\geq \| (A - a) \psi, (B - b) \psi \| \quad \text{Inequality 1}$$

$$\geq |\text{Im}((A - a) \psi, (B - b) \psi)| \quad \text{Inequality 2}$$

$$= \left| \frac{1}{2i} \left[ ((A - a) \psi, (B - b) \psi) - ((B - b) \psi, (A - a) \psi) \right] \right|$$

$$= \frac{1}{2} |c|.$$

The condition for $\psi$ to be a MUS for $A, B$ is that at both the places above (Inequality 1 and 2) the equality must be satisfied. The first one is satisfied if and only if there is a complex number $z$ such that

$$(A - a) \psi = z(B - b) \psi \quad (5)$$

where we assume $\Delta B = \| (B - b) \psi \| \neq 0$ to avoid the trivial case when both $\Delta A$ and $\Delta B$ are zero. By taking norm on both sides of the above equation we also note that

$$\Delta A = |z| \Delta B. \quad (6)$$

The second inequality (Inequality 2) becomes an equality if and only if the real part of $((A - a) \psi, (B - b) \psi)$ is zero. This happens if

$$(A - a) \psi, (B - b) \psi) + ((B - b) \psi, (A - a) \psi) = 0$$

which, in the light of $(A - a) \psi = z(B - b) \psi$ implies that $\text{Re} \ z = 0$. In other words, $z = i \lambda$ for a real number $\lambda$. The magnitude of $\lambda$ follows from (6) above as

$$|\lambda| = \frac{\Delta A}{\Delta B}. \quad (7)$$

To obtain the sign of $\lambda$ we proceed as follows. Write $z = i \lambda$ in (5) and calculate

$$\| (A - i \lambda B) \psi \|^2 = |a - i \lambda b|^2 = a^2 + \lambda^2 b^2. \quad (8)$$

The left hand side is

$$\| (A - i \lambda B) \psi \|^2 = ((A - i \lambda B) \psi, (A - i \lambda B) \psi) = (\psi, (A + i \lambda B)(A - i \lambda B) \psi),$$
and

$$(A + i\lambda B)(A - i\lambda B) = A^2 + \lambda^2 B^2 + \lambda C.$$  

Substituting these in (5) and using $(\Delta A)^2 = (\psi, A^2\psi) - a^2$, $\Delta A = |\lambda|\Delta B$ etc. we get,

$$2\lambda^2(\Delta B)^2 + \lambda c = 0$$

which shows that the sign of $\lambda$ must be opposite to that of $c$.

With the notation as above, we have proved the following theorem:

For $\psi$ to be a MUS for the pair $A, B$ (with $\Delta B \neq 0$) the necessary and sufficient condition is that

$$(A - a)\psi = i\lambda(B - b)\psi$$

where $\lambda$ is a real number whose magnitude is given by $|\lambda| = \Delta A/\Delta B$ and whose sign is opposite to that of $c$.

We see that the condition for MUS can also be written as

$$(A - i\lambda B)\psi = (a - i\lambda b)\psi,$$  

(9)

which means that $\psi$ must be an eigenvector of the non-hermitian operator $A - i\lambda B$ with the complex eigenvalue $a - i\lambda b$.

A well-known example of MUS is the gaussian wave-packets in one dimension:

$$\psi = \frac{1}{(2\pi \sigma^2)^{1/4}} \exp \left[ ikx - \frac{(x - x_0)^2}{4\sigma^2} \right],$$  

(10)

for the pair of operators $A = x$ and $B = -id/dx$. Here $a = x_0, b = k, \Delta A = \sigma$ and $\Delta B = 1/(2\sigma)$. Thus $|\lambda| = 2\sigma^2$, and because $c = 1 > 0$ we have $\lambda = -2\sigma^2$. One can check that the wave packet above is the eigenfunction of the operator

$$\left( x + 2\sigma^2 \frac{d}{dx} \right)$$

with complex eigenvalue $x_0 + 2i\sigma^2 k$.

**Acknowledgement**

I thank Pravabati Chingangbam and Tabish Qureshi for useful discussions on the uncertainty inequality in 1995 and 2011 respectively.