Edge Intersection Graphs of $L$-Shaped Paths in Grids

Kathie Cameron\textsuperscript{1} Steven Chaplick\textsuperscript{2} Chúň T. Hoàng\textsuperscript{3}

\textsuperscript{1} Department of Mathematics, Wilfrid Laurier University, Waterloo, On., Canada, N2L 3C5, e-mail: kcameron@wlu.ca
\textsuperscript{2} Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, On., Canada, N2L 3C5, e-mail: chaplick@cs.toronto.edu
\textsuperscript{3} Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, On., Canada, N2L 3C5, e-mail: choang@wlu.ca

Abstract. In this paper we continue the study of the edge intersection graphs of single bend paths on a rectangular grid (i.e., the edge intersection graphs where each vertex is represented by one of the following shapes: \(\langle, \rangle\), \(\langle, \rangle\)). These graphs, called $B_1$-EPG graphs, were first introduced by Golumbic et al (2009) \cite{4}. We focus on the class \([\langle\rangle]\) (the edge intersection graphs of $\langle\rangle$-shapes) and show that testing for membership in \([\langle\rangle]\) is NP-complete. We then give a characterization and polytime recognition algorithm for special subclasses of $\text{Split} \cap [\langle\rangle]$. We also consider the natural subclasses of $B_1$-EPG formed by the subsets of the four single bend shapes (i.e., \([\langle\rangle], [\langle, \rangle], [\langle, \rangle], [\langle, \rangle, \rangle]\) – note: all other subsets are isomorphic to these up to 90 deg rotation). We observe the expected strict inclusions and incomparability (i.e., \([\langle\rangle]\) $\subset$ \([\langle, \rangle]\), \([\langle, \rangle]\) $\subset$ \([\langle, \rangle, \rangle]\) $\subset$ \([\langle, \rangle, \rangle]\) $\subset$ \([\text{B}_1\text{-EPG}]$ and \([\langle, \rangle]\) is incomparable with \([\langle, \rangle]\)).

1 Introduction

A graph $G$ is called an $\text{EPG graph}$ if $G$ is the intersection graph of paths on a grid, where each vertex in $G$ corresponds to a path on the grid and two vertices are adjacent in $G$ iff the corresponding paths share an edge on the grid. EPG graphs were introduced by Golumbic et al \cite{4}. The motivation for studying these graphs comes from circuit layout problems \cite{2}. Golumbic and Jamison \cite{5} proved that the recognition problem for the edge intersection graphs of paths in trees (EPT) is NP-complete even when restricted to chordal graphs. Golumbic et al \cite{4} defined a $B_k$-EPG graph to be the edge intersection graph of paths on a grid where the paths are allowed to have at most $k$ bends (turns). The $B_0$-EPG graphs are exactly the well studied $\text{interval graphs}$ (the intersection graphs of intervals on a line).

Heldt et al \cite{6} proved that the recognition problem for $B_1$-EPG is NP-complete. A graph is $\text{chordal}$ if it does not contain a chordless cycle with at

* Research support by Natural Sciences and Engineering Research Council of Canada.
least four vertices as an induced subgraph. A graph is a split graph if its vertices can be partitioned into a clique and a stable set. Asinowski and Ries [1] characterized special subclasses of chordal $B_1$-EPG graphs.

Consider a $B_1$-EPG graph $G$ with a path representation on a grid. The paths can be of the following four shapes: $\sqcup$, $\sqcap$, $\sqcup$, $\sqcap$. In this paper, we study $B_1$-EPG graphs whose paths on the grid belong to a proper subset the four shapes. If $S$ is a subset of $\{\sqcup, \sqcap, \sqcup, \sqcap\}$, then $[S]$ denotes the class of graphs that can be represented by paths whose shapes belong to $S$. We are interested in the class $[\sqcup]$ of $B_1$-EPG graphs whose paths are of the type $\sqcup$. Our main results are:

- A proof of NP-completeness of recognition of $[\sqcup]$.
- Characterizations of, and recognition algorithms for gem-free split $[\sqcup]$-graphs and bull-free split $[\sqcup]$-graphs, where Split denotes the class of split graphs.
- Establishment of expected separation between the classes: $[\sqcup] \subsetneq [\sqcup, \sqcap], [\sqcup, \sqcup] \subsetneq [\sqcup, \sqcap, \sqcup, \sqcap] \subsetneq B_1$-EPG and the incomparability between $[\sqcup, \sqcap]$ and $[\sqcup, \sqcup]$.

In section 2 we discuss background results and establish some properties of $B_1$-EPG graphs. In section 3 we show that recognition of $[\sqcup]$ is an NP-complete problem. In section 4 we give polytime recognition algorithms for the classes $[\sqcup]$ of gem-free split graphs, and of bull-free split graphs. Finally, we conclude with some open questions in section 5.

## 2 Properties of $B_1$-EPG graphs

Let $\mathcal{P}$ be a collection of nontrivial simple paths on a grid $G$. The edge intersection graph $\text{EPG}(\mathcal{P})$ has a vertex $v$ for each path $P_v \in \mathcal{P}$ and two vertices are adjacent in $\text{EPG}(\mathcal{P})$ if the corresponding paths in $\mathcal{P}$ share an edge of $G$. For any grid edge $e$, the set of paths containing $e$ is a clique in $\text{EPG}(\mathcal{P})$; such a clique is called an edge-clique [4]. A claw in a grid consists of three grid edges meeting at a grid point. The set of paths which contain two of the three edges of a claw is a clique; such a clique is called a claw-clique [4] (see figure 1).

![Fig. 1. Left: An edge-clique. Right: A claw-clique.](image)

**Lemma 1 (4).** Consider a $B_1$-EPG representation on a grid of a graph $G$. Every clique in $G$ corresponds to either an edge-clique or a claw clique.

An asteroidal triple (AT) is a set of three vertices such that for every pair, there is a path between them which avoids the neighbourhood of the other vertex.

**Lemma 2 (AT Lemma [1], Theorem 9).** In a $B_1$-EPG graph, no vertex can have an AT in its neighbourhood.
Let \( C_4 \) denote the chordless cycle \( a, b, c, d, a \) on four vertices. Golumbic et al. [4] proved that any \( B_1 \)-EPG representation of \( C_4 \) corresponds to what they call a “true pie”, a “false pie”, or a “frame”. True and false pies require paths other than \( \cup \)'s. A frame is a rectangle in the grid \( G \) such that each corner is the bend-point for one of \( P_a, P_b, P_c \) and \( P_d \); \( P_a \cap P_b, P_b \cap P_c, P_c \cap P_d \), and \( P_d \cap P_a \) each contain at least one edge; and \( P_a \cap P_c \) and \( P_b \cap P_d \) each do not contain an edge. Consider the \( C_4 \) and its four representations shown in figure 2. The first two representations are frames, the third is a false pie, and the fourth is a true pie. It follows that:

\[
\text{Fig. 2. Left: } C_4 \text{ and its representations. Right: } K_{2,3} \text{ and its representations.}
\]

Lemma 3 (\( C_4 \) Lemma). In the \([\cup, \cap]\)-representation of a \( C_4 \) every \( \cup \) has a neighbour on both its vertical segment and its horizontal segment.

Observation 1 \( K_{2,3} \) is in \([\cup, \cap]\).

Proof. See figure 2 for a \([\cup, \cap]\)-representation of \( K_{2,3} \).

Lemma 4 (\( K_{2,3} \) Lemma). In an \([\cup, \cap]\)-representation of a \( K_{2,3} \) every \( \cup \) (and \( \cap \)) has a neighbour on both its vertical segment and on its horizontal segment.

Proof. Consider \( K_{2,3} \) to be the complete bipartite graph with bipartition \{\{a, b\}, \{c, d, e\}\}. Note that each of the following is a \( C_4 \): \( a, c, b, d, a \); \( a, c, b, e, a \); and \( a, d, b, e, a \). As noted above, any \( B_1 \)-EPG representation of \( C_4 \) corresponds to a “true pie”, a “false pie”, or a “frame”. True pies require paths all four types, but false pies and frames can be made from just \( \cup \)'s and \( \cap \)'s.

If an \([\cup, \cap]\)-representation of a \( C_4 \) corresponds to a frame, then every \( \cup \) (and \( \cap \)) has a neighbour on both its vertical segment and on its horizontal segment. Consider an \([\cup, \cap]\)-representation of a \( K_{2,3} \).

Suppose first that both \{\( a, c, b, d\)\} and \{\( a, c, b, e\)\} correspond to frames. Then \( P_d \) and \( P_e \) must have the same bend-point, and this bend-point must be an intersection point of \( P_a \) and \( P_b \). Since \( d \) and \( e \) are not adjacent, one of \( P_d \) and \( P_e \) is an \( \cup \) and the other is an \( \cap \). It follows that every \( \cup \) (and \( \cap \)) has a neighbour on both its vertical segment and on its horizontal segment.

Now suppose that \{\( a, c, b, d\)\} corresponds to a false pie. If \( P_a \) and \( P_b \) have the same bend-point, the bend-point must be an intersection point of \( P_a \) and \( P_b \), and then there is nowhere to place \( P_c \) so that it intersects both \( P_a \) and \( P_b \). So it must be that \( P_c \) and \( P_d \) have the same bend-point, which must be an intersection point of \( P_a \) and \( P_b \), say point \( p \). Then \( P_c \) must have bend-point at an intersection point of \( P_a \) and \( P_b \), but since \( e \) is not adjacent to \( c \) or to \( d \),
this must be a different intersection point from \( p \). So we have a configuration such as that in figure 2 and it follows that every \( \downarrow \) (and \( \uparrow \)) has a neighbour on both its vertical segment and on its horizontal segment. (Note that \( \{ a, c, b, e \} \) corresponds to a frame.)

It is not possible for both \( \{ a, c, b, d \} \) and \( \{ a, c, b, e \} \) to both correspond to false pies.

**Observation 2** \( K_{2,3} \) is in \( [\downarrow, \uparrow] \) but not in \( [\downarrow, \uparrow^\ast] \).

**Proof.** Again, recall that \( a, c, b, d, a \) and \( a, c, b, e, a \) are \( C_4 \)s in \( K_{2,3} \). True and false pies are not representable using just \( \downarrow \)s and \( \uparrow \)s. So both of these must be represented as frames. As argued above, \( P_d \) and \( P_e \) must have the same bend-point. But since \( d \) and \( e \) are not adjacent, if \( P_d \) is an \( \downarrow \), then \( P_e \) must be an \( \uparrow^\ast \) and vice versa. It follows that \( K_{2,3} \) is not in \( [\downarrow, \uparrow^\ast] \).

\[ \square \]

![Fig. 3. Left: 3-sun and its representation. Right: 4-wheel and its representation](image)

**Observation 3** The 3-Sun is in \( [\downarrow^\ast, \uparrow^\ast] \) but not in \( [\downarrow, \uparrow] \).

**Proof.** See figure 3 for the 3-sun and an \( [\downarrow^\ast, \uparrow^\ast] \)-representation of the 3-sun. To see that the 3-sun does not have an \( [\downarrow, \uparrow] \)-representation, recall that in a \( B_1 \)-EPG graph, every clique is an edge-clique or a claw-clique. The vertices of the 3-sun can be partitioned into a clique with vertices \( a, b, c \) and a stable set with vertices \( d, e, f \) with edges \( da, dc, ea, eb, fb, fc \). It is easy to see that if the clique \( \{ a, b, c \} \) is an edge-clique, then only two of \( d, e, f \) can be represented regardless of which types of 1-bend paths are used. So the clique \( \{ a, b, c \} \) is a claw-clique. But an \( \downarrow^\ast \) and \( \uparrow^\ast \) can not be together in a claw-clique.

\[ \square \]

**Observation 4** The 4-wheel is in \( [\downarrow^\ast, \uparrow^\ast, \uparrow^\ast] \) but not in \( [\downarrow, \uparrow] \) or \( [\downarrow, \uparrow^\ast] \).

**Proof.** See figure 3 for the 4-wheel and an \( [\downarrow^\ast, \uparrow^\ast, \uparrow^\ast] \)-representation of the 4-wheel. Lemma 3 in [1] shows that in a \( B_1 \)-representation of \( W_4 \), the \( C_4 \) corresponds to a true pie or false pie. Since the true pie requires four shapes, we may assume the \( C_4 \) of the \( W_4 \) is represented by a false pie. So, \( W_4 \) is not an \( [\downarrow, \uparrow] \)-graph. Consider the vertex \( u \) of \( W_4 \) that is adjacent to all vertices of the \( C_4 \). If \( P_u \) is of type \( \downarrow \) or \( \uparrow^\ast \), then \( P_u \) can not share an edge grid with all four paths of the \( C_4 \). So, the \( W_4 \) is not an \( [\downarrow, \uparrow] \)-graph.

\[ \square \]
3 NP-Hardness: Recognition of \([\cdot] \)

It is well-known that interval graphs (i.e., \(B_0\)-EPG graphs) can be recognized in polynomial time \([3]\). The complexity of the recognition problem for \(B_k\)-EPG \((k > 0)\) was given as an open problem in the paper introducing EPG graphs \([4]\). The recognition problem for \(B_1\)-EPG has been shown to be NP-complete in a recent paper \([6]\). In this section we consider the complexity of recognizing the simplest natural subclass of \(B_1\)-EPG which is a superclass of \(B_0\)-EPG; namely, \([\cdot]\). Specifically, we show that it is NP-complete to decide membership in \([\cdot]\).

**Theorem 5.** Deciding membership in \([\cdot]\) is NP-complete.

**Proof.** A given \([\cdot]\) model is easily verified, so \([\cdot]\) recognition is in NP. For NP-hardness we demonstrate a reduction from the usual 3-SAT problem (defined below). Our reduction is inspired by the NP-completeness proof for \(B_1\)-EPG \([6]\).

The essential ingredients of our construction are described in the following observations. In an \([\cdot]\)-representation \(H\) of a graph \(G\) with vertices \(u, v\), we say that \(v\) is an internal neighbor of \(u\) in \(H\) when: \(v\) is adjacent to \(u\), \(P_u\)'s bend-point is not contained in \(P_v\) and w.l.o.g. \(P_v\)'s horizontal contains \(P_v\)'s horizontal (see figure 4(i)). We also say that \(v\) is an external neighbor of \(u\) when \(v\) is adjacent to \(u\) but \(v\) is not an internal neighbor of \(u\). Notice that, in any \([\cdot]\)-representation of a graph a vertex can have at most four stable external neighbors (as depicted in figure 4(ii)). Additionally, if a vertex \(v\) is an internal neighbor of a vertex \(u\), then \(v\) can have at most two stable external neighbors which are not adjacent to \(u\) (see figure 4(iii)). Finally, we say that a vertex \(u\) is adjacent to a \(C_4\) when \(u\) is adjacent to exactly one vertex in an induced \(C_4\) (see figure 4(iv)). Now consider a graph \(G\) with a vertex \(u\) which is adjacent to a \(C_4\) and let \(v\) be \(u\)'s neighbor in this \(C_4\). Recall that, in any \([\cdot]\)-representation of an induced \(C_4\), every \([\cdot]\)-path has a neighbor with an edge intersection on its vertical and a neighbor with an edge intersection on its horizontal (by Lemma 3). Thus, in any \([\cdot]\)-representation of \(G\), \(v\) is necessarily an external neighbor of \(u\). With these observations in mind we can now describe the structure of our graph \(G_\Phi\). A 3-SAT formula \(\Phi\) is a boolean formula over variables \(x_1, \ldots, x_k\) where \(\Phi\) is a conjunction of \(t\) clauses \((D_1, D_2, \ldots, D_t)\), each clause \(D_i\) \((1 \leq i \leq t)\) is a disjunction of three literals.

![Fig. 4.](image-url)
\((\ell_1, \ell_2, \ell_3)\), and each literal \(\ell_{iq}\) (1 \(\leq q \leq 3\)) is either the negation or non-negation of some variable \(x_j\) (1 \(\leq j \leq k\)). Given a 3-SAT formula \(\Phi\), it is well known that it is NP-complete to decide whether there exists an assignment to the variables of \(\Phi\) that satisfies \(\Phi\) \[7\].

Given a 3-SAT formula \(\Phi\) we will construct a graph \(G_\Phi\) such that \(G_\Phi\) is an \([\land]\)-graph iff \(\Phi\) can be satisfied. \(G_\Phi\) consists of an induced subgraph \(G_{D_i}\) for each clause \(D_i\) of \(\Phi\) and a variable gadget to identify the clauses with their corresponding literals. The general form of these gadgets is given in figure 5. We begin by describing the structure of the \([\land]\)-representation of the variable gadget. Notice that the vertex \(X\) is adjacent to four \(C_4\)s. Thus, as we have observed, \(X\) will have four external neighbors in any \([\land]\)-representation of \(G_\Phi\). Furthermore, since the neighborhood of \(X\) is a stable set, the vertices \(C, x_1, x_2, ..., x_k, \overline{x_1}, \overline{x_2}, ..., \overline{x_k}\) are all internal neighbors of \(X\). Finally, suppose that \(x_j\) is an internal horizontal neighbor of \(X\). Since \(G_\Phi[\{X, x_j, \overline{x_j}, \overline{z_j}\}]\) is a \(C_4\), \(\overline{x_j}\) is necessarily an internal vertical neighbor of \(X\). Similarly, if \(x_j\) were to be an internal vertical neighbor of \(X\), \(\overline{x_j}\) would necessarily be an internal horizontal neighbor of \(X\) \[4\]. From these observations we depict the general structure of an \([\land]\)-representation of the subgraph of \(G_\Phi\) induced by \(\{X, x_1, ..., x_k, \overline{x_1}, ..., \overline{x_k}, z_1, ..., z_k\}\) and the \(C_4\)s adjacent to these vertices in figure 5.

Now, w.l.o.g., suppose that \(C\) is an internal horizontal neighbor of \(X\). Notice that \(C\) is adjacent to two \(C_4\)s, is an internal horizontal neighbor of \(X\), and the neighborhoods of \(X\) and \(C\) are disjoint. Thus, since the neighborhood of \(C\) is a stable set, the vertices \(c_1, ..., c_t\) are internal vertical neighbors of \(C\). Similarly, for each \(1 \leq i \leq t\), \(d_i\) is an internal horizontal neighbor of \(c_i\) since each \(c_i\) is an internal vertical neighbor of \(C\) and each \(c_i\) is adjacent to two \(C_4\)s. These observations provide the general structure of an \([\land]\)-representation of the subgraph of \(G_\Phi\).

\[4\] We will later use the location (i.e., as an internal horizontal or internal vertical neighbor of \(X\)) as a variable’s truth value.
induced by \( \{X, C, c_1, \ldots, c_t, d_1, \ldots, d_t\} \) and the \( C_4 \)'s adjacent to these vertices (as seen in figure 6). With the restricted structure of the variable gadget in mind, we

\[
\begin{array}{c}
\text{Left: The possible } [\cdot] \text{-representations of } G_{\Phi} \text{ induced by } \{X, x_1, \ldots, x_k, \overline{x}_1, \ldots, \overline{x}_k, z_1, \ldots, z_k\} \text{ and the } C_4 \text{'s adjacent to these vertices (note: } w_i \in \{x_i, \overline{x}_i\} \text{ and } \{w_i, w'_i\} = \{x_i, \overline{x}_i\}, \text{ and } \pi \text{ is a permutation on } \{1, \ldots, k\} \text{). Right: The possible } [\cdot] \text{-representations of } G_{\Phi} \text{ induced by } \{X, C, c_1, \ldots, c_t, d_1, \ldots, d_t\} \text{ and the } C_4 \text{'s adjacent to these vertices (note: } \rho \text{ is a permutation on } \{1, \ldots, t\} \text{).}
\end{array}
\]

Now turn our attention to the clause gadget of a clause \( D_i = (\ell_1, \ell_2, \ell_3) \). Notice that \( \{d_i, a_1, a_2, a_3\} \) is a clique (i.e., \( \{P_{d_i}, P_{a_1}, P_{a_2}, P_{a_3}\} \) form an edge-clique in any \([\cdot]\)-representation of \( G_{\Phi} \)). Furthermore, \( \{P_{d_i}, P_{a_1}, P_{a_2}, P_{a_3}\} \) form a vertical edge-clique since \( d_i \) is an internal horizontal neighbor of \( c_i \) and \( c_i \) is not adjacent to any of \( a_1, a_2, \) or \( a_3 \). Notice that, in a vertical edge-clique, there can be at most two \( \cup \)-paths which contain vertical grid edges that are not contained in the other \( \cup \)-paths of the edge-clique (i.e., the “top-most” and bottom-most \( \cup \)-paths – e.g., the clique \( \{d_i, a_1, a_2, a_3\} \) in figure 7). Thus, w.l.o.g., we suppose that \( P_{a_1} \) and \( P_{a_3} \) have this property. In particular, this means that \( P_{a_2} \) and \( P_{y_2} \) necessarily intersect via a horizontal grid edge and no vertical grid edge (since every grid edge contained in \( P_{a_2} \) is also contained in one of \( P_{a_1} \) or \( P_{a_3} \) and \( y_2 \) is adjacent to neither \( a_1 \) nor \( a_3 \)). Additionally, observe that, when \( \ell_3 \) is an internal vertical neighbor of \( X, y_q \) is necessarily an internal horizontal neighbor of \( \ell_q \) since \( \ell_q \) is adjacent to two \( C_4 \)'s. Similarly, when \( \ell_q \) is an internal horizontal neighbor of \( X, y_2 \) is an internal vertical neighbor of \( \ell_2 \). However, \( y_2 \) cannot be an internal horizontal neighbor of \( \ell_2 \) since \( \ell_2 \) is not adjacent to \( a_2 \) and \( P_{y_2} \) and \( P_{a_2} \) have a horizontal grid edge in common. Thus, it is not possible for all three literals to be internal vertical neighbors of \( X \). On the other hand, when at most two literals are internal vertical neighbors of \( X \), we can always construct the \([\cdot]\)-representation of the clause gadget. In particular, this can be done using one of the three templates depicted in figure 7. Note, to form an \([\cdot]\)-representation of \( G_{\Phi} \), the placement of the \([\cdot]\)-representations of the clause gadgets from figure 7 can be described as follows:

\[\text{Remember, } \ell_q \text{ is some } x_j \text{ or } \overline{x}_j \text{ (} 1 \leq j \leq k \text{).}\]
For type (i) and (ii) (i.e., at most one literal is an internal vertical neighbor of \(X\)), we place the \([\cdot]\)-representation of the clause gadget “below” \(P_{w_\pi(k)}\) and to the “left” of \(P_{w_\pi'(1)}\) (with respect to the depiction in figure 6).

For type (iii) (i.e., two literals \(\ell_1\) and \(\ell_3\) which are internal horizontal neighbors of \(X\)), we need to place the \([\cdot]\)-representation of the clause gadget “between” \(P_{\ell_1}\) and \(P_{\ell_3}\) and to the “left” of \(P_{w_\pi'(1)}\) (with respect to the depiction in figure 6).

![Diagram](image)

**Fig. 7.** \([\cdot]\)-representations of the clause gadget for a clause \((\ell_1, \ell_2, \ell_3)\) inside an \([\cdot]\)-representation of \(G_\Phi\). (i) \(\ell_1 = \ell_2 = \ell_3 = \text{true}\); (ii) \(\ell_1 = \text{false}\) and \(\ell_2 = \ell_3 = \text{true}\); (iii) \(\ell_1 = \ell_3 = \text{false}\) and \(\ell_2 = \text{true}\).

We can now see that a literal being an internal vertical neighbor of \(X\) corresponds to when that literal is \text{false} (since at most two literals can be internal vertical neighbors of \(X\)) and a literal being an internal horizontal neighbor of \(X\) corresponds to when that literal is \text{true}. Thus, since \(x_j\) and \(\overline{x_j}\) cannot both be internal vertical (or horizontal) neighbors of \(X\), the \([\cdot]\)-representations of the graph \(G_\Phi\) correspond to satisfying assignments of \(\Phi\). \(\square\)

An interesting observation regarding this proof is that it can be easily adapted to show the NP-completeness of recognition for \(\llbracket\cdot, \llbracket\) and \(\llbracket\cdot, \rrbracket\). For \(\llbracket\cdot, \rrbracket\), the same graph \(G_\Phi\) can be used, but one has to be careful about the structure of the \([\cdot, \llbracket]\)-representation of the clause gadget (since it need not be an edge clique). For \(\llbracket\cdot, \rrbracket\), we alter \(G_\Phi\) slightly. First, we replace each \(C_4\) adjacent to a vertex with a \(K_{2,3}\) adjacent to the same vertex (by Lemma 4, this forces the vertex to have an external neighbor for each \(K_{2,3}\), just as we had with the \(C_4\)s in \([\cdot]\)-representations). Second, for each \(1 \leq i \leq k\), we add a vertex \(z_i\) which is adjacent to \(x_i\) and \(\overline{x_i}\) (thus, turning the \(C_4\)s induced by \(\{X, x_i, \overline{x_i}, z_i\}\) into \(K_{2,3}\)s and preventing \(x_i\) and \(\overline{x_i}\) from both being internal vertical (horizontal) neighbors of \(X\)). These two changes to \(G_\Phi\) allow the proof to proceed as before.

We conjecture that similar adaptations can be performed to demonstrate the NP-completeness of \([\cdot, \llbracket, \rrbracket]\) and possibly even \(B_k\)-EPG for \(k > 1\).
4 Characterization and Recognition of $\text{Split} \cap [\land]$

Recall that recognizing chordal EPT graphs is NP-complete. We have just shown that recognizing $[\land]$-graphs is NP-complete. Thus, it is of interest to characterize the class $\text{Chordal} \cap [\land]$. A first step in this direction would be to study $\text{Split} \cap [\land]$, that is, the class of split $[\land]$-graphs. We split this discussion into three parts. In the first part, we establish some properties of split $[\land]$-graphs. In the latter two parts, we characterize two special subclasses of split $[\land]$-graphs.

4.1 Properties of $\text{Split} \cap [\land]$

In this section, we will establish some properties of the class $\text{Split} \cap [\land]$. We will pose a conjecture on the characterization of this class. First, we need to introduce a few definitions.

A vertex $x$ dominates a vertex $y$ if $N(y) \subseteq N(x) \cup \{x\}$, where $N(a)$ denotes the set of vertices adjacent to a vertex $a$. Vertex $x$ is comparable to vertex $y$ if $x$ dominates $y$, or vice versa. The domination relation is a partial order. A vertex is maximal if it is not dominated by another vertex. If vertex $x$ belongs to a set $X$ of vertices of $G$, we will say that $x$ is an $X$-vertex. We let $G - X$ denote the subgraph of $G$ induced by the vertices belonging to $V(G)$ but not to $X$. We let $N(X)$ denote the set of vertices outside $X$ that have some neighbors in $X$.

Two vertices $a, b$ are twins if $N(a) \cup \{a\} = N(b) \cup \{b\}$ or $N(a) = N(b)$. A split partition $(C, S)$ of a graph $G$ is a partition of its vertices into a clique $C$ and a stable set $S$. We will enumerate the vertices of $S$ as $\{s_1, ..., s_k\}$. In this section, we consider split graphs that admit $[\land]$-representations.

Let $G$ be an $[\land]$-graph with a split partition $(C, S)$. It follows from Theorem 1 that $C$ corresponds to an edge-clique. Consider an $[\land]$-representation of $G$ on the grid. We may assume without loss of generality that the edge of the grid that belongs to all of $C$ is vertical. The horizontal parts of paths of $C$ are called branches. The vertical part of $C$ below the first (top) branch is called the trunk. The vertical part of $C$ above the first branch is called the crown (see figure 8).

**Observation 6** The vertices of $S$ whose $\land$-paths lie on the same branch (or, the crown) are pairwise comparable. An $S$-vertex whose path lies on the trunk dominates all $S$-vertices whose paths lie below it in the representation. $\square$

See figure 8 for an illustration of Observation 6. The gem is the graph with vertices $a, b, c, d, e$, edges $ab, bc, cd, ea, eb, ec$. The bull is the graph with vertices $a, b, c, d, e$, edges $ab, bc, cd, eb, ec$.

**Observation 7** Let $G$ be a split graph with a split partition $(C, S)$. If $G$ admits an $[\land]$-representation and contains a gem, then exactly one of the gem’s $S$-vertices has its $\land$-path occurring on the crown of the representation.

**Proof.** Let the vertices of the gem be $c_1, c_2, c_3, s_1, s_2$ with $c_1, c_2, c_3 \in C, s_1, s_2 \in S$ and $s_1c_1, s_1c_2, s_2c_2, s_2c_3 \in E(G)$. Assume that both $P_{s_1}$ are not on the crown.
Since \( s_1 \) and \( s_2 \) are incomparable, by Observation 6 we may assume \( P_{s_2} \) lies on a branch. Since \( s_1 \) is adjacent to \( c_2 \), \( P_{s_1} \) must lie on the vertical segment of \( P_{c_2} \) in the representation. By our assumption, \( P_{s_2} \) must be on the trunk. By Observation 3 \( s_1 \) dominates \( s_2 \), a contradiction. Thus, we may assume \( P_{s_1} \) is on the crown. Since \( s_1 \) is incomparable with \( s_2 \), \( P_{s_2} \) cannot be on the crown. \( \Box \)

**Definition 1.** An S-bull is a bull such that the three vertices of degrees less than three in the bull are in \( S \).

In figure \( \{b, c, 1, 2, 3\} \) is an S-bull but \( \{a, b, c, 5, 6\} \) is not an S-bull even though it is a bull.

**Observation 8** Let \( G \) be a split graph with a split partition \((C, S)\). If \( G \) admits an \([\_]\)-representation and contains an S-bull, then some S-vertices of this bull have their paths occurring on either the crown or trunk of the representation. \( \Box \)

**Observation 9** Let \( G \) be a split graph with a split partition \((C, S)\). Suppose there is a vertex \( v \) in \( G \) with \( N(v) = C - \{v\} \). Then \( G \) is an \([\_]\)-graph iff \( G - s \) is.

Proof. Suppose \( G - v \) has an \([\_]\)-representation. First assume \( v \in S \). On the trunk, there is a vertical segment where all of \( C \) meets. We can place \( P_v \) there to get a representation of \( G \). Now, we may assume \( v \in C \). Thus, \( v \) has no neighbor in \( S \). We can place \( P_v \) on the grid edge at the base of the crown –and if necessary move the other paths of \( S \) on the crown up– to obtain a representation for \( G \). \( \Box \)

**Observation 10** Let \( G \) be a split graph with a split partition \((C, S)\). Suppose \( G \) contains twins \( a, b \). Then \( G \) is an \([\_]\)-graph iff \( G - a \) is.

Proof. Suppose \( a \) is adjacent to \( b \). Suppose further that \( a \) is in \( S \). Then \( b \) is in \( C \) and it follows that \( a \) is adjacent to all vertices of \( C \). So, we are done by Observation 9. Thus, both \( a \) and \( b \) are in \( C \). Consider an \([\_]\)-representation of \( G - a \). By making \( P_a \) an exact copy of \( P_b \), we obtain a representation for \( G \).

So, we may assume \( a \) is not adjacent to \( b \). Suppose both \( a \) and \( b \) are in \( S \). Consider an \([\_]\)-representation of \( G - a \). Then \( P_b \) occurs on a branch (trunk, crown). By placing \( P_a \) next to \( P_b \) on this branch (trunk, crown), we obtain a representation for \( G \) (see figure 8 for an illustration.) Now, we may assume \( a \) is in \( C \) and \( b \) is in \( S \). It follows that \( a \) has no neighbor in \( S \). But then we are done by Observation 9. \( \Box \)
Observation 11. Let $G$ be a split graph with a split partition $(C, S)$. Suppose there is a subset $D$ of $C$ such that the vertices of $X = N(D) \cap S$ are pairwise comparable and $N(S) \subseteq D$. Then $G$ is an $[\_\_\_]$-graph iff $G - (D \cup X)$ is.

Proof. Suppose there is an $[\_\_\_]$-representation of $G - (D \cup X)$. Vertices of $D$ will be represented by $[\_\_\_]-$paths with the same bend-point. Recall that the $[\_\_\_]-$paths of $C$ lie on the trunk. We will add the vertical segments of the $[\_\_\_]-$paths $D$ to the trunk. We can move the $[\_\_\_]-$paths of the $S$-vertices at the crown up so they do not intersect with the vertical parts of the paths of $D$ whose bend-point is placed just below the first branch. We can place the vertices (paths) of $X$ on this new branch. \end{proof}

Observation 12. Let $G$ be a split graph with a split partition $(C, S)$. Suppose some vertex $c \in C$ is such that all of its neighbors in $S$ have degree one. Then $G$ is an $[\_\_\_]$-graph iff $G - c$ is.

Observation 13. Let $G$ be a gem-free graph with a split partition $(C, S)$. Then any two vertices of $S$ with a common neighbor in $C$ are comparable.

Observation 14. Let $G$ be a gem-free graph with a split partition $(C, S)$. Let $s$ be a maximal vertex in $S$ and $s'$ be a vertex in $S$ with a common neighbor with $s$. Then $s$ dominates $s'$.

Consider the nine graphs shown in figure 9. We believe that they are the only minimal forbidden obstructions for a split graph to be an $[\_\_\_]$-graph.

**Fig. 9.** In $U_1$ and $U_2$, the vertex $u$ is adjacent to all remaining vertices.

**Lemma 5.** None of the nine graphs shown in figure 9 is an $[\_\_\_]$-graph.

**Proof.** By Lemma 1 the graphs $U_1$ and $U_2$ do not admit $[\_\_\_]-$representations. Consider the graph $S_1$ with the split partition $(C, S)$ where $C = \{c_1, c_2, c_3\}$, $S = \{s_1, s_2, s_3\}$, and $v_1c_i, v_i c_{i+1} \in E(G)$ with the subscripts taken modulo 3. Now, consider the gem $\{s_1, s_2, c_1, c_2, c_3\}$. By Observation 4 we may assume $P_{s_1}$ is on the crown and $P_{s_2}$ is not. The gem $\{s_1, s_3, c_1, c_2, c_3\}$ implies that $P_{s_3}$ is
not on the crown because $P_{s_1}$ already is. But the two $S$-vertices of the gem
\{ $s_2, s_3, c_1, c_2, c_3$ \} have no paths on the crown, a contradiction to Observation 7.
So, $S_1$ is not an [.] graph. Similar arguments show $S_2$, $S_3$, and $S_4$ are not [.] graphs.
Consider the graph $S_5$. Suppose $S_5$ admits an [.] representation. Let $B_1, B_2, B_3$ be the three $S$-bulls of $S_5$. By Observation 5 each $B_i$ contains an
$S$-vertex $s_i$ such that $P_{s_i}$ lies on the trunk or crown. Without loss of generality, we may assume the trunk contains $s_1$ and $s_2$. The fact that $s_1$ is incomparable with $s_2$ contradicts Observation 6. Similar arguments show that $S_6$ and $S_7$ are not [.] graphs. Finally, it is a routine but tedious matter to show that all proper induced subgraphs of the graphs in figure 9 are [.] graphs. ✷

We believe the nine graphs in figure 9 are the only minimal forbidden subgraphs for Split $\cap$ [.] We pose that as a conjecture.

**Conjecture 1.** A split graph is an [.] graph iff it does not contain any of the nine graphs in figure 9 as an induced subgraph.

Theorems 17 and 18 (proved in the next sections) can be seen as first steps in this direction.

The $k$-sun ($k \geq 3$) is the graph obtained by taking a cycle on $2k$ vertices and joining odd-indexed vertices by edges. So, a 3-sun is the graph $S_1$, a 4-sun is the graph $S_2$, and $S_3$ occurs in any $k$-sun with $k \geq 5$. A graph is strongly chordal if it is chordal and contains no $k$-sun. The following lemma follows from Lemma 5.

**Observation 15** Chordal $\cap$ [.] = Strongly Chordal $\cap$[.]. ✷

### 4.2 Split graphs without $S$-bulls

In this section, we give a characterization by forbidden induced subgraphs of split [.] graphs without $S$-bulls. This provides a polytime algorithm for recognizing split [.] graphs without $S$-bulls.

**Observation 16** Let $x_1, x_2$ be two incomparable vertices in $S$. If $G$ does not contain an $S$-bull, then no vertex $s \in S$ is adjacent to some vertex $x$ of $N(x_1) - N(x_2)$ and some vertex $y$ of $N(x_2) - N(x_1)$.

**Proof.** If such vertex $s$ exists, then $\{ s, x, y, x_1, x_2 \}$ induces a $S$-bull. ✷

**Theorem 17.** Let $G$ be a graph with a split partition $(C, S)$ and with no $S$-bull. Then $G$ admits an [.] representation iff $G$ does not contain $U_1$ or $S_4$, as an induced subgraph.

**Proof.** By induction on the number of vertices. We only need to prove the “if” part. Let $G$ be a graph with a split partition $(C, S)$ and with no $S$-bull, $U_1$, or $S_4$. Suppose no two incomparable $S$-vertices have a common neighbor (ie., $G$ is gem-free). Then, by Observation 16 there is a partition of $S$ into sets $X_1, X_2, \ldots, X_k$ ($k \geq 2$) such that the vertices of $X_i$ are pairwise comparable and $N(u) \cap N(v) = \emptyset$ for any $u \in X_i, v \in X_j, i \neq j$. By the induction hypothesis, the
graph $G' = G - (X_1 \cup N(X_1) \cap S)$ admits an \([\cdot]\)-representation. By Observation 11, $G$ admits an \([\cdot]\)-representation. (Note: this implies Corollary 1 below.)

So, there are two incomparable $S$-vertices with a common neighbor. Let $s_1, s_2 \in S$ be two incomparable vertices with a common neighbor such that $d(s_1) + d(s_2)$ is largest, where $d(x)$ denotes the degree of vertex $x$. The following two facts are easy to establish.

Let $s_3$ be an $S$-vertex with a neighbor in $N(s_1) \cup N(s_2)$. Then

\begin{equation}
\text{s}_3 \text{ is comparable to } \text{s}_1 \text{ or to } \text{s}_2.
\end{equation}

Suppose $s_3$ is incomparable to both $s_1$ and $s_2$. Vertex $s_3$ has no neighbors in $N(s_1) \cap N(s_2)$, for otherwise $G$ contains an universal AT, and thus an $S$-bull or $U_1$. Without loss of generality, we may assume $s_3$ has a neighbor $x$ in $N(s_1) - N(s_2)$. Now, there is a $S$-bull with vertices $s_1, s_2, s_3, x$, and some $y \in N(s_1) \cap N(s_2)$. We have established (1).

Define $C_0 = N(s_1) \cup N(s_2)$.

For any vertex $s_3 \in S$ with a neighbor in $C_0$, either $s_1$ or $s_2$ dominates $s_3$.

Consider a vertex $s_3 \in S$ with a neighbor in $C_0$. Suppose $s_3$ has a neighbor $y \notin C_0$. By (1), we may assume $s_3$ is comparable to $s_2$. The existence of $y$ implies $s_3$ dominates $s_2$. It follows that $s_3$ is comparable to $s_1$, for otherwise, $d(s_3) + d(s_1) > d(s_2) + d(s_1)$, contradicting our choice of $s_1$ and $s_2$. Thus, $s_3$ dominates $s_1$. By Observation 16 with $x_1 = s_1, x_2 = s_2, s_3 = s$, $G$ contains an $S$-bull, a contradiction. So, we have $N(s_3) \subseteq C_0$. By Observation 16 $s_3$ has no neighbor in $N(s_1) - N(s_2)$ or in $N(s_2) - N(s_1)$. Thus, (2) is established.

We show the placements of the paths of $N(C_0) \cap S$ on the crown and first branch. We place the paths of $S - N(C_0)$ on branches below that first branch. By (2), the vertices of $N(C_0) \cap S$ can be partitioned into two sets $D_1$ and $D_2$ such that $s_i$ is in $D_i$ and dominates every vertex in $D_i - s_i$. Now, we claim that

The vertices in each $D_i$ are pairwise comparable.

If some two vertices $x_1, x_2 \in D_i$ are incomparable, then by Observation 16 $G$ contains an $S$-bull. So, (3) holds. It follows that the vertices of $C_0$ are pairwise comparable in the subgraph of $G$ induced by $C_0 \cup D_1$ (and, $C_0 \cup D_2$). Vertices of $C_0$ will be represented by $\cup$-paths with the same bend-point. Place the paths representing $D_1$ at the crown with $P_x$ being above $P_y$ if $x$ is dominated by $y$. Place the paths representing $D_2$ on the first branch with $P_x$ to the right of $P_y$ if $x$ is dominated by $y$. For any two vertices $a, b$ of $D_1$ (resp., $D_2$), $a$ dominates $b$ in $D_1$ (resp., $D_2$), then every $\cup$-path of a $C$-vertex must pass through an edge of $P_a$ to reach $P_b$. This completes the description of the representation of $C_0 \cup (N(C_0) \cap S)$.

Define $C' = C - C_0$. By (2), there is no vertex in $S$ with a neighbor in $C'$ and one in $C_0$. The set $C' \cup (N(C') \cap S)$ contains no gem, for otherwise, $G$ contains $S_4$. It follows from Observations 13 and 14 that the set $C'$ can be partitioned into sets $C_1, C_2, \ldots, C_k$ ($k \geq 1$) such that, for each $i$, the vertices
in $N(C_i) \cap S$ are pairwise comparable, and no $S$-vertex has a neighbor in $C_i$ and one in $C_j$, for $i \neq j$ (in particular, for each $C_i$, there is a maximal $S$-vertex $s$ with $N(s) \cap C = C_i$). Define $X = N(C_i) \cap S$. By the induction hypothesis, $G - (C_i \cup X)$ is an $\lfloor . \rfloor$-graph. By Observation 11, $G$ is an $\lfloor . \rfloor$-graph.  

We note a polytime algorithm to construct an $\lfloor . \rfloor$-representation for the input graph can be extracted from the proofs above. The algorithm is certifying in the sense that it produces either an $\lfloor . \rfloor$-representation, or an obstruction. The proof of the theorem also implies the following corollary:

**Corollary 1.** All $S$-bull free, gem-free split graphs are $\lfloor . \rfloor$-graphs.  

4.3 Split graphs without gems

In this section, we give a characterization by forbidden induced subgraphs of split $\lfloor . \rfloor$-graphs without gems. This provides a polytime algorithm for recognizing split $\lfloor . \rfloor$-graphs without gems. First, we need to introduce a definition. Two vertex-disjoint $S$-bulls are incomparable if every $S$-vertex in one bull is incomparable with every $S$-vertex in the other bull.

**Lemma 6.** Let $G$ be a gem-free graph with a split partition $(C, S)$. Suppose $G$ does not contain two incomparable $S$-bulls. Then, there is an $\lfloor . \rfloor$-representation of $G$ with no $S$-vertices having their $\lfloor . \rfloor$-paths lying on the trunk.

**Proof.** By induction on the number of vertices. We may assume every vertex of $C$ has a neighbor in $S$, for otherwise we are easily done by the induction hypothesis. Let $s$ be a maximal vertex in $S$.

Suppose $N(s) = C$. By the induction hypothesis, there is an $\lfloor . \rfloor$-representation of $G - s$ with no paths of $S - s$ on the trunk. We can place $P_s$ at the lowest part of the crown to obtain the desired $\lfloor . \rfloor$-representation for $G$.

So, we have $C - N(s) \neq \emptyset$. By Observation 14, $C$ contains at least two proper subsets $A, B$ such that no $S$-vertex has some neighbors in $A$ and some in $B$. It follows $C$ contains a proper subset $C'$ such that $C' \cup S'$ contains no $S$-bull, where $S' = N(C') \cap S$. We may choose $C'$ such that the edges between $C'$ and $S'$ form a connected subgraph of $G$. It follows that the vertices of $S'$ are pairwise comparable. By the induction hypothesis, $G - (C' \cup S')$ admits a $\lfloor . \rfloor$-representation with no paths representing $S' - S'$ on the trunk. We now add the paths of $C'$ to the trunk making them share the same bend-point which is placed just below the first branch. By moving the paths of the $S$-vertices on the crown up, we can preserve the required adjacency between them and the new paths. We can place the paths of $S'$ on the new branch, introducing no new paths on the trunk.  

**Theorem 18.** Let $G$ be a gem-free graph with a split partition $(C, S)$. Then $G$ admits an $\lfloor . \rfloor$-representation iff $G$ does not contain $S_5$ as induced subgraphs.
Proof. By induction on the number of vertices. We only need to prove the “if” part. Let $G$ be a gem-free graph with a split partition $(C, S)$ and not containing $S_6$. By Observation[9] we may assume every vertex in $C$ has a neighbor in $S$. Let $x$ be a maximal vertex in $S$. By Observation[9], we may assume every vertex in $C$ has a neighbor in $S$. Let $x$ be a maximal vertex in $S$. By Observation[9], we may assume every vertex in $C$ has a neighbor in $S$. It follows from Observation[14] that $S$ contains vertices $x_1, x_2, \ldots, x_k$ ($k \geq 2$) such that no vertex $s \in S$ has a neighbor in $N(x_i)$ and one in $N(x_j)$, $i \neq j$. Define $C_i = N(x_i)$, $S_i = N(C_i) \cap S$, for $i = 1, 2, \ldots, k$. If for some $C_i$, the vertices of $S_i$ are pairwise comparable, then we are done by the induction hypothesis and Observation[11]. Therefore, for all $i$, $S_i$ must contain two incomparable vertices. That is, each set $C_i \cup S_i$ must contain an $S$-bull. It follows that $k = 2$, and we may assume furthermore that $G_1 = G[C_1 \cup S_1]$ does not contain two incomparable $S$-bulls, for otherwise, $G$ contains $S_5$. By Lemma[6], there is a $[\_]$-representation of $G_1$ with no vertices of $S_1$ on the trunk. By the induction hypothesis, the graph $G_2 = G - (C_1 \cup S_1)$ has a $[\_]$-representation. We place the branches of $G_2$ under those of of $G_1$ and extend the vertical segments of the paths of $C - C_1$ to the crown of $G_1$. The adjacency of $G$ is preserved because $G_1$ has no $S$-vertices on the trunk in the $[\_]$-representation. \(\square\)

5 Concluding Remarks and Open Problems

In this paper, we considered the edge intersection graphs of $[\_]$-shaped paths on a grid. We showed that recognizing such graphs is NP-complete. We considered the open problem of characterizing chordal $[\_]$-graphs. As first steps in solving this problem, we found characterizations of split gem-free $[\_]$-graphs and split $[\_]$-graphs without $S$-bulls (a class more general than split bull-free $[\_]$-graphs). Our characterizations imply polytime algorithms for recognizing these two classes of graphs. We posed a conjecture on the characterization of split $[\_]$-graphs. This conjecture would imply a polytime recognition theorem for split $[\_]$-graphs. The following open problems related to our works arise: (1) Extending the observations in section[4] to other subclasses of $B_1$-EPG graphs; (2) Find a polytime algorithm for Chordal $\cap [\_]$; (3) Establish NP-completeness for recognition of $B_k$-EPG graphs for every $k$ at least 2.

References

1. Asinowski, A., and Ries, B. Some properties of edge intersection graphs of single-bend paths on a grid. Discrete Mathematics 312, 2 (2012), 427–440.
2. Bandy, M., and Sarrafzadeh, M. Stretching a knock-knee layout for multilayer wiring. IEEE Transactions on Computers 39 (1990), 148151.
3. Booth, K. S., and Lueker, G. S. Testing for the consecutive ones property, interval graphs, and graph planarity using pq-tree algorithms. Journal of Computer and System Sciences 13, 3 (1976), 335–379.
4. Golumbic, M., Lipshteyn, M., and Stern, M. Edge intersection graphs of single bend paths on a grid. Networks 54 (2009), 130–138.
5. Golumbic, M. C., and Jamison, R. E. The edge intersection graphs of paths in a tree. J. Comb. Theory, Ser. B 38, 1 (1985), 8–22.
6. Heldt, D., Knauer, K., and Ueckerdt, T. Edge-intersection graphs of grid paths: the bend-number. unpublished manuscript, http://arxiv.org/abs/1009.2861.
7. Karp, R. Reducibility among combinatorial problems. In Complexity of Computer Computations, R. Miller and J. Thatcher, Eds. Plenum Press, 1972, pp. 85–103.