THE $k$-PROPERTY AND COUNTABLE TIGHTNESS OF FREE TOPOLOGICAL VECTOR SPACES

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Abstract. The free topological vector space $V(X)$ over a Tychonoff space $X$ is a pair consisting of a topological vector space $V(X)$ and a continuous map $i = i_X : X \to V(X)$ such that every continuous mapping $f$ from $X$ to a topological vector space $E$ gives rise to a unique continuous linear operator $\tilde{f} : V(X) \to E$ with $f = \tilde{f} \circ i$. In this paper the $k$-property and countable tightness of free topological vector space over some generalized metric spaces are studied. The characterization of a space $X$ is given such that the free topological vector space $V(X)$ is a $k$-space or the tightness of $V(X)$ is countable. Furthermore, the characterization of a space $X$ is also provided such that if the fourth level of $V(X)$ has the $k$-property or is of the countable tightness then $V(X)$ is too.

1. Introduction

The free topological group $F(X)$, the free abelian topological group $A(X)$ and the free locally convex space $L(X)$ over a Tychonoff space $X$ were introduced by Markov [22] and intensively studied over the last half-century, see for example [1, 5, 6, 7, 13, 14, 15, 27, 29, 30, 31]. Recently, in [7] S.S. Gabriyelyan and S.A. Morris introduced and studied the free topological vector space $V(X)$ over a Tychonoff space $X$. One surprising fact is that the free topological vector spaces in some respect behave better than the free locally convex spaces. For example, if $X$ is a $k_\omega$-space then $V(X)$ is also a $k_\omega$-space [7]; however, for a Tychonoff space $X$, the space $L(X)$ is a $k$-space if and only if $X$ is a countable discrete space [5]. Therefore, it is natural to consider the following question:

**Question 1.1.** If $V(X)$ is a $k$-space over a Tychonoff space $X$, is $V(X)$ a $k_\omega$-space?

Recently, S.S. Gabriyelyan proved that for a metrizable space $X$ the free locally convex $L(X)$ is of countable tightness if and only if $X$ is separable, see [5]. Hence it is is natural to consider the following question:

**Question 1.2.** Let $X$ be a metrizable space. Is the countable tightness of $V(X)$ equivalent to the separability of $X$?

In this paper, we shall give an affirmative answer to Question [1.1] if $X$ is a $k^*$-metrizable space, and an affirmative answer to Question [1.2] if $X$ is a paracompact $k$-and $\sigma$-space. Moreover, the characterization of a space $X$ is also given such that if the fourth level of $V(X)$ has the $k$-property or is of the countable tightness then $V(X)$ is too.

The paper is organized as follows. In Section 2, we introduce the necessary notation and terminologies which are used for the rest of the paper. In Section 3, we investigate the $k$-property and countable tightness on free topological vector spaces. In section 4, we pose some interesting questions about the free topological vector spaces which are still unknown to us.
2. Notation and Terminologies

In this section, we introduce the necessary notation and terminologies. Throughout this paper, all topological spaces are assumed to be Tychonoff and all vector spaces are over the field of real numbers $\mathbb{R}$, unless otherwise is explicitly stated. First of all, let $\mathbb{N}$ be the set of all positive integers and $\omega$ the first infinite ordinal. Let $X$ be a topological space and $A \subseteq X$ be a subset of $X$. The closure of $A$ in $X$ is denoted by $\overline{A}$; moreover, we always denote the set of all the non-isolated points of $X$ by $X'$. For undefined notation and terminologies, the reader may refer to [2], [4], [8] and [19].

**Definition 2.1.** Let $X$ be a space.

1. The space $X$ is **separable** if it contains a countable subset $B$ such that $\overline{B} = X$.
2. The space $X$ is **of countable tightness** if the closure of any subset $A$ of $X$ equals the union of closures of all countable subsets of $A$, and the countable tightness of a space $X$ is denoted by $t(X) \leq \omega$.
3. The space $X$ is called a **$k$-space** provided that a subset $C \subseteq X$ is closed in $X$ if $C \cap K$ is closed in $K$ for each compact subset $K$ of $X$.
4. The space $X$ is called a **$k$-$\omega$-space** if there exists a family of countably many compact subsets $\{K_n : n \in \mathbb{N}\}$ of $X$ such that each subset $F$ of $X$ is closed in $X$ provided that $F \cap K_n$ is closed in $K_n$ for each $n \in \mathbb{N}$.
5. A subset $P$ of $X$ is called a **sequential neighborhood** of $x \in X$, if each sequence converging to $x$ is eventually in $P$. A subset $U$ of $X$ is called **sequentially open** if $U$ is a sequential neighborhood of each of its points. A subset $F$ of $X$ is called **sequentially closed** if $X \setminus F$ is sequentially open. The space $X$ is called a **sequential space** if each sequentially open subset of $X$ is open.

Let $\kappa$ be an infinite cardinal. For each $\alpha \in \kappa$, let $T_\alpha$ be a sequence converging to $x_\alpha \notin T_\alpha$. Let $T = \bigoplus_{\alpha \in \kappa}(T_\alpha \cup \{x_\alpha\})$ be the topological sum of $\{T_\alpha \cup \{x_\alpha\} : \alpha \in \kappa\}$. Then $S_\kappa = \{x\} \cup \bigcup_{\alpha \in \kappa} T_\alpha$ is the quotient space obtained from $T$ by identifying all the points $x_\alpha \in T$ to the point $x$.

A space $X$ is called an **S$_2$-space** (Arens’ space) if

$$X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,m} : m, n \in \omega\}$$

and the topology is defined as follows: Each $x_{n,m}$ is isolated; a basic neighborhood of $x_n$ is $\{x_n\} \cup \{x_{n,m} : m > k\}$, where $k \in \omega$; a basic neighborhood of $\infty$ is

$$\{\infty\} \cup \left(\bigcup \{V_n : n > k \text{ for some } k \in \omega\}\right),$$

where $V_n$ is a neighborhood of $x_n$ for each $n \in \omega$.

**Definition 2.2.** Let $X$ be a space and $\mathcal{P}$ a family of subsets of $X$.

1. The family $\mathcal{P}$ is a **network** of $X$ if for each $x \in X$ and $x \in U$ with $U$ open in $X$, then $x \in P \subseteq U$ for some $P \in \mathcal{P}_x$. A regular space $X$ is called a **$\sigma$-space** (resp. cosmic space) if it has a $\sigma$-locally finite network (resp. countable network).
2. The family $\mathcal{P}$ is called a **cs-network** [10] at a point $x \in X$ if for every sequence $\{x_n : n \in \mathbb{N}\}$ converging to $x$ and an arbitrary open neighborhood $U$ of $x$ in $X$ there exist an $m \in \mathbb{N}$ and an element $P \in \mathcal{P}$ such that

$$\{x\} \cup \{x_n : n \geq m\} \subseteq P \subseteq U.$$ 

The space $X$ is called **csf-countable** if $X$ has a countable cs-network at each point $x \in X$.
3. The family $\mathcal{P}$ is called a **$k$-network** [23] if for every compact subset $K$ of $X$ and an arbitrary open set $U$ containing $K$ in $X$ there is a finite subfamily $\mathcal{P}' \subseteq \mathcal{P}$ such that $K \subseteq \bigcup \mathcal{P}' \subseteq U$. A regular space $X$ is called an **$\aleph_0$-space** (resp. $\aleph_0$-space) if it has a $\sigma$-locally finite $k$-network (resp. countable $k$-network).
4. Let $\mathcal{P}$ be a cover of $X$ such that (i) $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$; (ii) for each point $x \in X$, if $U, V \in \mathcal{P}_x$, then $W \subseteq U \cap V$ for some $W \in \mathcal{P}_x$; and (iii) for each point $x \in X$ and each open neighborhood $U$ of $x$ there is some $P \in \mathcal{P}_x$ such that $x \in P \subseteq U$. Then,
is called an **sn-network** [17] for \( X \) if for each point \( x \in X \), each element of \( \mathcal{P}_x \) is a sequential neighborhood of \( x \) in \( X \), and \( X \) is called **snf-countable** [17] if \( X \) has an **sn-network** \( \mathcal{P} \) and \( \mathcal{P}_x \) is countable for all \( x \in X \).

Clearly, the following implications follow directly from definitions:

\[
\text{first countable} \Rightarrow \text{snf-countable} \Rightarrow \text{csf-countable}.
\]

Note that none of the above implications can be reversed. It is well known that \( S_w \) is csf-countable but not snf-countable and \( S_{\omega_1} \) is not csf-countable.

A Hausdorff topological space \( X \) is \( k^* \)-metrizable if \( X \) is the image of a metrizable space \( M \) under a continuous map \( f : M \to X \) having a section \( s : X \to M \) that preserves precompact sets in the sense that the image \( s(K) \) of any compact set \( K \subseteq X \) has compact closure in \( X \).

In [3], T. Banakh, V.I. Bogachev and A.V. Kolesnikov introduced this concept of \( k^* \)-metrizable spaces, and systematically studied this class of \( k^* \)-metrizable spaces. The class of \( k^* \)-metrizable spaces is closed under many countable (and some uncountable) topological operations. Regular \( k^* \)-metrizable spaces can be characterized as spaces with \( \sigma \)-compact-finite \( k \)-network, see [3] Theorem 6.4. This characterization shows that the class of \( k^* \)-metrizable spaces is sufficiently wide and contains all Čech spaces (closed images of metrizable spaces), all \( \mathbb{R}_0 \)-spaces and all \( \mathbb{R} \)-spaces. \( k^* \)-metrizable spaces form a new class of generalized metric spaces and have various applications in topological algebra, functional analysis, and measure theory, see [3, 9]. From [3], we list some properties of \( k^* \)-metrizable spaces.

1. The \( k^* \)-metrizability is preserved by subspaces, countable product, box-product and topological sum;
2. Each sequentially compact subset of a \( k^* \)-metrizable space is metrizable;
3. A \( k^* \)-metrizable space is sequential if and only if it is a \( k \)-space.
4. A regular space is \( k^* \)-metrizable space if and only if it has a \( \sigma \)-compact-finite \( k \)-network;
5. A regular \( X \) is a \( k^* \)-metrizable space with countable network if and only if it is an \( \mathbb{R}_0 \)-space.

**Definition 2.3.** [7] The free topological vector space \( V(X) \) over a Tychonoff space \( X \) is a pair consisting of a topological vector space \( V(X) \) and a continuous map \( i = i_X : X \to V(X) \) such that every continuous mapping \( f \) from \( X \) to a topological vector space (tvs) \( E \) gives rise to a unique continuous linear operator \( \tilde{f} : V(X) \to E \) with \( f = \tilde{f} \circ i \).

The change of the word “topological vector space” to “abelian topological group” and “locally convex space” in the above definition gives the definition of the free abelian topological group \( A(X) \) and free locally convex space \( L(X) \) on \( X \) respectively.

For a space \( X \) and an arbitrary \( n \in \mathbb{N} \), we denote by \( sp_n(X) \) the following subset of \( V(X) \)

\[
sp_n(X) = \{ \lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_i \in [-n,n], x_i \in X, i = 1, \ldots, n \}.
\]

Then \( V(X) = \bigcup_{n \in \mathbb{N}} sp_n(X) \) and each \( sp_n(X) \) is closed in \( V(X) \), see [7] Theorem 2.3. Moreover, if \( v \in V(X) \), then \( v \) has a unique representation

\[
v = \lambda_1 x_1 + \cdots + \lambda_n x_n, \text{ where } \lambda_i \in \mathbb{R} \setminus \{0\} \text{ and } x_i \in X \text{ are distinct;}
\]

then the set \( \text{supp}(v) = \{ x_1, \ldots, x_n \} \) is called the **support** [7] of the element \( v \). For every \( n \in \mathbb{N} \), define the mapping \( T_n : [-n,n]^n \times X^n \to sp_n(X) \) by

\[
T_n((a_1, \cdots, a_n) \times (x_1, \cdots, x_n)) = a_1 x_1 + \cdots + a_n x_n
\]

for arbitrary \( (a_1, \cdots, a_n) \times (x_1, \cdots, x_n) \in [-n,n]^n \times X^n \).

### 3. Main Results

First of all, we give a characterization of a collectionwise normal \( \mathbb{N} \)-space \( X \) such that \( V(X) \) is an \( \mathbb{R}_0 \)-space, and obtain that \( sp_1(X) \) is csf-countable if and only if \( V(X) \) is an \( \mathbb{R}_0 \)-space if and only if \( X \) is separable. In order to prove this result, we need two lemmas and some concepts.
Lemma 3.1. If $X$ is Dieudonné-complete and $\phi$ is a compact set in $V(X)$, then there exists a compact set $Z \subseteq X$ and $n \in \mathbb{N}$ such that $\phi$ is the continuous image of some compact subspace in $[-n,n]^n \times Z^n$.

Proof. By Proposition 5.5, the set $Z = supp(\phi)$ is compact in $X$ and there exists $n \in \mathbb{N}$ such that $\phi \subseteq sp_n(X)$. Consider the mapping

$$
\sigma = T_n([-n,n]^n \times Z^n) \cap T_n^{-1}(\phi) \to \phi.
$$

Then $\sigma$ is a continuous onto mapping from a compact subspace $([-n,n]^n \times Z^n) \cap T_n^{-1}(\phi)$ to $\phi$, hence $\phi$ is the continuous image of some compact subspace in $[-n,n]^n \times Z^n$.

Lemma 3.2. If $X$ is an $\aleph_0$-space, then $V(X)$ is also an $\aleph_0$-space.

Proof. For each $n \in \mathbb{N}$, let $Y_n = [-n,n]^n \times X^n$ and fix a countable $k$-network $\mathcal{P}_n$ in $Y_n$ since $Y_n$ is an $\aleph_0$-space. Since $X$ is Dieudonné-complete, it follows from Lemma 3.1 that for each compact set $\phi \subseteq sp_n(X)$ there exists a compact set $\phi_1 \subseteq Y_n$ such that $T_n(\phi_1) = \phi$. For each $n \in \mathbb{N}$, since $T_n$ is continuous, it easily see that the family $\mathcal{F}_n = \{T_n(P) : P \in \mathcal{P}_n\}$ is a countable $k$-network of the subspace $sp_n(X)$. Therefore, it follows from Corollary 3.4 that the family $\mathcal{F} = \bigcup \mathcal{F}_n : n \in \mathbb{N} \}$ is a countable $k$-network of $V(X)$.

Let $\kappa$ be an infinite cardinal, let $\mathbb{V}_\kappa = \bigoplus_{i<\kappa} \mathbb{R}_i$ be the direct sum of $\kappa$ copies of $\mathbb{R}$, and let $\tau_\kappa$, $\nu_\kappa$ and $\mu_\kappa$ be the box topology, maximal locally convex vector topology and maximal vector topology on $\mathbb{V}_\kappa$, respectively. Obviously, $\tau_\kappa \subseteq \nu_\kappa \subseteq \mu_\kappa$ and $V(D) \cong (\mathbb{V}_\kappa, \mu_\kappa)$, where $D$ is a discrete space of cardinality $\kappa$. Indeed, let $D = \{x_\alpha : \alpha < \kappa\}$. Then define the mapping $f : V(D) \to (\mathbb{V}_\kappa, \mu_\kappa)$ by

$$
f(\lambda_1 x_{\alpha_1} + \ldots + \lambda_n x_{\alpha_n}) = (y_\beta)_{\beta < \kappa},
$$

where $y_\alpha = \lambda_\alpha, 1 \leq i \leq n$ and $y_\beta = 0$ if $\beta \notin \{\alpha_i : 1 \leq i \leq n\}$. Then $f$ is a topologically linear isomorphic between $V(D)$ and $(\mathbb{V}_\kappa, \mu_\kappa)$. Moreover, it follows from Theorem 1] that $\tau_\kappa = \nu_\kappa = \mu_\kappa$. However, if $\kappa$ is uncountable the situation changes [24 Theorem 1]. Furthermore, for each $n \in \mathbb{N}$, let

$$
\mathbb{V}_{\kappa,n} = \{x_\alpha : \alpha < \kappa, x_\alpha \in [-n,n]_\alpha, \alpha < \kappa\}.
$$

Obviously, each $\mathbb{V}_{\kappa,n}$ is closed subspace in $\mathbb{V}_\kappa$ within the topologies $\tau_\kappa$, $\nu_\kappa$ or $\mu_\kappa$.

For a subspace $Z$ of a space $X$, let $V(Z, X)$ be the vector subspace of $V(X)$ generated algebraically by $Z$. We recall the following fact from [24].

Fact: Let $\kappa$ be an infinite cardinal. For each $i \in \kappa$, choose some $\lambda_i \in \mathbb{R}^+_i, \lambda_i > 0$, and denote by $S_\kappa$ the family of all subsets $\mathbb{V}_\kappa$ of the form

$$
\bigcup_{i<\kappa} [\llbracket -\lambda_i, \lambda_i \rrbracket \times \prod_{j<\kappa, j \neq i} \{0\}].
$$

For every sequence $\{S_k\}_{k \in \omega}$ in $S_\kappa$, we put

$$
\sum_{k \in \omega} S_k = \bigcup_{k \in \omega} (S_0 + \ldots + S_k)
$$

and denote by $\mathcal{N}_\kappa$ the family of all subsets of $\mathbb{V}_\kappa$ of the form $\sum_{k \in \omega} S_k$. Then $\mathcal{N}_\kappa$ is a base at 0 for $(\mathbb{V}_\kappa, \mu_\kappa)$.

Theorem 3.3. Let $X$ be a collectionwise normal $\aleph_0$-space. Then the following statements are equivalent:

1. $sp_1(X)$ is csf-countable;
2. $V(X)$ is an $\aleph_0$-space;
3. $X$ is separable.

Proof. By Lemma 3.2 we have (2) $\Rightarrow$ (3). Moreover, (2) $\Rightarrow$ (1) is obvious. It suffice to prove (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2).

(3) $\Rightarrow$ (2). Suppose that $X$ is separable, it easily check that $X$ has a countable k-network. Hence $V(X)$ is an $\aleph_0$-space by Lemma 3.2.
(1) ⇒ (3). Assume that $sp_1(X)$ is $csf$-countable and $X$ is not separable. Since $X$ is a collectionwise normal $\aleph$-space, $X$ contains a closed discrete subspace $D$ with the cardinality of $\omega_1$. Then the subgroup $V(D,X)$ is group isomorphic to $V_{\omega_1}$. Then there exists a topology $\sigma$ on $V_{\omega_1}$ such that $V(D,X)$ is topologically isomorphic to $(V_{\omega_1},\sigma)$. Since $L(D)$ is topologically isomorphic to $L(D,X)$ [20] and the topology of $L(D,X)$ is coarser than $V(D,X)$, we have $\sigma$ is finer than the box topology $\tau_{\omega_1}$. For each $\alpha < \omega_1$ and $n \in \mathbb{N}$, let $a_{\alpha,n} = (x_{\beta})_{\beta < \omega_1}$, where $x_{\beta} = \frac{1}{n}$ if $\beta = \alpha$, and $x_{\beta} = 0$ if $\beta \neq \alpha$. Put

$$Y = \{0\} \cup \{a_{\alpha,n} : \alpha < \omega_1, n \in \mathbb{N}\}.$$ 

Then since $\sigma$ is finer than $\tau_{\omega_1}$, it is easy to see that $Y$ is a copy of $S_{\omega_1}$ in $(V_{\omega_1},\sigma|_{V_{\omega_1}})$. Hence $sp_1(X)$ contains a copy of $S_{\omega_1}$. However, $S_{\omega_1}$ is not $csf$-countable, which is a contradiction. □

Theorem 3.5 below gives a characterization of a space $X$ such that $V(X)$ is $snf$-countable, and obtain that $V(X)$ is $snf$-countable if and only if $X$ is finite.

**Proposition 3.4.** The space $V(\mathbb{N})$ is not $snf$-countable.

**Proof.** Since $V(\mathbb{N})$ is topologically isomorphic to $(\mathbb{V}_\omega,\mu_\omega)$, it suffices to prove that $(\mathbb{V}_\omega,\mu_\omega)$ is not $snf$-countable. For arbitrary $n, m \in \mathbb{N}$, let $x_{m,n} = (x_i)_{i \in \mathbb{N}}$, where $x_i = \frac{1}{n}$ if $i = m$, and $x_i = 0$ if $i \neq m$. Put

$$Y = \{0\} \cup \{x_{m,n} : m, n \in \mathbb{N}\}.$$ 

We claim that $Y$ is a copy of $S_{\omega_1}$ in $V(\mathbb{N})$. It suffices to prove that for an arbitrary infinite subset $B \subset \{x_{m,n} : m, n \in \mathbb{N}\}$ with $|B \cap \{x_{m,n} : n \in \mathbb{N}\}| < \omega$ for each $m \in \mathbb{N}$ we have $0 \notin B$. Indeed, there exists a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $B \cap \{x_{m,i} : i \geq \varphi(m)\} = \emptyset$ for each $m \in \mathbb{N}$. For each $k \in \omega$, let

$$S_k = \bigcup_{i < \omega} \left( \left[ -\frac{1}{2^{k+2}\varphi(i)}, -\frac{1}{2^{k+2}\varphi(i)} \right) \times \prod_{j < \omega, j \neq i} \{0\} \right).$$ 

Let $G = \sum_{k \in \omega} S_k$. The set $G$ is an open neighborhood of $0$ in $V(\mathbb{N})$ such that

$$G \subset \prod_{i \in \omega} \left[ -\frac{1}{2\varphi(i)}, \frac{1}{2\varphi(i)} \right] = U.$$ 

However, $U \cap B = \emptyset$. Hence $0 \notin B$. Since $S_{\omega_1}$ is not $snf$-countable, $(\mathbb{V}_\omega,\mu_\omega)$ is not $snf$-countable. □

**Theorem 3.5.** Let $X$ be a (Tychonoff) space. Then $V(X)$ is $snf$-countable if and only if $X$ is finite.

**Proof.** If $X$ is finite, then it is obvious. Assume that $V(X)$ is $snf$-countable and $X$ is infinite. Then it follows from [2] Theorem 4.1 that $V(X)$ contains a closed vector subspace which is topologically isomorphic to $V(\mathbb{N})$. Hence $V(\mathbb{N})$ is $snf$-countable, which is a contradiction with Proposition 3.4. □

We shall give a characterization of a $k^*$-metrizable space $X$ such that $V(X)$ is a $k$-space, which gives an affirmative answer to Question 3.3 if $X$ is a $k^*$-metrizable space. Moreover, the characterization of a non-metrizable $k^*$-metrizable space $X$ is also given such that if the $k$-property of $sp_2(X)$ implies the $k$-property of $V(X)$. First, we shall prove some results, which will be used in our proof.

**Theorem 3.6.** Let $X$ be a submetrizable space. Then $V(X)$ is submetrizable.

**Proof.** Since $X$ is submetrizable, there exists a metric space $M$ such that $i : X \to M$ is an one-to-one continuous mapping. Then the mapping $\hat{i} : V(X) \to V(M)$ is an one-to-one continuous mapping. Let $d$ be a metric of $M$, and $\hat{d}$ the Graev extension of $d$ in $L(M)$. Then $(L(M),\hat{d})$ is a metric space [27]. Hence $L(M)$ is submetrizable. Since the mapping $\hat{i} : V(X) \to V(M)$ is an one-to-one continuous mapping. Thus $V(X)$ is submetrizable. □
Corollary 3.7. Let $X$ be a submetrizable space. Then $V(X)$ is a $k$-space if and only if $V(X)$ is sequential.

Lemma 3.8. For any uncountable cardinal $\kappa$, the tightness of $(\mathbb{V}_\kappa,\mu_\kappa|\mathbb{V}_\kappa)$ is uncountable.

Proof. By [12 Theorem 20.2], we can find two families $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ and $\mathcal{B} = \{B_\alpha : \alpha \in \omega_1\}$ of infinite subsets of $\omega$ such that

(a) $A_\alpha \cap B_\beta$ is finite for all $\alpha, \beta < \omega_1$;
(b) for no $A \subset \omega$, all the sets $A_\alpha \setminus A$ and $B_\alpha \cap A$, $\alpha \in \omega_1$ are finite.

For arbitrary $\alpha, \beta \in \omega_1$ and $n \in \omega$, let

$$c_{\alpha,\beta,n} = (x_\gamma)_{\gamma < \kappa} \in \mathbb{V}_\kappa,$$

where $x_\gamma = \frac{1}{\gamma + n}$ if $\gamma = \alpha$ or $\beta$, and $x_\gamma = 0$ if $\gamma \notin \{\alpha, \beta\}$. Put

$$X = \{c_{\alpha,\beta,n} : \alpha, \beta \in \omega_1, n \in A_\alpha \cap B_\beta\}.$$ 

Obviously, we have $0 \notin X$. We shall prove that $0$ belongs to the closure of $X$ in $(\mathbb{V}_\kappa,\mu_\kappa|\mathbb{V}_\kappa)$, but $0$ does not belong to the closure of $B$ for any countable subset $B$ of $X$ in $(\mathbb{V}_\kappa,\mu_\kappa|\mathbb{V}_\kappa)$.

Since $(\mathbb{V}_\kappa,\mu_\kappa|\mathbb{V}_\kappa)$ is closed in $(\mathbb{V}_\kappa,\mu_\kappa)$ and $X \subset \mathbb{V}_\kappa$, we shall prove that $0$ belongs to the closure of $X$ in $(\mathbb{V}_\kappa,\mu_\kappa)$. Then it suffices to prove $X \cap \sum_{k \in \omega} S_k \neq \emptyset$ for an arbitrary sequence $\{S_k\}_{k \in \omega}$ in $S_\kappa$. Obviously, we can choose $S'_n \in S_\kappa$ such that $S'_n \subset S_n$ and $S'_n \subset S_1$. Then there exists a function $\varphi : \omega_1 \to \omega$ such that

$$W = \bigcup_{i < \kappa} \left( \left[ \left[ - \frac{1}{\varphi(i)} + 1, \frac{1}{\varphi(i)} + 1 \right] \times \prod_{j < \kappa, j \neq i} \{0\} \right] \subset S'_n. $$

Then $\sum_{k \in \omega} W_k \subset \sum_{k \in \omega} S_k$, where $W_0 = W_1 = W$ and $W_k = S_k$ for each $k > 1$. We claim that $X \cap \sum_{k \in \omega} W_k \neq \emptyset$, hence $\sum_{k \in \omega} S_k \cap X \neq \emptyset$. Indeed, for each $\alpha \in \omega_1$, put

$$A'_\alpha = \{n \in A_\alpha : n \geq \varphi(\alpha) + 1\} \text{ and } B'_\alpha = \{n \in B_\alpha : n \geq \varphi(\alpha) + 1\}.$$ 

It follows that there exist $\alpha, \beta \in \omega_1$ such that $A'_\alpha \cap B'_\beta \neq \emptyset$. If not, then we have

$$\bigcup\{A'_\alpha : \alpha \in \omega_1\} \cap \bigcup\{B'_\alpha : \alpha \in \omega_1\} = \emptyset.$$ 

Put $A = \bigcup\{A'_\alpha : \alpha \in \omega_1\}$. We get a contradiction with (b). Therefore, choose $n \in A'_\alpha \cap B'_\beta$.

Then it is obvious that

$$c_{\alpha,\beta,n} \in X \cap (W + W) \subset \sum_{k \in \omega} W_k \subset \sum_{k \in \omega} S_k.$$ 

Finally, we prove that for an arbitrary countable subset $B$ of $X$ the point $0$ does not belong to the closure of $B$ in $(\mathbb{V}_\kappa,\mu_\kappa)$. Take an arbitrary countable subset $B$ of $X$. Then there exists an cardinal $\eta < \omega_1$ such that

$$B \subset \{c_{\alpha,\beta,n} : \alpha, \beta < \eta, n \in \omega\}.$$ 

Without loss of generality, we may assume $\eta = \omega$ (otherwise order $\eta$ as $\omega$). Define a function $\psi : \omega_1 \to \omega$ by

$$\psi(m) = 2 + \max\{A_k \cap B_l : k, l \leq m\}$$ 

for each $m < \omega$ and $\psi(\alpha) = 1$ if $\omega \leq \alpha < \omega_1$. For each $k \in \omega$, put

$$G_k = \bigcup_{i < \kappa} \left( \left[ \left[ - \frac{1}{2k+1 + \psi(i)} + \frac{1}{2k+1 + \psi(i)} \right] \times \prod_{j < \kappa, j \neq i} \{0\} \right] \subset S_k.$$ 

Then $\sum_{k \in \omega} G_k \in \mathcal{N}_\kappa$ and $\sum_{k \in \omega} G_k \cap B = \emptyset$. Indeed, assume $c_{\alpha,\beta,n} \in B \cap \sum_{k \in \omega} G_k$, and then since $c_{\alpha,\beta,n} \in B$, we have $n \in A_\alpha \cap B_\beta$. Moreover, we can assume that $\alpha \leq \beta$. Obviously, $\psi(\beta) > \max(A_\alpha \cap B_\beta) + 1$, thus $\frac{1}{n+1} > \frac{1}{\psi(\beta)}$. Then it follows from the definition of each $G_k$ that $c_{\alpha,\beta,n} \notin \sum_{k \in \omega} G_k$. $\square$

Lemma 3.9. Let $D$ be an uncountable discrete space. Then the tightness of $sp_2(D)$ is uncountable. In particular, $V(D)$ is uncountable.
Proof. Let $D$ be the cardinality of $\kappa$. It is easy to see that $\text{sp}_2(D)$ homeomorphic to $(\forall \kappa, \omega_1, \mu_\kappa | \kappa_\omega)$. By Lemma 3.11 the tightness of $\text{sp}_2(D)$ is uncountable.

The following Lemma 3.11 improves a well-known result in [1]. First, we recall a concept. A space is called $\aleph_1$-compact if every uncountable subset of $X$ has a cluster point.

**Lemma 3.10.** Let $X$ be a $k^*$-metrizable space. Then $A(X)$ is a $k$-space if and only if $X$ is the topological sum of a $k_\omega$-space and a discrete space.

**Proof.** Clearly, it suffices to prove the necessity. Assume that $A(X)$ is a $k$-space, which also implies that $X^2$ is a $k$-space. Since $X$ is a $k^*$-metrizable space, $X$ has a compact-countable $k$-network. Then it follows from [21] Theorem 3.4 that $X$ is either first-countable or locally $k_\omega$. If $X$ is first-countable, then $X$ is metrizable since $X$ is a $k^*$-metrizable space. Then it follows from [1], that $X$ is locally compact and the set of all non-isolated points $X'$ of $X$ is separable. Therefore, it easily see that $X$ can be represented as $X = X_0 \bigoplus D$, where $X_0$ is a $k_\omega$-space and $D$ is a discrete space.

Assume that $X$ is locally $k_\omega$. Since $X$ is a $k^*$-metrizable space, then there exists a compact-countable $k$-network consisting of sets with compact closures, hence $X$ has a star-countable $k$-network. Then it follows from [24] Corollary 2.4 that $X$ is a paracompact $\sigma$-space, then $A(X)$ is also a paracompact $\sigma$-space by [2] Theorem 7.6.7. Therefore, $A(X)$ is a sequential space since $A(X)$ is a paracompact $\sigma$-space. We claim that the set $X'$ of all non-isolated points is $\aleph_1$-compact.

Suppose not, then there exists an uncountable closed discrete subset $\{x_\alpha : \alpha < \omega_1\}$ of $X'$. Since $X$ is paracompact, there exists a family of discrete open subsets $\{U_\alpha : \alpha < \omega_1\}$ such that $x_\alpha \in U_\alpha$ for each $\alpha < \omega_1$. Since $A(X)$ is a $\sigma$ and $k$-space, the space $X$ is sequential. Then for each $\alpha < \omega_1$ we can take a nontrivial convergent sequence $\{x_{\alpha,n} : n \in \mathbb{N}\} \subset U_\alpha$ with the limit point $x_\alpha$. Let $Y$ be the quotient space by identifying all the points $\{x_\alpha : \alpha < \omega_1\}$ to a point $\{\infty\}$. Then the natural mapping $A(X) \to A(Y)$ is quotient, hence $A(Y)$ is a sequential space. Then it is easy to check that $Y$ contains a closed copy of $S_\omega$, hence $A(Y)$ contains a closed copy of $S_\omega_1 \times S_\omega_1$, which is a contradiction with [21] Corollary 7.6.23. Therefore, the set $X'$ of all non-isolated points is $\aleph_1$-compact. Then $X'$ is a Lindelöf space. Therefore, it easily check that $X$ is the topological sum of a $k_\omega$-space with a discrete space since $X'$ is a Lindelöf space and $X$ and is locally $k_\omega$.

The following Lemma 3.11 improves a well-known result in [29].

**Lemma 3.11.** Let $X$ be a $k^*$-metrizable space. Then the following statements are equivalent:

1. $A_4(X)$ is a $k$-space;
2. each $A_n(X)$ is a $k$-space;
3. $X$ satisfies at least one of the following conditions:
   (a) $X$ is the topological sum of a $k_\omega$-space and a discrete space;
   (b) $X$ is metrizable and $X'$ is compact.

**Proof.** Obviously, we have (2) $\Rightarrow$ (1). It suffice to prove (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2).

(3) $\Rightarrow$ (2). If $X$ is metrizable and $X'$ is compact, then it follows from [29] Theorem 4.2 that each $A_n(X)$ is a $k$-space. Assume that $X$ is the topological sum of a $k_\omega$-space $Y$ and a discrete space $D$. Then $A(X)$ is topologically isomorphic to $A(Y) \times A(D)$. Then it follows from [2] Theorem 7.41 that each $A_n(X)$ is a $k$-space.

(1) $\Rightarrow$ (3). By [29] Theorem 4.2, it suffices to prove $X$ is metrizable or the topological sum of a $k_\omega$-space and a discrete space. Assume that $X$ is not metrizable. Then we shall prove that $X$ is the topological sum of a $k_\omega$-space and a discrete space. First, we claim that $X$ contains a closed copy of $S_\omega$ or $S_2$. Suppose not, since $X$ is a $k$-space with a point-countable $k$-network, it follow from [28] Lemma 8 and [18] Corollary 3.10 that $X$ has a point-countable base, and thus $X$ is metrizable since a paracompact $\sigma$-space with a point-countable base is metrizable [5], which is a contradiction with the assumption. Moreover, by [16] Lemma 4.7, $A_4(X)$ contains
a closed copy of $X^2$, hence $X^2$ is a $k$-space. Next, we shall prove that $X$ is the topological sum of a family of $k_ω$-spaces. We divide the proof into the following two cases.

**Case 1.1:** The space $X$ contains a closed copy of $S_ω$.

Since $S_ω × X$ is a closed subspace of $X^2$, the subspace $S_ω × X$ is a $k$-space. By [20 Lemma 4], the space $X$ has a compact-countable $k$-network consisting of sets with compact closures, hence $X$ has a compact-countable compact $k$-network $P$. Then $P$ is star-countable, hence it follows from [11] that we have

$$P = \bigcup_{α ∈ A} P_α,$$

where each $P_α$ is countable and $(\bigcup P_α) \cap (\bigcup P_β) = \emptyset$ for any $α \neq β ∈ A$. For each $α ∈ A$, put $X_α = \bigcup P_α$. Obviously, each $X_α$ is a closed $k$-subspace of $X$ and has a countable compact $k$-network $P_α$. Moreover, we claim that each $X_α$ is open in $X$. Indeed, fix an arbitrary $α ∈ A$. Since $X$ is a $k$-space, it suffices to prove that $\bigcup \{X_β : β ∈ A, β \neq α\} ∩ K$ is closed in $K$ for each compact subset $K$ in $X$. Take an arbitrary compact subset $K$ in $X$. Since $P$ is a $k$-network of $X$, there exists a finite subfamily $P' ⊂ P$ such that $K ⊂ \bigcup P'$. Then

$$\bigcup \{X_β : β ∈ A, β \neq α\} ∩ K = \bigcup \{X_β : β ∈ A, β \neq α\} ∩ K \bigcap \bigcup P' = K ∩ \{P : P ∈ P', P \notin P_α\}.$$ 

Since each element of $P'$ is compact, the set $\bigcup \{X_β : β ∈ A, β \neq α\} ∩ K$ is closed in $K$. Therefore, $X = \bigoplus_{α ∈ A} X_α$ and each $X_α$ is a $k_ω$-subspace of $X$. Thus $X$ is the topological sum of a family of $k_ω$-subspaces.

**Case 1.2:** The space $X$ contains a closed copy of $S_2$.

Obviously, $S_2 × X$ is a $k$-space. Since $S_ω$ is the image of $S_2$ under the perfect mapping and the $k_ω$-property is preserved by the quotient mapping, $S_ω × X$ is a $k$-space. By Case 1.1, $X$ is the topological sum of a family of $k_ω$-subspaces.

Therefore, $X$ is the topological sum of a family of $k_ω$-subspaces. Finally, it suffices to prove that $X'$ is $ω_1$-compact. Indeed, this fact is easily checked by using the well-known Theorem of Yamada [20 Theorem 3.4]. Then $X$ is the topological sum of a $k_ω$-space and a discrete space.

By Lemmas 3.10 and 3.11 we have the following theorem.

**Theorem 3.12.** Let $X$ be a non-metrizable $k^*$-metrizable space. Then the following statements are equivalent:

1. $A(X)$ is a $k$-space;
2. $A_4(X)$ is a $k$-space;
3. $X$ is the topological sum of a $k_ω$-space and a discrete space.

Now, we can prove some of our main results in this paper.

**Theorem 3.13.** Let $X$ be a $k^*$-metrizable space. Then the following statements are equivalent:

1. $V(X)$ is a $k$-space;
2. $V(X)$ is a $k_ω$-space;
3. $X$ is a $k_ω$-space.

**Proof.** By [7 Theorem 3.1], we have (3) ⇒ (2), and (2) ⇒ (1) is obvious. It suffices to prove (1) ⇒ (3).

(1) ⇒ (3). By Lemma 3.10, $X$ is the topological sum of a $k_ω$-space and a discrete space. Let $X = X_0 ⊕ D$, where $X_0$ is a $k_ω$-space and $D$ is a discrete space. By [7 Corollary 2.6], we have

$$V(X) \cong V(X_0) ⊕ V(D).$$

Hence $V(D)$ is a $k$-space. Then $V(D)$ is sequential by Corollary 3.7, which implies $D$ is countable by Lemma 3.9. Therefore, $X$ is a $k_ω$-space.
By Corollary 3.14, Theorems 3.13 and 3.15, we have the following corollary.

**Corollary 3.14.** For an arbitrary uncountable discrete space $D$, the space $sp_2(D)$ and $V(D)$ are all not $k$-spaces.

**Remark:** In [23, Theorem 5], the authors said that $(\forall \kappa, \mu_\kappa)$ is not sequential. However, the proof is wrong. Theorem 3.13 shows that $(\forall \kappa, \mu_\kappa)$ is not sequential.

The following theorem gives a characterization of a non-metrizable $k^*$-metrizable $X$ such that if $sp_4(X)$ is a $k$-space then $V(X)$ is also a $k$-space.

**Theorem 3.15.** Let $X$ be a non-metrizable $k^*$-metrizable space. The following statements are equivalent:

1. $V(X)$ is a $k$-space;
2. $V(X)$ is a $k_\omega$-space;
3. $sp_4(X)$ is a $k_\omega$-space;
4. $sp_4(X)$ is a $k$-space;
5. $X$ is a $k_\omega$-space.

**Proof.** Obviously, it suffices to prove that $X$ is a $k_\omega$-space if $sp_4(X)$ is a $k$-space. Since $A(X)$ is closed in $V(X)$ and an embedding by [7, Proposition 11.3], $A_4(X)$ is a closed subset of $V(X)$, hence $A_4(X)$ is a $k$-space. By Theorem 3.15, $X$ is the topological sum of a $k_\omega$-space and a discrete space. Then $D$ is countable by Lemma 3.16. Therefore, $X$ is a $k_\omega$-space.

Finally, we shall prove the last main results of this paper, and prove that for a paracompact $\sigma$- and $k$-space $X$, the tightness of $V(X)$ is countable if and only if $X$ is separable, which gives an affirmative answer to Question 1.2. First of all, we prove a lemma.

**Lemma 3.16.** Let $X$ be a paracompact, sequential space. If the tightness of $A_4(X)$ is countable, then the set of all non-isolated points $X$ is separable.

**Proof.** Assume that the set $X'$ of all non-isolated points $X$ is not separable. Then since $X'$ is closed in a paracompact space $X$, there exists an uncountable closed discrete subset $\{x_\alpha : \alpha < \omega_1\}$ of $X'$. Since $X$ is paracompact, there exists a family of discrete open subsets $\{U_\alpha : \alpha < \omega_1\}$ such that $x_\alpha \in U_\alpha$ for each $\alpha < \omega_1$. Since $X$ is a sequential space, for each $\alpha < \omega_1$ we can take a nontrivial convergent sequence $\{x_{\alpha,n} : n \in \mathbb{N}\} \subset U_\alpha$ with the limit point $x_\alpha$. For each $\alpha < \omega_1$, put $C_\alpha = \{x_\alpha\} \cup \{x_{\alpha,n} : n \in \mathbb{N}\}$. Put $Y = \bigoplus_{\alpha<\omega_1} C_\alpha$. It follows from [30, Theorem 4.2] that $A_4(Y)$ is not of countable tightness. However, since $X$ is paracompact and $Y$ is closed in $X$, it follows from [20] that $A(Y)$ is topologically isomorphic to $A(Y, X)$. Then $A_4(Y)$ is of countable tightness, which is a contradiction. Therefore, the set of all non-isolated points of $X$ is separable.

**Theorem 3.17.** Let $X$ be a paracompact $\sigma$-space. If $X$ is a $k$-space, then the following statements are equivalent:

1. the tightness of $V(X)$ is countable;
2. the tightness of $sp_4(X)$ is countable;
3. $X$ is separable.

**Proof.** Clearly, we have (1) $\Rightarrow$ (2). It suffices to prove (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Assume that $X$ is separable, then $X$ is a cosmic space. By Corollary 3.14, 5.20, $V(X)$ is a cosmic space, hence the tightness of $V(X)$ is countable.

(2) $\Rightarrow$ (3). Assume that tightness of $sp_4(X)$ is countable. Since $X$ is a $\sigma$-space, the $k$-property in $X$ is equivalent to the sequentiality of $X$. Since $A(X)$ is closed in $V(X)$ and an embedding by [7, Proposition 11.3], the tightness of $A_4(X)$ is countable. It follows from Lemma 3.16 that the set of all non-isolated points $X'$ is separable. Since $X'$ is a paracompact $\sigma$-space, $X'$ is cosmic space. Then $X'$ is $R_1$-compact. By Lemma 3.14, it is easy to see that for each neighborhood $U$ of $X'$ in $X$ we have $X \setminus U$ is countable, which also implies that $U$ is separable. Suppose not, there exists an open subset $U$ of $X'$ such that $U$ is not separable. Then $W$ is not $R_1$-compact.
since $U$ is also a $\sigma$-space. It easily check that $W$ contains an uncountable closed discrete subset $D$ of $X \setminus X'$, then $U \setminus D$ is an open neighborhood of $X'$. However, $U \setminus D$ is uncountable, which is a contradiction. Thus $X$ is separable. □

By Theorems 3.3 and 3.17, we have the following corollary.

**Corollary 3.18.** Let $X$ be a paracompact $\aleph$-space. If $X$ is a $k$-space, then the following statements are equivalent:

1. the tightness of $V(X)$ is countable;
2. the tightness of $sp_4(X)$ is countable;
3. $sp_4(X)$ is csf-countable;
4. $V(X)$ is an $\aleph_0$-space;
5. $X$ separable.

**Remark:** By Theorems 3.13 and 3.17, the tightness of $V(P)$ is countable; however, $V(P)$ is not a $k$-space, where the irrational number $P$ endowed with the usual topology.

4. **Open questions**

In this section, we pose some interesting questions about the free topological vector spaces, which are still unknown to us.

It is well-known that for a closed subset $Y$ of a metrizable space $X$, we have $F(Y)$, $A(Y)$ and $L(Y)$ are topologically isomorphic to some subgroups of $F(X)$, $A(X)$ and $L(X)$ respectively. Hence we have the following question:

**Question 4.1.** Let $Y$ be a closed subset of metrizable space $X$. Is $V(Y)$ topologically isomorphic to a vector subspace of $V(X)$? What if $Y$ is a closed discrete subspace?

By Theorems 3.15, 3.17 and Lemma 3.9, it is natural to pose the following two questions:

**Question 4.2.** Let $X$ be a metrizable space. If $sp_2(X)$ is a $k$-space, is $sp_4(X)$ a $k$-space?

**Question 4.3.** Let $X$ be a metrizable space. If $sp_4(X)$ is a $k$-space, is each $sp_n(X)$ a $k$-space?

Indeed, we have the following some partial answer to Question 4.3.

**Theorem 4.4.** Let $X$ be a metrizable space. If $sp_4(X)$ is a $k$-space, then the set $X'$ is compact and $X \setminus X'$ is countable.

**Proof.** Indeed, it is easy to see by Lemmas 3.9, 3.11 and the proof of (2) $\Rightarrow$ (3) in Theorem 3.17.

**Corollary 4.5.** Let $X$ be a locally compact metrizable space. Then the following statements are equivalent:

1. $V(X)$ is a $k$-space;
2. $V(X)$ is a $k_\omega$-space;
3. $sp_4(X)$ is a $k$-space;
4. $sp_2(X)$ is a $k_\omega$-space;
5. $X$ is a $k_\omega$-space.

**Question 4.6.** Let $X$ be a metrizable space. If $sp_2(X)$ is of countable tightness, is $sp_4(X)$ of countable tightness?

**Question 4.7.** Let $X$ be a non-metrizable $k^*$-metrizable space. If $sp_4(X)$ is a $k$-space, is $V(X)$ a $k$-space?
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