SOME IDENTITIES FOR THE BERNOULLI, THE EULER AND THE GENOCCHI NUMBERS AND POLYNOMIALS

TAEKYUN KIM

Division of General Education-Mathematics,
Kwangwoon University, Seoul 139-701, S. Korea
e-mail: tkkim@kw.ac.kr

ABSTRACT. The purpose of this paper is to give some new identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials.

1. Introduction

Let \( p \) be a fixed odd prime and let \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C} \) and \( \mathbb{C}_p \) denote the ring of \( p \)-adic rational integers, the field of \( p \)-adic rational numbers, the complex number field and the completion of algebraic closure of \( \mathbb{Q}_p \). The \( p \)-adic absolute value in \( \mathbb{C}_p \) is normalized so that \( |p|_p = \frac{1}{p} \). For \( f \in UD(\mathbb{Z}_p) \), let us start with the expression

\[
\sum_{0 \leq j < p^N} (-1)^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu(j + p^N \mathbb{Z}_p)
\]

representing analogue of Riemann’s sums for \( f \), cf.[1-10].

The fermionic \( p \)-adic invariant integral of \( f \) on \( \mathbb{Z}_p \) will be defined as the limit \((N \to \infty)\) of these sums, which it exists. The fermionic \( p \)-adic invariant integral of a function \( f \in UD(\mathbb{Z}_p) \) is defined in [1, 4, 5, 10] as follows:

\[
(1) \quad I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \sum_{0 \leq j < p^N} f(j) \mu(j + p^N \mathbb{Z}_p) = \lim_{N \to \infty} \sum_{0 \leq j < p^N} f(j)(-1)^j.
\]

2000 Mathematics Subject Classification 11S80, 11B68

Keywords and phrases: Euler number, \( p \)-adic invariant integrals, zeta function, \( p \)-adic fermionic integrals

Typeset by \LaTeX\
From (1), we note that

\[ I(f_1) + I(f) = 2f(0), \] where \( f_1(x) = f(x+1) \).

By using integral iterative method, we also easily see that

\[ I(f_n) + (-1)^n I(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l), \] where \( f_n(x) = f(x+n) \) for \( n \in \mathbb{N} \).

For \( d \) a fixed positive even integer with \((p, d) = 1\), let

\[ X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \]

\[ X^* = \bigcup_{0 < a < dp \atop (a,p) = 1} a + dp\mathbb{Z}_p, \]

\[ a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}, \]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \), (see [1-10]). Recently, several authors have studied the properties of \( p \)-adic invariant integral on \( \mathbb{Z}_p \) related to Euler numbers and polynomials (see [1, 2, 4, 5, 7, 8, 9, 10]). The integral equation (2) for \( n \equiv 1 \pmod{2} \) are useful to study the congruence of Euler numbers and polynomials (see [1, 2, 4, 5, 7, 8, 9, 10]). In this paper, we consider the equations of the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) for \( n \equiv 0 \pmod{2} \). From those equations of the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) for \( n \equiv 0 \pmod{2} \), we derive some interesting and valuable identities for the Euler, the Genocchi and the Bernoulli numbers and polynomials.

2. Some identities of the Bernoulli, the Euler and the Genocchi numbers and polynomials

If \( n \equiv 0 \pmod{2} \) in (2), then we see that

\[ I(f_n) - I(f) = 2 \sum_{l=0}^{n-1} (-1)^{l-1} f(l), \quad f_n(x) = f(x+n). \]

Let us take \( f(x) = e^{tx} \). Then we have

\[ \int_{\mathbb{Z}_p} e^{tx} d\mu(x) = \frac{2 \sum_{l=0}^{d-1} (-1)^{l-1} e^{lt}}{e^{dt} - 1}, \] for \( d \in \mathbb{N} \) with \( d \equiv 0 \pmod{2} \).
It is easy to see that
\[
(5) \quad \frac{2 \sum_{l=0}^{d-1} (-1)^{l-1} e^{lt}}{e^{dt} - 1} = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},
\]
where \( E_n \) are the \( n \)-th Euler numbers.

From (4), we can also derive
\[
\int_{\mathbb{Z}_p} e^{xt} d\mu(x) = \frac{2 \sum_{l=0}^{d-1} (-1)^{l-1} e^{lt}}{e^{dt} - 1} = \frac{2}{e^t + 1} \left( \frac{dt}{e^{dt} - 1} \right) e^{lt} \]
\[
= \frac{1}{dt} 2 \sum_{l=0}^{d-1} (-1)^{l-1} \left( \sum_{n=1}^{\infty} B_n \left( \frac{l}{d} \right) d^n \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} \left( 2d^n \sum_{l=0}^{d-1} (-1)^{l-1} B_{n+1} \left( \frac{l}{d} \right) \frac{t^n}{n+1} \right) \frac{t^n}{n!},
\]
where \( B_n(x) \) are the \( n \)-th Bernoulli polynomials.

Thus, we have
\[
(7) \quad \int_{\mathbb{Z}_p} x^n d\mu(x) = \frac{2d^n}{n+1} \sum_{l=0}^{d-1} (-1)^{l-1} B_{n+1} \left( \frac{l}{d} \right), \quad \text{where} \ d \in \mathbb{N} \text{ with } d \equiv 0 \pmod{2}.
\]

By (4) and (5), we also see that
\[
(8) \quad \int_{\mathbb{Z}_p} e^{tx} d\mu(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},
\]
where \( E_n \) are the \( n \)-th Euler numbers.

From (7) and (8), we obtain the following theorem.

**Theorem 1.** For \( n \in \mathbb{Z}_+ \), \( d \in \mathbb{N} \) with \( d \equiv 0 \pmod{2} \), we have
\[
\frac{E_n}{2} = \frac{d^n}{n+1} \sum_{l=0}^{d-1} (-1)^{l-1} B_{n+1} \left( \frac{l}{d} \right).
\]

It is easy to show that
\[
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{2 \sum_{l=0}^{d-1} (-1)^{l-1} e^{lt}}{e^{dt} - 1} e^{xt} = \frac{2}{dt} \sum_{l=0}^{d-1} (-1)^{l-1} \left( \frac{dt}{e^{dt} - 1} \right) e^{(l+x)t}
\]
\[
= 2 \sum_{l=0}^{d-1} (-1)^{l-1} \sum_{n=0}^{\infty} B_{n+1} \left( \frac{l+x}{d} \right) \frac{d^n t^n}{n+1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( 2d^n \sum_{l=0}^{d-1} (-1)^{l-1} B_{n+1} \left( \frac{l+x}{d} \right) \frac{t^n}{n+1} \right) \frac{t^n}{n!},
\]
From (3), we can easily derive the following equation (3).

\[
\int_{\mathbb{Z}_p} e^{(x+y)t} \, d\mu(y) = 2 \sum_{l=0}^{d-1} \frac{(-1)^{l-1} e^{lt}}{e^{dt} - 1} e^{xt} = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]

By (3), (9) and (10), we have

\[
E_n(x) = \frac{1}{2} \int_{\mathbb{Z}_p} (y + x)^n \, d\mu(y) = \frac{d^n}{n+1} \sum_{l=0}^{d-1} (-1)^{l-1} B_{n+1} \left( \frac{l + x}{d} \right),
\]

and

\[
\frac{1}{2} (E_n(d) - E_n) = \frac{d^n}{n+1} \left( \sum_{l=0}^{d-1} (-1)^{l-1} \left( B_{n+1} \left( \frac{l}{d} + 1 \right) - B_{n+1} \left( \frac{l}{d} \right) \right) \right) = \sum_{l=0}^{d-1} (-1)^{l-1} l^n.
\]

Therefore, we obtain the following theorem.

**Theorem 2.** For \( d \in \mathbb{N} \) with \( d \equiv 0 \pmod{2} \), \( n \in \mathbb{Z}_+ \), we have

\[
\sum_{l=0}^{d-1} (-1)^{l-1} \left( \frac{l}{d} \right)^n = \frac{1}{n+1} \left( \sum_{l=0}^{d-1} (-1)^{l-1} \left( B_{n+1} \left( \frac{l}{d} + 1 \right) - B_{n+1} \left( \frac{l}{d} \right) \right) \right),
\]

and

\[
\frac{1}{2} (E_n(d) - E_n) = \sum_{l=0}^{d-1} (-1)^{l-1} l^n.
\]

Let us define the \( n \)-th Genocchi polynomials as follows: for \( d \in \mathbb{N} \) with \( d \equiv 0 \pmod{2} \),

\[
t \int_{\mathbb{Z}_p} e^{(x+y)t} \, d\mu(y) = \frac{2t}{d} \sum_{l=0}^{d-1} \frac{(-1)^{l-1} e^{lt}}{e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}.
\]

Thus, we note that

\[
\frac{2t}{d} \sum_{l=0}^{d-1} \frac{(-1)^{l-1} e^{lt}}{e^{dt} - 1} e^{xt} = \frac{2}{d} \int \frac{dt}{e^{dt} - 1} \sum_{l=0}^{d-1} (-1)^{l-1} \sum_{n=0}^{\infty} B_n \left( \frac{l + x}{d} \right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( 2d^{n-1} \sum_{l=0}^{d-1} (-1)^{l-1} B_n \left( \frac{l + x}{d} \right) \right) \frac{t^n}{n!}.
\]

By comparing coefficients on the both sides of (11) and (12), we obtain the following theorem.
Theorem 3. For \( d \in \mathbb{N} \) with \( d \equiv 0 \pmod{2} \), we have
\[
\frac{G_n(x)}{2} = d^{n-1} \sum_{l=0}^{d-1} (-1)^{l-1} B_n\left(\frac{l+x}{d}\right), \quad \text{for } n \in \mathbb{Z}_+.
\]

Recently, several authors have studied the generalized Euler numbers and polynomials attached to the Dirichlet’s character with odd conductor (see [1, 2, 4, 5, 7, 8, 9, 10, 11, 12]). Now, we consider the generalized Euler polynomials attached to the Dirichlet’s character with even conductor. For \( d \in \mathbb{N} \) with \( d \equiv 0 \pmod{2} \), let \( \chi \) be the Dirichlet’s character with conductor \( d \). From (2), we note that
\[
\int_X \chi(y) e^{x+y} d\mu(y) = \left( 2 \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) e^{lt} \right) e^{xt}.
\]

Let us define the generalized Euler polynomials for the Dirichlet’s character with even conductor as follows: for \( d \in \mathbb{N} \) with \( d \equiv 0 \pmod{2} \),
\[
\sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!}.
\]

From (13) and (14), we can also derive the following equation:
\[
\int_X \chi(y)(x+y)^n d\mu(y) = E_{n,\chi}(x), \quad \text{for } n \in \mathbb{Z}_+.
\]

The \( n \)-th Genocchi polynomials are also defined by
\[
\sum_{n=0}^{\infty} G_{n,\chi}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,\chi}(x) \frac{t^n}{n!}, \quad \text{where } d \in \mathbb{N} \text{ with } d \equiv 0 \pmod{2}.
\]

By (15), we see that
\[
\sum_{n=0}^{\infty} G_{n,\chi}(x) \frac{t^n}{n!} = \frac{2}{d} \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) \frac{dt}{e^{dt} - 1} e^{(l+x)t} = \frac{2}{d} \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) \sum_{n=0}^{\infty} d^n B_n\left(\frac{l+x}{d}\right) \frac{t^n}{n!}.
\]

By comparing coefficients on the both sides of (16), we obtain the following theorem.
Theorem 4. Let \( d \equiv 0 \pmod{2} \). Then we have

\[
\frac{G_{n,\chi}(x)}{2} = d^{n-1} \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) R_n\left(\frac{l+x}{d}\right).
\]

Moreover, \( G_{0,\chi}(x) = 0 \), and \( E_{n,\chi}(x) = \frac{G_{n+1,\chi}(x)}{n+1} \).

It is not difficult to show that

\[
\int_X \chi(x)e^{nt} d\mu(x) - \int_X \chi(x) e^x d\mu(x) = \frac{2 \int_X \chi(x) e^x d\mu(x)}{\int e^{nt} d\mu(x)}
= \sum_{k=0}^{\infty} \left( 2 \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) t^k \right) \frac{t^k}{k!}.
\]

Let \( T_{k,\chi}(n) = \sum_{l=0}^{n-1} (-1)^{n-1} \chi(l) t^k \). From (17), we note that

\[
2 \int_X \chi(x) e^{xt} d\mu(x) \int_X e^{nt} d\mu(x) = \sum_{k=0}^{\infty} 2T_{k,\chi}(dn-1) \frac{t^k}{k!}, \text{ where } d \in \mathbb{N} \text{ with } d \equiv 0 \pmod{2}, n \in \mathbb{Z}_+,
\]

and

\[
\int_X \int_X e^{(w_1 x_1 + w_2 x_2) t} \chi(x_1) \chi(x_2) d\mu(x_1) d\mu(x_2)
\int_X e^{d t} d\mu(x)
= \left( \frac{2(e^{d w_1 w_2 t} - 1)}{(e^{w_1 d t} - 1)(e^{w_2 d t} - 1)} \right) \left( \sum_{a=0}^{d-1} \chi(a) e^{w_1 a t} (-1)^{a-1} \right) \left( \sum_{b=0}^{d-1} \chi(b) (-1)^{b-1} e^{w_2 b t} \right),
\]

where \( d \in \mathbb{N} \text{ with } d \equiv 0 \pmod{2}, \text{ and } w_1, w_2 \in \mathbb{N} \).

Let

\[
K(\chi; w_1, w_2 | x) = \frac{\int_X \int_X \chi(x_1) \chi(x_2) e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x) t} d\mu(x_1) d\mu(x_2)}{\int_X e^{d t} d\mu(x)}.
\]

By (18), we see that \( K(\chi; w_1, w_2 | x) \) is symmetric in \( w_1 \) and \( w_2 \). From (19) and Theorem 4, we obtain the following theorem.
Theorem 5. For $d \in \mathbb{N}$ with $d \equiv 0 \pmod{2}$, let $\chi$ be the Dirichlet’s character with conductor $d$. Then we have

$$
\sum_{i=0}^{l} \binom{l}{i} \frac{d^i}{i+1} \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) B_{i+1} \left( \frac{l + w_2 x}{d} \right) T_{l-i, \chi} (dw_1 - 1) w_1^i w_2^{l-i}
$$

$$
= \sum_{i=0}^{l} \binom{l}{i} \frac{d^i}{i+1} \sum_{l=0}^{d-1} (-1)^{l-1} \chi(l) B_{i+1} \left( \frac{l + w_1 x}{d} \right) T_{l-i, \chi} (dw_2 - 1) w_2^i w_1^{l-i}.
$$

References

1. T. Kim, Symmetry identities for the twisted generalized Euler polynomials, Adv. Stud. Contemp. Math. 19 (2009), 151-155.
2. M. Cenkci, The $p$-adic generalized twisted $(h, q)$-Euler-l-function and its applications, Adv. Stud. Contemp. Math. 15 (2007), 37-47.
3. Z. Zhang, H. Yang, Some closed formulas for generalizations of Bernoulli and Euler numbers and polynomials, Proceedings of the Jangjeon Mathematical Society 11 (2008), 191-198.
4. Y. H. Kim, K.-W. Hwang, Symmetry of power sum and twisted Bernoulli polynomials, Adv. Stud. Contemp. Math. 18 (2009), 127-133.
5. H. Ozden, I. N. Cangul, Y. Simsek, Remarks on $q$-Bernoulli numbers associated with Dahee numbers, Adv. Stud. Contemp. Math. 15 (2009), 41-48.
6. C.-P. Chen, L. Lin, An inequality for the generalized Euler constant function, Adv. Stud. Contemp. Math. 17 (2008), 105-107.
7. C. S. Ryoo, Calculating zeros of the twisted Genocchi polynomials, Adv. Stud. Contemp. Math. 17 (2008), 147-159.
8. M. Cenkci, Y. Simsek, V. Kurt, Multiple two-variable $p$-adic $q$-L-function and its behavior at $s = 0$, Russ. J. Math. Phys. 15 (2008), 447-459.
9. Y. Simsek, On $p$-adic twisted $q$-L-functions related to generalized twisted Bernoulli numbers, Russian J. Math. Phys. 13 (2006), 340-348.
10. T. Kim, Symmetry of power sum polynomials and multivariate fermionic $p$-adic integral on $\mathbb{Z}_p$, Russian J. Math. Phys. 16 (2009), 93-96.
11. M. Cenkci, M. Can, Some results on $q$-analogue of the Lerch zeta function, Adv. Stud. Contemp. Math. 12 (2006), 213-223.
12. M. Cenkci, M. Can, V. Kurt, $p$-adic interpolation functions and Kummer-type congruences for $q$-twisted and $q$-generalized twisted Euler numbers, Adv. Stud. Contemp. Math. 9 (2004), 203-216.