AN ARC GRAPH DISTANCE FORMULA FOR THE FLIP GRAPH

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Abstract. Using existing technology, we prove a Masur-Minsky style distance formula for flip-graph distance between two triangulations, expressed as a sum of the distances of the projections of these triangulations into arc graphs of the suitable subsurfaces of $S$.

1. INTRODUCTION

Let $S$ be a surface with at least one puncture and $\chi(S) < 0$, and write $\mathcal{F}(S)$ for the flip graph of $S$. This is the graph whose vertices are in a one-to-one correspondence with ideal triangulations, and whose edges connect triangulations that differ by a flip; see [DP14] and Figure 1. The purpose of this note is to prove the following formula estimating distance in $\mathcal{F}(S)$.

Theorem 1.1. Fix $S$, a connected, orientable, finite type, surface of non-positive Euler characteristic, with at least one puncture, and not a pair of pants. For any $k > 0$ sufficiently large, there exists $K \geq 1, C \geq 0$ so that for any two triangulations $T_1, T_2 \in \mathcal{F}(S)$ we have

$$d_{\mathcal{F}}(T_1, T_2) \xrightarrow{K,C} \sum_{Y \subseteq S} |d_{\mathcal{A}(Y)}(T_1, T_2)|_k.$$  

The distances on the right are arc graph distances in subsurfaces, $[x]_k$ is the cut-off function giving value $x$ if $x \geq k$ and 0 otherwise, and $x \xrightarrow{K,C} y$ is shorthand for the condition $\frac{1}{K}(x-C) \leq y \leq Kx+C$. See the next section for a precise statement.

Our theorem follows more-or-less directly from the Masur-Minsky distance formula [MM00] and the Masur-Schleimer distance formula [MS13], but seems worth making explicit since $\mathcal{F}(S)$ is an important, particularly tractable, geometric model for the mapping class group of $S$ (see e.g. [Har85, Hat91, Mos95, DPT1, Bel14]), while on the other side, the geometry of the arc graph has been greatly simplified in [HPW15]. Various distance formulas [MM99, MS13, Raf07] have been used extensively to understand the geometry of mapping class group, Teichmüller space, and homomorphisms (see e.g. [Bro03, Beh06, KLo08, Bow09, BDS11, BKMM12, CML12, Tno13, EMR14, BBF15]) and have motivated research in related areas (see e.g. [SS12, CP12, Tay13, Sis13, KKL14, BFT14, Tay14, HH15, Vog15]).

It would be interesting to find a proof of Theorem 1.1 that does not appeal to the previous distance formulas.

Acknowledgements. The authors thank Valentina Disarlo, Hugo Parlier, and Kasra Rafi for useful conversations. The second author was partially supported by NSF grant DMS 1510034.

2. THE PROOF

For a surface $S$ of genus $g$ with $n$ punctures, we write $\xi(S) = 3g - 3 + n$ (we do not distinguish between a puncture and a hole, and will only refer to punctures to avoid confusion later). All
surfaces we consider are orientable, have at least one puncture, and have $\xi > 0$, with one exception: we allow annuli (which have $\xi = -1$). In particular, we exclude three-punctured spheres in all of what follows. Arcs, curves, multiarcs, and multicurves are assumed essential and are considered up to isotopy. Multiarcs and multicurves have pairwise non-isotopic components. Ideal triangulations are multiarcs with a maximal number of components. Markings are complete clean markings (see [MM00]).

We write $C(Y)$ for the arc-and-curve graph of a surface $Y$, which is quasi-isometric to the curve graph (more precisely, the inclusion of the curve graph into the arc-and-curve graph is a quasi-isometry). Given any multiarc, multicurve, marking, or triangulation, $\alpha$ on a surface $S$ and subsurface $Y \subseteq S$ which is not an annulus, we let $\pi_Y(\alpha)$ denote the arc-and-curve projection: This is the union of the isotopy classes of arcs and curves of intersection of $\alpha$ with $Y$ (assuming they are in minimal position). For $Y$ an annulus, we use the usual projection to $A(Y)$ via the cover corresponding to $Y$; see [MM00] for details. We will write $d_{C(Y)}(\alpha, \beta) = \text{diam}(\pi_Y(\alpha) \cup \pi_Y(\beta))$ where the diameter is taken in $C(Y)$. When the projections are non-empty, for example if $\alpha$ is a marking or a triangulation, then $d_{C(Y)}$ satisfies a triangle inequality. If $\alpha$ is an arc or a triangulation, then $\pi_Y(\alpha)$ is in the arc graph, $A(Y)$, and so we can define $d_{A(Y)}(\alpha, \beta)$ similarly. We note that using the arc-and-curve graph projection, it follows that for any $X \subseteq Y \subseteq S$, we have $\pi_X \circ \pi_Y = \pi_X$, unless $X$ is an annulus.

As stated in the introduction, the flip graph $F(S)$ is the graph whose vertex set is the set isotopy classes of (ideal) triangulations. Two vertices in the graph share an edge if they are related by a flip, in other words, if they differ at most by an arc; see [DP14] and Figure 1.

For markings $\mu_1, \mu_2$ on $S$, we let $d_M(\mu_1, \mu_2)$ denote the distance in the marking graph $M(S)$; see [MM00]. The first distance formula we will need is due to Masur and Minsky:

**Theorem 2.1** ([MM00]). Fix $S$, a connected, orientable surface with $\xi(S) > 0$. For any $k > 0$ sufficiently large, there exists $K, C \geq 1$ so that for any two markings $\mu_1, \mu_2$ we have

$$d_M(\mu_1, \mu_2) \leq \sum_{Y \subseteq S} [d_{C(Y)}(\mu_1, \mu_2)]_k.$$ 

In this theorem, we note that $K, C$ can be chosen to depend monotonically on $k$. Indeed, the right-hand side becomes less efficient at estimating the left-hand side as $k$ increases, so at least coarsely, this monotonicity is necessary.
There is a distance formula for arc graphs due to Masur and Schleimer (see Lemma 7.2 and Theorems 5.10 and 13.1 of [MST]). To state this formula, we recall that given a surface $Y$, a hole for $\mathcal{A}(Y)$ is an essential subsurface $X \subseteq Y$ such that the punctures of $Y$ are also punctures of $X$, which we write as $\partial Y \subseteq \partial X$. We let $H(\mathcal{A}(Y))$ denote the set of holes for $\mathcal{A}(Y)$. For $Y$ an annulus, the only hole for $\mathcal{A}(Y)$ is $Y$, and $Y$ is not a hole for $\mathcal{A}(X)$, for any other surface $X$.

**Theorem 2.2 ([MST]).** Fix $S$, a connected, orientable surface with at least one puncture and $\xi(S) > 0$. Then for any $k > 0$ sufficiently large, there exists $K \geq 1, C \geq 0$ so that for any two arcs $\alpha_1, \alpha_2$, 

$$d_{\mathcal{A}(S)}(\alpha_1, \alpha_2) \leq K C \sum_{X \in H(\mathcal{A}(S))} [d_{\mathcal{C}(X)}(\alpha_1, \alpha_2)]_k.$$ 

The proof of Theorem 2.1 also requires the following elementary observation.

**Lemma 2.3.** Fix a surface $S$. For any essential subsurface $X \subseteq S$, there are at most $2^{\xi(X)} \leq 2^{\xi(S)}$ subsurfaces $Y$ such that $X$ is a hole for $\mathcal{A}(Y)$.

**Proof.** An essential subsurface $X$ is a component of the complement of an essential multicurve that we denote $\partial_0 Y$. If $X$ is a hole for $\mathcal{A}(Y)$, then observe that $Y$ is the component of the complement of $\partial_0 Y$ containing $X$. Therefore $Y$ is determined by $X$ and the multicurve $\partial_0 Y \subseteq \partial_0 X$. There are $2^{|\partial_0 X|}$ submulticurves of $\partial_0 Y$, and $|\partial_0 Y| \leq \xi(X)$, and hence at most this many $Y \subseteq S$ such that $X$ is a hole for $\mathcal{A}(Y)$. \hfill \Box

**Proof of Theorem 1.1** Fix $S$. For every ideal triangulation $T$, we choose a marking $\mu(T)$ so that $i(T, \mu(T))$ is minimized (here we simply take the sum of intersection numbers of components of $T$ and $\mu(T)$). Because the mapping class group $\text{Mod}(S)$ has only finitely many orbits on $\mathcal{F}(S)$, this intersection number is uniformly bounded, independent of $T$. Consequently, there exists $\delta_0 > 0$ such that for each triangulation $T$ of $S$ and every subsurface $Y \subseteq S$ we have

$$d_{\mathcal{C}(Y)}(\mu(T), T) < \delta_0.$$ 

Furthermore, we claim that $T \mapsto \mu(T)$ is coarsely $\text{Mod}(S)$–equivariant. More precisely, for every $g \in \text{Mod}(S)$ and $T \in \mathcal{F}(S)$, we claim that $d_{\mathcal{M}}(\mu(gT), g\mu(T))$ is uniformly bounded. This follows from Theorem 2.1 since (1) and the triangle inequality imply 

$$d_{\mathcal{C}(Y)}(\mu(gT), g\mu(T)) \leq d_{\mathcal{C}(Y)}(\mu(gT), gT) + d_{\mathcal{C}(Y)}(gT, g\mu(T)) = d_{\mathcal{C}(Y)}(\mu(gT), gT) + d_{\mathcal{C}(g^{-1}Y)}(T, \mu(T)) \leq 2\delta_0.$$ 

Since $\text{Mod}(S)$ acts cocompactly by isometries on the proper geodesic spaces $\mathcal{F}(S)$ and $\mathcal{M}(S)$, the Milnor–Svarc Lemma implies $T \mapsto \mu(T)$ is a quasi-isometry. Thus, for $T_1, T_2 \in \mathcal{F}(S)$ and $\mu_i = \mu(T_i)$, for $i = 1, 2$ we have

$$d_{\mathcal{F}}(T_1, T_2) \simeq d_{\mathcal{M}}(\mu_1, \mu_2).$$ 

Let ($K_0, C_0$) be the implicit constants in this coarse equation.

Next, we choose constants $0 < k_1 < k_2 < k_3 < \infty$ large enough so that for all $T_1, T_2 \in \mathcal{F}(S)$:

(i) If $X$ is a hole for $\mathcal{A}(Y)$ and $d_{\mathcal{C}(X)}(T_1, T_2) \geq k_3$, then $d_{\mathcal{A}(Y)}(T_1, T_2) \geq k_2$; and

(ii) if $d_{\mathcal{A}(Y)}(T_1, T_2) \geq k_2$, then 

$$d_{\mathcal{A}(Y)}(T_1, T_2) \simeq \sum_{X \in H(\mathcal{A}(Y))} [d_{\mathcal{C}(X)}(T_1, T_2)]_{k_1}.$$
where the implicit constants in this coarse equation are $(K_1, 0)$. For (ii), this means that when the arc graph distance is at least $k_2$, the sum with cut-off function $k_1$ is correct with only a multiplicative error. To see that we can find such $k_1, k_2, K_3$ and $K_1$, we first appeal to Theorem 1.2 to find $k_1, k_2, K_3$ so that (ii) holds. This is possible since once the the arc-graph distance is bigger than twice the additive constant, say, then by doubling the multiplicative constant, we may remove the additive error. Appealing to Theorem 1.2 again guarantees that for $k_3$ sufficiently large (i) also holds. For reasons that will become clear later, we will also assume that $k_1 \geq 10\delta$ and that $k_1 - 2\delta_0$ is above the threshold for Theorem 1.1 to hold.

For $T_1, T_2 \in \mathcal{F}(S)$, let $\Omega(T_1, T_2, k_2)$ be the set of subsurfaces $Y \subseteq S$ so that $d_{\mathcal{A}(Y)}(T_1, T_2) \geq k_2$. Then we have

$$
\sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_{k_2} = \sum_{Y \in \Omega(T_1, T_2, k_2)} d_{\mathcal{A}(Y)}(T_1, T_2) \approx \sum_{Y \in \Omega(T_1, T_2, k_2)} \sum_{X \in H(Y)} [d_{\mathcal{C}(X)}(T_1, T_2)]_{k_1}.
$$

The implicit constants in the coarse equation are again $(K_0, 0)$ by (ii).

Let $\mathcal{H} = \mathcal{H}(T_1, T_2, k_1, k_2, k_3)$ be the set of all $X$ which appear with nonzero contribution in the sum on the right-hand side of the above coarse equation. We note that $\mathcal{H}$ does not keep track of how many times such an $X$ appears. By Lemma 2.3 any $X \in \mathcal{H}$ appears at most $2^{\ell(S)}$ times in the sum. Therefore we have

$$
\sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(T_1, T_2) \approx \sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_{k_2}.
$$

(3) Here the implicit constants can be taken to be $(2^{\ell(S)} K_0, 0)$.

By definition, for each $X \in \mathcal{H}$, $d_{\mathcal{C}(X)}(T_1, T_2) \geq k_1$. On the other hand, if $d_{\mathcal{C}(X)}(T_1, T_2) \geq k_3$, then $X \in \mathcal{H}$. Thus $\mathcal{H}$ contains all subsurfaces with distance at least $k_3$ and some subsurfaces with distance at least $k_1$. Since $d_{\mathcal{C}(X)}(\mu_1, T_1) \leq \delta_0$, it follows that if $X \in \mathcal{H}$, then $d_{\mathcal{C}(X)}(\mu_1, \mu_2) \geq k_1 - 2\delta_0$, and if $d_{\mathcal{C}(X)}(\mu_1, \mu_2) \geq k_3 + 2\delta_0$, then $X \in \mathcal{H}$. By the monotonicity of the constants in Theorem 2.1 we have

$$
\sum_{X \in \mathcal{H}} d_{\mathcal{A}(Y)}(\mu_1, \mu_2) \approx \sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(\mu_1, \mu_2).
$$

(4) Here the implicit constants $(K_2, C_2)$ in the coarse equation are the same as those in Theorem 2.1 for threshold $k_3 + 2\delta_0$. Finally, since $k_1 \geq 10\delta_0$, we have

$$
\sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(\mu_1, \mu_2) \approx \sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(T_1, T_2)
$$

(5) and one can check that the implicit constant is $(\frac{n}{10}, 0)$ (since each term on the left differs from the corresponding term on the right by an additive error which is small compared to it size).

Setting $k = k_2$, and combining 2, 4, 5, and 3

$$
d_{\mathcal{F}}(T_1, T_2) \approx \sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_{k_2}
$$

where the implicit constants in the coarse equation depend on all the above constants. This completes the proof. □

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