Defective 2-colorings of planar graphs without 4-cycles and 5-cycles

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Abstract

Let $G$ be a graph without 4-cycles and 5-cycles. We show that the problem to determine whether $G$ is $(0, k)$-colorable is NP-complete for each positive integer $k$. Moreover, we construct non-$(1, k)$-colorable planar graphs without 4-cycles and 5-cycles for each positive integer $k$. Finally, we prove that $G$ is $(d_1, d_2)$-colorable where $(d_1, d_2) = (4, 4), (3, 5), \text{and} (2, 9)$.

1 Introduction

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A $k$-vertex, a $k^+$-vertex, and $k^-$-vertex are a vertex of degree $k$, at least $k$, and at most $k$, respectively. The similar notation is applied for faces. A $(d_1, d_2, \ldots, d_k)$-face $f$ is a face of degree $k$ where all vertices on $f$ have degree $d_1, d_2, \ldots, d_k$. If $v$ is not on a 3-face $f$ but $v$ is adjacent to some 3-vertex on $f$, then we call $f$ a pendant face of a vertex $v$ and $v$ is a pendant neighbor of a 3-vertex $v$. A 3-face (respectively, 2-vertex) incident to a 2-vertex (respectively, 3-face) is called a bad 3-face (respectively, bad 2-vertex). Otherwise, it is a good 3-face (respectively, good 2-vertex).

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A \textit{k-coloring} $c$ (not necessary proper) is a function $c : V(G) \rightarrow \{1, \ldots, k\}$. Define $V_i := \{v \in V(G) : c(v) = i\}$. We call $c$ a $(d_1, d_2, \ldots, d_k)$-coloring if $V_i$ is an empty set or the induced subgraph $G[V_i]$ has the maximum degree at most $d_i$ for each $i \in \{1, \ldots, k\}$. A graph $G$ is called $(d_1, d_2, \ldots, d_k)$-colorable if $G$ admits a $(d_1, d_2, \ldots, d_k)$-coloring. Thus the four color theorem \cite{2,3} can be restated as every planar graphs is $(0, 0, 0, 0)$-colorable. For improper 3-colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is $(2, 2, 2)$-colorable \cite{10}. Eaton and Hull \cite{11} proved that $(2, 2, 2)$-colorability is optimal by showing non-$(k, k, 1)$-colorable planar graphs for each $k$.

Grötzsch \cite{12} showed that every planar graph without 3-cycles is $(0, 0, 0, 0)$-colorable. The famous Steinberg’s conjecture proposes that every planar graph without 4-cycles and 5-cycles is also $(0, 0, 0)$-colorable. Recently, this conjecture is disproved by Cohen-Addad et al \cite{1}. One way to relax the conjecture is allowing some color classes to be improper. For every planar graph $G$ without 4-cycles and 5-cycles, Xu, Miao, and Wang \cite{17} proved that $G$ is $(1, 1, 0)$-colorable, and Chen et al. \cite{8} proved that $G$ is $(2, 0, 0)$-colorable.

Many papers investigate $(d_1, d_2)$-coloring of planar graphs in various settings. Montassier and Ochem \cite{14} constructed planar graphs of girth 4 that are not $(i, j)$-colorable for each $i, j$. Borodin, Ivanova, Montassier, Ochem, and Raspaud \cite{4} constructed planar graphs of girth 6 that are not $(0, k)$-colorable for each $k$. On the other hand, for every planar graph $G$ of girth 5, Havet and Seren \cite{13} showed that $G$ is $(2, 6)$-colorable and $(4, 4)$-colorable, and Choi and Raspaud \cite{9} showed that $G$ is $(3, 5)$-colorable.

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph $G$ is called $(d_1, d_2, \ldots, d_k)$-colorable if $V(G)$ can be partitioned into sets $V_1, V_2, \ldots, V_k$ such that the induced subgraph $G[V_i]$ for $i \in [k]$ has the maximum degree at most $d_i$. Thus the four color theorem \cite{2,3} can be restated as every planar graphs is $(0, 0, 0, 0)$-colorable. For improper 3-colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is $(2, 2, 2)$-colorable \cite{10}. Eaton and Hull \cite{11} and Škrekovski \cite{15} prove that $(2, 2, 2)$-colorability is optimal by showing non-$(k, k, 1)$-colorable planar graphs for each $k$.

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planar graph $G$ of girth 5, Havet and Seren [13] showed that $G$ is $(2, 6)$-colorable and $(4, 4)$-colorable, and Choi and Raspaud [9] showed that $G$ is $(3, 5)$-colorable. Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed planar graphs of girth 6 that are not $(0, k)$-colorable for each $k$. Montassier and Ochem [14] constructed planar graphs of girth 4 that are not $(i, j)$-colorable for any $i, j$.

There are many papers [4, 6, 13, 7, 5, 14] that investigate $(d_1, d_2)$-colorability for graphs with girth length of $g$ for $g \geq 6$; see [14] for the rich history. For example, Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] constructed a graph in $g_6$ (and thus also in $g_5$) that is not $(0, k)$-colorable for any $k$. The question of determining if there exists a finite $k$ where all graphs in $g_5$ are $(1, k)$-colorable is not yet known and was explicitly asked in [14]. On the other hand, Borodin and Kostochka [6] and Havet and Sereni [13], respectively, proved results that imply graphs in $g_5$ are $(2, 6)$-colorable and $(4, 4)$-colorable.

Let $G$ be a graph without 4-cycles and 5-cycles. We show that the problem to determine whether $G$ is $(0, k)$-colorable is NP-complete for each positive integer $k$. Moreover, we construct non-$(1, k)$-colorable planar graphs without 4-cycles and 5-cycles for each positive integer $k$. Finally, we prove that $G$ is $(d_1, d_2)$-colorable where $(d_1, d_2) = (4, 4), (3, 5), \text{ and } (2, 9)$.

## 2 NP-completeness of $(0, k)$-colorings

**Theorem 1.** [14] Let $g_{k,j}$ be the largest integer $g$ such that there exists a planar graph of girth $g$ that is not $(k, j)$-colorable. The problem to determine whether a planar graph with girth $g_{k,j}$ is $(k, j)$-colorable for $(k, j) \neq (0, 0)$ is NP-complete.

**Theorem 2.** The problem to determine whether a planar graph without 4-cycles and 5-cycles is $(0, k)$-colorable is NP-complete for each positive integer $k$.

**Proof.** We use a reduction from the problem in Theorem [14] to prove that $(0, k)$-coloring for planar graph without 4-cycles and 5-cycles. From [14], $6 \leq g_{0,1} \leq 10$. Let $G$ be a graph of girth $g_{0,1}$. Take $k - 1$ copies of 3-cycles $v_i v'_i v''_i$ ($i = 1, \ldots, k - 1$) for each vertex $v$ of $G$. The graph $H_k$ is obtained from $G$ by identifying $v_i$ (in a 3-cycle $v_i v'_i v''_i$) to $v$ for each vertex $v$. The resulting graph $H_k$ has neither 4-cycles nor 5-cycles.

Suppose $G$ has a $(0, 1)$-coloring $c$. We extend a coloring to $c(v'_i) = 1$ and $c(v''_i) = 2$ for each vertex $v$ and each $i = 1, \ldots, k - 1$. One can see that $c$ is a $(0, k)$-coloring of $H_k$. Suppose $H_k$ has a $(0, k)$-coloring $c$. Consider $v \in V(G)$ with $c(v) = 2$. By construction, $v$ has at least
$k - 1$ neighbors with the same color in $V(H_k) - V(G)$. Thus $v$ has at most one neighbor
with the same color in $V(H_k) - V(G)$. It follows that $c$ with restriction to $V(G)$ is a $(0, 1)$-
coloring of $G$. Hence $G$ is $(0, 1)$-colorable if and only if $H_k$ is $(0, k)$-colorable. This completes
the proof. $\square$

3 Non-$(1, k)$-colorable planar graphs without 4-cycles
and 5-cycles

We construct a non-$(1, k)$-colorable planar graph $G$ without 4-cycles and 5-cycles. Con-
sider the graph $H_{u,v}$ shown in Figure 1.

A non-$(k, 1)$-colorable planar graph $G$ without 4-cycles and 5-cycles The vertices $a, b, c,$
and $d$ cannot receive the same color 1. Now, we construct the graph $S_z$ as follows. Let $z$
be a vertex and $x_1x_2x_3$ be a path. Take $2k + 1$ copies $H_{u_i,v_j}$ of $H_{u,v}$ with $1 \leq i \leq 2k + 1$
and $1 \leq j \leq 3$. Identify every $u_i$ with $z$ and identify $v_j$ with $x_j$. Finally, we obtain $G$
from three copies $S_{z_1}, S_{z_2},$ and $S_{z_3}$ by adding the edges $z_1z_2$ and $z_2z_3$. In every $(1, k)$-coloring of
$G$, the path $z_1z_2z_3$ contains a vertex $z$ with color 2. In the copy of $S_z$ corresponding to $z$, the
path $x_1x_2x_3$ contains a vertex $x$ with color 2. Since each of $z$ and $x$ has at most $k$ neighbors
colored 2, one of $2k + 1$ copies of $H_{u,v}$ between $z$ and $x$, does not contain a neighbor of $z$
and $x$ colored 2. This copy is not $(1, k)$-colorable, and thus $G$ is not $(1, k)$-colorable.
4 Helpful Tools

Now, we investigate \((d_1, d_2)\) such that \(G\) is \((d_1, d_2)\)-colorable for every graph \(G\) without 4-cycles and 5-cycles. From two previous sections, we have that \(d_1, d_2 \geq 2\). First, we present useful proposition and lemmas about a minimal planar graph \(G\) that is not \((d_1, d_2)\)-colorable where \(d_1 \leq d_2\).

Proposition 1. (a) Each vertex \(v\) of \(G\) is a \(2^+\)-vertex.
(b) If \(v\) is a \(k\)-vertex has \(\alpha\) incident 3-faces, \(\beta\) adjacent good 2-vertices, and \(\gamma\) pendant 3-faces, then \(\alpha \leq \lfloor \frac{k}{2} \rfloor\) and \(2\beta + \alpha + \gamma \leq k\)

Lemma 2. \[9\] Let \(G\) be \((d_1, d_2)\)-colorable where \(d_1 \leq d_2\).
(a) If \(v\) is a \(3^-\)-vertex, then at least two neighbors of \(v\) are \((d_1 + 2)^+\)-vertices one of which is a \((d_2 + 2)^+\)-vertex.
(b) If \(v\) is a \((d_1 + d_2 + 1)^-\)-vertex, then at least one neighbor of \(v\) is a \((d_1 + 2)^+\)-vertex.

Lemma 3. If a 2-vertex \(v\) is on a bad 3-face \(f\), then the other face \(g\) which is incident to \(v\) is a \(7^+\)-face.

Proof. Suppose that a face \(g\) is a \(6^-\)-face. Let a face \(f = uvw\). By condition of \(G\), a face \(g\) is neither 4, 5-face nor 3-face, otherwise \(G\) contains \(C_4\). Now we suppose a face \(g\) is a 6-face and let \(g = u_1u_2u_3uvw\). Since \(u\) is adjacent to \(w\), there is a 5-cycle = \(u_1u_2u_3uw\), a contradiction.

Lemma 4. Let \(f\) be a \(k\)-face where \(k \geq 7\). Then, \(f\) has at most \(k - 6\) incident bad 2-vertices.

Proof. By proof of Lemma [9] if a face \(f\) is incident to \(m\) bad 2-vertices, then there is a cycle \(C_{k-m}\) since we can add some edge to \(f\) to obtain a new cycle that has the length least than a face \(f\).

Lemma 5. Let \((u, v, w)\) be a bad 3-face \(f\) where \(d(u) = 2\). Then at least one of following statements is true.
(S1) A vertex \(v\) is a \((d_1 + 3)^+\)-vertex which has at least two \((d_2 + 2)\)-neighbors.
(S2) A vertex \(w\) is a \((d_2 + 3)^+\)-vertex which has at least two \((d_1 + 2)\)-neighbors.
(S3) A vertex \(v\) or a vertex \(w\) is a \((d_1 + d_2 + 2)^+\)-vertex.

Proof. Assume \(c\) is a \((d_1, d_2)\)-coloring in \(G - u\). If two neighbors of \(u\) share the same color, then we can color \(u\) by \(\{1, 2\} - \{c(v)\}\). So \(c(v) \neq c(w)\). By symmetry let \(c(v) = 1\) and
\[ c(w) = 2. \] By Lemma 2, we have a vertex \( v \) is a \((d_1 + 2)\) and a vertex \( w \) is a \((d_2 + 2)\). Then \( v \) has \( d_1 \) neighbors of color 1 to forbid \( u \) from being colored by 1 and \( w \) has \( d_2 \) neighbors of color 2 to forbid \( u \) from being colored by 2. Next, to avoid recoloring \( v \) by 2 and \( w \) by 1.

Then \( v \) has one neighbor with color 2 which has \( d_2 \) neighbors of color 2 or \( v \) has \( d_1 \) neighbors with color 2. Otherwise, \( w \) has one neighbor with color 1 which has \( d_1 \) neighbors of color 1 or \( w \) has \( d_1 \) neighbors with color 1.

\[ \square \]

## 5 \((4, 4)\)-coloring

**Theorem 3.** If \( G \) is a planar graph without cycles of length 4 or 5, then \( G \) is \((4, 4)\)-colorable.

**Proof.** Suppose that \( G \) is a minimal counterexample. The discharging process is as follows.

Let the initial charge of a vertex \( u \) in \( G \) be \( \mu(u) = 2d(u) - 6 \) and the initial charge of a face \( f \) in \( G \) be \( \mu(f) = d(f) - 6 \). Then by Euler’s formula \(|V(G)| - |E(G)| + |F(G)| = 2\) and by the Handshaking lemma, we have

\[
\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.
\]

Now, we establish a new charge \( \mu^*(x) \) for all \( x \in V(G) \cup F(G) \) by transferring charge from one element to another and the summation of new charge \( \mu^*(x) \) remains \(-12\). If the final charge \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), then we get a contradiction and the prove is completed.

The discharging rules are

(R1) Every \( 6^+ \)-vertex sends charge 1 to each adjacent good 2-vertex.
(R2) Every \( 6^+ \)-vertex sends charge 2 to each incident 3-face.
(R3) Every \( 6^+ \)-vertex sends charge 1 to each adjacent pendant 3-face.
(R4) Every \( 7^+ \)-face sends charge 1 to each incident bad 2-vertex.
(R5) Every 4-vertex or 5-vertex sends charge 1 to each incident 3-face.
(R6) Every bad 3-face sends charge 1 to each incident 2-vertex.

It remains to show that resulting \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \).

It is evident that \( \mu^*(x) = \mu(x) = 0 \) if \( x \) is a 3-vertex or a 6-face.

Now, let \( v \) be a \( k \)-vertex.

For \( k = 2 \), a vertex \( v \) has two \( 6^+ \)-neighbors by Lemma 2. If \( v \) is a good 2-vertex, then \( \mu^*(v) \geq \mu(v) + 2 \cdot 1 = 0 \) by (R1). If \( v \) is a bad 2-vertex, then \( v \) is incident to a \( 7^+ \)-face by Lemma 3. Thus \( \mu^*(v) \geq \mu(v) + 1 + 1 = 0 \) by (R4) and (R6).
For \( k = 4, 5 \), by Proposition 1(b), a vertex \( v \) is incident to at most two 3-faces. By (R5), \( \mu^*(v) \geq \mu(v) - 2 \cdot 1 \geq 0 \).

Consider \( k = 6^+ \). Let \( v \) have \( \alpha \) incident 3-faces, \( \beta \) adjacent good 2-vertices, and \( \gamma \) pendant 3-faces. By Proposition 1(b), we have \( 2\alpha + \beta + \gamma \leq d(v) \). Moreover, \( \mu(v) = 2d(v) - 6 \geq d(v) \) if \( d(v) \geq 6 \). Thus, by (R1), (R2), and (R3), we have \( \mu^*(v) = \mu(v) - (2\alpha + \beta + \gamma) \geq 0 \).

Now let \( f \) be a \( k \)-face.

For \( k = 7^+ \), by Lemma 4, a \( k \)-face \( f \) has at most \( k - 6 \) incident bad 2-vertices. By (R4), \( \mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0 \).

Consider \( k = 3 \). If \( f \) is a bad 3-face, then we have \( f = (2, 6^+, 6^+)-face \) by Lemma 2. Then by (R2) and (R6), \( \mu^*(f) \geq \mu(f) + 2 \cdot 2 - 1 = 0 \). Now, It remains to consider a good 3-face. If \( f \) is incident to a \( 4^+ \)-vertex and a \( 6^+ \)-vertex, then \( \mu^*(f) \geq \mu(f) + 2 + 1 \geq 0 \) by (R2) and (R5). If \( f \) is a \( (3, 3, 6^+)-face \), then the pendant neighbor of a 3-vertex is a \( 6^+ \)-vertex by Lemma 2. Thus \( \mu^*(f) \geq \mu(f) + 2 + 1 + 1 \leq 0 \) by (R2) and (R3). Finally, if \( f \) is a \( (4^+, 4^+, 4^+)-face \), then \( \mu^*(f) \geq \mu(f) + 3 \cdot 1 \leq 0 \) by (R5).

Since \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), this completes the proof.

6 \((3, 5)\)-coloring

**Theorem 4.** If \( G \) is a planar graph without cycles of length 4 or 5, then \( G \) is \((3, 5)\)-colorable.

**Proof.** Suppose that \( G \) is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex \( u \) in \( G \) be \( \mu(u) = 2d(u) - 6 \) and the initial charge of a face \( f \) in \( G \) be \( \mu(f) = d(f) - 6 \). Then by Euler’s formula \(|V(G)| - |E(G)| + F(G) = 2 \) and by the Handshaking lemma, we have

\[
\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.
\]

Now, we establish a new charge \( \mu^*(x) \) for all \( x \in V(G) \cup F(G) \) by transferring charge from one element to another and the summation of new charge \( \mu^*(x) \) remains \(-12 \). If the final charge \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), then we get a contradiction and the prove is completed.

The discharging rules are

(R1) Every 5-vertex sends charge \( \frac{4}{5} \) to each adjacent good 2-vertex.
(R2) Every 5-vertex sends charge \( \frac{8}{5} \) to each incident 3-face.
(R3) Every 5-vertex sends charge \( \frac{4}{5} \) to each adjacent pendant 3-face.
(R4) Every 6-vertex sends charge 1 to each adjacent good 2-vertex.
(R5) Every 6-vertex or 7-vertex sends charge 2 to each incident 3-face.
(R6) Every 6-vertex sends charge 1 to each adjacent pendant 3-face.
(R7) Every 7+ vertex sends charge $\frac{6}{5}$ to each adjacent good 2-vertex.
(R8) Every 8+ vertex sends charge $\frac{12}{5}$ to each incident 3-face.
(R9) Every 7+ vertex sends charge $\frac{6}{5}$ to each adjacent pendant 3-face.
(R10) Every 7+ face sends charge 1 to each incident bad 2-vertex.
(R11) Every 4-vertex sends charge 1 to each incident 3-face.
(R12) Every bad 3-face sends charge 1 to each incident 2-vertex.

Next, we show that the final charge $\mu^*(u)$ is nonnegative.

It is evident that $\mu^*(x) = \mu(x) = 0$ if $x$ is a 3-vertex or a 6-face.

Now, let $v$ be a $k$-vertex.

For $k = 2$, a vertex $v$ has two $5^+$ neighbors one of which is a $7^+$-neighbor by Lemma 2. If $v$ is a good 2-vertex, then $\mu^*(v) \geq \mu(v) + \frac{4}{5} + \frac{4}{5} = 0$ by (R1) and (R7). If $v$ is a bad 2-vertex, then $v$ is incident to a $7^+$-face by Lemma 3. Thus $\mu^*(v) \geq \mu(v) + 1 + 1 = 0$ by (R10) and (R12).

For $k = 4$, by Proposition 1(b), a vertex $v$ is incident to at most two 3-faces. By (R11), $\mu^*(v) \geq \mu(v) - 2 \cdot 1 \geq 0$.

Consider $k = 5$. Let $v$ have $\alpha$ incident 3-faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3-faces. By Proposition 1(b), $2 \alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\frac{8}{5} \alpha + \frac{4}{5} \beta + \frac{4}{5} \gamma = \frac{4}{5}(2 \alpha + \beta + \gamma) \leq \frac{4}{5}d(v)$ and $\mu(v) = 2d(v) - 6 = \frac{4}{5}d(v)$ if $d(v) = 5$. Thus by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (\frac{8}{5} \alpha + \frac{4}{5} \beta + \frac{4}{5} \gamma) \geq 0$.

Consider $k = 6$. Let $v$ have $\alpha$ incident 3-faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3-faces. By Proposition 1(b), we have $2 \alpha + \beta + \gamma \leq d(v)$. Moreover, $\mu(v) = 2d(v) - 6 = d(v)$ if $d(v) = 6$. Thus, by (R4), (R5), and (R6), we have $\mu^*(v) = \mu(v) - (2 \alpha + \beta + \gamma) = 0$.

Consider $k = 7$. If $v$ is not incident to a 3-face, then we have $\mu^*(v) = \mu(v) - 6 \cdot \frac{6}{5} \geq 0$ by Lemma 2 (R7), and (R9). If $v$ is incident to one 3-face, then we have $\mu^*(v) = \mu(v) - (2 + 5 \cdot \frac{6}{5}) = 0$ by (R5), (R7), and (R9). If $v$ is incident to two 3-faces, then we have $\mu^*(v) = \mu(v) - (2 \cdot 2 + 3 \cdot \frac{6}{5}) \geq 0$ by (R5), (R7), and (R9). Finally, if $v$ is incident to three 3-faces, then we have $\mu^*(v) = \mu(v) - (3 \cdot 2 + \frac{6}{5}) \geq 0$ by (R5), (R7) and (R9).

Consider $k = 8^+$. Let $v$ have $\alpha$ incident 3-faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3-faces. By Proposition 1(b), $2 \alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\frac{12}{5} \alpha + \frac{6}{5} \beta + \frac{6}{5} \gamma = 0$. Thus by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (\frac{12}{5} \alpha + \frac{6}{5} \beta + \frac{6}{5} \gamma) \geq 0$.

Consider $k = 9$. Let $v$ have $\alpha$ incident 3-faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3-faces. By Proposition 1(b), $2 \alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\frac{18}{5} \alpha + \frac{9}{5} \beta + \frac{9}{5} \gamma = 0$. Thus by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (\frac{18}{5} \alpha + \frac{9}{5} \beta + \frac{9}{5} \gamma) \geq 0$.

Consider $k = 10$. Let $v$ have $\alpha$ incident 3-faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3-faces. By Proposition 1(b), $2 \alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\frac{24}{5} \alpha + \frac{12}{5} \beta + \frac{12}{5} \gamma = 0$. Thus by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (\frac{24}{5} \alpha + \frac{12}{5} \beta + \frac{12}{5} \gamma) \geq 0$.
\[ \frac{5}{3}(2\alpha + \beta + \gamma) \leq \frac{6}{3}d(v) \] and \( \mu(v) = 2d(v) - 6 \geq \frac{6}{3}d(v) \) if \( d(v) \geq 8 \). Thus by (R7), (R8), and (R9), we have \( \mu^*(v) = \mu(v) - (\frac{12}{3}\alpha + \frac{6}{3}\beta + \frac{6}{3}\gamma) \geq 0 \).

Now let \( f \) be a \( k \)-face.

For, \( k = 7^+ \). By Lemma \([\text{I}]\) a \( k \)-face \( f \) has at most \( k - 6 \) incident bad 2-vertices. By (R11), \( \mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0 \).

Consider \( k = 3 \). If \( f \) is a bad 3-face, then we have \( f \) is a \((2, 6^+, 6^+)\)-face or \( f \) is a \((2, 5^+, 8^+)\) by Lemma \([3]\). Then by (R2), (R5), (R8), and (R12), \( \mu^*(f) \geq \mu(f) + 2 \cdot 2 - 1 = 0 \) or \( \mu^*(f) \geq \mu(f) + \frac{8}{5} + \frac{12}{5} - 1 = 0 \). Now, it remains to consider a good 3-face. If \( f \) is incident to a 4\(^+\)-vertex and a 6\(^+\)-vertex, then \( \mu^*(f) \geq \mu(f) + 2 + 1 \geq 0 \) by (R5) and (R11). If \( f \) is a \((3, 3, 7^+)\)-face, then the pendant neighbor of a 3-vertex is a 5\(^+\)-vertex by Lemma \([2]\) Thus \( \mu^*(f) \geq \mu(f) + 2 \cdot \frac{4}{5} + 2 \geq 0 \) by (R3) and (R5). If \( f \) is a \((3, 3, 5^+)\)-face, then the pendant neighbor of a 3-vertex is a 7\(^+\)-vertex by Lemma \([2]\). Thus \( \mu^*(f) \geq \mu(f) + 2 \cdot \frac{6}{5} + \frac{8}{5} \geq 0 \) by (R2) and (R7). Finally, if \( f \) is a \((4^+, 4^+, 4^+)\)-face, then \( \mu^*(f) \geq \mu(f) + 3 \cdot 1 \leq 0 \) by (R11).

Since \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), this completes the proof. \( \square \)

7 \((2, 9)\)-coloring

**Theorem 5.** If \( G \) is a planar graph without cycles of length 4 or 5, then \( G \) is \((2, 9)\)-colorable.

**Proof.** Suppose that \( G \) is a minimal counterexample. The discharging process is as follows. Let the initial charge of a vertex \( u \) in \( G \) be \( \mu(u) = 2d(u) - 6 \) and the initial charge of a face \( f \) in \( G \) be \( \mu(f) = d(f) - 6 \). Then by Euler’s formula \( |V(G)| - |E(G)| + F(G) = 2 \) and by the Handshaking lemma, we have

\[
\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.
\]

Now, we establish a new charge \( \mu^*(x) \) for all \( x \in V(G) \cup F(G) \) by transferring charge from one element to another and the summation of new charge \( \mu^*(x) \) remains \(-12\). If the final charge \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), then we get a contradiction and the prove is completed.

The discharging rules are

(R1) Every \( k \)-vertex for \( 4 \leq k \leq 10 \) sends charge \( \frac{1}{2} \) to each adjacent good 2-vertex.

(R2) Every 4-vertex sends charge 1 to each incident 3-face.

(R3) Every \( k \)-vertex for \( 4 \leq k \leq 10 \) sends \( \frac{1}{2} \) to each adjacent pendant 3-face.
(R4) Every $k$-vertex for $5 \leq k \leq 10$ sends charge $\frac{2}{k}$ to each incident 3-face.
(R5) Every 11-vertex sends charge $\frac{5}{2}$ to each incident 3-face.
(R6) Every $11^+$-vertex sends charge $\frac{3}{2}$ to each adjacent good 2-vertex.
(R7) Every $12^+$-vertex sends charge 3 to each incident 3-face.
(R8) Every $11^+$-vertex sends charge $\frac{3}{2}$ to each adjacent pendant 3-face.
(R9) Every $7^+$-face sends charge 1 to each incident bad 2-vertex.
(R10) Every bad 3-face sends charge 1 to each incident 2-vertex.

Next, we show that the final charge $\mu^*(u)$ is nonnegative.

It is evident that $\mu^*(x) = \mu(x) = 0$ if $x$ is a 3-vertex or a 6-face.

Now, let $v$ be a $k$-vertex. 

For $k = 2$, a vertex $v$ has two $4^+$-neighbors one of which is a 11$^+$-neighbor by Lemma 2. If $v$ is a good 2-vertex, then $\mu^*(v) \geq \mu(v) + \frac{1}{2} + \frac{3}{2} = 0$ by (R1) and (R6). If $v$ is a bad 2-vertex, then $v$ is incident to a 7$^+$-face by Lemma 3. Thus $\mu^*(v) \geq \mu(v) + 1 + 1 = 0$ by (R9) and (R10).

Consider $k = 4$. Let $v$ have $\alpha$ incident 3-faces, $\beta$ adjacent good 2-vertices, and $\gamma$ pendant 3-faces. By Proposition 1 (b), $2\alpha + \beta + \gamma \leq d(v)$. Moreover, we have $\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = \frac{1}{2}(2\alpha + \beta + \gamma) \leq \frac{1}{2}d(v)$ and $\mu(v) = 2d(v) - 6 = \frac{1}{2}d(v)$ if $d(v) = 4$. Thus by (R1), (R2), and (R3), we have $\mu^*(v) = \mu(v) - (\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma) \geq 0$.

Consider $k$ for $5 \leq k \leq 10$. By (R1), (R3), and (R4), we show only the case that $v$ has $\lfloor \frac{d(v)}{2} \rfloor$ incident 3-faces because this case has final charge less than the other cases. Consider $\frac{3}{2} \cdot \frac{d(v)}{2} \leq 2d(v) - 6$, then we have $d(v) \geq 5$ because two times charge in (R1) or (R3) is less than charge in (R4). Thus we have $\mu^*(v) \geq 0$.

Consider $k = 11$. By (R5), (R6), and (R8), we show only the case that $v$ is not incident to 3-face because this case has final charge less than the other cases. we have $\mu^*(v) = 16 - 10(\frac{3}{2}) \geq 0$. If there is one 3-face, then $\mu^*(v) = 16 - (9(\frac{3}{2}) + \frac{3}{2}) = 0$.

Now let $f$ be a $k$-face.

For $k = 7^+$. By Lemma 4 a $k$-face $f$ has at most $k - 6$ incident bad 2-vertices. By (R9), $\mu^*(f) = \mu(f) - (k - 6) \cdot 1 = 0$.

Consider $k = 3$. If $f$ is a bad 3-face, then we have $f$ is a(2, 4$^+$, 12$^+$)-face or $f$ is a (2, 5$, 11^+$) by Lemma 5. Then by (R2), (R4), (R5), and (R7), $\mu^*(f) \geq \mu(f) + 1 + 3 - 1 = 0$ or $\mu^*(f) \geq \mu(f) + \frac{3}{2} + \frac{5}{2} - 1 = 0$. Now, it remains to consider a good 3-face. Consider $f$ is incident to exactly one 3-vertex. If $f$ is not incident to a 11$^+$-vertex, then pendant neighbor of a 3-vertex is a 11$^+$-vertex by Lemma 2. Thus $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{1}{2} + \frac{5}{2} \geq 0$ by (R2) and (R8). If $f$ is incident to a 4$^+$-vertex and a 11$^+$-vertex, then $\mu^*(f) \geq \mu(f) + \frac{1}{2} + \frac{5}{2} \geq 0$.
by (R2) and (R5). If $f$ is a $(3, 3, 11^+)$-face, then the pendant neighbor of a 3-vertex is a $4^+$-vertex by Lemma 2. Thus $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{1}{2} + \frac{5}{2} \geq 0$ by (R3) and (R5). If $f$ is a $(3, 3, 4^+)$-face, then the pendant neighbor of a 3-vertex is a $11^+$-vertex by Lemma 2. Thus $\mu^*(f) \geq \mu(f) + 2 \cdot \frac{3}{2} + 1 \geq 0$ by (R2) and (R8). Finally, if $f$ is a $(4^+, 4^+, 4^+)$-face, then $\mu^*(f) \geq \mu(f) + 3 \cdot 1 \geq 0$ by (R2).

Since $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, this completes the proof. 

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