Khasminskii-Type Theorem and LaSalle-Type Theorem for Stochastic Evolution Delay Equations

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Abstract

In this paper we study the well-known Khasminskii-Type Theorem, i.e. the existence and uniqueness of solutions of stochastic evolution delay equations, under local Lipschitz condition, but without linear growth condition. We then establish one stochastic LaSalle-type theorem for asymptotic stability analysis of strong solutions. Moreover, several examples are established to illustrate the power of our theories.

Keywords: stochastic evolution delay equation; Khasminskii-type theorem; LaSalle-type theorem; asymptotic stability; exponential stability.

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1 Introduction

The study of stochastic evolution delay equations is motivated by the fact that when one wants to model some evolution phenomena arising in mechanical, economic, physics, biology, engineering, etc., some hereditary characteristic such as after-effect, time-lag, time-delay can appear in the variables (see, for example, Liu \cite{7}, Mohammed \cite{14} and Wu \cite{18}). One the
other hand, some of the important and interesting aspects in existence-and-uniqueness theories and stability analysis for strong solutions have been greatly developed over the past few years. Here, we refer to Caraballo et al. \cite{1,2,3}, Liu \cite{7,8}, Real \cite{16} and references therein. For most of papers mentioned, the coefficients of stochastic evolution delay equations require the global Lipschitz and linear growth conditions to guarantee the existence and uniqueness, and analyze asymptotic stability for strong solutions. However, there are many stochastic evolution delay equations which do not satisfy linear growth condition, for example:

\[
\begin{cases}
dy(t, x) = \frac{\partial^2}{\partial x^2} y(t, x) dt - (y^2(t - \tau, x) - y^3(t, x)) dt + y^2(t - \tau, x) dB(t), & t \geq 0, \ x \in (0, \pi), \\
y(t, x) = \phi(t, x), & 0 \leq x \leq \pi, \ t \in [-\tau, 0]; \ y(t, 0) = y(t, \pi) = 0, & t \geq 0,
\end{cases}
\]

where \( \phi \in C^2([0, \pi] \times [-\tau, 0]; R) \), \( \tau \), positive constant, and \( B(t), t \geq 0 \), is a real standard Brownian motion. Moreover, for such stochastic evolution delay equations, to show existence-and-uniqueness results and analyze asymptotic stability for strong solutions, unfortunately, there are not results available for us to apply. That is, we have no alternative but to put forward new arguments to overcome the difficulties brought by the nonlinear growth.

For finite dimensional cases when the drift and diffusion coefficients of stochastic differential delay equations satisfy local Lipschitz condition, Mao \cite{13} established an existence-and-uniqueness theorem of Khasminskii type. Subsequently, many scholars generalize the classical result to cover more general stochastic differential delay equations with Markovian switching and neutral stochastic differential delay equations, e.g., Mao, Shen and Yuan \cite{10}, Yuan and Glover \cite{19} and Yuan and Mao \cite{20}. In particular, it is worth pointing out that \cite{11} by Mao and Rassias gave some Khasminskii-type theorems for highly nonlinear stochastic differential delay equations and discussed moment estimations.

On the basis of Khasminskii-type theorems, Mao \cite{9} established some stochastic LaSalle-type asymptotic convergence theorems, and applied to establish sufficient criteria for the stochastically asymptotic stability of stochastic differential delay equations. Then, there are extensive literatures which generalize these stochastic LaSalle-type theorems, see, e.g., Mao, Shen and Yuan \cite{10}, Mao \cite{12,13} and Yuan and Mao \cite{20}.

However, for stochastic evolution delay equations in infinite dimensions, as we stated before, in general, the existing existence-and-uniqueness results and asymptotic stability analysis for strong solutions are done under the global Lipschitz and linear growth conditions. Motivated by these papers, we shall intend to establish one stochastic Khasminskii-type theorem for existence-and-uniqueness theory and one stochastic LaSalle-type theorem for asymptotic stability analysis of strong solutions to stochastic evolution delay equations in infinite dimensions under local Lipschitz condition, but without linear growth condition. As we shall see in Section 3 and Section 4, our established theories have greatly improve some existing results. To the best of our knowledge to date, there are few literatures concerned with our problems, therefore, we aim to close a gap.

The contents of this paper will be arranged as follows: In section 2 we collect some preliminaries; In section 3, under local Lipschitz condition, but without linear growth condition, one Khasminskii-type theorem is established for stochastic evolution delay equations and one example is constructed to illustrate the established theory; On the basis of the established
Khasminskii-type theorem, we then investigate almost surely asymptotic stability for strong solutions, which is called the LaSalle-type theorem, exponential stability is also discussed, and two examples are provided to explain our theories in the last section.

2 Preliminaries

First of all, we introduce the framework in which our analysis is going to be carried out. Let $V$ be a Banach space and $H, K$ real, separable Hilbert spaces such that

$$ V \hookrightarrow H \equiv H^* \hookrightarrow V^*, $$

where $V^*$ is the dual of $V$ and the injections are continuous, dense and compact. We denote by $\| \cdot \|_V, \| \cdot \|_H$ the norms in $V^*, V$ and $H$, respectively, by $\langle \cdot, \cdot \rangle$ the duality product between $V^*, V$ and by $\langle \cdot, \cdot \rangle_H$ the scalar product in $H$. Furthermore, assume that for some $\beta > 0$

$$ \beta \| u \|_H \leq \| u \|, \quad \forall u \in V. $$

Assume that $B(t), t \geq 0$, is a $K$-valued Wiener process defined on a certain probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{ \mathcal{F}_t \}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets), with covariance operator $Q \in \mathcal{L}(K) = \mathcal{L}(K, K)$. By [15, Proposition 4.1, p.87],

$$ E\langle B(t), x \rangle_K \langle B(s), y \rangle_K = (t \wedge s) \langle Qx, y \rangle_K, \quad \forall x, y \in K, $$

where $Q$ is a positive, self-adjoint, trace class operator. In addition, we assume that $e_k, k \in \mathbb{N}$, is an orthonormal basis of $K$ consisting of eigenvectors of $Q$ with corresponding eigenvalues $\lambda_k \geq 0, k \in \mathbb{N}$, numbered in decreasing order, and then, according to the representation theorem of $Q$-Wiener process [15, Proposition 4.1, p.87],

$$ B(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \geq 0, $$

where $\beta_k(t), k \in \mathbb{N}$ is a sequence of real valued standard Brownian motions mutually independent on the probability space $\{ \Omega, \mathcal{F}, \mathcal{P} \}$. For an operator $G \in \mathcal{L}(K, H)$, the space of all bounded linear operators from $K$ into $H$, we denote by $\|G\|_2$ its Hilbert-Schmit norm, i.e.

$$ \|G\|_2^2 = \text{trace}(GQG^*). $$

In this paper we investigate stochastic evolution delay equation in the form:

$$ dx(t) = [A(t, x(t)) + f(t, x(t), x(t-\tau))]dt + g(t, x(t), x(t-\tau))dB(t), \quad t \geq 0 $$

with $\tau > 0$ and initial datum $x(\theta) = \psi(\theta) \in C_{\mathcal{F}_0}^b([-\tau, 0]; V) \cap C_{\mathcal{F}_0}^b([-\tau, 0]; H)$, the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; V) \cap C([-\tau, 0]; H)$-valued random variables.
**Assumption 2.1.** For any $T > 0$ let $A(t, \cdot): V \to V^*$ be a family of (nonlinear) operators on $t \in [0, T]$ satisfying $A(t, 0) = 0$ and $p \geq 2$:

(A.1) (Monotonicity and Coercivity) $\exists \alpha > 0, \lambda \in R$ such that

$$2(A(t, x) - A(t, y), x - y) \leq -\alpha \|x - y\|^p + \lambda \|x - y\|^2_H, \quad \forall x, y \in V;$$

(A.2) (Measurability) $\forall x \in V$, the map $t \in [0, T] \to A(t, x) \in V^*$ is Lebesgue measurable;

(A.3) (Hemicontinuity) The map

$$\theta \in R \to \langle A(t, x + \theta y), z \rangle \in R$$

is continuous for arbitrary $x, y, z \in V$ and $t \in [0, T]$;

(A.4) (Boundedness) $\exists \gamma > 0$ such that for $t \in [0, T]$

$$\|A(t, x)\|_* \leq \gamma\|x\|^{p-1}, \quad \forall x \in V.$$

**Assumption 2.2.** Let $f(t, \cdot, \cdot): H \times H \to H$ and $g(t, \cdot, \cdot): H \times H \to \mathcal{L}(K, H)$ be the families of nonlinear operators defined for $t \in [0, T]$ and satisfy:

(B.1) (Measurability) For any $x, y \in H$, the maps

$$\forall t \in [0, T] \to f(t, x, y) \in H \quad \text{and} \quad g(t, x, y) \in \mathcal{L}(K, H)$$

are Lebesgue-measurable, respectively.

(B.2) (Boundedness)

$$M := \sup_{t \geq 0}\{\|f(t, 0, 0)\|_H \vee \|g(t, 0, 0)\|_H\} < \infty.$$  \hspace{1cm} (2.3)

**Assumption 2.3.** (Local Lipschitz Condition) For each $h > 0$, $\exists L_h > 0$ such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\|_H \vee \|g(t, x_1, y_1) - g(t, x_2, y_2)\|_2 \leq L_h(\|x_1 - y_1\|_H + \|x_2 - y_2\|_H)$$  \hspace{1cm} (2.4)

for $t \in [0, T]$ and $x_1, x_2, y_1, y_2 \in H$ with $\|x_1\|_H \vee \|x_2\|_H \vee \|y_1\|_H \vee \|y_2\|_H \leq h$;

Let $I^p([-\tau, T]; V)$ denotes the space of all $V$-valued processes $x(t)$, which are $\mathcal{F}_t$-measurable from $[-\tau, T] \times \Omega$ to $V$ and satisfy $E\int_T^\infty \|x(t)\|^p dt < \infty$, and $L^2(\Omega; C([-\tau, T]; H))$ denote $H$-valued processes from $[-\tau, T] \times \Omega$ to $H$ such that $E\sup_{-\tau \leq t \leq T} \|x(t)\|_H^2 < \infty$. Let’s recall the definition of strong solutions.

**Definition 2.1.** For any initial datum $x(\theta) = \psi(\theta) \in C^0([-\tau, 0]; V) \cap C^0_{\mathcal{F}_0}([-\tau, 0]; H)$, a stochastic process $x(t), t \in [0, T]$, is said to be a strong solution of (2.2) if the following conditions are satisfied:

(a) $x(t) \in I^p([-\tau, T]; V) \cap L^2(\Omega; C([-\tau, T]; H));$
Theorem 2.2. Suppose

\[
U(t,x) = U(0,\psi(0)) + \int_0^t [A(s,x(s)) + f(s,x(s),x(s-\tau))]ds + \int_0^t g(s,x(s),x(s-\tau))dB(s)
\]

with initial condition \(x(\theta) = \psi(\theta) \in C_{\mathcal{F}_0}^b([-\tau,0]; V) \cap C_{\mathcal{F}_0}^b([-\tau,0]; H)\).

If \(T\) is replaced by \(\infty\), \(x(t), t \geq 0\), is called a global strong solution of (2.2).

In what follows, we shall also need the following global Lipschitz condition.

Assumption 2.4. There exists a constant \(L > 0\) such that, for \(t \in [0,T]\) and \(x_1, x_2, y_1, y_2 \in H\),

\[
\|f(t, x_1, y_1) - f(t, x_2, y_2)\|_H + \|g(t, x_1, y_1) - g(t, x_2, y_2)\|_2 \leq L(\|x_1 - y_1\|_H + \|x_2 - y_2\|_H). \quad (2.5)
\]

Under the global Lipschitz condition (2.5), the following existence-and-uniqueness result can be found in [3, Theorem 3.1].

Theorem 2.1. Assume that Assumption 2.1, Assumption 2.2 and Assumption 2.4 hold. Then, for each \(\psi \in C_{\mathcal{F}_0}^b([-\tau,0]; V) \cap C_{\mathcal{F}_0}^b([-\tau,0]; H)\) there exists a unique strong solution of (2.2) in \(\Pi([\rho, \tau]; V) \cap L^2(\Omega; C([-\tau, T]; H))\).

Remark 2.1. In general, Assumption 2.1–2.3 will only guarantee a unique maximal local strong solution to (2.2) for any given initial data \(\psi\). However, the additional conditions imposed in one of our main results, Theorem 3.1, will guarantee that this maximal local strong solution is in fact a unique global strong one.

Now we recall the Itô formula which will play key role in what follows. Let \(R_+\) be non-negative real number, and \(C^{1,2}(R_+ \times H; R_+)\) denote the space of all real valued non-negative functions \(U\) on \(R_+ \times H\) with properties:

(i) \(U(t,x)\) is once differential in \(t\) and twice (Fréchet) differentiable in \(x\);

(ii) \(U_x(t,x)\) and \(U_{xx}(t,x)\) are both continuous in \(H\) and \(L(H)\), respectively.

Theorem 2.2. Suppose \(U \in C^{1,2}(R_+ \times H; R_+)\) and \(x(t), t \geq 0\), is a strong solution to (2.2), then

\[
U(t,x(t)) = U(0,\psi(0)) + \int_0^t \mathcal{L}U(s,x(s),x(s-\tau))ds + \int_0^t \langle U_x(s,x(s)), g(s,x(s),x(s-\tau))dB(s) \rangle_H,
\]

where \(\mathcal{L}\) is the associated diffusion operator defined by, for any \(t \geq 0\) and \(x,y \in V\),

\[
\mathcal{L}U(t,x,y) = \frac{\partial U(t,x)}{\partial t} + \langle A(t,x) + f(t,x,y), U_x(t,x) \rangle + \frac{1}{2} \text{trace}(U_{xx}(t,x)g(t,x,y)Qg^*(t,x,y)).
\]
We will also need the following useful semimartingale convergence theorem established by Lipster and Shiryayev [5, Theorem 7, p.139].

**Theorem 2.3.** Let $A_1(t)$ and $A_2(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A_1(0) = A_2(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let $\zeta$ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $E\zeta < \infty$. Define for $t \geq 0$

$$X(t) := \zeta + A_1(t) - A_2(t) + M(t).$$

If $X(t)$ is nonnegative, then a.s.

$$\left\{ \lim_{t \to \infty} A_1(t) < \infty \right\} \subset \left\{ \lim_{t \to \infty} X(t) < \infty \right\} \cap \left\{ \lim_{t \to \infty} A_2(t) < \infty \right\},$$

where $C \subset D$ a.s. means $P(C \cap D^c) = 0$. In particular, if $\lim_{t \to \infty} A_1(t) < \infty$ a.s., then, with probability one,

$$\lim_{t \to \infty} X(t) < \infty, \quad \lim_{t \to \infty} A_2(t) < \infty$$

and

$$-\infty < \lim_{t \to \infty} M(t) < \infty.$$

That is, all of the three processes $X(t), A_2(t)$ and $M(t)$ converge to finite random variables.

## 3 Khasminskii-Type Theorem

In this section, under local Lipschitz condition, but without linear growth condition, we shall establish one Khasminskii-type theorem for existence-and uniqueness theory for stochastic evolution delay equations in infinite dimensions.

**Theorem 3.1.** Let Assumption 2.1,2.2 hold. Assume further that there are functions $U \in C^{1,2}(R_+ \times H; R_+), W \in C(R_+ \times H; R_+)$, and positive constants $\lambda_1$ and $\lambda_2$ such that

$$LU(t, x, y) \leq \lambda_1 [1 + U(t, x) + U(t - \tau, y) + W(t - \tau, y)] - \lambda_2 W(t, x), \quad (t, x, y) \in R_+ \times V \times V,$$

and

$$\lim_{\|x\| \to \infty} \inf_{0 \leq t < \infty} U(t, x) = \infty, \quad x \in V. \quad (3.2)$$

Then, for any initial data $x(\theta) = \psi(\theta) \in C^b_{\mathcal{F}_0}(\theta \in [-\tau, 0]; V) \cap C^b_{\mathcal{F}_0}(\theta \in [-\tau, 0]; H), (2.2)$ admits a unique global solution.

**Proof.** For any integer $k \geq b$, bound of $\psi$, and $x, y \in H$, define

$$f_k(t, x, y) = f \left( t, \frac{\|x\| \wedge k}{\|x\|} x, \frac{\|y\| \wedge k}{\|y\|} y \right), \quad g_k(t, x, y) = g \left( t, \frac{\|x\| \wedge k}{\|x\|} x, \frac{\|y\| \wedge k}{\|y\|} y \right),$$

where we set $(\|x\| \wedge k/\|x\|) x = 0$ when $x = 0$. Then, by (2.1) and (2.3), for any $x, y \in H$ and $t \geq 0$ we observe that $f_k(t, x, y)$ and $g_k(t, x, y)$ satisfy the global Lipschitz condition and
linear growth condition. Hence, there exists by Theorem 2.1 a unique global solution $x_k(t)$ on $[-\tau, \infty)$ to the following stochastic evolution delay equation

$$dx_k(t) = [A(t, x_k(t)) + f_k(t, x_k(t), x_k(t - \tau))]dt + g_k(t, x_k(t), x_k(t - \tau))dB(t)$$

with initial data $x(\theta) = \psi(\theta) \in C^b_{F_\theta}([-\tau, 0]; V) \cap C^b_{F_0}([-\tau, 0]; H)$. Define the stopping time

$$\sigma_k = \inf\{t \geq 0 : \|x_k(t)\|_H \geq k\},$$

where we set $\inf \emptyset = \infty$ as usual. Clearly, for any $s \leq \sigma_k$, $\|x_k(s)\|_H \wedge \|x_k(s - \tau)\|_H \leq k$. Then, recalling the definition of $f_k$ and $g_k$, it is easy to see that, for any $0 \leq s \leq \sigma_k$,

$$f_{k+1}(s, x_k(s), x_k(s - \tau)) = f_k(s, x_k(s), x_k(s - \tau)) = f(s, x_{k+1}(s), x_{k+1}(s - \tau))$$

and

$$g_{k+1}(s, x_k(s), x_k(s - \tau)) = g_k(s, x_k(s), x_k(s - \tau)) = g(s, x_{k+1}(s), x_{k+1}(s - \tau)).$$

Consequently,

$$x_k(t \wedge \sigma_k) = \psi(0) + \int_0^{t \wedge \sigma_k} [A(s, x_k(s)) + f_k(s, x_k(s), x_k(s - \tau))]ds + \int_0^{t \wedge \sigma_k} g_k(s, x_k(s), x_k(s - \tau))dB(s)$$

which immediately gives

$$x_{k+1}(t) = x_k(t), \quad 0 \leq t \leq \sigma_k.$$

This further implies that $\sigma_k$ is increasing in $k$. So we can define $\sigma = \lim_{k \to \infty} \sigma_k$. The property above also enables us to define $x(t)$ for $t \in [-\tau, \sigma)$ as follows

$$x(t) = x_k(t), \quad -\tau \leq t \leq \sigma_k.$$

It is clear that $x(t)$ is a unique solution to (2.2) for $t \in [-\tau, \sigma_k)$. To complete the proof, we only need to show that $P(\sigma = \infty) = 1$. Indeed, to show the desired assertion we compute by the Itô formula and (3.1) that for any $t \in [0, \tau]$

$$EU(t \wedge \sigma_k, x(t \wedge \sigma_k)) = EU(0, \psi(0)) + E \int_0^{t \wedge \sigma_k} \mathcal{L}U(s, x(s), x(s - \tau))ds$$

$$\leq EU(0, \psi(0)) + E \int_0^\tau \lambda_1[1 + U(s - \tau, x(s - \tau)) + W(s - \tau, x(s - \tau))]ds$$

$$+ \lambda_1 E \int_0^{t \wedge \sigma_k} U(s, x(s))ds - \lambda_2 E \int_0^{t \wedge \sigma_k} W(s, x(s))ds$$

$$\leq C_1 + \lambda_1 E \int_0^t U(s \wedge \sigma_k, x(s \wedge \sigma_k))ds - \lambda_2 E \int_0^{t \wedge \sigma_k} W(s, x(s))ds,$$

(3.3)
where
\[ C_1 = EU(0, \psi(0)) + E \int_{-\tau}^{0} \lambda_1 [1 + U(s, \psi(s)) + W(s, \psi(s))] ds. \]

Therefore, for \( t \in [0, \tau] \) the Gronwall inequality yields
\[ EU(t \wedge \sigma_k, x(t \wedge \sigma_k)) \leq C_1 e^{\lambda_1 \tau}, \tag{3.4} \]
and, in addition to the definition of \( \sigma_k \),
\[ P(\sigma_k \leq \tau) \leq \frac{C_1 e^{\lambda_1 \tau}}{\inf_{t \geq 0, \|x\| \geq k} U(t, x)}. \]

Letting \( k \to \infty \) and then observing (3.2), we obtain
\[ P(\sigma \leq \tau) = 0. \]
Namely,
\[ P(\sigma > \tau) = 1. \tag{3.5} \]

Hence, in (3.4), letting \( k \to \infty \) gives, for any \( t \in [0, \tau] \),
\[ EU(t, x(t)) \leq C_1 e^{\lambda_1 \tau}. \tag{3.6} \]

However, by (3.3), for any \( t \in [0, \tau] \),
\[ \lambda_2 E \int_{0}^{t \wedge \sigma_k} W(s, x(s)) ds \leq C_1 + \lambda_1 E \int_{0}^{t} U(s \wedge \sigma_k, x(s \wedge \sigma_k)) ds. \]

So, letting \( k \to \infty \), together with (3.3) and (3.6), it could be deduced that
\[ E \int_{0}^{\tau} W(x(s)) ds \leq \frac{C_1 + \lambda_1 \tau C_1 e^{\lambda_1 \tau}}{\lambda_2} < \infty. \tag{3.7} \]

In the same manner as (3.3) was done, for any \( t \in [0, 2\tau] \)
\[ EU(t \wedge \sigma_k, x(t \wedge \sigma_k)) \leq C_2 + \lambda_1 E \int_{0}^{t} U(s \wedge \sigma_k, x(s \wedge \sigma_k)) ds - \lambda_2 E \int_{0}^{t \wedge \sigma_k} W(s, x(s)) ds, \tag{3.8} \]
where
\[ C_2 = EU(0, \psi(0)) + E \int_{0}^{2\tau} \lambda_1 [1 + U(s - \tau, x(s - \tau)) + W(s - \tau, x(s - \tau))] ds \]
\[ = EU(0, \psi(0)) + E \int_{0}^{\tau} \lambda_1 [1 + U(s, x(s)) + W(s, x(s))] ds \]
\[ + E \int_{0}^{\tau} \lambda_1 [1 + U(s, x(s)) + W(s, x(s))] ds \]
\[ < \infty. \tag{3.9} \]
Therefore, taking into account (3.2),

\[ EU(t \wedge \sigma_k, x(t \wedge \sigma_k)) \leq C_2e^{2\lambda_1\tau}. \] (3.10)

Therefore, taking into account (3.2),

\[ P(\sigma \leq 2\tau) = 0, \]

that is

\[ P(\sigma > 2\tau) = 1. \]

Next, by letting \( k \to \infty \) in (3.8) and (3.10), respectively, we derive that, for any \( t \in [0, 2\tau] \),

\[ EU(t, x(t)) \leq C_2e^{2\lambda_1\tau} \quad \text{and} \quad E\int_0^{2\tau} W(x(s))ds < \frac{C_2 + 2\lambda_1\tau C_2e^{2\lambda_1\tau}}{\lambda_2} < \infty. \]

By induction, for any integer \( k \geq 1 \) it follows easily that

\[ EU(t, x(t)) \leq C_k e^{k\lambda_1\tau} \]

whenever \( t \in [0, k\tau] \) and

\[ E\int_0^{k\tau} W(s, x(s))ds < \frac{C_k + k\lambda_1\tau C_k e^{k\lambda_1\tau}}{\lambda_2}. \]

So, we can conclude by (3.2) that

\[ P(\sigma < \infty) = 0, \]

and then (2.2) admits a unique global strong solution on \( t \geq 0 \).

Now we shall use Theorem 3.1 to analyze the example which appeared in the introduction section.

**Example 3.1.** Consider the following semilinear stochastic partial differential equation:

\[
\begin{aligned}
    dy(t, x) &= \frac{\partial^2}{\partial x^2} y(t, x)dt + (y^2(t, x) - y^3(t, x))dt + y^2(t - \tau, x)dB(t), \quad t \geq 0, \quad x \in (0, \pi), \\
    y(t, x) &= \phi(t, x), \quad 0 \leq x \leq \pi, \quad t \in [-\tau, 0]; \quad y(t, 0) = y(t, \pi) = 0, \quad t \geq 0,
\end{aligned}
\] (3.11)

where \( \phi \in C^2([0, \pi] \times [-\tau, 0]; \mathbb{R}), \tau, \text{ positive constant, and } B(t), t \geq 0, \) is a real standard Brownian motion. Take \( H = L^2([0, \pi]), V = H_0^1([0, \pi]), V^* = H^{-1}([0, \pi]), K = R, A(t, u) = \frac{\partial^2}{\partial x^2} u(x), f(t, u, v) = v^2(x) - u(x) \) and \( g(t, u, v) = v^2(x), u, v \in V \). Furthermore, the norms in \( H \) and \( V \) are defined as \( \|\xi\|_H = \left( \int_0^{\pi} (\xi^2(s)) ds \right)^{1/2} \) for \( \xi \in H \) and \( \|\xi\| = \left( \int_0^{\pi} \left( \frac{\partial \xi}{\partial s} \right)^2 ds \right)^{1/2} \) for \( \xi \in V \). Then, clearly, in (2.1), we can take \( \beta = 1 \). Setting \( U(t, x) = \|x\|_H^2 \) and recalling the definition of diffusion operator \( LU \), it follows easily that

\[ LU(t, x, y) = 2\langle A(t, x, x) + 2\langle f(t, x, y), x \rangle_H + \|g(t, x, y)\|_H^2 \\
= 2\langle A(t, x, x) + 2\langle y^2 - x^3, x \rangle_H + \|y\|_H^2 \\
\leq -2\|x\|_H^2 + 2\|x\|_H \|\phi\|_H^2 - 2\|x\|_H^4 + \|y\|_H^4 \\
\leq \|x\|_H^4 - 2\|x\|_H^4 + \frac{4}{3}\|y\|_H^4. \]
Hence, by Theorem 3.1, setting \( \lambda_1 = 1 \) and \( \lambda_2 = \frac{4}{3} \) and \( W(t, x) = \|x\|_H^4 \), we immediately conclude that (3.11) admits a global solution on \( t \geq 0 \).

**Remark 3.1.** Since \( f \) and \( g \) do not satisfy the linear growth condition, then [2, Theorem 3.1] certainly cannot apply to the above example. However, by our established theory we can deduce that (3.11) has a unique global solution on \( t \geq 0 \). Therefore, Theorem 3.1 covers many highly nonlinear stochastic evolution delay equations.

### 4 LaSalle-Type Theorem

On the basis of the established Khasminskii-type theorem, in what follows we shall analyze the asymptotic stability properties of strong solutions under some special conditions of Khasminskii type by using Lyapunov method. As we know, the Lyapunov method has been developed and applied by many authors during the past century. In 1968, (see [6], Hale and Lunel [4] and the reference therein), LaSalle used the the Lyapunov method to locate limit sets for ordinary nonautonomous systems, which is one of the important developments in this direction, and the theorem is called LaSalle theorem. After thirty years, Mao [9] established a stochastic version of the LaSalle theorem for stochastic differential equations in finite dimensional space. In this section, we shall extend the LaSalle theorem to the strong solutions of stochastic evolution delay equations. We shall see there are a lot of difficulties to overcome from finite dimensional cases to infinite dimensional cases.

Let \( L^1(R_+; R_+) \) denote the family of all functions \( \xi : R_+ \to R_+ \) such that \( \int_0^\infty \xi(s)ds < \infty \).

**Theorem 4.1.** Let Assumption 2.1-2.3 hold. Assume that there are functions \( U \in C^{1,2}(R_+ \times H; R_+) \), \( \gamma \in L^1(R_+; R_+) \) and \( w_1, w_2 \in C(H; R_+) \) such that

\[
LU(t, x, y) \leq \gamma(t) - w_1(x) + w_2(y), \quad \forall (t, x, y) \in R_+ \times V \times V, \tag{4.1}
\]

\[
w_1(0) = w_2(0) = 0, \quad w_1(x) > w_2(x), \tag{4.2}
\]

moreover

\[
\lim_{\|x\|_H \to \infty} \inf_{0 \leq t < \infty} U(t, x) = \infty \quad \text{and} \quad \lim_{\|x\| \to \infty} \inf_{0 \leq t < \infty} U(t, x) = \infty, \quad x \in V. \tag{4.3}
\]

Then the solution \( x(t) \) of (2.2) satisfies

\[
\lim_{t \to \infty} \sup U(t, x(t)) < \infty \quad \text{a.s.,} \tag{4.4}
\]

and

\[
\lim_{t \to \infty} w(x(t)) = 0 \quad \text{a.s.,} \tag{4.5}
\]

where \( w = w_1 - w_2 \), and moreover

\[
P \left( \lim_{t \to \infty} \|x(t)\|_H = 0 \right) = 1, \tag{4.6}
\]

that is, the solution of (2.2) is almost surely asymptotically stable.
Proof. First of all, by Theorem 3.1 it is easy to see (2.2) has a unique global solution for \( t \geq 0 \) under the conditions of Theorem 4.1. Applying the Itô formula to \( V(t, x) \) and solution \( x(t), t \geq 0 \), of (2.2), we derive that

\[
U(t, x(t)) = U(0, \psi(0)) + \int_0^t L U(s, x(s), x(s-\tau)) ds + \int_0^t \langle U_x(s, x(s)), g(s, x(s), x(s-\tau)) dB(s) \rangle_H.
\]

This, together with (4.1), implies that

\[
U(t, x(t)) \leq U(0, \psi(0)) + \int_0^t \gamma(s) - w_1(x(s)) + w_2(x(s-\tau)) ds
\]

\[
+ \int_0^t \langle U_x(s, x(s)), g(s, x(s), x(s-\tau)) dB(s) \rangle_H.
\]

(4.7)

Then, by (4.2) and Theorem 2.3 we obtain

\[
\lim_{t \to \infty} \sup U(t, x(t)) < \infty \quad \text{a.s.}
\]

(4.8)

Taking expectations on both sides of (4.7) and then letting \( t \to \infty \), one derives that

\[
E \int_0^\infty (w_1(x(s)) - w_2(x(s))) ds < \infty,
\]

(4.9)

which certainly implies

\[
\int_0^\infty (w_1(x(s)) - w_2(x(s))) ds < \infty, \quad \text{a.s.}
\]

(4.10)

Clearly, \( w \in C(H; R_+) \). It is straightforward to see from (4.10) that

\[
\lim_{t \to \infty} \inf w(x(t)) = 0 \quad \text{a.s.}
\]

(4.11)

In what follows, we intend to claim that

\[
\lim_{t \to \infty} w(x(t)) = 0 \quad \text{a.s.}
\]

(4.12)

By contradiction, if (4.12) is false, then

\[
P \left\{ \lim_{t \to \infty} \sup w(x(t)) > 0 \right\} > 0.
\]

(4.13)

Hence, there is a number \( \epsilon > 0 \) such that

\[
P(\Omega_1) \geq 3\epsilon,
\]

(4.14)
where
\[ \Omega_1 = \left\{ \lim_{t \to \infty} \sup w(x(t)) > 2\epsilon \right\}. \]

It is easy to observe from (4.8) and the continuity of both the solution \( x(t) \) and the function \( U(t, x) \) that
\[ \sup U(t, x(t)) < \infty \text{ a.s.} \]

Define \( \rho : R_+ \to R_+ \) by
\[ \rho(r) = \inf_{\|x\| \geq r, 0 \leq t < \infty} U(t, x). \]

Clearly, \( \rho(\|x(t)\|) \leq U(t, x(t)) \) so
\[ \sup_{0 \leq t < \infty} \rho(\|x(t)\|) \leq \sup_{0 \leq t < \infty} U(t, x(t)) < \infty \text{ a.s.} \]

While by (4.13)
\[ \rho(r) = \infty. \]

We therefore must have
\[ \sup_{0 \leq t < \infty} \|x(t)\| < \infty \text{ a.s.} \]  \hspace{1cm} (4.15)

Recalling the boundedness of the initial data we can then find a positive number \( h \), which depends on \( \epsilon \), sufficiently large for \( \|\psi(\theta)\| < h \) for all \( -\tau \leq \theta \leq 0 \) almost surely while
\[ P(\Omega_2) \geq 1 - \epsilon, \]  \hspace{1cm} (4.16)

where
\[ \Omega_2 = \left\{ \sup_{-\tau \leq t < \infty} \|x(t)\| < h \right\}. \]

It is easy to see from (4.14) and (4.16) that
\[ P(\Omega_1 \cap \Omega_2) \geq 2\epsilon. \]  \hspace{1cm} (4.17)

Let us now define a sequence of stopping times,
\[ \tau_h = \inf\{t \geq 0 : \|x(t)\| \geq h\}, \]
\[ \sigma_1 = \inf\{t \geq 0 : w(x(t)) \geq 2\epsilon\}, \]
\[ \sigma_{2k} = \inf\{t \geq \sigma_{2k-1} : w(x(t)) < \epsilon\}, \hspace{1cm} k = 1, 2, \ldots \]
\[ \sigma_{2k+1} = \inf\{t \geq \sigma_{2k} : w(x(t)) \geq 2\epsilon\}, \hspace{1cm} k = 1, 2, \ldots , \]

where throughout this paper we set \( \inf \emptyset = \infty \). Note from (4.11) and the definitions of \( \Omega_1 \) and \( \Omega_2 \) that
\[ \tau_h = \infty, \hspace{1cm} \sigma_k < \infty, \hspace{1cm} \forall k \geq 1 \]  \hspace{1cm} (4.18)
whenever \( \omega \in \Omega_1 \cap \Omega_2 \). By (4.13), we compute

\[
\infty > E \int_0^\infty w(t, x(t))dt \\
geq \sum_{k=1}^\infty E \left[ I_{\{\sigma_{2k-1} < \infty, \sigma_{2k} < \infty, \tau_h = \infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k}} w(t, x(t))dt \right] \\
geq \epsilon \sum_{k=1}^\infty E[I_{\{\sigma_{2k-1} < \infty, \tau_h = \infty\}}(\sigma_{2k} - \sigma_{2k-1})],
\]

where \( I_A \) is the indicator function of set \( A \) and we have noted from (4.11) that \( \sigma_{2k} < \infty \) whenever \( \sigma_{2k-1} < \infty \). On the other hand, by Itô’s formula and (A.1)

\[
E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \sup_{0 \leq t \leq T} \|x(\tau_h \wedge (\sigma_{2k-1} + t)) - x(\tau_h \wedge \sigma_{2k-1})\|^2_H \right] \\
+ \alpha E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \sup_{0 \leq t \leq T} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1} + t)} \|x(s)\|^2 ds \right] \\
\leq |\lambda| E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1} + T)} \|x(s)\|^2_H ds \right] \\
+ 2E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1} + T)} \langle A(s, x(s)), x(\tau_h \wedge \sigma_{2k-1}) \rangle ds \right] \\
+ 2E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1} + T)} \langle f(s, x(s), x(s - \tau)), x(s) - x(\tau_h \wedge \sigma_{2k-1}) \rangle_H ds \right] \\
+ E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1} + T)} \|g(s, x(s), x(s - \tau))\|^2_H ds \right] \\
+ 2E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \sup_{0 \leq t \leq T} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1} + t)} \langle x(s) - x(\tau_h \wedge \sigma_{2k-1}), g(s, x(s), x(s - \tau))dB(s) \rangle_H \right].
\]

(4.20)

Obviously, it follows easily from (2.1) that

\[
|\lambda| E \left[ I_{\{\tau_h \wedge \sigma_{2k-1} < \infty\}} \int_{\tau_h \wedge \sigma_{2k-1}}^{\tau_h \wedge (\sigma_{2k-1} + T)} \|x(s)\|^2_H ds \right] \leq \frac{|\lambda|Th^2}{\beta^2} := C_1T^{\frac{3}{2}}.
\]

(4.21)
Hence, putting (4.21)-(4.25) into (4.20), it holds

Next it could be deduced from (2.4) that

Now compute by (A.4) that

and

Furthermore, taking into account Burhold-Davis-Gundy inequality, together with (2.1),

Furthermore, taking into account Burhold-Davis-Gundy inequality, together with (2.1),

Hence, putting (4.21)-(4.25) into (4.20), it holds
Since \( w(x) \) is continuous in \( V \), it must be uniformly continuous in the closed ball \( \bar{S}_h = \{ x \in V : \| x \| \leq h \} \). We can therefore choose \( \delta = \delta(\epsilon) > 0 \) so small that

\[
|w(x) - w(y)| \leq \epsilon
\]  

whenever \( \| x - y \| \leq \delta \) with \( x, y \in \bar{S}_h \). We further choose \( T = T(\epsilon, \delta, h) > 0 \) sufficiently small for

\[
(C_1 + C_2 + C_3 + C_4 + C_5) T^{\frac{1}{2}} / \delta < \epsilon.
\]

By the Chebyshev inequality, (4.26) gives

\[
P \left( \{ \tau_h \land \sigma_{2k-1} < \infty \} \cap \left\{ \sup_{0 \leq t \leq T} \| x(\tau_h \land (\sigma_{2k-1} + t)) - x(\tau_h \land \sigma_{2k-1}) \|_H^2 \geq \delta \right\} \right) < \epsilon.
\]

Consequently,

\[
P \left( \{ \sigma_{2k-1} < \infty, \tau_h = \infty \} \cap \left\{ \sup_{0 \leq t \leq T} \| x(\sigma_{2k-1} + t) - x(\sigma_{2k-1}) \|_H^2 \geq \delta \right\} \right)
\]

\[
= P \left( \{ \tau_h \land \sigma_{2k-1} < \infty, \tau_h = \infty \} \cap \left\{ \sup_{0 \leq t \leq T} \| x(\tau_h \land (\sigma_{2k-1} + t)) - x(\tau_h \land \sigma_{2k-1}) \|_H^2 \geq \delta \right\} \right)
\]

\[
\leq P \left( \{ \tau_h \land \sigma_{2k-1} < \infty \} \cap \left\{ \sup_{0 \leq t \leq T} \| x(\tau_h \land (\sigma_{2k-1} + t)) - x(\tau_h \land \sigma_{2k-1}) \|_H^2 \geq \delta \right\} \right)
\]

\[
\leq \epsilon.
\]

Recalling (4.17) and (4.18), we further compute

\[
P \left( \{ \sigma_{2k-1} < \infty, \tau_h = \infty \} \cap \left\{ \sup_{0 \leq t \leq T} \| x(\sigma_{2k-1} + t) - x(\sigma_{2k-1}) \|_H^2 < \delta \right\} \right)
\]

\[
= P \left( \{ \sigma_{2k-1} < \infty, \tau_h = \infty \} \right)
\]

\[
- P \left( \{ \sigma_{2k-1} < \infty, \tau_h = \infty \} \cap \left\{ \sup_{0 \leq t \leq T} \| x(\sigma_{2k-1} + t) - x(\sigma_{2k-1}) \|_H^2 \geq \delta \right\} \right)
\]

\[
\geq 2\epsilon - \epsilon = \epsilon.
\]

Using (4.27), we derive that

\[
P \left( \{ \sigma_{2k-1} < \infty, \tau_h = \infty \} \cap \left\{ \sup_{0 \leq t \leq T} |w(x(\sigma_{2k-1} + t)) - w(x(\sigma_{2k-1}))| < \epsilon \right\} \right)
\]

\[
\geq P \left( \{ \sigma_{2k-1} < \infty, \tau_h = \infty \} \cap \left\{ \sup_{0 \leq t \leq T} \| x(\sigma_{2k-1} + t) - x(\sigma_{2k-1}) \| < \delta \right\} \right)
\]

\[
\geq \epsilon.
\]

Set

\[
\Omega_k = \left\{ \sup_{0 \leq t \leq T} |w(x(\sigma_{2k-1} + t)) - w(x(\sigma_{2k-1}))| < \epsilon \right\}.
\]
Noting
\[ \sigma_{2k}(\omega) - \sigma_{2k-1}(\omega) \geq T \]
whenever \( \omega \in \{ \sigma_{2k-1} < \infty, \tau_h = \infty \} \cap \Omega_k \), we derive from (4.19) and (4.28) that
\[
\infty > \epsilon \sum_{k=1}^{\infty} E[I_{\{\sigma_{2k-1} < \infty, \tau_h = \infty \}}(\sigma_{2k} - \sigma_{2k-1})] \\
\geq \epsilon \sum_{k=1}^{\infty} E[I_{\{\sigma_{2k-1} < \infty, \tau_h = \infty \}} \cap \Omega_k (\sigma_{2k} - \sigma_{2k-1})] \\
\geq \epsilon T \sum_{k=1}^{\infty} \epsilon = \infty
\]
which is a contradiction. So (4.12) must hold. We observe from (4.12) and (4.15) that there is an \( \Omega_0 \subset \Omega \) with \( P(\Omega_0) = 1 \) such that for all \( \omega \in \Omega_0 \)
\[
\lim_{t \to \infty} w(x(t), \omega) = 0 \quad \text{and} \quad \sup_{0 \leq t < \infty} \|x(t, \omega)\| < \infty. \quad (4.29)
\]
We shall now show that for any \( \omega \in \Omega_0 \)
\[
\lim_{t \to \infty} \|x(t, \omega)\| = 0. \quad (4.30)
\]
If this is false, then there is some \( \bar{\omega} \in \Omega_0 \) such that
\[
\limsup_{t \to \infty} \|x(t, \bar{\omega})\| > 0.
\]
Whence there is a subsequence \( \{x(t_k, \bar{\omega})\}_{k \geq 1} \) of \( \{x(t, \bar{\omega})\}_{t \geq 0} \) such that
\[
\|x(t_k, \bar{\omega})\| \geq \rho, \quad k \geq 1
\]
for some \( \rho > 0 \). Since \( \{x(t_k, \bar{\omega})\}_{k \geq 1} \) is bounded so there must be an increasing subsequence \( \{\bar{t}_k\}_{k \geq 1} \) such that \( \{x(\bar{t}_k, \omega)\}_{k \geq 1} \) converges to some \( z \in V \) with \( \|z\| \geq \rho \). Hence
\[ w(z) = \lim_{k \to \infty} w(x(\bar{t}_k, \omega)). \]
However, by (4.29), \( w(z) = 0 \). This is a contradiction and hence (4.30) must hold. Therefore,
\[
P\left( \lim_{t \to \infty} \|x(t)\|_H = 0 \right) = 1.
\]
by using (2.1). That is, the solution \( x(t) \) of (2.2) is almost surely asymptotically stable, and the proof is therefore complete. \( \square \)

**Remark 4.1.**

Now one example is constructed to illustrate our theory.
Example 4.1. Consider the following semilinear stochastic partial differential equation with delay:
\[
\begin{align*}
\frac{dy(t, x)}{dt} &= \frac{\partial^2}{\partial x^2}y(t, x) + (y^3(t, x) + y(t, x))dt + y(t, x)\sin t dB(t), \quad t \geq 0, \quad x \in (0, \pi), \\
y(t, x) &= \phi(t, x), \quad 0 \leq x \leq \pi, \quad t \in [-\tau, 0];
\end{align*}
\]
where \(\phi \in C^2([0, \pi] \times [-\tau, 0] ; R)\), \(\tau > 0\) is a positive constant, and \(B(t)\), \(t \geq 0\), is a real standard Brownian motion.

We can set this problem in our formulation by taking \(H = L^2([0, \pi])\), \(V = H_0^1([0, \pi])\), \(V^* = H^{-1}([0, \pi])\), \(K = R\), \(A(t, u) = \frac{\partial^2}{\partial x^2}u(x)\), \(f(t, u, v) = -[u^3(x) + u(x)]\) and \(g(t, u, v) = v(x)\sin t\).

Furthermore, the norms in \(H\) and \(V\) are defined as \(\|\xi\|_H = (\int_0^\pi \xi^2(s)ds)^{\frac{1}{2}}\) for \(\xi \in H\) and \(\|\xi\| = (\int_0^\pi (\frac{\partial \xi}{\partial s})^2 ds)^{\frac{1}{2}}\) for \(\xi \in V\), respectively. Setting \(U(t, x) = \|x\|_H^2\) and recalling the definition of diffusion operator \(LV\), it follows easily that
\[
LU(t, x, y) = 2\langle A(t, x), x \rangle + 2\langle f(t, x, y), y \rangle_H + \|g(t, x, y)\|_H^2
\]
\[
= 2\langle A(t, x), x \rangle + 2\langle x^3, x \rangle_H + \|y \sin t\|_H^2
\]
\[
\leq -2\|x\|^2 - 2(\|x\|_H^4 + \|x\|_H^2) + \|y\|_H^2
\]
\[
\leq -2(\|x\|_H^2 + 2\|x\|_H^2) + \|y\|_H^2.
\]

Taking \(\gamma(t) = 0, w_1(x) = 2(\|x\|_H^2 + 2\|x\|_H^2)\), \(w_2(x) = \|x\|_H^2\) and then applying Theorem 4.1, the solution of (4.31) is almost surely asymptotically stable.

Now, we further take into account another Khasminskii-type condition to give a powerful criterion for exponential stability of stochastic evolution delay equation, especially for highly nonlinear cases.

Theorem 4.2. Let Assumption 2.1 and Assumption 2.2 hold. Assume that there are functions \(U \in C^{1,2}(R_+ \times H; R_+)\), \(W_1 \in C(R_+ \times H; R_+)\), \(\gamma(t), t \in R_+\), nonnegative continuous function, and constants \(\beta_1 > 0, \beta_2 > 0, \alpha_1 > \alpha_2 \geq 0, \alpha_3 > \alpha_4 > 0, \mu > 0\) such that
\[
\beta_1 \|x\|_H^2 \leq U(t, x) \leq \beta_2 \|x\|_H^2, \quad \forall (t, x) \in R_+ \times V
\]
and
\[
LU(t, x, y) \leq \gamma(t) - \alpha_1 U(t, x) + \alpha_2 U(t, y) - \alpha_3 W_1(t, x) + \alpha_4 W_1(t, y), \quad (t, x, y) \in R_+ \times V \times V,
\]
where \(\gamma(t)\) satisfies \(\int_0^\infty \gamma(t)e^{\mu t} dt < \infty\). Then
\[
\limsup_{t \to \infty} \frac{1}{t} \log(E\|x(t)\|_H^2) \leq - (\mu \wedge \epsilon),
\]
where \(\epsilon = \epsilon_1 \wedge \epsilon_2\) while \(\epsilon_1 > 0\) and \(\epsilon_1 > 0\) are the unique roots to the following equations
\[
\alpha_1 = \epsilon_1 + \alpha_2 e^{\epsilon_1 \tau} \quad \text{and} \quad \alpha_3 = \alpha_4 e^{\epsilon_2 \tau}.
\]
In other words, the global strong solution of (2.2) is mean square exponential stability and the Lyapunov exponent should not be greater than \(- (\mu \wedge \epsilon)\).
Proof. Noting that (4.32) and (4.33) imply (3.1) and (3.2), respectively, therefore, (2.2) has a unique global strong solution on \( t \geq 0 \). To show the desired assertion (4.34), compute by Itô’s formula and (4.33) that for \( t \geq 0 \)

\[
E(e^{\epsilon t}U(t, x(t)))
= EU(0, x(0)) + \epsilon E \int_0^t e^{\epsilon s}U(s, x(s))ds + E \int_0^t e^{\epsilon s}LU(s, x(s), x(s) - \tau))ds
\leq EU(0, x(0)) + \int_0^t \gamma(s)e^{\epsilon s}ds + \epsilon E \int_0^t e^{\epsilon s}U(s, x(s))ds - \alpha_1 E \int_0^t e^{\epsilon s}U(s, x(s))ds
+ \alpha_2 E \int_0^t e^{\epsilon s}U(s - \tau, x(s - \tau))ds - \alpha_3 E \int_0^t e^{\epsilon s}W_1(s, x(s))ds + \alpha_4 E \int_0^t e^{\epsilon s}W_1(s - \tau, x(s - \tau))ds.
\]

(4.36)

Observing that
\[
\alpha_2 \int_0^t e^{\epsilon s}U(s - \tau, x(s - \tau))ds = \alpha_2 \int_{-\tau}^{t-\tau} e^{\epsilon s}U(s, x(s))ds
\leq \alpha_2 \int_{-\tau}^0 e^{\epsilon s}U(s, x(s))ds + \alpha_2 e^{\epsilon \tau} \int_0^t e^{\epsilon s}U(s, x(s))ds.
\]

Similarly,
\[
\alpha_4 \int_0^t e^{\epsilon s}W_1(s - \tau, x(s - \tau))ds = \alpha_4 \int_{-\tau}^{t-\tau} e^{\epsilon s}W_1(s, x(s))ds
\leq \alpha_4 \int_{-\tau}^0 e^{\epsilon s}W_1(s, x(s))ds + \alpha_4 e^{\epsilon \tau} \int_0^t e^{\epsilon s}W_1(s, x(s))ds.
\]

Hence, in (4.36)

\[
E(e^{\epsilon t}U(t, x(t))) \leq C_3 + \int_0^t \gamma(s)e^{(k+\mu-(\mu\wedge\alpha))s}ds - (\alpha_1 - \alpha_2 e^{\epsilon \tau} - \epsilon) E \int_0^t e^{\epsilon s}U(s, x(s))ds
- (\alpha_3 - \alpha_4 e^{\epsilon \tau}) E \int_0^t e^{\epsilon s}W_1(s, x(s))ds,
\]

(4.37)

where
\[
C_3 = EU(0, x(0)) + \alpha_2 e^{\epsilon \tau} E \int_{-\tau}^0 e^{\epsilon s}U(s, x(s))ds + \alpha_4 e^{\epsilon \tau} E \int_{-\tau}^0 e^{\epsilon s}W_1(s, x(s))ds.
\]

Furthermore, by (4.33)
\[
E(e^{\epsilon t}U(t, x(t))) \leq C_3 + C_4 e^{(k-(\mu\wedge\alpha))t},
\]

where \( C_4 = \int_0^t \gamma(s)e^{\mu s}ds < \infty \). This certainly implies that
\[
E(U(t, x(t))) \leq e^{-\epsilon t}C_3 + C_4 e^{-(\mu\wedge\alpha)t} \leq (C_3 \vee C_3)e^{-(\mu\wedge\alpha)t},
\]

and then the desired assertion follows from (4.32). \( \square \)
Example 4.2. Consider the following stochastic delay differential equation:

$$\begin{align*}
\begin{cases}
    dy(t, x) &= \frac{\partial}{\partial x} \left( a(t, x) \frac{\partial y(t, x)}{\partial x} \right) dt + y(t, x)(a + by(t - \tau, x) - y^2(t, x))dt \\
    &+ cy(t, x)(t - \tau, x)dB(t), \quad t \geq 0, \quad x \in (0, \pi), \\
    y(t, x) &= \phi(t, x), \quad 0 \leq x \leq \pi, \quad t \in [-\tau, 0]; \quad y(t, 0) = y(t, \pi) = 0, \quad t \geq 0,
\end{cases}
\end{align*}$$

(4.38)

where $\phi \in C^2([0, \pi] \times [-\tau, 0]; R)$, $\tau > 0$ is positive constant, $B(t), t \geq 0$, is a real standard Brownian motion.

Indeed, define $A(t, u) = \frac{\partial}{\partial x} \left( a(t, x) \frac{\partial u}{\partial x} \right)$, where $a(t, x)$ is measurable in $[0, \infty) \times [0, \pi]$ and satisfy $0 < \nu \leq a(t, x) \leq \alpha$ on $[0, \infty) \times [0, \pi]$ and let $H = L^2([0, \pi]), V = H^1_0([0, \pi]), V^* = H^{-1}([0, \pi])$, with the usual norms in the spaces $H$ and $V$ defined as $\|\xi\|_H = \left( \int_0^\pi \xi^2(s)ds \right)^{\frac{1}{2}}$ for $\xi \in H$ and $\|\xi\| = \left( \int_0^\pi \left( \frac{\partial \xi}{\partial x} \right)^2 ds \right)^{\frac{1}{2}}$ for $\xi \in V$, respectively. Setting $U(t, x) = \|x\|^2_H$, then it could be derived that

$$\mathcal{L}U(t, x, y) = 2\langle A(t, x), x \rangle + 2\langle f(t, x, y), x \rangle_H + \|g(t, x, y)\|_H^2$$

$$= 2\langle A(t, x), x \rangle + 2\langle x(a + by - x^2), x \rangle_H + \|cxy\|_H^2$$

$$\leq -2\nu\|x\|^2 + 2a\|x\|_H^2 + \frac{1}{2}\|x\|_H^4 + 2b^2\|y\|_H^2 - 2\|x\|_H^4 + \frac{1}{2}\|x\|_H^4 + \frac{1}{2}\|x\|_H^4 + \frac{1}{2}c^4\|y\|_H^4$$

$$\leq -2(\nu - a)\|x\|_H^2 + 2b^2\|y\|_H^2 - \|x\|_H^4 + \frac{1}{2}c^4\|y\|_H^4.$$ 

Hence, if $\nu - a > b^2 > 0$ and $c^4 < 2$, by Theorem 4.2, the global strong solution to (4.26) is exponential stability.

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