Asymptotic analysis and numerical modeling of mass transport in tubular structures

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October 30, 2009

Abstract

In the paper the flow in a thin tubular structure is considered. The velocity of the flow stands for a coefficient in the diffusion-convection equation set in the thin structure. An asymptotic expansion of solution is constructed. This expansion is used further for justification of an asymptotic domain decomposition strategy essentially reducing the memory and the time of the code. A numerical solution obtained by this strategy is compared to the numerical solution obtained by a direct FEM computation.

Keywords: asymptotic expansion, partial asymptotic decomposition of domain, stokes equation, diffusion-convection equation.

MSC (2000): 35B27, 35Q30, 76M45, 65N55

1 Introduction

The paper is devoted to the strategy of numerical implementation of the asymptotic partial decomposition of the domain for the tubular structures of a complicated geometry. We will consider the Stokes flow in this structure and the convection-diffusion and sorption process for some diluted substance. First we consider an asymptotic expansion of the solution. We emphasize the importance of the boundary layers in the neighborhood of some special structural elements of the tubular domain, such as the bifurcations of canals and "stenosis areas". That is why some multiscale strategy should be applied to the analysis of the convection-diffusion process: the 1D limit description in the canals will be coupled with some 2D zooms in these special structural elements.

In section 2 we define a tubular structure as a union of thin rectangles connected by some domains of small diameters. The Stokes equation and the diffusion-convection equation are set in this domain. For the Stokes equation the Dirichlet conditions are respected at the lateral boundary with some given inflow and outflow. For the diffusion-convection equation we pose the Robin type condition at the
lateral boundary with some given inflow and outflow concentrations. The viscosity and the diffusion are constant out of some "stenosis area" where they may have variations. The varying viscosity can be used for the modeling of a clot in the blood circulation process. Indeed, if at some part of the domain the viscosity is great, then, applying the idea of the fictitious domain method, we can exclude this part of the domain from the flow area (see Remark 1).

In section 3 we consider the Stokes equation in tubular structure. The asymptotic expansion of the solution for constant viscosity has been obtained in [1]. In this section we construct the boundary layer correctors for the varying viscosity in the stenosis areas.

In section 4 the convection-diffusion equation is considered. First we construct the asymptotic expansion in an infinite tube (subsection 4.1). Then using this expansion as a regular ansatz we add the boundary layer correctors in the stenosis zones (subsection 4.2) in a bifurcation area and in the entrance/exit elements (subsections 4.3 and 4.4). In the subsection 4.5 the leading term of the asymptotic expansion is presented. The justification of the asymptotic expansion follows the scheme: estimate for the residuals and application of the a priori estimates for the initial problem.

Section 5 describes one version of the partial asymptotic domain decomposition strategy for the mass transport problem in a tubular structure.

Finally, section 6 develops the numerical experiment comparing the direct numerical solution of the 2D problem and the asymptotic solution of the partially decomposed problem. The results of this experiment confirm good coincidence of the exact solution and the approximate solution obtained by the method of asymptotic partial decomposition of domain.

2 Geometry of tubular structure and setting of the problem

We will introduce the tubular domain which consists of three types of structural elements: canals, bifurcations and stenosis areas. This tubular structure is similar to the rod structures introduced in [2] and the tube structures or pipe structures introduced in [3]; we consider a new element that is, the stenosis area, simulated by varying coefficients of the equation (viscosity and diffusion coefficients) and not by geometric singularity.

Let us remind the definition of a tube structure.

Let $e_1, ..., e_n$ be n closed segments in $\mathbb{R}^2$ which have a single common point 0 (i.e. the origin of the coordinate system) and let it be the common end point of all these segments. Let $\theta_1, ..., \theta_n \in (0, 1)$ be n positive numbers. Making a change of variables (rotation) such that the new axis $x_1$ denoted $x_1^{e_i}$ contains the segment $e_i$ and the second new axis $x_2^{e_i}$ is orthogonal to $e_i$, we define

$$B_i^\varepsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^{e_i} \in (0, |e_i|), x_2^{e_i} \in \left( -\frac{\theta_i \varepsilon}{2}, \frac{\theta_i \varepsilon}{2} \right) \right\}.$$  

![Fig. 1](image_url)

Let $\gamma_0$ be a bounded domain containing $O$, $\gamma_i$ be a bounded domain containing $O_i$, the end point of $e_i$ (different from O). We assume for simplicity that $\text{diam}(\gamma_i), \text{diam}(\gamma_0) < 2$. Consider the homothetic
contraction of $\gamma_0$ in $\frac{1}{\varepsilon}$ times with the center of the homothety in $O$ and denote $\gamma_O^\varepsilon$ the image of $\gamma_0$ by this homothety, i.e.

$$\gamma_O^\varepsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \in \gamma_0 \right\}.$$ 

In the same way we consider

$$\gamma_i^\varepsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} + O_i \in \gamma_i \right\}$$

the homothetic contraction of $\gamma_i$ in $\frac{1}{\varepsilon}$ times with the center of the homothety in $O_i$.

Define the one bundle tube structure

$$B^\varepsilon = \left( \bigcup_{i=1}^{n} B_i^\varepsilon \right) \cup \left( \bigcup_{i=1}^{n} \gamma_i^\varepsilon \right)$$

such that it is a connected domain with the $C^2$-smooth boundary.

In a more general case, we will consider several one-bundle structures $B_{\varepsilon 1}, B_{\varepsilon 2}, ..., B_{\varepsilon m}$ such that every of these structures is associated to some segments:

$$e_{11}, \ldots, e_{1n_1} \text{ for } B_{\varepsilon 1},$$

$$e_{21}, \ldots, e_{2n_2} \text{ for } B_{\varepsilon 2},$$

$$\ldots$$

$$e_{m1}, \ldots, e_{mn_n} \text{ for } B_{\varepsilon m}.$$ 

Assume that if two of these segments have common point then it is an end point for each of these segments; assume that the union $\bigcup_{q=1}^{m} B_{eq}$ is a connected domain with $C^2$-smooth boundary. In this case we will call this union $\bigcup_{q=1}^{m} B_{eq}$ a multi-bundle tube structure. If an end point of some segment $e_{ij}$ is not an end point for all other segments then such end point will be called solitary.

We will consider the Stokes equation and the convection-diffusion equation in such tube structure. The boundary condition is the vanishing velocity for the Stokes equation everywhere except of some special parts of the boundary (entrance and exit). These parts are some connected parts $\Gamma_i$ of the boundary of smoothing domains $\gamma_i$. We assume that the end points of segments corresponding to this $\gamma_i$ are solitary end points. Let $\Gamma_1, \ldots, \Gamma_r$ be these parts of the boundary.

We consider the Stokes equation in such a tube structure with the varying viscosity coefficient $\mu$:

$$\text{div} \left( \mu(x) \left( \nabla u\varepsilon + (\nabla u\varepsilon)^T \right) - p\varepsilon I \right) = f\varepsilon (x), \quad (1)$$

where the divergence is taken with respect to the elements of each line of the matrix $\mu(x) \left( \nabla u\varepsilon + (\nabla u\varepsilon)^T \right) - p\varepsilon I$, and the convection-diffusion equation

$$- \text{div} \left( K\varepsilon (x) \nabla c\varepsilon \right) + u\varepsilon (x) \cdot \nabla c\varepsilon = g(x_{1i}). \quad (2)$$

Assume that $g = 0$ in some neighborhood of the end points of the segments, $g \in C^{k+2}(e_i)$ for all segments $e_i$.

We will assume that $\mu\varepsilon$ and $K\varepsilon$ are positive constants $\mu$ and $\kappa$ respectively, everywhere except some "stenosis areas" where they have a form:

$$\mu\varepsilon (x) = \mu + M \left( \frac{x - x_{\varepsilon s}}{\varepsilon} \right) \quad \text{and} \quad K\varepsilon (x) = \kappa + K \left( \frac{x - x_{\varepsilon s}}{\varepsilon} \right) \quad \text{(3)}$$
where $M$ and $K$ are measurable bounded function having a finite support inside the ball $B\left(\frac{\pi_s}{\varepsilon}, 2\right)$ with the center $\frac{\pi_s}{\varepsilon}$ and the radius 2, such that,

$$\exists \kappa_1 > 0 : \mu_\varepsilon(x), K_\varepsilon(x) \geq \kappa_1.$$

Here $\pi_s$ are some points belonging to the segments $e_i$ of the graph of the structure, they are different from the end points and are independent of $\varepsilon$.

The sorption will be modeled by the boundary condition for the diffusion-convection equation, i.e. let us consider the boundary conditions:

$$u_\varepsilon = 0 \text{ on the lateral boundary } \partial B_\varepsilon \setminus \left(\bigcup_{t=1}^r \Gamma_t\right), \quad (4)$$
$$K_\varepsilon(x) \frac{\partial c_\varepsilon}{\partial n} = \varepsilon \beta c_\varepsilon \text{ on the lateral boundary } \partial B_\varepsilon \setminus \left(\bigcup_{t=1}^r \Gamma_t\right), \quad (5)$$
$$u_\varepsilon = G\left(\frac{x - x_{b_t}}{\varepsilon}\right) \text{ on } \Gamma_t \quad (6)$$
$$c_\varepsilon = q_t = \text{const} \text{ on } \Gamma_t, \ t = 1, \ldots, r \quad (7)$$

where $x_{b_t}$ is an end point, inside $\gamma_t$, of a corresponding segment, $n$ is an outer normal, $G \in C_0^2(\gamma_t)$ and $\sum_t \int_{\Gamma_t} n \cdot G\left(\frac{x - x_{b_t}}{\varepsilon}\right) ds = 0$.

**Remark 1** The varying viscosity and diffusion coefficients can be used for the modelling of a clot in the blood circulation process. Let us remind the fictitious domain method. Consider an example: the Poisson equation $\Delta u = f$ posed in a bounded domain $G$ with the boundary condition $u|_{\partial G} = 0$; $f \in L^2(G)$. The fictitious domain method reduces this problem to the problem set in a larger rectangular $R \supset G$:

$$\text{div} (K_\omega(x) \nabla u_\omega) = F(x), \quad x \in R,$$
$$u_\omega|_{\partial R} = 0$$

where

$$K_\omega(x) = \begin{cases} 1, & x \in G, \\ \omega, & x \in R \setminus G, \end{cases} \quad F(x) = \begin{cases} f(x), & x \in G, \\ 0, & x \in R \setminus G. \end{cases}$$

For smooth $\partial G$, one can prove that, as $\omega \to +\infty$

$$u_\omega \to \begin{cases} u(x), & x \in G, \\ 0, & x \in R \setminus G, \end{cases} \text{ in } H^1(R).$$

So the field $u_\omega$ vanishes in the fictitious part of the domain. The same effect holds for the Stokes equation, where the varying viscosity can be used to modelling the absence of flow in the fictitious part of domain occupied by the clot.
3 The Stokes equation

We will consider separately the problem for the Stokes equation and the convection-diffusion one. First we apply the results of [3] and get the asymptotic expansion of the solution. At the second stage we assume that the velocity \( u_\varepsilon \) is known and consider the convection-diffusion equation with the velocity coefficient corresponding to the first term of the asymptotic approximation.

The asymptotic solution of the Stokes problem was considered in [1]. The only difference is related to the "stenosis areas" where the boundary layers are constructed as follows.

A stenosis area can be simulated by a varying viscosity in some close neighborhood of the origin of the coordinate system. Then we can consider one channel parallel to the \( O x_1 \) axis, i.e. \( \tilde{G}_\varepsilon = (-1, 1) \times \left( -\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right) \). We will not take care of the ends of this channel because these ends are supposed to be junction points with some other channels and the construction of the bifurcation boundary layers is described in [1]. So we will try to construct a solution of problem [1] stabilizing to a Poiseuille solution as \( \varepsilon \rightarrow +\infty \).

Let equations [1] be considered in this channel \( \tilde{G}_\varepsilon \) and let the viscosity coefficient \( \mu_\varepsilon \) have a structure

\[
\mu_\varepsilon (x) = \mu + M \left( \frac{x}{\varepsilon} \right)
\]

where \( \mu > 0 \) is constant and \( M \) has a support inside the ball \( B(0, 2) = \{\xi \in \mathbb{R}^2 : \xi_1^2 + \xi_2^2 < 4\} \) such that \( M(\xi) \geq 0 \).

Then if the right-hand side \( f_\varepsilon \) is equal to zero, the asymptotic solution out of the boundary layer zone (at some finite distance from 0) is a Poiseuille flow:

\[
\pi_p = c_1 \left( \frac{1}{2\pi} \left( \frac{x^2 - \varepsilon^2}{0} \right) \right), \quad \pi(x) = c_1 x_1 + c_2,
\]

where \( c_1, c_2 \) are some constants.

Then the boundary layer corrector has a form \((\varepsilon^2 U(\xi), \varepsilon P(\xi))\) and \((U, P)\) is a solution of the following problem

\[
\text{div}_\xi \left( (\mu + M \left( \frac{x}{\varepsilon} \right)) \left( \nabla_\xi U + (\nabla_\xi U)^T \right) - P(\xi) I \right) = -c_1 \text{div}_\xi \left( M(\xi) \begin{pmatrix} 0 & \frac{1}{\mu} \\ \frac{1}{\mu} & 0 \end{pmatrix} \xi_2 \right),
\]

\[
\text{div}_\xi U = 0, \quad \xi \in (-\infty, +\infty) \times \left( -\frac{1}{2}, \frac{1}{2} \right),
\]

\[
U|_{\xi_2=\pm \frac{1}{2}} = 0,
\]

where the right hand side

\[
-c_1 \text{div}_\xi \left( M(\xi) \begin{pmatrix} 0 & \frac{1}{\mu} \\ \frac{1}{\mu} & 0 \end{pmatrix} \xi_2 \right)
\]

has a support inside \( B(0, 2) \). It is well known ([3, 5]) that this problem has a unique solution \((U, P)\) stabilizing to \((0, \text{const})\) at the infinity. Then \((\pi_p + \varepsilon^2 U \left( \frac{x}{\varepsilon} \right), P(x) + \varepsilon P \left( \frac{x}{\varepsilon} \right))\) satisfies equation [1] and flow coincides with the Poiseuille flow at a finite distance from zero with an exponentially small error \( O(\varepsilon^{-\alpha}) \), \( \alpha > 0 \).

If \( P(\xi) \rightarrow 0 \) for \( \xi_1 \rightarrow -\infty \) and \( P(\xi) \rightarrow c_+ \) for \( \xi_1 \rightarrow +\infty \), then we have to proceed the following special gluing of pressure at zero:

1) we redefine a new (discontinuous)

\[
\overline{\pi}(x) = \begin{cases} 
 c_1 x_1 + c_2, & \text{for } x_1 < 0, \\
 c_1 x_1 + c_2 + \varepsilon c_+, & \text{for } x_1 > 0;
\end{cases}
\]

2) we redefine a new (discontinuous) (denoted by the same letter):

\[
P(\xi) := \begin{cases} 
 P(\xi), & \text{for } \xi_1 < 0, \\
 P(\xi) - c_+, & \text{for } \xi_1 > 0.
\end{cases}
\]
Mention that the sum \( \overline{p}(x) + \varepsilon P\left(\frac{\xi}{\varepsilon}\right) \) is still smooth and this new \( P(\xi) \to 0 \) for \( \xi_1 \to \pm \infty \), i.e. it has a standard boundary layer shape. The justification of this expansion follows laterally [1].

4 The convection-diffusion equation

Consider the diffusion-convection problem (2), (5) and (7) in a tube structure \( B_\varepsilon \), where the coefficient \( K_\varepsilon \) is given by formula (3) and \( u_\varepsilon \) in (2) is replaced by the given vector-valued function \( V_\varepsilon \) having the following structure:

\[
V_\varepsilon(x) = \begin{cases} 
V_i\left(\frac{x-O_i}{\varepsilon}\right) & \text{for } x \in B_\varepsilon : \left|\frac{x-O_i}{\varepsilon}\right| < 2, \ i = 0, \ldots, n, \\
V_s\left(\frac{x}{\varepsilon}\right) & \text{for } x \in B_\varepsilon : \left|\frac{x}{\varepsilon}\right| < 2, \\
V_p\left(\frac{x}{\varepsilon}\right) & \text{for } x \in B_\varepsilon : x_2^s \in e_j, \left|\frac{x-O}{\varepsilon}\right| \geq 2, \left|\frac{x}{\varepsilon}\right| \geq 2, \text{ for all } i \text{ and } s,
\end{cases}
\]

where \( V_p(\xi^2) = -\left((\xi^2 - \frac{1}{4})\right) \) and \( V_s(\xi) \) and \( V_i(\xi) \) are some given smooth vector-valued functions with finite support in the ball \( B(0,2) \), and \( \overline{\tau}_s \) are the "stenosis nodes" and \( O_i \) are the ends of the segments \( \varepsilon i \).

Really, the structure of the velocity field \( u_\varepsilon \) is more complicated: out of the balls of radius \( 2\varepsilon \) surrounding the nodes \( O_i \) and the stenosis points \( \overline{\tau}_s \) the velocity differs from the Poiseuille flow by some exponentially decaying boundary layer functions. Here we simplify the structure of the velocity field replacing \( u_\varepsilon \) by \( V_\varepsilon \). So we consider here the problem:

\[
- \text{div} (K_\varepsilon(x) \nabla c_\varepsilon) + V_\varepsilon(x) \cdot \nabla c_\varepsilon = g(x_1^i)
\]

with the boundary conditions

\[
K_\varepsilon(x) \frac{\partial c_\varepsilon}{\partial n} = \varepsilon \beta c_\varepsilon \quad \text{on the lateral boundary } \partial B_\varepsilon \setminus \left( \bigcup_{t=1}^r \Gamma_t \right),
\]

\[
c_\varepsilon = q_t = \text{const} \quad \text{on } \Gamma_t, \ t = 1, \ldots, r.
\]

We assume that \( \beta \) is a constant and \( \varepsilon \leq 0 \). It means that the sorption takes place: the outflow \( -\varepsilon \frac{\partial c_{\varepsilon}}{\partial n} \) is positive. It is well known that if at least one of given concentrations \( c_s \) is equal to zero, then the Poincaré-Friedrichs inequality for \( c_s \) holds with a constant independent of \( \varepsilon \). Then it could be shown that there exists a constant \( u_0 \), independent of \( \varepsilon \), such that, if \( \|u_\varepsilon\|_{L^\infty(B_\varepsilon)} \leq u_0 \) then there exists a unique solution \( c_\varepsilon \) of problem (12) - (13). It satisfies an a priori estimate \( \|c_\varepsilon\|_{H^1(B_\varepsilon)} \leq c\left(\|g\|_{L^2(B_\varepsilon)} + \|G\|_{H^1(B_\varepsilon)}\right) \), where \( c \) is independent of \( \varepsilon \) (see [3]).

4.1 The asymptotic expansion in a channel

Consider the equation

\[
- \text{div}(\varepsilon \nabla c_{\varepsilon}) + \mathbf{V}\left(\frac{x_2}{\varepsilon}\right) \cdot \nabla c_\varepsilon = g(x_1),
\]

in the infinite channel \( G_\varepsilon = (-\infty, +\infty) \times \left(-\varepsilon, \varepsilon\right) \), where \( \mathbf{V}(\xi_2) = \left(\begin{smallmatrix} V_p(\xi_2) \\ 0 \end{smallmatrix}\right), V_p(\xi_2) = -\left(\xi_2^2 - \frac{1}{4}\right) \); we consider the boundary conditions

\[
\varepsilon \frac{\partial c_\varepsilon}{\partial n} = \varepsilon \beta c_\varepsilon \quad \text{if } x_2 = \pm \frac{\varepsilon}{2}.
\]

We will construct a function

\[
c_\varepsilon^{(k)}(x_1, \xi_2) = \sum_{j=0}^{k} \varepsilon^j c_j(x_1, \xi_2)
\]
such that it satisfies (12), (13) up to the terms of order $\varepsilon^k$ if $g \in C^{k+2}(\mathbb{R})$. Substituting (14) into (12), (13) we get

$$
\sum_{j=0}^{k} \varepsilon^{j-2} \left( \varepsilon^j \frac{\partial^2 c_j}{\partial \xi_2^2} + \varepsilon^j \frac{\partial^2 c_{j-2}}{\partial x_1^2} - V_p (\xi_2) \frac{\partial c_{j-2}}{\partial x_1} \right) = g (x_1) + R^{(k)}_\varepsilon
$$

where $R^{(k)}_\varepsilon$ is a discrepancy

$$
R^{(k)}_\varepsilon = \sum_{j=k-1}^{k} \varepsilon^j \left( \varepsilon^j \frac{\partial^2 c_j}{\partial \xi_2^2} - V_p (\xi_2) \frac{\partial c_j}{\partial x_1} \right),
$$

and if $\xi_2 = \pm \frac{1}{2}$,

$$
\pm \sum_{j=0}^{k} \varepsilon^{j-1} \frac{\partial c_j}{\partial \xi_2} (x_1, \xi_2) \bigg|_{\xi_2 = \pm \frac{1}{2}} = \sum_{j=0}^{k-1} \varepsilon^{j-1} \beta c_{j-1} (x_1, \xi_2) + \varepsilon^k \beta c_k (x_1, \xi_2).
$$

Equating the terms of the same power of $\varepsilon$, we get

$$
\varepsilon \frac{\partial^2 c_j}{\partial \xi_2^2} = -\varepsilon \frac{\partial^2 c_{j-2}}{\partial x_1^2} + V_p (\xi_2) \frac{\partial c_{j-2}}{\partial x_1} + g (x_1) \delta_{j2}
$$

and for $\xi_2 = \pm \frac{1}{2}$

$$
\pm \varepsilon \frac{\partial c_j}{\partial \xi_2} \bigg|_{\xi_2 = \pm \frac{1}{2}} = \beta c_{j-2} (x_1, \xi_2).
$$

The necessary and sufficient condition of existence of $c_j$: the condition

$$
\varepsilon \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial^2 c_j}{\partial \xi_2^2} d\xi_2 = \varepsilon \frac{\partial c_j}{\partial \xi_2} \left( x_1, \frac{1}{2} \right) - \varepsilon \frac{\partial c_j}{\partial \xi_2} \left( x_1, -\frac{1}{2} \right)
$$

implies

$$
-\varepsilon \frac{\partial^2 \langle c_{j-2} \rangle}{\partial x_1^2} + \langle V_p (\xi_2) \frac{\partial c_{j-2}}{\partial x_1} \rangle + g (x_1) \delta_{j1} = \beta \left( c_{j-2} \left( x_1, \frac{1}{2} \right) + c_{j-2} \left( x_1, -\frac{1}{2} \right) \right)
$$

where $\langle \cdot \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi_2$.

Each function $c_j$ is sought as a sum $c_j (x_1, \xi_2) = \overline{c}_j (x_1) + \tilde{c}_j (x_1, \xi_2)$, $\langle \tilde{c}_j \rangle = 0$ and $\langle c_j \rangle = \overline{c}_j$. So for $\overline{c}_{j-2}$ we get the equation:

$$
-\varepsilon \frac{\partial^2 \overline{c}_{j-2}}{\partial x_1^2} + \langle V_p \rangle \frac{\partial \overline{c}_{j-2}}{\partial x_1} + g (x_1) \delta_{j2} = 2 \beta \overline{c}_{j-2} \left( x_1, \frac{1}{2} \right) + \tilde{c}_{j-2} \left( x_1, -\frac{1}{2} \right).
$$

So, we get an algorithm of the successive determination of $\overline{c}_j$, $\tilde{c}_j$:

$$
\begin{cases}
\varepsilon \frac{\partial^2 \overline{c}_0}{\partial \xi_2^2} = 0, & \xi_2 \in (-\frac{1}{2}, \frac{1}{2}) \\
\varepsilon \frac{\partial \overline{c}_0}{\partial \xi_2} \bigg|_{\xi_2 = \pm \frac{1}{2}} = 0, & \langle \overline{c}_0 \rangle = 0
\end{cases}
$$

and so, $\overline{c}_0 = 0$:

$$
-\varepsilon \frac{\partial^2 \overline{c}_0}{\partial x_1^2} + \langle V_p \rangle \frac{\partial \overline{c}_0}{\partial x_1} + g (x_1) = 2 \beta \overline{c}_0;
$$

(15)
Consider the convection-diffusion equation in the channel $G_\varepsilon$ with the modified coefficients $\varkappa$ and $V$:

For example, $c_j$ and $c_{j-2}$ are replaced by the functions $\varkappa c_j (x_1, \xi_2)$ and $V P (\xi_2) \partial c_{j-2} / \partial x_1 + g (x_1) \delta_{j1}$, $\xi_2 \in (-\frac{1}{2}, \frac{1}{2})$

\[
\begin{align*}
\varkappa \partial^2 c_j / \partial x_1^2 - \varkappa \partial^2 c_{j-2} / \partial x_1^2 + V_P (\xi_2) \partial c_{j-2} / \partial x_1 + g (x_1) \delta_{j1}, & \quad \xi_2 \in (-\frac{1}{2}, \frac{1}{2}) \\
\pm \varkappa \partial c_j / \partial \xi_2 |_{\xi_2=\pm \frac{1}{2}} &= \beta c_{j-2} (x_1, \xi_2), \quad \langle c_j \rangle = 0.
\end{align*}
\]

Remark $\tilde{c}_1 = 0$, and find $\tilde{c}_j$:

\[-\varkappa \partial^2 \tilde{c}_j / \partial x_1^2 + (V_P) \partial \tilde{c}_j / \partial x_1 + \left\langle V_P \partial \tilde{c}_j / \partial x_1 \right\rangle = 2 \beta \hat{c}_j (x_1, \frac{1}{2}) + \tilde{c}_j (x_1, -\frac{1}{2}) \]

Lemma 2 Each $\tilde{c}_j (x_1, \xi_2)$ is a polynomial function in $\xi_2$ of degree 2$j$.

**Proof.** By induction we prove that $\tilde{c}_j$ is a polynomial function of the degree 2$j$. Indeed, $V_P$ is a quadratic function and so the right hand side $[16]$ is a polynomial of order 2$(j - 2) + 2$. And so, after two integrations of $[16]$ we check that $\tilde{c}_j$ is a polynomial of order 2$(j - 2) + 2 + 2 = 2j$.

So we have constructed an asymptotic approximation $[16]$ which satisfies equation $[12]$ up to the remainder $R^{(k)}$ and conditions $[13]$ up to the remainder $\varepsilon^k \beta c_k (x_1, \xi_2) |_{\xi_2=\pm \frac{1}{2}}$.

Let us eliminate this remainder in the boundary conditions. To this end we will add a corrector $\varepsilon^{k+1} \hat{c}_{k+1} (x_1, \xi_2)$ such that

\[\pm \varkappa \partial \varepsilon^{k+1} \hat{c}_{k+1} / \partial x_2 = \varepsilon^{k+2} \beta \hat{c}_{k+1} (x_1, \frac{\xi_2}{\varepsilon}), \quad \text{for } x_2 = \pm \frac{\varepsilon}{2}, \]

i.e.

\[\pm \varkappa \partial \varepsilon^{k+1} \hat{c}_{k+1} / \partial \xi_2 = \varepsilon^{k+2} \beta \hat{c}_{k+1} (x_1, \xi_2), \quad \text{for } \xi_2 = \pm \frac{1}{2} \quad (17)\]

For example,

\[\hat{c}_{k+1} = \varkappa^{-1} \left\{ (\xi_2 - \frac{1}{2}) (\xi_2 + \frac{1}{2})^2 \beta c_k (x_1, \frac{1}{2}) - (\xi_2 - \frac{1}{2}) \beta c_k (x_1, -\frac{1}{2}) \right\}
\]

satisfies $[17]$. Then $c_{k+1}^{(k)} - \varepsilon^{k+1} \hat{c}_{k+1}$ denoted by $c_{k+1}^{(k)}$ satisfies the boundary conditions $[13]$ exactly and the equation $[12]$ up to the remainder $R^{(k)} + \hat{R}^{(k)}$, where

\[\hat{R}^{(k)} = \left\{- \text{div} (\varkappa \nabla \hat{c}_{k+1}) - V_P (\xi_2) \partial \hat{c}_{k+1} / \partial x_1 \right\} \varepsilon^{k+1}.
\]

4.2 Structural element ”stenosis area”.

Consider the convection-diffusion equation in the channel $G_\varepsilon$ with the modified coefficients $\varkappa$ and $V$:

they are replaced by the functions

\[K_\varepsilon (x) = \varkappa + K \left( \frac{x}{\varepsilon} \right) \quad \text{and} \quad V_\varepsilon (x) = \left( \frac{V_P}{\varepsilon} \right) + \nabla \left( \frac{x}{\varepsilon} \right)
\]

such that $K_\varepsilon (x) \geq \chi_1 > 0$ for all $x$, $\text{div} \nabla \left( \frac{x}{\varepsilon} \right) = 0$ and the support of the functions $K (\xi)$ and $\nabla (\xi)$ belongs to the ball $B (O, 2) = \{\xi \in \mathbb{R}^2 : |\xi| < 2\}$.

Consider the convection-diffusion equation

\[- \text{div} (K_\varepsilon (x) \nabla \varepsilon c_\varepsilon) + V_\varepsilon \left( \frac{x}{\varepsilon} \right) \cdot \nabla \varepsilon c_\varepsilon = g_\varepsilon (x_1), \quad \text{in } G_\varepsilon
\]

(18)
with the boundary conditions

\[ K_\varepsilon (x) \frac{\partial c_\varepsilon}{\partial n} = \varepsilon \beta c_\varepsilon, \quad x_2 = \pm \frac{\varepsilon}{2}. \]  

(19)

Let us construct a function \( c_{\varepsilon}^{(k)} \) satisfying equation (18) and conditions (19) up to remainder of order \( \varepsilon^k \). To this end we will consider the asymptotic approximation \( c_{\varepsilon}^{(k)} \) of the previous section completed by the special boundary layer corrector \( c_{\varepsilon}^{(k)_{sbl}} \), having the form

\[ c_{\varepsilon}^{(k)_{sbl}} = \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right) \]

such that

\[ |U_j (\xi)| \leq c_1 e^{-c_2|\xi|}. \]

Denote

\[ L_\varepsilon = \text{div} (K_\varepsilon \nabla) - \mathbf{V}_\varepsilon \cdot \nabla. \]

Define

\[ \rho (t) = \begin{cases} 1 & \text{for } |t| > 2; \\ 0 & \text{for } |t| < 1; \end{cases} \]

such that \( \rho \) and \( \rho' \) are bounded. Consider the following asymptotic approximation

\[ c_{\varepsilon}^{(k)} = \tilde{c}_{\varepsilon}^{(k)} \rho \left( \frac{x_1}{\varepsilon} \right) + \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right) . \]

Then

\[ L_\varepsilon \tilde{c}_{\varepsilon}^{(k)} - g (x_1) = -L_\varepsilon \left( \tilde{c}_{\varepsilon}^{(k)} \left( 1 - \rho \left( \frac{x_1}{\varepsilon} \right) \right) \right) - g + \left( 1 - \rho \left( \frac{x_1}{\varepsilon} \right) \right) \left( L_\varepsilon \tilde{c}_{\varepsilon}^{(k)} - g \right) + \\
\hspace{1cm} + L_\varepsilon \left( \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right) \right) + \overline{R}_k, \]

where \( \overline{R}_k = \rho \left( \frac{x_1}{\varepsilon} \right) \left( L_\varepsilon \tilde{c}_{\varepsilon}^{(k)} - g \right) \) and \( |\overline{R}_k| \leq \varepsilon^k \text{const.} \)

Expand \( \tilde{c}_{\varepsilon}^{(k)} \) in powers of \( \varepsilon \xi_1 \):

\[ \tilde{c}_{\varepsilon}^{(k)} = \sum_{j=0}^{k} \varepsilon^j \sum_{l=0}^{k} \frac{\varepsilon^l}{l!} \xi_1^l \frac{\partial^l c_\varepsilon}{\partial x_1^l} (0, \xi_2) + \overline{R}_k = \sum_{r=0}^{k} \varepsilon^r \sum_{l=0}^{r} \frac{\xi_1^l}{l!} \frac{\partial^l c_{\varepsilon r} - l}{\partial x_1^l} (0, \xi_2) + \overline{R}_k. \]

Let us denote \( K (\xi) \) the function \( \varphi + \overline{K} (\xi) \) and \( \mathbf{V} (\xi) \) the function \( \left( \frac{\varphi}{\xi} \right) + \overline{\mathbf{V}} (\xi) \). We have

\[ L_\varepsilon \left( \tilde{c}_{\varepsilon}^{(k)} \left( 1 - \rho \left( \frac{x_1}{\varepsilon} \right) \right) \right) = \sum_{j=0}^{k} \varepsilon^{j-2} \text{div}_\varepsilon \left( K (\xi) \nabla_\xi \left( \sum_{l=0}^{j} \frac{\xi_1^l}{l!} \frac{\partial^l c_{\varepsilon j} - l}{\partial x_1^l} (0, \xi_2) \times (1 - \rho (\xi_1)) \right) \right) - \\
- \sum_{j=0}^{k} \varepsilon^{j-1} \mathbf{V} (\xi) \cdot \nabla_\xi \left( \sum_{l=0}^{j} \frac{\xi_1^l}{l!} \frac{\partial^l c_{\varepsilon j - l}}{\partial x_1^l} (0, \xi_2) \times (1 - \rho (\xi_1)) \right). \]

So, for \( U_j \) we get equation

\[ \text{div}_\varepsilon \left( K (\xi) \nabla_\xi U_j (\xi) \right) - \mathbf{V} (\xi) \cdot \nabla_\xi U_{j-1} (\xi) + T_j (\xi) = 0, \quad \xi_1 \in \mathbb{R}, \quad \xi_2 \in \left( -\frac{1}{2}, \frac{1}{2} \right), \]

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where

\[ T_j (\xi) = \text{div}_\xi \left( K (\xi) \nabla_\xi \left( \sum_{l=0}^{j} \frac{\xi^l}{l!} \frac{\partial^l c_j}{\partial x_1^l} (0, \xi_2) \times (1 - \rho (\xi_1)) \right) \right) - \]

\[ - V (\xi) \cdot \nabla_\xi \left( \sum_{l=0}^{j-1} \frac{\xi^l}{l!} \frac{\partial^l c_j}{\partial x_1^l} (0, \xi_2) \times (1 - \rho (\xi_1)) \right). \]

For the boundary conditions, in the same way:

\[ \pm \sum_{j=0}^{k} \varepsilon^j K (\xi) \frac{\partial U_j}{\partial \xi_2} \pm \sum_{j=0}^{k} \varepsilon^j K (\xi) \frac{\partial c_j}{\partial \xi_2} (\rho (\xi_1) - 1) = \varepsilon \beta \left( \sum_{j=0}^{k} \varepsilon^j U_j + \sum_{j=0}^{k} \varepsilon^j c_j (\rho (\xi_1) - 1) \right), \quad \xi_2 = \pm \frac{1}{2} \]

and so

\[ \pm K (\xi) \frac{\partial U_j}{\partial \xi_2} = S_j (\xi), \quad \xi_2 = \pm \frac{1}{2} \]

where

\[ S_j (\xi) = - \left\{ \pm \sum_{l=0}^{j} K (\xi) \frac{\xi^l}{l!} \frac{\partial^{l+1} c_j}{\partial \xi^l_2 \partial x_1^l} (0, \xi_2) (1 - \rho (\xi_1)) - \beta \sum_{l=0}^{j-2} \frac{\xi^l}{l!} \frac{\partial^l c_j}{\partial x_1^l} (0, \xi_2) (1 - \rho (\xi_1)) - \beta U_{j-2} \right\}. \]

Necessary and sufficient condition of existence of a bounded solution \( U_j \):

\[ \{ T_j (\xi) - V (\xi) \cdot \nabla_\xi U_{j-1} \} = \{ S_j \}_+ + \{ S_j \}_- \]

where

\[ \{ \cdot \} = \int_{(-\infty, 0) \times (-\frac{1}{2}, \frac{1}{2})} \cdot d\xi + \int_{(0, +\infty) \times (-\frac{1}{2}, \frac{1}{2})} \cdot d\xi \]

and

\[ \{ \cdot \}_\pm = \int_{(-\infty, 0)} \mid_{\xi_2 = \pm \frac{1}{2}} d\xi_1 + \int_{(0, +\infty)} \mid_{\xi_2 = \pm \frac{1}{2}} d\xi_1. \]

This condition gives one interface condition for \( \frac{\partial \sigma_{j-1}}{\partial x_1} (0^+) \) and \( \frac{\partial \sigma_{j-1}}{\partial x_1} (0^-) \):\n
\[ \{ T_j \} = \int_{-\frac{1}{2}}^{\frac{1}{2}} K (\xi) \, d\xi_2 \frac{\partial \sigma_{j-1}}{\partial x_1} (0^-) - \int_{-\frac{1}{2}}^{\frac{1}{2}} K (\xi) \, d\xi_2 \frac{\partial \sigma_{j-1}}{\partial x_1} (0^+) + \]

\[ + \int_{-\frac{1}{2}}^{\frac{1}{2}} K (\xi) \, d\xi_2 \frac{\partial \sigma_{j-1}}{\partial x_1} (0^-, \xi_2) d\xi_2 - \int_{-\frac{1}{2}}^{\frac{1}{2}} K (\xi) \, d\xi_2 \frac{\partial \sigma_{j-1}}{\partial x_1} (0^+, \xi_2) d\xi_2 + \]

\[ + \text{all other terms of } T_j \text{ except } l = 1 \text{ for } \text{div}. \]

So, if we pose \( Q_j = \int_{-\frac{1}{2}}^{\frac{1}{2}} K (\xi) \, d\xi_2 \frac{\partial \sigma_{j-1}}{\partial x_1} (0^-, \xi_2) d\xi_2 - \int_{-\frac{1}{2}}^{\frac{1}{2}} K (\xi) \, d\xi_2 \frac{\partial \sigma_{j-1}}{\partial x_1} (0^+, \xi_2) d\xi_2 \), we get condition:

\[ - \int_{-\frac{1}{2}}^{\frac{1}{2}} K (0, \xi_2) \, d\xi_2 \left[ \frac{\partial \sigma_{j-1}}{\partial x_1} \right] = g_j \]

(20)

where

\[ g_j = - \{ T_j (\xi) - V (\xi) \cdot \nabla_\xi U_{j-1} \} - Q_j + \{ S_j \}_+ + \{ S_j \}_- , \]

where in \( T_j (\xi) \) there are not the term corresponding to \( l = 1 \) for \( \text{div} \).
Now we solve the problem on $U_j$ exponentially stabilizing to some constants. We choose this constant equal to $0$ at $-\infty$. At $+\infty$ we have $\tilde{U}_j \to \tilde{q}_j$. To make $U_j \to 0$ as $\xi_1 \to +\infty$, we subtract this constant for all $\xi_1 > 0$; we set

$$U_j = \begin{cases} 
\tilde{U}_j & \text{for } \xi_1 < 0 \\
\tilde{U}_j - \tilde{q}_j & \text{for } \xi_1 > 0.
\end{cases}$$

This function $U_j$ is exponentially decaying at $\infty$. Then we put the compensating condition for $\tau_j$ at $x_1 = 0 : [\tau_j] = \tilde{q}_j$.

Another interface condition for $\tau_j$ is (20).

### 4.3 The boundary layer in a "bifurcation area".

Consider the convection-diffusion equation in a one bundle structure with the common point $O$ (i.e. the origin of the coordinate system). We will construct an asymptotic expansion in the neighborhood of the point $O$ (that will be good in all the channels $B^ij$). First we construct a regular expansion as in section 4.1 for every branch $B^ij$ making the local change of variables by rotation in such a way that new $x_1$ axis (denoted $x^ej$) contains the segment $e_i$. Let us denote the approximation $\hat{c}^{(k)e_i}_{cr}$ for $e_i$ as

$$\hat{c}^{(k)e_i}_{cr} (x) = \sum_{j=0}^{k} \varepsilon^j C_j \left( x^ej, x^ej \theta_j \varepsilon \right) - \varepsilon^{k+1} \hat{c}_{k+1}^{(k)e_i} \left( x^ej, x^ej \theta_j \varepsilon \right).$$

Here $(x^ej, x^ej)$ are new local variables related to the segment $e_i$, and $\theta_j \varepsilon$ is the thickness of the channel $B^ij$. Then, as in section 4.2, we construct the boundary layer corrector

$$c_{eb}^{(k)} = \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right)$$

such that

$$|U_j (\xi)| \leq \overline{c}_1 e^{-\overline{c}_2 |\xi|}, \quad \overline{c}_1, \overline{c}_2 > 0.$$

Consider an asymptotic approximation

$$c_{eb}^{(k)} = \sum_{i=1}^{n} \hat{c}_{cr}^{(k)} \left( x^ei, x^ei \theta_i \varepsilon \right) \chi_i (x) \rho \left( \frac{\alpha x^ei}{\varepsilon} \right) + \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right), \quad (21)$$

where $\chi_i$ is a characteristic function of $B^ij$ ($\chi_i (x) = 1$ if $x \in B^ij$, and $0$ if not); $\alpha$ is a number such that each point of the circle of a radius $\alpha \varepsilon$ with center $O$ belongs to at most one rectangle $B^ij$ $(i = 1, ..., n)$ and it does not intersect the domain $\gamma^O \overline{c}_1, \overline{c}_2 > 0$.

![Fig. 3](image-url)
Assume that the right hand side is equal to zero in some neighborhood of \( O \). Then for every \( i = 1, ..., n \)
for \( x \in B_i^\varepsilon \)
\[
L_\varepsilon c_{ib}^{(k)c_i} - g(x_i^\varepsilon) = -L_\varepsilon \left( \frac{\alpha x_i^\varepsilon}{\varepsilon} \right) + \left( 1 - \rho \left( \frac{\alpha x_i^\varepsilon}{\varepsilon} \right) \right) \left( L_\varepsilon c_{ib}^{(k)c_i} - g \right) + L_\varepsilon \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right) + \overline{R}_k,
\]
where \( \overline{R}_k = \rho \left( \frac{\alpha x_i^\varepsilon}{\varepsilon} \right) \left( L_\varepsilon c_{ib}^{(k)c_i} - g \right) \). Inside the circle \( B(0; \alpha \varepsilon) \) we get: \( g = 0, \rho = 0 \) and so
\[
L_\varepsilon c_{ib}^{(k)c_i} - g = L_\varepsilon \left( \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right) \right).
\]
Here the terms \( \left( 1 - \rho \left( \frac{\alpha x_i^\varepsilon}{\varepsilon} \right) \right) \left( L_\varepsilon c_{ib}^{(k)c_i} - g \right) \) and \( \rho \left( \frac{\alpha x_i^\varepsilon}{\varepsilon} \right) \left( L_\varepsilon c_{ib}^{(k)c_i} - g \right) \) are bounded by \( \varepsilon^k \text{const.} \)
So we will define \( U_j \) in such a way that out of the circle \( B(0; \alpha \varepsilon) \) we have
\[
L_\varepsilon \left( \sum_{j=0}^{k} \varepsilon^j U_j \left( \frac{x}{\varepsilon} \right) \right) - \sum_{i=1}^{n} L_\varepsilon \left( e_{ib}^{(k)c_i} \chi_i \left( 1 - \rho \left( \frac{\alpha x_i^\varepsilon}{\varepsilon} \right) \right) \right) = 0.
\]
Expand each \( e_{ib}^{(k)c_i} \) in powers of \( \varepsilon x_i^\varepsilon \):
\[
e_{ib}^{(k)c_i} = \sum_{j=0}^{k} \varepsilon^j \sum_{l=0}^{j} \frac{\varepsilon^j}{l!} (\xi_i^\varepsilon)^l \frac{\partial c_{ib}^{(k)c_i}}{\partial (x_i^\varepsilon)} (0, \xi_2^\varepsilon) + \overline{R}_k = \sum_{r=0}^{k} \varepsilon^r \sum_{l=0}^{r} \frac{\xi_i^\varepsilon}{l!} \frac{\partial c_{ib}^{(k)c_i}}{\partial (x_i^\varepsilon)} (0, \xi_2^\varepsilon) + \overline{R}_k, \xi_2^\varepsilon = \frac{x_i^\varepsilon}{\varepsilon}.
\]
Consider domain \( \Omega_0 = \gamma_0 \cup \bigcup_{i=1}^{n} \Omega_i \), where \( \Omega_i \) is the half-strip \( \{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^\varepsilon > 0, |\xi_2^\varepsilon| < \frac{\theta_i}{2} \} \). So for \( U_j \) we get equation:
\[
\kappa \Delta U_j (\xi) - \nabla \xi U_j - (\xi) + T_j (\xi) = 0, \xi \in \Omega_0,
\]
where
\[
T_j (\xi) = \begin{cases} T_j^{(k)} (\xi) & \text{if } \xi \in \Omega_i \text{ out of the circle } B(0, \alpha), \\ 0 & \text{inside the circle } B(0, \alpha), \end{cases}
\]
\[
T_j^{(k)} (\xi) = \kappa \Delta \xi \left( \sum_{i=0}^{j} \frac{(\xi_i^\varepsilon)^l}{l!} \frac{\partial c_{ib}^{(k)c_i}}{\partial (x_i^\varepsilon)} (0, \xi_2^\varepsilon) \times (1 - \rho (\alpha \xi_i^\varepsilon)) \right) - \nabla \xi \left( \sum_{i=0}^{j} \frac{(\xi_i^\varepsilon)^l}{l!} \frac{\partial c_{ib}^{(k)c_i}}{\partial (x_i^\varepsilon)} (0, \xi_2^\varepsilon) \right).
\]
For the boundary conditions on $\partial\Omega_0$:

$$
\pm \sum_{j=0}^{k} \varepsilon^{j-1} \frac{\partial U_j}{\partial \xi_2} \pm \sum_{j=0}^{k} \varepsilon^{j-1} \frac{\partial e_j^\xi}{\partial \xi_2} (\alpha \xi_1^\xi) (\rho (\alpha \xi_1^\xi) - 1) = \\
= \varepsilon \beta \sum_{j=0}^{k} \varepsilon^{j} U_j + \sum_{j=0}^{k} \varepsilon^{j} e_j^\xi (\rho (\alpha \xi_1^\xi) - 1), \text{ out of the circle } B(0, \alpha)
$$

and

$$
\pm \sum_{j=0}^{k} \varepsilon^{j-1} \frac{\partial U_j}{\partial \xi} = \varepsilon \beta \sum_{j=0}^{k} \varepsilon^{j} U_j, \text{ inside the circle } B(0, \alpha)
$$

up to the terms of order $\varepsilon^k$. This gives

$$
\frac{\partial U_j}{\partial \xi} = S_j(\xi), \quad \xi \in \partial\Omega_0,
$$

where

$$
S_j(\xi) = \\
\left\{ \begin{array}{ll}
- \left( \pm \sum_{l=0}^{j} \alpha^l \xi_1^l \partial^{l+1} e_1^{-1} \frac{\partial}{\partial \xi_2} (0, \xi_1^\xi) (1 - \rho (\alpha \xi_1^\xi)) - \\
- \beta \sum_{l=0}^{j} \alpha^l \xi_1^l \partial^{l+1} e_1^{-1} \frac{\partial}{\partial (x_1^\xi)} (0, \xi_1^\xi) (1 - \rho (\alpha \xi_1^\xi)) - \beta U_{j-2}, & \text{ out of the circle } B(0, \alpha) \text{ on } \partial\Omega_i, \\
+ \beta U_{j-2}, & \text{ inside the circle } B(0, \alpha) \text{ on } \partial\Omega_0
\end{array} \right.
$$

Necessary and sufficient condition of existence of a bounded solution $U_j$:

$$
\{ T_j(\xi) - V(\xi) \cdot \nabla q U_{j-1} \}_{\Omega_0} = \{ S_j \}_{\partial \Omega_0},
$$

where

$$
\{ \cdot \}_{\Omega_0} = \int_{\Omega_0} \cdot d\xi \quad \text{and} \quad \{ \cdot \}_{\partial \Omega_0} = \int_{\partial \Omega_0} \cdot ds.
$$

This condition gives one interface condition for $\frac{\partial \bar{\tau}^\xi}{\partial x_1^\xi} (0^\xi)$, the limit value of the derivative $\frac{\partial \bar{\tau}^\xi}{\partial x_1^\xi}$ on $e_i$ in the origin of $e_i$:

$$
\{ T_j \}_{\Omega_0} = - \sum_{i=1}^{n} \alpha^l \theta \frac{\partial \bar{\tau}^\xi_{j-1}}{\partial x_1^\xi} (0^\xi) - \sum_{i=1}^{n} \alpha^l \theta \frac{\partial \bar{\tau}^\xi_{j-1}}{\partial x_1^\xi} (0^\xi, \xi_1^\xi) + \text{all other terms of } T_j \text{ except } l = 1 \text{ for } \Delta^\xi.
$$

So, we get Kirchoff type condition:

$$
- \sum_{i=1}^{n} \theta \frac{\partial \bar{\tau}^\xi_{j-1}}{\partial x_1^\xi} (0^\xi) = g_j
$$

where $g_j$ depends on $\tau_0, ..., \tau_{j-2}$.

Now we solve the problem on $U_j$ in $\Omega_0$; it stabilizes exponentially to some constants $q_{ji}$ at every branch $\Omega_i$. This solution $U_j$ corresponds to some values of $\tau_j^\xi (0^\xi)$ that enter in the expression for $T_j^\xi$ (when $l = 0$). If we change these values adding $q_{ji}$ then the solution $U_j$ will be transformed into $U_j - \rho (\alpha \xi_1^\xi) q_{ji}$ which tend to zero as $|\xi| \to +\infty$.

It means that for every $e_i$, Kirchoff condition for $\frac{\partial \bar{\tau}^\xi_{j-1}}{\partial x_1^\xi}$ in 0 and continuity condition for $\tau_j^\xi$ in 0, solve the problem for $U_j$ and then modify the values $\bar{\tau}_j^\xi (0^\xi)$ in such a way that $U_j \to 0$ as $|\xi| \to +\infty$. 

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4.4 The entrance/exit element

In this case the boundary layer is constructed in the following way. It satisfies equation (12) boundary conditions everywhere except the part \( \Gamma_s \), where \( U_j = q_j \delta_{oj} \). There is no more necessary and sufficient condition for the existence of a bounded solution. This bounded solution exponentially stabilizes to some constant. Then we redefine the value \( \tau_j (x_{bs}) \) in such a way that \( U \to 0 \) as \( \xi^i_j \to + \infty \). In particular, \( \tau_0 (x_{bs}) = q_s \) and \( U_0 = (1 - \rho) q_s \).

4.5 Algorithm of assembling of an asymptotic solution

Let us describe more precisely the algorithm of the assembling of the different structural elements in the construction of the asymptotic expansion.

For \( \tau_0 \) we get from the construction of the subsection 4.1 the 1D convection diffusion equation

\[
x \theta_i \frac{\partial^2 \tau_0}{\partial (x_1^e)^2} - \langle V_p \rangle_{t_i} \frac{\partial \tau_0}{\partial x_1^e} + \theta_i g (x_1) = 2 \beta \tau_0^{e_i}, \quad \text{for every } e_i, \tag{22}
\]

at every stenosis point \( x_s \) we get the interface conditions

\[
\left[ x \frac{\partial \tau_0^{e_i}}{\partial x_1} \right] = 0 \quad \text{and} \quad \left[ \tau_0^{e_i} \right] = 0. \tag{23}
\]

At every bifurcation point \( x_b \) that is an end point of segments \( e_1, ..., e_n \) we get

\[
x \sum_{i=1}^{n} \theta_i \frac{\partial \tau_0^{e_i}}{\partial x_1^e} (x_b) = 0 \tag{24}
\]

and \( \tau_0^{e_i} (x_b) = \tau_0^{e_i} (x_b) \; \forall \; i, j \in \{1, ..., n\} \) (continuity).

At every entrance/exit point \( x_t \) we set \( \tau_0^{e_i} (x_t) = q_t, \; t = 1, ..., r \), where \( x_t \) is an end-point corresponding to \( \Gamma_t \).

Then for every \( j \) we get analogous problem for \( \tau_j \) with some right hand sides depending on previous approximations \( \tau_0, ..., \tau_{j-1} \). We solve the problem for \( \tau_j \) and then for \( U_j \). We define the interface values \( \tau_j^{e_i} (x_b) \) for all segments \( e_i \) having \( x_b \) as an end point in such a way that \( U_j \to 0 \) for \( | \xi | \to + \infty \).

Remark 3 Every function \( \tau_0^{e_i} \) related to \( e_i \) depends on the local variable \( x_1^{e_i} \) such that \( x_1^{e_i} = 0 \) in one of the end points of \( e_i \). Therefore we get two local variables \( x_1^{e_i}, \; \text{"starting" from each of two ends of the segment } e_i; \) therefore, if we denote \( x_1^{e_i+} \) and \( x_1^{e_i-} \) these two different local variables, we get the relation for these variables \( x_1^{e_i+} = |e_i| - x_1^{e_i-} \). Let us give more details. Every segment \( e_i \) having two end points \( O_{i1} \) and \( O_{i2} \) can be associated to two possible local coordinate systems: one of them (denoted by \( Ox^{e_i} \)) has its origin in \( O_{i1} \) and another in \( O_{i2} \) (denoted by \( Ox^{e_i+} \)). Therefore for every differential equation (22), or for every junction condition this system should be chosen and fixed. The evident change variable relation for any function \( f \) defined on the segment \( e_i \) is:

\[
f (x_1^{e_i}) = f (|e_i| - x_1^{e_i}),
\]

where \( |e_i| \) is the length of the segment, in the left side we use the first variable of the system \( Ox^{e_i} \) and in the right side – the first variable of the system \( Ox^{e_i+} \). Consequently, for \( x = O_{i1} \), we have

\[
\frac{\partial f}{\partial x_1^{e_i}} (0) = - \frac{\partial f}{\partial x_1^{e_i+}} (|e_i|).
\]
4.6 Justification: draft.

Substituting the asymptotic solution of a form \( y(x) \) we satisfy the convection-diffusion equation with a discrepancy \( O(\varepsilon^k) \) in \( L^2 \) norm. The boundary conditions are satisfied with the same accuracy. Then applying the Poincaré-Friedrichs inequality for rod structures (see [1]) and the a priori estimate derived from the variational formulation, we get that

\[
\left\| c_\varepsilon - c_\varepsilon^{(k)} \right\|_{H^1(B_\varepsilon)} \leq C \varepsilon^{k/2},
\]

where \( C \) does not depend on \( \varepsilon \).

In particular the leading term \( c_0 \) satisfies the following estimate

\[
\| c_\varepsilon - c_0 \|_{L^2(B_\varepsilon)} \leq C \varepsilon.
\]

5 The partial asymptotic domain decomposition

The constructed above asymptotic expansions of the solutions of the Stokes problem (1), (4), (6) and the diffusion-convection-sorption problem (2), (5), (7) allow us to apply the idea of the partial asymptotic domain decomposition [7, 10]. We will cut off the two-dimensional subdomains of \( B_\varepsilon \) containing the "stenosis areas", bifurcations and eventually (for the Stokes problem) the entrance and exit elements by the lines orthogonal to the rectangles \( B_{\varepsilon j} \) at the distance \( \delta = K \varepsilon |\ln \varepsilon| \) from the nodes (bifurcation points) and from the nodal points of the stenosis areas. We will call these subdomains the 2D zoom zones. Here \( K \) is independent of \( \varepsilon \) and will be defined later.

Then we pass to the 1D description out of these subdomains. It means that we pass to the projection of the variational formulation of the Stokes problem on the Sobolev subspace of vector-valued functions having the Poiseuille "parabolic" shape out of these subdomains (see [8, 9]).

For the diffusion-convection-sorption problem we apply the projection on the Sobolev space of functions having vanishing derivatives of order greater than \( 2k \) in the direction orthogonal to \( B_{\varepsilon j} \) (also out of these 2D zoom zones). This choice of the projection space is motivated by Lemma 2 and formula (14). This gives us a variational formulation of the partially decomposed diffusion-convection problem and according to the general theory of the error estimate for the method of partial asymptotic domain decomposition ([9, 10]) we get the estimates:

- for the difference of \( u_\varepsilon \) and \( u_{\varepsilon,\delta}^{dec} \) (solution of the partially decomposed problem for Stokes equation) we get as in [3] that for any \( N \) there exists \( K \) independent of \( \varepsilon \) such that, if \( \delta = K \varepsilon |\ln \varepsilon| \) then

\[
\| u_\varepsilon - u_{\varepsilon,\delta}^{dec} \|_{H^1(B_\varepsilon)} \leq O(\varepsilon^N) \sqrt{\varepsilon};
\]

- for the difference of \( c_\varepsilon \) and \( c_{\varepsilon,\delta}^{dec} \) (solution of the partially decomposed problem for the diffusion-convection equation) we get that there exists \( K \) independent of \( \varepsilon \) such that, if \( \delta = K \varepsilon |\ln \varepsilon| \) then

\[
\| c_\varepsilon - c_{\varepsilon,\delta}^{dec} \|_{H^1(B_\varepsilon)} \leq O(\varepsilon^k) \sqrt{\varepsilon}.
\]

(25)

These estimates justify the application of the MAPDD.

Remark 4 The interface conditions between the 2D parts of the domain and the 1D parts follows from the variational formulation for the partially decomposed problem by integrating by parts.

Remark 5 Although for \( k = 0 \) estimate (25) is not too precise, we will hold below a numerical experiment comparing the difference between the exact solution and the solution of partially decomposed problem in this simplest case, when the projection space consists of functions with vanishing first transversal derivative out of the 2D zoom zones.
6 Numerical experiments

Here we will compare the solution of the leading term 1D equation (22)-(24) to the numerical FEM solution of the coupled 2D flow-diffusion problem (1), (2), (5), (6), (7) in a thin rectangle $(0, 1) \times (0, \varepsilon)$. We will trace the boundary layer zones.

![Fig. 4: Thin rectangle ("straight channel geometry")](image)

In the second part of this section we will discuss the numerical solution attained by the MAPDD for a one bundle tube structure corresponding to three segments $\epsilon_1, \epsilon_2, \epsilon_3$.

![Fig. 5: One bundle tube structure](image)

The finite element discretization of the Stokes equations (1) is based on the classical P2/P1 lagrangian finite element test functions in combination with the triangular finite element mesh. More precisely, the velocity field is approximated by quadratic lagrangian test functions while the pressure field is approximated with linear lagrangian test functions. As it is well known, this finite element flow formulation satisfies the classical Babuska-Brezzi [11] compatibility condition and consequently produces numerically stable and adequate solution strategy for Stokes and Navier-Stokes problems. The concentration field $c$ is approximated by quadratic lagrangian test functions. Finally, we would like to mention that for the studied flow conditions, no specific divergence problems were encountered.

6.1 Straight channel geometry

Before analyzing the case of a 2D bifurcation problem, we will present some results concerning the simple straight channel rectangle geometry $B_\varepsilon = (0, \varepsilon) \times (0, 1)$. The main goal is to compare the predictions of our two methods (complete 2D and asymptotic 1D model) in this simple flow conditions: $\varepsilon$ is taken equal to 0.05, viscosity $\mu = 1$. 

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In this case, the velocity distribution is described by a planar Poiseuille flow given by formula (8). The entry concentration is fixed as $q_0 = 1$, while the exit concentration is maintained to be $q_1 = 0.5$. We calculate the solutions for $\beta = 0.4$ and three different values of the diffusion coefficient $\kappa$. The comparison of 2D solution and the 1D asymptotic solution is presented at Fig. 6.

These variations of the diffusion coefficient are taken in order to find the limits of the asymptotic approximation. This asymptotic analysis was applied under the hypothesis of absence of other small parameters in the model. Indeed, when the diffusion coefficient becomes a second small parameter then the asymptotic analysis taking into account only one small parameter may be not too precise.

Fig. 6: Distribution of the velocity field and distribution of the concentration for values

(a) $\kappa = 0.01$, (b) $\kappa = 0.1$, (c) $\kappa = 1$.

Steady state concentration distribution: color map shows the concentration distribution; the arrows show that the total flux density for the concentration distribution (i.e., diffusive plus convective flux).

At low diffusion (Fig. 6(a) and 6(b)), the concentration distribution seems to be essentially 2D and therefore one can expect some differences between the corresponding 1D and 2D approaches. On the other hand, when the diffusion increases, the concentration iso-levels become more and more planar and, consequently, we can expect better performance from the 1D approach. This analysis is further confirmed by the direct comparison between the predictions of the 1D and the 2D approaches presented in Fig. 7.
Concentration distribution: full line stands for the average over a cross-section of concentration distribution for the 2D geometry; dashed line and dots stand for the 1D concentration distribution.

As we can see from Fig. 7, some differences between 1D and 2D predictions exist only for the value of $\kappa = 0.01$ when the diffusion is 5 times smaller than $\varepsilon$. So the numerical experiment confirms the great precision of the asymptotic solution (even in the case of small diffusion coefficient!).

6.2 2D bifurcation geometry

In this part, we extend our study to a more complex 2D bifurcation geometries. The 2D flow geometry is presented in Fig. 8(a). As it is seen from this figure, each channel could have different thickness and could be expressed in terms of so called streamline function $\psi$ according to the following definition:

$$\frac{\partial \psi}{\partial x} = -U_y; \quad \frac{\partial \psi}{\partial y} = U_x$$

(26)

which is calculated as a solution of the following differential equation

$$\Delta \psi = \frac{\partial U_x}{\partial y} - \frac{\partial U_y}{\partial x}.$$  

(27)

The flow kinematics around the bifurcation point is illustrated in Fig. 8(a). The corresponding pressure distribution is given in Fig. 8(b). As it is predicted by the asymptotic analysis, the pressure gradient in each arm is constant and naturally depends on the channel thickness and flow rate distribution.
Fig. 8: Model of the Stokes flow and the convection/diffusion process in a one bundle tube structure: (a) flow kinematics, (b), (c) pressure distribution in the bifurcation zone, (d) and (e) iso-lines of concentration for two values of $\kappa$: $\kappa = 0.25$ (d) and $\kappa = 0.10$ (e), (f) comparison between the 2D and MAPDD solutions for the diffusion $\kappa = 0.1, 1$ and 10.
It is important to emphasize that each geometry corner represents a singular point for the pressure, which can be seen at Fig. 8(c). These peaks are predicted by the corner singularities analysis ([4] [13] [14] [15]). Two typical solutions representing the cases of lower ($\kappa = 0.25$) and higher ($\kappa = 10$) diffusivity are given in Fig. 8(d) and 8(e). Like in the previously analyzed geometries, there is a critical diffusivity value, which ensures the validity of the simplified 1D approach. In Fig. 8(f), we have given the direct comparison between 1D and 2D predictions. As it is seen from these figures, the critical diffusivity value is around 1.

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