Massive CP$^1$ theory from a microscopic model for doped antiferromagnets

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A path-integral for the $t$-$J$ model in two dimensions is constructed based on Dirac quantization, with an action found originally by Wiegmann (Phys. Rev. Lett. 60, 821 (1988); Nucl. Phys. B323, 311 (1989)). Concentrating on the low doping limit, we assume short range antiferromagnetic order of the spin degrees of freedom. Going over to a local spin quantization axis of the dopant fermions, that follows the spin degree of freedom, staggered CP$^1$ fields result and the constraint against double occupancy can be resolved. The staggered CP$^1$ fields are split into slow and fast modes, such that after a gradient expansion, and after integrating out the fast modes and the dopant fermions, a CP$^1$ field-theory with a massive gauge field is obtained that describes generically incommensurate coplanar magnetic structures, as discussed previously in the context of frustrated quantum antiferromagnets. Hence, the possibility of deconfined spinons is opened by doping a collinear antiferromagnet.

I. INTRODUCTION

High temperature superconductivity (HTS) remains an unresolved problem in spite of an enormous research effort over more than twenty years [1]. On the theoretical side, a number of phenomenological theories were proposed [2, 3, 4] that may describe certain aspects of the experimental findings, however, they rely on fundamentally different assumptions, so that it is, at this point, difficult to assess their validity. Also approaches based on microscopic models like the $t$-$J$ one were advanced [5], based on the so-called slave boson formulation, where electrons are split into separated spin (spinon) and charge (holons) degrees of freedom. Such a separation introduces a local gauge invariance, that renders mean-field like approximations particularly troublesome. A common feature of the above mentioned approaches is the difficulty to connect in a controlled way the states proposed for the doped case to the undoped one.

In fact, the theoretical description of the evolution from a Néel state in the parent compounds towards the state of a doped quantum antiferromagnet, lies at the heart of a theoretical understanding of HTS. A phenomenological description of doped antiferromagnets was first given by Shraiman and Siggia in a series of seminal papers [6, 7, 8, 9]. The picture emerging from a semiclassical treatment of mobile holes in an antiferromagnet corresponds to a coplanar twist of the spin background that gives rise to a dipolar field centered on the dopant holes. Similar coplanar structures are also expected in frustrated quantum antiferromagnets [10, 11, 12].

The field-theoretic treatment of frustrated quantum antiferromagnets showed on the one-hand, that an O(4) symmetry is dynamically generated, and in two dimensions at temperature $T = 0$ long-range order can set in either through a first order transition or a second order one with exponents corresponding to an O(4) non-linear $\sigma$ model [10, 11]. On the other hand, when the effective theory is formulated in terms of CP$^1$ fields, the presence of deconfined spinons can be induced from the fact that the corresponding gauge field becomes massive [12, 13, 14]. Yet, an explicit connection between doped antiferromagnets and the effective theories for frustrated quantum antiferromagnets is missing.

The experimental observation of incommensurate peaks in neutron scattering experiments on La$_{2-x}$Sr$_x$CuO$_4$ (LSCO) [15] and on YBa$_2$Cu$_3$O$_7$ (YBCO) [16, 17, 18, 19] revived the interest on coplanar magnetic configurations in cuprates. A number of theoretical works [20, 21, 22, 23, 24] focused on material specific descriptions of the magnetic structures at low doping, mostly based on the phenomenology developed by Shraiman and Siggia. Although these works were very successful in interpreting the experimental results, demonstrating that the used phenomenology describes very well the generic features at low doping, they did not make an explicit connection to a microscopic model.

An alternative way to describe the low energy physics of doped antiferromagnets is the development of an effective field-theory based on the symmetries of the system (global SU(2) and U(1) in the case of a doped quantum antiferromagnet) and the possible spontaneously broken symmetry phases. Such a path, reminiscent of chiral perturbation theory was followed starting with the Hubbard model, as a representative one for doped antiferromagnets [25]. At the moment the predictive power of the derived effective action is not clear.

The most direct approach for treating a microscopic model like the Hubbard [26] or $t$-$J$ [27] models encountered until now a number of difficulties. While on the one hand, a mean-field treatment of the $t$-$J$ model based on the slave boson approach [5] led to a qualitative understanding of various experimental results, like the existence of a
pseudo-gap, it is difficult to assess their reliability due to the uncontrolled nature of the mean-field approximation. The inclusion of fluctuations of the gauge fields remains until now difficult, since in this case their coupling to the matter fields is strong. Although their treatment led to a qualitative description of the doped phase, still progress would be desirable on a more quantitative basis [30]. On the other hand, numerically exact results could be only obtained by diagonalization in rather small clusters [28] so that their interpretation remains inconclusive when contrasted to the low energy behavior in HTS. For larger sizes, variational Monte Carlo techniques indicate that the $t$-$J$ model supports superconductivity [20]. Still, in spite of the accuracy of the method, further insight in the system is still missing. Unfortunately, large scale quantum Monte Carlo simulations are hindered by the so-called minus sign problem that affects the simulation of doped antiferromagnets.

In view of the described situation, a controlled analytical treatment of a microscopic model is desirable, in particular with the possibility of examining the low doping regime, such that the change from an antiferromagnet with long-range order at zero doping to a doped situation can be followed in detail. A first step in that direction was made by Wiegmann [30], who obtained an action for the $t$-$J$ model based on the coherent-states method [31]. The same action was found by using a supersymmetric version of the Faddeev-Jackiw symplectic formalism [32] applied to the $t$-$J$ model [33], with explicit expressions for the measure of the path integral.

In the present work we use alternatively the well known procedure of Dirac quantization for constrained systems [34] to set up the path integral, recovering the results in Ref. [33], as shown in Sec. II. There we shortly review the $t$-$J$ model and its representation in terms of Hubbard $X$-operators [35, 36, 37], that fulfill the graded Lie algebra $\text{Sp}(2,1)$ [30, 31, 33]. Introducing a series of primary constraints and on the basis of the action proposed by Wiegmann [30, 31], all the constraints of the theory are determined. It turns out that only second class constraints appear, such that they can be solved by inverting the matrix of constraints, and the Dirac brackets reproduce the algebra of the $X$-operators. On this basis, the path integral can be set up. Passing from the representation by $X$-fields to real vector fields for the spin degrees of freedom and Grassmann variables for the dopant holes [30, 33], we arrive at the action that will be the starting point for a long-wavelength expansion. In order to proceed further, we restrict ourselves to the low doping limit and neglect terms quadratic in the density of dopant holes. The long-wavelength expansion is performed in Sec. III where a staggered $C^1$ representation is introduced for the spin-fields. Using the $C^1$ representation it is possible to resolve exactly the constraint against double occupancy [30], that is in general the stumbling block for a controlled treatment of the model. Slow and fast modes of the $C^1$ fields are identified in the same spirit as done for vector fields in quantum antiferromagnets [39]. The effective action for the magnetic degrees of freedom is reached after integrating out the fast $C^1$ modes and the fermions. The resulting effective field-theory corresponds to a $C^1$ model with a massive gauge field, as was generally discussed in the context of frustrated quantum antiferromagnets [12, 13, 14]. In the present work, however, we obtained the explicit doping dependence of the coupling constants. In Sec. VII we discuss the obtained results that open the possibility of having deconfined spinons by doping. Some intermediate results are presented in the appendices that may be helpful for readers interested in reproducing our results.

II. DIRAC QUANTIZATION OF THE $t$-$J$ MODEL

We introduce first the $t$-$J$ model and its representation in terms of so-called $X$-operators that operate only in the subspace without doubly occupancy. After discussing shortly the algebra they fulfill, we delineate the procedure of Dirac quantization.

A. The $t$-$J$ model and $X$-operators

The $t$-$J$ model is defined by the following Hamiltonian in second quantization:

$$H_{t-J} = -t \sum_{\langle i,j \rangle} \tilde{c}_{i\sigma}^\dagger \tilde{c}_{j\sigma} + \frac{J}{2} \sum_{\langle i,j \rangle} \left( S_i \cdot S_j - \frac{1}{4} \tilde{n}_i \tilde{n}_j \right) - \mu \sum_i \tilde{n}_i ,$$

with $\tilde{c}_{i\sigma}^\dagger = (1 - n_{i-\sigma}) c_{i\sigma}^\dagger$, $\tilde{n}_i = \sum_{\sigma} \tilde{c}_{i\sigma}^\dagger \tilde{c}_{i\sigma}$, $S_i = \sum_{\sigma,\sigma'} c_{i\sigma}^\dagger \sigma_{\sigma\sigma'} c_{i\sigma'}$, where the operators $c_{i\sigma}^\dagger$ and $c_{i\sigma}$ denote canonical creation and annihilation operators, respectively, for electrons at site $i$ and spin indices $\sigma = \pm$. The operators $\tilde{c}_{i\sigma}^\dagger$ and $\tilde{c}_{i\sigma}$ project out doubly occupied states. Then, $t$ gives the hopping amplitude, $J$ is the antiferromagnetic exchange coupling and $\mu$ the chemical potential. The symbol $\langle i,j \rangle$ restricts the sums to nearest neighbors. The Hamiltonian [11] was obtained from a multiband model [27] and represents the minimal model for cuprates.
Introducing so-called $X$-operators $[35,36,37]$ defined as

$$X_i^{\alpha\beta} = | \alpha_i > < \beta_i | , $$

with $\alpha_i = 0, \sigma$ for site $i$, the Hamiltonian becomes a bilinear form in such operators.

$$H_{i-J} = -t \sum_{\sigma} \hat{X}_i^{\sigma 0} \hat{X}_j^{\sigma 0} + \frac{J}{4} \sum_{\sigma} \left( \hat{X}_i^{\sigma \bar{\sigma}} \hat{X}_j^{\bar{\sigma} \sigma} - \hat{X}_i^{\sigma \bar{\sigma}} \hat{X}_j^{\bar{\sigma} \sigma} \right) - \mu \sum_{i,\sigma} \hat{X}_i^{\sigma 0} \hat{X}_i^{0\sigma} .$$

The $X$-operators fulfill the following graded algebra

$$[\hat{X}_i^{\alpha\beta}, \hat{X}_j^{\gamma\delta}] = \delta_{ij} \left( \delta^{\beta\gamma} \hat{X}_i^{\alpha\delta} \pm \delta^{\alpha\delta} \hat{X}_i^{\beta\gamma} \right) ,$$

with $-$ ($+$) corresponding to a commutator (anticommutator). Anticommutation relations appear only when both operators are fermionic. Furthermore, we have the completeness condition

$$\sum_{\alpha} \hat{X}_i^{\alpha\alpha} = 1 .$$

A further insight into the graded algebra above can be gained by considering the commutation and anticommutation relations of the even (bosonic) and odd (fermionic) parts. Denoting the even generators by $Q_m$, $m = 1,2,3$ and $B$, and the odd ones by $U_i$, $i = 1, \ldots, 4$, the commutation relations of the Spl(2,1) algebra are $[38]

$$[Q_m, Q_n] = i\varepsilon_{mnp}Q_p , \quad [Q_m, B] = 0 ,$$

$$[Q_m, U_i] = \frac{1}{2} \delta_{m\alpha}U_\beta , \quad [B, U_\alpha] = \frac{1}{2} \hat{\epsilon}_{\alpha\beta}U_\beta ,$$

$$\{U_\alpha, U_\beta\} = \left( \hat{C} \hat{\sigma}^m \right)_{\alpha\beta} Q_m - \left( \hat{C} \hat{\epsilon} \right)_{\alpha\beta} B ,$$

where the $4 \times 4$ matrices $\hat{\sigma}^m$, $\hat{C}$, and $\hat{\epsilon}$ are defined as follows,

$$\hat{\sigma}^m = \begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} , \quad \hat{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} , \quad \hat{\epsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

with $\sigma^m$, $m = 1,2,3$ the Pauli matrices and $C = i\tau^2$. In the irreducible representation corresponding to the $X$-operators, the generators above look as follows. For the even sector we have

$$Q_1 = \frac{1}{2} (\hat{X}^{++} + \hat{X}^{--}) , \quad Q_2 = -\frac{i}{2} (\hat{X}^{+-} - \hat{X}^{-+}) ,$$

$$Q_3 = \frac{1}{2} (\hat{X}^{++} - \hat{X}^{--}) , \quad B = \frac{1}{2} (\hat{X}^{++} + \hat{X}^{--}) - 1 .$$

The first three generators are those of SU(2) while the last one corresponds essentially to particle number. Here and in the following we eliminate $\hat{X}^{00}$ using the completeness relation $[39]$. For the odd sector we have

$$U_1 = \hat{X}^{+0} , \quad U_2 = \hat{X}^{-0} , \quad U_3 = \hat{X}^{0-} , \quad U_4 = -\hat{X}^{0+} .$$

As shown in Ref. $[38]$, the Casimir operator quadratic in the generators is given by

$$K_2 = Q^2 - B^2 + \frac{1}{2} U \hat{C} U ,$$

that in terms of the $X$-operators looks as follows.

$$K_2 = \frac{1}{2} (\hat{X}^{++} \hat{X}^{--} + \hat{X}^{+-} \hat{X}^{-+}) + \frac{1}{4} (\hat{X}^{++} - \hat{X}^{--})^2 - \left[ \frac{1}{2} (\hat{X}^{++} + \hat{X}^{--}) - 1 \right]^2$$

$$+ \frac{1}{2} (-\hat{X}^{+0} \hat{X}^{0+} - \hat{X}^{-0} \hat{X}^{0-} + \hat{X}^{0-} \hat{X}^{-0} + \hat{X}^{0+} \hat{X}^{+0}) .$$

Then, it is easily seen, that this Casimir operator has eigenvalue zero in the present irreducible representation.
B. Dirac quantization for the $t-J$ model

In the following we consider a classical system with the Lagrangian found by Wiegmann [30, 31], expressed in terms of $X$-fields [32], with, as usual, complex fields corresponding to bosonic operators in Sec. [1A] and Grassmann fields for fermionic ones.

$$L(X, \dot{X}) = -\frac{i}{2} \sum_{i} \left( \frac{1 + \rho_i}{2 - v_i} \right) u_i u_i - \frac{1}{4} \left( X_{i}^{+} \dot{X}_{i}^{+} - X_{i}^{-} \dot{X}_{i}^{-} \right) + \frac{i}{2} \sum_{i, \sigma} \left( X_{i}^{\sigma} \dot{X}_{i}^{\sigma} + X_{i}^{\sigma} X_{i}^{\sigma} \right) - H(X), \quad (12)$$

where the following definitions were introduced

$$\rho_i \equiv X_{i}^{0+} X_{i}^{+0} + X_{i}^{0-} X_{i}^{-0}, \quad u_i \equiv X_{i}^{++} X_{i}^{--} - X_{i}^{+} X_{i}^{-}, \quad v_i \equiv X_{i}^{++} + X_{i}^{--}.$$  

Furthermore, we take into account the set of primary constraints found in the symplectic formality introduced by Faddeev and Jackiw [33], where we omit the site indices.

$$\phi^{(1)} = X^{++} + X^{--} + \rho - 1, \quad \phi^{(2)} = X^{++} - X^{--} + \frac{1}{4} u^2 - \left( 1 - \frac{1}{2} v \right)^2 + \rho, \quad \phi^{(7)} = X^{00} - (X^{0+} X^{+0} + X^{0-} X^{-0}) \quad \phi^{(9)} = \frac{X^{0+} X^{++} - X^{0-}}{X^{++}} - X^{0-}, \quad \phi^{(10)} = X^{00} X^{++} - X^{0-} X^{++}.$$  

The constraints are imposed by setting $\phi^{(a)} = 0$ and the order of the labels $a$ is such that the supermatrix of constraints has a normal form [40, 41]. The special choice of $\phi^{(9)}$ was made in order to have a simple expression for the measure of the path integral obtained at the end. The constraint $\phi^{(1)}$ (together with $\phi^{(7)}$) corresponds to the completeness relation [3], while $\phi^{(2)}$ comes from the Casimir operator [11] having eigenvalue zero. Furthermore, $\phi^{(7)}$ relates the empty sites with fermionic holes, and the rest of the constraints above define new product rules instead of the ones obeyed by the operators in Sec. [1A] [33].

There are other nine primary constraints resulting from considering the canonical momenta

$$\Pi^{\alpha \beta} = \frac{\partial L}{\partial \dot{X}^{\alpha \beta}}, \quad (15)$$

where $\partial L/\partial \dot{X}^{\alpha \beta}$ refers to the right derivative, the $X^{\alpha \beta}$ and its derivatives being elements of the Berezin algebra [40] (see Appendix A). The additional nine constraints are listed in Appendix A.

Once the constraints are determined, a Hamiltonian

$$H = \sum_{\alpha \beta} \Pi^{\alpha \beta} X^{\alpha \beta} - L \quad (16)$$

can be in principle obtained, where we have to take into account the constraints. This can be done introducing Lagrange multipliers $\lambda$ that enter in the equations of motion for any observable $f(X, \Pi)$. In particular, if the primary constraints are required to apply at any time, we should require that

$$\dot{\phi}^{(a)} = \left\{ H, \phi^{(a)} \right\} + \lambda_b \left\{ \phi^{(b)}, \phi^{(a)} \right\} = 0,$$  

where the curly brackets denote now Poisson brackets and summation over repeated indices is assumed. It is understood that the constraints are applied after the derivatives in the Poisson brackets are calculated. For the Poisson brackets among the constraints we have the general form [40]

$$\left\{ \phi^{(a)}, \phi^{(b)} \right\} = \sum_{k, \alpha, \beta} \left[ \frac{\partial \phi^{(a)}}{\partial X^{\alpha \beta}_k} \frac{\partial \phi^{(b)}}{\partial \Pi^{\alpha \beta}_k} - (-1)^{P_{\phi^{(a)}} P_{\phi^{(b)}}} \frac{\partial \phi^{(a)}}{\partial X^{\alpha \beta}_k} \frac{\partial \phi^{(b)}}{\partial \Pi^{\alpha \beta}_k} \right], \quad (18)$$

where $P_{\phi^{(a)}}$ is the parity of $\phi^{(a)}$, and $\partial \phi^{(b)}/\partial \Pi^{\alpha \beta}_k$ refers to the left derivative [40], that is defined in a similar way as the right derivative in [15] (see Appendix A). If the matrix of constraints $\left\{ \phi^{(a)}, \phi^{(b)} \right\}$ is not singular, then the constraints are second class [34].
In our case, the matrix of Poisson brackets can be written as a supermatrix of the form

\[ \{ \phi^{(a)}, \phi^{(b)} \} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \]  

(19)

where \( A \) is an \( 8 \times 8 \), \( B \) an \( 8 \times 6 \), \( C \) a \( 6 \times 8 \), and \( D \) a \( 6 \times 6 \) matrix. Matrix \( A \) has the form

\[ A = A^{(0)} + A^{(1)}, \]  

(20)

where \( A^{(0)} \) contains only bosonic fields and \( A^{(1)} \) is proportional to \( \rho \). The explicit form of the matrices above is given in Appendix [A]. In order to see whether the matrix is singular, we have to consider the superdeterminant [40, 41]

\[ \text{sdet} \{ \phi^{(a)}, \phi^{(b)} \} = \det A \left[ \det (D - CA^{-1}B) \right]^{-1}. \]  

(21)

Here we have

\[ \det A = -(1 + 2\rho), \]  

(22)

and

\[ \det (D - CA^{-1}B) = 1, \]  

(23)

such that the matrix of constraints is not singular. Hence we have only second class constraints.

Since \( \{ \phi^{(a)}, \phi^{(b)} \} \) is not singular, it is possible to obtain the Lagrange multipliers from eq. (17) by considering its inverse

\[ \lambda_a = -\left\{ H, \phi^{(b)} \right\} \left\{ \phi^{(b)}, \phi^{(a)} \right\}^{-1}. \]  

(24)

Then, the equation of motion for an observable can be written in terms of the Dirac bracket [34]

\[ \dot{f} = \{ H, f \}_D, \]  

(25)

where

\[ \{ f, g \}_D = \{ f, g \} - \left\{ f, \phi^{(a)} \right\} \left\{ \phi^{(a)}, \phi^{(b)} \right\}^{-1} \left\{ \phi^{(b)}, g \right\}. \]  

(26)

In order to obtain the inverse of the supermatrix of the constraints we determine first the inverses of \( A^{(0)} \) and \( D \), and form the following matrix

\[ N = \begin{bmatrix} (A^{(0)})^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}, \]  

(27)

Multiplying the matrix of the constraints (19) by \( N \), we have

\[ \left\{ \phi^{(a)}, \phi^{(b)} \right\} N = 1 - R, \]  

(28)

where

\[ R = -\begin{bmatrix} A^{(1)} (A^{(0)})^{-1} BD^{-1} \\ C (A^{(0)})^{-1} 0 \end{bmatrix}. \]  

(29)

Then, since \( R \) contains Grassmann fields, the inverse of the matrix of constraints is achieved with a finite number of powers of \( R \):

\[ \left\{ \phi^{(a)}, \phi^{(b)} \right\}^{-1} = N \left( 1 + R + R^2 + R^3 + R^4 \right). \]  

(30)
Having the inverse of the matrix of constraints, it can be readily seen that the commutations relations \([14]\) are reproduced by the Dirac brackets: \(\left[ X_i^{\alpha \beta}, X_j^{\gamma \delta} \right]_\pm = i \left\{ X_i^{\alpha \beta}, X_j^{\gamma \delta} \right\}_D \). Hence, the Lagrangian \([12]\) and the set of constraints \([14]\) lead to the algebra \([14]\).

The path integral quantization can be performed starting with the Hamiltonian, and integrating over the canonical fields and the corresponding momenta. As shown by Senjanovic \([40, 42]\), in the case of second class constraints, the path integral is as follows,

\[
Z = \int \mathcal{D}X \mathcal{D}\Pi \prod_a \delta \left[ \phi^{(a)} \right] \text{sdet}^{1/2} \left\{ \phi^{(a)}, \phi^{(b)} \right\} e^{i \int dt \left[ \Pi X - H \right]}.
\]

In the present case it is possible to integrate over the momenta, leading to

\[
Z = \int \mathcal{D}X \prod_i \delta \left[ \phi^{(1)}_i \right] \delta \left[ \phi^{(2)}_i \right] \delta \left[ \phi^{(7)}_i \right] \delta \left[ \phi^{(9)}_i \right] \delta \left[ \phi^{(10)}_i \right] (1 + 2 \rho_i)^{\frac{1}{2}} e^{-S},
\]

where the action in imaginary time is given by

\[
S = \int_0^\beta d\tau \left\{ -\sum_i \frac{(1 + \rho_i) u_i - 1}{(2 - v_i)^2 - 4 \rho_i - u_i^2} \left( X_i^{++} \dot{X}_i^{++} - X_i^{+-} \dot{X}_i^{+-} \right) + \frac{1}{2} \sum_{i, \sigma} \left( X_i^{\sigma 0} \dot{X}_i^{0 \sigma} + X_i^{0 \sigma} \dot{X}_i^{\sigma 0} \right) + H(X) \right\},
\]

with \(\beta = 1/k_B T\) the inverse temperature. The term coming from the superdeterminant gives just a shift of the chemical potential, and can be ignored. Since \(X^{00}\) does not enter in the action, we can integrate over it, eliminating the constraint \(\phi^{(7)}\).

As a final step, we perform the change of variables introduced in \([30, 33]\) that will be useful for further considerations:

\[
\begin{align*}
X^{++} &= (1 - \rho) (1 + \Omega_z) / 2, & X^{+-} &= (1 - \rho) (1 - \Omega_z) / 2, \\
X^{--} &= (1 - \rho) (\Omega_x + i \Omega_y) / 2, & X^{-+} &= (1 - \rho) (\Omega_x - i \Omega_y) / 2, \\
X^{0+} &= \psi^+, & X^{-0} &= \psi^-, \\
X^{0+} &= \psi^+, & X^{0-} &= \psi^-,
\end{align*}
\]

where we introduced Grassmann field \(\psi^+\) and \(\psi^-\). Accordingly, we have \(\rho = \psi^+ \psi^+ + \psi^- \psi^-\). Since we have only 7 new variables, chosen in such a way that \(\phi^{(1)}\) is automatically satisfied, we integrate over \(X^{--}\) taking care of the constraint \(\phi^{(1)}\) before performing the change of variables. After the change of variables, the remaining constraints look as follows:

\[
\begin{align*}
\phi^{(2)} &= \frac{1}{4} (1 - \rho)^2 \left( \Omega^2 - 1 \right), \\
\phi^{(9)} &= \psi^+ \frac{\Omega_x - i \Omega_y}{1 + \Omega_z} - \psi^- , \\
\phi^{(10)} &= \psi^+ \frac{\Omega_x + i \Omega_y}{1 + \Omega_z} - \psi^- (1 + \Omega_z).
\end{align*}
\]

For \(\phi^{(2)}\) the factor \((1 - \rho)^2\) can be absorbed in the chemical potential, such that it reduces to \(\phi^{(2)} \rightarrow \tilde{\phi}^{(2)} = (\Omega^2 - 1)\).

After making the change of variables, we finally have

\[
S = \int_0^\beta \left\{ -\frac{i}{2} \sum_i \frac{\Omega_x \dot{\Omega}_y - \Omega_y \dot{\Omega}_x}{1 + \Omega_z} + \sum_{i, \sigma} \psi^*_i \psi_i + t \sum_{<i,j>} \psi_i \psi_j \sigma \\
+ \frac{J}{8} \sum_{<i,j>} (1 - \rho_i) (1 - \rho_j) \left( \Omega_i \cdot \Omega_j - 1 \right) - \mu \sum_i \rho_i \right\},
\]

with constraints \(\tilde{\phi}^{(2)}, (30),\) and \((37)\). For the undoped case, the action above reduces to the one corresponding to a quantum Heisenberg antiferromagnet, as obtained e.g. using coherent states \([39]\).
III. STAGGERED CP\textsuperscript{1} REPRESENTATION

Here we concentrate on the limit of low doping, in order to study the consequences of doping on the antiferromagnetic state present in the undoped case. In such a situation, we can assume a large correlation length for spins, such that a long-wavelength expansion is justified. In this limit we can also neglect terms $\sim \rho_i \rho_j$ obtaining thus a bilinear form in the fermionic degrees of freedom. Then, the action has the following form

$$S = S_S + S_F ,$$

where

$$S_S = \int_0^\beta d\tau \left\{ -\frac{i}{2} \sum_i A_{\Omega i} \cdot \partial_\tau \Omega_i + \frac{J}{8} \sum_{<i,j>} \Omega_i \cdot \Omega_j \right\} ,$$

is the action of a pure Heisenberg model. $A$ is the vector potential of a magnetic monopole, $(\nabla \times A) \cdot \Omega = 1$, where derivatives are taken in the $\Omega$-space. In this case it is given by

$$A = \left[ -\frac{\Omega_y}{(1 + i \Omega_z)}, \frac{\Omega_x}{(1 + i \Omega_z)}, 0 \right] .$$

The fermionic part is given by

$$S_F = \int_0^\beta d\tau \left\{ \sum_{i,\sigma} \psi_{i\sigma}^* \partial_\tau \psi_{i\sigma} + t \sum_{<i,j>} \psi_{i\sigma}^* \psi_{j\sigma} + t' \sum_{<<i,j>>} \psi_{i\sigma}^* \psi_{j\sigma} + t'' \sum_{<<<i,j>>>} \psi_{i\sigma}^* \psi_{j\sigma} - \frac{J}{4} \sum_{<i,j>} \psi_{i\sigma}^* \psi_{i\sigma} \Omega_i \cdot \Omega_j + \mu \sum_{i,\sigma} \psi_{i\sigma}^* \psi_{i\sigma} \right\} ,$$

where we allowed hopping to 2nd. nearest neighbors along the diagonals of the square lattice, denoted by $<<i,j>>$, with hopping amplitude $t'$, and to 2nd. nearest neighbors along the principal axis, denoted by $<<<i,j>>>$, with hopping amplitude $t''$. Such hopping terms have been taken into account, since there is consensus that they are present in the real materials \textsuperscript{43}. Furthermore, we have to take into account the constraints $\tilde{\phi}^{(2)}$, (36) and (37).

A. Rotating reference frame and staggered CP\textsuperscript{1} representation

In order to take into account the constraints (36) and (37), and in order to work with smoothly varying fields, we consider now the fact that the field $\Omega$ is staggered on nearest neighbors and define a local quantization axis for the fermions with a rotation that fulfills

$$U_i^\dagger \Omega_i \cdot \sigma U_i = (-1)^i \sigma^z ,$$

where $\sigma^a$, with $a = x, y, \text{ or } z$ are the Pauli matrices. For $i$ even the condition above is accomplished by $U \in SU(2)$,

$$U = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix} ,$$

where $\bar{z} z = 1$, and

$$\Omega^a = \bar{z} \sigma^a z .$$

For $i$ odd, on the other hand, we take rotation matrices (44) and

$$\Omega^a = z_\alpha \sigma_{\alpha\beta}^y \sigma_{\beta\gamma}^y \sigma_{\gamma\delta}^y z_\delta^* .$$

This ensures that (43) is fulfilled.
1. Constraints in the rotating reference frame

Here we discuss the transformation of the constraints (36) and (37) on going to the rotating reference frame introduced above. For simplicity of notation, we redefine them as follows.

\[
\varphi_{F, 2}^* = -\psi^* \frac{\Omega_x - i\Omega_y}{1 + \Omega_z} + \psi^* , \\
\varphi_{F, 2} = -\psi_+ (\Omega_x + i\Omega_y) + \psi_- (1 + \Omega_z) ,
\]

(47)

Since \(\varphi_{F, 2}^*\) is a fermionic constraint, we can use a corresponding \(\delta\)-function for Grassmann variables \(\xi\) and \(\xi'\):

\[
\delta (\xi - \xi') = -(\xi - \xi') ,
\]

(48)
such that

\[
\delta (\varphi_{F, 2}^*) = \frac{1}{1 + \Omega_z} \delta (\tilde{\varphi}_{F, 2}^*) ,
\]

(49)

where we defined \(\tilde{\varphi}_{F, 2}^* \equiv -\psi_+^* (\Omega_x - i\Omega_y) + \psi_-^* (1 + \Omega_z)\). Furthermore, going back to the original formulation in terms of the \(X\)-fields, we can rewrite \(\phi^{(9)}\) in (14) as follows. Insertion of \(\phi^{(1)}\) into \(\phi^{(2)}\) leads to

\[
\phi^{(2')} = X^+ - X^- - X^{++} X^{--} ,
\]

(50)
such that

\[
X^{++} = \frac{X^+ - X^-}{X^{--}} ,
\]

(51)

and \(\phi^{(9)}\) can be brought to the form

\[
\phi^{(9')} = X^0_+ X^{--} - X^0_-- X^{+} ,
\]

(52)

that after the change of variables (54) we can write as

\[
\tilde{\varphi}_{F, 1}^* = \psi_+^* (1 - \Omega_z) - \psi_-^* (\Omega_x + i\Omega_y) .
\]

(53)

Then, we can consider the first constraint in (47) as part of

\[
\tilde{\varphi}_F^* = \psi^* (1 - \Omega \cdot \sigma) = 0 .
\]

(54)

We can introduce now new fermions \(\chi = U^\dagger \psi\) and \(\chi^* = \psi^* U\), such that

\[
\tilde{\varphi}_F^* U = \chi^* U^\dagger (1 - \Omega \cdot \sigma) U = \chi^* (1 \mp \sigma^z) = 0 ,
\]

(55)

where the upper (lower) sign is for even (odd) sites.

In the same way, we have for the second constraint in (47)

\[
U^\dagger \varphi_F = U^\dagger (1 - \Omega \cdot \sigma) U \chi = (1 \mp \sigma^z) \chi = 0 .
\]

(56)

However, since the constraints are actually given solely in terms of \(\tilde{\varphi}_F^*\) and \(\varphi_F\), we introduce

\[
\theta^* \equiv \tilde{\varphi}_F^* U , \quad \theta \equiv U^\dagger \varphi_F ,
\]

(57)

such that for the original form of the constraints we have

\[
\delta (\varphi_{F, 2}^*) \delta (\varphi_{F, 2}) = \frac{1}{1 + \Omega_z} \delta (\tilde{\varphi}_{F, 2}^*) \delta (\varphi_{F, 2})
\]

\[
= \frac{1}{1 + \Omega_z} \delta (\theta_1^* z_2^* + \theta_2^* z_1) \delta (z_2 \theta_1 + z_1 \theta_2) .
\]

(58)

We consider now the action of the constraints on even and on odd sites.
i) Even sites.

\[
\theta_1^* = \theta_1 = 0 , \quad \theta_2^* = 2\chi_+ , \quad \theta_2 = 2\chi_.
\]

Furthermore, we have

\[
1 + \Omega_z = 2 \mid z_1 \mid ^2 ,
\]

such that finally,

\[
\delta (\varphi_{F,2}^* \delta (\varphi_{F,2}) = \frac{1}{2} \mid z_1 \mid ^2 \delta (\theta_2^* z_1) \delta (z_2^* \theta_2) = 2\delta (\chi_+) \delta (\chi_-) .
\]

i) Odd sites.

\[
\theta_1^* = 2\chi_+ , \quad \theta_1 = 2\chi_+ , \quad \theta_2^* = \theta_2 = 0 .
\]

In this case we have due to (46),

\[
1 + \Omega_z = 2 \mid z_2 \mid ^2 ,
\]

such that finally,

\[
\delta (\varphi_{F,2}^* \delta (\varphi_{F,2}) = \frac{1}{2} \mid z_2 \mid ^2 \delta (\theta_1^* z_2) \delta (z_2 \theta_1) = 2\delta (\chi_+) \delta (\chi_+).
\]

The constraints above lead to \(\chi_- = 0\) on even sites, whereas \(\chi_+ = 0\) on odd sites. We can therefore work with spinless fermions \(\chi_A\) on even sites and \(\chi_B\) on odd sites, where \(A\) and \(B\) denote the two sublattices, such that the constraints on fermions are exactly taken into account.

2. Slow CP\(^3\) variables in the rotating reference frame

After introducing cells \(j\), each one containing one even \((A)\) and one odd \((B)\) site, we can define new fields

\[
\tilde{z}_j = \frac{1}{2} (z_B^j + z_A^j) , \quad a\tilde{z}_j = \frac{1}{2} (z_B^j - z_A^j) ,
\]

where \(a\) is the original lattice constant. Due to the constraints \(|z_A^j|^2 = |z_B^j|^2 = 1\), the new fields are subjected to the constraint

\[
\tilde{z} \tilde{\zeta} + \tilde{\zeta} \tilde{z} = 0 .
\]

This condition can be used to fix the phase of \(\zeta\) with respect to that of \(z\), such that they change by the same amount under a gauge transformation. This will be discussed in more detail in Sec. VI A. Equation (66) also implies that both fields are subjected to the constraint

\[
\tilde{z} \tilde{\zeta} + a^2 \tilde{\zeta} \tilde{z} = 1 .
\]

We then introduce new fields

\[
\tilde{\zeta} = \tilde{z} \sqrt{1 - a^2 \tilde{\zeta} \tilde{z}} ,
\]

such that the constraint (67) is satisfied with \(\tilde{z} z = 1\), and the constraint (66) translates into

\[
\tilde{\zeta} \zeta + \tilde{\zeta} z = 0 .
\]

The fact that \(z_B^j - z_A^j\) is of \(O(a)\) can also be seen by going back to the vector representation, where we have \(\Omega^A - \Omega^B \sim \tilde{z} \sigma z\) while \(\Omega^A + \Omega^B \sim \tilde{\zeta} \sigma z + \tilde{z} \sigma \zeta\), i.e. the field \(\zeta\) is directly related to ferromagnetic fluctuations within the unit cell.
IV. LONG-WAVELENGTH EXPANSION

Once we have identified smoothly varying fields and their slow modes, we perform an expansion of the action in powers of the lattice constant $a$ up to second order, after a transformation to the rotating reference frame. For clarity of the presentation we deal first with the action $S_S$ in eq. (40) and then with $S_F$ in eq. (42).

A. Spin action in the staggered CP\(^1\) representation

Here we consider the pure Heisenberg model as given by (40). We pass to CP\(^1\) variables using (45) for even sites and (46) for odd sites.

For the Berry phase we have

$$-\sum_i \frac{i}{2} A[\Omega_i] \cdot \partial_\tau \Omega_i = 2a \sum_j \left[ \tilde{\zeta}_j \partial_\tau \zeta_j + \bar{\zeta}_j \partial_\tau \tilde{\zeta}_j \right] + \mathcal{O}(a^4) \ .$$

Terms containing a total time derivative were discarded due to periodic boundary conditions in imaginary time.

For the interaction term we first discuss our convention in defining new coordinates for the units cells containing sites $A$ and $B$. On passing to the new coordinate system we choose

$$x' = \frac{1}{\sqrt{2}} (x + y) \ , \quad y' = \frac{1}{\sqrt{2}} (y - x) \ ,$$

such that in the new coordinate system the basis vectors for sublattice A and B are, respectively,

$$x_A = (0, 0) , \quad x_B = \left( a/\sqrt{2}, -a/\sqrt{2} \right) .$$

Introducing the notation

$$G_j \equiv 2i z_j \sigma_\alpha^\beta \zeta_j^\beta \ , \quad F_j^\mu \equiv i z_j \sigma_\alpha^\beta \partial_\mu \tilde{\zeta}_j^\beta \ ,$$

where $\mu = x$ or $y$, we obtain after a lengthy but straightforward calculation,

$$\sum_{i < j >} \Omega_i \cdot \Omega_j = 8 \int d\xi^2 \left\{ 2 \left( G_j^* G_j^* + F_{jy}^* F_{jy}^* + F_{jx}^* F_{jx}^* \right) - (F_{jx}^* F_{jy}^* + F_{jy}^* F_{jx}^*) \right\} + \sqrt{2} \left[ (F_{jy} - F_{jx}) G_j^* + G_j (F_{jy}^* - F_{jx}^*) \right] ,$$

where constants terms were discarded.

B. Fermionic part in the staggered CP\(^1\) representation

Here we consider the action (42) in the rotating reference frame, i.e. with fermions as defined in Sec. II A.1

$$\chi = U^\dagger \psi \ , \quad \chi^* = \psi^* U \ .$$

After applying the constraints (36) and (37), we can define a new spinor per unit cell

$$\chi = \begin{pmatrix} \chi_A \\ \chi_B \end{pmatrix} ,$$

following the discussion in Sec. II A.1.
1. Temporal derivatives

Defining

\[ A_{j,\mu} = -i \bar{z}_j \partial_\mu z_j, \quad C_{j,\mu} \equiv \bar{z}_j \partial_\mu \zeta_j + \bar{\zeta}_j \partial_\mu z_j, \]

we have

\[ \sum_{i,\sigma} \psi_{i\sigma}^* \partial_\tau \psi_{i\sigma} = \sum_j \chi_j^* (D_\tau - a C_\tau) \chi_j, \]

where we introduced

\[ D_\mu \equiv \partial_\mu + i\sigma^2 A_\mu. \]

At this point we can check gauge invariance. Since \( \chi_A \) and \( \chi_B \) have opposite charges, a gauge transformation leads to

\[ \chi_A \rightarrow e^{+i\phi} \chi_A, \quad \chi_B \rightarrow e^{-i\phi} \chi_B, \quad z \rightarrow e^{+i\phi} z, \quad \zeta \rightarrow e^{+i\phi} \zeta. \]

It can be easily seen that the term with the covariant derivative is invariant under the gauge transformations above, and that \( C_\mu \) is gauge invariant due to the constraint (66):

\[ C_\mu \rightarrow \bar{z} \partial_\mu \zeta + \bar{\zeta} \partial_\mu z + \partial_\mu \phi \left( \bar{z} \zeta + \bar{\zeta} z \right). \]

2. Kinetic term

For the kinetic term of the fermions we have the contributions proportional to \( t, t', \) and \( t'' \), that we treat separately in the following. We consider first contributions from nearest neighbor hopping, where, after Fourier transformation, we have

\[ -t \sum_{<i,k>} \psi_{i\sigma}^* \psi_{k\sigma} \rightarrow -t \sum_{k_1, k_2} \chi_{k_1}^* \Xi(k_1, k_2) \chi_{k_2}, \]

where

\[ \Xi(k_1, k_2) \equiv \left[ \sum_{i=1}^4 \Xi_{AB}^{(i)}(k_1, k_2) \right] \sigma^+ + \left[ \sum_{i=1}^4 \Xi_{BA}^{(i)}(k_1, k_2) \right] \sigma^-, \]

with \( \sigma^\pm = (\sigma^z \pm i\sigma^y)/2 \). The explicit expressions for \( \Xi^{(i)}(k_1, k_2) \) are given in Appendix B. They contain only contributions up to \( O(a) \), since, as seen above, terms proportional to \( t \) are off-diagonal, and hence, no contribution linear in \( \Xi \) appears in the final action, as shown in Sec. V. They enter in a quadratic form after integrating fermions out.

Next we consider contributions from second nearest neighbor hopping along the diagonal. In this case we have

\[ -t' \sum_{<<i,k>>} \psi_{i\sigma}^* \psi_{k\sigma} \rightarrow \sum_{k_1, k_2} \chi_{k_1}^* \Psi(k_1, k_2) \chi_{k_2}, \]

where

\[ \Psi(k, k') = \epsilon_1(k) \delta_{k, k'} 1 + \left[ \sum_{i=1}^4 \Psi_{AA}^{(i)}(k, k') \right] \gamma^+ + \left[ \sum_{i=1}^4 \Psi_{BB}^{(i)}(k, k') \right] \gamma^-, \]

with

\[ \epsilon_1(k) = -2t' \left[ \cos \left( \sqrt{2} k_x a \right) + \cos \left( \sqrt{2} k_y a \right) \right], \]
and $\gamma^\pm = (1 \pm \sigma^z) / 2$. Again, explicit expressions for $\Psi(i) (k, k')$ are given in Appendix B. Contrary to the previous case, we consider here terms up to $O \left( a^2 \right)$, since terms proportional to $t'$ connect sites within a sublattice, giving rise to diagonal contributions to the self-energy of the fermions, as seen above.

Finally, we have the contributions from second nearest neighbor hopping along the principal axis:

$$
-t'' \sum_{\ll \ll \ll \ll} \psi^*_i \psi_k \phi_{k_1} \phi_{k_2} \chi_{k_1} \phi(k_1, k_2) \chi_{k_2}, \quad (87)
$$

where

$$
\phi(k, k') = \epsilon_2(k) \delta_{k, k'}^1 \left[ \sum_{l=1}^{4} \Phi_A(k, k') \right] \gamma^+ + \left[ \sum_{l=1}^{4} \Phi_B(k, k') \right] \gamma^-, \quad (88)
$$

with

$$
\epsilon_2(k) = -4t'' \cos \left( \sqrt{2} k_x a \right) \cos \left( \sqrt{2} k_y a \right), \quad (89)
$$

and $\Phi(i) (k, k')$ as given in Appendix B. Here again, contributions up to $O \left( a^2 \right)$ have to be taken into account.

3. Spin interaction dressed with fermions

For such contributions we have

$$
\frac{J}{4} \sum_{\ll \ll \ll \ll} \psi^*_i \psi_k \Omega_i \cdot \Omega_k = \frac{J}{4} \sum_{k_1, k_2} \chi^*_k \phi(k_1, k_2) \chi_{k_2} - J \sum_k \chi^*_k \chi_k, \quad (90)
$$

where

$$
\phi(k_1, k_2) = \phi^+ + \phi^- \sigma^z, \quad (91)
$$

with $\phi^+$ and $\phi^-$ displayed in Appendix B.

V. INTEGRATION OF FERMIONIC DEGREES OF FREEDOM

Once the gradient expansion was performed, we integrate out in a first step the fermionic degrees of freedom, that are considered at all wavelengths. In a second step we integrate out the magnetic fast degrees of freedom in order to obtain the effective action for the slow magnetic modes. Collecting all results obtained in Sec. IV B we have the following form for the action

$$
S_F = - \sum_{k, k'} \chi^*_k \left[ G_0^{-1}(k, k') - \Sigma(k, k') \right] \chi_{k'}, \quad (92)
$$

where $k = (i \nu_n, k)$, $\nu_n$ being Matsubara frequencies for fermions, and we defined

$$
G_0^{-1}(k, k') = \frac{1}{\beta} \{ \epsilon(k) + J + \mu \} \delta_{k, k'}, \quad (93)
$$

i.e. the free propagator for fermions in the present theory. The self-energy $\Sigma$ is given by

$$
\Sigma(k, k') = i \sigma^z A_\tau(k, k') - a C_\tau(k, k') + t \Xi(k, k') + t' \Psi(k, k') + t'' \phi(k, k') - \frac{J}{4} \phi(k, k'). \quad (94)
$$

The dispersion relation for the holes is given by the ones in (81) and (82).

$$
\epsilon(k) = \epsilon_1(k) + \epsilon_2(k) = -2t' \left[ \cos \left( \sqrt{2} k_x a \right) + \cos \left( \sqrt{2} k_y a \right) \right] - 4t'' \cos \left( \sqrt{2} k_x a \right) \cos \left( \sqrt{2} k_y a \right). \quad (95)
$$
This dispersion is obtained for the magnetic Brillouin zone. By rotating to the original Brillouin zone, we have
\[ k'_x = \frac{1}{\sqrt{2}} (k_x - k_y) , \quad k'_y = \frac{1}{\sqrt{2}} (k_x + k_y) , \] (96)
and the dispersion is given by
\[ \epsilon (k') = -4t' \cos (k_x a) \cos (k_y a) - 2t'' [\cos (2k_x a) + \cos (2k_y a)] . \] (97)
It should be noticed, that the free dispersion of the dopant holes is determined here entirely by \( t' \) and \( t'' \). This is to be contrasted with calculations of the single hole dispersion of the pure \( t-J \) model in the self-consistent Born approximation [44, 45, 46] that are in very good agreement with quantum Monte Carlo simulations [47], where a dispersive band is found also in the case \( t' = t'' = 0 \). Instead, in our case, on passing to the continuum limit, such a dispersion can be obtained only by explicitly introducing finite values for those hopping amplitudes. In fact, the dispersion of the single hole found previously can be reproduced by (97) for appropriate values of \( t' \) and \( t'' \) should be expected to be generated even if such operators are initially missing.

Arranging terms according to powers of the UV cutoff \( a \), we have
\[ \Sigma (k, k') = a \Sigma^{(1)}(k, k') + a^2 \Sigma^{(2)}(k, k') + \mathcal{O} (a^3) , \] (98)
such that after integrating out fermions and keeping contributions up to \( \mathcal{O} (a^2) \), the fermionic part of the action goes over into
\[ S_F \rightarrow -\text{Tr} \left( G_0 \Sigma + \frac{1}{2} G_0 \Sigma G_0 \Sigma \right) + \mathcal{O} (a^3) . \] (99)
The terms entering (99) can be regrouped as
\[ \Sigma^{(i)} = \Sigma^{(i)}_0 1 + \Sigma^{(i)}_+ \sigma^+ + \Sigma^{(i)}_- \sigma^- , \] (100)
with \( i = 1, 2 \). This leads for the terms linear in \( \Sigma \) to
\[ \text{Tr} G_0 \Sigma = \text{Tr} G_0 \left[ a \Sigma^{(1)}_0 + a^2 \Sigma^{(2)}_0 \right] + \mathcal{O} (a^3) , \] (101)
and for the terms quadratic in \( \Sigma \) to
\[ \text{Tr} G_0 \Sigma G_0 \Sigma = a^2 \text{Tr} \left[ G_0 \Sigma^{(1)}_0 G_0 \Sigma^{(1)}_0 + G_0 \Sigma^{(1)}_+ G_0 \Sigma^{(1)}_- + 2 G_0 \Sigma^{(1)}_0 G_0 \Sigma^{(1)}_- \right] + \mathcal{O} (a^3) . \] (102)
Introducing indices \( \tau, t, t' \), and \( t'' \) for the temporal, and different hopping processes, respectively, we arrange the different contributions in (99) as follows:
\[ \Sigma^{(i)}_{0, \tau}(k, k') = \Sigma^{(i, \tau)}_{0, \tau} + \Sigma^{(i, t')}_{0, \tau} + \Sigma^{(i, t'')}_{0, \tau} , \] (103)
while \( \Sigma^{(1)} \) contains only contributions proportional to \( t \).

An explicit evaluation using the results in Appendix B shows that there are no contributions in first order in \( a \). The contributions in \( \mathcal{O} (a^2) \) coming from \( \text{Tr} G_0 \Sigma \) are as follows:
\[ \text{Tr} G_0 \Sigma^{(2, \tau)} = \frac{4 \tilde{\rho}}{a} \int_0^\beta d\tau \int d^2 x \left( \bar{\zeta} \partial_\tau \zeta + \bar{\zeta} \partial_\tau \zeta \right) , \]
\[ \text{Tr} G_0 \Sigma^{(2, t')} = -16 t'' \tilde{\rho}_1 \int d\tau \int d^2 x \left( \partial_x \bar{\zeta} \partial_x \zeta + \partial_y \bar{\zeta} \partial_y \zeta \right) , \]
\[ \text{Tr} G_0 \Sigma^{(2, t'')} = -16 t'' \tilde{\rho}_2 \int d\tau \int d^2 x \left( \partial_x \bar{\zeta} \partial_x \zeta + \partial_y \bar{\zeta} \partial_y \zeta \right) , \]
\[ \text{Tr} G_0 \Sigma^{(2, J)} = -8 J \tilde{\rho} \int d\tau \int d^2 x \left\{ 2 G \bar{G}^* + 2 F_x F_y^* + 2 F_x F_y^* - (F_x F_y^* + F_y F_x^*) \right\} , \] (104)
where \( \hat{\rho} \) is the density of holes, i.e.

\[
\hat{\rho} = \frac{1}{N} \sum_k n(k),
\]

(105)

\( \hat{\rho}_1 \) is defined as

\[
\hat{\rho}_1 = \frac{1}{N} \sum_k n(k) \cos (\sqrt{2}k_x a)
\]

(106)

and

\[
\hat{\rho}_2 = \frac{1}{N} \sum_k n(k) \cos (\sqrt{2}k_x a) \cos (\sqrt{2}k_y a).
\]

(107)

In the expressions above, and in the following, \( n(k) \) is the Fermi distribution function

\[
n(k) = \frac{1}{\exp \{\beta [\epsilon(k) - \mu]\} + 1}.
\]

(108)

We focus now on the contributions quadratic in \( \Sigma \), given in eq. (102). The different terms from the expressions containing \( \Sigma^{(1)}_0 \) are:

\[
\text{Tr} G_0 \Sigma_0^{(1)} G_0 \Sigma_0^{(1)} = \text{Tr} G_0 \Sigma_0^{(1,\tau)} G_0 \Sigma_0^{(1,\tau)} + 2\text{Tr} G_0 \Sigma_0^{(1,\tau)} G_0 \left[ \Sigma_0^{(1,\tau')} + \Sigma_0^{(1,\tau'')} \right] + \text{Tr} G_0 \Sigma_0^{(1,\tau')} G_0 \Sigma_0^{(1,\tau')} + \text{Tr} G_0 \Sigma_0^{(1,\tau')} G_0 \Sigma_0^{(1,\tau')}.
\]

(109)

The same expression results for the terms containing \( \Sigma^{(1)}_z \).

In the following we consider the different nonvanishing contributions. From the temporal part we have

\[
\text{Tr} G_0 \Sigma_0^{(1,\tau)} G_0 \Sigma_0^{(1,\tau)} + \text{Tr} G_0 \Sigma_0^{(1,\tau)} G_0 \Sigma_0^{(1,\tau)} = \frac{4\hat{\rho} \kappa}{d^2} \int_0^\beta d\tau \int d^2 x (\vec{z} \partial_x z)^2,
\]

(110)

where the electronic compressibility is defined as usually,

\[
\kappa \equiv \frac{1}{\hat{\rho}} \frac{\partial \hat{\rho}}{\partial \mu}.
\]

(111)

From terms proportional to \( t \) we have

\[
\text{Tr} G_0 \Sigma_0^{(1,\tau)} G_0 \Sigma_0^{(1,\tau)} = 16t^2 \hat{\rho} \int_0^\beta d\tau \int d^2 x \left[ 2\kappa_1 G^* G + \sqrt{2}\kappa_1 G^* (F_y - F_x) + \sqrt{2}\kappa_1 (F^*_y - F^*_x) G \\
+ (\kappa_2 F^*_y F_y - \kappa_1 F^*_y F_x - \kappa_1 F^*_x F_y + \kappa_2 F^*_x F_x) \right],
\]

(112)

where we defined

\[
\kappa_1 \equiv \frac{1}{\hat{\rho}} \frac{\partial}{\partial \mu} \frac{1}{N} \sum_k n(k) \cos^2 \left( \frac{\sqrt{2}}{2} k_x a \right) \cos^2 \left( \frac{\sqrt{2}}{2} k_y a \right),
\]

(113)

and

\[
\kappa_2 \equiv \frac{1}{\hat{\rho}} \frac{\partial}{\partial \mu} \frac{1}{N} \sum_k n(k) \cos^2 \left( \frac{\sqrt{2}}{2} k_x a \right).
\]

(114)

The terms proportional to \( t' \) lead to

\[
\text{Tr} G_0 \Sigma_0^{(1,\tau')} G_0 \Sigma_0^{(1,\tau')} + \text{Tr} G_0 \Sigma_0^{(1,\tau')} G_0 \Sigma_0^{(1,\tau')} = -32t'^2 a^2 \hat{\rho} \kappa_3 \sum_j \int_0^\beta d\tau \left[ (\vec{z} \partial_x z)^2 + (\vec{z} \partial_y z)^2 \right],
\]

(115)
where we defined
\[
\kappa_3 = \frac{1}{\tilde{\rho} \partial \mu N} \sum_k n(k) \sin^2 \left( \sqrt{2} k_x a \right). \tag{116}
\]

From the terms proportional to \( t'' \) we obtain
\[
\text{Tr} \, G_0 \Sigma_0^{(1,t'')} G_0 \Sigma_0^{(1,t'')} + \text{Tr} \, G_0 \Sigma_z^{(1,t'')} G_0 \Sigma_z^{(1,t'')} = -64 t'' a^2 \tilde{\rho} \kappa_4 \int_0^\beta d\tau \left[ (\bar{z} \partial_x z)^2 + (\bar{z} \partial_y z)^2 \right], \tag{117}
\]
with
\[
\kappa_4 = \frac{1}{\tilde{\rho} \partial \mu N} \sum_k n(k) \sin^2 \left( \sqrt{2} k_x a \right) \cos^2 \left( \sqrt{2} k_y a \right). \tag{118}
\]

Finally, there is a contribution proportional to \( t' t'' \) of the form
\[
\text{Tr} \, G_0 \Sigma_0^{(1,t')} G_0 \Sigma_0^{(1,t')} + \text{Tr} \, G_0 \Sigma_z^{(1,t')} G_0 \Sigma_z^{(1,t')} = -64 t' t'' a^2 \tilde{\rho} \kappa_5 \int_0^\beta d\tau \left[ (\bar{z} \partial_x z)^2 + (\bar{z} \partial_y z)^2 \right], \tag{119}
\]
where
\[
\kappa_5 = \frac{1}{\tilde{\rho} \partial \mu N} \sum_k n(k) \left[ \sin^2 \left( \sqrt{2} k_x a \right) \cos \left( \sqrt{2} k_y a \right) + \sin^2 \left( \sqrt{2} k_y a \right) \cos \left( \sqrt{2} k_x a \right) \right], \tag{120}
\]

Collecting all contributions after integrating fermions out, we have
\[
S_F \to S_F^{(z)} + S_F^{(c)}, \tag{121}
\]
where
\[
S_F^{(z)} = \int_0^\beta d\tau \int d^2 x \left\{ 16 \left( t' \tilde{\rho}_1 + t'' \tilde{\rho}_2 \right) (\partial_x \bar{z} \partial_x z + \partial_y \bar{z} \partial_y z) - \frac{2 \tilde{\rho} \kappa_5}{a^2} (\bar{z} \partial_x z)^2 \right. \nonumber \\
+ 16 \tilde{\rho} \left( J - t^2 \kappa_2 \right) \left( F_{x}^{*} F_{x} + F_{y}^{*} F_{y} \right) - 8 \tilde{\rho} \left( J - 2 t^2 \kappa_1 \right) \left( F_{x}^{*} F_{y} + F_{y}^{*} F_{x} \right) \\
+ 16 \tilde{\rho} \left( t^2 \kappa_3 + 2 t'' \kappa_4 + 4 t' t'' \kappa_5 \right) \left[ (\bar{z} \partial_x z)^2 + (\bar{z} \partial_y z)^2 \right] \right\}, \tag{122}
\]
contains only contributions with the \( z \)-field, and
\[
S_F^{(c)} = \int_0^\beta d\tau \int d^2 x \left\{ - \frac{4 \tilde{\rho}}{a} (\bar{z} \partial_x \zeta + \bar{\zeta} \partial_x z) \right. \nonumber \\
+ 8 \tilde{\rho} \left( J - 2 t^2 \kappa_1 \right) \left\{ 2 G^{*} G + \sqrt{2} \left[ G^{*} (F_{y} - F_{x}) + (F_{y}^{*} - F_{x}^{*}) G \right] \right\}, \tag{123}
\]
contains contributions with \( \zeta \)-fields.

VI. INTEGRATION OF FAST MAGNETIC MODES

The effective action at this stage contains only the slow modes described by the \( z \)-fields and the fast modes corresponding to the \( \zeta \)-fields. The next step is to integrate out the \( \zeta \)-fields. However, due to the gauge freedom introduced by the CP\(^1\) fields, a gauge fixing is necessary.
A. Gauge fixing

In order to integrate out the \( \zeta \)-field we have to discuss the gauge-fixing for the \( z \)-fields and its consequences on the \( \zeta \)-fields. As already mentioned in Sec. [III A 2] the condition (60) can be used to fix a global phase of the \( \zeta \)-field with respect to the one of the \( z \)-field. In order to see this, we can use the following parametrization of the \( z \)-field:

\[
\mathbf{z} = \begin{bmatrix} \cos \left( \frac{\theta}{2} \right) \exp \left\{ -i \left( \frac{\theta}{2} - \Lambda \right) \right\} \\ \sin \left( \frac{\theta}{2} \right) \exp \left\{ +i \left( \frac{\theta}{2} + \Lambda \right) \right\} \end{bmatrix},
\]

where \( 0 \leq \theta \leq \pi \). For \( \zeta \) we use a similar parametrization

\[
\mathbf{\zeta} = \begin{bmatrix} \rho_1 \exp \left\{ -i \left( \frac{\theta}{2} - \Gamma \right) \right\} \\ \rho_2 \exp \left\{ +i \left( \frac{\theta}{2} + \Gamma \right) \right\} \end{bmatrix}.
\]

Then, the condition (66) can be fulfilled with \( \Gamma = \Lambda + (2m + 1)\pi /2 \), with \( m \) integer, that is equivalent to

\[
\frac{\zeta_1}{|\zeta_1|} \frac{\zeta_2}{|\zeta_2|} = e^{2i\Gamma} = e^{2i\Lambda} = -\frac{z_1}{|z_1|} \frac{z_2}{|z_2|},
\]

that can be translated into

\[
1 - \frac{|z_1||\zeta_1|}{|z_2||\zeta_2|} = 0.
\]

Then, the condition (66) can be enforced as follows:

\[
\delta \left( \bar{z} \zeta + \bar{\zeta} z \right) = \frac{|z_2||\zeta_2|}{|z_1|(|\zeta_2^* z_2 + z_2^* \zeta_2|)} \delta \left( \frac{|z_2||\zeta_2|}{|z_1|} - |\zeta_1| \right),
\]

where we solved the constraint (127) in favor of \(|\zeta_1|\).

Once we enforced the constraint (66), we impose the following gauge fixing for the \( z \) field

\[
z_1 + z_1^* = 0,
\]

as normally done in a CP\(^1\) theory. The implementation of the constraint (128) in the part of the action containing \( \zeta \)-fields (123) is given in Appendix [C]

B. Change of measure

We discuss here the change of variables from the real vectors \( \mathbf{\Omega}^{A,B} \) to the complex fields \( z \) and \( \zeta \), taking into account the Jacobian of the transformation, together with the constraints for the \( z \)- and \( \zeta \)-fields already discussed in Sec. [VI A] For this purpose we first consider (45) and (46).

\[
\mathbf{\Omega}^A_j = \left( \bar{z}_j \sqrt{1 - a^2 \bar{\zeta}_j \zeta_j} - a \zeta_j \right) \sigma_j \left( z_j \sqrt{1 - a^2 \bar{\zeta}_j \zeta_j} - a \zeta_j \right),
\]

\[
\mathbf{\Omega}^B_j = \left( z_j \sqrt{1 - a^2 \bar{\zeta}_j \zeta_j} + a \zeta_j \right) \sigma^y \sigma^y \left( z_j^* \sqrt{1 - a^2 \bar{\zeta}_j \zeta_j} + a \zeta_j^* \right).
\]

Then, we have

\[
|\mathbf{\Omega}^A_j|^2 = \left( \bar{z}_j z_j \left( 1 - a^2 \bar{\zeta}_j \zeta_j \right) + a^2 \bar{\zeta}_j \zeta_j - a \left( \bar{z}_j \zeta_j + \bar{\zeta}_j z_j \right) \sqrt{1 - a^2 \bar{\zeta}_j \zeta_j} \right)^2,
\]

\[
|\mathbf{\Omega}^B_j|^2 = \left( \bar{z}_j z_j \left( 1 - a^2 \bar{\zeta}_j \zeta_j \right) + a^2 \bar{\zeta}_j \zeta_j + a \left( \bar{z}_j \zeta_j + \bar{\zeta}_j z_j \right) \sqrt{1 - a^2 \bar{\zeta}_j \zeta_j} \right)^2.
\]

Therefore,

\[
\delta \left( |\mathbf{\Omega}^A| - 1 \right) \delta \left( |\mathbf{\Omega}^B| - 1 \right) \simeq \left( 1 + \frac{3}{2} a^2 \bar{\zeta} \zeta \right) \delta (\bar{z} z - 1) \delta (\bar{\zeta} z + \bar{z} \zeta).
\]
The transformation of the last constraint due to gauge fixing was already discussed in (128). On the other hand, we have due to (127)

\[ \bar{\zeta} \zeta = \frac{|\zeta_2|^2}{|z_1|^2}, \]  

(134)

where we used \( \bar{z}z = 1 \).

Due to gauge fixing, and without imposing the constraints on the moduli of the vector fields \( \Omega^A \) and \( \Omega^B \), since we take them into account with (133), we have independent variables \( \text{Im} z_1, \text{Re} z_1, \text{Im} z_2, |\zeta_1|, \text{Re} \zeta_2, \) and \( \text{Im} \zeta_2 \). The Jacobian going from vector fields to \( \mathbb{C}P^1 \) variables is in an expansion in powers of \( a \),

\[ J = a^3 (J_0 + a^2 J_2) + O(a^6), \]  

(135)

where

\[ J_0 = 32 \frac{i \text{Im} z_1 (2 |z_1|^2 - 1) (\zeta_2^* z_2 + z_2^* \zeta_2)}{|z_1| |z_2| |\zeta_2|}, \]  

(136)

and

\[ J_2 = -48 \frac{i \text{Im} z_1 (2 |z_1|^2 - 1) (\zeta_2^* z_2 + z_2^* \zeta_2)}{|z_1| |z_2| |\zeta_2|}, \]  

(137)

such that we finally have

\[ J = a^3 32 \frac{i \text{Im} z_1 (2 |z_1|^2 - 1) (\zeta_2^* z_2 + z_2^* \zeta_2)}{|z_1| |z_2| |\zeta_2|} \left(1 - \frac{3}{2} a^2 |\zeta_2|^2 \right). \]  

(138)

Taking into account the Jacobian above, the constraints in Sec. VI A the transformation of the constraints (133) together with (134), the measure changes as follows

\[ \int d\Omega^A d\Omega^B \delta(|\Omega^A| - 1) \delta(|\Omega^B| - 1) \rightarrow \int \text{dIm} z_1 \text{dRe} z_2 \text{dIm} z_2 \delta(\bar{z}z - 1) \frac{i \text{Im} z_1 (2 |z_1|^2 - 1)}{|z_1|^2} \]  

\[ \times \int \text{dRe} \zeta_2 \text{dIm} \zeta_2. \]  

(139)

C. Effective field theory for the magnetic properties of the t-J model

The final step is to integrate out the \( \zeta \)-field. Therefore, we concentrate on the corresponding integrals.

\[ \int D\text{Re} \zeta_2 D\text{Im} \zeta_2 e^{-S_{\text{eff}}(\zeta)} \]  

\[ = \int D\text{Re} \zeta_2 D\text{Im} \zeta_2 \exp \left\{ - \int_0^3 \frac{d\tau}{2} \int d^2x \left[ \zeta_2^2 \Delta \zeta_2 + \Lambda^* \zeta_2^2 + \Lambda \zeta_2^2 + (\Xi - \Gamma) \zeta_2^2 - (\Xi^* + \Gamma^*) \zeta_2 \right] \right\}, \]  

(140)

where we introduced the following notation:

\[ \bar{J} = [J + 8 \rho (J - 2t^2 \kappa_1)] , \]  

\[ \Delta = 8J \frac{|z_2|^4 + |z_1|^4}{|z_1|^2} , \]  

\[ \Lambda = 8 \bar{J} z_2^2 , \]  

\[ \Xi = \left[ \frac{2}{a} \frac{(1 - 2\rho)}{a} \right] z_2 \left( \frac{1}{|z_1|^2} z_1 \partial_\tau z_1^* + \frac{1}{|z_2|^2} z_2^* \partial_\tau z_2 \right) , \]  

\[ \Gamma = 2 \sqrt{2} \bar{J} z_2 \left[ \frac{z_2^* z_2^*}{|z_2|^2} (F_y - F_x) + (F_y^* - F_x^*) \frac{z_2^* z_2^*}{|z_1|^2} \right]. \]  

(141)
We go now over to real and imaginary parts \( \zeta_2 = \eta + i\xi \), such that for eq. (140) we have

\[
\int D\text{Re\zeta}_2 D\text{Im\zeta}_2 e^{-\mathcal{S}(\zeta)} \propto \int D\eta D\xi \exp \left[ -\int_0^\beta d\tau \int d^2 x \left( A\eta^2 + B\xi^2 + 2i\mathcal{C}\eta\xi + D\eta + i\mathcal{E}\xi \right) \right], \tag{142}
\]

with \( A \equiv \Delta + \Lambda + \Lambda, \quad B \equiv \Delta - (\Lambda^* + \Lambda), \quad C \equiv \Lambda^* - \Lambda, \quad D \equiv \Xi - \Xi^* - (\Gamma^* + \Gamma), \quad \text{and} \quad \mathcal{E} \equiv - (\Xi + \Xi^*) + (\Gamma - \Gamma^*). \)

Performing the integrals over the \( \zeta \)-fields leads to the effective action

\[
\tilde{S}_{\text{eff}} = \int d\tau \sum_j \left\{ 1 \over 2 \ln (AB + C^2) + \left( AB^2 - 2CD\mathcal{E} - BD^2 \right) \right\} \tag{143}
\]

Next we proceed to evaluate the different contributions to the action. We consider first the term that affects the measure

\[
AB + C^2 = \left( {8 \over |z_1|^2} \left[ J + 8\hat{\rho} \left( J - 2t^2\kappa_1 \right) \right] \left( 1 - 2 |z_1|^2 \right) \right)^2 \tag{144}
\]

Then,

\[
\exp \left[ -1 \over 2 \ln (AB + C^2) \right] = \left\{ 1 \over \left[ J + 8\hat{\rho} \left( J - 2t^2\kappa_1 \right) \right] \left( 1 - 2 |z_1|^2 \right) \right\}^{-1} \tag{145}
\]

such that the measure and constraints of the path-integral are now

\[
Z \rightarrow \int \mathcal{D}\bar{z} \mathcal{D}z \delta (\bar{z}z - 1) \delta (z_1 + \bar{z}_1) (z_1 - \bar{z}_1) \cdots \tag{146}
\]

that are the ones usually appearing in the CP\(^1\) model.

After a lengthy but straightforward calculation we have

\[
\left( AB^2 - 2CD\mathcal{E} - BD^2 \right) \over 4(AB + C^2) = \left\{ 1 \over 2J \right\} \left[ (1 - 2\hat{\rho}) \over a \right]^2 \left[ \partial_t \bar{z} \partial_t z + (\bar{z}\partial_t z)^2 \right]
\]

\[
+ \left\{ 1 \over \sqrt{2} \right\} \left[ (1 - 2\hat{\rho}) \over a \right] \left[ \partial_t \bar{z} \left( \bar{z}\partial_y z - \bar{z}\partial_z z \right) - \partial_t z \left( \partial_y \bar{z} - \partial_z \bar{z} \right) \right]
\]

\[
- \tilde{J} \left[ \partial_x \bar{z} \partial_t z + \partial_y \bar{z} \partial_y z + (\bar{z}\partial_x z)^2 + (\bar{z}\partial_y z)^2 \right.
\]

\[
\left. - \partial_x \bar{z} \partial_y z - \partial_y \bar{z} \partial_x z - 2 (\bar{z}\partial_x z) (\bar{z}\partial_y z) \right\} \tag{147}
\]

Taking into account the contribution to the action in (122), that remains unaffected by the integration over the \( \zeta \)-fields, we arrive at the effective action for the \( z \)-fields

\[
S = \int d\tau d^2 x \sum_\mu {1 \over g_\mu} \left[ \partial_\mu \bar{z} \partial_\mu z + \gamma_\mu (\bar{z}\partial_\mu z)^2 \right] \tag{148}
\]

with \( \mu = \tau, x, y, \) and

\[
g_\tau = {2\tilde{J}a^2 \over (1 - 2\hat{\rho})},
\]

\[
g_x = g_y = \left[ J \left( 1 \over 1 + \hat{\rho} \right) + 16\hat{\rho} \left( \kappa_1 - \kappa_2 \right) + 16 \left( t^\prime \hat{\rho}_1 + t^\prime t^\prime \hat{\rho}_2 \right) \right]^{-1},
\]

\[
\gamma_\tau = 1 - 4\hat{\rho} \tilde{J} \over (1 - 2\hat{\rho}) \tag{149},
\]

\[
\gamma_x = \gamma_y = \left[ J \left( 1 \over 1 + \hat{\rho} \right) + 16\hat{\rho} \left[ t^2 \left( \kappa_1 - \kappa_2 \right) + t^2 \kappa_3 + 2t^\prime t^\prime \kappa_4 + 4t^\prime t^\prime t^\prime \kappa_5 \right] \right. \over \left[ J \left( 1 \over 1 + \hat{\rho} \right) + 16\hat{\rho} \left( \kappa_1 - \kappa_2 \right) + 16 \left( t^\prime \hat{\rho}_1 + t^\prime t^\prime \hat{\rho}_2 \right) \right],
\]

where \( \tilde{J} \) was defined in (141). Equation (148) together with (149) are the main result of this work.
VII. DISCUSSION OF THE RESULTS AND CONCLUSIONS

The action [148] at which we arrived, is of the form previously analyzed in the context of frustrated quantum antiferromagnets [12, 13, 14]. In the absence of doping, i.e. setting \( \hat{\rho} = \hat{\rho}_1 = \hat{\rho}_2 = 0 \), we have \( \gamma_x = \gamma_y = 1 \), such that the model reduces to the CP\(^1\) model appropriate for a Heisenberg antiferromagnet, making thus explicitly the connection with the undoped case. As is well known, the excitations in that case correspond to bosons describing the transverse fluctuations of a vector field pertaining to the O(3) non-linear \( \sigma \)-model. From the point of view of the CP\(^1\) model this corresponds to a phase where the z-fields are confined.

However, in the presence of doping, when the couplings \( \gamma_\mu \neq 1 \), the model does not correspond any more to a collinear antiferromagnet but describes in general coplanar incommensurate quantum antiferromagnets [12, 13]. This can be seen by constructing \( R \in \text{SO}(3) \) out of matrices \( g \in \text{SU}(2) \), as follows:

\[
R_{ab}(g) = \frac{1}{2} \text{Tr} \sigma^a g^b g^d, \tag{150}
\]

where \( g \) is given in terms of the z-fields like in [144]. Then, it can be readily shown that the action [148] can be expressed in terms of SO(3) fields:

\[
S = \int d\tau dx^2 \sum_\mu \frac{1}{8g_\mu} \text{Tr}(\partial_\mu R \partial_\mu R^{-1} - \gamma_\mu \partial_\mu R \partial_\nu R \partial_\nu R^{-1}) , \tag{151}
\]

where \( Q = \text{diag}(1,1,-1) \). Expressing the matrix \( R \) as \( R = (n_1, n_2, n_3) \), where the vectors \( n_\mu \) fulfill \( n_\mu^2 = 1 \) for \( i = 1,2,3 \) and \( n_i \cdot n_j = 0 \) for \( i \neq j \), the action [151] takes the form

\[
S = \int d\tau dx^2 \sum_\mu \frac{1}{4g_\mu} \left[ \sum_{i=1}^2 \partial_\mu n_i \cdot \partial_\mu n_i - (1 + \gamma_\mu)(n_1 \cdot \partial_\mu n_2)^2 \right] , \tag{152}
\]

showing that in general a coplanar configuration is favored. In the case \( \gamma_\mu = 1 \), the action above reduces as expected to the O(3) non-linear \( \sigma \)-model by virtue of the relation

\[
\partial_\mu n_3 \cdot \partial_\mu n_3 = \partial_\mu n_1 \cdot \partial_\mu n_1 + \partial_\mu n_2 \cdot \partial_\mu n_2 - 2(n_1 \cdot \partial_\mu n_2)^2 . \tag{153}
\]

On the other hand, for the general case \( \gamma_\mu \neq 1 \), assuming for simplicity that \( \gamma_\mu = \gamma \) and \( g_\mu = g \) for \( \mu = \tau, x, y \), it can be seen that, following Ref. [13], the gauge fields responsible for confinement in the case \( \gamma = 1 \), acquire a mass

\[
M^2 = \frac{2}{g} \left( 1 - \frac{1}{\gamma} \right) . \tag{154}
\]

Moreover, it is expected that in the infrared limit, the behavior is dominated by an O(4) fixed point [12, 13], i.e. it is expected that the coupling \( \gamma \) scales to zero, raising the mass of the gauge fields. Further arguments in Cavour of deconfinement were advanced by showing that for \( \gamma \neq 1 \), the mass of the gauge field is inversely proportional to the square root of the temperature [14], and therefore, a confinement-deconfinement transition as a function of temperature can be expected when \( \gamma \) departs from 1. Hence, we see that the possibility of deconfined (bosonic) spinons is opened due to doping of a collinear antiferromagnet.

Finally, we would like to remark, that the present results show some differences from those obtained previously by one of the authors [48] and collaborators, dealing with a doped antiferromagnet modeled by the so-called spin-fermion model. This model is the starting point that leads to the \( t-J \) model in the limit where the exchange coupling between the dopant hole, that mainly resides on the oxygen orbitals, and the copper hole form the Zhang-Rice singlet [27]. The gradient expansion for the spin-fermion model based on the assumption of a short-range antiferromagnetic order led to an O(3) non-linear \( \sigma \)-model, that as a function of doping had a transition to the corresponding quantum disordered phase [44]. Hence, under the same assumption as in the present work, no hint to deconfinement of spinons was obtained. An SO(3) non-linear \( \sigma \)-model can be obtained from the spin-fermion model only under the assumption of an incommensurate coplanar short-range order in the microscopic model [51]. Therefore, although considering the symmetries present in a doped antiferromagnet, both models appear equivalent, we conclude on the basis of the present results that imposing the constraint against double occupancy leads to a richer picture of the possible phases of a doped antiferromagnet, with a minimal number of assumptions.

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Appendix A: Set of Constraints for the \( t-J \) Model

We give here the explicit expressions of the constraints arising from considering the canonical momenta of the system, and the matrix of constraints. First we recall the definitions of right and left derivatives in order to reach a self-contained presentation.

Given generators \( z^a \) of a Berezin algebra with \( a = 1, \ldots, k \), the right derivative is defined as

\[
\frac{\partial_r}{\partial z^a} z^{a_1} \ldots z^{a_k} = \sum_{i=1}^k (-1)^{i+k} P(a_i) P(a_j) \delta_{a_i a_j} z^{a_1} \ldots \hat{z}^{a_i} \ldots z^{a_k},
\]

where \( P(a) = 0,1 \) depending on the parity of the generator (even or odd, respectively). For later use, we introduce also left derivatives defined as

\[
\frac{\partial_l}{\partial z^a} z^{a_1} \ldots z^{a_k} = \sum_{i=1}^k (-1)^{i+k} P(a_i) P(a_j) \delta_{a_i a_j} z^{a_1} \ldots \hat{z}^{a_i} \ldots z^{a_k}.
\]

In both cases it is understood that when the index \( j \) runs beyond the interval \([1,k]\), then \( P(a_j) = 0 \). Both derivatives are equivalent to ordinary derivatives when dealing with even generators \( z^a \).

With the rules above we obtain the following constraints from \([13]\).

\[
\phi^{(3)} = \Pi^{++}, \quad \phi^{(4)} = \Pi^{+-} + i \frac{(1+\rho)u-1}{2-\rho-u^2} X^+, \quad \phi^{(5)} = \Pi^{--} - i \frac{(1+\rho)u-1}{2-\rho-u^2} X^-,
\]

\[
\phi^{(6)} = \Pi^{--}, \quad \phi^{(8)} = \Pi^{00}, \quad \phi^{(11)} = \Pi^{0+} - \frac{1}{2} X^{0+}, \quad \phi^{(14)} = \Pi^{0-} - \frac{1}{2} X^{0-}.
\]

The matrix of constraints \([13]\) has the following components

\[
A^{(0)} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & X^{--} & -1 & X^{--} & X^+ & X^+ & X^+ & X^+ & 0 \\
0 & -X^{--} & 0 & 4X^+ & -\frac{i}{4} & 0 & 0 & 0 & 0 \\
0 & -X^{--} & 0 & \frac{i}{4} & 0 & 0 & 0 & 0 & 0 \\
-1 & X^{++} & 0 & 0 & 0 & 4X^{++} & 4X^{++} & 4X^{++} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad (A4)
\]

\[
A^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i(2X^{++}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i(2X^{++}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (A5)
\]

The matrix \( B \) contains only single Grassmann generators.

\[
B = \begin{pmatrix}
0 & 0 & X^{0+} & X^{0-} & -X^{0+} & -X^{0-} \\
0 & 0 & X^{0+} & X^{0-} & -X^{0+} & -X^{0-} \\
B_{31} & X^{--} & 0 & 0 & 0 & 0 \\
B_{43} & 0 & B_{44} & B_{45} & B_{46} & 0 \\
B_{53} & 0 & B_{54} & B_{55} & B_{56} & 0 \\
0 & 0 & -X^{0+} & -X^{0-} & X^{0+} & X^{0-} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (A6)
\]
where

\[
B_{31} = \frac{X^0+X^+}{(X^+)^2}, \quad B_{41} = -\frac{X^0+}{X^+},
\]

\[
B_{43} = -\frac{i(1+X^-+X^++X^-+X^-)X^0+}{4X^+X^-}, \quad B_{44} = -\frac{i(1+X^-+X^++X^-+X^-)X^0-}{4X^+X^-},
\]

\[
B_{45} = \frac{i(1+X^-+X^++X^-+X^-)X^0+}{4X^+X^-}, \quad B_{46} = \frac{i(1+X^-+X^++X^-+X^-)X^0-}{4X^+X^-},
\]

\[
B_{53} = \frac{i(1+X^-+X^++X^-+X^-)X^0+}{4X^+X^-}, \quad B_{54} = \frac{i(1+X^-+X^++X^-+X^-)X^0-}{4X^+X^-},
\]

\[
B_{55} = -\frac{i(1+X^-+2X^++X^-)X^0+}{4X^+X^-}, \quad B_{56} = -\frac{i(1+X^-+2X^++X^-)X^0-}{4X^+X^-}.
\]

Furthermore, \(C = -B^T\), and

\[
D = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{X^+}{X^+} & -1 \\
0 & 0 & X^- & -X^+ & 0 & 0 \\
0 & X^- & 0 & 0 & -i & 0 \\
0 & -X^+ & 0 & 0 & 0 & -i \\
\frac{X^-}{X^+} & 0 & -i & 0 & 0 & 0 \\
-1 & 0 & 0 & -i & 0 & 0
\end{pmatrix}.
\]

**APPENDIX B: FERMIONIC TERMS IN THE STAGGERED CP\(^1\) REPRESENTATION**

We give here the explicit expressions for the fermionic contributions up to \(O(a^2)\). Defining vectors

\[
x^{(1)} = (0, 0), \quad x^{(2)} = (0, \sqrt{2}a), \quad x^{(3)} = \sqrt{2}a(-1, 1), \quad x^{(4)} = \sqrt{2}a(-1, 0),
\]

we have for the fermionic terms \(\Xi^{(i)}_{AB}(k_1, k_2)\) with \(i = 1, \ldots, 4\)

\[
\Xi^{(i)}_{AB}(k_1, k_2) = -\frac{2a}{N} \exp \left\{ i \left[ k_1 \cdot x_A - k_2 \cdot (x_B + x^{(i)}) \right] \right\} \sum_j \exp \left[ i (k_1 - k_2) \cdot x_j \right] \tilde{U}^{(i)}_{j,AB},
\]

where \(\tilde{U}^{(i)}_{j,AB}\) denotes the matrix element of the product of the SU(2) matrices in nearest neighbor sites, connecting sublattices \(A\) and \(B\) within the unit cell \(j\). The index \(i\) denotes the four nearest neighbors starting from sublattice \(A\). Using the notation introduced in (A3), the gradient expansion of the products of the SU(2) matrices up to \(O(a)\) have the following form

\[
\tilde{U}^{(1)}_{j,AB} = G^*_j, \quad \tilde{U}^{(2)}_{j,AB} = G^*_j + \sqrt{2}F^*_{jy}, \quad \tilde{U}^{(3)}_{j,AB} = G^*_j - \sqrt{2}(F^*_{jx} - F^*_{yx}), \quad \tilde{U}^{(4)}_{j,AB} = G^*_j - \sqrt{2}F^*_{jx}.
\]

For the fermionic terms \(\Xi^{(i)}_{BA}(k_1, k_2)\) we have

\[
\Xi^{(i)}_{BA}(k_1, k_2) = -\frac{2a}{N} \exp \left\{ i \left[ k_1 \cdot x_B - k_2 \cdot (x_A - x_1) \right] \right\} \sum_j \exp \left[ i (k_1 - k_2) \cdot x_j \right] \tilde{U}^{(i)}_{j,BA},
\]

with \(\tilde{U}^{(i)}_{j,BA} = \tilde{U}^{(i)*}_{j,AB}\).

Next we list the terms originating from contributions proportional to \(t'\). For \(\Psi^{(i)}_{AA}(k_1, k_2), i = 1, \ldots, 4\), we have

\[
\Psi^{(1,3)}_{AA}(k_1, k_2) = \frac{2}{N} \exp \left[ \pm ik_2 \cdot x_4 \right] \sum_j \exp \left[ i (k_1 - k_2) \cdot x_j \right] \tilde{U}^{(1,3)}_{j,AA},
\]

\[
\Psi^{(2,4)}_{AA}(k_1, k_2) = \frac{2}{N} \exp \left[ \mp ik_2 \cdot x_2 \right] \sum_j \exp \left[ i (k_1 - k_2) \cdot x_j \right] \tilde{U}^{(2,4)}_{j,AA},
\]

where \(\tilde{U}^{(i)}_{j,AA}\) denotes the product of SU(2) matrices on second nearest neighbor sites on sublattice \(A\) along the diagonals, starting from the unit cell \(j\), and \(i\) numbers the four possibilities. They have the following expansion in
powers of $a$:

\[
U^{(1)}_{j,AA} = \sqrt{2} a \bar{z}_j \partial_x z_j + a^2 \left[ \bar{z}_j \partial^2_x z_j - \sqrt{2} \left( \bar{z}_j \partial_x \bar{c}_j + \bar{c}_j \partial_x z_j \right) \right] ,
\]

\[
U^{(2)}_{j,AA} = \sqrt{2} a \bar{z}_j \partial_y z_j + a^2 \left[ \bar{z}_j \partial^2_y z_j - \sqrt{2} \left( \bar{z}_j \partial_y \bar{c}_j + \bar{c}_j \partial_y z_j \right) \right] ,
\]

\[
U^{(3)}_{j,AA} = -\sqrt{2} a \bar{z}_j \partial_x z_j + a^2 \left[ \bar{z}_j \partial^2_x z_j + \sqrt{2} \left( \bar{z}_j \partial_x \bar{c}_j + \bar{c}_j \partial_x z_j \right) \right] ,
\]

\[
U^{(4)}_{j,AA} = -\sqrt{2} a \bar{z}_j \partial_y z_j + a^2 \left[ \bar{z}_j \partial^2_y z_j + \sqrt{2} \left( \bar{z}_j \partial_y \bar{c}_j + \bar{c}_j \partial_y z_j \right) \right] .
\]  

(B7)

For $\Psi^{(i)}_{BB}$, we have the same phase factors as for $\Psi^{(i)}_{AA}$ but we have to replace $U^{(i)}_{j,AA}$ by $U^{(i)}_{j,BB}$ that are as follows:

\[
U^{(1)}_{j,BB} = \sqrt{2} a z_j \partial_x \bar{z}_j + a^2 \left[ z_j \partial^2_x \bar{z}_j + \sqrt{2} \left( z_j \partial_x \bar{c}_j + \bar{c}_j \partial_x \bar{z}_j \right) \right] ,
\]

\[
U^{(2)}_{j,BB} = \sqrt{2} a z_j \partial_y \bar{z}_j + a^2 \left[ z_j \partial^2_y \bar{z}_j + \sqrt{2} \left( z_j \partial_y \bar{c}_j + \bar{c}_j \partial_y \bar{z}_j \right) \right] ,
\]

\[
U^{(3)}_{j,BB} = -\sqrt{2} a z_j \partial_x \bar{z}_j + a^2 \left[ z_j \partial^2_x \bar{z}_j - \sqrt{2} \left( z_j \partial_x \bar{c}_j + \bar{c}_j \partial_x \bar{z}_j \right) \right] ,
\]

\[
U^{(4)}_{j,BB} = -\sqrt{2} a z_j \partial_y \bar{z}_j + a^2 \left[ z_j \partial^2_y \bar{z}_j - \sqrt{2} \left( z_j \partial_y \bar{c}_j + \bar{c}_j \partial_y \bar{z}_j \right) \right] .
\]  

(B8)

For the contributions proportional to $l''$ we have for processes involving the sublattice $A$,

\[
\Phi^{(1,3)}_{AA} (k_1, k_2) = \frac{2}{N} \exp \left[ \pm i k_2 \cdot x_3 \right] \sum_j \exp \left[ i (k_1 - k_2) \cdot x_j \right] U^{(1,3)}_{j,AA} ,
\]

(B9)

\[
\Phi^{(2,4)}_{AA} (k_1, k_2) = \frac{2}{N} \exp \left[ \mp i k_2 \cdot (x_2 - x_4) \right] \sum_j \exp \left[ i (k_1 - k_2) \cdot x_j \right] U^{(2,4)}_{j,AA} ,
\]

(B10)

where $U^{(i)}_{j,AA}$ denotes the product of SU(2) matrices on second nearest neighbor sites on sublattice $A$ along the principal axes, starting from the unit cell $j$, and $i$ numbers the four possibilities. They have the following expansion in powers of $a$:

\[
U^{(1)}_{j,AA} = \sqrt{2} a \left( \bar{z}_j \partial_x z_j - \bar{z}_j \partial_y z_j \right)
\]

\[
+ a^2 \left\{ \bar{z}_j \partial^2_x z_j - 2 \bar{z}_j \partial_x \partial_y z_j + \bar{z}_j \partial^2_y z_j - \sqrt{2} \left[ \bar{z}_j \partial_x \bar{c}_j + \bar{c}_j \partial_x z_j \right] \right\} ,
\]

\[
U^{(2)}_{j,AA} = \sqrt{2} a \left( \bar{z}_j \partial_x z_j + \bar{z}_j \partial_y z_j \right)
\]

\[
+ a^2 \left\{ \bar{z}_j \partial^2_x z_j + 2 \bar{z}_j \partial_x \partial_y z_j + \bar{z}_j \partial^2_y z_j - \sqrt{2} \left[ \bar{z}_j \partial_x \bar{c}_j + \bar{c}_j \partial_x z_j \right] \right\} ,
\]

\[
U^{(3)}_{j,AA} = -\sqrt{2} a \left( \bar{z}_j \partial_x z_j - \bar{z}_j \partial_y z_j \right)
\]

\[
+ a^2 \left\{ \bar{z}_j \partial^2_x z_j - 2 \bar{z}_j \partial_x \partial_y z_j + \bar{z}_j \partial^2_y z_j + \sqrt{2} \left[ \bar{z}_j \partial_x \bar{c}_j + \bar{c}_j \partial_x z_j \right] \right\} ,
\]

\[
U^{(4)}_{j,AA} = -\sqrt{2} a \left( \bar{z}_j \partial_x z_j + \bar{z}_j \partial_y z_j \right)
\]

\[
+ a^2 \left\{ \bar{z}_j \partial^2_x z_j + 2 \bar{z}_j \partial_x \partial_y z_j + \bar{z}_j \partial^2_y z_j + \sqrt{2} \left[ \bar{z}_j \partial_x \bar{c}_j + \bar{c}_j \partial_x z_j \right] \right\} .
\]  

(B11)
For $\Phi_{BB}^{(i)}$, we have the same phase factors as for $\Phi_{AA}^{(i)}$ but we have to replace $\bar{U}_{j,AA}^{(i)}$ by $\bar{U}_{j,BB}^{(i)}$ that are as follows:

\[
\begin{align*}
\bar{U}_{j,BB}^{(1)} &= \sqrt{2} a (z_j \partial_z \bar{z}_j - z_j \partial_y \bar{z}_j) \\
&\quad + a^2 \left\{ z_j \partial_y^2 \bar{z}_j - 2 z_j \partial_y \bar{y}_j \bar{z}_j + z_j \partial_y^2 \bar{y}_j + \sqrt{2} \left[ z_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) + \zeta_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) \right] \right\}, \\
\bar{U}_{j,BB}^{(2)} &= \sqrt{2} a (z_j \partial_z \bar{z}_j + z_j \partial_y \bar{z}_j) \\
&\quad + a^2 \left\{ z_j \partial_y^2 \bar{z}_j + 2 z_j \partial_y \bar{y}_j \bar{z}_j + z_j \partial_y^2 \bar{y}_j + \sqrt{2} \left[ z_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) + \zeta_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) \right] \right\}, \\
\bar{U}_{j,BB}^{(3)} &= -\sqrt{2} a (z_j \partial_z \bar{z}_j - z_j \partial_y \bar{z}_j) \\
&\quad + a^2 \left\{ z_j \partial_y^2 \bar{z}_j - 2 z_j \partial_y \bar{y}_j \bar{z}_j + z_j \partial_y^2 \bar{y}_j - \sqrt{2} \left[ z_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) + \zeta_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) \right] \right\}, \\
\bar{U}_{j,BB}^{(4)} &= -\sqrt{2} a (z_j \partial_z \bar{z}_j + z_j \partial_y \bar{z}_j) \\
&\quad + a^2 \left\{ z_j \partial_y^2 \bar{z}_j + 2 z_j \partial_y \bar{y}_j \bar{z}_j + z_j \partial_y^2 \bar{y}_j - \sqrt{2} \left[ z_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) + \zeta_j \left( \partial_z \bar{\zeta}_j - \partial_y \bar{\zeta}_j \right) \right] \right\}. \\
\end{align*}
\] (B12)

Finally, we list below the contributions from spin interactions dressed with fermions. We defined

\[
\mathcal{F}^{(e)} \equiv \frac{1}{2} \left[ \mathcal{F}^{(e)}(k_1, k_2) \pm \mathcal{F}^{(o)}(k_1, k_2) \right],
\] (B13)

with

\[
\mathcal{F}^{(e,o)}(k_1, k_2) = \frac{4 a^2}{N} \exp \left[ i (k_1 - k_2) \cdot x_A, B \right] \sum_j \exp \left[ i (k_1 - k_2) \cdot x_j \right] \Upsilon_j,
\] (B14)

where

\[
\Upsilon_j = 4 \left\{ 2 G_j G^*_{jy} + 2 F_{jy}^{} F^*_{jy} + 2 F_{jx}^{} F^*_{jx} - (F_{jy}^{} F^*_{jx} + F_{jx}^{} F^*_{jy}) + \sqrt{2} \left[ (F_{jy}^{} - F_{jx}^{}) G_j^* + G_j^{} (F_{jy}^{} - F_{jx}^{}) \right] \right\},
\] (B15)

contains the contributions from $\Omega_i \cdot \Omega_j$.

**APPENDIX C: EFFECTIVE ACTION $S_{\text{eff}}^{(c)}$**

We display here, in order to facilitate a reproduction of our results, the different contributions to the part of the action containing the $\zeta$-fields [123] after imposing the constraint [128]. We consider first the part containing temporal derivatives in [123]

\[
\bar{\zeta} \partial_t \zeta + \bar{\zeta} \partial_t \zeta = \frac{G^*_j \zeta_j}{|z_1|^2} z_1 \partial_t z^*_1 - \frac{z^2_j \zeta_j}{|z_1|^2} z^*_1 \partial_t z_1 + z^*_2 \partial_t \zeta_2 - z_2 \partial_t \zeta^*_2. \] (C1)

We consider next the interaction part in [123]. Here we have

\[
G = -2 z_1 \left( \frac{z^2_2 \zeta^*_2}{|z_1|^2} + \zeta_2 \right). \] (C2)

Then, for the different terms entering [123] we have,

\[
\begin{align*}
G^* G &= 4 \left( \frac{|z_2|^4 + |z_1|^4}{|z_1|^2} |\zeta_2|^2 + z^2_2 \zeta^*_2 + z^2_2 \zeta^*_2 + z^2_2 \zeta^*_2 \right), \\
G^* F_\alpha &= -2 z_1 \left( \frac{z^2_2 \zeta^*_2}{|z_1|^2} + \zeta_2 \right) F_\alpha, \\
F_\alpha^* G &= -2 F_\alpha^* z_1 \left( \frac{z^2_2 \zeta^*_2}{|z_1|^2} + \zeta_2 \right). \] (C3)

