Infinitely many local minima of sequentially weakly lower semicontinuous functionals and applications

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I am here concerned with an overview of some applications of Theorem 1 below that have been obtained in the last two years.

THEOREM 1 ([14]). - Let $X$ be a non-empty sequentially weakly closed set in a reflexive real Banach space, and let $\Phi, \Psi : X \to ]-\infty, +\infty]$ be two sequentially weakly lower semicontinuous functionals. Assume also that $\Psi$ is (strongly) continuous. Denote by $I$ the set of all real numbers $\rho > \inf X \Psi$ such that $\Psi^{-1}(-\infty, \rho]$ is bounded and intersects the domain of $\Phi$. Assume that $I \neq \emptyset$. For each $\rho \in I$, put

$$\varphi(\rho) = \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - \inf_{\Psi^{-1}(-\infty, \rho]} \Phi}{\rho - \Psi(x)},$$

where $(\Psi^{-1}(-\infty, \rho])_w$ is the closure of $\Psi^{-1}(-\infty, \rho]$ in the relative weak topology of $X$. Furthermore, set

$$\gamma = \liminf_{\rho \to (\sup I)^-} \varphi(\rho)$$

and

$$\delta = \liminf_{\rho \to (\inf X \Psi)^+} \varphi(\rho).$$

Then, the following conclusions hold:

(a) For each $\rho \in I$ and each $\mu > \varphi(\rho)$, the restriction of the functional $\Phi + \mu \Psi$ to $\Psi^{-1}(-\infty, \rho]$ has a global minimum.

(b) If $\gamma < +\infty$, then, for each $\mu > \gamma$, the following alternative holds: either the restriction of $\Phi + \mu \Psi$ to $\Psi^{-1}(-\infty, \sup I]$ has a global minimum, or there exists a sequence $\{x_n\}$ of local minima of $\Phi + \mu \Psi$ such that $\Psi(x_n) < \sup I$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \Psi(x_n) = \sup I$.

(c) If $\delta < +\infty$, then, for each $\mu > \delta$, there exists a sequence $\{x_n\}$ of local minima of $\Phi + \mu \Psi$, with $\lim_{n \to \infty} \Psi(x_n) = \inf X \Psi$, which weakly converges to a global minimum of $\Psi$.

A first consequence of conclusion (a) of Theorem 1 is as follows:

THEOREM 2 ([14]). - Let $E$ be a reflexive real Banach space, $X$ a closed, convex, unbounded subset of $E$, and $\Phi, \Psi : X \to \mathbb{R}$ two convex functionals, with $\Phi$ lower semicontinuous and $\Psi$ continuous and satisfying $\lim_{x \in X, \|x\| \to +\infty} \Psi(x) = +\infty$. Put

$$\lambda^* = \inf_{\rho > \inf X \Psi} \inf_{x \in \Psi^{-1}(-\infty, \rho]} \frac{\Phi(x) - \inf_{\Psi^{-1}(-\infty, \rho]} \Phi}{\rho - \Psi(x)}.$$
Then, for each $\lambda > \lambda^*$, the functional $\Phi + \lambda \Psi$ has a global minimum in $X$. Moreover, if $\lambda^* > 0$, for each $\mu < \lambda^*$, the functional $\Phi + \mu \Psi$ has no global minima in $X$.

**REMARK 1.** - The second conclusion of Theorem 2 does not hold, in general, if $\lambda^* = 0$. To see this, consider, for instance, the case when $\Phi(x) = \|x\|^2$ and $\Psi(x) = \|x\|$.

Let us recall that a Gâteaux differentiable functional $J$ on a real Banach space $X$ is said to satisfy the Palais-Smale condition if each sequence $\{x_n\}$ in $X$ such that $\sup_{n \in \mathbb{N}} |J(x_n)| < +\infty$ and $\lim_{n \to +\infty} \|J'(x_n)\|_{X^*} = 0$ admits a strongly converging subsequence.

A joint application of conclusion (a) of Theorem 1 with the main result of [13] gives the following two critical points theorem:

**THEOREM 3 ([17]).** - Let $X$ be a reflexive real Banach spaces and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functionals. In addition, assume that, for each $\lambda > 0$, the functional $\Phi + \lambda \Psi$ satisfies the Palais-Smale condition.

Then, for each $\rho > \inf_X \Psi$ and each $\mu$ satisfying

$$
\mu > \inf_{x \in \Psi^{-1}([-\infty, \rho])} \frac{\Phi(x) - \inf_{y \in \Psi^{-1}([-\infty, \rho])} \Phi(y)}{\rho - \Psi(x)},
$$

the following alternative holds: either the functional $\Phi + \mu \Psi$ has a strict global minimum which lies in $\Psi^{-1}([-\infty, \rho])$, or the same functional has at least two critical points one of which lies in $\Psi^{-1}([-\infty, \rho])$.

We also recall that if $X$ is a real Hilbert space, an operator $A : X \to X$ is said to be a potential operator if there exists a a Gâteaux differentiable functional $P$ on $X$ (which is called a potential of $A$) such that $P' = A$.

Applying Theorem 1, we get the following result about fixed points of potential operators in real Hilbert spaces:

**THEOREM 4 ([14]).** - Let $X$ be a real Hilbert space, and let $A : X \to X$ be a potential operator, with sequentially weakly upper semicontinuous potential $P$. For each $\rho > 0$, put

$$
\varphi(\rho) = \inf_{\|x\|^2 < \rho} \sup_{\|y\|^2 \leq \rho} \frac{P(y) - P(x)}{\rho - \|x\|^2}.
$$

Furthermore, set

$$
\gamma = \liminf_{\rho \to +\infty} \varphi(\rho)
$$

and

$$
\delta = \liminf_{\rho \to 0^+} \varphi(\rho).
$$

Then, the following conclusions hold:
(a) If there is $\rho > 0$ such that $\varphi(\rho) < \frac{1}{2}$, then the operator $A$ has a fixed point whose norm is less than $\sqrt{\rho}$.

(b) If $\gamma < \frac{1}{2}$, then the following alternative holds: either the functional $x \to \frac{1}{2}\|x\|^2 - P(x)$ has a global minimum, or the set of all fixed points of $A$ is unbounded.

(c) If $\delta < \frac{1}{2}$, then the following alternative holds: either $0$ is a local minimum of the functional $x \to \frac{1}{2}\|x\|^2 - P(x)$, or there exists a sequence of pairwise distinct fixed points of $A$ which strongly converges to $0$.

In particular, we have

**THEOREM 5 ([14]).** Let $X$ be a real Hilbert space, and let $A : X \to X$ be a potential operator, with sequentially weakly upper semicontinuous potential $P$ satisfying

$$
\liminf_{r \to +\infty} \sup_{\|x\| \leq r} \frac{P(x)}{r^2} < \frac{1}{2} < \limsup_{r \to +\infty} \sup_{\|x\| \leq r} \frac{P(x)}{r^2}.
$$

Then, the set of all fixed points of $A$ is unbounded.

¿From now on, $\Omega$ is an open bounded subset of $\mathbb{R}^n$, with smooth boundary, and (for $p > 1$) $W^{1,p}(\Omega)$, $W^{1,p}_0(\Omega)$ are the usual Sobolev spaces, with norms

$$
\|u\| = \left( \int_{\Omega} |\nabla u(x)|^p \, dx + \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}}
$$

and

$$
\|u\| = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}}
$$

respectively.

Let $p > 1$, and let $A : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function.

Recall that a weak solution of the Dirichlet problem

$$
\begin{cases}
-\text{div}(\|u|^{p-2}\nabla u) = f(x, u) & \text{in } \Omega \\
u|_{\partial \Omega} = 0
\end{cases}
$$

is any $u \in W^{1,p}_0(\Omega)$ such that

$$
\int_{\Omega} |\nabla u(x)|^{p-2}\nabla u(x) \nabla v(x) \, dx - \int_{\Omega} f(x, u(x))v(x) \, dx = 0 \quad (\ast)
$$

for all $v \in W^{1,p}_0(\Omega)$. While, a weak solution of the Neumann problem

$$
\begin{cases}
-\text{div}(\|u|^{p-2}\nabla u) = f(x, u) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
$$
ν being the outer unit normal to ∂Ω, is any \( u \in W^{1,p}(Ω) \) satisfying identity (\( \ast \)) for all \( v \in W^{1,p}(Ω) \).

The following is one of the most classical existence results on the Dirichlet problem for nonlinear elliptic equations:

**THEOREM A ([1]).** - Assume that:

1. there are two positive constants \( a, q \), with \( q < \frac{n+2}{n-2} \) if \( n \geq 3 \), such that
   \[
   |f(x, ξ)| \leq a(1 + |ξ|^q)
   \]
   for all \((x, ξ) \in Ω \times \mathbb{R}\);
2. there are constants \( r \geq 0 \) and \( c > 2 \) such that
   \[
   0 < c \int_0^ξ f(x, t)dt \leq ξ f(x, ξ)
   \]
   for all \((x, ξ) \in Ω \times \mathbb{R} \) with \( |ξ| \geq r \);
3. one has
   \[
   \lim_{ξ \to 0} \frac{f(x, ξ)}{ξ} = 0
   \]
   uniformly with respect to \( x \).

Then, the problem

\[
\begin{cases}
-Δu = f(x, u) & \text{in } Ω \\
u|_{∂Ω} = 0
\end{cases}
\]

has a non zero weak solution.

What can be said whether, in Theorem A, condition (3) is removed at all? Using Theorem 3, one obtains the following

**THEOREM 6 ([17]).** - Assume that conditions (1) and (2) hold. Then, for each \( ρ > 0 \) and each \( μ \) satisfying

\[
μ > \inf_{u \in B_ρ} \sup_{v \in B_ρ} \frac{\int_Ω \left( \int_0^{v(x)} f(x, ξ)dξ \right) dx - \int_Ω \left( \int_0^{u(x)} f(x, ξ)dξ \right) dx}{ρ - \int_Ω |∇u(x)|^2dx}, \quad (**)
\]

where

\[
B_ρ = \left\{ u \in W_0^{1,2}(Ω) : \int_Ω |∇u(x)|^2dx < ρ \right\},
\]

the problem

\[
\begin{cases}
-Δu = \frac{1}{2μ}f(x, u) & \text{in } Ω \\
u|_{∂Ω} = 0
\end{cases}
\]
has at least two weak solutions one of which lies in $B$.  

The following problem naturally arises in connection with Theorems A and 6.

PROBLEM 1. - Under conditions (1) and (2) of Theorem A, is there some $\rho > 0$ such that the infimum appearing in (***) is less than $1/2$?

Clearly, if the answer to this problem was positive, then Theorem 6 would be a proper improvement of Theorem A.

Another result proved using Theorem 1 is the following:

THEOREM 7 ([18]). - Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two Carathéodory functions. Assume that:

(i) there is $s > 1$ such that

$$
\limsup_{\xi \to 0^+} \frac{\sup_{x \in \Omega} |f(x, \xi)|}{\xi^s} < +\infty ;
$$

(ii) there is $q \in ]0, 1[$ such that

$$
\limsup_{\xi \to 0^+} \frac{\sup_{x \in \Omega} |g(x, \xi)|}{\xi^q} < +\infty ;
$$

(iii) there are a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive measure such that

$$
\limsup_{\xi \to 0^+} \frac{\inf_{x \in B} \int_0^\xi g(x, t) dt}{\xi^2} = +\infty , \quad \liminf_{\xi \to 0^+} \frac{\inf_{x \in D} \int_0^\xi g(x, t) dt}{\xi^2} > -\infty .
$$

Then, for some $\lambda^* > 0$ and for each $\lambda \in ]0, \lambda^*[,$ the problem

$$
\begin{cases}
-\Delta u = f(x, u) + \lambda g(x, u) & \text{in } \Omega \\
u|_{\partial \Omega} = 0 ,
\end{cases} 
\quad (P_\lambda)
$$

admits a non-zero, non-negative weak solution $u_\lambda \in C^1(\Omega)$. Moreover, one has

$$
\limsup_{\lambda \to 0^+} \frac{\|u_\lambda\|_{C^1(\Omega)}}{\lambda^{\frac{q}{2}} < +\infty
$$

and the function

$$
\lambda \to \frac{1}{2} \int_\Omega |\nabla u_\lambda(x)|^2 dx - \int_\Omega \left( \int_0^{u_\lambda(x)} f(x, \xi) d\xi \right) dx - \lambda \int_\Omega \left( \int_0^{u_\lambda(x)} g(x, \xi) d\xi \right) dx
$$
is negative and decreasing in \(0, \lambda^\ast\). If, in addition, \(f, g\) are continuous in \(\Omega \times [0, +\infty]\) and

\[
\liminf_{\xi \to 0^+} \frac{\inf_{x \in \Omega} g(x, \xi)}{\xi |\log \xi|^2} > -\infty,
\]

then \(u_\lambda\) is positive in \(\Omega\).

REMARK 2. - Observe that Theorem 7 is bifurcation result. In fact, it ensures, in particular, that \(\lambda = 0\) is a bifurcation point for problem \((P_\lambda)\), in the sense that \((0, 0)\) belongs to the closure in \(C^1(\Omega) \times \mathbb{R}\) of the set

\[
\{(u, \lambda) \in C^1(\Omega) \times ]0, +\infty[ : u \text{ is a weak solution of } (P_\lambda), \ u \neq 0, \ u \geq 0\}.
\]

In view of [2], where problem \((P_\lambda)\) is studied for particular nonlinearities, the following problems arises:

PROBLEM 2. - Under the assumptions of Theorem 7, does problem \((P_\lambda)\) admit a non-zero, non-negative, minimal solution for each \(\lambda > 0\) small enough?

REMARK 3. - Again applying Theorem 1, some bifurcations theorem for Hammerstein nonlinear integral equations (in the spirit of Theorem 7) have been obtained by F. Faraci in [10].

Note, in particular, the following corollary of Theorem 7:

COROLLARY 1 ([18]). - Let \(0 < q < 1 < s\) and let \(\alpha, \beta\) be two Hölder continuous functions on \(\overline{\Omega}\). Assume that

\[
0 \leq \inf_{\Omega} \beta, \ 0 < \sup_{\Omega} \beta.
\]

Then, for some \(\lambda^\ast > 0\) and for each \(\lambda \in ]0, \lambda^\ast[\), the problem

\[
\begin{aligned}
-\Delta u &= \alpha(x)u^s + \lambda \beta(x)u^q \quad \text{in } \Omega \\
u|_{\partial \Omega} &= 0
\end{aligned}
\]

admits a positive classical solution \(u_\lambda\). Moreover, one has

\[
\limsup_{\lambda \to 0^+} \frac{\|u_\lambda\|_{C^1(\Omega)}}{\lambda^{\frac{q}{q-s}}} < +\infty,
\]

and the function

\[
\lambda \to \frac{1}{2} \int_{\Omega} |\nabla u_\lambda(x)|^2 dx - \frac{1}{s+1} \int_{\Omega} \alpha(x)|u_\lambda(x)|^{s+1} dx - \frac{\lambda}{q+1} \int_{\Omega} \beta(x)|u_\lambda(x)|^{q+1} dx
\]
is negative and decreasing in \([0, \lambda^*]\).

In the next theorems, \(\lambda\) denotes a function in \(L^\infty(\Omega)\), with \(\text{ess inf}_\Omega \lambda > 0\). They have been obtained in [16] as applications of conclusions (b) and (c) of Theorem 1.

**THEOREM 8 ([16]).** - Assume \(p > n\). Let \(f : \mathbb{R} \to \mathbb{R}\) be a continuous function, and \(\{a_k\}, \{b_k\}\) two sequences in \(\mathbb{R}^+\) satisfying

\[
a_k < b_k \quad \forall k \in \mathbb{N}, \quad \lim_{k \to \infty} b_k = +\infty, \quad \lim_{k \to \infty} \frac{a_k}{b_k} = 0,
\]

\[
\max \left\{ \sup_{\xi \in [a_k, b_k]} \int_{a_k}^{\xi} f(t) dt, \sup_{\xi \in [-b_k, -a_k]} \int_{-a_k}^{\xi} f(t) dt \right\} \leq 0 \ \forall k \in \mathbb{N}
\]

and

\[
\limsup_{|\xi| \to +\infty} \frac{\int_{0}^{\xi} f(t) dt}{|\xi|^p} = +\infty.
\]

Then, for every \(\alpha, \beta \in L^1(\Omega)\), with \(\min\{\alpha(x), \beta(x)\} \geq 0\) a.e. in \(\Omega\) and \(\alpha \neq 0\), and for every continuous function \(g : \mathbb{R} \to \mathbb{R}\) satisfying

\[
\sup_{\xi \in \mathbb{R}} \int_{0}^{\xi} g(t) dt \leq 0
\]

and

\[
\liminf_{|\xi| \to +\infty} \frac{\int_{0}^{\xi} g(t) dt}{|\xi|^p} > -\infty,
\]

the problem

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

admits an unbounded sequence of weak solutions in \(W^{1,p}(\Omega)\).

A typical example of application of Theorem 8 is as follows:

**EXAMPLE 1 ([16]).** - Let \(p > n\). Then, for each \(\eta \in L^1(\Omega)\) with \(\text{ess inf}_\Omega \eta > 0\), the problem

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) = \eta(x) \left( \sum_{k=1}^{\infty} (\text{dist}(u, \mathbb{R} \setminus [k!k, (k+1)!]))^p - |u|^{p-2}u \right) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

admits an unbounded sequence of weak solutions in \(W^{1,p}(\Omega)\).
THEOREM 9 ([16]). - Assume $p > n$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and \{a_k\}, \{b_k\} two sequences in $\mathbb{R}^+$ satisfying

$$a_k < b_k \quad \forall k \in \mathbb{N}, \quad \lim_{k \to \infty} b_k = 0, \quad \lim_{k \to \infty} \frac{a_k}{b_k} = 0$$

$$\max \left\{ \sup_{\xi \in [a_k, b_k]} \int_{a_k}^{\xi} f(t) dt, \sup_{\xi \in [-b_k, -a_k]} \int_{-a_k}^{\xi} f(t) dt \right\} \leq 0 \quad \forall k \in \mathbb{N}$$

and

$$\limsup_{\xi \to 0} \frac{\int_{0}^{\xi} f(t) dt}{|\xi|^p} = +\infty.$$ 

Then, for every $\alpha, \beta \in L^1(\Omega)$, with $\min\{\alpha(x), \beta(x)\} \geq 0$ a.e. in $\Omega$ and $\alpha \neq 0$, and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\sup_{\xi \in \mathbb{R}} \int_{0}^{\xi} g(t) dt \leq 0$$

and

$$\liminf_{\xi \to 0} \frac{\int_{0}^{\xi} g(t) dt}{|\xi|^p} > -\infty,$$

the problem

$$\begin{cases}
-\text{div}(|\nabla u|^{p-2} \nabla u) + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega
\end{cases}$$

admits a sequence of non zero weak solutions which strongly converges to 0 in $W^{1,p}(\Omega)$.

REMARK 4. - Again using Theorem 1, Theorems 8 and 9 have been extended to the setting of elliptic variational-hemivariational inequalities by D. Motreanu and S. A. Marano in [12]. While P. Candito, in [8], extended them to the case of discontinuous nonlinearities. Further various applications of Theorem 1 can be found in [3], [4], [5], [9], [11] and [15]. See also the related papers [6] and [19].

PROBLEM 3. - In Theorems 8 and 9, when $g = 0$, are the conclusions still valid without the assumption

$$\lim_{k \to \infty} \frac{a_k}{b_k} = 0 ?$$

A partial answer to this problem has recently been provided by G. Anello and G. Cordaro in [7].
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