Conformal superspaces, projectivity and quantization

María A. Lledó
Departament de Física Teòrica, Universitat de València and IFIC (CSIC-UVEG).
C/Dr. Moliner, 50. E-46100 Burjassot (València), Spain.
E-mail: maria.lledo@ific.uv.es

Abstract. In this paper we study the complex conformal superspace in $D = 4$ for $N = 1$ supersymmetry as a projective superspace. Among the different superspaces that carry a transitive action of the conformal supergroup, the superflag $Fl(2|0, 2|1, 4|1)$ is distinguished as the minimal one that has a reality condition preserved by the real conformal supergroup. As in the non super case, the different flag supermanifolds are related to parabolic subalgebras and imposing the constraint that the reduced manifold must be the Grassmannian $Gr(2, 4)$, the complex conformal space, leaves us with five choices, corresponding to the five non isomorphic Borel subalgebras. We identify the ones that are used in physics. As it is well known, not all superflags have a projective embedding, but in the cases of interest one has a super Plücker embedding, so they are projective super varieties. For the superflag $Fl(2|0, 2|1, 4|1)$, this is followed by a super Segre embedding, which then provides a method for its quantization.

1. Introduction

The results that we describe here appeared essentially in Refs. [1] and [2]. One starts with the idea that, when complexified, the conformal compactification of the Minkowski space can be seen as the space of two planes inside a four dimensional space, that is, the Grassmannian $Gr(2, 4)$. Here the four dimensional space is not spacetime but the twistor space of Penrose [3]. On the twistor space the group $SL(4, \mathbb{C})$ acts with the fundamental representation. We regard this group as the complexification of $SU(2, 2)$, the spin group of the conformal group in dimension four, $SO(2, 4)$, and Minkowskian signature. The Grassmannian is then an homogeneous space

$$Gr(2, 4) = SL(4, \mathbb{C})/P$$

where $P$ is a parabolic subgroup that we will describe explicitly in Section 2.

Moreover, the (complex) Minkowski superspace can be identified with the big cell of $Gr(2, 4)$. This will be done by finding that the subgroup in $SL(4, \mathbb{C})$ that leaves the big cell invariant is the(complex) Poincaré group times dilations. Here one has to understand the Poincaré group as the semidirect product of the spin group of the Lorentz group $SO(1, 3)$, that is, $SL(2, \mathbb{C})_\mathbb{R}$ and the translation abelian subgroup. When complexified, $(SL(2, \mathbb{C})_\mathbb{R})^\mathbb{C} \approx SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

This is the basic structure that we will use in the generalization to the super category by extending $SL(4, \mathbb{C})$ to the $N = 1$ conformal supergroup $SL(4|1)$. This program will be carried out in Section 3.

Furthermore, we will be able to find a quantization of super conformal space by upgrading the supergroup to a quantum supergroup $SL_q(4|1)$. This will be done in Section 4, where we will give a presentation of the non commutative algebra of the quantum superconformal space in terms of a quantum superline bundle related (in its classical version) to the projective embedding.
2. Projective geometry

Let \( G \) be a semisimple Lie group, \( \mathfrak{g} = \text{Lie}(G) \), \( \mathfrak{h} \) a Cartan subalgebra and \( \Delta = \Delta^+ + \Delta^- \) the set of (positive and negative) roots. Let

\[
\mathfrak{n}_+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha,
\]

then

\[
\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.
\]

The subalgebras \( \mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm \) are Borel subalgebras of \( \mathfrak{g} \). All Borel subalgebras of \( \mathfrak{g} \) are related by conjugation.

A parabolic subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) is a subalgebra that contains the Borel subalgebra but it is not \( \mathfrak{g} \) itself.

Example 2.1. For \( \text{SL}_4(\mathbb{C}) \) we consider the subalgebras

\[
\begin{pmatrix}
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & *
\end{pmatrix},
\begin{pmatrix}
* & * & * & * \\
& * & * & * \\
& & 0 & 0 \\
& & & 0
\end{pmatrix}.
\]

They are a Borel and a parabolic subalgebra, respectively.

Theorem 2.2. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, \( \mathfrak{p} \) a parabolic subalgebra, \( G \) a connected Lie group with \( \text{Lie}(G) = \mathfrak{g} \) and \( P \) a parabolic subgroup of \( G \) with \( \text{Lie}(P) = \mathfrak{p} \). Then, the generalized flag manifold \( G/P \) is a compact Kähler manifold and a projective algebraic variety. Moreover, \( P \) is parabolic if and only if \( G/P \) is a projective algebraic variety.

This theorem basically settles the problem of the existence of a projective embedding. We will then need to find an explicit one. For such purpose, one may remember that embeddings of a variety into projective spaces are in one to one correspondence with very ample line bundles, that is, bundles that have enough global sections so to span, at each point, the fiber. These global sections are then used as projective coordinates. It is always instructive to see how this works in the case of projective space itself.

Example 2.3. Projective space. \( \mathbb{P}^n = \{ \text{Lines in } \mathbb{C}^{n+1} \} \).

The order one polynomials \( (x^0, \ldots, x^n) \) span globally a line bundle, since they are not simultaneously 0. It is a very ample line bundle.

For every parabolic subalgebra \( \mathfrak{p} \) there exists a \( k \)-grading of \( \mathfrak{g} \)

\[
\mathfrak{g} = \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \cdots
\]

such that the parabolic subalgebra becomes \( \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \cdots \). One also denotes as \( \mathfrak{g}_\pm = \mathfrak{g}_{\pm 1} \oplus \mathfrak{g}_{\pm 2} \oplus \cdots \).

We consider now \( G/P \) and \( \pi : G \rightarrow G/P \). With the help of the above decomposition we are going to describe the line bundle. The sheaf of regular functions on \( G/P \) can be given in terms of the sheaf of functions over \( G \) satisfying an invariance condition. So, for \( U \subset \text{open } G/P \)

\[
\mathcal{R}_{G/P}(U) \approx \{ f \in \mathcal{O}_G(\pi^{-1}(U)) \mid f(gp) = f(g) \quad \forall g \in \pi^{-1}(U), \quad p \in P \}.
\]
The Lie subgroup $G_0 \subset G$ whose Lie algebra is $\mathfrak{g}_0$ is the *Levi subgroup* of $G$ with respect to this grading. We consider the one dimensional representation

$$
G_0 \xrightarrow{\chi} \mathbb{C}
$$

$$
g \longrightarrow |\det(\text{Ad}_\chi(g))|^{-\frac{1}{2}}
$$

(we leave for the moment being the constant $d$ undetermined). This representation of $G_0$ one can be extended to $P$ just by letting it act trivially on $\mathfrak{g}_+$. Then, by the method of induced representations, we extend it to the full $G$. As the representation space we shall consider the set of sections of the line bundle $G \times_P \mathbb{C}$ over $G/P$, defined by

$$
\Gamma(G \times_P \mathbb{C}) = \{ f \in \mathcal{O}(G) \otimes \mathbb{C} \mid f(gp) = \chi(p)^{-1}f(g) \quad \forall g \in G, \ p \in P \}.
$$

The *complexified, conformal space* is the Grassmannian

$$
G(2,4) = \text{SL}_4(\mathbb{C})/P = \{2\text{-planes in } \mathbb{C}^4\}
$$

where

$$
p = \left\{ \begin{pmatrix} 1 & q \\ 0 & r \end{pmatrix} \right\},
$$

with grading

$$
\mathfrak{g}_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \right\}, \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{+1} = \left\{ \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \right\}.
$$

The big cell of $G(2,4)$ is the complexified Minkowsi space. It can be given in its standard form in terms of two column vectors as

$$
\text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{pmatrix} \right\} = \text{span} \left\{ \mathbb{I} \right\},
$$

where the entries of $A$ are unconstrained. In terms of the Pauli matrices one recovers the standard coordinates in Minkowski space,

$$
A = x^\mu \sigma_\mu,
$$

and the Minkowski metric is just

$$
\det A = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.
$$

The subgroup of $\text{SL}(4,\mathbb{C})$ that leaves the big cell invariant is

$$
\mathcal{P} = \left\{ \begin{pmatrix} L & 0 \\ N & R \end{pmatrix} \mid \det L \cdot \det R = 1 \right\}.
$$

It acts on the big cell as

$$
A \longrightarrow RAL^{-1} + N,
$$

so we can identify it with the action of the complex Poincaré group times dilations, $(\text{SL}(2,\mathbb{C}) \times \text{SL}(2,\mathbb{C}) \times \mathbb{C}^\times) \times \text{M}_2(\mathbb{C})$, on the complex Minkowski space $\sim M_2(\mathbb{C})$.

The Levi subgroup is $G_0 = \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \times \mathbb{C}^\times$. It acts on the twistor space and the corresponding induced line bundle is called the bundle of *conformal densities*. 
Proposition 2.4. The bundle of conformal densities of weight 1 over $\text{SL}_4(\mathbb{C})/P$, defined by the character $\det L$ of $P$ is very ample.

In fact, let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}(4, \mathbb{C}), \quad p = \begin{pmatrix} L & Q \\ 0 & R \end{pmatrix} \in P.$$ 

Then, the determinants formed by minors corresponding to the first two columns of $g$ are equivariant functions on $\text{SL}_4(\mathbb{C})$:

$$gp = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} L & Q \\ 0 & R \end{pmatrix} = \begin{pmatrix} AL & AQ + BR \\ CL & CQ + DR \end{pmatrix}$$

and they cannot all be simultaneously 0.

The embedding defined by this very ample line bundle is exactly the Plücker embedding of $G(2, 4)$ into $\mathbb{P}(\wedge^2 \mathbb{C}^4) = \mathbb{P}^5$. In order to see this, one can take the first two columns of $g$ as two vectors spanning a plane in $G(2, 4)$. The minors formed with the columns 1, 2 and rows $i, j$, denoted as $d_{ij}$, $i < j = 1, \ldots, 4$ are homogeneous coordinates for $\mathbb{P}^5$.

As it is well known, the image of the embedding is given by the Klein quadric

$$d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23} = 0.$$

Remark 2.5. When taking the physical real form, the conformal group is $\text{SU}(2|2)$ and the real space on which it acts is NOT a real Grassmannian.

3. Super projective geometry

There is no generalization of Theorem 2.2 to the super case. There are superflag manifolds that are not projective, as it was shown by Manin [4] for the Grassmannian $G(1|1, 2|2)$ by computing its sheaf cohomology. In fact this is the generic situation, and the property of admitting a projective embedding is rare.

Still, for the ones that are projective, there is a one to one correspondence between super projective embeddings and very ample superline bundles (rank $1|0$ bundles).

Example 3.1. Projective superspace. We consider homogeneous polynomials in

$$\mathbb{C}[x_0, \ldots, x_n; \xi_0, \ldots, \xi_n].$$

The projective coordinates $(x^0, \ldots, x^n; \xi_0, \ldots, \xi_n)$ form a basis of the set of global sections of a super line bundle.

As in the classical case, superflags are associated with parabolic subgroups of the semisimple supergroup $\text{SL}(4|1)$. Let us denote its super Lie algebra as

$$\mathfrak{g} = \mathfrak{sl}(4|1) = \left\{ \begin{pmatrix} l & q & \nu \\ p & r & \alpha \\ \mu & \beta & s \end{pmatrix} \mid \text{tr } l + \text{tr } r = s \right\}.$$ 

As in the classical case, one can define Borel and parabolic subalgebras in terms of the root systems, but one has usually more than one root system for a given semisimple superalgebra [5, 6]. For $\mathfrak{sl}(4|1)$ we have five classes of equivalence of such systems and to them they correspond five different Borel subalgebras and five different complete superflags. We are interested in parabolic subalgebras whose even part has as a factor the subalgebra (1). In Refs. [1, 2] we sort out the five parabolic subalgebras satisfying this condition. Her we consider three examples to which we can attach a physical meaning.
• $\text{Gr}_1 := \text{Gr}(2|0, 4|1)$. It corresponds to the algebra of antichiral superfields. It does not admit a real form compatible with the real conformal supergroup $\text{SU}(2, 2|1)$, so it is a genuinely complex superspace.

$$p_1 = \left\{ \begin{pmatrix} l & q & \nu \\ 0 & r & \alpha \\ 0 & \beta & s \end{pmatrix} \right\},$$

with grading

$$g_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix} \right\}, \quad g_0 = \left\{ \begin{pmatrix} l & 0 & 0 \\ 0 & r & \alpha \\ 0 & \beta & s \end{pmatrix} \right\}, \quad g_{+1} = \left\{ \begin{pmatrix} 0 & q & \nu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

• $\text{Gr}_2 := \text{Gr}(2|1, 4|1)$. It corresponds to the algebra of chiral superfields. As for $\text{Gr}_1$, it is a complex superspace.

$$p_2 = \left\{ \begin{pmatrix} l & q & \nu \\ 0 & r & 0 \\ \mu & \beta & s \end{pmatrix} \right\},$$

with grading

$$g_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad g_0 = \left\{ \begin{pmatrix} l & 0 & \nu \\ 0 & r & 0 \\ \mu & 0 & s \end{pmatrix} \right\}, \quad g_{+1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \right\}.$$

• $F := F(2|0, 2|1, 4|1) \subset \text{Gr}_1 \times \text{Gr}_2$. This admits a real form compatible with the action of $\text{SU}(2, 2|1)$.

$$p_u = p_1 \cap p_2 = \left\{ \begin{pmatrix} l & q & \nu \\ 0 & r & 0 \\ \mu & \beta & s \end{pmatrix} \right\},$$

with grading

$$g_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad g_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ \mu & 0 & 0 \end{pmatrix} \right\}, \quad g_0 = \left\{ \begin{pmatrix} l & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{pmatrix} \right\},$$

$$g_{+2} = \left\{ \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad g_{+1} = \left\{ \begin{pmatrix} 0 & 0 & \nu \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \right\}.$$

We now provide the projective embedding explicitly, starting with $\text{Gr}_1$. This is better understood in the functor of points approach. For this purpose, let $E = \wedge^2(C^4|1) \approx C^{7|4}$ and $A$ an arbitrary superalgebra. We consider the map

$$\text{Gr}_1(A) \xrightarrow{P_A} P(E)(A)$$

$$W_1(A) = \text{span} \{a_1, a_2\} \xrightarrow{\cdot} [a_1 \wedge a_2].$$

We ask then when a generic even vector $w = d + \delta \wedge \mathcal{E}_5 + d_{55} \mathcal{E}_5 \wedge \mathcal{E}_5$, with

$$d := d_{12} e_1 \wedge e_2 + d_{13} e_1 \wedge e_3 + d_{14} e_1 \wedge e_4 + d_{23} e_2 \wedge e_3 + d_{24} e_2 \wedge e_4 + d_{34} e_3 \wedge e_4,$$

$$\delta := \delta_{15} e_1 + \delta_{25} e_2 + \delta_{35} e_3 + \delta_{45} e_4.$$
is in the image of this \( p_A \), which means that it has to be decomposable. This gives the following algebraic relations:

\[
\begin{align*}
    d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23} &= 0, \\
    d_{ij}d_{\delta} - d_{ik}d_{\delta} + d_{jk}d_{\delta} &= 0, \\
    \delta_{ij}d_{\delta} &= d_{55}d_{ij},
\end{align*}
\]

(classical Plücker relation) \( 1 \leq i < j < k \leq 4 \), \( 1 \leq i < j \leq 4 \).

They are the super Plücker relations. They generate an homogeneous ideal in \( \mathbb{C}[d_{ij}, d_{55}, \delta_{ij}] \).

There is still another description of the coordinate ring of \( \text{Gr}_1 \) as a subring of the coordinate ring of \( \text{SL}(4|1) \). This point is crucial for the quantization. Let

\[
\begin{pmatrix}
g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} \\
g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} \\
g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} \\
g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} \\
\gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & \gamma_{55}
\end{pmatrix}
\]

\( \text{SL}(4|1)(A). \)

Let now

\[
\begin{align*}
    e_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
    e_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
    ge_1 &= \begin{pmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{41} \\ \gamma_{51} \end{pmatrix}, \\
    ge_2 &= \begin{pmatrix} g_{12} \\ g_{22} \\ g_{32} \\ g_{42} \\ \gamma_{52} \end{pmatrix},
\end{align*}
\]

be two vectors of the standard basis \( e_1 \) and \( e_2 \) and the result of applying \( g \) to them. This selects the first two columns of \( g \), which are the two independent vectors generating the subspace \( W_1(A) \). The coordinate ring of \( \text{SL}(4|1) \) is

\[
\mathbb{C}[\text{SL}(4|1)] = \mathbb{C}[g_{ij}, \gamma_{ij}, \gamma_{5i}]/(\text{Ber} \, g - 1),
\]

were we are now interpreting the entries as (even and odd) indeterminates. One can show that \( \mathbb{C}[\text{Gr}_1] \) is the subring of \( \mathbb{C}[\text{SL}(4|1)] \) generated by the \( 2 \times 2 \) determinants

\[
d_{ij} := g_{1i}g_{j2} - g_{12}g_{ij}, \quad \delta_{ij} := g_{1i}\gamma_{52} - g_{i2}\gamma_{51}, \quad d_{55} = \gamma_{51}^2\gamma_{52},
\]

with \( i, j = 1, \ldots, 4 \). This projective embedding is realized in terms of bundles by taking the one dimensional representation \( d_{12} \) of \( G_0 \).

For \( \text{Gr}_2 \) one uses a duality relation. The Grassmannian of \( 2|1 \)-planes on \( \mathbb{C}^{4|1} \) is isomorphic to the Grassmannian of \( 2|0 \)-planes on \( (\mathbb{C}^{4|1})^* \). One just has to substitute \( g \) by \( (g^{-1})^T \). The super Plücker relations are then

\[
\begin{align*}
    d_{12}^*d_{34}^* - d_{13}^*d_{24}^* + d_{14}^*d_{23}^* &= 0, \\
    d_{ij}^*d_{\delta}^* - d_{ik}^*d_{\delta}^* + d_{jk}^*d_{\delta}^* &= 0, \\
    \delta_{ij}d_{\delta}^* &= d_{55}d_{ij}^*,
\end{align*}
\]

(classical Plücker relations) \( 1 \leq i < j < k \leq 4 \), \( 1 \leq i < j \leq 4 \).

These relations generate an homogeneous ideal in \( \mathbb{C}[d_{ij}^*, d_{55}^*, \delta_{ij}^*] \), and the line bundle is constructed with \( d_{12}^* \).

We have then proven that \( \text{Gr}_1 \times \text{Gr}_2 \mathbb{P}^{6|4} \times (\mathbb{P}^{6|4})^* \). The flag supervariety \( F \) is a subset of this product. One just has to impose the condition that the \( 2|0 \) subspace is inside the \( 2|1 \) subspace. This gives the incidence relations

\[
\sum_{j=1}^{5} d_{ij}^*d_{jk}^* = 0, \quad \forall \, i, k = 1, \ldots, 5,
\]

which are an homogeneous ideal in \( \mathbb{C}[d_{ij}, \delta_{ij}, d_{55}, d_{ij}^*, \delta_{ij}^*, d_{55}^*] \). We then have
Theorem 3.2. There is an embedding of the superflag

\[ F \subset \text{Gr}_1 \times \text{Gr}_2 \subset \mathbb{P}(E) \times \mathbb{P}(E^*), \]

with \( E = \wedge^3(T) \cong \mathbb{C}^{|4|4}, E^* = \wedge^3(T^*) \cong \mathbb{C}^{|4|4} \) and \( T \cong \mathbb{C}^{|4|1} \). With respect to such embedding, the coordinate ring of \( F \) is given by

\[ \mathbb{C}[F] := \mathbb{C}[d_{ij}, \delta_{15}, d_{55}, d_{ij}^*, \delta_{15}^*, d_{55}^*] / (I_{\text{Gr}_1} + I_{\text{Gr}_2} + I_{\text{inc}}) \]

where \( I_{\text{inc}} \) is the ideal generated by the 25 incidence relations

\[ \text{with } I_{\text{inc}} , \text{is the ideal generated by the 25 incidence relations} \]

We will use now a super Segre embedding in order to have the flag embedded into one projective space. This will give us also the information on the line bundle.

The standard Segre embedding maps the product of two projective spaces into one super projective space:

\[ \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^d(\mathbb{C}) \quad \overset{\psi}{\longrightarrow} \quad \mathbb{P}^n(\mathbb{C}) \]

\[ ([x_0, \ldots, x_n], [y_0, \ldots, y_d]) \quad \mapsto \quad [x_0y_0, x_0y_1, \ldots, x_iy_j, \ldots, x_ny_d] \]

with \( i = 0, \ldots, n, \ j = 0, \ldots, d \) and \( N = (n+1)(d+1) - 1 \).

Let \( z_{ij} \) be the homogeneous coordinates of \( \mathbb{P}^N(\mathbb{C}) \). As for the Plücker embedding, one can show that the image is an algebraic projective variety given as the zero locus of the 2x2 minors of the matrix

\[ \begin{pmatrix}
  z_{00} & z_{02} & \cdots & z_{0d} \\
  z_{20} & \ddots & \cdots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  z_{n0} & \cdots & \cdots & z_{nd}
\end{pmatrix} \]

The line bundle is given by the character \( d_1d_2 \).

The Segre embedding can be generalized to the super case. Composing with it, we will embed \( F \) into \( \mathbb{P}^{|M|N} \), where \( M \mid N = 64 \mid 56 \). Explicitly, we get:

\[ \mathbb{P}(E)(A) \times \mathbb{P}(E^*)(A) \quad \overset{\psi}{\longrightarrow} \quad \mathbb{P}^{|M|N}(A) \]

\[ ([z_{ij}, z_{55} | \zeta_{15}, [z_{ij}^*, z_{55}^* | \zeta_{15}^*]) \quad \mapsto \quad [z_{ij}z_{55}^*, z_{55}z_{55}^*, z_{ij}z_{55}^*, z_{55}z_{55}^*, \zeta_{15}\zeta_{15}^* | \]

\[ z_{ij}\zeta_{k5}, z_{55}\zeta_{k5}, \zeta_{15}\zeta_{k5}, \zeta_{15}\zeta_{k5}, ] . \]

Let us denote \( I, K = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) \): Then, we can organize the coordinates on \( \mathbb{C}^{|M+1|N} \) in matrix form:

\[ \begin{pmatrix}
  z_{1z}z_{K} & z_{55}z_{K}^* \\
  z_{1z}z_{55}^* & z_{55}z_{55}^* \\
  \zeta_{15}z_{K}^* & \zeta_{15}z_{55}^* \\
  \zeta_{15}z_{K} & \zeta_{15}z_{55}
\end{pmatrix} . \]

The image of this map is a projective algebraic variety in the generators

\[ \begin{pmatrix}
  Z_{IK} & Z_{5K} & \Lambda_{ik} \\
  Z_{15} & Z_{55} & \Lambda_{5k} \\
  \Gamma_{iK} & \Gamma_{i5} & T_{ik}
\end{pmatrix} , \]

satisfying homogeneous polynomial relations.
As in the non super case, the global sections of the superline bundle on $G/P$ can be thought as elements of $\mathcal{O}(G)$ with an the equivariance condition. Since $V = \mathbb{C}$ we identify $\mathcal{O}(G) \otimes V \cong \mathcal{O}(G)$. If $I(P)$ is an ideal in $\mathcal{O}(G)$ defining $P$ we have $\pi : \mathcal{O}(G) \to \mathcal{O}(P) = \mathcal{O}(G)/I(P)$.

Let $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ be the coproduct in $\mathcal{O}(G)$. Then the equivariant condition becomes

$$O(G/P)_1 = \left\{ f \in O(G) \mid (\mathbb{I} \otimes \pi) \Delta(f) = f \otimes S(\chi) \right\}.$$ 

Let $t \in O(G)$ such that $t = \pi(\chi)$. If $\mathcal{L}$ is very ample we have the following result [7].

**Proposition 3.3.** Let the supervariety $G/P$ be embedded into some projective superspace via the line bundle $\mathcal{L}$ defined by $\chi$. Let $\Delta$ be the coproduct in $\mathcal{O}(G)$ and $\pi : \mathcal{O}(G) \to \mathcal{O}(P) = \mathcal{O}(G)/I(P)$. Then, there exists an element $t \in \mathcal{O}(G)$, with $\pi(t) = \chi$, such that

$$(\mathbb{I} \otimes \pi) \circ \Delta(t) = t \otimes \pi(t), \quad \pi(t^m) = \pi(t^n), \quad \forall m \neq n \in \mathbb{N},$$

$$O(G/P)_n = \left\{ f \in O(G) \mid (\mathbb{I} \otimes \pi) \Delta(f) = f \otimes \pi(t^n) \right\},$$

$$O(G/P) = \bigoplus_{n \in \mathbb{N}} O(G)_n,$$

and $O(G/P)$ is generated in degree 1, namely by $O(G/P)_1$.

We call $t$ the classical section associated to the super line bundle $\mathcal{L}$. The following are the relevant examples:

- For $\text{Gr}_1$, $t = d_{12} \in \mathcal{O}(\text{SL}(4|1))$.
- For $\text{Gr}_2$, $t = d_{12}^* \in \mathcal{O}(\text{SL}(4|1))$.
- For $F$ (super Segre embedding), $t = d_{12}d_{12}^* \in \mathcal{O}(\text{SL}(4|1))$.

We have then achieved a description of the coordinate ring of the projective embedding of $F$ in $\mathbf{P}^{64}_{456}$ as a (graded) subring of $\mathcal{O}(\text{SL}(4|1))$. Notice that we do not know a priori if this ring is isomorphic to the one appearing in Theorem 3.2, where the embedding was in the product of projective spaces $\mathbf{P}^{6}_{4} \times \mathbf{P}^{4}_{4}$. But this way of realizing the superflag has the advantage that one can proceed to its quantization.

### 4. Quantization

We have managed to give a presentation of the ring of the flag manifold in terms of generators $(d_{ij}, d_{ij}^*)$ and relations (super Plücker and incidence relations).

Now we want to replace $\text{SL}(4|1)$ by the quantum group $\text{SL}_q(4|1)$. We use the Manin approach [8], so we first define the bialgebra of quantum supermatrices:

$$M_q(m|n) = \text{def } \mathbb{C}_q(a_{ij})/I_M,$$

$$i, j = 1, \ldots, n,$$

where $\mathbb{C}_q(a_{ij})$ denotes the free algebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the homogeneous variables $a_{ij}$ and the ideal $I_M$ is generated by the Manin relations:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})}q^{-1}r_{(i)l+1}a_{il}a_{ij}, \quad \text{if } j < l$$

$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})}q^{-1}r_{(k)l+1}a_{kj}a_{ij}, \quad \text{if } i < k$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})}a_{kl}a_{ij}, \quad \text{if } i < k, j > l$$

$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})}a_{kl}a_{ij} = \eta(q^{-1} - q)a_{kj}a_{il}, \quad \text{if } i < k, j < l$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})}a_{kl}a_{ij} = \eta(q^{-1} - q)a_{kj}a_{il}, \quad \text{if } i < k, j < l$$

8
where

\[ i, j, k, l = 1, \ldots, m + n, \quad \eta = (-1)^{p(k)p(l)+p(j)p(l)+p(k)p(j)}, \]
\[ p(i) = 0 \text{ if } 1 \leq i \leq m, \quad p(i) = 1 \text{ if } m + 1 \leq i \leq n + m \quad \text{and} \]
\[ \pi(a_{ij}) = p(i) + p(j). \]

The determinants \( d_{ij} \) become quantum determinants \( D_{qij} \) and one can manage with difficulty, since it involves long calculations to show that they generate, inside the quantum supergroup, a subalgebra that is a non commutative version (deformation) of the algebra of \( Gr_1 \) and \( Gr_2 \). So we can safely define a quantum Grassmannians \( Gr_{1q}, Gr_{2q} \).

The commutation relations among \( D_{qij} \) and \( D_{qij}^* \) as well as the incidence relations involve both types of coordinates and then are extremely difficult to compute (if possible). We have then resorted to another strategy: we will give a quantum super line bundle realizing a quantum super Segre embedding with character \( D_{q12}D_{q12}^* \).

The quantum projective supervariety so obtained is a deformation of \( F(2|0, 2|1, 4|1) \): the quantum superflag \( F_q(2|0, 2|1, 4|1) \). We can call it the quantum super conformal space.

**Definition 4.1.** Let \( L \) be the super line bundle on \( G/P \) given by the classical section \( t \). A quantum section or quantization of \( t \) is an element \( d \in \mathcal{O}_q(G) \) such that

1. \( (\mathbb{I} \otimes \pi)\Delta(d) = d \otimes \pi(d) \).
2. \( t = d \mod (q-1)\mathcal{O}_q(G) \).

**Definition 4.2.** Let \( d \) be a quantum section of \( L \). We define

\[ \mathcal{O}_q(G/P) := \oplus_{n \in \mathbb{N}} \mathcal{O}_q(G/P)_n, \quad \text{where} \]
\[ \mathcal{O}_q(G/P)_n := \{ f \in \mathcal{O}_q(G) \mid (\mathbb{I} \otimes \pi)\Delta(f) = f \otimes \pi(d^n) \} \].

One can then prove that \( \mathcal{O}_q(G/P) \) is a projective, quantum supervariety which is an homogeneous space under the coaction of \( \mathcal{O}_q(G) \)

**Proposition 4.3.** The element \( d = D_{12}D_{12}^* \in SL_q(4|1) \) is a quantum section, with respect to the super line bundle \( L \) on \( SL(4|1)/P_u \) given by \( t = D_{12}D_{12}^* \).

**Corollary 4.4.** The \( \mathbb{Z} \)-graded subalgebra

\[ C_q := \mathcal{O}_q(G/P) \subset SL_q(4|1), \quad G = SL(4|1), \quad P = P_u \]

defined by the quantum section \( d = D_{12}D_{12}^* \) is a quantum deformation of the graded subalgebra of \( SL_q(4|1) \) obtained via the classical section \( t = d_{12}d_{12}^* \).

Furthermore, \( C_q \) has a natural coaction of the supergroup \( SL_q(4|1) \). Therefore it is a quantum homogeneous superspace.

**Acknowledgements**

This work has been supported in part by grants FIS2011-29813-C02-02, FIS2014-57387-C3-1 and SEV-2014-0398 of the Ministerio de Economía y Competitividad (Spain) and European Funds for Regional Development, European Union - A way to construct Europe and by Generalitat Valenciana through the project SEJI/2017/042.
References

[1] Fioresi R, Latini E and Lledó M A 2017 The Segre embedding of the quantum conformal superspace
Preprint arXiv:1709.03075. To appear in ATMP
[2] Fioresi R and Lledó M A 2015 The Minkowski and conformal superspaces: the classical and the quantum
pictures (Singapore: World Scientific Publishing)
[3] Penrose R 1967 Twistor algebra JMP 8 pp 345-366
[4] Manin Y 1988 Gauge field theory and complex geometry. (Berlin Heidelberg: Springer Verlag) (Original
Russian edition in 1984).
[5] Kac V 1997 Lie superalgebras Adv Math 26 pp 8-96
[6] Ivanova N I, Onishchik A L 2008 Parabolic subalgebras and gradings of reductive Lie superalgebras J of
Math Sc 152 pp 1-60
[7] Fioresi R 2015 Quantum homogeneous superspaces and quantum duality principle Banach Center
Publications 106 pp 59-72
[8] Manin Y (1989) Multiparametric quantum deformation of the general linear supergroup Commun Math
Phys 123 pp 163-17