History dependence of mechanical properties in granular systems

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We study the history dependence of the mechanical properties of granular media by numerical simulations. We perform a compaction of frictional disk packings in a two-dimensional system by controlling the area of the domain with various strain rates. We then find the strain rate dependence of the critical packing fraction above which the pressure becomes finite. The observed behavior makes a contrast with the well-studied jamming transitions for frictionless disk packings. We also observe that the elastic constants of the disk packings depend on the strain rate logarithmically. This result provides an experimental test for the history dependence of granular systems.

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Granular materials have attracted the attentions of scientists for a long time [1]. Despite the extensive efforts, our understanding has not yet been deepened enough to construct a fundamental law. One major difficulty arises from the process dependence of the macroscopic behaviors. It is rather surprising to know that the mechanical properties of even static granular media cannot be determined from all the information of the constituents. An example is given by a so-called stress dip, which is a local minimum of a vertical pressure distribution just under the apex of a sandpile. This stress dip appears by local deposition of grains with a point source whereas it does not appear by homogeneous deposition with an extended source [2]. Such process dependence is often called 'memory' which sounds mysterious. A sandpile is merely a collection of sand grains between which the interaction is rather simple; they interact each other only when they are in contact. What attracts us is that only granules store information on how they are piled up. See Refs. [3, 4, 5] as other related memory effects.

Granular materials have also been of interest as a jamming system [6, 7]. As an idealized case of jamming transitions, the packing problem has been studied intensively with frictionless particles [8, 9, 10, 11]. The important result is that the pressure of disk packings is not zero above the random close packing [8]. On the other hand, with frictional particles, the transition point appears between the random loose packing and the random close packing [8, 12]. It might be naturally conjectured that the packing fraction of the transition depends on the packing process. However, such a dependence has not been well studied. We aim to get insight into the history dependence of jamming transitions. Furthermore, we explore a possibility to describe the history dependence of mechanical properties of frictional disk packings.

In this Letter, for the purpose of illuminating the nature of the history dependence, we propose a frictional disk packing in a square box without gravity as the simplest system. We characterize the packing process only by its time scale, which is precisely given by the inverse of the strain rate of the shrinkage in the packing process. By varying this time scale in a systematic manner, we attempt to observe the process dependence of mechanical properties such as macroscopic elastic constants.

Here, it should be noted that our numerical experiments are carried out with identical constituents. In order to examine history dependence in an explicit way, it is of the essence to investigate a characteristic feature of granular media with persisting with identical constituents. This makes a contrast with previous studies investigating how macroscopic elastic constants depend on the material properties of the particles such as the stiffness, the friction coefficient, and the shape [14, 15, 16]. The main discovery in the present work is an existence of two time scales which are separated over two digits. Interestingly, between the two time scales, macroscopic elastic properties are expressed as logarithmic functions of the strain rate.

Model: We consider $N$ frictional circular disks in a two-dimensional square box. We assume that the diameter of disk $i$, denoted by $d_i$, is either $d$ or $d/1.4$ with an equal ratio, and that its mass $m_i$ is given by $m_i = \pi d_i^2 \rho/4$. The interaction between disks appears only when two disks are in contact. Suppose disk $i$ at position $r_i$ with velocity $v_i$ and angular velocity $\omega_i$, contacts with disk $j$ at $r_j$ with $v_j$ and $\omega_j$. Using the relative position $r_{ij} = r_i - r_j$ and the relative velocity $v_{ij} = v_i - v_j$, the normal and tangential parts of the relative velocity of contacting points, $v_{ij}^n$ and $v_{ij}^t$, are given as $v_{ij}^n = v_{ij} \cdot n$ and $v_{ij}^t = v_{ij} \cdot \mathbf{t} - (d_i \omega_i + d_j \omega_j)/2$, respectively, with the normal unit vector $n = r_{ij}/|r_{ij}|$ and the tangential unit vector $\mathbf{t}$. Let $k_n$ and $k_t$ be normal and tangential elastic constants, $\eta_n$ and $\eta_t$ be normal and tangential viscous coefficients, and $\mu$ be the Coulomb friction coefficient for sliding friction [17]. Then, the normal force $F_{ij}^n$ and the tangential force $F_{ij}^t$ acting on disk $i$ from disk $j$ are given
We investigate cases with several values of \( \delta \tau \) with \( \delta W = 5.0 \times 10^{-2} \) fixed. We confirmed that our claim in this Letter is independent of the value of \( \delta W \) if it is sufficiently small. We stop moving the walls at certain time \( t = \tau \) and we wait until the total kinetic energy of disks becomes smaller than \( 10^{-14} \) so that we can assume it to be in a static state. This compaction process is characterized by the strain rate measured from the initial state:

\[
\dot{\epsilon} = \frac{W(t = 0) - W(t = \tau)}{W(t = 0) \tau}.
\]

Note that since the first contact occurs immediately after we start a compaction, we assume the initial configuration as a reference state to determine the strain rate. Below we will regard the strain rate \( \dot{\epsilon} \) as the control parameter of the problem we consider.

**Evidence of the process dependence:** We first study the process dependence of the final pressure with the packing fraction fixed. In this study, we stop the shrinkage of the walls when the packing fraction reaches a prescribed value.

For comparison, we consider the case \( \mu = 0.0 \). As shown in the inset of Fig. 2 the results with four different strain rates yield one pressure curve as a function of packing fraction. Here, the packing fraction of the system is measured in the domain away from the boundary at the distance of \( 2d \) in order to remove boundary effects. We observe that the pressure starts increasing at a critical packing fraction \( \phi_c \approx 0.84 \), which corresponds to the so-called jamming transition point \([11]\). It is a remarkable property that the critical packing fraction for frictionless particles is determined uniquely irrespective of packing processes.

In contrast, under the existence of the tangential friction with the friction coefficient of \( \mu = 0.4 \), the pressure curve strongly depends on the packing process as shown in Fig. 2. In particular, as we pack slower, the pressure curve approaches the one in the case \( \mu = 0.0 \) (displayed by the triangle symbols). In order to extract a quantitative relation of the process dependence, we measured the critical packing fraction \( \phi_c \) as a function of \( \dot{\epsilon} \). As shown in Fig. 3 our result suggests

\[
\phi_c(\mu = 0.4, \dot{\epsilon}) - \phi_c(\mu = 0) \approx -\alpha \log \frac{\dot{\epsilon}}{\epsilon_c} \tag{4}
\]

in the regime \( \dot{\epsilon}_c \leq \dot{\epsilon} \leq \dot{\epsilon}_c' \) with \( \dot{\epsilon}_c \ll \dot{\epsilon}_c' \), where \( \dot{\epsilon}_c \) and \( \dot{\epsilon}_c' \) correspond to the inverse of two cross-over time scales \( \tau_c \) and \( \tau_c' \), respectively.

We shall present a phenomenological argument to understand Eq. (4). Let us fix \( \rho \) of the final state. When the shrinkage is faster than the change of contact network and the force balance is realized in this network, the packing fraction in the final state does not seriously depend on the strain rate. This provides the first cross-over time scale \( \tau_c' \). When the shrinkage is slower than

We employ the Adams method with a time step of 2

\[
\frac{d^2 x}{dt^2} = -\frac{k}{m} x + \frac{1}{m} \sum F_i.
\]

\[
F_i = \begin{cases} -k \left( |x_i| - d \right) & \text{if } |x_i| > d, \\ 0 & \text{otherwise.} \end{cases}
\]

with \( x_i = r_i \) and \( r_i \) is the position of the disk. The interaction between a disk and the walls is calculated in the same way with the one for the interaction between disks.

The parameters in our simulations are converted into dimensionless quantities so that \( d = 1, k_n = 1, \) and \( \rho = 1 \). Then, the parameter values we used are as follows. \( d = 1, \) \( k_n = 0.4, k_t = 0.1, \) and \( \eta_n = \eta_t = 0.35 \). Note that the normal restitution coefficient is approximately estimated as 0.6. We restrict our investigation to the systems with \( N = 1000 \). For a reference to laboratory experiments, for example, in the case of disks with maximum diameter of 10mm, stiffness of 2.5MPa, thickness of 6mm, and mass density of 1.6 \( \times \) \( g/cm^3 \), the time unit in our simulations corresponds to about \( 3.1 \times 10^{-5} \)s.

The time evolution of the position, velocity and rotation angle of disk \( i \) is described by the equation of motion with using the forces described above. In numerical simulations, we solve a set of equations of motion employing the Admas method with a time step of \( 2.2 \times 10^{-2} \).
this time scale, the disks have enough time to search for a more dense network. The searching time \( \tau \) for a configuration with \( \phi \) might be proportional to the number of force balance configurations \( \Sigma \) for a given \( \phi \). Here, due to the combinatorial complexity of force balance states, \( \Sigma \simeq \exp(aN) \) in the large \( N \) limit with a constant \( a \) that characterizes the number of local configurations satisfying force balance conditions. Since it is reasonable to assume that \( a \) can be expanded in \( \phi \) near the packing fraction \( \phi_c' \), at which the shrinkage rate dependence starts increasing, \( \Sigma \) depends on \( \phi \) in an exponential manner. The consideration leads us to

\[
\tau \simeq \tau_c' \exp(A(\phi - \phi_c')), \tag{5}
\]

where \( A \) is a constant. This tendency ceases when the packing fraction \( \phi_c \) is close to \( \phi_c(\mu = 0) \), which provides the time scale \( \tau_c \) as \( \tau_c \approx \tau_c' \exp(A(\phi_c(\mu = 0) - \phi_c')) \) in the case \( \rho \simeq 0 \). Since the searching time \( \tau \) is related to \( 1/\dot{\varepsilon} \) linearly, we obtain Eq. (4).

![FIG. 2: Pressure \( p \) versus packing fraction \( \phi \) for the system with \( \mu = 0.4 \) and various values of strain rate \( \dot{\varepsilon} \).](image)

**FIG. 2:** Pressure \( p \) versus packing fraction \( \phi \) for the system with \( \mu = 0.4 \) and various values of strain rate \( \dot{\varepsilon} \). Cross symbol, \( 3.56 \times 10^{-3} \) (Plus symbol), \( 1.78 \times 10^{-3} \) (Asterisk symbol), and \( 4.45 \times 10^{-5} \) (Square symbol). For comparison, the result in the case \( \mu = 0 \) and the strain rate of \( 4.45 \times 10^{-5} \) is superposed as the triangle symbols. Inset: Results in the case \( \mu = 0.0 \), with various strain rates. The symbols coincide with those in the main frame, but all the symbols are one curve. The average is taken over 20 samples.

**Experimental method:** In many laboratory experiments, the packing fraction is not easily measurable. We thus propose a method by which the process dependence can be detected without observing the packing fraction. The key idea here is to focus our attention on mechanical properties of the packing. Concretely, we fix the pressure \( p \) in the packing. That is, we stop the compaction when the final pressure becomes an prescribed value. In the present study, we consider the case \( p = 0.025 \). As is seen from Fig. 2, the packing fraction might depend on the strain rate of the shrinkage, and therefore it is naturally expected that Young’s modulus \( E \) and Poisson’s ratio \( \nu \) of the packings also depend on the strain rate \( \dot{\varepsilon} \).

![FIG. 3: Deviation of the critical packing fraction \( \phi_c(\mu = 0.4) \) from \( \phi_c(\mu = 0.0) \) as a function of the strain rate of the shrinkage, where \( \phi_c(\mu = 0.0) = 0.842 \) is assumed. Inset: Strain rate independence of critical packing fraction \( \phi_c \) with \( \mu = 0.0 \).](image)

These elastic constants of disk packings can be measured by performing a bi-axial compression test [18] with the final states of the packing process. We measure increments of stress, \( \delta \sigma_{ij} \), of the system when the strain, \( \delta\varepsilon_{yy} = 0.01 \) is given while \( \delta\varepsilon_{x} = 0.0 \). Then, assuming the linear elastic theory, we estimate the values of \( E \) and \( \nu \). Note that we carefully performed the compression test so that the quasi-static condition is satisfied during the compression.

The results are summarized in Fig. 4. We find that the largest Young’s modulus is about 1.54 times larger than the smallest one. This fact clearly indicates that the elastic properties depend on the packing process despite the material properties of constituents are identical. Furthermore, the shapes of curves in Fig. 4 suggests

\[
E = E_0 - \alpha E \log(\dot{\varepsilon}), \tag{6}
\]

\[
\nu = \nu_0 + \alpha_\nu \log(\dot{\varepsilon}), \tag{7}
\]

in the regime \( \dot{\varepsilon}_c \leq \dot{\varepsilon} \leq \dot{\varepsilon}_c' \).

![FIG. 4: Young’s modulus \( E \) and Poisson’s ratio \( \nu \) (inset) as functions of the strain rate \( \dot{\varepsilon} \). Young’s modulus \( E = 1 \) corresponds to the dimensionless normal stiffness of the constituent itself. 40 samples are taken for each \( \tau \).](image)

Here we wish to remark on a possibility of realizing
laboratory experiments with our idea. For example, for the experimental system we consider in the paragraph Model, the regime $10^{-6} \leq \dot{\epsilon} \leq 10^{-4}$ can be realized by reducing the size of a box from 46cm to 38cm with compaction time $\tau^{\text{exp}}$ satisfying $0.3(s) \leq \tau^{\text{exp}} \leq 30(s)$. This operation is accessible in laboratory experiments. Our simple setting will provide us a great advantage to investigate history dependence of granular media. We expect that the logarithmic dependence of $E$ and $\nu$ on $\dot{\epsilon}$ will be examined by laboratory experiments.

**Conclusion and discussion:** To conclude, we have found the history dependence of the jamming transition points. For this phenomenon, the tangential friction is found to be essential. Furthermore, we have present a clear evidence for the packing process dependence of the elastic properties of disk packings with identical constituents.

![Graph](image)

**FIG. 5:** Bond-orientational order parameter $\psi_6$ as a function of the strain rate $\dot{\epsilon}$.

The most important question left to be answered is to uncover the universal nature of the logarithmic dependence of $E$ and $\nu$ on $\dot{\epsilon}$. As discussed above, the simple argument suggests the logarithmic dependence of the packing fraction on $\dot{\epsilon}$ under the condition that the pressure $\rho$ in the final state is fixed. Then, one naïvely conjectures that the state with larger packing fraction with $\rho$ fixed corresponds to a state closer to crystal in random packings. In order to confirm this conjecture, we measure the extent of crystallization of the packings with the six-fold order parameter of interbond angle defined by

$$
\psi_6 = \frac{1}{N_{\text{bond}}} \sum_i \left| \sum_{\langle i j k \rangle} \exp(6i\theta_{jk}) \right|
$$

(8)

where $\langle i j k \rangle$ represents a pair of disks which are in contact with disk $i$ next to each other, $\theta_{jk}$ is the interbond angle between bond $ij$ and $ik$, and $N_{\text{bond}}$ is the total number of such interbond angles. Here, bond $ij$ means the line segment connecting the centers of disk $i$ and disk $j$. Note that $\sum_{\langle i j k \rangle} \theta_{jk} = 2\pi$ for each $i$. As shown in Fig. 5 the extent of crystallization decreases rapidly between the two cross-over time scales. Therefore, the understanding of the tendency to crystallization in slower operations might be a heart of future problems. We will study it from several viewpoints.

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