Noncommutative Instantons on $\mathbb{C}P^n$

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Abstract

We construct explicit solutions of the Hermitian Yang-Mills equations on the noncommutative space $\mathbb{C}_\theta^n$. In the commutative limit they coincide with the standard instantons on $\mathbb{C}P^n$ written in local coordinates.
1 Introduction and summary

Noncommutative deformations of gauge field theory provide a controlled theoretical framework beyond locality [1]. Of particular importance are noncommutative instantons (see e.g. [2, 3] and references therein), which are BPS configurations in four dimensions solving the Yang-Mills self-duality equations. In the string context, these solutions describe arrangements of noncommutative branes (see e.g. [4] and references therein).

Natural BPS-type equations for gauge fields in more than four dimensions [5, 6] appear in superstring compactification as the conditions for the survival of at least one supersymmetry [7]. Various solutions to these first-order equations were found e.g. in [8, 9], and their noncommutative generalizations have been considered e.g. in [10, 11]. For $U(n)$ gauge theory on a Kähler manifold these BPS-type equations specialize to the Hermitian Yang-Mills equations [6].

In this Letter we consider the noncommutative space $\mathbb{C}^{n}_\theta$ and construct an explicit $u(n)$-valued solution of the Hermitian Yang-Mills equations. In the commutative limit our configuration coincides with the instanton solution on $\mathbb{C}P^n$ given in local coordinates on a patch $\mathbb{C}^n$ of $\mathbb{C}P^n$. We also describe a noncommutative deformation of a local form of the Abelian configuration on $\mathbb{C}P^n$.

2 Noncommutative space $\mathbb{R}^{2n}_\theta$

Classical field theory on the noncommutative deformation $\mathbb{R}^{2n}_\theta$ of the space $\mathbb{R}^{2n}$ may be realized in a star-product formulation or in an operator formalism [1]. While the first approach alters the product of functions on $\mathbb{R}^{2n}$ the second one turns these functions $f$ into operators $\hat{f}$ acting on the $n$-harmonic-oscillator Fock space $\mathcal{H}$. The noncommutative space $\mathbb{R}^{2n}_\theta$ may then be defined by declaring its coordinate functions $\hat{x}^\mu$ with $\mu = 1, \ldots , 2n$ to obey the Heisenberg algebra relations

$$[\hat{x}^\mu , \hat{x}^\nu ] = i \theta^{\mu\nu}$$

with a constant antisymmetric tensor $\theta^{\mu\nu}$. The coordinates can be chosen in such a way that the matrix $(\theta^{\mu\nu})$ will be block-diagonal with non-vanishing components

$$\theta^{2a-1}{}^{2a} = -\theta^{2a}{}^{2a-1} =: \theta^a \quad \text{for} \quad a = 1, \ldots , n .$$

(2.2)

We assume that all $\theta^a \geq 0$; the general case does not hide additional complications. Both approaches are related by the Moyal-Weyl map [1].

For the noncommutative version of the complex coordinates

$$y^a = x^{2a-1} + i x^{2a} \quad \text{and} \quad \bar{y}^{\bar{a}} = x^{2a-1} - i x^{2a}$$

(2.3)

we have

$$[\bar{y}^{\bar{a}} , y^a] = 2\delta^{\bar{a}a} \theta^a =: \theta^{\bar{a}a} \geq 0 .$$

(2.4)

The Fock space $\mathcal{H}$ is spanned by the basis states

$$|k_1, k_2, \ldots , k_n \rangle = \prod_{a=1}^n (2\theta^a k_a!)^{-1/2} \langle \tilde{y}^{\bar{a}}(k_a) \rangle_{k_a} |0, \ldots , 0 \rangle \quad \text{for} \quad k_a = 0, 1, 2, \ldots ,$$

(2.5)
which are connected by the action of creation and annihilation operators subject to
\[ \left[ \frac{\hat{y}^a}{\sqrt{2\theta^a}}, \frac{\hat{y}^b}{\sqrt{2\theta^b}} \right] = \delta^{ab}. \] (2.6)
For simplicity we consider the case \( \theta^a = \theta \) for all \( a \) and drop the hats from now on.

3 Flat \( u(n+1) \)-connection on \( \mathbb{C}^n_\theta \)

We begin by collecting the coordinates into
\[ Y := \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} \quad \text{and} \quad Y^\dagger = (\bar{y}^1, \ldots, \bar{y}^n), \] (3.1)
so that
\[ Y^\dagger Y = \bar{y}^a y^a = \gamma^2 - 1 - n\theta \] (3.2)
with the definition
\[ \gamma := \sqrt{x^\mu x^\mu + 1} = \sqrt{\bar{y}^a y^a + 1 + n\theta}. \] (3.3)
As this is an invertible operator, we may also introduce the \( n \times n \) matrix
\[ \Lambda := 1_n - Y \frac{1}{\gamma (\gamma + \sqrt{1+n\theta})} Y^\dagger, \] (3.4)
which obeys
\[ \Lambda Y = Y \frac{\sqrt{1+n\theta}}{\gamma}, \quad Y^\dagger \Lambda = \frac{\sqrt{1+n\theta}}{\gamma} Y^\dagger \quad \text{and} \quad \Lambda^2 = 1_n - Y \frac{1}{\gamma^2} Y^\dagger. \] (3.5)
Since all matrix entries are operators acting in the Fock space \( \mathcal{H} \), their ordering is essential, in contrast to the commutative case. In the present section and the following one, all objects are operator-valued in this sense.

Basic for our construction are the \((n+1) \times (n+1)\) matrices
\[ V = \begin{pmatrix} \sqrt{1+n\theta} \gamma^{-1} & -\gamma^{-1} Y^\dagger \\ Y \gamma^{-1} \Lambda \end{pmatrix} \quad \text{and} \quad V^\dagger = \begin{pmatrix} \sqrt{1+n\theta} \gamma^{-1} & \gamma^{-1} Y^\dagger \\ -Y \gamma^{-1} \Lambda \end{pmatrix}. \] (3.6)
With the help of the identities (3.5), one can show that
\[ V^\dagger V = 1_{n+1} = V V^\dagger, \quad \text{i.e.} \quad V \in U(n+1). \] (3.7)
Using \( V \), we build a connection one-form
\[ \mathcal{A} = V^\dagger dV, \] (3.8)
which defines the zero curvature
\[ \mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = dV^\dagger \wedge dV + V^\dagger dV \wedge V^\dagger dV = 0 \] (3.9)
on the free module \( \mathbb{C}^{n+1} \otimes \mathcal{H} \) over \( \mathbb{C}^n_\theta \).

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\(^1\)Here, \( \dagger \) means Hermitian conjugation.
4 Nontrivial $u(1)$ and $u(n)$ gauge fields

Let us rewrite $A$ of (3.8) in the block form

$$A = \begin{pmatrix} a & -\phi^\dagger \\ \phi & A \end{pmatrix} \quad \text{with} \quad a \in u(1) \quad \text{and} \quad A \in u(n) ,$$

(4.1)

Clearly, $\phi$ is an $n \times 1$ matrix and $\phi^\dagger$ its hermitian conjugate. From the definition (3.8) we find that

$$a = \gamma d\gamma^{-1} + \gamma^{-1}Y^\dagger(dY)\gamma^{-1} ,$$

(4.2)

$$A = Y \gamma^{-1}(d\gamma^{-1})Y^\dagger + Y \gamma^{-2}dY^\dagger + \Lambda d\Lambda ,$$

(4.3)

$$\phi = \Lambda(dY)\gamma^{-1} = (dY - Y(\gamma^2 + \gamma \sqrt{1+n\theta})^{-1}Y^\dagger dY)\gamma^{-1} ,$$

(4.4)

$$\phi^\dagger = \gamma^{-1}(dY^\dagger)\Lambda = \gamma^{-1}(dY^\dagger - (dY^\dagger)Y(\gamma^2 + \gamma \sqrt{1+n\theta})^{-1}Y^\dagger) .$$

(4.5)

Introducing the components $\phi^a$ of the column $\phi = (\phi^a)$, the last two equations read

$$\phi^a = (dy^a - y^a(\gamma^2 + \gamma \sqrt{1+n\theta})^{-1}\delta_{bc}\bar{y}^b dy^c)\gamma^{-1} ,$$

(4.6)

$$\bar{\phi}^a = \gamma^{-1}(dy^a - dy^c\delta_{bc} y^b(\gamma^2 + \gamma \sqrt{1+n\theta})^{-1}\bar{y}^a) .$$

(4.7)

The $(1,0)$-forms $\phi^a$ and the $(0,1)$-forms $\bar{\phi}^b$ constitute a basis for the forms of type $(1,0)$ and $(0,1)$, respectively.

Substituting (4.1) into (3.9), we obtain

$$F_{u(1)} := da + a \wedge a = \phi^\dagger \wedge \phi = \delta_{ab}\bar{\phi}^a \wedge \phi^b = \bar{\phi}^1 \wedge \phi^1 + \ldots + \bar{\phi}^n \wedge \phi^n ,$$

(4.8)

$$F_{u(n)} := dA + A \wedge A = \phi \wedge \phi^\dagger = (\phi^a \wedge \bar{\phi}^b) = \begin{pmatrix} \phi^1 \wedge \bar{\phi}^1 & \ldots & \phi^1 \wedge \bar{\phi}^n \\ \vdots & \ddots & \vdots \\ \phi^n \wedge \bar{\phi}^1 & \ldots & \phi^n \wedge \bar{\phi}^n \end{pmatrix} ,$$

(4.9)

as well as

$$0 = d\phi + \phi \wedge a + A \wedge \phi \quad \text{and} \quad 0 = d\phi^\dagger + a \wedge \phi^\dagger + \phi^\dagger \wedge A .$$

(4.10)

From (4.8) and (4.9) one sees that the gauge fields $F_{u(1)}$ and $F_{u(n)}$ have vanishing $(2,0)$ and $(0,2)$ components, i.e. they are of type $(1,1)$. Moreover, (4.9) expresses $F_{u(n)}$ in the basis $\{\phi^a \wedge \bar{\phi}^b\}$ of $(1,1)$-forms as

$$F_{u(n)} = F_{ab}\phi^a \wedge \bar{\phi}^b \quad \implies \quad F_{ab} = 0 = F_{\bar{a}b} \quad \text{and} \quad F_{ab} = e_{ab} = -F_{ba} ,$$

(4.11)

where the basis matrix $e_{ab}$ has a unit entry in the $(ab)$ position and is zero elsewhere.

It is apparent that the operator-valued components of the $u(n)$-valued gauge field $F_{u(n)}$ satisfy the Hermitian Yang-Mills equations\(^2\)

$$F_{ab} = 0 = F_{\bar{a}b} \quad \text{and} \quad F_{1\bar{1}} + \ldots + F_{n\bar{n}} = I_n .$$

(4.12)

\(^2\)Their general form for the structure group $U(k)$ reads $F_{ab} = 0 = F_{\bar{a}b} , F_{1\bar{1}} + \ldots + F_{n\bar{n}} = \tau I_k$, where $\tau$ is a constant.
In the commutative case these equations are the conditions of stability for a holomorphic vector bundle over $\mathbb{C}P^n$ with finite characteristic classes [6]. In the star-product formulation obtained by the inverse Moyal-Weyl transform, the gauge field (4.9) describes a smooth Moyal deformation of the instanton-type gauge field configuration given in local coordinates on a patch $\mathbb{C}^n$ of $\mathbb{C}P^n$. This is why we call the configuration (4.3) and (4.9) the ‘noncommutative U($n$) instanton on $\mathbb{C}P^n$’. Likewise, the Abelian field strength (4.8) with components $f_{ab} := -\delta_{ab}$ satisfies the Hermitian Maxwell equations

$$f_{ab} = 0 = f_{\bar{a}b} \quad \text{and} \quad f_{1\bar{1}} + \ldots + f_{n\bar{n}} = -n,$$

whence the configuration (4.2) and (4.8) is the ‘noncommutative U(1) instanton on $\mathbb{C}P^n$’.

5 Commutative limit

In the commutative limit, $\theta \to 0$, the gauge potential $A$ defining $F_{u(n)}$ coincides with the instanton-type canonical connection on $\mathbb{C}P^n$, which is described as follows [12]. Consider the group U($n+1$), its Grassmannian subset $\mathbb{C}P^n = \text{U}(n+1)/\text{U}(1)\times\text{U}(n)$ and the fibration

$$\text{U}(n+1) \overset{\text{U}(1)\times\text{U}(n)}{\longrightarrow} \mathbb{C}P^n,$$

with fibres $\text{U}(1)\times\text{U}(n)$. For $g \in \text{U}(n+1)$ the canonical one-form $\Omega = g^\dagger dg$ on $\text{U}(n+1)$ takes values in the Lie algebra $u(n+1)$ and satisfies the Maurer-Cartan equation

$$d\Omega + \Omega \wedge \Omega = 0. \quad (5.2)$$

The matrix $V$ from (3.6) defines a local section of the bundle (5.1) over a patch $\mathbb{C}^n \subset \mathbb{C}P^n$, viz. the embedding of $\mathbb{C}P^n$ into $\text{U}(n+1)$. For such an embedding the one-form $\Omega$ coincides with the flat connection $A$ given by (3.8). It follows that (3.9) is the Maurer-Cartan equation (5.2) reduced to $\mathbb{C}^n \subset \mathbb{C}P^n$, and the block form (4.1) results from the splitting of $A$ into components $\phi$ and $\phi^\dagger$ tangent$^3$ to $\mathbb{C}P^n$ and into one-forms $a$ and $A$ on $\mathbb{C}P^n$ with values in the tangent space $u(1)\oplus u(n)$ to the fibre $\text{U}(1)\times\text{U}(n)$ of the bundle (5.1).

By construction, the one-form $A$ from (4.3) is the canonical connection on the Stiefel bundle

$$\text{U}(n+1)/\text{U}(1) \overset{\text{U}(n)}{\longrightarrow} \mathbb{C}P^n,$$

given by [12]

$$A = S^\dagger dS,$$

where $S$ is an $(n+1)\times n$ matrix-valued section of the bundle (5.3) such that $S^\dagger S = 1_n$. In our case it is chosen as

$$S = \begin{pmatrix} -\gamma^{-1}Y^\dagger \\ \Lambda \end{pmatrix}, \quad (5.4)$$

i.e. as the $(n+1)\times n$-part of the matrix $V$ from (3.6). Similarly, the one-form $a$ from (4.2) in the commutative limit coincides with the canonical Abelian connection

$$a = s^\dagger ds \quad (5.5)$$

$^3$They are basis one-forms on $\mathbb{C}P^n$ taking values in the complexified tangent bundle of $\mathbb{C}P^n$. 

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on another Stiefel bundle:

\[ S^{2n+1} = U(n+1)/U(n) \xrightarrow{U(1)} \mathbb{C}P^n. \]  

(5.6)

In our case, \( s = \left( \frac{1}{\chi} \right) \gamma^{-1} \) is the \((n+1) \times 1\) matrix complementing \( S \) inside the matrix \( V \). Moreover, the Abelian gauge field \( F_{u(1)} = -\delta_{ab} \phi^a \wedge \phi^b \) is proportional to the two-form

\[ \omega = \frac{i}{2} \delta_{ab} \phi^a \wedge \phi^b, \]  

(5.7)

which is the canonical Kähler two-form on \( \mathbb{C}P^n \).

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