Existence of solution for an optimal control problem associated to the Ginzburg-Landau system in superconductivity

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Abstract

This article develops a global existence result for the solution of an optimal control problem associated to the Ginzburg-Landau system. This main result is based on standard tools of analysis and functional analysis, such as the Friedrichs Curl Inequality and the Rellich-Kondrashov Theorem. In the concerning model, we consider the presence of an external magnetic field and the control variable is a complex function acting on the super-conducting sample boundary. Finally the state variables are the Ginzburg-Landau order parameter and the magnetic potential, defined on domains properly specified.

1 Introduction

This work develops an existence result for an optimal control problem closely related to the Ginzburg-Landau system in superconductivity. First, we recall that about the year 1950 Ginzburg and Landau introduced a theory to model the super-conducting behavior of some types of materials below a critical temperature \( T_c \), which depends on the material in question. They postulated the free density energy may be written close to \( T_c \) as

\[
F_s(T) = F_n(T) + \frac{\hbar}{4m} \int_{\Omega} |\nabla \psi|^2 \, dx + \frac{\alpha(T)}{4} \int_{\Omega} |\psi|^4 \, dx - \frac{\beta(T)}{2} \int_{\Omega} |\psi|^2 \, dx,
\]

where \( \psi \) is a complex parameter, \( F_n(T) \) and \( F_s(T) \) are the normal and super-conducting free energy densities, respectively (see [2] for details). Here \( \Omega \subset \mathbb{R}^3 \) denotes the super-conducting sample with a boundary denoted by \( \partial \Omega = \Gamma \). The complex function \( \psi \in W^{1,2}(\Omega; \mathbb{C}) \) is intended to minimize \( F_s(T) \) for a fixed temperature \( T \).
Denoting $\alpha(T)$ and $\beta(T)$ simply by $\alpha$ and $\beta$, the corresponding Euler-Lagrange equations are given by:

$$
\begin{cases}
-\frac{\hbar}{2m} \Delta^2 \psi + \alpha |\psi|^2 \psi - \beta \psi = 0, & \text{in } \Omega \\
\frac{\partial \psi}{\partial n} = 0, & \text{on } \partial \Omega.
\end{cases}
$$

This last system of equations is well known as the Ginzburg-Landau (G-L) one. In the physics literature is also well known the G-L energy in which a magnetic potential here denoted by $A$ is included. The functional in question is given by:

$$
J(\psi, A) = \frac{1}{8\pi} \int_{\mathbb{R}^3} (\text{curl } A - B_0)^2 \, dx + \frac{\hbar^2}{4m} \int_{\Omega} \left| \nabla \psi - \frac{2ie}{\hbar c} A \psi \right|^2 \, dx + \frac{\alpha}{4} \int_{\Omega} |\psi|^4 \, dx - \frac{\beta}{2} \int_{\Omega} |\psi|^2 \, dx
$$

Considering its minimization on the space $U$, where

$$
U = W^{1,2}(\Omega; \mathbb{C}) \times W^{1,2}(\mathbb{R}^3; \mathbb{R}^3),
$$

through the physics notation the corresponding Euler-Lagrange equations are:

$$
\begin{cases}
\frac{1}{2m} \left(-i\hbar \nabla - \frac{2e}{c} A\right)^2 \psi + \alpha |\psi|^2 \psi - \beta \psi = 0, & \text{in } \Omega \\
(i\hbar \nabla \psi + \frac{2e}{c} A \psi) \cdot n = 0, & \text{on } \partial \Omega,
\end{cases}
$$

and

$$
\begin{cases}
\text{curl } (\text{curl } A) = \text{curl } B_0 + \frac{4\pi}{c} \tilde{J}, & \text{in } \Omega \\
\text{curl } (\text{curl } A) = \text{curl } B_0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega},
\end{cases}
$$

where

$$
\tilde{J} = \frac{ie\hbar}{2m} \left( \psi^* \nabla \psi - \psi \nabla \psi^* \right) - \frac{2e^2}{mc} |\psi|^2 A.
$$

and $B_0 \in L^2(\mathbb{R}^3; \mathbb{R}^3)$

is a known applied magnetic field.

Existence of a global solution for a similar problem has been proved in [3].

### 2 An existence result for a related optimal control problem

Let $\Omega \subset \mathbb{R}^3$, $\Omega_1 \subset \mathbb{R}^3$ be open, bounded and connected sets with Lipschitzian boundaries, where $\overline{\Omega} \subset \Omega_1$ and $\Omega_1$ is convex. Let $\phi_d : \Omega \to \mathbb{C}$ be a known function in $L^4(\Omega; \mathbb{C})$ and consider the problem of minimizing

$$
|||\phi|^2 - |\phi_d|^2||^2_{0,2,\Omega}
$$

with $(\phi, A, u)$ subject to the satisfaction of the Ginzburg-Landau equations, indicated in [5] and [6] in the next lines.
For such a problem, the control variable is \( u \in L^2(\partial \Omega; \mathbb{C}) \) and the state variables are the Ginzburg-Landau order parameter \( \phi \in W^{1,2}(\Omega, \mathbb{C}) \) and the magnetic potential \( A \in W^{1,2}(\Omega_1, \mathbb{R}^3) \).

Our main existence result is summarized by the following theorem.

**Theorem 2.1.** Consider the functional

\[
J(\phi, A, u) = \frac{\varepsilon}{2} \|\nabla \phi\|_{0,2,\Omega}^2 + K_1 \|\phi\|^2 - |\mu d|^2_{0,2,\Omega} + K_2 \|u\|^2_{0,2,\partial \Omega},
\]

subject to \((\phi, A, u) \in \mathcal{C}\), where

\[
\mathcal{C} = \{ (\phi, A, u) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega_1, \mathbb{R}^3) \times L^2(\partial \Omega; \mathbb{C}) : \text{ such that (5) and (6) hold} \},
\]

where

\[
\begin{aligned}
\frac{1}{2m} \left( -i \hbar \nabla - \frac{2e}{c} A \right)^2 \phi + \alpha |\phi|^2 \phi - \beta \phi &= 0, \quad \text{in } \Omega, \\
(i \hbar \nabla \phi + \frac{2e}{c} A \phi) \cdot n &= u, \quad \text{on } \partial \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
\text{curl curl } A &= \text{curl } B_0 + \frac{4\pi}{c} \tilde{J}, \quad \text{in } \Omega, \\
\text{curl curl } A &= \text{curl } B_0, \quad \text{in } \Omega \setminus \Omega_1, \\
\text{div } A &= 0, \quad \text{in } \Omega_1, \\
A \cdot n &= 0, \quad \text{on } \partial \Omega_1
\end{aligned}
\]

where,

\[
\tilde{J} = -\frac{i \hbar c}{2m} (\phi^* \nabla \phi - \phi \nabla \phi^*) - \frac{2e^2}{mc} |\phi|^2 A,
\]

and where \( \varepsilon > 0 \) is a small parameter, \( K_1 > 0 \) and \( K_2 > 0 \).

Under such hypotheses, there exists \((\phi_0, A_0, u_0) \in \mathcal{C}\) such that

\[
J(\phi_0, A_0, u_0) = \min_{(\phi, A, u) \in \mathcal{C}} J(\phi, A, u).
\]

**Proof.** Let \( \{(\phi_n, A_n, u_n)\} \) be a minimizing sequence (such a sequence exists from the existence result for \( u = 0 \) in \([3]\), and from the fact that \( J \) is lower bounded by 0).

Thus, such a sequence is such that

\[
J(\phi_n, A_n, u_n) \to \eta = \inf_{(\phi, A, u) \in \mathcal{C}} J(\phi, A, u).
\]

From the expression of \( J \), there exists \( K > 0 \) such that

\[
\|\nabla \phi_n\|_{0,2,\Omega} \leq K, \\
\|\phi_n\|_{0,4,\Omega} \leq K, \\
\|\phi_n\|_{0,2,\Omega} \leq K,
\]

and

\[
\|u_n\|_{0,2,\partial \Omega} \leq K, \quad \forall n \in \mathbb{N}
\]
so that, from the Rellich-Kondrashov Theorem, there exists a not relabeled subsequence, \( \phi_0 \in W^{1,2}(\Omega, \mathbb{C}) \) and \( u_0 \in L^2(\Omega, \mathbb{C}) \) such that

\[
\phi_n \rightharpoonup \phi_0, \text{ weakly in } W^{1,2}(\Omega, \mathbb{C}),
\]

\[
\phi_n \to \phi_0, \text{ in norm in } L^2(\Omega, \mathbb{C}) \text{ and } L^4(\Omega, \mathbb{C}),
\]

\[
u_n \rightharpoonup u_0, \text{ weakly in } L^2(\partial \Omega, \mathbb{C}), \text{ as } n \to \infty.
\]

On the other hand, we have from (6), from the generalized Hölder inequality and for constants 
\( \gamma = \frac{4\pi}{c} \left| \frac{ie\hbar}{2m} \right| > 0 \) and \( \gamma_1 = \frac{4\pi 2\pi^2}{mc} > 0 \) that

\[
0 = \rho_{1,n} \equiv \langle \text{curl } A_n, \text{curl } A_n \rangle_{L^2(\Omega; \mathbb{R}^3)}
- \langle \text{curl } A_n, B_0 \rangle_{L^2(\Omega; \mathbb{R}^3)}
+ \frac{4\pi}{c} \left\langle \frac{ie\hbar}{2m} (\phi^* \nabla \phi - \phi \nabla \phi^*) + \frac{2e^2}{mc} |\phi|^2 A_n, A_n \right\rangle_{L^2(\Omega; \mathbb{R}^3)}
\geq \langle \text{curl } A_n, \text{curl } A_n \rangle_{L^2(\Omega; \mathbb{R}^3)}
- \| \text{curl } A_n \|_{0,2,\Omega_1}^2 \| B_0 \|_{0,2,\Omega_1} - \gamma \| A_n \|_{0,4,\Omega_1} \| \nabla \phi_n \|_{0,2,\Omega_1}
+ \gamma_1 \langle |\phi|^2, A_n \cdot A_n \rangle_{L^2(\Omega; \mathbb{R}^3)}. \tag{7}
\]

From the Friedrichs Inequality (see [3] for details) and the Sobolev Imbedding Theorem for appropriate constants indicated, we obtain

\[
\| A_n \|^2_{0,4,\Omega_1} \leq K_3 \| A_n \|^2_{1,2,\Omega_1} \leq K_4 (\| \text{div } A_n \|_{0,2,\Omega_1} + \| \text{curl } A_n \|_{0,2,\Omega_1}^2)
\]

\[
= K_4 \| \text{curl } A_n \|^2_{0,2,\Omega_1}, \tag{8}
\]

since from the London Gauge assumption,

\[
\text{div } A_n = 0, \text{ in } \Omega_1, \forall n \in \mathbb{N}.
\]

Summarizing, we have obtained, for some appropriate \( K_5 > 0 \),

\[
0 = \rho_{1,n} \geq K_5 \| A_n \|^2_{0,4,\Omega_1} + \frac{1}{2} \| \text{curl } A_n \|^2_{0,2,\Omega_1}
- \| \text{curl } A_n \|_{0,2,\Omega_1} \| B_0 \|_{0,2,\Omega_1} - \gamma \| A_n \|_{0,4,\Omega_1} K^2
+ \gamma_1 \langle |\phi|^2, A_n \cdot A_n \rangle_{L^2(\Omega; \mathbb{R}^3)}
\equiv \rho_{2,n}. \tag{9}
\]

Now, suppose to obtain contradiction there exists a subsequence \( \{n_k\} \subset \mathbb{N} \) such that

\[
\| A_{n_k} \|_{0,4,\Omega_1} \to \infty, \text{ as } k \to \infty.
\]

From (9) we obtain

\[
\rho_{2,n_k} \to \infty, \text{ as } k \to \infty,
\]

which contradicts

\[
\rho_{2,n} \leq 0, \forall n \in \mathbb{N}.
\]
Hence, there exists $K_6 > 0$ such that
\[ \| A_n \|_{0, 4, \Omega_1} < K_6, \]
and
\[ \| A_n \|_{0, 2, \Omega_1} < K_6, \forall n \in \mathbb{N}. \]

From this and (7) we have,
\[
0 = \rho_{1, n} \geq \| \text{curl} A_n \|_{0, 2, \Omega_1}^2 - \| \text{curl} A_n \|_{0, 2, \Omega_1} \| B_0 \|_{0, 2, \Omega_1} - \gamma K_6 K^2 + \gamma_1 \langle |\phi|^2, A_n \cdot A_n \rangle_{L^2(\Omega, \mathbb{R}^3)} \\
\equiv \rho_{3, n}. \tag{10}
\]

Suppose to obtain contradiction there exists a subsequence \( \{n_k\} \subset \mathbb{N} \) such that
\[ \| \text{curl} A_{n_k} \|_{0, 2, \Omega_1} \to \infty, \text{ as } k \to \infty. \]

From (10) we obtain
\[ \rho_{3, n_k} \to \infty, \text{ as } k \to \infty, \]
which contradicts
\[ \rho_{3, n} \leq 0, \forall n \in \mathbb{N}. \]

Hence, there exists $K_7 > 0$ such that
\[ \| \text{curl} A_n \|_{0, 2, \Omega_1} < K_7, \forall n \in \mathbb{N} \]
so that from this, the Friedrichs inequality and the London Gauge hypothesis, we obtain $K_8 > 0$ such that
\[ \| A_n \|_{1, 2, \Omega_1} < K_8. \]

So from such a result and the Rellich-Kondrashov Theorem there exists a not relabeled subsequence and $A_0 \in W^{1, 2}(\Omega_1, \mathbb{R}^3)$ such that
\[ A_n \rightharpoonup A_0 \text{ weakly } \in W^{1, 2}(\Omega_1, \mathbb{R}^3) \]
\[ A_n \to A_0 \text{ in norm in } L^2(\Omega_1, \mathbb{R}^3) \text{ and } L^4(\Omega_1, \mathbb{R}^3). \]

Moreover from the Sobolev Imbedding Theorem, there exist real constants $\hat{K} > 0, \hat{K}_1 > 0$ such that
\[ \| \phi_n \|_{0, 6, \Omega} \leq \hat{K} \| \phi_n \|_{1, 2, \Omega} < \hat{K}_1, \forall n \in \mathbb{N}. \]

Thus, from this and the first equation in (5), there exist real constants $\hat{K}_2 > 0, \ldots, \hat{K}_6 > 0$, such that
\[
\| \nabla^2 \phi_n \|_{0, 2, \Omega} < \hat{K}_2 \| A_n \|_{0, 2, \Omega} \| \nabla \phi_n \|_{0, 2, \Omega} \\
+ \hat{K}_3 \| A_n \|_{0, 4, \Omega} \| \phi_n \|_{0, 2, \Omega} + \hat{K}_4 \| \phi_n \|_{0, 6, \Omega}^3 + \hat{K}_5 \| \phi_n \|_{0, 2, \Omega}^3 \\
\leq \hat{K}_6, \forall n \in \mathbb{N}. \tag{11}
\]
From this, up to a subsequence, we get
\[ \nabla^2 \phi_n \rightharpoonup \nabla^2 \phi_0 \text{ weakly in } L^2(\Omega; \mathbb{C}). \]

Let \( \varphi \in C_c^\infty(\Omega, \mathbb{C}) \), \( \varphi_1 \in C_c^\infty(\Omega, \mathbb{R}^3) \) and \( \varphi_2 \in C_c^\infty(\Omega_1 \setminus \Omega, \mathbb{R}^3) \).

From the last results, we may easily obtain the following limits
1. \( \langle \nabla^2 \phi_n, \varphi \rangle_{L^2} \to \langle \nabla^2 \phi_0, \varphi \rangle_{L^2} \),
2. \( \langle \nabla \phi_n, \nabla \varphi \rangle_{L^2} \to \langle \nabla \phi_0, \nabla \varphi \rangle_{L^2} \),
3. \( \langle A_n \cdot \nabla \phi_n, \varphi \rangle_{L^2} \to \langle A_0 \cdot \nabla \phi_0, \varphi \rangle_{L^2} \),
4. \( \langle |A_n|^2 \phi_n, \varphi \rangle_{L^2} \to \langle |A_0|^2 \phi_0, \varphi \rangle_{L^2} \),
5. \( \langle |\phi_n|^2 \varphi \rangle_{L^2} \to \langle |\phi_0|^2 \varphi \rangle_{L^2} \),
6. \( \langle \nabla \phi_n, \phi_1 \rangle_{L^2} \to \langle \nabla \phi_0, \phi_1 \rangle_{L^2} \),
7. \( \langle \phi_n^* \nabla \phi_n, \phi_1 \rangle_{L^2} \to \langle \phi_0^* \nabla \phi_0, \phi_1 \rangle_{L^2} \),
8. \( \langle \phi_n \nabla \phi_n^*, \phi_1 \rangle_{L^2} \to \langle \phi_0 \nabla \phi_0^*, \phi_1 \rangle_{L^2} \),
9. \( \langle |\phi_n|^2 A_n, \phi_1 \rangle_{L^2} \to \langle |\phi_0|^2 A_0, \phi_1 \rangle_{L^2} \).

For example, for (4), for an appropriate real \( \tilde{K} > 0 \) we have

\[
|\langle |A_n|^2 \phi_n, \varphi \rangle_{L^2} - \langle |A_0|^2 \phi_0, \varphi \rangle_{L^2}| \\
\leq |\langle |A_n|^2 - |A_0|^2 \rangle \phi_n, \varphi \rangle_{L^2} + |\langle |A_0|^2 \rangle \phi_n - \phi_0, \varphi \rangle_{L^2}| \\
\leq |\langle |A_n| - |A_0| \rangle |A_n| + |A_0| \rangle \phi_n, \varphi \rangle_{L^2} + |\langle |A_0|^2 \phi_n - \phi_0, \varphi \rangle_{L^2}| \\
\leq |\langle |(A_n| + |A_0|)\rangle_{0,4,\Omega} |A_n| - |A_0| \rangle |\phi_n|_{0,2,\Omega} \varphi \rangle_{L^2} + |\langle |A_0|^2 \phi_n - \phi_0, \varphi \rangle_{L^2}| \\
\leq K(\|A_n| - |A_0|\|_{0,4,\Omega} \|A_n| - |A_0|\|_{0,2,\Omega} \|\phi_n - \phi_0, \varphi \|_{L^2}) \\
\to 0, \text{ as } n \to \infty. \quad (12)
\]

The other items may be proven similarly.

Now let \( \varphi \in C^\infty(\overline{\Omega}, \mathbb{C}) \). Observe that
\[ \langle u_n, \varphi \rangle_{L^2(\partial \Omega, \mathbb{C})} = \left\langle \left( i \hbar \nabla \phi_n + \frac{2e}{c} A_n \phi_n \right) \cdot n, \varphi \right\rangle_{L^2(\partial \Omega, \mathbb{C})} \]
\[ = \left\langle i \hbar \nabla \phi_n + \frac{2e}{c} A_n \phi_n, \nabla \varphi \right\rangle_{L^2(\Omega, \mathbb{C}^3)} + \left\langle \text{div} \left( i \hbar \nabla \phi_n + \frac{2e}{c} A_n \phi_n \right), \varphi \right\rangle_{L^2(\Omega, \mathbb{C})} \]
\[ \to \left\langle i \hbar \nabla \phi_0 + \frac{2e}{c} A_0 \phi_0, \nabla \varphi \right\rangle_{L^2(\Omega, \mathbb{C}^3)} + \left\langle \text{div} \left( i \hbar \nabla \phi_0 + \frac{2e}{c} A_0 \phi_0 \right), \varphi \right\rangle_{L^2(\Omega, \mathbb{C})} \]
\[ = \left\langle \left( i \hbar \nabla \phi_0 + \frac{2e}{c} A_0 \phi_0 \right) \cdot n, \varphi \right\rangle_{L^2(\partial \Omega, \mathbb{C})}. \tag{13} \]

From this and from
\[ \langle u_n, \varphi \rangle_{L^2(\partial \Omega, \mathbb{C})} \to \langle u_0, \varphi \rangle_{L^2(\partial \Omega, \mathbb{C})}, \]
we have
\[ \left\langle \left( i \hbar \nabla \phi_0 + \frac{2e}{c} A_0 \phi_0 \right) \cdot n - u_0, \varphi \right\rangle_{L^2(\partial \Omega, \mathbb{C})} = 0, \ \forall \varphi \in C^\infty(\overline{\Omega}, \mathbb{C}), \]
so that in such a distributional sense,
\[ \left( i \hbar \nabla \phi_0 + \frac{2e}{c} A_0 \phi_0 \right) \cdot n = u_0, \ \text{on} \ \partial \Omega. \]

The other boundary condition may be dealt similarly. Thus, from these last results we may infer that in the distributional sense,
\[ \begin{cases} 
\frac{1}{2m} \left( -i \hbar \nabla - \frac{2e}{c} A_0 \right)^2 \phi_0 + \alpha |\phi_0|^2 \phi_0 - \beta \phi_0 = 0, & \text{in} \ \Omega, \\
(i \hbar \nabla \phi_0 + \frac{2e}{c} A_0 \phi_0) \cdot n = u_0, & \text{on} \ \partial \Omega, \end{cases} \tag{14} \]
and
\[ \begin{cases} 
\text{curl} \ \text{curl} \ A_0 = \text{curl} \ B_0 + \frac{4\pi}{c} \tilde{J}_0, & \text{in} \ \Omega, \\
\text{curl} \ \text{curl} \ A_0 = \text{curl} \ B_0, & \text{in} \ \Omega_1 \setminus \Omega, \\
\text{div} \ A_0 = 0, & \text{in} \ \Omega_1, \\
A_0 \cdot n = 0, & \text{on} \ \partial \Omega_1 \end{cases} \tag{15} \]
where,
\[ \tilde{J}_0 = -\frac{i e \hbar}{2m} (\phi_0^* \nabla \phi_0 - \phi_0 \nabla \phi_0^*) - \frac{2e^2}{mc} |\phi_0|^2 A_0. \]
Hence \((\phi_0, A_0, u_0) \in C\).

Finally, from \(\phi_n \to \phi_0\) in \(L^2\) and \(L^4\), \(\phi_n \rightharpoonup \phi_0\) weakly in \(W^{1,2}\), \(u_n \to u_0\) weakly in \(L^2(\partial\Omega)\), by continuity in \(\phi\) and the convexity of \(J\) in \(\nabla \phi\) and \(u\), we have,

\[
\eta = \liminf_{n \to \infty} J(\phi_n, A_n, u_n) \geq J(\phi_0, A_0, u_0).
\]

The proof is complete.

3 Conclusion

In this article we have developed a global existence result for a control problem related to the Ginzburg-Landau system in superconductivity. We emphasize the control variable \(u\) acts on the super-conducting sample boundary, whereas the state variables, namely, the order parameter \(\phi\) and the magnetic potential \(A\) are defined on \(\Omega\) and \(\Omega_1\), respectively. The problem has non-linear constraints but the cost functional is convex. Finally, we highlight the London Gauge assumption and the Friedrichs Inequality have a fundamental role in the establishment of the main results.

References

[1] R.A. Adams and J.F. Fournier, Sobolev Spaces, 2nd edn. (Elsevier, New York, 2003).
[2] J.F. Annet, Superconductivity, Superfluids and Condensates, 2nd edn. (Oxford Master Series in Condensed Matter Physics, Oxford University Press, New York, Reprint, 2010)
[3] F. Botelho, A Classical Description of Variational Quantum Mechanics and Related Models, Nova Science Publishing, New York, 2017.
[4] F. Botelho, Functional Analysis and Applied Optimization in Banach Spaces, (Springer Switzerland, 2014).
[5] F. Botelho, Real Analysis and Applications, (Springer Switzerland, 2018).
[6] B. Hall, Quantum Theory for Mathematicians (Springer, New York 2013).
[7] L.D. Landau and E.M. Lifschits, Course of Theoretical Physics, Vol. 5- Statistical Physics, part 1. (Butterworth-Heinemann, Elsevier, reprint 2008).
[8] B. Schweizer, On Friedrichs Inequality, Helmholtz Decomposition, Vector Potentials, and the div-curl Lemma. Trends in Applications of Mathematics to Mechanics, INdAM Series, Springer, Berlin, 2018.