Bivariate Burr Type III Distribution: Estimation and Prediction

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Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Abstract

In this paper, a bivariate Burr Type III distribution is constructed and some of its statistical properties such as bivariate probability density function and its marginal, joint cumulative distribution and its marginal, reliability and hazard rate functions are studied. The joint probability density function and the joint cumulative distribution are given in closed forms. The joint expectation of this distribution is proposed. The maximum likelihood estimation and prediction for a future observation are derived. Also, Bayesian estimation and prediction are considered under squared error loss function. The performance of the proposed bivariate distribution is examined using a simulation study. Finally, a data set is analyzed under the proposed distribution to illustrate its flexibility for real-life application.

Keywords: Bivariate distributions; compounding; maximum likelihood estimation; maximum likelihood prediction; Bayesian estimation; Bayesian prediction.

1 Introduction

In this paper, a bivariate Burr Type III (BBIII) distribution is considered by compounding inverted Weibull (IW) distribution with the gamma distribution. Among the features of the bivariate distributions their
importance on both theoretical and applied grounds; bivariate analysis was applied to a variety of disciplines.

In the recent statistical literature several methodologies of constructing bivariate and multivariate distributions based on marginal and conditional distributions are proposed by Arnold et al. [1,2], Kotz et al. [3] and Balakrishnan and Lai [4] among others. An important field of research focuses on the study of new classes of univariate distributions which contain the classical proposals, also allowing for more flexibility in fitting data.

Domma [5] extended some results related to the dependence structure of the BBIII distribution; proposed by Rodriguez [6] using copula representations of bivariate distributions. Headrick et al. [7] presented a method for simulating univariate and multivariate BBIII and Burr Type XII distributions with specified correlation matrices. Ismail and Khalid [8] used some copulas as Ali-Mikhail-Haq, Clayton and Gumbel based on uncensored data to joint specific BBIII and Burr Type XII distributions by using theorem and algorithm of construction. Capitani et al. [9] showed a bivariate Burr Type III copula to the trivariate case. Ogana et al. [10] considered Frank and Plackett copulas; the two copulas were evaluated on seven distributions models using data from temperate and tropical forests where one of the seven distributions is BBIII distribution. Azizi and Sayyareh [11] constructed bivariate BBIII using Marshall and Olkin [12] technique as they presented some properties and estimated the parameters using maximum likelihood (ML) method. AL-Hussaini and Ateya [13] considered a class of multivariate distributions using a special case of Takahasi [14] technique. One can apply the same technique suggested by AL-Hussaini and Ateya [13] to construct bivariate distributions as special case of their technique.

One of the objectives of this paper is to construct BBIII distribution; based on Takahasi [14]. He obtained multivariate Burr Type XII distribution, by compounding IW distribution which have a simple algebraic cumulative distribution function with gamma distribution as a compounder. One can apply Takahasi technique to construct multivariate Burr distributions where these distributions will have some nice properties compared to other different multivariate distributions. It could be useful in studying reliability maintainability of complicated systems. The same technique can be used to introduce and construct BBIII distribution, also some properties can be studied.

The rest of the paper is organized as follows: In Section 2, a construction of BBIII distribution is introduced, also some properties of the distribution are obtained. Maximum likelihood estimation and prediction are considered in Section 3. In Section 4, a numerical illustration for the ML estimation and prediction is given. In Section 5, Bayesian estimation and two-sample prediction is considered. In Section 6, a numerical illustration for Bayesian estimation and prediction is presented.

2 Construction of Bivariate Burr Type III Distribution

This section aims to construct BBIII distribution by using the method suggested by Takahasi [14] considering the bivariate case as special case.

2.1 The joint probability density function and the joint cumulative density function

Assuming that $T_1, T_2$ have conditional independent IW distributions upon a common scale parameter $b$; with probability density function (pdf) given by,

$$f(t|b) = \left[ \prod_{i=1}^{2} f(t_i|b) \right], \quad \text{where} \quad t = (t_1, t_2).$$

Where, the pdf and cumulative distribution function (cdf) of $T_i$~IW distribution (where $i = 1,2$) can be written, respectively, as follows:
and that the parameter $b$ has a gamma distribution with pdf

$$g(b) = \frac{a^k}{\Gamma(k)} b^{k-1} \exp(-ab), \quad b > 0, \quad (k, a > 0).$$

For more details see AL-Hussaini and Ateya [13].

Then the joint density function of $T_1, T_2$ is given by,

$$f(t_1, t_2) = \int_0^\infty f(t_1 | b) g(b) db$$

$$= \int_0^\infty \prod_{i=1}^2 c_i b t_i^{-(c_i+1)} \exp(-b t_i^{-c_i}) \left[ \frac{a^k}{\Gamma(k)} b^{k-1} \exp(-ab) \right] db.$$

Hence,

$$f(t_1, t_2) = \frac{a^k \Gamma(k+2)}{\Gamma(k)} \left( a + \sum_{i=1}^2 t_i^{-c_i} \right)^{-(k+2)} \prod_{i=1}^2 \left( c_i t_i^{-(c_i+1)} \right)$$

$$= \frac{k(k+1)}{a^2} c_1 c_2 t_1^{-(c_1+1)} t_2^{-(c_2+1)} \left[ 1 + \frac{1}{a} t_1^{-c_1} + \frac{1}{a} t_2^{-c_2} \right]^{-(k+2)} , \quad 0 < t_2, t_2 < \infty, \quad i = 1, 2. \quad (4)$$

Equation (4) implies that $f(t_1, t_2; c_1, c_2, a, k)$ is the 2-dimensional pdf of compound IW (2-dimensional direct product) distribution with the gamma distribution as a compounder. Then (4) is the pdf for BBIII distribution, the joint pdf can be rewritten as follows:

$$f(t_1, t_2) = \begin{cases} \frac{k(k+1)}{a^2} \left( \prod_{i=1}^2 c_i t_i^{-(c_i+1)} \right) \left( 1 + \frac{1}{a} \sum_{i=1}^2 t_i^{-c_i} \right)^{-(k+2)} , & t_i > 0, \quad (i = 1, 2) \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

The contour plots of the joint pdf of the BBIII distribution for different parameter values are presented in Fig. 1.

From Fig. 1, one can observe that the contour plots of BBIII distribution are increasing and decreasing, hence BBIII distribution is flexible and has many applications in life testing.
Fig. 1. The contour plots of the joint pdf of the BBIII distribution for different parameter values:

(1.a) \((k = 0.5, \alpha = 0.75, c_1 = 2, c_2 = 1)\), (1.b) \((k = 1.5, \alpha = 1, c_1 = 2, c_2 = 1)\),

(1.c) \((k = 3, \alpha = 3, c_1 = 3, c_2 = 3)\) and (1.d) \((k = 4, \alpha = 2, c_1 = 2, c_2 = 2)\)

The joint cdf of the BBIII distribution is given by

\[
F(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} f(t_1, t_2) dt_2 dt_1
= \int_0^{t_1} \int_0^{t_2} \frac{k(k + 1)}{a^2} c_1 c_2 t_1^{-(c_1+1)} t_2^{-(c_2+1)} \left[ 1 + \frac{1}{a} t_1^{-c_1} + \frac{1}{a} t_2^{-c_2} \right]^{-(k+2)} dt_2 dt_1
= 1 - \left[ 1 + \frac{1}{a} t_1^{-c_1} \right]^{-k} - \left[ 1 + \frac{1}{a} t_2^{-c_2} \right]^{-k} + \left[ 1 + \frac{1}{a} t_1^{-c_1} + \frac{1}{a} t_2^{-c_2} \right]^{-k}
\]

(6)

The marginals pdf and cdf of the BBIII distribution can be written, respectively, as
2.2 The joint reliability function and the joint hazard rate function

The joint reliability function (rf) of BBIII distribution is given by,

\[ R(t_1, t_2) = P(T_1 > t_1, T_2 > t_2) = 1 - F(t_1) - F(t_2) + F(t_1, t_2) = \left[ 1 + \frac{1}{a} t_1^{-c_1} + \frac{1}{a} t_2^{-c_2} \right]^{-k}. \]  

(9)

The joint hazard rate function (hrf) can be defined as,

\[ h(t_1, t_2) = \frac{f(t_1, t_2)}{R(t_1, t_2)} = \frac{k(k + 1)}{a^2} c_1 c_2 t_1^{-(c_1+1)} t_2^{-(c_2+1)} \left[ 1 + \frac{1}{a} t_1^{-c_1} + \frac{1}{a} t_2^{-c_2} \right]^{-2}. \]  

(10)

If \((T_1, T_2) \sim\) BBIII, then for all values of \(t_1 > 0, t_2 > 0\) both components of \(h(t_1, t_2)\) are decreasing function of \(t_1\) and \(t_2\). The contour plots of the joint hrf of the BBIII distribution for different parameter values are presented in Fig. 2.

From Fig. 2, one can observe that the joint hrf is decreasing function, then the BBIII is flexible for modeling decreasing survival data.

2.3 The joint moments of bivariate Burr III distribution

If the vector \( \mathbf{T} = (t_1, t_2) \) has the pdf in (4), then the joint moments of order \( p = \sum_{i=1}^{2} p_i \) (bivariate case) for BBIII distribution can be written as

\[ E\left( t_1^{p_1}, t_2^{p_2} \right) = \sum_{i=1}^{2} \left( k + i - 1 \right) \left( \frac{p_i}{c_i} \right) B \left( k + i - 1 + \sum_{j=1}^{i} \frac{p_j}{c_j}, 1 - \frac{p_i}{c_i} \right). \]  

(11)

Proof:

By integrating the right hand side of the previous equation with respect to \( t_2 \), one obtains

\[ E\left( t_1^{p_1}, t_2^{p_2} \right) = k(k + 1) \int_0^\infty t_1^{p_1} c_1 \left( \frac{1}{a} t_1^{-(c_1+1)} \left( 1 + \frac{1}{a} t_1^{-c_1} \right)^{-(k+1)} \right) I_2 \, dt_1, \]  

(12)

Where,

\[ I_2 = \int_0^\infty t_2^{p_2} c_2 \left( \frac{1}{a} t_2^{-(c_2+1)} \left( 1 + D_2 t_2^{-c_2} \right)^{-(k+2)} \right) \, dt_2 \]
Using the transformation \( z_2 = \left[1 + D_2 t_2^{-c_2}\right]^{-1} \), then \( t_2 = \left(\frac{1-z_2}{d_2 z_2}\right)^{-\frac{1}{c_2}} \) and \( D_2 = \frac{a^{-1}}{1+\frac{1}{a}^{-c_1}} \)

\[
dt_2 = \frac{1}{c_2} \left(\frac{1-z_2}{d_2 z_2}\right)^{-\frac{1}{c_2}} \left(\frac{1}{d_2 z_2^2}\right) dz_2,
\]

hence,

\[
l_2 = \left[1 + \frac{1}{a} t_2^{-c_1}\right] \left(\frac{1}{d_2}\right) \frac{p_2}{c_2} B \left(k + 1 + \frac{p_2}{c_2}, 1 - \frac{p_2}{c_2}\right).
\]

Fig. 2. Contour plots of the joint hazard of the BBIII distribution for different parameter values:

(2.a) \( k = 1, \alpha = 1, c_1 = 1, c_2 = 0.5 \),
(2.b) \( k = 0.8, \alpha = 0.8, c_1 = 0.8, c_2 = 0.8 \),
(2.c) \( k = 1.5, \alpha = 0.5, c_1 = 2, c_2 = 1 \) and
(2.d) \( k = 0.5, \alpha = 0.75, c_1 = 2, c_2 = 1 \)
Substituting (13) in (12), Equation (11) is proved.

The joint moments of order \( p = \sum_{i=1}^{2} p_{i} \) for BBIII distribution can be obtained, using (11)

\[
E(t_{1}^{p_{1}}, t_{2}^{p_{2}}) = k(k + 1) \left( \frac{1}{a} \right)^{p_{1}/c_{1} + p_{2}/c_{2}} B \left( k + \frac{p_{1}}{c_{1}} + \frac{p_{2}}{c_{2}}, 1 - \frac{p_{1}}{c_{1}} \right) B \left( k + 1 + \frac{p_{2}}{c_{2}}, 1 - \frac{p_{2}}{c_{2}} \right),
\]

where,

\[
B(a, b) = \int_{0}^{1} z^{a-1}(1 - z)^{b-1} dz \text{ is the beta function. For univariate case, the moments are}
\]

\[
E(t_{i}^{p_i}) = k \left( \frac{1}{a} \right)^{p_i/c_1} B \left( k + \frac{p_i}{c_1}, 1 - \frac{p_i}{c_1} \right). \tag{15}
\]

The correlation coefficient \( \rho(t_1, t_2) \) can be computed using (14) and (15) with the appropriate choices of \( p_1 \) and \( p_2 \).

### 3 Maximum Likelihood Estimation and Prediction

In this section, the ML estimation and prediction for the BBIII distribution will be derived.

#### 3.1 Maximum likelihood estimation

Instead of obtaining the likelihood function by a direct use of the pdf in (4), the advantage of its construction by compounding \( \prod_{i=1}^{k} f(t_i | \omega) \) and \( g(b) \) is applied; which makes the estimation quite easier. The likelihood function of BBIII distribution is given by

\[
L(\omega; t_1, t_2, b) = \prod_{j=1}^{n} f(t_{1j}, t_{2j}, b_j) = \prod_{j=1}^{n} \left( \prod_{i=1}^{2} f(t_{ij}) g(b_j) \right)
\]

\[
= \frac{a^{nk}}{(\Gamma(k))^n} \left( \prod_{j=1}^{n} b_j \right)^{k+1} c_1^n c_2^n \left( \prod_{j=1}^{n} t_{1j} \right)^{-(c_1+1)} \left( \prod_{j=1}^{n} t_{2j} \right)^{-(c_2+1)}
\]

\[
\times \exp \left[ -\sum_{j=1}^{n} b_j (t_{1j}^{-c_1} + t_{2j}^{-c_2} + a) \right], \tag{16}
\]

where, \( \omega = (a, k, c_1, c_2) \).

The log likelihood function is given by

\[
\ell \equiv \ell(\omega; t_1, t_2, b)
\]

\[
= nk \ln a - n \ln \Gamma(k) + (k + 1) \left( \sum_{j=1}^{n} \ln b_j \right) + n \ln c_1 + n \ln c_2
\]
The ML estimators of the parameters are obtained by differentiating (17) with respect to the parameters, setting to zero and then solving the resulting non-linear system of likelihood equations. Hence,

\[
\frac{\partial \ell}{\partial \hat{a}} = \frac{n \hat{k}}{\hat{a}} - \sum_{j=1}^{n} b_j = 0,
\]

\[
\frac{\partial \ell}{\partial \hat{c}_1} = \frac{n}{\hat{c}_1} - \sum_{j=1}^{n} \ln t_{1j} + \sum_{j=1}^{n} b_j t_{1j}^{\hat{c}_1} \ln t_{1j} = 0,
\]

and,

\[
\frac{\partial \ell}{\partial \hat{c}_2} = \frac{n}{\hat{c}_2} - \sum_{j=1}^{n} \ln t_{2j} + \sum_{j=1}^{n} b_j t_{2j}^{\hat{c}_2} \ln t_{2j} = 0,
\]

where, \(\psi(\hat{k}) = \Gamma(\hat{k})/\Gamma(\hat{k})\).

The invariance property of the ML estimators can be applied to obtain the ML estimators for \(R(t_1, t_2)\) and \(h(t_1, t_2)\) by replacing the parameters in (9) and (10) by their ML estimators

\[
\hat{R}(t_1, t_2) = \left[1 + \frac{1}{\hat{a}^{\hat{c}_1}} + \frac{1}{\hat{a}^{\hat{c}_2}}\right]^{-\hat{k}},
\]

and,

\[
\hat{h}(t_1, t_2) = \frac{\hat{k} + 1}{\hat{a}^{\hat{c}_1} \hat{c}_1 t_1^{\hat{c}_1 + 1} t_2^{\hat{c}_2 + 1}} \left[1 + \frac{1}{\hat{a}^{\hat{c}_1}} + \frac{1}{\hat{a}^{\hat{c}_2}}\right]^{-2}.
\]

\(\hat{R}(t_1, t_2)\) and \(\hat{h}(t_1, t_2)\) can be evaluated numerically.

### 3.2 Two-sample maximum likelihood prediction

Considering two-sample prediction, the first variable in the vector of the bivariate distribution is the ordered observation and the second variable is its concomitant. Therefore the joint pdf of the ordered observations and the concomitants is needed to obtain the joint predictive density function for the future ordered observations and their concomitants. For a future bivariate sample of size \(m\), the joint pdf of the future \(s\)-th ordered observation and its \(s\)-th concomitant denoted by \((y_{1(s;m)}, y_{2(s;m)})\), \(s = 1, 2, ..., m\), has the joint pdf which is given in (4) after replacing \(t_1\) by \(y_{1(s;m)}\) and \(t_2\) by \(y_{2(s;m)}\). For simplicity, \((y_{1(s)}, y_{2(s)})\) can be written instead of \((y_{1(s;m)}, y_{2(s;m)})\). Then the joint pdf of \((y_{1(s)}, y_{2(s)})\) can be derived as follows:
using the binomial expansion to simplify the last term in the previous equation, one gets

\[ [1 - F(y_{1(s)}, y_{2(s)})]^{m-s} = \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j [F(y_{1(s)}, y_{2(s)})]^j. \]

Thus, the joint pdf of \((y_{1(s)}, y_{2(s)})\) is as follows:

\[ f_{x,m}(y_{1(s)}, y_{2(s)}; \omega) = f(y_{1(s)}, y_{2(s)}; \omega) \sum_{j=0}^{m-s} C_{m,s,j} [F(y_{1(s)}, y_{2(s)})]^{s+j-1}, \quad (20) \]

where,

\[ C_{m,s,j} = \frac{m!}{(s-1)! (m-s-j)! (j)!} (-1)^j. \quad (21) \]

Substituting \(f(t_1, t_2)\) given in (4) and \(F(t_1, t_2)\) in (6) after replacing \(t_1\) by \(y_{1(s)}\) and \(t_2\) by \(y_{2(s)}\) then, the joint ML predictive density of ordered observations and their concomitants is given by

\[ f(y_{1(s)}, y_{2(s)}; \hat{\omega}_{ML}) = \frac{k(k+1)}{\hat{a}^2} \frac{\hat{c}_1 \hat{c}_2}{\hat{y}_{1(s)} \hat{y}_{2(s)}} \left[ \frac{1}{\hat{a}} y_{1(s)}^{-\hat{c}_1} + \frac{1}{\hat{a}} y_{2(s)}^{-\hat{c}_2} \right]^{-(k+2)} \times \sum_{j=0}^{m-s} C_{m,s,j} \left[ 1 + \frac{1}{\hat{a}} y_{1(s)}^{-\hat{c}_1} \right]^{-\hat{k}} \left[ 1 + \frac{1}{\hat{a}} y_{2(s)}^{-\hat{c}_2} \right]^{-\hat{k}} [F(y_{1(s)}, y_{2(s)})]^{s+j-1}, \quad \left( y_{1(s)}, y_{2(s)} > 0, \omega > 0 \right). \quad (22) \]

The point predictors of the future ordered observation and their concomitants \((Y_{1(s)}, Y_{2(s)})\),

\[ s = 1, 2, ..., m, \] can be obtained as given below

\[ \hat{Y}_1 = E(Y_{1(s)}; \hat{\omega}_{ML}) = \int_{y_{1(s)}=0}^{\infty} Y_{1(s)} \int_{y_{2(s)}=0}^{\infty} f(Y_{1(s)}, Y_{2(s)}; \hat{\omega}_{ML}) dY_{2(s)} dY_{1(s)}. \quad (23) \]

and,

\[ \hat{Y}_2 = E(Y_{2(s)}; \hat{\omega}_{ML}) = \int_{y_{2(s)}=0}^{\infty} Y_{2(s)} \int_{y_{1(s)}=0}^{\infty} f(Y_{1(s)}, Y_{2(s)}; \hat{\omega}_{ML}) dY_{1(s)} dY_{2(s)}. \quad (24) \]

From (23) and (24) it is clear that the point predictors \(\hat{Y}_1\) and \(\hat{Y}_2\) cannot be obtained in closed forms. Then, the joint point predictors of the future ordered observations are

\[ \hat{Y}_1, \hat{Y}_2 = E(Y_{1(s)}, Y_{2(s)}; \hat{\omega}_{ML}) = \int_{0}^{\infty} \int_{0}^{\infty} Y_{1(s)} Y_{2(s)} f(Y_{1(s)}, Y_{2(s)}; \hat{\omega}_{ML}) dY_{2(s)} dY_{1(s)}, \quad (25) \]

which can be evaluated numerically.
4 Numerical Illustration

This section aims to investigate the precision of the theoretical results of estimation and prediction on the basis of simulated data and example data set.

4.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates based on generated data from BBIII distribution. The ML averages of the parameters, rf and hrf are computed. Moreover, confidence intervals (CIs) of the parameters, rf and hrf are calculated. Simulation studies are performed using Mathematica 11 for illustrating the obtained results.

The steps of the simulation procedure are as follows:

a) For given values of $\omega = (a, k, c_1, c_2)$, random samples of size $n$ are generated from the BBIII distribution.

b) For each sample size the $t_{ij}$'s are sorted, such that $(t_{ij}, t_{i21}), (t_{i12}, t_{i22}), \ldots, (t_{i1n}, t_{i2n})$.

c) Repeat the previous two steps $N$ times where $N$ represents a fixed number of simulated samples.

d) The Newton-Raphson method is used to obtain the ML averages and the CIs for the parameters. Also, the rf, hrf and their CIs are calculated using the ML averages of the parameters.

e) Evaluating the performance of the estimates is considered through some measurements of accuracy. In order to study the precision and variation of the estimates, it is convenient to use.

the relative absolute bias (RAB) $= \frac{|\text{bias (estimate)}|}{\text{true value}}$, 

and the estimated risk (ER) $= \frac{\sum_{i=1}^{N} (\text{estimate} - \text{true value})^2}{N}$.

f) Simulation results of the ML estimates are displayed in Tables 1 and 2, where $N = 10000$ is the number of repetitions and samples of size $n=30, 50, 100$, where the population parameter values are $(k = 0.34, a = 0.48, c_1 = 1.03, c_2 = 0.67)$ and $(k = 1.25, a = 0.34, c_1 = 0.28, c_2 = 0.38)$.

g) Table 1 and 2 give the ML averages, RABs, ERs and CIs for the unknown parameters. While Tables 3 and 4 present ML averages, RABs, ERs and CIs of the rf and hrf for different values of time $t_{01}, t_{02}$. The two-sample ML predictors are presented in Table 8.

4.2 Example data set

This subsection aims to demonstrate how the proposed model can be used in practice. A lifetime data set of size $n=30$ is applied for this purpose. It was given by Sankaran and Kundu [15] as follows:

$$(0.252, 8.400), (1.105, 0.458), (0.427, 1.602), (12.491, 2.383), (0.260, 0.106), (0.240, 1.769), (4.888, 0.758), (0.870, 0.572), (0.036, 0.254), (1.537, 0.023), (1.508, 0.535), (0.239, 1.4120), (0.173, 0.011), (1.090, 1.278), (6.002, 0.017), (0.897, 2.032), (0.690, 0.138), (1.883, 0.398), (0.960, 0.257), (0.561, 0.573), (5.370, 0.325), (0.167, 0.260), (13.602, 0.364), (3.922, 0.938), (0.132, 0.547), (0.603, 0.102), (0.226, 0.481), (0.143, 0.779), (0.643, 0.071), (0.349, 1.586).$$
• The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values are given, respectively, by 0.2391, 0.0709. The p value showed that the model fits the data very well.

• Tables 5 displays the ML estimates and standard errors (SE) for the unknown parameters. While Table 6 presents the ML estimates and SEs of the $r_f$ and $h_f$ for the time $t_{01}, t_{02}$. Table 8 presents the ML two-sample predictors for the future sample of the data set.

4.3 Concluding remarks

1. It is noticed, from Tables 1 and 2 that the ML averages are very close to the population parameter values as the sample size increases. Also, RABs and ERs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the true parameter values as the sample size increases.

2. The lengths of the CIs of the parameters become narrower as the sample size increases.

3. The ML average for the $r_f$ and $h_f$ performs better as the sample size increases [see Tables 3 and 4].

4. The length of the CI of the first future order statistic is smaller than the length of the CI of the last future order statistic [Tables 7 and 8].

5. The ML interval includes the estimates [between the lower limit (LL) and upper limit (UL)].

5 Bayesian Estimation and Prediction

In this section, Bayesian estimation and prediction of the vector of the parameters $\omega = (a, k, c_t, c_z)$ for the BBIII distribution will be considered.

5.1 Bayesian estimation

Assuming that $(k, a)$ and $(c_t, c_z)$ are independent, then the prior density function of $\omega$, suggested by AL-Hussaini and Ateya [13], is given by

$$\pi(\omega) \propto \pi_1(k, a) \pi_2(c_t, c_z),$$

$$\pi_1(k, a) = \pi_{11}(k|a)\pi_{12}(a), \quad k|a \sim \text{Gamma}(r_1, a) \text{ and } a \sim \text{Gamma}(r_2, r_3),$$

and,

$$\pi_2(c_t, c_z) = \pi_{21}(c_t|c_z)\pi_{22}(c_z), \quad c_t|c_z \sim \text{Gamma}(r_4, c_z) \text{ and } c_z \sim \text{Gamma}(r_5, r_6),$$

hence,

$$\pi_1(k, a) \propto a^{r_1+r_2-1}k^{r_1-1}e^{-a(r_3+k)},$$

and,

$$\pi_2(c_t, c_z) \propto c_t^{r_4+r_5-1}c_z^{r_4-1}e^{-c_z(r_6+c_t)}.$$
Table 1. ML averages, relative absolute biases, estimated risks and 95% confidence intervals for the parameters (N = 10000, k = 0.34, α = 0.48, c₁ = 1.03, c₂ = 0.67)

| n | Parameters | Averages | RAB | ER | UL   | LL   | Length |
|---|------------|----------|-----|----|------|------|--------|
| 30| k          | 0.4053   | 0.1921 | 0.0415 | 0.7837 | 0.0269 | 0.7567 |
|   | a          | 0.7598   | 0.5829 | 0.6549 | 2.2481 | 0.0000 | 2.2481 |
|   | c₁         | 1.1088   | 0.0765 | 0.0562 | 1.5469 | 0.6705 | 0.8764 |
|   | c₂         | 0.7212   | 0.0764 | 0.0238 | 1.0006 | 0.4362 | 0.5701 |
|   | k          | 0.3818   | 0.1229 | 0.0293 | 0.7073 | 0.0563 | 0.6509 |
| 50| a          | 0.6583   | 0.3715 | 0.4446 | 1.9176 | 0.0000 | 1.9176 |
|   | c₁         | 1.0619   | 0.0309 | 0.0128 | 1.2745 | 0.8493 | 0.8495 |
|   | c₂         | 0.6907   | 0.0308 | 0.00541| 0.8291 | 0.5524 | 0.5524 |
|   | k          | 0.3502   | 0.0299 | 0.0042 | 0.4759 | 0.2244 | 0.2515 |
| 100| a         | 0.5358   | 0.1162 | 0.0599 | 1.0028 | 0.0688 | 0.9340 |
|    | c₁         | 1.0377   | 0.0075 | 0.0088 | 1.2216 | 0.8538 | 0.3679 |
|    | c₂         | 0.6750   | 0.0074 | 0.0038 | 0.7947 | 0.5554 | 0.2393 |

Table 2. ML averages, relative absolute biases, estimated risks and 95% confidence intervals for the parameters (N = 10000, k = 1.25, α = 0.34, c₁ = 0.28, c₂ = 0.38)

| n | Parameters | Averages | RAB | ER | UL   | LL   | Length |
|---|------------|----------|-----|----|------|------|--------|
| 30| k          | 1.8603   | 0.48827| 0.4908 | 2.5344 | 1.1863 | 1.3480 |
|   | a          | 0.5873   | 0.75334| 0.0819 | 0.8520 | 0.3227 | 0.5292 |
|   | c₁         | 0.2374   | 0.1521 | 0.0023 | 0.2817 | 0.1932 | 0.0885 |
|   | c₂         | 0.3179   | 0.1520 | 0.0042 | 0.3772 | 0.2587 | 0.1185 |
|   | k          | 1.6079   | 0.2864 | 0.1799 | 2.0543 | 1.1616 | 0.8927 |
| 50| a          | 0.5043   | 0.5054 | 0.0387 | 0.6996 | 0.3091 | 0.3906 |
|   | c₁         | 0.2618   | 0.0647 | 0.00059| 0.2938 | 0.2299 | 0.0639 |
|   | c₂         | 0.3507   | 0.0646 | 0.0011 | 0.3936 | 0.3079 | 0.0857 |
|   | k          | 1.7623   | 0.4098 | 0.2962 | 2.1223 | 1.4023 | 0.7200 |
| 100| a         | 0.5681   | 0.69588| 0.0592 | 0.70556| 0.4307 | 0.2749 |
|    | c₁         | 0.2552   | 0.0887 | 0.0007 | 0.2809 | 0.2294 | 0.0515 |
|    | c₂         | 0.3418   | 0.0886 | 0.0014 | 0.3763 | 0.3072 | 0.0691 |

Table 3. ML averages, relative absolute biases, estimated risks and 95% confidence intervals for the reliability and hazard rate functions (N = 10000, k = 0.34, α = 0.48, c₁ = 1.03, c₂ = 0.67, t₀₁ = 2, t₀₂ = 3)

| n | rf and hrf | Averages | RAB | ER | UL   | LL   | Length |
|---|------------|----------|-----|----|------|------|--------|
| 30| R(t₀₁, t₀₂) | 0.7346   | 0.0695 | 0.0002 | 0.8597 | 0.6098 | 0.2501 |
|   | h(t₀₁, t₀₂) | 0.0071   | 0.2097 | 0.0531 | 0.0202 | 0.0000 | 0.0202 |
| 50| R(t₀₁, t₀₂) | 0.7273   | 0.0588 | 0.0001 | 0.8440 | 0.6106 | 0.2334 |
|   | h(t₀₁, t₀₂) | 0.0059   | 0.0069 | 0.0521 | 0.0153 | 0.0000 | 0.0153 |
| 100| R(t₀₁, t₀₂) | 0.6634   | 0.0342 | 0.0001 | 0.7506 | 0.5761 | 0.1745 |
|    | h(t₀₁, t₀₂) | 0.0102   | 0.7432 | 0.0432 | 0.0164 | 0.0040 | 0.0124 |
Table 4. ML averages, relative absolute biases, estimated risks and 95% confidence intervals for the reliability and hazard rate functions 
(N = 10000, k = 1.25, a = 0.34, c1 = 0.28, c2 = 0.38, t01 = 2, t02 = 4)

| n   | rf and hrf | Averages | RAB    | ER     | UL     | LL     | Length |
|-----|------------|----------|--------|--------|--------|--------|--------|
| 30  | R(t_{01}, t_{02}) | 0.0832   | 0.3216 | 0.0002 | 0.1371 | 0.0293 | 0.1078 |
|     | h(t_{01}, t_{02}) | 0.0122   | 0.4986 | 0.0006 | 0.0200 | 0.0044 | 0.0156 |
| 50  | R(t_{01}, t_{02}) | 0.0838   | 0.3168 | 0.0002 | 0.1208 | 0.0467 | 0.0741 |
|     | h(t_{01}, t_{02}) | 0.0122   | 0.5013 | 0.0005 | 0.0188 | 0.0057 | 0.0131 |
| 100 | R(t_{01}, t_{02}) | 0.0821   | 0.2940 | 0.0001 | 0.1116 | 0.0526 | 0.0590 |
|     | h(t_{01}, t_{02}) | 0.0126   | 0.4501 | 0.0005 | 0.0183 | 0.0069 | 0.0113 |

Table 5. ML estimates and standard errors of the parameters for a data set

| Parameters | Estimates | SE  |
|------------|-----------|-----|
| k          | 0.5475    | 0.0431 |
| a          | 0.9112    | 0.1859 |
| c1         | 1.1534    | 0.0152 |
| c2         | 0.8986    | 0.05224 |

Table 6. ML estimates and standard errors of the reliability and hazard rate functions

| rf and hrf | Estimates | SE  |
|------------|-----------|-----|
| R(t_{01}, t_{02}) | 0.6886 | 0.29534 |
| h(t_{01}, t_{02}) | 0.0055 | 0.15597 |

Table 7. Averages of the ML predictor and bounds for the future observation under two-sample prediction (n = 30, k = 0.3, a = 0.5, c1 = 0.7, c2 = 2)

| s   | \(\hat{y}_1(s)\) | Averages | UL    | LL    | Length |
|-----|-----------------|----------|-------|-------|--------|
| 1   | 0.0008          | 0.0066   | 0.0000| 0.0066|        |
| 12  | 0.0429          | 0.1723   | 0.0001| 0.1722|        |
| 18  | 0.0068          | 0.0489   | 0.0000| 0.0489|        |

Table 8. Estimates of the ML predictor and bounds of the future observation for a data set under two-sample prediction

| s   | \(\hat{y}_1(s)\) | Estimates | UL    | LL    | Length |
|-----|-----------------|-----------|-------|-------|--------|
| 1   | 0.0751          | 0.3045    | 0.0251| 0.2794|        |
|     | 0.1418          | 0.4440    | 0.0695| 0.3745|        |
| 12  | 0.1645          | 0.6301    | 0.0829| 0.5373|        |
| 18  | 0.2491          | 0.7216    | 0.0516| 0.6700|        |
| 18  | 0.2980          | 0.9683    | 0.0000| 0.9683|        |
| 18  | 0.3884          | 0.9782    | 0.0000| 0.9782|        |
Substituting (27) and (28) in (26), and using the likelihood function in (16) the posterior density function is the product of two posteriors, which are given below

\[ \pi_1^*(k,a|t_3,t_2,b) \propto \frac{a^{nk+r_1+r_2-1}}{(1-k)^{n-k}} \left( \prod_{j=1}^{n} b_j \right)^{k+1} \exp \left[ -a \left( \sum_{j=1}^{n} b_j + r_3 + k \right) \right], \]  

(29)

and,

\[ \pi_2^*(c_1,c_2|t_3,t_2,b) \propto c_2^{r_4+n_1+n_2-1} c_1^{r_4+n_1-1} \left( \prod_{j=1}^{n} t_{1j} \right)^{-(c_1+1)} \left( \prod_{j=1}^{n} t_{2j} \right)^{-(c_2+1)} \times \exp \left[ -\left( c_2 r_6 + c_1 \right) + \sum_{j=1}^{n} \left( t_{1j}^{c_1} + t_{2j}^{c_2} \right) \right], \]  

(30)

from (29) and (30), the posterior density function is given by

\[ \pi^*(\omega|t_3,t_2,b) \propto \pi_1^*(k,a|t_1,t_2,b) \pi_2^*(c_1,c_2|t_3,t_2,b) \]  

(31)

The Bayes estimators of the \( R(t_1,t_2) \) and \( h(t_1,t_2) \) can be obtained using (9), (10) and (31), respectively, as follows:

\[ R_{SE}^*(t_1,t_2) = E(R(t_1,t_2)|\omega) = \int_{\omega} R(t_1,t_2) \pi^*(\omega|t_1,t_2) d\omega, \]  

(32)

and,

\[ h_{SE}^*(t_1,t_2) = E(h(t_1,t_2)|\omega) = \int_{\omega} h(t_1,t_2) \pi^*(\omega|t_1,t_2) d\omega. \]  

(33)

Equations (31)-(33) can be evaluated numerically to obtain the Bayes estimates of the parameters, \( r_f \) and \( h_{rf} \) based on squared error loss (SEL) function.

### 5.2 Two-sample Bayesian prediction

The Bayes predictive density of the ordered observations and their concomitants can be obtained by substituting (4) and (6) in (20) as

\[ \left( y_{1(s)}, y_{2(s)} | \omega \right) = \frac{k(k+1)}{a^{2(s)}} c_{12} y_{1(s)}^{c_1+1} y_{2(s)}^{c_2+1} \left[ 1 + \frac{1}{a} y_{1(s)}^{c_1} + \frac{1}{a} y_{2(s)}^{c_2} \right]^{-(k+2)} \times \sum_{j=0}^{m-2} \sum_{s=0}^{n-2} c_{m,s} \left[ 1 - \left( 1 + \frac{1}{a} y_{1(s)}^{c_1} + \frac{1}{a} y_{2(s)}^{c_2} \right)^{-k} + \left[ 1 + \frac{1}{a} y_{1(s)}^{c_1} + \frac{1}{a} y_{2(s)}^{c_2} \right]^{-k} \right]^{m+j-1}, \]

(34)

Hence, the joint Bayes predictive density of the ordered observations and their concomitants is given by
where,
\[
\int_{\omega} = \int_{k}^{a} \int_{c_1}^{c_2} \int_{e_1}^{e_2}, \text{ and } d\omega = dkdadc_2dc_1.
\]  
(36)

Substituting (31) and (34) in (35), yields the joint Bayesian predictive density of \((y_{1(s)}, y_{2(s)})\) as
\[
h(y_{1(s)}, y_{2(s)}|\omega) = \int_{\omega} I_1 I_2 I_3 I_4 I_5 d\omega,
\]  
(37)

where,
\[
I_1 = \frac{(k+1)k^k}{(\Gamma(k))^{k+1}} a^n a^n+r_3+n^r_4+n^r_5+n^r_6+n^r_7+n^r_8+n^r_9+n^r_10, \\
I_2 = e^{-a\sum_{j=1}^{n} b_j+\sum_{j=1}^{n} b_j}, \\
I_3 = \left[1 + \frac{1}{a} y_{1(s)}^{c_1} + \frac{1}{a} y_{2(s)}^{c_2}\right]^{(k+2)} \left[\prod_{j=1}^{n} b_j\right]^{k+1}, \\
I_4 = \left[\prod_{j=1}^{n} y_{1(s)} \prod_{j=1}^{n} y_{2(s)}\right] e^{-\sum_{j=1}^{n} b_j(y_{1(s)}^{c_1} + y_{2(s)}^{c_2})}, \\
I_5 = \sum_{j=0}^{m-s} C_{n,s,j} \left[1 - \left[1 + \frac{1}{a} y_{1(s)}^{c_1}\right]^{-k} - \left[1 + \frac{1}{a} y_{2(s)}^{c_2}\right]^{-k} + \left[1 + \frac{1}{a} y_{1(s)}^{c_1} + \frac{1}{a} y_{2(s)}^{c_2}\right]^{-k}\right]^{(s+j-1)}
\]  
(38)

The point predictors of the future ordered observation and their concomitants \((Y_{1(s)}, Y_{2(s)})\), \(s = 1, 2, ..., m\), under SEL function can be obtained as follows:
\[
Y_1^* = E(y_{1(s)}|\omega) = \int_{y_{1(s)=0}}^{\infty} y_{1(s)} h(y_{1(s)}, y_{2(s)}|\omega) dy_{2(s)} dy_{1(s)}, 
\]  
(39)

and,
\[
Y_2^* = E(y_{2(s)}|\omega) = \int_{y_{2(s)=0}}^{\infty} y_{2(s)} h(y_{1(s)}, y_{2(s)}|\omega) dy_{1(s)} dy_{2(s)},
\]  
(40)

From (39) and (40), the point predictors \(Y_1^*\) and \(Y_2^*\) cannot be obtained in closed forms. The joint Bayes point predictors of the future ordered observations are
\[
Y_1^*, Y_2^* = E(y_{1(s)}, y_{2(s)}|\omega) = \int_{0}^{\infty} \int_{0}^{\infty} y_{1(s)} y_{2(s)} h(y_{1(s)}, y_{2(s)}|\omega) dy_{1(s)} dy_{2(s)},
\]  
(41)
6 Numerical Illustration

This section aims to investigate the precision of the theoretical results of Bayesian estimation and prediction on the basis of simulated data and an example of data set.

6.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes estimates on the basis of generated data from the BBIII distribution. Bayes averages of the parameters, rf and hrf are computed. Moreover, credible intervals of the parameters, rf and hrf are calculated, two sample Bayes point predictors for a future observation from the BBIII distribution are computed. All simulation studies are performed using R programming language.

6.1.1 Simulation algorithm

1. Several data sets are generated from the BBIII distribution for a combination of the population parameter values of \( \omega = (a, k, c_1, c_2) \).

2. Also, for samples of size (30, 50 and 100) using \( N=10000 \) replications for each sample size.

3. The population parameter values of \( a, k, c_1 \) and \( c_2 \) used in this simulation study are \((k = 0.8, a = 0.5, c_1 = 0.9, c_2 = 0.3)\) and \((k = 1.6, a = 1.0, c_1 = 1.8, c_2 = 0.6)\).

4. If \( \omega^*_j \) is an estimate of \( \omega \); based on sample \( j, j = 1,2,\ldots,N \), then the average estimates over the samples are calculated by \( \bar{\omega}^*_j = \frac{1}{N} \sum_{j=1}^{N} \omega^*_j \).

5. The RABs and ERs of \( \omega^* \), over the \( N \) samples are computed.

- Tables 9 and 10 display the Bayes averages, RABs, ERs and credible intervals based on samples of size \( n \) and \( N=10000 \) repetitions with informative prior.

- Tables 11 and 12 present the Bayes averages, ER and credible intervals of rf and hrf for different values of the time \( t_{01}, t_{02} \) based on informative priors.

- The averages of the Bayes predictors using informative priors is presented in Table 15.

6.2 Example data set

The data set was given in Subsection 4.2 to illustrate the theoretical results.

- Tables 13 and 14 display the Bayes estimates of the parameters; rf and hrf, for the example data under informative prior, also standard errors are calculated.

- Table 16 presents two-sample Bayes predictors.
Table 9. Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the parameters of BBIII using informative prior (N = 10000, k = 0.8, α = 0.5, c_1 = 0.9, c_2 = 0.3)

| n   | Parameters | Averages | RAB    | ER     | UL   | LL   | Length |
|-----|------------|----------|--------|--------|------|------|--------|
| 30  | k          | 0.8025   | 0.0031 | 8.8298e-06 | 0.8049 | 0.7999 | 0.0050 |
|     | a          | 0.4989   | 0.0020 | 2.3773e-06 | 0.5006 | 0.4968 | 0.0038 |
|     | c_1        | 0.8978   | 0.0024 | 7.4181e-06 | 0.9001 | 0.8948 | 0.0053 |
|     | c_2        | 0.3026   | 0.0029 | 7.3948e-06 | 0.3053 | 0.2998 | 0.0037 |
|     | k          | 0.7989   | 0.0013 | 1.5478e-06 | 0.7998 | 0.7973 | 0.0025 |
| 50  | a          | 0.5008   | 0.0017 | 1.9581e-06 | 0.50279 | 0.4991 | 0.0037 |
|     | c_1        | 0.9016   | 0.0018 | 4.1999e-06 | 0.90319 | 0.8993 | 0.0038 |
|     | c_2        | 0.2989   | 0.0012 | 1.7172e-06 | 0.2999 | 0.2966 | 0.0033 |
|     | k          | 0.7997   | 0.0003 | 3.2371e-06 | 0.80044 | 0.7989 | 0.0018 |
| 100 | a          | 0.5002   | 0.0003 | 5.8005e-07 | 0.5016 | 0.4989 | 0.0003 |
|     | c_1        | 0.8999   | 0.0001 | 2.7199e-07 | 0.9007 | 0.8989 | 0.0018 |
|     | c_2        | 0.3007   | 0.0007 | 8.2478e-07 | 0.3017 | 0.2994 | 0.0024 |

Table 10. Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the parameters of BBIII using informative prior (N = 10000, k = 1.6, α = 1.0, c_1 = 1.8, c_2 = 0.6)

| n   | Parameters | Averages | RAB    | ER     | UL   | LL   | Length |
|-----|------------|----------|--------|--------|------|------|--------|
| 30  | k          | 1.5981   | 0.0011 | 5.6291e-06 | 1.6005 | 1.5956 | 0.0049 |
|     | a          | 0.9982   | 0.0018 | 4.7798e-06 | 1.0001 | 0.9964 | 0.0037 |
|     | c_1        | 1.7975   | 0.0014 | 7.8964e-06 | 1.7997 | 1.7956 | 0.0041 |
|     | c_2        | 0.6012   | 0.0007 | 2.0126e-06 | 0.6024 | 0.5996 | 0.0028 |
|     | k          | 1.5987   | 0.0008 | 2.5007e-06 | 1.6999 | 1.5969 | 0.0029 |
| 50  | a          | 0.9989   | 0.0011 | 3.5025e-06 | 1.0009 | 0.9963 | 0.0045 |
|     | c_1        | 1.8009   | 0.0005 | 1.7233e-06 | 1.8022 | 1.07996 | 0.0026 |
|     | c_2        | 0.5989   | 0.0002 | 1.5929e-06 | 0.6002 | 0.5975 | 0.0027 |
|     | k          | 1.5995   | 1.6006 | 7.2853e-07 | 1.6006 | 1.5979 | 0.0027 |
| 100 | a          | 1.0009   | 1.002  | 1.3011e-06 | 1.0017 | 0.9993 | 0.0024 |
|     | c_1        | 1.8002   | 1.8012 | 2.7273e-07 | 1.8013 | 1.7993 | 0.0020 |
|     | c_2        | 0.6009   | 0.0006 | 1.1998e-06 | 0.6018 | 0.5999 | 0.0018 |

Table 11. Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the reliability and hazard rate functions, using informative prior (N = 10000, k = 0.8, α = 0.5, c_1 = 0.9, c_2 = 0.3, t_{o1} = 2, t_{o2} = 4)

| n   | rf and hrf | Averages | RAB    | ER     | UL   | LL   | Length |
|-----|------------|----------|--------|--------|------|------|--------|
| 30  | R(t_{o1}, t_{o2}) | 0.3869   | 0.0050 | 5.6898e-06 | 0.3890 | 0.3847 | 0.0044 |
|     | h(t_{o1}, t_{o2}) | 0.0327   | 0.0435 | 2.4666e-06 | 0.0336 | 0.0313 | 0.0023 |
| 50  | R(t_{o1}, t_{o2}) | 0.3903   | 0.0035 | 2.1480e-06 | 0.3911 | 0.3887 | 0.0024 |
|     | h(t_{o1}, t_{o2}) | 0.0349   | 0.0217 | 8.9055e-07 | 0.0358 | 0.03367 | 0.0022 |
| 100 | R(t_{o1}, t_{o2}) | 0.3885   | 0.0010 | 3.2160e-07 | 0.3892 | 0.3876 | 0.0016 |
|     | h(t_{o1}, t_{o2}) | 0.0337   | 0.0139 | 3.2691e-07 | 0.0342 | 0.0328 | 0.0014 |
Table 12. Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the reliability and hazard rate functions, using informative prior $(N = 10000, k = 1.6, \alpha = 1.0, c_1 = 1.8, c_2 = 0.6, t_{01} = 2, t_{02} = 3)$

| n  | rf and hrf | Averages | RAB | ER     | UL     | LL     | Length |
|----|------------|----------|-----|--------|--------|--------|--------|
| 30 | $R(t_{01}, t_{02})$ | 0.3782   | 0.0047  | 6.8458e-06 | 0.3809 | 0.3754 | 0.0054 |
|    | $h(t_{01}, t_{02})$ | 0.0069   | 0.1688  | 1.5211e-06 | 0.0085 | 0.0058 | 0.0027 |
| 50 | $R(t_{01}, t_{02})$ | 0.3745   | 0.0050  | 4.6043e-06 | 0.3760 | 0.3726 | 0.0034 |
|    | $h(t_{01}, t_{02})$ | 0.0051   | 0.1390  | 9.4281e-07 | 0.0060 | 0.0042 | 0.0018 |
| 100| $R(t_{01}, t_{02})$ | 0.3890   | 0.0003  | 2.2692e-07 | 0.3881 | 0.3881 | 0.0019 |
|    | $h(t_{01}, t_{02})$ | 0.0335   | 0.0183  | 4.9983e-07 | 0.0327 | 0.0327 | 0.0015 |

Table 13. Bayes estimates and standard errors for the parameters using informative prior

| Parameters | Estimates | SE |
|------------|----------|----|
| $k$        | 0.7997   | 0.0007 |
| $\alpha$   | 0.5004   | 0.0004 |
| $c_1$      | 0.8979   | 0.0008 |
| $c_2$      | 0.3005   | 0.0005 |

Table 14. Bayes estimates and standard errors of the reliability and hazard rate functions

| rf and hrf | Estimate | SE |
|------------|----------|----|
| $R(t_{01}, t_{02})$ | 0.3914 | 0.0009 |
| $h(t_{01}, t_{02})$ | 0.0344 | 0.0007 |

Table 15. Averages of the Bayes predictors and bounds (informative prior), relative absolute biases, estimated risks for the future observation $(N = 10000, k = 0.8, \alpha = 0.5, c_1 = 0.9, c_2 = 0.3)$

| n  | s | $\hat{y}(s)$ | Averages | RAB | ER     | UL     | LL     | Length |
|----|---|-------------|----------|-----|--------|--------|--------|--------|
| 30 | 1 | $\hat{y}_{1(s)}$ | 3.9993   | 1.8723e-04 | 1.347e-06 | 4.0010 | 3.9978 | 0.0032 |
|    | 12| $\hat{y}_{2(s)}$  | 7.0001   | 2.0680e-05 | 3.6387e-07 | 7.0011 | 6.9987 | 0.0024 |
| 50 | 1 | $\hat{y}_{1(s)}$  | 4.0009   | 0.0002  | 2.2281e-06 | 4.0027 | 3.9989 | 0.0038 |
|    | 12| $\hat{y}_{2(s)}$  | 6.9993   | 0.0001  | 1.8146e-06 | 7.0009 | 6.9968 | 0.0041 |
| 18 | 1 | $\hat{y}_{1(s)}$  | 3.9978   | 0.0006  | 6.4075e-06 | 3.9980 | 3.9959 | 0.0038 |
|    | 12| $\hat{y}_{2(s)}$  | 7.0032   | 0.0005  | 1.2724e-05 | 7.0040 | 7.0000 | 0.0049 |
| 100| 1 | $\hat{y}_{1(s)}$  | 4.0004   | 9.1879e-05 | 7.6209e-07 | 4.0016 | 3.9986 | 0.0030 |
|    | 12| $\hat{y}_{2(s)}$  | 6.9991   | 1.1572e-04 | 8.4898e-07 | 6.9999 | 6.9978 | 0.0021 |
| 50 | 1 | $\hat{y}_{1(s)}$  | 3.9988   | 0.0003  | 2.5261e-06 | 4.0000 | 3.9969 | 0.0031 |
|    | 12| $\hat{y}_{2(s)}$  | 7.0004   | 5.0516e-05 | 4.1645e-07 | 7.0012 | 6.9990 | 0.0022 |
| 18 | 1 | $\hat{y}_{1(s)}$  | 3.9988   | 2.9660e-04 | 3.6059e-06 | 4.0007 | 3.9960 | 0.0047 |
|    | 12| $\hat{y}_{2(s)}$  | 7.0012   | 0.0002  | 2.7990e-06 | 7.0035 | 6.9998 | 0.0038 |
| 100| 1 | $\hat{y}_{1(s)}$  | 4.0004   | 8.8721e-05 | 2.9684e-07 | 4.0011 | 3.9996 | 0.0015 |
|    | 12| $\hat{y}_{2(s)}$  | 6.9998   | 3.3927e-05 | 3.3168e-07 | 7.0007 | 6.9987 | 0.0020 |
| 18 | 1 | $\hat{y}_{1(s)}$  | 3.9991   | 2.1631e-04 | 1.2559e-06 | 4.0003 | 3.9978 | 0.0024 |
|    | 12| $\hat{y}_{2(s)}$  | 6.9999   | 5.6636e-06 | 2.4937e-07 | 7.0009 | 6.9989 | 0.0021 |
| 18 | 1 | $\hat{y}_{1(s)}$  | 3.9977   | 6.7391e-05 | 1.5545e-06 | 4.0013 | 3.9975 | 0.0038 |
|    | 12| $\hat{y}_{2(s)}$  | 6.9993   | 9.5371e-05 | 7.8542e-07 | 7.0000 | 6.9972 | 0.0029 |
Table 16. Estimates of the Bayes predictors for the future observation and standard errors for a data set

| s  | $\hat{y}_1(s)$ | Estimates  | SE  |
|----|----------------|------------|-----|
| 1  | 3.9999         | 7.0001     | 0.0001 |
| 12 | 4.0005         | 7.0001     | 0.0006 |
| 18 | 4.0010         | 7.0037     | 0.0016 |

7 Concluding Remarks

- A simulation study is conducted to illustrate the performance of the Bayes estimates based on the basis of generated data from BBIII distribution. One example data set is used to demonstrate how the proposed method can be used in practice. All the calculations in this paper are performed using Mathematica 11 and R programming language.
- The Bayes averages for the parameters, rf and hrf from the BBIII distributions is obtained based on complete data, under the SEL function (as a symmetric loss function). Point estimation is considered for the parameters, rf and hrf.
- The ML and Bayes two-sample predictor of the future observations for the BBIII distribution is derived. The point prediction is considered. The Bayesian prediction is derived under SEL loss function. Numerical illustration is given and some interesting comparisons are presented.

Both the simulation study and the real data indicate that:

1. The variance of the estimates is inversely proportional to the sample size and the variance of an estimate tends to zero as the sample size tends to infinity.
2. The lengths of the CIs for the parameters become narrower as the sample size increases.
3. The Bayes averages for the rf and hrf performs better as sample size increases. Also ER is decreasing when the sample size is increasing.
4. It is interesting to notice that if the variables of the prior density are independent and if the likelihood function factors out with respect to these variables then the variables of the posterior given data are also independent.
5. Instead of obtaining the likelihood function by a direct use of the pdf in (4), the advantage of its construction by compounding $\prod_{i=1}^{k} f(t_i | \omega)$ and $g(b)$ is applied; which makes the ML or Bayesian estimation quite easier.

That if $\pi(\omega_1, ..., \omega_k) = \prod_{i=1}^{k} \pi(\omega_i)$ and if $L(\omega_1, ..., \omega_k | t) = \prod_{i=1}^{k} L(\omega_i | t)$, then $\pi(\omega_1, ..., \omega_k | t) \propto \pi(\omega_1, ..., \omega_k)L(\omega_1, ..., \omega_k | t) = \prod_{i=1}^{k} \pi(\omega_i) L(\omega_i | t) \propto \prod_{i=1}^{k} \pi(\omega_i | t) \Rightarrow (\omega_1 | t, ..., \omega_k | t)$ is independent, where the analysis will be easier.

6. All the results become better as the informative sample size gets larger.
7. One can observe that the point predictors of the first, medium and last (i.e, 20-th) ordered predicted observations and their concomitants in the independent future sample of size $n=20$ under different sample sizes ($n=30, 50, 100$). The Bayes point predictors were obtained based on SEL function.
8. In most cases the first predicted pair is smaller than the predicted pair of the second order; the second predicted pair is smaller than the predicted third order.
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Competing Interests
Authors have declared that no competing interests exist.

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