Nonextensive Pythagoras’ Theorem

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Abstract. Kullback-Leibler relative-entropy, in cases involving distributions resulting from relative-entropy minimization, has a celebrated property reminiscent of squared Euclidean distance: it satisfies an analogue of the Pythagoras’ theorem. And hence, this property is referred to as Pythagoras’ theorem of relative-entropy minimization or triangle equality and plays a fundamental role in geometrical approaches of statistical estimation theory like information geometry. Equivalent of Pythagoras’ theorem in the generalized nonextensive formalism is established in (Dukkipati at el., Physica A, 361 (2006) 124-138 [1]). In this paper we give a detailed account of it.
1. Introduction

Apart from being a fundamental measure of information, Kullback-Leibler relative-entropy or KL-entropy plays a role of ‘measure of the distance’ between two probability distributions in statistics. Since it is not a metric, at first glance, it might seem that the geometrical interpretations that metric distance measures provide usually might not be possible at all with the KL-entropy playing a role as a distance measure on a space of probability distributions. But it is a pleasant surprise that it is possible to formulate certain geometric propositions for probability distributions, with the relative-entropy playing the role of squared Euclidean distance. Some of these geometrical interpretations cannot be derived from the properties of KL-entropy alone, but from the properties of “KL-entropy minimization”; restating the previous statement, these geometrical formulations are possible only when probability distributions resulting from ME-prescriptions of KL-entropy are involved.

As demonstrated by Kullback [2], minimization problems of relative-entropy with respect to a set of moment constraints find their importance in the well known Kullback’s minimum entropy principle and thereby play a basic role in the information-theoretic approach to statistics [3, 4]. They frequently occur elsewhere also, e.g., in the theory of large deviations [5], and in statistical physics, as maximization of entropy [6, 7].

Kullback’s minimum entropy principle can be considered as a general method of inference about an unknown probability distribution when there exists a prior estimate of the distribution and new information in the form of constraints on expected values [8]. Formally, one can state this principle as: given a prior distribution \( r \), of all the probability distributions that satisfy the given moment constraints, one should choose the posterior \( p \) with the least relative-entropy. The prior distribution \( r \) can be a reference distribution (uniform, Gaussian, Lorentzian or Boltzmann etc.) or a prior estimate of \( p \). The principle of Jaynes maximum entropy is a special case of minimization of relative-entropy under appropriate conditions [9].

Many properties of relative-entropy minimization just reflect well-known properties of relative-entropy but there are surprising differences as well. For example, relative-entropy does not generally satisfy a triangle relation involving three arbitrary probability distributions. But in certain important cases involving distributions that result from relative-entropy minimization, relative-entropy results in a theorem comparable to the Pythagoras’ theorem cf. [10] and [11, § 11]. In this geometrical interpretation, relative-entropy plays the role of squared distance and minimization of relative-entropy appears as the analogue of projection on a sub-space in a Euclidean geometry. This property is also known as triangle equality [8].

The main aim of this paper is to study the possible generalization of Pythagoras’ theorem to the nonextensive case. Before we take up this problem, we present the properties of Tsallis relative-entropy minimization and present some differences with the classical case. In the representation of such a minimum entropy distribution, we highlight the use of the \( q \)-product (\( q \)-deformed version of multiplication), an operator
that has been introduced recently to derive the mathematical structure behind the Tsallis statistics. Especially, $q$-product representation of Tsallis minimum relative-entropy distribution will be useful for the derivation of the equivalent of triangle equality for Tsallis relative-entropy. We mention here that a general class of relative-entropy functionals which satisfy Pythagorean relation is established by Grünwald and Dawid [12]. Recently a Pythagoras’ theorem for a version of Rényi relative entropy is reported by Sundaresan [13].

Before we conclude this introduction on geometrical ideas of relative-entropy minimization, we make a note on the other geometric approaches. One approach is that of Rao [14], where one looks at the set of probability distributions on a sample space as a differential manifold and introduce a Riemannian geometry on this manifold. This approach is pioneered by Čencov [11] and Amari [15] who have shown the existence of a particular Riemannian geometry which is useful in understanding some questions of statistical inference. This Riemannian geometry turns out to have some interesting connections with information theory and as shown by Campbell [16], with the minimum relative-entropy. In this approach too, the above mentioned Pythagoras’ Theorem plays an important role [17, pp.72].

The other idea involves the use of Hausdorff dimension [18, 19] to understand why minimizing relative-entropy should provide useful results. This approach was begun by Eggleston [20] for a special case of maximum entropy and was developed by Campbell [21]. For an excellent review on various geometrical aspects associated with minimum relative-entropy one can refer to [22].

The structure of this paper is organized as follows. We present the necessary background in §2, where we discuss properties of relative-entropy minimization in the classical case. In §3 we present the ME prescriptions of Tsallis relative-entropy and discuss its differences with the classical case. Finally, the derivation of Pythagoras’ theorem in the nonextensive case is presented in §4.

Regarding the notation, we define all the information measures on the measurable space $(X, \mathcal{M})$. The default reference measure is $\mu$ unless otherwise stated. For simplicity in exposition, we will not distinguish between functions differing on a $\mu$-null set only; nevertheless, we can work with equations between $\mathcal{M}$-measurable functions on $X$ if they are stated as being valid only $\mu$-almost everywhere ($\mu$-a.e or a.e). Further we assume that all the quantities of interest exist and also assume, implicitly, the $\sigma$-finiteness of $\mu$ and $\mu$-continuity of probability measures whenever required. Since these assumptions repeatedly occur in various definitions and formulations, these will not be mentioned in the sequel. With these assumptions we do not distinguish between an information measure of pdf $p$ and that of the corresponding probability measure $P$ – hence when we give definitions of information measures for pdfs, we also use the corresponding definitions of probability measures as well, wherever convenient or required – with the understanding that $P(E) = \int_E p\,d\mu$, and the converse holding as a result of the Radon-Nikodym theorem, with $p = \frac{dP}{d\mu}$. In both the cases we have $P \ll \mu$.

Note that though results presented in this paper do not involve major measure
theoretic concepts, we write all the integrals with respect to the measure $\mu$, as a convention; these integrals can be replaced by summations in the discrete case or Lebesgue integrals in the continuous case.

2. Relative-Entropy Minimization in the Classical Case

Kullback’s minimum entropy principle can be stated formally as follows. Given a prior distribution $r$ with a finite set of moment constraints of the form

$$\int_X u_m(x)p(x)\,d\mu(x) = \langle u_m \rangle, \ m = 1, \ldots, M, \tag{1}$$

one should choose the posterior $p$ which minimizes the relative-entropy

$$I(p\|r) = \int_X p(x) \ln \frac{p(x)}{r(x)} \,d\mu(x). \tag{2}$$

In (1), $\langle u_m \rangle$, $m = 1, \ldots, M$ are the known expectation values of $\mathfrak{M}$-measurable functions $u_m : X \to \mathbb{R}$, $m = 1, \ldots, M$ respectively.

With reference to (2) we clarify here that, though we mainly use expressions of relative-entropy defined for pdfs in this paper, we use expressions in terms of corresponding probability measures as well. For example, when we write the Lagrangian for relative-entropy minimization below, we use the definition of relative-entropy

$$I(P\|R) = \begin{cases} \int_X \ln \frac{dP}{dR} \,dP & \text{if } P \ll R, \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

for probability measures $P$ and $R$, corresponding to pdfs $p$ and $r$ respectively. This correspondence between probability measures $P$ and $R$ with pdfs $p$ and $r$, respectively, will not be described again in the sequel.

2.1. Canonical Minimum Entropy Distribution

To minimize the relative-entropy (2) with respect to the constraints (1), the Lagrangian turns out to be

$$\mathcal{L}(x, \lambda, \beta) = \int_X \ln \frac{dP}{dR}(x) \,dP(x) + \lambda \left( \int_X dP(x) - 1 \right) + \sum_{m=1}^M \beta_m \left( \int_X u_m(x) \,dP(x) - \langle u_m \rangle \right), \tag{4}$$

where $\lambda$ and $\beta_m$, $m = 1, \ldots, M$ are Lagrange multipliers. The solution is given by

$$\ln \frac{dP}{dR}(x) + \lambda + \sum_{m=1}^M \beta_m u_m(x) = 0,$$

and the solution can be written in the form of

$$\frac{dP}{dR}(x) = \frac{e^{-\sum_{m=1}^M \beta_m u_m(x)}}{\int_X e^{-\sum_{m=1}^M \beta_m u_m(x)} \,dR}. \tag{5}$$
Finally, from (5) the posterior distribution \( p(x) = \frac{dp}{d\mu} \) given by Kullback’s minimum entropy principle can be written in terms of the prior \( r(x) = \frac{dR}{d\mu} \) as

\[
p(x) = \frac{r(x)e^{-\sum_{m=1}^{M} \beta_m u_m(x)}}{\hat{Z}},
\]

where

\[
\hat{Z} = \int_X r(x)e^{-\sum_{m=1}^{M} \beta_m u_m(x)} d\mu(x)
\]

is the partition function.

Relative-entropy minimization has been applied to many problems in statistics [2] and statistical mechanics [23]. The other applications include pattern recognition [24], spectral analysis [25], speech coding [26], estimation of prior distribution for Bayesian inference [27] etc. For a list of references on applications of relative-entropy minimization see [9] and a recent paper [28].

Properties of relative-entropy minimization have been studied extensively and presented by Shore [8]. Here we briefly mention a few.

The principle of maximum entropy is equivalent to relative-entropy minimization in the special case of discrete spaces and uniform priors, in the sense that, when the prior is a uniform distribution with finite support \( W \) (over \( E \subset X \)), the minimum entropy distribution turns out to be

\[
p(x) = \frac{e^{-\sum_{m=1}^{M} \beta_m u_m(x)}}{\int_E e^{-\sum_{m=1}^{M} \beta_m u_m(x)} d\mu(x)},
\]

which is in fact, a maximum entropy distribution of Shannon entropy with respect to the constraints (1).

The important relations to relative-entropy minimization are as follows. Minimum relative-entropy, \( I \), can be calculated as

\[
I = -\ln \hat{Z} - \sum_{m=1}^{M} \beta_m \langle u_m \rangle,
\]

while the thermodynamic equations are

\[
\frac{\partial}{\partial \beta_m} \ln \hat{Z} = -\langle u_m \rangle, \quad m = 1, \ldots M,
\]

and

\[
\frac{\partial I}{\partial \langle u_m \rangle} = -\beta_m, \quad m = 1, \ldots M.
\]

2.2. Pythagoras’ Theorem

The statement of Pythagoras’ theorem of relative-entropy minimization can be formulated as follows [10].
**Theorem 2.1.** Let $r$ be the prior, $p$ be the probability distribution that minimizes the relative-entropy subject to a set of constraints

$$
\int_X u_m(x)p(x)\,d\mu(x) = \langle u_m \rangle, \quad m = 1, \ldots, M,
$$

with respect to $\mathcal{M}$-measurable functions $u_m : X \to \mathbb{R}$, $m = 1, \ldots M$ whose expectation values $\langle u_m \rangle$, $m = 1, \ldots M$ are (assumed to be) a priori known. Let $l$ be any other distribution satisfying the same constraints (12), then we have the triangle inequality

$$
I(l\|r) = I(l\|p) + I(p\|r).
$$

**Proof.** We have

$$
I(l\|r) = \int_X l(x) \ln \frac{l(x)}{r(x)} \,d\mu(x)
= \int_X l(x) \ln \frac{l(x)}{p(x)} \,d\mu(x) + \int_X l(x) \ln \frac{p(x)}{r(x)} \,d\mu(x)
= I(l\|p) + \int_X l(x) \ln \frac{p(x)}{r(x)} \,d\mu(x)
$$

From the minimum entropy distribution (6) we have

$$
\ln \frac{p(x)}{r(x)} = - \sum_{m=1}^M \beta_m u_m(x) - \ln \hat{Z}.
$$

By substituting (15) in (14) we get

$$
I(l\|r) = I(l\|p) + \int_X l(x) \left\{ - \sum_{m=1}^M \beta_m u_m(x) - \ln \hat{Z} \right\} \,d\mu(x)
= I(l\|p) - \sum_{m=1}^M \beta_m \left\{ \int_X l(x) u_m(x) \,d\mu(x) \right\} - \ln \hat{Z}
= I(l\|p) - \sum_{m=1}^M \beta_m \langle u_m \rangle - \ln \hat{Z} \quad \text{(By hypothesis)}
= I(l\|p) + I(p\|r). \quad \text{(By (12))}
$$

A simple consequence of the above theorem is that

$$
I(l\|r) \geq I(p\|r)
$$

since $I(l\|p) \geq 0$ for every pair of pdfs, with equality if and only if $l = p$.

Detailed discussions on the importance of Pythagoras’ theorem of relative-entropy minimization can be found in [8] and [17, pp. 72]. For a study of relative-entropy minimization without the use of Lagrange multiplier technique and corresponding geometrical aspects, one can refer to [10].

Pythagorean relation of relative-entropy minimization not only plays a fundamental role in geometrical approaches of statistical estimation theory [11] and information
geometry [15, 29] but is also important for applications in which relative-entropy minimization is used for purposes of pattern classification and cluster analysis [24].

3. Tsallis Relative-Entropy Minimization

Unlike the generalized entropy measures, ME of generalized relative-entropies is not much addressed in the literature. Here, one has to mention the work in [30], where the minimum relative-entropy distribution of Tsallis relative-entropy with respect to the constraints in terms of $q$-expectation values is given.

In this section, we study several aspects of Tsallis relative-entropy minimization. First we derive the minimum entropy distribution in the case of $q$-expectation values (see (18)) and then in the case of normalized $q$-expectation values (see (35)). We propose an elegant representation of these distributions by using $q$-deformed binary operator called $q$-product $\otimes_q$. This operator is defined in [31] along similar lines as $q$-addition $\oplus_q$ and $q$-subtraction $\ominus_q$. Since $q$-product plays an important role in nonextensive formalism, we include a detailed discussion on the $q$-product in this section. Finally, we study properties of Tsallis relative-entropy minimization and its differences with the classical case.

3.1. Generalized Minimum Relative-Entropy Distribution

To minimize Tsallis relative-entropy

$$I_q(p\|r) = -\int_X p(x) \ln_q \frac{r(x)}{p(x)} \, d\mu(x)$$

(17)

with respect to the set of constraints specified in terms of $q$-expectation values

$$\int_X u_m(x)p(x)^q \, d\mu(x) = \langle u_m \rangle_q \quad m = 1, \ldots, M,$$

(18)

the concomitant variational principle is given as follows: Define

$$\mathcal{L}(x, \lambda, \beta) = \int_X \ln_q \frac{r(x)}{p(x)} \, dP(x) - \lambda \left( \int_X dP(x) - 1 \right) - \sum_{m=1}^M \beta_m \left( \int_X p(x)^{q-1} u_m(x) \, dP(x) - \langle u_m \rangle_q \right)$$

(19)

where $\lambda$ and $\beta_m$, $m = 1, \ldots, M$ are Lagrange multipliers. Now set

$$\frac{d\mathcal{L}}{dP} = 0.$$  

(20)

The solution is given by

$$\ln_q \frac{r(x)}{p(x)} - \lambda - p(x)^{q-1} \sum_{m=1}^M \beta_m u_m(x) = 0.$$  

(21)
which can be rearranged by using the definition of \( q \)-logarithm \( \ln_q x = \frac{x^{1-q} - 1}{1-q} \) as
\[
p(x) = \frac{r(x)^{1-q} - (1-q) \sum_{m=1}^{M} \beta_m u_m(x)}{(\lambda(1-q) + 1)^{1/q}}.
\]
Specifying the Lagrange parameter \( \lambda \) via the normalization \( \int_X p(x) \, d\mu(x) = 1 \), one can write Tsallis minimum relative-entropy distribution as \[30\]
\[
p(x) = \frac{r(x)^{1-q} - (1-q) \sum_{m=1}^{M} \beta_m u_m(x)}{\hat{Z}_q}.
\]
where the partition function is given by
\[
\hat{Z}_q = \int_X \left[ r(x)^{1-q} - (1-q) \sum_{m=1}^{M} \beta_m u_m(x) \right]^{1/q} \, d\mu(x).
\]
The values of the Lagrange parameters \( \beta_m, m = 1, \ldots, M \) are determined using the constraints \[18\].

3.2. \( q \)-Product Representation for Tsallis Minimum Entropy Distribution

Note that the generalized relative-entropy distribution \[21\] is not of the form of its classical counterpart \[6\] even if we replace the exponential with the \( q \)-exponential. But one can express \[21\] in a form similar to the classical case by invoking \( q \)-deformed binary operation called \( q \)-product.

In the framework of \( q \)-deformed functions and operators a new multiplication, called \( q \)-product defined as
\[
x \otimes_q y \equiv \begin{cases} 
(x^{1-q} + y^{1-q} - 1)^{1/q} & \text{if } x, y > 0, \\
0 & \text{if } x^{1-q} + y^{1-q} - 1 > 0
\end{cases}
\]
This is first introduced in \[32\] and explicitly defined in \[31\] for satisfying the following equations:
\[
\ln_q(x \otimes_q y) = \ln_q x + \ln_q y, \quad (24)
\]
\[
e_{q}^{x \otimes_q y} = e_{q}^{x+y}. \quad (25)
\]
The \( q \)-product recovers the usual product in the limit \( q \to 1 \) i.e., \( \lim_{q \to 1}(x \otimes_q y) = xy \). The fundamental properties of the \( q \)-product \( \otimes_q \) are almost the same as the usual product, and the distributive law does not hold in general, i.e.,
\[
a(x \otimes_q y) \neq ax \otimes_q y \quad (a, x, y \in \mathbb{R}).
\]
Further properties of the \( q \)-product can be found in \[32, 31\].
One can check the mathematical validity of the $q$-product by recalling the expression of the exponential function $e^x$

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$  

Replacing the power on the right side of (26) by $n$ times the $q$-product $\otimes_q$:

$$x^{\otimes_q^n} = x \otimes_q \ldots \otimes_q x,$$

one can verify that

$$e^x_q = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{\otimes_q^n}.$$  

Further mathematical significance of $q$-product is demonstrated in [34] by discovering the mathematical structure of statistics based on the Tsallis formalism: law of error, $q$-Stirling’s formula, $q$-multinomial coefficient and experimental evidence of $q$-central limit theorem.

Now, one can verify the non-trivial fact that Tsallis minimum entropy distribution (21) can be expressed as [35],

$$p(x) = \frac{r(x) \otimes_q e_q^{\sum_{m=1}^M \beta_m u_m(x)}}{\hat{Z}_q},$$

where

$$\hat{Z}_q = \int_X r(x) \otimes_q e_q^{-\sum_{m=1}^M \beta_m u_m(x)} d\mu(x).$$

Later in this paper we see that this representation is useful in establishing properties of Tsallis relative-entropy minimization and corresponding thermodynamic equations.

It is important to note that the distribution in (21) could be a (local/global) minimum only if $q > 0$ and the Tsallis cut-off condition specified by Tsallis maximum entropy distribution is extended to the relative-entropy case i.e., $p(x) = 0$ whenever

$$\left[r(x)^{1-q} - (1-q) \sum_{m=1}^M \beta_m u_m(x)\right] < 0.$$  

The latter condition is also required for the $q$-product representation of the generalized minimum entropy distribution.

In this case, one can calculate minimum relative-entropy $I_q$ as

$$I_q = -\ln_q \hat{Z}_q - \sum_{m=1}^M \beta_m \langle u_m \rangle_q.$$  

To demonstrate the usefulness of $q$-product representation of generalized minimum entropy distribution we present the verification (31). By using the property of $q$-multiplication (25), Tsallis minimum relative-entropy distribution (29) can be written as

$$p(x) \hat{Z}_q = e_q^{-\sum_{m=1}^M \beta_m u_m(x) + \ln_q r(x)}.$$
By taking $q$-logarithm on both sides, we get
\[
\ln_q p(x) + \ln_q \widehat{Z}_q + (1-q) \ln_q p(x) \ln_q \widehat{Z}_q = - \sum_{m=1}^M \beta_m u_m(x) + \ln_q r(x)
\]
By the property of $q$-logarithm \( \ln_q \left( \frac{x}{y} \right) = y^{q-1}(\ln_q x - \ln_q y) \), we have
\[
\ln_q \left( \frac{r(x)}{p(x)} \right) = p(x)^{q-1} \left\{ \ln_q \widehat{Z}_q + (1-q) \ln_q p(x) \ln_q \widehat{Z}_q + \sum_{m=1}^M \beta_m u_m(x) \right\}.
\] (32)

By substituting (32) in Tsallis relative-entropy (17) we get
\[
I_q = - \int_X p(x)^q \left\{ \ln_q \widehat{Z}_q + (1-q) \ln_q p(x) \ln_q \widehat{Z}_q + \sum_{m=1}^M \beta_m u_m(x) \right\} \, d\mu(x).
\]
By (18) and expanding $\ln_q p(x)$ one can write $I_q$ in its final form as in (31).

It is easy to verify the following thermodynamic equations for the minimum Tsallis relative-entropy:
\[
\frac{\partial}{\partial \beta_m} \ln_q \widehat{Z}_q = -\langle u_m \rangle_q, \quad m = 1, \ldots, M,
\] (33)
\[
\frac{\partial I_q}{\partial \langle u_m \rangle_q} = -\beta_m, \quad m = 1, \ldots, M,
\] (34)
which generalize thermodynamic equations in the classical case.

3.3. The Case of Normalized $q$-Expectations

In this section we discuss Tsallis relative-entropy minimization with respect to the constraints in the form of normalized $q$-expectations
\[
\frac{\int_X u_m(x)p(x)^q \, d\mu(x)}{\int_X p(x)^q \, d\mu(x)} = \langle \langle u_m \rangle_q \rangle, \quad m = 1, \ldots, M.
\] (35)

The variational principle for Tsallis relative-entropy minimization in this case is as below. Let
\[
\mathcal{L}(x, \lambda, \beta) = \int_X \ln_q \left( \frac{r(x)}{p(x)} \right) \, dP(x) - \lambda \left( \int_X \, dP(x) - 1 \right)
- \sum_{m=1}^M \beta_m^{(q)} \left\{ \int_X p(x)^{q-1} \left( u_m(x) - \langle \langle u_m \rangle_q \rangle \right) \, dP(x) \right\},
\] (36)
where the parameters $\beta_m^{(q)}$ can be defined in terms of the true Lagrange parameters $\beta_m$ as
\[
\beta_m^{(q)} = \frac{\beta_m}{\int_X p(x)^q \, d\mu(x)}, \quad m = 1, \ldots, M.
\] (37)
This gives minimum entropy distribution as

\[ p(x) = \frac{1}{\hat{Z}_q} \left[ r(x)^{1-q} - (1 - q) \frac{\sum_{m=1}^{M} \beta_m \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right)}{\int_X p(x)^q \, d\mu(x)} \right]^{\frac{1}{1-q}} \]  

(38)

where

\[ \hat{Z}_q = \int_X \left[ r(x)^{1-q} - (1 - q) \frac{\sum_{m=1}^{M} \beta_m \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right)}{\int_X p(x)^q \, d\mu(x)} \right]^{\frac{1}{1-q}} \, d\mu(x). \]

Now, the minimum entropy distribution (38) can be expressed using the \( q \)-product (23) as

\[ p(x) = \frac{1}{\hat{Z}_q} \left\{ r(x)^{\otimes_q} \exp_q \left( \frac{\sum_{m=1}^{M} \beta_m \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right)}{\int_X p(x)^q \, d\mu(x)} \right) \right\}. \]

(39)

Minimum Tsallis relative-entropy \( I_q \) in this case satisfies

\[ I_q = -\ln_q \hat{Z}_q, \]

(40)

while one can derive the following thermodynamic equations:

\[ \frac{\partial}{\partial \beta_m} \ln_q \hat{Z}_q = -\langle \langle u_m \rangle \rangle_q, \quad m = 1, \ldots, M, \]

(41)

\[ \frac{\partial I_q}{\partial \langle \langle u_m \rangle \rangle_q} = -\beta_m, \quad m = 1, \ldots, M, \]

(42)

where

\[ \ln_q \hat{Z}_q = \ln_q \hat{Z}_q - \sum_{m=1}^{M} \beta_m \langle \langle u_m \rangle \rangle_q. \]

(43)

4. Nonextensive Pythagoras’ Theorem

With the above study of Tsallis relative-entropy minimization, in this section, we present our main result, Pythagoras’ theorem or triangle equality (Theorem 2.1) generalized to the nonextensive case. To present this result, we shall discuss the significance of triangle equality in the classical case. We restate Theorem 2.1 which is essential for the derivation of the triangle equality in the nonextensive framework.
4.1. Pythagoras’ Theorem Restated

Significance of the triangle equality is evident in the following scenario. Let $r$ be the prior estimate of the unknown probability distribution $l$, about which, the information in the form of constraints

$$\int_X u_m(x)l(x)\,d\mu(x) = \langle u_m \rangle, \quad m = 1, \ldots, M$$

is available with respect to the fixed functions $u_m$, $m = 1, \ldots, M$. The problem is to choose a posterior estimate $p$ that is in some sense the best estimate of $l$ given by the available information i.e., prior $r$ and the information in the form of expected values (44). Kullback’s minimum entropy principle provides a general solution to this inference problem and provides us the estimate (45) when we minimize relative-entropy $I(p\|r)$ with respect to the constraints

$$\int_X u_m(x)p(x)\,d\mu(x) = \langle u_m \rangle, \quad m = 1, \ldots, M.$$  

This estimate of posterior $p$ by Kullback’s minimum entropy principle also offers the relation (Theorem 2.1)

$$I(l\|r) = I(l\|p) + I(p\|r),$$

from which one can draw the following conclusions. By (46), the minimum relative-entropy posterior estimate of $l$ is not only logically consistent, but also closer to $l$, in the relative-entropy sense, that is the prior $r$. Moreover, the difference $I(l\|r) - I(l\|p)$ is exactly the relative-entropy $I(p\|r)$ between the posterior and the prior. Hence, $I(p\|r)$ can be interpreted as the amount of information provided by the constraints that is not inherent in $r$.

Additional justification to use minimum relative-entropy estimate of $p$ with respect to the constraints (45) is provided by the following expected value matching property [8]. To explain this concept we restate our above estimation problem as follows.

For fixed functions $u_m$, $m = 1, \ldots, M$, let the actual unknown distribution $l$ satisfy

$$\int_X u_m(x)l(x)\,d\mu(x) = \langle w_m \rangle, \quad m = 1, \ldots, M,$$

where $\langle w_m \rangle$, $m = 1, \ldots, M$ are expected values of $l$, the only information available about $l$ apart from the prior $r$. To apply minimum entropy principle to estimate posterior estimation $p$ of $l$, one has to determine the constraints for $p$ with respect to which we minimize $I(p\|r)$. Equivalently, by assuming that $p$ satisfies the constraints of the form (45), one has to determine the expected values $\langle u_m \rangle$, $m = 1, \ldots, M$.

Now, as $\langle u_m \rangle$, $m = 1, \ldots, M$ vary, one can show that $I_q(l\|p)$ has the minimum value when

$$\langle u_m \rangle = \langle w_m \rangle, \quad m = 1, \ldots, M.$$
The proof is as follows [8]. Proceeding as in the proof of Theorem 2.1, we have

\[
I(l\|p) = I(l\|r) + \sum_{m=1}^{M} \beta_m \left\{ \int_{X} l(x) u_m(x) \, d\mu(x) \right\} + \ln \hat{Z}
\]

\[
= I(l\|r) + \sum_{m=1}^{M} \beta_m \langle w_m \rangle + \ln \hat{Z} \quad \text{(By (17))} \tag{49}
\]

Since the variation of \(I(l\|p)\) with respect to \(\langle u_m \rangle\) results in the variation of \(I(l\|p)\) with respect to \(\beta_m\) for any \(m = 1, \ldots, M\), to find the minimum of \(I(l\|p)\) one can solve

\[
\frac{\partial}{\partial \beta_m} I_q(l\|p) = 0, \quad m = 1, \ldots M,
\]

which gives the solution as in (48).

This property of expectation matching states that, for a distribution \(p\) of the form (6), \(I(l\|p)\) is the smallest when the expected values of \(p\) match those of \(l\). In particular, \(p\) is not only the distribution that minimizes \(I(p\|r)\) but also minimizes \(I(l\|p)\).

We now restate the Theorem 2.1 which summarizes the above discussion.

**Theorem 4.1.** Let \(r\) be the prior distribution, and \(p\) be the probability distribution that minimizes the relative-entropy subject to a set of constraints

\[
\int_{X} u_m(x) p(x) \, d\mu(x) = \langle u_m \rangle, \quad m = 1, \ldots, M. \tag{50}
\]

Let \(l\) be any other distribution satisfying the constraints

\[
\int_{X} u_m(x) l(x) \, d\mu(x) = \langle w_m \rangle, \quad m = 1, \ldots, M. \tag{51}
\]

Then

(i) \(I_1(l\|p)\) is minimum only if (expectation matching property)

\[
\langle u_m \rangle = \langle w_m \rangle, \quad m = 1, \ldots M. \tag{52}
\]

(ii) When (52) holds, we have

\[
I(l\|r) = I(l\|p) + I(p\|r) \tag{53}
\]

By the above interpretation of triangle equality and analogy with the comparable situation in Euclidean geometry, it is natural to call \(p\), as defined by (6) as the projection of \(r\) on the plane described by (51). Csiszár [10] has introduced a generalization of this notion to define the projection of \(r\) on any convex set \(E\) of probability distributions. If \(p \in E\) satisfies the equation

\[
I(p\|r) = \min_{s \in E} I(s\|r),
\]

then \(p\) is called the projection of \(r\) on \(E\). Csiszár [10] develops a number of results about these projections for both finite and infinite dimensional spaces. In this paper, we will not consider this general approach.
4.2. The Case of q-Expectations

From the above discussion, it is clear that to derive the triangle equality of Tsallis relative-entropy minimization, one should first deduce the equivalent of expectation matching property in the nonextensive case.

We state below and prove the Pythagoras theorem in nonextensive framework established by Dukkipati et al. [1].

**Theorem 4.2.** Let \( r \) be the prior distribution, and \( p \) be the probability distribution that minimizes the Tsallis relative-entropy subject to a set of constraints

\[
\int_X u_m(x)p(x)^q \, d\mu(x) = \langle u_m \rangle_q, \quad m = 1, \ldots, M. \tag{55}
\]

Let \( l \) be any other distribution satisfying the constraints

\[
\int_X u_m(x)l(x)^q \, d\mu(x) = \langle w_m \rangle_q, \quad m = 1, \ldots, M. \tag{56}
\]

Then

(i) \( I_q(l\|p) \) is minimum only if

\[
\langle u_m \rangle_q = \frac{\langle w_m \rangle_q}{1 - (1 - q)I_q(l\|p)}, \quad m = 1, \ldots, M. \tag{57}
\]

(ii) Under \( \text{(57)} \), we have

\[
I_q(l\|r) = I_q(l\|p) + I_q(p\|r) + (q - 1)I_q(l\|p)I_q(p\|r). \tag{58}
\]

**Proof.** First we deduce the equivalent of expectation matching property in the nonextensive case. That is, we would like to find the values of \( \langle u_m \rangle_q \) for which \( I_q(l\|p) \) is minimum. We write the following useful relations before we proceed to the derivation.

We can write the generalized minimum entropy distribution \( \text{(29)} \) as

\[
p(x) = \frac{e^{\ln_q r(x)} \otimes_q e_q^{-\sum_{m=1}^M \beta_m u_m(x)}}{\widehat{Z}_q} = \frac{e_q^{-\sum_{m=1}^M \beta_m u_m(x) + \ln_q r(x)}}{\widehat{Z}_q}, \tag{59}
\]

by using the relations \( e_q^{\ln_q x} = x \) and \( e_q^x \otimes_q e_q^y = e_q^{x+y} \). Further by using

\[
\ln_q(xy) = \ln_q x + \ln_q y + (1 - q) \ln_q x \ln_q y
\]

we can write \( \text{(59)} \) as

\[
\ln_q p(x) + \ln_q \widehat{Z}_q + (1 - q) \ln_q p(x) \ln_q \widehat{Z}_q = -\sum_{m=1}^M \beta_m u_m(x) + \ln_q r(x). \tag{60}
\]

By the property of \( q \)-logarithm

\[
\ln_q \left( \frac{x}{y} \right) = y^{q-1}(\ln_q x - \ln_q y), \tag{61}
\]

and by \( q \)-logarithmic representations of Tsallis entropy,

\[
S_q = -\int_X p(x)^q \ln_q p(x) \, d\mu(x),
\]
One can verify that

\[ I_q(p||r) = - \int_X p(x)^q \ln_q r(x) \, d\mu(x) - S_q(p) \]  

(62)

With these relations in hand we proceed with the derivation. Consider

\[ I_q(l||p) = - \int_X l(x) \ln_q \frac{p(x)}{l(x)} \, d\mu(x) . \]

By (61) we have

\[ I_q(l||p) = - \int_X l(x)^q \left[ \ln_q p(x) - \ln_q l(x) \right] \, d\mu(x) = I_q(l||r) - \int_X l(x)^q \left[ \ln_q p(x) - \ln_q r(x) \right] \, d\mu(x) . \]  

(63)

From (60), we get

\[ I_q(l||p) = I_q(l||r) + \sum_{m=1}^M \beta_m \langle w_m \rangle_q + \ln_q \widehat{Z}_q \int_X l(x)^q \, d\mu(x) \]

\[ + (1-q) \ln_q \widehat{Z}_q \int_X l(x)^q \ln_q p(x) \, d\mu(x) . \]  

(64)

By using (56) and (62),

\[ I_q(l||p) = I_q(l||r) + \sum_{m=1}^M \beta_m \langle w_m \rangle_q + \ln_q \widehat{Z}_q \int_X l(x)^q \, d\mu(x) \]

\[ + (1-q) \ln_q \widehat{Z}_q \left[ - I_q(l||p) - S_q(l) \right] , \]  

(65)

and by the expression of Tsallis entropy \( S_q(l) = \frac{1}{q-1} \left[ 1 - \int_X l(x)^q \, d\mu(x) \right] \), we have

\[ I_q(l||p) = I_q(l||r) + \sum_{m=1}^M \beta_m \langle w_m \rangle_q + \ln_q \widehat{Z}_q - (1-q) \ln_q \widehat{Z}_q I_q(l||p) . \]  

(66)

Since the multipliers \( \beta_m, \; m = 1, \ldots, M \) are functions of the expected values \( \langle u_m \rangle_q \), variations in the expected values are equivalent to variations in the multipliers. Hence, to find the minimum of \( I_q(l||p) \), we solve

\[ \frac{\partial}{\partial \beta_m} I_q(l||p) = 0 . \]  

(67)

By using thermodynamic equation (33), solution of (67) provides us with the expectation matching property in the nonextensive case as

\[ \langle u_m \rangle_q = \frac{\langle w_m \rangle_q}{1 - (1-q) I_q(l||p)} , \; m = 1, \ldots, M . \]  

(68)

In the limit \( q \to 1 \) the above equation gives \( \langle u_m \rangle_1 = \langle w_m \rangle_1 \) which is the expectation matching property in the classical case.
Now, to derive the triangle equality for Tsallis relative-entropy minimization, we substitute the expression for $\langle w_m \rangle_q$, which is given by (68), in (66). And after some algebra one can arrive at (58).

Note that the limit $q \to 1$ in (58) gives the triangle equality in the classical case (53).

The two important cases which arise out of (58) are,

\begin{align}
I_q(l\|r) & \leq I_q(l\|p) + I_q(p\|r) \quad \text{when } 0 < q \leq 1 , \\
I_q(l\|r) & \geq I_q(l\|p) + I_q(p\|r) \quad \text{when } 1 < q .
\end{align}

We refer to Theorem 4.2 as nonextensive Pythagoras' theorem and (58) as nonextensive triangle equality, whose pseudo-additivity property is consistent with the pseudo additivity of Tsallis relative-entropy

\begin{equation}
I_q(X_1 \times Y_1 \parallel X_2 \times Y_2) = I_q(X_1 \parallel X_2) + I_q(Y_1 \parallel Y_2) \\
\quad + (q - 1) I_q(X_1 \parallel X_2) I_q(Y_1 \parallel Y_2),
\end{equation}

where $X_1, X_2$ and $Y_1, Y_2$ are r.v.s such that $X_1$ and $Y_1$ are independent, and $X_2$ and $Y_2$ are independent respectively; hence is a natural generalization of triangle equality in the classical case.

4.3. In the Case of Normalized $q$-Expectations

In the case of normalized $q$-expectation too, the Tsallis relative-entropy satisfies nonextensive triangle equality with modified conditions from the case of $q$-expectation values [1, 36].

**Theorem 4.3.** Let $r$ be the prior distribution, and $p$ be the probability distribution that minimizes the Tsallis relative-entropy subject to the set of constraints

\begin{equation}
\frac{\int_X u_m(x)p(x)^q \, d\mu(x)}{\int_X p(x)^q \, d\mu(x)} = \langle \langle u_m \rangle_q \rangle, \quad m = 1, \ldots, M.
\end{equation}

Let $l$ be any other distribution satisfying the constraints

\begin{equation}
\frac{\int_X u_m(x)l(x)^q \, d\mu(x)}{\int_X l(x)^q \, d\mu(x)} = \langle \langle w_m \rangle_q \rangle, \quad m = 1, \ldots, M.
\end{equation}

Then we have

\begin{equation}
I_q(l\|r) = I_q(l\|p) + I_q(p\|r) + (q - 1) I_q(l\|p) I_q(p\|r),
\end{equation}

provided

\begin{equation}
\langle \langle u_m \rangle_q \rangle = \langle \langle w_m \rangle_q \rangle, \quad m = 1, \ldots, M.
\end{equation}
Proof. From Tsallis minimum entropy distribution \( p \) in the case of normalized \( q \)-expected values (39), we have
\[
\ln_q r(x) - \ln_q p(x) = \ln_q \hat{Z}_q + (1 - q) \ln_q p(x) \ln_q \hat{Z}_q + \frac{\sum_{m=1}^{M} \beta_m (u_m(x) - \langle \langle u_m \rangle \rangle_q)}{\int_X p(x)^q \, d\mu(x)} .
\] (76)

Proceeding as in the proof of Theorem 4.2, we have
\[
I_q(l\|p) = I_q(l\|r) - \int_X l(x)^q \left[ \ln_q p(x) - \ln_q r(x) \right] \, d\mu(x) .
\] (77)

From (76), we obtain
\[
I_q(l\|p) = I_q(l\|r) + \ln_q \hat{Z}_q \int_X l(x)^q \, d\mu(x)
+ (1 - q) \ln_q \hat{Z}_q \int_X l(x)^q \ln_q p(x) \, d\mu(x)
+ \frac{1}{\int_X p(x)^q \, d\mu(x)} \sum_{m=1}^{M} \beta_m \int_X l(x)^q \left( u_m(x) - \langle \langle u_m \rangle \rangle_q \right) \, d\mu(x) .
\] (78)

By (73) the same can be written as
\[
I_q(l\|p) = I_q(l\|r) + \ln_q \hat{Z}_q \int_X l(x)^q \, d\mu(x)
+ (1 - q) \ln_q \hat{Z}_q \int_X l(x)^q \ln_q p(x) \, d\mu(x)
+ \frac{\int_X l(x)^q \, d\mu(x)}{\int_X p(x)^q \, d\mu(x)} \sum_{m=1}^{M} \beta_m \left( \langle \langle w_m \rangle \rangle_q - \langle \langle u_m \rangle \rangle_q \right) .
\] (79)

By using the relations
\[
\int_X l(x)^q \ln_q p(x) \, d\mu(x) = -I_q(l\|p) - S_q(l) ,
\]
and
\[
\int_X l(x)^q \, d\mu(x) = (1 - q)S_q(l) + 1 ,
\]
(79) can be written as
\[
I_q(l\|p) = I_q(l\|r) + \ln_q \hat{Z}_q - (1 - q) \ln_q \hat{Z}_q I_q(l\|p)
+ \frac{\int_X l(x)^q \, d\mu(x)}{\int_X p(x)^q \, d\mu(x)} \sum_{m=1}^{M} \beta_m \left( \langle \langle w_m \rangle \rangle_q - \langle \langle u_m \rangle \rangle_q \right) .
\] (80)

Finally using (40) and (75) we have the nonextensive triangle equality (74). \( \square \)
Note that in this case the minimum of $I_q(l\|p)$ is not guaranteed. Also the condition (75) for nonextensive triangle equality here is the same as the expectation value matching property in the classical case.

5. Conclusions

Pythagoras’ theorem of relative-entropy plays an important role in geometrical approaches of statistical estimation theory like information geometry. In this paper we presented Pythagoras’ theorem in the nonextensive case i.e., for Tsallis relative-entropy minimization. In our opinion, this result is yet another remarkable and consistent generalization shown by the Tsallis formalism.

Now, equipped with the nonextensive Pythagoras’ theorem in the generalized case of Tsallis, it is interesting to know the resultant geometry when we use generalized information measures and role of entropic index in the geometry.

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