A New $N = 4$ Superconformal Algebra

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Abstract

It is shown that the previously known $N = 3$ and $N = 4$ superconformal algebras can be contracted consistently by singular scaling of some of the generators. For the later case, by a contraction which depends on the central term, we obtain a new $N = 4$ superconformal algebra which contains an $SU(2) \times U(1)^4$ Kac-Moody subalgebra and has nonzero central extension.
It has been known for long time that the contraction of $SU(2)$ Lie algebra by singular scaling of two of its generators gives rise to the algebra of $E(2)$. Similar contractions exist for other groups and are known as Inönu-Wigner contractions. It should be emphasized that this contraction procedure is different from a finite scaling of generators which does not change the algebra. Due to this characteristic difference, there are strong restrictions on the choice of the generators that can be scaled singularly.

In a recent paper, Majumdar generalized the contraction procedure to the Kac-Moody algebras by dividing the generators $J^a (a = 1, \ldots, D)$ into two sets, $J^\alpha (\alpha = 1, \ldots, d)$ and $J^i, (i = d + 1, \ldots, D)$ where $J^i$ are scaled as

$$J^i \to \frac{1}{f(\epsilon)} J^i,$$

with $f(0) = 0$, $f(1) = 1$ and $J^\alpha$ are left unscaled. The contracted algebra corresponds to the limit $\epsilon \to 0$. For the Kac-Moody case, for a consistent contraction to exist, one gets the condition that the structure constants $f^{\alpha \beta i} = 0$. It comes from the fact that a higher order singularity in the right hand side of a commutator with respect to the one in the left has to be avoided for a meaningful reinterpretation of the algebra. The contracted algebra is then given by,

$$[J^\alpha_m, J^\beta_n] = i f^{\alpha \beta \gamma} J^\gamma_{m+n} + \frac{k}{2} n \delta^{\alpha \beta} \delta_{m+n,0},$$

$$[J^\alpha_m, J^i_n] = i f^{\alpha ij} J^j_{m+n},$$

$$[J^i_m, J^j_n] = 0.$$
It is therefore observed that for the unscaled generators, $J^\alpha$, the contraction procedure allows a central term. But no such term is allowed for the scaled generators $J^i$'s.

In this paper, we analyze the effect of singular scalings of the type described above for $N = 3$ \cite{3} and Sevrin et al’s $N = 4$ \cite{4} superconformal algebras. The $N=3$ algebra contains an $SU(2)$ subalgebra. We find that when the $SU(2)$ generators are scaled as in \cite{2}, the consistency of the full algebra requires the scaling of at least one of the three superconformal generators. We present a consistent set of scaling and the corresponding contracted algebra.

Contraction for the $N = 4$ algebra of ref.\cite{4} is much more interesting. In this case we find that the resulting algebra is a new $N = 4$ superconformal algebra with an underlying $SU(2) \times U(1)^4$ Kac-Moody subalgebra-with all the central terms surviving. For this case, the scaling is different from the ones used in previous paragraphs, since it is dependent on an in-built parameter of the original algebra which determines the central terms.

To illustrate our basic procedure and for algebraic simplicity, we first present the contraction of the $N = 3$ algebra and later on go to the more interesting case of $N = 4$. We start by writing down the $N = 3$ superconformal algebra:

\begin{align}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{5} \\
[L_m, \phi_n] &= [(d\phi - 1)m - n]\phi_{m+n}, \tag{6}
\end{align}
where \( \phi_n \in \{ G^a_r, J^a_r, \Gamma_r \} \), corresponding \( d_\phi \in \{ \frac{3}{2}, 1, \frac{1}{2} \} \) and \( a \in \{ 1, 2, 3 \} \).

\[
\{ G^a_r, G^b_s \} = i \epsilon^{abc}(r-s)J^c_{r+s}, \quad (a \neq b),
\]

(7)

\[
[J^a_m, G^a_r] = m \Gamma_{m+r}, \quad \text{(no sum on } a),
\]

(8)

\[
[J^a_m, \Gamma_r] = 0,
\]

(9)

\[
[J^a_m, J^b_n] = i \epsilon^{abc} J^c_{m+n} + \frac{c}{3} m \delta^{ab} \delta_{m+n},
\]

(10)

\[
\{ G^a_r, G^a_s \} = 2L_{r+s} + \frac{c}{3} (r^2 - \frac{1}{4}) \delta_{r+s,0}, \quad \text{(no sum on } a),
\]

(11)

\[
\{ \Gamma_r, \Gamma_s \} = \frac{c}{3} \delta_{r+s,0},
\]

(12)

\[
\{ \Gamma_r, G^a_s \} = J^a_{r+s},
\]

(13)

\[
[J^a_m, G^b_r] = i \epsilon^{abc} G^c_{m+r}, \quad (a \neq b).
\]

(14)

For the contraction of this algebra the Kac-Moody generators \( J^a \) are scaled in the same manner as in [2] with \( J^a \equiv J^3, J^i \equiv (J^1, J^2) \). As a result, the modified algebra for these generators becomes,

\[
[J^3_m, J^3_n] = \frac{c}{3} m \delta_{m+n,0},
\]

(15)

\[
[J^3_m, J^i_n] = i \epsilon^{ij} J^j_{m+n},
\]

(16)

\[
[J^i_m, J^j_n] = 0.
\]

(17)

By observing eqns. (7) and (13), we conclude that a consistent contraction of the \( N = 3 \) algebra then requires the singular scaling of one of the two pairs of generators, \( (G^3, \Gamma) \) or \( (G^1, G^2) \). We discuss here the former case, although the later one is also analogous. The scaling of the four generators \( J^1, J^2, G^3 \) and \( \Gamma \) is done by the same factor \( \frac{1}{f(c)} \). Then eqns.(3)-(9) remain
unchanged after scaling. The rest of the algebra, eqns.(11)-(14), is modified to the following form:

\[ \{ G^3_r, G^3_s \} = 0, \quad (18) \]

\[ \{ \Gamma_r, \Gamma_s \} = 0, \quad (19) \]

and

\[ \{ G^i_r, G^i_s \} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \text{(no sum on } i), \]

\[ \{ \Gamma_r, G^i_s \} = J^i_{r+s}, \quad (i = 1, 2), \]

\[ \{ \Gamma_r, G^3_s \} = 0, \]

\[ [J^i_m, G^2_r] = 0, \]

\[ [J^i_m, G^i_r] = i\epsilon^{ij}G^3_{m+r}, \]

\[ [J^3_m, G^i_r] = i\epsilon^{ij}G^j_{m+r}. \quad (20) \]

We have explicitly verified that after the above modifications, all the Jacobi identities are satisfied. Therefore the modified algebra (3)-(8), (15)-(20) is a consistent superconformal algebra. As in ref.[2] a difficulty in the physical interpretation of the above algebra is the presence of vanishing (anti-) commutators in eqns.(17), (18), (19). It causes problem for a free field realization of the algebra and thus for an explicit construction of the Hilbert space of such theories. Moreover the vanishing anticommutator (18) implies that the operator \( G^3 \) can not be interpreted as the usual superconformal generator. Similar problems also occur if one tries to contract \( N = 1 \) and 2
superconformal algebras. We now discuss a contraction of the $N = 4$ algebra of ref.\[3\], where both of these problems can be avoided.

To work out the contraction of the $N = 4$ algebra and to be self contained, we start by writing down the algebra in \[4\]. This algebra has sixteen generators, namely, $L_m$, $G^a_r$, ($a = 1, \ldots, 4$), $A_{m}^{\pm i}$ ($i = 1, 2, 3$), $Q^a_r$ ($a = 1, \ldots, 4$) and $U_m$. Their conformal weights $d_\phi$ are $2, \frac{3}{2}, 1, \frac{1}{2}$ and $1$ respectively. The algebra is written as,

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \]
\[ [L_m, \phi_n] = [(d_\phi - 1)m - n]\phi_{m+n}, \quad \phi_n \in \{G^a_r, A^{\pm i}_m, U_m, Q^a_r\}, \]
\[ [A^{\pm i}_m, A^{-j}_n] = 0, \]
\[ [A^{i}_m, Q^a_n] = \alpha^{i+}_a Q^b_{m+n}, \]
\[ [U_m, G^a_n] = mQ^a_{m+n}, \]
\[ [U_m, Q^a_n] = 0, \]
\[ [U_m, A^{\pm i}_m] = 0, \quad (21) \]

and

\[ \{G^a_m, G^b_n\} = 2\delta^{ab}L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta^{ab}\delta_{m+n,0} + 4(n - m)[\gamma \alpha^{+i}_a A^{+i}_{m+n} + (1 - \gamma)\alpha^{-i}_a A^{-i}_{m+n}], \]
\[ [A^{+i}_m, G^a_n] = \alpha^{+i}_a [G^b_{m+n} - 2(1 - \gamma)mQ^b_{m+n}], \]
\[ [A^{-i}_m, G^a_n] = \alpha^{-i}_a [G^b_{m+n} + 2\gamma mQ^b_{m+n}], \]
\[ [A^{\pm i}_m, A^{\pm j}_n] = \epsilon^{i+} A^{\pm k}_{m+n} - m\frac{c}{12\gamma}\delta^{ij}\delta_{m+n,0}. \]
\[
[A^{-i}, A^{-j}] = \epsilon^{ijk} A^{-k}_{m+n} - m \frac{c}{12(1 - \gamma)} \delta^{ij} \delta_{m+n,0},
\]
\[
\{Q^a_m, Q^b_n\} = 2(\alpha^{+i}_{ab} A^{+i}_{m+n} - \alpha^{-i}_{ab} A^{-i}_{m+n}) + \delta^{ab} U_{m+n},
\]
\[
[A^{-i}, Q^a_n] = \alpha_{ab}^{-i} Q^b_{m+n},
\]
\[
\{Q^a_m, Q^b_n\} = -\frac{c}{12\gamma(1 - \gamma)} \delta^{ab} \delta_{m+n,0},
\]
\[
[U_m, U_n] = -m \frac{c}{12\gamma(1 - \gamma)} \delta_{m+n,0},
\]

(22)

with

\[
\alpha^{\pm i}_{jk} = \frac{1}{2} \epsilon_{ijk}; \quad \alpha^{\pm i}_{j4} = -\alpha^{\pm i}_{4j} = \pm \frac{1}{2} \delta_{ij}; \quad \alpha^{\pm i}_{44} = 0,
\]

\[
[\alpha^{\pm i}, \alpha^{\pm j}] = -\epsilon^{ijk} \alpha^{\pm k}, \quad \{\alpha^{\pm i}, \alpha^{\pm j}\} = -\frac{1}{2} \delta^{ij}; \quad [\alpha^{+i}, \alpha^{-j}] = 0.
\]

It is noticed that the central extension of the above algebra is parameterized by two parameters \(c\) and \(\gamma\). This fact is crucial for our contraction, since the scaling function \(f(\epsilon)\) for this case depends on \(\gamma\). More precisely, we choose \(\epsilon = 1 - \gamma > 0\), \(f(\epsilon) = \sqrt{\epsilon}\) with \(\gamma \to 1\), and scale eight of the generators \(A^{-i}, Q^a, U\) of the \(N = 4\) algebra by the same scaling function \(\frac{1}{f(\epsilon)}\). For \(\gamma > 1\) one can choose \(\epsilon = \gamma - 1\) and obtain similar results. For our choice, a close observation shows that this is a consistent scaling for the algebra in eqns.(21) and (22). After this singular scaling one gets an algebra where the (anti-) commutators in eqns.(21) are left unchanged. The rest of the algebra, eqn.(22), is modified to
\[
\{G^a_m, G^b_n\} = 2\delta^{ab}L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta^{ab}\delta_{m+n,0}
+ 4(n-m)\alpha^{+i}_{ab} A^{+i}_{m+n},
\]

\[
[A^+_i, G^a_n] = \alpha^{i}_{ab} G^a_{m+n},
\]

\[
[A^-_i, G^a_n] = 2m\alpha^{-i}_{ab} Q^b_{m+n},
\]

\[
[A^+_i, A^+_j] = \epsilon^{ijk} A^+_{k,m+n} - m c \frac{\delta^{ij}}{12} \delta_{m+n,0},
\]

\[
[A^-_i, A^-_j] = -m c \frac{\delta^{ij}}{12} \delta_{m+n,0},
\]

\[
\{Q^a_m, G^b_n\} = -2\alpha^{-i}_{ab} A^-_{m+n} + \delta^{ab} U_{m+n},
\]

\[
[A^-_i, Q^a_n] = 0,
\]

\[
\{Q^a_m, Q^b_n\} = -\frac{c}{12} \delta^{ab} \delta_{m+n,0},
\]

\[
[U_m, U_n] = -m c \frac{\delta_{m+n,0}}{12}.
\] (23)

In this case also we have explicitly verified that the commutation relations in eqns. (21) and (23) satisfy all the Jacobi identities among these generators. Hence they form a consistent algebra.

It is noticed that the generators \(A^-i\)'s in the contracted algebra satisfy the commutation relations of affine \(U(1)\) generators since the structure constants for these have disappeared. This, in fact, is a general property of the contraction since it was already observed in eqns. (4) that the structure constants in the commutator of the scaled generators \(J^i\)'s vanish. Another aspect of our contraction is that the central terms in all the (anti-) commutators of the original algebra, eqns. (21)-(22) survive and therefore one can
hope to obtain its free field realization.

We have thus obtained a new $N = 4$ superconformal algebra which is distinguished from the previously known $N = 4$ algebras of [3] and [4] by the underlying $SU(2) \times U(1)^4$ Kac-Moody symmetry. It is also observed that the the contracted algebra, eqns. (21) and (23), contains, as a subalgebra, the Ademollo et al $N = 4$ algebra [3] which is satisfied by the generators, $L_m, G^a,$ and $A^{+i}$. Although the existence of Ademollo et al $N = 4$ subalgebra was also noticed in [4], however, for their case the choice $\gamma = 1$ implied $c = 0$ and the algebra became centerless. Here we have shown that an algebra with nonzero central extension can be obtained for $\gamma \to 1$ by the singular scaling of eight of the generators $A^{-i}, Q^a,$ and $U$.

Finally, another contraction of the algebra in [4] can also be done by scaling $A^{+i}$ instead of $A^{-i}$ and taking the limit $\gamma \to 0$ with $\epsilon = \gamma$. It will be interesting to obtain a free field realization of the new $N = 4$ superconformal algebra presented in this paper. This is crucial for obtaining the Hilbert space and the unitary representations of this algebra.
References

[1] E. Inönü and E. Wigner, Proc. Nat. Acad. Sci.(USA)39, 510(1953); R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications*, Wiley, New York (1974).

[2] P. Majumdar, Inönü-Wigner contraction of Kac-Moody algebras, Matscience preprint, IMSc/92-26, 1992.

[3] M. Ademollo et al, Phys. Lett. B62, 105(1976).

[4] A. Sevrin, W. Troost, and A. Van Proeyen, Phys. Lett. B208, 447(1988).