Further Extended Theories of Gravitation: Part I*

by M.Di Mauro, L.Fatibene, M.Ferraris, M.Francaviglia

Abstract: We shall here propose a class of relativistic theories of gravitation, based on a foundational paper of Ehlers Pirani and Schild (EPS). All “extended theories of gravitation” (also known as $f(R)$ theories) in Palatini formalism are shown to belong to this class. In a forthcoming paper we shall show that this class of theories contains other more general examples. EPS framework helps in the interpretation and solution of these models that however have exotic behaviours even compared to $f(R)$ theories.

1. Introduction

Once one starts to generalize GR in order to include new observational data it is not clear whether there is a natural border that physically reasonable models of gravitation should not cross. Ehlers, Pirani and Schild (EPS) proposed in 1972 some axioms to deduce GR from observational structures; see [1]. Their subtle analysis was based on physical and mathematical axioms and turned into the understanding that gravity and causality require on spacetime both an affine and a metric structure a priori independent but related by suitable compatibility requirements. The program was never completed in a fully satisfactory way, in the sense that they could not finally prove that one needs to use as a connection the Levi-Civita connection of the metric structure on spacetime.

However, EPS framework turns out to be a natural framework for the interpretation of Palatini extended theories of gravitation in general, and in particular for $f(R)$ models. These models have been recently used with the hope that they could account for effects attributed to dark matter and energy; [2], [3], [4], [5], [6], [7] and references quoted therein. A number of criticisms have been variously raised on Palatini framework; see e.g. [3], [8] and references quoted therein. However, in view of EPS framework it appears as the most natural setting for gravitational theories, at least from a foundational viewpoint; see also [9], [10].

This paper is divided into two parts. In this first part we shall review the EPS, define the class of further extended theories of gravitation (FETG) and show that $f(R)$ theories are particular cases of FETG.

In the second part we shall consider some examples of FETG other than standard $f(R)$ theories. We shall show that FETG allow projective structures (i.e., connections) that are not necessarily metric. Moreover, a specific example will be considered that reproduces the behaviour of purely metric $f(R)$ theories starting from a Palatini framework. This is particularly interesting since it overcomes some of the recent criticisms against Palatini framework based on problems in modelling polytropic stars; see [8].

Hereafter, spacetime $M$ is assumed to be a 4-dimensional, connected, paracompact manifold which allows global Lorentzian metrics.

---

* This paper is published despite the effects of the Italian law 133/08 (http://groups.google.it/group/scienceaction). This law drastically reduces public funds to public Italian universities, which is particularly dangerous for free scientific research, and it will prevent young researchers from getting a position, either temporary or tenured, in Italy. The authors are protesting against this law to obtain its cancellation.
2. EPS Axioms and Compatibility

Ehlers, Pirani and Schild (EPS) proposed in the seventies an axiomatic construction of General Theory of Relativity (see [1]). They showed that spacetime geometry can be obtained from few assumptions about two observational quantities: the worldlines of light rays and free falling mass particles. Light rays and particles are to be understood in a classical sense: light rays are “small wave packets” and particles are material balls with negligible extension.

In order to introduce a differential structure on $M$, EPS defined the radar coordinates: given an event $e \in M$ and two particles (closed enough to $e$) one can consider echoes from the particles on the event $e$. Radar coordinates are in particular defined by the following map:

$$\Phi_{pp} : e \mapsto (u, v, u', v') \quad (2.1)$$

where $(u, v, u', v')$ are the leaving and arriving parameters of echoes on the two particles.

Then EPS introduced a function $g$ (see Axiom $L_1$ in [1]) considering a particle $P$ passing through an event $e$ and light cone $v_e$ emitted from an event $p$ external to $P$. The function $g$ is defined by $g = t(e1)t(e2)$, where $t(e1)$ and $t(e2)$ are parameters on $P$ relative to the encounter of $v_e$ with the particle $P$.

The Hessian matrix (at $e$) of this map $g$ defines a tensor $g_{\mu\nu}$ with the following properties:

$$g_{\mu\nu} T^\mu T^\nu = 0 \quad g_{\mu\nu} L^\mu L^\nu = 2 \quad (2.2)$$

for any vector $T$ tangent to light rays passing through $e$ and any vector $L$ tangent to particles passing through $e$. Unfortunately, the tensor $g_{\mu\nu}$ depends on the parametrization fixed along the particles which are unphysical; they in fact correspond to fix a conventional clock (see [11]). One can show that physically one can single out a class of tensors which are defined modulo a conformal factor (which in fact does not affect light cones defined by $g$).

One can easily check that EPS axioms are coherent. For example one can start with $M = \mathbb{R}^4$ and two families of straight lines and show that in Cartesian coordinates the tensor $g_{\mu\nu}$ turns out to be

$$g_{\mu\nu} = \frac{2}{\|u\|^2} \eta_{\mu\nu} \quad (2.3)$$

where $(\partial_0, u)$ are the 4-velocities of the particles involved in the definition of radar coordinates and for $\eta_{\mu\nu}$ is the standard Minkowski metric. So we have found that the tensor $g_{\mu\nu}$ verifies (2.2). Moreover $\det(g_{\mu\nu}) \neq 0$ so it defines in fact a spacetime metric which exhibits $M$ as equivalent to Minkowski spacetime.

In order to introduce an object that is invariant with respect to particles reparametrizations (i.e. conformal transformations on $g$) EPS considered the conformal metric

$$g'_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt{\left|\det g\right|}} \quad (2.4)$$

which is a tensor density of weight $\frac{1}{2}$.

The quantity $g'_{\mu\nu}$ is invariant for conformal transformations $g'_{\mu\nu} = \varphi \cdot g_{\mu\nu}$; in fact

$$g'_{\mu\nu} = \frac{g'_{\mu\nu}}{\sqrt{g'}} = \frac{\varphi \cdot g_{\mu\nu}}{\sqrt{\varphi g \cdot g}} = \frac{g_{\mu\nu}}{\sqrt{g}} = g_{\mu\nu} \quad (2.5)$$
For simplicity let us set $\sqrt[4]{g} := \sqrt[4]{|\text{det}g|}$. The conformal metric $g_{\mu\nu}$ defines a conformal structure on $M$. The equation $g_{\mu\nu}T^\mu T^\nu = 0$ represents light cones emitted from (or received at) an event $e$ so it allows to distinguish among timelike, spacelike and lightlike vectors through $e$. The conformal structure is equivalent to assigning the light cones at each point. Hereafter in this Section, spacetime indices will be lowered and raised by the conformal metric $g$.

Then EPS axiom $P_2$ relates to an infinitesimal version of the law of inertia in projective coordinates $\tilde{x}^\alpha$, namely

$$\frac{d^2 \tilde{x}^\alpha}{d\bar{u}^2} = 0$$

Changing parameters and coordinates the previous equation becomes:

$$\tilde{x}^\alpha + \Pi^\alpha_{\mu\nu} \tilde{x}^\mu \bar{x}^\nu = \lambda \tilde{x}^\alpha \quad (2.6)$$

The coefficients $\Pi^\alpha_{\mu\nu}$ can be chosen without loss of generality to obey the properties $\Pi^\alpha_{[\mu\nu]} = 0$ and $\Pi^\alpha_{\mu\mu} = 0$; they are called projective coefficients. On the other hand one can fix parametrization so to have $\lambda = 0$.

They do not directly refer to a connection because in view of the traceless condition they transform differently according to the following transformation rules:

$$\Pi^\alpha_{\beta\mu} = J^\alpha_\lambda \left( \Pi^\lambda_{\rho\sigma} \bar{J}^\rho_{\beta\mu} + \bar{J}^\lambda_{\beta\mu} \right) + \frac{1}{3} \delta^\alpha_{\beta\mu} \bar{J}^\rho_{\beta\mu} \partial^\rho \ln J \quad (2.7)$$

The projective coefficients define thence a geometric object which is called a projective connection and a projective structure on $M$. Any curve that satisfies equation (2.6) is said geodesic (trajectory). In particular, particles follow geodesics.

Once the conformal and the projective structure of spacetime have been introduced, EPS proposed a compatibility axiom (see [1] Axiom C):

*a solution of equations (2.6) is a particle iff its initial 4-velocity at $e$ is contained in the interior of the light cone $u_e$.*

This axiom allows to express an analytical relation between the conformal and the projective structure. They introduced $\{g\}_\mu^\alpha$, the “Christoffel symbols” of $g_{\mu\nu}$:

$$\{g\}_\mu^\alpha := \{g\}_\mu^\alpha + \frac{1}{8} \left( g^{\gamma\alpha} g_{\mu\nu} - 2 \delta^\alpha_{\mu} \delta^\gamma_{\nu} \right) \partial_\gamma \ln g \quad (2.8)$$

where $g$ is any representative of the conformal class identified by $g$. The coefficients $\{g\}_\mu^\alpha$ transform as:

$$\{g\}_\beta^\alpha = J^\alpha_\lambda \left( \{g\}_{\rho\sigma}^\lambda \bar{J}^\rho_{\beta\mu} + \bar{J}^\lambda_{\beta\mu} \right) - \frac{1}{4} J^\alpha_\lambda \left( g^{\nu\lambda} g_{\rho\sigma} - 2 \delta^\alpha_{\rho} \delta^\nu_{\sigma} \right) \bar{J}^\rho_{\beta\mu} \partial^\nu \ln J \quad (2.9)$$

The difference $\Delta^\alpha_{\beta\mu} = \Pi^\alpha_{\beta\mu} - \{g\}_\beta^\alpha$ has the following two properties $\Delta^\alpha_{[\beta\mu]} = 0$ and $\Delta^\alpha_{\beta\mu} = 0$. Then EPS obtained the relation for $\Delta_{\alpha\beta\mu} := \Pi^\alpha_{\beta\mu} \Delta^\lambda_{\beta\mu}$:

$$\Delta_{\alpha\beta\gamma} = \Delta_{(\alpha\beta\gamma)} + \frac{1}{2} \left( p_\alpha \theta_{\beta\gamma} - p_\alpha (\theta_{\beta\gamma} - \alpha_{\beta\gamma}) \right) + L_{\alpha(\beta\gamma)} \quad (2.10)$$

where we set $L_{\alpha(\beta\gamma)} = \frac{4}{3} \Delta_{(\alpha(\beta\gamma)} - p_\alpha \theta_{\beta\gamma]}$ and $p_\alpha = \frac{8}{3} \Delta_{(\alpha\lambda)} \lambda$. (In [1] the definition of $p_\alpha$ was given with a different sign probably due to a typo. With our definition the quantity $L_{\alpha(\beta\gamma)}$ has the
following properties $L^\alpha{}_{\beta\alpha} = 0$, $L_{[\alpha\beta\gamma]} = 0$ and $L_{(\alpha\beta)\gamma} = 0$ while with the original EPS definition the first one does not hold true; see Appendix A).

Equation (2.10) can be recasted into the following form

$$\Delta^\alpha_{\beta\mu} = L^\alpha_{(\beta\mu)} + 5\tilde{q}^\alpha g_{\beta\mu} - 2\delta^\alpha_{(\beta} \tilde{q}_{\mu)}$$

(2.11)

by setting $\tilde{q}^\alpha = \frac{1}{18} \Delta^\alpha_{\beta\mu} g^{\beta\mu}$.

Now consider a sequence $\{P_n : n \in \mathbb{N}\}$ of particles passing through an event $e$. For Axiom C all $P_n$ are internal to the light cone $v_e$. $P_n$ are particles (hence projective geodesics). Let us consider a sequence determined by initial velocities that tend to a light velocity $c^\mu$. Let $P$ be the light ray with initial velocity $c^\mu$. Since $P_n \to P$, the 4-velocity of particles can be as near as one wishes to $c^\mu$; accordingly, $P$ is a projective geodesic and a conformal geodesics. Hence Axiom C can be equivalently expressed in this way:

null projective-geodesic are identical to null conformal-geodesics.

Conformal geodesics are obtained as solution of the following equation:

$$\ddot{x}^\alpha + \{g\}^{\alpha}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \nu \dot{x}^\alpha$$

(2.12)

EPS were at this point able to give a formal expression to the compatibility condition as

$$\Delta^\alpha_{\mu\nu} = 5\tilde{q}^\alpha g_{\mu\nu} - 2\delta^\alpha_{(\mu} \tilde{q}_{\nu)}$$

(2.13)

Another equivalent form of Axiom C is:

the set of particles is identical with the set of conformal-timelike projective-geodesics.

Finally, EPS introduced a connection $\Gamma^\alpha_{\beta\mu}$ given in terms of the projective and conformal structures

$$\Gamma^\alpha_{\beta\mu} := \{g\}^{\alpha}_{\beta\mu} + \left(g^{\alpha\sigma} g_{\beta\mu} - 2\delta^\alpha_{(\beta} \tilde{q}^\sigma_{\mu)}\right) q_{\nu}$$

(2.14)

where we set $q_{\nu} := 5\tilde{q}_{\nu}$.

This can be introduced as a generic linear combination

$$\Gamma^\alpha_{\beta\mu} = a \{g\}^{\alpha}_{\beta\mu} + b q^\alpha g_{\beta\mu} - 2c \delta^\alpha_{(\beta} \tilde{q}_{\mu)}$$

(2.15)

and then fixing the coefficients so that $\Gamma$ transforms as a connection. In view of (2.7) and (2.9) one can check that $\Gamma$ transforms in fact as a connection

$$\Gamma^\nu_{\rho\sigma} = a \{g\}^{\nu}_{\rho\sigma} + b q^\nu g_{\rho\sigma} - 2c \delta^\nu_{(\rho} \tilde{q}_{\sigma)} =$$

$$= \partial_\lambda \left( \frac{b^\lambda}{20} - \frac{a}{2} \right) \delta^\nu_{\rho\sigma} + \left( \frac{b^\nu}{20} - \frac{a}{2} \right) \delta^\rho_{\langle\nu} \delta^\sigma_{\rangle\nu} \right) q_{\nu}$$

(2.16)

This transforms as a connection iff we set $a = 1$ and $b = c = 5$. Accordingly we have (2.14).
3. Non-metric EPS Connections

EPS showed that the most general connection $\Gamma^\alpha_{\beta\mu}$ compatible (in EPS sense) with a metric structure $g_{\mu\nu}$ is in the form (2.14). By tracing the EPS connection we obtain

$$
\Gamma^\alpha_{\beta\mu} = \{g\}^\alpha_{\beta\mu} + (g^{\alpha\upsilon}g_{\alpha\mu} - \delta^\alpha_\beta \delta^\upsilon_\mu) q_\upsilon = \{g\}^\alpha_{\beta\mu} + (\delta^\alpha_\beta - 4\delta^\alpha_\mu - \delta^\alpha_\mu) q_\epsilon = \{g\}^\alpha_{\beta\mu} - 4q_\mu.
$$

(3.1)

If $h_{\mu\nu} = \varphi \cdot g_{\mu\nu}$ we have moreover

$$
\{h\}^\alpha_{\beta\mu} = \frac{1}{2} \delta^\alpha_\beta \left( -\partial_\mu \varphi + \partial_\nu \varphi + \partial_\nu \partial_\mu \varphi \right) = (g)_{\beta\mu} + \frac{1}{8} \left( \frac{1}{2} \partial_\lambda \ln \varphi + \frac{1}{4} \left( \partial_\gamma \varphi + 2 \partial_\gamma \varphi + \partial_\gamma \varphi \right) \right) = \{g\}_{\beta\mu} + \frac{1}{8} \left( \frac{1}{2} \partial_\lambda \ln \varphi + \frac{1}{4} \left( \partial_\gamma \varphi + 2 \partial_\gamma \varphi + \partial_\gamma \varphi \right) \right)
$$

(3.2)

The case $h_{\mu\nu} = g_{\mu\nu} = \frac{\partial_\mu \varphi}{\varphi}$ is a special case for $\varphi = g^{-1/4}$,

$$
\{g\}_{\beta\mu} = \{g\}_{\beta\mu} + \frac{1}{8} \left( g^{\alpha\beta} g_{\beta\mu} - 2 \delta^{\alpha}_\beta \delta^\mu_\rho \right) \partial_\lambda \ln \varphi
$$

(3.3)

Notice the similarity with (2.14) for $q_\epsilon = \frac{1}{2} \partial_\lambda \ln \varphi$. The trace is:

$$
\{g\}_{\alpha\mu} = \{g\}_{\alpha\mu} + \frac{1}{8} \left( g^{\alpha\beta} g_{\beta\mu} - 2 \delta^{\alpha}_\beta \delta^\mu_\rho \right) \partial_\lambda \ln \varphi = \frac{1}{2} \delta^\alpha_\beta \left( -\partial_\mu \varphi + \partial_\nu \varphi + \partial_\nu \partial_\mu \varphi \right) \partial_\lambda \ln \varphi
$$

(3.4)

Thence we have

$$
q_\mu = -\frac{1}{4} \Gamma^\alpha_{\alpha\mu}
$$

(3.5)

Moreover, using (3.3) we have

$$
\Gamma^\alpha_{\beta\mu} = \{g\}_{\beta\mu} + \left( g^{\alpha\upsilon} g_{\beta\mu} - 2 \delta^{\alpha}_\beta \delta^\upsilon_\mu \right) \left( q_\epsilon + \frac{1}{8} \partial_\lambda \ln g \right)
$$

(3.6)

If we define a conformal metric $h_{\mu\nu} = \varphi \cdot g_{\mu\nu}$ we also have

$$
\Gamma^\alpha_{\beta\mu} = \{h\}_{\beta\mu} + \left( h^{\alpha\upsilon} h_{\beta\mu} - 2 \delta^{\alpha}_\beta \delta^\upsilon_\mu \right) \left( q_\epsilon + \frac{1}{8} \partial_\lambda \ln \varphi \right) = \{h\}_{\beta\mu} + \left( h^{\alpha\upsilon} h_{\beta\mu} - 2 \delta^{\alpha}_\beta \delta^\upsilon_\mu \right) \left( q_\epsilon + \frac{1}{8} \partial_\lambda \ln \varphi \right)
$$

(3.7)

i.e. the characterization of EPS-connections is conformally invariant.

As it is well-known, the functions $q_\epsilon$ parametrize the connections EPS-compatible to $g_{\mu\nu}$. When $q_\epsilon = \frac{1}{8} \partial_\lambda \ln \varphi$ for some function $\varphi$ then we get a metric connection $\Gamma^\alpha_{\beta\mu} = \{\varphi \cdot g\}_{\beta\mu}$ for a conformal factor $\varphi$. On the other hand when $q_\epsilon$ has no potential then the connection $\Gamma^\alpha_{\beta\mu}$ is non-metric.

In fact if the connection $\Gamma$ is the Levi-Civita connection of some metric $\gamma$ then one has

$$
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \gamma^{\alpha\lambda} \left( -\partial_\lambda \gamma_\beta + \partial_\beta \gamma_\lambda + \partial_\gamma \gamma_\lambda \right) = \frac{1}{2} \gamma^{\alpha\lambda} \partial_\beta \gamma_\lambda = \frac{1}{2} \gamma \partial_\beta \gamma = \partial_\beta \ln \sqrt{\gamma}.
$$

(3.8)

On the other hand if the same connection $\Gamma$ is EPS-compatible with the metric $g$ then $\Gamma^\alpha_{\beta\gamma} = -4q_\beta$; hence we have the relation

$$
q_\beta = \frac{1}{8} \partial_\beta \ln \sqrt{\gamma}
$$

(3.9)

Then a connection $\Gamma$ which is EPS-compatible with a metric $g$ is metric or, equivalently, iff it is the Levi-Civita connection of a metric $h$ conformal to $g$ iff $q_\epsilon$ has a potential. It is non-metric otherwise.
4. EPS-Compatibility Condition

We need a condition to express EPS-compatibility in differential form. We shall then look for relativistic theories in which field equations imply EPS-compatibility.

Let us start by considering the quantity \( \sqrt{h} \) when \( \Gamma \) is EPS-compatible with \( h \) (or any other metric \( g \) conformal to \( h \)); we have

\[
\frac{\Gamma}{\Gamma} \left( \sqrt{h}h^{\alpha\beta} \right) = h_{\mu} \left( \sqrt{h}h^{\alpha\beta} \right) + K_{\epsilon\mu} \sqrt{h}h^{\epsilon\beta} + K_{\epsilon\mu} \sqrt{h}h^{\alpha\epsilon} - K_{\epsilon\mu} \sqrt{h}h^{\alpha\beta} =
\]

\[
= - \sqrt{h} \left( h_{\mu\lambda}h^{\alpha\beta} - 2\delta^{(\alpha}_{\lambda} \delta^{\beta)}_{\rho} \right) h_{\rho\lambda} K_{\epsilon\mu} \tag{4.1}
\]

where we set \( K_{\epsilon\mu} := \Gamma_{\epsilon\mu} - \{ h \}_{\epsilon\mu} \); if \( \Gamma \) is EPS-compatible with \( h \) then we have

\[
K_{\epsilon\mu} = \left( h^{\lambda\sigma} h_{\epsilon\mu} - 2\delta^{(\lambda}_{\epsilon} \delta^{\sigma)}_{\mu} \right) \left( q_{\sigma} + \frac{1}{2} \partial_{\sigma} \ln h \right) \tag{4.2}
\]

and substituting back into (4.1) we obtain

\[
\frac{\Gamma}{\Gamma} \left( \sqrt{h}h^{\alpha\beta} \right) = - \sqrt{h} \left( h_{\mu\lambda}h^{\alpha\beta} - 2\delta^{(\alpha}_{\lambda} \delta^{\beta)}_{\rho} \right) h_{\rho\lambda} \left( h^{\lambda\sigma} h_{\epsilon\mu} - 2\delta^{(\lambda}_{\epsilon} \delta^{\sigma)}_{\mu} \right) \left( q_{\sigma} + \frac{1}{2} \partial_{\sigma} \ln h \right) =
\]

\[
= - \sqrt{h} \left( h_{\mu\lambda}h^{\alpha\beta} - 2\delta^{(\alpha}_{\lambda} \delta^{\beta)}_{\rho} \right) \left( h^{\lambda\sigma} h_{\epsilon\mu} - 2\delta^{(\lambda}_{\epsilon} \delta^{\sigma)}_{\mu} \right) \left( q_{\sigma} + \frac{1}{2} \partial_{\sigma} \ln h \right) = \tag{4.3}
\]

\[
= - \sqrt{h} \left( h^{\alpha\beta} h_{\epsilon\mu} - 4h^{\alpha\beta} \delta^{\epsilon}_{\mu} - h^{\alpha\beta} \delta^{\epsilon}_{\mu} - \delta^{(\beta}_{\mu} h^{\alpha)}_{\epsilon} + h^{\alpha\beta} \delta^{\epsilon}_{\mu} - h^{\alpha\beta} \delta^{\epsilon}_{\mu} + \delta^{(\alpha}_{\mu} \delta^{\beta)}_{\epsilon} + h^{\alpha\beta} \delta^{\epsilon}_{\mu} - h^{\alpha\beta} \delta^{\epsilon}_{\mu} + \right.
\]

\[
\left. + h^{\beta\epsilon} \delta^{\alpha}_{\mu} + h^{\alpha\epsilon} \delta^{\beta}_{\mu} \right) \left( q_{\epsilon} + \frac{1}{2} \partial_{\epsilon} \ln h \right) = 2\sqrt{h} h^{\alpha\beta} \left( q_{\mu} + \frac{1}{2} \partial_{\mu} \ln h \right)
\]

Hence the necessary condition for a connection \( \Gamma \) to be EPS-compatible with the metric \( h \) (or any other metric \( g \) conformal to \( h \)) is that

\[
\frac{\Gamma}{\Gamma} \left( \sqrt{h}h^{\alpha\beta} \right) = \alpha_{\mu} \sqrt{h} h^{\alpha\beta} \tag{4.4}
\]

Vice versa, if (4.4) holds true then we can define

\[
q_{\mu} := \frac{1}{2} \alpha_{\mu} - \frac{1}{2} \partial_{\mu} \ln h \quad \Gamma_{\beta\mu} := \{ h \}_{\beta\mu} + \left( h^{\alpha\sigma} h_{\beta\mu} - 2\delta^{(\alpha}_{\beta} \delta^{\sigma)}_{\mu} \right) \left( q_{\sigma} + \frac{1}{2} \partial_{\sigma} \ln h \right) \tag{4.5}
\]

Then one has of course that also for this connection the following holds true

\[
\frac{\Gamma}{\Gamma} \left( \sqrt{h}h^{\alpha\beta} \right) = \alpha_{\mu} \sqrt{h} h^{\alpha\beta} \tag{4.6}
\]

Let us now define the tensor \( \tilde{H}_{\beta\mu}^{\alpha} := \Gamma_{\beta\mu}^{\alpha} - \Gamma_{\beta\mu}^{\alpha} \); by subtracting (4.4) and (4.6) we obtain

\[
\tilde{H}_{\epsilon\mu}^{\alpha} h^{\epsilon\beta} + \tilde{H}_{\epsilon\mu}^{\beta} h^{\epsilon\alpha} - \tilde{H}_{\epsilon\mu}^{\alpha} h^{\epsilon\beta} = 0 \tag{4.7}
\]

By tracing this last equation with \( h_{\alpha\beta} \) we obtain

\[
\tilde{H}_{\epsilon\mu}^{\alpha} + H_{\epsilon\mu}^{\beta} - 4H_{\epsilon\mu}^{\sigma} = 0 \quad \Rightarrow \tilde{H}_{\epsilon\mu}^{\alpha} = 0 \tag{4.8}
\]

and substituting back into (4.7)

\[
\tilde{H}_{\epsilon\mu}^{\alpha} h^{\epsilon\beta} = 0 \tag{4.9}
\]
One can now choose coordinates so that at a point \( x \in M \) one has \( h_{\mu\nu}(x) = \eta_{\mu\nu} \); one can prove algebraically that (4.9) implies \( H_{\mu\nu}^\alpha(x) = 0 \). This can be done at any point \( x \in M \) and since equation (4.9) is tensorial then \( \tilde{H}_{\mu\nu}^\alpha = 0 \) holds everywhere and in any coordinate system. Hence one has necessarily \( \Gamma^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} \); in other words the connection \( \Gamma^\alpha_{\beta\mu} \) that obeys (4.4) is necessarily in the form:

\[
\Gamma^\alpha_{\beta\mu} := \{ h \}^\alpha_{\beta\mu} + \frac{1}{2} \left( h^{\alpha\epsilon} h_{\beta\mu} - 2 \delta^\alpha_{(\beta} \delta^\epsilon_{\mu)} \right) \alpha, \tag{4.10}
\]

We are hence able to define:

**Definition:** a Further Extended Gravitational Theory (FEGT) as a metric-affine relativistic theory (i.e. a Lagrangian theory on a triple \( (M, g, \Gamma) \) with \( g \) a Lorentzian metric and \( \Gamma \) a linear connection a priori independent of \( g \)) defined by an action

\[
L = L(g, R(\Gamma)) + L_m(g, \Gamma, \phi) \tag{4.11}
\]

such that field equations imply EPS-compatibility, i.e. that

\[
\nabla_\mu \left( \sqrt{h} h^{\alpha\beta} \right) = \alpha_\mu \sqrt{h} h^{\alpha\beta} \tag{4.12}
\]

for some metric \( h = \varphi(\phi) \cdot g \) conformal to \( g \) and some 1-form \( \alpha = \alpha_\mu(g, \phi) dx^\mu \) functions of \( g \) and the matter field \( \phi \) (possibly together with their derivatives up to some finite order).

Notice that this encompasses usual \( f(R) \) theories (in Palatini formulation) in which the matter Lagrangian is usually assumed to be independent of the connection \( \Gamma \). We shall hereafter show that there are in fact non-trivial FEGT, i.e. FEGT other that the usual \( f(R) \) theories.

In such a kind of FEGT the connection \( \Gamma \) is defined as in (4.10). If the differential form \( \alpha := \alpha_\mu dx^\mu \) is not closed then the theory is non-trivial and its connection is non-metrical though it turns out to be EPS-compatible with the original metric structure \( g \) (or equivalently to any conformal metric structure \( h \)).

In such theories the lightlike geodesics coincide with the metric geodesics, but the timelike geodesics (i.e. worldlines of matter points) are exotic. This situation has been considered in literature (see [12]) though not in connection with EPS criteria. Of course, further investigations should be devoted to the possibility that such exotic dynamics can model anomalous rotation curves in galaxies and/or cosmological acceleration (as is already proven with \( f(R) \) theories) or be detected by solar system experiments.

### 5. A “Trivial” Example: \( f(R) \) Theories

We shall hereafter investigate the class of FETG. One class of examples is well-known: in any \( f(R) \) theory, in Palatini formulation one has a connection \( \Gamma^\alpha_{\beta\mu} = \{ h \}^\alpha_{\beta\mu} \) which is the Levi-Civita connection of a conformal metric \( h_{\mu\nu} = f'(R(\Gamma, g)) \cdot g_{\mu\nu} = f'(T) \cdot g_{\mu\nu} \), where \( T = T_{\mu\nu} g^{\mu\nu} \); see [13]. Thus \( \Gamma^\alpha_{\beta\mu} \), in view of (3.2), has the following expression:

\[
\Gamma^\alpha_{\beta\mu} = \{ f' g \}^\alpha_{\beta\mu} = \{ g \}^\alpha_{\beta\mu} - \frac{1}{2} \left( g^{\alpha\nu} g_{\mu\nu} - 2 \delta^\alpha_{(\beta} \delta^\epsilon_{\mu)} \right) \partial_\epsilon \ln f' \tag{5.1}
\]
Now we calculate the trace $\Gamma^\alpha_{\mu\nu}$:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \partial_\mu g_{\lambda\nu} - \frac{1}{2} (1 - 5) \partial_\mu \ln f' = \frac{1}{2} \partial_\mu \ln g + 2 \partial_\mu \ln f' = \partial_\mu \ln (f'' \sqrt{g}) = \partial_\mu \ln \sqrt{h}$$  \hspace{1cm} (5.2)

and thence $q_\epsilon = - \frac{1}{8} \partial_\epsilon \ln h$.

The connection $\Gamma_{\mu\nu}^\alpha$ is in fact compatible in the sense of EPS with the conformal structure identified by $g$.

To check this one can simply check that the identity (3.6)

$$\Gamma_{\beta\mu}^\alpha = \{g\}_{\beta\mu}^\alpha + \left( g^{\alpha\nu} g_{\beta\mu} - 2 \delta_\beta^\alpha \delta_\mu^\nu \right) \left( q_\epsilon + \frac{1}{8} \partial_\epsilon \ln g \right)$$  \hspace{1cm} (5.3)

holds true.

Hence, by comparing (5.1) and (5.3), one has to check that

$$- \frac{1}{2} \partial_\epsilon \ln f' = q_\epsilon + \frac{1}{8} \partial_\epsilon \ln g = - \frac{1}{8} \partial_\epsilon \ln h + \frac{1}{8} \partial_\epsilon \ln g = - \frac{1}{8} \partial_\epsilon \ln f' - \frac{1}{8} \partial_\epsilon \ln g + \frac{1}{8} \partial_\epsilon \ln g$$  \hspace{1cm} (5.4)

We can also check that compatibility condition (4.4) holds true directly; we have that

$$H_{\beta\mu}^\alpha = \Gamma_{\beta\mu}^\alpha - \{g\}_{\beta\mu}^\alpha = - \frac{1}{2} \left( g^{\alpha\nu} g_{\beta\mu} - 2 \delta_\beta^\alpha \delta_\mu^\nu \right) \partial_\epsilon \ln f'$$  \hspace{1cm} (5.5)

Thus we have

$$\nabla_\alpha (\sqrt{g} g^{\mu\nu}) = \nabla_\alpha (\sqrt{g} g^{\mu\nu}) + H_{\lambda\alpha}^\mu \sqrt{g} g^{\lambda\nu} + H_{\lambda\alpha}^{\lambda} \sqrt{g} g^{\mu\nu} - H_{\lambda\alpha}^{\lambda} \sqrt{g} g^{\mu\nu} =$$

$$= \frac{\sqrt{g}}{2} \left( - g^{\mu\nu} \delta_\alpha^\nu + \delta_\alpha^\nu g^{\mu\nu} + \delta_\alpha^\nu g^{\mu\nu} + (- g^{\mu\nu} \delta_\alpha^\nu + \delta_\alpha^\nu g^{\mu\nu} + \delta_\alpha^\nu g^{\mu\nu}) +$$

$$+ (\delta_\alpha^\nu g^{\mu\nu} - 4 \delta_\alpha^\nu g^{\mu\nu} - \delta_\alpha^\nu g^{\mu\nu}) \right) \partial_\epsilon \ln f' = - \sqrt{g} g^{\mu\nu} \partial_\epsilon \ln f' = \alpha_\alpha \sqrt{g} g^{\mu\nu}$$  \hspace{1cm} (5.6)

which is in fact in the expected form with $\alpha = - \partial_\epsilon \ln f' dx^\alpha = d(- \ln f')$. The form $\alpha$ is exact, in view of the fact that the connection $\Gamma$ is metric by construction.

6. Conclusions and Perspectives

We here reviewed EPS framework. In particular we corrected a misprint in the original paper and we proved identity (2.10) proving also that it is essentially unique (modulo two inessential parameters). We also find an equivalent, covariant characterization of EPS compatibility which will be expected to be consequence of field equations in FETG. In this models the connection is initally independent of the spacetime metric, but it is guaranteed to be on-shell EPS compatible with the metric.

EPS compatibility is used in two ways: first it enhances a physical interpretation in terms of observational quantities such as light rays and free falling mass particles. On the other hand, EPS compatibility allows to determine the connection in terms of metric and matter fields.

We finally showed that $f(R)$ are examples of FETG. We called this example as “trivial” FETG. In the second part (see [14]) we shall present and discuss non-trivial examples of FETG.
Appendix A. EPS Compatibility

EPS introduced in [1] relation (2.10) for $\Delta_{\mu
u}^\alpha$ using $L_{\alpha\beta\gamma} = \frac{4}{3} \Delta_{[\alpha\beta] \gamma} - p_{[\alpha} g_{\beta] \gamma}$, and $p_{\alpha} = -\frac{8}{3} \Delta_{(\alpha)} \lambda^\lambda$. Moreover they assert that:

$$L_{\lambda\beta\alpha} = 0 \quad L_{[\alpha\beta] \gamma} = 0 \quad L_{(\alpha\beta) \gamma} = 0 \quad (A.1)$$

which are used later on to deduce equation (2.13) which is important for deducing the compatibility condition (2.14). However, considering that:

$$L_{\beta\lambda\alpha} = \frac{2}{3} (\Delta_{\nu\beta\gamma} - \Delta_{\nu\gamma\beta}) + \frac{2}{3} (\Delta_{(\alpha} \lambda \beta \gamma - \Delta_{(\beta} \lambda \alpha \gamma}) \quad (A.2)$$

we find that the first relation of (A.1) is not verified:

$$L_{\beta\lambda\alpha} = \frac{2}{3} (\Delta_{\nu\beta\gamma} - \Delta_{\nu\gamma\beta}) + \frac{2}{3} (\Delta_{(\alpha} \lambda \beta \gamma - \Delta_{(\beta} \lambda \alpha \gamma}) = -\frac{1}{3} \Delta_{\nu\beta} \lambda^\lambda \neq 0 \quad (A.3)$$

Thus in order to obtain the correct definitions of the objects involved we started from general linear combinations

$$\tilde{L}_{\alpha\beta\gamma} = 2h_{\lambda\mu\nu} L_{\alpha\beta\gamma} + 2e_{\mu\nu} \tilde{g}_{\beta\gamma} \quad (A.4)$$

and we set $\tilde{p}_{\alpha} = 2a \Delta_{(\alpha)} \lambda^\lambda = a \Delta_{\alpha}$ and $\tilde{\Delta}_{\alpha} = \Delta_{\alpha\nu\mu} \tilde{g}^{\nu\mu}$. Then we determined the unknown coefficients $(a, b, c)$ so that the required properties (A.1) hold true.

The second and third property in (A.1) are trivially satisfied; for the first one we have:

$$\tilde{L}_{\beta\lambda\alpha} = b(\Delta_{\nu\beta\gamma} - \Delta_{\nu\gamma\beta}) + ac(\Delta_{(\alpha} \lambda \beta \gamma - \Delta_{(\beta} \lambda \alpha \gamma}) = -(3ac + b) \Delta_{\beta} \lambda^\lambda \equiv 0 \quad \Rightarrow \quad b = -3ac \quad (A.5)$$

So the general expression of $\tilde{L}_{\alpha\beta\gamma}$ and $\tilde{p}_{\alpha}$ that satisfies (A.1) is:

$$\tilde{L}_{\alpha\beta\gamma} = -2ac(3\Delta_{(\alpha\beta\gamma)} - \Delta_{(\alpha\beta\gamma}) \tilde{p}_{\alpha} = a \Delta_{\alpha} \quad (A.6)$$

These objects are defined in order to prove the identity (2.10). Let us consider a linear combination

$$\Delta_{\alpha\beta\gamma} = 3A \Delta_{(\alpha\beta\gamma)} + B \tilde{p}_{\alpha} g_{\beta\gamma} + 2C g_{\alpha(\beta} \tilde{p}_{\gamma)} + 2D L_{\alpha(\beta\gamma)} \quad (A.7)$$

and we search for the most general set of coefficients $(A, B, C, D)$ for which this is an identity.

By using (A.6) one finds:

$$\Delta_{\alpha\beta\gamma} = (A - 6acD) \Delta_{\alpha\beta\gamma} + (A + 3acD) \Delta_{\beta\gamma\alpha} + (A + 3acD) \Delta_{\gamma\alpha\beta} +$$

$$+ a(B + 2cD) \Delta_{\alpha} g_{\beta\gamma} + a(C - cD) \Delta_{\alpha} g_{\beta\gamma} + a(C - cD) \Delta_{\beta} g_{\gamma\alpha} \quad (A.8)$$

Choosing $(\alpha, \nu)$ as free parameters, the general solution is

$$\begin{cases}
A = \frac{1}{3} \\
c = -\frac{1}{3aD}
\end{cases} \quad \begin{cases}
C = -\frac{1}{3a} \\
B = \frac{2}{3a}
\end{cases} \quad (A.9)$$

and the identity is found in the form

$$\Delta_{\alpha\beta\gamma} = \Delta_{(\alpha\beta\gamma)} + \frac{2}{3a} (\tilde{p}_{\alpha} g_{\beta\gamma} - g_{\alpha(\beta} \tilde{p}_{\gamma)}) + 2 DL_{\alpha(\beta\gamma)} \quad (A.10)$$

where $(a, D)$ are free parameters. Above in the paper we chose the particular solution $a = \frac{1}{9}$, $D = \frac{1}{2}$ and hence

$$\begin{cases}
A = \frac{1}{3} \\
c = -\frac{1}{2} \\
C = -\frac{1}{4} \\
B = \frac{2}{3}
\end{cases} \quad (A.11)$$
Acknowledgments

We wish to thank G. Magnano for useful discussions. This work is partially supported by MIUR: PRIN 2005 on *Leggi di conservazione e termodinamica in meccanica dei continui e teorie di campo*. We also acknowledge the contribution of INFN (Iniziativa Specifica NA12) and the local research funds of Dipartimento di Matematica of Torino University.

References

[1] J. Ehlers, F. A. E. Pirani, A. Schild, *The Geometry of Free Fall and Light Propagation*, in General Relativity, ed. L. O'Raifeartaigh (Clarendon, Oxford, 1972).
[2] S. Capozziello, M. De Laurentis, V. Faraoni *A bird’s eye view of f(R)-gravity* (2009); arXiv:0909.3672
[3] T. P. Sotiriou, V. Faraoni, *f(R) theories of gravity*, (2008); arXiv: 0805.1720v2
[4] T. P. Sotiriou, S. Liberati, *Metric-affine f(R) theories of gravity*, Annals Phys. 322 (2007) 935-966; gr-qc/0604006
[5] T. P. Sotiriou, *Modified Actions for Gravity: Theory and Phenomenology*, Ph.D. Thesis; gr-qc/0710.4438
[6] S. Capozziello, M. De Laurentis, M. Francaviglia, S. Mercadante, *First Order Extended Gravity and the Dark Side of the Universe: the General Theory* Proceedings of the Conference “Universe Invisible”, Paris June 29 July 3, 2009 - to appear in 2010
[7] S. Capozziello, M. De Laurentis, M. Francaviglia, S. Mercadante, *First Order Extended Gravity and the Dark Side of the Universe II: Matching Observational Data*, Proceedings of the Conference “Universe Invisible”, Paris June 29 July 3, 2009 to appear in 2010
[8] E. Barausse, T. P. Sotiriou, J. C. Miller, *A no-go theorem for polytropic spheres in Palatini f(R) gravity*, Class. Quant. Grav. 25 (2008) 062001; gr-qc/0703132
[9] S. Capozziello, M. Francaviglia, *Extended Theories of Gravity and their Cosmological and Astrophysical Applications*, Journal of General Relativity and Gravitation 40 (2-3), (2008) 357-420.
[10] S. Capozziello, M. F. De Laurentis, M. Francaviglia, S. Mercadante, *From Dark Energy and Dark Matter to Dark Metric*, Foundations of Physics 39 (2009) 1161-1176 gr-qc/0805.3642v4
[11] V. Perlick, *Characterization of standard clocks by means of light rays and freely falling particles* General Relativity and Gravitation, 19(11) (1987) 1059-1073
[12] T. P. Sotiriou, *f(R) gravity, torsion and non-metricity*, Class. Quant. Grav. 26 (2009) 152001; gr-qc/0904.2774
[13] G. Magnano, L. M. Sokolowski, *On Physical Equivalence between Nonlinear Gravity Theories* Phys.Rev. D50 (1994) 5039-5059; gr-qc/9312008
[14] L. Fatibene, M. Ferraris, M. Francaviglia, S. Mercadante, *Further Extended Theories of Gravitation: Part II*, arXiv:0911.2842