On possible Chern Classes of stable Bundles on Calabi-Yau threefolds

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Abstract

Supersymmetric heterotic string models, built from a Calabi-Yau threefold $X$ endowed with a stable vector bundle $V$, usually lead to an anomaly mismatch between $c_2(V)$ and $c_2(X)$; this leads to the question whether the difference can be realized by a further bundle in the hidden sector. In math.AG/0604597 a conjecture is stated which gives sufficient conditions on cohomology classes on $X$ to be realized as the Chern classes of a stable reflexive sheaf $V$; a weak version of this conjecture predicts the existence of such a $V$ if $c_2(V)$ is of a certain form. In this note we prove that on elliptically fibered $X$ infinitely many cohomology classes $c \in H^4(X, \mathbb{Z})$ exist which are of this form and for each of them a stable $SU(n)$ vector bundle with $c = c_2(V)$ exists.

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1 Introduction

To get \( N = 1 \) heterotic string models in four dimensions one compactifies the tendimensional heterotic string on a Calabi-Yau threefold \( X \) which is furthermore endowed with a polystable holomorphic vector bundle \( V' \). Usually one takes \( V' = (V, V_{hid}) \) with \( V \) a stable bundle considered to be embedded in (the visible) \( E_8 \) (\( V_{hid} \) plays the corresponding role for the second hidden \( E_8 \)); the commutator of \( V \) gives the unbroken gauge group in four dimensions.

The most important invariants of \( V \) are its Chern classes \( c_i(V) \), \( i = 0, 1, 2, 3 \). We consider in this note bundles with \( c_0(V) = rk(V) = n \) and \( c_1(V) = 0 \); more specifically we will consider \( SU(n) \) bundles. The net number of generations of chiral particle multiplets in the fourdimensional effective theory is given by \( N_{gen}(V) = c_3(V)/2 \). On the other hand the second Chern class is important to assure anomaly freedom of the construction: this is encoded in the integrability condition for the existence of a solution to the anomaly cancellation equation

\[
c_2(X) = c_2(V) + W. \tag{1.1}
\]

Here \( W \), as it stands, has just the meaning to indicate a possible mismatch for a certain bundle \( V \); it can be understood either as the cohomology class of (the compact part of the world-volume of) a fivebrane, or as second Chern class of a further stable bundle \( V_{hid} \) in the hidden sector. Furthermore in the first case the class of \( W \) has to be effective for supersymmetry to be preserved.

Often one will argue just from the data provided by the Chern classes, say to secure a certain phenomenologically favored generation number, and so has to make sure that a corresponding \( SU(n) \) bundle with suitably prescribed Chern class \( c_3(V) \) actually exists. On the other hand, following the route to solve (1.1) described, one has the same problem for \( c_2(V_{hid}) = W := c_2(X) - c_2(V_{vis}) \) concerning the hidden bundle.

In [4] it has been shown that whenever the topological constraint can be satisfied with \( W = 0 \) then \( X \) and \( V \) can be deformed to a solution of the anomaly equation even already on the level of differential forms (a solution to the system involving the three-form field-strength \( H \), investigated first in [5], exists).

This leads to the general question to give sufficient conditions for the existence of stable bundles with prescribed Chern classes \( c_2(V) \) and \( c_3(V) \). Concerning this the following conjecture is put forward in [1] by Douglas, Reinbacher and Yau (DRY) (actually we use the particular case of the conjecture with \( c_1(V) = 0 \)).
**DRY-Conjecture.** On a Calabi-Yau threefold $X$ of $\pi_1(X) = 0$ a stable reflexive sheaf $V$ of rank $n$ and $c_1(V) = 0$ with prescribed Chern classes $c_2(V)$ and $c_3(V)$ will exist if, for an ample class $H \in H^2(X, \mathbb{R})$, these can be written as (where $C := 16\sqrt{2}/3$)

\[ c_2(V) = n \left( H^2 + \frac{c_2(X)}{24} \right) \tag{1.2} \]

\[ c_3(V) < C nH^3. \tag{1.3} \]

Note that the conjecture just predicts the existence of a stable reflexive sheaf; in our examples below $V$ will be a vector bundle.

We will also formulate a weaker version of the conjecture, which is implied by the proper (strong) form and concentrates on the existence of $V$ given that just its (potential) $c_2(V)$ fulfills the relevant condition. To refer more easily to the notions involved, we make first the following definitions. We restrict to the case of $V$ being a vector bundle. We will consider rank $n$ bundles of $c_1(V) = 0$ and treat actually the case of $SU(n)$ vector bundles.

**Definition.** Let $X$ be a Calabi-Yau threefold of $\pi_1(X) = 0$ and $c \in H^4(X, \mathbb{Z})$,

i) $c$ is called a Chern class if a stable $SU(n)$ vector bundle $V$ on $X$ exists with $c = c_2(V)$

ii) $c$ is called a DRY class if an ample class $H \in H^2(X, \mathbb{R})$ exists (and an integer $n$) with

\[ c_2(V) = n \left( H^2 + \frac{c_2(X)}{24} \right). \tag{1.4} \]

With these definitions we can now state the weak DRY conjecture, in the framework as we will use it, as follows:

**Weak DRY-Conjecture.** On a Calabi-Yau threefold $X$ of $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbb{Z})$ is a Chern class.

Here it is understood that the integer $n$ occurring in the two definitions is the same.

The paper has three parts. In section 2 we give criteria for a class to be a DRY class. In section 3 we present some bundle constructions and show that their $c_2(V)$ fulfill these criteria for infinitely many $V$. In section 4 we give an application in a physical set-up.

# 2 DRY classes on elliptic Calabi-Yau threefolds

To test these conjectures we choose $X$ to be elliptically fibered over the base surface $B$ with section $\sigma : B \to X$ (we will also denote by $\sigma$ the embedded subvariety $\sigma(B) \subset X$ and its cohomology class in $H^2(X, \mathbb{Z})$), a case particularly well studied. The typical examples for $B$ are rational surfaces like a Hirzebruch surface $F_k$ (where we consider
the following cases \( k = 0, 1, 2 \) as only for these bases exists a smooth elliptic \( X \) with Weierstrass model, a del Pezzo surface \( dP_k \) \((k = 0, \ldots, 8)\) or the Enriques surface (or suitable blow-ups of these cases). We will consider specifically bases \( B \) for which \( c_1 := c_1(B) \) is ample. This excludes in particular the Enriques surface and the Hirzebruch surface\(^3 \) \( F_2 \). (The classes \( c_1^2 \) and \( c_2 := c_2(B) \) will be considered as (integral) numbers.)

On the elliptic Calabi-Yau space \( X \) one has according to the general decomposition \( H^4(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \sigma \oplus H^4(B, \mathbb{Z}) \) the decompositions (with \( \phi, \rho \in H^2(X, \mathbb{Z}) \))

\[
\begin{align*}
c_2(V) &= \phi \sigma + \omega \\
c_2(X) &= 12c_1 + c_2 + 11c_1^2
\end{align*}
\]

where \( \omega \) is understood as an integral number (pullbacks from \( B \) to \( X \) will be usually suppressed).

One now solves for \( H = a \sigma + \rho \) (using the decomposition \( H^2(X, \mathbb{Z}) \cong \mathbb{Z} \sigma \oplus H^2(B, \mathbb{Z}) \)), given an arbitrary but fixed class \( c = \phi \sigma + \omega \in H^4(X, \mathbb{Z}) \), and has then to check that \( H \) is ample. The conditions for \( H \) to be ample are (cf. appendix)

\[
H \text{ ample } \iff a > 0, \quad \rho - ac_1 \text{ ample.}
\tag{2.3}
\]

Inserting \( H \) into equ. (1.4) one gets the following relations

\[
\begin{align*}
\phi &= 2an\rho + n\left(\frac{1}{2} - a^2\right)c_1 \\
\omega &= \rho^2 + \frac{5}{12}c_1^2 + \frac{1}{2}
\end{align*}
\tag{2.4}
\tag{2.5}
\]

(note that \( \sigma^2 = -c_1 \sigma \), cf. [2]) where in (2.5) Noethers relation \( c_2 + c_1^2 = 12 \) for the rational surface \( B \) has been used. This implies

\[
\rho = \frac{1}{2an}\left(\phi - n\left(\frac{1}{2} - a^2\right)c_1\right)
\tag{2.6}
\]

As we assumed that \( c_1 \) is ample one finds that the condition that the class

\[
\rho - ac_1 = \frac{1}{2an}\left(\phi - n\left(\frac{1}{2} + a^2\right)c_1\right)
\tag{2.7}
\]

is ample leads also to an upper bound on the positive real number \( a^2 \)

\[
0 < a^2 < b
\tag{2.8}
\]

This bound will be specified in an example below explicitly. Furthermore equ. (2.7) shows that \( \phi - \frac{a}{2}c_1 \) must necessarily be ample.

\(^3\)as \( c_1 \cdot b = (2b + 4f) \cdot b = 0 \), using here the notations from footn. 6
Having solved by now (2.4) in terms of $\rho$ one now has to solve the following equation

$$\omega = \frac{1}{4a^2n^2}\left(\phi - n\left(\frac{1}{2} - a^2\right)c_1\right)^2 + \frac{5}{12}c_1^2 + \frac{1}{2}$$

(2.9)

in terms of $a$. Actually the only non-trivial point will be that $a$ is real and satisfies (2.8). Concretely one gets a quadratic equation $a^4 + pa^2 + q = 0$ in $a^2$ with

$$p = -\frac{4}{c_1^2}(\omega - r)$$

(2.10)

$$q = \frac{4}{(c_1^2)^2}s^2$$

(2.11)

where we used the abbreviations

$$r := \frac{1}{2n}\phi c_1 + \frac{1}{6}c_1^2 + \frac{1}{2}$$

(2.12)

$$s := \frac{1}{2n}\sqrt{c_1^2}\sqrt{\left(\phi - \frac{n}{2}c_1\right)^2}$$

(2.13)

Now one has three conditions which have to be satisfied by at least one solution $a^2_*$ of this equation, namely$^4$

$$i) \quad a^2_* \in \mathbb{R} \iff p^2 \geq 4q \iff (\omega - r)^2 \geq s^2$$

(2.14)

$$ii) \quad a^2_* > 0 \iff -p > 0 \iff \omega > r$$

(2.15)

$$iii) \quad a^2_* \leq b \iff \begin{cases} -p < b + \frac{q}{b} & \text{for } +\sqrt{\text{ and } b \geq -\frac{q}{2}} \\ \text{arbitrary} & \text{for } -\sqrt{\text{ and } b \geq -\frac{q}{2}} \\ -p > b + \frac{q}{b} & \text{for } -\sqrt{\text{ and } b < -\frac{q}{2}} \end{cases}$$

(2.16)

Concerning ii) note that necessarily $q > 0$, cf. the remark after (2.8); furthermore the evaluation of the condition is independent of the question which sign for the square root is taken. Note that in iii) the case where $+\sqrt{\text{ is taken and } b < -\frac{q}{2}}$ is excluded.

Concerning condition iii) note that for $b \geq -p/2$ one gets no further restriction and has to pose in total just the first two conditions, i.e. $\omega \geq r + s$. By contrast for $b < -p/2$ the condition $-p > b + \frac{q}{b}$, or equivalently $\omega > \omega_0(\phi; b) := r + \frac{c_1^2}{4}(b + \frac{q}{b})$, implies i) and ii).

As $p$ (and thus $\omega$ itself) occurs in the domain restrictions on $b$ one has to rewrite these conditions slightly. Let us consider first the regime $b < -\frac{p}{2}$ which means explicitly $\omega > r + \frac{b^2c_1^2}{2}$. Now one has to distinguish again two cases: the ensuing condition $\omega > \omega_0(\phi; b)$ makes sense as an (additional) condition (which would be to be required besides the domain restriction $\omega > r + \frac{b^2c_1^2}{2}$) only if $r + \frac{b^2c_1^2}{2} < \omega_0(\phi; b)$, i.e. for $b < \sqrt{q}$; on the other hand for $b \geq \sqrt{q}$ one just has to demand $\omega > r + \frac{b^2c_1^2}{2}$.

$^4$here it is understood that ii) assumes i) satisfied and iii) assumes i) and ii) satisfied
In the second regime $b \geq -p/2$, or equivalently $\omega \leq r + \frac{b}{2}c_1^2$, one has the condition $\omega \geq r + s$ (note that these two conditions are compatible just for $s \leq \frac{b}{2}c_1^2$, i.e. $b \geq \sqrt{q}$). Thus in total one can make an $\omega$-independent regime distinction for $b$ according to $b < \sqrt{q}$ (with the demand $\omega > \omega_0$) or $b \geq \sqrt{q}$ (where one should have either $\omega > r + \frac{b}{2}c_1^2$ or $\omega \leq r + \frac{b}{2}c_1^2$ but in that latter case one has to demand $\omega \geq r + s$; but, as $r + s \leq r + \frac{b}{2}c_1^2$ in the present $b$-regime, one just has to demand that $\omega \geq r + s$).

Note in this connection that, for $\phi$ held fixed, $\omega_0(\phi; b)$ becomes large for small and large $b$ and the intermediate minimum is achieved at $b_{\text{min}} = \sqrt{q}$. As the condition $\omega > \omega_0(\phi; b)$ is relevant only for $b < \sqrt{q}$ whereas for $b > \sqrt{q}$ one gets the condition $\omega > r + s$ and as one has $\omega_0(\phi, b_{\text{min}}) = r + s$ there is a smooth transition in the conditions; and furthermore, all in all, $\omega \geq r + s$ is a necessary condition for a class to be a DRY class.

Therefore we get the following theorem

**Theorem on DRY classes.** For a class $c = \phi\sigma + \omega \in H^4(X, \mathbb{Z})$ to be a DRY class one has the following conditions (where $b$ is some $b \in \mathbb{R}^\geq$ and $\omega \in H^4(B, \mathbb{Z}) \cong \mathbb{Z}$):

a) **necessary and sufficient**: $\phi - n(\frac{1}{2} + b)c_1$ is ample and

$$\begin{cases} 
\omega \geq r + s & \text{for } b \geq \sqrt{q} \\
\omega > \omega_0(\phi; b) & \text{for } b < \sqrt{q}
\end{cases}$$

b) **sufficient**: $\phi - \frac{n}{2}c_1$ is ample and $\omega$ sufficiently large

c) **necessary**: $\phi - \frac{n}{2}c_1$ is ample and $\omega \geq r + s$.

Here part b) follows immediately from a) as the ample cone of $B$ is an open set. So in particular the condition on $\omega$ can be fulfilled in any bundle construction which contains a (discrete) parameter $\mu$ in $\omega$ such that $\omega$ can become arbitrarily large if $\mu$ runs in its range of values (this strategy will be used for spectral and extension bundles).

Let us discuss further the conditions on $\phi$ and $\omega$ given in the theorem for $c = \phi\sigma + \omega$ to be a DRY class. As the notion of $c$ being a DRY class does not involve any $b$ one should compare these conditions for different $b$. One then realises that as $b$ becomes larger the condition on $\phi$ becomes stronger and stronger; on the other hand as $b$ increases from 0 to $\sqrt{q}$ (for a fixed $\phi$) the condition on $\omega$ becomes weaker first, and then, from $\sqrt{q}$ on, remains unchanged. From this consideration one learns that it is enough to use $b$’s in the interval $0 < b \leq \sqrt{q}$ as test parameters. That is the set of DRY classes is the union of allowed ranges of $\phi$ and $\omega$ for all these $b$.

**Remark:** Note that, although for $b_{\text{min}}$ the condition on $\omega$ is as weak as possible, $\phi - n(\frac{1}{2} + b_{\text{min}})c_1$ might not be ample (cf. the example of the tangent bundle given below); nevertheless the ampleness condition on $\phi$ might be satisfied for a $b$ where the bound $\omega \geq \omega_0(\phi; b)$ turns out to be more stringent (cf. again the example).\(^5\)

\(^5\)Note also the following property of $b_{\text{min}}(\phi)$: the zero class lies in the boundary of the ample cone;
3 Examples for the DRY-Conjecture

In this section we give examples of cohomology classes which are of DRY-form and appear as second Chern classes of stable $SU(n)$ vector bundles.

3.1 The tangent bundle

Let us see whether the cohomology class given by the second Chern class of the tangent bundle $T_X$ is detected by the weak DRY-Conjecture as a Chern class. For this we apply the Theorem above to see whether $c^2(X)$ is a DRY class. The minimum of $\omega_0(\phi; b)$ is taken at $b_{\min} = 7/2$ but one finds $0 < b < 7/2$ as the allowed range for $b$ ($c_1$ was assumed ample); so although $\omega_{T_X} = 10c_1^2 + 12 \geq \omega_0(12c_1, b_{\min}) = \frac{47}{12}c_1^2 + \frac{1}{2}$ is fulfilled one has to take another $b$ which makes the bound $\omega \geq \omega_0(12c_1; b)$ more stringent; but $b = 3$, say, where the $\omega_0$ becomes $\frac{49}{24} + \frac{11}{2}$, will do. So $c_2(X)$ is a DRY class and the weak DRY-conjecture is fulfilled; as $c_3(X) = -60c_1^2$ is negative actually even the (proper) DRY-conjecture is true.

3.2 Spectral bundles

In case of spectral cover bundles [2] one has the following expression for $\omega$

$$\omega = (\lambda^2 - \frac{1}{4}) \frac{n}{2} \phi(\phi - nc_1) - \frac{n^3 - n}{24} c_1^2$$

(3.1)

Here $\phi$ is an effective class in $B$ with $\phi - nc_1$ also effective and $\lambda$ is a half-integer satisfying the following conditions: $\lambda$ is strictly half-integral for $n$ being odd; for $n$ even an integral $\lambda$ requires $\phi \equiv c_1$ (mod 2) while a strictly half-integral $\lambda$ requires $c_1$ even. (In addition one has to assume that the linear system $|\phi|$ is base point free.)

Often one assumes, as we will do here, that $\phi - nc_1$ is not only effective but even ample in $B$. Then equ. (2.7) shows that we can take $b = 1/2$ as upper bound on $a$.

One has now to check whether the three conditions on $a^2$ given above can be fulfilled. According to part b) of the theorem in section 2 one learns that this is the case as $\omega$ increases to arbitrarily large values when the parameter $\lambda$ is increasing.

so, if the condition on $\phi$ is considered for this limiting case, one finds $\phi - \frac{c_1}{2} = bnc_1$ such that $b = b_{\min}(\phi)$.

6 a base point is a point common to all members of the system $|\phi|$ of effective divisors which are linearly equivalent to the divisor $\phi$ (note that on $B$ the cohomology class $\phi$ specifies uniquely a divisor class); on $B$ a Hirzebruch surface $F_k$ with base $P^1$ $b$ and fibre $P^1$ $f$ this amounts to $\phi \cdot b \geq 0$
**Theorem.** i) On $X$ an elliptic Calabi-Yau threefold the class $c_2(V) = c = \phi\sigma + \omega$ for $V$ a spectral bundle (of discrete bundle parameters $\eta \in H^2(B, \mathbb{Z})$ and $\lambda \in \frac{1}{2}\mathbb{Z}$) satisfies the assumptions of the weak DRY-Conjecture on $c$ for all but finitely many values of the parameter $\lambda$.

ii) For the infinitely many classes $c \in H^4(X, \mathbb{Z})$ described in i) the weak DRY-Conjecture is true.

iii) For the classes in ii) with negative $\lambda$ the (proper) DRY-Conjecture is true.

Here part ii) follows of course just from reversing the perspective: whereas in part i) one started from a given spectral bundle $V$ and found a condition ($\lambda^2$ sufficiently large) that its $c_2(V)$ fulfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has trivially confirmed the existence of a stable bundle for a $c = c_2(V)$ which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as $c_3(V) = 2\lambda\phi(\phi - nc_1)$ is negative for $\lambda$ negative as $\phi \neq 0$ is effective and $\phi - nc_1$ was assumed ample, so $\phi(\phi - nc_1)$ is positive (this argument underlies of course already part ii) as well).

### 3.3 Extension bundles

Stable vector bundles built as an extension of given stable bundles have been constructed on elliptic Calabi-Yau threefolds in [3]. Let $E$ be a rank $r$ $H_B$-stable vector bundle on the base $B$ of the Calabi-Yau space with Chern classes $c_1(E) = 0$ and $c_2(E) = k$. The pullback bundle $\pi^*E$ is then shown to be stable on $X$ with respect to the ample class $J = z\sigma + H_B$ where $H_B = hc_1$ (with $h \in \mathbb{R}_{>0}$) [3]. The bundle extension

$$0 \to \pi^*E \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(rD) \to 0 \tag{3.2}$$

with $D = x\sigma + \alpha$ defines a stable rank $n = r + 1$ vector bundle if the numerical condition equ. (3.4) is satisfied. We consider here the case $x = -1$ for simplicity. For this bundle $c = \phi\sigma + \omega$ is given by

$$\phi = (n - 1)\frac{n}{2}(2\alpha + c_1), \quad \omega = k - (n - 1)\frac{n}{2}\alpha^2 \tag{3.3}$$

As in the spectral case one now has to check whether the three conditions on $\alpha^2$ given in section 2 can be fulfilled. This is the case according to part b) of the theorem in section 2 if $\alpha$ is chosen such that $2(n - 1)\alpha + (n - 2)c_1$ is ample and $k$ is chosen sufficiently large. Note that this is in agreement with the condition that the extension can be chosen nonsplit if

$$\frac{n - 1}{2} \left[n^2(\alpha(\alpha + c_1) + \frac{c_1^2}{3}) - c_1(2\alpha + \frac{c_1}{3}) + 1\right] - k < 0 \tag{3.4}$$
As above in the spectral bundle case we get here the following result.

**Theorem.** i) On $X$ an elliptic Calabi-Yau threefold the class $c_2(V) = c = \phi \sigma + \omega$ for $V$ an extension bundle (of discrete bundle parameters $\alpha \in H^2(B, \mathbb{Z})$ and $k \in \mathbb{Z}$) satisfies the assumptions of the weak DRY-Conjecture on $c$ for all but finitely many values of the parameter $k$.

ii) For the infinitely many classes $c \in H^4(X, \mathbb{Z})$ described in i) the weak DRY-Conjecture is true.

iii) For infinitely many classes $c \in H^4(X, \mathbb{Z})$ the (proper) DRY-Conjecture is true.

As above section 3.2, part ii) follows from reversing the perspective: whereas in part i) one started from a given extension bundle $V$ and found a condition ($k$ sufficiently large) that its $c_2(V)$ fullfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has now trivially confirmed the existence of a stable bundle for a $c = c_2(V)$ which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as $c_3(V) = -(n-1)(n-2)/3 (c_1^2 + 3\alpha(\alpha + c_1)) - 2k < 0$ for $k$ sufficiently large.

4 Application

Let us finally come back to the motivating question from the introduction. We will take a stable bundle in the visible sector $V_{vis}$ of the heterotic string and want to supplement this by a stable bundle $V_{hid}$ of rank $n_h$ such that the anomaly condition $c_2(V_{vis}) + c_2(V_{hid}) = c_2(X)$ is satisfied. To assure the existence of $V_{hid}$ we will assume the weak DRY conjecture. So, concretely we will check whether $c := c_2(X) - c_2(V_{vis})$ is a DRY class.

Let us take $V_{vis} = \pi^*E$ where $E$ on $B$ is a bundle with $c_2(E) = k$, stable with respect to the ample class $H_B$ on $B$. Thus in this case we have

$$\phi = 12c_1, \quad \omega = 10c_1^2 + 12 - k$$

and furthermore one gets the explicit expression for the bound

$$\omega_0 = \left[\frac{6}{n_h} + \frac{1}{6} + \frac{b}{4} + \frac{(12 - \frac{n_h}{2})^2}{4b^2n_h^2}\right]c_1^2 + \frac{1}{2}.$$  

(4.2)

We will use part b) of the theorem of section 2. We get $12 - n_h(\frac{1}{2} + b) > 0$ from the amplenes condition on $\phi$ and $\omega \geq \omega_0$ as further condition; here we assume that we are in the regime $b < \sqrt{q} = \frac{12 - \frac{n_h}{2}}{n_h}$. Note further that the DRY conjecture does not specify a polarization with respect to which $V_{hid}$ will be stable; so $V_{vis}$ should be stable with respect
to an arbitrary ample class; this is true in our case $V_{vis} = \pi^*E$ on $B = \mathbb{P}^2$ according to Lemma 5.1 of [3].

Thus for example, for $n_h = 4$ and $b = \frac{1}{2}$ one finds that $c$ is for $k \leq 11$ a DRY class.

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A Ample classes on elliptic Calabi-Yau threefolds

Let $H = a\sigma + \rho \in H^2(X, \mathbb{R}) \cong \mathbb{R}\sigma + H^2(B, \mathbb{R})$ be a class on the elliptic Calabi-Yau threefold $X$. Then one has if $c_1$ is ample

$$H \text{ ample } \iff a > 0, \rho - ac_1 \text{ ample.} \quad (A.1)$$

Consider first the $\implies$ direction: one has $a = H \cdot F > 0$ according to the Nakai-Moishezon criterion that $H$ is ample just if $H^3 > 0, H^2 \cdot S > 0, H \cdot C > 0$ for all irreducible surfaces $S$ and irreducible curves $C$ in $X$; here this is applied to the fibre $F$. Furthermore, if $c$ is an irreducible curve in $B$ one has $(\rho - ac_1) \cdot c = H \cdot c\sigma > 0$; and one also has $(\rho - ac_1)^2 = H^2 \cdot \sigma > 0$, such that by the same criterion, applied now on $B$, indeed the class $\rho - ac_1$ is ample.

Consider now the $\impliedby$ direction: the class of an irreducible curve $C$ in $X$ is built from the class $F$ and non-negative linear combinations of classes of the form $c\sigma$, where $c$ is now the class of an irreducible curve in $B$; therefore, turning the previous arguments around, one ends up indeed with $H \cdot C > 0$. The classes of irreducible surfaces are in a similar way built from $\sigma$ and the $\pi^*c$; for $H^2 \cdot \sigma$ one can again turn around the previous argument; this is not so however for $H^2 \cdot \pi^*c = ac(2\rho - ac_1)$; in this case we adopt the additional assumption that $c_1$ is ample, which implies that $\rho$, and therefore $2\rho - ac_1$ too, is also ample to get the required conclusion. Similarly one concludes for $H^3 = a[\rho^2 + (\rho - ac_1)(2\rho - ac_1)]$.

B Examples of one-parameter Calabi-Yau spaces

Although we treat in the main body of the paper the case of elliptic Calabi-Yau spaces $X$ let us briefly comment here on the simpler case where $X$ is a one-parameter space, i.e., $h^{1,1}(X) = 1$. 

10
In this case one has the representations (with \(k, t \in \mathbb{Z}\))

\[
c = kJ^2 \quad \text{(B.1)}
\]

\[
c_2(X) = tJ^2 \quad \text{(B.2)}
\]

where \(J\) is a generating element of \(H^2(X, \mathbb{Z})\); for the ample class \(H\) one has \(H = hJ\) with \(h \in \mathbb{R}_{>0}\).

The condition for a class \(c\) to have DRY form becomes here

\[
k = n\left(h^2 + \frac{t}{24}\right) \quad \text{(B.3)}
\]

This amounts to the condition

\[
k > \frac{n}{24}t \quad \text{(B.4)}
\]

whereas the necessary Bogomolov inequality \(c \cdot J > 0\) gives just \(k > 0\) (for example on the quintic one gets the stronger condition \(k > \frac{n}{24}t\)). Note that the second Chern class of the tangent bundle always has DRY-form; thus for this cohomology class the weak DRY-conjecture is satisfied, and for negative Euler number even the (proper) DRY-conjecture.

Some examples are provided by the complete intersection spaces \(\mathbb{P}^4(5), \mathbb{P}^5(2, 4), \mathbb{P}^5(3, 3), \mathbb{P}^6(2, 2, 3), \mathbb{P}^7(2, 2, 2, 2)\) with \(t = 10, 7, 6, 5, 4\) and Euler numbers \(-200, -176, -144, -144, -128\). (similarly one can discuss the one parameter cases \(\mathbb{P}_{2,1,1,1,1}(6), \mathbb{P}_{4,1,1,1,1}(8), \mathbb{P}_{5,2,1,1,1,1}(10)\)).

On the quintic one has some further bundles, occurring in the list in [6], with \(c_2(V) = c_2(X)\) with some of them (the first five examples) shown to be stable in [7], which have the same \(t\) as \(TX\) and also negative \(c_3(V)\); thus these provide further examples of the weak DRY-conjecture and actually even of the (proper) DRY-Conjecture.

Physically one has to demand in addition anomaly cancellation. Thus one gets then in total the condition

\[
n\frac{t}{24} < k \leq t \quad \text{(B.5)}
\]

(note that one has here \(k_{hid} > 0\) for a potential hidden bundle from the Bogomolov inequality).

For the generation number one gets, in the framework of the assumptions of the DRY conjecture, the bound

\[
N_{\text{gen}} < C\frac{n}{2}\left(\frac{k}{n} - \frac{t}{24}\right)^{3/2}. \quad \text{(B.6)}
\]
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On possible Chern Classes of stable Bundles on Calabi-Yau threefolds

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Abstract

Supersymmetric heterotic string models, built from a Calabi-Yau threefold $X$ endowed with a stable vector bundle $V$, usually lead to an anomaly mismatch between $c_2(V)$ and $c_2(X)$; this leads to the question whether the difference can be realized by a further bundle in the hidden sector. In math.AG/0604597 a conjecture is stated which gives sufficient conditions on cohomology classes on $X$ to be realized as the Chern classes of a stable reflexive sheaf $V$; a weak version of this conjecture predicts the existence of such a $V$ if $c_2(V)$ is of a certain form. In this note we prove that on elliptically fibered $X$ infinitely many cohomology classes $c \in H^4(X, \mathbb{Z})$ exist which are of this form and for each of them a stable $SU(n)$ vector bundle with $c = c_2(V)$ exists.

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1 Introduction

To get $N = 1$ heterotic string models in four dimensions one compactifies the tendimensional heterotic string on a Calabi-Yau threefold $X$ which is furthermore endowed with a polystable holomorphic vector bundle $V'$. Usually one takes $V' = (V, V_{hid})$ with $V$ a stable bundle considered to be embedded in (the visible) $E_8$ ($V_{hid}$ plays the corresponding role for the second hidden $E_8$); the commutator of $V$ gives the unbroken gauge group in four dimensions.

The most important invariants of $V$ are its Chern classes $c_i(V)$, $i = 0, 1, 2, 3$. We consider in this note bundles with $c_0(V) = \text{rk}(V) = n$ and $c_1(V) = 0$; more specifically we will consider $SU(n)$ bundles. The net number of generations of chiral particle multiplets in the fourdimensional effective theory is given by $N_{\text{gen}}(V) = c_3(V)/2$. On the other hand the second Chern class is important to assure anomaly freedom of the construction: this is encoded in the integrability condition for the existence of a solution to the anomaly cancellation equation

$$c_2(X) = c_2(V) + W. \quad (1.1)$$

Here $W$, as it stands, has just the meaning to indicate a possible mismatch for a certain bundle $V$; it can be understood either as the cohomology class of (the compact part of the world-volume of) a fivebrane, or as second Chern class of a further stable bundle $V_{hid}$ in the hidden sector. Furthermore in the first case the class of $W$ has to be effective for supersymmetry to be preserved.

Often one will argue just from the data provided by the Chern classes, say to secure a certain phenomenologically favored generation number, and so has to make sure that a corresponding $SU(n)$ bundle with suitably prescribed Chern class $c_3(V)$ actually exists. On the other hand, following the route to solve (1.1) described, one has the same problem for $c_2(V_{hid}) = W := c_2(X) - c_2(V_{vis})$ concerning the hidden bundle.

In [4] it has been shown that whenever the topological constraint can be satisfied with $W = 0$ then $X$ and $V$ can be deformed to a solution of the anomaly equation even already on the level of differential forms (a solution to the system involving the three-form field-strength $H$, investigated first in [5], exists).

This leads to the general question to give sufficient conditions for the existence of stable bundles with prescribed Chern classes $c_2(V)$ and $c_3(V)$. Concerning this the following conjecture is put forward in [1] by Douglas, Reinbacher and Yau (DRY) (actually we use the particular case of the conjecture with $c_1(V) = 0$).
**DRY-Conjecture.** On a Calabi-Yau threefold $X$ of $\pi_1(X) = 0$ a stable reflexive sheaf $V$ of rank $n$ and $c_1(V) = 0$ with prescribed Chern classes $c_2(V)$ and $c_3(V)$ will exist if, for an ample class $H \in H^2(X, \mathbb{R})$, these can be written as (where $C := 16\sqrt{2}/3$)

$$c_2(V) = n \left( H^2 + \frac{c_2(X)}{24} \right)$$  \hspace{1cm} (1.2)

$$c_3(V) < C \cdot n H^3.$$  \hspace{1cm} (1.3)

Note that the conjecture just predicts the existence of a stable reflexive sheaf; in our examples below $V$ will be a vector bundle.

We will also formulate a weaker version of the conjecture, which is implied by the proper (strong) form and concentrates on the existence of $V$ given that just its (potential) $c_2(V)$ fulfills the relevant condition. To refer more easily to the notions involved, we make first the following definitions. We restrict to the case of $V$ being a vector bundle. We will consider rank $n$ bundles of $c_1(V) = 0$ and treat actually the case of $SU(n)$ vector bundles.

**Definition.** Let $X$ be a Calabi-Yau threefold of $\pi_1(X) = 0$ and $c \in H^4(X, \mathbb{Z})$,

i) $c$ is called a Chern class if a stable $SU(n)$ vector bundle $V$ on $X$ exists with $c = c_2(V)$

ii) $c$ is called a DRY class if an ample class $H \in H^2(X, \mathbb{R})$ exists (and an integer $n$) with

$$c_2(V) = n \left( H^2 + \frac{c_2(X)}{24} \right).$$ \hspace{1cm} (1.4)

With these definitions we can now state the weak DRY conjecture, in the framework as we will use it, as follows:

**Weak DRY-Conjecture.** On a Calabi-Yau threefold $X$ of $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbb{Z})$ is a Chern class.

Here it is understood that the integer $n$ occurring in the two definitions is the same.

The paper has three parts. In section 2 we give criteria for a class to be a DRY class. In section 3 we present some bundle constructions and show that their $c_2(V)$ fulfill these criteria for infinitely many $V$.

## 2 DRY classes on elliptic Calabi-Yau threefolds

To test these conjectures we choose $X$ to be elliptically fibered over the base surface $B$ with section $\sigma : B \rightarrow X$ (we will also denote by $\sigma$ the embedded subvariety $\sigma(B) \subset X$ and its cohomology class in $H^2(X, \mathbb{Z})$), a case particularly well studied. The typical examples for $B$ are rational surfaces like a Hirzebruch surface $F_k$ (where we consider
the following cases $k = 0, 1, 2$ as only for these bases exists a smooth elliptic $X$ with Weierstrass model, a del Pezzo surface $dP_k$ ($k = 0, \ldots, 8$) or the Enriques surface (or suitable blow-ups of these cases). We will consider specifically bases $B$ for which $c_1 := c_1(B)$ is ample. This excludes in particular the Enriques surface and the Hirzebruch surface$^3 F_2$. (The classes $c_1^2$ and $c_2 := c_2(B)$ will be considered as (integral) numbers.)

On the elliptic Calabi-Yau space $X$ one has according to the general decomposition $H^4(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})\sigma \oplus H^4(B, \mathbb{Z})$ the decompositions (with $\phi, \rho \in H^2(X, \mathbb{Z})$)

\begin{align}
c_2(V) &= \phi \sigma + \omega \\
c_2(X) &= 12c_1\sigma + c_2 + 11c_1^2
\end{align}

where $\omega$ is understood as an integral number (pullbacks from $B$ to $X$ will be usually suppressed).

One now solves for $H = a\sigma + \rho$ (using the decomposition $H^2(X, \mathbb{Z}) \cong \mathbb{Z}\sigma \oplus H^2(B, \mathbb{Z})$), given an arbitrary but fixed class $c = \phi\sigma + \omega \in H^4(X, \mathbb{Z})$, and has then to check that $H$ is ample. The conditions for $H$ to be ample are (cf. appendix)

\[ H \text{ ample } \iff a > 0, \ \rho - ac_1 \text{ ample.} \quad (2.3) \]

Inserting $H$ into equ. (1.4) one gets the following relations

\begin{align}
\phi &= 2an\rho + n\left(\frac{1}{2} - a^2\right)c_1 \\
\frac{1}{n}\omega &= \rho^2 + \frac{5}{12}c_1^2 + \frac{1}{2}
\end{align}

(note that $\sigma^2 = -c_1\sigma$, cf. [2]) where in (2.5) Noethers relation $c_2 + c_1^2 = 12$ for the rational surface $B$ has been used. This implies

\[ \rho = \frac{1}{2an}\left(\phi - n\left(\frac{1}{2} - a^2\right)c_1\right) \quad (2.6) \]

As we assumed that $c_1$ is ample one finds that the condition that the class

\[ \rho - ac_1 = \frac{1}{2an}\left(\phi - n\left(\frac{1}{2} + a^2\right)c_1\right) \quad (2.7) \]

is ample leads also to an upper bound on the positive real number $a^2$

\[ 0 < a^2 < b \quad (2.8) \]

This bound will be specified in an example below explicitly. Furthermore equ. (2.7) shows that $\phi - \frac{a}{2}c_1$ must necessarily be ample.

\[ ^3 \text{as } c_1 \cdot b = (2b + 4f) \cdot b = 0, \text{ using here the notations from footn. 6} \]
Having solved by now (2.4) in terms of $\rho$ one now has to solve the following equation

$$\frac{1}{n} \omega = \frac{1}{4a^2n^2} \left( \phi - n \left( \frac{1}{2} - a^2 \right)c_1 \right)^2 + \frac{5}{12}c_1^2 + \frac{1}{2}$$  \hspace{1cm} (2.9)

in terms of $a$. Actually the only non-trivial point will be that $a$ is real and satisfies (2.8). Concretely one gets a quadratic equation $a^4 + pa^2 + q = 0$ in $a^2$ with

$$p = -\frac{4}{c_1^2} \left( \frac{1}{n} \omega - r \right)$$  \hspace{1cm} (2.10)

$$q = \frac{4}{(c_1^2)^2} s^2$$  \hspace{1cm} (2.11)

where we used the abbreviations

$$r := \frac{1}{2n} \phi c_1 + \frac{1}{6} c_1^2 + \frac{1}{2}$$  \hspace{1cm} (2.12)

$$s := \frac{1}{2n} \sqrt{c_1^2} \sqrt{\left( \phi - \frac{n}{2} c_1 \right)^2}$$  \hspace{1cm} (2.13)

Now one has three conditions which have to be satisfied by at least one solution $a^2_*$ of this equation, namely

1) $a^2_* \in \mathbb{R} \iff p^2 \geq 4q \iff \left( \frac{1}{n} \omega - r \right)^2 \geq s^2$  \hspace{1cm} (2.14)

2) $a^2_* > 0 \iff -p > 0 \iff \frac{1}{n} \omega > r$  \hspace{1cm} (2.15)

3) $a^2_* \leq b \iff \begin{cases} -p < b + \frac{q}{b} & \text{for } + \sqrt{ } \text{ and } b \geq -\frac{p}{2} \\ \text{arbitrary} & \text{for } - \sqrt{ } \text{ and } b \geq -\frac{p}{2} \\ -p > b + \frac{q}{b} & \text{for } - \sqrt{ } \text{ and } b < -\frac{p}{2} \end{cases}$  \hspace{1cm} (2.16)

Concerning ii) note that necessarily $q > 0$, cf. the remark after (2.8); furthermore the evaluation of the condition is independent of the question which sign for the square root is taken. Note that in iii) the case where $+ \sqrt{ }$ is taken and $b < -\frac{p}{2}$ is excluded.

Concerning condition iii) note that for $b \geq -p/2$ one gets no further restriction and has to pose in total just the first two conditions, i.e. $\frac{1}{n} \omega \geq r + s$. By contrast for $b < -p/2$ the condition $-p > b + \frac{q}{b}$, equivalently $\frac{1}{n} \omega > \omega_0(\phi; b) := r + \frac{c_1^2}{4}(b + \frac{q}{b})$), implies i) and ii).

As $p$ (and thus $\omega$) occurs in the domain restrictions on $b$ one has to rewrite these conditions slightly. Consider first the regime $b < -\frac{p}{2}$, or explicitly $\frac{1}{n} \omega > r + \frac{q}{2}c_1^2$, and distinguiosh two cases: the ensuing condition $\frac{1}{n} \omega > \omega_0(\phi; b)$ makes sense as an additional condition (required besides the domain restriction $\frac{1}{n} \omega > r + \frac{q}{2}c_1^2$) only if $r + \frac{b}{2}c_1^2 < \omega_0(\phi; b)$, i.e. for $b < \sqrt{q}$; on the other hand for $b \geq \sqrt{q}$ one just has to demand $\frac{1}{n} \omega > r + \frac{b}{2}c_1^2$.

\hspace{1cm} 4\text{here it is understood that ii) assumes i) satisfied and iii) assumes i) and ii) satisfied}
In the second regime $b \geq -p/2$, or equivalently $\frac{1}{n} \omega \leq r + \frac{b}{2} c_1^2$, one has the condition $\frac{1}{n} \omega \geq r + s$ (note that these two conditions are compatible just for $s \leq \frac{b}{2} c_1^2$, i.e. $b \geq \sqrt{q}$). Thus in total one can make an $\omega$-independent regime distinction for $b$ according to $b < \sqrt{q}$ (with the demand $\frac{1}{n} \omega > \omega_0$) or $b \geq \sqrt{q}$ (where one should have either $\frac{1}{n} \omega > r + \frac{b}{2} c_1^2$ or $\frac{1}{n} \omega \leq r + \frac{b}{2} c_1^2$ but in that latter case one has to demand $\frac{1}{n} \omega \geq r + s$; but, as $r + s \leq r + \frac{b}{2} c_1^2$ in the present $b$-regime, one just has to demand that $\frac{1}{n} \omega \geq r + s$).

Note in this connection that, for $\phi$ held fixed, $\omega_0(\phi; b)$ becomes large for small and large $b$ and the intermediate minimum is achieved at $b_{\text{min}} = \sqrt{q}$. As the condition $\frac{1}{n} \omega > \omega_0(\phi; b)$ is relevant only for $b < \sqrt{q}$ whereas for $b > \sqrt{q}$ one gets the condition $\frac{1}{n} \omega > r + s$ and as one has $\omega_0(\phi, b_{\text{min}}) = r + s$ there is a smooth transition in the conditions; and, all in all, $\frac{1}{n} \omega \geq r + s$ is a necessary condition for a class to be a DRY class.

Therefore we get the following theorem

**Theorem on DRY classes.** For a class $c = \phi \sigma + \omega \in H^4(X, \mathbb{Z})$ to be a DRY class one has the following conditions (where $b$ is some $b \in \mathbb{R}^{>0}$, $b < \sqrt{q}$, and $\omega \in H^4(B, \mathbb{Z}) \cong \mathbb{Z}$):

a) **necessary and sufficient:** $\phi - n(\frac{1}{2} + b)c_1$ is ample and $\frac{1}{n} \omega > \omega_0(\phi; b)$

b) **sufficient:** $\phi - \frac{n}{2} c_1$ is ample and $\omega$ sufficiently large

c) **necessary:** $\phi - \frac{n}{2} c_1$ is ample and $\frac{1}{n} \omega \geq r + s$.

Here part b) follows immediately from a) as the ample cone of $B$ is an open set. So in particular the condition on $\omega$ can be fulfilled in any bundle construction which contains a (discrete) parameter $\mu$ in $\omega$ such that $\omega$ can become arbitrarily large if $\mu$ runs in its range of values (this strategy will be used for spectral and extension bundles).

Note that as the notion of $c$ being a DRY class does not involve any $b$ one should compare these conditions for different $b$. One then realises that as $b$ becomes larger the condition on $\phi$ becomes stronger and stronger; on the other hand as $b$ increases from 0 to $\sqrt{q}$ (for a fixed $\phi$) the condition on $\omega$ becomes weaker. Note that, assuming that $\phi - \frac{n}{2} c_1 = A + b N c_1$ with an ample class $A$ on $B$, one has $(\phi - \frac{n}{2} c_1)^2 > b^2 N^2 c_1^2$, which is $q > b^2$. So it is enough to use $b$'s in the interval $0 < b < \sqrt{q}$ as test parameters. That is the set of DRY classes is the union of allowed ranges of $\phi$ and $\omega$ for all these $b$.

**Remark:** Note that, although for $b_{\text{min}}$ the condition on $\omega$ is as weak as possible, $\phi - n(\frac{1}{2} + b_{\text{min}})c_1$ might not be ample (cf. the example of the tangent bundle given below); nevertheless the ampleness condition on $\phi$ might be satisfied for a $b$ where the bound $\omega \geq \omega_0(\phi; b)$ turns out to be more stringent (cf. again the example).\(^5\)

\(^5\)Note also the following property of $b_{\text{min}}(\phi)$: the zero class lies in the boundary of the ample cone; so, if the condition on $\phi$ is considered for this limiting case, one finds $\phi - \frac{n}{2} c_1 = b_{\text{min}} c_1$ such that $(\phi$ is proportional to $c_1$ and) $b = b_{\text{min}}(\phi)$.
3 Examples for the DRY-Conjecture

In this section we give examples of cohomology classes which are of DRY-form and appear as second Chern classes of stable $SU(n)$ vector bundles.

3.1 The tangent bundle

Let us see whether the cohomology class given by $c_2(TX)$ is detected by the weak DRY-Conjecture as a Chern class. For this we apply the Theorem above to see whether $c_2(X)$ is a DRY class. The minimum of $\omega_0(\phi; b)$ is assumed at $b_{\text{min}} = \sqrt{q} = 7/2$ but one finds $0 < b < 7/2$ as the allowed range for $b$ ($c_1$ was assumed ample); furthermore one has

$$\frac{1}{3} \omega_{TX} = \frac{10}{3} c_1^2 + 4 \geq \omega_0(12c_1, b_{\text{min}}) = r + s = \frac{47}{12} c_1^2 + \frac{1}{2}$$

only for $c_1^2 \leq 6$, but one has in any case to take a smaller $b$ which makes the bound $\omega \geq \omega_0(12c_1; b)$ even more stringent. It suffices however to take $b$ minimally smaller (which is also optimal for the bound $\frac{1}{n} \omega \geq \omega_0(12c_1, b)$ as $\omega_0(12c_1, b)$ becomes minimally greater for $b$ becoming minimally smaller), such that one gets $c_1^2 < 6$ as precise condition for $c_2(TX)$ to be a DRY-class; i.e., in these cases the weak DRY-conjecture is fulfilled (as $c_3(X) = -60c_1^2$ is negative actually even the (proper) DRY-conjecture is true); by contrast the cases $c_1^2 \geq 6$ illustrate that being a DRY-class is only a sufficient condition for a class to be realised as Chern class of a stable bundle, but not a necessary one.

3.2 Spectral bundles

In case of spectral cover bundles [2] one has the following expression for $\omega$

$$\omega = (\lambda^2 - \frac{1}{4}) \frac{n}{2} \phi(\phi - nc_1) - \frac{n^3 - n}{24} c_1^2$$

(3.1)

Here $\phi$ is an effective class in $B$ with $\phi - nc_1$ also effective and $\lambda$ is a half-integer satisfying the following conditions: $\lambda$ is strictly half-integral for $n$ being odd; for $n$ even an integral $\lambda$ requires $\phi \equiv c_1 \pmod{2}$ while a strictly half-integral $\lambda$ requires $c_1$ even. (In addition one has to assume that the linear system $|\phi|$ is base point free$^6$.)

Often one assumes, as we will do here, that $\phi - nc_1$ is not only effective but even ample in $B$. Then equ. (2.7) shows that we can take $b = 1/2$ as upper bound on $a$.

---

$^6$ a base point is a point common to all members of the system $|\phi|$ of effective divisors which are linearly equivalent to the divisor $\phi$ (note that on $B$ the cohomology class $\phi$ specifies uniquely a divisor class); on $B$ a Hirzebruch surface $F_k$ with base $P^1$ $b$ and fibre $P^1$ $f$ this amounts to $\phi \cdot b \geq 0$
One has now to check whether the three conditions on \(a^2\) given above can be fulfilled. According to part b) of the theorem in section 2 one learns that this is the case as \(\omega\) increases to arbitrarily large values when the parameter \(\lambda\) is increasing.

**Theorem.**

i) On \(X\) an elliptic Calabi-Yau threefold the class \(c_2(V) = c = \phi \sigma + \omega\) for \(V\) a spectral bundle (of discrete bundle parameters \(\eta \in H^2(B, \mathbb{Z})\) and \(\lambda \in \frac{1}{2}\mathbb{Z}\)) satisfies the assumptions of the weak DRY-Conjecture on \(c\) for all but finitely many values of the parameter \(\lambda\).

ii) For the infinitely many classes \(c \in H^4(X, \mathbb{Z})\) described in i) the weak DRY-Conjecture is true.

iii) For the classes in ii) with negative \(\lambda\) the (proper) DRY-Conjecture is true.

Here part ii) follows of course just from reversing the perspective: whereas in part i) one started from a given spectral bundle \(V\) and found a condition (\(\lambda^2\) sufficiently large) that its \(c_2(V)\) fulfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has trivially confirmed the existence of a stable bundle for a \(c = c_2(V)\) which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as \(c_3(V) = 2\lambda \phi (\phi - nc_1)\) is negative for \(\lambda\) negative as \(\phi \neq 0\) is effective and \(\phi - nc_1\) was assumed ample, so \(\phi (\phi - nc_1)\) is positive (this argument underlies of course already part ii) as well).

### 3.3 Extension bundles

Stable vector bundles built as an extension of given stable bundles have been constructed on elliptic Calabi-Yau threefolds in [3]. Let \(E\) be a rank \(r\) \(H_B\)-stable vector bundle on the base \(B\) of the Calabi-Yau space with Chern classes \(c_1(E) = 0\) and \(c_2(E) = k\). The pullback bundle \(\pi^*E\) is then shown to be stable on \(X\) with respect to the ample class \(J = z\sigma + H_B\) where \(H_B = hc_1\) (with \(h \in \mathbb{R}^{>0}\)) [3]. The bundle extension

\[
0 \to \pi^*E \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(rD) \to 0 \quad (3.2)
\]

with \(D = x\sigma + \alpha\) defines a stable rank \(n = r + 1\) vector bundle if the numerical condition equ. (3.4) is satisfied. We consider here the case \(x = -1\) for simplicity. For this bundle \(c = \phi \sigma + \omega\) is given by

\[
\phi = (n - 1)\frac{n}{2}(2\alpha + c_1), \quad \omega = k - (n - 1)\frac{n}{2}\alpha^2 \quad (3.3)
\]

As in the spectral case one now has to check whether the three conditions on \(a^2\) given in section 2 can be fulfilled. This is the case according to part b) of the theorem in section
2 if \(\alpha\) is chosen such that 
\[
2(n - 1)\alpha + (n - 2)c_1 \text{ is ample and } k \text{ is chosen sufficiently large.}
\]
Note that this is in agreement with the condition that the extension can be chosen nonsplit if
\[
\frac{n - 1}{2} \left[ n^2 (\alpha + c_1) + \frac{c_1^2}{3} - c_1 (2\alpha + c_1 + 1) \right] - k < 0
\] (3.4)

As above in the spectral bundle case we get here the following result.

**Theorem.** i) On \(X\) an elliptic Calabi-Yau threefold the class \(c_2(V) = c = \phi \sigma + \omega\) for \(V\) an extension bundle (of discrete bundle parameters \(\alpha \in H^2(B, \mathbb{Z})\) and \(k \in \mathbb{Z}\)) satisfies the assumptions of the weak DRY-Conjecture on \(c\) for all but finitely many values of the parameter \(k\).

ii) For the infinitely many classes \(c \in H^4(X, \mathbb{Z})\) described in i) the weak DRY-Conjecture is true.

iii) For infinitely many classes \(c \in H^4(X, \mathbb{Z})\) the (proper) DRY-Conjecture is true.

As above section 3.2, part ii) follows from reversing the perspective: whereas in part i) one started from a given extension bundle \(V\) and found a condition \((k\text{ sufficiently large})\) that its \(c_2(V)\) fulfills the assumption of the weak DRY-Conjecture, one then turns around the perspective in part ii), where one has now trivially confirmed the existence of a stable bundle for a \(c = c_2(V)\) which satisfies the assumptions of the weak DRY-Conjecture.

Part iii) follows as \(c_3(V) = -\frac{(n-1)(n-2)}{3} (c_1^2 + 3\alpha (\alpha + c_1)) - 2k < 0\) for \(k\text{ sufficiently large.}\)

### 3.4 A further Example

Let us finally come back to the motivating question from the introduction. We will take a stable bundle in the visible sector \(V_{vis}\) of the heterotic string and want to supplement this by a stable bundle \(V_{hid}\) of rank \(n_h\) such that the anomaly condition \(c_2(V_{vis}) + c_2(V_{hid}) = c_2(X)\) is satisfied. To assure the existence of \(V_{hid}\) we will assume the weak DRY conjecture. So, concretely we will check whether \(c := c_2(X) - c_2(V_{vis})\) is a DRY class.

Let us take \(V_{vis} = \pi^*E\) where \(E\) on \(B\) is a bundle with \(c_2(E) = k\), stable with respect to the ample class \(H_B\) on \(B\). Thus in this case we have
\[
\phi = 12c_1, \quad \omega = 10c_1^2 + 12 - k
\] (3.5)

and furthermore one gets the explicit expression for the bound
\[
\omega_0 = \left[ \frac{6}{n_h} + \frac{1}{6} + \frac{b}{4} + \frac{(12 - \frac{n_h}{2})^2}{4bn_h^2} \right] c_1^2 + \frac{1}{2}.
\] (3.6)
Let us consider part a) of the theorem of section 2. We get \(12 - n_h(\frac{1}{2} + b) > 0\) from the ampleness condition (so we are in the regime \(b < \sqrt{q = \frac{12 - k}{n_h}}\)) on \(\phi\) and \(\frac{1}{n_h}\omega \geq \omega_0\) as further condition. Note further that the DRY conjecture does not specify a polarization with respect to which \(V_{hid}\) will be stable; so, to get a polystable bundle in total, \(V_{vis}\) should be stable with respect to an arbitrary ample class; this is true in our case \(V_{vis} = \pi^*E\) only for \(B = \mathbb{P}^2\) (where \(H^{1,1}(B)\) is onedimensional) according to Lemma 5.1 of [3]. This restriction is however in contradiction with the necessary condition \(\frac{1}{n_h}\omega \geq r + s\) from which one finds \(c_1^2 \leq \frac{12-k-n_h/2}{2-n_h/12}\). Thus, for this (rather special) example of \(V_{vis}\) one does not succeed in complementing (in the sense of satisfying the anomaly equation) \(V_{vis}\) by a hidden bundle. In many more relevant examples for \(V_{vis}\), however, this strategy succeeds [8].

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A Ample classes on elliptic Calabi-Yau threefolds

Let \(H = a\sigma + \rho \in H^2(X, \mathbb{R}) \cong \mathbb{R}\sigma + H^2(B, \mathbb{R})\) be a class on the elliptic Calabi-Yau threefold \(X\). Then one has if \(c_1\) is ample

\[
H \text{ ample } \iff a > 0, \ rho - ac_1 \text{ ample.} \tag{A.1}
\]

Consider first the “\(\Rightarrow\)” direction: one has \(a = H \cdot F > 0\) according to the Nakai-Moishezon criterion that \(H\) is ample just if \(H^3 > 0, H^2 \cdot S > 0, H \cdot C > 0\) for all irreducible surfaces \(S\) and irreducible curves \(C\) in \(X\); here this is applied to the fibre \(F\). Furthermore, if \(c\) is an irreducible curve in \(B\) one has \((\rho - ac_1) \cdot c = H \cdot c\sigma > 0\); and one also has \((\rho - ac_1)^2 = H^2 \cdot \sigma > 0\), such that by the same criterion, applied now on \(B\), indeed the class \(\rho - ac_1\) is ample.

Consider now the “\(\Leftarrow\)” direction: the class of an irreducible curve \(C\) in \(X\) is built from the class \(F\) and non-negative linear combinations of classes of the form \(c\sigma\), where \(c\) is now the class of an irreducible curve in \(B\); therefore, turning the previous arguments around, one ends up indeed with \(H \cdot C > 0\). The classes of irreducible surfaces are in a similar way built from \(\sigma\) and the \(\pi^*c\); for \(H^2 \cdot \sigma\) one can again turn around the previous argument; this is not so however for \(H^2 \cdot \pi^*c = ac(2\rho - ac_1)\); in this case we adopt the additional assumption that \(c_1\) is ample, which implies that \(\rho\), and therefore \(2\rho - ac_1\) too, is also ample to get the required conclusion. Similarly one concludes for \(H^3 = a[\rho^2 + (\rho - ac_1)(2\rho - ac_1)]\).
Examples of one-parameter Calabi-Yau spaces

Although we treat in the main body of the paper the case of elliptic Calabi-Yau spaces $X$ let us briefly comment here on the simpler case where $X$ is a one-parameter space, i.e., $h^{1,1}(X) = 1$.

In this case one has the representations (with $k, t \in \mathbb{Z}$)

$$c = kJ^2 \quad (B.1)$$

$$c_2(X) = tJ^2 \quad (B.2)$$

where $J$ is a generating element of $H^2(X, \mathbb{Z})$; for the ample class $H$ one has $H = hJ$ with $h \in \mathbb{R}^{>0}$.

The condition for a class $c$ to have DRY form becomes here

$$k = n\left(h^2 + \frac{t}{24}\right) \quad (B.3)$$

This amounts to the condition

$$k > n\frac{t}{24} \quad (B.4)$$

whereas the necessary Bogomolov inequality $c \cdot J > 0$ gives just $k > 0$ (for example on the quintic one gets the stronger condition $k > \frac{n}{12}h$). Note that the second Chern class of the tangent bundle always has DRY-form; thus for this cohomology class the weak DRY-conjecture is satisfied, and for negative Euler number even the (proper) DRY-conjecture.

Some examples are provided by the complete intersection spaces $\mathbb{P}^4(5)$, $\mathbb{P}^5(2, 4)$, $\mathbb{P}^5(3, 3)$, $\mathbb{P}^6(2, 2, 3)$, $\mathbb{P}^7(2, 2, 2, 2)$ with $t = 10, 7, 6, 5, 4$ and Euler numbers $-200, -176, -144, -144, -128$. (similarly one can discuss the one parameter cases $\mathbb{P}_{2,1,1,1,1}(6)$, $\mathbb{P}_{4,1,1,1,1,1}(8)$, $\mathbb{P}_{5,2,1,1,1}(10)$).

On the quintic one has some further bundles, occurring in the list in [6], with $c_2(V) = c_2(X)$ with some of them (the first five examples) shown to be stable in [7], which have the same $t$ as $TX$ and also negative $c_3(V)$; thus these provide further examples of the weak DRY-conjecture and actually even of the (proper) DRY-Conjecture.

Physically one has to demand in addition anomaly cancellation. Thus one gets then in total the condition

$$\frac{n}{24}t < k \leq t \quad (B.5)$$

(note that one has here $k_{hid} > 0$ for a potential hidden bundle from the Bogomolov inequality).
For the generation number one gets, in the framework of the assumptions of the DRY conjecture, the bound

\[ N_{gen} < C \frac{n}{2} \left( \frac{k}{n} - \frac{t}{24} \right)^{3/2}. \] (B.6)

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