Almost universal codes for MIMO wiretap channels
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Abstract

Despite several works on secrecy coding for fading and MIMO wiretap channels from an error probability perspective, the construction of information theoretically secure codes over such channels remains as an open problem. In this paper, we consider a fading wiretap channel model where the transmitter has only partial statistical channel state information. We extend the flatness factor criterion from the Gaussian wiretap channel to fading and MIMO wiretap channels, and propose concrete lattice codes with a vanishing flatness factor to achieve information theoretic security. These codes are built from algebraic number fields with constant root discriminant for the single-antenna fading wiretap channel, and from division algebras centered at such number fields for the MIMO wiretap channel, respectively. The proposed lattice codes achieve strong secrecy and semantic security over any ergodic stationary fading/MIMO wiretap channel with sufficiently fast decay of time correlations, for all secrecy rates $R < C_b - C_e - \kappa$, where $C_b$ and $C_e$ are Bob and Eve’s channel capacities respectively, and $\kappa$ is an explicit constant gap. Moreover, these codes are almost universal in the sense that a fixed code is good for secrecy for a wide range of fading models.

Index Terms

algebraic number theory, division algebras, fading wiretap channel, information theoretic security, lattice coding, MIMO wiretap channel, statistical CSIT.

I. INTRODUCTION

The wiretap channel model was introduced by Wyner [2], who showed that secure and reliable communication can be achieved simultaneously over noisy channels even without the use of secret keys. In the information theory community, the most widely accepted secrecy metric is Csiszár’s strong secrecy: the mutual information $I(M; Z^k)$ between the confidential message $M$ and the channel output $Z^k$ should vanish when the code length $k$ tends to infinity.

While in the information theory community confidential messages are often assumed to be uniformly distributed, this assumption is not accepted in cryptography. A cryptographic treatment of the wiretap channel was proposed in [3] to combine the requirements of the two communities, establishing that achieving semantic security in the cryptographic sense is equivalent to achieving strong secrecy for all distributions of the message. This equivalence holds also for continuous channels [4].

In the case of Gaussian wiretap channels, [4] considered the problem of designing lattice codes which achieve strong secrecy and semantic security. Following an approach by Csiszár [5, 6], strong secrecy is guaranteed if the output distributions of the eavesdropper’s channel corresponding to two different messages are indistinguishable in the sense of variational distance. To this aim, the flatness factor of a lattice was proposed in [4] as a fundamental criterion which implies that conditional outputs are indistinguishable. Using random coding arguments, it was shown that there exist families of lattice codes which are good for secrecy, meaning that their flatness factor is vanishing, and achieve semantic security for rates up to $1/2$ nat from the secrecy capacity.

In this paper, we consider a multi-input multi-output (MIMO) fading wiretap channel model where the transmitter has only access to partial statistical channel state information (CSI), while the legitimate receiver has perfect...
knowledge of its own channel, and the eavesdropper has perfect knowledge of both channels. A special case is the single-antenna fading wiretap channel, firstly studied in [7]. Despite several works using an error probability criterion [7–10], the construction of wiretap codes with strong secrecy on fading and MIMO channels remains elusive. In this work, we extend the criterion based on the flatness factor to the case of fading and MIMO channels and propose a family of concrete lattice codes from algebraic number fields satisfying this criterion. A recent paper [11] also adopted the flatness factor as a design criterion in MIMO wiretap channels, yet it is unclear whether their approach achieves strong secrecy. Intuitively, a vanishing flatness factor, to be defined precisely in our paper, implies that the output distributions of the eavesdropper’s channel corresponding to different messages converge to the same distribution (which depends on the eavesdropper’s channel). Hence no information is leaked to the eavesdropper asymptotically, even if she knows her channel as well as the legitimate user’s channel. For this purpose, we construct wiretap lattice codes from a particular sequence of algebraic number fields with constant root discriminant. In [12, 13], it was shown that these lattice codes are “almost universal” in the sense that they achieve a constant gap to channel capacity over any ergodic stationary fading channel. The underlying multiplicative structure and constant root discriminant property guarantee that the received lattice after fading has a good minimum distance when the channel is not in outage.

Coincidentally, the sequences of number fields that we consider are also used in cryptography. The flatness factor is related to the smoothing parameter in lattice-based cryptography, which is used to show that if certain lattice problems are hard to solve in the worst case, they are also hard to solve in the average case within a polynomial factor of the input size. This is known as the worst-case to average-case reduction of hard lattice problems [14]. In the case of number fields with constant root discriminants, the connection factor between the worst case and average case is of the order $O(\log n)$ in the size of the input, which is the best known yet.

In this paper, we show that these lattices also achieve strong secrecy and semantic security in fading and MIMO wiretap channels. The key feature is that the dual of the faded lattice has a good minimum distance, so that the flatness factor of the faded lattice vanishes unless the channel is in outage. In particular, for the Gaussian wiretap channel this suggests a simple design criterion where the packing density of the lattice and its dual should be maximized simultaneously. Compared to [13], we also improve the coding rate by replacing spherical shaping with a discrete Gaussian distribution over the infinite lattice as in [4]. As a consequence, our nested lattice codes achieve strong secrecy and semantic security for all secrecy rates $R < C_b - C_e - \kappa$, where $C_b$ and $C_e$ are Bob and Eve’s channel capacities respectively, and $\kappa$ is an explicit constant gap. Moreover, these codes are almost universal in the sense that given $C_b$ and $C_e$, the same code is good for secrecy for a wide range of fading models. More precisely, all static and ergodic fading models are allowed for the main channel. For the eavesdropper’s channel, strong secrecy and semantic security hold for static channels, i.i.d. fading channels, and stationary ergodic channels with faster than linear decay of correlations; for general stationary ergodic channels with slow decay of time correlations, only weak secrecy is guaranteed.

Organization of the paper: To make the paper reader-friendly, we present our methodology firstly for single-antenna fading wiretap channels, then for MIMO wiretap channels, since the latter requires division algebras which are more technical. The rest of the paper is accordingly organized as follows. In Section II, we give preliminaries of lattice codes, in particular lattice Gaussian distribution, the flatness factor, and number field lattices. Section III is devoted to code construction and security proofs for single-antenna fading wiretap channels. The proposed lattice codes can be generalized to the MIMO case using the multi-block matrix lattices from division algebras in [13]. This is accomplished in Section IV (which reviews multi-block codes for MIMO channels and derives their flatness factor) and Section V which a reader may skip in the first reading. In Section VI, we discuss the implications of our results in terms of code design criteria. Finally, Section VII concludes the paper and presents some open problems.

II. Preliminaries

A. Basic lattice definitions

In this section we recall some basic notions about lattices and define the corresponding notations.

We note that the dual code also plays a role in the design of wiretap codes for discrete memoryless channels, such as LDPC codes for binary erasure wiretap channels [15].
Consider $\mathbb{C}^k$ as a $2k$-dimensional real vector space with the real inner product

$$\langle x, y \rangle = \Re(x^T y).$$
(1)

This inner product naturally defines a metric on the space $\mathbb{C}^k$ by setting $||x|| = \sqrt{\langle x, x \rangle}$. With this inner product, we can identify $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ with the canonical real inner product, through the isometry

$$\phi(z_1, \ldots, z_k) = (\Re(z_1), \ldots, \Re(z_k), \Im(z_1), \ldots, \Im(z_k)).$$
(2)

An $n$-dimensional lattice $\Lambda$ is a discrete subgroup of $\mathbb{R}^n$ defined by

$$\Lambda = \{Gx : x \in \mathbb{Z}^n\},$$

where the columns of the generator matrix $G \in M_n(\mathbb{R})$ are linearly independent.

We consider lattices of even dimension $n = 2k$ in the Euclidean space $\mathbb{R}^{2k}$, which is identified with the complex space $\mathbb{C}^k$ through (2). Given a lattice $\Lambda \subset \mathbb{C}^k$, we define the dual lattice as

$$\Lambda^* = \{x \in \mathbb{C}^k | \forall y \in \Lambda, \langle x, y \rangle \in \mathbb{Z}\}.$$

A fundamental region of the lattice $\Lambda$ is a measurable set $\mathcal{R}(\Lambda) \subset \mathbb{R}^n$ such that $\mathbb{R}^n$ is the disjoint union of the translates of $\mathcal{R}(\Lambda)$, i.e. $\mathbb{R}^n = \bigcup_{\lambda \in \Lambda} (\mathcal{R}(\Lambda) + \lambda)$. We denote by $V(\Lambda)$ the volume of any fundamental region of $\Lambda$.

We denote by $\lambda_1(\Lambda)$ the minimum distance of the lattice, i.e. the smallest norm of a non-zero vector:

$$\lambda_1(\Lambda) = \min_{\lambda \in \Lambda \setminus \{0\}} ||\lambda||.$$

### B. Flatness factor and discrete Gaussian distribution

In this section, we define some fundamental lattice parameters that will be used in the rest of the paper. For more background about the smoothing parameter and the flatness factor in information theory and cryptography, we refer the reader to [16, 4, 17].

Let $f_{\sqrt{\Sigma},c}(z)$ denote the $k$-dimensional circularly symmetric complex normal distribution with mean $c$ and covariance matrix $\Sigma$:

$$f_{\sqrt{\Sigma},c}(z) = \frac{1}{\pi^k \det(\Sigma)} e^{-\langle z-c, \Sigma^{-1}(z-c) \rangle} \quad \forall z \in \mathbb{C}^k.$$

We use the notation $f_{\sigma,c}(z)$ for $f_{\sigma I,c}(z)$ and $f_{\sqrt{\Sigma}}$ for $f_{\sqrt{\Sigma},0}$.

Given a lattice $\Lambda \subset \mathbb{C}^k$, we consider the $\Lambda$-periodic function

$$f_{\sqrt{\Sigma},\Lambda}(z) = \sum_{\lambda \in \Lambda} f_{\sqrt{\Sigma},\lambda}(z), \quad \forall z \in \mathbb{C}^k.$$

Note that the restriction of $f_{\sqrt{\Sigma},\Lambda}(z)$ to any fundamental region $\mathcal{R}(\Lambda)$ is a probability distribution.

**Definition 2.1:** Given a complex lattice $\Lambda \subset \mathbb{C}^k$ and a positive definite matrix $\Sigma \in M_n(\mathbb{C})$, the flatness factor $\epsilon_\Lambda(\sqrt{\Sigma})$ is defined as the maximum deviation of $f_{\sqrt{\Sigma},\Lambda}$ from the uniform distribution over a fundamental region $\mathcal{R}(\Lambda)$ of $\Lambda$, with volume $V(\Lambda)$:

$$\epsilon_\Lambda(\sqrt{\Sigma}) = \max_{z \in \mathcal{R}(\Lambda)} \left| V(\Lambda) f_{\sqrt{\Sigma},\Lambda}(z) - 1 \right|.$$

Compared to [4], in this paper we use an extended version of the flatness factor for correlated Gaussians, related to the extended notion of the smoothing parameter in [17]. We also extend the definition to the case of complex lattices. In the case of scalar matrices we write $\epsilon_\Lambda(\sigma) = \epsilon_\Lambda(\sigma I)$.

Note that correlations can be absorbed by the lattice in the sense that $\epsilon_\Lambda(\sqrt{\Sigma}) = \epsilon_{\sqrt{\Sigma}^{-1},\Lambda}(I)$, and that $\epsilon_\Lambda(\sqrt{\Sigma_1}) \leq \epsilon_\Lambda(\sqrt{\Sigma_2})$ if $\Sigma_1$ and $\Sigma_2$ are two positive definite matrices with $\Sigma_1 \succeq \Sigma_2$.

**Definition 2.2:** Given a lattice $\Lambda$ and $\varepsilon > 0$, the smoothing parameter $\eta_\varepsilon(\Lambda)$ is the smallest $s$ such that

$$\sum_{\lambda \in \Lambda \setminus \{0\}} e^{-\frac{s^2}{2} ||\lambda||^2} \leq \varepsilon,$$

where $\Lambda^*$ is the dual lattice.

Note that we define the smoothing parameter per complex dimension, which differs by a factor $\sqrt{2}$ from the definition in [16]. We have adjusted the bounds on $\eta_\varepsilon(\Lambda)$ accordingly.
For scalar covariance matrices the smoothing parameter is related to the flatness factor as follows [4]:
\[ \sqrt{2\pi\sigma} = \eta_\varepsilon(\Lambda) \quad \text{if and only if} \quad \epsilon_\Lambda(\sigma I) = \varepsilon. \]

More generally, for \( \Sigma \succeq 0 \) we can say that
\[ \sqrt{2\pi\Sigma} \succeq \eta_\varepsilon(\Lambda) \quad \text{if} \quad \epsilon_\Lambda(\sqrt{\Sigma}) \leq \varepsilon. \]

The smoothing parameter is upper bounded by the minimum distance of the dual lattice [16]. More precisely, we have the following corollary of a result by Banaszczyk [18]:

\begin{equation}
\text{Lemma 2.3:} \quad \text{Suppose that} \quad \Lambda \text{ is an } n\text{-dimensional lattice, and consider two constants } c > \frac{1}{\sqrt{2\pi}}, \quad C = c\sqrt{2\pi}e^{-\pi c^2} < 1.
\end{equation}

If \( \tau > \frac{\sqrt{n\varepsilon}}{\lambda_1(\Lambda)} \), then
\[ \sum_{\lambda \in \Lambda \setminus \{0\}} e^{-\tau^2 \|\lambda\|^2} \leq \frac{C^n}{1 - C^n}. \]

Therefore the smoothing parameter of the dual lattice is bounded as follows:
\[ \eta_\varepsilon(\Lambda^*) \leq \frac{\sqrt{n\varepsilon}}{\lambda_1(\Lambda)} \quad \text{for} \quad \varepsilon = \frac{C^n}{1 - C^n}. \]

Equivalently, in terms of the flatness factor,
\[ \epsilon_{\Lambda^*} \left( \frac{\sqrt{n\varepsilon}}{\sqrt{\pi} \lambda_1(\Lambda)} \right) \leq \frac{C^n}{1 - C^n}. \]

\textbf{Proof:} Let \( B \) be the open unit ball, and \( \rho(A) = \sum_{x \in A} e^{-\pi x^2} \). From Lemma 1.5 in [18] we have that
\[ \forall c \geq \frac{1}{\sqrt{2\pi}}, \quad \rho(\Lambda \setminus c\sqrt{n}B) < C^n \rho(\Lambda), \]
where \( C = c\sqrt{2\pi}e^{-\pi c^2} \). Then we can write
\[ \rho(\Lambda \setminus c\sqrt{n}B) < C^2 \rho(\Lambda) = C^n \rho(\Lambda \setminus c\sqrt{n}B) + C^n \rho(\Lambda \cap c\sqrt{n}B) \]
\[ \Rightarrow \quad \rho(\Lambda \setminus c\sqrt{n}B) < \frac{C^n}{1 - C^n} \rho(\Lambda \cap c\sqrt{n}B). \]

Now suppose that \( \tau > \frac{\sqrt{n\varepsilon}}{\lambda_1(\Lambda)} \) and consequently \( \tau \Lambda \setminus c\sqrt{n}B = \tau \Lambda \setminus \{0\} \). We have
\[ \sum_{\lambda \in \Lambda \setminus \{0\}} e^{-\tau^2 \|\lambda\|^2} = \sum_{\lambda \in \tau \Lambda \setminus \{0\}} e^{-\pi \|\tau \lambda\|^2} = \rho(\tau \Lambda \setminus \{0\}) = \rho(\tau \Lambda \setminus c\sqrt{n}B) < \frac{C^n}{1 - C^n} \rho(\Lambda \cap c\sqrt{n}B) \]
\[ = \frac{C^n}{1 - C^n} \rho(\{0\}) = \frac{C^n}{1 - C^n}. \]

The second tool that we need to define our lattice coding schemes is the notion of discrete Gaussian distribution. Given \( c \in \mathbb{C}^k \) and \( \Sigma \succeq 0 \), the \textit{discrete Gaussian distribution} over the (shifted) lattice \( \Lambda - c \subseteq \mathbb{C}^k \) is the following discrete distribution taking values in \( \Lambda - c \):
\[ D_{\Lambda - c, \sqrt{\Sigma}}(\lambda - c) = \frac{f_{\sqrt{\Sigma}}(\lambda - c)}{\sum_{\lambda' \in \Lambda} f_{\sqrt{\Sigma}}(\lambda' - c)}. \]

The following result is a generalization of Regev’s lemma [19, Claim 3.9] (see also [4, Lemma 8]) to correlated Gaussian distributions. The proof is given in Appendix [ Lemma]

\textbf{Lemma 2.4:} Let \( X_1 \) be sampled according to the discrete Gaussian distribution \( D_{\Lambda + c, \sqrt{\Sigma}_1} \) and \( X_2 \) be sampled according to the continuous Gaussian \( f_{\sqrt{\Sigma}_2} \). Let \( \Sigma_0 = \Sigma_1 + \Sigma_2 \) and \( \Sigma^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1} \). Denote by \( g(x) \) the density of the random variable \( X = X_1 + X_2 \). If
\[ \epsilon_\Lambda(\sqrt{\Sigma}) \leq \varepsilon \leq \frac{1}{2}, \]
then
\[ \epsilon_\Lambda(\sqrt{\Sigma}) \leq \varepsilon \leq \frac{1}{2}. \]
then the $L^1$ distance $\mathbb{V}(,)$ between the distributions $g$ and $f_{\sqrt{\sigma}}$ is bounded as follows:

$$\mathbb{V}(g, f_{\sqrt{\sigma}}) \leq 4\varepsilon.$$ 

We will also need a basic result concerning linear transformations of discrete Gaussian distributions, which is proven in Appendix [I-B]

**Lemma 2.5:** Let $X$ be sampled according to the $k$-dimensional discrete Gaussian distribution $D_{\Lambda+\epsilon,\sqrt{\Sigma}}$, and let $A \in M_k(\mathbb{C})$ an invertible matrix. Then the distribution of $Y = AX$ is $D_{A(\Lambda+\epsilon)\sqrt{\Sigma},A^r}.$

Finally, we introduce subgaussian random variables, whose tails behave similarly to the Gaussian tail distributions:

**Definition 2.6:** A random vector $z$ taking values in $\mathbb{C}^k$ is $\delta$-subgaussian with parameter $\sigma$ if $\forall t \in \mathbb{C}^k, \mathbb{E}[e^{R(t^t z)}] \leq e^{\sigma^2 ||t||^2}.$

Note that for a complex Gaussian vector $Z \sim N_C(0, \Sigma)$, $\mathbb{E}[e^{R(t^t z)}] = e^{\frac{1}{2}t^t \Sigma t}$.

The following result holds (see also [20, Lemma 2.8]):

**Lemma 2.7:** Let $x \sim D_{\Lambda+\epsilon,\sigma}$ be a $k$-dimensional discrete complex Gaussian random variable, and let $A \in M_k(\mathbb{C})$. Suppose that $\epsilon_{A}(\sigma) < 1$. Then $\forall t \in \mathbb{C}^k$,

$$\mathbb{E}[e^{R(t^t Ax)}] \leq \left(\frac{1 + \epsilon_{A}(\sigma)}{1 - \epsilon_{A}(\sigma)}\right) e^{\frac{1}{2}||A^t t||^2}.$$ 

The proof can be found in Appendix [I-C]

### C. Ideal lattices from number fields with constant root discriminant

Let us first formalize some properties of algebraic number fields that are relevant for our construction of algebraic lattice codes in the single-antenna case. We refer the reader to [21] for the relevant notions about number fields.

Let $F$ be a totally complex number field of degree $[F : \mathbb{Q}] = 2k$, with ring of integers $\mathcal{O}_F$. We denote by $d_F$ the discriminant of the number field. The *relative canonical embedding* of $F$ into $\mathbb{C}^k$ is given by

$$\psi(x) = (\sigma_1(x), \ldots, \sigma_k(x)),$$

where $\{\sigma_1, \ldots, \sigma_k\}$ is a set of $\mathbb{Q}$-embeddings $F \rightarrow \mathbb{C}$ such that we have chosen one from each complex conjugate pair.

Assume that $\mathcal{I}$ is a fractional ideal of $F$, that is, there exists some integer $a$ such that $a\mathcal{I}$ is a proper ideal of $\mathcal{O}_F$.

Then $\Lambda = \psi(\mathcal{I})$ is a $2k$-dimensional lattice in $\mathbb{C}^k$. In particular, $\psi(\mathcal{O}_F)$ is a lattice.

We define the codifferent of $F$ as

$$\mathcal{O}_F^c = \{x \in F : \text{Tr}_{F/\mathbb{Q}}(x\mathcal{O}_F) \subseteq \mathbb{Z}\}.$$ 

The codifferent is a fractional ideal, and its algebraic norm is the inverse of the discriminant:

$$N(\mathcal{O}_F^c) = 1/d_F.$$ (8)

The codifferent embeds as the complex conjugate of the dual lattice:

$$\Lambda^* = 2\overline{\psi(\mathcal{O}_F^c)}.$$ (9)

Using Lemma 2.3, equation (5), we have that $\forall c > \frac{1}{\sqrt{2\pi}}$

$$\eta_c(\Lambda) \leq \frac{\sqrt{4kc}}{\lambda_1(\Lambda^*)} = \frac{\sqrt{k}}{\lambda_1(\psi(\mathcal{O}_F^c))}.$$ (10)

where $\varepsilon = \frac{c^{2k}}{1-c^{2k}} \rightarrow 0$ as $k \rightarrow \infty$.

Due to the arithmetic mean – geometric mean inequality, for any fractional ideal $\mathcal{I}$ of $\mathcal{O}_F$, $\lambda_1(\psi(\mathcal{I})) \geq \sqrt{k}(N(\mathcal{I}))^{\frac{1}{2k}}.$ In particular, from (8) we get

$$\lambda_1(\psi(\mathcal{O}_F^c)) = \lambda_1(\psi(\mathcal{O}_F^c)) \geq \sqrt{k}d_F^{\frac{1}{2k}}.$$ (11)

A similar result is shown in [14, Lemma 6.2] for $\varepsilon = 2^{-2k}$. In this paper we prefer to consider general $\varepsilon$ in order to get the best possible secrecy rates.
Combining equations (10) and (11), we find that the smoothing parameter of \( \Lambda \) is upper bounded by the root discriminant:

\[
\eta_c(\Lambda) \leq c \sqrt{|d_F|} \text{ for } \varepsilon_k = \frac{C^{2k}}{1 - C^{2k}}.
\] (12)

Note that as long as \( c > \frac{1}{\sqrt{2\pi}} \), we have \( C < 1 \) and \( \varepsilon_k \to 0 \) exponentially fast, but the rate of convergence will get slower if \( C \) is very close to 1.

In order to have small smoothing parameter when the dimension \( k \) is large, we need the discriminant \( |d_F| \) to grow as slowly as possible with \( k \).

The following theorem by Martinet [22] proves the existence of infinite towers of totally complex number fields with constant root discriminant:

**Theorem 2.8:** There exists an infinite tower of totally complex number fields \( \{F_k\} \) of degree \( 2^k = 5 \cdot 2^t \), such that

\[
|d_{F_k}|^{1/2k} = G,
\] (13)

for \( G \approx 92.368 \).

**Remark 2.9:** Although in principle the number fields in the family \( \{F_k\} \) can be computed explicitly for fixed degree \( k \), at present an efficient algorithm to do so is not available; see the discussion in [13].

We denote by \( \{\Lambda^{(k)}\} = \{\psi(O_{F_k})\} \) the corresponding sequence of lattices in \( \mathbb{C}^k \), with volume

\[
V(\Lambda^{(k)}) = 2^{-k} \sqrt{|d_F|} = 2^{-k} G^k.
\] (14)

### D. Ideal lattices and normalized product distance

Given an element \( x = (x_1, \ldots, x_k) \in \mathbb{C}^k \) we will use the notation \( p(x) = \prod_{i=1}^k |x_i| \), and define

\[
p(\Lambda) = \inf_{x \in \Lambda \setminus \{0\}} p(x).
\]

A classically used parameter to design lattices for the Rayleigh fast fading channel [23] is the normalized product distance

\[
\text{Np}(\Lambda) = \frac{p(\Lambda)}{V(\Lambda)^{2/k}}.
\]

The proof of the following will be given in Appendix D.1-D.3.

**Lemma 2.10:** Let \( F/\mathbb{Q} \) be a totally complex extension of degree \( 2k \) and let \( \psi \) be the relative canonical embedding and \( \mathcal{I} \) a fractional ideal of \( F \). Then

\[
\text{Np}(\psi(\mathcal{I})) \geq \frac{2^{\frac{k}{2}}}{|d_F|^{\frac{1}{2k}}}, \quad \text{Np}(\psi(\mathcal{I})^*) \geq \frac{2^{\frac{k}{2}}}{|d_F|^{\frac{1}{2k}}}.
\]

In other words, given a fixed number field \( F \), the product distances of all its ideal lattices and their duals are lower bounded by the same value \( 2^{\frac{k}{2}}/|d_F|^{\frac{1}{2k}} \), which only depends on the size of the discriminant of the field \( F \).

This property of number fields immediately implies a result concerning the euclidean distance of lattice points in ideal lattices.

A 2k-dimensional lattice \( \Lambda \) in \( \mathbb{C}^k \), its **Hermite invariant** is defined as

\[
h(\Lambda) = \inf_{x \in \Lambda \setminus \{0\}} \frac{||x||^2}{V(\Lambda)^{2/k}} = \frac{\lambda_1(\Lambda)^2}{V(\Lambda)^{2/k}}.
\]

Using the arithmetic – geometric mean inequality, we have for all 2k-dimensional lattices that

\[
(Np(\Lambda))^2 \leq \frac{h(\phi(\Lambda))^k}{k^k}.
\] (15)

Therefore, given a fixed number field \( F \), for any ideal \( \mathcal{I} \) we have that

\[
h(\psi(\mathcal{I})) \geq \frac{2k}{|d_F|^{1/2k}}, \quad h(\psi(\mathcal{I})^*) \geq \frac{2k}{|d_F|^{1/2k}}.
\] (16)

In other words, given a number field with small discriminant, then all the ideal lattices and their duals have large Hermite invariants.
III. SINGLE-ANTENNA FADING WIRETAP CHANNEL

A. Channel model

We consider the single-antenna ergodic fading channel model illustrated in Figure 1, where the outputs $y$ and $z$ at Bob and Eve’s end are given by

$$
\begin{align*}
y_i &= h_{b,i}x_i + w_{b,i}, \\
z_i &= h_{e,i}x_i + w_{e,i},
\end{align*}
$$

where $w_{b,i}, w_{e,i}$ are i.i.d. complex Gaussian vectors with zero mean and variance $\sigma_b^2, \sigma_e^2$ per complex dimension. A confidential message $M$ and an auxiliary message $M'$ with rate $R$ and $R'$ respectively are encoded into $x$. We denote by $\hat{M}$ the estimate of the confidential message at Bob’s end. We define $H_e = \text{diag}(h_{e,1},\ldots,h_{e,k})$, $H_b = \text{diag}(h_{b,1},\ldots,h_{b,k})$. The input $x$ satisfies the average power constraint

$$
\frac{1}{k} \sum_{i=1}^{k} |x_i|^2 \leq P.
$$

We suppose that $h_{b,i}, h_{e,i}$ are isotropically invariant channels such that the channel capacities $C_b$ and $C_e$ are well-defined. All rates are expressed in nats per complex channel use.

We assume that the weak law of large numbers (LLN) holds for Bob’s channel: $\forall \delta > 0$,

$$
\lim_{k \to \infty} \mathbb{P} \left\{ \frac{1}{k} \sum_{i=1}^{k} \ln \left( 1 + \frac{P|h_{b,i}|^2}{\sigma_b^2} \right) - C_b > \delta \right\} = 0,
$$

We note that this general setting includes the Gaussian channel as well as ergodic fading channels.

Moreover, we require a stricter condition for Eve’s channel, i.e. the asymptotic rate of convergence in the LLN should be faster than $o\left(\frac{1}{k}\right)$: $\forall \delta' > 0$,

$$
\lim_{k \to \infty} k \mathbb{P} \left\{ \frac{1}{k} \sum_{i=1}^{k} \ln \left( 1 + \frac{P|h_{e,i}|^2}{\sigma_e^2} \right) - C_e > \delta' \right\} = 0
$$

This condition is satisfied for static channels, i.i.d. fading channels, and ergodic channels whose decay of correlations is vanishing with rate $o\left(\frac{1}{k}\right)$ \cite{24}. Sufficient conditions for (20) to hold are discussed in detail in Appendix III\textsuperscript{4}.

We suppose that Bob has perfect CSI of his own channel, and Eve has perfect CSI of both channels. Alice has no instantaneous CSI, apart from partial knowledge of channel statistics. More precisely, the knowledge of $C_b$ and $C_e$ and of the properties (19) and (20) is sufficient for Alice.

**Definition 3.1:** A coding scheme achieves strong secrecy if

$$
\lim_{k \to \infty} \mathbb{P}\{\hat{M} \neq M\} = 0,
$$

$$
\lim_{k \to \infty} \mathbb{I}(M; z, H_e) = 0.
$$

\textsuperscript{4}This condition was missing in the conference version of this paper \cite{1}, where it was stated that Theorem 3.6 holds whenever Eve’s channel is ergodic. Actually ergodicity is not sufficient with the current approach. Here we make that statement more precise.
**Definition 3.2:** A coding scheme achieves weak secrecy if
\[
\lim_{k \to \infty} P\{M \neq M\} = 0, \\
\lim_{k \to \infty} \frac{1}{k} I(M; z, H_e) = 0.
\]

**Remark 3.3:** Note that even if Eve knows Bob’s channel \(H_b\), and even though \(H_e\) and \(H_b\) are possibly correlated, the leakage can still be expressed as \(I(M; z, H_e)\). In fact, the Markov chain \(z - H_e - H_b\) always holds, and using the chain rule for mutual information twice we get
\[
I(M; z | H_e, H_b) = I(M, H_b; z | H_e) - I(H_b; z | H_e) = I(M, H_b; z | H_e) - I(H_b; z | M, H_e)
\]

\[= I(M; z | H_e).
\]

**Remark 3.4:** To the best of our knowledge, in the case of statistical CSIT only, for general channels the strong and weak secrecy capacities \(C_s\) and \(C_s^w\) are not known. In [25] the equality \(C_s^w = C_b - C_e\) was shown in the case of i.i.d. Rayleigh fading channels where Bob and Eve’s channels are independent. In [26, Lemma 2], it was shown that \(C_s^w \geq C_b - C_e\) for arbitrary wiretap channels. In [6] (Corollary 2 and remarks about Theorem 3) it was noted that this result extends to the strong secrecy metrics for i.i.d. channels provided that exponential convergence holds in the Chernoff bound.

**B. Lattice wiretap coding**

Let \(\Lambda_e \subset \Lambda_b\) be a pair of nested lattices in \(\mathbb{C}^k\) such that \(|\Lambda_b/\Lambda_e| = e^{kR_e}\), and let \(\mathcal{R}(\Lambda_e)\) be a fundamental region of \(\Lambda_e\). We consider the secrecy scheme in [4], where each confidential message \(m \in \mathcal{M} = \{1, \ldots, e^{kR_e}\}\) is associated to a coset leader \(\lambda_m \in \Lambda_b \cap \mathcal{R}(\Lambda_e)\). To transmit the message \(m\), Alice samples \(x \in \Lambda_b\) from the discrete Gaussian \(D_{\Lambda_e + \lambda_m, \sigma^2}\) with \(\sigma^2 = P\). We denote this lattice coding scheme by \(\mathcal{C}(\Lambda_b, \Lambda_e)\).

It follows from [4, Lemma 6] that as \(k \to \infty\), the variance per complex dimension of \(x\) tends to \(P\) provided that
\[
\lim_{k \to \infty} \epsilon_{\Lambda_e}(\sqrt{P}) = 0. \tag{21}
\]

From [4, Lemma 7], the information rate \(R'\) of the auxiliary message \(M'\) (corresponding to the choice of a point in \(\Lambda_e\)) is \(\epsilon\)
\[
R' \approx \ln(\pi e P) - \frac{1}{k} \ln V(\Lambda_e).
\]

Therefore, we have
\[
V(\Lambda_e) \approx \frac{(\pi e P)^k}{e^{kR_e}}, \quad V(\Lambda_b) \approx \frac{(\pi e P)^k}{e^{k(R_e + R')}} \tag{22}
\]

**Coding scheme based on number fields with constant root discriminant.** Let \(\{\Lambda^{(k)}\}\) be the lattice sequence defined in Section [II-C] We consider scaled versions \(\Lambda_b = \alpha_b \Lambda^{(k)}\), \(\Lambda_e = \alpha_e \Lambda^{(k)}\) such that (22) holds.

Since the choice of \(R\) and \(R'\) determines the scaling factors \(\alpha_b\) and \(\alpha_e\), we will denote the corresponding lattice coding scheme by \(\mathcal{C}(\Lambda^{(k)}, R, R')\).

**C. Achievable secrecy rates**

We now state our main result, which will be proven in sections [III-D] and [III-E]

**Theorem 3.5:** Consider the wiretap scheme \(\mathcal{C}(\Lambda_b, \Lambda_e)\) in Section [II-B] and suppose that there exist positive constants \(t_b, t_e\) such that
\[
N_p(\Lambda_b)^{2/k} \geq t_b, \quad N_p(\Lambda_e)^{2/k} \geq t_e. \tag{23}
\]

5Note that the weak secrecy capacity is an upper bound for the strong secrecy capacity.

6More precisely, [4, Lemma 7] requires \(\epsilon_{\Lambda_e}(c\sqrt{P}) < 1\) for some constant \(c\) that can be taken arbitrarily close to 1, although the approximation becomes less precise as \(c \to 1\). See [4, Remark 6].
If the main channel and the eavesdropper’s channel verify the conditions (19) and (20), then the codes $C(\Lambda_b, \Lambda_e)$ achieve strong secrecy for any message distribution $p_M$, and thus they achieve semantic security, if

$$R' > C_e + \ln \left( \frac{e}{\pi} \right) - \ln t_e,$$

$$R + R' < C_b - \ln \left( 4 \frac{\pi e}{\pi} \right) + \ln t_b.$$

Thus, any strong secrecy rate

$$R < C_b - C_e - 2 \ln \left( \frac{e}{\pi} \right) + \ln t_b t_e$$

is achievable with the proposed lattice codes.

Then, we can state the following Corollary.

**Corollary 3.6:** If the main channel and the eavesdropper’s channel verify the conditions (19) and (20) respectively, then the wiretap coding scheme $C(\Lambda^{(k)}; R, R')$ achieves strong secrecy and semantic security if

$$R' > C_e + \ln \left( \frac{Ge}{2\pi} \right), \quad R + R' < C_b - \ln \left( 2G \frac{\pi e}{\pi} \right).$$

Thus, any strong secrecy rate

$$R < C_b - C_e - 2 \ln \left( \frac{G}{\pi} \right)$$

is achievable with the proposed lattice codes.

**Proof of the Corollary:** By using the definition of normalized product distance and Lemma 2.10, we find that for the number field lattices $C(\Lambda^{(k)}; R, R')$ we have $\text{NP}(\Lambda_e) \geq 2/G$ and $\text{NP}(\Lambda_e^*) \geq 2/G$.

**Remark 3.7:** Let $S(C_b, C_e)$ denote the set of all ergodic stationary isotropically invariant fading processes $\{(H_b, H_e)\}$ such that (19) and (20) hold. The proposed codes are *almost universal* in the sense that a fixed coding scheme $C(\Lambda^{(k)}; R, R')$ with rates satisfying (24) achieves strong secrecy and semantic security over *all* channels in the set $S(C_b, C_e)$. Moreover, it is clear from the statement of Theorem 3.6 that this fixed code will also achieve secrecy over all fading processes in $S(C_b'; C_e')$ for all $C_b' \geq C_b$ and for all $C_e' \leq C_e$.

Although a rate of convergence of the order $o \left( \frac{1}{k} \right)$ in the law of large numbers for Eve’s channel seems to be necessary for strong secrecy, any rate of convergence is enough to guarantee weak secrecy:

**Proposition 3.8:** Suppose that (23) holds for the wiretap scheme $C(\Lambda_b, \Lambda_e)$. If the condition (19) holds for the main channel and $\forall \delta > 0$ we have

$$\lim_{k \to \infty} \mathbb{P} \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \ln \left( 1 + \frac{P |h_{k,0}^2|}{\sigma_e^2} \right) - C_e \right| > \delta \right\} = 0$$

for the eavesdropper’s channel, then $C(\Lambda_b, \Lambda_e)$ achieves weak secrecy for all rates (24). In particular, any weak secrecy rate $R < C_b - C_e - 2 \ln \left( \frac{G}{\pi} \right)$ is achievable with the codes $C(\Lambda^{(k)}; R, R')$.

A sketch of the proof of Proposition 3.8 can be found in Appendix I-F.

**Remark 3.9:** At least in the settings in which the secrecy capacity is known and is equal to $C_s = C_b - C_e$, the proposed lattice schemes incur a gap to secrecy capacity of $2 \ln(G/\pi)$ nats per channel use, i.e. approximately 6.76 nats (or 9.76 bits) per channel use.

### D. Proof of Theorem 3.5: Secrecy

Let $x \in \Lambda_b$ be the lattice point sampled by Alice from the discrete Gaussian $D_{\lambda_b, \lambda_M, \sigma_e}$. Then, the received signal $z$ at Eve’s end is $z = H_e x + w_e$. Since the message $M$ and the channel $H_e$ are independent, the leakage can be expressed as follows:

$$\mathbb{I}(M; z, H_e) = \mathbb{I}(M; H_e) + \mathbb{I}(M; z|H_e) = \mathbb{I}(M; z|H_e) = \mathbb{E}_{H_e} \left[ \mathbb{I}(P_M|H_e; P_{z|H_e}) \right] = \mathbb{E}_{H_e} \left[ \mathbb{I}(P_M; P_{z|H_e}) \right].$$

We want to show that the average leakage with respect to the fading is small. In order to do so, we will show that for any confidential message $m$, the output distributions $P_{z|H_e, M=m}$ are close to a Gaussian distribution with
high probability. For a fixed channel realization $H_e$ and a fixed message $M = m$, from Lemma 2.5 we have that $H_e \sim D_{H_{e,0} + H_e \lambda_m, \sqrt{H_e} \mathcal{N} \sqrt{P}}$. Using Lemma 2.4 with $\Sigma_1 = H_e H_e^\dagger P$, $\Sigma_2 = \sigma_e^2 I$, we have
\[
\mathbb{V}(p_{H_e, M = m}, f_{\sqrt{\Sigma}}) \leq \varepsilon
\] (28)
provided that
\[
\epsilon_{H_e, \Lambda_e} (\sqrt{\Sigma}) \leq \varepsilon \leq \frac{1}{2},
\]
where we define
\[
\Sigma_0 = H_e H_e^\dagger P + \sigma_e^2 I, \quad \Sigma^{-1} = \frac{(H_e H_e^\dagger)}{P} + \frac{I}{\sigma_e^2}.
\]
If (28) holds, then it follows from [4, Lemma 2] that
\[
\mathbb{I}(p_M; p_{H_e}) \leq 8k \varepsilon R - 8 \varepsilon \log 8 \varepsilon.
\] (29)
Note that $\epsilon_{H_e, \Lambda_e} (\sqrt{\Sigma}) = \epsilon_{\sqrt{\Sigma^{-1} H_e \Lambda_e}} (1)$. Recalling the upper bound (3) in Lemma 2.3, we have that for any $c > \frac{1}{\sqrt{2 \pi}}$, $C = c \sqrt{2 \pi e - \pi e^2}$, $\varepsilon_k = \frac{C c_k}{1 - e^2}$.

Replacing in (30), we find that for $\varepsilon_k = \frac{C c_k}{1 - e^2}$,
\[
\eta_k (\sqrt{\Sigma^{-1} H_e \Lambda_e}) \leq \frac{2c \sqrt{k}}{\lambda_1 (\sqrt{\Sigma^{-1} H_e \Lambda_e})} = \frac{2c \sqrt{k}}{\lambda_1 (\sqrt{\Sigma (H_e^\dagger)^{-1} \Lambda_e^\dagger})}.
\] (30)
Using (9) and the arithmetic mean – geometric mean inequality,
\[
\lambda_1 (\sqrt{\Sigma (H_e^\dagger)^{-1} \Lambda_e^\dagger}) \geq \sqrt{k} \prod_{i=1}^{k} \left( \frac{P \sigma_e^2}{\sigma^2 + P |h_{e,i}|^2} \right)^{\frac{1}{2}} \min_{x \in \Lambda_e \setminus \{0\}} \prod_{i=1}^{k} |x_i| \leq \sqrt{k} \prod_{i=1}^{k} \left( \frac{P \sigma_e^2}{\sigma^2 + P |h_{e,i}|^2} \right)^{\frac{1}{2}} p(\Lambda_e^\dagger)^{\frac{1}{k}}
\]
for fixed fading $H_e$. Given $\delta > 0$, the law of large numbers (20) implies that
\[
\mathbb{P} \left\{ \prod_{i=1}^{k} \left( 1 + \frac{P}{\sigma^2} |h_{e,i}|^2 \right)^{\frac{1}{k}} > e^{C_e + \delta} \right\} \to 0.
\] (31)
Now suppose that
\[
\frac{2ce^{-C_e + \frac{1}{2}}}{p(\Lambda_e^\dagger)^{\frac{k}{2}} \sqrt{2\pi P}} \leq 1.
\] (32)
We can bound the leakage as follows:
\[
\mathbb{E}_{H_e} \left[ \mathbb{I}(p_M; p_{H_e}) \right] \leq \mathbb{E}_{H_e} \left[ \mathbb{I}(p_M; p_{H_e}) \right] \leq \mathbb{P} \left\{ \prod_{i=1}^{k} \left( 1 + \frac{P}{\sigma^2} |h_{e,i}|^2 \right)^{\frac{1}{k}} > e^{C_e + \delta} \right\} (kR) + \mathbb{E}_{H_e} \left[ \mathbb{I}(p_M; p_{H_e}) \right] \leq e^{C_e + \delta}
\] (33)
The first term vanishes when $k \to \infty$ due to the condition (20). Now consider the second term. Under the hypothesis that $\prod_{i=1}^{k} \left( 1 + \frac{P}{\sigma^2} |h_{e,i}|^2 \right)^{\frac{1}{k}} \leq e^{C_e + \delta}$, we have
\[
\epsilon_{H_e, \Lambda_e} (\sqrt{\Sigma}) = \epsilon_{\sqrt{\Sigma^{-1} H_e \Lambda_e}} (1) \leq \epsilon_{\sqrt{\Sigma^{-1} H_e \Lambda_e}} \left( \frac{2ce^{-C_e + \frac{1}{2}}}{p(\Lambda_e^\dagger)^{\frac{k}{2}} \sqrt{2\pi P}} \right) \leq \epsilon_{\sqrt{\Sigma^{-1} H_e \Lambda_e}} \left( \frac{2ce^{-C_e + \frac{1}{2}}}{p(\Lambda_e^\dagger)^{\frac{k}{2}} \sqrt{2\pi P}} \right) \leq \varepsilon_k \to 0
\] (34)
when $k \to \infty$. Using (29), the second term in (33) is also vanishing and the lattice coding scheme achieves strong secrecy over Eve’s channel.

Recalling the definition of normalized product distance and the scaling condition (22), we have

$$N_p(\Lambda_e^*) = \frac{p(\Lambda_e^*)}{\sqrt{V(\Lambda_e^*)}} = \frac{p(\Lambda_e^*)}{\sqrt{\frac{\pi e P}{e R'/2}}}.$$ 

Thus we can rewrite the condition (32) as

$$\frac{2ec^2 e^{C_e + \delta}}{N_p(\Lambda_e^*)^2 e R'} \leq 1.$$

In particular if the bound (23) holds for $N_p(\Lambda_e^*)^{2/k}$, this condition will be guaranteed if

$$\frac{2ec^2 e^{C_e + \delta}}{e R' \sqrt{\frac{\pi e P}{t_e e R'}}} \leq 1,$$

or equivalently if

$$R' > C_e + \delta + \ln \left( \frac{e}{\pi} \right) - \ln t_e.$$

Since this is true for any $\delta > 0$ and any $c > \frac{1}{\sqrt{2 \pi}}$, we find that a rate

$$R' > C_e + \ln \left( \frac{e}{\pi} \right) - \ln t_e \quad (35)$$

is required for strong secrecy.

**Entropy of auxiliary message.** Note that if (35) holds, then the condition (21) required for the correct approximation of the information rate $R'$ is automatically satisfied. In fact, we have

$$\eta_e(\Lambda_e) \leq \frac{2c \sqrt{k}}{\lambda_1(\Lambda_e^*)}.$$

By the arithmetic–geometric mean inequality,

$$\lambda_1(\Lambda_e^*) \geq \sqrt{k} p(\Lambda_e^*)^{\frac{1}{k}} = \sqrt{k} N_p(\Lambda_e^*)^{\frac{1}{k}} \sqrt{\frac{\pi e P}{e R' /2}} \geq \sqrt{k} \sqrt{t_e} e^{R'/2} \sqrt{\frac{\pi e P}{2}}.$$

Therefore $\varepsilon_{\Lambda_e}(\sqrt{P}) = \varepsilon_k \to 0$ provided that

$$\sqrt{P} \geq \frac{2c \sqrt{\pi e P}}{\sqrt{t_e} e^{R'/2} \sqrt{2 \pi}},$$

or equivalently $R' \geq \ln(2c^2 e) - \ln t_e$. For $c \to \frac{1}{\sqrt{2 \pi}}$, we find the condition

$$R' \geq \ln \left( \frac{e}{\pi} \right) - \ln t_e \quad (36)$$

which is weaker than (35).

**Remark 3.10:** Note than in equation (35), we improve the gap compared to the conference version of this paper, due to considering general $c > \frac{1}{\sqrt{2 \pi}}$ rather than $c = 1$.

**Remark 3.11:** In this proof we’re only using the fact that the probability to have a good channel for Eve is vanishing faster than $\frac{1}{k}$. Consequently, in the case when Alice does not know Eve’s channel capacity $C_e$ but only knows an upper bound $\bar{C}_e \geq C_e$ such that

$$\lim_{k \to \infty} k \mathbb{P} \left\{ \frac{1}{k} \sum_{i=1}^{k} \ln \left( 1 + \frac{P |h_e,i|^2}{\sigma_e^2} \right) > \bar{C}_e + \delta' \right\} = 0 \quad (37)$$

holds, strong secrecy is still guaranteed provided that $R' > \bar{C}_e + \ln \left( \frac{e}{\pi} \right) - \ln t_e$. 


E. Proof of Theorem 3.3: Reliability

Recall that to transmit the message $m$, Alice samples $x$ from the discrete Gaussian $D_{\Lambda_e + \lambda_m, \sigma_e}$. Let $y = H_b x + w_b$ be the received signal at Bob. Note that if Bob correctly decodes $x$, he can also identify the right coset of $\Lambda_e$ in $\Lambda_b$, and consequently the confidential message $m$.

We suppose that Bob performs MMSE-GDFE preprocessing as in [27]: let $\rho_b = \frac{P}{\sigma_b^2}$, and consider the QR decomposition

$$
\tilde{H}_b = \left( \frac{H_b}{\sqrt{\rho_b}} I \right) = \left( \begin{array}{c} Q_1 \\ Q_2 \end{array} \right) R,
$$

where $R, Q_1 \in M_k(C)$. Observe that $\tilde{H}_b^\dagger \tilde{H}_b = H_b^\dagger H_b + \frac{L}{\rho_b} = R^\dagger R$, and

$$
\|y - H_b x\|^2 + \frac{1}{\rho_b} \|x\|^2 = x^\dagger H_b H_b x + y^\dagger H_b x - x^\dagger H_b^\dagger y + y^\dagger y + \frac{x^\dagger x}{\rho_b} = x^\dagger R^\dagger Rx - y^\dagger Q_1 Rx - x^\dagger R^\dagger Q_1^\dagger y + y^\dagger y = \|Q_1^\dagger y - Rx\|^2 + C,
$$

where $C$ is a constant which does not depend on $x$.

Since the distribution of $x$ is not uniform, MAP decoding is not equivalent to ML. However, similarly to [4, Theorem 5], for fixed $H_b$, which is known at the receiver, the result of MAP decoding can be written as

$$
\hat{x}_{\text{MAP}} = \arg\max_{x \in \Lambda_b} p(x|y) = \arg\max_{x \in \Lambda_b} (p(x)p(y|x)) = \arg\max_{x \in \Lambda_b} \left( e^{-\frac{1}{2\rho_b} \|y - H_b x\|^2} \right)
$$

$$
= \arg\min_{x \in \Lambda_b} \left( \frac{1}{\rho_b} \|x\|^2 + \|y - H_b x\|^2 \right) = \arg\min_{x \in \Lambda_b} \|Q_1^\dagger y - Rx\|^2.
$$

Thus, Bob can compute

$$
y' = Q_1^\dagger y = Rx + v,
$$

where $v = Q_1^\dagger w_b - \frac{1}{\rho_b} (R^{-1})^\dagger x$ [27].

Clearly, the error probability for the original system model with optimal (MAP) decoding is upper bounded by the ML error probability for the system model [38].

The noise $v$ is the sum of a discrete Gaussian and of a continuous Gaussian. We will show that its tails behave similarly to a Gaussian random variable.

Suppose that a fixed message $m$ has been transmitted, so that $x \sim D_{\Lambda_e + \lambda_m, \sqrt{P}}$. It follows from Lemma 2.7 that $x$ is $\delta$-subgaussian with parameter $\sqrt{P}$ for $\delta = \ln \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)$ provided that

$$
\varepsilon = \varepsilon_{\Lambda_e}(\sqrt{P}) < 1,
$$

which is guaranteed by (30). This is weaker than the condition (35) we have already imposed for secrecy, so it doesn’t affect the achievable secrecy rate. Consequently, for the equivalent noise $v$,

$$
\mathbb{E}[e^{R(t^\dagger v)}] = \mathbb{E} \left[ e^{R(t^\dagger Q_1^\dagger w_b)} \right] \mathbb{E} \left[ e^{-R(\frac{1 + \varepsilon}{\rho_b} (R^{-1})^\dagger x)} \right] \leq \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) e^{\frac{\varepsilon^2}{1 - \varepsilon} \|Q_1^\dagger Q_1 + \frac{1}{\rho_b} (R^{-1})^\dagger R^{-1}\| t^\dagger t} = \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) e^{\frac{\varepsilon^2}{1 - \varepsilon} \|t\|^2}
$$

since

$$
Q_1^\dagger Q_1 + \frac{1}{\rho_b} (R^{-1})^\dagger R^{-1} = (R^{-1})^\dagger \left( H_b^\dagger H_b + \frac{1}{\rho_b} I \right) R^{-1} = I.
$$

Therefore, $v$ is $\delta$-subgaussian with parameter $\sigma_b$.

For fixed $R$, from the union bound for the error probability we get

$$
P_e(R) \leq \sum_{x \in \Lambda_b, x' \neq x} \mathbb{P}\{ x \rightarrow x'| R \}$$
Note that we have
\[ P \{ \mathbf{x} \to \mathbf{x}' | R \} = P \left\{ \| \mathbf{v} - R(\mathbf{x} - \mathbf{x}') \|^2 \leq \| \mathbf{v} \|^2 \right\} = P \left\{ 2 \langle R(\mathbf{x} - \mathbf{x}'), \mathbf{v} \rangle \geq \| R(\mathbf{x} - \mathbf{x}') \|^2 \right\} \]
where \( a = \Re \left( \frac{(R(\mathbf{x} - \mathbf{x}'))^\dagger \mathbf{v}}{\| R(\mathbf{x} - \mathbf{x}') \|^2} \right) \) is a real scalar random variable with zero mean. By subgaussianity of \( \mathbf{v} \), \( \forall t > 0 \)
\[ E[e^{ta}] \leq e^{\delta e^{4\tau^2/\delta t}}. \]
Using the Chernoff bound, we find that \( \forall t > 0 \)
\[ P \left\{ a \geq \frac{1}{2} \| R(\mathbf{x} - \mathbf{x}') \| \right\} \leq E[e^{ta}] e^{-\frac{1}{4} \| R(\mathbf{x} - \mathbf{x}') \|^2} \leq e^{\delta e^{4\tau^2/\delta t} e^{-\frac{1}{4} \| R(\mathbf{x} - \mathbf{x}') \|^2}}. \]
The tightest bound is obtained for \( t = \| R(\mathbf{x} - \mathbf{x}') \| / \sigma_b^2 \) and yields
\[ P \left\{ \mathbf{x} \to \mathbf{x}' | R \right\} \leq e^{\delta e^{\| R(\mathbf{x} - \mathbf{x}') \|^2 / 4\tau}}. \]
Therefore we find
\[ P_e(R) \leq e^{\delta} \sum_{\lambda \in R\Lambda_b \setminus \{0\}} e^{-\frac{1}{4} \lambda_1^2 \tau^2 / \lambda_1^2}. \]
Due to Lemma 2.3 equation (4), \( P_e(R) \to 0 \) as long as
\[ \tau^2 = \frac{1}{4\pi \sigma_b^2} > \frac{2c^2k}{\lambda_1(R\Lambda_b)^2}. \]
The minimum distance in the received lattice is lower bounded as follows using the arithmetic – geometric mean inequality:
\[ \lambda_1(R\Lambda_b)^2 = \min_{\mathbf{x} \in \Lambda_b \setminus \{0\}} \sum_{i=1}^k |R_i x_i|^2 \geq k \prod_{i=1}^k \left( \frac{1}{\rho_b} + |h_{b,i}|^2 \right)^{\frac{1}{2}} p(\Lambda_b)^2. \]
From the scaling condition (22), we have
\[ p(\Lambda_b) = N p(\Lambda_b) \sqrt{\langle \Lambda_b \rangle} = N p(\Lambda_b) \frac{\sqrt{\pi} e^{P_b}}{e^{(R + R')/2}}. \]
Replacing in (40), we find that \( P_e(R) \to 0 \) when \( k \to \infty \) as long as
\[ e^{R + R'} < \frac{N p(\Lambda_b) \sqrt{\langle \Lambda_b \rangle}}{8c^2} \prod_{i=1}^k \left( 1 + \frac{P}{\sigma_b^2} |h_{b,i}|^2 \right)^{\frac{1}{2}}. \]
Using the assumption (23), a sufficient condition is that
\[ e^{R + R'} < \frac{t_b e}{8c^2} \prod_{i=1}^k \left( 1 + \frac{P}{\sigma_b^2} |h_{b,i}|^2 \right)^{\frac{1}{2}}. \]
By the law of total probability, \( \forall \eta > 0 \),
\[ P_e \leq P \left\{ \prod_{i=1}^k \left( 1 + \frac{P}{\sigma_b^2} |h_{b,i}|^2 \right)^{1/k} < e^{C_k - \eta} \right\} + P \left\{ \hat{\mathbf{x}} \neq \mathbf{x} \prod_{i=1}^k \left( 1 + \frac{P}{\sigma_b^2} |h_{b,i}|^2 \right)^{1/k} \geq e^{C_k - \eta} \right\}. \]
The first term vanishes when \( k \to \infty \) due to the law of large numbers (19). The second term tends to 0 if
\[ R + R' < C_b - \eta - \ln \left( \frac{8c^2}{e} \right) + \ln t_b. \]
Since $\eta$ and $c > \frac{1}{\sqrt{2\pi}}$ are arbitrary, any rate

$$R + R' < C_b - \ln \left( \frac{4}{\pi e} \right) + \ln t_b$$

is achievable for Bob.

From equations (35) and (41), the proposed coding scheme achieves strong secrecy for any message distribution (and thus semantic security) for any secrecy rate

$$R < C_b - C_e - 2 \ln \left( \frac{2}{\pi} \right) + \ln t_b t_e.$$

This concludes the proof of Theorem 3.6.

Remark 3.12: In the conference version of this paper [1], the error probability estimate was based on the sphere bound, while in this paper it is based on the union bound. Both approaches give the same gap to Bob’s capacity.

Remark 3.13: Note that in this proof we only need the one-sided law of large numbers

$$\lim_{k \to \infty} P \left\{ \frac{1}{k} \sum_{i=1}^{k} \ln \left( 1 + \frac{P}{\sigma_b^2} |h_{e,i}|^2 \right) < C_b - \delta \right\} = 0.$$

Therefore if Alice does not know Bob’s capacity $C_e$ but only knows an upper bound $\tilde{C}_b \leq C_b$, reliability holds provided that $R + R' < \tilde{C}_b - \ln \left( \frac{4}{\pi e} \right) + \ln t_b$.

Remark 3.14: From Remarks 3.11 and 3.13, we can conclude that if Alice does not know the exact capacities $C_b$ and $C_e$ but is provided with a lower bound $\tilde{C}_b \leq C_b$ and an upper bound $\tilde{C}_e \geq C_e$ such that (37) holds, the scheme can still achieve strong secrecy rates $R < \tilde{C}_b - \tilde{C}_e - 2 \ln \left( \frac{2}{\pi} \right) + \ln t_b t_e$.

F. Gaussian wiretap channel

Although in our proofs we used the product distance properties of the lattices $\Lambda_b$ and $\Lambda_e$, if we assume that the channels under consideration are Gaussian, we only need to know that the Hermite invariants of $\Lambda_b$ and $\Lambda_e$ are large.

Consider the special case of the channel model (17) where $h_{b,i}, h_{e,i}$ are constant and equal to 1 for all $i = 1, \ldots, k$:

$$\begin{align*}
    y_i &= x_i + w_{b,i}, \\
    z_i &= x_i + w_{e,i},
\end{align*}$$

(42)

Proposition 3.15: Consider the wiretap scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ in Section III-B and suppose that

$$h(\Lambda_b) \geq kh_b^2, \quad h(\Lambda_e) \geq kh_e^2$$

for some positive constants $h_b, h_e$. Then the codes $\mathcal{C}(\Lambda_b, \Lambda_e)$ achieve strong secrecy and semantic security if

$$R' > \ln \left( 1 + \frac{P}{\sigma_e^2} \right) + \ln \left( \frac{e}{\pi} \right) - \ln h_e, \quad R + R' < \ln \left( 1 + \frac{P}{\sigma_b^2} \right) - \ln \left( \frac{4}{\pi e} \right) + \ln h_b.$$

Thus, any strong secrecy rate

$$R < \ln \left( 1 + \frac{P}{\sigma_b^2} \right) - \ln \left( 1 + \frac{P}{\sigma_e^2} \right) - 2 \ln \left( \frac{2}{\pi} \right) + \ln h_b h_e$$

is achievable with the proposed lattice codes.

The proof of Proposition 3.15 is very similar to the proof of Theorem 3.5. A sketch is provided in Appendix I-G.
IV. ALGEBRAIC LATTICE CONSTRUCTIONS FOR MULTI-ANTENNA CHANNELS

In this section, we will recall the algebraic constructions of lattice codes for multiple antenna wireless channels, which will be needed for the wiretap coding scheme in the MIMO case.

A. Matrix lattices

The space $M_{nk \times n}(\mathbb{C})$ is a $2n^2k$-dimensional real vector space endowed with a real inner product
\[
\langle X, Y \rangle = \Re(\text{Tr}(X^\dagger Y)),
\]
where $\text{Tr}$ is the matrix trace. This inner product defines a metric on the space $M_{nk \times n}(\mathbb{C})$ by setting $\lVert X \rVert = \sqrt{\langle X, X \rangle}$.

**Remark 4.1:** Consider the function $\xi : M_{nk \times n}(\mathbb{C}) \to \mathbb{C}^{n^2k}$ which vectorizes each matrix by stacking its columns. Note that $\xi$ is an isometry between $M_{nk \times n}(\mathbb{C})$ with the previously defined inner product and $\mathbb{C}^{n^2k}$ with the inner product (43).

Given $H \in M_{nk \times nk}(\mathbb{C})$ and $X \in M_{nk \times n}(\mathbb{C})$, we have
\[
\xi(HX) = H\xi(X), \quad H = H \otimes I_n.
\]

Given a matrix $X \in M_{nk \times n}(\mathbb{C})$ of the form
\[
X = \begin{pmatrix}
X_1 \\
\vdots \\
X_k
\end{pmatrix},
\]
we introduce the notation
\[
X^h = \begin{pmatrix}
X_1^\dagger \\
\vdots \\
X_k^\dagger
\end{pmatrix}.
\]

We also define the **product determinant** as follows:
\[
p\det(X) = \prod_{i=1}^{k} \det(X_i). \tag{46}
\]

**Remark 4.2:** For $X$ of the form (45), we have
\[
\lVert X \rVert^2 = \sum_{i=1}^{k} \lVert X_i \rVert^2 \overset{(a)}{=} n \sum_{i=1}^{k} \lvert \det(X_i) \rvert^2 \overset{(b)}{=} nk \prod_{i=1}^{k} \lvert \det(X_i) \rvert^{\frac{2}{nk}} = nk \lvert \prod_{i=1}^{k} \det(X_i) \rvert^{\frac{2}{nk}}. \tag{47}
\]

Here (a) follows from the inequality $\lVert A \rVert^n \geq \lvert \det(A) \rvert n^{n/2}$ for any $A \in M_n(\mathbb{C})$, and (b) follows from the arithmetic-geometric mean inequality.

**Definition 4.3:** A matrix lattice $L \subseteq M_{nk \times n}(\mathbb{C})$ has the form
\[
L = \mathbb{Z}B_1 \oplus \mathbb{Z}B_2 \oplus \cdots \oplus \mathbb{Z}B_r,
\]
where the matrices $B_1, \ldots, B_r$ are linearly independent over $\mathbb{R}$, i.e., form a lattice basis, and $r$ is called the **rank** or the **dimension** of the lattice.

The **Gram matrix** of an $r$-dimensional lattice $L \subseteq M_{nk \times n}(\mathbb{C})$ is defined as
\[
G(L) = ((X_i, X_j))_{1 \leq i, j \leq r},
\]
where $\{X_i\}_{1 \leq i \leq r}$ is a basis of $L$. The volume of the fundamental parallelepiped of $L$ is then given by
\[
V(L) = \sqrt{|\det(G(L))|}.
\]

**Definition 4.4:** Given a lattice $L$ in $M_{nk \times n}(\mathbb{C})$, the **dual lattice** is defined as
\[
L^* = \{X \in M_{nk \times n}(\mathbb{C}) \mid \forall Y \in L, \langle X, Y \rangle \in \mathbb{Z}\}.\]
We also define the product determinant and normalized minimum determinant of the matrix lattice \( L \subset M_{kn \times n}(\mathbb{C}) \) as follows:

\[
pdet(L) = \min_{X \in L \setminus \{0\}} pdet(X),
\]
\[
\delta(L) = \frac{\text{pdet}(L)}{V(L)^{\frac{n}{kn}}},
\]

**B. MIMO lattices from division algebras**

We will first recall the construction of single-block space-time codes from cyclic division algebras (see for example [28]). Due to space constraints, we refer the reader to [29] for algebraic definitions.

**Definition 4.5:** Let \( F \) be an algebraic number field of degree \( 2k \) and assume that \( E/F \) is a cyclic Galois extension of degree \( n \) with Galois group \( \text{Gal}(E/F) = \langle \sigma \rangle \). We can define an associative \( F \)-algebra

\[
A = (E/F, \sigma, \gamma) = E \oplus uE \oplus u^2E \oplus \cdots \oplus u^{n-1}E,
\]

where \( u \in A \) is an auxiliary generating element subject to the relations \( xu = u\sigma(x) \) for all \( x \in E \) and \( u^n = \gamma \in F \setminus \{0\} \).

We call the resulting algebra a cyclic algebra. Here \( F \) is the center of the algebra \( A \).

**Definition 4.6:** We call \( \sqrt{|A : F|} \) the degree of the algebra \( A \). It is easily verified that the degree of \( A \) is equal to \( n \).

We consider \( A \) as a right vector space over \( E \). Every element \( a = x_0 + ux_1 + \cdots + u^{n-1}x_{n-1} \in A \), with \( x_i \in E \) for all \( i = 0, \ldots, n-1 \), has the following representation as a matrix:

\[
\phi(a) = \begin{pmatrix}
x_0 & \gamma \sigma(x_{n-1}) & \gamma \sigma^2(x_{n-2}) & \cdots & \gamma \sigma^{n-1}(x_1) \\
x_1 & \sigma(x_0) & \gamma \sigma^2(x_{n-1}) & \cdots & \gamma \sigma^{n-1}(x_2) \\
x_2 & \sigma(x_1) & \sigma^2(x_0) & \cdots & \gamma \sigma^{n-1}(x_3) \\
\vdots & \sigma(x_2) & \sigma^2(x_1) & \cdots & \cdots \\
x_{n-1} & \sigma(x_{n-2}) & \sigma^2(x_{n-3}) & \cdots & \sigma^{n-1}(x_0)
\end{pmatrix}
\]

The mapping \( \phi \) is called the left regular representation of \( A \) and allows us to embed any cyclic algebra into \( M_n(\mathbb{C}) \). Under such an embedding \( \phi(A) \) forms an \( 2kn^2 \)-dimensional \( \mathbb{Q} \)-vector space.

We are particularly interested in algebras \( A \) for which \( \phi(a) \) is invertible for all non-zero \( a \in A \).

**Definition 4.7:** A cyclic \( F \)-algebra \( D \) is a division algebra if every non-zero element of \( D \) is invertible.

In order to code over several fading blocks, we will next define a multi-block lattice construction based on a cyclic division algebra. A multi-block embedding was constructed in [30, 31] for division algebras whose center \( F \) contains an imaginary quadratic field. In this paper we consider a more general multi-block embedding proposed in [32], which applies to any totally complex center \( F \).

Let \( F \) be totally complex of degree \( [F: \mathbb{Q}] = 2k \). \( F \) admits \( 2k \) \( \mathbb{Q} \)-embeddings \( \alpha_i : F \hookrightarrow \mathbb{C} \) in complex conjugate pairs: \( \alpha_i = \overline{\alpha_{i+k}} \), for \( 1 \leq i \leq k \). Each \( \alpha_i \) can be extended to an embedding \( E \hookrightarrow \mathbb{C} \). Given \( a \in D \), consider the mapping \( \psi : D \hookrightarrow M_{nk \times n}(\mathbb{C}) \) given by

\[
\psi(a) = \begin{pmatrix}
\alpha_1(\phi(a)) \\
\vdots \\
\alpha_k(\phi(a))
\end{pmatrix},
\]

where each \( \alpha_i \) is extended to an embedding \( \alpha_i : M_n(E) \hookrightarrow M_n(\mathbb{C}) \).

**Remark 4.8:** For all \( x \in D \),

\[
\text{pdet}(\psi(a)) = \prod_{i=1}^{k} \det(\alpha_i(\phi(a))) = \prod_{i=1}^{k} \alpha_i(\text{det}(\phi(a))) = (N_{F/\mathbb{Q}}(N_D/F(a)))^{\frac{1}{2}} = (N_{D/\mathbb{Q}}(a))^{\frac{1}{2}},
\]

where (a) follows from the fact that the \( \alpha_i \) are ring homomorphisms, and (b) follows from the definition of the reduced norm.

In order to obtain a matrix lattice, we will consider a suitable discrete subset of the algebra called an order.
Definition 4.9: A \( \mathbb{Z} \)-order \( \Gamma \) in \( \mathcal{D} \) is a subring of \( \mathcal{D} \) having the same identity element as \( \mathcal{D} \), and such that \( \Gamma \) is a finitely generated module over \( \mathbb{Z} \) which generates \( \mathcal{D} \) as a linear space over \( \mathbb{Q} \).

The following result was proven in [32, Proposition 5]:

Proposition 4.10: Let \( \Gamma \) be a \( \mathbb{Z} \)-order in \( \mathcal{D} \) and \( \psi \) the previously defined embedding. Then \( \psi(\Gamma) \) is a \( 2kn^2 \)-dimensional lattice in \( M_{nk \times n}(\mathbb{C}) \) which satisfies

\[
\min_{a \in \Gamma \setminus \{0\}} |\text{pdet}(\psi(a))| = 1, \quad V(\psi(\Gamma)) = 2^{-kn^2} \sqrt{|d(\Gamma/\mathbb{Z})|}.
\]

Here \( d(\Gamma/\mathbb{Z}) \) is a non-zero integer called the \( \mathbb{Z} \)-discriminant of the order \( \Gamma \). We refer the reader to [29] for the relevant definitions.

C. Dual lattice and codifferent

Let \( \Gamma \) be a \( \mathbb{Z} \)-order in \( \mathcal{D} \). We define the codifferent of \( \Gamma \) as

\[
\Gamma^\vee = \{ x \in \mathcal{D} : \text{tr}_{\mathcal{D}/\mathbb{Q}}(x\Gamma) \subseteq \mathbb{Z} \},
\]

where \( \text{tr}_{\mathcal{D}/\mathbb{Q}} \) is the reduced trace.

The codifferent is an ideal of \( \mathcal{D} \), and its reduced norm is related to the discriminant as follows [29]:

\[
N_{\mathcal{D}/\mathbb{Q}}(\Gamma^\vee) = \frac{1}{d(\Gamma/\mathbb{Z})^{\frac{n}{2}}}. \tag{50}
\]

Similarly to the commutative case, the codifferent of \( \Gamma \) embeds as the complex conjugate of the dual lattice.

Lemma 4.11: \( \psi(\Gamma)^* = 2\psi(\Gamma^\vee)^h \).

This Lemma is proven in Appendix I-E.

D. Orders with small discriminants and dense matrix lattices

A family of division algebras with orders having particularly small discriminants was constructed in [13]. These orders yield dense lattices as shown in Proposition 4.10.

First, we need the following Theorem [33, Theorem 6.14]:

Theorem 4.12: Let \( F \) be a number field of degree \( 2k \) and \( P_1 \) and \( P_2 \) be two prime ideals of \( F \). Then there exists a degree \( n \) division algebra \( \mathcal{D} \) having an order \( \Gamma \) with discriminant

\[
d(\Gamma/\mathbb{Z}) = (N_{F/\mathbb{Q}}(P_1)N_{F/\mathbb{Q}}(P_2))^{n(n-1)}(d_F)^{n^2}. \tag{51}
\]

Thanks to this property, a suitable family of division algebras can be chosen in two steps.

First, we should choose an infinite sequence of centers \( \{F_k\} \) with small discriminants, such as Martinet’s sequence (Theorem 2.8). Furthermore, one can choose suitable ideals in these number fields [13, Lemma 7.9]:

Lemma 4.13: Every number field \( F_k \) in the Martinet family has ideals \( P_1 \) and \( P_2 \) such that

\[
N_{F/\mathbb{Q}}(P_1) \leq 23^k/10 \quad \text{and} \quad N_{F/\mathbb{Q}}(P_2) \leq 23^k/10.
\]

This leads us to the main result in [13]:

Theorem 4.14: Given \( n \), there exists a sequence of totally complex number fields \( \{F_k\} \) of degree \( 2k \) and a sequence of division algebras \( \mathcal{D}_k \) of index \( n \) over \( F_k \) having an order \( \Gamma_k \) with discriminant

\[
d(\Gamma_k/\mathbb{Z}) \leq \beta^{2kn(n-1)}G^{2kn^2},
\]

where \( G \approx 92.368 \) and \( \beta = 23^{2k/3} \). Consequently, \( \{\Lambda^{(n,k)}\} = \{\psi(\Gamma_k)\} \) is a sequence of \( 2n^2k \)-dimensional lattices with

\[
\text{pdet}(\Lambda^{(n,k)}) = 1, \quad V(\Lambda^{(n,k)}) \leq \beta^{kn(n-1)} \left( \frac{G}{2} \right)^{n^2k}.
\]
E. Flatness factor of multi-block matrix lattices from division algebras

Remark 4.15: Due to the isometry $\xi$ between $M_{nk \times n}(\mathbb{C})$ and $\mathbb{C}^{n^2 k}$ (Remark 4.1), the definitions of flatness factor, smoothing parameter and discrete Gaussian distribution extend in a natural way for matrix lattices in $M_{nk \times n}(\mathbb{C})$. Given a lattice $\Lambda \subset M_{nk \times n}(\mathbb{C})$, a multi-block matrix $\tilde{X} \in M_{nk \times n}(\mathbb{C})$ and a positive definite matrix $\Sigma \in M_{nk \times nk}(\mathbb{C})$, we define

$$
\epsilon_\Lambda(\sqrt{\Sigma}) \doteq \epsilon_{\xi(\Lambda)}(\sqrt{\Sigma} \otimes I_n),
$$

$$
\eta_\epsilon(\Lambda) \doteq \eta_{\epsilon}(\xi(\Lambda)),
$$

$$
D_{\Lambda-X,\Sigma}(X-X) \doteq D_{\xi(\Lambda-X),\Sigma \otimes I_n}(\xi(X-X)) \quad \forall X \in \Lambda.
$$

Note that these definitions are consistent with the previous ones: for example,

$$
\epsilon_{\sqrt{\Sigma^{-1} \Lambda}}(I) = \epsilon_{\xi(\sqrt{\Sigma^{-1} \Lambda})}(I) = \epsilon_{\xi(\Lambda)}(\sqrt{\Sigma^{-1} \Lambda}) = \epsilon_{\xi(\Lambda)}(\sqrt{\Sigma} \otimes I) = \epsilon_\Lambda(\sqrt{\Sigma}).
$$

We now focus on the sequence of $n^2 k$-dimensional multi-block matrix lattices $\Lambda^{(n,k)} = \psi(\Gamma_k) \subset M_{nk \times n}(\mathbb{C})$ in Theorem 4.14

Let $\frac{c}{\sqrt{2\pi}} \cdot C = \frac{c}{\sqrt{2\pi}} \cdot C e^{-\pi c^2 e}$, $\varepsilon = \frac{c^{2n^2 k}}{1-C^{2n^2 k}}$.

From (5) and Lemma 4.11 we obtain

$$
\eta_\epsilon(\Lambda^{(n,k)}) \leq \frac{n \sqrt{\varepsilon c}}{\lambda_1(\psi(\Gamma_k)^h)}.
$$

(52)

V. MIMO WIRETAP CHANNEL

A. Channel model

We consider a MIMO fading channel model where Alice is equipped with $n$ antennas, while Bob and Eve have $n_b$ and $n_e$ antennas respectively. In this paper, we always assume that $n_b \geq n$ and $n_e \geq n$.

Transmission takes place over $k$ quasi-static fading blocks of delay $T = n$, and the transmitted codeword is of the form (45), where the matrix $X_i \in M_n(\mathbb{C})$ is sent during the $i$-th block.

The outputs $Y$ and $Z$ at Bob and Eve’s end respectively are given by

$$
\begin{align*}
Y &= H_b X + W_b, \\
Z &= H_e X + W_e,
\end{align*}
$$

(53)

where the channel matrices $H_e = \text{diag}(H_{e,1}, \ldots, H_{e,k}) \in M_{nk \times nk}(\mathbb{C})$, $H_b = \text{diag}(H_{b,1}, \ldots, H_{b,k}) \in M_{nk \times nk}(\mathbb{C})$ are (possibly rectangular) block diagonal matrices. The coefficients of the noise matrices $W_b$ and $W_e$ are i.i.d. circularly symmetric complex Gaussian with zero mean and variance $\sigma_b^2$, $\sigma_e^2$ per complex dimension. The input $X$ satisfies the average power constraint (per channel use)

$$
\frac{1}{nk} \sum_{i=1}^{k} \|X_i\|^2 \leq P.
$$

(54)

The average power per symbol is $\sigma_s^2 = \frac{P}{n}$. We denote by $\rho_b = \frac{\sigma_b^2}{\sigma_s^2}$ and $\rho_e = \frac{\sigma_e^2}{\sigma_s^2}$ the signal-to-noise ratios for Bob and Eve respectively.

We suppose that $\{H_{b,i}\}, \{H_{e,i}\}$ are isotropically invariant channels such that the channel capacities $C_b$ and $C_e$ are well-defined and $\forall \gamma, \gamma' > 0$,

$$
\lim_{k \to \infty} \mathbb{P} \left\{ \frac{1}{k} \sum_{i=1}^{k} \ln \det \left( I + \rho_b H_{b,i}^\dagger H_{b,i} \right) - C_b > \gamma \right\} = 0
$$

(55)

and

$$
\lim_{k \to \infty} k \mathbb{P} \left\{ \frac{1}{k} \sum_{i=1}^{k} \ln \det \left( I + \rho_e H_{e,i}^\dagger H_{e,i} \right) - C_e > \gamma' \right\} = 0.
$$

(56)

We suppose that Alice has no instantaneous CSI, Bob has perfect CSI of his own channel, and Eve has perfect CSI of her channel and of Bob’s.
A confidential message $M$ and an auxiliary message $M'$ with rate $R$ and $R'$ respectively are encoded into the multi-block codeword $X$.

As in the single-antenna case (Remark 3.3), we have that $\mathbb{I}(M; Z|H_b, H_e) = \mathbb{I}(M; Z|H_e)$, i.e. the leakage is given by $\mathbb{I}(M; Z|H_e)$.

**Remark 5.1:** For general channels the strong secrecy capacity is not known in this setting (see Remark 3.4 for the SISO case). In [34] it was shown that the weak secrecy capacity

$$C^w = C_b - C_e$$

for i.i.d. fading wiretap channels such that Bob and Eve’s fadings are independent.

### B. Multi-block lattice wiretap coding

Let $\Lambda_e \subset \Lambda_b$ be a pair of nested multiblock matrix lattices in $M_{nk \times n}(\mathbb{C})$ such that $\Lambda_e \subset \Lambda_b$ and $|\Lambda_b/\Lambda_e| = e^{nkR}$. Each message $m \in \mathcal{M} = \{1, \ldots, e^{nkR}\}$ is mapped to a coset leader $X^{(m)} \in \Lambda_b \cap R(\Lambda_e)$, where $R(\Lambda_e)$ is a fundamental region of $\Lambda_e$. In order to transmit the message $m$, Alice samples $X$ from the discrete Gaussian $D_{\Lambda_e + X^{(m)}, \sigma_e}$, where $\sigma_e^2 = \frac{E}{n}$. We denote this coding scheme by $\mathcal{C}(\Lambda_b, \Lambda_e)$.

It follows from [4, Lemma 6] that the variance per complex dimension of $X$ tends to $\sigma_e^2$ as $k \to \infty$ if

$$\lim_{k \to \infty} \epsilon_{\Lambda_e}(\sigma_e) = 0. \quad (57)$$

Using [4, Lemma 7], we can compute the information rate per complex symbol of the auxiliary message:

$$\frac{R'}{n} \approx \ln(\pi e \sigma_e^2 ) - \frac{1}{n^2 k} \ln V(\Lambda_e)$$

Given rates $R, R'$ for the confidential and auxiliary message respectively, we have the scaling

$$V(\Lambda_e) \approx \left(\frac{\pi e \sigma_e^2}{e^{nkR}}\right)^{n^2 k} \quad V(\Lambda_b) \approx \left(\frac{\pi e \sigma_e^2}{e^{nk(R+R')}}\right)^{n^2 k} \quad (58)$$

**Coding scheme based on division algebras with constant root discriminant.** Let $\{\Lambda^{(n,k)} = \{\psi(\Gamma_k)\}$ be the sequence of $n^2 k$-dimensional multi-block matrix lattices in $M_{nk \times n}(\mathbb{C})$ from Theorem 4.14. We consider scaled versions $\Lambda_0 = \alpha_0 \Lambda^{(n,k)}$, $\Lambda_e = \alpha_e \Lambda^{(n,k)}$ such that $\Lambda_e \subset \Lambda_b$ and $|\Lambda_b/\Lambda_e| = e^{nkR}$. Given rates $R, R'$, we denote the corresponding multi-block lattice coding scheme by $\mathcal{C}(\Lambda^{(n,k)}, R, R')$.

### C. Achievable secrecy rates

We now state the main result for MIMO wiretap channels, which will be proven in Sections V-D and V-E.

**Theorem 5.2:** Consider the multi-block wiretap coding scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ in Section V-B and suppose that

$$\delta(\Lambda_e^*) \geq d_e, \quad \delta(\Lambda_b^{*}) \geq d_b \quad (59)$$

for some positive constants $d_e$, $d_b$.

If the main channel and the eavesdropper’s channel verify the conditions (55) and (56) respectively, then $\mathcal{C}(\Lambda_b, \Lambda_e)$ achieves strong secrecy for any message distribution $p_M$ (and thus semantic security) if

$$R' > C_e + n \ln \left(\frac{M e}{\pi}\right) - \ln d_e, \quad (60)$$

$$R + R' < C_b - n \ln \left(\frac{4n}{\pi e}\right) + \ln d_b. \quad (61)$$

Thus, any strong secrecy rate

$$R < C_b - C_e - 2n \ln \left(\frac{2n}{\pi}\right) + \ln d_b d_e \quad (62)$$

is achievable with the proposed lattice codes.
Corollary 5.3: If the main channel and the eavesdropper’s channel verify the conditions (55) and (56) respectively, then the multi-block wiretap coding scheme $C(\Lambda^{(n,k)}, R, R')$ achieves strong secrecy and semantic security if

$$R' > C_e + n \ln \left( \frac{ne\beta^{n+1}G}{2\pi} \right), \quad R + R' < C_b - n \ln \left( \frac{2n\beta^{n+1}G}{\pi e} \right).$$

Thus, any strong secrecy rate

$$R < C_b - C_e - 2n \ln \left( \frac{nG\beta^{n+1}}{\pi} \right)$$

is achievable with the proposed lattice codes.

Proof of the Corollary: From Theorem 4.14 we get

$$\delta(\Lambda_b)^2 = \delta(\Lambda^{(n,k)})^2 = \frac{2^n}{\beta(n-1)G^n}.$$  \[5\]

On the other side, for the dual lattice we have

$$pDet((\Lambda^{(n,k)})^*) \overset{(a)}{=} \sqrt{\frac{N_D}{q(2\psi(\Gamma_k^\vee))}} \overset{(b)}{=} \frac{1}{d(\Gamma_k/Z)^{1/n}} \overset{(c)}{=} \frac{2^{nk}}{\beta^k(n-1)G^k n},$$

where (a) follows from (49), (b) follows from (50) and (c) from Theorem 4.14. The normalized minimum determinant of $\Lambda_e^*$ is

$$\delta(\Lambda_e^*) = \delta((\Lambda^{(n,k)})^*) = \frac{pDet((\Lambda^{(n,k)})^*)}{V((\Lambda^{(n,k)})^*)} = \frac{pDet((\Lambda^{(n,k)})^*)V(\Lambda^{(n,k)})^{1/2}}{\beta^{k(n-1)/2}G^{k/2}} = \frac{2^{nk}}{\beta^k(n-1)G^n},$$

and so we find that

$$\delta(\Lambda_e^*)^2 = \frac{2^n}{\beta(n-1)G^n}.$$  \[\Box\]

Remark 5.4: Let $\mathcal{S}(C_b, C_e)$ denote the set of all ergodic stationary isotropically invariant fading processes $\{(H_b, H_e)\}$ such that (55) and (56) hold. Similarly to the single antenna case, a fixed lattice code sequence $C(\Lambda^{(n,k)}, R, R')$ with rates satisfying (60) and (61) universally achieves strong secrecy and semantic security over all channels in the set $\mathcal{S}(C_b', C_e')$ for all $C_b' \geq C_b$ and for all $C_e' \leq C_e$.

Finally, the condition (56) can be relaxed if only weak secrecy is required:

Proposition 5.5: If the condition (55) holds for the main channel and $\forall \gamma > 0$ we have

$$\lim_{k \to \infty} P \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \ln \text{det} \left( I + \rho_e H_{e,i}^H H_{e,i} \right) - C_e \right| > \gamma \right\} = 0$$

for the eavesdropper’s channel, then the wiretap coding scheme $C(\Lambda_b, \Lambda_e)$ achieves weak secrecy if (60) and (61) hold. In particular, any weak secrecy rate $R < C_b - C_e - 2n \ln \left( \frac{nG\beta^{n+1}}{\pi} \right)$ is achievable with the lattice codes $C(\Lambda^{(n,k)}, R, R')$.

The proof of Proposition 5.5 is very similar to the proof of Theorem 5.2 and is omitted.

D. Proof of Theorem 5.2: Secrecy

The proof follows the same steps as in the single antenna case (Section III-D). We distinguish two cases: the symmetric case ($n_e = n$) and the asymmetric case ($n_e > n$).
a) Case $n_e = n$. The received signal at Eve’s end is $Z = H_e X + W_e$. Similarly to equation (27), the leakage can be written as

$$\mathbb{I}(M;Z,H_e) = \mathbb{E}_{H_e} \left[ \mathbb{I}(p_M;p_Z|H_e) \right]$$

For a fixed realization $H_e = \text{diag}(H_{e,1},\ldots,H_{e,k})$, we have

$$H_e X \sim D_{H_{e,\Lambda_e} + H_e X(m), \sqrt{H_e \Lambda_e} \sigma_s}$$

recalling the notation in Remark 4.15. Using Lemma 2.4 with $\Sigma_1 = H_e H_e^\dagger \sigma_s^2$, $\Sigma_2 = \sigma_s^2 I_{nk}$, we have

$$\forall (p_Z|H_e,M=m), f(\Sigma) \leq \varepsilon$$

provided that $\epsilon_{H_e,\Lambda_e}(\sqrt{\Sigma}) \leq \varepsilon \leq \frac{1}{2}$, where we define

$$\Sigma_0 = H_e H_e^\dagger \sigma_s^2 + \sigma_s^2 I_{nk}, \quad \Sigma^{-1} = \frac{(H_e H_e^\dagger)^{-1}}{\sigma_s^2} + \frac{I_{nk}}{\sigma_s^2}.$$ 

In particular, $\Sigma = \sigma_s^2 \sigma_e^2 (\sigma_s^2 I_{nk} + \sigma_s^2 H_e H_e^\dagger)^{-1} H_e H_e^\dagger$.

If (63) holds, then it follows from [4, Lemma 2] that

$$\mathbb{I}(p_M;p_Z|H_e) \leq 8n^2 k \varepsilon R - 8 \varepsilon \log 8\varepsilon.$$  

From the upper bound (5), for $\varepsilon_k = \frac{C_{2n^2 k}}{1-C_{2n^2 k}}$

$$\eta_e(\sqrt{\Sigma^{-1}} H_e \Lambda_e) \leq \frac{2c n \sqrt{k}}{\lambda_1(\sqrt{\Sigma}(H_e^\dagger)^{-1}(\Lambda_e)^*)}.$$  

Using Remark 4.2 we find

$$\lambda_1(\sqrt{\Sigma}(H_e^\dagger)^{-1}\Lambda_e^*) \geq nk \prod_{i=1}^k \frac{(\sigma_e^2)^{\frac{1}{k}}}{\det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{nk}} \text{pdet}(\Lambda_e^*)}^{\frac{1}{nk}}$$

Replacing in the bound (65) for $\varepsilon = \frac{C_{2n^2 k}}{1-C_{2n^2 k}}$ we have

$$\eta_e(\sqrt{\Sigma^{-1}} H_e \Lambda_e) \leq \frac{2c n \sqrt{k}}{\text{pdet}(\Lambda_e^*)^{\frac{1}{nk}} \sigma_s} \prod_{i=1}^k \det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{nk}}.$$ 

Equivalently,

$$\frac{\varepsilon \sqrt{\Sigma^{-1}} H_e \Lambda_e}{\text{pdet}(\Lambda_e^*)^{\frac{1}{nk}} \sqrt{2\pi \sigma_s}} \left( \frac{2c \sqrt{n} \prod_{i=1}^k \det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{nk}}}{\text{pdet}(\Lambda_e^*)^{\frac{1}{nk}} \sqrt{2\pi \sigma_s}} \right) \leq \varepsilon.$$

Due to the law of large numbers (56), $\forall \gamma > 0$

$$\mathbb{P}\left\{ \prod_{i=1}^k \det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{nk}} > e^{C_{\gamma} + \frac{1}{n}} \right\} \to 0.$$ 

Suppose that

$$\frac{2c \sqrt{n} e^{C_{\gamma} + \frac{1}{n}}}{\text{pdet}(\Lambda_e^*)^{\frac{1}{nk}} \sqrt{2\pi \sigma_s}} \leq 1.$$  

The average leakage is bounded as follows:

$$\mathbb{E}_{H_e} \left[ \mathbb{I}(p_M;p_Z|H_e) \right] \leq \mathbb{P}\left\{ \prod_{i=1}^k \det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{nk}} > e^{C_{\gamma} + \frac{1}{n}} \right\} (n^2 k R) + \mathbb{E}_{H_e} \left[ \mathbb{I}(p_M;p_Z|H_e) \prod_{i=1}^k \det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{nk}} \leq e^{C_{\gamma} + \frac{1}{n}} \right].$$

(67)
The first term vanishes when $k \to \infty$ due to the condition (56). Supposing that $\prod_{i=1}^{k} \left(1 + \frac{P}{\sigma^2} |h_{e,i}|^2\right)^{\frac{1}{2}} \leq e^{C_e + \gamma}$, we have
\[
\epsilon \sqrt{\Sigma^{-1} H_e \Lambda_e} \left(\frac{2c \sqrt{ne} \epsilon^{-\frac{1}{2n}}}{\ln(n^e \epsilon^{-\frac{1}{2n}})} \right) \leq \epsilon \sqrt{\Sigma^{-1} H_e \Lambda_e} \left(\frac{2c \sqrt{n} \prod_{i=1}^{k} \det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{2n}}}{\det(\Lambda_e^{\frac{1}{2n}} \sqrt{2\pi \sigma_s})}\right)
\leq \epsilon_k = \frac{C^{2n^2k}}{1 - C^{2n^2k}}.
\]
Using (64), the second term in (67) tends to zero and the scheme achieves strong secrecy.

Recalling the definition of normalized minimum determinant and the scaling condition (58), we have
\[
p\det(\Lambda_e^{\frac{1}{2n}}) = \left(\delta(\Lambda_e) V(\Lambda_e^{\frac{1}{2n}}) \right)^{\frac{1}{n^e}} = \delta(\Lambda_e^{\frac{1}{2n}}) V(\Lambda_e^{\frac{1}{2n}}) \approx \frac{\delta(\Lambda_e^{\frac{1}{2n}})}{n^e \epsilon^{\frac{1}{2n}} \sqrt{\pi \sigma_s}}.
\]

The sufficient condition (66) for secrecy can be written as
\[
\frac{c \sqrt{2ne} \epsilon^{-\frac{1}{2n}}}{\delta(\Lambda_e^{\frac{1}{2n}})} \leq 1.
\]
In particular if the bound (59) holds for $\delta(\Lambda_e)$, the sufficient condition (66) for secrecy is satisfied if
\[
\frac{c \sqrt{2ne} \epsilon^{-\frac{1}{2n}}}{\det(\Lambda_e^{\frac{1}{2n}})} \leq 1.
\]
or equivalently
\[
R' > C_e + \gamma - \ln d_e + 2n \ln(c \sqrt{2ne}).
\]
Since $\gamma > 0$ and $c > \frac{1}{\sqrt{2\pi}}$ are arbitrary, any rate
\[
R' > C_e - \ln d_e + n \ln \left(\frac{n^e}{\pi}\right)
\]
is sufficient for strong secrecy.

b) Case $n_e > n$. As before, the received signal is $Z = H_{e}X + W_{e} \in H_{e} \Lambda_{e} + H_{e}X^{(m)} + W_{e}$. If $H_{e}$ is full rank, the lattice $H_{e} \Lambda_{e}$ is a $2n^2k$-dimensional lattice contained in a $2nn_{e}k$-dimensional space. Consider the QR decomposition
\[
H_{e} = Q_{e} R_{e}
\]
where $Q_{e} \in M_{n_{e}k \times nk}(\mathbb{C})$ is unitary and $R_{e} \in M_{nk \times nk}(\mathbb{C})$ is upper triangular. We have $Q_{e} = [Q'_{e}, Q''_{e}]$, where $Q'_{e} \in M_{n_{e}k \times nk}(\mathbb{C})$ is such that $(Q'_{e})^\dagger Q'_{e} = I_{nk}$, and $R_{e} = \left[ \begin{array}{c} R'_{e} \\ 0 \end{array} \right]$, $R'_{e} = \textrm{diag}(R'_{e,1}, \ldots, R'_{e,k}) \in M_{nk \times nk}(\mathbb{C})$.

Multiplying Eve’s channel equation in (53) by $Q_{e}^\dagger$, we obtain
\[
Q_{e}^\dagger Z = R_{e} X + Q_{e}^\dagger W_{e} = \left[ \begin{array}{c} R'_{e} X + (Q'_{e})^\dagger W_{e} \\ (Q''_{e})^\dagger W_{e} \end{array} \right] = \left[ \begin{array}{c} Z' \\ Z'' \end{array} \right]
\]
Therefore, the second component is pure noise and contains no information about the message. Since $Q''_{e}$ is unitary, $W'_{e} = (Q''_{e})^\dagger W_{e}$ has the same distribution as $W_{e}$ and is independent of $X$ and $H_{e}$. Consequently, we can rewrite the leakage as
\[
\| (M; Z | H_{e}) = \| (M; Z' | H_{e}) = \| (M; Z' | R'_{e})
\]
The rest of the proof then proceeds exactly as in the case $n_{e} = n$, by replacing $Z$ with $Z'$ and $H_{e}$ with $R'_{e}$. Observe that $H'_{e} H_{e} = (R'_{e})^\dagger R'_{e}$ and so
\[
\prod_{i=1}^{k} \det(I + \rho_e (R'_{e,i})^\dagger R'_{e,i})^{\frac{1}{2}} = \det(I + \rho_e (R'_{e})^\dagger R'_{e})^{\frac{1}{2}} = \det(I + \rho_e (H_{e})^\dagger H_{e})^{\frac{1}{2}}
\]

\[
= \prod_{i=1}^{k} \det(I + \rho_e (H'_{e,i})^\dagger H_{e,i})^{\frac{1}{2}}.
\]
Entropy of auxiliary message. We still need to check that (57) holds. Using Remark 4.2, we have

$$\lambda_1(\Lambda^*_e) \geq \sqrt{n}k \ln \det(\Lambda^*_e) = \frac{\sqrt{n}k \delta(\Lambda^*_e)}{\pi \sigma_s} \approx \frac{\sqrt{n}k \delta(\Lambda^*_e) e^{\frac{b_0}{2n}}}{\sqrt{\pi \epsilon \sigma_s}} \geq \frac{\sqrt{n}k \epsilon d^2 e^{\frac{b_0}{2n}}}{\sqrt{\pi \epsilon \sigma_s}}$$

Then we have

$$\eta_e(\Lambda^*_e) \leq \frac{2cn\sqrt{k}}{\lambda_1(\Lambda^*_e)} \leq \frac{2c\sqrt{n} \pi \epsilon \sigma_s}{d^2 e^{\frac{b_0}{2n}}}$$

Then $\epsilon_{\Lambda_e}(\sigma_s) \to 0$ provided that

$$\sigma_s \geq \frac{2c\sqrt{n} \pi \epsilon \sigma_s}{d^2 e^{\frac{b_0}{2n}} (\sqrt{\pi} \sqrt{2})}$$

or equivalently

$$R' \geq n \ln(2n c^2) - \ln d_e.$$

In particular when $c \to \frac{1}{\sqrt{2\pi}}$ we obtain the condition

$$R' \geq n \ln \left(\frac{ne}{\pi}\right) - \ln d_e,$$

which is weaker than (69).

E. Proof of Theorem 5.2: Reliability

Recall that the received signal at Bob is $Y = H_b X + W_b$. Let $\rho_b = \frac{\sigma_s^2}{\sigma_b^2}$, and consider the “thin” QR decomposition

$$\tilde{H}_b = \left( \begin{array}{c} H_b \\ \frac{1}{\sqrt{\rho_b}} I_{nk} \end{array} \right) = QR_b = \left( \begin{array}{c} Q_1 \\ Q_2 \end{array} \right) R_b,$$

where $\tilde{H}_b, Q \in M_{k(n_k+n) \times kn}$. Note that $Q$ has orthonormal columns, $R_b \in M_{kn}$ is upper triangular and square block-diagonal, and

$$R_b^1 R_b = \tilde{H}_b^1 \tilde{H}_b = H_b^1 H_b + \frac{1}{\rho_b} I.$$

For the sake of simplicity, we consider the vectorized version of the received message: let $x = \xi(X)$, $y = \xi(Y)$, $w_b = \xi(W_b)$. Then

$$y = H_b x + w_b,$$

where $H_b = H_b \otimes I_n$. Note that if we set $Q_1 = Q_1 \otimes I_n$, $R = R_b \otimes I_n$, we also have $H_b = Q_1 R$.

Similarly to the single antenna case (Section III-E), Bob can compute

$$y' = Q_1^T y = Rx + v,$$

where $v = Q_1^T w_b - \frac{1}{\rho_b} (R^{-1})^T x$ [27].

Recall that $x$ is sampled from $P_{\xi(X)} \otimes \xi(X)$, $\sigma_s$. Using Lemma 2.7 $x$ is $\delta$-subgaussian with parameter $\sigma_s$ for $\delta = \ln \left(\frac{1+\epsilon}{1-\epsilon}\right)$ provided that $\epsilon = \epsilon_{\Lambda_e}(\sigma_s) < 1$, which is guaranteed by (69). With the same argument as in Section III-E we can show that the equivalent noise $v$ is $\delta$-subgaussian with parameter $\sigma_b$.

Following the same steps as in Section III-E, we have the union bound on the error probability for fixed $R$:

$$P_e(R) \leq e^\delta \sum_{\lambda \in R \Lambda_b \setminus \{0\}} e^{-\frac{\lambda \Lambda^2}{2\epsilon^2}}.$$

Using Lemma 2.3 $P_e(R) \to 0$ if

$$\tau^2 = \frac{1}{4\pi \sigma_b^2} > \frac{2c^2 n^2 k}{\lambda_1(\mathbb{R} \Lambda_b)^2}.$$  

(70)
Note that the minimum distance in the received lattice is lower bounded as follows:

\[ \lambda_1(R \Lambda_b)^2 = \min_{X \in \Lambda_b \setminus \{0\}} \| R \xi(X) \|^2 = \min_{X \in \Lambda_b \setminus \{0\}} \| R_b X \|^2 \geq \min_{X \in \Lambda_b \setminus \{0\}} nk \prod_{i=1}^k \| \det(R_{b,i} \tilde{X}_i) \|^{\frac{2}{nk}} \]

where (a) follows from Remark 4.2. From the scaling condition (58), we get

\[ \text{pdet}(\Lambda_b)^{\frac{2}{nk}} \approx \frac{\delta(\Lambda_b)^{\frac{2}{nk}} e \sigma_x^2}{e^{\frac{\eta d_b}{\pi nk}}}. \]

Replacing in the condition (70), we have that \( P_e(R) \to 0 \) when \( k \to \infty \) if

\[ e^{\frac{\eta d_b}{\pi nk}} < \prod_{i=1}^k \det \left( I + \rho_b H_{b,i}^\dagger H_{b,i} \right)^{\frac{1}{nk}} \delta(\Lambda_b)^{\frac{2}{nk}} e \sigma_x^2 \frac{1}{8c^2 n \sigma_b^2}. \]

In particular, recalling the assumption (59), a sufficient condition is

\[ e^{\frac{\eta d_b}{\pi nk}} < \prod_{i=1}^k \det \left( I + \rho_b H_{b,i}^\dagger H_{b,i} \right)^{\frac{1}{nk}} \frac{d_b}{8c^2 n}. \]

By the law of total probability, \( \forall \eta > 0, \)

\[ P_e \leq \mathbb{P} \left\{ \prod_{i=1}^k \det \left( I + \rho_b H_{b,i}^\dagger H_{b,i} \right)^{\frac{1}{nk}} < e^{\frac{\eta d_b}{nk}} \right\} + \mathbb{P} \left\{ \tilde{x} \neq x \mid \prod_{i=1}^k \det \left( I + \rho_b H_{b,i}^\dagger H_{b,i} \right)^{\frac{1}{nk}} \geq e^{\frac{\eta d_b}{nk}} \right\}. \]

Due to the law of large numbers (55), the first term vanishes when \( k \to \infty \). The second term tends to 0 if

\[ R + R' < C_b - \eta - n \ln \left( \frac{8c^2 n}{e} \right) + \ln d_b. \]

Since this result holds for any choice of \( \eta > 0 \) and \( c > \frac{1}{\sqrt{2\pi}} \), any rate

\[ R + R' < C_b - \eta - n \ln \left( \frac{4n}{\pi e} \right) + \ln d_b. \]

is achievable for Bob.

From equations (68) and (71), the proposed coding scheme achieves strong secrecy and semantic security rates

\[ R < C_b - C_e - 2n \ln \left( \frac{2n}{\pi} \right) + \ln d_b d_e. \]

This concludes the proof of Proposition 5.2.

VI. CODE DESIGN CRITERIA FOR FADING AND MIMO WIRETAP CHANNELS

We will now discuss the implications of our results in terms of design of wiretap lattice codes.

A. Single antenna fading and Gaussian wiretap channels

Although in Corollary 3.6 we focused on a particular sequence of nested lattices \( \Lambda_e \subset \Lambda_b \) that were scaled versions of the same lattice \( \Lambda^{(k)} \), Theorem 3.3 suggests a more general design criterion for building promising lattice codes for fading channels. Namely, we should consider pairs of nested lattices \( \Lambda_e \subset \Lambda_b \) for which the product

\[ \text{Np}(\Lambda_b) \text{Np}(\Lambda_e^*) \]

is maximized. As shown earlier, ideals from number fields with small discriminants give us promising candidates.
Here the term $t_b = \text{NP}(\Lambda_b)^{2k}$ can be seen as providing reliability for the communication between Alice and Bob while $t_e = \text{NP}(\Lambda_e^*)^{2k}$ provides security against the wiretapper.

While we mainly targeted general fading channels in this work, we also gained some intuition on code design in Gaussian wiretap channels. Proposition 3.15 suggests that in the Gaussian case one should maximize the product of the Hermite invariants

$$h(\Lambda_b) h(\Lambda_e^*).$$

(72)

Rather than using number field lattices, in this particular case one might optimize (72) for example by considering the densest self dual lattices.

B. Code design for MIMO wiretap channels

An analogous code design criterion can be given also in the MIMO case using the concept of normalized minimum determinant $\delta(\Lambda)$ of a matrix lattice, which was defined in Section IV-A.

Using this concept, Theorem 5.2 suggests that for MIMO channels we should maximize $\delta(\Lambda_e^*) \delta(\Lambda_b)$.

C. Comparison with earlier code design

The earliest work on lattice code design for the AWGN channel is based on an error probability approach [35]. The main criterion for maximizing the confusion of the eavesdropper is that the theta function of $\Lambda_e$ should be minimized. As this function is hard to analyze, the authors discussed a simplified criterion where one should maximize the Hermite invariant of $\Lambda_e$ [35, eq. (48)].

In comparison, our criterion differs in two ways. First, we prove that following our design principles the information leakage will be minimized. Second, our study emphasizes that the code design criterion for secrecy should be stated in terms of $\Lambda_e^*$ and not of $\Lambda_e$.

The work [4] concentrates on achieving strong secrecy over the Gaussian wiretap channel. Its results suggest that the theta function of $\Lambda_e^*$ should be minimized for secrecy. Maximizing the Hermite invariant of $\Lambda_e^*$ can be seen as a first-order approximation of this criterion, which we now make rigorous in Proposition 3.15.

Lattice code design for the fading wiretap channel was pioneered in [7] and [8] where the error probability approach led the authors to consider certain inverse determinant sums over the lattice $\Lambda_e$; both of these works suggest the use of number fields for wiretap coding. Similar conditions were derived also in [11, 36] to minimize the information leakage.

Compared to earlier works on fading wiretap channels, our criterion is the first which guarantees positive strong secrecy rates, in the sense that we prove that by maximizing $\text{NP}(\Lambda_e^*)$ or $\delta(\Lambda_e^*)$ one can indeed push the leaked information to zero. Also similarly to the Gaussian case it seems to be better to state the design criterion for $\Lambda_e^*$ instead of $\Lambda_e$.

Remark 6.1: We point out that in the derivation of the code design criterion for $\Lambda_e$ in [35, p. 5706] the authors first obtain a condition for the theta function of $\Lambda_e^*$ and only after using Poisson summation they end up with a condition for $\Lambda_e$. So the authors could also have stated their criterion for $\Lambda_e^*$.

Obviously for lattices that are isodual or even self-dual it is irrelevant whether the condition is given for $\Lambda_e$ or for $\Lambda_e^*$. It is interesting to note that the authors in [35] were concentrating on the analysis of iso-dual or self dual lattices with large Hermite invariants. For such lattices our criterion agrees with theirs. In the fading case, [7] and [8] focused on number field and division algebra lattices. Therefore their code design principles automatically lead to lattices for which $\delta(\Lambda_e^*)$ is non zero.

VII. CONCLUSIONS AND PERSPECTIVES

In this work, we have shown that algebraic lattice constructions based on number fields and division algebras can achieve strong secrecy and semantic security universally over a wide range of fading and MIMO wiretap channels. Universality is a very desirable property for practical applications in which the eavesdropper’s channel is not known at the transmitter.

We note, however, that several improvements are needed before our construction can be implemented in practice. In particular, although the proposed families of lattices are deterministic, their construction is not explicit since
it requires the computation of Hilbert class fields of high degree, for which efficient algorithms are currently not available.

Moreover, our construction incurs a large gap to the secrecy capacity. This gap might be reduced by improving the nested lattice construction, for example by taking suitable ideals of the ring of integers in the number field case, or ideals of orders in the division algebra case, in order to optimize the code design according to the criteria proposed in Section VI.

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APPENDIX I
PROOFS OF TECHNICAL LEMMAS

A. Proof of Lemma 2.4

We need the following elementary fact characterizing the product of two Gaussian functions (see, e.g., [17, Fact 1]):

Let $\Sigma_1, \Sigma_2 \succ 0$ be positive definite matrices, let $\Sigma_0 = \Sigma_1 + \Sigma_2 \succ 0$ and $\Sigma^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1} \succ 0$, let $X, c_1, c_2 \in \mathbb{C}^k$ be arbitrary, and let $c_3 \in \mathbb{C}^k$ such that $\Sigma^{-1} c_3 = \Sigma_1^{-1} c_1 + \Sigma_2^{-1} c_2$. Then $\forall x \in \mathbb{C}^k$,

$$f_{\sqrt{\Sigma_1}}(x - c_1) f_{\sqrt{\Sigma_2}}(x - c_2) = f_{\sqrt{\Sigma_0}}(c_1 - c_2) f_{\sqrt{\Sigma}}(x - c_3)$$

(73)

Now, we are ready to generalize Regev’s lemma to correlated Gaussian distributions. Let $c_3 = \Sigma \Sigma_2^{-1} x$. We have

$$g(x) = \sum_{x_1 \in \Lambda + e} \frac{f_{\sqrt{\Sigma_1}}(x_1)}{f_{\sqrt{\Sigma}}(\Lambda + e)} f_{\sqrt{\Sigma_2}}(x - x_1)$$

$$\overset{(a)}{=} \sum_{x_1 \in \Lambda + e} \frac{f_{\sqrt{\Sigma_0}}(x)}{f_{\sqrt{\Sigma}}(\Lambda + e)} f_{\sqrt{\Sigma}}(x_1 - c_3)$$

$$= f_{\sqrt{\Sigma_0}}(x) \frac{\Sigma_{x_1 \in \Lambda + e} f_{\sqrt{\Sigma}}(x_1 - c_3)}{f_{\sqrt{\Sigma}}(\Lambda + e)}$$

$$= f_{\sqrt{\Sigma_0}}(x) \frac{f_{\sqrt{\Sigma}}(\Lambda + e - c_3)}{f_{\sqrt{\Sigma}}(\Lambda + e)}$$

$$\overset{(b)}{=} f_{\sqrt{\Sigma_0}}(x) \left[ 1 - \varepsilon, 1 + \varepsilon \right]$$

$$\overset{(c)}{=} f_{\sqrt{\Sigma_0}}(x) \left[ 1 - 4\varepsilon, 1 + 4\varepsilon \right]$$

where $(a)$ is due to (73), $(b)$ follows from the definition of the flatness factor for correlated Gaussian distributions, and $(c)$ is because $\varepsilon \leq \frac{1}{2}$. More precisely, since $\sqrt{\Sigma_0} \succeq \eta(\varepsilon) \sqrt{\Sigma_0}$, $f_{\sqrt{\Sigma_0}}(\Lambda + e - c_3) \in \left[ \frac{1 - \varepsilon}{\sqrt{\eta(\varepsilon)}}, \frac{1 + \varepsilon}{\sqrt{\eta(\varepsilon)}} \right]$; moreover, since $\Sigma_1 \succ \Sigma_3$, we also have $f_{\sqrt{\Sigma}}(\Lambda + e) \in \left[ \frac{1 - \varepsilon}{\sqrt{\eta(\varepsilon)}}, \frac{1 + \varepsilon}{\sqrt{\eta(\varepsilon)}} \right]$.

B. Proof of Lemma 2.5

Let $\mu \in A(\Lambda + e)$. Then

$$P\{Y = \mu\} = P\{X = A^{-1}\mu\} = \frac{f_{\sqrt{\Sigma}}(A^{-1}\mu)}{f_{\sqrt{\Sigma}}(\Lambda + e)} = \frac{e^{-\mu \cdot (A^{-1})^\dagger \Sigma^{-1} A^{-1}\mu}}{\sum_{z \in \Lambda + e} e^{-z \cdot \Sigma^{-1} z}}$$

The thesis follows since by definition

$$D_{A(\Lambda + e), \sqrt{\Sigma} A^{-1}}(\mu) = \frac{f_{\sqrt{\Sigma} A^{-1}}(\mu)}{\sum_{\mu' \in A(\Lambda + e)} f_{\sqrt{\Sigma} A^{-1}}(\mu')} = \frac{e^{-\mu \cdot (A^{-1})^\dagger \Sigma^{-1} A^{-1}\mu}}{\sum_{z \in \Lambda + e} e^{-z \cdot \Sigma^{-1} z}}$$
C. Proof of Lemma 2.7

We have
\[ E \left[ e^{2R(t^\dagger Ax)} \right] = \sum_{x \in \Lambda + c} D_{\Lambda+c,c}(x) e^{2R(t^\dagger Ax)} = \sum_{x \in \Lambda + c} f_\sigma(x) e^{2R(t^\dagger Ax)}. \]

Therefore we can write
\[ f_\sigma(A + c) E \left[ e^{2R(t^\dagger Ax)} \right] = \sum_{x \in \Lambda + c} \frac{1}{(\pi \sigma^2)^k} e^{-\frac{|x|^2}{\sigma^2} + 2R(t^\dagger Ax)}. \]

Using the identity
\[ \left| \frac{x}{\sigma} - \sigma A^\dagger t \right|^2 = \frac{|x|^2}{\sigma^2} - 2R(t^\dagger Ax) + \sigma^2 \left| A^\dagger t \right|^2, \]
we can rewrite the last expression as
\[ \sum_{x \in \Lambda + c} \frac{1}{(\pi \sigma^2)^k} e^{-\frac{|x|^2}{\sigma^2} + 2R(t^\dagger Ax) + \sigma^2 \left| A^\dagger t \right|^2} = e^{\sigma^2 \left| A^\dagger t \right|^2} f_\sigma(A + c - \sigma^2 A^\dagger t). \]

Thus we have
\[ E \left[ e^{2R(t^\dagger Ax)} \right] = e^{\sigma^2 \left| A^\dagger t \right|^2} \frac{f_\sigma(A + c - \sigma^2 A^\dagger t)}{f_\sigma(A + c)}. \]

Adapting [4, Lemma 4] to the complex case, we find that \( \forall c \in \mathbb{C}^k \)
\[ \frac{f_{\sigma,c}(\Lambda)}{f_{\sigma}(\Lambda)} \in \left[ \frac{1 - \epsilon_\Lambda(\sigma)}{1 + \epsilon_\Lambda(\sigma)}, 1 \right]. \]

Replacing \( t \) by \( t/2 \), we obtain
\[ E \left[ e^{R(t^\dagger Ax)} \right] = \frac{1 + \epsilon_\Lambda(\sigma)}{1 - \epsilon_\Lambda(\sigma)} e^{\frac{\sigma^2}{4} \left| A^\dagger t \right|^2}. \]

D. Proof of Lemma 2.10

Before giving the proof we need some notation.

Given an ideal \( I \) of \( F \), the complementary ideal of \( I \) is defined as \( I^\vee = \{ x \in F : \text{Tr}_{F/Q}(xI) \subseteq \mathbb{Z} \} \). It is always an ideal of \( F \).

With this notation we have that
\[ \psi(I)^* = \overline{\psi(I^\vee_F)}, \] (74)
where overline means complex conjugation element wise.

Proof: Let us first assume that \( \mathcal{I} \) is an integral ideal. In this case a classical result from algebraic number theory states that
\[ V(\psi(\mathcal{I})) = |\mathcal{O}_F : \mathcal{I}| 2^{-k} \sqrt{|d_F|}. \]

Noticing that \( \sqrt{|N_{F/Q}(x)|} = |p(\psi(x))| \) and using the definition of the product distance we have that
\[ \delta(\psi(\mathcal{I})) = \frac{2^{\frac{k}{2}}}{|d_F|^{\frac{1}{2}}} \min(I), \]
where \( \min(I) := \min_{x \in \mathcal{I} \setminus \{0\}} \sqrt{|N_{F/Q}(x)|} / N(\mathcal{I}) \) and \( N(\mathcal{I}) = |\mathcal{O}_F : \mathcal{I}| \) is the norm of the ideal \( \mathcal{I} \). From basic algebraic number theory we have that for any element of \( a \in \mathcal{I}, |N_{F/Q}(a)| \ | N(\mathcal{I}) \) and the first claim follows.

Let us now assume that \( \mathcal{I} \) is a genuine fractional ideal. In this case we can choose an integer \( n \) such that \( n \mathcal{I} \) is an integral ideal. The extension to fractional ideals now follows as for any lattice \( \Lambda \) we have \( \delta(n\Lambda) = \delta(\Lambda) \).

In (74) we saw that \( \psi(\mathcal{I})^* \) is just a complex conjugated version of fractional ideal lattice \( 2\psi(\mathcal{I}^\vee) \). Therefore the last claim follows from the first one. \( \square \)

\(^7\text{This result is well known but we do prove a more general version of it in Appendix E.} \)
E. Proof of Lemma 4.11

Let \( x, y \in \mathcal{D} \). Then we have

\[
\text{tr}_{\mathcal{D}/\mathcal{Q}}(xy) = \text{tr}_{\mathcal{F}/\mathcal{Q}}(\text{tr}_{\mathcal{D}/\mathcal{F}}(xy)) = \text{tr}_{\mathcal{F}/\mathcal{Q}}(\text{Tr}(\phi(xy))) = \sum_{i=1}^{2k} \alpha_i(\text{Tr}(\phi(xy))) = \sum_{i=1}^{2k} \text{Tr}(\alpha_i(\phi(xy))) = \text{Tr}\left( \sum_{i=1}^{2k} \alpha_i(\phi(xy)) \right) = 2\Re \text{Tr}\left( \sum_{i=1}^{k} \alpha_i(\phi(x))\alpha_i(\phi(y)) \right) = 2\Re \text{Tr}(\psi(x)^{\dagger}\psi(y)).
\]

By definition,

\[
\psi(\Gamma)^* = \{ X \in M_{nk \times n}(\mathbb{C}) : \forall y \in \Gamma, \; \Re(\text{Tr}(X^\dagger\psi(y))) \in \mathbb{Z} \},
\]

and so \( 2\psi(\Gamma)^h \subseteq \psi(\Gamma)^* \). We would like to show that \( 2\psi(\Gamma)^h = \psi(\Gamma)^* \).

The trace form \( \text{tr}_{\mathcal{D}/\mathcal{Q}}: \mathcal{D} \times \mathcal{D} \to \mathcal{Q} \) is a non-degenerate bilinear form on the \( \mathbb{Q} \)-vector space \( \mathcal{D} \). Then, any full \( \mathbb{Z} \)-module in \( \mathcal{D} \) has a dual basis in \( \mathcal{D} \) [21]. In particular, if \( \{w_1, \ldots, w_{2n^2k}\} \) is a basis of \( \mathcal{D} \) as \( \mathbb{Z} \)-module, then there exists a dual basis \( \{w_1', \ldots, w_{2n^2k}'\} \) in \( \mathcal{D} \) such that \( \forall i, j \in \{1, \ldots, 2n^2k\} \), we have \( \text{tr}_{\mathcal{D}/\mathcal{Q}}(w_i'w_j) = \delta_{ij} \). Therefore,

\[
\Re \text{Tr}(2(\psi(w_i')^{\dagger}\psi(w_j))) = \text{tr}_{\mathcal{D}/\mathcal{Q}}(w_i'w_j) = \delta_{ij}
\]

and by definition of the codifferent, this implies that \( \psi(w_i')^h \in \psi(\Gamma)^h \). Since \( 2\psi(\Gamma)^h \subseteq \psi(\Gamma)^* \) and it contains a dual basis for \( \psi(\Gamma)^* \), we can conclude that \( 2\psi(\Gamma)^h = \psi(\Gamma)^* \). \( \square \)

F. Sketch of the proof of Proposition 3.15

The proof follows the same steps as the proof of Theorem 3.6. Note that equation (33) still holds and

\[
\mathbb{E}_{H_e}\left[ \frac{1}{k} \| (p_M; p_x|H_e) \| \leq R \mathbb{P}\left\{ \prod_{i=1}^{k} \left( 1 + \frac{P|h_{e,i}|^2}{\sigma_e^2} \right)^\frac{1}{2} > e_{C,e,\delta} \right\} + \frac{1}{k} \mathbb{E}_{H_e}\left[ \| (p_M; p_x|H_e) \| \prod_{i=1}^{k} \left( 1 + \frac{P|h_{e,i}|^2}{\sigma_e^2} \right)^\frac{1}{2} \leq e_{C,e,\delta} \right] \right]
\]

The first term vanishes because of (26) and the second term vanishes using (29) and (34) as before. The proof of reliability is unchanged. \( \square \)

G. Sketch of the proof of Proposition 3.15

Secrecy. With the same scaling as in equation (22), we have

\[
\eta_e(\Lambda_e) \leq \frac{2\sqrt{ke}}{\lambda_1(\Lambda_e)} = \frac{2\sqrt{ke}}{h(\Lambda_e^0)V(\Lambda_e^0)\frac{\pi}{2}} \approx \frac{2\sqrt{ke\pi}}{h(\Lambda_e^0)\frac{\pi}{2}} \leq \frac{2e\sqrt{\pi eP}}{h_e\frac{\pi}{2}}.
\]

With the same notation as in Section III-E, we have \( \Sigma = \frac{P\sigma_e}{1+\sigma_e^2} \) and \( \varepsilon_{\Lambda_e}(\sqrt{\Sigma}) \to 0 \) as long as

\[
\sqrt{\Sigma} = \frac{\sqrt{P}\sigma_e}{\sqrt{P+\sigma_e^2}} > \frac{2e\sqrt{\pi eP}}{h_e\sqrt{2\pi e \frac{\sigma_e^2}{2}}}.
\]

This condition is equivalent to

\[
R' > \ln \frac{e}{\pi h_e} + \ln \left( 1 + \frac{P}{\sigma_e^2} \right).
\]

Reliability. With the same notation as in Section III-E, we have \( R = \sqrt{\frac{1+\rho_b}{\rho_b}}I \). With the scaling (22), the error probability tends to zero if (40) holds, that is

\[
\frac{1}{4\pi \sigma_b^2} > \frac{2e^2k\rho_b}{(1+\rho_b)\lambda_1(\Lambda_b)^2} = \frac{2e^2k\rho_b}{h(\Lambda_b)^2V(\Lambda_b)\frac{\pi}{2}} \approx \frac{2e^2\rho_b e^{R+R'}}{(1+\rho_b)h_b\pi eP}.
\]

Recalling that \( \rho_b = P/\sigma_b^2 \), after elementary calculations we find

\[
R + R' < \ln \left( 1 + \frac{P}{\sigma_b^2} \right) - \ln \left( 4 \pi e \right) + \ln h_b.
\]

\( \square \)
Appendix II

Rate of Convergence in the Law of Large Numbers and Decay of Correlations

In this section we discuss a sufficient condition for equation (20) to hold for the eavesdropper’s channel. We first recall some notions from the ergodic theory of stochastic processes (see also Section V.B in [13]).

A. Correlation and large deviations

We consider a real-valued random process \( X^Z = \{X_i\}_{i \in \mathbb{Z}} \) on a probability space \( (\Omega, \mathcal{B}, \mathbb{P}) \). We can define a probability measure \( \mu \) on the sequence space \( \mathcal{X} = \mathbb{R}^\mathbb{Z} \) with the Borel sigma-algebra \( \mathcal{B}(\mathcal{X}) \) as follows:

\[
\mu(A) = \mathbb{P} \left\{ \omega : X^Z(\omega) \in A \right\}, \quad \forall A \in \mathcal{B}(\mathcal{X}). \tag{75}
\]

**Definition II.1:** The process \( \{X_i\} \) is **stationary** if \( \forall t, k \in \mathbb{N}, \forall i_1, i_2, \ldots, i_k \in \mathbb{Z}, \) the joint distribution of \( (X_{i_1}, X_{i_2}, \ldots, X_{i_k}) \) is the same as that of \( (X_{i_1+t}, X_{i_2+t}, \ldots, X_{i_k+t}) \).

In the stationary case, the shift map \( T : \mathcal{X} \to \mathcal{X} \) such that \( T(\{x_i\}) = \{x_{i+1}\} \) preserves the measure \( \mu \), i.e. \( \forall A \in \mathcal{B}(\mathcal{X}), \mu(T^{-1}(A)) = \mu(A) \).

**Definition II.2:** The process \( \{X_i\} \) is **ergodic** if \( \forall A \in \mathcal{B}(\mathcal{X}) \) such that \( T^{-1}(A) = A, \) \( \mu(A) \) is equal to 0 or 1.

We follow the notation in [24].

**Definition II.3:** Let \( \varphi, \psi \in L^\infty(\mu) \). The \( k \)-th correlation coefficient of the observables \( \varphi \) and \( \psi \) with respect to \( T \) is defined as

\[
\text{Cor}_k(\varphi, \psi) = \frac{1}{\|\varphi\| \|\psi\|} \left| \int \varphi(\psi \circ T^k) d\mu - \int \varphi d\mu \int \psi d\mu \right|.
\]

Given \( \delta > 0 \), we define the **large deviation**

\[
\text{LD}(\varphi, \delta, k) = \mu \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \varphi \circ T^i - \int \varphi d\mu \right| > \delta \right\}.
\]

If \( \mu \) is ergodic, Birkhoff’s theorem implies the law of large numbers, and in particular we have \( \text{LD}(\varphi, \delta, k) \to 0 \) as \( k \to 0 \). Moreover, the rate of convergence in the law of large numbers is determined by the corresponding decay of correlation for \( L^\infty \) observables [24, Theorem D]:

**Theorem II.4:** Let \( T : \mathcal{X} \to \mathcal{X} \) preserving an ergodic probability measure \( \mu \), and let \( \varphi \in L^\infty(\mu) \)

1. Let \( \beta > 0 \) and suppose that \( \forall \psi \in L^\infty(\mu) \) we have \( \text{Cor}_k(\varphi, \psi) \leq c_T k^{-\beta} \) for some constant \( c_T > 0 \). Then \( \forall \delta > 0 \) there exists \( M = M(\varphi, \delta) \) such that

\[
\text{LD}(\varphi, \delta, k) \leq Mk^{-\beta}. \tag{76}
\]

2. Let \( \theta, \tau > 0 \) and suppose that \( \forall \psi \in L^\infty(\mu) \) we have \( \text{Cor}_k(\varphi, \psi) \leq c_T e^{-\tau k^\theta} \) for some constant \( c_T' \). Then \( \forall \delta > 0 \) there exists \( M = M(\varphi, \delta), \tau' = \tau'(\tau, \varphi, \delta) \) such that

\[
\text{LD}(\varphi, \delta, k) \leq Me^{-\tau' k^{\theta/(\theta+2)}}. \tag{77}
\]

Theorem II.4 can be extended to \( L^1(\mu) \) functions as follows:

**Corollary II.5:** Under the same hypotheses as in Theorem II.4, the large deviations bounds (76) and (77) hold as well for all \( \varphi \in L^1(\mu) \).

**Proof:** We focus on the proof for the bound (76). Let \( \delta \) and \( k \) be fixed. Since \( \mu \) is a probability measure over \( \mathcal{X} \), the compactly supported continuous functions \( C_c(\mathcal{X}) \) are dense in \( L^1(\mu) \).

Therefore, given \( \varphi \in L^1(\mu) \) there exists a sequence \( \{\varphi_l\} \) of \( L^\infty \) functions such that for \( l \) large enough, we have \( \int |\varphi - \varphi_l| d\mu < \frac{\delta}{2} k^{-\beta} \). By the triangle inequality, \( \forall x \in \mathcal{X} \),

\[
\left| \frac{1}{k} \sum_{i=1}^{k} (\varphi \circ T^i)(x) - \int \varphi d\mu \right| \leq \left| \frac{1}{k} \sum_{i=1}^{k} ((\varphi - \varphi_l) \circ T^i)(x) \right| + \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi_l \circ T^i)(x) - \int \varphi_l d\mu \right| + \left| \int (\varphi - \varphi_l) d\mu \right|. \tag{78}
\]

\(^8\)Note that although the statement of [24, Theorem D] requires \( T \) to be non-singular, this condition is automatically satisfied if \( T \) is measure-preserving.
Note that if \( \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi \circ T^i)(x) - \varphi d\mu \right| \geq \delta \) and \( \left| \frac{1}{k} \sum_{i=1}^{k} ((\varphi - \varphi_l) \circ T^i)(x) \right| \geq \frac{\delta}{3} \), then from (78) we get
\[
\left| \frac{1}{k} \sum_{i=1}^{k} (\varphi_l \circ T^i)(x) - \varphi_l d\mu \right| \geq \frac{\delta}{3}.
\]
Then by the law of total probability we can write
\[
\text{LD}(\varphi, \delta, k) = \mu \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi \circ T^i) - \int \varphi d\mu \right| \geq \delta \right\}
\leq \mu \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi - \varphi_l) \circ T^i \right| \geq \frac{\delta}{3} \right\} + \mu \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi_l \circ T^i) - \int \varphi_l d\mu \right| \geq \frac{\delta}{3} \right\}.
\]
We can bound the average of the first term as follows:
\[
\mathbb{E}_{\mu} \left[ \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi - \varphi_l) \circ T^i \right| \right] \leq \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}_{\mu} \left[ |(\varphi - \varphi_l) \circ T^i| \right] = \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}_{\mu} \left[ |\varphi - \varphi_l| \right] \leq \frac{\delta}{3} k^{-\beta},
\]
where (a) follows from the fact that \( T \) is measure-preserving. By the Markov inequality,
\[
\mu \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi - \varphi_l) \circ T^i \right| \geq \frac{\delta}{3} \right\} \leq k^{-\beta}.
\]
Then from (79) and (76) we get \( \text{LD}(\varphi, \delta, k) \leq k^{-\beta} + M(\varphi, \delta/3)k^{-\beta} \). This concludes the proof for the polynomial bound (76); the argument for the exponential bound (77) is identical.

**B. Large deviations for fading channels**

If the fading process \( \{H_i\} \) is stationary and ergodic, the random process \( \{X_i\} = \log \det \left( I + H_i^\dagger H_i \right) \) is also stationary and ergodic and preserves the measure \( \mu \).

In particular, the projection \( \Pi : \mathbb{R}^Z \rightarrow \mathbb{R} \) on the first coordinate is \( L^1(\mu) \) if and only if \( \mathbb{E} \left[ \log \det \left( I + H_i^\dagger H_i \right) \right] < \infty \). Under this hypothesis, Birkhoff’s theorem implies the law of large numbers:
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} X_i = \int_{\mathcal{X}} \Pi(\{x_i\}) d\mu(\{x_i\}) = \int_{\Omega} X_1 d\mathbb{P} = \mathbb{E}[X]
\]
almost everywhere. Equivalently
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \log \det(I + H_i^\dagger H_i) = \mathbb{E}_H \left[ \log \det(I + H^\dagger H) \right] = C
\]
almost everywhere, where \( C \) is the channel capacity, and in particular
\[
\text{LD}(\Pi, \delta, k) = \mathbb{P} \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \log \det \left( I + H_i^\dagger H_i \right) - C \right| > \delta \right\} \rightarrow 0.
\]
If the decay of correlations for the fading channel is \( o(k^{-\beta}) \), then Corollary II.3 guarantees that
\[
\mathbb{P} \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \log \det \left( I + H_i^\dagger H_i \right) - C \right| > \delta \right\} \leq Mk^{-\beta}.
\]
