A Search for Spreads of Hermitian Unitals

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Abstract

A spread of a Hermitian unital in $PG(2,q^2)$ is a set of $q^2 - q + 1$ pairwise disjoint blocks that partition the points of the unital. In this paper, we discuss the results of an exhaustive computer search for spreads of Hermitian unitals of small orders, namely the Hermitian unitals in $PG(2,16)$, $PG(2,25)$ and $PG(2,49)$.

1 Introduction

Let $q$ be a prime power, and let $\Pi$ be the projective plane over $GF(q)$. A Hermitian unital is the set of absolute points of a unitary polarity in $\Pi$. A Hermitian unital $U$ has the property that any line of $\Pi$ meets $U$ in either 1 or $q + 1$ points, and these lines are called tangents and secants, respectively. Considering the intersections of $U$ and its secants as blocks, the Hermitian unital in $PG(2,q^2)$ is a $2 - (q^3 + 1, q + 1, 1)$ design. When no confusion can arise, we will refer to the set of lines of $\Pi$ whose intersections with $U$ form a spread of $U$ as a spread as well.

An interesting question with any design is whether or not one can partition the pointset of the design into pairwise disjoint blocks, creating a spread. Spreads of many different types of spaces have been studied, and in [3], the author investigated spreads of Ree unitals.

The obvious method for creating a spread of a Hermitian unital $U$ in $\Pi$ requires one to start with a point $P \in \Pi$ outside of $U$. The set of secants to $U$ through $P$, together with $P^\perp$, the polar of $P$ under the unitary polarity associated with $U$, form a spread of $U$. We refer to such a spread as being regular.

In addition to the regular spread, Baker, et. al. [1] provide several examples of spreads of Hermitian unitals. The focus in that work is on finding spreads which are orthogonally divergent, i.e., spreads such that no line of the spread contains the pole of any other line of the spread, with respect to the polarity associated with the Hermitian unital.

The first type of spread from [1] is a cyclic spread, i.e., a spread which can be obtained by taking the orbit of one of its members under a cyclic group of
order $q^2 - q + 1$. Baker, et. al., prove that a cyclic spread of a Hermitian unital in $PG(2, q^2)$ exists if and only if $q$ is even.

The second type of spread given in [1] involves an interesting partition of the Hermitian unital. Let $U$ be the Hermitian unital defined by the equation $x^{q+1} + y^{q+1} + z^{q+1} = 0$, where $x$, $y$, and $z$ are homogeneous coordinates for $\Pi = PG(2, q^2)$. Let $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$ be the vertices of a self-polar triangle with respect to $U$. Then $U$ is partitioned by the sides of the triangle $\triangle PQR$ and the sets

$$T_a = \{(1, y, z) : y^{q+1} = -(1 + a), z^{q+1} = a\},$$

for each $a \in GF(q) \setminus \{0, -1\}$. Each of the $T_a$’s is triply-ruled, in that there are sets of $q + 1$ secants to $U$ through each of $P$, $Q$ and $R$ whose intersection with $U$ exactly cover $T_a$. These sets are:

$$H_a = \{[u, 0, 1] : u^{q+1} = a\}$$

$$V_a = \{[v, 1, 0] : v^{q+1} = -(1 + a)\}$$

$$D_a = \{[0, w, 1] : w^{q+1} = -a/(1 + a)\}$$

Hence one can create a spread of $U$ by selecting one of the three ruling families $H_a$, $V_a$, or $D_a$ for each $T_a$ and adjoining the three sides of the triangle $\triangle PQR$.

Based on the data generated in the search described here, the author [4] provides a generalization of this construction to create a family of subregular spreads of the Hermitian unital. Using an analogy with the spreads of $PG(3, q)$, we call the spreads generated in Baker, et. al. [1] an André spread, noting that there exist subregular spreads of the Hermitian unital which are not André spreads.

### 2 Search Procedure

Our basic algorithm for searching for spreads is a standard backtrack, where we sequentially attempt to add lines to a partial spread until we either get to a full spread or have no lines remaining that are disjoint from our partial spread, in which case we backtrack. Despite multiple attempts to increase the efficiency of this backtrack, we felt it unlikely that it would be able to complete in a reasonable amount of time in $PG(2, 49)$. Moreover, this backtrack produces an enormous number of isomorphic spreads which must then be sorted.

To improve the performance of this procedure, we perform an aggressive isomorph rejection phase prior to backtracking to determine a number of “starter” configurations that exhaustively cover the search space as follows. We set up our initial data structures using Magma [2], including the unital, defined to be $U = \{(x, y, z) : x^{q+1} + y^{q+1} + z^{q+1} = 0\}$, and we fix the secant $[1, 0, 0]$. Since the automorphism group of the Hermitian unital is transitive on secants, we may assume that $[1, 0, 0]$ is in our spread.
We define a starter configuration $S$ to be a pair $(S_i, S_c)$ of sets such that $S_i$ is the set of secants to the unital which are assumed to be in the spread we will be constructing, and $S_c$ is the set of secants specifically excluded from this potential spread. Thus our initial starter configuration is $((\{1,0,0\}, \emptyset))$.

From this initial starter configuration $S = (\{1,0,0\}, \emptyset)$, we may generate a set of new starter configurations which also cover the search space as follows. Compute the stabilizer $L$ within the automorphism group of the unital of the set $\{1,0,0\}$. Let $\{O_j : 1 \leq j \leq n\}$ be the orbits under $L$ of those secants which do not meet $[1,0,0]$. Letting $\ell_j$ be an orbit representative of $O_j$, it is clear that every spread of $U$ contains a set of lines isomorphic to $\{1,0,0\}, \ell_j$ for some $1 \leq j \leq n$. Thus we can partition our search space into $n$ new search spaces, namely $\{(\{1,0,0\}, \ell_j), \emptyset\}$ for each $1 \leq j \leq n$.

Partitioning the search space in this way does not give us maximum advantage. Suppose there exists a spread $S$ of $U$ that contains the lines $\{1,0,0\}, m_j, m_k$ where $m_j \in O_j$ and $m_k \in O_k$ with $j \neq k$. Then our search will find spreads isomorphic to $S$ in each of the search spaces with starters $(\{1,0,0\}, \ell_j), \emptyset$ and $(\{1,0,0\}, \ell_k), \emptyset$.

To be more efficient, we could have the search space with starter $(\{1,0,0\}, \ell_j), \emptyset$ exclude the blocks from the orbit $O_k$ and still find a spread isomorphic to $S$. Therefore, we will instead partition our search space $((\{1,0,0\}, \emptyset))$ into the search spaces with starter configurations:

\[
(\{1,0,0\}, \ell_j), \bigcup_{k=j+1}^n O_k
\]

We can iterate this procedure with each of the new starters to further decompose our search space. Let $(S_i, S_c) = ((\{1,0,0\}, \ell_j), \bigcup_{k=j+1}^m O_k)$ be one of our new starter configurations. Again compute the setwise stabilizer $L'$ of $S_i$ in the unital $U$. We look at the orbits $O_k'$, $1 \leq k \leq m$ of secants under $L'$. As before we need not consider those orbits whose secants meet a secant of $S_i$ in a point of $U$. However, we may also exclude from consideration all of those orbits which contain a secant in $S_c$, even if those orbits contain secants which are not in $S_c$.

To see why this exclusion is valid, suppose there exists a spread $S'$ which contains the secants $[1,0,0], \ell_j$, and $\ell'$, where $\ell' \notin S_c$, but there exists a line $\ell'' \in S_c$ in $\ell'$’s orbit under $L'$. Then there exists an automorphism of $U'$ that maps $\{1,0,0\}, \ell_j$ onto itself and maps $\ell'$ onto $\ell''$, and thus the spread $S'$ onto a new spread $S''$ containing $\{1,0,0\}, \ell_j, \ell''$. Since $\ell''$ is in $S_c$, it must be in an orbit $O_m$ with $m > j$, which means that a spread isomorphic to $S''$, and hence $S'$ will be found in the search space with starter configuration $((\{1,0,0\}, \ell_m), \bigcup_{k=m+1}^n O_k)$.

We use Magma to iterate this procedure as long as one of the starter configurations has an initial partial spread with a nontrivial stabilizer in the automorphism group of $U$, or the number of starter configurations becomes too unwieldy. At the end of the isomorph rejection phase, the searcher will be left with a number of starter configurations which are the seeds to our backtracking algorithm.
Using Magma, we set up the data structures for our C backtracking algorithm. For each starter configuration \((S_i, S_e)\), we create the initial partial spread from \(S_i\) and a list of all candidate secants that can be used to complete the initial partial spread to a full spread. This set consists of those secants which neither meet an element of \(S_i\) in a point of the unital nor are in the set of excluded secants, \(S_e\).

After all of the backtrack searches are completed, we use a number of Perl scripts to marshall the spreads together, leveraging Magma to compute several combinatorial invariants as an initial isomorphism sort. We then pass completely back into Magma to execute a full isomorphism check and produce a comprehensive list of projectively inequivalent spreads.

Using a custom-built Pentium 4 3.0 GHz machine with 2GB of RAM running RedHat Linux 9.0, the isomorph rejection phase of the search ran almost instantaneous in PG(2,16), took a few minutes for PG(2,25), and lasted approximately 20 hours for PG(2,49). The backtracking phase was again almost instantaneous for PG(2,16), less than a minute for PG(2,25), and lasted approximately 10 weeks for PG(2,49). In the PG(2,49) case, we searched a total of 20,846 starter configurations, many trivial due to large numbers of excluded secants. As none of the individual backtrack runs took more than 26 hours to complete, the highly parallelizable nature of our search suggests that an exhaustive search for spreads of the Hermitian unital in PG(2,64) is not beyond reach with more robust hardware.

3 \(PG(2, 16)\)

Let \(w\) be a primitive element of \(GF(16)\) with minimal polynomial \(x^4 + x + 1\) over \(GF(2)\). Our search found exactly three spreads of the Hermitian unital \(U = \{(x, y, z) : x^{q+1}+y^{q+1}+z^{q+1} = 0\}\) in \(PG(2, 16)\), all of which were previously known, namely the regular spread, as well as the cyclic spread and a spread from the triply-ruled partition as constructed in Baker, et. al. \[1\].

Constructions and properties of these spreads are presented in Table 1. In order to report the lines of each spread, we start with the regular spread \(S\) which consists of the secants to \(U\) through \((1, 0, 0)\) together with the line \([1, 0, 0]\). Each reported spread \(S'\) is described by the set of lines in \(S' \setminus S\). Thus, \(S'\) can be reconstructed by starting with the lines of \(S\), discarding any line of \(S\) that meets one of the reported lines, then adding the reported lines.

We represent the line \([x, y, z]\) in the form \((a, b, c)\) where \(a\) (resp. \(b, c\)) is the discrete logarithm of \(x\) (resp. \(y, z\)) with respect to \(w\). * will be used to represent the 0 of the field. In addition to a representative spread, we also present a number of invariants of the spread, including:

- \(|G|\) = order of the stabilizer of the spread within the automorphism group of the Hermitian unital.

- Orbit = orbit structure of the stabilizer of the spread on its elements, reported as a set of pairs \((i, j)\) where \(j\) is the number of orbits of size \(i\).
Pairs for which \( j = 0 \) are not shown.

- **Type** = set of pairs \((i, j)\) where \( j \) is the number of points outside of the unital \( U \) that lie on exactly \( i \) secants of the spread. Pairs for which \( j = 0 \) are not shown.

- **K =** class of spread if generated from a known construction. Regular spreads are denoted with an “\( r \)”, cyclic spreads with a “\( c \)”, the André spreads of Baker, et. al. \([1]\) with an “\( a \)”, and non-André subregular spreads, as defined in \([4]\), with an “\( s \)”.

### Table 1: The spreads of the Hermitian unital in \( \mathbb{P}G(2, 16) \)

| Idx | Spread | \([G]\) | Orbit | Type | K |
|-----|--------|--------|-------|------|---|
| 1   | \((0,1), (0,13,12), (0,13,*), (0,1,*), (0,0,3), (0,6,8), (0,3,14), (0,2,8), (0,12,6), (0,9,9), (0,214)\) | 156   | 13,1  | \((0,130), (2,78)\) | c |
| 2   | \((0,12,10), (0,6,4), (0,3,1), (0,0,13), (0,9,7)\) | 100   | 1,1, 2,1, 10,1 | \((0,100), (170), (2,36), (7,2)\) | a |
| 3   | \((0,12,10), (0,6,4), (0,3,1), (0,0,13), (0,9,7)\) | 1200  | 1,1, 12,1 | \((0,75), (1,120), (2,12), (12,1)\) | r |

To classify our spreads into the known types, we can use the following rules. Regular spreads are distinguished by having a point off the unital lying on \( q^2 - q \) lines of the spread. From Baker, et. al. \([1]\) cyclic spreads exist only when \( q \) is even, so we need only check spreads for even \( q \) with a transitive automorphism group. An algorithm to identify subregular spreads is given in Dover \([4]\), as is an algorithm to distinguish within this class the spreads obtainable from the triply-ruled \( T_a \’s \).

### 4 \( \mathbb{P}G(2, 25) \)

Let \( w \) be a primitive element of \( \text{GF}(25) \) with minimal polynomial \( x^2 + 4x + 2 \) over \( \text{GF}(5) \). Our search found ten nonisomorphic spreads of the Hermitian unital, given in Table 2.

### Table 2: The spreads of the Hermitian unital in \( \mathbb{P}G(2, 25) \)

| Idx | Spread | \([G]\) | Orbit | Type | K |
|-----|--------|--------|-------|------|---|
| 1   | \((0,1), (0,13,12), (0,13,*), (0,1,*), (0,0,3), (0,6,8), (0,3,14), (0,2,8), (0,12,6), (0,9,9), (0,214)\) | 1440  | (1,1), (20,1) | \((0,144), (1,360), (2,20), (20,1)\) | r |

\( \text{Continued on next page.} \)
| Idx | Spread                  | |G| | Orbit | Type                  | K |
|-----|-------------------------|---|---|---|------------------------|---|
| 2   | (0,4,0), (0,20,16), (0,12,8), (0,16,12), (0,8,4), (0,20,0) | 72 | (1,3), (6,3) | (0,216), (1,216), (2,91), (8,1), (14,1) | a |
| 3   | (0,10,18), (0,2,10), (0,14,22), (0,6,14), (0,22,6), (0,18,2) | 72 | (1,3), (6,3) | (0,216), (1,216), (2,91), (8,1), (14,1) | a |
| 4   | (0,18,5), (0,2,13), (0,14,1), (0,10,21), (0,6,17), (0,22,9) | 72 | (1,3), (6,3) | (0,216), (1,216), (2,91), (8,1), (14,1) | a |
| 5   | (0,15,*), (0,3,*), (0,*17), (0,*1), (0,*21), (0,*13), (0,19,*), (0,*9), (0,7,*), (0,11,*) | 216 | (3,1), (18,1) | (0,252), (1,144), (2,126), (8,3) | a |
| 6   | (0,16,1), (0,16,5), (0,8,17), (0,20,9), (0,12,1), (0,0,13), (0,20,5), (0,0,9), (0,4,17), (0,4,13), (0,8,21), (0,12,21) | 24 | (1,1), (2,2), (4,1), (12,1) | (0,240), (1,190), (2,68), (3,24), (7,2), (8,1) | s |
| 7   | (0,5,10), (0,1,6), (0,6,17), (0,17,22), (0,10,21), (0,2,13), (0,21,2), (0,22,9), (0,14,1), (0,9,14), (0,18,5), (0,13,18) | 432 | (3,1), (18,1) | (0,216), (1,252), (2,18), (3,36), (8,3) | a |
| 8   | (0,5,12), (0,13,14), (0,17,0), (0,17,18), (0,1,2), (0,21,4), (0,9,16), (0,0,10), (0,1,8), (0,21,22), (0,13,20), (0,5,6) | 144 | (1,1), (2,1), (6,1), (12,1) | (0,216), (1,252), (2,18), (3,36), (8,3) | a |
| 9   | (0,23,3), (0,19,11), (0,10,11), (0,3,19), (0,*21), (0,18,19), (0,22,11), (0,13,19), (0,5,5), (0,2,3), (0,14,3), (0,7,11), (0,6,19), (0,15,19), (0,11,3) | 432 | (9,1), (12,1) | (0,216), (1,252), (2,36), (4,9), (5,12) | a |
| 10  | (0,17,4), (0,15,3), (0,16,5), (0,*4), (0,15,15), (0,17,16), (0,16,*), (0,4,*), (0,*5), (0,3,3), (0,4,5), (0,3,15), (0,4,17), (0,*16), (0,5,16), (0,16,17) | 336 | (21,1) | (0,224), (1,252), (3,28), (4,21) | a |

Only two of the spreads found in $PG(2, 25)$ do not belong to known families, namely spreads 9 and 10. Spread 10 admits a transitive automorphism group on the spread which contains an index 2 copy of $PSL(2, 7)$. Based on the interesting group structure, this spread has been previously constructed by Durante and Penttila [6]. While a very interesting spread, it seems unlikely that it will generalize to an infinite family of spreads for general $q$.

On the other hand, spread 9 seems capable of generalization. Spread 9 has the interesting property that its automorphism group has the same order as spread 7, which is obtained from the triply-ruled construction of Baker, et al. [1] via

$$\{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \cup V_3 \cup H_1 \cup D_2,$$
using the notation of Equation 1. Moreover this automorphism group is significantly larger than the automorphism groups for other spreads obtainable from this construction, primarily due to the presence of automorphisms that permute the lines \([1, 0, 0], [0, 1, 0], [0, 0, 1]\), as opposed to leaving each of them fixed. It is possible that there is some connection between these two spreads.

5 \(PG(2, 49)\)

Let \(w\) be a primitive element of \(GF(49)\) with minimal polynomial \(x^2 + 6x + 3\) over \(GF(7)\). Our search found eighty-one nonisomorphic spreads of the Hermitian unital in \(PG(2, 49)\). Surprisingly, all but two of these spreads are obtainable from known constructions: 1 is regular, 42 are André, and 36 are subregular, but not André. The two remaining spreads are listed as numbers 58 and 81 in Table 3.

| Idx | Spread                                                                 | \(|G|\) | Orbit | Type | K |
|-----|------------------------------------------------------------------------|-------|-------|------|---|
| 1   | \((0.25, 0.30), (0.12, 1), (0.26, 15), (0.1, 6), (0.44, 3), (0.14, 3), (0.6, 43), (0.31, 36), (0.2, 0.9), (0.18, 7), (0.36, 25), (0.42, 31), (0.37, 42), (0.7, 12), (0.8, 45), (0.2, 39), (0.38, 27), (0.30, 19), (0.0, 37), (0.32, 21), (0.43, 0), (0.13, 18), (0.19, 24), (0.24, 13)\) | 128   | \((1, 3), (8, 5)\) | (0.832), (1, 848), (2, 360), (3, 64), (10, 1), (18, 2) | a |
| 2   | \((0.9, 0.44), (0.3, 0.38), (0.15, 2), (0.10, 29), (0.23, 10), (0.41, 28), (0.45, 32), (0.35, 22), (0.42, 33), (0.34, 5), (0.16, 35), (0.29, 16), (0.17, 4), (0.27, 14), (0.11, 46), (0.40, 11), (0.5, 40), (0.46, 17), (0.21, 8), (0.39, 26), (0.47, 34), (0.28, 47), (0.33, 20), (0.22, 41)\) | 128   | \((1, 3), (8, 5)\) | (0.832), (1, 848), (2, 360), (3, 64), (10, 1), (18, 2) | a |
| 3   | \((0.36, 7), (0.10, 29), (0.0, 19), (0.4, 23), (0.34, 5), (0.16, 35), (0.10, 45), (0.24, 43), (0.16, 3), (0.46, 33), (0.22, 9), (0.34, 21), (0.40, 11), (0.4, 39), (0.46, 17), (0.12, 31), (0.40, 27), (0.6, 25), (0.28, 15), (0.42, 13), (0.30, 1), (0.28, 47), (0.22, 41), (0.18, 37)\) | 128   | \((1, 3), (8, 5)\) | (0.832), (1, 848), (2, 360), (3, 64), (10, 1), (18, 2) | a |
| 4   | \((0.0, 19), (0.30, 1), (0.46, 33), (0.0, 7, 42), (0.43, 9), (0.10, 45), (0.43, 30), (0.19, 6), (0.16, 3), (0.12, 31), (0.34, 21), (0.25, 12), (0.28, 15), (0.6, 25), (0.40, 27), (0.22, 9), (0.37, 24), (0.42, 13), (0.13, 0), (0.31, 18), (0.18, 37), (0.24, 43), (0.1, 36), (0.36, 7)\) | 128   | \((1, 3), (8, 5)\) | (0.832), (1, 848), (2, 360), (3, 64), (10, 1), (18, 2) | a |
| 5   | \((0.46, 33), (0.39, 10), (0.10, 45), (0.43, 9), (0.11, 46), (0.16, 3), (0.21, 40), (0.34, 21), (0.15, 34), (0.17, 4), (0.3, 22), (0.41, 28), (0.35, 22), (0.33, 4), (0.27, 46), (0.28, 15), (0.47, 34), (0.40, 27), (0.45, 16), (0.29, 16), (0.22, 9), (0.9, 28), (0.23, 10), (0.5, 40)\) | 128   | \((1, 3), (8, 5)\) | (0.832), (1, 848), (2, 360), (3, 64), (10, 1), (18, 2) | a |

Continued on next page.
| Idx | Spread | Orbit | Type | K |
|-----|--------|-------|------|---|
| 6   | (0,0.19), (0.30,1), (0.46,33), (0.39,10), (0.4,39), (0.10,45), (0.16,3), (0.21,40), (0.12,31), (0.34,21), (0.15,34), (0.3,22), (0.33,4), (0.27,46), (0.28,15), (0.6,25), (0.40,27), (0.45,16), (0.22,9), (0.42,13), (0.9,28), (0.18,37), (0.24,43), (0.36,7) | 128 | (1,3), (8,5) | (0.832), (1.848), (2,360), (3,64), (10,1), (18,2) |
| 7   | (0.26,20), (0.27,45), (0.20,38), (0.14,8), (0.15,33), (0.21,39), (0.20,14), (0.32,2), (0.38,32), (0.39,9), (0.38,8), (0.45,15), (0.2,20), (0.14,32), (0.8,26), (0.8,2), (0.26,44), (0.32,26), (0.2,44), (0.9,27), (0.44,38), (0.3,21), (0.33,3), (0.44,14) | 128 | (1,3), (8,5) | (0.832), (1.848), (2,360), (3,64), (10,1), (18,2) |
| 8   | (0.46,11), (0.21,34), (0.26,7), (0.16,29), (0.4,17), (0.14,43), (0.20,1), (0.8,37), (0.27,40), (0.28,41), (0.10,23), (0.33,46), (0.44,25), (0.15,28), (0.45,10), (0.38,19), (0.9,22), (0.39,4), (0.32,13), (0.3,16), (0.22,35), (0.2,31), (0.34,47), (0.4,40) | 128 | (1,3), (8,5) | (0.832), (1.848), (2,360), (3,64), (10,1), (18,2) |
| 9   | (0.38,30), (0.20,12), (0.32,24), (0.26,18), (0.4,36), (0.3,35), (0.9,41), (0.46,30), (0.21,5), (0.39,23), (0.22,6), (0.14,6), (0.44,36), (0.2,42), (0.45,29), (0.4,24), (0.16,0), (0.33,17), (0.15,47), (0.10,42), (0.27,11), (0.8,0), (0.28,12), (0.34,18) | 128 | (1,3), (8,5) | (0.832), (1.848), (2,360), (3,64), (10,1), (18,2) |
| 10  | (0.23,35), (0.29,41), (0.9,21), (0.21,33), (0.39,3), (0.11,47), (0.47,11), (0.41,29), (0.17,5), (0.27,39), (0.5,41), (0.45,9), (0.47,35), (0.15,27), (0.11,23), (0.29,17), (0.17,29), (0.35,23), (0.23,11), (0.35,47), (0.3,15), (0.33,45), (0.5,17), (0.41,5) | 128 | (1,3), (8,5) | (0.832), (1.848), (2,360), (3,64), (10,1), (18,2) |
| 11  | (0.34,10), (0.40,16), (0.4,28), (0.46,46), (0.36,12), (0.4,40), (0.22,46), (0.42,18), (0.22,22), (0.18,42), (0.16,40), (0.16,16), (0.28,4), (0.46,22), (0.6,30), (0.28,28), (0.24,0), (0.12,36), (0.4,4), (0.10,34), (0.34,34), (0.0,24), (0.10,10), (0.30,6) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1.040), (2.168), (3.128), (10,1), (18,2) |
| 12  | (0.27,32), (0.39,44), (0.25,30), (0.9,14), (0.26,15), (0.1,6), (0.44,33), (0.14,3), (0.31,36), (0.29,9), (0.45,2), (0.37,42), (0.7,12), (0.15,20), (0.8,45), (0.21,26), (0.2,39), (0.38,27), (0.33,38), (0.3,8), (0.32,21), (0.43,0), (0.13,18), (0.19,24) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1.040), (2.168), (3.128), (10,1), (18,2) |

Continued on next page.
| Idx | Spread | [G] | Orbit | Type | K  |
|-----|--------|-----|-------|------|----|
| 13  | (0.29,40), (0.14,25), (0.11,22), (0.41,4), (0.17,28), (0.20,15), (0.38,1), (0.23,34), (0.26,21), (0.8,19), (0.32,27), (0.47,10), (0.38,33), (0.35,46), (0.8,3), (0.44,7), (0.5,16), (0.20,31), (0.14,9), (0.32,43), (0.2,13), (0.2,45), (0.44,39), (0.26,37) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1,1040), (2,168), (3,128), (10,1), (18,2) | a |
| 14  | (0.4,46), (0.38,8), (0.32,2), (0.46,40), (0.10,4), (0.8,26), (0.9,27), (0.26,44), (0.16,10), (0.3,21), (0.22,16), (0.20,38), (0.21,39), (0.44,14), (0.14,32), (0.45,15), (0.34,28), (0.28,22), (0.33,3), (0.39,9), (0.27,45), (0.15,33), (0.40,34), (0.2,20) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1,1040), (2,168), (3,128), (10,1), (18,2) | a |
| 15  | (0.16,33), (0.22,23), (0.47,16), (0.46,47), (0.35,4), (0.16,17), (0.40,41), (0.29,46), (0.28,29), (0.46,15), (0.34,3), (0.23,40), (0.10,11), (0.17,34), (0.40,9), (0.28,45), (0.5,22), (0.34,35), (0.22,39), (0.41,10), (0.4,5), (0.10,27), (0.4,21), (0.11,28) | 128 | (1,3), (8,5) | (0.768), (1,1040), (2,168), (3,128), (10,1), (18,2) | a |
| 16  | (0.31,36), (0.19,24), (0.43,0), (0.14,3), (0.8,45), (0.20,9), (0.32,21), (0.38,27), (0.2,39), (0.25,30), (0.7,12), (0.1,6), (0.13,18), (0.37,42), (0.26,15), (0.44,33) | 128 | (1,3), (8,5) | (0.768), (1,976), (2,296), (3,64), (10,2), (26,1) | a |
| 17  | (0.10,29), (0.41,28), (0.23,10), (0.35,22), (0.4,23), (0.34,5), (0.16,35), (0.29,16), (0.17,4), (0.11,46), (0.40,11), (0.5,40), (0.46,17), (0.47,34), (0.28,47), (0.22,41) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1,976), (2,296), (3,64), (10,2), (26,1) | a |
| 18  | (0.8,3), (0.39,2), (0.2,45), (0.45,8), (0.45,8), (0.27,38), (0.14,9), (0.44,39), (0.20,15), (0.33,44), (0.32,27), (0.21,32), (0.9,20), (0.3,14), (0.26,21), (0.38,33), (0.15,26) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1,976), (2,296), (3,64), (10,2), (26,1) | a |
| 19  | (0.8,27), (0.26,45), (0.32,3), (0.23,10), (0.41,28), (0.35,22), (0.14,33), (0.29,16), (0.17,4), (0.11,46), (0.2,21), (0.5,40), (0.44,15), (0.20,39), (0.38,9), (0.47,34) | 128 | (1,3), (8,5) | (0.768), (1,976), (2,296), (3,64), (10,2), (26,1) | a |
| 20  | (0.37,14), (0.39,32), (0.21,14), (0.7,32), (0.1,26), (0.19,44), (0.9,2), (0.25,2), (0.31,8), (0.15,8), (0.45,38), (0.27,20), (0.3,44), (0.43,20), (0.13,38), (0.33,26) | 128 | (1,3), (8,5) | (0.768), (1,976), (2,296), (3,64), (10,2), (26,1) | a |
| 21  | (0.3,22), (0.10,45), (0.16,3), (0.46,33), (0.27,46), (0.22,9), (0.9,28), (0.34,21), (0.4,39), (0.40,27), (0.45,16), (0.28,15), (0.21,40), (0.15,34), (0.39,10), (0.33,4) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1,976), (2,296), (3,64), (10,2), (26,1) | a |
| 22  | (0.5,21), (0.23,39), (0.17,33), (0.47,15), (0.9,17), (0.29,45), (0.35,3), (0.3,11), (0.33,41), (0.11,27), (0.21,29), (0.45,5), (0.41,9), (0.15,23), (0.39,47), (0.27,35) | 256 | (1,1), (2,1), (8,1), (16,2) | (0.768), (1,976), (2,296), (3,64), (10,2), (26,1) | a |

Continued on next page.
| Idx | Spread   | [G] | Orbit   | Type | K     |
|-----|----------|-----|---------|------|-------|
| 23  | (0,30,10), (0,24,4), (0,45,15), (0,27,45), (0,39,9), (0,6,34), (0,15,33), (0,12,40), (0,9,27), (0,18,46), (0,33,3), (0,21,39), (0,42,22), (0,3,21), (0,28), (0,21,39), (0,42,22), (0,3,21), (0,28) | 32 | (1,1), (2,7), (4,3), (16,1) | K | (0,768), (1,976), | (2,296), (3,64), (10,2), (26,1) |
| 24  | (0,26,20), (0,27,45), (0,14,8), (0,15,33), (0,21,39), (0,20,14), (0,38,32), (0,39,9), (0,45,15), (0,8,2), (0,32,26), (0,2,44), (0,9,27), (0,3,21), (0,44,38), (0,33,3) | 128 | (1,3), (8,5) | a | (0,768), (1,976), (2,296), (3,64), (10,2), (26,1) |
| 25  | (0,21,34), (0,26,7), (0,14,43), (0,20,1), (0,8,37), (0,27,40), (0,33,46), (0,44,25), (0,15,28), (0,45,10), (0,38,19), (0,9,22), (0,39,4), (0,32,13), (0,3,16), (0,2,31) | 128 | (1,3), (8,5) | a | (0,768), (1,976), (2,296), (3,64), (10,2), (26,1) |
| 26  | (0,46,45), (0,10,41), (0,28,11), (0,34,33), (0,22,5), (0,34,17), (0,40,39), (0,28,27), (0,40,23), (0,22,21), (0,4,35), (0,10,9), (0,16,15), (0,46,29), (0,4,3), (0,16,47) | 128 | (1,3), (8,5) | a | (0,768), (1,976), (2,296), (3,64), (10,2), (26,1) |
| 27  | (0,1,6), (0,25,39), (0,13,18), (0,37,42), (0,19,24), (0,43,0), (0,7,12), (0,31,36) | 128 | (1,3), (8,5) | a | (0,640), (1,1168), (2,297), (10,1), (34,1) |
| 28  | (0,35,22), (0,11,46), (0,5,40), (0,29,16), (0,17,4), (0,23,10), (0,47,34), (0,41,28) | 128 | (1,3), (8,5) | a | (0,640), (1,1168), (2,297), (10,1), (34,1) |
| 29  | (0,28,15), (0,46,33), (0,16,3), (0,34,21), (0,10,45), (0,4,39), (0,40,27), (0,22,9) | 128 | (1,3), (8,5) | a | (0,640), (1,1168), (2,297), (10,1), (34,1) |
| 30  | (0,28,16), (0,46,34), (0,16,4), (0,40,28), (0,10,46), (0,34,22), (0,22,10), (0,4,40) | 128 | (1,3), (8,5) | a | (0,640), (1,1168), (2,297), (10,1), (34,1) |
| 31  | (0,45,15), (0,27,45), (0,39,9), (0,15,33), (0,9,27), (0,33,3), (0,21,39), (0,3,21) | 128 | (1,3), (8,5) | a | (0,640), (1,1168), (2,297), (10,1), (34,1) |
| 32  | (0,27,32), (0,39,44), (0,25,30), (0,9,14), (0,1,6), (0,31,36), (0,45,2), (0,37,42), (0,7,12), (0,15,20), (0,21,26), (0,33,38), (0,3,8), (0,43,0), (0,13,18), (0,19,24) | 128 | (1,3), (8,5) | a | (0,768), (1,912), (2,425), (18,1), (26,1) |
| 33  | (0,9,44), (0,3,38), (0,15,2), (0,23,10), (0,41,28), (0,45,32), (0,35,22), (0,29,16), (0,17,4), (0,27,14), (0,11,46), (0,5,40), (0,21,8), (0,39,26), (0,47,34), (0,33,20) | 128 | (1,3), (8,5) | a | (0,768), (1,912), (2,425), (18,1), (26,1) |
| Idx | Spread | | [G] | Orbit | Type | K |
|-----|--------|-----------------|-----|-------|------|---|
| 34  | (0.23,10), (0.41,28), (0.35,22), (0.10,45), (0.29,16), (0.17,4), (0.46,33), (0.16,3), (0.11,46), (0.22,9), (0.34,21), (0.4,39), (0.5,40), (0.40,27), (0.28,15), (0.47,34) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 35  | (0.9,44), (0.3,38), (0.15,2), (0.45,32), (0.10,45), (0.16,3), (0.27,14), (0.46,33), (0.22,9), (0.34,21), (0.4,39), (0.40,27), (0.21,8), (0.28,15), (0.39,26), (0.33,20) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 36  | (0.17,45), (0.9,37), (0.47,27), (0.27,7), (0.41,21), (0.15,43), (0.21,1), (0.35,15), (0.39,19), (0.33,13), (0.5,33), (0.11,39), (0.45,25), (0.3,31), (0.29,9), (0.23,3) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 37  | (0.6,24), (0.24,42), (0.36,6), (0.27,45), (0.15,33), (0.21,39), (0.3,31), (0.5,33), (0.45,29), (0.3,31), (0.9,27), (0.3,21), (0.33,3) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 38  | (0.29,23), (0.9,3), (0.41,35), (0.39,33), (0.17,11), (0.5,47), (0.23,17), (0.21,15), (0.3,45), (0.11,5), (0.45,39), (0.47,41), (0.33,27), (0.35,29), (0.27,21), (0.15,9) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 39  | (0.20,38), (0.27,45), (0.15,33), (0.21,39), (0.32,2), (0.39,9), (0.38,8), (0.45,15), (0.2,20), (0.14,32), (0.8,26), (0.26,44), (0.9,27), (0.3,21), (0.33,3), (0.44,14) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 40  | (0.33,14), (0.26,7), (0.9,38), (0.14,43), (0.20,1), (0.8,37), (0.15,44), (0.44,25), (0.3,32), (0.38,19), (0.45,26), (0.32,13), (0.2,31), (0.27,8), (0.39,20), (0.21,2) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 41  | (0.7, *), (0.16, *), (0.1, *), (0.25, *), (0.19, *), (0.28, *), (0.22, *), (0.31, *), (0.40, *), (0.13, *), (0.4, *), (0.46, *), (0.34, *), (0.10, *), (0.37, *), (0.43, *) | 128 | (1,3), (8,5) | (0.768), (1.912), (2.425), (18,1), (26,1) | a |
| 42  | (0.31,36), (0.19,24), (0.13,30), (0.43,0), (0.37,6), (0.43,12), (0.31,0), (0.7,24), (0.19,36), (0.1,18), (0.1,6), (0.25,30), (0.7,12), (0.25,42), (0.13,18), (0.37,42) | 32 | (1,1), (2,5), (4,4), (16,1) | (0.784), (1.928), (2.344), (3.48), (10,2), (26,1) | s |
| 43  | (0.16,4), (0.39,2), (0.4,40), (0.10,46), (0.33,44), (0.22,10), (0.15,26), (0.9,20), (0.45,8), (0.46,34), (0.28,16), (0.3,14), (0.21,32), (0.27,38), (0.34,22), (0.40,28) | 32 | (1,1), (2,7), (4,3), (16,1) | (0.784), (1.928), (2.344), (3.48), (10,2), (26,1) | s |
| 44  | (0.41,5), (0.39,2), (0.45,8), (0.27,38), (0.11,23), (0.33,44), (0.17,29), (0.21,32), (0.9,20), (0.23,35), (0.29,41), (0.3,14), (0.35,47), (0.47,11), (0.15,26), (0.5,17) | 16 | (1,9), (2,9), (8,2) | (0.784), (1.928), (2.344), (3.48), (10,2), (26,1) | s |

Continued on next page.
| Idx | Spread | [G] | Orbit | Type | K |
|-----|--------|-----|-------|------|---|
| 45  | (0.17,29), (0.41,5), (0.33,28), (0.5,17), (0.11,23), (0.27,22), (0.21,16), (0.23,35), (0.15,10), (0.35,47), (0.47,11), (0.3,46), (0.45,40), (0.9,4), (0.39,34), (0.29,41) | 32  | (1,1), (2,7), (4,3), (16,1), (0.78,4), (1.92,8), (2.34,4), (3,48), (10,2), (26,1) |
| 46  | (0.26,7), (0.27,45), (0.15,33), (0.14,43), (0.20,1), (0.8,37), (0.21,39), (0.39,9), (0.45,15), (0.44,25), (0.38,19), (0.32,13), (0.2,31), (0.9,27), (0.3,21), (0.33,3) | 32  | (1,3), (2,4), (4,4), (16,1), (0.78,4), (1.92,8), (2.34,4), (3,48), (10,2), (26,1) |
| 47  | (0.21,37), (0.22,16), (0.16,10), (0.34,28), (0.33,1), (0.3,19), (0.28,22), (0.15,31), (0.46,40), (0.9,25), (0.46,2), (0.27,43), (0.45,13), (0.10,4), (0.39,7), (0.40,34) | 16  | (1,5), (2,11), (8,2), (0.78,4), (1.92,8), (2.34,4), (3,48), (10,2), (26,1) |
| 48  | (0.26,6), (0.44,24), (0.31,39), (0.37,45), (0.43,3), (0.32,12), (0.19,27), (0.13,21), (0.19), (0.38,18), (0.25,33), (0.2,30), (0.8,36), (0.20,0), (0.14,42), (0.7,15) | 16  | (1,5), (2,11), (8,2), (0.78,4), (1.92,8), (2.34,4), (3,48), (10,2), (26,1) |
| 49  | (0.7,42), (0.11,46), (0.43,30), (0.8,27), (0.19,6), (0.20,39), (0.26,45), (0.17,4), (0.41,28), (0.35,22), (0.14,33), (0.25,12), (0.38,9), (0.47,34), (0.29,16), (0.37,24), (0.13,0), (0.31,18), (0.23,10), (0.1,36), (0.32,3), (0.2,21), (0.5,40), (0.44,15) | 128 | (1,3), (8,5), (0.896), (1.656), (2.552), (10,1), (18,2) |
| 50  | (0.1,30), (0.37,2), (0.27,40), (0.43,8), (0.7,36), (0.13,42), (0.25,6), (0.31,44), (0.37,18), (0.21,34), (0.9,22), (0.19,0), (0.13,26), (0.15,28), (0.43,24), (0.3,16), (0.25,38), (0.7,20), (0.1,14), (0.39,4), (0.31,12), (0.33,46), (0.45,10), (0.19,32) | 128 | (1,3), (8,5), (0.896), (1.656), (2.552), (10,1), (18,2) |
| 51  | (0.15,19), (0.3,7), (0.33,37), (0.41,28), (0.23,10), (0.35,22), (0.29,16), (0.17,4), (0.11,46), (0.39,33), (0.45,1), (0.5,40), (0.21,25), (0.47,34), (0.9,13), (0.27,31) | 32  | (1,1), (2,5), (4,4), (16,1), (0.80,0), (1.894), (2.362), (3,48), (9,2), (26,1) |
| 52  | (0.36,7), (0.0,19), (0.10,45), (0.24,43), (0.16,3), (0.46,33), (0.22,9), (0.34,21), (0.4,39), (0.12,31), (0.40,27), (0.6,25), (0.28,15), (0.42,13), (0.3,0,1), (0.18,37) | 128 | (1,3), (8,5), (0.832), (1.784), (2.488), (10,2), (26,1) |
| 53  | (0.46,11), (0.26,7), (0.16,29), (0.4,17), (0.14,43), (0.20,1), (0.8,37), (0.28,41), (0.28,15), (0.42,13), (0.3,0,1), (0.18,37) | 128 | (1,3), (8,5), (0.832), (1.784), (2.488), (10,2), (26,1) |
| 54  | (0.8,34), (0.2,45), (0.16,4), (0.3,47), (0.14,9), (0.20,15), (0.44,39), (0.28,4), (0.22,4), (0.9,5), (0.15,11), (0.45,41), (0.32,27), (0.40,4), (0.21,17), (0.4,4), (0.46,4), (0.34,4), (0.27,23), (0.33,29), (0.38,33), (0.26,21), (0.10,4), (0.39,35) | 16  | (1,5), (2,7), (8,1), (16,1), (0.864), (1.814), (2.345), (3,64), (4,16), (9,2), (10,1), (18,1) |
| Idx | Spread | | | | | K |
|-----|--------|-----|-----|-----|-----|-----|
| 55  | (0.39,2), (0.34,44), (0.20,15), (0.20,1), (0.26,7), (0.15,26), (0.26,21), (0.32,27), (0.9,20), (0.45,8), (0.38,33), (0.8,3), (0.14,9), (0.3,14), (0.14,43), (0.38,19), (0.21,32), (0.44,25), (0.32,13), (0.27,38), (0.8,37), (0.2,45), (0.2,31), (0.44,39) | 32 | (1.3), (2.4), (4.2), (8.1), (16.1) | (0.864), (1.814), (2.345), (3.64), (4.16), (9.2), (10.1), (18.1) |
| 56  | (0.9,4), (0.39,34), (0.21,16), (0.27,22), (0.23,35), (0.3,46), (0.15,10), (0.29,41), (0.30,46), (0.47,11), (0.42,10), (0.0,16), (0.36,4), (0.6,22), (0.33,28), (0.12,28), (0.11,23), (0.45,40), (0.18,34), (0.17,29), (0.35,47), (0.5,17), (0.24,40), (0.41,5) | 16 | (1.7), (2.6), (8.1), (16.1) | (0.864), (1.814), (2.345), (3.64), (4.16), (9.2), (10.1), (18.1) |
| 57  | (0.8,3), (0.2,45), (0.3,47), (0.14,9), (0.20,15), (0.44,39), (0.9,5), (0.15,11), (0.45,41), (0.32,27), (0.21,17), (0.27,23), (0.33,29), (0.38,33), (0.26,21), (0.39,35) | 16 | (1.5), (2.11), (8.2) | (0.784), (1.935), (2.329), (3.56), (9.1), (10.1), (26.1) |
| 58  | (0.6,20), (0.3,4), (0.20,42), (0.32,6), (0.15,16), (0.28,45), (0.40,9), (0.16,3), (0.9,28), (0.4,39), (0.44,18), (0.40,27), (0.28,15), (0.45,16), (0.42,8), (0.27,28), (0.30,44), (0.21,40), (0.4,21), (0.39,40), (0.16,33), (0.18,32), (0.33,4), (0.8,30) | 192 | (1.1), (6.1), (12.1), (24.1) | (0.864), (1.792), (2.420), (5.24), (6.6), (18.1) |
| 59  | (0.21,34), (0.17,45), (0.47,27), (0.27,40), (0.41,21), (0.33,46), (0.35,15), (0.45,10), (0.15,28), (0.39,4), (0.5,33), (0.9,22), (0.11,39), (0.3,16), (0.29,9), (0.23,3) | 5376 | (1.1), (42.1) | (0.384), (1.1680), (2.42), (42.1) |
| 60  | (0.46,9), (0.26,7), (0.34,45), (0.40,3), (0.14,43), (0.10,21), (0.20,1), (0.8,37), (0.16,27), (0.44,25), (0.38,19), (0.28,39), (0.32,13), (0.4,15), (0.2,34), (0.22,33) | 16 | (1.5), (2.11), (8.2) | (0.792), (1.911), (2.353), (3.48), (9.1), (10.1), (26.1) |
| 61  | (0.39,2), (0.26,7), (0.45,8), (0.27,38), (0.14,43), (0.2,21), (0.5,37), (0.33,44), (0.21,32), (0.9,20), (0.44,25), (0.38,19), (0.28,39), (0.32,13), (0.2,31), (0.3,14), (0.15,26) | 16 | (1.7), (2.10), (8.2) | (0.792), (1.911), (2.353), (3.48), (9.1), (10.1), (26.1) |
| 62  | (0.9,41), (0.46,11), (0.27,11), (0.33,17), (0.16,29), (0.4,17), (0.39,23), (0.28,41), (0.10,23), (0.15,47), (0.21,5), (0.22,35), (0.34,47), (0.3,35), (0.45,29), (0.40,5) | 16 | (1.5), (2.11), (8.2) | (0.792), (1.911), (2.353), (3.48), (9.1), (10.1), (26.1) |
| 63  | (0.4,15), (0.39,2), (0.22,33), (0.16,27), (0.33,44), (0.26,7), (0.20,1), (0.15,26), (0.40,3), (0.9,20), (0.45,8), (0.3,14), (0.34,45), (0.14,43), (0.38,19), (0.21,32), (0.44,25), (0.10,21), (0.32,13), (0.27,38), (0.28,39), (0.8,37), (0.46,9), (0.2,31) | 16 | (1.7), (2.6), (8.3) | (0.848), (1.815), (2.377), (3.64), (10.1), (17.1), (18.1) |

Continued on next page.
| Idx | Spread | \(|G|\) | Orbit | Type | K |
|-----|--------|------|--------|------|---|
| 65  | \((0.9, 4), (0.4, 15), (0.39, 34), (0.21, 16), (0.27, 22), (0.3, 46), (0.15, 10), (0.22, 33), (0.16, 27), (0.26, 7), (0.20, 1), (0.40, 3), (0.33, 28), (0.45, 40), (0.34, 45), (0.14, 43), (0.38, 19), (0.10, 21), (0.44, 25), (0.32, 13), (0.28, 39), (0.8, 37), (0.46, 9), (0.231)\) | 32  | \((1.3, 0.664), (1.822, 0.848), (2.321, 0.948), (3.88, 0.48), (9.2, 0.816), (10.1, 1.811)\) | s   |
| 66  | \((0.47, 15), (0.39, 8), (0.15, 32), (0.35, 3), (0.16, 41), (0.23, 39), (0.40, 17), (0.10, 35), (0.29, 45), (0.17, 33), (0.11, 27), (0.41, 9), (0.34, 11), (0.28, 5), (0.5, 21), (0.42, 9), (0.2, 6), (0.45, 14), (0.22, 47), (0.46, 23), (0.9, 26), (0.45, 14), (0.22, 47), (0.3, 20), (0.21, 38), (0.3, 22), (0.27, 44)\) | 8   | \((1.19, 0.864), (1.822, 0.848), (2.321, 0.948), (3.88, 0.48), (9.2, 0.816), (10.1, 1.811)\) | s   |
| 67  | \((0.8, 36), (0.30, 12), (0.24, 6), (0.14, 42), (0.33, 47), (0.0, 30), (0.3, 3), (0.18, 0), (0.27, 41), (0.6, 36), (0.32, 12), (0.26, 6), (0.39, 5), (0.36, 18), (0.12, 42), (0.42, 24), (0.20, 0), (0.45, 11), (0.3, 17), (0.9, 23), (0.38, 18), (0.15, 29), (0.44, 24), (0.21, 35)\) | 8   | \((1.19, 0.864), (1.822, 0.848), (2.321, 0.948), (3.88, 0.48), (9.2, 0.816), (10.1, 1.811)\) | s   |
| 68  | \((0.42, 22), (0.26, 7), (0.36, 16), (0.14, 46), (0.27, 45), (0.15, 33), (0.14, 43), (0.2, 6), (0.80, 37), (0.21, 39), (0.30, 10), (0.6, 34), (0.24, 4), (0.39, 9), (0.45, 15), (0.44, 25), (0.38, 19), (0.32, 13), (0.2, 31), (0.12, 40), (0.9, 27), (0.3, 21), (0.33, 2), (0.27, 45), (0.15, 29), (0.44, 24), (0.21, 35)\) | 16  | \((1.7, 0.848), (1.848, 0.848), (2.327, 0.948), (3.64, 0.816), (4.16, 0.816), (10.3, 1.811)\) | s   |
| 69  | \((0.37, 2), (0.6, 34), (0.42, 22), (0.43, 8), (0.12, 40), (0.9, 27), (0.30, 10), (0.31, 44), (0.3, 21), (0.24, 4), (0.13, 26), (0.21, 39), (0.45, 15), (0.36, 18), (0.25, 38), (0.18, 46), (0.33, 3), (0.39, 9), (0.28, 0), (0.27, 45), (0.7, 20), (0.1, 14), (0.15, 33), (0.19, 32)\) | 48  | \((1.1, 0.816), (1.960, 0.816), (2.183, 0.816), (3.144, 0.816), (10.3, 1.811)\) | s   |
| 70  | \((0.27, 33), (0.24, 16), (0.22, 42), (0.36, 28), (0.1, 19), (0.15, 21), (0.33, 39), (0.16, 36), (0.9, 15), (0.10, 30), (0.30, 22), (0.21, 27), (0.46, 18), (0.0, 40), (0.39, 45), (0.28, 0), (0.12, 4), (0.4, 24), (0.18, 10), (0.42, 34), (0.40, 12), (0.6, 46), (0.45, 3), (0.36, 46)\) | 96  | \((1.1, 0.816), (1.960, 0.816), (2.183, 0.816), (3.144, 0.816), (10.3, 1.811)\) | s   |
| 71  | \((0.46, 10), (0.4, 0.4), (0.30, 46), (0.16, 28), (0.22, 34), (0.2, 0.15), (0.42, 10), (0.0, 16), (0.34, 46), (0.36, 46), (0.26, 21), (0.32, 27), (0.6, 22), (0.38, 33), (0.12, 28), (0.8, 3), (0.18, 34), (0.14, 9), (0.10, 22), (0.24, 40), (0.4, 16), (0.2, 45), (0.28, 40), (0.44, 39)\) | 16  | \((1.7, 0.856), (1.846, 0.846), (2.297, 0.948), (3.96, 0.948), (4.8, 1.811), (10.1, 1.811)\) | s   |

Continued on next page.
| Idx | Spread | [G] | Orbit | Type | K |
|-----|--------|-----|-------|------|---|
| 72  | (0,1,33), (0,8,36), (0,31,15), (0,11,40), (0,14,42), (0,2,30), (0,5,34), (0,13,45), (0,43,27), (0,47,28), (0,23,12), (0,41,22), (0,26,6), (0,29,10), (0,17,46), (0,47,28), (0,23,12), (0,41,22), (0,26,6), (0,29,10), (0,17,46), (0,43,27), (0,47,28) | (16,1) | (1.5), (2.7), (8.1), (16.1) | 16 | (0,856), (1,846), (2,207), (3,96), (4,8), (9.2), (10.1), (18.1) |
| 73  | (0,4,15), (0,46,10), (0,22,33), (0,16,27), (0,40,4), (0,30,46), (0,16,28), (0,22,34), (0,42,10), (0,16,28), (0,22,34), (0,42,10), (0,0,16), (0,34,46), (0,36,4), (0,6,22), (0,40,3), (0,12,28), (0,18,34), (0,34,45), (0,10,22), (0,10,21), (0,24,40), (0,28,39), (0,4,16), (0,4,16), (0,24,40), (0,28,39), (0,4,16), (0,4,16), (0,24,40) | 8 | (1.19), (8.3) | (0,872), (1,798), (2,345), (3,80), (4,8), (9.2), (10.1), (18.1) |
| 74  | (0,36,1), (0,8,36), (0,5,45), (0,0,13), (0,11,3), (0,14,42), (0,23,15), (0,35,27), (0,41,33), (0,2,30), (0,12,25), (0,6,19), (0,32,12), (0,29,21), (0,26,6), (0,30,43), (0,17,9), (0,18,31), (0,47,39), (0,20,0), (0,24,40), (0,28,39), (0,4,16), (0,4,16), (0,24,40), (0,28,39), (0,4,16), (0,4,16), (0,24,40) | 16 | (1.5), (2.7), (8.1), (16.1) | (0,872), (1,798), (2,345), (3,80), (4,8), (9.2), (10.1), (18.1) |
| 75  | (0,18,34), (0,42,22), (0,6,22), (0,36,16), (0,18,46), (0,24,40), (0,27,45), (0,42,10), (0,30,46), (0,15,33), (0,21,39), (0,36,4), (0,30,10), (0,6,34), (0,12,28), (0,24,4), (0,39,9), (0,45,15), (0,0,16), (0,12,40), (0,9,27), (0,3,21), (0,33,3), (0,28) | 32 | (1.3), (2.4), (4.2), (8.1), (16.1) | (0,912), (1,672), (2,471), (3,48), (10.3), (18.1) |
| 76  | (0,4,46), (0,16,4), (0,39,2), (0,46,40), (0,4,40), (0,10,4), (0,10,46), (0,33,44), (0,22,10), (0,15,26), (0,9,20), (0,16,10), (0,22,16), (0,45,8), (0,46,34), (0,28,16), (0,34,28), (0,3,14), (0,28,22), (0,21,32), (0,27,38), (0,34,22), (0,40,34), (0,40,28) | 48 | (1.1), (3.4), (6.1), (24.1) | (0,864), (1,816), (2,327), (3,96), (10.3), (18.1) |
| 77  | (0,9,4), (0,8,36), (0,39,34), (0,21,16), (0,27,22), (0,23,35), (0,3,46), (0,15,10), (0,29,41), (0,14,42), (0,2,30), (0,47,11), (0,32,12), (0,26,6), (0,33,28), (0,11,23), (0,45,40), (0,17,29), (0,20,0), (0,35,47), (0,5,17), (0,38,18), (0,41,5), (0,44,24) | 48 | (1.1), (3.4), (6.1), (24.1) | (0,864), (1,816), (2,327), (3,96), (10.3), (18.1) |
| 78  | (0,4,46), (0,39,2), (0,23,35), (0,29,41), (0,46,40), (0,10,4), (0,33,44), (0,47,11), (0,15,26), (0,9,20), (0,16,10), (0,22,16), (0,45,8), (0,11,23), (0,34,28), (0,3,14), (0,17,29), (0,21,32), (0,28,22), (0,35,47), (0,27,38), (0,5,17), (0,40,34), (0,41,5) | 16 | (1.5), (2.7), (8.1), (16.1) | (0,856), (1,832), (2,327), (3,80), (4,8), (10.3), (18.1) |

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### Conclusion

There are several potentially interesting directions for further research based upon the results here. As mentioned previously, it is likely possible to port the existing search algorithms to more robust hardware and complete a similar search for spreads of the Hermitian unital in $\mathbb{P}G(2,64)$. Such a search would likely provide new examples of spreads for even $q$.

In addition to spread problems, the problem of finding resolutions of Hermitian unitals, namely partitions of the blocks of a unital into pairwise disjoint spreads, is of some geometric interest. As one application, the author [5] provides a construction of a fan of the three-dimensional Hermitian variety from a resolution of the Hermitian unital. Currently, only the obvious resolution of the Hermitian unital into regular spreads whose starting points are all collinear is known.

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