INVARINANCE PRINCIPLE FOR ADDITIVE FUNCTIONALS OF MARKOV CHAINS

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Abstract. We consider a sequence of additive functionals \( \{\phi_n\} \), set on a sequence of Markov chains \( \{X_n\} \) that weakly converges to a Markov process \( X \). We give sufficient condition for such a sequence to converge in distribution, formulated in terms of the characteristics of the additive functionals, and related to the Dynkin’s theorem on the convergence of \( W \)-functionals. As an application of the main theorem, the general sufficient condition for convergence of additive functionals in terms of transition probabilities of the chains \( X_n \) is proved.

1. Introduction

Let a sequence of processes \( X_n = X_n(\cdot) \) be given, converging in distribution (in some sense, e.g., in a sense of convergence of finite-dimensional distributions, distributions in spaces \( C \) or \( D \), etc.) to a limit process \( X = X(\cdot) \). Also let the family of functionals \( \phi_n \) of the processes \( X_n \) be given. Assume that they are additive in an appropriate sense with respect to time variable. The general question, considered in the present paper, is what an information about the limit behavior of the distributions of functionals \( \phi_n \) can be obtained in a situation where the processes \( X_n, X \) possess certain Markov properties. The starting point in our considerations is provided by the comparatively simple, but important particular case of the problem outlined above, in which all the processes \( X_n \) coincide. In this situation, \( \phi_n \) are functionals of the same process \( X \), and if \( X \) is Markov process and \( \phi_n \) are \( W \)-functionals (see [1], Chapter 6), then their limit behavior, according to the well known theorem by E.B.Dynkin ([1], Theorem 6.4), is determined by the limit behavior of their characteristics (that is, their expectations).

In the present paper we consider the processes \( X_n \) that differ one from another. The class of sequences of processes \( X_n \), considered in the framework of our approach, contains sequences of Markov chains with appropriately normalized time, embedded into \( C \) or \( D \) (for example, by means of standard operations of linearization or construction of graduated processes), and weakly convergent to Markov process \( X \). Important partial case is provided by random broken lines (or random step functions) \( X_n \), constructed by a random walk in \( \mathbb{R}^d \) and weakly convergent to a homogenous stable process \( X \) (particularly, to the Brownian motion).

We show that, under some structural assumption about processes \( X_n, X \) (the condition is that the sequence \( \{X_n\} \) provides Markov approximation for the process \( X \)), the full analogue of the Dynkin’s theorem takes place: if the characteristics of functionals \( \phi_n \) converge weakly to the characteristics of \( W \)-functional \( \phi \) of the limit process \( X \), then the distributions of \( \phi_n \) converge to the distribution of \( \phi \). Our method of proof is based on \( L_2 \)-estimates for the distance between additive functionals, similar to those given in Lemma 6.5 [1]. The proof of these estimates is concerned with a preliminary construction of processes \( X_n, X \) on one probability space in such a way, that the functionals \( \phi_n, \phi \), associated initially with a different processes, are interpreted as a functionals of one two-component process. The (some kind of) Markov property of the two-component process is essential for the estimates, analogous to those given in Lemma 6.5 [1]; the structural assumption mentioned above is just the claim for such a property to hold true in an appropriate form.

The method, proposed by authors, allows one to reduce the problem of studying of asymptotic behavior of the distributions of additive functionals to a priori more simple problem of studying of their means. In our opinion, it provides a good addition to the available methods of studying the limit behavior of additive functionals both for the important partial case of random walks (we do not give the detailed review here,
referring the reader to monographs [2, 3, 4], papers [5, 6] and reviews there), and for general Markov chains. Among the latter, it is necessary to mention the method that is based on the passing to the limit in the difference equations that describe characteristic functions of additive functionals of Markov chains, and ascends to the works of I.Gikhman at 50-ies (see [7, 8], also [9] and the survey paper [10]).

The structure of the article is following. In Chapter 2, we introduce the notion of Markov approximation and give examples that illustrate it. In Chapter 3, the main theorem of the article is introduced and proved. In Chapters 4,5, the two elementary examples of application of this theorem are given. In Chapter 6, the main theorem is applied to the proof of a general sufficient condition for weak convergence of additive functionals, set on the sequence of Markov chains, that is formulated in terms of transition probabilities of the chains.

2. Markov approximation.

Further we assume that the processes $X_n, X$ are defined on $\mathbb{R}^+$ and have a locally compact metric phase space $(\mathbb{X}, \rho)$. We say that the process $X$ possesses the Markov property at the time moment $s \in \mathbb{R}^+$ w.r.t. filtration $\{G_t, t \in \mathbb{R}^+\}$, if $X$ is adapted to this filtration and for each $k \in \mathbb{N}$, $t_1, \ldots, t_k > s$ there exists a stochastic kernel $\{P_{st_1 \ldots t_k}(x,A), x \in \mathbb{X}, A \in \mathcal{B}(\mathbb{X})\}$ such that

$$E[I_A((X(t_1), \ldots, X(t_k)))|G_s] = P_{st_1 \ldots t_k}(X(s), A) \text{ a.s., } A \in \mathcal{B}(\mathbb{X}).$$

The measure $P_{st_1 \ldots t_k}(x, \cdot)$ has a natural interpretation as the finite-dimensional distribution of $X$ at the points $t_1, \ldots, t_k$, conditioned by $\{X(s) = x\}$; we denote below $P_{st_1 \ldots t_k}(x, \cdot) = P((X(t_1), \ldots, X(t_k)) \in \cdot|X(s) = x)$.

Remark 1. In some cases, (2.1) implies the following functional analogue of (2.1):

$$E[I_A((X(s^\infty))|G_s] = E[I_A((X(s^\infty))|X(s)],$$

where $X|s^\infty$ denotes the trajectory of the process $X$ on the time interval $[s, +\infty)$, considered as an element of appropriate functional space. For instance, if the Kolmogorov’s sufficient condition for existence of continuous modification holds true both for unconditional and conditional distributions of $X$, then (2.2) holds with $X|s^\infty$ considered as an element of $C([s, +\infty), \mathbb{X})$.

Everywhere below we assume that the process $X$ possesses the Markov property w.r.t. its canonic filtration at every point $s \in \mathbb{R}^+$ and for the processes $X_n$ the same property holds true at every point of the type $\frac{i}{n}, i \in \mathbb{Z}^+$ (the choice of the denominator here is quite arbitrary: it is possible to put any expression $N(n) \to \infty, n \to \infty$ instead of $n$, but we avoid to do this in order to shorten the notation).

The next definition is introduced in [11].

Definition 1. The sequence $\{X_n\}$ provides Markov approximation for the process $X$, if for arbitrary $\gamma > 0, T < +\infty$ there exists $K(\gamma, T) \in \mathbb{N}$ and a sequence of two-componential processes $\{\hat{Y}_n = (X_n, \hat{X}^n)\}$, defined on another probability space, such that

(i) $\hat{X}_n \overset{d}{=} X_n, \hat{X}^n \overset{d}{=} X$;

(ii) the process $\hat{Y}_n$, together with the processes $\hat{X}_n, \hat{X}^n$, possesses the Markov property at the points $\frac{iK(\gamma, T)}{n}$, $i \in \mathbb{N}$ w.r.t. filtration $\{\hat{F}_t^\gamma = \sigma(\hat{Y}_n(s), s \leq t)\}$;

(iii) $\limsup_{n \to +\infty} P\left(\sup_{t \leq i\frac{K(\gamma, T)}{n}} \rho(\hat{X}_n\left(\frac{iK(\gamma, T)}{n}\right), \hat{X}^n\left(\frac{iK(\gamma, T)}{n}\right)) > \gamma\right) < \gamma$.

Remark 2. Condition (ii) implies that, for $i, k \in \mathbb{N}, t_1, \ldots, t_k > \frac{iK(\gamma, T)}{n}, (x, y) \in \mathbb{X}^2$, the marginal distributions $P\left(\hat{Y}_n(t_1), \ldots, \hat{Y}_n(t_k) \in \cdot|\hat{Y}_n\left(\frac{iK(\gamma, T)}{n}\right) = (x, y)\right)$ are equal to $P\left((X_n(t_1), \ldots, X_n(t_k)) \in \cdot|X_n\left(\frac{iK(\gamma, T)}{n}\right) = x\right)$ and $P\left((X(t_1), \ldots, X(t_k)) \in \cdot|X\left(\frac{iK(\gamma, T)}{n}\right) = y\right)$ respectively.

Let us give some examples that illustrate Definition 1.
Example 1. Let \( \{\xi_k\} \) be a sequence of i.i.d random vectors in \( \mathbb{R}^d \) with \( E[\|\xi_k\|_p]^{-\delta} < \infty \) for some \( \delta > 0 \). Assume \( \{\xi_k\} \) to have zero mean and identity for their covariance matrix. Let us introduce the sequence of processes \( X_n \) ("random broken lines") on \( \mathbb{R}^+ \) by

\[
X_n(t) = \frac{S_{k-1}}{\sqrt{n}} + (nt - k + 1) \left[ \frac{S_k}{\sqrt{n}} - \frac{S_{k-1}}{\sqrt{n}} \right], \quad t \in \left[ \frac{k-1}{n}, \frac{k}{n} \right], \quad k \in \mathbb{N},
\]

where \( S_n = \sum_{k=1}^n \xi_k \). Then \( X_n \) converge by distribution in \( C(\mathbb{R}^+, \mathbb{R}^d) \) to the Brownian motion \( X \) in \( \mathbb{R}^d \).

It is shown in [11] that the sequence \( \{X_n\} \) provides Markov approximation for the process \( X \) (part I. of Theorem 1 [11]). On the other hand, in the same paper (part II. of the same Theorem) the following effect is revealed. Let us denote by \( K(\gamma, T) \) the minimal constant \( K(\gamma, T) \) such that there exists a process \( \hat{Y}_n \) satisfying conditions (i)-(iii) of Definition [11]. Then, in all the cases except one trivial case \( \xi_k \sim N(0, I) \), for each fixed \( T > 0 \) the convergence \( K(\gamma, T) \to \infty, \gamma \to 0^+ \) takes place. In other words, while the accuracy of approximation of the Brownian motion \( X \) by the random walk \( X_n \) becomes better (this accuracy is described by the parameter \( \gamma \)), the Markov properties of the pair of processes \( (X, X_n) \) necessarily become worse (these properties are characterized by \( K(\gamma, T) \)).

Example 2. Let \( \{\xi_k\} \) be i.i.d random variables, belonging to the normal domain of attraction for \( \alpha \)-stable distribution \( L, \alpha \in (0, 2) \). By the definition, this means that

\[
n^{-\frac{1}{\alpha}}[S_n - a_n] \Rightarrow L, \quad a_n = \begin{cases} 0, & \alpha \in (0, 1) \\ nE\xi_1, & \alpha \in (1, 2) \\ n^2E\sin\frac{\alpha}{n}, & \alpha = 1 \end{cases}
\]

([12], Chapter XVII.5). In order to shorten the notation, we assume that \( a_n \equiv 0 \) and consider processes \( X_n \) on \( \mathbb{R}^+ \) of the type

\[
X_n(t) = n^{-\frac{1}{\alpha}}S_{k-1} + (nt - k + 1) \left[ n^{-\frac{1}{\alpha}}S_k - n^{-\frac{1}{\alpha}}S_{k-1} \right], \quad t \in \left[ \frac{k-1}{n}, \frac{k}{n} \right], \quad k \in \mathbb{N}.
\]

Then \( X_n \) converge by distribution in \( \mathbb{D}(\mathbb{R}^+) \) to the homogeneous process with independent increments \( X \) in \( \mathbb{R} \), for which \( X(1) = X(0) \overset{d}{=} L \) (we call such process a process an \( \alpha \)-stable one).

It is shown in [11] (Theorem 2) that the sequence \( \{X_n\} \) provides Markov approximation for the process \( X \). Furthermore, in this situation, on the contrary to the previous example, \( K(\gamma, T) = 1 \) for all \( \gamma, T \). This means that, in this case, the Markov properties do not become worse while accuracy of approximation improves.

Remark 3. The last example shows that the property of Markov approximation does imply, in general, the convergence of distributions of the processes \( X_n \) to the distribution of \( X \) in \( \mathcal{C} = C(\mathbb{R}^+, \mathcal{X}) \) even if \( X_n \) has continuous trajectories. The same can be said about convergence in \( \mathbb{D} = \mathbb{D}(\mathbb{R}^+, \mathcal{X}) \) (we omit the corresponding example).

Let us remark that the approach, introduced in the present paper, is closely related to the Skorokhod’s method of embedding of random walk into Wiener process by means of of appropriate sequence of stopping moments ([13]), widely used in literature. The basic idea is the same: we have to construct two processes on the same probability space, with the pair keeping Markov or martingale properties. However, the Skorokhod’s method, while being quite efficient for one-dimensional random walks that approximate Wiener process, is much less appropriate in a multi-dimensional situation or for stable domain of attraction. Examples 1 and 2 show that the claim for the Markov approximation to hold true is not restrictive, at least for all basic classes of random walks with no regard to the dimension of the phase space or to the type of limit distribution.

The following example shows that the property of Markov approximation is "stable" in the following sense. This property is preserved under construction of a new pair \( (Z_n, Z) \) from the pair \( (X_n, X) \), possessing this property, in some regular way (e.g., as a solution of a family of stochastic equations).
Example 3. Let $X_n, X$ be as in Example 1, functions $a : \mathbb{R}^m \to \mathbb{R}^m, b : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be Lipschitz and $b^*(x)b(x) > 0, x \in \mathbb{R}^m$ (the sign * denotes the operation of taking of the adjoint matrix). Define

\begin{equation}
Z_n \left( \frac{k+1}{n} \right) = Z_n \left( \frac{k}{n} \right) + a \left( Z_n \left( \frac{k}{n} \right) \right) \frac{1}{n} + b \left( Z_n \left( \frac{k}{n} \right) \right) \Delta X_n \left( \frac{k}{n} \right), \quad Z_n(0) = z,
\end{equation}

$\Delta X_n \left( \frac{k}{n} \right) \equiv \left[ X_n \left( \frac{k+1}{n} \right) - X_n \left( \frac{k}{n} \right) \right]$. Then (12), (15) $Z_n$ converge by distribution in $C([0, \infty), \mathbb{R}^m)$ to the process $Z$, defined by SDE

\begin{equation}
dZ(t) = a(Z(t))dt + b(Z(t))dX(t), \quad Z(0) = z,
\end{equation}

where $X$ is the Brownian motion in $\mathbb{R}^d$. It is natural to call the sequence $Z_n$ the difference approximation of the diffusion process $Z$.

Let us show that the sequence $\{Z_n\}$ provides Markov approximation for the process $Z$. For arbitrary $\gamma, T$, we construct a pair $(\hat{X}_n, \hat{X}_n)$, corresponding to processes $X_n, X$ and satisfying conditions of Definition 1 (such construction is possible due to Example 1).

Let us construct the processes $\tilde{Z}_n, \tilde{Z}_n$ as the functionals of the processes $\hat{X}_n, \hat{X}_n$ by equalities (2.5), (2.6) with $X_n$ replaced by $\hat{X}_n$, and $X$ replaced by $\hat{X}_n$ (note that (2.6) has unique strong solution, hence this procedure is correct). By the construction, the pair $(\tilde{Z}_n, \tilde{Z}_n)$ satisfies condition (i) of Definition 1. It is easy to verify that the Markov condition (ii) for the pair $(\hat{X}_n, \hat{X}_n)$ holds in the functional form (2.2) with $Y_n|_\infty$ considered as an element of $C([s, \infty), \mathbb{R}^d \times \mathbb{R}^d)$ (see Remark 1). Hence, the pair $(\tilde{Z}_n, \tilde{Z}_n)$ also satisfies condition (ii) of Definition 1. Let us write

\begin{equation}
\Delta(\gamma) = \lim_{n \to +\infty} \sup \left[ \rho \left( Z_n \left( \frac{iK(\gamma, T)}{n} \right), \hat{Z}_n \left( \frac{iK(\gamma, T)}{n} \right) \right) > \gamma \right],
\end{equation}

and show that

\begin{equation}
\Delta(\gamma) \to 0+, \quad \gamma \to 0+.
\end{equation}

Note that (16) immediately implies Markov approximation: for arbitrary $\delta > 0$ we chose, using (16), $\gamma = \gamma(\delta)$ such that inequalities $\gamma < \delta$ and $\Delta(\gamma) < \delta$ hold. Then the pair $(\tilde{Z}_n, \tilde{Z}_n)$, constructed by the scheme described above, satisfy Definition 1 with the constant $\gamma$ replaced by $\delta$ (note that, under this construction, the value $K(\delta, T) = K_Z(\delta, T)$ for the pair $(\tilde{Z}_n, \tilde{Z}_n)$ is expressed through the same value for the pair $(\hat{X}_n, \hat{X}_n)$ by $K_Z(\delta, T) = K_X(\gamma(\delta), T)$).

Now assume that (16) does not hold, then there exist constant $c > 0$ and sequence $\gamma_k \to 0+, n_k \to +\infty$ such that

\begin{equation}
\frac{K(\gamma_k, T)}{n_k} \to 0, \quad P \left[ \rho \left( Z_n \left( \frac{iK(\gamma_k, T)}{n_k} \right), \hat{Z}_n \left( \frac{iK(\gamma_k, T)}{n_k} \right) \right) > \gamma_k \right] > c.
\end{equation}

Consider the sequence of four-component processes $(\hat{X}_{n_k}, \hat{X}_{n_k}, \tilde{Z}_{n_k}, \tilde{Z}_{n_k})$. Every component of this sequence is weakly compact in $C([0, \infty), \mathbb{R}^d)$ or $C([s, \infty), \mathbb{R}^m)$, hence the whole sequence is also weakly compact in $C([0, \infty), \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m)$. Consider arbitrary limit point $(\hat{X}_s, \hat{X}_s, \tilde{Z}_s, \tilde{Z}_s)$ (in a sense of convergence by distribution) of this sequence. It follows from (18) that

\begin{equation}
P(\tilde{Z}_s \neq \hat{X}_s) > 0.
\end{equation}

It follows from Theorem 2.2 [15] (see also Chapter 9.5 [14]) that the processes $Z_s, Z^*$ satisfy SDE (2.6) with $X$ replaced by $X_s, X^*$. However, the SDE (2.6) possesses the property of pathwise uniqueness (see [16]), and the property (iii) of the pair $(X_{n_k}, \hat{X}_{n_k})$ implies that the processes $X_s, X^*$ coincide a.s. Therefore, the processes $Z_s, Z^*$ also coincide a.s., that contradicts to (19) and show that our assumption that $\Delta(\gamma) \neq 0, \gamma \to 0+$ is false.
The examples given above show that the claim for the Markov approximation to hold is not very restrictive, and is provided in a typical situation. On the other hand, this claim is strong enough to provide one the opportunity to obtain an analog of the Dynkin’s theorem; this will be shown in the next chapter.

3. MAIN THEOREM

We consider the functionals of the type

\[
\phi_n^{s,t}(Y) \equiv \sum_{k:s \leq k/n < t} F_n \left( Y \left( \frac{k}{n} \right), Y \left( \frac{k+1}{n} \right), \ldots, Y \left( \frac{k+L-1}{n} \right) \right), \quad 0 \leq s < t,
\]

where the functions \( F_n(\cdot) \) are nonnegative, \( L \) is a fixed integer. Together with the functionals \( \phi_n \), that are "stepwise" functions w.r.t. every time variable, we consider random broken lines, related to these functions:

\[
\psi^{s,t}_n = \varphi_n^{s,t} \cdot 1 \bigg\{ n(s-k) = j \bigg\} + (nt - k)\varphi_n^{s,t} \cdot 1 \bigg\{ n(s-k) = k \bigg\}, \quad s \in \left[ \frac{j-1}{n}, \frac{j}{n} \right), \quad t \in \left[ \frac{k-1}{n}, \frac{k}{n} \right).
\]

We interpret the random broken lines \( \psi_n \) as a random elements in space \( C(T, \mathbb{R}^+) \), where \( T \equiv \{(s, t)|0 \leq s \leq t\} \).

If process \( Y \) possesses Markov property w.r.t. the filtration, associated with this process, at the points of the type \( s = \frac{i}{n}, i \in \mathbb{Z}_+ \), then, for functional \( \phi_n \), its characteristic \( f_n \) is naturally defined by the formula

\[
f_n^{s,t}(x) \equiv \mathbb{E}[\phi_n^{s,t}(Y)|Y(s) = x], \quad s = \frac{i}{n}, i \in \mathbb{Z}_+, t > s, x \in \mathcal{X}.
\]

Note that the functional \( (3.1) \) is a function of values of \( Y \) at finite number of time moments, thus the mean value in \( (3.2) \) is well defined as the integral over the family \( \{P_{st,i\ell}(x, \cdot), t_1, \ldots, t_k > s, k \in \mathbb{N} \} \), which is conditional finite-dimensional distributions of the process \( Y \).

The main result of this chapter is given in the following theorem.

**Theorem 1.** Assume that there exist the sequence \( X_n \) that provides Markov approximation for the homogeneous Markov process \( X \) and the sequence \( \{\phi_n \equiv \phi_n(X_n)\} \) of the functionals of the type \( (3.1) \). Let the following conditions hold true:

1. The functions \( F_n(\cdot) \) are bounded and uniformly tend to zero:

\[
\delta(F_n) \equiv \sup \{F_n(x_1, \ldots, x_L)|x_1, \ldots, x_L \in \mathcal{X}\} \to 0, \quad n \to \infty.
\]

2. There exists a function \( f \), that appears to be a characteristics (in a sense of Chapter 6 [1]) of some \( W \)-functional \( \phi = \phi(X) \) of the limiting Markov process \( X \), such that, for each \( T \),

\[
\sup_{s=t \in (s,T)} \|f_n^{s,t}(\cdot) - f_{s,t}(\cdot)\| \to 0, \quad n \to \infty,
\]

where \( \|g(\cdot)\| \equiv \sup_{x \in \mathcal{X}} |g(x)| \).

3. The limiting function \( f \) is uniformly continuous with respect to variable \( x \), that is, for arbitrary \( T \)

\[
\sup_{0 \leq s \leq t < T} \|f^{s,t}(x') - f^{s,t}(x'')\| \to 0, \quad |x' - x''| \to 0.
\]

Then

\[
\psi_n(X_n) \Rightarrow \phi(X) \equiv \{\phi^{s,t}(X), (s, t) \in T\},
\]

where \( \psi_n \) are the random broken lines corresponding to the functionals \( \phi_n \), and convergence is understood in a sense of \( C(T, \mathbb{R}^+) \).

**Remark 4.** Conditions 1,2 are analogous to those of the Dynkin’s theorem: condition 2 is exactly the condition for the characteristics to converge, condition 1 corresponds to the assumption that the prelimin functionals are \( W \)-functionals. In the present situation, of course, we can not say that \( \phi_n \) are \( W \)-functionals, particularly, \( \phi_n \) are not continuous with respect to temporary variable. Condition 1 means exactly that the values of jumps
are negligible while \( n \to \infty \). Condition 3, though not very restrictive, is specific, and is caused by necessity to consider functionals, set over different processes.

**Remark 5.** If \( X_n \Rightarrow X \) in \( \mathbb{C} \) or in \( \mathbb{D} \) (this condition is not provided by the conditions of the Theorem, see Remark 3), then, as one can easily see from the proof, \((X_n, \psi_n(X_n)) \Rightarrow (X, \phi(X)) \) in \( \mathbb{C} \times C(T, \mathbb{R}^+) \) or in \( \mathbb{D} \times C(T, \mathbb{R}^+) \), respectively.

Note that the result of the theorem also holds for the Markov process \( X \) that is not homogeneous w.r.t. time variable; the claim for the limit Markov process to be homogeneous is imposed in order to shorten the notation only. This remark concerns also the most of the results stated below.

**Proof of the theorem.** The general scheme of the proof is close to the one, proposed in [17] in order to prove the analogue of the Dynkin’s theorem for the family of functionals of a single Markov process, for which the properties of additivity, continuity and homogeneity may fail, but the violations become negligible while \( n \to \infty \).

First let us show that the finite-dimensional distributions of \( \phi_n \) converge to the corresponding distributions of \( \phi \). Let the constants \( \gamma, T \) be fixed and \( \hat{X}_n, \hat{X} \) be processes satisfying conditions (i)-(iii) of Definition 1 with these constants. For these processes, one can consider the functionals \( \phi_n(\hat{X}_n), \phi(\hat{X}) \); obviously, their distributions and characteristics coincide with those for \( \phi_n(X_n), \phi(X) \). In order to shorten notation, we denote further \( \phi_n = \phi_n(\hat{X}_n), \phi = \phi(\hat{X}), K = K(\gamma, T), \mathcal{F}_s = \mathcal{F}^n_s = \sigma(\hat{X}_n(s), \hat{X}_n(s), s \leq t) \).

It follows from the condition (iii) and the definition of characteristics that, for arbitrary \( t \in \left( \frac{iK}{n}, T \right] \),

\[
E \left[ f_n^{\Delta^{s,t}_n} \mid \mathcal{F}_{\Delta^{s,t}_n} \right] = f_n^{\Delta^{s,t}_n} \left( \hat{X}_n \left( \frac{Ki}{n} \right) \right), \quad E \left[ f_n^{\Delta^{s,t}_n} \mid \mathcal{F}_{\Delta^{s,t}_n} \right] = f_n^{\Delta^{s,t}_n} \left( \hat{X}_n \left( \frac{Ki}{n} \right) \right)
\]

almost surely.

**Lemma 1.** For \( 0 \leq s \leq t \leq T \), the following estimate holds:

\[
\lim_{n \to \infty} E \left( \phi_n^{s,t}(\hat{X}_n) - \phi^{s,t}(\hat{X}) \right)^2 \leq 4 \left\| f^{0,T} \right\| G(f, \gamma, T) + 4 \sqrt{2} \left\| f^{0,T} \right\|^2,
\]

where \( G(f, \gamma, T) = \sup_{0 \leq s \leq t \leq T, \left| x' - x'' \right| < \gamma} \left| f^{s,t}(x') - f^{s,t}(x'') \right| \).

**Proof.** We will prove the statement of lemma for \( s = 0, t = T \); in general case the proof is exactly the same. Consider the partition of the axis \( \mathbb{R}^+ \) by points of the type \( \frac{iK}{n} \), \( i \in \mathbb{N} \). Denote \( M_n = \left\lfloor \frac{T}{\Delta^{s,t}_n} \right\rfloor + 1 \),

\[
\Delta^{s}_{i,n} \overset{\text{def}}{=} \phi_n^{(i-1)K,n \Delta^{s}_{i,n}}, \quad \Delta^{n}_i \overset{\text{def}}{=} \phi^{(i-1)K,n \Delta^{n}_i}, \quad i = 1, \ldots, M_n.
\]

We have that

\[
(\phi^{0,T}_n - \phi^{0,T})^2 = \left( \sum_{i=1}^{M_n} \Delta^{n}_i - \Delta^{n}_i \right)^2 = \left( \sum_{i=1}^{M_n} \Delta^{n}_i \right)^2 + \left( \sum_{i=1}^{M_n} \Delta^{n}_i \right)^2 - 2 \sum_{i=1}^{M_n} \sum_{j=1}^{M_n} \Delta^{n}_i \Delta^{n}_j = \Sigma_1^n + 2 \Sigma_2^n,
\]

where

\[
\Sigma_1^n \overset{\text{def}}{=} \sum_{i=1}^{M_n} (\Delta^{n}_i)^2 + \sum_{i=1}^{M_n} (\Delta^{n}_i)^2 - 2 \sum_{i=1}^{M_n} \Delta^{n}_i \Delta^{n}_i,
\]

\[
\Sigma_2^n \overset{\text{def}}{=} \left[ \sum_{1 \leq i < j \leq M_n} \Delta^{n}_i \Delta^{n}_j - \sum_{1 \leq j < k \leq M_n} \Delta^{n}_j \Delta^{n}_k \right] + \left[ \sum_{1 \leq j < k \leq M_n} \Delta^{n}_j \Delta^{n}_k - \sum_{1 \leq j < k \leq M_n} \Delta^{n}_j \Delta^{n}_k \right].
\]

Let us estimate the expectations \( \Sigma_1^n, \Sigma_2^n \) separately. Since the increments \( \Delta^{n}_i, \Delta^{n}_j \) are non-negative, the first sum can be estimated by the sum of the first two terms:

\[
\Sigma_1^n \leq \sum_{i=1}^{M_n} (\Delta^{n}_i)^2 + \sum_{i=1}^{M_n} (\Delta^{n}_i)^2.
\]
The expectation of the first term in (3.4) can be estimated via the definition of $\phi_n$:

$$E \sum_{i=1}^{M_n} (\Delta_i^n)^2 \leq E \left( \sup_{i=1,M_n} \Delta_i \right) M_n \sum_{i=1}^{M_n} \Delta_i^n \leq K \delta_n f_{0,T}^n \left( \hat{X}_n(0) \right) \leq K \delta_n \left\| f_{0,T}^n \right\| \to 0, \quad n \to +\infty,$$

where $\delta_n \equiv \delta(F_n)$. Convergence to zero of the expectation of the second term in (3.4) is provided by the arguments, analogous to those used in [1] Chapter 6: on the one hand, by the continuity of functional $\phi$, $\sum_{i=1}^{M_n} (\Delta_i^n)^2 \to 0$ by probability; on the other hand, $\sum_{i=1}^{M_n} (\Delta_i^n)^2$ is dominated by the variable $(\phi_{0,T}^n)^2$; the expectation of this variable, due to Lemma 6.4 [1], does not exceed $2 \left\| f_{0,T}^n \right\|^2 / 2$. Therefore, $E \sum_{i=1}^{M_n} (\Delta_i^n)^2 \to 0$ due to the Lebesgue theorem on dominated convergence. Hence, $\lim_{n \to -\infty} E \Sigma_n^2 \leq 0$.

The expectation of $\Sigma_n^2$ is equal

$$E \Sigma_n^2 = E \left[ \sum_{1 \leq i < l \leq M_n} \Delta_i^n \Delta_l^n - \sum_{1 \leq i \leq l \leq M_n} \Delta_i^n \Delta_l^n \right] + E \left[ \sum_{1 \leq j < k \leq M_n} \Delta_j^n \Delta_k^n - \sum_{1 \leq j < l \leq M_n} \Delta_j^n \Delta_l^n \right] =$$

$$= E \sum_{i=1}^{M_n-1} \Delta_i^n \left( \phi_{n,T}^i - \phi_{n,T}^n \right) - E \sum_{i=1}^{M_n-1} \Delta_i^n \left[ \phi_{n,T}^i - \phi_{n,T}^n \right] | \mathcal{F}_{K_i^n} \leq$$

$$\leq E \sum_{i=1}^{M_n-1} \Delta_i^n \left( f_{n,T}^i \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) - f_{n,T}^n \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) \right) \leq$$

$$\leq E \sum_{i=1}^{M_n-1} \Delta_i^n \left( f_{n,T}^i \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) - f_{n,T}^n \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) \right) \leq$$

$$\leq \left\| f_{0,T} \right\| \sup_{s=\frac{1}{n}, t \in (s,T)} \left\| f_{n,t}^s \left( \cdot \right) - f_{n,t}^s \left( \cdot \right) \right\| + E \sum_{i=1}^{M_n-1} \Delta_i^n \left| f_{n,T}^i \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) - f_{n,T}^n \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) \right| \leq$$

$$\leq \left\| f_{0,T} \right\| \left( \sup_{s=\frac{1}{n}, t \in (s,T)} \left\| f_{n,t}^s \left( \cdot \right) - f_{n,t}^s \left( \cdot \right) \right\| + E \sum_{i=1}^{M_n-1} \Delta_i^n \left| f_{n,T}^i \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) - f_{n,T}^n \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) \right| \right|_{\mathcal{F}_{K_i^n}} \leq$$

$$\leq \left\| f_{0,T} \right\| G(f, \gamma, T) \mathbf{1}_{\Omega, \gamma, T} + \sum_{i=1}^{M_n-1} \Delta_i^n \left| f_{n,T}^i \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) - f_{n,T}^n \left( \hat{X}_n \left( \frac{K_i^n}{n} \right) \right) \right| \mathbf{1}_{\Omega, \gamma, T} \leq$$

$$\leq \left\| f_{0,T} \right\| E \phi_{0,T} G(f, \gamma, T) \mathbf{1}_{\Omega, \gamma, T} \leq \left\| f_{0,T} \right\| \left[ E \left( \phi_{0,T}^n \right)^2 \right]^{\frac{1}{2}} \left[ P(\Omega, \gamma, T) \right]^{\frac{1}{2}} \leq \left\| f_{0,T} \right\|^2 \sqrt{2\gamma}.$$
assume that \( \limsup \) is impossible here, since the variable \( \Delta_n \) is not measurable with respect to \( \mathcal{F}_{\bar{X}_n} \). Without loss of generality, one can assume that \( K \geq L \) (otherwise one can make the same procedure with the constant \( K \) replaced by \( K \cdot L \)). Then the variable \( \Delta_n \) is measurable with respect to \( \mathcal{F}_{\bar{X}_n} \). The functionals \( \phi_n, \phi \) are additive at points of the type \( \frac{t}{n} \). Applying (3.3) and condition 1 of the Theorem, we obtain the following relation

\[
E \sum_{i=1}^{M_n-1} \Delta_i^n \left[ \phi_n^{K_i/T} - \phi_n^{K_{i+1}/T} \right] = E \sum_{i=1}^{M_n-1} \Delta_i^n \left[ \phi_n^{K_i+1/n} - \phi_n^{K_{i+1}+1/n} \right] + \\
+ E \sum_{i=1}^{M_n-1} \Delta_i^n \left[ f_n^{K(i+1)/n} \left( \bar{X}_n \left( \frac{K(i+1)}{n} \right) \right) - f_n^{K(i+1)/n} \left( \bar{X}_n \left( \frac{K(i+1)}{n} \right) \right) \right] 
\]

Now, let us proceed with the estimation of the first term in (3.9). Straightforward use of the property (3.3) is impossible here, since the variable \( \Delta_i^n \) is a functional of values of the process \( \bar{X}_n \) at the points \( \frac{K_i}{n}, \frac{K_{i+1}}{n}, \ldots, \frac{K_{i+L}}{n} \), that is, it is not measurable with respect to \( \mathcal{F}_{\bar{X}_n} \). Without loss of generality, one can assume that \( K \geq L \) (otherwise one can make the same procedure with the constant \( K \) replaced by \( K \cdot L \)). Then the variable \( \Delta_i^n \) is measurable with respect to \( \mathcal{F}_{\bar{X}_n} \). The functionals \( \phi_n, \phi \) are additive at points of the type \( \frac{t}{n} \). Applying (3.3) and condition 1 of the Theorem, we obtain the following relation

Now, we can complete the proof of the convergence of finite-dimensional distributions of \( \phi_n \) to those of \( \phi \). In order to shorten notation we consider the one-dimensional distributions only; in general case considerations are completely the same.

Take arbitrary \( s, t, s < t \). In order to prove weak convergence \( \phi_n^{s,t}(X_n) \) to \( \phi^{s,t}(X) \), it is sufficient to show that, for arbitrary bounded Lipschitz function \( g \),

\[
\limsup_{n \to \infty} |E g(\phi_n^{s,t}(X_n)) - E g(\phi^{s,t}(X))| = 0.
\]
Let \( g \) be fixed, consider a pair of processes \( \hat{X}_n, \hat{X}^n \), corresponding (in a sense of Definition 1) to \( T = t \) and given positive \( \gamma \). By construction, \( \phi^{s,t}_n(X_n) \triangleq \phi^{s,t}_n(\hat{X}_n), \phi^{s,t}(X) \triangleq \phi^{s,t}(\hat{X}^n) \). Applying Lemma 1 we obtain that
\[
\limsup_{n \to \infty} \left| E g(\phi^{s,t}_n(X_n)) - E g(\phi^{s,t}(X)) \right| \leq \limsup_{n \to \infty} E \left| g(\phi^{s,t}_n(\hat{X}_n)) - \phi^{s,t}(\hat{X}^n) \right| \leq \text{Lip}(g) \limsup_{n \to \infty} E \left| \phi^{s,t}_n(\hat{X}_n) - \phi^{s,t}(\hat{X}^n) \right| \leq 2\text{Lip}(g) \sqrt{\|f_0^n\| \|G(f, \gamma, t) + \sqrt{2\gamma} \|f_0^n\|^2},
\]
here \( \text{Lip}(g) \) denotes the Lipshits constant for \( g \). Condition 3 of the Theorem provides that \( G(f, \gamma, t) \to 0, \gamma \to 0^+ \). Therefore, since \( \gamma > 0 \) is arbitrary, (3.12) follows from the estimate given above.

Since \( \sup_{s,t} |\phi^{s,t}_n - \phi^{s,t}| \leq \delta_n \to 0 \), the finite-dimensional distributions of \( \phi_n \) converge to corresponding distributions of \( \phi \). Thus, the only thing left to show in order to prove the Theorem, is that the family of distributions of \( \psi_n \) is dense in \( C(T, \mathbb{R}^+) \). The values of the functions \( \psi_n \) at the points \( s, t \) differ from the values at the closest knots of partition \( s_*, t_* \in \frac{1}{n} \mathbb{Z}_+ \) at most on \( \delta_n \), and \( \psi_n \) are monotone as the functions of the time variables. Hence, in order to prove the required statement, it is sufficient to show that, for arbitrary sequence of partitions \( \{S_n = \{s^n_0 = 0 < s^n_1 < \cdots < s^n_k < \cdots\} \subset \frac{1}{n} \mathbb{Z}_+, n \in \mathbb{N}\} \) with \( \sigma_n \equiv \max_k (s^n_k - s^n_{k-1}) \to 0, n \to +\infty \) and arbitrary \( T \in \mathbb{R}^+ \),
\[
E \sum_{k:s_k \leq T} \left[ \psi^{s^n_k - 1, s^n_k}_n \right]^2 \to 0, \quad n \to +\infty.
\]
Set \( \gamma_{n,T} = \sup_{0 < t-s < \sigma_n, t \in T} \|f_0^n\| \), note, that \( \gamma_{n,T} \to 0, n \to +\infty \) due to continuity of the limit characteristics \( f \) and uniform convergence of \( f_n \Rightarrow f \). In the same way with (3.10) we obtain the estimate
\[
E \left[ \phi^{s^n_k - 1, s^n_k}_n \right]^2 \leq \{(2K + 1)\delta_n + 2\gamma_{n,T}\} E \phi^{s^n_k - 1, s^n_k}_n.
\]
Summing up the estimates (3.13) w.r.t. \( k \) (recall that \( \phi^{s,t}_n = \psi^{s,t}_n \) when \( s, t \in \frac{1}{n} \mathbb{Z}_+ \)), we obtain
\[
E \sum_{k:s_k \leq T} \left[ \psi^{s^n_k - 1, s^n_k}_n \right]^2 \leq \{(2K + 1)\delta_n + 2\gamma_{n,T}\} \|f_0^n\|^2 \to 0, \quad n \to +\infty,
\]
what was to be proved. The theorem is proved.

Let us make one remark. For the random walks, the Skorohod’s method is well-known, allowing one to reduce the investigation of the sums of the type (3.1) to the case \( L = 1 \). This method can be applied in the context of current paper, also. Namely, the reasoning, similar to the one used in the proof of Theorem 1, Chapter 5.3 [2], provides the following result (the proof is omitted).

**Proposition 1.** Let the sequence of functionals \( \{\phi_n = \phi_n(X_n)\} \) of the type (3.1) be given, and, for every \( n \), the process \( X_n \) possesses the Markov property at the time moments \( \frac{k}{n}, i \in \mathbb{Z}_+ \). Consider the functionals
\[
\chi^{s,t}_n(X_n) \triangleq \sum_{k:s_k \leq k/n < t} \Psi_{n,k} \left( X_n \left( \frac{k}{n} \right) \right), \quad 0 \leq s < t,
\]
where
\[
\Psi_{n,k}(x) \equiv E \left[ F_n \left( x, X_n \left( \frac{k+1}{n} \right), \ldots, X_n \left( \frac{k+L-1}{n} \right) \right) \right] | X_n \left( \frac{k}{n} \right) = x, \quad x \in \mathcal{X}.
\]
Let functions \( F_n(\cdot) \) be non-negative and satisfy condition 1 of Theorem [4] then the functionals \( \phi^{s,t}_n \) have a limit distribution if and only if the functionals \( \chi^{s,t}_n \) have a limit distribution, and the limit distributions of the functionals \( \phi^{s,t}_n, \chi^{s,t}_n \) are equal as soon as they exist.

It is worth to note that the Proposition [4] does not lead to simplification of the initial problem in the context of current paper. The number of values of process \( X_n \), contained in a one summand for the functional \( \phi_n \) (that is, number \( L \)), is not involved significantly into the proof of the main theorem. We will see later that the main problem in the application of the Theorem consists in verification of the condition 2 of uniform convergence of characteristics; the characteristics of the functionals \( \phi_n \) and \( \chi_n \), obviously, coincide.

In the following two chapters, the examples of application of Theorem [4] are given.
4. The local time of a random walk at a point.

Let the processes \( X_n \) be constructed w.r.t. one-dimensional random walk that belongs to the normal domain of attraction of an \( \alpha \)-stable law, \( \alpha \in (1, 2] \) (see Examples 1, 2). We assume the centering sequence \( a_n \) to be equal to zero, and set the random broken lines \( X_n \) by equality (2.3).

Consider, for arbitrary \( z_* \in \mathbb{R} \), the functionals \( \phi_n = \phi_n(X_n) \) of the type (5.1) with \( L = 2, F_n(x, y) = \frac{1}{n |y-x|^2} \left[ I_{(x-z_*)}(y-z_*)<0 + \frac{1}{2}(I_{x \neq z_*, y=z_*} + I_{x=z_*, y \neq z_*}) \right] \). For every \( s < t, s, t \in \{ \frac{i}{n}, j \in \mathbb{Z}_+ \} \), with probability 1 the following equality takes place

\[ (4.1) \phi^{n, t}_\alpha(X_n) = \lim_{\epsilon \to 0^+} \frac{1}{2 \epsilon} \int_s^t 1_{X_n(r) \in (z_*-\epsilon, z_*+\epsilon) \setminus \{ z_* \}} \, dr, \quad 0 \leq s < t. \]

Therefore the functionals \( \phi_n \) can be naturally interpreted as the censored local times for the broken lines \( X_n \) at the point \( z_* \) (the censoring operation consists in removing horizontal parts of the broken lines). Theorem 3.1 allows one to obtain the following limit result.

**Proposition 2.** Let the distribution of the jump \( \xi_1 \) of the random walk be concentrated on \( \mathbb{Z} \) and aperiodic. Then the conditions of Theorem 7 hold true and \( \phi^{n, t}_\alpha(X_n) \) converge by distribution to \( \phi^{s,t}(X) = P(\xi_1 \neq 0) \cdot L^{s,t}(X, z_*) \), where \( L(X, z_*) \) is the local time of the limit \( \alpha \)-stable process \( X \) at the point \( z_* \).

**Proof.** The condition for \( X_n \) to provide Markov approximation for \( X \) holds true (see Example 2). Condition 1 of the Theorem holds with \( \delta_n = 2n^{\frac{1}{\alpha}-1} \) since either the increment of the process \( X_n \) in the neighboring knots is equal to zero or the absolute value of this increment is not less than \( n^{-\frac{1}{\alpha}} \). Let us show that the characteristics of functionals \( \phi_n \) converge uniformly to the function

\[ (4.2) f^{s,t}(x) = P(\xi_1 \neq 0) \int_0^{t-s} p_r(z_* - x) \, dr, \]

where \( p_r(\cdot) \) is the density of distribution \( X(r) \) under condition \( X(0) = 0 \); this provides conditions 2, 3 of the Theorem.

In order to shorten notation we take \( z_* = 0 \). Denote \( P^k_i = P(S_k = i), P^1_j = P(\xi_1 = j), i, j \in \mathbb{Z} \). We have that

\[ (4.3) \varepsilon_k \equiv \sup_{i \in \mathbb{Z}} \left| k^{\frac{1}{\alpha}} p^k_i - p_1 \left( \frac{i}{k^{\frac{1}{\alpha}}} \right) \right| \to 0, \quad k \to +\infty. \]

Hence

\[ (4.4) f^{s,t}_n(x) = \frac{1}{n} \sum_{s \leq \frac{i}{n} < t} \left[ \sum_{j \neq 0} P_i^j \left( \sum_{i \in (x_n^{\frac{1}{\alpha}} - j, x_n^{\frac{1}{\alpha}})} \left( \sum_{n \in \mathbb{Z}} p_1 \left( \frac{n}{k^{\frac{1}{\alpha}}} \right) \right) \left( \sum_{j \in \mathbb{Z}} p_1 \left( \frac{j}{k^{\frac{1}{\alpha}}} \right) \right) \right) \right] + \Xi_n(x), \]

where

\[ (4.5) |\Xi_n(x)| \leq \frac{1}{n} \sum_{k=1}^{[nt]} \left( \frac{n}{k} \right)^{\frac{1}{\alpha}} \varepsilon_k, \]

and \( \Xi_n \Rightarrow 0, n \to +\infty \) via the Toeplitz’s theorem.
The density $p_1$ is uniformly continuous over $\mathbb{R}$, hence, using the same arguments, one can show that, up to a submanifold that uniformly converges to zero, the value of $f_{n,t}^s(x)$ equals

\[
\frac{1}{n} \sum_{s \leq \frac{t}{n} < t} \left[ \frac{1}{n} \sum_{j \neq 0} |j| \left( \sum_{i \in \{nx_1, nx_2, \ldots, nx_r\}} \left( \frac{n}{k} \right)^{\frac{1}{k}} p_1 \left( \frac{nx_1^{1/k}}{k^{1/k}} \right) + \left( \frac{n}{k} \right)^{\frac{1}{k}} \right) \right] = \frac{P(\xi_1 \neq 0)}{n} \sum_{s \leq \frac{t}{n} < t} \left( \frac{n}{k} \right)^{\frac{1}{k}} p_1 \left( \frac{nx_1^{1/k}}{k^{1/k}} \right)
\]

in the latter equality, we have used that the process $X$ is self-similar, that is, $p_r(x) = r^{-\frac{d}{2}}p_1(r^{-\frac{d}{2}}x)$, $r > 0$. The sum in the right hand part of (4.6) is exactly the integral sum for the integral in the right hand part of (4.2), the functions $\{p_r(\cdot), r \geq r_0\}$ are uniformly continuous for arbitrary $r_0 > 0$ and $\text{sup}_x p_r(x) \leq Cr^{-\frac{d}{2}}$. This immediately provides the required uniform convergence of $f_n$ to $f$. The proposition is proved.

The similar result can be proved for $\xi_k$ with non-lattice distribution, for which there exists a bounded distribution density of $S_{n_0}$ for some $n_0$ (the proof is omitted).

The result of Proposition 2 and its analog for non-lattice random walks is not essentially new; one can obtain it applying either Proposition 1 and the technique, exposed in §III.2, III.3 [3], or the reasonings, similar to those used in the proof of Theorem 3 [9]. Our reason to give this example consists, on the one hand, in describing the way of application of Theorem 1 in a simple situation where an appropriate local limit theorem is available, and on the other hand, in emphasizing the following interesting fact, that is not reflected in a literature available for us. For a "good" random walks (lattice or essentially non-lattice), their local times at the point, defined by the natural equality (4.1), converge by distribution exactly to the local time of the limit process at the same point, as soon as the broken lines corresponding to the random walk does not contain horizontal sections.

5. Difference approximations of diffusion processes.

Consider the sequence $\{Z_n\}$ of difference approximations of diffusion process $Z$ (see Example 3 equalities (2.5), (2.9)). The sequence $\{Z_n\}$ provides Markov approximation for $Z$, that allows one to apply Theorem 1 while considering the question on the limit behavior of the functionals of type (4.1) for $\{Z_n\}$.

One of possible way to proceed here is to apply the estimates based on an appropriate local limit theorem, like it was made in the previous chapter. In order to make this paper reasonably short, we do not give the detailed exposition of this subject here (see the separate paper [19]). In this chapter, we give a simple corollary of Theorem 1 that provides invariance principle for certain "canonic" additive functionals, that are related to the Doob’s decomposition of $|Z_n(\cdot)|$.

Let us consider the objects introduced in Example 3 with $m = d = 1$ and $a, b, \{\xi_n\}$ satisfying conditions introduced there. Put

\[
\phi_{n,t}^s (Z_n) \equiv \sum_{k \in \{nx_1, nx_2, \ldots, nx_r\}} |Z_n \left( \frac{k}{n} \right)| \cdot \left[ 2 \mathbf{1}_{Z_n \left( \frac{k-1}{n} \right) \cdot Z_n \left( \frac{k}{n} \right) < 0} + \mathbf{1}_{Z_n \left( \frac{k-1}{n} \right) = 0} \right],
\]

$\psi_n$ are corresponding broken lines.

**Proposition 3.** The processes $\psi_n$ converge by distribution in $C(\mathbb{T}, \mathbb{R})$ to the local time

\[
\phi_{n,t}^s \equiv \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{|Z(r)| < \varepsilon} b^2(Z(r)) \, dr
\]

of the diffusion process $Z$ at the point 0.

**Proof.** Since the diffusion coefficient is non-degenerate, $Z$ possesses continuous transition density $p_t(x, y)$ and the standard estimate $\text{sup}_x p_t(x, y) \leq \frac{C(y)}{\sqrt{t}}$ holds true. This implies existence of the local time of $Z$ at
the point 0. This local time is a \( W \)-functional with the characteristics \( f^{0,t}(x) = b^2(0) \int_0^t p_s(x, 0) \, ds \), that is, condition 3 of Theorem 1 holds. Straightforward calculations prove the equality
\[
(5.2) \quad |Z_n(t)| - |Z_n(s)| = \phi_n^{0,t}(Z_n) + \sum_{k=n}^{[nt]-1} \left[ a \left( \frac{Z_n}{k/n} \right) \frac{1}{n} + b \left( \frac{Z_n}{k/n} \right) \Delta X_n \left( \frac{k}{n} \right) \right] \text{sign} \left( \frac{Z_n}{k/n} \right),
\]
where \( s \in \frac{1}{n} \mathbb{Z}_+ \), \( \text{sign} (0) = 0 \). This provides that
\[
f_n^{s,t}(x) = E \left[ |Z_n(t)||Z_n(s) = x| - |x| - \frac{1}{n} E \left[ \sum_{k=0}^{[nt]-1} a \left( \frac{Z_n}{k/n} \right) \text{sign} \left( \frac{Z_n}{k/n} \right) |Z_n(s) = x| \right] \right].
\]
Processes \( Z_n \) converge weakly to \( Z \), function \( a(x) \text{sign} (x) \) has unique jump at point \( x = 0 \) and \( P(Z(r) = 0) = 0 \) for every \( r > 0 \). Hence the standard reasonings provide that (we omit the details)
\[
(5.3) \quad f_n^{s,t}(x) = E \left[ |Z(t)||Z(s) = x| - |x| - E \left[ \int_s^t a(Z_r) \text{sign} (Z_r) \, dr \right] |Z(s) = x| \right].
\]
This proves condition 2 of Theorem 1, since the right hand side of (5.3) is exactly the characteristics of the local time \( \phi \) due to Ito-Tanaka formula.

In order to provide condition 1, let us, for a while, suppose additionally that the coefficients \( a, b \) are bounded. We apply the standard "cutting" procedure: on each step of approximation, together with the process \( Z_n \), we consider the process \( \tilde{Z}_n \), constructed by the same scheme from a sequence of i.i.d.r.v. \( \{\xi_k\} \), satisfying conditions \( ||\xi_n|| \leq n^{\frac{1}{2^k+2}} \) and \( \xi_1 = \xi_2 = \ldots = \xi_n = 1/2 \) for \( ||\xi_n|| \leq n^{\frac{1}{2^k+2}} \). For such \( \tilde{Z}_n \), condition 1 of theorem holds with
\[
\delta(F_n) \leq n^{-1} \max_x |a(x)| + n^{-\frac{1}{2^k+2}} \max_x |b(x)|,
\]
and the other conditions of theorem for \( \tilde{Z}_n \) remain to hold true. This proves the statement of Proposition 3 for \( \{\tilde{Z}_n\} \). On the other hand, for arbitrary \( T \in \mathbb{R}^+ \)
\[
P \left( Z_n|_{[0,T]} \neq \tilde{Z}_n|_{[0,T]} \right) = O \left( n^{1-\frac{2k+1}{2^k+2}} \right) = o(1), \quad n \to +\infty,
\]
and therefore the statement of Proposition 3 holds true for \( \{Z_n\} \). At last, the additional assumption that the coefficients \( a, b \) are bounded, can be removed via a standard localization procedure. The proposition is proved.

**Remark 6.** Let \( a = 0, b = 1, P(\xi_k = \pm 1) = \frac{1}{2} \) (that is, \( Z_n \) corresponds to the Bernoulli’s random walk), then functional (5.1) can be represented at the form
\[
\tilde{\phi}_n^{s,t} = \frac{1}{\sqrt{n}} \# \{ k \in [sn, tn]: Z_n(k) = 0 \}.
\]
The functional (5.4) is widely used in a literature as the difference analogue of the local time at the point zero for lattice random walks. Proposition 3 shows that the functional (5.1) is a natural difference analogue of the local time both for random walks and, more generally, for difference approximations of diffusion processes without any restrictions on the distribution of the sequence \( \{\xi_k\} \).

6. **Invariance principle for additive functionals of Markov chains**

In previous two chapters we have considered more or less particular examples illustrating possible ways to provide the main condition of Theorem 1 (condition 2). In this chapter we introduce general sufficient condition of weak convergence of additive functionals, constructed on the sequence of Markov chains, that is formulated in terms of the transition probabilities of these chains and the functions \( F_n \) involved in representation (5.1). This condition is obtained as an application of Theorem 1 and the main assumption here is that the local limit theorem (condition 4 of Theorem 2 below) takes place in an appropriate form. For recurrent Markov chains this condition, together with a natural condition of weak convergence of "symbols" of additive functionals (exact formulation is given below), is sufficient for convergence of characteristics, and the estimates here are
similar to (4.4) – (4.6) (see Theorem 3 below). For transient chains these estimates are not powerful enough, since in this case the estimate (4.5) does not provide that \( \Xi_n \) is negligible. One possible way to overcome this difficulty is to apply a more strong version of local limit theorem, for instance, to claim explicitly the rate of convergence \( \varepsilon_k \to 0 \) in (4.6).

We assume that a σ finite measures \( \nu, \nu_n \) on \( \mathcal{X} \) are given such that

\[
P(X(t) \in dy|X(s) = x) = p_{n-s}(x,y)\nu(dy), \quad 0 \leq s < t, x, y \in \mathcal{X},
\]

\[
P \left( X_n \left( \frac{i+k}{n} \right) \in dy | X_n \left( \frac{i}{n} \right) = x \right) = p_{n,k}(x,y)\nu_n(dy), \quad i \in \mathbb{Z}_+, k \in \mathbb{N}, x, y \in \mathcal{X}.
\]

The measurable functions \( p_t, p_{n,k} \) are interpreted as the transition probability densities for \( X, X_n \) w.r.t. measures \( \nu, \nu_n \).

We assume the W-functional \( \phi = \phi(X) \) with the characteristics \( f \) to be given. It is known (see [1], Chapter 6) that

\[
\phi^{s,t} = L_2 - \lim_{\varepsilon \to 0+} \int_s^t \frac{1}{\varepsilon} f_{0,\varepsilon}(X(r)) \, dr,
\]

and therefore

\[
f^{s,t}(x) = \lim_{\varepsilon \to 0+} \int_s^t \int_{\mathcal{X}} p_r(x,y) \frac{1}{\varepsilon} f_{0,\varepsilon}(y) \nu(dy) \, dr.
\]

We assume that, as \( \varepsilon \to 0+ \), the measures \( \frac{1}{\varepsilon} f_{0,\varepsilon} \, d\nu \) converge weakly (i.e., on every bounded continuous function) to a finite measure \( \mu \), the characteristics \( f \) can be represented in the form

\[
f^{s,t}(x) = \int_0^{t-s} \int_{\mathcal{X}} p_r(x,y) \mu(dy) \, dr, \quad \text{and} \quad \int_0^T \left[ \sup_{x \in \mathcal{X}} \int_{\mathcal{X}} p_r(x,y) \mu(dy) \right] \, dr < +\infty, \quad T \in \mathbb{R}^+.
\]

We also consider the sequence of the functionals \( \phi_n = \phi_n(X_n) \) of the type (6.1) with \( L = 1 \) and \( F_n = \frac{1}{n} g_n \) (the case \( L > 1 \) can be considered similarly and we omit it in order to shorten notation). The characteristics of \( \phi_n \) have the form

\[
f^{s,t}_n(x) = \frac{1}{n} \sum_{s \leq \frac{k}{n} < t} \int_{\mathcal{X}} p_{n,k}(x,y) \mu_n(dy), \quad 0 \leq s < t, x \in \mathcal{X},
\]

where \( \mu_n(dy) \equiv g_n(y)\nu_n(dy) \) are the "symbols" of the functionals \( \phi_n \).

**Theorem 2.** Assume the following conditions to hold true.

1. Trajectories of the processes \( X_n \) are continuous, and the sequence \( \{X_n\} \) possesses Markov approximation of \( X \).
2. \( \frac{1}{n} \sup_x g_n(x) \to 0, n \to +\infty \).
3. For arbitrary \( t_0 > 0 \), the function \( (t, x, y) \mapsto p_t(x,y) \) is uniformly continuous on \( [t_0, +\infty) \times \mathcal{X}^2 \), and for arbitrary \( y \in \mathcal{X} \)

\[
\sup_{x \in B(y,R)} p_t(x,y) \to 0, \quad R \to +\infty
\]

(here and below \( B(x, R) \equiv \{ x \in \mathcal{X} | \rho(x,y) < R \} \)). Furthermore, there exist constants \( \gamma > 0, C_\gamma > 0 \) such that

\[
\sup_{x,y \in \mathcal{X}} p_t(x,y) \leq C_\gamma t^{-\gamma}, \quad t > 0.
\]

4. There exist sequences \( \{\alpha_n\}, \{\beta_n\} \subset \mathbb{R}^+ \) tending to zero, such that

\[
\sup_{x,y \in \mathcal{X}} |p_{n,k}(x,y) - p_{n,k}(x,y)| \leq (\alpha_n + \beta_k) \left( \frac{n}{k} \right)^\gamma, \quad n, k \in \mathbb{N}.
\]
There exist constants $\delta > 0, C_\delta > 0$ such that, for arbitrary $T > 0$,

$$
\sup_{x \in \mathcal{X}, n \in \mathbb{N}} E \left( \sup_{t,s \in [0,T],|t-s| \geq \frac{1}{n}} \frac{\rho(X_n(t), X_n(s))}{|t-s|^\delta} \right) C_\delta |X(0) = x| < +\infty.
$$

Measures $\mu_n$ are finite and converge weakly to measure $\mu$. There exist constants $\theta > 0, C_\theta, c_\theta > 0$ such that

$$
\mu_n(B(x, R)) \leq c_\theta R^\theta, \quad x \in \mathcal{X}, n \in \mathbb{N}, R > c_\theta n^{-\delta}
$$

(note that the latter condition provides that $\mu(B(x, R)) \leq c_\theta R^\theta, x \in \mathcal{X}, R > 0$).

The constants $\gamma, \delta, \theta, C_\delta$ satisfy the relations

$$
\delta \theta + 1 > \gamma, \quad C_\delta > 2 \theta + 2.
$$

Then $(X_n, \psi_n(X_n)) \Rightarrow (X, \phi(X))$ in a sense of convergence in distribution in $C(\mathbb{R}^+, \mathcal{X}) \times C(\mathbb{T}, \mathbb{R}^+)$ ($\psi_n$ are the random broken lines corresponding to the functionals $\phi_n$).

**Proof.** In order to prove the Theorem, it is sufficient to show that, for every $T \in \mathbb{R}^+$,

$$
(6.2) \quad f^{s,t}_n(x) \to f^{s,t}(x), \quad n \to +\infty.
$$

Indeed, the sequence $\{X_n\}$ provides Markov approximation for $X$ (condition 1), and condition 1 of Theorem is provides by condition 2 of Theorem. Having (6.2) proved, we provide condition 2 of Theorem. Condition 3 of this theorem is provided by (6.1) and uniform continuity of the density $p$. At last, condition 5 of Theorem provides weak convergence of $X_n$ to $X$ in $C(\mathbb{R}^+, \mathcal{X})$, that allows one to apply Theorem and Remark.

Before proving (6.2), let us make some auxiliary estimates. Denote

$$
H_{\delta,n}^{s,t}(X_n) = \sup_{v,w \in [s,t],|v-w| \geq \frac{1}{n}} \frac{\rho(X_n(v), X_n(w))}{|v-w|^\delta},
$$

$$
D_{\delta,n}^{s,t} = \left\{ X_n(r) \in B(X_n(s), A(r-s)^\theta) : r \in \left[ s + \frac{1}{n}, t \right] \right\},
$$

note that $\{H_{\delta,n}^{s,t}(X_n) < A\} \subset D_{\delta,n}^{s,t}$. Also denote $\alpha = \max_n \alpha_n, \beta = \max_k \beta_k, \delta_n = \frac{\sup_n |g_n(x)|}{n}, B_1 = \max_n \delta_n, B_2(T) = \frac{C_\delta(C_n + \alpha + \beta)}{1 + \delta_n + \gamma} T^{1+\theta-\gamma}$. For arbitrary $A > c_\theta, T \in \mathbb{R}^+$, consider the functionals $\phi_{n,A}^{s,t} = \phi_{n,A}^{s,t}1_{H_{\delta,n}^{s,t}(X_n) < A}, s \leq t \leq T$.

**Lemma 2.** 1. $E \left[ \phi_{n,A}^{s,t} | X_n(s) = x \right] \leq B_1 + B_2(T)A^\theta$.

2. $E \left( \phi_{n,A}^{s,t} \right)^2 | X_n(s) = x \leq 3B_1(B_1 + B_2(T)A^\theta) + 2(B_1 + B_2(T)A^\theta)^2$.

3. Let $p \in \left( 1, \frac{2C_\delta - 2}{C_\delta + 2\theta} \right)$ (recall that $1 < \frac{2C_\delta - 2}{C_\delta + 2\theta}$ due to condition 7 of the Theorem). Then

$$
\sup_{x \in \mathcal{X}, n \in \mathbb{N}, s \leq t \leq T} E \left[ (\phi_{n,A}^{s,t})^p | X_n(s) = x \right] < +\infty.
$$

**Proof.** Using condition 4 of the Theorem and then condition 6, we obtain, for $t, s \in \frac{1}{n} \mathbb{Z}_+$, the estimate

$$
E \left[ \phi_{n,A}^{s,t} | X_n(s) = x \right] \leq E \left[ \phi_{n,A}^{s,t} \mathbf{1}_{D_{\delta,n}^{s,t}} | X_n(s) = x \right] + \frac{1}{n} \sum_{k=1}^{n(t-s)-1} \int B(x,A(k)) \rho_n,k(x,y) \mu_n(dy) \leq \delta_n + \frac{C_\gamma + \alpha + \beta}{n} \sum_{k=1}^{n(t-s)-1} \left( \frac{n}{k} \right)^\gamma \mu_n \left( B \left( x, A \left( \frac{k}{n} \right) \delta \right) \right) \leq \delta_n + \frac{C_\gamma + \alpha + \beta}{n} \sum_{k=1}^{n(t-s)-1} \left( \frac{n}{k} \right)^{\gamma-\delta}\theta,$$
that immediately proves the first statement of the Lemma. The second statement can be obtained from the first one via the estimate similar to (3.10) with the use of the inequality

$$I_{H_{k,n}^s(X_n) < A} \leq I_{H_{k,n}^s(X_n) < A} I_{H_{k,n}^r(X_n) < A},$$

that holds true for arbitrary $r \in (s, t)$.

Applying statement 2 and Hölder inequality we obtain

$$E \left[ (\phi^{s,t}_n)^p | X_n(s) = x \right] = \sum_{N=1}^\infty E \left[ (\phi^{s,t}_n)^p I_{H_{k,n}^s(X_n) \in [N-1,N]} | X_n(s) = x \right] \leq$$

$$\leq \sum_{N=1}^\infty E \left[ (\phi^{s,t}_n)^2 I_{H_{k,n}^s(X_n) < N} | X_n(s) = x \right] \frac{p}{2} \left[ P(H_{k,n}^s(X_n) \geq N-1) \right]^{\frac{2-p}{p}} \leq$$

$$\leq \sum_{N=1}^\infty \left[ B_3(T) + B_4(T) N^{2\theta} \right] \frac{p}{2} B_5(T) \left[ (N-1) \vee 1 \right]^{-\frac{2-p}{p}} C_5,$$

here and below $B_i(T), i = 3, 4, \ldots$ denotes a constant, that can be expressed explicitly through $T$ and the constants introduced in the formulation of the Theorem, but an explicit expression is not needed in our consideration. Since $\theta p - \frac{2-p}{p} C_5 < -1$ by the choice of $p$, this proves the statement 3. The lemma is proved.

Let us proceed with the proof of (6.2). Choose non-increasing Lipschitz function $\Psi : \mathbb{R}^+ \to [0, 1]$ such that $\Psi([0, 1]) = \{1\}, \Psi([2, +\infty)) = \{0\}$, and set

$$\Psi_r(x,y) = \Psi(r^{-1} \cdot \rho(x,y)), \quad r > 0, x, y \in \mathcal{X}, \quad \Psi_0 \equiv 1.$$

Note that, for arbitrary $r_0 > 0$, the function $(r, x, y) \mapsto \Psi_r(x,y)$ is uniformly continuous on $[r_0, +\infty) \times \mathcal{X}^2$.

For fixed $s \leq t \leq T, A \in \mathbb{R}^+$ we decompose $\phi^{s,t}_n$ as $\phi^{s,t}_n = \eta^{s,t}_{n,A} + \zeta^{s,t}_{n,A}$, where

$$\eta^{s,t}_{n,A} = \frac{1}{n} \sum_{s \leq k < t} g_n \left( X_n \left( \frac{k}{n} \right) \right) \Psi_A \left( \frac{k}{n} - s \right) \left( X_n(s), X_n \left( \frac{k}{n} \right) \right).$$

We have that, on the set $D^{s,t}_{n,A}$, for $k$ such that $s \leq \frac{k}{n} < t$,

$$\rho \left( X_n(0), X_n \left( \frac{k}{n} \right) \right) \leq A \left( \frac{k}{n} - s \right) \Rightarrow \Psi_A \left( \frac{k}{n} - s \right) \left( X_n(s), X_n \left( \frac{k}{n} \right) \right) = 1,$n

hence $\{\phi^{s,t}_n = \eta^{s,t}_{n,A}\} \supset D^{s,t}_{n,A}$ and

$$\{\zeta^{s,t}_{n,A} \neq 0\} \subset \Omega \setminus D^{s,t}_{n,A} \subset \{H_{\delta,n} \geq A\}.$$

Let $p$ be the same as in statement 3 of Lemma 2. Then it follows from (6.3) and inequality $0 \leq \zeta^{s,t}_{n,A} \leq \phi^{s,t}_n$ that

$$E \left[ \zeta^{s,t}_{n,A} | X_n(s) = x \right] \leq E \left[ (\phi^{s,t}_n)^p | X_n(s) = x \right] \frac{p}{2} \left[ P(H_{\delta,n}^{s,t} \geq A | X_n(s) = x) \right]^{\frac{2-p}{p}} \leq B_6(T) A^{-\delta} \frac{1}{p}.$$

Similarly, one can write $\phi^{s,t}_n = \eta^{s,t}_A + \zeta^{s,t}_A$, where $\eta^{s,t}_A = \int_s^t \Psi_A(x-s)(X(s), X(r))d\phi^{s,r}$,

$$E \left[ \zeta^{s,t}_A | X(s) = x \right] \leq B_6(T) A^{-\delta} \frac{1}{p}.$$

We have

$$\left| E \left[ \eta^{s,t}_A | X_n(s) = x \right] - E \left[ \eta^{s,t}_A | X(s) = x \right] \right| =$$

$$= \frac{g_n(x)}{n} + \frac{1}{n} \sum_{k=1}^{\lfloor n(t-s) \rfloor} \int_{\mathcal{X}} p_{k,n}(x, y) \Psi_A \left( \frac{k}{n} - s \right) (x, y) \mu_n(dy) - \int_0^{t-s} \int_{\mathcal{X}} p_r(x, y) \Psi_A(x-s)(x, y) \mu(dy) dr \leq$$

$$\leq \delta_n + \Delta^1_n(x, A, s, t) + \Delta^2_n(x, A, s, t) + \Delta^3_n(x, A, s, t),$$
where \(|z|\equiv \min\{N \in \mathbb{Z}, N \geq z\},
\[
\Delta_1^j(x,A,s,t) = \left| \frac{1}{n} \sum_{k=1}^{[n(t-s)]-1} \int_X \left[ p_k(x,y) - p_{k,n}(x,y) \right] \Psi_{A}(\frac{x}{n})^s(x,y) \mu_n(dy) \right|,
\]
\[
\Delta_2^j(x,A,s,t) = \left| \frac{1}{n} \sum_{k=1}^{[n(t-s)]-1} \int_X p_k(x,y) \Psi_{A}(\frac{x}{n})^s(x,y) \mu_n(dy) - \int_0^{t-s} \int_X p_r(x,y) \Psi_{A^r}(x,y) \mu_n(dy) \, dr \right|,
\]
\[
\Delta_3^j(x,A,s,t) = \left| \int_0^{t-s} \int_X p_r(x,y) \Psi_{A^r}(x,y) \mu_n(dy) - \mu(dy) \, dr \right|.
\]
Denote \(\Delta^j_i(A,T) = \sup_{x \in X, s \leq t \leq T} \Delta^j_i(x,A,s,t), i = 1, 2, 3.\) Since \(\Psi_r(x,y) \in [0,1] \) and \(\{\Psi_r(x,y) \neq 0\} \subset \{y \in B(x,2r)\},\)
\[
\Delta_1^j(A,T) \leq \frac{1}{n} \sum_{k=1}^{[nT]-1} (\alpha_n + \beta_k) \left( \frac{k}{n} \right)^{\gamma} \mu_n \left( B \left( x, 2A \left( \frac{k}{n} \right)^{\delta} \right) \right) \leq C_0 (2A)^\theta \cdot \frac{1}{n} \sum_{k=1}^{[nT]-1} (\alpha_n + \beta_k) \left( \frac{k}{n} \right)^{\delta \theta - \gamma} \to 0, \quad n \to +\infty
\]
by Toeplitz theorem.

The function \((r, x, y) \mapsto p_r(x,y)\Psi_r(x,y)\) is uniformly continuous over \([r_0, +\infty) \times X^2\) for any \(r_0 > 0,\) therefore an estimate analogous to (6.6) provides that
\[
\sup_{x \in X, s \leq t \leq T} \left| \frac{1}{n} \sum_{k=[rn]+1}^{[nT]-1} \int_X p_k(x,y) \Psi_{A}(\frac{x}{n})^s(x,y) \mu_n(dy) - \int_0^{t-s} \int_X p_r(x,y) \Psi_{A^r}(x,y) \mu_n(dy) \, dr \right| \to 0
\]
(note that \(\max_n \mu_n(X) < +\infty\) since \(\mu_n\) weakly converge to \(\mu\).) The same arguments provide that
\[
\limsup_{n \to +\infty} \Delta_2^j(A,T) \leq \limsup_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{[rn]} C_\gamma \left( \frac{n}{k} \right)^\gamma C_0 \left( 2A \left( \frac{k}{n} \right)^{\delta} \right) + \int_0^{r_0} C_{r^\gamma} C_0 \left( 2A^r \right)^\theta \, dr \right| = B_T(A,T)(r_0)^{\delta \theta - \gamma + 1}.
\]
Since \(r_0 > 0\) is arbitrary, this implies that
\[
\Delta_2^j(A,T) \to 0, \quad n \to +\infty.
\]
At last, the weak convergence of \(\mu_n\) to \(\mu\) and the first part of condition 3 provide that, for every \(t,\)
\[
I_n(A,t) \equiv \sup_{x \in X} \left| \int_X p_t(x,y) \Psi_{A^r}(x,y) \mu_n(dy) - \mu(dy) \right| \to 0, \quad n \to +\infty.
\]
Since \(I_n(A,t) \leq C_{\gamma} t^{-\gamma} \cdot C_0 (2A^\gamma)^\theta,\) the Lebesgue theorem of dominated convergence provides that
\[
\Delta_3^j(A,T) \to 0, \quad n \to +\infty.
\]
It follows from the estimates (6.4) – (6.8) that
\[
\limsup_{n \to +\infty} \sup_{x \in X, s \leq t \leq T} \left| f_n(x) - f^A_t(x) \right| \leq 2B_6(T) A^{-\frac{\delta \theta - \gamma + 1}{\gamma}}, \quad A > c_0.
\]
Taking \(A \to +\infty\) we obtain (6.2), that completes the proof. The theorem is proved.

In order to make our exposition complete, let us formulate a version of Theorem 2 for the recurrent case.

Theorem 3. Let conditions 1 – 4 of Theorem 3 hold true and \(\gamma < 1.\) Also let \(\mu_n\) converge weakly to \(\mu,\) and \(X_n\) converge to \(X\) by distribution in \(C(\mathbb{R}^+, \mathcal{F}).\)

Then \((X_n, \psi_n(X_n)) \Rightarrow (X, \phi(X))\) in a sense of convergence in distribution in \(C(\mathbb{R}^+, \mathcal{F}) \times C(\mathbb{T}, \mathbb{R}^+).\)
The proof, with slight changes, repeats the proof of Theorem 3 and is omitted. Note that, under conditions of Theorem 3, the convergence of finite-dimensional distributions of \( \phi_n \) can be provided with the use of the technique, mentioned in the Introduction, that was proposed by I.I.Gikhman and is based on studying of limit behavior of difference equations for characteristic functions of \( \phi_n^{n.t} \) (see for instance the proof of Theorem 3 [9]). In the transient case, treated in Theorem 2 this technique can not be applied since the uniform estimates, analogous to (4.3) – (4.6), are not available in this case.

At last, let us give an example of application of Theorem 2. To shorten exposition we omit the proofs of some technical details.

**Example 4.** Let \( \mathcal{X} = \mathbb{R}^d, d \geq 2 \) and \( X_n, X \) be as in Example 1. Let \( K \subset \mathbb{R}^d \) be a compact set, for which the surface measure \( \lambda_K \) is well defined by equality

\[
\lambda_K(\cdot) \equiv w - \lim_{\varepsilon \to 0^+} \frac{\lambda(\cdot \cap K_{\varepsilon})}{\lambda(K_{\varepsilon})},
\]

where \( w - \lim \) means the limit in the sense of weak convergence of measures, \( \lambda^d \) is Lebesgue measure on \( \mathbb{R}^d \), \( K_{\varepsilon} \equiv \{ x | \text{dist}(x, K) \leq \varepsilon \} \). Assume that the condition

\[
(6.9) \quad \lambda^d(K_{\varepsilon}) \geq \text{const} \cdot \varepsilon^\beta, \quad \varepsilon > 0
\]

holds with some \( \beta < 2 \). In particular, the set \( K \) can be smooth (or, more generally, Lipschitz) surface of codimension \( 1 \) or fractal with its Hausdorff-Besikovich dimension greater then \( 2 - d \).

It not hard to verify that \( \mu \equiv \lambda_K \) is \( W \)-measure (see [1], Chapter 8.1 for the terminology), and therefore corresponds to some \( W \)-functional \( \phi \) of the Wiener process \( X \). This functional is naturally interpreted as the local time of Wiener process at the set \( K \), and can be written as \( \phi^{n.t} = \int_0^t \lambda_K(X_r) \, dr \).

We consider the functionals \( \phi_n(X_n) \) of the form

\[
\phi_n^{t.a} = \frac{1}{n\lambda^d(K_{\varepsilon})} \sum_{k \in [s_n, t_n]} \mathbb{I}_{\{ X_n \in K_{\varepsilon} \}},
\]

and apply Theorem 2 in order to prove convergence of the distributions in \( C(\mathbb{R}^+, \mathbb{R}^d) \times C(T, \mathbb{R}^+) \)

\[
(6.10) \quad (X_n, \psi_n(X_n)) \Rightarrow (X, \phi(X))
\]

(\( \psi_n \) are the broken lines corresponding to \( \phi_n \)).

Condition 1 holds true due to Example 1, condition 2 is provided by condition (6.9) (by this condition, \( \sup \gamma \), \( g_n(x) \leq \text{const} \cdot n^{\frac{\beta}{d}} \)). Condition 3 holds with \( p_t(x, y) = (2\pi t)^{-\frac{d}{2}} \exp\left\{-\frac{1}{2} \| y - x \|^2 \right\} \) and \( \gamma = \frac{d}{2} \). Condition (6.9) implies condition 6 with \( \theta = d - \beta \).

We assume that the random walk \( S_n \) is either aperiodic on some lattice \( h\mathbb{Z}^d \) or is strongly non-lattice (i.e., \( S_n \) has bounded distribution density for some \( n_0 \)). Under this assumption, condition 4 holds with \( \alpha_n \equiv 0, \nu = \lambda^d \) and \( \nu_n \) equal to counting measures on \( \frac{h}{\sqrt{n}} \mathbb{Z}^d \) in lattice case or \( \lambda^d \) in strongly non-lattice case.

It remains to provide conditions 5, 7. We have \( \frac{\gamma - 1}{\alpha} = \frac{d - 2}{2(d - \beta)} < \frac{1}{2} \). Choose some \( \delta \in \left( \frac{\gamma - 1}{\alpha}, \frac{1}{2} \right) \) and consider \( \alpha > 0 \) such that \( \frac{\gamma - 1}{\alpha} > \delta \) and \( \alpha > 2\theta + 2 \). Suppose that

\[
(6.11) \quad E\| \xi_k \|_{\mathbb{R}^d}^\alpha < +\infty.
\]

Then applying Burkholder inequality we obtain that

\[
(6.12) \quad E\| X_n(t) - X_n(s) \|_{\mathbb{R}^d}^{\alpha \theta} \leq \text{const} \cdot |t - s|^{\frac{\alpha \theta}{\alpha}}, \quad |t - s| \geq \frac{1}{\sqrt{n}}, x \in \mathbb{R}^d.
\]

Repeating the standard proof of the Kolmogorov’s theorem on existence of continuous modification (see, for instance [20], p. 44,45), one can deduce from (6.12) that, for \( \varsigma < \alpha, \vartheta < \frac{\gamma - 1}{\alpha} \),

\[
\sup_n E \left[ \sup_{t,s \in [0,T], |t-s| \geq \frac{1}{n}} \frac{\| X_n(t) - X_n(s) \|_{\mathbb{R}^d}}{|t - s|^{\vartheta}} \right]^{\varsigma} < +\infty.
\]
Finally, choosing $\vartheta = \delta, \varsigma > 2\theta + 2$ we obtain that conditions 5,7 hold with $C_\theta = \varsigma$. Applying Theorem 2 we obtain weak convergence (6.10) under additional moment condition (6.11). One can remove this condition using the "cutting" procedure, described in the proof of the Proposition 3.

Let us remark that for the lattice random walks the result, exposed in Example 4, was obtained in [5] by a technique, essentially different from the one proposed here. Convergence (6.10) in continuous case, as far as it is known to authors, is a new result.

References

[1] Dynkin E.B. Markov processes, M.: Fizmatgiz, 1963 (in Russian).
[2] Skorokhod A.V., Slobodensuk M.P. Limit theorems for random walks, Kiev: Naukova dumka, 1970 (in Russian).
[3] Borodin A.N., Ibragimov I.A. Limit theorems for the functionals of random walks, Proc. of the Mathematical Institute of R. Acad. Sci, vol. 195. St.-P.: Nauka, 1994 (in Russian).
[4] Revesz P. Random walk in random and nonrandom environments, World Sci. Publ. Co., Inc., Teaneck, NJ, 1990.
[5] Bass R.F., Khoshnevisan D. Local times on curves and uniform invariance principles, Prob. Theory Rel. Fields 92, 1992, p. 465 – 492.
[6] Cherny A.S., Shiryayev A.N., Yor M. Limit behavior of the "horizontal-vertical" random walk and some extensions of the Donsker-Prokhorov invariance principle. Probability theory and its applications, vol. 47, 3, 2002, p. 498 – 517.
[7] Gikhman I.I. Some limit theorems for the number of intersections of a boundary of a given domain by a random function, Sci. notes of Kiev Un-ty, 1957, vol. 16, 10, p. 149 – 164 (in Ukrainian).
[8] Gikhman I.I. Asymptotic distributions for the number of intersections of a boundary of a domain by a random function, Visnyk of Kiev Un-ty, serie astron., athem and mech., 1958, v. 1, 1, p. 25 – 46 (in Ukrainian).
[9] Portenko N.I. Integral equations and limit theorems for additive functionals of Markov processes, robability theory and its applications, 1967, v. 12, 3, p. 551 – 558 (in Russian).
[10] Portenko N.I. The development of I.I.Gikhman’s idea concerning the methods for investigating local behavior of diffusion processes and their weakly convergent sequences, Probab. Theory and Math. Stat., 1994, 50, p. 7 – 22.
[11] Kulik A.M. Markov Approximation of stable processes by random walks, vol.12(28) 2006, 1-2, p. 87 – 93.
[12] Feller W. An introduction to probability theory and its applications, Vol II, M.: Mir, 1984 (Russian, translated from W.Feller, An introduction to probability theory and its applications, John Wiley & Sons, New York, 1971).
[13] Skorokhod A.V. Studies in theory of stochastic processes, Kiev, Kiev Univ-ty publishing house, 1961 (in Russian).
[14] Jacod J., Shiryaev A. Limit theorems for stochastic processes, Springer, Berlin, 1987.
[15] Kartz T.G., Protter Ph. Weak limit theorems for stochastic integrals and SDE’s, Annals of Probability, 1991, vol. 19, 3, p. 1035 – 1070.
[16] Yamada T., Watanabe S. On the uniqueness of solutions of stochastic differential equations, J. Math. Kyoto Univ., 1971, vol. 11, p. 156 – 167.
[17] Androshchuk T.O., Kulik A.M. Limit theorems for oscillatory functionals of a Markov process. Theory of stochastic processes, vol. 11(27), p. 3 – 13.
[18] Ibragimov I.A., Linnik Yu.V. Linnik, Independent and stationary related variables, M.: Nauka, 1965 (in Russian).
[19] Kulik A.M. Difference approximation for local times of multidimensional diffusions, [arXiv:math/0702175]
[20] Skorokhod A.V. Lections on theory of stochastic processes, Kyiv: Lybid, 1990 (in Ukrainian).

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