Scalar Curvature Splittings II: Removal of Singularities
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1 Introduction

We resume our discussion of general inductive scalar curvature splittings, involving singular minimal hypersurfaces, from the first part [L1]. We also adopt the notations, concepts and results from [L1]. In this paper we establish splitting schemes with built-in regularizations to treat problems in scalar curvature geometry and general relativity in arbitrary dimensions.

1.1 Statement and Discussion of Results

The splitting approach involves the use of singular minimal hypersurfaces. In [L1] we have seen that any singular compact area minimizer \((H^n, d_H)\) in a scal > 0-manifold \(M^{n+1}\) admits a conformal deformation to some minimal factor geometry \((H^n, d_S)\) that shares many properties with \((H^n, d_H)\) like the Ahlfors regularity, the validity of Poincaré inequalities and the presence of, in this case, scal > 0-tangent cones equipped with their minimal factor geometry. These geometries are amenable to surgery style arguments to eliminate singular sets stepwise. The basic building blocks we get are splitting with boundary theorems.

**Theorem 1** Let \(H^n, n \geq 2,\) be a compact area minimizer, with singular set \(\Sigma,\) in a scal > 0-manifold \(M^{n+1}\). Then there are arbitrarily small neighborhoods \(U\) of \(\Sigma,\) so that \(H \setminus U\) is conformal to some scal > 0-manifold \((X_U, g_X)\) with minimal boundary \(\partial X_U = \partial U\).

For \(\Sigma_H = \emptyset\) we choose \(U = \emptyset, \partial U = \emptyset.\) If \(\Sigma \neq \emptyset,\) then it suffices if \(M^{n+1}\) has scal \(\geq 0.\) We actually get a smooth (generally non-complete) open scal > 0-manifold \(Y^o_U\) that extends \(X_U,\) i.e. \(\overline{X_U} \subset Y_U,\) so that \(\partial X_U \subset Y_U\) is locally area minimizing.
Inductive Removal of Singularities Theorem 1 replaces the singular set \( \Sigma_H \) of \((\mathcal{H}^n, d_S)\) by a minimal boundary. The point is that although, in general, the boundary \( \partial X_U = \partial U \) is also singular, its singular set has a lower dimension than \( \Sigma_H \). The minimality of \( \partial X_U \) then allows us to iteratively shift singular problems to lower dimensions before they disappear in dimension 7. We describe some basic implementations of this idea.

The straightforward way to apply Theorem 1 is to use \( X^n_U \) as a replacement for \((\mathcal{H}^n, d_S)\) for small \( U \). Following the classical splitting recipe, we consider a local area minimizer \( L^{n-1} \subset X^n_U \). Since \( X^n_U \) is not complete we observe that \( L^{n-1} \) satisfies an additional constraint: \( L^{n-1} \) is an area minimizer with obstacle \( \partial U \) that keeps \( L^{n-1} \) in \( X^n_U \). This is a standard situation in geometric measure theory, cf. [Gi, Th. 1.20, Rm. 1.22], but it is rather uncommon in the context of scalar curvature splittings.

- However, since \( \partial U \subset Y_U \) is an unconstrained area minimizer by itself, the strict maximum principle [Si] applies and it shows, componentwise, that either \( L^{n-1} \equiv \partial U \) or \( L^{n-1} \cap \partial U = \emptyset \). This means \( L^{n-1} \) always is an unconstrained area minimizer in \( Y^n_U \).

- \( L^{n-1} \) could be singular, but when \( L^{n-1} \) is compact we can apply Theorem 1 to \( L^{n-1} \subset Y^n_U \), in dimension \( n-1 \), since \( Y^n_U \) is a smooth scal \( > 0 \)-manifold. In this sense Theorem 1 allows us to stay in the smooth category. Then we are ready for the next loop in the dimensional descent. Once we reach \( n = 7 \) all further area minimizers are regular.

Applications We indicate some typical applications (addressed in separate accounts). We start from a compact scal \( > 0 \)-manifold \( M^{n+1} \) and classes \( \alpha[1], \ldots, \alpha[m] \in H^1(\mathcal{M}^{n+1}, \mathbb{Z}) \), \( m \leq n-1 \). The scheme above gives us the means to represent \( \alpha[k] \cap \cdots \cap \alpha[1] \cap [M] \), \( k \leq m \), by area minimizers \( L^{n+1-k} \) in smooth scal \( > 0 \)-manifolds of dimension \( n + 2 - k \), provided we broaden our playground and include complete spaces with periodic ends and coarse homologies, cf. [R] and [L5] for a survey (also of underlying techniques and applications).

To explain the idea, we let \( H^n \) represent \( \alpha[1] \cap [M] \) and apply Theorem 1 to get a conformal deformation to a scal \( > 0 \)-geometry with minimal boundary \( \partial U \) for some neighborhood \( U \) of \( \Sigma_H \). When \( L^{n-1} = L^{n-1}(U) \equiv \partial U \), the class \( \alpha[2] \cap \alpha[1] \cap [M] \) is trivial, since \( L^{n-1} \) is null-homologous in \( H^n \setminus U \). For non-trivial \( \alpha[2] \cap \alpha[1] \cap [M] \), and small \( U \), we have two possible scenarios for \( L^{n-1} \), (i) and (ii), illustrated in Fig. 1.

(i) In many cases one can choose a setup so that the class contains hypersurfaces disjoint from \( \Sigma \) and thus from \( \overline{U} \), when \( U \) is small enough. Then we find some representing compact and unconstrained area minimizer \( L^{n-1} \subset H^n \setminus \overline{U} \). A typical situation where this happens is that where \( M^{n+1} \) has geometrically large components, like nearly flat torus summands. Examples are the positive mass/energy theorems where we get such components from a compactification of asymptotically flat ends [L6, Ch. 6], [L7].

![Figure 1: \( \Sigma_H \) has an empty intersection (i) with some hypersurface in \( \alpha[2] \cap \alpha[1] \cap [M] \) or not (ii).](image-url)
(ii) Otherwise the minimal factor geometry \((H^n, d_S)\) allows us to get *intrinsically complete* local area minimizers \(L^{n-1}(U)\) asymptotically approaching \(\partial U\). For shrinking \(U\), the \(L^{n-1}(U)\) either approximate or represent \(\alpha[2] \cap \alpha[1] \cap [M]\) in coarse homology, annihilating the infinite spinning of \(L^{n-1}\) towards the compact \(\partial U\). We apply Theorem 1 to \(\partial U\) and periodically transfer the outcome to the end of \(L^{n-1}\). (The deformation in Theorem 1 are assembled from local deformations and we can combine the transferred deformations from \(\partial U\) with those on the compact part of \(L^{n-1}\) away from these ends.) As a concrete application, we note that this matches the scheme in [GL1, Ch. 12] and it shows that *enlargeable manifolds* cannot admit \(\text{scal} > 0\)-metrics, cf. [L8].

Another way of applying this scheme is that of creating a smooth replacement \(H^n\) for the singular \(H^n\) that admits a \(\text{scal} > 0\)-metric. This does not change the scope of the method, but it fits into the layout of some classical arguments for the lower dimensional cases.

**Variations and Extensions** We extend Theorem 1 in two directions. We start with the case of manifolds and area minimizers with already given boundaries. This scenario is used, for instance, in arguments in the style of classical comparison geometry, [GL1], Ch. 12:

**Theorem 2** Let \(H^n\) be a compact area minimizer with boundary \(\partial H \cap \Sigma = \emptyset\) in a \(\text{scal} > 0\)-manifold \(M^{n+1}\). Then there are arbitrarily small neighborhoods \(U \cap \partial H = \emptyset\) of \(\Sigma\), so that \(H \setminus U\) is *conformal* to some \(\text{scal} > 0\)-manifold \(X_U\) with disjoint boundary components \(\partial X_0\) diffeomorphic to \(\partial H\) and *minimal* \(\partial X_1\).

The methods equally apply to larger classes of *almost* minimizers, like \(\mu\)-bubbles, levels sets of various geometric flows or horizons of black holes from Lorentzian geometry. The admissible classes \(H\) and \(G\) of almost minimizers are discussed and specified in [L1, Ch. 1.2] and [L4, Ch. 3.1]. We have the following generalization of Theorem 1.

**Theorem 3** Let \(H^n \subset M^{n+1}\) be a compact almost minimizer with singular set \(\Sigma_H\) with \(S\)-adapted conformal Laplacian \(L_H\). Then there are arbitrarily small neighborhoods \(U\) of \(\Sigma\), so that \(H \setminus U\) is *conformal* to a \(\text{scal} > 0\)-manifold \(X_U\) with *minimal* boundary \(\partial X_U\).

The \(S\)-adaptedness of \(L_H\) means that the principal eigenvalue of \(\langle A \rangle^{-2} \cdot L_H\) is positive and, thus, \(H\) can be conformally deformed into a \(\text{scal} > 0\)-minimal factor \((H, d_S)\) by an eigenfunction of \(\langle A \rangle^{-2} \cdot L_H\). There is also a boundary version similar to Theorem 2.

**1.2 Overview of the Argument**

**Basic Strategy** The deformations in our Theorems are similar to those in classical \(\text{scal} > 0\)-preserving surgeries [GL2], [SY]: for a \(\text{scal} > 0\)-manifold \(M\) and a compact submanifold \(N \subset M\) of codimension \(\geq 3\), there are small neighborhoods \(U \subset V\) of \(N\) so that \(M \setminus U\) can be conformally deformed within \(V\) to a \(\text{scal} > 0\)-metric so that \(\partial U\) is a local area minimizer. This variant is taken from [L5, Ch. 4]. For the subsequent surgery/gluing processes in [GL2], [SY], which we do not need in our context, one appends (non-conformal) deformations of \(V \setminus U\) to transform \(\partial U\) into a totally geodesic boundary.

In our case, the assumptions in our Theorems mean that the singular (almost) minimizer
$H^n \subset M^{n+1}$ admits a conformal deformation to a still singular but well-behaved scal $0$-minimal factor geometry $(H, d_S)$ from [L1]. We take $(H, d_S)$ as our initial scal $0$-geometry and let $\Sigma \subset (H, d_S)$ play the rôle of $N \subset M$ in the classical smooth theory above. We construct small neighborhoods $U \subset V$ of $\Sigma$ and a conformal deformation, this time of $d_S$, supported in $V \setminus \Sigma$, to another scal $0$-metric on $H \setminus U$ so that $\partial U$ is a local area minimizer. Since $\Sigma$ can be rather complicated and $H \setminus \Sigma$ degenerates towards $\Sigma$, the construction of this deformation is broken into simpler local bump deformations of $(H, d_S)$. Each of these local bumps creates a cavity oriented along some part of $\Sigma$. This gives rise to a local area minimizer $L$ that shields all of $\Sigma$, proving the Theorems.

Figure 2: In this illustration, the initial minimal factors $(C, d_S)$ are entirely flat planes. The singularities are marked as bluish spots and lines. After bump deformations we get local area minimizers $L^{n-1} = \text{the red rubber bands.}$ In other words, the deformations shield (parts of) $\Sigma$ from intrusions of $L^{n-1}$.

A Glance at Technical Details We work with the metric measure space $(H, d_S, \mu_S)$ associated to the metric completion $(H, d_S)$ of $(H \setminus \Sigma, \Psi_H^{4/(n-2)} \cdot g_H)$, where $\Psi_H > 0$ is a $C^{2,\alpha}$-(super)solution, for an $\alpha \in (0,1)$, with minimal growth towards $\Sigma_H$, of

$$L_{H,\lambda}(\phi) := L_H(\phi) - \lambda \cdot \langle A \rangle^2 \cdot \phi := -\Delta \phi + \frac{n-2}{4(n-1)} \cdot \text{scal}_H \cdot \phi - \lambda \cdot \langle A \rangle^2 \cdot \phi = 0 \text{ on } H \setminus \Sigma,$$

for some subcritical eigenvalue $\lambda \in (0, \lambda_H^{(A)})$, where $\lambda_H^{(A)}$ is the principal eigenvalue of $\langle A \rangle^{-2} \cdot L_H$, to get a well-controlled geometry $(H, d_S)$. Equation (1) and the value of $\lambda$ remain invariant under scalings. Under blow-up in any $p \in \Sigma_H$, $\Psi_H$ induces the unique solution $^1$ $\Psi_\sigma > 0$ of $L_{C,\lambda} \phi = 0$ with minimal growth towards $\Sigma_C$ on each of its tangent cones $C$ [L1, Th. 1.5 and 1.6]. From this, the completion $(C, d_S)$ of $(C, \Psi_\sigma^{4/(n-2)} \cdot g_C)$ is again a cone and it is a scal $0$-tangent cone of $(H, d_S)$: for large $\tau \gg 1$, it locally approximates $(H, \tau \cdot d_H)$ similarly to ordinary tangent cones for the original geometry $(H, \tau \cdot d_H)$. This allows us to construct local bumps on $(H, d_S)$ from cone reduction arguments we indicate in the following.

Dimension 7: Isolated Singularities We start in dimension 7 since all almost minimizers are regular in lower dimensions. The singular set $\Sigma_H$ of a compact singular almost minimizer $H^7 \subset M^8$ contains at most finitely many singular points $p_1, ..., p_m$ and in each $p_i \in \Sigma_H$ we have tangent cones $C^g_i$ approximating $\tau \cdot H$ near $B_1(p_i) \subset \tau \cdot H$ for large $\tau \gg 1$. That is, for each $i$, there is a canonical $C^{2,\alpha}$-diffeomorphism $\text{ID}_i : B_1(0) \setminus B_{\epsilon}(0) \cap C^g_i \to$

\footnote{unique$^\#$ is our abbreviation for unique up to multiplication by a positive constant.}
\( B_1(p_i) \setminus B_r(p_i) \subset \tau \cdot H \), up to minor modifications near the boundary. \( \text{ID}_i|_{(B_1(0) \setminus B_r(0))} \) is nearly an isometry when \( \tau \) is large enough, cf. [L1, Ch. 1.3.D.2.], and this allows us the transfer of local deformations between \( C_i \) and \( \tau \cdot H \). The \( C_i^7 \) are singular only in 0 and there is some \( \lambda(H) \in (0, \lambda_H^{(A)}) \) so that for \( (\omega, r) \in (\partial B_1(0) \cap C_i^7) \times \mathbb{R}^> \cdot \mathbb{R}^> = C_i^7 \setminus \{0\} \), cf. [L4, Prop. 4.6]:

\[
(2) \quad \Psi_\circ(\omega, r) = \psi(\omega) \cdot r^{\alpha_\circ}, \text{ with } 0 > \alpha_\circ \geq -\left(1 - \sqrt{\frac{2}{3}}\right) \cdot \frac{7-2}{2} > -\frac{7-2}{2}
\]

and for the antagonist of \( \Psi_\circ \), the solution \( \Psi_\bullet \) with \text{maximal growth} towards \( \{0\} \):

\[
(3) \quad \Psi_\bullet(\omega, r) = \psi(\omega) \cdot r^{\alpha_\bullet}, \text{ with } -\frac{7-2}{2} > -(1 + \sqrt{\frac{2}{3}}) \cdot \frac{7-2}{2} \geq \alpha_\bullet \geq -(7 - 2).
\]

For balls \( B_\theta(0) \subset C_i^7, \theta > 0 \), the mean curvature of \( \partial B_\theta \) is negative\(^2\). The critical exponent \( \beta \) for a conformal deformation \((r^\beta)^{4/7-2} \cdot g_{C_i^7}\) to flip the sign of the mean curvature is \( \beta = -\frac{7-2}{2} \) where this deformation yields a scalar > 0-cylinder making \( \partial B_\theta \) totally geodesic. Thus the mean curvature of \( \partial B_\theta \) relative to \( \Psi_\circ^{4/7-2} \cdot g_{C_i^7} \) is still \text{negative}, but relative to \( \Psi_\bullet^{4/7-2} \cdot g_{C_i^7} \), it is \text{positive}. We merge these metrics to a new scalar > 0-metric, the \text{bumped metric} \( g_\Omega = (\Phi_\Delta \cdot \Psi_\circ)^{4/7-2} \cdot g_{C_i^7}, \text{ for some } C^{2,\alpha}-\text{regular } \Phi_\Delta \geq 1, \text{ cf. the left image of Fig. 2, so that:}

- there is a radius \( r \in (0, 1) \) so that \( \text{supp } |\Phi_\Delta - 1| \subset B_1(0) \setminus B_r(0), \)
- there is a \( \theta \in (r, 1) \) so that \( L^{-1} := \partial B_\theta(0) \) is \text{locally area minimizing} relative to \( g_\Omega \).

We call \( \Delta_{C_i^7} := (\Phi_\Delta - 1) \cdot \Psi_\circ)^{4/7-2} \cdot g_{C_i^7} \) a \text{local bump pseudo metric}\(^3\). Now we use the maps \( \text{ID}_i \), for \( \tau \) large enough, and transplant such local bumps from tangent cones to the balls \( B_1(p_i) \) in \((H, \tau \cdot d_S)\) adding the \( \text{ID}_i\)-push-forward of the pseudo metric from \( C_i \) to \( \tau \cdot H \) and get Theorem 1 in dimension 7.

\textbf{Dimensions} \( \geq 8 \): \textbf{Localization via Ahlfors regularity and Isoperimetry} As before we want use local bumps. The practical hurdle is that for a general area minimizer \( H^n, n \geq 8 \), the singular set is no longer a discrete set. A typical example is a product cone \( C^{n-1} \times \mathbb{R} \) singular in \( \{0\} \times \mathbb{R} \) with an \( \mathbb{R} \)-invariant bump pseudo metric that shields \( \Sigma_{C^{n-1} \times \mathbb{R}} \), cf. the middle of Fig. 2. These deformations have \text{non-compact} support whereas tangent cones smoothly approximate \((H, \tau \cdot d_S)\) only on subsets with \text{compact closure} in \( C^n \setminus \Sigma_C \). This obstructs the transfer of such deformations from \((C, d_S)\) to \((H, \tau \cdot d_S)\) as a whole. We trim these pseudo metrics on \( C^n \setminus \Sigma_C \) to deformations with compact support keeping scal > 0 to define a \text{local bump}. (This is indicated in the right image of Fig. 2. The proper definition needs a bit more work illustrated in Fig. 5. We give a separate overview in the beginning of Ch. 2.) The trade-off is that the (support of) local bumps no longer topologically separate small neighborhoods of the singularities from large regular complementary parts of \( H \). In view of the singular nature of the underlying space this raises the question whether such a trimming ruins the shielding effect of the bumps.

At this point the \textbf{Ahlfors regularity} and the \textbf{isoperimetric inequality} for \((H, d_S, \mu_S) \) [L1, Th. 1.11] enter the game. They compensate this shortcoming and keep the area minimizer \( L \) from entering the cavity surrounded by local bumps. This shielding property of local bumps

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\(^2\)Our sign convention is that \( \partial B_1(0) \subset (\mathbb{R}^n, g_{\text{Eucl}}) \) has \text{negative} mean curvature relative to \( \mathbb{R}^n \setminus B_1(0) \).

\(^3\)In this paper a pseudo metric is a symmetric \text{positive semi-definite} \text{bilinear form}.
is stable enough to survive the transfer from cones to the more general $H$. We use a suitable covering scheme to place such bumps along $\Sigma_H$ while keeping their support disjoint. As a direct consequence of this disjointness and the stable shielding property of local bumps the resulting global bump on $H$ shields all of $\Sigma_H$.

**Organization of the Paper** We focus on Theorem 3 and assume $\Sigma_H \neq \emptyset$. Theorems 1 and 2 are special cases.

In Ch. 1.3, we specify the admissible class of bumped metrics we use in this paper and we discuss results from [L1] and [L4] that we may apply to such metrics.

In Ch. 2.1, we construct simple bump pseudo metrics on product cones $\mathbb{R}^{n-k} \times C^k$ shielding the axis $\mathbb{R}^{n-k} \times \{0\}$. Then, in Ch. 2.2-2.4, we apply localizing modifications to this basic pseudo metric and assemble local bumps on $\mathbb{R}^{n-k} \times C^k$ shielding a cube $Q^{n-k}$ in $\mathbb{R}^{n-k} \times \{0\}$.

In Ch. 3.1-3.2, we prepare the transfer of local bumps to general (almost) minimizers and, in Ch. 3.3 we place families of local bumps so that the union of the shielded subsets covers $\Sigma_H$.

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1.3 Area Minimizers in $\Lambda$-Bumped Spaces

We summarize some of the results from [L2]-[L4] and extend them to the class of controllably bumped metrics we work with. Throughout the paper we exploit particularities of the conformal Laplacians $L_H$ and $\langle A \rangle^{-2} \cdot L_H$ for subcritical but positive eigenvalues where we have a good control over the analysis near the singular set.

For the sake of consistency we oftentimes state results for all almost minimizers $H \in \mathcal{G}$. As in [L1] this comes with the caveat that for regular $H \in \mathcal{G}$ or $H = \mathbb{R}^n \in \mathcal{H}^\mathbb{R}$ some statements become trivial/void because there is no singular set or no subcritical eigenfunction.

**Minimal Factors** The singular almost minimizer $H \in \mathcal{G}$ in Theorem 3 has a positive principal eigenvalue $\lambda_{H}^{(A)} > 0$. This means that for any $\lambda < \lambda_{H}^{(A)}$ and $L_{H,\lambda} := L_H - \lambda \cdot \langle A \rangle^2$ there are positive solutions of $L_{H,\lambda} \phi = 0$, cf. [L3]. The coefficients are locally Lipschitz regular, hence any such solution is $C^{2,\alpha}$-regular for an $\alpha \in (0,1)$. For some subcritical eigenvalue $\lambda \in (0, \lambda_{H}^{(A)})$, we set, following [L1]:

**Definition 1.1 (Minimal Factor Metrics)** For $H \in \mathcal{G}$, let $\Psi > 0$ be a $C^{2,\alpha}$-supersolution of $L_{H,\lambda} \phi = 0$ on $H \setminus \Sigma_H$ so that in the case

- $H \in \mathcal{G}^c_n$: $\Psi$ is a solution in a neighborhood of $\Sigma$ with minimal growth towards $\Sigma$.
- $H \in \mathcal{H}^\mathbb{R}_n$: $\Psi$ is a solution on $H \setminus \Sigma_H$ with minimal growth towards $\Sigma$.

We call the scalar $> 0$-metrics $\Psi^{(n-2)} \cdot g_H$ on $H \setminus \Sigma$ the minimal factor metrics.

For non-totally geodesic $H \in \mathcal{H}^\mathbb{R}_n$ there is a unique such solution $\Psi_\phi$ of $L_{H,\lambda} \phi = 0$ for subcritical $\lambda > 0$ [L3, Theorem 3]. To any singular $H \in \mathcal{G}$ we assign one fixed supersolution $\Psi_H$.

**Choice of Subcritical Eigenvalues** Now we select a subcritical eigenvalue $\lambda^*_{H}$ of $\langle A \rangle^{-2} \cdot L_H$ to gain the needed control over the growth rate of a supersolution $\Psi$, as in
Def. 1.1, towards $\Sigma_H$. To this end, we consider tangent cones $C^n$ obtained from blow-ups around a singular point of $H^n$. The blow-up process can be iterated in singular points outside 0. After each blow-up the resulting minimal cone acquires an additional Euclidean factor. After $n - k$ blow-up steps we reach a singular cone $C^n = \mathbb{R}^{n-k} \times C^k \subset \mathbb{R}^{n+1}$, $k \geq 7$. When $\Sigma_{C^k} = \{0\}$ the blow-up process terminates.

From [L4, Theorem 3] we know that $\Psi$ induces the unique# solution $\Psi_\circ = \Psi_\circ[C^n]$ of $(L_{C^n})\phi = 0$ on $C^n \setminus \Sigma_C$ with minimal growth towards $\Sigma_C$. This uniqueness implies that $\Psi_\circ$ shares the $\mathbb{R}^{n-k}$-translation symmetry with the underlying space $\mathbb{R}^{n-k} \times C^k$. This also means that $(\mathbb{R}^{n-k} \times C^k, d_S)$ is again a cone invariant under $\mathbb{R}^{n-k}$-translation and this means that $\Psi_\circ|_{\{0\} \times C^k}$ satisfies the equation $(L_{C^k,n})\Psi_\circ(\omega, r)|_{\{0\} \times C^k} = 0$ where we have set

$$L_{C^k,n} := -\Delta - \frac{n-2}{(n-1)} \cdot |A|^2$$

for $n \geq k$ and $(L_{C^k,n})\lambda = -(L_{C^k,n})\lambda = L_{C^k,n} - \lambda \cdot \langle A \rangle^2$.

$L_{C^k,n}$ is a dimensionally shifted conformal Laplacian on $C^k$. $\Delta$, $|A|^2 = -\text{scal}$ and $\langle A \rangle^2$ are defined relative to $C^k$. The dimension shift comes from using $\frac{n-2}{(n-1)}$ in place of $k-2$.

From $\langle A \rangle(\omega, \rho) = \langle A \rangle(\omega, 1) \cdot \rho^{-1}$ for $x = (\omega, \rho) \in S_{C^k} \setminus \Sigma_{C^k} \times \mathbb{R}^>0 \cong C^k \setminus \{0\}$, where $S_{C^k} := \partial B_1(0) \cap C^k \setminus \Sigma_C$, we have

$$\langle A \rangle(\omega, 1) = \langle A \rangle(\omega, 1) \cdot \rho^{-1} \cdot |A|^2(\omega, 1) = \lambda \cdot \langle A \rangle^2(\omega, 1) \cdot |A|^2$$

We recall [L4, Prop. 4.6]:

**Proposition 1.2 (Growth Estimates)** Let $C^k \subset \mathbb{R}^{k+1}$ be a singular area minimizing cone and $n \geq k \geq 7$ where $C^k$ can also be singular outside $\{0\}$. Then there is a constant $\lambda_k^* > 0$ such that for any $n \geq k$ and $\lambda \in (0, \lambda_k^*]$ there are two distinguished solutions of $(L_{C^k,n})\phi = 0$:

$$(L_{C^k,n})\lambda = \lambda_\circ[C^n](\omega, \rho) = \psi(\omega) \cdot \rho^{\alpha_\circ}$$

and $\lambda_\star[C^n](\omega, \rho) = \psi(\omega) \cdot \rho^{\alpha_\star}$, for $x = (\omega, \rho) \in S_{C^k} \times \mathbb{R}^>0$

$\Psi_\circ$ and $\Psi_\star$ are the unique# solutions with minimal growth towards $\Sigma_{C^k}$ and $\Sigma_{C^k} \setminus \{0\}$ and we have some $\vartheta_\lambda^* > 0$ depending only on $\lambda$ and $k$, but not on $n$, so that

$$(\lambda^* - (k-2)) \cdot \vartheta_\lambda^* > \alpha_\circ \geq -(1 - \sqrt{\frac{2}{3}}) \cdot \frac{k-2}{2} > -\frac{k-2}{2} > -(1 + \sqrt{\frac{2}{3}}) \cdot \frac{k-2}{2} \geq \alpha_\star > \vartheta_\lambda^* - (k-2)$$

We interpret $\Psi_\circ$, $\Psi_\star$ as solutions $\Psi_\circ[R^{n-k} \times C^k](x, y) = \Psi_\circ(x)$ and $\Psi_\star[R^{n-k} \times C^k](x, y) = \Psi_\star(x)$ of $(L_{R^{n-k} \times C^k})\lambda = 0$ for $(x, y) \in R^{n-k} \times C^k \setminus \Sigma_{R^{n-k} \times C^k}$.

The spherical component $\psi(\omega)$ and the radial growth rates $\alpha_\circ$ of $\Psi_\circ$ and $\alpha_\star$ of $\Psi_\star$ in (7) are related through a spherical eigenvalue equation [L4, Theorem 4.4] to the (non-weighted) principal eigenvalue $\mu_{C^k, L_\lambda^*}$ of the associated operator $L_\lambda^*$ on $S_{C^k}$ for $\psi(\omega) > 0$ with $L_\lambda^* \psi = \mu_{C^k, L_\lambda^*} \psi$, on $S_{C^k} \setminus \Sigma_{S_{C^k}}$. In particular we have $\alpha_\circ + \alpha_\star = -(k-2)$.

We also recall from [L4, Lemma 3.9 and Theorem 4.5] that for any tangent cone $C$ in some singular point of $H$, we have $\lambda_H(A) \leq \lambda_C(A)$ and there is a constant $\lambda_k^* > 0$ so that $\lambda_C(A) \geq \lambda_k^*$ for any singular area minimizing cone $C^k$. This suggests the following choice for a subcritical
eigenvalue that respects all constraints and yields the uniform growth estimates of (8). Let $H^n \in G^c_n$ be a singular almost minimizer so that $(\lambda)^{-2} \cdot L_H$ has principal eigenvalue $\lambda^*_H > 0$. Then we define our **standard eigenvalue**

$$\lambda^*_H := 1/2 \cdot \min \{\lambda^*_H, \lambda^*_7, \ldots, \lambda^*_n, \lambda^*_7, \ldots, \lambda^*_n\} > 0.$$  

For the **rest of this paper**, on a given $H^n$, we choose $\lambda := \lambda^*_H > 0$ and a fixed supersolution of $(L_{H^n})_\lambda \phi = 0$ as in Def. 1.1, also written as $\Psi_\phi$, and we use this $\lambda$ and the induced unique solutions $\Psi_\phi$ of $(L_{C^n})_\lambda \phi = 0$ on $C^n \setminus \Sigma_C$ on any of its (iterated) tangent cones $C^n$. Since we keep these choices unchanged, we generally do not mention $\lambda$ and $\Psi_\phi$ explicitly.

**Controlled Bumps** The metric completion $(H, d_H)$ of $(H \setminus \Sigma, \Psi_o^{(n-2)} \cdot g_H)$ can be augmented to a metric measure space $(H, d_S, \mu_S)$. As indicated in Ch. 1.2 above, our plan is to add disjointly supported local bump deformations to $(H, d_S)$. This means that all bump deformations in this paper remain **$\Lambda$-equivalent** to $(H, d_S)$, in the sense of **$\Lambda$-bumped spaces** in the following result. From this, many of the results for $(H, d_S, \mu_S)$, in [L1], remain valid for the bumped cases. We define, for some fixed bumping constant $\Lambda \geq 1$, the class of admissible $\Lambda$-bounded conformal deformation functions

$$C[H^n, \Lambda] := \{ f : H^n \to [1, \Lambda^{(n-2)/2}] \mid f \in C^0(H^n) \cap C^{2,\alpha}(H^n \setminus \Sigma_H), \text{ for any } \alpha \in (0, 1) \}. $$

In the following we simplify the notations and only use the index $\Lambda$ for quantities actually depending on $G \in C[H, \Lambda]$ since the estimates only depend on $\Lambda$.

**Theorem 1.3 (\Lambda-Bumped Spaces)** For $\Lambda \geq 1$, $H^n \in G_n$, and $G \in C[H, \Lambda]$ we have:

- $(G \cdot \Psi_o)^{4/(n-2)} : h_H$ on $H \setminus \Sigma$ can be completed to a geodesic metric space $(X^n, d_X)$. The space $(X^n, d_X)$ is homeomorphic to $(H, d_H)$ with singular set $\Sigma_X \equiv \Sigma_H$ of Hausdorff dimension $\leq n - 7$ relative to $(X^n, d_X)$. From this we can write $(X^n, d_X)$ as $(H, d_S)$.

- The volume $(G \cdot \Psi_o)^{2n/(n-2)} : \mu_H$ on $H \setminus \Sigma_H$ extends to a Borel measure $\mu_S^\Lambda (E)$ defined by $\int_{E \setminus \Sigma_H} (G \cdot \Psi_o)^{2n/(n-2)} : d\mu_H$, for any Borel set $E \subset H$, and we get the metric measure space $(H, d_S^\Lambda, \mu_S^\Lambda)$ and we call it a **$\Lambda$-bumped space** originating from $(H, d_S, \mu_S)$.

- $(H, d_S^\Lambda, \mu_S^\Lambda)$ is **Ahlfors $n$-regular**: there are constants $A, B(H, \Psi_o, \Lambda) > 0$ so that:

$$A \cdot r^n \leq \mu_S^\Lambda(B_r(q), d_S^\Lambda) \leq B \cdot r^n, \text{ for any } r \in [0, \text{diam}(H, d_S^\Lambda)) \text{ and any } q \in H.$$

For $H \in H_n^R$ the constants only depend on the dimension and $\Lambda$.

- For any $H \in G$, there is constant $C_0(H, \Psi_o, \Lambda) > 0$, depending only on $n$ and $\Lambda$ for $H \in H_n^R$, so that for any ball $B \subset (H, d_S)$ we have a **Poincaré inequality**: for any function $u$ on $H$, integrable on bounded balls, and any upper gradient $w$ of $u$:

$$\int_B |u - u_B| \cdot d\mu_S^\Lambda \leq C_0 \cdot \text{diam}(B) \cdot \int_B w \cdot d\mu_S^\Lambda.$$

The assertions readily follow from their counterparts for minimal factor geometries in [L1] and the $\Lambda$-equivalence of the original $(H, d_S, \mu_S)$ and the bumped spaces $(H, d_S^\Lambda, \mu_S^\Lambda)$. From (11) and (12) we also get an isoperimetric inequality $(H, d_S, \mu_S)$ and the following growth estimates. Alternatively, one can use the $\Lambda$-equivalence of the metrics to get such applications from the estimates we have for $(H, d_S, \mu_S)$ from [L1, Cor. 3.16 and Prop. 3.17].
Corollary 1.4 For \((H, d_S^\Lambda, \mu_S^\Lambda)\), some open subset \(\Omega \subset H\) and an oriented minimal boundary \(L^{n-1} \subset \Omega\) bounding an open set \(L^+ \subset \Omega\), there are \(\kappa, \kappa^+ (H, \Psi_o, \Lambda) > 0\), so that for \(p \in L\):
\[
\kappa \cdot r_{n-1} \leq \mu_S^\Lambda(L \cap B_r(p)) \quad \text{and} \quad \kappa^+ \cdot r_{n-1} \leq \mu_S^\Lambda(L^+ \cap B_r(p)),
\]
for \(r \in [0, (A/B)^{1/n} \cdot \text{dist}(p, \partial \Omega)/4]\) and where \(0 < A < B\) are the Ahlfors constants. For \(H \in \mathcal{H}_r^n, \kappa, \kappa^+ > 0\) depend only on \(n\) and \(\Lambda\).

From (11) and (12) the BV-theory of Ambrosio [A] and Miranda [M] applies to \((H, d_S^\Lambda, \mu_S^\Lambda)\). The coarea formula [M, Prop. 4.2] and the Ahlfors \(n\)-regularity yield growth estimates for \(\mu_S^\Lambda(\partial B_r(q))\) in terms of a density result for the Lebesgue measure \(\mu_1\) on \(\mathbb{R}\), cf. [M, Ch. 5.2]:

Corollary 1.5 For any \(q \in H\), we define the set of radii where \(\mu_S^\Lambda(\partial B_r(q))\) exceeds \(c \cdot r_{n-1}\) by \(J(q, a, b, c) := \{r \in [a, b] \mid \mu_S^\Lambda(\partial B_r(q)) > c \cdot r_{n-1}\}, 0 \leq a < b\) and \(c > 0\). We have
\[
\mu_1(J(q, a, b, c)) \leq (B \cdot b - A \cdot a + (n-1) \cdot B \cdot (b-a))/c.
\]
That is, for large \(c > 0\) it is increasingly likely that \(\mu_S^\Lambda(\partial B_r(q)) \leq c \cdot r_{n-1}\) since, choosing a small \(\varepsilon > 0\) and \(c = n \cdot B/\varepsilon\), we have \(\mu_1(J(q, 0, b, c)) \leq \varepsilon \cdot b = \varepsilon \cdot \mu_1([0, b])\). Using balls with radii \(\notin J(q, 0, b, c)\) we may exploit the low Hausdorff dimension of \(\Sigma \subset (H, d_S^\Lambda)\) to get:

Corollary 1.6 For \(H^n \in \mathcal{G}_n\) and any \(\varepsilon > 0\) and \(R > 0\) there is a locally finite cover of \(\Sigma_H\) by balls \(B_{r_i}(p_i), p_i \in \Sigma_H, r_i \in (0, R), i \in I,\) in \((H, d_S^\Lambda, \mu_S^\Lambda)\) so that for \(c \geq 2 \cdot n \cdot B\):
\[
\sum_{i \in I} \mu_S^\Lambda(\partial B_{r_i}(p_i)) \leq c \cdot \sum_{i \in I} r_i^{n-1} < \varepsilon.
\]
Similarly, 1.5 shows the otherwise non-obvious finiteness of the perimeter when we look for area minimizers in \((H, d_S^\Lambda, \mu_S^\Lambda)\). The lower semi-continuity of perimeters [M, Prop. 3.6] and the compactness of the BV-function space in the \(L^1\)-function space [M, Prop. 3.7] yield the existence of oriented minimal boundaries in \((H, d_S^\Lambda, \mu_S^\Lambda)\) as in [Gi, Theorem 1.20].

Proposition 1.7 (Plateau Problems in \((H, d_S^\Lambda, \mu_S^\Lambda)\)) Let \(\Omega \subset H\) be a bounded open and orientable set and let \(A \subset H\) be a possibly empty \textbf{Caccioppoli set}, i.e., a set of locally finite perimeter. Then there exists a set \(E \subset H\) coinciding with \(A\) outside \(\Omega\) and such that
\[
\mu_S^\Lambda(\partial E \cap \Omega) \leq \mu_S^\Lambda(\partial F \cap \Omega)
\]
for any Borel set \(F \subset H\) with \(F = A\) outside \(\Omega\). We say that \(E\) has \(\Omega\)-boundary value \(A\), and for open \(E \cap \Omega\), we call \(\partial E\) a \textbf{Plateau solution} with \(\Omega\)-boundary value \(\partial A\).

2 Constructions on Cones

In this chapter we construct local bumps on product cones. We start with a more detailed description of the key steps of these deformations and their intended use.

(1) Product Bumps In Ch. 2.1 we are starting from some singular cone \(C^n = \mathbb{R}^{n-k} \times C^k\) with its scal \(> 0\)-minimal factor metric \(\Psi_0^{4/(n-2)} \cdot g_{C^k}\). We emphasize that \(C^k\) can also be singular outside \(\{0\}\). We make an elementary deformation \((F \cdot \Psi_o)^{4/(n-2)} \cdot g_{C^n}, \) with \(F(x, y) = f(|y|),\) for some smooth \(f \geq 1\) with \(\supp f = [r-1] \subset [r, 1]\), for some \(1 > r > 0\), \((F \cdot \Psi_o)^{4/(n-2)} \cdot g_{C^n}\) keeps scal \(> 0\) and it turns \(\mathbb{R}^{n-k} \times \partial B_\rho(0),\) for some \(\rho \in (r, 1)\), into a local area minimizer. \((F \cdot \Psi_o)^{4/(n-2)} \cdot g_{C^n}\) is the source material for all further steps where we trim and rearrange this deformation to build larger bump structures.
(2) Regular Support For the first trimming process, in Ch. 2.2, we define a cut-off function 
\( \phi_\beta : C \setminus \Sigma_C \to [0,1] \) with \( \phi_\beta = 0 \) on \( \delta^{-1}_{(A)} (\mathbb{R}^{\leq \beta}) \) and \( \phi_\beta = 1 \) on \( \delta^{-1}_{(A)} (\mathbb{R}^{\geq \beta}) \), for some \( \delta(\beta) \to 0 \), for \( \beta \to 0 \). We set \( F_\beta = \phi_\beta \cdot (F - 1) + 1 \) for usage in the new metric 
\( (F_\beta \cdot \Psi_\circ)^{4/(n-2)} \cdot g_{C^n} \) and observe that \( \text{supp} [F_\beta - 1] \cap \Sigma_C = \emptyset \). We think of \( \delta^{-1}_{(A)} (\mathbb{R}^{\leq \beta}) \), where we have trimmed \( F \) to zero, as a tunnel that surrounds the singularities. This tunnel passes (and removes) part of the support of \( (F \cdot \Psi_\circ)^{4/(n-2)} \cdot g_{C^n} \) reaching from \( \mathbb{R}^{n-k} \times \partial B_1(0) \) to \( \mathbb{R}^{n-k} \times \partial B_2(0) \), cf. Fig. 3. We choose \( \delta_{(A)} \) in place of the metric distance since it commutes with the convergence of the underlying spaces as needed to transfer the bumps between different spaces and it gives us a sufficiently good control to carry this out keeping scal \( (F_\beta \cdot \Psi_\circ)^{4/(n-2)} \cdot g_{C^n} > 0 \).

(3) Tightness of Regular Bumps For \( (F_\beta \cdot \Psi_\circ)^{4/(n-2)} \cdot g_{C^n} \) the boundary \( \mathbb{R}^{n-k} \times \partial B_\delta(0) \) is no longer minimal, but in Ch. 2.3 we show that for small \( \beta > 0 \), there is an area minimizer \( T_\beta \), in this metric which is Hausdorff close to \( \mathbb{R}^{n-k} \times \partial B_\delta(0) \). In particular, \( T_\beta \cap \delta^{-1}_{(A)} (\mathbb{R}^{\leq \beta}) \) remains close to \( \delta^{-1}_{(A)} (\mathbb{R}^{\leq \beta}) \cap \mathbb{R}^{n-k} \times \partial B_\delta(0) \) and we will see that this is true for any area minimizer relative to \( (F_\beta \cdot \Psi_\circ)^{4/(n-2)} \cdot g_{C^n} \) within a certain threshold distance from \( \mathbb{R}^{n-k} \times \partial B_\delta(0) \). In this sense, the tunnel is tight and \( (F_\beta \cdot \Psi_\circ)^{4/(n-2)} \cdot g_{C^n} \) still shields a neighborhood of \( \mathbb{R}^{n-k} \times \{0\} \) against intrusions of these area minimizers.

(4) Compact Support In Ch. 2.4 we apply an additional cut-off, now along the \( \mathbb{R}^{n-k} \)-axis, to \( (F_\beta \cdot \Psi_\circ)^{4/(n-2)} \cdot g_{C^n} \), to make the support compact, cf. the right image of Fig. 2. We call the resulting deformation of \( \Psi_\circ^{4/(n-2)} \cdot g_{C^n} \) an \((n-k)\)-bump element and they are the elementary particles of our bump construction. In this step we can preserve scal > 0, but it is difficult to prevent minimizers from shrinking along the \( \mathbb{R}^{n-k} \)-axis.

(5) Tight Configurations To resolve this issue, we inductively place \((m)\)-bump elements along the \((m)\)-dimensional faces of the unit cube \( Q_1^{n-k} \times \{0\} \subset \mathbb{R}^{n-k} \times C^k \). (We get an \((m)\)-bump element writing \( C^m = \mathbb{R}^m \times (\mathbb{R}^{n-k-m} \times C^k) \).) In this process, the cut-off ends of \((m)\)-elements belong to subsets shielded from \((q)\)-elements for \( q < m \), cf. Fig. 5. The tightness (3) ensures that these bump configurations shield \( Q_1^{n-k} \) against intrusion of area minimizers. We call them local bumps. They deform \( \Psi_\circ^{4/(n-2)} \cdot g_{C^n} \) to scal > 0-metrics containing open sets \( U \supset Q_1^{n-k} \times \{0\} \) with locally minimal boundary \( \partial U \).

(6) Stable Shielding The bump elements in local bumps (and in all larger configurations we build until we reach global bumps) have disjoint supports, but the generated minimizing boundaries intersect the support of other bump elements and they change whenever we add further bump elements. In general, any such addition produces several new locally minimizing boundaries. Some of them jump far away and they become useless for our purposes, but the central tightness result in Ch. 2.3 also shows that there is always at least one minimizing boundary only slightly perturbed compared to the initial one.

2.1 Deformations on Product Cones

The main building blocks for our deformations are blends of the two extremal metrics associated to \( \Psi_\circ \) and \( \Psi_\bullet \) on \( \mathbb{R}^{n-k} \times C^k \):

\[
g^\circ_{C^n} := \Psi_\circ^{4/(n-2)} \cdot g_{C^n} \quad \text{and} \quad g^\bullet_{C^n} := \Psi_\bullet^{4/(n-2)} \cdot g_{C^n}.
\]

We compute the mean curvature of distance tubes of \( \mathbb{R}^{n-k} \times \{0\} \) relative to these two metrics.
Lemma 2.1 (Bending Effects) We consider \( u = f(\omega) \cdot r^\beta \), for \( \omega \in S_{C^k} := \partial B_1(0) \cap C^k \), \( r(x) := \text{dist}(x, \mathbb{R}^{n-k} \times \{0\}) \), for \( x \in \mathbb{R}^{n-k} \times C^k \) and some positive \( f \in C_{\text{loc}}^2(\mathbb{S}^k \setminus \Sigma S_{C^k}, \mathbb{R}) \). The mean curvature \( tr A_T^e \) in regular points of the boundary \( T \) of distance tubes \( \mathbb{R}^{n-k} \times B_{\rho}(0) \cap C^k \subset C^n \) of \( \mathbb{R}^{n-k} \times \{0\} \), of radius \( \rho > 0 \) relative to \( g_{C^n} \), is given by

\[
(18) \quad u^{4/(n-2)} \cdot tr A_T^e(u^{4/(n-2)} \cdot g_{C^n}) = -((k-1) + 2 \cdot (n-1) \cdot \beta/(n-2)) \cdot \rho^{-1}.
\]

In particular, for the two scal \( > 0 \)-metrics \( g^0_{C^n} \) and \( g^\bullet_{C^n} \) on \( C^n \setminus \Sigma C^n \), we get for any \( \rho > 0 \):

\[
tr A_T^e(g^0_{C^n}) < 0 \quad \text{and} \quad tr A_T^e(g^\bullet_{C^n}) > 0.
\]

For the superposition, we get some \( \rho_0(\alpha, n, k) > 0 \) which continuously depends on \( \alpha \) so that

\[
(19) \quad tr A_T^e((\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot g_{C^n}) > 0 \quad \text{for} \quad \rho < \rho_0, = 0 \quad \text{for} \quad \rho = \rho_0 \quad \text{and} \quad < 0 \quad \text{for} \quad \rho > \rho_0.
\]

\( T_{\rho_0} \) is an area minimizer in the metric measure space associated to the completion of \( (C^n \setminus \Sigma C^n, (\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot g_{C^n}) \). For any bounded open set \( \Omega \subset C^n \), \( T_{\rho_0} \cap \Omega \) is the unique oriented minimal boundary with \( \Omega \)-boundary value \( T_{\rho_0} \) bounding an open set containing \( \mathbb{R}^{n-k} \times \{0\} \).

**Proof** The second fundamental form \( A_L(g) \) of a submanifold \( L \) with respect to a metric \( g \) transforms under conformal deformations \( g \to u^{4/(n-2)} \cdot g \) as follows [Be, 1.163, p. 60]:

\[
(20) \quad A_L(u^{4/(n-2)} \cdot g)(v, w) = A_L(g)(v, w) - \frac{2}{n-2} \cdot \mathcal{N}(\nabla u/u) \cdot g(v, w),
\]

where \( \mathcal{N}(\nabla u/u) \) is the normal component of \( \nabla u/u \). \( C^k \subset \mathbb{R}^{k+1} \) is a cone and \( \mathbb{R}^{n-k} \times \{z\} \subset T_\rho \), \( z \in C^k \), is totally geodesic. Thus we have \( tr A_T^e(g) = -(k-1)/\rho \). The trace of the second summand is \( 2/(n-2) \cdot \beta/\rho \) multiplied by \( n-1 \). This yields:

\[
(21) \quad (\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot tr A_T^e((\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot g_{C^n}) =
\]

\[
(22) \quad -((k-1) + 2 \cdot (n-1) \cdot \alpha_o/(n-2)) \cdot \rho^{-1} < -((k-2)/2 \cdot \rho^{-1} < 0,
\]

since \( k \geq 7 \) and, in particular, \( k \geq 3 \). For \( (\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot g_{C^n} \) we get

\[
(21) \quad (\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot tr A_T^e((\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot g_{C^n}) =
\]

\[
(22) \quad -((k-1)/\rho + 2(n-1)/(n-2) \cdot (\alpha_o \cdot g^{\alpha_o-1} + \alpha^\bullet \cdot g^{\alpha^\bullet-1}))/g^{\alpha_o} + g^\bullet).
\]

The mean curvature is constant on each \( T_\rho \). To understand its sign, we recall that

- \( \alpha_o + \alpha^\bullet = -(k-2) \) and \( 2 \cdot \alpha_o + 1 + (k-2) > 0 \),

- \( |2(n-1)/(n-2) \cdot \alpha_o| < k-1 \) and \( |2(n-1)/(n-2) \cdot \alpha^\bullet| > k-1 \).

We multiply (22) by \( (g_{\alpha_o} + g^\bullet) \) and \( g^{\alpha_o+1+(k-2)} \) and get:

\[
-((k-1) + 2(n-1)/(n-2) \cdot \alpha_o) \cdot g^{2\alpha_o+1+(k-2)} - ((k-1) + 2(n-1)/(n-2) \cdot \alpha^\bullet).
\]

From this we see that there is a unique \( \rho_0(\alpha, n, k) > 0 \) with

\[
tr A_T^e((\Psi_o + \Psi^\bullet)^{4/(n-2)} \cdot g_{C^n}) > 0 \quad \text{for} \quad \rho < \rho_0, = 0 \quad \text{for} \quad \rho = \rho_0 \quad \text{and} \quad < 0 \quad \text{for} \quad \rho > \rho_0.
\]

Hence, the radial projection in the \( C^k \)-factor \( \pi_{s, \rho} \) : \( T_s \to T_{\rho_0} \) strictly decreases the hypersurface area element for any \( \rho \not= \rho_0 \). From this we see that \( T_{\rho_0} \) is the unique oriented minimal
boundary with boundary value $T_{60}$ bounding an open set containing $\mathbb{R}^{n-k} \times \{0\}$. This carries over to the metric completion of $(C^n \setminus \Sigma_C, (\Psi_0 + \Psi_*)^{(n-2)/2} \cdot g_{C^n})$ since the Hausdorff dimension of $\Sigma_C \setminus \mathbb{R}^{n-k} \times \{0\}$ is still $\leq n-7$ from Theorem 1.3. \hfill \Box

We use these estimates in our basic bump deformation of the minimal factor geometry $(C^n, d_S)$. The construction uses two particularities of $(C^n, d_S)$ while we approach $\mathbb{R}^{n-k} \times \{0\}$.

- A first transition from $\Psi_0$ to $\Psi_*$ uses, in the guise of $tr A_{\gamma}(g_{C^n}) > 0$, that the codimension of the singular set is $\geq 3$ matching the condition in the scal $> 0$-preserving surgeries of [GL2], [SY]. This deformation yields a local area minimizer $T_{\gamma}$ with $T_{\gamma} \cap \mathbb{R}^{n-k} \times \{0\}$.

- Then we append a second transition back from $\Psi_*$ to $\Psi_0$ and return to the geometry $(C^n, d_S)$ near $\mathbb{R}^{n-k} \times \{0\} \subset \Sigma_C$. This step has no classical counterpart in the mentioned scal $> 0$-preserving surgeries. It uses the scaling invariance of the scal $> 0$-cone $(C, d_S)$.

**Proposition 2.2 (Basic Deformations)** For any singular cone $C^n = \mathbb{R}^{n-k} \times C^k \in \mathcal{SC}_n$ there are a smooth function $f[C] : \mathbb{R}^2 \to \mathbb{R} \geq 1$ and a core radius $r[n] \in (0, 1/100)$, such that for $z = (x, y) \in C^n \setminus \Sigma_{C^n} = \mathbb{R}^{n-k} \times (C^k \setminus \Sigma_C)$ and $F(x, y) := f(|y|)$, the metric

$$g_{C^n}^{n-k} := (F \cdot \Psi_0)^{(n-2)/2} \cdot g_{C^n} = (f(|y|) \cdot \Psi_0(x, y))^{(n-2)/2} \cdot g_{C^n}$$

has the following properties:

1. $F = f[C]$ depends continuously on $C \in \mathcal{SC}_n$: for a converging sequence $C_i \to C$ in $\mathcal{SC}$ we have a compact $C^2_k$-convergence $F[C_i] \circ \text{Id} \to F[C]$ on $C \setminus \Sigma_C$.
2. $\text{supp} \{f - 1\} \subset [r, 1], f \leq \Lambda^{(n-2)/2}$, for some $\Lambda |n| \geq 1$, i.e. $F \in \mathcal{C}[H, \Lambda |n|]$.
3. $L_{C^n}(F \cdot \Psi_0) \geq \lambda/4 \cdot \langle 2 \cdot F \cdot \Psi_0 \rangle$ and, in particular, $\text{scal}(g_{C^n}^{n-k}) > 0$,
4. there are constants $\theta_i[C] \in (r, 1), i = 1, 2$, with $\theta_1 < \theta_2$ and some $\varphi \in (\theta_1, \theta_2)$, all depending continuously on $C$, so that $T_{\gamma} = \mathbb{R}^{n-k} \times \partial B_{\varphi}(0)$ is a local area minimizer. For any bounded open set $\Omega \subset C^n, T_{\gamma} \cap \Omega$ is the unique oriented minimal boundary of some Caccioppoli set in $\mathbb{R}^{n-k} \times \overline{B_{\theta_2}(0)} \setminus B_{\theta_1}(0)$ with $\Omega$-boundary value $T_{\gamma}$.

We call $\Delta_{n,k}^{n-k} := (F - 1 \cdot \Psi_0)^{(n-2)/2} \cdot g_{C^n}$ the associated bump pseudo metric and $\mathbb{R}^{n-k} \times \overline{B_{r[n]}(0)} \subset \mathbb{R}^{n-k} \times C^k$ its core.

**Proof** We deform $\Psi_0$ on $C$ consecutively in 3 steps while we gradually approach the axis $\mathbb{R}^{n-k} \times \{0\}$. To simplify the definition of the deformations we work also with large radii, but since the setup is scaling invariant we can finally rescale the steps to compose them seamlessly.

**Step 1 ($\Psi_0 \to \Psi_*$) We first pass from $\Psi_0$ to $\Psi_0 + \Psi_*$. We choose a cut-off $\phi \in C^\infty(\mathbb{R}, [0, 1])$ with $\phi \equiv 1$ on $\mathbb{R}^{\leq 0}, \phi \equiv 0$ on $\mathbb{R}^{\geq 1}$ and $\phi' \leq 0$. For large $D > 0$, we set $h_D(z) := \phi_D(\rho) := \phi(\rho - D)$, where $\rho = \text{dist}(z, \mathbb{R}^{n-k} \times \{0\})$ and get (usually omitting to write the coordinate $z$)

$$\begin{align*}
(\Psi_0(z) + h_D(z) \cdot \Psi_*(z))^{(n+2)/(n-2)} \cdot \text{scal} \left( (\Psi_0(z) + h_D(z) \cdot \Psi_*(z))^{(n-2)/2} \cdot g_{C^n} \right) &= \\
- (\Delta \Psi_0 + h_D \cdot \Delta \Psi_*) + \left( \frac{(n-2)}{4(n-1)} \right) \cdot \text{scal}(g_{C^n}) \cdot (\Psi_0 + h_D \cdot \Psi_*) - (\Delta h_D \cdot \Psi_0) - (\Delta h_D \cdot \Psi_*) + 2 \cdot \langle \nabla h_D, \nabla \Psi_* \rangle \ \\
&= \lambda \cdot \langle A \rangle^2 \cdot (\Psi_0 + h_D \cdot \Psi_*) - (\Delta h_D \cdot \Psi_0) + 2 \cdot \langle \nabla h_D, \nabla \Psi_* \rangle
\end{align*}
$$

(24)
(25) \[ \geq \psi_C(\omega) \cdot \left( \lambda \cdot \rho^{-2} \cdot (\rho^{\alpha_0} + \phi_D \cdot \rho^{\alpha_{**}}) - (\phi_D'' + (k-1) / \rho \cdot \phi_D') \cdot \rho^{\alpha_{**}} - 2 \cdot \alpha_\star \cdot \phi_D' \cdot \rho^{\alpha_{**}-1} \right), \]

from \( \langle A \rangle(z) \geq 1 / \text{dist}(z, \Sigma) \) and \(|\phi_D'|, |\phi_D''| \leq c(\phi) \), for some \( c(\phi) > 0 \), independent of \( D \). For large \( D(n, k, \alpha_0, \alpha_\star) > 0 \), independent of the chosen cone the expression (25) is positive, since \( \lambda > 0 \), \( \alpha_0 - \alpha_\star \geq 2 \cdot \sqrt{2/3} \cdot \frac{\alpha_0}{2} \geq 3 \) and, hence, \( \rho^{\alpha_{**}-2} / \rho^{\alpha_0} \rightarrow \infty \), for \( \rho \rightarrow \infty \).

Now we pass from \( \Psi_\omega + \Psi_\star \) to \( \Psi_\star \). For \( E > 0 \), \( h_E(z) := \phi_E(\rho) := 1 - \phi(\rho / E - 1) \) we have

\[ (h_E(z) \cdot \Psi_\omega(z) + \Psi_\star(z))^{(n+2)/(n-2)} \cdot \text{scal} \left( (h_E(z) \cdot \Psi_\omega(z) + \Psi_\star(z))^{4/(n-2)} \cdot g_{C^n} \right) \geq \]

(26) \[ \psi_C(\omega) \cdot \left( \lambda \cdot \rho^{-2} \cdot (\phi_E \cdot \rho^{\alpha_0} + \rho^{\alpha_{**}}) - (\phi_E'' + (k-1) / \rho \cdot \phi_E') \cdot \rho^{\alpha_0} - 2 \cdot \alpha_0 \cdot \phi_E' \cdot \rho^{\alpha_0-1} \right). \]

For small \( E(n, k, \alpha_0, \alpha_\star) > 0 \) this is positive. Namely, \( |\phi_E'| \cdot E, |\phi_E''| \cdot E^2 \leq c^*(\phi) \), for some \( c^*(\phi) > 0 \) and \( \alpha_0 - \alpha_\star > 0 \) and, hence, for \( \rho \leq E : \rho^{\alpha_{**}-2}/(\rho^{\alpha_0} \cdot E^{-2}) \geq \rho^{\alpha_{**}-2}/\rho^{\alpha_0-2} \rightarrow \infty \), for \( \rho \rightarrow 0 \). Summarizing we have some \( D > 10 \), an \( E \in (0, 1/10) \) and a smooth function \( \Psi > 0 \) so that \( \text{scal}(\Psi^{4/(n-2)} \cdot g_{C^n}) > 0 \) on \( C^n \setminus \Sigma \), with \( \Psi \equiv \Psi_\omega \) for \( \rho \geq D + 1 \) and \( \Psi \equiv \Psi_\star \) for \( \rho \leq E \). We observe that \( D \) and \( E \) can be chosen depending continuously on \( \alpha_0, \alpha_\star \) and, hence, on \( C \).

**Step 2** (\( \Psi_\star \mapsto c \cdot \Psi_\omega \)) Now we pass from \( \Psi_\star \) back to \( c \cdot \Psi_\omega \), for some \( c > 0 \). Since

- \( \psi_C(\omega) \cdot \rho^{\alpha_{**}} \) and \( c \cdot \psi_C(\omega) \cdot \rho^{\alpha_0} \) solve \( L_{(n)} C^n \phi = \lambda \cdot \langle A \rangle^2 \cdot \phi \)
- \( (E/8)^{\alpha_{**}-\alpha_0} \cdot \rho^{\alpha_0}|_{\rho = E} = \rho^{\alpha_0}|_{\rho = E} \) and \( (E/8)^{\alpha_{**}-\alpha_0} \cdot (\rho^{\alpha_0})'|_{\rho = E} > (\rho^{\alpha_0})'|_{\rho = E} \),

we can find a smooth interpolation \( F > 0 \), continuously depending on \( \alpha_0, \alpha_\star \), such that \( F \leq \min\{(E/8)^{\alpha_{**}-\alpha_0} \cdot \rho^{\alpha_0}, \rho^{\alpha_{**}}\} \), \( F(\rho) = E^{\alpha_{**}-\alpha_0} \cdot \rho^{\alpha_0} \) for \( \rho < 99/100 \cdot (E/8), F(\rho) = \rho^{\alpha_{**}} \) for \( \rho > 101/100 \cdot (E/8), \)

\[ L_{(n)} C^n (\psi_C(\omega) \cdot F(r)) = -\psi_C(\omega) \cdot (F'' + (k-1) / \rho \cdot F') + L_{\lambda} \psi(\omega) \cdot F(r) = \]

(27) \[ \psi_C(\omega) \cdot \left( -F'' + (k-1) / \rho \cdot F' + \mu_{C^n, L_{\lambda}} \cdot F \right) \geq \lambda / 2 \cdot \langle A \rangle^2 \cdot \psi_C(\omega) \cdot F(r). \]

**Step 3** (\( c \cdot \Psi_\omega \mapsto \Psi_\omega \)) In Step 2 we returned to \( E^{\alpha_{**}-\alpha_0} \cdot \Psi_\omega \). To reach \( \Psi_\omega \), we now show how to pass from \( c \cdot \Psi_\omega \), for \( c := E^{\alpha_{**}-\alpha_0} > 1 \), to \( \Psi_\omega \), while keeping \( \text{scal} > 0 \). For \( d \in \mathbb{Z} \), we set \( h_d^*(z) := \phi_d^*(\rho) := \phi(2^d \cdot (\rho - 2^{-d})) \). For \( \zeta \in (-1/2, 0) \), we have:

\[ ((1 + \zeta \cdot h_d^*) \cdot \Psi_\omega)^{(n+2)/(n-2)} \cdot \text{scal} \left( ((1 + \zeta \cdot h_d^*) \cdot \Psi_\omega)^{4/(n-2)} \cdot g_{C^n} \right) \]

(28) \[ = \lambda \cdot \langle A \rangle^2 \cdot (1 + \zeta \cdot h_d^*) \cdot \Psi_\omega - \zeta \cdot (\Delta h_d^* \cdot \Psi_\omega + 2 \cdot \langle \nabla h_d^*, \nabla \Psi_\omega \rangle) \]

Now we write \( \Psi_\omega(\omega, \rho) = \psi(\omega) \cdot \rho^{\alpha_0} \) and recall that \( \langle A \rangle(\omega, \rho) = \langle A \rangle(\omega, 1) \cdot \rho^{-1} \) and, since \(|\delta_A(p) - \delta_A(q)| \leq d_H(p, q) \), we have \( \langle A \rangle(\omega, 1) \geq 1 \). Thus (28) is lower bounded by

(29) \[ \psi(\omega) \cdot \left( \lambda \cdot \rho^{-2} \cdot (1 + \zeta \cdot \phi_d^*) - |\zeta| \cdot \rho^{\alpha_0} \cdot \phi_d'' + (k-1) / \rho \cdot (\phi_d')' \right) - 2 \cdot |\zeta| \cdot \alpha_0 \cdot \rho \cdot (\phi_d')' \cdot \rho^{\alpha_0}. \]

From (8) we have negative upper and lower bounds on \( \alpha_0 \), independent of \( C \), and, hence, when \(|\zeta| \leq \zeta_0(d) \), for some small \( \zeta_0(d) > 0 \), (28) is positive. \( \zeta_0(d) \) can actually be chosen independently of \( d \), i.e. \( \zeta_0 = \zeta_0(n, k) \). Namely, since \( \Psi_\omega \) is unique#, scaling by \( 2^d \) transforms

(30) \[ (1 + \zeta \cdot h_d^*)^{4/(n-2)} \cdot g_{C^n} \text{ into } (1 + \zeta \cdot h_0^*)^{4/(n-2)} \cdot \Psi_\omega^{4/(n-2)} \cdot g_{C^n}. \]
supp up to multiplication by a positive constant. (More explicitly, the scaling effects $(2^{-d} \cdot \rho)^{-2} = 2^{2d} \cdot \rho^{-2}$, $(\phi^i_d)^{(2^{-d} \cdot \rho^{-1} = 2^{2d} \cdot (\phi^i_d)'(z) \cdot \rho^{-1}$ and $(\phi^i_d)''(2^{-d} \cdot \rho) = 2^{2d} \cdot (\phi^i_d)''(z)$ are all the same.) The interiors of supp $|\nabla h_{d_1}^*|$ and supp $|\nabla h_{d_2}^*|$ are disjoint when $d_1 \neq d_2$ and, hence, we can iteratively multiply by $1 + \zeta \cdot h_\alpha^*$ keeping $\text{scal} > 0$ when $|\zeta| \leq \zeta_0(n, k)$:

$$\text{scal}(\prod_{d=1}^j (1 + \zeta \cdot h_\alpha^*) \cdot \Psi_0)^{4(n-2)} \cdot g_{C^n} > 0, \text{ for any } j \geq 1.$$ 

In turn, we observe that for any fixed $\zeta \in (-1/2, 0)$ and $\eta > 0$ there is some $j > 0$ so that $\prod_{d=1}^j (1 + \zeta \cdot h_\alpha^*) \leq \eta$ on $\mathbb{R}^{2n \cdot j}$. Therefore, there are some $\zeta < 0$ and $j \geq 1$ so that (31) holds, $\prod_{d=1}^j (1 + \zeta \cdot h_\alpha^*) = 1$, for $\rho \geq 1$, and $= 1/c$, for $\rho \leq 2^{-j}$. Since $c = E^{\alpha_\ast - \alpha_0} > 1$ is upper bounded, there is a large $j(i, n, k)$, independent of $C$, so that we can choose $\zeta < 0$ depending continuously on $c$ and thus on $C$.

**Conclusion** We scale the constructions in steps 1-3 individually. In step 1 and 2 we rescale $C^n$ by $(D + 1)^{-1}$ and multiply $\Psi_0$ by $(D + 1)^{-\alpha_0}$. Then the deformation starts from $T_0$. To append step 3, we rescale $C^n$ by $E/10$ and multiply $\Psi_0$ by $(E/10)^{-\alpha_0}$. This defines $F(C) \cdot \Psi_0(C)$ as (i). The various radii and $A$ depend on $n$ and $k$, but since there are only finitely many $k \leq n$ we can choose $k$ depending on $\rho$ choosing common estimates. From the compactness of $SC_n$ we find some lower bound $r(n) > 0$ so that supp $|f(C) - 1| \subset [r, 1] and some upper bound $f \leq A^{(n-2)/2}$ for any $C \in SC_n$ as claimed in (ii). We observe that each step still works, with adjusted parameters and function $F$, where we replace the dominating term $\lambda \cdot \langle A \rangle^2 \cdot \phi$ for $\lambda/2 \cdot \langle A \rangle^2 \cdot \phi$ in (24), (26) and (28). Then we finally add $\lambda/4 \cdot \langle A \rangle^2 \cdot F \cdot \Psi_0$ and get $L_{C^n}(F \cdot \Psi_0) \geq \lambda/4 \cdot \langle A \rangle^2 \cdot F \cdot \Psi_0$ and (iii). (iv) follows from Step 1 and Lemma 2.1.  

2.2 Removal of Intersections with $\Sigma_C$

The **metric distance** $\text{dist} (\cdot, \Sigma)$ has allowed us to find the area minimizer $T_0$ in Prop. 2.2 and we have supp $g_{n-k}^{n-k} \cap \mathbb{R}^n \times \{0\} = \emptyset$. In general, $C^k$ is singular also outside $\{0\}$ and supp $g_{n-k}^{n-k} \cap \mathbb{R}^n \times (\Sigma_C \cap B_1(0) \setminus B_r(0)) \neq \emptyset$. An important case is that of $\Delta_{m,n}$, for $0 \leq m < n - k$, on $\mathbb{R}^{n-k} \times C^k$ considered as $\mathbb{R}^m \times (\mathbb{R}^{n-k-m} \times C^k)$, cf. Fig. 3.

We use the $S$-**distance** $\delta_{(A)} = 1/\langle A \rangle$ to trim such bump pseudo metrics so that their support no longer intersects $\Sigma_C$ while keeping $\text{scal} > 0$. This is a preparation to be able to combine and transfer local bumps to general hypersurfaces. Since $\delta_{(A)}(z)$ is merely Lipschitz regular, we also use its Whitney smoothing $\delta_{(A)}^\ast \cdot \delta_{(A)}^\ast \cdot \delta_{(A)}^\ast \cdot \delta_{(A)}^\ast \cdot \delta_{(A)}^\ast$ has the same coarse geometric properties as $\delta_{(A)}$, cf. [L1, Appendix B]. In normal coordinates $\partial/\partial x_i$, $i = 1, ..., n$, in $z \in C \setminus \Sigma_C$, we have:

$$1/c^\ast \cdot \delta_{(A)}^\ast (z) \leq \delta_{(A)}^\ast (z) \leq c^\ast \cdot \delta_{(A)}^\ast (z) \text{ and } |\partial^\beta \delta_{(A)}^\ast /\partial x^\beta|(z) \leq c^\ast \cdot \delta_{(A)}^\ast (z)$$

for constants $c^\ast(n) \geq 1$, $c^\ast(n, \beta) > 0$. $\beta$ is a multi-index for derivatives. In particular,

$$|\nabla \delta_{(A)}^\ast (z) \leq c^\ast \text{ and } |\Delta \delta_{(A)}^\ast (z) \leq c^\ast \cdot \delta_{(A)}^\ast (z) = c^\ast \cdot \langle A \rangle (z), \text{ for } C^\ast(n), c^\ast(n) \geq 1.$$ 

Now we introduce the secondary or $\beta$-trimming deformation.

**Proposition 2.3 ($\beta$-Trimming)** There is a $\beta_0(n) \in (0, 1)$ so that for $\beta \in [0, \beta_0)$ there is a smooth $F_{\beta} \in C \cap C^2 \alpha$-norm, with $F \geq F_{\beta} \geq 1$ and $F_{\beta} \in C[C^n, C^2[n]]$ and some $\delta(\beta, n) > \beta$, with $\delta(\beta) \to 0$ for $\beta \to 0$, so that:
(i) $g_{C_n}^{n,n-k|\beta} = (F_\beta \cdot \Psi_o)^{4/(n-2)} \cdot g_{C_n}^{[n,n-k]}$ equals $g_{C_n}^{[n,n-k]}$ for $\delta(A)^{\ast}(z) \geq \delta$, $g_{C_n}^{\circ}$ for $\delta(A)^{\ast}(z) \leq \beta$, i.e. the $\beta$-trimmed bump $\Delta^{n,n-k|\beta}$ is defined as $((F_\beta - 1) \cdot \Psi_o(x,y))^{4/(n-2)} \cdot g_{C_n}$ is supported on a subset of $\delta(A)^{-1}(\mathbb{R}^{\geq \beta})$ and $g_{C_n}^{n,n-k|0} = g_{C_n}^{n,n-k}$.

(ii) $L_{C_n}(F_\beta \cdot \Psi_o) \geq \lambda/8 \cdot \langle A \rangle^2 \cdot F_\beta \cdot \Psi_o$, in particular, $g_{C_n}^{n,n-k|\beta}$ has scal $> 0$.

We call $\delta(A)^{-1}(\mathbb{R}^{\leq \beta}) \cap \text{supp} \ g_{C_n}^{n,n-k}$ a $\beta$-tunnel.

**Proof of 2.3** The idea of the $\beta$-trimming process is the same as in Step 3 of 2.2 where we have deformed $c \cdot \Psi_o$, for some $c > 1$, to $\Psi_o$ near $\mathbb{R}^{n-k} \times \{0\}$. That process can be interpreted as a trimming of the support of $(c - 1) \cdot \Psi_o$. Here we trim $\Delta^{n,n-k}$ towards the remainder of $\Sigma_C$, i.e., we gradually deform $F - 1$ to zero keeping scal $> 0$. We exploit again that we are using the unique solution $\Psi_o > 0$ of the equation $-\Delta \Psi_o - \frac{n-2}{4(n-1)} \cdot |A|^2 \cdot \Psi_o = \lambda \cdot \langle A \rangle^2 \cdot \Psi_o$.

For a cut-off function $\phi \in C^\infty(\mathbb{R}, [0,1])$ with $\phi \equiv 1$ on $\mathbb{R}^{\leq 0}$, $\phi \equiv 0$ on $\mathbb{R}^{>0}$, we define

\begin{align*}
\delta_{A}^\ast &:= \phi \cdot (\delta(A)^{\ast}(z)) := \phi(2^d \cdot (\delta(A)^{\ast}(z) - 2^d)) \quad \text{for } d \in \mathbb{Z} \text{ and } z \in C \setminus \Sigma_C. \\
\text{For } P_i^j(\zeta) &:= \prod_{d=1}^i (1 + \zeta \cdot h_d^{\ast}), \text{ where } \zeta < 0 \text{ with norm } |\zeta| < 1/2, \text{ we have}
\end{align*}

\begin{align*}
|\nabla P_i^j(\zeta)||z| &\leq |\zeta| \cdot n \cdot 2^d \cdot c_\zeta \cdot c_\phi, \text{ on } \text{supp} \nabla h_d^{\ast}, \\
|\Delta P_i^j(\zeta)||z| &\leq |\zeta| \cdot n \cdot c_\phi \cdot (2^2 \cdot c_\zeta^2 + 2^d \cdot c_\zeta \cdot \langle A \rangle(z)), \text{ on } \text{supp} \nabla h_d^{\ast},
\end{align*}

and for any integers $i \leq a$ and any $\eta > 0$ there is an $j(i, \eta, \zeta) > a$ with

\begin{align*}
0 < P_i^j(\zeta)(z) &\leq \eta, \text{ for } z \in C \setminus \Sigma_C \text{ with } \delta(A)^{\ast}(z) \leq 2^{-j}.
\end{align*}

**Step 1 (Cut-Off adapted to $\delta(A)^{\ast}$)** For $\zeta \in (-1/2, 0)$ and integers $i < j$, we define

\begin{equation}
g^{n,n-k|\beta}(i,j) := ((P_i^j(\zeta) \cdot (F - 1) + 1) \cdot \Psi_o)^{4/(n-2)} \cdot g_{C_n}.
\end{equation}
We note that $g_{C_n}^{n,n-k}(i, j) = (F \cdot \Psi_o)^{4/(n-2)}(z) \cdot g_{C_n}$ for $\delta_{(A)^\cdot}(z) \geq 2^{1-i}$ and that $g_{C_n}^{n,n-k}(i, j)$ asymptotically approaches $\Psi_o^{4/(n-2)} \cdot g_{C_n}$ towards $\Sigma$, from (37). We claim that there is an $\zeta_0(n, k) \in (0, 1/2)$ so that, for any $\zeta < 0$ with $|\zeta| \leq \zeta_0(n, k)$ and any pair $i < j$, we have $\text{scal}(g_{C_n}^{n,n-k}(i, j)) > 0$ on $C \setminus \Sigma_C$. As before, we start from the scal-transformation law:

$$\left(\left(P_i^j(\zeta) \cdot (F - 1) + 1\right) \cdot \Psi_o\right)^{\frac{n^2}{n-2}} \cdot \text{scal}(g_{C_n}^{n,n-k}(i, j)) =$$

$$-\Delta(\left(P_i^j(\zeta) \cdot (F - 1) + 1\right) \cdot \Psi_o) - \frac{(n-2)}{4(n-1)} \cdot |A|^2 \cdot \left(\left(P_i^j(\zeta) \cdot (F - 1) + 1\right) \cdot \Psi_o\right)$$

Using the brackets $[,]$ and $[[ ]$ to group the terms the second line of this equation becomes

$$[-\Delta P_i^j(\zeta) \cdot F \cdot \Psi_o] + [-2 \cdot \langle \nabla(P_i^j(\zeta), \nabla(F \cdot \Psi_o)\rangle] + \left[-P_i^j(\zeta) \cdot \Delta(F \cdot \Psi_o)\right] + ...$$

$$[-\Delta(1 - P_i^j(\zeta) \cdot \Psi_o)] + [-2 \cdot \langle \nabla(1 - P_i^j(\zeta), \nabla \Psi_o)\rangle] + \left[-(1 - P_i^j(\zeta)) \cdot \Delta \Psi_o\right] + ...$$

$$\left[-\frac{(n-2)}{4(n-1)} \cdot |A|^2 \cdot P_i^j(\zeta) \cdot F \cdot \Psi_o\right] + \left[-\frac{(n-2)}{4(n-1)} \cdot |A|^2 \cdot (1 - P_i^j(\zeta)) \cdot \Psi_o\right]$$

- For the sum of the $-$ terms we use $\nabla(F \cdot \Psi_o) = F \cdot \nabla \Psi_o + \Psi_o \cdot \nabla F$ and (34)-(36). The definition of $F$ in Prop. 2.2 shows that $F, |\nabla F|, |\Delta F| \leq c_F$ on $C \setminus \Sigma_C$, for some $c_F(n, k) > 0$. This gives us the following lower estimate on $\delta_{(A)^\cdot}(\{2^{-a}, 2^{-1-a}\})$:

$$|\zeta| \cdot n \cdot c_\phi \cdot \left(2^{2a} \cdot c_\phi^2 + 2^a \cdot c_\Delta \cdot \langle A(\zeta)\rangle\right) \cdot c_F \cdot \Psi_o - 2 \cdot n \cdot |\zeta| \cdot 2^a \cdot c_V \cdot c_\phi \cdot c_F \cdot \Psi_o - ...$$

$$2 \cdot n \cdot |\zeta| \cdot 2^a \cdot c_V \cdot c_\phi \cdot c_F \cdot |\nabla \Psi_o| - |\zeta| \cdot n \cdot c_\phi \cdot \left(2^{2a} \cdot c_\phi^2 + 2^a \cdot c_\Delta \cdot \langle A(\zeta)\rangle\right) \cdot \Psi_o - 2 \cdot n \cdot |\zeta| \cdot 2^a \cdot c_V \cdot c_\phi \cdot |\nabla \Psi_o|.$$

- The sum of the $[[ ]$ and $[ ]$-terms can be lower estimated from 2.2(iii) and $F \geq 1$:

$$\lambda \cdot \langle A \rangle^2 \cdot (1 - P_i^j(\zeta)) \cdot \Psi_o + \lambda/4 \cdot \langle A \rangle^2 \cdot P_i^j(\zeta) \cdot F \cdot \Psi_o \geq \lambda/4 \cdot \langle A \rangle^2 \cdot \Psi_o.$$

**Step 2 (Elliptic Estimates relative to $\delta_{(A)^\cdot}$)** To compare (39) with (40), we estimate $|\nabla \Psi_o|$ in terms of $|\Psi_o|$. $\text{supp} \nabla h_{a}^\cdot \subset \delta_{(A)^\cdot}(\{2^{-a}, 2^{-1-a}\})$ is geometrically well-controlled: for $a = 1$ we have $\delta_{(A)^\cdot}(\{1/2, 1\}) \subset \delta_{(A)^\cdot}(\{c_\phi^{-1}/2, c_\phi\})$ and thus

$$|A|(z) \leq 2 \cdot c_\phi, \text{ for any } z \in \delta_{(A)^\cdot}(\{1/2, 1\}) \text{ from } \langle A \rangle \geq |A|.$$

From Gauß equations this yields a uniform control over the geometry of $\delta_{(A)^\cdot}(\{1/2, 1\})$ even on the neighborhood $V := \bigcup \{B_{1/(2(A)(\rho))(p)} \mid p \in \delta_{(A)^\cdot}([c_\phi^{-1}/2, c_\phi])\}$ from [L1, Lemma B.2]:

$$|\langle A \rangle(p)/\langle A \rangle(q) - 1| \leq |\langle A \rangle(p) \cdot d_H(p, q)|, \text{ for } q \in B_{1/(2(A)(\rho))(p)}.$$
where we have also used that $\Psi_0 > 0$ is the unique\# solution on $C \setminus \Sigma_C$. Since $\delta_{\langle A \rangle^c,H} = c \cdot \delta_{\langle A \rangle}$, for any $c > 0$, and thus $c \cdot c_{\ast}^{1} \cdot \delta_{\langle A \rangle^c,H} \leq \delta_{\langle A \rangle^c,H} \leq c \cdot c_{\ast} \cdot \delta_{\langle A \rangle^c,H}$, we have

$$
\delta_{\langle A \rangle^c}^{-1}([2^{-a} \cdot 2^{1-a}]) \subset \delta_{\langle A \rangle^{-1}}([c_{\ast}^{-1} \cdot 2^{-a} \cdot c_{\ast} \cdot 2^{1-a}]) = 2^{-a} \cdot V.
$$

From (43) we find some $\zeta_0(1) \in (0,1/2)$ so that for $\zeta < 0$ with $|\zeta| \leq \zeta_0$, the norm of (39) is smaller than $\lambda \cdot \langle A \rangle^2 \cdot \Psi_0$ on $V$. As in 2.2, we observe that, due to the uniqueness of $\Psi_0$, we have, under scaling by $2^n$, that, up to a constant multiple

$$
\Psi_0 \text{ and } h_0^\delta \text{ on } 2^{-a} \cdot V \text{ transform into } \Psi_0 \text{ and } h_1^\delta \text{ on } V.
$$

From (44), (43) and $\langle A \rangle_{2^{-n},H} = 2^a \cdot \langle A \rangle_H$ we therefore see that, for $a > 0$ and $|\zeta| \leq \zeta_0$, the norm of (39) is smaller than $\lambda \cdot 100 \cdot \langle A \rangle^2 \cdot \Psi_0$ on $2^{-a} \cdot V$; i.e. $\zeta_0$ can be chosen to depend on $n$ and $k$ and not on $a$. This shows that for $\zeta < 0$ with $|\zeta| \leq \zeta_0$ and $0 < i \leq j$ on $C \setminus \Sigma_C$:

$$
L_{C^n}(P_i^j(\zeta) \cdot (F - 1 + 1) \cdot \Psi_0 \geq \lambda/6 \cdot (P_i^j(\zeta) \cdot (F - 1 + 1) \cdot \Psi_0.
$$

Step 3 (Iterated Cut-Offs) We use this $\zeta_0(n,k)$ and some $\zeta < 0$ with small enough $|\zeta| \leq \zeta_0$ and, from (37), some sufficiently large $j$ so that the trimmed pseudo metric

$$
(((1 - h_j^\delta) \cdot P_i^j(\zeta) \cdot (F - 1 + 1) \cdot \Psi_0)^{1/(n-2)} \cdot g_{C^n}
$$

has scalar $> 0$ on $C \setminus \Sigma_C$. Namely, we consider the transformation under scaling by $2^j$:

$$
P_i^j(\zeta) \text{ on } 2^{-j} \cdot V \text{ transform into } P_{i-(j-1)}^1(\zeta)(1 + \zeta \cdot h_j^\delta) \text{ on } V
$$

and this uniformly $C^2$-converges to zero on $V$, when $j \to \infty$. From this a suitably large $j(i,n,k)$ can be chosen so that the metric (47) has scalar $> 0$ on $C \setminus \Sigma_C$.

To choose the parameter $\delta(\beta) > 0$, we start with $i = 1$ and $j(1)$ and set $\beta_0(n,k) := 2^{-j(1)}$. For any $\beta \in (0, \beta_0(n,k))$, we therefore have some $\delta(\beta) \leq 1$ as claimed. For $2^{-j(i+1)} \leq \beta < 2^{-j(i)}$, we choose $\delta(\beta) := 2^{-i}$ and we see that $\delta(\beta) \to 0$ for $\beta \to 0$. In turn, we associate these $i$ and $j(i+1)$ to any $\beta$ with $2^{-j(i+1)} \leq \beta < 2^{-j(i)}$ and we set

$$
F_\beta := ((1 - h_j^\delta) \cdot P_i^j(\zeta) \cdot (F - 1 + 1).
$$

As before, we can drop the dependency on $k$ to simplify our notations. From the discussion above we see that for $F_\beta$ we have $L_{C^n}(F_\beta \cdot \Psi_0) \geq \lambda/8 \cdot \langle A \rangle^2 \cdot F_\beta \cdot \Psi_0$ and it satisfies the other asserted properties directly from the construction.

\[ \square \]

2.3 Tightness of Cavities

We show that there are area minimizers $\mathcal{T}_\beta$ relative to the trimmed bump metric $g_{C^n}^{n,n-k} \cdot 1 = F_\beta \cdot \Psi_0)^{1/(n-2)} \cdot g_{C^n}$ and, as a part of the argument, we also prove that $\mathcal{T}_\beta$ Hausdorff converges to $\mathcal{T}_\delta$, for $\delta \to 0$, where $\mathcal{T}_\delta$ is the minimizer relative to $g_{C^n}^{n,n-k}$ of Prop. 2.2.

In our assembly of global bumps, we superpose disjointly supported families of trimmed bumps to $(C,d_S)$ and later to $(H,d_S)$, for $H \in \mathcal{G}$, cf. Fig. 5. Therefore we consider a more general $\Lambda$-bumped situation and include the following type of additional deformations:

$$
(G_{\beta} \cdot F_\beta \cdot \Psi_0)^{1/(n-2)} \cdot g_{C^n} \text{ with support } |G_\beta - 1| \cap \text{supp } |F_\beta - 1| = \emptyset \text{ and } G_\beta \in C[C^n, \Lambda[n]],
$$

where $\Lambda[n]$ is a $\Lambda$-ball in $C^n$. A $\Lambda$-ball $B$ is a Lipschitz map $B : C^n \to \mathbb{R}$ such that $\Lambda[n] = \{ x \in C^n : B(x) < \frac{1}{2} \}$. The $\Lambda$-bumped function $G_\beta$ is a constant $1$ on $B$. The $\Lambda$-bump $G_\beta$ is a Lipschitz function $G_\beta : C^n \to \mathbb{R}$ such that $\text{supp } G_\beta \subseteq B$. For any $\Lambda$-bumped function $G_\beta$, we define $\text{supp } G_\beta \subseteq B$ as the support of $G_\beta$ on $B$.
The $G_\beta$ are the placeholders for further local bumps. They do not need to be translation $\mathbb{R}^{n-k}$-invariant and we note from supp $|G_\beta - 1| \cap$ supp $|F_\beta - 1| = \emptyset$ that $G_\beta \cdot F_\beta \in C[\mathbb{C}^n, \Lambda[n]]$. We denote the associated hypersurface and volume measures by $\mu_S^{\Lambda,n-1}[\beta, G_\beta]$ and $\mu_S^{\Lambda}[\beta, G_\beta]$, where $[\beta, G_\beta] = [0, 1]$ means the case of the untrimmed metric $(F \cdot \Psi_\rho)^{4/(n-2)} \cdot gc^n$.

Local versus Global Minimizers Now we consider local area minimizers in the metric measure space $(C, d_S^\Lambda, \mu_S^\Lambda)$, of Theorem 1.3, associated to $(G_\beta \cdot F_\beta \cdot \Psi_\rho)^{4/(n-2)} \cdot gc^n$. The attribute local means that these minimizers are area minimizing under, not quantitatively specified, small perturbations. The hypersurfaces $T_\varrho$ in Prop. 2.2 or the minimal boundaries in our main Theorems belong to this class. In some cases we can specify an open test set $\Omega \subset (H, d_S^\Lambda, \mu_S^\Lambda)$ so that the area minimizer becomes a global area minimizer when compared to all other hypersurfaces within $\Omega$ and with the same (Plateau) boundary data along $\partial \Omega$. In the BV-approach, $\partial \Omega$ also serves as an obstacle that keeps interior points of these minimizers to stay in $\overline{\Omega}$, cf. [Gi, Rm. 1.22]. We choose the auxiliary

$$\Omega := Q_{\ell}^{n-k} \times C^k, Q_{\ell}^{n-k} := (-\ell/2, \ell/2)^{n-k}, \ell \geq 1 \text{ and } O_\rho := \mathbb{R}^{n-k} \times B_\rho(0), \rho > 0,$$

to get existence results and, in a second step, we check that the minimizers actually avoid $\partial \Omega$ away from their Plateau boundary. Concretely, for small $\beta$ we get the existence of global minimizers in a suitably chosen $\Omega$ to all other hypersurfaces within $\Omega$ and with the same (Plateau) boundary data along $\partial \Omega$. In the BV-approach, $\partial \Omega$ also serves as an obstacle that keeps interior points of these minimizers to stay in $\overline{\Omega}$, cf. [Gi, Rm. 1.22].

Proposition 2.4 (Tightness of $\beta$-Tunnels) There is some $\beta^*[n, \ell, k] \in (0, 1)$, so that for $\beta \in [0, \beta^*]$, $F_\beta$ from Prop. 2.3, and any $G_\beta \in C[\mathbb{C}^n, \Lambda[n]]$ with supp $|G_\beta - 1| \cap$ supp $|F_\beta - 1| = \emptyset$, we have for $\vartheta_\ell[n, C] \in (r, 1)$, $\varrho \in (\vartheta_1, \vartheta_2)$ from Prop. 2.2 and $\zeta = 1/4 \cdot \min\{|\varrho - \vartheta_1|, |\varrho - \vartheta_2|\}$:

(i) there is an open Caccioppoli set $U[n - k, \ell, G_\beta] \subset C^n$ with $O_{\varrho - \zeta} \subset U \subset O_{\varrho + \zeta}$ so that $\mathcal{T}[n - k, \ell, G_\beta] := \partial U$ and $\mathcal{T} \cap \Omega$ is area minimizing with $\Omega$-boundary value $\mathcal{T}_\varrho$, relative to $(C, d_S^\Lambda, \mu_S^\Lambda)$ associated to $(G_\beta \cdot F_\beta \cdot \Psi_\rho)^{4/(n-2)} \cdot gc^n$,

(ii) for any such oriented minimal boundary $\mathcal{T}_\bullet := \partial U_\bullet \subset O_{\varrho + 2\zeta} \setminus O_{\varrho - 2\zeta}$ with the obstacle $\partial (O_{\varrho - 2\zeta} \setminus O_{\varrho + 2\zeta})$, we already have $\mathcal{T}_\varrho \subset O_{\varrho + \zeta} \setminus O_{\varrho - \zeta}$. We write $\mathcal{P}[n - k, \ell, G_\beta]$ for the class of all such Plateau solutions $\mathcal{T}_\varrho$.

In the case $k = n$, $\ell$ is a void parameter, but we keep writing it for a consistent notation. In turn, for known dimension of the Euclidean factor we also drop writing $n - k$.

Proof We consider the auxiliary obstacle problem for open Caccioppoli sets $U$

$$\mathbb{R}^{n-k} \times B_{\vartheta_1}(0) \subset \mathbb{R}^{n-k} \times B_{\varrho - 2\zeta}(0) \subset U \subset \mathbb{R}^{n-k} \times B_{\varrho + 2\zeta}(0) \subset \mathbb{R}^{n-k} \times B_{\varrho_2}(0).$$

To be able to use convergence arguments, we ensure compactness properties for the obstacles. For $n = k$ any such $\partial U \subset B_{\vartheta_1}(0) \setminus B_{\vartheta_1}(0)$ is compact. For $n > k$ we choose the compact set $\overline{Q_{\ell}^{n-k}} \setminus B_{\vartheta_1}(0) \setminus B_{\vartheta_1}(0) \subset \Omega$. We let $\mathcal{T} \subset \Omega$ be an oriented minimal boundary $\mathcal{T} = \partial U$ with $\Omega$-boundary value $\mathcal{T}_\varrho$ solving the obstacle problem (52). We observe that $Q_{\ell}^{n-k}$ is an intersection of Euclidean halfspaces and thus $\partial \Omega$ is locally outer minimizing, that is, its area increases under local outward deformations, cf. the argument of 2.8. That is, a free Plateau solution with $\Omega$-boundary value $\mathcal{T}_\varrho$ does not leave $\Omega$. We claim that $\mathcal{T}$ is also a free solution
in the radial direction if $\beta > 0$ is small enough and we even show that $T[\ell, G_{\beta}] \subset O_{e+\zeta} \setminus O_{e-\zeta}$.

**Step 1** We define the $\Omega$-flat norm as the $\mu^A_S[0,1]$-volume of the difference set $(U \Delta O_{\varrho}) \cap \Omega$. In this step we show that for the oriented minimal boundaries $T = \partial U$,

\begin{equation}
T[\ell, G_{\beta}] \rightarrow T_0 \text{ in } \Omega\text{-flat norm, for } \beta \rightarrow 0,
\end{equation}

uniformly for all singular area minimizing cones $C^n = \mathbb{R}^{n-k} \times C^k$ and for all admissible $G_{\beta}$. From the naturality of $\langle A \rangle_H$ on $G$ and its properness on $H \setminus \Sigma_H$, for any non-totally geodesic $H \in \mathcal{G}$, we get some constant $\nu(\beta, \Omega) > 0$, independent of $\mathbb{R}^{n-k} \times C^k$, so that

\begin{equation}
\mu^A_S[0,1](\delta_{(A)}(\mathbb{R}^{\leq \delta(\beta)} \cap T_0 \cap \Omega) \leq \nu \text{ and } \nu(\beta, \Omega) \rightarrow 0 \text{ for } \beta \rightarrow 0.
\end{equation}

The minimality of the $\mathcal{T}$ and of $T_0$ and $G_{\beta} \geq 1$ show that

\begin{equation}
\mu^A_S[0,1](T_0 \cap \Omega) \leq \mu^A_S[0,1]([\beta, G_{\beta}](T \cap \Omega) \leq \mu^A_S[0,1](\delta_{(A)}(\mathbb{R}^{\geq \delta}) \cap T_0 \cap \Omega) + \Lambda^{n-1} \cdot \nu.
\end{equation}

Since $\mu^A_S[0,1]([\beta, G_{\beta}](T \cap \Omega) \geq \mu^A_S[0,1](T \cap \Omega)$, we infer that for $\beta \rightarrow 0$,

\begin{equation}
\mu^A_S[0,1]([\beta, G_{\beta}](T[\ell, G_{\beta}] \cap \Omega) \text{ and } \mu^A_S[0,1](T[\ell, G_{\beta}] \cap \Omega) \rightarrow \mu^A_S[0,1](T_0 \cap \Omega).
\end{equation}

From this, the BV-compactness relative to $\mu^A_S[0,1]$, [M, Prop. 3.7], shows that for any sequence $\beta_i \rightarrow 0$, for $i \rightarrow \infty$, there is a $\Omega$-flat norm converging subsequence of the $T[\ell, G_{\beta_i}] \cap \Omega$. From the lower semicontinuity of the perimeter [M, Prop. 3.6] and the uniqueness of $T_0$ we infer that the limit is $T_0$. Another loop of the argument shows the convergence of the entire sequence $T[\ell, G_{\beta_i}] \cap \Omega$,

\begin{equation}
\mu^A_S[0,1]((U[\ell, G_{\beta_i}] \Delta O_{\varrho}) \cap \Omega) \rightarrow 0 \text{ for } i \rightarrow \infty.
\end{equation}

From compactness results for minimal factor geometries on cones and from the continuous dependence of $F_{\beta}$ and $\varrho$ on $C$, this convergence is uniform for all $C^n = \mathbb{R}^{n-k} \times C^k$ and $G_{\beta}$.

**Step 2** We apply the $\mu^A_S$-growth estimate of Corollary 1.4 (13) to the difference set to upgrade the flat norm convergence from step 1 to Hausdorff convergence. If $U[\ell, G_{\beta_i}] \cap \mathbb{R}^{n-k} \times \partial B_{e+\zeta}(0) \neq \emptyset$ for all $i$, then there is a $p_i \in (U[\ell, G_{\beta_i}] \Delta O_{\varrho}) \cap \mathbb{R}^{n-k} \times \partial B_{e+\zeta/2}(0)$. We have $B_{\zeta/4}(p_i) \cap \mathbb{R}^{n-k} \times \partial B_{e}(0) = \emptyset, p_i \in U[\ell, G_{\beta_i}]$ and the $T[\ell, G_{\beta_i}]$ are (global) area minimizers relative to the test set $B_{\zeta/4}(p_i) \subset \Omega$. Thus, Cor. 1.4 (13) shows

\begin{equation}
\kappa^+ \cdot (\zeta/8)^n \leq \mu^A_S[\beta, G_{\beta}](U[\ell, G_{\beta_i}] \cap B_{\zeta/8}(p_i)) \leq \Lambda[n]^{n-1} \cdot \mu^A_S[0,1]((U[\ell, G_{\beta_i}] \Delta O_{\varrho}) \cap \Omega).
\end{equation}

This contradicts (57). For $U[\ell, G_{\beta_i}] \cap \mathbb{R}^{n-k} \times \partial B_{e-\zeta}(0) = \emptyset$ we argue similarly. Since only $1 \leq G_{\beta} \leq \Lambda$ for $G_{\beta}$ were used, compactness arguments for the space of all singular cones $C^n = \mathbb{R}^{n-k} \times C^k$ with their minimal factor geometry show that there is some $\beta^*[n, \ell] > 0$, independent of $C^n$ and $G_{\beta}$, so that for any $\beta \in [0, \beta^*]: T[\ell, G_{\beta}] \subset O_{e+\zeta} \setminus O_{e-\zeta}$ proving claim (i). The argument also implies (ii) since it applies to all such area minimizers. \qed
2.4 Local Bumps and Shields

The $\beta$-trimmed bump pseudo metrics $\Delta^{n,n-k}|\beta$ are supported away from $\Sigma_{C}$. This is one of the requirements to transfer such bumps from $C$ to other hypersurfaces. Since tangent cone approximations generally only work on bounded subsets of the cone, we additionally need to trim the bump pseudo metrics so that their support becomes compact in $C \setminus \Sigma_{C}$.

For this we inductively use the $\Delta^{n,q}|\beta$ for all $q$ starting from $q = 0$ up to $q = n - k$, on $C^{n} = \mathbb{R}^{q} \times (\mathbb{R}^{n-k-q} \times C^{k}) = \mathbb{R}^{n-k} \times C^{k}$. All of the $\Delta^{n,q}|\beta$ are defined on the same cone $\mathbb{R}^{n-k} \times C^{k}$, but for $q < n - k$, the definition of $\Delta^{n,q}|\beta$ ignores the $\mathbb{R}^{n-k}-q$-symmetry and treats $\mathbb{R}^{n-k-q} \times C^{k}$ as the cone factor. $\Delta^{n,0}|\beta$ already has compact support.

![Diagram](image)

**Figure 4:** The illustration shows $g_{\Sigma_{C}}^{n,0}|\beta$ for $\Sigma_{C} = \mathbb{R}^{n-k} \times \{0\}$ where the $\beta$-trimming ignores the $\mathbb{R}^{n-k}$-translation symmetry of $C^{n}$. By contrast, $g_{\Sigma_{C}}^{n,n-k}|\beta$ looks like the middle image of Fig. 2 without $\beta$-tunnels.

**Definition 2.5 (Bump Elements)** For $q = 0, \ldots, n - k$, $\ell \geq 1$ and $\phi \in C^{\infty}(\mathbb{R}, [0, 1])$ with $\phi \equiv 1$ on $\mathbb{R}^{\leq 0}$, $\phi \equiv 0$ on $\mathbb{R}^{\geq 1}$, we set $F_{\beta}^{\phi,Q_{\ell}^{q}} := \phi \left( \text{dist} (z, Q_{\ell}^{q} \times \mathbb{R}^{n-k-q} \times C^{k}) \right) \cdot (F_{\beta} - 1) + 1$:

- $\Delta^{n,q}|\beta,\ell := (F_{\beta}^{\phi,Q_{\ell}^{q}} - 1)^{4/(n-2)} \cdot g_{C^{n}}$ is the $(q)$-bump element aligned to $Q_{\ell}^{q}$,
- $F_{\beta}^{\phi,Q_{\ell}^{q}} \cdot \Psi_{\circ}$ is the associated bumped metric,
- $\text{cut} \Delta^{n,q}|\beta,\ell := \text{supp} \Delta^{n,q}|\beta,\ell \cap \{ z \in C^{n} \mid 0 < \text{dist} (z, Q_{\ell}^{q}) < 1 \}$ is the cut-off region.

For our purposes we can think of dist as a smooth function. As in Ch. 2.2, we may choose Whitney smoothings of dist whenever needed.

**Remark 2.6 (scal on cut $\Delta$)** On $C \setminus (\text{cut} \cup \Sigma_{C})$ the bumped metric coincides with $g_{\Sigma_{C}}^{n,n-k}|\beta$ respectively with $\Psi_{\circ}^{\phi/(n-2)} \cdot g_{C^{n}}$. In both case we have $L_{C^{n}}(F_{\beta}^{\phi,Q_{\ell}^{q}} \cdot \Psi_{\circ}) \geq \lambda/8 \cdot (A)^{2} \cdot F_{\beta}^{\phi,Q_{\ell}^{q}} \cdot \Psi_{\circ}$. In turn, we anticipate that cut $\Delta$ will be inductively shielded by other bump elements, i.e. it belongs to the finally deleted neighborhood $U$ of $\Sigma$ with minimal boundary $\partial U$ of $H \setminus U$ we described in our theorems. This makes knowing scal on cut $\Delta$ dispensable. Even so, to avoid exceptional subsets in our further discussion we indicate how to ensure that on cut $\Delta$:

$$L_{C^{n}}(F_{\beta}^{\phi,Q_{\ell}^{q}} \cdot \Psi_{\circ}) \geq \lambda/10 \cdot (A)^{2} \cdot F_{\beta}^{\phi,Q_{\ell}^{q}} \cdot \Psi_{\circ}$$

We observe that $F_{\beta} \cdot \Psi_{\circ}$ and $\Psi_{\circ}$ satisfy the relation $L_{C^{n}} F \geq \lambda/8 \cdot (A)^{2} \cdot F$. When we replace $\phi$ by $\phi_{\tau}$, $\tau > 0$, with $\phi_{\tau}(t) := \phi(\tau \cdot t)$, we have $|\phi'_{\tau}|, |\phi''_{\tau}| \to 0$, for $\tau \to 0$. Thus the terms involving $|\phi'_{\tau}|$ and $|\phi''_{\tau}|$ are majorized by $\phi_{\tau} \cdot L_{C^{n}}(F_{\beta} \cdot \Psi_{\circ}) + (1 - \phi_{\tau}) \cdot L_{C^{n}} \Psi_{\circ}$ and we get, for small $\tau > 0$, that $L_{C^{n}}(F_{\beta}^{\phi,Q_{\ell}^{q}} \cdot \Psi_{\circ}) \geq \lambda/10 \cdot (A)^{2} \cdot F_{\beta}^{\phi,Q_{\ell}^{q}} \cdot \Psi_{\circ}$. For fixed $\phi$ this $\tau$ depends only on $n$ and $\beta$. To avoid non-essential parameters we henceforth assume that $\tau = 1$. \hfill $\Box$
Local Bumps For $0 \leq q \leq n - k$, cut $\Delta^{n,q|\beta,\ell}$ is compact. For $q = 0$, this leaves $\Delta^{n,0|\beta}$ unchanged, but for the sake of consistency we keep the parameter $\ell$. For $(0)$-bump elements, Prop. 2.4 shows that for small $\beta > 0$ there are minimizing boundaries $\mathcal{T}^\beta = \partial U$ for an open $U$ with $O_r \subset U \subset O_1$. For $m > 0$ we inductively and disjointly place scaled bump elements along all lower dimensional faces of $Q_1^{n-k}$, so that the cut-off regions of the $(q)$-bump elements belong to the union of the ($\phi$-trimmings of the) cores $\mathbb{R}^q \times B_{r[n]}^{n-k-q}(0) \subset \mathbb{R}^q \times (\mathbb{R}^{n-k-q} \times C^k)$, from 2.2, of the already assigned $(q)$-bump elements, for $q < m$, cf. Fig. 5. For each $(q)$-dimensional face $Q_1^{qj} \subset Q_1^{n-k}, f \in F_q$ for some index set $F_q$, we choose an affine transformation

$$T[\eta, f] : \mathbb{R}^{n-k} \times C^k \rightarrow \mathbb{R}^{n-k} \times C^k \text{ with } (x, y) \mapsto \eta \cdot (x, A_f(y) + v_f),$$

where $\eta \in (0, \ell^{-1}], A_f \in O(n - k), v_f \in \mathbb{R}^{n-k}$, that maps the model $(q)$-cube $Q_\ell^q$ to $Q_1^{qj}$ so that $T[\ell^{-1}, f](Q_\ell^q) = Q_1^{qj}$. From Prop.1.2 we have $\Psi_\omega(\omega, \rho) = \psi(\omega) \cdot \rho^{\alpha_\omega}$ and hence

$$T[\eta, f],(\Psi_0) = \Psi_0 \circ T[\eta, f]^{-1} = \eta^{-\alpha_\omega} \cdot \Psi_0.$$  

The parameters $\beta, \ell$ and $\eta$ are determined inductively in the following construction. The choices anticipate additions of $(q)$-bump elements for $q \geq 1$, as shown in Fig. 5, and of local bumps when we form larger configurations finally reaching global bumps. To this end, we recall from 2.4 that any area minimizer $\mathcal{T}_* \in \mathcal{P}[q, \ell, G_\beta]$ with $\Omega$-boundary value $\mathcal{T}_\omega$, for $\Omega_q = Q_\ell^q \times \mathbb{R}^{n-k-q} \times C^k$ and $O_\rho = \mathbb{R}^{n-k} \times B_{\rho}(0), \rho > 0$, satisfies

$$\mathcal{T}_* \cap \Omega \subset (O_{e+\zeta} \setminus O_{e-\zeta}) \cap \Omega \subset (O_1 \setminus \overline{O_r}) \cap \Omega,$$

where $r = r[n] \in (0, 1)$ is the core radius of $(F \cdot \Psi_\omega)^{4/(n-2)} : \text{gc}$ in Prop. 2.2.

0-faces For $\eta_0 = 1/10$, we place $\eta_0$-scaled $(0)$-bump elements $\Delta^{n,0|\beta,\ell}$ in each 0-dimensional face (=vertex) $v_f := p_f \cdot Q_1^{n-k}$. For $\beta_1[n] \in (0, 1)$ from 2.4, $\beta_0 \in (0, \beta_1^*[n]), \xi \in [r[n], 1]$, we set

$$\Box^{\beta_0, \eta_0, \ell_0} := \sum_{f \in F_0} T[\eta_0, f],(\Delta^{n,0|\beta_0,\ell_0}), \quad \Box_{0, \xi} : = \bigcup_{f \in F_0} O^{0, \ell_0}_{f, \xi} := \bigcup_{f \in F_0} T[\eta_0, f](O_\xi)$$

for the void parameter $\ell_0 = 1$. We observe that

$$\text{supp } T[\eta_0, f_1],(\Delta^{n,0|\beta_0,\ell_0}) \cap \text{supp } T[\eta_0, f_2],(\Delta^{n,0|\beta_0,\ell_0}) = \emptyset \text{ for } f_1 \neq f_2 \in F_0.$$

q-faces We continue inductively for $q$ with $1 \leq q \leq n - k$. We choose a large $\ell_q \geq 1$, some $\eta_q \in (0, \min\{\eta_0, \beta_0/10, \ldots, \eta_{q-1} \cdot \beta_{q-1}/10, \ell^{-1}\})$ and an $\beta_q \in [0, \beta_1^*[n, \ell_q])$, so that after placing the $\eta_q$-scaled $(q)$-bump elements $\Delta^{n,q|\beta_q,\ell_q}$ along the $(q)$-faces of the cube $Q_1^{n-k}$, the support of any two $(q)$-bump elements and of any $(q)$-bump element and the already placed $(m)$-bump element, $0 \leq m < q$ are disjoint, cf. (i) and (ii) below. The cut-off region of each $(q)$-bump element belongs to the union of the cores of the $(m)$-bump elements for $0 \leq m < q$, cf. (iii).

(i) $\text{supp } T[\eta_q, f_1],(\Delta^{n,q|\beta_q,\ell_q}) \cap \text{supp } T[\eta_q, f_2],(\Delta^{n,q|\beta_q,\ell_q}) = \emptyset$, for $f_1 \neq f_2 \in F_q$,

(ii) $\text{supp } T[\eta_q, f],(\Delta^{n,q|\beta_q,\ell_q}) \cap \bigcup_{k=0 \ldots q-1} \text{supp } \Box^{\beta_q, \eta_q, \ell_q} = \emptyset$, for $f \in F_q$,

(iii) $\bigcup_{f \in F_q} T[\eta_q, f](\text{cut } \Delta^{n,q|\beta_q,\ell_q}) \subset \bigcup_{m=0 \ldots q-1} \Box^{\eta_q, \ell_q, m} \subset \bigcup_{m=0 \ldots r[n]} \Box^{\eta_q, \ell_q, m}.$

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and for the induction step we set for $q$ the following unions of bump elements and cylinders:

\[
\square_{\beta_q, \eta_q, \ell_q} := \sum_{f = F_q} T[\eta_q, f] (\Delta^n q | \beta_q) \quad \text{and}
\]

\[
\bigcirc_{\eta_q, \ell_q} := \bigcup_{f \in F_q} O_{f, q, \eta_q, \ell_q} := \bigcup_{f \in F_q} T[\eta_q, f] (O_q \cap \Omega_q).
\]

$\bigcup_{q = 0}^{n-k} \bigcirc_{\eta_q, \ell_q}$ is an open neighborhood of $Q_1^{n-k} \times \{0\}$, for any $\xi \in [r[n], 1]$, and from Prop. 2.2 and Prop. 2.4, we notice that the inductively chosen parameters

\[
\ell_q \geq 1, \eta_q > 0 \quad \text{and} \quad \beta_q > 0,
\]

are independent of $C^n = \mathbb{R}^{n-k} \times C^k \in \mathcal{S}_C$. We make one fixed choice of these parameters and keep it for the rest of this paper.

\[
\bigcirc_{C_{n-k}} := \bigcup_{q = 0}^{n-k} \bigcirc_{\eta_q, \ell_q}
\]

the core of the local bump.

Figure 5: An illustration of the support of a local bump shielding $Q_1^{n-k} \times \{0\}$.

Definition 2.7 (Local Bumps) We call the pseudo metric

\[
\square(Q_1^{n-k}) := \sum_{q = 0}^{n-k} \square_{\beta_q, \eta_q, \ell_q}
\]

an \textit{(n-k)-local bump} on $C^n = \mathbb{R}^{n-k} \times C^k$.

We define the associated \textit{(n-k)-local bump metric}

\[
(\Phi \cdot \Psi) \cdot g_{C^n} := \left( \left( \prod_{q = 0}^{n-k} \prod_{f = F_q} (F_{\beta_q}^{Q_{1}^{n-k}} \circ T[\eta_q, f]^{-1}) \right) \cdot \Psi \right)^{4/(n-2)} \cdot g_{C^n}
\]

and we call $\bigcirc_{n-k} := \bigcup_{q = 0}^{n-k} \bigcirc_{\eta_q, \ell_q}$ the core of the local bump.

$\Phi$ is the multiplicative representation of a local bump with $\Phi \in \mathcal{C}[C^n, \Lambda[n]]$ and we have supp $|\Phi - 1| = \sup \square(Q_1^{n-k})$

\[
L_{C^n} (\Phi \cdot \Psi) \geq \lambda/10 \cdot (A)^2 \cdot \Phi \cdot \Psi.
\]

This follows from 2.6 (59) and (61) since the supp $|F_{\beta_q}^{Q_{1}^{n-k}} \circ T[\eta_q, f]^{-1} - 1|$ for any two different $(q)$-bump elements are disjoint and the relation (67) remains invariant under translations and scalings. We use Prop. 2.4 to understand the effect of local bumps.
Proposition 2.8 (Shielding) For \( G \in \mathcal{C}[C^n, \Lambda[n]] \) with \( \text{supp} |G - 1| \cap \text{supp} |\Phi_{\square} - 1| = \emptyset \) and the \( \ell_q, \eta_q, \beta_q \) from (64), we choose for any \( (q) \)-face \( f \in F_q \) some
\[
(68) \quad \mathcal{T}_f = \partial U_f \in \mathcal{P}[q, \ell_q, G_{\beta_q}] \text{ for an open } O_{\ell_q} \cap \Omega_q \subset U_f \cap \Omega_q \subset O_1 \cap \Omega_q
\]
where \( G_{\beta_q} := G \circ \mathcal{T}[\eta_q, f] \) on \( O_1 \cap \Omega \). Combining these \( U_f \) we get a local shield for \( Q_1^{n-k} \):
\[
(69) \quad U(Q_1^{n-k}) := \bigcup_{q=0}^{n-k} U_{q, \eta_q, f} := \bigcup_{q=0}^{n-k} \bigcup_{f \in F_q} \mathcal{T}[\eta_q, f] \big( U_f \cap \Omega_q \big) \supset Q_{n-k} \times \{0\},
\]
\[
(70) \quad \partial U \text{ is a local inner area minimizer relative to } U \text{ and } (G \cdot \Phi_{\square} \cdot \Psi_{\circ})^{4/(n-2)} \cdot g_{C^n}.
\]
That is, \( \mu_S^{\Lambda,n-1}(\partial U) \leq \mu_S^{\Lambda,n-1}(\partial (U \cup A)) \) for sufficiently small open sets \( A \subset C \).

Proof From Prop. 2.4 we see that \( U \) is an open neighborhood of \( Q_1^{n-k} \times \{0\} \). For (70) we note that the union of two locally inner area minimizing \( U \) and \( V \) is again locally inner area minimizing: for an open \( A \) we first use this inner minimality of \( U \), for \( A \setminus V \), and then that of \( V \), for \( A \cap V \), and write \( (U \cup V) \setminus A = ((U \cup V) \setminus (A \setminus V)) \setminus A \cap V \).

This applies to \( \partial U \) since, dropping \( \Omega_q \) for \( q = 0 \), \( \partial U \cap \bigcup_{q=0}^{n-k} \bigcup_{f \in F_q} \mathcal{T}[\eta_q, f] \big( U_f \cap \Omega_q \big) = \emptyset \). We only need to consider \( A \cap \bigcup_{q=0}^{n-k} \bigcup_{f \in F_q} \mathcal{T}[\eta_q, f] \big( U_f \cap \Omega_q \big) = \emptyset \) and we argue by induction over all \( \mathcal{T}[\eta_q, f] \big( U_f \cap \Omega_q \big) \) in the union \( U \).

Groupings We also need bumps shielding arbitrarily large cubes \( Q_1^{n-k} = Q_{a_1, a_2, \ldots, a_{n-k}} = [-a_1, a_1] \times \cdots \times [-a_{n-k}, a_{n-k}] \times \{0\} \subset \mathbb{R}^{n-k} \times C^k \), for some \( a_i > 0 \). To control the interplay with other bumps we must keep the size of both, the (support of) local bumps in the \( C^k \)-directions and of the tunnel parameters \( \beta_q \), unchanged. This means, we cannot just scale the bump construction. Alternatively, when we try to adjust the parameters in (65) to shield a larger cube \( Q \), we find from 2.4 that, starting from an initial \( \beta_n \)-tunnel for the \( (0) \)-faces and \( (0) \)-bump elements, the subsequent \( \beta_q \) have to be chosen the smaller the larger \( Q \) becomes.

The way out are groupings of local bumps, each shielding a translated copy of the unit cube \( Q_1^{n-k} \). Concretely, we consider disjoint unions \( \bigcup_{v \in S} (Q_1^{n-k} + v) \subset Q_{L,\ldots,L} \subset \mathbb{R}^{n-k} \), for finite, otherwise arbitrary subsets \( S \subset \mathbb{Z}^{n-k} \) and some \( L = L(S) > 0 \) large enough.

We repeat the inductive process we have used to define local bumps and place \( (m) \)-bump elements, \( 0 \leq m \leq n-k \), along the \( (m) \)-faces of the cubes in this union. We count all faces in this union with multiplicity one even when they belong to more than one cube \( Q_1^{n-k} + v \).

Applying 2.7 and 2.8 to this periodic version of the local bump construction yields:

Corollary 2.9 (Periodic Groupings) We get periodic bumps and shields from the obvious extensions of the definitions (65) and (69):
\[
(71) \quad \Box \big( \bigcup_{v \in S} (Q_1^{n-k} + v) \big), \text{ the bumped metric } (\Phi_{\square} \cdot \Psi_{\circ})^{4/(n-2)} \cdot g_{C^n} \text{ and } \mathcal{U} \big( \bigcup_{v \in S} (Q_1^{n-k} + v) \big),
\]
and we define the core of the periodic bump by
\[
(72) \quad \mathcal{O}_{n-k} \big( \bigcup_{v \in S} (Q_1^{n-k} + v) \big) := \bigcup_{v \in S} \Big( \bigcup_{q=0}^{n-k} \mathcal{O}_{\ell_q}^{a_q, f_q} + v \Big).
\]
For \( G \in \mathcal{C}[C^n, \Lambda[n]] \) with \( \text{supp} |G - 1| \cap \text{supp} |\Phi_{\square} - 1| = \emptyset \), and the \( r \)-distance neighborhood \( U_r \) in \( C^n \) of the family \( S \) of cubes, we have:
(i)  $\bigcup(\bigcup_{v \in S}(Q_1^{n-k} + v)) \supseteq O_{n-k}^C(\bigcup_{v \in S}(Q_1^{n-k} + v)) \supseteq \bigcup_{v \in S}(Q_1^{n-k} + v)$,

(ii) $\partial \mathbb{U}$ is locally inner area minimizing relative to $\mathbb{U}$, i.e. $\mathbb{U}$ shields $O_{n-k}^C(\bigcup_{v \in S}(Q_1^{n-k} + v))$,

(iii) $L_Cn(\Phi_\square \cdot \Psi_o) \geq \lambda/10 \cdot (\mathbb{A})^2 \cdot \Phi_\square \cdot \Psi_o$.

These results extend to $\bigcup_{v \in S}(Q_1^{n-k} + v) \subseteq Q_{L_1, \ldots, L} \subseteq \mathbb{R}^{n-k}$ after scaling by $2^a$, $a \in \mathbb{Z}^{2^1}$, where we interpret $2^a \cdot Q_1^{n-k}$ as a union of $2^a(n-k)$ appropriately translated copies of $Q_1^{n-k}$. Rescaling the resulting bump by $2^{-a}$ yields the $2^a$-rescaled periodic bumps $\square[a]$ and shields with

$$
\mathbb{U}[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) \supseteq O_{n-k}^C[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) \supseteq \bigcup_{v \in S}(Q_1^{n-k} + v),
$$

with $\mathbb{U}[a] := 2^{-a} \cdot \mathbb{U}(2^a \cdot \bigcup_{v \in S}(Q_1^{n-k} + v))$ and $O_{n-k}^C[a] := 2^{-a} \cdot O_{n-k}^C(2^a \cdot \bigcup_{v \in S}(Q_1^{n-k} + v))$.

**Proof** We note that, unless to faces of different cubes coincide, all bump elements have again disjoint support. From this we can repeat the same bump element-wise argument as in the proof of Prop. 2.8 for the same parameters with the same conclusions.  

**Remark 2.10 (Estimates for $\text{supp} \square$)** We set $r^\circ[n] := \min\{\beta_0 \cdot \eta_0, \ldots, \beta_{n-1} \cdot \eta_{n-1}\} \cdot r[n]/4$ as a lower bound for all radii, independent of the dimension of the factor $\mathbb{R}^{n-k}$, and note that

$$
U_{2^{-a}}(\bigcup_{v \in S}(Q_1^{n-k} + v)) \setminus U_{2^{-a},r^\circ[n]}(\mathbb{R}^{n-k} \times \{0\}) \supseteq \text{supp} \square[a](\bigcup_{v \in S}(Q_1^{n-k} + v)).
$$

Estimates in terms of $\delta_{(A)}$ are compatible with transfers between $H$ and tangent cones. From (2.3)(i) and $\delta_{(A)}(z) \leq \text{dist}(z, \Sigma_C)$, we get for any $p \in \Sigma$

$$
\frac{B_{2^{-a},r^\circ[n]}(p) \cap \text{supp} \square[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) \subset \ldots
$$

$$
\{ z \in C \setminus \Sigma_C \mid \delta_{(A)}(z) \leq 2^{-a} \cdot r^\circ[n] \} \cap \text{supp} \square[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) = \emptyset.
$$

and from (74) complementary to (75):

$$
\{ z \in C \setminus \Sigma_C \mid \delta_{(A)}(z) \geq 2^{-a} \} \cap \text{supp} \square[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) = \emptyset.
$$

**Example 2.11 (Shielding Balls)** The rescaling of periodic bumps shrinks the distance of the support to $\bigcup_{v \in S}(Q_1^{n-k} + v)$ but it also improves the control over the shielded subset. For $B_{1}^{n-k}(0) \subseteq \mathbb{R}^{n-k} \times \{0\}$ we interpret the rescaling as a partition of the cubes into $2^a(n-k)$-subcubes of side length $2^{-a}$, $a \in \mathbb{Z}^{2^1}$, and consider the minimal set $S_{a,n-k} \subset 2^{-a} \cdot \mathbb{Z}^{n-k}$ of such cubes with $B_1^{n-k}(0) \subset \bigcup_{v \in S_{a,n-k}}(Q_2^{n-k} + v)$. We have from $\text{diam}(Q_1^{n-k}) = \sqrt{n-k}$:

$$
B_1^{n-k}(0) \supseteq \bigcup_{v \in S_{a,n-k}}(Q_2^{n-k} + v) \supseteq B_1^{n-k}(0).
$$

From this and (74) we have for some $a_n \in \mathbb{Z}^{2^1}$ large enough and any $m \in \mathbb{Z}^{2^1}$

- $B_2^a(0) \supseteq B_{1+2^{-a\cdot a_n}}^{a}(0) \supseteq \text{supp} \square[\kappa_n \cdot (a_n + m)](\bigcup_{v \in S_{a,n-k}}(Q_2^{n-k} + v))$,

- $\text{supp} \square[\kappa_n \cdot a_n](\bigcup_{v \in S_{a,n-k}}(Q_2^{n-k} + v)) \cap \text{supp} \square[\kappa_n \cdot (a_n + m)](\bigcup_{v \in S_{a,n-k}}(Q_2^{n-k} + v)) = \emptyset$,

where $\kappa_n := [\text{smallest integer } \geq \lfloor \ln(r^\circ[n]) \rfloor]$, for the radius $r^\circ[n] > 0$ in 2.10.
3 From Local to Global Deformations

We turn to a general hypersurface $H \in G_n$ and gradually scale $H$ around some $p \in \Sigma_H$, by large $\tau \gg 1$. Eventually there is some not necessarily unique area minimizing cone $C$ that Hausdorff approximates $\tau \cdot H$ on the unit ball. On compact parts of $C \setminus \Sigma_C$ this can be improved to a smooth norm convergence, cf. [L2, Appendix A]. This smooth approximation comes with a uniquely determined smooth map, which we call an ID-map, locally parameterizing $\tau \cdot H$ as a section of the normal bundle over $C$, we can use to transfer local bump configurations on $C$ to $H \setminus \Sigma_H$. The challenge to build global bumps along $\Sigma_H$, from local transfers, is to compensate the differences between the singular sets of $H$ and such cones.

(1) Basic Idea We would like to have ball covers of $\Sigma_H$ with (i) common ball radii and controlled intersection numbers $c(n)$, depending only on $n$ and so that (ii) each of the balls admits a good cone approximation to place cone bumps on $H \setminus \Sigma_H$ and to estimate their effect. Yet, we cannot accomplish both goals (i) and (ii) at the same time. For any $p \in \Sigma_H$ and $\varepsilon > 0$ there is a critical radius, the $\varepsilon$-accuracy radius $R_\varepsilon(p)$. Balls of radius $r \in (0, R_\varepsilon(p))$, suitably rescaled, admit fine approximations by cones of a (yet to define) accuracy $\varepsilon$. One hurdle is that $R_\varepsilon(p)$ does not depend continuously on $p \in \Sigma$.

(2) Self-Similar Covers To resolve this discontinuity issue, we define a hierarchy on $\Sigma$, the accuracy decomposition $\Sigma = \bigcup_{i \geq 0} \Sigma_i$, with $\Sigma_0 = \{p \in \Sigma | R_\varepsilon(p) > 1/\varepsilon\}$ and $\Sigma_i = \{p \in \Sigma | s^{i-1}/\varepsilon \geq R_\varepsilon(p) > s^i/\varepsilon\}$, for $i \geq 1$ and some $s \in (0, 1/100)$, in Ch. 3.1. For each $\Sigma_i$ we can find covers by balls of radius $s^i$ with the two conditions (i) and (ii) in (1). Scaling by $s$ shifts the family index by 1 and makes the cover self-similar under scaling by $s$ in the sense that properties we derive for one $\Sigma_i$ carry over to the other families. For small $s > 0$, the bumps placed on ball families for different $i$ do not interfere.

(3) Auto-Alignment For the placement of bumps on families of equally sized balls we observe that singular and highly curved directions in any two neighboring balls align like compass needles in a common field. This controls not only the relative position of the singular set in these balls but also the positioning of the balls themselves.

3.1 ID-Maps and Ball Covers

Around any $q \in \Sigma_H$ in some $H \in G_n$, we have tangent cones of $H$: $H_i = \tau_i \cdot H$, for some sequence $\tau_i \to \infty$, $i \to \infty$, flat norm subconverges to some (generally non-unique) area minimizing tangent cone $C^m \subset \mathbb{R}^{s+1}$. Allard theory shows that this convergence can be upgraded.

**ID-maps** For $B_R(q) \cap C \setminus \Sigma_C$, $R > 0$, large $i$ and $B_R(q_i) \cap H_i$ for suitable $q_i \in H \setminus \Sigma$, there is a local $C^8$-section $\Gamma_i : B_R(q) \cap C \to B_R(q_i) \cap H_i \subset \nu$ of the normal bundle $\nu$ of $B_R(q) \cap C$ up to minor adjustments near $\partial B_R(q)$. For $i \to \infty$, $\Gamma_i$ converges in $C^k$-norm to the zero section, which we identify with $B_R(q) \cap C$. We call $\textbf{ID} := \Gamma_i$ the asymptotic identification map, briefly the **ID-map**. Since they are assigned canonically, we generally omit writing indices. We also recall the definition of the $S$-pencil $\mathbb{P}(p, \omega)$ pointing to $p \in \Sigma$, for some $\omega > 0$: $\mathbb{P}(p, \omega) := \{x \in H \setminus \Sigma_H | \delta_{\nu|\mu}(x) > \omega \cdot d_H(x, p)\}$. We consider the ID-maps on bounded subsets of $\mathbb{P}(p, \omega)$, the **truncated $S$-pencils** $\mathbb{T}\mathbb{P}$.

\begin{equation}
(78) \quad \mathbb{T}\mathbb{P}_H(p, \omega, R, r) := B_R(p) \setminus B_r(p) \cap \mathbb{P}(p, \omega) \subset H.
\end{equation}
For increasingly large \( \tau > 0 \), \( \tau \cdot \mathbb{P}_H(p, \omega, R/r, r/\tau) \) is gradually better \( C^5 \)-approximated by the truncated \( S \)-pencil in some tangent cone, i.e., the twisting of \( \mathbb{P}(p, \omega) \) slows down as \( \tau \to \infty \), it asymptotically freezes. More precisely, we have [L4, Prop. 3.10]:

**Proposition 3.1 (S-Freezing)** Let \( H \in \mathcal{G}_n \) and \( p \in \Sigma_H \). We pick an \( \varepsilon > 0 \), some \( R > 1 > r > 0 \) and an \( \omega \in (0, 1) \). Then there is a smallest \( \tau^* = \tau^*(\varepsilon, \omega, R, r, p) \geq 1 \) so that for any \( \tau > \tau^* \) there is some tangent cone \( C^*_p \) so that for the identity map to \( \tau \cdot H \):

- \( |\text{ID} - \text{id}_{C^*_p}|_{C^0(\mathbb{T}_p, (0, \omega, R, r))} \leq \varepsilon \), where we view the zero section of \( n \) as the identity \( \text{id} \),
- \( \Psi_H \) induces the uniquely\# determined solution \( \Psi_0 > 0(C^*_p) \) on \( C^*_p \) from Prop. 1.2 and we have, normalizing \( \Psi_0 \) appropriately, \( |\Psi_H \circ \text{ID}/\Psi_0 - 1|_{C^{2, \alpha}(\mathbb{T}_p, (0, \omega, R, r))} \leq \varepsilon \).

We call \( \varepsilon \) the **accuracy** of the \( \text{ID} \)-map and \( R_\varepsilon(H, p) := 1/\tau^*(\varepsilon, \omega, 1/\varepsilon, \varepsilon, p) \) the **\( \varepsilon \)-accuracy radius** of \( p \in \Sigma_H \). That is, any ball \( B_\rho(p) \subset H \), \( 0 < \rho \leq R_\varepsilon \), rescaled by \( (\varepsilon \cdot \rho)^{-1} \) admits an \( \varepsilon \)-accurate \( \text{ID} \)-map over \( \mathbb{T}_p(0, \varepsilon, 1/\varepsilon, \varepsilon) \subset B_{1/\varepsilon}(0) \setminus B_\varepsilon(0) \cap C \), for some tangent cone \( C \).

**Remark 3.2** For Euclidean area minimizers, the flat norm \( d_H \)-convergence implies Hausdorff \( d_H \)-convergence from volume growth estimates [Gi, Prop. 5.14]. We already used this argument in Step 2 of 2.4 for area minimizers in \( (C, d^A_S, \mu^A_S) \) based on Corollary 1.4 (13).

The spaces \( \mathcal{C}_n \) and \( \mathcal{H}_n^R \) are compact under compact convergence in the flat norm topology. \( \mathcal{K}_n \) is compact in flat norm topology. \( \mathcal{S}_n \subset \mathcal{C}_n \) and \( \mathcal{S}_+ \subset \mathcal{H}_+^R \subset \mathcal{H}_n^R \) are closed and, hence, compact. Consequently \( \bar{\mathcal{T}}_H \subset \mathcal{S}_n \), also we have \( \bar{\mathcal{T}}_p = \mathcal{T}_p \). \( \square \)

We also recall a variant for singular cones, [L4, Cor. 3.11], where we use the Hausdorff distance on \( \mathcal{S}_n \) measured in terms of the distance of the unit spheres \( S_C = \partial B_1(0) \cap C \).

**Corollary 3.3 (Cone Transitions)** Let \( \alpha \in (0, 1) \), \( \varepsilon, \omega > 0 \) and \( R > 1 > r > 0 \). Then there is a \( \zeta(\varepsilon, \omega, R, r) > 0 \) so that for any \( C \) and \( C' \in \mathcal{S}_n \) with \( d_H(S_C, S_{C'}) \leq \zeta \):

- \( |\text{ID} - \text{id}_C|_{C^0(\mathbb{T}_C, (0, \omega, R, r))} \leq \varepsilon \),
- \( \Psi_0(C') \) on \( C' \) induces \( \Psi_0(C) \) on \( C \) with \( |\Psi_0(C')/\Psi_0(C) - 1|_{C^{2, \alpha}(\mathbb{T}_p, (0, \omega, R, r))} \leq \varepsilon \).

**Self-Similar Ball Covers** We write \( \Sigma \) as a disjoint union of subsets \( \Sigma_i, i \in \mathbb{Z}^{\geq 1} \), sorted according to their \( \varepsilon \)-accuracy radii. Then we cover these subsets by balls with radii \( s^i \), for some small \( s > 0 \). This makes the scenario self-similar under scaling by powers of \( 1/s \).

**Definition 3.4 (Accuracy Decomposition)** For \( H \in \mathcal{G}_n \), some \( \varepsilon > 0 \) and \( s \in (0, s_0) \) we define the **\( \varepsilon \)-accuracy decomposition** of \( \Sigma = \bigcup_{i \geq 0} \Sigma_i : \)

\[
\Sigma_0 = \{ p \in \Sigma \mid R_\varepsilon(p) > 1/\varepsilon \} \text{ and } \Sigma_i = \{ p \in \Sigma \mid s^{i-1}\varepsilon \geq R_\varepsilon(p) > s^i/\varepsilon \}.
\]

In general, the \( \Sigma_i \) will be uncountable, but each of them contains a **countable** and **dense** subset \( \Sigma^*_i \subset \Sigma_i \) and we set \( \Sigma^* := \bigcup_{i \geq 0} \Sigma^*_i \subset \Sigma \). We write the elements of \( \Sigma^*_i \) accordingly as \( \alpha_{i,1}, \alpha_{i,2}, \ldots \). We assign a ball \( B_s(p) \) to any of the points \( p \in \Sigma^*_i \) and set

\[
B[i] := \{ B_s(p) \mid p \in \Sigma^*_i \} \text{ and } B := \bigcup_{i \geq 0} B[i].
\]

We note that \( B[i] \) is a cover of \( \Sigma \) and, hence, \( B \) is a cover of \( \Sigma \) since for any \( x \in \Sigma_i \) there is some \( p_x \in \Sigma^*_i \) with \( d_H(x, p_x) < s^i/2 \) and thus \( x \in B_{s^i}(p_x) \).
Proposition 3.5 (Self-Similar Covers) For \( H \in G_n \), with \( \Sigma_H \neq \emptyset \), some accuracy \( \varepsilon \in (0, 10^{-5}) \), \( R \in [10^2, \varepsilon^{-1}/10^2] \) and any self-similarity factor \( s \in (0, s_0) \), the following holds:

(i) For each \( i \geq 0 \), there is a locally finite family \( \mathcal{A}[i] \subset \mathcal{B}[i] \), from (80), of closed balls

\[
\mathcal{A}[i] = \{ B_s(p) \subset H \mid p \in A[i] \}, \text{ for some discrete set } A[i] \subset \Sigma^*_i.
\]

(ii) For any \( k \geq 0 \) we have: \( \bigcup_{i \leq k} \mathcal{A}[i] \) covers \( \bigcup_{i \leq k} \Sigma_i \).

(iii) \( q \notin B_s(p), p \notin B_s(q) \) for any two different \( B_s(p) \in \mathcal{A}[i], B_s(q) \in \mathcal{A}[k] \).

For \( i \geq 0 \), there is a subset \( \mathcal{A}^*[i] \subset A[i] \) and a family \( \mathcal{A}^*[i] = \{ B_{R_s}(p) \subset H \mid p \in A^*[i] \} \) with

\[
\bigcup_{x \in A^*[i]} B_s(x) \subset \bigcup_{z \in \mathcal{A}^*[i]} B_{R_s}(z) \quad \text{and} \quad z^* \notin B_{(R-1)s}(z), \quad \text{for } z \neq z^*, z, z^* \in A[i]
\]

and so that, for some constant \( c(H) \), \( \mathcal{A}^*[i] \) splits into disjoint families \( \mathcal{A}^*[i, 1], ..., \mathcal{A}^*[i, c] \) with

\[
B_{10R_s}(p) \cap B_{10R_s}(q) = \emptyset, \quad \text{for } B_{R_s}(p), B_{R_s}(q) \text{ in the same family } \mathcal{A}^*[i, j].
\]

We call the \( \mathcal{A}^*[i] \) the layers of the ball cover \( \mathcal{A}^* := \bigcup_{i \geq 1} \mathcal{A}^*[i] \) of \( \Sigma \) around the points of \( \mathcal{A}^* := \bigcup_{i \geq 1} \mathcal{A}^*[i] = \bigcup_{i \geq 1, c \geq 1} \mathcal{A}^*[i, c] \). The \( \mathcal{A}^*[i, j] \) are the sublayers of the \( \mathcal{A}^*[i] \).

Remark 3.6 1. To explain the meaning of 3.5 we anticipate that in the assembly of global bumps in Ch.3 we transfer a periodic bump to \( B_{R_s}(p) \in \mathcal{A}^*[i] \) that shields all balls \( B_s(x) \subset B_{R_s}(p) \) with \( B_s(x) \in A[i] \). The balls in \( \mathcal{A}^*[i] \) may intersect but we can ensure that the support of the bumps in the at most \( c(n, R) \) different sublayers \( \mathcal{A}^*[i] \) are disjoint. For balls in \( \mathcal{A}^*[i], \mathcal{A}^*[j] \), with \( i \neq j \), the bumps are disjoint when \( s > 0 \) is sufficiently small.

2. The covering number \( c(H) \) depends only on \( n \) if \( \varepsilon \cdot R \leq r_{H,q} \), for \( r_{H,q} > 0 \) in \([L1, Th. 3.4] \), or when \( H \in \mathcal{H}_{H_n} \), since, cf. (85) below, in both cases the Ahlfors constants depend only on \( n \). From (82) we see that for any \( t \in (0, 10) \) and any point \( z \in H \):

\[
\#\{ p \in \mathcal{A}^*[i] \mid z \in B_{tR_s}(p) \} \leq c.
\]

3. After scaling the balls in \( \mathcal{A}^*[i] \) by \( (\varepsilon \cdot s^i / \varepsilon)^{-1} = s^{-i} \), \( B_s(p) \) transforms to \( B_1(p) \) and \( B_{R_s}(p) \) becomes \( B_R(p) \) with a tangent cone approximating with accuracy \( \varepsilon \) over \( \mathbb{T}_C(0, \varepsilon, 1, \varepsilon) \) respectively over \( \mathbb{T}_C(0, \varepsilon, \varepsilon, \varepsilon) \), since \( R \leq \varepsilon^{-1}/10 \). We observe that \( R_n(p) < \text{diam}(H, d_q) \), for any \( p \in H \). Thus, the Ahlfors regularity estimates (11) apply to the balls in these covers. \( \Box \)

Proof We inductively define two different selection maps \( \kappa \) on \( \Sigma^*_i \). In a first step we select \( A \subset \Sigma^*_i \) for the small balls \( B_s \) and then \( \mathcal{A}^* \subset A \) for the large balls \( B_{R_s} \).

Step 1 (Small Cover \( A \)) We define \( \kappa_i : \Sigma^*_i \rightarrow \mathbb{Z}_2 \). When \( \kappa_i(p) = 0 \), then we delete \( B_s(p) \), otherwise, we keep it. To start the (double) induction over \( i \), we define \( \kappa_0 \) on the first non-empty \( \Sigma^*_1 \):

Start on \( \Sigma^*_i \): For \( a_{i,1} \in \Sigma^*_i \), we set \( \kappa_i(a_{i,1}) := 1 \).

Step on \( \Sigma^*_i \): We assume \( \kappa_i \) has been defined for \( a_{i,j}, j < m \). Then we set \( \kappa_i(a_{i,m+1}) := 0 \), if \( a_{i,m+1} \in \bigcup_{j \leq m} \{ B_s(a_{i,j}) \mid a_{i,j} \in \Sigma^*_i, \kappa_i(a_{i,j}) = 1 \} \), otherwise \( \kappa_i(a_{i,m+1}) := 1 \).
We observe that $\kappa_f$ finitely of $\kappa_i$ has been defined for any $a \leq i$ we continue with the definition of $\kappa_{i+1}$ on $\Sigma^*_i$. We start with $a_{i+1,1} \in \Sigma^*_i$ and set $\kappa_{i+1}(a_{i+1,1}) := 0$ when

$$a_{i+1,1} \in \bigcup_{l \leq i} \{ B_s(a_{l,j}) \mid a_{l,j} \in \Sigma^*_l, \kappa_l(a_{l,j}) = 1 \}$$

and $\kappa_{i+1}(a_{i+1,1}) := 1$ otherwise.

**Step on $\Sigma^*_i$:** We assume $\kappa_{i+1}$ has been defined for $a_{i+1,j}, j \leq m$. When

$$a_{i+1,m+1} \in \bigcup_{j \leq m} \{ B_s(a_{i+1,j}) \mid a_{i+1,j} \in \Sigma^*_i, \kappa_{i+1}(a_{i+1,j}) = 1 \} \cup \bigcup_{l \leq i} \{ B_s(a_{l,j}) \mid a_{l,j} \in \Sigma^*_l, \kappa_l(a_{l,j}) = 1 \},$$

we set $\kappa_{i+1}(a_{i+1,m+1}) := 0$ and otherwise, $\kappa_{i+1}(a_{i+1,m+1}) := 1$. Now we define for $i \geq 0$ the following sets of balls and of center points of these balls

- $A[i] := \{ B_s(p) \mid p \in \Sigma^*_i, \kappa_i(p) = 1 \}$, $A := \bigcup_{i \geq 0} A[i]$.
- $A[i] := \{ p \in \Sigma^*_i \mid \kappa_i(p) = 1 \}$, $A := \bigcup_{i \geq 0} A[i]$.

We observe that $B_{s/3}(x) \cap B_{s/3}(y) = \emptyset$, for $x \neq y, x \in A[i], y \in A[j]$. From this the Ahlfors $n$-regularity, [L1, Remark 3.7], of $(H, d_H, \mu_H)$ on $H \in G_n$ shows that in a given ball $B \subset H$ there are at most finitely many balls belonging to $A[i]$, for any given $i \geq 0$. We infer that

$$\bigcup_{i \leq k} A[i] \text{ covers } \bigcup_{i \leq k} \Sigma_i \text{ for any } k \geq 0.$$

Otherwise, there were a $q \in \Sigma_i \setminus \bigcup_{p \in U_{i \leq k} A[i]} B_s(p)$, for some $l \leq k$. The local finiteness of each $A[i]$ shows that $\bigcup_{p \in U_{i \leq k} A[i]} B_s(p)$ is a closed subset of $H$ and, thus, $q$ belongs to the open complement. Since $\Sigma_t \subset \Sigma_i$ is dense, there is some $q' = a_{l,m} \in \Sigma_t \setminus \bigcup_{p \in U_{i \leq k} A[i]} B_s(p)$. For $\kappa[l]_1(a_{l,m}) = 0$, $a_{l,m} \in \bigcup_{p \in A[i]} B_s(p) \subset \bigcup_{p \in U_{i \leq k} A[i]} B_s(p)$. Alternatively, for $\kappa[l]_1(a_{l,m}) = 1$, we have $a_{l,m} \in B_s(a_{l,m}) \subset \bigcup_{p \in U_{i \leq k} A[i]} B_s(p)$. That is, both cases lead to contradictions.

**Step 2 (Large Cover $A^*$):** Now we select subfamilies $A^*[i]$ of $\{ B_{R,s^i}(p) \subset H \mid p \in A[i] \}$ with (81) and (82). We define another selection map $\kappa^*_i : A[i] \to \mathbb{Z}^0$, writing the elements of $A[i]$ as $p_{i,1}, p_{i,2}, \ldots$ When $\kappa_i^*(p) = 0$, then we delete $B_{R,s^i}(p)$, otherwise, we keep it.

**Start:** For $p_{i,1} \in A[i]$, we set $\kappa_i^*(p_{i,1}) := 1$.

**Step:** We assume $\kappa^*_i$ has been defined for $p_{i,j}, j \leq m$. We set

$$\kappa^*_i(p_{i,j+1}) := 0,$$

when $B_s(p_{i,j+1}) \subset \bigcup_{k \leq j} \{ B_{R,s^i}(p_{i,k}) \mid \kappa^*_i(p_{i,k}) \geq 1 \},$ and otherwise

$$\kappa^*_i(p_{i,j+1}) := \min \{ \{ k \leq j \mid \kappa^*_i(p_{i,k}) \geq 1 \text{ with } d(p_{i,j+1}, p_{i,k}) > 20 \cdot R \cdot s^i \} \cup \{ j + 1 \} \}.$$

For $j \geq 1$ we define the following sets of balls and of center points of these balls

- $A^*[i,j] := \{ B_{R,s^i}(p) \mid p \in A[i], \kappa_i^*(p) = j \}$, $A^* := \bigcup_{i \geq 0} A^*[i] := \bigcup_{i \geq 0, j \geq 1} A^*[i,j]$,
- $A^*[i,j] := \{ p \in A[i] \mid \kappa_i^*(p) = j \}$, $A^* := \bigcup_{i \geq 0} A^*[i] := \bigcup_{i \geq 0, j \geq 1} A^*[i,j]$.
From the induction step we notice that \( \bigcup_{x \in A[i]} B_{s}(x) \subset \bigcup_{z \in A^*[i]} B_{R-s}(z) \) and \( z^* \notin B_{(R-1)-s}(z) \), for \( z \neq z^*, z, z^* \in A^*[i] \). From Ahlfors \( n \)-regularity (11) we have for any \( z \in H \):

\[
\mu_{H}(B_{R/4-s}(z)) \geq A \cdot (R/4 \cdot s)^n \quad \text{and} \quad \mu_{H}(B_{20 \cdot R-s}(z)) \leq B \cdot (20 \cdot R \cdot s)^n,
\]

Now we set \( c := \text{the smallest integer} \geq 100^n \cdot B/A \) and we claim \( A^*[i, j] = \emptyset \) for \( j > c \).

Otherwise, we had some \( B_{R-s}(p) \in A^*[i, c + 1] \). Then there are at least \( c \) different \( p_m \in A^*[i, j_m] \) with \( B_{10 \cdot R-s}(p) \cap B_{10 \cdot R-s}(p_m) \neq \emptyset \). Since the \( B_{R/4-s}(x_m) \) are pairwise disjoint we get \( \mu_{H}(B_{20 \cdot R-s}(p)) \geq 100^n \cdot B/A \cdot A \cdot (R/4 \cdot s)^n \). But this contradicts the upper estimate for \( \mu_{H}(B_{20 \cdot R-s}(p)) \), and, hence, we have \( A^*[i, c + 1] = \emptyset \).

\[\Box\]

### 3.2 Transfer of Local Bumps

We cannot transfer shields on cones \( C \) via \( \text{ID} \)-maps to a general singular \( H \in \mathcal{G} \) since they intersect \( \Sigma_{C} \). Instead we transfer periodic bumps from cones and use them to reproduce shields on \( H \). (But we use the shields on \( C \) to derive estimates for shields on \( H \) in 3.9 below.) We consider \( \bigcup_{v \in S}(Q_{1}^{n-k} + v) \subset Q_{L, \ldots, L} \subset \mathbb{R}^{n-k} \), for some \( L > 0 \), and the periodic bump \( \square(\bigcup_{v \in S}(Q_{1}^{n-k} + v)) \) on \( \mathbb{R}^{n-k} \times C^{k} \in \mathcal{S}_{C} \) for \( \omega_{0}(n, L), R_{0}(n, L), r_{0}(n, L) \) with \( \omega \in (0, \omega_{0}) \), \( R > R_{0} \), \( r \in (0, r_{0}) \) and some \( \beta_{\star}(\omega, r, R) \in (0, \beta_{n-k} \cdot \eta_{n-k}) \):

\[
\text{supp} \square(\bigcup_{v \in S}(Q_{1}^{n-k} + v)) \subset \delta_{(A)}^{-1}(\mathbb{R}^{\geq \beta \star}) \cap B_{R}(0) \subset \text{TP} \mathbb{R}^{n-k} \times C^{k}(0, \omega, R, r)
\]

**Corollary 3.7 (Periodic Bumps on \( H \))** For \( H \in \mathcal{G} \), some \( p \in \Sigma_{H} \) and a tangent cone \( (\mathbb{R}^{n-k} \times C^{k}, 0) \) in \( p \) so that the \( \text{ID} \)-map over \( \text{TP} \mathbb{R}^{n-k} \times C^{k}(0, \omega, R, r) \subset C^{n} \), for \( \omega \in (0, \omega_{0}) \), \( R > R_{0} \), \( r \in (0, r_{0}) \), with \( \Psi_{\omega} \) appropriately normalized, satisfies, for some \( \varepsilon > 0 \):

\[
|\text{ID} - id_{C}|_{C^{1}(\text{TP} \mathbb{R}^{n-k}(0, \omega, R, r))} \leq \varepsilon \quad \text{and} \quad |\Psi_{\omega} \circ \text{ID}/\Psi_{\omega} - 1|_{C^{2, \alpha}(\text{TP} \mathbb{R}^{n-k}(0, \omega, R, r))} \leq \varepsilon.
\]

For some small \( \varepsilon_{0}(n) \in (0, 1/10), \varepsilon \in [0, \varepsilon_{0}] \), (86) shows that \( \Phi_{H} := \Phi_{\square} \circ \text{ID}^{-1} \) is well-defined and extends to \( H \setminus \Sigma_{H} \) with

\[
L_{H}^{\ast}(\Phi_{H} \cdot \Psi_{H}) \geq \lambda/16 \cdot (A)^{2} \cdot \Phi_{\square} \cdot \Psi_{H} \text{ on } H \setminus \Sigma_{H}.
\]

We call \( (\Phi_{H} \cdot \Psi_{H})^{4/(n-2)} \cdot g_{H} \) a periodic bump metric and the pseudo metric \( ((\Phi_{H} - 1) \cdot \Psi_{H})^{4/(n-2)} \cdot g_{H} \) a periodic bump.

**Remark 3.8** We have \( \Phi_{H} \in C(H^{n}, \Lambda[n]) \), \( \text{supp} |\Phi_{H} - 1| = \text{supp}(\square \circ \text{ID}^{-1}) \). When we add \( \square := \square \circ \text{ID}^{-1} \) to \( \Psi_{H}^{4/(n-2)} \cdot g_{H} \) this may change the conformal class since \( \text{ID} \) need not to be conformal. Using the multiplicative local bump \( \Phi_{\square} \) resolves this issue. The two resulting deformations coincide in the limit of \( \varepsilon \rightarrow 0 \) and we continue speaking of added local bumps but technically we use the multiplicative version.

**Tightness of Bumps on \( H \)** We use the Ahlfors \( n \)-regularity of \( (H, d_{S}^{1}, \mu_{S}^{1}) \) to extend the tightness result 2.4 to the \( \text{ID} \)-images of bump elements in \( H \in \mathcal{G} \), by some perturbation argument when \( H \) is close to a product tangent cone in the sense that (87) of Lemma 3.7 holds for \( \varepsilon \in [0, \varepsilon_{1}] \), for some \( \varepsilon_{1}[n, \ell] \in (0, \varepsilon_{0}[n, \ell]) \). In general \( \Sigma_{H} \) has no local product structure.
For a given product tangent cone \( \mathbb{R}^{n-k} \times C^k \) we can, however, use induced structures when \( H \) is (locally) closely approximated by \( \mathbb{R}^{n-k} \times C^k \) in a common ambient space \( \mathbb{R}^{n+1} \).

(89) \[ \Omega_\ell := Q^{n-k}_\ell \times \mathbb{R}^{k+1}, \] matching (51) from \( Q^{n-k}_\ell \times C^k = \mathbb{R}^{n-k} \times C^k \cap Q^{n-k}_\ell \times \mathbb{R}^{k+1}, \]

(90) \[ O_\rho^c := \mathbb{R}^{n-k} \times (C^k \cap B^{k+1}_\rho(0)) \) and \( O_\rho^H := H \cap \mathbb{R}^{n-k} \times B^{k+1}_\rho(0), \)

keeping in mind that \( O_\rho^H \) depends on the chosen approximation by \( \mathbb{R}^{n-k} \times C^k \) in a common ambient space \( \mathbb{R}^{n+1} \). For \( \varepsilon > 0 \) small enough so that for \( \vartheta_i[n,C] \in (0,1), \varrho \in (\vartheta_1, \vartheta_2) \) and \( \zeta = 1/4 \cdot \min\{|\varrho - \vartheta_1|, |\varrho - \vartheta_2|\} \) of 2.4, we have for \( \Psi_\varrho \) appropriately normalized:

(91) \[ |\text{ID} - \text{id}_C|_{C^{2,\alpha}(\delta_\varrho^{-1}(\mathbb{R}^{2^*}) \cap O_\rho^c, \Omega_{\ell+1})} \leq \varepsilon, \]

(92) \[ |\Psi_H \circ \text{ID}/\Psi_\varrho - 1|_{C^{2,\alpha}(\delta_\varrho^{-1}(\mathbb{R}^{2^*}) \cap O_\rho^c, \Omega_{\ell+1})} \leq \varepsilon, \]

and after small adjustments of the \( \text{ID} \)-map near the boundary in \( C^{2,\alpha} \)-norm for small \( \varepsilon > 0 \):

(93) \[ \text{ID}(\delta_\varrho^{-1}(\mathbb{R}^{2^*}) \cap O_\rho^c, \Omega_{\ell+1}) + \delta_\varrho^{-1}(\mathbb{R}^{2^*}) \cap O_\rho^c, \Omega_{\ell+1}) = \delta_\varrho^{-1}(\mathbb{R}^{2^*}) \cap O_\rho^c, \Omega_{\ell+1}) \leq \varepsilon. \]

As a counterpart of \( T^C_\varrho \) in 2.4 we set \( T^H_\varrho := \Omega_{\ell+1} \cap \partial O_\rho^H \). For almost every \( \rho > 0 \), \( \Omega_{\ell+1} \cap O_\rho^H \) is a Caccioppoli set. Since \( T^C_\varrho \subset \mathbb{R}^{n-k} \times C^k \) is the level set of the radial distance function \( d \) with \( |\nabla d| = 1 \), \( T^H_\varrho \cap \delta_\varrho^{-1}(\mathbb{R}^{2^*}) \) is smooth for small \( \varepsilon > 0 \) and we have a smooth convergence

(94) \[ \text{ID}^{-1}(T^H_\varrho \cap \delta_\varrho^{-1}(\mathbb{R}^{2^*})) \cap \Omega_{\ell+1} \to T^C_\varrho \cap \delta_\varrho^{-1}(\mathbb{R}^{2^*}) \cap \Omega_{\ell+1} \subset C^n, \text{ for } \varepsilon \to 0. \]

In general, \( T^H_\varrho \) is not a minimal hypersurface. We use \( T^H_\varrho \) to prescribe an appropriate Plateau boundary data in the following extension of 2.4 for \( \zeta := 102/100 \cdot \zeta \).

**Proposition 3.9 (Tightness and ID-Maps)** There is an \( \varepsilon_1[n,\ell] \in (0, \varepsilon_0[n,\ell]) \) so that for \( \varepsilon \in [0,\varepsilon_1] \) in 3.7, \( \beta \in [0, \beta^*] \), \( G^H_\beta \subset C[H^n, \Lambda] \) with

(95) \[ \sup \{|G^H_\beta - 1| \cap \text{supp} |F_\beta \circ \text{ID}^{-1} - 1| \cap O_\rho^H \cap \Omega_{\ell+1} \} \leq \emptyset. \]

(i) there is a Caccioppoli set \( U^H[l, n, k, H^n] \subset H \cap \Omega_{\ell+1} \cap U^\\beta \subset O_\rho^H \cap \Omega_{\ell+1} \) so that \( \mathcal{T}^H_{\varrho} \subset O^H_{\rho+c} \cap \Omega_{\ell+1} \) and \( \mathcal{T}^H_{\varrho} \cap \Omega_{\ell+1} \) is area minimizing with \( \Omega_{\ell+1} \)-boundary value \( \mathcal{T}^H_{\varrho} \), relative to \( (H, d^{\lambda}_{\varrho}, \mu^{\lambda}_{\varrho}) \) associated to \( (G^H_\beta, F_\beta \circ \text{ID}^{-1}, \Psi_H)^4(n-2) \cdot g_{H^n} \)

(ii) for any such oriented minimal boundary \( \mathcal{T}^H_{\varrho} \subset O^H_{\rho+c} \cap \Omega_{\ell+1} \), allowing intersections with the obstacle \( \partial(O^H_{\rho+c} \cap \Omega_{\ell+1}) \), we already have \( \mathcal{T}^H_{\varrho} \subset O^H_{\rho+c} \cap \Omega_{\ell+1} \). We write \( \mathcal{P}_H[n-k,\ell, G^H_\beta] \) for the class of all such Plateau solutions \( \mathcal{T}^H_{\varrho} \).

**Proof** We assume there is no such approximation threshold \( \varepsilon_1[n,\ell] > 0 \), then we have sequences \( H_i \in \mathcal{G}_n \) and of singular minimal cones \( \mathbb{R}^{n-k} \times C^k \), so that the pairs \( H_i \subset \mathbb{R}^{n-k} \times C^k \) satisfy (91) and (92) for some \( \varepsilon_i \to 0 \), when \( i \to \infty \), violating (i). (As in 2.4, claim (ii) follows from the argument for (i).) Since the cones subconverge to another such cone \( \mathbb{R}^{n-k} \times C^k \), we may assume that \( \mathbb{R}^{n-k} \times C^k = \mathbb{R}^{n-k} \times C^k = C^n \). We consider an area minimizer \( \mathcal{T}^H_{\varrho} \) solving the obstacle problem for open Caccioppoli sets \( U_i \) with \( O^H_{\rho+c} \cap \Omega_{\ell+1} \subset U_i \cap \Omega_{\ell+1} \subset O^H_{\rho+c} \cap \Omega_{\ell+1} \) so...
that $\partial U_1$ has $\Omega_\epsilon$-boundary value $\mathcal{H}$. As in the cone case 2.4 we can assume that $\partial \Omega_\epsilon \cap \mathcal{H}$ is locally outer minimizing. Namely, for $i$ large enough, we can replace the intersection of the hyperplanes (in the ambient space) with $\mathcal{H}_i \subset \mathcal{H}_i$ by Hausdorff approximating area minimizing hypersurfaces in $\mathcal{H}_i$, where we use the hyperplanes to define the boundary value. This follows from Cor.1.4 (13) using a tightness argument as in 2.4. In the following we set $\Omega = \Omega_\epsilon$.

**Step 1 (Using $C$ for Estimates in $\mathcal{H}$)** The first step is to use $\mathcal{H}$ to upper estimate the area of $\mathcal{H}$. For such a comparison, we use $\mathcal{H}$-maps to embed $\mathcal{H}$ and $\mathcal{H}$ into the same space $\mathcal{H}$. These maps are, however, controllably defined only away from the singular set, typically on $\mathcal{H}$, convergence shows that for $x \in \mathcal{H}$, $\mathcal{H}$ also holds for the corresponding balls, up to a factor $\mathcal{H}$. From 1.6 we may assume (after generic changes of the radii in $\mathcal{H}$) for $\mathcal{H}$, $\mathcal{H}$ follows from Cor.1.4 (13) using a tightness argument as in 2.4. In the following we set $\Omega = \Omega_\epsilon$.

**Step 2 (Compact Convergence in $\mathcal{H}$)** Now we exchange the roles of $\mathcal{H}$ and $\mathcal{H}$ and consider $\mathcal{H}$-ball extensions of $\mathcal{H}$-maps to embed $\mathcal{H}$ and $\mathcal{H}$ into the same space $\mathcal{H}$. These maps are, however, controllably defined only away from the singular set, typically on $\mathcal{H}$, convergence shows that for $x \in \mathcal{H}$, $\mathcal{H}$ also holds for the corresponding balls, up to a factor $\mathcal{H}$. From 1.6 we may assume (after generic changes of the radii in $\mathcal{H}$) for $\mathcal{H}$, $\mathcal{H}$ follows from Cor.1.4 (13) using a tightness argument as in 2.4. In the following we set $\Omega = \Omega_\epsilon$. 

**Step 3 (Using $C$ for Estimates in $\mathcal{H}$)** The first step is to use $\mathcal{H}$ to upper estimate the area of $\mathcal{H}$. For such a comparison, we use $\mathcal{H}$-maps to embed $\mathcal{H}$ and $\mathcal{H}$ into the same space $\mathcal{H}$. These maps are, however, controllably defined only away from the singular set, typically on $\mathcal{H}$, convergence shows that for $x \in \mathcal{H}$, $\mathcal{H}$ also holds for the corresponding balls, up to a factor $\mathcal{H}$. From 1.6 we may assume (after generic changes of the radii in $\mathcal{H}$) for $\mathcal{H}$, $\mathcal{H}$ follows from Cor.1.4 (13) using a tightness argument as in 2.4. In the following we set $\Omega = \Omega_\epsilon$.
write the open sets they bound as \( W_i \). From (100) we get a flat norm subconvergence to some area minimizer, we write it again as \( \mathcal{T}_C^\delta \cap \Omega \) bounding \( U_C \). We claim this implies compact Hausdorff convergence on \( C \setminus \Sigma_C \). Assume that there is a ball \( B_r(p) \in C \setminus \Sigma_C \) so that \( B_r(p) \cap \mathcal{T}_C^\delta = \emptyset \) and \( p \in W_i \Delta U_C \) for all \( i \). Then we consider the \( \text{ID} \)-map image where \( \mathcal{T}_{H_i}^\delta \) is an area minimizer to argue from Cor. 1.4 (13) that \( \mu_S^\delta C(W_i \Delta U_C) > c \), for some \( c > 0 \) independent of \( i \), contradicting the flat norm convergence.

**Step 3 (Growth Estimates in \( H_i \))** We return to \( H_i \) to derive (i) from a contradiction. From step 2 we know after applying the \( \text{ID} \)-map that

\[
\mathcal{T}_{H_i}^\delta \cap \delta_{(A)}^{-1}((\mathbb{R}^\omega) \cap \Omega) \to \text{ID}(\mathcal{T}_C^\delta \cap \delta_{(A)}^{-1}((\mathbb{R}^\omega) \cap \Omega)) \quad \text{for} \quad i \to \infty
\]

in Hausdorff-norm. We consider \( \xi \)-ball extensions of \( \mathcal{T}_{H_i}^\delta \cap \delta_{(A)}^{-1}((\mathbb{R}^\omega) \cap \Omega) \) and assume that for any \( i \) there is some \( p_i \in \mathcal{T}_{H_i}^\delta \cap \delta_{(A)}^{-1}((\mathbb{R}^\omega) \cap \Omega) \cap \partial(O_{e+\zeta}^H_i \setminus O_{e-\zeta}^{H_i}) \), for \( \zeta = 101/100 \cdot \zeta \). Then we get from Corollary 1.4 (13): \( \mu_S^{k,n-1}[\beta, G_\beta](\mathcal{T}_{H_i}^\delta \cap B_{1/100}(p_i)) > c \) for some \( c > 0 \) independent of \( i \). This contradicts the Hausdorff closeness of \( \mathcal{T}_{H_i}^\delta \cap \delta_{(A)}^{-1}((\mathbb{R}^\omega) \cap \Omega) \to \text{ID}(\mathcal{T}_C^\delta \cap \delta_{(A)}^{-1}((\mathbb{R}^\omega) \cap \Omega)) \) for large \( i \) from (101), for sufficiently small \( \xi > 0 \).

To formulate the generalization of Cor. 2.9 we think of \( H \) as being *locally* closely approximated by \( \mathbb{R}^{n-k} \times C^k \) in a common ambient space \( \mathbb{R}^{n+1} \). We consider \( O_{q,\xi}^{\eta_1,\xi_1} \subset \mathbb{R}^{n-k} \times C^k \) defined from (63). \( O_{q,\xi}^{\eta_1,\xi_1} \) is union of trimmed cylinders we can write as intersections of trimmed cylinders \( Z_f^{n+1} \subset \mathbb{R}^{n+1} \) surrounding the faces \( f \) of \( Q_1^{n-k} \times \{0\} \subset \mathbb{R}^{n-k} \times C^k \subset \mathbb{R}^{n+1} \):

\[
(102) \quad T[\eta_1, f](O_\xi \cap \Omega_q) =: Z_f^{n+1} \cap (\mathbb{R}^{n-k} \times C^k) \subset \mathbb{R}^{n-k} \times C^k
\]

We use the intersection of \( Z_f^{n+1} \) with \( H \) to define the *core* of the periodic bump in \( H \) we get for \( \bigcup_{v \in S}(Q_1^{n-k} + v) \subset \mathbb{R}^{n-k} \subset \mathbb{R}^{n-k} \cap \mathbb{R}^{n-k} \times C^k \).

\[
(103) \quad O_{q,\xi}^{\eta_1,\xi_1} \oplus v := (Z_f^{n+1} + v) \cap \Omega_q \subset \mathbb{R}^{n-k} \subset \mathbb{R}^{n-k} \times C^k
\]

where we assume an accuracy \( \varepsilon \in [0, \varepsilon_1] \), as in 3.7 and 3.9. We transfer the original construction on cones for each \((q)\)-face viewed as a subset of \( \mathbb{R}^q \times (\mathbb{R}^{n-k-q} \times C^k) \) with the assumptions and the notations of Cor. 3.9, where we redefine \( \zeta := 1021/1000 \cdot \zeta \).

**Corollary 3.10 (Periodic Shields on \( H \))** There is an \( \varepsilon_2(n) > 0 \) so that for an approximation with \( \varepsilon \in [\varepsilon_2, \varepsilon_1] \) as in (3.7), \( G \in \mathcal{C}(H^n, \Lambda[n]) \) with \( \supp |G - 1| \cap \supp |\Phi H| = \emptyset \) and the \( \ell_q, \eta_q, \beta_q \) from 2.8 we have for any \((q)\)-face \( f \in F_q \) of \( Q_1^{n-k} \times \{0\} \) and \( v \in S \) some

\[
(104) \quad T_{f+v} = \partial U_{f+v} \in \mathcal{P}_H[\eta, \ell_q, G_{\beta_q}] \quad \text{for an open} \quad O_{q,\xi_1,\xi_1}^{\eta_1,\xi_1} \oplus v \subset U_{f+v} \subset O_{q,\xi_1,\xi_1}^{\eta_1,\xi_1} \oplus v,
\]

where \( T_{f+v} \) is an area minimizer of 3.9 for the \( \text{ID} \)-map image of a bump element assigned to the face \( f + v \) for some \( G_{\beta_q} \in \mathcal{C}(\mathbb{R}^n, \Lambda[n]) \) with \( G_{\beta_q} := G \circ \text{ID} \circ (T[\eta_1, f] + v) \) on \( \delta_{(A)}^{-1}((\mathbb{R}^\omega) \cap \Omega_{e+2\zeta}) \cap \Omega_q \). From these \( U^H \) we get a periodic shield:

\[
(105) \quad \mathbb{U} := \bigcup_{q=0}^{n-k} \mathbb{U}_{q,\xi_1,\xi_1} := \bigcup_{q=0}^{n-k} \bigcup_{f \in F_q} U_f^H,
\]
with \( U \supset \Omega_{n-k}(\bigcup_{v \in S}(Q_1^{n-k} + v)) \supset \Sigma_H \cap \Omega_{n-k}^H(\bigcup_{v \in S}(Q_1^{n-k} + v)) \) so that

\[
(106) \quad \partial U \text{ is a local inner area minimizer relative to } U \text{ and } (G \cdot \Phi^H \cdot \Psi_o)^{4/n-2} \cdot g_H.
\]

For any \( a \in \mathbb{Z}^2 \) there is some \( \varepsilon^*_a(n, a) \in (0, \varepsilon_2(n)] \) so that after rescaling the periodic bump on \( \mathbb{R}^{n-k} \times C^k \), described in 2.9 and 2.10, we get (up to uniform factors for the radii, when compared to the cone case, converging to 1, for \( \varepsilon \to 0 \), we omit for the sake of readability):

(i) \( \text{ID}(U_{2^{-a}}(\bigcup_{v \in S}(Q_1^{n-k} + v)) \setminus U_{2^{-a} \cdot r^o[n]}(\mathbb{R}^{n-k} \times \{0\})) \supset \text{supp} \square_H(\bigcup_{v \in S}(Q_1^{n-k} + v)) \),

(ii) \( \{ z \in C \setminus \Sigma_C | 2^{-a} \geq \delta(A)(z) \geq 2^{-a} \cdot r^o[n] \} \supset \text{supp} \square[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) \),

(iii) \( \bigcup[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) \supset \Omega_{n-k}^H[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) \supset \Sigma_H \cap \Omega_{n-k}^H[a](\bigcup_{v \in S}(Q_1^{n-k} + v)) \),

where \( \bigcup[a] \) and \( \Omega_{n-k}^H[a] \) are the counterparts of the rescaled shields and shielded cores in 2.9 for the scaled faces \((q)-\text{face} f \in F_q \) of \( Q_1^{n-k} \times \{0\} \) and their ID-map transfer to \( H \).

**Proof** We define \( \varepsilon_2(n) \) to be the minimum of the \( \varepsilon_1[n, \ell_q] \), which we multiply by \( \eta_q \) to compensate the scaling of \((q)\)-bump elements in Cor. 3.9. With this choice, the same bump element-wise argument as in Prop. 2.8, based on the tightness result 3.9, applies. \( \Box \)

**Remark 3.11 (Almost Tangent Cones)** In the results 3.7 - 3.10 on the transfer and evaluation of periodic bumps we only mentioned the typical case where \( (\mathbb{R}^{n-k} \times C^k, 0) \) is a tangent cone in a singular point \( p \in H \). We can slightly relax the coupling to \( H \) since we only used the presence of a quantitatively controlled approximation by \( \mathbb{TP}_{\mathbb{R}^{n-k} \times C^k}(0, \omega, R, r) \subset (\mathbb{R}^{n-k} \times C^k, 0) \). The results remain valid provided the ID-map over \( \mathbb{TP}_{\mathbb{R}^{n-k} \times C^k}(0, \omega, R, r) \) satisfies the conditions of (3.7) for some \( \varepsilon \in (0, \varepsilon_2] \). In this case we call \((\mathbb{R}^{n-k} \times C^k, 0)\) an \( \varepsilon \)-almost tangent cone. \( \Box \)

### 3.3 Global Bumps and Alignments

When assembling a global bump we use self-similar ball covers to suitably place almost periodic bumps along the singular set \( \Sigma_H \subset H \). An essential observation, used to transfer the bumps, is that singular directions in any two tangent cones associated to neighboring balls align in a simple way. We start with a model situation that will be the limit case we use to understand fine tangent cone approximations for self-similar covers of high accuracy.

**Lemma 3.12 (Linear Auto-Alignments)** For \( H \in H_n^\mathbb{R} \) and \( p_1, \ldots, p_m \in \Sigma_H \), let \( C_{p_1}, \ldots, C_{p_m} \) be tangent cones with basepoints \( p_1, \ldots, p_m \) so that \( H = C_{p_1} = \ldots = C_{p_m} \subset \mathbb{R}^{n+1} \). When the affine span of \( \{p_1, \ldots, p_m\} \) is \( q \)-dimensional we have (up to a common rotation and translation)

\[
(107) \quad H = \mathbb{R}^q \times C^{n-q} \text{ and } p_1, \ldots, p_m \in \mathbb{R}^q \times \{0\}, \text{ for some cone } C^{n-q} \in SC_{n-q},
\]

in particular, we note that \( q \leq n - 7 \).

**Proof** We may assume that \( p_1 = 0 \). \( C_{p_1} \) is scaling invariant and, thus, from scaling \( C_{p_1} \) by \( \lambda > 0 \) around 0 we infer that for \( C_{p_2} \) with basepoint \( p_2 \neq 0 \), using \( H = C_{p_1} \) in the first equality and \( H = C_{p_2} \) in the second and last one:

\[
H = \lambda \cdot H = \lambda \cdot C_{p_2} = C_{p_2} - (1 - \lambda) \cdot p_2 = H - (1 - \lambda) \cdot p_2.
\]

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From this we see that $H$ is translation invariant in $p_2$-direction. After suitable rotation and translation this means $H^n = \mathbb{R} \times C^{n-1}$, with $p_1, p_2 \in \mathbb{R} \times \{0\}$, for some cone $C^{n-1} \in SC_{n-1}$. For $p_3, \ldots, p_m$ we continue inductively and note that the step is non-trivial only when the newly added point does not belong to the span of its predecessors.

We can rewrite (107) as $B_1(p_k) \subset O^{R^q \times C^{n-q}}_1 \subset \mathbb{R}^q \times C^{n-q}$, for any $k = 1, \ldots, m$. In the following we show that a completely similar inclusion locally holds for any layer $A[i]$ in a self-similar cover of some sufficiently high accuracy.

**Proposition 3.13 (Auto-Alignments)** For $H \in C^\varepsilon_n$, $R > 10^2$ and $\eta \in (0, \varepsilon_2)$, so that any self-similar cover of accuracy $\varepsilon \in (0, \varepsilon_3)$, $s \in (0, s_0)$ and $p \in A^*$ there is an $\eta$-almost tangent cone $C_p = \mathbb{R}^m \times C^{n-m}$, $m \in \mathbb{Z}^0$ in $p$ with

$$B_1(z) \subset O^{s^{-1}H}_2$$ for any $B_1(z) \subset B_R(p) \subset s^{-i} \cdot H$ with $z \in A[i]$ when $p \in A^*[i]$.

In general, and even in a fixed point $p$, $m$ is not uniquely determined.

**Proof** Otherwise we had some $\eta \in (0, \varepsilon_2)$ and a sequence $\varepsilon_k \rightarrow 0$, for $k \rightarrow \infty$, so that there are self-similar covers of accuracy $\varepsilon_k$ and points $p_k \in A^*[i_k]$, so that

$$\bigcup_{z \in A[i_k], B_1(z) \subset B_R(p_k)} B_1(z) \not\subset O^{s^{-1}H}_2$$

for any $\eta$-almost tangent cone $C$ in $p_k$ recalling from (90) that $O^H_\rho$ is defined relative to $C$.

- The number of balls $B_1(q_k)$ with $B_1(q_k) \subset B_R(p_k) \subset s^{-i_k} \cdot H$ and $q_k \in A[i_k]$ is uniformly upper estimated depending only on $n$ and $R$ from the Ahlfors $n$-regularity of $(H, \mu_H)$, giving lower and upper volume bounds for the $B_{s^{i_k}/2}(q_k), B_{R-s^{i_k}}(p_k) \subset H$ as in Prop.3.5 since $B_{s^{i_k}/2}(q_k) \cap B_{R-s^{i_k}}(p_k) = \emptyset$, for $q_k \neq q_k^* \in A[i_k]$.

- Since $H$ is compact, we observe that $i_k \rightarrow \infty$, for $k \rightarrow \infty$. After selecting a subsequence the pairs

$$((1/s^{i_k} \cdot H, p[i_k]), (1/s^{i_k} \cdot (B_{(R-1),s^{i_k}}(p_k) \cap A[i_k])))$$

of pointed spaces and of finite subsets, of upper bounded cardinality, converge to some limit $((H_\infty, p_\infty), B_{R-1}(p_\infty) \cap A_\infty)$, where $H_\infty \in H^0_n$ and $B_{R-1}(p_\infty) \cap A_\infty \subset \Sigma_{H_\infty}$, from Allard theory. In turn, for $k \rightarrow \infty$, the refining $\varepsilon_k$-accuracy and the compactness of the cone space $SC_n$ show that $H_\infty$ actually is a cone $C_\infty$ and any sequence of $\varepsilon_k$-accurately approximating tangent cones $C_{p_k}$, with basepoint $p_k$, subconverges to $C_\infty$. From this $C_\infty$ is an $\eta_k$-almost tangent cone of $(1/s^{i_k} \cdot H, p_k)$, for some $\eta_k \rightarrow 0$, when $k \rightarrow \infty$.

- For further subsequences there are $\varepsilon_k$-accurately approximating tangent cones in the points $1/s^{i_k} \cdot (B_{(R-1),s^{i_k}}(p_k) \cap A[i_k])$ converging to tangent cones of $C_\infty$ in corresponding points of $A_\infty$. Each of the limit cones coincides with $C_\infty$ since $\varepsilon_k \rightarrow 0$. That is, we reach a configuration as in 3.12 and thus we have $C_\infty = \mathbb{R}^m \times C^{n-m}$, for some $m \in \mathbb{Z}^0$ and $B_{R-1}(p_\infty) \cap A_\infty \subset O^{\mathbb{R}^m \times C^{n-m}}_1$.

For large enough $k$, $C_\infty$ is an $\eta_k$-almost tangent cone with $\eta_k \leq \min\{\eta, \varepsilon_2\}$ and the convergence of points in $A[i_k]$ to corresponding points in $A_\infty$ eventually contradicts (109) for $(1/s^{i_k} \cdot H, p_k)$. This proves the claim for $H \in C^\varepsilon_n$, that is, we get an $\varepsilon_3(H, \eta, R)$ as asserted.
In the argument we do not need that the hypothetical sequence of points \( p_k \) belongs to a fixed \( H \in G_n^c \), that is, after an appropriate adjustment of the \( i_k \), we even get an \( \varepsilon_3(n, \eta, R) \) independent of \( H \).

\[ \text{(110)} \]

\[ \text{supp} \Box_b \subset \{ z \in B_{3^{-k}R_n}(0) \mid \gamma_b^+ \geq \delta_{(A)}(z) \geq \gamma_b^- \} \]

for \( \gamma_b^\pm = 2m_b^\pm \), for some \( m_b^\pm \in \mathbb{Z}^{\geq 1} \), depending only on \( n \), with \( \gamma_b^+ > \gamma_b^- \geq 4 \) and \( \gamma_{b+1} \geq 2 \cdot \gamma_b^+ \), and so that each of the \( \Box_b \) provides a shield \( \cup_b^C \) for \( B_2^{n-k}(0) \), that is,

\[ \{ z \in B_{3^{-k}R_n}(0) \mid \gamma_b^+ \geq \delta_{(A)}(z) \} \supset \cup_b^C \supset B_2^{n-k}(0). \]

For the transfer to \( H \in G_n^c \) we choose an \( \varepsilon_4 \in (0, \varepsilon_3] \) so that \( R_n \in [10^2, \varepsilon^{-1}/10^2] \), for any \( \varepsilon \in (0, \varepsilon_4] \), and we consider a self-similar cover \( A^* \) of accuracy \( \varepsilon \).

\[ \text{Figure 6: Disjoint placement of periodic bumps } \Box_1 \text{ and } \Box_2 \text{ for intersecting balls in different sublayers of } \mathcal{A}^*[i]: \mathcal{A}^*[i,1] = \text{dashed lines and } \mathcal{A}^*[i,2] = \text{straight lines.} \]
Step 2 (Transfer to \(H\)) For any of the sublayers \(A^*[i, j]\) of \(A^*[i]\), we consider the balls \(B_{R_n,i,s}(q) \in A^*[i, j]\) and, after scaling by \(s^{-1}\), we transfer \(\square\) via \(\text{ID}\)-map from some \(\eta\)-almost tangent cone as provided by Prop.3.13 to \(B_{R_n,i,s}(q)\). From Prop.3.10 and Prop.3.13 we can choose the accuracy \(\varepsilon \in (0, \varepsilon_3)\) so that for \(\eta \in (0, \varepsilon_2)\) the transfer of the periodic bumps to \(B_{R_n,i,s}(q)\) yields a periodic shield \(\cup_j \subset B_{R_n,i,s}(q)\) that contains all \(B_{s^2}(z) \in B_{R_n,i,s}(q)\) with \(z \in A[i]\). We repeat this for any \(j\) and any ball in \(A^*[i, j]\) and, hence, in \(A^*[i]\) and notice that the periodic shields are disjointly supported and contain all balls in \(A[i]\).

Step 3 (Global Shield) We repeat this bump placement for all layers \(A^*[k]\) of \(A^*\) with the same estimates, up to appropriate scaling, for the bumps and shields. From estimate (75), which carries over to \(H\) and \(p \in \Sigma_H\), we observe that for \(s > 0\) small enough we get disjointly supported periodic bumps. We recall from Prop.3.5(ii) that \(\bigcup_{i \leq k} A[i]\) covers \(\bigcup_{i \leq k} \Sigma_i\). From Cor.3.10 each of the, at first infinitely many, periodic bumps we associated to the balls in \(A^*\) shields a core independent of the placeholder function \(G \in \mathcal{C}[H, \Lambda]\) we use for that particular bump. These cores form an open cover of the compact set \(\Sigma_H\). We take a finite subcover.

We define the global bump metric from adding the periodic bumps associated to the cores in this finite subcover to \((H, d_\Sigma, \mu_\Sigma)\). Now, we consider each of these periodic bumps \(\square\) separately, and get a shield \(\cup_{\square}\) relative to the global bump metric. The union \(\cup_{\square}\) of all these \(\cup_{\square}\) is bounded by locally inner minimizer \(\partial \cup_{\square}\) from the same argument as in the proof of Prop.2.8. For suitably small \(\varepsilon > 0\) we can also make sure that \(\cup_{\square} \subset W\).

Finally, we show how 3.14 implies our main splitting result.

**Theorem 3.15 (Splitting with Boundary)** Let \(H^n \subset M^{n+1}\) be an almost minimizer \(H \in \mathcal{G}^c\) with singular set \(\Sigma_H\), so that the conformal Laplacian \(\langle A \rangle^{-2} \cdot L_H\) has a positive principal eigenvalue \(\lambda^A_H > 0\). Then there are arbitrarily small neighborhoods \(U\) of \(\Sigma\) so that \(H \setminus U\) is conformal to a scal > 0-manifold \(X_U\) with minimal boundary \(\partial X_U\).

This is Theorem 3 from the introduction. It contains Theorem 1 as a special case from [L4, Theorem 2(i)]. For Theorem 2, where \(H\) has a boundary \(\partial H \cap \Sigma = \emptyset\), we note that the assembly of a global bump and the proof of its shielding effect only use properties close to \(\Sigma\). This leaves the argument unchanged, cf. [L1, Rm. 1.13] and [L4, Rm. 3.10], and, hence, the following argument equally applies to Theorem 2.

**Proof** For \(\varepsilon > 0\) we consider the \(10 \cdot \varepsilon\)-neighborhood \(U_{10\varepsilon}\) of \(\Sigma_H\) in \((H, d_\Sigma, \mu_\Sigma)\). Using smooth approximations of the distance function we may assume that \(\partial U_{10\varepsilon}\) is smooth. Now we make an auxiliary deformation that, however, becomes invisible towards the end of the argument. We deform \(H\) in an \(\varepsilon\)-distance tube of \(\partial U_{10\varepsilon}\) so that \(\partial U_{10\varepsilon}\) becomes positively mean curved relative to \(U_{10\varepsilon}\). Then \(\partial U_{10\varepsilon}\) is locally inner minimizing relative to \(H \setminus U_{10\varepsilon}\).

We claim that for any \(\eta > 0\) there is a neighborhood \(\eta \subset U_{10\varepsilon}\) of \(\Sigma_H\) with

\[
\text{dist}(\partial V, \partial U_{10\varepsilon}) > 5\varepsilon \quad \text{and} \quad \mu_\Sigma^{\Lambda_{10\varepsilon}}(\partial V) \leq \eta.
\]

To check (112) we use Cor. 1.6: for any \(\eta > 0\) and \(r \in (0, 1)\) there is some ball cover \(B_{r_i}(p_i)\), \(p_i \in \Sigma, i \in I\), of \(\Sigma\) with \(r_i \leq r\) so that \(\sum_{i \in I} \mu_\Sigma^{\Lambda_{10\varepsilon}}(\partial B_{r_i}(p_i), d_\Sigma) \leq b^\circ \cdot \sum_{i \in I} r_i^{\Lambda_{10\varepsilon}} < \eta.\) For sufficiently small \(r \in (0, \varepsilon)\), \(V_\varepsilon := \bigcup_{i \in I} B_{r_i}(p_i)\) satisfies (112). From 3.14 we can choose a global bump \(\square\) supported in \(V_\varepsilon\). Since \(\partial U(\square)\) is locally inner
minimizing relative to $U$, we get an open Caccioppoli set $W_\eta$ with $U \subset W_\eta \subset U_{10\cdot \varepsilon}$ so that $\partial W_\eta$ is area minimizing under all such $W$ and thus

$$(113) \quad \mu_S^{\Lambda,n-1}(\partial W_\eta) \leq \mu_S^{n-1}(\partial V_\eta) \leq \eta.$$ 

In turn, we get from Cor. 1.4 (13) some $a > 0$, so that for any ball $B_\varepsilon(p)$ for some $p \in U_{10\cdot \varepsilon}$ with $\text{dist}(p, \partial U_{10\cdot \varepsilon}) = 3 \cdot \varepsilon$ and any area minimizer $L^{n-1} \subset (H, d_S, \mu_S)$ passing through $p$ we have $\mu_S^{n-1}(L^{n-1} \cap B_\varepsilon(p)) = \mu_S^{\Lambda,n-1}(L^{n-1} \cap B_\varepsilon(p)) > a$ and, hence, for $\eta > 0$ small enough, $\text{dist}(\partial W_\eta, \partial U_{10\cdot \varepsilon}) \geq 2 \cdot \varepsilon$. That is, $W_\eta$ is disjoint from the support of the auxiliary deformation of $(H, d_S, \mu_S)$ in the $\varepsilon$-neighborhood of $\partial U_{10\cdot \varepsilon}$. Summarizing, we find for any $\varepsilon > 0$ some $\eta > 0$ and some global bump $\otimes$, with support in $V_\eta$, so that $(\Phi^H_\otimes \cdot \Psi_H)^{4/(n-2)} \cdot g_H$ and $U := W_\eta$ have the asserted properties. □

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