Regular level sets of Lyapunov graphs of nonsingular Smale flows on 3-manifolds

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Abstract: In this paper, we first discuss the regular level set of a nonsingular Smale flow (NSF) on a 3-manifold. The main result about this topic is that a 3-manifold $M$ admits an NSF flow which has a regular level set homeomorphic to $(n + 1)T^2$ ($n \in \mathbb{Z}, n \geq 0$) if and only if $M = M'\sharp nS^1 \times S^2$. Then we discuss how to realize a template as a basic set of an NSF on a 3-manifold. We focus on the connection between the genus of the template $T$ and the topological structure of the realizing 3-manifold $M$.

Keywords: Lyapunov graphs, nonsingular Smale flows, three manifolds, templates.

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1.Introduction

The qualitative study of a class of smooth flows on a manifold is often done by understanding the behavior of the flow on the basic sets and describing the disposition of the basic sets on the manifold.

Smale flow is an important class of flow. There are many works about Smale flows on 3-manifolds. F.Béguin and C.Bonatti [BB] described the behavior of the neighborhood of a basic set of an Smale flow on a 3-manifold. M.Sullivan [Su] used template and knot theory to describe more embedding information of a special type of nonsingular Smale flows on $S^3$. J.Franks ([F3], [F4]) used homology to describe some embedding information of nonsingular Smale flows (NSF) on 3-manifolds.

In particular, J.Franks [F1] introduced Lyapunov graph to give a global picture of how the basic sets of an NSF on $S^3$ are situated on $S^3$. Following the idea of J.Franks, K. de Rezende [R] used Lyapunov graph to classify Smale flows on $S^3$. For a smooth flow $\phi_t : M \rightarrow M$ with a Lyapunov function $f : M \rightarrow R$, the Lyapunov graph $L$ is a rather natural object. The idea is to construct an oriented graph by identifying to a point each component of $f^{-1}(c)$ for each $c \in R$. For Smale flow, the weight of a edge of the Lyapunov graph is defined to be the genus of the regular set of the edge, see [R].

It is interesting to generalize J.Franks’ work to all 3-manifolds. This amounts to determine the necessary and sufficient condition when an abstract L graph is associated with an NSF on a 3-manifold. N.Oka made an attempt on this topic in his paper [Ok].

To generalize J.Franks’ work, we should study how the topology of $M$ is related to the Lyapunov graph $L$ of a Smale flow on $M$. In particular, we are interested in studying: (1) the relation between the topology of $M$ and the topology of $L$; (2) the relation between the topology of $M$ and the weights of edges of $L$. 
For $M = S^3$, J. Franks proved that the Lyapunov graph of any NSF on $S^3$ must be a tree and the weight of any edge of $L$ is 1, i.e., the regular level sets of any NSF on $S^3$ must be tori. See [F1]. For general manifold $M$, R.N. Cruz and K. de Rezende [CR] studied the relation between the topology of $M$ and the topology of $L$. In particular, they showed that the cycle-rank $r(L)$ of the Lyapunov graph $L$ of any Smale flow on a 3-manifold $M$ satisfies: $r(L) \leq g(M)$. Here $g(M)$ is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension one sub-manifolds that do not disconnect $M$.

In the first part of this paper, we will discuss the connection between the weights of edges of $L$ and the topological structure of the 3-manifold. By [F1] and [R], for any integer $n \geq 0$ and any 3-manifold $M$, there exists a Smale flow such that the genus of a regular level set of a Lyapunov graph associated with the flow is $n$. Therefore in some sense the interesting part is NSF. The following theorem is related to this topic.

**Theorem 1** Suppose a closed orientable 3-manifold $M$ admits a nonsingular Smale flow $\varphi_t$ with Lyapunov function $f$. Let $L$ be the Lyapunov graph of $f$. Then $M$ admits an NSF flow which has a regular level set homeomorphic to $(n+1)T^2$ ($n \in \mathbb{Z}, n \geq 0$) if and only if $M = M'_\sharp nS^1 \times S^2$. Here $(n+1)T^2$ means the connected sum of $n+1$ tori and $M'$ is any closed orientable 3-manifold. The cyclic number of a Lyapunov graph of the NSF, $r(L) \geq n$.

Another interesting problem is to discuss how to realize 1 dimensional basic sets in an NSF on a 3-manifold. This will be the topic of the second part of this paper.

If the basic sets are described by a suspension of a subshift of finite type (SSFT), the realization problem is considered in [PS] and [F2]. In particular, by the result of [F1] and Proposition 6.1 of [F4], it is easy to show that any SSFT can be realized as a basic set of an NSF on any closed orientable 3-manifold.

If the basic sets are described by template, Theorem 3.5.1 in Meleshuk [Me] describes an algorithm to decide whether an embedded template can be realized as a basic set of an NSF on $S^3$. Meleshuk [Me] also proved (see [Me] Theorem 3.4.1) that for any template there exists some 3-manifold with an NSF on it having a basic set modeled by the template. M. Sullivan [Su] and the author [Yu] considered how to realize an NSF with three basic sets: a repelling orbit $r$, an attracting orbit $a$, and a non-trivial saddle set modeled by a Lorenz template and a Lorenz like template on 3-manifolds. G. Frank [Fr] gave some obstruction to the realization of certain templates in a homology 3-sphere. The first statement of Theorem 2 is a generalization of Theorem A in [Fr]. The second statement is a direct corollary of the first one and Theorem 3.4.1 of [Me]. But we will give a constructive proof of the second statement in section 4. Actually our construction also proves that for any template there exists some 3-manifold with an NSF on it having a basic set modeled by the template.

**Theorem 2** Let $T$ be a template.

(1). If a closed orientable 3-manifold $M$ admits an NSF with a basic set modeled by $T$, then $M = M'_\sharp g(T)S^1 \times S^2$. Moreover, $r(L) \geq g(T)$, where $r(L)$ is the cyclic number of the Lyapunov graph $L$ of the NSF.

(2). There exists a closed orientable 3-manifold $M'$ such that $M = M'_\sharp g(T)S^1 \times S^2$ admits an NSF with a basic set modeled by $T$. 

In Theorem 2, $g(T)$ is a number which describes the basic set modeled by $T$, see Definition 2.7. In Theorem 1 and Theorem 2 above, $S^1 \times S^2$ plays an important role. The similar phenomenon appears in the study of nonsingular Morse-Smale flows on 3-manifolds, see M. Saito’s paper [Sa].

The paper is organized as follows. In Section 2, we give some definitions and detailed background knowledge. Theorem 1 and Theorem 2 are proved in Section 3 and Section 4 respectively. In Section 5, we use thickened template and some surgeries to give a visualization of an NSF with $2T^2$ regular level set on $S^1 \times S^2$. In Section 6, we discuss possible development of this work by asking two questions.

2. Preliminaries

For the standard material of hyperbolic dynamical systems, we refer the reader to the book written by C. Robinson [Ro]. For a more detailed account of Smale flows, see [F3].

Definition 2.1 A smooth flow $\phi_t : M \to M$ on a compact manifold is called a Smale flow if:

1. the chain recurrent set $R$ has hyperbolic structure;
2. the $\text{dim}(R) \leq 1$;
3. $\phi_t$ satisfies the transverse condition.

If a Smale flow $\phi_t$ has no singular point, we call $\phi_t$ nonsingular Smale flow (NSF).

Definition 2.2 A Lyapunov graph $L$ for a flow $\phi_t : M \to M$ and a Lyapunov function $f : M \to R$ is obtained by taking the quotient complex of $M$ by identifying to a point each component of a level set of $f$. Denote the cyclic number (or the first Betti number) of $L$ by $r(L)$.

Definition 2.3 An abstract Lyapunov graph is a finite, connected and oriented graph $L$ satisfying the following two conditions:

1. $L$ possesses no oriented cycles;
2. each vertex of $L$ is labeled with a chain recurrent flow on a compact space.

Theorem 2.4 (Bowen [Bo]) If $\phi_t$ is a flow with hyperbolic chain recurrent set and $\Lambda$ is a 1-dimensional basic set, then $\phi_t$ restricted to $\Lambda$ is topologically equivalent to the suspension of a basic subshift of finite type (i.e., a subshift associated to an irreducible matrix).

This theorem enables us to label a vertex of the (abstract) Lyapunov graph that represent the 1-dimensional basic sets of a Smale flow with the suspension of an SSFT $\sigma(A)$. For simplicity we will label the vertex with the nonnegative integer irreducible matrix $A$. For a vertex labeled with matrix $A = (a_{ij})$, let $B = (b_{ij})$, where $b_{ij} \equiv a_{ij}(\text{mod}2)$ and $k = \text{dim ker}((I - B) : F^m_2 \to F^m_2)$, $F_2 = \mathbb{Z}/2$. The number of incoming (outgoing) edges is denoted by $e^+ (e^-)$. Denote the weight of an edge of the Lyapunov graph by the genus of the regular level set of the edge. Let $g^+_j (g^-_j)$ be the weight on an incoming (outgoing) edge of the vertex. In this paper, we always denote the initial vertex and the terminal vertex of an oriented edge $E$ by $i(E)$ and $t(E)$ respectively. Whenever we talk about the Lyapunov graph $L$ of an NSF on a 3-manifold $M$, $L$ is always associated with a Lyapunov function $f : M \to R$ and a map $h : M \to L$ such that $f = \pi \circ h$. Here $\pi : L \to R$ is the natural projection. By the work of F.Béguin and C.Bonatti [BB], it is easy to show that there is a unique Lyapunov graph $L$ of a Smale flow on a 3-manifold.
The following classification theorem is due to K. de Rezende [R]:

**Theorem 2.5** Given an abstract Lyapunov graph $L$ whose sink (source) vertices are each labeled with an attracting (repelling) periodic orbit or an index 0 (index 3) singularity, then $L$ is associated with a Smale flow $\phi_t$ and a Lyapunov function $f$ on $S^3$ if and only if the following conditions hold:

1. The underlying graph $L$ is a tree with exactly one edge attached to each sink or source vertex.
2. If a vertex is labeled with an SSFT with matrix $A_{m \times m}$, then we have
   
   $\sum_{i=1}^{e^-} g_i^- ≤ k + 1 - G^- ≤ e^+ ≤ k + 1$, with $G^- = \sum_{i=1}^{e^-} g_i^-$ and
   $k + 1 - G^+ ≤ e^- ≤ k + 1$, with $G^+ = \sum_{j=1}^{e^+} g_j^+$.

3. All vertices must satisfy the Poincare-Hopf condition. Namely, if a vertex is labeled with a singularity of index $r$, then
   
   $$(-1)^r = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

   and if a vertex is labeled with a suspension of an SSFT or a periodic orbit, then

   $$0 = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

Template theory was first introduced to dynamics by R.F. Williams and J. Birman in their papers [BW1], [BW2]. [GHS] is a monograph on this subject. As a model of a basic set of an NSF, it provides more information than SSFT.

**Definition 2.6** A Template $(T, \phi)$ is a smooth branched 2-manifold $T$, constructed from two types of charts, called *joining charts* and *splitting charts*, together with a semi-flow. A semi-flow is the same as flow except that one cannot back up uniquely. In Figure 1 the semi-flows are indicated by arrows on charts. The gluing maps between charts must respect the semi-flow and act linearly on the edges.

(a) joining chart  
(b) splitting chart

Figure 1
If we extend a template in the direction perpendicular to its surface, we have a thickened template. The inverse limit of the semi-flow on the template produces a flow on the thickened template \( T \). \( \partial T \) is composed of entrance sets \( X \), exit sets \( Y \) and dividing curves. Let \( X \) be a union of \( s \) components: \( X_1, ..., X_s \) and \( Y \) be a union of \( t \) components: \( Y_1, ..., Y_t \). Suppose the genus of \( X_i \) is \( n_i^+ \) and the genus of \( Y_j \) is \( n_j^- \). Reindexing if necessary, we may assume

\[
\begin{align*}
&\begin{cases}
n_i^+ > 1, & i = 1, ..., s_0; \\
n_i^+ = 0, & i = s_0 + 1, ..., s_0 + s_1; \\
n_i^+ = 1, & i = s_0 + s_1 + 1, ..., s.
\end{cases}
\]

and

\[
\begin{align*}
&\begin{cases}
n_j^- > 1, & j = 1, ..., t_0; \\
n_j^- = 0, & j = t_0 + 1, ..., t_0 + t_1; \\
n_j^- = 1, & j = t_0 + t_1 + 1, ..., t.
\end{cases}
\]

\[
\sum I_p = \frac{1}{2}(X(\partial M^+) - X(\partial M^-)) \text{ where the summation is taken over all singularities in } M, I_p \text{ is the index of the singularity } p \text{ and } X \text{ denotes the Euler characteristic.}
\]

\[
\sum I_p = \frac{1}{2}(X(\partial M^+) - X(\partial M^-)) \text{ where the summation is taken over all singularities in } M, I_p \text{ is the index of the singularity } p \text{ and } X \text{ denotes the Euler characteristic.}
\]

\[
\text{Definition 2.7} \text{ The genus of } T, g(T) \text{ is defined by } g(T) = \max\{\sum_{i=1}^{s_0} n_i^+, \sum_{j=1}^{t_0} n_j^- - t_0\}.
\]

3. The proof of Theorem 1

The following lemma is Theorem 4.7 in [R]. It is proved by using Poincare-Hopf formula.

\[
\text{Lemma 3.1} \text{ Suppose } \phi_t \text{ is a smooth flow on an odd-dimensional manifold } M \text{ which transverses outside to } \partial M^- \text{ and transverses inside to } \partial M^+, \text{ where } \partial M = \partial M^+ \cup \partial M^- . \text{ Then }
\]

\[
\sum I_p = \frac{1}{2}(X(\partial M^+) - X(\partial M^-)) \text{ where the summation is taken over all singularities in } M, I_p \text{ is the index of the singularity } p \text{ and } X \text{ denotes the Euler characteristic.}
\]

\[
\text{Lemma 3.2} \text{ Let } v \text{ be a vertex of } L, \text{ which is the Lyapunov graph of an NSF on a 3-manifold } M, \text{ then } e^+ - e^- - \sum g_i^+ + \sum g_i^- = 0.
\]

\[
\text{Proof:} \text{ To the inverse image (for the map } h : M \to L) \text{ of a small neighborhood of } v, \text{ }
\]

\[
\sum I_p = 0, \partial M^+ = 2e^+ - 2 \sum g_i^+ \text{ and } \partial M^- = 2e^- - 2 \sum g_i^- . \text{ By Lemma 3.1, we have } 0 = \frac{1}{2}(2e^+ - 2 \sum g_i^+ - 2e^- + 2 \sum g_i^-). \text{ Therefore } e^+ - e^- - \sum g_i^+ + \sum g_i^- = 0. \text{ Q.E.D.}
\]

\[
\text{Lemma 3.3} \text{ For a Lyapunov graph } L \text{ associated with an NSF on a closed orientable 3-manifold } M, \text{ if there exists an edge } E \subset L \text{ such that its weight is } (n + 1), n \geq 0, \text{ then there exists weight 0 edges } E_1, ..., E_n \subset L \text{ such that } L \text{ is connected if we cut } L \text{ along } E_1, ..., E_n . \text{ In particular, } r(L) \geq n , \text{ where } r(L) \text{ is the cyclic number of } L.
\]
Proof: Suppose $L(E)$ is composed of $x \in L$ which satisfies the condition that there exists an oriented arc starting at $t(E)$ and ending at $x$, which is a union of a sequence of oriented edges of nonzero weight. Obviously $L(E)$ is a connected subgraph of $L$. An edge of weight 0 is said to be a vanishing weight 0 edge if its terminal vertex is in $L(E)$. We denote the set of all vanishing weight 0 edges by $F$.

For a 3-manifold, if a one dimensional basic set is a hyperbolic attractor, it must be a closed orbit attractor. So the terminal edges of $L$ (also $L(E)$) are weight 1 edges. By Lemma 3.2, for any vertex, $\sum g_j^+ - e^+ = \sum g_j^- - e^-$. By this formula and the fact that a terminal edge of $L$ must be 1, we have the edge number of $F$, $\sharp F$, is no less than $n$.

We obtain a new graph $L'$ if we cut $L$ along all vanishing weight 0 edges. We claim that $L'$ is connected. Otherwise, let $L' = L_1 \sqcup L_2$. Here $L_1$ is the connected component containing the subgraph $L(E)$. The boundaries of $L_2$ are exit sets. Let $M_1$ and $M_2$ be $h^{-1}(L_1)$ and $h^{-1}(L_2)$ respectively. $M_2$ admits an NSF which transverses outside to $\partial M_2$. Since we cut $L$ along weight 0 edges, $\partial M_2$ is homeomorphic to a union of $\sharp F$ disjoint 2-spheres. Hence $X(\partial M_2) = 2\sharp F$. By Lemma 3.1, we have $X(\partial M_2) = 0$. It is a contradiction, therefore $L'$ is connected.

Since $\sharp F \geq n$, we can choose $E_1, ..., E_n \subset L$ such that $L$ is connected if we cut $L$ along $E_1, ..., E_n$. In particular, the cyclic number of $L$, $r(L)$, is no less than $n$. Q.E.D.

The proof of necessity in Theorem 1

Proof: By Lemma 3.3, there exist weight 0 edges $E_1, ..., E_n \subset L$ such that $L$ is connected if we cut $L$ along $E_1, ..., E_n$.

For any given $p_i \in \text{int}(E_i), i = 1, ..., n$, $h^{-1}(p_i) \cong S^2$. Denote $S_i^2 = h^{-1}(p_i), i = 1, ..., n$. Obviously they are unparallel. $M$ is connected if we cut $M$ along these 2-spheres. There exists a neighborhood $W$ of $S_i^2$ which is homeomorphic to $S^2 \times [0, 1]$. Let $V = \overline{M - W}$. $V$ is connected and $\partial V \cong S^2 \sqcup S^2$. Fix two points $p_1$ and $p_2$ in each of the two boundary components of $V$ respectively. There exists a curve $c$ in $V$ connecting $p_1$ and $p_2$. Choose a cylinder neighborhood $N(c)$ of $c$. Let $W' = W \sqcup N(c), V' = M - W'$. Then $\partial W' = \partial V' \cong S^2$. Let $M_1 = V' \cup_\partial D^3, W' \cup_\partial D^3 \cong S^1 \times S^2$, so $M = M' \sharp S^1 \times S^2$. Here $D^3$ is a 3-ball. We can repeat the procedure for each $S_i^2$, to show that $M = M' \sharp nS^1 \times S^2$, where $M'$ is a closed orientable 3-manifold. Furthermore $r(L) \geq n$. Q.E.D.

The proof of sufficiency in Theorem 1 uses some constructions which rely on Theorem 2.5. We first prove some lemmas and propositions.

Lemma 3.4 For any $n \in Z, n \geq 1$, there exists a Smale flow $\varphi_t$ on $S^3$ such that $\varphi_t$ satisfies:

(1) Its singularities are composed of $n$ singular attractors and $n$ singular repellers;

(2) There exists an $(n + 1)T^3$ regular level set of the Lyapunov graph of $\varphi_t$.

Proof: It is sufficient to construct a Lyapunov graph $L$ satisfies: (1) There exist $n$ singular attractors vertexes and $n$ singular repellers vertexes in $L$. (2) There are no more vertices labeled with singularities in $L$. (3) There exists a weight $(n + 1)$ edge on $L$. (4) $L$ satisfies the conditions of Theorem 2.5.

We start with an oriented graph $E$ with only one edge. Suppose the weight of $E$ is $n + 1$. To vertex $i(E)$ ($t(E)$), we add two oriented edges $E_n (E^n), F_n (F^n)$ such that $t(E_n) = i(E), i(F_n) = i(E), i(E^n) = t(E), t(F^n) = t(E))$. Here $E_n (E^n)$ is a weight $n$ edge and $F_n (F^n)$ is a weight 0 edge.

Let $A_1^{n1} = (a_{ij}^1_1), A_2^{n2} = (a_{ij}^2_1)$ be the nonnegative integer irreducible matrices associated with $i(E), t(E)$ respectively. Let $B_l = (b_{ij}^l)$, where $b_{ij}^l = a_{ij}^l (mod 2)$ and $l = 1, 2$. Let $k_l = \text{dim ker}((I -$
$B_l : F_{2l}^m \to F_{2l}^m$), where $F_2 = \mathbb{Z}/2$ and $l = 1, 2$. By Theorem 2.5, $k_1, k_2$ satisfy

$$\begin{cases}
k_1 + 1 - n \leq 2 \leq k_1 + 1 \\
k_1 + 1 - (n + 1) \leq 1 \leq k_1 + 1 \\
k_1 + 1 - (n + 1) \leq k_1 \leq k_1 + 1 \\
k_2 + 1 - n \leq 2 \leq k_2 + 1 \\
k_2 + 1 - (n + 1) \leq k_1 \leq k_2 + 1 \\
k_2 + 1 - n \leq 2 \leq k_2 + 1 \\
\end{cases}$$

(1)

We choose $k_1 = k_2 = n$. Obviously they satisfy (1) and (2) and there exist nonnegative integer irreducible matrices $A_1, A_2$ such that $k_1 = k_2 = n$.

Let $n_1 = n \geq 1$. If $n_1 > 1$, let $n_2 = n_1 - 1$ and we add two oriented edges $E_{n_2}$ ($E_{n_2}^n$), $F_{n_2}$ ($F_{n_2}^n$) such that $t(E_{n_2}) = i(E_{n_1}), i(F_{n_2}) = i(E_{n_1})$ ($i(E_{n_2}) = t(E_{n_1}), t(F_{n_2}) = t(E_{n_1})$). Here $E_{n_2}$ ($E_{n_2}^n$) is a weight $n_2$ edge and $F_{n_2}$ ($F_{n_2}^n$) is a weight 0 edge. Similar to above, we can associate regular matrices to $i(E_{n_1})$ and $t(E_{n_1})$. We can repeat the procedure for $n_k$ until $n_k = 1$. Thus we have constructed a Lyapunov graph $L$ on $S^3$. It is easy to check that it satisfies all conditions in the first paragraph of the proof. Figure 3 is an example for $n = 2$. Q.E.D.

![Figure 3](image)

**Proposition 3.5** For any $n \in \mathbb{Z}, n \geq 1$, there exists an NSF $\varphi_l$ on $nS^1 \times S^2$ such that there exists an $(n + 1)T^2$ regular level set of the Lyapunov graph of $\varphi_l$.

**Proof:** For any $n \in \mathbb{Z}, n \geq 0$, we have constructed a Lyapunov graph $L$ in the proof of Lemma 3.4. Obviously there is only one weight $n$ edge on $L$. Suppose the outgoing (incoming) weight 0 edges in $L$ are $e_1, \ldots, e_n$ ($e_1^1, \ldots, e_n^m$). Let $x_i \in \text{int}(e_i)$ and $y_i \in \text{int}(e_i^i), i = 1, \ldots, n$.

Cut $L$ along $x_i, y_i, i = 1, \ldots, n$. Denote by $L'$ the connected component which contains the weight $n$ edge after cutting. Let $M = h^{-1}(L')$ and $X_i = h^{-1}(x_i), Y_i = h^{-1}(y_i), i = 1, \ldots, n$. Since $e_i, e_i^i, i = 1, \ldots, n$ are weight 0, $h^{-1}(x_i), h^{-1}(y_i), i = 1, \ldots, n$ are homeomorphic to $S^2$. $\varphi_l |_M$ transverses outside (inside) to $X_i (Y_i), i = 1, \ldots, n$.

Pasting $X_i$ to $Y_i, i = 1, \ldots, n$, we obtain $nS^1 \times S^2$. After this surgery, with some permutations of $\varphi_l |_M$ in a small neighborhood of the pasted surfaces, we obtain an NSF $\varphi_l$ on $nS^1 \times S^2$. Since there exists an $(n + 1)T^2$ regular level set of the Lyapunov graph of $\varphi_l$ which is disjoint with a small neighborhood of the pasted surfaces in $nS^1 \times S^2$, there exists an $(n + 1)T^2$ regular level set of the Lyapunov graph of $\varphi_l$. Q.E.D.

In Figure 4, $G$ is a Lyapunov graph. The two vertexes in $G$ represent a saddle closed orbit and a closed orbit attractor. The proof of Lemma 3.7 below shows that $G$ is realizable. $L_1, L_2$ are Lyapunov graphs. $e_1, e_2$ are weight 1 edges and $v_1, v_2$ denote two closed orbit attractors.
We cut \( L_1, L_2 \) along \( e_1, e_2 \) respectively. Connecting the components of \( L_1 \) and \( L_2 \) which don’t contain \( v_1 \) and \( v_2 \) after cutting with \( G \), we obtain a new Lyapunov graph \( L \) denoted by \( L = L_{1\#(e_1,e_2)}L_2 \). See Figure 4.

**Lemma 3.6** Let \( M, N \) be two closed orientable 3-manifolds with NSF \( \varphi_1, \varphi_2 \) respectively and \( L_1, L_2 \) are the Lyapunov graphs of \( \varphi_1, \varphi_2 \) respectively. Then there exists an NSF \( \phi_t \) on \( M\#N \) such that \( L = L_{1\#(e_1,e_2)}L_2 \) is the Lyapunov graph of \( \phi_t \), where \( e_1, e_2 \) are edges of \( L_1, L_2 \) respectively which terminate at closed orbit attractor vertexes.

**Proof:** We will use a surgery method due to Morgan \([Mo]\) in the proof of this lemma. Before we start the proof, we recall some basic constructions in \([Mo]\). Let \( W_1 \) and \( W_2 \) be two solid tori as shown in Figure 5. There is a flow on \( W_1 \) such that the center of \( W_1 \) is an attractor and the flow transverse inside to \( \partial W_1 \). \( W_1 \) with such a flow is called a round 0-handle. There is another flow on \( W_2 \) such that the center of \( W_2 \) is a saddle closed orbit and the flow transverse inside to \( B_1, B_2 \) and outside to \( \partial M_2 - \overline{B_1 \cup B_2} \). \( W_2 \) is one type of the so called round 1-handles.

\[ A_1, A_2, B_1, B_2 \] are annuluses as shown in Figure 5. If we attach \( A_1, A_2 \) to \( B_1, B_2 \) respectively, we obtain a 3-manifold \( W \) with an NSF \( \psi_t \). \( W \cong S^1 \times D^2\#S^1 \times D^2 \) and \( \psi_t \) transverse outside to \( \partial(S^1 \times D^2\#S^1 \times D^2) \). It is easy to see that \( G \) is a Lyapunov graph of the flow \( \psi_t \).

Now, let’s turn to the proof of this lemma. Since \( \varphi_1^1, \varphi_2^2 \) are NSF on the closed orientable 3-manifolds \( M, N \), there exist closed orbit repellers in \( M, N \). We cut out two round 2-handles of \( M, N \) respectively and call the rest parts \( M', N' \) respectively. We paste \( W \) to \( M, N \) along their boundaries suitably such that the new 3-manifold is homeomorphic to \( M\#N \). One can check that, \( \varphi_1, \varphi_2 \) and \( \psi_t \) form an NSF \( \phi_t \) on \( M\#N \) with Lyapunov graph \( L = L_{1\#(e_1,e_2)}L_2 \). Q.E.D.
Different from nonsingular Morse-Smale flow (NMSF) and Anosov flow (For NMSF and Anosov flow, see [Mo] and [Fe]), all closed orientable 3-manifolds admit NSF.

**Proposition 3.7** Every closed orientable 3-manifold admits NSF.

**Proof (sketch):** Proposition 6.1 in [F4] shows that for any smooth link \( K \) in \( S^3 \) there exists an NSF on \( S^3 \) such that \( K \) is the set of attractors of the NSF.

A famous theorem of Lickorish and Wallace (see, for example, [Rol]) shows that every closed orientable 3-manifold may be obtained by surgery (cut solid tori and paste solid tori) on a link on \( S^3 \).

By using the two facts above and some easy combinatorial arguments, one can show that every closed orientable 3-manifold admits NSF. Q.E.D.

**The proof of the sufficiency in Theorem 1**

**Proof:** For any closed orientable 3-manifold \( M \), let \( M = M_n^S \times S^2 \) \( (n \in \mathbb{N}) \). By Proposition 3.7, there exists an NSF \( \phi \) on \( M \) with a Lyapunov graph \( L_1 \). By Proposition 3.5 there exists an NSF \( \psi \) on \( S^1 \times S^2 \) with a Lyapunov graph \( L_2 \) such that there exists an \((n+1)T^2\) regular level set and \( r(L_2) \geq n \). By Lemma 3.6 we have an NSF \( \phi_t \) on \( M = M_n^S \times S^2 \) such that the Lyapunov graph of \( \phi_t \) is \( L = L_1 \times (e_1,e_2) L_2 \). Furthermore, there exists an \((n+1)T^2\) regular level set in \( M \). Since \( r(L_2) \geq n, r(L) \geq n \). Q.E.D.

4. The proof of Theorem 2

**Lemma 4.1** For a template \( T \), \( \sum_{i=1}^{s_0} n_i^+ - s_0 - s_1 = \sum_{j=1}^{s_0} n_j^- - t_0 - t_1 \)

**Proof:** Let \( \overline{T} = D \times I \), where \( D \) is a disc and \( I = [0,1] \). Pasting some copies of \( \overline{T} \) to \( T \) along dividing curves, we have a 3-manifold \( M \). The NSF on \( T \) can be extended to \( M \) in a standard way as an NSF. By Lemma 3.1, \( \sum_{i=1}^{s_0} (2 - 2n_i^+) + \sum_{s_0+1}^{s_1} (2 - 0) = \sum_{j=1}^{s_0} (2 - 2n_j^-) + \sum_{t_0+1}^{t_1} (2 - 0) \). Therefore \( \sum_{i=1}^{s_0} n_i^+ - s_0 - s_1 = \sum_{j=1}^{t_0} n_j^- - t_0 - t_1 \). Q.E.D.

**Proposition 4.2** Suppose \( \phi \) is an NSF on a closed orientable 3-manifold \( M \) such that a basic set of \( \phi \) can be modeled by \( T \). Then we can choose a map \( h : M \to L \) with respect to \( \phi \) such that every \( X_i \) (\( Y_j \)) is a subset of a regular level set of \( L \). Here \( L \) is the Lyapunov graph of \( \phi \).

**Proof:** By the construction of thickened template (See [GHS]), \( T \) can be regarded as the suspension of Markov partition. Then there exists \( N(T) \cap R(\phi) = \Lambda \). Here \( \Lambda \) is the basic set modeled by \( T \) and \( R(\phi) \) is the chain recurrent set of \( \phi \). By Theorem 0.3 of [BB], \( N(T) \) is unique up to topological equivalence.

[BB] also implies that \( N(T) \) can be obtained from \( T \) by attaching discs and annuluses along dividing curves. We can construct a map \( h : M \to L \) with respect to \( \phi \) such that \( \partial N(T) \) are regular level sets of \( L \). Obviously, \( X_i(Y_j) \subseteq \partial N(T) \). Q.E.D.

Assume that \( \sum_{i=1}^{s_0} n_i^+ - s_0 \geq \sum_{j=1}^{t_0} n_j^- - t_0 \). Under this assumption, \( g(T) = \sum_{i=1}^{s_0} n_i^+ - s_0 \).

**Lemma 4.3** \( s \leq s_0 \) and \( \sum_{i=1}^{s} g_i - s \geq \sum_{i=1}^{s_0} n_i^+ - s_0 = g(T) \).
Proof: By Proposition 4.2, there exists a map \( P : \{X_i\} \to \{e_j\}, i = 1, \ldots, s_0; j = 1, \ldots, s. \) Since \( g_j > 1 \) \((i = 1, \ldots, s)\), \( P \) must be a surjection. Hence \( s \leq s_0 \). Since \( s \leq s_0 \) and \( \sum_{i=1}^{s} g_i \geq \sum_{i=1}^{s_0} n_i^+ \) \((P \text{ is a surjection})\), \( \sum_{i=1}^{s} g_i - s \geq \sum_{i=1}^{s_0} n_i^+ - s_0 = g(T). \) Q.E.D.

Lemma 4.4 There exist weight 0 edges \( E_1, \ldots, E_{g(T)} \subseteq L \) such that \( L \) remains connected if we cut \( L \) along \( E_1, \ldots, E_{g(T)} \). In particular, \( r(L) \geq g(T) \), where \( r(L) \) is the cyclic number of \( L \).

Proof: The proof is similar to the proof of Lemma 3.3.

Let \( L_1 \) be the set of \( x \in L \) which satisfies the condition that there exists an arc composed of a sequence of oriented edges of nonzero weight starting at \( x \) and terminating at some vanishing weight 0 edges by \( F \). We denote the set of all vanishing weight 0 edges by \( F \).

In 3-manifold, if a 1 dimensional basic set is an attractor, it must be the standard closed orbit attractor. So those edges of \( L_1 \) (also \( L_2 \)) which are adjacent to a vertex modeling an attractor are weight 1 edges. By Lemma 3.2. For any vertex, \( \sum g_i^+ - e^+ = \sum g_i^- - e^- \). Due to this formula and the fact that the weight of the edges above must be 1, we have \( \sharp(F) \geq \sum_{i=1}^{s} g_i - s \geq g(T) \) (Lemma 4.3), where \( \sharp(F) \) is the edge number of \( F \).

The rest of the proof are the same as the last two paragraphs in the proof of Lemma 3.3. Q.E.D.

The proof of (1) in Theorem 2

Proof: The proof is the same as the proof of necessity of Theorem 1 except that we use Lemma 4.4 instead of Lemma 3.3. Q.E.D.

Given \( n \in \mathbb{N} \), we cut Lyapunov graph \( L \) in the proof of Lemma 3.4 along the weight \((n + 1)\) edge of \( L \). \( L \) is divided to two Lyapunov graphs: \( L_1, L_2 \). \( L_1 \) \( L_2 \) has weight \((n + 1)\) outside (inside) edge. Let \( S^3 = M_1 \cup M_2 \) such that \( M_i = h^{-1}(L_i), i = 1, 2 \). \( M_1 \) and \( M_2 \) are two genus \((n + 1)\) handlebodies. See [F1] and [Me]. Before we prove part (2) of Theorem 2, we construct for each \( n(n \in \mathbb{N}) \) two handlebodies \( H_1 \) and \( H_2 \) as follows. Cutting small 3-ball neighborhoods of the singular points in \( M_1, M_2 \), we obtain two genus \((n + 1)\) handlebodies with \( n \) holes, denoted by \( H_1 \) and \( H_2 \) respectively.

The proof of (2) in Theorem 2

Proof: Assume that \( \sum_{i=1}^{s_0} n_i^+ - s_0 \geq \sum_{j=1}^{t_0} n_j^- - t_0 \). It is easy to show that \( X_i \) is not a disk for any \( i \in \{1, \ldots, s_0\} \). We use some annuluses to connect all \( X_i, i > s_0 + s_1 \) to get a connected surface \( X \). Here the boundaries of the annuluses are attached to some dividing curves of \( \overline{T} \). Obviously the genus of \( X \) is 0. We use an annulus to connect \( X \) along a connected component of \( \partial X \) and a connected component of \( \partial X_1 \). We denote the new surface by \( X'_1 \). The genus of \( X'_1 \) is equivalent to the genus of \( X_1, n_1^+ \). We attach discs to the boundary of \( X_1 \) and \( X_i, 2 \leq i \leq s_0 + s_1 \). Then we thicken all attached annuluses and disks to get a new 3-manifold denoted by \( W \). The connected component of \( \partial W \) containing \( X'_1 \) is still denoted by \( X'_1 \), while the connected component of \( \partial W \) containing \( X_i \) \((2 \leq i \leq s_0 + s_1)\) is denoted by \( X'_i \). Then we extend flow on \( \overline{T} \) to \( W \) such that the flow is an NSF. Define the flow on \( W \) by \( \varphi_t \). For more detail on how to extend the flow, see Section 3.2 of [Me]. Obviously \( \varphi_t \) is transverse to the boundary of \( W \).

For any \( X'_i, i > s_0 \), we attach a round 2-handle to \( W \) along \( X'_i \). For any \( X'_i, i \leq s_0 \), if the genus of \( X'_i \) is \( n + 1 \), we attach \( H_1 \) to \( X'_i \). Let \( Y'_1, \ldots, Y'_t \) be all the outside boundaries of \( W \) such that the genus of anyone of them is larger than 0. We attach \( H_2 \) and round 0-handles to \( W \) along \( Y'_1, \ldots, Y'_t \).
as above. Therefore, we obtain a 3-manifold $V$ with an NSF $\psi_t$. $\partial V = 2g(T)S^2$, $\psi_t$ is transverse inside to one copy of $g(T)S^2$ and outside to the other.

Attaching all inside $g(T)S^2$ to all outside $g(T)S^2$ one by one, we obtain a 3-manifold $M$ with an NSF $\phi_t$. $M = M'g(T)S^1 \times S^2$. Here $M'$ is a closed orientable 3-manifold. Q.E.D.

5. A visualization of an NSF on $S^1 \times S^2$

In the discussion of Theorem 1 and Theorem 2, the existence of NSF is always proved by some constructions. These constructions are included in the proofs of the main theorems in [R] and [F1]. [BB] implies that any NSF on a 3-manifold can be constructed from thickened templates by some surgeries. In this section, we use template to construct an NSF on $S^1 \times S^2$ with $2T^2$ regular level set.

Denote the template in Figure 6 by $T$. Suppose $\overline{T}$ is the thickened $T$.

![Figure 6](image)

The boundary of $\overline{T}$ is composed of entrance sets $X_1, X_2$ and exit set $Y$. $X_1 \cap Y = c_1$ and $X_2 \cap Y = \{c_2, c_3, c_4\}$ are dividing curves. See Figure 7.

![Figure 7](image)

We attach three copies of thickened discs $D^2 \times [0,1]$ to $\partial \overline{T}$ along $c_2, c_3, c_4$ (add 2-handles). We paste thickened punctured torus to $\partial \overline{T}$ along $c_1$. Then we get a new 3-manifold $W$ with an NSF $\varphi_t$. $\partial W = X'_1 \cup X'_2 \cup Y'$, $X'_1 \cong 2T^2$, $X'_2 \cong S^2$ and $Y' \cong T^2$. $X_i \subset X'_i, i = 1,2$ and $Y \subset Y'$. $\varphi_t$ is transverse inside to $X'_1 \cup X'_2$ and outside to $Y'$. $W \cong (T^2 \times [0,1]) \cup_{D_1=D_2} (S^1 \times D^2 - \text{int}(D^3))$. Here $D^i$ is an i-ball. $D_1, D_2$ are two discs and $D_1 \subset \partial T^2 \times [0,1]$, $D_2 \subset \partial (S^1 \times D^2)$. See Figure 8.
The Lyapunov graph of \((W, \varphi_t)\) is \(G\) (Figure 9). Let \((V, \psi_t)\) be the same flow as \((W, \varphi_{-t})\) with an inverse direction. We add a round 0 handle to \(W\) along \(Y'\) such that the new 3-manifold \(W'\) is a genus 2 handlebody with a hole. Similarly, we add a round 2 handle to \(V\) along \(Y'\) such that the new 3-manifold \(V'\) is a genus 2 handlebody with a hole. Now we attach \(W'\) to \(V'\) along their genus 2 surfaces such that the new 3-manifold \(M'\) is \(S^3\) with two holes. \(\partial M' = S^2 \sqcup S^2\). Attaching \(M'\) to itself along its two boundary components, we obtain a 3-manifold \(M \cong S^1 \times S^2\) with an NSF \(\phi_t\). \(L\) (Figure 9) is the Lyapunov graph of \(\phi_t\). Obviously there exists a 2\(T^2\) regular level set on \(\phi_t\).

Actually, if we don’t restrict how to attach \(W'\) to \(V'\), we can get any 3-manifold \(M \cong M'\sharp S^1 \times S^2\). Here \(M'\) is any closed orientable 3-manifold whose Heegaard genus isn’t larger than 2. Obviously, \(M\) admits an NSF and \(L\) (Figure 9(2)) is the Lyapunov graph of the flow.

6. Conclusion

Theorem 1 discusses the regular level set of an NSF on a 3-manifold. The motivation is to generalize J.Franks’ work [F1]. The ultimate goal is to answer,

**Question 6.1** How to determine the necessary and sufficient condition when an abstract Lyapunov graph is associated with an NSF on a 3-manifold?

Theorem 2 gives some rough description about an NSF on a 3-manifold with a basic set modeled by a given template \(T\). Suppose \(\overline{T}\) is the thickened \(T\). Adding 2-handles along all dividing curves of \(\overline{T}\), we obtain a 3-manifold \(N(T)\). [BB] proved that if an NSF on a 3-manifold admits a basic set modeled by a given template \(T\), then there exists a neighborhood of the basic set which can be
obtained by a sequence of standard surgeries (see [BB]) in \( N(T) \). So in some sense understanding the topological structure of \( N(T) \) is not only useful to our question but also important to describing NSF on 3-manifolds. Therefore, we ask,

**Question 6.2** What can we say about the topological structure of \( N(T) \)? For example, given a template \( T \), is \( N(T) \) irreducible?

Since the dividing curves of \( T \) are difficult to control, describing the topological structure of \( N(T) \) systematically is difficult. So discussing whether \( N(T) \) is irreducible may be more suitable.

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