New Spin-Two Gauged Sigma Models and General Conformal Field Theory

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Abstract

Recently, we have studied the general Virasoro construction at one loop in the background of the general non-linear sigma model. Here, we find the action formulation of these new conformal field theories when the background sigma model is itself conformal. In this case, the new conformal field theories are described by a large class of new spin-two gauged sigma models. As examples of the new actions, we discuss the spin-two gauged WZW actions, which describe the conformal field theories of the generic affine-Virasoro construction, and the spin-two gauged $g/h$ coset constructions. We are able to identify the latter as the actions of the local Lie $h$-invariant conformal field theories, a large class of generically irrational conformal field theories with a local gauge symmetry.
1 Introduction

The general affine-Virasoro construction [1–3] describes the general conformal stress tensor
\[ T = L^{ab} : J_a J_b : , \quad a, b = 1, \ldots, \dim g \] (1.1)
in the background of the WZW action on Lie \( g \), where \( J_a \) are the currents of affine Lie \( g \) [4–6], and the coefficients \( L^{ab} \) satisfy the Virasoro master equation. The conformal field theories (CFTs) described by (1.1) have generically irrational central charge, even when the theories are unitary. The generic affine-Virasoro action [7–9] is a large set of spin-two gauged WZW actions which describe the generic CFT whose stress tensors have the form (1.1). The spin-two nature of the generic theory is a consequence of \( K \)-conjugation covariance [6, 10, 11, 1], which tells us that each \( T \) comes with a commuting \( K \)-conjugate partner \( \tilde{T} \)
\[ K_g : \quad T_g = T + \tilde{T} \] (1.2)
such that \( T \) and \( \tilde{T} \) sum to the affine-Sugawara construction [6, 10, 12, 13] \( T_g \) on \( g \). See [3] for a review of the general affine-Virasoro construction and irrational conformal field theory.

Recently, the first part of this program has been extended [14, 15] from the WZW background to the general non-linear sigma model. In particular, the general conformal stress tensor
\[ T \sim L_{ij} \partial_+ x^i \partial_+ x^j , \quad i, j = 1, \ldots, \dim M \] (1.3)
was studied at one loop in the background of the general non-linear sigma model, including the dilaton, and a unified Einstein-Virasoro master equation was obtained for the coefficient \( L_{ij} \). It remains therefore to find the actions of the new CFTs associated to these more general stress tensors.

In this paper, we confine ourselves to those background sigma models which are themselves conformal, in which case the more general stress tensors (1.3) also exhibit \( K \)-conjugation covariance in the form
\[ K_G : \quad T_G = T + \tilde{T} \] (1.4)
where \( T_G \) is the conformal stress tensor of the background sigma model.

Following [7], we find that the generic conformal field theory of this type is described by a very large class of generically new spin-two gauged sigma models, including the general non-linear sigma model itself, the Polyakov-gauged general non-linear sigma model, and the spin-two gauged WZW actions as special cases.
As another special case, we study the spin-two gauged coset constructions, for which $K$-conjugation takes the form

$$K_{g/h} : \quad T_{g/h} = T + \tilde{T}$$

where $T_{g/h}$ is the stress tensor of the $g/h$ coset construction [6, 10, 11]. We are able to identify the spin-two gauged coset constructions as the actions of the local Lie $h$-invariant CFTs [16], which are known to exhibit $K$-conjugation through the coset constructions. The set of local Lie $h$-invariant CFTs is the large class of generically irrational CFTs in the Virasoro master equation with an extra Lie $h$ gauge symmetry, including the coset constructions as the simplest case. Because of their extra Lie $h$ gauge symmetry, the actions of this class of theories were previously unknown. With the answer in hand, however, we find that these theories may equivalently be described as spin-two and spin-one gauged WZW actions.

More generally, we expect that the class of CFTs described by the spin-two gauged sigma models is vast: one hint in this direction is the observation of many other kinds [16] of $K$-conjugation covariance beyond the examples $K_g$ and $K_{g/h}$ discussed explicitly here.

## 2 Spin-Two Symmetries of the General Sigma Model

In this section, we review the classical form of the general non-linear sigma model and its spin-two symmetries, following the development and notation of [14, 15].

On any manifold $M$, the Minkowski-space form of the general non-linear sigma model is

$$S_G = \int d^2 \xi \mathcal{L}_G \quad \text{(2.1a)}$$

$$\mathcal{L}_G = \frac{1}{8\pi\alpha'} (G_{ij} + B_{ij}) \partial_+ x^i \partial_- x^j \quad \text{(2.1b)}$$

$$d^2 \xi = d\tau d\sigma, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma \quad \text{(2.1c)}$$

$$H_{ijk} = \partial_\tau B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \quad \text{(2.1d)}$$

where $0 \leq \sigma \leq 2\pi$, $x^i$, $i = 1, \ldots, \text{dim} \ M$ are coordinates on $M$ and $\alpha'$ is the string tension. The fields $G_{ij}$ and $H_{ijk}$ are respectively the metric and the torsion field on $M$. The equations of motion can be written in two equivalent forms

$$\partial_+ \partial_- x^i + \partial_\pm x^j \partial_\mp x^k \hat{\Gamma}^{\pm i}_{jk} = 0 \quad \text{(2.2a)}$$

$$\hat{\Gamma}^{\pm i}_{jk} = \Gamma_{jk}^{i} \pm \frac{1}{2} H_{jk}^{i} \quad \text{(2.2b)}$$
where \( \hat{\Gamma}^\pm \) are the generalized Christoffel connections with torsion.

We also introduce the vielbein \( e^i_a, a = 1, \ldots, \dim M \) on \( M \),

\[
G_{ij} = e^i_a G_{ab} e^b_j
\]

\[
\hat{\nabla}^\pm_i e^j_a = \partial_i e^j_a - \hat{\Gamma}^\pm_k e^j_a + e^j_b (\hat{\omega}^\pm_i)_b^a = 0
\]

\[
(\hat{\omega}^\pm_i)^b_a = (\omega_i)^b_a \pm \frac{1}{2} H_{ia}^b
\]

where \( G_{ab} \) is the tangent-space metric and \( \hat{\omega}^\pm \) are the generalized spin connections with torsion. This gives the tangent-space form of the equations of motion

\[
\partial^\mp J^\pm_a = (\hat{\omega}^\pm c)^b_a J^\pm_b \quad J^\pm_a = G^a_{cb} \partial_{\pm} x^b
\]

in terms of the currents \( J^\pm \).

To obtain the Hamiltonian formulation, we need the Poisson brackets of the coordinates with the momenta \( p_i = \partial L_G/\partial \dot{x}^i \),

\[
[x^i(\sigma), p_j(\sigma')] = i \delta^i_j \delta(\sigma - \sigma')
\]

\[
J^\pm_a = e^i_a (4 \pi \alpha' p_i + (B_{ij} \pm G_{ij}) \partial_\sigma x^j)
\]

and the equal-time current algebra [14]

\[
[J^+_a(\sigma), J^+_b(\sigma')] = 8 \pi i \alpha' G_{ab} \partial_\sigma \delta(\sigma - \sigma') + 4 \pi i \alpha' \delta(\sigma - \sigma') [J^-_c \hat{\omega}^+_{ab} - J^+_c \hat{\tau}^+_{ab}]
\]

\[
[J^-_a(\sigma), J^-_b(\sigma')] = -8 \pi i \alpha' G_{ab} \partial_\sigma \delta(\sigma - \sigma') + 4 \pi i \alpha' \delta(\sigma - \sigma') [J^+_c \hat{\omega}^-_{ab} - J^-_c \hat{\tau}^-_{ab}]
\]

\[
[J^+_a(\sigma), J^-_b(\sigma')] = 4 \pi i \alpha' \delta(\sigma - \sigma') [\hat{\omega}^+ c J^+_c - \hat{\omega}^- c J^-_c]
\]

\[
(\hat{\tau}^\pm)_{cab} \equiv \omega_{cab} + \omega_{abc} + \omega_{bca} \pm \frac{1}{2} H_{cab}
\]

which follows from (2.5). The sigma model Hamiltonian is

\[
H_G = \int_0^{2 \pi} d\sigma \mathcal{H}_G
\]

\[
\mathcal{H}_G = T^G_++T^G_-
\]

\[
T^G_\pm = \frac{1}{8 \pi \alpha'} L^a_{G b} J^\pm_a, \quad L^a_{G b} = \frac{G^a_{ab}}{2}
\]

\[
\hat{A} = i[H_G, A]
\]

where \( G_{ab} \) is the inverse of the tangent space metric.
The stress tensors $T_G^{++}$ and $T_G^{--}$ are respectively chiral and antichiral

$$\partial_\pm T_G^{\pm\pm} = 0$$

(2.8)

and satisfy the commuting Virasoro algebras

$$[T_{\pm\pm}(\sigma), T_{\pm\pm}(\sigma')] = \pm i[T_{\pm\pm}(\sigma) + T_{\pm\pm}(\sigma')] \partial_{\sigma} \delta(\sigma - \sigma')$$

$$[T_{++}(\sigma), T_{--}(\sigma')] = 0$$

(2.9)

at equal time. The semiclassical limit of the central charge of $T_G^{\pm\pm}$ is $c_G = \dim M$.

We turn now to the construction of new chiral and antichiral stress tensors in the background of the sigma model. With [14, 15], we suppose that the manifold $M$ supports two covariantly-constant second-rank symmetric tensors $L$ and $\bar{L}$

$$\hat{\nabla}_i^\pm L^{ab} = \hat{\nabla}_i^- \bar{L}^{ab} = 0$$

(2.10a)

$$L^{ab} = 2L^{ac}G_{cd}L^{db}, \quad \bar{L}^{ab} = 2\bar{L}^{ac}G_{cd}\bar{L}^{db}$$

(2.10b)

called the (inverse) inertia tensors on $M$. The relations (2.10b) are familiar as the high-level or semiclassical form [17, 18, 7] of the Virasoro master equation, and are easily solved as

$$L_a^b = \frac{P_a^b}{2}, \quad \bar{L}_a^b = \frac{\bar{P}_a^b}{2}$$

(2.11)

where $P$ and $\bar{P}$ are orthogonal projectors. The relations (2.10a) are called the covariant-constancy conditions; necessary and sufficient conditions for the existence of solutions to these conditions are known [14, 15], and examples will be discussed below.

For each such pair of inertia tensors on $M$, one has the associated chiral and antichiral stress tensors

$$T_{++} = \frac{1}{8\pi\alpha'} L^{ab} J_a^+ J_b^+, \quad T_{--} = \frac{1}{8\pi\alpha'} \bar{L}^{ab} J_a^- J_b^-$$

$$\partial_\pm T_{\pm\pm} = 0$$

(2.12a)

which also satisfy the two commuting Virasoro algebras in (2.9).

The $K$-conjugate [6, 10, 11, 1] stress tensors

$$\tilde{T}_{++} = \frac{1}{8\pi\alpha'} \tilde{L}^{ab} J_a^+ J_b^+, \quad \tilde{T}_{--} = \frac{1}{8\pi\alpha'} \tilde{\bar{L}}^{ab} J_a^- J_b^-$$

$$\tilde{L}^{ab} = L^{ab} - L^{ab}, \quad \tilde{\bar{L}}^{ab} = \bar{L}^{ab} - \bar{L}^{ab}$$

$$\partial_\pm \tilde{T}_{\pm\pm} = 0$$

(2.13a)

(2.13b)

(2.13c)
are also respectively chiral and antichiral and satisfy the Virasoro algebra (2.9) because the corresponding relations

\[ \hat{\nabla}_i^+ \tilde{L}^{ab} = \hat{\nabla}_i^- \tilde{L}^{ab} = 0 \]  

\[ \tilde{L}^{ab} = 2 \tilde{L}^{ac} G_{cd} \tilde{L}^{db}, \quad \tilde{L}^{ab} = 2 \tilde{L}^{ac} G_{cd} \tilde{L}^{db}, \]  

(2.14a)  

follow from (2.10a), (2.10b) and (2.13b).

Counting the \( K \)-conjugate stress tensors, this gives us a total of four commuting Virasoro generators

\[ T_{++}, \tilde{T}_{++}, T_{--}, \tilde{T}_{--} \]  

(2.15)

which sum in \( K \)-conjugate pairs to the sigma model stress tensors \( T^G_{\pm\pm} \)

\[ K_G : \quad T^G_{\pm\pm} = T_{\pm\pm} + \tilde{T}_{\pm\pm}. \]  

(2.16)

The interpretation of this structure, well-known from the general affine-Virasoro construction, is that the background sigma model \( T^G_{\pm\pm} \) factorizes into two commuting \( K \)-conjugate conformal field theories

\[ \begin{align*} 
L \text{ theory} & : \quad T_{++}, T_{--} \\
\tilde{L} \text{ theory} & : \quad \tilde{T}_{++}, \tilde{T}_{--}
\end{align*} \]  

(2.17)

each of which possesses its own chiral and antichiral stress tensors.

The four Virasoro generators \( T_{++}, \tilde{T}_{++}, T_{--}, \tilde{T}_{--} \) correspond to spin-two symmetries of the sigma model action \( S_G \) in (2.1a) and these generators can be obtained from \( S_G \) by Noether’s theorem, using the symmetries

\[ \begin{align*} 
\delta x^i & = 2 \xi (\tau^+) L^i_j \partial_+ x^j \\
\tilde{\delta} x^i & = 2 \tilde{\xi} (\tau^+) \tilde{L}^i_j \partial_+ x^j \\
\tilde{\delta} x^i & = 2 \tilde{\xi} (\tau^+) \tilde{L}^i_j \partial_- x^j \\
\tilde{\tilde{\delta}} x^i & = 2 \tilde{\tilde{\xi}} (\tau^+) \tilde{\tilde{L}}^i_j \partial_- x^j \\
\tau^\pm & = \tau \pm \sigma \\
\delta S_G & = \tilde{\delta} S_G = \tilde{\delta} S_G = \tilde{\tilde{\delta}} S_G = 0
\end{align*} \]  

(2.18)

where \( \xi, \tilde{\xi}, \tilde{\xi}, \tilde{\tilde{\xi}} \) are the infinitesimal parameters of the transformations.

In [14], the one-loop quantum corrections to this classical discussion were considered in detail, including the dilaton \( \Phi \). It was shown there that this picture, including the
$K$-conjugate pairs of Virasoro operators, survives at one loop\(^1\) as long as the background sigma model is itself conformal ($G, B$ and $\Phi$ satisfy the Einstein equations). In what follows, the discussion applies only to this case. We also emphasize that the classical inverse inertia tensors $L, \bar{L}, \tilde{L}, \tilde{\bar{L}}$ are only the semiclassical limits of the one-loop inertia tensors of [14], but this is all that is needed [7] to construct the classical Hamiltonian and action formulation of the new CFTs. The semiclassical limits of the central charges of the various stress tensors are

\[
\begin{align*}
  c(T_{++}) &= \text{rank } L, \quad c(T_{--}) = \text{rank } \bar{L} \\
  c(\tilde{T}_{++}) &= \text{rank } \tilde{L}, \quad c(\tilde{T}_{--}) = \text{rank } \tilde{\bar{L}} \\
  c_G &= \dim M = c(T_{++}) + c(\tilde{T}_{++}) = c(T_{--}) + c(\tilde{T}_{--}).
\end{align*}
\]

See [13] for further details at the one-loop level.

3 Hamiltonian and Action Formulations of the New CFT’s

In [14] and Section 2, the discussion was limited to the construction of the stress tensors of the new CFTs, and we now want to find the classical Hamiltonian and action formulations of the new theories, following the development of [7] for the generic affine-Virasoro action. In what follows, we focus on the $L$ theory, with stress tensors $T_{\pm\pm}$, but the corresponding constructions for the $\tilde{L}$ theory follow by $K$-conjugation

\[
L \leftrightarrow \tilde{L}, \quad \bar{L} \leftrightarrow \tilde{\bar{L}}, \quad T_{\pm\pm} \leftrightarrow \tilde{T}_{\pm\pm}
\]

at any stage of the discussion.

3.1 Hamiltonian Formulation

We begin with the basic Hamiltonian of the $L$ theory

\[
\begin{align*}
  H_0 &= \int d\sigma \mathcal{H}_0 \\
  \mathcal{H}_0 &= T_{++} + T_{--} \\
  T_{++} &= \frac{1}{8\pi\alpha'} L^{ab} J_a^+ J_b^+, \quad T_{--} = \frac{1}{8\pi\alpha'} \bar{L}^{ab} J_a^- J_b^- \\
  J_a^\pm &= e_a^i (4\pi\alpha' \dot{p}_i + (B_{ij} \pm G_{ij}) \partial_{\sigma} x^j) \\
  \dot{A} &= i[H_0, A] \\
  \partial_{\pm} T_{\pm\pm} &= 0
\end{align*}
\]

\(^1\)The classical picture is of course exact in the quantum theory when the background sigma model is the WZW action, since this case corresponds to the general affine-Virasoro construction.
where the metric \( G_{ij} \), vielbein \( e^a_i \) and antisymmetric tensor field \( B_{ij} \) are those of the background conformal sigma model \( S_G \). Note that the currents \( J^\pm \) are now defined by their canonical construction (3.2d), which guarantees the general current algebra (2.6a), (2.6b), although the relation (2.4b) no longer holds (except for the special case \( L^{ab} = \bar{L}^{ab} = L^G_{ab} \), which returns us to the original sigma model). As a consequence, all four stress tensors \( T_{\pm\pm}, \tilde{T}_{\pm\pm} \) satisfy commuting Virasoro algebras (as they did in the original sigma model) and we find in particular that the \( K \)-conjugate stress tensors (associated to the \( \bar{L} \) theory) are local symmetries of the basic Hamiltonian

\[
[H_0, \bar{T}_{++}(\sigma)] = [H_0, \bar{T}_{--}(\sigma)] = 0. \quad (3.3)
\]

This identifies the system as a spin-two gauge theory, and we may follow Dirac to construct the full Hamiltonian of the \( L \)-theory

\[
H = \int d\sigma \mathcal{H} \quad (3.4a)
\]
\[
\mathcal{H} = H_0 + v \cdot \bar{T} \quad (3.4b)
\]
\[
= T_{++} + T_{--} + v \bar{T}_{++} + \bar{v} \bar{T}_{--} \quad (3.4c)
\]
\[
= \mathcal{H}_G + \frac{1}{8\pi \alpha} [(v - 1) \bar{L}^{ab} J^+_a J^+_b + (\bar{v} - 1) \bar{L}^{ab} J^-_a J^-_b] \quad (3.4d)
\]
\[
\dot{A} = i[H, A] \quad (3.4e)
\]
\[
\partial_{\mp} T_{\pm\pm} = 0 \quad (3.4f)
\]

in which the \( L \)-theory is gauged by its \( K \)-conjugate partner, the \( \bar{L} \) theory. Here, \( v \) and \( \bar{v} \) are Lagrange multipliers which form a world-sheet spin-two gauge field [7]

\[
\sqrt{-\bar{h}h^{mn}} = \frac{2}{v + \bar{v}} \left( \begin{array}{cc}
-1 & \frac{1}{2}(v - \bar{v}) \\
\frac{1}{2}(v - \bar{v}) & v\bar{v}
\end{array} \right), \quad m = (\tau, \sigma) \quad (3.5)
\]

called the \( K \)-conjugate metric. The Hamiltonian (3.2a) is correct for the generic \( L \) theory, but must be further gauged if \( H_0 \) possesses further local symmetries (see Sections 5.3 and 7).

This system possesses a \( \text{Diff} S^1 \times \text{Diff} S^1 \) symmetry generated by the \( K \)-conjugate stress tensors

\[
\partial A = i[\int d\sigma \epsilon(\sigma)\bar{T}_{++}(\sigma) + \bar{\epsilon}(\sigma)\bar{T}_{--}(\sigma), A] \quad (3.6a)
\]
\[
\delta v = \epsilon \overset{\leftrightarrow}{\partial_\sigma} v, \quad \delta \bar{v} = -\bar{\epsilon} \overset{\leftrightarrow}{\partial_\sigma} \bar{v} \quad (3.6b)
\]
\[
\delta x^i = e^i_a (\epsilon \bar{L}^{ab} J^+_b + \bar{\epsilon} \bar{L}^{ab} J^-_b) \quad (3.6c)
\]
\[
\delta H = \delta H_0 = 0 \quad (3.6d)
\]
which is extended to Diff $S^2$ on passage to the non-linear form of the action in the usual manner [7]. We shall return to the non-linear form of the action below.

Here we follow [8], going directly to the linear form of the action via the introduction of the auxiliary fields $B, \bar{B}$,

$$
\mathcal{H}' = \mathcal{H}_G + \frac{1}{4\pi\alpha'} [\alpha \tilde{L}^{ab} B_a B_b + \frac{1}{2} (B_a - J^+_a) G^{ab} (B_b - J^+_b) + \bar{\alpha} \tilde{L}^{ab} \bar{B}_a \bar{B}_b + \frac{1}{2} (\bar{B}_a - J^-_a) G^{ab} (\bar{B}_b - J^-_b) ]
$$

(3.7a)

$$
\alpha = \frac{1 - v}{1 + v}, \quad \bar{\alpha} = \frac{1 - \bar{v}}{1 + \bar{v}}
$$

(3.7b)

$$
\dot{A} = i [H', A]
$$

(3.7c)

$$
\mathcal{H}'_{(B, \bar{B})_+} = \mathcal{H}.
$$

(3.7d)

The new Hamiltonian $H' = \int d\sigma \mathcal{H}'$ is equivalent to the Hamiltonian $H$, as shown, after using the $(B, \bar{B}) = (B, \bar{B})^*$ equations of motion.

### 3.2 Action Formulation: The New Spin 2 Gauged Sigma Models

Defining $\dot{x}^i$ by (3.7c) as usual leads to the useful identities

$$
J_a^+ + J_a^- = 2[B_a + \bar{B}_a + G_{ab} e^b_i \dot{x}^i]
$$

(3.8a)

$$
J_a^+ - J_a^- = 2 e^i_a G_{ij} \partial_\sigma x^j
$$

(3.8b)

$$
J_a^\pm = B_a + \bar{B}_a - G_{ab} e^b_i \partial_{\pm} x^i
$$

(3.8c)

and then, with $\mathcal{L}' = \dot{x}^i p_i - \mathcal{H}'$, we find the linear form of the action of the $L$-theory

$$
S' = \int d^2 \xi \mathcal{L}'
$$

(3.9a)

$$
\mathcal{L}' = \mathcal{L}_G + \frac{1}{4\pi\alpha'} [\alpha \tilde{L}_{ij} B^i B^j + \bar{\alpha} \tilde{L}_{ij} \bar{B}^i \bar{B}^j - (B^i - \partial_+ x^i) G_{ij} (\bar{B}^j - \partial_- x^j) ]
$$

(3.9b)

as a generically new spin-two gauged sigma model. This is the central result of this paper. In fact, the construction describes a very large class of CFTs, one family for each pair $\tilde{L}, \bar{L}$ of covariantly constant inertia tensors

$$
\hat{\nabla}_i^+ \tilde{L}_j^k = \hat{\nabla}_i^- \bar{L}_j^k = 0
$$

(3.10a)

$$
\tilde{L}_i^j = 2 \bar{L}_i^k \bar{L}_k^j, \quad \bar{L}_i^j = 2 \tilde{L}_i^k \tilde{L}_k^j
$$

(3.10b)
in the background of the original sigma model $S_G$ (recall that (2.10) is equivalent to (3.10)). Necessary and sufficient conditions for the solution of (3.10) are given in [14, 15] and these references also discuss the conformal stress tensors of these theories at the one-loop level including the dilaton.

This action exhibits a spin-two gauge symmetry, or $\text{Diff} S^2$ invariance

$$
\delta \alpha = -\partial_+ \xi^i + \xi^i \partial_+ \alpha \quad (3.11a)
$$

$$
\delta x^i = \xi^i = 2\bar{\xi} \bar{L}^i_j B^j \quad (3.11b)
$$

$$
\delta B^i = \partial_+ \xi^i - (B^i - \partial_+ x^j)\xi^k(\hat{\Gamma}^+)_{jk}^i \quad (3.11c)
$$

$$
\delta \bar{B}^i = -\xi^i B^k(\hat{\Gamma}^-)_{jk}^i \quad (3.11d)
$$

$$
\bar{\delta} \bar{\alpha} = -\partial_- \bar{\xi} + \bar{\xi} \partial_- \bar{\alpha} \quad (3.12a)
$$

$$
\bar{\delta} x^i = \bar{\xi}^i = 2\xi \bar{L}^i_j B^j \quad (3.12b)
$$

$$
\bar{\delta} B^i = -\bar{\xi}^i B^k(\hat{\Gamma}^+)_{jk}^i \quad (3.12c)
$$

$$
\bar{\delta} \bar{B}^i = \partial_- \bar{\xi}^i - (\bar{B}^i - \partial_- x^j)\bar{\xi}^k(\hat{\Gamma}^-)_{jk}^i \quad (3.12d)
$$

$$
\delta S' = \bar{\delta} S' = 0 \quad (3.13)
$$

associated to the $\bar{L}$ theory, where $\xi$ and $\bar{\xi}$ are the infinitesimal parameters of the transformation. The $\alpha, \bar{\alpha}$ transformations in (3.11a) and (3.12a) can be related to the usual [7] $\text{Diff} S^2$ transformations of $v$ and $\bar{v}$ by

$$
\xi = \frac{1 + \alpha}{2} \epsilon \quad \leftrightarrow \quad \delta v = \dot{\epsilon} + \epsilon \partial_\sigma v \quad (3.14a)
$$

$$
\bar{\xi} = \frac{1 + \bar{\alpha}}{2} \bar{\epsilon} \quad \leftrightarrow \quad \bar{\delta} \bar{v} = \dot{\bar{\epsilon}} + \bar{\epsilon} \partial_\sigma \bar{v} \quad (3.14b)
$$

It will also be useful to have the tangent-space form of these invariances

$$
\delta x^i = \xi^a e^i_a, \quad \xi^a = 2\xi \bar{L}^a_b B^b \quad (3.15a)
$$

$$
\delta B^a = \partial_+ \xi^a - \xi^b B^c(\hat{\omega}^-)_{bc}^a + \xi^b(\partial_+ x)^c(\hat{\omega}^+)_{cb}^a \quad (3.15b)
$$

$$
\delta \bar{B}^a = -\xi^b \bar{B}^c(\hat{\omega}^-)_{bc}^a \quad (3.15c)
$$

$$
\bar{\delta} x^i = \bar{\xi}^a e^i_a, \quad \bar{\xi}^a = 2\bar{\xi} \bar{L}^a_b \bar{B}^b \quad (3.16a)
$$

$$
\bar{\delta} B^a = -\bar{\xi}^b B^c(\hat{\omega}^+)_{bc}^a \quad (3.16b)
$$

$$
\bar{\delta} \bar{B}^a = \partial_- \bar{\xi}^a - \bar{\xi}^b \bar{B}^c(\hat{\omega}^-)_{bc}^a + \bar{\xi}^b(\partial_- x)^c(\hat{\omega}^-)_{cb}^a \quad (3.16c)
$$
where \((\partial_{\pm} x)^a = \partial_{\pm} x^i e_i^a\). The semiclassical central charges of the spin-two gauged sigma models are \((c_L, c_R) = (\text{rank } L, \text{rank } \tilde{L})\).

The original sigma model action \(S_G\) is contained in the action \(S'\) in two independent ways:

1. \(L^{ab} = \bar{L}^{ab} = L_G^{ab}, \quad \tilde{L} = \tilde{\tilde{L}} = 0\) \hspace{2cm} (3.17)
2. \(v = \bar{v} = 1, \quad \alpha = \bar{\alpha} = 0, \quad \tilde{h}^{mn} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\) \hspace{2cm} (3.18)

after trivially integrating out \(B\) and \(\bar{B}\). The first is the choice to return to \(S_G\), while the second is the conformal gauge of the new spin-two gauged sigma models. The Polyakov-gauged general non-linear sigma model is also contained as the special case \(L = \tilde{L} = 0, \quad \bar{L} = \bar{\tilde{L}} = L_G\) of these spin-two gauged sigma models (see Section 3.3).

### 3.3 Non-Linear Form of the Spin 2 Gauged Sigma Models

To go to the non-linear form of the action, we must integrate out the auxiliary fields \(B, \bar{B}\) of \(S'\) using the \(B, \bar{B}\) equations of motion

\[
2\alpha \bar{L}^i_j B^j - \bar{B}^i + \partial_- x^i = 0 \hspace{2cm} (3.19a)
\]
\[
2\bar{\alpha} \tilde{L}^i_j \bar{B}^j - \bar{B}^i + \partial_+ x^i = 0. \hspace{2cm} (3.19b)
\]

These can be solved for \(B, \bar{B}\) as

\[
B = (1 - 4\alpha \tilde{L})^{-1}(\partial_+ x + 2\bar{\alpha} \bar{L} \partial_- x) \hspace{2cm} (3.20a)
\]
\[
\bar{B} = (1 - 4\alpha \tilde{L})^{-1}(\partial_- x + 2\bar{\alpha} \bar{L} \partial_+ x) \hspace{2cm} (3.20b)
\]

and substitution into \(S'\) gives the non-linear form of the new spin 2 gauged sigma models

\[
S = \int d^2 \xi \mathcal{L} \hspace{2cm} (3.21a)
\]
\[
\mathcal{L} = \frac{1}{8\pi \alpha'} \left[ (-G_{ij} + B_{ij}) \partial_+ x^i \partial_- x^j + m^T N m \right] \hspace{2cm} (3.21b)
\]

where

\[
m = \begin{pmatrix} \partial_+ x \\ \partial_- x \end{pmatrix}, \quad N = \begin{pmatrix} 2\alpha \tilde{L}(1 - 4\alpha \tilde{L} \tilde{L})^{-1} & (1 - 4\alpha \tilde{L} \tilde{L})^{-1} \\ (1 - 4\alpha \tilde{L} \tilde{L})^{-1} & 2\alpha \tilde{L}(1 - 4\alpha \tilde{L} \tilde{L})^{-1} \end{pmatrix}. \hspace{2cm} (3.22)
\]

The Diff \(S^2\) or spin-two gauge invariance of the non-linear action

\[
\delta x = 2\xi \tilde{L}(1 - 4\alpha \tilde{L} \tilde{L})^{-1}(\partial_+ x + 2\bar{\alpha} \bar{L} \partial_- x)
\]
\[
\bar{\delta} x = 2\xi \bar{L}(1 - 4\alpha \tilde{L} \tilde{L})^{-1}(\partial_- x + 2\bar{\alpha} \bar{L} \partial_+ x) \hspace{2cm} (3.23)
\]
(together with (3.11a) and (3.12a) is obtained by substitution of (3.20a), (3.20b) in (3.11b) and (3.12b). As in [7], these nonlinear realizations of Diff $S^2$ are generically new (see also (3.28)-(3.30)).

As in the linearized version, the original sigma model action $S_G$ is contained in the nonlinear action $S$ in the same two independent ways (3.18), including the conformal gauge choice $\alpha = \bar{\alpha} = 0$. In this connection, we note that the traceless covariantly-conserved symmetric stress tensor

$$\Theta^{mn} = \frac{2}{\sqrt{-\tilde{h}}} \frac{\delta S}{\delta \tilde{h}^{mn}}$$

reduces in the conformal gauge to

$$\Theta_{00} = \Theta_{11} = \tilde{T}_{++} + \tilde{T}_{--}$$
$$\Theta_{01} = \Theta_{10} = \tilde{T}_{++} - \tilde{T}_{--}.$$  

(3.25)

(3.26)

This identifies the $K$-conjugate metric $\tilde{h}_{mn}$ as the world-sheet metric of the $\tilde{L}$ theory, as expected from the generic affine-Virasoro action [7, 8].

Using the conformal gauge, we can also check directly that, in the new actions, we have correctly gauged the $K$-conjugate part ($\tilde{T}_{\pm\pm}$) of the spin-two symmetries (2.18) of the original sigma model $S_G$. According to (3.11a) and (3.12a), the conformal gauge has a residual symmetry which leaves $\alpha = \bar{\alpha} = 0$,

$$\partial_- \xi = \partial_+ \bar{\xi} = 0$$

$$\delta x^i = 2\xi (\tau^+ \tilde{L}_{ij} \partial_+ x^j) \equiv \tilde{\delta} x^i$$

$$\tilde{\delta} x^i = 2\bar{\xi} (\tau^- \tilde{\bar{L}}_{ij} \partial_- x^j) \equiv \tilde{\bar{\delta}} x^i$$

(3.27a)

(3.27b)

(3.27c)

and (3.27b) and (3.27c) are identical to our earlier results in (2.18).

We finally note that the Polyakov-gauged general non-linear sigma model

$$S_{GP} = \frac{1}{8\pi \alpha'} \int d^2 \xi (-\sqrt{-\tilde{h}} \tilde{h}^{mn} G_{ij} \partial_m x^i \partial_n x^j + B_{ij} \partial_+ x^i \partial_- x^j)$$

(3.28)

(see (3.5)) is included as the special case

$$L = \tilde{L} = 0, \quad \tilde{\bar{L}} = \tilde{L} = L_G$$

(3.29)

in this set of spin-two gauged sigma models. As in [7], this is the only case where the spin-two gauge transformations (3.23)

$$(\delta + \tilde{\delta}) x = \epsilon \partial_+ x + \bar{\epsilon} \partial_- x$$

$$\left( \begin{array}{c} \epsilon \\ \bar{\epsilon} \end{array} \right) = \frac{1}{1 - \alpha \bar{\alpha}} \left( \begin{array}{cc} 1 & \alpha \\ \bar{\alpha} & 1 \end{array} \right) \left( \begin{array}{c} \xi \\ \bar{\xi} \end{array} \right)$$

(3.30a)

(3.30b)
are ordinary diffeomorphisms and the K-conjugate metric $\tilde{h}^{mn}$ is the world-sheet metric of the theory.

4 Example: The Spin-Two Gauged WZW Actions

In this section we study the spin-two gauged WZW actions as an explicit example of the spin-two gauged sigma models (3.9) and (3.10), and identify this set of actions as the generic affine-Virasoro action [7–9].

To begin we choose the background sigma model $S_G$ to be the sigma model form of the WZW action $S_{WZW}$ on simple Lie $g$, whose explicit form is given in Appendix A. The sigma model data for the WZW action is [14]

\begin{align}
  y^i &= \frac{1}{\sqrt{\alpha'}} x^i, \quad i = 1, \ldots, \dim g \tag{4.1a} \\
  G_{ab} &= k \eta_{ab}, \quad a, b = 1, \ldots, \dim g \tag{4.1b} \\
  e_i &= -ig^{-1} \frac{\partial}{\partial y^i} g = e_i^a T_a, \quad \bar{e}_i = -ig \frac{\partial}{\partial y^i} g^{-1} = \bar{e}_i^a T_a \tag{4.1c} \\
  \text{Tr}(T_a T_b) &= y G_{ab} \tag{4.1d} \\
  \hat{\omega}_{ab}^+ &= 0, \quad \hat{\omega}_{ab}^- = -\frac{1}{\sqrt{\alpha'}} f_{ab}^c. \tag{4.1e}
\end{align}

Here $y^i$ are the dimensionless coordinates of the WZW action, $g \in G$ is the group element, $\eta_{ab}$ and $f_{ab}^c$ are respectively the Killing metric and structure constants of $g$ and $k$ is the (high) level of affine $g$. The asymmetry of the spin connections $\hat{\omega}^\pm$ in (4.1e) follows because we have identified the sigma model vielbein with the left-invariant vielbein $e_i^a$ on the group manifold.

In this case, the stress tensors $T_{g \pm \pm}$ of the background sigma model are the (high-level forms of) the affine-Sugawara constructions $T_{g \pm \pm}$ on $g$, and the $K$-conjugation relations (2.16) read

\begin{align}
  T_{g \pm \pm}^q &= T_{\pm \pm} + \tilde{T}_{\pm \pm} \tag{4.2a} \\
  K_g : \quad L_{g \pm \pm}^{ab} &= L^{ab} + \tilde{L}^{ab} \tag{4.2b} \\
  L_{g \pm \pm}^{ab} &= \frac{G_{ab}}{2} = \frac{\eta^{ab}}{2k} \tag{4.2c}
\end{align}

in this case. With $L_{g \pm \pm}^{ab} = \eta^{ab}/(2k + Q_g)$ and the exact solutions $L, \tilde{L}$ of the Virasoro master equation, these relations are exact at the quantum level in the general affine-Virasoro construction. It follows that the corresponding spin-two gauged form (3.9) of the WZW action must be equivalent to the generic affine-Virasoro action, which describes
a very large class of irrational conformal field theories. As we shall see, this is not difficult
to check.

To realize the spin-two gauged WZW action (3.9), we need the solutions of the co-
variant constancy conditions (3.10a) for the WZW background (4.1). These were given
in [14],

\[ L^{ab} = \text{constant} \quad (4.3a) \]
\[ \bar{L}^{ab} = \bar{L}^{cd} \Omega^a_{\ c} \Omega^b_{\ d} = \text{constant} \quad (4.3b) \]
\[ \frac{\partial}{\partial y} \Omega^a_{\ b} + e_i^d \Omega^c_{\ a} \Omega^b_{\ c} = 0 \quad (4.3c) \]

where \( \Omega \) is the adjoint action of \( g \). The \( K \)-conjugate form of these solutions

\[ \tilde{L}^{ab} = \text{constant} \quad (4.4a) \]
\[ \tilde{\bar{L}}^{ab} = \tilde{\bar{L}}^{cd} \Omega^a_{\ c} \Omega^b_{\ d} = \text{constant} \quad (4.4b) \]

then follows from eq (4.2b). The following associated definitions

\[ B'_a = \frac{1}{\sqrt{\alpha'}} B_a, \quad \bar{B}'_a = -\frac{1}{\sqrt{\alpha'}} (\Omega^{-1})_a^b \bar{B}_b \quad (4.5a) \]
\[ B' = B'^a T_a, \quad \bar{B}' = \bar{B}'^a T_a \quad (4.5b) \]
\[ \xi' = 2 \xi, \quad \bar{\xi}' = 2 \bar{\xi} \quad (4.5c) \]
\[ \partial = \frac{1}{2} \partial_+, \quad \bar{\partial} = \frac{1}{2} \partial_- \quad (4.5d) \]

will also be useful.

With the WZW results (4.1), (4.3) and (4.4) and the definitions (4.5), we find that
the spin-two gauged WZW action (3.9) can be written as

\[ S'_{AV} = S_{WZW} + \int d^2 \xi \Delta \mathcal{L}_B \quad (4.6a) \]
\[ \Delta \mathcal{L}_B = \frac{\alpha}{\pi y^2} \tilde{L}^{ab} \text{Tr}(T_a B') \text{Tr}(T_b B') + \frac{\bar{\alpha}}{\pi y^2} \tilde{\bar{L}}^{ab} \text{Tr}(T_a \bar{B'}) \text{Tr}(T_b \bar{B'}) - \frac{1}{\pi y} \text{Tr}(\bar{D} g D g^{-1}) \quad (4.6b) \]
\[ D = \partial + i B', \quad \bar{D} = \bar{\partial} + i \bar{B}' \quad (4.6c) \]
\[ \tilde{L}^{ab} = 2 \tilde{L}^{ac} G_{cd} \tilde{L}^{db}, \quad \tilde{\bar{L}}^{ab} = 2 \tilde{\bar{L}}^{ac} G_{cd} \bar{L}^{db} \quad (4.6d) \]

where \( g \in G \) is the group element of the WZW action \( S_{WZW} \). Moreover, the tangent-space
form (3.15) of the Diff $S^2$ invariance of the spin-two gauged WZW action can be put in the form

$$\delta \alpha = -\bar{\partial} \xi' + \xi' \bar{\partial} \alpha, \quad \delta \bar{\alpha} = -\partial \xi' + \xi' \partial \bar{\alpha} \quad (4.7a)$$

$$\delta g = g i \lambda - i \bar{\lambda} g \quad (4.7b)$$

$$\delta B' = \partial \lambda + i [B, \lambda], \quad \delta \bar{B} = \bar{\partial} \bar{\lambda} + i [\bar{B}, \bar{\lambda}] \quad (4.7c)$$

$$\lambda = \lambda^a T_a, \quad \bar{\lambda} = \bar{\lambda}^a \bar{T}_a \quad (4.7d)$$

$$\lambda^a = 2 \xi' \tilde{L}^{ab} B'_b, \quad \bar{\lambda}^a = 2 \bar{\xi}' \tilde{\bar{L}}^{ab} B'_b \quad (4.7e)$$

If we also make the special choice\(^1\)

$$\tilde{L}_{ab} = L_{ab} \equiv \bar{L}_{\infty}^{ab}$$

$$\tilde{\bar{L}}_{ab} = \bar{L}_{ab} \equiv \bar{\bar{L}}_{\infty}^{ab} \quad (4.8a)$$

we see that (4.6) and (4.7) are precisely the generic affine-Virasoro action [7–9] (in the form given in [9]) and its spin-two gauge invariance. Without the choice (4.8a), the spin-two gauged WZW actions (4.6) offer a mild generalization of the affine-Virasoro action, which may describe non-diagonal constructions of the corresponding irrational conformal field theories.

We emphasize that the generic affine-Virasoro action describes only those generic constructions in the Virasoro master equation which have no larger local symmetry than the Diff $S^2$ associated to the $K$-conjugate theory. In particular, the generic action does not describe the local Lie $h$-invariant CFTs [16]. This is the large, generically irrational set of all CFTs with an additional local Lie $h$ invariance, including the coset constructions as the simplest case. The actions for the local Lie $h$-invariant CFTs will be obtained in the following section.

5 Example: The Spin-Two Gauged Coset Constructions

In this section, we study another special case of the spin-two gauged sigma models, namely the spin-two gauged $g/h$ coset constructions. This class of examples includes the spin-two gauged WZW actions in the formal limit when $g \supset h \to 0$.

\(^1\)In the language of [7], the subscript $\infty$ means the high-level limit $L_{\infty}^{ab} = P^{ab}/2k$ of any high-level smooth solution of the Virasoro master equation, in agreement with (4.6d). More generally, the classical inverse inertia tensors $L, \tilde{L}, \bar{L}, \tilde{\bar{L}}$ of the spin-two gauged sigma models are the semiclassical limit of the one-loop inverse inertia tensors of [14].
In this case, the background sigma model $S_G$ is the sigma model description $S_{g/h}$ of the coset constructions, that is, the sigma model form of the spin-one gauged WZW actions (see Appendix A). The $K$-conjugation relations read

$$K_{g/h} : \quad T_{g/h}^{\pm \pm} = T_{\pm \pm} + \tilde{T}_{\pm \pm}$$

(5.1a)

$$L_{g/h}^{ab} = L_g^{ab} - L_h^{ab} = L^{ab} + \tilde{L}^{ab} = \bar{L}^{ab} + \tilde{\bar{L}}^{ab}$$

(5.1b)

in this case, where $T_{g/h}^{\pm \pm}$ are the (high-level forms of the) stress tensors of the coset constructions. $K$-conjugation through the coset constructions, as seen in (5.1a), is known to be exact at the quantum level for the local Lie $h$-invariant CFTs [16] of the Virasoro master equation, so we expect and will find that our action (3.9) for the general spin-two gauged coset construction is the action for this large class of generically irrational conformal field theories.

### 5.1 Inverse Inertia Tensors in the Coset Backgrounds

To realize the spin-two gauged coset actions (3.9) in this case, our first task is to solve the covariant-constancy conditions (3.10a) in the background of the general $g/h$ coset construction. For this, we will need the following data, derived in Appendix A, for the sigma model description of the coset construction,

\[ y^i = \frac{1}{\sqrt{\alpha'}} x^i \]

(5.2a)

\[ G_{ij} = e^a e_{ij} G_{ab} e^b = \tilde{e}^a e_{ij} \tilde{G}_{ab} \tilde{e}^b \]

(5.2b)

\[ e_i^a = (1 + M)^a \mu L_i^\mu \]

(5.2c)

\[ \tilde{e}_i^a = -e_i^b \Lambda_b^a = -L_i^\mu ((1 + M)\Omega)^a_\mu \]

(5.2d)

\[ \Lambda = P_{g/h}(1 + M)\Omega P_{g/h}, \quad N = (1 - P_h \Omega P_h)^{-1} \]

(5.2e)

\[ (\hat{\omega}^+)^{ab} = \frac{1}{\sqrt{\alpha'}} L_i^\mu M^A_\mu f_{Aa}^b \]

(5.2f)

\[ (\hat{\omega}^-)^{ab} = \Lambda_a^c \hat{\omega}^-(\Lambda)_{c}^d (\Lambda^{-1})_{d}^{b} + \partial_i \Lambda_a^c (\Lambda^{-1})_{c}^b \]

(5.2g)

\[ \hat{\omega}^-(\Lambda)_{a}^b = -\frac{1}{\sqrt{\alpha'}} N^A_B L_i^B f_{Aa}^b \]

(5.2h)

\[ i, j = 1, \ldots, \text{dim } g/h \quad (\text{curved}) \]

\[ a, b = 1, \ldots, \text{dim } g/h \quad (\text{flat}) \]

\[ \mu, \nu = 1, \ldots, \text{dim } g \quad (\text{flat}) \]

\[ A, B = 1, \ldots, \text{dim } h \quad (\text{flat}). \]

(5.2i)

Here $P_h$ and $P_{g/h}$ are projectors onto $h$ and $g/h$, $\Omega$ is the adjoint action of $g$, and $L_i^\mu$ is the restriction to $i = 1, \ldots, \text{dim } g/h$ of the left invariant vielbein $L$ on the group manifold.
The quantities $\Lambda$, $e$ and $\bar{e}$ in (5.2) are the analogues on the coset of the adjoint action $\Omega$ and the left and right invariant vielbeins $L$ and $R$ on the group manifold (called $e$ and $\bar{e}$ in Section 4). Indeed, as a check, we note the agreement with the WZW data (4.1),

$$
\Lambda \rightarrow \Omega, \quad e_i^a \rightarrow L_i^a, \quad \bar{e}_i^a \rightarrow R_i^a = -L_i^\mu \Omega_\mu^a
$$

$$
(\hat{\omega}_a^+) c \rightarrow 0, \quad (\hat{\omega}_a^-) c \rightarrow -\frac{1}{\sqrt{\alpha'}} f_{ab}^c
$$

in the formal limit $h \rightarrow 0$, $g/h \rightarrow g$.

As in the case of the WZW background, $\hat{\omega}^-$ is more complicated than $\hat{\omega}^+$, but $\hat{\omega}^-$ is a gauge transformation of $\hat{\omega}^-(\Lambda)$, whose form is similar to $\hat{\omega}^+$. We may bring both covariant-constancy conditions into similar form

$$
\hat{\nabla}_i^-(\hat{\omega}^+) L_{ab}^i = \hat{\nabla}_i^-(\hat{\omega}^-(\Lambda)) \bar{L}_{ab}^i = 0 \quad (5.4a)
$$

$$
\bar{L}_{ab}^i \equiv \bar{L}^{cd} \Lambda_c^a \Lambda_d^b
$$

(5.4b)

by the definition in (5.4b). In (5.4a) we have explicitly indicated the spin connections in the gradients $\hat{\nabla}_i^\pm$.

In both problems (5.4a) the relevant spin connections $\hat{\omega}^\pm \sim \hat{\omega}$ are proportional to the structure constants, and we take them in the block-diagonal form

$$
\hat{\nabla}_i(\hat{\omega}) L_{ab}^i = 0
$$

(5.5a)

$$
(\hat{\omega}_i)^a_b = R_i^A f_{Aa}^b = \begin{pmatrix}
B_1^i & & \\
& \ddots & \\
B_r^i & & 0
\end{pmatrix}
$$

(5.5b)

$$
L_{a}^b = 2L_a^c L_c^b
$$

(5.5c)

where the blocks $B_i^a$ are irreps of $h \subset g$. The general solution to the system (5.5) has been discussed in [15]. The result is

$$
L_{a}^b = \frac{1}{2} \begin{pmatrix}
1_1 \theta_1 & & \\
& 1_2 \theta_2 & \\
& & \ddots \\
1_r \theta_r & & 0
\end{pmatrix}
$$

(5.6a)

$$
\theta_i = 0, 1; \quad \theta^2 = \theta
$$

(5.6b)
where \( \mathbf{1}_s \) are unit matrices and the projectors \( \theta_r \), \( \theta \) are constants, so that (5.6) solves (5.5) in the form
\[
\partial_i L^{ab} = L^{c(a} f_{cA}^{b)} = 0.
\] (5.7)
Here \((ab)\) means symmetrization with respect to the indices \( a \) and \( b \).

In summary, we have shown that a large class of inverse inertia tensors exist in the coset backgrounds:
\[
L^{ab} = \text{constant} \quad (5.8a)
\]
\[
\bar{L}^{ab} = \bar{L}^{cd} \Lambda^a_c \Lambda^b_d = \text{constant} \quad (5.8b)
\]
\[
L^{ab} = 2 L^{ac} G_{cd} L^{db}, \quad \bar{L}^{ab} = 2 \bar{L}^{ca} G_{cd} \bar{L}^{db} \quad (5.8c)
\]
\[
L^{c(a} f_{cA}^{b)} = \bar{L}^{c(a} f_{cA}^{b)} = 0 \quad (5.8d)
\]
where the constant solutions of (5.8c), (5.8d) have the form (5.6). The same results with \( L \to \bar{L}, L \to \bar{L} \) follow for the \( K \)-conjugate inverse inertia tensors. The spin-two gauged coset actions (3.9) are realized for any solution of (5.8).

5.2 Identification with the Local Lie \( h \)-Invariant CFTs

We now turn to the identification of the new spin-two gauged coset actions as the actions of the local Lie \( h \)-invariant CFTs [16].

The Lie \( h \)-invariant CFTs are found in the Virasoro master equation on \( g \supset h \), where the general inverse inertia tensor \( L \) is labelled as
\[
L^{\mu\nu} = \text{constant}, \quad \mu = (A, a), \quad \nu = (B, b) \quad (5.9a)
\]
\[
L^{\mu\nu} = 2 L^{\mu\rho} G_{\rho\sigma} L^{\sigma\nu} + \mathcal{O}(k^{-2}) \quad (5.9b)
\]
in the present notation (see (5.2)). The relation in (5.9b) is the semiclassical or high-level form of the Virasoro master equation on \( g \), which is all we will need for this discussion (see also Appendix B). The necessary and sufficient condition that the CFT is Lie \( h \)-invariant is that the inverse inertia tensor is invariant under infinitesimal \( h \) transformations
\[
L^{\rho(\mu} f_{\rhoA}^{\nu)} = 0, \quad A = 1, \ldots, \dim h. \quad (5.10)
\]
The distinction between global and local Lie \( h \)-invariant theories is that the \( h \)-currents \( J_A \) are respectively \((1, 0)\) and \((0, 0)\) operators of \( T \)
\[
\text{global} : \quad [T(m), J_A(n)] = -n J_A(m + n) \quad (5.11)
\]
\[
\text{local} : \quad [T(m), J_A(n)] = 0 \quad (5.12)
\]
when $T = L^{\mu \nu} : J^+_{\mu} J^+_{\nu} :$ is a Lie $h$-invariant construction. The coset constructions are the simplest examples of local Lie-$h$-invariant CFTs, but both classes are vast, with generically irrational central charge.

The local Lie $h$-invariant CFTs always occur in $K$-conjugate pairs

$$K_{g/h} : T_{g/h} = T + \tilde{T}$$

(5.13)

where the conjugation is through the coset constructions.

Eq. (5.12) also tells us that no $h$-currents are present in the stress tensor of a local Lie $h$-invariant CFT at high level

$$L^{ab} = O(k^{-1}), \quad L^{AB} = O(k^{-2}), \quad L^{Aa} = O(k^{-2})$$

(5.14)

and hence that the coset-valued part of $L$ is semiclassically dominant for the local Lie $h$-invariant CFTs. In this case, (5.9b) and (5.10) reduce to

$$L^{ab} = 2L^{ac} G_{cd} L^{db} + O(k^{-2})$$

(5.15a)

$$L^{c(a f_{cA} b)} = O(k^{-2})$$

(5.15b)

and the same equations hold for the $K$-conjugate theory $L \to \tilde{L}$ because both $T$ and $\tilde{T} = T_{g/h} - T$ are local Lie $h$-invariant CFTs under conjugation through $g/h$.

The semiclassical conditions (5.15) are the same conditions (5.8c), (5.8d) we found for the allowed classical inverse inertia tensors in the coset backgrounds, and this completes the identification of our spin-two gauged coset actions as the actions of the local Lie $h$-invariant CFTs.

### 5.3 Spin 2 and Spin 1 Gauged Form of the New Actions

In this section, we find spin 2 and spin 1 gauged actions which are equivalent descriptions of the spin 2 gauged coset constructions/local Lie $h$-invariant CFTs of the previous section.

Although the generic affine-Virasoro action (4.6) is not applicable to the local Lie $h$-invariant CFTs, our new spin 2 gauged coset actions should be equivalent to a spin 1 gauging (by $h \subset g$) of the affine-Virasoro action for local Lie $h$-invariant CFTs:

$$S_{2,1} = S'_{AV|_{\text{local Lie } h}} + \frac{1}{\pi y} \int d^2 \xi \ Tr(A\bar{A} - AB' - \bar{A}B')$$

(5.16a)

$$S'_{AV|_{\text{local Lie } h}} = S_{WZW} + \int d^2 \xi [-\frac{1}{\pi y} Tr(\bar{D} g D g^{-1})]$$

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\[ + \frac{\alpha}{\pi y^2} \tilde{L}^{ab} \text{Tr}(T_a B', \text{Tr}(T_b B')) \]
\[ + \frac{\tilde{\alpha}}{\pi y^2} \tilde{\tilde{L}}^{tab} \text{Tr}(\tilde{T}_a \tilde{B}', \text{Tr}(T_b \tilde{B}')) \] (5.16b)

\[ \tilde{L}^{ab} = \text{constant} \] (5.16c)

\[ \tilde{\tilde{L}}^{ab} = \tilde{\tilde{L}}^{cd} \Lambda_c^a \Lambda_d^b = \text{constant} \] (5.16d)

\[ \tilde{L}^{ab} = 2 \tilde{L}^{ac} G_{cd} \tilde{L}^{db}, \quad \tilde{\tilde{L}}^{ab} = 2 \tilde{L}^{fc} G_{cd} \tilde{\tilde{L}}^{db} \] (5.16e)

Here \( S'_{AV}|_{\text{local Lie } h} \) is nothing but the generic affine-Virasoro action \( S'_{AV} \) on \( g \) in (4.6), now evaluated for any \( \tilde{L}, \tilde{\tilde{L}}' \) which are local Lie \( h \)-invariant, as shown in (5.16f). The indices \( a, b \) in (5.16) take the values \( 1, \ldots, \dim g/h \), because \( \tilde{L}, \tilde{\tilde{L}}' \) are dominated by their values on the coset, and the auxiliary fields \( B^i, \tilde{B}^i \) are \( g \)-valued. On the other hand, the spin 1 gauge fields \( A, \tilde{A} \) are valued on \( h \). This action has the following spin 2 and spin 1 gauge symmetries

\[ \delta \alpha = -\partial \xi' + \xi' \hat{\partial} \alpha, \quad \delta \tilde{\alpha} = -\partial \tilde{\xi}' + \tilde{\xi}' \hat{\partial} \tilde{\alpha} \] (5.17a)

\[ \delta g = g i \lambda - i \tilde{\lambda} g \] (5.17b)

\[ \delta B' = \partial \lambda + i [B', \lambda], \quad \delta \tilde{B}' = \partial \tilde{\lambda} + i [\tilde{B}', \tilde{\lambda}] \] (5.17c)

\[ \delta A = \partial \epsilon_H + i [A, \epsilon_H], \quad \delta \tilde{A} = \partial \epsilon_H + i [\tilde{A}, \tilde{\epsilon}_H] \] (5.17d)

\[ \lambda = \lambda^a T_a + \epsilon_H, \quad \lambda = \tilde{\lambda}^a T_a + \epsilon_H \] (5.17e)

\[ \lambda^a = 2 \xi' \tilde{L}^{ab} B'_b, \quad \tilde{\lambda}^a = 2 \tilde{\xi}' \tilde{L}^{ab} B'_b \] (5.17f)

where \( \epsilon_H \), which is is \( h \)-valued, is the parameter for the spin 1 symmetry, and \( \xi', \tilde{\xi}' \) are the parameters for the spin 2 symmetry.

From the viewpoint of stress tensors in the Virasoro master equation on \( g \), the action (5.16) describes the local Lie \( h \)-invariant stress tensor \( T_{lh} = T(L) \), which appears in two \( K \)-conjugation relations [16]

\[ T_g = T_{lh} + T_{gh} \] (5.18a)

\[ T_{g/h} = T_{lh} + \tilde{T}_{lh} \] (5.18b)

where \( \tilde{T}_{lh} = T(\tilde{L}) \) is also local Lie \( h \)-invariant and \( T_{gh} \) is a global Lie \( h \)-invariant theory. Combining these relations, we find that

\[ \tilde{T}_{lh} = T_{gh} - T_h \] (5.19a)

\[ [T_{lh}, \tilde{T}_{lh}] = [T_{lh}, J_A] = [\tilde{T}_{lh}, J_A] = 0 \] (5.19b)
where $A$ labels the $h$-currents. The commuting operators $\tilde{T}_{lh}$ and $J_A$ generate the local spin-two and spin-one symmetries by which the theory $T_{lh} = T(L)$ is gauged in (5.16). For the special case of the cosets themselves ($T_{lh} = T_{g/h}$), we see that $\tilde{T}_{lh} = \tilde{L} = 0$ and the spin-two gauging decouples in (5.16), leaving the ordinary spin-one gauged WZW action [22–25].

As a check, we integrate out $A$ and $\bar{A}$ in (5.16) to obtain

$$S' = S_{WZW} + \int d^2 \xi \Delta L'_{B}$$

$$\Delta L'_{B} = \frac{\alpha}{\pi y^2} \bar{L}_{ab} \text{Tr}(T_a B') \text{Tr}(T_b B')$$

$$+ \frac{\bar{\alpha}}{\pi y^2} \bar{L}_{ab} \text{Tr}(T_a \bar{B}') \text{Tr}(T_b \bar{B}')$$

$$- \frac{1}{\pi y} \text{Tr}((\partial + i \bar{B}') g (\partial + i B') g^{-1})$$

$$- \frac{1}{\pi} B_A' G^{AB} B_B'$$

(5.20b)

where $\Delta L'_{B}$ differs from $\Delta L_{B}$ in (4.6a) only by the last term in (5.21). This action still enjoys the spin 2 and spin 1 symmetries given in (5.17) except for (5.17d). Finally, we have shown by gauge fixing the spin 1 symmetry and integrating out $B_A'$, $\bar{B}_A'$ that we get exactly the sigma model form (3.9) of the spin 2 gauged coset constructions, with the identifications

$$y^i = \frac{1}{\sqrt{\alpha'}} x^i$$

(5.21a)

$$B_a' = \frac{1}{2 \sqrt{\alpha'}} B_a$$

(5.21b)

$$\bar{B}_a' = - \frac{1}{2 \sqrt{\alpha'}} (\Lambda^{-1})_a^b \bar{B}_b$$

(5.21c)

$$\bar{L}_{ab} = \text{constant}$$

(5.21d)

$$\bar{L}'_{ab} = \bar{L}'_{cd} \Lambda^a_c \Lambda^b_d = \text{constant.}$$

(5.21e)

It follows that the spin 2 and spin 1 gauged actions (5.16) or (5.20) are equivalent to the spin 2 gauged coset actions (3.9), and any of these are equivalent descriptions of the local Lie $h$-invariant CFTs.

6 The Doubly-Gauged Actions

As a final topic, we briefly discuss the doubly-gauged sigma model actions, which include the doubly-gauged WZW actions of Refs. [7–9]. In these actions the background
sigma model is gauged by both the $L$ and the $\tilde{L}$ theories, resulting in a $\text{Diff} S^2 \times \text{Diff} S^2$ symmetry with two spin-two gauge fields.

Following the references, one begins with the doubly-gauged Hamiltonian

$$H_2 = u \cdot T + v \cdot \tilde{T}$$

$$= u T_{++} + \bar{u} T_{--} + v \tilde{T}_{++} + \bar{v} \tilde{T}_{--}$$

$$= \frac{1}{8\pi\alpha'[4(1 + \alpha)(1 + \bar{\alpha})]}[\left(\alpha \tilde{L}^{ij} \beta L^{ij}\right)B^i B^j$$

$$+ (\bar{\alpha} \tilde{L}^{ij} \beta \bar{L}^{ij})\tilde{B}^i \tilde{B}^j$$

$$- (B^i - \partial_+ x^i)G_{ij}(\tilde{B}^j - \partial_- x^j)]$$

(6.1)

which exhibits a $(\text{Diff} S^1)^4$ symmetry generated by all four stress tensors $T_{\pm \pm}$ and $\tilde{T}_{\pm \pm}$. The extra multipliers $u$ and $\bar{u}$ form a second spin-two gauge field $h_{mn}$ of the form (3.5) with $v, \bar{v} \rightarrow u, \bar{u}$ which can be identified, as in (3.24), (3.25) as the world-sheet metric of the $L$ theory.

Adding the usual auxiliary fields $B$ and $\bar{B}$, one finds the doubly-gauged sigma models

$$\mathcal{L}_2 = \mathcal{L}_G + \frac{1}{4\pi\alpha'}[\left(\alpha \tilde{L}^{ij} \beta L^{ij}\right)B^i B^j$$

$$+ (\bar{\alpha} \tilde{L}^{ij} \beta \bar{L}^{ij})\tilde{B}^i \tilde{B}^j$$

$$- (B^i - \partial_+ x^i)G_{ij}(\tilde{B}^j - \partial_- x^j)]$$

(6.2a)

$$\alpha = \frac{1 - v}{1 + v}, \quad \bar{\alpha} = \frac{1 - \bar{v}}{1 + \bar{v}}, \quad \beta = \frac{1 - u}{1 + u}, \quad \bar{\beta} = \frac{1 - \bar{u}}{1 + \bar{u}}$$

(6.2b)

where $\mathcal{L}_G$ is the sigma model Lagrange density in (2.1). The conditions (3.10) (and the equivalent conditions (2.10)) apply here as well. In these actions, the $(\text{Diff} S^1)^4$ symmetry of the Hamiltonian (6.1) is promoted to a $\text{Diff} S^2 \times \text{Diff} S^2$ invariance

$$\delta S_2 = \delta \bar{S}_2 = 0$$

(6.3)

where the explicit form of the invariance is

$$\delta \alpha = -\partial_- \xi + \xi \chi_+ \alpha, \quad \delta \bar{\alpha} = -\partial_+ \bar{\zeta} + \bar{\zeta} \chi_+ \bar{\alpha}, \quad \delta \beta = -\partial_- \zeta + \zeta \chi_+ \beta$$

(6.4a)

$$\delta x^i = \xi^i = 2(\xi \tilde{L}^i_j + \zeta L^i_j)B^j$$

(6.4b)

$$\delta B^i = \partial_+ \xi^i - (B^j - \partial_+ x^j)\xi^k(\tilde{\Gamma}^+)^j_{ki}$$

(6.4c)

$$\delta \tilde{B}^i = -\xi^i \tilde{B}^k(\tilde{\Gamma}^-)^j_{ki}$$

(6.4d)

$$\bar{\delta} \alpha = -\partial_+ \bar{\xi} + \bar{\xi} \chi_- \bar{\alpha}, \quad \bar{\delta} \bar{\alpha} = -\partial_+ \bar{\zeta} + \bar{\zeta} \chi_- \bar{\alpha}, \quad \bar{\delta} \beta = -\partial_- \bar{\zeta} + \bar{\zeta} \chi_- \beta$$

(6.4e)

$$\bar{\delta} x^i = \bar{\xi}^i = 2(\bar{\xi} \tilde{L}^i_j + \bar{\zeta} L^i_j)\bar{B}^j$$

(6.4f)

$$\bar{\delta} B^i = -\bar{\xi}^i \bar{B}^k(\bar{\Gamma}^+)^j_{ki}$$

(6.4g)

$$\bar{\delta} \tilde{B}^i = -\partial_- \bar{\xi}^i - (\bar{B}^j - \partial_- x^j)\bar{\xi}^k(\bar{\Gamma}^-)^j_{ki}.$$
With the discussion of Section 4, we see that the doubly-gauged WZW action [7–9] is included in (6.2) when the background sigma model is the WZW action. Moreover, using the results of Section 5, we see that the doubly-gauged coset actions are included in (6.2) for each \(K\)-conjugate pair of local Lie \(h\)-invariant CFTs. Using the two world-sheet metrics \(h_{mn}\) and \(\tilde{h}_{mn}\) of the \(K\)-conjugate theories, one application for such constructions is discussed in [9].

In the doubly gauged actions (6.2), the pair \(L, \tilde{L}\) of \(K\)-conjugate CFTs are included symmetrically: To describe the CFT \(L\), one views \(h_{mn}\) as a fixed world-sheet metric and integrates out the \(K\)-conjugate metric \(\tilde{h}_{mn}\), and vice-versa to describe the CFT \(\tilde{L}\).

An alternative procedure is to integrate out both spin-two gauge fields \(h_{mn}\) and \(\tilde{h}_{mn}\), which defines a new class of string theories where the physical states are simultaneously primary under the \(K\)-conjugate pairs of commuting Virasoro generators \((T_{++}, T_{--}, \tilde{T}_{++}\) and \(\tilde{T}_{--})\). The first example of this kind of string theory was the “spin-orbit” model of [6] (see also [3]) and this new class of string theories may also be related to the models of [19]. We note in particular that Virasoro biprimary fields [26, 3] have arisen naturally in both contexts.

### 7 Conclusions and Discussion

We have obtained the action formulation of a large class of new CFTs whose stress tensors were recently constructed [14, 15] at the one-loop level in the background of the general conformal non-linear sigma model. The actions are generically new spin-two gauged sigma models

\[
S' = \int d^2 \xi L'
\]

\[
L' = L_G + \frac{1}{4\pi \alpha'}[\alpha L_{ij}B^iB^j + \alpha \tilde{L}_{ij}\tilde{B}^i\tilde{B}^j - (B^i - \partial_+ x^i)G_{ij}(\tilde{B}^j - \partial_- x^j)]
\]

\[
\hat{\nabla}^+ L_j^k = \hat{\nabla}^- \tilde{L}_j^k = 0
\]

\[
\tilde{L}_i^j = 2\tilde{L}_i^k\tilde{L}_k^j, \quad \tilde{L}_i^j = 2\tilde{L}_i^k\tilde{L}_k^j
\]

where \(\int d^2 \xi L_G\) is the action of the general non-linear sigma model. The spin-two gauge symmetry of these actions is associated with \(K\)-conjugation \(K_G\) through the background
conformal sigma models. The spin-two gauged sigma models contain (at least) the following special cases

- The general non-linear sigma model
- The Polyakov-gauged general non-linear sigma model
- The spin-two gauged WZW actions
- The spin-two gauged $g/h$ coset actions

which we have discussed in some detail: The spin-two gauged WZW actions (associated to $K_g$ conjugation through the affine-Sugawara construction) describe the generic CFT in the Virasoro master equation, and these actions are nothing but the sigma model form of the generic affine-Virasoro action [7–9]. The spin-two gauged coset actions (associated to $K_{g/h}$ conjugation through the coset constructions) describe the local Lie $h$-invariant CFTs [16]. This is the set of all CFTs in the Virasoro master equation with an extra $h$ gauge symmetry, including the coset constructions as the simplest case.

Beyond these examples, new CFTs are obtained for every solution of the conditions (7.1c), (7.1d) in the background of the conformal sigma model. The necessary and sufficient conditions for the solutions of these conditions is discussed in [14, 15]. It is expected that the class of CFTs described by the spin-two gauged sigma models is vast, one hint being the observation of many other $K$-conjugation covariances [16] in the Virasoro master equation, beyond $K_g$ and $K_{g/h}$ discussed here. These include $K$-conjugation through $g + h$ and the general affine-Sugawara nests $g/h_1/\ldots/h_n$. Beyond this, it is reasonable to expect many new $K_G$-covariances not associated to group manifolds.

Another direction for generalization is as follows. The spin-two gauged sigma models describe only generic CFTs whose local symmetry is associated only to $K$-conjugation. On the other hand, there will be special cases of higher symmetry (for example a $W_3$ symmetry) for which one needs to include a higher-spin gauging as well. If the higher symmetry is generated by holomorphic/antiholomorphic polynomials $P_r(x^i, \partial_+ x^i)$, $\bar{P}_r(x^i, \partial_- x^i)$, the action has the form (7.1b),

\begin{align}
S &= \int d^2 \xi \mathcal{L} \\
\mathcal{L} &= \mathcal{L}_G + \frac{1}{4\pi \alpha'} \left[ \sum_r \alpha_r P_r(x^i, B^i) + \sum_r \bar{\alpha}_r \bar{P}_r(x^i, \bar{B}^i) \right. \\
& \quad \left. - (B^i - \partial_+ x^i)G_{ij}(\bar{B}^j - \partial_- x^j) \right] 
\end{align}  

(7.2a)

(7.2b)

where $\alpha \bar{L}_{ij} B^i B^j$ has been replaced by $\sum_r \alpha_r P_r(x^i, B^i)$, and similarly for the term involving $\bar{\alpha}$. These actions include the spin-two gauged sigma models (7.1) when the generating
where the detailed form of $\delta \alpha_r$ and $\delta \bar{\alpha}_r$ depends on the Poisson bracket algebra formed by $P_r$ and $\bar{P}_r$. Adding to (7.2) an additional $B\bar{B}$-term, one can also include a vector gauging of non-abelian spin 1 symmetries.

It is quite remarkable that by introducing auxiliary fields this large class of actions can be brought to this simple polynomial form. Integrating out the auxiliary fields yields the non-linear form of these actions, which are also non-local in the general case.

Acknowledgements

We thank N. Obers, K. Sfetsos and A. Tseytlin for helpful discussions. This research is supported in part by NSF grant PHY-95-14797 and DOE grant DE-AC03-76SF00098. JdB is a fellow of the Miller Institute for Basic Research in Science.
Appendices

A  Sigma Model Form of the Spin-One Gauged WZW Action

The results of this appendix were worked out with K. Sfetsos.

The WZW action [20, 21] is

\[
S_g = -\frac{1}{8\pi y} \int d^2 \xi \text{Tr}(g^{-1} \partial_+ gg^{-1} \partial_- g) - \frac{1}{12\pi y} \int \text{Tr}(g^{-1} \text{dg})^3
\]

\[
\text{Tr}(T_a T_b) = y G_{ab}, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma, \quad d^2 \xi = d\tau d\sigma
\]  

(A.1)

and the gauged WZW action [22–25], which describes the \( g/h \) coset constructions [6, 10, 11], is

\[
S_{g/h} = S_g + \frac{1}{4\pi y} \int d^2 \xi \text{Tr}[ig^{-1} \partial_+ g A_- - iA_+ \partial_- gg^{-1} - g^{-1} A_+ g A_- + A_+ A_-].
\]

(A.2)

The equations of motion of the gauge fields \( A_\pm \) are

\[
g^{-1} D_+ g|_h = D_- gg^{-1}|_h = 0
\]

\[
D_\pm g = \partial_\pm g + i[A_\pm, g]
\]

(A.3)

and the matter equations of motion can be broken apart to read

\[
F_- = \partial_- A_+ - \partial_+ A_- + i[A_-, A_+] = 0
\]

\[
D_-(g^{-1} D_+ g) = 0.
\]

(A.4)

To go to the sigma model form of the coset actions, we gauge fix the \( h \) invariance of (A.2), integrate out \( A_\pm \) and compare to the sigma model (2.1) and its equations of motion (2.4). This gives the coset metric, vielbein and spin connections in (5.2).

Moreover, we find for the gauge fields that

\[
(A_+)_a^b = A_+^A(-i f_{Aa}^b) = i\partial_+ x^i (\dot{\omega}^-_i(A))_a^b
\]

\[
(A_-)_a^b = A_-^A(-i f_{Aa}^b) = i\partial_- x^i (\dot{\omega}^+_i)_a^b
\]

(A.5)

where \( \dot{\omega}^+ \) and \( \dot{\omega}^-(A) \) are given in (5.2g) and (5.2i). Using these relations, the flatness condition in (A.4) can be expressed as

\[
\hat{R}^+_{ca} b + \hat{\nabla}_d^+ \Phi_{ca}^b = 0
\]

\[
(\Phi_i)_a^b = (\dot{\omega}^+_i)_a^b - \dot{\omega}^-_i(A)_a^b = (1 + M + M^T) A_\mu L_i^\mu f_{Aa}^b
\]

(A.6)
where the matrix $M$ is given in (5.2f) and $\hat{R}_{cda}^{\phantom{cda}b}$ is the generalized Riemann tensor with torsion (see [14] and Section 2). The sigma model Einstein equations are

$$\hat{R}^+_i = 2\hat{\nabla}^+_i \hat{\nabla}^+_j \Phi$$

where $\Phi$ is the dilaton. Comparing (A.6) and (A.7) gives the known [27] form of the coset dilaton

$$\Phi = \frac{1}{2} \log \det(P_hNP_h) = -\frac{1}{2} \log \det(P_h(1 - P_h \Omega P_h)P_h)$$

(A.8a)

$$2\partial_i \Phi = (\hat{\omega}^+_a - \hat{\omega}^-_a (\Lambda))_i^a = e_i^a M^A_b f_A^{\phantom{A}b}$$

(A.8b)
in a new way. We found that the identities

$$(1 + M^T)P_{g/h}(1 + M) = 1 + M + M^T$$

(A.9)

$$M^a_A M^B_b f_B^{\phantom{B}a} = f_B^{\phantom{B}A} M^B_D$$

(A.10)

were helpful in the algebra above.

**B Semiclassical Limit of the Lie $h$-Invariant CFTs**

Here, we review and extend some facts [16] about the local Lie $h$-invariant CFTs.

The general affine-Virasoro construction on simple $g$ is [1,2]

$$T = L^{ab} : J_a J_b :$$

(B.1a)

$$L^{ab} = 2L^{ac}G_{cd}L^{db} - L^{cd} L^{ef} f_{ce}^a f_{df}^b - L^{cd} f_{ce}^a f_{df}^{(a} L^{b)e}$$

(B.1b)

$$c = 2G_{ab} L^{ab}$$

(B.1c)

$$G_{ab} = k\eta_{ab}$$

(B.1d)

$$a, b, c = 1, \ldots, \dim g$$

(B.1e)

where $a, b \rightarrow \mu, \nu$ in the text, and (B.1b) is the Virasoro master equation. The high-level smooth solutions of the master equation have the semiclassical form

$$L^{ab} = \frac{P^{ab}}{2k} + O(k^{-2}), \quad P_a^c P_c^b = \delta_a^b$$

(B.2)

where $P_a^b = G_{ac} P^{cb}$.

The Lie $h$-invariant CFTs on $g$, with $g/h$ a reductive coset space, satisfy

$$a = (A, I), \quad A = 1, \ldots, \dim h, \quad I = 1, \ldots, \dim g/h$$

(B.3a)

$$G_{AI} = f_{AB}^I = f_{AI}^B = 0$$

(B.3b)

$$L^{(a} f_{cA}^{b)} = 0$$

(B.3c)
where $I,J \rightarrow a,b$ in the text. The Lie $h$-invariant CFTs fall into two classes, global and local, according to the realization (5.11), (5.12) of the $h$ symmetry.

It was observed [16] in the graph-theory ansatz on $g = SO(n)$ that the local Lie $h$-invariant CFTs have the semiclassical behavior

$$L^{IJ} = O(k^{-1}), \quad L^{AB} = O(k^{-2}), \quad L^{AI} = O(k^{-2})$$

(B.4) so that the coset-valued $L$ is semiclassically dominant. It was argued in the text that this is true in general for all local Lie $h$-invariant CFTs.

Then the master equation (B.1b) gives the leading semiclassical form of the general local Lie $h$-invariant CFT:

$$L^{IJ} = \frac{p^{IJ}}{2k} + O(k^{-2}), \quad P^{IK}P_K^J = \delta^J_I$$

(B.5a)

$$P^{K(I}f_{KA}^J) = 0$$

(B.5b)

$$c = \text{rank}(P^{IJ}) + O(k^{-1})$$

(B.5c)

$$L^{AB} = -\frac{1}{4k^2}P^{IJ}P^{KL}f_{IK}^Af_{JL}^B + O(k^{-3})$$

(B.5d)

$$L^{AI}(\delta^J_I - P_{AJ}) = -\frac{1}{4k^2}P^{MN}(P^{JK}f_{JM}^Af_{KN}^I + f_{MJI}f_{NJ}^AP^{IJ}) + O(k^{-3}).$$

(B.5e)

It would be interesting to study the local Lie $h$-invariant CFTs in the one-loop unified Einstein-Virasoro master equation of [14], where the dilaton must simulate the $L^{AB}$ and $L^{AI}$ contributions which are missing in the sigma model description.

References

[1] M.B. Halpern and E. Kiritsis, Mod. Phys. Lett. A4 (1989) 1373; Erratum ibid. A4 (1989) 1797.

[2] A.Yu Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, Int. J. Mod. Phys. A5 (1990) 803.

[3] M.B. Halpern, E. Kiritsis, N.A. Obers and K. Clubok, “Irrational Conformal Field Theory”, Physics Reports 265 (1996) 1.

[4] V.G. Kač, Funct. Anal. App. 1 (1967) 328.

[5] R.V. Moody, Bull. Am. Math. Soc. 73 (1967) 217.

[6] K. Bardakçı and M.B. Halpern, Phys. Rev. D3 (1971) 2493.

[7] M.B. Halpern and J.P. Yamron, Nucl. Phys. B351 (1991) 333.

[8] J. de Boer, K. Clubok and M.B. Halpern, Int. J. Mod. Phys. A9 (1994) 2451.
[9] K. Clubok and M.B. Halpern, The generic world-sheet action of irrational conformal field theory, in: “Strings'95”, I. Bars et al., eds., World Scientific, Singapore (1996).

[10] M.B. Halpern, Phys. Rev. D4 (1971) 2398.

[11] P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88.

[12] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83.

[13] G. Segal, unpublished.

[14] J. de Boer and M.B. Halpern, Int. J. Mod. Phys. A12 (1997) 1551.

[15] J. de Boer and M.B. Halpern, “Unification of the General Non-Linear Sigma Model and the Virasoro Master Equation”, to appear in the Proceedings of the NATO Workshop “New Developments in Quantum Field Theory”, Zakopane, June 1997, Plenum Press, N.Y.

[16] M.B. Halpern, E. Kiritsis and N.A. Obers, in: “Infinite Analysis”; Int. J. Mod. Phys. A7, [Suppl. 1A] (1992) 339.

[17] M.B. Halpern, E. Kiritsis, N.A. Obers, M. Porrati and J.P. Yamron, Int. J. Mod. Phys. A5 (1990) 2275.

[18] M.B. Halpern and N.A. Obers, Nucl. Phys. B345 (1990) 607.

[19] I. Bars and C. Kounnas, Phys. Rev. D56 (1997) 3664.

[20] S.P. Novikov, Usp. Math. Nauk. 37 (1982) 3.

[21] E. Witten, Commun. Math. Phys. 92 (1984) 455.

[22] K. Bardakç, E. Rabinovici and B. Säring, Nucl. Phys. B299 (1988) 151; D. Altschuler, K. Bardakç and E. Rabinovici, Comm. Math. Phys. 118 (1988) 241.

[23] K. Gawedski and A. Kupainen, Phys.Lett. B215 (1988) 119; Nucl. Phys. B320 (1989) 625.

[24] D. Karabali, Q-H. Park, H.J. Schnitzer and Z. Yang, Phys. Lett. B216 (1989) 307.

[25] D. Karabali and H.J. Schnitzer, Nucl. Phys. B329 (1990) 649.

[26] M.B. Halpern, Ann. of Phys. 194 (1989) 247.

[27] I. Bars and K. Sfetsos, Phys. Rev. D48 (1993) 844.