EXTREMAL SEQUENCES FOR A PROBLEM RELATED TO THE HARDY OPERATOR

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Abstract: We prove a characterization of the extremal sequences of the extremal problem for the Hardy operator, related to the Bellman function of the dyadic maximal operator. In fact we prove that they behave approximately like eigenfunctions of the Hardy operator for a specific eigenvalue.

Keywords: Hardy operator, dyadic, maximal.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$\mathcal{M}_d\phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(y)| dy : x \in Q, Q \subseteq \mathbb{R}^n \text{ in a dyadic cube} \right\}$$

for every $\phi \in L^1_{loc}(\mathbb{R}^n)$, where the dyadic cubes are those formed by the grids

$$2^{-N}\mathbb{Z}^n, \text{ for } N = 0, 1, 2, \ldots .$$

It is well known that it satisfies the following weak type $(1,1)$ inequality:

$$(1.2) \quad |\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\mathcal{M}_d\phi > \lambda} |\phi(u)| du,$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$.

$$(1.2)$$

as it is easily seen implies the following $L^p$-inequality

$$\|\mathcal{M}_d\phi\|_p \leq \frac{p}{p-1}\|\phi\|_p,$$

It is also easy to see that the weak type inequality $(1.2)$ is best possible while $(1.3)$ is also sharp.

(See [1], [2] for general martingales and [3] for dyadic ones).

Further study of the dyadic maximal operator leads us to introduce the following function of two variables, defined by

$$B_p(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (\mathcal{M}_d\phi)^p : \phi \geq 0, \frac{1}{|Q|} \int_Q \phi = f, \frac{1}{|Q|} \int_Q \phi^p = F \right\},$$

where $Q$ is a fixed dyadic cube and $0 < f^p \leq F$. 

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The function (1.4), which is called the Bellman function of two variables of the dyadic maximal operator, is in fact independent of the cube $Q$ and its value has been given in [3]. More precisely it is proved there that

$$B_p(f, F) = F \omega_p(f^p / F)^p,$$

where $\omega_p : [0, 1] \to \left[1, \frac{p}{p - 1}\right]$ denotes the inverse function $H_p^{-1}$ of $H_p$ which is defined by

$$H_p(z) = -(p - 1)z^p + pz^{p-1}, \text{ for } z \in \left[1, \frac{p}{p - 1}\right].$$

As a matter of fact this evaluation has been done in a much more general setting where the dyadic sets are given now as elements of a tree $T$ on a non-atomic probability space $(X, \mu)$. Then the associated dyadic maximal operator is defined by:

$$(1.5) \quad M_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\},$$

Additionally the inequalities (1.1) and (1.3) remain true and sharp even in this setting. Moreover, if we define

$$(1.6) \quad B'_{p,T}(f, F) = \sup \left\{ \int_X (M_T \phi)^p d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F \right\},$$

for $0 < f^p \leq F$, then $B'_{p,T}(f, F) = B_p(f, F)$.

In particular the Bellman of the dyadic maximal operator is independent of the structure of the tree $T$.

Another approach for finding the value of $B_p(f, F)$ is given in [4] where the following function of two variables has been introduced:

$$(1.7) \quad S_p(f, F) = \sup \left\{ \int_0^1 \left( \frac{1}{t} \int_0^t g \right)^p dt : g : (0, 1] \to \mathbb{R}^+ : \text{non-increasing}, \right. \left. \right. \right.$$  

$$\left. \left. \text{continuous} \text{ and } \int_0^1 g = f, \int_0^1 g^p = F \right\}. \right.$$  

The first step, as it can be seen in [4], is to prove that $S_p(f, F) = B_p(f, F)$. This can be viewed as a symmetrization principle of the dyadic maximal operator with respect to the Hardy operator. The second step is to prove that $S_p(f, F)$ has the value that was mentioned above.

Now the proof of the fact that $S_p = B_p$ can be given in an alternative way as can be seen in [7]. In fact it is proved there the following result.

**Theorem A.** Given $g, h : (0, 1] \to \mathbb{R}^+$ non-increasing integrable functions and a non-decreasing function $G : [0, +\infty) \to [0, +\infty)$ the following equality holds:

$$\sup \left\{ \int_K G[(M_T \phi)^*]h(t)dt : \phi^* = g, \ K \text{ measurable subset of } [0, 1] \right\} = \int_0^k G \left( \frac{1}{t} \int_0^t g \right) h(t)dt,$$

where $|K| = k$. 

for any $k \in (0, 1]$ where $\phi^*$ denotes the equimeasurable decreasing rearrangement of $\phi$. □

It is obvious that Theorem A implies the equation $S_p = B_p$, and gives an immediate connection of the dyadic maximal operator with the Hardy operator.

An interesting question that arises now is the behaviour of the extremal sequences of functions for the quantities (1.6) and (1.7). The problem concerning (1.6) has been solved in [5] where it is proved the following:

**Theorem B.** If $\phi_n : (X, \mu) \to \mathbb{R}^+$ be such that $\int_X \phi_n d\mu = f$, $\int_X \phi_n^p d\mu = F$, then the following are equivalent

1. $\lim_{n} \int_X (M_T \phi_n)^p d\mu = F \omega_p(f^p/F)^p$
2. $\lim_{n} \int_X |M_T \phi_n - c\phi_n|^p d\mu = 0$, where $c = \omega_p(f^p/F)$. □

Now it is interesting to search for the opposite problem concerning (1.7). In fact we will prove the following:

**Theorem 1.** Let $g_n : (0, 1] \to \mathbb{R}^+$ be a sequence of non-increasing functions continuous such that $\int_0^1 g_n(u) du = f$ and $\int_0^1 g_n^p(u) du = F$, for every $n \in \mathbb{N}$. Then the following are equivalent

1. $\lim_{n} \frac{1}{\mu(I)} \int_I (\frac{1}{t} \int_0^t g_n) dt = F \omega_p(f^p/F)^p$
2. $\lim_{n} \frac{1}{\mu(I)} \int_I |g_n - cg_n(t)|^p dt = 0$

where $c = \omega_p(f^p/F)$. □

The proof is based on the proofs of Theorem A and B.

Concerning now the problem (1.6) it can be easily seen that extract functions do not exist (when the tree $T$ differentiates $L^1(X, \mu)$). That is for every $\phi \in L^p(X, \mu)$ with $\phi \geq 0$ and $\int_X \phi d\mu = f$, $\int_X \phi^p d\mu = F$ we have the strict inequality $\int_X (M_T \phi)^p d\mu < F \omega_p(f^p/F)^p$.

This is because a self-similar property that is mentioned in [6] and states that for every extremal sequence $\phi_n$ for (1.6) the following is true:

$$\lim_{n} \frac{1}{\mu(I)} \int_I \phi_n d\mu = f \quad \text{while} \quad \lim_{n} \frac{1}{\mu(I)} \int_I \phi_n^p d\mu = F.$$  \hfill (1.8)

So, if $\phi$ is an extremal function for (1.6), then we must have that $\frac{1}{\mu(I)} \int_I \phi d\mu = f$ and $\frac{1}{\mu(I)} \int_I \phi^p d\mu = F$ and if the tree $T$ differentiates $L^1(X, \mu)$ then we must have that $\mu$-a.e
the following equalities hold $\phi(x) = f$ and $\phi^p(x) = F$, that is $f^p = F$ which is the trivial case.

It turns out that the above doesn’t hold for the extremal problem (1.7). That is there exist extremal functions for (1.7). We state it as:

**Theorem 2.** There exist $g : (0, 1] \to \mathbb{R}^+$ non-increasing and continuous with
\[
\int_0^1 g(u) du = f \quad \text{and} \quad \int_0^1 g^p(u) du = F
\]
such that
\[
\int_0^1 \left( \frac{1}{t} \int_0^t g \right)^p dt = F \omega_p(f^p/F)^p.
\]

(1.9)

As it is expected due to Theorem 1 it satisfies the following equality $\frac{1}{t} \int_0^t g(u) du = \omega_p(f^p/F)g(t)$ for every $t \in (0, 1]$ which gives immediately gives (1.9).

After proving Theorem 2 we will be able to prove the following

**Theorem 3.** Let $g_n$ be as in Theorem 1. Then the following are equivalent

i) $\lim_n \int_0^1 \left( \frac{1}{t} \int_0^t g_n \right)^p dt = F \omega_p(f^p/p)^p$

ii) $\lim_n \int_0^1 |g_n - g|^p dt = 0$, where $g$ is the function constructed in Theorem 2.

In this way we complete the discussion about the characterization of the extremal functions for this problem related to the Hardy operator.

The paper is organized as follows:

We prove Theorem 1 and 2 and 3 in Sections 3 and 4 and 5 respectively while in Section 2 we give some preliminary definitions and results.

2. Preliminaries

Let $(X, \mu)$ be a non-atomic probability measure space. A set $\mathcal{T}$ of measurable subsets of $X$ will be called a tree if it satisfies conditions of the following

**Definition 2.1.**

i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.

ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that
   a) the elements of $C(I)$ are pairwise disjoint subsets of $I$
   b) $I = \bigcup C(I)$.

iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}(m)$ where $\mathcal{T}(0) = \{X\}$ and $\mathcal{T}(m+1) = \bigcup_{I \in \mathcal{T}(m)} C(I)$.

iv) We have that $\lim_{m \to \infty} \sup_{I \in \mathcal{T}(m)} \mu(I) = 0$. \hfill $\square$
Examples of trees are given in [3]. The most known is the one given by the family of all dyadic subcubes of \([0,1]^n\).

The following has been proved in [3].

**Lemma 2.1.** For every \(I \in \mathcal{T}\) and every \(a\) such that \(0 < a < 1\) there exists a subfamily \(\mathcal{F}(I) \subseteq \mathcal{T}\) consisting of disjoint subsets of \(I\) such that
\[
\mu \left( \bigcup_{J \in \mathcal{F}(I)} J \right) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1 - a)\mu(I).
\]

□

We will also need the following fact obtained in [7].

**Lemma 2.2.** Let \(\phi : (X, \mu) \to \mathbb{R}^+\) and \((A_j)_j\) a measurable partition of \(X\) such that \(\mu(A_j) > 0\) \(\forall j\). Then if \(\int_X \phi d\mu = f\) there exists a rearrangement of \(\phi\), say \(h(h^* = \phi^*)\) such that
\[
\frac{1}{\mu(A_j)} \int_{A_j} g d\mu = f, \text{ for every } j.
\]

□

Now given a tree on \((X, \mu)\) we define the associated dyadic maximal operator as follows
\[
\mathcal{M}_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\varphi| d\mu : x \in I \in \mathcal{T} \right\}, \ \phi \in L^1(X, \mu)
\]

We will also the following from [7].

**Lemma 2.3.** Let \(k \in (0,1]\) and \(K\) measurable subset of \(X\) with \(\mu(K) = k\). Then the following inequality holds
\[
\int_K G[\mathcal{M}_T \phi] d\mu \leq \int_0^k G \left( \frac{1}{t} \int_0^t g(u) du \right) dt
\]
where \(g = \phi^*\), \(\phi \in L^1(X, \mu)\) and \(G : [0, +\infty) \to [0, +\infty)\) is a non-decreasing function. □

3. **Proof of Theorem 1**

We will prove Theorem 1 by repeating the proof of Theorem A and by using Theorem B.

We begin with a sequence \((g_n)_n\) of non-increasing continuous functions \(g_n : (0,1] \to \mathbb{R}^+\) such that \(\int_0^1 g_n(u) du = f\) and \(\int_0^1 g_n^p(u) du = F\) where \(0 < f^p \leq F\). We set \(c = \omega_p(f^p/F)\) and we suppose that \((g_n)_n\) is extremal for \([17]\), that is
\[
\lim_n \int_0^1 \left( \frac{1}{t} \int_0^t g_n \right)^p dt = F \omega_p(f^p/F)^p = F \cdot c^p.
\]

Our aim is to prove that
\[
\lim_n \int_0^1 \left| \frac{1}{t} \int_0^t g_n - cg_n(t) \right|^p dt = 0.
\]
For this reason it is enough to prove that

\[(3.1) \quad \int_{\{t: \frac{1}{t} \int_0^t g_n > cg_n(t)\}} \left[ \frac{1}{t} \int_0^t g_n - cg_n(t) \right]^p dt = I_{1,n} \to 0, \quad \text{and} \]

\[\int_{\{t: \frac{1}{t} \int_0^t g_n < cg_n(t)\}} \left[ cg_n(t) - \frac{1}{t} \int_0^t g_n \right]^p dt = I_{2,n} \to 0, \quad \text{as} \quad n \to \infty.\]

We consider the quantity in (3.1) and similarly we work in the second. Set \( A_n = \left\{ t \in (0, 1] : \frac{1}{t} \int_0^t g_n > cg_n(t) \right\} \) so we need to prove that

\[\int_{A_n} \left[ \frac{1}{t} \int_0^t g_n - cg_n(t) \right]^p dt \to 0, \quad \text{as} \quad n \to \infty.\]

Since \((x - y)^p < x^p - y^p\), for \(x > y > 0\) and \(p > 1\) it is enough to prove that

\[II_n = \int_{A_n} \left( \frac{1}{t} \int_0^t g_n \right)^p dt - \varepsilon \int_{A_n} g_n^p \to 0, \quad n \to \infty.\]

For each \( A_n \), which is an open set of \((0, 1]\) we consider its connected components \( I_{n,i}, i = 1, 2, \ldots \). So \( A_n = \bigcup_{i=1}^{\infty} I_{n,i} \), with \( I_{n,i} \) open intervals in \((0, 1]\) with \( I_{n,i} \cap I_{n,j} = \emptyset \) for \( i \neq j \).

Let \( \varepsilon > 0 \). For every \( n \in \mathbb{N} \) choose \( i_n \in \mathbb{N} \) such that

\[|III_n - III_{1,n}| < \varepsilon \quad \text{and} \quad |IV_n - IV_{1,n}| < \varepsilon\]

where

\[III_n = \int_{A_n} \left( \frac{1}{t} \int_0^t g_n \right)^p dt, \quad III_{1,n} = \int_{F_n} \left( \frac{1}{t} \int_0^t g_n \right)^p dt, \quad IV_n = \varepsilon^p \int_{A_n} g_n^p, \quad IV_{1,n} = \varepsilon^p \int_{F_n} g_n^p,\]

and \( F_n = \bigcup_{i=1}^{i_n} I_{n,i} \).

It is clear that such choice of \( i_n \) exists. Then \( |II_n - II_{1,n}| < 2\varepsilon \) where

\[II_{1,n} = \int_{F_n} \left( \frac{1}{t} \int_0^t g_n \right)^p dt - \varepsilon^p \int_{F_n} g_n^p.\]

We need to find a \( n_0 \in \mathbb{N} \) such that \( II_{1,n} < \varepsilon, \forall n \geq n_0 \). Fix now a \( g_n =: g \). We prove the following

**Lemma 3.1.** There exists a family \( \phi_a : (X, \mu) \to \mathbb{R}^+ \) of rearrangements of \( g \) (\( \phi_a^* = g \) for each \( a \in (0, 1) \)) such that for each \( \gamma \in (0, 1] \) there exists a family of measurable subsets of \( X \), \( S_{\alpha}^{(\gamma)} \) satisfying the following:

\[\lim_{a \to 0^+} \int_{S_{\alpha}^{(\gamma)}} [M_T(\phi_a)]^p d\mu = \int_0^\gamma \left( \frac{1}{t} \int_0^t g \right)^p dt\]

and \( \lim_{a \to 0^+} \mu(S_{\alpha}^{(\gamma)}) = \gamma. \) Moreover \( S_{\alpha}^{(\gamma)} \subseteq S_{\alpha}^{(\gamma')} \) for each \( \alpha \gamma < \gamma' \leq 1 \) and \( \alpha \in (0, 1) \). \( \square \)
Proof. We follow [7]. Let \( a \in (0,1) \). By using Lemma 2.1 we choose for every \( I \in \mathcal{T} \) a family \( \mathcal{F}(I) \subseteq \mathcal{T} \) of disjoint subsets of \( I \) such that

\[
\sum_{J \in \mathcal{F}(I)} \mu(J) = (1-a)\mu(I).
\]

We define \( S = S_a \) to be the smallest subset of \( \mathcal{T} \) such that \( X \in S \) and for every \( I \in S \), \( \mathcal{F}(I) \subseteq S \). We write for \( I \in S \), \( A_I = I \setminus \bigcup_{J \in \mathcal{F}(I)} J \). Then if \( a_I = \mu(A_I) \) we have because of (3.2) that \( a_I = a\mu(I) \). It is also clear that for every \( I \)

\[
S_a = \bigcup_{m \geq 0} S_a(m), \quad \text{where } S_a(0) = \{X\} \quad \text{and } S_a(m+1) = \bigcup_{I \in S_a(m)} \mathcal{F}(I).
\]

We define also for \( I \in S_a \), \( \text{rank}(I) = r(I) \) to be the unique integer \( m \) such that \( I \in S_a(m) \).

Additionally, we define for every \( I \in S_a \) with \( r(I) = m \)

\[
\gamma(I) = \gamma_m = \frac{1}{a(1-a)^m} \int_{(1-a)^{m+1}} (1-a)^m g(u)du.
\]

We also set for \( I \in S_a \)

\[
b_m(I) = \sum_{S \supseteq I \subseteq I} \mu(J).
\]

We easily then see inductively that

\[
b_m(I) = (1-a)^m \mu(I).
\]

It is also clear that for every \( I \in S_a \)

\[
I = \bigcup_{S_a \ni J \subseteq I} A_J.
\]

At last we define for every \( m \) the measurable subset of \( X \), \( S_m = \bigcup_{I \in S_a(m)} I \).

Now, for each \( m \geq 0 \), we choose \( \tau_a^{(m)} : S_m \setminus S_{m+1} \to \mathbb{R} \) such that

\[
[\tau_a^{(m)}]^* = \left( g/((1-a)^{m+1}, (1-a)^m) \right)^*.
\]

This is possible since \( \mu(S_m \setminus S_{m+1}) = \mu(S_m) - \mu(S_{m+1}) = b_m(X) - b_{m+1}(X) = (1-a)^m - (1-a)^{m+1} = a(1-a)^m \). It is obvious now that \( S_m \setminus S_{m+1} = \bigcup_{I \in S_a(m)} A_I \) and that

\[
\int_{S_m \setminus S_{m+1}} \tau_a^{(m)} d\mu = \int_{(1-a)^m} (1-a)^m g(u)du \Rightarrow \frac{1}{\mu(S_m \setminus S_{m+1})} \int_{S_m \setminus S_{m+1}} \tau_a d\mu = \gamma_m.
\]

Using now Lemma 2.2 we see that there exists a rearrangement of \( \tau_a | S_m \setminus S_{m+1} = \tau_a^{(m)} \) called \( \phi_a^{(m)} \) for which \( \frac{1}{a_I A_I} \int_{A_I} \phi_a^{(m)} = \gamma_m \), for every \( I \in S_a(m) \).

Define now \( \phi_a : X \to \mathbb{R}^+ \) by \( \phi_a(x) = \phi_a^{(m)}(x) \), for \( x \in S_m \setminus S_{m+1} \). Of course \( \phi_a^* = g \).
Let now $I \in S_{a,(m)}$. Then
\[
Av_I(\phi_a) = \frac{1}{\mu(I)} \int_I \phi_a d\mu = \frac{1}{\mu(I)} \sum_{s \ni J \ni I} \int_{A_J} \phi_a d\mu
\]
\[
= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{s \ni J \ni I \atop r(J) = r(I) + \ell} \int_{A_J} \phi_a d\mu
\]
\[
= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{s \ni J \ni I} \gamma_{m+\ell} a_J
\]
\[
= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{s \ni J \ni I} a_{\mu(J)} \frac{1}{a(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}} g(u) du
\]
\[
= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}} g(u) du \cdot \sum_{s \ni J \ni I \atop r(J) = m+\ell} \mu(J)
\]
\[
= \frac{1}{(1-a)^m} \sum_{\ell \geq 0} \int_{(1-a)^{m+\ell+1}} g(u) du
\]
\[
= \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u) du.
\]
\[
(3.3)
\]
Now for $x \in S_m \setminus S_{m+1}$, there exists $I \in S_{a,(m)}$ such that $x \in I$ so
\[
(3.4) \quad \mathcal{M_T}(\phi_a)(x) \geq Av_I(\phi_a) = \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u) du =: \theta_m,
\]
Since $\mu(S_m) = (1-a)^m$, for every $m \geq 0$ we easily see from the above that we have
\[
[\mathcal{M_T}(\phi_a)]^*(t) \geq \theta_m, \quad \text{for every} \quad t \in \left((1-a)^{m+1}, (1-a)^m\right].
\]
For any $a \in (0,1]$ we now choose $m = m_a$ such that $(1-a)^{m+1} \leq \gamma < (1-a)^m$. So we have $\lim_{a \to 0^+} (1-a)^{m_a} = \gamma$.

Then using Lemma 2.3 we have that
\[
(3.5) \quad \limsup_{a \to 0^+} \int_{\cup S_{a,(m_a)}} [\mathcal{M_T}(\phi_a)]^p d\mu \leq \int_0^\gamma \left( \frac{1}{t} \int_0^t g \right)^p dt < +\infty,
\]
where $\cup S_{a,(m,a)}$ denotes the union of the elements of $S_{a,(m_a)}$. This is $S_{m_a} = \bigcup_{I \in S_{a,(m_a)}} I$.

This is true since $\mu(S_{m_a}) \to \gamma$, as $a \to 0^+$. 

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Then
\[
\int_{S_m \cap S_{\ell+1}} (M_T \phi)^p d\mu = \sum_{\ell \geq m} \int_{S_{\ell} \cap S_{\ell+1}} (M_T \phi)^p d\mu \\
\geq \sum_{\ell \geq m} \left( \frac{1}{1-a} \right)^\ell \int_0^{1-a} g(u) du \mu(S_{\ell} \cap S_{\ell+1}) \\
\geq \sum_{\ell \geq m} \left( \frac{1}{1-a} \right)^\ell \int_0^{1-a} g(u) du \left\{ \left( (1-a)^{\ell+1} \right) \right\}
\]
(3.6)
\[
(1-a)^m \to \gamma \text{ and the right hand side of (3.6) expresses a Riemann sum of the} \\
\int_0^{1-a} \left( \frac{1}{t} \int_0^t g \right)^p dt \text{ we conclude that} \\
\limsup_{\ell \to 0^+} \int_{S_m \cap S_{\ell+1}} (M_T \phi)^p d\mu \geq \int_0^{\gamma} \left( \frac{1}{t} \int_0^t g \right)^p dt.
\]
(3.7)

Then by (3.5) we have equality on (3.7).

We thus constructed the family \((\phi_a)_{a \in (0,1)}\) we need for which we easily see that if
\[0 < \gamma < \gamma' \leq 1\] then \(S_\gamma(a) \subseteq S_{\gamma'}(a)\) for each \(\ell \in (0,1)\).
\[\square\]

**Remark 3.1.** It is not difficult to see by the proof of Lemma 3.1 that for every \(\ell \in \mathbb{N}\), of \(a \in (0,1)\) the following holds \(h = g/(0,(1-a)^\ell]\), where \(h\) is defined by \(h := \left( \phi_a/S_{a,\ell} \right)\) on \((0,(1-a)^\ell]\).

We now return to the proof of Theorem 1.

We remind that
\[II_{1,n} = \int_{F_n} \left( \frac{1}{t} \int_0^t g_n \right)^p dt - \left( \gamma \right)^p \int_{F_n} g_n^p = III_{1,n} - IV_{1,n}\]
with \(F_n = \bigcup_{i=1}^{i_n} I_{n,i} = \bigcup_{i=1}^{i_n} (a_{n,i},b_{n,i})\), which is a disjoint union. Thus
\[III_{1,n} = \sum_n \left[ \int_{0}^{b_{n,i}} \left( \frac{1}{t} \int_0^t g_n \right)^p dt - \int_{0}^{a_{n,i}} \left( \frac{1}{t} \int_0^t g_n \right)^p dt \right].
\]

Now, for every \(n \in \mathbb{N}\) we consider the corresponding to \(g_n\), family \((\phi_{a,n})_{a \in (0,1)}\) and the respective subsets of \(X\) \(S_{a,n}^{(a_{n,i})}\), \(S_{a,n}^{(b_{n,i})}\), \(a \in (0,1), i = 1,2,\ldots,n\) for which
\[
\mu(S_{a,n}^{(a_{n,i})}) \to a_{n,i} \text{ and } \mu(S_{a,n}^{(b_{n,i})}) \to b_{n,i}, \text{ as } a \to 0^+.
\]

We can also suppose that
\[a_{n,i} < b_{n,i} \leq a_{n,i+1} < b_{n,i+1}, \text{ } i = 1,2,\ldots,i_n - 1.\]

Then we also have that
\[S_{a,n}^{(a_{n,i})} \subseteq S_{a,n}^{(b_{n,i})} \subseteq S_{a,n}^{(a_{n,i+1})}\] and of course...
Additionally, we can suppose because of the relation 

\[ V_n = \sum_{i=1}^{n} \left[ \int_{S_{n,i}} (M_T \phi_{n,i})^p d\mu - \int_{S_{n,i}^c} (M_T \phi_{n,i})^p d\mu \right] \]

and since \( g_n \) is extremal for the problem (1.7) that \( a_{0,n} \) can be chosen such that for every \( n \in \mathbb{N} \)

\[ \left| \int_X (M_T \phi_{n,i})^p d\mu - F \omega_p(f^p/F)^p \right| \leq \frac{1}{n}, \quad \text{for every } n \in \mathbb{N}. \]

Choose \( a'_{n} \in (0, a_n) \) and from the sequence

\[ \phi_{a'_{n}, n} =: \phi_n. \]

Then, because of (3.9) and since \( \phi_n^* = g_n \), we have that \( \phi_n \) is extremal for (1.6).

Because of the Remark, after the proof of Lemma 3.1 we have now for every \( \ell \in \mathbb{N} \), each \( n \in \mathbb{N} \) and \( a \in (0,1) \), that

\[ \left( \phi_{a,n}/S_{a,1} \right)^*: (0, \mu(S_t) = (1-a)^\ell) \to \mathbb{R}^+ \]

is equal to \( g_n/(0, (1-a)^\ell) \). Since \( \lim_{a \to 0^+} \mu(A_n^{(a)}) = |F_n| \), for every \( n \in \mathbb{N} \) we can suppose that \( a_{0,n} \) satisfies the following

\[ |\mu(A_n^{(a)}) - |F_n|| < \frac{1}{n}, \quad \text{for every } a \in (0, a_{0,n}) \]

so if \( L_n = L_n^{(a')} \) we must have additionally, since \( \phi_{a'_{n}, n} = \phi_n \), that

\[ \left| \int_{F_n} \left( \frac{1}{t} \int_0^t g_n \right)^p dt - \int_{A_n} (M_T \phi_n)^p d\mu \right| \leq \frac{1}{n} \]

and that \( |\mu(L_n) - |F_n|| < \frac{1}{n}, \quad \text{for every } n \in \mathbb{N}. \)

It is also easy to see because of the above relations and the Remark 3.1 and the form of \( L_n \) (passing to a subsequence if necessary) that

\[ \lim_n \int_{L_n} \phi_{n}^p = \lim_n \int_{F_n} g_{n}^p. \]

We now take advantage of Theorem B.
Since \( \phi_n \) is extremal for (1.6) we must have that \( \int_X |M_T \phi_n - c\phi_n|^p \, d\mu \to 0 \), as \( n \to \infty \) where \( c = \omega_p(f^p/F)^p \). This implies:

\[
\int_{A_n \cap \{M_T \phi_n \geq c\phi_n\}} (M_T \phi_n - c\phi_n)^p \, d\mu \to 0, \quad \text{as} \quad n \to \infty \quad \text{or}
\]

\[
\int_{A'_n} (M_T \phi_n - c\phi_n)^p \, d\mu \to 0, \quad \text{as} \quad n \to \infty, \quad \text{where} \quad A'_n = A_n \cap \{M_T \phi_n \geq c\phi_n\}.
\]

Since

\[
\left[ \int_{A'_n} (M_T \phi_n)^p \right]^{1/p} \leq \left[ \int_{A'_n} (M_T \phi_n - c\phi_n)^p \right]^{1/p} + \left[ \int_{A'_n} (c\phi_n)^p \right]^{1/p}
\]

we must have, because of the definition of \( A'_n \) that:

\[
\lim_n \int_{A'_n} (M_T \phi_n)^p = c^p \lim_n \int_{A'_n} \phi_n^p.
\]

In the same way we prove that:

\[
\lim_n \int_{A_n \setminus A'_n} (M_T \phi_n)^p = c^p \lim_n \int_{A_n \setminus A'_n} \phi_n^p, \quad \text{so}
\]

\[
\lim_n \int_{A_n} (M_T \phi_n)^p \, d\mu = c^p \lim_n \int_{A_n} \phi_n^p \, d\mu.
\]

Because now of (3.10) we have that

\[
\lim_n \int_{F_n} \left( \frac{1}{t} \int_0^t g_n \right)^p \, dt = \lim_n c^p \int_{F_n} g_n^p,
\]

and from the choice of \( F_n \) we see that we must have that \( II_n < 2\varepsilon \), for \( n \geq n_0 \), for a suitable \( n_0 \in \mathbb{N} \). And this is what we wanted to prove. \( \square \)

4. Uniqueness of extremal functions

In this section we will prove that there exists unique \( g_0 : (0, 1] \to \mathbb{R}^+ \) with

\[
\int_0^1 g_0(u) \, du = f, \quad \int_0^1 g_0^p(u) \, du = f^p \quad \text{and}
\]

\[
\int_0^1 \left( \frac{1}{t} \int_0^t g_0(u) \, du \right)^p \, dt = F \omega_p(f^p/F)^p.
\]

This is the statement of Theorem 2.

**Proof of Theorem 2.** By Theorem 1 it is obvious that if such a function \( g_0 \) exists, it must satisfies

\[
(4.1) \quad \frac{1}{t} \int_0^t g_0(u) \, du = cg_0(t), \quad \text{a.e on} \quad (0, 1], \quad \text{where} \quad c = \omega_p(f^p/p).
\]

Because of the continuity of \( g_0 \) we must have equality on (4.1) in all \( (0, 1] \).
So, in order that $g_0$ satisfies (4.1) we need to set $g_0(t) = kt^{-1 + \frac{1}{c}}$, $t \in (0, 1]$, and search for a constant $k$ such that (by solving the respective first order linear differential equation)

$$\int_0^1 g_0(u)du = f \quad \text{and} \quad \int_0^p g_0^p(u)du = f.$$  

The first equation becomes

$$\int_0^1 k t^{-1 + \frac{1}{c}} dt = f \Leftrightarrow kc = f \Leftrightarrow k = f/c.$$  

Se, we ask if $g_0$ for this $k$ satisfies the second equation. This is

$$\int_0^1 g_0^p(u)du = F \Leftrightarrow \frac{k^p}{(-p + 1 + \frac{1}{c})} = F \Leftrightarrow f^p/F = \left[ (-p + 1) + \frac{p}{c} \right] e^p \Leftrightarrow$$

$$-(p - 1)c + p c^{p-1} = f^p/F.$$  

But this is true because of the choice of $c = \omega_p(f^p/F)$ and $\omega_p = H_p^{-1}$ where

$$H_p(z) = -(p - 1)z^p + pz^{p-1}, \text{ for } t \in \left[ 1, \frac{p}{p - 1} \right].$$

Because now of the form of $g_0 : (0, 1] \to \mathbb{R}^+$ we have that

$$\frac{1}{t} \int_0^t g_0(u)du = cg_0(t), \ \forall t \in (0, 1] \Rightarrow \int_0^1 \left( \frac{1}{t} \int_0^t g_0(u)du \right)^p du = F \omega_p(f^p/F)^p.$$  

So $g_0$ is the only extremal function in $(0, 1]$.  

5. **Uniqueness of extremal sequences**

We are now able to prove Theorem 3.

The direction ii)$\Rightarrow$i) is obvious from the conditions that $g$ satisfies.

We now proceed to ii)$\Rightarrow$i)

We suppose that we are given $g_n : (0, 1] \to \mathbb{R}^+$ non-increasing, continuous, such that

$$\frac{1}{t} \int_0^t g_n(u)du = f, \int_0^p g_n^p(u)du = F \text{ and}$$

$$\lim_{n \to \infty} \int_0^1 \left( \frac{1}{t} \int_0^t g_n(u)du \right)^p dt = F \omega_p(f^p/F)^p.$$  

Using Theorem 2 we conclude that

$$
\lim_{n \to \infty} \int_0^1 \left| \frac{1}{t} \int_0^t g_n - cg_n(t) \right|^p dt.
$$

Thus there exists a subsequence $(g_{k_n})_n$ such that if

$$F_n(t) = \frac{1}{t} \int_0^t g_n - cg_n(t), \ t \in (0, 1], \ n \in \mathbb{N},$$

then $F_{k_n} \to 0$ almost everywhere (with respect to Lesbesgue measure). By a well known theorem in measure theory we have because of the finiteness of the measure space that $F_{k_n} \to 0$ uniformly almost every where on $(0, 1]$. This means that there exists a
sequence of Lesbesgue measurable subsets of \((0,1]\), say \((H_n)_n\) such that \(H_{n+1} \subseteq H_n\), \(|H_n| \leq \frac{1}{n}\) satisfying the following condition

\[
\left| \frac{1}{t} \int_0^t g_{k_n} - cg_{k_n}(t) \right| = |F_{k_n}(t)| \leq \frac{1}{n}, \quad \forall \ t \in (0,1] \setminus H_n.
\]

Additionally, we can suppose that \(H_n\) is the closure of an open set in \((0,1]\) therefore a disjoint union of closed intervals on \((0,1]\). Let now \(t, t' \in [a,1] \setminus H_{k_n}\), where \(a\) is a fixed element of \((0,1]\).

Then the following hold \((c = \omega_p(f^p/F))\)

\[
|cg_{k_n}(t) - cg_{k_n}(t')| \leq |cg_{k_n}(t) - \frac{1}{t} \int_0^t g_{k_n}| + \left| \frac{1}{t} \int_0^t g_{k_n} - \frac{1}{t} \int_0^{t'} g_{k_n} \right| + \left| \frac{1}{t'} \int_0^{t'} g_{k_n} - cg_{k_n}(t') \right| = I + II + III.
\]

Then \(I \leq \frac{1}{k_n}\) since \(t \notin H_n\). Similarly for III.

We look now at the second quantity II.

We also suppose that \(t' > t\), so \(t' = t + \delta\) for some \(\delta > 0\). Then

\[
II = \frac{1}{tt'}\left| \int_0^{t'} g_{k_n} - \int_0^t g_{k_n} \right|
\]

\[
\leq \frac{1}{at} \left( (t + \delta) \int_0^t g_{k_n} - t \int_0^t g_{k_n} - t \int_t^{t'} g_{k_n} \right)
\]

\[
= \frac{1}{at} \left| \frac{\delta}{a} \int_0^t g_{k_n} - t \int_t^{t'} g_{k_n} \right|
\]

\[
\leq \frac{\delta}{a^2} f + \frac{1}{a} \int_t^{t'} g_{k_n},
\]

(5.1)

where \(f = \frac{1}{t} \int_0^{t'} g_{k_n}\). Now by Holder’s inequality we have that

\[
\int_t^{t'} g_{k_n} \leq \left( \int_t^{t'} g_{k_n}^p \right)^{1/p} |t' - t|^{1 - \frac{1}{p}} = F \delta^{1 - \frac{1}{p}}.
\]

Thus \(II \leq \frac{\delta f}{a} + \frac{1}{a} \delta^{1 - \frac{1}{p}} F\).

We consequently have that for a given \(\varepsilon > 0\) there exists \(\delta = \delta_{a,\varepsilon} > 0\) for which the following implication holds

\[
t, t' \in [a,1] \setminus H_{k_n}, \quad |t - t'| < \delta \Rightarrow |g_{k_n}(t) - g_{k_n}(t')| < \varepsilon, \quad \text{for every } n \in \mathbb{N}.
\]

Thus \((g_{k_n})_n\) has a property of type of equicontinuity on a certain set that depends on \(a\). We consider now an enumeration of the rationals in \((0,1]\), let \(\{q_1,q_2,\ldots,g_k,\ldots\} = Q \cap (0,1]\).
For every \( q \in Q \cap (0, 1) \) we have that \( (g_{k_n}(q))_n \) is a bounded sequence of real numbers, because \( g_{k_n} \) is a sequence of non-negative, non-increasing functions on \((0, 1]\) satisfying
\[
\sup_n \int_0^1 g_{k_n} < +\infty.
\]

By a diagonal argument we produce a subsequence which we denote again by \( g_{k_n} \) such that \( g_{k_n}(q) \to \lambda_q, \ n \to \infty \) where \( \lambda_q \in \mathbb{R}^+ \), \( q \in Q \cap (0, 1] \).

Let \( H = \bigcap_{n=1}^{\infty} H_{k_n} \), which is a set of Lesbesgue measure zero, and suppose that \( x \in (a, 1) \setminus \hat{H} \). Then \( x > a \), and there exist \( n_0 \in \mathbb{N} \) such that \( x \notin H_{k_{n_0}} \), so that \( x \notin H_{n_k}, \ \forall \ n \geq n_0 \). Additionally, choose a sequence \((p_k)_k\) of rationals on \((a, 1) \setminus H_{n_0}\) such that \( p_k \to x \). This is possible because the set \((q, 1) \setminus H_{n_0}\) is an open set. Thus, we have that \( p_k > a \) and \( p_k \notin H_{k_n} \), \( n \geq n_0, \ k \in \mathbb{N} \).

Let now \( k_0 \in \mathbb{N} : |p_k - x| < \delta, \ \forall \ k \geq k_0, \) where \( \delta \) is one given in \((5.2)\).

We then have that \( |g_{k_n}(x) - g_{k_n}(p_{k_0})| < \varepsilon, \) for every \( n \in \mathbb{N} \). Thus, for every such \( x \), and every \( n, m \in \mathbb{N} \) we have that
\[
|g_{k_n}(x) - g_{k_m}(x)| \leq |g_{k_n}(x) - g_{k_n}(p_{k_0})| + |g_{k_n}(p_{k_0}) - g_{k_m}(p_{k_0})| + |g_{k_m}(p_{k_0}) - g_{k_m}(x)| < 2\varepsilon + |g_{k_n}(p_{k_0}) - g_{k_m}(p_{k_0})|.
\]

But \((g_{k_n}(p_{k_0}))_n\) is convergent sequence, thus Cauchy. Then \((g_{k_n}(x))_n\) is a Cauchy sequence for every \( x \in (a, 1) \setminus \hat{H} \) for every \( a \in (0, 1] \).

Thus \((g_{k_n}(x))_n\) is a Cauchy sequence in all \((0, 1] \setminus \hat{H} \).

As a consequence there exists \( g'_0 : (0, 1] \to \mathbb{R}^+ \) such that
\[
(5.3) \quad g_{k_n} \to g'_0 \text{ a.e. on } (0, 1] \Rightarrow g_{k_n} \to g'_0 \text{ uniformly a.e. on } (0, 1].
\]

But this easily implies since \( F_{k_n}(t) \to 0 \), a.e. that
\[
(5.4) \quad \frac{1}{t} \int_0^t g'_0(u)du = c g'_0(t), \text{ a.e. on } (0, 1].
\]

Since \((5.3)\) holds we have that \( g'_0 \in L^p((0, 1]) \) and that \( \frac{1}{t} \int_0^t g'_0 = f \) and \( \frac{1}{t} \int_0^t (g'_0)^p = F. \)

Also, since the function \( t \mapsto \int_0^t g'_0 \) is continuous on \((0, 1]\) we must have that \( g'_0 \) can be considered continuous with equality on \((5.4)\) everywhere on \((0, 1]\).

This gives us that \( g'_0 \) is the function constructed in Theorem 2.

Additionally we obtain
\[
(5.5) \quad \int_0^1 |g_{k_n}' - g_0'|^p dt \to 0, \text{ as } u \to \infty
\]

because of \((5.3)\) and the fact that
\[
(5.6) \quad \lim_{\delta \to 0^+} \left( \sup \left\{ \int_A |g_{k_n}'(u)du : n \in \mathbb{N}, \ A \subseteq (0, 1] \text{ with } |A| = \delta \right\} \right) = 0.
\]

The validity of \((5.4)\) can be concluded from the following
Remark 5.2. In \cite{3} it can be seen that the following is true
\[
\sup_{K} \left\{ \int_{K} (\mathcal{M}_{T} \phi)^{p} d\mu : \phi \geq 0, \int_{X} \phi d\mu = f, \int_{X} \phi^{p} d\mu = F, \mu(k) = k \right\} \to 0
\]
as \( k \to 0 \), for \( 0 < f^{p} \leq F \).

This implies because of the symmetrization principle (Theorem A) that
\[
\sup_{B_{p}(f,F,k)} \left\{ \int_{0}^{1} \left( \frac{1}{t} \int_{0}^{t} g \right)^{p} dt; g : (0,1] \to \mathbb{R}^{+} \text{ non-increasing, continuous,} \int_{0}^{1} g_{n} = f, \int_{0}^{1} g_{1} = F \right\}, \text{ as } k \to 0.
\]
Then the supremum is (5.6) is bounded above \( B_{p}(\ell,F,\delta) \) for every \( \delta \in (0,1] \).

Thus, if we work on every subsequence of \((g_{n})_{n}\) which is again extremal we produce a subsequence of it for which (5.5) is satisfied. Therefore the proof of Theorem 3 is completed. \( \square \)

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