RADIAL REGULAR AND RUPTURE SOLUTIONS FOR A MEMS MODEL WITH FRINGING FIELD

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ABSTRACT. We investigate radial solutions for the problem

\[
\begin{aligned}
-\Delta U &= \frac{\lambda + \delta |\nabla U|^2}{1-U}, & U > 0 & \text{in } B, \\
U &= 0 & & \text{on } \partial B,
\end{aligned}
\]

which is related to the study of Micro-Electromechanical Systems (MEMS). Here, \(B \subset \mathbb{R}^N (N \geq 2)\) denotes the open unit ball and \(\lambda, \delta > 0\) are real numbers. Two classes of solutions are considered in this work: (i) regular solutions, which satisfy \(0 < U < 1\) in \(B\) and (ii) rupture solutions which satisfy \(U(0) = 1\), and thus make the equation singular at the origin. Bifurcation with respect to parameter \(\lambda > 0\) is also discussed.

1. Introduction and the main results

In this paper we are concerned with the problem

\[
(1.1) \quad \begin{aligned}
-\Delta U &= \frac{\lambda + \delta |\nabla U|^2}{1-U}, & U > 0 & \text{in } B, \\
U &= 0 & & \text{on } \partial B,
\end{aligned}
\]

where \(B \subset \mathbb{R}^N (N \geq 2)\) denotes the open unit ball and \(\lambda, \delta > 0\) are real numbers. The study of (1.1) is motivated by the more general problem

\[
(1.2) \quad \begin{aligned}
-\Delta U &= \frac{\lambda + \delta |\nabla U|^2}{(1-U)^p}, & U > 0 & \text{in } B, \\
U &= 0 & & \text{on } \partial B,
\end{aligned}
\]

where \(p \geq 1\). The case \(p = 2\) in (1.2) was discussed in [18]. As the authors in [18] emphasized, their approach is suitable to treat (1.2) for all \(p > 1\). The problem [18] with \(p = 2\) arises in the mathematical modelling of the Micro-Electromechanical Systems (MEMS). In such a context, \(\lambda\) represents the applied voltage while \(\delta |\nabla u|^2\) is related to the effect of a fringing electrostatic field (see, e.g., [10]).

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In this paper we are interested in the study of two classes of radial solutions to (1.1) namely

- **regular solutions**, which satisfy $0 < U < 1$ in $B$.
- **rupture solutions** which satisfy $U(0) = 1$, and thus make the main equation in (1.1) singular at the origin.

The results for radial solutions to (1.2) obtained in [18] show that:

- There exists $\Lambda > 0$ such that:
  - (i) problem (1.2) has no regular solutions for $\lambda > \Lambda$.
  - (ii) problem (1.2) has a unique regular solution if $\lambda = \Lambda$.
  - (iii) problem (1.2) has at least two regular solutions for $0 < \lambda < \Lambda$.
- Problem (1.2) has no rupture solutions for all $\lambda > 0$.

In this paper, our results for (1.1) reveal a striking difference to those for (1.2). More precisely, we show that:

- Problem (1.1) has infinitely many regular solutions for suitable $\lambda$ and $\delta$ (see Theorem 1.1 (ii) below).
- If $0 < \delta < N/2$ then problem (1.1) has exactly one rupture solution (see Theorem 1.1 below).
- If $\delta \geq N/2$ then, for $\lambda > 0$ small, problem (1.1) has infinitely many rupture solutions (see Theorem 1.4 below).

Finally, let us mention that non-radial rupture solutions for the problem

\begin{equation}
(1.3) \quad \begin{cases}
-\Delta U = \frac{\lambda(1 + |\nabla U|^2)}{(1 - U)^2}, & U > 0 \quad \text{in } \Omega, \\
U = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^2$ is a smooth domain, are discussed in [3]. It is obtained in [3] that (1.3) admits a solution which develops an isolated rupture as $\lambda \to 0$. Furthermore, if $\Omega$ is not simply connected, then, for any $m \geq 1$ problem (1.3) admits a solution with $m$ isolated ruptures as $\lambda \to 0$.

Radial regular solutions of (1.1) satisfy

\begin{equation}
(1.4) \quad \begin{cases}
U'' + \frac{N-1}{r} U' + \frac{\lambda + \delta(U')^2}{1 - U} = 0 & \text{for } 0 < r < 1, \\
U(0) = \alpha \in (0,1), \\
U'(0) = 0, \quad U(1) = 0.
\end{cases}
\end{equation}

In our approach we will first try to use a change of unknown in order to reduce the gradient term $|\nabla U|^2$ in (1.1). Thus, taking $U = 1 - \phi(u)$ in (1.1) we see that $u$ fulfills

\begin{equation}
(1.5) \quad \begin{cases}
\Delta u = \frac{\lambda}{\phi(u)\phi'(u)}, & u > 0 \quad \text{in } B, \\
u = a \geq 0 & \text{on } \partial B,
\end{cases}
\end{equation}

provided $\phi$ satisfies

\begin{align*}
\phi'(u) &= \phi(u)^\delta \quad \text{and} \quad \phi(a) = 1.
\end{align*}
Solving (1.5) we are led to three distinct shapes of \( \phi \) according to the cases
\( 0 < \delta < 1, \delta = 1 \) and \( \delta > 1 \). We shall see that problem (1.1) (and also (1.4)) is equivalent to (3.1), (3.3) or (3.4) which features a more convenient form to study. In particular, by the classical result in [6], any regular solution \( U \) of (1.1) must be radial and thus, it satisfies (1.4) for some \( \alpha > 0 \).

Furthermore, for fixed \( \delta > 0 \), the bifurcation diagram is given by the graph
\[ C := \{ (\alpha, \lambda(\alpha)) : 0 < \alpha < 1 \} \]
and \( C \) determines the solution structure. We easily see that \( \lim_{\alpha \to 0} \lambda(\alpha) = 0 \). In order to describe the shape of the graph \( \lambda(\alpha) \) let us introduce the following definition.

**Definition.** Let \( C \) be the bifurcation curve of (1.4).

(i) We call \( C \) of Type I if there exists \( \lambda^* > 0 \) such that \( \lambda(\alpha) \to \lambda^* \) (\( \alpha \to 1 \)) and \( \lambda(\alpha) \) is strictly increasing.

(ii) We call \( C \) of Type II if there exists \( \lambda^* > 0 \) such that \( \lambda(\alpha) \to \lambda^* \) (\( \alpha \to 1 \)) and \( \lambda(\alpha) \) oscillates around \( \lambda^* \).

We will see that for \( 0 < \delta < \frac{N}{2} \) the above \( \lambda^* \) is given by \( \lambda^* = N - 1 - \delta. \)

Our main result concerning regular solutions of (1.1) is stated below.

**Theorem 1.1.** Assume \( N \geq 2 \) and \( 0 < \delta < \frac{N}{2} \). The following hold true.

(i) If \( N \geq 3 \) and \( \delta \leq (N - 2 - 2\sqrt{N - 1})/2 \), then the bifurcation diagram of (1.1) is of Type I. Furthermore, (1.1) has a unique rupture solution \((\lambda^*, U^*)\) given by

\[
(\lambda^*, U^*) = (N - 1 - \delta, 1 - r)
\]
and (1.1) has exactly one radial regular solution for each fixed \( \lambda \in (0, \lambda^*) \).

(ii) If \( N \geq 3 \) and \( \delta > (N - 2 - 2\sqrt{N - 1})/2 \) or if \( N = 2 \), then the bifurcation diagram of (1.1) is of Type II. Furthermore, (1.1) has a unique rupture solution \((\lambda^*, U^*)\) given by (1.6) and (1.1) with \( \lambda = \lambda^* \) has infinitely many radial regular solutions. In particular, this occurs for \( 3 \leq N \leq 6 \) (since \( (N - 2 - 2\sqrt{N - 1})/2 < 0 \)).

Also, from the above result we deduce that if \( \delta = 1 \) and \( N \geq 3 \) then the following hold:

• If \( 3 \leq N \leq 9 \), then the bifurcation curve is of Type II.
• If \( N \geq 10 \), then the bifurcation curve is of Type I.

Fixing \( \delta > 0 \) we are next interested in the range of \( \lambda > 0 \) for which problem (1.1) has regular solutions. We obtain the following result.

**Theorem 1.2.** Let \( N \geq 2 \).

(i) Assume \( \delta > 0 \). If (1.1) has a regular solution then

\[
\lambda < \min \left\{ \frac{\mu_1}{4}, \frac{\mu_1}{\delta} \right\},
\]

where \( \mu_1 > 0 \) is the first eigenvalue of the Dirichlet Laplacian on \( B \).
Assume $\delta \geq N/2$. Then, there exists $\bar{\lambda} > 0$ such that (1.1) has exactly two radial regular solutions for each $\lambda \in (0, \bar{\lambda})$, exactly one radial regular solution for $\lambda = \bar{\lambda}$ and no radial regular solution for $\lambda > \bar{\lambda}$. Moreover,

$$\bar{\lambda} \geq N \left( \frac{2}{\delta + 1} \right)^{(\delta+1)/(\delta-1)}.$$  

Assume $\delta = N/2$. Then all the radial regular solutions of (1.1) can be described as

$$\lambda(U(r)) = (2N\alpha(1-\alpha), \alpha(1-r^2)), \ 0 < \alpha < 1.$$  

In particular, (1.1) has exactly two radial regular solutions for each $\lambda \in (0, N/2)$, exactly one radial regular solution for $\lambda = N/2$, and no radial regular solution for $\lambda > N/2$.

Remark. A parameter $\bar{\lambda} > 0$ is called the extremal value for (1.1) if

- (1.1) has a regular solution for $\lambda < \bar{\lambda}$.
- (1.1) has no regular solution for $\lambda > \bar{\lambda}$.

The solution for $\lambda = \bar{\lambda}$ is called the extremal solution of (1.1). By Theorem 1.1 and Theorem 1.2 we easily deduce the following.

Corollary 1.3. The following hold true.

(i) If $0 < \delta \leq (N - 2 - 2\sqrt{N-1})/2$, then $\bar{\lambda} = \lambda^*$ and the extremal solution is singular;

(ii) If $(N - 2 - 2\sqrt{N-1})/2 < \delta < N/2$, then

$$\lambda^* < \bar{\lambda} < \min \left\{ \frac{\mu_1}{4}, \frac{\mu_1}{\delta} \right\}$$

and the extremal solution is regular;

(iii) If $\delta \geq N/2$, then

$$N \left( \frac{2}{\delta + 1} \right)^{(\delta+1)/(\delta-1)} \leq \bar{\lambda} < \min \left\{ \frac{\mu_1}{4}, \frac{\mu_1}{\delta} \right\}.$$  

We turn next to the study of rupture solutions to (1.1). From Theorem 1.1 above we see that if $0 < \delta < N/2$ then (1.1) has a unique rupture solution given by (1.6). The result below presents our findings for $\delta \geq N/2$.

Theorem 1.4. Assume $\delta \geq N/2$. The following hold true.

(i) If $N = 2$, then there exists $\lambda^{**} > 0$ such that for each $\lambda \in (0, \lambda^{**})$, (1.1) has infinitely many rupture solutions. Furthermore, if $\delta = 1$ we have $\lambda^{**} = 1$.

(ii) If $N \geq 3$ and $N/2 \leq \delta < N - 1$ then, for any

$$0 < \lambda < \lambda^{***} := \frac{\delta(N - 1 - \delta)}{\delta - 1},$$

problem (1.1) has infinitely many rupture solutions $U(r)$ satisfying the following:
(ii1) For $N/2 < \delta < N - 1$, then

\[ U(r) = 1 - \sqrt{\frac{\lambda}{N - 1 - \delta}} r(1 + o(1)) \quad \text{as} \quad r \to 0. \]

(ii2) For $\delta = N/2$, then there exists $c > 1$ such that

\[ 1 - \sqrt{2\lambda} N c r \leq U(r) < 1 - \sqrt{2\lambda} N r \quad \text{for} \quad 0 < r < 1. \]

(iii) If $N \geq 3$ and $\delta \geq N - 1$, then there exists $\lambda^{****} > 0$ such that for each $\lambda \in (0, \lambda^{****})$, problem (1.1) has infinitely many rupture solutions. We see that for $N/2 \leq \delta < N - 1$ we have

\[ \lambda^* = N - 1 - \delta < \lambda^{****} = \delta(N - 1 - \delta)/(\delta - 1) \]

and that the rupture solution (1.6) is included in both the cases (ii1) and (ii2) in Theorem 1.4 above.

The results in Theorems 1.1-1.4 can be summarized in the tables below.

| $\delta$            | (0, 1)          | 1            | (1, $\infty$) |
|---------------------|-----------------|--------------|---------------|
| regular solutions   | $(0, \lambda^*)$| $(0, 1]$     | $(0, \lambda]$|
| rupture solutions    | $\lambda^*$     | $(0, 1]$     | $(0, \lambda^{**})$|

Table 1. Case $N = 2$; The range of $\lambda$ for which problem (1.1) has a regular/rupture solution.

| $\delta$            | $(0, \frac{N}{2})$| $\frac{N}{2}$| $(\frac{N}{2}, N - 1)$| $[N - 1, \infty)$ |
|---------------------|--------------------|---------------|----------------------|------------------|
| regular solutions   | $(0, \lambda^*)$   | $(0, \frac{N}{2})$| $(0, \lambda]$       | $(0, \lambda]$   |
| rupture solutions    | $\lambda^*$        | $(0, \frac{N}{2})$| $(0, \lambda^{***})$ | $(0, \lambda^{****})$|

Table 2. Case $N \geq 3$; The range of $\lambda$ for which problem (1.1) has a regular/rupture solution.

We see that in the study of regular solutions to (1.1) we identify two critical parameters $\lambda^*$ and $\bar{\lambda}$ while in the study of rupture solutions we identify four critical parameters $\lambda^*, \lambda^{**}, \lambda^{***}$ and $\lambda^{****}$. Also $\lambda^* < \lambda^{***}$ (see (1.12)).

2. Some preliminary results

In this section we collect some useful results in our approach to prove Theorems 1.1-1.4.
Proposition 2.1 (see [8] [9]). Consider the problem

\[
\begin{cases}
  w'' + \frac{N-1}{r}w' + \lambda(w+1)^p = 0 & \text{for } 0 < r < 1, \\
  w'(0) = 0, \\
  w(1) = 0,
\end{cases}
\]

(2.1)

where \(1 < p \leq (N + 2)/(N - 2)\) if \(N \geq 3\), and \(1 < p < \infty\) if \(N = 2\). Then, there exists \(\lambda_* > 0\) such that:

- Problem (2.1) has exactly two regular solutions for \(0 < \lambda < \lambda_*\);
- Problem (2.1) has exactly one regular solution for \(\lambda = \lambda_*\);
- Problem (2.1) has no regular solutions for \(\lambda > \lambda_*\).

The case \(N \geq 3\) in the above result is discussed in [8, Section X] while the case \(N = 2\) follows from [9, Theorem 2.6].

Proposition 2.2 (see [2]). Let \(f : [0, \infty) \times \mathbb{R} \to \mathbb{R}\) and \(g : [0, \infty) \times [0, \infty) \to [0, \infty)\) be continuous functions such that:

(i) \(|f(t, z)| \leq g(t, |z|)\) for all \(z \in \mathbb{R}\), \(t \geq 0\);

(ii) the mapping \([0, \infty) \ni z \mapsto -\int_0^t g(t, z)\) is nondecreasing for all \(t \geq 0\).

Then, for every \(m > 0\) for which

\[
\int_0^\infty g(t, 2mt)dt < m,
\]

(2.2)

there exists a global solution \(z : [0, \infty) \to \mathbb{R}\) of

\[
\begin{cases}
  z''(t) + f(t, z(t)) = 0, & \text{for all } t > 0, \\
  z(0) = 0,
\end{cases}
\]

such that \(z(t)/t \to m\) as \(t \to \infty\).

3. Proof of Theorem 1.1

Let \(U\) be a solution of (1.4). The proof will be divided into three cases.

Case 1: \(0 < \delta < 1\). Let \(u(r) := 1 - (1 - U(r))^{1-\delta}\) and \(\tilde{\lambda} := (1 - \delta)\lambda\). Then (1.4) reads

\[
\begin{cases}
  u'' + \frac{N-1}{r}u' + \frac{\tilde{\lambda}}{(1-u)^p} = 0 & \text{for } 0 < r < 1, \\
  u(0) = 1 - (1 - \alpha)^{1-\delta}, \\
  u'(0) = 0, \\
  u(1) = 0,
\end{cases}
\]

(3.1)

where \(p := (1 + \delta)/(1 - \delta) > 1\). This is equivalent to problem (1.2) with \(\delta = 0\) and has been studied by many authors. The existent results in \([5, 11]\) for (3.1) can be summarized as follows:

- If \(p > 1\), then

\[
\tilde{\lambda} \to \tilde{\lambda}^* := \frac{2}{p+1} \left( N - 2 + \frac{2}{p+1} \right) \quad \text{as } u(0) \to 1.
\]

(3.2)
• If $p > 1$, then Eq. (3.1) has a unique rupture solution $(\tilde{\lambda}^*, 1 - r^{2/(p+1)})$.
• If $p \leq p_c$, then the bifurcation curve is of Type I.
• If $p > p_c$, then the bifurcation curve is of Type II.

Here,
\[ p_c := \begin{cases} \infty & \text{if } N \geq 10, \\ -1 + \frac{4}{4-N+2\sqrt{N-1}} & \text{if } 2 \leq N < 10. \end{cases} \]

The bifurcation was discussed in [11, Theorem 1.2] while the uniqueness of the rupture solution and the convergence (3.2) was recently obtained in [5]. Note that $\lambda = \tilde{\lambda} \rightarrow N - 1 - \delta = \lambda^*$. Since $p = (1 + \delta)/(1 - \delta)$, we see that if $(N - 2 - 2\sqrt{N-1})/2 < \delta < 1$ (resp. $\delta \leq (N - 2 - 2\sqrt{N-1})/2$), then $p > p_c$ (resp. $p \leq p_c$). The assertions (i) and (ii) follow from the above bifurcation result for (3.1).

**Case 2:** $\delta = 1$. Let $u(r) := -2 \log(1 - U(r))$ and $\tilde{\lambda} = 2\lambda$. Then, $u$ satisfies
\[
\begin{cases}
    u'' + \frac{N-1}{r} u' + \tilde{\lambda} u = 0 & \text{for } 0 < r < 1, \\
    u(0) = -2 \log(1 - \alpha) > 0, \\
    u'(0) = 0, \\
    u(1) = 0.
\end{cases}
\]

Classical results [8, 12, 14] for (3.3) yield:
• If $N > 2$, then $\tilde{\lambda} \rightarrow \tilde{\lambda}^* := 2(N - 2)$ as $u(0) \rightarrow \infty$.
• If $N > 2$, then (3.3) has a unique singular solution $(\tilde{\lambda}^*, -2 \log r)$.
• If $2 < N < 10$, then the bifurcation curve $\tilde{\lambda}(u(0))$ oscillates around $\tilde{\lambda}^*$ as $u(0) \rightarrow \infty$.
• If $N \geq 10$, then the bifurcation curve $\tilde{\lambda}(u(0))$ is strictly increasing in $u(0)$.

We refer the reader to [8, Section IX] for the bifurcation result and the convergence, to [12] for the uniqueness of the singular solution as well as to [14] for the uniqueness and convergence. Note that $\lambda = \frac{1}{2} \rightarrow N - 2$. The assertions (i) and (ii) in Theorem 1.1 follow from the above results on (3.3).

**Case 3:** $1 < \delta < N/2$, $N \geq 3$. Let $u(r) := (1 - U(r))^{-(\delta-1)} - 1$ and $\tilde{\lambda} = (\delta - 1)\lambda$. Then (1.4) becomes
\[
\begin{cases}
    u'' + \frac{N-1}{r} u' + \tilde{\lambda}(u + 1)^p = 0 & \text{for } 0 < r < 1, \\
    u(0) = (1 - \alpha)^{-(\delta-1)} - 1 > 0, \\
    u'(0) = 0, \\
    u(1) = 0,
\end{cases}
\]

where $p = (\delta + 1)/(\delta - 1) > 1$. Classical results [8, 13, 15] related to (3.4) show that:
\begin{itemize}
\item If \( p > p_S \), then
\[
\tilde{\lambda} \rightarrow \tilde{\lambda}^* := \frac{2}{p-1} \left( N - 2 - \frac{2}{p-1} \right) \quad \text{as} \quad u(0) \rightarrow 1.
\]
\item If \( p > p_S \), then \((\tilde{\lambda}^*, r^{2/(p-1)} - 1)\) is the unique singular solution of (3.4).
\item If \( p_S < p < p_{JL} \), then the bifurcation curve \( \tilde{\lambda}(u(0)) \) oscillates around \( \tilde{\lambda}^* \) as \( u(0) \rightarrow \infty \).
\item If \( p \geq p_{JL} \), then the bifurcation curve \( \tilde{\lambda}(u(0)) \) is strictly increasing in \( u(0) \).
\end{itemize}

Here,
\[
p_S := \begin{cases} 
\infty & \text{if } N \leq 2, \\
\frac{N+2}{N-2} & \text{if } N > 2,
\end{cases}
p_{JL} := \begin{cases} 
\infty & \text{if } N < 11, \\
1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{if } N \geq 11.
\end{cases}
\]

The above bifurcation and the convergence results are discussed in [8, Section X]; the uniqueness of the singular solutions follows from [15]. The uniqueness and the convergence is also obtained in [13]. Note that \( \lambda = \frac{\lambda}{\delta - 1} \rightarrow N - 1 - \delta = \lambda^* \). Since \( p = (\delta + 1)/(\delta - 1) \), we see that if \( (N - 2 - 2\sqrt{N-1})/2 < \delta < N/2 \) (resp. \( 1 < \delta \leq (N - 2 - 2\sqrt{N-1})/2 \)), then \( p_S < p < p_{JL} \) (resp. \( p \geq p_{JL} \)). The assertions (i) and (ii) follow from the above bifurcation result for (3.4).

4. Proof of Theorem 1.2

(i) Let \( U \) be a regular solution of problem (1.1) and denote by \( \varphi_1 \) the first eigenfunction of the Dirichlet Laplacian on \( B \) such that \( \varphi_1 > 0 \) in \( B \). Since \( U \) is superharmonic and nonconstant we have \( U > 0 \) in \( B \). From this one easily obtains that \( \frac{1}{U} \geq 4U \) in \( B \), so \( U \) satisfies
\[
-\Delta U > 4\lambda U \quad \text{in} \quad B,
\]
and strict inequality holds on a set with positive Lebesgue measure. We multiply by \( \varphi_1 \) in the above inequality and integrate over \( B \). We find
\[
\mu_1 \int_B \varphi_1 U dx = - \int_B \Delta \varphi_1 U dx = - \int_B \varphi_1 \Delta U dx > 4\lambda \int_B \varphi_1 U dx,
\]
which yields \( \mu_1 > 4\lambda \). Let us now establish the inequality \( \lambda < \mu_1/\delta \). Since \( 0 < U < 1 \) in \( B \) we have
\[
(4.1) \quad -\Delta U > \lambda + \delta |\nabla U|^2 \quad \text{in} \quad B.
\]
Let \( V = e^{\delta U} - 1 \) which from (4.1) and \( U = 0 \) on \( \partial B \) satisfies
\[
-\Delta V \geq \lambda \delta (1 + V) \quad \text{in} \quad B
\]
and \( V = 0 \) on \( \partial B \). We next multiply the above inequality by \( \varphi_1 \), integrate over \( B \) and proceed as before to deduce \( \lambda \delta < \mu_1 \).
Remark. The above method works also for problem (1.2). The relevant inequalities that one employs are \( \frac{1}{(1-U)^p} \geq 4U \) and \( \frac{1}{(1-U)^p} > 1 \) in \( B \), which hold for all \( p \geq 1 \).

(ii) Let \( u(r) := (1-U(r))^{-(\delta-1)} - 1, \lambda = (\delta - 1)\lambda \) and \( p := (\delta + 1)/(\delta - 1) \). Then \((\lambda, u)\) is a classical solution of

\[
\begin{align*}
\Delta u + \lambda(u + 1)^p &= 0 \quad \text{in } B, \\
u &= 0 \quad \text{on } \partial B,
\end{align*}
\]

if and only if \((\lambda, U)\) is a regular solution of (1.1). Since \( \delta \geq N/2 \), we see that \( 1 < p \leq (N + 2)/(N - 2) \). By Proposition 2.1 we see that the first assertion of (ii) holds.

It follows from [4, Theorem 12] that if \( \tilde{\lambda} \leq 2N(p-1)^{p-1}/p^p \), then (4.2) has a classical solution. Since \( p = (\delta + 1)/(\delta - 1) \), we see that if

\[
0 < \lambda \leq \frac{1}{\delta - 1} 2N(p-1)^{p-1}/p^p = N\left(\frac{2}{\delta + 1}\right)^{(\delta+1)/(\delta-1)},
\]

then (1.1) has a solution. By Proposition 2.1 we see that (1.7) holds.

(iii) Observe that the family (1.8) satisfies (1.1). The conclusion follows by part (ii) above.

5. Proof of Theorem 1.4

(i) Let us first assume \( \delta = 1 \) and prove that (1.1) has infinitely many solutions for any \( \lambda \in (0, 1) \). Let \( v(r) = \log 2\lambda - 2\log(1 - U(r)) \). Then (1.1) becomes

\[
\begin{align*}
\Delta v + e^v &= 0 \quad \text{in } B, \\
v &= 0 \quad \text{on } \partial B.
\end{align*}
\]

By [17, Theorem 1.1 (ii)] problem (5.1) has a two-parameter family of singular solutions

\[
v(r) = \log a + (b - 2) \log r - 2 \log \left(1 + \frac{a}{2b^2} r^b\right) \quad \text{for } a > 0 \text{ and } 0 < b < 2.
\]

Since \( U(1) = 0 \), we obtain

\[
\lambda = \frac{2ab^4}{(a + 2b^2)^2} \quad \text{and} \quad U(r) = 1 - \frac{ar^b + 2b^2}{a + 2b^2} r^{(2-b)/2} \quad \text{for } a > 0 \text{ and } 0 < b < 2.
\]

We easily see that for each \( \lambda \in (0, 1) \) the above solutions consist of a one-parameter family of rupture solutions of (1.1). Thus, the assertion (i) holds.

Assume now \( \delta > 1 \). It is easy to see that if \( U \) is a solution of (1.1), then \( v(r) = (1 - U(r))^{1-\delta} \) satisfies

\[
\begin{align*}
v'' + \frac{N-1}{r} v' + \lambda(\delta - 1)v^p &= 0 \quad \text{for } 0 < r < 1, \\
v(1) &= 1,
\end{align*}
\]

(5.2)
where \( p = (\delta + 1)/(\delta - 1) > 1 \). Further, let \( t = -\ln r \in [0, \infty) \) and \( w(t) = v(r) \). From (5.2) we obtain that \( w \) satisfies

\[
\begin{align*}
\begin{cases}
  w_{tt} + \lambda(\delta - 1)e^{-2t}w^p = 0 & \text{for } 0 < t < \infty, \\
  w(0) = 1.
\end{cases}
\end{align*}
\]

Let \( z(t) := w(t) - 1 \). Then (5.3) becomes

\[
\begin{align*}
\begin{cases}
  z_{tt} + f(t, z) = 0 & \text{for } 0 < t < \infty, \\
  z(0) = 0,
\end{cases}
\end{align*}
\]

where \( f(t, z) := \lambda(\delta - 1)e^{-2t}|z + 1|^p \). We have

\[
f(t, z) \leq 2^{p-1}\lambda(\delta - 1)e^{-2t} + 2^{p-1}\lambda(\delta - 1)e^{-2t}|z|^p =: g(t, z).
\]

Then, \( g(t, z) \) is increasing in \( z > 0 \). In order to apply Proposition 2.2 we need to check that condition (2.2) is fulfilled for small \( m > 0 \). Indeed, we have

\[
\int_0^\infty g(t, 2mt)dt = 2^{p-2}\lambda(\delta - 1) + 2^{p-1}\lambda(\delta - 1)m^pa,
\]

where \( a := \int_0^\infty t^pe^{-2t}ds \). For each small \( \lambda > 0 \), there is an interval \( I \subset (0, \infty) \) such that \( \int_0^\infty g(t, 2mt)dt < m \) for all \( m \in I \), since

\[
\frac{2^{p-2}\lambda(\delta - 1)}{m} + 2^{p-1}\lambda(\delta - 1)am^{p-1} < 1 \quad \text{for } m \in I.
\]

By Proposition 2.2 problem (5.4) has a solution \( z(t) \) such that \( z > 0 \) on \((0, \infty)\) and \( z(t)/t \to m \) as \( t \to \infty \). Hence, \( w = z + 1 \geq 1 \) is a solution of (5.3) and thus, for each small \( \lambda > 0 \), (5.2) has infinitely many solutions.

(ii) Let \( v(r) := (\delta - 1)(\delta - 1/2)^{(1 - U(r))^{-(\delta - 1)}} \). Then

\[
\begin{align*}
\begin{cases}
  v'' + \frac{N-1}{r}v' + \lambda v^p = 0 & \text{for } 0 < r < 1, \\
  v(1) = (\delta - 1)^{(\delta - 1)/2},
\end{cases}
\end{align*}
\]

where \( p := (\delta + 1)/(\delta - 1) \). Let

\[
x(t) := \frac{v(r)}{((\delta - 1)(N - 1 - \delta)\lambda^{-1})^{(\delta - 1)/2r - \delta + 1}} \quad \text{and} \quad t := -\log r.
\]

Then \( x(t) \) satisfies

\[
\begin{align*}
\begin{cases}
  x' = y, \\
  y' = (N - 2\delta)y + (\delta - 1)(N - 1 - \delta)(x - x^p)
\end{cases}
\]

for \( t > 0 \). Suppose that \((x(t), y(t))\) satisfies (5.6). Let

\[
E(x, y) := \frac{1}{2}y^2 - (\delta - 1)(N - 1 - \delta)\left(\frac{x^2}{2} - \frac{x^{p+1}}{p+1}\right).
\]
Since $N/2 \leq \delta < N - 1$, we have
\begin{equation}
\frac{d}{dt} E(x(t), y(t)) = y \left\{ y' - (\delta - 1)(N - 1 - \delta)(x - x^p) \right\} = -(2\delta - N)y^2 \leq 0
\end{equation}
and hence $E$ is a Lyapunov function. It is well known that $\Omega := \{(x, y); x > 0, E(x, y) < 0\}$ is a tear-shaped region, and it follows from (5.7) that if $(x(t_0), y(t_0)) \in \Omega$, then $(x(t), y(t)) \in \Omega$ for all $t \geq t_0$. If $\delta = N/2$, then the orbit $\{(x(t), y(t))\}$ is periodic, and a simple calculation shows that there is $c_0 > 0$ such that $c_0 \leq x(t) < \delta$, where $x_\delta := \{(\delta/(\delta - 1))^{(\delta - 1)/2}$. Then $v(r) \to \infty$ as $r \to 0$. The corresponding solution $v(r)$ is a singular solution, and hence the corresponding solution $U(r)$ is a rupture solution. When $N/2 < \delta < N - 1$, it follows from (5.7) that there is no periodic orbit. Since $\Omega$ is bounded, by Poincaré-Bendixon theorem the orbit $\{(x(t), y(t))\}$ converges to an equilibrium point in $\Omega$, which is $(1, 0)$. Therefore, $x(t) \to 1$ as $t \to \infty$. The corresponding solution $v(r)$ is a singular solution, and the corresponding solution $U(r)$ is a rupture solution.

It is enough to show that for each $\lambda \in (0, \lambda_0)$, there are infinitely many initial data $(x(0), y(0)) \in \Omega$ such that
\begin{equation}
v(1) = \{(\delta - 1)(N - 1 - \delta)\lambda^{-(\delta - 1)/2}x(0) = (\delta - 1)^{(\delta - 1)/2}.
\end{equation}
Here, the boundary condition in (5.5) is satisfied if (5.8) holds. Let
\begin{equation}
y_{\delta, x} := \sqrt{2(\delta - 1)(N - 1 - \delta) \left( \frac{x^2}{2} - \frac{x^{p+1}}{p+1} \right)}.
\end{equation}
Now, let $\lambda \in (0, \lambda^{**})$ be fixed, where $\lambda^{**} > 0$ is given in (1.9) and let $x(0) = \left( \frac{\lambda}{N-1-\delta} \right)^{(\delta - 1)/2}$. Then, $0 < x(0) < \delta$ and (5.8) holds. Let $|y(0)| < y_{\delta, x}(0)$. Then $(x(0), y(0)) \in \Omega$. Hence, the orbit $\{(x(t), y(t))\}$ with initial data $(x(0), y(0))$ is in $\Omega$ for $t > 0$. Moreover, the orbit is uniformly away from $(0, 0)$, since $E(x(t), y(t)) \leq E(x(0), y(0))$ for $t > 0$. Hence the corresponding solution $U$ of (1.1) is a rupture solution.

The asymptotic expansion (1.10) follows from the fact that $x(t) \to 1$ ($t \to \infty$), and (1.11) follows from the fact that $c_0 \leq x(t) < x_\delta$.

(iii) Letting $v = (1 - U)^{1-\delta}$, we see that $v$ satisfies (5.2) where $p = (\delta + 1)/(\delta - 1)$ fulfills $1 < p \leq N/(N - 2)$. The rest of the proof follows from the result below.

**Lemma 5.1.** Let $N \geq 3, 1 < p \leq N/(N - 2)$ and $a > 0$. Then, there exists $\Lambda_0 > 0$ such that for all $\lambda \in (0, \Lambda_0)$ the problem
\begin{equation}
\begin{cases}
v'' + \frac{N-1}{r}v' + \lambda v^p = 0 & \text{for } 0 < r < 1, \\
v(1) = a,
\end{cases}
\end{equation}
has infinitely many singular solutions.

**Proof.** We shall analyse separately the cases $p = N/(N - 2)$ and $1 < p < N/(N - 2)$ for which we provide different arguments.
Case 1: \( p = N/(N-2) \). By the proof of Theorem 1(i) in [16], there exists \( \alpha^* > 0 \) such that for any \( \alpha \in (0, \alpha^*) \) the problem

\[
\begin{cases}
v'' + \frac{N-1}{r} v' + v^{N/(N-2)} = 0 & \text{for } 0 < r < 1, \\
v(1) = 0, v'(1) = -\alpha, 
\end{cases}
\]

has a singular solution \( v_\alpha \). (More precisely, \( \alpha^* = (N-2)\alpha_0 \), where \( \alpha_0 \) appears in the proof in [16] Theorem 1(i), page 664.) By the result in [1, Theorem A] we have

\[
\lim_{r \to 0} \frac{v_\alpha(r)}{r^2 - N (\log \frac{1}{r})^{-\frac{N-2}{2}}} = \left( \frac{N-2}{\sqrt{2}} \right)^{N-2}.
\]

Fix \( \alpha \in (0, \alpha^*) \) and let \( a > 0 \). Take \( \lambda > 0 \) small such that

\[
\max_{0 \leq \rho \leq 1} \rho^{N-2} v_\alpha(\rho) > \lambda \frac{N-2}{a}.
\]

Note that by the asymptotic behavior of \( v_\alpha \) given by (5.11), the above maximum is finite and is achieved inside of the interval \((0, 1)\). By the continuous dependence of solutions \( v_\alpha \) on \( \alpha \), one can find a small interval \( I \subset (0, \alpha^*) \) centred at \( \alpha \) such that

\[
\max_{0 \leq \rho \leq 1} \rho^{N-2} v_\beta(\rho) > \lambda \frac{N-2}{a},
\]

for any \( \beta \in I \). Thus, for any \( \beta \in I \) there exists \( \rho_\beta \in (0, 1) \) such that

\[
\rho_\beta^{N-2} v_\beta(\rho_\beta) = \lambda \frac{N-2}{a}.
\]

Then, the family \( \{V_\beta\}_{\beta \in I} \) given by

\[
V_\beta(r) = \rho_\beta^{N-2} \lambda \frac{N-2}{a} v_\beta(\rho_\beta r) \quad \text{for } 0 < r \leq 1,
\]

consists of singular solutions of (5.9). Let us note that if \( \beta_1, \beta_2 \in I, \beta_1 \neq \beta_2 \) then \( V_{\beta_1} \neq V_{\beta_2} \). Indeed, if this was not true, by the local uniqueness of solutions to regular ODE, \( V_{\beta_1}(r) = V_{\beta_2}(r) \) for all \( 0 < r \leq \min\{1/\rho_{\beta_1}, 1/\rho_{\beta_2}\} \) which contradicts the definition of \( v_{\beta_1} \) and \( v_{\beta_2} \) as solutions of (5.10) with \( \alpha = \beta_1 \) and \( \alpha = \beta_2 \) respectively. Thus, we provided an infinite family of singular solutions to (5.9) which proves Theorem 1.4(iii) in the case \( p = N/(N-2) \).

Case 2: \( 1 < p < N/(N-2) \). It is proved in [7] that if \( a = 0 \) then (5.9) has infinitely many solutions for any \( \lambda > 0 \). For \( a > 0 \) we provide a different approach based on Proposition 2.2. Let \( a > 0 \) and \( w(t) = v(r), t = r^{2-N} \). Then \( w \) satisfies

\[
\begin{cases}
wtt + \frac{\lambda}{(N-2)^2} t^{-2(N-1)/(N-2)} w^p = 0 & \text{for } t > 1, \\
w(1) = a.
\end{cases}
\]
Consider the problem
\begin{align}
  z_{tt} + \frac{\lambda}{(N-2)^2} (t + 1)^{-2(N-1)/(N-2)} (z + a)^p &= 0 \quad \text{for } t > 0, \\
  z(0) &= 0.
\end{align}

It is not hard to see that for any $1 < p < N/(N - 2)$, the function $g(t, z) = \frac{\lambda}{(N-2)^2} (t + 1)^{-2(N-1)/(N-2)} (z + a)^p$ satisfies condition (2.2) provided $\lambda > 0$ is small enough. By Proposition 2.2 it follows that (5.13) has infinitely many positive solutions with $z(t) \to \infty$ as $t \to \infty$. Letting now $w(t) = z(t-1) + a$, it follows that (5.12) has infinitely many solutions with $w(t) \to \infty$ as $t \to \infty$ provided $\lambda > 0$ is small. This implies in turn that for any small $\lambda > 0$, problem (5.9) has infinitely many singular solutions and the proof of Lemma 5.1 is now complete.

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