On the number of limit cycles for Bogdanov-Takens system under perturbations of piecewise smooth polynomials

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Abstract In this paper, we study the bifurcate of limit cycles for Bogdanov-Takens system ($\dot{x} = y$, $\dot{y} = -x + x^2$) under perturbations of piecewise smooth polynomials of degree 2 and $n$ respectively. We bound the number of zeros of first order Melnikov function which controls the number of limit cycles bifurcating from the center. It is proved that the upper bounds of the number of limit cycles with switching curve $x = y^2m$ ($m$ is a positive integral) are $(39m + 36)n + 77m + 21(m \geq 2)$ and $50n + 52(m = 1)$ (taking into account the multiplicity). The upper bounds number of limit cycles with switching lines $x = 0$ and $y = 0$ are 11 (taking into account the multiplicity) and it can be reached.

Key Words limit cycle; Abelian integral; bifurcation.

§1. Introduction and the main results

The determination of limit cycles is one important problem in the qualitative theory of planar differential systems. Stimulated by non-smooth phenomena in the real world such as control system [1], economics [12], nonlinear oscillations [20], and biology [5], [14], the investigation of limit cycles for piecewise smooth differential systems has attracted many attentions.

Many scholars have studied the number of limit cycles for piecewise smooth differential systems. In [11] and [17], the authors studied the expressions of the first order Melnikov function for the piecewise Hamiltonian systems under the piecewise perturbations.

For the piecewise smooth differential systems with the switching lines, there have been a lot of results for example [4,16,18,23,26]. Nowadays many scholars begin to pay attention to the study of piecewise smooth differential systems with the nonlinear switching curves (see [2,6,19,21,22,24,27-29]).

It is shown by Horozov and Iliev in [9] that any cubic Hamiltonian can be transformed into the following normal form

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3,$$

where $a$, $b$ are parameters lying in the region

$$G = \left\{(a, b) : -\frac{1}{2} \leq a \leq 1, 0 \leq b \leq (1 - a)(1 + 2a)^{1/2}\right\}.$$

Moreover, their respective vector fields $X_H$ are degenerate if $(a, b) \in \partial G$. If $(a, b) \in \partial G$, then in suitable coordinates (see [13]) all respective basic dynamics of $X_H$ can be classified into eight types that contain Bogdanov-Takens system with the fist integral

$$H(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{3}x^3 = h, \ h \in (0, \frac{1}{6}). \quad (1.1)$$

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Motivated by [23-25], in the present paper, we study the upper bounds of the number of limit cycles bifurcating from the period annuluses of Bogdanov-Takens system when it are perturbed inside any discontinuous polynomial differential systems. Concretely, we consider the following systems ($0 < |\varepsilon| \ll 1$)

$$
\begin{cases}
\dot{x} = y + \varepsilon p^+(x, y), & x > y^{2m},
\dot{y} = -x + x^2 + \varepsilon q^+(x, y), & x > y^{2m},
\end{cases}
$$

(1.2)

where $m$ is a positive integral and

$$
p^\pm(x, y) = \sum_{i+j=0}^{n} a_{i,j}^\pm x^i y^j, \quad q^\pm(x, y) = \sum_{i+j=0}^{n} b_{i,j}^\pm x^i y^j,
$$

and

$$
\begin{cases}
\dot{x} = y + \varepsilon p^+(x, y), & x > 0, y > 0,
\dot{y} = -x + x^2 + \varepsilon q^+(x, y), & x > 0, y < 0,
\end{cases}
$$

(1.3)

where

$$
p^\pm(x, y) = \sum_{i+j=0}^{2} p_{i,j}^\pm x^i y^j, \quad q^\pm(x, y) = \sum_{i+j=0}^{2} q_{i,j}^\pm x^i y^j,
$$

$$
\tilde{p}^\pm(x, y) = \sum_{i+j=0}^{2} \tilde{p}_{i,j}^\pm x^i y^j, \quad \tilde{q}^\pm(x, y) = \sum_{i+j=0}^{2} \tilde{q}_{i,j}^\pm x^i y^j.
$$

Let $H(n)$ be the number of limit cycles for system (1.2) and (1.3) bifurcating from period annulus (taking into account the multiplicity). The main results are as follows.

**Theorem 1.1.** Consider system (1.2), by using the first order of Melnikov function in $\varepsilon$, the upper bounds of the number of limit cycles (taking into account the multiplicity) bifurcating from period annuli are

(i) If the switching curve is $y = x^2$, then $H(n) \leq 50n + 52$.

(ii) If the switching curve is $y = x^{2m}(m \geq 2)$, then $H(n) \leq (39m + 36)n + 77m + 21$.

**Theorem 1.2.** Consider system (1.3), by using the first order of Melnikov function in $\varepsilon$, the upper bounds of the number of limit cycles (taking into account the multiplicity) bifurcating from period annuli are 11, and the upper bounds can be reached for some $p_{i,j}^\pm(\tilde{p}_{i,j}^\pm)$ and $q_{i,j}^\pm(\tilde{q}_{i,j}^\pm)$ ($i, j = 0, 1, 2$).
Remark 1.3. B. Li et al. [15] considered respectively systems (1.2)_{ε=0} under continuous perturbations of arbitrary polynomials with degree \( n \). It is proved that for perturbed system (1.2)_{ε=0}, the exactly upper bound of the first order Melnikov function (Abelian integral) is \( n-1 \), and the exactly upper bound of the second order Melnikov function is \( 2n-2 \) (\( n \) is even) or \( 2n-3 \) (\( n \) is odd) when the first order Melnikov function vanishes. S. Sui et al. [23] and W. Cui et al. [3] considered respectively systems (1.1) under discontinuous perturbations of arbitrary polynomials with degree \( n \) with switching line \( y = 0 \) and \( x = 0 \). It is proved that the upper bound of number of the isolated zeros of Abelian integrals for perturbed Bogdanov-Takens system are \( 12n + \left\lceil \frac{n}{2} \right\rceil + 5 \) and \( 16n + \left\lceil \frac{n}{2} \right\rceil - 10 \) respectively.

§2. Preliminaries

Next, we shall introduce the first order Melnikov function of discontinuous differential systems. For \( 0 < |ε| \ll 1 \), we consider the following Near-Hamilton system:

\[
(\dot{x}, \dot{y}) = \begin{cases}
(H^+(y, x) + εp^+(y, x), -H^+_x(y, x) + εq^+(x, y)), & x \geq ψ(y), \\
(H^-_y(y, x) + εp^-(y, x), -H^-_x(y, x) + εq^-(x, y)), & x < ψ(y),
\end{cases}
\]

where \( ψ(x) \) is analytic with \( ψ(0) = 0 \), and \( p^±(x, y) \) and \( q^±(x, y) \) are polynomials with degree \( n \). System (2.1) has two sub-systems:

\[
\begin{align*}
\dot{x} &= H^+_y(y, x) + εp^+(y, x), & x \geq ψ(y), \\
\dot{y} &= -H^+_x(y, x) + εq^+(x, y),
\end{align*}
\]

and

\[
\begin{align*}
\dot{x} &= H^-_y(y, x) + εp^-(y, x), & x < ψ(y), \\
\dot{y} &= -H^-_x(y, x) + εq^-(x, y).
\end{align*}
\]

Suppose that (2.1)_{ε=0} has a family of periodic orbits around the origin and satisfies the following assumptions.

Assumption (I). There exists an open interval \( Σ \) such that for each \( h \in Σ \), there are two points \( A(h) \) and \( B(h) \) on the curve \( x = ψ(y) \) with \( A(h) = (ψ(a(h)), a(h)) \) and \( B(h) = (ψ(b(h)), b(h)) \) satisfying

\[
H^+(A(h)) = H^+(B(h)) = h, \quad H^-(A(h)) = H^-(B(h)), \quad a(h) < 0 < b(h).
\]

Assumption (II). The subsystem (2.2)_{ε=0} has an orbital arc \( L^+_h \) starting from \( A(h) \) and ending at \( B(h) \) defined by \( H^+(x, y) = h \ (x \geq ψ(y)) \). The subsystem (2.3)_{ε=0} has an orbital arc \( L^-_h \) starting from \( B(h) \) and ending at \( A(h) \) defined by \( H^-(x, y) = h \ (x < ψ(y)) \).

Assumption (III). For each \( h \in Σ \),

\[
H^+_x(x, y)ψ'(y) + H^+_y(x, y) \neq 0 \quad \text{at points } A(h) \text{ and } B(h).
\]

This means that the orbital arcs \( L^+_h \) are not tangent to curve \( x = ψ(y) \) at points \( A(h) \) and \( B(h) \).

Under the assumptions Assumption (I),(II) and (III), system (2.1)_{ε=0} has a family of non-smooth periodic orbits \( L_h = L^+_h \cup L^-_h \ (h \in Σ) \). For definiteness, we assume that the
orbits $L_h$ orientate clockwise. For $x = \psi(x)$, the authors [24] established the bifurcation function $F(h, \epsilon)$ for (2.1). Let $F(h, 0) = M(h)$. In [24], the authors obtained the following results.

**Lemma 2.1.**[24] Under the assumptions **Assumption (I)**, **(II)** and **(III)**, we have

(i) If $M(h)$ has $k$ zeros in $h$ on the interval $\Sigma$ with each having an odd multiplicity, then (2.1) has at least $k$ limit cycles bifurcating from the period annulus for $0 < |\epsilon| \ll 1$.

(ii) If $M(h)$ has at most $k$ zeros in $h$ on the interval $\Sigma$, taking into account the multiplicity, then there exist at most $k$ limit cycles of (2.1) bifurcating from the period annulus.

(iii) The first order Melnikov function $M(h)$ of system (2.1) can be expressed as

$$M(h) = \int_{L_h^+} q^+ dx - p^+ dy + \frac{H_x^+(A)\psi''(a(h)) + H_y^+(A)}{H_x^-(A)\psi''(a(h)) + H_y^-(A)} \int_{L_h^-} q^- dx - p^- dy.$$  \hspace{1cm} (2.4)

**Definition 2.2.**[7] We say that $\mathcal{V}$ is a **Chebyshev space**, provided that each non-zero function in $\mathcal{V}$ has at most $\dim(\mathcal{V}) - 1$ zeros, counted with multiplicity.

Let $\mathcal{S}$ be the solution space of a second order linear analytic differential equation

$$x'' + a_1(t)x' + a_2(t)x = 0$$  \hspace{1cm} (2.5)

on an open interval $I$.

**Lemma 2.3.**[7] The solution space $\mathcal{S}$ of (2.5) is a Chebyshev space of the interval $I$ if and only if there exists a nowhere vanishing solution $x_0(t) \in \mathcal{S}(x_0(t) \neq 0, \forall t \in I)$.

**Lemma 2.4.**[7] Suppose the solution space of the homogeneous equation (2.5) is a Chebyshev space and let $R(t)$ be an analytic function on $I$ having $l$ zeros (counted with multiplicity). Then every solution $x(t)$ of the non-homogeneous equation

$$x'' + a_1(t)x' + a_2(t)x = R(t)$$

has at most $l + 2$ zeros on $I$.

In this section we first introduce some results for determining the numbers of isolated zeros of a function.

**Definition 2.5.**[8] Let $\mathcal{F} = \{f_0(x), f_1(x), ..., f_n(x)\}$ be an ordered set of $C^\infty$ functions on an open interval $J \subset \mathbb{R}$. The ordered set $\{f_0(x), f_1(x), ..., f_n(x)\}$ is said to be an ECT-system on $J$ if, for all $k = 1, 2, ..., n, n + 1$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k - 1$ isolated zeros on $J$ counted with multiplicities.

**Lemma 2.6.**[8] The ordered set $\{f_0(x), f_1(x), ..., f_n(x)\}$ is an ECT-system on $J$ if and only if, for each $k = 1, 2, ..., n + 1$,

$$W(f_0, f_1, ..., f_{k-1}) \neq 0$$

for all $x \in J$, where $W(f_0, f_1, ..., f_{k-1})$ is the Wronskian of functions $f_0(x), f_1(x), ..., f_{k-1}(x)$.

§3. Proof of Theorem 1.1
This section we consider system (1.2). System (1.2) has two singular points. There are a center \(O(0,0)\) corresponding to \(h = 0\), a saddle \(S(1,0)\) corresponding to \(h = \frac{1}{6}\). For \(h \in (0, \frac{1}{6})\), by Lemma 2.1, we have

\[
M(h) = \int_{L^+_h} q^+(x, y)dx - p^+(x, y)dy + \int_{L^-_h} q^-(x, y) - p^-(x, y)dy, \tag{3.1}
\]

where

\[
L^+_h(L^-_h) = \{(x, y)\mid H(x, y) = h, x > y^2(x < y^{2m})\}.
\]

Suppose that \(H(x, y) = h\) and \(y = 0\) intersects at \((\bar{x}(h), 0)\) and \((\bar{x}(h), 0)\) \((\bar{x} < 0 < \bar{x})\).

\(x = y^{2m}\) and \(H(x, y) = h\) intersects at points \(A(u(h)^{2m}, u(h))\) and \(B(u(h)^{2m}, -u(h))\). In the following we denote \(u(h), I_{i,j}(h)\) and \(I_{i,j}(h)\) as \(u, J_{i,j}\) and \(I_{i,j}\). Hence we have

\[
\frac{1}{2}u^2 + \frac{1}{2}u^{4m} - \frac{1}{3}u^{6m} = h. \tag{3.2}
\]

For \(i, j \geq 0\), let

\[
J_{i,j}(h) = \int_{L^+_h} x^iy^jdx, \quad I_{i,j}(h) = \int_{L^-_h} x^iy^jdx.
\]

Without loss of generality, we only consider the case of \(m = 1\). For \(m \geq 2\) it can be proved similarly. For polynomial \(f(h)\), we denote it as \(f(u)\) if we substitute (3.2) into \(f(h)\).

**Lemma 3.1.** Consider system (1.2) for \(h \in (0, \frac{1}{6})\). Then \(M(h)\) can be expressed as

\[
M(h) = \alpha(h)J_{0,1} + \beta(h)J_{1,1} + \gamma(h)I_{0,1} + \eta(h)I_{1,1} + \Phi(u), \tag{3.3}
\]

where \(\deg \alpha(h), \gamma(h) \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \deg \beta(h), \eta(h) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\), and \(\Phi(u) = \sum_{i=0}^{3\left\lfloor \frac{n-1}{2} \right\rfloor + 1} c_iu^{2i+1}(c_i\text{ are constants})\).

**Proof.** First we assert that

\[
M(h) = \sum_{i+j+1=0}^{n} \rho^+_i\rho^-_{i,j+1}J_{i,j+1} + \sum_{i+j+1=0}^{n} \rho^-_i\rho^+_{i,j+1}I_{i,j+1} + \sum_{i+j=0}^{n} \rho^+_i\rho^-_{i,j}I_{i,j} + \sum_{i+j=0}^{n} (a^+_i - a^-_i) \left( \frac{2}{2j+1} - \frac{1}{2j+1} \right) u^{2i+2j+1}. \tag{3.4}
\]

Using Green’s formula we have

\[
\int_{L^+_h} x^iy^jdy = -\frac{i}{j+1} \left( \int_{L^+_h} x^{i-1}y^{j+1}dx + \int_{\partial \Delta} x^{i-1}y^{j+1}dx \right) - \int_{\partial \Delta} x^iy^jdx
\]

\[
= -\frac{i}{j+1} J_{i-1,j+1} - \frac{1 + (-1)^j}{j+1} u^{2i+j+1}.
\]

Similarly, we can obtain

\[
\int_{L^-_h} x^iy^jdy = -\frac{i}{j+1} J_{i-1,j+1} + \frac{1 + (-1)^j}{j+1} u^{2i+j+1}.
\]

Thus,

\[
M(h) = \sum_{i+j=0}^{n} \rho^+_i\rho^-_{i,j}J_{i,j} + \sum_{i+j=0}^{n} \rho^-_i\rho^+_{i,j}I_{i,j} + \sum_{i+j=0}^{n} (a^+_i - a^-_i) \left( \frac{1 + (-1)^j}{j+1} \right) u^{2i+j+1},
\]

\[\text{(3.3)}\]
where \( u = u(h), \rho_{i,j}^\pm = b_{i,j}^\pm + i^{i+1}a_{i+1,j-1}^\pm (j \geq 1) \) and \( \rho_{i,0}^\pm = b_{i,0}^\pm \).

Noticing the symmetry of \( H(x,y) \), we have \( J_{i,2j} - I_{i,2j} = 0 \). Therefore we can get (3.4).

Next, differentiating (1.1) with respect to \( x \), we obtain

\[
x + y \frac{\partial y}{\partial x} - x^2 = 0. \tag{3.5}
\]

Multiplying (1.1) and (3.5) by \( x^iy^jdx \) and integrating over \( L_h^+ \), we have

\[
J_{i+1,j} - \frac{i}{j+1}J_{i-1,j+2} - J_{i+2,j} - \frac{1 + (-1)^{j+1}}{j+2}u^{2i+j+2} = 0, \tag{3.6}
\]

\[
\frac{1}{2}J_{i+2,j} + \frac{1}{2}J_{i,j+2} - \frac{1}{3}J_{i+3,j} = hJ_{i,j}. \tag{3.7}
\]

Elementary manipulations reduce Eqs. (3.6) and (3.7) to

\[
J_{i,j} = \frac{6j}{2i+3j+2}hJ_{i,j-2} - \frac{j}{2i+3j+2}J_{i+2,j-2} - \frac{2(1 + (-1)^{j-1})}{2i+3j+2}u^{2i+j+2}, \tag{3.8}
\]

\[
J_{i,j} = -\frac{6(i-2)}{2i+3j+2}hJ_{i-3,j} + \frac{3(i+j)}{2i+3j+2}J_{i-1,j} - \frac{3(1 + (-1)^{j+1})}{2i+3j+2}u^{2i+j-2}. \tag{3.9}
\]

Similarly, we have

\[
I_{i,j} = \frac{6j}{2i+3j+2}hI_{i,j-2} - \frac{j}{2i+3j+2}I_{i+2,j-2} + \frac{2(1 + (-1)^{j-1})}{2i+3j+2}u^{2i+j+2}, \tag{3.10}
\]

\[
I_{i,j} = -\frac{6(i-2)}{2i+3j+2}hI_{i-3,j} + \frac{3(i+j)}{2i+3j+2}I_{i-1,j} + \frac{3(1 + (-1)^{j+1})}{2i+3j+2}u^{2i+j-2}. \tag{3.11}
\]

By (3.6), let \( i = 0, j = 1 \), we can obtain

\[
J_{2,1} = J_{1,1} - \frac{2}{3}u^3. \tag{3.12}
\]

Similarly, we can obtain

\[
I_{2,1} = I_{1,1} + \frac{2}{3}u^3. \tag{3.13}
\]

Then using (3.4) and (3.8)–(3.13) we can obtain (3.3). This ends the proof.

**Lemma 3.2.** Let \( V_1(h) = (J_{0,1}, J_{1,1})^T, V_2(h) = (I_{0,1}, I_{1,1})^T \), and \( \sigma(u) = 1/(1+2u^2-2u^4) \).

Then the vector functions \( V_1(h) \) and \( V_2(h) \) satisfy respectively the following Picard-fuchs equations:

\[
V_1(h) = (B_1h + C_1)V_1'(h) + W(u), \tag{3.14}
\]

\[
V_2(h) = (B_1h + C_1)V_2'(h) - W(u), \tag{3.15}
\]

where

\[
B_1h + C_1 = \left( \begin{array}{cc}
\frac{6}{5}h & -\frac{1}{5} \\
\frac{2}{5}h - \frac{4}{5}
\end{array} \right),
\]

\[
W(u) = \left( \begin{array}{c}
\frac{2}{5}u + \frac{8}{15}u^3 \\
\frac{12}{35}u + \frac{18}{35}u^3 + \frac{8}{5}u^5
\end{array} \right) \cdot \sigma(u).
\]
Similarly, we have
\begin{equation}
\frac{\partial x}{\partial h} = \frac{1}{x(1-x)}, \quad \frac{\partial u}{\partial h} = \frac{1}{u + 2u^3 - 2u^5}.
\end{equation}

Through the analysis of the singular points of the system, we have \(|\frac{\partial x}{\partial h}(h)| < \infty\). Notice \(\frac{\partial u}{\partial h} = 1/y\), we can obtain that
\begin{equation}
J'_{i,2j+1} = (2j + 1)J_{i,2j-1} - 4\sigma(u)u^{2i+2j+1}.
\end{equation}

Similarly, we have
\begin{equation}
I'_{i,2j+1} = (2j + 1)I_{i,2j-1} + 4\sigma(u)u^{2i+2j+1}.
\end{equation}

Thus we have
\begin{equation}
J_{i,2j+1} = \frac{1}{2j+3} \left( J'_{i,2j+3} + 4\sigma(u)u^{2i+2j+3} \right),
\end{equation}
\begin{equation}
I_{i,2j+1} = \frac{1}{2j+3} \left( I'_{i,2j+3} - 4\sigma(u)u^{2i+2j+3} \right).
\end{equation}

Combining (3.8)–(3.13), we can obtain (3.14) and (3.15).

**Lemma 3.3.** Let \(J_0 = J_{0,1} + I_{0,1}\), \(J_1 = J_{1,1} + I_{1,1}\) and \(D(h) = h(6h - 1)\). Then \(J_0\) and \(J_1\) satisfy
\begin{equation}
J'_0 = \frac{1}{D(h)} [k_{0,0}(h)J_0 + k_{0,1}(h)J_1],
\end{equation}
\begin{equation}
J'_1 = \frac{1}{D(h)} [k_{1,0}(h)J_0 + k_{1,1}(h)J_1],
\end{equation}
where \(k_{0,0}(h) = 5h - 1\), \(k_{0,1}(h) = 6/7\), \(k_{1,0}(h) = -h\) and \(k_{1,1}(h) = 7h\).

**Proof.** By (3.14), we have
\begin{equation}
\det(B_1 h + C_1)V_1'(h) = (B_1 h + C_1)^* (V_1(h) - W(u)).
\end{equation}
Combining \(\frac{1}{7}u^2 + \frac{1}{7}u^4 - \frac{1}{3}u^6 = h\), we can obtain
\begin{equation}
J'_{0,1} = \frac{1}{D(h)} [k_{0,0}(h)J_{0,1} + k_{0,1}(h)J_{1,1} + w_1(u)],
\end{equation}
\begin{equation}
J'_{1,1} = \frac{1}{D(h)} [k_{1,0}(h)J_{0,1} + k_{1,1}(h)J_{1,1} + w_2(u)],
\end{equation}
where
\begin{equation}
w_1(u) = -\sigma(u) \left( \frac{38}{35}u^5 + \frac{4}{7}u^7 - \frac{16}{35}u^9 \right),
\end{equation}
\begin{equation}
w_2(u) = -\sigma(u) \left( \frac{6}{35}u^3 + \frac{12}{35}u^5 + \frac{26}{35}u^7 + \frac{4}{7}u^9 - \frac{16}{35}u^{11} \right).
\end{equation}
Similarly, we have
\[ I_0' = \frac{1}{D(h)} [k_{0,0}(h) I_{0,1} + k_{0,1}(h) I_{1,1} - w_1(u)], \]
\[ I_1' = \frac{1}{D(h)} [k_{1,0}(h) I_{0,1} + k_{1,1}(h) I_{1,1} - w_2(u)]. \]

Therefore, we can get (3.19) and (3.20) by \( J_0 = J_{0,1} + I_{0,1} \) and \( J_1 = J_{1,1} + I_{1,1} \). This ends the proof.

**Lemma 3.4.** Let \( \phi_1(h) = \alpha(h) J_0 + \beta(h) J_1 \). Then for \( h \in (0, \frac{1}{6}) \), there exist polynomials \( P_i(h)(i = 0, 1, 2) \) such that \( L(h)\phi_1(h) = 0 \), where
\[ L(h) = P_2(h) D(h) \frac{d^2}{dh^2} + P_1(h) D(h) \frac{d}{dh} + P_0(h), \] (3.22)
where \( \deg P_2(h) \leq n_1 \), \( \deg P_1(h) \leq n_1 - 1 \), \( \deg P_0(h) \leq n_1 - 2 \), and
\[ n_1 = [(n - 1)/2] + [n/2] + 2. \]

In addition, we have
\[ M_1(h) := L(h) M(h) = \tilde{\gamma}(h) I_{0,1} + \tilde{\eta}(h) I_{1,1} + \tilde{\Phi}(u), \] (3.23)
where \( \deg \tilde{\gamma}(h) \leq 2 \left[ \frac{n-1}{2} \right] + \frac{n}{2} + 2 \), \( \deg \tilde{\eta}(h) \leq [ \frac{n-1}{2} ] + 2 \left[ \frac{n}{2} \right] + 1 \) and \( \tilde{\Phi}(u) \) is a rational fraction of \( u \).

**Proof.** Differential both sides (3.21) and combining (3.14), we have
\[ J''_{0,1} = \frac{1}{D(h)} \left( -\frac{5}{6} J_{0,1}' + w_1'(u) \right), \]
\[ J''_{1,1} = \frac{1}{D(h)} \left( -J_{0,1}' + \frac{7}{6} J_{1,1}' + w_2'(u) \right), \]
where
\[ w_1'(u) = -\frac{2}{35} u (7 - 39u^2 - 72u^4 - 46u^6 - 24u^8 + 32u^{10}) \sigma(u)^3, \]
\[ w_2'(u) = -\frac{2}{35} u (3 + 3u^2 + 77u^4 + 176u^6 + 22u^8 - 120u^{10} + 32u^{12}) \sigma(u)^3. \]

Similarly, we have
\[ I''_{0,1} = \frac{1}{D(h)} \left( \frac{5}{6} I_{0,1}' - w_1'(u) \right), \]
\[ I''_{1,1} = \frac{1}{D(h)} \left( -I_{0,1}' + \frac{7}{6} I_{1,1}' - w_2'(u) \right). \]

Hence,
\[ J_0'' = -\frac{5}{6D(h)} J_0, \quad J_1'' = \frac{1}{D(h)} \left( -J_0 + \frac{7}{6} J_1 \right). \] (3.24)

Suppose that
\[ P_2(h) = \sum_{k=0}^{s} p_{2,k} h^k, \quad P_1(h) = \sum_{k=0}^{s-1} p_{1,k} h^k, \quad P_0(h) = \sum_{k=0}^{s-2} p_{0,k} h^k. \]
Using (3.19), (3.20) and (3.24), we can obtain

\[ L(h)\phi_1(h) = P_2(h)D(h)\phi_1''(h) + P_1(h)D(h)\phi_1'(h) + P_0(h)\phi_1(h) \]

\[ = X(h)J_0 + Y(h)J_1, \]

where \( X(h) \) and \( Y(h) \) are polynomials with degree no more than \( 2[(n-1)/2] + \lfloor n/2 \rfloor + 2 \) and \( [(n-1)/2] + 2\lfloor n/2 \rfloor + 1 \) respectively. Let

\[ X(h) = \sum_{i=0}^{\text{deg}X} x_i h^i, \quad Y(h) = \sum_{j=0}^{\text{deg}Y} y_j h^j, \]

\( x_i \) and \( y_j \) are expressed by \( p_{2,k}, p_{1,k} \) linearly. So \( L(h)\phi_1(h) = 0 \) is satisfied if we let

\[ x_i = 0, \quad y_j = 0, \quad (0 \leq i \leq \text{deg}X, \ 0 \leq j \leq \text{deg}Y). \] (3.25)

System (3.25) is a homogeneous linear equation with \( 3\lfloor (n-1)/2 \rfloor + 3\lceil n/2 \rceil + 5 \) equations about \( 3\lfloor (n-1)/2 \rfloor + 3\lceil n/2 \rceil + 6 \) variables of \( p_{2,k}, p_{1,k} \) and \( p_{0,k} \). It follows that from the theory of linear algebra that exist \( p_{2,k}, p_{1,k} \) and \( p_{0,k} \) such that the result holds. Let

\[ \Phi_1(u) = \Phi'(u)D(u) \frac{\partial u}{\partial h} + w_1(u)(\alpha(u) - \gamma(u)) + w_2(u)(\beta(u) - \eta(u)), \] (3.26)

\[ \Phi_2(u) = \left( \Phi'(u) \frac{\partial u}{\partial h} \right)' \frac{\partial u}{\partial h} D(u) + 2w_1(\alpha' - \gamma') + 2w_2(\beta' - \eta') + w_1'(\alpha - \gamma) + w_2'(\beta - \eta). \] (3.27)

By the same progress, we can obtain

\[ L(h)M(h) = P_2(h)D(h)M'(h) + P_1(h)D(h)M'(h) + P_0(h)M(h) \]

\[ = X(h)J_{0,1} + Y(h)J_{1,1} + \tilde{\gamma}(h)I_{0,1} + \tilde{\eta}(h)I_{1,1} + \Phi(u), \]

where

\[ \text{deg}\tilde{\gamma}(h) \leq 2\left[ \frac{n-1}{2} \right] + \left[ \frac{n}{2} \right] + 2, \quad \text{deg}\tilde{\eta}(h) \leq \left[ \frac{n-1}{2} \right] + 2\left[ \frac{n}{2} \right] + 1, \]

and

\[ \Phi(u) = P_2(u)\Phi_2(u) + P_1(u)\Phi_1(u) + P_0(u)\Phi(u). \] (3.28)

This ends the proof.

Let

\[ \tilde{\Phi}_1(u) = \tilde{\Phi}'(u)D(u) \frac{\partial u}{\partial h} - w_1(u)\tilde{\gamma}(u) - w_2(u)\tilde{\eta}(u), \] (3.29)

\[ \tilde{\Phi}_2(u) = \left( \tilde{\Phi}'(u) \frac{\partial u}{\partial h} \right)' \frac{\partial u}{\partial h} D(u) - 2w_1\tilde{\gamma}' - 2w_2\tilde{\eta}' - w_1'\tilde{\gamma} - w_2'\tilde{\eta}. \] (3.30)

Similar to the proof of Lemma 3.4, we can obtain the following Lemma.

**Lemma 3.5.** Let \( \phi_2(h) = \tilde{\gamma}(h)J_0 + \tilde{\eta}(h)J_1 \). Then for \( h \in (0, \frac{1}{6}) \), there exist polynomials \( \tilde{P}_i(h)(i = 0, 1, 2) \) such that \( \tilde{L}(h)\phi_2(h) = 0 \), where

\[ \tilde{L}(h) = \tilde{P}_2(h)D(h) \frac{d^2}{dh^2} + \tilde{P}_1(h)D(h) \frac{d}{dh} + \tilde{P}_0(h), \] (3.31)
and
\[ M_2(h) := \tilde{L}(h)M_1(h) = \tilde{\Phi}(u), \] (3.32)
where \( \deg \tilde{P}_2(h) \leq n_2, \deg \tilde{P}_1(h) \leq n_2 - 1, \deg \tilde{P}_0(h) \leq n_2 - 2, \)
\[ \Phi(u) = \tilde{P}_2(u)\Phi_2(u) + \tilde{P}_1(u)\Phi_1(u) + \tilde{P}_0(u)\Phi(u), \] (3.33)
and
\[ n_2 = 3([\frac{n-1}{2}] + [\frac{n}{2}]) + 6. \]

**Proof for the Theorem 1.1.** Denote \( \mathcal{L}(u_1, u_2, u_3, ..., u_{n-1}, u_n) \) is the linear combination of function \( u_1, u_2, u_3, ..., u_{n-1}, u_n \). First, we will analysis the structure of \( \Phi(u) \) and \( \Phi(u) \).

Through direct computation we can get for polynomial \( f(h) \) with degree no more than \( n_0 \), it can become the linear combination of 1, \( u^2, u^4, ..., u^{2n-2}, u^{2n} (n \leq 3n_0) \) by \( h = \frac{1}{2}u^2 + \frac{1}{2}u^4 - \frac{1}{3}u^6 \). Thus, from the expression of \( \Phi(u) \), \( w_i(u) \), \( w_i^\ast(u) \) \( (i = 1, 2) \), we can obtain
\[ \Phi_1(u) = \sigma(u) \cdot \mathcal{L}(u, u^3, u^5, ..., u^6\left[\frac{n-1}{2}\right]+11, u^6\left[\frac{n-1}{2}\right]+13), \]
\[ \Phi_2(u) = \sigma(u)^3 \cdot \mathcal{L}(1, u^2, u^4, ..., u^{2k_1-2}, u^{2k_1})/u. \]

Therefore, we have
\[ \tilde{\Phi}(u) = P_2(h)\Phi_2(u) + P_1(h)\Phi_1(u) + P_0(h)\Phi(u) \]
\[ = \sigma(u)^3 \cdot \mathcal{L}(1, u^2, u^4, ..., u^{2k_1-2}, u^{2k_1})/u, \]
with \( k_1 = 6\left[\frac{n-1}{2}\right] + 3\left[\frac{n}{2}\right] + 14 \). Similarly, we can get
\[ \tilde{\Phi}(u) = \tilde{P}_2(u(h))\Phi_2(u) + \tilde{P}_1(u(h))\Phi_1(u) + \tilde{P}_0(u(h))\Phi(u) \]
\[ = \sigma(u)^7 \cdot \mathcal{L}(1, u^2, u^4, ..., u^{2k_2-2}, u^{2k_2})/u^3. \]

with \( k_2 = 15\left[\frac{n-1}{2}\right] + 12\left[\frac{n}{2}\right] + 41 \). Suppose \( u_\ast \) satisfies \( \frac{1}{2}u^2 + \frac{1}{2}u^4 - \frac{1}{3}u^6 = \frac{1}{6} \), then the number of zeros of function \( f(h) \) are same with the number of zeros of function \( f(u) \) in \( h \in (0, \frac{1}{6}) \) and \( u \in (0, u_\ast) \). Notice \( 1 + 2u^2 - 2u^4 \neq 0 \) and function 1, \( u, u^2, ..., u^i \) are ECT-system in \( u \in (0, u_\ast) \), we have
\[ \# \left\{ \tilde{\Phi}(u) = 0, u \in (0, u_\ast) \right\} \leq k_1, \quad \# \left\{ \tilde{\Phi}(u) = 0, u \in (0, u_\ast) \right\} \leq k_2. \]

Next, we analysis the number of zeros of function \( \phi_1(h) \) and \( \phi_2(h) \). Since
\[ J_0(h) = J_{0,1}(h) + I_{0,1}(h) = \int_{L_h^+ \cup L_h^-} ydx = \int \int_{L_h^+ \cup L_h^-} dxdy \neq 0, \]
we let
\[ S(h) = J_1(h)/J_0(h), \quad F(h) = \frac{\phi_1(h)}{J_0(h)} = \alpha(h) + \beta(h)S(h). \]

Then \( F(h) \) satisfy the following Riccati equation
\[ D(h)\beta(h)F'(h) = -k_{0,1}\beta(h)F(h)^2 + N_1(h)F(h) + N_2(h), \]
with \( N_1(h) \) and \( N_2(h) \) are polynomials with degree no more than \( \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor - 1 \) and \( 2 \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor - 1 \) respectively. Thus we have

\[
\# \{ \phi_1(h) = 0, h \in (0, 1/6) \} \leq \# \{ D(h) \beta (h) = 0, h \in (0, 1/6) \} + \# \{ N_2(h) = 0, h \in (0, 1/6) \} + 1 \\
\leq 2 \left( \left\lceil (n - 1)/2 \right\rceil + \left\lfloor n/2 \right\rfloor \right) + 1.
\]

Similarly, we get

\[
\# \{ \phi_2(h) = 0, h \in (0, 1/6) \} \leq 6 \left( \left\lceil (n - 1)/2 \right\rceil + \left\lfloor n/2 \right\rfloor \right) + 9.
\]

Therefore By Definition 2.2 and Lemma 2.3–2.4, we can obtain

\[
\# \{ M_1(h) = 0, h \in (0, 1/6) \} \leq \# \left\{ \tilde{\Phi}(u) = 0, u \in (0, u_*) \right\} \\
+ 3 \left( \# \left\{ \tilde{P}_2(h) D(h) = 0, h \in (0, 1/6) \right\} + \# \{ \phi_2(h) = 0, h \in (0, 1/6) \} \right) + 2 \\
\leq 42 \left( \left\lceil (n - 1)/2 \right\rceil + 39 \left\lfloor n/2 \right\rfloor + 94 \right),
\]

\[
\# \{ M(h) = 0, h \in (0, 1/6) \} \leq \# \{ M_1(h) = 0, h \in (0, 1/6) \} \\
+ 3 \left( \# \left\{ P_2(h) D(h) = 0, h \in (0, 1/6) \right\} + \# \{ \phi_1(h) = 0, h \in (0, 1/6) \} \right) + 2 \\
\leq 51 \left( \left\lceil (n - 1)/2 \right\rceil + 48 \left\lfloor n/2 \right\rfloor + 111 \right).
\]

§4. Proof of Theorem 1.2

This section we consider system (1.3). Similar to the Lemma 2.1, we can obtain

\[
M(h) = \int_{L^+_h} q^+(x, y) dx - p^+(x, y) dy + \int_{L^-_h} q^-(x, y) - p^-(x, y) dy \\
+ \int_{\tilde{L}^+_h} q^+(x, y) dx \tilde{p}^+(x, y) dy + \int_{\tilde{L}^-_h} q^-(x, y) - \tilde{p}^-(x, y) dy,
\]

where

\[
L^+_h(L^-_h) = \{(x, y) | H(x, y) = h, x > 0, y > 0(x < 0, y > 0) \},
\]

\[
\tilde{L}^+_h(\tilde{L}^-_h) = \{(x, y) | H(x, y) = \tilde{h}, x > 0, y < 0(x < 0, y < 0) \}.
\]

Let

\[
J_{i,j}(h) = \int_{L^+_h} x^i y^j dx, \quad I_{i,j}(h) = \int_{L^-_h} x^i y^j dx,
\]

\[
\tilde{J}_{i,j}(h) = \int_{\tilde{L}^+_h} x^i y^j dx, \quad \tilde{I}_{i,j}(h) = \int_{\tilde{L}^-_h} x^i y^j dx.
\]

Then we have \( \tilde{J}_{i,j}(h) = (-1)^{j+1} J_{i,j}(h), \tilde{I}_{i,j}(h) = (-1)^{j+1} I_{i,j}(h) \). Then we have

\[
M(h) = (\rho_{0,0}^+ - \rho_{0,0}^-) J_{0,0} + (\rho_{1,0}^+ - \rho_{1,0}^-) J_{1,0} + (\rho_{0,1}^+ + \rho_{0,1}^-) J_{0,1} \\
+ (\rho_{2,0}^+ - \rho_{2,0}^-) J_{2,0} + (\rho_{1,1}^+ + \rho_{1,1}^-) J_{1,1} + (\rho_{0,2}^+ - \rho_{0,2}^-) J_{0,2} \\
+ (\rho_{0,0}^- - \rho_{0,0}^+) I_{0,0} + (\rho_{1,0}^- - \rho_{1,0}^+) I_{1,0} + (\rho_{0,1}^- + \rho_{0,1}^+) I_{0,1} \\
+ (\rho_{2,0}^- - \rho_{2,0}^+) I_{2,0} + (\rho_{1,1}^- + \rho_{1,1}^+) I_{1,1} + (\rho_{0,2}^- - \rho_{0,2}^+) I_{0,2} \\
:= p_{0,0} J_{0,0} + p_{1,0} J_{1,0} + p_{0,1} J_{0,1} + p_{2,0} J_{2,0} + p_{1,1} J_{1,1} + p_{0,2} J_{0,2} \\
+ q_{0,0} I_{0,0} + q_{1,0} I_{1,0} + q_{0,1} I_{0,1} + q_{2,0} I_{2,0} + q_{1,1} I_{1,1} + q_{0,2} I_{0,2}
\]

(4.2)
where $\rho_{i,j}^\pm(\tilde{\varphi}_{i,j}^\pm) = q_{i,j}^\pm(\tilde{q}_{i,j}^\pm) + \frac{j+1}{j} \rho_{i+1,j-1}^\pm(\tilde{\varphi}_{i+1,j-1}^\pm)$ ($j \geq 1$) and $\rho_{i,0}^\pm(\tilde{\varphi}_{i,0}^\pm) = p_{i,0}^\pm(\tilde{\varphi}_{i,0}^\pm)$. By direct computation, we have

\[
J_{0,0} = x(h), \quad I_{0,0} = -\bar{x}(h), \quad J_{1,0} = \frac{1}{2} \bar{x}(h)^2, \quad I_{1,0} = -\frac{1}{2} \bar{x}(h)^2, \quad J_{2,0} = \frac{1}{3} \bar{x}(h)^3. \quad (4.3)
\]

\[
I_{2,0} = -\frac{1}{3} \bar{x}(h)^3, \quad J_{0,2} = 2h\bar{x}(h) - \frac{1}{3} \bar{x}(h)^3 + \frac{1}{6} \bar{x}(h)^4, \quad I_{0,2} = -2h\bar{x}(h) + \frac{1}{3} \bar{x}(h)^3 - \frac{1}{6} \bar{x}(h)^4, \quad (4.4)
\]

where

\[
\bar{x}(h) = \cos \left( \frac{\pi}{3} + \frac{1}{3} \arccos(12h - 1) \right) + \frac{1}{2}, \quad \bar{x}(h) = -\cos \left( \frac{1}{3} \arccos(12h - 1) \right) + \frac{1}{2}.
\]

Let $u = x(\frac{1}{2} - \frac{1}{3}x)^{1/2}$, then for $x > 0$ small enough, by the implicit theorem we have

\[
x = \phi(u) = -\sqrt{2u} - \frac{\sqrt{2}}{3} u^2 - \frac{11}{54} \sqrt{2} u^3 - \frac{\sqrt{2}}{8} u^4 - \frac{379}{3240} \sqrt{2} u^5 - \frac{565}{5832} \sqrt{2} u^6 - \frac{751}{9072} \sqrt{2} u^7 - \frac{1687}{23328} \sqrt{2} u^8 - \frac{161809}{2519424} \sqrt{2} u^9 - \frac{727783}{12597120} \sqrt{2} u^{10} - \frac{8730965}{166281984} \sqrt{2} u^{11} + o(u^{11}). \quad (4.5)
\]

Therefore

\[
J_{0,0} = \int_0^{\bar{x}(h)} \sqrt{2h + \frac{2}{3} x^3 - x^2} dx = \int_0^{h^{1/2}} \sqrt{2h - 2u^2} \phi'(u) du \\
= \int_0^{h^{1/2}} \sqrt{2h - 2(h^{1/2}u)^2} \phi'(h^{1/2}u) d(h^{1/2}u) = \sqrt{2h} \int_0^{1} \sqrt{1 - 2u^2} \phi'(h^{1/2}u) du \\
= \sqrt{2h} \left( -\frac{1}{4} \sqrt{2\pi} - \frac{2}{9} \sqrt{2h^{1/2}} - \frac{11}{288} \sqrt{2\pi} - \frac{1}{15} \sqrt{2h^{1/2}} - \frac{379}{20736} \sqrt{2\pi} h^{1/2} - \frac{226}{5103} \sqrt{2h^{1/2}} - \frac{3755}{331776} \sqrt{2\pi} h^{3/2} - \frac{964}{32805} \sqrt{2h^{1/2}} - \frac{1132663}{143327232} \sqrt{2\pi} h^{5/2} - \frac{207938}{9743085} \sqrt{2h^{1/2}} - \frac{61116755}{10319560704} \sqrt{2\pi} h^5 + o(h^5) \right), \quad (4.6)
\]

\[
I_{0,0} = \int_0^{\bar{x}(h)} \sqrt{2h - \frac{2}{3} x^3 - x^2} dx = \int_{-h^{1/2}}^0 \sqrt{2h - 2u^2} \phi'(u) du = \sqrt{2h} \int_{-1}^0 \sqrt{1 - u^2} \phi'(h^{1/2}u) du \\
= \sqrt{2h} \left( -\frac{1}{4} \sqrt{2\pi} + \frac{2}{9} \sqrt{2h^{1/2}} - \frac{11}{288} \sqrt{2\pi} + \frac{1}{15} \sqrt{2h^{1/2}} - \frac{379}{20736} \sqrt{2\pi} h^{1/2} + \frac{226}{5103} \sqrt{2h^{1/2}} - \frac{3755}{331776} \sqrt{2\pi} h^{3/2} + \frac{964}{32805} \sqrt{2h^{1/2}} - \frac{1132663}{143327232} \sqrt{2\pi} h^{5/2} + \frac{207938}{9743085} \sqrt{2h^{1/2}} - \frac{61116755}{10319560704} \sqrt{2\pi} h^5 + o(h^5) \right). \quad (4.7)
\]
Similarly, we can obtain

\[
J_{1,1}(I_{1,1}) = \sqrt{2h} \left( \pm \frac{2}{3} h^{\frac{3}{2}} + \frac{1}{8} \pi h \pm \frac{112}{405} h^{\frac{3}{2}} + \frac{625}{10368} \pi h^2 \pm \frac{20924}{127575} h^{\frac{5}{2}} \\
+ \frac{65863}{1492992} \pi h^3 \pm \frac{319442}{2679075} h^{\frac{7}{2}} + \frac{5919829}{179159040} \pi h^4 \pm \frac{218941144}{2387055825} h^{\frac{9}{2}} \right). 
\]

In addition, for \( h > 0 \) sufficiently small

\[
\bar{x}(h) = \sqrt{2h} \left( \pm \frac{2}{3} h^{\frac{3}{2}} + \frac{2}{3} h + \frac{5}{9} \sqrt{2} h^{\frac{5}{2}} + \frac{32}{27} h^2 + \frac{77}{54} \sqrt{2} h^{\frac{7}{2}} + \frac{896}{243} h^3 + \frac{2431}{486} \sqrt{2} h^{\frac{9}{2}} + \frac{10240}{729} h^4 \\
+ \frac{1062347}{52488} \sqrt{2} h^2 + \frac{1171456}{19683} h^5 + \frac{14003665}{157464} \sqrt{2} h^{\frac{11}{2}} + \frac{47710208}{177147} h^6 + o(h^6), \right)
\]

\[
\bar{x}(h) = -\sqrt{2h} \left( \pm \frac{2}{3} h - \frac{5}{9} \sqrt{2} h^{\frac{5}{2}} + \frac{32}{27} h^2 - \frac{77}{54} \sqrt{2} h^{\frac{7}{2}} + \frac{896}{243} h^3 - \frac{2431}{486} \sqrt{2} h^{\frac{9}{2}} + \frac{10240}{729} h^4 \\
- \frac{1062347}{52488} \sqrt{2} h^2 + \frac{1171456}{19683} h^5 - \frac{14003665}{157464} \sqrt{2} h^{\frac{11}{2}} + \frac{47710208}{177147} h^6 + o(h^6). \right) 
\]

Therefore, combining (4.3)–(4.4) and (4.6)–(4.10), we can get

\[
M(h) = \sum_{i \geq 1} \delta_i h^i, 
\]

where

\[
\delta_1 = \sqrt{2} (p_{0,0} + q_{0,0}), \\
\delta_2 = -\frac{2}{3} (q_{0,0} - p_{0,0}) - \frac{\pi}{2} (q_{0,1} + p_{0,1}) - (q_{1,0} - p_{1,0}), \\
\delta_3 = \frac{5}{9} \sqrt{2} (q_{0,0} + p_{0,0}) + \frac{4}{9} (q_{0,1} - p_{0,1}) - \frac{4}{3} \sqrt{2} (q_{0,2} - p_{0,2}) + \frac{2}{3} \sqrt{2} (q_{1,0} + p_{1,0}) \\
- \frac{2}{3} \sqrt{2} (q_{1,1} - p_{1,1}) + \frac{2}{3} \sqrt{2} (q_{2,0} + p_{2,0}), \\
\delta_4 = -\frac{32}{27} (q_{0,0} - p_{0,0}) - \frac{11}{144} \pi (q_{0,1} + p_{0,1}) + \frac{2}{3} (q_{0,2} + p_{0,2}) - \frac{4}{3} (q_{1,0} + p_{1,0}) \\
+ \frac{1}{8} \sqrt{2} (q_{1,1} + p_{1,1}) - \frac{4}{3} (q_{2,0} - p_{2,0}), \\
\delta_5 = \frac{77}{54} \sqrt{2} (q_{0,0} + p_{0,0}) + \frac{2}{15} (q_{0,1} - p_{0,1}) + \frac{4}{9} \sqrt{2} (q_{0,2} + p_{0,2}) + \frac{14}{9} \sqrt{2} (q_{1,0} + p_{1,0}) \\
- \frac{112}{405} \sqrt{2} (q_{1,1} - p_{1,1}) + \frac{14}{9} \sqrt{2} (q_{2,0} + p_{2,0}), \\
\delta_6 = \frac{896}{243} (q_{0,0} - p_{0,0}) - \frac{379}{10368} \pi (q_{0,1} + p_{0,1}) - \frac{64}{81} (q_{0,2} - p_{0,2}) - \frac{320}{81} (q_{1,0} - p_{1,0}) \\
+ \frac{625}{10368} \sqrt{2} (q_{1,1} + p_{1,1}) - \frac{320}{81} (q_{2,0} - p_{2,0}), 
\]
\[\delta_7 = \frac{2431}{486} \sqrt{2}(q_{0,0} + p_{0,0}) + \frac{452}{5103}(q_{0,1} - p_{0,1}) + \frac{22}{27}\sqrt{2}(q_{0,2} + p_{0,2}) + \frac{143}{27}\sqrt{2}(q_{1,0} + p_{1,0})
- \frac{20924}{127575}\sqrt{2}(q_{1,1} - p_{1,1}) + \frac{143}{27}\sqrt{2}(q_{2,0} + p_{2,0}),
\]
\[\delta_8 = -\frac{10240}{729}(q_{0,0} - p_{0,0}) - \frac{3755}{16588}\pi(q_{0,1} + p_{0,1}) - \frac{448}{243}(q_{0,2} - p_{0,2}) - \frac{3584}{243}(q_{1,0} - p_{1,0})
+ \frac{65863}{1492992}\sqrt{2}\pi(q_{1,1} + p_{1,1}) - \frac{3584}{243}(q_{2,0} - p_{2,0}),
\]
\[\delta_9 = \frac{1062347}{52488}\sqrt{2}(q_{0,0} + p_{0,0}) + \frac{1928}{32805}(q_{0,1} - p_{0,1}) + \frac{4862}{2187}\sqrt{2}(q_{0,2} + p_{0,2})
+ \frac{46189}{2187}\sqrt{2}(q_{1,0} + p_{1,0}) - \frac{319442}{2679075}\sqrt{2}(q_{1,1} - p_{1,1}) + \frac{46189}{2187}\sqrt{2}(q_{2,0} + p_{2,0}),
\]
\[\delta_{10} = -\frac{19683}{45056}(q_{0,0} - p_{0,0}) - \frac{71663616}{1132663}\pi(q_{0,1} + p_{0,1}) - \frac{4096}{729}(q_{0,2} - p_{0,2})
- \frac{179159040}{729}(q_{1,0} - p_{1,0}) + \frac{5919829}{179159040}\sqrt{2}\pi(q_{1,1} + p_{1,1}) - \frac{4096}{729}(q_{2,0} - p_{2,0}),
\]
\[\delta_{11} = \frac{14003665}{157464}\sqrt{2}(q_{0,0} + p_{0,0}) + \frac{415876}{9743085}\pi(q_{0,1} + p_{0,1}) + \frac{96577}{13122}\sqrt{2}(q_{0,2} + p_{0,2})
+ \frac{244425}{26244}\sqrt{2}(q_{1,0} + p_{1,0}) - \frac{218941144}{2387055825}\sqrt{2}(q_{1,1} - p_{1,1}) + \frac{244425}{26244}\sqrt{2}(q_{2,0} + p_{2,0}),
\]
\[\delta_{12} = -\frac{177147}{16400384}(q_{0,0} - p_{0,0}) - \frac{3584}{336369143}(q_{0,1} + p_{0,1}) - \frac{1171456}{336369143}\pi(q_{0,2} - p_{0,2})
+ \frac{16400384}{59049}(q_{1,0} - p_{1,0}) + \frac{12899450880}{59049}\sqrt{2}(q_{1,1} + p_{1,1}) - \frac{16400384}{59049}(q_{2,0} - p_{2,0}),
\]
and
\[\delta_i = \mathcal{L}(\delta_1, \delta_2, \delta_3, ..., \delta_{12}), \ i \geq 13. \quad (4.12)\]

Then we calculate that
\[
\frac{\partial (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}, \delta_{12})}{\partial (p_{0,0}, p_{1,0}, p_{0,1}, p_{2,0}, p_{1,1}, p_{0,2}, q_{0,0}, q_{1,0}, q_{0,1}, q_{2,0}, q_{1,1}, q_{0,2})} = \frac{5988673995065166529703225049}{15759296625811548028684506048}\pi^2, \quad (4.13)
\]
which means that \(\delta_1, \delta_2, ..., \delta_{12}\) can be chosen arbitrary. On the other hands, by (4.12) we have
\[M(h) = h^{\frac{1}{2}}\delta_1 (1 + P_1(h, \delta)) + h\delta_2 (1 + P_2(h, \delta)) + h^{\frac{3}{2}}\delta_3 (1 + P_3(h, \delta)) + \cdots + h^{6}\delta_{12} (1 + P_{12}(h, \delta)),\]
where \(\delta = (\delta_1, \delta_2, \delta_3, ..., \delta_{12})\) and
\[P_i \in C^\infty, \ P_i(0, \delta) = 0, \ (i = 1, 2, ..., 12).\]

Therefore, \(M(h) \equiv 0\) when \(\delta_1 = \delta_2 = \cdots = \delta_{12} = 0\). By [11], we can get the Theorem 1.2 combining (4.13). This ends the proof.

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