Minimally-disturbing Heisenberg-Weyl symmetric measurements using hard-core collisions of Schrödinger particles

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Abstract

In a previous paper we have presented a general scheme for the implementation of symmetric generalized measurements (POVMs) on a quantum computer. This scheme is based on representation theory of groups and methods to decompose matrices that intertwine two representations. We extend this scheme in such a way that the measurement is minimally disturbing, i.e., it changes the state vector $|\Psi\rangle$ of the system to $\sqrt{\Pi}|\Psi\rangle$ where $\Pi$ is the positive operator corresponding to the measured result.

Using this method, we construct quantum circuits for measurements with Heisenberg-Weyl symmetry. A continuous generalization leads to a scheme for optimal simultaneous measurements of position and momentum of a Schrödinger particle moving in one dimension such that the outcomes satisfy $\Delta x \Delta p \geq \hbar$.

The particle to be measured collides with two probe particles, one for the position and the other for the momentum measurement. The position and momentum resolution can be tuned by the entangled joint state of the probe particles which is also generated by a collision with hard-core potential. The parameters of the POVM can then be controlled by the initial widths of the wave functions of the probe particles. We point out some formal similarities and differences to simultaneous measurements of quadrature amplitudes in quantum optics.
1 Introduction

The question of how to implement quantum measurements is an important issue of quantum information theory. Even though the standard model of quantum computers uses only one-qubit measurements in the computational basis at the end or during the computation,\(^1\) other measurements are also relevant for quantum information for several reasons.

In quantum computing, models have been proposed where collective measurements on more than one qubit are necessary.\(^2\) In Ref. 3 a quantum algorithm is described which uses even more general measurements than the usual von Neumann measurements, i.e., they are not described by a family of mutually orthogonal projections but by a so-called positive operator-valued measure (POVM).

In non-computing applications of quantum information theory, like future nanoscience, it may, for instance, be useful to implement approximative simultaneous measurements of observables which are actually incompatible when measured accurately. An important example would be the position and the momentum of a Schrödinger particle.

The implementation of generalized measurements is not trivial since this is also true for the smaller class of von-Neumann measurements.\(^4\) Even though it is known that it is in principle possible to reduce every POVM measurement to a von Neumann measurement on an extended quantum system it is little known so far about how to realize the required transformation by physical processes, particularly when the implementation should disturb the quantum state in a minimal way. In Refs. 5 and 6 the class of POVMs is described which can be measured using linear optics. Some special POVM measurements on low-dimensional spaces are described in Refs. 7–12.

In Ref. 13 we have described a general design principle to implement symmetric POVMs on a quantum register where universal quantum computation capabilities are available. However, these implementations are not minimally-disturbing.

Here we describe an extension of the theory of Ref. 13 such that the symmetric measurements disturb the state in a minimal way and apply it to the Heisenberg-Weyl group in finite dimensions. Since the latter actually defines a family of groups it is desirable to have an implementation which is efficient in the sense that its running time scales polynomially in the logarithm of the dimension, i.e., with the number of qubits. We show that this can indeed be achieved for Heisenberg-Weyl groups with a power of two as dimension.

We then adapt the implementation scheme to the continuous situation. The corresponding POVM provides a good example for a measurement where the feature of minimal disturbance makes sense: Given the motivation to measure position and momentum of a particle in order to monitor its motion one would clearly try to avoid disturbance as far as possible.
The general idea of the paper is to show that the finite dimensional circuits provide a paradigm for the continuous variable implementation. By replacing the finite dimensional gates with appropriate analogues, we obtain also a possible measurement scheme even though it is not a priori clear how the required “gates” could be implemented physically. However, we show that a modification of the gate sequence could in principle be realized by three hard-core scattering processes. This kind of idealized scattering in not unphysical since hard-core potential can be a useful approximation in many real collision processes.

We proceed as follows. In the next section we recapitulate the definition of POVMs and recall a general scheme for the implementation of general measurements by orthogonal measurements. In Sec. 3 we define the symmetry of POVMs and present a method for designing measurement algorithms for symmetric POVMs. In Sec. 4 we consider the implementation for two special classes of POVMs to illustrate the latter. Explicitly, we consider POVMs on qubits with cyclic symmetry groups and POVMs on $d$-dimensional quantum systems with Heisenberg-Weyl symmetry. In Sec. 5 we convert the implementation to quantum systems with Hilbert spaces of infinite dimension and describe a potential realization by scattering processes on an abstract level. In Sec. 6 we compare this scheme to a quantum optical implementation of simultaneous measurements for the quadrature amplitudes.

2 Minimally-disturbing implementation by von-Neumann measurements

In this section we briefly outline a general scheme\(^1\) for the minimally-disturbing implementation of a POVM. Consider a quantum system with Hilbert space $\mathbb{C}^d$. A POVM consists of $n$ operators $\Pi_j \in \mathbb{C}^{d \times d}$ with $\Pi_j \geq 0$ and $\sum_j \Pi_j = I_d$ where $I_d$ denotes the identity matrix of size $d \times d$. A definition for POVMs on infinite dimensional quantum systems and an infinite number of results can be found in Ref. 14. In Sec. 5 we use this more general definition but here we start with finite POVMs since we consider implementation schemes on quantum computers at first. Following Refs. 15 and 16 we define:

**Definition 1 (Minimally-disturbing measurement)** Let $(\Pi_j)$ be a POVM. Then a measurement is called minimally-disturbing if it changes the state vector according to

$$|\Psi\rangle \mapsto \frac{\sqrt{\Pi_j}|\Psi\rangle}{\|\sqrt{\Pi_j}|\Psi\rangle\|},$$

given that the measurement result is $j$. 

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The motivation for this definition is given by a theorem in Ref. 15 stating that the above type of measurements maximizes the average fidelity between the input and the output state if the input is drawn from a uniformly distributed ensemble of pure states.

The following lemma\(^1\) reduces the implementation of this kind of measurements to von-Neumann measurements (with Lüder’s projection postulate) in the standard basis.

**Lemma 2 (Reduction of a POVM to a von-Neumann measurement)** Let \( P \) be a POVM with the \( n \) operators \( \Pi_j \in \mathbb{C}^{d \times d} \). Furthermore, let \( \mathbb{C}^n \) be the Hilbert space of an ancilla that is initialized with \( |0\rangle \). Then, a minimally-disturbing measurement of \( P \) can be achieved by a measurement in the standard basis of the ancilla after the implementation of a unitary \( U \in \mathbb{C}^{dn \times dn} \) satisfying the equation

\[
U(|0\rangle \otimes |\Psi\rangle) = \sum_{j=0}^{n-1} |j\rangle \otimes \sqrt{\Pi_j} |\Psi\rangle. \tag{1}
\]

Eq. (1) states that \( U \) is a unitary extension of the matrix

\[
M := \sum_j |j\rangle \otimes \sqrt{\Pi_j} = (\sqrt{\Pi_1}, \sqrt{\Pi_2}, \ldots, \sqrt{\Pi_n})^T \in \mathbb{C}^{dn \times d} \tag{2}
\]

which is defined by \( P \). In the following section we consider this extension for symmetric POVMs. In some cases, the following observations help to show that a unitary \( U \) implements a POVM in a minimally disturbing way.

**Lemma 3 (Linear assignment of Kraus operators)** Let \( U \in \mathbb{C}^{dn \times dn} \) be a unitary operating on a bipartite system which is initialized with \( |\Phi\rangle \otimes |\Psi\rangle \) where \( |\Phi\rangle \in \mathbb{C}^n \) and \( |\Psi\rangle \in \mathbb{C}^d \). Let the first component be measured in the standard basis after the joint system has been subjected to the unitary \( U \). Then the conditional post-measurement state is pure.

Let \( A_{U,\Phi,j} \) be the Kraus operator describing the corresponding state change

\[
|\Psi\rangle \mapsto \frac{A_{U,\Phi,j} |\Psi\rangle}{\|A_{U,\Phi,j} |\Psi\rangle\|}
\]

where \( j \) is the measurement result. For each \( U \) and \( j \), the mapping \( |\Phi\rangle \mapsto A_{U,\Phi,j} \) is linear.

The proof is straightforward since the projected state of the composed system is a product state and the map given by the partial trace is linear. We find:
Corollary 4 (Minimally disturbing Kraus operators) Let \( P \) be a POVM with operators \( \Pi_j \). Furthermore, let \( U \), \(|\Phi\rangle\), and \( A_{U,\Phi,j} \) be as defined in Lemma 3. If the equation
\[
A_{U,\Phi,j} = \sqrt{\Pi_j}
\]
holds for all \( j \) then \( U \) gives rise to a minimally-disturbing measurement of \( P \). In other words, whenever \( A_{U,\Phi,j} \) is positive for each \( j \), it defines a minimally-disturbing measurement for the POVM given by
\[
\Pi_j := A_{U,\Phi,j}^2.
\]

3 Implementation of symmetric POVMs

In this section we analyze how the symmetry of a POVM can be used for the implementation scheme of Lemma 2. Here, we follow the approach of Ref. 13 where we have obtained a general implementation scheme for POVMs without consideration of the disturbance of the measurement process. This implementation scheme also relies on the unitary extension of a matrix that is defined by the POVM operators. It turned out, that the symmetry of the POVM leads to a symmetry of the matrix which can be exploited for the extension. In this section we show that a similar construction is possible for the minimally-disturbing implementation of POVMs.

To begin with, we define the symmetry of POVMs:

**Definition 5 (Symmetric POVMs)** Let \( \sigma : G \to \mathbb{C}^{d \times d} \) be a unitary representation of a finite group \( G \). A POVM with operators \( \Pi_0, \ldots, \Pi_{n-1} \) is called \((\sigma, \pi)\)-symmetric if there is a permutation representation \( \pi : G \to S_n \) of the indices such that
\[
\sigma(g) \Pi_j \sigma(g)^\dagger = \Pi_{\pi(g)j}.
\]
Here, \( S_n \) denotes the symmetric group consisting of all permutations of \( n \) objects.

As mentioned above, the symmetry of a matrix is a useful tool for the implementation of POVMs. Here we define the symmetry of a matrix as in Refs. 17–19.

**Definition 6 (Matrices with symmetry and intertwining spaces)** Let \( G \) be a finite group and \( \sigma : G \to \mathbb{C}^{m \times m} \) as well as \( \tau : G \to \mathbb{C}^{n \times n} \) be unitary representations. A matrix \( A \in \mathbb{C}^{m \times n} \) is \((\sigma, \tau)\)-symmetric if it satisfies
\[
\sigma(g)A = A\tau(g)
\]
for all \( g \in G \). We also write \( \sigma M = M\tau \) for the \((\sigma, \tau)\)-symmetry. We call the set
\[
\text{Int}(\sigma, \tau) = \{ A \in \mathbb{C}^{m \times n} : \sigma A = A\tau \}
\]
of all such matrices the intertwining space of \( \sigma \) and \( \tau \).
The structure of the intertwining space of two representations can be easily specified if both representations are decomposed into a direct sum of irreducible representations of the group as the following lemma\textsuperscript{19} shows:

**Lemma 7 (Structure of intertwining space)** Let $\sigma$ and $\tau$ be decomposed into the direct sums

$$\sigma = \bigoplus_j (I_{m_j} \otimes \kappa_j) \quad \text{and} \quad \tau = \bigoplus_j (I_{n_j} \otimes \kappa_j)$$

of different irreducible representations $\kappa_j$ of the group $G$. Then

$$\text{Int}(\sigma, \tau) = \bigoplus_j \left( \mathbb{C}^{m_j \times n_j} \otimes I_{\deg(\kappa_j)} \right)$$

where $\deg(\kappa_j)$ denotes the degree of $\kappa_j$. For $m_j = 0$ and $n_j = 0$ we insert $n_j \deg(\kappa_j)$ zero columns or $m_j \deg(\kappa_j)$ zero rows, respectively.

The key observation used for the extension of the matrix $M$ from Eq. (2) to a unitary is that the symmetry of a POVM leads to a matrix $M$ with symmetry. This is summarized in the following lemma which can be proved by direct calculation.

**Lemma 8 (Symmetry of a POVM and its matrix)** If the POVM with operators $\Pi_1, \ldots, \Pi_n$ is $(\sigma, \pi)$-symmetric then the corresponding matrix $M$ is $(\sigma \otimes \sigma, \sigma')$-symmetric with the permutation matrix representation $\sigma'(g) = \sum_j |\pi(g)j\rangle \langle j|$.

The following theorem explicitly shows how the $(\sigma \otimes \sigma, \sigma')$-symmetry of $M$ can be extended to a $(\sigma \otimes \sigma, \sigma \oplus \tilde{B}^\dagger \sigma' \tilde{B})$-symmetry of $U$ where $\sigma'$ is an appropriate representation and $\tilde{B}$ a unitary.

**Theorem 9 (Implementation of symmetric POVMs)** Let $M$ be the matrix of Eq. (3) for a $(\sigma, \pi)$-symmetric POVM with symmetry group $G$. Let $A$ and $B$ be transformations that decompose $\sigma \otimes \sigma$ and $\sigma$ into irreducible representations, respectively. Then there is a representation $\sigma'$ of $G$ such that $B \sigma B^\dagger \oplus \sigma'$ is equal to $A(\sigma \otimes \sigma)A^\dagger$ up to a permutation of the irreducible components. Furthermore, there is a transformation $W \in \text{Int}(A(\sigma \otimes \sigma)A^\dagger, B \sigma B^\dagger \oplus \sigma')$ which is a unitary extension of $AMB^\dagger$. Then

$$U := A^\dagger W(B \oplus \tilde{B})$$

implements the POVM for every unitary $\tilde{B}$. The unitary $U$ is $(\sigma \otimes \sigma, \sigma \oplus \tilde{B}^\dagger \sigma' \tilde{B})$-symmetric.

Proof: We decompose $\sigma \otimes \sigma$ and $\sigma$ with the unitaries $A \in \mathbb{C}^{dn \times dn}$ and $B \in \mathbb{C}^{d \times d}$, i.e., we obtain the equations

$$A(\sigma \otimes \sigma)A^\dagger = \bigoplus_j (I_{m_j} \otimes \kappa_j) \quad \text{and} \quad B \sigma B^\dagger = \bigoplus_j (I_{n_j} \otimes \kappa_j) .$$

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Therefore, the equation
\[
\left( \bigoplus_j (I_{m_j} \otimes \kappa_j) \right) A M B^\dagger = A M B^\dagger \left( \bigoplus_j (I_{n_j} \otimes \kappa_j) \right)
\]
holds. Following Lemma 7 the matrix \( N := A M B^\dagger \) has the decomposition
\[
N = \bigoplus_j (A_j \otimes I_{d_j})
\]
with \( A_j \in \mathbb{C}^{m_j \times n_j} \) and \( d_j := \deg(\kappa_j) \). From Th. 5 of Ref. 13 it follows that \( B \sigma B^\dagger \) can be extended to \( A(\sigma \otimes \sigma)A^\dagger \), i.e., the representations \( A(\sigma \otimes \sigma)A^\dagger \) and \( B \sigma B^\dagger \otimes \sigma' \) with
\[
\sigma' = \bigoplus_j (I_{m_j-n_j} \otimes \kappa_j)
\]
are equal up to a permutation of the irreducible components. We choose a unitary extension \( W \in \text{Int}(A(\sigma \otimes \sigma)A^\dagger, B \sigma \tilde{B} \otimes \sigma') \) of \( N \). This extension can be achieved by appending appropriate columns to the right side of \( N \) since the matrix \( N \) has orthogonal columns. We can write \( W = (N|\tilde{N}) \) for this extension if we denote the new columns by \( \tilde{N} \). With this matrix we obtain for an arbitrary unitary \( \tilde{B} \in \mathbb{C}^{n(d-1) \times n(d-1)} \) the unitary extension
\[
A^\dagger(N|\tilde{N})(B \oplus \tilde{B}) = (M|A^\dagger \tilde{N} \tilde{B})
\]
of \( M \). □

As shown in the following section, the unitary \( W \) of Th. 9 can be chosen to be sparse for some POVMs. Within the standard model of quantum computing, this can be used for obtaining efficient decompositions into elementary gates for the cases discussed in the next section. Furthermore, there are methods known to decompose the transformations \( A \) and \( B \oplus \tilde{B} \) into products of simpler matrices.\(^{17–19}\)

4 Examples

In this section we explicitly construct quantum circuits and implementation schemes for the minimally-disturbing implementation of two families of symmetric POVMs. First, we introduce the following notations: For \( m \in \mathbb{N} \) define \( \omega_m := \exp(-2\pi i/m) \) and let
\[
X_m := \sum_{j=0}^{m-1} |(j + 1) \mod m \rangle \langle j| \in \mathbb{C}^{m \times m}
\]
be the cyclic shift of the basis vectors of an \( m \)-dimensional space. Furthermore, define the diagonal phase matrix
\[
Z_m := \sum_{j=0}^{m-1} \omega_m^j |j\rangle\langle j| \in \mathbb{C}^{m \times m}
\]
and the Fourier transform
\[
F_m := \sqrt{\frac{1}{m}} \sum_{j,k=0}^{m-1} \omega_m^{jk} |j\rangle\langle k| \in \mathbb{C}^{m \times m}.
\]

We obtain the equalities \( F_m X_m F_m^\dagger = Z_m \) and \( Z_m X_m = \omega_m X_m Z_m \) which we will use in the following without proof.

### 4.1 Cyclic groups

Simple examples for our implementation scheme are POVMs operating on a qubit with a cyclic symmetry. Measurements with cyclic symmetry can, for instance, provide an estimation of time when applied to a dynamical quantum system. The reason for this is that the time evolution of a quantum system with energy eigenvalues being rational multiples of each other is periodic and the dynamics is therefore a unitary representation of \( \text{SO}(2) \). This leads naturally to the finite cyclic groups after discretization.

Fix \( n \geq 2 \). We consider the cyclic group \( C_n = \langle r : r^n = 1 \rangle \) with \( n \) elements, the unitary matrix representation \( \sigma : C_n \to \mathbb{C}^{2 \times 2} \) with \( \sigma(j) = R_n^j \) for
\[
R_n := \begin{pmatrix} 1 & 0 \\ 0 & \omega_n \end{pmatrix} \in \mathbb{C}^{2 \times 2},
\]
and the orbit of the vector \( \sqrt{1/n}(1,1)^T \in \mathbb{C}^2 \) with respect to this representation of \( C_n \). We have the POVM operators
\[
\Pi_j := \frac{1}{n} \begin{pmatrix} 1 & 0 \\ 0 & \omega_n^j \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \omega_n^{-j} \end{pmatrix} = \frac{1}{n} \begin{pmatrix} 1 & \omega_n^j \\ \omega_n^{-j} & 1 \end{pmatrix}
\]
for \( j \in \{0, \ldots, n-1\} \).

Applying the methods discussed in Sec. 3 we obtain the following unitary for the minimally-disturbing implementation of the POVM with cyclic symmetry.

**Theorem 10 (Implementation of POVMs with cyclic symmetry)** The POVM with the operators of Eq. (4) can be implemented by the unitary
\[
U := (F_n^\dagger \otimes I_2) X_{2n}^\dagger (I_n \otimes F_2) K^\dagger \in \mathbb{C}^{2n \times 2n}
\]
where $K$ denotes the permutation matrix which is defined by

$$K|2j\rangle = |j\rangle \quad \text{and} \quad K|2j + 1\rangle = |n + j\rangle.$$  

The rather technical proof can be found in the appendix. The idea is the following. The transformation $A$ must diagonalize the cyclic shift $X_n \otimes I_2$. This can be achieved by $F_n^\dagger \otimes I_2$. Furthermore, we need the cyclic shift $X_2^\dagger n$ to obtain the correct order of the irreducible representations. The transformation $B$ is trivial since $\sigma$ is already diagonal. The remaining transformation $(I_n \otimes F_2)K^\dagger$ is the the sparse matrix in the intertwining space.

If $n$ is a power of 2 the ancilla system can be a qubit register and the unitary of Th. 10 can be implemented efficiently as the following corollary states.

**Corollary 11 (Circuits for cyclic POVM)** For $n = 2^m$, $m \geq 1$, the unitary $U$ of Th. 10 can be implemented efficiently with the circuit of Fig. 1.

Proof: Since Fourier transforms can be implemented with a polynomial number$^{1,21}$ of elementary gates, i.e., one and two-qubit gates, $F_n^\dagger \otimes I_2$ can be implemented efficiently. Furthermore, the cyclic shift $X_2^\dagger$ can be written as $X_2^\dagger = F_2^\dagger Z_2^\dagger F_2$ with

$$Z_2^\dagger = R_{2n}^{-n} \otimes R_{2n}^{-n/2} \otimes \ldots \otimes R_{2n}^{-1}.$$  

The unitary $K$ is only a cyclic shift of qubits. □
### 4.2 Heisenberg-Weyl groups

The operators $X_d$ and $Z_d$ are discrete analogues of translations in position and momentum space. If $\mathbb{C}^d$ denotes the $d$ possible positions of a particle on a cyclic chain, the eigenvectors of $Z_d$ can be interpreted as positions eigenstates and the eigenvectors of $X_d$ as eigenvectors of crystal momentum. Like for continuous quantum systems these observables are incompatible and it can be desirable to have approximative simultaneous measurements such that the result can be interpreted as a point in $2d$-dimensional “phase space”. In Sec. 5 we discuss the continuous analogue.

The basis of simultaneous measurements of position and momentum are POVMs with Heisenberg-Weyl symmetry. For all $d \geq 2$, the Heisenberg-Weyl group is given by $G = \langle X_d, Z_d \rangle$ and has order $d^3$. For a positive operator $\mu$ with $\text{tr}(\mu) = 1/d$ we consider the POVM with the $d^2$ operators

$$Z_d^k X_d^j \mu X_d^{-j} Z_d^{-k} \quad \text{for} \quad k, j = 0, \ldots, d-1.$$  

The following theorem shows how to implement this type of POVMs.

**Theorem 12 (POVMs with Heisenberg-Weyl symmetry)** Given the Heisenberg-Weyl POVM $P$ with the $d^2$ operators from Eq. (5) with $\mu := |\alpha\rangle \langle \alpha|/d$ for some state vector $|\alpha\rangle$. Then $P$ can be implemented by the circuit in Fig. 2 where the inputs of the ancillas are given by $|\alpha\rangle \otimes |\overline{\alpha}\rangle$ with the complex conjugated wave function $\overline{\alpha}$. For general $\mu$ the ancilla input has to be replaced with the state vector $|\gamma\rangle := \sqrt{d} \sum_{j,k=0}^{d-1} \sqrt{\mu_{jk}} |j\rangle \otimes |k\rangle \in \mathbb{C}^d$, where $\sqrt{\mu_{jk}}$ denotes the entry of $\sqrt{\mu}$ in the $j$th row and $k$th column.

The proof of the theorem can be found in the appendix. In the following we briefly sketch the main points of the proof. For the decomposition of $\sigma_\pi \otimes \sigma$ we observe that the permutation $\pi$ given by the action of the Heisenberg-Weyl group on the operators is a translation in the finite plane $(\mathbb{Z}/d\mathbb{Z})^2$. This translation is diagonalized by the inverse Fourier transform $F_d^\dagger \otimes F_d^\dagger \otimes I_d$ at the end of the circuit in Fig. 2. This transformation already block-diagonalizes $\sigma_\pi \otimes \sigma$. However, the irreducible components are only equivalent, but not equal, to $\sigma$. We apply the controlled $Z$ and controlled $X^\dagger$ operations to obtain equality. Hence, these operations correspond to the matrix $A$. The matrix $B$ is trivial since $\sigma$ is an irreducible representation. The unitary extension $W$ used in Th. 12 is decomposed into two components. One component is given by the first two gates of the circuit in Fig. 2 the other is absorbed into the preparation procedure for the initial state.
As already stated, we can efficiently implement $F_d$ by elementary gates on a qubit register. We also obtain efficient implementations of controlled $X$ and $Z$ gates by concatenations of controlled $R_d$ gates as defined in Eq. (3). Hence, we can implement the POVM with initial operator $\mu = \vert \alpha \rangle \langle \alpha \vert /d$ efficiently if the same is true for the preparation of the states $\vert \alpha \rangle$ and $\vert \alpha \rangle$. 

5 Continuous measurements

Here we want to address how to implement Heisenberg-Weyl symmetric POVMs for continuous quantum systems such that we have also minimal disturbance. For a detailed mathematical description of such POVMs we refer also to Refs. 22, 23.

The continuous degree of freedom can either be a Schrödinger wave of a quantum particle moving on a line (where the Heisenberg-Weyl group formalizes translations in position and momentum space) or a quantum optical light mode (where the translations shift the quadrature amplitudes). The most natural representation of the Hilbert space of a particle in one dimension is $\mathcal{H} := L^2(\mathbb{R})$, the space of square integrable functions over the real line. For a light mode, it is often more appropriate to choose the isomorphic Hilbert space $l^2(\mathbb{N}_0)$ of square-summable sequences. In this section, we will focus on Schrödinger particles since simultaneous measurements of quadrature amplitudes in quantum optics have already been implemented.\textsuperscript{24} We will compare our implementation to the latter in the next section.

We first describe the continuous analogues of the “gates” in Fig. 2 and show that their concatenation leads indeed to a correct implementation. Later we will discuss a modification of the scheme which can be implemented by hard-core scattering processes. The description below refers to the Schrödinger representation where the
position operator $X$, defined on a dense subspace of $\mathcal{H}$, is the multiplication operator

$$X\psi(x) := x\psi(x).$$

The momentum operator is

$$P\psi(x) := -i\frac{d}{dx}\psi(x),$$

where we have chosen the units such that $\hbar = 1$. Following Sec. 3.4 of Ref. 14 (with a slight modification of the sign) we introduce a family $(U_{s,t})$ of unitaries

$$(U_{s,t}\psi)(x) := e^{-ixs}\psi(x-t),$$

which formalize shifts in momentum and position space. These unitaries define a measurement by the positive operators

$$\Pi_{s,t} := \frac{1}{2\pi} U_{s,t}|\alpha\rangle\langle\alpha| U_{s,t}^\dagger,$$

where $|\alpha\rangle \in \mathcal{H}$ is a wave function which is sufficiently localized in momentum and position space. The probability density for the result $(s,t)$ is $\text{tr}(\rho\Pi_{s,t})$ if the system state is described by the density operator $\rho$. The outcome $(s,t)$ is interpreted as momentum $s$ and position $t$ of the particle in a “coarse grained phase space”. We can clearly generalize the POVM above by replacing $|\alpha\rangle\langle\alpha|/(2\pi)$ with any operator $\mu$ having trace $1/(2\pi)$.

In agreement with the discussions of finite POVMs in the preceding sections we want to implement the POVM in such a way that the state changes according to

$$\rho \mapsto \frac{\sqrt{\Pi_{s,t}\rho\sqrt{\Pi_{s,t}}}}{\text{tr}(\Pi_{s,t}\rho)},$$

given that the measurement outcome is $(s,t)$.

Now we describe how to find a continuous analogue of the circuit in Fig. 2. The system Hilbert space $(\mathbb{C}^d)^{\otimes 3}$ is replaced by $\mathcal{H}^{\otimes 3}$, i.e., in additional to the particle to be measured one uses two particles in one dimension as ancilla system. The final von Neumann measurement is a position measurement on both ancillas$^a$.

The continuous analogues of the required gates are as follows. The discrete Fourier transform (whose inverse is occurring three times in Fig. 2) is replaced with the continuous unitary Fourier transform

$$(F\psi)(x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-ixy}\psi(y)dy.$$  

(6)

$^a$One could also use the remaining two dimensions of a particle in three dimensions as ancilla system.
The controlled cyclic shift is replaced by a unitary $Y$ describing controlled translations on the real line. It acts on the wave function $\psi$ of two particles according to

$$(Y \psi)(x, y) = \psi(x, y - x),$$  \hspace{2cm} (7)

since this transformation would correspond to the transformation

$$|x\rangle \otimes |y\rangle \mapsto |x\rangle \otimes |x + y\rangle,$$

if such position eigenstates $|x\rangle$ and $|y\rangle$ existed. Conjugating $Y$ with the Fourier transform on the second tensor component makes it more apparent that it is indeed a unitary map since we obtain then the multiplication operator

$$V := \left((I \otimes F)Y(I \otimes F^\dagger)\right)(x, y) = e^{-ixy}\psi(x, y).$$  \hspace{2cm} (8)

Here $I$ denotes the identity operator on $\mathcal{H}$. The unitary in Eq. (8) is the straightforward generalization of the controlled phase-shift operation that is the fourth gate in Fig. 2. The following theorem shows that the above described replacements provide in fact the desired measurement procedure:

**Theorem 13** Replace the gates in Fig. 2 with their continuous analogues as follows:

1. Set the inverse of the continuous unitary Fourier transform given by Eq. (6) instead of $F^\dagger_d$

2. Set the inverse of $Y$ given in Eq. (7) instead of the controlled-$X^\dagger_d$ gate.

3. Set $V$ as given by Eq. (8) instead of the controlled-$Z_d$ gate.

Let $\mu$ be an arbitrary positive operator with $\text{tr}(\mu) = 1/(2\pi)$ and the two ancilla systems be in the state $\gamma$ with

$$|\gamma\rangle := \sum_{j=0}^{\infty} \sqrt{\lambda_j}|\alpha_j\rangle \otimes |\pi_j\rangle,$$

where $|\alpha_j\rangle$ is an eigenvector basis of $\mu$ such that

$$2\pi\mu = \sum_j \lambda_j |\alpha_j\rangle \langle \alpha_j|.$$

Then the resulting transformation on $\mathcal{H}^{\otimes 3}$ implements a minimally-disturbing measurement for the POVM

$$\Pi_{s,t} := U_{s,t}\mu U^\dagger_{s,t},$$

when followed by position measurements on both ancillas at the end and interpreting the position of the first particle in Fig. 2 as $t$ and the position of the second as $s$. 

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Proof: Due to Lemma 3 and its corollary it is sufficient to restrict the attention to rank-one operators \( \mu := |\alpha\rangle\langle \alpha|/(2\pi) \) and show that the unnormalized output state coincides with the desired state. The linearity argument holds also if \( \mu \) is an infinite series since one can check that the map

\[
\Phi \mapsto A_{U,\Phi,j}
\]

is continuous with respect to the topologies induced by the Hilbert space norm and the operator norm, respectively. This is seen from

\[
\|\Phi\|^2 = \text{tr}(A^\dagger_{U,\Phi,j}A_{U,\Phi,j}) \leq \|A^\dagger_{U,\Phi,j}A_{U,\Phi,j}\| = \|A_{U,\Phi,j}\|^2.
\]

The whole “circuit” creates some wave function \( \Psi \in \mathcal{H}^\otimes 3 \). After measuring \( t \) and \( s \) we obtain an unnormalized conditional state vector given by the wave function

\[
z \mapsto \tilde{\Psi}(t, s, z) := \tilde{\Psi}_{t, s}(z).
\]

We want to show that it satisfies

\[
|\tilde{\Psi}_{t, s}\rangle = \sqrt{\Pi_{s,t}}|\Psi\rangle = \frac{1}{\sqrt{2\pi}} U_{s,t}|\alpha\rangle \langle \alpha| U^\dagger_{s,t}\Psi\rangle.
\]

This means explicitly that

\[
\tilde{\Psi}_{t, s}(z) = \sqrt{\frac{1}{2\pi}} e^{-izs} \alpha(z - t) \int_{-\infty}^{\infty} \alpha(u - t) e^{ius}\Psi(u)du.
\]

Now we calculate the effect of the circuit starting with the joint state

\[
\alpha(x)\bar{\alpha}(y)\Psi(z),
\]

where \( \Psi \) is the wave function of the measured particle. First, we apply the controlled inverse translation and obtain

\[
\alpha(x)\bar{\alpha}(y + x)\Psi(z).
\]

The inverse Fourier transform changes this state to

\[
\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{iux}\alpha(u)\bar{\alpha}(y + u)\Psi(z)du.
\]

The second controlled inverse shift followed by the controlled phase yields

\[
\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{iux}\alpha(u)\bar{\alpha}(y + u)e^{-izx}\Psi(z + y)du.
\]
After applying the inverse Fourier transform to both ancilla registers we obtain
\[
\sqrt{\frac{1}{8\pi^3}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iyw} e^{ixv} e^{iuv} \alpha(u) \overline{\alpha}(w + u) \Psi(z + w) du dv dw.
\]
We simplify this term into
\[
\sqrt{\frac{1}{8\pi^3}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iyw} e^{i(x-z+u)v} \alpha(u) \overline{\alpha}(w + u) \Psi(z + w) du dv dw.
\]
The integral over \(v\) is only non-vanishing for \(x - z + u = 0\). Hence, we obtain
\[
\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{iyu} \alpha(z-x) \overline{\alpha}(z + w - x) \Psi(z + w) dw.
\]
With the substitution \(u := z + w\) we get
\[
\sqrt{\frac{1}{2\pi}} e^{-iyz} \alpha(z-x) \int_{-\infty}^{\infty} e^{iyu} \overline{\alpha}(u - x) \Psi(u) du.
\]
The conditional state given that we obtain the result \(x = t\) and \(y = s\) coincides with Eq. (10).

In order to realize the transformation in Th. 13 by a physical process we first observe that scattering processes realize quantum gates which are close to the controlled phase shift in Eq. (7): Consider two particles interacting with hard-core potential, i.e., the interaction energy is zero whenever their distance is larger than some \(a > 0\) and infinite if the distance is smaller than \(a\). In Ref. 25 we have discussed the state change caused by such a scattering provided that the considered time scale is small compared to the time scale on which the width of wave packets grows by dispersion. We will first explain the scattering process in momentum space since the change of momenta of classical particles provide a good intuition about the quantum case. The momentum \(p_2\) of the light particle obtains a sign change since it is reflected. Due to the conservation of total momentum, the heavy particle acquires an additional momentum \(2p_2\). The vector of momenta of both particles is therefore changed according to a linear transformation \(N\) given by
\[
N \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1 + 2p_2 \\ -p_2 \end{pmatrix}.
\]
Neglecting irrelevant translations in position space, the corresponding linear transformation \(M\) in position space is already given by the requirement that the \(4 \times 4\) matrix
transformation $M \oplus N$ acting on the two positions and the two momenta has to be symplectic. We have therefore $M = (N^T)^{-1}$ and obtain in agreement with Ref. 25

$$M = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$  

The scattering process $S$ acts therefore on the wave function in position space by multiplying the coordinate vector with $M$, i.e.,

$$(S\psi)(\mathbf{x}) := \psi(M\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2.$$  

We obtain

$$(S\psi)(x, y) = \psi(x, -y + 2x).$$  

In order to understand the relation to the gates in Th. 13 we may represent this operation by the circuit in Fig. 3. The “reflection” gate $R$ corresponds to a change of the wave function according to

$$(R\psi)(x) := \psi(-x).$$

Elementary calculations show that Fig. 2 is equivalent to the circuit in Fig. 4 where we have absorbed the Fourier transform on the wire in the middle by replacing an $X$-measurement with a $P$-measurement.

After we have converted the desired circuit into the equivalent one in Fig. 4 that avoids controlled phase gates we still have the problem that it requires controlled-$X$
gates and its inverse instead of a controlled-$X^2$ gate which implements the shift twice. However, we observe that we may convert these gates into each other by conjugating them with the unitary squeezing operator

$$S_2|x\rangle := |2x\rangle$$

combined with reflections when needed.\(^b\) We will see later that we do not have to worry about the physical realization of $S_2$ since we need this gate and its inverse only at the end or at the beginning of the first or the second wire. Hence, they can either be absorbed into the preparation procedure or into the measurement by reinterpreting the result.

We will furthermore modify the entangling operation on ancilla 1 and 2, i.e., the first gate of the circuit in Fig. 4 for the following reason. An important feature of the circuits in Figs. 2 and 4 is that the POVM consists of rank-one operators if the input ancilla state is the product state $|\alpha\rangle \otimes |\bar{\alpha}\rangle$ and entangled inputs lead to POVM operators of higher rank. The preparation of these entangled states was not considered in Subsection 4.2. Here we also want to describe how to entangle ancilla 1 and 2 when POVMs of higher rank are desired. The goal is therefore to change the operation on ancilla 1 and 2 preceding the interaction with the system to be measured such that a family of product input states allow the implementation of POVMs of higher rank. In other word, we want to tune the achieved information and the caused disturbance of the measurements by plugging different product states into the circuit.

After subsequently replacing the gates in Fig. 4 with scattering processes combined with squeezing operations and reflections and modifying the entangling operation between the ancillas, we found that an interesting class of POVMs can indeed be implemented by three scattering processes as depicted in Fig. 5 when the initial ancilla states are Gaussian wave packets.

To understand the effect of the “circuit” in Fig. 5 we shall compute a $3 \times 3$-matrix which describes the effect of the whole circuit on the three position coordinates. For doing so, we recall (see Fig. 5) that the masses of the particles satisfy

$$m_1 \ll m_3 \ll m_2.$$  

First, we implement a collision between particle 1 and 2. Here, the position of particle 2 controls the position of particle 1. In analogy to the remarks above we describe the scattering and reflection by matrices that acts on the vector of position coordinates of the three particles. The scattering processes with the pairs $(2, 1)$, $(2, 3)$, and $(3, 1)$

\(^b\)Note that the described reduction of controlled-$SC$ to controlled-$X$ is also possible in finite dimensions $d$. The definition $R|x\rangle := |-x\rangle$ is always possible and $|x\rangle \mapsto |2x \mod d\rangle$ is bijective if $d$ is odd. Then the ring $\mathbb{Z}/d\mathbb{Z}$ allows division by 2.
Figure 5: Implementation of minimally-disturbing simultaneous measurement of position and momentum by three scattering processes and one reflection. The masses \( m_1 \) and \( m_2 \) of the two ancilla particles are extremely small or extremely large compared to the mass \( m_3 \) of the particle to be measured.

correspond to the matrices

\[
S_{21} := \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{23} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}, \quad S_{31} := \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The scatterings are followed by a reflection of the \( z \)-coordinate (the \( R \) gate). Taking into account that we have to concatenate the effect on the coordinates from the left to the right, the complete transformations in position coordinate space is given by

\[
A := S_{31} S_{23} S_{21} R_3 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.
\]

Let the initial state of the three particles be given by the wave function

\[
\alpha(x)\beta(y)\psi(z).
\]

After subjecting the arguments to \( A \) we obtain

\[
\alpha(x + 2y + 2z)\beta(y)\psi(2y + z).
\]

In order to reduce a momentum measurement on the second wire to a position measurement we apply a Fourier transform to the state (11) and obtain

\[
\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \alpha(x + 2\tilde{w} + 2z) \beta(\tilde{w}) \psi(2\tilde{w} + z) e^{-i\tilde{w}y} d\tilde{w}.
\]

With \( w := 2\tilde{w} + z \) we get

\[
\sqrt{\frac{1}{8\pi}} \int_{-\infty}^{\infty} \alpha(x + w + z) \beta((w - z)/2) \psi(w) e^{i\frac{w}{2}(w-z)} dw.
\]
We define the integral kernel
\[ k_{x,y}(z, w) := \sqrt{\frac{1}{8\pi}} \alpha(x + w + z) \beta\left(\frac{(w - z)}{2}\right) e^{i \frac{\pi}{2}(w-z)}. \]

It defines for fixed \( x, y \) an operator \( K_{x,y} \) on \( H \) by
\[
(K_{x,y} \psi)(z) := \int_{-\infty}^{\infty} k_{x,y}(z, w) \psi(w) \, dw.
\]

Note that the Kraus operators \( K_{x,y} \) describe the unnormalized output state \( K_{x,y} |\psi\rangle \) of particle 3 given that we have measured \( x \) and \( y \) on the first and second particle, respectively (in straightforward analogy to the Kraus operators in Lemma 3 for the discrete setting).

Now we show that \( K_{x,y} \) can be obtained by subjecting \( K_{0,0} \) to the Heisenberg-Weyl group elements by
\[
K_{x,y} = U_{-x/2, -y} K_{0,0} U_{-x/2, -y}^\dagger.
\]
To see this, we observe that the translation by \(-x/2\) in position space changes the integral kernel \( k_{0,0}(z, w) \) into \( k_{0,0}(z + x/2, w + x/2) \) and the additional translation in momentum space by \(-y/2\) changes it into
\[
k_{0,0}(z + x, w + x) e^{i \frac{\pi}{2}(w-z)} = k_{x,y}(z, w).
\]
This shows that the process in Fig. 5 implements a measurement for the POVM
\[
\Pi_{s,t} := U_{s,t} K_{0,0}^\dagger K_{0,0} U_{s,t}
\]
when reinterpreting the measurement outcomes \( x, y \) on particle 1 and 2 as \( t = -x/2 \) and \( s = -y \), respectively. In order to obtain a minimally-disturbing implementation, we have to ensure that \( K_{0,0} \) is positive (in straightforward generalization of Corollary 4 to the continuous setting) because it can then be interpreted as \( \sqrt{\mu} \). If \( \alpha \) and \( \beta \) are real and \( \beta \) is an even function, i.e., \( \beta(-y) = \beta(y) \), \( K_{0,0} \) is self-adjoint due to
\[
k_{0,0}(z, w) = k_{0,0}(w, z).
\]
The integral kernel of \( K_{0,0} \) is explicitly given by
\[
k_{0,0}(z, w) = \frac{1}{\sqrt{2\pi}} \alpha(w + z) \beta\left(\frac{(w - z)}{2}\right).
\]
Now we assume that \( \alpha \) and \( \beta \) are both real Gaussian wave functions with widths \( \sigma_1 \) and \( \sigma_2 \), respectively, i.e.,
\[
\alpha(x) := \frac{1}{\sqrt{\sigma_1^2 \pi^{1/4}}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right) \quad \text{and} \quad \beta(y) := \frac{1}{\sqrt{\sigma_2^2 \pi^{1/4}}} \exp\left(-\frac{y^2}{2\sigma_2^2}\right).
\]
Under these conditions, $k_{0,0}$ defines a positive operator whenever $\sigma_1 \geq 2\sigma_2$. This follows from the following lemma after replacing $a$ and $b$ with $1/(2\sigma_1^2)$ and $1/(8\sigma_2^2)$, respectively.

**Lemma 14** The operator given by the integral kernel

$$k(x, y) := d e^{-a(x+y)^2 - b(x-y)^2}$$

with $d > 0$ is for all $b > a \geq 0$ positive.

Proof: Rewrite the kernel as

$$k(x, y) = d e^{-2ax^2} e^{-(b-a)(x-y)^2} e^{-2ay^2}. \quad (12)$$

It is known that the integral kernel

$$\tilde{k}(x, y) := d e^{-c(x-y)^2}$$

defines for all positive $c, d$ a positive operator\(^{26}\) which we shall denote by $\tilde{K}$. Then the operator $K$ given by the kernel \(^{12}\) can be written as $K = D\tilde{K}D$ where $D$ is the multiplication operator

$$(D\psi)(x) := e^{-2ax^2} \psi(x).$$

Hence, $K$ is also positive.  \(\Box\)

Since we have now described sufficient conditions for which $K_{0,0}$ is positive, we would like to better understand the POVM operator $\mu = K_{0,0}^2$. As simple computations show, it is (up to the normalization factor $2\pi$) given by the reduced state of one particle in a two-particle system, if the latter is described by the wave function

$$\phi(x, y) := \sqrt{2\pi} k_{0,0} = \alpha(x + y) \beta((x - y)/2). \quad (13)$$

It can be obtained from the state $|\alpha\rangle \otimes |\beta\rangle$ by a linear mapping of the wave function arguments according to

$$\left( \begin{array}{c} x \\ y \end{array} \right) \mapsto \left( \begin{array}{cc} 1 & 1 \\ 1/2 & -1/2 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) =: G \left( \begin{array}{c} x \\ y \end{array} \right).$$

Such a linear operation transforms the initial Gaussian state into a entangled Gaussian state. Since Gaussian states are completely determined by their covariance matrix\(^{27}\) we will compute the latter for the state in Eq. (13).
For doing so, we must describe the linear transformation corresponding to $G$ that acts on the arguments of the wave function in momentum space. According to the remarks at the beginning of this section, it is given by the transposed inverse:
\[
\begin{pmatrix}
P_x \\
P_y
\end{pmatrix} \mapsto \left( G^T \right)^{-1} \begin{pmatrix}
P_x \\
P_y
\end{pmatrix}.
\]

The covariance matrix of a two-particle state $\rho$ consists of the entries
\[
\text{tr}(\rho X_i X_j) - \text{tr}(\rho X_i) \text{tr}(\rho X_j) \quad \text{with} \quad i, j = 1, \ldots, 4
\]

where $X_1, X_2$ denote the position operators and $X_3, X_4$ the momentum operators of particle 1 and 2, respectively. For the state $|\alpha\rangle \otimes |\beta\rangle$ it is given by (see Ref. 27)
\[
\sigma = \begin{pmatrix}
\frac{\sigma_1^2}{2} & 0 & 0 & 0 \\
0 & \frac{\sigma_2^2}{2} & 0 & 0 \\
0 & 0 & 1/(2\sigma_1^2) & 0 \\
0 & 0 & 0 & 1/(2\sigma_2^2)
\end{pmatrix}.
\]

If the coordinate vector in the position wave function is subjected to some area-preserving linear map $G$ and the coordinates of the momentum wave function to $(G^T)^{-1}$, the covariance matrix transforms in the following way:
\[
\sigma' := \begin{pmatrix}
G^{-1} & 0 \\
0 & G^T
\end{pmatrix} \sigma \begin{pmatrix}
(G^T)^{-1} & 0 \\
0 & G
\end{pmatrix}
= \frac{1}{8} \begin{pmatrix}
\sigma_1^2 + 4\sigma_2^2 & \sigma_1^2 - 4\sigma_2^2 & 0 & 0 \\
\sigma_1^2 - 4\sigma_2^2 & \sigma_1^2 + 4\sigma_2^2 & 0 & 0 \\
0 & 0 & \frac{4}{\sigma_1^2} + \frac{1}{\sigma_2^2} & \frac{4}{\sigma_1^2} - \frac{1}{\sigma_2^2} \\
0 & 0 & \frac{4}{\sigma_1^2} - \frac{1}{\sigma_2^2} & \frac{4}{\sigma_1^2} + \frac{1}{\sigma_2^2}
\end{pmatrix},
\]

as simple computations show. The covariance matrix of the reduced state of each particle is given by the $2 \times 2$ sub-matrices that refer to its position and momentum. Due to the symmetry of our state, it is for both particles given by
\[
\frac{1}{8} \begin{pmatrix}
\sigma_1^2 + 4\sigma_2^2 & 0 \\
0 & \frac{4}{\sigma_1^2} + \frac{1}{\sigma_2^2}
\end{pmatrix}.
\]

It is known\textsuperscript{27} that such a state is pure if and only if the determinant is $1/4$. This is given for $\sigma_1 = 2\sigma_2$. One can rewrite a Gaussian state of a single mode having diagonal covariance matrix as a thermal state of a harmonic oscillator\textsuperscript{c} with frequency $\omega$, mass

\textsuperscript{c}In quantum optics, one would also need squeezing transformations to obtain a general diagonal Gaussian state. But here the product of frequency and mass of the oscillator provides an additional free parameter.
and average phonon number $N$. It is explicitly given by

$$\rho_{N,\nu} = (1 - e^{-1/N}) \sum_{n=0}^{\infty} e^{-n/N} |n\rangle \langle n|,$$

where $|n\rangle$ with $n \in \mathbb{N}_0$ denotes the $n$th energy eigenstate of the oscillator. We will first use dimensionless position and momentum variables

$$X' := \frac{1}{\sqrt{2}} (a + a^\dagger) = \sqrt{m\omega} X$$

(15)

and

$$P' := \frac{1}{i\sqrt{2}} (a - a^\dagger) = \frac{1}{\sqrt{m\omega}} P,$$

(16)

with creation operator $a^\dagger$ and annihilation operator $a$. In these coordinates, the covariance matrix of the thermal state with average phonon number $N$ is the identity matrix times $(N + 1)/2$, this follows, e.g., from Eqs. (2.16) in Ref. 27. In natural units, we have therefore the covariance matrix

$$\frac{N + 1}{2} \left( \begin{array}{cc} \frac{1}{m\omega} & 0 \\ 0 & m\omega \end{array} \right).$$

(17)

Comparing Eq. (14) to Eq. (17) we obtain

$$(m\omega)^2 = \frac{4/\sigma_1^2 + 1/\sigma_2^2}{\sigma_1^2 + 4\sigma_2^2}$$

and

$$(N + 1)^2 = \frac{1}{4} \left( \frac{\sigma_1^2}{\sigma_1^2 + 4\sigma_2^2} \right) \left( \frac{1}{\sigma_1^2} + \frac{1}{4\sigma_2^2} \right).$$

Hence

$$N = \frac{1}{2} \sqrt{2 + \frac{4\sigma_2^2}{\sigma_1^2} + \frac{\sigma_1^2}{4\sigma_2^2}} - 1.$$

For $\sigma_1 = 2\sigma_2$ one obtains $N = 0$, i.e., the ground state of the oscillator which corresponds to a rank-one operator $\mu$. We rephrase the findings implied by the above discussion as a theorem:

**Theorem 15** Given three particles such that their masses satisfy

$$m_1 \ll m_3 \ll m_2.$$
Let the first and the second particle be in Gaussian states with real wave functions such that their widths satisfy $\sigma_1 \geq 2\sigma_2$. Then the sequence of scattering processes depicted in Fig. 5 implements the Heisenberg-Weyl POVM $(U_{s,t} \mu U_{s,t}^\dagger)_{s,t}$ in a minimally-disturbing way when the position of particle 1 and the momentum of particle 2 is measured and the result $(x, y)$ is interpreted as $t = -x/2$ and $s = -y$. The initial operator $\mu$ of the POVM is given by

$$\mu = \frac{1}{2\pi} \rho_{N,m\omega},$$

where $\rho_{N,m\omega}$ is the thermal equilibrium state of a harmonic oscillator with mass $m$ and frequency $\omega$ when the temperature is chosen such that the average phonon number is $N$. The parameters $N$ and $m\omega$ are determined by the widths $\sigma_1$ and $\sigma_2$ according to

$$(m\omega)^2 = \frac{4/\sigma_1^2 + 1/\sigma_2^2}{\sigma_1^2 + 4\sigma_2^2},$$

and

$$N = \frac{1}{2} \sqrt{2 + \frac{4\sigma_2^2}{\sigma_1^2} + \frac{\sigma_1^2}{4\sigma_2^2}} - 1.$$  

We want to briefly explain qualitatively how the measured POVM is tuned by the parameters $\sigma_1$ and $\sigma_2$. The ratio of both determine the purity of $\mu$, for $\sigma_1 = 2\sigma_2$ we obtain a rank-one measurement. By increasing or decreasing both we can achieve a better resolution in momentum space or in position space: Small values $\sigma_1, \sigma_2$ lead to good position measurements for the cost of having large errors in the momentum measurement. If $\sigma_1 \gg 2\sigma_2$ both position and momentum measurements are bad and we obtain a measurement with small disturbance.

For detailed discussions on the disturbance and accuracy of the measurements we refer also to Refs. 22, 23. It is shown that the outcomes for position and momentum in POVMs of the above type satisfy the inequality $\Delta x \Delta p \geq \hbar$ in contrast to the Heisenberg uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

### 6 Comparison to quantum optics implementations

There are meanwhile several methods known to measure the quadrature amplitudes of a light mode simultaneously (see Ref. 24). We will consider the scheme shown in Fig. 6 which has some nice similarities to the continuous analogue of our circuit in Fig. 2. In the following, we will use the dimensionless formal position and momentum operators as given in Eqs. (15) and (16) which generate the momentum and position translations in the Heisenberg-Weyl group and define furthermore implicitly...
a Schrödinger representation of a single mode state as wave function $\psi \in L^2(\mathbb{R})$ in position state.

The method in Fig. 6 uses also two ancilla modes. As in our proposal, the entanglement of the two modes tunes the POVM operator. One part of an entangled two-mode state (wire 2 and 3 in Fig. 6) interferes with the input state (wire 1) in a beam splitter. One of its output modes is subjected to a position measurement, the other to a momentum measurement. The results of the measurement determine furthermore displacements performed on the second component of the entangled input. The idea behind the scheme is to perform a teleportation using a non-maximally entangled bipartite state (a maximally entangled state does not exist anyway in continuous variables) as resource. Then the transfer of quantum information is not perfect but the measurements performed during the “bad” teleportation provide some information on the input state. Similar to our scheme, the more entangled the joint state, the less information provides the measurement and the less disturbance on the output state will be observed.

One technical difference to our scheme is that the input and output are not on the same wire. The main difference is, however, that the interaction between input and entangled ancilla is given by a beam-splitter, whereas we use scattering processes. This is geometrically the difference between a rotation or a shear in coordinate space (for details see Ref. 25). Note, however, that the effect of the controlled displacements of Fig. 6 could be mimicked by a controlled-$X$ gate from wire 1 to 3 and a controlled-$Z$ gate from wire 2 to 3 if the latter gate is conjugated by a Fourier transform on wire 2. The reason is that it does not make a difference whether the controlled operation is performed before the measurement or afterwards. The scheme contains therefore quite similar elements as ours.

In Ref. 24 we did not find an explicit remark saying that their implementation scheme is minimally-disturbing in the sense considered here since the authors use the term “minimally-disturbing” in a different sense. Furthermore, the attention was restricted to Gaussian states for both the input as well as for the ancilla states. We have observed that the implementation is also applicable for non-Gaussian states and non-Gaussian POVM operators:

**Theorem 16** The scheme of Ref. 6 can in principle be used for a minimally-disturbing implementation of any Heisenberg-Weyl symmetric POVM

$$U_{s,t} \mu U_{s,t}^\dagger$$

by preparing the ancilla state

$$\sum_j \sqrt{\lambda_j} |\alpha_j\rangle \otimes |\alpha_j\rangle,$$
Figure 6: Measurement scheme of Ref. 24. The box at the beginning of mode 2 and 3 indicate the entangled input on these two modes. The entanglement tunes the POVM. In the limit of infinite entanglement the output coincides with the input state and no information is gained. If mode 2 and 3 start in a product state, a rank-one POVM is implemented. The input interferes with mode 2 in a balanced beam-splitter were one output mode is subjected to a position measurement and the other to a momentum measurement. The results determine the displacements in position and momentum the output is subjected to.
where the $|\alpha_j\rangle$ denote the eigenvector basis for $\mu$ and $\lambda_j/(2\pi)$ the corresponding eigenvalues.

Proof: Due to the linearity argument in Lemma 3 we may prove our statement for the case that the two ancillas are in a product state. The initial three mode wave function is then given by
\[ \phi(x, y, z) = \psi(x) \alpha(y) \beta(z), \]
where $\psi$ is the wave function of the mode to be measured. The beam splitter transfers it to the wave function
\[ \psi\left(\frac{x+y}{\sqrt{2}}\right) \alpha\left(\frac{x-y}{\sqrt{2}}\right) \beta(z). \]
We simulate the momentum measurement by an inverse Fourier transform followed by a position measurement. Conditioned on the measurement result $(x, y)$ we obtain therefore a one-mode wave function (having $z$ as argument) which is given by
\[
\sqrt{\frac{1}{2\pi}} \left\{ \int_{-\infty}^{\infty} \psi\left(\frac{x+w}{\sqrt{2}}\right) \alpha\left(\frac{-x+w}{\sqrt{2}}\right) e^{iwy} \, dw \right\} \beta(z).
\]
After obtaining the measurement results $x$ and $y$ on wire 1 and 2, respectively, the conditioned displacement of position and momentum by $\sqrt{2}x$ and $\sqrt{2}y$, respectively, leads to
\[
\sqrt{\frac{1}{2\pi}} \left\{ \int_{-\infty}^{\infty} \psi(w) \alpha(-\sqrt{2}x + w) e^{i\sqrt{2}y} \, dw \right\} \beta(z - \sqrt{2}x) e^{-iz\sqrt{2}y}.
\]
This shows that the unnormalized state vector of the third particle, given that $x, y$ was measured, reads
\[ U_{\sqrt{2}y, \sqrt{2}x} |\beta\rangle \langle \alpha| U_{\sqrt{2}y, \sqrt{2}x}^\dagger |\psi\rangle. \]
After taking into account that the quantum optics convention for position and momentum differs from the canonical definition of Eq. (15) and (16) by the factor $\sqrt{2}$ (see Ref. 27), this is exactly the desired output state.

By choosing the input $|\alpha\rangle \otimes |\alpha\rangle$ we have therefore $\mu = |\alpha\rangle \langle \alpha| = \sqrt{\mu}$. Similarly we can obtain operators $\mu$ with higher rank by choosing entangled input states. 

Note that the calculations which show that the scheme does indeed implement a minimally-disturbing POVM is very similar to the calculations in Sec. 5 which shows the close formal analogy of both methods.
7 Conclusions

We have presented a general scheme to implement minimally-disturbing symmetric measurements by quantum circuits. By applying it to the Heisenberg-Weyl group, we obtain circuits for simultaneous measurements of position and momentum of a particle moving on a discrete cyclic chain. We show that an infinite dimensional generalization of this circuit leads to a well-defined measurement process on a Schrödinger particle moving on the real line using two probe particles. The “circuit” for this continuous variable quantum system can in principle be obtained by particle collisions with hard-core potential. The whole measurement process on the three particles shows some analogies but also differences to simultaneous measurements of the quadrature amplitudes in quantum optics using two ancilla modes.

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Appendix

Proof of Th. 10

The \((2n \times 2)\)-matrix \(M\) of Eq. (2) has the \((\sigma_\pi \otimes \sigma, \sigma)\)-symmetry that is defined by

\[(X_n \otimes R_n)M = MR_n,
\]
as straightforward computation shows, i.e., we have \((\sigma_\pi \otimes \sigma)(j) = X_n^j \otimes R_n^j\) and \(\sigma(j) = R_n^j\). To find the diagonalizing operations \(A\) and \(B\) of Th. 9 we observe that \(\sigma\) is already decomposed into irreducible representations and \(B\) is therefore trivial. To decompose \(\sigma_\pi \otimes \sigma\) we diagonalize \(\sigma_\pi\) by the Fourier transform \(F_n\). The eigenvalues of \((F_n X_n F_n^\dagger) \otimes R_n\) are in the order 1, \(\omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\). We apply the cyclic shift \(X_{2n}\) to group them into a sequence of pairs \((\omega_n^j, \omega_n^j)\) as in Lemma 6. Therefore, we have \(A = X_{2n}(F_n \otimes I_2)\). Following Th. 9 we only have to find \(W \in \text{Int}(A(\sigma_\pi \otimes \sigma)A^\dagger, B\sigma B^\dagger + \sigma')\) which is a unitary extension of \(N := A^\dagger M B\). Hence, we can choose

\[
W := \sqrt{\frac{1}{2}} \begin{pmatrix}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1
\end{pmatrix},
\]

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since the first two columns of this matrix coincide with $N$. This is verified by straightforward computations, too. One can also easily check that $W$ can be written as $W = (I_n \otimes F_2) K^\dagger$ where $K$ is defined in Th. 10.

**Proof of Th. 12**

To define $M$ as in Eq. (2) we have to define a correspondence between ancilla basis states and POVM operators. Since our ancilla system is a tensor product of two $d$-dimensional systems, this correspondence is canonical and we obtain

$$M = \sum_{j, k=0}^{d-1} |j\rangle \otimes |k\rangle \otimes Z_d^j X_d^j \sqrt{\mu} X_d^{-j} Z_d^{-k} \in \mathbb{C}^{d^3 \times d^3}.$$  \hspace{1cm} (18)

The symmetry $(\sigma_\pi \otimes \sigma) M = M \sigma$ of $M$ is defined by

$$(I_d \otimes X_d \otimes Z_d) M = M Z_d \text{ and } (X_d \otimes I_d \otimes X_d) M = M X_d,$$

as straightforward computations show. Following Th. 9 we decompose the representation on the left side into a direct sum of irreducible representations. First of all, we diagonalize the shifts $X_d$ in the first and second tensor components by the Fourier transform. We obtain

$$(I_d \otimes Z_d \otimes Z_d) (F_d \otimes F_d \otimes I_d) M = (F_d \otimes F_d \otimes I_d) M Z_d$$

and

$$(Z_d \otimes I_d \otimes X_d) (F_d \otimes F_d \otimes I_d) M = (F_d \otimes F_d \otimes I_d) M X_d.$$

The matrices on the left side can be written as

$$(I_d \otimes Z_d \otimes Z_d) = \bigoplus_{j=0}^{d^2-1} \omega_d^{j \text{mod } d} Z_d \text{ and } (Z_d \otimes I_d \otimes X_d) = \bigoplus_{j=0}^{d^2-1} \omega_d^{j \text{div } d} X_d.$$  

Therefore, the representation is decomposed into a direct sum of representations that are equal to $\sigma$ up to phase factors. We now eliminate these factors. To simplify notation we define the block diagonal matrices

$$X_{\text{mod}} := \bigoplus_{j=0}^{d^2-1} X_d^{j \text{mod } d} \text{ and } Z_{\text{div}} := \bigoplus_{j=0}^{d^2-1} Z_d^{j \text{div } d}.$$  

Using $Z_d^{\dagger} X_d Z_d = \omega_d^{-1} X_d$ and $X_d Z_d X_d^{\dagger} = \omega_d^{-1} Z_d$ we obtain

$$X_{\text{mod}} \left( \bigoplus_{j=0}^{d^2-1} \omega_d^{j \text{mod } d} Z_d \right) X_{\text{mod}}^{\dagger} = \bigoplus_{j=0}^{d^2-1} Z_d.$$
and

\[ Z_{\text{div}}^\dagger \left( \bigoplus_{j=0}^{d^2-1} \omega_d^j \text{div} X_d \right) Z_{\text{div}} = \bigoplus_{j=0}^{d^2-1} X_d. \]

Using both equations we can write

\[ X_{\text{mod}} Z_{\text{div}}^\dagger (I_d \otimes Z_d \otimes Z_d) Z_{\text{div}} X_{\text{mod}}^\dagger = (I_d \otimes I_d \otimes Z_d) \]

and

\[ X_{\text{mod}} Z_{\text{div}}^\dagger (Z_d \otimes I_d \otimes X_d) Z_{\text{div}} X_{\text{mod}}^\dagger = (I_d \otimes I_d \otimes X_d), \]

where we have no phase factors. Consequently, we obtain

\[ (I_d \otimes I_d \otimes Z_d) X_{\text{mod}} Z_{\text{div}}^\dagger (F_d \otimes F_d \otimes I_d) M = X_{\text{mod}} Z_{\text{div}}^\dagger (F_d \otimes F_d \otimes I_d) M Z_d \]

and

\[ (I_d \otimes I_d \otimes X_d) X_{\text{mod}} Z_{\text{div}}^\dagger (F_d \otimes F_d \otimes I_d) M = X_{\text{mod}} Z_{\text{div}}^\dagger (F_d \otimes F_d \otimes I_d) M X_d. \]

We can rewrite this as

\[ (I_d \otimes I_d \otimes \sigma) N = N \sigma \]

with \( N = X_{\text{mod}} Z_{\text{div}}^\dagger (F_d \otimes F_d \otimes I_d) M \). Hence, using the notation of Th. 9 we have

\[ A := X_{\text{mod}} Z_{\text{div}}^\dagger (F_d \otimes F_d \otimes I_d) \quad \text{and} \quad B = I_d \]

since \( \sigma \) is an irreducible representation. The matrix \( N \) is an element of the intertwining space \( \text{Int}(\bigoplus_{j=0}^{d^2-1} \sigma, \sigma) \). Following Lemma 7 it has the decomposition

\[ N = \Phi_1 \otimes I_d \in \mathbb{C}^{d^3 \times d^2} \]

with \( \Phi_1 \in \mathbb{C}^{d^2} \). Elementary but cumbersome computations\(^d\) show

\[ |\Phi_1\rangle = (F_d^\dagger \otimes I_d) \left( \sum_{q=0}^{d-1} |q\rangle \otimes X_d^{-q} \right) \left( \sqrt{d} \sum_{j,k=0}^{d-1} \mu_{jk} |j\rangle \otimes |k\rangle \right). \]

We extend the representation \( \sigma \) on the right side of Eq. (19) to the direct sum of \( d^2 \) copies of \( \sigma \). The matrix \( W \) of the resulting intertwining space has the decomposition \( C \otimes I_d \) with \( C \in \mathbb{C}^{d^2 \times d^2} \). Therefore, we extend \( \{|\Phi_1\rangle, |\Phi_2\rangle, \ldots, |\Phi_{d^2}\rangle\} \) to an orthonormal basis \( \{|\Phi_1\rangle, |\Phi_2\rangle, \ldots, |\Phi_{d^2}\rangle\} \) of \( \mathbb{C}^{d^2} \). We can define the unitary

\[ U := A^\dagger W (B \oplus \bar{B}) = A^\dagger \left( |\Phi_1\rangle |\Phi_2\rangle \ldots |\Phi_{d^2}\rangle \right) \otimes I_d \]

\(^d\)Write \( \sqrt{\mu} = \sum_{j=0}^{d-1} X_d^j \Delta_j \) with appropriate diagonal matrices \( \Delta_j \) and powers of the shift \( X_d \).
that extends $M$ with $\tilde{B} := I_{(n-1)d}$. Now we show how to simplify the implementation by preparing an appropriate ancilla state. We have

$$U(|0\rangle \otimes |\Psi\rangle) = A^\dagger \left((|\Phi_1\rangle|\Phi_2\rangle \ldots |\Phi_d\rangle) \otimes I_d \right) (|0\rangle \otimes |\Psi\rangle) = A^\dagger (|\Phi_1\rangle \otimes |\Psi\rangle).$$

Hence, we can omit the implementation of $W$ if we initialize the ancilla with $|\Phi_1\rangle$ of Eq. (20). In summary, we have to implement the unitary

$$(F_d^\dagger \otimes F_d^\dagger \otimes I_d) Z_{\text{div}} X_{\text{mod}}(F_d^\dagger \otimes I_d \otimes I_d) \left(\sum_{q=0}^{d-1} |q\rangle \langle q| \otimes X_{d}^{-q} \otimes I_d \right)$$

after we have initialized the ancillas with the state vector

$$|\gamma\rangle := \sqrt{d} \sum_{j,k=0}^{d-1} \sqrt{\mu_{jk}} |j\rangle \otimes |k\rangle \in \mathbb{C}^{d^2}. \quad (21)$$

As a special case consider the initial operator $\mu = |\alpha\rangle \langle \alpha| / d$ with $|\alpha\rangle \in \mathbb{C}^d$ and $\langle \alpha| \alpha \rangle = 1$. In this case we have $\sqrt{\mu} = |\alpha\rangle \langle \alpha| / \sqrt{d}$. Furthermore, we have

$$|\gamma\rangle = |\alpha\rangle \otimes |\bar{\alpha}\rangle.$$

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