Union-Closed vs Upward-Closed Families of Finite Sets

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Abstract

A finite family $\mathcal{F}$ of subsets of a finite set $X$ is union-closed whenever $f, g \in \mathcal{F}$ implies $f \cup g \in \mathcal{F}$. These families are well known because of Frankl’s conjecture [10]. In this paper we developed further the connection between union-closed families and upward-closed families started in [18] using rising operators. With these techniques we are able to obtain tight lower bounds to the average of the length of the elements of $\mathcal{F}$ and to prove that the number of joint-irreducible elements of $\mathcal{F}$ can not exceed $2^{\left\lfloor \frac{n}{2} \right\rfloor} + \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$ where $|X| = n$.

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1 Introduction

Consider a finite set $X = \{a_1, \ldots, a_n\}$ formed by $n \geq 1$ elements. A family of subsets $F$ of the powerset $2^X$ such that for any $f, g \in F$, $f \cup g \in F$ is called union-closed (briefly $\cup$-closed). Without loss of generality we can assume that $X = \bigcup_{f \in F} f$ and for the rest of the paper, when it is not differently stated, $F$ will denote a $\cup$-closed family on $X = \{a_1, \ldots, a_n\}$ with $X = \bigcup_{f \in F} f$. In 1979, Frankl stated the following conjecture

**Conjecture 1.** For all union-closed families $F$, there exists an $a \in X$ such that $|\{f \in F : a \in f\}| \geq \frac{|F|}{2}$.

Although many attempts to solve this simple-sounding conjecture have been made, this remains open and has become known as the union-closed conjecture or Frankl’s conjecture. A simple argument in [14] shows that there is an $a \in X$ which is contained in at least $\frac{|F|}{\log_2(|F|)}$ elements of $F$. In [22], this bound is improved by a multiplicative constant. The conjecture holds if $|F| < 40$ (see [15, 20]) or $|X| \leq 11$ (see [6, 16]) or $|F| > \frac{52}{8}|X|$ (see [7, 8, 9]) or $F$ contains some collection of small sets (see [6, 16]).

The family $F$ is a semilattice with respect to the union operation, furthermore, since $F$ is finite we can endow $F \cup \{\emptyset\}$ with a structure of lattice. In this direction it is possible to give another formulation of Frankl’s conjecture in the framework of lattice theory. Let $(L, \lor, \land)$ be a finite lattice, we denote by $J(L)$ the set of join-irreducible elements, i.e., the elements $z \in L$ such that if $g = x \lor y$ then $x = z$ or $y = z$. Denoting by $V_x = \{y \in L : y \leq x\}$ the principal filter generated by $x$, Frankl’s conjecture is equivalent to the following lattice theoretic conjecture

**Conjecture 2.**

$$\frac{1}{|L|} \min\{|V_x| : x \in J(L)\} \leq \frac{1}{2}.$$

This approach has received a significant amount of attention (see [1, 2, 3, 4, 7, 8, 9, 11, 17, 19, 21]). Although much research has been done on union-closed families, it seems there is no general tool to tackle this problem.

In a different direction, Reimer in [18] developed a connection between $\cup$-closed sets and upward-closed sets by using a repeated application of rising
operators. From this connection he provided a lower bound on the average of the length of the elements of \( \mathcal{F} \) showing

\[
\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{\log_2(|\mathcal{F}|)}{2}.
\]

The aim of this paper is to develop further the correspondence introduced by Reimer and study more deeply the consequences and some of the results that can be achieved from this point of view. The hope is to give an approach to the study of \( \cup \)-closed families of sets that can be helpful to give some insight to a possible solution of the Conjecture \( \text{(1)} \).

The paper is organized as following: in Section 2 we give some definitions and we fix the notation, in Sections 3 and 4 we extend the results of [18], in Section 5 we use this approach to obtain some lower bounds on the localized average of the length of the elements of \( \mathcal{F} \). More precisely, given \( S \subseteq \mathcal{F} \), we provide lower bounds to the quantity

\[
\frac{1}{|\{f \in \mathcal{F} : \exists z \in S, z \subseteq f\}|} \sum_{f \in \mathcal{F} : \exists z \in S, z \subseteq f} |f|
\]

Finally in Section 6 we use these techniques to prove that the number of joint-irreducible elements of \( \mathcal{F} \subseteq 2^X \) is at most \( 2^{\left(\binom{n}{\lfloor n/2 \rfloor}\right)} + \left(\binom{n}{\lfloor n/2 \rfloor} + 1\right) \) where \( |X| = n \).

## 2 Preliminaries

For an element \( t \in 2^X \) we denote the cardinality of \( t \) by \( |t| \). Let us fix a subset \( S \subseteq 2^X \) (note that the cardinality of \( S \) is also denoted by \( |S| \)). Let \( a \in X \), \( S \) is partitioned into two subsets \( S_a, S_{\neg a} \) of the elements of \( S \) containing \( a \), not containing \( a \), respectively. We can see \( S \) endowed with the order induced by the relation \( \subseteq \) as a poset \((S, \subseteq)\), thus an antichain \( A \subseteq S \) is a non-empty subset such that any pair of elements of \( A \) is incomparable. We denote the set of minimal (maximal) elements of \( S \) by \( \text{min}(S) \) (\( \text{max}(S) \)). Both \( \text{min}(S), \text{max}(S) \) are clearly antichains. Let us denote the set of all \( f^c \) for \( f \in S \), where \( f^c = X \setminus f \) is the complement set of \( f \), by \( S^c \). Given two different elements \( f, g \in S \) we say that \( g \) covers \( f \), written \( f \preceq g \) if there is no \( h \in S \) such that \( f \subseteq h \subseteq g \).

An upward-closed family (also called upset or filter, see [5]) is a subset \( \mathcal{F} \subseteq 2^X \) such that if \( f \subseteq g \) for some \( f \in \mathcal{F}, g \in 2^X \), then \( g \in \mathcal{F} \), and a downward-closed family (also downset or simplicial complex) is defined
analogously. Note that an upward-closed family is a \( \cup \)-closed set. Let us consider a \( \cup \)-closed family \( \mathcal{F} \) of \( 2^X \). An ideal of \( \mathcal{F} \) is a subset \( I \subseteq \mathcal{F} \) such that \( I \cup g \subseteq I \) for all \( g \in \mathcal{F} \). Let \( z \in 2^X \), the principal ideal of \( z \) denoted by \( \mathcal{F}[z] \), is the (possibly empty) set of all the elements of \( \mathcal{F} \) containing \( z \). Clearly if \( z \in \mathcal{F} \), then \( \mathcal{F}[z] \) is the principal ideal generated by \( z \), i.e., \( \mathcal{F}[z] = z \cup \mathcal{F} = \{z \cup f, f \in \mathcal{F}\} \). This definition can be extended to any sub-family \( S \subseteq \mathcal{F} \), thus the principal ideal generated by \( S \) is the set \( \mathcal{F}[S] = \cup_{z \in S} \mathcal{F}[z] \). It is straightforward to see that \( \mathcal{F}[\min(\mathcal{F})] = \mathcal{F} \).

In the particular case \( \mathcal{F} = 2^X \), we use the shorter notation \( S^\uparrow \) for the set \( 2^X[S] \).

An element \( g \in \mathcal{F} \) is called irreducible whenever \( g = h \cup t \) implies \( h = g \) or \( t = g \). We denote by \( J(\mathcal{F}) \) the set of irreducible elements of \( \mathcal{F} \). It is evident that \( \min(\mathcal{F}) \subseteq J(\mathcal{F}) \). This set plays an important role since it is the minimal set of generators of the semilattice \( (\mathcal{F}, \cup) \). This set is \( \cup \)-independent, in the following sense. Given \( S \subseteq 2^X \), we say that \( S \) is \( \cup \)-independent whenever no element \( z \in S \) can be written as a union of elements in \( S \setminus \{z\} \).

### 3 Union-closed and upward-closed families

In this section we further explore the connection between \( \cup \)-closed families and upward-closed families. This connection has been already established in [18] using the concept of rising function, a well-known operator used also by Frankl in [13]. We briefly recall such operator. Given an element \( a \in X \) and a family (not necessarily \( \cup \)-closed) of subsets \( S \subseteq 2^X \), the rising function \( \varphi_{S,a} : S \to 2^X \) is the function defined for all \( z \in S \) by

\[
\varphi_{S,a}(z) = \begin{cases} 
    z \cup \{a\} & \text{if } z \cup \{a\} \notin S, \\
    z & \text{otherwise}; 
\end{cases}
\]

This is a one-to-one function \( \varphi_{S,a} : S \to 2^X \) and the image \( \varphi_{S,a}(S) \) is called the \( a \)-rising of \( S \). In [13], the author iterates these rising functions in the following way. Let \( X = \{a_1, \ldots, a_n\} \) and let \( \varphi_0 \) be the identity function on \( 2^X \), and \( S_0 = S \), then for all \( 1 \leq j \leq n \) let

\[
S_j = \varphi_j(S_{j-1}), \quad \varphi_j = \varphi_{S_{j-1},a_j} \circ \varphi_{j-1}
\]

We call \( \varphi_w \) the rising function with respect to the word \( w = a_1a_2\ldots a_n \) of the family \( S \) and we denote it by \( \varphi_w \) to underline the dependency of this map from the order used to perform these iterations. We call each \( S_i \) the
\(i\)-section and for any \(z \in S\) the elements \(z_i = \varphi_i(z)\) for \(i = 0, \ldots, n\) is called the trajectory of \(z\) through the iterated application of the rising functions.

We immediately note that this definition depends on the order in which the rising functions are iterated. Indeed consider the set \(X = \{a, b, c\}\) and the \(\cup\)-closed family \(\mathcal{F} = \{\{a\}, \{a, b, c\}\}\), it is evident that \(\varphi_w(\mathcal{F}) \neq \varphi_{w'}(\mathcal{F})\) when \(w = abc, w' = acb\). We denote by \(\mathfrak{S}_X\) the permutation group of the set of objects \(X = \{a_1, \ldots, a_n\}\), and for a word \(w = \alpha_1 \ldots \alpha_n\), we use the notation \(w\theta = \theta(\alpha_1) \ldots \theta(\alpha_n)\).

There is an evident action of \(\mathfrak{S}_X\) on the set \(\{\varphi_{w\theta}(g) : g \in \mathcal{F}, \theta \in \mathfrak{S}_X\}\) given by \(\sigma \cdot \varphi_{w\theta}(g) = \varphi_{w\theta \sigma}(g)\). In Section 4 we characterize the orbits of such action and we explore some consequences.



Lemma 1. With the notation above, if there are two different elements \(z, z' \in S\) such that \(\varphi_i(z) = \varphi_i(z') \cup \{a_i+1\}\), then \(a_i+1 \in z \setminus \varphi_w(z')\).



Proof. If \(a_i+1 \notin z\), then \(a_i+1 \notin \varphi_k(z)\) for all \(k \leq i\), but this contradicts \(\varphi_i(z) = \varphi_i(z') \cup \{a_i+1\}\), thus \(a_i+1 \in z\). Since \(\varphi_i\) is a bijection and \(\varphi_i(z') = \varphi_i(z') \cup \{a_i+1\}\) with \(z \neq z'\), we get \(a_i+1 \notin \varphi_i(z')\) and so \(a_i+1 \notin \varphi_w(z')\).

If we add the \(\cup\)-closed condition, we have the following lemma:

Lemma 2. [18] Let \(\mathcal{F}\) be a \(\cup\)-closed family of subsets of \(X = \{a_1, \ldots, a_n\}\). For each \(0 \leq i \leq n\) the \(i\)-section \(\mathcal{F}_i\) is a \(\cup\)-closed family.

The following lemma is a consequence of [18] Lemma 3.3, but for the sake of completeness we present here with proof.

Lemma 3. Let \(\mathcal{F}\) be a \(\cup\)-closed family of sets and let \(f \in \mathcal{F}\). Consider the rising function \(\varphi_w\) with respect to the word \(w = a_1a_2 \ldots a_n\) of the family \(\mathcal{F}\). Then if \(t\) belongs to the \(i\)-section \(\mathcal{F}_i\), for some \(1 \leq i \leq n\), then also \(t \cup f \in \mathcal{F}_i\).

Proof. We prove it by induction on the index \(i\). Suppose that \(i = 0\), since \(\mathcal{F}_0 = \mathcal{F}\) is a \(\cup\)-closed family, if \(t \in \mathcal{F}\) then, since \(f \in \mathcal{F}\), \(t \cup f \in \mathcal{F} = \mathcal{F}_0\).

Suppose that the statement of the theorem is true for \(i\) and let us prove it for \(i+1\). Suppose \(t \in \mathcal{F}_{i+1}\) and let \(\overline{t} = \varphi_{\mathcal{F}_{i+1}}^{-1}(t) \in \mathcal{F}_i\). By the inductive hypothesis \(\overline{t} \cup f \in \mathcal{F}_i\), we consider several cases.

Case 1. Suppose \(a_{i+1} \in \overline{t}\). Thus \(a_{i+1} \in \overline{t} \cup f \in \mathcal{F}_i\), hence \(\overline{t} = \varphi_{\mathcal{F}_{i+1}}(\overline{t}) = t\) and \(\varphi_{\mathcal{F}_{i+1}}(\overline{t} \cup f) = \overline{t} \cup f = t \cup f\) and so \(t \cup f \in \mathcal{F}_{i+1}\).

Case 2. Suppose \(a_{i+1} \notin \overline{t}\). We consider two further subcases.
• $a_{i+1} \in t$, hence necessarily by definition of the rising function $\varphi_{F, a_{i+1}}$, $\overline{t} \cup \{a_{i+1}\} \notin F$. Therefore $\varphi_{F, a_{i+1}}(\overline{t}) = \overline{t} \cup \{a_{i+1}\} = t$ and, if $\overline{t} \cup f \cup \{a_{i+1}\} \notin F$ then $\varphi_{F, a_{i+1}}(\overline{t} \cup f) = \overline{t} \cup f \cup \{a_{i+1}\} = t \cup f \in F_{i+1}$. Otherwise $\overline{t} \cup f \cup \{a_{i+1}\} \in F_i$, thus $t \cup f \in F_i$, hence $\varphi_{F, a_{i+1}}(t \cup f) = t \cup f \in F_{i+1}$.

• $a_{i+1} \notin t$, hence necessarily $\overline{t} \cup a_{i+1} \in \mathcal{F}_i$ and $t = \overline{t}$. Thus, if $a_{i+1} \in \overline{t} \cup f$ then $\varphi_{F, a_{i+1}}(\overline{t} \cup f) = \overline{t} \cup f = t \cup f \in F_{i+1}$. Otherwise $a_{i+1} \notin \overline{t} \cup f$.

Since $\overline{t} \cup a_{i+1} \in F_i$, then by the inductive hypothesis $\overline{t} \cup a_{i+1} \cup f \in F_i$ whence $\varphi_{F, a_{i+1}}(\overline{t} \cup f) = \overline{t} \cup f = t \cup f \in F_{i+1}$.

\[ \square \]

The following theorem establishes an interesting property of the associate upward-closed family $\mathcal{F} = \varphi_w(\mathcal{F})$.

**Theorem 1.** For each $f \in \mathcal{F}$, $\varphi_w$ is a bijection between the principal ideals

$\varphi_w : \mathcal{F}[f] \to \mathcal{F}[f]$

**Proof.** Since $\varphi_w$ is a one to one function it is sufficient to prove that $\varphi_w : \mathcal{F}[f] \to \mathcal{F}[f]$ is also surjective. Since $f \in \mathcal{F}$, then $\varphi_w(\mathcal{F}[f]) \subseteq \mathcal{F}[f]$, in particular $\mathcal{F}[f]$ is non-empty. Consider an element $\eta \in \mathcal{F}[f]$ and let $\eta^* = \varphi_w^{-1}(\eta)$. We claim that $f \subseteq \eta^*$ and so $\eta^* \in \mathcal{F}[f]$. Suppose, contrary to our claim, that $f \notin \eta^*$. Let $\eta_0 = \eta^*$ and $\eta_i = \varphi_i(\eta_0)$ for $i = 1, \ldots, n$ be the trajectory of $\eta^*$. Since $f \subseteq \eta = \eta_n$, there is a minimal index $j \leq n$ such that $f \subseteq \eta_j$ and $j > 0$ ($f \notin \eta_0$). By the minimality of $j$, $f \notin \eta_{j-1}$. Since $f \subseteq \eta_j$, we have $a_j \notin \eta_{j-1}$ and so $a_j \in f$. By Lemma 3 since $\eta_{j-1} \in \mathcal{F}_{j-1}$, we get also $\eta_{j-1} \cup f \in \mathcal{F}_{j-1}$. Therefore, since $a_j \in f$, $f \subseteq \eta_{j-1} \cup a_j$ and so $\eta_{j-1} \cup a_j = \eta_{j-1} \cup f \in \mathcal{F}_{j-1}$, hence $f \notin \eta_j = \eta_{j-1}$, a contradiction. \[ \square \]

We have the following corollary:

**Corollary 1.** For each $S \subseteq \mathcal{F}$, $\varphi_w$ is a bijection between the principal ideals

$\varphi_w : \mathcal{F}[S] \to \mathcal{F}[S]$

Moreover the inverse of $\varphi_w : \mathcal{F} \to \mathcal{F}$ is given by

$\varphi_w^{-1}(\eta) = \bigcup_{\{f \in \mathcal{F} : f \subseteq \eta\}} f$
Proof. Since $\varphi_w$ is injective, it is sufficient to prove that it is also surjective. Thus, consider $\eta \in F[S]$, then there is an $f \in S$ such that $f \subseteq \eta$, and so $\eta \in F[f]$. Therefore, by Theorem 1 $\varphi_w^{-1}(\eta) \in F[f] \subseteq F[S]$. Let us prove the last statement, so consider an element $\eta \in F$. By the previous statement $F = F[\min(F)]$. Hence the set $\{f \in F : f \subseteq \eta\}$ is non-empty and so, since $F$ is union-closed:

$$\bigcup_{\{f \in F : f \subseteq \eta\}} f = \eta^* \in F$$

By Theorem 1 and $\eta^* \subseteq \eta$, we get $\varphi_w^{-1}(\eta) \subseteq \eta$ and $\eta^* \subseteq \varphi_w^{-1}(\eta)$. Therefore we get $\varphi_w^{-1}(\eta) \subseteq \eta^* \subseteq \varphi_w^{-1}(\eta)$, i.e. $\eta^* = \varphi_w^{-1}(\eta)$.

We give a lemma useful in the sequel.

**Lemma 4.** The map $\psi(z) = z \cup \{a\}$ is an embedding

$$\psi : F_\pi \hookrightarrow \varphi_w(F_a)$$

Proof. Since $\psi : F_\pi \rightarrow 2^X$ is already injective, it is sufficient to prove $\psi(F_\pi) \subseteq \varphi_w(F_a)$. Suppose, contrary to our claim, that there is $\eta \in F_\pi$ such that $z = \varphi_w^{-1}(\eta \cup \{a\})$ is not in $F_a$. Since $a \notin z$ and $z \subseteq \eta \cup \{a\}$, then $z \subseteq \eta$. Let $z' = \varphi_w^{-1}(\eta)$, since $z \subseteq \eta$, then by Theorem 1 we get $z \subseteq z'$. On the other side, since $z' \subseteq \eta \subseteq \eta \cup \{a\}$ then by Theorem 1 $z' \subseteq \varphi_w^{-1}(\eta \cup \{a\}) = z$, whence $z = z'$ which implies $\eta = \eta \cup \{a\}$, a contradiction.

We say that $g \in F$ is fixed by $\varphi_w$ whenever $\varphi_w(g) = g$ holds. The following proposition characterized the elements of $F$ with this property.

**Proposition 1.** $\varphi_w(g) = g$ if and only if $g \cup a \in F$ for all $a \in X$. Moreover if $S \subseteq F$ then $F \cap S$ is the set of elements of $S$ fixed by $\varphi_w$.

Proof. Using the definition of $\varphi_w$ and Lemma it is straightforward to check that if $g \cup a \in F$ for all $a \in X$ then $\varphi_w(g) = g$. Conversely, suppose that $\varphi_w(g) = g$ and let us prove that $g \cup a_i \in F$. Since $\varphi_w(g) = g$, then if $g_i$ is the trajectory of $g$ in the rising process, then $g_i = g$ for all $i = 1, \ldots, n$. In particular $g \cup a_{i+1} \in F_i$ by definition of the rising function $\varphi_{F_i,a_{i+1}}$. By Lemma $a_{i+1} \in \varphi_{i+1}^{-1}(g \cup a_{i+1}) \setminus g$ and by Corollary $g \subseteq \varphi_i^{-1}(g \cup a_{i+1})$, whence $g \cup a_{i+1} \subseteq \varphi_i^{-1}(g \cup a_{i+1}) \subseteq g \cup a_{i+1}$, i.e. $g \cup \{a_{i+1}\} = \varphi_i^{-1}(g \cup a_{i+1}) \in F$. The proof of the last statement of the lemma is also a consequence of Corollary and it is left to the reader.
The last proposition shows that all the upward-closed families of sets are leaved unchanged by the rising operator \( \varphi_w \).

We remind that if \( A \subseteq B \) are two subsets of \( X \) then the interval \([A, B]\) is defined by \( \{D \subseteq X : A \subseteq D \subseteq B\} \). In [13] Lemma 1.3 (ii) the author shows that if \( g \neq f \) are two distinct elements of \( \mathcal{F} \) then \([g, \varphi_w(g)] \cap [f, \varphi_w(f)] = \emptyset \). This facts is independent from the order with which we rise the set, indeed we have the following proposition.

**Proposition 2.** Let \( f, g \in \mathcal{F} \) and \( \sigma, \theta \in S_X \). Then \( f \neq g \) if and only if \([f, \varphi_w(\sigma f)] \cap [g, \varphi_w(\theta g)] = \emptyset \).

**Proof.** Suppose that \( z \in [f, \varphi_w(\sigma f)] \cap [g, \varphi_w(\theta g)] \neq \emptyset \). By Corollary [1] and \( f \subseteq c \subseteq \varphi_w(g) \) we get \( g = \bigcup_{h \in \mathcal{F}, h \subseteq \varphi_w(g)} h \supseteq f \). Changing \( g \) with \( f \) we obtain the other inclusion \( g \subseteq f \), whence \( g = f \). The other side of the implication is trivial.

4 The invariant upward-closed family associated to a union-closed family

In this section we introduce an upward-closed family associated to \( \mathcal{F} \) which do not depend on a parameter like the case obtained using the rising functions in Section [3]. From Theorem [1] we have that \( \varphi_w(\mathcal{F}) \) is an upward-closed family, moreover since the union of upward-closed families is still an upward-closed family, we can associate to \( \mathcal{F} \) the upward-closed family

\[
\mathcal{U}(\mathcal{F}) = \bigcup_{\vartheta \in S_X} \varphi_w(\vartheta(\mathcal{F}))
\]

where \( w = a_1 \ldots a_n \). We call \( \mathcal{U}(\mathcal{F}) \) the *invariant upward-closed family* associated to \( \mathcal{F} \). We have already noted in Section [3] that there is an action of \( S_X \) on this set given by \( \beta \cdot \varphi_w(\vartheta(g)) = \varphi_w(\vartheta(\beta g)) \). So it seems natural to characterize the orbits \( S_X \cdot \varphi_w(g) = \{\varphi_w(\vartheta(g)), \vartheta \in S_X\} \). Before giving this characterization we need first some definitions. The rising function \( \varphi_w \) depends on the parameter \( w \), however by Corollary [1] the inverse of \( \varphi_w \) does not. Moreover, by the same Corollary, \( \varphi_w(\mathcal{F}) \subseteq \min(\mathcal{F})^\uparrow \) and so \( \mathcal{U}(\mathcal{F}) \subseteq \min(\mathcal{F})^\uparrow \). For this reason it is important to extend the map \( \varphi_w^{-1} \) to an operator

\[
\circ^* : \min(\mathcal{F})^\uparrow \to \mathcal{F}
\]

which associates to each element \( z \in \min(\mathcal{F})^\uparrow \) the element

\[
z^* = \bigcup_{\{h \in \mathcal{F}, h \subseteq z\}} h
\]
Using the fact that \( \mathcal{F} \) is \( \cup \)-closed and the domain is \( \min(\mathcal{F})^\uparrow \), it is immediate to see that this operator is well defined. Moreover \( \sigma^* \) preserves inclusion, i.e. if \( z \subseteq y \) then \( z^* \subseteq y^* \) and it is clearly surjective, thus we can define the fiber of each \( g \in \mathcal{F} \) as

\[
\text{Fib}(g) = \{ h \in \min(\mathcal{F})^\uparrow : h^* = g \}.
\]

The following proposition characterizes the union-closed families in term of the operator \( \sigma^* \).

**Proposition 3.** Let \( \mathcal{H} \) be a family of subsets of \( X \) and consider the operator

\[
\sigma^* : \min(\mathcal{H})^\uparrow \to 2^X
\]

defined by sending each \( z \in \min(\mathcal{H})^\uparrow \) into \( z^* = \bigcup_{\{h \in \mathcal{F}, h \subseteq z\}} h \). Then \( \mathcal{H} \) is a \( \cup \)-closed family if and only if the image of \( \sigma^* \) is contained in \( \mathcal{H} \).

**Proof.** As we have already noticed before if \( \mathcal{H} \) is \( \cup \)-closed then \( \sigma^* \) is well defined map \( \sigma^* : \min(\mathcal{H})^\uparrow \to \mathcal{H} \). Conversely, let \( g, h \in \mathcal{H} \) and let \( \mathcal{H}' \subseteq \mathcal{H} \) be the image of \( \mathcal{H} \) by means of \( \sigma^* \). The element \( g \cup h \in \min(\mathcal{H})^\uparrow \) and so \( g \cup h \in \text{Fib}(t) \) for some \( t \in \mathcal{H}' \). Since \( \text{Fib}(t) \) is formed by the elements \( z \) such that \( z^* = t \) and \( t \in \mathcal{H}' \) we have that \( t \subseteq z \) for all \( z \in \text{Fib}(t) \), in particular \( t \subseteq g \cup h \). On the other hand \( g, h \subseteq g \cup h \) and so \( g, h \subseteq (g \cup h)^* = t \), whence \( g \cup h \subseteq t \) and so \( g \cup h = t \in \mathcal{H}' \). \( \square \)

Given a word \( u = w \theta = a_{i_1} \ldots a_{i_n} \) for some \( \theta \in \mathcal{S}_X \) and a subset \( \gamma \subseteq X = \{a_1, \ldots, a_n\} \) we say that \( \gamma \) is contained in a prefix of \( u \) (or \( u \) has a prefix containing \( \gamma \)) whenever either \( \gamma \) is empty or there is a prefix \( u' = a_{i_1} \ldots a_{i_l} \) of \( u \) for some \( l \) with \( n \geq l \geq 1 \) with \( \gamma = \{a_{i_1}, \ldots, a_{i_l}\} \).

**Lemma 5.** Let \( \mathcal{F} \) be a \( \cup \)-closed family of sets of \( X = \{a_1, \ldots, a_n\} \), let \( g \in \mathcal{F} \) and \( \eta \in \max(\text{Fib}(g)) \). Then for any word \( u = a_{i_1} \ldots a_{i_n} \) having a prefix containing \( \eta \) \( \setminus \eta^* \) we have \( \varphi_u(g) = \eta \).

**Proof.** Suppose \( \eta \setminus \eta^* \neq \emptyset \) (the empty case can be treated analogously) and let \( u = a_{i_1} \ldots a_{i_n} \) be a word with the property of the statement and so there is some \( l \) with \( n \geq l \geq 1 \) such that \( \eta \setminus \eta^* = \{a_{i_1}, \ldots, a_{i_l}\} \). Let \( \eta_0 = g \) and \( \eta_j = \varphi_j(g) \) for \( j = 1, \ldots, n \) be the trajectory of \( g \) through the iterated application of the rising functions with respect to \( u \) and let \( \mathcal{F}_j \) be the associated sections. Suppose that there is an integer \( s \) with \( 0 \leq s < l \) such that \( \eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s \) and let us suppose without loss of generality that such \( s \) is minimum between the integers with this property. Since \( \eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s \) there is an element \( f \in \mathcal{F} \) with \( f \neq g \) such that \( \eta_s \cup a_{i_{s+1}} = \varphi_s(f) \). Thus,
since $g \subseteq \eta_s$ we get $g \subseteq \eta_s \cup a_{i+1} = \varphi_s(f) \subseteq \varphi_w(f)$ and so by Corollary \ref{corollary1} we have $g \subseteq f$. Since $a_1 \ldots a_i$ is a prefix of $u$ and $\{a_1, \ldots, a_i\} = \eta \setminus \eta^*$, $g = \eta_0 \subseteq \eta$ then $\eta_s \subseteq \eta$, moreover since $s < l$ then $a_{i+1} \in \eta$, hence $\eta_s \cup a_{i+1} \subseteq \eta$. Therefore we have the contradiction:

$$g = \eta^* \supseteq (\eta_s \cup a_{i+1}^*) = f \supseteq g$$

since by Corollary \ref{corollary1} $f = \varphi_u(f) \supseteq (\eta_s \cup a_{i+1}^*) \supseteq f$. Hence we can suppose that for all $0 \leq s < l$ we have $\eta_s \cup a_{i+1} \notin \mathcal{F}_s$ and so we have $\eta_l = \eta$. Thus $\eta \subseteq \varphi_u(g)$. Let us prove that actually $\eta = \varphi_u(g)$. Suppose on the contrary that $\eta \subseteq \varphi_u(g)$, since by Corollary \ref{corollary1} $g = (\varphi_u(g))^*$ then $\varphi_u(g) \in Fib(g)$, however $\eta \subseteq \varphi_w^*(g)$ contradicts the maximality of $\eta$, hence $\eta = \varphi_u(g)$.

The following theorem characterizes the orbits of $U(\mathcal{F})$.

**Theorem 2.** Let $\mathcal{F}$ be a $\cup$-closed family of sets of $X = \{a_1, a_2, \ldots, a_n\}$ and let $w = a_1 a_2 \ldots a_n$. Let $g \in \mathcal{F}$ then:

$$\mathcal{G}_X \cdot \varphi_w(g) = \max(Fib(g))$$

**Proof.** The inclusion $\max(Fib(g)) \subseteq \mathcal{G}_X \cdot \varphi_w(g)$ is a consequence of Lemma \ref{lemma3}. On the other hand, let $\varphi_w^*(g) \in \mathcal{G}_X \cdot \varphi_w(g)$ for some $w' = w\theta$, $\theta \in \mathcal{G}_X$. By Corollary \ref{corollary1} $(\varphi_w^*(g))^* = g$, thus we have $\varphi_w^*(g) \in Fib(g)$. Suppose, contrary to the statement of the lemma, that $\varphi_w^*(g)$ is not maximal in $Fib(g)$ and so let $\eta' \in Fib(g)$ such that $\varphi_w^*(g) \subseteq \eta'$. Since $\varphi_w(g)$ is an upward-closed set and $\varphi_w^*(g) \in \varphi_w^*(\mathcal{F})$ with $\varphi_w^*(g) \subseteq \eta'$, then we get $\eta' \in \varphi_w^*(\mathcal{F})$. However, by Corollary \ref{corollary1} we have the contradiction $g \subseteq (\eta')^* = g$. Hence $\varphi_w^*(g) \in \max(Fib(g))$ and so $\mathcal{G}_X \cdot \varphi_w(g) \subseteq \max(Fib(g))$.

Note that Theorem\ref{theorem2} together with the fact that $Fib(g) \cap Fib(f) = \emptyset$ iff $g \neq f$, implies Proposition \ref{proposition2} in particular we have

$$Fib(g) = \bigcup_{\vartheta \in \mathcal{G}_X} [g, \varphi_{w\vartheta}(g)]$$

Using the invariant upward-closed family $U(\mathcal{F})$ we can give tights upper and lower bounds to $|\mathcal{F}|$ depending on $rk(\mathcal{F}) = \min\{|\eta| : \eta \in \min(U(\mathcal{F}))\}$. We have the following proposition:

**Proposition 4.**

$$2^{n - rk(\mathcal{F})} \leq |\mathcal{F}| \leq \sum_{i \geq rk(\mathcal{F})} \binom{n}{i}$$

and these bounds are tights.
Proof. Let \( z \in \min(U(\mathcal{F})) \) with \( |z| = rk(\mathcal{F}) \), then by Theorem \( \[ \] \) \( z = \varphi_w^\theta(g) \) for some \( \theta \in \mathcal{S}_X \), thus \( z^\uparrow \subseteq \varphi_w^\theta(\mathcal{F}) \). Thus \( |F| = |\varphi_w^\theta(\mathcal{F})| \geq |z^\uparrow| = 2n - rk(\mathcal{F}) \). This bound is attained considering the \( \cup \)-closed family \( \{z\}^\uparrow \). By Proposition \( \[ \] \) \( \varphi_w^\theta(\mathcal{F}) = \{z\}^\uparrow \) for all \( \theta \in \mathcal{S}_X \), thus \( U(\mathcal{F}) = \{z\}^\uparrow \) and so \( rk(\mathcal{F}) = |z| \). The upper bound is obtained in a similar way and its proof is left to the reader. \( \square \)

Let \( x \in U(\mathcal{F}) \), \( \mathcal{G}_x \) denotes the stabilizer subgroup of \( x \) and as usual by \( U(\mathcal{F})^\theta \) the set of elements of \( U(\mathcal{F}) \) fixed by an element \( \theta \in \mathcal{S}_X \). As a consequence of Theorem \( \[ \] \) and Burnside’s Lemma we have the following corollary:

**Corollary 2.**

\[
|F| = \frac{1}{n!} \sum_{x \in U(\mathcal{F})} |\mathcal{G}_x|
\]

In particular we have the following inequality:

\[
\frac{1}{|F|} \sum_{f \in \mathcal{F}} \sum_{x \in \max(Fib(f))} \frac{1}{(n^{|x\setminus x^*|})} \leq 1
\]

Proof. Using Burnside’s Lemma

\[
|U(\mathcal{F})/\mathcal{G}_X| = \frac{1}{|\mathcal{G}_X|} \sum_{\theta \in \mathcal{S}_X} |U(\mathcal{F})^\theta| = \frac{1}{n!} \sum_{x \in U(\mathcal{F})} |\mathcal{G}_x|
\]

by Theorem \( \[ \] \) the set of orbits \( U(\mathcal{F})/\mathcal{G}_X \) is in one to one correspondence with \( \mathcal{F} \), thus \( |U(\mathcal{F})/\mathcal{G}_X| = |F| \) and so the equality of the corollary is proved. To prove the inequality we give a lower bound to \( |\mathcal{G}_x| \) for \( x \in \max(Fib(f)) \). By Lemma \( \[ \] \) we have that for any word \( u = a_{i_1} \ldots a_{i_n} \) having a prefix containing \( x \setminus x^* \), \( \varphi_u(g) = x \). There are \( |x \setminus x^*|!(n - |x \setminus x^*|)! \) such words and so \( |\mathcal{G}_x| \geq |x \setminus x^*|!(n - |x \setminus x^*|)! \). Therefore by Theorem \( \[ \] \) we have

\[
1 = \frac{1}{|F|n!} \sum_{x \in U(\mathcal{F})} |\mathcal{G}_x| \geq \frac{1}{|F|} \sum_{x \in U(\mathcal{F})} \frac{(|x \setminus x^*|!(n - |x \setminus x^*|)!)}{n!}
\]

\[
\geq \frac{1}{|F|} \sum_{f \in \mathcal{F}} \sum_{x \in \max(Fib(f))} \frac{1}{(n^{|x\setminus x^*|})}
\]

\( \square \)

The following lemma characterizes the elements not containing an \( a \in X \) for which in the rising process, for some order of rising, the elements will also not contain \( a \).
Lemma 6. Let \( g \in \mathcal{F}_\pi \) and \( \eta \in \max(\text{Fib}(g)) \). Then \( a \notin \eta \) if and only if there is \( h \in \mathcal{F}_a \) such that \( h \subseteq \eta \cup \{ a \} \) and \( g < h \).

Proof. Suppose that \( a \notin \eta \) and let \( h' = (\eta \cup \{ a \})^\ast \). Since \( \eta \in \text{Fib}(g) \), then \( g \subseteq \eta \) and so \( g \subseteq h' \). We claim \( h' \setminus \{ a \} \) is \( g \). By Lemma 4 and Theorem 2 we get \( h' \in \mathcal{F}_a \), and by definition of the operator \( \circ^\ast \), \( h' \subseteq \eta \cup \{ a \} \). Since \( g \subseteq h' \) and \( g \in \mathcal{F}_\pi \), then \( g \subseteq (h' \setminus \{ a \}) \) and \( \eta = g \) and so the claim \( (h' \setminus \{ a \})^\ast = g \). Reasoning by contradiction suppose that there is a \( g' \in \mathcal{F}_\pi \) such that \( g < g' \subseteq h' \). Thus \( g' \subseteq (h' \setminus \{ a \}) \) and so we get the contradiction \( g < g' \subseteq (h' \setminus \{ a \})^\ast = g \), whence there is an \( h \in \mathcal{F}_a \) such that \( g < h \subseteq h' \).

On the other side, suppose, contrary to the statement of the lemma, that \( a \in \eta \). Thus \( h \subseteq \eta \), hence we have \( h \subseteq \eta^\ast = g \). However \( h \in \mathcal{F}_a \) and \( g \in \mathcal{F}_\pi \), a contradiction. \( \square \)

The following lemma characterizes the elements of \( \mathcal{F}_\pi \) that have at least one maximal element in their fiber that do not contain the element \( a \).

Lemma 7. Let \( a \in X \) and let \( g \in \mathcal{F}_\pi \). Then there is an \( \eta \in \max(\text{Fib}(g)) \) with \( a \notin \eta \) if and only if there is an \( h \in \mathcal{F}_a \) such that \( g < h \).

Proof. Suppose that there is an \( \eta \in \max(\text{Fib}(g)) \) with \( a \notin \eta \). By Lemma 6 there is an \( h \in \mathcal{F}_a \) such that \( g < h \). We prove the other side of the equivalence using an argument similar to the one in Lemma 6. Indeed consider the permutation \( (a_{i_1}, \ldots, a_{i_n}) \) of \( X \) with \( g = \{ a_{i_1}, \ldots, a_{i_k} \} \), \( h = \{ a_{i_1}, \ldots, a_{i_l} \} \), \( a_{i_l} = a \) for some \( n \geq l \geq k \). Consider the word \( w' = a_{i_1}, \ldots, a_{i_n} \), put \( \eta_0 = g \) and \( \eta_j = \varphi_j(\eta_0) \) for \( j = 1, \ldots, n \) be the trajectory of \( g \) through the iterated application of the rising functions with respect to \( w' \) and let \( \mathcal{F}_j \) be the associated sections. We claim that \( \varphi_{l-1}(g) = g \cup \{ a_{i_{k+1}}, \ldots, a_{i_{l-1}} \} \). Clearly \( \eta_k = g \) and suppose, contrary to our claim, that there is an integer \( s \) with \( k \leq s < l - 1 \) such that \( \eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s \) and let us suppose that \( s \) is the minimum between the integers with this property. Since \( \eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s \) there is an element \( g' \in \mathcal{F} \) with \( g' \neq g \) such that \( \eta_s \cup a_{i_{s+1}} = \varphi_s(g') \). Thus, since \( g \subseteq \eta_s \) we get \( g \subseteq \eta_s \cup a_{i_{s+1}} = \varphi_s(g) \subseteq \varphi_{w'}(g') \) and so by Corollary 4 we have \( g \subseteq g' \). Since \( a_{i_1}, \ldots, a_{i_l} \) is a prefix of \( w' \), \( h = \{ a_{i_1}, \ldots, a_{i_l} \} \), \( \eta_0 \subseteq h \), \( s < l - 1 \) and \( a_{i_s} = a \) then \( \eta_s \cup a_{i_{s+1}} \subseteq h \setminus \{ a \} \). Since \( g < h \) and \( g \in \mathcal{F}_\pi \) it is straightforward to check that \( g = (h \setminus \{ a \})^\ast \) and so we have the contradiction:

\[
g = (h \setminus \{ a \})^\ast \supseteq (\eta_s \cup a_{i_{s+1}})^\ast = g' \supseteq g
\]

since by Corollary 4 we have \( g' = \varphi_{w'}(g')^\ast \supseteq (\eta_s \cup a_{i_{s+1}})^\ast \supseteq g' \). Therefore \( \eta_s \cup a_{i_{s+1}} \notin \mathcal{F}_s \) for all \( k \leq s < l - 1 \) and so \( \varphi_{l-1}(g) = g \cup \{ a_{i_{k+1}}, \ldots, a_{i_{l-1}} \} \). Since \( h = \varphi_{l-1}(h) \in \mathcal{F}_{l-1} \) and \( \varphi_{l-1}(g) \cup \{ a_{i_k} \} = h \) we have \( \varphi_{l-1}(g) \cup \{ a_{i_k} \} \in \mathcal{F}_{l-1} \).
F_{l-1}$ hence $\varphi_l(g) = \varphi_{l-1}(g) = h \setminus a$ $(a = a_i)$ and so $a \notin \eta_m$ for all $m \geq l$. In particular $a \notin \varphi_w'(g)$, whence by Theorem 2, $\varphi_w'(g) \in \max(\text{Fib}(g))$ is the element $\eta$ satisfying the condition of the lemma. 

In view of Lemma 7 we say that $g \in \mathcal{F}_a$ is covered in $a$ if there is an $h \in \mathcal{F}_a$ such that $g \subsetneq h$. In this case we say that $h$ covers $g$ in $a$. The following proposition gives an equivalent formulation of this definition.

**Proposition 5.** $g \in \mathcal{F}_a$ is covered in $a$ iff there is an $h \in \mathcal{F}_a$ such that $(h \setminus \{a\})^* = g$.

**Proof.** Suppose that $h \in \mathcal{F}_a$ such that $g \subsetneq h$, then it is straightforward to see that $(h \setminus \{a\})^* = g$. Conversely suppose that there is an $h \in \mathcal{F}_a$ such that $(h \setminus \{a\})^* = g$. Arguing by contradiction suppose that $g$ is not covered in $a$ and so for any $t \in \mathcal{F}_a$ there is a $g' \in \mathcal{F}_a$ such that $g \subsetneq g' \subseteq t$. In particular this occurs for $h$, hence there is a $g' \in \mathcal{F}_a$ with $g \subsetneq g' \subseteq h$. Thus we have the contradiction $g' \subseteq (h \setminus \{a\})^* = g \subsetneq g'$.

From this proposition we have that the set

$$\text{Cov}_a(g) = \{h \in \mathcal{F}_a : (h \setminus \{a\})^* = g\}$$

is non-empty iff $g$ is covered in $a$.

## 5 Some results around Frankl’s conjecture

The connection between upward-closed families and $\cup$-closed families that we have established in the previous two sections can be useful to try to tackle Frankl’s conjecture. The aim of this section is to introduce some subsets which are related to this conjecture. In particular in the first part we fix a word $w$ and we introduce these sets using the rising function $\varphi_w$, in the second part we draw some consequences of this approach giving some lower bounds on the quantity $\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f|$, for any $S \subseteq \mathcal{F}$, and in the last part we consider the invariant case.

### 5.1 Some useful subsets

**Definition 1.** Let $\mathcal{H}$ be a family of sets of $X = \{a_1, \ldots, a_n\}$ and let $a \in X$. We denote by $S(\mathcal{H}, a)$ the set of all the elements $z \in \mathcal{H}$ such that $z \cup \{a\} \notin \mathcal{H}$. Dually we put $P(\mathcal{H}, a)$ as the set of all the elements $z \in \mathcal{H}$ such that $z \setminus \{a\} \notin \mathcal{H}$.
Note that $P(\mathcal{H}, a)$ is non-empty since $\min\{\mathcal{H}\}_a \subseteq P(\mathcal{H}, a)$. We have the following proposition.

**Proposition 6.** Let $\mathcal{H}$ be a family of sets of $X$, then for any $a \in X$:

$$|\mathcal{H}_a| - |\mathcal{H}_\varnothing| = |P(\mathcal{H}, a)| - |S(\mathcal{H}, a)|$$

Moreover if $\mathcal{H}$ is $\cup$-closed, then Frankl’s conjecture holds for $\mathcal{H}$ if and only if there is some $a \in X$ such that

$$|P(\mathcal{H}, a)| \geq |S(\mathcal{H}, a)|$$

**Proof.** It is straightforward to check that the function $\psi$ from the set $\{f \in \mathcal{H}_a : f \setminus \{a\} \in \mathcal{H}\}$ onto the set $\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \in \mathcal{H}\}$ defined by $\psi(z) = z \setminus \{a\}$ is a bijection. Furthermore $\{f \in \mathcal{H}_a : f \setminus \{a\} \in \mathcal{H}\}$ is in bijection with the set $\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \in \mathcal{H}\}$ and so $|\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \in \mathcal{H}\}| = |\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \in \mathcal{H}\}|$, whence

$$|\mathcal{H}_a| - |\mathcal{H}_\varnothing| = |\mathcal{H}_a| - |\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \in \mathcal{H}\}| = |\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \in \mathcal{H}\}|$$

and so the statement follows from $|\mathcal{H}_a| = |\mathcal{H}_\varnothing|$, $|\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \notin \mathcal{H}\}| = |\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \notin \mathcal{H}\}| = |\{f \in \mathcal{H}_\varnothing : f \cup \{a\} \notin \mathcal{H}\}| = |\mathcal{H}_a| - |\mathcal{H}_\varnothing|$. The last claim of the proposition is a consequence of $2|\mathcal{H}_a| - |\mathcal{H}_\varnothing| = |\mathcal{H}_a| - |\mathcal{H}_\varnothing|$. \(\blacksquare\)

Therefore the study of the sets $S(\mathcal{H}, a)$ and $P(\mathcal{H}, a)$ seems important in a possible proof of the Frankl’s conjecture. Let us fix a $\cup$-closed family $\mathcal{F}$ on $X$, let $\mathcal{F} = \varphi_w(\mathcal{F})$ be the associated upward-closed family for some fixed word $w$. We introduce now two analogous sets which are important to give a lower bound to the quantity $\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f|$, for any $S \subseteq \mathcal{F}$ and which are somehow related to $S(\mathcal{H}, a)$ and $P(\mathcal{H}, a)$.

**Definition 2.** Let $a \in X$, the set $\sigma_w(\mathcal{F}, a) = \{\eta \in \varphi_w(\mathcal{F}) : a \in \eta \setminus \varphi_w^{-1}(\eta)\}$ is called the set of spurious elements of $\mathcal{F}$ with respect to $a$. The set $\pi_w(\mathcal{F}, a) = \{\eta \in \varphi_w(\mathcal{F}_a) : \eta \setminus \{a\} \notin \varphi_w(\mathcal{F})\}$ is called the set of pure elements of $\mathcal{F}$ with respect to $a$.

Let $\eta \in \mathcal{F}$, the set of pure elements of $\eta$, denoted by $\pi_w(\mathcal{F}, \eta)$, is the set $\{a \in X : \eta \in \pi_w(a)\}$ and analogously the set of spurious elements of $\eta$ is the set $\sigma_w(\mathcal{F}, \eta) = \{a \in X : \eta \in \sigma_w(a)\}$. 

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When the $\bigcup$-closed set $\mathcal{F}$ is clear from the context, we drop $\mathcal{F}$ from $\sigma_w(\mathcal{F},a), \sigma_w(\mathcal{F},\eta), \pi_w(\mathcal{F},a), \pi_w(\mathcal{F},\eta)$ and we use instead $\sigma_w(a), \sigma_w(\eta), \pi_w(a), \pi_w(\eta)$. We have the following lemma.

**Lemma 8.** The two sets $\varphi_w(\mathcal{F}_a), \sigma_w(a)$ form a partition of $\mathcal{F}_a$. In turn $\varphi_w(\mathcal{F}_a)$ is partitioned by $\pi_w(a)$, $\psi(\mathcal{F}_a)$, where $\psi(z) = z \cup \{a\}$. Moreover $\sigma_w(a) \cup \pi_w(a) = \{z \in \mathcal{F} : z \setminus \{a\} \notin \mathcal{F}\}$ and

$$|\mathcal{F}_a| = |\mathcal{F}_a| + |\pi_w(a)| + |\sigma_w(a)|. $$

**Proof.** Since $\sigma_w(a) \subseteq \mathcal{F}_a$ and $\varphi_w(\mathcal{F}_a) \subseteq \mathcal{F}_a$, then $\mathcal{F}_a \setminus \varphi_w(\mathcal{F}_a)$ is formed by elements $z \in \mathcal{F}_a$ for which $a$ is a spurious element of $z$, i.e. $\mathcal{F}_a \setminus \varphi_w(\mathcal{F}_a) = \sigma_w(a)$. By Lemma 4 $\psi(\mathcal{F}_a) \subseteq \varphi_w(\mathcal{F}_a)$ and if $z \in \varphi_w(\mathcal{F}_a) \setminus \psi(\mathcal{F}_a)$ then $z \setminus \{a\} \notin \mathcal{F}$, otherwise $z = \psi(z \setminus \{a\})$. Therefore $\pi_w(a) = \varphi_w(\mathcal{F}_a) \setminus \psi(\mathcal{F}_a)$.

By the previous statements it is also evident that:

$$\sigma_w(a) \cup \pi_w(a) = \mathcal{F}_a \setminus \psi(\mathcal{F}_a) = \{z \in \mathcal{F} : z \setminus \{a\} \notin \mathcal{F}\}$$

Since $\mathcal{F}_a$ is partitioned into the two sets and $\sigma_w(a), \varphi_w(\mathcal{F}_a)$ which in turn is partition by the two sets $\pi_w(a), \psi(\mathcal{F}_a)$, and $\psi$ is an injective map we have:

$$|\mathcal{F}_a| = |\psi(\mathcal{F}_a)| + |\sigma_w(a)| + |\pi_w(a)| = |\mathcal{F}_a| + |\sigma_w(a)| + |\pi_w(a)|$$

and this completes the proof of the lemma. $\square$

The following proposition gives an alternative formulation of Frankl’s conjecture which is the analogous of Proposition 6.

**Proposition 7.** For any $a \in X$

$$|\pi_w(a)| - |\sigma_w(a)| = |P(\mathcal{F},a)| - |S(\mathcal{F},a)|$$

and so Frankl’s conjecture holds for $\mathcal{F}$ if and only if $|\pi_w(a)| \geq |\sigma_w(a)|$ for some $a \in X$. Moreover $|\pi_w(a)| \leq |P(\mathcal{F},a)|, |\sigma_w(a)| \leq |S(\mathcal{F},a)|$.

**Proof.** It is not difficult to check that $|\mathcal{F}_a| - |\sigma_w(a)| = |\mathcal{F}_a|$ and by Lemma 4 we have $|\mathcal{F}_a| = |\mathcal{F}_a| + |\sigma_w(a)|$. Thus by the same Lemma 4 we get

$$|\mathcal{F}_a| = |\mathcal{F}_a| + |\pi_w(a)| - |\sigma_w(a)|$$

and so, by Proposition 6 we get the statement $|P(\mathcal{F},a)| - |S(\mathcal{F},a)| = |\mathcal{F}_a| - |\mathcal{F}_a| = |\pi_w(a)| - |\sigma_w(a)|$.

Let us prove the last statement showing that $\sigma_w(a) \subseteq \varphi_w(S(\mathcal{F},a))$. Let $\eta \in \sigma_w(a)$. Reasoning by contradiction, suppose that $z = \varphi_w^{-1}(\eta) \notin S(\mathcal{F},a)$
and so \(z \cup \{a\} \in \mathcal{F}\). Since \(z \cup \{a\} \subseteq \eta\), by Corollary 1 we get \(\eta \in \mathcal{F}[z \cup \{a\}] \subseteq \mathcal{F}[z \cup \{a\}]\) and so \(a \in z \cup \{a\} \subseteq \varphi^{-1}(\eta) = z\) which contradicts \(\eta \in \sigma_w(a)\).

The statement \(|\pi_w(a)| \leq |P(\mathcal{F}, a)|\) is a consequence of \(|\pi_w(a)| - |\sigma_w(a)| = |P(\mathcal{F}, a)| - |S(\mathcal{F}, a)|\) and \(|\sigma_w(a)| \leq |S(\mathcal{F}, a)|\).

In view of Proposition 7 it is interesting to give a lower bound to the set \(|\pi_w(a)|\). The following proposition gives a partial answer, we recall that \(\circ^*\) is the operator introduced in Section 4.

**Proposition 8.** For any \(a \in X\) we have:

\[
\{\varphi_w(g) : g \in \mathcal{F}_a, (g \setminus \{a\})^* = \emptyset\} \subseteq \pi_w(a)
\]

**Proof.** Let \(g \in \mathcal{F}_a, (g \setminus \{a\})^* = \emptyset\) and suppose, contrary to the statement, that \(\varphi_w(g) \setminus \{a\} \in \mathcal{F}\). By Corollary 1 we have

\[
\varphi^{-1}_w(\varphi_w(g) \setminus \{a\}) = \bigcup_{f \subseteq \varphi_w(g) \setminus \{a\}} f \subseteq \bigcup_{f \subseteq g \setminus \{a\}} f = (g \setminus \{a\})^*
\]

whence \((g \setminus \{a\})^* \neq \emptyset\), a contradiction. \(\square\)

We remark that the set \(\{g \in \mathcal{F}_a : (g \setminus \{a\})^* = \emptyset\}\) is non-empty since it contains \(\min(\mathcal{F}_a)\).

The subsets \(\pi_w(\eta), \sigma_w(\eta)\) introduced in Definition 2 are the “local” version of \(\pi_w(a), \sigma_w(a)\) in the following sense:

\[
\sum_{a \in X} |\pi_w(a)| = \sum_{\eta \in \mathcal{F}} |\pi_w(\eta)|, \quad \sum_{a \in X} |\sigma_w(a)| = \sum_{\eta \in \mathcal{F}} |\sigma_w(\eta)|
\]

We also note that by Lemma 8 \(\pi_w(\eta), \sigma_w(\eta)\) are two disjoint subsets of \(\eta\) and in particular by the definition we get \(\sigma_w(\eta) = \eta \setminus \varphi^{-1}_w(\eta)\). The interest in introducing such subsets is given by the following characterization:

**Proposition 9.** For any \(\eta \in \mathcal{F}\) we have:

\[
\sigma_w(\eta) = \bigcap_{\xi \subseteq \eta} \sigma_w(\xi), \quad \pi_w(\eta) = \bigcap_{\xi \subseteq \eta} \xi \cap \varphi^{-1}_w(\eta)
\]

**Proof.** By Lemma 8 \(\sigma_w(a) \cup \pi_w(a) = \{z \in \mathcal{F} : z \setminus \{a\} \notin \mathcal{F}\}\), thus it is straightforward to check

\[
\pi_w(\eta) \cup \sigma_w(\eta) = \{a \in X : \eta \setminus \{a\} \notin \mathcal{F}\} = \bigcap_{\xi \subseteq \eta} \xi
\]
Since $\pi_w(\eta) \subseteq \varphi^{-1}_w(\eta)$ and $\sigma_w(\eta) = \eta \setminus \varphi^{-1}_w(\eta)$ then
\[
\pi_w(\eta) = \bigcap_{\xi \subseteq \eta} \xi \cap \varphi^{-1}_w(\eta), \quad \sigma_w(\eta) = \bigcap_{\xi \subseteq \eta} \xi \cap \sigma_w(\eta)
\]
We claim that if $\xi \subseteq \eta$ then $\sigma_w(\eta) \subseteq \sigma_w(\xi)$ from which it follows $\sigma_w(\eta) = \bigcap_{\xi \subseteq \eta} \sigma_w(\xi)$. Indeed by Lemma 4 for all $b \in \eta \setminus \xi$, $b \in \varphi^{-1}_w(\xi \cup \{b\})$. Thus, since $\xi \cup \{b\} \subseteq \eta$, by Theorem 1 we also get $\varphi^{-1}_w(\xi) \subseteq \varphi^{-1}_w(\eta)$. By Theorem 1 we also get $\varphi^{-1}_w(\xi) \subseteq \varphi^{-1}_w(\eta)$, thus $\eta \setminus \xi \cup \varphi^{-1}_w(\xi) \subseteq \varphi^{-1}_w(\eta)$ from which we obtain $\sigma_w(\eta) \subseteq \sigma_w(\xi)$.

If $f \subseteq g$ for some $f, g \in \mathcal{F}$, then in general $\sigma_w(\varphi_w(f)) \subseteq \sigma_w(\varphi_w(g))$ do not hold. However if we keep the freedom to choose the order of the rising we can have this property. With the notation of Section 4 we have the following:

**Lemma 9.** Let $f, g \in \mathcal{F}$ with $f \subseteq g$ and let $\eta \in \text{max}(\text{Fib}(g))$, then there is a word $w' = a_{i_1} \ldots a_{i_n}$ such that $\eta = \varphi_w(g)$ and
\[
\sigma_{w'}(\varphi_{w'}(g)) \subseteq \sigma_{w'}(\varphi_{w'}(f))
\]

**Proof.** Let us prove that $\eta' = (\eta \setminus g) \cup f \in \text{Fib}(f)$. It is obvious that $f \subseteq \eta'$, a let us assume, contrary to our claim, that there is $h \in \mathcal{F}$ such that $h \subseteq \eta'$ with $f \not\subseteq h$. Thus $(h \setminus f) \cap (\eta \setminus g) \neq \emptyset$. Since $\eta' \subseteq \eta$, then $h \subseteq \eta$ and so $h \subseteq \eta^* = g$. In particular we have $(h \setminus f) \subseteq g$ which contradicts $(h \setminus f) \cap (\eta \setminus g) \neq \emptyset$. Therefore $\eta' \in \text{Fib}(f)$, and let $\nu \in \text{max}(\text{Fib}(f))$ such that $\eta' \subseteq \nu$. Then we have
\[
\eta \setminus \eta^* = \eta \setminus g = \eta' \setminus f \subseteq \nu \setminus f = \nu \setminus \nu^*
\]
(2)

If we prove that there is a word $w'$ such that $\eta = \varphi_{w'}(g)$, $\nu = \varphi_{w'}(f)$ then we have proved the statement of the lemma since (2) holds and $\sigma_{w'}(\eta) = \eta \setminus \eta^*$, $\sigma_{w'}(\nu) = \nu \setminus \nu^*$. Since $\eta \setminus \eta^* \subseteq \nu \setminus \nu^*$, then we can find a word $w'$ such that both $\eta \setminus \eta^*$ and $\nu \setminus \nu^*$ are contained in a prefix of $w'$, hence by Lemma 5 we have $\eta = \varphi_{w'}(g), \nu = \varphi_{w'}(f)$.

**5.2 The average length**

The average of the length of the elements of $\mathcal{F}$, simply the average of the family $\mathcal{F}$, is the integer $\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f|$, this number is important because the following well known equality holds
\[
\sum_{\sigma \in \mathcal{X}} \frac{|\mathcal{F}_\sigma|}{|\mathcal{F}|} \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f|
\]

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For instance the averaged Frankl’s property \( \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{n}{2} \) implies that Frankl’s conjecture is true for \( \mathcal{F} \). Unfortunately the converse is not true, indeed it is a well known fact that many union-closed families fail to satisfy the averaged Frankl’s property (see [7, 8]). However the average of \( \mathcal{F} \) is still an interesting parameter at least because any lower bound on it gives rise to a lower bound of \( \max_{a \in X} \{|\mathcal{F}_a|/|\mathcal{F}|\} \). In [18] Reimer shows that

\[
\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{1}{2} \log_2(|\mathcal{F}|)
\]

and in [12] the bound is improved in the case of a separating family. What we consider here is the localized version of the average of \( \mathcal{F} \), given a subfamily \( S \subseteq \mathcal{F} \), the average of \( \mathcal{F} \) localized on \( S \) is defined by

\[
\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f|
\]

and gives the average of the length of the elements contained in the principal ideal of \( \mathcal{F} \) generated by \( S \). Our aim is to provide lower bounds to such quantity. Note that we can assume without loss of generality that \( S \) is an antichain. We fix the notation and for the rest of the section \( \mathcal{F} \) denotes a \( \cup \)-closed family of sets of \( X = \{a_1, \ldots, a_n\} \), \( S \subseteq \mathcal{F} \) is an antichain, and \( \mathcal{F} = \varphi_w(\mathcal{F}) \) is the upward-closed family associate to \( \mathcal{F} \) with respect to the word \( w = a_1a_2\ldots a_n \).

**Proposition 10.** The following bound holds:

\[
\sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n}{2} |\mathcal{F}[S]| + \frac{1}{2} \sum_{a \in X} |\pi_w(a) \cap S| - |\sigma_w(a) \cap S|.
\]

with equality if \( S = \min(\mathcal{F}) \).

**Proof.** By Lemma 8 there is a partition \( \mathcal{F}_a = \psi(\mathcal{F}_\pi) \cup \pi_w(a) \cup \sigma_w(a) \), hence:

\[
\mathcal{F}_a[S] = (\psi(\mathcal{F}_\pi) \cap S^\uparrow) \cup (\pi_w(a) \cap S^\uparrow) \cup (\sigma_w(a) \cap S^\uparrow) \tag{3}
\]

We have \( \psi(\mathcal{F}_\pi \cap S^\uparrow) \subseteq \psi(\mathcal{F}_\pi) \cap S^\uparrow \) with equality if \( S = \min(\mathcal{F}) \), whence \( \sum_a |\psi(\mathcal{F}_\pi) \cap S^\uparrow| \geq \sum_a |\psi(\mathcal{F}_\pi \cap S^\uparrow)| = \sum_{\eta \in \mathcal{F}[S]} (n - |\eta|) \). Thus summing all the equalities (3) on the index \( a \in X \), we get

\[
2 \sum_{\eta \in \mathcal{F}[S]} |\eta| \geq n|\mathcal{F}[S]| + \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| + |\sigma_w(a) \cap S^\uparrow|.
\]

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By Theorem 1, $\sigma_w(a) \cap S^\uparrow = \{ \varphi_w(f), f \in \mathcal{F}[S], a \in \varphi_w(f) \setminus f \}$, and so:
\[
\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|)
\]
Moreover by Theorem 1 we also get
\[
\sum_{f \in \mathcal{F}[S]} |f| = \sum_{\eta \in \mathcal{F}[S]} |\varphi_w^{-1}(\eta)| = \sum_{\eta \in \mathcal{F}[S]} |\eta| - |\eta \setminus \varphi_w^{-1}(\eta)| = \\
\sum_{\eta \in \mathcal{F}[S]} |\eta| - \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) = \sum_{\eta \in \mathcal{F}[S]} |\eta| - \sum_{a \in X} |\sigma_w(a) \cap S^\uparrow|
\]
Therefore using (1) and $\mathcal{F}[S] \simeq \mathcal{F}[S]$ (Corollary 1) we get
\[
\sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n}{2} |\mathcal{F}[S]| + \frac{1}{2} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| - |\sigma_w(a) \cap S^\uparrow|
\]
with equality if $S = \min(\mathcal{F})$.

We have the following corollary on the local average in the case $\min(\mathcal{F})$ is a maximal antichain and the elements are uniformly bounded by some integer.

**Corollary 3.** Let $\mathcal{F}$ be a $\cup$-closed family of sets such that $\mathcal{G} = \min(\mathcal{F})$ is a maximal antichain of $2^X$ and there is a positive integer $k$ such that for all $g \in \mathcal{G}$, $|g| \leq k$, then:
\[
\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n - k}{2} + \frac{1}{2 |\mathcal{F}[S]|} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow|.
\]

*Proof.* We have already noted in the proof of Proposition 10 that:
\[
\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) = \sum_{f \in \mathcal{F}[S]} |\varphi_w(f) \setminus f|
\]
by the same proposition it is sufficient to prove that $|\varphi_w(f) \setminus f| \leq k$. Since $\mathcal{G}$ is a maximal antichain, then for any $f \in \mathcal{F}[S]$ there is a $g \in \mathcal{G}$ such that either $g \subseteq \varphi_w(f) \setminus f$ or $\varphi_w(f) \setminus f \subseteq g$. We prove that only $\varphi_w(f) \setminus f \subseteq g$ can occur, and so $|\varphi_w(f) \setminus f| \leq k$. Indeed, if $g \subseteq \varphi_w(f) \setminus f$, then $g \subseteq \varphi_w(f)$ and so, by Theorem 1 $g \subseteq f$, a contradiction. \qed
Observe that Corollary 3 also holds if we assume the existence of a maximal antichain $A \subseteq \mathcal{F}$ such that $|g| \leq k$ for all $g \in A$.

The following corollary is the analogous of Corollary 3 in the case we drop the maximality condition. Let $S \subseteq \mathcal{F}$ we define $\sigma(S) = \max\{\sigma_w(f) : f \in S, \theta \in \Theta_X\}$.

**Corollary 4.**

$$
\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n - \sigma(S)}{2} + \frac{1}{2\mathcal{F}[S]} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow|.
$$

**Proof.** Like in the proof of Corollary 3 and by Proposition 10 it is sufficient to show $|\varphi_w(f) \setminus f| = |\sigma_w(\varphi_w(g))| \leq \sigma(S)$ for all $g \in \mathcal{F}[S]$. Consider any $g \in \mathcal{F}[S]$, and let $f \in S$ such that $f \subseteq g$. By Lemma 9 there is a word $w'$ such that

$$
\sigma_w(\varphi_w(g)) = \varphi_w(g) \setminus g = \varphi_w'(g) \setminus g \subseteq \varphi_w'(f) \setminus f = \sigma_w'(\varphi_w'(f))
$$

and so the claim $|\sigma_w(\varphi_w(g))| \leq \sigma(S)$.

The following theorem gives a lower bound of the average localized on $S$ depending on the parameter $|S^\uparrow|$.

**Theorem 3.**

$$
\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n}{2} + \frac{1}{2|\mathcal{F}[S]|} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| - \frac{1}{2} \log_2 \left\{ \frac{|S^\uparrow|}{|\mathcal{F}[S]|} \right\}
$$

and the bound is attained when $S = \min(\mathcal{F})$ and when $\mathcal{F}$ is upward-closed.

**Proof.** By Proposition 10 it is enough to give an upper bound to the quantity $\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow|$. Following a similar argument in [13], we use Jensen’s inequality to upper bound $\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|)$. Indeed, we have

$$
\exp_2 \left\{ \frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) \right\} \leq \frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} 2^{\varphi_w(f) - |f|}.
$$

By Proposition 2 $f \neq g$ implies $[f, \varphi_w(f)] \cap [g, \varphi_w(g)] = \emptyset$, hence since $|[f, \varphi_w(f)]| = 2^{\varphi_w(f) - |f|}$ we get

$$
\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) \leq \log_2 \left\{ \frac{|C(F, S)|}{|\mathcal{F}[S]|} \right\}.
$$
where \( C(\mathcal{F}, S) = \bigcup_{f \in \mathcal{F}[S]} [f, \varphi_w(f)] \). The statement of the theorem thus follows from \( C(\mathcal{F}, S) \subseteq S^\uparrow \).

If \( S = \min(\mathcal{F}) \) we have the equality in the bound of Proposition 10; moreover if \( \mathcal{F} \) is upward-closed, then \( \mathcal{F} = \mathcal{F} \). Thus \( \sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = 0 \) which is equal to \( \frac{1}{2} \log_2(|S^\uparrow|/|\mathcal{F}[S]|) \) since \( \mathcal{F} \) is upward-closed and so \( S^\uparrow = \mathcal{F}[S] \). Therefore the lower bound in the statement is reached for \( S = \min(\mathcal{F}) \) and the class of upward-closed families.

Remark 1. In Theorem 3, Corollaries 4,3, we can give a lower bound to the quantity \( \frac{1}{2^{|\mathcal{F}[S]|}} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| \). Indeed, by Proposition 8 and Theorem 7 it is not difficult to see that

\[
\sum_{a \in X} |\pi_w(a) \cap S^\uparrow| \geq \sum_{g \in \mathcal{F}[S]} |\{a \in g : (g \setminus \{a\})^* = \emptyset\}|
\]

and the equality is attained if \( S = \min(\mathcal{F}) \) and when \( \mathcal{F} \) is upward-closed.

5.3 The invariant case

In this section we obtain some lower bounds on the average of \( \mathcal{F} \) localized on \( S \) using the invariant upward-closed set associated to \( \mathcal{F} \). The following definition can be considered as the analogous of the spurious and pure elements of Definition 2 in the invariant case.

Definition 3. Let \( U(\mathcal{F}) \) be the invariant upward-closed set associated to \( \mathcal{F} \) and let \( a \in X \) the set

\[ \Sigma(\mathcal{F}, a) = \{g \in \mathcal{F}_a : \forall \eta \in \max(Fib(g)), a \in \eta\} \]

is called the set of hyper-spurious elements. The local version of this set is \( \Sigma(\mathcal{F}, g) = \{a \in X \setminus g : \forall \eta \in \max(Fib(g)), a \in \eta\} \). The elements of the set

\[ \Pi(\mathcal{F}, a) = \{g \in \mathcal{F}_a : \forall \eta \in \max(Fib(g)), \eta \setminus \{a\} \notin U(\mathcal{F})\} \]

are called hyper-pure. The local version of this set is \( \Pi(\mathcal{F}, g) = \{a \in g : \forall \eta \in \max(Fib(g)), \eta \setminus \{a\} \notin U(\mathcal{F})\} \).

The connection between these sets and the spurious, pure sets introduced in Definition 2 is given by the following proposition.

Proposition 11. The following equalities hold:

\[ \Sigma(\mathcal{F}, g) = \bigcap_{\eta \in \max(Fib(g))} \eta \setminus g = \bigcap_{\theta \in \in X} \sigma_w(\mathcal{F}, \varphi_w(\theta)) \] (5)
\[ \Pi(\mathcal{F}, g) = \bigcap_{\theta \in \mathcal{G}_X} \pi_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g)) \]  

(6)

**Proof.** The first equality in (5) is a consequence of the definition, the second one of Theorem 2. Let us prove (6). Let \( b \in \Pi(\mathcal{F}, g) \), then for any \( \eta \in \max(Fib(g)) = \mathcal{G}_X \cdot \varphi_w(g) \) (by Theorem 2) we have \( \eta \setminus \{ b \} \notin U(\mathcal{F}), \) hence for any \( \theta \in \mathcal{G}_X, \varphi_{w\theta}(g) \setminus \{ b \} \notin \mathcal{G}_X \cdot \varphi_w(g), \) i.e. \( b \in \bigcap_{\theta \in \mathcal{G}_X} \pi_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g)). \)

On the other side, let \( b \in \bigcap_{\theta \in \mathcal{G}_X} \pi_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g)). \) To obtain a contradiction suppose that there is \( \eta \in \max(Fib(g)) \), for some \( g \in \mathcal{F}, \) such that \( \eta \setminus \{ b \} \in U(\mathcal{F}), \) say \( \eta \setminus \{ b \} \in U(\mathcal{F}). \) Since by Theorem 2 \( \max(Fib(h)) = \mathcal{G}_X \cdot \varphi_{w\theta}(h) \), there is a \( \vartheta \in \mathcal{G}_X \) such that \( \eta \setminus \{ b \} = \varphi_{w\theta}(h). \)

Since \( \eta \setminus \{ b \} \subseteq \eta \), we have \( \eta \in \varphi_{w\theta}(\mathcal{F}), \) in particular, since \( \varphi_{w\theta}(\eta) = \eta^* = g, \) we have \( \eta = \varphi_{w\theta}(g). \) However \( b \in \mathcal{F} \) implies \( b \in \varphi_{w\theta}(\varphi_{w\theta}(g)), \) which contradicts \( \eta \setminus \{ b \} \subseteq \varphi_{w\theta}(\varphi_{w\theta}(g)). \)

We recall that at the end of Section 4 we have introduced the set

\[ Cov_a(g) = \{ h \in \mathcal{F}_a : (h \setminus \{ a \})^* = g \} \]

we have the following proposition:

**Proposition 12.**

\[ \Sigma(\mathcal{F}, g) = \{ b \in X \setminus g : Cov_b(g) = \emptyset \} = X \setminus \bigcup_{\{ h : g \leq h \}} h \]

\[ |\mathcal{F}_a| - |\Sigma(\mathcal{F}, a)| \leq \sum_{g \in \mathcal{F}_{a \setminus \Sigma(\mathcal{F}, a)}} |\max(Cov_a(g))| \leq |\mathcal{F}_a| - |\Pi(\mathcal{F}, a)|, \forall a \in X \]

**Proof.** By Proposition 5 and Lemma 4 we have that \( Cov_a(g) = \emptyset \) if and only if \( a \in \Sigma(\mathcal{F}, g). \) The second equality is also a consequence of Proposition 5 and the definitions. Let us prove the inequalities. We first claim that for any \( h \in \max(Cov_a(g)) \) with \( g \in \mathcal{F}_a \setminus \Sigma(\mathcal{F}, a) \) and for any \( \eta \in \max(Fib(h)), \) \( \eta \setminus \{ a \} \in U(\mathcal{F}). \) Since \( (h \setminus \{ a \})^* = g \) and \( (h \setminus \{ a \}) \subseteq \eta \setminus \{ a \} \) we have \( g \subseteq (\eta \setminus \{ a \})^*. \) On the other hand, let \( (\eta \setminus \{ a \})^* = g'. \) Since \( g' \subseteq \eta \setminus \{ a \} \) and \( \eta \subseteq (h \setminus \{ a \})^* = g \) from which we have the equality \( (\eta \setminus \{ a \})^* = g. \) Therefore \( (\eta \setminus \{ a \}) \in Fib(g), \) and so there is a \( \nu \in \max(Fib(g)) \) with \( (\eta \setminus \{ a \}) \subseteq \nu. \) Consider \( \nu \cup \{ a \} \) and let us prove that \( \nu \cup \{ a \} \in \max(Fib(h)). \) Clearly \( \nu \cup \{ a \} \in \max(Fib(h')) \) for some \( h', \) we observe that since \( h \subseteq \eta \subseteq \nu \cup \{ a \} \) we have \( h \subseteq (\nu \cup \{ a \})^* = h'. \) If we prove that \( h' \in Cov_a(g), \) then by the maximality of \( h, \) we get \( h = h'. \) Suppose, contrary to our claim, that \( h' \notin Cov_a(g). \) Thus, if we put \( g' = (h' \setminus \{ a \})^*, \)
we have $g = (h \setminus \{a\})^* \subseteq (h' \setminus \{a\})^* = g'$. However, we also have $g' = (h' \setminus \{a\})^* \subseteq \nu^* = g$, a contradiction. Therefore, the claim is true and so we can deduce the following inclusion:

$$\bigcup_{g \in \mathcal{F}_a \setminus \Sigma(a)} \max(Cov_a(g)) \subseteq \mathcal{F}_a \setminus \Pi(\mathcal{F}, a)$$

Thus, the inequality easily follows from this inclusion and the following facts: $\text{Cov}_a(g) \neq \emptyset$ if $g \in \mathcal{F}_a \setminus \Sigma(a)$ and $\text{Cov}_a(g) \cap \text{Cov}_a(g') = \emptyset$ for $g \neq g'$. □

The following theorem is the analogous of Theorem \ref{23} in the invariant case.

**Proposition 13.**

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{n}{2} - \frac{1}{2|\mathcal{F}|} \sum_{f \in \mathcal{F}} |\Sigma(\mathcal{F}, g)| + \frac{1}{2|\mathcal{F}|} \sum_{a \in X} |\Pi(\mathcal{F}, a) \cap S^\uparrow|$$

this bound is attained for $S = \min(\mathcal{F})$ and when $\mathcal{F}$ is upward-closed.

**Proof.** Proposition \ref{12} can be easily adapt to prove that for all $a \in X$:

$$|\mathcal{F}_a[S]| - |\Sigma(\mathcal{F}, a) \cap S^\uparrow| \leq |\mathcal{F}_a[S]| - |\Pi(\mathcal{F}, a) \cap S^\uparrow|$$

The statement can be thus proved summing all these inequalities with $a$ running on $X$ and using the equalities $\sum_{a \in X} |\mathcal{F}_a[S]| = \sum_{f \in \mathcal{F}} |f|$, $\sum_{a \in X} |\mathcal{F}_a[S]| = \sum_{f \in \mathcal{F}_a[S]} (n - |f|)$, $\sum_{a \in X} |\Sigma(\mathcal{F}, a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}_a[S]} |\Sigma(\mathcal{F}, f)|$. By Theorem \ref{23} and Remark \ref{11} in the case of an upward-closed family $\mathcal{F}$ and $S = \min(\mathcal{F})$, we have

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| = \frac{n}{2} - \frac{1}{2|\mathcal{F}|} \sum_{a \in X} |\{f : (g \setminus \{a\})^* = \emptyset\}|$$

On the other hand by Proposition \ref{12} it is not difficult to check that in the case $\mathcal{F}$ is an upward-closed family $\Sigma(\mathcal{F}, g) = \emptyset$ and $\Pi(\mathcal{F}, g) = \{a \in g : (g \setminus \{a\})^* = \emptyset\}$ and so the bound is attained in this case. □

Using the first equality of Proposition \ref{12} and Proposition \ref{13} we can rewrite the bound of Proposition \ref{13} as

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{1}{2|\mathcal{F}|} \sum_{f \in \mathcal{F}} |\bigcup_{h : f < h} |h| + \frac{1}{2|\mathcal{F}|} \sum_{a \in X} |\Pi(\mathcal{F}, a) \cap S^\uparrow|$$

We observe that by Propositions \ref{14} \ref{23} similarly to Remark \ref{14} we also have the following lower bound

$$\sum_{a \in X} |\Pi(\mathcal{F}, a) \cap S^\uparrow| \geq \sum_{g \in \mathcal{F}[S]} |\{a \in g : (g \setminus \{a\})^* = \emptyset\}|$$
6 Upper bounds for the join-irreducible elements of a union-closed family

Let $\mathcal{F}$ be a $\cup$-closed family of sets of $2^X$ with $X = \{a_1, a_2, \ldots, a_n\}$, in this section we use the techniques obtained in Section 3 to give an upper bound to the number of join-irreducible elements of $\mathcal{F}$. We remark that if $m \in J(\mathcal{F})$ then $\mathcal{F} \setminus \{m\}$ is again a $\cup$-closed family of $2^X$. Therefore it is interesting and quite natural studying the effect of erasing an irreducible element from $\mathcal{F}$ in the rising process. For this reasons we will denote by $\varphi_w$, $\varphi'_w$ the rising function with respect to the word $w = a_1 a_2 \ldots a_n$ respectively of $\mathcal{F}$, $\mathcal{F}' = \mathcal{F} \setminus \{m\}$. We recall that the rising function at the $i$-th step is defined by

$$\varphi_{\mathcal{F},a_{i+1}}(z) = \begin{cases} z \cup \{a_{i+1}\} & \text{if } z \cup \{a_{i+1}\} \notin \mathcal{F}_i, \\ z & \text{otherwise;} \end{cases}$$

Here we simplify the cumbersome notation and we write $\varphi_{a_{i+1}}, \varphi'_{a_{i+1}}$ for $\varphi_{\mathcal{F},a_{i+1}}, \varphi'_{\mathcal{F},a_{i+1}}$, respectively. With this notation, the rising function with respect to $w = a_1 \ldots a_n$ is the last function $\varphi_n$ of the sequence of functions defined inductively by $\varphi_i = \varphi_{a_i} \circ \varphi_{i-1}$ for $i = 1, \ldots n$ where $\varphi_0$ is the identity function on $2^X$. We have the following lemma.

**Lemma 10.** With the above notation, for each $i \in \{0,1,\ldots,n\}$ there is an element $\mu_i \in \mathcal{F}_i$ such that $\mathcal{F}'_i = \mathcal{F}_i \setminus \{\mu_i\}$. Moreover we have two possibilities

1. if there is no $z \in \mathcal{F}'_i$ such that $z \cup a_{i+1} = \mu_i$, then $\mathcal{F}'_{i+1} = \mathcal{F}_{i+1} \setminus \{\mu_{i+1}\}$ with $\mu_{i+1} = \varphi_{a_{i+1}}(\mu_i)$ and $\varphi_{a_{i+1}}(z) = \varphi'_{a_{i+1}}(z)$ for all $z \in \mathcal{F}'_i$,

2. if there is $z \in \mathcal{F}'_i$ such that $z \cup a_{i+1} = \mu_i$, then $\mathcal{F}'_{i+1} = \mathcal{F}_{i+1} \setminus \{\mu_{i+1}\}$ with $\mu_{i+1} = z = \varphi_{a_{i+1}}(z)$, $\varphi'_{a_{i+1}}(z) = \mu_i$ and $\varphi'_{a_{i+1}}(y) = \varphi_{a_{i+1}}(y)$ for all $y \in \mathcal{F}'_i \setminus \{z\}$.

**Proof.** We prove the statement by induction on the index $i$. The statement is true for $i = 0$, since $\mathcal{F}'_0 = \mathcal{F}' = \mathcal{F} \setminus \{m\} = \mathcal{F}_0 \setminus \{m\}$. So, putting $\mu_0 = m$, we can suppose that the statement is true for $i > 0$ and let us prove it for $i + 1$. By induction, there is an element $\mu_i \in \mathcal{F}_i$ such that $\mathcal{F}'_i = \mathcal{F}_i \setminus \{\mu_i\}$. Let $z \in \mathcal{F}'_i$, we have the following cases:

i) $z \cup \{a_{i+1}\} \notin \mathcal{F}_i$ and so also $z \cup \{a_{i+1}\} \notin \mathcal{F}'_i$ which implies $\varphi_{a_{i+1}}(z) = \varphi'_{a_{i+1}}(z) = z \cup \{a_{i+1}\}$.

ii) $z \cup \{a_{i+1}\} \in \mathcal{F}'_i \subseteq \mathcal{F}_i$ and so $\varphi_{a_{i+1}}(z) = \varphi'_{a_{i+1}}(z) = z$. 

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iii) \( z \cup \{a_{i+1}\} \in \mathcal{F}_i \setminus \mathcal{F}_i' = \{\mu_i\}, \) and so \( z \cup \{a_{i+1}\} = \mu_i. \) Hence \( \varphi'_{a_{i+1}}(z) = z \cup \{a_{i+1}\} = \mu_i = \varphi_{a_{i+1}}(\mu_i) \) and \( \varphi_{a_{i+1}}(z) = z. \)

Thus, if condition \( z \cup \{a_{i+1}\} = \mu_i \) do not hold for any \( z \in \mathcal{F}_i, \) then i), ii) hold and so condition 1. is true. Otherwise if there is \( z \in \mathcal{F}_i' \) such that \( z \cup a_{i+1} = \mu_i, \) then iii) holds and so \( z = \varphi_{a_{i+1}}(z) \) is missing in \( \mathcal{F}_i' \), whence \( \mathcal{F}_i' = \mathcal{F}_{i+1} \setminus \{\mu_{i+1}\} \) with \( \mu_{i+1} = z \) and \( \varphi'_{a_{i+1}}(z) = \mu_i. \) For any \( y \in \mathcal{F}_i' \setminus \{z\} \) either condition i) or ii) holds and so \( \varphi'_{a_{i+1}}(y) = \varphi_{a_{i+1}}(y), \) and this concludes the proof of statement 2.

The previous Lemma shows that in each \( i \)-section \( \mathcal{F}_i' \) there is exactly one missing element belonging to \( \mathcal{F}_i \setminus \mathcal{F}_i' \). This element plays an important role in the way the raising function changes. For this reason we call \( \mu_i \) of Lemma 10 the missing element at the \( i \)-th section. The next lemma gives a more precise description of the way the raising function changes.

**Lemma 11** (swapping lemma). With the notation of Lemma 10, there are \( k + 1 \) different elements \( m_i \in \mathcal{F}, \) for \( i = 0, \ldots, k \) such that \( m_0 = m \) and an increasing sequence of \( k \) integers \( 1 \leq i_1 < \ldots < i_k < n \) such that for all \( 0 < j \leq k \)

\[
\varphi'_t(m_j) = \begin{cases} 
\varphi_t(m_j) & \text{if } t < i_j \\
\varphi_t(m_{j-1}) & \text{otherwise;}
\end{cases}
\]

while \( \varphi'_t(z) = \varphi_t(z) \) for all \( 1 \leq t \leq n \) and \( z \in \mathcal{F} \setminus \{m_0, \ldots, m_k\} \). For any \( 0 \leq i \leq n \) the missing element is \( \mu_i = \varphi_i(m_s) \) where \( 0 < s \leq k \) satisfies \( i_s \leq i < i_{s+1} \) if \( s < k \) or \( i_{k} \leq i < n \) if \( s = k \). Moreover for all \( 1 \leq j \leq k \), \( \varphi'_{i_{j-1}}(m_j) \cup \{a_{i_j}\} = \varphi_{i_{j-1}}(m_{j-1}) \).

**Proof.** By Lemma 10 we have \( \varphi'_{a_i}(z) = \varphi_{a_i}(z) \) for all \( z \in \mathcal{F} \) and the missing element is \( \mu_i = \varphi_i(m_0) \) for all \( 1 \leq t < i_1 \leq n \) and \( i_1 \) is the first integer such that there is an element \( \varphi'_{i_{1-1}}(m_1) \in \mathcal{F}_{i_{1-1}}' \), for some \( m_1 \in \mathcal{F} \) with \( m_1 \neq m_0 \), satisfying \( \varphi'_{i_{1-1}}(m_1) \cup \{a_{i_1}\} = \mu_{i_{1-1}} = \varphi_{i_{1-1}}(m_0) \). Therefore by Lemma 10 we have that \( \varphi'_{i_{1}}(m_1) = \varphi_{i_{1}}(m_0) = \varphi_{i_{1}}(m_0) \) and the missing element becomes \( \mu_{i_1} = \varphi_{i_1}(m_1) \). Moreover, since \( \varphi'_{i_{1-1}}(m_1) \) and \( \varphi'_{i_{1-1}}(m_1) \cup \{a_{i_1}\} = \mu_{i_{1-1}} = \varphi_{i_{1-1}}(m_0) \in \mathcal{F}_{i_{1-1}}' \), by Lemma 10 we get \( a_{i_1} \notin \varphi_{i_1}(m_1) = \mu_{i_1}. \) In this way we have proved the base case of the following property

\( P_h: \) There is a sequence of integers \( 1 \leq i_1 < \ldots < i_j \leq h < n \) and \( j + 1 \) different elements \( m_0, \ldots, m_j \in \mathcal{F} \) such that for all \( 0 < l \leq j \) and for all \( t \leq h \)

\[
\varphi'_t(m_l) = \begin{cases} 
\varphi_t(m_l) & \text{if } t < i_l \\
\varphi_t(m_{l-1}) & \text{otherwise;}
\end{cases}
\]
Hence the proof we have to show becomes
\[
\text{since } a_{i_1} = \varphi_i(m_j) \text{ for } i_j \leq i \leq h. \text{ Moreover for all } 0 < s \leq j, \varphi_{i_s - 1}(m_s) \cup \{a_i\} = \varphi_{i_s - 1}(m_{s-1}) \text{ and } a_{i_1}, \ldots, a_{i_s} \notin \mu_{i_s}.
\]

Let us prove this property by induction. By Lemma 10 it is clear that if for any \( z \in \mathcal{F}_h \) the condition \( z \cup \{a_{h+1}\} = \mu_h \) does not occur, then \( P_{h+1} \) is true. Suppose that \( z \cup \{a_{h+1}\} = \mu_h \). Let us prove that there is an element \( m_{j+1} \in \mathcal{F} \) different from \( m_l \) for all \( l \leq j \) such that \( z = \varphi_h(m_{j+1}) \). Suppose, contrary to our claim, that \( m_{j+1} = m_s \) for some \( s \leq j \). We first claim that \( a_{i_s} \in \varphi_h(m_s) \setminus \mu_h \). Indeed conditions \( a_{i_1}, \ldots, a_{i_j} \notin \mu_i \) and \( s \leq j \) yields to \( a_{i_s} \notin \mu_i = \varphi_i(m_j) \) and so, since \( \mu_h = \varphi_h(m_j) \) and \( i_s \leq i_j \leq h \), we get \( a_{i_s} \notin \mu_h \). Since \( \varphi_{i_s - 1}(m_s) \cup \{a_i\} = \varphi_{i_s - 1}(m_{s-1}) \), by Lemma 1 \( a_{i_s} \in \varphi_{i_s - 1}(m_{s-1}) = \varphi_{i_s}(m_{s-1}) \), hence by property \( P_h \), we get \( a_{i_s} \notin \varphi_{i_s - 1}(m_{s-1}) = \varphi_{i_s}(m_{s-1}) = \varphi_{i_s}(m_s) \). Thus \( a_{i_s} \in \varphi_h(m_s) \) and so, with \( a_{i_s} \notin \mu_h \), we get the claim \( a_{i_s} \in \varphi_h(m_s) \setminus \mu_h \). However this contradicts \( \varphi_{i_s}(m_s) \cup \{a_{i_s}\} = z \cup \{a_{h+1}\} = \mu_h \).

Therefore we can suppose that there is a \( m_{j+1} \in \mathcal{F} \) different from \( m_0, \ldots, m_j \) such that \( \varphi_h(m_{j+1}) \cup \{a_{h+1}\} = \mu_h \). Therefore by induction we get \( \varphi_i(m_{j+1}) = \varphi_i(m_{j+1}) \) for all \( 0 \leq h \leq i_{j+1} - 1 \). Putting \( i_{j+1} = h + 1 \) we get, by Lemma 10 and \( P_h \)
\[
\varphi_{i_{j+1}}(m_{j+1}) = \mu_{i_{j+1} - 1} = \mu_h = \varphi_h(m_j) = \varphi_{h+1}(m_j) = \varphi_{i_{j+1}}(m_j)
\]
since \( a_{i_{j+1}} \in \mu_h \) and so \( \mu_h = \varphi_h(m_j) = \varphi_{h+1}(m_j) \). Moreover we also have \( \varphi_{i_{j+1}}(m_{j+1}) \cup \{a_{i_{j+1}}\} = \varphi_{i_{j+1} - 1}(m_j) \).

Since \( \varphi_h(m_{j+1}) \cup \{a_{h+1}\} = \mu_h \in \mathcal{F}_h \) and \( \varphi'(m_{j+1}) = \varphi_h(m_{j+1}) \) we have that \( \varphi_h(m_{j+1}) = \varphi_{h+1}(m_{j+1}) \) and so by Lemma 10 the missing element becomes
\[
\mu_{i_{j+1}} = \varphi_h(m_{j+1}) = \varphi_{h+1}(m_{j+1}) = \varphi_{i_{j+1}}(m_{j+1})
\]
Hence \( \mu_{i_{j+1}} \cup \{a_{h+1}\} = \mu_h \) and so, by Lemma 1 \( a_{h+1} \notin \mu_{i_{j+1}} \). To conclude the proof we have to show \( a_{i_1}, \ldots, a_{i_{j+1}} \notin \mu_{i_{j+1}} \). By induction \( a_{i_1}, \ldots, a_{i_j} \notin \mu_i \), hence \( a_{i_1}, \ldots, a_{i_j} \notin \mu_h \). Therefore, from \( \mu_{i_{j+1}} \cup \{a_{h+1}\} = \mu_h \) and \( a_{h+1} \notin \mu_{i_{j+1}} \) we get \( a_{i_1}, \ldots, a_{i_j}, a_{i_{j+1}} \notin \mu_{i_{j+1}} \).

We remark that the above swapping Lemma holds for a general family of subsets \( \mathcal{F} \) since the hypothesis of \( \cup \)-closure is never used in the proof. As a consequence of the previous swapping Lemma we have the following proposition.
Proposition 14. Let $\mathcal{F}$ be a family of subsets of $X = \{a_1, \ldots, a_n\}$ and let $m \in \mathcal{F}$. Consider the rising functions $\varphi_w$, $\varphi'_w$ with respect to the word $w = a_1a_2 \ldots a_n$ respectively of $\mathcal{F}$, $\mathcal{F}' = \mathcal{F} \setminus \{m\}$. There are $k + 1$ different elements $m_i \in \mathcal{F}$, for $i = 0, \ldots, k$ such that $m_0 = m$ and for all $z \in \mathcal{F} \setminus \{m_0, \ldots, m_k\}$ we have $\varphi'_w(z) = \varphi_w(z)$ while for all $0 < j \leq k$

$$\varphi'_w(m_j) = \varphi_w(m_{j-1})$$

in particular $\varphi'_w(\mathcal{F}') = \varphi_w(\mathcal{F}) \setminus \{\varphi_w(m_k)\}$ and $\varphi_w(m_k) \in \min(\varphi_w(\mathcal{F}))$. Moreover $a_{i_j} \in m_{j-1} \setminus \varphi_w(m_j)$ for all $1 \leq j \leq k$.

Proof. The first claim is an immediate consequence of Lemma 1 when $t = n$. In particular the missing element $\mu_n = \varphi_w(m_k)$ and so $\varphi'_w(\mathcal{F}') = \varphi_w(\mathcal{F}) \setminus \{\varphi_w(m_k)\}$. Moreover since both $\varphi'_w(\mathcal{F}')$, $\varphi_w(\mathcal{F})$ are upward-closed sets, then it is straightforward to prove that necessarily the missing elements must be minimal, otherwise $\varphi'_w(\mathcal{F}) \setminus \{\varphi_w(m_k)\}$ would not be upward-closed, whence $\varphi_w(m_k) \in \min(\varphi_w(\mathcal{F}))$.

From Lemma 11 we have that for all $1 \leq j \leq k$, $\varphi'_{i_j-1}(m_j) \cup \{a_{i_j}\} = \varphi_{i_j-1}(m_{j-1})$ and $\varphi'_{i_j-1}(m_j) = \varphi_{i_j-1}(m_j)$, whence

$$\varphi_{i_j-1}(m_j) \cup \{a_{i_j}\} = \varphi_{i_j-1}(m_{j-1})$$

and so, by Lemma 11 we get for all $1 \leq j \leq k$, $a_{i_j} \in m_{j-1} \setminus \varphi_w(m_j)$. \qed

We now assume $\mathcal{F}$ is $\cup$-closed and we consider the situation when we take away an irreducible element $m \in J(\mathcal{F})$. In this case we have a limitation on the number of possible swappings, indeed the following proposition holds.

Proposition 15. Let $\mathcal{F}$ be a $\cup$-closed family of subsets of a set $X = \{a_1, \ldots, a_n\}$ and let $m \in J(\mathcal{F})$. Consider the rising functions $\varphi_w$, $\varphi'_w$ with respect to the word $w = a_1a_2 \ldots a_n$ respectively of $\mathcal{F}$, $\mathcal{F}' = \mathcal{F} \setminus \{m\}$ and denote by $\mathcal{F} = \varphi_w(\mathcal{F})$, $\mathcal{F}' = \varphi'_w(\mathcal{F}')$. There are two possibilities:

1. $\varphi_w(m) \in \min(\mathcal{F})$, $\mathcal{F}' = \mathcal{F} \setminus \{\varphi_w(m)\}$ and for all $z \in \mathcal{F}'$ $\varphi'_w(z) = \varphi_w(z)$.

2. The set $\overline{m} = \cup \{f \in \mathcal{F} : f \subseteq \varphi_w(m)\}$ is non-empty. For all $z \in \mathcal{F} \setminus \{m, \overline{m}\}$ we have $\varphi'_w(z) = \varphi_w(z)$ and $\varphi'_w(\overline{m}) = \varphi_w(m)$. Moreover $\varphi_w(\overline{m}) \in \min(\mathcal{F})$ and $\mathcal{F}' = \mathcal{F} \setminus \{\varphi_w(\overline{m})\}$.

Proof. Using the notation of Proposition 14 suppose $k \geq 2$. Therefore, there are two distinct elements $m_1, m_2$ different from $m$ such that $\varphi'_w(m_2) = \varphi_w(m_1)$ and $\varphi'_w(m_1) = \varphi_w(m)$. We claim

$$\{f \in \mathcal{F} : f \subseteq \varphi_w(m_1)\} = \{f \in \mathcal{F}' : f \subseteq \varphi_w(m_1)\}$$

(7)
Clearly \( \{ f \in \mathcal{F}' : f \subseteq \varphi_w(m_1) \} \subseteq \{ f \in \mathcal{F} : f \subseteq \varphi_w(m_1) \} \) and to prove the other inclusion it is sufficient to prove that \( m \not\subseteq \varphi_w(m_1) \). Suppose on the contrary that actually \( m \subseteq \varphi_w(m_1) \), however by Proposition 14 \( a_i, a \in m \setminus \varphi_w(m_1) \), a contradiction. Thus \( \triangledown \) holds.

Since \( m \in J(\mathcal{F}) \), then \( \mathcal{F}' = \mathcal{F} \setminus \{ m \} \) is a \( \cup \)-closed family and so by Corollary 11 equality \( \triangledown \) and \( \varphi'_w(m_2) = \varphi_w(m_1) \) we get

\[
m_1 = \varphi_w^{-1}(\varphi_w(m_1)) = \bigcup_{f \in \mathcal{F} : f \subseteq \varphi_w(m_1)} f = \bigcup_{f \in \mathcal{F} : f \subseteq \varphi_w(m_1)} f = \varphi_w^{-1}(\varphi'_w(m_2)) = m_2
\]

a contradiction. Therefore we have two possibilities either \( k = 0 \) or \( k = 1 \).

Applying Proposition 14 to the case \( k = 0 \) we get for all \( z \in \mathcal{F} \setminus \{ m \} \)

\[
\varphi_w(z) = \varphi_w(z) \quad \text{and} \quad \mathcal{F}' = \mathcal{F} \setminus \{ \varphi_w(m) \} \quad \text{and} \quad \varphi_w(m) \in \min(\mathcal{F}).
\]

Consider the case \( k = 1 \). We prove that in this case \( m_1 = \overline{m} \) where \( \overline{m} = \bigcup_{f \in \mathcal{F} : f \subseteq m} f \).

Since \( \varphi'_w(m_1) = \varphi_w(m) \) then \( m_1 \not\subseteq \varphi_w(m) \) and so by Theorem 11 \( m_1 \subseteq m, \) hence \( \overline{m} \neq \emptyset \). Since \( \varphi'_w(m_1) = \varphi_w(m) \) then

\[
\{ f \in \mathcal{F}' : f \subseteq \varphi'_w(m_1) \} = \{ f \in \mathcal{F}' : f \subseteq \varphi_w(m) \}
\]

moreover \( \{ f \in \mathcal{F} : f \not\subseteq m \} \subseteq \{ f \in \mathcal{F}' : f \subseteq \varphi_w(m) \} \) and by Theorem 14 it is not difficult to check that \( \{ f \in \mathcal{F}' : f \subseteq \varphi_w(m) \} \subseteq \{ f \in \mathcal{F} : f \not\subseteq m \} \) also holds. Hence by equality \( \triangledown \) we have \( \{ f \in \mathcal{F}' : f \subseteq \varphi'_w(m_1) \} = \{ f \in \mathcal{F} : f \not\subseteq m \} \) and so by Corollary 11

\[
m_1 = \varphi'_w(m_1) = \bigcup_{f \in \mathcal{F} : f \subseteq \varphi'_w(m_1)} f = \bigcup_{f \in \mathcal{F} : f \subseteq \varphi'_w(m_1)} f = \overline{m}
\]

The other properties are consequences of Proposition 14.

We have the following theorem.

**Theorem 4.** Let \( \mathcal{F} \) be a \( \cup \)-closed family of sets of \( 2^X \) with \( X = \{ a_1, \ldots, a_n \} \). Consider the rising function \( \varphi_w \) with respect to the word \( w = a_1a_2 \ldots a_n \) and let \( \mathcal{F} = \varphi_w(\mathcal{F}) \), then

\[
|J(\mathcal{F})| \leq 2|\min(\mathcal{F})| + |\min(\mathcal{F} \setminus \min(\mathcal{F}))|
\]

**Proof.** \( J(\mathcal{F}) \) can be partitioned into two subsets \( J_1, J_2 \) respectively of the elements \( m \in J(\mathcal{F}) \) such that \( \varphi_w(m) \in \min(\mathcal{F}) \) and the elements \( m \) for which condition 2 of Proposition 15 holds but \( \varphi_w(m) \notin \min(\mathcal{F}) \) (conditions...
In view of Proposition 15, we consider the restriction \( \iota_\mathcal{F} : J(\mathcal{F}) \to \mathcal{F} \) taking an element \( m \) into

\[
\iota_\mathcal{F}(m) = \bigcup_{f \in \mathcal{F}, f \subseteq m} f
\]

It is straightforward to check that whenever it is defined: \( \iota_\mathcal{F}(m) \subseteq m \) (it can not be equal since \( m \) is irreducible) and if \( m' \subseteq m \), then \( m' \subseteq \iota_\mathcal{F}(m) \).

In view of Proposition 15, we consider the restriction \( \iota_\mathcal{F} : J_2 \to \mathcal{F} \) which is a function. Thus for a \( \overline{m} \in \iota_\mathcal{F}(J_2) \), the set \( \iota_\mathcal{F}^{-1}(\overline{m}) \) is clearly non-empty and let \( \iota_\mathcal{F}^{-1}(\overline{m}) = \{m_1, \ldots, m_k\} \) for some \( k \geq 1 \). We observe that for all \( i \neq j \), \( m_i \nsubseteq m_j \) since, otherwise \( m_i \subseteq m_j \) would imply the contradiction \( \overline{m} \subseteq m_i \subseteq \iota_\mathcal{F}(m_j) = \overline{m} \). Therefore for all \( i \neq j \)

\[
\iota_\mathcal{F}(m_j) = \iota_\mathcal{F}(J_2 \setminus \{m_i\})(m_j)
\]

We claim that for all \( i \neq j \) we have that at least one between \( \varphi_w(m_i), \varphi_w(m_j) \) is minimal in \( \mathcal{F}' = \mathcal{F} \setminus \{\varphi_w(\overline{m})\} \). This is a consequence of the application of Proposition 15 twice. Indeed, consider \( \mathcal{F} \setminus \{m_i\} \) and let \( \varphi'_w \) be the rising function of this set with respect to \( w \). By Proposition 15 we have \( \varphi_w(\overline{m}) \in \min(\mathcal{F}) \), \( \varphi'_w(\overline{m}) = \varphi_w(m_i) \) and \( \varphi'_w(m_j) = \varphi_w(m_j) \). It is evident that \( m_j \in J(\mathcal{F} \setminus \{m_i\}) \) and so consider the \( \cup \)-closed set \( (\mathcal{F} \setminus \{m_i\}) \setminus \{m_j\} \). Let \( \varphi'_w \) be the rising function of this set with respect to \( w \). By Proposition 15 we have two possibilities: either \( \varphi'_w(m_j) = \varphi_w(m_j) \) is minimal in \( \mathcal{F} \setminus \{\varphi_w(\overline{m})\} \), or by (10), we have that

\[
\varphi'_w(\iota_\mathcal{F}(J_2 \setminus \{m_i\})(m_j)) = \varphi'_w(\iota_\mathcal{F}(m_j)) = \varphi'_w(\overline{m}) = \varphi_w(m_i)
\]

is minimal in \( \mathcal{F} \setminus \{\varphi_w(\overline{m})\} \). Therefore, it is straightforward to prove that all the \( m_i \) except at most one, say \( m_k \), are minimal in \( \mathcal{F} \setminus \{\varphi_w(\overline{m})\} \). Hence, denoting by \( J'_2 \) the set of elements \( m \in J_2 \) such that \( \varphi_w(m) \) is minimal in \( \mathcal{F} \setminus \{\varphi_w(\iota_\mathcal{F}(m))\} \), we get that there is an injection of \( J_2 \setminus J'_2 \) into \( \iota_\mathcal{F}(J_2 \setminus J'_2) \) which is in one to one correspondence with the elements of \( \varphi_w(\iota_\mathcal{F}(J_2 \setminus J'_2)) \) (being \( \varphi_w \) injective) which is in turn a subset of \( \min(\mathcal{F}) \) (by definition of the set \( J_2 \) and Proposition 15), whence:

\[
|J_2 \setminus J'_2| \leq |\min(\mathcal{F})|
\]
We now prove that $\varphi_w(J'_2) \subseteq \min(\mathcal{F} \setminus \min(\mathcal{F}))$. Since $J'_2 \subseteq J_2$, then, by definition of $J_2$, we have that $\varphi_w(m) \notin \min(\mathcal{F})$ for all $m \in J'_2$. Thus $\varphi_w(J'_2) \subseteq \mathcal{F} \setminus \min(\mathcal{F})$. Moreover, if $m \in J'_2$, then $\varphi_w(m)$ is minimal in $\mathcal{F} \setminus \{\varphi_w(\iota_{\mathcal{F}}(m))\}$ and since $\varphi_w(\iota_{\mathcal{F}}(m)) \in \min(\mathcal{F})$ we have

$$\mathcal{F} \setminus \min(\mathcal{F}) \subseteq \mathcal{F} \setminus \{\varphi_w(\iota_{\mathcal{F}}(m))\}$$

hence $\varphi_w(m)$ is also minimal in $\mathcal{F} \setminus \min(\mathcal{F})$, and so the claim $\varphi_w(J'_2) \subseteq \min(\mathcal{F} \setminus \min(\mathcal{F}))$. Therefore $|J'_2| \leq |\min(\mathcal{F} \setminus \min(\mathcal{F}))|$, and so by (9), (11) we obtain the upper bound of the statement

$$|J(\mathcal{F})| = |J_1| + |J_2 \setminus J'_2| + |J'_2| \leq 2|\min(\mathcal{F})| + |\min(\mathcal{F} \setminus \min(\mathcal{F}))|$$

As an immediate consequence of the previous theorem and Sperner’s Theorem we have $|J(\mathcal{F})| \leq 3\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$. This bound is not the best that can be obtained from Theorem 4. Indeed, we devote Subsection 6.1 to prove Theorem 5 showing that for an upward-closed family $\mathcal{F}$ on a set $X$ with $|X| = n$ we have $2|\min(\mathcal{F})| + |\min(\mathcal{F} \setminus \min(\mathcal{F}))| \leq 2\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right) + \left(\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1\right)$ and this bound is tight. Therefore we have the following corollary.

**Corollary 5.** Let $\mathcal{F}$ be a $\cup$-closed family of sets of $2^X$ with $X = \{a_1, \ldots, a_n\}$, then

$$|J(\mathcal{F})| \leq 2\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right) + \left(\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1\right)$$

In particular any family $S \subseteq 2^X$ with $|S| > 2\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right) + \left(\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1\right)$ is not $\cup$-independent.

A natural question that arises from this corollary is the precise upper bound of the quantity

$$J(n) = \max\{|J(\mathcal{F})| : \mathcal{F} \text{ is a } \cup \text{-closed family on a set } X \text{ with } |X| = n\}$$

Although we are not able to answer to this question we can easily give a lower bound to $J(n)$. Indeed, consider the $\cup$-closed family of $2^X$ consisting of elements whose cardinality is greater than or equal to $\lfloor \frac{n}{2} \rfloor$. The set of joint-irreducible elements consists of the subsets of cardinality exactly $\lfloor \frac{n}{2} \rfloor$, whence we can bound the function $J(n)$ as

$$\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right) \leq J(n) \leq 2\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right) + \left(\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1\right)$$
6.1 An extremal problem

In this section we study the extremal problem of maximizing the quantity 
\[2|\min(F)| + |\min(F \setminus \min(F))|\] where \(F\) is an upward-closed set on the set \(X\). We can restate this problem in the following way. Given an antichain \(\mathcal{A}\) of \(2^X\), we want to maximize the quantity \(2|\mathcal{A}| + |\min(\mathcal{A}^\uparrow \setminus \mathcal{A})|\). Before studying this problem more in detail we give some definitions. For an integer \(0 < k \leq n\) we denote by \(A_k = \{A \in \mathcal{A} : |A| = k\}\), in general a family of \(k\)-subsets \(\mathcal{B}\) is a collection of sets of \(X\) with cardinality \(k\). We recall that the \textit{shade} (see [5]) of \(A_k\) is defined by 
\[\nabla(A_k) = \{B \in 2^X : |B| = k + 1, A \subseteq B\ for\ some\ A \in A_k\}\]
Similarly the \textit{shadow} of \(A_k\) is defined by 
\[\Delta(A_k) = \{B \in 2^X : |B| = k - 1, B \subseteq A\ for\ some\ A \in A_k\}\]
We can extend these definitions to the whole set \(\mathcal{A}\) by taking \(\nabla(\mathcal{A}) = \bigcup_{k=1}^n \nabla(A_k)\) and \(\Delta(\mathcal{A}) = \bigcup_{k=1}^n \Delta(A_k)\). Note that \(\min(\mathcal{A}^\uparrow \setminus \mathcal{A}) = \min(\nabla(\mathcal{A}))\). Thus, it makes sense defining the \textit{first upward level} of an antichain \(\mathcal{A}\) as the set \(\nabla(\mathcal{A}) = \min(\nabla(\mathcal{A}))\). The operator \(\nabla\) is also interesting because \(\mathcal{A}^\downarrow\) can be partitioned into “foils”, where the \(i\)-th foil for \(i \geq 1\) is given by \(\nabla^i(\mathcal{A}) = \nabla(\nabla^{i-1}(\mathcal{A}))\) and \(\nabla^0(\mathcal{A}) = \mathcal{A}\). We state some useful properties whose proofs are left to the reader.

Lemma 12. Let \(\mathcal{A}, \mathcal{B}\) be two antichains, then:

1. \(\nabla(\mathcal{A} \cup \mathcal{B}) = \nabla(\mathcal{A}) \cup \nabla(\mathcal{B}), \Delta(\mathcal{A} \cup \mathcal{B}) = \Delta(\mathcal{A}) \cup \Delta(\mathcal{B}), \mathcal{A} \subseteq \nabla(\Delta(\mathcal{A})), \mathcal{A} \subseteq \Delta(\nabla(\mathcal{A})).\)

2. Assume \(\mathcal{A} \subseteq \mathcal{B}\), then for any \(g \in \nabla(\mathcal{A})\) there is a \(g' \in \nabla(\mathcal{B})\) such that \(g' \subseteq g\).

3. \(\nabla(\mathcal{A}) \subseteq \nabla(\mathcal{A})\), moreover \(g \in \nabla(\mathcal{A}) \setminus \nabla(\mathcal{A})\) iff there is \(g' \in \nabla(\mathcal{A})\) such that \(g' \subseteq g\).

4. If \(\mathcal{A} \cup \mathcal{B}\) is an antichain and for all \(g \in \nabla(\mathcal{A})\) there is no \(g' \in \nabla(\mathcal{B})\) such that \(g' \subseteq g\), then \(\nabla(\mathcal{A}) \subseteq \nabla(\mathcal{A} \cup \mathcal{B}).\)

5. Assume \(\mathcal{A} \subseteq \mathcal{B}\), if for any \(g \in \nabla(\mathcal{A})\) there is a \(g' \in \nabla(\mathcal{B} \setminus \mathcal{A})\) such that \(g' \subseteq g\), then \(\nabla(\mathcal{B} \setminus \mathcal{A}) = \nabla(\mathcal{B}).\)

We devote the rest of the paper to the proof of following theorem.
Theorem 5. Let $A$ be an antichain of $2^X$ with $|X| = n$, then

$$2|A| + |\nabla(A)| \leq 2 \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$$

and this bound is tight.

Note that if $n$ is odd the theorem can be easily proved. Indeed, both $|A|$ and $|\nabla(A)|$ are antichains, hence by Sperner’s theorem we get $2|A| + |\nabla(A)| \leq 3 \left( \left\lfloor \frac{n}{2} \right\rfloor \right) = 2 \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + \left( \left\lfloor \frac{n}{2} \right\rfloor \right)$ since $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1$. It is not difficult to check that this bound is attained when $A$ consists of all the $n - 1$ subsets of $X$. Therefore, in the sequel we can assume that $n$ is even. We prove the theorem using an augmentation argument. More precisely, we define two maps $\alpha^+, \alpha^-$, called respectively the upward-augmenting, lower-augmenting map, with the property of transforming $A$ into the antichains $\alpha^+(A)$, $\alpha^-(A)$ with

$$2|A| + |\nabla(A)| \leq 2|\alpha^+(A)| + |\nabla(\alpha^+(A))|$$
$$2|A| + |\nabla(A)| \leq 2|\alpha^-(A)| + |\nabla(\alpha^-(A))|$$

then we repetitively apply these operators to obtain an antichain formed by $k$-subsets of $X$ with $k = \frac{n}{2}, \frac{n}{2} - 1$. However we define these maps only for particular classes of antichains that we are going to introduce, first we need some preliminary definitions. Given a family $B \subseteq 2^X$ we denote the maximum (minimum) of the lengths of the elements of $B$ by $\|B\|_M (\|B\|_m)$, and we put $\text{Max}(B) = \{B \in B : |B| = \|B\|_M\}$, $\text{Min}(B) = \{B \in B : |B| = \|B\|_m\}$. The following lemma shows that we can restrict our attention to a particular class of antichains.

Lemma 13. Let $A'$ be an antichain in $2^X$, then

1) for any $h \in \text{Min}(A')$ and $a \in X \setminus h$ we have $h \cup \{a\} \in \nabla(A')$.

Moreover there is an antichain $A \supseteq A'$ such that $|A| = \|A'\|_M$, $|A| = \|A'\|_m$, $2|A'| + |\nabla(A')| \leq 2|A| + |\nabla(A)|$ and with the following property:

2) let $k = \|\nabla(A)\|_M$, then either $\cup_{i \geq k} A_i \neq \emptyset$ or for any $h \in \text{Max}(\nabla(A))$ and $a \in h$ we have $h \setminus \{a\} \in A$.

Proof. Let us prove Condition 1). To obtain a contradiction suppose that there is $h \in \text{Min}(A')$ and $a \in X \setminus h$ such that $h \cup \{a\} \not\in \nabla(A)$. Thus $h \cup \{a\} \in \nabla(A') \setminus \nabla(A')$, and so by property 3 of Lemma 12 there is a
\[ g \in \nabla(A') \text{ and } h' \in A' \text{ with } h' \subsetneq g \subseteq h \cup \{a\}, \text{ whence } |h'| < |g| < |h| + 1. \]
Thus \(|h'| < |h|\) which contradicts the minimality of \(|h|\).

The second statement is proved if we show that given an antichain \(B'\) with \(k = \|B'\|_M\) either \(\bigcup_{i \geq k} B'_i \neq \emptyset\) or if there is an \(h \in \text{Max}(\nabla(B'))\) and \(a \in h\) such that \(h \setminus \{a\} \not\subseteq B'\), then the family \(B = B' \cup \{h \setminus \{a\}\}\) is an antichain with \(2|B'| + |\nabla(B')| = 2|B| + |\nabla(B)|\). Indeed starting from \(A'\) by repetitively adding elements for which condition 2) does not hold, we eventually end with an antichain \(A\) satisfying property 2). Suppose that \(\bigcup_{i \geq k} B'_i = \emptyset\), otherwise we have done. It is easily seen that \(\|B\|_M = \|B'\|_M, \|B\|_m = \|B'\|_m\). We now prove that \(B\) is an antichain. Note first that \(h\) cannot be a singleton, thus to reach a contradiction suppose \(B\) is not an antichain. Since \(B'\) is an antichain, there is a \(g \in B'\) such that either \(g \subsetneq h \setminus \{a\}\) or \(h \setminus \{a\} \subsetneq g\). Suppose that \(g \subsetneq h \setminus \{a\}\), hence there is a \(h' \in \nabla(B')\) such that \(h' \subsetneq g \cup \{a\} \subsetneq h\) which contradicts the fact that \(\nabla(B')\) is an antichain. On the other hand suppose that \(h \setminus \{a\} \subsetneq g\). Thus, \(|g| \geq |h| = k\) which implies \(g \in \bigcup_{i \geq k} B'_i = \emptyset\), a contradiction. Therefore \(B = B' \cup \{h \setminus \{a\}\}\) is an antichain. We now prove that \(\nabla(B') \subseteq \nabla(B)\) from which, with \(B = B' \cup \{h \setminus \{a\}\}\), implies our claim \(2|B'| + |\nabla(B')| = 2|B| + |\nabla(B)|\). Suppose, contrary to our claim, that there is a \(t \in \nabla(B') \setminus \nabla(B)\) \(\neq \emptyset\). It is straightforward to check that there is a \(t' \in \nabla(B)\) with \(t' \subsetneq t\). It follows easily that \(h \setminus \{a\} \subsetneq t' \subsetneq t\) (otherwise we would have the contradiction \(t' \in \nabla(B')\)). Thus we have \(|t| > |t'| \geq |h| = k\), against \(\|\nabla(B')\|_M = k\). \(\square\)

An antichain \(A\) satisfying the two properties in Lemma 13 is called augmentable. We now define the lower-augmenting, upward-augmenting map on the set of augmentable antichains over \(X\). Given an augmentable antichain \(A\) with \(k = \|\nabla(A)\|_M, k' = \|A\|_M, s = \|A\|_m\), the lower-augmenting map is defined by

\[
\alpha^- (A) = \begin{cases} 
(A \setminus A_{k'}) \cup \Delta(A_{k'}), & \text{if } k' \geq k, k' \geq \frac{s}{2} + 1 \\
(A \setminus A_{k-1}) \cup \Delta(A_{k-1}), & \text{if } k' < k, k > \frac{s}{2} + 1 \\
A & \text{otherwise.}
\end{cases}
\]

and the upward-augmenting map by

\[
\alpha^+ (A) = \begin{cases} 
(A \setminus A_s) \cup \nabla(A_s), & \text{if } s < \frac{k}{2} - 1 \\
A & \text{otherwise.}
\end{cases}
\]

The following lemma shows that \(\alpha^+(A), \alpha^-(A)\) are antichains.

**Lemma 14.** Let \(A\) be an antichain and let \(M = \|A\|_M, m = \|A\|_m\), then

\[ A \setminus A_m \cup \nabla(A_m), A \setminus A_M \cup \Delta(A_M) \]
are antichains with $\mathcal{A} \setminus \mathcal{A}_m \cap \Delta(\mathcal{A}_m) = \emptyset$, $\mathcal{A} \setminus \mathcal{A}_M \cap \Delta(\mathcal{A}_M) = \emptyset$.

**Proof.** Suppose, contrary to our claim, that there is $g \in \mathcal{A} \setminus \mathcal{A}_m \cap \Delta(\mathcal{A}_m) \neq \emptyset$. Thus $g \in \Delta(\mathcal{A}_m)$ implies that there is a $g' \in \mathcal{A}_m$ such that $g' \subsetneq g$ which contradicts the fact that $\mathcal{A}$ is an antichain. Similarly, $\mathcal{A}_M \cap \Delta(\mathcal{A}_M) \neq \emptyset$ contradicts the fact that $\mathcal{A}$ is an antichain. Let us prove that $\mathcal{A} \setminus \mathcal{A}_m \cup \Delta(\mathcal{A}_m)$ is an antichain. Since the two terms of the union are disjoint antichains, to reach a contradiction, we can suppose that there is a $g \in \mathcal{A} \setminus \mathcal{A}_m$ and $g' \in \Delta(\mathcal{A}_m)$ such that either $g \subsetneq g'$ or $g' \subsetneq g$. Since $m$ is the minimum of the length of the elements of $\mathcal{A}$, then $g \in \mathcal{A} \setminus \mathcal{A}_m$ implies $|g| \geq m + 1$, while $g' \in \Delta(\mathcal{A}_m)$ implies $|g'| = m + 1$. Thus only $g' \subsetneq g$ can occur. However $g' \in \Delta(\mathcal{A}_m)$ implies $g'' \subsetneq g'$, for some $g'' \in \mathcal{A}_m$. Hence $g'' \subsetneq g'$ or $g' \subsetneq g$ which contradicts the fact that $\mathcal{A}$ is an antichain. Hence $\mathcal{A} \setminus \mathcal{A}_m \cup \Delta(\mathcal{A}_m)$ is an antichain. Let us prove that $\mathcal{A} \setminus \mathcal{A}_M \cup \Delta(\mathcal{A}_M)$ is also an antichain. Suppose, contrary to our claim, that $\mathcal{A} \setminus \mathcal{A}_M \cup \Delta(\mathcal{A}_M)$ is not an antichain. Similarly to the above situation, we can assume that only $g \subsetneq g'$ for $g \in \mathcal{A} \setminus \mathcal{A}_M$ and $g' \in \Delta(\mathcal{A}_M)$ can occur. However, $g' \in \Delta(\mathcal{A}_M)$ implies that there is a $g'' \in \mathcal{A}_M$ with $g' \subsetneq g''$, hence $g \subsetneq g' \subsetneq g''$ contradicts the fact that $\mathcal{A}$ is an antichain. Therefore $\mathcal{A} \setminus \mathcal{A}_M \cup \Delta(\mathcal{A}_M)$ is an antichain and this concludes the proof of the lemma. \qed

**Lemma 15.** Let $\mathcal{A}$ be an augmentable antichain, then $\alpha^+(\mathcal{A})$ is antichain with $\| \alpha^+(\mathcal{A}) \|_m > \| \mathcal{A} \|_m$ if $\| \mathcal{A} \|_m < \| \mathcal{A} \|_M$ then $\| \alpha^+(\mathcal{A}) \|_M = \| \mathcal{A} \|_M$, moreover:

$$2|\mathcal{A}| + |\Delta(\mathcal{A})| \leq 2|\alpha^+(\mathcal{A})| + |\Delta(\alpha^+(\mathcal{A}))|$$

**Proof.** Let $s = \| \mathcal{A} \|_m$, it is evident that $\alpha^+$ substitutes $\text{Min}(\mathcal{A})$ with $\Delta(\text{Min}(\mathcal{A}))$. Thus $\| \alpha^+(\mathcal{A}) \|_m > \| \mathcal{A} \|_m$. Moreover if $s < \| \mathcal{A} \|_M$, then, since we just add elements of cardinality $s + 1$, it is also immediate that $\| \alpha^+(\mathcal{A}) \|_M = \| \mathcal{A} \|_M$.

By Lemma 14 $\alpha^+(\mathcal{A})$ is an antichain with:

$$\mathcal{A} \setminus \mathcal{A}_s \cap \Delta(\mathcal{A}_s) = \emptyset$$

(12)

Let us prove the inequality $2|\mathcal{A}| + |\Delta(\mathcal{A})| \leq 2|\alpha^+(\mathcal{A})| + |\Delta(\alpha^+(\mathcal{A}))|$. We first claim that

$$\nabla(\alpha^+(\mathcal{A})) \supseteq \nabla(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \cup \Delta(\nabla(\mathcal{A}_s))$$

(13)

where $\nabla(\mathcal{A}_s) \subseteq \nabla(\mathcal{A})$ and

$$\nabla(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \cap \Delta(\nabla(\mathcal{A}_s)) = \emptyset$$

(14)
By property 1) of an augmentable chain $A$ we have $\nabla(A_s) \subseteq \nabla(A)$. Let us prove (14). Suppose that (14) do not hold and let $h \in \nabla(A) \setminus \nabla(A_s) \cap \nabla(\nabla(A_s))$. Thus $h = g \cup \{a,b\}$ for some $g \in A_s$ and $a, b \notin g$, since $g' = g \cup \{a\} \in \nabla(A)$ we have $g' \subseteq h$ for $g', h \in \nabla(A)$, a contradiction. Let us prove (13). We split the proof of (13) by showing first $\nabla(A) \setminus \nabla(A_s) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$ and then $\nabla(\nabla(A_s)) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$.

- Case $\nabla(A) \setminus \nabla(A_s) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$. Since $\nabla(A)_{s+1} \subseteq \nabla(A_s) \subseteq \nabla(A)_{s+1}$, then $\nabla(A)_{s+1} = \nabla(A)$. Thus $\nabla(A) \setminus \nabla(A)_{s+1} = \nabla(A \setminus A_s)$, and so, by property 1) of Lemma 12 we get $\nabla(A) \setminus \nabla(A_s) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$. Suppose, contrary to our claim, that there is a $g \in \nabla(A \setminus A_s) \cap \nabla(\nabla(A_s))$ such that $g \in \nabla(A \setminus A_s \cup \nabla(A_s)) \setminus \nabla(A \setminus A_s \cup \nabla(A_s))$. Therefore, by properties 3), 1) of Lemma 12 there is a $g'' \in \nabla(A \setminus A_s \cup \nabla(A_s))$ such that $g'' \subseteq g$. We consider two cases, either $g'' \in \nabla(A \setminus A_s)$ or $g'' \in \nabla(\nabla(A_s))$. Suppose that $g'' \in \nabla(A \setminus A_s)$, then by property 2) of Lemma 12 there is a $g'' \in \nabla(A) \setminus A_s$ such that $g'' \subseteq g$. Also in this case we consider the two cases either $g'' \in \nabla(A)$ or $g'' \in \nabla(\nabla(A_s))$. Since $s = ||A||_m$, then $|g''| \geq s + 2$, while $g'' \in \nabla(\nabla(A_s))$ implies $|g| = s + 2$ which contradicts $g'' \subseteq g$. In the other case, if $g'' \in \nabla(\nabla(A_s))$, then $g'' \in \nabla(\nabla(A_s))$ and $g'' \subseteq g$ contradict the fact that $\nabla(A \setminus A_s)$ is an antichain. Hence there is a $g'' \in \nabla(\nabla(A_s)) \subseteq \nabla(A)$ such that $g'' \subseteq g' \subseteq g \in \nabla(A)$ which again contradicts the fact that $\nabla(A)$ is an antichain. Hence we conclude $\nabla(A) \setminus \nabla(A_s) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$.

- Case $\nabla(\nabla(A_s)) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$. It is evident by property 2) of Lemma 12 that $\nabla(\nabla(A_s)) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$. Suppose, contrary to our claim, that there is a $g \in \nabla(\nabla(A_s))$ such that $g \in \nabla(A \setminus A_s \cup \nabla(A_s)) \setminus \nabla(A \setminus A_s \cup \nabla(A_s))$. Therefore, by properties 3), 1) of Lemma 12 there is a $g' \in \nabla(A \setminus A_s \cup \nabla(A_s)) = \nabla(A \setminus A_s \cup \nabla(A_s))$ such that $g' \subseteq g$. Also in this case we consider the two cases either $g' \in \nabla(A) \setminus A_s$ or $g' \in \nabla(\nabla(A_s))$. Suppose that $g' \in \nabla(A) \setminus A_s$. Since $s = ||A||_m$, then $|g'| \geq s + 2$, while $g \in \nabla(\nabla(A_s))$ implies $|g| = s + 2$ which contradicts $g' \subseteq g$. In the other case, if $g' \in \nabla(\nabla(A_s))$, then $g' \in \nabla(\nabla(A_s))$ and $g' \subseteq g$ contradict the fact that $\nabla(\nabla(A_s))$ is an antichain. Hence $\nabla(\nabla(A_s)) \subseteq \nabla(A \setminus A_s \cup \nabla(A_s))$ and this completes the proof of (13).

Let us complete the proof of the lemma showing the inequality in the statement. Since $s < \frac{2}{\lambda - 1}$, then by [5] Corollary 2.1.2, $|\nabla(A_s)| - |A_s| \geq 0$ and $|\nabla(\nabla(A_s))| - |\nabla(A_s)| \geq 0$. By (12), (13), (14) we have $2|\alpha^+(A)| + |\nabla(\alpha^+(A))| \geq 2|A| - 2|A_s| + 2|\nabla(A_s)| + |\nabla(A) \setminus \nabla(A_s)| + |\nabla(\nabla(A_s))|$. Furthermore, using $\nabla(A_s) \subseteq \nabla(A)$, $|\nabla(A_s)| - |A_s| \geq 0$ and $|\nabla(\nabla(A_s))| - 35$
We now prove the inequality of the statement. We claim
\[ \kappa \] is an antichain with:
\[ k \]
Consider two cases: either
\[ k \]
We first prove the following
\[ \kappa \]
Lemma 16. Let \( \mathcal{A} \) be an augmentable antichain, then \( \alpha^{-}(\mathcal{A}) \) is an antichain with:
\[ || \alpha^{-}(\mathcal{A}) ||_{M} < || \mathcal{A} ||_{M}, \text{ if } || \mathcal{A} ||_{m} < || \mathcal{A} ||_{M} \text{ then } || \alpha^{-}(\mathcal{A}) ||_{m} = || \mathcal{A} ||_{m}, \text{ moreover:} \]
\[ 2|\mathcal{A}| + |\nabla(\mathcal{A})| \leq 2|\alpha^{-}(\mathcal{A})| + |\nabla(\alpha^{-}(\mathcal{A}))| \]
Proof. Let \( k = || \nabla(\mathcal{A}) ||_{M}, k' = || \mathcal{A} ||_{M}. \) For this operator we need to consider two cases: either \( k' \geq k \) and \( k' \geq \frac{n}{2} + 1, \) or \( k' < k \) and \( k > \frac{n}{2} + 1. \)
Note that in the case \( k' < k, \) since \( \mathcal{A} \) is augmentable, then by property 2) of Lemma 13, we have \( k' = k - 1. \) In both cases, the map \( \alpha^{-} \) substitutes \( \text{Max}(\mathcal{A}) \) with \( \Delta(\text{Max}(\mathcal{A})), \) thus \( || \alpha^{-}(\mathcal{A}) ||_{M} < || \mathcal{A} ||_{M} \text{ holds and if } || \mathcal{A} ||_{m} < || \mathcal{A} ||_{M} \text{ then it is also obvious that } || \alpha^{-}(\mathcal{A}) ||_{m} = || \mathcal{A} ||_{m}. \)
Consider now the case \( k' \geq k. \) By Lemma 14, \( \alpha^{-}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{k'}) \cup \Delta(\mathcal{A}_{k'}) \) is an antichain with:
\[ (\mathcal{A} \setminus \mathcal{A}_{k'}) \cap \Delta(\mathcal{A}_{k'}) = \emptyset \]
We now prove the inequality of the statement. We claim
\[ \nabla(\alpha^{-}(\mathcal{A})) \supseteq \nabla(\mathcal{A}) \]
We first prove that \( \nabla(\mathcal{A} \setminus \mathcal{A}_{k'}) = \nabla(\mathcal{A}). \) Since \( k = || \nabla(\mathcal{A}) ||_{M} \) and \( k' \geq k, \)
then any element in \( \nabla(\mathcal{A}_{k'}) \) contains some element in \( \nabla(\mathcal{A} \setminus \mathcal{A}_{k'}). \) Thus by property 5 of Lemma 12, we have the claim \( \nabla(\mathcal{A} \setminus \mathcal{A}_{k'}) = \nabla(\mathcal{A}). \) If we show that for any \( g \in \nabla(\mathcal{A}) \) there is no \( g' \in \nabla(\Delta(\mathcal{A}_{k'})) \) such that \( g' \subseteq g, \) then the conclusion follows from property 4 of Lemma 12 and \( \nabla(\mathcal{A} \setminus \mathcal{A}_{k'}) = \nabla(\mathcal{A}). \) Indeed suppose, contrary to our claim, that there are \( g \in \nabla(\mathcal{A}), g' \in \nabla(\Delta(\mathcal{A}_{k'})) \) such that \( g' \subseteq g. \) Since \( \nabla(\mathcal{A}) \) is formed by elements of cardinality less or equal to \( k \) and \( \nabla(\Delta(\mathcal{A}_{k'})) \) of elements whose cardinality is \( k' \geq k \) we have the contradiction \( k \geq |g| > |g'| \geq k \) and this concludes the proof of 16. We now prove the inequality in the statement of the lemma. By 15 and inclusion 16 we have
\[ 2|\alpha^{-}(\mathcal{A})| + |\nabla(\alpha^{-}(\mathcal{A}))| = 2|\mathcal{A}| - 2|\mathcal{A}_{k'}| + 2|\Delta(\mathcal{A}_{k'})| + |\nabla(\alpha^{-}(\mathcal{A}))| \geq 2|\mathcal{A}| + 2(|\Delta(\mathcal{A}_{k'})| - |\mathcal{A}_{k'}|) + |\nabla(\mathcal{A})|. \]
Since \( k' \geq \frac{n}{2} + 1, \) then by 5 Corollary 2.1.2 we have \( |\Delta(\mathcal{A}_{k'})| - |\mathcal{A}_{k'}| \geq 0, \)
whence
\[ 2|\alpha^{-}(\mathcal{A})| + |\nabla(\alpha^{-}(\mathcal{A}))| \geq 2|\mathcal{A}| + |\nabla(\mathcal{A})|. \]
Consider now the other case \( k' < k \) and \( k > \frac{n}{2} + 1. \) Therefore, by Lemma 14, \( \alpha^{-}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}) \) is an antichain with:
\[ (\mathcal{A} \setminus \mathcal{A}_{k-1}) \cap \Delta(\mathcal{A}_{k-1}) = \emptyset \]
We now prove the inequality of the statement. We first prove the following inclusion:
\[ \nabla((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1})) \supseteq \nabla(\mathcal{A}) \setminus \nabla(\mathcal{A})_{k} \cup \Delta(\nabla(\mathcal{A})_{k}) \]
with \(\nabla(A) \setminus \nabla(A)_k \cap \Delta(\nabla(A)_k) = \emptyset\). Let us prove first this last property. Suppose on the contrary that there is an \(h \in \nabla(A) \setminus \nabla(A)_k \cap \Delta(\nabla(A)_k) \neq \emptyset\). Since \(A\) is augmentable and \(k' < k\), then by property 2) of Lemma 13 we have

\[
\Delta(\nabla(A)_k) \subseteq A_{k-1}
\]  

(19)

Therefore we have \(h \in A_{k-1} \cap \nabla(A) \subseteq A \cap \nabla(A) = \emptyset\), a contradiction. Hence the two terms in the right part of the inclusion (18) are disjoint. We divide the proof of (18) into two cases. We first prove \(\Delta(\nabla(A)_k) \subseteq A_{k-1}\) and then \(\Delta(\nabla(A)) \subseteq A \setminus A_{k-1} \cup A_k \setminus A_{k-1}\).

- Case \(\nabla(A) \setminus \nabla(A)_k \subseteq \nabla((A \setminus A_{k-1}) \cup \Delta(A_{k-1}))\). It is evident that \(\nabla(A) \setminus \nabla(A)_k \subseteq \nabla(A \setminus A_{k-1})\), thus it is sufficient to show \(\nabla(A \setminus A_{k-1}) \subseteq \nabla((A \setminus A_{k-1}) \cup \Delta(A_{k-1}))\) and to prove this inclusion we use property 1) of Lemma 12. Indeed consider \(g \in \nabla(A \setminus A_{k-1})\) and \(g' \in \Delta(A_{k-1})\), then \(|g| \leq k - 1\), \(|g'| = k - 1\). Therefore the inclusion \(g' \subsetneq g\) can not occur and so \(\nabla(A) \setminus \nabla(A)_k \subseteq \nabla(A \setminus A_{k-1}) \subseteq \nabla(\nabla(A \setminus A_{k-1}) \cup \Delta(A_{k-1}))\).

- Case \(\Delta(\nabla(A)_k) \subseteq \nabla((A \setminus A_{k-1}) \cup \Delta(A_{k-1}))\). Using (19) and property 1) of Lemma 12 we have

\[
\Delta(\nabla(A)_k) \subseteq A_{k-1} \subseteq \nabla(\Delta(A_{k-1})) \subseteq \nabla((A \setminus A_{k-1}) \cup \Delta(A_{k-1}))
\]

To reach a contradiction suppose that there is a \(g \in \Delta(\nabla(A)_k)\) such that \(g \in \nabla((A \setminus A_{k-1}) \cup \Delta(A_{k-1})) \setminus \nabla((A \setminus A_{k-1}) \cup \Delta(A_{k-1}))\). By property 3) of Lemma 12 there is a \(g' \in \nabla((A \setminus A_{k-1}) \cup \Delta(A_{k-1})) = \nabla(A \setminus A_{k-1}) \cup \Delta(A_{k-1})\) with \(g' \subsetneq g\). We consider two cases, either \(g' \in \nabla(A \setminus A_{k-1})\) or \(g' \in \Delta(A_{k-1})\). If \(g' \in \nabla(A \setminus A_{k-1})\), then there is a \(g'' \in A \setminus A_{k-1}\) such that \(g'' \subsetneq g' \subsetneq g\), a contradiction since \(g'' \in A, g \in \Delta(\nabla(A)_k) \subseteq A_{k-1} \subseteq A\) and \(A\) is an antichain. On the other hand, if \(g' \in \Delta(\nabla(A))\) then \(|g'| = k - 1\), moreover, since \(g \in \Delta(\nabla(A)_k) \subseteq A_{k-1}\), then \(|g| = k - 1\) which contradicts \(g' \subsetneq g\) and this concludes the proof of inclusion (18).

We can now conclude the proof of the lemma showing the inequality in the statement. By (17) and (18) we have

\[
2|\alpha^{-}(A)| + |\nabla(\alpha^{-}(A))| = 2|A| + 2(|\Delta(A_{k-1})| - |A_{k-1}|) + |\nabla(\alpha^{-}(A))| \geq 2|A| + |\nabla(A)| + 2(|\Delta(A_{k-1})| - |A_{k-1}|) + \Delta(\nabla(A)_k) - |\nabla(A)_k|
\]

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Since \( k > \frac{n}{2} + 1 \) we have by [5, Corollary 2.1.2]
\[
|\Delta(A_{k-1})| - |A_{k-1}| \geq 0, \quad |\Delta(\nabla(A)_k)| - |\nabla(A)_k| \geq 0
\]
from which it follows \( 2|\alpha^-(A)| + |\nabla(\alpha^-(A))| \geq 2|A| + |\nabla(A)| \) and this concludes the proof of the lemma. \( \square \)

We are now in position to prove Theorem 5.

**Proof of Theorem 5**. The bound is clearly attained when the antichain consists of all the \( \frac{n}{2} \)-subsets of \( X \). Let us now prove the bound. Starting from \( A_0 = A \) by Lemma 13 we suppose without loss of generality that \( A_0 \) is augmentable, then applying for instance the upward-augmenting map we obtain a new antichain \( A_1 \) for which, by Lemmas 15, \( 2|A_0| + |\nabla(A_0)| \leq 2|A_1| + |\nabla(A_1)| \) and \( ||A_i||_{m+1} > ||A_i||_{m} \). Furthermore by Lemma 13 we can suppose that \( A_1 \) is also augmentable. In this way, by a repeated application of Lemmas 13, 15, 16 we can find a sequence of augmentable antichains \( A_i \) such that \( 2|A_{i-1}| + |\nabla(A_{i-1})| \leq 2|A_i| + |\nabla(A_i)| \) and either \( ||A_i||_{m} > ||A_{i-1}||_{m} \) and \( ||A_i||_{M} = ||A_{i-1}||_{M} \), or \( ||A_i||_{M} > ||A_{i-1}||_{M} \) and \( ||A_i||_{m} = ||A_{i-1}||_{m} \). This process stops when it is reached an augmentable antichain \( A_j \) with \( \frac{n}{2} \geq ||A_j||_{M} \geq ||A_j||_{m} \geq \frac{n}{2} - 1 \). If \( ||A_j||_{M} = ||A_j||_{m} \), \( A_j \) consists of either \( \frac{n}{2} \)-subsets or \( \frac{n}{2} \)-1-subsets and the statement of the theorem clearly holds. Thus we can assume \( ||A_j||_{M} > ||A_j||_{m} \) and let \( B_1 = \min(A_j), B_2 = \max(A_j) \). Since \( A_j \) is augmentable, by property 1) of Lemma 13 \( \nabla(B_1) \subseteq \nabla(A_j) \). Thus, putting \( C_1 = \nabla(B_1) \), we can decompose \( \nabla(A_j) = C_1 \cup C_2 \) where \( C_2 \subseteq \nabla(B_2) \). Let \( b_i = |B_i|, c_i = |C_i| \), for \( i = 1, 2 \). Since \( \nabla(A_j) \cap A_j = \emptyset \), then \( C_1 \cap B_2 = \emptyset \), moreover since both \( B_2 \) and \( C_1 \) are \( \frac{n}{2} \)-subsets of \( X \) we get \( b_2 + c_1 \leq \left( \frac{n}{2} + 1 \right) \). Furthermore, since \( A_j \) is an antichain we also get \( b_1 + b_2 \leq \left( \frac{n}{2} \right) \). Hence, since \( 2|A| + |\nabla(A)| = 2(b_1 + b_2) + (c_1 + c_2) \), we have: \( 2|A| + |\nabla(A)| \leq 2\left( \frac{n}{2} \right) + (b_1 + c_2) \). Thus to prove the theorem, it is enough to show \( b_1 + c_2 \leq \left( \frac{n}{2} \right) \). Note that \( B_1 \) is formed by \( \frac{n}{2} \)-1-subsets of \( X \), while the elements of \( C_2 \) are \( \frac{n}{2} \)-1-subsets. We claim that \( B_1 \cup C_2 \) is an antichain. Indeed, if there is a \( z \in B_1 \) and \( z' \in C_2 \) with \( z \not\subseteq z' \), then, since \( |z'| = \frac{n}{2} + 1 \) and \( |z| = \frac{n}{2} - 1 \) there is a \( a \in X \) with \( z \cup \{a\} \not\subseteq z' \). However \( z \cup \{a\} \in \nabla B_1 = C_1 \) and \( z' \in C_2 \) contradict the fact that \( \nabla(A_j) \) is an antichain. Therefore \( B_1 \cup C_2 \) is an antichain. Let \( A_1, \ldots, A_{\ell} \) be a symmetric chains decomposition of the set of subsets of \( X \) (see [5, Section 3.2]). We define the map \( \varphi : B_1 \to 2^X \) which associates to each \( z \in B_1 \) with \( z \in A_i \), for some \( i \in \{1, \ldots, \ell\} \), the “specular” set \( \varphi(z) \) in \( A_i \) with \( |z| + |\varphi(z)| = n \). Note that \( \varphi \) is clearly injective, furthermore it sends \( \frac{n}{2} \)-1-subsets into \( \frac{n}{2} \)-1-subsets.

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Thus, to prove $b_1 + c_2 \leq \left(\frac{n}{2} + 1\right)$, it is enough to show $\varphi(B_1) \cap C_2 = \emptyset$. Indeed, suppose, contrary to our claim, that there is $z' \in \varphi(B_1) \cap C_2$, then we can find a $z \in B_1$ with $\varphi(z) = z'$. Since $z, z'$ belong to the same symmetric chain, we get $z \subsetneq z'$ which contradicts the fact that $B_1 \cup C_2$ is an antichain and this concludes the proof of the theorem.

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