THE $\mathcal{H}$-FLOW TRANSLATING SOLITONS IN $\mathbb{R}^3$ AND $\mathbb{R}^4$

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ABSTRACT. We solve the prescribed Hoffman-Osserman Gauss map problem for translating soliton surfaces to the mean curvature flow in $\mathbb{R}^4$. Our solution is inspired by Ilmanen’s correspondence between translating soliton surfaces and minimal surfaces.

The recent decades admit intensive research devoted to the study of solitons [8] to the mean curvature flow ($\mathcal{H}$-flow for short). The simplest example is the grim reaper $y = \ln(\cos x)$ which moves by downward translation under the $\mathcal{H}$-flow. As known in [3, 9, 18, 21], there exist fascinating geometric dualities between the $\mathcal{H}$-flow solitons and minimal submanifolds.

A surface is a translator [21] when its mean curvature vector field agrees with the normal component of a constant Killing vector field. Translators arise as Hamilton’s convex eternal solutions and Huisken-Sinestrari’s Type II singularities for the $\mathcal{H}$-flow, and become natural generalization of minimal surfaces. The eight equivalent definitions of minimal surfaces illustrated in [12] show the richness of the minimal surfaces theory. However, even in $\mathbb{R}^3$, only few non-minimal translators are known:

Altschuler and Wu [1] showed the existence of the convex, rotationally symmetric, entire graphical translator. Clutterbuck, Schnürer and Schulze [4] constructed the winglike bigraphical translators, which are analogous to catenoids. Halldorsson [5] proved the existence of helicoidal translators. Nguyen [16] used Scherk’s minimal towers to desingularize the intersection of a grim reaper cylinder and a plane, and obtained a complete embedded translator. See also her generalization [17].

Our main goal is to adopt the splitting of the generalized Gauss map of oriented surfaces in $\mathbb{R}^4$ to construct an explicit Weierstrass type representation for translators in $\mathbb{R}^3$. We first introduce the complexification of the generalized Gauss map. Inside the complex projective space $\mathbb{CP}^3$, we take the variety

$$Q_2 = \{ [\zeta] = [\zeta_1 : \cdots : \zeta_4] \in \mathbb{CP}^3 : \zeta_1^2 + \cdots + \zeta_4^2 = 0 \},$$

which becomes a model for the Grassmannian manifold $G_{2,2}$ of oriented planes in $\mathbb{R}^4$. Reading the biholomorphic map from $Q_2$ to $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a splitting of $Q_2$, Hoffman and Osserman [6, 7] defined the generalized Gauss map of a conformal immersion $X : \Sigma \to \mathbb{R}^4$, $z \mapsto X(z)$ as follows:

$$G(z) = \left[ \frac{\partial X}{\partial z} \right] = [ 1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2) ] \in Q_2 \subset \mathbb{CP}^3.$$

We call the induced pair $(g_1, g_2)$ the complexified Gauss map of the immersion $X$. 

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Lemma 1 (Poincaré’s Lemma). Let $\xi : \Omega \to \mathbb{C}$ be a function on a simply connected domain $\Omega \subset \mathbb{C}$. If we have $\frac{\partial}{\partial z} \xi(z) \in \mathbb{R}$ for all $z \in \Omega$, then there exists a function $x : \Omega \to \mathbb{R}$ such that $\frac{\partial}{\partial z} x(z) = \xi(z)$.

Theorem 2 (Correspondence from null curves in $\mathbb{C}^4$ to translators in $\mathbb{R}^4$). Let $(g_1, g_2)$ be a pair of nowhere-holomorphic $C^2$ functions from a simply connected domain $\Omega \subset \mathbb{C}$ to the open unit disc $\mathbb{D} := \{ w \in \mathbb{C} \mid |w| < 1 \}$ satisfying the compatibility condition

$$0 = (g_1)_z^2 + \left( \frac{g_2}{1 - g_1 g_2} - \frac{g_1}{1 + |g_1|^2} \right) (g_1)_z (g_1)_\bar{z} + \frac{g_1 + g_2}{1 - g_1 g_2} \frac{|g_1|^2}{1 + |g_1|^2} |(g_1)_\bar{z}|^2,$$

Then, we obtain the following statements.

(a) Both (0.2) and (0.3) hold. (Assuming (0.1), we claim that (0.2) is equivalent to (0.3).)

(b) The complex curve $\phi := (\phi_1, \phi_2, \phi_3, \phi_4) : \Omega \to \mathbb{C}^4$ defined by

$$\phi = f \left( 1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i (g_1 + g_2) \right), \quad f := -2i \mathcal{F}$$

fulfills the three properties on the domain $\Omega$:

(b1) nullity $\phi \cdot \phi = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 0$,

(b2) non-degeneracy $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 > 0$,

(b3) integrability $\frac{\partial}{\partial \bar{z}} \phi = \left( \frac{\partial \phi_1}{\partial \bar{z}}, \frac{\partial \phi_2}{\partial \bar{z}}, \frac{\partial \phi_3}{\partial \bar{z}}, \frac{\partial \phi_4}{\partial \bar{z}} \right) \in \mathbb{R}^4$.

(c) Integrating the complex null immersion $\phi : \Omega \to \mathbb{C}^4$ yields a translator $\Sigma$ in $\mathbb{R}^4$.

(c1) There exists a conformal immersion $X = (x_1, x_2, x_3, x_4) : \Omega \to \mathbb{R}^4$ satisfying $X_z = \phi$.

(c2) The induced metric $ds^2$ on the $z$-domain $\Omega$ by the immersion $X$ reads

$$ds^2 = \frac{16 |(g_1)_\bar{z}|^2}{|1 - g_1 g_2|^2} \frac{1 + |g_2|^2}{1 + |g_1|^2} |dz|^2 = \frac{16 |(g_2)_z|^2}{|1 - g_1 g_2|^2} \frac{1 + |g_1|^2}{1 + |g_2|^2} |dz|^2.$$

(c3) The pair $(g_1, g_2)$ is the complexified Gauss map of the surface $\Sigma = X(\Omega)$. In other words, the generalized Gauss map of the conformal immersion $X$ reads

$$[X_z] = [1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i (g_1 + g_2)] \in \mathbb{Q}_2 \subset \mathbb{C}P^3.$$

(c4) The surface $\Sigma$ becomes a translator with the translating velocity $-e_4 = (0, 0, 0, -1)$.

Proof. Step A. For the proof of (a), we first set up the notations

$$\mathcal{L} := (g_1)_z^2 + \left( \frac{g_2}{1 - g_1 g_2} - \frac{g_1}{1 + |g_1|^2} \right) (g_1)_z (g_1)_\bar{z} + \frac{g_1 + g_2}{1 - g_1 g_2} \frac{|g_1|^2}{1 + |g_1|^2} |(g_1)_\bar{z}|^2,$$

$$\mathcal{R} := (g_2)_z^2 + \left( \frac{g_1}{1 - g_1 g_2} - \frac{g_2}{1 + |g_2|^2} \right) (g_2)_z (g_2)_\bar{z} + \frac{g_1 + g_2}{1 - g_1 g_2} \frac{|g_2|^2}{1 + |g_2|^2} |(g_2)_\bar{z}|^2.$$
We first assume only (0.1). Taking the conjugation in (0.1) yields

\[ \mathcal{F} = \frac{(g_2)_z}{(1 - g_1 g_2^*) (1 + |g_2|^2)} = \frac{(g_1)_z}{(1 - \overline{g_1} g_2) (1 + |g_1|^2)}. \]

Taking into account this, we deduce

\[
\begin{align*}
\frac{\mathcal{F}_z}{\mathcal{F}} &= \frac{(g_1)_z}{(g_1)_z^*} + \left( \frac{\overline{g_2}}{1 - g_1 g_2^*} - \frac{\overline{g_1}}{1 + |g_1|^2} \right) (g_1)_z + \left( \frac{1 + |g_2|^2}{1 - \overline{g_1} g_2} - 1 \right) - \frac{g_1}{1 + |g_1|^2} (g_1)_z \\
&= \frac{\mathcal{L}}{(g_1)_z^*} - \mathcal{F} \left[ g_1 \left(1 - |g_2|^2\right) + g_2 \left(1 - |g_1|^2\right) \right]
\end{align*}
\]

and

\[
\begin{align*}
\frac{\mathcal{F}_z}{\mathcal{F}} &= \frac{(g_2)_z}{(g_2)_z^*} + \left( \frac{\overline{g_1}}{1 - g_1 g_2^*} - \frac{\overline{g_2}}{1 + |g_2|^2} \right) (g_2)_z + \left( \frac{1 + |g_1|^2}{1 - \overline{g_1} g_2} - 1 \right) - \frac{g_2}{1 + |g_2|^2} (g_2)_z \\
&= \frac{\mathcal{R}}{(g_2)_z^*} - \mathcal{F} \left[ g_1 \left(1 - |g_2|^2\right) + g_2 \left(1 - |g_1|^2\right) \right].
\end{align*}
\]

These two equalities thus show the equality

\[ \frac{\mathcal{L}}{(g_1)_z^*} = \frac{\mathcal{R}}{(g_2)_z^*}, \]

which means the desired implications: (0.2) \(\leftrightarrow\) \(\mathcal{L} = 0 \leftrightarrow\) \(\mathcal{R} = 0 \leftrightarrow\) (0.3).

**Step B.** We deduce several equalities which will be used in the proof of (b) and (c). According to (a), from now on, we assume that both (0.2) and (0.3) hold. Since both \(\mathcal{L}\) and \(\mathcal{R}\) vanish, the previous equalities imply

\[ \mathcal{F}_z = -|\mathcal{F}|^2 \left[ g_1 \left(1 - |g_2|^2\right) + g_2 \left(1 - |g_1|^2\right) \right]. \]

Conjugating this and using the definition \(f = -2i\mathcal{F}\), we arrive at the equality

\[ (0.4) \quad f_\tau = \frac{1}{2} |f|^2 \left[ \overline{g_1} \left(1 - |g_2|^2\right) + \overline{g_2} \left(1 - |g_1|^2\right) \right]. \]

The compatibility condition (0.1) can be written in terms of \(f = -2i\mathcal{F}\):

\[ (0.5) \quad \overline{\mathcal{F}} = \frac{2i (g_1)_z}{(1 - g_1 g_2^*) (1 + |g_1|^2)} = \frac{2i (g_2)_z}{(1 - \overline{g_1} g_2) (1 + |g_2|^2)}. \]

It immediately follows from (0.4) and (0.5) that

\[ (0.6) \quad (f g_1)_\tau = f_\tau g_1 + (g_1)_\tau f = -\frac{i}{2} |f|^2 \left(1 - 2g_1 \overline{g_2} + |g_1|^2 |g_2|^2\right) \]

and

\[ (0.7) \quad (f g_2)_\tau = f_\tau g_2 + (g_2)_\tau f = -\frac{i}{2} |f|^2 \left(1 - 2\overline{g_1} g_2 + |g_1|^2 |g_2|^2\right). \]

Another computation taking into account (0.5) and (0.6) shows

\[ (f g_1 g_2)_\tau = (f g_1)_\tau g_2 + (g_2)_\tau f g_1 = -\frac{i}{2} |f|^2 \left[ g_1 \left(1 - |g_2|^2\right) + g_2 \left(1 - |g_1|^2\right) \right]. \]

**Step C.** Our aim here is to establish the claims in (b) on the complex curve

\[ \phi = (\phi_1, \phi_2, \phi_3, \phi_4) = f \left(1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2)\right). \]
First, the equality in (b1) is obvious. Next, by the assumptions on $g_1$ and $g_2$, we see that $f = -2(\mathcal{F})$ never vanish. Then, the assertion (b2) follows from the equality

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 = 2|f|^2 \left(1 + |g_1|^2\right) \left(1 + |g_2|^2\right).$$

We employ the equalities in Step B to show the assertion (b3). Joining the equalities in (0.4), (0.6), (0.7), and (0.8) and the definition of $\phi$, we reach

$$(0.10) \begin{align*}
\begin{cases}
(\phi_1)_x = |f|^2 \left[1 - |g_2|^2\right] \text{Im} \, g_1 + \left(1 - |g_1|^2\right) \text{Im} \, g_2, \\
(\phi_2)_x = -|f|^2 \left[1 - |g_2|^2\right] \text{Re} \, g_1 + \left(1 - |g_1|^2\right) \text{Re} \, g_2, \\
(\phi_3)_x = 2|f|^2 \text{Im} \, (\overline{g_1}g_2), \\
(\phi_4)_x = -|f|^2 \left[1 - 2\text{Re} \, (\overline{g_1}g_2) + |g_1|^2|g_2|^2\right].
\end{cases}
\end{align*}$$

These four equalities guarantee the integrability condition $\left(\frac{\partial \phi_1}{\partial x}, \frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_3}{\partial x}, \frac{\partial \phi_4}{\partial x}\right) \in \mathbb{R}^4$.

Step D. We prove the claims (c1), (c2), and (c3). Thanks to (b3), we can integrate the curve $\phi$. Since $\Omega$ is simply connected, applying Lemma 1 to the complex curve $\phi$, we see the existence of the function $X = (x_1, x_2, x_3, x_4) : \Omega \rightarrow \mathbb{R}^4$ satisfying

$$X_z = \phi = f \left(1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2)\right).$$

This and the nullity of $\phi$ guarantee that the mapping $X$ is conformal. Using (0.9), one then find that the induced metric $ds^2 = \Lambda^2|dz|^2$ by the immersion $X$ reads

$$(0.11) \quad ds^2 = \Lambda^2|dz|^2 = 4|f|^2 \left(1 + |g_1|^2\right) \left(1 + |g_2|^2\right)|dz|^2.$$

Since $f$ never vanish, this completes the proof of (c1). Also, joining (0.5) and (0.11) imply the equality in (c2). The integrability $X_z = \phi$ and the definition of $\phi$ gives

$$[X_z] = \left[1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2)\right],$$

which completes the proof of (c3).

Step E. Finally, we prove the claim (c4). First, we find the normal component of the vector field $-e_4 = (0, 0, 0, -1)$ in terms of $g_1$ and $g_2$. We compute

$$(-e_4)^\perp = -e_4 - \frac{1}{\Lambda} \left[\left(\frac{X_u}{\Lambda} \cdot (-e_4)\right) \frac{X_u}{\Lambda} + \left(\frac{X_v}{\Lambda} \cdot (-e_4)\right) \frac{X_v}{\Lambda}\right]$$

$$= -e_4 + \frac{2}{\Lambda^2} \left[\left(X_u \cdot e_4\right) X_z + \left(X_v \cdot e_4\right) X_z\right]$$

$$= \begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix} + \frac{4}{\Lambda^2} \begin{bmatrix}
\text{Re} \left(\phi_1 \overline{\phi_4}\right) \\
\text{Re} \left(\phi_2 \overline{\phi_4}\right) \\
|\phi_4|^2
\end{bmatrix}.$$

Combining this, (0.10), and (0.11) yields

$$(-e_4)^\perp = \frac{1}{\left(1 + |g_1|^2\right) \left(1 + |g_2|^2\right)} \left[\begin{bmatrix}
\left(1 - |g_2|^2\right) \text{Im} \, g_1 + \left(1 - |g_1|^2\right) \text{Im} \, g_2 \\
-\left(1 - |g_2|^2\right) \text{Re} \, g_1 + \left(1 - |g_1|^2\right) \text{Re} \, g_2 \\
-2 \text{Im} \, (\overline{g_1}g_2) + |g_1|^2|g_2|^2
\end{bmatrix}
\right].$$
Second, we find the mean curvature vector \( \mathcal{H} = \Delta_{dc} \mathbf{X} = \frac{1}{\zeta} \frac{\partial}{\partial \zeta} \left( \frac{1}{\zeta} \mathbf{X} \right) = \frac{1}{\zeta} \phi \) on our surface \( \Sigma = \mathbf{X}(\Omega) \). Now, joining this, (0.10), and (0.11), we can write the mean curvature vector \( \mathcal{H} \) in terms of \( g_1 \) and \( g_2 \):

\[
\mathcal{H} = \frac{1}{(1 + |g_1|^2)(1 + |g_2|^2)} \left[ \begin{array}{c}
\left( 1 - |g_2|^2 \right) \text{Im} g_1 + \left( 1 - |g_1|^2 \right) \text{Im} g_2 \\
\left( 1 - |g_2|^2 \right) \text{Re} g_1 + \left( 1 - |g_1|^2 \right) \text{Re} g_2 \\
2 \text{Im} (\overline{g_1}g_2) \\
- \left[ 1 - 2 \text{Re} (\overline{g_2}) + |g_1|^2 |g_2|^2 \right]
\end{array} \right].
\]

We therefore conclude that \( \mathcal{H} = (-e_4)^{\perp} \).

\[\square\]

Remark 1 (Ilmanen’s correspondence). Theorem 2 generalizes the classical Weierstrass construction from holomorphic null immersions in \( C^3 \) to conformal minimal immersions in \( \mathbb{R}^3 \). The key ingredient behind Theorem 2 is the Ilmanen correspondence between translators and minimal surfaces. (See [9] and [21]). We deform the flat metric of \( \mathbb{R}^4 \) conformally to introduce the four-dimensional Riemannian manifold

\[\mathcal{I}^4 = \left( \mathbb{R}^4, e^{-x_4} (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) \right).\]

Any conformal immersion \( \mathbf{X} : \Omega \to \mathbb{R}^4 \) of a downward translator with the translating velocity \(-e_4 = (0, 0, 0, -1)\) in Euclidean space \( \mathbb{R}^4 \) then can be identified as a conformal minimal immersion \( \mathbf{X} : \Omega \to \mathcal{I}^4 \). However, it is not easy to find Riemannian manifolds which admit explicit representations for their minimal surfaces.

Example 3 (The Hamiltonian stationary Lagrangian translator in \( C^2 \)). Recently, interesting Lagrangian translators in the complex plane \( C^2 \) are discovered in [2, 10, 13]. In 2010, Castro and Lerma [2, Corollary 2] classified all Hamiltonian stationary Lagrangian translators in \( C^2 \). Locally, they are unique up to dilations (except for the totally geodesic ones) [2, Corollary 3]. The point of this example is to explicitly recover the Hoffman-Osserman Gauss map of the Castro-Lerma translator in \( \mathbb{R}^4 = C^2 \).

We first notice that Theorem 2 still holds when we regard the prescribed Gauss map \((g_1, g_2)\) as a pair of functions from a simply connected domain \( \Omega \) to the complex plane (not just the unit disc). However, in this case, the induced mapping \( \mathbf{X} : \Omega \to \mathbb{R}^4 \) of the translator may admit the branch points where \( \overline{g_1}g_2 = 1 \) (or equivalently, \( g_1\overline{g_2} = 1 \)).

Imposing the additional condition \(|g_1| = 1\) produces Lagrangian translators with the velocity \(-e_4 = (0, 0, 0, -1)\). Then, our integrability condition in (c1) for downward translators can be re-written as

\[
\mathbf{X}_z = \left( (x_1)_z, (x_2)_z, (x_3)_z, (x_4)_z \right) = -\theta z \left( \frac{1 + g_1g_2}{g_1 - g_2}, \frac{1 - g_1g_2}{g_1 - g_2}, 1, \frac{-1}{g_1 + g_2} \right),
\]

where \( \theta \) denotes the Lagrangian angle with \( ig_1 = e^{i\theta} \). The third terms \((x_3)_z = -\theta z\) can be viewed as [2, Proposition 1], [10, Proposition 2.5] and [15, Proposition 2.1].

We consider a complexified Gauss map of the form, for some \( \mathbb{R} \)-valued function \( G \),

\[
(g_1(z), g_2(z)) = \left( e^{iz}, G(u)e^{iu} \right), \quad z = u + iv \in \mathbb{R} + i\mathbb{R}
\]

and want to solve the system (0.1) and (0.2). First, the compatibility condition (0.1) induces the ordinary differential equation

\[
\frac{1}{2} = \frac{1}{1 + G^2} \left( G - \frac{dG}{du} \right).
\]
and a canonical solution is given by $G(u) = \frac{u + i}{u - i}$. One can easily check that

\[(g_1(z), g_2(z)) = (e^{i\varphi}, G(u)e^{i\varphi}) = \left( e^{iu}, \frac{u + 1}{u - 1}e^{iu} \right)\]

satisfies the integrability condition (0.2). Then, the induced Lagrangian translator $\Sigma$ with the velocity $-e_4$ admits the conformal parametrization

\[X(u, v) = \left( u \sin v, -u \cos v, -v, -\frac{1}{2}u^2 \right).\]

Since the induced metric on $\Sigma$ reads $ds^2 = (1 + u^2)(du^2 + dv^2)$, the Lagrangian angle function $\theta(u, v) = \frac{\pi}{2} + v$ with $ig_1 = e^{i\theta}$ is harmonic on $\Sigma$. We find that this Hamiltonian stationary Lagrangian translator $\Sigma$ with the velocity $(0, 0, 0, -1)$ coincides with the Castro-Lerma translator [2, Corollary 2] with the velocity $(1, 0, 0, 0)$ by a suitable change of coordinates.

**Theorem 4 (Correspondence from null curves in $\mathbb{C}^3$ to translators in $\mathbb{R}^3$).** When a nowhere-holomorphic $C^2$ function $G : \Omega \to \mathbb{D}$ from a simply connected domain $\Omega \subset \mathbb{C}$ to the open unit disc $\mathbb{D} := \{w \in \mathbb{C} \mid |w| < 1\}$ satisfying the translator equation

\[G_{,\varphi} + 2\frac{G|G|^2}{1 - |G|^4}G_{,\varphi} + 2\frac{G}{1 - |G|^4}|G_{,\varphi}|^2 = 0, \quad z \in \Omega, \quad (0.12)\]

we associate a complex curve $\phi = \phi_G = (\phi_1, \phi_2, \phi_3) : \Omega \to \mathbb{C}^3$ as follows:

\[\phi = \frac{2Gz}{|G|^4} \left( 1 - G^2, i(1 + G^2), 2G \right)\]

(a) Then, the complex curve $\phi$ fulfills the three properties on the domain $\Omega$:

(a1) nullity $\phi \cdot \phi = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0$,

(a2) non-degeneracy $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0$,

(a3) integrability $\frac{\partial \phi}{\partial \varphi} = \left( \frac{\partial \phi_1}{\partial \varphi}, \frac{\partial \phi_2}{\partial \varphi}, \frac{\partial \phi_3}{\partial \varphi} \right) \in \mathbb{R}^3$.

(b) Also, integrating $X_\varphi = \phi$ on $\Omega$ yields a downward translator $\Sigma = X(\Omega)$ with the velocity $-e_3 = (0, 0, -1)$ in $\mathbb{R}^3$. The prescribed map $G$ becomes the complexified Gauss map of the induced surface $\Sigma = X(\Omega)$ via the stereographic projection from the north pole. The induced metric $ds^2$ by the immersion $X$ reads $ds^2 = \frac{16|G|^2}{(|G|^2 - 1)^2}|dz|^2$.

**Proof.** We take $(g_1, g_2) = (iG, iG)$ in Theorem 2. \qed

**Example 5 (Downward grim reaper cylinder as an analogue of Scherk’s surface).**

(a) An application of our representation formula in Theorem 4 to the solution

\[G(z) = G(u + iv) = \tanh u \in (-1, 1), \quad u + iv \in \mathbb{C}\]

of the translator equation (0.12) yields the conformal immersion $X : \mathbb{R}^2 \to \mathbb{R}^3$

\[X(u, v) = (x_1, x_2, x_3) = (-2 \tan^{-1}(\tanh u), 2v, -\ln(\cosh(2u)))\]

(b) It represents the graphical translator with the translating velocity $-e_3$:

\[x_3 = F_3(x_1, x_2) = \ln(\cos x_1), \quad (x_1, x_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}.\]
Its height function $F : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \to \mathbb{R}$ is a Jenkins-Serrin type solution of

\begin{equation}
\nabla \cdot \left( \frac{1}{\sqrt{1 + |\nabla F|^2}} \nabla F \right) + \frac{1}{\sqrt{1 + |\nabla F|^2}} = 0
\end{equation}

and has $-\infty$ boundary values. Our graph $x_3 = F(x_1, x_2)$ becomes a cylinder over the downward grim reaper on the $x_1x_3$-plane. It can be viewed as an analogue of the classical Jenkins-Serrin type minimal graph, discovered by Scherk in 1834,

$$x_3 = \ln (\cos x_1) - \ln (\cos x_2), \quad (x_1, x_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Its height function takes the values $\pm \infty$ on alternate sides of the square domain.

Remark 2 (Jenkins-Serrin type problem for graphical translators). A beautiful theory for infinite boundary value problems of minimal graphs is developed by Jenkins and Serrin [11]. Moreover, Spruck [19] obtained a Jenkins-Serrin type theory for constant mean curvature graphs. It would be very interesting to investigate a similar Dirichlet problem for graphical translators. In the following Example 6, we prove that, for any $l \geq \pi$, there exists a downward unit-speed graphical translator that its height function is defined over an infinite strip of width $l$ and takes the values $-\infty$ on its boundary. We propose a conjecture that the lower bound $\pi$ is a critical constant in the sense that, for any $l \in (0, \pi)$, there exists no downward unit-speed graphical translator defined over an infinite strip of width $l$ approaching $-\infty$ on its boundary.

Example 6 (Deformations of grim reaper cylinder). Let $\theta \in \mathbb{R}$ be a constant.

(a) We begin with the following solution $G = G^\theta(z)$ of the translator equation (0.12):

$$G(z) = G(u + iv) = \frac{\cosh \theta \sinh(2u) + i \sin \theta}{1 + \cosh \theta \cosh(2u)}, \quad u + iv \in \mathbb{C}.$$  

Theorem 4 then induces the conformal immersion $X^\theta = (x_1, x_2, x_3) : \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{cases}
x_1(u, v) = -2 \cosh \theta \tan^{-1} (\tanh u), \\
x_2(u, v) = \sinh \theta \ln (\cosh(2u)) + 2v, \\
x_3(u, v) = -\ln (\cosh(2u)) + 2v \sin \theta.
\end{cases}$$

The downward translator $G^\theta = X^\theta (\mathbb{R}^2)$ has the translating velocity $(0, 0, -1)$.

(b) Using the patch $X^\theta$, one can easily check that Gauss map of the translator $G^\theta$ lies on a half circle. Let us introduce a new linear coordinate

$$x_0 = \frac{1}{\cosh \theta} x_2 + \frac{\sin \theta}{\cosh \theta} x_3$$
and then prepare an orthonormal basis
\[ \mathcal{U}_1 = (1, 0, 0), \mathcal{U}_2^\theta = \left( 0, -\frac{\sinh \theta}{\cosh \theta}, \frac{1}{\cosh \theta} \right), \mathcal{U}_3^\theta = \left( 0, \frac{1}{\cosh \theta}, \frac{\sinh \theta}{\cosh \theta} \right). \]

It is easily shown that our surface \( G^\theta \) admits a new geometric patch
\[ (x_1, x_2, x_3) = \tilde{X}^\theta (x_1, x_0) = x_1 \mathcal{U}_1 + T^\theta (x_1) \mathcal{U}_2^\theta + x_0 \mathcal{U}_3^\theta. \]

Here, \( T^\theta (\cdot) = \cosh \theta \ln \left( \cos \left( \frac{\theta}{\cosh \theta} \right) \right) \) is a parabolic rescaling of the downward unit-speed grim reaper function. The patch \( \tilde{X}^\theta \) says that the surface \( G^\theta \) becomes a cylinder over a parabolically rescaled grim reaper curve in the plane spanned by \( \mathcal{U}_1 \) and \( \mathcal{U}_2^\theta \).

(c) Our one-parameter family \( \{ G^\theta \}_{\theta \in \mathbb{R}} \) of cylinders with the same translating velocity admits a simple geometric description. Applying a suitable rotation in the ambient space \( \mathbb{R}^3 \) to the grim reaper cylinder \( G^0 \) with velocity \( -\mathcal{U}_2^0 = (0, 0, -1) \):
\[ (x_1, x_0) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \mathbb{R} \mapsto \tilde{X}^\theta (x_1, x_0) = x_1 \mathcal{U}_1 + T^\theta (x_1) \mathcal{U}_2^\theta + x_0 \mathcal{U}_3^\theta, \]
we obtain the congruent cylinder parametrized by
\[ (x_1, x_0) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \mathbb{R} \mapsto x_1 \mathcal{U}_1 + T^\theta (x_1) \mathcal{U}_2^\theta + x_0 \mathcal{U}_3^\theta, \]
which translates with the rotated velocity \( -\mathcal{U}_2^\theta \) under the \( \mathcal{H} \)-flow. However, we observe that this rotated cylinder also can be viewed as a translator with new velocity \( -\cosh \theta \mathcal{U}_2^0 = (0, 0, -\cosh \theta) \). Employing the appropriate parabolic rescaling as a speed-down action, we meet our cylinder \( G^\theta \) parametrized by
\[ (x_1, x_0) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \cos \theta \right) \times \mathbb{R} \mapsto \tilde{X}^\theta (x_1, x_0) = x_1 \mathcal{U}_1 + T^\theta (x_1) \mathcal{U}_2^\theta + x_0 \mathcal{U}_3^\theta, \]
which translates with velocity \( -\mathcal{U}_2^0 = (0, 0, -1) \) under the \( \mathcal{H} \)-flow.

(d) We prove the claim in Remark 2. We are able to view the downward unit-speed translator \( G^\theta \) as the graph of the function \( \mathcal{F}^\theta : \left( -\frac{\pi}{2} \cosh \theta, \frac{\pi}{2} \cosh \theta \right) \times \mathbb{R} \rightarrow \mathbb{R} \):
\[ x_3 = \mathcal{F}^\theta (x_1, x_2) = \cosh \theta T^\theta (x_1) + \sinh \theta x_2. \]

Its height function \( \mathcal{F}^\theta \) solves the PDE (0.13) over the strip of width \( l = \pi \cosh \theta \geq \pi \).

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