OHM’S LAW FOR A RELATIVISTIC PAIR PLASMA

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[published in Phys. Rev. Lett. 71, 3481, (1993)].

ABSTRACT

We derive the fully relativistic Ohm’s law for an electron-positron plasma. The absence
of non-resistive terms in Ohm’s law and the natural substitution of the 4-velocity for the
velocity flux in the relativistic bulk plasma equations do not require the field gradient
length scale to be much larger than the lepton inertial lengths, or the existence of a frame
in which the distribution functions are isotropic.
For a plasma, Ohm’s law describes the relation between the induced current and the plasma electric field. For an ion-electron plasma, the field depends on resistive, inertial, and Hall effect contributions. The result is usually derived for the non-relativistic limit in the Boltzmann picture\textsuperscript{1}. Ohm’s law plays a direct role in the magnetic induction equation used in the description of bulk plasma dynamics.

Relativistic plasma models have been effective in explaining the observations of relativistic astrophysical jets and winds\textsuperscript{2}. Such models have generally employed the relativistic continuity equation as a vanishing 4-divergence of the bulk 4-velocity, and Ohm’s law as a simple bulk 4-vector generalization of the non-relativistic equation\textsuperscript{2,3}.

We shall see that for a two-component relativistic plasma composed of different mass particles, the natural use of these magnetohydrodynamic (MHD) forms for the continuity equation and Ohm’s law requires the existence of a reference frame in which both distribution functions are isotropic in momentum. The constraint results from the non-linear relation between momenta and velocities in the relativistic regime.

This requirement is non-trivial because distribution function isotropy also requires the plasma under study to be microinstability saturated; otherwise microinstabilities could grow because of distribution function anisotropy. Yet, evidence for the presence of anisotropies and microinstabilities in relativistic winds has come from observations of the Crab Nebula\textsuperscript{4}. In particular, the Weibel instability has been suggested to explain the presence of wisps downstream from relativistic shocks\textsuperscript{5}. Anisotropies downstream from relativistic shocks are likely present in jets as well.\textsuperscript{6}
Depending on density and temperature conditions, anisotropies in relativistic plasmas may also arise from anisotropic radiation onto a plasma, such as in coronae of stars or in models active galactic nuclei (AGN) for which a pair atmosphere forms. The impinging winds and jets onto ambient media also produces anisotropies.

In this paper we show that in contrast to an ion-electron plasma: 1) only a resistive contribution to Ohm’s law for a relativistic $e^+ - e^-$ plasma is relevant under quite general conditions, and 2) that anisotropy in the distribution functions need not affect the form of the unperturbed relativistic bulk equations for the pair plasma. Thus, relativistic bulk dynamics for pair plasmas which exhibit evidence for microinstabilities are appropriately described by the relativistic MHD formalism, whereas ion-electron plasmas which exhibit such instabilities are not.

The Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + v^i \partial_i f + F^i \partial_{p_i} f = [\Delta_t f]_{\text{coll.}} + [\Delta_t f]_{\text{radiat.}} + [\Delta_t f]_{\text{creation}} + [\Delta_t f]_{\text{annihilation}}$$ (1)

where $v^i$ is the particle velocity, $p^i$ is the particle momentum, $t$ is the time, $F^i$ is the electromagnetic force (since we ignore gravity), $f = f(x, p, t)$ is the scalar distribution function, and the terms on the right are schematic. We shall assume that collisional losses dominate synchrotron radiation losses, which is acceptable for

$$d|p|/dt|_{\text{synch}}/|evxB| << 10^{-16} \gamma^2 B sin\phi,$$ (2)

where $\gamma$ is the particle Lorentz factor, $\phi$ is the pitch angle, $e$ is the positron charge, and $B$ is the magnetic field measured in Gauss.
Define the plasma quantities: number density

$$n_\pm = \int f_\pm(x, p, t) d^3p,$$

and velocity flux density

$$\phi^i_\pm = \int v^i_\pm f_\pm(x, p, t) d^3p = n_\pm \langle v^i_\pm \rangle.$$  \hspace{1cm} (4)

Adopting the signature $(+, -,-,-)$, we then have the flux density 4-vector $\phi^\mu_\pm = (cn_\pm, \phi^i_\pm)$, and current density 4-vector $j^\mu = e(\phi^\mu_+ - \phi^\mu_-)$. The energy and momentum densities are the $\langle 00 \rangle$ and $\langle i0 \rangle$ components of the symmetric kinetic tensor given by

$$K^i_\pm^{jk} = \int v^i_\pm v^j_\pm f_\pm d^3p = n_\pm \langle v^i_\pm v^k_\pm \rangle$$  \hspace{1cm} (5)

and

$$K^0_\pm^{\mu} = (\epsilon_\pm, c\Pi^i_\pm),$$  \hspace{1cm} (6)

where $\Pi^i_\pm$ is the momentum density of either positrons or electrons.

Quantities without the $\pm$ shall refer to the plasma as a whole— the sum of the component contributions. We require the existence of a proper frame moving with 4-velocity $U_\mu$ in which the charge density, the velocity flux density, and the momentum density vanish. We denote quantities in this frame with a superscript *. For an $e^+ - e^-$ plasma this means

$$j_0^* = e(n^*_+ - n^*_-) = 0,$$  \hspace{1cm} (7a)

$$\phi^{i*} = 0,$$  \hspace{1cm} (7b)

and

$$\Pi^{i*} = m_e n^*_+/2(\langle \gamma_+ v^i_+ \rangle^* + \langle \gamma_- v^i_- \rangle^*) = 0.$$  \hspace{1cm} (7c)
Since $K_{\mu\nu}$ has 10 independent components, we can write

$$K^*_{\mu\nu} = P^*_{\mu\nu} + A^*_{\mu\nu}, \quad (8)$$

where the symmetric tensors $P_{\mu\nu}$ and $A_{\mu\nu}$ satisfy

$$P^*_{ij} = P\delta_{ij}, \quad P_{0i} = 0, \quad P_{00} = \epsilon^*, \quad (9)$$

and

$$A^*_{0\mu} = 0. \quad (10)$$

In (8) – (10), $P$ is the scalar pressure and $A^*_{ij}$ has 5 independent components that measure anisotropy.

Finally, define the 4-vector $H^\mu$ by

$$H^\mu = \left[2/(m_+ + m_-)\right][\rho^*U^\mu - (m_+\phi_+^\mu + m_-\phi_-^\mu)]. \quad (11)$$

where $\rho^*$ is the proper frame rest mass density. For a pair plasma this becomes

$$H_{\mu}^{\text{pair}} = (1/m_e)[\rho^*U^\mu - m_e(\phi^\mu)]. \quad (12)$$

Thus from (7b), $H^*_{\mu}^{\text{pair}}$ vanishes. But $H^0_{\mu}^{\text{pair}} = 0$ by definition, so $H^{\mu*}_{\mu} = 0$. Since $H^\mu_{\mu}^{\text{pair}}$ is a 4-vector, the vanishing of $H^\mu_{\mu}^{\text{pair}}$ implies that $H^\mu_{\mu}^{\text{pair}} = 0$ in all frames. Note that $H^\mu_{\mu}^{*}$ measures the heat flux density per unit mass in the proper frame, so we shall call $H^\mu$ the heat flux density 4-vector.

The procedure used to derive Ohm’s law for a relativistic pair plasma is as follows:

(i) First we obtain a “resistive” type collision term appropriate for a nearly collisionless
relativistic pair plasma. (ii) Second, we relate this to the current density. (iii) Finally, a subtraction of the momentum density equations for the positrons and electrons yields the desired result.

We can find the value of the momentum density in any frame by Lorentz transforming the stress-energy tensor. The result is

\[ c\Pi^i = c^2 m_e (n_+ \langle u^i_+ \rangle + n_- \langle u^i_- \rangle) = \gamma_V^2 (P + \epsilon^*) V_i c^{-1} + A^{0i}, \]  

(13)

where \( u^i \) is a spatial component of the particle 4-velocity, \( V^i \) is a component of the bulk 3-velocity, \( \gamma_V \) is the bulk Lorentz factor, and the anisotropy term is given by

\[ A_{0i} = A^*_{ij} U^j + U^k U^l A^*_{kl} U^i / (\gamma_V + 1). \]  

(14)

Note that since the proper frame 4-vector \( A^{\nu\nu} U^\nu_* = 0 \), we know that it is zero in all frames. Thus \( A_{i0} = A_{ik} U^k \) and (13) can be written

\[ c\Pi^i = c^2 m_e (n_+ \langle u^i_+ \rangle + n_- \langle u^i_- \rangle) = \gamma_V (P + \epsilon^*) U^i + A^{ij} U_j. \]  

(15)

Inverting (15) we obtain

\[ \gamma_V V^i c^{-1} = U^i = [c^2 m_e (n_+ \langle u^i_+ \rangle + n_- \langle u^i_- \rangle) - A^{ij} U_j] / [\gamma_V (P + \epsilon^*]]. \]  

(16)

The average 4-momentum gains from collisions in the proper frame are given by

\[ \Delta p^\mu_+ = -\Delta p^\mu_- = (m_e/2) (\langle u^\mu_+ \rangle - \langle u^\mu_- \rangle). \]  

(17)

The approximate proper frame pair plasma collision term is then

\[ (n^* v^*_c / 4)(\Delta p^\mu_+^*) = -(n^* v^*_c / 4)(\Delta p^\mu_-^*) \equiv P^\mu_+ = -P^\mu_-^*, \]  

(18)
where \( \nu^*_c \) is the proper frame collision frequency.

Equations (12) and (16) give

\[
m_e c^3 (n_+ u^+_i + n_- u^-_i) - c A^{ij} U_j = \gamma_V (P + \epsilon^*) n^{*-1} (\phi^-_i + \phi^+_i).
\]  

(19)

Now in the electron frame, we have

\[
c^2 \Pi^+_i (\gamma^* -1) = \gamma^*_V (P + \epsilon^*) n^{*-1} \phi^+_i,
\]

(20)

and in the positron frame

\[
c^2 \Pi^-_i (\gamma^* -1) = \gamma^*_V (P + \epsilon^*) n^{*-1} \phi^-_i.
\]

(21)

Transforming (20) and (21) to the proper frame, subtracting, and using (7) and (18) gives

\[
\nu^*_c (\Pi^+_i - \Pi^-_i) = 2 P^i = n^* \eta_r j^i,
\]

(22)

where the effective resistivity \( \eta_r \) is given by

\[
\eta_r = \nu^*_c (n^* e)^{-1} c^{-2} \{(P + \epsilon^*) (n^* e) \gamma^* c (\gamma^* c + 1)^{-1} j_\mu j^\mu (en^* c)^{-2} + \gamma^*_c + 1
\]

\[-\gamma^*_c \epsilon^* (en^*)^{-1} - 2 \nu^*_c \gamma^* c (\gamma^* c + 1)^{-1} (n^* e)^{-2} j_\mu j^\mu (en^* c)^{-2} + \gamma^*_c + 1
\]

(23)

and \( \gamma^*_c \) is the Lorentz factor corresponding to the velocity

\[
(n^*/2)^{-1} \phi^+_i = -(n^*/2)^{-1} \phi^-_i = j^i / (en^*).
\]

(24)

We have used the 4-vector indices in \( \eta_r \) since \( j^0 = 0 \).

Equation (22) suggests that we subtract the momentum equations for the electrons and the positrons to obtain Ohm’s law. This is standard in the non-relativistic case, but is only fruitful in the relativistic case because \( H^\mu_{\text{pair}} = 0 \).
Under the conditions of (7), conservation of energy and momentum implies that the contributions of the annihilation and creation terms of (1) to the energy and momentum equation cancel for each plasma component in the proper frame. Thus, by covariance, the contributions to the 1st moments in all frames vanish, and we do not consider them further.

The $i$th component of the proper frame relativistic energy momentum tensor for positrons, as obtained from the 1st moment of the Boltzmann equation is given by

$$\partial_0 \Pi_{++}^i = -\partial_k K_{++}^{ki} - e(n^*/2)E_{++}^i - [e\langle v_+/c \rangle \times B]^i + P_{+}^i.$$  \hspace{1cm} (25)

Subtracting the analogous equation for electrons, and using (8), (18), and (22) we get

$$E_{++}^i = \eta_r j_{++}^i - m_e c (2 e \rho^*)^{-1} \partial_k (A_{++}^{ki} - A_{--}^{ki}) - \partial_0 [\eta_r / \nu_c^*] j_{++}^i].$$ \hspace{1cm} (26)

The requirements (7a) and (7b) which led to (26), are assumptions about the 0th and 1st moments of the Boltzmann equation. If we further assume that

$$A_{++}^{ij} = A_{--}^{ij},$$ \hspace{1cm} (27)

the second term on the right hand side of (26) would vanish. Then, in the steady state, we would be left with

$$E_{++}^i = \eta_r j_{++}^i.$$ \hspace{1cm} (28)

Note that the dependence on the current in (28), results from (22). The latter follows for example, even in the Fokker-Planck approximation if, as in our case, the charge density
vanishes in the proper frame. In addition, note that none of the assumptions that led to (28) require the existence of a reference frame in which the distribution functions of positrons and electrons are isotropic.

In a general frame, (28) becomes

\[
F_{\mu\nu}U_{\nu} = \eta_r (j^\mu + j^\tau U_{\tau} U^\mu),
\]

where we have added a convection term. Equation (29) resembles the magnetofluid relation \(^{11}\). Here however, the effective resistivity \(\eta_r\) depends on a scalar function of the current and momentum density of the plasma components. This dependence is eliminated when we choose the coordinate axes such that the proper frame velocity lies on a principal axis. In addition, \(\gamma_c^*\) factors out of (23) and we have

\[
\eta_r \to \eta_r^{(pr)} = \nu_c e^{-2} (n^* e)^{-2} \{2(P + \epsilon^*) - \epsilon^*\} = \nu_c e^{-2} (n^* e)^{-2} (2P + \epsilon^*),
\]

where the superscript \((pr)\) indicates that the current flows along a principal axis.

Note that unlike the Ohm’s law for an ion-electron plasma, there are no Hall effect or pressure contributions to (29). The vanishing of the bulk Hall effect term is the result of an effectively zero net particle gyration frequency; the sum of the electron and positron contributions vanish due to opposite streaming of positrons and electrons around the field lines. The vanishing of the pressure term results because our assumptions have appealed to the mass symmetry of the problem, eliminating diffusion. The Hall effect and pressure terms are negligible for an ion-electron plasma only when \(\lambda_{fg} \gg \lambda_i\), and \(\lambda_{fg} \gg \lambda_i \beta\), respectively, where \(\lambda_{fg}\) is the length scale of the field gradients, \(\lambda_i\) is the ion-inertial length, and \(\beta\) is the ratio of thermal to magnetic pressure.
The vanishing of the heat flux, which led to (22), is also the condition which allows substitution of the bulk 4-velocity components for the bulk 4-flux components in the plasma continuity equation. The point is that the usual integration of (1) over momentum gives \( \partial_\mu \phi^\mu = 0 \), so substitution of the 4-velocity for the 4-flux \( \phi \) requires \( H^\mu = 0 \). If \( H^\mu \neq 0 \), then this substitution would produce an inhomogeneous equation. Because \( m_+ \gg m_- \), for a relativistic ion-electron plasma, \( H^\mu \) only naturally vanishes if both distribution functions are isotropic in the proper frame. This is true whether or not the proton component has a relativistic temperature.

Note that \( H^\mu \) vanishes for any two component plasma in the non-relativistic limit. To see this note that

\[
H^i \propto -m_+ n_+^* (v_+^i)^* + m_- n_-^* (v_-^i)^* = m_+ n_+^* (\gamma_+ v_+^i - v_+^i)^* + m_- n_-^* (\gamma_- v_-^i - v_-^i)^*,
\]

where the second equality follows since the proper frame momentum density vanishes by definition. Thus when \( n_+^* = n_-^* \), (31) vanishes when \( \gamma_+ = \gamma_- = 1 \). Therefore \( H^\mu \) vanishes by the argument that follows equation (12).

Finally, note that our proper frame result is not the full nonrelativistic limit, because of the relativistic temperature. In the non-relativistic limit, we have the additional result that the pressure drops out of (23) and that \( \epsilon^* = n^* m_e c^2 \) so

\[
\eta_c^{(pr)} \rightarrow \eta = (\nu_c n^* m_e c^2)/(n^* e c)^2
\]

\[
= m_e \nu_c/(n^* e^2) = 2 \mu_{\text{pair}} \nu_c/(n^* e^2),
\]

where \( \eta \) is the non-relativistic resistivity, and \( \mu_{\text{pair}} \equiv m_e/2 \) is the reduced mass for the pair
plasma. Equation (33) is the usual non-relativistic result. In the case of a non-relativistic ion-electron plasma, \( \mu_{\text{pair}} \) is replaced by \( \mu_{\text{ie}} \equiv \frac{m_i m_e}{(m_i + m_e)} \sim m_e \).

We would like to thank A. Loeb for discussion.

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