Wheeler–De Witt equation and AdS/CFT correspondence

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Abstract

The radial Wheeler–De Witt equation on the asymptotically AdS spacetime proposed in [10] has as its semiclassical solution the wave function that asymptotically satisfies the conformal Ward identity, exemplifying the AdS/CFT correspondence. In this paper we show that this results holds also in the case of a complete quantum solution of the radial Wheeler–De Witt equation. It turns out that if the wavefunction is expanded in the parameter ρ with ρ → 0 defines the asymptotic boundary of the spacetime, the quantum loop corrections to the semiclassical wave are of subleading order.

Introduction

The AdS/CFT correspondence is a deep and beautiful relation between quantum gravity theory in the bulk of an asymptotically Anti de Sitter space and conformal field theory on the boundary of this space. It has been first conjectured in the context of string theory [1] on AdS$_5 \times$ S$^5$ background, but it has soon been realized that it might be a generic property of any quantum gravity theory. This interpretation of AdS/CFT is implicit in the seminal Witten’s paper [2] (see also [3]) and has been further supported by semiclassical investigations reported in the series of papers [4], [5], [6], [7], [8].

One of the most interesting aspects of these investigations was the discovery of a direct relation between the radial flow off the boundary in the bulk with the renormalization group flow in the boundary theory [5]. This relation was established by using the Hamilton-Jacobi theory that makes it possible to cast the flow equation of (super) gravity into the form of the
Callan-Symanzik equations. It was already suggested in [5] that a natural generalization of its results would be to consider the full Wheeler-De Witt equation, especially in view of the formal similarity between the Wheeler-De Witt equation and the right hand side of the Polchinski’s exact renormalization group equation [9].

This research program has started with an interesting paper [10], in which it was shown that AdS/CFT correspondence can be established in the case of the “wave function of the universe”, a solution of the radial Wheeler-De Witt equation [11], and that such solutions provide a simple and systematic way to derive the local action in the asymptotic expansion. As shown in that paper in the case of an asymptotically AdS space the radial Wheeler–De Witt equation is formally exactly the same as the standard one (see e.g., [12] for review). The difference in the physical interpretation of these two equations is that while the standard Wheeler–De Witt equation is related to the Hamiltonian constraint of general relativity, related with evolution in time, the radial equation describes a constraint associated with radial expansion of a “constant space-time radius surface.” Thus, by solving this latter equation, and taking the limit of infinite radius one can investigate the property of asymptotic wave function. It has been argued in [10] that in such a limit the wave function indeed possesses desired properties, satisfying the correct conformal Ward identity.

In particular, the correspondence reads

$$
\Psi_\rho (\frac{\gamma}{\rho^2}) = e^{\frac{i}{\rho} S^{(d)}(\frac{\gamma}{\rho^2})} Z_+ + e^{-\frac{i}{\rho} S^{(d)}(\frac{\gamma}{\rho^2})} Z_-,
$$

(1)

\(\Psi_\rho(\gamma)\) being the wave function on the \(D\)-dimensional spatial slice \(\Sigma_\rho\) having metric \(\gamma_{ab}\), while the functionals \(Z_{\pm}\) are solutions of the (nonanomalous) Ward identities associated with conformal invariance on \(\Sigma_\rho\). The local action \(S^{(d)}(\gamma)\) can be expanded in the limit of small AdS radius \(\rho\) and for \(\rho \to 0\) it has the following properties (at least for \(d < 5\)):

- the terms with negative powers of \(\rho\) coincide with minus the counterterms \(S_{ct}(\gamma)\) adopted in holographic renormalization to regularize the gravitational action [7];

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1. This result found an explanation in [13], where the correspondence between the terms responsible for the finiteness of the action and the Hamilton-Jacobi formulation has been elucidated.
• the term of the $\rho^0$ order gives the anomaly $A_d$ of the conformal theory on $\Sigma_\rho$.

Therefore, one can write

$$e^{\frac{i}{\kappa} S_{d}(\varpi)} Z = e^{-\frac{i}{\kappa} S_{ct}(\varpi)} \tilde{Z},$$

in which $\tilde{Z}$ satisfies the conformal anomalous Ward identity.

One of the new features of [10] is that the correspondence holds between a quantum gravity model and two conformal theories. This is due to the fact that the Wheeler-De Witt equation is of second order and a solution is specified by two coefficients $Z_{\pm}$. One finds out the standard AdS/CFT correspondence by fixing $Z_{-} = 0$ and by performing a semiclassical limit for $\Psi_\rho$, i.e.

$$\Psi_\rho(\gamma) = \int_{g_{|\Sigma_\rho}=\gamma} Dg \ e^{\frac{i}{\kappa} S_{d}(g)} \sim e^{\frac{i}{\kappa} S_{d}(\gamma)} \tilde{Z}_{+},$$

$S_{d}(\gamma)$ being the gravitational action evaluated at the classical solution having as initial condition the metric $\gamma$ on $\Sigma_\rho$. This way, one can rewrite (1) as follows

$$e^{\frac{i}{\kappa} S_{d}(\varpi)} = e^{-\frac{i}{\kappa} S_{ct}(\varpi)} \tilde{Z}_{+},$$

and by moving to the left-hand side the factor $e^{-\frac{i}{\kappa} S_{ct}}$ one gets the standard correspondence between the regularized gravitational action and the solution of the Ward identity for a (possibly anomalous) conformal theory.

It is should be stressed that the approach presented in [10] automatically accounts for both holographic renormalization [4], [5], [6], [7], [8] and the anomaly of the conformal theory, which, in the standard AdS/CFT paradigm, are unrelated.

The argument of [10] is however not complete since renormalization effect are not taken into account there. Such effects necessarily arise because the Wheeler–De Witt operator contains two functional derivatives acting at the same spacetime point, whose action produce terms proportional to the $n$th derivative delta function at zero, $\delta^{(n)}(0)$, which should be carefully renormalized [14]. It is not clear therefore if the results of [10] stands when the presence of such terms is taken into account. This is the question we would like to address in this paper.

Some time ago in the paper [15] we showed that there is a way to regularize Wheeler–De Witt equation and to renormalize its action on wave functions.
The method of that paper can be readily applied in the present context. But let us start reviewing briefly the argument presented in [10] showing the formal coincidence between the radial Wheeler–De Witt equation and the standard one.

In a $D + 1$-dimensional space-time manifold $\mathcal{M}$ endowed with metric $g_{\mu\nu}$ and with boundary $\Sigma$ having the induced metric $\gamma_{ab}$, the action for gravity which leads to the well-posed variational principle reads

$$S(g) = -\frac{1}{2\kappa} \int_\mathcal{M} d^{D+1}x \sqrt{g} (R(g) - 2\Lambda) + \frac{\epsilon}{\kappa} \int_\Sigma d^Dx \sqrt{\gamma} K,$$

where $R$ denotes the scalar curvature, and $K = \gamma^{ab} K_{ab}$ is the trace of the extrinsic curvature on $\Sigma$. The cosmological constant $\Lambda$ is related to the cosmological length scale $\ell$ as follows

$$\Lambda = -\epsilon \frac{D(D-1)}{2\ell^2},$$

$\epsilon$ being $-1$ ($+1$) for $\Sigma$ spacelike (timelike) and $\kappa = 8\pi G$.

The on-shell variation of the action (5) under a bulk diffeomorphism reads

$$\delta_\xi S(g) = -\frac{1}{2\kappa} \int_\Sigma d^Dx \sqrt{\gamma} \xi \left[ R(\gamma) - 2\Lambda + \epsilon (K^2 - K^a_b K_b^a) \right]$$

$\xi$ being the infinitesimal parameter of the transformation. The expression inside square brackets in the equation above coincides formally with the superhamiltonian, the only difference being the presence of the factor $\epsilon$ (which in ADM formalism is fixed to $-1$ corresponding to the spacelike boundary). Hence, the Ward identity associated with bulk diffeomorphisms invariance tell us that the associated operator $H$ vanishes when acting on the wave function $\Psi(\gamma) = \int_{g|\Sigma=\gamma} Dg \ e^{\sqrt{\epsilon} S(g)}$. Thus, the radial evolution is dictated by an equation which is essentially identical to the Wheeler-De Witt one.

Following [10] we therefore write the radial Wheeler–De Witt operator $H$ as follows

$$H = -\kappa^2 \hat{\Pi} \circ \hat{\Pi}(x) + R[\gamma](x) + \frac{D(D-1)}{\ell^2},$$

with

$$\hat{\Pi} \circ \hat{\Pi}(x) = \hat{\Pi}_a^b(x) \hat{\Pi}_b^a(x) - \frac{\hat{\Pi}(x) \hat{\Pi}(x)}{D-1}.$$
In this equation $\hat{\Pi}$ is the momentum operator

$$\hat{\Pi}_b^a(x) \equiv \frac{2}{\sqrt{\gamma(x)}} \gamma_{bc} \frac{\delta}{\delta \gamma_{ac}(x)}, \quad \hat{\Pi} \equiv \hat{\Pi}_a^a, \quad (10)$$

$\gamma_{ab}$ is the metric on $D$ dimensional space $\Sigma$ of constant “radius” $\rho$. Notice that in writing down (8) we made use of a particular ordering defined by (10), resolving therefore the ordering ambiguity. This particular ordering has the virtue that, as we will see below, when $H$ acts on the volume $V = \int_{\Sigma^\rho} \sqrt{\gamma} d^Dx$ the result is finite and does not require renormalization.

As it stands, eq. (8) is meaningless because it involves the product of two functional derivatives defined at the same point, and clearly requires regularization. Following the idea of [15] we define the regularized Wheeler–De Witt operator $H^{reg}$ with the help of heat kernel, to wit

$$H^{reg} = -\kappa^2 \hat{\Pi} \circ \hat{\Pi}(x) + R[\gamma](x) + \frac{D(D-1)}{\ell^2} \quad (11)$$

with

$$\hat{\Pi} \circ \hat{\Pi}(x) = \int_{\Sigma^\rho} d^Dy K(x, y; t) \sqrt{\gamma(x)} \left( \hat{\Pi}_b^a(x) \hat{\Pi}_a^b(y) - \frac{\hat{\Pi}(x)\hat{\Pi}(y)}{D-1} \right)$$

The heat kernel satisfies the equation

$$\frac{\partial}{\partial t} K(x, y; t) = \nabla^2_{(x)} K(x, y; t) + \xi R(x) K(x, y; t) \quad (12)$$

with the initial condition

$$\lim_{t \to 0} K(x, y; t) = \frac{\delta^{(3)}(x - y)}{\sqrt{\gamma(x)}} \quad (13)$$

Equation (12) can be solved perturbatively in powers of $t$. To the order which will be of interest in the present context, the solution reads [16] (see [17] for a recent discussion)

$$K(x, y; t) = \exp \left( \left( -\frac{1}{4t} \gamma_{ab}(x) - \frac{1}{2t} R_{ab}(x) \right) \Delta^a \Delta^b \right) \left\{ 1 + t \left( \xi - \frac{1}{6} \right) R(x) + t^2 \left[ \frac{1}{6} \left( \xi - \frac{1}{6} \right) \nabla^2 R(x) + \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 R^2(x) + \frac{1}{60} R_{ab}(x) R^{ab}(x) - \frac{1}{180} R^2(x) \right] + O(t^3) \right\}$$

(14)
with $\Delta^a = x^a - y^a$.

Then, the coincidence limit $t \to 0$ is performed, such that all the terms with positive powers of $t$ are avoided. However, some infinities generically arise because of the presence of negative powers of $t$. These terms can be treated according with the procedure defined in [14] via analytic continuation and they define renormalized (dimensionful) constant.

Having established the form of Wheeler–De Witt radial operator we define the wave function $\Psi(\gamma)$ on a radial slice parametrized by radius $\rho$ to be a solution of Wheeler–De Witt equation

$$H\Psi = 0 \quad (15)$$

It should be stressed that $\Psi$ encodes all the information on quantum space-time and in particular, as argued in [10] its asymptotic properties can be read off from the infinitely rescaled wave function $\Psi(\rho^{-2}\gamma)$ in the limit when $\rho$ is going to zero.

To investigate this asymptotic behavior of the solutions of Wheeler–De Witt equation it is convenient to rescale variables

$$\gamma_{ab} \to \rho^{-2}\gamma_{ab} \quad (16)$$

so that the radius $\rho$ is explicit in the Wheeler–De Witt operator:

$$H^\text{reg}_\rho = -\kappa^2 \rho^2 \tilde{\Pi} \circ_{\kappa_\rho} \tilde{\Pi}(x) + \rho^2 R[\gamma](x) + \frac{D(D-1)}{\ell^2} \quad (17)$$

with the rescaled wave function $\Psi(\rho^{-2}\gamma) \equiv \Psi_\rho(\gamma)$ being a solution of

$$H_\rho \Psi_\rho = 0 \quad (18)$$

The crucial difference between (16) and the analogous formula in [10] is that now the heat kernel acquires the dependence on the radius $\rho$. Let us investigate if this would lead to any modifications of the results presented there.

It is quite remarkable that the rescaled heat kernel has almost the same form as the original one. In fact, one can rescale the heat kernel time $t \to t' = t\rho^2$ to find

$$K_\rho(x, y; t) = \rho^D K(x, y, t') \quad (19)$$

so that (17) becomes

$$H^\text{reg}_\rho = -\kappa^2 \rho^{2D} \tilde{\Pi} \circ_{\kappa} \tilde{\Pi}(x) + \rho^2 R[\gamma](x) + \frac{D(D-1)}{\ell^2}, \quad (20)$$
where the expansion of the kernel is given in (14).

The asymptotic behavior of the solution (18) can be investigated by writing $\Psi_\rho = \Psi_\rho^{(0)} \Psi_\rho^{(1)} \Psi_\rho^{(2)} \ldots$ and by performing an expansion of $H_\rho \Psi_\rho$ in powers of $\rho$.

**D=2**

Let us use the expression (20) in the case of the of AdS$_3$/CFT$_2$. Following [10] we use the ansatz \[
\Psi_\rho(\gamma) = e^{iS^0} Z_+(\gamma) + e^{-iS^0} Z_-(\gamma), \quad S^0 = \frac{1}{\kappa \ell \rho^2} \int_{\Sigma_\rho} \sqrt{\gamma} \tag{21}\]
so that $e^{\pm iS^0}$ satisfies the equation

\[
\kappa^2 \rho^4 \hat{\Pi} \circ \hat{\Pi}(x) e^{\pm iS^0} = \frac{2}{\ell^2} e^{\pm iS^0}.
\]

This is easy to see noticing that as a result of the ordering we use there is no second derivative of $S^0$ term resulting from the left hand side of this expression. Then $Z_\pm$ satisfy the equation

\[
\left\{-\kappa^2 \rho^2 \hat{\Pi} \circ \hat{\Pi}(x) \pm 2i \frac{\kappa}{\ell} \hat{\Pi}(x) \right\} Z_\pm = - R[\gamma](x) Z_\pm \tag{22}\]

If we neglect the first term for a moment we find the conformal Ward identity

\[
\hat{\Pi} Z_\pm = \pm i \frac{\ell}{2\kappa} Z_\pm, \tag{23}\]

with central charge $c = 12\ell/\kappa = 3\ell/2G$.

There are two ways to write the solution of the equation above. The first one is discussed also in [10] and it reads

\[
Z_\pm(e^{2\phi \hat{\gamma}}) = e^{\pm i \frac{\ell}{2\kappa} S_L(\phi, \hat{\gamma})} Z(\hat{\gamma}), \quad S_L(\phi, \hat{\gamma}) = \int_{\Sigma} \sqrt{\gamma}(\phi \nabla^2 \phi + \phi R[\gamma]) d^2x, \tag{24}\]

---

\[\text{It is worth noting that } S_0 \text{ and the analogous terms appearing at higher orders of the asymptotic expansions are actually infinite since the integration extends over the non compact space } \Sigma_\rho, \text{ which for AdS}_D \text{ is } R \times S_{D-2}.\]
in which $\gamma_{ab} = e^{2\phi} \hat{\gamma}_{ab}$ and $\det \hat{\gamma} = 1$.

Equivalently the solution of Eq. (23) can be written in terms of the non-local Polyakov action \cite{18}, i.e.

$$S_{Pol} = \int d^2 x d^2 x' (\sqrt{\gamma} R(x)) G(x, x')(\sqrt{\gamma} R)(x'),$$  \hspace{1cm} (25)

in which the propagator $G(x, x')$ is the inverse of the Laplacian operator:

$$\sqrt{\gamma(x)} \nabla^2 G(x, x') = \delta(x - x').$$  \hspace{1cm} (26)

The solution reads

$$Z_{\pm}(\gamma) = e^{\mp i \frac{4}{3 \kappa} S_{Pol}}.$$  \hspace{1cm} (27)

It is easy to see that in both cases asymptotically the contribution to the equation (23) resulting from the first term in (22), including the loop corrections, is negligible. Indeed $Z_{\pm}$ does not depend on $\rho$ and therefore the first term in (22) is negligible compared to other two in the limit $\rho \to 0$. Thus there is no correction to $Z_{\pm}$ in the leading order resulting from this term, in particular from those that result from the coincidence limit of the heat kernel, that we will call `loop corrections' (see below.) Such corrections will be, of course present, if the terms of higher order in $\rho$ are taken into account, but unfortunately, due to the complexity of such term we have not be able to calculate their explicit form.

D=3

Let us now turn to the $D = 3$ case, which corresponds to the physical (at least at low energies) four dimensional, asymptotically AdS spacetime, i.e., to the AdS$_4$/CFT$_3$ correspondence. In this case we use the ansatz

$$\Psi_{\rho}(\gamma) = e^{i(S^0 + S^1)} Z_+(\gamma) + e^{-i(S^0 + S^1)} Z_-(\gamma),$$  \hspace{1cm} (28)

where $S^0$ and $S^1$ are the leading and next-to-leading order solutions of the radial Wheeler–De Witt equation

$$H_{\rho} \Psi_{\rho}(\gamma) = 0,$$

$$H_{\rho} = -\kappa^2 \rho^6 \hat{\Pi} \circ K \hat{\Pi}(x) + \rho^2 R[\gamma](x) + \frac{6}{\ell^2}. $$  \hspace{1cm} (29)
Similarly to the $D = 2$ case we assume that $e^{iS^0}$ satisfies the equation

$$\left\{-\kappa^2 \rho \hat{\Pi} \circ \hat{\Pi} (x) + \frac{6}{\ell^2}\right\} e^{iS^0} = 0,$$

and we find that

$$S^0 = \frac{2}{\kappa \ell \rho^2} \int_{\Sigma_\rho} \sqrt{\gamma} \, d^3 x.$$

(30)

Substituting this to (29) we get

$$\left[\kappa^2 \rho^4 \hat{\Pi} \circ \hat{\Pi} (x) - R[\gamma](x) \pm 2 \frac{i \kappa \rho}{\ell} \hat{\Pi} (x)\right] e^{\pm iS^1} Z_\pm = 0.$$

(31)

The first term in (31) seems again to be negligible in the limit $\rho \to 0$, so let us use the ansatz

$$\left(R[\gamma](x) \pm 2 \frac{i \kappa \rho}{\ell} \hat{\Pi} (x)\right) e^{\pm iS^1} = 0,$$

from which we get

$$S^1 = \frac{\ell}{2 \kappa \rho} \int_{\Sigma_\rho} \sqrt{\gamma} \, R[\gamma] d^3 y.$$

(32)

Then, still neglecting the first term in (31), we find that

$$\hat{\Pi} Z_\pm = 0,$$

(33)

which expresses the conformal invariance of $Z_\pm$. A solution of this equation is given be the Chern-Simons action [19]

$$Z_\pm = \int_{\Sigma_\rho} d^3 y \, e^{abc} \left(\frac{1}{2} \Gamma^i_{aj} \partial_i \Gamma^j_{ci} + \frac{1}{3} \Gamma^i_{aj} \Gamma^j_{bk} \Gamma^k_{ci}\right)$$

(34)

This expression can be checked to be diffeomorphism invariant, because its variation with respect to $\gamma$ produces the Cotton tensor, which is symmetric, traceless, with vanishing covariant divergence.

We can now easily check that this solution is self-consistent. Indeed letting the first term in (31) act on $e^{\pm iS^1} Z_\pm$ we obtain terms of order of $\rho^3$ and $\rho^2$ which are small in the limit $\rho \to 0$. 

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For $D = 4$ the correspondence is AdS$_5$/CFT$_4$. The Wheeler-De Witt equations is now given by

$$H_{\rho} = -\kappa^2 \rho^8 \hat{\Pi} \circ_{K} \hat{\Pi}(x) + \rho^2 R[\gamma](x) + \frac{12}{\ell^2},$$

(35)

and the wave function can still be written as in Eq.(28), with

$$S^0 = \frac{3}{\kappa \ell^4} \int_{\Sigma_{\rho}} \sqrt{\gamma} d^3 x.$$  

(36)

The analogous of Eq.(31) now reads

$$\left[ \kappa^2 \rho^6 \hat{\Pi} \circ_{K} \hat{\Pi}(x) - R[\gamma](x) \mp \frac{2i\kappa \rho^2}{\ell} \hat{\Pi}(x) \right] e^{\pm i S^1} Z_\pm = 0.$$

(37)

In order to cancel the $R$-term, let us make the following ansatz

$$S^1 = \frac{\ell}{4 \kappa \rho^2} \int_{\Sigma_{\rho}} \sqrt{\gamma} R[\gamma] d^3 y.$$  

(38)

The first term in Eq.(37) cannot be neglected anymore and the resulting equation for $Z_\pm$ now reads

$$\frac{\ell^2}{4} \left( R^a_b(x) R^b_a(x) - \frac{1}{3} R^2(x) \right) Z_\pm \mp \frac{2i\kappa}{\ell} \hat{\Pi} Z_\pm = 0,$$

(39)

which coincides with the conformal Ward identity in $D = 4$ for proper values of the central charges. An explicit solution for the equation above has been given in [10] and it takes the same form as $Z_\pm$ in Eq.(24) with the Liouville action given by

$$S_L(\phi, \gamma) = \int_{\Sigma_{\rho}} d^3 x \left( \frac{1}{2} \phi \nabla^2 \phi + \phi Q_4(\gamma) \right),$$

(40)

in which

$$\nabla^2 = \nabla^2 + \nabla_a (2 P \gamma^{a b} - 4 P^a_b) \nabla_b,$$  

$$Q_4(\gamma) = \frac{1}{2} \nabla^2 P + P^2 - P^a_b P_a^b,$$

(41)

with $P_{ab} = \frac{1}{2} (R_{ab} - \frac{1}{3} \gamma_{ab} R)$. 

10
Loop corrections

In what follows, we are going to investigate the asymptotic behavior of those terms which are absent in the un-regularized scheme presented in [10] and which are peculiar of the adopted regularization procedure. Since they are purely quantum corrections just like loop corrections in ordinary quantum field theory, we will call them ‘loop corrections’.

For a generic dimension $D > 2$ the expansion given in [10] is consistent. In fact, the zero order term reads

$$\Psi^{(0)}_{\rho}(\gamma) = \exp\left(\pm \frac{i}{\kappa \rho^D} \frac{D-1}{\ell} \int_{\Sigma_{\rho}} \sqrt{\gamma(x)} d^D x \right), \quad (42)$$

such that

$$\kappa^2 \rho^{2D} \hat{\Pi} \circ \hat{\Pi}(x) \Psi^{(0)}_{\rho} = \frac{D(D-1)}{\ell^2} \Psi^{(0)}_{\rho}. \quad (43)$$

The first order expansion of Eq.(18) reads

$$\pm \frac{2\kappa}{\ell} \rho^D \hat{\Pi}(x) \Psi^{(1)}_{\rho} + \rho^2 R[\gamma] \Psi^{(1)}_{\rho} = \kappa^2 \rho^{2D} \hat{\Pi} \circ \hat{\Pi}(x) \Psi^{(1)}_{\rho}, \quad (44)$$

and since the right-hand side can be neglected in the limit $\rho \to 0$, a proper solution is given by

$$\Psi^{(1)}_{\rho}(\gamma) = \exp\left(-i \frac{1}{\kappa} \rho^{2-D} \frac{\ell}{2(D-2)} \int_{\Sigma_{\rho}} \sqrt{\gamma(x)} R[\gamma] d^D x \right). \quad (45)$$

The right-hand side of Eq.(18) enters next orders of the asymptotic expansion and it gives

$$\kappa^2 \rho^{2D} \hat{\Pi} \circ \hat{\Pi}(x) \Psi^{(1)}_{\rho} = \frac{\ell^2}{(D-2)^2} \rho^4 \left( R^0_{\rho}(x) - \frac{D}{4(D-1)} R^2(x) \right) \Psi^{(1)}_{\rho} -$$

$$- \frac{i \kappa \ell}{(D-2)} \rho^{2+D} \frac{D^2 - 2}{D-1} K(x, x; t) R(x) \Psi^{(1)}_{\rho} +$$

$$+ \frac{i \kappa \ell}{D-2} (2 - D(D-1)) \rho^{2+D} [\nabla^2_{(y)} K(x, y, t)]_{y=x} \Psi^{(1)}_{\rho}. \quad (46)$$

The first term coincides with the one found in [10], while the additional contributions are corrections due to renormalization scheme. In fact, in the
coincidence limits $K(x, x; t)$ and $[\nabla_y^2 K(x, y; t)]_{y=x}$ the singularities encountered as $t \to 0$ give rise to the finite renormalization constants $[14], [15].$

At the next order, the Wheeler–De Witt equation reads

$$\pm i \frac{2 \kappa}{\ell} \rho^D \hat{\Pi}(x) \Psi^{(2)}_\rho = \left( \Psi^{(1)}_\rho \right)^{-1} \left[ \kappa^2 \rho^{2D} \hat{\Pi} \circ \hat{\Pi}(x) \Psi^{(1)}_\rho \right] \Psi^{(2)}_\rho =$$

$$= \frac{\ell^2}{(D-2)^2} \rho^4 \left( R^a_b(x) R^b_a(x) - \frac{D}{4(D-1)} R^2(x) \right) \Psi^{(2)}_\rho + O(\rho^5),$$

(47)

and the loop corrections are negligible being $O(\rho^{2+D})$. This is the case for all the relevant orders as $\rho \to 0$. In fact, for a generic term of order $n$ in the expansion one has the following equation in the un-regularized case

$$\pm i \frac{2 \kappa}{\ell} \rho^D \hat{\Pi}(x) \Psi^{(n)}_\rho \propto \rho^{2n} \Psi^{(n)}_\rho + O(\rho^{2n+1}).$$

(48)

The leading corrections coming from the regularization procedure are $O(\rho^{D+2})$ [16], thus they are relevant for $2n \geq D + 2$. At such orders of the asymptotic expansion one has

$$\hat{\Pi}(x) \Psi^{(n)}_\rho \propto \rho^{2n-D} \Psi^{(n)}_\rho, \quad 2n - D \geq 2,$$

(49)

and being the dependence from $\rho$ through a positive power, the right-hand side of the equation above can be neglected as $\rho \to 0$. Hence, the terms of the expansion for which loop corrections are relevant turn out to be negligible in the asymptotic limit.

**Conclusions**

In this work, we presented a regularization scheme for the Wheeler-De Witt operator describing the radial evolution of the gravitational field in $D + 1$-dimensional spacetime. This way, we could investigate the role of loop corrections on the asymptotic behavior of the wave function in an asymptotically AdS space time. This analysis was motivated by the results presented in [10], which elucidated the correspondence between the bulk wave function and the partition function of a boundary CFT via the asymptotic expansion of the solution of the radial Wheeler-De Witt equation. We found that all the corrections coming from renormalization are negligible in the asymptotic limit. Therefore, the asymptotic form of the wave-function $\Psi$ is insensitive
to loop corrections and the results of \cite{10} are well-grounded. Although this result is not surprising in view of \cite{13}, where it was argued that the solution of the WKB expansion gets asymptotical contributions only from the long-distance divergencies of the gravitational action, it is encouraging to derive it explicitly.

In this paper we considered only pure (quantum) gravity in the bulk and we showed that the asymptotic wave function is conformally invariant (up to the possible conformal anomaly). This result holds for an arbitrary quantum gravity theory described, at least in some regime, by Wheeler-De Witt equation, irrespectively of its possible UV completion. If so the theory should somehow notice the renormalizability problem of pure quantum gravity. This problem should disappear, of course, in the UV complete theory, like string theory, in which the short distance pathologies are automatically taken care of. However, as explained above, even in pure quantum gravity the renormalizability problem arises only when we move away from the boundary, and the terms of higher order in $\rho$ cannot be neglected anymore. Since the radial flow in the bulk is associated with the renormalization group flow of the dual theory \cite{6, 3, 22} (see \cite{23} for recent developments), the running of the couplings of the conformal theory should be sensitive to loop corrections in the bulk. The connection between the radial flow of the bulk wave function and the renormalization group flow of the theory on the boundary will be the subject of our future investigations.

It should be mentioned that Wheeler-De Witt equation has already been used in \cite{20} to evaluate the $1/N^2$ corrections to the conformal anomaly of the boundary $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory in AdS$_5$/CFT$_4$. The results coincided with the ones obtained in \cite{21} in the Schrodinger representation. It is worth noting the crucial role of the heat-kernel expansion in such calculations (the anomaly corrections are proportional to the coefficient $a_2(x,x)$ in the De Witt-Schwinger proper time representation). However, \cite{20} discusses the case of gravity coupled to $N$ massive scalar fields and the authors use the heat kernel to regularize the kinetic part of the scalar field only, which makes it hard to directly compare their results with ours.

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References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[2] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[4] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP 9807, 023 (1998) [arXiv:hep-th/9806087].

[5] J. de Boer, E. P. Verlinde and H. L. Verlinde, “On the holographic renormalization group,” JHEP 0008 (2000) 003 [hep-th/9912012].

[6] J. de Boer, “The Holographic renormalization group,” Fortsch. Phys. 49 (2001) 339 [hep-th/0101026].

[7] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” Commun. Math. Phys. 217, 595 (2001) [arXiv:hep-th/0002230].

[8] I. Papadimitriou and K. Skenderis, “AdS / CFT correspondence and geometry,” [arXiv:hep-th/0404176]

[9] J. Polchinski, “Renormalization and Effective Lagrangians,” Nucl. Phys. B 231 (1984) 269.

[10] L. Freidel, “Reconstructing AdS/CFT,” arXiv:0804.0632 [hep-th].

[11] B. S. DeWitt, “Quantum Theory of Gravity. 1. The Canonical Theory,” Phys. Rev. 160, 1113 (1967).

[12] C. J. Isham, “Structural issues in quantum gravity,” arXiv:gr-qc/9510063.

[13] I. Papadimitriou, “Holographic renormalization as a canonical transformation,” JHEP 1101 014 (2010) [arXiv:1007.4592]
[14] P. Mansfield, “Continuum strong coupling expansion of Yang-Mills theory: Quark confinement and infrared slavery” Nucl. Phys. B 418 113 (1994)

[15] J. Kowalski-Glikman and K. A. Meissner, “A Class of Exact Solutions of the Wheeler – De Witt Equation,” Phys. Lett. B 376, 48 (1996) [arXiv:hep-th/9601062].

[16] L. Parker and D. J. Toms, “New Form For The Coincidence Limit Of The Feynman Propagator, Or Heat Kernel, In Curved Space-Time,” Phys. Rev. D 31, 953 (1985).

[17] A. Codello and O. Zanusso, “On the non-local heat kernel expansion,” J. Math. Phys. 54, 013513 (2013)

[18] A. M. Polyakov, “Quantum Gravity in Two-Dimensions,” Mod. Phys. Lett. A 2 (1987) 893

[19] R. Jackiw and S. Y. Pi, “Chern-Simons modification of general relativity,” Phys. Rev. D 68 (2003) 104012 [gr-qc/0308071].

[20] T. Kubota, T. Ueno and N. Yokoi, “Wheeler-DeWitt equation in AdS / CFT correspondence” Phys. Lett. B 579, 200 (2004) [arXiv:hep-th/0310109]

[21] P. Mansfield, D. Nolland and T. Ueno, “Order 1 / N² test of the Maldacena conjecture: 2. The full bulk one loop contribution to the boundary Weyl anomaly” Phys. Lett. B 565, 207 (2003) [arXiv:hep-th/0208135]

[22] E. T. Akhmedov, “A Remark on the AdS / CFT correspondence and the renormalization group ow,” Phys. Lett. B 442 (1998) 152 [hep-th/9806217]; V. Balasubramanian and P. Kraus, “Space-time and the holographic renormalization group,” Phys. Rev. Lett. 83 (1999) 3605 [hep-th/9903190]

[23] I. Heemskerk and J. Polchinski, “Holographic and Wilsonian Renormalization Groups,” JHEP 1106 (2011) 031 [arXiv:1010.1264]; T. Faulkner, H. Liu, and M. Rangamani, “Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm,” JHEP 1108 (2011) 051 [arXiv:1010.4036] [hep-th]