Dobiński-type relations and the Log-normal distribution

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Abstract.
We consider sequences of generalized Bell numbers $B(n)$, $n = 0,1,\ldots$ which can be represented by Dobiński-type summation formulas, i.e. $B(n) = \frac{1}{C} \sum_{k=0}^{\infty} \frac{[P(k)]^n}{D(k)}$, with $P(k)$ a polynomial, $D(k)$ a function of $k$ and $C = \text{const}$. They include the standard Bell numbers ($P(k) = k, D(k) = k!, C = e$), their generalizations $B_{r,r}(n)$, $r = 2,3,\ldots$ appearing in the normal ordering of powers of boson monomials ($P(k) = \frac{(k+r)!}{k!}, D(k) = k!, C = e$), variants of “ordered” Bell numbers $B^{(p)}(n)$ ($P(k) = k, D(k) = (p+1)^{p}, C = 1 + p, p = 1,2,\ldots$), etc. We demonstrate that for $\alpha, \beta, \gamma, t$ positive integers ($\alpha, t \neq 0$), $\left[B(\alpha n^2 + \beta n + \gamma)\right]^t$ is the $n$-th moment of a positive function on $(0, \infty)$ which is a weighted infinite sum of log-normal distributions.
In a recent investigation [1] we analysed sequences of integers which appear in the process of normal ordering of powers of monomials of boson creation $a^\dagger$ and annihilation $a$ operators, satisfying the commutation rule $[a, a^\dagger] = 1$. For $r, s$ integers such that $r \geq s$, we define the generalized Stirling numbers of the second kind $S_{r,s}(n, k)$ as

$$[(a^\dagger)^r a^s]^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k)(a^\dagger)^k a^k$$

and the corresponding Bell numbers $B_{r,s}(n)$ as

$$B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n, k).$$

In [1] explicit and exact expressions for $S_{r,s}(n, k)$ and $B_{r,s}(n)$ were found. In a parallel study [2] it was demonstrated that $B_{r,s}(n)$ can be considered as the $n$-th moment of a probability distribution on the positive half-axis. In addition, for every pair $(r, s)$ the corresponding distribution can be explicitly written down. These distributions constitute the solutions of a family of Stieltjes moment problems, with $B_{r,s}(n)$ as moments. Of particular interest to us are the sequences with $r = s$, for which the following representation as an infinite series has been obtained:

$$B_{r,r}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{(k + r)!}{k!} \right]^{n-1}$$

$$= \frac{1}{e} \sum_{k=0}^{\infty} \frac{[k(k+1)\ldots(k+r-1)]^n}{(k+r-1)!}, \quad n > 0. \tag{4}$$

Eqs.(3) and (4) are generalizations of the celebrated Dobiński formula ($r = 1$) [3]:

$$B_{1,1}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!} k^n, \quad n \geq 0, \tag{5}$$

which expresses the conventional Bell numbers $B_{1,1}(n)$ as a rapidly convergent series. Its simplicity has inspired combinatorialists such as G.-C. Rota [4] and H.S. Wilf [5]. Eq.(5) has far-reaching implications in the theory of stochastic processes [6], [7], [8].

The probability distribution whose $n$-th moment is $B_{r,r}(n)$ is an infinite ensemble of weighted Dirac delta functions located at a specific set of integers (a so-called Dirac comb):

$$B_{r,r}(n) = \int_0^{\infty} x^n \left\{ \frac{1}{e} \sum_{k=0}^{\infty} \delta(x - k(k+1)\ldots(k+r-1)) \right\} dx, \quad n \geq 0. \tag{6}$$

For $r = 1$ the discrete distribution of Eq.(6) is the weight function for the orthogonality relation for Charlier polynomials [9]. In contrast we emphasize that for $r \neq s$ the $B_{r,s}(n)$ are moments of continuous distributions [2].

In this note we wish to point out an intimate relation between the formulas of Eqs. (3), (4), (5) and the log-normal distribution [10], [11]:

$$P_{\sigma,\mu}(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln(x) - \mu)^2}{2\sigma^2}}, \quad x \geq 0, \quad \sigma, \mu > 0. \tag{7}$$
First we quote the standard expression for its $n$-th moment:

$$M_n = \int_0^\infty x^n P_{\sigma, \mu}(x) dx = e^{n(\mu + n\sigma^2)}, \quad n \geq 0,$$

which can be reparametrized for $k > 1$ as

$$M_n = k^\alpha n^2 + \beta n,$$

with

$$\mu = \beta \ln(k),$$

$$\sigma = \sqrt{2\alpha \ln(k)} > 0.$$

Given three integers $\alpha, \beta, \gamma$ (where $\alpha > 0$), we wish to find a weight function $W_{1,1}(\alpha, \beta, \gamma; x) > 0$ such that

$$B_{1,1}(\alpha n^2 + \beta n + \gamma) = \int_0^\infty x^n W_{1,1}(\alpha, \beta, \gamma; x) dx.$$  

Eqs.(5), (7) and (9) provide an immediate solution:

$$W_{1,1}(\alpha, \beta, \gamma; x) = \frac{1}{e} \left[ \delta(x - 1) + \sum_{k=2}^\infty k\gamma \exp \left( \frac{\ln(x) - \beta \ln(k)^2}{4\alpha \ln(k)} \right) \right],$$

which is an infinite sum of weighted log-normal distributions supplemented by a single Dirac peak of weight $e^{-1}$ located at $x = 1$. Thus it is a superposition of discrete and continuous distributions. Virtually the same approach can be adopted for the sequences $B_{r,r}(n), \ r > 1$. In this case the $k = 1$ term in the numerator of Eq.(3) is larger than one.
and so there will be no Dirac peak in the formula. Then the function $W_{r,r}(\alpha, \beta, \gamma; x) > 0$ defined by ($\alpha, \beta, \gamma$ integers, $\alpha, \gamma > 0$):

$$B_{r,r}(\alpha n^2 + \beta n + \gamma) = \int_0^\infty x^n W_{r,r}(\alpha, \beta, \gamma; x) dx,$$

is a purely continuous probability distribution given again by an infinite sum of weighted log-normal distributions:

$$W_{r,r}(\alpha, \beta, \gamma; x) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{[(k+r)!]}{k!} \gamma^{-1} \exp \left( - \frac{[\ln(x) - \beta \ln\left(\frac{(k+r)!}{k!}\right)]^2}{4\alpha \ln\left(\frac{(k+r)!}{k!}\right)} \right).$$

The solutions of the moment problems of Eqs.(9), (12) and (14) are not unique. More general solutions may be obtained by the method of the inverse Mellin transform, see [12].

Several other types of combinatorial sequences have properties exemplified by Eqs.(12) and (14). We quote for example the so-called “ordered” Bell numbers $B_\omega(n)$ defined as [5]:

$$B_\omega(n) = \sum_{k=1}^{n} S(n, k) k!,$$

where $S(n, k)$ are the Stirling numbers of the second kind, $S_{1,1}(n, k)$ in our notation.
These ordered Bell numbers satisfy the following Dobiński-type relation:

\[ B_o(n) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k}, \]

from which a formula analogous to Eq.(13) readily follows. A more general identity of type (17) is [13]:

\[ B_o^{(p)}(n) = \frac{1}{p+1} \sum_{k=1}^{\infty} k^n \left( \frac{p}{p+1} \right)^k = \sum_{k=0}^{n} S(n, k) k! p^k, \quad p = 2, 3, \ldots. \]

We will not discuss other types of sequences but rather observe that the relations of Eqs. (3), (4), (5), (17), (18) naturally imply that any power of these numbers also satisfies a Dobiński-type relation. As an example we give explicitly the simplest case of Eq.(5). For integer \( t > 0 \):

\[ [B_{1,1}(n)]^t = \frac{1}{e^t} \sum_{k_1, k_2, \ldots, k_t=0}^{\infty} \frac{(k_1 k_2 \ldots k_t)^n}{k_1! k_2! \ldots k_t!}, \]

with correspondingly more complicated formulas of a similar nature for powers of \( B_{r,r}(n) \), \( B_o(n) \) and \( B_o^{(p)}(n) \). For combinatorial applications of Eq.(19) see [14], [15] and [16].

We conclude that for any sequences of the type \( B(n) \) specified above \([B(\alpha n^2 + \beta n + \gamma)]^t\) is always given as an \( n \)-th moment of a positive function on \((0, \infty)\) expressible by sums of weighted log-normal distributions. We illustrate such a function for \( B_{1,1}(n) \) in Fig.(1). The application to \( B_{r,r}(n) \) for \( r = 2, 3, 4 \) is presented in Fig.(2). The area under every curve is equal to 1 on extrapolating to large \( x \) (not displayed). In both examples we have chosen \( \alpha = \beta = \gamma = 1 \). Observe the exceedingly slow decrease of these probabilities for \( x \to \infty \). This is confirmed by the fact that the moment sequences \([B(\alpha n^2 + \beta n + \gamma)]^t\) are extremely rapidly increasing. In the simplest case \( \alpha = \beta = \gamma = t = 1 \) we find \( B_{1,1}(n^2 + n + 1) = 1, 5, 877, 27644437, 474869816156751 \) for \( n = 0 \ldots 4 \).

The circumstance that we can determine the positive solutions of the Stieltjes moment problem with both \( B(n) \) (discrete distribution) and \([B(\alpha n^2 + \beta n + \gamma)]^t\) (continuous distribution) is a very specific consequence of the existence of Dobiński-type expansions. To our knowledge it has no equivalent in standard solutions of the moment problem. For instance, if the moments are \( n! \) the solution \( e^{-x} \) does not give any indication as how one might obtain the solution for the moments equal to \((n^2)!\).

The strict positivity of \( W_{r,r}(\alpha, \beta, \gamma; x) \), for \( r = 1, 2 \ldots \), suggests their use in the construction of coherent states, which satisfy the resolution of identity property [17], [18], [19], [20]. This can be done by the substitution \( n! \to B_{r,r}(\alpha n^2 + \beta n + \gamma) \) in the definition of standard coherent states. More precisely for a complete and orthonormal set of wave functions \(|n\rangle\) such that \( \langle n|n'\rangle = \delta_{n,n'} \) and complex \( z \) we define the normalized coherent state as

\[ |z; \alpha, \beta, \gamma\rangle = \frac{1}{\mathcal{N}^{1/2}(\alpha, \beta, \gamma; |z|^2)} \sum_{n=0}^{\infty} \sqrt{B_{r,r}(\alpha n^2 + \beta n + \gamma)} |n\rangle, \]

(20)
with the normalization

\[ N(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \frac{x^n}{B_{r,r}(\alpha n^2 + \beta n + \gamma)}, \tag{21} \]

which is a rapidly converging function of \( x \) for \( 0 \leq x < \infty, \ x = |z|^2 \). Then, using the procedure of [18] we can demonstrate that the states of Eq.(20) along with Eq.(14) automatically satisfy the resolution of unity

\[ \int \int_{\mathbb{C}} d^2z |\alpha, \beta, \gamma; z|^2 \tilde{W}_{r,r}(\alpha, \beta, \gamma; |z|^2) \langle z; \alpha, \beta, \gamma | = I = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{22} \]

with

\[ W_{r,r}(\alpha, \beta, \gamma; |z|^2) = \frac{\pi}{N(\alpha, \beta, \gamma; |z|^2)} \tilde{W}_{r,r}(\alpha, \beta, \gamma; |z|^2). \tag{23} \]

We are currently investigating the quantum-optical properties of states defined in Eq.(20).

We close by quoting from [7] that, “the idea of representing the combinatorially defined numbers by an infinite sum or an integral, typically with a probabilistic interpretation, has proved to be a very fruitful one”. In our particular case it has allowed us to reveal quite an unexpected relation between the Dobiński-type summation relations, which by themselves are reflections of boson statistics, and the log-normal distribution.

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