Review of Reliability-Based Design Optimization Approach and Its Integration with Bayesian Method

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Abstract. A lot of uncertain factors lie in practical engineering, such as external load environment, material property, geometrical shape, initial condition, boundary condition, etc. Reliability method measures the structural safety condition and determine the optimal design parameter combination based on the probabilistic theory. Reliability-based design optimization (RBDO) is the most commonly used approach to minimize the structural cost or other performance under uncertainty variables which combines the reliability theory and optimization. However, it cannot handle the various incomplete information. The Bayesian approach is utilized to incorporate this kind of incomplete information in its uncertainty quantification. In this paper, the RBDO approach and its integration with Bayesian method are introduced.

1. Reliability-based Design Optimization
Various uncertainties lie in the practical engineering process, which have a great influence on the design optimization results. Some types of constraints such as initial condition and load capacity may be violated when uncertainties are taken into consideration. Under the circumstance, reliability-based design optimization (RBDO) is developed to solve the design optimization problem with reliability constraints. The RBDO process deals with two optimization models, simultaneously. The first one is a design optimization model, which search the feasible solution in an original random space. The other one is a reliability analysis model, which transfer the probability distribution into nonlinear mapping and finds the optimal solution in a standard normal random space. Hence the double-loop strategy is most often used to deal with RBDO problem [1, 2, 3].

The complexity and large calculating quantity of RBDO model is also a difficult question to be addressed. A single-loop strategy was proposed to improve computation efficiency. By decoupling the design optimization model and reliability analysis model, the computational process is greatly simplified [4, 5]. The RBDO model was initially developed by defining a probabilistic constraint as the reliability, a so-called reliability index approach (RIA). Then, an inverse approach was proposed by formulating RBDO with a probabilistic performance measure (PMA). And RIA and PMA have been the mainstream of RBDO.

In the system parameter design process, the RBDO model [6] can be generally defined as
Minimize \( \text{Cost}(d) \)
subject to \( P(G_i(X) \leq 0) - \Phi(-\beta_i) \leq 0, \ i = 1,2,\cdots, \text{NP} \)
\[
d^L \leq d \leq d^U, \ d \in \mathbb{R}^{n_{\text{adv}}} \text{ and } X \in \mathbb{R}^{n_{\text{rev}}}
\] (1)

Where \( d = \mu(X) \) is the design vector, \( X \) is the random vector, and the probabilistic constraints are described by the performance function \( G_i(X) \) with \( G_i(X) < 0 \) indicates failure, their probabilistic models, and their prescribed confidence level \( \beta_i \).

Performance function failure is statistically defined by a cumulative distribution function \( F_{G_i}(0) \) as
\[
P(G_i(X) \leq 0) = F_{G_i}(0) = \int_{G_i(X) \leq 0} \cdots \int f_X(x) dx \leq \Phi(-\beta_i)
\] (2)

In Eq. (2), \( F_{G_i}(0) \) is a joint probability density function, which needs to be integrated. The first-order reliability method (FORM) is a widely-used integration method, which combines the approximate probability and reliability measure approach. The approach has proven to be efficient and accurate. By inverse transformation, the probabilistic in Eq. (2) can be further expressed in two distinct forms as:
\[
\beta_S_i = (-\Phi^{-1}(F_{G_i}(0))) \geq \beta_t
\] (3)
\[
G_{p_{i}} = F_{G_i}^{-1}(\Phi(-\beta_i)) \geq 0
\] (4)

Where \( \beta_S_i \) and \( G_{p_{i}} \) represent the safety reliability index and the probabilistic performance measure for the \( i^{th} \) probabilistic constraint, respectively. Using the safety reliability index, Eq. (3) is then employed to describe the probabilistic constraint in Eq. (1), i.e., the so-called reliability index approach (RIA). Similarly, Eq. (4) can replace the probabilistic constraint in Eq. (1) with the performance measure, referred to as the performance measure approach (PMA).

### 1.1 First-Order Reliability Analysis in RIA

In RIA, the first-order safety reliability index \( \beta_S_i \) is obtained by solving the optimization problem formulated in FORM, with an implicit equality constraint in \( U \)-space defined as the limit state function:
\[
\text{minimize } \|U\| \text{ subject to } G(U) = 0
\] (5)

The optimum point on the failure surface is referred to as the most probable failure point (MPFP) \( u_{G(U)=0} \). Any MPFP search algorithm specifically developed for FORM or general optimizer can be used to solve Eq. (5).

### 1.2 First-Order Reliability Analysis in PMA

The first-order reliability analysis in PMA can be formulated as the inverse of the first-order reliability analysis in RIA. The first-order probabilistic performance measure \( G_{p,\text{FORM}} \) is obtained from a nonlinear optimization problem with an \( n \)-dimensional explicit sphere constraint in \( U \)-space, defined as
\[
\text{minimize } G(U), \text{ subject to } \|U\| = \beta_t
\] (6)

The optimum point on a target reliability surface is identified as the most probable point (MPP) \( u_{\beta=\beta_t} \).

An advanced mean value (AMV) method can be used to numerically solve the inverse PMA problem.
However, it is found that AMV method exhibits poor behavior for concave constraint functions, although it is effective for convex constraint functions. To overcome difficulties in the AMV method, a conjugate mean value (CMV) method is proposed for the concave constraint function in PMA [3]. The HMV method [3] combines both the CMV and AMV methods - the CMV method is used for concave constraint functions and the AMV method is used for convex constraint functions. The HMV method has been shown to be very robust and efficient.

1.3 Comparison Between RIA and PMA
As mentioned above, in RBDO model the reliability is represented as probabilistic constraints with RIA. But problems do exist with this approach. There is defect of slow speed of convergence due to various reasons. PMA is used to simplify the constraints, which improves the convergence speed by solving an inverse problem for the FORM. Although the cost function becomes more complicated, the searching space is largely reduced.

Compared to RIA, PMA has the following advantages: (1) PMA is more robust and effective whether the probabilistic constrains are feasible or not. (2) PMA can always find the local optimal solution, nevertheless, the solution infeasibility may arise in RIA due to a variety of reasons such as Gumbel or uniform distributions. (3) PMA is more effective than RIA when RBDO applies response surface method (RSM) to conduct reliability analysis and design optimization. However, when combining with Bayesian approach, RIA is much more convenient and straightforward in formulation.

2. Improved RBDO Approach with Bayesian Method
To accurately infer the probability distributions, we have to get a large number of experimental data from actual design cases. One of the most common ways is to obtain sample data at some interval from historical data or from field measurements. However, collecting more samples is not only time-consuming but also very expensive. In terms of its information content, quantification of uncertainty as a set of finite samples can be considered as a medium level of information. In this situation, much of this valuable information is discarded. There's been a lot of research over the past decade to integrate the incomplete information with uncertainty quantification. Du, et. al., focused on reliability-based optimization under a mixture of both random and interval variables [7]. This method is quite successful in accounting for either end of the spectrum although it is not applicable to problems with finite samples. In some approaches the probability density function of the uncertain quantities is determined based on the sampled data, which makes the RBDO algorithms more applicable [8] [9].

The maximum entropy principle [10] is also used to derive pdf based on the given data. However, the process of deriving the pdf will produce an error in the data and it is difficult to propagate the error in the prediction. Another popular research direction is the use of fuzzy sets to quantify the amount of uncertainty based on the possibility of the method [11] [12]. This method includes likelihood of evidence theory [13]. Fuzzy sets can capture different levels of information by controlling the reduction of membership functions. However, it is difficult to build a membership function because it involves "expert opinions", and each expert may be different and may even be in conflict. [14] Tried to combine the possibility of quantification with probability to obtain the advantages of both representations.

In conclusion, RBDO problems with incomplete information are still not well solved in practical design situations. Our objective in this article is to develop a reliability-based optimization method for problems with incomplete information. More specifically, we utilize Bayesian statistics to solve problems for which a mixture of finite samples and probability density functions of the uncertain quantities is known.

2.1 Bayesian Inference Method
Consider a Bernoulli process, such as a coin toss, whose probabilities of “success” and “failure” are p and (1-p) respectively. Given N independent trials, the probability of having r successes out of these trials follows a Binomial distribution: \( r \sim \text{bin}(N,p) \) In a Binomial distribution, the probability of success p of each trial is known, and we want to predict the outcome of a trial from it. An inference of this
process seeks to calculate \( p \) based on the outcomes of the trials. Given \( r \) successes out of \( N \) trials, the probability distribution of \( p \) can be calculated using Bayes’ theorem shown in Eq. (7)

\[
f(p | r) = \frac{f(p) \times f(r | p)}{\int_0^1 f(p) \times f(r | p) \, dp}
\]

In this equation \( f(p) \) is the prior distribution of \( p \), \( f(p | r) \) is the posterior distribution of \( p \), and \( f(r | p) \) is the likelihood of \( r \) given \( p \). The integral in the denominator is a normalizing factor to make the probability distribution proper.

The posterior distribution is the distribution of interest. It is the estimate of \( p \) based on the outcome of the trials. The prior distribution is our knowledge about \( p \) before the information from the trials. Many types of prior distribution have been proposed, but it has been shown that the analysis is insensitive to the choice of prior if the number of trials is large enough. In this article we assume a uniform prior for \( p \). This assumption implies that any values of \( p \) between 0 and 1 are equally probable. It is a conservative choice based on the maximum ignorance principle. The only information we have a priori is that \( p \) must be within [0, 1], and a uniform distribution can be fully defined based solely on this information. The likelihood function is the probability that there will be \( r \) successes out of \( N \) trials if the probability of success is \( p \), which is the binomial distribution of \( r \). Using a uniform prior and a Binomial likelihood function in Eq. (7) results in a Beta posterior distribution, Eq. (8), where \( \alpha = r+1 \) and \( \beta = (N-r) + 1 \)

\[
f(p | r) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}
\]

In other words, \( p \) is distributed according to a Beta distribution whose two parameters depend on the outcome of the trials; \( p \sim \text{Beta}(r+1,N-r+1) \).

One very important feature of Bayes’ theorem is that it facilitates an updating scheme to account for additional information. Suppose that after the \( N_1 \) initial trials, we conduct \( N_2 \) additional trials and observe \( r_2 \) more successes. In Bayes’ theorem, the posterior distribution from the \( N_1 \) trials can be used as the prior distribution for the \( N_2 \) trials, thus creating a chain of analysis based on additional information. Using \( \text{Beta}(\alpha_1, \beta_1) \) for the prior distribution in Eq.(3), the posterior distribution of \( p \) after the additional \( N_2 \) trials is as shown in Eq.(6) Here \( \alpha_2 = \alpha_1 + r_2 \) and \( \beta_2 = \beta_1 + (N_2 - r_2) \).

It can be immediately seen that the new posterior is also a Beta distribution with the two parameters equal to the prior parameters updated by the new information. This is a conjugacy feature of a Beta distribution when used in Bayes’ theorem. This conjugacy is not limited to just two successive trials. If there is a sequence of \((N_1, N_2, \ldots, N_k)\) trials, then the posterior of the \((k-1)\) th trial becomes the prior of the \( k \) th trial, except for \( k=1 \) whose prior is Beta \((1,1)\) (mathematically, a uniform distribution is equivalent to a Beta\((1,1)\) distribution). In general, the posterior distribution of \( p \) at the \( k \) th trial is Beta \((\alpha_k, \beta_k)\) where \( \alpha_k = \alpha_{k-1} + r_k \) and \( \beta_k = \beta_{k-1} + (N_k - r_k), k \geq 1 \).

### 2.2 Bayesian Reliability Estimate

In an optimization problem, an inequality constraint \( g_j \) divides the design space into two domains: feasible and infeasible. Suppose for now that the pdf’s of all \( X \) and \( P \) are unknown: \([X, P] = \varnothing\), and that there are only \( N \) samples available. Each of these samples is a realization of \((X, P)\) in the design space, so each sample corresponds to either a feasible or an infeasible realization. Let \( r \) and \((N-r)\) be the number of feasible and infeasible realizations, respectively. The goal is to estimate \( F_{g_j}(0) \) given these
r and (N-r) realizations. This \( F_{g_j(0)} \) estimation problem is analogous to the Bayesian inference problem discussed previously. If we use a coin toss as an analogy, then \( g_j \) is the “coin” with only two mutually exclusive outcomes, the N samples are the “coin tosses” or trials, and \( F_{g_j(0)} \) is the “probability of head,” where we have designated a feasible realization as a “head.” The problem of inferring \( F_{g_j(0)} \) given r and (N-r) realizations is then equivalent to the problem of inferring p, the probability of heads, given that there are r heads out of N coin tosses. From the above discussion we obtain
\[
F_{g_j(0)} \sim \text{beta}(\alpha, \beta),
\]
where \( \alpha = r+1 \) and \( \beta = (N-r)+1 \).

A Beta distribution is strictly bounded within [0,1]. This makes the estimate proper because \( F_{g_j(0)} \) is a cdf. The shape of a Beta distribution depends on the number of feasible-infeasible realizations. If there are many feasible realizations, it will be skewed to the right towards one; if there are many infeasible realizations, it will be skewed to the left towards zero (Fig.2). This is also a proper behavior for a \( F_{g_j(0)} \) estimate. If a design has many feasible realizations, we expect its reliability to be high, and vice versa.

This estimate of \( F_{g_j(0)} \) assumes that the pdfs of all X and P are unknown. When some of the pdfs are known \( ([X_t, P_j] \neq \emptyset) \), the coin toss analogy still holds provided that this pdf information is accounted for. The difference between the two cases lies in the calculation of the feasible-infeasible realization of a design. In the case where all pdfs are unknown, one \((X_s, P_s)\) sample gives a distinct feasible or infeasible realization, either to the left or right of \( gj = 0 \) in the \( gj \) axis. In contrast, when some of the pdfs are known, one sample results in a distribution of \( gj \) values (Fig.3).

Since each \((X_s, P_s)\) sample now has a distribution of \( gj \) values, it is not possible to simply count the number of feasible or infeasible realizations. What we can calculate instead is the probability that \( gj \) is feasible given the kth sample of \((X_s, P_s)\):
\[
\Pr[g_j(X_t, P_t) \leq 0 \mid (X_s, P_s)_k]
\]
This probability is the expected feasible realization of one sample. The sum of the probabilities of all samples is then the expected total number of feasible realizations of the design
\[
E(r) = \sum_{k=1}^{N} \Pr[g_j(X_t, P_t) \leq 0 \mid (X_s, P_s)_k]
\]
(9)
Using $E(r)$ as the expected number of heads, we can use the coin toss analogy again to estimate $F_{g_j}(0)$. This new estimate is given by a Beta distribution with $\alpha = E(r) + 1$ and $\beta = (N - E(r)) + 1$. Notice that Eq. (9) is valid for both when only $(X_s, P_s)$ samples are available and when there is a mix of $(X_s, P_s)$ samples and $(X_t, P_t)$ pdf’s. In the case where there is no known pdf, each probability in Eq. (9) becomes an indicator function where $I_k = 1$ if $g_j((X_s, P_s)_k) \leq 0$ and $I_k = 0$ otherwise.

Under incomplete information, $F_{g_j}(0)$ can be estimated with a Beta distribution. The precision of this estimate (graphically depicted as the spread of the distribution) depends on the number of samples. As N increases, the estimate becomes more precise. In the extreme that $N \rightarrow \infty$, the distribution converges to a Dirac delta function meaning that $F_{g_j}(0)$ is known exactly, which is the same as knowing the pdf’s of all $X$ and $P$. This link between $N$ and precision of the estimate is captured by the Bayesian update. Following previous discussion, adding N2 samples to an existing N1 samples revises the $F_{g_j}(0)$ estimate from a Beta(1,1) to a Beta(2,2), where $\alpha_2 = \alpha_1 + E(r_2)$ and $\beta_2 = \beta_1 + (N - E(r_2))$.

2.3 Bayesian Approach Integrated with Reliability-Based Optimization

Due to the incompleteness of the random variable information, we can only obtain the distribution of the design reliability rather than the exact value. Therefore, in design optimization, we cannot determine whether the reliability of the design is less than or greater than the target value. There is not enough information to make such an exact statement. Then, the optimization requires additional measures to determine whether the design can be considered to meet the reliability requirements. We define a quantity we call confidence to be such a measure. For a design, the confidence of that design with respect to the jth inequality constraint is defined to be the probability that it will meet or exceed the reliability target.

\[
\zeta_j(\mu_X) = \Pr(F_{g_j}(0) | \mu_X \geq R_j) \quad j=1, 2, ..., J
\]

$\zeta_j = 0$ means that the design is certainly not reliable, while $\zeta_j = 1$ means that the design certainly meets or exceeds the target. Since the $F_{g_j}(0)$ estimate is a Beta distribution, the confidence can also be written as

\[
\zeta_j(\mu_X) = 1 - \Phi_{B_j}(R_j)
\]

where $\Phi_{B_j}()$ is the cdf of the jth Beta distribution, for all $j=1, ..., J$.

When we have complete information regarding the random variables/parameters, optimization seeks a design with the best objective value and with reliability greater than or equal to $R_j$. There is a trade-off between minimizing $f$ and satisfying the reliability constraints. To account for this trade-off, the problem must be reformulated into a multi-objective optimization problem where the objectives are to minimize $f$ and to maximize the $\zeta_j$’s. There is a J number of $\zeta_j$’s; however, in reliability-based optimization a design is considered feasible if all probabilistic constraints are satisfied. Following this convention, the j’s can be lumped into:

\[
\zeta_s(\mu_X) = \min_{j=1,...,J} \zeta_j(\mu_X)
\]

, a quantity called the overall confidence of a design. Using this measure the multi-objective problem then becomes as shown in Eq. (11).
\[
\begin{align*}
\min \quad & f(\mu_X, \mu_P) \\
\max \quad & \zeta_s(\mu_X) \\
\text{subject to:} \quad & 0 \leq \zeta_s(\mu_X) \leq 1
\end{align*}
\] (11)

Solving equation (8) usually produces a set of Pareto optimal solution sets, rather than the optimal design. This is the result of incomplete information based on X and P: it is not possible to determine exactly which design is optimal and reliable. On the contrary, we can only get a set of designs, each design has a certain degree of probability to become a true optimal and reliable design (corresponding to the complete information).

The multi-objective formula in the formula (11) allows the Pareto solution to be increased or reduced according to the target f to the true best reliable design. This is because the past is not feasible for the design of probability constraints. If the distance probability boundary is not too far, there may be a non-zero. If the problem is finite and local monotonic, the relaxation of the active probability constraint improves the optimal f-value. As a result, the design farther away from the active boundary to the infeasible domain, the better. This trade-off design is part of the forefront of Pareto, whose value is beyond the true best and reliable design, which in turn is established. The farther away from active borders to viable areas, the better the design is, and the f becomes farther. Some of the Pareto boundaries formed by these designs impact the true and reliable design of the f-value.

If the information of the random quantities is only available as samples, i.e., [Xt,Pt]=∅, the Pareto frontier of Eq.(11) will be necessarily discrete. This property can be explained as follows. The Beta distribution of the \( F_{g,j}(0) \) estimate is created from updating a uniform prior distribution with the number of feasible-infeasible realizations. For N samples, the possible r values are discrete from a minimum of 0 to a maximum of N. So there are only a total of (N+1) different Beta distributions possible, and this translates to only (N+1) discrete possible values of s. Since s is one of the two objectives in Eq. (11), the Pareto frontier can only contain these particular values, and hence is discrete. If the pdf’s of some of X and P are known, the discreteness of s disappears. However, the Pareto frontier may be discrete or continuous depending on the continuity of f.

3. Multi-objective Algorithm for RBDO

In single-objective optimization, the best possible design or decision is usually the global minimum or global maximum, depending on whether the optimization problem is minimized or maximized. PSO and GA algorithms have been widely used in single target RBDO problems. In recent years, some scholars have developed a hybrid GA-PSO method for solving the RBDO redundancy assignment problem. On the other hand, there may not be a solution to the best (global minimum or maximum) of all targets for multiple targets. In multi-objective optimization, there is a set of solutions that, when considering all targets, the remaining solutions in the search space outperform other solutions in the space in one or more goals (not all). These solutions are called Pareto optimal or non-dominant solutions (Srinivas and Deb (1994)), and the rest of the solution is called the dominant solution. Because any solution in a non-dominant collection cannot be considered to be superior to the other, any of them is an acceptable solution. Since the reliability of each component is an interval value, the reliability of the system will be differentiated. MOMS-GA-based methods have been proposed to solve multi-objective and multi-state reliability optimization with interval targets [15]. In addition, the multi-objective optimization problem based on reliability has been transformed into a single target unconstrained optimization problem by using penalty function techniques [16].

4. Multi-state Algorithm for RBDO

Reliability is defined as the probability that the device or system can satisfactorily perform its intended function within a specified time under specified conditions. However, the traditional reliability method
assumes that the system and its components can only be in a fully operational state or a complete failure state, that is, the intermediate state is not allowed. This assumption helps to develop a powerful and broad theory to analyze the performance of the system. However, in some cases, the traditional theory of reliability cannot represent the real behavior of the system. This situation may be a major drawback when the system has a series of intermediate states that the traditional reliability estimates do not take into account. In order to describe the satisfactory performance of the equipment or system, we may need to use multi-level satisfaction, for example, excellent, average and poor. Some studies have proposed multi-state reliability as a supplemental theory to deal with traditional reliability theory and model problems of systematic analysis. Then, in a multi-state system, the system and its components are allowed to experience two or more possible states, such as full work, partial or partial failure, and complete failure [15].

5. Conclusion
In this paper, the reliability-based design optimization model is introduced in detail. With the integration of reliability theory and optimization methods, RBDO can deal with engineering problems under uncertainties. As the most commonly used approach, RIA represents the reliability as probabilistic constraints, while PMA simplifies the constraints. PMA is more effective and robustness than RIA. RBDO Approach can also be improved by combining Bayesian Method when information is incomplete. This kind of problem can be converted into multi-objective and multi-state reliability optimization with interval targets, which meets the practical requirements more.

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