A NEW WEAK GRADIENT FOR THE STABILIZER FREE WEAK GALERKIN METHOD WITH POLYNOMIAL REDUCTION

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Abstract. The weak Galerkin (WG) finite element method is an effective and flexible general numerical technique for solving partial differential equations. It is a natural extension of the classic conforming finite element method for discontinuous approximations, which maintains simple finite element formulation. Stabilizer free weak Galerkin methods further simplify the WG methods and reduce computational complexity. This paper explores the possibility of optimal combination of polynomial spaces that minimize the number of unknowns in the stabilizer free WG schemes without compromising the accuracy of the numerical approximation. A new stabilizer free weak Galerkin finite element method is proposed and analyzed with polynomial degree reduction. To achieve such a goal, a new definition of weak gradient is introduced. Error estimates of optimal order are established for the corresponding WG approximations in both a discrete $H^1$ norm and the standard $L^2$ norm. The numerical examples are tested on various meshes and confirm the theory.

Key words. weak Galerkin, finite element methods, weak gradient, second-order elliptic problems, stabilizer free.

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1. Introduction. The weak Galerkin finite element method is an effective and flexible numerical technique for solving partial differential equations. The main idea of weak Galerkin finite element methods is the use of weak functions and their corresponding discrete weak derivatives in algorithm design. The WG method was first introduced in [13, 14] and then has been applied to solve various partial differential equations [2, 4, 3, 5, 7, 8, 9, 10, 6, 11, 12, 15].

A stabilizing/penalty term is often essential in finite element methods with discontinuous approximations to enforce connection of discontinuous functions across element boundaries. Removing stabilizers from discontinuous finite element methods simplifies finite element formulations and reduces programming complexity. Stabilizer free WG finite element methods have been studied in [16, 1, 20]. The idea is increasing the connectivity of a weak function across element boundary by raising the degree of polynomials for computing weak derivatives. In [16], we have proved that a stabilizer can be removed from the WG finite element formulation for the WG element $(P_k(T), P_k(e), [P_j(T)]^d)$ if $j \geq k + n - 1$, where $n$ is the number of edges/faces of an element. The condition $j \geq k + n - 1$ has been relaxed in [1]. Stabilizer free DG methods have also been developed in [17, 18].

For simplicity, we demonstrate the idea by using the second order elliptic problem that seeks an unknown function $u$ satisfying

\begin{align}
-\nabla \cdot (a \nabla u) &= f \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega,
\end{align}

where $\Omega$ is a polytopal domain in $\mathbb{R}^d$, $\nabla u$ denotes the gradient of the function $u$, and $a$ is a symmetric $d \times d$ matrix-valued function in $\Omega$. We shall assume that there exist

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two positive numbers $\lambda_1, \lambda_2 > 0$ such that

\begin{equation}
\lambda_1 \xi^T \xi \leq \xi^T a \xi \leq \lambda_2 \xi^T \xi, \quad \forall \xi \in \mathbb{R}^d.
\end{equation}

Here $\xi$ is understood as a column vector and $\xi^T$ is the transpose of $\xi$.

The goal of this paper is to propose and analyze a stabilizer free weak Galerkin method for (1.1)-(1.2) by using less number of unknowns than that of [16] without compromising the order of convergence. The WG scheme will use the configuration of $(P_k(T), P_{k-1}(e), P_j(T)^d)$. For the WG element $(P_k(T), P_{k-1}(e), P_j(T)^d)$, the stabilizer free WG method with the standard definition of weak gradient only produces suboptimal convergence rates in both energy norm and the $L^2$ norm, shown in Table 1.1 [19].

### Table 1.1

| $P_k(T)$ | $P_{k-1}(e)$ | $[P_{k+1}(T)]^d$ | $r_1$ | $r_2$ |
|----------|----------|-----------------|------|------|
| $P_0(T)$ | $P_0(e)$ | $[P_2(T)]^d$ | 0 | 0 |
| $P_2(T)$ | $P_1(e)$ | $[P_3(T)]^d$ | 2 | 2 |
| $P_3(T)$ | $P_2(e)$ | $[P_4(T)]^d$ | 3 | 3 |

The standard definition for a weak gradient $\nabla_w v$ of a weak function $v = \{v_0, v_b\}$ is a piecewise vector valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w v \in [P_{k+1}(T)]^2$ satisfies (13) [14] [8]

\begin{equation}
(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla \cdot \mathbf{q})_T + (v_b, \mathbf{q} \cdot \mathbf{n})_{\partial T}, \quad \forall \mathbf{q} \in [P_2(T)]^2.
\end{equation}

A new way of defining weak gradient is introduced in (2.3) for the WG element $(P_k(T), P_{k-1}(e), [P_j(T)]^d)$ such that the corresponding stabilizer free WG approximation converges to the true solution with optimal order convergence rates, shown in Table 1.2.

### Table 1.2

| $P_k(T)$ | $P_{k-1}(e)$ | $[P_{k+1}(T)]^d$ | $r_1$ | $r_2$ |
|----------|----------|-----------------|------|------|
| $P_0(T)$ | $P_0(e)$ | $[P_2(T)]^d$ | 1 | 2 |
| $P_2(T)$ | $P_1(e)$ | $[P_3(T)]^d$ | 3 | 3 |
| $P_3(T)$ | $P_2(e)$ | $[P_4(T)]^d$ | 4 | 4 |

We also prove the optimal convergence rates theoretically for the stabilizer free WG approximation in an energy norm and in the $L^2$ norm. The numerical examples are tested on different finite element partitions.

### 2. Weak Galerkin Finite Element Schemes

Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [14]. Denote by $\mathcal{E}_h$ the set of all edges/faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$ be the set of all interior edges/faces. For simplicity, we will use term edge for edge/face without confusion.

For a given integer $k \geq 1$, let $V_h$ be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows

\begin{equation}
V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_{k-1}(e), e \subset \partial T, T \in \mathcal{T}_h\}
\end{equation}
and its subspace $V_h^0$ is defined as
\begin{equation}
V_h^0 = \{ v : v \in V_h, \, v_b = 0 \text{ on } \partial \Omega \}.
\end{equation}

We would like to emphasize that any function $v \in V_h$ has a single value $v_b$ on each edge $e \in \mathcal{E}_h$.

For $v = \{ v_0, v_b \} \in V_h \cup H^1(\Omega)$, a weak gradient $\nabla_w v$ is a piecewise vector valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w v|_T \in [P_j(T)]^d$ satisfies
\begin{equation}
(\nabla_w v, q)_T = (\nabla v_0, q)_T + \langle Q_b(v_b - v_0), q \cdot n \rangle_{\partial T} \quad \forall q \in [P_j(T)]^d,
\end{equation}
where $j > k$ depends on the shape of the elements and will be determined later. In the above equation, we let $v_0 = v$ and $v_b = v$ if $v \in H^1(\Omega)$.

For simplicity, we adopt the following notations,
\begin{align*}
(v, w)_{T_h} & = \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T vw dx, \\
(v, w)_{\partial T_h} & = \sum_{T \in \mathcal{T}_h} \langle v, w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} vw ds.
\end{align*}

Let $Q_0$ and $Q_b$ be the two element-wise defined $L^2$ projections onto $P_k(T)$ and $P_{k-1}(e)$ on each $T \in \mathcal{T}_h$, respectively. Define $Q_h u = \{ Q_0 u, Q_b u \} \in V_h$. Let $Q_h$ be the element-wise defined $L^2$ projection onto $[P_j(T)]^d$ on each element $T \in \mathcal{T}_h$.

**Weak Galerkin Algorithm 1.** A numerical approximation for (1.1)-(1.2) can be obtained by seeking $u_h = \{ u_0, u_b \} \in V_h$ satisfying $u_b = Q_b g$ on $\partial \Omega$ and the following equation:
\begin{equation}
(a \nabla_w u_h, \nabla_w v)_{T_h} = (f, v_0) \quad \forall v = \{ v_0, v_b \} \in V_h^0.
\end{equation}

The following lemma reveals a nice property of the weak gradient.

**Lemma 2.1.** Let $\phi \in H^1(\Omega)$, then on any $T \in \mathcal{T}_h$, we have
\begin{equation}
\nabla_w \phi = Q_h \nabla \phi.
\end{equation}

**Proof.** The definition of weak gradient (2.3) implies that for any $q \in [P_j(T)]^d$
\begin{align*}
(\nabla_w \phi, q)_T & = (\nabla \phi, q)_T + \langle Q_b(\phi - \phi), q \cdot n \rangle_{\partial T} \\
& = (Q_h \nabla \phi, q)_T,
\end{align*}
which implies (2.5). \hfill \Box

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [14] for details):
\begin{equation}
\| \varphi \|_{\partial T}^2 \leq C \left( h_T^{-1} \| \nabla \varphi \|_{T}^2 + h_T \| \nabla \varphi \|_{T}^2 \right).
\end{equation}

**3. Well Posedness.** For any $v \in V_h \cup H^1(\Omega)$, define two semi-norms
\begin{align*}
\| v \|_1^2 & = (\nabla_w v, \nabla_w v)_{T_h}, \\
\| v \|_2^2 & = (a \nabla_w v, \nabla_w v)_{T_h}.
\end{align*}
It follows from \((1.3)\) that there exist two positive constants \(\alpha\) and \(\beta\) such that
\[
(3.3) \quad \alpha \|v\| \leq \|v\|_1 \leq \beta \|v\|.
\]

We introduce a discrete \(H^1\) semi-norm as follows:
\[
(3.4) \quad \|v\|_{1,h} = \left( \sum_{T \in T_h} \left( \|\nabla v_0\|_T^2 + h_T^{-1}\|Q_b(v_0 - v_b)\|_{\partial T}^2 \right) \right)^{\frac{1}{2}}.
\]

It is easy to see that \(\|v\|_{1,h}\) defines a norm in \(V_h^0\).

Next we will show that \(\|\cdot\|\) also defines a norm for \(V_h^0\) by proving the equivalence of \(\|\cdot\|\) and \(\|\cdot\|_{1,h}\) in \(V_h\). First we need the following lemma.

**Lemma 3.1.** \((\text{[18]})\) Let \(T\) be a convex, shape-regular \(n\)-polygon/polyhedron of size \(h_T\). Let \(e \in \partial T\) be an edge/face-polygon of \(T\), of size \(Ch_T\). Let \(v_h \in V_h\) and \(v_h = \{v_0, v_b\}\) on \(T\). Then there exists a polynomial \(q \in [P_j(T)]^d\), \(j = n + k - 1\), such that
\[
(3.5) \quad -(\nabla v_0, q)_T = 0,
\]
\[
(3.6) \quad \langle Q_b(v_0 - v_b), q \cdot n \rangle_e = \|Q_b(v_0 - v_b)\|_e^2 \quad \forall e \subset \partial T,
\]
\[
(3.7) \quad \|q\|_T^2 \leq Ch_T \|Q_b(v_0 - v_b)\|_e^2.
\]

**Lemma 3.2.** There exist two positive constants \(C_1\) and \(C_2\) such that for any \(v = \{v_0, v_b\} \in V_h\), we have
\[
(3.8) \quad C_1 \|v\|_1 \leq \|v\| \leq C_2 \|v\|_{1,h}.
\]

**Proof.** For any \(v = \{v_0, v_b\} \in V_h\), it follows from the definition of weak gradient \((2.3)\) and integration by parts that on each \(T \in T_h\)
\[
(3.9) \quad (\nabla_w v, q)_T = (\nabla v_0, q)_T + \langle Q_b(v_b - Q v_0), q \cdot n \rangle_{\partial T} \quad \forall q \in [P_j(T)]^d.
\]

By letting \(q = \nabla v_0|_T\) in \((3.9)\) we arrive at
\[
(\nabla_w v, \nabla v_0)_T = (\nabla v_0, \nabla v_0)_T + \langle Q_b(v_b - Q v_0), \nabla v_0 \cdot n \rangle_{\partial T}.
\]

Letting \(q = \nabla v_0|_T\) in \((3.9)\) implies
\[
(3.10) \quad (\nabla_w v, \nabla v_0)_T = (\nabla v_0, \nabla v_0)_T + \langle Q_b(v_b - v_0), \nabla v_0 \cdot n \rangle_{\partial T}.
\]

From the trace inequality \((2.6)\) and the inverse inequality we have
\[
\|\nabla v_0\|_T^2 \leq \|\nabla v_0\|_T \|\nabla v_0\|_T + \|Q_b(v_b - v_0)\|_{\partial T} \|\nabla v_0\|_{\partial T}
\leq \|\nabla v_0\|_T \|\nabla v_0\|_T + C h_T^{-1/2} \|Q_b(v_b - v_0)\|_{\partial T} \|\nabla v_0\|_T,
\]
which implies
\[
\|\nabla v_0\|_T \leq C \left( \|\nabla v_0\|_T + h_T^{-1/2} \|Q_b(v_b - v_0)\|_{\partial T} \right),
\]

\[
\|\nabla v_0\|_T \leq C \left( \|\nabla v_0\|_T + h_T^{-1/2} \|Q_b(v_b - v_0)\|_{\partial T} \right),
\]

\[
\|v_0\|_{1,h} = \left( \sum_{T \in T_h} \left( \|\nabla v_0\|_T^2 + h_T^{-1}\|Q_b(v_0 - v_b)\|_{\partial T}^2 \right) \right)^{\frac{1}{2}}.
\]
and consequently
\[ \|v\| \leq C_2 \|v\|_{1,h}. \]

Next we will prove \( C_1 \|v\|_{1,h} \leq \|v\| \). First we need to prove
\[ (3.11) \quad h^{-1/2}_e |||v|||_e \leq C \|\nabla w v\|_T. \]
For \( e \in E_h \) and \( T \in T_h \) with \( e \subset \partial T \), it has been proved in Lemma 3.1 that there exists \( q_0 \in [P_j(T)]^d \) such that
\[ (3.12) \quad (\nabla v_0, q_0)_T = 0, \quad \langle Q_b(v_b - v_0), q_0 \cdot n \rangle_{\partial T} = \|Q_b(v_b - v_0)\|^2_{\partial T}, \]
and
\[ (3.13) \quad \|q_0\|_T \leq C h^{-1/2}_T \|Q_b(v_b - v_0)\|_{\partial T}. \]
Substituting \( q_0 \) into (3.9), we get
\[ (3.14) \quad (\nabla w v, q_0)_T = \|Q_b(v_b - v_0)\|^2_{\partial T}. \]
It follows from Cauchy-Schwarz inequality and (3.13) that
\[ \|Q_b(v_b - v_0)\|^2_{\partial T} \leq C \|\nabla w v\|_T \|q_0\|_T \leq C h^{-1/2}_T \|\nabla w v\|_T \|Q_b(v_b - v_0)\|_{\partial T}, \]
which implies
\[ (3.15) \quad h^{-1/2}_T \|Q_b(v_b - v_0)\|_{\partial T} \leq C \|\nabla w v\|_T. \]
It follows from (3.10), the trace inequality, the inverse inequality and (3.15),
\[ \|\nabla v_0\|^2_T \leq \|\nabla w v\|_T \|\nabla v_0\|_T + C h^{-1/2}_T \|Q_b(v_b - v_0)\|_{\partial T} \|\nabla v_0\|_T \leq C \|\nabla w v\|_T \|\nabla v_0\|_T, \]
which implies
\[ \|\nabla v_0\|_T \leq C \|\nabla w v\|_T. \]
Combining the above estimate and (3.15), we prove the lower bound of (3.8) and complete the proof of the lemma. \( \Box \)

**Lemma 3.3.** The weak Galerkin finite element scheme (2.4) has a unique solution.

**Proof.** If \( u_h^{(1)} \) and \( u_h^{(2)} \) are two solutions of (2.4), then \( \varepsilon_h = u_h^{(1)} - u_h^{(2)} \in V^0_h \) would satisfy the following equation
\[ (a \nabla w \varepsilon_h, \nabla w v)_T = 0, \quad \forall v \in V^0_h. \]
Then by letting \( v = \varepsilon_h \) in the above equation and (3.3), we arrive at
\[ \|\varepsilon_h\|^2 = (a \nabla w \varepsilon_h, \nabla w \varepsilon_h) = 0. \]
It follows from (3.8) that \( \|\varepsilon_h\|_{1,h} = 0 \). Since \( \|\cdot\|_{1,h} \) is a norm in \( V^0_h \), one has \( \varepsilon_h = 0 \). This completes the proof of the lemma. \( \Box \)
4. Error Analysis. The goal of this section is to establish error estimates for the weak Galerkin finite element solution $u_h$ arising from (2.1). For simplicity of analysis, we assume that the coefficient tensor $a$ in (1.1) is a piecewise constant matrix with respect to the finite element partition $T_h$. The result can be extended to variable tensors without any difficulty, provided that the tensor $a$ is piecewise sufficiently smooth.

4.1. Error Equation. Let $e_h = u-u_h$ and $\epsilon_h = Q_h u-u_h \in V_h$. In this section, we derive an error equation that $e_h$ satisfies.

**Lemma 4.1.** For any $v \in V^0_h$, the following error equation holds true

\begin{equation}
(a \nabla_e e_h, \nabla_w v)_{T_h} = e_1(u, v) + e_2(u, v),
\end{equation}

where

\begin{align*}
e_1(u, v) &= (a(\nabla u - Q_h \nabla u) \cdot n, Q_b v_0 - v_b)_{\partial T_h},
\end{align*}

\begin{align*}
e_2(u, v) &= (a \nabla u \cdot n, v_0 - Q_b v_0)_{\partial T_h}.
\end{align*}

**Proof.** For $v = \{v_0, v_b\} \in V^0_h$, testing (1.1) by $v_0$ and using the fact that $(a \nabla u \cdot n, v_0)_{\partial T_h} = 0$, we arrive at

\begin{equation}
(a \nabla u, \nabla v_0)_{T_h} - (a \nabla u \cdot n, v_0 - v_b)_{\partial T_h} = (f, v_0).
\end{equation}

Obviously, we have

\begin{equation}
(a \nabla u \cdot n, v_0 - v_b)_{\partial T_h} = (a \nabla u \cdot n, Q_b v_0 - v_b)_{\partial T_h} + (a \nabla u \cdot n, v_0 - Q_b v_0)_{\partial T_h}.
\end{equation}

Combining (4.2) and (4.3) gives

\begin{equation}
(a \nabla u, \nabla v_0)_{T_h} - (a \nabla u \cdot n, Q_b v_0 - v_b)_{\partial T_h} = (f, v_0) + e_2(u, v).
\end{equation}

It follows from (2.3) and (2.5) that

\begin{align*}
(a \nabla u, \nabla v_0)_{T_h} &= (a Q_h \nabla u, \nabla v_0)_{T_h} \\
&= (a Q_h \nabla u, \nabla v_0)_{T_h} + (Q_b v_0 - v_b, a Q_h \nabla u \cdot n)_{\partial T_h} \\
&= (a \nabla u, \nabla v_0)_{T_h} + (Q_b v_0 - v_b, a Q_h \nabla u \cdot n)_{\partial T_h}.
\end{align*}

Using (4.4) and (4.5), we have

\begin{equation}
(a \nabla u, \nabla v_0)_{T_h} = (f, v_0) + e_1(u, v) + e_2(u, v).
\end{equation}

Subtracting (2.4) from (4.6) yields,

\begin{equation}
(a \nabla_e e_h, \nabla_w v)_{T_h} = e_1(u, v) + e_2(u, v) \quad \forall v \in V^0_h.
\end{equation}

This completes the proof of the lemma. \qed

4.2. Error Estimates in Energy Norm. Optimal convergence rate of the WG approximation in energy norm will be obtained in this section. First we will bound the two terms $e_1(u, v)$ and $e_2(u, v)$ in the error equation (4.1).

**Lemma 4.2.** For any $w \in H^{k+1}(\Omega)$ and $v = \{v_0, v_b\} \in V^0_h$, we have

\begin{align*}
|e_1(w, v)| &\leq C h^k |w|_{k+1} \|v\|, \\
|e_2(w, v)| &\leq C h^k |w|_{k+1} \|v\|.
\end{align*}
Proof. Using the Cauchy-Schwarz inequality, the trace inequality \(2.6\), \(1.3\) and \(3.8\), we have

\[
|e_1(w, v)| = \left| \sum_{T \in T_h} \langle a(\nabla w - Q_h \nabla w) \cdot n, Q_b v_0 - v_b \rangle_{\partial T} \right|
\]

\[
\leq C \sum_{T \in T_h} \| \nabla w - Q_h \nabla w \|_{\partial T} \| Q_b v_0 - v_b \|_{\partial T}
\]

\[
\leq C \left( \sum_{T \in T_h} h_T \| (\nabla w - Q_h \nabla w) \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \| Q_b v_0 - v_b \|_{\partial T}^2 \right)^{\frac{1}{2}}
\]

\[
\leq Ch^k |w|_{k+1} |||v|||.
\]

Let \(Q_{k-1}\) be the element-wise defined \(L^2\) projection onto \([P_{k-1}(T)]^d\) on each \(T \in T_h\). Using the Cauchy-Schwarz inequality, the trace inequality \(2.6\), \(1.3\) and the inverse inequality, we have

\[
|e_2(w, v)| = \left| \sum_{T \in T_h} \langle a \nabla w \cdot n, v_0 - Q_b v_0 \rangle_{\partial T} \right|
\]

\[
= \left| \sum_{T \in T_h} \langle a(\nabla w - Q_{k-1} \nabla w) \cdot n, v_0 - Q_b v_0 \rangle_{\partial T} \right|
\]

\[
\leq C \sum_{T \in T_h} \| \nabla w - Q_{k-1} \nabla w \|_{\partial T} \| v_0 - Q_b v_0 \|_{\partial T}
\]

\[
\leq C \left( \sum_{T \in T_h} h_T \| (\nabla w - Q_{k-1} \nabla w) \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} h_T^2 \| v_0 - Q_b v_0 \|_{\partial T}^2 \right)^{\frac{1}{2}}
\]

\[
\leq Ch^k |w|_{k+1} |||v|||.
\]

We have proved the lemma. □

**Lemma 4.3.** Let \(u \in H^{k+1}(\Omega)\), then

\begin{equation}
(4.9) \quad \| u - Q_h u \| \leq Ch^k |u|_{k+1}.
\end{equation}

Proof. It follows from \(2.3\) and \(2.6\),

\[
|\langle \nabla_w (u - Q_h u), q \rangle_T| = |\langle \nabla (u - Q_0 u), q \rangle_T + \langle Q_h (Q_0 u - Q_h u), q \cdot n \rangle_{\partial T}| = |\langle \nabla (u - Q_0 u), q \rangle_T + \langle Q_0 (Q_0 u - u), q \cdot n \rangle_{\partial T}| \leq \| \nabla (u - Q_0 u) \|_T \| q \|_T + Ch^{-1/2} \| Q_0 u - u \|_{\partial T} \| q \|_T \leq Ch^k |u|_{k+1,T} |||q|||_T.
\]

Letting \(q = \nabla_w (u - Q_h u)\) in the above equation and taking summation over \(T\), we have

\[
\| u - Q_h u \| \leq Ch^k |u|_{k+1}.
\]
We have proved the lemma. □

**Theorem 4.4.** Let $u_h \in V_h$ be the weak Galerkin finite element solution of \((2.4)\). Assume the exact solution $u \in H^{k+1}(\Omega)$. Then, there exists a constant $C$ such that

\[ \| u - u_h \| \leq Ch^k |u|_{k+1}. \]

**Proof.** It is straightforward to obtain

\[ \| e_h \|^2 = (a \nabla w e_h, \nabla w e_h)_{\mathcal{T}_h} \]
\[ = (a(\nabla w u - \nabla w u_h), \nabla w e_h)_{\mathcal{T}_h} \]
\[ = (a(\nabla w Q_h u - \nabla w u_h), \nabla w e_h)_{\mathcal{T}_h} + (a(\nabla w u - \nabla w Q_h u), \nabla w e_h)_{\mathcal{T}_h} \]
\[ = (a \nabla w e_h, \nabla w e_h)_{\mathcal{T}_h} + (a \nabla w(u - Q_h u), \nabla w e_h)_{\mathcal{T}_h}. \]

We will bound the two terms in (4.11). Letting $v = \epsilon_h \in V_h$ in (4.1) and using (4.7), (4.8) and (4.9), we have

\[ |(a \nabla w e_h, \nabla w e_h)_{\mathcal{T}_h}| = |e_1(u, \epsilon_h) + e_2(u, \epsilon_h)| \leq Ch^k |u|_{k+1} \| \epsilon_h \| \leq Ch^k |u|_{k+1} \| Q_h u - u_h \| \leq Ch^k |u|_{k+1} (\| Q_h u - u \| + \| u - u_h \|) \leq Ch^{2k} |u|_{k+1}^2 + \frac{1}{4} \| \epsilon_h \|^2. \]

The estimate (4.9) implies

\[ |(\nabla w(u - Q_h u), \nabla w e_h)_{\mathcal{T}_h}| \leq C \| u - Q_h u \| \| \epsilon_h \| \leq Ch^{2k} |u|_{k+1}^2 + \frac{1}{4} \| \epsilon_h \|^2. \]

Combining the estimates (4.12) and (4.13) with (4.11), we arrive

\[ \| e_h \| \leq Ch^k |u|_{k+1}, \]

which completes the proof. □

The estimates (4.9) and (4.10) imply

\[ \| Q_h u - u_h \| \leq Ch^k |u|_{k+1}. \]

**4.3. Error Estimates in $L^2$ Norm.** The standard duality argument is used to obtain $L^2$ error estimate. Recall $e_h = \{e_0, \epsilon_h\} = u - u_h$ and $\epsilon_h = \{e_0, \epsilon_h\} = Q_h u - u_h$. The dual problem seeks $\Phi \in H_0^1(\Omega)$ satisfying

\[ -\nabla \cdot a \nabla \Phi = e_0 \quad \text{in } \Omega. \]

Assume that the following $H^2$-regularity holds

\[ \| \Phi \|_2 \leq C \| e_0 \|. \]

**Theorem 4.5.** Let $u_h \in V_h$ be the weak Galerkin finite element solution of (2.4). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and (4.10) holds true. Then, there exists a constant $C$ such that

\[ \| Q_0 u - u_0 \| \leq Ch^{k+1} |u|_{k+1}. \]
Proof. By testing (4.15) with \( \epsilon_0 \) we obtain
\[
\|\epsilon_0\|^2 = -(\nabla \cdot (a \nabla \Phi), \epsilon_0)
\]
\[
= (a \nabla \Phi, \nabla \epsilon_0)_\Omega - (a \nabla \Phi \cdot \epsilon_0 - \epsilon_0)_{\partial \Omega}
\]
\[
= (a \nabla \Phi, \nabla \epsilon_0)_\Omega - (a \nabla \Phi \cdot \epsilon_0, \epsilon_0)_{\partial \Omega} + (Q_b \epsilon_0 - \epsilon_0, \epsilon_0)_{\partial \Omega}
\]
\[
(4.18)
\]
where we have used the fact that \( \epsilon_0 = 0 \) on \( \partial \Omega \). Setting \( u = \Phi \) and \( v = \epsilon_h \) in (4.15) yields
\[
(4.19)
\]
Substituting (4.19) into (4.18) gives
\[
\|\epsilon_0\|^2 = (a \nabla_w \epsilon_h, \nabla_w \Phi)_\Omega - e_1(\Phi, \epsilon_h) - e_2(\Phi, \epsilon_h)
\]
\[
= (a \nabla_w \epsilon_h, \nabla_w \Phi)_\Omega + (Q_b \epsilon_0 - \epsilon_0, \nabla_w \Phi)_{\partial \Omega} - e_1(\Phi, \epsilon_h) - e_2(\Phi, \epsilon_h)
\]
\[
= (a \nabla_w \epsilon_h, \nabla_w Q_b \Phi)_{\partial \Omega} + (Q_b \epsilon_0 - \epsilon_0, \nabla_w (\Phi - Q_b \Phi)_{\partial \Omega}
\]
\[
+ (Q_b \epsilon_0 - \epsilon_0, \nabla_w \Phi)_{\partial \Omega} - e_1(\Phi, \epsilon_h) - e_2(\Phi, \epsilon_h)
\]
\[
= e_1(u, Q_b \Phi) + e_2(u, Q_b \Phi) - e_1(\Phi, \epsilon_h) - e_2(\Phi, \epsilon_h)
\]
\[
(4.20)
\]
Let us bound all the terms on the right hand side of (4.20) one by one. Using the Cauchy-Schwarz inequality and the definition of \( Q_b \), we obtain
\[
|e_1(u, Q_b \Phi)| = \left| \sum_{T \in \bar{T}_h} (a(\nabla u - Q_h \nabla u) \cdot \epsilon_0, Q_b Q_0 \Phi - Q_b \Phi)_{\partial T} \right|
\]
\[
= \sum_{T \in \bar{T}_h} (a(\nabla u - Q_h \nabla u) \cdot \epsilon_0, Q_b (Q_0 \Phi - \Phi))_{\partial T}
\]
\[
\leq C \sum_{T \in \bar{T}_h} \|\nabla u - Q_h \nabla u\|_{\partial T} \|Q_0 \Phi - \Phi\|_{\partial T}
\]
\[
(4.21)
\]
From the trace inequality (2.6) we have
\[
\left( \sum_{T \in \bar{T}_h} \|Q_0 \Phi - \Phi\|_{\partial T}^2 \right)^{1/2} \leq C h^{1/2} \|\Phi\|_2
\]
and
\[
\left( \sum_{T \in \bar{T}_h} \|a(\nabla u - Q_h \nabla u)\|_{\partial T}^2 \right)^{1/2} \leq C h^{k-1/2} \|u\|_{k+1}.
\]
Combining the above two estimates with (4.21) gives
\[
|e_1(u, Q_b \Phi)| \leq C h^{k+1} \|u\|_{k+1} \|\Phi\|_2.
\]
(4.22)
Using the Cauchy-Schwarz inequality, the trace inequality \((2.6)\) and \((1.3)\), we have

\[
|e_2(u, Q_h \Phi)| = \left| \sum_{T \in T_h} \langle a \nabla u \cdot n, Q_0 \Phi - Q_b Q_0 \Phi \rangle_{\partial T} \right| \\
= \left| \sum_{T \in T_h} \langle a(\nabla u - Q_{k-1} \nabla u) \cdot n, Q_0 \Phi - Q_b Q_0 \Phi \rangle_{\partial T} \right| \\
\leq C \sum_{T \in T_h} \| \nabla u - Q_{k-1} \nabla u \|_{\partial T} \| Q_0 \Phi - Q_b Q_0 \Phi \|_{\partial T} \\
\leq Ch^{k-1/2} |u|_{k+1} h^{3/2} \| \Phi \|_2 \\
\leq Ch^{k+1} |u|_{k+1} \| \Phi \|_2.
\]

It follows from \((4.7)\), \((4.8)\) and \((4.14)\) that

\[
|e_1(\Phi, \epsilon_h)| \leq Ch|\Phi|_2 \| \epsilon_h \| \leq Ch^{k+1} |u|_{k+1} \| \Phi \|_2,
\]

and

\[
|e_2(\Phi, \epsilon_h)| \leq Ch|\Phi|_2 \| \epsilon_h \| \leq Ch^{k+1} |u|_{k+1} \| \Phi \|_2.
\]

The estimates \((4.9)\) and \((4.10)\) imply

\[
|(a \nabla \epsilon_h, \nabla_w (\Phi - Q_h \Phi))_{T_h}| \leq C \| \epsilon_h \| \| \Phi - Q_h \Phi \| \leq Ch^{k+1} |u|_{k+1} \| \Phi \|_2.
\]

To bound the term \((a \nabla_w (Q_h u - u), \nabla_w \Phi)_{T_h}\), we define a \(L^2\) projection element-wise onto \([P_0(T)]^d\) denoted by \(Q_0\). Then it follows from the definition of weak gradient \((2.3)\) and integration by parts,

\[
\begin{align*}
\langle \nabla_w (Q_h u - u), aQ_0 \nabla \Phi \rangle_T &= \langle \nabla (Q_0 u - u), aQ_0 \nabla \Phi \rangle_T + \langle (Q_b(Q_h u - u - (Q_0 u - u)), Q_0 \nabla \Phi \cdot n)_{\partial T} \\
&= \langle \nabla (Q_0 u - u), aQ_0 \nabla \Phi \rangle_T + \langle Q_b u - Q_0 u, aQ_0 \nabla \Phi \cdot n \rangle_{\partial T} \\
&= -\langle Q_0 u - u, \nabla \cdot aQ_0 \nabla \Phi \rangle_T + \langle Q_b u - Q_0 u, aQ_0 \nabla \Phi \cdot n \rangle_{\partial T} \\
&= \langle Q_b u - u, aQ_0 \nabla \Phi \cdot n \rangle_{\partial T} = 0.
\end{align*}
\]

Using the equation above, \((2.5)\) and \((4.9)\) and the definition of \(Q_0\), we have

\[
\begin{align*}
|\langle a \nabla_w (Q_h u - u), \nabla_w \Phi \rangle_{T_h}| &= |\langle a \nabla_w (Q_h u - u), Q_h \nabla \Phi \rangle_{T_h}| \\
&= |\langle \nabla_w (Q_h u - u), a \nabla \Phi \rangle_{T_h}| \\
&= |\langle \nabla_w (Q_h u - u), a \nabla \Phi - Q_0 \nabla \Phi \rangle_{T_h}| \\
&\leq Ch^{k+1} |u|_{k+1} \| \Phi \|_2.
\end{align*}
\]

Combining all the estimates above with \((4.20)\) yields

\[
\| \epsilon_0 \|^2 \leq Ch^{k+1} |u|_{k+1} \| \Phi \|_2.
\]

Using the regularity assumption \((4.16)\) and the estimate above, we derived \((4.17)\). \(\square\)

5. Numerical Experiments.
5.1. Example 1. Consider problem (1.1) with $\Omega = (0,1)^2$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The source term $f$ and the boundary value $g$ are chosen so that the exact solution is

$$u(x,y) = \sin(x)\sin(\pi y).$$

(5.1)

We use uniform triangular meshes as shown in Figure 5.1. The error and the order of convergence are listed in Table 5.1, where we have optimal order of convergence for $k \geq 1$ in both $L^2$ norm and $H^1$-like triple-bar norm.

![Figure 5.1. Example 1. The first three triangular grids.](image)

**Table 5.1**

Example 1. The $P_k - P_{k-1} - [P_{k+1}]^2$ element, on triangular grids shown in Figure 5.1

| $k$ | $T_l$ | $\|Q_hu - u_h\|$ | Rate | $\|Q_hu - u_h\|$ | Rate |
|-----|------|------------------|------|------------------|------|
| 6   | 0.3871E-01 | 1.00 | 0.3306E-03 | 1.99 |
| 1   | 7     | 0.1937E-01 | 1.00 | 0.8279E-04 | 2.00 |
| 8   | 9.685E-02 | 1.00 | 0.2070E-04 | 2.00 |
| 2   | 6     | 0.4131E-03 | 1.98 | 0.1783E-05 | 2.95 |
| 7   | 1.038E-03 | 1.99 | 0.2268E-06 | 2.97 |
| 8   | 0.2602E-04 | 2.00 | 0.2859E-07 | 2.99 |
| 3   | 5     | 0.2925E-04 | 2.99 | 0.1515E-06 | 3.98 |
| 6   | 0.3665E-05 | 3.00 | 0.9518E-08 | 3.99 |
| 7   | 0.4587E-06 | 3.00 | 0.5963E-09 | 4.00 |
| 4   | 5     | 0.4091E-06 | 3.99 | 0.1592E-08 | 4.97 |
| 6   | 0.2568E-07 | 3.99 | 0.5026E-10 | 4.99 |
| 7   | 0.1608E-08 | 4.00 | 0.1610E-11 | 4.96 |

5.2. Example 2. Consider problem (1.1) with $\Omega = (0,1)^2$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The source term $f$ and the boundary value $g$ are chosen so that the exact solution is

$$u(x,y) = \sin(\pi x)\cos(\pi y).$$

(5.2)

We use triangular meshes as shown in Figure 5.2 for Example 2. The error and the order of convergence are listed in Table 5.2, where we have optimal order of convergence for $k \geq 1$ in both $L^2$ norm and $H^1$-like triple-bar norm.

5.3. Example 3. Consider problem (1.1) with $\Omega = (0,1)^2$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The source term $f$ and the boundary value $g$ are chosen so that the exact solution is

$$u(x,y) = e^{\pi x}\cos(\pi y).$$

(5.3)
Example 2. The first three triangular grids.

Table 5.2

| $k$ | $T_i$ | $\|Q_h u - u_h\|$ | Rate | $\|Q_h u - u_h\|$ | Rate |
|-----|-------|-----------------|------|-----------------|------|
| 1   | 6     | 0.120E-02       | 2.01 | 0.141E-01       | 2.00 |
|     | 7     | 0.300E-03       | 2.00 | 0.354E-02       | 2.00 |
|     | 8     | 0.749E-04       | 2.00 | 0.885E-03       | 2.00 |
| 2   | 6     | 0.5734E-03      | 2.00 | 0.1482E-05      | 3.00 |
|     | 7     | 0.1434E-03      | 2.00 | 0.1852E-06      | 3.00 |
|     | 8     | 0.3584E-04      | 2.00 | 0.2314E-07      | 3.00 |
| 3   | 5     | 0.3645E-04      | 3.00 | 0.1360E-06      | 3.99 |
|     | 6     | 0.4559E-05      | 3.00 | 0.8517E-08      | 4.00 |
|     | 7     | 0.5699E-06      | 3.00 | 0.5326E-09      | 4.00 |
| 4   | 5     | 0.4222E-06      | 4.00 | 0.9119E-09      | 5.01 |
|     | 6     | 0.2639E-07      | 4.00 | 0.2846E-10      | 5.00 |
|     | 7     | 0.1650E-08      | 4.00 | 0.1110E-11      | 4.68 |

We use rectangular meshes as shown in Figure 5.3 for Example 3. The error and the order of convergence are listed in Table 5.3, where we have one order supconvergence for $k = 1$ in $H^1$-like triple-bar norm, one order supconvergence for $k = 2$ in both $L^2$ norm and $H^1$-like triple-bar norm, and one order supconvergence for $k = 3$ in both $L^2$ norm and $H^1$-like triple-bar norm.

Fig. 5.3. Example 3. The first three rectangular grids.

5.4. Example 4. Consider problem (1.1) with $\Omega = (0, 1)^2$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The source term $f$ and the boundary value $g$ are chosen so that the exact solution is

\[ u(x, y) = e^{2x-1}(y - y^3). \]
Example 3. The $P_k - P_{k-1} - [P_{k+1}]^2$ element, on rectangular grids shown in Figure 5.3.

| $k$ | $T_l$ | $\|Q_h u - u_h\|$ Rate | $\|Q_h u - u_h\|$ Rate |
|-----|-------|-----------------|-----------------|
| 6   | 1     | 0.141E-01 2.00  | 0.120E-02 2.01  |
| 7   | 1     | 0.354E-02 2.00  | 0.300E-03 2.00  |
| 8   | 1     | 0.885E-03 2.00  | 0.749E-04 2.00  |

We use polygonal meshes (mixing dodecagons(12 sided) and heptagons(7 sided)) as shown in Figure 5.4 for Example 4. The error and the order of convergence are listed in Table 5.4, where we have optimal order convergence for all $k \geq 1$ in both $L^2$ norm and $H^1$-like triple-bar norm.

Fig. 5.4. Example 4. The first three polygonal grids.

Table 5.4
Example 4. The $P_k - P_{k-1} - [P_{k+2}]^2$ element, on polygonal grids shown in Figure 5.4.

| $k$ | $T_l$ | $\|Q_h u - u_h\|$ Rate | $\|Q_h u - u_h\|$ Rate |
|-----|-------|-----------------|-----------------|
| 6   | 1     | 0.3735E-01 1.00  | 0.7856E-04 2.00  |
| 7   | 1     | 0.1868E-01 1.00  | 0.1966E-04 2.00  |
| 8   | 1     | 0.9339E-02 1.00  | 0.4916E-05 2.00  |

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