Introduction to molecular dynamics simulations

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We provide an introduction to molecular dynamics simulations in the context of the Kob-Andersen model of a glass. We introduce a complete set of tools for doing and analyzing the results of simulations at fixed NVE and NVT. The modular format of the paper allows readers to select sections that meet their needs. We start with an introduction to molecular dynamics independent of the programming language, followed by introductions to an implementation using Python and then the freely available open source software package LAMMPS. We also describe analysis tools for the quick testing of the program during its development and compute the radial distribution function and the mean square displacement using both Python and LAMMPS.

I. INTRODUCTION

Computer simulations are a powerful approach for addressing questions which are not accessible by theory and experiments. Simulations give us access to analytically unsolvable systems, and contrary to laboratory experiments, there are no unknown “impurities” and we can work with a well defined model. In this paper we focus on the simulation of many particle systems using molecular dynamics, which models a system of classical particles whose dynamics is described by Newton’s equations and its generalizations.

Our goal is to provide the background for those who wish to use and analyze molecular dynamics simulations. This paper may also be used in a computer simulation course or for student projects as part of a course.

Many research groups no longer write their own molecular dynamics programs, but use instead highly optimized and complex software packages such as LAMMPS. To understand the core of these software packages and how to use them wisely, it is educational for students to write and use their own program before continuing with a software package. One intention of this paper is to guide students through an example of a molecular dynamics simulation and then implement the same task with LAMMPS. A few examples are given to illustrate the wide variety of possibilities for analyzing molecular dynamics simulations and hopefully to lure students into investigating the beauty of many particle systems.

Although we discuss the Python programming language, the necessary tools are independent of the programming language and are introduced in Sec. II. Those who prefer to start programming with minimal theoretical background may start with Sec. III and follow the guidance provided on which subsection of Sec. II is important for understanding the corresponding subsection in Sec. III.

Throughout the paper we refer to problems that are listed in Sec. VII. Answers to Problems (1)–(9) are given in the text immediately following their reference. These suggested problems are intended to encourage active engagement with the paper by encouraging readers to work out sections of the paper by themselves.

FIG. 1. Flow chart of a molecular dynamics simulation program.

II. MOLECULAR DYNAMICS SIMULATION

A. Introduction

Molecular dynamics simulates a classical system of \( N \) particles. The core of most simulations is to start with the initial positions and velocities of all particles and to then repeatedly apply a “recipe” to update each particle’s position and velocity from time \( t \) to time \( t + \Delta t \) (see Fig. 1). The dynamics is governed by Newton’s second law

\[
\mathbf{F}_i = m_i \mathbf{a}_i, \tag{1}
\]

where \( \mathbf{a}_i \) is the acceleration of particle \( i \).

In Sec. II B we define the net force \( \mathbf{F}_i \) which is used in this paper and discuss in Sec. II D the update rules for the positions \( \mathbf{r}_i \) and velocities \( \mathbf{v}_i \).
B. Model

The model is specified by the net force \( \boldsymbol{F}_i \) on each particle of mass \( m_i \). The force \( \boldsymbol{F}_i \) can be due to all other particles and/or additional interactions such as effective drag forces or interactions with a wall or an external field. In the following we will consider only conservative forces which are due to all the other particles. We also assume pair-wise interactions given by a potential

\[
V = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} V_{ij}. \tag{2}
\]

Specifically, we use the Lennard-Jones potential

\[
V_{ij} = 4\epsilon \left( \frac{\sigma}{r_{ij}} \right)^{12} - \left( \frac{\sigma}{r_{ij}} \right)^{6}, \tag{3}
\]

where \( r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j| \) is the distance between particle \( i \) at position \( \boldsymbol{r}_i \) and particle \( j \) at position \( \boldsymbol{r}_j \).

The advantage of the Lennard-Jones potential is that it can be used to simulate a large variety of systems and scenarios. For example, each particle may represent an atom, a colloid, or a monomer of a polymer.\(^2\)–\(^5\) Depending on parameters such as the temperature, density, and shear stress the particles may form a gas, liquid, or solid (crystal or glass).\(^6\)–\(^7\) The reason for this wide variety of applications is that the Lennard-Jones potential incorporates two major effective forces: a strong repulsive force for short distances and an attractive force for intermediate distances. The attractive term of the Lennard-Jones potential \( \propto (\sigma/\rho)^6 \) is the van der Waals interaction due to mutual polarization of two particles.\(^8\) The repulsive part \( \propto (\sigma/\rho)^{12} \) is proportional to a power of \( (\sigma/\rho)^6 \) and thus simplifies the computation of the force. Note that the Lennard-Jones potential is short-range. For long-range interactions (gravitational and Coulomb) more advanced techniques are necessary.\(^9\) See Refs. 8–10 for an overview of further applications of the Lennard-Jones potential and other particle interactions and additional contributions to \( \boldsymbol{F} \).

In this paper we illustrate how to simulate a glass forming system. We use the binary Kob-Anderson potential,\(^3\)\(^,\)\(^11\)\(^,\)\(^12\) which has been developed as a model for the Ni\(_{80}\)P\(_{20}\) alloy,\(^11\) and has become one of the major models for studying supercooled liquids, glasses, and crystallization. Examples are discussed in Refs. 3, 11–18 and references cited in Refs. 3, 17, and 18. The Kob-Anderson model is an 80:20 mixture of particles of type \( A \) and \( B \). The Lennard-Jones potential in Eq. (3) is modified by the dependence of \( \epsilon \) and \( \sigma \) on the particle type \( \alpha, \beta \in \{A, B\} \) of particles \( i \) and \( j \).

\[
V_{ij} = V_{\alpha\beta}(r_{ij}) = 4\epsilon_{\alpha\beta} \left[ \left( \frac{\sigma_{\alpha\beta}}{r_{ij}} \right)^{12} - \left( \frac{\sigma_{\alpha\beta}}{r_{ij}} \right)^6 \right]. \tag{4}
\]

We use units such that \( \sigma_{AA} = 1 \) (length unit), \( \epsilon_{AA} = 1 \) (energy unit), \( m_A = 1 \) (mass unit), and \( k_B = 1 \) (the temperature unit is \( \epsilon_{AA}/k_B \)). The resulting time unit is \( \sqrt{m_A\sigma_{AA}^4/\epsilon_{AA}} \). With these units, the Kob-Anderson parameters are \( \sigma_{AA} = 1.0, \epsilon_{AA} = 1.0, \sigma_{AB} = 0.8, \epsilon_{AB} = 1.5, \sigma_{BB} = 0.88, \epsilon_{BB} = 0.5, \) and \( m_A = m_B = 1.0 \). To save computer time, the potential is truncated and shifted at \( r_{ij} = r_{\alpha\beta}^{cut} = 2.5\sigma_{\alpha\beta} \)

\[
V_{ij}^{cut} = \begin{cases} 
V_{\alpha\beta}(r_{ij}) - V_{\alpha\beta}(r_{\alpha\beta}^{cut}) & r_{ij} < r_{\alpha\beta}^{cut} \\
0 & (\text{otherwise})
\end{cases}. \tag{5}
\]

For the truncated and shifted KA-LJ system, \( V \) is given by Eq. (2) by replacing \( V_{ij} \) with \( V_{ij}^{cut} \).

The force is given by \( \boldsymbol{F}_i = -\nabla_i V \). The \( x \)-component of the force on particle \( i \) is given by (see Problem 1)

\[
F_{ix} = 48 \sum_{\text{neighbors } j} \epsilon_{\alpha\beta} \left( \frac{\sigma_{\alpha\beta}^{12}}{r_{ij}^{14}} - 0.5 \frac{\sigma_{\alpha\beta}^6}{r_{ij}^8} \right) (x_i - x_j). \tag{6}
\]

and similarly for \( F_{iy} \) and \( F_{iz} \). The sum is only over particles \( j \) for which \( j \neq i \) and \( r_{ij} < 2.5\sigma_{\alpha\beta} \).

C. Periodic boundary conditions and the minimum image convention

To determine the neighbors we need to specify the boundaries of the system. We will assume that the goal of the simulation is to model the structure and dynamics of particles in a very large system \((N = 10^{23})\) far from the boundaries. However, most molecular dynamics simulations contain on the order of \( 10^3 \)–\( 10^6 \) particles. To minimize the effect of the boundaries, we use periodic boundary conditions as illustrated in Fig. 2 for a two-dimensional system of linear dimension \( L \). The system, framed by thick lines, is assumed to be surrounded by periodic images (framed by thin lines). For particle \( i \) the neighboring particles within a distance \( r_{cut}^{\alpha\beta} \) are the particles inside the (blue) large circle. To determine the distance \( r_{ij} \) between particles \( i \) and \( j \), we use the “minimum image convention.” For example, the distance between \( i \) and particle \( j = 18 \) would be \( r_{ij} > r_{cut}^{\alpha\beta} \) without using periodic images because particle 18 in the left bottom corner of the system is outside the (blue) large circle. But with periodic images \( r_{ij} < r_{cut}^{\alpha\beta} \) because the nearest periodic image of particle 18 is above particle 18 within the blue circle. For particles \( i \) and \( j = 20 \) we use the direct distance between the two particles within the system (thick frame), because this distance is less than the distance to any of the periodic images of \( j = 20 \).

D. Numerical integration

We next specify the core of a molecular dynamics program, that is, the numerical integration of the \( d \times N \) coupled differential equations represented by Eq. (1) (see
Fig. 2. Sketch of a binary two-dimensional system illustrating periodic boundary conditions and the minimum image convention. To identify the neighbors \( j \) of particle \( i \), the position \( \mathbf{r}_j \) of particle \( j \) is chosen from the position of \( j \) within the system (within thick frame) and the positions of \( j \)'s periodic images (within boxes framed with thin lines), such that \( r_{ij} \) is a minimum. Neighbors of \( i \) satisfy \( r_{ij} < r_{\text{cut}}^{ij} \) and are within the blue large circle.

Fig. 1). We will use the velocity Verlet algorithm

\[
\begin{align*}
\mathbf{r}_i(t + \Delta t) &= \mathbf{r}_i(t) + \mathbf{v}_i(t) \Delta t + \frac{1}{2} \mathbf{a}_i(t) (\Delta t)^2 \\
\mathbf{v}_i(t + \Delta t) &= \mathbf{v}_i(t) + \frac{1}{2}[\mathbf{a}_i(t) + \mathbf{a}_i(t + \Delta t)] \Delta t.
\end{align*}
\]

The velocity Verlet algorithm is commonly used because it is energy drift free and second order in the velocity and third order in the position.\(^9\) For other numerical integration techniques we refer readers to Refs. 9, 19–21. Note that the velocity update in Eq. (8) is directly applicable only if \( \mathbf{a}_i(t + \Delta t) \) does not depend on \( \mathbf{v}_i(t + \Delta t) \); that is, \( \mathbf{a}_i(t) \) depends only on the positions of the particles.

By using Eqs. (7) and (8) we obtain the flow chart (see Problem 2) in Fig. 3. Most of the computational time is used to determine the accelerations. Note that for each time step \( \Delta t \) the accelerations need to be determined only once for \( \mathbf{a}_i(t + \Delta t) \) (instead of twice for \( \mathbf{a}_i(t) \) and \( \mathbf{a}_i(t + \Delta t) \)). Thus, only one array for the accelerations is needed, but one must have the correct order of updates within the time step.

Fig. 3. Flow chart for molecular dynamics with the velocity Verlet algorithm.

**E. Temperature bath**

So far we have used Newton’s second law, Eq. (1), and numerical integration to determine the dynamics, which corresponds to simulating a system at constant energy. We also assumed that the number of particles and the volume are constant, and therefore we have described the NVE or microcanonical ensemble.\(^22–24\) In experiments the temperature \( T \) and the pressure \( P \) are controlled rather than \( E \) and \( V \). Many algorithms have been developed for NVT and NPT simulations, including generalizations which allow box shapes to vary during the simulation. For an overview of these algorithms we recommend Refs. 9, 19, and 25. In this section we focus on fixed NVT. We discuss in Sec. II E 1 an algorithm that also can be used to obtain the initial velocities and then discuss in Sec. II E 2 the Nosé-Hoover algorithm. A generalization of the latter is the default NVT algorithm in LAMMPS and is used in Sec. IV C.

1. Statistical temperature bath

The canonical ensemble corresponds to a system that can exchange energy with a very large system at constant temperature \( T \). In equilibrium the probability of a microstate \( s \) is proportional to the Boltzmann factor

\[
P(s) \propto e^{-E(s)/k_BT},
\]
where \( k_B \) is Boltzmann’s constant. Equation (9) applies to any system. The microstate \( s \) is specified by the position and velocity of each particle, \( \{r_i, v_i\} \), and the system energy is

\[
E(\{r_i, v_i\}) = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2 + V(\{r_i\}).
\]

(10)

From Eqs. (9) and (10) it follows that the probability distribution for the \( x \)-component of the velocity of particle \( i \) is given by the Maxwell-Boltzmann distribution

\[
P(v_{i,x}) = \frac{1}{\sqrt{2\pi} \sigma_i} e^{-v_{i,x}^2/(2\sigma_i^2)},
\]

(11)

with the standard deviation

\[
\sigma_i = \sqrt{\frac{k_B T}{m_i}}.
\]

(12)

The probability distributions for the \( y \)- and \( z \)-components of the velocity are obtained by replacing in Eq. (11) \( v_{i,x} \) by \( v_{i,y} \) and \( v_{i,z} \), respectively. It is straightforward to show that (see Problem 3)

\[
\langle E_{\text{kin}} \rangle = \left\langle \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \right\rangle = \frac{3N}{2} k_B T.
\]

(13)

To achieve simulations at fixed NVT Andersen\(^{26}\) incorporated particle collisions with a temperature bath by choosing the particle velocities from the Maxwell-Boltzmann distribution.\(^9,19\) We will use a slight modification to the Andersen algorithm introduced by Andrea et al.\(^{27}\) At periodic intervals (approximately every 50 time steps) all velocities are newly assigned by giving each particle a velocity component \( v_{i,\mu} (\mu \in \{x, y, z\}) \) chosen from the Maxwell-Boltzmann distribution in Eqs. (11) and (12).

Figure 4 shows the flow chart for creating a Maxwell-Boltzmann distribution for the velocities of the particles. Step 1 can be done in any programming language with either already defined functions (see Sec. III C for Python and Sec. IV B for LAMMPS) or with functions (or subroutines), from for example, Ref. 21. Step 2 can be skipped when \( m_i \) is the same for all particles. Step 3 ensures that the center of mass of the system does not drift, and step 4 rescales all the velocities to achieve the desired temperature.

The computer code illustrated by the flow chart of Fig. 4 is inserted into the code described by the flow chart of Fig. 3 with a conditional statement (e.g., \( \textbf{if} \)) after the (blue) “Time Step” box and within the (green) “Loop Over Time Steps.” We can also apply steps 1—4 to set the initial velocities \( \{v_i(t_0)\} \) as part of the “Initialization” box in Fig. 3.

2. Nosé-Hoover algorithm

Another way to implement a constant temperature bath, which is used in LAMMPS,\(^{28}\) is given by the Nosé-Hoover style algorithm. The key concept for most of the advanced algorithms is that we no longer use Newton’s second law for the equations of motion, but instead use an “extended system” with additional parameters. For the Nosé-Hoover algorithm\(^{29}\) Eq. (1) is replaced by

\[
\dot{\mathbf{r}}_i = \frac{\mathbf{F}_i}{m_i} - \xi_i \dot{\mathbf{r}}_i,
\]

(14)

\[
\frac{d^2 \ln s}{dt^2} = \dot{\xi} = \frac{1}{Q} \left( \sum_{i=1}^{N} m_i \ddot{r}_i^2 - dNk_B T \right),
\]

(15)

where \( d \) is the spatial dimension. The idea is to introduce a fictitious dynamical variable \( \xi \) that plays the role of a friction which changes the acceleration until the temperature equals the desired value. The parameter \( Q \) is the mass of the temperature bath. Equations (14) and (15) follow from generalizations of Hamiltonian mechanics\(^{30}\) and can be written as first-order differential equations for \( \mathbf{r}_i, \dot{\mathbf{r}}_i, \) and \( \xi.\)\(^{25,29,31–35}\) A simplified derivation of these equations is given in Appendix A for readers who are familiar with Hamiltonian mechanics.

A generalization of Eqs. (14) and (15) is the Nosé-Hoover chain method, which includes variables for several temperature baths, corresponding to more accurate dynamics in cases with more constraints than the example presented in this paper.\(^{25,31,34,36,37}\)

Given the equations of motion, our task is to convert them to appropriate difference equations so that we can use numerical integration. We would like to use the ve-
velocity Verlet algorithm in Eqs. (7) and (8). However, Eq. (14) for \( \mathbf{a}_i \) depends on the velocity \( \dot{\mathbf{r}}_i \), which means that the right side of Eq. (8) also depends on \( \mathbf{v}_i(t + \Delta t) \). Fox and Andersen suggested a velocity-Verlet numerical integration technique that can be applied when the equations of motion are of the form

\[
\dot{x}(t) = f[x(t), \dot{x}(t), y(t), \dot{y}(t)] \quad (16)
\]
\[
\dot{y}(t) = g[x(t), \dot{x}(t), y(t)]. \quad (17)
\]

The Fox-Andersen integration technique and its application to the Nosé-Hoover equations of motion, Eqs. (14) and (15), are discussed in Appendix B. The resulting update rules are given in Eqs. (B14)–(B16), and (B19). For more advanced integration techniques see Ref. 39.

F. Initialization of positions and velocities

As shown in Figs. 1 and 3, a molecular dynamics simulation starts with the initialization of every particle’s position and velocity. (For more complicated systems, further variables such as angular velocities need to be initialized.) As noted in Ref. 19, Sec. 8.6, “An appropriate choice of the initial conditions is more difficult than might first appear.” We therefore discuss a few options in detail. The most common options for the initialization of the particle positions include (1) using the positions resulting from a previous simulation of the same system, (2) choosing uniformly distributed positions at random, or (3) starting with positions on a lattice. For simplicity, the latter may be on a cubic lattice and/or a crystalline structure.

The advantage of the first option is that the configuration might correspond to a well equilibrated system at the desired parameters and the updates do not need extra precautions as in options 2 and 3. Even if the available configuration is not exactly for the desired parameters, that option is always the best. Option 2 is the most common initialization of velocities because it corresponds to velocities of a well equilibrated system (see Sec. II E 1). Option 3 is an option for \( T = 0 \) simulations, which we will not discuss further.

III. IMPLEMENTATION OF MD SIMULATIONS WITH PYTHON

In this section, we assume that readers know how to write basic Python programs. For the newcomer without Python experience, we recommend the first few chapters of Ref. 20, which are available online, and/or other online resources. (Reference 40 includes external links.)

A. Initialization of positions

At the beginning of the simulation the initial configuration needs to be set. We use arrays for \( \mathbf{r} \) and \( \mathbf{v} \) of size \( N = N_A + N_B \). For \( N_A = 800 \) and \( N_B = 200 \) the Python commands are

```python
import numpy as np
global Na
global Nb
global N
Na=800
Nb=200
N=Na+Nb
x = np.zeros(N,float)
y = np.zeros(N,float)
z = np.zeros(N,float)
```

If the positions are available, we read them from a file. We assume the filename \texttt{initpos} contains \( N \) lines each with three columns for \( r_{i,x}, r_{i,y}, \) and \( r_{i,z} \) and assign the positions using the statement

\[
x, y, z = \text{sp.loadtxt('initpos', dtype='float', unpack=True)}
\]
To choose positions at random and to ensure reproducible results we set the seed once at the beginning of the program.

```python
import scipy as sp
sp.random.seed(15)
```

For a system of linear dimension \( L = 9.4 \) we set the positions with

\[
L = 9.4 \\
x, y, z = sp.random.uniform(low=0.0, high=L, size=(3, N))
\]

For simplicity, we use the simple cubic lattice to place the particles on lattice sites (see Problem 4) as in the following:

```python
defines sitesx = int(round(pow(N,(1.0/3.0))))
sitesx = L / float(sitesx)
for ni in range(sitesx):
    tmpz = (0.5 + ni)*sitesx
for nj in range(sitesx):
    tmpy = (0.5 + nj)*sitesx
for nk in range(sitesx):
    tmpx = (0.5 + nk)*sitesx
i=nk+nj*sitesx+ni*(sitesx**2)
x[i]=tmpx
y[i]=tmpy
z[i]=tmpz
```

The arrays \( x, y, z \) use the indices 0, 1, …, \( N_A - 1 \) to store the positions of the \( A \) particles. Because the lattice positions are assigned successively to the lattice, the code places all \( A \) particles on one side and all \( B \) particles on the other side. This arrangement is not what is intended for a glassy or supercooled system. Therefore, in addition to assigning lattice sites, we next swap each \( B \) particle’s position with a randomly chosen \( A \) particle’s position:

```python
for i in range(Na,N):
    j = sp.random.randint(Na)
    x[i], x[j] = x[j], x[i]
y[i], y[j] = y[j], y[i]
z[i], z[j] = z[j], z[i]
```

B. Plotting and visualization tools

Long programs should be divided into many smaller tasks and each task tested. Before we continue with the implementation of molecular dynamics, we use a few printing and plotting tools to check if the program is working as expected.

We can save the positions into a file with a name such as `initposcheck`

```python
sp.savetxt('initposcheck', (sp.transpose(sp.vstack((x,y,z)))))
```

and then check the numbers in the file or look at the positions visually. To plot the positions we can use either plotting commands such as `gnuplot`, `xmgrace`, or Python-plotting tools. For simplicity, we use the latter. The Python commands for making a two-dimensional scatter plot of \( r_{i,z} \) and \( r_{i,x} \) which distinguishes \( A \) and \( B \) particles by color and size are

```python
import matplotlib as mpl
import matplotlib.pyplot as plt
plt.figure()
plt.scatter(x[:Na],z[:Na],s=150,color='blue')
plt.scatter(x[Na:],z[Na:],s=70,color='red')
```

```python
plt.xlim(0,L)
plt.xlabel('$x$')
plt.ylabel('$z$')
plt.show()
```

Figure 5(a) shows the resulting scatter plot for the case where the initial positions are on a lattice. To make a three-dimensional scatter plot we use at the beginning of the program the same `import` commands and add the following line:

```python
from mpl_toolkits.mplot3d import Axes3D
fig3d = plt.figure()
fax = fig3d.add_subplot(111, projection='3d')
fax.scatter(x[:Na],y[:Na],z[:Na], marker="o",s=150,facecolor='blue')
fax.scatter(x[Na:],y[Na:],z[Na:], marker="o",s=70,facecolor='red')
```

and then

```python
plt.xlim(0,L)
plt.xlabel('\$x\$')
plt.ylabel('\$z\$')
plt.show()
```
FIG. 5. Visualizations of the particle positions, which were assigned initially to a cubic lattice.

```python
def maxwellboltzmannvel(temp):
    global vx
    global vy
    global vz
    nopart=len(vx)
    sigma=np.sqrt(temp) #sqrt(kT/m)
    vx=np.random.normal(0.0,sigma,nopart)
    vy=np.random.normal(0.0,sigma,nopart)
    vz=np.random.normal(0.0,sigma,nopart)
    # make sure that center of mass does not drift
    vx -= sum(vx)/float(nopart)
    vy -= sum(vy)/float(nopart)
```

C. Initialization of velocities

If the positions and velocities are already available, they can be read from a file with six columns, similar to our earlier example. As described in Sec. II.E.1 a temperature bath can be achieved by periodically resetting all velocities from the Maxwell-Boltzmann distribution at the desired temperature. We therefore define a function for this task.
vz -= sum(vz)/float(nopart)
# make sure that temperature is exactly wanted temperature
scalefactor = np.sqrt(3.0*temp*nopart/sum(vx*vx+vy*vy+vz*vz))
vx *= scalefactor
vy *= scalefactor
vz *= scalefactor

This function is used in the example using the Python command maxwellboltzmannvel(0.2) with \( T = 0.2 \). To check the resulting velocities we save them in a file, plot them, and/or visualize them in VPython with

\[
\text{arrow(pos=vector(tx,ty,tz), axis=vector(txv,tyv,tvz),color=color.green)}
\]

where \( tx,ty,tz \) correspond to the positions and \( txv,tyv,tvz \) correspond to the velocities. The implementation for this section is in KALJ_initposvel.py.\(^1\)

### D. Accelerations

As shown in Fig. 3, the accelerations \( a_i \) are determined after the initialization and at each time step. A user-defined function for determining the accelerations is therefore recommended.(see Problem 5). For the KALJ system, \( m_i = 1 \) and \( F_{i,x} \) as given in Eq. (6) (similarly \( F_{i,y} \) and \( F_{i,z} \)). We store the accelerations in arrays \( ax, ay, az \). To determine \( a_i \) for all \( i = 1, \ldots, N \) (in Python particle index \( i=0, \ldots, N-1 \)), we need a loop over \( i \) and implement the sum over neighbors using an inner loop over \( j \). From Newton’s third law, \( F_{ij} = -F_{ji} \), we can reduce this double sum by a half.

```python
def acceleration(x,y,z):
    ax=sp.zeros(N)
    ay=sp.zeros(N)
    az=sp.zeros(N)
    for i in range(N-1):
        ... for j in range(i+1,N):
            ... ax[i] += ...
            ... ax[j] -= ...
    ...

To determine \( r_{ij}^2 \) we use \( r_{ij}^2 \),

```python
def acceleration(x,y,z):
    ax=sp.zeros(N)
    ... for i in range(N-1):
        xi=x[i]
        yi=y[i]
        zi=z[i]

    rijto2 = xij*xij + yij*yij + zij*zij

In addition, we need to implement the minimum image convention as described in Sec. II C.

```python
def acceleration(x,y,z):
    global L
    global Ldiv2
    ...
    xij=xi-x[j]
    yij=yi-y[j]
    zij=zi-z[j]
    # minimum image convention
    if xij > Ldiv2: xij -= L
    elif xij < - Ldiv2: xij += L
    if yij > Ldiv2: yij -= L
    elif yij < - Ldiv2: yij += L
    if zij > Ldiv2: zij -= L
    elif zij < - Ldiv2: zij += L
    rijto2 = xij*xij + yij*yij + zij*zij
    ...
```

Here \( L/2.0 \), and we assume the particle positions satisfy \( 0 < x, y, z < L \), which means that \( x, y, z \) are updated with periodic boundary conditions to stay within the central simulation box. Because the sum is only over neighbors within \( 2.5\sigma \) of a given particle, we add an if statement:

```python
def acceleration(x,y,z):
    ...
    rijto2 = xij*xij + yij*yij + zij*zij
    if(rijto2 < rcutto2):
        onedivrijto2 = 1.0/rijto2
        fmagtmp= eps*(sigmato12*onedivrijto2**7 - 0.5*sigmato6*onedivrijto2**4)
        ax[i] += fmagtmp*xij
```
We avoided the additional costly computational determination of \( r_{ij} = \sqrt{r_{ij}^2} \), and the factor 48 was multiplied only once via matrix multiplication. This acceleration function is called in the main program with the statement

\[
ax, ay, az = \text{acceleration}(x, y, z)
\]

The variables \( r_{cut2} = (r_{cut})^2 \), \( \epsilon \), \( \sigma_{12} \), and \( \sigma_6 \) are particle type dependent for the binary Kob-Andersen model. We include this dependence with conditional statements

\[
\begin{align*}
\text{if } i < Na: & \\
& \text{if } j < Na: \quad \# AA \\
& \quad r_{cut2} = r_{cutAAto2} \\
& \quad \sigma_{12} = \sigma_{AAto12} \\
& \quad \sigma_6 = \sigma_{AAto6} \\
& \quad \epsilon = \epsilon_{AA} \\
& \text{else:} \quad \# AB \\
& \quad r_{cut2} = r_{cutABto2} \\
& \quad \sigma_{12} = \sigma_{ABto12} \\
& \quad \sigma_6 = \sigma_{ABto6} \\
& \quad \epsilon = \epsilon_{AB} \\
& \text{else:} \quad \# BB
\end{align*}
\]

These conditional statements cost computer time and can be avoided by replacing the i, j loops with three separate i, j loops.

# AA interactions

\[
\begin{align*}
\text{for } i \text{ in range}(Na-1): \\
& \quad \text{for } j \text{ in range}(i+1, Na): \\
& \quad \quad r_{ijto2} = x_{ij}x_{ij} + y_{ij}y_{ij} + z_{ij}z_{ij} \\
& \quad \quad \text{if}(r_{ijto2} < r_{cutAAto2}): \\
& \quad \quad \quad \text{onedivrijto2} = 1.0/r_{ijto2} \\
& \quad \quad \quad \text{fmagtmp} = \epsilon_{AA}(\sigma_{AAto12} \cdot \text{onedivrijto2}^{**7} - 0.5 \cdot \sigma_{AAto6} \cdot \text{onedivrijto2}^{**4})
\end{align*}
\]

# AB interactions

\[
\begin{align*}
\text{for } i \text{ in range}(Na): \\
& \quad \text{for } j \text{ in range}(Na, N): \\
& \quad \quad r_{ijto2} = x_{ij}x_{ij} + y_{ij}y_{ij} + z_{ij}z_{ij} \\
& \quad \quad \text{if}(r_{ijto2} < r_{cutABto2}): \\
& \quad \quad \quad \text{onedivrijto2} = 1.0/r_{ijto2} \\
& \quad \quad \quad \text{fmagtmp} = \epsilon_{AB}(\sigma_{ABto12} \cdot \text{onedivrijto2}^{**7} - 0.5 \cdot \sigma_{ABto6} \cdot \text{onedivrijto2}^{**4})
\end{align*}
\]

# BB interactions

\[
\begin{align*}
\text{for } i \text{ in range}(Na, N-1): \\
& \quad \text{for } j \text{ in range}(i+1, N): \\
& \quad \quad \text{return } 48 \cdot ax, 48 \cdot ay, 48 \cdot az
\end{align*}
\]

E. NVE molecular dynamics simulation

We are now equipped to implement a NVE molecular dynamics simulation (see Problem 6). We follow the flow chart of Fig. 3 and after the initialization of \( \{r_i\} \), \( \{v_i\} \), and \( \{a_i\} \), add a loop over time steps and update the positions and velocities within this loop. To update \( r_i \) and \( v_i \) for all \( i \), we use matrix operations instead of for loops, because matrix operations are computationally faster in
Python. We need to ensure periodic boundary conditions, which we implement assuming that we start with \(0 < x_i, y_i, z_i \leq L\) and that during each time step no particle moves further than \(L\).

```python
for tstep in range(1,nMD+1):
    # update positions
    x += vx*Deltat + 0.5*ax*Deltatto2
    y += vy*Deltat + 0.5*ay*Deltatto2
    z += vz*Deltat + 0.5*az*Deltatto2
    # periodic boundary conditions:
    for i in range(N):
        if x[i] > L: x[i] -= L
        elif x[i] <= 0: x[i] += L
        if y[i] > L: y[i] -= L
        elif y[i] <= 0: y[i] += L
        if z[i] > L: z[i] -= L
        elif z[i] <= 0: z[i] += L
    # update velocities
    vx += 0.5*ax*Deltat
    vy += 0.5*ay*Deltat
    vz += 0.5*az*Deltat
    ax,ay,az = acceleration(x,y,z)
    vx += 0.5*ax*Deltat
    vy += 0.5*ay*Deltat
    vz += 0.5*az*Deltat
```

Here \(nMD = n_{MD}\) is the number of time steps, \(Deltat = \Delta t\), and \(Deltatto2 = (\Delta t)^2\).

The most time consuming part of the time loop is the determination of \(\{a_i\}\), because it includes the double loop over \(i\) and \(j\). In more optimized MD programs, the loop over \(j\) would be significantly sped up by looping only over neighbors of particle \(i\) (instead of over all particles \(j\)) via a neighbor list. However, for our purpose of becoming familiar with MD, we will do without a neighbor list.

### F. Python: More analysis and visualization tools

As a check of the program, we may either plot the trajectories of a specified particle (see Fig. 6) or make a scatter plot of the initial and final configuration as shown in Fig. 7 for \(z(x)\) after \(nMD = 50\) time steps with \(\Delta t = 0.005\).

For Fig. 6 we used the plot tools as described in Sec. III B. To plot the \(y\)-component of particle 8, an array for these values was defined before the time loop

```python
yiplotarray = np.zeros(nMD,float)
```

This array was updated within the time loop after the time step

```python
yiplotarray[tstep-1]=y[7]
```

Note that particle 8 corresponds to index 7. After the time loop the following plotting commands are used:

```python
tarray = np.arange(Deltat,(nMD+1)*Deltat,Deltat)
plt.figure()
plt.rcParams['ytick.labelsize']=11
plt.plot(tarray,yiplotarray,color='blue')
plt.xlabel('$t$',fontsize=15)
plt.ylabel('$y_8$',fontsize=15)
plt.show()
```

To make Fig. 7 we stored the initial configuration using

```python
x0 = np.copy(x)
y0 = np.copy(y)
z0 = np.copy(z)
```

FIG. 6. The \(y\)-component of particle 8 as a function of time \(t\) (in LJ time units \(\sqrt{m_A\sigma_{AA}^2/\epsilon_{AA}}\)) for the Kob-Andersen model of \(N = 1000\) particles starting with the initial configuration `initposvel`, and for 50 time steps with \(\Delta t = 0.005\). (The largest time is \(50 \times 0.005 = 0.25\).)

FIG. 7. \(z(x)\) for the initial configuration in black and for the final configuration at \(t = 0.25\) (in LJ time units \(\sqrt{m_A\sigma_{AA}^2/\epsilon_{AA}}\)) in blue.
and used `plt.scatter` as described in Sec. III B.

We can also make an animation using VPython. Before the time loop we create spheres (particles at their positions) and arrows (velocities) as

\[
\begin{align*}
\text{s} &= \text{np.empty}(N,\text{sphere}) \\
\text{ar} &= \text{np.empty}(N,\text{arrow}) \\
\text{for } i \text{ in range}(N):
\begin{cases}
\text{if } i < \text{Na} : \\
\text{s}[i] &= \text{sphere}(\text{pos=vector}(x[i],y[i],z[i]), \text{radius}=0.5, \text{color}=\text{color.blue}) \\
\text{else:} \\
\text{s}[i] &= \text{sphere}(\text{pos=vector}(x[i],y[i],z[i]), \text{radius}=0.2, \text{color}=\text{color.red}) \\
\text{ar}[i] &= \text{arrow}(\text{pos=vector}(x[i],y[i],z[i]), \text{axis=vector}(vx[i],vy[i],vz[i]), \\
\text{color}=\text{color.green})
\end{cases}
\end{align*}
\]

Within the time loop we update the spheres and arrows as

\[
\begin{align*}
\text{rate}(30) \\
\text{for } i \text{ in range}(N): \\
\text{s}[i].\text{pos} &= \text{vector}(x[i],y[i],z[i]) \\
\text{ar}[i].\text{pos} &= \text{vector}(x[i],y[i],z[i]) \\
\text{ar}[i].\text{axis} &= \text{vector}(vx[i],vy[i],vz[i])
\end{align*}
\]

We can also plot the kinetic energy per particle \( E_{\text{kin}}/N \), potential energy per particle \( V/N \), and the total energy per particle \( E_{\text{tot}}/N = (E_{\text{kin}} + V)/N \) as a function of time. To include the minimum image convention and the cases AA, AB, and BB, a user-defined function can be written similar to acceleration of Sec. III D. The Python program for this section, `KALJnve.py`, including the accelerations in Sec. III D, is available in Ref. 41.

\[
\text{FIG. 8. The total energy per particle } E_{\text{tot}}/N = (E_{\text{kin}} + V)/N \text{ as a function of time (in LJ units) for a NVE simulation with } n_{\text{MD}} = 50 \text{ and with } \Delta t = 0.005. \text{ The value } -6.161 \text{ on the top left indicates that the tick labels on the vertical axis are } E_{\text{tot}}/N + 6.151. \text{ The initial positions and velocities were read in from the file } \text{initposvel} (T = 0.5).
\]

As shown in Fig. 8, \( E_{\text{tot}}/N \) exhibits very small variations about a constant as expected for the NVE simulation.

G. NVT molecular dynamics simulation

We now implement a statistical temperature bath as described in Sec. II E 1, which will allow readers to do simulations at desired temperatures (see Problem 7).

We need to update all the velocities using the Maxwell-Boltzmann distribution specified in Eqs. (11) and (12). Because we already have implemented the Maxwell-Boltzmann distribution in Sec. III C with the user-defined function `maxwellboltzmannvel`, we can implement the temperature bath with only two lines in the time loop after the time step. For the case of computing new velocities every \( n_{\text{stepBoltz}} \) time steps:

\[
\text{if } (tstep \% n_{\text{stepBoltz}}) == 0 : \\
\text{maxwellboltzmannvel}(\text{temperature})
\]

You can test your program by plotting the temperature as a function of time. The temperature can be determined by solving Eq. (13) for \( T \). Figure 9 shows an example of a MD simulation using positions initially equilibrated at \( T = 0.5 \), and then run at \( T = 0.2 \). The Python program, `KALJnvt.py`, for this section is available at Ref. 41.

IV. IMPLEMENTATION OF MD SIMULATIONS WITH LAMMPS

Readers might have noticed that the simulation we have discussed was for only 1000 particles and 50 time steps and took painfully long. (The pain level depends on the computer.) To shorten the computation time, there exist optimization techniques such as nearest neighbor lists, as well as coding for multiple processors. We now introduce the free and open source software, LAMMPS. LAMMPS (Large-scale Atomic/Molecular Massively Parallel Simulator) allows a very wide range of simulation techniques and physical systems. The
A. Introduction to LAMMPS

In LAMMPS the user chooses the simulation technique, system, particle interactions, and parameters, all via an input file. The main communication between the user and LAMMPS occurs via the input file. To run LAMMPS with parallel code, the simulation is started with commands such as

```
mpirun -np 16 lmp_mpi < inKALJ_nve > outLJnve
```

where `-np 16` specifies the number of cores, `lmp_mpi` is the name of the LAMMPS executable (which might have a different name depending on the computer), `outLJnve` is the output file (see the following for a description on what information is written into this file), and `inKALJ_nve` is the input file. Becoming familiar with LAMMPS mainly requires learning the commands in this input file. Further documentation can be found at Ref. 1. A set of input file examples is available at Ref. 42. Appendix C describes how to run LAMMPS on a shared computer using a batch system.

B. NVE simulation with LAMMPS

To run at fixed NVE the input file, `inKALJ_nve`, contains

```plaintext
#KALJ NVE, read data
atom_style atomic
boundary p p p #periodic boundary cond. in each direction
read_data initconf_T05eq.data #read data file (incl.mass)
pair_style lj/cut 2.5 # Define interaction potential
pair_coeff 1 1 1.0 1.0 2.5 #type type eps sigma rcut
pair_coeff 1 2 1.5 0.80 2.0 #typeA typeB epsAB sigmaAB rcutAB=2.5*0.8=2.0
pair_coeff 2 2 0.5 0.88 2.2 #typeB typeB epsBB sigmaBB rcutBB=2.5*0.88=2.2
timestep 0.005 #Delta t
neighbor 0.3 bin
neigh_modify every 1 delay 0 check yes
dump mydump all custom 50 confdump.*.data id type x y z vx vy vz
dump_modify mydump sort id
# set numerical integrator
fix nve1 all nve # NVE; default is velocity verlet
run 100
```

Comments start with `. The statement `atom_style atomic` specifies the type of particle, and `boundary p p p` implements periodic boundary conditions. Not included in this sample input file are the two possible commands,

```
units lj
dimension 3
```

because they are the default settings.

Initial positions and velocities are read from the file `initconf_T05eq.data`. For the LAMMPS `read_data` command, the specified file (here `initconf_T05eq.data`) contains

```plaintext
#bin. KALJ data file T=0.5
1000 atoms
2 atom types
0 9.4 xlo xhi
0 9.4 ylo yhi
```
The first few lines specify the type of system, \( N = 1000 \) atoms with \( A \) and \( B \) particles, the box length, \( L = 9.4 \), and the masses \( m_A = m_B = 1 \). The 1000 lines following \texttt{Atoms} specify the particle index, \( i = 1, 2, \ldots, 1000 \) in the first column, the particle type in the second column; that is, 1 for particles 1, 2, \ldots, 800 (\( A \)-particles) and 2 for particles 801, \ldots, 1000 (\( B \)-particles). Columns three, four, and five are \( x_i, y_i, z_i \) respectively. The lines following \texttt{Velocities} contain \( i, v_{x,i}, v_{y,i}, \) and \( v_{z,i} \).

In the input file \texttt{inKALJ_nve} the particle interactions are defined by the commands \texttt{pair_style} and \texttt{pair_coeff}. Note that \texttt{lj/cut} corresponds to the forces of Eq. (6). However, the potential energy excludes the term \( V_{\alpha\beta}(r_{\alpha\beta}^{cut}) \) of Eq. (5). In LAMMPS there is also the option of the truncated and force shifted Lennard-Jones interactions \texttt{lj/smooth/linear}. We chose \texttt{lj/cut} to allow for the direct comparison of the Python and LAMMPS simulations. In the file \texttt{inKALJ_nve} the line \texttt{timestep 0.005} sets \( \Delta t = 0.005 \). The commands \texttt{neighbor} and \texttt{neigh_modify} are parameters for the neighbor list. The LAMMPS commands \texttt{dump} and \texttt{dump_modify} periodically save snapshots of all atoms. In our example, every 50 time steps (starting with \( t = 0 \)) a file is written with file name \texttt{confdump.0.data}, \texttt{confdump.50.data}, \texttt{confdump.100.data}, and the content of each written file has columns \( i, \) particle type (1 or 2), \( x_i, y_i, z_i, v_{x,i}, v_{y,i}, \) and \( v_{z,i} \). In the \texttt{dump} command \texttt{mdump} is the LAMMPS-ID for this \texttt{dump} command. It can be replaced with any name the reader chooses. The ID allows further specifications for this dump as used in the command \texttt{dump_modify mdump sort id}, which ensures that the lines in the dump files are sorted by particle index \( i \).

The integration technique is set by the command \texttt{fix nve1 all nve}; \texttt{nve1} is an ID for this \texttt{fix} command, all means that this integration step is applied to all particles, and \texttt{nve} specifies the NVE time step which is the velocity Verlet integration step by default. The command \texttt{run100} means that the simulation is run for 100 time steps under these specified conditions.

The input file \texttt{inKALJ_nve} assumes that the initial positions and velocities are available. For a small system such as \( N = 1000 \) the initial positions may be generated by doing a simulation with Python. However, for simulations with significantly more particles, the initial positions and velocities may not be available. If we instead initialize with uniformly randomly distributed positions and velocities from the Maxwell-Boltzmann distribution, we replace in \texttt{inKALJ_nve} the \texttt{read_data} command with the following LAMMPS commands

\begin{verbatim}
region my_region block 0 9.4 0 9.4 0 9.4
create_box 2 my_region
create_atoms 1 random 800 229609 my_region
create_atoms 2 random 200 691203 my_region
mass 1 1
mass 2 1
velocity all create 0.5 92561 dist gaussian
\end{verbatim}

The first two commands create the simulation box for two types of atoms, the \texttt{create_atoms} commands initialize the atom positions randomly drawn from a uniform distribution and random number generator seeds 229609 and 691203 (any positive integers), and the last command initializes the velocities of all particles with the Maxwell-Boltzmann distribution for temperature \( T = 0.5 \) and random number seed 92561.

As noted in Sec. II F, random positions can lead to very large forces. These can be avoided by adding in \texttt{inKALJ_nve} before the \texttt{fix} command the line

\begin{verbatim}
minimize 1.0e-4 1.0e-6 1000 1000
\end{verbatim}

The files (\texttt{inKALJ_nve, inKALJ_nve_rndposvel, initconf_T05eq.data}, and \texttt{runKALJ_slurm.sh}) for this section and the previous two sections are available. \textendnote{41}{C. NVT simulations}

The default NVT simulation in LAMMPS uses the Nosé-Hoover algorithm (see Sec. II E 2, and Appendices A and B). \textendnote{25}{25,28,31,34,36,37} To implement this temperature bath in LAMMPS, we replace the command \texttt{fix nve1 all nve} in the input file with

\begin{verbatim}
fix nose all nvt temp 0.2 0.2 $(100.0*dt)
\end{verbatim}

As described in Ref. 28, \texttt{nose} is the ID chosen by the user for this \texttt{fix} command, all indicates that this \texttt{fix} is applied to all atoms, \texttt{nvt temp} 0.2 0.2 sets the constant temperature to \( T = 0.2 \), and the last parameter sets the damping parameter as recommended in Ref. 28 to 100\( \Delta t \).

Another way of achieving a temperature bath is to implement the statistical temperature bath as described in Sec. II E 1. We use an implementation similar to that used in Python in Sec. III C. To compute random velocities periodically in time, we replace the command \texttt{run 100}. To compute new velocities every 10 time steps at temperature \( T = 0.2 \) the replacement line is

\begin{verbatim}
...
run 100 pre no every 10 "velocity all create 0.2 ${rnd} dist gaussian"

where \( \{\text{rnd}\} \) is a user-defined LAMMPS variable corresponding to a random reproducible integer; \( \text{rnd} \) needs to be defined before the modified \text{run} \ command by

\[
\text{variable rnd equal floor(random(1,100000,3259))}
\]

where we used the LAMMPS function \text{random} \ (see Ref. 43). To test the program, readers may plot the measured temperature as a function of time, \( T_{\text{meas}}(t) \) (similar to Fig. 9) and \( E_{\text{tot}} \) as a function of time (similar to Fig. 8). Such time dependent functions can be computed and saved with \text{thermo_style},

\[
\text{thermo_style custom step temp pe ke etotal}
\]

\[
\text{gawk 'NF==5 && !/[a-zA-Z]/ \{print $1,$2\}' outLJnvt}
\]

The resultant output can be redirected into a file or directly piped into a plotting tool, e.g., by adding to \text{gawk} \ at the end \ | \text{xmgrace -pipe}. To obtain \( E_{\text{tot}}/N \) as a function of the number of time steps, we replace the \text{gawk} \ command \$2 \ by \$5.

The LAMMPS input files, \text{inKALJ\_nvt\_stat} \ and \text{inKALJ\_nvt\_Nose}, \ are available at Ref. 41.

\[\text{VI. SIMULATION RUN SEQUENCE}\]

Readers can now run molecular dynamics simulations with Python or LAMMPS. To illustrate what a simulation sequence entails, we give a few examples of simulations for the Kob-Andersen model.

The first set of papers on the Kob-Andersen model were on the equilibrium properties of supercooled liquids.\textsuperscript{3,11,12} As described in Ref. 11, the system was first equilibrated at \( T = 5.0 \) and then simulated at successively lower temperatures \( T = 4.0, 3.0, 2.0, \ldots, 0.475, 0.466 \). For each successive temperature, a configuration was taken from the previously equilibrated temperature run, the temperature bath (stochastic in this study) was applied for \( t_{\text{equi}} \) time units, followed by an NVE simulation run also for \( t_{\text{equi}} \) time units, and then followed by an NVE production run during which the dynamics and structure of the system were determined. This sequence of reaching successively lower temperatures was applied to eight independent initial configurations.

Another example for a simulation sequence is to apply a constant cooling rate as was done in Ref. 13 with an NPT algorithm.

References 44–46 studied the Kob-Andersen model out of equilibrium by first equilibrating the system at a high temperature \( T_i \) and then quenching instantly to a lower temperature \( T_i \). That is, a well equilibrated configuration from the simulation at \( T_i \) was taken to be the initial configuration for an NVT simulation at \( T_i \). During the run at \( T_i \) the structure and dynamics of the system depend on the waiting time, which is the time elapsed since the temperature quench.\textsuperscript{14–46}

\[\text{VI. ANALYSIS}\]

In this section we discuss the analysis of molecular dynamics simulations. To give readers a taste of the wide variety of analysis tools, we focus on two commonly studied quantities: the radial distribution function and the mean square displacement.

\[\text{A. Radial distribution function}\]

The radial distribution function, \( g(r) \), is an example of a structural quantity and is a measure of the density of particles \textit{j} at a distance \textit{r} from a particle \textit{i}, where \( r = r_{ij} = |\textbf{r}_i - \textbf{r}_j| \) and radial symmetry is assumed. For a binary system \( g_{AA}(r), g_{BB}(r) \), and \( g_{AB}(r) \) are defined as

\[
g_{aa}(r) = \frac{V}{N_a (N_a - 1)} \left\langle \sum_{i=1}^{N_a} \sum_{j=1 \neq i}^{N_a} \delta(r - |\textbf{r}_i - \textbf{r}_j|) \right\rangle ,
\]

where \( \alpha \in \{A, B\} \), and (see Refs. 9 and 11)

\[
g_{AB}(r) = \frac{V}{N_A N_B} \left\langle \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \delta(r - |\textbf{r}_i - \textbf{r}_j|) \right\rangle .
\]
Equations (18) and (19) include sums over particle pairs $(i,j)$ of the types specified. The Dirac delta function $\delta(x)$ is the number density for a point particle at $x = 0$. The number density of $(i,j)$ pairs with distance $r = r_{ij}$ is normalized by the global density. Therefore $g(r)$ characterizes the distribution of particle distances. The average $\langle \ldots \rangle$ in Eqs. (18) and (19) can be taken either by averaging over independent simulation runs and/or via a time average by averaging measurements at different times $t$. For the measurement of $g(r)$ in equilibrium, the system needs to be first equilibrated, and therefore $t > t_{\text{equil}}$ for all measurements. For more advanced readers the generalization of the radial distribution function is the van Hove correlation function $G(r,t)$.\textsuperscript{6,11}

1. Radial distribution function with Python

To determine a histogram of the $r_{ij}$ distances, we use an array as illustrated for $g_{AA}$ in Fig. 10. Before determining the histogram we set to zero the arrays $\text{gofrAAhist}$, $\text{gofrBBhist}$, and $\text{gofrABhist}$. For each measurement, we loop over all unique particle combinations ($j > i$), determine the minimum image distance (see Sec. III D), and add to the counter of the corresponding bin.\textsuperscript{9}

```python
for i in range(0,N-1):
    xi=x[i]
    ...
    for j in range(i+1,N):
        xij=xi-x[j]
        ...
        #minimum image convention
        if xij > Ldiv2: xij -= L
        ...
        rjto2 = xij*xij + yij*yij + zij*zij
        rij=sp.sqrt(rjto2)
        grbin=int(rij/grdelta)
        if(grbin < grnbinmax):
            if(i < Na):
                if (j < Na): #AA
                    histgofrAA[grbin] += 2
                else: #AB
                    histgofrAB[grbin] += 1
            else: #BB
                histgofrBB[grbin] += 2
Here grdelta = $\Delta r$ is the bin size (see Fig. 10). If the average is a time average, taken via measurements (assume with user defined function histmeas) after tequil time steps every nstepgofr time steps, we set the arrays histgofrAA etc. to zero before the time loop, and add the conditional statement
```

```python
if (tstep > tequil) and ((tstep % nstepgofr)==0):
    histmeas(x,y,z)
```

within the time loop and after the time step, that is in the flow chart of Fig. 3 after the (blue) “Time Step” box and within the “Loop Over Time Steps,” so before the (green) time loop repeats. We can then save the resulting radial distribution functions into a file of name $\text{gofrAABBAB.data}$ by adding to the program after the time loop

```python
fileoutgofr= open("gofrAABBAB.data",mode='w')
for grbin in range(grnbinmax):
    rinner = grbin*grdelta
    router = rinner + grdelta
    shellvol = (4.0*sp.pi/3.0)*(router**3 - rinner**3)
    gofrAA = (L**3/(Na*(Na-1)))*histgofrAA[grbin]/(shellvol*nmeas)
    gofrBB = (L**3/(Nb*(Nb-1)))*histgofrBB[grbin]/(shellvol*nmeas)
    gofrAB = (L**3/(Na*Nb))*histgofrAB[grbin]/(shellvol*nmeas)
    rmid = rinner + 0.5*grdelta
    print(rmid,gofrAA,gofrBB,gofrAB,file=fileoutgofr)
```
FIG. 11. Radial pair distribution functions for the Kob-Andersen model with $N = 1000$, $L = 9.4$ at $T = 0.5$. The green vertical arrows indicate peak positions of $g_{AA}(r)$.

We have assumed that $n_{\text{meas}}$ measurements of the histogram were taken, and the variables are as shown in Fig. 10 with $gr_{\text{delta}} = \Delta r$. For the normalization we determine the shell volume/area, which is sketched in Fig. 10 as the cyan (gray) shaded area enclosed by the two large circles drawn with thick lines. In three dimensions the shell volume is $(4\pi/3)(r_{\text{outer}}^3 - r_{\text{inner}}^3)$.

The resulting radial distribution functions are shown in Fig. 11 for which we ran the NVT simulation with $N = 1000$ at $T = 0.5$, starting with a well equilibrated configuration at $T = 0.5$, running the simulation for 200 time steps and measuring the histogram every $\text{nstepgofr}=25$ time steps with $\Delta r = 0.1$. To measure distances up to $L/2$, we set $gr_{\text{bimmax}} = \text{int}(\text{Ldiv2}/gr_{\text{delta}})$. The Python program $\text{KALJ_nvt_gofr.py}$ for this section is in Ref. 41.

2. Interpretation of radial distribution function

Because the repulsive interaction $V(r_{ij} \to 0) \to \infty$ prevents the complete overlap of two particles, $g(r) = 0$. The first peak of $g(r)$ corresponds to the most likely radius of the first shell of neighboring particles surrounding particle $i$. The second peak of $g(r)$ corresponds to the second nearest neighbor shell, etc. (see Fig. 12). With increasing $r$ the peaks become less and less pronounced, because the system has, contrary to a crystal, no long range order. The peak positions of $g_{AA}(r)$, $g_{BB}(r)$, and $g_{AB}(r)$ in Fig. 11 are consistent with the results of Kob and Andersen (Fig. 9 of Ref. 11). For their more quantitative study, they used longer simulation runs, several independent simulation runs, and a smaller (and probably more than one) value of $\Delta r$.

FIG. 12. Sketch of particle arrangements to illustrate the interpretation of the radial distribution function. The first, second, third, and forth neighbor shells are indicated with dark green circles (thick lines). The radii of these shells correspond to the peak positions of the radial distribution function as indicated with dark green vertical arrows in Fig. 11 for the example of $g_{AA}(r)$.

3. Radial distribution function with LAMMPS

We can determine the radial distribution functions with LAMMPS by adding to the input file before the run command the lines

```
compute rdfAABBAB all rdf 25 1 1 2 2 1 2
fix myrdf all ave/time 25 8 200 c_rdfAABBAB[*] file gofrAABBAB.data mode vector
```

The `compute` command defines the measurements, which are done during the run:

1. `rdfAABBAB` is the user defined ID for this `compute` command. This ID is used in the `fix` command with `c_rdfAABBAB`, which means the ID is like a variable name.
2. all applies the command to all atoms.
3. rdf computes the radial distribution function.
4. \( 25 = N_{\text{bin}} \) specifies \( \Delta r \) to be \( r^{\text{cut}}/25 \). The following numbers specify the particle type combinations, that is, \( g_{AA}, g_{BB}, \) and \( g_{AB} \).
5. fix ave/time defines the time averaging. As described in Ref. 47 the three numbers in our example specify that the histogram is measured every \( \text{Nevery} = 25 \) time steps, \( \text{Nrepeat} = 8 \) measurements are averaged (in Sec. VI A 1 \( \text{mneas} \)), and \( \text{Nfreq} = 200 \) is the interval of time steps at which the time average is printed. That is, if the simulation run is \( n_{\text{MD}} = 600 \) then the averages of \( g(r) \) are printed out three times, the first by averaging measurements taken at time step 200, 175, 150, \ldots, 25, and the last one at time steps 600, 575, 550, \ldots, 425. Constraints on the choice of \( \text{Nevery}, \text{Nrepeat}, \) and \( \text{Nfreq} \) are given in Ref. 47. In addition, compatible times need to be chosen, if the LAMMPS command \text{run every} \text{ } \text{is} \text{ } \text{used}, \text{ } \text{which} \text{ } \text{we} \text{ } \text{used} \text{ } \text{for} \text{ } \text{the} \text{ } \text{statistical} \text{ } \text{temperature} \text{ } \text{bath} \text{ } \text{in} \text{ } \text{Sec. IV C}.
6. \([\cdot]\) takes time averages for each of the variables of the \text{compute} \text{ } \text{rdf} \text{ } \text{command}
7. file gofrAABBAB.data specifies that the results are saved in the file \text{gofrAABBAB.data}.
8. mode vector is necessary, because \( g_{AA}(r) \) etc. are vectors instead of scalars, with indices for the \( N_{\text{bin}} \) bins \( r = [0, \Delta r), [\Delta r, 2\Delta r), \ldots \). The entries in the file \text{gofrAABBAB.data} are for each average time (here 200, 400, 600) starting with one line specifying the print time in time steps, 200 etc., and \text{Nevery}, followed by \( N_{\text{bin}} \) lines, each with columns for the bin number, \( r, g_{AA}, g_{AA}, g_{BB}, c_{BB}, g_{AB}, c_{AB} \), where \( c_{AA} \) etc. are coordination numbers.

The LAMMPS input file inKALJ_T05_gofr is in Ref. 41.

### B. Mean square displacement

We next study how the system evolves as a function of time. The mean square displacement\(^9,11\) captures how far each particle moves during a time interval \( t \):

\[
\text{msd} = \langle r^2(t) \rangle = \left\langle (\mathbf{r}(t) - \mathbf{r}(0))^2 \right\rangle \tag{20}
\]

where \( \langle \ldots \rangle \) corresponds to an average over particles and may also include an average over independent simulation runs. In the following discussion on the implementation of the mean square displacement with Python and mean square displacement with LAMMPS, we average only over particles of one type

\[
\langle r^2_\alpha(t) \rangle = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} |\mathbf{r}_i(t) - \mathbf{r}_i(0)|^2, \tag{21}
\]

where \( \alpha \in \{A, B\} \) is the particle type.

A generalization of Eq. (20) is

\[
\langle r^2(t, t + t) \rangle = \left\langle |\mathbf{r}(t) - \mathbf{r}(t)|^2 \right\rangle \tag{22}
\]

If the system is in equilibrium, \( \langle r^2(t, t + t) \rangle \) is independent of starting time \( t \) and the average \( \langle \ldots \rangle \) may include an average over \( t \).

1. Mean square displacement with Python

It is suggested that readers do Problem 8 before reading the following. We use arrays to store the positions at \( t = 0 \) after the initialization of the positions with \( x0 = np.copy(x) \ldots \). We cannot use periodic boundary conditions to determine the mean square displacement and instead use unwrapped coordinates and define the additional arrays \( xu, yu, \) and \( zu \) which are initially also copied from \( x \), etc. These arrays are updated in the time step loop as in Sec. III E.

\[
xu += vx*Deltat + 0.5*ax*Deltatto2
\]
\[
yu += vy*Deltat + 0.5*ay*Deltatto2
\]
\[
zu += vz*Deltat + 0.5*az*Deltatto2
\]

Periodic boundary conditions are not applied to \( xu, yu, \) and \( zu \).

To save the results into the file \text{msd.data}, we add before the time loop the statement \text{fileoutmsd = open("msd.data",mode='w')}. Measurements of the mean square displacements are done within the time loop and after the time step.

\[
\text{msdA} = 0.0
\]

for \( i \) in range(\( Na \)):

\[
dx = xu[i]-x0[i]
\]
\[
dy = yu[i]-y0[i]
\]
\[
dz = zu[i]-z0[i]
\]

\[
\text{msdA} += dx*dx+dy*dy+dz*dz
\]

\[
\text{msdA} /= \text{float}(Na)
\]

\[
\text{msdB} = 0.0
\]

for \( i \) in range(\( Na, N \)):

\[
dx = xu[i]-x0[i]
\]
\[
dy = yu[i]-y0[i]
\]
\[
dz = zu[i]-z0[i]
\]

\[
\text{msdB} += dx*dx+dy*dy+dz*dz
\]

\[
\text{msdB} /= \text{float}(Nb)
\]

print(tstep*Deltat,msdA,msdB,file=fileoutmsd)

Figure 13 shows the resultant mean square displacement as a function of time. After a steep increase for very small times, \( \langle r^2 \rangle \) reaches a plateau. The plateau value is larger for the smaller \( B \) particles. For significantly longer times \( \langle r^2 \rangle \) increases again. To quantify \( \langle r^2(t) \rangle \) we need to record every time step for short times and longer and longer time intervals for larger times so that the data points on the horizontal axis are evenly spaced on a log-log plot of \( \langle r^2(t) \rangle \) as shown in Fig. 14 for
The mean square displacement $\langle r^2 \rangle$ as function of time $t$ (in LJ units) for the Kob-Andersen model. The initial configuration is equilibrated at $T = 0.5$. The results are for a NVE simulation for $N = 1000$ and for 1000 time steps (with a Python program, recording $\langle r^2 \rangle$ every time step. After a steep increase for very small times, $\langle r^2 \rangle$ reaches a plateau. For the interpretation of this figure see Sec. VI B 3.

For $k_{\text{max}}$ print times, we solve Eq. (23) for $A$ for the case of $k = k_{\text{max}}$, when $(t_{k_{\text{max}}} / \Delta t) = n_{\text{MD}}$ is the total number of time steps. 

$$A = \left( \frac{n_{\text{MD}}}{(t_0/\Delta t)} \right)^{1/k_{\text{max}}} \quad (24)$$

The parameters needed for the calculation of $\langle r^2(t) \rangle$ can be set in Python for the example of $n_{\text{MD}} = 1000$, $(t_0/\Delta t) = 1.0$, and $k_{\text{max}} = 60$ with the following lines before the time step loop:

$$\text{kmsdmax} = 60$$
$$\text{t0msd} = 1.0$$
$$A = (\text{float(nMD)}/\text{t0msd})^{1.0/\text{float(kmsdmax)}}$$

The resulting files can be used to make a figure. However, as will become clear in Sec. VI B 3, if long simulation runs of the order of $10^7$ time steps are desired, saving in logarithmic time becomes necessary (see Sec. VI B 1). Logarithmic printing can be achieved by using the function$^{43,48}$ logfreq3 to define the print times.

The Python programs, KALJ_nve_msd_lin.py and KALJ_nve_msd_log.py, for this section are available in Ref. 41.

2. Mean square displacement with LAMMPS

The determination of the mean square displacement requires a computation during the simulation run. This computation can be done in LAMMPS with the compute command. (Another example of the compute command is in Sec. VI A 3 for the computation of $g(r)$.) In Eq. (21) the sum is over only $A$ or $B$ particles. Thus, in the LAMMPS input script we need to define these groups of atoms, which we then use for the following compute commands:

```plaintext
group A type 1
group B type 2
compute msdA A msd
compute msdB B msd
```

If we wish to save the mean square displacement every time step, or more generally with linear time averaging, we can use the fix ave/time command as described in Sec VI A 3. To write $\langle r_A^2 \rangle$ and $\langle r_B^2 \rangle$ for every time step into files msdA.data and msdB.data, respectively, we use the commands

```plaintext
fix msdAfix A ave/time 1 1 1 c_msdA[4] file msdA.data
fix msdBfix B ave/time 1 1 1 c_msdB[4] file msdB.data
```
tmsd with the variable command and then by using thermo_style and thermo (see Sec. IV C) to print the mean square displacements into the output file together with other scalar quantities which depend on time. The previous fix ave/time commands are replaced by

\[
\text{variable tmsd equal logfreq3(1,200,10000000)}
\]
\[
\text{variable tLJ equal step*dt}
\]
\[
\text{thermo_style custom v_tLJ c_msdA[4] c_msdB[4] pe etotal}
\]
\[
\text{thermo v_tmsd}
\]

We also defined the variable tLJ for the printing of \(t\) in LJ units (instead of time steps).49

Another way to obtain information logarithmic in time is to print all unwrapped particle positions during the LAMMPS simulation,

\[
\text{variable tmsd equal logfreq3(1,200,10000000)}
\]
\[
\text{dump msddump all custom 5000 posudump.*.data id x}
\]
\[
\text{dump_modify msddump sort id every v_tmsd}
\]

and then analyze the resulting posudump* files with Python or another programming language. The LAMMPS input files, inKALJ_nve_msd_lin, inKALJ_nve_msd_log, and inKALJ_nve_msd_logdumps, for this section are in Ref. 41.

3. Interpretation of mean square displacement

Figure 14 shows \(\langle r^2_A \rangle\) and \(\langle r^2_B \rangle\) obtained with thermo_style as described in Sec. VI B 2.

For very short times \(t\), we can approximate \(r_i(t) = r_i(0) + v_i(0) t\) and write Eq. (21) for small \(t\) as

\[
\langle r^2_A \rangle = \frac{1}{N_A} \sum_{i=1}^{N_A} |v_i(0) t|^2 = C t^2.
\]

(25)

We see that \(\ln \langle r^2_A \rangle = \ln C + 2 \ln t\), corresponding to a line with slope 2 as indicated by the dashed line at short times in Fig. 14.

For intermediate times \(\langle r^2_A \rangle\) reaches a plateau. This plateau is typical for glass formers at high enough density at which each particle is trapped in a cage formed by its neighboring particles. The smaller \(B\) particles reach a higher plateau. For long enough times each particle escapes its cage of neighbors and therefore \(\langle r^2_A \rangle\) increases. At very large times the dynamics of many successive escape events can be modeled as a random walk. For a random walk in \(d\) dimensions of step size \(a\) and an equal probability to step right or left, we have after \(N_{\text{step}}\) steps\(^{19}\)

\[
\langle (\Delta r)^2 \rangle = da^2 N_{\text{step}}.
\]

(26)

Equation (26) implies that \(\langle r^2_A \rangle \propto t\) and therefore a log-log plot yields a line of slope 1 as indicated by a blue dashed line at long times in Fig. 14.

VII. SUGGESTED PROBLEMS

1. Determine \(F_{i,x} = -dV/dx_i\).

2. Sketch the flow chart for the molecular dynamics simulation in Fig. 1 in more detail, specifying the order of the determination of positions, velocities, accelerations, and the application of periodic boundary conditions.

3. To derive Eq. (13), first determine \(\langle \frac{1}{2} m_i v_{i,x}^2 \rangle = \int_{-\infty}^{\infty} \frac{1}{2} m_i v_{i,x}^2 P(v_{i,x}) \, dv_{i,x}\) (the result is a special case of the equipartition theorem), and then obtain Eq. (13).

4. Write a program that places the \(N\) particles on lattice sites of a simple cubic lattice.

5. Outline the implementation of the acceleration function and program it with Python.
6. Use Fig. 3 to add to your Python program the loop over time steps and update positions and velocities.

7. Use Sec. II E 1 to add to your Python program the stochastic temperature bath algorithm.

8. Add to your NVE Python program the determination of the mean square displacement and save the results in a file.

9. For the $(3N + 1)$ generalized coordinates $\mathbf{q} = \{ (\mathbf{r}_i, s) \}$ determine the conjugate momenta $\mathbf{p}_i$ and $p_s$ and then the Hamiltonian.

10. To simulate a system of sheared bubbles, Durian$^{50,51}$ introduced a model such that bubble $i$ of radius $R_i$ interacts with bubbles $j$ of radius $R_j$ as

$$ V_{ij} = \frac{F_0}{2} \left[ 1 - \frac{r_{ij}}{(R_i + R_j)} \right]^2 $$

for all $j$ with $r_{ij} \leq (R_i + R_j)$. Determine the force $\mathbf{F}_{ij}$ on particle $i$ due to particle $j$. The solution is Eq. (1) in Ref. 51.

11. Compute the radial distribution functions $g_{\alpha\beta}$ for (a) temperatures $0.1 \leq T \leq 3.0$ and (b) densities $0.1 \leq N/L^3 \leq 2.0$. For each parameter set, first equilibrate before measuring $g_{\alpha\beta}$. Choose $\Delta t \leq 0.05$. In Python you can include a parameter in the name of the output file. For example, to use the temperature in the name we can write

```python
fileoutgofr= open("gofrAABBAB"+str(temperature)+".data",mode='w').
```

Interpret your results. Reference 11 includes $g_{\alpha\beta}$ for $0.466 \leq T \leq 5.0$ in Fig. 9 as well as a discussion of its behavior.

12. Compute the mean square displacement given in Eq. (22). Average over each type of particle separately, that is, compute

$$ \langle r_{\alpha}^2(t_w, t_w + t) \rangle = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} |\mathbf{r}_i(t_w + t) - \mathbf{r}_i(t_w)|^2. \quad (28) $$

Use several values of $t_w$. For example in a run with $n_{MD} = 1000$, use $t_w = 0, 10, 20, 50, 100, 500 \Delta t$. First do a NVE simulation as done in Sec. VIB. Use as initial configuration the provided file `initposvel` (for Python) or `initconf.T05seq.data` (for LAMMPS), which is well equilibrated at $T = 0.5$. Make a plot of $\langle r_{\alpha}^2(t_w, t_w + t) \rangle$ as a function of the time difference $t$ for different values of $t_w$. Interpret your results. Then, using the same initial configuration file, do a NVT simulation at $T = 0.2$. Make again a plot of $\langle r_{\alpha}^2(t_w, t_w + t) \rangle$ as a function of the time difference $t$ for different values of $t_w$. Compare your plots for the NVE run ($T = 0.5$) and the NVT run at $T = 0.2$ and interpret your results.

13. Determine the mean square displacement $\langle r_{\alpha}^2(t) \rangle$ of the KA-LJ system for $N = 1000$, $L = 9.4$ at temperatures $3.0, 2.0, 1.0, 0.8, 0.6, 0.55, 0.5$, and $0.475$ (a subset of the temperatures studied by Kob and Andersen.$^{11}$ Be sure to equilibrate the system sufficiently at each investigated temperature. As in Ref. 11 start at $T = 3.0$, apply a temperature bath for $n_{equi,T=3}$ time steps, continue with a NVE simulation run of $n_{equi,T=3}$ time steps, use the resulting configuration as initial configuration for the production run at $T = 3.0$ and also as initial configuration of the next lower temperature, $T = 2.0$. Apply the temperature bath at $T = 2.0$ for $n_{equi,T=2}$ time steps, followed by a NVE simulation run of $n_{equi,T=2}$ time steps, etc. For $\Delta t = 0.0025$ we recommend $n_{equi,T} = 10^5$ time steps for $T \geq 0.8$, $n_{equi,T} = 2 \times 10^6$ for $0.6 \geq T \geq 0.55$, and $n_{equi,T} = 5 \times 10^6$ for $0.5 \geq T \geq 0.475$. When doing a sequence of NVT and NVE runs, use the LAMMPS command `unfix` before applying the next `fix` command. To be able to apply logarithmic printing of the mean square displacement as in Sec. VIB2, you may also use the LAMMPS command `reset_timestep 0`. To ensure that the neighbor list is updated sufficiently frequently, use the LAMMPS command `neighbor 0.2 bin` (instead of `neighbor 0.3 bin`). If the Nosé temperature bath is used, we recommend for $T \leq 0.5$ to scale the velocities after the NVT run such that the total energy per particle of the NVE run is equal to the average total energy per particle $\langle E_{tot}/N \rangle$ during the NVT run with the (time) average taken near the end of the NVT run. Velocity scaling can be achieved with the LAMMPS command `velocity all scale $\{scaleEn\}`, where `$\{scaleEn\}$` corresponds to the temperature corresponding to the time averaged total energy per particle obtained for example with the LAMMPS commands.
14. Simulate the binary Lennard-Jones system in two dimensions. Compare your results with Fig. 2 of Ref. 11 and check your fits with a log-log plot of $r^2(t)$. Fitting can be done for example with Python or gnuplot. For each fit check the goodness of the fit by eye by plotting your data and the fitting curve. To ensure that Eq. (29) is a good approximation adjust the $t$-range used for the fitting accordingly. Use the resulting fit parameters to obtain $D_A(T)$ and $D_B(T)$. As done in Ref. 11, fit the predictions from mode-coupling theory

$$D_\alpha = A (T - T_c)^{\alpha}, \quad \text{for } \alpha \in \{A, B\}$$

and check your fits with a log-log plot of $D_\alpha$ as function of $(T - T_c)$. Another prediction for $D_\alpha(T)$ is the Vogel-Fulcher law

$$D_\alpha = C \exp[-B/(T - T_{VF})].$$ (31)

Compare your results with Fig. 3 of Ref. 11 which shows $6D_\alpha$ (not $D_\alpha^{\text{fit}}$) with diffusion unit $\sigma_{AA}/m_A$ (instead of $\sigma_{AA}/c_{AA}/m_A$ as for your results). The results are discussed in Ref. 11.

These commands need to be before the run command of the NVT run. Interpret the resulting mean square displacements and compare your results with Fig. 2 of Ref. 11, keeping in mind, that we use the time unit $\sqrt{m_A\sigma_{AA}^2/c_{AA}}$, whereas Kob and Andersen use the time unit $\sqrt{m_A\sigma_{AA}^2/48\epsilon_{AA}}$. For large times $t$ the mean square displacement depends on $t$ as (see Ref. 19)

$$\langle r^2_\alpha(t) \rangle = 2d_\alpha t,$$ (29)

where $D_\alpha$ is the diffusion constant for particles $\alpha \in \{A, B\}$. Determine $D_\alpha(T)$ by fitting Eq. (29) to $\langle r^2_\alpha(t) \rangle$. Fitting can be done for example with Python or gnuplot. For each fit check the goodness of the fit by eye by plotting your data and the fitting curve. To ensure that Eq. (29) is a good approximation adjust the $t$-range used for the fitting accordingly. Use the resulting fit parameters to obtain $D_A(T)$ and $D_B(T)$. As done in Ref. 11, fit the predictions from mode-coupling theory

$$D_\alpha = A (T - T_c)^{\alpha}, \quad \text{for } \alpha \in \{A, B\}$$

and check your fits with a log-log plot of $D_\alpha$ as function of $(T - T_c)$. Another prediction for $D_\alpha(T)$ is the Vogel-Fulcher law

$$D_\alpha = C \exp[-B/(T - T_{VF})].$$ (31)

Compare your results with Fig. 3 of Ref. 11 which shows $6D_\alpha$ (not $D_\alpha^{\text{fit}}$) with diffusion unit $\sigma_{AA}/m_A$ (instead of $\sigma_{AA}/c_{AA}/m_A$ as for your results). The results are discussed in Ref. 11.

14. Simulate the binary Lennard-Jones system in two dimensions. Choose the same density $N/L^2 = 1.204$ with $N = 1000$ and $L = 28.82$. Either start with random positions and velocities from a Maxwell-Boltzmann distribution, or use the input file configurations initposvel_2d_lammps.data (for LAMMPS) or initposvel_2d_python.data (for Python). Both are a result of simulations at $T = 0.2$ and are provided in Ref. 41. Do an NVE or NVT simulation for $T < 0.5$. Remember to replace 3 by 2 in Eq. (13) and adjust the variable shellvol in the Python computation of the radial distribution function. For LAMMPS follow the instructions in Ref. 53: you may set $z_i = v_{i,z} = 0.0$ with the LAMMPS command set atom 1000 z 0.0 vz 0.0. Make a scatter plot of the resulting particle positions and compute the radial distribution function. Compare with the three-dimensional results. An interpretation of your results is given in Ref. 54, which introduced the two-dimensional Kob-Andersen Lennard-Jones model with the particle ratio $N_A : N_B = 65 : 35$ instead of $80 : 20$.

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Appendix A: Hamiltonian Formalism for Nosé-Hoover Thermostat

We motivate Eqs. (14) and (15) using the Hamiltonian formalism. We follow the derivation given in Chapter 6 of Ref. 31 and present a shortened version here for simplicity. For a complete derivation see Refs. 29, 31, and 33.

We start with the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2} m_i (s \dot{r}_i)^2 - V(\{r_i\}) + \frac{1}{2} Q s^2 - X k_B T \ln s,$$ (A1)

where $X = dN$ (see Problem 9). The momenta are $p_{i,x} = \partial \mathcal{L}/\partial \dot{r}_{i,x} = m_i s^2 \dot{r}_{i,x}$ and similarly for $p_{i,y}$ and $p_{i,z}$. Therefore

$$p_i = m_i s^2 \dot{r}_i.$$ (A2)

Similarly, $p_s = \partial \mathcal{L}/\partial \dot{s} = Q \dot{s}$. We apply Hamiltonian mechanics using as generalized coordinates $q_k$ for $k = 1, 2, \ldots, 3N + 1$, where the first $3N$ values of $k$ label $q_k = r_{i,\mu}$ for particles $i = 1, \ldots, N$ and $\mu \in \{x, y, z\}$ and
We also define $q_{3N+1} = s$. The corresponding Hamiltonian is

$$H = \sum_{k=1}^{3N+1} \dot{q}_k p_k - \mathcal{L}$$

$$= \sum_{i=1}^{N} \frac{p_i^2}{2m_i s^2} + V (\{s\}) + \frac{p_s^2}{2Q} + X k_B T \ln s \quad (A3)$$

We follow Frenkel and Smit and switch to “real variables” $\dot{r}_i, \dot{p}_i, \dot{s}, \dot{p}_s$, $d\tilde{t}$, corresponding to a rescaling of the time:

$$\dot{r}_i = r_i$$

$$\dot{p}_i = \frac{p_i}{s} \quad (A9)$$

$$\dot{p}_s = \frac{p_s}{s}$$

$$\dot{s} = s$$

$$\dot{\tilde{t}} = \frac{dt}{d\tilde{t}}$$

We also define

$$\xi = \frac{ds}{d\tilde{t}} = \frac{d\ln \dot{s}}{d\tilde{t}} \quad (A14)$$

By using Eq. (A4) the Hamiltonian in terms of real variables can be expressed as

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V (\{r_i\}) + \frac{Q}{2} \dot{\xi}^2 + X k_B T \ln \dot{s} \quad (A18)$$

For the equations of motion (A15)–(A17) the constant of motion is given by Eq. (A18). Note that Eqs. (A16) and (A17) are the same as Eqs. (14) and (15) by replacing in Eqs. (A16) and (A17) $\dot{r}_i, \dot{s}, \dot{p}_i, \dot{p}_s$ with $r_i, s, p_i, p_s$; that is, we do a (confusing) change of notation for the sake of simplicity in Sec. II E 2.

**Appendix B: Fox-Anderson Integration of the Nosé-Hoover Equations**

We cannot directly apply the velocity-Verlet algorithm of Eqs. (7) and (8) to numerically integrate Eqs. (14) and (15) because the acceleration $a_i(t + \Delta t)$ depends on the velocity $v_i(t + \Delta t)$. We use instead the more general velocity Verlet integration technique of Fox and Andersen and apply it to the NVT Nosé-Hoover equations of motion. As described in Appendix A of Ref. 38 this technique is applicable when the form of the equations of motion is

$$\dot{x}(t) = f[x(t), \dot{x}(t), y(t), \dot{y}(t)] \quad (B1)$$

$$\dot{y}(t) = g[x(t), \dot{x}(t), y(t)] \quad (B2)$$

These equations can be expressed as [see Ref. 38, Eq. (A.4)]
\[ x(t + \Delta t) = x(t) + \dot{x}(t) \Delta t + 0.5 \dot{x}(t, x(t), y(t), \dot{y}(t)) (\Delta t)^2 \] (B3)

\[ y(t + \Delta t) = y(t) + \dot{y}(t) \Delta t + 0.5 g(x(t), x(t), y(t)) (\Delta t)^2 \] (B4)

\[ \dot{y}^{\text{approx}}(t + \Delta t) = \dot{y}(t) + 0.5 \left\{ g[x(t), x(t), y(t)] + g[x(t + \Delta t), x(t), y(t + \Delta t)] \right\} \Delta t \] (B5)

\[ \dot{x}(t + \Delta t) = \dot{x}(t) + 0.5 \left\{ f[x(t), \dot{x}(t), y(t), \dot{y}(t)] + f[x(t + \Delta t), \dot{x}(t + \Delta t), y(t + \Delta t), \dot{y}(t + \Delta t)] \right\} \Delta t \] (B6)

\[ \dot{y}(t + \Delta t) = \dot{y}(t) + 0.5 \left\{ g[x(t), \dot{x}(t), y(t)] + g[x(t + \Delta t), \dot{x}(t + \Delta t), y(t + \Delta t)] \right\} \Delta t. \] (B7)

As Fox and Andersen note, Eq. (B6) contains \( \dot{x}(t + \Delta t) \) on both sides. For the case of Nosé-Hoover equations, Eq. (B6) can be solved for \( \dot{x}(t + \Delta t) \). We write

\[ x = \{ r_i \} \] (B8)

\[ y = \ln s \] (B9)

\[ \dot{y} = \xi \] (B10)

\[ \ddot{y} = \dot{\xi} \] (B11)

\[ f = \left\{ \frac{F_i}{m_i} - \xi \dot{r}_i \right\} \] (B12)

\[ g = \frac{1}{Q} \left( \sum_{i=1}^{N} m_i \dot{r}_i^2(t) - X k_B T \right). \] (B13)

and see that Eqs. (B3)-(B5) correspond to

\[ r_i(t + \Delta t) = r_i(t) + r_i(t) \Delta t + 0.5 \left[ \frac{F_i(t)}{m_i} - \xi r_i(t) \right] (\Delta t)^2 \] (B14)

\[ \ln s(t + \Delta t) = \ln s(t) + \xi(t) \Delta t + \frac{1}{2Q} \left( \sum_{i=1}^{N} m_i \dot{r}_i^2(t) - X k_B T \right) (\Delta t)^2 \] (B15)

\[ \xi^{\text{approx}}(t + \Delta t) = \xi(t) + \frac{\Delta t}{Q} \left[ \sum_{i=1}^{N} m_i \dot{r}_i^2(t) - X k_B T \right]. \] (B16)

Equation (B6) corresponds to

\[ \dot{r}_i(t + \Delta t) = \dot{r}_i(t) + \frac{1}{2} \left\{ \left[ \frac{F_i(t)}{m_i} - \xi(t) \dot{r}_i(t) \right] + \left[ \frac{F_i(t + \Delta t)}{m_i} - \xi^{\text{approx}}(t + \Delta t) \dot{r}_i(t + \Delta t) \right] \right\} \Delta t, \] (B17)

which can be solved for \( \dot{r}_i(t + \Delta t) \)

\[ \dot{r}_i(t + \Delta t) = \left\{ \dot{r}_i(t) + \frac{\Delta t}{2} \left[ \frac{F_i(t)}{m_i} - \xi(t) \dot{r}_i(t) + \frac{F_i(t + \Delta t)}{m_i} \right] \right\} \left[ 1 + \frac{\Delta t}{2} \xi^{\text{approx}}(t + \Delta t) \right]^{-1}. \] (B18)

We use a Taylor series and keep terms up to order \( (\Delta t)^2 \) and obtain Eq. (A9) of Ref. 55:

\[ \dot{r}_i(t + \Delta t) = \dot{r}_i(t) + \frac{\Delta t}{2} \left\{ \frac{F_i(t) + F_i(t + \Delta t)}{m_i} - [\xi(t) + \xi^{\text{approx}}(t + \Delta t)] \dot{r}_i(t) \right\} \times \left[ 1 - \frac{\Delta t}{2} \xi^{\text{approx}}(t + \Delta t) \right]. \] (B19)

**Appendix C: Batch System**

This appendix is necessary only if the reader uses a supercomputer with a batch system. Often, supercom-
running of programs. Instead a batch system is used to provide computing power to many users who run many and/or long (hours–months) simulations. In this case an extra step is needed. That is, the user writes a batch-script, which contains the `mpirun` command, and submits a run via this script. Some of these script commands are supercomputer specific.

An example of a slurm batch-script is

```
#!/bin/bash
#SBATCH -p short # partition (queue)
#SBATCH -n 16 # number of cores

mpirun -np 16 lmp_mpi < inKALJ_nve > outLJnve
```

We can look at submitted jobs using `squeue` and if necessary, kill a submitted job with `scancel`.

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