I. INTRODUCTION

Superconducting quantum processors are one of the leading platforms in the race to achieve fault-tolerant quantum computation. Several important performance figures such as the qubit coherence times, gate and measurement fidelities have been steadily improving in the last two decades and they reached the error thresholds as required by the quantum error-correction protocols. However keeping the same performance for the individual components while scaling the circuits up remains a big engineering challenge.

Superconducting qubits are made of Josephson junctions which are lossless two-terminal circuit elements. Josephson junctions provide the non-linearity needed to obtain the qubit modes by allowing the “supercurrent” to flow between their terminals by tunneling while introducing minimal loss. The most popular superconducting qubit Transmon is obtained by shunting the Josephson junction with a relatively large capacitor. As such the Transmon qubit is a nonlinear oscillator with small anharmonicity to make it insensitive to charge fluctuations that cause dephasing. Transmon qubits are often modeled as multi-level quantum Duffing oscillators and they are designed to interact with each other and with their environment in the circuit-QED architecture.

In the circuit-QED architecture qubit interactions are mediated by the linear and passive microwave environment that the Josephson junctions are embedded in. Qubits are typically coupled to each other and to the control/measurement electronics by the help of microwave components constructed out of CPW transmission lines and their bare interactions with the internal modes of these structures are of the exchange energy type. When the strength of these interactions is smaller than the detuning of the qubits from the internal modes the system is said to be operated in the dispersive regime. It was shown that in the dispersive limit multi-mode circuit-QED systems can be described by an effective Hamiltonian of Duffing oscillators whose interactions with each other and with the drive lines are directly related to the entries of the impedance matrix defined between the qubit ports.

However when one reduces the effective Hamiltonian of the circuit to a qubit Hamiltonian by eliminating the higher levels of the Duffing oscillators non-zero ZZ-interaction (Ising type) terms are generated due to the finite and small anharmonicities of the qubits in addition to the main exchange interaction terms. These ZZ-interaction terms might be a nuisance for some two-qubit gate schemes such as the CR-gate which is the most popular microwave activated gate that creates the entanglement between the qubits via the ZX-interaction. A non-zero ZZ-term in the qubit Hamiltonian will cause spurious phase accumulations in the presence of spectator qubits and will lead to the loss of the gate fidelity.

The suppression of the ZZ-term in the qubit Hamiltonians has recently been studied actively to improve two-qubit gate fidelities. In a coupler design is proposed that consists of two arms one of which is frequency tunable and the coupler suppresses the ZZ-term by the interference of the interaction paths through each arm. It is shown that the system can be tuned to a point where the ZZ-interaction becomes zero but the effective exchange interaction $J$ remains finite. Originally a similar topology was used to make exchange interaction zero. Other approaches include the use of qubits of opposite anharmonicities and of tunable qubits to cancel the ZZ-interaction. More recently with a circuit topology similar to but using non-tunable elements only showed that it is possible to suppress the ZZ-term over a relatively large band while keeping a finite $J$-coupling rate that is useful for running the CR-gate. Although a source of cross-talk for the CR-gate ZZ-interactions can also mediate controlled-phase (CZ) gates.

In this paper we develop a method for the accurate estimation of the ZZ rates between Transmon type low anharmonicity qubits in the multi-mode circuit-QED. The frequency dependence of the ZZ rates is captured by the impedance entries connecting the qubit ports hence the high computational cost of diagonalizing multi-mode Hamiltonians is avoided making the microwave engineering of multi-mode quantum couplers streamlined.
In Section (III) we start with the summary of the theory [1] that our method is based on. The method is described in Section (III). The predictions of our theory are validated with numerical simulations on the example circuits in Section (IV). In Section (V) we compare the experimental data collected from multi-qubit devices to the ZZ values calculated with our method.

II. THE EFFECTIVE HAMILTONIAN

We start with an overview of the results in [1] as our calculations for the estimation of the ZZ rates in the next section will be performed in the reference frame given by the effective Hamiltonian derived in [1]. We assume that the quantum device under study consists of Transmon qubits connected to each other and to the read-out/control lines with the help of linear and passive microwave components such as transmission lines. For such systems the following effective Hamiltonian is derived in [1] in the dispersive limit of the circuit-QED:

\[ \hat{H} = \hat{H}_Q + \hat{H}_x + \hat{H}_R \]

(1)

\( \hat{H} \) is obtained by block-diagonalizing the initial system Hamiltonian in Eq. (17) of [1] by applying a Schrieffer-Wolff transformation. \( \hat{H}_Q \) collects the terms corresponding to the qubit subspace:

\[ \hat{H}_Q = \hat{H}_Q^D + \hat{H}_Q^I + \hat{H}_Q^V \]

where \( \hat{H}_Q^D \) is the diagonal part:

\[ \hat{H}_Q^D = \sum_{i=1}^{N} \omega_i \hat{b}_i \hat{b}_i + \frac{\delta_i}{2} \hat{b}_i \hat{b}_i \hat{b}_i \hat{b}_i - 1 \]

(2)

which is the quantum Hamiltonian of \( N \) Duffing oscillators of frequencies \( \omega_i \)'s, anharmonicities \( \delta_i \)'s and annihilation(creation) operators \( \hat{b}_i (\hat{b}_i)^\dagger \)'s for \( 1 \leq i \leq N \). The anharmonicity \( \delta_i \) of the qubit \( i \) is given by [1]

\[ \delta_i = -E_C^{(i)} \left( \frac{\omega_i}{\omega_0} \right)^2 = -\frac{E_C^{(i)}}{1 - 2E_C^{(i)}/\omega_i} \]

(3)

where \( E_C^{(i)} \) is the charging energy of the \( i \)-th qubit given by \( E_C^{(i)} = \frac{e^2}{2C_i} \), \( C_i \) is the total Transmon shunting capacitance of the qubit \( i \); for \( 1 \leq i \leq N \) and \( \omega_{J_i} = 1/\sqrt{L_{J_i}C_i} \); \( L_{J_i} \), being the bare junction inductance corresponding to the qubit \( i \).

Exchange couplings between qubits is given by the term \( \hat{H}_Q^I \)

\[ \hat{H}_Q^I = \sum_{i,j} J_{ij} (\hat{b}_i \hat{b}_j + \hat{b}_i^\dagger \hat{b}_j^\dagger) \]

(4)

where the following formula is derived in [1] for the exchange coupling \( J_{ij} \) between qubit modes \( i \) and \( j \)

\[ J_{ij} = -\frac{1}{4} \sqrt{\frac{\omega_i \omega_j}{L_i L_j}} \im \left[ \frac{Z_{ij}(\omega_i)}{\omega_i} + \frac{Z_{ij}(\omega_j)}{\omega_j} \right] \]

(5)

where \( Z_{ij} \) is the impedance entry connecting the qubit port \( i \) to the qubit port \( j \). Qubit ports are defined across the Josephson junctions of the Transmon qubits: port currents are the currents flowing through and the port voltages are the voltages developed across the Josephson junctions. \( L_i \) is the inductance of the \( i \)-th qubit and is related to the bare junction inductance \( L_{J_i} \) by \( L_i = L_{J_i}/(1 - \frac{2E_C^{(i)}}{\hbar \omega_i}) \).

The term \( \hat{H}_Q^V \) couples the qubits to the voltage drives and is given by

\[ \hat{H}_Q^V = \sum_{i=1}^{N} \sum_{d=1}^{N_D} \varepsilon_{id}(\hat{b}_i - \hat{b}_i^\dagger) V_d \]

(6)

where \( \varepsilon_{id} \) is the impedance entry (evaluated at the frequency \( \omega_i \) of the qubit \( i \)) connecting the qubit port \( i \) to the drive port \( p(d) \) corresponding to the voltage source \( V_d \). \( C(p(d)) \) is the capacitance shunting the drive port \( p(d) \) and \( Z_0 \) is the characteristic impedance of the drive line. \( \omega_d \) is the frequency of the voltage source \( V_d \) which is assumed to be a pure sinusoidal signal for simplicity. \( \theta_d = \frac{\pi}{2} - \arctan(\omega_d Z_0 C(p(d))) \) is the phase factor corresponding to \( V_d \). See [1] for the details about the circuit model (the multiport Cauer network) used to define the qubit and drive ports.

\( \hat{H}_R \) collects the terms corresponding to the modes of the linear passive environment that the qubits are embedded in

\[ \hat{H}_R = \hat{H}_R^D + \hat{H}_R^I + \hat{H}_R^V \]

(8)

where \( \hat{H}_R^D \) is the diagonal part given by

\[ \hat{H}_R^D = \sum_{k=1}^{M} \omega_{R_k} \hat{a}_k^\dagger \hat{a}_k + \frac{\chi_{R_k}^{(R)}}{2} \hat{a}_k^\dagger \hat{a}_k (\hat{a}_k^\dagger \hat{a}_k - 1) \]

(9)

where it is assumed that there are \( M \) internal modes with the annihilation(creation) operators \( \hat{a}_k^\dagger \)'s and frequencies \( \omega_{R_k} \)'s. We note that the internal modes have acquired anharmonicities \( \chi_{R_k}^{(R)} \)'s generated by the junction
non-linearities and described by the self-Kerr terms in the Eq. (9) above.

Similar to the case for the qubit modes the term $\hat{H}_R^l$ in Eq. (8) gives the exchange coupling $J_{kk'}$ between the internal modes $k$ and $k'$

$$\hat{H}_R^l = \sum_{k,k'} J_{kk'}(\hat{a}_k^\dagger \hat{a}_{k'}^\dagger + \hat{a}_k \hat{a}_{k'})$$

and the term $\hat{H}_R^V$ couples the internal modes to the voltage drives

$$\hat{H}_R^V = \sum_{k=1}^M \sum_{d=1}^{N_D} \varepsilon_{kd}(\hat{a}_k - \hat{a}_k^\dagger)V_d$$

$\hat{H}_x$ holds the Kerr-type coupling terms left after the block-diagonalization. In particular the terms generating the qubit state dependent frequency shifts $\chi_{ik}$'s in the readout resonator frequencies are contained in $\hat{H}_x$ given by

$$\hat{H}_x = \sum_{i=1}^N \sum_{k=1}^M \chi_{ik} \hat{b}_i^\dagger \hat{b}_i \hat{a}_{k}^\dagger \hat{a}_k$$

### III. Calculation of the ZZ-Interaction Rate

The exchange coupling $J_{ij}$ in Eq. (5) between qubits $i$ and $j$ can be diagonalized to get the following qubit Hamiltonian $[14]$ (with $\hbar = 1$)

$$\hat{H}_Q = -\frac{(\omega_{10} + \omega_{ZZ})/2}{2}\hat{Z}\hat{I} - \frac{(\omega_{01} + \omega_{ZZ})/2}{2}\hat{I}\hat{Z} + \omega_{ZZ}\hat{Z}\hat{Z}$$

where $\omega_{10} = \omega_i + J_{ij}^2/\Delta_{ij}$ and $\omega_{01} = \omega_j - J_{ij}^2/\Delta_{ij}$ are the dressed qubit frequencies, $\Delta_{ij} = \omega_i - \omega_j$ the detuning between the qubits $i$ and $j$ and the ZZ-interaction rate $\omega_{ZZ}$ is shown to be [14]

$$\omega_{ZZ} = \omega_{11} - \omega_{10} - \omega_{01}$$

$$= -\frac{2J_{ij}^2(\delta_i + \delta_j)}{(\Delta_{ij} + \delta_j)(\delta_j - \Delta_{ij})}$$

However the accuracy of this formula can be improved significantly if one inspects more closely the higher order terms that are often dropped with a Rotating-Wave Approximation and include in the treatment the terms that are rotating at much slower frequencies. Such terms will bring corrections to the couplings between the second excited states of the qubits $|20\rangle$, $|02\rangle$ and the $|11\rangle$ state as shown in the state diagram in Fig. 1. Note that we have $J_{3i} = J_{\delta i}$ in the formula in Eq. (15) since the higher order corrections are not taken into account in its derivation. To find the higher order corrections we borrow here the fourth order expansion of the cosine potentials of the Josephson junctions used in Eq. (115) of [14] after normal-ordering as

$$\hat{H}_\beta = -\sum_{pp'qq'}\beta_{pp'qq'}(6\hat{a}_{p}^\dagger \hat{a}_{p'}^\dagger \hat{a}_q \hat{a}_{q'} + 4\hat{a}_{p}^\dagger \hat{a}_{p'}^\dagger \hat{a}_q \hat{a}_{q'} + 4\hat{a}_p \hat{a}_{p'} \hat{a}_q \hat{a}_{q'})$$

This expansion was originally given in [20] in a frame different than our block-diagonal frame. Anharmonicity terms in the diagonal Duffing Hamiltonian in Eq. (2) and the $\chi$-shift term $\hat{H}_x$ in the Hamiltonian in Eq. (14) are both generated by $\hat{H}_\beta$ in Eq. (15). To obtain the corrections to the couplings between $|20\rangle$, $|02\rangle$ and $|11\rangle$ states as shown in Fig. 1 we need to consider the terms $\hat{b}_i^\dagger \hat{b}_i \hat{b}_j^\dagger \hat{b}_j$ and their Hermitian conjugates $\hat{b}_i^\dagger \hat{b}_i \hat{b}_j^\dagger \hat{b}_j$ respectively in Eq. (16) above ($\hat{b}$ denotes qubit operators in Eq. (16)). Note that these terms are rotating (if one goes to the interaction picture for example) at the frequency of qubit detuning $\Delta_{ij}$ which is much slower compared to the other terms. The weights of these terms are $\delta_i \sqrt{\omega_i/\omega_j} \alpha_{ij}$ and $\delta_j \sqrt{\omega_j/\omega_i} \alpha_{ji}$ respectively with

$$\alpha_{ij} = \frac{Z_{ji}^{-1}}{2(\omega_{10}^2 - \omega_{j0}^2)} \text{Im} [(\omega_i^2 - \omega_j^2)Z_{ij}(\omega_j) + \omega_i \omega_j Z_{ij}(\omega_i)]$$

where we defined the “cross characteristic impedance” $Z_{ij} = \sqrt{L_i/C_j}$. The expression above is an update to the expression for $\alpha_{ij}$ originally given in Eq. (121) of [1] and is calculated using the total coordinate transformation $\alpha = T \exp(S)$ of [1]. See Appendix VII A for the details of the derivation of the expression for $\alpha_{ij}$ in Eq. (17).
Finally we introduce another parameter that was originally defined in Eq. (120) of [1] due to $\alpha_{ii}$'s into our treatment all we need to do is to replace the charging energy $E_{C}^{(i)}$ in the expression for the anharmonicity in Eq. (26) with $\alpha_{ii}^{2}E_{C}^{(i)}$: i.e. we need to update the anharmonicity $\delta_{i}$ of the qubit $i$ given in Eq. (25) as follows

$$
\delta_{i} = \frac{\alpha_{ii}^{2}E_{C}^{(i)}}{1 - 2\alpha_{ii}^{2}E_{C}^{(i)}/\omega_{i}}
$$

The updated formula above for the anharmonicity $\delta_{i}$ allows us to capture the frequency dependent changes due to the existence of the high frequency modes coupled to the qubits. Refer to the Appendix (VII B) for the derivation of the expressions given in Eqs. (26) - (27) for $\alpha_{ii}$ and $\delta_{i}$.

IV. NUMERICAL EXAMPLES

We apply the method developed in the previous section for the direct calculation of the ZZ-rates to some simple circuits and compare the results to the exact diagonalization of the circuit Hamiltonians.

A. Single Mode Coupler

We start with the simple circuit shown in Fig. (2) consisting of two Transmon qubits coupled via a single mode shunt $LC$ resonator bus. With $C_{q} = 60 \text{ fF}$, $C_{c} = 5 \text{ fF}$ we fix the qubits at 5.0 GHz and 5.2 GHz by adjusting the values of $L_{j_{1}}$ and $L_{j_{2}}$ accordingly and plot the ZZ-interaction rate as a function of the bus resonator frequency $f_{b}$ in Fig. (3) using three different methods. The "naive" method of calculating the ZZ-rate is to plug-in the value of $J_{12}$ calculated using Eq. (11) into (10). However we observe in Fig. (3) that there is significant discrepancy between this method (green curve labeled "Naive" in the legend) and the exact value of the ZZ-rate obtained with the numerical diagonalization of the circuit Hamiltonian (blue curve). And this discrepancy stays at a significant level even if we go deep into the dispersive region; i.e. as the bus frequency increases. However the accuracy is improved considerably when we apply the updated formula for the ZZ-rate in (24) by using the values

$$
\omega_{ZZ} = 2 \frac{J_{i}^{2} (\delta_{j} - \Delta_{ij}) + J_{j}^{2} (\delta_{i} + \Delta_{ij})}{(\Delta_{ij} + \delta_{i})(\Delta_{ij} - \delta_{j})}
$$

Figure 2. Single Mode Bus Circuit Diagram.
of $J_1$ and $J_2$ defined in Eqs. 18, 19. This is plotted as “Z-method-0” in Fig. 3 in orange color. The inset plot informs us about the exchange coupling strength $J_{12}$ between the qubits as a function of the bus frequency calculated using the impedance formula in Eq. 19 for the same set of circuit parameters.

The accuracy of our calculation of the ZZ-rate can be improved further by adding the “cross-Kerr” contribution in Eq. 25 and the corrections due to the $\alpha_{ii}$ coefficients in Eqs. 26-27 into our treatment. The results are shown in Fig. 4 where we observe that the ZZ-rate calculated with the addition of the cross-Kerr term (brown curve) underestimates slightly the exact ZZ-values (blue) whereas with the addition of $\alpha_{ii}$ corrections we obtain a very good agreement (dashed red) with true ZZ-values (blue) all the way down to $f_b = 5.6$ GHz which is only 400 MHz away from one of the qubits.

### B. Two-Mode Coupler with Two $J_{12}$ Zeros

Here we apply our method to a coupler consisting of two finite frequency modes. We start by artificially creating the following trans-impedance response

$$Z_{12}(\omega) = \frac{A(\omega_{p1}^2 - \omega^2)(\omega_{p2}^2 - \omega^2)}{\omega(\omega_{p1}^2 - \omega^2)(\omega_{p2}^2 - \omega^2)}$$  \hspace{1cm} (28)

with three poles at DC, $\omega_{p1} = 4.0$ GHz, $\omega_{p2} = 6.25$ GHz and two zeros at $\omega_{z1} = 4.5$ GHz and $\omega_{z2} = 5.5$ GHz. The coefficient $A$ is set to the value of $-7.97 \times 10^{10}$. Assuming Transmon shunt capacitances of 65 fF we obtain the results in Fig. 5.

### C. Multi-Mode ZZ Cancellation Coupler

In this section we study an example of a multi-mode coupler designed to cancel the ZZ-interaction rate while keeping a finite $J$-coupling strength [20]. The coupler topology consists of two branches: one direct coupling branch of the form of a short segment of $\lambda/2$ CPW transmission line resonator and another branch of a short segment of CPW shunted to ground in the middle through a...
Circuit parameters are \( C_{12} = 36 \, \text{fF}, C_{1g} = C_{2g} = 46 \, \text{fF}, C_{1c}^{(1)} = 8 \, \text{fF}, C_{1c}^{(2)} = 12 \, \text{fF} \). CPW transmission lines have lengths \( l_1 = 1.0 \, \text{mm}, l_2 = 0.5 \, \text{mm}, l_3 = 3.75 \, \text{mm} \) and center traces of width 10 \, \text{um} with 6 \, \text{um} gap to ground which gives the characteristic impedances \( Z_1 = Z_2 = Z_3 = 50 \, \Omega \).

In Fig. (7) we plot ZZ-interaction rate for qubits coupled by the circuit in Fig. (6). ZZ-rate admits two zeros at \( \sim 4.75 \, \text{GHz} \) and \( \sim 5.04 \, \text{GHz} \) and remains small in magnitude in between which is the typical qubit band.

\( \lambda/2 \) CPW resonator. These two branches run in parallel and are connected to the same Transmon qubit pads as shown in Fig. (6). The \( \lambda/4 \) section generates a mode at \( \sim 6.3 \, \text{GHz} \). In Fig. (7) we plot ZZ-interaction rate for qubits coupled by the circuit in Fig. (6). ZZ-rate admits two zeros at \( \sim 4.75 \, \text{GHz} \) and \( \sim 5.04 \, \text{GHz} \) and remains small in magnitude in between which is the typical qubit band.

\begin{align*}
\text{ZZ vs } f_2 & \quad \text{(Impedance Formula)} \\
\text{Exact ZZ} & \\
\text{ZZ (Z-method)} & \end{align*}

\begin{align*}
\|Z_{12}\| (\text{Impedance Formula}) & \\
\text{Exact ZZ} & \\
\text{ZZ (Z-method)} & \end{align*}

Figure 8. Scatter plot of the measured ZZ values in the multi-qubit chip F609. C where qubit frequencies are spread over the band 4.8 – 5.0 GHz. The least squares fit in orange line has equation \( y = 1.015x_2 - 0.388 \). The standard deviation is \( \sigma = 3.7 \, \text{kHz} \). For each data point \( x \)-value corresponds to the measured value whereas the \( y \)-value is the ZZ-rate calculated using the impedance method. ZZ-rates measured from both directions are included in the plot for each pair.

V. EXPERIMENTAL RESULTS

In this section we test the validity of the analytical methods we developed in the previous sections on the measurement data collected from real devices. In Figs. (8) and (9) we compare the ZZ values predicted by our method to the measured values from two different multi-qubit devices with qubit frequencies spread over two different frequency bands: on chip F609. C qubit frequencies fall in the band 4.8 – 5.0 GHz whereas qubits on chip F608. C lie between 5.0 – 5.2 GHz. Each chip contains 27 Transmon qubits connected by the ZZ-cancellation type couplers introduced in [20]. We studied this type of coupler in Section (IV.C) as an example to compare the results obtained by our analytical method to the exact ZZ-rate values computed by the full-diagonalization of the circuit Hamiltonian. Note that the ZZ-cancellation effect is not observed since the cancellation band is missed in these chips. The coupler topology stays the same across the chip in the sense that all couplers consist of two short \( \lambda/2 \) CPW arms in parallel with one of the arms shorted to ground in the middle with a \( \lambda/4 \) resonator CPW section. However, the length and the characteristic impedance of the arms and the resonance frequency of the \( \lambda/4 \) resonator differ creating 6 different coupler responses which allows us to validate of our analytical methods over a larger ensemble with the good agreement seen in Figs. (8) and (9). The calculated ZZ-values are obtained using the \( Z_{12} \) output of the 2.5D microwave simulations evaluated at the qubit frequencies. Self-capacitances \( C_{J_i}'s \) of the Josephson junctions are extracted (from the anharmonicity data) to be 3.0 \, \text{fF} \) for the chip F609. C and 2.2 \, \text{fF} \) for the chip F608. C as the chip F609. C had junctions with larger area.
Figure 9. Scatter plot of the ZZ-values for the Chip F608_C. ZZ-values are larger by almost an order of magnitude compared to the chip F609_C in Fig. 5 since qubit frequencies are higher (in the band 5.0 - 5.2 GHz). The least squares fit line is given by \( y = 0.994x + 3.71 \). The standard deviation is \( \sigma = 28.5 \text{ kHz} \). Again ZZ-rates measured from both directions are included in the plot for each pair when available.

VI. CONCLUSION

We described a method for the accurate calculation of the ZZ-rates between superconducting qubits in the multi-mode circuit-QED. By relating the ZZ-interaction rates directly to the impedance entries connecting the qubit ports our method allows streamlined analysis and design of the qubit-qubit couplers with the help of microwave simulations. In particular this opens a path for the design of higher-order multi-pole ZZ-cancellation couplers by avoiding computationally intensive multi-mode quantum Hamiltonian diagonalization.

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VII. APPENDIX

A. Derivation of the expressions for \( \alpha_{ij} \), \( \alpha_{ii} \) and \( \delta_i \)

The expressions given in Eqs. (18) and (20) for the coefficients \( \alpha_{ij} \) and \( \alpha_{ii} \) can be derived by expanding the total coordinate transformation \( \mathbf{\alpha} = \mathbf{T} \exp(\mathbf{S}) \) given in [1] to second-order in \( \mathbf{S} \). Here \( \mathbf{\alpha} \), \( \mathbf{T} \) and \( \mathbf{S} \) are all \((N + M) \times (N + M)\) matrices where \( N \) is the number of qubits and \( M \) is the number of the internal modes. The coordinate transformation \( \mathbf{T} \) is defined in Eq. (21) of [1] as

\[
\mathbf{T} = \left( \begin{array}{cc} \mathbf{C}_0^{1/2} \mathbf{R}^T & \mathbf{0}_{M \times N} \\ \mathbf{0}_{N \times M} & \mathbf{1}_{M \times M} \end{array} \right)
\]  

where \( \mathbf{C}_0 \) is the diagonal matrix of Transmon capacitances \( (C_1, \ldots, C_N) \) and \( \mathbf{R} = [r_{ik}] \) is the \( N \times M \) turns-ratio matrix of the multiport Belevitch transformers in the canonical multiport Cauer network used in [1]. The matrix \( \mathbf{S} \) defines the Schrieffer-Wolff transformation \( \exp(\mathbf{S}) \) that block-diagonalizes the matrix \( \mathbf{M}_1 \) defined in Eq. (23) of [1] by

\[
\mathbf{M}_1 = \mathbf{T} \mathbf{C}_0^{-1/2} \mathbf{M}_0 \mathbf{C}_0^{-1/2} \mathbf{T}^T = \left( \begin{array}{cc} \Omega_j^2 & \Omega_j \mathbf{C}_0^{-1/2} \mathbf{R}^T \\ \mathbf{R} \mathbf{C}_0^{-1/2} \Omega_j^2 & \mathbf{C}_0^{-1/2} \mathbf{R}^T \end{array} \right) \]

where \( \Omega_j \) is the \( N \times N \) diagonal matrix holding the qubit frequencies whereas the \( M \times M \) matrix \( \mathbf{R}_N \) corresponds to the subspace of the internal modes. \( \mathbf{M}_0 \) is the diagonal matrix with entries \((1/L_1, \ldots, 1/L_N, 1/L_{R1}, \ldots, 1/L_{R_M})\) where \( L_i \)'s are qubit inductances for \( 1 \leq i \leq N \) and \( L_{R_k} = 1/\omega_R^2 \) are the inductances of the internal modes for \( 1 \leq k \leq M \).

After expanding \( \exp(\mathbf{S}) \) to second-order in \( \mathbf{S} \) using for example Eqs. (B.12) and (B.15) in [27] we obtain the expressions in Eqs. (19) and (20) for the coefficients \( \alpha_{ij} \) and \( \alpha_{ii} \) from the \((i,j)\) and \((i,i)\) entries, respectively, of the upper-left \( N \times N \) subsector of \( \mathbf{T} \exp(\mathbf{S}) \) corresponding to the qubit subspace.

The updated expression in Eq. (27) for the anharmonicity \( \delta_i \) can be derived by noting that the capacitance re-scaling performed by the diagonal matrix \( \mathbf{C}_0^{-1/2} \) can be lumped into the total coordinate transformation \( \mathbf{\alpha} \) which would re-normalize the diagonal capacitance \( C_i \) by \( 1/\alpha_{ii}^2 \) hence the charging energy \( E_{C}^{(i)} \) gets updated by the prefactor \( \alpha_{ii}^2/E_{C}^{(i)} \), i.e. \( E_{C}^{(i)} \rightarrow \alpha_{ii}^2 E_{C}^{(i)} \). One then obtains

\[
\delta_i = \frac{\alpha_{ii}^2 E_{C}^{(i)}}{1 - 2 \alpha_{ii}^2 E_{C}^{(i)} / \omega_i}
\]

for the anharmonicity using the updated charging energy in the formula given in Eq. (3).

B. Derivation of the expressions for \( J_{ij} \) and \( J_{ii} \)

The expression given in Eq. (13) for \( J_{ij} \) is obtained by noting that the term \( \hat{b}_i^{**} \hat{b}_j^\dagger \) and its Hermitian conjugate \( \hat{b}_j^\dagger \hat{b}_i \) which have weight \( \delta_i \sqrt{2} \alpha_{ij} \) in the expansion \( \hat{H}_B \) given in Eq. (10) (\( \hat{b} \) denotes qubit operators in Eq. (10) ) couple the state \(|20\rangle \) to the state \(|11\rangle \) with strength \( \sqrt{2} \alpha_{ij} \). This brings an additive correction to the main coupling \( \sqrt{2} J_{ij} \) between the states \(|20\rangle \) and \(|11\rangle \) generated by the exchange coupling \( J_{ij} \) given by the expression in Eq. (5) between qubits \( i \) and \( j \); i.e. \( J_{ij} = J_{ij} + \delta_i \sqrt{2} \alpha_{ij} \). After re-arrangement we arrive at the expression given in Eq. (15) for \( J_{ij} \). A similar analysis applied to the terms \( \hat{b}_j^{**} \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_i \) (with weight \( \delta_j \sqrt{2} \alpha_{ij} \)) for the coupling between the states \(|02\rangle \) and \(|11\rangle \) gives the expression in Eq. (19) for \( J_{ij} \).
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