Dynamic universal relations at nonequilibrium

Shi-Guo Peng

State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics,
Innovation Academy for Precision Measurement Science and Technology,
Chinese Academy of Sciences, Wuhan 430071, China
(Dated: May 25, 2022)

We show that many-body systems driven away from equilibrium obey dynamic virial theorem, which imposes a universal intrinsic constrain on energy dynamics. Its straightforward application to ideal quantum liquids gives rise to dynamic pressure relation, that describes a simple dynamic relation between the pressure and energy. The out-of-equilibrium dynamics of quantum correlations is entirely governed by Tan's contact. Of particular interest, they capture key nonequilibrium features of many-body systems, such as the energy conversion and particle flow. These universal relations are exemplified in several dynamic processes of cold atoms, including free expansion, Efimovian expansion, and monopole oscillation. Observable consequences are discussed, for instance, the evolution of cloud size and profound scale-invariant dynamics. Our results provide fundamental understanding of generic behaviors of interacting many-body systems at nonequilibrium, and are readily examined in experiments with ultracold atoms.

Introduction—Investigating nonequilibrium dynamics of strongly-interacting many-body systems is of fundamental importance and remains an open challenge in modern physics, while it is broadly relevant to many phenomena in universe, ranging from the evolution of neutron stars, procedure of chemical reactions to complex living systems in biology. Discovering universal features in quantum systems at nonequilibrium, irrelevant to the microscopic detail of objects studied, is a long-standing challenge to date and attracts a great deal of attention in both theory and experiment. Ultracold atoms, as a clean, controllable and versatile quantum systems, provide an unprecedented platform for exploration of a wide variety of phenomena in many-body physics, ranging from thermodynamic equilibrium to out-of-equilibrium dynamics [1–6]. Remarkably, ultracold atomic systems take unique advantages for studying nonequilibrium dynamics within experimentally resolved intrinsic time scales (typically milliseconds), holding promising opportunities to test universality of many-body dynamics far from equilibrium [7–13].

A set of universal relations in cold atoms, connected by a simple contact parameter, have been discovered [14–16]. These relations simply follow from the short-range correlation of two-body physics and provide remarkable understanding of profound properties of interacting many-body systems. To date, an impressive amount of experimental and theoretical efforts have been devoted to confirm universal relations and explore their important consequences [17–24]. While most of them are focused on properties of many-body systems at equilibrium, universal relations at nonequilibrium are rarely studied and still remain elusive.

In this letter, two dynamic universal relations are found for interacting many-body systems driven away from equilibrium by either time-dependent external potential or time-dependent interatomic interactions. These are favorite ways to study out-of-equilibrium dynamics in experiments of cold atoms, such as expansion dynamics [25, 26] and interaction quench process [7–10, 13]. One of universal relations at nonequilibrium is the dynamic virial theorem, which reveals a deep insight into the precise energy conversion relation

\[
E(t) - 2E_{ho}(t) = \frac{1}{4} \frac{d^2 I(t)}{dt^2} - \frac{\hbar^2 C(t)}{8\pi ma(t)},
\]

that imposes an intrinsic constrain on energy dynamics. Here, \(m\) is the atomic mass, \(\hbar\) is Planck’s constant, and \(a(t)\) is the scattering length generally varying in time. \(E_{ho}(t)\) is the energy corresponding to the external harmonic potential, \(I(t) \equiv \langle mv^2 \rangle\) is the moment of inertia of systems, and \(C(t)\) is Tan’s contact [14–16]. In the classical (or high-temperature) limit, the kinetic energy dominates the internal energy. The dynamic virial theorem immediately implies an interesting non-damping monopole oscillation with twice the trapping frequency in a perfectly spherical trap, and systems never reach thermal equilibrium. This long-predicted phenomenon by Boltzmann’s equation has recently been observed by JILA’s group in a thermal Bose gas [27]. In the quantum (or low-temperature) limit, the interaction between particles comes into play, and shifts the frequency of monopole oscillation [28–31]. In the unitarity limit, the systems display scale invariance. The dynamic virial theorem requires a unitary gas to follow exactly the same dynamics of an ideal gas [28–32]. At equilibrium, the dynamic virial theorem simply recovers that of [16] at finite interaction strength, and further reduces to the well-known equilibrium result of \(E = 2E_{ho}\) for unitary and ideal gases [33].

Another universal relation found in this work is the pressure relation, which naturally connects the microscopic and macroscopic properties of interacting many-body systems. It provides a fundamental relation between instantaneous pressure \(P(t)\) and internal energy
for ideal quantum liquids without viscosity at nonequilibrium,

\[ P(t) = \frac{2}{3} E_{\text{internal}}(t) + \frac{\hbar^2 C(t)}{12\pi ma(t)}, \tag{2} \]

where \( E_{\text{internal}}(t) = E(t) - E_{\text{ho}}(t) - E_{\text{flow}}(t) \) is the temporal internal energy, \( E_{\text{flow}}(t) = \langle \frac{mv^2}{2} \rangle \) is the energy corresponding to the atom flow with velocity field \( v(r,t) \), and \( P(t) \) is the total pressure defined as the integral of local pressure over space, i.e., \( P(t) = \int d\mathbf{r} p(r,t) \). Although the form of dynamic pressure relation \([2]\) seems quite similar to that for equilibrium systems \([10]\), they have remarkable distinctions. An additional energy results from the atom flow comes into the relation, which characterizes the crucial dynamic feature of pressure relation at nonequilibrium. Besides, the pressure and all kinds of energies defined here are total ones by integrating corresponding local quantities over the space, regarding the inhomogeneity of systems driven out-of-equilibrium even if in the absence of external potential. In the classical limit, in which the interatomic interactions are negligible, Eq. (2) simply reduces to the form for a classical fluid \([3]\). The dynamic pressure relation gives rise to a generic dynamic feature for ideal quantum fluids by taking into account of interactions. At resonant interactions, the evolution of systems shows the scale invariance and leads to \( P(t) = 2E_{\text{internal}}(t)/3 \). The scale-symmetry-breaking behavior away from resonance is obviously governed by contact, i.e., \( \Delta P(t) \propto C(t)/a(t) \), which has already been observed in a two-component Fermi gas during free expansion \([23]\). At equilibrium, the dynamic pressure relation simply recovers that of \([10]\) with vanishing flow. It further reduces to the well-known result \( P = 2E_{\text{internal}}/3 \) for ideal and unitary gases, which has been verified experimentally to high precision \([32]\).

**Dynamic virial theorem** —To prove dynamic virial theorem, let us consider a system consisting of \( N \) atoms (either bosons or fermions) in a harmonic trap. The Hamiltonian takes the form of \( \hat{H} = \hat{H}_0 + \hat{V}_{\text{int}} \), in which \( \hat{H}_0 = \sum_{j=1}^{N} \left[ \frac{\mathbf{p}_j^2}{2m} + \hat{V}_{\text{ho}}(\mathbf{r}_j; t) \right] \) is the single-particle Hamiltonian, and \( \hat{V}_{\text{int}} = \sum_{ij} \hat{U} \left( \mathbf{r}_{ij}; a(t) \right) \) is the interatomic interactions with \( \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \). Here, we consider the most general case that the harmonic potential \( \hat{V}_{\text{ho}} \) and interatomic interactions both vary in time, i.e., with time-dependent trapping frequency \( \omega(t) \) and s-wave scattering length \( a(t) \). The evolution of moment of inertia \( I(t) = \sum_{j} m_0 r_j^2 \) is governed by Heisenberg equation. Then we have \( \dot{I} = 2 \langle \dot{\mathbf{D}} \rangle \), where \( \dot{I} \equiv dI/dt \), and \( \dot{\mathbf{D}} = \sum_{j=1}^{N} (\mathbf{r}_j \cdot \mathbf{p}_j + \mathbf{p}_j \cdot \mathbf{r}_j)/2 \) is the generator of scale transformation named the dilatation operator \([22, 36, 37]\). By further taking the second derivative with respect to time, i.e., \( \ddot{I} = 2 \left\langle \left[ \dot{D}, \dot{H} \right] \right\rangle /i\hbar \), we obtain

\[ \frac{d^2 I}{dt^2} = 4 \left\langle \dot{H} \right\rangle - 8 \left\langle \dot{V}_{\text{ho}} \right\rangle - 4 \left\langle \dot{V}_{\text{int}} \right\rangle + \frac{2}{i\hbar} \left\langle \left[ \dot{D}, \dot{V}_{\text{int}} \right] \right\rangle. \tag{3} \]

To proceed, we notice that the two-body interaction \( \hat{U} \left( \mathbf{r}_{ij}; a \right) \), under the scale transformation, has the property of \( e^{-i\epsilon \hat{D}/\hbar} \hat{U} \left( \mathbf{r}_{ij}; a \right) e^{i\epsilon \hat{D}/\hbar} = e^{2i\epsilon} \hat{U} \left( \mathbf{r}_{ij}; e^a \right) \) for infinitesimal \( \epsilon \). Expanding both sides of this equation up to the first-order term of \( \epsilon \), we obtain the commutation relation of \( \left[ \dot{D}, \dot{U} \right] \), which in turn gives

\[ \frac{1}{i\hbar} \left\langle \left[ \dot{D}, \dot{V}_{\text{int}} \right] \right\rangle = 2 \left\langle \dot{V}_{\text{int}} \right\rangle + a \left\langle \frac{\partial V_{\text{int}}}{\partial a} \right\rangle. \tag{4} \]

Inserting Eq. (4) into Eq. (3) and combining with Hellmann-Feynman theorem and Tan’s relation \([15]\), i.e.,

\[ \left\langle \frac{\partial V_{\text{int}}}{\partial a} \right\rangle = \frac{\partial E}{\partial a} = \frac{\hbar^2 C}{4\pi ma^2}. \tag{5} \]

we finally arrive at Eq. (1) with \( E(t) = \left\langle \hat{H} \right\rangle \) and \( E_{\text{ho}}(t) = \left\langle \hat{V}_{\text{ho}} \right\rangle \).

**Pressure relation** —The dynamic pressure relation \([22]\) is a straightforward deduction of dynamic virial theorem. For an ideal quantum liquid without viscosity, its dynamic behavior is well described by hydrodynamic theory, which is based on the equation of continuity

\[ \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{6} \]

and on the Euler equation

\[ m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} m \mathbf{v}^2 + \mu (n) + V_{\text{ho}} \right) = 0, \tag{7} \]

where \( n (r,t) \) and \( \mathbf{v} (r,t) \) are respectively the atom number density and velocity field, and \( \mu (n) \) is the local chemical potential in a harmonic trap \( V_{\text{ho}} = m\omega^2 r^2/2 \). From these two hydrodynamic equations, we have

\[ \frac{\partial^2 n}{\partial t^2} = \nabla \cdot \left\{ \mathbf{v} \nabla \cdot (n\mathbf{v}) + \frac{n}{m} \nabla \left[ \frac{1}{2} m \mathbf{v}^2 + \mu (n) + V_{\text{ho}} \right] \right\}. \tag{8} \]

Multiplying Eq. (8) by \( m \mathbf{v}^2 \) and integrating over the whole space, we obtain

\[ \frac{d^2 I}{dt^2} + 2\omega^2 I - 2 \left( \frac{m}{2} \mathbf{v}^2 \right) - 6 \int d\mathbf{r} p(r,t) = 0, \tag{9} \]

where \( p(r,t) \) is the local pressure related to the local chemical potential by Gibbs-Duhem relation \( dp = nd\mu \). Using the dynamic virial theorem and defining the total pressure \( P(t) = \int d\mathbf{r} p(r,t) \), we finally arrive at the pressure relation \([22]\) with \( E_{\text{flow}} = \langle m\mathbf{v}^2/2 \rangle \), corresponding to the energy of atom flow at nonequilibrium.
In classical mechanics—Let us first discuss the corresponding dynamic universal relations in classical mechanics, where the interaction effect is negligible, and thus only a single-particle problem is relevant. The moment of inertia of a classical particle is defined as $I(\mathbf{r}) = m\mathbf{r}^2$, and then we have $\dot{I} = 2(\dot{\mathbf{p}} \cdot \mathbf{r} + \mathbf{p} \cdot \dot{\mathbf{r}})$. As we have $p = F$, the force acting on the particle, and the kinetic energy $E_{kin} = p \cdot \dot{r}/2$, we obtain $\dot{I} = 2(F \cdot r + 2E_{kin})$.

In harmonic potentials, the force acting on particle is the negative gradient of potential energy, i.e., $F = -\nabla E_{ho}$, which immediately gives $F \cdot r = -2E_{ho}$. Therefore, we obtain $E(t) - 2E_{ho}(t) = I(t)/4$, the dynamic virial theorem that depicts the instantaneous energy dynamics of a classical harmonic oscillator. The pressure relation, i.e., $P(t) = 2E_{internal}(t)/3$, is then easily derived for an ideal classical fluid, following a similar route as that in hydrodynamic regime. Comparing with Eqs. (1) and (2), we find that the dynamics of correlations for a quantum liquid is fully governed by universal contact parameter.

Free expansion of quantum gases—Free expansion, simply releasing atoms from the trap, provides crucial information on both equilibrium and dynamic properties of ultracold atomic gases. It is widely used in experiments of cold atoms, such as the time of flight. One of direct applications of dynamic virial theorem is to calculate the release energy, i.e., $E_{rel} = E(t)$. Since the release energy is well understood in weakly-interacting and unitary gases, it is inversely convenient to verify the validity of dynamic virial theorem. To this end, the knowledge of evolution of Tan’s contact as well as that of moment of inertia is required. As an example, let us consider free expansion of a two-component Fermi gas, initially prepared at equilibrium in the trap. We evaluate the evolution of Tan’s contact and moment of inertia of the system in the framework of hydrodynamic theory. To solve the hydrodynamic equations, we adopt the scaling form of time-dependent density profile, i.e., $n(\mathbf{r}, t) = n_0(\mathbf{r})/b^3(t)$, where $n_0(\mathbf{r})$ is the equilibrium density profile initially in the trap. The time dependence of $n(\mathbf{r}, t)$ is entirely determined by the scaling factor $b(t)$ [38, 39]. Here, we consider a spherically trapped Fermi gas before release for simplicity, and the approach below is easily generalized to that in anisotropic traps. The atomic cloud size $\langle r^2 \rangle_0(t)$ during expansion is related to the initial size $\langle r^2 \rangle_0$ by scaling factor $b(t)$ as $\langle r^2 \rangle(t) = b^2(t) \langle r^2 \rangle_0$, which in turn gives

$$\frac{d^2 I}{dt^2} = \frac{4}{\omega_0^2} b^2(b^2 + b b') E_{ho}^{(0)},$$

and $E_{ho}^{(0)}$ is the initial potential energy. From the hydrodynamic equations [31] and (7), we find that the scaling factor $b(t)$ obeys

$$\dot{b} - \frac{\omega_0^2}{b^4} \int dm r^2 n_0(\mathbf{r}) \frac{\partial \mu_b}{\partial \mu_b} \big|_{n=n_0} = 0,$$

(11)

where $\omega_0$ is the initial trapping frequency and $\partial \mu_b = \partial \mu/\partial n$.

In the weakly-interacting Bardeen-Cooper-Schrieffer (BCS) limit, Eq. (11) becomes [38]

$$\dot{b} - \frac{\omega_0^2}{b^4} b^3 + \frac{3}{2} \chi (g) \omega_0^2 (b^3 - b^{-4}) = 0,$$

(12)

where $\chi (g) = E_{int}^{(0)} / E_{ho}^{(0)}$ is the ratio of interaction energy $E_{int}^{(0)}$ and potential energy $E_{ho}^{(0)}$ initially in the trap, and $g = 4\pi \hbar^2 a / m$ is the interaction strength. The contact $C(t)$ during expansion may be evaluated according to the local density approximation (LDA) by making the adiabatic ansatz [10], i.e., $C(t) = \int d\mathbf{r} I(\mathbf{r}, t)$ with local contact density $I(\mathbf{r}, t) \approx 4\pi n^2(\mathbf{r}, t) \alpha^2$ [12], which in turn gives $C(t) = 4\pi m a E_{int}^{(0)} / h^2 b^3$. With all these results in hands, we easily obtain the release energy according to the dynamic virial theorem, i.e., $E_{rel} = E_{ho}^{(0)} \left[ \frac{\omega_0^2}{2} (b^2 + b b') - \chi (g) b^{-3}/2 \right]$. From Eq. (12), we find $\omega_0^2 (b^2 + b b') = 1 - \chi (g) (1 - b^{-3})/2$ [41], thus the release energy becomes $E_{rel} = [1 - \chi (g)/2] E_{ho}^{(0)}$ in the weakly-interacting limit, and further reduces to the well-known result $E_{rel} = E_{ho}^{(0)}$ in the BCS limit [2].

In the Bose-Einstein-Condensation (BEC) limit, Eq. (11) becomes a simple form of $\dot{b} - \omega_0^2 b^{-4} = 0$, which is nothing than that of weakly-interacting bosons [12, 13]. The evolution of contact is again determined by $C(\mathbf{t}) = \int d\mathbf{r} I(\mathbf{r}, t)$ with local contact density $I(\mathbf{r}, t) \approx 4\pi n(\mathbf{r}, t) / \alpha + \pi^2 n^2(\mathbf{r}, t) a m_\alpha$ [13], where $a_m \approx 0.6 a$ is the scattering length between molecules [14]. Then we have $C(t) = C_0 + C_m b^{-3}$, in which $C_0 = 4\pi N / a$ is the contribution from binding energy of tightly bound molecules that is not released during expansion. $C_m = \int d\mathbf{r} \pi n^2(\mathbf{r}) a m_\alpha$ is the initial contact corresponding to the interaction energy between molecules, i.e., $E_{m,int}^{(0)} = \hbar^2 C_m / 4\pi m a$. From dynamic virial theorem, the release energy is easily calculated, i.e., $E_{rel} \equiv E + N c_b / 2 = (b^{-3} + 2) E_{ho}^{(0)} / 3 - b^{-3} E_{m,int}^{(0)} / 2$, where $c_b = \hbar^2 / ma^2$ is the binding energy of molecules. By further using the equilibrium virial theorem for weakly-interacting bosons initially in the trap $E_{m,int}^{(0)} = 2 E_{ho}^{(0)} / 3$ [42], we obtain the release energy $E_{rel} = 2 E_{ho}^{(0)} / 3$, the well-known result in the BEC limit [2].

In the unitarity limit, the release energy is rather easy to calculate, since the system obeys an exact scale-invariant evolution, which yields $b(t) = \sqrt{1 + \omega_0^2 t^2}$ [32, 37]. Besides, the second term on the right-hand side of Eq. (11) vanishes in the unitarity limit. Combining
with Eq. (10), the dynamic virial theorem simply gives the well-known result $E_{\text{rel}} = E_{\text{ho}}^{(0)} \frac{P}{I}$.

One another application of free expansion is to test scale invariance of strongly-interacting quantum gases as well as important consequences resulted from scale-symmetry breaking $[25, 40, 47]$. From the dynamic universal relations, we find that the scale-symmetry-breaking behaviors are entirely controlled by contact, for instance, $\Delta P(t) \propto C(t)/a$. In the noninteracting limit, the contact tends to zero as $a \to \pm \infty$, we have $\Delta P = 0$ as well. In both cases, the pressure follows the same dynamic behavior $P = 2E_{\text{internal}}/3$ as pointed out in $[25, 33]$. Such scale-invariant dynamics can be tested according to the evolution of cloud size, which is governed by dynamic virial theorem. If defining $\tau^2 (t) \equiv m \left( \langle r^2 \rangle - \langle r^2 \rangle_0 \right)/2E_{\text{ho}}^{(0)}$, we easily obtain the equation satisfied by $\tau^2 (t)$

$$\frac{d^2}{dt^2} \tau^2 (t) = 2 + \frac{\hbar^2 |C (t) - C_0|}{4\pi maE_{\text{ho}}^{(0)}},$$

where $C_0$ is the initial contact before release. In $[25]$, $\tau^2 (t)$, as an identification of scale invariance, is measured in the free expansion and obeys $\tau^2 (t) = t^2$ for a scale-invariant Fermi gas. We find that the behavior resulted from the breaking of scale symmetry is governed by the dynamics of contact during expansion. In the weakly-interacting limit, we obtain $d^2 \tau^2 / dt^2 \approx 2 + 0.29278k_{F\text{I}} \alpha \left[ (1 + \omega_0^2 t^2)^{-3/2} - 1 \right]$, while $d^2 \tau^2 / dt^2 \approx 2 + 2.4668 \left( k_{F\text{I}} \alpha \right)^{-1} \left[ (1 + \omega_0^2 t^2)^{-1/2} - 1 \right]$ in the strongly-interacting limit, where $k_{F\text{I}}$ is the Fermi wave number at trap center for an ideal gas. Generally, an accurate estimation of contact is needed to depict the evolution of $\tau^2 (t)$ in the free expansion, for example, by using high-temperature virial expansion $[25, 48, 49]$.

**Efimovian expansion**—Scale-invariant quantum gases display a fancy scaling expansion dynamics in time-dependent harmonic traps, i.e., $\omega (t) \sim 1/\sqrt{I}$ with variation rate $\lambda$. This phenomenon is termed as “Efimovian expansion” $[26]$. Here, we show that such profound expansion dynamics is inherently governed by dynamic virial theorem. By further taking the derivative of dynamic virial theorem with respect to $t$, we obtain

$$\frac{1}{4} \frac{d^3 I}{dt^3} = \frac{dE}{dt} - 2 \frac{dE_{\text{ho}}}{dt} + \frac{\hbar^2}{8\pi ma} \frac{dC}{dt}.$$

According to Hellmann-Feynman theorem, we have $dE/dt = -I/\lambda t^3$. Combining with $E_{\text{ho}} = I/2\lambda t^2$, we arrive at

$$\frac{d^3 I}{dt^3} + \frac{4}{\lambda t^2} \frac{dI}{dt} - \frac{4}{\lambda^3 t^3} I = \frac{\hbar^2}{2\pi ma} \frac{dC}{dt}.$$

For noninteracting and resonant-interacting (scale-invariant) Fermi gases, the right-hand side of Eq. (15) vanishes, which recovers that of $[26, 50]$. The evolution of cloud size demonstrates a temporal scaling expansion behavior. Away from the scale-invariant point at finite interaction strength, the expansion dynamics resulted from scale-symmetry breaking is simply characterized by the evolution of contact $[51, 53]$. In the strongly-interacting limit, the contact can approximately be estimated as before, i.e., $C (t) \approx C_0 / b (t)$ with $b^2 (t) = I / I_0$. Here, $I_0$ is the initial moment of inertia and $C_0 = 256\pi \alpha N k_{F\text{I}} / 35\xi_B^{1/4}$ is the initial contact $[48, 49]$, where $N$ is the total atom number, $\alpha \approx 0.12$ and $\xi_B \approx 0.37$ are universal parameters. The evolution of cloud size (or moment of inertia) during Efimovian expansion near resonant interaction is shown in Fig. 1.

**Monopole oscillation**—The study of low-energy elementary excitations is a subject of primary importance in many-body physics. It is achieved, for example, by abruptly disturbing the external trap and exerting the oscillation of system around its equilibrium. In the follows, let us discuss the monopole oscillation of cold atoms in a spherical trap, which we find is governed by dynamic virial theorem. In the classical (high-temperature) limit, the interaction effect is negligible. The dynamic virial theorem immediately provides a constraint on the evolution of cloud size, i.e.,

$$\frac{d^2}{dt^2} \langle r^2 \rangle + 4\omega_0^2 \left( \langle r^2 \rangle - \langle r^2 \rangle_0 \right) = 0,$$

where $\langle r^2 \rangle_0 = E/m_0 \omega_0^2$ is the initial size of cloud at equilibrium. We find that the cloud size follows a simple monopole oscillation with twice the trapping frequency around its equilibrium position $\langle r^2 \rangle_0$. This long-predicted phenomenon by Boltzmann’s equation in classical systems has recently been observed in a thermal spherically trapped Bose gas $[27]$. 

![Figure 1.](image-url)
In the quantum (low-temperature) limit, the interactions between atoms come into play. Let us take a two-component Fermi gas as an example. At equilibrium before exerting the oscillation, the total energy of system is simply given by virial theorem, and we have $E = -\frac{\hbar^2 c}{12 m a} \chi$. When the BCS limit, we have $E = 2E^{(0)}_{ho}$ in the unitarity limit, and $E \approx 5/3E^{(0)}_{ho} - N\epsilon_b/2$ in the BEC limit and $\epsilon_b$ is the binding energy of molecules. At time $t = 0$, the monopole oscillation is excited due to a slight disturbing on trap. We easily acquire the evolution of cloud size according to the dynamic virial theorem. In the BCS limit, we have

$$\frac{d^2 y}{dt^2} + 4\omega_0^2 y - \omega_0^2 \chi (g) y^{-3/2} - 2\omega_0^2 E^{(0)}_{ho} = 0,$$  \hfill (17)

where $y \equiv \langle r^2 \rangle / \langle r^2 \rangle_0$ is the ratio of temporal cloud size to initial size at equilibrium. By linearizing Eq.(17) around equilibrium, i.e., $y = 1 + \delta y$, we obtain

$$\frac{d^2 \delta y}{dt^2} + \left[ 4 + 3\chi (g) \right] \omega_0^2 \delta y = 0.$$  \hfill (18)

We find that the frequency of monopole oscillation is shifted to $\omega_M \approx \omega_0 \sqrt{4 + 3\chi (g)}/2$ by interatomic interactions. It further reduces to $\omega_M = 2\omega_0$ in the non-interacting limit. Following a similar procedure, we obtain the frequency of monopole oscillation $\omega_M = 2\omega_0$ in the unitarity limit and $\omega_M = \sqrt{5}\omega_0$ in the BEC limit, consistent with the well-known results of unitary gases and Bose gases in the presence of mean-field interactions.

**Conclusions** —We have obtained the dynamic virial theorem for interacting many-body systems at out-of-equilibrium. For ideal quantum liquids, a universal relation between the pressure and energy has been discovered, namely the dynamic pressure relation. Meanwhile the dynamic universal relations have been exemplified in several dynamic processes of cold atoms, in which observable consequences are discussed, such as the evolution of cloud size and scale-invariant dynamics. Complementary to those at equilibrium, they present a complete set of fundamental relations that elegantly characterize both out-of-equilibrium dynamics and equilibrium properties of interacting many-body systems (as being summarized in Table I). These dynamic universal relations provide a fundamental understanding of generic behaviors of interacting many-body systems at nonequilibrium, and are readily examined in experiments with ultracold atoms.

We would like to thank Ran Qi for inspiring discussion on the derivation of dynamic virial theorem. We are particularly grateful to Xi-Wen Guan for his helpful suggestions. This work is supported by the National Natural Science Foundation of China under Grant No.11974384 and the Natural Science Foundation of Hubei Province under Grant No. 2021CFA027.

|                | Equilibrium |
|----------------|-------------|
| Sweep theorem  | $\frac{dE}{dt} = \frac{\hbar^2 c}{12 m a} \chi$ |
| Virial theorem | $E = -\frac{\hbar^2 c}{12 m a} \chi$ |
| Pressure relation | $P = \frac{\hbar^2 c}{12 m a} \chi$ |

Table I. Comparison of universal relations for interacting many-body systems at equilibrium and at nonequilibrium.

---

*[pengshiguowipm.ac.cn](mailto:pengshiguowipm.ac.cn)*

[1] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Reviews of Modern Physics 80, 885 (2008).

[2] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of ultracold atomic fermi gases, Reviews of Modern Physics 80, 1215 (2008).

[3] I. Bloch, J. Dalibard, and S. Nascimbene, Quantum simulations with ultracold quantum gases, Nature Physics 8, 267 (2012).

[4] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, Reviews of Modern Physics 83, 863 (2011).

[5] T. Langen, R. Geiger, and J. Schmiedmayer, Ultracold atoms out of equilibrium, Annual Review of Condensed Matter Physics 6, 201 (2015).

[6] J. Eisert, M. Friesdorf, and C. Gogolin, Quantum many-body systems out of equilibrium, Nature Physics 11, 124 (2015).

[7] P. Makotyn, C. E. Klauss, D. L. Goldberger, E. A. Cornell, and D. S. Jin, Universal dynamics of a degenerate unitary Bose gas, Nature Physics 10, 116 (2014).

[8] C. Eigen, J. A. P. Glieden, R. Lopes, N. Navon, Z. Hadzibabic, and R. P. Smith, Universal scaling laws in the dynamics of a homogeneous unitary Bose gas, Physical Review Letters 119, 250404 (2017).
[9] C. Eigen, J. A. P. Glidden, R. Lopes, E. A. Cornell, R. P. Smith, and Z. Hadzibabic, Universal prethermal dynamics of Bose gases quenched to unitarity, Nature 563, 221 (2018).

[10] M. Prüfer, P. Kunkel, H. Strobel, S. Lammt, D. Linne-mann, C. M. Schmied, J. Berges, T. Gasenzer, and M. K. Oberthaler, Observation of universal dynamics in a spinor Bose gas far from equilibrium, Nature 563, 217 (2018).

[11] B. Ko, J. W. Park, and Y. Shin, Kibble-Zurek universality in a strongly interacting Fermi superfluid, Nature Physics 15, 1227 (2019).

[12] S. Dyke, A. Hogan, I. Herrera, C. C. N. Kuhn, S. Hoinka, B. Ko, J. W. Park, and Y. Shin, Kibble-Zurek universality in an isolated one-dimensional Bose gas far from equilibrium, Nature 563, 225 (2018).

[13] M. A. Baranov and D. S. Petrov, Low-energy collective excitations in a superfluid trapped Fermi gas, Physical Review A 62, 041601 (2000).

[14] D. Guery-Odelin, Mean-field effects in a trapped gas, Physical Review A 66, 033613 (2002).

[15] F. Werner and Y. Castin, Unitary gas in an isotropic harmonic trap: Symmetry properties and applications, Physical Review A 74, 053604 (2006).

[16] J. E. Thomas, J. Kinast, and A. Turlapov, Virial theorem and universality in a unitary Fermi gas, Physical Review Letters 95, 120402 (2005).

[17] R. P. Feynman, R. B. Leighton, and M. Sands, The Feynman lectures on physics (Volume II), (Addison-Wesley, United States, 2005).

[18] M. J. H. Ku, A. T. Sommer, L. W. Cheuk, and M. W. Zwierlein, Revealing the superfluid lambda transition in the universal thermodynamics of a unitary Fermi gas, Science 335, 563 (2012).

[19] L. P. Pitaevskii and A. Rosch, Breathing modes and hidden symmetry of trapped-atom in two dimensions, Physical Review A 55, R853 (1997).

[20] J. Maki and F. Zhou, Quantum many-body conformal dynamics: Symmetries, geometry, conformal tower states, and entropy production, Physical Review A 100, 023601 (2019).

[21] C. Menotti, P. Pedri, and S. Stringari, Expansion of an interacting Fermi gas, Physical Review Letters 89, 250402 (2002).

[22] H. Hu, A. Minguzzi, X. J. Liu, and M. P. Tosi, Collective modes and ballistic expansion of a Fermi gas in the BCS-BEC crossover, Physical Review Letters 93, 190403 (2004).

[23] C. L. Qu, L. P. Pitaevskii, and S. Stringari, Expansion of harmonically trapped interacting particles and time dependence of the contact, Physical Review A 94, 063635 (2016).

[24] We may set \( y = \tilde{b} \) and have \( \dot{\tilde{b}} = \dot{y} = ydy/d\theta \). Then Eq. (12) becomes a differential equation of \( \theta \) with respect to \( \tilde{b} \), which can easily be solved by using initial conditions \( \tilde{b}(\theta = 1) = 0 \).

[25] Y. Castin and R. Dum, Bose-Einstein condensates in time dependent traps, Physical Review Letters 77, 5315 (1996).

[26] Y. Kagan, E. L. Surkov, and G. V. Shlyapnikov, Evolution of a Bose-condensed gas under variations of the confining potential, Physical Review A 54, R1753 (1996).

[27] D. S. Petrov, C. Salomon, and G. V. Shlyapnikov, Weakly bound dimers of fermionic atoms, Physical Review Letters 93, 090404 (2004); D. S. Petrov, C. Salomon, and G. V. Shlyapnikov, Scattering properties of weakly bound dimers of fermionic atoms, Physical Review A 71, 012708 (2005).

[28] L. P. Pitaevskii and S. Stringari, Bose-Einstein condensation and superfluidity, Oxford University Press (2016).

[29] R. Saint-Jalm, P. C. M. Castillo, E. Le Cerf, B. Bakkali-Hassani, J. L. Ville, S. Nascimbene, J. Beugnon, and J. Dalibard, Dynamical symmetry and breathers in a two-dimensional Bose gas, Physical Review X 9, 021035 (2019).

[30] J. Maki, S. Z. Zhang, and F. Zhou, Dynamics of strongly interacting Fermi gases with time-dependent interactions: Consequence of conformal symmetry, Physical Review Letters 128, 040401 (2022).

[31] X. J. Liu, H. Hu, and P. D. Drummond, Virial expansion for a strongly correlated Fermi gas, Physical Review...
Letters 102, 160401 (2009).

[49] X. J. Liu, Virial expansion for a strongly correlated fermi system and its application to ultracold atomic fermi gases, Physics Reports-Review Section of Physics Letters 524, 37 (2013).

[50] The moment of inertia $I(t)$ defined here is related to the cloud size defined in [28],

$$\langle \hat{R}^2 \rangle (t) = \left\langle \sum_{i=1}^{N} r_i^2 / N \right\rangle (t)$$

with total atom number $N$, as $I(t) = mN \langle \hat{R}^2 \rangle (t)$.

[51] S. E. Gharashi and D. Blume, Broken scale invariance in time-dependent trapping potentials, Physical Review A 94, 063639 (2016).

[52] R. Qi, The Efimovian Expansion in Scale Invariant Quantum Gases (The First Beijing-TokyoWorkshop on Ultra-cold Atomic Gases, Beijing, China, April 13, 2016).

[53] Z.-Y. Shi, Efimovian Expansion in Scale Invariant Quantum Gases (International Conference on Few-body Physics in Cold Atomic Gases, Beijing, China, April 16, 2016).