Perfect extensions of de Morgan algebras

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Dedicated to the memory of Dr. Milan Demko (1963–2021).

Abstract. An algebra \( A \) is called a perfect extension of its subalgebra \( B \) if every congruence of \( B \) has a unique extension to \( A \). This terminology was used by Blyth and Varlet [1994]. In the case of lattices, this concept was described by Grätzer and Wehrung [1999] by saying that \( A \) is a congruence-preserving extension of \( B \). Not many investigations of this concept have been carried out so far. The present authors in another recent study faced the question of when a de Morgan algebra \( M \) is perfect extension of its Boolean subalgebra \( B(M) \), the so-called skeleton of \( M \). In this note a full solution to this interesting problem is given. The theory of natural dualities in the sense of Davey and Werner [1983] and Clark and Davey [1998], as well as Boolean product representations, are used as the main tools to obtain the solution.

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1. Introduction

Blyth and Varlet introduced MS-algebras as algebras abstracting de Morgan and Stone algebras in [2] (see also [3]). In [1] we presented, for a special class of MS-algebras, an analogue of Grätzer’s problem [8, Problem 57], which was formulated for distributive p-algebras. Compared to the solution for the distributive p-algebras by Katriňák in [10], we provided in [1] a much simpler though not very descriptive solution, and a short and elegant proof of it. That is why we then considered in [1] an analogue of the Grätzer problem in the...
special case when the MS-algebra $L$ we investigated was a perfect extension of its largest Stone subalgebra. This case was subsequently reduced to the property that a de Morgan subalgebra $M$ of $L$ was a perfect extension of its Boolean skeleton $B(M)$.

The description of general de Morgan algebras $M$ which are perfect extensions of their Boolean subalgebras $B(M)$ turns out to be an interesting problem in its own right, and the answer has not, until now, been known. In this paper we give a satisfactory solution to this problem. We show that a de Morgan algebra $M$ is a perfect extension of its Boolean skeleton $B(M)$ if and only if $M$ is a Boolean product of copies of the four-element subdirectly irreducible de Morgan algebra and its three- and two-element subalgebras, and that this is equivalent to a simple and nice condition on the natural dual space of $M$.

2. Preliminaries

By a de Morgan-Stone algebra (or MS-algebra) we mean an algebra $L = (L; ∨, ∧, ^0, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; ∨, ∧, 0, 1)$ is a bounded distributive lattice and $^0$ is a unary operation such that for all $x, y \in L$, $x \leq x^{00}$, $(x \land y)^0 = x^0 \lor y^0$, $1^0 = 0$.

An important subvariety of MS-algebras is that of de Morgan algebras, which satisfy the additional identity $x = x^{00}$. For a de Morgan algebra $L$ its skeleton is the Boolean subalgebra $B(L) = \{x \in L | x \lor x^0 = 1\}$.

We also mention other distinguished subvarieties of MS-algebras are those of Kleene algebras, which are de Morgan algebras satisfying the identity $(x \land x^0) \lor y \lor y^0 = y \lor y^0$, the subvariety of Stone algebras characterized by the identity $x \land x^0 = 0$, and the subvariety of Boolean algebras determined by the identity $x \lor x^0 = 1$.

It is also appropriate to recall here some other concepts. An algebra $A$ satisfies the Congruence Extension Property (briefly (CEP)) if for every subalgebra $B$ of $A$ and every congruence $\theta$ of $B$, $\theta$ extends to a congruence of $A$. An algebra $A$ is said to be a perfect extension of its subalgebra $B$, if every congruence of $B$ has a unique extension to $A$ (see [3, page 30]). We notice that in the literature such an $A$ is also called a congruence-preserving extension of $B$; we refer to a paper by Grätzer and Wehrung [9] where this concept is used in case of lattices. Of course, if $A$ is a perfect extension of $B$, then the congruence lattices $\text{Con}(A)$ of $A$ and $\text{Con}(B)$ of $B$ are isomorphic.

The classes of distributive lattices and of MS-algebras are known to satisfy the (CEP) [8, 3]. However, very little seems to be known when in these or other classes of algebras, an algebra $A$ is a perfect extension of its subalgebra $B$. In [9], Grätzer and Wehrung proved that every lattice with more than one element has a proper congruence-preserving extension. In [1] we showed that a so-called principal MS-algebra $L$ is a perfect extension of its largest Stone subalgebra $L_S = \{x \in L | x^0 \lor x^{00} = 1\}$ if and only if the de Morgan subalgebra $L^{00} = \{x \in L | x = x^{00}\}$ of $L$ is a perfect extension of its Boolean skeleton $B(L)$. 
This leads to a natural problem, which will be formulated and addressed in the next section.

3. Problem of perfect extensions for de Morgan algebras

In this section we focus on the following problem:

**Problem.** Describe de Morgan algebras $M$ which are perfect extensions of their Boolean skeletons $B(M)$.

An answer to this problem has not been known. The main result of our paper is a satisfactory solution to this problem. We characterize de Morgan algebras that are perfect extensions of their Boolean skeletons both algebraically and via dual spaces. For the latter we find it convenient to employ the theory of natural dualities as described in [7, 5].

It is well known that the variety of de Morgan algebras is equal to $\text{ISP}(M_1)$, where $M_1$ is the four-element subdirectly irreducible de Morgan algebra, and I, S and P denote the well-known operators of isomorphic copies, subalgebras and products, respectively (for example, see [5, 4.3.15]). In the natural duality for the variety $\text{ISP}(M_1)$ based on $M_1$, the dual spaces $X = (X; f, \preceq, \tau)$ are the usual Priestley spaces $(X; \preceq, \tau)$ endowed with an order-reversing homeomorphism $f$ of order two [6]. Every de Morgan algebra is isomorphic to the set of all morphisms from its dual space $X$ into the alter ego of the algebra $M_1$, which is the structure $\tilde{M}_1 = (\{0, a, b, 1\}; f, \preceq, \tau)$ (see [5, Theorem 4.3.16]). The alter ego is the ordered set $(\{0, a, b, 1\}; \preceq)$ with $a$ and $b$ being the top and the bottom elements, respectively, and the two atoms 0, 1 (see Figure 1). The homeomorphism $f$ is defined by $f(0) = 0$, $f(1) = 1$, $f(a) = b$, $f(b) = a$, and $\tau$ is the discrete topology. The set of morphisms inherits the de Morgan algebra structure from the power algebra $M_1^X$.

In our characterization we also use the concept of a Boolean product (see [4]). A Boolean product of an indexed family $(A_y)_{y \in Y}$ of algebras, for a non-empty set $Y$, is a subdirect product $A \leq \prod_{y \in Y} A_y$, where the set $Y$ can be endowed with a Boolean space topology so that

(i) the set $E(a, b) = \{y \in Y \mid a(y) = b(y)\}$ is clopen for every $a, b \in A$;
(ii) if $a, b \in A$ and $K \subseteq Y$ is clopen, then $a \upharpoonright K \cup b \upharpoonright (Y \setminus K) \in A$.

Figure 1. The de Morgan algebra $M_1$ and its alter ego.
The condition (ii) is sometimes called the patchwork property.

To shorten the proof of the main theorem, we first prove the following technical lemma.

**Lemma 3.1.** Let $\mathbb{X} = (X; f, \preceq, \tau)$ be a Priestley space endowed with an order-reversing homeomorphism $f$ of order two. Choose a subset $Y \subseteq X$ satisfying the following conditions:

(a) every $x \in X$ with $f(x) = x$ belongs to $Y$;
(b) if $x \neq f(x)$, then exactly one of these two elements belongs to $Y$.

Then the system

$$\sigma = \{Z \subseteq Y \mid Z \cup f(Z) \text{ is open in } \mathbb{X}\}$$

defines a Boolean topology on $Y$.

**Proof.** We leave it for the reader to check that $\sigma$ is indeed a topology on $Y$. (But notice that $Y$ is not a topological subspace of $\mathbb{X}$.) Now we check the equality

$$(Y \setminus Z) \cup f(Y \setminus Z) = X \setminus (Z \cup f(Z))$$

for every $Z \subseteq Y$. First, let $t \in Y \setminus Z$. We need to show that $t \in X \setminus (Z \cup f(Z))$, that is $t \notin f(Z)$. For a contradiction, let $t = f(u)$, $u \in Z$. Then $u, f(u) \in Y$, which is only possible if $u = f(u) = t$, which contradicts $t \notin Z$. Next, let $t \in f(Y \setminus Z)$. Then $t = f(u)$ for some $u \in Y \setminus Z$. The injectivity of $f$ implies $t \notin f(Z)$. If $t \in Z$, then $u, f(u) \in Y$, hence $u = f(u) = t \in Z$, a contradiction. So, $t \in X \setminus (Z \cup f(Z))$.

Conversely, let $t \in X \setminus (Z \cup f(Z))$. Then $t \notin f(Z)$ and $t = f(f(t))$ imply $f(t) \notin Z$. If $t \in Y$, then clearly $t \in Y \setminus Z$. If $t \notin Y$, then $f(t) \in Y \setminus Z$, so $t \in X \setminus (Z \cup f(Z))$.

As a consequence we obtain that $Z \subseteq Y$ is closed (clopen) if and only if $Z \cup f(Z)$ is closed (clopen) in $\mathbb{X}$. Let us check that the space $Y$ is Boolean. Let $x, y \in Y$, $x \neq y$. Then also $x \neq f(y)$, as either $f(y) = y$ or $f(y) \notin Y$.

There exists a clopen set $U \subseteq X$ with $x \in U$ and $y, f(y) \notin U$. Since $f$ is a topological homeomorphism, the sets $f(U)$ and $U \cup f(U)$ are also clopen. Let $V := Y \cap (U \cup f(U))$. Then $V \cup f(V) = U \cup f(U)$.

Indeed, from $V \subseteq U \cup f(U)$ it follows $f(V) \subseteq f(U \cup f(U)) = f(U) \cup U$. Conversely, if $u \in U \cup f(U)$, then either $u \in Y$ and hence $u \in V$, or $f(u) \in Y$ and hence $f(u) \in V$ and $u \in f(V)$.

Hence, $V$ is clopen in $Y$ and $x \in V$. Since $f(y) \notin U$, we immediately have $y \notin U \cup f(U)$, so $y \notin V$. Thus, every two points of $Y$ can be separated by a clopen set. To check the compactness of $Y$, let $\{Z_i \mid i \in I\}$ be an open cover of $Y$. Then $\{Z_i \cup f(Z_i) \mid i \in I\}$ is an open cover of $\mathbb{X}$. Since this space is compact, there exists a finite subset $I_0 \subseteq I$ such that $\{Z_i \cup f(Z_i) \mid i \in I_0\}$ covers $\mathbb{X}$. Then it is easy to check that $\{Z_i \mid i \in I_0\}$ covers $Y$.

**Theorem 3.2.** Let $\mathbf{M}$ be a de Morgan algebra. The following are equivalent:

1. $\mathbf{M}$ is a perfect extension of its Boolean skeleton $B(\mathbf{M})$.
2. $\mathbf{M}$ is a Boolean product of copies of $\mathbf{M}_1$ and its subalgebras $\{0, a, 1\}$ and $\{0, 1\}$. 
(3) In the dual space $\mathbb{X}$ of $\textbf{M}$, $x \preceq y$ implies $x = y$ or $x = f(y)$ for all $x, y \in X$.

Proof. (2)$\implies$(1) Let $\textbf{M} \leq \Pi_{y \in Y} \textbf{A}_y$ be a Boolean product, where each $\textbf{A}_y$ is equal to $\textbf{M}_1$ or $\{0, a, 1\}$ or $\{0, 1\}$. The elements of $\textbf{M}$ have the form $u = (u(y))_{y \in Y}$. The skeleton $B(\textbf{M})$ consists exactly of those elements of $\textbf{M}$ whose every coordinate is 0 or 1. For every clopen set $N \subseteq Y$ we define

$$s_N(y) = \begin{cases} 0 & \text{if } y \in N, \\ 1 & \text{if } y \notin N. \end{cases}$$

Since constant 0 and constant 1 are elements of $\textbf{M}$, the patchwork property implies that $s_N \in M$ for every $N$. Clearly, $s_N \in B(\textbf{M})$, so let $t_N$ denote its complement. We need to prove that every congruence of $\textbf{M}$ is determined by its restriction to $B(\textbf{M})$.

For every $u, v \in \textbf{M}$, the set $E(u, v) = \{y \in Y \mid u(y) = v(y)\}$ is clopen. Considering the cases $v = 0$ and $v = 1$ we obtain that the sets $\{y \in Y \mid u(y) = 0\}$, $\{y \in Y \mid u(y) = 1\}$, and consequently, $\{y \in Y \mid u(y) \in \{a, b\}\}$, are clopen.

Let $\theta \in \text{Con}(\textbf{M})$. We claim that

$$(u, v) \in \theta \quad \text{if and only if} \quad (0, s_{E(u \land v, u \lor v)}) \in \theta.$$ 

It suffices to show it for $u \leq v$. Consider the sets

$$J := \{y \in Y \mid u(y) = 0, \, v(y) = 1\},$$

$$K := \{y \in Y \mid u(y) = 0, \, v(y) \in \{a, b\}\},$$

$$L := \{y \in Y \mid u(y) \in \{a, b\}, \, v(y) = 1\}.$$ 

These sets are clopen and $J \cup K \cup L$ is the complement of $E(u, v)$, so it follows that $t_{J \cup K \cup L} = s_{E(u, v)}$.

Assume that $(u, v) \in \theta$. Then $(0, t_J) = (u \land t_J, v \land t_J) \in \theta$. Further, $u \land t_K = 0$, so $(0, v \land t_K) \in \theta$, which implies that $(1, v^0 \lor t_K) \in \theta$. Taking the meet with $t_K$ we obtain that $(1, v^0 \lor t_K) \land t_K = (t_K, v^0 \lor t_K) \in \theta$. Clearly, $v^0 \land t_K = v \land t_K$, whence we get $(0, t_K) \in \theta$. Similarly, $(u \land t_L, v \land t_L) = (u \land t_L, t_L) \in \theta$, which implies $(u^0 \lor t^0_L, t^0_L) \in \theta$. Taking the meet with $t_L$ and using $u^0 \land t_L = u \land t_L$ we obtain that $(0, u \land t_L) \in \theta$, hence also $(0, t_L) \in \theta$. Consequently, $(0, s_{E(u, v)}) = (0, t_J \lor t_K \lor t_L) \in \theta$.

Conversely, let $(0, s_{E(u, v)}) \in \theta$. It is easy to see that $(0 \lor u) \land v = u$ and $(s_{E(u, v)} \lor v) \land v = v$. Hence $(u, v) \in \theta$.

(1)$\implies$(3) Assume that $\textbf{M}$ is equal to the set of all morphisms from the dual space $\mathbb{X}$ into $\textbf{M}_1$ and that (3) is not satisfied. Hence, we have $x \preceq y$ such that $x \neq y$ and $x \neq f(y)$. Then $\{x, f(x)\}$ and $\{y, f(y)\}$ are distinct closed substructures of $\mathbb{X}$, so they determine distinct congruences on $\textbf{M}$, namely

$$\alpha = \{(\varphi, \psi) \in M^2 \mid \varphi \upharpoonright \{x, f(x)\} = \psi \upharpoonright \{x, f(x)\}\},$$

$$\beta = \{(\varphi, \psi) \in M^2 \mid \varphi \upharpoonright \{y, f(y)\} = \psi \upharpoonright \{y, f(y)\}\}.$$ 

We claim that $\alpha$ and $\beta$ coincide on $B(\textbf{M})$. To show this, take morphisms $\varphi, \psi \in B(\textbf{M})$, hence we have for their ranges $\text{rng}(\varphi), \text{rng}(\psi) \subseteq \{0, 1\}$. Since the maps $\varphi, \psi$ preserve $\preceq$, we have $\varphi(x) \preceq \varphi(y)$ and $\psi(x) \preceq \psi(y)$. For elements
in \( \{0,1\} \) this means \( \varphi(x) = \varphi(y) \) and \( \psi(x) = \psi(y) \). The morphisms commute with \( f \), so \( \varphi(f(x)) = f(\varphi(x)) = f(\varphi(y)) = \varphi(f(y)) \) and similarly \( \psi(f(x)) = \psi(f(y)) \). Hence, \( (\varphi, \psi) \in \alpha \) if and only if \( (\varphi, \psi) \in \beta \). This proves that \( M \) is not a perfect extension of \( B(M) \).

(3) \( \implies \) (2) As in the previous part, assume that \( M \) is equal to the set of all morphisms from the dual space \( X \) into \( M_1 \). Let (3) be satisfied. Choose a subset \( Y \subseteq X \) satisfying the conditions (a) and (b) from Lemma 3.1.

The set \( M' := \{ \varphi \mid Y \mid \varphi \in M \} \) clearly forms a subalgebra of \( M^Y_1 \). The assignment \( \varphi \mapsto \varphi \mid Y \) is a surjective homomorphism \( M \rightarrow M' \). It is also injective: if \( \varphi \not= \psi \), then \( \varphi(x) \neq \psi(x) \) for some \( x \in X \). If \( x \not\in Y \), then \( f(x) \not\in Y \) and \( \varphi(f(x)) \neq \psi(f(x)) \). Hence, \( M \) is isomorphic to \( M' \) and we will prove that \( M' \) is a Boolean product.

By Lemma 3.1, the space \( Y \) is Boolean. For every \( y \in Y \), the set \( A_y = \{ \varphi(y) \mid \varphi \in M \} \) forms a subalgebra \( A_y \) of \( M_1 \), so it is equal to \( \{0,1\} \), \( \{0,a,1\} \), \( \{0,b,1\} \) or \( M_1 \). Since \( \{0,a,1\} \) is isomorphic to \( \{0,b,1\} \), the algebra \( M' \) is a subdirect product of algebras isomorphic to \( \{0,1\} \), \( \{0,a,1\} \) and \( M_1 \). It remains to check the conditions from the definition of the Boolean product. First we will check the conditions concerning the equalizers. Let \( \varphi, \psi \in M \) and let \( K = \{ y \in Y \mid \varphi(y) = \psi(y) \} \). We claim that \( K \cup f(K) = \{ x \in X \mid \varphi(x) = \psi(x) \} \). Indeed, if \( x = f(y) \) for some \( y \in K \), then \( \varphi(x) = \varphi(f(y)) = f(\varphi(y)) = f(\psi(y)) = \psi(f(y)) = \psi(x) \). For the reverse set containment, if \( \varphi(x) = \psi(x) \), then \( x \in Y \) (and hence \( x \in K \)) or \( f(x) \in Y \) (hence \( f(x) \in K \) and \( x \in f(K) \)). The set \( \{ x \in X \mid \varphi(x) = \psi(x) \} \) is clopen in \( \mathbb{X} \), because it is a union of sets \( \{ x \in X \mid \varphi(x) = c \} \cap \{ x \in X \mid \psi(x) = c \} \) for \( c \in \{0,1,a,b\} \). (And these sets are clopen because \( \varphi \) and \( \psi \) are continuous.) By the proof of Lemma 3.1, the set \( K \) is clopen in \( Y \).

Now let \( K \subseteq Y \) be clopen in \( Y \) and \( \varphi, \psi \in M \). Then the set \( K \cup f(K) \) is clopen in \( \mathbb{X} \). We define \( \rho : X \rightarrow M_1 \) as follows:

\[
\rho(x) = \begin{cases} 
\varphi(x) & \text{if } x \in K \cup f(K), \\
\psi(x) & \text{if } x \notin K \cup f(K).
\end{cases}
\]

Since both \( K \cup f(K) \) and its complement are clopen, closed under \( f \), and any pair of elements \( x \in K \cup f(K) \) and \( y \notin K \cup f(K) \) is incomparable with respect to \( \preceq \), the mapping \( \rho \) is a morphism, hence \( \rho \in M \). Clearly, \( \rho \) coincides with \( \varphi \) on \( K \) and with \( \psi \) on \( Y \setminus K \).

For finite \( Y \), the Boolean products are the usual direct products, so we have the following result as an immediate consequence:

**Corollary 3.3.** A finite de Morgan algebra \( M \) is a perfect extension of \( B(M) \) if and only if it is a direct product of finitely many copies of \( \{0,1\} \), \( \{0,a,1\} \) and \( M_1 \).
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