NIEMEIER LATTICES AND K3 GROUPS

Dedicated to Professor I. Dolgachev on the occasion of his sixtieth birthday

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Abstract. In this note, we consider $K3$ surfaces $X$ with an action by the alternating group $A_5$. We show that if a cyclic extension $A_5.C_n$ acts on $X$ then $n = 1, 2, \text{ or } 4$. We also determine the $A_5$-invariant sublattice of the $K3$ lattice and its discriminant form.

Introduction

We work over the complex numbers field $\mathbb{C}$. A $K3$ surface $X$ is a simply connected projective surface with a nowhere vanishing holomorphic 2-form $\omega_X$. In this note, we will consider finite groups in $\text{Aut}(X)$. An element $h \in \text{Aut}(X)$ is symplectic if $h$ acts trivially on the 2-form $\omega_X$. A group $G_N \subseteq \text{Aut}(X)$ is symplectic if every element of $G_N$ is symplectic.

According to Nikulin [Ni1], Mukai [Mu1] and Xiao [Xi], there are exactly 80 abstract finite groups which can act symplectically on $K3$ surfaces. Among these 80, there are exactly three non-abelain simple groups $A_5$, $L_2(7)$ and $A_6$.

To be more precise, as in (1.0) below, for every finite group $G$ acting on a $K3$ surface $X$, the symplectic elements of $G$ (i.e., those $h$ acting trivially on the non-zero 2-form $\omega_X$) form a normal subgroup $G_N$ such that $G/G_N \cong \mu_I$ (the cyclic group of order $I$ in $\mathbb{C}^*$). Namely, we have $G = G_N.\mu_I$ (see Notation below). The natural number $I = I(G)$ is determined by $G$ and called the transcendental value of $G$.

It is proved in [OZ3] and [KOZ1, 2] that when $G_N$ is either one of the two bigger simple groups above, the transcendental value $I(G) \neq 3$. As expected or unexpected, the same is
true for the smaller (indeed the smallest non-abelian simple group) \( A_5 \):

**Theorem A.** Suppose that \( G = A_5.\mu_I \) acts faithfully on a \( K3 \) surface (assuming that \( G_N = A_5 \)). Then \( G = A_5 : \mu_I \) (semi-direct product) and \( I = 1, 2, \) or 4.

In general, for a group of the form \( G = A_5.C_n \) acting on a \( K3 \) surface (here \( G_N \) might be bigger than \( A_5 \); and \( C_n \) an abstract cyclic group of order \( n \)), we have a similar result:

**Theorem B.** Suppose that a group of the form \( G = A_5.C_n \) acts faithfully on a \( K3 \) surface. Then \( G = A_5 : C_n \) and \( I = 1, 2, \) or 4. Moreover, \( G_N = A_5 \) (and hence \( C_n = \mu_n \) in the notation above or (1.0)) unless \( G = G_N = S_5 \).

We can determine the \( A_5 \)-invariant sublattice of the \( K3 \) lattice in the result below, which has application in helping determine the transcendental lattice \( T_X \) and hence the surface itself (when rank \( T_X = 2 \)).

**Theorem C.** Suppose that \( A_5 \) acts faithfully on a \( K3 \) surface \( X \). Then we have:

1. The \( A_5 \)-invariant sublattice \( L^A_5 \) of the \( K3 \) lattice \( L = H^2(X, \mathbb{Z}) \) has rank 4. The \( A_5 \)-invariant sublattice \( S^A_X \) of the Neron Severi lattice \( S_X \) has rank equal to 1 or 2.
2. The discriminant group \( A_{L^A_5} = \text{Hom}(L^A_5, \mathbb{Z})/L^A_5 \) equals one of the following (see Theorem (2.1) for the corresponding intersection forms):
   \[
   \mathbb{Z}/(30) \oplus \mathbb{Z}/(30), \quad \mathbb{Z}/(30) \oplus \mathbb{Z}/(10), \quad \mathbb{Z}/(20) \oplus \mathbb{Z}/(20),
   \]
   \[
   \mathbb{Z}/(60) \oplus \mathbb{Z}/(20), \quad \mathbb{Z}/(60) \oplus \mathbb{Z}/(20) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2).
   \]

**Remark D.**

1. In [Z2], it is proved that there is no faithful action of \( A_5.\mu_4 \) on a \( K3 \) surface. So the \( I \) in Theorems A and B can only be 1 or 2.
2. Theorem C is used in [Z2, Lemma 3.5]. The proofs of Theorems A, B and C here are independent of the paper [Z2].
3. Theorem C is applicable in the following situation: Suppose in addition that a non-symplectic involution \( g \in \text{Aut}(X) \) commutes with every element in \( A_5 \) and that the fixed
locus $X^g$ is a union of a genus $\geq 2$ curve $C$ and $s \geq 1$ smooth rational curves $D_i$. Then $S^A_5 X$ contains $L_0 = \mathbb{Z}[C, \sum_{i=1}^s D_i]$ as a sublattice of finite index $d_1$. Note that $L^A_5$ contains $S^A_5 X \oplus T_X$ as a sublattice of finite index $d$. So $|L_0||T_X| = d_1^2|S^A_5 X||T_X| = d_1^2d^2|L^A_5|$ while $-|L^A_5| = 30^2, 3 \times 10^2, 20^2, 3 \times 20^2,$ or $3 \times 40^2$ as given in Theorem C. This is a restriction imposed on $|T_X|$. In [Z2], we determined $d_1, d$, and $|T_X|$ using the existence of the extra $\mu_4$ in (the impossible case:) $A_5, \mu_4$ where $T_X$ then has the intersection form diag$[2m, 2m]$ for some $m \geq 1$.

(4) The same construction in [OZ3, Appendix] shows that there is a smooth non-isotrivial family of $K3$ surfaces $f : X \to \mathbb{P}^1$ such that all fibres admit $A_6$ actions and infinitely many of them are algebraic $K3$ surfaces. So, the symplectic part alone can not determine the surface uniquely, and the study of transcendental value is needed.

The main tools of the paper are the Lefschetz fixed point formula (both the topological version and vector bundle version due to Atiyah-Segal-Singer [AS2, 3]), the representation theory on the $K3$ lattice and the study on automorphism groups of Niemeier lattices (in connection with Golay binary or ternary codes) where the latter is much inspired by Conway-Sloane [CS], Kondo [Ko1] and Mukai [Mu2].

We believe that the way of combining different very powerful machinaries to deduce results as done in the paper should be applicable to the study of other problems. Our humble paper also demonstrates the powerfulness and depth of these algebraic results in the study of geometry.

Notation.

1. $S_n$ is the symmetric group in $n$ letters, $A_n$ ($n \geq 3$) the alternating group in $n$ letters, $\mu_I = \langle \exp(2\pi \sqrt{-1})/I \rangle$ the multiplicative group of order $I$ in $\mathbb{C}^*$ and $C_n$ an abstract cyclic group of order $n$.

2. For a group $G$, we write $G = A.B$ if $A$ is normal in $G$ so that $G/A = B$. We write $G = A : B$ if assume further that $A$ is normal in $G$ and $B$ is a subgroup of $G$ such that the composition $B \to G \to G/A = B$ is the identity (we say then that $G$ is a semi-direct product of $A$ and $B$).
3. For groups $H \leq G$ and $x \in G$ we denote by $c_x : H \to G$ ($h \mapsto c_x(h) = x^{-1}hx$) the conjugation map.

4. For a $K3$ surface $X$, we let $S_X$ and $T_X$ be the Neron-Severi and transcendental lattices. So the $K3$ lattice $H^2(X, \mathbb{Z})$ contains $S_X \oplus T_X$ as a sublattice of finite index.

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§1. Preliminary Results

(1.0). In this section, we will prepare some basic results to be used late. Let $X$ be a $K3$ surface with a non-zero 2-form $\omega_X$ and let $G \subseteq \text{Aut}(X)$ be a finite group of automorphisms. For every $h \in G$, we have $h^*\omega_X = \alpha(h)\omega_X$ for some scalar $\alpha(h) \in \mathbb{C}^*$. Clearly, $\alpha : G \to \mathbb{C}^*$ is a homomorphism. A fact in basic group theory says that $\alpha(G)$ is a finite cyclic group, so $\alpha(G) = \mu_I = \langle \exp(2\pi\sqrt{-1}/I) \rangle$ for some $I \geq 1$. This natural number $I = I(G)$ is called the transcendental value of $G$. It is known that $I = I(G)$ for some $G$ if and only if that the Euler function $\varphi(I) \leq 21$ and $I \neq 60$ [MO].

Set $G_N = \text{Ker}(\alpha)$. Then we have the basic exact sequence below:

$$1 \longrightarrow G_N \longrightarrow G \overset{\alpha}{\longrightarrow} \mu_I \longrightarrow 1.$$

For the $G$ in the basic exact sequence, we write $G = G_N.\mu_I$, though there is no guarantee that $G = G_N : \mu_I$ (a semi-direct product).

1.0A. If $G$ is a finite perfect group, i.e., the commutator group $[G, G] = G$ (especially, if $G$ is a non-abelian simple group like $A_5$), then $G = G_N$.

1.0B. $G_N$ acts trivially on the transcendental lattice $T_X$ (Lefschetz theorem on $(1,1)$-classes).
1.0C. If a subgroup $H \leq G_N$ fixes a point $P$, then $H < SL(T_X, P) \cong SL_2(C)$ [Mu1, (1.5)]. The finite subgroups of $SL_2(C)$ are listed up in [Mu1, (1.6)]. These are cyclic $C_n$, binary dihedral (or quaternion) $Q_{4n}$ ($n \geq 2$), binary tetrahedral $T_{24}$, binary octahedral $O_{48}$ and binary icosahedral $I_{120}$.

**Lemma 1.1.** Suppose that $G := A_5 : \mu_I$ acts faithfully on a $K3$ surface $X$.

1. The Picard number $\rho(X) \geq 19$, and $I = 1, 2, 3, 4, 6$. Moreover, $\rho(X) = 20$ if $I \geq 3$.

2. We have $G = A_5 : \mu_I$, i.e., a semi-product of a normal subgroup $A_5$ and a subgroup $\mu_I$ of $G$. Moreover, $G = A_5 \times \mu_I$ if $I = 3$.

**Proof.** (1) In notation of [Xi, the list], $\rho(X) = \text{rank } S_X \geq c + 1 = 19$. Also the Euler function $\varphi(I)$ divides $\text{rank } T_X = 22 - \rho(X)$ by [Ni1, Theorem 0.1]. So (1) follows.

(2) Let $g \in G$ such that $\alpha(g)$ is a generator of $\mu_I$. Since $\text{Aut}(A_5) = S_5 > A_5$ and the conjugation homomorphism $A_5 \to \text{Aut}(A_5)$ ($x \mapsto c_x$) is an isomorphism onto $A_5$, the conjugation map $c_g$ equals $c_{(12)}a$ or $c_a$ on $A_5$ for some $a \in A$. Replacing $g$ by $ga^{-1}$, we may assume that $c_g = c_{(12)}$ or $c_{\text{id}}$. Thus $g^2$ commutes with every element in $A_5$. If $2|I$, then $g^I \in \text{Ker}(\alpha) = A_5$ is in the centre of $A_5$ (which is trivial) and hence $\text{ord}(g) = I$; thus $G = A_5 : \mu_I$. If $I = 3$, then $\gcd(3, \text{ord}(g)/3) = 1$ as proved in [IOZ] or [Og, Proposition 5.1]; so replacing $g$ by $g^\ell$ with $\ell = \text{ord}(g)/3$ (or $2\text{ord}(g)/3$), we have $G = A_5 \times \langle g \rangle = A_5 \times \mu_3$.

The second result below [Ni1, §5] is crucial in classifying symplectic groups in [Mu1]. For the first, see [Ni2], [Z1] or [Z2, Lemma 1.2], where the Hodge index theorem is also used here.

**Lemma 1.2.** (1) Let $h$ be a non-symplectic involution on a $K3$ surface $X$. Then $X^h$ is a disjoint union of $s$ smooth curves $C_i$ with $0 \leq s \leq 10$. To be precise, $X^h$ (if not empty) is either a disjoint union of a genus $\geq 2$ curve $C$ and a few $\mathbb{P}^1$’s, or a disjoint union of a few elliptic curves and $\mathbb{P}^1$’s, or a disjoint union of a few $\mathbb{P}^1$’s.

(2) If $\delta$ is a non-trivial symplectic automorphism of finite order on a $K3$ surface $X$, then $\text{ord}(\delta) \leq 8$ and $X^\delta$ is a finite set. To be precise, if $\text{ord}(\delta) = 2, 3, 4, 5, 6, 7, 8$, then $|X^\delta| = 8, 6, 4, 4, 2, 3, 2$, respectively. In particular, if $A_5 \subseteq \text{Aut}(X)$ then $\sum_{\delta \in A_5} \chi_{\text{top}}(X^\delta) = 360$.
For an automorphism $h$ on a smooth algebraic surface $Y$, we split the pointwise fixed locus as the disjoint union of 1-dimensional part and the isolated part: $Y^h = Y^h_{1-\dim} \bigsqcup Y^h_{\text{isol}}$.

The proof of (1) below is similar to that for (1) in (1.2).

1.3. (1) $Y^h_{1-\dim}$ (if not empty) is a disjoint union of smooth curves.

(2) The Euler number $\chi_{\text{top}}(Y^h_{1-\dim}) = \sum_C (2 - 2g(C)) = 2n_h$ for some integer $n_h$, where $C$ runs in the set $Y^h_{1-\dim}$ of curves.

(3) The Euler number $\chi_{\text{top}}(Y^h) = m_h + 2n_h$, where $m_h = |Y^h_{\text{isol}}|$.

The results of [IOZ] below follow from the application of Lefschetz fixed point formula to the trivial vector bundle in Atiyah-Segal-Singer [AS2, AS3, pages 542 and 567]. For a proof, see [OZ1, Lemma 2.3] and [Z2, Proposition 1.4].

**Lemma 1.4.** Let $X$ be a K3 surface and let $h \in \text{Aut}(X)$ be of order $I$ such that $h^*\omega_X = \eta_I\omega_X$ for some primitive $I$-th root $\eta_I$ of 1.

(1) Suppose that $I = 3$. Then $m_h = 3 + n_h$ and hence $\chi_{\text{top}}(X^h) = 3(1 + n_h)$. Moreover, $-3 \leq n_h \leq 6$.

(2) Suppose that $I = 3$. If $\delta \in \text{Aut}(X)$ is symplectic of order 5 and commutes with $h$. Then $|X^{h\delta}| = 4$.

The following result can be found in [Ni1, Theorem 0.1], [MO, Lemma (1.1)], or [OZ3, Lemma (2.8)].

**Lemma 1.5.** Suppose that $X$ is a K3 surface of Picard number $\rho(X) = 20$ and $g$ an order-3 automorphism such that $g^*\omega_X = \eta_3\omega_X$ with a primitive 3rd root $\eta_3$ of 1. Then we can express the transcendental lattice $T_X$ as $T_X = \mathbb{Z}[t_1, t_2]$ so that $t_2 = g^*(t_1)$, $g^*(t_2) = -(t_1 + t_2)$. In particular, for some $m \geq 1$, the intersection form $(t_i, t_j) = \begin{pmatrix} 2m & -m \\ -m & 2m \end{pmatrix}$.

Now we assume that $G = G_N \cdot \mu_I$ (with $I = I(G)$) acts on a K3 surface $X$. When $G_N = A_5$, we will determine the action of $G_N$ on the Neron Severi lattice $S_X$ of $X$: 

(see (1.0A)).
Lemma 1.6. (1) Suppose that $A_5$ acts on a K3 surface $X$, and $\text{rank } S_X = 20$ (this is true if $I \geq 3$ by (1.1)). Then we have the irreducible decomposition below (in the notation of Atlas for the characters of $A_5$), where $\chi_1$ (the trivial character), $\chi_4$ and $\chi_5$ have dimensions 1, 4 and 5, respectively, where $\chi_i'$ is a copy of $\chi_i$:

$$S_X \otimes \mathbb{C} = \chi_1 \oplus \chi_1' \oplus \chi_4 \oplus \chi_4' \oplus \chi_5 \oplus \chi_5' .$$

(2) For conjugacy class $nA$ (and $nB$) of order $n$ in $A_5$ and the characters $\chi_i$ of $A_5$, we have the following Table 1 from [Atlas], where $Z$ is respectively $1A$, $2A$, $3A$, $5A$ or $5B$:

$$[\chi_1, \chi_2, \chi_3, \chi_4, \chi_5](Z) = [1, 3, 3, 4, 5], \quad [1, -1, -1, 0, 1], \quad [1, 0, 0, 1, -1],$$
$$[1, (1 - \sqrt{5})/2, (1 + \sqrt{5})/2, -1, 0], \quad [1, (1 + \sqrt{5})/2, (1 - \sqrt{5})/2, -1, 0].$$

Proof. The assertion(1) appeared in [Z2]. For the readers’ convenience, we reprove it here. Applying the Lefschetz fixed point formula to the action of $A_5$ on $H^*(X, \mathbb{Z}) = \bigoplus_{i=0}^{4} H^i(X, \mathbb{Z})$ and noting that $H^2(X, \mathbb{Z})$ contains $S_X \oplus T_X$ as a finite index sublattice, we obtain (see also (1.0A-B) and (1.2)):

$$2 + \text{rank } T_X + \text{rank } (S_X)^{A_5} = \text{rank } H^*(X, \mathbb{Z})^{A_5} = \frac{1}{|A_5|} \sum_{a \in A_5} \chi_{\text{top}}(X^a) = 360/60 = 6 .$$

Thus $\text{rank } S_X^{A_5} = 2$. So the irreducible decomposition is of the following form, where $a_i$ are non-negative integers:

$$S(X) \otimes \mathbb{C} = 2 \chi_1 \oplus a_2 \chi_2 \oplus a_3 \chi_3 \oplus a_4 \chi_4 \oplus a_5 \chi_5 .$$

Using the topological Lefschetz fixed point formula, the fact that $\text{rank } T(X) = 2$ and (1.0B), we have, for $a \in A_5$, that:

$$\chi_{\text{top}}(X^a) = 2 + \text{rank } T_X + \text{Tr}(a^*|S(X))$$

Running $a$ through the five conjugacy classes and calculating both sides, using (1.2) and
the character Table 1 in (2), we obtain the following system of equations:

\[
\begin{align*}
20 &= 2 + 3(a_2 + a_3) + 4a_4 + 5a_5, \\
4 &= 2 - (a_2 + a_3) + a_5, \\
2 &= 2 + a_4 - a_5, \\
0 &= 2 + \frac{1 - \sqrt{5}}{2}a_2 + \frac{1 + \sqrt{5}}{2}a_3 - a_4, \\
0 &= 2 + \frac{1 + \sqrt{5}}{2}a_2 + \frac{1 - \sqrt{5}}{2}a_3 - a_4.
\end{align*}
\]

Now, we get the result by solving this system of Diophantine equations.

The two results below are either easy or well known and will be frequently used in the arguments of the subsequent sections.

**Lemma 1.7.** Let \( f : A_5 \to S_r \) \((r \geq 2)\) be a homomorphism.

1. If \( r = 2, 3, \) or \( 4, \) then \( f \) is trivial.
2. If \( \text{Im}(f) \) is a transitive subgroup of the full symmetry group \( S_r \) in \( r \) letters \( \{1, 2, \ldots, r\} \)
   (whence \( r \geq 5 \) by (1)), then \( r|A_5| \) with \( |A_5|/r \) equal to the order of the subgroup of \( A_5 \)
   stabilizing the letter 1, so \( r \in \{5, 6, 10, 12, 15, 20, 30\} \).

**Lemma 1.8.** (1) \( \text{Aut}(\mathbb{P}^1) \) includes \( A_5 \) but not \( S_5 \) [Su, Theorem 6.17]. The action of \( A_5 \)
on \( \mathbb{P}^1 \) is unique up to isomorphisms.

1. Every \( A_5 \) in \( PGL_3(\mathbb{C}) \) is the image of an \( A_5 \) in \( SL_3(\mathbb{C}) \).
2. The conjugation by \( (12) \in S_5 \) switches the two 3-dimensional characters \( \chi_2 \) and \( \chi_3 \) of \( A_5 \) [Atlas].
3. If \( f \neq \text{id} \in \text{Aut}(\mathbb{P}^1) \) is an automorphism of finite order, then \( f \) fixes exactly two point
   of \( \mathbb{P}^1 \) (by the diagonalization of a lifting of \( f \) to \( GL_2(\mathbb{C}) \)).
4. If \( f_r \) \((r = 2 \text{ or } 3)\) is an order–\( r \) automorphism of an elliptic curve \( E \), then either \( f_r \)
   acts freely on \( E \), or the fix locus satisfies \( |X^{f_r}| = 4 \) (resp. \( = 3 \)) if \( r = 2 \) (resp. \( r = 3 \)) (by
   the Hurwitz formula).

**Proof.** (1) For the uniqueness of the action of \( A_5 \) on \( \mathbb{P}^1 \), one may assume the representation
of $D_{10} = \langle \gamma = (12345), \sigma = (14)(23) \rangle$ is given by $\gamma : z \to \eta z$ with $\eta$ a primitive 5-th root of 1 and $\sigma : z \to \alpha/z$. Note that $A_5 = \langle \gamma, \varepsilon \rangle$ with $\varepsilon = (12)(34)$. If one lets $\varepsilon : z \to (az + b)/(cz + d)$ be in $\text{Aut}(\mathbb{P}^1)$, then one can check that $d = -a$ because $\text{ord}(\varepsilon) = 2$, and also $b = -c\alpha$ because $\varepsilon$ commutes with $\sigma$. So $\varepsilon : z \to (z - \alpha e)/(e z - 1)$ with $e = c/a$. The commutativity of $\varepsilon\sigma\varepsilon = (12)(45)$ with $\sigma\gamma^{-1} = (15)(24)$ implies that $e^2\alpha = \eta + \eta^{-1} - 1$. Now let $\rho : z \to \varepsilon \sigma\varepsilon z$ be in $\text{Aut}(\mathbb{P}^1)$. Then $\rho^{-1}\gamma\rho : z \to \eta^{-1}z$, $\rho^{-1}\sigma\rho : z \to e^2\alpha/z$ and $\rho^{-1}\varepsilon\rho : z \to (z - e^2\alpha)/(z - 1)$. Hence the action of $A_5$ on $\mathbb{P}^1$ is unique modulo isomorphisms.

(2) For an $A_5$ in $SL_3(\mathbb{C})$, see [Bu, §232]. The inverse $\tilde{A}_5 \subset SL_3(\mathbb{C})$ of an $A_5 \subset PGL_3(\mathbb{C})$ is of the form $\tilde{A}_5 = A_5 : \mu_3$ (indeed, a direct product) because the Schur multiplier $M(A_5) = 2$, coprime to 3 [Atlas]. So (2) follows.

§2. Alternating groups actions on the Niemeier lattices

For a $K3$ surface $X$, denote by $L = H^2(X, \mathbb{Z})$ the $K3$ lattice, $S_X = \text{Pic} X$ (now) the Neron-Severi lattice and $T_X$ the transcendental lattice. So $T_X = S_X^\perp$ in $L$ and $L$ contains a finite-index sublattice $S_X \oplus T_X$.

(2.0) Suppose that $G_N = A_5$ acts faithfully on $X$. In this section we shall prove Theorem C which is part of (2.1) below. Indeed, by the proof of (1.6), we have rank $L_{G_N} = \text{rank} T_X + \text{rank} S_X^{G_N} = 4$, so (rank $T_X$, rank $S_X$, rank $S_X^{G_N}$) = (2, 20, 2) or (3, 19, 1).

Denote by $L_{G_N} := \{ x \in L \mid g^* x = x \text{ for all } g \in G_N \}$ and its orthogonal $L_{G_N} := (L_{G_N} \perp = \{ x \in L \mid (x, y) = 0 \text{ for all } y \in L_{G_N} \}$. Then $L_{G_N}$ contains $S_X^{G_N} \oplus T_X$ as a sublattice of finite index by (1.0A-B).

By [Ko1, Lemmas 5 and 6], there are a (non-Leech) Niemeier lattice $N(Rt)$, a primitive embedding $A_1 \oplus L_{G_N} \subset N(Rt)$ and a faithful action of $G_N$ on $N(Rt)$ such that $L_{G_N} = N(Rt)_{G_N}$, and the action of $G_N$ on the summand $A_1$ is trivial and stabilizes a Weyl chamber (one of whose codimension one faces corresponds to this $A_1$). Moreover, $G_N \leq S(N(Rt)) := O(N(Rt))/W(N(Rt)) \leq O(Rt)/W(N(Rt))(= : \text{Sym}(Rt))$, where Sym($Rt$) is
the full symmetry group of the Coxeter-Dynkin diagram $R_t$. Note that rank $N(R_t)^{G_N} = 2 + \operatorname{rank} L^{G_N} = 6$ and the discriminant groups satisfy:

$$(*) \quad A_L^{G_N} \cong A_{L_{G_N}}(-1) = A_{N(R_t)^{G_N}}(-1) \cong A_{N(R_t)^{G_N}}.$$ 

Now $N(R_t)^{G_N}$ is a rank 6 lattice generated by $e_1, \ldots, e_6$ say. Denote by $M = (e_i.e_j)$ the intersection matrix and $M^{-1} = (f_1, \ldots, f_6)$ with $f_j$ column vectors and set $e_i^* = (e_1, \ldots, e_6)f_i$. Then $(N(Rt)^{G_N})^\vee = \operatorname{Hom}(N(Rt)^{G_N}, \mathbb{Z})$ has the dual basis $\{e_1^*, \ldots, e_6^*\}$ with the intersection matrix $(e_i^*.e_j^*)_{1 \leq i,j \leq 6} = M^{-1}$. The discriminant groups satisfy $A_L^{G_N} \cong A_{N(Rt)^{G_N}} = \mathbb{Z}[e_1^*, \ldots, e_6^*]/\mathbb{Z}[e_1, \ldots, e_6]$.

In this section, we shall prove the following result (much inspired by [Ko1]), which (and the proof of which) should be useful in studying $\operatorname{Aut}(X)$ from the $K3$ lattice point of view. This result is used in [Z2, Lemma 3.5].

**Theorem 2.1.** Suppose that $G_N = A_5$ acts faithfully on a $K3$ surface $X$.

(1) We have $Rt = 24A_1$ or $Rt = 12A_2$. The lattice $N(Rt)^{G_N}$ is of rank 6 and generated by $e_1, \ldots, e_6$ say. Denote by $M = (e_i.e_j)$ the intersection matrix and write $M^{-1} = (f_1, \ldots, f_6)$ with $f_j$ column vectors and set $e_i^* = (e_1, \ldots, e_6)f_i$.

(2) If $Rt = 24A_1$, then the orbit decomposition of the $G_N$-action on the 24 simple roots is either one of

(i) $[1, 1, 5, 5, 6, 6]$, (ii) $[1, 1, 1, 5, 6, 10]$, (iii) $[1, 1, 1, 1, 5, 15]$, (iv) $[1, 1, 1, 1, 10, 10]$.

If $Rt = 12A_2$, then the orbit decomposition of the $G_N$-action on the 24 simple roots is either one of

(v) $[1, 1, 1, 1, 10, 10]$, (vi) $[1, 1, 5, 6, 6]$,

where in (v) (resp. (vi)) 10$A_2$ (resp. 5$A_2$, or 6$A_2$) is split into two orbits with 10 (resp. 5, or 6) disjoint roots each.
The discriminant group (cf. (2.0), $A_{L^{G_N}} \cong A_{N(Rt)^{G_N}} = \text{Hom}(N(Rt)^{G_N}, \mathbb{Z})/N(Rt)^{G_N} \cong \mathbb{Z}/(30) \oplus \mathbb{Z}/(30)$ and is generated by cosets $\tau_1^*$ and $\tau_2^* + \tau_3^* + \tau_4^*$ with intersection form:

\[
\begin{pmatrix}
(\tau_1^*)^2 & \tau_1^* \cdot (\tau_2^* + \tau_3^* + \tau_4^*) \\
(\tau_2^* + \tau_3^* + \tau_4^*) & (\tau_2^* + \tau_3^* + \tau_4^*)^2
\end{pmatrix} = \begin{pmatrix}
-23/30 & -1/5 \\
-1/5 & -35/30
\end{pmatrix}.
\]

(4) For Case(2ii), the intersection matrix $M_2 = (e_i.e_j)$ and its inverse $M_2^{-1}$ are respectively:

\[
\begin{pmatrix}
-2 & 0 & 0 & -1 & -1 & 1 \\
0 & -2 & 0 & -1 & -1 & 1 \\
0 & 0 & -2 & -1 & 0 & 0 \\
-1 & -1 & -1 & -4 & -1 & 1 \\
-1 & -1 & 0 & -1 & -4 & 1 \\
-1 & -1 & 0 & -1 & -1 & 6
\end{pmatrix} \cdot \begin{pmatrix}
-11/15 & -7/30 & -1/10 & 1/5 & 1/6 & 1/10 \\
-7/30 & -11/15 & -1/10 & 1/5 & 1/6 & 1/10 \\
-1/10 & -1/10 & -3/5 & 1/5 & 0 & 0 \\
1/5 & 1/5 & 1/5 & 0 & 0 & 0 \\
1/6 & 1/6 & 0 & 0 & -1/3 & 0 \\
1/10 & 1/10 & 0 & 0 & 0 & -1/5
\end{pmatrix}.
\]

The discriminant group $A_{N(Rt)^{G_N}}$ is isomorphic to $\mathbb{Z}/(30) \oplus \mathbb{Z}/(10)$ and generated by the cosets $\tau_1^*$ and $\tau_3^*$ with intersection form:

\[
\begin{pmatrix}
(\tau_1^*)^2 & \tau_1^* \cdot \tau_3^* \\
\tau_1^* \cdot \tau_3^* & (\tau_3^*)^2
\end{pmatrix} = \begin{pmatrix}
-11/15 & -1/10 \\
-1/10 & -3/5
\end{pmatrix}.
\]

(5) For Case(2iii), the intersection matrix $M_3 = (e_i.e_j)$ and $M_3^{-1}$ are respectively:

\[
\begin{pmatrix}
-2 & 0 & 0 & 0 & -1 & 0 \\
0 & -2 & 0 & 0 & -1 & 0 \\
0 & 0 & -2 & 0 & -1 & 0 \\
0 & 0 & 0 & -2 & 0 & -1 \\
-1 & -1 & -1 & 0 & -4 & 0 \\
0 & 0 & 0 & -1 & 0 & -8
\end{pmatrix} \cdot \begin{pmatrix}
-3/5 & -1/10 & -1/10 & 0 & 1/5 & 0 \\
-1/10 & -3/5 & -1/10 & 0 & 1/5 & 0 \\
-1/10 & -1/10 & -3/5 & 0 & 1/5 & 0 \\
0 & 0 & 0 & -8/15 & 0 & 1/15 \\
1/5 & 1/5 & 1/5 & 0 & -2/5 & 0 \\
0 & 0 & 0 & 1/15 & 0 & -2/15
\end{pmatrix}.
\]

The discriminant group $A_{N(Rt)^{G_N}}$ is isomorphic to $\mathbb{Z}/(30) \oplus \mathbb{Z}/(10)$ and generated by the cosets $\tau_2^*$ and $\tau_1^* + \tau_4^*$ with intersection form:

\[
\begin{pmatrix}
(\tau_2^*)^2 & \tau_2^* \cdot (\tau_1^* + \tau_4^*) \\
\tau_2^* \cdot (\tau_1^* + \tau_4^*) & (\tau_1^* + \tau_4^*)^2
\end{pmatrix} = \begin{pmatrix}
-3/5 & -1/10 \\
-1/10 & 13/15
\end{pmatrix}.
\]
(6) For Case(2iv), the intersection matrix $M_{4} = (e_{i}, e_{j})$ and $M_{4}^{-1}$ are respectively:

\[
\begin{pmatrix}
-2 & 0 & 0 & 0 & -1 & 0 \\
0 & -2 & 0 & 0 & -1 & 0 \\
0 & 0 & -2 & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & -6 & 0 \\
0 & 0 & -1 & -1 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
-11/20 & -1/20 & 0 & 0 & 1/10 & 0 \\
-1/20 & -11/20 & 0 & 0 & 1/10 & 0 \\
0 & 0 & -11/20 & -1/20 & 0 & 1/10 \\
0 & 0 & -1/20 & -11/20 & 0 & 1/10 \\
1/10 & 1/10 & 0 & 0 & -1/5 & 0 \\
0 & 0 & 1/10 & 1/10 & 0 & -1/5
\end{pmatrix}.
\]

The discriminant group $A_{N(Rt)}^{G_{N}}$ is isomorphic to $\mathbb{Z}/(20) \oplus \mathbb{Z}/(20)$ and generated by the cosets $\bar{e}_{1}$ and $\bar{e}_{3}$ with intersection form:

\[
\begin{pmatrix}
\mathbf{(e_{1}^{i})}^{2} & \mathbf{e_{1}^{i}} \cdot \mathbf{e_{3}} \\
\mathbf{e_{1}^{i}} \cdot \mathbf{e_{3}} & \mathbf{(e_{3})}^{2}
\end{pmatrix} = \begin{pmatrix}
-11/20 & 0 \\
0 & -11/20
\end{pmatrix}.
\]

(7) For Case(2v), the intersection matrix $M_{5} = (e_{i}, e_{j})$ and its inverse $M_{5}^{-1}$ are respectively:

\[
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & -1 \\
0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & -1 \\
0 & 0 & 0 & 0 & -20 & 0 \\
0 & -1 & 0 & -1 & 0 & -8
\end{pmatrix},
\begin{pmatrix}
-41/60 & -11/30 & -1/60 & -1/30 & 0 & 1/20 \\
-11/30 & -11/15 & -1/30 & -1/15 & 0 & 1/10 \\
-1/60 & -1/30 & -41/60 & -11/30 & 0 & 1/20 \\
-1/30 & -1/15 & -11/30 & -11/15 & 0 & 1/10 \\
0 & 0 & 0 & 0 & -1/20 & 0 \\
1/20 & 1/10 & 1/20 & 1/10 & 0 & -3/20
\end{pmatrix}.
\]

The discriminant group $A_{N(Rt)}^{G_{N}}$ is isomorphic to $\mathbb{Z}/(60) \oplus \mathbb{Z}/(20)$ and generated by the cosets $\bar{e}_{1}$ and $\bar{e}_{5}$ with intersection form:

\[
\begin{pmatrix}
\mathbf{(e_{1}^{i})}^{2} & \mathbf{e_{1}^{i}} \cdot \mathbf{e_{5}} \\
\mathbf{e_{1}^{i}} \cdot \mathbf{e_{5}} & \mathbf{(e_{5})}^{2}
\end{pmatrix} = \begin{pmatrix}
-41/60 & 0 \\
0 & -1/20
\end{pmatrix}.
\]

(8) For Case(2vi), the intersection matrix $M_{6} = (e_{i}, e_{j})$ and $M_{6}^{-1}$ are respectively:

\[
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & -1 \\
0 & 0 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & -12 & 0 & 0 \\
0 & -1 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & -4
\end{pmatrix},
\begin{pmatrix}
-7/10 & -2/5 & 0 & 0 & 1/10 & 0 \\
-2/5 & -4/5 & 0 & 0 & 1/5 & 0 \\
0 & 0 & -1/10 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/12 & 0 & 0 \\
1/10 & 1/5 & 0 & 0 & -3/10 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/4
\end{pmatrix}.
\]

The discriminant group $A_{N(Rt)}^{G_{N}} = \mathbb{Z}/(60) \oplus \mathbb{Z}/(20) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) = \mathbb{Z}/(10) \oplus \mathbb{Z}/(10) \oplus \mathbb{Z}/(12) \oplus \mathbb{Z}/(4)$ and the latter is generated by the cosets $\bar{e}_{j}^{i}$ ($j = 1, 3, 4, 6$).

(9) In both of the cases of $M_{2}$ and $M_{3}$, the discriminant group $A_{N(Rt)}^{G_{N}}$ is isomorphic to the group $(\bar{e}_{1}, \bar{e}_{2}) \cong \mathbb{Z}/(30) \oplus \mathbb{Z}/(10)$ with the intersection matrix $(\bar{e}_{i}, \bar{e}_{j}) = \begin{pmatrix}
1/15 & 1/30 \\
1/30 & 1/15
\end{pmatrix}$. 

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We now prove (2.1). Since rank $N(Rt)^{G_N} = 6$, the $G_N$-action on the 24 simple roots of $Rt$ has exactly 6 orbits.

We argue as in the proof of [Ko1, Theorem 4]. The fact that $G_N = A_5 < S(N(Rt))$ implies that $Rt$ is one of the following: $24A_1, 12A_2, 6A_4, 6D_4$.

If $Rt = 6A_4$, then $S(N(Rt)) = 2.PGL_2(5) (< 2.S_6)$ [CS, Ch 16, §1], where the order 2 element acts as a symmetry of order 2 on each connected component of Dynkin type $A_4$, and $PGL_2(5)$ acts on the set (identified with $\{0,1,2,3,4,\infty\}$) of 6 components of $Rt$ as permutations in a natural way. Since $A_5$ is simple, the composition of homomorphisms below is an injection: $A_5 \subset S(N(Rt)) \rightarrow PGL_2(5)$, so we may assume that $A_5 < PGL_2(5)$.

Since $G_N = A_5$ fixes one simple root of $Rt$ by the construction, our $A_5$ is a subgroup of the stabilizer subgroup of $PGL_2(5)$ and this stabilize is of order $|PGL_2(5)|/6 = 20$. This is impossible because $|A_5| = 60 > 20$.

If $Rt = 6D_4$, then $S(N(Rt)) = 3.S_6$, where the order 3 element acts as a symmetry of order 3 on each connected component of Dynkin type $D_4$, and $S_6$ acts on the set of 6 connected components of $Rt$ as permutations. As above, the simplicity of $G_N$ implies that the subgroup $G_N$ of $S(N(Rt))$ is indeed a subgroup of $S_6$. Since $G_N = A_5$ fixes one simple root of $Rt$, our group $A_5$ is a subgroup ($=[S_5,S_5]$) of the stabilizer subgroup $S_5$ of $S_6$. So this $A_5$ acts transitively on the remaining 5 connected components of $Rt$ and hence the $G_N$-action on the 24 simple roots has exactly 8 orbits, noting that one connected component of $Rt$ is component wise fixed by $G_N$, a contradiction.

Suppose that $Rt = 12A_2$. Then $S(N(Rt)) = 2.M_{12}$, where the order 2 element acts as a symmetry of order 2 on each connected component of Dynkin type $A_2$, and the Mathieu group $M_{12}$ acts on the set of 12 connected components of $Rt$ as permutations. Let $r_{2k-1} + r_{2k} \ (1 \leq k \leq 12)$ be the 12 connected components of $Rt$ with $r_j$ the 24 simple roots. Every non-trivial element of $N(Rt)/Rt$ is of the form $\sum_{i \in H} \pm (r_{2i-1} + 2r_{2i})/3$ where $H$ is an element of the ternary Golay code and $|H| = 6,9,12$ [CS, Ch 3, §2.8.5]. Since the group $G_N = A_5$ is simple and fixes one simple root of $Rt$, this $G_N$ is a subgroup of $M_{12}$ and indeed, a subgroup of the stabilizer subgroup $M_{11}$ of $M_{12}$. Suppose the $G_N$-orbit decomposition on the 12 connected components is $1 + a + b$. Then the $G_N$-orbit decomposition of the 24 simple roots is $[1,1,a,a,b,b]$ (so $a + b = 11$), where $aA_2$ (resp.
$bA_2$) is split into two $G_N$-orbits with $a$ (resp. $b$) disjoint simple roots each. Thus Case (2v) or (2vi) occurs by (1.7).

Suppose that $Rt = 24A_1$. Then $S(N(Rt)) = M_{24}$. The elements of $N(Rt)/Rt$ form the binary Golay code. Since $G_N = A_5$ fixes one simple root of $Rt$, our group $G_N$ is the stabilizer subgroup $M_{23}$ of $M_{24}$. Let the $G_N$-orbit decomposition of the 24 simple roots be $\{1, a, b, c, d, e\}$ with $a \leq b \leq c \leq d \leq e$ (so $a + b + c + d + e = 23$). By (1.7), all $a, b, c, d, e$ are in $\{1, 5, 6, 10, 12, 15, 20\}$ and hence Cases (2i) - (2vi) occur.

(4) According to the ordering of $[1, 1, 1, 5, 6, 10]$, we label the orbits as $O_1 = \{r_1\}, O'_1 = \{r_2\}, O''_1 = \{r_3\}, O_5 = \{r_4, \ldots, r_8\}, O_6 = \{r_9, \ldots, r_{14}\}, O_{10} = \{r_{15}, \ldots, r_{24}\}$, where $r_j$ are the 24 simple roots. We claim that $O_1 + O'_1 + O''_1 + O_5$ (to be precise, after divided by 2) is an octad, and $O_1 + O'_1 + O_6$ is also an octad (after relabelling $O_1, O'_1, O''_1$). So $e_i = r_i (1 \leq i \leq 3), e_4 = \frac{1}{2}(O_1 + O'_1 + O''_1 + O_5), e_5 = \frac{1}{2}(O_1 + O'_1 + O_6), e_6 = \frac{1}{2}(O_1 + O'_1 + O_{10})$

form a basis of $N(Rt)^{G_N}$, noting that the last dodecad is the complement of the symmetric sum (a dodecad) of the two octads above and that except for the above-mentioned two octads and two dodecads, there is no any other octad or dodecad which is a union of orbits. Indeed, let $Oct_1$ be the unique octad containing $O_5$. Note that the cycle type in $M_{24}$ of an order-5 element $\gamma$ in $A_5$ is $(5^4)$, Appendix B, Table 5.1. So $\gamma$ is of type $(5^2)$ (resp. $(5)$) on $O_{10}$ (resp. on $O_5$ and $O_6$). Since $\gamma(Oct_1) \cap Oct_1$ contains $O_5$, we have $\gamma(Oct_1) = Oct_1$. If $Oct_1$ contains an element of $O_{10}$ then it contains the five images in $O_{10}$ by the action of $\langle \gamma \rangle$, so $|Oct_1| \geq 10$, absurd. If $Oct_1$ contains an element $r_j$ in $O_6$ we may choose $\gamma$ not fixing $r_j$ (note that the stabilizer subgroup of $A_5$, regarded as a subgroup of $\text{Sym}(O_6) = S_6$ and fixing an element ($\neq r_j$) in $O_6$, has order 10 and hence gives rise to such $\gamma$). Then we will get a similar contradiction. Thus $Oct_1 = O_1 + O'_1 + O''_1 + O_5$ as claimed.

Let $Oct_2$ be the unique octad containing the first 5 elements in $O_6$. Let $\gamma$ be an order-5 element in $A_5$ fixing the last element in $O_6$. Then $\gamma(Oct_2) = Oct_2$. As above, this implies that $Oct_2$ is disjoint from $O_5$ and $O_{10}$. So either $Oct_2 = O_1 + O'_1 + O_6$ after relabelling the 1-element orbits, or $Oct_2$ is the union of the 5 elements in $O_6$ and the three 1-element orbits (this leads to that the symmetric sum of $Oct_1$ and $Oct_2$ is a 10-word Golay code, absurd).
(3) For the orbit decomposition \([1, 1, 5, 5, 6, 6]\), we label the orbits as \(O_1 = \{r_1\}, O'_1 = \{r_2\}, O_5 = \{r_3, \ldots, r_7\}, O'_5 = \{r_8, \ldots, r_{12}\}, O_6 = \{r_{13}, \ldots, r_{18}\}, O'_6 = \{r_{19}, \ldots, r_{24}\}\). As in (4), we can prove that both \(O_1 + O'_1 + O_6\) and \(O_1 + O'_1 + O'_6\) are octads. Thus \(N(Rt)^{G_N}\) has a basis below, noting that except for the two octads, the symmetric sum (a dodecad) of the two octads and the complement (another dodecad) of this dodecad, there is no other octad or dodecad which is the union of orbits:

\[
e_i = r_i(i = 1, 2), e_3 = O_5, e_4 = \frac{1}{2}(O_1 + O'_1 + O_6), e_5 = \frac{1}{2}(O_1 + O'_1 + O'_6), e_6 = \frac{1}{2}(O_1 + O'_1 + O_5 + O'_5)
\]

(5) For the orbit decomposition \([1, 1, 1, 1, 5, 15]\), we label the orbits as \(O_1 = \{r_1\}, O'_1 = \{r_2\}, O''_1 = \{r_3\}, O'''_1 = \{r_4\}, O_5 = \{r_5, \ldots, r_9\}, O_{15} = \{r_{10}, \ldots, r_{24}\}\). As in (4), we may assume that \(O_1 + O'_1 + O''_1 + O_5\) is an octad after relabelling the 1-element orbits and that there is no any other octad or dodecad which is a union of orbits. Thus \(N(Rt)^{G_N}\) has a basis:

\[
e_i = r_i(i = 1, 2, 3, 4), e_5 = \frac{1}{2}(O_1 + O'_1 + O''_1 + O_5), e_6 = \frac{1}{2}(O'''_1 + O_{15})
\]

(6) For the orbit decomposition \([1, 1, 1, 1, 10, 10]\), we label the orbits as \(O_1 = \{r_1\}, O'_1 = \{r_2\}, O''_1 = \{r_3\}, O'''_1 = \{r_4\}, O_{10} = \{r_5, \ldots, r_{14}\}, O'_{10} = \{r_{15}, \ldots, r_{24}\}\). Take an order-5 element \(\gamma\) of \(A_5\). So \(O_{10}\) splits into two 5-element subsets on each of which \(\gamma\) acts transitively. Let \(Oct_j\) \((j = 1, 2)\) be the unique octad containing the first (resp. second) 5-element subset. As in (4), we can show that each \(Oct_j\) is the union of the 5-element subset and three 1-element orbits. The symmetric sum of \(Oct_1\) and \(Oct_2\) is a dodecad which may be assumed to be \(O_1 + O'_1 + O_{10}\); its complement is also a dodecad. Except for these two dodecads, there is no any other dodecad which is a union of orbits. Thus \(N(Rt)^{G_N}\) has a basis:

\[
e_i = r_i(i = 1, 2, 3, 4), e_5 = \frac{1}{2}(O_1 + O'_1 + O_{10}), e_6 = \frac{1}{2}(O''_1 + O'''_1 + O'_{10})
\]

(8) For \(Rt = 12A_2\) and the orbit decomposition \([1, 1, 5, 5, 6, 6]\), we label the orbits as \(O_1 = \{r_1\}, O'_1 = \{r_2\}, O_5 = \{r_3, r_5, \ldots, r_{11}\}, O'_5 = \{r_4, r_6, \ldots, r_{12}\}, O_6 = \{r_{13}, r_{15}, \ldots, r_{23}\}\)
$O_6' = \{ r_{14}, r_{16}, \ldots, r_{24} \}$, where $r_{2k-1} + r_{2k}$ ($1 \leq k \leq 12$) are the 12 connected components of $Rt$. Every non-trivial element of the group $N(Rt)/Rt$ is represented by some $\gamma_H = \sum_{i \in H} \pm (r_{2i-1} + 2r_{2i})/3$ where $H$ is an element of the ternary Golay code (so $|H| = 6, 9, 12$) which is also the Steiner system $St(5, 6, 12)$ [Atlas]. Let $H_i$ with $i = 1$ (resp. $i = 2$) be the unique element of the ternary Golay code with $|H_i| = 6$ such that $\gamma_{H_1} = \frac{1}{3} \sum_{i=2}^{6} (r_{2i-1} + 2r_{2i}) \pm \frac{1}{3} (r_{2j_1-1} + 2r_{2j_1})$ for some $j$ (resp. $\gamma_{H_2} = \frac{1}{3} \sum_{i=1}^{11} (r_{2i-1} + 2r_{2i}) \pm \frac{1}{3} (r_{2j_2-1} + 2r_{2j_2})$ for some $j$); such Golay code can also be constructed from the binary Golay code = Steiner system $St(5, 8, 24)$ where such $H_i$ is the intersection of a fixed dodecad and an octad. Using the fact that an order-5 element in $A_5$ has cycle type $(5^2)$ in $M_{12}$ [EDM], as in the case of $Rt = 24A_1$, we can prove that $N(Rt)^{G_N}$ has a basis:

$$e_i = r_i (i = 1, 2), e_3 = \sum_{k=2}^{6} r_{2k-1}, e_4 = \sum_{k=7}^{12} r_{2k-1}, e_5 = \frac{1}{3} \sum_{k=1}^{6} (r_{2k-1} + 2r_{2k}), e_6 = \frac{1}{3} \sum_{k=7}^{12} (r_{2k-1} + 2r_{2k}).$$

(7) For $Rt = 12A_2$ and the orbit decomposition $[1, 1, 1, 10, 10]$, we label the orbits as $O_1 = \{ r_1 \}, O_1' = \{ r_2 \}, O_1'' = \{ r_3 \}, O_1''' = \{ r_4 \}, O_{10} = \{ r_5, r_7, \ldots, r_{23} \}, O_{10}' = \{ r_6, r_8, \ldots, r_{24} \}$, where $r_{2k-1} + r_{2k}$ ($1 \leq k \leq 12$) are the 12 connected components of $Rt$. As in (8), we can prove that $N(Rt)^{G_N}$ has a basis:

$$e_i = r_i (1 \leq i \leq 4), e_5 = \sum_{k=3}^{12} r_{2k-1}, e_6 = \frac{1}{3} \sum_{k=1}^{12} (r_{2k-1} + 2r_{2k}).$$

(9) follows from the direct calculation. Indeed, in the case of $M_2$, the isomorphism $\varphi_2 : \langle \tilde{t}_1, \tilde{t}_2 \rangle \rightarrow A_{N(Rt)}^{G_N}$ is given by $(\varphi_2(\tilde{t}_1), \varphi_2(\tilde{t}_2)) = (\tilde{e}_1, \tilde{e}_3) \left( \begin{array}{cc} 2 & 7 \\ 1 & 0 \end{array} \right)$. In the case of $M_3$, the isomorphism $\varphi_3 : \langle \tilde{t}_1, \tilde{t}_2 \rangle \rightarrow A_{N(Rt)}^{G_N}$ is given by $(\varphi_3(\tilde{t}_1), \varphi_3(\tilde{t}_2)) = (\tilde{e}_2, \tilde{e}_1 + \tilde{e}_4) \left( \begin{array}{cc} 1 & 7 \\ 1 & -4 \end{array} \right)$. This proves (2.1).

§3. The proof of Theorems A and B

In this section, we shall prove Theorems A and B. We prove first the result below which includes Theorem A.

**Theorem 3.1.**
(1) There is no faithful group action of the form $A_5, \mu_3$ (see (1.0)) on a $K3$ surface.

(2) If $G = A_5, \mu_I$ acts faithfully on a $K3$ surface. Then $G = A_5 : \mu_I$ and $I = 1, 2, 4$. (It is proved in [Z2] that $I = 4$ is impossible.)

(2) is a consequence of (1) and (1.1). Indeed, if $I = 6$, then the subgroup $H = \alpha^{-1}(\mu_3)$ of $G = A_5, \mu_6$ is of the form $H = A_5, \mu_3$ which is impossible by (1). To prove (1), we need the following result first.

**Lemma 3.2.** Suppose that $G = A_5, \mu_3$ acts on a $K3$ surface $X$. Let $\zeta_3 = \exp(2\pi \sqrt{-1}/3)$.

(1) We have $G = A_5 \times \mu_3$. Moreover, a generator $g$ of $\mu_3$ can be chosen so that $g^*|S_X \otimes \mathbb{C} = \text{diag}[1, 1, \zeta_3I_4, \zeta_3^{-1}I_4, \zeta_3I_5, \zeta_3^{-1}I_5]$, where the decomposition here is compatible with that in (1.6) in the sense that $g^*|\chi_4 \oplus \chi_4' = \text{diag}[\zeta_3I_4, \zeta_3^{-1}I_4]$ and $g^*|\chi_5 \oplus \chi_5' = \text{diag}[\zeta_3I_5, \zeta_3^{-1}I_5]$. In particular, $\chi_{\text{top}}(X^g) = -6$.

(2) We have $S_X^6 = S_X^5 = S_X^{A_5} = H^0(X, \mathbb{Z})^g$. This lattice is of rank 2 (whose $\mathbb{C}$-extension is $\chi_1 \oplus \chi_1'$) and its discriminant group is 3-elementary.

(3) We have $S_X^{A_5} = U = U(1), \text{ or } U(3)$, where $U(n) = \mathbb{Z}[u_1, u_2]$ is a rank 2 lattice with $u_1^2 = 0$ and $u_1u_2 = n$.

**Proof.** (1) The first part is from (1.1). For a generator $g$ of $\mu_3$, since $o(g) = 3$ and by the form of the decomposition in (1.6), each $\chi_i$ ($i = 4, 5$) is $g$-stable. Since the order-3 element $g$ acts on the rank-2 lattice $S_X^{A_5}$ (which is defined over $\mathbb{Z}$ and whose $\mathbb{C}$-extension is $\chi_1 \oplus \chi_1'$), it has at least one eigenvalue equal to 1 because $G = \langle A_5, g \rangle$ stabilizes an ample line bundle (the pull back of an ample line bundle on $X/G$). So $g^*|S_X^{A_5} = \text{id}$. The commutativity of $g$ with all elements in $A_5$ implies that $g^*|\chi_i$ is a scalar multiple, by Schur’s lemma.

Thus we can write $g^*|S_X \otimes \mathbb{C} = \text{diag}[1, 1, \zeta_3I_4, \zeta_3^{-1}I_4, \zeta_3I_5, \zeta_3^{-1}I_5]$, where the ordering is the same as in (1.6). Let $a \in A_5$. Then $(ga)^*|T_X = g^*|T_X$ and the latter can be diagonalized as $\text{diag}[\zeta_3, \zeta_3^{-1}]$, noting that rank $T_X = 22 - \text{rank} S_X = 2$ [Ni1, Theorem 0.1], (1.0A-B). So $\text{Tr}(ga)^*|T_X = -1$. As in the proof of (1.8), the topological Lefschetz fixed point formula implies that $\chi(X^{ga}) = 2 + \text{Tr}(ga)^*|T_X + \text{Tr}(ga)^*|S_X = 1 + \text{Tr}(ga)^*|S_X = 3 + \zeta^b \text{Tr}(a^*|\chi_4) + \zeta^c \text{Tr}(a^*|\chi_4') + \zeta^d \text{Tr}(a^*|\chi_5) + \zeta^e \text{Tr}(a^*|\chi_5')$. So for $a = \text{id, 2A, 3A, 5A with...
nA denoting an element of order n in A₅, we have:

\[
\chi_{\text{top}}(X^g) = 3 + 4(\zeta_3^b + \zeta_3^c) + 5(\zeta_3^d + \zeta_3^e),
\]
\[
\chi_{\text{top}}(X^{g2A}) = 3 + \zeta_3^d + \zeta_3^e,
\]
\[
\chi_{\text{top}}(X^{g3A}) = 3 + \zeta_3^b + \zeta_3^c - \zeta_3^d - \zeta_3^e,
\]
\[
\chi_{\text{top}}(X^{g5A}) = 3 - \zeta_3^b - \zeta_3^c.
\]

The fact \(\chi(X^{g5A}) = 4\) in (1.4) implies that \((\zeta_3^b, \zeta_3^c) = (\zeta_3, \zeta_3^{-1})\) after switching \(\chi_4\) with \(\chi'_4\) if necessary. Since \(\chi(X^{g3A}) = 0\) is in \(R\) (in \(Z\), indeed), we may assume that \((\zeta_3^d, \zeta_3^e) = (\zeta_3, \zeta_3^{-1})\), or \((1, 1)\). If the former case occurs then the lemma is true.

Suppose that the latter case occurs. Then \(\chi_{\text{top}}(X^g) = 9\), whence \(n_g = 2\) and \(|X^g_{\text{isol}}| = m_g = n_g + 3 = 5\) by (1.4). Since \(g\) commutes with every element in \(A_5\), our \(A_5\) acts on the 5-point set \(X^g_{\text{isol}}\). By (1.7), \(A_5\) either fixes a point \(P_1\) of the set (and hence \(A_5 < SL(T_X, P_1)\), contradicting (1.0C)), or acts transitively as a subgroup (= \([S_5, S_5]\)) of \(S_5\), on the set with an order-12 stabilizer (of a point \(P_1\)) subgroup \(A_4 < A_5\), so \(A_4 < SL(T_X, P_1)\), contradicting (1.0C). This proves the assertion (1).

(2) The first part follows from (1), that \(g^*|T_X \otimes \mathbb{C} = \text{diag}[\zeta_3, \zeta_3^{-1}]\) w.r.t. to a suitable basis by [Ni1, Theorem 0.1] and that all lattices in (2) are primitive (of the same rank as they turn out to be) in \(L := H^2(X, \mathbb{Z})\). We still have to show that the discriminant group \(A_{L^g} = \text{Hom}(L^g, \mathbb{Z})/L^g\) of \(L^g\) is 3-elementary. Let \(L_g = (L^g)\perp\) be the orthogonal of \(L^g\) in \(L\). Then \(g^*|L_g\) has only \(\zeta_3^\pm\) as eigenvalues. Now arguing as in [OZ2, Lemma (1.3)] (for the finite index sublattice \(L^g \oplus L_g\) of \(L\), instead of \(S_X \oplus T_X\)), we can show that \(A_{L^g}\) is 3-elementary.

(3) follows from (2). See [CS, Table 15.2a].

The fixed locus \(X^g\) can be determined:

**Lemma 3.3.** (1) With the assumption and notation in (3.1) and (3.2), either \(X^g = C \bigsqcup R\) is a disjoint union of a genus-5 curve \(C\) and a curve \(R (\cong \mathbb{P}^1)\) (so \(C^2 = 8\), and \(S_X^g = U \supset \mathbb{Z}[C, R]\)), or \(X^g\) equals a single genus-4 curve \(C\) (so \(C^2 = 6\)).
(2) In the former case, \( \Phi|_C : X \to P^5 \) is a degree-2 morphism onto either the Veronese-embedded \( P^2 \) in \( P^5 \) or the normal cone \( \Sigma_4 \) over a rational normal twisted quartic in \( P^4 \).

**Proof.** Since \( \chi(X^g) = -6 \) by (3.2), we have \( n_g = -3 \) and \( m_g = 0 \) in notation of (1.4). \( n(g) < 0 \) infers that \( X^g \) is a disjoint union of a smooth curve \( C \) of genus \( \geq 2 \) and \( t \) of \( P^1 \)'s with \( -6 = 2 - 2g(C) + 2t \) (see (1.2)). The fact that \( \operatorname{rank} S_X^g = 2 \) in (3.2) implies that either \( t = 0 \) (so \( g(C) = 4 \)), or \( t = 1 \) (so \( g(C) = 5 \)) so that the two curves in \( X^g \) give rise to two linearly independent classes of \( S_X^g \).

If \( S_X^g = U(3) \), then \( C^2 = 0 \pmod{3} \) because \( C \) is in \( S_X^g \), whence \( C^2 = 6 \). This proves the first assertion of the lemma, by virtue of (3.2).

Consider the case \( X^g = C \bigsqcup R \). By [SD, Theorem 3.1], \( |C| \) is base point free and we have a morphism \( \varphi := \Phi|_C : X \to P^5 \). Now \( 8 = C^2 = \deg(\varphi) \cdot \deg(\operatorname{Im} \varphi) \), where \( \deg(\operatorname{Im} \varphi) \geq 5 - 1 \). Thus either \( \varphi \) is an embedding modulo the curves in \( C^\perp \), or \( \varphi \) is a degree-2 map as described in (3.3) [SD, Theorem 5.2, Propositions 5.6 and 5.7].

Write \( S_X^g = \mathbb{Z}[u_1, u_2] \) with \( u_i^2 = 0 \) and \( u_1 \cdot u_2 = 1 \). Express \( C = a_1 u_1 + a_2 u_2 \). Then \( 8 = 2a_1 a_2 \) and we may assume that \( (a_1, a_2) = (2, 2) \) or \( (4, 1) \) (after replacing \( u_i \) by \(-u_i \) or switching \( u_1 \) with \( u_2 \) if necessary). So \( C. u_i > 0 \) and hence the Riemann-Roch theorem implies that \( \dim |u_1| \geq 1 \). Write \( |u_1| = |M| + F \) with \( |M| \) the movable part. Then \( 0 < C.M \leq C.u_1 = a_2 \leq 2 \). If \( \varphi \) is birational then \( \varphi(M) \) is a plane conic or a line, whence \( M \cong P^1 \), \( M^2 = -2 \) and \( |M| \) is not movable, a contradiction. This proves the lemma.

We now prove (3.1) (1). Consider the case in (3.3), where \( X^g = C \bigsqcup R \) and \( \varphi = \Phi|_C : X \to P^5 \) is a degree-2 morphism onto the Veronese-embedded \( P^2 \) in \( P^5 \). Since \( C \) (and hence \( |C| \)) is \( G \)-stable, there is an induced action of \( G \) on \( P^5 \) (and hence also an action of \( G \) on the image \( \varphi(X) = P^2 \)) so that the map \( \varphi \) is \( G \)-equivariant. The \( G = A_5 \times \mu_3 \) action on the image is also faithful because \( A_5 \) is simple and \( \deg(\varphi) = 2 \) is coprime to \( 3 \) (\( = |\mu_3| \)).

The action of \( A_5 \) on the image is via \( A_5 \subset SL_3(\mathbb{C}) \subset PGL_3(\mathbb{C}) \) and is given in Burnside [Bu, §232, or §266] (1.8). In particular, the commutativity of \( g \) with the two generators (order 5 and 2) of \( A_5 \) in [Bu, §266] shows that \( g \) is a scalar and acts trivially on the image \( \varphi(X) = P^2 \), a contradiction.
Consider the case in (3.3), where \(X^g = C \bigsqcup R\) in (3.3) and \(\varphi = \Phi_{|C|} : X \to \mathbf{P}^5\) is a degree-2 morphism onto the cone \(\Sigma_4\). Note that the minimal resolution \(\Sigma_4\) of \(\Sigma_4\) is the Hirzebruch surface of degree 4. As in the previous case, there is a faithful action of \(G\) on \(\Sigma_4\) such that \(\varphi\) is \(G\)-equivariant. Note that the image \(\varphi(C)\) is a hyperplane section away from the singularity and with \(\varphi(C)^2 = 4\). Let \(\ell\) be a generating line of the cone \(\Sigma_4\). Then \(\varphi(C) \sim 4\ell\) as Weil divisors. This gives rise to a \(\mathbf{Z}/(4)\)-cover \(\pi : Y = \text{Spec} \oplus_{i=0}^3 \mathcal{O}_{\Sigma_4}(-i\ell) \to \Sigma_4\) which is (totally) ramified exactly over \(\varphi(C)\). One sees that \(Y \cong \mathbf{P}^2\) and \(\pi^*\varphi(C) = 4L\) with \(L\) a line. Clearly, \(A_5 \subsetneq G\) stabilizes the divisorial sheaves \(\mathcal{O}(-i\ell)\) and fixes the defining equation of \(\varphi(C)\), so there is an induced faithful \(A_5\)-action on \(Y = \mathbf{P}^2\) so that \(\pi\) is \(A_5\)-equivariant (see (1.7)). Now \(L\) is stabilized by \(A_5\) (because so is \(\varphi(C)\)). So the defining equation \(F_1 = 0\) of \(L\) is semi \(A_5\)-invariant (and hence \(A_5\)-invariant because of the simplicity of the group \(A_5\)). But every \(A_5\)-invariant form is of even degree by [Bu, §266], noting also that the action of \(A_5\) on \(\mathbf{P}^2\) is via \(A_5 \subset SL_3(\mathbf{C}) \to PGL_3(\mathbf{C})\) by (1.8). We reach a contradiction.

Consider the case \(X^g = C\) in (3.3). Let \(f : X \to Y = X/\langle g \rangle\) be the quotient map. There is an induced faithful action \(A_5\) on \(Y\) so that \(f\) is \(A_5\)-equivariant. Then by the ramification divisor formula, \(0 \sim K_X = f^*(K_Y) + 2C\). Pushing down by \(f_*\), one obtains \(0 \sim 3K_Y + 2B\) with \(B = f_*C = f(C) \cong C\) and \(f^*B = 3C\), so \(B^2 = 3C^2 = 18\). Solving, one obtains \(B = (-3/2)K_Y\) and \(K_Y^2 = 8\). Thus the smooth ruled surface \(Y\) equals a Hirzebruch surface \(\Sigma_d\) of degree \(d\). The irreducibility of \(B\) (being a \(\mathbf{Z}\)-divisor) implies that \(d = 0, 2\) [Ha, Ch V, Cor. 2.18].

Suppose that \(d = 2\). Then the (-2)-curve \(M\) on \(Y\) is disjoint from \(B = (-3/2)K_Y\) and hence \(f^*M = \coprod_{i=1}^3 M_i\) is a disjoint union of three (-2)-curves not intersecting \(C\). Since \(M\) is clearly \(A_5\)-stable, the set \(\coprod M_i\) is also \(A_5\)-stable, whence each \(M_i\) is \(A_5\)-stable (1.7). An order-5 element \(5A\) in \(A_5\) acts on each \(M_i\) faithfully by (1.2) and has exactly two fixed points by (1.8). But according to (1.2), \(4 = |X^{5A}| \geq \sum_{i=1}^3 |M_i^{5A}| = 6\), a contradiction.

Thus \(d = 0\). Clearly, the simple group \(A_5\) stabilizes each ruling and there is an induced action \(\rho_i : A_5 \times \mathbf{P}^1 \to \mathbf{P}^1\) with \(i = 1, 2\) for the \(i\)-th \(\mathbf{P}^1\) in \(Y = \Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1\) so that \(\rho_1 \times \rho_2\) is the given \(A_5\) action on \(Y\). Changing coordinates suitably we may assume that the \(A_5\) action on \(Y\) commutes with the involution \(\iota\) of \(Y\) switching the two \(\mathbf{P}^1\)'s in \(Y\), i.e.,
\( \rho_1 = \rho_2 \) as actions of \( A_5 \) on the same \( \mathbb{P}^1 \) (1.8).

Let \( j : Y \to Z = Y/\langle \iota \rangle = \mathbb{P}^2 \) be the quotient map. Then there is an induced faithful action of \( A_5 \) on \( \mathbb{P}^2 \) such that \( j \) is \( A_5 \)-equivariant. Now \( \iota(B) \) is an irreducible curve with \((\iota(B))^2 = B^2/2 = 9\). It is a cubic curve and \( A_5 \)-stable because so is \( B = f(C) \). The action of \( A_5 \) on \( Y/\langle \iota \rangle = \mathbb{P}^2 \) is via \( SL_3(\mathbb{C}) \to PGL_3(\mathbb{C}) \) (1.8). The defining equation \( F_3 \) of \( \iota(B) \) is then a cubic form and semi \( A_5 \)-invariant (and hence \( A_5 \)-invariant by the simplicity of the group \( A_5 \)). However, Burnside [Bu, §266] shows that every \( A_5 \)-invariant form is of even degree, a contradiction. This completes the proof of (3.1) (1) and also of (3.1).

We now prove Theorem B. Suppose that \( G = A_5.C_n \) acts faithfully on a \( K3 \) surface \( X \). By (1.0A), \( A_5 \leq G_N \). So \( G_N = A_5, S_5, A_6 \) or \( M_{20} = C_2^\oplus 4 : A_5 \) by [Xi, the list]. In notation of (1.0), for some \( m \mid n \), we have \( G_N = Ker(\alpha) = A_5.C_m \) and \( G/G_N = \mu_I \), where \( n = mI \).

By the same proof of (1.1), we have \( I = 1, 2, 3, 4, \) or \( 6 \). Let \( h \in G \) such that the coset of \( h \) is a generator of \( G/A_5 = C_n \). Then \( h^* \omega_X = \eta_I \omega_X \) for some primitive \( I \)-th root \( \eta_I \) of 1.

Note that \( n \mid ord(h) \) and \( h^I \in G_N \), whence \( ord(h^I) \leq 8 \) by (1.2). Thus \( ord(h) = I \ord(h^I) \) and \( m \mid \ord(h^I) \). In particular, \( |G_N| = m|A_5| \leq 8|A_5| \). Hence \( G_N \neq M_{20} \).

If \( G_N = A_5.C_m = A_6 \), then \( m = 6 \) and \( A_6 \) includes \( \langle h^I \rangle \geq C_6 \), which is impossible. If \( G_N = A_5 \), then \( C_n = \mu_I \) in notation of (1.0), and Theorem B follows from (3.1).

Consider the case \( G_N = S_5 \). Then \( m = 2 \) and \( n = 2I \). Moreover, \( G_N = \langle A_5, h^I \rangle \). So \( h^I \in S_5 - A_5 \). Since \( S_5 \to Aut(S_5) (x \mapsto c_s) \) is an isomorphism, we have \( c_h = c_s \) for some \( s \in S_5 \). Set \( g = hs^{-1} \). Then \( g \) commutes with every element in \( S_5 \) and also \( \alpha(g) = \alpha(h) \) is a generator of \( \text{Im}(\alpha) = \mu_I \). Now \( g^I \in Ker(\alpha) = G_N = S_5 \) is in the centre of \( S_5 \) (which is (1)). So \( ord(g) = I \) and \( G = G_N \times \langle g \rangle = S_5 \times \mu_I > A_5 \times \mu_I \). By (3.1), we have \( I = 1, 2 \) or \( 4 \). The \( S_5 \times \mu_I \) should have an element \( h \) such that \( h^I \in S_5 - A_5 \) (i.e., \( h^I \) is not an even permutation). Thus, \( I \neq 2, 4 \). Therefore, \( I = 1 \) and \( G = G_N = S_5 \). This completes the proof of Theorem B.

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