INVERSE PROBLEM FOR KIRCHHOFF-LOVE PLATE EQUATION

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Abstract. We consider the two-dimensional Kirchhoff-Love plate equation in the context of elasticity modeling the stresses and deformations in thin plates subjected to forces and moments. We establish global recovery of the material parameters like bending stiffness, Poisson coefficient, Lamé parameters from the associated boundary Cauchy data of the equation.

1. Introduction and the statement of the main result

This article studies an inverse problem of recovering elasticity parameters like stiffness, Lamé parameters of a smooth elastic domain \( \Omega \subset \mathbb{R}^2 \) under consideration. Motivated by the idea that a mid-surface plane can be used to represent a three-dimensional plate \( (\Omega \times [-\frac{h}{2}, \frac{h}{2}], \text{where } h > 0 \text{ small is the width of the domain) in two-dimensional form, the Kirchhoff-Love theory of plates offers such two-dimensional mathematical model to determining the stresses and deformations in thin plates subjected to forces and moments. Considering the Kirchhoff’s plate equation, it is formally given by a fourth order variable coefficient elliptic operator where the coefficients depict the elasticity properties of the material. We would like to see whether such material parameters inside the domain can be determined from the associated boundary Cauchy data of the Kirchoff-Love plate equation. To make things precise let us begin with the direct problem.

1.1. Direct Problem. The Kirchhoff-Love plate (KL) operator is formally given by

\[
K_{B,q} v(x,y) := \text{div} \left( \text{div} \left( B(1-q) \nabla^2 v + Bq \Delta v I \right) \right), \text{ in } \Omega. \tag{1.1}
\]

Here \( B(x,y) \in C^\infty(\Omega) \) is the bending stiffness and \( q(x,y) \in C^\infty(\Omega) \) is the Poisson’s coefficient in \( \Omega \). The bending stiffness \( B(x,y) \) and the Poisson coefficient \( q \) are given as

\[
B(x,y) := \frac{h^3 E(x,y)}{12(1-q^2(x,y))}, \quad q(x,y) := \frac{\lambda(x,y)}{2(\lambda(x,y) + \mu(x,y))}, \tag{1.2}
\]

where \( \lambda, \mu \) are the Lamé parameters and \( E(x,y) \) is the Young’s modulus given as

\[
E(x,y) = \frac{\mu(x,y) [2\mu(x,y) + 3\lambda(x,y)]}{\lambda(x,y) + \mu(x,y)}. \]

We make the strong convexity assumption on the Lamé parameters as

\[
\mu(x,y) \geq \alpha_0 > 0, \quad 2\mu(x,y) + 3\lambda(x,y) \geq \beta_0 > 0, \text{ in } \Omega. \tag{1.3}
\]

As a consequence of the strong convexity conditions we obtain that \( B(x,y) \neq 0 \) and \( q \neq \pm 1 \) everywhere in \( \Omega \).

In general, \( K_{B,q} \) can be written as the perturbed (upto third order) bi-harmonic operator as \((-\Delta)^2 + \sum_{|\alpha|=3,2} c_\alpha D^\alpha \). We refer \([1, 13, 18, 39, 14]\) for an overall details on fourth order elliptic equation including this one.

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Let us now consider the Dirichlet boundary value problem
\begin{align}
\mathcal{K}_{B,q}u &= 0 \quad \text{in } \Omega, \\
(u, \partial_u) &= (f_0, f_1) \quad \text{on } \partial \Omega,
\end{align}
(1.4)
with \((f_0, f_1) \in H^2(\partial \Omega) \times H^2(\partial \Omega)\). The solution \(u(x, y)\) represents the transversal displacement, lies in \(H^4(\Omega)\). The smoothness assumptions on the coefficients \(B, q\) and the domain \(\Omega\), gives the strong solution in \(H^4(\Omega)\) following the standard regularity results, see [1, 14, 18]. The solution operator is continuous, that is
\[ S_{B,q} : H^2(\partial \Omega) \times H^2(\partial \Omega) \to H^4(\Omega), \quad (f_0, f_1) \mapsto u \]
is bounded
Taking into account that the domain of \(\mathcal{K}_{B,q}\), \(\mathcal{D}(\mathcal{K}_{B,q}) := \{w \in H^4(\Omega) : w|_{\partial \Omega} = \partial_\nu w|_{\partial \Omega} = 0\}\), and the above map is injective or the solution becomes unique when 0 is not an eigenvalue for the operator \(\mathcal{K}_{B,q}\) in \(\mathcal{D}(\mathcal{K}_{B,q})\).

Let us define the boundary Cauchy data \(\mathcal{C}_{B,q}\) in \(H^2(\partial \Omega) \times H^2(\partial \Omega) \times H^2(\partial \Omega) \times H^2(\partial \Omega)\) corresponding to the above problem (1.4) as
\[ \mathcal{C}_{B,q} := \left( u|_{\partial \Omega}, \partial_\nu u|_{\partial \Omega}, (-\Delta)u|_{\partial \Omega}, \partial_\nu (-\Delta)u|_{\partial \Omega} \right). \]
(1.5)
\[ \] 1.2. Inverse Problem. We would like to ask whether the boundary Cauchy data \(\mathcal{C}_{B,q}\) can determine \(B, q\) uniquely in \(\Omega\) and further the unique determinations of the Lamé parameters \(\lambda\) and \(\mu\). This is the inverse problem we study here. It falls into the category of Calderón type inverse problem, which goes back to 1980s [11]. In Calderón type inverse problems one uses static voltage and current measurements at the boundary of an object to know about its internal conductivity. If one assumes the conductivity is isotropic and regular then we do recover it from the voltage current measurements at the boundary. See the seminal works of Sylvester and Uhlmann [41] in dimension three and higher and Nachmann [32] for the two dimensional case. If the conductivity is anisotropic, then the unique recovery assertion fails [42]. We refer readers to the survey article [43] and references there in.

Here in this paper, we work with the isotropic parameters \(B\) and \(q\) (cf. (1.2)) and show that it is possible to determine them from the Cauchy data \(\mathcal{C}_{B,q}\) measured on the boundary. Our result also include the global recovery of the Lamé parameters \(\lambda\) and \(\mu\) as well.

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^2\) be a bounded domain with smooth connected boundary. Let \(\mathcal{K}_{B,q}(x, D)\) and \(\mathcal{K}_{\tilde{B}, \tilde{q}}(x, D)\) be two operators defined as in (1.1) with the coefficients \((B, q)\) and \((\tilde{B}, \tilde{q})\) respectively. We assume all the coefficients are \(C^\infty(\overline{\Omega})\) smooth and \(B = \tilde{B}\) on \(\partial \Omega\). Let \(\mathcal{C}_{B,q} = \mathcal{C}_{\tilde{B}, \tilde{q}}\) on \(\partial \Omega\), where the Cauchy data \(\mathcal{C}_{B,q}\), \(\mathcal{C}_{\tilde{B}, \tilde{q}}\) are defined as in (1.5) for the pair of coefficients \((B, q)\) and \((\tilde{B}, \tilde{q})\). Then \(B = \tilde{B}\) and \(q = \tilde{q}\) in \(\Omega\).

Moreover, if the corresponding the Lamé moduli are given by \(\lambda, \mu \in C^\infty(\overline{\Omega})\) and \(\tilde{\lambda}, \tilde{\mu} \in C^\infty(\overline{\Omega})\) respectively with satisfying (1.3), then it further implies \(\lambda = \tilde{\lambda}\) and \(\mu = \tilde{\mu}\) in \(\Omega\).

As we pointed out above that the KL operator falls into the category of perturbed biharmonic operators (fourth order) with lower order perturbations upto third order, in particular of the form:
\[ \Delta^2 + W(x, y) \cdot \nabla \circ \Delta + A(x, y) : \nabla^2, \quad \text{in } \Omega, \]
where \(W\) is a vector field and \(A\) is a symmetric matrix.

The novelty of this article is that we are able to come out with a theory of determining higher order perturbed coefficients of a fourth order operator in the interior by measuring the Cauchy data.
on the boundary of the plane. A study of the unique continuation (strong) from boundary for the KL operator and perturbed biharmonic operator in plane has been done in [2]. We also refer recent studies [30, 31] addressing doubling inequalities, propagation of smallness for the KL operator. However, such studies from the point of view of solving inverse problem previously only known in the dimensions 3 and higher, and that too quite restrictive in terms of allowing all the lower order perturbations and whether they (perturbed terms) are given by the lower order tensors or not. For biharmonic operator with only first order perturbation on a bounded domain in \( \mathbb{R}^n, n \geq 3 \), we refer [28, 21, 27, 16, 15, 3, 4, 7, 8] and others, where recently in [8] the authors showed the recovery of a general second order perturbation of a biharmonic operator, given by a symmetric matrix. The main method of the proof lies on performing various Carleman estimates for the principal part of the operator, which is bi-Laplacian in our case. In this two dimensional setup we take the advantage of a stronger Carleman estimate, in order to recover even a third order perturbation of a biharmonic operator, which was out of the scope for [8]. We will briefly present our method soon. At this point, let us mention the work [29] for a detailed understanding and wide references on the related topics of biharmonic operator.

**A brief discussion on the techniques.** A general approach to solve Calderón type inverse problems follows from the pioneering work of [41, 26]. In two-dimensional setup it includes [32, 5, 9, 22, 24, 23]. We consider two sets of Dirichlet problems having two sets of parameters under observation with the same Cauchy data at the boundary. Using integrations by parts formula and the equality of the Cauchy data at the boundary, we obtain an integral equation concerning the difference of the unknown parameters and the solutions of the two operators:

\[
\int_{\Omega} \left[ \left( B - \tilde{B} \right) \left( \nabla^2 \tilde{u} : \nabla^2 u \right) - \left( Bq - \tilde{Bq} \right) \left( \nabla^2 \tilde{u} : \nabla^2 u \right) \right] \, dx \, dy + \int_{\Omega} (Bq - \tilde{Bq}) \Delta \tilde{u} \Delta u \, dx \, dy = 0
\]

where \( \tilde{u}, u \in H^4(\Omega) \) satisfies \( K_{\tilde{B},q} \tilde{u} = 0 \) in \( \Omega \), and \( K_{B,q}u = 0 \) in \( \Omega \) respectively.

Now, to recover the unknown parameters from the integral equation, we intend to produce enough number of solutions of both the KL operators. To show existence of an ample amount of solutions of the KL operators we use the method of constructing Complex Geometric Optics (CGO) solutions of the form \( e^{i\varphi(x,y) + r(x,y)} \), where \( \varphi \) is harmonic and \( \psi \) its harmonic conjugate, and the amplitude function \( a \) is needed to be determined from the transport equation given below:

\[
\partial_z^2 a(x,y) + P(x,y)\partial_x a(x,y) - Q(x,y)a(x,y) = 0, \quad \text{in } \Omega
\]

where \( \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \), and \( P, Q \) are smooth coefficients. The remainder term \( r \) is small and goes to 0 in appropriate semiclassical norm as \( \hbar \to 0 \). We take advantage of the two dimensional set up of using any harmonic functions as the phase function or as the Carleman weight, which is certainly not possible in dimensions 3 or higher. See Lemma 2.1 for the Carleman estimate we use. To construct the amplitudes solving the above transport equation, we seek another CGO ansatz for it as \( e^{i\xi + \eta} \left( a_0(x,y) + \rho(x,y) \right) \), where \( \xi \) is harmonic and \( \eta \) its harmonic conjugate, \( a_0 \) is any smooth function, and the remainder term here \( \rho \) goes to 0 in appropriate semiclassical norm as \( \tau \to 0 \). We further use Carleman estimates on \( d \)-bar operator to execute this, see Lemma 2.4. By constructing the CGO solutions in this two step fashion, we get access to a large class of solutions for the KL operator and finally obtain the uniqueness of the stiffness and Poisson coefficients in \( \Omega \).

Let us also mention that our technique does not work in solving Calderón problem in plane. There one is interested to determine \( V \in L^\infty(\Omega) \) from the boundary Cauchy data associated with this Schrödinger equation \( (-\Delta + V)u = 0 \) in \( \Omega \subset \mathbb{R}^2 \). By using the Carleman estimate [17, Lemma
proves the Theorem 1.1 using the constructions given in the previous section. In Section 3, we show the recovery of the Lamé parameters \( \lambda \) and \( \mu \) in \( \Omega \). It was soon followed by a series of works determining other material parameters from the boundary measurements, see [36, 38, 20, 40, 6, 10] etc.

Let us end our discussion here by mentioning the plan of the rest of the article. It is divided into two major parts, given as Sections 2 and 3, where Section 2 is purely constructive and Section 3 proves the Theorem 1.1 using the constructions given in the previous section. In Section 2 we construct CGO solutions of the KL plate operator \( K_{B,q} \) in \( \Omega \). As we mentioned before the construction of CGOs is divided into two major parts where the later part includes the amplitudes which comes from the first part. We introduce them in Sections 2.1 and 2.2 respectively, and complete the discussion in Section 2.3. Next in Section 3, from the integral identity we determine the parameters i.e. \( B = \tilde{B} \) and \( q = \tilde{q} \) in \( \Omega \), and finally, in Section 3.2 we show the recovery of the Lamé parameters \( \lambda \) and \( \mu \) in \( \Omega \).

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2. Carleman estimate and C.G.O. solutions

In this section we construct complex geometric optics (CGO) type solutions of \( K_{B,q}(x,D)u = 0 \) in \( \Omega \). To construct complex geometric optics solutions we need certain solvability result for the correction term, with desired decay estimates. We use the method of Carleman estimates to derive suitable weighted norm estimates.
Let $h > 0$ be a small parameter, we define the semiclassical Fourier transform $\mathcal{F}_h$ on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ as

$$\hat{f}_h(\xi) = \mathcal{F}_h f(\xi) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{-\frac{i}{h}(x,\xi)} f(x) dx, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^2).$$

We define the semiclassical Sobolev space $H^s_{sc}(\mathbb{R}^2)$, $s \in \mathbb{R}$, via the semiclassical Fourier transform as

$$H^s_{sc}(\mathbb{R}^2) := \{v \in L^2(\mathbb{R}^2) : (1 + |\xi|^2)\hat{v}_h(\xi) \in L^2(\mathbb{R}^2)\}.$$

For a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, we define the semiclassical Sobolev space $H^s_{sc}(\Omega)$ as the restriction of $H^s_{sc}(\mathbb{R}^2)$ in $\Omega$. For $s = m$ a positive integer we get

$$\|u\|_{H^m_{sc}(\Omega)} \simeq \sum_{|\alpha| \leq m} \|hD^\alpha u\|_{L^2(\Omega)},$$

where $\simeq$ denotes equivalence in the two norms on both sides of the above relation. We define $H^{s,0}_{sc}(\Omega)$ to be closure of the $C_c^\infty(\Omega)$ in $H^s_{sc}(\Omega)$ for any $s > 0$. For $s < 0$ one can realize $H^s_{sc}(\Omega)$ to be the dual of the space $H^{-s}_{sc,0}(\Omega)$.

First let us observe that

$$K_{B,q}(x,y,D)v = \text{div} \left( \text{div} \left( B(1-q)\nabla^2 v + Bq\Delta v I \right) \right) = B\Delta^2 v + 2\nabla B \cdot \nabla \Delta v + E(x,D)v,$$

where $E(x,y,D) = E_{B,q}(x,D)$ is a second order differential operator given as

$$E_{B,q}(x,D)v := \Delta(qB)\Delta v + (\partial_x^2 ((1-q)B)) \partial_x^2 v + (\partial_y^2 ((1-q)B)) \partial_y^2 v + 2(\partial_x \partial_y ((1-q)B)) \partial_x \partial_y v.$$  \hfill (2.1)

Owing to the fact that $B \neq 0$ anywhere in $\Omega$ (see (1.3)), we write

$$Lv := \frac{1}{B}K_{B,q}(x,y,D) = \Delta^2 v + 2\nabla \log B \cdot \nabla \Delta v + \frac{1}{B}E(x,y,D)v. \quad \text{(2.2)}$$

Let us assume that $q, \tilde{q}, B, \tilde{B} \in C^\infty(\overline{\Omega})$ such that $\partial_a^\alpha B = \partial_a^\alpha \tilde{B}$ on $\partial \Omega$ for $|\alpha| = 0, 1$.

Here let us define the component wise matrix multiplication as $M : N$ where $M, N$ are two $2 \times 2$ complex matrices and $M : N$ is given as

$$M : N := \sum_{j,k=1}^2 M_{jk} N_{jk} \in \mathbb{C}.$$  

Therefore, expanding the operator $L$ we get

$$L_{W,A}v = \Delta^2 v + W(x,y) \cdot \nabla (\Delta v) + A(x,y) : \nabla^2 v, \quad \text{(2.3)}$$

where $W$ is a vector field, $A$ is a symmetric matrix given as

$$W = \nabla \log B, \quad A = \frac{1}{B} \begin{pmatrix} \Delta(qB) + \partial_x^2 ((1-q)B) & \partial_x \partial_y ((1-q)B) \\ \partial_x \partial_y ((1-q)B) & \Delta(qB) + \partial_y^2 ((1-q)B) \end{pmatrix}. \quad \text{(2.4)}$$
2.1. Carleman Estimates. In order to construct a solution of the operator $L$ and thus a solution of $K_{B,q}$ we first prove a Carleman estimate for the principal operator $\Delta^2$.

Lemma 2.1. Let $\Omega \subset \mathbb{R}^2$ be a regular bounded open set. Let $\varphi(x,y)$ be a harmonic function in $\Omega$ such that $|\nabla \varphi| > c$ for some $c > 0$ in $\Omega$. Then, for all real valued $w \in C_0^\infty(\Omega)$, and for small positive parameters $h > 0$ we have

$$
\|e^{-\frac{h^2}{2}}(-h^2\Delta)^2 e^\frac{h^2}{2} w\|_{L^2} \geq Ch \|w\|_{H^3_{sc}(\Omega)},
$$

(2.5)

where $C > 0$ is independent of $h > 0$ and $w$.

Proof. Let $\varphi$ be harmonic in $\Omega \subset \mathbb{R}^2$. Let $\psi \in C^\infty(\Omega)$ be the harmonic conjugate of $\varphi$ satisfying the set of Cauchy Riemann equations

$$
\partial_x \varphi = -\partial_y \psi \quad \text{and} \quad \partial_y \varphi = \partial_x \psi, \quad \text{in} \quad \Omega.
$$

Note that, since we have assumed $|\nabla \varphi| > c > 0$, therefore, we must have $|\nabla \psi| > c > 0$ in $\Omega$. From a direct calculation, using the fact that $\varphi$ and $w$ are real valued, we observe that

$$
e^{-\frac{h^2}{2}}h(\pm \partial_x + i \partial_y)(e^\frac{h^2}{2} w) = e^{\frac{h^2}{2}}h(\pm \partial_x + i \partial_y)(e^{\frac{h^2}{2}} w).
$$

(2.6)

Using a similar technique as in [17, Lemma 3.4] we obtain

$$
\left\| e^{\frac{h^2}{2}}h(\pm \partial_x + i \partial_y)(e^{\frac{h^2}{2}} w) \right\|_{L^2}^2 = \| h \partial_y w + w \partial_x \psi \|_{L^2}^2 + \| h \partial_y w + w \partial_x \psi \|_{L^2}^2.
$$

(2.7)

Now, using Poincaré inequality and accounting (2.6) we obtain

$$
\left\| e^{-\frac{h^2}{2}}h(\pm \partial_x + i \partial_y)(e^\frac{h^2}{2} w) \right\|_{L^2} \geq \| h \nabla w \|_{L^2(\Omega)} \geq Ch \|w\|_{L^2(\Omega)}, \quad \forall w \in C_c^\infty(\Omega).
$$

(2.8)

Now, note that

$$
e^{-\frac{h^2}{2}}(-h^2\Delta)^2(e^\frac{h^2}{2} w) = \left(e^{-\frac{h^2}{2}}h^2(\partial_x + i \partial_y)^2 e^\frac{h^2}{2}\right) \left(e^{-\frac{h^2}{2}}h^2(\partial_x + i \partial_y)^2 e^\frac{h^2}{2}\right) w.
$$

Let us denote $P_{\varphi,\pm} := (e^{-\frac{h^2}{2}}h(\pm \partial_x + i \partial_y)e^\frac{h^2}{2})$ and $\langle hD \rangle := (1 + |h\nabla|^2)^\frac{1}{2}$. Observe that

$$
\| \langle hD \rangle a \|_{H^{s}_{sc}} \simeq \|a\|_{H^{s+1}_{sc}}, \quad \forall s \geq 0, \quad \forall a \in C^\infty(\overline{\Omega}).
$$

Using (2.7), (2.8) repeatedly, for $w \in C_0^\infty(\Omega)$ we see

$$
\left\| e^{-\frac{h^2}{2}}h^4 \Delta^2 e^\frac{h^2}{2} w \right\|_{L^2} \geq Ch \|P_{\varphi,+}(P_{\varphi,-}^2 w)\|_{L^2} \geq Ch \|P_{\varphi,-}^2 w\|_{H^1_{sc}} \geq Ch \|w\|_{H^3_{sc}}.
$$
Here the last inequality follows from the fact that

\[
\|P_{\varphi,-}^2w\|_{H^s_{\text{sc}}} \simeq \|\langle hD \rangle P_{\varphi,-}^2w\|_{L^2} \\
\geq \|P_{\varphi,-}^2(\langle hD \rangle)w\|_{L^2} - Ch \|E_0w\|_{L^2} \\
\geq \|P_{\varphi,-}^2(\chi \langle hD \rangle)w\|_{L^2} - \|P_{\varphi,-}^2(1 - \chi)\langle hD \rangle)w\|_{L^2} - Ch \|w\|_{H^s_{\text{sc}}} \\
\geq \|P_{\varphi,-}^2(\chi \langle hD \rangle)w\|_{H^s_{\text{sc}}} - \|P_{\varphi,-}^2\langle hD \rangle(1 - \chi)w\|_{L^2} - Ch \|E_1w\|_{L^2} - Ch \|w\|_{H^s_{\text{sc}}} \\
\geq \|\langle hD \rangle P_{\varphi,-}^2(\chi \langle hD \rangle)w\|_{L^2} - Ch \|w\|_{H^s_{\text{sc}}} \\
\geq \|\langle hD \rangle^3\chi w\|_{L^2} - Ch \|E_2w\|_{H^s_{\text{sc}}} - Ch \|w\|_{H^s_{\text{sc}}} \\
\geq \|w\|_{H^s_{\text{sc}}} - Ch \|w\|_{H^s_{\text{sc}}} \geq \frac{1}{2} \|w\|_{H^s_{\text{sc}}}, \quad \text{for small enough } h > 0,
\]

where $E_0, E_1, E_2$ are Pseudodifferential Operators ($\Psi$DOs) of order 2 and $\chi \in C^\infty_0(\mathbb{R}^2)$ is a cutoff function having $\chi = 1$ in $\Omega$ and $0 \leq \chi \leq 1$ in $\mathbb{R}^2$.

Let us assume that $v = \chi \langle hD \rangle^{-s}w$ for $s = 1, 2, \ldots$, where $\langle hD \rangle := (1 + |hD|^2)^{\frac{1}{2}}$ and $\chi \in C^\infty_0(\tilde{\Omega})$ such that $\chi = 1$ in $\Omega$, where $\tilde{\Omega}$ is an open neighbourhood of $\Omega$ in $\mathbb{R}^2$. From a direct calculation setting $s = 3$ we get

\[
C \|v\|_{H^s_{\text{sc}}(\mathbb{R}^2)} \geq \|\langle hD \rangle^s\chi \langle hD \rangle^{-s}w\|_{L^2(\mathbb{R}^2)} \\
\geq \|\chi w\|_{L^2(\mathbb{R}^2)} - Ch \|E_{s-1}\langle hD \rangle^{-s}w\|_{L^2(\mathbb{R}^2)} \geq C \|w\|_{L^2(\Omega)} - Ch \|w\|_{L^2(\Omega)},
\]

for $h > 0$ small, where $E_{s-1}$ is a $\Psi$DO of order $s - 1$ in $\mathbb{R}^2$. On the other hand, we have

\[
\left|e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}v\right|_{L^2(\mathbb{R}^2)} \leq C \left|\chi e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}w\right|_{L^2(\mathbb{R}^2)} + Ch \|E_1\langle hD \rangle^{-s}w\|_{L^2(\mathbb{R}^2)} \\
\leq C \left|\langle hD \rangle^s\chi e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}w\right|_{H^s_{\text{sc}}(\mathbb{R}^2)} + Ch \|w\|_{L^2(\mathbb{R}^2)} \\
\leq C \left|e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}w\right|_{H^s_{\text{sc}}(\mathbb{R}^2)} + Ch \|E_{s-1}\langle hD \rangle^{-s}w\|_{H^s_{\text{sc}}(\mathbb{R}^2)} + Ch \|w\|_{L^2(\mathbb{R}^2)} \\
\leq C \left|e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}w\right|_{H^{s-1}_{\text{sc}}(\mathbb{R}^2)} + Ch \|E_{s-1}\langle hD \rangle^{-s}w\|_{H^{s-1}_{\text{sc}}(\mathbb{R}^2)} + Ch \|w\|_{L^2(\mathbb{R}^2)}. 
\]

Now, combining (2.9) and (2.10) we obtain

\[
C \|w\|_{L^2(\Omega)} - Ch \|w\|_{L^2(\Omega)} \leq C \|v\|_{H^s_{\text{sc}}(\mathbb{R}^2)} \leq \left|e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}v\right|_{L^2(\mathbb{R}^2)} \\
\leq C \left|e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}w\right|_{H^s_{\text{sc}}(\mathbb{R}^2)} + Ch \|w\|_{L^2(\mathbb{R}^2)} \\
\implies C \|w\|_{L^2(\Omega)} \leq C \left|e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}w\right|_{H^{s-1}_{\text{sc}}(\mathbb{R}^2)} + Ch \|w\|_{H^s_{\text{sc}}(\mathbb{R}^2)}.
\]

Therefore, using the estimate (2.11) for $s = 3$ we obtain

\[
\left|e^{-\frac{\pi}{h}\langle hD \rangle}e^{\frac{\pi}{h}\langle hD \rangle}w\right|_{H^{-3}_{\text{sc}}(\mathbb{R}^2)} \geq Ch \|w\|_{L^2(\Omega)}. 
\]
Now, adding lower order perturbations in the estimate we obtain

\[ C h \| w \|_{L^2(\Omega)} \leq \left\| e^{-\frac{i}{h} x^4} W, A e^{i\varphi} w \right\|_{H^{-3}(\mathbb{R}^2)} - 2 \left\| e^{-\frac{i}{h} x^4} W \cdot \nabla \Delta \left( e^{i\varphi} w \right) \right\|_{H^{-3}(\mathbb{R}^2)} \]

\[ - \left\| e^{-\frac{i}{h} x^4} A : \nabla^2 \left( e^{i\varphi} w \right) \right\|_{H^{-3}(\mathbb{R}^2)}. \tag{2.13} \]

With a straightforward calculation similar in [7, 8] we can show that

\[ \left\| e^{-\frac{i}{h} x^4} W \cdot \nabla \Delta \left( e^{i\varphi} w \right) \right\|_{H^{-3}(\mathbb{R}^2)} \leq C W h^2 \| w \|_{L^2(\Omega)}, \tag{2.14} \]

\[ \left\| e^{-\frac{i}{h} x^4} A : \nabla^2 \left( e^{i\varphi} w \right) \right\|_{H^{-3}(\mathbb{R}^2)} \leq C W h^2 \| w \|_{L^2(\Omega)}. \tag{2.15} \]

where \( C_W, C_{W,A} \) are constants independent of \( h \) but depending on \( W, A \) and their derivatives respectively. Using the bounds (2.14), (2.15), in (2.13) we readily obtain

\[ C h \| w \|_{L^2(\Omega)} \leq \left\| e^{-\frac{i}{h} x^4} W, A e^{i\varphi} w \right\|_{H^{-3}(\mathbb{R}^2)}, \text{ for small enough } h > 0. \tag{2.16} \]

**Proposition 2.2.** Let \( F \in L^2(\Omega) \) then there exists \( r \in H^3_{\text{scf}}(\Omega) \) such that

\[ e^{-\frac{i}{h} x^4} W, A e^{i\varphi} r(x, y; h) = F(x, y), \text{ in } \Omega, \tag{2.17} \]

with the decay estimate

\[ \| r(x, y; h) \|_{H^3_{\text{scf}}(\Omega)} \leq C h^{-1} \| F \|_{L^2(\Omega)} \text{ in } \Omega. \tag{2.18} \]

The proof follows from a functional analysis argument and the Carleman estimate (2.16), which is standard in the literature. See [7, 8] for instance.

### 2.2. Construction of C.G.O. solutions.

Here we construct complex geometric optics type solutions for the operator \( L_{W,A} \). Let us denote \((x, y)\) to be the coordinates in \( \Omega \subset \mathbb{R}^2 \). Let \( \varphi \in C^\infty(\overline{\Omega} : \mathbb{R}) \) such that \( |\nabla \varphi| \neq 0 \) in \( \Omega \). We propose ansatz of \( L_{W,A} u = 0 \) the form

\[ u(x, y; h) = e^{(\varphi + i\psi)/h}(a(x, y) + r(x, y; h)), \tag{2.19} \]

where \( 0 < h \ll 1 \), \( a(x, y) \) is an amplitudes to be determined and \( r(x, y; h) \) is the correction term vanishing in \( \Omega \) as \( h \to 0 \). The real valued phase function \( \psi \in C^\infty(\overline{\Omega}) \) is chosen to be the complex conjugate of \( \varphi \), that is it satisfies the Cauchy-Riemann equations

\[ \partial_x \varphi = -\partial_y \psi, \quad \partial_y \varphi = \partial_x \psi. \tag{2.20} \]

Note that, by our choices of the phase functions, we have

\[ \nabla(\varphi \pm i\psi) \cdot \nabla(\varphi \pm i\psi) = 0 = \Delta(\varphi \pm i\psi), \text{ in } \Omega. \tag{2.21} \]

Now, to calculate \( L_{W,A} u(x, y, h) \), let us denote the transport operator

\[ T = [(\nabla \varphi + i \nabla \psi) \cdot \nabla] = 4 \left( \overline{\partial_z \varphi} \right) \partial_{\bar{z}}, \]
where $\partial_z = \frac{1}{2} (\partial_x - i \partial_y)$ and $\overline{\partial}_z = \frac{1}{2} (\partial_x + i \partial_y)$. We expand the conjugated operator as

$$
e^{-\frac{(\varphi + i \psi)}{h}} h^4 L_{W,A} e^{-\frac{(\varphi + i \psi)}{h}} = (-h^2 \Delta - 2hT)^2$$

$$+ \left[ 3h^2 (W \cdot D(\varphi + i \psi)) D(\varphi + i \psi) \cdot D + 6h^3 (W \cdot D(\varphi + i \psi)) \Delta \right.$$  

$$+ h^4 (W \cdot D) \Delta]$$  

$$+ \sum_{j,k=1}^2 h^2 A_{jk} (D_j(\varphi + i \psi)D_k(\varphi + i \psi) + 2hD_j(\varphi + i \psi)D_k + h^2 D_jD_k).$$

(2.22)

We take amplitudes $a(x, y) \in H^4(\Omega)$ solving

$$16\partial_2^2 a(x, y) + 6 (W_1 - iW_2) \partial_x a(x, y) - (A_{11} - A_{22} - 2iA_{12}) a(x, y) = 0, \quad \text{in } \Omega.$$  

(2.23)

Such choice makes the $O(h^2)$ to be zero in (2.22). Thus, from (2.22) we obtain equation for $r$ as

$$e^{-\frac{(\varphi + i \psi)}{h}} h^4 L_{W,A} \left( e^{-\frac{(\varphi + i \psi)}{h}} r(x, h) \right) = G(x, y, h)a(x, y),$$  

(2.24)

where

$$G := -h^4 L_{W,A} - h^4 (W \cdot D) \Delta - 6h^3 (W \cdot D(\varphi + i \psi)) \Delta.$$  

$$\simeq O(h^3), \quad \text{in the sense of } L^2(\Omega) \text{ norm}.$$

Thanks to Proposition 2.2, for $h > 0$ small enough, there exists a solution $r \in H^3_{sc}(\Omega)$ of (2.24) satisfying the estimate

$$\|r\|_{H^3_{sc}(\Omega)} \leq C h^{-1}\|Ga(x, y)\|_{L^2(\Omega)} \quad \Rightarrow \quad \|r\|_{H^3_{sc}(\Omega)} \leq C h^2,$$

(2.25)

where $C > 0$ is independent of $h > 0$.

Observe that, from the relations (2.2), (2.4), thus we obtain a complex geometric optics solution $u$ for the operator $K_{B,q}$ in $\Omega$. Summing up, we have the following result.

**Proposition 2.3.** For all $0 < h \ll 1$, there exists a solution $u \in H^4(\Omega)$ of $K_{B,q}u = 0$ in $\Omega$, of the form

$$u(x, y, h) = e^{\frac{\varphi(x, y) + i \psi(x, y)}{h}} (a(x, y) + r(x, y, h))$$  

(2.26)

where $\varphi$ and $\psi$ are smooth as in (2.20). Here $a \in C^\infty(\overline{\Omega}) \setminus \{0\}$ solving (2.23) and $r \in H^3_{sc}(\Omega)$ satisfies the estimate $\|r\|_{H^3_{sc}(\Omega)} = O(h^2)$.

2.3. **Construction of the CGO amplitudes.** Here we will prove existence of $a(x, y) \in H^4(\Omega)$ solving (2.23), viz.

$$T a(x, y) := \partial_2^2 a(x, y) + P(x, y) \partial_x a(x, y) - Q(x, y) a(x, y) = 0, \quad \text{in } \Omega,$$

where we denote $P = \frac{3}{8} (W_1 - iW_2)$ and $Q = -\frac{1}{10} (A_{11} - A_{22} - 2iA_{12})$ in $\Omega$. We consider a complex geometric optics form for the solution $a(x, y)$, given by the ansatz

$$a(x, y; \tau) = e^{\frac{\xi(x, y)}{h}} (a_0(x, y) + \rho(x, y; \tau)),$$

(2.27)

where $\tau > 0$ is a small parameter, $\xi(x, y) \in C^\infty(\overline{\Omega})$ is a real valued harmonic function in $\Omega$ and $\eta(x, y) \in C^\infty(\overline{\Omega})$ is the complex harmonic conjugate of $\xi$, satisfying the Cauchy-Riemann equations

$$\partial_x \xi = -\partial_y \eta \quad \text{and} \quad \partial_y \xi = \partial_x \eta, \quad \text{in } \Omega.$$  

(2.28)

For the real valued harmonic weight function $\xi(x, y)$ we prove a necessary Carleman estimate as follows.
Lemma 2.4. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. Let $\xi(x, y)$ be a real valued harmonic function in $\Omega$ such that $|\nabla \xi| > c$ for some $c > 0$ in $\Omega$. Define the convexified weight $\tilde{\xi} := \xi - \frac{\tau}{2\varepsilon} \xi^2$ where $0 < h < \varepsilon < 1$. Then, for all real valued $w \in C_0^\infty(\Omega)$, and for small positive parameters $\tau > 0$ and $\varepsilon > \tau > 0$ we have

$$\|e^{-\frac{\tilde{\xi}}{\varepsilon}} \tau^2 T e^{\tilde{\xi}} w\|_{H^{-1}_s(\Omega)} \geq C \frac{\tau}{\sqrt{\varepsilon}} \|w\|_{L^2(\Omega)}, \quad (2.29)$$

where $C > 0$ is independent of $\tau > 0$, $\varepsilon > \tau$ and $w$.

Proof. We divide the proof into the following two steps. First we prove an a priori estimate for the principal part of the conjugated operator $\tau^2 e^{-\tilde{\xi}/\varepsilon} T e^{\tilde{\xi}/\tau}$, and in the next step we add the lower order terms in $T$ to obtain the Carleman estimate (2.29).

**Step 1:** We start the proof with obtaining an a priori estimate for the principal part of the operator $\tau^2 e^{-\tilde{\xi}/\varepsilon} T e^{\tilde{\xi}/\tau}$, given by $e^{-\tilde{\xi}/\varepsilon} \tau^2 \partial_x^2 e^{\tilde{\xi}/\tau}$. Let $\xi(x, y)$ be a real valued harmonic function in $\Omega \subset \mathbb{R}^2$. Let the convexified weight be $\tilde{\xi} = \xi - \frac{\tau}{2\varepsilon} \xi^2$ where $0 < \tau < \varepsilon < 1$. From a direct calculation, using the fact that $\tilde{\xi}$ and $w$ are real valued, we observe that

$$\left| e^{-\tilde{\xi}} \tau (\partial_x - i \partial_y) (e^{\tilde{\xi}} w) \right|^2 = \left| w (\partial_x \tilde{\xi}) + \tau \partial_x w \right|^2 - i \left( w (\partial_y \tilde{\xi}) + \tau \partial_y w \right)^2 \geq \tau^2 |\nabla w|^2 + \left| \nabla \tilde{\xi} |w|^2 + \tau \left[ (\partial_y \tilde{\xi}) (\partial_y w^2) + (\partial_x \tilde{\xi}) (\partial_x w^2) \right] \right|^2.$$

Hence, using $w \in C_0^\infty(\Omega)$ and doing integration by parts, we get

$$\left\| e^{-\tilde{\xi}} \tau (\partial_x - i \partial_y) (e^{\tilde{\xi}} w) \right\|^2_{L^2} = \tau^2 \|\nabla w\|^2_{L^2} + \left\| \nabla \tilde{\xi} |w|^2 + \tau \langle w(-\Delta) \tilde{\xi}, w \rangle_{L^2}. \quad (2.30)$$

Observe that, since $\Delta \xi = 0$ in $\Omega$ we have

$$\left\langle w(-\Delta) \tilde{\xi}, w \right\rangle_{L^2} = \frac{\tau}{\varepsilon} \|\nabla \xi |w|^2_{L^2}. \quad (2.31)$$

Therefore, from (2.30) we obtain

$$\left\| e^{-\tilde{\xi}} \tau (\partial_x - i \partial_y) (e^{\tilde{\xi}} w) \right\|^2_{L^2} = \left\| |\nabla \xi |w|^2_{L^2} + \|\tau \nabla w\|^2_{L^2} + \frac{\tau^2}{\varepsilon} \|\nabla \xi |w|^2_{L^2}. \quad (2.32)$$

Thus, using the fact that $|\nabla \xi| \geq c > 0$ in $\Omega$, from (2.31) we obtain

$$\left\| e^{-\tilde{\xi}} \tau \partial_x (e^{\tilde{\xi}} w) \right\|^2_{L^2} \geq C \|w\|^2_{H^{-1}_s(\Omega)} + \frac{c^2 \tau^2}{\varepsilon} \|w\|^2_{L^2}. \quad (2.33)$$

We can view the estimate (2.33) as a dichotomy where we can either choose to estimate the derivatives of $w$ or we can have a bound on the decay of order $\frac{\tau}{\sqrt{\varepsilon}}$ in the right hand side. We iterate (2.33) twice to obtain an estimate for the conjugated $(\tau \partial_x)^2$ operator, where first we use the decay bound of order $\frac{\tau}{\sqrt{\varepsilon}}$ and then use the bound of the derivatives and finally obtain the following estimate

$$\left\| e^{-\tilde{\xi}} \tau^2 \partial_x^2 e^{\tilde{\xi}} w \right\|^2_{L^2} \geq C \left( \frac{\tau}{\sqrt{\varepsilon}} \right) \left\| e^{-\tilde{\xi}} \tau \partial_x e^{\tilde{\xi}} w \right\|^2_{L^2} \geq C \left( \frac{\tau}{\sqrt{\varepsilon}} \right) \|w\|_{H^{-1}_s(\Omega)}^2, \quad \forall w \in C_0^\infty(\Omega, \mathbb{R}).$$

**Step 2:** In the next step we adopt a similar analysis done in (2.9) and (2.10) with $s = 1$ and obtain

$$\left\| e^{-\tilde{\xi}} \tau^2 \partial_x^2 e^{\tilde{\xi}} w \right\|_{H^{-1}_s(\mathbb{R}^2)} \geq C \left( \frac{\tau}{\sqrt{\varepsilon}} \right) \|w\|_{L^2(\Omega)}. \quad (2.33)$$
we get existence

Here we use that fact that the choices of \( \xi, \eta \)

A straightforward calculation similar to (2.14), (2.15) we get

where \( C_P, C_Q \) are constants independent of \( \tau, \varepsilon \) but depending on \( \| P \|_{W^1, \infty(\Omega)} \) and \( \| Q \|_{L^\infty(\Omega)} \). Using the bounds (2.35), (2.36), in (2.34) we readily obtain

\[
\frac{C \tau}{\sqrt{\varepsilon}} \| w \|_{L^2(\Omega)} \leq \left\| e^{-\frac{\varepsilon}{\tau^2}} T e^{\frac{\varepsilon}{\tau^2}} w \right\|_{H^{-1}_{scl}(\mathbb{R}^2)} \text{, for small enough } \tau > 0 \text{ and } \varepsilon > \tau > 0.
\]

Remark 2.5. Since \( 1 < e^{\frac{1}{2\varepsilon}} \frac{\varepsilon}{\tau^2} < C \) and \( \frac{1}{2} \leq 1 + \frac{\varepsilon}{\tau^2} \xi \leq \frac{3}{2} \), fixing \( \varepsilon > 0 \) suitably we can replace \( \tilde{\xi} \) by \( \xi \) in (2.29) and obtain

\[
\left\| e^{-\frac{\xi}{\tau^2}} T e^{\frac{\xi}{\tau^2}} w \right\|_{H^{-1}_{scl}(\mathbb{R}^2)} \geq C\tau \| w \|_{L^2(\Omega)}.
\]

Proposition 2.6. Let \( F(x, y, \tau) \in L^2(\Omega) \) then there exists \( \rho \in H^1_{scl}(\Omega) \) such that

\[
e^{-\frac{\xi}{\tau^2}} T e^{\frac{\xi}{\tau^2}} \rho(x, y, \tau) = F(x, y, \tau), \quad \text{in } \Omega,
\]

with the decay estimate

\[
\| \tau^2 \rho(x, y, h) \|_{H^1_{scl}(\Omega)} \leq C\tau \| F \|_{L^2(\Omega)} \quad \text{in } \Omega,
\]

where \( C > 0 \) is independent of \( \tau > 0 \).

Proof. The proof follows from a standard functional analysis argument, using Hahn-Banach Theorem and the Carleman estimate (2.37). \( \square \)

Now, let us recall the proposed form of \( a(x, y, \tau) \) as in (2.27) and observe that

\[
T a(x, y, \tau) = 0 \quad \implies \quad e^{-\frac{s \eta}{\tau^2}} T e^{\frac{s \eta}{\tau^2}} \rho = \tau^2 T a_0(x, y).
\]

Here we use that fact that the choices of \( \xi, \eta \) in (2.28) implies \( \partial_2(\xi + i\eta) = 0 \) in \( \Omega \). Now, let \( a_0(x, y) \in C^\infty(\overline{\Omega}) \) be any non zero function. As a consequence of Proposition 2.6 we get existence of \( \rho \in H^1_{scl}(\Omega) \) solving

\[
e^{-\frac{s \eta}{\tau^2}} T e^{\frac{s \eta}{\tau^2}} \rho = \tau^2 T a_0(x, y), \quad \text{in } \Omega,
\]

with the decay \( \| \rho \|_{H^1_{scl}(\Omega)} \leq C\tau \).

Furthermore, from (2.40) and the fact that \( \partial_2(\xi + i\eta) = 0 \) in \( \Omega \) we see that

\[
\partial_2^2 \rho(x, y; \tau) = T a_0(x, y) - P(x, y) \partial_2 \rho(x, y; \tau) + Q(x, y) \rho(x, y; \tau) \in L^2(\Omega), \quad \text{in } \Omega.
\]

Since, \( \| \rho \|_{H^1_{scl}(\Omega)} \leq C\tau \), hence we have \( \| \rho \|_{L^2(\Omega)}, \| \nabla \rho \|_{L^2(\Omega)} \simeq \mathcal{O}(1) \) and thus (2.42) implies

\( \| \partial_2^2 \rho \|_{L^2(\Omega)} \simeq \mathcal{O}(1) \).
Remark 2.7. Observe that \( a(x, y; \tau) = e^{\frac{i\pi}{2m}}(a_0(x, y) + \rho(x, y; \tau)) \) with \( a_0(x, y) \in C^\infty(\Omega) \) and \( \|\rho\|_{L^2(\Omega)}, \|\nabla \rho\|_{L^2(\Omega)} \simeq O(1) \), satisfies (2.23) that is
\[
\partial_z^2 a(x, y) + P(x, y) \partial_z a(x, y) - Q(x, y) a(x, y) = 0, \quad \text{in } \Omega.
\]
Thus, one can easily see that
\[
\partial_z^2 a(x, y) = -P(x, y) \partial_z a(x, y) + Q(x, y) a(x, y) \in L^2(\Omega).
\] (2.43)
Similarly, using (2.43) iteratively with the fact \( P, Q \) are smooth, one obtains
\[
\partial_z^k a(x, y) \in L^2(\Omega), \quad \text{for } k = 3, 4, \ldots .
\] (2.44)
This helps to justify that \( a \in H^m(\Omega) \) for \( m \geq 1 \).

Remark 2.8. Since \( \|\rho\|_{H^1_{cd}(\Omega)} \simeq O(\tau) \), therefore from (2.42) we have
\[
\partial_z^2 \rho(x, y; \tau) = -\partial_z^2 a_0(x, y) - P(x, y) \partial_z a(x, y) + Q(x, y) a(x, y) \simeq O(1)
\]
in the sense of \( L^2(\Omega) \), providing \( \|\partial_z^2 \rho\|_{L^2(\Omega)} \simeq O(1) \) as \( \tau \to 0 \).

3. Determination of the coefficients

In this section we will be using the special form of the solution \( u \) (cf. (2.26)) to determine the smooth coefficients \( B \) and \( q \) in \( \Omega \). Let us assume that \( \mathcal{K}_{B,q} \) and \( \mathcal{K}_{\tilde{B},\tilde{q}} \) be two operators given as (1.1) corresponding to the parameters \( B, q \) and \( \tilde{B}, \tilde{q} \) respectively. Let \( \mathcal{C}_{B,q} \) be the set of Cauchy data, given in (1.5), corresponding to the operator \( \mathcal{K}_{B,q} \). In this section we will show that if \( \mathcal{C}_{B,q} = \mathcal{C}_{\tilde{B},\tilde{q}} \) on \( \partial \Omega \) then \( B = \tilde{B} \) and \( q = \tilde{q} \) in \( \Omega \).

3.1. Integral identity involving the coefficients. We recall that
\[
\mathcal{K}_{B,q}(x, D) = \text{div} \left( \text{div} (B(1 - q) \nabla^2 v + Bq \Delta v I) \right),
\]
where the second order differential operator \( E \) is defined as in (2.1). We have the following integral identity
\[
\int_{\Omega} (\mathcal{K}_{B,q}(x, D)w) \frac{v(x, y)}{y} \, dx - \int_{\Omega} w(x, y) \left( \mathcal{K}_{B,q}(x, D)v \right) \, dx = 0, \quad \forall w \in H^4_0(\Omega), v \in H^4(\Omega). \tag{3.1}
\]
where \( H^4_0(\Omega) \) is the closure of \( C^\infty_0(\Omega) \) functions in \( H^4(\Omega) \) norm and \( (\mathcal{K}_{B,q})^* = \overline{\mathcal{K}_{B,q}} = \overline{\mathcal{K}_{B,q}} \). Let \( u, \tilde{u} \in H^4(\Omega) \) solves
\[
\mathcal{K}_{B,q}(x, D)u = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathcal{K}_{\tilde{B},\tilde{q}}(x, D)\tilde{u} = 0 \quad \text{in } \Omega, \tag{3.2}
\]
and we assume (cf. Theorem 1.1)
\[
\mathcal{C}_{B,q} \, u = \mathcal{C}_{\tilde{B},\tilde{q}} \, \tilde{u} \quad \text{on } \partial \Omega. \tag{3.3}
\]
Let \( v \in H^4(\Omega) \) satisfies \( (\mathcal{K}_{B,q})^* v = 0 \) in \( \Omega \), then from the integral identity (3.1) we get
\[
\langle (\mathcal{K}_{B,q}(\tilde{u} - u), v \rangle_{L^2(\Omega)}
\]
\[
= \int_{\Omega} \left[ \left( B - \tilde{B} \right) \left( \nabla^2 \tilde{u} : \nabla^2 v \right) - \left( Bq - \tilde{B}q \right) \left( \nabla^2 \tilde{u} : \nabla^2 v \right) + (Bq - \tilde{B}q) \Delta \tilde{u} \Delta v \right] \, dx \, dy = 0. \tag{3.4}
\]
Next we choose $\tilde{u}$ and $v$ to be the CGO type solutions constructed in Section 2.2. Note that $(\mathcal{K}_{\mathcal{B},q})^* = \mathcal{K}_{\mathcal{B},q}$ and thus for $h > 0$ small enough, we can set the solutions of the form

$$\begin{aligned}
\begin{cases}
\tilde{u}(x, y; h) = e^{\frac{2 \pi i x y}{\lambda}} (\tilde{a}(x, y) + \tilde{r}(x, y; h)) & \quad \text{in } \Omega, \\
v(x, y; h) = e^{\frac{2 \pi i x y}{\lambda}} (a(x, y) + r(x, y; h)) & \quad \text{in } \Omega.
\end{cases}
\end{aligned} \tag{3.5}$$

The amplitudes $\tilde{a}(x, y), a(x, y)$ are solving the equations

$$\begin{aligned}
\begin{cases}
\partial_x^2 \tilde{a}(x, y) + P(x, y) \partial_y \tilde{a}(x, y) - Q(x, y) \tilde{a}(x, y) = 0, \\
\partial_y^2 a(x, y) + P(x, y) \partial_x a(x, y) - Q(x, y) a(x, y) = 0,
\end{cases}
\end{aligned} \quad \text{in } \Omega,$$

where $P = \frac{3}{8} (W_1 - i W_2)$ and $Q = -\frac{1}{16} (A_{11} - A_{22} - 2 i A_{12})$ in $\Omega$.

The correction term $\tilde{r}(x, y; h), r(x, y; h)$ satisfies decay $\|\tilde{r}\|_{H^3_{\text{ccl}}}, \|r\|_{H^3_{\text{ccl}}} = O(h^2)$.

We substitute the form of $\tilde{u}, v$ from (3.5) in (3.4) with $\varphi$ and $\psi$ to be linear weights. Multiplying (3.4) with $h^2$ we get

$$\begin{aligned}
0 &= 16 \int_\Omega (\partial_z \varphi)^2 \left( B - \tilde{B} \right) \left[ (a + r) \partial_x^2 (\tilde{a} + \tilde{r}) + (\tilde{a} + \tilde{r}) \partial_x^2 (a + r) - 2 (\partial_z (\tilde{a} + \tilde{r})) (\partial_z (a + r)) \right] \, dx \, dy \\
&\quad - 16 \int_\Omega (\partial_z \varphi)^2 \left( B q - \tilde{B} q \right) \left[ (a + r) \partial_x^2 (\tilde{a} + \tilde{r}) + (\tilde{a} + \tilde{r}) \partial_x^2 (a + r) + 2 (\partial_z (\tilde{a} + \tilde{r})) (\partial_z (a + r)) \right] \, dx \, dy \\
&\quad + 8 h \int_\Omega \left( B - \tilde{B} \right) \left[ (\partial_x (\tilde{a} + \tilde{r}) \partial_x - \partial_y (\tilde{a} + \tilde{r}) \partial_y) \partial_z (a + r) \right. \\
&\quad \left. - (\partial_x (a + r) \partial_x - \partial_y (a + r) \partial_y) \partial_z (a + r) \right] \, dx \, dy \\
&\quad + 8 h \int_\Omega \left( B q - \tilde{B} q \right) \left[ (\partial_x (\tilde{a} + \tilde{r}) i \partial_y - \partial_y (\tilde{a} + \tilde{r}) \partial_x) \partial_z (a + r) \right. \\
&\quad \left. - (\partial_x (a + r) i \partial_y - \partial_y (a + r) \partial_x) \partial_z (a + r) \right] \, dx \, dy \\
&\quad + h^2 \int_\Omega \left( B - \tilde{B} \right) \left[ \partial_x^2 (\tilde{a} + \tilde{r}) \partial_x^2 (a + r) + \partial_y^2 (\tilde{a} + \tilde{r}) \partial_y^2 (a + r) + 2 \partial_x \partial_y (\tilde{a} + \tilde{r}) \partial_x \partial_y (a + r) \right] \, dx \, dy \\
&\quad + h^2 \int_\Omega \left( B q - \tilde{B} q \right) \left[ \partial_x^2 (\tilde{a} + \tilde{r}) \partial_y^2 (a + r) + \partial_y^2 (\tilde{a} + \tilde{r}) \partial_x^2 (a + r) - 2 \partial_x \partial_y (\tilde{a} + \tilde{r}) \partial_x \partial_y (a + r) \right] \, dx \, dy.
\end{aligned} \tag{3.6}$$

Here let us use the decays of $r, \tilde{r}$ given as $\|\tilde{r}\|_{H^3_{\text{ccl}}(\Omega)}, \|r\|_{H^3_{\text{ccl}}(\Omega)} \lesssim O(h^2)$ and take $h \to 0$ to obtain

$$\begin{aligned}
0 &= \int_\Omega (\partial_z \varphi)^2 \left( B - \tilde{B} \right) \left[ \tilde{a} \partial_x^2 \tilde{a} + \tilde{a} \partial_x^2 \tilde{a} - 2 \partial_z \tilde{a} \partial_z \tilde{a} \right] \, dx \, dy \\
&\quad - \int_\Omega (\partial_z \varphi)^2 \left( B q - \tilde{B} q \right) \left[ \tilde{a} \partial_y^2 \tilde{a} + \tilde{a} \partial_y^2 \tilde{a} + 2 \partial_z \tilde{a} \partial_z \tilde{a} \right] \, dx \, dy.
\end{aligned} \tag{3.7}$$

Since the amplitudes are of the form (cf. (2.27))

$$\begin{aligned}
\left\{ \begin{array}{l}
\tilde{a}(x, y; \tau) = e^{\frac{2 \pi i x y}{\lambda}} (\tilde{a}_0(x, y) + \tilde{p}(x, y; \tau)), \\
\tilde{a}(x, y; \tau) = e^{\frac{2 \pi i x y}{\lambda}} (\tilde{a}_0(x, y) + \tilde{p}(x, y; \tau)),
\end{array} \right.
\end{aligned} \tag{3.8}$$
we get from (3.7):

\[
0 = \int_\Omega (\partial_\varphi)^2 \left( \tilde{B} - B \right) \left[ (\tilde{a}_0 + \tilde{\rho}) \partial_z^2 (\tilde{a}_0 + \tilde{\rho}) + (\tilde{a}_0 + \tilde{\rho}) \partial_z^2 (\tilde{a}_0 + \tilde{\rho}) - 2 \partial_z (\tilde{a}_0 + \tilde{\rho}) \partial_z (\tilde{a}_0 + \tilde{\rho}) \right] \, dx \, dy
\]
\[
- \int_\Omega (\partial_\varphi)^2 \left( Bq - \tilde{B}q \right) \left[ (\tilde{a}_0 + \tilde{\rho}) \partial_z^2 (\tilde{a}_0 + \tilde{\rho}) + (\tilde{a}_0 + \tilde{\rho}) \partial_z^2 (\tilde{a}_0 + \tilde{\rho}) + 2 \partial_z (\tilde{a}_0 + \tilde{\rho}) \partial_z (\tilde{a}_0 + \tilde{\rho}) \right] \, dx \, dy.
\]

(3.9)

Let us denote

\[
\mathcal{I} = \int_\Omega (\partial_\varphi)^2 \left( \tilde{B} - B \right) (\partial_z \tilde{\rho}) (\partial_z \bar{\rho}) \, dx \, dy.
\]

(3.10)

Since we have assumed (see the statement of Theorem 1.1) that \( (\tilde{B} - B) |_{\partial \Omega} = 0 \), thus using an integration by parts formula to obtain

\[
-\mathcal{I} = \int_\Omega (\partial_\varphi)^2 \left( \tilde{B} - B \right) \tilde{\rho} (\partial_z \bar{\rho}) \, dx \, dy + \int_\Omega \partial_z \left[ (\partial_\varphi)^2 \left( \tilde{B} - B \right) \right] \tilde{\rho} (\partial_z \bar{\rho}) \, dx \, dy.
\]

Now we have \( \| \rho \|_{H_0^1(\Omega)} \simeq \mathcal{O}(\tau) \), and from Remark 2.8 since \( \| \partial_z^2 \rho \|_{L^2(\Omega)} \simeq \mathcal{O}(1) \) as \( \tau \to 0 \). Therefore by taking limit \( \tau \to 0 \) to (3.9) we get

\[
0 = \int_\Omega (\partial_\varphi)^2 \left( \tilde{B} - B \right) \left[ \tilde{a}_0 \partial_z^2 \tilde{a}_0 + \bar{a}_0 \partial_z^2 \bar{a}_0 - 2 (\partial_z \tilde{a}_0) (\partial_z \bar{a}_0) \right] \, dx \, dy
\]
\[
- \int_\Omega (\partial_\varphi)^2 \left( Bq - \tilde{B}q \right) \left[ \tilde{a}_0 \partial_z^2 \tilde{a}_0 + \bar{a}_0 \partial_z^2 \bar{a}_0 + 2 (\partial_z \tilde{a}_0) (\partial_z \bar{a}_0) \right] \, dx \, dy.
\]

(3.11)

Let us now fix \( \tilde{a}_0(x, y) = 1 \), in \( \Omega \), then we have

\[
\int_\Omega (\partial_\varphi)^2 \left[ \left( \tilde{B} - B \right) - \left( Bq - \tilde{B}q \right) \right] \partial_z^2 \tilde{a}_0 \, dx \, dy = 0.
\]

(3.12)

Let \( b(x, y) \in C^\infty(\bar{\Omega}) \) be any function and let us take \( \tilde{a}_0(x, y) \in C^\infty(\Omega) \) such that \( \partial_z^2 \tilde{a}_0(x, y) = b(x, y) \) in \( \Omega \). Then we get

\[
\int_\Omega (\partial_\varphi)^2 \left[ \left( \tilde{B} - B \right) - \left( Bq - \tilde{B}q \right) \right] b(x, y) \, dx \, dy = 0, \quad \forall b \in C^\infty(\bar{\Omega}).
\]

(3.13)

Thus using \( (\partial_z \varphi) \neq 0 \) everywhere in \( \Omega \), from (3.13) we obtain

\[
(\tilde{B} - B) = (Bq - \tilde{B}q), \quad \text{in } \Omega.
\]

(3.14)

Now, substituting (3.14) to (3.11) we obtain

\[
\int_\Omega (\partial_\varphi)^2 (B - \tilde{B}) (\partial_z \tilde{a}_0) (\partial_z \bar{a}_0) \, dx \, dy = 0.
\]

(3.15)

Now, for the choice \( \tilde{a}_0(x, y) = x \) in \( \Omega \), (3.15) reads

\[
\int_\Omega (\partial_\varphi)^2 (B - \tilde{B}) \partial_z \bar{a}_0 \, dx \, dy = 0, \quad \forall a_0 \in C^\infty(\bar{\Omega}).
\]

(3.16)

Since for any \( b(x, y) \in C^\infty(\bar{\Omega}) \) there exists \( a_0(x, y) \in C^\infty(\bar{\Omega}) \) such that \( \partial_z a_0(x, y) = b(x, y) \) in \( \Omega \), so from (3.16) we obtain

\[
\int_\Omega (\partial_\varphi)^2 (B - \tilde{B}) b(x, y) \, dx \, dy = 0, \quad \forall b \in C^\infty(\bar{\Omega}).
\]

(3.17)
This implies $B = \tilde{B}$ in $\Omega$. Thus combining with (3.14) we conclude

$$B = \tilde{B} \quad \text{and} \quad q = \tilde{q}, \quad \text{in} \ \Omega.$$

### 3.2. Recovering the Lamé moduli.

In the previous part we have already established that

$$B(x, y) = \tilde{B}(x, y) \quad \text{and} \quad q(x, y) = \tilde{q}(x, y) \quad \text{in} \ \Omega.$$  \hfill (3.18)

Furthermore, we can untangle the expression of $B, q$ in $\mathcal{K}_{B,q}$ to obtain uniqueness of the Lamé parameters $\lambda, \mu$ in $\Omega$ from (3.18). Recall the form of $B, q$ given in (1.2) as

$$B(x, y) := \frac{k^3 E(x, y)}{12(1 - q^2(x, y))}, \quad q(x, y) := \frac{\lambda(x, y)}{2(\lambda(x, y) + \mu(x, y))}.$$  

Let us denote $c_P = \lambda + \mu$ and $c_S = \mu$ and $\tilde{c}_P, \tilde{c}_S$ accordingly in $\Omega$. In terms of Lamé parameters in classical elasticity, $c_P$ and $c_S$ are the square of the compressional and the shear wave speeds respectively. Now, $q = \tilde{q}$ implies

$$\frac{c_P - c_S}{c_P} = \frac{\tilde{c}_P - \tilde{c}_S}{c_P} \quad \implies \quad \frac{c_S}{c_P} = \frac{\tilde{c}_S}{c_P}, \quad \text{in} \ \Omega. \quad (3.19)$$

Now, from $B = \tilde{B}$ and $q = \tilde{q}$ we obtain

$$\frac{c_P}{c_S} (3c_P - c_S) = \frac{\tilde{c}_P}{c_S} (3\tilde{c}_P - \tilde{c}_S) \quad \implies \quad 3(c_P - \tilde{c}_P) = c_S - \tilde{c}_S, \quad \text{in} \ \Omega. \quad (3.20)$$

Now, since (3.19) implies $\tilde{c}_P = c_P \frac{\tilde{c}_S}{c_S}$, from (3.20) we obtain

$$3c_P \left( 1 - \frac{c_S}{c_S} \right) = c_S - \tilde{c}_S \quad \implies \quad \frac{(3c_P - c_S)}{c_S} (c_S - \tilde{c}_S) = 0, \quad \text{in} \ \Omega. \quad (3.21)$$

Now, note that $c_S = \mu > 0$ and $3\lambda + 2\mu = 3c_P - c_S > 0$ in $\Omega$. Therefore, (3.21) implies $c_S = \tilde{c}_S$ in $\Omega$ and subsequently $c_P = \tilde{c}_P$ from (3.20). Thus we obtain $\lambda = \tilde{\lambda}$ and $\mu = \tilde{\mu}$ in $\Omega$.

This completes the discussion of the proof of Theorem 1.1.

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