$k$ variables are needed to define $k$-Clique in first-order logic

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Abstract

In an early paper, Immerman raised a proposal on developing model-theoretic techniques to prove lower bounds on ordered structures, which represents a long-standing challenge in finite model theory. An iconic question standing for such a challenge is how many variables are needed to define $k$-Clique in first-order logic on the class of finite ordered graphs? If $k$ variables are necessary, as widely believed, it would imply that the bounded (or finite) variable hierarchy in first-order logic is strict on the class of finite ordered graphs. In 2008, Rossman made a breakthrough by establishing an optimal average-case lower bound on the size of constant-depth unbounded fan-in circuits computing $k$-Clique. In terms of logic, this means that it needs greater than $\lceil \frac{k}{4} \rceil$ variables to describe the $k$-Clique problem in first-order logic on the class of finite ordered graphs, even in the presence of arbitrary arithmetic predicates. It follows, with an unpublished result of Immerman, that the bounded variable hierarchy in first-order logic is indeed strict. However, Rossman’s methods come from circuit complexity and a novel notion of sensitivity by himself. And the challenge before finite model theory remains there. In this paper, we give an alternative proof for the strictness of bounded variable hierarchy in FO using pure model-theoretic toolkit, and answer the question completely for first-order logic, i.e. $k$-variables are indeed needed to describe $k$-Clique in this logic. In contrast to Rossman’s proof, our proof is purely constructive. Then we embed the main structures into a pure arithmetic structure to show a similar result where arbitrary arithmetic predicates are presented. Finally, we discuss its application in circuit complexity.

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1 Introduction

The Clique problem has been studied extensively in theoretical computer science, not only in classical computational complexity, but also in parameterized complexity. In particular, in circuit complexity, some remarkable lower bounds for $k$-Clique in restricted models have been established (for a short survey, cf., e.g., Rossman [24]). Following [20], Rossman [21] showed that, on average, no constant-depth unbounded fan-in circuits of size $O(n^{\frac{k}{4}})$ can recognize $k$-Clique. It represents a significant breakthrough in circuit complexity, because it is the first unconditional lower bound that cleverly breaks out of the traditional size-depth tradeoff, a well-known barrier to progress in the study of constant-depth circuits model for many years [9]. We note that Rossman has achieved a stronger result in circuit complexity. But in the context of logic, his result is roughly the following: on the class of finite ordered graphs, any first-order logic sentence that defines $k$-Clique in the average case needs greater than $\lfloor \frac{k}{4} \rfloor$ variables, even when arbitrary arithmetic predicates are available. In addition, a study of $k$-pebble games over random graphs was introduced in [23] based on this result. Note that his lower bound is tight in the average case on a certain natural distribution [1]. Rossman [25] also gave the tight average-case lower bound for $k$-Clique on the class of finite ordered graphs, without arithmetic predicates other than the built-in linear order, by showing a tight upper bound $\frac{k}{4} + O(1)$ on the number of variables that is needed for defining $k$-Clique in the average case. In logic, Rossman’s lower bound implies that the bounded variable hierarchy (also called the finite variable hierarchy in the literature) in first-order logic (FO, for short) will not collapse. Together with an unpublished result of Immerman (cf. Rossman [24]), it follows that the bounded variable hierarchy in FO is strict, i.e. for any $k$, there is a property that can be defined by $k$ variables but is not definable by $k - 1$ variables in FO, thereby solved a long-standing question going back to Immerman [15]. As mentioned in [9], this question on bounded variable hierarchy is “deceptively simple” in the appearance, but very hard to answer. Therefore, its settling by Rossman also represents “one of the most significant breakthroughs in the field of finite model theory in many year” [9].

It is not surprising that a result in the field of circuit complexity would have such an impact on the field of finite model theory. See an explanation of such a connection in [8]. The connection between computational complexity and finite model theory started by an early work of Fagin [12], which showed the equivalence of NP and the class of properties that can be expressed by the existential second-order logic. Such kind of research was then carried
on under the name of descriptive complexity, where Immerman has been playing a crucial role by making numerous fundamental contributions. To have a grasp of this field, cf. the monograph [19].

The study of finite variable fragments of FO may be traced back to the 19th century [2]. Its important role in model theory is well-known, cf. e.g. [9, 13, 15]. Immerman began the study of syntactically uniform sequences of first-order formulas with constant number of variables. It is important because most of the well-known complexity classes can be characterized by such sequences. Moreover, the number of variables correspond to amount of hardware, e.g. the number of processors in CRAM. In particular, it is well-known that $\text{DSPACE}[n^k] = \text{VAR}[k + 1]$, which elegantly connects the number of variables in descriptions with the number of memory locations in deterministic Turing machines [13]. It means that $k$ variables roughly correspond to $n^k$ memory locations.

In 1982, Immerman raised a proposal on developing techniques to prove lower bounds on finite structures with linear orders (cf. [15], p.97), which stands for a challenge in finite model theory for decades. It is very important because all the known logical characterizations of well-known complexity classes inside NP rely on a built-in linear order over input finite structures. For NP and beyond, we can guess an order and rely on this order to simulate computation of Turing machines. In this sense, we still need linear orders in such cases, although implicitly. In short, computation needs orders to carry out. Note that his full proposal is very general and beyond our goal. Immerman considered syntactically uniform sequences of first-order formulas that define properties, whereas we only consider the most uniform one, i.e. all the formulas in a sequence are the same, which means the formulas in such a sequence have a constant size. In other words, we only consider the expressiveness of FO, which has very limited expressive power. If we have techniques for uniform sequences of first-order formulas without such restriction, as Immerman asked for, this would lead to many profound results, including a possible settling of the well-known open problem on P vs. NP if we allow the formulas to have polynomial sizes in terms of the length of inputs. It is for this reason that we restrict our concern on FO.

The strictness of bounded variable hierarchy was studied in various logics, such as modal logics and temporal logics (cf. [2, 5, 9]). It is not obvious to tell how many variables are needed in FO. For example, it is known that three variables suffice to describe any first-order property of linear orders with unary relations only [21].

Using similar ideas, we can show that it

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1 There is an alternative proof in the context of temporal logic, cf. p.4 of [5] for a brief
holds even in finite pure arithmetic structures (cf. Remark 53). In fact, it turns out to be an extremely hard problem to answer, when the structures are finite ordered graphs. It was observed that the \( k \)-Clique question may be the key to solve this problem \[8\]. Intuitively, linear orders seem useless in reducing the number of variables that are needed to define \( k \)-Clique, which strongly suggests that precisely \( k \) variables are needed to define \( k \)-Clique in FO (cf. \[8\] p.23, \[22\] p.10, \[24\] p.71). If this is true, then FO needs arbitrary number of variables. The conjecture that bounded variable hierarchy over finite ordered graphs is strict in FO was first explicitly presented in \[8\] (cf. p.3, Conjecture 2, the stronger version), which can go back to the proposal raised in Immerman’s early paper \[15\], because the hierarchy is strict if \( k \) variables are needed to define \( k \)-Clique over finite ordered graphs in FO. In circuit complexity, this hierarchy corresponds to the size hierarchy. Rossman’s result \[22\] implies that this size hierarchy is infinite. It was later sharpened by Amano who showed that this size hierarchy is strict \[1, 24\].

Here we should note that it is the finiteness and, in particular, the linear orders that make Immerman’s proposal, even in terms of FO, very hard to take up, because it is “notoriously difficult” (cf. \[8\], p.23) to apply the standard tool in finite model theory on finite ordered structures (also cf. \[19\]). For example, it is very hard to show that \( k \) variables are needed to define \( k \)-Clique on finite ordered graphs. But, if either “finite” or “order” is dropped off, the statement turns out to be easy to prove. Therefore, on finite ordered graphs, showing that \( k \) variables are needed to define \( k \)-Clique, as well as showing the strictness of the bounded variable hierarchy in FO, is an iconic problem that represents such a challenge \[9\]. Hence, in this paper we shall refer to this particular question (i.e. how many variables are needed in FO to define \( k \)-Clique on the class of finite ordered graphs) when we mention “Immerman’s question” (originated from the proposal). Although it was proved that the bounded variable hierarchy is strict in FO, it was solved mainly by techniques from circuit complexity and a novel notion called clique-sensitive core by Rossman. And the challenge before finite model theory remains there. In this paper, based on standard finite model-theoretic toolkit, we develop novel notions and techniques to give an alternative proof for the strictness of bounded variable hierarchy in FO, and fully answer Immerman’s question. Note that our proof is completely constructive. Compared with existential proofs which merely show the existence of pursued mathematical objects, a constructive proof need show every bit of the objects clearly: in terms of our study, the structures should be constructed explic-
ity and the strategies should be fully revealed. It justifies the length of our proof and allows a straightforward application of the result to answer a related important question. That is, precisely $k$ variables are needed for $k$-Clique in FO(BIT). At the end of this paper, we discuss its application in circuit complexity.

For more background on the $k$-Clique problem and the linear order issue in finite model theory, the readers can confer the survey paper [8] which collected related issues, connections, observations and results. In particular, Dawar showed that existential first-order formulas (and even existential infinitary logic formulas) require $k$ variables to define $k$-Clique on the class of finite ordered graphs, which we will introduce in section 3.

2 Preliminaries

Let $\mathbb{Z}, \mathbb{N}_0$ and $\mathbb{N}^+$ be the set of integer numbers, non-negative integer numbers and positive integer numbers respectively. By default, we assume that all the numbers mentioned in this paper are in $\mathbb{N}_0$. Assume that $k$ is a fixed integer number where $k > 1$. In this paper, we use semicolons to mean “AND” in definitions of sets. We use $|X|$ to denote the cardinality of the set $X$ and use $\wp(X)$ to denote the power set of $X$. A tuple is a multiset whose elements are ordered. Hence we can compute the intersection or union of a set and a tuple (redundant elements are omitted). For a tuple $\bar{c}$, we use $\bar{c}(i)$ to denote the $i$-th element of $\bar{c}$. Let $|\bar{c}|$ be the length of $\bar{c}$. If $|\bar{c}| \neq 0$, we use $\bar{c} \subseteq \bar{d}$ to denote that $\bar{c}(i) = \bar{d}(i)$ for $0 \leq i < |\bar{c}|$. If $|\bar{c}| = 0$, $\bar{c} \subseteq \bar{d}$ for any $\bar{d}$. By convention, we use “$\lfloor x \rfloor$” to denote the floor functions $\lfloor x \rfloor$, i.e. $\lfloor x \rfloor = \max \{n \in \mathbb{Z} \mid n \leq x\}$, for any real number $x$. For $n \in \mathbb{N}^+$, we let $[n]$ be the set $\{0, \ldots, n - 1\}$ and let $[1, n]$ be the set $\{1, \ldots, n\}$. And for $n_0, n_1 \in \mathbb{N}^+$ we let $[n_0] \times [n_1]$ be the Cartesian product $[n_0] \times [n_1]$. Henceforth, we use a pair of integer to denote a point in a two dimension coordinate plane. For a sequence of $n$ elements, a right circular shift of this sequence is a permutation $\sigma$ such that the element in the $i$-th position is moved to the $\sigma(i)$-th position of the sequence, where $\sigma(i) = (i + 1) \mod n$.

In the following of this section we introduce some standard concepts and well-known results in finite model theory. Although we try to make it self-contained, the readers are assumed to have some elementary knowledge of first-order logic.
2.1 Logic and structures

Let $R_i$ be a relation symbol and $c_i$ a constant symbol. Let $\sigma = \langle R_1, \ldots, R_\ell, c_1, \ldots, c_n \rangle$ be a relational signature, a $\sigma$-structure $\mathfrak{A}$ consists of a universe $A$ together with an interpretation of $R_i$ and $c_i$ over $A$:

- each $s$-ary relation symbol $R_i \in \sigma$ as a $s$-ary relation on $A$, usually written $R_i^A$;
- each constant symbol $c_i \in \sigma$ as an element in $A$, usually written $c_i^A$.

The structure $\mathfrak{A}$ is a finite structure if $A$ is a finite set.

For any $D \subseteq A$, a $\sigma$-substructure of the $\sigma$-structure $\mathfrak{A}$ induced by $D$, denoted $\mathfrak{A}[D]$, is a $\sigma$-structure, wherein each constant symbol is interpreted as it is in $\mathfrak{A}$, and each $j$-ary relation symbol $R_i$ is interpreted by $R_i^A \cap D^j$.

For example, a finite digraph $G_d$ is a finite $\langle E \rangle$-structure where $E$ is a binary relation symbol; and a finite graph $\mathcal{G}$ is a digraph where $E^\mathcal{G}$ is symmetric. By convention, the universe of a graph is called vertex set, denoted $V$, and the set of pairs (i.e. 2-tuples) that are used to interpret $\langle E \rangle$ are called edges, usually denoted by $E$ instead of $E^\mathcal{G}$. The fact that two vertices $a$ and $b$ are joined by an edge can be denoted by $E_{ab}$, $E(a,b)$ or $(a,b) \in E$. Or we can say $a$ is adjacent to $b$. A graph $G' = \langle V', E' \rangle$ is a subgraph of a graph $G = \langle V, E \rangle$ if $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$; $G'$ is an induced subgraph of $G$, denoted by $G[V']$, if $G'$ is a subgraph of $G$ and $E' = E \cap (V' \times V')$. We also use $|G|$ to denote the set of vertices of $G$.

Let $\sigma' \subseteq \sigma$. The $\sigma'$-reduct of $\mathfrak{A}$, denoted $\mathfrak{A}|_{\sigma'}$, is obtained from $\mathfrak{A}$ by “forgetting” $\sigma \setminus \sigma'$, i.e. leaving all the symbols in $\sigma \setminus \sigma'$ uninterpreted.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures over the same finite signature $\sigma$. Say that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic (or, there is an isomorphism between them), if there is a bijection $f : A \to B$ such that

$$
\begin{align*}
  f(c_i^A) &= c_i^B, & \text{for any constant symbol } c_i \in \sigma; \\
  R_i^A(a_1, \ldots, a_j) &\Leftrightarrow R_i^B(f(a_1), \ldots, f(a_j)), & \text{for any } j\text{-ary relation symbol } R_i \in \sigma.
\end{align*}
$$

For example, two graphs are isomorphic if they are the same after renaming of their vertices.

Let $\mathcal{L}$ be a logic, e.g. FO. Let $\mathfrak{A}$ be a $\sigma$-structure and $\psi$ be an $\mathcal{L}$-sentence. We use $\mathfrak{A} \models \psi$ to denote that $\psi$ is true in $\mathfrak{A}$, and we call $\mathfrak{A}$ a model for $\psi$. Let $\text{Mod}(\psi)$ be the set of models of $\psi$. A property $Q$ over $\sigma$ is a set of $\sigma$-structures closed under isomorphism. Say that $Q$ is expressible,
or definable, in a logic $\mathcal{L}$ if there is a sentence $\varphi$ in $\mathcal{L}$ such that for every $\mathfrak{A}$, $\mathfrak{A} \in \text{Mod}(\varphi)$ iff $\mathfrak{A} \in Q$.

Square lattice, denoted $\mathbb{Z}^2$, is the lattice in the two-dimensional Euclidean space whose lattice points are pairs of integers. A finite upright square lattice is a finite square lattice that is isomorphic to the set of isolated vertices $[a] \times [b]$ for some $a, b \in \mathbb{N}^+$. In the sequel, when we mention “square lattice”, we mean finite upright square lattice by default. Here, $a$ is the width of the square lattice, and $b$ is the height of the square lattice. Hence we can talk about the width and height of a graph structure whose universe is a square lattice. And we can talk about a row of this graph, which is composed of the vertices of the same second coordinate. We call the bottom row as the 0-th row of the graph.

A linear order is a binary relation that is transitive, antisymmetric and total. We usually use the infix notation $\leq$ to denote a linear order. For example, we use $a \leq b$ to stand for $(a, b) \in \leq^\mathfrak{A}$ when the structure $\mathfrak{A}$ is clear from the context. We use $a < b$ to denote $a \leq b$ and $a \neq b$. Any linear order induces a natural distance measure, i.e. we can talk about $|a - b|$ when $a, b$ are two elements of a linear order.

A $k$-clique of $\mathcal{G}$ is a complete subgraph of $\mathcal{G}$, containing $k$ vertices. That is, there is an edge between each pair of vertices in the $k$-clique. The $k$-Clique problem asks whether there is a $k$-clique in a given graph. By default, the graphs are ordered finite graphs.

For any $x, i \in \mathbb{N}_0$, let $\text{BIT}(x, i) \in \{0, 1\}$ be 1 iff the $i$-th bit of the binary representation of $x$ is 1. Here we assume that the rightmost bit is the 0-th bit. A pure arithmetic structure is a structure whose signature contains only arithmetic predicates. It is finite by default. In particular, we assume that the signature contains the binary predicate $\text{BIT}$, which can be used to define arbitrary arithmetic predicates, including linear orders.

The quantifier rank of a formula $\phi \in \text{FO}$, written $qr(\phi)$, is the maximum depth of nesting of its quantifiers. Formally, it is defined inductively as follows:

1. If $\phi$ is atomic, then $qr(\phi) = 0$.
2. $qr(\phi_1 \lor \phi_2) = qr(\phi_1 \land \phi_2) = \max\{qr(\phi_1), qr(\phi_2)\}$.
3. $qr(\neg \phi) = qr(\phi)$.
4. $qr(\exists x \phi) = qr(\forall x \phi) = qr(\phi) + 1$.

Let $\mathcal{L}^k$ be the fragment of FO whose formulas have at most $k$ distinct variables, free or bound. Infinitary logic, written $\mathcal{L}_{\infty\omega}$, is the closure of
first-order logic with infinitary conjunctions and disjunctions. $L^k_{\infty\omega}$ is the fragment of $L_{\infty\omega}$ whose formulas have at most $k$ distinct variables, free or bound. $\exists L^k_{\infty\omega}$ is the fragment of $L^k_{\infty\omega}$ in which no universal quantifiers appear in the formulas and in which all the existential quantifiers are within the scope of even number of negations.

2.2 Pebble games

A game board consists of a pair of structures, e.g. $(\mathfrak{A}, \mathfrak{B})$. An $m$-round $(k-1)$-pebble game over the game board $(\mathfrak{A}, \mathfrak{B})$, written $\nabla^k_{m-1}(\mathfrak{A}, \mathfrak{B})$, is defined as the following.

There are two players in the game, called Spoiler and Duplicator. There are $k-1$ pairs of pebbles, say $(e_1, f_1), \cdots, (e_k, f_k)$, available for the players, which are off the board at the beginning of the game. In each round, a pair of pebbles, say $(e_i, f_i)$, will be put on the structures wherein $e_i$ is put on an element of $\mathfrak{A}$ and $f_i$ is put on an element of $\mathfrak{B}$. Spoiler first selects a structure and puts a pebble on one element of the selected structure; then Duplicator puts the other pebble in the same pair (matching pebble, for short) on one element of the other structure. If there is no pebble off the board, Spoiler can move a pebble to a new element; then Duplicator should move the matching pebble to some element in the other structure.

In the $\ell$-th round of the game, let $\overline{c_\mathfrak{A}} = (a_1, a_2, \ldots, a_n)$, where $n \leq k-1$, includes all the elements in $\mathfrak{A}$ that have pebbles on them; let $\overline{c_\mathfrak{B}} = (b_1, b_2, \ldots, b_n)$ includes all the pebbled elements in $\mathfrak{B}$. And assume that, for any $i$, $a_i$ and $b_i$ are the positions of $e_j$ and $f_j$ for some $j$ in this round.

Sometimes we use $((\mathfrak{A}, \overline{c_\mathfrak{A}}), (\mathfrak{B}, \overline{c_\mathfrak{B}}))$ to denote the game board in this round. But it does not mean that the game board is changed (we will mention a sort of imaginary games that allow changes of boards later). Only that some elements are pebbled. $((\mathfrak{A}, \overline{c_\mathfrak{A}}), (\mathfrak{B}, \overline{c_\mathfrak{B}}))$ is in partial isomorphism or $\overline{c_\mathfrak{A}} \overline{c_\mathfrak{B}}$ defines an partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, if $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ defines an isomorphism between $\mathfrak{A}^{\overline{c_\mathfrak{A}}}$ and $\mathfrak{B}^{\overline{c_\mathfrak{B}}}$.

Spoiler wins the game if the game board is not in partial isomorphism in some round; otherwise, Duplicator wins the game. If Duplicator can guarantee a win after $m$ rounds of such $(k-1)$-pebble game, we say Duplicator has a winning strategy in the $m$-round $(k-1)$-pebble game, denoted by $\mathfrak{A} \equiv_m^{k-1} \mathfrak{B}$.

The following holds in a $(k-1)$-pebble game.

**Fact 1.** In each round of a $(k-1)$-pebble game, either Duplicator can win this round by mimicking, or, there are at most $k-2$ pairs of pebbles on the game board at the start of this round.
It is because that moving a pebble from one element to another element consists of two steps: first pick up the pebble off the board; afterwards put this pebble on the new element. We can always turn a game into such a game without losing anything. The first step will not violate the partial isomorphism of the game board, therefore when we talk about winning strategies we can skip such rounds safely and always assume that there are at most \( k - 2 \) pairs of pebbles on the game board at the start of a round.

It is well-known that pebble games characterize the expressive power of finite variable logics.

**Theorem 1.** The following statements are equivalent.

- For any \( \phi \in \mathcal{L}^{k-1} \) with \( qr(\phi) \leq m \), \( \mathfrak{A} \models \phi \iff \mathfrak{B} \models \phi \).
- \( \mathfrak{A} \equiv_{m}^{k-1} \mathfrak{B} \).

Therefore, if for any \( m \) we can find a pair of structures, e.g. \((\mathfrak{A}, \mathfrak{B})\), such that \( \mathfrak{A} \) satisfies some property while \( \mathfrak{B} \) doesn’t, and \( \mathfrak{A} \equiv_{m}^{k-1} \mathfrak{B} \), then this property is not expressible in \( \mathcal{L}^{k-1} \).

To shorten description, usually we also say that a player picks a vertex if the player puts a pebble on this vertex. If in some round of the game element \( e \) has a pebble on it, we say \( e \) is pebbled in this round. Sometimes, we also use the verb “pick” (a pebbled vertex) to mean “remove” (the pebble from the vertex). Let \( e \) and \( f \) be a pair of vertices picked in some round of the game \( \mathcal{O}_{m}^{k-1}(\mathfrak{A}, \mathfrak{B}) \), with \( e \) picked in \( \mathfrak{A} \) and \( f \) picked in \( \mathfrak{B} \). We use \( e \models f \) to denote it. And for any two sets \( X \) and \( Y \), if there is a bijection \( \eta : X \to Y \) such that for any \( e \in X \) \( \models \eta(e) \), then we use \( X \models Y \) to denote it. If the sets are ordered, i.e. they are tuples, then the bijection simply maps the \( i \)-th element (or called item) of \( X \) to the \( i \)-th element of \( Y \). Sometimes we also use \( e \models f \) to denote the pair of vertices \( e \) and \( f \), where \( e \models f \).

If Spoiler can only put pebbles on elements of \( \mathfrak{A} \) and can play for arbitrary number of rounds, then such \((k-1)\)-pebble games characterize exactly the expressive power of \( \exists \mathcal{L}_{\infty}^{k-1} \). On the other hand, if the players can use arbitrary number of pebbles in the games, such games are called \( (m\text{-round}) \) Ehrenfeucht-Fraissé games, written \( \mathcal{O}_{m}(\mathfrak{A}, \mathfrak{B}) \).

### 3 Bounded variable hierarchy in \( \exists \mathcal{L}_{\infty}^{k} \)

To help understand some bits of the main proof, we first consider a simpler problem and a simple structure \( \mathcal{B}_{k} \).
In a two dimension coordinate plane, the *coordinate congruence number* (or, coordinate residue class number) of a vertex \((x, y)\) in the plane, denoted by \(cc(x, y)\), is defined as the following:

\[
cc(x, y) := x + y \mod k - 1 \quad (3.1)
\]

Note that there are \(k - 1\) different values for coordinate congruence numbers.

**Definition 2.** \(B_k\) is an ordered graph over the universe \([k - 1] \times [k]\) and the linear order is defined by the lexicographic ordering on the Cartesian product \([k] \times [k - 1]\). That is, \((x_i, y_i) \leq (x_j, y_j)\) if \(y_i < y_j\) or \(y_i = y_j \land x_i \leq x_j\). A vertex \((x_i, y_i)\) is adjacent to another vertex \((x_j, y_j)\) if and only if \(y_i \neq y_j\) and \(cc(x_i, y_i) \neq cc(x_j, y_j)\).

It is easy to see that \(B_k\) has no \(k\)-clique, by pigeonhole principle.

In the following we introduce a result by Dawar (cf. [8], p27) and use this chance to introduce some bits of the ideas that are used in our main proofs.

**Theorem 3.** For each \(k\), there is a formula of \(\exists L^k\) that is not equivalent to any formula of \(\exists L^{k - 1}\) on ordered graphs.

**Proof.** Our tool is the variant of \((k - 1)\)-pebble games for \(\exists L^{k - 1}_{\omega \omega}\), where the game board is \((A_k, B_k)\). Here \(A_k\) is a \(k\)-clique that is composed of the vertices \((0, 0), (0, 1), \ldots, (0, k - 1)\) with a linear order defined as that of \(B_k\). Recall that Spoiler is required to pick only in \(A_k\). Observe that Duplicator is able to ensure that the subgraph of \(B_k\) induced by the pebbles in \(B_k\) is a complete graph in each round. In particular, to ensure a \((k - 1)\)-clique in \(B_k\) Duplicator needs only pick those vertices such that for any two picked vertices \((x_i, y_i), (x_j, y_j)\), \(cc(x_i, y_i) \neq cc(x_j, y_j)\) if \(y_i \neq y_j\). The main point is that, in each round, Duplicator can always find a vertex \((x, y)\) for any \(y\) such that \(cc(x, y)\) is different from that of all the pebbled vertices, if there are no more than \(k - 2\) pebbles on a structure (cf. Fact 1).}

We introduce this proof instead of others (e.g. a proof of algebraic flavor) because the ideas presented in this proof can shed some light on Lemma 36, which will be used in the proof of the main Lemma 37 (cf. Strategy 2).

Note that \(B_k\) is the same as the structure introduced by Dawar, if we circular shift the vertices of the \(i\)-th row \(i\) times to the right. Moreover, we shall see that such right circular shifts can prevent Duplicator from the so-called "boundary checkout strategy" of Spoiler (cf. p. 31). Nevertheless, to help the readers understand the intuitions behind the constructions and to make the proofs as less involved as possible, in the following sections we first introduce the original structures, then shift the vertices afterwards.
4 Outline of the remainder of the paper

The rest of the paper is organized as follows. We first introduce a proof for the special case where \( k = 3 \) in section 5.1. It is a good place to bring forward a key notion called “(structural) abstraction”. We index the vertices of our graphs and view the graphs in different scales, each of which is a distinctive abstraction. A higher abstraction characterizes some key feature of lower abstractions. And Duplicator uses strategies over abstractions to decide her picks in the original games. In this viewpoint, we reduce the original games to games over abstractions. That is, the players are also playing a game over some specific abstraction in each round, in addition to the original game: each pick are projected to this abstraction and Duplicator need only ensure partial isomorphisms over this abstraction to win this round.

In section 5.2, we will introduce a pair of graphs \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \). Before this, we introduce a notion called board history, which characterizes reasonable evolutions of a game board, and “embed” it into every vertex of \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \) to construct a pair of ordered graphs \( \mathfrak{A}_{k,m} \) and \( \mathfrak{B}_{k,m} \), for the game board. In the process of creating \( \mathfrak{B}_{k,m}^* \), we need a notion called congruence label, based on which the key notion type label is defined, which roughly tells us how a vertex of some label is connected to another vertex of other label. An element in the definition of type label is a set \( \Omega \), based on which we forbid some sort of edges in \( \mathfrak{B}_{k,m}^* \). And such missing of edges characterizes some global feature of some subgraphs of \( \mathfrak{B}_{k,m}^* \), thereby distinguishing one subgraph from another. The notion “abstraction” is somehow based on such features. The pair of main structures are \( \tilde{\mathfrak{A}}_{k,m} \) and \( \tilde{\mathfrak{B}}_{k,m} \), which are obtained from \( \mathfrak{A}_{k,m} \) and \( \mathfrak{B}_{k,m} \) by right circular shifting of vertices.

In section 6, we use \( \tilde{\mathfrak{A}}_{k,m} \) and \( \tilde{\mathfrak{B}}_{k,m} \) for the game board to prove, by a simultaneous induction, that Duplicator has a winning strategy in the game, thereby \( k \)-variables are needed for \( k \)-Clique in FO. But instead of studying it over the main structures directly, we study it over \( \tilde{\mathfrak{A}}_{k,m}^* \) and \( \tilde{\mathfrak{B}}_{k,m}^* \), and classify the vertices of these two graphs into \( m \) sets (the \((i-1)\)-th set \( X_{i-1}^* \) subsumes the \( i \)-th set \( X_i^* \)), each of which induces a graph, i.e. an abstraction, that resembles \( \mathcal{B}_k \) to some extent. The \( i \)-th abstraction is an induced subgraph of the \((i - 1)\)-th abstraction. We shall see that the Duplicator has a winning strategy in the original game if she has a winning strategy in a so called associated game over abstractions and changing board. In such an associated game, if Duplicator is not able to respond Spoiler by picking a vertex in the \( i \)-th abstraction, then Duplicator resorts to the \((i - 1)\)-th abstraction for a solution. In the games, Duplicator can *force* the games
played over some specific abstraction, which, when necessary, enables herself to find a solution in the closest lower abstraction in each round.

In section 7, based on a not well-known but still reasonable assumption, we show that $n^{k-1}$ gates not suffice to compute $k$-Clique on $DLOGTIME$-uniform families of constant-depth unbounded fan-in circuits. In section 7.1 we first show that $k$-variables are needed to define $k$-Clique in FO($\text{BIT}$), by embedding the main structures (cf. section 5.2) in a pure arithmetic structure introduced by Schweikardt and Schwentick [28]. Afterwards, in section 7.2 we translate this result in logic to a size lower bound in circuit complexity. We first show that $O(n^{k-3})$ gates not suffice to compute $k$-Clique, using the standard translation and an observation that the bounded variable hierarchy in FO collapses to FO$^3$ in the pure arithmetic. Afterwards, based on an assumption, we get the believed tight lower bound via a notion called succinct regular circuits, whose structures respect the “logical structure” of first-order formulas.

In section 8, we summarize our main results and discuss future work and open questions.

5 The structures

5.1 Vertex index, structural abstractions and games over abstractions: the case where $k = 3$

In this section we prove that our main result holds in the special case where $k = 3$. That is, 3 variables are needed to define 3-Clique in FO. In other words, Duplicator has a winning strategy in the 2-pebble games of arbitrary finite rounds. The case where $k = 3$ is quite different from other cases: it is much simpler than the cases where $k > 3$ (see the subsequent sections), but much more difficult than the case where $k = 2$. Note that it is trivial when $k = 2$. For any $m \in \mathbb{N}^+$, $\mathcal{B}_{2,m}$ is simply a graph of two isolated vertices with arbitrary order on them. $\mathcal{A}_{2,m}$ is built from $\mathcal{B}_{2,m}$ by adding one edge between these two vertices. Duplicator simply mimics Spoiler, which is a winning strategy in an $m$-round 1-pebble game over the game board $(\mathcal{A}_{2,m}, \mathcal{B}_{2,m})$, for arbitrary $m$.

For the special case where $k = 3$, we introduce a proof that is most suitable to cast light on some of the concepts and ideas that will be used in the subsequent sections. In particular, we introduce a key concept called “(structural) abstraction”, as well as pebble games over abstractions, which is also crucial in the subsequent sections. In addition to giving a proof for this special case, we hope this can offer some intuition for the following
more technical constructions and proofs. Note that almost all the lemmas introduced in this section will be used in section 5.2 and section 6.

Firstly, we construct a structure $\mathcal{B}_3', m$ via a process that can be called “(iterative) structural expansion”\(^2\). Instead of a formal definition, which is easy to give, we explain it briefly by the following example. We first construct a structure, called $\mathcal{B}_3', m[\mathcal{X}_m^*]$, whose universe is a square lattice and whose width is $\gamma_0^*$. Then, we use it as a “skeleton” or “blueprint” to build a larger structure, called $\mathcal{B}_3', m[\mathcal{X}_m^*-1]$. The basic “brick” we shall use to build base on the blueprint can be anything. But here the brick we use is similar to $\mathcal{B}_3', m[\mathcal{X}_m^*]$ itself. More precisely, we “expand”, or replace, every vertex by a successive vertices. Hence any “path” (not necessary connected) of the “skeleton” that is from the bottom to the top corresponds to a set of vertices of $\mathcal{B}_3', m[\mathcal{X}_m^*-1]$, which is isomorphic to a square lattice. We call such a square lattice (not necessary upright) a “brick”. Such bricks are either isomorphic or very similar. Once we get $\mathcal{B}_3', m[\mathcal{X}_m^*-1]$, whose width is $\gamma_1^*$, we take it as a new “skeleton” and use it to build $\mathcal{B}_3', m[\mathcal{X}_m^*-2]$, and so on, until we get $\mathcal{B}_3', m[\mathcal{X}_1^*]$, i.e. the structure $\mathcal{B}_3', m$ we want, whose width is $\gamma_{m-1}^*$. Once $\mathcal{B}_3', m$ is obtained, we create a new structure $\mathcal{B}_3, m$, as well as $\tilde{\mathcal{B}}_3, m$, based on it. In the following we define $\mathcal{B}_3', m$ formally.

For any $m, i \in \mathbb{N}^+$, where $m \geq 3$ and $0 < i < m$, let
\begin{align*}
\gamma_0^* & := 4m \\
\gamma_i^* & := 4(m-i)\gamma_{i-1}^*
\end{align*}
\hspace{1cm} (5.1)

For $x \in [\gamma_{m-1}^*]$ and $1 \leq i \leq j \leq m$, let
\begin{align*}
\beta_{m-j}^{m-i} & := \frac{\gamma_{m-i}^*}{\gamma_{m-j}^*} \\
[x]_i & := [x/\beta_{m-j}^{m-1}] \\
\langle x \rangle_i & := [x/\beta_{m-j}^{m-1}] + \frac{1}{2} \sum_{1 < \ell \leq i} \beta_{m-\ell}^{m-1}
\end{align*}
\hspace{1cm} (5.3)

\hspace{1cm} (5.4)

\hspace{1cm} (5.5)

Note that $\beta_{m-j}^{m-i} = \prod_{m-j \leq \ell < m-i} \frac{\gamma_{\ell+1}^*}{\gamma_{\ell}^*} = 4^{j-i} \times \frac{(j-1)!}{(i-1)!}$. By convention, $0! = 1! = 1$. Hence $\gamma_{m-1}^* = \gamma_0^* \beta_{0}^{m-1} = m! \times 4^m$.

Obviously, the structure $\mathcal{B}_3', m$ is big. So we put a remark in the appendix to illustrate some essence of the notion structural expansion. Cf. Remark \(^\text{13}\).

\(^2\)Note that it is different from the concepts “expansion” and “extension” in model theory, as defined in the classical textbook by Chang and Keisler.
The readers should be aware of the difference between the notations \([x]_i\) and \([x]\). The latter is seldom used in our paper, which most often appears in the Cartesian product when we define the universe of a structure. \(^3\)

Let

\[ X_i^* := [\gamma_{m-1}^*] \times [3]. \] (5.6)

For \(1 < i \leq m\), let

\[ X_i^* := \{(x, y) \in X_i^* \mid x = \langle i \rangle_i\}. \] (5.7)

The structure \(\mathcal{B}_{3,m}'\) is an ordered graph over the universe \(X_1^*\), wherein the linear order is defined as the lexicographic ordering over \([3] \times [\gamma_{m-1}^*]\). And for any pair of vertices \((x_i, y_i)\) and \((x_j, y_j)\), if \((x_i, y_i) \in X_p^*\) implies \((x_j, y_j) \in X_p^*,\) and \(\ell\) is the maximum in \([1, m]\) s.t. \((x_i, y_i) \in X_p^*\), then \((x_i, y_i)\) is adjacent to \((x_j, y_j)\) if and only if \(y_i \neq y_j\) and \(\text{cc}([x_i]_\ell, y_i) \neq \text{cc}([x_j]_\ell, y_j)\). \(^4\)

We can regard the universe of \(\mathcal{B}_{3,m}'\) as a square lattice, whose width is \(\gamma_{m-1}^*\) and whose height is 3. Its lattice points are the set of elements of \(X_1^*\). We can define the \(i\)-th abstraction of the structure as the induced graph \(\mathcal{B}_{3,m}'[X_i^*]\), whose universe is a square (sub)lattice of \(\mathcal{B}_{3,m}'\) and whose width is \(\gamma_{m-i}^*\). For instance, the \(m\)-th abstraction is \(\mathcal{B}_{3,m}'[X_m^*]\), who has width \(\gamma_0^*\) and height 3. We take it that the \(i\)-th abstraction is a higher abstraction relative to the \(j\)-th abstraction if \(i > j\). Note that, for any \(0 \leq j \leq i \leq m-1\), we can regard \(\beta_j^i\) successive vertices in the \((m-i)\)-th abstraction as one “vertex” in the \((m-j)\)-th abstraction. More precisely, we can take it that each row of \(\mathcal{B}_{3,m}'[X_{m-i}^*]\) is divided evenly into \(\gamma_{m-j}^*\) intervals of the same length \(\beta_j^i\), where each vertex in \(X_{m-j}^*\) is roughly in the middle of some interval that is composed of vertices in \(X_{m-j}^*\).

So far we regard \(\mathcal{B}_{3,m}'[X_{m-j}^*]\) as an abstraction of \(\mathcal{B}_{3,m}'[X_{m-i}^*]\). Besides “(structural) abstraction”, we may also consider the dual concept, i.e. “(structural) expansion”, which describes the reverse side. That is, the following two statements are equivalent:

- \(\mathcal{B}_{3,m}'[X_{t}^*]\) is an abstraction of \(\mathcal{B}_{3,m}'[X_{t-1}^*]\); \(\beta_{m-i}^{m-i+1}\) vertices in the \((i-1)\)-th abstraction are encapsulated into one vertex in the \(i\)-th abstraction;

\(^3\)By Wikipedia, Gauss introduced the notation \([x]\) for the floor function in 1808, which remained the standard until 1962 when there is need to distinguish the notation of ceiling functions from that of floor functions. In our paper, no ceiling functions are involved. Moreover, we need a notation to distinguish it from the standard notation \([x]\). Hence we adopt and alter Gauss’s notation here, i.e. using \([x]\) to denote a special kind of floor functions, as defined in \([x]\).

\(^4\)We shall see that \(X_{t+1}^* \subseteq X_t^*\) for any \(t\), due to Lemma \([x]\). We will define a concept called “vertex index” (cf. Definition \([x]\)) and we shall see that \((x, y) \in X_i^* - X_{i+1}^*\) if and only if the index of \((x, y)\) is \(t\), for \(1 \leq t < m\).
• $\mathcal{B}_3^t[X_i^*]$ is an expansion of $\mathcal{B}_3^t[X_i^*]$; every vertex of $\mathcal{B}_3^t[X_i^*]$ is replaced by $\beta_{m-i+1}^m$ successive vertices.

Note that, for any $(x,y) \in X_i^*-X_i^*$, we can regard $[x]_i$ as a sort of “abstraction”, which tells us the “(relative) position” of $(x,y)$ in $\mathcal{B}_3^t[X_i^*]$. So we call $[x]_i$ the “$i$-th relative first coordinate of $(x,y)$” and $([x]_i,y)$ the “$i$-th relative position of the vertex $(x,y)$”.

And for any $(x,y) \in X_i^*$, where $1 \leq j < i \leq m$, the vertex $([x]_i,y)$, which is a lattice point of $\mathcal{B}_3^t[X_i^*]$, can be regarded as the projection of $(x,y)$ (a lattice point of $\mathcal{B}_3^t[X_j^*]$) in the $i$-th abstraction, because $([x]_i,y) \in X_i^*$ and $([x]_i)_i = [x]_i$, which is unique for $(x,y)$ by the following lemma.

**Lemma 4.** Let $1 \leq i,j \leq m$. For any $(x,y) \in X_j^*$ and $(x',y) \in X_i^*$, if $[x']_i = [x]_i$, then $x' = [x]_i$.

From now on, we call $([x]_i,y)$ the projection of $(x,y)$ in the $i$-th abstraction. This lemma says that, if $(x',y)$ is a lattice point of $\mathcal{B}_3^t[X_i^*]$, and $(x,y)$ has the same $i$-th relative position as $(x',y)$, then $(x',y)$ is the projection of $(x,y)$ in the $i$-th abstraction.

The following lemma says that $X_i^*$ subsumes $X_j^*$ if $i \leq j$. Hence a vertex is in lower abstractions if it is in some higher abstraction. That is, for the square lattice $\mathcal{B}_3^t$, a lattice point of the square (sub)lattice $\mathcal{B}_3^t[X_j^*]$ is also a lattice point of the square (sub)lattice $\mathcal{B}_3^t[X_i^*]$.

**Lemma 5.** For any $i$ where $1 \leq i \leq m$, if $(x,y) \in X_i^*$, then $(x,y) \in X_j^*$ for any $1 \leq j \leq i$.

In other words, $(x,y) \in X_i^*$ implies that $x = [x]_j$ for $1 \leq j \leq i$. Because of this lemma, it is meaningful to introduce the following important concept, by which we can index the vertices of $\mathcal{B}_3^t$.

**Definition 6.** The index of $(x,y) \in X_i^*$, written $\text{idx}(x,y)$, is the maximum $i$, where $1 \leq i \leq m$, such that $(x,y) \in X_i^*$.

By Lemma 4, $(x,y)$ has index $i$ if and only if $(x,y) \in X_i^*-X_{i+1}^*$, for $1 \leq i < m$; and $\text{idx}(x,y) \geq j$ if and only if $(x,y) \in X_j^*$, for $1 \leq j \leq m$.

Note that $\mathcal{B}_3^t$ has many 3-cliques, i.e. triangles. We can index these triangles such that the index of a triangle is the smallest index of its vertices. We can generalize this concept to arbitrary $k$ as the following.

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5 Imagining that, if we look at a picture from far away, then many vertices in a row might seem as one.
Definition 7. A $k$-clique $C_k$, where $|C_k| \subset X^*_1$, has index $i$ if $i$ is the maximum in the range $[1, m]$ such that $|C_k| \subset X^*_i$.

By definitions, we have the following easy observations, whose proofs are straightforward.

Lemma 8. For any vertex $(x, y)$ of index $i$ and $j \leq i$, we have

$$x = \langle x \rangle_j.$$ 

By lemma 5, if $(x, y)$ has index $i$ and $j \leq i$, then $(x, y)$ is already a vertex in $X^*_j$. Therefore, by definition, the projection of $(x, y)$ in the $j$-th abstraction is itself.

Lemma 9. For any $(x, y) \in X^*_i$ and $i \leq j$, we have

1. $[\langle x \rangle_i]_j = [x]_j.$
2. $\langle [x]_i \rangle_j = \langle x \rangle_j.$

In particular, by (1), we have $[\langle x \rangle_i]_i = [x]_i.$ (2) says that the projection of $(x, y)$ to the $j$-th abstraction can be regarded as a process wherein we first project $(x, y)$ to the $(i + 1)$-th abstraction, then to the $(i + 2)$-th abstraction, and so on, until we project $(\langle x \rangle_{j-1}, y)$ to the $j$-th abstraction, i.e. projecting to $(\langle x \rangle_j, y)$.

We immediately have the following observation, as a corollary of (2) of Lemma 9.

Lemma 10. For any $(x, y) \in X^*_i$ and $i \in [1, m]$, we have

$$\text{idx}(\langle x \rangle_i, y) \geq i.$$ 

This lemma says that the projection of $(x, y)$ in the $i$-th abstraction is a vertex in $X^*_i$, which is obvious. To prove it, we need only show that $(\langle x \rangle_i, y) \in X^*_i$, i.e. $\langle x \rangle_i = \langle \langle x \rangle_i \rangle_i.$

Fact 2. For any vertex $(x, y)$ of index $i$ where $1 < i \leq m$, there are exactly $\beta_{m-i+1}^m - 1$ vertices of index $i - 1$ such that the projections of these vertices in the $i$-th abstraction is exactly $(x, y)$. Moreover, these vertices, together with $(x, y)$, are successive which form an interval. And $(x, y)$ is in the middle of this interval.
Proof of Fact:

The first part of this claim is obvious: these vertices are exactly the set of vertices \([x_i]_\beta^{m-i+1} + \ell, y\) for any \(0 \leq \ell < \frac{1}{2}\beta^{m-i+1}_m\) or \(\frac{1}{2}\beta^{m-i+1}_m < \ell < \beta^{m-i+1}_m\). For any other vertex, it is easy to verify that its projection in the \(i\)-th abstraction is either less than or greater than \((x, y)\) with respect to the linear order.

To prove the second part of this claim, we observe that the (relative) position of \((x, y)\), i.e. \([x_i/\beta^{m-1}_{m-i+1}]\), in the \((i-1)\)-th abstraction is \(\lfloor x_i/\beta^{m-1}_{m-i+1} \rfloor\). Note that \(\lfloor x_i/\beta^{m-1}_{m-i+1} \rfloor = [x_i^\ast]_\beta^{m-j+1}_m\). Hence, there are \(\frac{1}{2}\beta^{m-i+1}_m\) successive vertices of index \(i-1\) that are on the left side of \((x, y)\) in the \(y\)-th row, and there are \(\frac{1}{2}\beta^{m-i+1}_m-1\) successive vertices of index \(i-1\) that are on the right side of \((x, y)\). This concludes the claim.

Q.E.D. of Fact.

The following is a direct corollary of this fact.

**Fact 3.** For any vertex \((x, y)\) of index \(i\) where \(1 < i \leq m\), there are exactly \(\beta^{m-1}_m - 1\) vertices of index up to \(i-1\) such that the projections of these vertices in the \(i\)-th abstraction is exactly \((x, y)\). Moreover, these vertices, together with \((x, y)\), are successive which form an interval. And \((x, y)\) is roughly in the middle of this interval.

Proof of Fact:

Just note that

\[\beta^{m-1}_{m-i} = \prod_{1 < j \leq i} \beta^{m-j+1}_{m-j}\]

Q.E.D. of Fact.

Assume that \(0 \leq j < i \leq m-1\). For any vertex \((x^\ast, y)\) of index \(m-j\), we call those \(\beta^{j-1}_j\) vertices, whose indices are greater than or equal to \(m-i\) but less than \(m-j\) and whose projections in the \((m-j)\)-th abstraction are \((x^\ast, y)\), the vertices in \(X^\ast_{m-i}\) that surround \((x^\ast, y)\). For example, for any \((x, y)\) where \(\text{idx}(x, y) = m-i\), \((x, y)\) is a vertex that surrounds \((x^\ast, y)\) if \([x]_{m-j} = [x^\ast]_{m-j}\). And each vertex of index \(m-j\) is surrounded by \(\beta^{m-j+1}_j\) vertices of lower abstractions, i.e. the vertices in \(X^\ast_1\). Therefore, each vertex of index \(m\) is surrounded by \(\beta^{m-1}_0\) vertices of index \(m-1\), where this vertex of index \(m\) is in the middle of the interval that is composed of these surrounding vertices in \(X^\ast_{m-1}\); each vertex in \(X^\ast_{m-1}\), i.e. a vertex of index \(m-1\) or \(m\), is also surrounded by \(\beta^2_1-1\) vertices of index \(m-2\), and so on.

A direct corollary of Fact 3 is the following fact.
Fact 4. For any \((x, y), (x', y) \in X_i^*,\) we have
\[|x - x'| = c_δ^{m-1}_{m-i},\]
for some \(c \in \mathbb{N}_0.\)

We shall introduce pebble games over abstractions. The following observation is crucial for such games.

Lemma 11. For any \(1 < \xi \leq m\) and \(a, a' \in \gamma_{m-1}\), if \(a - \langle a \rangle_\xi = a' - \langle a' \rangle_\xi,\) then the following hold:

1. \(\langle a \rangle_\xi - \langle a \rangle_{\xi-1} = \langle a' \rangle_\xi - \langle a' \rangle_{\xi-1}\)
2. \(a - \langle a \rangle_{\xi-1} = a' - \langle a' \rangle_{\xi-1}.\)

For \(i \in [k],\) let
\[tr(i) := (i \mod k - 1) \times \sum_{1 \leq p \leq m} \beta^{m-1}_{m-p}.\]  
\[\text{(5.8)}\]

The structure \(\tilde{B}_{3,m}\) is constructed from \(B'_{3,m}\) by

1. removing a set of edges: for any vertex \((x, 1) \in X^*_\ell - X^*_{\ell+1}\) (i.e. \(\text{id}_x(x, 1) = \ell < m\)) where \([x]_\ell\) is even, we delete the following edges between \((x, 1)\) and any vertex in \(\Omega_x\) where
\[\Omega_x := \{(u, v) \in X^*_{\ell+1} | (u, v) \text{ is not adjacent to } (\langle x \rangle_{\ell+1}, 1)\};\]  
\[\text{(5.9)}\]
2. circular shifting the vertices of the \(i\)-th row for \(tr(i)\) times to the right.

We call the structure constructed from \(B'_{3,m}\) after the first step (i.e. before the circular shifts) \(B_{3,m}\). Note that, all the Lemmas mentioned so far continue to work with such adaption. In the following we are mainly interested in this structure and all the results are created for this structure. The shifts will be met and used only when we discuss “\(4^o\)” in the proof of Lemma 18 (cf. p. 30) and a strategy of Spoiler called “boundary checkout strategy” (cf. p. 31).

In Fig. 1, \(c\) is even, i.e. \([x]_\ell\) is even (\(\because (x^*, 1) \in X^*_{\ell+1}\); we shall see it shortly). Then, by (5.9), \((x, 1)\) is not adjacent to \((u, 2)\) since \((x^*, 1)\) is not adjacent to \((u, 2)\). Similarly, \((x', 1)\) is not adjacent to \((u, 2)\) since \((x, 1)\) is
Figure 1: From $B'_{3,m}$ to $B_{3,m}$: some edges are forbidden. Here $x^* = ([x]_{\ell+1}$. Suppose $\text{idx}(x, 1) = \ell$, $\text{idx}(x', 1) = \ell - 1$, and $(u, 2), (x^*, 1) \in X^*_{\ell+1}$. Assume $c$ and $d$ are even.

not adjacent to $(u, 2)$ and $d$ is even. Hence, the missing of an edge in higher abstraction (e.g. the one between $(x^*, 1)$ and $(u, 2)$) will propagate to lower abstractions (e.g. the one between $(x', 1)$ and $(u, 2)$).

By Fact 3, for any vertex $(x, y)$ of index $\ell$ where $1 < \ell \leq m$, there are exactly $\beta_{m-\ell}^{m-1}$ vertices of index up to $\ell - 1$ such that the projections of these vertices in the $\ell$-th abstraction is exactly $(x, y)$. Moreover, these vertices, together with $(x, y)$, are successive which form an interval (i.e. the dashed rectangle in Fig. 1). And $(x, y)$ is roughly in the middle of this interval.

Note that, for the sake of convenience, here we regard the leftmost vertex of the $\ell$-th row of $\tilde{B}_{3,m}$ as $(\gamma_{m-1}^* - tr(\ell), \ell)$ instead of $(0, \ell)$\(^6\). Because both $\gamma_{m-1}^*$ and $\beta_{m-\ell}^{m-1}$ are divisible by $(k - 1)\beta_{m-i}^{m-1}$ for any $i < p$, we have the following observation.

**Lemma 12.** For any $\ell \in [k]$ and $1 \leq i \leq m$,

$$cc([\gamma_{m-1}^* - tr(\ell)]_i, \ell) = 0.$$ 

It implies that, for any $i$ and $0 < \ell < k - 1$ (i.e. $\ell \not\equiv 0 \pmod{k - 1}$),

$$[\gamma_{m-1}^* - tr(\ell)]_i \not\equiv 0 \pmod{k - 1}. \quad (5.10)$$

\(^6\) In this viewpoint, we regard "$(x, y)$" as a name or label for the associated vertex. Then we can preserve the definitions, such as (3.1). A shortcoming of such treatment is that we have to be cautious when computing the distance of two vertices in a row. It is possibly no more the difference of the “first coordinates”\(^7\). Fortunately, most often we can think of $\mathfrak{B}_{3,m}$ instead of $\tilde{B}_{3,m}$. Only when we discuss the reason “4” can be ensured” (cf. p. 30) or when we meet “boundary checkout strategy” (cf. p. 31) we should switch to $\tilde{B}_{3,m}$.

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Only when $\ell = 0$ or $\ell = k - 1$, we have that, for any $i$,
\[
[\gamma_{m-1}^* - tr(\ell)]_i \equiv 0 \pmod{k-1}.
\tag{5.11}
\]

By Fact 2 and Fact 3, we know that the index of a vertex in $X_i^*$ should be $i$, if it is a boundary vertex (i.e. either the leftmost or the rightmost) of the $i$-th abstraction. Similarly, we have the following observation.

**Lemma 13.** For any $1 \leq i \leq m$ and $\ell \in [k],$
\[
\text{idx}([\gamma_{m-1}^* - tr(\ell)]_i, \ell) = i.
\]

**Proof.** The reason is simple. Just observe that, in $B_{k,m}$, $(\gamma_{m-1}^* - tr(\ell))$ is the rightmost vertex whose index is $i$, and that the distance between $(\gamma_{m-1}^* - tr(\ell))_i, \ell)$ is only $\ell$, much less than $\frac{1}{2}\beta_{m-i-1} - 1$. As a consequence, the index of $(\gamma_{m-1}^* - tr(\ell))_i, \ell)$ cannot be $i+1$. On the other hand, by Lemma 10, $(\gamma_{m-1}^* - tr(\ell))_i, \ell) \in X_i^*$. This concludes the claim. \hfill \qed

Let
\[
\text{mid} := 2m\beta_0^{m-1} + \frac{1}{2} \sum_{1 < j \leq m} \beta_{m-j}^{m-1}.
\tag{5.12}
\]

By the definition, we know that $\text{mid} = (\text{mid})_m$, thereby $(\text{mid}, y) \in X_m^*$ for any $y$. Note that mid is roughly half of $\gamma_{m-1}^*$.\footnote{\footnote{Just observe that $\gamma_{m-1}^* = 4m\beta_0^{m-1}$ and that $\beta_j^{m-1}$ is much smaller than $\beta_0^{m-1}$ when $1 < j < m.$}} The structure $\mathfrak{A}_{3,m}$ is built from $\mathfrak{B}_{3,m}$ by adding an edge between $(\text{mid}, 0)$ and $(\text{mid}, 2)$. Call the endpoints of this edge critical points.

The structure $\tilde{\mathfrak{A}}_{3,m}$ is obtained from $\mathfrak{A}_{3,m}$ by the same circular shifts as the way we obtain $\mathfrak{B}_{3,m}$. In other words, $\tilde{\mathfrak{A}}_{3,m}$ is also obtained from $\mathfrak{B}_{3,m}$ by adding an edge between $(\text{mid}, 0)$ and $(\text{mid}, 2)$.

For each set $X_i^*$, we define the $i$-th abstraction of the structure $\mathfrak{A}_{3,m}$ ($\mathfrak{B}_{3,m}$ resp.) by $\mathfrak{A}_{3,m}[X_i^*]$ ($\mathfrak{B}_{3,m}[X_i^*]$ resp.). For any $(u, v)$, call $cc([u]_{\ell}, v)$ the coordinate congruence number of $(u, v)$ in the $\ell$-th abstraction.

The following statement is a straightforward but important observation. Recall that $k = 3$ in this section.

**Lemma 14.** If $1 \leq j < i$ and $(x, y) \in X_i^*$, then $[x]_j \equiv 0 \pmod{k-1}$.

In other words, $cc([x]_j, y) = y \mod{k-1}$. That is, for any vertex in higher abstraction, its coordinate congruence number in lower abstractions is completely determined by its second coordinate. Hence, in the case where
Remark 15. For any vertex \((x, y) \in X_\ast\) and a number \(i\) where \(i < \text{idx}(x, y) = t\), we call the set of successive vertices surrounding \((x, y)\) the \(i\)-th complete expansion of \((x, y)\), denoted \(cex(x, y, i)\), if their relative positions in the \(t\)-th abstraction are the same as that of \((x, y)\). More precisely, \(cex(x, y, i) = \{(u, y) \in X_i \mid \text{idx}(u, y) \leq i; [u]_t = x\} \cup \{(x, y)\}\). We can regard \(cex(x, y, i)\) as one object that contains \((x, y)\). For example, the object \(cex(mid, 0, i)\) contains the critical point \((\text{mid}, 0)\) and those vertices whose indices are no more than \(i\) and their relative position in the \(m\)-th abstraction is \((\text{mid}, 0)\). The reason we regard \(B_{3,m}[X_i]\) as an “abstraction” of \(B_{3,m}\) is not only because we can regard \(\beta_{m-1}^{m-1}\) elements in the first abstraction as one element in the \(i\)-th abstraction, but also because whether two vertices are adjacent in the \(i\)-th abstraction will determine the adjacency of some of the vertices in the lower abstractions, due to the missing of edges in the process we produce \(B_{3,m}\) from \(B_{3,m}'\) (cf. (5.9), the definition of \(\Omega_\ast\) when \(k > 3\), cf. “\(\Omega^\ast\)” in Definition 25). What’s more, for any \((x_0, y_0), (x_1, y_1), (x'_0, y_0), (x'_1, y_1) \in X_i - X_{i+1}\), the subgraph induced by \(cex(x_0, y_0, t - 1)\) and \(cex(x_1, y_1, t - 1)\) is isomorphic to the subgraph induced by \(cex(x'_0, y_0, t - 1)\) and \(cex(x'_1, y_1, t - 1)\) if and only if the adjacency between \((x_0, y_0)\) and \((x_1, y_1)\) is the same as that between \((x'_0, y_0)\) and \((x'_1, y_1)\). Here we give a brief intuitive explanation. A strict proof is very verbose, which can be found in Remark 57. The isomorphism is defined by the bijection \(h\) such that \(h(x_i) = x'_i\), where \(i \in \{0, 1\}\), and \(u - x_i = h(u) - h(x_i)\) for any \((u, y_i)\) in the subgraph induced by \(cex(x_0, y_0, t - 1)\) and \(cex(x_1, y_1, t - 1)\). From Fact 2 and Fact 3, we can see that \(\text{idx}(u, y_i) = \text{idx}(h(u), y_i)\). We confess that this is word-of-mouth. Suppose that \(\text{idx}(u, y_i) = \ell\). By Lemma 14 and the fact that \((u - x_i)/\beta_{m-1}^{m-1} = (h(u) - h(x_i))/\beta_{m-1}^{m-1}\), we have \(cc([u]_\ell, y_i) = cc([h(u)]_\ell, y_i)\). Moreover, we can show that \(cc([u]_j, y_i) = cc([h(u)]_j, y_i)\) for \(\ell < j \leq t\). For a strict proof, cf. Remark 57. These facts justify the following observations. Firstly, observe that the subgraph induced by \(cex(x_0, y_0, t - 1)\) and \(cex(x_1, y_1, t - 1)\) is isomorphic to the subgraph induced by \(cex(x'_0, y_0, t - 1)\) and \(cex(x'_1, y_1, t - 1)\), without considering the missing of edges due to the process we create \(B_{3,m}\) from \(B_{3,m}'\). Secondly, by definition (5.9), we know that the nonadjacency of \((x_0, y_0)\) and \((x_1, y_1)\) will propagate to lower abstractions. Assume that \((x_0, y_0)\) is not adjacent to \((x_1, y_1)\) and so are \((x'_0, y_0)\) and \((x'_1, y_1)\). Let \((u, y_i)\) be a vertex of index \(\ell = t - 1\) such that

\[k = 3, \text{ for any } (u, v) \in X_{i+1}, \text{ } cc([u]_\ell, v) = v \mod 2 \text{ since } [u]_\ell \text{ is even}.\]
We prove it by contradiction. Assume that there are triangles in $L_{cc,adj}$ adjacent to $(x,0)$. Similarly, $(h(u),y_1)$ is not adjacent to $(h(x_0),y_0)$ iff either $[h(u)]_{t-1}$ is even or $cc([h(u)]_{t-1},y_1) = cc([h(u)]_{t-1},y_0)$. That is, the adjacency of $(x_0,y_0)$ and $(u,y_1)$ is the same as that of $(h(x_0),y_0)$ and $(h(u),y_1)$: the missing of edges is propagated from the $t$-th abstraction to the $(t-1)$-th abstraction. Such propagations will be the same in two isomorphic structures when they are toward lower abstractions. In summary, the adjacency of $(x_0,y_0)$ and $(x_1,y_1)$ determines the unique feature of the subgraph induced by $ccex(x_0,y_0,t-1)$ and $ccex(x_1,y_1,t-1)$. We can generalize it by introducing more vertices in $X^*_t$.

For example, assume that the vertices $(x,0), (x,1)$ and $(x,2)$ have the same adjacency as the vertices $(x',0), (x',1)$ and $(x',2)$; and assume that they are vertices in $X^*_t - X^*_{t+1}$. Then the subgraph induced by $ccex(x,0,t-1)$, $ccex(x,1,t-1)$ and $ccex(x,2,t-1)$ is isomorphic to the subgraph induced by $ccex(x',0,t-1)$, $ccex(x',1,t-1)$ and $ccex(x',2,t-1)$. This somehow justifies the notion “abstraction”. For any $i,j,\ell$ where $i > j > p$, $B_{3,m}[X^*_j]$ is an “abstraction” of $B_{3,m}[X^*_j]$, which is also an “abstraction” of $B_{3,m}[X^*_j]$. Hence “abstraction” is relative. And each abstraction is a sketch of the structure with respect to some “scale”.

A strict argument for the above observation is verbose and involved, cf. the proofs of Lemma 58, 59 and Lemma 60 for insights, which are used in a more general and complicated setting. But this perhaps helps: the way we construct the structure via iterative structural expansion enforces the isomorphism of neighbourhoods of vertices of identical index.

Clearly, $A_{3,m}$ has triangles, which implies that $\bar{A}_{3,m}$ has triangles. In particular, $A_{3,m}$ has a triangle formed by the set of vertices $$\{(mid,0), (mid,1), (mid,2)\},$$ because all the vertices have index $m$, which implies that $\Omega_{mid} = \emptyset$, and $cc([mid]_m,i) = i \text{ mod } 2$, which implies that both $(mid,0)$ and $(mid,2)$ are adjacent to $(mid,1)$. In contrast, we have the following observation.

**Fact 5.** $B_{3,m}$ has no triangle.

**Proof of Fact:**

We prove it by contradiction. Assume that there are triangles in $B_{3,m}$ and $C_3$ is such a triangle that has the maximum index, say $t$. Note that
cannot be \( m \), for otherwise there are two vertices that have the same coordinate congruence number in the \( m \)-th abstraction by the pigeonhole principle. Similarly, \( C_3 \) must contain both vertices in \( X_t^* - X_{t+1}^* \) and vertices in \( X_{t+1}^* \), due to the pigeonhole principle. Let \( |C_3| = \{(a, 0), (b, 1), (c, 2)\} \), inasmuch as the second coordinates of the vertices of \( C_3 \) must be different.

Let \( P = \{(x, y) ∈ X_t^* - X_{t+1}^* | (x, y) ∈ |C_3|\}. And let \( Q = \{(x, y) ∈ X_{t+1}^* | (x, y) ∈ |C_3|\}. Note that \( P \cap Q = \emptyset \). By Lemma 5, the set of vertices of \( C_k \) is exactly \( P ∪ Q \).

Let \( cC_3 := \{cc([x]_t, y) | (x, y) ∈ |C_3|\}. Since there are 3 elements in \( C_3 \) and \(|cC_3| ≤ 2\), by pigeonhole principle, there are two vertices such that their coordinate congruence numbers in the \( t \)-th abstraction are the same. If one of them is in \( P \), then there is no edge between them, by definition. Therefore, to have a triangle, both of them should be in \( Q \). Recall that, by Lemma 14, \( cc([x]_t, y) = y \mod 2 \) for any \((x, y) ∈ X_{t+1}^* \). Therefore, their coordinate congruence numbers in the \( t \)-th abstraction should be 0. In other words, these two vertices are \((a, 0)\) and \((c, 2)\). Note that \((b, 1) ∈ P\) since \( P \neq \emptyset \) and \( cc([b]_t,1) \) should be 1, for otherwise \((b, 1)\) is not adjacent to both \((a, 0)\) and \((c, 2)\). In other words, \([b]_t\) is even. Note that \((\emptyset)_{t+1}, 1)\) is either not adjacent to \((a, 0)\) or not adjacent to \((c, 2)\), for otherwise there is a \( k \)-clique whose index is greater than \( t \). That is, either \((a, 0)\) or \((c, 2)\) is in \( Ω_b \). Therefore, either \((a, 0)\) or \((c, 2)\) is not adjacent to \((b, 1)\). A contradiction occurs.

**Q.E.D. of Fact.**

As a direct corollary, we have

\[ \mathfrak{B}_{3,m} \text{ has no triangle.} \] (5.13)

Note that the universes of \( \mathfrak{A}_{3,m} \) and \( \mathfrak{B}_{3,m} \) are square lattices that have the width of \( γ_{m-1}^* \) and the height of 3. For each row of a lattice, there is a linear order that is a segment of the original linear order of the universe of the structure. There are three such linear orders in a structure or an abstraction, corresponding to three distinct rows. We can use three intervals to describe these orders. For \( 0 ≤ i ≤ 2 \), we use \([0, i), (γ_{m-j}^*, i)\) to describe the linear orders in the \( j \)-th abstraction.

Since we can view the structures from different scales, which correspond to different abstractions, it is important to know how the linear orders in different abstractions are related to each other. The following lemma says that an object is ahead of another one in the \((i - 1)\)-abstraction if it is ahead of that object in the \( i \)-th abstraction. Recall that the following lemmas
are about the structures $\mathfrak{A}_{3,m}$ and $\mathfrak{B}_{3,m}$, not about $\bar{\mathfrak{A}}_{3,m}$ and $\bar{\mathfrak{B}}_{3,m}$ unless explicitly stated.

**Lemma 16.** For $2 \leq i \leq m$ and any $x_1, x_2 \in X^*_1$, we have

$$[x_1]_i < [x_2]_i \Rightarrow [x_1]_{i-1} < [x_2]_{i-1}.$$  

The intuition is simple. Imagine that we have a graph which is drawn on a grid. Making the cells of the grid smaller by introducing more columns, which evenly divides a cell into a bunch of smaller cells, will not change the order of two points in the graph. Assume that $[x_1]_i < [x_2]_i$. Each row of the $(i-1)$-th abstraction can be divided evenly into $\gamma_{m-i}^* - 1$ intervals of length $\beta_{m-i}^* + 1$, and each interval has a single vertex of index $i$, which is roughly in the middle of this interval. We can regard each interval as a bucket. These buckets have a natural linear order induced from the original one. Then it is clear that a vertex in a “bucket” $B_{1}$ is ahead of a vertex in another “bucket” $B_{2}$ in the original linear order if $B_{1}$ is ahead of $B_{2}$ in the induced linear order.

It implies that an object is ahead of another one in some lower abstraction if it is ahead of that object in any higher abstraction. So a direct corollary of Lemma 16 is that

$$[x_1]_i < [x_2]_i \Rightarrow x_1 < x_2. \quad (5.14)$$

Recall that $k = 3$.

**Lemma 17.** For any $(x, y), (x', y) \in X^*_1$ and for any $p$ where $1 \leq p \leq m$, if

(1) $x - \langle x \rangle_p = x' - \langle x' \rangle_p$,

(2) $\langle x \rangle_p \equiv \langle x' \rangle_p \pmod{k-1}$, where $q \leq \min\{\text{id}(\langle x \rangle_p, y), \text{id}(\langle x' \rangle_p, y)\}$,

then, for any $i$ where $1 \leq i \leq q$ and any $j$ where $1 \leq j \leq p$,

$$\langle x \rangle_{i,j} \equiv \langle x' \rangle_{i,j} \pmod{k-1}. \quad (5.15)$$

In particular, when $i = j$, (5.15) is equivalent to the following:

$$[x]_i \equiv [x']_i \pmod{k-1}. \quad (5.15)$$

Recall that the leftmost vertex of the $i$-th row of $\bar{\mathfrak{B}}_{3,m}$ is $(\gamma_{m-1}^* - tr(i), i)$. Cf. Footnote 6. Let $\mathfrak{A}^+_{3,m}$ and $\mathfrak{B}^+_{3,m}$ be built from $\mathfrak{A}_{3,m}$ and $\mathfrak{B}_{3,m}$ respectively by adding a set of constants

$$\{(a, b) \mid a = \gamma_{m-1}^* - tr(b) \text{ or } a = \gamma_{m-1}^* - tr(b) - 1; b \in [0, 2]\}. \quad (5.16)$$

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We can take it that the constants are interpreted as extra immovable “pebbles” on the boundaries of rows of the structures. Call them boundary constants. It is easy to see that 
\[
\widetilde{A}^+_3 \equiv_m 2^m \widetilde{B}^+_3 \text{ implies } \widetilde{A}_3 \equiv_m 2^m \widetilde{B}_3.
\] (5.17)

In the following we prove the main result of this section.

**Lemma 18.** For any \( m \geq 3 \),
\[
\widetilde{A}^+_3 \equiv_m 2^m \widetilde{B}^+_3.
\]

In each round, replying Spoiler’s pick by a vertex of the same row is a basic element in the strategy of Duplicator. Assume that Spoiler picks \((x, y)\) in some structure and Duplicator responds with \((x', y)\) in the other structure. Say that the game (and the board) is over the \(i\)-th abstraction, if \(u-L_i u M_i = u'-L_i u' M_i\) for any pair of pebbled vertices \((u, v) \models (u', v)\), and the projections of pebbled vertices in the \(i\)-th abstraction define a partial isomorphism.

**Claim 1.** For any \( i \in [2, m] \), if the game board is over the \(i\)-th abstraction, then it is also over the \((i-1)\)-th abstraction.

**Proof of Claim:**

The argument is simple. First, by Lemma 11, we have that \(u-L_i u M_i-1 = u'-L_i u' M_i-1\) for any pair of pebbled vertices \((u, v) \models (u', v)\). Second, by Remark 15, we have that the projections of pebbled vertices in the \((i-1)\)-th abstraction define a partial isomorphism w.r.t edges. Moreover, it is also easy to see that partial isomorphism w.r.t order can also be preserved.

Q.E.D. of Claim.

Duplicator’s strategy works over abstractions. That is, in each round Duplicator plays a related game over some specific abstraction and uses it to decide her pick in the original game. We use \(\xi\) to remind Duplicator in which abstraction she should play in the current round of the game over abstractions. More precisely, \(\xi\) is the maximum \(i\) such that the game board is over the \(i\)-th abstraction. At the beginning, \(\xi = m\), i.e., in Duplicator’s mind, the players are playing in the highest abstraction in the first round of the related game. We use \(\theta\) to denote how many rounds are left at the start of the current round. At the beginning, \(\theta = m\). After each round, \(\theta\) decreases by one automatically and the game, both the original and the one above abstractions, moves to the next round. In each round, \(\xi\) remains
unchanged if Duplicator can respond properly such that the game board is still over the \( \xi \)-th abstraction. However, if Duplicator cannot do so, she tries to seek a solution in the closest lower abstraction, which will be explained in page 28.

Occasionally, we say that “Duplicator picks an object”. By this we mean that she picks a vertex, and this vertex is in this object, by default in the \( \xi \)-th abstraction (cf. p. 21, Remark 15).

To prevent from violating partial isomorphism due to linear orders, in each round Duplicator should ensure the following requirements in the first place, when she makes her picks. Assume that the current round is the \( \ell \)-th round. Although the game board is \( (LxM_\xi, y) \), the following is stated w.r.t. \( (L_{3,m}^+, B_{3,m}^+) \). Recall that, the circular shifts are introduced only to tackle 4\( ^\circ \) (cf. p. 30) and so called “boundary checkout strategy” of Spoiler (cf. p. 31).

(1) If \([x]_\xi < m - \ell \) or \( \gamma^*_{m-\xi} - [x]_\xi < m - \ell\), then \([x']_\xi = [x]_\xi\);

(2) If \(m - \ell \leq [x]_\xi \leq \gamma^*_{m-\xi} - m + \ell\), then \(m - \ell \leq [x']_\xi \leq \gamma^*_{m-\xi} - m + \ell\).

Call the above the requirement of linear order over abstractions, or abstraction-order-condition (for \( k = 3 \)).

We shall show that Duplicator can ensure this requirement in the first place, meanwhile the game board is in partial isomorphism, after each round.

So far, we defaultly assume that Spoiler picks a vertex in the \( \xi \)-th abstraction in the current round of the original game, wherein two types of games, i.e. the original game and the corresponding game over abstractions, are coincided. However, suppose that Spoiler tries to break this assumption by picking a vertex in the \( i \)-th abstraction where \( i < \xi \), in the original game. In such case, Duplicator regards it as if \((Lx_\xi, y)\) were picked, and responds with \((x', y)\) such that \((Lx'_\xi, y)\) is the vertex she will pick to respond the picking of \((Lx_\xi, y)\) using her strategy that works in the \( \xi \)-th abstraction (or, when she cannot do it, responds \((Lx_{\xi-1}, y)\) using her strategy that works in the \(( \xi - 1 \))-th abstraction, wherein a solution is ensured); meanwhile, Duplicator ensures that

\[
x' - (Lx'_\xi) = x - (Lx_\xi) \quad \text{(or } x' - (Lx'_{\xi-1}) = x - (Lx_{\xi-1})\text{, in the other case). (5.18)}
\]

Duplicator is a copycat in the sense of (5.18), which is called horizontal-
residue-copycat (hr-copycat, in short). In other words, Duplicator resorts to the game over abstractions to determine her pick in the original game. In this way, Duplicator can reduce the game to a game over some specific abstraction in each round, i.e. either the ξ-th abstraction or the (ξ − 1)-th abstraction. We shall see that Duplicator can win the games over abstractions in each round, provided that \( \theta < \xi \). And we shall see that \( \theta < \xi \) is preserved throughout the game after the first round, which ensures that Duplicator is able to resort to the closest lower abstraction for a solution when necessary.

In the following we explain Duplicator’s strategy in more detail, using a simultaneous induction as follows, and show that it is a winning strategy. Note that, whenever we say that “(Duplicator) wins this round”, we mean that she not only wins in the original pebble games, but also wins in the corresponding pebble games over the ξ-th abstraction.

Proof. This proof is by induction, wherein we show that the follows are preserved after each round. (recall that we always assume that Spoiler picks \((x, y)\) and Duplicator responds with \((x', y)\) in the current round)

1° \( x - \langle x \rangle_\xi = x' - \langle x' \rangle_\xi \).

2° The abstraction-order-condition holds.

3° The board, without considering the (projections of) boundary constants (cf. (5.16)), is in partial isomorphism over the ξ-th abstraction w.r.t. edges.

4° 3° holds even if the boundaries of rows of the ξ-th abstraction are occupied with extra immovable pebbles.

5° \( \theta < \xi \) after the first round.

6° The game board is in partial isomorphism.

We shall see that 1°–4° implies 6°, according to Remark 15. Moreover, although all of the conditions should be ensured simultaneously, in the game Duplicator will first try to ensure 2°, then 3°, then 4°, and then 1° and 5°.

9° Observation that \( x' - \langle x' \rangle_\xi = x - \langle x \rangle_\xi \) if and only if \( x' - \langle x' \rangle_\xi \equiv x - \langle x \rangle_\xi \) (mod \( \beta_m^{-1} \)). As a consequence, we give this name. Similar thing can be find in Remark 47.

10° They are not counted in the \( k - 1 \) pairs of pebbles.

11° In other words, Duplicator has a winning strategy in the original game if she has a winning strategy in the game over abstractions, provided that she is a hr-copycat in each round.

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In any round, Duplicator will first try to pick \((x', y)\) such that \(1^\circ\sim 4^\circ\) hold\(^{12}\). If she cannot find such a vertex, she resorts to the \((\xi - 1)\)-th abstraction. In the following of this section, whenever we say “\(\text{Duplicator resorts to the } (\xi - 1)\text{-th abstraction}\)” we mean that Duplicator tries to ensure \(1^\circ\sim 4^\circ\), wherein \(\xi\) is replaced by \(\xi - 1\) in these requirements; and \(\xi := \xi - 1\) at the end of this round\(^{13}\). Note that in this case Duplicator regards it as if Spoiler picked “\((\lfloor x \rfloor_{\xi - 1}, y)\)” in current round and she replies in such a way that the projection of her pick in the \((\xi - 1)\)-th abstraction is her response over this abstraction. We shall see that, \(4^\circ\) can be ensured if if \([x']_{\xi - 1} \equiv 0 \pmod 2\) when Duplicator resorts to the \((\xi - 1)\)-th abstraction: in this case \([x]_{\xi - 1} \equiv [x']_{\xi - 1} \pmod 2\), which meets (2) of Lemma\(^{17}\).

**Basis:** In the first round, Duplicator simply mimics. Clearly, she wins this round. \(\xi\) is unchanged, whereas \(\theta := \theta - 1\). Therefore, \(\theta < \xi\) after the first round, i.e. \(5^\circ\) holds in the following rounds if \(\xi\) is to be decreased by at most one in each round. Obviously, the abstraction-order-condition holds at the end of the first round, and the other conditions hold.

**Induction Step:** Suppose that Duplicator can win the first \(\ell - 1\) rounds where \(1 < \ell \leq m\), and \(1^\circ\sim 5^\circ\) hold, we prove that she can also win the \(\ell\)-th round, i.e. \(6^\circ\) holds, and \(1^\circ\sim 5^\circ\) are also preserved. Recall that we assume that Spoiler picks \((x, y)\) and Duplicator picks \((x', y)\) in the \(\ell\)-th round, i.e. the current round. Moreover, we assume that there is one pair of pebbles on the board at the start of the \(\ell\)-th round. If there is no such a pair, Duplicator simply mimics Spoiler in this round, as in the first round. Assume that \((a, b), (a', b)\) are the pair of pebbles on the board at the start of the \(\ell\)-th round, and that \((a, b), (x, y)\) are in the same structure. By induction hypothesis, \(a - \lfloor a \rfloor_{\xi} = a' - \lfloor a' \rfloor_{\xi}\).

**Assume that** 
\((x, y) \in X^*_\xi\), which implies that \(x - \lfloor x \rfloor_{\xi} = 0\). In such case Duplicator will first try to reply with \((x', y) \in X^*_\xi\) such that \(2^\circ\sim 4^\circ\) hold. If she can do it, \(1^\circ\) is also ensured since \(x' - \lfloor x' \rfloor_{\xi} = 0\). And so is \(5^\circ\).

Suppose that \(y = b\). It is clear that \(3^\circ\) holds. Moreover, we have the following observation.

**Claim 2.** On condition that \(2^\circ\) and \(5^\circ\) hold at the start of the \(\ell\)-th round, \(2^\circ\) can be preserved after this round, at the price of decreasing \(\xi\) by at most one.

\(^{12}\)Because of \(1^\circ\), Duplicator will first try to make it such that \((x', y) \in X^*_\xi\) if \((x, y) \in X^*_\xi\).

\(^{13}\)Such a treatment makes it possible for a discussion involving both “\(\xi\)” and “\(\xi - 1\)” without introducing additional symbols.

Note that \(\text{idx}(\lfloor x' \rfloor_{\xi - 1}, y) = \xi - 1\): by Lemma\(^{10}\) \(\text{idx}(\lfloor x' \rfloor_{\xi - 1}, y) \geq \xi - 1\); by Lemma\(^9\) \(\lfloor x' \rfloor_{\xi - 1} = \lfloor x' \rfloor_{\xi}\); then, by definition \(^{5.7}\) \(\lfloor x' \rfloor_{\xi - 1} = \lfloor x' \rfloor_{\xi} \text{ if } \text{idx}(\lfloor x' \rfloor_{\xi - 1}, y) > \xi - 1\).
Proof of Claim:

By her strategy, if Duplicator cannot pick a vertex in the \( \xi \)-th abstraction that satisfies \( 2^\circ \), then she resorts to the \( (\xi - 1) \)-th abstraction, where she can \textit{always} find a solution. Here is a brief argument. By definition, \( \theta + \ell = m + 1 \). Hence \( \xi > \theta = m - \ell + 1 \). Observe that \((\langle a \rangle_{\xi-1}, b)\) and \((\langle a' \rangle_{\xi-1}, b)\) satisfy (2) of the requirement: the number of vertices in the \( (\xi - 1) \)-th abstraction, which surround a vertex of index \( \xi \), is \( \beta_{m-\xi}^{m-\xi+1} = 4(\xi - 1) > 4(m - \ell) \). Therefore, by Fact 2 there are at least \( 2(m - \ell) \) vertices of index \( \xi - 1 \) that are on the left side (right side, resp.) of the leftmost (rightmost, resp.) vertex of index \( \xi \) in any row of \( \mathcal{B}_{3,m} \). That is, any pebbled vertex in \( X_{\xi}^\ast \) is away from the leftmost vertex or the rightmost vertex in the \( (\xi - 1) \)-th abstraction, thereby satisfying (2) of the requirement, provided that Duplicator picks a vertex of index \( \xi - 1 \) such that it is away from both boundaries of a row in the \( (\xi - 1) \)-th abstraction.

Q.E.D. of Claim.

By induction hypothesis, \( 2^\circ \) also holds for \( a \) and \( a' \), if \( x \) is substituted with \( a \) and \( x' \) is substituted with \( a' \). Therefore it is easy for Duplicator to ensure that \( [x]_\xi \leq [a]_\xi \) iff \( [x']_\xi \leq [a'_j]_\xi \); if \( [x]_\xi \leq [a]_\xi \) iff \( [x']_\xi \geq [a'_j]_\xi \). and \( [a]_\xi < m - \ell \) then Duplicator picks \( (x', y) \) s.t. \( [x']_\xi = [x]_\xi \); otherwise \( [a]_\xi \) and \( [a']_\xi \) are far away from the two boundaries of the \( i \)-th row of the \( \xi \)-th abstractions. In the former case, \( 2^\circ \) clearly holds after the \( \ell \)-th round. However, in the latter case, \( 2^\circ \) not necessarily holds after the \( \ell \)-th round. For example, there are more than \( m - \ell + 1 \) vertices of index \( \xi \) on the left side of \((\langle a \rangle_{\xi}, b)\), whereas there are exactly \( m - \ell + 1 \) such vertices on the left side of \((\langle a' \rangle_{\xi}, b)\). If Spoiler picks \( (x, y) \) s.t. \( [x]_\xi = [a]_\xi - 1 \), then no matter how Duplicator responds, \( 2^\circ \) will never hold in the \( \xi \)-th abstraction. Nevertheless, in such case, if Duplicator resorts to the \( (\xi - 1) \)-th abstraction, then \( 2^\circ \) can be ensured in the \( (\xi - 1) \)-th abstraction after the \( \ell \)-th round. That is, \( [x]_{\xi-1} \leq [a]_{\xi-1} \) iff \( [x']_{\xi-1} \leq [a']_{\xi-1} \) and (2) of abstraction-order-condition is met by Claim 2. If \( [x]_{\xi-1} \neq [a]_{\xi-1} \), then by (5.1.4), either \( x < a \) and \( x' < a' \) or \( x > a \) and \( x' > a' \). If \( [x]_{\xi-1} = [a]_{\xi-1} \), then \( [x']_{\xi-1} = [a']_{\xi-1} \). By induction hypothesis, \( a - \langle a \rangle_{\xi} = a' - \langle a' \rangle_{\xi} \) which also implies that \( a - \langle a \rangle_{\xi} = a' - \langle a' \rangle_{\xi} \). Duplicator can ensure that \( x - \langle x \rangle_{\xi} = x' - \langle x' \rangle_{\xi} \) or \( x - \langle x \rangle_{\xi} = x' - \langle x' \rangle_{\xi} \) depending on whether Duplicator has to resort to the \( (\xi - 1) \)-th abstraction for a solution. That is, \( 1^\circ \) can be ensured. Therefore, in both of the cases,

\[ \text{We also take this vertex into the account of the number.} \]
\[ \text{If } (x', y) \in X_{\xi}^\ast, \text{ then } x - \langle x \rangle_{\xi} = x' - \langle x' \rangle_{\xi} = 0; \text{ if } (x', y) \in X_{\xi-1}^\ast, \text{ then } x - \langle x \rangle_{\xi-1} = \]

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Moreover, $4^\circ$ should be ensured. If it cannot, again, Duplicator resorts to the $(\xi-1)$-th abstraction, picking $(x', y) \in X_{\xi-1}^+ - X_{\xi}^+$ such that $[x']_{\xi-1} \equiv 0 \pmod{2}$. By Lemma 9, it implies that $[[x']_{\xi-1}][\xi-1]_1 \equiv 0 \pmod{2}$. Then by Lemma 14, which tells us that $[x]_{\xi-1} \equiv 0 \pmod{2}$ and hence $[[x]_{\xi-1}][\xi-1]_1 \equiv 0 \pmod{2}$, and by Lemma 17, $[[x]_{\xi-1}][\xi-1]_i \equiv 0 \pmod{2}$, for $1 \leq i \leq \xi-1$ and $1 \leq j \leq \xi - 1$. Let $n := \xi - 1$. It means that once $[[x]_{\xi-1}][\xi-1]_n \equiv [[x']_{\xi-1}][\xi-1]_n \pmod{2}$ holds, it still holds for them in the following rounds, despite of how much $\xi$ and $n$ are decreased. It implies that, for $i < \xi$ and any $(c, d)$ where $[[c]_i, d]$ is a boundary vertex (either the leftmost or the rightmost) of the $i$-th abstraction, $(c, d) \in \Omega_\xi$ if and only if $(c, d) \in \Omega_{x'}$ if $y = 1$ and $d \neq y$. Moreover, by Lemma 12, $cc([c]_i, d) = 0 \pmod{2}$ for any $t$; and by Lemma 14, $cc([[x]_{\xi-1}[p], y) = cc([[x']_{\xi-1}[p], y) = y \pmod{2}$, if $p = idx(c, d) < \xi - 1$. Therefore, $4^\circ$ holds. Then, by definition, $(c, d)$ is adjacent to $(x, y)$ if and only if $(c, d)$ is adjacent to $(x', y)$ in the other structure.

In all the cases, $\theta$ is decreased by one, whereas $\xi$ is at most decreased by one. Hence $5^\circ$ is preserved. To summarize, when $(x, y) \in X_{\xi}^+$, Duplicator has a winning strategy in the case $y = b$, and $1^\circ$-$5^\circ$ hold.

Now assume that $b \neq y$. By definition, $(a, b) < (x, y)$ iff $b < y$, and $(a', b) < (x', y)$ iff $b < y$.

Firstly, suppose that Spoiler picks the vertex $(x, y)$ in $X_{\xi}^+$. Duplicator first tries to find all the vertices, whose index is in the range $[\xi, idx(x, y)]$, that can ensure the abstraction-order-condition. These vertices are the candidates that Duplicator will possibly pick. Then she chooses the subset of vertices of them such that $3^\circ$ and $4^\circ$ hold (i.e. $(x', y)$ is in this subset). Note that $3^\circ$ says that $([[x']_\xi, y)$ is adjacent to $([[a']_\xi, b)$ if and only if $([[x]_\xi, y)$ is adjacent to $([[a]_\xi, b)$. And $4^\circ$ says that this holds even if extra pebbles are put on the boundaries of rows of the $\xi$-th abstraction (recall that when we talk about $4^\circ$, the structures involved are $\mathcal{A}_m^+$ and $\mathcal{B}_m^+$ instead of $\mathcal{A}_m^+$ and $\mathcal{B}_m^+$). If Duplicator can respond in this way, then she obviously wins this round, provided that $1^\circ$ holds, and $\xi$ remains unchanged. Hence $\theta < \xi$

$x' - [x']_{\xi-1} = 0$. To see the latter, we need only show that $x - [x]_{\xi-1} = 0$, which is obvious. Recall that $(x, y) \in X_{\xi}^+$. By Lemma 5, $(x, y) \in X_{\xi-1}$. Therefore, $x - [x]_{\xi-1} = 0$.

Because $(x, y) \in X_{\xi-1}$, by Lemma 5, $[x]_{\xi-1} = x$.

To see it, we need analyze two cases. Note that it is easier to do it when $k = 3$. But the following arguments works even when $k > 3$, wherein the structures are generated in the similar way as we construct $\mathcal{B}_m^+$. If $0 < d < k - 1$, $(c, d) \notin \Omega_x$ and $(c, d) \notin \Omega_{x'}$: it is trivial when $d = 1$; if $d \neq 1$ (i.e. $k > 3$), it is because of $(5, 10)$. Suppose that $d = 0$ or $d = k - 1$. By definition, $\Omega_x \cap \mathcal{T} = \Omega_{x'} \cap \mathcal{T} = \emptyset$ where $\mathcal{T}$ is the set of vertices $(x, 0)$ or $(x, k - 1)$ that has index $\xi - 1$, which implies that $(\langle c \rangle_{\xi-1}, d) \notin \Omega_x$ and $(\langle c \rangle_{\xi-1}, d) \notin \Omega_{x'}$, by Lemma 13.
is preserved. Note that 1° holds because \((x, y), (x', y) \in X^*_\xi\) which implies that \(x' - (x')_\xi = x - (x)_\xi = 0\).

**Secondly, suppose that she cannot do so.** Then Duplicator resorts to the \((\xi - 1)\)-th abstraction (considering the projections of pebbled vertices in the \((\xi - 1)\)-th abstraction), wherein she can always do it in this way because \(\theta < \xi\). In such case, Duplicator picks a vertex of index \(\xi - 1\), i.e. \(\text{idx}(x', y) = \xi - 1\). 1° holds since \(x' - (x')_{\xi - 1} = x - (x)_{\xi - 1} = 0 \quad (\because (x, y) \in X^*_{\xi - 1}\) by Lemma 5). Recall that she will ensure 2° in the first place, which is easy. It remains to show that Duplicator has a strategy to ensure 3° in the same time. Since Duplicator cannot respond properly in the \(\xi\)-th abstraction, it means that one of the following two cases holds (note that \(x = (x)_\xi\) if \((x, y) \in X^*_\xi\), if \((x', y) \in X^*_\xi\):

- \((a)_\xi, y\) is adjacent to \((a)_\xi, b\), whereas \((x')_\xi, y\) is not adjacent to \((a')_\xi, b\);
- \((a)_\xi, y\) is not adjacent to \((a)_\xi, b\), whereas \((x')_\xi, y\) is adjacent to \((a')_\xi, b\).

It happens when Duplicator has to pick such a vertex due to (1) of the abstraction-order-condition (recall that she will first try to ensure 2° before ensuring 3°). In particular, Spoiler can choose the leftmost (or rightmost) vertex of a row in the \(\xi\)-th abstraction. For example, Spoiler can simply pick the leftmost vertex of a row in a structure, and Duplicator has to pick the leftmost vertex of the same row in the other structure. We call this strategy of Spoiler “boundary checkout strategy”. Clearly such strategy will not violate 4° since we have shown that, in the worst case, Duplicator can resort to the \((\xi - 1)\)-th abstraction such that 4° can be ensured (cf. p. 30). Moreover, in the following we show that 2° and, in particular, 3° can also be ensured if Duplicator resorts to the \((\xi - 1)\)-th abstraction.

Assume that \([(x)]_\xi, y\) is not the leftmost vertex or the rightmost vertex of the \(y\)-th row of the \(\xi\)-th abstraction. Recall that Duplicator has to resort to the \((\xi - 1)\)-th abstraction and that \(\frac{1}{2}\beta_{m-\xi+1} > 2(m - \ell)\). In this case, there are so many vertices of index \(\xi - 1\), which surround \((x, y)\), that \((x, y)\) (i.e. the vertex \([(x)]_{\xi - 1}, y\)) is away from the two boundaries of the \(y\)-th row of the \((\xi - 1)\)-th abstraction. That is, there are at least \(m - \ell\) vertices of index \(\xi - 1\) that are between the vertex \([(x)]_{\xi - 1}, y\) and a boundary vertex of \(y\)-th row of \((\xi - 1)\)-th abstraction. It implies that the abstraction-order-condition can be met only if the same thing holds for \([(x')]_{\xi - 1}, y\) (cf. (2) of the abstraction-order-condition, where the structures are \(\mathfrak{A}_{3,m}\) and \(\mathfrak{B}_{3,m}\) instead of \(\tilde{\mathfrak{A}}_{3,m}\) and \(\tilde{\mathfrak{B}}_{3,m}\)), which is easy.
Suppose that \( \text{idx}(\langle a' \rangle_{\xi}, b) = t \). By Lemma 10, \( t \geq \xi \). Recall that Duplicator has to resort to the \((\xi - 1)\)-th abstraction, wherein Spoiler uses the boundary checkout strategy. That is, in all the cases she picks \((x', y)\) such that \( \text{idx}(x', y) = \xi - 1 \) and \([x']_{\xi - 1} \equiv 0 \pmod{2} \). By Lemma 9, \([\langle x' \rangle_{\xi - 1}]_{\xi - 1} = [x']_{\xi - 1} \). It implies that \( 4^\circ \) holds, as have been explained (cf. p. 30). In the following we show that \( 3^\circ \) also can be ensured. Note that \( \langle x' \rangle_{\xi - 1}, y \rangle \) is adjacent to \( \langle a' \rangle_{\xi - 1}, b \rangle \) if and only if \( \langle x \rangle_{\xi - 1}, y \rangle \) is adjacent to \( \langle a \rangle_{\xi - 1}, b \rangle \). In other words, by Lemma 8, Duplicator need to show that

\[
(x', y) \text{ is adjacent to } (\langle a' \rangle_{\xi - 1}, b) \iff (x, y) \text{ is adjacent to } (\langle a \rangle_{\xi - 1}, b). \tag{5.19}
\]

We explain it case by case.

(I) Both \( y \mod 2 = 0 \) and \( b \mod 2 = 0 \).

By definition, \( \Omega_x = \Omega_{x'} = \Omega_a = \Omega_{a'} = \emptyset \). Moreover, \([\langle a' \rangle_{\xi - 1}]_{\xi - 1} \equiv [\langle a \rangle_{\xi - 1}]_{\xi - 1} \equiv [x]_{\xi - 1} \equiv 0 \pmod{2} \), by Lemma 14. By Lemma 17, \([\langle a' \rangle_{\xi - 1}]_{\xi - 1} \equiv [\langle a \rangle_{\xi - 1}]_{\xi - 1} \pmod{2} \).

(II) \( y = 1 \) and \( b \mod 2 = 0 \).

- \((x, y)\) is adjacent to \( (\langle a \rangle_{\xi - 1}, b)\):

  We shall see shortly that Duplicator can pick \((x', y)\) such that, for any \( i \) where \( \xi \leq i \leq m \),

  \[
  \text{cc}(\langle x' \rangle_{i}, y) \neq \text{cc}(\langle a' \rangle_{i}, b) \text{ and } \text{idx}(\langle x' \rangle_{i}, y) = i. \tag{5.20}
  \]

  Provided that \(\text{(5.20)}\) holds, in the following we show that

  \[
  (\langle x' \rangle_{\xi}, y) \text{ is adjacent to } (\langle a' \rangle_{\xi}, b), \tag{5.21}
  \]

  and use this to show that

  \[
  (x', y) \text{ is adjacent to } (\langle a' \rangle_{\xi - 1}, b). \tag{5.22}
  \]

  Firstly, note that \( \text{idx}(\langle x' \rangle_{i}, y) = i \) for \( \xi - 1 \leq i \leq m \), because \( \text{idx}(\langle x' \rangle_{\xi - 1}, y) = \xi - 1 \) (since \( \text{idx}(x', y) = \xi - 1 \), \( (x', y) \in X'_{\xi - 1} \)). It means that \( x' = \langle x' \rangle_{\xi - 1} \); moreover, \( \text{idx}(\langle a' \rangle_{i}, b) \geq i = \text{idx}(\langle x' \rangle_{i}, y) \).

  Secondly, by Lemma 9, \( \text{cc}(\langle x' \rangle_{i}, y) \neq \text{cc}(\langle a' \rangle_{i}, b) \) implies that \( \text{cc}(\langle x' \rangle_{i}, y) \neq \text{cc}(\langle a' \rangle_{i}, b) \). Also by Lemma 9, for any \( i \) where \( \xi \leq i \leq m \), \( \langle a' \rangle_{i} = (a')_{i} \). Therefore,

  \[
  \text{cc}(\langle x' \rangle_{i}, y) \neq \text{cc}(\langle a' \rangle_{\xi - 1}, b),
  \]

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for \( \xi \leq i \leq m \). It means that \( \langle [a']_{\xi}, b \rangle \notin \Omega_{[a']_{\xi}} \).

As a consequence, (5.21) holds since \( \text{cc}([x']_{\xi}, y) \neq \text{cc}([a']_{\xi}, b) \) and \( ([a']_{\xi}, b) \notin \Omega_{[a']_{\xi}} \). This implies that \( ([a']_{\xi}, b) \notin \Omega_{x'} \).

Recall that \( \text{idx}([a']_{\xi}, b) \geq \xi \). Hence, \( \text{cc}([a']_{\xi-1}, b) = 0 \), by Lemma 14. Recall that \( [x']_{\xi-1} \equiv 0 \pmod{2} \). As a consequence, \( \text{cc}([a']_{\xi-1}, b) \neq \text{cc}([x']_{\xi-1}, y) \). Hence, \((x', y)\) is adjacent to \( ([a']_{\xi}, b) \).

Recall that \( \text{idx}([a']_{\xi-1}, b) \geq \xi - 1 \). Suppose that \( \text{idx}([a]_{\xi-1}, b) = \xi - 1 \). Then \( ([a]_{\xi-1}, b) \equiv 0 \pmod{2} \), for otherwise \((x, y)\) is not adjacent to \( ([a]_{\xi-1}, b) \). Then, similarly, \( ([a']_{\xi-1}, b) \equiv 0 \pmod{2} \), by Lemma 14. Recall that \( \text{idx}([a']_{\xi-1}, b) = \xi - 1 \) and \( [x']_{\xi-1} \equiv 0 \pmod{2} \). Hence \( \text{cc}([a']_{\xi-1}, b) \neq \text{cc}([x']_{\xi-1}, y) \) determines the adjacency of \((x', y)\) and \( ([a']_{\xi-1}, b) \), which implies that (5.22) holds.

Now suppose that \( \text{idx}([a]_{\xi-1}, b) > \xi - 1 \). Because \( a - [a]_{\xi} = a' - [a']_{\xi} \), by Lemma 14, \( a' - [a']_{\xi-1} = a - [a]_{\xi-1} \neq 0 \). Hence \( \text{idx}([a']_{\xi-1}, b) > \xi - 1 \). Therefore, by Lemma 14, \( \text{cc}([a]_{\xi-1}, b) \neq \text{cc}([x']_{\xi-1}, y) \), which implies that (5.22) holds.

We get the desired result on condition that (5.20) holds. Now we give a process, by which Duplicator does can choose a vertex for \((x', y)\) to satisfy (5.20), meanwhile satisfying 1\(^\circ\), 2\(^\circ\) and 4\(^\circ\). Duplicator first chooses a vertex of index \( m \), say \( (x'_m, y) \), such that \( \text{cc}([x'_m]_{m}, y) \neq \text{cc}([a']_{m}, b) \). Afterwards, she chooses a vertex of index \( m - 1 \), say \( (x'_{m-1}, y) \), from \( \text{cc}([x'_m]_{m-1}, y) \neq \text{cc}([a']_{m-1}, b) \). Then she chooses a vertex of index \( m - 2 \), say \( (x'_{m-2}, y) \), from \( \text{cc}([x'_{m-2}]_{m-2}, y) \neq \text{cc}([a']_{m-2}, b) \), and so on. Finally, she chooses \((x', y) \in X^x_{\xi} - X^y_{\xi} \) from the object \( \text{cc}([x']_{\xi}, y, \xi - 1) \) such that \( \text{cc}([x']_{\xi}, y, \xi - 1) \neq \text{cc}([a']_{\xi}, b) \). Note that this final step implies that \( [x']_{\xi-1} \equiv 0 \pmod{2} \) in the process Duplicator can

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Note that \( \text{cc}([a]_{\xi-1}, b) = 1 \) if \([a]_{\xi-1} \equiv 1 \pmod{2} \). On the other hand, by Lemma 14, \( \text{cc}([x]_{\xi-1}, y) = 1 \), because \((x, y) \in X^x_{\xi} \). Therefore, \((x, y)\) is not adjacent to \( ([a]_{\xi-1}, b) \), since \( \text{cc}([x]_{\xi-1}, y) = \text{cc}([a]_{\xi-1}, b) \).

Recall that \( a - [a]_{\xi} = a' - [a']_{\xi} \). Let \( \Delta := a - [a]_{\xi} \). By definition, \( [a]_{\xi-1} - [a]_{\xi} = [\Delta/\beta_{m-1}^{\xi-1}] \) because it is easy to see that \([a]_{\xi-1} \in N^+ \) (cf. (5.5)). Similarly, \( [a']_{\xi-1} - [a']_{\xi} = [\Delta/\beta_{m-1}^{\xi-1}] \). Hence \( [a]_{\xi-1} - [a]_{\xi} = [a']_{\xi-1} - [a']_{\xi} \). By Lemma 10, \( ([a]_{\xi}, b), ([a']_{\xi}, b) \in X^x_{\xi} \). Hence, \([a]_{\xi-1} \equiv [a']_{\xi-1} \equiv 0 \pmod{2} \), by Lemma 14. Hence, \( [a']_{\xi} \equiv [a]_{\xi} \pmod{2} \). By Lemma 9, \( [a']_{\xi-1} \equiv [a]_{\xi-1} \pmod{2} \).

To see it, just note that only when \([x']_{\xi-1} \equiv 0 \pmod{2} \) it is possible that \( \text{cc}([x']_{\xi}, y) \neq \text{cc}([a']_{\xi}, b) \): \( \text{cc}([x']_{\xi}, y) = 1 \) and \( \text{cc}([a']_{\xi}, b) = 0 \) by Lemma
choose a vertex that is at least one vertex away from the boundaries of the $y$-th row of the $i$-th abstraction where $\xi \leq i \leq m$, i.e., $0 < [x'_i] \leq \gamma^{*}_{m-i} - 1$ (for the sake of convenience, suppose that we are talking about $\mathfrak{A}_{3,m}$ and $\mathfrak{B}_{3,m}$ instead of $\mathfrak{A}_{i,m}$ and $\mathfrak{B}_{i,m}$). It implies that the abstraction-order-condition can be ensured because $\frac{1}{2} \beta^{m-\xi+1} > 2(m - \ell)$. By definition (cf. Remark 15 for the definition of \textquoteleft \textquoteleft cex$(x, y, i)$\textquoteright\textquoteright), $[x'_i] = [x'_i]_i$ for $\xi - 1 \leq i \leq m$.

Then by Lemma 14, we have $[x'_i]_i = x'_i$. As a consequence, Duplicator can pick $(x', y)$ to satisfy (3.20). Moreover, in the process Duplicator has the freedom to pick a vertex that is not a critical point or abstractions of a critical point, i.e. $[x'_i]_i \neq [mid]_i$ (note that the current round is not the first round).

• $(x, y)$ is not adjacent to $([a]_{\xi-1}, b)$:

By Lemma 10, $\text{idx}([a]_{\xi-1}, b) \geq \xi - 1$. Assume that

$$\text{idx}([a]_{\xi-1}, b) = \xi - 1.$$  

Then we have $cc([a]_{\xi-1}, b) = 1$, for otherwise $(x, y)$ is adjacent to $([a]_{\xi-1}, b)$: by Lemma 14, we have $cc([a]_{\xi-1}, y) = 1$; moreover, $([a]_{\xi-1}, b) \notin \varpi_{x}$ since $\text{idx}(x, y) > \text{idx}([a]_{\xi-1}, b)$. By Remark 15 (also cf. Fact 2 and Fact 3), we have $\text{idx}([a]_{\xi-1}, y) = \text{idx}([a]_{\xi-1}, y)$. By Footnote 19, we have $[a]_{\xi-1} \equiv [a]_{\xi-1} \pmod{2}$. Hence, $cc([a]_{\xi-1}, b) = 1$. Note that $cc([x']_{\xi-1}, y) = cc([x]_{\xi-1}, y) = 1$. Therefore, $(x', y)$ is not adjacent to $([a]_{\xi-1}, b)$ since $cc([x']_{\xi-1}, y) = cc([a]_{\xi-1}, b)$. Now suppose that

$$\text{idx}([a]_{\xi-1}, b) = t' > \xi - 1.$$  

Duplicator first finds a vertex $(x'_{t'}, y)$ such that its index is $t'$ and $cc([x'_{t'}], y) = cc([a]_{\xi-1}, b)$. The vertex $(x'_{t'}, y)$ can be one that is neither the leftmost nor the rightmost vertex of index $t'$.

Then she chooses $(x'_{t'-1}, y)$ from $\text{cex}(x'_{t'}, y, t' - 1)$ such that its index is $t' - 1$ and $[x'_{t'-1}]_{t'-1} \equiv 0 \pmod{2}$; and so on, until she chooses $(x'_{t'}, y)$ from $\text{cex}(x'_{t'-1}, y, \xi)$ such that its index is $\xi$ and $[x'_{t'}]_{\xi} \equiv 0 \pmod{2}$. Finally, she chooses $(x', y)$ from $\text{cex}(x'_{t'}, y, \xi - 1)$ such that $\text{idx}(x', y) = \xi - 1$ and $[x']_{\xi-1} \equiv 0 \pmod{2}$. By definition and Lemma 4, $([a]_{\xi-1}, b) \in \varpi_{x}$. Therefore, $(x', y)$ is not adjacent to $([a]_{\xi-1}, b)$.

Footnote 14. It means that $[x]_{\xi-1} \equiv [x]_{\xi-1} \equiv 0 \pmod{2}$, which ensures $4^\circ$. Moreover, $1^\circ$ holds since $x' - [x]_{\xi-1} = x - [x]_{\xi-1} = 0$.  

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(III) $y \mod 2 = 0$ and $b = 1$.

If $(x,y)$ is not adjacent to $(\langle a \rangle_{\xi-1}, b)$, Duplicator can use the same process as introduced in (II) to pick a vertex for $(x',y)$ such that $(x',y)$ is not adjacent to $(\langle a' \rangle_{\xi-1}, b)$. In the following we assume that $(x,y)$ is adjacent to $(\langle a \rangle_{\xi-1}, b)$.

Note that $cc((x')_{\xi-1}, y) = 0$. The following arguments are similar to that in (II), cf. page 33 and related footnotes, i.e. Footnote 18 and Footnote 19. Suppose that $\text{id}x((\langle a \rangle_{\xi-1}, b) = \xi - 1$. Then \[\text{id}x((\langle a' \rangle_{\xi-1}, b) = 0 \mod 2\] Hence, \[cc((\langle a' \rangle_{\xi-1}, b) = 1 \neq cc([x']_{\xi-1}, y)\]. Now suppose that $\text{id}x((\langle a \rangle_{\xi-1}, b) > \xi - 1$. It implies that $\text{id}x((\langle a' \rangle_{\xi-1}, b) > \xi - 1$. Therefore, Lemma 14, $cc((\langle a' \rangle_{\xi-1}, b) \neq cc([x']_{\xi-1}, y)$. Moreover, $(x', y) \notin \Omega_{\{a\}_{\xi-1}}$ because, by Lemma 10, $\text{id}x((\langle a \rangle_{\xi-1}, b) \geq \text{id}x(x', y) = \xi - 1$. It implies that $(x', y)$ is adjacent to $(\langle a' \rangle_{\xi-1}, b)$ if $(x,y)$ is adjacent to $(\langle a \rangle_{\xi-1}, b)$.

By Lemma 11, $a - \langle a \rangle_{\xi} = a' - \langle a' \rangle_{\xi}$ implies that $a - \langle a \rangle_{\xi-1} = a' - \langle a' \rangle_{\xi-1}$.

Note that all the vertices $(x,y)$, $(x',y)$, $(\langle a \rangle_{\xi-1}, b)$, and $(\langle a' \rangle_{\xi-1}, b)$ are in $X_{\xi-1}$. Then by Remark 15, we have that $(x', y)$ is adjacent to $(a', b)$ if and only if $(x,y)$ is adjacent to $(a,b)$.

**Thirdly,** if Spoiler picks a vertex $(x,y)$ in $X_{\xi-1} - X_{\xi}$. The ideas are very similar to the last argument, i.e. Duplicator resorts to the $(\xi-1)$-th abstraction for a solution. Just note that Duplicator need ensure that $\text{id}x(x', y) = \xi - 1$, which means that it is not a critical point, and $cc([x']_{\xi-1}, y) = cc([x]_{\xi-1}, y)$.

**Fourthly,** so far we assume that Spoiler picks a vertex $(x,y)$ in $X_{\xi}$. Now suppose that he picks a vertex in $X_{1} - X_{\xi-1}$, i.e. a vertex of index less than $\xi - 1$. Duplicator has a simple strategy that we have mentioned briefly before. That is, she regards it as if $(\langle x \rangle_{\xi})$, or $(\langle x \rangle_{\xi-1}, y)$, were picked, and responds with $(x',y)$ such that.

(i) if she can respond properly to the picking of $(\langle x \rangle_{\xi})$ in the $\xi$-th abstraction:

$(\langle x' \rangle_{\xi}, y)$ is the vertex she would pick to respond to the “picking” of $(\langle x \rangle_{\xi})$; meanwhile, she let $x' - \langle x \rangle_{\xi} = \langle x \rangle_{\xi}$;

(ii) otherwise, she resorts to the $(\xi-1)$-th abstraction for a solution:

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Footnote 19. Suppose that $\text{id}x((\langle a \rangle_{\xi-1}, b) = \xi - 1$. Then \[\text{id}x((\langle a' \rangle_{\xi-1}, b) = 0 \mod 2\]. Hence, \[cc((\langle a' \rangle_{\xi-1}, b) = 1 \neq cc([x']_{\xi-1}, y)\]. Now suppose that $\text{id}x((\langle a \rangle_{\xi-1}, b) > \xi - 1$. It implies that $\text{id}x((\langle a' \rangle_{\xi-1}, b) > \xi - 1$. Therefore, Lemma 14, $cc((\langle a' \rangle_{\xi-1}, b) \neq cc([x']_{\xi-1}, y)$. Moreover, $(x', y) \notin \Omega_{\{a\}_{\xi-1}}$ because, by Lemma 10, $\text{id}x((\langle a \rangle_{\xi-1}, b) \geq \text{id}x(x', y) = \xi - 1$. It implies that $(x', y)$ is adjacent to $(\langle a' \rangle_{\xi-1}, b)$ if $(x,y)$ is adjacent to $(\langle a \rangle_{\xi-1}, b)$.

Footnote 11. $a - \langle a \rangle_{\xi} = a' - \langle a' \rangle_{\xi}$ implies that $a - \langle a \rangle_{\xi-1} = a' - \langle a' \rangle_{\xi-1}$.

Note that all the vertices $(x,y)$, $(x',y)$, $(\langle a \rangle_{\xi-1}, b)$, and $(\langle a' \rangle_{\xi-1}, b)$ are in $X_{\xi-1}$. Then by Remark 15, we have that $(x', y)$ is adjacent to $(a', b)$ if and only if $(x,y)$ is adjacent to $(a,b)$.

**Thirdly,** if Spoiler picks a vertex $(x,y)$ in $X_{\xi-1} - X_{\xi}$. The ideas are very similar to the last argument, i.e. Duplicator resorts to the $(\xi-1)$-th abstraction for a solution. Just note that Duplicator need ensure that $\text{id}x(x', y) = \xi - 1$, which means that it is not a critical point, and $cc([x']_{\xi-1}, y) = cc([x]_{\xi-1}, y)$.

**Fourthly,** so far we assume that Spoiler picks a vertex $(x,y)$ in $X_{\xi}$. Now suppose that he picks a vertex in $X_{1} - X_{\xi-1}$, i.e. a vertex of index less than $\xi - 1$. Duplicator has a simple strategy that we have mentioned briefly before. That is, she regards it as if $(\langle x \rangle_{\xi})$, or $(\langle x \rangle_{\xi-1}, y)$, were picked, and responds with $(x',y)$ such that.

(i) if she can respond properly to the picking of $(\langle x \rangle_{\xi})$ in the $\xi$-th abstraction:

$(\langle x' \rangle_{\xi}, y)$ is the vertex she would pick to respond to the “picking” of $(\langle x \rangle_{\xi})$; meanwhile, she let $x' - \langle x \rangle_{\xi} = \langle x \rangle_{\xi}$;

(ii) otherwise, she resorts to the $(\xi-1)$-th abstraction for a solution:

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Footnote 21. In fact, (i) and (ii) include all the cases needed to discuss because we can take it that the first case (cf. p. 30) $b \neq y$, i.e. Spoiler picks a a vertex $(x,y)$ in $X_{\xi}$ and Duplicator can reply properly with $(x',y) \in X_{\xi}$, is a special case of (i) where $x' - \langle x \rangle_{\xi} = \langle x \rangle_{\xi} = 0$. 

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\((\langle \! \langle x \! \rangle \! \rangle_{\xi-1}, y)\) is the vertex she would pick to respond the “picking” of \((\langle x \rangle_{\xi-1}, y)\), and \(x' - \langle \! \langle x' \! \rangle \! \rangle_{\xi-1} = x - \langle x \rangle_{\xi-1}\).

Observe that such strategy implies that \(\text{idx}(x, y) = \text{idx}(x', y)\) in this case (cf. Fact 2 and Fact 3 or Remark 15). In the case (i), Duplicator’s strategy can help her win this round: as have been explained in Remark 15 it means that \((x, y)\) is adjacent to \((a, b)\) if and only if \((x', y)\) is adjacent to \((a', b)\) because \((\langle x \rangle_{\xi}, y)\) is adjacent to \((\langle a \rangle_{\xi}, b)\) if and only if \((\langle x' \rangle_{\xi}, y)\) is adjacent to \((\langle a' \rangle_{\xi}, b)\), by her strategy.

Similarly, recall that Duplicator has a winning strategy over the \((\xi - 1)\)-th abstraction, if we regard it as if \((\langle x \rangle_{\xi-1}, y)\) instead of \((x, y)\) were picked in this round (cf. p. 31). That is, the projection of all the pebbled vertices to the \((\xi - 1)\)-th abstraction, including \((\langle x \rangle_{\xi-1}, y)\) and \((\langle x' \rangle_{\xi-1}, y)\), will form a new game board that is in partial isomorphism. Recall that \(x - \langle x \rangle_{\xi-1} = x' - \langle x' \rangle_{\xi-1}\). According to Remark 15 \((x, y)\) is adjacent to \((u, v)\) if and only if \((x', y)\) is adjacent to \((u', v)\).

All in all, her strategy is a winning strategy in such 2-pebble games. \(\square\)

From Duplicator’s strategy we can see that, at the end of the current round, \(\xi\) is the maximum number in \([1, m]\) that makes \(1^\circ\) hold.

By \((5.17)\), we have
\[
\mathfrak{A}_{3,m} \equiv_m \mathfrak{B}_{3,m}.
\]

As a corollary, it is easy to see that it needs and only needs 3 variables to define 3-clique in FO on finite ordered graphs (cf. the proof of Theorem 38 in Section 5 for the details).

**Could we make the structures smaller**, such that we can have a feeling of how the structures look like? The answer is yes, at the price of complicating the arguments a little bit. With further thinking, we can remove all the vertices of index 1 from \(\mathfrak{A}_{3,m}\) and \(\mathfrak{B}_{3,m}\) and the result still holds. It is because Duplicator can ensure that \(\xi = m\) after the first two rounds.

Essentially there are only two changes in the structures. Firstly, there are \(m - 1\) abstractions in a structure. Secondly, there are \(4(m - 1)\) vertices in the \(m\)-th abstraction.

For any \(m, i \in \mathbb{N}^+\), where \(m \geq 3\) and \(0 < i < m - 1\), let
\[
\gamma_0^* := 4(m - 1) \\
\gamma_i^* := 4(m - i - 1)\gamma_{i-1}^*
\]
For $x \in [\gamma_{m-2}]$ and $2 \leq i \leq j \leq m$, let

$$\beta_{m-j} := \frac{\gamma_{m-i}}{\gamma_{m-j}}$$

$$[x]_i := \lfloor x/\beta_{m-i} \rfloor$$

$$\langle x \rangle_i := [x]_i \beta_{m-i} + \frac{1}{2} \sum_{2 \leq \ell \leq i} \beta_{m-\ell}$$

Now $\beta_{m-j} = \prod_{m-j \leq \ell < m-i} \frac{\gamma_{\ell+1}}{\gamma_{\ell}} = 4^{j-i} \times \frac{(j-2)!}{(i-2)!}$. And $\gamma_{m-2} = \gamma_0^{m-2} = (m-1)! \times 4^{m-1}$.

Recall that Duplicator simply mimics Spoiler in the first round. In the following rounds, Duplicator continues mimicking until both of $cex(mid, 0, m-1)$ and $cex(mid, 2, m-1)$ have pebbled vertices in one structure and it is Spoiler who picks one of them in this round. Recall that we always assume that Spoiler picks $(x, y)$ in this round (i.e. current round). Suppose w.l.o.g. that $y = 2$. In this “icebreaking” round, if $(x, y)$ is a vertex of $\mathfrak{A}_{3, m}$, Duplicator need only ensure that

- $cc([x'], y) \neq cc([mid]_m, 0) = 0$;
- $x' - \langle x \rangle_m = x - \langle x \rangle_m$;
- $[x']_m \neq 0$ and $[x']_m \neq \gamma_0^* - 1$ (that is, $(\langle x \rangle_m, y)$ is away from the boundaries of the $y$-th row of the $m$-th abstraction).

Note that there are more than one vertex that Duplicator can choose to satisfy these conditions.

If $(x, y)$ is a vertex of $\mathfrak{B}_{3, m}$, Duplicator need only ensure that

- $cc([x'], y) = cc([mid]_m, 0) = 0$;
- $\langle x \rangle_m \neq mid$;
- $x' - \langle x \rangle_m = x - \langle x \rangle_m$;
- $[x']_m \neq 0$ and $[x']_m \neq \gamma_0^* - 1$.

Recall that we right circular shift the middle row such that $4^*$ holds. In the previous analysis, we assume that immovable “pebbles” are put on the boundaries of rows at the start of games. Now we take away such assumption and study directly the game $\mathcal{C}^2_m(\mathfrak{A}_{3, m}, \mathfrak{B}_{3, m})$ instead of the game
\(2^m(\mathfrak{A}^+_3, \mathfrak{B}^+_3)\). As a consequence, \(\xi = m\) after the first two rounds. Therefore, the game (over abstractions) will never go into the first abstraction. Such treatment saves one more round for Duplicator, i.e. she can win \(i+1\) rounds in \(2^m(\mathfrak{A}_3, \mathfrak{B}_3)\) if she can win \(i\) rounds in \(2^m(\overline{\mathfrak{A}}_3, \overline{\mathfrak{B}}_3)\).

Moreover, the time Duplicator stops mimicking is the time when those two pebbles are put in different rows: one is on a vertex of the bottom row and the other is on a vertex of the top row in a structure. Hence we can make the length of a row a bit smaller. It is especially useful when we try to draw a picture for the structures. It is for these two reasons that we can change the previous definitions a little bit, while almost all the arguments remain the same, except that

- we substitute \(\beta^{m-1}_m, X^*_m, \gamma^*_m\) with \(\beta^{m-2}_m, X^*_2, \gamma^*_2\);
- we need to take the second round into account when proving the induction basis, as have just been introduced;
- the abstraction-order-condition is adapted as follows:
  1. If \([x]_\xi < m - 1 - \ell \) or \(\gamma^*_m - \xi - [x]_\xi < m - 1 - \ell\), then \([x']_\xi = [x]_\xi\);
  2. If \(m - 1 - \ell \leq [x]_\xi \leq \gamma^*_m - m + 1 + \ell\), then \(m - 1 - \ell \leq [x']_\xi \leq \gamma^*_m - m + 1 + \ell\).

\[\text{mid} := 2(m - 1)\beta^0_m + \frac{1}{2} \sum_{2 < j < m} \beta^0_{m-j}\]

Note that, now \(\xi > \theta + 1 = m - \ell + 2\) for \(2 < \ell \leq m\) since \(\xi = m\) after the first two rounds, therefore Claim 2 still holds.

However, the structures are still a bit big even for the simplest cases. Fig. 2 shows \(\mathfrak{A}_{3,3}\) on the left side and the third abstraction \(\mathfrak{A}_{3,3}[X^*_3]\) on the right side, both of which are rotated 90° counterclockwise. Note that the

\(\text{EXAMPLE}\) for the first two rounds of the games. In the first round, Spoiler picks \((\text{mid}, 0)\); Duplicator replies with \((\text{mid}, 0)\) in the other structure. In the second round, Spoiler picks \((\text{mid}, 2)\) in \(\mathfrak{A}_{k,m}\); Duplicator responds by picking \((x, 2)\) in \(\mathfrak{B}_{3,m}\) where \((x, 2) \in X^*_m, \text{cc}(x, 2) = 1\) and \([x]_m \neq 0, \gamma^*_0 - 1\). Therefore, after these two rounds, \(\xi\) is still \(m\). Although \(\Delta \) no more holds if there are additional immovable pebbles on the boundaries of rows, Spoiler can “show” it only when he picks \((c, 1)\) where \(c\) is the projection of the leftmost vertex in the \(m\)-th abstraction, because there are no such additional pebbles on the game board \((\mathfrak{A}_{3,3}, \mathfrak{B}_{3,3})\). This will use one more round. And Duplicator can resort to the \((m - 1)\)-th abstraction for a solution. Note that \((x, 2)\), as well as \((\text{mid}, 2)\), is adjacent to \((c', 1)\) where \(c'\) is the projection of the leftmost vertex in the \((m - 1)\)-th abstraction.
vertex with a label “mid0” is just the vertex \((\text{mid}, 0)\), and the vertex with a label “mid1” is the vertex \((\text{mid}, 2)\).

The black nodes in the graph represent the vertices of the third abstraction of \(A_{3,3}\), whereas the grey nodes represent those vertices in the second abstraction. We group the vertices of each row of the structure \(A_{3,3}\) by blue dashed rectangles. The vertices in the same rectangle have the same “position” in the third abstraction. That is, for any \((u, v)\) and \((u', v)\), \([u]_3 = [u']_3\) if they are in the same rectangle. To simplify the picture, we only show the blue dashed rectangles in the highest row of \(A_{3,3}\) and omit all the others.

By a simple counting, we know that there are 8 triangles in \(A_{3,3}\). If we remove the red edge of \(A_{3,3}\), we obtain the structure \(B_{3,3}\), which is triangle-free by Fact 5. Obviously, for any \(m \geq 3\), the girth of \(A_{3,3}\) is 3 and the girth of \(B_{3,3}\) is 4. See, e.g., the shortest cycle that consists of the vertices \((\text{mid}, 0)\), \((\text{mid}, 1)\), \((\text{mid}, 2)\) and \((2, 1)\) in \(B_{3,3}\).

It is easy to see that Duplicator has a winning strategy in \(\mathcal{G}^2_3(\tilde{A}_{3,3}, \tilde{B}_{3,3})\). In other words, in this special case we don’t have to right circular shift the mid row (the 1-th row) of the structures. Nevertheless, Fig. 2 is still not easy to handle directly insomuch as edges crisscross one another in a fashion that deters “observation”, let alone to play a game over this game board. Hence it is better to study a piece of the structures to understand the key concept “abstraction”, as shown in Fig. 3.

In Fig. 3, we not only give a small piece of \(A_{3,3}\), but also give its third abstraction and overlap them in a way that delivers a bit intuition: we can watch the graph from different scales. Compared with the higher abstractions, the lower abstractions show more details of the original structure, thereby in the finer scales. Note that we use braces instead of dashed rectangles in Fig. 3. The subgraph induced by \(cex(14, 0, 2)\) and \(cex(mid, 1, 2)\) is isomorphic to that induced by \(cex(22, 0, 2)\) and \(cex(mid, 1, 2)\), for both \((14, 0)\) and \((22, 0)\) are not adjacent to \((\text{mid}, 1)\). By contrast, The subgraph induced by \(cex(14, 0, 2)\) and \(cex(mid, 1, 2)\) is not isomorphic to that induced by \(cex(mid, 0, 2)\) and \(cex(mid, 1, 2)\), because \((\text{mid}, 0)\) is adjacent to \((\text{mid}, 1)\) while \((14, 0)\) is not. In general, the adjacency of vertices of the \(i\)-th abstraction determines the adjacency of all the middle row vertices of the \((i - 1)\)-th

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23There are four vertices, whose index is 3, whose coordinate congruence number is 0 in the third abstraction, and whose second coordinate is 1: that is, the vertices \((2, 1)\), \((10, 1)\) and \((18, 1)\) and \((26, 1)\). The two vertices with labels “mid0”, “mid1”, and any one of these four vertices can form a triangle. In addition, there are four vertices, whose index is 2, whose coordinate congruence number is 0 in the second abstraction, and whose second coordinate is 1: that is, the vertices \((0, 1)\), \((8, 1)\), \((16, 1)\) and \((24, 1)\).
Figure 2: \( \mathcal{A}_{3,3} \) (left side) and its third abstraction \( \mathcal{A}_{3,3}[X_3] \). Removing the red edge from \( \mathcal{A}_{3,3} \), we obtain the structure \( \mathcal{B}_{3,3} \).
Figure 3: A piece of $\mathfrak{A}_{3,3}$. To simplify the figure, in this small piece we only show the edges between $(\text{mid}, 2)$ (i.e. the vertex with label “mid1”) and those vertices whose second coordinate is 1 and the edges between $(\text{mid}, 1)$ and those vertices whose second coordinate is 0. For example, the red edge between $(\text{mid}, 2)$ and $(\text{mid}, 0)$ is omitted in this figure.
abstraction whose \((i - 1)-th\) relative first coordinates are even.

**Further Remark**

We create the structures \(A_{3,3}\) and \(B_{3,3}\) to ensure that Duplicator has a winning strategy in a 3-round 2-pebble game. We can further simplify the structures. In fact, the structure on the right side of Fig. 2 (the third abstraction) can be taken as the structure \(A_{3,3}\). What’s more, we even don’t have to right circular shift the mid row of the structures in this very special case. It is not only because the additional immovable pebbles on the boundaries are not necessary a part of the games, but also based the simple observation that, in the last round, Duplicator need only ensure that the game board is in partial isomorphism, either w.r.t. edges or w.r.t. orders.

It is possible to make the structures even smaller. For example, in the case where \(k = m = 3\), there exists a pair of structures \(A_{3,3}\) and \(B_{3,3}\), each of which only has 12 vertices. See Fig. 4. Duplicator’s strategy in this special case is similar to the one we introduced before. Duplicator simply mimics Spoiler in the first round. And in the following rounds Duplicator continues mimicking until both of \((\text{mid}, 0)\) and \((\text{mid}, 2)\) are “pebbled” in one structure and it is Spoiler who picked one of them in this round. Suppose that in this icebreaking round Spoiler picked a critical point in \(A_{3,3}\). In such case, Duplicator can pick either \((1, 0)\) or \((1, 2)\) in \(B_{3,3}\), depending on which critical point is picked by Spoiler. Clearly Duplicator wins this round. Hence there is at most one round left, which is easy for Duplicator to handle.

For example, assume that both of \((\text{mid}, 0)\) and \((\text{mid}, 2)\) are pebbled in \(A_{3,3}\) and \((1, 2)\) is pebbled in \(B_{3,3}\). Then, if Spoiler moves the pebble on \((\text{mid}, 0)\) to \((0, 1)\) in \(A_{3,3}\), Duplicator need only move the pebble on \((\text{mid}, 0)\) to \((1, 1)\) in \(B_{3,3}\).

Similarly, we can construct \(A_{3,4}\) and \(B_{3,4}\), each of which only contains 45 vertices. However, on the one hand it is still a bit complicated; on the other hand, it is hard to generalize. Therefore we do not put the picture here.

In conclusion, it is possible to make the structures smaller in some special cases. Nevertheless, we shall not consider it in the following more general cases, to make the arguments as simple as possible.

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\(^{24}\)The “adjacency of a vertex” tells us how the vertex is connected to the other vertices.
Figure 4: $A_{3,3}$ (left side) and $B_{3,3}$ (right side). Note that $(x, 0)$ is not adjacent to $(x, 2)$ for any $x$, which is exemplified by a dotted red arc in $A_{3,3}$. We omit all other such dotted arcs in the figure. This game board is simplified to the extent that permits the readers to actually play and check the outcome of the 3-round 2-pebble game over it.
5.2 Type labels, board histories, and structures with a structure

Since we have already proved the special cases where \( k = 2, 3 \) in the last section, henceforth we assume that \( k \geq 4 \). Compared with the special cases, including the case where Spoiler is only allowed to pick in \( A_k \) (cf. Section 3), the general cases suddenly become complicated for the following reasons.

(1*) The two graphs in the game board should be sufficiently similar so that the choosing of structure in each round will not immediately help Spoiler find the difference. By contrast, the graphs we constructed for the special case wherein Spoiler only picks in \( A_k \) is so different that Spoiler can find the difference immediately if he is allowed to choose \( B_k \).

(2*) In the case where \( k = 3 \), there are two pebbles in total, which can only tell us whether two pebbled vertices are adjacent or not. With more pebbles, the adjacency of a vertex to its neighbours that can be detected is greatly complicated.

(3*) In the case where Spoiler only picks in \( A_k \), each vertex in \( A_k \) has the same type and the pebbled vertices induce only cliques. That is, the newly picked vertex is adjacent to all the pebbled vertices. In the general case, as we have many different types of vertices, pebbled vertices may induce different subgraphs, possibly not in favor of Duplicator.

(4*) With more “types” of vertices, it is much easier for Spoiler to find the difference simply by picking in a relative “small” range (w.r.t. the linear order) of a row persistently, when the graphs are ordered. It is particularly for this reason that makes the linear order issue notoriously unfavorable for explicit constructions.

As a consequence, we need to create some new concepts and techniques to deal with the difficulty. For (1*), we forbidd some edges to ensure that no \( k \)-clique (cf., e.g., RngNum(\( \cdot, \cdot \)) and sgn(\( \cdot, \cdot \)) functions) exists in one of the structure. For (2*), we introduce complicated notion of “type” labels (cf. Definition 28). For (3*), we can first find a vertex that is adjacent to (the projection of) all the pebbled vertices (in some specific abstraction) (cf. Lemma 36); afterwards, with “small” adjustment, we can find the vertex that has the adjacency to the pebbled vertices in the way we want, cf. Strategy 2, (6.16). For (4*), we introduce novel concepts and techniques, e.g. board histories, to tackle this issue.
Note that, the concepts and lemmas introduced in the last section will continue to be used in the following proofs.

For the sake of strictness, in the following we define some numbers that will be used later (cf. Definition 25, 27, 28, 29). The readers may choose to skip these definitions temporarily and recall them only when necessary. The intuition behind these somewhat strange and elaborated definitions is that we want to use the values of $x, y$ (especially the value of $x$) of a vertex $(x, y)$ to determine the type label of $(x, y)$, which tells us how $(x, y)$ is connected to the other vertices.

In the following we define $U^*_i, \eta^*_i, \gamma^*_i, \beta_{i,j}, [x], (y)$, and $X^*_i$ using simultaneous induction, which means that some notations will possibly be used before they are defined.

We use a $U^*_i$-tuple to denote one unit in the $i$-th abstraction of the structures to be constructed. Let $U^*_m := k - 1$. Note that the factor "$k - 1$" is the number of distinct coordinate congruence numbers. And we usually regard every $k - 1$ successive vertices as one object.

For any $1 \leq i < m$, let
\begin{align*}
\eta^*_i &:= (k - 1) \times c_{i+1}, \quad \text{where} \quad c_{i+1} = \left( 2k \cdot \gamma_{m-i-1}^* + \sum_{j=1}^{k-2} (k \cdot \gamma_{m-i-1})^j \right); \\
\mathcal{U}^*_i &:= 3 \times 2^{(k-2)} \times \eta^*_i.
\end{align*}

To understand what the value $c_{i+1}$ stands for, cf. page 51. And cf. page 47 for the role of $\mathcal{U}^*_i$. Cf. page 52 for an explanation of the factors $2^{(k-2)}$, and page 51 for an explanation of the constant "3" in (5.25).

Let $\gamma^*_i$ ($0 \leq i < m$) be defined as the following:
\begin{align*}
\gamma^*_0 &:= 2^m \times U^*_m; \\
\gamma^*_i &:= 2^{m-i} \times U^*_{m-i} \times \gamma^*_{i-1}. \\
X^*_i &:= [\gamma^*_{m-i}] \times [k].
\end{align*}

Note that $2^m$, as well as $2^{m-i}$, comes from Fact 6. It helps to ensure that Spoiler cannot win the game simply by exploiting pure linear orders.\footnote{Later we shall see that Duplicator will resort to the $(m-i)$-th abstraction if she cannot respond properly without violating a linear order requirement in the $(m-i+1)$-th abstraction. From the definition (5.31), we can see that $2^{m-i}$ objects (each object contains $U^*_{m-i}$ objects in the $(m-i)$-th abstraction, which include all the different types of vertices of index $m-i$). Cf. Definition 6 for the concept vertex index are “generated (or expanded)” from one object of the $(m-i+1)$-th abstraction. As a consequence, Duplicator will find
Compared with the corresponding definitions 5.1, 5.2 (cf. the special case where \( k = 3 \)), where \( 4m \) suffice to make it and \( U_j^* = 1 \) for any \( j \), in the general cases we need \( 2^m \) and \( U_j^* \) needs to be sufficiently large to take account of all types of vertices.

We shall define a notion called board history in Definition 20. In the following we use \( bh^\# \) to denote the number of all possible board histories (including invalid ones).

\[
bh^\# := m(k \cdot \gamma_{m-1}^* + 1)^{k-1}.
\]

(5.29)

Similarly we can define \( \gamma_i \) and \( X_1 \) as follows.

\[
\gamma_0 := 2^m \times U_m^* \times m \times bh^\#;
\]

(5.30)

\[
\gamma_i := 2^{m-i} \times U_{m-i}^* \times \gamma_{i-1}^* \times m \times bh^\#, \text{ where } 0 < i < m.
\]

(5.31)

\[
X_1 := [\gamma_{m-1}^*] \times [k] \times [m] \times [bh^\#].
\]

(5.32)

By the definition, we have that \( \gamma_i = m \times bh^\# \times \gamma_i^* \).

The notations \( \beta_j^i, \), [\( x \)], and \( (x) \), are defined in (5.3) – (5.5). We copy them here to remind the readers.

For any \( 0 \leq j \leq i \leq m - 1 \),

\[
\beta_j^i := \frac{\gamma_i^*}{\gamma_j^*}.
\]

Note that \( \frac{\gamma_i^*}{\gamma_{i-1}^*} = 2^{m-\ell} \times U_{m-\ell}^* \) for any \( 1 \leq \ell < m \), by (5.31). That is, \( \frac{\gamma_i^*}{\gamma_{i-1}^*} \in N^+ \) since, obviously, \( U_{m-\ell}^* \in N^+ \). Therefore, \( \beta_j^i \in N^+ \) because \( \beta_j^i = \prod_{j \leq \ell < i} \frac{\gamma_{i-1}^*}{\gamma_{i-\ell}^*} \).

For any \( 0 \leq x < \gamma_{m-1}^* \) and \( 1 \leq i \leq m \), let

\[
[x]_i := \lfloor x/\beta_{m-i}^m \rfloor
\]

\[
(\langle x \rangle)_i := [x]_i \beta_{m-i}^m + \frac{1}{2} \sum_{1 < j \leq i} \beta_{m-j}^m
\]

that every interval (delimited by pebbles or boundaries of the structures) in the \((m-i)\)-th abstraction is sufficiently large even if the interval is of length one in the \((m-i+1)\)-th abstraction. It is for this reason that we use \( 2^{m-1} \) instead of \( 2^m \) in definition (5.31).

\footnote{By definition, \( U_m^* \in N^+ \). \( \cdots \gamma_0^* \in N^+ \). Note that \( \gamma_0^* \) is the width of the \( m \)-th abstraction of the structure \( \mathcal{B}_{k,m}^\# \) we are going to construct. \( U_{m-1}^* \) is then defined based on \( \gamma_0^* \), which is in \( N^+ \). Then \( \gamma_1^* \) is defined based on \( U_{m-1}^* \). Then based on \( \gamma_1^* \), we can define \( U_{m-2}^* \). Afterwards, \( \gamma_2^* \) is defined based on \( U_{m-2}^* \), which is in \( N^+ \). And so on. In this process, \( \gamma_{m-i-1}^* \in N^+ \) for any \( i \), \( \cdots \gamma_{i+1}^* \in N^+ \); \( \cdots \eta_i^* \in N^+ \). As a consequence, \( U_{m-\ell}^* \in N^+ \) for any \( \ell \).}
For $2 \leq i \leq m$, let

$$X^*_i := \{(x, y) \in X^*_1 \mid x = [x]_i\}.$$  \hfill (5.33)

Note that $|X^*_i| = k \times \gamma^*_m / \beta^m - 1 = k \times \gamma^*_m i$. That is, $\gamma^*_m i$ describes the width of the finite upright square lattice whose lattice points are the set of elements of $X^*_i$. We will define a pair of structures $A^*_k, m$ and $B^*_k, m$ in Definition 29 whose universe are $X^*_1$. Hence $\gamma^*_m i$ is the width of the universe of $A^*_k, m$ and $B^*_k, m$. These two structures have $m$ abstractions as $A_3, m$ and $B_3, m$ do. In some sense, $B^*_k, m[X^*_m]$ is similar to $B^*_3, m[X^*_m]$, except that we forbid some edges (for the definition of $B^*_k, m$, cf. Definition 29). Each vertex of $B^*_k, m$, as well as $A^*_k, m$, has an index that is defined in the same way as Definition 6. From the viewpoint of iterative structural expansion, we can construct $B^*_k, m[X^*_m]$ from $B^*_k, m[X^*_m]$ and so on, until we obtain $B^*_k, m[X^*_1]$, i.e. $B^*_k, m$. But now the situation is much more complicated than the special case: in the $i$-th abstraction, we need to create all types of vertices in this abstraction by forbidding some edges, and put these varied distinct types of vertices (in the same row) into one “unit” or object, i.e. an interval of $U^*_i$ vertices. Note the resemblance between (5.30) and (5.32) if we regard every successive sequence of $U^*_i$ vertices as one object in the $i$-th abstraction of the structures. For instance, we can regard every $U^*_m$ successive vertices in a row of the $m$-th abstraction of $A^*_k, m$ as one object, or one unit. We can call it an $U^*_m$-tuple. We shall see, in Definition 28, how the value $[x]_i \mod U^*_i$ determines the way $(x, y)$ is connected to the other vertices, provided that $\operatorname{idx}(x, y) = i$.

For any $(x, y) \in X^*_p$, we use $([x]_p, y)$ to denote the $U^*_p$-tuple of vertices $\{(x_{\min}, y), (x_1, y), \ldots, (x_{U^*_p - 1}, y)\}$ wherein, for any $i \in [1, U^*_p - 1]$,

1. $(x_{\min}, y) \in X^*_p$;
2. $[x_{\min}]_p / U^*_p = [x]_p / U^*_p = [x_{\min}]_p / U^*_p$;
3. $\operatorname{idx}(x, y) = p$;
4. $[x]_p = [x_{\min}]_p + i$.

(1) means that $\operatorname{idx}(x_{\min}, y) \geq p$, (2) implies that $\gamma^*_m - p$ can be divided by $U^*_p$, and $[x_{\min}]_p$ and $[x]_p$ is in the same $U^*_p$-tuple where $[x_{\min}]_p$ is the first element of this tuple; (4) means that, in the $p$-th abstraction, the distance between $(x, y)$ and $(x_{\min}, y)$ is $i$. We use $[x]_p^{\min}$ to denote $[x_{\min}]_p$. We introduce $([x]_p, y)$ to denote such an interval that contains all the different types of vertices of index $p$. 

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Once $A^*_k,m$ and $B^*_k,m$ are constructed, whose universes are $X^*_1$, another pair of structures $A_{k,m}$ and $B_{k,m}$ can be constructed based on them, whose universes are $X_1$. In the sequel, we define a notation $x^b$ for any $x \in [\gamma^*_m-1]$ as follows

$$x^b := x \mod (\gamma^*_m - 1 \times k) \quad (5.34)$$

We associate a vertex $(x,y)$ in $A_{k,m}$ with a vertex $(x^b,y)$ in a “flat” structure $A^*_{k,m}$ that “forgets” board history information (cf. Definition 19 and Definition 20) of vertices of $A_{k,m}$. We can take it that each vertex $(x,y)$ in $A_{k,m}$ is a pair $((x^b,y),h_{xy})$ where $(x^b,y)$ is a vertex in $A^*_{k,m}$ and $h_{xy}$ describes the board history associated with $(x^b,y)$. Later, we shall see that all the vertices associated with the same board history are arranged together w.r.t. the first coordinate.

For $1 < i \leq m$, let

$$X_i := \{(x,y) \in X_1 | (x^b,y) \in X^*_i\} \quad (5.35)$$

We can also assign an index for each vertex $(x,y)$ of $A_{k,m}$, as well as $B_{k,m}$, with $\text{idx}(x^b,y)$. Therefore, $\text{idx}(x,y) = i$ if and only if $(x,y) \in X_i - X_{i+1}$, provided that $i < m$.

**Definition 19.** A **board configuration** is a $(k-1)$-tuple $\{X^*_1 \cup \{(\ast,\ast)\}\}^{k-1}$.

A valid board configuration is a board configuration where “$(\ast,\ast)$”s can only appear in the tail of the tuple. Let $\mathcal{BC}$ be the set of all valid configurations. For any $(x,y) \in X_1$, we use $(x,y)[BC]$ to denote the board configuration associating with $(x,y)$. Moreover, for any valid board configuration $Z$, we use $|Z|$ to denote the number of elements of $Z$ that is not $(\ast,\ast)$. $Z$ is an **empty board configuration**, denoted by $BC_\emptyset$, if $|Z| = 0$.

In this paper, when we talk about a board configuration, it is a valid one by default. We use $(x,y)[BC] \circ (u,v)$, where $(u,v) \in X_1$, to denote that the first “$(\ast,\ast)$” in the tuple $(x,y)[BC]$, if there is a $(\ast,\ast)$ and $(w,v)$ is not in the tuple, is replaced by $(w,v)$. $(x_i,y_i)[BC]$ describes the setting when exactly those vertices $(u,v)$ in $(x_i,y_i)[BC]$ are “supposed to” be pebbled, though not necessary true. A configuration $(x_i,y_i)[BC]$ can evolve to another configuration $(x_j,y_j)[BC]$ in one round of a reasonable game over board $(A^*_{k,m},B^*_{k,m})$, denoted $(x_i,y_i) \triangleright (x_j,y_j)$, if and only if either $(x_i,y_i)[BC] \circ

---

*An alternative definition is the following. A board configuration is a $(k-1)$-tuple $\{(X^*_1,n) \cup \{(\ast,\ast),0\}\}^{k-1}$, where $n \in [1,k-1]$, such that the sum of the second item of the pairs is no more than $k - 1$. This version of definition tells us how many pebbles are put on the same vertex.
($x_i, y_i$) = ($x_j, y_j$)[$BC$], or ($x_i, y_i$)[$BC$] = ($x_j, y_j$)[$BC$] $\circ$ ($x_i, y_i$). That is, it is either ($x_i, y_i$)[$BC$] = ($x_j, y_j$)[$BC$], or ($x_j, y_j$)[$BC$] evolves from ($x_i, y_i$)[$BC$] by adding or removing ($x_i^0, y_i$). And if $| (x_i, y_i) |$ = $k$ − 1, then ($x_i, y_i$)[$BC$] could only evolve to ($x_j, y_j$)[$BC$] by removing ($x_i^0, y_i$); if $| (x_i, y_i) |$ = 0, then ($x_i, y_i$)[$BC$] could only evolve to ($x_j, y_j$)[$BC$] by adding ($x_i^0, y_i$).

For any board configuration $Z$, we can use $Z - \{(*,*)\}$ to denote the tuple that is obtained from $Z$ by removing all the “(*,*)”. For convenience, we also use ($x, y$)BC to denote ($x, y$)BC − {(*,*)}. Hence such truncated board configurations may have less than $k − 1$ elements.

In the following we define an important concept that reflects intrinsic “logic” of evolution of games, but not the real evolution of games.

**Definition 20.** For any ($x, y$) $\in \mathbb{X}_1$, the board history of ($x, y$), denoted by $\chi(x, y)$ $\upharpoonright$ BH, consists of a sequence of $m$ board configurations, written (BC$_0, \ldots, BC_{m-1}$). A valid board history need satisfy the following requirements.

- BC$_i$ is a valid board configuration, for any $i$;
- let $i_{\mathbb{C}^y}^{x,y} := \chi(x, y) \upharpoonright bc + 1$ and ($x_{i_{\mathbb{C}^y}^{x,y}}, y_{i_{\mathbb{C}^y}^{x,y}}$) := ($x, y$); for $0 \leq j < i_{\mathbb{C}^y}^{x,y}$, BC$_j$ = ($x_{j+1}, y_{j+1}$)[$BC$], for some ($x_{j+1}, y_{j+1}$) $\in \mathbb{X}_1$; in particular,
  - (i) $\chi(x, y) \upharpoonright$ BH($i_{\mathbb{C}^y}^{x,y} - 1$) = ($x, y$)[$BC$];
  - (ii) BC$_0$ = BC$_0$;
- for $1 \leq j < i_{\mathbb{C}^y}^{x,y}$, ($x_j, y_j$) $\triangleright$ ($x_{j+1}, y_{j+1}$); henceforth, for any $i$ we use ($x, y$)$_H^i$ to denote ($x_i, y_i$), e.g. ($x, y$) = ($x, y$)$_H^{i_{\mathbb{C}^y}^{x,y}}$;
- for $i_{\mathbb{C}^y}^{x,y} \leq j \leq m - 1$, BC$_j$ = {(*,*)}, ... {(*,*)}.

For convenience, the denotation “$\chi(x, y)$ $\upharpoonright$ BH” coincides with that of type label, a notion which will be introduced later (cf. Definition 25). By default, a board history is a valid board history. The (actual) length of the board history of ($x, y$) is $\chi(x, y)$ $\upharpoonright$ bc, i.e. $i_{\mathbb{C}^y}^{x,y} - 1$. Say that $\chi(x, y)$ | BH is a void board history, if all board configurations in $\chi(x, y)$ | BH are empty, i.e. $\chi(x, y)$ $\upharpoonright$ bc = 0. We use $\chi(x, y)$ | BH $\circ$ BC$_l$, where BC$_l$ is a nonempty board configuration (i.e. not a sequence of {(*,*)}, ... {(*,*)}), to denote that the board history of ($x, y$) is extended by one more board configuration, i.e. the first {(*,*)}, ... {(*,*)}) in the tail is replaced by BC$_l$. For convenience, we also use $\chi(x, y)$ | BH to denote the board history of ($x, y$) wherein all the empty board configurations in the tail are removed. Hence such board histories may have less than $m$ elements.
A board history χ(x_i, y_i) ∣ BH can legally evolve from another board history χ(x_j, y_j) ∣ BH in one step, denoted (x_i, y_i) \xrightarrow{\text{con.}}_{BC} (x_j, y_j), if and only if

(1) i_{\text{cur}}^x = i_{\text{cur}}^x + 1; and

(2) χ(x_i, y_i) ∣ BH ⊆ χ(x_j, y_j) ∣ BH; and

(3) (x_i, y_i) ↑ (x_j, y_j).

If χ(x_p, y_p) ∣ BH can evolve from χ(x_i, y_i) ∣ BH using l steps, and (x_p, y_p) \xrightarrow{\text{con.}}_{BC} (x_j, y_j), then χ(x_j, y_j) ∣ BH can evolve from χ(x_i, y_i) ∣ BH using l + 1 steps. We use (x_i, y_i) \xrightarrow{\text{con.}}_{BC} (x_j, y_j) to denote that χ(x_j, y_j) ∣ BH can evolve from χ(x_i, y_i) ∣ BH using i_{\text{cur}}^x - i_{\text{cur}}^x + y_{\text{cur}} steps.

For valid configurations (x_i, y_i) ∣ [BC] and (x_j, y_j) ∣ [BC], if (x_i, y_i) \xrightarrow{\text{con.}}_{BC} (x_j, y_j) ∧ (x_i, y_i) ∣ [BC] ⊆ (x_j, y_j) ∣ [BC], (x_j, y_j) \xrightarrow{\text{con.}}_{BC} (x_i, y_i) ∣ [BC] ⊆ (x_j, y_j) ∣ [BC], we say that the board histories are in continuity (in the default direction28), written (x_i, y_i) \xrightarrow{\text{con.}}_{BC} (x_j, y_j) or (x_j, y_j) \xrightarrow{\text{con.}}_{BC} (x_i, y_i) respectively. Note that only valid board histories can be in continuity. By the transitivity of the binary relations \xrightarrow{\text{con.}}_{BC} and ⊆, the binary relation \xrightarrow{\text{con.}}_{BC} is also transitive.

The initial segment of board history χ(x, y) ∣ BH of length ℓ, written by χ(x, y) ∣ IBH[ℓ], is composed of the first ℓ + 1 board configurations in χ(x, y) ∣ BH. We use (x, y)[ℓ| \xrightarrow{\text{con.}}_{BC} (x', y) to denote that χ(x, y) ∣ BH[ℓ] can evolve to and is in continuity with χ(x', y) ∣ BH.

In the following we introduce some constructions that explain the numbers c_l^m in (5.24). We can create a row of vertices that is isomorphic to \mathcal{B}_{k,m}^*[\mathcal{X}_1^*]|(≤). We denote such a tuple by \mathcal{L}_i. Note that the elements of \mathcal{L}_i have the same second coordinate, whereas those of \mathcal{B}_{k,m}^*[\mathcal{X}_1^*]|(≤) have k different second coordinates. Now we make a larger tuple of vertices in a row, denoted \mathcal{L}_i^+, by concatenating two copies of \mathcal{L}_i with k − 2 more lists \mathcal{L}_j, where 1 ≤ j ≤ k − 2 and the productions are Cartesian products. Note that |\mathcal{L}_i^+| = c_l^m. That is, we want to use a number modulo c_l^m to encode game boards with up to k − 2 “pebbled” vertices over the i-th abstraction. And we shall see what the number means in Definition 25. Note that, \mathcal{L}_i^+ is only a small piece of an \mathcal{U}_i^*-tuple in \mathcal{B}_{k,m}^*.

We define a function RngNum(·, ·) as the following.

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28That is, χ(x_i, y_i) ∣ BH can be evolved from χ(x_j, y_j) ∣ BH if χ(x_j, y_j)bc > χ(x_i, y_i)bc, and vice versa. Moreover, we can also define a notion “∪∗" akin to ⊂ and give an alternative definition for continuity based on it.
For any $1 \leq l \leq m$ and any $(x_i, y_i) \in X_1$,

$$RngNum(x_i^b, l) := \left\lfloor \frac{[x_i^b]_l \mod \mathcal{U}_l^*}{\frac{1}{3} \mathcal{U}_l^*} \right\rfloor - 1. \quad (5.36)$$

Note that there are exactly 3 different values for $RngNum$. It explains the constant "3" that appears in (5.25).

Because $[x_i^b]_l \mod \mathcal{U}_l^* = 0$ if $l < \text{idx}(x_i, y_i)$, we immediately have the following observation.

**Lemma 21.** If $\text{idx}(x_i, y_i) > l$, then

$$RngNum(x_i^b, l) = -1.$$  

Likewise, for any $(x_i, y_i) \in X_1^*$, we can define $RngNum$ in the similar way except that the superscript "b" should be removed from the definition.

Assume that, like $(x_i, y_i)$, $(x_j, y_j)$ is also in $X_1$ and that $\min\{\text{idx}(x_i^b, y_i), \text{idx}(x_j^b, y_j)\} = t$ and $y_i \neq y_j$, we define a function $\text{sgn} : X_1^* \times X_1^* \mapsto \{0, 1\}$ as follows. This function will be used in Definition 27. Note that, when $y_i, y_j \in [1, k - 2]$, the value of $\text{sgn}((x_i^b, y_i), (x_j^b, y_j))$ is meaningless since it will not be used. From here on we assume that either $y_i \equiv 0 \mod k - 1$ or $y_j \equiv 0 \mod k - 1$. Let $\text{sgn}((x_i^b, y_i), (x_j^b, y_j)) = \text{sgn}((x_j^b, y_j), (x_i^b, y_i))$, where

- $\text{idx}(x_i^b, y_i) = \text{idx}(x_j^b, y_j) = t$:
  $$\text{sgn}((x_i^b, y_i), (x_j^b, y_j)) = 0.$$

- $\text{idx}(x_i^b, y_i) > t = \text{idx}(x_j^b, y_j)$ (it is symmetric when $\text{idx}(x_j^b, y_j) > t$):
  $$\text{sgn}((x_i^b, y_i), (x_j^b, y_j)) = 1 \text{ if and only if one of the following holds}$$
  - $k - 1 = y_i > y_j > 0$ and $RngNum(x_j^b, t) = 0$;
  - $k - 1 > y_j > y_i = 0$ and $RngNum(x_j^b, t) = 1$.

The following observation is direct.

**Lemma 22.** Suppose that $\min\{\text{idx}(x_i^b, y_i), \text{idx}(x_j^b, y_j)\} = t$ and $RngNum(x_i^b, t) = RngNum(x_j^b, t) = -1$. Then

$$\text{sgn}((x_i^b, y_i), (x_j^b, y_j)) = 0.$$

By definition, we know that, for any $(x, y)$ where $0 < y < k - 1$, \(\text{idx}(x, y) = i \) and $RngNum(x, i) \neq -1$, $(x, y)$ is either not adjacent to
Note that in this case RngNum\((a, i)\) the ÎΣ\(k - x, y\) to encode SW\((g)\). There are several ways to encode SW\((x, y)\). Example 1. Let \(X : \{0, \ldots, \gamma^a_{m-1} - 1\} \times \{1, \ldots, k - 2\}\). As a matter of fact, this is not the most standard one. Hence we give an example in the following. For any \((x, y) \in X^*_t - X^*_t \oplus 1\), \(g(x) := 0\) if \([x]_i\) mod \(\eta^x_t < \frac{1}{3}\eta^x_t\); \(g(x) := \lfloor[x]_i/\eta^x_t \rfloor\) mod \(2^{(k-2)}\), otherwise. Let 

\[
\text{SW}\left((x^i_t, y^i_t), (x^j_t, y^j_t)\right) := \left(\left|g(x^i_t) - g(x^j_t)\right|\right)_{2:(k-2)} \tag{5.37}
\]

\(\text{SW}\left((x^i_t, y^i_t), (x^j_t, y^j_t)\right)\) is undefined if \(\text{idx}(x^i_t, y^i_t) \neq \text{idx}(x^j_t, y^j_t)\).

For any \((x, y), (x', y') \in X^*_t\) where \(0 < y < y' < k - 1\) and \(\text{idx}(x, y) = \text{idx}(x', y')\), we use a \((k - 2)\)-by\(-(k - 2)\) symmetric 0-1 matrix, say \(M = (a_{i,j})\), to encode \(\text{SW}((x, y), (x', y'))\). succintly. That is, only the entry \(a_{i,j}\) where \(k - 2 \geq j > k - 1 - i \geq 1\) has a valid value. Let \(\hat{q}(y, y') := y' - y + \frac{\gamma^y_t - 2}{(k - 3 - s)}\) if \(y < y'\); undefined, otherwise. We use \(a_{k-1-y, y'}\) to denote the \((\hat{q}(y, y') - 1)\)-th bit of \(\text{SW}((x, y), (x', y'))\). In other words, we use \(a_{k-2, 2}\) to denote the 0-th bit of \(\text{SW}((x, y), (x', y'))\), and \(a_{k-2, 3}\) for the 1-th bit, \(\ldots\), and \(a_{k-3, 3}\) for the \((k - 3)\)-th bit, and so on. Note that \(a_{2, k-2}\) is the leftmost bit of \(\text{SW}((x, y), (x', y'))\). We use \(a_{k-1-y, y'} \in \{0, 1\}\) to tell whether the edge between \((x, y)\) and \((x', y')\) should be “switched off” or not (cf. Definition \(27\) 2 (e)).

**Example 1.** There are several ways to encode \(\text{SW}((, , ))\). Perhaps our choice is not the most standard one. Hence we give an example in the following.

Assume that \(k = 7\) and that the pair of vertices \((x, 2), (x', 4) \in X^*_t\), where \(\text{idx}(x, 2) = \text{idx}(x', 4)\), satisfy that

\[
\text{SW}((x, 2), (x', 4)) = 1011100011.
\]

The matrix \((a_{i,j})\) is shown in the following.

\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
Note that, the topmost “1” in the position of the second row from the top and the fifth column from the left is the value of $a_{2,5}$.

By definition, $\hat{q}(2, 4) = 6$. From the matrix, we can see that the 5-th bit of $\text{SW}((x, 2), (x', 4))$ is 1.

Hence $\text{BIT} (\text{SW}((x, 2), (x', 4)), \hat{q}(2, 4)) = 1$.  

We will define the structures $\mathfrak{A}_{k,m}^*$ and $\mathfrak{B}_{k,m}^*$ in Definition 29. We will use $E_{\mathfrak{A}}$ to denote the set of edges of $\mathfrak{A}_{k,m}^*$ and $E_{\mathfrak{B}}$ to denote that of $\mathfrak{B}_{k,m}^*$. Moreover, we use $\mathcal{E}^*$ to denote the set of edges of either $\mathfrak{A}_{k,m}^*$ or $\mathfrak{B}_{k,m}^*$.

We associate every vertex of $\mathfrak{B}_{k,m}^*$ with a label, called congruence label, that is related to the coordinate congruence number in the $i$-th abstraction where $i$ is the index of this vertex. Assume that $m \geq k$.

**Definition 23.** The set of congruence labels of $\mathfrak{B}_{k,m}^*$, denoted by $\text{Cl}_i$, are defined as follows. Note that we use underline to stand for a string. For example, $\underline{x}$ stands for some sort of encoding of $x$.

$$\text{Cl}_m := \{n,j;m;R;\emptyset \mid n \in [k-1]; j \in [k]; R \in \{-1,0,1\}\};$$  \hspace{1cm} (5.38)

For $1 < i \leq m$,

$$\text{Cl}_{i-1} := \{n,j;i-1;R;M \mid n \in [k-1]; j \in [k]; \hspace{1cm}$$

$$R \in \{-1,0,1\}; M \subseteq \text{Cl}_i \} \cup \text{Cl}_i. \hspace{1cm} \text{(5.39)}$$

**Remark 24.** In (5.38), $n$ is intended to denote the coordinate congruence number of a vertex in the $m$-th abstraction; while $j$ is the second coordinate of the vertex and $m$ is its index. $R$ is intended to denote a value for $\text{RngNum} (\cdot, \cdot)$. In a moment we shall introduce a related notion, i.e. a congruence label associating with a vertex, cf. Definition 26 and Definition 28. Every vertex is associated with a congruence label. For example, in $\mathfrak{B}_{k,m}^*$, if a vertex $(x, y)$ has a congruence label of $1, 2; m; -1; \emptyset$, it implies that $\text{id}x(x, y) = m, y = 2, \text{cc}(\lfloor x \rfloor, y) = 1$ and $\text{RngNum}(x, m) = -1$. In (5.39), “$i - 1$” denotes the index of a vertex, and we use $M$ to denote that the vertex is not adjacent to any vertex whose congruence label is in $M$. Note that, the index of any vertex, whose congruence label is in $\text{Cl}_i$, is greater than or equal to $i$.  

Note that the size of $\text{Cl}_i$ is completely determined by $k$, which is not equivalent to $cl_{i}^*$ or $cl_i$. 

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In the following, for the sake of conciseness, in the first place we define \( \mathfrak{B}_{k,m} \) (cf. Definition 27), and closely related concepts "type label" and "congruence label" for elements in \( X_1 \). But the readers can choose to read and understand Definition 28 and Definition 29 first, which are relatively more simple, and come back here only when necessary. Although reading them will require an understanding of Definition 25—Definition 27, the readers can ignore the parts related to board histories at the moment. After understanding these notions and structures, the readers should continue to read and understand Definition 25—Definition 27. Note that we define the congruence label and type label of a vertex, as well as the (edges of) structure \( \mathfrak{B}^*_{k,m} \), simultaneously. Cf. page 51 for the definition of \( L^+_i \).

**Definition 25.** For any \((x, y) \in X_1\), the type label of \((x, y)\), denoted by \( \chi(x, y) \), is defined as follows.

- Assume that \( \text{id}_x(x^b, y) = i \). Let \( \chi(x, y) := a; y; \beta; \Omega \) where
  - \( a = \text{cc}(\lfloor x^b_{i+1} \rfloor); \) (5.40)
  - BH (board history) is the \( j^\prime \)-th element of \( J^+_i \) where \( j^\prime = \lfloor x^b_{i+1} / (k - 1) \rfloor \mod bh^\#; \) (5.41)
  - bc is the (actual) length of the board history of \((x, y)\) where \( bc = \lfloor x^b_{i+1} / (k \times bh^\#) \rfloor \mod m; \) (5.42)
  - if \( i = m \) then \( S = \Omega = \emptyset \); otherwise \( 1 \leq i < m), \)
  - \( \Omega \) is defined as follows:
    - \( y \in \{0, k - 1\}: \Omega = \emptyset; \)
    - \( y \in [1, k - 2]: \Omega = \{ (u^a, v) \in X^*_i \mid (u, v) \in X_i; v \neq y; |x^b| \equiv |u^a|_i \congruent 0 \mod (k - 1); (\langle u^a \rangle_i, v), (\langle x^b \rangle_{i+1}, y) \notin E_s; v \in \{0, k - 1\} \rightarrow (u^a, v) \in X^*_i \chi_{i+1} \land \text{sgn}(\langle x^b \rangle_{i+1}, y), (\langle u^a \rangle_{i+1}, v)) = 0 \}; \)
  - \( S \subseteq \text{Cl}_{i+1} \) is determined by the \( j \)-th element of \( \mathfrak{B}^+_i \) where
    - \( j = \lfloor x^b_{i+1} / (k - 1) \rfloor \mod cl^*_{i+1}; \) (5.43)
    - if \( j < \gamma^*_{m-i-1} \) or \( j \geq cl^*_{i+1} - \gamma^*_{m-i-1} \) then \( S := \emptyset \); otherwise, assume that this \( j \)-th element is a \( d \)-tuple (note that \( 1 \leq d \leq k - 2 \)) \( ((u_1, v_1), \cdots, (u_d, v_d)) \in (X^*_i)^d \), then
      \[ S := \bigcup_{1 \leq i \leq d} \{ \text{cl}(u_i, v_i) \}. \] (5.44)
In Definition 27 where \( B_{k,m} \) is defined, we shall see that type labels determine whether a vertex having some type label is NOT adjacent to another vertex that has another type label. Henceforth, we use \( \chi(x,y) \upharpoonright S \) to denote \( S \) in \( \chi(x,y) \), and similarly for \( \chi(x,y) \upharpoonright \Omega \), \( \chi(x,y) \upharpoonright \text{BH} \), and \( \chi(x,y) \upharpoonright \text{bc} \). Note that, \( (5.41) \) says that, in a row of the structure, every successive sequence of vertices of length \( \gamma_{m-1}^* \times k \) have been associated with the same board history and \( j' \) determines the board history associated with \((x,y)\); \( (5.42) \) tells us about the length of board history that is associated with \((x,y)\). Moreover, in Definition 27 we shall see how edges are forbidden based on \( \chi(x,y) \upharpoonright \Omega \), \( \chi(x,y) \upharpoonright S \) and \( \chi(x,y) \upharpoonright \text{BH} \).

**Definition 26.** In the structure \( B_{k,m} \), for any \((x,y) \in X_1\) and any \( 1 \leq i \leq m \), let \( \text{idx}(x^b, y) = i \). The congruence label of \((x,y)\), denoted by \( \text{cl}(x,y) \), is defined as the following:

\[
\text{cl}(x,y) := \text{cc}([x^b]_i, y) \mid R_i \mid \chi(x,y) \upharpoonright S \quad \text{where} \quad R_i = \text{RngNum}(x^b, i). \tag{5.45}
\]

The readers can cf. Example 3 for an explanation on what a congruence label is meant in some structure (i.e. \( B_{k,m}^* \)) that will be defined soon.

We use \((u,v) \rightarrow (x,y)\) to denote the formula \( \psi \), where

\[
\psi = (u,v) \xrightarrow{\text{con}}_{BC} (x,y) \land (u^b, v) \in (x,y)[BC] \tag{5.46}
\]

Note that \( \rightarrow \) is a strict preorder, i.e. a transitive relation.

We use \((x_j, y_j) \leadsto (x_i, y_i)\) to denote the formula \( \varphi \), where

\[
\varphi = (x_i, y_i) \rightarrow (x_j, y_j) \land \text{cl}(x_i, y_i) \in \chi(x_j, y_j)[S \land [x_j^b, y_j] \notin (x_i, y_i)[BC]] \tag{5.47}
\]

Now we introduce the pair of important structures, a sort of structures with (temporal) “structures”.

**Definition 27.** \( B_{k,m} \) is a \( \langle E, \leq \rangle \)-structure, i.e. ordered graph, over the universe \( X_1 \) and the linear order is defined as the follows. For any vertices \((x_i, y_i)\) and \((x_j, y_j)\) of \( B_{k,m} \), \((x_i, y_i) < (x_j, y_j)\) if \( y_i < y_j \). Suppose that \( y_i = y_j \). For any vertex \((x,y)\) of \( B_{k,m} \), the value \( x/\gamma_{m-1}^* \times k \) mod \( bh# \) is supposed to determine the board history (cf. Definition 25, \( (5.41) \)) associated with the vertex \((x,y)\). Hence, we need an ordering of different board histories. Here, we define the ordering as the following: firstly we can define a lexicographic ordering on the game configurations: BC1 is less than
BC₂ if |BC₁| < |BC₂| (hence the empty game board is the minimal element in the order); now for any invalid board history \( h₁ \) and any valid one \( h₂ \), \( h₁ < h₂ \); for valid board histories in \( N₁ × \cdots × Nₘ \) where \( Nᵢ \) stands for the set of game configurations, the linear order is defined by the lexicographic ordering on the Cartesian product \( N₁ × \cdots × Nₘ \). Note that we only take the actual board histories into account. Hence a shorter board history is ahead of a longer board history.

Then, on condition that \((xᵢ, yᵢ)\) and \((xⱼ, yⱼ)\) have different board histories, \((xᵢ, yᵢ) < (xⱼ, yⱼ)\) if \( χ(xᵢ, yᵢ)|BH < χ(xⱼ, yⱼ)|BH \).

Suppose that \( χ(xᵢ, yᵢ)|BH = χ(xⱼ, yⱼ)|BH \)\(^{29}\) then \((xᵢ, yᵢ) ≤ (xⱼ, yⱼ)\) if \( xᵢ \mod γᵢ₋₁ × k \) is no more than \( xⱼ \mod γⱼ₋₁ × k \).

The edge set \( E^B \) of \( B_{k,m} \) is defined as follows.

For any \( 1 ≤ t ≤ m \) and for any vertices \((xᵢ, yᵢ)\) and \((xⱼ, yⱼ)\)

1. \((xᵢ, yᵢ)\) is not adjacent to \((xⱼ, yⱼ)\) if \( yᵢ = yⱼ \).

   In the following assume that \( yᵢ \neq yⱼ \).

2. If \( \min\{\text{idx}(xᵢ, yᵢ), \text{idx}(xⱼ, yⱼ)\} = t \), then \((xᵢ, yᵢ), (xⱼ, yⱼ) \in E^B \) if and only if all the following conditions hold:

   a. \(((xᵢ, yᵢ)(xⱼ, yⱼ)) \in E^B \) (cf. Definition \(^{29}\));

   b. Either \((xᵢ, yᵢ) \rightarrow (xⱼ, yⱼ)\), or \((xⱼ, yⱼ) \rightarrow (xᵢ, yᵢ)\);

   c. \((xⱼ, yⱼ) \notin \{(u, v) ∈ X₁ | (xᵢ, yᵢ) \sim (u, v)\}\), and

   \((xᵢ, yᵢ) \notin \{(u, v) ∈ X₁ | (u, v) \sim (xⱼ, yⱼ)\}\);

   d. \((xᵢ, yᵢ) \notin χ(xᵢ, yᵢ)|Ω \) and \((xⱼ, yⱼ) \notin χ(xⱼ, yⱼ)|Ω \);

   e. If either \( yᵢ \notin \{0, k − 1\} \) or \( yⱼ \notin \{0, k − 1\} \):

      \[ cc([xᵢ], yᵢ) \neq cc([xⱼ], yⱼ) \] and \( \text{sgn}((xᵢ, yᵢ), (xⱼ, yⱼ)) = 0; \)

If \( yᵢ, yⱼ \in [1, k − 2] \):

   - If \( \text{idx}(xᵢ, yᵢ) ≠ \text{idx}(xⱼ, yⱼ) \):

      \[ cc([xᵢ], yᵢ) ≠ cc([xⱼ], yⱼ); \]

   - If \( \text{idx}(xᵢ, yᵢ) = \text{idx}(xⱼ, yⱼ) = t; \)

      \[ (cc([xᵢ], yᵢ) - cc([xⱼ], yⱼ)) × (yᵢ - yⱼ) \]

      \[ × (-1)^{\text{BIT}(SW((xᵢ, yᵢ). (xⱼ, yⱼ)). δ(yᵢ, yⱼ))} > 0. \]

\(^{29}\)In other words, \([xᵢ/(γᵢ₋₁ × k)] \mod bh^# = [xⱼ/(γⱼ₋₁ × k)] \mod bh^# .\)

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\(\mathfrak{A}_{k,m}\) is constructed from \(\mathfrak{B}_{k,m}\) by adding a set \(E^+\) of edges where \(E^+ = \{((x_i, y_i), (x_j, y_j)) \mid (x_i^0, y_i), (x_j^0, y_j) \in \mathfrak{X}^*_m; y_i, y_j = 0 \text{ or } k - 1; y_i \neq y_j; [x_i^0]_m \equiv [x_j^0]_m \equiv 0 \pmod{k - 1}; (x_i, y_i) \rightarrow (x_j, y_j) \text{ or } (x_j, y_j) \rightarrow (x_i, y_i)\} \). *

As the way we define type label, congruence label and \(\mathfrak{B}_{k,m}\) over the universe \(\mathfrak{X}_1\), we define the dual notions over \(\mathfrak{X}_1\) simultaneously.

**Definition 28.** In the structure \(\mathfrak{B}_{k,m}^*,\) for any \((x, y) \in \mathfrak{X}_1^*\), the congruence label of \((x, y),\) also denoted \(\text{cl}(x, y),\) is defined as the dual one in Definition 26, except that "\(\flat\)"s are removed. The readers are suggested to confer Example 3 in the appendix.

For any \((x, y) \in \mathfrak{X}_1^*\), the type label of \((x, y),\) also denoted \(\chi(x, y),\) is defined as follows.

Assume that \(\text{idx}(x, y) = i.\) Let \(\chi(x, y) := a, y; i; S; \Omega\) where 30

- \(a = \text{cc}([x], y);\)
- If \(i = m\) then \(S = \Omega = \emptyset;\) otherwise (i.e. \(1 \leq i < m),\)
  - if \(y \in \{0, k - 1\}\) then \(\Omega = \emptyset;\) otherwise, \(\Omega = \{(u, v) \in \mathfrak{X}_1^* \mid [u]_i \equiv 0 \pmod{k - 1}; (\langle u \rangle_{i+1}, v), (\langle x \rangle_{i+1}, y) \notin E_s; v \in \{0, k - 1\} \rightarrow (u, v) \in \mathfrak{X}_1^* \wedge \text{sgn}(\langle u \rangle_{i+1}, v), (\langle u \rangle_{i+1}, v) = 0\};\)
  - \(S \subseteq \text{Cl}_{i+1}\) is determined by the \(j\)-th element of \(\mathfrak{L}_{i+1}^*\) where \(j = [\langle x \rangle_i/(k - 1)]\) mod \(\text{cl}_{i+1}\); if \(j < \gamma_{m-i-1}\) or \(j \geq \text{cl}_{i+1}^* - \gamma_{m-i-1}\) then \(S := \emptyset;\) otherwise, assume that this \(j\)-th element is \((u_{i1}, v_{i1}), \cdots, (u_{i1}, v_{id}) \in \mathfrak{X}_{i+1}^d,\) then \(S := \bigcup_{1\leq i \leq d} \{\text{cl}(u_i, v_i)\}.\)

Note that, \(\text{cl}(u, v) = \text{cl}(u^0, v)\) and \(\chi(x, y) \mid S = \chi(x^0, y) \mid S\) for a vertex \((x, y) \in \mathfrak{X}_1.\) In fact, the two version of "type label" introduced in Definition 25 and Definition 28 are also very similar except that the former one also take into account the board history associating with a vertex.

**Definition 29.** \(\mathfrak{B}_{k,m}^*\) is defined similarly as \(\mathfrak{B}_{k,m}\), except that

- \(\mathfrak{X}_i,\) for any \(i,\) is replaced by \(\mathfrak{X}_i^*;\)
- remove 2 (a), (b) in Definition 27
- revise 2 (c) as the following:
  - \((x_j, y_j) \notin \{(u, v) \in \mathfrak{X}_1^* \mid \text{cl}(u, v) \in \chi(x_i, y_i) \mid S\};\)
  - \((x_j, y_j) \notin \{(u, v) \in \mathfrak{X}_1^* \mid \text{cl}(x_i, y_i) \in \chi(u, v) \mid S\};\)

30 Here \(a, y; i; S\) represents a congruence label. Also confer Remark 24.
• remove all the superscripts "f" in Definition 27

\( \mathfrak{A}_{k,m}^* \) is constructed from \( \mathfrak{B}_{k,m}^* \) by adding a set \( E^+ \) of edges where

\[ E^+ = \{(x_i,y_i), (x_j,y_j) \mid x_i, y_i, x_j, y_j \in X^*_m; y_i = 0 \text{ or } k-1; y_i \neq y_j; [x_i]_m \equiv [x_j]_m \equiv 0 \pmod{k-1}\}. \]

We call endpoints of such edges critical points.

For each \( t \) in \([1, m]\) we define a function \( \mathfrak{f}_t^e : |\mathfrak{B}_{k,m}^*| \rightarrow \mathbb{B}_C \), whose concrete definition will be decided later (cf. page 66). In addition, we define \( \mathfrak{B}_{k,m}^{*e} \) similarly as \( \mathfrak{B}_{k,m}^* \), except that we revise 2) c) as the following:

\[ (x_j, y_j) \notin \{(u,v) \in X^*_1 \mid (u,v) \in \mathfrak{f}_t^e(x_i, y_i) \land \chi(u,v) \in \chi(x_i, y_i) \} \]

Similarly, we can define \( \mathfrak{f}_t^a \) and \( \mathfrak{A}_{k,m}^{*a} \).

Note that \( \mathfrak{f}_t^e(x, y) \) is different from \( (x, y)[BC] \). The latter is a notion defined w.r.t. \( \mathfrak{A}_{k,m} \) or \( \mathfrak{B}_{k,m} \).

We shall find in section 6 that \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \), as well as \( \mathfrak{A}_{k,m}^{*e} \) and \( \mathfrak{B}_{k,m}^{*e} \), are the structures that we will study in the pebble games, which are the “associated” structures of \( \mathfrak{A}_{k,m} \) and \( \mathfrak{B}_{k,m} \). In fact, it is a good idea to understand the definition of them before reading Definition 27.

Definition 30. For any \( 1 \leq t \leq m \), the \( t \)-th abstraction of \( \mathfrak{B}_{k,m} \) (\( \mathfrak{A}_{k,m} \) resp.), denoted \( \mathfrak{B}^{(t)} \) (\( \mathfrak{A}^{(t)} \) resp.), is \( \mathfrak{B}_{k,m}[X_t] \) (\( \mathfrak{A}_{k,m}[X_t] \) resp.).

Note that the graph \( \mathfrak{A}_{k,m} \) and \( \mathfrak{B}_{k,m} \) are just \( \mathfrak{A}^{(1)}_{k,m} \) and \( \mathfrak{B}^{(1)}_{k,m} \) respectively.

We use \( X^*_t \) to denote the set \( X_t \), \( E \) of \( \mathfrak{A}_{k,m} \), and use \( X_t \), \( E \) to denote the set \( X_t \), \( E \) of either \( \mathfrak{A}_{k,m} \) or \( \mathfrak{B}_{k,m} \). The abstractions of \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \) can be defined similarly. We call \( \text{cc}([x]_t) \) the “coordinate congruence number in the \( t \)-th abstraction”, for any \( (x, y) \in X_t \).

We get a pair of structures \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \) from \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \), just like the way we obtain \( \mathfrak{A}_{3,m}^* \) and \( \mathfrak{B}_{3,m}^* \) from \( \mathfrak{A}_{3,m}^* \) and \( \mathfrak{B}_{3,m}^* \), by circular shifting the vertices of the \( t \)-th row for \( tr(i) \) times to the right (cf. p. 18). Similarly, we can define \( \mathfrak{B}_{k,m}^{*e} \) etc. Recall that, for \( i \in [k] \), \( tr(i) = (i \mod k-1) \times \sum_{1 \leq p \leq m} \gamma_{m-p}^{m-1} \). We obtain a pair of structures \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \) from \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \), by moving each vertex \((x, y)\) to \((x', y)\) where \( x' = (x + tr(y)) \mod (\gamma_{m-1} \times k) \). They are the pair of main structures that form the game board. We shall meet these newly created structures in the next section, i.e. Section 6. Abuse of denotations, we still call the edge sets of \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \) as \( E^B \) and \( E^B \) respectively.

Recall that we regard every \( \mathfrak{A}^{(t)} \)-tuple of vertices as one object in \( \mathfrak{A}_{k,m}^* \), and such an object includes all types of vertices.
Lemma 31. Let \( 1 \leq r < i \leq m \). For any \((e,f) \in \mathbb{X}_r^*\), \( a \in \{k-1\}, \ell \in \{-1,0,1\}\), and \( w \in \varphi(\text{Cl})\), there is \((e',f) \in \mathbb{X}_r^*\) in the \( \mathcal{U}_r^*\)-tuple \([ [e]_r, f \) such that

\[
\text{cl}(e', f) = a, f; r; \ell; w.
\]

We shall see in the Main Lemma 37 (cf. Strategy 2, (2-5)) that Lemma 31 gives Duplicator the freedom to ensure her picked vertex satisfying some conditions: Lemma 31 allows Duplicator to choose \( \text{cl}(e', f) \) freely. Lemma 31 is also used in the proof of Lemma 36.

It is clear that \( \mathfrak{A}_{k,m} \) contains \( k \)-cliques. In particular, the following lemma says that there is a \( k \)-clique roughly in the middle of the structure with respect to the first coordinate.

Lemma 32. The subgraph of \( \mathfrak{A}_{k,m} \) induced by a set of vertices \((x_0,0), \ldots, (x_{k-1},k-1)\) is a \( k \)-clique, where \( x_i^k = \frac{1}{2} \gamma_{m-1}^k + \frac{1}{2} \sum_{1<j\leq m} \beta_{m-j}^{m-1} \) for any \( i \), and \((x_0,0) \rightarrow (x_1,1) \rightarrow \cdots \rightarrow (x_{k-1},k-1)\).

Proof. First, we show that, for any \( i, (x_i^k, i) \in \mathbb{X}_m^* \) and \( \text{cc}([x_i^k]_m, i) = i \mod k - 1 \).

It is because \([x_i^k]_m = [(\frac{1}{2} \gamma_{m-1}^k + \frac{1}{2} \sum_{1<j\leq m} \beta_{m-j}^{m-1} \gamma_0^k \beta_0^{m-1} + \frac{1}{2} \sum_{1<j\leq m} \beta_{m-j}^{m-1}] / \beta_0^{m-1} = \frac{1}{2} \gamma_{m-1}^k / \beta_0^{m-1} + \frac{1}{2} \sum_{1<j\leq m} \beta_{m-j}^{m-1}] = x_i^k \). The last equation is based on the easy observation that \( \frac{1}{2} \gamma_{m-1}^k \), which equals \( \frac{1}{2} \gamma_0^k \beta_0^{m-1} \), is divisible by \( \beta_0^{m-1} \), since \( \gamma_0^k \) is even. Therefore, \((x_i^k, i) \in \mathbb{X}_m^* \). Moreover, \([x_i^k]_m = \frac{1}{2} \gamma_0^k \), which is divisible by \( k - 1 \). Hence, \( \text{cc}([x_i^k]_m, i) = i \mod k - 1 \).

Second, \( g(x_i) = 0 \), for \([x_i^k]_m \mod \mathcal{U}_m^* = \frac{1}{2} \gamma_0^k \mod \mathcal{U}_m^* = 2^{m-1} \mathcal{U}_m^* \mod \mathcal{U}_m^* = 0 < 2^{(k-2)} \times \eta_m^* \). Therefore, for any \( i, j \), \( \text{SW}((x_i^k, i), (x_j^k, j)) = 0 \). If \( i, j \in [1,k-2] \), then the condition e) of 2) in Definition 27 clearly holds. Assume that either \( i \) or \( j \) is either 0 or \( k-1 \). In this case, we need only notice that \( \text{RngNum}(x_i^k, m) = \text{RngNum}(x_j^k, m) = -1 \) because \([x_i^k]_m \mod \mathcal{U}_m^* = [x_j^k]_m \mod \mathcal{U}_m^* = 0 < 2^{(k-2)} \times \eta_m^* \).

Third, for any \( i, j \), \( \text{cl}(x_i^k, i) \notin (x_j^k, j) \)\( \bigcup \) and \( (x_i^k, i) \notin (x_j^k, j) \)\( \bigcup \), since we’ve already proved that \((x_i^k, i), (x_j^k, j) \in \mathbb{X}_m^* \) for any \( i, j \). Note that \( \text{cl}(x_i^k, i) \notin (x_j^k, j) \)\( \bigcup \) means that the formula \((x_j^k, j) \sim (x_i^k, j) \) does not hold.

Fourth, the binary relation \( \rightarrow \) is transitive. Therefore, \((x_0,0) \rightarrow (x_1,1) \rightarrow \cdots \rightarrow (x_{k-1},k-1) \) implies that \((x_i, i) \rightarrow (x_j, j) \) for any \( 0 \leq i < j \leq k - 1 \).

Then by Definition 29, the set of vertices \((x_0^k,0), \ldots, (x_{k-1}^k, k-1)\) is a \( k \)-clique in \( \mathfrak{A}_{k,m}^* \). As a consequence, by Definition 27, the set of vertices \((x_0,0), \ldots, (x_{k-1},k-1)\) is also a \( k \)-clique in \( \mathfrak{A}_{k,m}^* \), since \((x_i, i) \rightarrow (x_j, j) \) for any \( 0 \leq i < j \leq k - 1 \).
Note that $(x_0, 0) \rightarrow (x_1, 1) \rightarrow \cdots \rightarrow (x_{k-1}, k-1)$ implies that $\chi(x_0, 0)\uparrow$ BH is a void board history, and $\chi(x_i, i)\uparrow$ BH consists of $i$ nonempty board configurations where $\chi(x_i, i)\uparrow$ BH$(j) = \chi(x_i, i)\uparrow$ BH$(j-1) \circ (x_{j-1}, j-1)$ for $0 < j \leq i$.

Lemma 32 implies that $\mathfrak{A}_{k,m}$ has $k$-cliques.

Although it is relatively easy to see that $\mathfrak{B}_{k,m}$ has a k-clique or not. The following lemma answers this question.

**Lemma 33.** $\mathfrak{B}_{k,m}$ has no $k$-clique.

**Proof.** Suppose that $\mathfrak{B}_{k,m}^*$ has no $k$-clique, then $\mathfrak{B}_{k,m}$ also has no $k$-clique, by virtue of 2) a) in Definition 27. Therefore, we need only prove that $\mathfrak{B}_{k,m}^*$ has no $k$-clique.

Assume for the purpose of a contradiction that there are $k$-cliques in $\mathfrak{B}_{k,m}^*$ and $C_k$ is such a $k$-clique that has the maximum index, say $t$, among all the $k$-cliques. $t$ cannot be $m$, for otherwise there are two vertices that have the same coordinate congruence number in the $m$-th abstraction, by the pigeonhole principle. According to the definition, the second coordinates of the vertices of $C_k$ must be different. That is, for each $i \in [0, k-1]$, there is a unique vertex whose second coordinate is $i$. Let $P = \{(x, y) \in |C_k| \ | (x, y) \in X_t \}$. And let $Q = \{(x, y) \in |C_k| \ | (x, y) \in X_{t+1}^*\}$. Note that, $P \cap Q = \emptyset$, and the set of vertices of $C_k$ is exactly $P \cup Q$.

Let $cC_k := \{cc([x]_t, y) \ | (x, y) \in |C_k|\}$. Since there are $k$ elements in $C_k$ and $|cC_k| \leq k - 1$, by pigeonhole principle, there are two vertices $(a^*, b^*)$, $(c^*, d^*)$ such that $cc([a^*]_t, b^*) = cc([c^*]_t, d^*)$. If $(a^*, b^*) \in P$ or $(c^*, d^*) \in P$, then by Definition 27, there is no edge between these two vertices. Therefore, to have a $k$-clique, both $(a^*, b^*)$ and $(c^*, d^*)$ should be in $Q$. Recall Lemma 14 for any $(x, y) \in Q$, $cc([x]_t, y) = y \mod k-1$. Therefore, $cc([a^*]_t, b^*) = cc([c^*]_t, d^*) = 0$. In other words, $\{b^*, d^*\} = \{0, k-1\}$.

Assume without loss of generality that

$$b^* = 0 \text{ and } d^* = k-1. \quad (5.48)$$

There are three cases need to consider.

(1) $Q = \emptyset$: As have just explained, we have $(a^*, b^*)$ in $Q$. Hence a contradiction occurs.

(2) $Q \neq \emptyset$, and for any vertex $(x, y)$ of $C_k$, $[x]_t \equiv 0 \mod k-1$:

Let $P' = \{(u, v) \in X_{t+1}^* \ | \exists (u', v) \in P \text{ s.t. } [u']_{t+1} = [u]_{t+1}\}$. The second coordinates of the $k$ vertices of $P' \cup Q$, all of which are in $X_{t+1}^*$, are
different. Since $C_k$ is the $k$-clique that has the maximum index, hence $\mathcal{B}_{k,m}[P' \cup Q]$ cannot be a $k$-clique and there are two vertices, say $(a, b)$ and $(c, d)$, of $P' \cup Q$ such that $((a, b)(c, d)) \notin E^B_k$.

i) $(a, b), (c, d)$ are vertices of $C_k$: Straightforward contradiction.

ii) $(a, b) \notin |C_k|$ while $(c, d) \notin |C_k|$ (the case when $(c, d) \notin |C_k|$ while $(a, b) \notin |C_k|$ is symmetric): Because $(c, d) \notin |C_k|$, $(c, d) \notin Q$. Then by (5.48), $d \in [1, k - 2]$. Let $(c', d) \in P$ where $|c'|_{t+1} = |c|_{t+1}$. By Lemma 14, $|a|_t \equiv |c|_t \equiv 0 \pmod{k-1}$. Note that $|a|_{t+1} = a$. By the definition of $|\Omega| \Gamma$ and Definition 27, $(a, b) \notin \chi(c', d)!\Omega$ (cf. Definition 25), which means that $(a, b)$ is not adjacent to $(c', d)$. A contradiction occurs.

iii) $(a, b), (c, d) \notin |C_k|$: By (5.48), we have $b, d \in [1, k - 2]$. Let $(a', b), (c', d) \in P$ such that $|a'|_{t+1} = |a|_{t+1}$ and $|c'|_{t+1} = |c|_{t+1}$. By definition, either $(a', b) \notin \chi(c', d)!\Omega$ or $(c', d) \notin \chi(a', b)!\Omega$. In other words, $(a', b)$ is not adjacent to $(c', d)$. We arrive at a contradiction.

(3) $Q \neq \emptyset$ and there exists a vertex $(x, y) \in P$ such that $[x]_t \not\equiv 0 \pmod{k-1}$:

Recall (5.48) that if $cc([a^*]_t, b^*) = cc([c^*]_t, d^*)$ then $(a^*, b^*), (c^*, d^*)$ are in $Q$ and $\{b^*, d^*\} = \{0, k - 1\}$. Hence $0 < y < k - 1$. Now consider the vertices in $P$. Their second coordinates are in the range $[1, k - 2]$. Imagine that we have a sequence of slots numbered by $0, 1, \ldots, k - 1$, some of which are already occupied with billiard balls, i.e. a ball with a number $i$ is filled in the $i$-th slot. In particular, the 0-th and $(k - 1)$-th slots are filled. And we want to fill the left slots with balls in the same way, i.e. we want to fill the $i$-th empty slot with a ball labelled with $i$. If we put a ball to a slot in some wrong way, then we can find two slots whose balls are in disorder: there are $l_1, l_2, s_1, s_2 \in [1, k - 2]$ such that $(l_1 - l_2)(s_1 - s_2) < 0$ and a ball with label $l_1$ is filled in the $s_1$-th slot and a ball with label $l_2$ is filled in the $s_2$-th slot. In our context, a vertex $(u, v) \in P$ is a “ball”, and the number $cc([u]_t, v)$ is the label on it. The $i$-th row of $\mathcal{B}_{k,m}$ is the $i$-th slot. Since there is a vertex $(x, y) (0 < y < k - 1)$ which is not in the right “slot”, i.e. $[x]_t \not\equiv 0 \pmod{k - 1}$ (hence $(x, y)$ must be a vertex in $P$), the sequence of the vertices of $P$ is in disorder. It means that there is another vertex $(x', y') \in |C_k|$ $(0 < y' < k - 1)$ such that $[x']_t \equiv 0 \pmod{k - 1}$ (hence $(x', y')$) must be a vertex in $P$, and $(x', y')$ is also not in the right “slot” and $(cc([x]_t, y) - cc([x']_t, y'))(y - y') < 0$. If SW$((x, y), (x', y')) = 0$ then by Definition 27, there is no edge between $(x, y)$ and $(x', y')$. So we arrive at a contradiction to the assumption.
that $C_k$ is a clique. Hence we assume that $\text{SW}((x, y), (x', y')) \neq 0$, which implies that either $\text{RngNum}(x, t) \neq -1$ or $\text{RngNum}(x', t) \neq -1$. Assume without loss of generality that $\text{RngNum}(x, t) \neq -1$. Therefore, either $(a^*, b^*)$ or $(c^*, d^*)$ is not adjacent to $(x, y)$, by definition.

Lemma 33 implies that $\mathfrak{B}_{k,m}$ has no $k$-clique.

Note that, in the proof we show that $\mathfrak{B}_{k,m}^*$ contains no $k$-clique even if we do not consider the missing of edges defined in 2) c) of Definition 29, which is important for the following observation.

**Corollary 34.** For any $t$ in $[1, m]$ and any $z_t^b$, $\mathfrak{B}_{k,m}^{z_t^b}$ has no $k$-clique.

### 6 $k$-Clique needs $k$ variables in FO: virtual games and associated games over changing board

In this section we introduce our main result, i.e. Duplicator has a winning strategy in the $(k-1)$-pebble game over the game board $(\mathfrak{A}_{k,m}, \mathfrak{B}_{k,m})$ (cf. Lemma 37). As have explained, most of the time we can study the game board $(\mathfrak{A}_{k,m}, \mathfrak{B}_{k,m})$ instead of $(\mathfrak{A}_{k,m}, \mathfrak{B}_{k,m})$.

We are able to prove the following lemma, which will be used shortly to prove the next crucial observation, i.e. Lemma 36.

**Lemma 35.** Assume that $P \subset X_r^* - X_{r+1}^*$ and $|P| \leq k - 2$, and for any $(u_i, v_i), (u_j, v_j)$, $v_i \neq v_j$ if $(u_i, v_i) \in P$ and $(u_j, v_j) \in P$. Then for any string $w_1w_2\cdots w_l \in \{0, 1\}^l$ and any $y \in [1, k - 2]$, where $y \neq v_i$ for any $(u_i, v_i) \in P$, there is $(x, y) \in X_r^* - X_{r+1}^*$ such that for any $(u_i, v_i) \in P$,

$$\text{BIT}(\text{SW}((x, y), (u_i, v_i)), \hat{q}(y, v_i)) = w_i. \quad (6.1)$$

The following observation, together with Lemma 31, gives Duplicator the freedom in the scenario when she uses Strategy 2 (cf. the following main Lemma 37). Recall that, in the proof of Theorem 3, Duplicator is always able to pick a vertex that is adjacent to all the pebbled vertices in $B_k$.

The following lemma (in particularly (3)) roughly says the similar thing. It allows Duplicator to first choose a vertex that is adjacent to all the pebbled vertices. Afterwards, Duplicator can adjust and make her pick by looking for a proper one around this vertex.
Lemma 36. Let $1 < t \leq m$. For any multiset $H$ of $k-2$ vertices $(x_1, y_1), \ldots, (x_{k-2}, y_{k-2}) \in X_t^k$ and $y \in \{0, \ldots, k-1\} - \{y_i \mid 1 \leq i \leq k-2\}$, the following hold.

(1) If there is $0 \leq c \leq k-1$ such that $c \neq y \mod k-1$ and $c \neq y_i \mod k-1$ for any $1 \leq i \leq k-2$, then for any $(x^*, y) \in X_{t-1}^k$, there is a vertex $(x^*, y) \in ([x^*]_{t-1}, y)$ such that $(x^*, y, (x_i, y_i)) \in E_*$ for any $(x_i, y_i) \in H$.

(2) For any $(x', y) \in X_{t-1}^k$ where $(x', y) \mid S = \emptyset$, if $\langle (x')_t, y \rangle$ is adjacent to every vertex in $H$, then there is a vertex $(x'', y) \in ([x']_{t-1}, y)$ s.t. $\idx(x'', y) = t - 1$, and $(x'', y, (x_i, y_i)) \in E_*$ for any $(x_i, y_i) \in H$.

(3) On condition that there are $(x_i, y_i), (x_j, y_j) \in H$ s.t. $x_i \neq x_j$ and $y_i = y_j$, there is $(x, y) \in X_{t-2}^k - X_{t-1}^k$ s.t. $(x, y, (x_i, y_i)) \in E_*$ for any $(x_i, y_i) \in H$, $[x]_{t-2} \equiv 0 \mod k-1$, $g(x) = 0$ and $\RNum(x, t-2) = -1$.

(4) On condition that $y_i \neq y_j$ for any $(x_i, y_i) \neq (x_j, y_j)$, there is $(x, y) \in X_{t-1}^k - X_t^k$ such that $(x, y, (x_i, y_i)) \in E_*$ for any $(x_i, y_i) \in H$, $[x]_{t-1} \equiv 0 \mod k-1$, $g(x) = 0$ and $\RNum(x, t-1) = -1$.

Now we introduce our main lemma, which asserts that Duplicator has a winning strategy in any $m$-round $(k-1)$-pebble game over the game board $(\mathfrak{A}_{k,m}, \mathfrak{B}_{k,m})$.

Lemma 37. $\mathfrak{A}_{k,m} \equiv^{k-1}_m \mathfrak{B}_{k,m}$, for $4 \leq k$ and $(k-1)(k-2) < m$.

At each round of the game, Duplicator’s strategy first works in some specific abstraction of the associated structures $\mathfrak{A}_{k,m}^*$ and $\mathfrak{B}_{k,m}^*$, which will be explained soon in the proof. Suppose that in the current round the players are playing in the $\xi$-th abstraction of the structures. That is, for any pebbled vertex $(u, v)$, Duplicator regards $\langle (u)_\xi, v \rangle$, instead of $(u, v)$, as been pebbled. As in the proof of Theorem 3, an indispensable component of a strategy of Duplicator is to ensure that the players pick a pair of pebbles in the same row of the structures in each round. For each $i$, the $i$-th row of the structures consists of several intervals delimited by pebbled vertices. When we talk about intervals, they are not overlapped. Note that, at the beginning of the game, there is only one interval $[(0, i), (\gamma_{m-\xi}^* - 1, i)]$ for the $i$-th row.\footnote{\textsuperscript{31} Such “interval” does not really exists since there is no pebble or delimiter that marks its boundary. It is an imaginary interval that exists in Duplicator’s mind.} In each round of the game, Duplicator ensures that $\mathfrak{A}_{k,m}^{(\xi)}$ and $\mathfrak{B}_{k,m}^{(\xi)}$ have the same number of intervals in the same row. And if Spoiler
puts a pebble in the $j^*$-th interval of a row, so does Duplicator in the other structure in the same row, for any $j^*$.

In the following we introduce the basic ideas that will be used to deal with the linear orders. By a folklore knowledge (cf. Remark 15), we know that it is impossible for Spoiler to find the difference between two linear orders if their lengths are large enough.

**Fact 6.** For any $m \geq m' \geq 0$, if $\mathcal{O}_a, \mathcal{O}_b$ are linear orders of length greater than or equal to $2^m - 1$, then $\mathcal{O}_a \equiv_m^m \mathcal{O}_b$.

Let $\ell_c$ be some number in $[1, m]$. In the $\ell_c$-th round of the game over abstractions where $\xi > m - \ell_c$, recall that all the picked vertices are “projected” in $X_\xi^*$, and assume that Spoiler “picks” vertex $(c, y)$ in the interval $[(a, y), (b, y)]$, where $c = \langle x \rangle_\xi$ and $(x, y)$ is the actually picked vertex in the associated game, thus splitting the interval $[(a, y), (b, y)]$ into two smaller intervals $[(a, y), (c, y)]$ and $[(c, y), (b, y)]$. And assume that the corresponding interval in the other structure is $[(a', y), (b', y)]$. Note that all these vertices mentioned, e.g. $(a, y)$, are in $X_\xi^*$. Let $l_\xi := \mathcal{U}_\xi \cdot \beta_{m-\xi}$. If we regard every $l_\xi$ successive vertices as one object, we get a linear order $\preceq^\xi$ induced from the original linear orders of the structures: $u \preceq^\xi u'$ if and only if $[u/l_\xi] \leq [u'/l_\xi]$, for any $u, u'$ in $X_\xi^*$. In the $\ell_c$-th round, Duplicator picks $(x', y)$ to respond Spoiler. Let $c' := \langle x' \rangle_\xi$. Duplicator needs to ensure that $(c', y)$ is in the interval $[(a', y), (b', y)]$ such that the following condition, called abstraction-order-condition, holds: for any $1 \leq i \leq \xi$,

a) if $0 < [c/l_i] - [a/l_i] < 2^{m-\ell_c} - 1$ then $[c'/l_i] - [a'/l_i] = [c/l_i] - [a/l_i]$; otherwise,

b) if $0 < [b/l_i] - [c/l_i] < 2^{m-\ell_c} - 1$ then $[b'/l_i] - [c'/l_i] = [b/l_i] - [c/l_i]$; otherwise,

c) if $[c/l_i] - [a/l_i] = 0$ or $[b/l_i] - [c/l_i] = 0$, then $[c]_i - [a]_i = [c']_i - [a']_i$ or $[b]_i - [c]_i = [b']_i - [c']_i$; respectively; otherwise,

d) $[c'/l_i] - [a'/l_i] \geq 2^{m-\ell_c} - 1$ and $[b'/l_i] - [c'/l_i] \geq 2^{m-\ell_c} - 1$.

Note that, this strategy implies that, if Spoiler puts a pebble on the vertex that is already pebbled, so does Duplicator; and if Spoiler picks a new vertex, so does Duplicator in the other structure. We call $[[c/l_\xi], [a/l_\xi]]$ “unabridged interval” in the $\xi$-th abstraction, for any $(c, y), (a, y) \in X_\xi^*$. Note that, such concepts like induced linear orders and unabridged intervals allow Duplicator takes care of the linear orders first, meanwhile leave space for the considerations for the partial isomorphism issue with respect to edges.
If the abstraction-order-condition can be preserved, then Duplicator can
win the game over the pair of (pure) induced linear orders. If this require-
ment cannot be satisfied, then Duplicator looks for the \( (\xi-1) \)-th abstraction
for a solution: assume that the length of any unabridged interval in the \( \xi \)-th
abstractions, e.g. \([c/l_\xi], [a/l_\xi] \), is at least 1, then the following always
holds:

\[
[c'/l_{\xi-1}] - [a'/l_{\xi-1}] \geq 2^{m-\ell_c};
[b'/l_{\xi-1}] - [c'/l_{\xi-1}] \geq 2^{m-\ell_c}.
\]  (6.2)

By Fact 6, since the linear orders are large enough now, it allows Duplicator
to respond Spoiler properly for one more round, at the price that
\( \xi \) decreases
by 1. By Lemma 16, if distinct pebbles are put on distinct objects of the
induced linear orders, then the orders are preserved in the lower abstractions.
In Remark 46, we discuss the order issue in more detail along this line.

However, Spoiler still has a way to win the game via linear orders, if
there are two pairs of vertices \((u_1, v), (u_1', v), (u_2, v), (u_2', v)\) in the \( \xi \)-th
abstraction of the structures, such that

- \((u_1, v) \models (u_1', v)\) and \((u_2, v) \models (u_2', v)\);
- \([u_1/l_\xi] = [u_2/l_\xi]\);
- \(u_1 < u_2 \iff u_2' < u_1'\).  (\(\ast\))

We use “virtual games” to denote the kinds of pebble games wherein
the players play the games in their mind (following the usual rules) without
really putting pebbles on the board. Now we introduce a sort of imaginary
games over changing boards. That is, the game board can be different
in each round. Certainly, this is not a precise definition for it doesn’t tell
us how the game board changes. In the following we introduce a specific
kind of such games. Moreover, they are a sort of virtual games. Recall that,
in \( A_{k,m} \) and \( B_{k,m} \), every vertex is associated with a board history, which is
supposed to reflect reasonable “evolution logic” of the game. If Spoiler picks
\((x, y)\) in a structure, say \( A_{k,m} \), Duplicator uses virtual games to determine
the board history of the vertex \((x', y)\) she is going to pick. Note that, from
here on we assume that \((x, y), (x', y) \in X_1\) (in this proof). For simplicity,
here we assume that no vertex is pebbled before Spoiler picking \((x, y)\).  

\[\text{Recall that } (a, y), (c, y), (a', y), (c', y) \in X_1. \text{ Because } [c/l_\xi] \neq [a/l_\xi], \text{ by induction hypothesis we have } [c'/l_\xi] \neq [a'/l_\xi]. \text{ Note that } \xi > m - \ell_c. \text{ It implies that } [c'/l_{\xi-1}] - [a'/l_{\xi-1}] \geq \frac{m-\xi+1}{\ell_{\xi-1}} = 2^{\xi-1} \geq 2^{m-\ell_c}.\]

\[\text{In case when some vertices are pebbled, the reader can cf. the proof of Claim 3 because we need to take account of the order of the board histories of pebbled vertices when playing the virtual games.}\]
virtual game in such a simple setting consists of $i_{x,y}^j - 1$ “virtual rounds”. No vertex is pebbled at the beginning of this virtual game. Spoiler “picks” according to $\chi(x,y)\upharpoonright BH(j)$, for $j = 1$ to $i_{x,y}^j - 1$, and Duplicator “replies” in the other structure, i.e. in $\mathfrak{B}_{k,m}$. In the following we define the virtual game board at the beginning of the $j$-th round, for any $j$ in $[1, i_{x,y}^j]$. Let $\mathbb{Z}_{xy}^j := \chi(x,y)\upharpoonright BH(j - 1)$ and $\mathbb{Z}_{x'y}^j := \chi(x',y)\upharpoonright BH(j - 1)$. Firstly, for any vertex $(e,f) \in \mathcal{X}_1^j$, $\mathbf{I}_j^j(e,f) = \mathbb{Z}_{xy}^j$ if $(e,f) \notin \mathbb{Z}_{xy}^j - \{(*,*)\}$; otherwise, $\mathbf{I}_j^j(e,f) = \mathbb{Z}_{xy}^\ell$ where $\ell = \max\{i \in [2,j] \mid (e,f) \in \mathbb{Z}_{xy}^i \land (e,f) \notin \mathbb{Z}_{xy}^{i-1}\}$. Similarly, $\mathbf{J}_j^j(e,f) = \mathbb{Z}_{xy}^j$ if $(e,f) \notin \mathbb{Z}_{xy}^j - \{(*,*)\}$; otherwise, $\mathbf{J}_j^j(e,f) = \mathbb{Z}_{x'y}^\ell$ where $\ell = \max\{i \in [2,j] \mid (e,f) \in \mathbb{Z}_{x'y}^i \land (e,f) \notin \mathbb{Z}_{x'y}^{i-1}\}$. For example, $\mathbf{I}_1^1(e,f) = \mathbf{I}_1^j(e,f) = BC_{\emptyset}$, for any $(e,f) \in \mathcal{X}_1^1$. At the start of the $j$-th round, the game board is $((\mathcal{A}_{k,m}^j,\mathbb{Z}_{xy}^j), (\mathcal{B}_{k,m}^j,\mathbb{Z}_{x'y}^j))$. Duplicator’s “virtual responses” determine the board history of the vertex that she should actually pick. Recall that the players do not really use pebbles in such virtual games. The strategy Duplicator uses in such virtual games will be introduced soon, cf. Strategy 1 - Strategy 3

Proof. We prove that Duplicator has a winning strategy in an $m$-round $(k-1)$-pebble game. Let $\overline{A}$ be the set of pebbled vertices in $\mathcal{A}_{k,m}$ at the start of the current round and $\overline{B}$ be the set of pebbled vertices in $\mathcal{B}_{k,m}$, both in the natural order. As explained in Fact 1 we assume that the lengths of $\overline{A}$ and $\overline{B}$ are less than or equal to $k - 2$.

Soon, we shall define a condition (5) in page 69. Say that the game (and the board) is over the $i$-th abstraction, if (5) holds for any pair of pebbled vertices, e.g. $(x^i,y) \vdash (x^\emptyset,y)$, and the projections of pebbled vertices in the $i$-th abstraction define a partial isomorphism. And $\xi$ is the maximum number in $[1,m]$ that makes (5) hold at the end of the current round.

We use $\xi$ to remind Duplicator which abstraction of the structures she should take care of at the start of the current round. At the start of the game, let $\xi := m$.

And we use $\theta$ to record how many rounds are still available to Spoiler at the start of the current round. In other words, the current round is the $(m + 1 - \theta)$-th round. Let $\ell_c := m + 1 - \theta$. Note that $\theta := \theta - 1$ after each round by default.

From now on, in all the cases, we assume that Spoiler picks $(x,y)$ in $\mathfrak{A}_{k,m}$ such that $\text{idx}(x^\emptyset,y) = t$ for some $t \in [1,m]$. The case when he picks in $\mathfrak{B}_{k,m}$ is similar. Correspondingly, in all the cases, assume that Duplicator picks $(x',y)$ in $\mathfrak{B}_{k,m}$. Moreover, assume that $\text{idx}(x'^\emptyset,y) = t'$ for some $t' \in [1,m]$. 

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Firstly, we define some ordered sets that are used in defining Duplicator’s strategy. In the sequel we use $\overrightarrow{c}_A$ and $\overrightarrow{c}_B$ to denote the following (ordered) sets (in the natural picking order).

$$
\overrightarrow{c}_A := \{ (u, v) | (u, v) \in (x, y)|BC]; v \neq y \};
\overrightarrow{c}_B := \{ (u, v) | (u, v) \in (x', y)|BC]; v \neq y \}.
$$

(6.3)

Note that $|\overrightarrow{c}_A| = |\overrightarrow{c}_B| \leq k - 2$ because of Fact [1]. Now, we define the following ordered sets, whose orders inherit $\overrightarrow{c}_A$ and $\overrightarrow{c}_B$.

$$
Z := \{ (u, v) \in \overrightarrow{c}_A | \text{idx}(u, v) = q \geq t; \text{cc}([u]_t, v) = \text{cc}([x^b], y) ;
\text{either } (x^b, y) \text{ or } (u, v) \text{ is not a critical point of } \mathfrak{A}^{*}_{k, m} \} \cup
\{ (u, v) \in \overrightarrow{c}_A | \text{idx}(u, v) = q < t; \text{cc}([u]_q, v) = \text{cc}([x^b], y) \};
$$

$$
Z^{\geq \xi} := Z \cap X^{\xi}; Z^{< \xi} := Z - Z^{\geq \xi};
$$

$$
U := \{ (u, v) \in \overrightarrow{c}_A | y, v \in [1, k - 2]; \text{idx}(u, v) = t; (\text{cc}([u]_t, v) - \text{cc}([x^b], y)) \times
(v - y) \times (-1)^{\text{BIT}(\text{SW}((u, v), (x^b, y)), q(v, y)) < 0}\}
$$

$$
R := \{ (u, v) \in \overrightarrow{c}_A | \text{cl}(u, v) \in \chi(x, y)|S\};
\text{R}^{\geq \xi} := R \cap X^{\xi}; R^{< \xi} := R - R^{\geq \xi}.
$$

$$
D := \{ (u, v) \in \overrightarrow{c}_A | (u, v) \in \chi(x^b, y)| \Omega \text{ or } (x^b, y) \in \chi(u, v)| \Omega \};
\text{D}^{\geq \xi} := D \cap X^{\xi}; D^{< \xi} := D - D^{\geq \xi};
$$

$$
T := \{ (u, v) \in \overrightarrow{c}_A | v \text{ or } y \in \{0, k - 1\}; v \neq y; \text{sgn}((u, v), (x^b, y)) = 1\}.
$$

Correspondingly,

$$
Z' := \{ (u, v) \in \overrightarrow{c}_B | \text{idx}(u, v) = q < t'; \text{cc}([u]_q, v) = \text{cc}([x^b], y) \} \cup
\{ (u, v) \in \overrightarrow{c}_B | \text{idx}(u, v) = q \geq t'; \text{cc}([u]_t', v) = \text{cc}([x^b], y) \};
$$

$$
Z'^{\geq \xi} := Z' \cap X^{\xi}; Z'^{< \xi} := Z' - Z'^{\geq \xi}.
$$

Similarly, if we replace $\overrightarrow{c}_A$, $t$ and $x$ by $\overrightarrow{c}_B$, $t'$ and $x'$ respectively, we obtain $U'$, $R'$, $D'$, $T'$ and $R'^{\geq \xi}$ etc.

For each set $X$ that is just defined, we define an associated set $X^{(\xi)}$ by substituting $u$ with $\langle u \rangle_\xi$ and $x$ with $\langle x \rangle_\xi$. For example, we define $D^{(\xi)}$ as the following set.

$$
D^{(\xi)} := \{ \langle (u, v) \rangle_\xi | (u, v) \in \overrightarrow{c}_A; \langle (u, v) \rangle_\xi \in \chi(\langle x^b \rangle_\xi)| \Omega \text{ or } (\langle x^b \rangle_\xi, y) \in \chi(\langle u \rangle_\xi, v)| \Omega \}.
$$

The definition is equivalent if we replace $\chi(x, y)|S$ by $\chi(x^b, y)|S$. 
}

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Similarly, we can define \( Z^{(\xi)} \), \( U^{(\xi)} \), \( R^{(\xi)} \), and \( T^{(\xi)} \). And correspondingly we can define the due sets \( Z^{(\xi)} \) etc.

Recall that \( \bar{c}_A \) is the set of pebbled vertices in \( \bar{v}_{k,m} \) at the start of the current round. Let

\[
\begin{align*}
\bar{c}_A := \{ (u,v) \in \bar{c}_A : (u,v) \rightarrow (x,y) \text{ or } (x,y) \rightarrow (u,v); v \neq y \}; \\
\bar{c}_B := \{ (u,v) \in \bar{c}_B : (u,v) \rightarrow (x',y) \text{ or } (x',y) \rightarrow (u,v); v \neq y \}. 
\end{align*}
\] (6.4)

Note that, by definition, \( (u,v) \rightarrow (x,y) \) implies that \( (u^\flat, v) \in (x,y) \)[BC]. Therefore, The set \( \{ (u^\flat, v) \mid (u,v) \in \bar{c}_A \} \) is a subset of \( \bar{c}_A \). Similarly, \( \bar{c}_B \) subsumes \( \{ (u^\flat, v) \mid (u,v) \in \bar{c}_B \} \).

For any set \( X \) in \( \{ Z, U, \ldots, T \} - \{ R \} \), we define a related set \( \bar{X} \), by substituting \( (u,v) \in \bar{c}_A \) with \( (u',v) \in \bar{c}_A \), and substituting \( (u,v) \) with \( (u^\flat,v) \) in the defining part (the statements behind "\( \rightarrow \)"), and substituting \( \bar{X} \) with \( \bar{X} \). For example, we define \( \bar{D} \) as the following set.

\[
\bar{D} := \{ (u,v) \in \bar{c}_A : (u,v) \rightarrow (x,y) \in \chi(x^\flat,y) \} \Omega \text{ or } (x^\flat,y) \in \chi(u^\flat,v) \} \Omega \}.
\]

And, \( \bar{D}^{2\xi} := \bar{D} \cap \bar{X}^{2\xi}; \bar{D}^{\leq \xi} := \bar{D} - \bar{D}^{2\xi} \).

In addition, we define \( \bar{R} \) as the following set.

\[
\bar{R} := \{ (u,v) \in \bar{c}_A : (x,y) \rightarrow (u,v) \text{ or } (u,v) \rightarrow (x,y) \}. 
\] (6.5)

Moreover, \( \bar{R}^{2\xi} := \bar{R} \cap \bar{X}^{2\xi}; \bar{R}^{\leq \xi} := \bar{R} - \bar{R}^{2\xi} \).

Correspondingly, we can define \( \bar{Z} \) etc.

We can also define \( \bar{Z}^{(\xi)} \) in the way like \( Z^{(\xi)} \).

Note that \( \bar{c}_A \Vdash \bar{c}_B \), where "\( \Vdash \)" is defined in page 6. Now we define a similar denotation "\( \Vdash \)" as the follows. For any vertex \( (u,v) \in \bar{X}_i \) picked in a game, \( (\|u\|_i,v) \Vdash_i (\|u^\flat\|_i,v) \) if \( (u,v) \Vdash (u',v) \). Similarly, for two sest \( X \) and \( X' \) of vertices we can define \( X \Vdash_i X' \) in the usual way.

Similar to the proof of Lemma 18, this proof is also by simultaneous induction, wherein we show that the follows are preserved after each round. Let \( S_i^A := \{ (u,v) \in \bar{c}_A \mid \chi(\|u\|_i,v) \in \chi(\|x\|_i,y) \} \} \) and \( S_i^B := \{ (u,v) \in \bar{c}_B \mid \chi(\|u\|_i,v) \in \chi(\|x\|_i,y) \} \} \).

(1°) \( \theta < \xi \) (after the first round);

(2°) The abstraction-order condition holds; moreover, Duplicator’s choice can prevent \( (\check{z}) \) from occurring (cf. page 65 for “(\check{z})”);
(3°) Duplicator can win this round in the corresponding associated games over the \( \xi \)-th abstractions. That is,

\[
Z^{(\xi)} \cup U^{(\xi)} \cup R^{(\xi)} \cup D^{(\xi)} \cup T^{(\xi)} \models_{\xi} Z'^{(\xi)} \cup U'^{(\xi)} \cup R'^{(\xi)} \cup D'^{(\xi)} \cup T'^{\xi};
\]

(4°) Duplicator’s choice makes the “boundary checkout strategy” (cf. p. [31]) ineffective in the next round. In other words, it means that the game board over the \((\xi - 1)\)-th abstraction is in partial isomorphism w.r.t. edges even if we take it as if the boundaries of rows of the \((\xi - 1)\)-th abstraction were occupied with extra immovable pebbles.

(5°) If \( t, t' < \xi - 1 \) then \( t \leq i < \xi \), on condition that \( \langle x^i \rangle_i \neq \langle x^i \rangle_{\xi} \),

(i) \( \text{idx}(\langle x^i \rangle_i, y) = \text{idx}(\langle x^o \rangle_i, y) \);

(ii) \( \text{cc}(\langle x^i \rangle_i, y) = \text{cc}(\langle x^o \rangle_i, y) \);

(iii) \( \langle \langle x^i \rangle_i \rangle_i - \langle \langle x^o \rangle_i \rangle_i = \langle \langle x^o \rangle_i \rangle_i - \langle \langle x^o \rangle_i \rangle_i \), \( \text{mod} \beta_{m-i-1} \);

\( \langle \langle x^i \rangle_i \rangle_i \leq \langle \langle x^o \rangle_i \rangle_i \); \( \text{iff} \langle \langle x^o \rangle_i \rangle_i \leq \langle \langle x^o \rangle_i \rangle_i \);

(iv) \( S_{t}^A \models S_{t}^B \);

(v) \( \text{RngNum}(\langle x^i \rangle_i, t) = \text{RngNum}(\langle x^o \rangle_i, t) \);

(vi) \( g(\langle x^i \rangle_i) = g(\langle x^o \rangle_i) \);

(vii) For any vertex \( (u, y) \in c_A \) and \( (u', y) \in c_B \) where \( (u, y) \models (u', y) \),

\( \langle \langle u^i \rangle_i \rangle_i = \langle \langle u^o \rangle_i \rangle_i \); \( \text{iff} \langle \langle u^o \rangle_i \rangle_i \leq \langle \langle u^o \rangle_i \rangle_i \);

(viii) \( \langle a', b \rangle \leq \langle \langle a \rangle_i \rangle_i \); \( \text{iff} (a, b) \leq \langle \langle a \rangle_i \rangle_i \).

(6°) The associated game board is still in partial isomorphism after picking \( (x^i, y) \) and \( (x^o, y) \). That is,

\[
Z \cup U \cup R \cup D \cup T \models Z' \cup U' \cup R' \cup D' \cup T'. \tag{6.6}
\]

\( \text{By definition of vertex index, we have} \ x^i - \langle x^i \rangle_{\xi} = x^o - \langle x^o \rangle_{\xi} = 0 \text{ if } t, t' \geq \xi; \)
\( x^i - \langle x^i \rangle_{t-1} = x^o - \langle x^o \rangle_{t-1} = 0 \text{ if } t = t' = \xi - 1. \)

\( \text{Note that it is equivalent to the condition that} \ x^i \leq \langle x^i \rangle_{\xi} \text{ iff } x^o \leq \langle x^o \rangle_{\xi}. \)

\( \text{That is,} \ (a', b) \text{ is the} \ j \text{-th item in the tuple encoding} \ \chi(\langle u^i \rangle_i, v) \mid S \text{ if} (a, b) \text{ is the} \ j \text{-th item in the tuple encoding} \ \chi(\langle u^i \rangle_i, v) \mid S. \text{ We shall see that it is possible because} \ \chi(\langle u^i \rangle_i, v) \mid S = \chi(\langle u^o \rangle_i, v) \mid S \text{ if} \ \text{idx}(u^i, v) < t < \xi - 1. \)
Call (1) a *winning-condition-set* of Duplicator. (2) and (3) ensure partial isomorphism in the games over the $\xi$-th abstraction (the former takes care of the linear order, while the latter takes care of edges). And we use (4) to ensure that, if the associated game board is in partial isomorphism over the $\xi$-th abstraction at the start of the current round, then it also holds over the $(\xi - 1)$-th abstraction (on condition that Duplicator uses the auxiliary games (6.12) in Strategy 1 and (6.21) in Strategy 2 and (6.23) in Strategy 3 cf. Remark 52). Note that (iii) of (4) implies that $x^b - \langle x^b \rangle^\xi \xi$ roughly equals $x^b - \langle x^b \rangle^\xi$. Hence (5) is also called the *approximate hr-copycat condition*. 

Later on, when we describe and discuss Duplicator’s strategy, we shall delay the discussion of (4) to the end. We treat it in this way to avoid unnecessary repetition of arguments.

Moreover, most of the time the readers can take $((\mathcal{A}_{k,m}, c_A), (\mathcal{B}_{k,m}, c_B))$ as the game board instead of $((\tilde{\mathcal{A}}_{k,m}, c_{\tilde{A}}), (\tilde{\mathcal{B}}_{k,m}, c_{\tilde{B}}))$. Only when we discuss (4) should we switch back to the real game board $((\mathcal{A}_{k,m}, c_A), (\mathcal{B}_{k,m}, c_B))$.

**Basis:** At the beginning of the game, $c_A$ and $c_B$ are empty. Hence $\tilde{c}_A$ and $\tilde{c}_B$ are empty. Recall that Spoiler picks $(x, y)$ and Duplicator replies $(x', y)$. In the first round, suppose that $\chi(x, y) | bc = 0$, Duplicator simply mimics Spoiler’s picking. Since the signature contains no unary relation symbol and the graphs contain no self-loop, the board is in partial isomorphism and Duplicator wins the first round. The value of $\xi$ is still $m$ at the end of the first round. But $\theta := \theta - 1 = m - 1$. Hence the winning-condition-set can be ensured.

In this situation Duplicator is a copycat. Certainly she is also an exact hr-copycat (recall p. 27 for the notion “hr-copycat”). Note that this simple copycat strategy also works in the (associated) games over changing boards wherein a game board constitutes a pair of “flat” structures which have no pebbled vertices at the start of the first round.

However, Spoiler can “cheat” by picking the first vertex with an arbitrary board history at the beginning. In this case, Duplicator is no more an exact hr-copycat. Instead, using Strategy 1/Strategy 3 which will be introduced soon, Duplicator first plays a *virtual game* that determines the board history of $(x', y)$ (cf. page 66); then she replies Spoiler in the associated structures, using Strategy 1/Strategy 3.

**Induction Step:** Assume that Duplicator wins the first $m - \theta$ rounds.  

\[38\text{It is because a unit of difference in higher abstraction means a huge difference in lower abstractions. This observation is also used in the proof of Lemma 60.}\]
We prove that she can also win the \((m - \theta + 1)\)-th round, and the winning-condition-set is preserved. In all the cases, soon we shall see that Duplicator follows some supplementary basic strategies. Henceforth, we always assume that \((u, v) \in \tau_A\) and \((u, v) \vdash (u', v)\).

B-1 Duplicator gives the abstraction-order-condition the highest priority.\(^{39}\)

B-2 Duplicator always ensures that
\[
cl(u, v) \neq cl(x, y) \iff cl(u', v) \neq cl(x', y).
\]

B-3 Duplicator uses virtual games to determine the board histories of her picked vertices.

Note that Duplicator can ensure B-1 because of (6.2). Based on it, Duplicator can know the approximate position for her pick. Confer Remark 50 for the reason that Duplicator should follow B-2.

We are able to prove the following claim: if an edge is forbidden in one structure due to discontinuities, so is the corresponding edge in the other structure.

**Claim 3.** If Duplicator stick to B-3, she has a way to ensure that, for any \((u, v), (u', v)\) where \((u, v) \vdash (u', v)\),

(i) \( (u, v) \rightarrow (x, y) \) if and only if \((u', v) \rightarrow (x', y)\).

(ii) If \(y = v\), then \(\lfloor x/(\gamma_{m-1}^* \times k) \rfloor \mod bh^\# \leq \lfloor u/(\gamma_{m-1}^* \times k) \rfloor \mod bh^\# \) if and only if \(\lfloor x'/\gamma_{m-1}^* \times k \rfloor \mod bh^\# \leq \lfloor u'/\gamma_{m-1}^* \times k \rfloor \mod bh^\#\).

Claim (i) says that the board history of \((x, y)\) is less than (or equals to) that of \((u, v)\) (in the order explained in page 56) if and only if the board history of \((x', y)\) is less than (or equals to) that of \((u', v)\). Note that in the case when equal holds, the order of the picked vertices is taken care of by the abstraction-order-condition introduced before. Moreover, from the proof of Claim 3, we can see that board histories of pebbled vertices satisfy(apx-1) (cf. Remark 45) if we regard board histories as objects of

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\(^{39}\)Duplicator first finds the “allowed” positions for picking in accordance with the abstraction-order-condition. Usually such positions are assembled into intervals rather than isolated points. Generally such intervals have vertices of all the necessary type labels. Hence, Duplicator can first use it to determine the approximate position for her pick, then selects a vertex of appropriate type label using Strategy 1/Strategy 3, which will be introduced soon.
the linear orders. Hence, for example, we know that $[x'/(\gamma_{m-1}^* \times k)] \mod bh^# = [u'/(\gamma_{m-1}^* \times k)] + 1 \mod bh^#$ if and only if $[x/(\gamma_{m-1}^* \times k)] \mod bh^# = [u/(\gamma_{m-1}^* \times k)] + 1 \mod bh^#$. Because of Claim 3 (i), we can safely assume that $(x, y)[BC]$ is a valid board configuration, for otherwise $(x, y)$ and $(x', y)$ are isolated vertices in respective structures; and we need only focus on the set of vertices that are in continuity with $(x, y)$ and $(x', y)$, in the following case by case discussion of Duplicator’s strategy. Note that, by this claim, $\widetilde{c}_A \models \widetilde{c}_B$. Moreover, $\widetilde{c}_A$ has $n$ distinctive vertices if and only if $\widetilde{c}_B$ does, where $0 \leq n \leq k - 1$.

We describe Duplicator’s strategy using simultaneous induction, in addition to B-1~B-3. But instead of studying the game directly, it is more convenient to study the game by a specific associated game over changing board wherein the game board is $(\widetilde{\mathfrak{A}}_{k,m}, \widetilde{\mathfrak{B}}_{k,m})$ at the start. Recall that $\mathfrak{I}^\ell_c(e, f) = \mathfrak{I}^\ell_c(e, f) = BC_\gamma$, for any $(e, f) \in \mathfrak{X}^2_1$. Cf. page 66 for the definition of $\mathfrak{I}^\ell_c$ and $\mathfrak{I}^\ell_b$. The game board at the start of the $\ell_c$-th round consists of a pair of structures $\mathfrak{A}_{k,m}$ and $\mathfrak{B}_{k,m}$; in this round the players pick $(x^\dagger, y) \in \mathfrak{X}^1_1$ in the associated game $\mathcal{C}_{k-1}^{\ell}((\mathfrak{A}_{k,m}, (x, y))[BC]), (\mathfrak{B}_{k,m}, (x', y))[BC]$. If they pick $(x, y)$ in the original game $\mathcal{C}_{k-1}^{\ell}((\mathfrak{A}_{k,m}, (x, y)), (\mathfrak{B}_{k,m}, (x', y))[BC])$, they pick $(x, y)$ in the original game $\mathcal{C}_{k-1}^{\ell}((\mathfrak{A}_{k,m}, (x, y)), (\mathfrak{B}_{k,m}, (x', y))[BC])$.

Claim 3 (i) tells us that we don’t have to consider the board histories when we study the original game restricted to edges; and Claim 3 (ii) tells us that we don’t have to be worry about the order issue if we ignore the board histories when we study the original game restricted to linear orders, which is relatively clear. It implies that it is possible for Duplicator to have a winning strategy in the original game $\mathcal{C}_m^{k-1}((\mathfrak{A}_{k,m}, (x, y)), (\mathfrak{B}_{k,m}, (x', y))[BC])$ if she has a winning strategy in the associated game over changing board. Soon we shall see the justification in Strategy 1. Therefore, we can talk about something like “Spoiler picks $(x^\dagger, y)$ in $\mathfrak{A}_{k,m}$” instead of “Spoiler picks $(x, y)$ in $\mathfrak{A}_{k,m}$”. And we shall see that the following strategy is a winning strategy for Duplicator in the specific associated game over changing board. Note that, games over changing boards are a sort of auxiliary games. That is, we do not play them alone. But they are the basis of the original game and the virtual games. We reduce the original game to the corresponding associated game and virtual games over changing board. And in both of these two kinds of imaginary games, Duplicator relies on the following strategy, i.e. Strategy 1. And we shall see the justification in Strategy 1. Note that, in the virtual games for bord histories, the imaginary “pebbled” vertices (not the vertices really picked) form the game configuration associated with the vertex to be “picked” in the virtual round.
Note that, by induction hypothesis, the game board is in partial isomorphism over the $\xi$-th abstraction at the start of the current round. We can show that this also holds over the $(\xi - 1)$-th abstraction.

The readers can choose to read a brief sketch (Remark 49, appendix) of the following strategy before diving directly into the details.

In the following, “strategy i” stands for the shorthand of “strategy for the case i”. As we have discussed the order issue in details in Remark 46, which will be used by Duplicator to stick to B-1, we use Strategy 1 kidnapped Strategy 3 explained in the following, to mainly determine the type label of $(x', y)$, i.e. how this vertex is adjacent to other vertices.

1. (Strategy 1) Suppose that Spoiler picks a vertex $(x, y) \in \mathcal{X}_\xi^A$ (recall that $\text{idx}(x^\flat, y) = t$; in other words, we assume $t \geq \xi$ in this case). And suppose that Duplicator can pick $(x', y) \in \mathcal{X}_\xi^B$ such that $(2\Diamond), (3\Diamond), (4\Diamond)$, and $(6\Diamond)$ hold.

Note that, the set $\widetilde{Z} \cup \widetilde{U} \cup \widetilde{R} \cup \widetilde{D} \cup \widetilde{T}$ is precisely the set of vertices that are not adjacent to $(x, y)$ in the current round, if do not consider the missing of edges due to 2) b) of Definition 27. Indeed, by Claim 3, we don’t have to consider it.

In the following we explain why $(3\Diamond)$ and $(6\Diamond)$ matter.

If $(x, y) \rightarrow (u, v)$ for some pebbled vertex $(u, v)$, then Duplicator simply let $(x', y)$ be $(u, v)|_{\mathcal{X}_\xi^A\upharpoonright_h}$, and we are able to show that $(6\Diamond)$ and $(3\Diamond)$ hold (cf. Remark 51). Henceforth, we assume that $\neg((x, y) \rightarrow (u, v))$ holds for any pebbled $(u, v)$. By Claim 3, $\neg((x', y) \rightarrow (u', v))$ holds for any pebbled $(u', v)$. That is, one of the following two cases holds:

(i) $(u, v) \rightarrow (x, y)$: by Claim 3, $(u', v) \rightarrow (x', y)$;

(ii) $\neg((u, v) \rightarrow (x, y))$: by Claim 3, $\neg((u', v) \rightarrow (x', y))$, which implies that $((x, y), (u, v)) \notin E_A \land ((x', y), (u', v)) \notin E_B$, due to 2) b) of Definition 27.

Therefore, we need only consider (i). As a consequence, henceforth we can assume that $\neg((u, v) \sim (x, y)) \land \neg((u', v) \sim (x', y))$ holds. That is, in this case

$$\widetilde{R}^{<\xi} = \widetilde{R}'^{<\xi} = \emptyset. \quad (6.8)$$

\[\text{Cf. Remark 52. The readers are suggested to read the arguments a bit later, i.e. after reading Strategy 3. It implies that (6\Diamond) is entailed by the other conditions, in particular (3\Diamond).}\]
It implies that we need only consider the case that
\[ \tilde{R} = \tilde{R}^{\geq \xi} = \{(u, v) \in \tilde{c}_A \mid (x, y) \leadsto (u, v)\}. \] (6.9)

That is, in this case (i), \( \tilde{R} = \{(u, v) \in \tilde{c}_A \mid (u, v) \leadsto (x, y) \land \text{cl}(u, v) \in \chi(x, y) | S\} \).

Therefore, by Claim 3, Duplicator need only consider (6.9) in the associated game to ensure the following in the original game, in the case (i):
\[ \tilde{Z} \cup \tilde{U} \cup \tilde{R} \cup \tilde{T} \models \tilde{Z}' \cup \tilde{U}' \cup \tilde{R}' \cup \tilde{T}'. \] (6.10)

In other words, if Duplicator can ensure (6.9), then Spoiler cannot win this round in the original game. Similarly, if Duplicator can ensure (6.9'), then Spoiler cannot win this round over the \( \xi \)-th abstraction in the original game, which in turn helps to ensure (6.10).

By convention, we use \( \circ \) to denote the concatenation of two tuples. Let \( \tilde{c}_A^{\xi} \) be an ordered set defined as the following:
\[ \tilde{c}_A^{\xi} := \{([a]_\xi, b) \mid (a, b) \in (x, y)[BC]\} \circ \{([e]_\xi, f) \mid (a, b) \in (x, y)[BC]; \ \text{idx}(a, b) < \xi; (e, f) \in \chi(a, b) | S\}. \] (6.11)

Likewise, \( \tilde{c}_B^{\xi} \) is defined in the similar way except that \( (x, y)[BC] \) is replaced by \( (x', y)[BC] \). The elements of \( \tilde{c}_A^{\xi} \) and \( \tilde{c}_B^{\xi} \) are in the natural order inheriting from \( (x, y)[BC] \) and \( (x', y)[BC] \). Note that \( |\tilde{c}_A^{\xi}| = |\tilde{c}_B^{\xi}| < m \). Therefore, by Remark 45, Duplicator can ensure that she wins the following one round game wherein she picks \( (x', y) \) to respond to Spoiler’s pick of \( (x, y) \), in accordance with the abstraction-order-condition:
\[ \forall_1((\tilde{R}^{*}_{k, m}[\tilde{X}^*_\xi]), (\tilde{c}_A^{\xi}), (\tilde{R}^{*}_{k, m}[\tilde{X}^*_\xi]), (\tilde{c}_B^{\xi})). \] (6.12)

Furthermore, \( \theta := \theta - 1 \) at the end of this round, and \( \xi \) remains unchanged. That is, (1) holds.

If Duplicator’s pick can ensure (2), (3'), (1'), and (6'), then she not only wins this round, but also wins it over the \( \xi \)-th abstraction. Otherwise, Duplicator uses Strategy 2 if she cannot ensure these conditions.

Since \( \text{idx}(x, y) \geq \xi \), it is easy to see that \( \{(u, v) \in \tilde{c}_A \mid (x, y) \leadsto (u, v)\} \subseteq \tilde{R}^{\geq \xi} \) because \( \text{cl}(u, v) \in \chi(x, y) | S \) implies that \( \text{idx}(u, v) > \text{idx}(x, y) \geq \xi \). On the other hand, for any \( (u, v) \in \tilde{c}_A \), \( (u, v) \leadsto (x, y) \) would imply that \( (x, y) \leadsto (u, v) \).
2. (Strategy 2) Spoiler picks a vertex in $X^A_\xi$ and Duplicator cannot pick an appropriate vertex in the $\xi$-th abstraction that satisfies (2\(^{3}\)) and (3\(^{3}\)). It implies that $cl(x,y) \neq cl(u,v)$ for any pebbled $(u,v)$ in $A_{\xi,m}$, for otherwise Duplicator can simply follow B-2, and the conditions (2\(^{3}\)), (3\(^{3}\)) and (6\(^{3}\)) can be ensured. In such a case, Duplicator picks a vertex $(x',y) \in X_{\xi-1} - X_\xi$ s.t. $[x^0]_{\xi-1} \equiv 0 \pmod{m}$ (hence B-2 is followed), and resorts to the $(\xi-1)$-th abstraction for a solution. Note that, by definition,
\[
R^{<\xi-1} = R^{<\xi-1} = \emptyset. \tag{6.13}
\]

In the following we show that Duplicator can ensure (3\(^{3}\)) and (6\(^{3}\)). Let $M_A := Z^{(\xi-1)} \cup R^{(\xi-1)} \cup D^{(\xi-1)} \cup U^{(\xi-1)} \cup T^{(\xi-1)}$ and $M_B$ be such a set that $M_A \parallel_{\xi-1} M_B$. \(^{42}\) Note that, in this case where $\text{idx}(x',y) \geq \xi$, $M_A = Z^{(\xi-1)} \cup R^{(\xi-1)} \cup D^{(\xi-1)} \cup U \cup T^{(\xi-1)}$, because $\text{idx}(u^0, v) = \xi$ implies that $\text{idx}(u^0, v) = \xi$.

(2.1) By Lemma 35 Duplicator can ensure that $U' = \emptyset$.

(2.2) By virtue of Lemma 36 Duplicator can ensure that:
\begin{itemize}
  \item $Z^{(\xi)} \cup R^{(\xi)} \cup D^{(\xi)} = \emptyset$,
  \item $[x^0]_{\xi-1-1} \equiv 0 \pmod{m}$,
  \item $\text{RngNum}(x^0, \xi-1) = -1$.
\end{itemize}

Soon we shall see in Strategy 3 that (5\(^{3}\)) is ensured. Then by Lemma 58 and Lemma 60 we have
\[
Z^{(\xi-1)} \cup R^{(\xi-1)} \cup D^{(\xi-1)} = \emptyset. \tag{6.14}
\]

By Lemma 10, $\text{idx}(u^0, v) \geq \xi - 1$. If $\text{idx}(u^0, v) = \xi - 1$, then $T' = \emptyset$, by definition. Now assume that $\text{idx}(u^0, v) > \xi$.

\(^{42}\) In other words, $M_D = \{(u^0,_{\xi-1}, b) \mid (a, b) \in C_\xi; (a, b) \parallel (a', b); ((a)_\xi, b) \in M_A\}$.

\(^{43}\) By Lemma 8 if $\text{idx}(u^0, v) = \xi - 1$, then $\text{idx}(u^0, v) = \xi - 1$; if $\text{idx}(u^0, v) > \xi$, then $\text{idx}(u^0, v) = \text{idx}(u^0, v) > \xi$.

\(^{44}\) Note that the above holds only when all the pebbled vertices are in different rows. If some of them are in the same row, then Duplicator need to resort to the $(\xi - 2)$-th abstraction due to (3) of Lemma 36 which seems that $2m$ abstractions are needed for a structure instead of $m$ abstractions. However, even in this case, we can show that $m$ abstractions suffice for our purpose. The simplest way for Duplicator is to ensure that all the pebbled vertices in the same row have distinct indices. This is possible if she always resorts to the $(\xi - 1)$-th abstraction whenever the row, in which she is going to put a pebble, already has a pebble. Note that in such case (3) of Lemma 36 can be revised s.t. “$t - 2$” (“$t - 1$” resp.) is replaced by $t - 1$ ($t$ resp.), and the argument for it is similar to (4) of Lemma 36.
By Lemma $21$, we have $\text{RngNum}(\{u^\flat\}_{x-1}, x - 1) = -1$. Therefore, by Lemma $22$, $T' = \emptyset$ since $\text{RngNum}(x^\flat, x - 1) = -1$. Therefore, we have

$$Z'\geq x - 1 \cup R'\geq x - 1 \cup D'\geq x - 1 \cup T'\geq x - 1 = \emptyset. \quad (6.15)$$

(2.3) So far, the vertex Duplicator selected usually does not satisfy $\ast$. Therefore, Duplicator need fine-tune $(x', y)$ a little bit: rename the currently selected vertex as $(x^\ast, y)$, and find a new value for $x'$ such that $\ast$ holds. By Lemma $31$, for any $S' \in \varnothing(C_{x - 1})$ and any $(x^\ast, y) \in X_{x - 1}$, the sequence of $U_{x - 1}$ successive vertices $([x^\ast], y)_{x - 1}$ contains at least one vertex $(x^\flat, y)$ where $\text{idx}(x^\flat, y) = x - 1$ and $\chi(x^\flat) \mid S' \ni \{\text{cl}(\{a\}_B, v) \mid (a, b) \in E^B_B\} = S'$. Hence Duplicator can simply pick $(x', y) = (x^\flat, y)$ to ensure that

$$S' = \{\text{cl}(e, f) \mid (e, f) \in M_B\}. \quad (6.16)$$

By $(6.16)$, for any $(a, b) \in M_A$ and $(a, b) \mid x - 1 (c, d), ((a, b), (x^\ast, y)) \notin E^B$ if and only if $((c, d), (x^\flat, y)) \notin E^B$. In other words, $\ast$ holds if we replace all the $(\xi)$ by $(\xi - 1)$ in the superscripts of the sets. Therefore,

$$Z'\geq x - 1 \cup U' \cup R'\geq x - 1 \cup D'\geq x - 1 \cup T'\geq x - 1 \mid x - 1 \cup D'\geq x - 1 \cup T'\geq x - 1. \quad (6.17)$$

(2.4) Because of Lemma $14$ and $(\xi)$ (ii),

$$Z'\leq x - 1 = Z'\leq x - 1. \quad (6.18)$$

(2.5) Because of Strategy 3, Duplicator is approximately a hr-copycat (cf. $(\xi)$ (iii)). By Lemma $59$, for any $(u, v) \in C_{x - 1}$ where $\text{idx}(u^\flat, v) = i^\flat \leq x - 1$, $[u^\flat]_i^{x - 1} = [u^\flat]_i^{x - 1}$. By Strategy 3, $\text{idx}(u^\flat, v) = i^\flat$ and $\text{cc}(u^\flat)_1, v) = \text{cc}(u^\flat)_1, v)$; by Lemma $14$, $[x^\flat]_i = [x^\flat]_i \equiv (\text{mod } k - 1)$ for any $i < x - 1$; then by Lemma $60$ and the definition of “$\Omega$”, missing of an edge between $(x^\flat, y)$ and $(\{u^\flat\}_{x - 1}, v)$, if there is one, would propagate downward to lower abstractions coincidently with missing of an edge between $(x^\flat, y)$ and $(\{u^\flat\}_{x - 1}, v)$ that behaves alike. Therefore, we have

$$D'\leq x - 1 = D'\leq x - 1. \quad (6.19)$$
Because of (5) (v), for any pair of pebbled vertices \((u, v) \vdash (u', v),\) where \(i^* = \text{idx}(u, v) = \text{idx}(u', v) < \xi - 1,\) we have 
\[ \text{RngNum}(u^b, i^*) = \text{RngNum}(u'^b, i^*). \]
Recall that \(\text{idx}(x^b, y) \geq \xi > \text{idx}(x'^b, y) = \xi - 1.\) Therefore, by definition,
\[ T < \xi - 1 \vdash T' < \xi - 1. \] (6.20)

Putting the observations, i.e. (6.17), (6.8), (6.18), (6.19), and (6.20), all together, it implies that (6) holds for this newly selected vertex \((x', y).\) After this round, \(\xi := \xi - 1\) and \(\theta := \theta - 1.\) Hence (1) still holds. Because Duplicator resorts to lower abstraction, thereby the first part of (2) holds, due to (6.2). And (\(\kappa)\) will not occur. Suppose on the contrary that it occurs, and \((x_0, y) \vdash (x'_0, y)\) are the pair of vertices that make it happen. Note that \(\text{idx}(x_0, y), \text{idx}(x'_0, y) < \xi.\) If \(x^b \neq \langle x'_0 \rangle_{\xi},\) then, by induction hypothesis, Duplicator can choose to pick \((x', y)\) such that \(\langle x' \rangle_{\xi} \neq \langle x'_0 \rangle_{\xi}.\) Hence \([x^b / \ell_{\xi - 1}] \neq [x'_0 / \ell_{\xi - 1}].\) A contradiction occurs. If \(x^b = \langle x'_0 \rangle_{\xi},\) then Duplicator simply pick \((x', y)\) such that \(x^b = \langle x'_0 \rangle_{\xi}\) and she wins this round because she wins the last round by induction hypothesis. But in this case \((x^b, y)\) is a vertex in \(X^*_\xi.\) That is, she doesn’t have to resort to the \((\xi - 1)\)-th abstraction for a solution, thereby no need to use Strategy 2. We arrive at a contradiction again. In short, Duplicator can ensure that (1) hold in this case.

In the arguments, we haven’t taken the boundary vertices into account yet, which is mainly handled in the discussion of (4). Recall that we delay such discussion to the end of this proof. Here we only mention one thing. The decision of \((x^b, y)|S\) has not considered the boundary vertices, despite that it should. Nevertheless, it is not a big issue, because Duplicator can adapt her pick in the following simple way when necessary: if the projection of a boundary vertex in the \((\xi - 1)\)-th abstraction is in \((x^b, y)|S,\) then she add it in \((x^b, y)|S\) too. Note that Duplicator has the freedom to do it.

Last but not least, the following is easy to observe: Duplicator can win the following one round game wherein she picks \((x^b, y)\) to reply the pick of \((x', y),\) in accordance with the abstraction-order-condition:
\[ \bigcup_1 \left( (\mathfrak{A}_{k,m}[X^*_{\xi - 1}])(\leq), \overline{c_{A^*_{\xi}}}^{-1}, \right), (\mathfrak{B}_{k,m}[X^*_{\xi - 1}])(\leq), \overline{c_{B^*}}^{-1} \right). \] (6.21)

\footnote{It is similar to the situation where she picks a “pebbled” vertex if Spoiler picks the corresponding “pebbled” vertex in the game over the \(\xi\)-th abstraction. Here “pebbled” vertex can be the projection of a really pebbled vertex in the \(\xi\)-th abstraction.}
Here, \( cA_s^x \) is defined similar to (6.11), except that \( \xi \) is replaced by \( \xi - 1 \). \( cB_s^\xi \) is defined likewise. While playing the game (6.21), we can first regard each \( U_{n-1}^e \)-tuple as a unit.

We’ve shown that Duplicator can win this round if she resorts to the \((\xi - 1)\)-th abstraction to respond the picking of \((x', y) \in X_\xi^*\). In fact, this strategy also works if \( \text{id}(x^\beta, y) = \xi - 1 \). The argument is very similar to the one just introduced. Duplicator picks \((x', y)\) such that \( \text{id}(x^\beta, y) = \xi - 1 \). In addition, she ensures that \( \text{cc}([x^\beta]_{\xi - 1}, y) = \text{cc}([x^\theta]_{\xi - 1}, y) \), \( g(x^\beta) = g(x^\theta) \) and \( \text{RngNum}(x^\theta, \xi - 1) = \text{RngNum}(x^\beta, \xi - 1) \). Also cf. the corresponding case (the third case) introduced in the proof of Lemma 18 in page 55.

3. (Strategy 3) Spoiler picks a vertex \((x, y)\) in \( A^{*}_{k,m} \) where \( \text{id}(x^\beta, y) = t < \xi - 1 \). Recall that, in the associated game, Spoiler also picks a vertex \((x^\beta, y)\) in \( A^{*}_{k,m} \). Duplicator regards it as if \(((x^\beta, y), \xi)\) or \(((x^\theta, y), \xi - 1, y)\) resp. is also picked at the same time, and picks a vertex \((x^\theta, y)\) whose index is also \( t \) in the other structure such that \(((x^\beta, y), \xi)\) or \(((x^\theta, y), \xi - 1, y)\) resp. is the vertex Duplicator will pick to respond the picking of \(((x^\beta, y), \xi)\) or \(((x^\theta, y), \xi - 1, y)\) resp. using strategy 1 or strategy 2 (cf. the last paragraph). It means that, if strategy 1 and, in particular, strategy 2 work well as claimed, for any pair of pebbles \((u^\omega, v)\) and \((u^\theta, v)\) on the board, the following holds for \( s = \xi \) or \( \xi - 1 \) depending on which strategy is adopted.

\[
((\langle x^\beta \rangle_s, \langle u^\beta \rangle_s, v), ((\langle x^\theta \rangle_s, y), (\langle u^\theta \rangle_s, v))) \in E^A_s \iff ((\langle x^\beta \rangle_s, y), (\langle u^\beta \rangle_s, v)) \in E^R_s (6.22)
\]

That is, (6) holds.

We can assume that \( s = \xi \). It is similar when \( s = \xi - 1 \). By Remark 57, the neighbourhood of \(((x^\beta, y), \xi)\) is the same as that of \(((x^\theta, y), \xi)\): they contain a lot of vertices of the same indices and the same coordinate congruence numbers in the same abstractions; moreover, a unit of difference in higher abstraction means a huge difference in lower abstractions w.r.t. distance of first coordinates.

Duplicator uses the following process to pick \((x^\beta, y)\) to meet (5). She first finds a vertex, say \((z_{\xi - 1}, y) \in X_1^*\), such that \(((z_{\xi - 1}, y), \xi)\) is the ver-

\[\text{Note that, to this end, we need to adapt Lemma 36 a little bit, which is very easy. The point is that, by Lemma 14, } \text{cc}([u^\omega]_{\xi}, v) = \text{cc}([u^\theta]_{\xi}, v) \text{ iff } \text{cc}([u^\omega]_{\xi}, [u^\theta]_{\xi}, v) = \text{cc}([u^\omega]_{\xi}, v) \text{ and } \text{cc}([u^\theta]_{\xi}, [u^\theta]_{\xi}, v) = \text{cc}([u^\omega]_{\xi}, v). \text{ Therefore, the lemma can be adapted to take care of the situation where } \text{cc}([u^\omega]_{\xi}, v) = \text{cc}([u^\theta]_{\xi}, v) \text{ and } \text{cc}([u^\theta]_{\xi}, [u^\theta]_{\xi}, v) = \text{cc}([u^\omega]_{\xi}, v). \text{ Likewise, it is easy to see that the values of } \text{RngNum}([\cdot, \cdot]) \text{ and } g(\cdot) \text{ will not cause a problem to 6.17.}\]
tex she will pick to render the picking of \((x^b)_\xi, y\), and \(\text{id}(z_{\xi-1}, y) = \text{id}(\langle x^b \rangle_{\xi-1}, y)\) if \(\text{id}(\langle x^b \rangle_{\xi-1}, y) < \xi\), and a special variation of \((5^\dagger)\) is met where \(i\) is fixed to \(\xi - 1\) and \(\langle x^b \rangle_{\xi-1}\) is replaced by \(\langle z_{\xi-1} \rangle\). Moreover, \(z_{\xi-1} = \langle z_{\xi-1} \rangle_{\xi} \) if \(\langle x^b \rangle_{\xi-1} = \langle x^b \rangle_{\xi}\). We are able to show that she can find such a vertex in a \(\mathcal{U}_{\xi-1}\)-tuple. Afterwards she finds a vertex, say \((z_{\xi-2}, y) \in \mathcal{X}_t\), such that \(\langle z_{\xi-2} \rangle_{\xi-1} = z_{\xi-1}, \text{id}(z_{\xi-2}, y) = \text{id}(\langle x^b \rangle_{\xi-2}, y)\) if \(\text{id}(\langle x^b \rangle_{\xi-2}, y) < \xi\), and a special variation of \((5^\dagger)\) is met where \(i\) is fixed to \(\xi - 2\) and \(\langle x^b \rangle_{\xi-2}\) is replaced by \(\langle z_{\xi-2} \rangle\). Moreover, \(z_{\xi-2} = \langle z_{\xi-2} \rangle_{\xi-1}\) if \(\langle x^b \rangle_{\xi-2} = \langle x^b \rangle_{\xi-1}\). And so on. Note that, once \((z_{\xi-2}, y)\) is chosen, the special variation of \((5^\dagger)\) is also met where \(i\) is fixed to \(\xi - 1\) and \(\langle x^b \rangle_{\xi-1}\) is replaced by \(\langle z_{\xi-1} \rangle\) (i.e. \(\langle z_{\xi-1} \rangle\)). Finally, she picks the vertex \((x^b, y)\) such that \(\langle x^b \rangle_{t+1} = z_{t+1}, \text{id}(x^b, y) = \text{id}(\langle x^b \rangle_{t+1}, y)\), and a special variation of \((5^\dagger)\) is met where \(i\) is fixed to \(t\). Moreover, \(x^b = z_{t+1}\) if \(x^b = \langle x^b \rangle_{t+1}\). In short, Duplicator can use this process to pick the vertex \((x^b, y)\) such that \(\langle x^b \rangle_{i} = z_{i}\), which implies that \((5^\dagger)\) is met\(^{47}\).

In the following, we use \(\langle i = t\rangle\) as an example to explain how to find a vertex that satisfies the special variation of \((5^\dagger)\) where \(i\) is fixed to \(t\), provided that \((z_{t+1}, y)\) is already determined. Henceforth, when we mention \((5^\dagger)\), we mean this special variation unless otherwise specified. The arguments for other variations are very similar. At the same time, we show that \((2^\dagger)\) and \((5^\dagger)\) can be met. Observe that, Duplicator has the freedom to pick a vertex to ensure that \((5^\dagger)\) (ii), \((5^\dagger)\) (iii), \((5^\dagger)\) (v) and \((5^\dagger)\) (vi) hold simultaneously. \((5^\dagger)\) (i) is already met since \(t' = t\). Note that, \(\langle x^b \rangle_{t} = x^b \) and \(\langle x^b \rangle_{t} = x^b \) since \(\text{id}(x^b, y) = \text{id}(\langle x^b \rangle_{t}, y) = t\).

\((5^\dagger)\) (v) implies that \(\text{sgn}((x^b, y), (u^p, v)) = \text{sgn}((x^b, y), (u^p, v))\) for any pebbled pair of vertices \((u, v) \vdash (u', v)\). It means that \(T \models T'\). Note that \((5^\dagger)\) (ii) implies that \([x^b]_t \equiv [x^b]_t \pmod{k - 1}\). Therefore, Duplicator can ensure that \(Z \models Z'\) (cf. Lemma \[14\] and \(U \models U'\) (cf. \((5^\dagger)\) (vi)).

Lemma \[14\] and \((5^\dagger)\) (ii) imply that any pair of pebbled vertices in \(\mathcal{X}_t\) in respective structures have the same coordinate congruence number in the \(t\)-th abstraction. Together with \((6.22)\) and \((5^\dagger)\) (i) (the full version), as well as the full version of \((5^\dagger)\) (ii), it implies that \(D^{2t} \models\)

\(^{47}\)Note that, if \(S_i^t \models S_i^p\) and \(S_i^t = S_i^p\) for any \(t \leq i < \xi\), then \((5^\dagger)\) (iii) implies that \([x^b]_t^{\text{min}} = [x^b]_\xi^{\text{min}}\) s.t. \([x^b]_t^{\text{min}} - [x^b]_\xi^{\text{min}} = [x^b]_\xi^{\text{min}} - [x^b]_\xi^{\text{min}}\). By Remark \[47\] this in turn implies that \((5^\dagger)\) (i), (ii) hold. With a little more thought, we know that \((5^\dagger)\) (v) (vi) also hold, and (iv) also holds provided that \(i > t\). Therefore, by Remark \[47\] in this special case Duplicator can also simply pick \((x^b, y)\) s.t. \([x^b]_t^{\text{min}} - [x^b]_\xi^{\text{min}} = [x^b]_\xi^{\text{min}} - [x^b]_\xi^{\text{min}}\).
$D^{\geq t}$ holds, since missing of edges in higher abstractions propagates to lower abstractions coincidently in these two structures. For the similar reason, $D^{< t} \models D^{< t}$ also holds. Hence, we have $D \models D'$.

In the following we give the justification that (5) (viii) can be ensured. In short, it is because that Duplicator has a winning strategy in the games over sufficiently large pure linear orders. Let $(a, b)$ be defined as in page 69. First, she can make it by ensuring that $(\langle a', t \rangle_t^{|t|}, b) < (\langle a \rangle_t^{|t|}, b)$ if $(\langle a']_t^{|t|}, b) < (\langle a \rangle_t^{|t|}, b)$, or $(\langle a']_t^{|t|}, b) > (\langle a \rangle_t^{|t|}, b)$. Duplicator can achieve this by an auxiliary game over pure linear orders to determine the value for $(\langle a']_t^{|t|}, b)$.

It means that Duplicator is able to win the following game, wherein Spoiler picks $(x^b, y)$ and she replies with $(x^a, y)$.

$$\mathcal{O}_1(\langle 3 \choose a, m \rangle \langle x_1^b \rangle^{|t|} \langle \leq \rangle, c_{A_b}^{|t|}, \langle 3 \choose a, m \rangle \langle x_1^b \rangle^{|t|} \langle \leq \rangle, c_{B_b}^{|t|})).$$

(6.23)

Here, $c_{A_b}^{|t|}$ is defined similar to (6.11), except that $\xi$ is replaced by $t$. $c_{B_b}^{|t|}$ is defined likewise. Note that, $|c_{A_b}^{|t|}| = |c_{B_b}^{|t|}| < m$, and that $\gamma^*_t$, as well as $\gamma^*_t / \xi^*_t$, is much greater than $2^m$. Therefore, by Remark 45, Duplicator has a winning strategy. Afterwards, Duplicator resorts to the (virtual) game (6.24) to determine the type label of $(x^b, y)$, which will be introduced soon. Second, assume that $(\langle a']_t^{|t|}, b) = (\langle a \rangle_t^{|t|}, b)$ and $(\langle a']_t^{|t|}, b) = (\langle a \rangle_t^{|t|}, b)$. It implies that $b = y$, as well as $\langle a]_t^{|t|} = \langle a \rangle_t^{|t|}$ and $\langle a']_t^{|t|} = \langle a \rangle_t^{|t|}$. Moreover, Duplicator can make it that $|\chi(\langle a]_t^{|t|}, b) | S| = |\chi(\langle a \rangle_t^{|t|}, b) | S| = |\chi(x^b, y) | S|$, provided that $\text{id}(u^x, v) < \xi - 1$ and $\text{id}(x^y, y) < \xi - 1$. Clearly, (5) (vii) holds if $|\chi(x^b, y) | S| = |\chi(\langle a]_t^{|t|}, b) | S| = 0$. If $|\chi(x^b, y) | S| = |\chi(\langle a]_t^{|t|}, b) | S| = 0$, Duplicator resorts to the (virtual) game (6.24), which not only ensures (5) (viii) but also determines the type label of $(x^b, y)$. Finally, note that (5) (viii) is easy to ensure if at least one of $|\chi(\langle a]_t^{|t|}, b) | S| = |\chi(x^b, y) | S| = 0$.

(5) (vii) prevents (x) from occurring. Its self is not difficult to ensure. The problem is whether Duplicator can ensure it without violating (5) (iv). In other words, Duplicator need find a way to satisfy (5) (vii) and (5) (iv) simultaneously. Recall that, we don’t have to consider the cases where there is $(a, b) \in c_{A_b}$ and the length of the board history of $(a, b)$ is greater than that of $(x, y)$, because in such cases Duplicator can.

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48By Lemma 10, $\text{id}(\langle a]_t^{|t|}, b) > t$. If $\text{id}(\langle a]_t^{|t|}, b) > t$, it implies that $|\chi(\langle a]_t^{|t|}, b) | S| = |\chi(\langle a']_t^{|t|}, b) | S| = 0$, which means that $(a, b) < (x^y, y)$ and $(a', b) < (x^y, y)$. If $\text{id}(\langle a]_t^{|t|}, b) = t$, then, by definition, $(a, b) < (x^y, y)$ if $0 < |\chi(\langle a]_t^{|t|}, b) | S| < |\chi(x^y, y) | S|$, and $(a, b) > (x^y, y)$ if $|\chi(\langle a]_t^{|t|}, b) | S| > |\chi(x^y, y) | S| > 0$. 

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resort to the board history of \((a', b)\), where \((a, b) \vdash (a', b)\), to determine \((x', y)\). Therefore, in the following we assume that \((a, b) \rightarrow (x, y)\) for any \((a, b) \in \mathcal{A}\).

Now we show that Duplicator can ensure (5) (vii) and (5) (iv) simultaneously. Note that (5) (vii) will not occur if \(\lfloor x^n_{\text{min}} \rfloor \neq \lfloor x^n_{\text{min}} \rfloor\), or \(\lfloor x^n_{\text{min}} \rfloor = \lfloor x^n_{\text{min}} \rfloor\) but \(|\chi(x^0, y) \mid S| = |\chi(u^1, v) \mid S|\). Therefore, we assume that \(\lfloor u^n_{\text{min}} \rfloor = \lfloor x^n_{\text{min}} \rfloor\) and \(|\chi(x^0, y) \mid S| = |\chi(u^1, v) \mid S|\).

Assume that \(|\chi(x^0, y) \mid S| = p\). If \(p = 0\), Duplicator simply plays the game over abstractions (resort to strategy 1 or 2) so that \(x^0 = \langle x^0 \rangle_{t+1} = x^0 - \langle x^0 \rangle_{t+1}\) (assume that the game is over the \(\xi\)-th abstraction; it is similar if the game is over the \((\xi - 1)\)-th abstraction). So (5) (vii) holds and \(|\chi(x^0, y) \mid S| = 0\). The latter, i.e. \(|\chi(x^0, y) \mid S| = |\chi(x, y) \mid S| = 0\), implies that (5) (iv) holds. Henceforth assume that \(p \geq 1\). It means that the value of \(j^* = \lfloor x^n_{\text{min}} \rfloor / (k - 1)\) mod \(cl_{t+1}^*\) (recall that \(cl_{t+1}^* = 2\lfloor X_{t+1}^* \rfloor + \Sigma_{i=1}^{k-2} |X_{t+1}^*|\)) falls in the range \(\lfloor (X_{t+1}^* + \Sigma_{i=1}^{p-1} |X_{t+1}^*|) / (X_{t+1}^* + \Sigma_{i=1}^{p} |X_{t+1}^*|) \rfloor - 1\). In other words, \(j^* = (X_{t+1}^* + \Sigma_{i=1}^{p-1} |X_{t+1}^*|) / (X_{t+1}^* + \Sigma_{i=1}^{p} |X_{t+1}^*|)\) encodes a p-tuple \((x_1, y_1), \ldots, (x_p, y_p)\) in \(|X_{t+1}^*|^p\). Duplicator resorts to virtual game to determine her pick. She will pick \((x', y)\) such that \(\lfloor x^n_{\text{min}} \rfloor / (k - 1)\) mod \(cl_{t+1}^*\) - \(\lfloor X_{t+1}^* \rfloor + \Sigma_{i=1}^{p-1} |X_{t+1}^*|\) encodes a p-tuple \((x'_1, y_1), \ldots, (x'_{p}, y_{p})\) in \(|X_{t+1}^*|^p\). Suppose that \((u_1, y)\ldots(u_r, y)\) are those pebbled vertices s.t. \(\lfloor u^n_{\text{min}} \rfloor = \lfloor x^n_{\text{min}} \rfloor\) and \(|\chi(u, y) \mid S| = p\). Similarly, \((u'_1, y)\ldots(u'_{r}, y)\) are the corresponding vertices where \((u_i, y) \vdash (u'_i, y)\) in the original game, which implies that \(|\chi(u'_i, y) \mid S| = p\) due to Strategy 3. Clearly, \(0 \leq r \leq k - 2\).

Note that \(\exists\) occurs only if \(r > 0\).

Here we assume that \(r > 0\). The case \(r = 0\) is similar. Suppose that \(\text{id}(x_j, y_j) = t_j\). For any \(l \in [1, r]\), let \(\lfloor u^n_{\text{min}} \rfloor / (k - 1)\) mod \(cl_{t+1}^*\) - \(\lfloor X_{t+1}^* \rfloor + \Sigma_{i=1}^{p-1} |X_{t+1}^*|\) encodes a p-tuple \((u_{1l}, v_{1l}), \ldots, (u_{pl}, v_{pl})\) in \(|X_{t+1}^*|^p\). Let

\[ H_{xy}^S := \{ i \mid (x_i, y_i) \in (x, y)[BC] \text{ and } (x_i, y_i) \text{ is in the p-tuple} \} \tag{49} \]

We can play the following pairs of games. The first one in a pair is an Ehrenfeucht-Fraïssé game over pure linear orders, which is used to determine the possible positions that is in accordance with the abstraction-order-condition; the second one is a 1-round \((k - 1)\)-pebble game wherein the order is “ignored” temporarily and Spoiler picks

\[ (x_1, y_1), \ldots, (x_p, y_p) \].
(x_j, y_j) and she replies with (x'_j, y_j) in the j-th round, for j ∈ [1, p] – H^S_{xy}. Note that the second game is used to determine the type label of (x^b, y). Moreover, for the games over pure linear orders, they are played successively\(^{50}\) by contrast, for the pebble games, they are played independently. By combining these two games Duplicator can decide what should (x'_j, y_j) be. More precisely, in the first game, Duplicator uses the following strategy.

- If t_j ≥ ξ, then Duplicator resorts to (6.12) or (6.21) to determine the unabridged interval where (x'_j, y_j) should reside.
- If t_j < ξ, then Duplicator first uses the game 6.23 to determine (|[x'_j]_{t_j}^{min}, y_j).

In the second game, Duplicator follows the following strategy.

- If t_j ≥ ξ, then Duplicator resorts to Strategy 1 and Strategy 2 to determine the type label of (x'_j, y_j). Cf., in particular, (6.16).
- If t_j < ξ, then Duplicator uses the following one round (virtual) game 6.24 to determine the type label of (x'_j, y_j).

\[\mathcal{D}_1^{k-1}(\tilde{A}^*_{k,m}, (x, y)[BC]), (\tilde{B}^*_{k,m}, (x', y)[BC])\]  

(6.24)

Note that in this case it may incur recursive calls. In the recursion a pair of games will be played to decide one pick, as just described. It is a sort of game reductions from lower abstraction to higher abstraction because idx(x'_j, y_j) > t, and will finally return a valid type label for (x'_j, y_j) since idx(x'_j, y_j) cannot be greater than m. Note that, in the recursions, “(x, y)[BC]”, as well as “(x', y)[BC]”, is fixed in (6.24).

Moreover, because the game boards are in partial isomorphism at the end of the games, it means that the following holds when j \notin H^S_{xy}:

u_{ij} ≤ x_j if and only if u'_{ij} ≤ x'_j.  

(6.25)

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\(^{50}\)By default, once a vertex is picked, a pebble is on it unless the players lift the pebble. As a consequence, the picking of (x_{j_1}, y_{j_1}) will influence the picking of (x_{j_2}, y_{j_2}), if j_1, j_2 ∈ [1, p] – H^S_{xy} and (x_{j_2}, y_{j_2}) is picked later. Note that, the purpose of playing the games successively is that Duplicator need to ensure that (u'_{j_1}, v_{i_1}) ≤ (u'_{j_2}, v_{i_2}) if (x_{j_1}, y_{j_1}) ≤ (x_{j_2}, y_{j_2}) where j_1, j_2 ∈ [1, p] and (x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}) are in the p-tuple associated to \(\chi(x', y)\) | S. Therefore, the order with which these games are played successively is not important.
Note that \((x_j, y_j) \not\in R\) and \((x'_j, y_j) \not\in R'\), since \(j \not\in H_{xy}^S\).

For any \(j \in H_{xy}^S\), Duplicator can determine \((x'_j, y_j)\) such that \((x_j, y_j) \parallel (x'_j, y_j)\) in the virtual game that she uses to determine \((x', y)[BC]\). It implies that (5) (iv) holds, since \((x_j, y_j) \in R\) iff \((x_j', y_j) \in R'\) for \(j \in [1, p]\). To see that (5) (vii) also holds, we need only show that (6.25) also holds when \(j \in H_{xy}^S\). If both of \((u_{ij}, v_{ij})\) and \((x_j, y_j)\) are in \((x, y)[BC]\), then it is determined by the fact that Duplicator wins the virtual game that determines the board history. Now suppose that \((u_{ij}, v_{ij}) \not\in (x, y)[BC]\) and \((x_j, y_j) \in (x, y)[BC]\). There are two cases. First, assume that \((x_j, y_j)\) is picked before \((u_i, v_i)\) in the virtual game that determines the board history of \((x', y)\). Obviously, (6.25) holds when \((u_i, v_i)\) is picked in the original game: the point is that \((x_j, y_j)\) belongs to \((u_i, v_i)[BC]\). Second, assume that \((x_j, y_j)\) is picked after \((u_i, v_i)\) in the virtual game. Recall that \(j \in H_{xy}^S\) and that \(\text{idx}(x_j, y_j) = t_j\). If \(t_j \geq \xi - 1\), then (6.25) holds, due to (6.12) or (6.21). Suppose that \(t_j < \xi - 1\). Then \(\text{idx}(x'_j, y_j)\) should be \(t_j\), due to Strategy 3. In such case (6.25) still holds because (5) (viii) can be ensured.

With this somewhat sophisticated argument we have shown that (2) can be ensured, in accordance with (5), in particular (iv), and with (6).

Note that, after this round, \(\xi\) is either unchanged or decreased by one. And, as usual, \(\theta := \theta - 1\). Hence (1) holds.

Now we need to check whether (1) can be ensured. Recall that we need switch back to the game board \((\mathfrak{A}_{k,m}, (\mathfrak{B}_{k,m}, \mathfrak{C})_\mathcal{B})\) for the discussion. To simplify the following discussion, we can safely assume that Duplicator’s picks ensure that \(\text{idx}(x^\alpha, y) \leq \text{idx}(x, y)\), and \(\text{idx}(x^\alpha, y) = \xi - 1\) if \(\text{idx}(x^\alpha, y) < \text{idx}(x, y)\). Recall that \(\text{idx}(x', y) = t\) and \(\text{idx}(x^\alpha, y) = t'\). Note that \([x^\alpha]_{\xi - 1} \equiv 0 \pmod{k - 1}\) if \(t \geq \xi\). Firstly, assume that \(t > t' = \xi - 1\). The case where \(\text{idx}(x', y) = \text{idx}(x, y) \geq \xi\) and \(\text{idx}(x', y) = \text{idx}(x, y) = \xi - 1\) are similar. There are several cases need to discuss. Suppose that \(0 < y < k - 1\). Then, for any boundary vertex \((a, b) \in \mathcal{X}_1\) where \(0 < b < k - 1\),

\[
(\langle a^\beta \rangle_{\xi - 1}, b) \not\in \chi(x^\beta, y) \mid \Omega \quad \text{and} \quad (\langle a^\beta \rangle_{\xi - 1}, b) \not\in \chi(x^\beta, y) \mid \Omega,
\]

\[\text{(6.26)}\]

\[\text{Note that, in the applying of (5)}\) (viii) and its argument, “\((x^\alpha, y)\)” and “\((x^\beta, y)\)” in (5) (viii) should be replaced by \((x_i, y_j)\) and \((x'_j, y_j)\) respectively; and “\((u, v)\)” should be replaced by \((u_i, y)\) for some \(i \in [1, r]\).

\[\text{In the latter case, just observe that the values of RngNum(\cdot, \cdot), g(\cdot) \mid S\] will not cause a problem to (4), provided Duplicator uses Strategy 2 (cf. the last paragraph of it).
simply because \([\langle a^b \rangle_{\xi-1} \rangle_{\xi-1} \equiv 0 \pmod{k-1}\) (recall that \(a^b = \gamma_{m-1} - tr(b)\) and \([\langle a^b \rangle_{\xi-1} \rangle_{\xi-1} = [a^b]_{\xi-1} \equiv 0 \pmod{k-1}\); cf. (5.10)). If \(b = 0\) or \(b = k-1\), again by definition, (6.26) holds simply because \(\text{idx}(\langle a^b \rangle_{\xi-1}, b) = \xi - 1 = t' < t\) (cf. Lemma 13). In addition, by definition, \(\text{sgn}(\langle a^b \rangle_{\xi-1}, b, (x^b, y)) = 1\) and \(\text{sgn}(\langle a^b \rangle_{\xi-1}, b, (x^b, y)) = 1\). By Lemma 14 \(\text{cc}(\langle a^b \rangle_{\xi-1}, y) = y \pmod{(k - 1)}\), which equals \(\text{cc}(\langle a^b \rangle_{\xi-1}, y)\). According to Strategy 3 (cf. p. 78) \((\langle a^b \rangle_{\xi-1}, b) \in \chi(x^b, y) \Leftrightarrow (\langle a^b \rangle_{\xi-1}, b) \in \chi(x^b, y)\). Furthermore, \(g(\langle a^b \rangle_{\xi-1}) = 0\) and \(g(x^b)\) can be chosen to 0 (cf. Lemma 36), which implies that \(\text{SW}(\langle a^b \rangle_{\xi-1}, b, (x^b, y)) = 0\). It means that \(\text{cc}(\langle a^b \rangle_{\xi-1}, b) = \text{cc}(\langle a^b \rangle_{\xi-1}, y) \times (b - y) \times (-1)^{\text{BIT}(\text{SW}(\langle a^b \rangle_{\xi-1}, b, (x^b, y), \gamma(b, y)))} > 0\). Therefore, \((\langle a^b \rangle_{\xi-1}, b)\) is adjacent to \((x^b, y)\) if and only if \((\langle a^b \rangle_{\xi-1}, b)\) is adjacent to \((x^b, y)\). Now suppose that \(y = 0\) or \(k - 1\). Because \(\text{cc}(\langle a^b \rangle_{\xi-1}, y) = \text{cc}(\langle a^b \rangle_{\xi-1}, y) = \text{cc}(\langle a^b \rangle_{\xi-1}, b) = 0\), \((\langle a^b \rangle_{\xi-1}, b)\) is not adjacent to \((x^b, y)\) and \((x^b, y)\).

Secondly, assume that \(t = t' < \xi\) (cf. Strategy 3). The arguments are similar except one point. By the last arguments, now we can ensure that either (i) \((\langle a^b \rangle_{\xi-1}, b)\) is adjacent to \((\langle x^b \rangle_{\xi-1}, y)\) if and only if \((\langle a^b \rangle_{\xi-1}, b)\) in the other structure, is adjacent to \((\langle x^b \rangle_{\xi-1}, y)\), or (ii) similar to (i) but \(\xi\) is substituted with \(\xi - 1\). Suppose that (i) holds. The other case is similar. The point is that, if \(b = 0\) or \(k - 1\), we need to show that \((\langle a^b \rangle_{\xi-1}, b) \in \chi(x^b, y)\) is if and only if \((\langle a^b \rangle_{\xi-1}, b) \in \chi(x^b, y)\). But this is clear now, the argument needed is similar to the one that we show that \(D \vdash D'\) (cf. page 80).

In summary, the boundary checkout strategy is not effective for Spoiler over the “flat” game board \(((\mathfrak{A}_{k,m}^*, (x, y)[BC]), (\mathfrak{B}_{k,m}^*, (x', y)[BC]))\) or over the “flat” changing boards. It implies that the boundary checkout strategy is not effective for Spoiler over the game board \(((\mathfrak{A}_{k,m}^*, \mathfrak{A}_{k,m}^* A), (\mathfrak{B}_{k,m}^*, \mathfrak{B}_{k,m}^* B))\) as well.

All in all, Duplicator can ensure that \(\text{(1)'}\sim\text{(6)'}\) hold throughout the game.

Note that the game board may change for each round in a virtual game. Indeed, if game board change is allowed in a normal game, then there is no need to introduce \((\mathfrak{A}_{k,m}, \mathfrak{B}_{k,m})\) at all. The strategy described in the proof of Lemma 37 work over the “flat” associated board \((\mathfrak{A}_{k,m}^*, \mathfrak{B}_{k,m}^*)\) directly if board change is allowed. In other words, games over changing bord is the basis of all the other games. In such games, we use some auxiliary games to help Duplicator make her decision. In particular, in Strategy 3, we have used a technique, called game reductions, to prevent Spoiler from winning the associated game simply by picking continuously inside a \(\mathfrak{U}_n^i\)-tuple for some \(i\) (cf. (6.23) and (6.24)). We also use them to determine Duplicator’s
picks in the virtual games wherein Strategy 2 applies.

Fig. 5 gives some hints on a way Spoiler can use to detect the difference between structures \( \tilde{A}_{k,m}^* \) and \( \tilde{B}_{k,m}^* \), wherein Duplicator has to resort to game reductions to reply properly. In the figure, a node is either black or red representing un-pebbled (in black) and pebbled (in red) vertices respectively. We assume that the vertex “a” is picked in the last round and the vertex “b” is picked in the current round. The dotted (in blue) and dashed (in red) arrows together indicate the set “\( \mathcal{S} \)” associated with a vertex. Moreover, we use dashed arrows to indicate the set \( R \). Here we regard the vertex b as the vertex “\((x^*, y)\)”. Note that, although dotted arrows and dashed arrows are used to indicate the edges forbidden in \( \tilde{A}_{k,m}^* \) or \( \tilde{B}_{k,m}^* \), they are not necessary forbidden in the changed boards, wherein only the dashed arrow have to be forbidden. For instance, from the figure we know that both \( \text{cl}(c) \) and \( \text{cl}(d) \) are in \( \chi(b)|S \), or \( \chi(x^*, y)|S \); and \( (x, y) \rightsquigarrow c^* \) where \( c^* \) is in \( \mathcal{S}^* \) such that c is the corresponding vertex in \( \mathcal{S}^* \). After playing the auxiliary virtual games over the linear orders, Duplicator is able to determine \( b^* \), i.e. \( (x^*, y) \), which she should pick to respond the picking of \( b \) by Spoiler. That is, we use the auxiliary games to determine the picks of Duplicator in the game over changing board that is associated to the original game.

Spoiler can make it that a and b be very closed. For example, \( b = a - 1 \). In such case we have \( b^* = a' - 1 \), provided that Duplicator responds properly, i.e. preserving the abstraction-order-condition. Note that, in the case \( b = a - 1 \), the result of virtual games should witness that \( b^* = a' - 1 \).
Now we are able to prove our main result of this paper, using Lemma 37.

**Theorem 38.** For any $k$, $k$ variables are necessary and sufficient to describe $k$-Clique in FO on finite ordered graphs.

**Proof.** Suppose that there is a $L^{k'}$ formula, where $k' < k$, to describe $k$-Clique, and assume that its quantifier rank is $m$. We can safely assume that $k' < k \leq m$ and $(k - 2)^2 < m$ if $k > 3$. If it is not true, we can define another logically equivalent $L^{k-1}$ formula by artificially increasing the quantifier rank of the formula. Consequently, Spoiler has a winning strategy in the game $\varphi_m^{k-1}(\mathcal{A}_{k,m}, \mathcal{B}_{k,m})$, which is in contradiction to Lemma 18 and Lemma 37. Therefore, $k$ variables are needed to define $k$-Clique over finite ordered graphs.

On the other hand, the following first-order formula describes $k$-Clique: 
$$\exists x_1 \ldots \exists x_k \bigwedge_{i \neq j} (\neg(x_i = x_j) \land E(x_i, x_j)).$$
Here we use $x_i$ to denote a vertex. Therefore, $k$ variables are sufficient to define $k$-Clique.

As a direct corollary of Theorem 38, we have the following well-known result, which was first proved by Rossman [22].

**Corollary 39.** The bounded variable hierarchy in FO is strict.

That is, for any $k$, over the finite ordered graphs there is a property that is expressible by $k$ variables, but not expressible by $k - 1$ variables in FO. In other words, first order logic needs infinite many variables. Hence we have given an alternative proof for this important result in finite model theory, based on pure finite model-theoretic tools.

## 7 Worst-case lower bound of $k$-Clique on constant-depth circuits

Recall that an ordered graph is a graph with a linear order in the background. In this section, we fist show that precisely $k$ variables are needed to define $k$-Clique on the class of graphs with arbitrary arithmetic background relations, a result akin to Theorem 38 except that the graphs are equipped with BIT in the background. It is another well-known challenge to play pebble games on such kind of structures, which has its root in circuit complexity [19]. Note that BIT predicate can be used to define arbitrary arithmetic predicates.

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53 Recall that, the simple case where $k = 2$ is already proved at the start of section 5.1.
54 When infinite ordered structures are concerned, it was proved by Venema [30].
including linear orders (for a survey, cf. [27]). Surprisingly, due to a work of Schweikardt and Schwentick [28], it turns out that this challenge is very similar to the challenge caused by linear orders. In section 7.1 we embed the main structures in section 5.2 in the pure arithmetic structure in [28] to show that \( k \) variables are needed to define \( k \)-Clique in \( \text{FO}(<, \preceq) \). Afterwards, in section 7.2 we show that this result implies a worst-case lower bound of \( k \)-Clique on constant-depth unbounded fan-in circuits.

Here we assume that the readers are familiar with the ideas and notations in the paper [28], wherein a sort of clever construction is presented to show that \textbf{BIT} can be replaced by two special linear orders. Because of its constructive nature, it offers a tool that can bridge the gap between pure linear orders and pure arithmetic. Abuse of notations, we also use \( \preceq \) to denote one of the linear orders, as in [28]. The readers should not confuse it with the induced linear order \( \preceq^\xi \) introduced in the main Lemma 37.

Another linear order is \( < \).

In the following, we briefly sketch related basic ideas of [28]. The paper [28] introduced a pure arithmetic structure whose elements are ordered in a specific way. We can also regard it as a set of isolated vertices ordered and organized as an isosceles right triangle (cf. Fig 1 of [28]) in a two dimension coordinate plane. Note that, there is a bijection between these two universes. For any vertices \((x_1, y_1)\) and \((x_2, y_2)\), \((x_1, y_1) < (x_2, y_2)\) if \(x_1 < x_2\) or \((x_1 = x_2\) and \(y_1 < y_2\)). This is called the bottom-to-top, left-to-right, column major order. On the other hand, we define \( \preceq_0 \) as the left-to-right, bottom-to-top, row major order: \((x_1, y_1) < (x_2, y_2)\) if \(y_1 < y_2\) or \((y_1 = y_2\) and \(x_1 < x_2)\). Confer p3 of [28]. Furthermore, we introduce two unary relations \( C \) and \( Q \). We use \( C \) to encode a binary representation of \(x + 1\), and use \( Q \) to encode a binary representation of \(\left(\frac{x+2}{2}\right)\). Schweikardt and Schwentick showed that \( \text{FO}(\preceq, \preceq_0, C, Q) \) has the same expressive power as \( \text{FO}(<, \preceq) \).

Then it is shown that \( \text{FO}(\preceq, \preceq_0, C, Q) \) is equivalent to \( \text{FO}(\prec, \preceq) \) in terms of expressive power, where \( \prec \) is a special linear order that can be used to encode the two unary relations. More precisely, if \( \ell \) is sufficiently large, we can use the order \( \prec \) on every complete interval (cf. p10 of [28] for this important concept) \( \{(x, y+1), \ldots, (x, y+\ell-1)\} \) to encode \( C, Q \) on the elements \((x, y), (x, y+1), \ldots, (x, y+3\ell-1)\). Note that an order corresponds to a permutation, say \( \pi_i \), where \( i \) is represented by a 0-1 string of length \( 6\ell \). This string can be used to encode a unary relation on the \( 6\ell \) elements. Hence, for any \((x, y)\), we are able to know whether \((x, y) \in C \) or \((x, y) \in Q \).

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7.1 \( k \) variables are needed to define \( k \)-Clique in \( \text{FO(BIT)} \)

We first briefly outline the ideas. Suppose that we have a pair of sufficiently large isomorphic arithmetic structures as described in [28]. Here, by “sufficiently large” we mean that \( \ell \) is such a big number that we can embed our main structure \( \tilde{A}_{k,m} \) or \( \tilde{B}_{k,m} \) (cf. section 5.2) into a piece of a complete interval where those two linear orders, i.e. \( < \) and \( \prec \), coincide: we embed a copy of the structure into it in the similar way we make the list \( L \) (cf. p. 51) with a difference: instead of fixing the second coordinates, now we fix the first coordinates to a constant.

Due to Stirling’s formula, we assume that
\[
n! = e^{c_n \cdot n \ln n} = 2^{(\log_2 e) c_n \cdot n \ln n},
\]
for some \( c_n \in \mathbb{R}^+ \). Note that, \( c_n \to 1 \) if \( n \to \infty \). In fact, \( c_n \approx \frac{9}{10} \) when \( n \geq e^{10} \). We choose a sufficiently large natural number \( \ell \) such that \( \ell > \max \left\{ \frac{5}{e^{5e^{-1}} + 1}, \gamma_{m-1} \right\} \). Recall that \( \gamma_{m-1} \) is a number depending only on \( k \) and \( m \). As a consequence, \( (\ell - 1)! > 2^{(\log_2 e) e^{5e^{-1}} (\ell - 1) \ln e^{\frac{5}{e^{5e^{-1}}}} = 2^{5(\log_2 e)} (\ell - 1) > 26\ell \), and \( \ell > \gamma_{m-1} \).

Therefore, these two structures are roughly a (huge) set of isolated vertices in the form of an isosceles right triangle except that, in a pair of corresponding complete intervals, there lies the pair of twisted structures isomorphic to \( \tilde{A}_{k,m} \) and \( \tilde{B}_{k,m} \) respectively. To make it easier, we assign the permutaion of this pair of complete intervals, say \( \pi_i \) for some \( i \in \{0, 1\}^{6\ell} \), to an order that is isomorphic to the natural order of an initial segment of \( \mathbb{N}_0 \). Note that, the linear orders defined in \( \tilde{A}_{k,m} \) and \( \tilde{B}_{k,m} \) are also isomorphic to the natural order of an initial segment of \( \mathbb{N}_0 \), thereby isomorphic to this order. It allows us to embed \( \tilde{A}_{k,m} \) and \( \tilde{B}_{k,m} \) into this pair of complete intervals without rearranging the vertices.

It remains to show that Duplicator has a winning strategy over this pair of sparse structures. To this end, we refer to a simple strategy composition as follows. Observe that the embedded structures \( \tilde{A}_{k,m} \) and \( \tilde{B}_{k,m} \) are disjoined with other parts of the structures. As a consequence, when Spoiler picks in an embedded structure, Duplicator resorts to the strategy introduced in Section 6 since \( < \) and \( \prec \) coincide (hence reduced to one order); when he picks in other parts of the structure, Duplicator is simply a copycat. Obviously, such a composed strategy works for Duplicator in the \( m \)-round \( k \)-pebble games. Afterwards, using an argument similar to that of Theorem 38 we arrive at our claim.

**Theorem 40.** \( k \) variables are needed for \( k \)-Clique in \( \text{FO(BIT)} \).

Note that this lower bound is optimal, as the one given by Theorem 38.
**Further Remark**

Could we reduce the size of the astonishing huge structures just introduced? Or what is the tight lower bound on such a size? Although we have proved our claim constructively, we believe that there exist smaller constructions. To have such a construction coincide with the ideas of [28], in our viewpoint, it would require an understanding of the patterns of binary encodings of $C$ and $Q$, which is not trivial. Obviously, it is preferable if the patterns are periodical. To have an impression of how mysterious would such patterns be, the readers could read, for example, a paper by Rowland [26].

### 7.2 Size lower bound of $k$-Clique on constant-depth circuits

We assume the readers know standard concepts and notations in circuit complexity. It is well-known that FO is closely related to constant-depth circuits [3, 8, 10, 14, 16, 17, 19]. In particular, the introducing of BIT as a background relation allows a suitable form of uniformity for circuit families, thereby establishing the equivalence between FO(BIT) and DLOGTIME-uniform $AC^0$ (or FO-uniform $AC^0$). In this section we always assume the presence of BIT. Moreover, we always assume that $k \geq 5$ and that $n$ is the cardinality of the vertex set of input graph. Since we resort frequently to [10] for inspiration and insights, we assume that the readers are familiar with the notions introduced in these papers. In particular, we will use a key notion called regular circuit (cf. [10], p237), which is defined as follows. By DeMorgan’s law, we can assume that the negations only appear in the input level of circuits. It will not influence the size of a circuit significantly.

A circuit takes an (ordinary) encoding of a structure as its input. Here we only talk about graphs, and by “ordinary” we mean the usual binary encodings of graphs. The order of a graph is the cardinality of its vertex set. The order of a circuit is the order of the input graph. Usually we use $C_n$ to denote a circuit of order $n$. A circuit $C_n$ is formatted w.r.t. $n$ and $\langle E \rangle$. That is, in the context of our concern, there is a surjection from the inputs to atoms $E(a,b)$, $a = b$, BIT$(a,b)$ or their negations where $a, b$ are vertices of the input graph. Note that, those arithmetic literals, e.g. $a = b$ and BIT$(a,b)$, can be replaced by two constant inputs 0 and 1, because their values are independent of the inputs.

A circuit $C_n$ is regular if the following hold.

1. Its structure is symmetric (satisfying some conditions such that the circuit structure completely respects the syntactic structure of some
first-order sentence as well as its evaluations on assignments, cf. [10], p.236–p.237); hence its wires can be labeled in a way that reflects the syntactic structure of the sentence (cf. (2)). It implies that the following hold.

(a) Its gates (without considering inputs) induce a tree where the output gate is the root of this tree;

(b) Each of its inner nodes of the tree, if we do not regard the inputs as leaves, has either \( n \) children or two children, depending on whether it corresponds to a quantifier or a logical operator, i.e. \( \land \) or \( \lor \);

(2) The wires are labelled as the follows (cf. [10] page 236). Let \( \Gamma_n = \{0, 1, \ldots, n - 1, \#_L, \#_R\} \). Let \( f_w \) be a permutation of \( \{0, 1, \ldots, n - 1\} \), which can be extended to a permutation (also called \( f_w \)) of \( \Gamma_n^* \) by setting \( f_w(\#_L) = \#_L, f_w(\#_R) = \#_R, \) and \( f_w(c_1c_2\ldots c_\ell) = c'_1c'_2\ldots c'_\ell, \) where \( f_w(c_i) = c'_i \) for each \( i \). Let \( \bar{x}, \bar{y} \in \Gamma_n^* \) and \( \bar{z} \in [n]^* \). Assume that \( \bar{x} \) is a wire of \( C_n \). Then for any \( f_w \), the following hold.

(i) \( f_w(\bar{x}) \) is a wire of \( C_n \);
(ii) \( \bar{x} \) and \( f(\bar{x}) \) are outputs from gates of the same type or are both input wires;
(iii) \( f_w(\bar{y}) \) is a child of \( f_w(\bar{x}) \) if \( \bar{y} \) is a child of \( \bar{x} \);
(iv)* if \( \bar{x} \) is an input wire whose formula label is an atomic formula \( P\bar{z} \) then the formula label of \( f_w(\bar{x}) \) is \( Pf_w(\bar{z}) \), provided that the predicate \( P \) is neither \( \leq \) nor \( \text{BIT} \).

A regular circuit has order \( n \) if the cardinality of the universe of the structure for the input is \( n \).

Note that we have given a slightly different symmetric in the labelling, i.e. (2) (iv)*, from the original definition. The original definition of regular circuits are the circuits that completely respect the syntactic structures of formulas and semantic requirement of logical queries to ensure closure under isomorphism, while we require that the closure under isomorphism holds when ignoring arithmetic predicates. By [10], a family of circuits, if there is any, recognize a first-order graph property invariant to the permutations of vertices only if there is a family of regular circuits, wherein every circuit is symmetric to ensure this property to be closed under isomorphism. Nevertheless, since we are talking about ordered graphs, requirement of symmetry
should be tailored because ordered graphs are isomorphic if and only if they are the same [19]. In this case it corresponds to “general expression” in [3]. Fortunately, k-Clique is order-invariant, which allows a variant of the symmetry in the labelling, i.e. (2) (iv).

Note that regular circuits, or general expressions, are not “space efficient” (for it is essentially a tree) so that it does not rule out the possibility that a family of much more succinct circuits can recognize the same property. Therefore, we relax the requirement that such a circuit should be a tree (without considering inputs). That is, for any such circuit $C_n$ (succinct regular circuit, for short) in this family, albeit still retaining (2) (cf. the definition of “regular circuit”), its wires may be succinctly arranged, i.e. the output of a gate can be many. For, and only for, convenience, we also require that all the children of a gate are “uniform” in case that these children corresponding to a quantified variable: the structures of the subcircuits (ignore the inputs), whose outputs are these children, are isomorphic. It makes the succinct regular circuit look more like a regular circuit. Call this uniform-children condition. Note that such condition is implicitly a part of the definition of regular circuits.

Succinct regular circuits reduce logically equivalent subformulas, thereby giving more succinct representations. Obviously regular circuits are special kind of succinct regular circuits. Note that, for any succinct circuits $C$, a regular circuit can be obtained from $C$ by straightforward “unraveling”, a process of producing copies of subcircuits. Although the size of the regular circuit may explode, the number of variables it needs for translating a circuit into a formula will not be changed.

Recall a well-known result of Barrington et al. [3], which connects first-order definable uniformity to $DLOGTIME$-uniform.

**Fact 7.** The following are equivalent.

1. $L$ is first-order definable.

2. $L$ is recognized by a $DLOGTIME$-uniform family of constant-depth, unbounded fan-in, polynomial-size circuits.

3. $L$ is recognized by a first-order definable family of such circuits.

4. $L$ is recognized by a $DLOGTIME$-uniform family of constant-depth, polynomial-size general expression.

5. $L$ is recognized by a first-order definable family of such expressions.
Fig. 6 summarizes the relationships between graph properties, general expressions, ordinary circuits and regular circuits. Soon we shall see how to convert a FO-uniform family of constant-depth circuits into a FO-uniform family of succinct regular circuits, which is indicated by \( \mapsto \). Note that, we haven’t given a formal definition for first-order definable family of (succinct) regular circuits yet. But it is clear and similar to those in FO-uniform AC\(^0\).

Here we adopt a slightly different notion of regular circuits wherein no such labels are explicitly put in the circuits: a circuit is regular if there exists a labelling such that those conditions are met.

In the following we prove that \( n^{k-1} \) gates not suffice to compute \( k \)-Clique for \( k \geq 5 \), based on Theorem 40 and an assumption. To show it, the following is needed and sufficient.

**Proposition 41.** For any graph property, it is describable by a formula in FO(BIT) using at most \( k \) variables if it is recognizable by a first-order definable family of constant-depth, unbounded fan-in circuits of size \( O(n^k) \), provided that \( k \geq 5 \).

Here a graph property refers to a set of graphs, defined in the usual way. We do not talk about other variations such as hypergraphs. Assume that the circuit depth is bounded by \( m \) where \( m \geq k \geq 5 \). We make a first try using a natural idea from \cite{3,19}. The basic idea is that, using a first-order sentence, we can simulate the function of a circuit by describing the structure of this circuit, which is defined by the first-order query we used to produce this family of circuits. Hence the sentence is true in the input graph if and only if the circuit accepts the encoding of the graph. Assume that there is a first-order query that maps a string \( 0^n \) to a circuit \( C_n \) of size \( O(n^k) \). Then the query may use \( 2(k+1) \) variables to describe the structure of circuits. In addition, it needs to describe the relations that
associate a node with a label in \{\land, \lor, \neg, \text{input}, \text{output}\}, which also needs \(2(k+1)\) variables. Moreover, we are able to show that the bounded variable hierarchy collapses to \(\text{FO}^3\) on pure arithmetic structures (cf. Remark \[53\]). By Corollary \[25\] it implies that the query can use at most \(2k+5\) variables to do the work, by reusing the variables. Corollary \[56\] gives us a slightly sharper lower bound, i.e. \(2(k+1)\) variables.

As usual, we can associate a node of circuit with a unique \((k+1)\)-tuple, and translate a circuit into a formula. It means that it needs at most \(2(k+1)\) variables to simulate the circuit when we use the method introduced in \[19\].

As have been seen, we still have a big gap between \(2(k+1)\) and \(k\) variables. It only shows that \(O(n^{0.5k-1.5})\) gates not suffice to compute \(k\)-Clique if \(k \geq 5\). Indeed, the first-order sentence, obtained from describing circuit structures via the first-order query defining this circuit family, will have to use more than \(k\) variables. Hence we need alternative ideas, which allow us to handle with more “regular” circuits, or circuits in some “normal form”, whose structures helps us to know something about the property. It is for this reason we resort to succinct regular circuits. We introduce a new technique called regularization to reduce the gap between an ordinary circuit and a succinct regular circuit. Nevertheless, we should note that the following proof is not constructive and is based on a reasonable but not well-known assumption. Recall that all the circuits in discourse are formatted w.r.t. \(n\) and \(\langle E \rangle\), and we do not take the inputs as gates. The assumption essentially says that any first-order query, which defines a family of constant-depth unbounded fan-in circuits, implicitly defines the syntactical structure of some first-order sentence. Here we adopt a variant that considers the uniform-children condition.

**Assumption 1.** For any first-order query \(I_0\), which defines a family of constant-depth unbounded fan-in circuits \(\{C_n\}\) of size \(O(n^k)\), there exists a first-order query \(I_1\), which defines a uniform-children family of constant-depth circuits of size \(O(n^k)\) defining the same property, implicitly defines the syntactical structure of some first-order sentence: for every \(C_n\), its gates are either those whose children correspond to the assignment of a block of (or several block of) relativized quantified variables, or those whose children correspond to distinct subformulas.

Here we briefly explain why this assumption is reasonable. An instance of a first-order formula is a propositional logic sentence where every variable of the first-order formula is replaced by a value assigned to it. An ordinary circuit can be divided into several pieces, each of which computes a first-order formula or an instance of it. However, the children of a gate
$x$ (for quantifiers), may stand for instances of distinct but logically equivalent formulas. In such case, we can select the simplest one w.r.t. the size of the (sub)circuit that compute it, and replace all the other (sub)circuits computing the same function with this subcircuit. Therefore, the number of gates will not increase. Such a process can make the circuits much more “regular” and easier to define. At the end, we piece up all the instances of formulas to obtain the sentence that defines the property. The case is a little more complicated when the children of a gate (for quantifiers) stand for instances of distinct formulas that are not equivalent. In such case, we should only note that the number of distinct formulas are finite, for otherwise it is not first-order definable, and that the query essentially express that some sort of children (determined by some first-order formula) compute some function, and some sort of children compute another function, and so on. Therefore, we can replace the subcircuit, whose output gate is the gate $x$, with a slightly different but equivalent one: we “split” the gate $x$ with several gates (depend on the number of distinct sorts of children), each of which computes a distinct function, and take these gates as the children of a new gate. Then, for each of these subcircuits, we “regularize” it using the method introduced in the last case. Note that this replacement will not increase the size of circuit significantly. So far we show that a family of FO-uniform circuits can be converted into a first-order uniform-children family of circuit without significantly increasing w.r.t. circuit size. The reason we think that the assumption is resonable also relies on the fact that a gate of $n$-children (corresponding to an expression in propositional logic) can be “interpreted” as a quantifier in first-order logic. Although the number of children may vary, it should be definable by a first-order formula, for, otherwise, the circuit structure is not first-order definable. Similarly, somehow we can also “interpret” the functions of some sorts of gates as first-order formulas (e.g. cf. Example 2).

It seems that our argument is constructive. But it isn’t. According to Trakhtenbrot’s Theorem \cite{29}, whether two first-order formulas are equivalent or not is not decidable. It implies that the argument involving replacements in the last paragraph is not constructive. Similarly, the following proof of Proposition 41 is also not constructive.

**Proof.** Recall that an inner gate of a succinct regular circuit has either two children or $n$ children, standing for either a logical operator $\land$ (or $\lor$) or a quantifier. In the latter case, say that a wire is quantifiered by $x$ if it is among one of the $n$ children that correspond to distinct values of the quantified variable $x$ in the sentence.
For any first-order sentence $\phi$, let $f_v(\phi)$ be the number of distinct variables in $\phi$. If $\phi = \psi_1 \land \psi_2$, then clearly $f_v(\phi) = \max\{f_v(\psi_1), f_v(\psi_2)\}$. Similarly, $f_v(\phi) = \max\{f_v(\psi_1), f_v(\psi_2)\}$ if $\phi = \psi_1 \lor \psi_2$. Note that, once we have a family of succinct regular circuits, it is easy to read their structures and get the corresponding first-order sentence. Suppose that $\phi$ is the sentence that correspond to this family of circuits. It implies that any directed path in a succinct regular circuit (of this family) contains wires that are quantified by at most $k$ distinct variables if the circuits has at most $O(n^k)$ gates. Therefore, $f_v(\phi) \leq k$.

The difference between ordinary circuits and succinct regular circuits are the follows.

(1*) Ordinary circuits can be more “succinct” than the so called succinct regular circuits in the representation of quantifier structures. That is, in an ordinary circuit a gate can use $n^i$ children to denote a block or even several blocks of quantifiers.

(2*) Even if the fan-in corresponds to one quantified variable, the number of children of a node can be varied, i.e. doesn’t have to be $n$. Therefore, in general, a gate can have $\ell$ children where $n^i < \ell < n^{i+1}$ for some $i$.

For example, consider a very simple 1-ary first-order query that defines a family of small circuits. Assume that $y$ is an $\land$-gate. Suppose that $x$ is a child of $y$ if $x$ and $y$ satisfy some first-order definable property, say defined by $\eta(z_1, z_2)$. That is, the following is a part of the first-order formula that defines the structure of the circuit.

$$\forall x (\eta(x, y) \rightarrow E(x, y)) \quad (7.1)$$

In this case the number of children of $y$ is determined by $\eta(z_1, z_2)$. The question is that $f_v(\eta(z_1, z_2))$ could be arbitrary. It may be much greater than $k$.

(3*) In an ordinary circuit, we can represent the conjunction or disjunction of several subformulas succinctly, whereas in a succinct regular circuit an $\land$-gate only has two children in such case. For example, given a quantifier-free subformula $\varphi_1 \land \varphi_2 \land \varphi_3$, in an ordinary circuit we can use one $\land$-gate with three children to describe it, whereas in a succinct regular circuit we need two $\land$-gates connected in the obvious way, with the subcircuits computing $\varphi_i$ as their children.

(4*) Ordinary circuits can use the arithmetic literals in an implicit way, whereas regular circuits have to use them explicitly. For instance, in
an ordinary circuit we don’t have to represent explicitly that \( x_i \neq x_j \). It can be encoded in the way the children of a gate are distributed (when the number of children is not exactly \( n \)).

(5\(^*\)) There are no labels in ordinary circuits. In contrast, regular circuits are defined based on proper labelling of wires.

(6\(^*\)) In an ordinary circuit, the children of a gate (for quantifiers) can stand for instances of distinct subformulas (subcircuits). But in a regular one, all the children should compute the same subformula of distinct instances.

We shall see that succinct regular circuits are not very different from ordinary circuits, and we can convert an ordinary one to a regular one without significantly increasing the number of gates.

A regularization of an ordinary circuit is a process that makes the quantifier structure more explicit in the circuit and a schematic labelling of its wires consistently w.r.t. some first-order sentence as follows. It replaces a gate that has \( n^i \) children with \( \sum_{j=0}^{i-1} n^j = \frac{n^i - 1}{n - 1} \) gates, i.e. replacing the gate by a perfect \( n \)-ary tree of gates. Note that such a process will not increase the number of gates significantly compared with the size of the circuit, which is very important. In addition, when we regularize a circuit, we first add two constants to its inputs, i.e. 0 and 1. We can use these two constant inputs to represent all the arithmetic atoms, i.e. \( x = y \), \( x \leq y \) and \( \text{BIT}(x,y) \). Recall that we don’t have to label the wires explicitly in succinct regular circuits. We need only show the existence of such a valid, or consistent, labelling. As a consequence, (1\(^*\)), (4\(^*\)) and (5\(^*\)) can be handled easily. Nevertheless, to obtain a slightly better lower bound, here we adapt the definition of succinct regular circuits such that the succinct representation of a block of quantifiers is allowed. That is, in the new definition, we allow the number of children of a gate to be \( n^i \) for any \( i \).

(6\(^*\)) will not make a big difference. We have briefly explained it in the discourse of Assumption 1.

(2\(^*\)) will not be a problem if \( \eta(z_1,z_2) \) is logically equivalent to some formula using bounded number of variables. To this end, we show that \( \eta(z_1,z_2) \) is indeed equivalent to a formula in \( \text{FO}^5 \). Note that \( \eta(z_1,z_2) \) is a Boolean combinations of constant number of formulas that describe intervals or size of intervals, because the atom formulas consist only of \( = \), \( \leq \) and \( \text{BIT} \), each of which can be described by three variables. Cf. Remark 53 (Corollary 55) to see why this can be justified. Note that (7.1) is equivalent to

\[
\forall x (\neg \eta(x,y) \lor E(x,y)).
\]

(7.2)
Recall that, all the arithmetic atoms are evaluated directly: their values are represented by those two constant inputs 0 and 1. Therefore, the number of gates will not increase significantly.

In general, we need to handle the following situation to ensure that not too much new variables are introduced.

\[
\forall x_1 (\eta_1(x_0, x_1) \rightarrow \exists x_2 (\eta_2(x_0, x_1, x_2) \land \forall x_3 (\eta_3(x_0, x_1, x_2, x_3) \rightarrow \cdots) \cdots))
\]

(7.3)

But, obviously, our method used to deal with the last simpler case can be applied here. Furthermore, (7.3) can be rewritten in the following form, provided that there are \( k \) quantifiers and the last quantifier is \( \forall x_k \) (the following formula is similar when the last quantifier is an existential one).

\[
\forall x_1 \exists x_2 \forall x_3 \cdots, \forall x_k (\eta(x_0, x_1, x_2, \cdots, x_k) \rightarrow \cdots) \cdots)
\]

(7.4)

Recall that \( k \geq 5 \), by Corollary 56, \( \eta(x_0, x_1, x_2, \cdots, x_k) \) doesn’t add new variables to the number of variables needed in the query. Therefore, the number of gates will not increase. It is for this reason we allow the number of children of a gate (in a succinct regular circuit) to be \( n^i \) for any \( i \geq 1 \) in the subcircuit that computes the quantifier structure. It saves three variables.

Finally, \((3^*)\) is very easy to handle. We can replace a succinctly represented conjunction or disjunction by an equivalent subcircuit whose gates has two children. Just note that we only need to add constant number of new gates to deal with it, where the number is independent of \( n \).

Suppose that we are given a first-order definable family of ordinary constant-depth unbounded fan-in circuits \( \{C_n \mid n \geq 5\} \) where \( C_n \) has \( O(n^k) \) gates. From a valid regularization of \( \{C_n\} \) we can obtain a family of succinct regular circuits \( \{C'_n \mid n \geq 5\} \) that recognize the same first-order graph property, and along any path there are at most \( k \) wires that are quantified by distinct variables because there are at most \( O(n^k) \) gates in \( C'_n \). Then the graph property defined by this family of circuits can be uniformly defined by one first-order sentence with at most \( k \) variables. The uniformity of this family of succinct regular circuits comes from the fact that we use the same process to regularize the ordinary circuits based on the first-order query that defines this family of ordinary circuits. And if one scheme of regularization works for \( C_n \), it also works for \( C_{n+1} \) because these circuits are very similar except for the value of fan-in: \( n \) for the former and \( n + 1 \) for the latter. To summarize, in principle, we can regularize a first-order definable family of ordinary circuits, provided that Assumption \([1]\) holds.

In the following we give an example to illustrate how we can obtain a regularization from a first-order query. Different from the last argument,
it is constructive, based on the first-order query defined in the following example.

**Example 2.** It was mentioned in [20], and presented in [22], a simple, and maybe optimal, circuit algorithm that computes \( k \)-Clique using \( n^k \) gates: it simply enumerates all the sets (of vertices) of size \( k \). Such a circuit family is first-order definable. Here is a \((k + 1)\)-ary first-order query that defines this circuit family. It is important that such a query is independent of \( n \), the order of a circuit. It is for this reason that the following regularization can be achieved: on the one hand, it implies that the size of the circuit after regularization will not increase significantly because the length of the sentence defining the property is independent of \( n \); on the other hand, it implies that the circuits after regularization are very similar except for the values of fan-in (e.g. \( n \) for \( C_n \) and \( n + 1 \) for \( C_{n+1} \)).

Note that two variables suffice to define a constant number, in the presence of a linear order [6]. Hence, the numbers 0, 1 and 2 are first-order definable and we shall use them for free. Let \( \bar{x} := x_0, x_1, \ldots, x_k \).

Let \( \varphi_\text{∨} (\bar{x}) \) define the unique \( \forall \)-gate.

\[
\varphi_\text{∨}(\bar{x}) := x_0 = 2 \land \bigwedge_{i \in [1,k]} x_i = 0
\] (7.5)

Let \( \varphi_\text{∨} (\bar{x}) \) define the output gate as \( \varphi_\text{∨} (\bar{x}) \).

Let \( \varphi_\text{∧} (\bar{x}) \) define those \( \land \)-gates.

\[
\varphi_\text{∧}(\bar{x}) := x_0 = 1 \land \bigwedge_{i,j \in [1,k]; i \neq j} x_i \neq x_j
\] (7.6)

By convention, here we use \( x_i \neq x_j \) to denote \( \neg (x_i = x_j) \).

Let \( \varphi_\text{¬} (\bar{x}) \) define the \( \neg \)-gates.

\[
\varphi_\text{¬}(\bar{x}) := \text{FALSE}
\] (7.7)

Let \( \varphi_\text{in}(\bar{x}) \) define the inputs.

\[
\varphi_\text{in}(\bar{x}) := x_0 = 0 \land x_1 \neq x_2 \land \bigwedge_{i \in [3,k]} x_i = 0
\] (7.8)

Let \( \varphi_0 \) define the universe of a circuit structure.

\[
\varphi_0(\bar{x}) := \varphi_\text{∨}(\bar{x}) \lor \varphi_\text{∧}(\bar{x}) \lor \varphi_\text{in}(\bar{x})
\] (7.9)
Finally, we give the heart of the definition, i.e. those arrows that exhibit
the inputs and outputs of the gates (or the edge relation between the gates).
Let \( \bar{y} := y_0, y_1, \ldots, y_k \).

\[
\varphi_R(\bar{x}, \bar{y}) := \psi_0(\bar{x}, \bar{y}) \lor \psi_1(\bar{x}, \bar{y})
\]

where

\[
\psi_0(\bar{x}, \bar{y}) := \varphi_r(\bar{x}) \land \varphi_\land(\bar{y})
\]

\[
\psi_1(\bar{x}, \bar{y}) := \varphi_\land(\bar{x}) \land \varphi_\land(\bar{y}) \land \bigvee_{i,j \in [1,k], i \neq j} (x_i = y_1 \land x_j = y_2)
\]

Note that, from \( \psi_0(\bar{x}, \bar{y}) \) we can obtain the quantifier structure needed in the
regularization. Together with \( \varphi_r(\bar{x}) \) and \( \varphi_\lor(\bar{x}) \), it tells us that the children
\( \bar{y} \) of the output gate is any string of length \( k \) where every element is distinct.
So clearly these children correspond to a block of \( k \) relativized existential
quantified variables. That is, \( \psi_0(\bar{x}, \bar{y}) \) tells us that the sentence is begin with
an existential quantifier block in the form:

\[
\exists x_1 \exists x_2 \cdots \exists x_k \left( \left( \bigwedge_{i,j \in [1,k], i \neq j} x_i \neq x_j \right) \land \xi \right)
\]

Here, for the sake of simplicity, we have used "\( x_1, x_2, \ldots, x_k \)" as the names
of the quantified variables, inheriting from the definition of the first-order
query. Note that, along any path, which assigns values to the variables
\( x_1 x_2 \cdots x_k \), the subformula \( x_1 \neq x_2 \neq \cdots \neq x_k \) can be evaluated imme-
diately: every path leads to an \( \land \)-gate; one of its children is either 0 or 1
depending on the true value of \( x_1 \neq x_2 \neq \cdots \neq x_k \); the other child of this
\( \land \)-gate is the output gate of the subcircuit that computes the subfor-
mula \( \xi \). Therefore, we do not need to adding new gates to compute those
subformulas \( x_1 \neq x_2 \neq \cdots \neq x_k \).

From \( \varphi_R(\bar{x}, \bar{y}) \) we know that \( \psi_1(\bar{x}, \bar{y}) \) defines the quantifier-free subfor-
mulas of the sentence. There is only one quantifier-free subformula in this
example. The formula \( \varphi_\land(\bar{x}) \) tells us that the atom formulas are collected
in a conjunction. The formula \( \psi_1(\bar{x}, \bar{y}) \) tells us how to bind the variables
with inputs, a process that give names to the variables in a formula.\(^{55}\) It is
possible to work out the binding directly from the query. In particular, the
subformula \( \bigvee_{i,j \in [1,k], i \neq j} (x_i = y_1 \land x_j = y_2) \) tells us that the conjunction is in
the form "\( \bigwedge_{i,j \in [1,k], i \neq j} E(x_i, x_j) \)" if we give the quantified variables the name

\(^{55}\)Without this process, we don’t know the names of variables. Hence the atoms are
in the form like \( E(?, ?) \). That is, we could only know the structure of a sentence, not
the complete description of the sentence. Using this process, we can know the precise
description, up to renaming.
“$x_1, x_2, \ldots, x_k$” inheriting from the first-order query. We can also know the binding from a technique called “consistent reverse assignment”. Assume that $\ell$ is a constant greater than the length of the query. Having a picture of $C'_\ell$ (i.e. the regularized $C_\ell$) in our mind, the binding follows the following process. Along a directed path of the quantifier structure, we have a subcircuit standing for the quantifier-free subformula. We first guess a labelling of the wires in the quantifier structure. It is a map from a value in $[n]$ to the name of a variable. Call such a map reverse assignment. Then for each subcircuit we guess a consistent reverse assignment. Because the first-query defines a circuit family that express a first-order property, there exists a consistent guess that matches both the guess of the reverse assignment of the subcircuit that computes the quantifier structure and the guesses of the reverse assignment for all the subcircuits that compute the instances of the quantifier-free subformula. Such a consistent guess must exist, for otherwise it is in contradiction with Assumption 1. Note that this process is still constructive since we can enumerate all the possible guesses, for up to $k$ distinct variables. The reason we need just consider $k$ distinct variables is because the quantifier structure is regularized such that every gate in the circuit, except those whose children are inputs, has either $n^i$ (corresponding to $i$ quantifiers) or 2 children (corresponding to “$\land$” or “$\lor$”), and there are at most $O(n^k)$ gates can be reached from any gate following the arrows.

Last but not least, the family of regularized circuits are first-order definable. The uniformity is inherited from the uniformity of the original family of gates.

In summary, suppose that we are given a first-order definable family of constant-depth circuits of size $O(n^k)$, we can convert it to a first-order definable family of succinct regular circuits of size $O(n^k)$, whose wires are not necessary explicitly labelled, that computes the same first-order property. Then we show that such property can be defined using $k$ distinct variables because the number of children of a gate in a succinct regular circuit, which corresponds to one quantifier, is $n^i$. Moreover, in this example we can even obtain the sentence defining $k$-Clique straightforward from reading the

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56 It is so named because usually an assignment maps variables to values. Such kind of map has been used by Denenberg et al., cf. 10, p.238. But we handle it differently. That is, in general we do not map a value $b$ to $v_b$.

57 In this simple example, there is only one subcircuit that computes the unique quantifier structure. In more complicated cases, it could have several such subcircuits that compute different quantifier structures.

58 If it is a block or several block of quantifiers, the number of children could be $n^i$ for some $i$. 

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structure of $C'_\ell$, using consistent reverse assignments. Hence, it is completely constructive w.r.t. this first-order query.

The following is straightforward, due to Theorem 40 and Proposition 41 (with Assumption 1).

**Corollary 42.** On condition that $k \geq 5$ and Assumption 1 holds, $k$-Clique cannot be computed by any first-order definable family of constant-depth unbounded fan-in circuits of size $O(n^{k-1})$.

It conditionally answers a question raised in [24] (cf. p.71), [22] (cf. p.10), and also in [8] (cf. p.25), based on Assumption 1. Note that, by our method, $O(n^2)$ not suffice to compute $k$-Clique if $k = 4$, because $k + 1$ variables are needed instead of $k$ in Proposition 41, and $O(1)$ not suffice to compute $k$-Clique if $k = 3$, because $k + 2$ variables are needed. Obviously, this trivial lower bound for the case $k = 3$ tells us nothing. It is not clear what are the tight lower bounds in these two special cases.

## 8 Conclusions

The study of finite model theory has been mostly motivated by questions in computational complexity theory and database theory. It has been one of the major challenges in finite model theory for a long time to establish lower bounds on finite ordered graphs, which stands for a well-known barrier that is in the way when we try to solve open problems in computational complexity using finite model-theoretic toolkit. We developed novel concepts and techniques to this end, for worst-case lower bounds.

Based on a kind of pebble games (with changing boards) over abstractions and a notion called board history, we provide an alternative proof for the strictness of bounded variable hierarchy in FO, which was first proved by Rossman [22] using tools from circuit complexity. Note that our proof is purely constructive. That is, we construct a pair of extraordinary huge graphs explicitly, use them as the game board and demonstrate the winning strategies for Duplicator with full details.

Moreover, we use the explicit constructions to prove an optimal lower bound of $k$-Clique, which fully answers a question [8, 22, 24] that goes back to an early paper of Immerman [15], which represents the challenge. Contrary to popular opinion, we find that big size is not a big issue in the constructions. On the contrary, big size even helps: intuitively, logics tend to exhibit their difference on large enough structures.
This work is extended by introducing other arbitrary arithmetic predicates to \( \text{FO} \), other than linear orders. Recall that \( \text{BIT} \) predicate can be used to define arbitrary arithmetic predicates. We show that precisely \( k \) variables are necessary and sufficient to describe \( k \)-Clique in \( \text{FO} \) on the class of finite graphs with built-in \( \text{BIT} \).

Afterwards, we apply this result to circuit complexity, which is motivated by the question on the worst-case lower bounds of constant-depth circuits raised in \([20, 8, 22, 24]\). It is also related to the question of Immerman since there is a well-known connection between first-order logic and first-order definable families of constant-depth unbounded fan-in circuits. Recall that Rossman’s tight lower-bound in average-case is also a unconditional worst-case lower bound, which says \( O(n^k) \) gates not suffice to compute \( k \)-Clique on constant-depth circuits. We improve the state of the art by a unconditional worst-case lower bound \( O(n^{k-\frac{3}{2}}) \). Then, based on a not well-known but still reasonable assumption (i.e. Assumption \([1]\)), we give the tight worst-case lower bound \( O(n^{k-1}) \). Certainly, it is not completely satisfactory for it is not unconditional. A proof from circuit complexity using its own, maybe very novel, techniques is expected to show the same tight lower bound without any assumption.

We can also study the \( k \)-Clique problem or strictness of bounded variable hierarchy in other more expressive logics. Moreover, it is interesting to know whether the notions and techniques introduced here can help to improve the state of affairs of other questions in circuit complexity. For instance, could we get better lower bounds for more general problems, say subgraph isomorphism problem, on the \( \text{FO} \)-uniform \( \text{AC}^0 \) model? Possibly, it is within reach of the techniques introduced in this paper.
References

[1] K. Amano, “k-Subgraph isomorphism on AC<sup>0</sup> circuits,” Computational Complexity, 19(2), pp.183–210, 2010.

[2] H. Andréka, I. Németi, J. V. Benthem, “Modal languages and bounded fragments of predicate logic,” Journal of Philosophical Logic, 27(3), pp.217–274, 1998.

[3] D. M. Barrington, N. Immerman, and H. Straubing, “On uniformity within NC<sup>1</sup>,” Journal of Computer and System Sciences, 41(3), pp.274–306, 1990.

[4] P. Beame, “Lower bounds for recognizing small cliques on CRCW PRAM’s,” Discrete Applied Mathematics, 29(1), pp.3–20, 1990.

[5] D. Berwanger, E. Grädel, and G. Lenzi, “The variable hierarchy of the µ-calculus is strict,” Theory of Computing Systems, vol.40, pp.437–466, 2007.

[6] A. Dawar, S. Lindell and S. Weinstein, “First order logic, fixed point logic and linear order,” In Computer Science Logic '95: Lecture Notes in Computer Science vol. 1092, Springer-Verlag, pp.161–177, 1996.

[7] A. Dawar, K. Doets, S. Lindell, and S. Weinstein, “Elementary properties of finite ranks,” Mathematical Logic Quarterly, 44, pp.349–353, 1998.

[8] A. Dawar, “How many first-order variables are needed on finite ordered structures?,” In: We Will Show Them: Essays in Honour of Dov Gabbay, Vol 1., College Publications, pp.489–520, 2005.

[9] A. Dawar, J. A. Makowsky, and D. Niwinski, The Ackermann award 2011, Report of the Jury, CSL 2011.

[10] L. Denenberg, Y. Gurevich, and S. Shelah, “Definability by constant-depth polynomial-size circuits,” Information and Control, 70(2/3), pp.216–240, 1986.

[11] H-D. Ebbinghaus and J. Flum, Finite Model Theory. Springer, 1999.

[12] R. Fagin, “Generalized First-Order Spectra and Polynomial-Time Recognizable Sets,” Complexity of Computation, (ed. R. Karp), SIAM-AMS Proc., 7, pp.27–41, 1974.
[13] M. Grohe, “Finite variable logics in descriptive complexity theory,” Bulletin of Symbolic Logic, 4(4), pp.345–398, 1998.

[14] Y. Gurevich and H. R. Lewis, “A logic for constant-depth circuits,” Information and Control, 61, pp.65–74, 1984.

[15] N. Immerman, “Upper and lower bounds for first order expressibility,” Journal of Computer and System Sciences, 25(1), pp.76–98, 1982.

[16] N. Immerman, “Languages which capture complexity classes,” In: STOC 1983: Proceedings of the 15th Annual ACM Symposium on Theory of Computing, pp.347–354, 1983.

[17] N. Immerman, “Expressibility and Parallel Complexity,” SIAM Journal on Computing, 18, pp.625–638, 1989.

[18] N. Immerman, “$\text{DSPACE}[n^k]=\text{VAR}[k + 1]$.” Sixth IEEE Structure in Complexity Theory Symposium, pp.334–340, 1991.

[19] N. Immerman, Descriptive Complexity. Springer Graduate Texts in Computer Science, New York, 1999.

[20] J.F. Lynch, “A Depth-Size Tradeoff for Boolean Circuits with Unbounded Fan-In,” In Proceedings of the conference on Structure in Complexity Theory, Lecture Notes in Computer Science 223, Springer-Verlag, pp.234–248, 1986.

[21] B. Poizat, “Deux ou trois choses que je sais de $L_n$.” Journal of Symbolic Logic, 47(3), pp 641–658, 1982.

[22] B. Rossman, “On the constant-depth complexity of $k$-clique,” In STOC’08: Proceedings of the 40th Annual ACM Symposium on Theory of Computing, pp.721–730, 2008.

[23] B. Rossman, “Ehrenfeucht-Fraïssé games on random structures,” In WoLLIC’09: Proceedings of the 15th Workshop on Logic, Language, Information and Computation, pp. 350–364, 2009.

[24] B. Rossman, Average-Case Complexity of Detecting Cliques. PhD thesis. MIT, 2010.

[25] B. Rossman, “A Tight Upper Bound on the Number of Variables for Average-Case $k$-Clique on Ordered Graphs,” In WoLLIC’12: Proceedings of the 19th Workshop on Logic, Language, Information and Computation, pp.282–290, 2012.
[26] E. Rowland, “Regularity versus complexity in the binary representation of $3^n$,” Complex Systems, 18, pp.367–377, 2009.

[27] N. Schweikardt, “Arithmetic, first-order logic, and counting quantifiers,” ACM Transactions on Computational Logic, 6(3), pp.634–671, 2005.

[28] N. Schweikardt and T. Schwentick, “A note on the expressive power of linear orders,” Logical Methods in Computer Science, 7(4), pp.1–13, 2011.

[29] B. A. Trakhtenbrot, “The Impossibility of an algorithm for the decidability problem on finite classes,” Proceeding of the USSR Academy of Sciences (in Russian), 70 (4), pp.569–572, 1950.

[30] Y. Venema, “Expressiveness and completeness of an interval tense logic,” Notre Dame Journal of Formal Logic, 31, pp.529–547, 1990.
Appendix

We put some remarks and proofs in this appendix to help the readers understand and evaluate the ideas.

**Remark 43.** As it turns out, $\mathcal{B}'_{3,m}$ is quite large even for moderate $m$. Hence, to deliver some essence of the notion “structural expansion”, we use the small structure $\mathcal{B}_3$ as the start point of an expansion.

![Figure 7:](image)

In Fig. 7, the graph on the left side is $\mathcal{B}_3$. The graph on the right side is an expansion of it. The “bricks” are akin to $\mathcal{B}_3$ except that the adjacency between the three vertices in the first column can be different. For example, the brick formed by the vertices 0, 1, ..., 5 is isomorphic to $\mathcal{B}_3$; whereas the brick formed by 0, 1, 4, 5, 6 and 7 is akin to $\mathcal{B}_3$ except that the subgraph induced by $a$, $b$ and $c$ in $\mathcal{B}_3$ is different from that induced by 0, 1 and 6 in the other. In short, the bricks are bound in such a way that their first columns respect the structure (or adjacency) of the graph $\mathcal{B}_3$. Hence, in some sense we can call $\mathcal{B}_3$ an “abstraction” (or a skeleton, or a blueprint, whatsoever) of its expansion. Moreover, we can call the latter the first abstraction, and $\mathcal{B}_3$ the second abstraction. Note that the universes of $\mathcal{B}_3$ and its (first) expansion, as well as all the bricks, are isomorphic to upright square lattices. The width of the “bricks” is 2. It “corresponds” to $\beta^1_0$ in (5.3).

Obviously, the definition of $\mathcal{B}'_{3,m}$ is more complicated. But the ideas are similar. We intend to use $X^*_i$ to denote the $i$-th abstraction. And we can regard $X^*_i$ as an “abstraction” of $X^*_{i-1}$. Hence $X^*_m$ is the highest abstraction, on the top of the hierarchy of abstractions; whereas $X^*_1$ is the lowest one, on
the bottom of this hierarchy. The index of a vertex tells us at which stage it is created in the structural expansion. The value $\beta_{m-j}^m$ tells us the width of the “bricks” we will use to build the $i$-th abstraction based on the “skeleton” $X^*_j$. Suppose $(x, y)$ is a vertex of the $p$-th abstraction. The construction should ensure that it is also a vertex in the $q$-th abstraction for any $q < p$. The vertex $(\langle x \rangle, y)$ is the projection of $(x, y)$ in the $i$-th abstraction. If $x = \langle x \rangle$, this means that $(x, y)$ is already in the $i$-th abstraction, which in turn implies that the index of $(x, y)$ is at least $i$. The value $[x]_i$ tells us where $(x, y)$ is in the $y$-th row of the $i$-th abstraction, provided that $(x, y)$ is a vertex in the first abstraction.

Example 3. We give an example on the concept “congruence label”, which is easier to illustrate in the “flat” structures $A^*_k,m$ and $B^*_k,m$ that “forget” board histories of vertices.

From the definition, if $\text{cl}(x, y) = 0, 1; m - 2; 1; \{0, 3; m; 2; 0, 1, 0; m - 1, 0; 2, 2; m; -1; 0\}$, it means that $\text{cc}([x]_{m-2}, y) = 0; y = 1; \text{idx}(x, y) = m - 2$ (i.e. $(x, y) \in X^*_m - X^*_m$); $\text{RngNum}(x, m - 2) = 1$; and $(x, y)$ is not adjacent to any vertex whose congruence label is in $\{0, 3; m; 2; 0, 1, 0; m - 1, 0; 2, 2; m; -1; 0\}$. In other words, $(x, y)$ is not adjacent to any vertex $(u, 3)$ whose index is $m$, $\text{cc}([u]_m, 3) = 0$ and $\text{RngNum}(u, m) = 2$; and $(x, y)$ is not adjacent to any vertex $(e, 0)$ whose index is $m - 1$, $\text{cc}([e]_{m-1}, 0) = 1$, $\text{RngNum}(e, m - 1) = 0$, and $(e, 0)$ is not adjacent to any vertex $(e', 2)$ whose index is $m$, $\text{cc}([e']_m, 2) = 2$ and $\text{RngNum}(e', m) = -1$.

Remark 44. Our structures are defined in a natural way. The only seemingly artificial bits are the introducing of RngNum and SW functions, which deserve more explanation. Note that $B_k$ is very symmetric (cf. section 3). Indeed, it is so symmetric that the automorphism group of a $k$-clique is isomorphic to a subgroup of the automorphism group of $B_k$ (when “forgetting” the order), which is a cyclic group. We can also use this fact to prove Theorem 3. Note that $B^*_k,m[X^*_j]$ resembles $B_k$ to some extent. However, without SW function $B^*_k,m[X^*_j]$ is not as symmetric as $B_k$ because of missing of some edges: all the edges satisfying $(\text{cc}([x]_{t-1}, y_i) - \text{cc}([x]_{t-1}, y_j)) \times (y_i - y_j) < 0$ would be missing. Spoiler can use such asymmetry to win the pebble games. But with SW function in the definition, the structures are sufficiently symmetric and Spoiler cannot exploit the asymmetry anymore. Note that SW function alone will cause a problem: without RngNum function the structure $B^*_k,m$, as well as $B_k,m$, will have $k$-cliques. To fully understand SW
function, cf. Lemma 36. And to fully understand RngNum function, cf. case (3) in the proof of Lemma 33.

Remark 45. Fact 6 comes directly from the following folklore knowledge [11].

For any \( m > 0 \), if \( O_a, O_b \) be linear orders of length greater than or equal to \( 2^m - 1 \), then \( O_a \equiv_m O_b \).

The notion “\( \equiv_m \)” is related to the standard Ehrenfeucht-Fra"issé games, wherein the players can use arbitrary number of pebbles. At the beginning, there is one interval for a linear order, i.e. the linear order itself. Recall that, in this paper, whenever we talk about an interval, it is an empty interval (i.e. no pebble is inside the interval), except that it may contain the newly picked vertex. That is, all the intervals are not overlapped. Duplicator’s strategy in such games likes the following. In the \( i \)-th round, where \( i \leq m \), assume that an interval \([a, b]\) in \( O_a \) is split into two parts, say \([a, x]\) and \([x, b] \). Duplicator tries to ensure that, the corresponding interval in \( O_b \) is also split into two parts, say \([a', x']\) and \([x', b'] \), such that

- if \( x - a < 2^{m-i} - 1 \) then \( x - a = x' - a' \);
- if \( b - x < 2^{m-i} - 1 \) then \( b - x = b' - x' \);
- if \( x - a \geq 2^{m-i} - 1 \) and \( b - x \geq 2^{m-i} - 1 \) then \( x' - a' \geq 2^{m-i} - 1 \) and \( b' - x' \geq 2^{m-i} - 1 \). \((\text{apx-1})\)

By a simple induction, we can see that it is a winning strategy of Duplicator in such games. That is, she can ensure (apx-1) throughout the game, which implies that \( a_i \leq a_j \) if and only if \( b_i \leq b_j \), for any \( a_i, a_j \) in \( O_a \) and \( b_i, b_j \) in \( O_b \), where \( a_i \vdash b_i \) and \( a_j \vdash b_j \).

Remark 46. In this remark, we assume that the game board consists of \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \). Recall that the players are playing in the associated structures \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \). We show that Duplicator can preserve the abstraction-order-condition throughout the game, at the possible price that the value of \( \xi \) is decreased by one in a round. Moreover, we show that Duplicator is able to avoid picking any object\(^{60}\) that is in the \( (\xi - 1) \)-th abstraction and that contains a critical point except for the case wherein no object of this size containing a critical point is already “picked” (and the exceptions due

\(^{59}\)Cf. Example 2.3.6 on page 22 of the second edition of [11]. The optimal lower bound used here is from a course note of Dawar.

\(^{60}\)Here, the notion “object” (cf. page 21) is slightly different from the one introduced before (cf. page 21 Remark 15).
to her basic strategy B-2., cf. Lemma 37), and the case where this object is already “picked” (in the $\xi$-th abstraction or below).

Assume that Spoiler picks $(x, y)$ whose index is $t$, and Duplicator replies with $(x', y)$ whose index is $t'$. Both of the vertices are in $X^*_1$. Note that, in the main part of the proof of Lemma 37 (from page 66), $(x, y)$ and $(x', y)$ are vertices in $X_1$.

When we talk about “picking $(x', y)$”, we are interested in $(Lx'M_\xi, y)$ instead. Once $(Lx'M_\xi, y)$ is determined, then we can determine $(x', y)$ based on the approximate hr-copycat condition (cf. (5\⋄), page 69). In the game over abstractions, assume that $[(a, y), (b, y)]$ is the interval that contains $(Lx'M_\xi, y)$ and that $[(a', y), (b', y)]$ is the corresponding interval in the other structure. Note that $(a, y), (b, y) \in X^*_\xi$. Duplicator will pick a vertex $(x', y)$ such that $a' \leq Lx'M_\xi \leq b'$. In the first round, Duplicator simply mimics Spoiler. In the following rounds, if the abstraction-order-condition is always preserved in the $m$-th abstraction, then Duplicator is happy. In the sequel, we assume that Duplicator has to resort to lower abstractions and currently they are playing the $\ell_c$-th round where $\ell_c > 1$. Therefore,

$$m > \xi \geq m - \ell_c + 2.$$

Now, we summarize the ideas, and explain briefly how they work.

Firstly, note that all the unabridged intervals formed in the game either are sufficiently big, i.e. greater than or equal to $2^{m-\ell_c} - 1$, or are isomorphic. It ensures the basic requirement for pure (induced) linear orders.

To force Duplicator to pick a critical point, in the $j$-th round Spoiler should pick $(x, y)$ in $B^*_k,m$. Assume that the interval $[(a', y), (b', y)]$, where $(x', y)$ should be settled, contains a critical point. Consider the following cases.

1. Assume that $[(Lx')_\xi/l_\xi] - [a/l_\xi] \geq 2^{m-\ell_c} - 1$ and $[b/l_\xi] - [(Lx')_\xi/l_\xi] \geq 2^{m-\ell_c} - 1$. By induction hypothesis, we know that Duplicator can make it that $[(Lx')_\xi/l_\xi] - [a'/l_\xi] \geq 2^{m-\ell_c} - 1$ and $[b'/l_\xi] - [(Lx')_\xi/l_\xi] \geq 2^{m-\ell_c} - 1$. Then by (6.2), the abstraction-order-condition holds (for any $i$-th abstraction where $1 \leq i \leq \xi$). Moreover, if $[b'/l_\xi] - [a'/l_\xi] \geq 2^{m-\ell_c+1} - 1$, then by Fact 6, Duplicator can avoid picking any object containing a critical point in this round, since the (unabridged) interval is sufficiently big and more than enough. If $[b'/l_\xi] - [a'/l_\xi] = 2^{m-\ell_c+1} - 2$, then Duplicator has to let $[(Lx')_\xi/l_\xi] - [a'/l_\xi] = 2^{m-\ell_c} - 1$. In this case, she is still able to pick an object, but possibly in the $(\xi - 1)$th abstraction, that does not contain a critical point, by (6.2).
Figure 8: The case (ii) (b): $|b'/l_{\xi-1}| - |c^*/l_{\xi-1}| \geq 2^{m-\ell_c} - 1.$

(ii) Assume that either $|\langle x \rangle_\xi/l_\xi| - |a/l_\xi| < 2^{m-\ell_c}-1$ or $|b/l_\xi| - |\langle x \rangle_\xi/l_\xi| < 2^{m-\ell_c}-1.$ Then due to her strategy, Duplicator will pick $(x', y)$ such that $(|\langle x \rangle_\xi/l_\xi| - |a'/l_\xi| = |\langle x \rangle_\xi/l_\xi| - |a/l_\xi|)$ or $(|b'/l_\xi| - |\langle x \rangle_\xi/l_\xi| = |b/l_\xi| - |\langle x \rangle_\xi/l_\xi|)$ respectively. Consider the following cases.

(a) If $|\langle x \rangle_\xi/l_\xi| - |a'/l_\xi| > 1$ and $|b'/l_\xi| - |\langle x \rangle_\xi/l_\xi| > 1$, then by (6.2), Spoiler cannot force Duplicator to pick an object in the $(\xi-1)$-th abstraction that contains a critical point, and the abstraction-order-condition holds for the lower abstractions.

(b) Assume that $|b'/l_\xi| - |\langle x \rangle_\xi/l_\xi| = 1$ and $|b'/l_\xi| - |\langle x \rangle_\xi/l_\xi| - |a'/l_\xi| \geq 1$. The case when $|b'/l_\xi| - |\langle x \rangle_\xi/l_\xi| \geq 1$ and $|\langle x \rangle_\xi/l_\xi| - |a'/l_\xi| = 1$ is similar. Note that, $(c^*, y) \in X_{m}^*$ for any critical point $(c^*, y)$. By Lemma 3, $(c^*, y) \in X_{\xi}^*$. If $|c^*/l_\xi| \neq |x'/l_\xi|$, then obviously Duplicator avoids picking the critical point. Hence, assume that $|c^*/l_\xi| = |x'/l_\xi|$. Recall that $m > \xi$, it implies that $x' \geq c^*$, because $[c^*/l_\xi] = [[c^*]_{\xi}^{min}]$, which is easy to show (cf. p. 47 for the definition of $[[c^*]_{\xi}^{min}]$. Recall that $\xi \geq m - \ell_c + 2$. Therefore, for any $(u, y)$ where $u/l_\xi \neq |x'/l_\xi|$ and $u > x'$, $|u/l_{\xi-1}| - |c^*/l_{\xi-1}| \geq \left[ [l_{\xi}^m \xi - 0.1 c^*] \right] = \left[ \frac{1}{2} \beta_{m-\xi}^{m-1} \beta_{m-\xi}^{m-1} \sum_{i < \xi} \beta_{m-i}^{m-i-1} \right] > \left[ \frac{1}{2} \beta_{m-\xi}^{m-1} \beta_{m-\xi}^{m-1} \right] \left\lfloor \frac{2^{2-2+\gamma_{\xi-1}^{m-1}} - 1}{\gamma_{\xi-1}^{m-1}} \right\rfloor > 2^{\xi-2} - 1 \geq 2^{m-\ell_c} - 1. \right\rfloor$

See Fig. 3. Note that black vertices in the figure (the 2nd, 5th, 8th, and 11th vertices) are those in $X_{\xi}^*$ and grey vertices (the 1st, 3rd, 4th, 6th, 9th, 10th and 12th vertices) are those whose indices are 1. Therefore, $|b'/l_{\xi-1}| - |c^*/l_{\xi-1}| \geq 2^{m-\ell_c} - 1$.

On the other hand, $|c^*/l_{\xi-1}| - |a'/l_{\xi-1}| > \left[ \frac{\beta_{m-\xi}^{m-1}}{2 \xi_{\xi-1}^{m-1}} \right] = \left[ \frac{\beta_{m-\xi}^{m-1+1}}{2 \xi_{\xi-1}^{m-1+1}} \right] = 2^{\xi-2} \geq 2^{m-\ell_c}$. See Fig. 9. As a consequence, $|b'/l_{\xi-1}| - |a'/l_{\xi-1}| \geq 2^{m-\ell_c+1} - 1$. Therefore, in this case Duplicator has the freedom to avoid picking a critical point, since the (unabridged) interval is big enough.

Similarly, $|b/l_{\xi-1}| - |\langle x \rangle_\xi/l_{\xi-1}| \geq 2^{m-\ell_c} - 1$ and $|\langle x \rangle_\xi/l_{\xi-1}| - |a/l_{\xi-1}| \geq 2^{m-\ell_c}$. In other words, in such case, the abstraction-
order-condition holds (for any \(1 \leq i \leq \xi - 1\)).

(c) Assume that either \(\lfloor x'_{\xi} / l_{\xi} \rfloor = \lfloor a'/l_{\xi} \rfloor\) or \(\lfloor x'_{\xi} / l_{\xi} \rfloor = \lfloor b'/l_{\xi} \rfloor\).

Let \((c^*, y)\) be any critical point that is settled between \((a', y)\) and \((b', y)\). If \([a']_{\xi} = [c^*]_{\xi}\), it means that the pebbled vertex \((x_{a'}, y)\), where \(x_{a'} = a'\), is projected to \((c^*, y)\) in the \(\xi\)-th abstractions, i.e. \(x_{a'} = c^*\), because of Lemma 4 and \([a']_{\xi} = [c^*]_{\xi}\). See Fig. 10. If Duplicator is forced to pick \((c^*, y)\) due to the abstraction-order-condition, i.e. \((x', y)\) is \((c^*, y)\), then \(a' = x'\). Therefore, by the abstraction-order-condition, \(x = a\).

But, Duplicator wins this round since she wins the last round. Similarly, if \([b']_{\xi} = [c^*]_{\xi}\), Duplicator can also win this round. Therefore, we need only consider the case where \([a']_{\xi} \neq [c^*]_{\xi} \neq [b']_{\xi}\). See Fig. 11. Because \((a', y), (b', y) \in X_{\xi}\), we have \([b']_{\xi-1} - [a']_{\xi} \geq 2^\xi > 2^{m-\ell+1}\). Therefore, Duplicator has the freedom to avoid picking any object in the \((\xi - 1)\)-th abstraction that contains a critical point in such case.

In summary, in all the cases, the abstraction-order-condition holds for any abstractions below the \(\xi\)-th abstractions. Therefore, the abstraction-
order-condition works. Moreover, Duplicator can also avoid picking critical points in most cases.

Another issue, which is directly related to linear orders, needs to be mentioned briefly. Recall Strategy 2 in the proof of the main lemma 37, we defaultly assume that Lemma 36 applies in accordance with the abstraction-order-condition. This is, however, quite obvious, according to the same intuition as presented in the last arguments. That is, a unit of difference in higher abstraction equals huge difference in lower abstractions. Hence, once the game enters lower abstractions, abstraction-order-condition will be ensured automatically.

**Remark 47.** \([([x]_{\pi})_{r} - \pi_r, \pi_r] = [\pi_r]_{\pi} x \beta_{m-\xi} m r - \frac{1}{2} \sum_{r<j \leq \xi} \beta_{m-j} - [\pi_r]_{\pi_r} < \frac{x}{\beta_{m-\xi}} \beta_{m-\xi} m r + \frac{1}{2} \sum_{r<j \leq \xi} \beta_{m-j} - \frac{1}{2} \sum_{r<j \leq \xi} \beta_{m-j} + 1 < \frac{1}{2} \beta_{m-\xi} m r + \frac{1}{2} \beta_{m-\xi} m r \beta_{m-\xi}

On the other hand, \([([x]_{\pi})_{r} - \pi_r, \pi_r] > \frac{1}{2} \beta_{m-\xi} m r - \frac{1}{2} \sum_{r<j \leq \xi} \beta_{m-j} - \frac{1}{2} \sum_{r<j \leq \xi} \beta_{m-j} + \frac{1}{2} \beta_{m-\xi} m r \beta_{m-\xi}

In other words, \([([x]_{\pi})_{r} - \pi_r, \pi_r] < \beta_{m-\xi} m r. It implies that, we can usually omit “mod \(\beta_{m-\xi} m r\)”. For example, it is not difficult to see that

\[
[x]_{r} \equiv [x']_{r} \equiv \mod \beta_{m-\xi} m r

\]

**Remark 48.** Note that \(\mathfrak{A}_{k, m}^*\) is an ordinary “flat” graph. By contrast, \(\mathfrak{A}_{k, m}\) has a logical structure (or temporal structure) that reflects evolution of a game, thereby enforce the game evolves reasonably. If Spoiler try to breakout current evolution by picking a vertex associated with board configuration that is not in the progress, the game will split into two, each progresses in isolation, with its own evolution of board configurations, since any pair of vertices whose board histories are not in a succession (or one history cannot continue the other) are not adjacent. Two evolutions may converge, if their board configurations are almost the same except for one vertex. However, two board histories that once diverge will never converge again since the initial segments of the histories will never match again, which is formalized by the following Lemma.

**Fact 8.** For any \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathfrak{X}_1\), if \((x_1, y_1) \xrightarrow{*}{(x_3, y_3)}, (x_2, y_2) \xrightarrow{*}{(x_3, y_3)} \text{ and } \chi(x_1, y_1) \neq \chi(x_2, y_2)\) bc, then either \((x_1, y_1) \xrightarrow{bc}{(x_3, y_3)}\)
\[(x_2, y_2) \text{ or } (x_2, y_2) \xrightarrow{\ast}_{BC} (x_1, y_1), \text{ but not both.}\]

It is obvious by definition. Here we give a summary.

\[(x_1, y_1) \xrightarrow{\ast}_{BC} (x_3, y_3) \text{ and } (x_2, y_2) \xrightarrow{\ast}_{BC} (x_3, y_3)\] say that the initial parts of the board histories of \((x_1, y_1)\) and \((x_2, y_2)\) must be the same, thereby one of the history can evolve to the other. Also note that, by definition, evolution has a direction, akin to time. That is, if board history \(H_1\) can evolve to history \(H_2\), then \(H_2\) cannot evolve to \(H_1\).

**Remark 49.** The proof is involved. We sketch the ideas as the following. Assume that Spoiler picks \((x, y)\) in \(\overline{A}_{k,m}\), and Duplicator replies with \((x', y)\) in \(\overline{B}_{k,m}\). We use \((\overline{A}^*_{k,m}, \overline{B}^*_{k,m})\) to denote the associated game board of \((\overline{A}_{k,m}, \overline{B}_{k,m})\). Hence, if Spoiler picks \((x, y)\) in \(\overline{A}_{k,m}\), we take it that he also picks \((x^\flat, y)\) in \(\overline{A}^*_{k,m}\) in an associated game.

First of all, we introduce a sort of imaginary games called *pebble games over changing board*. It is similar to the usual pebble games except that the game board can be different in each round, i.e. the pair of graphs ("flat" structures) are continuously changing during the game. Such games are the basis for the so called virtual games. We use "virtual games" to denote the kinds of pebble games over changing board wherein the players play the games in their mind without really putting pebbles on the board. For simplicity, here we assume that no vertex is pebbled before Spoiler picks \((x, y)\). A virtual game in such a simple setting consists of \(i_{k,m}^{x,y} - 1\) "virtual rounds". No vertex is pebbled at the beginning of this virtual game. Spoiler “picks” in a structure akin to \(\overline{A}^*_{k,m}\) (varying in each round) according to \(\chi(x, y) | H(j), \text{ for } j = 1 \text{ to } i_{k,m}^{x,y} - 1\), and Duplicator “replies” in the other (changing) structure, i.e. in a structure akin to \(\overline{B}_{k,m}\). The point is that, in a changing board, for any vertex \((u, v) \in \mathbb{X}^*_1\), if \((u, v)\) is not adjacent to \((x^\flat, y)\) simply because it is in \(\chi(x^\flat, y) | S\), then it is already pebbled. Similarly, for any vertex \((u', v)\) in the other structure w.r.t. the adjacency to \((x', y)\), Duplicator uses virtual games to determine the board history of the vertex \((x', y)\) she is going to pick.

We are able to prove the following claim: Duplicator has a strategy such that, if an edge is forbidden in one structure due to discontinuities, so is the corresponding edge in the other structure; moreover, the orders of the board histories of the pebbled vertices can be properly taken care of by Duplicator.

Therefore, via the virtual games, we can reduce the original game to the associated game over changing board, wherein Duplicator uses strategy over abstractions. Note that the \(\xi\)-th abstraction is the abstraction that Duplicator would care about at the start of the current round. Suppose we have a
version of abstraction-order-condition that is similar to 1°–6° introduced in page 27. The point is that, if Duplicator can win this round in the game over the \(\xi\)-th abstraction, she can also win this round in lower abstraction. Particularly, it means she can win this round in the first abstraction, i.e. over the game board \((\tilde{A}_{k,m}^*, \tilde{B}_{k,m}^*)\). At the beginning of the game, \(\xi = m\). Suppose Spoiler picks \((x^0, y)\) and Duplicator replies \((x^0, y)\) in the current round. While playing the game, in each round Duplicator first finds the candidate positions that are in accordance with the abstraction-order-condition, and that usually form intervals containing vertices of necessary type labels (using an auxiliary game over linear orders); afterwards, she determines the type label of \((x^0, y)\) and makes the pick. The following strategy will help her decide the type label of \((x^0, y)\). It helps Duplicator keep the game board in partial isomorphism at the end of a round, not only in the original associated game, but also in the associated game over abstractions. The strategy (i.e. Strategy 1–Strategy 3) of Duplicator can be sketched briefly and roughly as follows. In short, she needs to ensure that the winning-condition-set is preserved throughout the game.

1. (Strategy 1) Assume that Spoiler picks \((x^b, y)\) in \(X^*_\xi\). Duplicator first *tries to* use the strategy that works for the \(\xi\)-th abstraction of the structures. That is, she plays the game over the \(\xi\)-th abstraction and pick a vertex in \(X^*_\xi\) s.t. the winning-condition-set holds.

2. (Strategy 2) Assume that Spoiler picks \((x^b, y)\) in \(X^*_\xi\). If Strategy 1 does not work, i.e. Duplicator cannot pick a vertex in \(X^*_\xi\) satisfying the winning-condition-set, then Duplicator resorts to the strategy that works for the \((\xi - 1)\)-th abstraction of the structures and pick a vertex in \(X^*_{\xi - 1} - X^*_\xi\). Note that, she can always find such a vertex that ensures a win for her in this round. More precisely, she first find a vertex of index \(\xi - 1\) such that it is adjacent to the projection of all the pebbled vertices in the \((\xi - 1)\)-th abstraction (cf. Lemma 36(4)); afterwards, she adjust her pick such that it satisfies the winning-condition-set.

This strategy also works if \(\text{idx}(x^b, y) = \xi - 1\). Duplicator will pick a vertex of index \(\xi - 1\). In addition, she ensures that \(\text{cc}([x^b]_{\xi - 1}, y) = \text{cc}([x^0]_{\xi - 1}, y), g(x^b) = g(x^0)\) and \(\text{RngNum}(x^b, \xi - 1) = \text{RngNum}(x^0, \xi - 1)\).

3. (Strategy 3) If Spoiler picks \((x^b, y)\) in \(X^*_t - X^*_t+1\), where \(t < \xi - 1\), then Duplicator regards it as if \((\lfloor x^0 \rfloor)_{\xi}, y)\) is picked, and replies with \((x^0, y)\) such that \((\lfloor x^0 \rfloor)_{\xi}, y)\) is the vertex she will pick to respond \((\lfloor x^0 \rfloor)_{\xi}, y)\)
using her strategy that works in the $\xi$-th abstraction (or responds with $(|x^0|_\xi_{-1}, y)$ using her strategy that works in the $(\xi - 1)$-th abstraction, if she cannot respond properly in the $\xi$-th abstraction). At the same time, Duplicator ensures that in the original game (5) holds, i.e. $x^b - (|x^0|_\xi)$ is roughly the same as $x^b - (|x^0|_\xi)$ (or $x^b - (|x^0|_{\xi-1})$ if she cannot respond properly in the $\xi$-th abstraction). Hence Duplicator is an approximate hr-copycat, which can ensure that lower abstractions are in partial isomorphism if so is some higher abstraction. Moreover, Duplicator resorts to a sort of game reduction from lower abstraction to higher abstraction to prevent Spoiler from finding difference via linear order (the auxiliary game over linear can only tell her the approximate (candidate) positions she should consider; it cannot avoid (x)). It also helps Duplicator decide the type label of $(x^b, y)$. In sum, Duplicator can ensure that, without exploring the difference in higher abstraction of the structures, Spoiler is not able to find difference between the structures by exploring the lower abstractions at some specific stage of the game.

**Remark 50.** It is obvious that the strategy introduced in the proof of Lemma 37 preserves (6.7) throughout the games. The reason we need (6.7) is that we depend on (6.16) to ensure that (6) holds in Strategy 2. Moreover, in Strategy 3 we also use it to ensure that (7) (vii) and (7) (iv) hold simultaneously, cf. the corresponding remark. We haven’t introduced the standard concept “types” yet. But, $\text{cl}(x^b, y) = \text{cl}(u^3, v)$ means that the vertices $(x^b, y)$ and $(u^3, v)$ are roughly the same, although they are usually different objects in the linear order, and they do not necessary have the same type. But all the critical points with the same second coordinate have the same type. We introduce the concept “type label” as an alternative of “type”, which gives us some flexibility in the constructions and proofs.

**Remark 51.** In Strategy 1 (cf. the proof of Lemma 37), we claim that “If $(x, y) \leadsto (u, v)$ for some pebbled vertex $(u, v)$, then Duplicator simply let $(x^*, y) = (u^*, v)^{\text{cl}}_{A}$, and we are able to show that (6) and (3) hold”. We first show that (6) holds. In other words, if $(x, y) \xrightarrow{\text{con}}_{\text{BH}} (u, v) \land (x^3, y) \in (u, v)[\text{BC}]$ for some pebbled vertex $(u, v)$, then $(x^3, y)$ is adjacent to a vertex in $\text{cl}_A$ if and only if $(x^b, y)$ is adjacent to the corresponding vertex in $\text{cl}_B$. Here we give more explanation.

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61 Just note that a unit of difference in higher abstraction means a huge difference in lower abstractions w.r.t. distance of first coordinates.
We can divide the pebbled vertices in a structure into two sets according to whether their associated board histories are in continuity with the board history of \((x,y)\). By 2 (b) of Definition 27, we know that, a vertex is not adjacent to \((x,y)\) if their associated histories are not in continuity. Recall that \((x,y) \vdash (x',y)\) and \((u,v) \vdash (u',v)\). By Claim 3 we know that if \((u,v)\) is not adjacent to \((x,y)\) because of this reason, so is \((u',v)\) to \((x',y)\).

Recall that the set of pebbled vertices, whose histories are in continuity with the history of \((x,y)\), is \(\tilde{c}_A\) and \(\tilde{c}_A \vdash \tilde{c}_B\). By assumption, \((u,v) \in \tilde{c}_A\).

Recall that Duplicator has a winning strategy in the virtual games over changing board that determine the board history of a vertex (this strategy is akin to Strategy 1 - Strategy 3 but the pebbled vertices at the start of a round may be different). Duplicator’s strategy only depends on the pair of board configurations at the start of the current virtual round.

By inductive hypothesis, Duplicator wins the virtual round wherein the pair of board configurations at the start are \((u,v)[BC]\) and \((u',v)[BC]\), and the players pick \((u,v), (u',v)\) in this virtual round. Therefore, \((x,y)\) is adjacent to \((u,v)\) if and only if \((x',y)\) is adjacent to \((u',v)\), according to the definition of virtual games, B-3 and the premise that \((x^b,y) \in (u,v)[BC]\).

Moreover, for any \((u^*,v^*) \in \tilde{c}_A\) where \((u^*,v^*) \vdash (u'^*,v'^*)\), if \((u^*,v^*) \rightarrow (x,y)\), then by Claim 3, \((u'^*,v'^*) \rightarrow (x',y)\), and by the transitivity of \(\rightarrow\), \((u^*,v^*) \rightarrow (u',v)\).

So is \((u^*,v^*) \rightarrow (u',v)\). By definition, \((x,y) \overset{\text{con.}}{\rightarrow} (u,v)\) and \((x,y)[BC] \subseteq (u,v)[BC]\). Therefore, \((u^*,v^*) \in (x,y)[BC] \subseteq (u,v)[BC]\).

Hence, \((u^*,v^*)[BC] \circ (u^*,v^*) \subseteq (u,v)[BC]\). In other words, both \((x^b,y)\) and \((u^*,v^*)\) are in \((u,v)[BC]\). Similarly, \((x^b,y),(u^*,v^*) \in (u',v')[BC]\). It implies that \((x^b,y)\) is adjacent to \((u^*,v^*)\) if and only if \((x^b,y)\) is adjacent to \((u^*,v^*)\), since Duplicator wins the virtual round wherein the pair of board configurations is made of \((u,v)[BC]\) and \((u',v')[BC]\) at the start of the round, and the players pick the pair of vertices \((u^b,v)\) and \((u^b,v)\). For any vertex that is not in \(\tilde{c}_A\), say \((a,b)\) where \((a,b) \vdash (a',b)\), we know that \((a,b),(x,y)) \notin E^A\) and \((a',b),(x,y)) \notin E^B\), by Claim 3 and Definition 27.

All in all, we have shown that the game board is in partial isomorphism after the players pick \((x,y)\) and \((x',y)\), if \((x,y) \overset{\text{con.}}{\rightarrow} (u,v) \land (x^b,y) \in (u,v)[BC]\) for some pebbled vertex \((u,v)\).

In the above argument, we show that (6) holds, based on the assumption that Duplicator has a winning strategy (cf. Strategy 1 - Strategy 3) in the

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62 It means that, in the virtual rounds of a board history wherein Spoiler picks one vertex, e.g. \((x,y)\), in the pair \((x,y) \vdash (x',y)\), Duplicator always responds with the other vertex, i.e. \((x',y)\), in the pair, if the game board is in the same state, i.e. the pair of board configurations is the same.
virtual games. Note that this strategy is also a strategy over abstractions. In other words, \( \emptyset \) can also be ensured, using similar argument.

**Remark 52.** If the game board is in partial isomorphism over the \( \xi \)-th abstraction at the start of the current round, then we can show that this also holds over the \((\xi - 1)\)-th abstraction. Here we give a brief explanation, since many ideas have already been explained in Strategy 1–Strategy 3. Firstly, for any pebbled vertex \((u, v)\), we have \((\|u^b\|_\xi, v) \not\models ((\|u^b\|_{\xi - 1}, v) \not\models \Omega\). The readers can cf. Strategy 3 for the arguments needed (i.e. the arguments for \(D \models D'\)). The main point is that, if \(\text{idx}(\|u^b\|_\xi, v) < \xi\), it implies that Duplicator has used Strategy 3 in the round where \((u, v)\) is picked. Then by \(\emptyset\) (i) (ii), we have \(\text{idx}(\|u^b\|_{\xi - 1}, v) = \text{idx}(\|u^b\|_{\xi - 1}, v), \) say equal to \(i\), and \(\text{cc}(\|u^b\|_{\xi - 1}, i, v) = \text{cc}(\|u^b\|_{\xi - 1}, i, v)\) if \(i \neq \xi\). Second, for similar reason, for pebbled pairs \((u, v) \models (u', v)\) and \((e, f) \models (e', f)\), if either \(\text{idx}(\|u^b\|_{\xi - 1}, v) < \xi\) or \(\text{idx}(\|e^b\|_{\xi - 1}, f) < \xi\), then we can show that

\[
\text{cc}(\|u^b\|_{\xi - 1}, \ell, v) = \text{cc}(\|u^b\|_{\xi - 1}, \ell, v)
\]

\[
\text{cc}(\|e^b\|_{\xi - 1}, \ell, f) = \text{cc}(\|e^b\|_{\xi - 1}, \ell, f),
\]

where \(\ell = \min\{\text{idx}(\|u^b\|_{\xi - 1}, v), \text{idx}(\|u^b\|_{\xi - 1}, v)\}\). Third, due to \(\emptyset\) (iv), \(\|u^b\|_{\xi - 1}, v) = \mathcal{S}(\ell, (x, y))[\mathcal{BC}] = \mathcal{S}(\ell, (x', y))[\mathcal{BC}]\), if \(v \neq y\). Fourth, \(\text{BIT}(\mathcal{S}(\ell, i, y), (\|x^b\|_{\xi - 1}, y), \mathcal{q}(v, y))\) if \(\text{BIT}(\mathcal{S}(\ell, i, y), (\|x^b\|_{\xi - 1}, y), \mathcal{q}(v, y))\) if \(\text{idx}(\|u^b\|_{\xi - 1}, v) < \xi\), due to \(\emptyset\) (vi). Finally, we need explain one more thing: the adjacency determined by \(\text{sgn}(\|u^b\|, (\|e^b\|, f))\) will not cause a problem when \(\xi\) is decreased by 1, because of \(\emptyset\) (v). Note that, so far we have only considered the case when \(\text{idx}(\|u^b\|_{\xi - 1}, v) < \xi\) or \(\text{idx}(\|e^b\|_{\xi - 1}, f) < \xi\). If both \(\text{idx}(\|u^b\|_{\xi - 1}, v) \geq \xi\) and \(\text{idx}(\|e^b\|_{\xi - 1}, f) \geq \xi\), then obviously \(\|u^b\|_{\xi - 1}, v) is adjacent to \(\|e^b\|_{\xi - 1}, f)\) if \(\|u^b\|_{\xi - 1}, v) is adjacent to \(\|e^b\|_{\xi - 1}, f)\), because in such case \((\|u^b\|_{\xi - 1}, v) = (\|u^b\|, v)\) and \((\|u^b\|_{\xi - 1}, v) = (\|e^b\|, v)\). In short, the game bord is still in partial isomorphism w.r.t. the edges when \(\xi\) is decreased by 1.

Moreover, we can also show that the game bord is still in partial isomorphism w.r.t. the orders when \(\xi\) is decreased by 1. By Lemma 5, we have \(u^b = \|u^b\|, 1 \leq i \leq \text{idx}(u^b, v)\). Therefore, for pebbled pairs \((u, v) \models (u', v)\) and \((e, f) \models (e', f)\), if both \(\text{idx}(\|u^b\|_{\xi - 1}, v) \geq \xi\) and \(\text{idx}(\|e^b\|_{\xi - 1}, f) \geq \xi\), then \(\|u^b\|_{\xi - 1} \leq \|e^b\|_{\xi - 1}\) if \(\|u^b\|_{\xi - 1} \leq \|e^b\|_{\xi - 1}\). That is, the following holds:

\[
\|u^b\|_{\xi - 1} \leq \|e^b\|_{\xi - 1} \Leftrightarrow \|u^b\|_{\xi - 1} \leq \|e^b\|_{\xi - 1}
\]

Otherwise, if either \(\text{idx}(\|u^b\|_{\xi - 1}, v) \geq \xi\) or \(\text{idx}(\|e^b\|_{\xi - 1}, v) \geq \xi\) but not both, then clearly either \(\|u^b\|_{\xi - 1} < \|e^b\|_{\xi - 1}\) and \(\|u^b\|_{\xi - 1} < \|e^b\|_{\xi - 1}\), or, \(\|u^b\|_{\xi - 1} > \}
\( \langle e^b \rangle_{\xi-1} \) and \( \langle u^b \rangle_{\xi-1} > \langle e^a \rangle_{\xi-1} \). If \( \text{idx}(\langle u^b \rangle_{\xi-1}, v) < \xi \), then it is because \( \langle u^b \rangle_{\xi-1} \) roughly equals \( \langle u^v \rangle_{\xi-1} \) (modulo \( \beta^{m-\xi+1} \)).\(^{63}\) Now suppose that both \( \text{idx}(\langle u^b \rangle_{\xi-1}, v) < \xi \) and \( \text{idx}(\langle e^a \rangle_{\xi-1}, v) < \xi \). In such case \( \text{idx}(\langle u^b \rangle_{\xi-1}, v) = \text{idx}(\langle u^a \rangle_{\xi-1}, v) \) and \( \text{idx}(\langle e^b \rangle_{\xi-1}, v) = \text{idx}(\langle e^a \rangle_{\xi-1}, v) \). If \( \text{idx}(\langle u^b \rangle_{\xi-1}, v) \neq \text{idx}(\langle e^a \rangle_{\xi-1}, v) \), then \( \langle u^b \rangle_{\xi-1} \neq \langle e^a \rangle_{\xi-1} \). If \( \langle u^b \rangle_{\xi} = \langle e^a \rangle_{\xi} \), then, similar to the last case, \( \langle e^b \rangle_{\xi-1} \) clearly holds. If \( \langle u^b \rangle_{\xi} \neq \langle e^a \rangle_{\xi} \), then \( \langle e^b \rangle_{\xi-1} \) holds because a "unit" of difference in higher abstraction is huge in lower abstraction, i.e. a vertex in higher abstraction corresponds to a very big interval in lower abstractions.\(^{64}\) Suppose that \( \text{idx}(\langle u^b \rangle_{\xi-1}, v) = \text{idx}(\langle e^b \rangle_{\xi-1}, v) \). Similar to the last case, \( \langle e^b \rangle_{\xi-1} \) clearly holds if \( \langle u^b \rangle_{\xi-1} \neq \langle e^b \rangle_{\xi-1} \). Assume that \( \langle u^b \rangle_{\xi-1} = \langle e^b \rangle_{\xi-1} \), \( \langle e^b \rangle_{\xi-1} \) is easy to prove because of \( \langle 6.25 \rangle \).

In summary, the game board is in partial isomorphism over the \( (\xi-1) \)-th abstraction if it is in partial isomorphism over the \( \xi \)-th abstraction, at the start of the current round.

**Remark 53.** It is well-known that the bounded variable hierarchy collapses to \( \text{FO}^3 \) on coloured linear orders.\(^{21}\) Similarly, we can prove that it also collapses to \( \text{FO}^3 \) on pure arithmetic structures, using similar pebble game type argument (cf., e.g., [8], p.9–p.10, or [19] p.105–p.107).

For clarity and proofreading, we put this proof here. Note that the proof of Lemma 3 introduced in [8] implicitly relies on transitivity of linear orders. But it is not true for BIT. Therefore, we need adapt the lemma as well as the proof a little bit to ensure that the partial isomorphisms over pairs of small pieces of structures can be merged consistently into one partial isomorphism over a pair of bigger piece. That is, we need to show that Duplicator’s strategies in 3-pebble games can be merged to ensure one partial isomorphism that extends all the partial isomorphisms in the 3-pebble games. Recall that we assume the structures in discourse to be \( (\leq, \text{BIT}) \)-structures. That is, here we only consider pure arithmetic structures.

**Lemma 54.** Let \( s = (a_1, \ldots, a_\ell) \) and \( t = (b_1, \ldots, b_\ell) \) be \( \ell \)-tuples where \( a_i \in |\mathfrak{A}| \), \( b_i \in |\mathfrak{B}| \) and \( a_i \leq a_{i+1}, b_i \leq b_{i+1} \) for any \( i \). If \( (\mathfrak{A}, a_i, a_j) \equiv^3_m (\mathfrak{B}, b_i, b_j) \) for any \( 1 \leq i, j \leq \ell \), then \( (\mathfrak{A}, s) \equiv_m (\mathfrak{B}, t) \).

**Proof.** The proof is by induction on \( m \). We only focus on BIT. For the argument that takes care of linear orders, the readers can cf. e.g. [8] or [19].

**Basis:**

\(^{63}\) It is equivalent if we ignore a difference in distance up to \( u^c_{\xi-1} \).

\(^{64}\) More precisely, \( \beta^{m-\xi} \) is greater than both \( |\langle u^b \rangle_{\xi} - |\langle u^a \rangle_{\xi-1}| \) and \( |\langle u^a \rangle_{\xi} - |\langle u^b \rangle_{\xi-1}| \).
Let \( f(a_i) = b_i \). If \( m = 0 \), then it is easy to verify that the map \( f \) defines a partial isomorphism from \( \mathfrak{A} \) to \( \mathfrak{B} \).

We can take it that \((a_i, b_i)\) be the pair of elements that are pebbled in the same round.

**Induction step:**

Assume that the claim holds for \( m = d \) and that \((\mathfrak{A}, a_i, a_j) \equiv_{d+1}^3 (\mathfrak{B}, b_i, b_j)\) for any \( 1 \leq i, j \leq \ell \). We need to show that \((\mathfrak{A}, s) \equiv_{d+1} (\mathfrak{B}, t)\). In the \((d+1)\)-th round, if Spoiler picks a pebbled element \( u \), then Duplicator simply picks the other pebbled element in the pair containing \( u \) and by induction hypothesis she wins this round. Hence we assume that Duplicator picks a new element in this round. Suppose w.l.o.g. that, in the first round of the Ehrenfeucht-Fraïssé game \( \mathcal{D}_{d+1}( (\mathfrak{A}, s), (\mathfrak{B}, t) ) \), Spoiler picks \( a^* \) in \( \mathfrak{A} \). Moreover, assume that \( a_i \leq a^* \leq a_{i+1} \) for some \( i \) (The cases when \( a_i \leq a^* \) and \( a^* \leq a_1 \) are similar). Duplicator can resort to the strategy that works over \( \mathcal{D}_{d+1}( (\mathfrak{A}, a_i, a_{i+1}), (\mathfrak{B}, b_i, b_{i+1}) ) \). The point is that three variables are necessary and sufficient to simulate the Ehrenfeucht-Fraïssé game in the 3-pebble game over a piece of the structures s.t. its strategy can be extended to give an isomorphism on a bigger piece, where \( a_i \) and \( a_{i+1} \) can be regarded as either constants or pebbled vertices (the pebbles are from the three pairs of pebbles in the pebble game). Observe that she really has a family of strategies work well in this round, varying on \( \text{BIT}(a^*, a_j) \) (if \( a_j \leq a^* \)) or \( \text{BIT}(a_j, a^*) \) (if \( a^* \leq a_j \)) for any \( a_j \) different from \( a_i \) and \( a_{i+1} \), only if one of them works well. For example, suppose that \( \text{BIT}(a^*, a_i) \) is true and \( \text{BIT}(a_{i+1}, a^*) \) is false. By induction hypothesis, Duplicator can find \( b^* \) such that \( \text{BIT}(b^*, b_i) \) is true and \( \text{BIT}(b_{i+1}, b^*) \) is false. Then she has a family of strategies that are in accordance with this condition, but different in other aspects, e.g. she can choose the one that sets \( \text{BIT}(b^*, b_{i-1}) \) to true or she can choose the one that sets \( \text{BIT}(b^*, b_{i-1}) \) to false. Note that either way leads to a valid strategy, which is crucial for the following arguments.\(^{65}\)

Similarly, for any \( j', j \), we can find a family of strategies work for Duplicator over the pebble game \( \mathcal{D}_{d+1}( (\mathfrak{A}, a_j, a_{j'}), (\mathfrak{B}, b_j, b_{j'}) ) \). The point is that the intersection of these family of strategies is not empty only if one of them works. Therefore any strategy in the intersection works over any 3-pebble games mentioned. Hence, in the first round Duplicator need only choose a strategy in the intersection to respond Spoiler, and this strategy ensures that \((\mathfrak{A}, a^*, a_p) \equiv_{d}^3 (\mathfrak{B}, b^*, b_p)\) for any \( p \). Let \( s' \) be the ordered list that ex-

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\(^{65}\)It explains why this proof does not work if the signature of the structures contains one binary relation that is not fixed as \( \text{BIT} \) does. Usually we also call \( \text{BIT} \) a background relation.
tends \( s \) by inserting the element \( a^* \) in appropriate position and similarly for \( t' \) by inserting \( b^* \). Then by induction hypothesis, we have \((A, s') \equiv_d (B, t')\). Therefore, \((A, s) \equiv_{d+1} (B, t)\).

A diagram may help us to understand the computation of intersection of families of strategies. Here we give a small example to illustrate it. See Fig. 12. Here \( s = (a_1, a_2, a_3, a_4) \) and \( t = (b_1, b_2, b_3, b_4) \). In the 3-pebble games, We assume w.l.o.g. that Spoiler picks \( a^* \) and Duplicator responds with \( b^* \). The elements are listed according to the linear order. Hence \( a^* \) is the third element in the order. For each entry \((a_i, a_j)\) that has a value 0 or 1, we can see that \( a_j \leq a_i \). And this entry tells us whether \( \text{BIT}(a_i, a_j) \) is 0 or 1. Note that “*” in the entries stands for either 0 or 1, i.e. both values are allowed. Pink cells stand for the partial isomorphisms that should be fixed in the corresponding game (indicated on the right side).

Once Lemma 54 is proved, by the Theorem 4 in [8], we know that the following holds.

**Corollary 55.** The bounded variable hierarchy collapses to \( \text{FO}^3 \) on pure arithmetic structures.

With a careful analysis of the proof of Lemma 54, we get a variant of Corollary 55 as follows.

**Corollary 56.** For any \( k \geq 5 \), on pure arithmetic structures, any sentence \( Q_1x_1Q_2x_2 \cdots Q_kx_k \varphi(x_1, x_2, \cdots, x_k) \) is equivalent to a sentence in \( \text{FO}^k \), where \( Q_i \in \{\exists, \forall\} \) and \( \varphi \) is any first-order formula.

**Proof of Lemma 4**

**Proof.** Since \((x', y) \in X_i^*\), by definition, we have

\[
x' = \langle x' \rangle_i = [x']i\beta_{m-i} + \frac{1}{2} \sum_{1<p \leq i} \beta_{m-p}^{m-1}
\]

By definition, we also have

\[
\langle x \rangle_i = [x]i\beta_{m-i} + \frac{1}{2} \sum_{1<p \leq i} \beta_{m-p}^{m-1}
\]

Therefore, \( x' = \langle x \rangle_i \), insomuch as \( [x']i = [x]i \).

**Proof of Lemma 5**
Figure 12: This example illustrates that the intersection of the families of strategies is not empty, and Duplicator can compute the intersection to obtain a strategy that works for the game $\varnothing_m((A, s), (B, t))$. 
Proof. Since \((x, y) \in X^*_i\), by definition \(x = \langle x \rangle_i\). It is trivial when \(i = 1, 2\).
For any \(2 < i \leq m\), we show that \(x = \langle x \rangle_{i-1}\) if \(x = \langle x \rangle_i\).

By definition, and \(\beta_{m-i}^{m-1}/\beta_{m-i+1}^{m-1} = \gamma_{m-i+1}/\gamma_{m-i} > i - 2\), we have
\[
\sum_{1 < j \leq i-1} \beta_{m-j}^{m-1} < (i - 2)\beta_{m-i}^{m-1} < \beta_{m-i}^{m-1}.
\] (8.3)

By definition,
\[
\frac{1}{2}\beta_{m-i}^{m-1} \text{ is divisible by } \beta_{m-i+1}^{m-1}.
\] (8.4)

In the following we show that
\[
[x]_{i-1}\beta_{m-i}^{m-1} + \frac{1}{2}\beta_{m-i}^{m-1} = [x]_{i-1}\beta_{m-i+1}^{m-1}.
\] (8.5)

First, suppose for a contradiction that \([x]_{i-1}\beta_{m-i}^{m-1} + \frac{1}{2}\beta_{m-i}^{m-1} > [x]_{i-1}\beta_{m-i+1}^{m-1}.

Let \(\psi_1 := [x]_{i-1}\beta_{m-i}^{m-1} + \frac{1}{2}\beta_{m-i}^{m-1}.

By \(x = \langle x \rangle_i\) and \(i > 2\), we have
\[
x > \psi_1.
\]

Let \(\psi_2 := ([x]_{i-1} + 1)\beta_{m-i+1}^{m-1}.

Then by the assumption and (8.4), we know that
\[
x > \psi_1 \geq \psi_2.
\]

Note that
\[
\left[ \frac{x}{\beta_{m-i+1}^{m-1}} \right] \beta_{m-i+1}^{m-1} \leq x < \left( \left[ \frac{x}{\beta_{m-i+1}^{m-1}} \right] + 1 \right) \beta_{m-i+1}^{m-1} = \psi_2.
\]

Therefore,
\[
x > \psi_2 > x.
\]

A contradiction occurs.

Second, suppose that \([x]_{i-1}\beta_{m-i}^{m-1} + \frac{1}{2}\beta_{m-i}^{m-1} < [x]_{i-1}\beta_{m-i+1}^{m-1}.

Therefore,
\[
x \geq [x]_{i-1}\beta_{m-i+1}^{m-1}
\geq [x]_{i-1}\beta_{m-i}^{m-1} + \frac{1}{2}\beta_{m-i}^{m-1} + \beta_{m-i+1}^{m-1} \quad \text{[by (8.4)]}
\geq x + \frac{1}{2}\beta_{m-i+1}^{m-1} - \frac{1}{2}\sum_{1 < j \leq i-2} \beta_{m-j}^{m-1}. \quad \text{[\because x = \langle x \rangle_i]}
\]

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By (8.3), we have
\[ \sum_{1 < j \leq i - 2} \beta_{m-j}^{m-1} < \beta_{m-i+1}^{m-1}. \]

As a consequence, we have \( x > x \). A contradiction occurs again. Therefore, (8.5) holds.

Therefore,
\[ L_x = x + \frac{1}{2} \sum_{1 < j < i} \beta_{m-j}^{m-1} \]
[by definition]
\[ = x + \frac{1}{2} \sum_{1 < j < i} \beta_{m-j}^{m-1} \]
[by (8.5)]
\[ = x + \frac{1}{2} \sum_{1 < j < i} \beta_{m-j}^{m-1} \]
[by definition]
\[ = \lfloor x \rfloor. \]

As a consequence, the claim holds. \( \square \)

Proof of Lemma 9

Proof. By definition, \( [[x]]_j = \lfloor x \rfloor \beta_{m-i}^{m-1} + \frac{1}{2} \sum_{1 < j \leq i} \beta_{m-j}^{m-1} \beta_{m-j}^{m-1} \)
[by definition]
\[ = [x]_j \beta_{m-i}^{m-1} + \frac{1}{2} \sum_{1 < j < i} \beta_{m-j}^{m-1} \]
[by (8.5)]
\[ = [x]_j \beta_{m-i}^{m-1} + \frac{1}{2} \sum_{1 < j < i} \beta_{m-j}^{m-1} \]
[by definition]
\[ = \lfloor x \rfloor. \]

As a consequence, the claim holds. \( \square \)
Proof of Lemma 11

Proof. It comes from the intuition that one unit of difference in higher abstraction is huge in lower abstractions. Note that

\[ \beta_{m-\xi+1}^m > |a - \langle a \rangle_{\xi-1}| \quad \text{and} \quad \beta_{m-\xi+1}^m > |a' - \langle a' \rangle_{\xi-1}|. \]  

(8.8)

We shall see that, \( a - \langle a \rangle_{\xi} \neq a' - \langle a' \rangle_{\xi} \) if \( \langle a \rangle_{\xi} - \langle a \rangle_{\xi-1} \neq \langle a' \rangle_{\xi} - \langle a' \rangle_{\xi-1} \), hence a contradiction occurs.

For example, assume that \( \langle a \rangle_{\xi} - \langle a \rangle_{\xi-1} > \langle a' \rangle_{\xi} - \langle a' \rangle_{\xi-1} \) and \( a - \langle a \rangle_{\xi} < 0 \). The other cases are similar. Firstly, \( a - \langle a \rangle_{\xi} = (a - \langle a \rangle_{\xi-1}) - (\langle a \rangle_{\xi} - \langle a \rangle_{\xi-1}) \).

Note that \( a' - \langle a' \rangle_{\xi} < 0 \), for \( a - \langle a \rangle_{\xi} = a' - \langle a' \rangle_{\xi} \). By Lemma 10, both \( \langle a \rangle_{\xi} \) and \( \langle a \rangle_{\xi-1} \) are vertices of index greater than or equal to \( \xi - 1 \), which means that both of them are in \( X_{\xi}^* \).

By Fact 4, we have

\[ |\langle a \rangle_{\xi} - \langle a \rangle_{\xi-1}| \geq \beta_{m-\xi+1}^m > |a - \langle a \rangle_{\xi-1}|. \]  

(8.9)

Therefore, \( \langle a \rangle_{\xi} - \langle a \rangle_{\xi-1} > 0 \) since \( (a - \langle a \rangle_{\xi-1}) - (\langle a \rangle_{\xi} - \langle a \rangle_{\xi-1}) < 0 \), and so is \( \langle a' \rangle_{\xi} - \langle a' \rangle_{\xi-1} \). In this case observe that either \( a \leq \langle a \rangle_{\xi-1} < \langle a \rangle_{\xi} \) and \( a' \leq \langle a' \rangle_{\xi-1} < \langle a' \rangle_{\xi} \) or \( \langle a \rangle_{\xi-1} \leq a < \langle a \rangle_{\xi} \) and \( \langle a' \rangle_{\xi-1} \leq a' < \langle a' \rangle_{\xi} \). Suppose that the former holds. The other case is similar. Then, by (8.8), \( \beta_{m-\xi+1}^m > |a - \langle a \rangle_{\xi-1} - a' - \langle a' \rangle_{\xi-1}| = |(a - \langle a \rangle_{\xi-1}) - (a' - \langle a' \rangle_{\xi-1})| \).

Similar to (8.9), by Fact 4 \( |\langle a \rangle_{\xi} - \langle a \rangle_{\xi-1}| - \langle a' \rangle_{\xi-1}| \geq \beta_{m-\xi+1}^m \), since \( \langle a \rangle_{\xi} - \langle a \rangle_{\xi-1} > \langle a' \rangle_{\xi} - \langle a' \rangle_{\xi-1} \). Therefore, \( (a - \langle a \rangle_{\xi}) - (a' - \langle a' \rangle_{\xi}) = ((a - \langle a \rangle_{\xi-1}) - (a' - \langle a' \rangle_{\xi-1})) - ((\langle a \rangle_{\xi} - \langle a \rangle_{\xi-1}) - (\langle a' \rangle_{\xi} - \langle a' \rangle_{\xi-1})) < 0 \).

We arrive at a contradiction. This shows that (1) holds, which immediately implies that (2) holds.

\[ \square \]

Proof of Lemma 14

Proof. Since \( (x, y) \in X_i^* \), by definition, \( x = \langle x \rangle_i = [x]_i \beta_{m-i}^{m-1} + \frac{1}{2} \sum_{1<p\leq i} \beta_{m-p}^{m-1} \).
Then for any \( 1 \leq j < i \),

\[
[x]_j = \left[ \frac{x}{\beta_{m-j}^{m-1}} \right]
\]

\[
= \left[ x \beta_{m-i}^{m-1} + \frac{1}{2} \sum_{1 \leq p \leq i} \beta_{m-p}^{m-1} \right] \frac{1}{\beta_{m-j}^{m-1}}
\]

\[
= \left[ x \beta_{m-i}^{m-1} \beta_{m-j}^{m-1} + \frac{1}{2} \sum_{1 \leq p \leq i} \beta_{m-j}^{m-1} \beta_{m-p}^{m-1} \right] \frac{1}{\beta_{m-j}^{m-1}}
\]

\[
= [x]_i \beta_{m-i}^{m-1} + \frac{1}{2} \sum_{j < p \leq i} \beta_{m-p}^{m-1}
\]

By definition, both \( \beta_{m-j}^{m-1} \) and \( \frac{1}{2} \beta_{m-j}^{m-1} \) are divisible by \( k - 1 \) for any \( j < p \leq i \). Therefore, \([x]_j\) is divisible by \( k - 1 \), and \( cc([x]_j, y) = y \mod k - 1 \). □

**Proof of Lemma 17.**

*Proof.* We first show that, for any \( i \) where \( 1 \leq i \leq q \),

\[
([x]_p)_i \equiv ([x']_p)_i \mod (k - 1).
\] (8.10)

If \( q < \min \{\text{idx}(\langle x \rangle_p, y), \text{idx}(\langle x' \rangle_p, y)\} \), then (8.10) holds due to Lemma 14. Henceforth we assume that \( q = \min \{\text{idx}(\langle x \rangle_p, y), \text{idx}(\langle x' \rangle_p, y)\} \).

Let \( \text{idx}(\langle x \rangle_p, y) = \ell \) and \( \text{idx}(\langle x' \rangle_p, y) = \ell' \). W.l.o.g. we assume that \( \ell' \leq \ell \). By Lemma 10, \( p \leq \ell' \leq \ell \). If \( ([x]_p)_i \equiv ([x']_p)_i \mod (k - 1) \) then, by Lemma 14, \( ([x]_p)_i \equiv ([x']_p)_i \mod (k - 1) \) for \( 1 \leq i \leq \ell' \). Hence (8.10) holds.

Suppose that \( ([x]_p)_i \equiv ([x']_p)_i \mod (k - 1) \) for \( 1 \leq i \leq \ell' \).

It is easy to observe that (8.10) implies (5.15), provided that (1) holds throughout the game. Briefly speaking, it relies on an observation that the neighbourhoods of vertices of the same index are isomorphic. By definition, \( \text{idx}(\langle x \rangle_p, y) \geq q \). If \( \text{idx}(\langle x \rangle_p, y) > q \), then by Lemma 14 and (2), \( ([x]_p)_i \equiv ([x']_p)_i \equiv 0 \mod (k - 1) \) for \( 1 \leq i \leq q \). Now suppose that \( \text{idx}(\langle x \rangle_p, y) = q \). By Lemma 11, \( ([x]_p)_j - ([x]_p) = ([x']_j - ([x']_p) \). Hence,

\[
\frac{([x]_j - ([x]_p)}{\beta_{m-i}^{m-1}} = \frac{([x']_j - ([x']_p)}{\beta_{m-i}^{m-1}}.
\]
Therefore, (5.15) holds because of (8.10).

Proof of Lemma 16

Proof. Let \( a := [x_1]_i \) and \( b := [x_2]_i \). By the assumption \( [x_1]_i < [x_2]_i \), hence \( a + 1 \leq b \). Note that, by the definition of the floor functions, \( x_1 < (a + 1)\beta_{m-i}^{m-1} \leq b\beta_{m-i}^{m-1} \leq x_2 \).

Therefore,

\[
\frac{x_1}{\beta_{m-i+1}^{m-1}} < \frac{(a + 1)\beta_{m-i}^{m-1}}{\beta_{m-i+1}^{m-1}} = (a + 1)\beta_{m-i}^{m-1} \leq b\beta_{m-i}^{m-1}
\]

Note that, \( b \in \mathbb{N}^+ \) and \( \beta_{m-i+1}^{m-1} \in \mathbb{N}^+ \).

Therefore,

\[
[x_1]_{i-1} \leq \frac{x_1}{\beta_{m-i+1}^{m-1}} < \frac{b\beta_{m-i}^{m-1}}{\beta_{m-i+1}^{m-1}} \leq \frac{x_2}{\beta_{m-i}^{m-1}}
\]

That is,

\[
[x_1]_{i-1} < \frac{x_2}{\beta_{m-i+1}^{m-1}} = [x_2]_{i-1}.
\]

Proof of Lemma 31

Proof. By the modular arithmetic, we immediately have the following observation: for any \( a \in [k - 1] \), \((e', f)\) can be such a vertex that \([e']_r + f \equiv a \mod k - 1\). By the definition of \( X_r^* \) and Definition 6, there is at most one vertex in \((e'_r, f, \ell, w)\) whose index is greater than \( r \). And all the other vertices with index \( r \) encodes all the vertices in \( X_{r+1}^* \) via “\(|S|\)”. It explains why there must be such a vertex \((e', f)\) that \( cl(e', f) = (a, f); r; \ell; w \), by Definition 25.
and Definition 27. Indeed, the value of \( [e']_r \) mod \( \eta_r \) determines the values of \( a, r \) and \( w \). By definition, \( \ell = \left\lceil \frac{[e']_r \mod \eta_r}{\frac{1}{3} \eta_r} \right\rceil - 1 \). We can find a vertex \((e'', f)\) where \( |e'' - e'| \equiv 0 \pmod{\frac{1}{3} \eta_r} \). Then it is clear that \( \mathcal{O}(e'', f) \) is similar to \( \mathcal{O}(e', f) \) except that they may have different value for \( \text{RngNum}(\cdot, \cdot) \). In other words, we can choose \((e', f)\) properly such that \( \ell \) can be any element in \( \{-1, 0, 1\} \). Also note that there are many vertices satisfy the requirements other than \((e', f)\). That is, Lemma 31 can be ensured.

Proof of Lemma 35.

Proof. We use a binary string \( s \in \{0, 1\}^{(k-2)} \) to encode \( g(x) \mod 2^{(k-2)} \). We use \((s)_{10}\) to denote the value encoded by \( s \). On the other hand, recall that for a natural number \( n \), we use \((n)_{2,(k-2)}\) to denote the binary representation of \( n \), a 0-1 string of length \( \frac{k-2}{2} \). We use \( s \downarrow[i, j] \) to denote the string adjusted from \( s \) by turning every bit to 0 except for the \( i \)-th bit and the \( j \)-th bit, as well as the bits between them, which are unchanged.

Because \( v_i \neq v_j \), \( \hat{q}(y, v_i) \neq \hat{q}(y, v_j) \). We can give an order \( \prec \) to the element \((u_i, v_i)\) of \( P \) based on \( \hat{q}(y, v_i) \) such that \((u_i, v_i) \prec (u_j, v_j)\) if and only if \( \hat{q}(y, v_i) < \hat{q}(y, v_j) \). Assume that \((u_i^{\xi}, v_i^{\xi})\) are these elements of \( P \) that are in the given order, i.e. \((u_i^{\xi}, v_i^{\xi}) \prec (u_j^{\xi}, v_j^{\xi})\) if and only if \( i < j \). Note that this order is usually different from the linear orders of the structures. Let \( f_\sigma(u_i^{\xi}, v_i^{\xi}) = j \) if \( u_i^{\xi} = u_j \) and \( v_i^{\xi} = v_j \). Let \( P_j := \{(u_i^{\xi}, v_i^{\xi}) \mid 1 \leq i \leq j\} \). The main idea is that we can adjust the value of \( x \) gradually to satisfy the lemma where \( P = P_i \) for \( i = 1 \) to \( l \), step by step.

For any \( 1 \leq p \leq \left(\frac{k-2}{2}\right) \), let \( \text{trip}(p) \) be a 0-1 string of length \( \left(\frac{k-2}{2}\right) \) such that all the elements in the string is 0 except for the \( p \)-th element, called a "trip point", which is 1. Recall that the rightmost element of the string is the \( 0 \)-th element, i.e. the lowest order bit.

At the beginning, we choose an \( x \) such that \( g(x) = g(u_1^{\xi}) \). Then we adjust \( g(x) \) such that \( g(x) := g(x) \downarrow[0, \hat{q}(y, v_1^{\xi})] \). Afterwards, we adjust \( g(x) \) if and only if

\[ \text{BIT}(\text{SW}((x, y), (u_1^{\xi}, v_1^{\xi})), \hat{q}(y, v_1^{\xi})) \neq w_{f_\sigma}(u_1^{\xi}, v_1^{\xi}). \]

Assume that it is necessary to adjust \( g(x) \), i.e. adjust \( x \) when \( w_{f_\sigma}(u_1^{\xi}, v_1^{\xi}) = 1 \). We adjust \( x \) such that \( g(x) \) is decreased by \( (\text{trip}(\hat{q}(y, v_1^{\xi})))_{10} \) if \( g(x) \geq \frac{1}{3} \eta_r \). There are many choices. Such freedom is necessary for us to apply the lemma. For the purpose of proving this lemma, we can simply let \( x \) be the minimal one that makes \( g(x) = g(u_1^{\xi}) \) hold. In the following, we will talk about adjusting \( g(x) \). Such adjusting certainly involves changing the value of \( x \).
Obviously, if $g(x)$ is increased by the same amount, otherwise. Now it is straightforward to verify that

$$
\text{BIT}(\text{SW}((x, y), (u_1^\le, v_1^\le)), \hat{q}(y, v_1^\le)) = w_{f_o}(u_1^\le, v_1^\le).
$$

Note that $g(x) < (\text{trip}(\hat{q}(y, v_1^\le) + 1))_{10}$.

Assume that for some $c \in [1, l - 1]$ and for any $1 \le i \le c$,

$$
\text{BIT}(\text{SW}((x, y), (u_i^\le, v_i^\le)), \hat{q}(y, v_i^\le)) = w_{f_o}(u_i^\le, v_i^\le).
$$

(8.11)

Let bit$_{c+1}^x := \text{BIT}(\text{SW}((x, y), (u_{c+1}^\le, v_{c+1}^\le)), \hat{q}(y, v_{c+1}^\le))$.

We adjust $x$ not only to preserve (8.11) but also to ensure that

$$
\text{bit}_{c+1}^x = w_{f_o}(u_{c+1}^\le, v_{c+1}^\le).
$$

(8.12)

In other words, we adjust the value of $x$ to make one more vertex in $P$ satisfy (8.11), if necessary.

If (8.12) holds, then $x$ is unchanged. Otherwise, we fine-tune $x$ for several rounds to satisfy (8.12), meanwhile still preserve (8.11).

First, assume that $g(u_{c+1}^\le) \ge g(x) + (\text{trip}(\hat{q}(y, v_{c+1}^\le)))_{10}$.

Let $\delta_{c+1} := g(u_{c+1}^\le) - g(x)$, and let

$$
\delta_{c+1}^\pm := \delta_{c+1} + (\text{bit}_{c+1}^x - w_{f_o}(u_{c+1}^\le, v_{c+1}^\le)) \times (\text{trip}(\hat{q}(y, v_{c+1}^\le)))_{10}.
$$

Obviously, if $g(x)$ is increased by $\delta_{c+1}^\pm$, then (8.12) is ensured. Let $\Delta_{c+1} := (\delta_{c+1})_{2/((k-2)/2)}[0, \hat{q}(y, v_{c+1}^\le)]_{10}$. Let $g(x) := g(x) + \Delta_{c+1}$. Note that (8.12) still holds because adding $(\delta_{c+1})_{2/((k-2)/2)}[\hat{q}(y, v_{c+1}^\le) + 1, (k-2)/2 - 1]$ to the sum doesn’t influence the 0-th bit to the $\hat{q}(y, v_{c+1}^\le)$-th bit of the sum.

In the following we adjust $g(x)$ to satisfy (8.11) step by step. For $j = 1$ to $c$, in the $j$-th round, if (8.11) still holds when $i = j$, then go to the next round. Otherwise, we add or minus $(\text{trip}(\hat{q}(y, v_j^\le)))_{10}$ to $g(x)$ to enforce that (8.11) holds in case of $i = j$, depending on whether it will propagate to influence (8.12) or not. Note that one of the choices will not propagate to influence (8.12). Also note that, if $1 < j$, this will not influence (8.11) for $1 \le i < j$. Therefore, (8.11) holds for $1 \le i \le j$ at the end of the $j$-th round.

In short, both (8.11) and (8.12) can be ensured when $g(u_{c+1}^\le) \ge g(x) + (\text{trip}(\hat{q}(y, v_{c+1}^\le)))_{10}$. This inductive argument shows that (6.1) holds in this case.
Second, assume that
\[ g(u_{c+1}^x) < g(x) + (\text{trip}(\hat{q}(y, v_{c+1}^x)))_{10}. \]

If (8.12) is already satisfied, then \( x \) is unchanged. Otherwise, let \( g(x) := g(u_{c+1}^x) + (\text{trip}(\hat{q}(y, v_{c+1}^x)))_{10} \) if \( g(u_{c+1}^x) < \left(\frac{k-2}{2}\right) - (\text{trip}(\hat{q}(y, v_{c+1}^x)))_{10} \) and let \( g(x) := g(u_{c+1}^x) - (\text{trip}(\hat{q}(y, v_{c+1}^x)))_{10} \) otherwise. Then, similar to the last case, we adjust \( g(x) \) gradually to satisfy (8.11) step by step, minusing or adding \( (\text{trip}(\hat{q}(y, v_{c+1}^x)))_{10} \), depending on whether (8.12) is still satisfied and \( g(x) \) is still in the range \([0, (k-2)/2)\). 

In summary, we can gradually adjust the value of \( x \) to satisfy (6.1). \( \square \)

**Proof of Lemma 36.**

Proof. In the following arguments we choose such vertices \((x^x, y), (x'', y)\) and \((x, y)\) that
\[ \chi(x^x, y) | S = \chi(x'', y) | S = \chi(x, y) | S = \emptyset. \] (8.13)

The claim (1) is obvious according to Definition 27 because, by Lemma 31, \((x^x, y)\) can be such a vertex that

- \( \text{id}(x^x, y) = t - 1; \)
- \( \text{cc}([x^x]_{t-1}, y) = c. \) By Lemma 14, \( c \) is different from \( \text{cc}([x_i]_{t-1}, y_i) = y_i \mod k - 1 \) for any \( i \). Moreover, \([x^x]_{t-1} \mod k - 1 \) \( \neq 0 \) because \( \text{cc}([x^x]_{t-1}, y) = c \neq y \mod k - 1 \), which means that \( \chi(x^x, y) | \Omega \cap H = \emptyset. \) Note that \( \chi(x^x, y) | S \cap H = \emptyset \) because \( \chi(x^x, y) | S = \emptyset. \) And \( (x^x, y) \notin \chi(x_i, y_i) | S \) since \( \text{id}(x^x, y) = t - 1 < \text{id}(x_i, y_i) \) for any \( (x_i, y_i) \in H. \)

The claim (2) is also obvious according to Definition 27, we can choose \((x'', y)\) to be such a vertex that \( \text{id}(x'', y) = t - 1 \) and \([x'']_{t-1} \equiv 0 \) (mod \( k - 1 \)). The latter means that \( \text{cc}([x'']_{t-1}, y) = y \mod k - 1 \), which implies that \( \text{cc}([x'']_{t-1}, y) \neq \text{cc}([x_i]_{t-1}, y_i) = y_i \mod k - 1 \) by Lemma 14. Because \((x'', y)\) is adjacent to \((x_i, y_i)\) for any \((x_i, y_i) \in H, (x_i, y_i) \notin \chi(x'', y) | \Omega. \) Moreover, we can choose \((x'', y)\) to be such a vertex that \( \chi(x'', y) | S = \emptyset. \) Hence \((x'', y) \notin \chi(x_i, y_i) | S \) and \((x_i, y_i) \notin \chi(x'', y) | S \) for any \((x_i, y_i) \in H. \)

In the above arguments, we haven’t considered RngNum yet. Note that \( \text{RngNum}(x_i, t - 1) = -1 \) for any \( i, \) by Lemma 21. Therefore, we can let \( \text{RngNum}(x^x, t - 1) = \text{RngNum}(x'', t - 1) = -1. \)

To prove the claim (3), we first apply (1) to find \((x', y) \in \mathbb{X}_{i-1}^* - \mathbb{X}_{i-2}^*\) such that \((x', y)\) is adjacent to every vertex in \( H; \) then we apply (2) to
find \((x, y) \in X^*_{t-2} - X^*_{t-3}\) such that \([x]_{t-2} \equiv 0 \pmod{k-1}\), \(g(x) = 0\) and \(\text{RngNum}(x, t-2) = -1\).

To prove the claim (4), we choose \((x, y) \in X^*_{t-1} - X^*_{t}\) to be such a vertex that \(\text{cc}([x]_{t-1}, y) = y \pmod{k-1}\), i.e. \([x]_{t-1} \equiv 0 \pmod{k-1}\). By Lemma 14
\[
\text{cc}([x]_{t-1}, y) \neq \text{cc}([x_i]_{t-1}, y_i).
\] (8.14)

Moreover, we can make \((x, y)\) be such a vertex that \((x_i, y_i) \notin \chi(x, y)\Omega\) for any \((x_i, y_i) \in H\). Here is the process. We first find a vertex \((x^\dagger, y)\) of index \(t\) satisfying some conditions s.t. the vertex is adjacent to any vertex in \(H\). Afterwards, we find such a vertex \((x, y)\) that

(a) \(\text{idx}(x, y) = t - 1\);
(b) \([x]_{t-1} \equiv 0 \pmod{k-1}\), i.e. \(\text{cc}([x]_{t-1}, y) = y \pmod{k-1}\);
(c) \([x^\dagger]_t = [x]_t\). It means that \([x]_t = x^\dagger\), by Lemma 4
(d) \(g(x) = 0\);
(e) \(\text{RngNum}(x, t-1) = -1\);
(f) \(\chi(x, y)|S = \emptyset\).

The crucial point is to find such a vertex \((x^\dagger, y)\).

Let \(H_t := \{([x]_t, y) \mid (x_i, y_i) \in H\}\). Let \((x^\dagger, y)\) be such a vertex that for any \(t \leq j \leq m\) the following hold:

(i) \(\text{idx}([x^\dagger]_j, y) = j\);
(ii) \(\chi([x^\dagger]_j, y)|S = \emptyset\);
(iii) \(([x^\dagger]_j, y)\) is adjacent to \(([x_i]_j, y_i)\) for any \((x_i, y_i) \in H_j\);
(iv) \(\text{RngNum}([x^\dagger]_j, j) = -1\).

Note that, (i) (ii) is easy to satisfy. So is (iv). It is (iii) that needs some justification.

Firstly, we can ensure that
\[
\text{cc}([x^\dagger]_j, y) \neq \text{cc}([x_i]_j, y_i)
\] (8.15)
because of \(|H_j| \leq k - 2\) and Lemma \[31\] (it says that we can choose a value for \(cc([x^t]_j, y)\) freely). By Lemma \[9\] \(cc([x^t]_j, y) = cc([x^t]_j, y)\) and \(cc([x^t]_j, y) = cc([x^t]_j, y)\). Therefore,

\[
cc([x^t]_j, y) \neq cc([x^t]_j, y). \tag{8.16}
\]

Moreover, \(\text{id}_x([x^t]_j, y) \geq j\), by Lemma \[10\]. Consequently, by definition,

\[
\text{sng}(([x^t]_j, y), ([x^t]_j, y)) = 0. \tag{8.17}
\]

Therefore,

\[
([x^t]_j, y) \notin \chi([x^t]_j, y). \tag{8.18}
\]

Secondly, by Lemma \[14\] \(y \in \{1, \ldots, k - 2\} - \{cc([x]_{i-1}, y) = (x, y) \in H\}\) since \(y \in \{1, \ldots, k - 2\} - \{y \mid (x, y) \in H\}\). By Lemma \[35\] we can choose \((x^t, y)\) to be such a vertex that, for any \((u, v) \in H_j \cap (X_j^* - X_{j+1}^*)\) where \(v \in [1, k - 2]\),

\[
(cc([u]_j, y) - cc([x^t]_j, y)) \times (v - y) \\
(-1)^{\text{BIT}(\text{SW}((u, v), ([x^t]_j, y)), \hat{q}(v, y))} > 0. \tag{8.19}
\]

Third, we have chosen \((x^t, y)\) to be such a vertex that \(\chi(([x^t]_j, y)|S = \emptyset.\)

Hence

\[
\text{cl}(([x^t]_j, y)) \notin \chi(([x^t]_j, y)|S. \tag{8.20}
\]

Note that the cases can be satisfied simultaneously. Then by definition (iii) holds.

Note that \(([x^t]_t = x^t)\). Due to (iii), \((x, y) \notin \chi(x^t, y)|\Omega\) because \((([x^t]_j, y)\) is adjacent to \(([x^t]_j, y)\) for any \(j\). That is, \(\chi(x, y)|\Omega \cap H = \emptyset.\)

In summary, \((x^t, y)\) is adjacent to any vertex in \(H\). Then by the proof of (2), the claim (4) holds.

\[\Box\]

**Proof of Claim \[3\]**

**Proof.** Firstly, we show that (i) holds. Recall that Spoiler picks \((x, y)\) and Duplicator replies with \((x', y)\). If \(\chi(x, y) \mid BH\) is not a valid board history, then for any \((u, v)\), neither \((x, y) \xrightarrow{\text{bc}} (u, v)\) nor \((u, v) \xrightarrow{\text{bc}} (x, y)\).

In other words, \((x, y)\) is an isolated vertex in \(A_{k, m}\). Duplicator simply picks \((x', y)\) such that \(\chi(x', y) \mid BH\) is not a valid board history. Clearly the claim holds. Furthermore, since there are sufficiently many such isolated vertices, she can do it in accordance with the usual condition for winning the Ehrenfeucht-Fraïssé games over pure linear orders, i.e. (apx-1) (cf. Remark
Therefore, in the following argument we assume that both $\chi(x, y)|_\text{BH}$ and $\chi(x', y)|_\text{BH}$ are valid.

We first prove the following by induction. That is, $(u', v) \rightarrow (x', y)$ if $(u, v) \rightarrow (x, y)$, and $(x', y) \rightarrow (u', v)$ if $(x, y) \rightarrow (u, v)$. Note that if the players do not take off pebbles in a virtual game then $\rightarrow$ equals $\overset{\text{BC}}{\rightarrow}$. Recall that in the virtual game, for any $(a, b) \models (a', b)$, if Spoiler takes off one of the pebble, e.g. $(a, b)$, then Duplicator will take off the other one in the pair, e.g. $(a', b)$. Consequently, We need only prove that $(u', v) \overset{\text{BC}}{\rightarrow} (x', y)$ if $(u, v) \overset{\text{BC}}{\rightarrow} (x, y)$, and $(x', y) \overset{\text{BC}}{\rightarrow} (u', v)$ if $(x, y) \overset{\text{BC}}{\rightarrow} (u, v)$.

**Basis:** Assume that there is only one pair of pebbled vertices $(u, v)$, $(u', v)$ in the game board, and Spoiler picks $(x, y)$ in $\mathbb{A}_{k,m}$. Since Duplicator sticks to B-3, by the definitions, $(u', v) \overset{\text{BC}}{\rightarrow} (x', y)$ if $(u, v) \overset{\text{BC}}{\rightarrow} (x, y)$, and $(x', y) \overset{\text{BC}}{\rightarrow} (u', v)$ if $(x, y) \overset{\text{BC}}{\rightarrow} (u, v)$.

**Induction Step:** There are up to $k - 2$ pairs of pebbled vertices $(u_i, v_i)$, $(u'_i, v_i)$ in the game board, where the claim holds. Let $\ell := \max\{j \mid (u_j, v_j)_j \overset{\text{BC}}{\rightarrow} (x, y)\}$. Let $(u_c, v_c)$ be a vertex, not necessary pebbled, such that $i_{\text{cur}}^{u_c, v_c} = \ell$ and $(u_c, v_c) \overset{\text{BC}}{\rightarrow} (x, y)$. And let $(u^*, v^*)$ be one of these pebbled vertices, if there is one, such that $i_{\text{cur}}^{u^*, v^*} = \min\{i_{\text{cur}}^{u_c, v_c} \mid (x, y) \overset{\text{BC}}{\rightarrow} (u_i, v_i)\}$. Since the binary relation $\overset{\text{BC}}{\rightarrow}$ is transitive, $(u_c, v_c) \overset{\text{BC}}{\rightarrow} (u^*, v^*)$. Let $(u^*, v^*) \models (u^*, v^*)$. Moreover, let $(u'_c, v_c)$ be such a vertex that $(u'_c, v_c) \overset{\text{BC}}{\rightarrow} (u^*, v^*)$ and $i_{\text{cur}}^{u'_c, v_c} = i_{\text{cur}}^{u^*, v^*}$. Duplicator simply picks $(x, y)$ such that its associated board history mimics that of $(u', v^*)$ in the first $\chi(x, y)|_\text{bc}$ rounds. By definition, $(u'_c, v_c) \overset{\text{BC}}{\rightarrow} (x, y) \overset{\text{BC}}{\rightarrow} (u^*, v^*)$. Because the relation $\overset{\text{BC}}{\rightarrow}$ is transitive, the claim holds for other pebbled vertices, e.g. $(u', v)$.

If there is no such vertex $(u^*, v^*)$, Duplicator first search for the pebbled vertices in $\mathbb{A}_{k,m}$ (it is similar when Spoiler picks in $\mathbb{B}_{k,m}$ in this round) and if she can find a pebbled vertex $(a, b)$ such that $\chi(a, b)|_\text{IBH[i]}$ is equivalent to $\chi(u_c, v_c)|_\text{IBH[i]}$ for $i \leq \chi(u_c, v_c)|_\text{bc}$, then Duplicator picks $(x', y)$ such that $\chi(x', y)|_\text{IBH[i]}$ equals $\chi(a, b)|_\text{IBH[i]}$ where $(a, b) \models (a', b)$; afterwards, she

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67For example, assume that $(u, v) \models (u', v)$ and $(u, v) \overset{\text{BC}}{\rightarrow} (x, y)$, and that $\chi(u, v)|_\text{bc} < \chi(u_c, v_c)|_\text{bc}$. By Fact 8 $(u, v) \overset{\text{BC}}{\rightarrow} (u_c, v_c)$ holds. Then $(u, v) \overset{\text{BC}}{\rightarrow} (u', v_c)$ holds. Hence by the transitivity of $\overset{\text{BC}}{\rightarrow}$, we know that $(u', v) \overset{\text{BC}}{\rightarrow} (x', y)$. In short, $(u', v) \overset{\text{BC}}{\rightarrow} (x', y)$ if $(u, v) \overset{\text{BC}}{\rightarrow} (x, y)$. The case when $(u, v) \models (u', v)$ and $(x, y) \overset{\text{BC}}{\rightarrow} (u, v)$ is similar.
uses the virtual game to determine \( \chi(x', y) | \BH(j) \) for \( \iota_{\text{cur}}^{x, y} \leq j < \iota_{\text{cur}}^{x, y} \). In this case, let \( (u'_c, v_c) \) be the vertex that Duplicator will pick in the \( \iota_{\text{cur}}^{x, y} \)-th round of this virtual game. Thus we can get \( (u'_c, v_c) \). By definition, 
\[
( u'_c, v_c ) \xrightarrow{\text{BC}} (x', y) \text{ if } ( u_c, v_c ) \xrightarrow{\text{BC}} (x, y). \]
Then by the transitivity of \( \xrightarrow{\text{BC}} \), the claim holds for pebbled vertices.

Now we prove that \((u', v)\) and \((x', y)\) are not in continuity if \((u, v)\) and \((x, y)\) are not in continuity. Note that, to avoid verbos case analysis, we discuss \( \xrightarrow{\text{com}} (\text{continuity}) \) in the next paragraph instead of \( \xrightarrow{\text{BC}} \). But it is easy to see that similar argument holds in the latter case.

Suppose that \((u, v)\) and \((x, y)\) are not in continuity. If \( \chi(u, v) | bc = \chi(x, y) | bc \), then \( \chi(u', v) | bc = \chi(x', y) | bc \). Hence \((u', v)\) and \((x', y)\) are not in continuity. In the following we assume that \( \chi(u, v) | bc \neq \chi(x, y) | bc \).

There are two cases. First, suppose that, for some \( 0 < i < min\{\iota_{\text{cur}}^{x, y}, \iota_{\text{cur}}^{x, y}\} - 1 \),
\( \chi(x, y) | \IBH[j] = \chi(u, v) | \IBH[j] \) if \( 0 \leq j \leq i \), and \( \chi(x, y) | \BH(i+1) \neq \chi(u, v) | \BH(i+1) \). Duplicator can make it that \( \chi(x', y) | \IBH[j] = \chi(u', v) | \IBH[j] \) for \( 0 \leq j \leq i \). Moreover, Duplicator can ensure that \( \chi(x', y) | \BH(i+1) \neq \chi(u', v) | \BH(i+1) \). She need only ensure that in the games with the same board configuration she replies with different vertices in a structure if Spoiler picks different vertices in the other structure. This would be made more precise in the following argument for \((ii)\) wherein games over pure linear orders are introduced. The second case is very similar. Assume w.l.o.g. that \( \chi(u, v) | bc < \chi(x, y) | bc \) and \( \chi(x, y) | BH(j) = \chi(u, v) | BH(j) \) for \( 0 \leq j \leq \chi(u, v) | bc \). Then in the \( \iota_{\text{cur}}^{x, y} \)-th virtual round of the board history of \((x, y)\), a vertex that is different from \((u, v)\) is picked. Similarly, Duplicator need only ensure that in the \( \iota_{\text{cur}}^{x, y} \)-th virtual round of the board history of \((x', y)\), a vertex that is different from \((u', v)\) is picked. How to pick such a vertex will be introduced shortly. \([Q.E.D]\)

A concern may probably rise. That is, in the above argument we don’t know if this will cause problem w.r.t. the linear order. We shall see why it is not a problem in the following argument.

Recall that \( \overline{\kappa} \) is the tuple of pebbled vertices in \( \overline{\kappa}_{k, m} \) and \( \overline{\kappa}(i) \) is the \( i \)-th element in the tuple. We can construct a tree of board configurations where the root is the empty board configuration and the leaves are the board configurations associated with vertices in \( \overline{\kappa}(i) \); each branch of the tree corresponds to a board history associated with \( \overline{\kappa}(i) \) for some \( i \). \footnote{For some other vertices in \( \overline{\kappa} \), their board histories maybe correspond to a path from the root to some inner node. By Fact \[A\] such a tree is possible.} Hence this tree encodes the board histories of vertices in \( \overline{\kappa} \). Each arrow represents one step evolution of some board history. Such a representation tells us where
two board histories diverge in the tree. Clearly, the number of children of a node is bounded by \( k - 2 \) since there are up to \( k - 2 \) leaves (recall that \( |c_A| \leq k - 2 \)). Some of the children correspond to a board configuration that is obtained by adding a pebbled vertex to its farther board configuration. Some of them correspond to a board configuration that is obtained by removing a pebble from its farther (a board configuration). By the order defined in Definition 27, the children of the latter case are less than the children of the former case in the order. We can also construct a similar tree that represents the board histories of vertices in \( c_B \). Then by (i) and, in particular, by the strategy introduced in page 133 (the induction step), we can observe that these two trees are isomorphic, without considering the order. In the following we give the justification for (ii) in detail. That is, we show that the trees are isomorphic even in the presence of the order. As usual, we study it case by case. Recall that \( v = y \).

If \( \chi(u, v) \mid bc < \chi(x, y) \mid bc \), then by the definition of the order of histories, \( \lfloor u/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# < \lfloor x/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# \). Likewise, \( \lfloor u'/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# < \lfloor x'/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# \) since \( \chi(u', v) \mid bc < \chi(x', y) \mid bc \) (recall that \( \chi(u', v) \mid bc = \chi(u, v) \mid bc \) and \( \chi(x', y) \mid bc = \chi(x, y) \mid bc \)). Hence the claim holds. Now assume that \( \chi(u, v) \mid bc = \chi(x, y) \mid bc = \ell_u \).

Assume that \( \lfloor x/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# = \lfloor u/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# \). That is, \( \chi(u, v) \mid BH = \chi(x, y) \mid BH \). In such case, Duplicator can ensure that \( \lfloor x'/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# = \lfloor u'/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# \), by the strategy that Duplicator used (cf. page 133, the induction step).

Suppose that \( \lfloor x/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# > \lfloor u/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# \). Another case where \( \lfloor x/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# < \lfloor u/(\gamma_{m-1}^* \times k) \rfloor \mod bh^# \) is very similar. Recall that \( \chi(x, y) \mid BH \) and \( \chi(u, v) \mid BH \) have the same ancestor, i.e. the empty board history (the root of the tree). These two histories must be diverged at some point, i.e. some node in the tree. Suppose that this node is \( BC_0 \) (a board configuration) and that \( (a, b) \) and \( (e, b) \) are two vertices in \( BC_0 \). Moreover, assume that in the initial segment of the board history from the beginning empty board configuration to \( BC_0 \), the game is evolved into the \( \xi \)-th abstraction. Then, over the \( \xi \)-th abstraction, the length of the interval between \( (a, b) \) and \( (e, b) \) is \( |a/l_{\ell_\xi} - e/l_{\ell_\xi}| \). By (6.2) and Remark 46, we know that any interval over the \( \xi \)-th abstraction, whose length is greater than 1, is sufficiently large over the \( (\xi - 1) \)-th abstraction. Hence it is easy for Duplicator to win the Ehrenfeucht-Fraïssé games over such pure linear orders (the intervals over the \( (\xi - 1) \)-th abstraction) in up to \( k - 2 \) rounds, because \( m > k \) and the length of the orders are greater than \( 2^m \). The exceptions, where the length of an interval over the \( \xi \)-th abstraction is 0 or 1, can be handled easily. Note that in such cases the pair of intervals
in respective structures have the same length, i.e. either 0 or 1. By her strategy that deals with order, Duplicator should mimic Spoiler’s picking in such intervals. So far, the games are played over the $\xi$-th or the $(\xi - 1)$-th abstraction. However, the vertices picked can be in very low abstractions, e.g. in $X_1^\ast - X_2^\ast$. If, in the $\xi$-th abstraction, a vertex is picked again, then the corresponding vertex will also be picked again in this abstraction. Or more precisely, their projection in the $\xi$-th abstraction are picked more than once.

It remains to show that (vii) and (iv) hold simultaneously (when $\text{idx}(x^y, y) = \text{idx}(x^b, y) < \xi$). But this is already explained in the proof of Lemma 37 (cf. Strategy 3).

Fig. 13 illustrates two isomorphic trees that encode the board histories of the vertices in $c_A$ and $c_B$. The tree on the left is used to represent board histories associated with the vertices in $c_A$, whereas the tree on the right is for those board histories of the vertices in $c_B$. Here the board configurations are succinctly represented. The node Root, as well as Root, is the empty board configuration. There is an arrow from Root to the board configuration $(c_0, d0)$. It means that the latter can be evolved from the former in one step in the board history that goes through them. Assume that $v = y = b$. We use $BC_1$ to denote $(c_0, d0)(c_1, d1)(a, b)$, which is associated with a vertex in $c_A$. Likewise, $BC_5$ is for $(c0, d0)(c1, d1)(e, b)$ (associated with a vertex in $c_A$). Similarly, $BC'_1$ stand for the board configuration $(c0', d0)(c1', d1)(a', b)$ and $BC'_5$ is for $(c0', d0)(c1', d1)(e', b)$. They are associated with a vertex in $c_B$. We require that $(c0, d0) \models (c0', d0), \ldots, (e, b) \models (e', b)$. Here “$\models$” is determined by a virtual game that combines two sorts of virtual games. The first sort is the imaginary pebble game played on the changing game board (cf. page 66). The second sort is the Ehrenfeucht-Fra"issé games over the pure linear orders, e.g. the interval between $(a, b)$ and $(e, b)$ over the $\xi$-th abstraction or the $(\xi - 1)$-th abstraction. Note that $BC_i$ is surrounded by a rounded rectangle. We use such a rectangle to denote an $l_\xi$-tuple of successive board configurations that include $BC_i$ as a member, whereas no other board configuration in it is associated with a vertex in $c_A$. Note that the board configurations in an $l_\xi$-tuple can only be distinguished by the last vertex in their representation. In this example, we assume that the third item of $BC_i$ ($BC'_i$ resp.) is $(h_i, b)$ ($h'_i, b$ resp.), where $a = h_1 < h_2 = h_6 < h_3 = u < h_4 < h_5 = e$ ($a' = h'_1 < h'_2 = h'_6 < h'_3 = u' < h'_4 < h'_5 = e'$, resp.). Here we use the dashed arrow issued from $BC_6$ to mean that $BC_6$ have some children. From the figure, we know that, w.r.t. the linear order, the board history indicated by the path from Root to $BC_i$ is less than that indicated by the path from Root to $BC_j$ if $i < j$. Moreover, all the board histories that go through $BC_i$ share the initial segment of them (the first two virtual
Figure 13: Trees that encode the board histories of the vertices in $\overline{c_A}$ and $\overline{c_B}$. Each node is a board configuration.

rounds, indicated by the blue arrows). In this example, if Spoiler picks $(x, y)$ in $\overline{A}$ s.t. $\chi(x, y) | BH$ goes through $BC_5$, then Duplicator can pick $(x', y)$ in $\overline{B}$ s.t. $\chi(x', y) | BH$ goes through $BC'_5$.

We’ve shown that Duplicator is able to ensure that the orders of an initial segment of histories (up to $\ell_u$ rounds; associated with $\overline{c_A}$ and $\overline{c_B}$) are isomorphic at the point of divergence, by (6.2) and Remark 46. More precisely, for some $i \leq \ell_u$, $\chi(x, y) | IBH[i] > \chi(u, v) | IBH[i]$ if $\chi(x', y) | IBH[i] > \chi(u', v) | IBH[i]$, on condition that $\chi(x', y) | BH(j) = \chi(u', v) | BH(j)$ for $0 \leq j < i$ and $\chi(x', y) | BH(i) \neq \chi(u', v) | BH(i)$. Recall that we order the board histories based on lexicographic ordering. Provided that the above holds, it implies that the orders of these board histories (of the same length; associated with the pebbled vertices) are preserved throughout the game, i.e. $x'/\gamma_{m-1} \times k) \mod bh# > u'/\gamma_{m-1} \times k) \mod bh#$ if $x'/\gamma_{m-1} \times k) \mod bh# > u'/\gamma_{m-1} \times k) \mod bh#$, since $\chi(u, v) | bc = \chi(x, y) | bc$.

Recall that Duplicator uses Strategy 1/3 to choose the type label for $(x', y)$; the order issue, when ignoring board histories, is handled in Remark 46. Here we just introduced the idea that takes care of the orders of board histories (associated with the pebbled vertices) and that make up these strategies: it ensures that (ii) holds, in accordance with (i).

Remark 57. If the readers have already been confirmed by the intuition stated in Remark 15, then there is no need to read the following involved arguments. But, if the readers are still doubt about the claim that “the
subgraph induced by \( cex(x_0, y_0, t - 1) \) and \( cex(x_1, y_1, t - 1) \) is isomorphic to the subgraph induced by \( cex(x'_0, y_0, t - 1) \) and \( cex(x'_1, y_1, t - 1) \) if and only if the adjacency between \((x_0, y_0)\) and \((x_1, y_1)\) is the same as that between \((x'_0, y_0)\) and \((x'_1, y_1)\)”, then patience should be paid to getting through the following proofs, by which we show a more general result: it is true even for the structures \( \mathfrak{A}_{k,m}^* \) and \( \mathfrak{B}_{k,m}^* \) defined in section 5.2. It justifies the notion “abstraction” in the more general cases \((k \geq 4)\), just akin to the intuition explained in Remark 15.

Suppose that \( \text{idx}(x, y) = r \). \( \lfloor x \rfloor^\text{min}_r - \lfloor \langle x \rangle_\xi \rfloor_r \equiv \lfloor x' \rfloor^\text{min}_r - \lfloor \langle x' \rangle_\xi \rfloor_r \), roughly corresponds to the equation \( x^b - \langle x^b \rangle_\xi = x'^b - \langle x'^b \rangle_\xi \). On the other hand, note that \( x^b - \langle x^b \rangle_\xi = \lfloor x^b \rfloor^\text{min}_1 - \lfloor \langle x^b \rangle_\xi \rfloor_1 \) if \( \text{idx}(x^b, y) = 1 \). Note that \( \lfloor x \rfloor^\text{min}_r = \lfloor x \rfloor_r \) if \( k = 3 \) since in this case \( \mathfrak{U}_r^i = 1 \) for any \( i \). Therefore, \( x - \langle x \rangle_\xi = x' - \langle x' \rangle_\xi \) implies that \( \lfloor x \rfloor^\text{min}_r - \lfloor \langle x \rangle_\xi \rfloor_r = \lfloor x' \rfloor^\text{min}_r - \lfloor \langle x' \rangle_\xi \rfloor_r \).

The following Lemma roughly says that, modulo some amount, if two vertices are (approximately) in the same position in lower abstraction (i.e. finer scale), then so are they in higher abstraction (i.e. coarser scale). See Fig. 14. Here in the figure vertex \( a, b, a' \) and \( b' \) stand for \((x, y), (\lfloor x \rfloor^\text{min}_r, y), (x', y)\) and \((\lfloor x' \rfloor^\text{min}_r, y)\) respectively. Vertices of different colour stand for vertices of different indices: black vertices have index \( \xi \); the grey ones have index \( r \); while the blue ones have index strictly between \( \xi \) and \( r \). Vertices under one black brace stand for a \( \mathfrak{U}_r^i \)-tuple. The number of vertices above a red brace stand for \( \lfloor x \rfloor^\text{min}_r - \lfloor \langle x \rangle_\xi \rfloor_r \).

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Lemma 58. For any $1 \leq r < \xi \leq m$ and $(x, y), (x', y) \in X_r^* - X_{r+1}^*$, if 
$[x]_r^\text{min} - [x]_\xi \equiv [x']_r^\text{min} - [x']_\xi \pmod{\beta_{m-\xi}^m}$, then 
$[x]_i \equiv [x']_i \pmod{\beta_{m-\xi}^m}$ for any $r < i < \xi$.

Note that the condition “$[x]_r^\text{min} - [x]_\xi \equiv [x']_r^\text{min} - [x']_\xi \pmod{\beta_{m-\xi}^m}$” is similar to the condition 1° in the proof of Lemma 18. By remark 47, it is equivalent to “$[x]_r^\text{min} - [x]_\xi = [x']_r^\text{min} - [x']_\xi$”.

The following lemma says something similar: two distances (between vertices) are equivalent w.r.t. coarser scale if they are equivalent w.r.t. finer scale. It is similar to Lemma 11. See Fig. 15. Similar to the last figure, vertices $a, b, c, d, a', b', c'$ and $d'$ stand for $(x, y), ([x]_r^\text{min}, y), ([x]_\xi, y), ([x']_r, y), ([x']_r^\text{min}, y), ([x']_i, y)$ and $([x']_\xi, y)$ respectively. Vertices of different colour stand for vertices of different indices: black vertices have index $\xi$; the grey ones have index $r$; while the blue ones have index $i$ that is strictly between $\xi$ and $r$. Vertices of the same colour have the same index. The set of vertices under one black brace stand for a $U^*$-tuple. The number of vertices above a red brace stand for $[x]_r^\text{min} - [x]_\xi$. The number of vertices above a yellow brace is $[x]_r^\text{min} - [x]_i$. 

Lemma 59. For any $1 < r < \xi - 1 < m$ and $(x, y), (x', y) \in X_r^* - X_{r+1}^*$, if 
$[x]_r^\text{min} - [x]_\xi \equiv [x']_r^\text{min} - [x']_\xi \pmod{\beta_{m-\xi}^m}$, then 
$[x]_r^\text{min} - [x]_i \equiv [x']_r^\text{min} - [x']_i \pmod{\beta_{m-\xi}^m}$ for any $r < i < \xi$.

The following lemma says that, modulo some amount, if two vertices are
Lemma 60. For any $1 \leq r < \xi \leq m$ and $(x, y), (x', y) \in X_r^* - X_{r+1}^*$, if $[x]_r^{\text{min}} - [x']_r^{\text{min}} - [y]_r^{\text{min}} - [y']_r^{\text{min}} \neq 0 \pmod{\beta_m^{m-r}}$ then for any $i$ where $r < i < \xi$,

$$\text{idx}([x]_i, y) = \text{idx}([x']_i, y),$$

if $\text{idx}([x]_i, y) < \xi$ and $\text{idx}([x']_i, y) < \xi$.

In the following we give the proofs of these lemmas.

Proof of Lemma 58

Proof. By definition, $[x]_r^{\text{min}} - [y]_r^{\text{min}} - [x']_r^{\text{min}} - [y']_r^{\text{min}} \pmod{\beta_m^{m-r}}$ implies that

$$| [x]_r \mod \beta_m^{m-r} - [x']_r \mod \beta_m^{m-r} | < u^*_r.$$

Let $[x']_r = [x]_r + a \beta_m^{m-r} + b$ where $a, b \in \mathbb{Z}$ and $|b| < u^*_r$.

By definition, for any $x$ and $i$ (recall that $r < i < \xi$),

$$[x']_i = \left\lfloor \frac{x'}{\beta_m^{m-1}} \right\rfloor = \left\lfloor \frac{x'}{\beta_m^{m-1}} \cdot \beta_m^{m-1} \right\rfloor = \left\lfloor \frac{x'}{\beta_m^{m-1}} \cdot \beta_m^{m-1} \right\rfloor.$$

Note that $[x']_r \leq \frac{x'}{\beta_m^{m-1}} \leq [x']_r + 1$.

Hence, $\left\lfloor [x']_r \frac{\gamma_m^{m-1}}{\gamma_m} \right\rfloor \leq [x']_i \leq \left\lfloor ([x']_r + 1) \frac{\gamma_m^{m-1}}{\gamma_m} \right\rfloor$.

In other words,

$$\left\lfloor ([x]_r + a \beta_m^{m-r} + b) \frac{\gamma_m^{m-1}}{\gamma_m^{m-r}} \right\rfloor \leq [x']_i \leq \left\lfloor ([x]_r + a \beta_m^{m-r} + b + 1) \frac{\gamma_m^{m-1}}{\gamma_m^{m-r}} \right\rfloor.$$

Similarly, by definition,

$$[x]_r = \left\lfloor \frac{x}{\beta_m^{m-i}} \right\rfloor = \left\lfloor \frac{x}{\beta_m^{m-i}} \cdot \beta_m^{m-i} \right\rfloor = \left\lfloor \frac{x}{\beta_m^{m-i}} \right\rfloor.$$

Because of $[x]_i \leq \frac{x}{\beta_m^{m-i}} < [x]_i + 1$ and $\frac{\gamma_m^{m-r}}{\gamma_m^{m-r}} \in \mathbb{N}^+$,

$$[x]_i \frac{\gamma_m^{m-r}}{\gamma_m^{m-r}} \leq [x]_r < ([x]_i + 1) \frac{\gamma_m^{m-r}}{\gamma_m^{m-r}}.$$

Hence

$$\left\lfloor \left( [x]_i \frac{\gamma_m^{m-r}}{\gamma_m^{m-r}} + a \beta_m^{m-r} \right) \frac{\gamma_m^{m-1}}{\gamma_m^{m-1}} \right\rfloor \leq [x']_i \leq \left\lfloor \left( ([x]_i + 1) \frac{\gamma_m^{m-r}}{\gamma_m^{m-r}} + a \beta_m^{m-r} + b + 1 \right) \frac{\gamma_m^{m-1}}{\gamma_m^{m-1}} \right\rfloor.$$
Proof. Firstly, we show that \(\{x\}, \{x'\} \leq \xi\), for any \(r<i<\xi\). For any \(r<i<\xi\),

\[\psi := \{\{x\}_r - \{\{x'\}_r\}\} \equiv \{\{x'\}_r - \{\{x'\}_r\}\} \pmod{\beta_m^{m-r}} \quad (8.21)\]

By Lemma 58, we have

\[\{x\}_r = \{x'\}_r \mod \beta_m^{m-r+1}.\]

Let \([x]\) := \([x']\) for some \(a \in \mathbb{N^+}\). For any \(r<i\leq \xi\),

\[\{[x]\}_r = \left\lfloor \frac{[x] \beta_m^{m-r} + \frac{1}{2} \sum_{1<j<i} \beta_m^{m-r}}{\beta_m^{m-r}} \right\rfloor + \frac{1}{2} \sum_{r<j<i} \beta_m^{m-r} \quad (8.22)\]

Let \(\psi := ([x]_r - [x']_r) - ([x']_r - [x']_r).\)

We have

\[\psi = ([x]_r \beta_m^{m-r+1} - [x']_r \beta_m^{m-r+1}) - ([x']_r \beta_m^{m-r} - [x']_r \beta_m^{m-r}) \]
\[= \beta_m^{m-r+1} ([x]_r - [x']_r) - \beta_m^{m-r} ([x]_r - [x']_r) \]
\[= a \beta_m^{m-r+1} \beta_m^{m-r+1} \beta_m^{m-r} \quad (8.23)\]

Therefore, the claim (8.21) holds.

Secondly, we can prove that \(x, \{x\}_r, \{x'\}_r\) has the same order as \(x', \{x'\}_r, \{x'\}_r\); \(x, \{x\}_r, \{x'\}_r\) does. For example, \(x \geq \{x\}_r \geq \{x'\}_r\) if and only if \(x' \geq \{x'\}_r \geq \{x'\}_r\). Here, we only prove the special case \(x \geq \{x\}_r \geq \{x'\}_r\). The other cases are not very different.
By (8.22),
\[
([x]_{\xi})_r = [x]_{\xi} \beta_{m-\xi}^m + \frac{1}{2} \sum_{r<i \leq \xi} \beta_{m-i}^m
\]
\[
= \beta_{m-r-1} \left( [x]_{\xi} \beta_{m-\xi}^{m-1} + \frac{1}{2} \sum_{r<i \leq \xi} \beta_{m-i}^{m-1} \right)
\]
\[
= 2^r \mathcal{U}_r^r \left( [x]_{\xi} \beta_{m-\xi}^{m-1} + \frac{1}{2} \sum_{r<i \leq \xi} \beta_{m-i}^{m-1} \right)
\]
\[
= 2^r \mathcal{U}_r^r \left( [x]_{\xi} \beta_{m-\xi}^{m-1} + \frac{1}{2} \sum_{r+1<i \leq \xi} \beta_{m-i}^{m-1} \right) + 2^{r-1} \mathcal{U}_r^r
\]
(8.23)

Note that \(\beta_{m-\xi}^{m-1}\) is a natural number, since \(\xi > r+1\). Therefore, \([([x]_{\xi})_r\) is divisible by \(\mathcal{U}_r^r\). Hence, \([([x]_{\xi})_r]_{min} = ([x]_{\xi})_r\). Together with the assumption that \([x]_{\xi}^{min} - ([x]_{\xi})_r \equiv [x']_{\xi}^{min} - ([x']_{\xi})_r \) (mod \(\beta_{m-\xi}^m\)), it implies that \([x]_{\xi} \leq x\) if and only if \(([x']_{\xi})_r \leq x'\).

Assume for the purpose of a contradiction that \(([x']_{\xi})_r < ([x']_{\xi})_r\).

By definition, we have
\[
([x']_{\xi-1})_{\xi-1} = \left[ \frac{[x']_{\xi-1} \beta_{m-\xi+1}^m - \frac{1}{2} \sum_{1<j \leq \xi-1} \beta_{m-j}^m}{\beta_{m-\xi+1}} \right]
\]
\[
= [x']_{\xi-1}.
\]
Therefore, \([x']_{\xi-1} = ([x']_{\xi-1})_{\xi-1} \leq ([x]_{\xi})_r - 1 \leq [x']_{\xi-1}, \) by \(([x']_{\xi})_r < ([x']_{\xi})_r\). Now assume for the purpose of a contradiction that \(x' < ([x']_{\xi})_r\). Since \(x' \in \mathbb{X}_1\) and \(([x']_{\xi})_r \in \mathbb{X}_1\) (by Lemma 5), we have \([x']_r < ([x']_{\xi})_r\), by definition. Let \(\varphi := ([x]_{\xi}^{min} - ([x]_{\xi})_r) \) mod \(\beta_{m-\xi}^m\). Then we have
\[
\varphi_0 = ([x']_{\xi}^{min} - ([x']_{\xi})_r) \) mod \(\beta_{m-\xi}^m\)
\[
\leq ([x']_r - ([x']_{\xi})_r) \) mod \(\beta_{m-\xi}^m\)
\[
< ([([x']_{\xi-1})_{\xi-1} - ([x']_{\xi})_r) \) mod \(\beta_{m-\xi}^m\).
\]

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On the other hand, by an argument akin to (8.23), we have \([\langle x \rangle_{\xi-1}]_{r}^{\text{min}} = [\langle x \rangle_{\xi-1}]_{r}\). Therefore, we also have that

\[
\varphi_0 \geq ([\langle x \rangle_{\xi-1}]_{r} - [\langle x \rangle]_{r}) \mod \beta_{m-\xi}^r.
\]

By (8.21), \(\varphi_0 \geq [\langle x' \rangle_{\xi-1}]_{r} - [\langle x' \rangle]_{r} \mod \beta_{m-\xi}^r\). A contradiction occurs.

We have shown that

\[
x \geq \langle x \rangle_{\xi-1} \geq \langle x \rangle \iff x' \geq \langle x' \rangle_{\xi-1} \geq \langle x' \rangle.
\]  

(8.25)

Together with (8.21) and the assumption that \([x]_{r}^{\text{min}} - [\langle x \rangle]_{r} \equiv [x']_{r}^{\text{min}} - [\langle x' \rangle]_{r} \mod \beta_{m-\xi}^r\), the claim of the lemma holds. \(\square\)

**Proof of Lemma 60**

*Proof.* Let \(r^+ := r + 1\). By definitions, \((\langle x \rangle_{r^+}, y) \in X_{r^+}^\ast\), thereby \(\idx(\langle x \rangle_{r^+}, y) \geq r^+ \geq 2\).

The following proof is by contradiction. Suppose that for some \(b\) where \(r < b < \xi\),

- \(\idx(\langle x \rangle_b, y), \idx(\langle x' \rangle_b, y) < \xi\);
- \(\idx(\langle x \rangle_b, y) \neq \idx(\langle x' \rangle_b, y)\);
- for any \(t\) where \(b < t < \xi\), \(\idx(\langle x \rangle_t, y) = \idx(\langle x' \rangle_t, y)\), if \(\idx(\langle x \rangle_t, y) < \xi\) and \(\idx(\langle x' \rangle_t, y) < \xi\).

Recall that \(\idx(\langle x \rangle_b, y) \geq b\). Suppose that \(b = \xi - 1\). Because \(\idx(\langle x \rangle_b, y) \neq \idx(\langle x' \rangle_b, y)\), then either \(\idx(\langle x \rangle_b, y) \geq \xi\) or \(\idx(\langle x' \rangle_b, y) \geq \xi\), a contradiction. Therefore, \(b < \xi - 1\).

Let \(\idx(\langle x \rangle_b, y) = j\) and \(\idx(\langle x' \rangle_b, y) = j'\). Assume without loss of generality that \(j > j'\). The case wherein \(j < j'\) is symmetric.

For any \((u, v) \in X_{\xi}^\ast\) and any \(r + 2 \leq s \leq \xi\), assume that \(\idx(\langle u \rangle_{s-1}, v) = p\) and \(\idx(\langle u \rangle_s, v) = q > p\). Observe that, there is no \((u', v) \in X_{\xi}^\ast\) such that its index is \(p + 1\) and it is strictly between \(\idx(\langle u \rangle_{s-1}, v)\) and \(\idx(\langle u \rangle_s, v)\). Otherwise, \((u', v)\) would be \(\idx(\langle u \rangle_s, v)\), a contradiction to the assumption that \(u' \neq \langle u \rangle_s\). Since \((\langle u \rangle_s, \langle u \rangle_{s-1}) \in X_{p}^\ast\), by definition, for some \(l \in [1, \frac{1}{2} \beta_{m-p-1} \cdot \beta_{m-p} - \frac{1}{2}]\),

\[
|\langle u \rangle_{s+1} - \langle u \rangle_{s-1}|_{r^+} = l \cdot \beta_{m-p}^{r^+}.
\]

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Hence, the following holds:

\[
\beta_{m-r}^m - r + m - p \leq |[L_u M_s r] + - [L_u M_s - 1] r + | (8.26)
\]

\[
\leq \frac{1}{2} \beta_{m-p}^m - r + 1 - \frac{1}{2} \beta_{m-p}^m (8.27)
\]

We use the observation that one unit of difference in higher abstraction means a tremendous difference in a lower one (w.r.t. distance of first coordinates) to prove the following Claim. Before that, we first prove some observations.

**Fact 9.** If \( m > i > i' \geq r^+ \), then

\[
\beta_{m-i-1}^m - r + m - i - 1 - \beta_{m-i'}^m - r + m - i' - 1 > 0.
\]

**Proof of Fact:**

\[
\left( \beta_{m-i-1}^m - r + m - i - 1 \right) - \left( \beta_{m-i'}^m - r + m - i' - 1 \right)
\]

\[
> \beta_{m-i-1}^m - r + m - i - 1 - 2 \beta_{m-i}^m
\]

\[
= \beta_{m-i}^m (2 \beta_{m-i-1}^m - 2)
\]

\[
> 0.
\]

**Q.E.D. of Fact.**

**Fact 10.** If \([x_\xi r]^+ < [x] r^+\), then

\[
[x] r^+ \mod \beta_{m-\xi}^m - r^+ = [x] r^+ - [x_\xi r]^+ + \frac{1}{2} \sum_{r^+ < i \leq \xi} \beta_{m-i}^m - r^+.
\]

**Proof of Fact:**

By definitions,

\[
[x_\xi r]^+ = \left[ \frac{[x] \beta_{m-\xi}^m - 1 \sum_{1 < i \leq \xi} \beta_{m-i}^m - 1}{\beta_{m-r}^m} \right]
\]

\[
= [x] \beta_{m-\xi}^m + \frac{1}{2} \sum_{r^+ < i \leq \xi} \beta_{m-i}^m - r^+.
\]

Note that \([x_\xi] \neq x\), because \([x] r^+ - [x_\xi] r^+ \equiv [x] r^+ - [x] r^+ \neq 0 \mod \beta_{m-\xi}^m\) and \([x_\xi] r^+ = [x] r^+\). Hence, we assume that \(x = a \beta_{m-\xi}^m + \)
If \( \theta > b_1 \), then 

\[
\beta_{m-\xi}^m - \xi > b_1 \Rightarrow \beta_{m-\xi}^m - \xi \in \mathbb{N}^+,
\]

where \( 0 < \beta_{m-\xi}^m - \xi < \beta_{m-\xi}^m \). Therefore, 

\[
[x]_{r+} = \beta_{m-\xi}^m - \xi \Rightarrow [x]_{r+} = \beta_{m-\xi}^m - \xi + \frac{1}{2} \sum_{r+ < i < \xi} \beta_{m-\xi}^m - \xi - [h]_{r+}.
\]

Note that \( [x]_{r+} = [x]_{r+} \mod \beta_{m-\xi}^m = [h]_{r+} \). Therefore, the fact holds.

Q.E.D. of Fact.

Similarly, we can prove the following observation.

**Fact 11.** If \( ([x]_{\xi})_{r+} < [x]_{r+} \), then

\[
[x']_{r+} \mod \beta_{m-\xi}^m = [x']_{r+} - ([x']_{\xi})_{r+} + \frac{1}{2} \sum_{r+ < i < \xi} \beta_{m-\xi}^m - \xi.
\]

**Claim 4.** \( |([x]_{\xi})_{r+} - ([x]_{b+1})_{r+}| = |([x']_{\xi})_{r+} - ([x']_{b+1})_{r+}|. \)

**Proof of Claim:**

Assume that there are \( d' \) types of vertices, where \( 0 \leq d' < \xi - b - 2 \), between \( ([x]_{b+1}, y) \) and \( ([x]_{\xi}, y) \), whose indices are different and less than \( \xi \) and greater than \( \text{idx}([x]_{b+1}, y) \). And assume that the indices of these vertices (including \( ([x]_{b+1}, y) \) and \( ([x]_{\xi}, y) \) are \( \text{IDX}(i) \) where

- \( 1 \leq i \leq d = d' + 2; \)
- \( \text{IDX}(i') < \text{IDX}(i' + 1) \), where \( 1 \leq i' < d; \)
- \( \text{IDX}(1) = \text{idx}([x]_{b+1}, y); \)
- \( \text{IDX}(d) = \text{idx}([x]_{\xi}, y). \)

Note that \( \text{idx}([x]_{\text{IDX}(i)}, y) = \text{IDX}(i) \), for \( 1 < i < d \).

Assume for the purpose of a contradiction that

\[
|([x]_{\xi})_{r+} - ([x]_{b+1})_{r+}| \neq |([x']_{\xi})_{r+} - ([x']_{b+1})_{r+}|.
\]

Then there must be a \( c \), where \( 1 < c \leq d \), such that

\[
(1^\#) \quad |([x]_{\xi})_{r+} - ([x]_{\text{IDX}(c)})_{r+}| = |([x']_{\xi})_{r+} - ([x']_{\text{IDX}(c)})_{r+}|;
\]

\[
(2^\#) \quad |([x]_{\text{IDX}(c)})_{r+} - ([x]_{\text{IDX}(c-1)})_{r+}| \neq |([x']_{\text{IDX}(c)})_{r+} - ([x']_{\text{IDX}(c-1)})_{r+}|.
\]

Because \( \text{IDX}(1) \geq b + 1 > b \), and \( \text{IDX}(c - 1) \geq \text{IDX}(1) \), we have \( \text{IDX}(c - 1) > b \). Therefore, \( \text{idx}([x]_{\text{IDX}(c-1)}, y) = \text{idx}([x']_{\text{IDX}(c-1)}, y) \).

Assume that

\[
|([x]_{\text{IDX}(c)})_{r+} - ([x]_{\text{IDX}(c-1)})_{r+}| > |([x']_{\text{IDX}(c)})_{r+} - ([x']_{\text{IDX}(c-1)})_{r+}|. \quad (8.28)
\]

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The other case is symmetric.

Assume that \( \text{idx}([x]_\text{IDX}(c-1), y) = n \). Note that \( n \geq \text{IDX}(c-1) \).
Furthermore, first assume that

\[
\begin{align*}
[\{x\}_\text{IDX}(c)]_{r+} &> [\{x\}_\text{IDX}(c-1)]_{r+} \quad (8.29) \\
[\{x\}_c]_{r+} &\geq [\{x\}_\text{IDX}(c)]_{r+} \quad (8.30)
\end{align*}
\]

Let

\[
\zeta := \left( [\{x\}_\text{IDX}(c)]_{r+} - [\{x\}_\text{IDX}(c-1)]_{r+} \right) - \left( [\{x\}'_\text{IDX}(c)]_{r+} - [\{x\}'_\text{IDX}(c-1)]_{r+} \right).
\]

By (8.28) and (8.29), \( \zeta > 0 \).
Hence, by (8.26) and (8.28),

\[
\zeta \geq \beta^{m-r^+}.
\]

Hence,

\[
\zeta > 2 \times \frac{1}{2} \left( \beta^{m-r^+} - 1 \right)
\]

That is,

\[
\zeta > 2 \times \frac{1}{2} \sum_{i=r+2}^{n} \left( \beta^{m-i} - \beta^{m-r^+} \right)
\]

Therefore, by (8.27) and Fact 9, we have

\[
\zeta > \sum_{i=r+2}^{\text{IDX}(c-1)} \left| \{x\}_i \right|_{r+} - \left| \{x\}_i-1 \right|_{r+} + \sum_{i=r+2}^{\text{IDX}(c-1)} \left| \{x\}'_i \right|_{r+} - \left| \{x\}'_{i-1} \right|_{r+}.
\]

Therefore,

\[
\zeta > \left| \sum_{i=r+2}^{\text{IDX}(c-1)} \left( \{x\}_i \right)_{r+} - \left( \{x\}_i-1 \right)_{r+} \right| + \left| \sum_{i=r+2}^{\text{IDX}(c-1)} \left( \{x\}'_i \right)_{r+} - \left( \{x\}'_{i-1} \right)_{r+} \right|
\]

\[
= \left| [\{x\}_\text{IDX}(c-1)]_{r+} - \left| x \right|_{r+} \right| + \left| [\{x\}'_\text{IDX}(c-1)]_{r+} - \left| x' \right|_{r+} \right|. \quad (8.31)
\]

The above argument also tell us more about (1\#). That is,

\[
[\{x\}_c]_{r+} - [\{x\}_\text{IDX}(c)]_{r+} = [\{x\}'_c]_{r+} - [\{x\}'_\text{IDX}(c)]_{r+}.
\]

Let \( \psi := \left| \{x\}_c \right|_{r+} - \left| x \right|_{r+} \).
\[ \psi = \left( \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c-1) \right]_{r^+} \right) + \\
\quad \left( \left[ \langle x \rangle \text{IDX}(c-1) \right]_{r^+} - \left[ x \right]_{r^+} \right) + \left[ \langle x \rangle \xi \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+}. \]

Recall that, by (8.31),
\[ \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c-1) \right]_{r^+} > \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+}. \]

Therefore,
\[ \psi \geq \left( \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c-1) \right]_{r^+} \right) - \\
\quad \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+} \right) + \left[ \langle x \rangle \xi \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+}. \quad \text{by (8.30)} \]

Therefore,
\[ \psi > \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+} + \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c-1) \right]_{r^+} + \\
\quad \left[ \langle x \rangle \xi \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+} \right| \quad \text{by (8.31) and (1\#)} \]

Therefore,
\[ \psi > \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+} + \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c-1) \right]_{r^+} + \\
\quad \left[ \langle x \rangle \xi \right]_{r^+} - \left[ \langle x \rangle \text{IDX}(c) \right]_{r^+} \right| = \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+}. \]

Because of the assumption that \( \left[ x \right]_{r^+} - \left[ \langle x \rangle \xi \right]_{r^+} = \left[ \langle x \rangle \text{IDX}(c-1) \right]_{r^+} \neq 0 \) (mod \( \beta_{-m-\xi} \)), it is easy to see that \( \langle \langle x \rangle \rangle \geq x \iff \langle x \rangle \xi \geq x' \). Therefore, we have \( \left[ x \right]_{r^+} \text{ mod } \beta_{-m-\xi}^{m+r^+} \neq \left[ x' \right]_{r^+} \text{ mod } \beta_{-m-\xi}^{m+r^+} \), by Fact 10 and Fact 11. But by Lemma 58, \( \left[ x \right]_{r^+} = \left[ x \right]_{r^+} \text{ mod } \beta_{-m-\xi}^{m+r^+} \). A contradiction occurs.

In the last arguments, if some of the assumptions (8.28), (8.29) and (8.30) do not hold, then the arguments need to be revised a little bit, but are very similar. It turns out that, if even number of these three assumptions are violated, then \( \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+} > \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+} \); otherwise, \( \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+} < \left| \langle \langle x \rangle \rangle \text{IDX}(c-1) \right|_{r^+} - \left[ x \right]_{r^+} \).

In summary, we arrive at a contradiction in all the cases. Therefore, the claim holds.

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Q.E.D. of Claim.

Assume that
\[ \langle x \rangle_x \geq \langle x \rangle_{b+1} \geq \langle x \rangle_b \geq x. \]  \hspace{1cm} (8.32)

We have \( \langle x' \rangle_x \geq x' \), since \[ [x]^{\min}_r - [(x')_x]_r = [x']^{\min}_r - [(x')_x]_r \neq 0 \pmod{\beta_{m-x}^{m-r}}. \]
Note that, \[ [x]^{\min}_r - [(x')_x]_r = [x']^{\min}_r - [(x')_x]_r \pmod{\beta_{m-x}^{m-r}} \] implies that \[ [x]^{\min}_r - [(x')_x]_r = [x']^{\min}_r - [(x')_x]_r. \]

We show that \( x' \) cannot be strictly between \( (x')_{b+1} \) and \( (x')_x \). Assume for a contradiction that \( x' \) is between \( (x')_{b+1} \) and \( (x')_x \). Note that \( \text{idx}((x')_{b+1}, y) < x \) if \( (x')_{b+1} \neq (x')_x \). Then \( \text{idx}((x')_{b+1}, y) = \text{idx}((x')_{b+1}, y) \). Assume that \( \text{idx}((x')_{b+1}, y) = I_{b+1} \). Because \( x' \) is strictly between \( (x')_{b+1} \) and \( (x')_x \),
\[ \frac{1}{2} \beta_{m-I_{b+1}}^{m-r} + 1 \leq [(x')_x]_r - [x']_r \leq \beta_{m-I_{b+1}}^{m-r} - 1, \]
and \( u_r^* + \frac{1}{2} \beta_{m-I_{b+1}}^{m-r} \leq [(x')_x]_r - [x']_r \leq \beta_{m-I_{b+1}}^{m-r} - 1 \).

On the other hand, \( [(x')_x]_r - [x']_r \geq \beta_{m-I_{b+1}}^{m-r} + 1 \) and \( [(x')_x]_r - [x']_r \geq \beta_{m-I_{b+1}}^{m-r} + u_r^* \), because \( \text{idx}((x')_x, y) \in X_{b+1} \).
(Lemma 5). Therefore, \[ [x']^{\min}_r - [(x')_x]_r \neq [x']^{\min}_r - [(x')_x]_r. \]
We arrive at a contradiction. If \( (x')_x < (x')_{b+1} \), then \( (x')_{b+1} \) would be \( (x')_x \) because \( (x')_x, (x')_{b+1}, y) \) in \( X_{b+1}^* \), again a contradiction. Therefore,
\[ \langle x' \rangle_x \geq \langle x' \rangle_{b+1} \geq x'. \]

Moreover, if \( (x')_b > (x')_{b+1} \), then \( (x')_b = (x')_{b+1} \), since \( (x')_{b+1}, y) \in X_b^* \), still a contradiction. Therefore,
\[ \langle x' \rangle_x \geq \langle x' \rangle_{b+1} \geq \langle x' \rangle_b \geq x'. \]  \hspace{1cm} (8.33)

By (8.32),
\[ \psi \geq |[(x')_x]_r - [(x')_{b+1}]_r| + |[(x')_{b+1}]_r - [(x')_b]_r|. \]

Hence,
\[ \psi \geq |[(x')_x]_r - [(x')_{b+1}]_r| + \beta_{m-I_j}^{m-r} \]
\[ \geq |[(x')_x]_r - [(x')_{b+1}]_r| + \beta_{m-I_j}^{m-r} \]

Therefore,
\[ \psi > |[(x')_x]_r - [(x')_{b+1}]_r| + \sum_{r^+ \leq i \leq j'} (\beta_{m-I_j-1}^{m-r} - \beta_{m-I}^{m-r}) \]

By Claim 4 we have
\[ \psi > |[(x')_x]_r - [(x')_{b+1}]_r| + \sum_{r^+ \leq i \leq j'} (\beta_{m-I_j-1}^{m-r} - \beta_{m-I}^{m-r}) \]

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By (8.27) and Fact 9 we have

\[ \psi > \left| \lfloor x' \rfloor_{\xi} - \lfloor x' \rfloor_{b+1} \right| + \sum_{r^+ \leq i \leq b} \left( \lfloor x' \rfloor_{i+1} - \lfloor x' \rfloor_{i} \right) \]

\[ = \lfloor x' \rfloor_{\xi} - \lfloor x' \rfloor_{r^+} \quad \text{[by (8.33)]} \]

Finally, by Fact 10 and Fact 11, \( [x]_{r^+} \neq [x']_{r^+} \mod \beta_{m-\xi}^{m-r^+} \). We arrive at a contradiction.

If (8.32) does not hold, the last arguments need to be revised a little bit, but very similar. We need take care of the order of the vertices \( \lfloor x \rfloor_{\xi}, \lfloor x \rfloor_{b+1}, \lfloor x \rfloor_{b} \) and \( x \). That is, if the order of (8.32) is changed, then the order of (8.33) will be changed accordingly, using similar arguments. \( \square \)