ETINGOF-KAZHDAN QUANTIZATION OF LIE SUPERBIALGEBRAS

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ABSTRACT. For every semi-simple Lie algebra $g$ one can construct the Drinfeld-Jimbo algebra $U_h^{DJ}(g)$. This algebra is a deformation Hopf algebra defined by generators and relations. To study the representation theory of $U_h^{DJ}(g)$, Drinfeld used the KZ-equations to construct a quasi-Hopf algebra $A_g$. He proved that particular categories of modules over the algebras $U_h^{DJ}(g)$ and $A_g$ are tensor equivalent. Analogous constructions of the algebras $U_h^{DJ}(g)$ and $A_g$ exist in the case when $g$ is a Lie superalgebra of type $A$-$G$. However, Drinfeld’s proof of the above equivalence of categories does not generalize to Lie superalgebras. In this paper, we will discuss an alternate proof for Lie superalgebras of type $A$-$G$. Our proof utilizes the Etingof-Kazhdan quantization of Lie (super)bialgebras. It should be mentioned that the above equivalence is very useful. For example, it has been used in knot theory to relate quantum group invariants and the Kontsevich integral.

1. INTRODUCTION

Quantum groups were introduced independently by Drinfeld and Jimbo around 1984. One of the most important examples of quantum groups are deformations of universal enveloping algebras. These deformations are closely related to Lie bialgebras. In particular, every deformation of a universal enveloping algebra induces a Lie bialgebra structure on the underlying Lie algebra. In [3] Drinfeld asked if the converse of this statement holds: “Does there exist a universal quantization for Lie bialgebras?” Etingof and Kazhdan gave a positive answer to this question. In this paper we further this work by extending Etingof and Kazhdan’s work from Lie bialgebras to the setting of Lie superbialgebras. Moreover, we will generalize a theorem of Drinfeld’s from Lie algebras to Lie superalgebras of type $A$-$G$.

Given a semisimple Lie algebra $g$, Drinfeld [5] constructs the following algebras:

1. the Drinfeld-Jimbo quantization $U_h^{DJ}(g)$ of $g$ which is a deformation of the Hopf algebra $U(g)$,

2. a quasi-Hopf algebra $A_g$ which is isomorphic as a vector space to $U(g)[[h]]$.

These algebras are quite different in nature. $U_h^{DJ}(g)$ is a Hopf algebra which is defined algebraically by generators and relations. The non-trivial and complicated structure of $U_h^{DJ}(g)$ is encoded in these relations and the formulas defining the coproduct. On the other hand, the definition of $A_g$ is based on the theory of the Knizhnik-Zamolodchikov differential equations. Compared to $U_h^{DJ}(g)$, the algebra structure and the coproduct of $A_g$ are easy to define. The rich structure of $A_g$ is encoded in the fact that its coproduct is not coassociative.
Drinfeld was interested in the representation theory of the algebras $U^D_J(g)$ and $A_g$. Let $X$ be a topological algebra and let $X\text{-}Mod_{fr}$ be the category of topologically free $X$-modules of finite rank (see 2.2).

**Theorem 1 ([5]).** The categories $U^D_J(g)\text{-}Mod_{fr}$ and $A_g\text{-}Mod_{fr}$ are tensor equivalent.

This theorem allows one to play the differences of $U^D_J(g)$ and $A_g$ off of one another, leading to a deeper understanding of the category $U^D_J(g)\text{-}Mod_{fr}$. It turns out that Theorem 1 is also useful in knot theory. In particular, Le and Murakami used Theorem 1 to show that quantum group knot invariants arising from representations of Lie algebras can be studied through the Kontsevich integral.

The algebras $U^D_J(g)$ and $A_g$ can be constructed for some classes of Lie superalgebras. In §4 we will construct $A_g$ for a suitable Lie superalgebra $g$. The generalization of $U^D_J(g)$ to the setting of Lie superalgebras has been considered by many authors (see [10, 14, 19]). This generalization introduces defining relations (e.g. (19-20)) that are of a different form than the standard quantum Serre relations of $U^D_J(g)$. Let us call these additional relations the extra quantum Serre-type relations. Unlike the case for semi-simple Lie algebras, the (quantum) Serre-type relations are not well understood for all Lie superalgebras. For this reason we will consider the Lie superalgebras of type A-G. Yamen [19, 20] obtained (quantum) Serre-type for every Lie superalgebra of type A-G.

The proof of Theorem 1 does not have a straightforward generalization to the setting of Lie superalgebras. Drinfeld’s proof uses deformation theoretic arguments based on the fact that $H^i(g, U(g)) = 0$, $i = 1, 2$, for a semisimple Lie algebra. In general, this vanishing result is not true for Lie superalgebras (for example $\mathfrak{sl}(2|1)$, [18]). However, in §10 we will prove that Theorem 1 is true when $g$ is a Lie superalgebra of type A-G. Our proof is based on a different approach than Drinfeld’s, utilizing the quantization of Lie (super)bialgebras.

Our proof of Theorem 1 (when $g$ is a Lie superalgebra of type A-G) starts by generalizing the Etingof-Kazhdan quantization of Lie bialgebras to the setting of Lie superbialgebras. Note that it can be shown that $g$ can be given a natural structure of a Lie superbialgebra. Let $U_h(g)$ be the E-K quantization of $g$. By construction $U_h(g)$ is gauge equivalent to $A_g$. With sufficient hypotheses, Drinfeld showed that if two algebras are gauge equivalent then their module categories are equivalent. In a similar fashion, we will show that $U_h(g)\text{-}Mod_{fr}$ is tensor equivalent to $A_g\text{-}Mod_{fr}$. As one would expect the generalizations discussed in this paragraph are straightforward.

The proof is completed by constructing a Hopf algebra isomorphism between $U_h(g)$ and $U_h^D(g)$. This method is similar to [5] where it is shown that analogous result holds for any generalized Kac-Moody Lie algebra $\mathfrak{a}$. The proof of [5] shows that $U_h(\mathfrak{a})$ is given by generators and relations. In particular, the authors of [5] define a bilinear form and use results of Lusztig [15] to show that the quantum Serre-type relations are in the kernel of this form. Similar techniques apply in the case when $g$ is a Lie superalgebra of type A-G. However, as mentioned above, $U_h^D_J(g)$ has extra quantum Serre-type relations. In order to adapt the above methods we will extend results of Lusztig [15] to the setting of superalgebras and check directly that the extra quantum Serre-type relations are in the kernel of the appropriate bilinear form.
The Etingof-Kazhdan quantization \[6\] has two important properties that we use in this paper: the first being that it is functorial and second that it commutes with taking the double. With this in mind, we will next discuss the notion of the double of an object. Let \( g \) be a finite dimensional Lie superbialgebra. The double of \( g \) is the direct sum \( D(g) := g \oplus g^* \) with a natural structure of a (quasitriangular) Lie superbialgebra. Similarly, let \( A \) be a quantized universal enveloping (QUE) superalgebra and let \( A^* \) be its quantum dual, i.e. a QUE superalgebra which is dual (in an appropriate sense) to \( A \). The double of \( A \) of is the tensor product \( D(A) := A \otimes A^* \) with a natural structure of a quasitriangular QUE superalgebra.

By saying the E-K quantization commutes with taking the double we mean that \( D(U_h(g)) \cong U_h(D(g)) \) as quasitriangular QUE superalgebras.

We will now give an outline of this paper. There are several different quantization given in this paper which turn out to be isomorphic. We hope that following outline will help the reader understand why each quantization is important.

In \( \S 2 \) we will recall facts and definitions related to Lie superbialgebras, topologically free modules and QUE superalgebra. In \( \S 3 \) we will give the definition of a Lie superalgebra of type A-G and its associated D-J type quantization \( U_{DJ}^h(g) \). In \( \S 4 \) we will use the super KZ equations to define the quasi-Hopf superalgebra \( A_g \). We will also define the Drinfeld category.

In \( \S 5 \) we will extend the Etingof-Kazhdan quantization of finite dimensional Lie bialgebras, given in Part I of \( \S 6 \), to the setting of Lie superbialgebras. Let \( g \) be a finite dimensional Lie superbialgebra. Section 5 consists of three important parts: (1) the construction of a quantization \( H \) of the double \( D(g) \), (2) show that \( H \) has a Hopf sub-superalgebra \( U_h(g) \) which is a quantization of \( g \), (3) prove that \( H \) and \( U_h(g) \) are further related by the following isomorphism of quasitriangular Hopf superalgebras

\[
H \cong U_h(g) \otimes U_h(g)^*
\]

where \( U_h(g)^* \) is the quantum dual of \( U_h(g) \) and \( D(U_h(g)) := U_h(g) \otimes U_h(g)^* \).

In \( \S 6 \) we will construct the E-K quantization of quasitriangular Lie superbialgebras. Let us denote this quantization by \( U^{qt}_h(g) \) where \( g \) is a quasitriangular Lie superbialgebra. The quantization \( U^{qt}_h(g) \) is similar to the quantization of Lie superbialgebra of section 5. In particular, by construction

\[
U^{qt}_h(D(g)) = H
\]

for any finite dimensional Lie superbialgebra \( g \).

In \( \S 7 \) we will construct a second quantization of finite dimensional Lie superbialgebras. The importance of this quantization is that it is functorial. It turns out that it is isomorphic to the quantization given in section 5. For this reason we also denote it by \( U_h(g) \).

In \( \S 8 \) we will use the functoriality of the quantization to show that \( U^{qt}_h(g) \cong U_h(g) \) for any finite dimensional quasitriangular Lie superbialgebra \( g \). As noted above, the double of a Lie superbialgebra has a natural structure of a quasitriangular Lie superbialgebra. Therefore, we have

\[
U^{qt}_h(D(g)) \cong U_h(D(g))
\]
for any finite dimensional Lie superbialgebras $g$. We will close section §13 by combining §12, §21 and §22 to conclude that the E-K quantization commutes with taking the double.

In §13 we will prove that the E-K quantization $U_{h}(g)$ is isomorphic to the D-J type quantization $U^{DJ}_{h}(g)$, where $g$ is a Lie superalgebra of type A-G. The proof of this will rely heavily on the fact that the E-K quantization is functorial and commutes with taking the double.

In §14 we give a proof of Theorem §11 when $g$ is a Lie superalgebra of type A-G.

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2. Preliminaries

Let $k$ be a field of characteristic zero.

2.1. Superspaces and Lie super(bi)algebras. In this subsection we recall facts and definitions related to superspaces and Lie super(bi)algebras, for more details see §12, §10.

A superspace is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ over $k$. We denote the parity of a homogeneous element $x \in V$ by $\bar{x} \in \mathbb{Z}_2$. We say $x$ is even (odd) if $x \in V_0$ (resp. $x \in V_1$). Let $V$ and $W$ be superspaces. The space of linear morphisms $\text{Hom}_k(V,W)$ from $V$ to $W$ has a natural $\mathbb{Z}_2$-grading given by $f \in \text{Hom}_k(V,W)$ if $f(V_i) \subseteq W_{i+j}$ for $i,j \in \mathbb{Z}_2$. In particular, the dual space $V^* = \text{Hom}_k(V,k)$ is a vector superspace where $k$ is the one-dimensional superspace concentrated in degree 0, i.e. $k = k_0$. Throughout this paper the tensor product will have the natural induced $\mathbb{Z}_2$-grading. Let $\tau_{V,W} : V \otimes W \rightarrow W \otimes V$ be the linear map given by

$$\tau_{V,W}(v \otimes w) = (-1)^{\bar{v}\bar{w}}w \otimes v$$

for homogeneous $v \in V$ and $w \in W$. When it is clear what $V$ and $W$ are we will write $\tau$ for $\tau_{V,W}$. A linear morphism can be defined on homogeneous elements and then extended by linearity. When it is clear and appropriate we will assume elements are homogeneous. Throughout, all modules will be $\mathbb{Z}_2$-graded modules, i.e. module structures which preserve the $\mathbb{Z}_2$-grading (see §12).

A Lie superalgebra is a superspace $g = g_0 \oplus g_1$ with a superbracket $[,] : g^{\otimes 2} \rightarrow g$ that preserves the $\mathbb{Z}_2$-grading, is super-antisymmetric ($[x,y] = -(-1)^{\bar{x}\bar{y}}[y,x]$), and satisfies the super-Jacobi identity (see §12). A Lie superbialgebra is a Lie superalgebra $g$ with a linear map $\delta : g \rightarrow \Lambda^2 g$ that preserves the $\mathbb{Z}_2$-grading and satisfies both the super-coJacobi identity and cocycle condition (see §11). A triple $(g, g_+, g_-)$ of finite dimensional Lie superalgebras is a finite dimensional super Manin triple if $g$ has a non-degenerate super-symmetric invariant bilinear form $\langle , \rangle$, such that $g \cong g_+ \oplus g_-$ as superspaces, and $g_+$ and $g_-$ are isotropic Lie sub-superalgebras of $g$. There is a one-to-one correspondence between finite dimensional super Manin triple and finite dimensional Lie superbialgebra (see §11 Proposition 1).

Let $g$ be a Lie superalgebra. Let $r \in g \otimes g$ and let

$$CYB(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in g^3$$

be the classical Yang-Baxter element. A quasitriangular Lie superbialgebra is a triple $(g, [,], r)$ where $(g, [,])$ is a Lie superalgebra and $r$ is an even element of $g \otimes g$. 



such that $r + r'(r)$ is $g$-invariant, $CYB(r) = 0$ and $(g, [\cdot, \cdot], \partial r)$ is a Lie superbialgebra, where $\partial r(x) := [x \otimes 1 + 1 \otimes x, r]$.

Now we define the double of a finite dimensional Lie superbialgebra. Let $(g, [\cdot, \cdot], \delta)$ be a finite dimensional Lie superbialgebra and $(g_+, g_-, \Omega)$ its corresponding super Manin triple. Then $g := g_+ \oplus g_-$ has a natural structure of a quasitriangular Lie superbialgebra as follows. The bracket on $g$ is given by

$$
[x, y] = \begin{cases} 
[x, y]_{g_+} & \text{if } x, y \in g_+ \\
[x, y]_{g_-} & \text{if } x, y \in g_- \\
(ad^* x)y - (-1)^{\hat{g}}(1 \otimes y)\delta(x) & \text{if } x \in g_+, y \in g_-
\end{cases}
$$

(5)

where $ad^*$ is the coadjoint action of $g_+$ on $g_- \cong g_+^*$. Let $p_1, ..., p_n$ be a homogeneous basis of $g_+$. Let $m_1, ..., m_n$ be the basis of $g_-$ which is dual to $p_1, ..., p_n$, i.e. $<m_i, p_j> = \delta_{i,j}$. Define $r = \sum p_i \otimes m_i \in g_+ \otimes g_- \subset g \otimes g$. Then the triple $(g, [\cdot, \cdot], r)$ is a quasitriangular Lie superbialgebra (see [10]). We call $g$ the double of $g_+$ and denote it by $D(g_+)$. We can also define the Casimir element of $g$. Notice $m_1, ..., m_n, p_1, ..., p_n$ is a basis of $g$ that is dual to the basis $p_1, ..., p_n, (-1)^m_1m_1, ..., (-1)^m_nm_n$. Define the Casimir element to be

$$
\Omega = \sum p_i \otimes m_i + \sum (-1)^m_i m_i \otimes p_i = r + r'(r).
$$

(6)

An element $a \in g \otimes g$ is invariant (resp. super-symmetric) if $[x \otimes 1 + 1 \otimes x, a] = 0$ for all $x \in g$ (resp. $a = r(a)$). The element $\Omega$ is an even, invariant, super-symmetric element. Also, note that the element $\Omega$ is independent of the choice of basis $p_1, ..., p_n$.

2.2. Topologically free modules. Here we recall the notion of topologically free modules (for more detail see [13] [10]).

Let $K = k[[h]]$, where $h$ is an indeterminate and we view $K$ as a superspace concentrated in degree 0. Let $M$ be a module over $K$. Consider the inverse system of $K$-modules

$$p_n : M_n = M/h^nM \to M_{n-1} = M/h^{n-1}M.
$$

Let $\hat{M} = \varprojlim M_n$ be the inverse limit. Then $\hat{M}$ has the natural inverse limit topology (called the h-adic topology). We call $\hat{M}$ the $h$-adic completion of $M$.

Let $V$ be a $k$-superspace. Let $V[[h]]$ to be the set of formal power series. The superspace $V[[h]]$ is naturally a $K$-module and has a norm given by

$$||v_nh^n + v_{n+1}h^{n+1} + \cdots|| = 2^{-n} \text{ where } v_n \neq 0.
$$

The topology defined by this norm is complete and coincides with the $h$-adic topology. We say that a $K$-module $M$ is topologically free if it is isomorphic to $V[[h]]$ for some $k$-module $V$. Notice that if $f : M \to N$ is a $K$-linear map between topologically free modules then $f$ is continuous in the $h$-adic topology since $f(h^nM) \subseteq h^nN$ by $K$ linearity. For this reason we will assume that all $K$-linear maps are continuous.

Let $M, N$ be topologically free $K$-modules. We define the topological tensor product of $M$ and $N$ to be $M \otimes_K N$ which we denote by $M \otimes N$. This definition gives us the convenient fact that $M \otimes N$ is topologically free and that

$$V[[h]] \otimes W[[h]] = (V \otimes W)[[h]]
$$

for $k$-module $V$ and $W$. 

We say a (Hopf) superalgebra defined over $K$ is topologically free if it is topologically free as a $K$-module and the tensor product is the above topological tensor product.

2.3. Quantized Universal Enveloping Superalgebras. A quantized universal enveloping (QUE) superalgebra $A$ is a topologically free Hopf superalgebra over $\mathbb{C}[[h]]$ such that $A/hA$ is isomorphic as a Hopf superalgebra to $U(g)$ for some Lie superalgebra $g$. The following proposition was first given in the non-super case by Drinfeld [4] and latter proven in the super case by Andruskiewitsch [1].

**Proposition 2** ([4],[1]). Let $A$ be a QUE superalgebra: $A/hA \cong U(g)$. Then the Lie superalgebra $g$ has a natural structure of a Lie superbialgebra defined by

\[
\delta(x) = h^{-1}(\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})) \mod h, \quad x \in g
\]

where $\tilde{x} \in A$ is a preimage of $x$ and $\Delta^{op} := \tau_{U(g).U(g)} \circ \Delta$ (for the definition of $\tau$, see [4]).

**Definition 3.** Let $A$ be a QUE superalgebra and let $(g,[,],\delta)$ be the Lie superbialgebra defined in Proposition 2. We say that $A$ is a quantization of the Lie superbialgebra $g$.

Let $A$ be a Hopf superalgebra and let $R \in A \otimes A$ be an invertible homogeneous element. We say $(A,R)$ is a quasitriangular Hopf superalgebra if

\[
R\Delta = \Delta^{op}R,
\]

\[
(\Delta \otimes 1)(R) = R_{13}R_{23}, \quad (1 \otimes \Delta)(R) = R_{13}R_{12}.
\]

From relations (9) it follows that $(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1$ which implies that $R$ is even.

Let $A$ be a quantization of a quasitriangular Lie superbialgebra $(g,r)$ and let $R \in A \otimes A$ be an invertible homogeneous element. We say $(A,R)$ is a quasitriangular quantization of $(g,r)$ if $R$ satisfies [3], [3] and

\[
R \equiv 1 + hr \mod h^2.
\]

2.4. The quantum dual and the double. In this subsection we define the quantum dual and double of a QUE superalgebra. We will use these construction throughout the rest of the text. The definition of the quantum dual was first given by Drinfeld [4] in the non-super case. For more on quantum duals see [3],[3],[11].

Let $A$ be the symmetric tensor category of topologically free $k[[h]]$-modules, with the super commutativity isomorphism $\tau$ given in [4] and the canonical associativity isomorphism. Let $A$ be a QUE superalgebra and set $A^* = \text{Hom}_A(A,k[[h]])$. Then $A^*$ is a topological Hopf superalgebra where the multiplication, unit, coproduct, counit, and antipode are given by $f(g(x)) = (f \otimes g)\Delta(x)$, $\epsilon$, $\Delta f(x \otimes y) = f(xy)$, 1, and $S^*$ (respectively) for $f,g \in A^*$ and $x,y \in A$. Let $I^*$ be the maximal ideal of $A^*$ defined by the kernel of the linear map $A^* \to k$ given by $f \mapsto f(1) \mod h$. This gives a topology on $A^*$ where $\{(I^*)^n, n \geq 0\}$ is a basis of the neighborhoods of zero.

Here we give the definition of the quantum dual. Define $(A^*)^\vee$ to be the $h$-adic completion of the $k[[h]]$-module $\sum_{n \geq 0} h^{-n}(I^*)^n$. Then $(A^*)^\vee$ is a QUE superalgebra which denote by $A^*$. We call $A^*$ the quantum dual of $A$. Let $\delta_n : A \to A^{\otimes n}$ be
the linear map given by \( \delta_1(a) = a - \epsilon(a)1 \), \( \delta_2(a) = \Delta(a) - a \otimes 1 - 1 \otimes a + \epsilon(a)1 \otimes 1 \), etc. Define \( A' = \{ a \in A | \delta_n(a) \in \mathfrak{h}^n A^n \} \). Then as shown in [11] we have
\[
(A')' = A^*, \quad (A')^\vee = A.
\]

Now we define the notion of the double of \( A \). Let \( \{ x_i \}_{i \in I} \) be a basis of \( A \) and let \( \{ y_i \}_{i \in I} \) be the corresponding dual elements of \( A^* \), i.e. \( \langle y_i, x_j \rangle = \delta_{ij} \). From [11, §3.5] it follows that \( \hat{R} = \sum_{i \in I} x_i \otimes y_i \) is a well defined element of \( A \otimes A^* \).

The following proposition was first due to Drinfeld.

**Proposition 4.** Let \( A \) be a QUE superalgebra and \( A^{op} \) its dual QUE superalgebra with opposite coproduct \( (\Delta^{op} = \tau_{A, A} \circ \Delta) \). Let \( \hat{R} \) be the canonical element defined above. Then there exist a unique Hopf superalgebra structure on \( D(A) := A \otimes A^{op} \) such that

1. \( A \) and \( A^{op} \) Hopf sub-superalgebra of \( D(A) \).
2. The linear map \( A \otimes A^{op} \rightarrow D(A) \) given by \( a \otimes a' \mapsto aa' \) is a bijection.
3. \( \hat{R} \) is a quasitriangular structure for \( D(A) \).

**Proof.** The proof follows as in the pictorial proof of Proposition 12.1 in [9]. One only needs to notice that the corresponding pictures hold in the super case and account for the necessary signs in relation (12.4) and in the proof of Lemma 12.1. We call \( D(A) \) the quantum double of \( A \).

3. The Drinfeld-Jimbo type quantization of Lie superalgebras of type A-G

In this section we recall the defining relations of both a complex Lie superalgebra of type A-G and its quantum analogue. The relations defining these superalgebras are not easily obtained and are of a different nature than relations arising from Lie algebras. In this section we also show that a Lie superalgebra of type A-G has a natural structure of a Lie superalgebra. For the purposes of this paper Lie superalgebras of type A-G will be complex and include the Lie superalgebra \( D(2, 1, \alpha) \).

Any two Borel subalgebras of a semisimple Lie algebra are conjugate. It follows that semisimple Lie algebras are determined by their root systems or equivalently their Dynkin diagrams. However, not all Borel subalgebras of classical Lie superalgebras are conjugate. As shown by Kac [12] a Lie superalgebra can have more than one Dynkin diagram depending on the choice of Borel. However, using Dynkin diagrams Kac gave a characterization of Lie superalgebras of type A-G. Using the standard Borel sub-superalgebra, Floreanini, Leites and Vinet [10] were able to construct defining relations for some Lie superalgebras and their quantum analogues. Then Yamane [20] gave defining relations for each Dynkin diagram of a Lie superalgebra of type A-G. These relations are given by formulas which depend directly on the choice of Dynkin diagram. For this reason, we will restrict our attention to the simplest case and only consider root systems with at most one odd root.

3.1. Lie superalgebras of type A-G. Let \( \mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra of type A-G such that \( \mathfrak{g}_1 \neq \emptyset \). As mentioned above, Kac [12] showed that \( \mathfrak{g} \) is characterized by its associated Dynkin diagrams or equivalently Cartan matrices. A Cartan matrix associated to a Lie superalgebra is a pair consisting of a matrix \( M \) and a set \( \tau \) determining the parity of the generators. As shown by Kac [12], there exist simple root systems of \( \mathfrak{g} \) with exactly one odd root. Let \( \Phi = \{ \alpha_1, \ldots, \alpha_s \} \)
be such a simple root system and let \((A, \{m\})\) be its corresponding Cartan matrix where \(\alpha_m\) is the unique odd root. Note that all simple root systems with exactly one odd root are equivalent and lead to the same Cartan matrix (see [12] §2.5.4).

The Dynkin diagrams corresponding to such Cartan matrices are listed in Table VI of [12]. For notational convenience we set \(I = \{1, \ldots, s\} \).

**Theorem 5** ([19] 20). Let \(\mathfrak{g}\) be a Lie superalgebra of type A-G with associated Cartan matrix \((A = (a_{ij}), \tau)\) where \(\tau = \{m\}\) (as above) or \(\tau = \emptyset\) (purely even case). Then \(\mathfrak{g}\) is generated by \(h_i, e_i, f_i\) for \(i \in I\) (whose parities are all even except for \(e_i\) and \(f_i\), \(i \in \tau\), which are odd) where the generators satisfy the relations

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j \quad [e_i, f_j] = \delta_{ij} h_i
\]

and the “super classical Serre-type” relations

\[
[e_i, e_i] = [f_i, f_i] = 0 \quad \text{for } i \in \tau
\]

\[
(ad e_i)^{1+|a_{ij}|} e_j = (ad f_i)^{1+|a_{ij}|} f_j = 0, \quad \text{if } i \neq j, \text{ and } i \notin \tau
\]

\[
[e_m, [e_{m-1}, [e_m, e_{m+1}]]] = [f_m, [f_{m-1}, [f_m, f_{m+1}]]] = 0
\]

if \(m - 1, m, m + 1 \in I\) and \(a_{mm} = 0\),

\[
[[e_{m-1}, e_m] e_m], e_m] = [[[f_{m-1}, f_m] f_m], f_m] = 0
\]

if the Cartan Matrix \(A\) is of type B, \(\tau = \{m\}\) and \(s = m\).

where \([,]\) is the super bracket, i.e. \([x, y] = xy - (-1)^{\bar{x} \bar{y}} yx\).

For the rest of this paper when considering Lie superalgebras of type A-G we will assume that these Lie superalgebras are defined by the generators and relations given in Theorem 5.

**3.2. Lie superbialgebra structure.** In this subsection we will show that Lie superalgebras of type A-G have a natural Lie superbialgebra structure. The following results are straightforward generalizations of the non-super case.

Let \(\mathfrak{g}\) be a Lie superalgebra of type A-G with associated Cartan matrix \((A, \tau)\) (here we consider any Cartan matrix). Let \(\mathfrak{h} = \langle h_i \rangle_{i \in I}\) be the Cartan subalgebra of \(\mathfrak{g}\). Let \(\mathfrak{n}_\pm\) (resp., \(\mathfrak{n}_-\)) be the nilpotent Lie sub-superalgebra of \(\mathfrak{g}\) generated by \(e_i\)’s (resp., \(f_i\)’s). Let \(\mathfrak{b}_\pm := \mathfrak{n}_\pm \oplus \mathfrak{h}\) be the Borel Lie sub-superalgebra of \(\mathfrak{g}\).

Let \(\eta_\pm : \mathfrak{b}_\pm \to \mathfrak{g} \oplus \mathfrak{h}\) be defined by

\[
\eta_\pm(x) = x \oplus (\pm \bar{x}),
\]

where \(\bar{x}\) is the image of \(x\) in \(\mathfrak{h}\). Using this embedding we can regard \(\mathfrak{b}_+\) and \(\mathfrak{b}_-\) as Lie sub-superalgebras of \(\mathfrak{g} \oplus \mathfrak{h}\).

From Proposition 2.5.3 and 2.5.5 of [12] there exists a unique (up to constant factor) non-degenerate supersymmetric invariant bilinear form \((,)\) on \(\mathfrak{g}\). Moreover, the restriction of this form to the Cartan sub-superalgebra \(\mathfrak{h}\) is non-degenerate. Let \((,)_{\mathfrak{g} \oplus \mathfrak{h}} := \((,)_{\mathfrak{g} \oplus \mathfrak{h}}\) where \((,)_{\mathfrak{g}}\) is the restriction of \((,)\) to \(\mathfrak{h}\).

**Proposition 6.** \((\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)\) is a super Manin triple with \((,)_{\mathfrak{g} \oplus \mathfrak{h}}\).

**Proof.** Under the embedding \(\eta_\pm\), the Lie subsuperalgebra \(\mathfrak{b}_\pm\) is isotropic with respect to \((,)_{\mathfrak{g} \oplus \mathfrak{h}}\). Since \((,)\) and \((,)_{\mathfrak{g}}\) both are invariant super-symmetric nondegenerate bilinear forms then so is \((,)_{\mathfrak{g} \oplus \mathfrak{h}}\). Therefore the Proposition follows.
The Proposition implies that \(g \oplus \mathfrak{h}, \mathfrak{b}_+\) and \(\mathfrak{b}_-\) are Lie superbialgebras. Moreover, we have that \(\mathfrak{b}_+ \cong \mathfrak{b}_+^{op}\) as Lie superbialgebras, where \(\mathfrak{b}_{-}\) is the opposite cobracket.

A straightforward calculation from the definition (see 4.4.2 of [19]) shows that \(0 \oplus \mathfrak{h}\) is an ideal of the Lie superbialgebra \(g \oplus \mathfrak{h}\). Therefore, \(g \oplus \mathfrak{h}/(0 \oplus \mathfrak{h}) \cong g\) is a Lie superbialgebra. Now from Proposition 8 of [11], we have that \((g, r)\) is a quasitriangular Lie superbialgebra where \(r\) is the image of the canonical element \(r\) in \(D(\mathfrak{b}_+) \cong g \oplus \mathfrak{h}\) under the natural projection (for the definitions of \(r\) and \(D\) see 2.1).

3.3. The Drinfeld-Jimbo type superalgebra \(U^D_J(g)\). As mentioned above, Yamane defined a QUE superalgebra for any Cartan matrix associated to the superbialgebras of type A-G. In this subsection we will summarize his results for Cartan matrices coming from root systems with exactly one odd root.

Set 
\[
\begin{bmatrix} m + n \\ n \end{bmatrix} = \prod_{i=0}^{n-1} ((t^{m+n-i} - t^{-m-n+i})/(t^{i+1} - t^{-i-1})) \in \mathbb{C}[t].
\]

Let \(g\) be a Lie superalgebra of type A-G with associated Cartan matrix \((A, \tau)\) where \(\tau = \{m\}\) or \(\tau = \emptyset\). The matrix \(A\) is symmetrizable, i.e. there exists nonzero rational numbers \(d_1, \ldots, d_s\) such that \(d_ia_{ij} = d_ja_{ji}\). By rescaling, if necessary, we may and will assume that \(d_1 = 1\).

Let \(h\) be an indeterminate. Set \(q = e^{h/2}\) and \(q_i = q^{d_i}\).

**Definition 7** ([19][20]). Let \(U^D_J(g)\) be the \(\mathbb{C}[[h]]\)-superalgebra generated by the elements \(h_i, e_i\) and \(f_i, i \in I\) satisfy the relations:

\begin{align*}
(14) & \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \\
(15) & \quad [e_i, f_j] = \delta_{ij}q^{d_i}h_i - q^{-d_i}h_i, \\
(16) & \quad e_i^2 = 0 \quad \text{for} \quad i \in I \text{ such that } a_{ii} = 0, \\
(17) & \quad [e_i, e_j] = 0 \quad \text{for} \quad i, j \in I \text{ such that } a_{ij} = 0 \text{ and } i \neq j, \\
(18) & \quad \sum_{\nu=0}^{1+|a_{ij}|} (-1)^\nu \left[ 1 + \frac{|a_{ij}|}{\nu} \right] e_i^{1+|a_{ij}|-\nu} e_j e_i^\nu = 0 \quad \text{for} \quad 1 \leq i \neq j \leq s \text{ and } i \notin \tau, \\
(19) & \quad e_me_{m-1}e_m e_{m+1} + e_me_{m+1}e_m e_{m-1} + e_{m-1}e_m e_{m+1}e_m + e_{m+1}e_m e_{m-1}e_m \\
& \quad - (q + q^{-1})e_me_{m-1}e_m e_m = 0 \quad \text{if } m - 1, m, m + 1 \in I \text{ and } a_{mm} = 0, \\
(20) & \quad e_{m-1}e_m e_{m-1}e_m - (q + q^{-1}) - 1) e_me_{m-1}e_m e_m - (q - q^{-1}) e_{m+1}e_m e_{m-1}e_m + e_{m-1}e_m e_{m-1}e_m = 0 \\
& \quad \text{if the Cartan Matrix } A \text{ is of type B, } \tau = \{m\} \text{ and } s = m.
\end{align*}

and the relations \((16)-(20)\) with \(e\) replaced by \(f\). All generators are even except for \(e_i\) and \(f_i\) \((i \in \tau)\) which are odd.
We call the relations [16, 20] the quantum Serre-type relations.

Khoroshkin and Tolstoy [14] and Yamane [19, 20] used the quantum double notion (see [24]) to give $U^D_J(h)$ an explicit structure of a quasitriangular Hopf superalgebra. In the remainder of this subsection we recall some of their results which are needed in this paper. Let $U^D_J(b_+)$ be the Hopf sub-superalgebra of $U^D_J(h)$ generated $b_+$ and elements $e_i$, $i = 1, \ldots, n + m - 1$. By construction $U^D_J(b_+)$ is a QUE superalgebra. From Proposition 4 we have that $(\delta \otimes 1) \circ \Delta = \Phi(\Delta \circ 1) \otimes \Delta \Phi^{-1}$, \hspace{1cm} (21)

\Delta^{op} R = R \Delta, \hspace{1cm} (22)

$(1 \otimes \epsilon)(1) = 1 \otimes 1$, \hspace{1cm} (23)

\Phi_{1, 2, 34} \Phi_{12, 3, 4} = \Phi_{2, 3, 4} \Phi_{1, 23, 4} \Phi_{1, 2, 3}$, \hspace{1cm} (24)

and the hexagon relations

\hspace{1cm} (25) \hspace{1cm} (\Delta \otimes 1)(R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi, \hspace{1cm} (1 \otimes \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1}.
Relation (26) implies that $\Phi$ is even. Also, from relations (26) it follows that $(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1$ which implies that $R$ is even.

Recall the definition of the $\tau$ given in (21). Let $g$ be a finite dimensional Lie superalgebra and let $U(g)$ be its universal enveloping superalgebra. Let $t$ be an even invariant super-symmetric element of $g \otimes g$, i.e. an element $t = \sum_k g_k \otimes h_k$ such that $\tilde{g}_k = h_k$ for all $k$,

$$\tau(t) = t \text{ and } [g \otimes 1 + 1 \otimes g, t] = 0 \text{ for all } g \in g.$$

For $n \in \mathbb{N}$ define $t_{ij} \in U(g)^{\otimes n}$ for all $i < j$ (resp. $i > j$) by $t$ (resp. $\tau(t)$) acting on the $i^{th}$ and $j^{th}$ components of the tensor product $U(g)^{\otimes n}[[h]]$.

Consider the system of differential equations

$$(26) \quad \frac{1}{\hbar} \frac{\partial w}{\partial z} = \left( \frac{t_{21}^i}{z} + \frac{t_{23}^i}{z - 1} \right) w$$

where $\hbar = h/(2\pi \sqrt{-1})$. This system of equations has singularities at 0, 1 and $\infty$. It follows from the theory of differential equations that a analytic solution on $(0, 1)$ with a given initial value is unique. Let $F_0(z)$ and $F_1(z)$ be the solutions of (26) define on $(0, 1)$ which have the asymptotic behavior $F_0(z) \sim z^{ht_{12}}$ as $z \to 0$ and $F_1(z) \sim (1 - z)^{ht_{23}}$ as $z \to 1$.

Define $\Phi$ to be the invertible element such that $F_0(z) = F_1(z)\Phi$. We call $\Phi$ the super KZ associator.

**Theorem 9.** [2] The superalgebra $(U(g)[[h]], \Delta, \epsilon, \Phi, R := e^{ht/2})$ is a quasitriangular quasi-superalgebra.

**Proof.** In [2] Drinfeld defines a Lie algebra $a_n$ as the free Lie algebra with generators $X_{ij}, 1 \leq i \neq j \leq n$, module the relations

$$X_{ij} - X_{ji} = 0,$$

$$[X_{ij}, X_{kl}] = 0,$$

$$[X_{ij} + X_{ik}, X_{jk}] = 0,$$

for $i \neq j \neq k \neq l$. Replacing $ht_{ij}$ in the KZ-equation by $X_{ij}$, Drinfeld showed that relations (21) - (25) hold. Now let $g$ be a Lie algebra with an $g$-invariant symmetric two tensor $t$ in any symmetric linear tensor category. The morphism

$$(a_n) \rightarrow U(g)^{\otimes n}$$

given by $X_{ij} \rightarrow t_{ij}$, imposes relations analogous to relations (21) - (25) on $U(g)$. Thus, applying the above discussion to the symmetric linear tensor category of superspaces the result follows.

The quasi-superalgebroid $(U(g)[[h]], \Delta, \epsilon, \Phi)$ is a deformation of the quasi-Hopf superalgebra $U(g)$, i.e. $U(g)[[h]]/hU(g)[[h]]$ is isomorphic, as a quasi-Hopf superalgebra, to $U(g)$. As in the non-super case deformations of quasi-Hopf superalgebra are quasi-Hopf superalgebra (see [2]). Therefore, there exists a homomorphism $S : U(g)[[h]] \rightarrow U(g)[[h]]$ such that $(U(g)[[h]], \Delta, \epsilon, S, \Phi)$ is a quasi-Hopf superalgebra.

In summary, we have constructed a topologically free quasitriangular quasi-Hopf superalgebra $(U(g)[[h]], \Delta, \epsilon, \Phi, R, S)$ which we denote by $A_{g,t}$. 


4.2. The Drinfeld category. Let $g_+$ be a finite dimensional Lie superbialgebra over $k$ and let $g = Dg_+ = g_+ \oplus g_-$ be the Drinfeld double of $g_+$ (see [24]). Let $\Omega$ be the Casimir element defined in [9]. As noted $\Omega$ is an even, invariant, supersymmetric element of $g \otimes g$. Let $\Phi$ and $R = e^{ht/2}$ be the element arising from the pair $(g, t)$, where $t = \Omega$ (see [4.1]).

Let $M_g$ be the category whose objects are $g$-modules and whose morphisms are given by $\text{Hom}_{M_g}(V, W) = \text{Hom}_g(V, W)[[h]]$. For any $V, W \in M_g$, let $V \otimes W$ be the usual super tensor product. Let $\beta_{V, W} : V \otimes W \to W \otimes V$ be the morphism given by the action of $e^{ht/2}$ on $V \otimes W$ composed with the morphism $\tau_{V, W}$ which is defined in [9]. For $V, W, U \in M_g$, let $\Phi_{V, W, U}$ be the morphism defined by the action of $\Phi$ on $V \otimes W \otimes U$ regarded as an element of $\text{Hom}_{M_g}((V \otimes W) \otimes U, V \otimes (W \otimes U))$. The morphisms $\Phi_{V, W, U}$ and $\beta_{V, W}$ define a braided tensor structure on the category $M_g$ (see [13, Prop. XIII.1.4]), which we call the Drinfeld category.

5. The quantization of Lie superbialgebras, Part I

In this section we give the first of two quantizations of Lie superbialgebras. The quantization of this section is important because it commutes with taking the double. The second quantization given in (27) is important because it is functorial. In §4 we show that these two quantizations are isomorphic.

The outline of this section is as follows. Let $g_+$ be a finite dimensional Lie superbialgebra and $g$ its double (see 2.1). We use Verma modules $M_\pm$ over $g$ to define a forgetful functor $F$ from the Drinfeld category $M_g$ (see 4.2) to the category of topologically free $k[[h]]$-modules. We show that the endomorphisms of $F$ are isomorphic to a quasitriangular quantization $H$ of $g$. We then construct a Hopf sub-superalgebra $U_h(g_+)$ of $H$ that is a quantization of $g_+$. We conclude the section with the important result that the quantum double of $U_h(g_+)$ is isomorphic to $H$. The last result is the main step in showing the quantization commutes with taking the double. The results given in this section are straightforward generalizations of [10].

Throughout this section we use the notation of [4.2]. When defining maps from topologically free $U(g)[[h]]$-modules it is helpful to use the following isomorphism,

$$\text{Hom}_{U(g)[[h]]}(X[[h]], Y) \cong \text{Hom}_{U(g)}(X, Y)$$

(27) for any $U(g)$-module $X$ and topologically free $U(g)[[h]]$-module $Y$.

5.1. The Tensor Functor $F$. Let $M_+, M_- \in M_g$ be the induced Verma modules given by

$$M_+ = U(g) \otimes_{U(g_+)} c_+ \quad \quad M_- = U(g) \otimes_{U(g_-)} c_-$$

where $c_\pm$ is the 1-dimensional trivial $g_\pm$-module. The Poincare-Birkhoff-Witt Theorem implies that the linear homomorphisms $U(g_+) \otimes U(g_-) \to U(g)$ and $U(g_-) \otimes U(g_+) \to U(g)$ are isomorphisms. These isomorphisms imply that

$$M_\pm = U(g_\pm) 1_\pm$$

where $1_\pm \in M_\pm$ In particular, $M_\pm$ is a free $U(g_\pm)$-module.

**Lemma 10.** The designation $1 \mapsto 1_+ \otimes 1_-$ extends to a linear map $\phi : U(g) \to M_+ \otimes M_-$ which is an even isomorphism of $g$-modules.
Proof. By the universal property of $U(\mathfrak{g})$ the linear map
$$\mathfrak{g} \to M_+ \otimes M_- \text{ given by } 1 \mapsto x_1^+ \otimes 1_- + 1_+ \otimes x_1^-$$
extends to a $\mathfrak{g}$-module morphism $\phi : U(\mathfrak{g}) \to M_+ \otimes M_-$. By definition this morphism
is even. Moreover, it is easy to check (using the standard grading of universal enveloping superalgebras) that $\phi$ is an isomorphism. Define the functor $F : \mathcal{M}_g \to \mathcal{A}$ as
$$F(V) = \text{Hom}_{\mathcal{M}_g}(M_+ \otimes M_-, V)$$
where $\mathcal{A}$ is the category of topologically free $k[[h]]$-modules (see \[24\]). As stated in \[2,1\] the set of morphisms between superspaces is a superspace, we give $F(V)$ this
superspace structure. The isomorphism $\phi$ of Lemma \[10\] implies that the map
$$\Psi_V : F(V) \to V[[h]] \text{ given by } f \mapsto f(1_+ \otimes 1_-)$$
is an even isomorphism of superspaces.

We now show that the functor $F$ is a tensor functor, i.e. there exists a family of isomorphisms $(J_{V,W})_{V,W \in \mathcal{M}_g}$ such that
$$J_{U \otimes V, W} \circ (J_{U,V} \otimes 1) = J_{U,V \otimes W} \circ (1 \otimes J_{V,W}) \tag{30}$$
for all $U, V, W \in \mathcal{M}_g$. Let $i_\pm : M_\pm \to M_+ \otimes M_-$ be the “coproduct” on $M_\pm$
determined by $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$. As in \[6, Lemma 2.3\] the $\mathfrak{g}$-module morphism $i_\pm$
is coassociative, i.e. $(i_\pm \otimes 1) \circ i_\pm = (1 \otimes i_\pm) \circ i_\pm$ in $\text{Hom}_{\mathcal{M}_g}(M_\pm, M_\pm \otimes M_\pm)$.\[11\]

**Definition 11.** For each pair $V, W \in \mathcal{M}_g$ define $J_{V,W} : F(V) \otimes F(W) \to F(V \otimes W)$ by
$$J_{V,W}(v \otimes w) = (v \otimes w) \circ \Phi_{1,2,3}^{-1} \circ (1 \otimes \Phi_{2,3,4}^{-1}) \circ \beta_{23} \circ (1 \otimes \Phi_{2,3,4}^{-1}) \circ \Phi_{1,2,3} \circ (i_+ \circ i_-).$$

**Theorem 12.** The functor $F$ with the family $(J_{V,W})_{V,W \in \mathcal{M}_g}$ is a tensor functor.\[12\]

**Proof.** Proposition 19.1 in \[9\] is the analogous statement in the case of Lie bialgebras. The proof in \[9\] is pictorial. It relies on the pictorial representation of $i_\pm$ being coassociative. The same pictorial representation of the coassociativity holds in our case. The proof follows exactly as in \[9\] after reinterpreting the pictures in our case.

5.2. The quantization of $\mathfrak{g} = D(\mathfrak{g}_+)$. With the use of the isomorphism given in Lemma \[10\] the functor $F$ can be thought of as the forgetful functor $V \mapsto \text{Hom}_{\mathcal{M}_g}(U(\mathfrak{g}), V)$. The general philosophy of tensor categories says that every forgetful functor, which is a tensor functor, induces a bialgebra structure on the underlying algebra (see §18.2.3 of \[9\]). In this subsection, we will follow this philosophy and show that the tensor functor $F$ induces a superbialgebra structure on $U(\mathfrak{g})[[h]]$. We do this in three steps: (1) show that endomorphisms of $F$ are isomorphic to $U(\mathfrak{g})[[h]]$, (2) show that the family $(J_{V,W})_{V,W \in \mathcal{M}_g}$ is determined by an element $J \in U(\mathfrak{g})[[h]] \otimes 2$, (3) use $J$ to define a quasitriangular Hopf superalgebra structure on $U(\mathfrak{g})[[h]]$.

Let $\text{End}(F)$ be the algebra of natural endomorphisms of $F$. In other words, $\text{End}(F)$ is the algebra consisting of elements $\eta$, so that each $\eta$ is a collection of linear morphisms $\eta_V : F(V) \to F(V)$ such that for all $V, W \in \mathcal{M}_g$ and $f : V \to W$ we have $F(f) \circ \eta_V = \eta_W \circ F(f)$. We make $\text{End}(F)$ a superspace by defining $\eta \in (\text{End}(F))_i$ if the parity of $\eta_V$ is $i$ for all $V \in \mathcal{M}_g$. This makes $\text{End}(F)$ into a superalgebra.
Lemma 13. There is a canonical even superalgebra isomorphism

\[ \theta : U(\mathfrak{g})[[h]] \to \text{End}(F) \]

where \( x \vert_\mathcal{V} \) is \( x \) acting on the \( U(\mathfrak{g})[[h]] \)-module \( \mathcal{V}[[h]] \).

Proof. Using the even isomorphism (29) we can identify \( F(V) \) and \( V[[h]] \). Under this identification \( \theta(x) = x \vert_\mathcal{V} \in \text{End}(F) \) is the endomorphism given by the action of \( x \) on \( V[[h]] \). The morphism \( \theta \) is even since the action of a homogeneous element \( x \) on \( V[[h]] \) preserves the grading. If \( x \neq y \) then \( x \vert_{U(\mathfrak{g})} \neq y \vert_{U(\mathfrak{g})} \) implying \( \theta \) is one to one.

Next we will show that \( \theta \) is onto. Let \( \eta \in \text{End}(F) \), using the above isomorphism we will think of \( \eta \vert_\mathcal{V} \) as a map from \( V[[h]] \) to itself. Set \( x = \eta \vert_{U(\mathfrak{g})}(1) \). Let \( y \in U(\mathfrak{g}) \) and let \( r_y \) be the element of \( \text{End}(U(\mathfrak{g})) \) given by

\[ r_y(z) = (-1)^{\bar{z} \bar{y}} z y \]

for \( z \in U(\mathfrak{g}) \). Note that \( F(r_y) \) under the isomorphism \( F(U(\mathfrak{g})) \to U(\mathfrak{g})[[h]] \) is \( r_y \).

We have

\[ \eta \vert_{U(\mathfrak{g})}(y) = \eta \vert_{U(\mathfrak{g})}(r_y 1) = (-1)^{\bar{z} \bar{y}} \eta \vert_{U(\mathfrak{g})}(1) = (-1)^{\bar{z} \bar{y}} y x = xy. \]

Combining this calculation with (29), we have \( \eta \vert_{U(\mathfrak{g})} = l_x \) where \( l_x(z) = xz \) for \( z \in U(\mathfrak{g}) \). Similarly \( \eta \vert_\mathcal{V} = x \vert_\mathcal{V} \) for any free \( \mathfrak{g} \)-module \( V \). This shows that \( \theta \) is onto since every \( \mathfrak{g} \)-module is a quotient of a free module.

In the rest of this subsection we use properties of the tensor functor \( F \) and the isomorphism \( \theta \) to put algebraic structures on \( U(\mathfrak{g})[[h]] \).

Define the element \( J \in U(\mathfrak{g})\otimes[2][[h]] \) to be

\[ J = (\phi^{-1} \otimes \phi^{-1})(\Phi_{1,2,34}^{-1}(1 \otimes \Phi_{2,3,4})\beta_{23}(1 \otimes \Phi_{2,3,4}^{-1})\Phi_{1,2,34}(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-)) \]

where \( \phi \) is the isomorphism given in Lemma 10.

Lemma 14. Let \( \theta \) be the isomorphism of Lemma 13. Then \( \theta(J) = J_{V,W} \), i.e.

\[ J(v \otimes w) = \Psi_{V \otimes W}(J_{V,W}(\Psi_{V}^{-1}(v) \otimes \Psi_{W}^{-1}(w))) \]

for \( v \in V[[h]] \) and \( w \in W[[h]] \).

Proof. For each \( v \in V[[h]] \) let \( f_v \) to be the element of \( F(V) \) defined by \( f_v(x) = (-1)^{\bar{v} \bar{x}} v \) for \( x \in M_+ \otimes M_- \). Notice that the element \( f_v \) has parity \( \bar{v} \). From Lemma 10 we have \( f_v(1_+ \otimes 1_-) = v \) which implies that \( f_v = \Psi_{V}^{-1}(v) \). To simplify notation let

\[ \vartheta_1 \otimes \vartheta_2 = \Phi_{1,2,34}^{-1}(1 \otimes \Phi_{2,3,4})\beta_{23}(1 \otimes \Phi_{2,3,4}^{-1})\Phi_{1,2,34}(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-) \]

be the element of \( (M_+ \otimes M_-)^{\otimes 2}[[h]] \). Now we have the right side of (33) is

\[ (J_{V,W}(\Psi_{V}^{-1}(v) \otimes \Psi_{W}^{-1}(w)))(1_+ \otimes 1_-) = (f_v \otimes f_w)(\vartheta_1 \otimes \vartheta_2) = (-1)^{\bar{v} \bar{w}} f_v \vartheta_1 \otimes f_w \vartheta_2 \]

\[ = (-1)^{\bar{v} \bar{w} + \bar{v} \bar{w} + \bar{v} \bar{w} \phi^{-1}(\vartheta_1)} v \otimes \phi^{-1}(\vartheta_2) \]

On the other hand, the left side of (33) is

\[ \phi^{-1}(\vartheta_1) \otimes \phi^{-1}(\vartheta_2)(v \otimes w) = (-1)^{\bar{v} \bar{w}} \phi^{-1}(\vartheta_1) v \otimes \phi^{-1}(\vartheta_2) w \]

\[ = (-1)^{\bar{v} \bar{w} + \bar{v} \bar{w} + \bar{v} \bar{w} \phi^{-1}(\vartheta_1)} v \otimes \phi^{-1}(\vartheta_2) w, \]

where the last equality follows from the fact that \( \vartheta_1 \otimes \vartheta_2 \) is even, i.e. \( \bar{v} + \bar{w} = 0 \).
Lemma 15. \( J \equiv 1 + \frac{\hbar}{2} \mod \hbar^2 \).

Proof. Recall the definition of \( r \) given in (2.1) i.e. \( r = \sum p_i \otimes m_i \) where \( p_i \) and \( (m_i)_i \) are bases of \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) respectively. It follows that

\[
\tau(r)(1_- \otimes 1_+) = \sum (-1)^{\bar{m}_i} m_i \otimes p_i)(1_- \otimes 1_+),
\]

whereobilem 10.

From the hexagon relation (25) we have \( \Phi = \Phi \equiv 1 \mod \hbar^2 \) and \( \tau \equiv \tau \equiv 1 \mod \hbar^2 \). Thus,

\[
J \equiv (\phi^{-1} \otimes \phi^{-1})(e^{\hbar Q_{23}/2})(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-) \mod \hbar^2
\]

\[
= 1 + \hbar/2(\phi^{-1} \otimes \phi^{-1})(\tau(r)_{23} + \tau(r)_{23})(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-) \mod \hbar^2
\]

\[
= 1 + \hbar/2(\phi^{-1} \otimes \phi^{-1})(\sum 1_+ \otimes p_i 1_- \otimes m_i 1_+ \otimes 1_-) \mod \hbar^2
\]

\[
= 1 + \hbar/2 \sum (p_i \phi^{-1}(1_+ \otimes 1_-) \otimes m_i \phi^{-1}(1_+ \otimes 1_-)) \mod \hbar^2
\]

\[
= 1 + \hbar/2 \mod \hbar^2
\]

where the third equivalence follows from (28) and the fourth follows from (25).

Proposition 16. Let \( H = U(\mathfrak{g})[[\hbar]] \). Then \( H \) is a Hopf superalgebra whose coproduct, counit and antipode are given by

\[
\Delta = J^{-1} \Delta_0 J, \quad \epsilon = \epsilon_0
\]

\[
S = QS_0 Q^{-1}
\]

where \( Q = m(S_0 \otimes 1)(J) \) and \( \Delta_0, \epsilon_0 \) and \( S_0 \) are the usual coproduct, counit and antipode of \( U(\mathfrak{g})[[\hbar]] \).

Proof. First, \( \Delta \) and \( \epsilon \) are algebra morphisms since \( \Delta_0 \) and \( \epsilon_0 \) are algebra morphisms. From Lemma 15 we have that \( (\epsilon \otimes 1)J = (1 \otimes \epsilon)J = 1 \) which implies \( (\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta \). Theorem 12 and Lemma 14 imply that

\[
J_{12,3}(J \otimes 1) = J_{1,23}(1 \otimes J).
\]

We will now use equality (38) to show that \( \Delta \) is coassociative.

\[
(1 \otimes \Delta) \Delta(x) = (1 \otimes J^{-1} \Delta_0 J)(J^{-1} \Delta_0 J)\Delta(x)
\]

\[
= (1 \otimes \Delta^{-1})J_{1,23}^2(1 \otimes \Delta_0)\Delta_0(1 \otimes J)
\]

\[
= (J^{-1} \otimes 1)J_{12,3}(\Delta_0 \otimes 1)\Delta_0 J_{12,3}(J \otimes 1)
\]

\[
= (\Delta \otimes 1)\Delta(x)
\]

for all \( x \in H \). The compatibility conditions between \( S \) and \( \epsilon \) follow in a similar manner.

The isomorphism \( \theta \) of Lemma 13 induces a Hopf superalgebra on \( \text{End}(F) \). For the rest of this paper, we identify the Hopf superalgebra \( H \) with \( \text{End}(F) \) (using \( \theta \)). As we will see it is sometimes convenient to use the elements of \( H \) and other times endomorphisms of \( \text{End}(F) \).

Theorem 17. \( H \) is a quantization of the Lie superbialgebra \( \mathfrak{g} \).
Proof. By definition $H/hH$ is isomorphic to the Hopf superalgebra $U(g)$. To prove the theorem we show that relation (44) holds. From the definition of the coproduct $\Delta$ and Lemma 16 we have

\[(43) \quad \Delta(x) \equiv \Delta_0(x) + (h/2)[\Delta_0(x), r] \mod h^2 \]

for all $x \in g \subset H$. Thus,

\[(44) \quad h^{-1}(\Delta(x) - \Delta_{op}(x)) \equiv h^{-1}\Delta_0(x) + 1/2[\Delta_0(x), r] - h^{-1}\Delta_{op}^0(x) - 1/2[\Delta_{op}^0(x), \tau(r)] \mod h \]

\[\equiv 1/2[\Delta_0(x), r - \tau_{g,g}(r)] \mod h \]

since $t = r + \tau(r)$ is $g$-invariant and $\Delta_0(x) = \Delta_{op}^0(x)$ for all $x \in g$ (for the definition of $\tau$, see (4)). The proof is completed by recalling that the cobracket of $g$ is defined by $\partial r(x) := [\Delta_0(x), r].$

Define $R = (J_{op})^{-1} e^{\frac{\Delta}{h}} J \in H \otimes H$. We call $R$ the R-matrix.

Corollary 18. $(H, R)$ is a quasitriangular quantization of $(g, r)$.

Proof. Replacing the standard commutativity isomorphism with the super commutativity isomorphism, (i.e. substituting $\tau$ for $\sigma$) the proof follows just as in the purely even case [3, Corollary 19.1].

5.3. Quantization of $g_+$ and $g_-$. Here we construct a quantization of the Lie superbialgebra $g_{\pm}$, which is a Hopf sub-superalgebra of $H$. To this end, we continue following the work of Etingof and Kazhdan [6] and notice that $R$ is polarized, i.e. $R \in U_h(g_+) \otimes U_h(g_-)$. It is possible to show directly that $U_h(g_+)$ is closed under coproduct. However, in [6, 3] we use the polarization of $R$ to show that the quantization commutes with the double.

Using the even isomorphisms (29) and (31) we can identify the superalgebras $\text{End}(F)$ and $\text{End}(M_+ \otimes M_-)$. We will not make a distinction between these superalgebras. Define $U_h(g_+) = F(M_-)$ and embed it into $H$ using the map $i : F(M_-) \to \text{End}(M_+ \otimes M_-)$ given by

\[i(x) = (1 \otimes x) \circ \Phi \circ (i_+ \otimes 1) \]

for $x \in F(M_-)$.

The coassociativity of $i_+$ implies that for $x, y \in F(M_-)$

\[i(x) \circ i(y) = i(z) \]

where $z = x \circ (1 \otimes y) \circ \Phi \circ (i_+ \otimes 1) \in F(M_-)$. Using the embedding $i$, we consider $U_h(g_+)$ is a subsuperalgebra of $H$. Similarly, the map $F(M_-) \to \text{End}(M_+ \otimes M_-)$ given by $x \mapsto (x \otimes 1) \circ \Phi \circ (1 \otimes i_-)$ makes $U_h(g_-) := F(M_+)$ into a subsuperalgebra of $H$.

Theorem 19. $U_h(g_+)$ and $U_h(g_-)$ are Hopf sub-superalgebra of $H$. Moreover $U_h(g_{\pm})$ is a quantization of the Lie superbialgebra $g_{\pm}$.

Proof. As in [6] we needed the following lemma to prove the theorem.

Lemma 20. $R$ is polarized, i.e. $R \in U_h(g_+) \otimes U_h(g_-) \subseteq H \otimes H$. 

Proof. In [9] the analogous statement in the purely even case is proved using a pictorial proof. After representing the Hopf superalgebra structure of $H$ and functoriality of the braiding $\beta = \tau e^{\frac{\omega}{2}}$ pictorially the proof follow exactly as in Lemma 19.4 [9].

Let $p_+: U_h(\mathfrak{g}_-)^* \rightarrow U_h(\mathfrak{g}_+)$ and $p_- : U_h(\mathfrak{g}_+)^* \rightarrow U_h(\mathfrak{g}_-)$ be the even linear maps given by

$$p_+(f) = (1 \otimes f)(R) \quad \text{and} \quad p_-(f) = (f \otimes 1)(R)$$

for $f \in U_h(\mathfrak{g}_-)^* := \text{Hom}_{\mathbb{A}}(U_h(\mathfrak{g}_\pm), k[[h]])$. Let $\text{Im} p_{\pm}$ be the images of $p_{\pm}$. Let $\text{Im} p_{\pm}$ be the closer of the $k[[h]]$-superalgebra generated by $\text{Im} p$.

Lemma 21. $\widehat{\text{Im} p_{\pm} \otimes k[[h]]} k((h))$ is the h-adic completion of $U_h(\mathfrak{g}_\pm) \otimes k[[h]]$ where the tensor product is the tensor product in the h-adic completion.

Proof. Using the even graded linear map $p_\pm$ the proof is identical to the proof of Proposition 4.5 [6]. In particular, no new signs are introduced in the proof and grading is preserved since it is preserved by $p_\pm$. Now we prove the theorem. Relations (4) imply that $\text{Im} p_\pm$ is closed under coproduct. Therefore, by Lemma 21 we have that $U_h(\mathfrak{g}_\pm)$ is closed under coproduct. Moreover, since $H$ is a quasitriangular Hopf superalgebra we have

$$(S \otimes 1)R = R^{-1}$$

which implies that $U_h(\mathfrak{g}_\pm)$ is closed under the antipode. This proves $U_h(\mathfrak{g}_\pm)$ is a Hopf sub-superalgebra of $H$.

Next we show that $U_h(\mathfrak{g}_\pm)$ is a quantization of $\mathfrak{g}_\pm$. The isomorphism given in [29] implies that $U_h(\mathfrak{g}_\pm)$ is isomorphic, as a superspace, to $U(\mathfrak{g}_\pm)[[h]]$. Moreover, the Hopf superalgebra $U_h(\mathfrak{g}_\pm)/hU_h(\mathfrak{g}_\pm)$ is isomorphic to $U(\mathfrak{g}_\pm)$. Since $U_h(\mathfrak{g}_\pm)$ is a Hopf sub-superalgebra of $H$ we have that equivalencies (4) and (4) hold for all $x \in \mathfrak{g}_\pm$. Thus, $U_h(\mathfrak{g}_\pm)$ satisfies equivalence (4) and so is a quantization of $\mathfrak{g}_\pm$. We call $U_h(\mathfrak{g}_\pm)$ the Etingof-Kazhdan quantization of $\mathfrak{g}_\pm$.

5.4. The quantum dual of $U_h(\mathfrak{g}_+)$.

Recall the definitions of the quantum dual and double of a QUE superalgebra given in [24]. In this subsection we will show that the quantum dual of $U_h(\mathfrak{g}_-)^{\text{op}}$ is $U_h(\mathfrak{g}_+)$ and that the double of $U_h(\mathfrak{g}_+)$ is $H$. The former statement follows from the use of the linear map $p_+$ which arises from the polarization of $R$. In [4] we will use the results of this subsection to show that the quantization commutes with taking the double.

Proposition 22. The linear map $p_+$ ($p_-$) is a even injective homomorphism of topological Hopf superalgebras $(U_h(\mathfrak{g}_-)^{\text{op}})^* \rightarrow U_h(\mathfrak{g}_+)$ (resp. $U_h(\mathfrak{g}_+)^* \rightarrow U_h(\mathfrak{g}_-)^{\text{op}}$). Moreover, $\text{Im} p_{\pm} = U_h(\mathfrak{g}_\mp)^!$.

Proof. The proof follows as in the proof of Proposition 4.8 and Proposition 4.11 in [6].

Corollary 23. The quantum dual of the QUE superalgebra $U_h(\mathfrak{g}_+)$ is $U_h(\mathfrak{g}_-)^{\text{op}}$. Moreover, the quantization of $\mathfrak{g} = D(\mathfrak{g}_+)$ given in [16] and the quantum double of $U_h(\mathfrak{g}_+)$ are isomorphic as quasitriangular QUE superalgebras, i.e. $H \cong D(U_h(\mathfrak{g}_+))$ (for the definition of the quantum double see Proposition 4).

Proof. The first assertion follows from

$$U_h(\mathfrak{g}_+)^* := (U_h(\mathfrak{g}_-)^{\text{op}})^! \cong ((U_h(\mathfrak{g}_-)^{\text{op}})^!)^! = U_h(\mathfrak{g}_-)^{\text{op}}$$
where the isomorphism comes from Proposition 22 and the third equality follows from (11).

To prove the second assertion we will show that \( H \) satisfies the defining relations of the Hopf superalgebra structure on the double of \( U_h(\mathfrak{g}_+) \), then the result follows from the uniqueness of Proposition 4. By Theorem 19, \( U_h(\mathfrak{g}_+) \) and \( U_h(\mathfrak{g}_-) \) are Hopf sub-superalgebras of \( H \). The multiplication map \( U_h(\mathfrak{g}_+) \otimes U_h(\mathfrak{g}_-) \to H \) is a bijection, as it is modulo \( h \). In equation (55) we concluded that the map \( p_- \) induces an isomorphism between \( U_h(\mathfrak{g}_+)^* \) and \( U_h(\mathfrak{g}_-)^{op} \). Therefore, the definition of the quantum dual implies

\[
D(U_h(\mathfrak{g}_+)) \cong U_h(\mathfrak{g}_+) \otimes U_h(\mathfrak{g}_-).
\]

Recall that the map was defined \( p_- \) was defined using the R-Matrix \( R \) of \( H \). It follows that \( R \) corresponds to the canonical element \( \hat{R} \) of \( D(U_h(\mathfrak{g}_+)) \). Thus, the uniqueness of Proposition 4 implies \( D(U_h(\mathfrak{g}_+)) = H \).

6. Quantization of Quasitriangular Lie Superbialgebras

Let \( \mathfrak{g}_+ \) be a finite dimensional Lie superbialgebra. Recall that the double \( D(\mathfrak{g}_+) \) of \( \mathfrak{g}_+ \) is a quasitriangular Lie superbialgebra (see [21]). In this section we will construct a quantization of quasitriangular Lie superbialgebras. This quantization is similar to the quantization of Lie superbialgebra of [22]. In [23] we will show that for finite dimensional quasitriangular Lie superbialgebra the two quantizations are isomorphic. Moreover, by construction the quantization \( H \) given in [23] is the same as the quantization of \( D(\mathfrak{g}_+) \) given below. These facts are used in proving that the quantization commutes with taking the double.

Let \( (\mathfrak{g}, r) \) be a quasitriangular Lie superbialgebra. Set

\[
\mathfrak{g}_+ = \{(1 \otimes f)r \mid f \in \mathfrak{g}^*\} \quad \text{and} \quad \mathfrak{g}_- = \{(f \otimes 1)r \mid f \in \mathfrak{g}^*\}.
\]

Then \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) are finite dimensional Lie superbialgebras (see [22] Lemma 5.2). Moreover, \( \mathfrak{g}_- \cong \mathfrak{g}_+^* \) and there is a natural homomorphism of quasitriangular Lie superbialgebras \( \pi : D(\mathfrak{g}_+) \to \mathfrak{g} \) (see §5 of [22]). Let \( \mathcal{M}_\mathfrak{g} \) be the category whose objects are \( \mathfrak{g} \)-modules and whose morphisms are given by \( \text{Hom}_{\mathcal{M}_{\mathfrak{g}}}(V, W) = \text{Hom}_\mathfrak{g}(V, W)[[h]] \). As in [14, 22] let \( \mathcal{M}_{D(\mathfrak{g}_+)} \) be the Drinfeld category of the double \( D(\mathfrak{g}_+) \). From the homomorphism \( \pi \) we have

\[
\pi^* : \mathcal{M}_\mathfrak{g} \to \mathcal{M}_{D(\mathfrak{g}_+)}
\]

whose pull back gives a braided tensor structure on the category \( \mathcal{M}_\mathfrak{g} \). Let \( M_- \) and \( M_+ \) be the “Verma” modules in \( \mathcal{M}_{D(\mathfrak{g}_+)} \) (see [25]).

Let \( \mathcal{F} : \mathcal{M}_\mathfrak{g} \to \mathcal{A} \) be the functor given by

\[
\mathcal{F}(V) = \text{Hom}_{\mathcal{M}_{D(\mathfrak{g}_+)}}(M_+ \otimes M_-, \pi^*(V)).
\]

Then \( \mathcal{F} \) is a tensor functor with the isomorphism of functors \( J \) giving in Definition 14. As in [28] the map

\[
\mathcal{F}(V) \to V[[h]] \quad \text{given by} \quad f \mapsto f(1_+ \otimes 1_-)
\]

is an even isomorphism of superspaces. Using this isomorphism we construct the canonical isomorphism \( \theta : U(\mathfrak{g})[[h]] \to \text{End}(\mathcal{F}) \) of Lemma 13. The equations (36) and (47) define a Hopf superalgebra structure on \( U(\mathfrak{g})[[h]] \) which is equal to \( \text{End}(\mathcal{F}) \).
Finally, as in Corollary 18 we have that \((U(\mathfrak{g})[[h]], R)\) is a quasitriangular quantization of \((\mathfrak{g}, r)\), where the R-matrix is defined as in §5.2. We denote this quasitriangular QUE superalgebra by \(U_h^q(\mathfrak{g})\).

7. The quantization of Lie superbialgebras, Part II

Here we give the second quantization of Lie superbialgebras. As mentioned before, this quantization is isomorphic to the first quantization of the Lie superbialgebra constructed in §3. We denote the quantization of this section by \(U_h(\mathfrak{g})\).

In §8 we will see that the quantization of this section is functorial.

We follow the quantization of Lie bialgebras given in Part II of [6]. The results of [6] should generalize to the setting of all Lie superbialgebras. However, we will only check that the results hold for finite dimensional Lie superbialgebras.

In this section we consider topological superspaces. We need topology to deal with convergence issue involving duals of infinite dimensional space and tensor products of such spaces. In particular, we need modules to be equicontinuous (see [6, §7.3]). Since we are working with finite dimensional Lie superbialgebras all modules are over such superalgebras are equicontinuous. For this reason, we will assume that all modules are equicontinuous. We proceed in much the same way as in §5. In other words, given a finite dimensional Lie superbialgebra \(\mathfrak{g}_+\), we use Verma modules to define a tensor functor such that the set of endomorphisms of this functor is a quantization of the double of \(\mathfrak{g}_+\) which contains a quantization of \(\mathfrak{g}_+\).

7.1. Topological superspaces. Let \(k\) be a field of characteristic zero. We consider \(k\) as a topological superspace concentrated in degree 0 with discrete topology.

Let \(V\) be a topological superspace, i.e. a \(\mathbb{Z}_2\)-graded topological vector space. We say \(V\) is linear if open superspaces of \(V\) form a basis of neighborhoods of 0. The superspace \(V\) is separated (complete) if the natural map \(V \to \varprojlim V/U\) is a monomorphism (resp. epimorphism) where the limit runs over open sub-superspaces \(U\). Throughout this section we will only consider complete, separated topological superspaces, so when we use the phrase “topological superspace” we will mean “complete, separated, linear topological superspaces”.

Let \(V\) and \(W\) be topological superspaces. If \(U\) is an open sub-superspace of \(V\) then \(V/U\) is discrete. Using this we define the tensor product of two topological superspaces \(V\) and \(W\) to be

\[
V \hat{\otimes} W := \varprojlim V/V' \otimes W/W'
\]

where \(V'\) and \(W'\) run over open sub-superspaces of \(V\) and \(W\) respectively. Let \(V[[h]] = V \hat{\otimes} k[[h]]\) be the space of formal poser series in \(h\). We give the superspace \(\text{Hom}_k(V, W)\) of all continuous homomorphisms a topology, as follows. Let \(B\) be a topological basis of \(W\). For any \(n \geq 1\) let \(U_1, U_2, ..., U_n \in B\) and \(v_1, v_2, ..., v_n \in V\). Then the collection

\[
\{ f \in \text{Hom}_k(V, W) : f(v_i) \in U_i \text{ for } i = 1, ..., n\}_{U_1, U_2, ..., U_n, v_1, v_2, ..., v_n}
\]

is a basis for the topology on \(\text{Hom}_k(V, W)\). We call this topology the weak topology.

Note that if \(V\) is finite dimensional then the weak topology on \(V^* = \text{Hom}_k(V, k)\) is the discrete topology.
7.2. Topological \( \mathfrak{g} \)-modules. Let \( \mathfrak{g}_\pm \) be a finite dimensional Lie superbialgebra we give \( \mathfrak{g}_+ \) the discrete topology. In this section a \( \mathfrak{g}_+ \)-module will be a topological superspace \( M \) with a continuous homomorphism of topological Lie algebras

\[
\pi : \mathfrak{g}_+ \to \text{End}(M)
\]

such that \( \pi((\mathfrak{g}_+)_i) \subset \text{End}(M)_i \) for \( i = 0, 1 \).

Let \( \mathfrak{g} = D(\mathfrak{g}_+) \) be the Drinfeld double of \( \mathfrak{g}_+ \) (see §2.1). Given two topological \( \mathfrak{g} \)-modules \( V, W \) let \( \text{Hom}_\mathfrak{g}(V, W) \) be the topological superspace of all continuous \( \mathfrak{g} \)-modules homomorphisms. Let \( \mathcal{M}_\mathfrak{g}^t \) be the category whose objects are \( \mathfrak{g} \)-modules and morphism are given by

\[
\text{Hom}_{\mathcal{M}_\mathfrak{g}^t}(V, W) = \text{Hom}_\mathfrak{g}(V, W)[[h]]
\]

for \( V, W \in \mathcal{M}_\mathfrak{g}^t \).

Using the tensor produce \( \hat{\otimes} \) we define a braided tensor structure on \( \mathcal{M}_\mathfrak{g}^t \) as follows. Let \( \Omega \) be the Casimir element defined in §2.1 and \( \Phi \) be the associator constructed using \( \Omega \) (see §2.2). For \( V, W, U \in \mathcal{M}_\mathfrak{g}^t \), let \( \Phi_{V, W, U} \) be the element of \( \text{Hom}_{\mathcal{M}_\mathfrak{g}^t}((V \hat{\otimes} W) \hat{\otimes} U, V \hat{\otimes} (W \hat{\otimes} U)) \) given by the action of \( \Phi \) on \( V \hat{\otimes} W \hat{\otimes} U \) and let \( \beta_{V, W} := \tau_{V, W} e^{\hbar/2} \in \text{Hom}_{\mathcal{M}_\mathfrak{g}^t}(V \hat{\otimes} W, W \hat{\otimes} V) \) (for the definition of \( \tau \), see §1). The morphisms \( \Phi_{V, W, U} \) and \( \beta_{V, W} \) define a braided tensor structure on \( \mathcal{M}_\mathfrak{g}^t \).

Let \( \mathcal{A}^t \) be the category whose objects are \( k[[h]] \)-modules and morphisms are continuous \( k[[h]] \)-linear maps. \( \mathcal{A}^t \) is a symmetric tensor category where the tensor product \( V \hat{\otimes} W \) is the tensor product \( V \otimes W \) modulo the image of the operator \( h \otimes 1 - 1 \otimes h \).

Recall the definitions of \( M_\pm \) and \( i_\pm \) given in §5.1. We give \( M_\pm \) the discrete topology. For finite dimensional Lie bialgebras these topologies are the same as the topologies defined in §7.5. Let \( M_\mathfrak{g}^* \) be the superspace of all continuous linear functionals on \( M_\mathfrak{g} \). For any \( n \geq 0 \) let \( U(\mathfrak{g}_-)_n \) be the elements of \( U(\mathfrak{g}_-) \) with degree \( \leq n \). Then \( M_\mathfrak{g}^* \) is the projective limit of \( U(\mathfrak{g}_-)_n^* \). By giving \( U(\mathfrak{g}_-)_n^* \) the discrete topology, the superspace \( M_\mathfrak{g}^* \) inherits a natural structure of a topological superspace.

Let \( i_+^* : M_\mathfrak{g}^* \hat{\otimes} M_\mathfrak{g}^* \rightarrow M_\mathfrak{g}^* \) be the map defined by

\[
i_+^*(f \hat{\otimes} g)(x) := (f \otimes g)i_+(x)
\]

for \( f, g \in M_\mathfrak{g}^* \) and \( x \in M_\mathfrak{g} \). By definition of the topology on \( M_\mathfrak{g}^* \) the map \( i_+^* \) is continuous. Therefore, \( i_+^* \) extends to a morphism \( i_+^* : M_\mathfrak{g}^* \otimes M_\mathfrak{g}^* \rightarrow M_\mathfrak{g}^* \). The proof of Lemma 8.3 implies that the map \( i_+^* \) is associative, i.e. \( i_+^* \circ (i_+^* \otimes 1) \Phi^{-1} = i_+^* \circ (1 \otimes i_+^*) \).

7.3. The Tensor Functor \( \overline{\mathcal{F}} \). Define the functor \( \overline{\mathcal{F}} : \mathcal{M}_\mathfrak{g}^t \rightarrow \mathcal{A}^t \) as

\[
\overline{\mathcal{F}}(V) = \text{Hom}_{\mathcal{M}_\mathfrak{g}^t}(M_-, M_\mathfrak{g}^* \hat{\otimes} V).
\]

From the following Lemma 24 we have that \( \overline{\mathcal{F}} : V \rightarrow V[[h]] \) where \( V[[h]] \) is the topologically free \( k[[h]] \)-module associated to the graded vector space underlying \( V \). For any \( V \in \mathcal{M}_\mathfrak{g}^t \) define

\[
\Psi_V : \text{Hom}_\mathfrak{g}(M_-, M_\mathfrak{g}^* \hat{\otimes} V) \rightarrow V
\]

by \( f \rightarrow (1_+ \otimes 1)f(1_-) \) where \( (1_+ \otimes 1)(g \otimes v) := g(1_+)v \) for \( g \in M_\mathfrak{g}^* \) and \( v \in V \).

Lemma 24. \( \Psi_V \) is a even vector superspace isomorphism.
proof: The proof of the lemma follows from checking that the isomorphisms of [6, Lemma 8.1] preserve the $\mathbb{Z}_2$-grading. We define the these isomorphisms and see that they are even.

By Frobenius reciprocity the following maps

\[
\overline{\text{Hom}}_\theta(M_-, M_+^* \hat{\otimes} V) \to (M_+^* \hat{\otimes} V)^{-} \quad \text{given by } f \mapsto f(1_-),
\]
\[
\overline{\text{Hom}}_\theta(M_+, V) \to V \quad \text{given by } f \mapsto f(1_+)
\]

are isomorphism of topological vector spaces. Let

\[
(M_+^* \hat{\otimes} V)^{-} \to \overline{\text{Hom}}_\theta(M_+, V)
\]

be the map given by $f \otimes x \mapsto f_x$, where $f_x(y) := (-1)^{\overline{f}} f(y).$ This map is an isomorphism of topological vector spaces (see [6, Proof of Lemma 8.1]) where $\overline{\text{Hom}}_\theta(M_+, V)$ has the weak topology. Also by definition all of these maps are even isomorphisms of superalgebras. Composing the above maps we have the desired isomorphism

\[
\overline{\text{Hom}}_\theta(M_-, M_+^* \hat{\otimes} V) \to V \quad \text{which is given by } f \mapsto (1_+ \otimes 1) f(1_-).
\]

Definition 25. For each pair $V, W \in \mathcal{M}_\theta$ define $\overline{J}_{V,W} : \overline{T}(V) \hat{\otimes} \overline{T}(W) \to \overline{T}(V \hat{\otimes} W)$ by

\[
\overline{J}_{V,W}(v \hat{\otimes} w) = (i^*_+ \otimes 1 \otimes 1) \circ \Phi^{-1}_{1,2,3,4} \circ (1 \otimes \Phi_{2,3,4}) \circ \beta^{-1}_{23} \circ (1 \otimes \Phi^{-1}_{2,3,4}) \circ \Phi_{1,2,3,4} \circ (v \hat{\otimes} w) \circ i_-.
\]

Theorem 26. The collection $(\overline{J}_{V,W})_{V,W \in \mathcal{M}_\theta}$ defines a tensor structure on $\overline{T}$, i.e. $\overline{T}$ is a tensor functor.

Proof. Using the facts that $i_-$ is coassociative and $i^*_+$ is associative the proof follows exactly in the same way as the universal or pictorial proof of Proposition 19.1 [9].

Let $\text{End}(\overline{T})$ be the endomorphisms of $\overline{T}$ (see [4,2]). Using $\Psi_V$ to identify $\overline{T}(V)$ and $V[[h]]$ the proof of Lemma [34] shows that there exists a canonical even superalgebra isomorphism

\[
\theta : U(\mathfrak{g})[[h]] \to \text{End}(\overline{T}) \quad \text{given by } x \mapsto x|_V
\]

where $x|_V$ is $x$ acting on the $U(\mathfrak{g})[[h]]$-module $V[[h]]$. We use this isomorphism is to identify $\text{End}(\overline{T})$ and $U(\mathfrak{g})[[h]]$.

Next we will define an element $\overline{J} \in U(\mathfrak{g}) \hat{\otimes}^2 [[h]]$ whose action on $V[[h]] \hat{\otimes} W[[h]]$ determines $\overline{J}_{V,W}$. Recall the isomorphism $\Psi_V : \overline{\text{Hom}}_\theta(M_-, M_+^* \hat{\otimes} V) \to V$ of Lemma [23]. Let $\phi : M_- \to M_+^* \hat{\otimes} U(\mathfrak{g})$ be the even morphism given by $\phi = \Psi^{-1}_{U(\mathfrak{g})}(1).$ Given $g \in \overline{\text{Hom}}_\theta(V,W)$ denote the map $\hat{g} : V[[h]] \to W[[h]]$ by $\sum v_i h^i \mapsto \sum g(v_i) h^i$.

Define the element $\overline{J} \in U(\mathfrak{g}) \hat{\otimes}^2 [[h]]$ by

\[
\overline{J} = (1_+ \otimes 1) \left((i^*_+ \otimes 1 \otimes 1) \Phi^{-1}_{1,2,3,4} (1 \otimes \Phi_{2,3,4}) \beta^{-1}_{23} (1 \otimes \Phi^{-1}_{2,3,4}) \Phi_{1,2,3,4}(y)\right)
\]

where $y = \phi(1_-) \otimes \phi(1_-)$.

The following lemma shows that the map $\overline{J}_{V,W}$ is determined by the element $\overline{J}$.

Lemma 27. Let $\theta$ be the isomorphism given in [43]. Then $\theta(J) = \overline{J}_{V,W}$, i.e.

\[
\overline{J}(v \hat{\otimes} w) = \Psi_{V \hat{\otimes} W}(\overline{J}_{V,W}(\Psi^{-1}_{V} v \hat{\otimes} \Psi^{-1}_{W} w))
\]

for all $v \in V[[h]]$ and $w \in W[[h]]$. 
Proof. By (29) it is enough to check that (50) holds for \( v \in V \) and \( w \in W \). Let \( \phi(1_-) \) be the tensor \( \sum f_j \otimes x_i \in (M^2_+ \hat{\otimes} U(\mathfrak{g}))^{g_7} \), then we have \( \sum f_i(1+)x_i = 1 \). Let \( v \in V \) and consider the following calculation:

\[
(1_+ \otimes 1)( \sum f_i \otimes (x_i v)) = \sum f_i(1+ v x) = v.
\]

The above shows that \( \Psi^{-1}_V(v)(1_-) = \sum f_i \otimes (x_i v) \).

Let \( \vartheta \in U(\mathfrak{g}) \otimes \mathbb{Z}[\hbar] \) be given by:

\[
\vartheta := \Phi^{-1}_{1,2,3,q}(1 \otimes \Phi_{2,3,4}) \beta_{23}^{-1}(1 \otimes \Phi^{-1}_{1,2,3,q}) \Phi_{1,2,3,q}
\]

We use \( \vartheta \) to simplify notation. Represent \( \vartheta = \sum \vartheta_i h^i \) where \( \vartheta_i = \sum_k \vartheta_k^i \otimes \vartheta_k^4 \otimes \vartheta_k^3 \otimes \vartheta_k^4 \). Evaluating the right side of (50) with \( v \in V \) and \( w \in W \), we have:

\[
\hat{\Psi}_V \otimes \hat{\Psi}_W \left( J_{V,W} (\Psi_V^{-1} v \otimes \Psi_W^{-1} w) \right) = (1_+ \otimes 1) \left[ i^*_+ \circ \vartheta \circ \tau \circ \sum f_j \otimes (x_j v) \otimes \sum f_i \otimes (x_i w) \right] = (1_+ \otimes 1) \left[ i^*_+ \circ \vartheta \circ \tau \left( \sum f_j \otimes (x_j v) \otimes (x_i w) \right) \right] = \sum \varepsilon_i \left( -1 \right)^A f_j (\vartheta_k^4 f_i (\vartheta_k^2) \vartheta_k^4 x_j \otimes \vartheta_k^4 x_i \right) v \otimes w
\]

where \( A = x_j v f_1 + f_j (\vartheta_k^2 \vartheta_k^4 + \vartheta_k^4 \vartheta_k^2) + f_k (\vartheta_k^2 \vartheta_k^4 + \vartheta_k^4 \vartheta_k^2) + x_j v \vartheta_k^4 + f_j \vartheta_k^4 + f_k \vartheta_k^4 \).

Similarly evaluating the left side of (50) we have:

\[
J(v \otimes w) = (1_+ \otimes 1) \left[ i^*_+ \circ \vartheta \circ \tau \left( \varphi \otimes \phi(1_- \otimes 1_-) \right) \right] v \otimes w = \sum \varepsilon_i \left( -1 \right)^B f_j (\vartheta_k^4 f_i (\vartheta_k^2) \vartheta_k^4 x_j \otimes \vartheta_k^4 x_i \right) v \otimes w
\]

where \( B = A - v f_1 - x_j \vartheta_k^4 \vartheta_k^2 + \vartheta_k^2 \vartheta_k^4 \vartheta_k^2 = A - v f_1 - x_j \vartheta_k^4 \vartheta_k^2 - \vartheta_k^2 \vartheta_k^4 \vartheta_k^2 = A \). The last equality follows from the fact that \( \sum f_i \otimes x_i \) is even, i.e. \( f_i = x_i \). Thus we have showed that (50) holds, completing the proof.

7.4 The quantization of the double \( \mathfrak{g} = D(\mathfrak{g}_+) \). As in (34) we will now define a Hopf superalgebra structure on \( U(\mathfrak{g})[[\hbar]] \) and show it is a quantization of \( \mathfrak{g} \). After replacing \( J \) with \( \hat{\mathcal{J}} \) equations (30) and (37) define a Hopf superalgebra structure on \( U(\mathfrak{g})[[\hbar]] \). Let \( \hat{\mathfrak{g}} \) be this Hopf superalgebra.

**Theorem 28.** \( \hat{\mathfrak{g}} \) is a quantization of the Lie superbialgebra \( \mathfrak{g} \).

**Proof.** We need the following lemma.

**Lemma 29.** \( \hat{\mathcal{J}} \equiv 1 + \frac{\hbar}{12} \mod \hbar^2 \).
Proof of Lemma 29. Since $\Phi \equiv 1 \mod h^2$, we have
\[
\mathcal{J} \equiv (1_+ \otimes 1)[i_+^*(1 - \frac{t_{23}^h}{2}) \tau_{23}(\phi(1_-) \otimes \phi(1_-))] \mod h^2
\]
\[
\equiv (1_+ \otimes 1) \left[ i_+^* \left( 1 - \frac{(r_{23}+r_{23})^h}{2} \right) \left( \sum_{i,j} (-1)^{\bar{f}_j \bar{x}_i} f_i \otimes f_j \otimes x_i \otimes x_j \right) \right] \mod h^2
\]
\[
\equiv \sum_{i,j} (-1)^{\bar{f}_j \bar{x}_i} f_i(1_+) f_j(1_+) x_i \otimes x_j
\]
\[
- \frac{h}{2} \sum_{j,k} (-1)^{\bar{f}_j \bar{x}_i + \bar{f}_j \bar{m}_k + 1 + \bar{f}_j \bar{p}_k + \bar{p}_k \bar{m}_k} f_i(1_+) f_j(m_k) p_k x_i \otimes x_j \mod h^2
\]
(51)
\[
\equiv 1 - \frac{h}{2} \sum_{j,k} (-1)^{\bar{f}_j \bar{m}_k + 1 + \bar{f}_j \bar{p}_k + \bar{p}_k \bar{m}_k} f_j(m_k) p_k \otimes x_j \mod h^2
\]
(52)
\[
\equiv 1 + \frac{h}{2} \sum_{j,k} p_k \otimes (-1)^{\bar{m}_k} f_j(m_k) x_j \mod h^2
\]
(53)
\[
\equiv 1 + \frac{hr}{\mathcal{F}} \mod h^2
\]
where $r = \sum_k p_k \otimes m_k\in \mathfrak{g}_+ \otimes \mathfrak{g}_-$ is the canonical element of $D(\mathfrak{g}_+)$ defined in 2.4. The first three equivalences follow by definition. Equivalence (51) follows from the fact: $\bar{x}_i = \bar{f}_i$; if $f_i$ is odd, then $f_i(1_+) = 0$ and $\sum f_i(1_+ x_i) = 1$. Equivalence (52) follows from the fact that $r$ is even, (53) hold because of the identity $\sum (-1)^{\bar{m}_k} f_i(m_k) x_i = m_k$ (which follows from $\bar{m}_k \neq \bar{x}_i$ implies $f_i(m_k) = 0$). Thus we have proven the Lemma. From Lemma 29 it follows that the equivalences (52) and (54) hold for all $x \in \mathfrak{g} \subset \mathcal{F}$. The theorem is proved.

7.5. The quantization of the Lie bialgebra $\mathfrak{g}_+$. As in 5.5 we use $\theta$ to identify $\mathcal{F}$ and $\text{End}(\mathcal{F})$. In this subsection we will construct a Hopf subalgebra of $\mathcal{F}$ which will be a quantization of the Lie bialgebra $\mathfrak{g}_+$.

Consider the even superspace homomorphism $i: \mathcal{F}(M_-) \to \mathcal{F}$ given by $x \mapsto i(x)$ where
\[
\hat{i}(x) v = (-1)^{\bar{f}}(i_+^* \otimes 1) \circ (1 \otimes v) \circ x
\]
for any $V \in \mathcal{M}_\mathfrak{g}$ and $x \in \mathcal{F}(M_-)$ and $v \in \mathcal{F}(V)$. Next we will show that the map $i$ is injective. Recall the isomorphism $\hat{\Psi}: \mathcal{F}(M_-) \to U(\mathfrak{g}_+)[[h]]$ given by $f \mapsto (1_+ \otimes 1) f(1_-)$. For $x \in U(\mathfrak{g}_+)[[h]]$ let $f_x := \hat{\Psi}^{-1}(x)$. Now for $x \in U(\mathfrak{g}_+)$ and $v \in \mathcal{F}(V)$ we have $\hat{\Psi}(i(f_x) v) \equiv x \hat{\Psi}(v) \mod h$. Therefore, $i$ is injective. Set $\mathcal{U}_h(\mathfrak{g}_+) = \hat{i}(\mathcal{F}(M_-))$.

Theorem 30. $\mathcal{U}_h(\mathfrak{g}_+)$ is a quantization of the Lie superbialgebra $\mathfrak{g}_+$.

Proof. The following lemma implies that $\mathcal{U}_h(\mathfrak{g}_+)$ is a sub-superbialgebra of $\mathcal{F}$.

Lemma 31. $\mathcal{U}_h(\mathfrak{g}_+)$ is closed under multiplication and coproduct in $\mathcal{F}$.
Proof. For $x, y \in \mathcal{T}(M_-)$ and $v \in \mathcal{T}(V)$ the associativity of $i^*_+ \otimes i^*_-$ and relation (21) imply (to simplify notation set $w = (1 \otimes y)x$)

\[
i(x) \circ i(y)v = (-1)^{\bar{x} \cdot \bar{y} + \bar{y} \cdot \bar{x} + \bar{y}} (i^*_+ \otimes 1) \Phi^{-1} (1 \otimes i^*_+ \otimes 1) \Phi^{-1}_{2,3,4} (1 \otimes 1 \otimes v)(1 \otimes y)x
\]

\[
= (-1)^{\bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{x}} (i^*_+ \otimes 1) (i^*_+ \otimes 1 \otimes 1) \Phi^{-1}_{1,2,3} \Phi^{-1}_{1,2,3} \Phi^{-1}_{2,3,4} (1 \otimes 1 \otimes v)w
\]

\[
= (-1)^{\bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{x}} (i^*_+ \otimes 1) (i^*_+ \otimes 1 \otimes 1) \Phi^{-1}_{1,2,3,4} (1 \otimes 1 \otimes v)w
\]

\[
= (-1)^{\bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{x}} (i^*_+ \otimes 1) \Phi^{-1}_1 (1 \otimes v)(i^*_+ \otimes 1) \Phi^{-1}_1 (1 \otimes y)x
\]

\[
= (-1)^{\bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{x} + \bar{y}} (i^*_+ \otimes 1) \Phi^{-1}_1 (1 \otimes 1 \otimes v)z
\]

\[
i(z) = (1)^{\bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{x} + \bar{y}} \Phi^{-1}_1 (1 \otimes y) \circ x.
\]

Following the proof in [6] Chapter 9 we have

\[
\Delta(i(x)) = (i \otimes i)(\mathcal{T}^{-1}_{M_-, M_-}(1 \otimes i_-) \circ x)
\]

which completes the proof of the Lemma. Next we show that $\mathcal{T}_h(g_+)$ is a Hopf superalgebra. Consider the even superspace isomorphism

\[
\mu : U(g_+)[[h]] \rightarrow \mathcal{T}_h(g_+)
\]

where $f_x := \hat{\Psi}^{-1}(x)$. For $x, y \in U(g_+)[[h]]$ we have

\[
i(f_x) \circ i(f_y) \equiv i \left( (-1)^{\bar{y}} (i^*_+ \otimes 1) \circ (1 \otimes f_y) \circ f_x \right) \mod h^2
\]

\[
\equiv i(f_{xy}) \mod h^2
\]

i.e. $\mu(x) \circ \mu(y) \equiv \mu(xy) \mod h^2$. Similarly, we have $\mu(x) \circ \mu(y) \equiv \Delta(\mu(x)) \mod h$. Therefore, $\mathcal{T}_h(g_+)$ is isomorphic to $U(g_+)$ as a superbialgebra. This implies that $\mathcal{T}_h(g_+)$ has a Hopf superalgebra structure.

To finish the proof we need to show that the equivalence (49) holds. Recall the isomorphism $\theta : U(g)[[h]] \rightarrow \mathcal{T}$ given in (40). Then we have

\[
(\mathcal{T}_h(g_+)) \rightarrow \mathcal{T}_h(g_+)
\]

for all $x \in U(g_+)$. In other words, the image of $U(g_+)$ in $\mathcal{T}_h(g_+)$ and $\mathcal{T}$ is equal modulo $h^2$. From Theorem (28) we have that the equivalences (49) and (44) hold for all $x \in g_+ \subset \mathcal{T}$. Combining the last statement with (49) and the fact that $g_+$ is a Lie superalgebra, we have that the equivalences (49) and (44) hold for all $x \in g_+ \subset \mathcal{T}_h(g_+)$. Thus, $\mathcal{T}_h(g_+)$ is a quantization of $g_+$.

Theorem 32. Let $g_+$ be a finite dimensional Lie superbialgebra. The quantization of $g_+$ constructed in [4] isomorphic to the be the quantization of $g_+$ constructed in this section, i.e. $U_h(g_+) \equiv \mathcal{T}_h(g_+)$.

Proof. Let $g$ be the double of $g_+$ and let $\mathcal{M}_g$ be the category of discrete $g$-modules. Consider the functor $\hat{F} : \mathcal{M}_g \rightarrow \mathcal{A}$ given by

\[
\hat{F}(V) = \text{End}_{\mathcal{M}_g}(M_+ \otimes M_-, V).
\]

By definition $\text{End}(\hat{F}|\mathcal{M}_g)$ is the quantization $H$ of the double $g$, defined in (45). Since $\text{End}(\hat{F})$ and $H$ are both isomorphic to $U(g)[[h]]$, we have that the morphism $\zeta : \text{End}(\hat{F}) \rightarrow H$ given by the restriction of $\mathcal{M}_g$ to $\mathcal{M}_g$, is an isomorphism of Hopf superalgebras.
Let \( \chi : \bar{F} \to \bar{F} \) be the natural transformation of functors given by \( \chi_V(v) = (1 \otimes v) \circ (\sigma \otimes 1) \) where \( \sigma \) is the canonical element in \( \text{Hom}_{\mathcal{M}_k} (k, M_+^+ \otimes M_+) \). Using the properties of the braiding \( \beta \) one can follow the proof of Proposition 9.7 in \cite{22} to show that \( \chi \) is a natural isomorphism of tensor functors. Therefore, \( \chi \) induces an isomorphism between the Hopf superalgebras \( \text{End}(\bar{F}) \) and \( \text{End}(\bar{F}) \). Composing this isomorphism with \( \zeta^{-1} \) we have an isomorphism of Hopf superalgebras \( \kappa : H \to \bar{P} \).

By construction the image of the restriction of \( \kappa \) to the Hopf sub-superalgebra \( U_h(\mathfrak{g}_+^+) \) is \( \bar{U}_h(\mathfrak{g}_+) \). In other words, \( \kappa|_{U_h(\mathfrak{g}_+^+)} : U_h(\mathfrak{g}_+) \to \bar{U}_h(\mathfrak{g}_+) \) is an isomorphism of Hopf superalgebras. Using the isomorphism \( \kappa \) (resp. \( \kappa|_{U_h(\mathfrak{g}_+^+)} \)) given in the proof of theorem \( 32 \) we will identify \( H \) and \( \bar{P} \) (resp. \( U_h(\mathfrak{g}_+) \) and \( \bar{U}_h(\mathfrak{g}_+) \)). From this point on, we will make no distinctions between \( H \) and \( \bar{P} \) or \( U_h(\mathfrak{g}_+) \) and \( \bar{U}_h(\mathfrak{g}_+) \). We call \( U_h(\mathfrak{g}_+) \) the Etingof-Kazhdan quantization of \( \mathfrak{g}_+^+ \).

8. Functoriality of the Quantizations

In this section we show that the quantizations \( \mathcal{Q} \) and \( \mathcal{L} \) are functorial. Then we use this to show that the quantization commutes with taking the double.

Let \( \text{LSBA}(k) \) be the category of finite dimensional Lie superbialgebra over \( k \) and let \( \text{QUES}(K) \) be the category of QUE superalgebra over \( K = k[[h]] \).

**Theorem 33.** There exists a functor from \( \text{LSBA}(k) \) to \( \text{QUES}(K) \) such that \( a \in \text{LSBA}(k) \) is mapped to \( U_h(a) \) which is the quantization defined in \( \mathcal{Q} \).

**Proof.** Once one accounts for the necessary signs, the proof is identical to the classical case (c.f. Theorem 10.1 and 10.2, \cite{22}).

Let \( \text{QLSQBA}(k) \) be the category of quasitriangular Lie superbialgebra over \( k \) and let \( \text{QTSUES}(K) \) be the category of quasitriangular QUE superalgebra over \( K = k[[h]] \).

**Theorem 34.** There exists a functor from \( \text{QLSQBA}(k) \) to \( \text{QTSUES}(K) \) such that \( (\mathfrak{g}, r) \in \text{QLSQBA}(k) \) is mapped to \( (U^\text{qtr}_h(\mathfrak{g}), R) \) which is the quantization defined in \( \mathcal{Q} \).

**Proof.** The proof is a consequence of Theorem 1.2 (ii) of \( \mathcal{Q} \). Theorem 1.2 (ii) states that there is a “universal quantization functor” from the cyclic category of quasitriangular Hopf algebras to the closure cyclic category of quasitriangular Lie bialgebras (see \( \mathcal{Q} \)). By considering linear algebraic structures in the symmetric tensor category of superspaces this “universal quantization functor” gives rises to a functor from \( \text{QLSQBA}(k) \) to \( \text{QTSUES}(K) \) with the desired properties.

Next we use the functoriality to prove the following theorem which first appeared in \( \mathcal{Q} \) for the non-super case.

**Theorem 35.** Let \( \mathfrak{g}_+^+ \) be a finite dimensional quasitriangular Lie superbialgebra. Then the quantization of the quasitriangular Lie superbialgebra \( \mathfrak{g}_+^+ \) constructed in \( \mathcal{Q} \) is isomorphic to the quantization of the Lie superbialgebra \( \mathfrak{g}_+^+ \) of \( \mathcal{Q} \) i.e. \( U^\text{qtr}_h(\mathfrak{g}_+) \cong U_h(\mathfrak{g}_+) \) as Hopf algebras.

**Proof.** To prove the theorem we need the following lemma (which first appeared in \( \mathcal{Q} \) for the non-super case).

**Lemma 36.** Let \( (\mathfrak{g}_+, r) \) be quasitriangular Lie superbialgebra and \( g = D(\mathfrak{g}_+) \) be its double. Then there exist a quasitriangular Lie superbialgebra morphism \( g \to g_+^+ \), which is the identity when restricted to \( g_+^+ \).
Proof. Let \( v : \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_+^* \to \mathfrak{g}_+ \) be the linear map given by
\[
v(x + f) = -x - (1 \otimes f)r
\]
for \( x \in \mathfrak{g}_+ \) and \( f \in \mathfrak{g}_+^* \). We will show that \( v \) is a quasitriangular Lie superbialgebra morphism.

First we show it is a Lie superbialgebra morphism, i.e. \( v([a, b]) = [v(a), v(b)] \) for all \( a, b \in \mathfrak{g} \). This is clear if \( a, b \in \mathfrak{g}_+ \). Recall the definition of the bracket on the double given in (4). Then if \( x \in \mathfrak{g}_+ \) and \( f \in \mathfrak{g}_+^* \) we have
\[
v([x, f]) = v \left( (ad^* x)f - (-1)^{\bar{x}\bar{f}}(1 \otimes f)\delta(x) \right)
\]
\[
= -(1)^{\bar{x}\bar{f}}(1 \otimes f)[1 \otimes x, r] + (-1)^{\bar{x}\bar{f}}(1 \otimes f)[x \otimes 1 + 1 \otimes x, r]
\]
\[
= -(1)^{\bar{x}\bar{f}}(1 \otimes f)[x \otimes 1, r]
\]
\[
= [x, (1 \otimes f)r]
\]
\[
= [v(x), v(f)]
\]
Note that \((ad^* x)f\) is the linear functional \( y \mapsto (-1)^{\bar{x}\bar{f}}f \circ [x, y] \). Similarly, one shows that \( v([f, g]) = [v(f), v(g)] \) for \( f, g \in \mathfrak{g}_+^* \).

Finally, we need to show that \( v \) is a quasitriangular Lie superbialgebra morphism, i.e. preserves the \( r \)-matrix. Let \( \tilde{r} \) be the \( r \)-matrix of \( \mathfrak{g} \). Choose a basis \( x_i \) for \( \mathfrak{g}_+ \) and let \( f_i \) be the dual basis of \( \mathfrak{g}_+^* \), then \( \tilde{r} = \sum x_i \otimes f_i \). Therefore we have
\[
(v \otimes v)\tilde{r} = \sum v(x_i) \otimes v(f_i) = \sum x_i \otimes (1 \otimes f_i)r = r.
\]
Thus \( v \) is the desired morphism.

Now we prove the theorem. Recall that by construction \( U_h(\mathfrak{g}_+) \) is a subalgebra of \( H \). From the Lemma we have \( v : \mathfrak{g} \to \mathfrak{g}_+ \) such that \( v|_{\mathfrak{g}_+} = id_{\mathfrak{g}_+} \). The functoriality of the quantization implies that \( v \) induces a morphism of QTQUE superalgebras \( U_h^{qt}(\mathfrak{g}) \to U_h^{qt}(\mathfrak{g}_+) \). Restricting this morphism to the subalgebra \( U_h(\mathfrak{g}_+) \) we have a morphism \( U_h(\mathfrak{g}_+) \to U_h^{qt}(\mathfrak{g}_+) \), which is a isomorphism since it is modulo \( \hbar \).

We end this section with the following theorem.

Theorem 37. The quantization of a finite dimensional Lie superbialgebra \( \mathfrak{g}_+ \) commutes with taking the double, i.e. \( D(U_h(\mathfrak{g}_+)) \cong U_h(D(\mathfrak{g}_+)) \) (for the definitions of the doubles see [24] and Proposition 4).

Proof. From Corollary [23] we have \( D(U_h(\mathfrak{g}_+)) \cong H \), where \( H \) is the quantization of \( D(\mathfrak{g}_+) \) constructed in [5]. By construction \( U_h^{qt}(D(\mathfrak{g}_+)) = H \), where \( U_h^{qt}(D(\mathfrak{g}_+)) \) is quantization of \( D(\mathfrak{g}_+) \) given by \[1\]. By Theorem [25] we have \( U_h^{qt}(D(\mathfrak{g}_+)) \cong U_h(D(\mathfrak{g}_+)) \). Combining the above isomorphism we have the desired result.

9. THE ETINGOF-KAZHDAN QUANTIZATION OF LIE SUPERALGEBRAS OF TYPE A-G

In this section we will show that, for Lie superalgebras of type A-G, the E-K quantization is isomorphic to the Drinfeld-Jimbo quantization. We follow [8] which proves the result for generalized Kac-Moody algebras. However, we must take the new quantum Serre-type relations into consideration. As in [8] we will show that the E-K quantization is given by the desired generators and relations. In particular, we extend results of Lusztig [12] to the setting Lie superalgebras of type A-G and
check directly that the new quantum Serre-type relations are in the kernel of the appropriate bilinear form.

Here we recall some notation from [331] and [332]. Let \( g \) be a Lie superalgebra of type A-G. Let \( \Phi = \{ \alpha_1, \ldots, \alpha_s \} \) be a simple root system with at most one odd root and let \( (A, \tau) \) be the corresponding Cartan matrix where \( \tau = \{ m \} \) or \( \tau = \emptyset \). Let \( d_1, \ldots, d_s \) be the nonzero numbers such that \( d_id_{ij} = d_ja_{ij} \) and \( d_1 = 1 \). Let \( (, ) \) be the unique non-degenerate supersymmetric invariant bilinear form on \( g \). By rescaling if necessary we may assume that the restriction of \( (, ) \) to \( h \) is determined by \( (a, h_i) = d_i^{-1}\alpha_i(a) \) for all \( a \in h \) and \( i \in I = \{ 1, \ldots, s \} \).

Let \( \tilde{h} \) be the Lie superalgebra generated by \( e_i, f_i \) and \( h_i \) for \( i \in I \) satisfying [12] where all generators are even except for \( e_i \) and \( f_i \) when \( t \in \tau \) which are odd. Let \( \tilde{b}_\pm \) be the Borel sub-superalgebra of \( \tilde{g} \) generated by \( e_i, h_i \) and \( f_i, h_i \), respectively. Let \( q = h/2 \).

**9.1. Generators and relations for \( U_h(\tilde{b}_+) \).**

**Theorem 38.** The quantized universal enveloping superalgebra \( U_h(\tilde{b}_+) \) is isomorphic to the quantized enveloping superalgebra \( U_b \) generated over \( C[[h]] \) by the elements \( e_i, h_i, i \in I \) (where all generators are even except for \( e_i, t \in \tau \) which is odd) satisfying the relations

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j,
\]

with coproduct

\[
\Delta(h_i) = 1 \otimes h_i + h_i \otimes 1, \quad \Delta(e_i) = e_i \otimes q^{d_i h_i} + 1 \otimes e_i,
\]

for all \( i, j \in I \).

The theorem follows from the following two lemmas.

**Lemma 39.** The universal quantized enveloping superalgebra \( U_h(\tilde{b}_+) \) is isomorphic to the quantized enveloping superalgebra generated over \( C[[h]] \) by the elements \( e_i, h_i, i \in I \) (where all generators are even except for \( e_i, t \in \tau \) which is odd) satisfying the relations

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j,
\]

with coproduct

\[
\Delta(h_i) = 1 \otimes h_i + h_i \otimes 1, \quad \Delta(e_i) = e_i \otimes q^{\gamma_i} + 1 \otimes e_i,
\]

for all \( i, j \in I \) and suitable elements \( \gamma_i \in h[[h]] \).

**Proof.** After replacing the ordinary tensor product with the super-tensor product, the proof is identical to the proof of Proposition 3.1 of [3]. There are no new signs introduced. For the most part, this is true because the arguments of the proof are based on the purely even Cartan subalgebra \( h \).

**Lemma 40.** \( \gamma_i = d_i h_i \)

**Proof.** By definition we have the natural projection \( \tilde{b}_+ \to b_+ \). Then the functoriality of the quantization implies that there is an epimorphism of Hopf superalgebras \( U_h(\tilde{b}_+) \to U_h(b_+) \). Therefore \( U_h(b_+) \) is generated by \( h_i, e_i \) satisfying the relations of Lemma 39 (and possibly other relations). So it suffices to show that \( \gamma_i = d_i h_i \) in \( U_h(b_+) \).
Next we show that $U_h(\mathfrak{b}_+) \cong U_{-h}(\mathfrak{b}_+)^{\text{op}}$. From the definition of $\mathfrak{gl}(m|n)$ the Lie superbialgebra $\mathfrak{b}_+$ is self dual, i.e. $\mathfrak{b}_+ \cong \mathfrak{b}_+^*$. Again from functoriality we have that $U_h(\mathfrak{b}_+) \cong U_h(\mathfrak{b}_+^*)$. From Proposition 40 and Theorem 42 imply that $U_h(\mathfrak{b}_+) \cong U_h(\mathfrak{b}_+^*)$. Substituting $\mathfrak{b}_+^*$ for $\mathfrak{b}_+$ we have $U_h(\mathfrak{b}_+^*)^{\text{op}} \cong U_h(\mathfrak{b}_+^*)$. Finally from relation (7) it follows that $U_{-h}(\mathfrak{b}_+^*)^{\text{op}} \cong U_{-h}(\mathfrak{b}_+^*)$. Thus, we have shown that $U_h(\mathfrak{b}_+) \cong U_{-h}(\mathfrak{b}_+^*)^{\text{op}}$.

This isomorphism gives rise to the bilinear form $B : U_h(\mathfrak{b}_+) \otimes U_{-h}(\mathfrak{b}_+) \to \mathbb{C}((h))$ which satisfies the following conditions

$$B(xy, z) = B(x \otimes y, \Delta(z)),$$
$$B(x, yz) = B(\Delta(x), y \otimes z)$$

Let $a \in \mathfrak{h}$ and $i \in I$. Set $B_i = B(e_i, e_i)$, which is nonzero. Using (55) we have

$$B(e_i, q^b e_i) = B(e_i \otimes q^b + 1 \otimes e_i, q^a \otimes e_i) = B(e_i, q^a)B(q^b e_i) + B(1, q^a)B(e_i, e_i) = B_i$$

since $B(e_i, q^a) = 0$. Similarly, we have $B(e_i, q^a e_i q^{-a}) = B(e_i, q^a e_i)B(q^a, q^{-a})$ implying

$$B_i q^{(a, \gamma_i)} = B(e_i, q^a e_i q^{-a}).$$

To complete the proof we need the following relation:

$$q^a e_i q^{-a} = q^{\alpha_i(a)} e_i$$

This relation is equivalent to $q^{b_j} e_i q^{-b_j} = q^{\alpha_i(h_j)} e_i$ which follows from expanding $q = e^h$ and using the relation $[a, e_i] = \alpha_i(a)e_i$. From (55) and (57) we have

$$B_i q^{(a, \gamma_i)} = B(e_i, q^a e_i q^{-a}) = B(e_i, q^{\alpha_i(a)} e_i) = B_i q^{\alpha_i(a)}.$$

Thus, $(a, \gamma_i) = \alpha_i(a)$, but $\alpha_i(a) = d_i(a, h_i)$, and so $\gamma_i = d_i h_i$, which completes the proof.

9.2. The quantized universal enveloping superalgebra $U_h(\mathfrak{b}_+)$. In this subsection we show that there exist a bilinear form on $U_h(\mathfrak{b}_+)$ such that $U_h(\mathfrak{b}_+)$ modulo the kernel of the form is isomorphic to $U_h(\mathfrak{b}_+)$.  

**Theorem 41.** There exists a unique bilinear form on $U_h(\mathfrak{b}_+)$ which takes values in $\mathbb{C}((h))$ with the following properties

$$B(xy, z) = B(x \otimes y, \Delta(z)),$$
$$B(x, yz) = B(\Delta(x), y \otimes z)$$

$$B(q^a, q^b) = q^{-(a, b)} , a, b \in \mathfrak{h}.$$

$$B(e_i, e_j) = \begin{cases} (q_i - q_i^{-1})^{-1} & \text{if } i = j \neq m, \\ 1 & \text{if } i = j = m, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover $U_h(\mathfrak{b}_+) \cong \mathcal{U}_+ := \mathcal{U}_+/\text{Ker}(B)$ as QUE superalgebras.
Proof. The existence and uniqueness follows from the fact that the superalgebra generated by the $e_i$ is free.

We will show that there is a nondegenerate bilinear form on $U_h(b_+)$ with the same properties as $B$. From the proof of Lemma 40 we have that $U_h(b_+) \cong U_{-h}(b_+)^{op}$. But the even homomorphism $U_{-h}(b_+)^{op} \rightarrow U_h(b_+)$ given by conjugation by $q^{-\sum x_i^2/2}$, where $x_i$ is an orthonormal basis for $\mathfrak{h}$, is a isomorphism. Therefore we have a even isomorphism $U_h(b_+) \cong U_h(b_+)^*$. This isomorphism gives rise to the desired form on $U_h(b_+)$. So the form $B$ is the pull back of the form on $U_h(b_+)$. Implieding that the kernel of the form on $U_h(b_+)$ is contained in the image of the kernel of $B$ under natural projection.

But the kernel of the form on $U_h(b_+)$ is zero since the form is nondegenerate. Thus we have $U_h(b_+) \cong U_+/Ker(B)$.

9.3. The kernel of $B$. In this subsection we show that $Ker(B)$ is generated by the quantum Serre-type relations $[16, 20]$. We first show that the quantum Serre-type relations are contained in $Ker(B)$. To this end, we extend results of Lusztig [15]. The outline of this subsection is as follows. We start with the initial data: a free associative superalgebra $\mathcal{I}$ with unit and a Cartan matrix. Using the Cartan matrix we define a twisted multiplication on $\mathcal{I} \otimes \mathcal{I}$ (see (59)). Then we prove that there is a unique form $C$ on $\mathcal{I}$ whose kernel contains the quantum Serre-type relations. We end the subsection by showing that this implies that these relations are in $Ker(B)$. Intuitively, this construction is imposing the information of the Cartan matrix onto the twisted multiplication which in turn is imposing the relations on the kernel of $C$.

Let $q$ be an indeterminate. Recall the definitions of Cartan data $(\Phi, (A, \tau), \ldots)$ given at the beginning of this section. Let $\mathcal{I}$ be the free associative $\mathbb{C}(q)$-superalgebra with 1 generated by $\theta_i$, for $i \in I$, where the parity is 0 for all generators except for $\theta_i$, $i \in \tau$ which has parity 1.

For any $\nu = \sum \nu_i i \in \mathbb{N}[I]$, let $\mathcal{I}_\nu$ be the $\mathbb{C}(q)$-subspace of $\mathcal{I}$ spanned by the monomials $\theta_{i_1} \theta_{i_2} \ldots \theta_{i_k}$ so that for each $i \in I$, the number of times $i$ appears in the sequence $i_1, i_2, \ldots, i_k$ is equal to $\nu_i$. Notice that $\mathcal{I} = \oplus \nu \mathcal{I}_\nu$. We say $x \in \mathcal{I}$ is homogeneous if $x \in \mathcal{I}_\nu$, for such an $x$, set $|x| = \nu$. For homogeneous $x, x' \in \mathcal{I}$, let

\[
|<x, x'| >= <\sum_i d_i \nu_i i, \sum_j v'_j j> = \sum_{i,j} d_i \nu_i v'_j (h_i),
\]

where $|x| = \sum \nu_i i$ and $|x'| = \sum v'_j j$. Note that $<x, x'| >= (\sum \nu_i i, \sum v'_j j)$ where $(,)$ is the super-symmetric bilinear form on $\mathfrak{g}$ (see (32)).

We make $\mathcal{I} \otimes \mathcal{I}$ into an a superalgebra with the following multiplication:

\[
(x_1 \otimes x_2)(y_1 \otimes y_2) = (-1)^{\nu_1 \nu_2} q^{-\sum x_i |y_i|} x_1 y_1 \otimes x_2 y_2.
\]

where $x_1, x_2, y_1, y_2 \in \mathcal{I}$ are homogeneous.

Let $r : \mathcal{I} \rightarrow \mathcal{I} \otimes \mathcal{I}$ be the superalgebra map defined by $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$.

Proposition 42. There is a unique bilinear form $C$ on $\mathcal{I}$ with values in $\mathbb{C}(q)$ such that $C(1, 1) = 1$ and

\[
C(\theta_i, \theta_j) = \begin{cases} 
(q_i - q_i^{-1})^{-1} & \text{if } i = j \neq m, \\
1 & \text{if } i = j = m, \\
0 & \text{otherwise}
\end{cases}
\]
(2) $C(x, yz) = C(r(x), y \otimes z)$ for all $x, y, z \in 'f$.

(3) $C(xy, z) = C(x \otimes y, r(z))$ for all $x, y, z \in 'f$.

where the bilinear form on $'f \otimes 'f$ (also denoted by $C$) is given by

$$C(x_1 \otimes x_2, y_1 \otimes y_2) = (-1)^{s_1 s_2} C(x_1, y_1) C(x_2, y_2).$$

Proof. The proof is similar to the proof of Proposition 1.2.3 in [15]. Here we define $C$ and refer the reader to [15] for the rest of the proof. First, we define a superalgebra structure on $'f$.

For any $v, v' \in \mathbb{N}$, composing the map $r|_{f^v f^{v'}} : f^v f^{v'} \to f \otimes f$ with the projection $f \otimes f \to f^v f^{v'}$, we have the linear map $f^v f^{v'} \to f \otimes f$. Taking the dual, we obtain linear maps $f^v f^{v'} \to f^v f^{v'}$. This defines an associative superalgebra structure on $'f$. For each $i \in I$, let $\xi_i f_i'$ be given by

$$\xi_i(\theta_i) = \begin{cases} 
(q_i - q_i^{-1})^{-1} & \text{if } i \neq m, \\
1 & \text{if } i = m.
\end{cases}$$

Let $\phi : f \to \bigoplus_i f_i'$ be the unique superalgebra homomorphism preserving 1, such that $\phi(\theta_i) = \xi_i$ for all $i$. For homogeneous $x, y \in f$, set $C(x, y) = (-1)^{s_1 s_2} \phi(y)(x)$. Now (2) follows as $\phi$ is an algebra homomorphism. From the definition of $\phi$ we have

$$C(x, y) = 0 \text{ unless } |x| = |y|$$

for homogeneous $x, y \in f$.

After putting in appropriate signs coming from (59) and (60), the proof of (3) follows as in [15].

Let $\text{Ker}(C)$ be the kernel of the form $C$, then $\text{Ker}(C)$ is a homogeneous ideal of $f$. Let $f = f/\text{Ker}(C)$. From (61), the decomposition $f = \bigoplus_i f_i'$ gives a direct sum decomposition of $f = \bigoplus_i f_i'$, where $f_i'$ is the image of $f_i'$ under the projection $f \to f$.

**Proposition 43.** The relations (10)-(20) with $e$ replaced by $\theta$ hold in the superalgebra $f$. In particular,

$$\theta^2_m = 0 \text{ if } \tau = \{m\},$$

$$\theta_m \theta_{-1} \theta_m \theta_{m-1} + \theta_m \theta_{m+1} \theta_m \theta_{m-1} + \theta_{m-1} \theta_m \theta_{m+1} \theta_m + \theta_{m+1} \theta_m \theta_{m-1} \theta_m = (q + q^{-1}) \theta_m \theta_{m-1} \theta_m = 0 \text{ if } m - 1, m, m + 1 \in I \text{ and } a_{mm} = 0,$$

$$\theta_{m-1} \theta^3_m - (q + q^{-1}) \theta_m \theta_{m-1} \theta_m = 0 \text{ if } m - 1, m, m + 1 \in I \text{ and } a_{mm} = 0,$$

$$a_{ij} = (1 + (-1)^{d_{i,m}}) \delta_{i,j} - (-1)^{d_{i,m}} \delta_{i,j-1} - \delta_{i,j+1}$$

for $i, j \in \{m - 1, m, m + 1\}$. We also have $d_{m-1} = d_m = 1$ and $d_{m+1} = -1$.
Let \( l \) be the left side of relation (63). To show that relation (66) holds it is enough to show \( C(x, l) = 0 \) for all \( x \in \mathfrak{f}(m-1)+2(m)+1(m-1) \). By relation (62) the vector space \( \mathfrak{f}(m-1)+2(m)+1(m-1) \) is generated by
\[
\theta_m \theta_{m-1} \theta_m \theta_{m+1}, \theta_m \theta_m \theta_{m-1} \theta_{m+1},
\theta_{m-1} \theta_m \theta_{m+1} \theta_m, \theta_{m+1} \theta_m \theta_{m-1} \theta_m,
\theta_m \theta_{m-1} \theta_m \theta_{m+1}.
\]
Therefore it suffices to check that \( C(x, l) = 0 \), when \( x \) is any of the above generators. We will check this condition for \( \theta_{m+1} \theta_m \theta_{m-1} \theta_m \), the others follow similarly.

Let \( c_i = (q_i - q_i^{-1})^{-1} \). From (59) we have
\[(66) \quad (1 \otimes \theta_j)(\theta_j \otimes 1) = (-1)^{\theta_j} q^{d_{a,i}} (\theta_j \otimes \theta_i). \]

We use (60), (65), (66) and Proposition 42, part (3) to make the following calculations:
\[
a_1 := C(\theta_m \theta_{m+1} \theta_m \theta_{m-1} \theta_m, \theta_m \theta_m \theta_{m-1} \theta_{m+1}) = 0
\]
\[
a_2 := C(\theta_m \theta_{m+1} \theta_m \theta_{m-1} \theta_m, \theta_{m-1} \theta_m \theta_{m+1} \theta_m) = 0
\]
\[
a_3 := C(\theta_{m+1} \theta_m \theta_{m-1} \theta_m, \theta_{m+1} \theta_m \theta_{m-1} \theta_m) = 0
\]
\[
a_4 := C(\theta_{m+1} \theta_m \theta_{m-1} \theta_m, \theta_{m+1} \theta_m \theta_m \theta_{m-1} \theta_m) = -q^2 c_{m+1} c_m - q^{-2} c_{m+1} c_m
\]
\[
a_5 := C(\theta_{m+1} \theta_m \theta_m \theta_{m-1} \theta_m, \theta_{m+1} \theta_m \theta_{m-1} \theta_{m+1} \theta_m) = -q^2 c_{m+1} c_m - q^{-2} c_{m+1} c_m
\]
So
\[
C(\theta_{m+1} \theta_m \theta_m \theta_{m-1} \theta_m, l) = a_1 + a_2 + a_3 + a_4 + (q + q^{-1}) a_5 = 0.
\]

It is not hard to follow the above computation and show that (63) holds and so the Proposition follows.

One can continue to follow (164) and show that the Drinfeld-Jimbo type \( \mathbb{C}(q) \)-superalgebra (see [10, 14]) can be recovered from \( f \). This result is not essential for our purposes here. However, in order to shed some light on the larger picture we will now state the results without proof.

Let \( \mathcal{U} \) be the \( \mathbb{C}(q) \)-superalgebra generated by \( q^{h_i}, e_i \) and \( f_i \) for \( i \in I \) modulo the relations (134) and
\[(67) \quad q^{h_i} q^{h_j} = q^{h_j} q^{h_i}, \quad q^{h_i} e_j = q^{a_{i,j}} e_j q^{h_i}, \quad q^{h_i} f_j = q^{-a_{i,j}} f_j q^{h_i}. \]

Let \( \mathcal{U} \) be the associative \( \mathbb{C}(q) \)-superalgebra \( \mathcal{U} \) modulo the following relations: for any relation \( q(\theta_i) \in \text{Ker} (C) \) we have \( q(e_i) = 0 \) and \( q(f_i) = 0 \) in \( \mathcal{U} \). Let \( \mathcal{U}^0 \) (resp. \( \mathcal{U}_0^0 \)) be the sub-superalgebra of \( \mathcal{U} \) generated by \( q^{h_i} (i \in I) \) (resp. \( q^{h_i}, e_i (i \in I) \)).
f \to U (x \mapsto x^+) and f \to U (x \mapsto x^-) be the homomorphism such that \( e_i = \theta_i^+ \) and \( f_i = \theta_i^- \) for all \( i \in I \). As in \cite{13}, one can show that

\[
\begin{align*}
&f \otimes U^0 \otimes f \to U \text{ given by } u \otimes q^a \otimes w \mapsto u^- q w^+ \\
&U^0 \otimes 'f \to 'U_0^+ \text{ given by } q^a \otimes x \mapsto q^a x^+
\end{align*}
\]

are isomorphisms of vector spaces. From the first isomorphism and Proposition \cite{13} it follows that the superalgebra \( U \) is isomorphic to the D-J type \( \mathbb{C}(q) \)-superalgebra (which is the superalgebra \( 'U \) modulo \( \langle 14, 20 \rangle \)).

Now we are ready to prove the following theorem.

**Theorem 44.** The quantum Serre-type relations \( \langle 14, 20 \rangle \) are contained in \( \text{Ker}(B) \).

**Proof.** Recall the superalgebra \( \tilde{U}_+ \) of Theorem \cite{13}. By setting \( q \) to \( e^{h/2} \) one can obtain an injective superalgebra morphism \( 'U^0_0 \to \tilde{U}_+ \). Then the composition

\[
'f \mapsto U^0 \otimes 'f \to 'U_0^+ \to \tilde{U}_+
\]

is injective. The form \( C : 'f \otimes 'f \to 'f \) of Proposition \cite{12} corresponds (under the above composition) to the form \( B \). Therefore, Proposition \cite{13} implies that the quantum Serre-type relations are contained in \( \text{Ker}(B) \).

**Corollary 45.** \( \text{Ker}(B) \) is generated by the quantum Serre-type relations \( \langle 14, 20 \rangle \).

**Proof.** Let \( \mathcal{U}_+ = \tilde{U}_+ / \text{Ker}(B) \). By construction the superalgebra \( U_+(b_+) \) is isomorphic as a vector space to \( U(b_+)[[h]] \), implying \( \mathcal{U}_+ \cong U(b_+)[[h]] \). Combining this observation with Theorem \cite{13} and the fact that \( b_+ \) is the quotient of \( b_* \) by the classical super Serre-type relations \( \langle 13 \rangle \) we have that the \( \text{Ker}(B) \) is generated by the quantum Serre-type relations.

9.4. Generators and relations for \( U_h(g) \).

**Theorem 46.** Let \( g \) be a Lie superalgebra of type \( A-G \). The QUE superalgebra \( U_h(g) \) is isomorphic to the quotient of the double \( D(\mathcal{U}_+) \) by the ideal generated by the identification of \( h \subset \mathcal{U}_+ \) and \( h^* \subset \mathcal{U}_+^* \), i.e. the Etingof-Kazhdan quantization \( U_h(g) \) is isomorphic to the Drinfeld-Jimbo type superalgebra \( U^{DJ}_h(g) \) (see \cite{13}).

**Proof.** Recall from \cite{12} that the Lie superbialgebra structure of \( g \) comes from identifying \( h \) and \( h^* \subset g \oplus h = b_+ \oplus b^*_+ \). Also since the quantization commutes with the double we have

\[
U_h(D(b_+)) \cong D(U_h(b_+)) = U_h(b_+) \otimes U_h(b_+)^{\text{op}}.
\]

Therefore, we have \( U_h(g) \) is isomorphic to \( D(U_h(b_+)) = U_h(b_+) \otimes U_h(b_+)^{\text{op}} \) modulo the ideal generated by the identification of \( h \subset U_h(b_+) \) and \( h^* \subset U_h(b_+)^{\text{op}} \). But from Theorem \cite{11} we have that \( D(U_h(b_+)) \cong D(\mathcal{U}_+) \) and then Corollary \cite{15} implies result.

10. A theorem of Drinfeld’s

Recall the definition of \( A_{q,t} \) and \( U^{DJ}_h(g) \) given in \cite{13} and \cite{13} respectively. Here we use all the results of this paper to show that the categories of topologically free modules over \( A_{q,t} \) and \( U^{DJ}_h(g) \) are braided tensor equivalent. We do this in two steps: (1) we show that \( U_h(g) \) and \( A_{q,t} \) have equivalent module categories, (2) we use the fact the that \( U^{DJ}_h(g) \) and \( U_h(g) \) are isomorphic to prove the desired result. For more on braided tensor categories see \cite{9, 113}.
10.1. **The E-K quantization** $U_h(g)$ and $A_{g,t}$. In this subsection we show that $U_h(g)$ is the twist of $A_{g,t}$ by $J$. To this end we recall the following definitions.

Let $(A, \Delta, \epsilon, \Phi, R)$ be a quasitriangular quasi-superbialgebra (see §4.1.) An invertible element $J \in A \otimes A$ is a gauge transformation on $A$ if

$$\epsilon \otimes \text{id}(J) = (\text{id} \otimes \epsilon)(J) = 1.$$ 

Using a gauge transformation $J$ on $A$, one can construct a new quasitriangular quasi-superbialgebra $A_J$ with coproduct $\Delta_J$, R-matrix $R_J$ and associator $\Phi_J$ defined by

$$\Delta_J = J^{-1} \Delta J, \quad R_J = (J^\text{op})^{-1}RJ, \quad \Phi_J = J_{23}^{-1}(\text{id} \otimes \Delta)(J^{-1})\Phi(\Delta \otimes \text{id})(J)J_{12}.$$ 

As is the case of quasitriangular (quasi-)bialgebra, the category of modules over a quasitriangular (quasi-)superbialgebra is a braided tensor category.

**Theorem 47.** Let $A$ and $A'$ be a quasitriangular quasi-superbialgebra. Suppose that $J$ is a gauge transformation on $A'$ and $\alpha : A \to A'_J$ is an isomorphism of quasitriangular quasi-superbialgebra then $\alpha$ induces a equivalence between the braided tensor categories $A'-\text{Mod}$ and $A-\text{Mod}$.

**Proof.** Let $\alpha^* : A'-\text{Mod} \to A-\text{Mod}$ be the functor defined as follows. On objects, the functor $\alpha^*$ is defined by sending the module $W$ to the same underlying vector space with the action given via the isomorphism $\alpha$. For any morphism $f : W \to X$ in $A'-\text{Mod}$ let $\alpha^*(f)$ be the image of $f$ under the isomorphism

$$\text{Hom}_{A'}(W, X) \cong \text{Hom}_A(W, X).$$

A standard categorical argument shows that this functor is an equivalence of braided tensor categories (see §XV.3 of [13]).

Let $g$ be a Lie superalgebra of type A-G. Recall from §3.2 that $g$ has a unique non-degenerate supersymmetric invariant bilinear form. Let $t$ be the corresponding even invariant super-symmetric element of $g \otimes g$. Let $J$ be the element of $U(g)[[h]]^\otimes$ defined in (32). By definition of the coproduct and R-matrix of $U_h(g)$ (see §5.2) we have that $U_h(g) = (A_{g,t})_J$.

10.2. **Main theorem.** Let $X$ be a topological (quasi) Hopf superalgebra and let $X-\text{Mod}_{fr}$ of topologically free $X$-modules of finite rank (see §2.2). The following theorem was first due to Drinfeld [5] in the case of semi-simple Lie algebras.

**Theorem 48.** The braided tensor categories $A_{g,t}-\text{Mod}_{fr}$ and $U_h^{DJ}(g)-\text{Mod}_{fr}$ are equivalent.

**Proof.** As mentioned at the end of the last subsection $U_h(g) = (A_{g,t})_J$. Combining this fact with Theorem 46 we have that there exists an isomorphism of quasitriangular quasi-superbialgebra

$$\alpha : U_h^{DJ}(g) \to (A_{g,t})_J.$$ 

Now as a consequence of Theorem 47 we have that the categories $A_{g,t}-\text{Mod}_{fr}$ and $U_h^{DJ}(g)-\text{Mod}_{fr}$ are braided tensor equivalent.

**Remark 49.** Drinfeld’s proof of Theorem 48 in the case of semi-simple Lie algebras uses deformation theoretic arguments to show the existence of $\alpha$. Our proof constructs the isomorphism $\alpha$ explicitly.
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