Topological equivalence canonical forms for linear multivariable systems without control∗

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Abstract

In this paper, we discuss the classification problem for linear time-invariant multivariable systems without control. It turns out that the observability and stability are invariant for topological equivalent systems. Abstract results concerning system decomposition according to eigenvalues and observability are obtained. Finally, as concrete examples, the topological equivalence canonical forms for a three dimensional system equipped with a scalar observation are presented explicitly.

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1 Introduction and main results

This paper studies the classification problem for the following linear system governed by autonomous linear ordinary differential equations (ODEs) with observation

\[
\begin{align*}
\dot{x}(t) &= Ax(t), \quad t \geq 0, \\
w(t) &= Cx(t), \quad t \geq 0,
\end{align*}
\]

(1.1)

where \( \dot{x} = \frac{dx}{dt} \), \( x(t) \in \mathbb{R}^n \) (\( n \in \mathbb{N} \)) is the state variable, \( w(t) \in \mathbb{R}^p \) (\( p \in \mathbb{N} \)) is the observation variable, \( A \in \mathbb{R}^{n \times n} \) and \( C \in \mathbb{R}^{p \times n} \). Since (1.1) is uniquely determined by the pair of matrices \( A \) and \( C \), we denote it briefly by \( (A, C) \).

The motivation of investigating system (1.1) lies in the fact that (1.1) is a special case of the following general linear time-invariant multivariable system with both control and observation:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0, \\
w(t) &= Cx(t), \quad t \geq 0.
\end{align*}
\]

(1.2)

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Here \( u(t) \in \mathbb{R}^m \) (\( m \in \mathbb{N} \)) is the control variable and \( B \in \mathbb{R}^{n \times m} \). When \( B = 0 \), system (1.2) degenerates to system (1.1). We will discuss the classification problem for system (1.2) in a separate paper.

The linear and topological equivalence for linear ODEs was investigated in 1973 (see [7] [8]). The feedback equivalence for completely controllable control systems was studied by P. Brunovsky [2]. The topological equivalence for linear time-invariant control systems was discussed by J. C. Willems [16] (see also [4]). There exist some further works on the classification of linear multivariable systems by linear equivalence transformation (cf. [1] [10] [13]).

Consider the following two linear time-invariant multivariable systems with only observation

\[
\begin{align*}
  \dot{x}(t) &= A_1 x(t), & t \geq 0, \\
  w(t) &= C_1 x(t), & t \geq 0
\end{align*}
\]

and

\[
\begin{align*}
  \dot{y}(t) &= A_2 y(t), & t \geq 0, \\
  z(t) &= C_2 y(t), & t \geq 0.
\end{align*}
\]

Here, \( x(t), y(t) \in \mathbb{R}^n \) are the state variables, \( w(t), z(t) \in \mathbb{R}^p \) are the observation variables, \( A_i \in \mathbb{R}^{n \times n} \) and \( C_i \in \mathbb{R}^{p \times n} \) (\( i = 1, 2 \)). We introduce the following definition of linear and topological equivalence.

**Definition 1.1**

1) Systems (1.3) and (1.4) are called topologically equivalent if there exists a (vector-valued) function \( H(x) \in C(\mathbb{R}^n; \mathbb{R}^n) \) such that

i) \( H(x) \) is a homeomorphism on \( \mathbb{R}^n \) (henceforth we denote the inverse function of \( H(x) \) by \( H^{-1}(y) \));

ii) Transformation \((y(t), z(t)) = (H(x(t)), w(t))\) brings (1.3) to (1.4), and transformation \((x(t), w(t)) = (H^{-1}(y(t)), z(t))\) brings (1.4) to (1.3).

2) Systems (1.3) and (1.4) are called linearly equivalent if the above function \( H(x) \) is a linear isomorphism on \( \mathbb{R}^n \).

Several remarks are in order.

**Remark 1.1** In other words, Definition 1.1 says that systems (1.3) and (1.4) are topologically equivalent if and only if

i) ODEs \( \dot{x}(t) = A_1 x(t) \) and \( \dot{y}(t) = A_2 y(t) \) are topologically equivalent;

ii) \( w(t) \equiv z(t) \) for \( t \geq 0 \).

We refer the reader to Definition 2.1 in Section 2 for the definition of topological equivalence for pure ODEs.

**Remark 1.2** If vector-valued function \( H(x) \) is a homeomorphism on \( \mathbb{R}^n \), one can check that vector-valued function \( F(x, w) := (H(x), w) \) is a homeomorphism on \( \mathbb{R}^n \times \mathbb{R}^p \). We denote the inverse function of \( F(x, w) \) by \( F^{-1}(y, z) := (H^{-1}(y), z) \). “Transformation \((y(t), z(t)) = (H(x(t)), w(t))\) brings (1.3) to (1.4)” means that: if \( x(t) \) is the solution of (1.3) with initial datum \( x(0) \), and \( w(t) \) is the observation of (1.3), then by transformation \((y(t), z(t)) = F(x(t), w(t)) = (H(x(t)), w(t)) \), \( y(t) = H(x(t)) \) is the solution of (1.4) with initial datum \( y(0) = H(x(0)) \), and \( z(t) = w(t) \) is the observation of (1.4).
Remark 1.3 We call \((H(x), w)\) the equivalence transformation from (1.3) to (1.4). In addition, it is clear that

\[ \text{Linear equivalence } \Rightarrow \text{Topological equivalence.} \]

For any topological equivalence transformation \((H(x), w)\) from (1.3) to (1.4), we always assume \(H(0) = 0\) in the following discussion, for otherwise we can replace \((H(x), w)\) by \((H(x) - H(0), w)\).

Our first result concerns the linear equivalence for ODE systems with observation.

Proposition 1.1 Systems (1.3) and (1.4) are linearly equivalent if and only if there exists a nonsingular matrix \(P \in \mathbb{R}^{n \times n}\) such that

\[ A_2 = P^{-1}A_1P, \quad C_2 = C_1P. \]  \hspace{1cm} (1.5)

The proof of Proposition 1.1 is given in Section 3. From Proposition 1.1 and Remark 1.3, one can transform the original system (1.1) into the following form

\[ \begin{bmatrix} \begin{bmatrix} A^0 & 0 & 0 \\ 0 & A^+ & 0 \\ 0 & 0 & A^- \end{bmatrix} & \begin{bmatrix} C^0 & C^+ & C^- \end{bmatrix} \end{bmatrix} \] \hspace{1cm} (1.6)

by a suitable nonsingular matrix \(P\). Here the real parts of the eigenvalues of \(A^0\), \(A^+\) and \(A^-\) are zero, positive and negative respectively.

Definition 1.2 Assume that there exists a nonsingular matrix \(L\), such that system (1.1) is linearly equivalent to system (1.6).

1) Let \(n^0(A)\) (resp., \(n^+(A), n^-(A)\)) denote the dimension of matrix \(A^0\) (resp., \(A^+, A^-\)). Obviously, \(n = n^0(A) + n^+(A) + n^-(A)\).

2) \(k_{\text{obs}}(A, C) := \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \)

is called the Kalman rank of observability for system \((A, C)\). Besides \(k_{\text{obs}}^0(A, C) := k_{\text{obs}}(A^0, C^0)\) (resp., \(k_{\text{obs}}^+(A, C) := k_{\text{obs}}(A^+, C^+)\), \(k_{\text{obs}}(A, C) := k_{\text{obs}}(A^-, C^-)\)) is called the Kalman rank of observability for subsystem \((A^0, C^0)\) (resp., subsystem \((A^+, C^+)\), subsystem \((A^-, C^-)\)).

Remark 1.4 In Section 3, we will prove that \(k_{\text{obs}}^0(A, C)\), \(k_{\text{obs}}^+(A, C)\) and \(k_{\text{obs}}^-(A, C)\) given in Definition 1.2 are all well-defined. That is, the set of indices

\[ \{k_{\text{obs}}(A, C), k_{\text{obs}}^0(A, C), k_{\text{obs}}^+(A, C), k_{\text{obs}}^-(A, C)\} \]

is uniquely determined by the given system \((A, C)\). In addition, it holds that

\[ k_{\text{obs}}(A, C) = k_{\text{obs}}^0(A, C) + k_{\text{obs}}^+(A, C) + k_{\text{obs}}^-(A, C). \]

For the proof of above formula, we refer the reader to Lemma 3.3 in Section 3.
Corollary 1.1 For systems (1.3) and (1.4) only having the eigenvalues with zero real parts (i.e. $n^0(A_1) = n^0(A_2) = n$), the following relation holds:

Linear equivalence $\iff$ Topological equivalence.

Corollary 1.1 shows that, the linear and topological equivalence coincide for ODE systems only having the eigenvalues with zero real parts. The following theorem says that, for completely observable systems, two kinds of equivalence coincide too.

Theorem 1.1 For completely observable systems (1.3) and (1.4) (i.e. $k_{\text{obs}}(A_1, C_1) = k_{\text{obs}}(A_2, C_2) = n$), the following relation holds:

Linear equivalence $\iff$ Topological equivalence.

Example 1.1 Let the real number $a > 0$ and $a \neq 3$. Then systems

$$
\left( \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right), \left( \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \text{ and } \left( \begin{bmatrix} 3 & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right)
$$

are topologically equivalent to each other (see Lemma 4.3). However, by Proposition 1.1 any pair of them are not linearly equivalent.

Our main result is the following theorem.

Theorem 1.2 System (1.1) can be topologically equivalent to the following system:

$$
\left( \begin{bmatrix} \hat{N} & 0 & 0 \\ 0 & \hat{B} & 0 \\ 0 & 0 & \hat{E} \end{bmatrix}, \begin{bmatrix} \hat{K} & \hat{D} & 0 \end{bmatrix} \right).
$$

(1.7)

Here

$$
\hat{E} = \begin{bmatrix} 1 & & \cdot & \cdot & \cdot \\ & 1 & -1 & & \\ & & 1 & -1 & \cdot \\ & & & \cdot & \cdot \cdot \\ & & & & 1 \end{bmatrix}
$$

$$
\hat{N} \in \mathbb{R}^{n^0(A) \times n^0(A)}, \hat{K} \in \mathbb{R}^{p \times n^0(A)}, \hat{B} \in \mathbb{R}^{[k_{\text{obs}}(A,C) + k_{\text{obs}}^-(A,C)] \times [k_{\text{obs}}^+(A,C) + k_{\text{obs}}(A,C)]}, \hat{D} \in \mathbb{R}^{p \times [k_{\text{obs}}^+(A,C) + k_{\text{obs}}^-(A,C)]}. (\hat{N}, \hat{K}) \text{ stands for a linear equivalence canonical form for systems with the real parts of the eigenvalues being zero. (\hat{B}, \hat{D}) stands for a linear equivalence canonical form for completely observable systems with the real parts of the eigenvalues being nonzero.}
$$

In Theorem 1.2 if $k_{\text{obs}}^0(A, C) = n^0(A)$, then $\left( \begin{bmatrix} \hat{N} & 0 \\ 0 & \hat{B} \end{bmatrix} \right) \in \mathbb{R}^{k_{\text{obs}}(A,C) \times k_{\text{obs}}(A,C)}$ and $\left[ \hat{K} \hat{D} \right] \in \mathbb{R}^{p \times k_{\text{obs}}(A,C)}$ in (1.7). Besides, the subsystem $\left( \begin{bmatrix} \hat{N} & 0 \\ 0 & \hat{B} \end{bmatrix}, \begin{bmatrix} \hat{K} & \hat{D} \end{bmatrix} \right)$ is completely observable.

We can merge these two subsystems into a single one, and denote the new subsystem by $(\hat{L}, \hat{T})$. $(\hat{L}, \hat{T})$ may stand for any classical canonical form (for instance, Luenberger canonical forms, Wonham canonical forms or Brunovsky canonical forms) for completely observable systems.
Corollary 1.2 When $k^0_{\text{obs}}(A,C) = n^0(A)$, system (1.1) has the following canonical forms:

$$
\left( \begin{bmatrix} \hat{L} & 0 \\ 0 & \hat{T} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).
$$

Here $(\hat{L}, \hat{T})$ stands for a linear equivalence canonical form for completely observable systems.

Proposition 1.2 The set of indices

$$
\{n^0(A), n^+(A), n^-(A), k^0_{\text{obs}}(A,C), k^0_{\text{obs}}(A,C), k^+_{\text{obs}}(A,C), k^-_{\text{obs}}(A,C)\}
$$

defined in Definition 1.2 is invariant under any topological equivalence transformation introduced in Definition 1.1.

Proposition 1.2 implies that the topological equivalence keeps the observability and stability of system (1.1).

The rest of this paper is organized as follows. Several preliminary propositions concerning the equivalence for autonomous linear ODEs are presented in Section 2. Section 3 is devoted to the linear equivalence for system (1.1) and giving the proof of Proposition 1.1. In Section 4, the topological equivalence for system (1.1) is discussed and Theorem 1.1 is proved. In Section 5, Proposition 1.2 is proved and concrete canonical forms are given for three dimensional ODE systems with a scalar observation (i.e., a single output).

2 Topological equivalence for ODEs

2.1 Definition and invariants

In this subsection, we present some existing results in the literature concerning the classification problem of autonomous linear ODEs. Let us recall the following definition of linear and topological equivalence for autonomous linear ODEs (c.f. [7, 8, 14]).

Definition 2.1 Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ and $t \in \mathbb{R}$.

1) ODEs $\dot{x}(t) = A_1x(t)$ and $\dot{y}(t) = A_2y(t)$ are called topologically equivalent if there exists a (vector-valued) function $H(x) \in C(\mathbb{R}^n; \mathbb{R}^n)$ such that

i) $H(x)$ is a homeomorphism on $\mathbb{R}^n$ (henceforth we denote the inverse function of $H(x)$ by $H^{-1}(y)$);

ii) Transformation $y(t) = H(x(t))$ brings $\dot{x}(t) = A_1x(t)$ to $\dot{y}(t) = A_2y(t)$, that is, if $x(t)$ is the solution of $\dot{x}(t) = A_1x(t)$ with condition $x(0) = x_0$, then by transformation $y(t) = H(x(t))$, $y(t) = H(x(t))$ is the solution of $\dot{y}(t) = A_2y(t)$ satisfying condition $y(0) = H(x_0)$. Transformation $x(t) = H^{-1}(y(t))$ brings $\dot{y}(t) = A_2y(t)$ to $\dot{x}(t) = A_1x(t)$.

2) ODEs $\dot{x}(t) = A_1x(t)$ and $\dot{y}(t) = A_2y(t)$ are called linearly equivalent if the above function $H(x)$ is a linear isomorphism on $\mathbb{R}^n$.

Remark 2.1 Let $H(x) \in C(\mathbb{R}^n; \mathbb{R}^n)$ be any given homeomorphism which brings $\dot{x}(t) = A_1x(t)$ ($t \geq 0$) to $\dot{y}(t) = A_2y(t)$ ($t \geq 0$), and its inverse $H^{-1}(x)$ brings $\dot{y}(t) = A_2y(t)$ ($t \geq 0$) to $\dot{x}(t) = A_1x(t)$ ($t \geq 0$). Noting that these two time invariant ODEs are both time reversible, one can check that $H(x)$ actually brings $\dot{x}(t) = A_1x(t)$ ($t \in \mathbb{R}$) to $\dot{y}(t) = A_2y(t)$ ($t \in \mathbb{R}$), and its inverse $H^{-1}(x)$ brings $\dot{y}(t) = A_2y(t)$ ($t \in \mathbb{R}$) to $\dot{x}(t) = A_1x(t)$ ($t \in \mathbb{R}$).
Comparing Definition 1.1 and Definition 2.1 and noting Remark 2.1, one can obtain the following proposition, which reveals the relation between the topological equivalence for pure ODEs and the topological equivalence for ODE systems with observation.

**Proposition 2.1** 1) ODEs \( \dot{x}(t) = A_1 x(t) \) and \( \dot{y}(t) = A_2 y(t) \) are linearly (resp., topologically) equivalent in the sense of Definition 2.1 if and only if systems \((A_1, 0)\) and \((A_2, 0)\) are linearly (resp., topologically) equivalent in the sense of Definition 1.1.

2) If systems \((A_1, C_1)\) and \((A_2, C_2)\) are linearly (resp., topologically) equivalent in the sense of Definition 1.1 then ODEs \( \dot{x}(t) = A_1 x(t) \) and \( \dot{y}(t) = A_2 y(t) \) are linearly (resp., topologically) equivalent in the sense of Definition 2.1.

The following proposition is well-known in the literature (see e.g. [7, 8, 14]), which shows that the set of indices \( \{n^+(A), n^-(A), n^0(A)\} \) for pure ODE \( \dot{x}(t) = A x(t) \) is invariant under any topological transformation as given in Definition 2.1. As a consequence, any topological equivalence transformation brings a stable ODE (i.e., \( n^-(A) = n \)) to another stable ODE.

**Proposition 2.2** 1) ODEs \( \dot{x}(t) = A_1 x(t) \) and \( \dot{y}(t) = A_2 y(t) \) are linearly equivalent in the sense of Definition 2.1 if and only if matrices \( A_1 \) and \( A_2 \) are similar.

2) ODEs \( \dot{x}(t) = A_1 x(t) \) and \( \dot{y}(t) = A_2 y(t) \) are topologically equivalent in the sense of Definition 2.1 if and only if \((n^0(A_1), n^+(A_1), n^-(A_1)) = (n^0(A_2), n^+(A_2), n^-(A_2))\) and matrices \( A_1^0 \) and \( A_2^0 \) are similar.

**Proposition 2.3** The set of indices \( \{n^0(A), n^+(A), n^-(A)\} \) for system \((A, C)\) is invariant under any topological equivalence transformation introduced in Definition 2.1.

**Proof.** Proposition 2.2 shows that the set of indices \( \{n^0(A), n^+(A), n^-(A)\} \) for pure ODE \( \dot{x}(t) = A x(t) \) is invariant under any topological transformation. This fact combining Proposition 2.1 yields that the set of indices \( \{n^0(A), n^+(A), n^-(A)\} \) for system \((A, C)\) is also invariant under any topological transformation.

### 2.2 Topological equivalence transformations

In this subsection, we will present a property concerning the topological equivalence transformations, which is useful in Section 4.

Consider the following two autonomous linear ODEs

\[
\dot{x}(t) = \begin{bmatrix} A_1^+ & 0 \\ A_m^+ & A_m^+ \end{bmatrix} x(t), \quad t \in \mathbb{R} \tag{2.1}
\]

and

\[
\dot{y}(t) = \begin{bmatrix} A_o^+ & 0 \\ 0 & A_u^+ \end{bmatrix} y(t), \quad t \in \mathbb{R}. \tag{2.2}
\]

Here matrices \( A_o^+ \in \mathbb{R}^{\nu \times \nu} \), \( A_u^+ \in \mathbb{R}^{(n-\nu) \times (n-\nu)} \) and \( A_m^+ \in \mathbb{R}^{(n-\nu) \times \nu} \). The real parts of the eigenvalues of \( A_o^+ \) and \( A_u^+ \) are both positive.

**Proposition 2.4** There exists a homeomorphism \( H \) on \( \mathbb{R}^n \) such that

i) \( H \) is a topological equivalence transformation which brings (2.1) to (2.2), and the inverse of \( H \) brings (2.2) to (2.1);

ii) Transformation \( (y_1, \cdots, y_\nu) = H(x_1, \cdots, x_n) \) has the property

\[
y_i = x_i, \quad i = 1, 2, \cdots, \nu. \tag{2.3}
\]
Remark 2.2 Proposition 2.4 still holds for two autonomous linear ODEs

\[
\dot{x}(t) = \begin{bmatrix} A^-_o & 0 \\ A^-_m & A^-_u \end{bmatrix} x(t), \quad t \in \mathbb{R}
\]

and

\[
\dot{y}(t) = \begin{bmatrix} A^-_o & 0 \\ 0 & A^-_u \end{bmatrix} y(t), \quad t \in \mathbb{R}.
\]

Here matrices \( A^-_o \in \mathbb{R}^{\nu \times \nu}, A^-_m \in \mathbb{R}^{(n-\nu) \times (n-\nu)} \) and \( A^-_u \in \mathbb{R}^{(n-\nu) \times \nu} \). The real parts of the eigenvalues of \( A^-_o \) and \( A^-_u \) are both negative.

We omit the proof of Proposition 2.4 here. We refer the reader to [7, 8, 14] for various methods for construction the transformation between topologically equivalent ODEs.

3 Linear equivalence

Proof of Proposition 1.1.

“Condition (1.5) ⇒ Linear equivalence”:

It is easy to check that

\[
\begin{bmatrix} y \\ z \\ w \end{bmatrix} = \begin{bmatrix} H(x) \\ w \end{bmatrix} := \begin{bmatrix} P^{-1} & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}
\]

is the linear equivalence transformation from (1.3) to (1.4). Here \( E \) is the identity matrix. Therefore these two systems are linearly equivalent.

“Linear equivalence ⇒ Condition (1.5)”:

Assume that systems (1.3) and (1.4) are linearly equivalent, by Proposition 2.1 ODEs \( \dot{x}(t) = A_1 x(t) \) and \( \dot{y}(t) = A_2 y(t) \) are linearly equivalent. This together with Proposition 2.2 yields that \( A_2 = P^{-1} A_1 P \) for some nonsingular matrix \( P \). Noting that \( w(t) \equiv z(t) \) (i.e., \( C_1 x(t) \equiv C_2 y(t) \)) and \( y(t) \equiv P^{-1} x(t) \), one obtain that \( C_2 = C_1 P \). Hence systems (1.3) and (1.4) satisfy relation (1.5).

By the knowledge of linear algebra, one can prove that

Lemma 3.1 Suppose that the eigenvalues of \( S_1 \in \mathbb{R}^{m_1 \times m_1} \) are different from the eigenvalues of \( S_2 \in \mathbb{R}^{m_2 \times m_2} \), then Sylvester equation \( S_2 X - XS_1 = 0 \) admits a unique solution \( X = 0 \in \mathbb{R}^{m_2 \times m_1} \).

Corollary 3.1 1) The set of indices \( \{ k_{\text{obs}}(A, C), k^0_{\text{obs}}(A, C), k^+_{\text{obs}}(A, C), k^-_{\text{obs}}(A, C) \} \) is uniquely determined by \( (A, C) \).

2) The set of indices \( \{ k_{\text{obs}}(A, C), k^0_{\text{obs}}(A, C), k^+_{\text{obs}}(A, C), k^-_{\text{obs}}(A, C) \} \) is invariant under any linear equivalence transformation introduced in Definition 1.1.

Proof. 1) Obviously, by Definition 1.2 \( k_{\text{obs}}(A, C) \) is uniquely determined by \( (A, C) \). Suppose that systems

\[
\begin{align*}
\begin{bmatrix} A^0 & 0 & 0 \\ 0 & A^+ & 0 \\ 0 & 0 & A^- \end{bmatrix}, \begin{bmatrix} C^0 & C^+ & C^- \end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
\begin{bmatrix} \tilde{A}^0 & 0 & 0 \\ 0 & \tilde{A}^+ & 0 \\ 0 & 0 & \tilde{A}^- \end{bmatrix}, \begin{bmatrix} \tilde{C}^0 & \tilde{C}^+ & \tilde{C}^- \end{bmatrix}
\end{align*}
\]

(3.1)
are both linearly equivalent to \((A, C)\). Here the real parts of the eigenvalues of \(A^0, A^+\) and \(A^-\) are zero, positive and negative respectively. And the real parts of the eigenvalues of \(\tilde{A}^0, \tilde{A}^+\) and \(\tilde{A}^-\) are zero, positive and negative respectively.

We claim that
\[
\kappa_{obs}(A^0, C^0) = \kappa_{obs}(\tilde{A}^0, \tilde{C}^0), \quad \kappa_{obs}(A^+, C^+) = \kappa_{obs}(\tilde{A}^+, \tilde{C}^+), \quad \kappa_{obs}(A^-, C^-) = \kappa_{obs}(\tilde{A}^-, \tilde{C}^-).
\]
(3.2)

In fact, since the two systems in (3.1) are linearly equivalent, by Proposition 1.1, there exists a nonsingular matrix \(P \in \mathbb{R}^n\) such that
\[
\begin{bmatrix}
11 & 12 & 13 \\
21 & 22 & 23 \\
31 & 32 & 33
\end{bmatrix}
\begin{bmatrix}
A^0 & 0 & 0 \\
0 & A^+ & 0 \\
0 & 0 & A^-
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{A}^0 & 0 & 0 \\
0 & \tilde{A}^+ & 0 \\
0 & 0 & \tilde{A}^-
\end{bmatrix}
\begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
P_{21} & P_{22} & P_{23} \\
P_{31} & P_{32} & P_{33}
\end{bmatrix}
.
\]

Then \(P_{12}A^+ = \tilde{A}^0P_{12}\). Noting that the eigenvalues of \(A^+\) are different from the eigenvalues of \(A^0\), by Lemma 3.1, we conclude that \(P_{12} = 0\). Similarly, we can show that \(P_{ij} = 0 \ (i \neq j \text{ and } i, j = 1, 2, 3)\). Therefore, \(L\) actually takes the form of
\[
\begin{bmatrix}
P_{11} & 0 & 0 \\
0 & P_{22} & 0 \\
0 & 0 & P_{33}
\end{bmatrix}.
\]
As a consequence,
\[
A^0 = P_{11}^{-1}\tilde{A}^0P_{11}, \quad C^0 = \tilde{C}^0P_{11},
\]
\[
A^+ = P_{22}^{-1}\tilde{A}^+P_{22}, \quad C^+ = \tilde{C}^+P_{22},
\]
\[
A^- = P_{33}^{-1}\tilde{A}^-P_{33}, \quad C^- = \tilde{C}^-P_{33}.
\]
Thus, (3.2) is valid.

2) Suppose that systems \((A, C)\) and \((\tilde{A}, \tilde{C})\) are linearly equivalent. It is obviously that \(k_{obs}(A, C) = k_{obs}(\tilde{A}, \tilde{C})\). Next, we need to show that
\[
k_{obs}^0(A, C) = k_{obs}^0(\tilde{A}, \tilde{C}), \quad k_{obs}^+((A, C) = k_{obs}^+(\tilde{A}, \tilde{C}), \quad k_{obs}^-((A, C) = k_{obs}^-(\tilde{A}, \tilde{C}).
\]
(3.3)

In fact, by Proposition 1.1, there exists a nonsingular matrix \(P \in \mathbb{R}^{n \times n}\) such that
\[
A = P^{-1}\tilde{A}P, \quad C = \tilde{C}P.
\]
(3.4)

Through a suitable linear equivalence transformation, we can transform system \((A, C)\) into (1.6). That is, there exists another nonsingular matrix \(Q \in \mathbb{R}^{n \times n}\) such that
\[
\begin{bmatrix}
A^0 & 0 & 0 \\
0 & A^+ & 0 \\
0 & 0 & A^-
\end{bmatrix} = Q^{-1}AQ, \quad \begin{bmatrix}
C^0 & C^+ & C^-
\end{bmatrix} = CQ.
\]
(3.5)

Combining (3.4) and (3.5), we have
\[
\begin{bmatrix}
A^0 & 0 & 0 \\
0 & A^+ & 0 \\
0 & 0 & A^-
\end{bmatrix} = (PQ)^{-1}\tilde{A}(PQ), \quad \begin{bmatrix}
C^0 & C^+ & C^-
\end{bmatrix} = \tilde{C}(PQ).
\]

In other words, through another suitable linear equivalence transformation, we can transform system \((\tilde{A}, \tilde{C})\) into (1.6) as well. By the result of 1) and Definition 1.2 (3.3) is valid. \(\square\)
Lemma 3.2 Assume that $S_i \in \mathbb{R}^{m_i \times m_i}$ and $O_i \in \mathbb{R}^{p \times m_i}$ ($i = 1, 2$), and the eigenvalues of $S_1$ are different from the eigenvalues of $S_2$. Then

$$k_{obs}\left(\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}; [O_1 \quad O_2] \right) = k_{obs}(S_1, O_1) + k_{obs}(S_2, O_2),$$

that is

$$\text{rank}\left[\begin{bmatrix} O_1 & O_2 \\ O_1 S_1 & O_2 S_2 \\ \vdots & \vdots \\ O_1 S_1^{m_1+m_2-1} & O_2 S_2^{m_1+m_2-1} \end{bmatrix}\right] = \text{rank}\left[\begin{bmatrix} O_1 & O_2 \\ O_1 S_1 & O_2 S_2 \\ \vdots & \vdots \\ O_1 S_1^{m_1-1} & O_2 S_2^{m_2-1} \end{bmatrix}\right].$$

Proof. Let $n = m_1 + m_2$. The Cayley-Hamilton theorem shows that $\varphi(S_1) = 0$, where

$$\varphi(\lambda) = |\lambda I - S_1| = \lambda^{m_1} + a_{m_1-1}\lambda^{m_1-1} + \cdots + a_1\lambda + a_0$$

is the characteristic polynomial of $S_1$. Then through a series of generalized elementary row transformations, we have

$$\begin{bmatrix} O_1 & O_2 \\ O_1 S_1 & O_2 S_2 \\ \vdots & \vdots \\ O_1 S_1^{m_1-1} & O_2 S_2^{m_1-1} \end{bmatrix} \rightarrow \begin{bmatrix} O_1 & O_2 \\ O_1 S_1^{m_1-1} & O_2 S_2^{m_1-1} \\ \vdots & \vdots \\ 0 & O_2 S_2^{m_2-2}\varphi(S_2) \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} O_1 & O_2 \\ O_1 S_1^{m_1-1} & O_2 S_2^{m_1-1} \\ \vdots & \vdots \\ 0 & O_2 S_2^{m_2-2}\varphi(S_2) \end{bmatrix}.$$

Therefore, noting that $\varphi(S_2)$ is nonsingular, we deduce that

$$\text{rank}\left[\begin{bmatrix} O_1 & O_2 \\ O_1 S_1 & O_2 S_2 \\ \vdots & \vdots \\ O_1 S_1^{m_1-1} & O_2 S_2^{m_1-1} \end{bmatrix}\right] = \text{rank}\left[\begin{bmatrix} O_1 & O_2 \\ O_1 S_1 & O_2 S_2 \\ \vdots & \vdots \\ O_1 S_1^{m_1-1} & O_2 S_2^{m_1-1} \end{bmatrix}\right] + \text{rank}\left[\begin{bmatrix} O_2 S_2 \varphi(S_2) \\ \vdots \end{bmatrix}\right].$$
Corollary 3.2 Assume that $S_i \in \mathbb{R}^{m_i \times m_i}$ and $O_i \in \mathbb{R}^{p \times m_i}$ ($i = 1, \cdots, s$), and the eigenvalues of $S_i$ are different from the eigenvalues of $S_j$ ($i, j = 1, \cdots, s$ and $i \neq j$). Then

$$k_{\text{obs}}\left(\begin{bmatrix} S_1 & \cdots & \cdots & S_s \end{bmatrix}, [O_1 \cdots O_s]\right) = k_{\text{obs}}(S_1, O_1) + \cdots + k_{\text{obs}}(S_s, O_s).$$

Lemma 3.3 For system (1.1) (i.e. $(A, C)$), it holds that

$$k_{\text{obs}}(A, C) = k_{\text{obs}}^0(A, C) + k_{\text{obs}}^+(A, C) + k_{\text{obs}}^-(A, C).$$

Proof. Applying Corollary 3.2 (with $s = 3$) to system (1.6) directly, we can obtain the desired results of Lemma 3.3 by virtue of Definition 1.2.

Example 3.1 Consider the following two systems:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a \\ 1 & a \\ 1 & a \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}.$$  

Here $a \in \mathbb{R}$ and $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$. We claim that the above two systems are linearly equivalent if and only if:

i) $a = 2$;

ii) $c_2 \neq 0$ and $c_3 = c_4 = 0$.

Proof. Sufficiency. If items i) and ii) hold simultaneously, both systems are linearly equivalent to

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$  

Hence these two systems are linearly equivalent.

Necessity. By 2) of Corollary 3.1, the linear equivalence of these two systems implies that

$$k_{\text{obs}}\left(\begin{bmatrix} a \\ 1 & a \\ 1 & a \end{bmatrix}, [c_1 \ c_2 \ c_3 \ c_4]\right) = 2.$$  

Thus item ii) holds. By 2) of Proposition 2.1, matrices

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} a & a \\ 1 & a \\ 1 & a \end{bmatrix}$$

are similar. Hence item i) is valid.

4 Topological equivalence

The main task in this section is to prove Theorem 1.2 (i.e., to obtain the topological equivalence canonical form (1.7)). First, we introduce the strategy.
4.1 Steps towards the topological equivalence canonical forms

Step 1. Decompose system (1.1) into two independent subsystems according to the real parts of the eigenvalues of $A$, as we shall do in Section 4.2:

$$
\begin{pmatrix}
A^0 & 0 \\
0 & A^\pm
\end{pmatrix},
\begin{pmatrix}
[C^0 & C^\pm]
\end{pmatrix}.
$$

Here the real parts of the eigenvalues of $A^0$ are zero, and the real parts of the eigenvalues of $A^\pm$ are not zero.

Step 2. Apply the well-known Kalman decomposition (according to the observability) to subsystem $(A^\pm, C^\pm)$ to yield

$$
\begin{pmatrix}
A^0 & 0 & 0 \\
0 & A^0_{\pm} & 0 \\
0 & 0 & A^0_{\pm}
\end{pmatrix},
\begin{pmatrix}
[C^0 & C^0_{\pm} & 0]
\end{pmatrix}.
$$

Step 3. By virtue of the topological equivalence transformation introduced in Lemma 4.3 change the above system to the following form:

$$
\begin{pmatrix}
A^0 & 0 & 0 \\
0 & A^0_{\pm} & 0 \\
0 & 0 & \hat{E}
\end{pmatrix},
\begin{pmatrix}
[C^0 & C^0_{\pm} & 0]
\end{pmatrix}.
$$

Step 4. Find the linear equivalence canonical form (which is denoted by $(\hat{N}, \hat{K})$) for subsystem $(A^0, C^0)$. And find the linear equivalence canonical form (which is denoted by $(\hat{B}, \hat{D})$) for completely observable subsystem $(A^0_{\pm}, C^0_{\pm})$. Finally we obtain the topological equivalence canonical form (4.1) as shown in Theorem 4.2.

4.2 System decomposition according to the eigenvalues

Suppose that systems (1.3) and (1.4) are both in the form of (1.6) through a suitable linear transformation. Thus, systems (1.3) and (1.4) can be rewritten as

$$
\begin{align*}
\begin{cases}
\begin{pmatrix}
\dot{x}^0(t) \\
\dot{x}^+(t) \\
\dot{x}^-(t)
\end{pmatrix} =
\begin{pmatrix}
A^0_1 & 0 & 0 \\
0 & A^+_1 & 0 \\
0 & 0 & A^-_1
\end{pmatrix}
\begin{pmatrix}
x^0(t) \\
x^+(t) \\
x^-(t)
\end{pmatrix}, & t \geq 0,
\end{cases}
\end{align*}
\tag{4.1}
$$

and

$$
\begin{align*}
\begin{cases}
\begin{pmatrix}
\dot{y}^0(t) \\
\dot{y}^+(t) \\
\dot{y}^-(t)
\end{pmatrix} =
\begin{pmatrix}
A^0_2 & 0 & 0 \\
0 & A^+_2 & 0 \\
0 & 0 & A^-_2
\end{pmatrix}
\begin{pmatrix}
y^0(t) \\
y^+(t) \\
y^-(t)
\end{pmatrix}, & t \geq 0,
\end{cases}
\end{align*}
\tag{4.2}
$$

and

$$
\begin{align*}
\begin{cases}
\begin{pmatrix}
z(t) \\
\dot{z}(t)
\end{pmatrix} =
\begin{pmatrix}
C^0_1 & C^+_1 & C^-_1
\end{pmatrix}
\begin{pmatrix}
x^0(t) \\
x^+(t) \\
x^-(t)
\end{pmatrix}, & t \geq 0.
\end{cases}
\end{align*}
$$

Denote

$$
A^\pm_i := \begin{pmatrix} A^+_i & 0 \\ 0 & A^-_i \end{pmatrix}, \quad i = 1, 2,$$

$$
x^\pm := (x^+, x^-), \quad y^\pm := (y^+, y^-), \quad h^\pm(x^0, x^\pm) := (h^+(x^0, x^\pm), h^-(x^0, x^\pm)).$$
Lemma 4.1 Suppose that ODEs
\[
\begin{align*}
\dot{x}^0(t) &= \begin{bmatrix} A_0^0 & 0 & 0 \\ 0 & A_1^0 & 0 \\ 0 & 0 & A_1^0 \end{bmatrix} \begin{bmatrix} x^0(t) \\ x^+(t) \\ x^-(t) \end{bmatrix}, \quad t \in \mathbb{R} \\
\dot{x}^+(t) &= \begin{bmatrix} A_0^+ & 0 & 0 \\ 0 & A_1^+ & 0 \\ 0 & 0 & A_1^- \end{bmatrix} \begin{bmatrix} x^0(t) \\ x^+(t) \\ x^-(t) \end{bmatrix}, \quad t \in \mathbb{R}
\end{align*}
\]
and
\[
\begin{align*}
\dot{y}^0(t) &= \begin{bmatrix} A_0^0 & 0 & 0 \\ 0 & A_2^0 & 0 \\ 0 & 0 & A_2^- \end{bmatrix} \begin{bmatrix} y^0(t) \\ y^+(t) \\ y^-(t) \end{bmatrix}, \quad t \in \mathbb{R} \\
\dot{y}^+(t) &= \begin{bmatrix} A_0^+ & 0 & 0 \\ 0 & A_2^+ & 0 \\ 0 & 0 & A_2^- \end{bmatrix} \begin{bmatrix} y^0(t) \\ y^+(t) \\ y^-(t) \end{bmatrix}, \quad t \in \mathbb{R}
\end{align*}
\]
are topologically equivalent in the sense of Definition 2.1 and denote the corresponding topological equivalence transformation by
\[
(y^0, y^+, y^-) = H(x^0, x^+, x^-) =: (h^0(x^0, x^+, x^-), h^+(x^0, x^+, x^-), h^-(x^0, x^+, x^-)).
\]
Then
1) \( h^+(x^0, 0, 0) \equiv 0 \) and \( h^-(x^0, 0, 0) \equiv 0 \) for any \( x^0 \in \mathbb{R}^{n^0(A_1)} \);
2) \( h^0(0, x^+, 0) \equiv 0 \) and \( h^-(0, x^+, 0) \equiv 0 \) for any \( x^+ \in \mathbb{R}^{n^+(A_1)} \);
3) \( h^0(0, 0, x^-) \equiv 0 \) and \( h^+(0, 0, x^-) \equiv 0 \) for any \( x^- \in \mathbb{R}^{n^-(A_1)} \).

Proof. Since ODEs (4.3) and (4.4) are topologically equivalent, by Proposition 2.2 denote 
\( n^0 := n^0(A_1) = n^0(A_2), n^+ := n^+(A_1) = n^+(A_2) \) and \( n^- := n^-(A_1) = n^-(A_2) \). The rest proof is divided into four steps.

Step 1. Item 1) is equivalent to
\[
h^\pm(x^0, 0) \equiv 0, \quad \forall x^0 \in \mathbb{R}^{n^0}.
\]
We use the contradiction argument to show the validity of (4.5). Hypothesize that there exist two constant vectors \( \alpha \in \mathbb{R}^{n^0} \) and \( 0 \neq \beta \in \mathbb{R}^{n^+ + n^-} \) such that \( h^\pm(\alpha, 0) = \beta \). Denote \( \gamma := h^0(\alpha, 0) \). The solution of ODE (4.3) satisfying condition \( (x^0(0), x^+(0)) = (\alpha, 0) \) is \((x^0(t), x^+(t)) = (e^{A^0t}\alpha, 0) \). The corresponding solution of ODE (4.4) is
\[
(y^0(t), y^+(t)) = H(e^{A^0t}\alpha, 0) = (e^{A^0t}\gamma, e^{A^\pm t}\beta), \quad t \in \mathbb{R}.
\]
Noting that the real parts of the eigenvalues of \( A_2^\pm \) are not zero, one concludes that either
\[
\|e^{A^\pm t}\beta\| \to +\infty \quad \text{as} \quad t \to +\infty
\]
or
\[
\|e^{A^\pm t}\beta\| \to +\infty \quad \text{as} \quad t \to -\infty.
\]
However,
\[
\|e^{A^0t}\alpha\| = \|\alpha\|, \quad t \in \mathbb{R}.
\]
The above facts combing with (4.6) contradict with the fact that \( H(\cdot) \) is a homeomorphism on \( \mathbb{R}^n \). Therefore, formula (4.3) is valid.

Step 2. We show that \( h^0(0, x^+, 0) \equiv 0 \) for any \( x^+ \in \mathbb{R}^{n^+} \). We use the contradiction argument. Hypothesize that there exist two constant vectors \( \xi \in \mathbb{R}^{n^+} \) and \( 0 \neq \eta \in \mathbb{R}^{n^0} \) such that \( h^0(0, \xi, 0) = \eta \). Denote \( \zeta := h^\pm(0, \xi, 0) \). The solution of ODE (4.3) satisfying condition
The corresponding solution of ODE (4.4) is

\[(y^0(t), y^+(t)) = H(0, e^{A_1^+ t} \xi, 0) = (e^{A_2^0 t} \eta, e^{A_2^0 t} \zeta), \quad t \in \mathbb{R}.\]  

(4.7)

Noting that all the real parts of the eigenvalues of \(A_1^+\) are positive, one deduce that

\[\|e^{A_1^+ t} \xi\| \to 0 \quad \text{as} \quad t \to -\infty.\]

However,

\[\|e^{A_2^0 t} \eta\| \equiv \|\eta\| \neq 0, \quad t \in \mathbb{R}.\]

The above facts combing with (4.7) contradict with the fact that \(H(\cdot)\) is a homeomorphism on \(\mathbb{R}^n\). Therefore, \(h^0(0, x^+, 0) \equiv 0\) for any \(x^+ \in \mathbb{R}^n^+\).

**Step 3.** We show that \(h^- (0, x^+, 0) \equiv 0\) for any \(x^+ \in \mathbb{R}^n^+\). We use the contradiction argument. Hypothesize that there exist two constant vectors \(\xi \in \mathbb{R}^{n^+}\) and \(0 \neq \rho \in \mathbb{R}^n^-\) such that \(h^- (0, \xi, 0) = \rho\). Denote \(\iota := h^0(0, \xi, 0)\) and \(\nu := h^+(0, \xi, 0)\). The solution of ODE (4.3) with condition \((x^0(0), x^+(0), x^-(0)) = (0, \xi, 0)\) reads \((x^0(t), x^+(t), x^-(t)) = (0, e^{A_1^+ t} \xi, 0)\). The corresponding solution of ODE (4.4) is

\[(y^0(t), y^+(t), y^-(t)) = H(0, e^{A_1^+ t} \xi, 0) = (e^{A_2^0 t} \iota, e^{A_2^0 t} \nu, e^{A_2^0 t} \rho), \quad t \in \mathbb{R}.\]

Since

\[\|e^{A_1^+ t} \xi\| \to 0 \quad \text{as} \quad t \to -\infty\]

and

\[\|e^{A_2^0 t} \rho\| \to +\infty \quad \text{as} \quad t \to -\infty,\]

these lead to a contradiction. Therefore, \(h^- (0, x^+, 0) \equiv 0\) for any \(x^+ \in \mathbb{R}^n^+\).

**Step 4.** Similar to Step 2, we can show that \(h^0(0, 0, x^-) \equiv 0\) for any \(x^- \in \mathbb{R}^n^-\). Similar to Step 3, we can show that \(h^- (0, 0, x^-) \equiv 0\) for any \(x^- \in \mathbb{R}^n^-\).

The following proposition implies that the topological classification problem for the whole system can be reduced to three individual classification problems for each subsystem.

**Proposition 4.1** Systems (4.1) and (4.2) are topologically equivalent if and only if

i) Subsystems \((A_1^0, C_1^0)\) and \((A_2^0, C_2^0)\) are linearly equivalent;

ii) Subsystems \((A_1^+, C_1^+)\) and \((A_2^+, C_2^+)\) are topologically equivalent;

iii) Subsystems \((A_1^-, C_1^-)\) and \((A_2^-, C_2^-)\) are topologically equivalent.

**Proof. Sufficiency.** Assume that items i)-iii) hold spontaneously. By item i) and Proposition 2.1, we obtain that ODEs \(\dot{x}^0(t) = A_1^0 x^0(t)\) and \(\dot{y}^0(t) = A_2^0 y^0(t)\) are linearly equivalent in the sense of Definition 2.1. Hence, by Proposition 2.2, matrices \(A_1^0\) and \(A_2^0\) are similar. By item ii) and Proposition 2.1, ODEs \(\dot{x}^+(t) = A_1^+ x^+(t)\) and \(\dot{y}^+(t) = A_2^+ y^+(t)\) are topologically equivalent in the sense of Definition 2.1. From Proposition 2.2, we get \(n^+(A_1) = n^+(A_2)\). Similar, item iii) implies that \(n^-(A_1) = n^-(A_2)\). Therefore, from Proposition 2.2 again, pure ODEs (4.3) and (4.4) are topologically equivalent in the sense of Definition 2.1.

Item i) implies that \(w^0(t) := C_1^0 x^0(t) \equiv C_2^0 y^0(t) := z^0(t)\). Similarly, item ii) implies that \(w^+(t) := C_1^+ x^+(t) \equiv C_2^+ y^+(t) := z^+(t)\); and item iii) implies that \(w^-(t) := C_1^- x^-(t) \equiv C_2^- y^-(t) := z^-(t)\). Thus,

\[w(t) = w^0(t) + w^+(t) + w^-(t) \equiv z^0(t) + z^+(t) + z^-(t) = z(t).\]
Therefore, systems (4.1) and (4.2) are topologically equivalent in the sense of Definition 1.1.

**Necessity.** Assume that systems (4.1) and (4.2) are topologically equivalent, and \( F(x, w) = (H(x), w) \) is the corresponding topological equivalence transformation bringing system (4.1) to system (4.2). By Proposition 2.1 ODEs (4.3) and (4.4) are topologically equivalent in the sense of Definition 2.1. Transformation \( H(x) \) brings ODE (4.3) to ODE (4.4). By 2) of Proposition 2.2 we know that \((n^0(A_1), n^+(A_1), n^-(A_1)) = (n^0(A_2), n^+(A_2), n^-(A_2))\) and matrices \(A_1^0, A_2^0\) are similar.

First we claim that item i) holds. In fact, since \(n^0(A_1) = n^0(A_2)\) and matrices \(A_1^0, A_2^0\) are similar, by 1) of Proposition 2.2 ODEs \( \dot{x}^0(t) = A_1^0 x^0(t) \) and \( \dot{y}^0(t) = A_2^0 y^0(t) \) are linearly equivalent in the sense of Definition 2.1.

On the other hand, since ODEs (4.3) and (4.4) are topologically equivalent, from 1) of Lemma 4.1, we deduce that the topological equivalence transformation \( H(x) \) brings ODE (4.3) with initial data (4.9) to ODE (4.4) with initial data (4.8).

This fact shows that the topological equivalence transformation \( F(x, w) = (H(x), w) \) brings system (4.1) with initial data (4.8) and observation \( w(t) = C_1^0 x^0(t) \) to system (4.2) with initial data (4.9) and observation \( z(t) = C_2^0 y^0(t) \). In this situation,

\[
w^0(t) := C_1^0 x^0(t) = w(t) \equiv z(t) = C_2^0 y^0(t) := z^0(t).
\]

Thus, subsystems \((A_1^0, C_1^0)\) and \((A_2^0, C_2^0)\) are topologically equivalent in the sense of Definition 1.1.

Next we claim that item ii) is valid. In fact, since \(n^+(A_1) = n^+(A_2)\), by 2) of Proposition 2.2 ODEs \( \dot{x}^+(t) = A_1^+ x^+(t) \) and \( \dot{y}^+(t) = A_2^+ y^+(t) \) are topologically equivalent in the sense of Definition 2.1.

On the other hand, from 2) of Lemma 4.1 one can see that the topological equivalence transformation \( F(x, w) = (H(x), w) \) brings system (4.1) with initial data (4.8) and observation \( w(t) = C_1^+ x^+(t) \) to system (4.2) with initial data (4.9) and observation \( z(t) = C_2^+ y^+(t) \).

Therefore,

\[
w^+(t) := C_1^+ x^+(t) = w(t) \equiv z(t) = C_2^+ y^+(t) := z^+(t).
\]

Thus, subsystems \((A_1^+, C_1^+)\) and \((A_2^+, C_2^+)\) are linearly equivalent in the sense of Definition 1.1.
Similarly, we can show the validity of item iii) by Propositions [2.2 and 3] of Lemma 4.1.

\[ Q.E.D. \]

**Proof of Corollary 1.1.**
When \( n^0(A_1) = n^0(A_2) = n \), it has \((A_1, C_1) = (A_1^0, C_1^0)\) and \((A_2, C_2) = (A_2^0, C_2^0)\). Corollary 1.1 is a direct consequence of Proposition 1.1.

\[ Q.E.D. \]

### 4.3 Topological equivalence for completely observable systems

**Proof of Theorem 1.1.**
From Remark 1.3 and Proposition 1.1, we only need to show that for completely observable systems (1.3) and (1.4),

\[ \text{Topological equivalence } \Rightarrow \text{ Linear equivalence.} \]

Assume that completely observable systems \((A_1, C_1)\) and \((A_2, C_2)\) are topologically equivalent. Denote the corresponding topological equivalence transformation by \( H(\cdot) \). In order to show that \( H(\cdot) \) is actually a linear isomorphism, we will prove that

\[ H(kx_0) = kH(x_0), \quad \forall k \in \mathbb{R}, \ x_0 \in \mathbb{R}^n \]

(4.10)

and

\[ H(x_0 + \bar{x}_0) = H(x_0) + H(\bar{x}_0), \quad \forall x_0, \ \bar{x}_0 \in \mathbb{R}^n \]

(4.11)

separately by two steps.

**Step 1.** First, we will show the validity of (4.10). The solution of (1.3) with any initial data \( x(0) = x_0 \in \mathbb{R}^n \) reads

\[ x(t) = e^{A_1t}x_0, \quad t \geq 0. \]

Since transformation \( H(\cdot) \) brings (1.3) to (1.4), the solution of (1.4) with initial data \( y_0 = y(0) = H(x(0)) = H(x_0) \) satisfies

\[ H(e^{A_1t}x_0) = H(x(t)) = y(t) = e^{A_2t}H(x_0), \quad t \geq 0. \]

Noting \( w(t) \equiv z(t), t \geq 0 \) (see Definition 1.1), we have

\[ C_1e^{A_1tx_0} = w(t) = z(t) = C_2e^{A_2t}H(x_0), \quad t \geq 0. \]

(4.12)

Differentiating (4.12) with respect to \( t \) repeatedly, we get

\[ C_1A_1^ie^{A_1tx_0} = C_2A_2^ie^{A_2t}H(x_0), \quad t \geq 0 \]

for \( i = 1, 2, \cdots, n - 1 \). Hence, for any \( x_0 \in \mathbb{R}^n \), it holds that

\[ e^{A_1tx_0} = \begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^{n-1} \end{bmatrix}, \quad e^{A_2t}H(x_0) = \begin{bmatrix} C_2 \\ C_2A_2 \\ \vdots \\ C_2A_2^{n-1} \end{bmatrix}, \quad t \geq 0. \]

(4.13)

On one hand, by (4.13), for any \( k \in \mathbb{R} \), it has

\[ k \begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^{n-1} \end{bmatrix} e^{A_1tx_0} = k \begin{bmatrix} C_2 \\ C_2A_2 \\ \vdots \\ C_2A_2^{n-1} \end{bmatrix} e^{A_2t}H(x_0) = \begin{bmatrix} C_2 \\ C_2A_2 \\ \vdots \\ C_2A_2^{n-1} \end{bmatrix} e^{A_2tkH(x_0), \quad t \geq 0. \]
On the other hand, still by (4.13), for any $k \in \mathbb{R}$, it holds
\[
\begin{bmatrix}
C_1 \\
C_1A_1 \\
\vdots \\
C_1A_1^{n-1}
\end{bmatrix} e^{A_1 t} x_0 =
\begin{bmatrix}
C_1 \\
C_1A_1 \\
\vdots \\
C_1A_1^{n-1}
\end{bmatrix} e^{A_1 t} (k x_0) =
\begin{bmatrix}
C_2 \\
C_2A_2 \\
\vdots \\
C_2A_2^{n-1}
\end{bmatrix} e^{A_2 t} H(k x_0), \quad t \geq 0.
\]

The above two formulae imply that
\[
\begin{bmatrix}
C_2 \\
C_2A_2 \\
\vdots \\
C_2A_2^{n-1}
\end{bmatrix} e^{A_2 t} (H(k x_0) - k H(x_0)) = 0, \quad t \geq 0.
\]

Take $t = 0$, we obtain
\[
\begin{bmatrix}
C_2 \\
C_2A_2 \\
\vdots \\
C_2A_2^{n-1}
\end{bmatrix} (H(k x_0) - k H(x_0)) = 0. \tag{4.14}
\]

By the observability of system $(A_2, C_2)$ (that is, rank \[
\begin{bmatrix}
C_2 \\
C_2A_2 \\
\vdots \\
C_2A_2^{n-1}
\end{bmatrix} = n \]
), we know
\[
H(k x_0) - k H(x_0) = 0
\]
from (4.14). This is just (4.10).

Step 2. Next, we continue to show the validity of (4.11). Similar to (4.13), systems (4.3) and (4.4) (with initial data $x_0 \in \mathbb{R}^n$ and $H(x_0) \in \mathbb{R}^n$ respectively) satisfy
\[
\begin{bmatrix}
C_1 \\
C_1A_1 \\
\vdots \\
C_1A_1^{n-1}
\end{bmatrix} e^{A_1 t} x_0 =
\begin{bmatrix}
C_2 \\
C_2A_2 \\
\vdots \\
C_2A_2^{n-1}
\end{bmatrix} e^{A_2 t} H(x_0), \quad t \geq 0. \tag{4.15}
\]

From (4.13) and (4.15), one has
\[
\begin{bmatrix}
C_2 \\
C_2A_2 \\
\vdots \\
C_2A_2^{n-1}
\end{bmatrix} e^{A_2 t} H(x_0 + x_0) =
\begin{bmatrix}
C_1 \\
C_1A_1 \\
\vdots \\
C_1A_1^{n-1}
\end{bmatrix} e^{A_1 t} (x_0 + x_0) =
\begin{bmatrix}
C_2 \\
C_2A_2 \\
\vdots \\
C_2A_2^{n-1}
\end{bmatrix} e^{A_2 t} (H(x_0) + H(x_0)) \tag{4.16}
\]
for $t \geq 0$.

Similar to Step 1, the observability of system $(A_2, C_2)$ combining with (4.16) yields that
\[
H(x_0 + x_0) = H(x_0) + H(x_0).
\]

This is just the desired formula (4.11). \qed
4.4 Topological equivalence for general systems

Assume that system (1.1) is in the form of (1.6). Denote

\[ A^\pm := \begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix}, \quad C^\pm := \begin{bmatrix} C^+ & C^- \end{bmatrix}. \]

Thus, we rewrite system (1.6) as below:

\[ \left( \begin{bmatrix} A^0 & 0 \\ 0 & A^\pm \end{bmatrix}, \begin{bmatrix} C^0 \\ C^\pm \end{bmatrix} \right). \tag{4.17} \]

Recall the definition of \( n^+(A), n^-(A), n^0(A), k_{\text{obs}}(A, C), k_{\text{obs}}^+(A, C), k_{\text{obs}}^-(A, C) \) and \( k_{\text{obs}}^0(A, C) \) given in Definition 1.2. The following lemma is due to the well-known Kalman decomposition (c.f. [5] and [6]) with respect to the observability.

**Lemma 4.2** Subsystem \((A^\pm, C^\pm)\) in (4.17) can be linearly equivalent to the following system:

\[ \left( \begin{bmatrix} A_o^\pm & 0 \\ 0 & A_u^\pm \end{bmatrix}, \begin{bmatrix} C_o^\pm \\ 0 \end{bmatrix} \right). \tag{4.18} \]

Here \( A_o^\pm \in \mathbb{R}^{[k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)] \times [k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)]}, C_o^\pm \in \mathbb{R}^{[k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)]}, \) and \( A_u^\pm \in \mathbb{R}^{[n^+(A)+n^-(-A)+k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)] \times [n^+(A)+n^-(-A)+k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)]. \) Besides, \((A_o^\pm, C_o^\pm)\) is completely observable and its Kalman rank of observability is \( k_{\text{obs}}^+(A, C) + k_{\text{obs}}^-(A, C), \) that is

\[ \text{rank} \begin{bmatrix} C_o^\pm \\ C_o^\pm A_o^\pm \\ \vdots \\ C_o^\pm (A_o^\pm)^{k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)-1} \end{bmatrix} = k_{\text{obs}}^+(A, C) + k_{\text{obs}}^-(A, C). \]

**Lemma 4.3** Subsystem \((A^\pm, C^\pm)\) in (4.17) can be topologically equivalent to the following system:

\[ \left( \begin{bmatrix} A_o^\pm & 0 \\ 0 & E^\pm \end{bmatrix}, \begin{bmatrix} C_o^\pm \\ 0 \end{bmatrix} \right). \tag{4.19} \]

Here

\[ E^\pm = \begin{bmatrix} 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & -1 \\ \cdots & \cdots & \cdots \\ -1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} n^+(A) - k_{\text{obs}}^+(A, C) \\ \vdots \\ n^-(A) - k_{\text{obs}}^-(A, C) \end{bmatrix}, \]

\( A_o^\pm \in \mathbb{R}^{[k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)] \times [k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)]}, \) and \( C_o^\pm \in \mathbb{R}^{[k_{\text{obs}}^+(A, C)+k_{\text{obs}}^-(A, C)]. \) Besides, \((A_o^\pm, C_o^\pm)\) is completely observable.

**Proof.** By Lemma 4.2 subsystem \((A^\pm, C^\pm)\) in (4.17) can be linearly equivalent to (4.18). We continue to show that (4.18) can be topologically equivalent to (4.19). Note that

\[ \begin{bmatrix} A_o^\pm & 0 \\ 0 & E^\pm \end{bmatrix} \]

has no eigenvalues with the real parts being zero, so does \( \begin{bmatrix} A_o^\pm & 0 \\ 0 & E^\pm \end{bmatrix} \). In addition,

\[ n^+(\begin{bmatrix} A_o^\pm & 0 \\ A_m^\pm & A_u^\pm \end{bmatrix}) = n^+(\begin{bmatrix} A_o^\pm & 0 \\ 0 & E^\pm \end{bmatrix}) = n^+(A) \]
and

\[ n^-(\begin{bmatrix} A_o^+ & 0 \\ A_m^+ & A_u^+ \end{bmatrix}) = n^-(\begin{bmatrix} A_o^+ & 0 \\ 0 & E^+ \end{bmatrix}) = n^-(A). \]

By Proposition 2.2, ODE \( \dot{x}^\pm(t) = \begin{bmatrix} A_o^+ & 0 \\ A_m^+ & A_u^+ \end{bmatrix} x^\pm(t) \) can be topologically equivalent to ODE \( \dot{y}^\pm(t) = \begin{bmatrix} A_o^+ & 0 \\ 0 & E^+ \end{bmatrix} y^\pm(t) \). Denote the corresponding topological equivalence transformation between these two ODEs by \( H(\cdot) \). By Proposition 2.4 and Remark 2.2, one can require that the map of \( H(\cdot) \) restricted on the first \( k_{\text{obs}}^+(A,C) + k_{\text{obs}}^-(A,C) \) variables is an identity map. Thus the observations of these two ODEs are equal at any time. As a consequence, system (4.18) is topologically equivalent to system (4.19).

**Proof of Theorem 1.2.**

Through a suitable linear equivalence transformation, we assume that system (1.1) is transformed into system (4.17).

Let \((\hat{N}, \hat{K})\) denote the linear equivalence canonical form of subsystem \((A^0, C^0)\). By Lemma 4.3, subsystem \((A^\pm, C^\pm)\) can be topologically equivalent to \((\hat{N}, \hat{K})\). Let \((\hat{B}, \hat{D})\) denote the linear equivalence canonical form of subsystem \((A_o^\pm, C_o^\pm)\). Rename \( E^\pm \) as \( \hat{E} \).

Since \((A^0, C^0)\) is linearly equivalent to \((\hat{N}, \hat{K})\), and \((A^\pm, C^\pm)\) is topologically equivalent to \((\hat{B}, \hat{D})\), we conclude that the whole system (4.17) is topologically equivalent to system (4.19). The proof of Theorem 1.2 is complete.

**Example 4.1**

Consider the following two systems:

\[
\begin{pmatrix}
2 & 1 & 2 \\
1 & 2 & 1 \\
1 & 2 & 1
\end{pmatrix},
\begin{pmatrix}
3 & 4 & 0 & 0 \\
1 & 2 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
2 & 1 & 2 \\
1 & 2 & 1 \\
1 & 2 & 1
\end{pmatrix},
\begin{pmatrix}
c_1 & c_2 & c_3 & c_4
\end{pmatrix}.
\]

Here \(c_i \in \mathbb{R}\) for \(i = 1, 2, 3, 4\). We claim that the above two systems are topologically equivalent if \(c_2 \neq 0\) and \(c_3 = c_4 = 0\).

**Proof.** By Proposition 2.4, there exists a topological equivalence transformation \(H_2\) which brings

\[
\dot{x}(t) = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} x(t), \quad t \in \mathbb{R}
\]

(4.20)

to

\[
\dot{y}(t) = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} y(t), \quad t \in \mathbb{R}.
\]

(4.21)

and the inverse of \(H_2\) brings (4.21) to (4.20). Besides, transformation \((y_1, y_2, y_3, y_4) = H_2(x_1, x_2, x_3, x_4)\) has the property

\[
y_1 = x_1, \quad y_2 = x_2.
\]

(4.22)

Now consider system (4.20) with observation

\[
w(t) = \begin{bmatrix} 3 & 4 & 0 & 0 \end{bmatrix} [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^T, \quad t \geq 0
\]

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and system (1.21) with observation

\[ z(t) = \begin{bmatrix} 3 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}^\top, \quad t \geq 0. \]

From (4.22), it has

\[ z(t) = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = w(t). \]

Hence we conclude that transformation \((y, z) = F(x, w) := (H_2(x), w)\) brings the solution of (1.20) with initial data \(x(0) = x^0\) and observation \(w(t)\) to the solution of (1.21) with initial data \(y(0) = H_2(x^0)\) and observation \(z(t) = w(t)\). Therefore, systems (1.20) and (1.21) equipped with observations are topologically equivalent in the sense of Definition 1.1.

Noting that subsystems \((\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix})\) and \((\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix})\) are topologically equivalent, and subsystems \((\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix})\) and \((\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \end{bmatrix})\)

are linearly equivalent when \(c_2 \neq 0\), we arrive at the conclusion that

\[
\begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 0 & 0 \end{bmatrix}
\]

and

\[
\begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & 0 & 0 \end{bmatrix}
\]

are topologically equivalent when \(c_2 \neq 0\).

5 Topological invariants and examples of canonical forms

5.1 Topological equivalence invariants

Proof of Proposition 1.2

In view of Proposition 2.3 and Lemma 3.3, it remains to prove that the set of indices

\[ \{ k^0_{\text{obs}}(A, C), k^+_{\text{obs}}(A, C), k^-_{\text{obs}}(A, C) \} \]

is invariant under any topological equivalence transformation. Suppose that systems (1.3) and (1.4) (i.e. \((A_1, C_1)\) and \((A_2, C_2)\)) are topologically equivalent. We need to show that

\[ k^0_{\text{obs}}(A_1, C_1) = k^0_{\text{obs}}(A_2, C_2) \]  

(5.1)

and

\[ k^+_{\text{obs}}(A_1, C_1) = k^+_{\text{obs}}(A_2, C_2), \quad k^-_{\text{obs}}(A_1, C_1) = k^-_{\text{obs}}(A_2, C_2). \]  

(5.2)

Firstly, we claim that (5.1) holds. Indeed, through suitable linear transformations, systems (1.3) and (1.4) can be transformed into (4.1) and (4.2) respectively. The topological equivalence of (1.3) and (1.4) leads to the topological equivalence of (4.1) and (4.2). Then Proposition 4.1 gives the linear equivalence of subsystems \((A^0_1, C^0_1)\) and \((A^0_2, C^0_2)\). By 2) of Corollary 3.1 and Definition 1.2 one has

\[ k^0_{\text{obs}}(A_1, C_1) = k^0_{\text{obs}}(A^0_1, C^0_1) = k^0_{\text{obs}}(A^0_2, C^0_2) = k^0_{\text{obs}}(A_2, C_2). \]
Hence (5.1) is valid.

Secondly, we show the validity of (5.2). By virtue of Theorem 1.2, systems (1.3) and (1.4) can be topologically equivalent to

\[ \begin{bmatrix} \hat{N}_i & 0 & 0 \\ 0 & \hat{B}_i & 0 \\ 0 & 0 & \hat{E}_i \end{bmatrix}, \quad i = 1, 2. \]  

(5.3)

Rewrite (5.3) as below

\[ \begin{cases} \dot{x}_1(t) = A_1 x_1(t) + E_1 x_2(t), & t \geq 0, \\ \dot{x}_2(t) = 0, & t \geq 0, \\ w(t) = [C_1 0] x_1(t), & t \geq 0, \end{cases} \]  

(5.4)

and

\[ \begin{cases} \dot{y}_1(t) = A_2 y_1(t), & t \geq 0, \\ \dot{y}_2(t) = 0, & t \geq 0, \\ z(t) = [C_2 0] y_1(t), & t \geq 0. \end{cases} \]  

(5.5)

The topological equivalence of (1.3) and (1.4) implies the topological equivalence of (5.4) and (5.5). Let \( F(x, w) = (H(x), w) =: (h_1(x_1, x_2), h_2(x_1, x_2), w) \) be the topological equivalence transformation from (5.4) to (5.5).

Next, we use the contradiction argument to show that

\[ h_1(x_1, x_2) \equiv h_1(x_1, 0), \quad \forall x_1, x_2. \]  

(5.6)

As a matter of fact, hypothesize that there exist four vectors \( \alpha, \beta \neq 0, \gamma_1 \) and \( \gamma_2 \) such that

\[ h_1(\alpha, \beta) = \gamma_1 \neq \gamma_2 = h_1(\alpha, 0). \]

On one hand, transformation \((h_1(x_1, x_2), h_2(x_1, x_2), w)\) brings system (5.4) with initial data

\[ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \]

and observation \( w(t) = C_1 e^{A_1 t} \alpha \) to system (5.5) with initial data

\[ \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ * \end{bmatrix} \]

and observation \( z(t) = C_2 e^{A_2 t} \gamma_1 \). From \( z(t) \equiv w(t) \), we get \( C_2 e^{A_2 t} \gamma_1 \equiv C_1 e^{A_1 t} \alpha \). On the other hand, transformation \((h_1(x_1, x_2), h_2(x_1, x_2), w)\) brings system (5.4) with initial data

\[ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \]

and observation \( w(t) = C_1 e^{A_1 t} \alpha \) to system (5.5) with initial data

\[ \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \gamma_2 \\ * \end{bmatrix} \]
and observation \( z(t) = C_2e^{A_2t} \gamma_2 \). From \( z(t) \equiv w(t) \), we get \( C_2e^{A_2t} \gamma_2 \equiv C_1e^{A_1t} \alpha \). Therefore,

\[
C_2e^{A_2t} \gamma_1 \equiv C_1e^{A_1t} \alpha \equiv C_2e^{A_2t} \gamma_2.
\]

By the complete observability of subsystem \((A_2, C_2)\), we obtain \( \gamma_1 = \gamma_2 \). This contradicts with the fact \( \gamma_1 \neq \gamma_2 \). Thus (5.6) is valid.

By (5.6) and Remark 1.3 we have

\[
h_1(0, x_2(0)) = h_1(0, 0) = 0, \quad \forall x_2(0) \in \mathbb{R}^n.
\]

Then one can check that transformation \((h_1(x_1, x_2), h_2(x_1, x_2), w)\) brings (5.4) with initial data

\[
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix} = \begin{bmatrix} 0 \\ x_2(0) \end{bmatrix}
\]

and observation \( w(t) = C_1x_1(t) \equiv 0 \) to (5.5) with initial data

\[
\begin{bmatrix}
y_1(0) \\
y_2(0)
\end{bmatrix} = \begin{bmatrix} 0 \\ h_2(0, x_2(0)) \end{bmatrix}
\]

and observation \( z(t) = C_2y_1(t) \equiv 0 \). This means that \( h_2(0, x_2) \) brings ODE \( \dot{x}_2(t) = \dot{E}_1x_2(t) \) to ODE \( \dot{y}_2(t) = \dot{E}_2y_2(t) \).

Similarly, let \( F^{-1}(y, z) = (H^{-1}(y), z) = (\tilde{h}_1(y_1, y_2), \tilde{h}_2(y_1, y_2), z) \), then one can verify that \( \tilde{h}_2(0, y_2) \) brings ODE \( \dot{y}_2(t) = \dot{E}_2y_2(t) \) to ODE \( \dot{x}_2(t) = \dot{E}_1x_2(t) \). Hence, we obtain the topological equivalence of these two ODEs. Therefore \( \tilde{E}_1 = \tilde{E}_2 \) by Proposition 2.2.

By the definition of \( \tilde{E} \) in (1.7), \( \tilde{E}_1 = \tilde{E}_2 \) implies that

\[
n^+(A_1) - k^+_{obs}(A_1, C_1) = n^+(A_2) - k^+_{obs}(A_2, C_2), \quad n^-(A_1) - k^-_{obs}(A_1, C_1) = n^-(A_2) - k^-_{obs}(A_2, C_2).
\]

Since

\[
n^+(A_1) = n^+(A_2), \quad n^-(A_1) = n^-(A_2)
\]

by Proposition 2.3 we obtain

\[
k^+_{obs}(A_1, C_1) = k^+_{obs}(A_2, C_2), \quad k^-_{obs}(A_1, C_1) = k^-_{obs}(A_2, C_2).
\]

Thus (5.2) is valid.

5.2 Concrete examples

In this subsection, we present concrete examples concerning the topological equivalence canonical forms for a 3-D system with a scalar observation.

Example 5.1 If \( n = 3 \) and \( p = 1 \), then system (1.1) with all the real parts of the eigenvalues of \( A \) being zero is linearly (or topologically) equivalent to one of the following canonical forms:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
A being nonzero is topologically equivalent to one of the following canonical forms:

\[
\begin{pmatrix}
0 & 1 & 0 \\
\mu & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\([0 \ 1 \ 0 ] , [0 \ 0 \ 0 ] , [0 \ 0 \ 0 ] , [1 \ 0 \ 0 ] , [0 \ 1 \ 0 ] \),
\([0 \ 1 \ 0 ] , [0 \ 0 \ 0 ] , [0 \ 0 \ 0 ] , [1 \ 0 \ 0 ] , [0 \ 1 \ 0 ] \),
\([0 \ 1 \ 0 ] , [0 \ 0 \ 0 ] , [0 \ 0 \ 1 ] , [0 \ 0 \ 0 ] , [0 \ 1 \ 0 ] \),
\([0 \ 1 \ 0 ] , [0 \ 0 \ 0 ] , [0 \ 0 \ 0 ] , [1 \ 0 \ 0 ] , [0 \ 1 \ 0 ] \),
\([0 \ 1 \ 0 ] , [0 \ 0 \ 0 ] , [0 \ 0 \ 0 ] , [1 \ 0 \ 0 ] , [0 \ 1 \ 0 ] \).

Note that
\[
n = n^0(A) + [k_{ob}^+(A, B) + k_{ob}^-(A, B)] + [n^+(A) + n^-(A) - k_{ob}^+(A, B) - k_{ob}^-(A, B)].
\]
That is, the dimension $n$ ($n = 1, 2, 3$) can be divided into three parts: $n^0(A), k_{ob}^+(A, B) + k_{ob}^-(A, B)$ and $n^+(A) + n^-(A) - k_{ob}^+(A, B) - k_{ob}^-(A, B)$. $n^0(A) = 0$ means that the real parts of the eigenvalues of $A$ are not zero.

**Example 5.2** If $n = 3$ and $p = 1$, then system \(1\) with the real parts of the eigenvalues of $A$ being nonzero is topologically equivalent to one of the following canonical forms:

1. “3 = 0 + 3 0”: \[
\begin{pmatrix} 0 & 1 & 0 \\ \mu_3 & \mu_2 & \mu_1 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \mu_i \in \mathbb{R} \ (i = 1, 2, 3), \text{ and the real parts}
\]
   of the eigenvalues of matrix \[
\begin{pmatrix} 0 & 1 & 0 \\ \mu_3 & \mu_2 & \mu_1 \end{pmatrix}
\] are not zero;

2. “3 = 0 + 2 1”: \[
\begin{pmatrix} 0 & 1 & 0 \\ \mu_2 & \mu_1 & 0 \\ 0 & 0 & \iota \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \mu_i \in \mathbb{R} \ (i = 1, 2), \iota = 1 \text{ or } -1, \text{ and the}
\]
   real parts of the eigenvalues of matrix \[
\begin{pmatrix} 0 & 1 \\ \mu_2 & \mu_1 \end{pmatrix}
\] are not zero;

3. “3 = 0 + 1 2”: \[
\begin{pmatrix} \mu & 0 & 0 \\ 0 & \iota_1 & 0 \\ 0 & 0 & \iota_2 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \mu \neq 0, \iota_i = 1 \text{ or } -1 \ (i = 1, 2), \iota_1 \geq \iota_2;
\]

4. “3 = 0 + 0 3”: \[
\begin{pmatrix} \iota_1 & 0 & 0 \\ 0 & \iota_2 & 0 \\ 0 & 0 & \iota_3 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \iota_i = 1 \text{ or } -1 \ (i = 1, 2, 3), \iota_1 \geq \iota_2 \geq \iota_3.
\]

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