Irreducible pseudo 2–factor isomorphic cubic bipartite graphs

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Abstract
A bipartite graph is pseudo 2–factor isomorphic if all its 2-factors have the same parity of number of circuits. In [2] we proved that the only essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph of girth 4 is $K_{3,3}$, and conjectured [2, Conjecture 3.6] that the only essentially 4-edge-connected cubic bipartite graphs are $K_{3,3}$, the Heawood graph and the Pappus graph.

There exists a characterization of symmetric configurations $n_3$ due to Martinetti (1886) in which all symmetric configurations $n_3$ can be obtained from an infinite set of so called irreducible configurations [10]. The list of irreducible configurations has been completed by Boben [4] in terms of their irreducible Levi graphs.

In this paper we characterize irreducible pseudo 2–factor isomorphic cubic bipartite graphs proving that the only pseudo 2-factor isomorphic irreducible Levi graphs are the Heawood and Pappus graphs. Moreover, the obtained characterization allows us to partially prove the above Conjecture.
1 Introduction

All graphs considered are finite and simple (without loops or multiple edges). A graph with a 2–factor is said to be 2–factor hamiltonian if all its 2–factors are Hamilton circuits, and, more generally, 2–factor isomorphic if all its 2–factors are isomorphic. Examples of such graphs are $K_4$, $K_5$, $K_{3,3}$, the Heawood graph (which are all 2–factor hamiltonian) and the Petersen graph (which is 2–factor isomorphic). Several recent papers have addressed the problem of characterizing families of graphs (particularly regular graphs) which have these properties. It is shown in [3, 7] that $k$–regular 2–factor isomorphic bipartite graphs exist only when $k \in \{2, 3\}$ and an infinite family of 3–regular 2–factor hamiltonian bipartite graphs, based on $K_{3,3}$ and the Heawood graph, is constructed in [7]. It is conjectured in [7] that every 3–regular 2–factor hamiltonian bipartite graph belongs to this family. Faudree, Gould and Jacobsen in [6] determine the maximum number of edges in both 2–factor hamiltonian graphs and 2–factor hamiltonian bipartite graphs. In addition, Diwan [5] has shown that $K_4$ is the only 3–regular 2–factor hamiltonian planar graph.

Moreover, 2–factor isomorphic bipartite graphs are extended in [2] to the more general family of pseudo 2–factor isomorphic graphs i.e. graphs $G$ with the property that the parity of the number of circuits in a 2–factor is the same for all 2–factors of $G$. Example of these graphs are $K_{3,3}$, the Heawood graph $H_0$ and the Pappus graph $P_0$. Finally, it is proved in [2] that pseudo 2–factor isomorphic $2k$–regular graphs and $k$–regular digraphs do not exist for $k \geq 4$. Recently these results has been generalized in [1] for regular graphs which are not necessarily bipartite.

An incidence structure is linear if two different points are incident with at most one line. A symmetric configuration $n_k$ (or $n_k$ configuration) is a linear incidence structure consisting of $n$ points and $n$ lines such that each point and line is respectively incident with $k$ lines and points. Let $C$ be a symmetric configuration $n_k$, its Levi graph $G(C)$ is a $k$–regular bipartite graph whose vertex set are the points and the lines of $C$ and there is an edge between a point and a line in the graph if and only if they are incident in $C$. We will indistinctly refer to Levi graphs of configurations as their incidence graphs.

Let $G$ be a graph and $E_1$ be an edge-cut of $G$. We say that $E_1$ is a non-trivial edge-cut if all components of $G – E_1$ have at least two vertices. The graph $G$ is essentially 4–edge–connected if $G$ is 3–edge–connected and has no non-trivial 3–edge–cuts.

Conjecture 1.1 [2] Let $G$ be an essentially 4–edge–connected pseudo 2–factor isomorphic cubic bipartite graph. Then $G \in \{K_{3,3}, H_0, P_0\}$. 
Theorem 1.2 [2] Let $G$ be an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph. Suppose that $G$ contains a 4-circuit. Then $G = K_{3,3}$.

It follows from Theorem 1.2 that an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph of girth greater than or equal to 6 is the Levi graph of a symmetric configuration $n_3$. In 1886 V. Martinetti [10] characterized symmetric configurations $n_3$, showing that they can be obtained from an infinite set of so called irreducible configurations, of which he gave a list. Recently, Boben proved that Martinetti’s list of irreducible configurations was incomplete and completed it [4]. Boben’s list of irreducible configurations was obtained characterizing their Levi graphs, which he called irreducible Levi graphs (cf. Section 2).

In this paper, we characterize irreducible pseudo 2-factor isomorphic cubic bipartite graphs proving that the Heawood and the Pappus graphs are the only irreducible Levi graphs which are pseudo 2-factor isomorphic cubic bipartite. Moreover, the obtained characterization allows us to partially prove Conjecture 1.1, i.e. in the case of irreducible pseudo 2-factor isomorphic cubic bipartite graphs.

2 Symmetric Configurations $n_3$

In 1886, Martinetti [10] provided a construction for a symmetric configuration $n_3$ from a symmetric configuration $(n - 1)_3$, say $C$. Suppose that in $C$ there exist two parallel (non-intersecting) lines $l_1 = \{\alpha, \alpha_1, \alpha_2\}$ and $r_1 = \{\beta, \beta_1, \beta_2\}$ such that the points $\alpha$ and $\beta$ are not on a common line. Then a symmetric configuration $n_3$ is obtained from $C$ by deleting the lines $l_1, r_1$, adding a point $\mu$ and adding the lines $h_1 = \{\mu, \alpha_1, \alpha_2\}$, $h_2 = \{\mu, \beta_1, \beta_2\}$ and $h_3 = \{\mu, \alpha, \beta\}$. Not all symmetric configurations $n_3$ can be obtained using this method on some symmetric configuration $(n - 1)_3$. The configurations that cannot be obtained in this way are called irreducible configurations, while the others are reducible configurations. However, if all irreducible symmetric configurations $n_3$ are known, then all symmetric configurations $n_3$ can be constructed iteratively with Martinetti’s method. The list of irreducible configurations in [10] turned out to be incomplete and it has been recently completed by Boben in [4 Thm. 8].

Theorem 2.1 [4] All connected irreducible $n_3$ configurations are:

1. cyclic configurations with base line $\{0, 1, 3\}$;
2. the configurations with their incidence graphs $T_1(n)$, $T_2(n)$, $T_3(n)$, $n \geq 1$, each of them giving precisely one $(10n)_3$ configuration, and;
3. the Pappus configuration.

As mentioned before, Boben’s list was obtained by studying the Levi graphs of irreducible configurations, which are called irreducible Levi graphs. Such graphs turned out to be either the Pappus graph, or belong to one of four infinite families $D(n)$, $T_1(n)$, $T_2(n)$, $T_3(n)$, $n \geq 1$, which we now proceed to describe.

The $D(n)$ family: Let $C(n)$, $n \geq 1$, be the graph on $6n$ vertices, consisting of $n$ segments (6–circuits labeled as in Fig. 1), linked by the edges $v^1_{i-1}u^1_i$, $v^2_{i-1}u^4_i$, and $u^3_{i-1}u^2_i$, for $i \geq 2$.

![Figure 1: label for the 6–circuits of the definition of $C(n)$](image)

Let the graph $D(n)$, $n \geq 1$ be defined as follows:

For $n \equiv 0 \pmod{3}$ let $D(n)$ be the graph $C(m)$, $m = n/3$, with the edges $u^1_mv^1_m$, $u^4_mv^2_m$, and $u^3_mw^1_m$ added (cf. Fig. 2).

For $n \equiv 1 \pmod{3}$ let $D(n)$ be the graph $C(m)$, $m = (n - 1)/3$, with two vertices $w^1_m$, $w^2_m$ and the edges $u^1_mw^1_m$, $u^2_mw^2_m$, $u^3_mw^3_m$, $w^1_mu^3_m$, $w^2_mv^1_m$ added.

For $n \equiv 2 \pmod{3}$ let $D(n)$ be the graph $C(m)$, $m = (n - 2)/3$, with four vertices $w^1_m$, $w^2_m$, $w^3_m$, $w^4_m$ and the edges $v^1_mw^1_m$, $v^2_mw^2_m$, $v^3_mw^3_m$, $u^1_mw^1_m$, $u^4_mw^4_m$, $w^1_mu^4_m$, $u^1_mw^1_m$, $u^3_mw^3_m$, $w^1_mw^3_m$, $w^2_mw^3_m$, $w^3_mw^4_m$ added.

![Figure 2: The graph $D(9)$](image)
A cyclic configuration has \( Z_n = \{0, 1, \ldots, n - 1\} \) as set of points and \( \mathcal{B} = \{\{0, b, c\}, \{1, b + 1, c + 1\}, \ldots, \{n - 1, b + n - 1, c + n - 1\}\} \) as set of lines, where the operations are modulo \( n \), and the base line is \( \{0, b, c\} \) for \( b, c \in Z_n \).

Note that the graphs \( D(n) \) are the Levi graphs of the cyclic \( n_3 \) configurations with base line \( \{0, 1, 3\} \). In particular, for \( n = 7 \) the cyclic \( 7_3 \) configuration is the Fano plane and \( D(7) \) is the Heawood graph \( H_0 \).

The \( T_1(n) \), \( T_2(n) \) and \( T_3(n) \) families: Let \( T(n) \), \( n \geq 1 \), be the graph on \( 20n \) vertices consisting of \( n \) segments \( G_T \) shown in Fig. 3, linked by the edges \( v_{i-1}^1u_i^1, v_{i-1}^2u_i^2, v_{i-1}^3u_i^3 \), for \( i \geq 2 \).

Let \( T_1(n) \) be the graph obtained from \( T(n) \) by adding the edges \( u_1^1v_n^1, u_1^2v_n^2, u_1^3v_n^3 \). Let \( T_2(n) \) be the graph obtained from \( T(n) \) by adding the edges \( u_1^1v_n^1, u_1^2v_n^2, u_1^3v_n^3 \). Let \( T_3(n) \) be the graph obtained from \( T(n) \) by adding the edges \( u_1^1v_n^3, u_1^2v_n^1, u_1^3v_n^2 \). In \([4]\), Boben proved that for each fixed value of \( n \), no two of the graphs \( T_1(n), T_2(n), T_3(n) \) are isomorphic.

Note that \( T_1(1) \) is the Levi graph of Desargues’ configuration, and \( T_2(1), T_3(1) \) correspond to the Levi graphs of the configurations \( 10_3F \) and \( 10_3G \) respectively according to Kantor’s [9] notation for the ten \( 10_3 \) configurations.

The Pappus graph: Recall that the Levi graph of the Pappus \( 9_3 \) configuration is the following pseudo 2–factor isomorphic but not 2–factor isomorphic cubic bipartite graph \([2]\), called the Pappus graph \( P_0 \).
3 2–factors of Irreducible Levi Graphs

Let G be a graph and u, v be two vertices in G. Then a (u, v)–path is a path from u to v. Given two disjoint paths P = u_1, . . . , u_n and Q = u_{n+1}, . . . , u_{n+m} (except maybe for u_1 = u_{n+m}), the path PQ = u_1, . . . , u_{n+m} is the concatenation of P and Q together with the edge u_nu_{n+1}. Similarly, for a vertex v ∈ (G − P) ∪ {v_1}, the path Pv is composed by P, v and the edge u_nv If u_1 = u_{n+m} or u_1 = v we write (PQ) and (Pv) respectively, to emphasize that PQ and Pv are circuits.

**Theorem 3.1** The Heawood and the Pappus graphs are the only irreducible Levi graphs which are pseudo 2–factor isomorphic.

**Proof.** It is straightforward to show that the Heawood graph H_0 is 2–factor hamiltonian and hence pseudo 2-factor isomorphic (cf. [7]). We have already proved in [2, Proposition 3.3] that the Pappus graph is pseudo 2-factor isomorphic. We need to prove that all other irreducible Levi graphs are not pseudo 2–factor isomorphic and we will do so by finding two 2–factors with different parity of number of circuits in each of them.

The following paths will be used for constructing 2–factors in D(n), for n ≥ 8.

\[
L_1^i = u_1^2 u_2^3 u_3^4 v_i^1 \\
M_1 = u_1^2 u_2^4 u_3^2 u_4^3 v_1^2 \\
N_1 = u_1^2 u_2^1 v_1^1 - u_2^1 u_3^3 \\
C_1 = (u_1^2 u_2^3 u_3^2 u_4^1 v_1^1) \\
C_2 = (u_1^2 u_2^1 u_3^4 v_1^1) \\
C_3 = (v_1^2 u_2^4 u_3^2 u_4^1 v_1^1)
\]

Hamiltonian 2–factors in D(n) are

\[
\begin{cases}
(L_1 L_2 \cdots L_m u_1) & n \equiv 0 \mod 3 \\
(L_1 L_2 \cdots L_m w_1 u_1) & n \equiv 1 \mod 3 \\
(L_1 L_2 \cdots L_m w_1 w_2 u_1) & n \equiv 2 \mod 3
\end{cases}
\]

Disconnected 2–factors with exactly two circuits in D(n) are

\[
\begin{cases}
C_1 \cup (M_1 u_1^1) & n = 9 \\
C_1 \cup (M_1 L_2 \cdots L_m u_1^1) & n = 3m, m \geq 4 & \text{for } n \equiv 0 \mod 3 \\
C_2 \cup C_3 & n = 10 \\
C_1 \cup (M_1 M_2) & n = 13 & \text{for } n \equiv 1 \mod 3 \\
C_1 \cup (M_1 L_2 \cdots L_m^2 M_2) & n = 3m + 1, m \geq 5 \\
C_1 \cup (N_m w_m^2) & n = 8 & \text{for } n \equiv 2 \mod 3 \\
C_1 \cup (N_3 \cdots N_m u_3^1) & n = 3m + 2, m \geq 3
\end{cases}
\]
Now we need to find such pairs of 2-factors for the graphs $T_1(n)$, $T_2(n)$ and $T_1(n)$, $n \geq 1$. To this purpose we need to consider the following six paths in the segment graph $G_T$ from Fig 3.

$$P_i^1 = u_i^1, u_i^3, x_i^1, y_i^1, z_i^1, v_i^1$$
$$P_i^2 = u_i^1, u_i^2, w_i^1, x_i^1, y_i^1, z_i^1, v_i^1$$
$$(P_i^3) = (v_i^1, x_i^2, t_i^1, y_i^1, z_i^1, v_i^1)$$
$$Q_i^1 = u_i^1, u_i^2, u_i^1, u_i^3, x_i^2, y_i^1, z_i^1, v_i^2$$
$$Q_i^2 = u_i^1, u_i^2, u_i^1, u_i^3, x_i^1, y_i^1, z_i^1, v_i^1$$

The paths $P_i^1$ and $Q_i^1$ are hamiltonian $(u_i^2, v_i^2)$- and $(u_i^3, v_i^2)$-paths, respectively. The paths $P_i^2$ and $Q_i^2$ are $(u_i^2, v_i^2)$- and $(u_i^3, v_i^2)$-paths on 10 vertices, respectively. Finally, $(P_i^3)$ is a 10-circuit in $G_T - P_i^2 = G_T - Q_i^2$.

In $T_1(n)$ and $T_2(n)$ the hamiltonian 2-factor $F_1(n) = (P_i^1P_i^2 \cdots P_i^1u_i^2)$ and the disconnected 2-factor with exactly two circuits $F_2(n) = (P_i^2P_i^2 \cdots P_i^1u_i^2) \cup (P_i^3)$ (even) show that these graphs are not pseudo 2-factor isomorphic.

Similarly, in $T_3(n)$ the hamiltonian 2-factor $F'_1(1) = (Q_i^1P_i^1 \cdots P_i^1u_i^3)$ and the disconnected 2-factor with exactly two circuits $F'_2(1) = (Q_i^2P_i^2 \cdots P_i^1u_i^3) \cup (P_i^3)$ show that these graphs are not pseudo 2-factor isomorphic.

Note that Theorem 3.1 proves Conjecture 1.1 in the case of irreducible pseudo 2-factor isomorphic cubic bipartite graphs. In the next Section we show that Conjecture 1.1 cannot be directly proved from Theorem 3.1 by extending it to reducible Levi graphs.

## 4 2-factors in extensions and reductions of Levi graphs of $n_3$ configurations

Recall that a Martinetti extension can be described in terms of graphs as follows:

Let $G_1$ be the Levi graph of a symmetric configuration $n_3$ and suppose that in $G_1$ there are two edges $e_1 = x_1y_1$ and $e_2 = x_2y_2$ with no common neighbours, then
the graph $G := G_1 - \{e_1, e_2\} + \{u, v\} + \{ux_1, ux_2, vy_1, vy_2\}$ is the Levi graph of an $(n + 1)_3$ configuration.

Similarly the Levi graph $G$ of a symmetric configuration $(n+1)_3$ is Martinetti reducible if there is an edge $e = uv$ in $G$ such that either $G := G_1 - \{u, v\} + x_1 y_1 + x_2 y_2$ or $G := G_1 - \{u, v\} + x_1 y_2 + x_2 y_1$ is again the Levi graph of a symmetric configuration $n_3$, where $x_1, x_2, y_1, y_2$ are the neighbours of $u$ and $v$ as in the following figure:

![Figure 4: Martinetti Extension](image)

![Figure 5: Martinetti Reduction](image)

It is well known that the $7_3$ configuration, whose Levi graph is the Heawood graph, is not Martinetti extendible and that the Pappus configuration is Martinetti extendible in a unique way; it is easy to show that this extension is not pseudo 2–factor isomorphic.

Let $C$ be a symmetric configuration $n_3$ and $C'$ be a symmetric configuration $(n + 1)_3$ obtained from $C$ through a Martinetti extension. It can be easily checked that there are 2–factors in $C'$ that cannot be reduced to a 2–factor in $C$. For example if $C$ corresponds to the first option in Fig. 5 a 2–factor of $C'$ containing the path $x_1 uv y_2$ will not reduce to a 2–factor in $C$. Conversely, there might be 2–factors of $C$ for which the parity of number of circuits is not preserved when extended to a 2–factor in $C'$. For example, the graph $H_0 * H_0$ (the star product [8, p. 90] of the Heawood graph with itself) which is 2–factor hamiltonian and Martinetti reducible (only through the edges of the non–trival 3–edge–cut), has all Martinetti reductions which are no longer pseudo 2–factor isomorphic. Hence, we cannot directly prove Conjecture 1.1 by studying the 2–factors of reducible configurations from the set of 2–factors of their underlying irreducible ones.

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