Dualities in Field Theories and the Role of K-Theory

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Summary. It is now known (or in some cases just believed) that many quantum field theories exhibit dualities, equivalences with the same or a different theory in which things appear very different, but the overall physical implications are the same. We will discuss some of these dualities from the point of view of a mathematician, focusing on “charge conservation” and the role played by K-theory and noncommutative geometry. Some of the work described here is joint with Mathai Varghese and Stefan Mendez-Diez; the last section is new.

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1 Overview with Some Classical Examples

1.1 Structure of Physical Theories

Most physical theories describe fields, e.g., the gravitational field, electric field, magnetic field, etc. Fields can be

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scalar-valued functions (scalars),
- sections of vector bundles (vectors),
- connections on principal bundles (special cases of gauge fields),
- sections of spinor bundles (spinors).

In classical physics, the fields satisfy a variational principle — they are critical points of the action $S$, which in turn is the integral of a local functional $\mathcal{L}$ called the Lagrangian. This is called the principle of least action, and can be traced back to Fermat’s theory of optics in 1662. The Euler-Lagrange equations for critical points of the action are the equations of motion.

Example 1 (Yang-Mills Theory). Let $M$ be a 4-dimensional Riemannian or Lorentzian manifold, say compact. We fix a compact Lie group $G$ and a principal $G$-bundle over $M$. A Yang-Mills field is a connection $A$ on this bundle. The “field strength” $F$ is the curvature, a $g$-valued 2-form. The action is $S = \int_M \text{Tr} F \wedge \ast F$ (up to a constant involving the coupling constant $g_{YM}$ measuring the strength of the interactions). Note that the metric on $M$ is needed to define the Hodge $\ast$-operator $F \mapsto \ast F$.

Example 2 (General Relativity in Empty Space). For convenience, we consider the “Wick rotation” of the theory to Euclidean signature. Let $M$ be a 4-manifold, say compact. A field is a Riemannian metric $g$ on $M$. The (Einstein-Hilbert) action is $S = \int_M R \, d\text{vol}$, $R =$ scalar curvature. (Strictly speaking one should insert a coupling constant in front, $\frac{c^4}{16\pi G}$, where $G$ is Newton’s gravitational constant and $c$ is the speed of light, which we usually set equal to 1 in suitable units.) The associated field equation is Einstein’s equation.

Unlike classical mechanics, quantum mechanics is not deterministic, only probabilistic. The key property of quantum mechanics is the Heisenberg uncertainty principle, that observable quantities are represented by noncommuting operators $A$ represented on a Hilbert space $\mathcal{H}$. In the quantum world, every particle has a wave-like aspect to it, and is represented by a wave function $\psi$, a unit vector in $\mathcal{H}$. The phase of $\psi$ is not directly observable, only its amplitude, or more precisely, the state $\varphi_\psi$ defined by $\psi$:

$$\varphi_\psi(A) = \langle A\psi, \psi \rangle.$$

But the phase is still important since interference depends on it.

The quantization of classical field theories is based on path integrals. The idea (not 100% rigorous in this formulation) is that all fields contribute, not just those that are critical points of the action (i.e., solutions of the classical field equations). Instead, one looks at the partition function

$$Z = \int e^{iS(\varphi)/\hbar} \, d\varphi \text{ or } \int e^{-S(\varphi)/\hbar} \, d\varphi,$$

depending on whether one is working in Lorentz or Euclidean signature. By the principle of stationary phase, only fields close to the classical solutions
should contribute very much. *Expectation values* of physical quantities are given by

\[ \langle A \rangle = \left( \int A(\varphi) e^{iS(\varphi)/\hbar} \, d\varphi \right)/Z. \]

### 1.2 Dualities

A *duality* is a transformation between different-looking physical theories that, rather magically, have the same observable physics. Often, such dualities are part of a discrete group, such as $\mathbb{Z}/2$ or $\mathbb{Z}/4$ or $SL(2, \mathbb{Z})$.

**Example 3 (Electric-magnetic duality).** Let $E$ and $B$ be the electric and magnetic fields, respectively. There is a symmetry of Maxwell’s equations in free space

\[ \begin{align*}
\nabla \cdot E &= 0, \\
\nabla \cdot B &= 0, \\
\frac{\partial E}{\partial t} &= c \nabla \times B, \\
\frac{\partial B}{\partial t} &= -c \nabla \times E,
\end{align*} \tag{1} \]

given by $E \mapsto -B$, $B \mapsto E$. This is a duality of order 4.

**Example 4 (Configuration space-momentum space duality).** Another example from standard quantum mechanics concerns the quantum harmonic oscillator (say in one dimension). For an object with mass $m$ and a restoring force with “spring constant” $k$, the Hamiltonian is

\[ H = \frac{k}{2} x^2 + \frac{1}{2m} p^2, \tag{2} \]

where $p$ is the momentum. In classical mechanics, $p = m\dot{x}$. But in quantum mechanics (with $\hbar$ set to 1),

\[ [x, p] = i. \tag{3} \]

We obtain a duality of (2) and (3) via $m \mapsto \frac{1}{k}$, $k \mapsto \frac{1}{m}$, $x \mapsto p$, $p \mapsto -x$. This is again a duality of order 4, and is closely related to the *Fourier transform*.

- A big puzzle in classical electricity and magnetism is that while there are plenty of charged particles (electrons, etc.), no magnetically charged particles (*magnetic monopoles*) have ever been observed, even though their existence would not contradict Maxwell’s equations.
- Another problem with classical electricity and magnetism is that it doesn’t explain why charges appear to be quantized, i.e., only occur in units that are integral multiples of the charge of the electron (or of the charges of [down-type] quarks).

Dirac [21] proposed to solve both problems at once with a *quantum* theory of electricity and magnetism that in modern terms we would call a *$U(1)$ gauge theory.*

In Dirac’s theory, we assume spacetime is a 4-manifold $M$, say $\mathbb{R}^4 \setminus \mathbb{R} \cong \mathbb{R}^2 \times S^2$ (Minkowski space with the time trajectory of one particle taken out).
The (magnetic) vector potential \((A^1, A^2, A^3)\) and electric potential \(A^0 = \phi\) of classical electricity and magnetism are combined into a single entity \(A\), a (unitary) connection on a complex line bundle \(L\) over \(M\). Thus \(iA\) is locally a real-valued 1-form, and \(F = i\mu dA\), \(\mu\) a constant, is a 2-form encoding both of the fields \(E\) (via the \((0,j)\) components) and \(B\) (via the \((j,k)\) components, \(0 < j < k\)). The Chern class \(c_1(L) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}\) is an invariant of the topology of the situation. Of course, \(F\) should really be \(i\mu\) times the curvature of \(A\), and Chern-Weil theory says that the de Rham class \([F]\) is \(2\pi\mu\) times the image of \(c_1(L)\) in \(H^2(M, \mathbb{Z}) \cong \mathbb{Z}\). \(L\) is associated to a principal \(U(1)\)-bundle \(P \rightarrow M\), and Dirac identifies a section of this bundle with the phase of a wave function of a charged particle in \(M\).

In the above setup, if we integrate \(F\) over the \(S^2\) that links the worldline we removed, we get \(2\pi\mu c_1(L)\), and this is the flux of the magnetic field through \(S^2\). So the deleted worldline can be identified with that of a magnetic monopole of charge \(g = \mu c_1(L)\) in suitable units. Suppose we consider the motion of a test charge of electric charge \(q\) around a closed loop \(\gamma\) in \(M\). In quantum electricity and magnetism, by the Aharonov-Bohm effect \([2]\), the exterior derivative is replaced by the covariant derivative (involving the vector potential \(A\)). So the phase change in the wave function is basically the holonomy of \((P \rightarrow M, A)\) around \(\gamma\), or (taking \(\hbar = 1\)) \(\exp\left(q\mu \oint_{\gamma} A\right)\). Since \(M\) is simply connected, \(\gamma\) bounds a disk \(D\) and the integral is (by Stokes’ Theorem) \(\exp\left(-i\int_D F\right)\). Taking \(D\) in turn to be the two hemispheres in \(S^2\), we get two answers which differ by a factor of

\[
\exp\left(iq \int_{S^2} F\right) = e^{2\pi i q \mu c_1(L)}.
\]

Since this must be 1, we get Dirac’s quantization condition \(qg \in \mathbb{Z}\).

The upshot of this analysis is that we expect both electrical and magnetic charges to be quantized, but that the basic quanta of electrical and magnetic charge should be inversely proportional in size. In other words, the smallness of the fundamental electrical charge means that the charge of any magnetic monopole has to be large. In any event, we expect the electrical and magnetic charges \((q, g)\) to take values in an abelian charge group \(C\), in this case \(\mathbb{Z}^2\). It is also reasonable to expect there to be particles, usually called dyons, with both charges \(g\) and \(g\) non-zero.

Now think about the classical electric-magnetic duality (Example 3) that switches \(E\) and \(B\). The Montonen-Olive Conjecture \([36]\), for which there is now some tantalizing evidence, is that in a wide variety of cases this should extend to a duality of quantum theories, which would necessarily give an isomorphism of charge groups between a theory and its dual.

In Dirac’s theory, the quantization of magnetic charge and of electrical charge arise from different origins. The former is a purely topological phenomenon; it comes from the fact that the Chern classes live in integral cohomology. Quantization of electrical charge comes from the requirement that
the action (for the field associated to a charged particle moving in the background electromagnetic field of a monopole) be well-defined and not multi-valued, so this can be viewed as a version of anomaly cancellation. However, since Maxwell’s equations are invariant under electro-magnetic duality, we can imagine an equivalent dual theory in which electric charge is topological and magnetic charge is quantized to achieve anomaly cancellation.

1.3 A General Framework and the Role of $K$-Theory

Extrapolating from case above, we will be looking at the following set-up:

1. We have a collection $\mathcal{C}$ of “physical theories” on which a discrete duality group $G$ operates by “equivalences.” (More generally, $G$ might be replaced by a groupoid.)

2. Each theory in $\mathcal{C}$ has an associated charge group $C$. If $g \in G$ gives an equivalence between two theories in $\mathcal{C}$, it must give an isomorphism between the associated charge groups. In particular, the stabilizer of a fixed theory operates by automorphisms on $C$.

3. In many cases, the charge groups arise as topological invariants. We have already seen how $\text{Pic}(X) = H^2(X, \mathbb{Z})$ arises. (The notation $\text{Pic}(X)$ denotes the set of isomorphism classes of complex line bundles over $X$, which is a group under tensor product.) We will see how $K$-theory arises in some cases.

Many of the most interesting examples of duality (and of topological charge groups) arise in (supersymmetric) string theories. These are quantum field theories based on the idea of replacing point particles by strings or 1-manifolds (always compact, but maybe with boundary — contrary to mathematical usage, physicists call these “open strings”). For anomaly cancellation reasons, the spacetime manifold has to be 10-dimensional. The worldsheet traced out by a string in the spacetime $X$ is a compact 2-manifold $\Sigma$ (again, possibly with boundary), so we obtain fields that are maps $f : \Sigma \to X$, with the sigma-model action of the form

$$\int_{\Sigma} \|\nabla f\|^2 + \int_{\Sigma} f^*(B) + \text{(terms involving other fields)}. \quad (4)$$

Here $B$ is a 2-form on $X$ called the $B$-field (not the magnetic field). The term $\int_{\Sigma} f^*(B)$ is called the Wess-Zumino term. The terms involving the “other fields” depend on which of the five superstring theories (type I, which allows unoriented strings, types IIA and IIB, and the two types of heterotic theories) one is dealing with. They differ with regard to such issues as chirality and orientation conditions, and whether or not open strings are allowed.

In string theories, boundary conditions (of Dirichlet or Neumann type) must be imposed on the open string states. These are given by $D$-branes ($D$ for “Dirichlet”), submanifolds of the spacetime $X$ on which strings are allowed.
to “end.” If we forget certain complications and look at type II string theory, then $X$ is a 10-dimensional spin manifold and the stable D-branes are spin$^c$ submanifolds, of even dimension for type IIB and of odd dimensional for type IIA. (At least in the absence of twisting, $X$ is generally $\mathbb{R}^4$ times a Calabi-Yau 3-fold, and in the type IIB case, the stable D-branes are complex submanifolds, whereas in the IIA case, they are typically isotropic submanifolds for the symplectic structure.)

There is another piece of structure; each D-brane carries a Chan-Paton vector bundle that reflects a $U(N)$ gauge symmetry allowing for local exchanges between coincident D-branes.

The D-branes carry charges which are not just numbers but elements of the $K$-group $K(X)$ (in the type IIB theory), $K^{-1}(X)$ (in the type IIA theory), or $KO(X)$ (in the type I theory).

The idea that the D-brane charges should take values in $K$-theory comes from Minasian-Moore [35] and Witten [46], around 1997–1998, with further elaboration by other authors later. Motivation comes from several sources:

- compatibility with anomaly cancellation formulas;
- better functoriality;
- compatibility with analysis of decay of unstable branes;
- compatibility with what is known about string duality.

We will not attempt to go through these arguments (which the reader can find in [46, 47, 26] and [7, §6.2]) but will discuss some consequences.

For a D-brane $W \to X$ with Chan-Paton bundle $E \to W$, the $K$-theory charge is $\iota([E])$, where $[E]$ is the class of $E$ in $K(W)$, and $\iota$ is the Gysin map in $K$-theory (defined using the spin$^c$ structures). While string dualities do not have to preserve the diffeomorphism type, or even the dimension, of D-branes, they do have to give rise to an isomorphism of the $K$-groups in which the D-brane charges lie.

The most important kinds of string theory dualities are $T$-duality, an outgrowth of classical Fourier duality (“T” originally standing for “target space”), and $S$-duality, an outgrowth of classical electro-magnetic duality. The big difference between them is that $T$-duality preserves coupling strength and changes geometry, whereas $S$-duality (“S” standing for “strong-weak”) interchanges strong and weak coupling and preserves the geometry of spacetime, just as electro-magnetic duality inverts the magnitude of charges.

Much of the interest of these dualities is that they are non-perturbative, in other words, don’t depend on perturbation expansions. In some cases, a quantity which is difficult to compute in one theory can be computed by passage to a dual theory in which the quantity is easier to compute directly, or can be computed via a perturbation expansion.

$T$-duality replaces tori (of a fixed dimension $k$) in the spacetime manifold $X$ by their dual tori (quotients of the dual space by the dual lattice) in the dual spacetime $X^\sharp$, inverting the radii. If $k$ is odd, $T$-duality interchanges the theories of types IIA and IIB, so one gets an isomorphism $K(X) \cong K^{-1}(X^\sharp)$.
or $K^{-1}(X) \cong K(X^2)$. S-duality interchanges type I string theory with the $SO(32)$ heterotic string theory, and also maps type IIB string theory to itself.

In Sections 2 and 3 I will discuss T-duality and S-duality in more detail, and the way charge conservation in $K$-theory sheds more light on them.

2 Topological T-Duality

2.1 The H-flux and Twisted $K$-Theory

It's now time to correct a slight oversimplification in Section 1: the “B-field” in the sigma-model action (4) is not necessarily globally well-defined, though its field strength $H = dB$ does make sense globally. Properly normalized, one can show that $H$ defines an integral de Rham class in $H^3$. This can be refined to an actual class in $[H] \in H^3(X, \mathbb{Z})$. Thus the Wess-Zumino term in the path integral should really be defined using a gerbe, for example a bundle gerbe in the sense of Murray [37] with curving $B$ and Dixmier-Douady class $[H]$. We usually refer to $H$ (or to the associated class $[H] \in H^3(X, \mathbb{Z})$) as the H-flux. (For an exposition of how gerbes can be used to make sense of the Wess-Zumino term, see for example [25] or [40, §4.3].)

The association of $H$ with a Dixmier-Douady class is not an accident, and indeed indicates a deeper connection with noncommutative geometry. To set this up in the simplest way, choose a stable continuous-trace algebra $A = CT(X, [H])$ with $\hat{A} = X$ and with Dixmier-Douady class $[H]$. Thus $A$ is the algebra of continuous sections vanishing at $\infty$ of a bundle over $X$ with fibers $K$ (the compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$) and structure group $\text{Aut} K = PU(\mathcal{H}) \simeq K(\mathbb{Z}, 2)$.

There are several possible definitions of twisted $K$-theory (see [22, 29, 5, 6, 20]), but for our purposes we can define it as $K^{-i}(M, [H]) = K_i(A)$ with $A = CT(M, [H])$ as above. Up to isomorphism, this only depends on $X$ and the cohomology class $[H] \in H^3(X, \mathbb{Z})$.

In the presence of a topologically nontrivial H-flux, the $K$-theoretic classification of D-brane charges has to be modified. A D-brane $W \hookrightarrow X$ in type II string theory is no longer a Spin$^c$ manifold; instead it is Spin$^c$ “up to a twist,” according to the Freed-Witten anomaly cancellation condition [24] $W_3(W) = \nu^*([H])$. Here $W_3$ is the canonical integral lift of the third Stiefel-Whitney class, which is the obstruction to a Spin$^c$ structure. Accordingly, the D-brane charge will live in the twisted $K$-group $K(X, [H])$ (in type IIB) or in $K^{-1}(X, [H])$ (in type IIA). Accordingly, if we have a T-duality between string theories on $(X, H)$ and $(X^2, H^2)$, conservation of charge (for D-branes) requires an isomorphism of twisted $K$-groups of $(X, [H])$ and $(X^2, [H^2])$, with no degree shift if we dualize with respect to even-degree tori, and with a degree shift if we dualize with respect to odd-degree tori.

One might wonder what happened to the $K$-groups of opposite parity, viz., $K^{-1}(X, [H])$ (in type IIB) and $K(X, [H])$ (in type IIA). These still have
a physical significance in terms of Ramond-Ramond fields \[17\], so want these to match up under T-duality also.

## 2.2 Topological T-Duality and the Bunke-Schick Construction

**Topological T-duality** focuses on the topological aspects of T-duality. The first example of this phenomenon was studied by Alvarez, Alvarez-Gaumé, Barbón, and Lozano in 1993 [4], and generalized 10 years later by Bouwknegt, Evslin, and Mathai [11, 12]. Let’s start with the simplest nontrivial example of a circle fibration, where \( X = S^3 \), identified with \( SU(2) \). Then \( T \) is a maximal torus. Then \( T \) acts freely on \( X \) (say by right translation) and the quotient \( X/T \) is \( \mathbb{C}P^1 \cong S^2 \), with quotient map \( p \): \( X \to S^2 \) the Hopf fibration. Assume for simplicity that the \( B \)-field vanishes. We have \( X = S^3 \) fibered over \( Z = X/T = S^2 \).

Think of \( Z \) as the union of the two hemispheres \( Z^\pm \cong D^2 \) intersecting in the equator \( Z^0 \cong S^1 \). The fibration is trivial over each hemisphere, so we have \( p^{-1}(Z^\pm) \cong D^2 \times S^1 \), with \( p^{-1}(Z^0) \cong S^1 \times S^1 \). So the T-dual also looks like the union of two copies of \( D^2 \times S^1 \), joined along \( S^1 \times S^1 \).

However, we have to be careful about the *clutching* that identifies the two copies of \( S^1 \times S^1 \). In the original Hopf fibration, the clutching function \( S^1 \to S^1 \) winds once around, with the result that the fundamental group \( \mathbb{Z} \) of the fiber \( T \) dies in the total space \( X \). But T-duality is supposed to interchange “winding” and “momentum” quantum numbers. So the T-dual \( X^\sharp \) has no winding and is just \( S^2 \times S^1 \), while the winding of the original clutching function shows up in the \( H \)-flux of the dual.

In fact, following Buscher’s method [19] for dualizing a sigma-model, we find that the \( B \)-field \( B^\sharp \) on the dual side is different on the two copies of \( D^2 \times S^1 \); they differ by a closed 2-form, and so \( H^2 = dB^\sharp \), the \( H \)-flux of the dual, is nontrivial in de Rham cohomology (for simplicity of notation we delete the brackets from now on) but well defined.

Let’s check the principle of \( K \)-theory matching in the case we’ve been considering, \( X = S^3 \) fibered by the Hopf fibration over \( Z = S^2 \). The \( H \)-flux on \( X \) is trivial, so D-brane charges lie in \( K^*(S^3) \), with no twisting. And \( K^0(S^3) \cong K^1(S^3) \cong \mathbb{Z} \).

On the T-dual side, we expect to find \( X^\sharp = S^2 \times S^1 \), also fibered over \( S^2 \), but simply by projection onto the first factor. If the \( H \)-flux on \( X \) were trivial, D-brane changes would lie in \( K^0(S^2 \times S^1) \) and \( K^1(S^2 \times S^1) \), both of which are isomorphic to \( \mathbb{Z}^2 \), which is too big.

On the other hand, we can compute \( K^*(S^2 \times S^1, H^\sharp) \) for the class \( H^\sharp \) which is \( k \) times a generator of \( H^3 \cong \mathbb{Z} \), using the Atiyah-Hirzebruch Spectral Sequence. The differential is

\[
H^0(S^2 \times S^1) \xrightarrow{k} H^3(S^2 \times S^1),
\]

so when \( k = 1 \), \( K^*(S^2 \times S^1, H^\sharp) \cong K^*(S^3) \cong \mathbb{Z} \) for both \( * = 0 \) and \( * = 1 \).
Axiomatics for $n = 1$

This discussion suggests we should try to develop an axiomatic treatment of the topological aspects of T-duality (for circle bundles). Note that we are ignoring many things, such as the underlying metric on spacetime and the auxiliary fields. Here is a first attempt.

Axioms 1.

1. We have a suitable class of spacetimes $X$ each equipped with a principal $S^1$-bundle $X \to Z$. ($X$ might be required to be a smooth connected manifold.)
2. For each $X$, we assume we are free to choose any $H$-flux $H \in H^3(X, \mathbb{Z})$.
3. There is an involution (map of period 2) $(X, H) \mapsto (X^\# , H^\#)$ keeping the base $Z$ fixed.
4. $K^*(X, H) \cong K^{*+1}(X^4, H^2)$.

The Bunke-Schick Construction

Bunke and Schick [16] suggested constructing a theory satisfying these axioms by means of a universal example. It is known that (for reasonable spaces $X$, say CW complexes) all principal $S^1$-bundles $X \to Z$ come by pull-back from a diagram

$$
\begin{array}{ccc}
X & \longrightarrow & ES^1 \simeq \ast \\
\downarrow & & \downarrow \\
Z & \longrightarrow & BS^1 \simeq K(\mathbb{Z}, 2)
\end{array}
$$

Here the map $Z \longrightarrow K(\mathbb{Z}, 2)$ is unique up to homotopy, and pulls the canonical class in $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$ back to $c_1$ of the bundle.

Similarly, every class $H \in H^3(X, \mathbb{Z})$ comes by pull-back from a canonical class via a map $X \longrightarrow K(\mathbb{Z}, 3)$ unique up to homotopy.

Theorem 2 (Bunke-Schick [16]). There is a classifying space $R$, unique up to homotopy equivalence, with a fibration

$$
\begin{array}{ccc}
K(\mathbb{Z}, 3) & \longrightarrow & R \\
\downarrow & & \downarrow \\
K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), & & \end{array}
$$

and any $(X, H) \to Z$ as in the axioms comes by a pull-back

$$
\begin{array}{ccc}
X & \longrightarrow & E \\
\downarrow & & \downarrow p \\
Z & \longrightarrow & R
\end{array}
$$
with the horizontal maps unique up to homotopy and $H$ pulled back from a canonical class $h \in \pi_3(E, \mathbb{Z})$.

**Theorem 3 (Bunke-Schick [16])**. Furthermore, the $k$-invariant of the Postnikov tower § characterizing $R$ is the cup-product in

$$H^4(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z})$$

of the two canonical classes in $H^2$. The space $E$ in the fibration

$$\begin{array}{c}
S^1 \\
\downarrow p \\
R
\end{array}
$$

has the homotopy type of $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$.

**Corollary 1.** If $(X \to Z, H)$ and $(X^\sharp \to Z, H^\sharp)$ are a T-dual pair of circle bundles over a base space $Z$, then the bundles and fluxes are related by the formula

$$p^!(H) = [p^\sharp], \quad (p^\sharp)^!(H^\sharp) = [p].$$

Here $[p], [p^\sharp]$ are the Euler classes of the bundles, and $p^!, (p^\sharp)^!$ are the “integration over the fiber” maps in the Gysin sequences. Furthermore, there is a pullback diagram of circle bundles

$$\begin{array}{c}
Y \\
p^!(p^\sharp)^*(p) \\
\downarrow p^* \\
X \\
\downarrow p \\
X^\sharp \\
\downarrow p^\sharp \\
Z
\end{array}$$

in which $H$ and $H^\sharp$ pull back to the same class on $Y$.

**The Case $n > 1$**

We now want to generalize T-duality to the case of spacetimes $X$ “compactified on a higher-dimensional torus,” or in other words, equipped with a principal $\mathbb{T}^n$-bundle $p: X \to Z$. In the simplest case, $X = Z \times \mathbb{T}^n = Z \times S^1 \times \cdots S^1$.

We can then perform a string of $n$ T-dualities, one circle factor at a time. A single T-duality interchanges type IIA and type IIB string theories, so this $n$-dimensional T-duality “preserves type” when $n$ is even and switches it when $n$ is odd. In terms of our Axioms [1] for topological T-duality, we would therefore expect an isomorphism $K^*(X, H) \cong K^*(X^\sharp, H^\sharp)$ when $n$ is even and $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$ when $n$ is odd.
In the higher-dimensional case, a new problem presents itself: *it is no longer clear that the T-dual should be unique.* In fact, if we perform a string of \( n \) T-dualities, one circle factor at a time, it is not clear that the result should be independent of the order in which these operations are done. Furthermore, a higher-dimensional torus does not split as a product in only one way, so in principle there can be a lot of non-uniqueness.

The way out of this difficulty has therefore been to try to organize the information in terms of a *T-duality group*, a discrete group of T-duality isomorphisms potentially involving a large number of spacetimes and \( H \)-fluxes. We can think of this group as operating on some big metaspace of possible spacetimes.

Another difficulty is that there are some spacetimes with \( H \)-flux that would appear to have no higher-dimensional T-duals at all, at least in the sense we have defined them so far, e.g., \( X = T^3 \), viewed as a principal \( T^3 \)-bundle over a point, with \( H \) the generator of \( H^3(X, \mathbb{Z}) \cong \mathbb{Z} \).

### 2.3 The Use of Noncommutative Geometry

Here is the strategy of the Mathai-Rosenberg approach [31, 32, 33]. Start with a principal \( \mathbb{T}^n \)-bundle \( p: X \to Z \) and an “\( H \)-flux” \( H \in H^3(X, \mathbb{Z}) \). We assume that \( H \) is trivial when restricted to each \( \mathbb{T}^n \)-fiber of \( p \). This of course is no restriction if \( n = 2 \), but it rules out cases with no T-dual in any sense.

We want to lift the free action of \( \mathbb{T}^n \) on \( X \) to an action on the continuous-trace algebra \( A = CT(X, H) \). Usually there is no hope to get such a lifting for \( \mathbb{T}^n \) itself, so we go to the universal covering group \( \mathbb{R}^n \). If \( \mathbb{R}^n \) acts on \( A \) so that the induced action on \( \widehat{A} \) is trivial on \( \mathbb{Z}^n \) and factors to the given action of \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \) on \( \widehat{A} \), then we can take the crossed product \( A \rtimes \mathbb{R}^n \) and use Connes’ Thom Isomorphism Theorem to get an isomorphism between \( K^{*-n}(X, H) = K_{*-n}(A) \) and \( K_*(A \rtimes \mathbb{R}^n) \).

Under favorable circumstances, we can hope that the crossed product \( A \rtimes \mathbb{R}^n \) will again be a continuous-trace algebra \( CT(X^\sharp, H^\sharp) \), with \( p^\sharp: X^\sharp \to Z \) a new principal \( \mathbb{T}^n \)-bundle and with \( H^\sharp \in H^3(X^\sharp, \mathbb{Z}) \). If we then act on \( CT(X^\sharp, H^\sharp) \) with the dual action of \( \mathbb{R}^n \), then by Takai Duality and stability, we come back to where we started. So we have a topological T-duality between \( (X, H) \) and \( (X^\sharp, H^\sharp) \). Furthermore, we have an isomorphism

\[
K^{*-n}(X, H) \cong K^*(X^\sharp, H^\sharp),
\]

as required for matching of D-brane charges under T-duality in Axioms (as modified for \( n \geq 1 \)).

Now what about the problems we identified before, about potential non-uniqueness of the T-dual and “missing” T-duals? These can be explained either by non-uniqueness of the lift to an action of \( \mathbb{R}^n \) on \( A = CT(X, H) \), or else by failure of the crossed product to be a continuous-trace algebra.
A Crucial Example

Let’s now examine what happens when we try to carry out this program in one of our “problem cases,” $n = 2$, $Z = S^1$, $X = T^3$ (a trivial $\mathbb{T}^2$-bundle over $S^1$), and $H$ the usual generator of $H^3(T^3)$. First we show that there is an action of $\mathbb{R}^2$ on $CT(X, H)$ compatible with the free action of $\mathbb{T}^2$ on $X$ with quotient $S^1$. We will need the notion of an induced action. We start with an action $\alpha$ of $\mathbb{Z}^2$ on $C(S^1, K)$ which is trivial on the spectrum. This is given by a map $\mathbb{Z}^2 \to C(S^1, Aut K) = C(S^1, PU(L^2(\mathbb{T})))$ sending the two generators of $\mathbb{Z}^2$ to the maps

$$w \mapsto \text{multiplication by } z,$$

$$w \mapsto \text{translation by } w,$$

where $w$ is the coordinate on $S^1$ and $z$ is the coordinate on $\mathbb{T}$. (Of course $S^1$ and $\mathbb{T}$ are homeomorphic, but we use different letters in order to distinguish them, since they play slightly different roles. These two unitaries commute in $PU$, not in $U$.)

Now form $A = \text{Ind}_{\mathbb{Z}^2}^\mathbb{R} C(S^1, K)$. This is a $C^*$-algebra with $\mathbb{R}^2$-action $\text{Ind } \alpha$ whose spectrum (as an $\mathbb{R}^2$-space) is $\text{Ind}_{\mathbb{Z}^2}^\mathbb{R} S^1 = S^1 \times \mathbb{T}^2 = X$. We can see that $A \cong CT(X, H)$ via “inducing in stages”. Let $B = \text{Ind}_{\mathbb{Z}^2}^\mathbb{R} C(S^1, K(L^2(\mathbb{T})))$ be the result of inducing over the first copy of $\mathbb{R}$. It’s clear that $B \cong C(S^1 \times \mathbb{T}, K)$. We still have another action of $\mathbb{Z}$ on $\mathbb{B}$ coming from the second generator of $\mathbb{Z}^2$, and $A = \text{Ind}_{\mathbb{Z}^2}^\mathbb{R} B$. The action of $\mathbb{Z}$ on $B$ is by means of a map $\sigma : S^1 \times \mathbb{T} \to PU(L^2(\mathbb{T})) = K(\mathbb{Z}, 2)$, whose value at $(w, z)$ is the product of multiplication by $z$ with translation by $w$. Thus $A$ is a CT-algebra with Dixmier-Douady invariant $[\sigma] \times c = H$, where $[\sigma] \in H^2(S^1 \times \mathbb{T}, \mathbb{Z})$ is the homotopy class of $\sigma$ and $c$ is the usual generator of $H^1(S^1, \mathbb{Z})$.

Now that we have an action of $\mathbb{R}^2$ on $A = CT(X, H)$ inducing the free $\mathbb{T}^2$-action on the spectrum $X$, we can compute the crossed product to see what the associated “T-dual” is. Since $A = \text{Ind}_{\mathbb{Z}^2}^\mathbb{R} C(S^1, K)$, we can use the Green Imprimitivity Theorem to see that

$$A \rtimes_{\text{Ind } \alpha} \mathbb{R}^2 \cong \left( C(S^1, K) \rtimes_{\alpha} \mathbb{Z}^2 \right) \otimes K.$$  

Recall that $A_\theta$ is the universal $C^*$-algebra generated by unitaries $U$ and $V$ with $UV = e^{2\pi i \theta} VU$. So if we look at the definition of $\alpha$, we see that $A \rtimes_{\text{Ind } \alpha} \mathbb{R}^2$ is the algebra of sections of a bundle of algebras over $S^1$, whose fiber over $e^{2\pi i \theta}$ is $A_\theta \otimes K$. Alternatively, it is Morita equivalent to $C^* (\Gamma)$, where $\Gamma$ is the discrete Heisenberg group of strictly upper-triangular $3 \times 3$ integral matrices.

Put another way, we could argue that we’ve shown that $C^* (\Gamma)$ is a noncommutative T-dual to $(T^3, H)$, both viewed as fibering over $S^1$. So we have an explanation for the missing T-dual: we couldn’t find it just in the world of topology alone because it’s noncommutative. We will want to see how widely this
phenomenon occurs, and also will want to resolve the question of nonuniqueness of T-duals when $n > 1$.

Further analysis of this example leads to the following classification theorem:

**Theorem 4 (Mathai-Rosenberg [31]).** Let $T^2 \text{ act freely on } X = T^3$ with quotient $Z = S^1$. Consider the set of all actions of $\mathbb{R}^2$ on algebras $CT(X, H)$ inducing this action on $X$, with $H$ allowed to vary over $H^3(X,Z) \cong \mathbb{Z}$. Then the set of exterior equivalence classes of such actions is parametrized by $\text{Maps}(\mathbb{Z}, T)$. The winding number of a map $f: \mathbb{Z} \to T$ can be identified with the Dixmier-Douady invariant $H$. All these actions are given by the construction above, with $f$ as the “Mackey obstruction map.”

Consider a general $T^2$-bundle $X \to Z$. We have an edge homomorphism

$$p: H^3(X,Z) \to E^{1,2}_{\infty} \subseteq H^1(Z,H^2(T^2,Z)) = H^1(Z,Z)$$

which turns out to play a major role.

**Theorem 5 (Mathai-Rosenberg[31]).** Let $p: X \to Z$ be a principal $T^2$-bundle as above, $H \in H^3(X,Z)$. Then we can always find a “generalized T-dual” by lifting the action of $\mathbb{R}^2$ on $CT(X, H)$ and forming the crossed product. When $p^*H = 0$, we can always do this in such a way as to get a crossed product of the form $CT(X^\sharp, H^\sharp)$, where $(X^\sharp, H^\sharp)$ is a classical T-dual (e.g., as found through the purely topological theory). When $p^*H \neq 0$, the crossed product $CT(X, H) \rtimes \mathbb{R}^2$ is never locally stably commutative and should be viewed as a noncommutative T-dual.

### 2.4 Current Directions in Topological T-Duality

Here we just summarize some of the current trends in topological T-duality:

1. the above approach with actions of $\mathbb{R}^n$ on continuous-trace algebras. For $n \geq 2$, the lift of even a free action of $T^n$ on $X$ to an action of $\mathbb{R}^n$ on $CT(X, H)$ is usually not essentially unique, and a more detailed study of non-uniqueness is required. One would also like to extend the study of topological T-duality to cases where the action of $T^n$ has isotropy, as progress on the famous “SYZ conjecture” [43] will require study of torus bundles with some degeneration. These issues have been studied in [33], [17], and [38], for example.

2. the homotopy-theoretic approach of Bunke-Schick, extended to the higher-dimensional case. This has been studied by Bunke-Rumpf-Schick [15], by Mathai-Rosenberg [33], and by Schneider [41].

3. a fancier approach using duality of sheaves over the Grothendieck site of (suitable) topological spaces (Bunke-Schick-Spitzweck-Thom [18]).

4. a generalization of the noncommutative geometry approach using groupoids (Daenzer [20]).
5. algebraic analogues, in the world of complex manifolds, schemes, etc.,
using Mukai duality with gerbes (Ben-Basset, Block, Panve \[8\] and Block
and Daenzer \[9\]).

6. work of Bouwknegt and Pande \[14\] relating the noncommutative geometry
approach to Hull’s notion of T-folds \[28\], which are certain nongeometric
backgrounds well-known in string theory.

7. an approach of Bouwknegt and Mathai using duality for loop group bun-
dles \[13\].

As one can see, this is a very active subject going off in many different
directions, and it would take a much longer survey to go into these matters
in detail.

3 Problems Presented by S-Duality and Other Dualities

3.1 Type I/Type IIA Duality on $T^4/K3$

In this subsection I want to describe some joint work with Stefan Mendez-Diez
\[34\]. As we mentioned before, there is believed to be an S-duality relating type
I string theory to one of the heterotic string theories. There are also various
other dualities relating these two theories to type IIA theory. Putting these
together, we expect a (non-perturbative) duality between type I string theory
on $T^4 \times \mathbb{R}^6$ and type IIA string theory on $K3 \times \mathbb{R}^6$, at least at certain points
in the moduli space. (Here $K3$ denotes a K3 surface, a simply connected
closed complex surface with trivial canonical bundle. The name K3 stands for
"Kummer, Kähler, Kodaira." As a manifold, it has Betti numbers 1, 0, 22,
0, 1, and signature $-16$.) This duality is discussed in detail in \[42\].

On the face of it, this appears ridiculous: $KO(T^4 \times \mathbb{R}^6) = KO^{-6}(T^4)$ has
lots of 2-torsion, while $K^*(K3)$ is all torsion-free and concentrated in even
degree.

One side is easy compute. Recall that for any space $X$,

$$KO^{-j}(X \times S^1) \cong KO^{-j}(X) \oplus KO^{-j-1}(X).$$

Iterating, we get

$$KO^{-6}(T^4) \cong KO^{-6} \oplus 4KO^{-7} \oplus 6KO^{-8} \oplus 4KO^{-9} \oplus KO^{-10},$$

$$\cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/2) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^5.$$

The way we deal with the opposite side of the duality is to recall that a K3
surface can be obtained by blowing up the point singularities in $T^4/G$, where
$G = \mathbb{Z}/2$ acting by multiplication by $-1$ on $\mathbb{R}^4/\mathbb{Z}^4$. This action is semi-free
with 16 fixed points, the points with all four coordinates equal to 0 or \( \frac{1}{2} \) mod 2. If fact one way of deriving the (type I on \( T^4 \)) ↔ (type IIA on K3) duality explicitly uses the orbifold \( T^4/G \).

But what group should orbifold brane charges live in? \( K^*(T^4/G) \) is not quite right, as this ignores the orbifold structure. One solution that has been proposed is \( K_G^*(T^4) \), which Mendez-Diez and I computed. However, as we'll see, there appears to be a better candidate.

Let \( M \) be the result of removing a \( G \)-invariant open ball around each \( G \)-fixed point in \( T^4 \). This is a compact manifold with boundary on which \( G \) acts freely; let \( N = M/G \). We get a K3 surface back from \( N \) by gluing in 16 copies of the unit disk bundle of the tangent bundle of \( S^2 \) (known to physicists as the Eguchi-Hanson space), one along each \( \mathbb{RP}^3 \) boundary component in \( \partial N \).

**Theorem 6 (34).**

\[
H^i(N, \partial N) \cong H_{4-i}(N) \cong \begin{cases} 0, & i = 0 \\ \mathbb{Z}^{15}, & i = 1 \\ \mathbb{Z}^6, & i = 2 \\ (\mathbb{Z}/2)^5, & i = 3 \\ \mathbb{Z}, & i = 4 \\ 0, & \text{otherwise}. \end{cases}
\]

Recall \( N \) is the manifold with boundary obtained from \( T^4/G \) by removing an open cone neighborhood of each singular point.

**Theorem 7 (34).** \( K^0(N, \partial N) \cong K_0(N) \cong \mathbb{Z}^7 \) and \( K^{-1}(N, \partial N) \cong K_1(N) \cong \mathbb{Z}^{15} \oplus (\mathbb{Z}/2)^5 \).

Note that the reduced \( K \)-theory of \( (T^4/G) \mod (\text{singular points}) \) is the same as \( K^*(N, \partial N) \). Note the resemblance of \( K^{-1}(N, \partial N) \) to \( KO^{-6}(T^4) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^5 \). While they are not the same, the calculation suggests that the brane charges in type I string theory on \( T^4 \times \mathbb{R}^6 \) do indeed show up some way in type IIA string theory on the orbifold limit of K3.

Again let \( G = \mathbb{Z}/2 \). Equivariant \( K \)-theory \( K_G^* \) is a module over the representation ring \( R = R(G) = \mathbb{Z}[t]/(t^2 - 1) \). This ring has two important prime ideals, \( I = (t - 1) \) and \( J = (t + 1) \). We have \( R/I \cong R/J \cong \mathbb{Z}, I \cdot J = 0, I + J = (I, 2) = (J, 2), R/(I + J) = \mathbb{Z}/2 \).

**Theorem 8 (34).** \( K^0_G(T^4) \cong R^8 \oplus (R/J)^8 \), and \( K^{-1}_G(T^4) = 0 \). Also, \( K^0_G(M, \partial M) \cong (R/I)^7 \), \( K^{-1}_G(M, \partial M) \cong (R/I)^{10} \oplus (R/2I)^5 \).

Note that the equivariant \( K \)-theory calculation is a refinement of the ordinary \( K \)-theory calculation (since \( G \) acts freely on \( M \) and \( \partial M \) with quotients \( N \) and \( \partial N \), so that \( K_G^*(M) \) and \( K_G^*(\partial M) \) are the same as \( K^*(N) \) and \( K^*(\partial N) \) as abelian groups, though with the addition of more structure). While we don't immediately need the extra structure, it may prove useful later in matching brane charges from \( KO(T^4 \times \mathbb{R}^6) \) on specific classes of branes.
3.2 Other Cases of Type I/Type II Charge Matching

More generally, one could ask if there are circumstances where understanding of $K$-theory leads us to expect the possibility of a string duality between type I string theory on a spacetime $Y$ and type II string theory on a spacetime $Y'$. For definiteness, we will assume we are dealing with type IIB on $Y'$. (This is no great loss of generality since as we have seen in Section 2, types IIA and IIB are related via T-duality.) Matching of stable D-brane charges then leads us to look for an isomorphism of the form
\[ KO^*(Y) \cong K^*(Y'). \]

In general, such isomorphisms are quite rare, in part because of 2-torsion in $KO^{-1}$ and $KO^{-2}$, and in part because $KO$-theory is usually 8-periodic rather than 2-periodic.

But there is one notable exception: one knows [1, p. 206] that
\[ KO \wedge (S^0 \cup_\eta e^2) \simeq K, \]
where $S^0 \cup_\eta e^2$ is the stable cell complex obtained by attaching a stable 2-cell via the stable 1-stem $\eta$. This is stably the same (up to a degree shift) as $\mathbb{C}P^2$, since the attaching map $S^3 \to S^2 \cong \mathbb{C}P^1$ for the top cell of $\mathbb{C}P^2$ is the Hopf map, whose stable homotopy class is $\eta$. Thus one might expect a duality between type I string theory on $X^6 \times (\mathbb{C}P^2 \setminus \{pt\})$ and type IIB string theory on $X^6 \times \mathbb{R}^4$. We plan to look for evidence for this.

3.3 The AdS/CFT Correspondence

The AdS/CFT correspondence or holographic duality is a conjectured physical duality, proposed by Juan Maldacena [30], of a different sort, relating IIB string theory on a 10-dimensional spacetime manifold to a gauge theory on another space. In the original version of this duality, the string theory lives on $AdS^5 \times S^5$, and the gauge theory is $\mathcal{N} = 4$ super-Yang-Mills theory on Minkowski space $\mathbb{R}^{1,3}$. Other versions involve slightly different spaces and gauge theories. A good survey may be found in [3]. Notation:

- $\mathcal{N}$ is the standard notation for the supersymmetry multiplicity. In other words, $\mathcal{N} = 4$ means there are 4 sets of supercharges, and there is a $U(4)$ $R$-symmetry group acting on them.
- $AdS^5$, 5-dimensional anti-de Sitter space is (up to coverings) the homogeneous space $SO(4,2)/SO(4,1)$. Topologically, this homogeneous space is $\mathbb{R}^4 \times S^1$. It’s better to pass to the universal cover $\mathbb{R}^5$, however, so that time isn’t periodic.
Nature of the Correspondence

We have already explained that D-branes carry Chan-Paton bundles. In type IIB string theory, a collection of $N$ coincident D3 branes have $3 + 1 = 4$ dimensions and carry a $U(N)$ gauge theory living on the Chan-Paton bundle. This gauge theory is the holographic dual of the string theory, and the number $N$ can be recovered as the flux of the Ramond-Ramond (RR) field strength 5-form $G_5$ through an $S^5$ linking the D3 brane [3, equation (3.7)]. The rotation group $SO(6)$ of $\mathbb{R}^5$ is identified (up to coverings) with the $SU(4)_R$ symmetry group of the $\mathcal{N} = 4$ gauge theory.

The AdS/CFT correspondence looks like holography in that physics in the bulk of AdS space is described by a theory of one less dimension “on the boundary.” This can be explained by the famous Beckenstein-Hawking bound for the entropy of a black hole in terms of the area of its boundary, which in turn forces quantum gravity theories to obey a *holographic principle*.

Recall that the Montonen-Olive Conjecture (Section 1.2) asserts that classical electro-magnetic duality should extend to an exact symmetry of certain quantum field theories. 4-dimensional super-Yang-Mills (SYM) with $\mathcal{N} = 4$ supersymmetry is believed to be a case for which this conjecture applies. The Lagrangian involves the usual Yang-Mills term

$$-\frac{1}{4g_{YM}^2} \int \text{Tr}(F \wedge *F)$$

and the *theta angle* term (related to the Pontrjagin number or *instanton number*)

$$\frac{\theta}{32\pi^2} \int \text{Tr}(F \wedge F).$$

We combine these by introducing the *tau parameter*

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}.$$

The tau parameter measures the relative size of “magnetic” and “electric charges.” Dyons in SYM have charges $(m, n)$ living in the group $\mathbb{Z}^2$; the associated *complex charge* is $q + iq = q_0(m + n\tau)$. As in the theory of the Dirac monopole, quantization of magnetic charge is related to integrality of characteristic classes in topology, i.e., to the fact that the Pontrjagin number must be an integer.

The *electro-magnetic duality group* $SL(2, \mathbb{Z})$ acts on $\tau$ by linear fractional transformations. More precisely, it is generated by two transformations: $T$: $\tau \mapsto \tau + 1$, which just increases the $\theta$-angle by $2\pi$, and has no effect on magnetic charges, and by $S$: $\tau \mapsto -\frac{1}{\tau}$, which effectively interchanges electric and magnetic charge. By the Montonen-Olive Conjecture [36], the same group $SL(2, \mathbb{Z})$ should operate on type IIB string theory in a similar way, and $\theta$ should correspond in the string theory to the expectation value of the RR scalar field $\chi$. (See for example [27, 44, 45, 96].)
Puzzles About Charge Groups

An important constraint on variants of the AdS/CFT correspondence should come from the action of the $SL(2,\mathbb{Z})$ S-duality group on the various charges. For example, this group is expected to act on the pair $(H, G_3)$ of type IIB string theory field strengths in $H^3(X, \mathbb{Z}) \times H^3(X, \mathbb{Z})$ by linear fractional transformations. Here $G_3$ denotes the RR 3-form field strength, or more precisely, its cohomology class. But now we have some puzzles:

- The classes of RR fields are really supposed to live in $K^{-1}$, not cohomology, whereas the NS class $[H]$ is really expected to live in ordinary cohomology. (Fortunately, since the first differential in the Atiyah-Hirzebruch spectral sequence is $Sq^3$, there is no difference when it comes to classes in $H^3$, except when $H^3$ has 2-torsion. See [23, 10] for related discussions.)
- Since the S-duality group mixes the NS-NS and RR sectors, it is not clear how it should act on D-brane and RR field charges.
- It’s also not so clear what conditions to impose at infinity when spacetime is not compact. For example, it would appear that the H-flux and RR fields do not have to have compact support, so perhaps $K$-theory with compact support is not the right home for the RR field charges. This point seems unclear in the literature.

Example 5. Let’s look again at the example of type IIB string theory on $AdS^5 \times S^5$, compared with $\mathcal{N} = 4$ SYM on 4-space. How do the $K$-theoretic charge groups match up? Our spacetime is topologically $X = \mathbb{R}^5 \times S^5$, where $\mathbb{R}^5$ is the universal cover of $AdS^5$. We think of $\mathbb{R}^5$ more exactly as $\mathbb{R}^4 \times \mathbb{R}_+$, so that $\mathbb{R}^4 \times \{0\}$, Minkowski space, is “at the boundary.” The RR field charges should live in $K^{-1}(X)$, according to [47], but we see this requires clarification: the RR field strength $G_3$ should represent the number $N$ in $H^3(S^5)$ (since as we mentioned, $N$ is computed by pairing the class of $G_3$ with the fundamental class of $S^5$), so we need to use homotopy theoretic $K$-theory $K_h$ here instead of $K$-theory with compact support, which we’ve implicitly been using before. Indeed, note that $K^{-1}(X) \cong K^{-1}(\mathbb{R}^5) \otimes K^0(S^5) \cong H^0(\mathbb{R}^4)$, while $K_h^{-1}(X) \cong K_h^0(\mathbb{R}^5) \otimes K^{-1}(S^5) \cong H^3(S^5)$, which is what we want.

Now what about the D-brane charge group for the string theory? This should be $\mathbb{Z} \cong K^0(X) \cong K^0(\mathbb{R}^4 \times Y) \cong K^0(\mathbb{R}^4) \otimes K^0(Y)$, where $Y$ is the D5-brane $\mathbb{R} \times S^5$, which has $K^0(Y) \cong \mathbb{Z}$. Note that this is naturally isomorphic to $K^0(\mathbb{R}^4) = \tilde{K}^0(S^4)$, which is where the instanton number lives in the dual gauge theory. But what charge group on $X$ corresponds to the group of electric and magnetic charges in the gauge theory? (This should be a group isomorphic to $\mathbb{Z}^2$ containing the group $\mathbb{Z}$ classifying the instanton number.)

It is believed that the string/gauge correspondence should apply much more generally, to many type IIB string theories on spaces other than $AdS^5 \times S^5$, and to gauge theories with less supersymmetry than the $\mathcal{N} = 4$ theory that we’ve been considering. Analysis of the relevant charge groups on both
the string and gauge sides of the correspondence should give us a guide as to what to expect. Study of these constraints is still in a very early stage.

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Index

action 2
  least 2
  sigma-model 5
AdS/CFT correspondence 16
Aharonov-Bohm effect 4
anomaly cancellation 5, 7
anti-de Sitter space 16

B-field 5

Chan-Paton bundle 6, 17
charge group 5
continuous-trace algebra 7

D-brane 5, 7, 16–18
Dirac quantization condition 4
Dirac monopole 3, 17
Dixmier-Douady class 7
dual torus 6
duality 3
electric-magnetic 3, 6, 17
Fourier 3, 6
S- 6, 14
T- 6
noncommutative 12
topological 8, 13
dyon 4, 17

Eguchi-Hanson space 15
expectation value 3

field
  gauge 2
  Ramond-Ramond 8, 17

field (in physics) 1
field strength 2, 7
Fourier transform 3

gerbe 7
Gysin map 6
H-flux 7
Heisenberg uncertainty principle 2
holography 17

instanton number 17, 18
K3 surface 14, 15
magnetic monopole 3
Maxwell’s equations 3
Montonen-Olive Conjecture 4, 17
partition function 2
path integral 2
quantization
  charge 3
  Dirac condition 4
R-symmetry 16
S-duality 6, 14
sigma model 5
stationary phase 2
string theory 5
super-Yang-Mills theory 17
T-duality 6
  noncommutative 12
topological 8, 13
| Term                         | Page |
|------------------------------|------|
| T-duality group              | 11   |
| theta angle                  | 17   |
| twisted $K$-theory           | 7    |
| wave function                | 2    |
| Wess-Zumino term             | 5, 7 |