Partitions and Indivisibility Properties of Countable Dimensional Vector Spaces

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Abstract

We investigate infinite versions of vector and affine space partition results, and thus obtain examples and a counterexample for a partition problem for relational structures. In particular we provide two (related) examples of an age indivisible relational structure which is not weakly indivisible.

Key words and phrases: Ramsey theory, homogeneous relational structures, vector and affine spaces.

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1 Introduction

In the present paper we study the divisibility properties of some classical groups by studying the divisibility properties of infinite dimensional vector spaces, generalizing those obtained for finite dimensional spaces. Before we can state the main results, we review basic divisibility notions from structural Ramsey Theory.

A relational structure consists of a non empty base set along with a collection of finitary relations which are defined on it. A typical example is a graph consisting of vertices (the base set) together with edges (one binary relation). Let $\mathcal{R}$ be a relational structure with base set $\mathcal{R}$. The skeleton of $\mathcal{R}$ is the set of finite induced substructures of $\mathcal{R}$ and the age of $\mathcal{R}$, $\text{age}(\mathcal{R})$, is the class of finite relational structures isomorphic to an element of the skeleton. A local isomorphism of $\mathcal{R}$ is an isomorphism between two elements of the skeleton. We denote by $\text{Aut}(\mathcal{R})$ the automorphism group of $\mathcal{R}$. The relational structure $\mathcal{R}$ is homogeneous if every local isomorphism of $\mathcal{R}$ has an extension to an automorphism of $\mathcal{R}$. An embedding of $\mathcal{R}$ into $\mathcal{R}$ is an isomorphism of $\mathcal{R}$ to an induced substructure of $\mathcal{R}$. We denote by $\text{Emb}(\mathcal{R})$ the set of embeddings of $\mathcal{R}$ into $\mathcal{R}$. For a set $A$ (possibly not $\subseteq \mathcal{R}$) we denote by $\mathcal{R}|A$ the relational structure induced by $\mathcal{R} \cap A$. A relational structure $\mathcal{R}'$ is richer than the relational structure $\mathcal{R}$ if $\mathcal{R}'$ and $\mathcal{R}$ have the same base set and every relation of $\mathcal{R}$ is a relation of $\mathcal{R}'$.

We recall the following three notions from structural Ramsey theory (see for example the Appendix of [6]), together with the notion of a uniform partition which we believe to be new (at least in this form).

Definition 1. 1. A relational structure $\mathcal{R}$ is indivisible if for every partition $(P_0, P_1)$ of $\mathcal{R}$ there exists an element $\epsilon \in \text{Emb}(\mathcal{R})$ and an $i \in 2$ with $\epsilon[R] \subseteq P_i$. 

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2. A relational structure $R$ is said to have a uniform partition if there is a finite partition $(U_i : i \in n)$ of $R$ such that

(a) for all $\epsilon \in \text{Emb}(R)$ and all $i \in n$, $\text{age}(\epsilon[R] \upharpoonright U_i) = \text{age}(R)$,

(b) for every partition $(P_0, P_1)$ of $R$ and any $i \in n$, there exists an element $\epsilon \in \text{Emb}(R)$ and $j \in 2$ with $(\epsilon[R] \cap U_i) \subseteq P_j$.

3. The relational structure $R$ is weakly indivisible if for every partition $(P_0, P_1)$ of $R$ with $\text{age}(R \upharpoonright P_0) \neq \text{age}(R)$ there exists an element $\epsilon \in \text{Emb}(R)$ with $\epsilon[R] \subseteq P_1$.

4. The relational structure $R$ is age indivisible if for every partition $(P_0, P_1)$ of $R$, there is an $i \in 2$ with $\text{age}(R \upharpoonright P_i) = \text{age}(R)$.

Each of the above properties easily implies the one following it. The more familiar notion of a canonical partition is defined as in 2 above by simply replacing the first condition by the weaker requirement that any copy of $R$ meets every block of the partition (see [13]):

**Definition 2.** A relational structure $R$ is said to have a canonical partition if there is a finite partition $(U_i : i \in n)$ of $R$ such that

1. for all $\epsilon \in \text{Emb}(R)$ and all $i \in n$, $\epsilon[R] \cap U_i \neq \emptyset$

2. for every partition $(P_0, P_1)$ of $R$ and any $i \in n$, there exists an element $\epsilon \in \text{Emb}(R)$ and $j \in 2$ with $(\epsilon[R] \cap U_i) \subseteq P_j$.

A uniform partition is clearly a canonical partition, and conversely a canonical partition together with the weakly indivisible property implies that it is a uniform partition. However the existence of a canonical partition alone does not suffice; indeed there are countable homogeneous structures having a canonical partition but not age indivisible, namely the structure consisting of two disjoint copies of the rationals.

More generally it is known that most of the above implications are non-reversible. Indeed there are countable homogeneous divisible (meaning not indivisible) structures having uniform partition; these will be further described in detail in a forthcoming paper (see See [14]). There are examples (see [12]) of countable weakly indivisible homogeneous divisible structures. We will see an example of an homogeneous structure below which is weakly indivisible but does not have a canonical partition, and therefore not a uniform one, and therefore is divisible.
We were not aware of any example of a countable age indivisible homogeneous structure which is not weakly indivisible. See [14] for the fact that the standard age indivisible examples of countable homogeneous relational structures are also weakly indivisible. Actually, there was no example of an age indivisible structure (homogeneous or not) which is not weakly indivisible. We provide two examples below obtained from vector spaces. They are unfortunately not as simple as one would intend in the sense that one has infinitely many relations. Although the other has a single relation, it is not homogeneous. It remains open whether to produce such an example of a countable homogeneous structure having only finitely many relations.

One of the standard tools to prove age indivisibility of relational structures is the Hales-Jewett Theorem [10]; weak indivisibility seems then related to an infinite version of that Theorem at least in cases similar to the vector space situation discussed in the present paper, and we shall therefore be interested in infinite dimensional vector spaces. In the case of finite dimensional vector spaces over a finite field \( \mathbb{F}_q \), Graham, Leeb and Rothschild, proved the following.

**Theorem 1.** [7] For all \( d, k, t \geq 0 \), there exists \( n = GLR^t(d, k) \) with the property that for any \( n \)-dimensional vector space \( \mathbb{F}_q \) over \( \mathbb{F}_q \) and any colouring of all \( t \)-dimensional (affine) subspaces into \( k \) colours, there exists a \( d \)-dimensional (affine) subspace \( U \subset \mathbb{V} \) such that all its \( t \)-dimensional (affine) subspaces have the same colour.

The reason for writing the adjective “affine” in parenthesis is that the above result where the notion of subspaces is interpreted as “affine” is equivalent, as proved by Graham and Rothschild, to the corresponding one using the normal sense of subspace. In that latter sense of usual vector subspace, the Theorem for \( t = 0 \) has no content. But for \( t = 1 \) (or equivalently its affine version for \( t = 0 \)) is already powerful, it implies in particular the following particular case which will be used in the proof of Theorem 13:

**Corollary 2.** [\( t=1 \)] For all \( d, k \geq 0 \), there exists \( n = GLR(d, k) \) such that for any \( n \)-dimensional vector space \( \mathbb{V} \) over \( \mathbb{F}_q \) and any colouring of the lines of \( V \) into \( k \) colours, there exists a \( d \)-dimensional subspace \( W \subset \mathbb{V} \) all of whose lines are the same colour.

In the affine case we get the following.

**Corollary 3.** [\( t=0 \) (affine)] For all \( d, k \geq 0 \), there exists \( n \) such that for any \( n \)-dimensional vector space \( \mathbb{V} \) over \( \mathbb{F}_q \) and any colouring of \( V \) into \( k \) colours, there exists a monochromatic \( d \)-dimensional affine subspace \( W \subset \mathbb{V} \).
A central motivation for this research is to investigate infinite versions of this result. To do so, we shall interpret a vector space as a relational structure, in which the Ramsey partition properties described above correspond to affine version of the usual vector subspace and related notions, and we assume this notion for the remainder of the paper unless specifically mentioned otherwise. In particular we will be interested in the affine transformations of $V$ onto $V$, forming a group called the (Inhomogeneous) General Linear group of $V$, and denoted by $IGL(V)$.

Hindman showed in [11] that a vector space of countable dimension over $F_2$ is indivisible. On the other hand we provide a proof in Theorem 7 the well known fact that a vector space of countable dimension over any other field is divisible, in fact does not have a uniform partition. We also prove in Theorem 13 that over any finite field, a vector space of countable dimension is weakly indivisible. Over an infinite field, we shall show that a vector space of countable dimension is not weakly indivisible, in fact it can be divided into two parts such that none of the parts contains an affine line (see Theorem 14). On the other hand, all infinite dimensional vector spaces are age indivisible. So a countable dimensional vector space over the rationals provides an example of an age indivisible, not weakly indivisible relational structure (with infinitely many relations).

It is known and we will provide a proof in Lemma 15 that if $V$ is a vector space over the rationals $\mathbb{Q}$ and $\epsilon : V \to V$ an injection with $\epsilon(\frac{a+b}{2}) = \frac{\epsilon(a)+\epsilon(b)}{2}$ for all $a, b \in V$ then $\epsilon \in Emb(V)$, that is $\epsilon$ is an affine transformation. It follows that if $M_V$ is the relational structure with base set $V$ and ternary relation $\mu(a, b, c)$ if and only if $b = \frac{a+c}{2}$ then $Emb(V) = Emb(M_V)$ and in particular that $Aut(M_V) = IGL(V)$. We prove in Theorem 19 that $M_V$ does constitute another example of an age indivisible but not weakly indivisible relational structure. Unfortunately $M_V$ is not homogeneous: for take any $n$ element sequence on an affine line no three of which in the midpoint relation $R$. Then there is a local isomorphism $\alpha$ to any other such $n$ element sequence on any other line, even if $\alpha$ is not an affine transformation, in which case $\alpha$ can not be extended to an element of $IGL(V)$.

It is worth noting that the above Ramsey properties for homogeneous structures are properties of the automorphism group seen as a permutation group. Conversely, given a permutation group closed in the product topology, there exist homogeneous structures with the given group as automorphism group (see [6] for some discussion on this). That is, those divisibility properties can be studied as permutation group properties.
2 The Affine Space Structure

An affine transformation $\alpha$ of a vector space $V$ onto another vector space $W$ is a function of $V$ to $W$ for which there exists an element $w \in W$ and an invertible linear transformation $\rho : V \rightarrow W$ so that $\alpha(v) = w + \rho(v)$ for all $v \in V$. The affine transformations of $V$ to $V$ form a group $\text{IGL}(V)$, the Inhomogeneous General Linear group of $V$, and we denote the set of corresponding affine embeddings of $V$ into $V$ by $\text{Emb}(V)$.

A sequence $< v_i : i \in \mathbb{N} >$ of affinely dependent elements of $V$ is called an affine cycle of $V$ if no proper subsequence is affinely dependent. Two affine cycles $< v_i : i \in \mathbb{N} >$ and $< v'_i : i \in \mathbb{N} >$ of $V$ are equivalent if there is an invertible affine transformation $\tau$ of the affine space generated by $\{v_i : i \in \mathbb{N}\}$ to the affine space generated by $\{v'_i : i \in \mathbb{N}\}$ with $\tau(v_i) = v'_i$ for all $i \in \mathbb{N}$. Interpreting every equivalence class of affine cycles of $V$ as a relation on $V$ yields a homogeneous relational structure $\mathcal{V}$. Under this situation it turns out that $\text{Aut}(V) = \text{IGL}(V)$, and that the set of affine embeddings of $V$ is equal to the set of embeddings of $\mathcal{V}$. The relational structure $\mathcal{V}$ is what we call the affine cycle structure of $V$. Every element of the skeleton of $V$ generates affinely a finite dimensional affine subspace of $V$ and hence every element of the age of $\mathcal{V}$ can be affinely embedded into a finite dimensional affine subspace of $V$. Of course when the field is finite, then every finite dimensional affine subspace of $V$ is also part of the skeleton. As a consequence we obtain the following translation of the structural Ramsey properties described above applied to $\mathcal{V}$. Thus we shall say by abuse of terminology that a vector space $V$ is indivisible if for every partition $(P_0, P_1)$ of $V$ there exists an affine embedding $\epsilon$ of $V$ and an $i \in 2$ such that $\epsilon[V] \subseteq P_i$. The vector space $V$ has a uniform partition if there is a finite partition $(U_i : i \in \mathbb{N})$ of $V$ such that (1) for all affine embeddings $\epsilon$ of $V$ and all $i \in \mathbb{N}$, $\text{age}(\epsilon[V] \upharpoonright U_i) = \text{age}(V)$, (2) for every partition $(P_0, P_1)$ of $V$ and any $i \in \mathbb{N}$, there exists an affine embedding $\epsilon$ of $V$ and $j \in 2$ with $(\epsilon[V] \cap U_i) \subseteq P_j$. The vector space $V$ is weakly indivisible if for every partition $(P_0, P_1)$ of $V$ with $\text{age}(V) \neq \text{age}(V \upharpoonright P_0)$, there exists an affine embedding $\epsilon$ of $V$ with $\epsilon[V] \subseteq P_1$. A vector space $V$ is age indivisible if for every partition $(P_0, P_1)$ of $V$, there is an $i \in 2$ with $\text{age}(V \upharpoonright P_i) = \text{age}(V)$.

Using a standard compactness argument one can show that a relational structure $\mathcal{R}$ is age indivisible if and only if the age of $\mathcal{R}$ is a Ramsey family. That is, if for every element $\mathcal{A}$ in the age of $\mathcal{R}$ with base $A$ there exists an element $\mathcal{B} \in \text{age}(\mathcal{R})$ with base set $B \supseteq A$ so that for every partition $(P_0, P_1)$ of $B$ there exists an embedding $\epsilon$ and an $i \in 2$ with $\epsilon[A] \subseteq P_i$. It follows readily from this that two relational structures with the same age
are either both age indivisible or neither is. This characterization is also
useful in proving age indivisibility; indeed if the age of a relational structure
\( R \) is closed under products (for an appropriate definition of product) as for
vector spaces in the case of this paper, then it follows from the Hales-Jewett
Theorem (see [10]), that the age of \( R \) has the Ramsey property and hence
that \( R \) is age indivisible. In the case of vector spaces over a finite field
\( \mathbb{F}_q \), one can use \( \mathbb{F}_q \) itself as the required alphabet and choose a sufficiently
large product and observe that a combinatorial line is an affine line, and
then more generally that a combinatorial space is an affine space. It follows
from this discussion that the affine cycle structure of a vector space is age
indivisible.

We conclude this section by reviewing some basic notions and notation
for vector spaces that will be used throughout this paper. If \( \lambda \) denotes the
dimension of a vector space \( V \) over a field \( \mathbb{F} \), then we identify \( V \) with
\( \mathbb{F}[^\lambda] \) as the set of functions \( h : \lambda \to \mathbb{F} \) taking non-zero values only finitely many
times. The support of such a function \( h \) is the set \( \text{supp}(h) := \{ \alpha \in \lambda : h(\alpha) \neq 0 \} \). We denote by \( \overline{0} \in V \) the constant sequence with value \( 0 \in \mathbb{F} \),
with the understanding that \( \text{supp}(\overline{0}) = \emptyset \). We write \( \maxsup(h) \) for the
largest element of \( \text{supp}(h) \) if \( h \neq \overline{0} \), and \( \maxsup(\overline{0}) := -\infty \); finally, we set
\( \hat{h} := h(\maxsup(h)) \) if \( h \neq \overline{0} \) and \( \hat{0} := 0 \). Similarly we write \( \minsup(h) \) for
the smallest element of \( \text{supp}(h) \) if \( h \neq \overline{0} \), and \( \minsup(\overline{0}) := -\infty \), and set
\( \check{h} := h(\minsup(h)) \) if \( h \neq \overline{0} \) and \( \check{0} := 0 \). More generally the support \( \text{supp}(A) \)
of a subset \( A \) of \( \mathbb{F}[^\lambda] \) is the set \( \bigcup_{h \in A} \text{supp}(h) \).

For two finite subsets \( X \) and \( Y \) of \( \lambda \), we write \( X \ll Y \) if the maximum
of \( X \) is strictly smaller than the minimum of \( Y \). We extend this notation
to \( f, g \in \mathbb{F}[^\lambda] \) by writing \( f \ll g \) if \( \text{supp}(f) \ll \text{supp}(g) \), and to \( A \ll B \) for
subsets \( A \) and \( B \) of \( \mathbb{F}[^\lambda] \) if \( f \ll g \) for all \( f \in A \) and \( g \in B \).

A subset \( A \) of \( V \) is an affine subspace if there is a (unique) subspace \( W \)
and an element \( v \) of \( V \) with \( A = v + W := \{ v + w : w \in W \} \). The dimension
of the affine subspace \( A \) is the dimension of \( W \). A (affine) line is a (affine)
subspace of \( V \) of dimension one, and we denote by \( L \) the set of lines of \( V \).
Every line of \( V \) contains exactly one element \( f \) with \( \hat{f} = 1 \), and conversely
every \( f \in V \) with \( f = 1 \) generates a line \( \{ af : a \in \mathbb{F} \} \) which we denote by
\( \langle f \rangle \); that is we name a line by its unique element \( f \) with \( \hat{f} = 1 \).

We shall be mostly interested in the countable case \( \lambda = \omega \), but many of
the results presented generalize to vector space of an arbitrary dimension \( \lambda \).
3 Vector spaces of countable dimension

For this section, fix a vector space $V$ of countable dimension over an arbitrary field $F$, and as described above we may assume that $V = F[\omega]$. We begin by producing a manageable and interesting structure for an infinite dimensional (affine) subspace.

**Lemma 4.** Let $W$ be an infinite dimensional subspace of $V$ and $x \in \omega$. Then there exists an infinite dimensional subspace $U$ of $W$ so that $x \ll \text{supp}(f)$ for all $f \in U$.

**Proof.** By a repeated application, it suffices to prove that for each $x \in \omega$, there is a nonzero vector $f \in W$ with $x \ll \text{supp}(f)$.

Since $W$ is infinite dimensional, there must be two linearly independent vectors $f, g \in W$ such that $f \upharpoonright x$ and $g \upharpoonright x$ are linearly dependent. That is $af \upharpoonright x + bg \upharpoonright x = \overline{0}$ for some $a, b \in F$ not both zero. But then $x \ll af + bg \neq \overline{0}$. 

An iterated application of Lemma 4 allows to construct the following structure for an infinite dimensional affine subspace of $V$.

**Proposition 5.** Every infinite dimensional affine subspace $v + W$ of $V$ contains an infinite sequence $(v + f_i : i \in \omega)$ such that

$$v \ll f_i \ll f_{i+1}$$

for all $i \in \omega$.

### 3.1 Indivisibility

In [11], Hindman proved that a vector space of countable dimension over $F_2$ is indivisible. This fact is an immediate consequence of (and equivalent to) Hindman’s finite union Partition Theorem using Proposition 5.

**Theorem 6.** [11] If $V$ is a vector space over $F_2$ of countable dimension, then $V$ is indivisible.

### 3.2 Uniform Partitions

It is a well known folklore result that every countable dimensional vector space over any other field than $F_2$ is divisible, in fact does not contain a uniform or even a canonical partition.
**Theorem 7.** (Folklore) If $\mathbb{F} \neq \mathbb{F}_2$, then any countable dimensional vector space $V$ over $\mathbb{F}$ is divisible. In fact $V$ does not have a uniform and even a canonical partition.

**Proof.** For $f \in V = \mathbb{F}[\omega]$, define $\text{osc}(f)$ as the number of times that $f$ changes from a nonzero value to a different nonzero value as we cover the support of $f$. That is, if $\text{supp}(f) = \{x_i : i \in \mathbb{n}\}$ is listed in increasing order, then

$$\text{osc}(f) = |\{i \in n - 1 : f(x_i) \neq f(x_{i+1})\}|.$$  

Observe that if $f \ll g$, then $\text{osc}(f + g) \geq \text{osc}(f) + \text{osc}(g)$, with equality iff the last value of $f$ equals the first value of $g$ (if those values are different, $\text{osc}(f + g) = \text{osc}(f) + \text{osc}(g) + 1$).

Now consider a sequence $< f_i : i \in \mathbb{n} >$ from a subspace $W$ of $V$ such that $f_i \ll f_{i+1}$. By an appropriate scalar multiplication, we may assume that $\hat{f}_i = \hat{f}_{i+1}$ for all $i \in n - 1$, and therefore $s := \text{osc}(\sum_{i \in \mathbb{n}} f_i) = \sum_{i \in \mathbb{n}} \text{osc}(f_i)$. Since $\mathbb{F} \neq \mathbb{F}_2$, choose for each $i \in n - 1$ an $a_i \neq 0 \in \mathbb{F}$ such that $a_i \neq a_{i+1}$.

But now observe that for each $j \in \mathbb{n}$,

$$\text{osc} \left( \sum_{i=0}^{j} a_i f_i + a_{j+1} \sum_{i=j+1}^{n-1} f_i \right) = s + j + 1.$$

This means that on any infinite dimensional subspace or even affine subspace of $\mathbb{F}[\omega]$, the range of the oscillation function contains arbitrarily long intervals. Hence, there cannot be any canonical partition. 

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### 3.3 Weak Indivisibility

#### 3.3.1 Weak Indivisibility over Finite Fields

Let $\mathbb{F}_q$ be the finite field of $q$ elements. We shall prove that $V_q = \mathbb{F}_q[\omega]$ is weakly indivisible.

**Lemma 8.** Let $k \in \omega$, $W$ a subspace of $V_q$ of dimension at least $k + 1$, and $x \in \omega$ arbitrary. Then there exists a $k$-dimensional subspace $U$ of $W$ such that $x \notin \text{supp}(U)$.

**Proof.** Let $\{f_i : i \in k + 1\}$ be linearly independent elements of $W$, and let $X = \{i : f_i(x) = 0\}$. If $X$ has size at least $k$, then it generates a subspace $U$ as desired.

Else for $i \in (k + 1) \setminus X$, we may assume that $f_i(x) = 1$ by replacing $f_i$ if necessary by $(f_i(x))^{-1} f_i$. Then fixing any $i_0 \in (k + 1) \setminus X$, the set $\{f_i : i \in X \} \cup \{f_j - f_{i_0} : j \in (k + 1) \setminus (X \cup \{i_0\})\}$ is a basis for $U$ as desired. 

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**Corollary 9.** Let $k \in \omega$, $W$ be a subspace of $V_q$ of dimension at least $k+1$, and $x \in \text{supp}(W)$. Then for any $a \in \mathbb{F}_q$, there exists an affine $k$-dimensional subspace $A$ of $W$ so that $f(x) = a$ for all $f \in A$.

**Proof.** Let $w \in W$ such that $w(x) \neq 0$, and by Lemma 8 let $U$ a $k$-dimensional subspace of $W$ such that $f(x) = 0$ for all $f \in U$. Then $A = \{a(w(x))^{-1}w + f : f \in U\}$ is the desired affine $k$-dimensional subspace. \qed

A colouring of a subset $W$ of $V_q$ is called end-determined if, for every $a \in \mathbb{F}_q \setminus 0$, the set $\{f \in W : \hat{f} = a\}$ is monochromatic.

**Lemma 10.** Let $k \in \omega$, $W$ be a subspace of $V_q$ of dimension at least $k+1$, and $v \in V_q$ with $\text{supp}(v) \ll \text{supp}(W)$. Let $\Delta$ be a colouring of the affine space $v + W$ into red and blue elements so that $\Delta(v) = \text{blue}$, and so that $v + W$ does not contain a monochrome red affine subspace of dimension $k$. If the colouring $\Delta$ is end-determined on $v + W$, then every element of $v + W$ is blue.

**Proof.** Assume for a contradiction that $h \in v + W$ is red, and let $a = \hat{h}$. Then every $g \in v + W$ with $\hat{g} = a$ is red since $\Delta$ is end-determined on that space. Hence may assume without loss of generality that $\text{maxsup}(h) = \text{maxsup}(W)$.

According to Corollary 9, there exists an affine $k$-dimensional subspace $A$ of $W$ so that $\hat{f} = a$ for every $f \in A$. But then $v + A$ is an affine red subspace of dimension $k$, a contradiction. \qed

**Lemma 11.** Let $k \leq d \in \omega$, $v \in V_q$, and $V$ a $\text{GLR}(d+1,2^{q-1})$-dimensional subspace of $V_q$ with $v \ll V$. Let $\Delta$ be a colouring of the affine space $v + V$ into red and blue elements so that $\Delta(v) = \text{blue}$ and $v + V$ does not contain a monochrome red affine subspace of dimension $k$.

Then there exists a $d$-dimensional subspace $U$ of $V$ so that every element of the affine space $v + U$ is blue.

**Proof.** Colour every line $L = \langle f \rangle = \{af : a \in \mathbb{F}_q\}$ (where $\hat{f} = 1$) of $V$ with the function $\gamma_L : \mathbb{F}_q \setminus \mathbb{F}_q \rightarrow \{\text{red, blue}\}$ given by $\gamma_L(a) = \Delta(v + af)$; that is with one of $2^{q-1}$ possible colours. Denote by $\Gamma$ this colouring of the set of lines in $V$ with $2^{q-1}$ colours.

Then by Theorem 1 there exists a $d+1$-dimensional subspace $W$ of $V$ and a function $\gamma : \mathbb{F}_q \setminus \mathbb{F}_q \rightarrow \{\text{red, blue}\}$ so that $\Gamma(\langle f \rangle) = \gamma$ for every line $\langle f \rangle \in W$.

But this means that the colouring $\Delta$ is end-determined on the affine subspace $v + W$, which is therefore by assumption and Lemma 10 monochrome blue. \qed
Let $\Delta$ be a colouring of $V$ into red and blue elements so that there is no monochrome red affine subspace of dimension $k$, and so that $\Delta$ is monochrome blue on $v + A$.

Then there exists a $d$-dimensional subspace $W$ of $V$ with $v + A \ll W$ and so that every element in $(v + A) + W$ is blue.

Proof. List the elements of $v + A$ as $f_{n-1}, f_{n-2}, f_{n-3}, \ldots, f_0$, and using Lemma 4, let $W_{n-1}$ be a $\Pi_n(d)$-dimensional subspace of $V$ with $v + A \ll W_{n-1}$.

Then $v + f_{n-1} + W_{n-1}$ is an affine space with $v + f_{n-1}$ blue. According to Lemma 11 there there exists a $\Pi_{n-1}(d)$-dimensional subspace $W_{n-2}$ of $W_{n-1}$ so that $(v + f_{n-1}) + W_{n-2}$ is monochrome blue.

More generally, assume that we have $1 \leq i < n$ and a $\Pi_{n-i}(d)$-dimensional subspace $W_{n-(i+1)}$ of $W_{n-i}$ so that the space $(v + f_{n-i}) + W_{n-(i+1)}$ is monochrome blue. Then $v + f_{n-(i+1)} + W_{n-(i+1)}$ is an affine space with $v + f_{n-(i+1)}$ blue. According to Lemma 11 there there exists a $\Pi_{n-(i+1)}(d)$-dimensional subspace $W_{n-(i+2)}$ of $W_{n-(i+1)}$ so that the space $(v + f_{n-(i+1)}) + W_{n-(i+2)}$ is monochrome blue.

We continue and for $i = n$ obtain a $d$-dimensional subspace $U$ of $W_{n-1}$ so that for every $v + f \in v + A$ and every $g \in U$ the element $v + f + g$ is blue.

We now come to the main result of this section: $V_q$ is weakly indivisible.

Theorem 13. Let $V$ be a countable dimensional subspace of $V_q$, $k \in \omega$, and $\Delta$ a colouring of $V$ into red and blue elements so that $V$ contains no monochrome red affine $k$-dimensional subspace. Then there exists a monochrome blue affine subspace of $V$ of infinite dimension.

Proof. The space $V$ must contain at least one blue element $v$. Then $\{v\}$ is a 0-dimensional subspace which is monochrome blue. We obtain the blue affine subspace of infinite dimension by repeated applications of Lemma 12.
3.3.2 Weak Indivisibility over Infinite Fields

If the field $\mathbb{F}$ is infinite, Theorem 7 has the following strengthening, namely that $\mathbb{F}[\omega]$ is not weakly indivisible.

**Theorem 14.** Every countable dimensional vector space $V$ over an infinite field $\mathbb{F}$ is not weakly indivisible.

In fact $V$ can be divided into two parts so that neither part contains an affine line.

**Proof.** If $\mathbb{F}$ is a field of infinite size $\kappa$, then since the space is of countable dimension we can enumerate the affine lines of $V$ as $< L_\alpha : \alpha \in \kappa >$. The intended set $A$ will be constructed recursively as a sequence $< a_\alpha : \alpha \in \kappa > \subseteq V$ such that for every $\alpha \in \kappa$, $a_\alpha \in L_\alpha$ and such that no affine line intersects $A_\alpha := \{a_\beta : \beta \in \alpha \}$ in more than two points. To do so, we pick $a_0 \in L_0$ arbitrary. Having defined $A_\alpha$, let $L_\alpha$ be the set of affine lines containing two distinct points of $A_\alpha$, if any. If $L_\alpha$ already intersects $A_\alpha$, let $a_\alpha \in (L_\alpha \cap A_\alpha)$. If not, observe that since any two distinct affine lines intersect in at most one point, $L_\alpha$ must have size less than $\kappa$, and therefore we can choose $a_\alpha \in L_\alpha \setminus (\bigcup L_\alpha)$.

In [1], Baumgartner proved the analog result for a vector space of any dimension over the field of rational numbers.

Thus any countable dimensional vector space over an infinite field provides an example of an age indivisible but not weakly indivisible homogeneous relational structure.

4 Midpoint Structure

In this section, we shall produce a somewhat simpler example of a countable age indivisible but not weakly indivisible relational structure. The structure will have a single ternary relation, but is not homogeneous.

Before we proceed, let $M$ be a commutative monoid. An arithmetic progression of length $n$ in $M$ is a sequence of the form $(a + ix)_{i \in n}$ for some $a \in M$ and $x \in M \setminus \{0\}$. An infinite arithmetic progression is defined similarly. Clearly, an arithmetic progression of length 3 is a sequence of three elements $a_0, a_1, a_2$ of $M$ such that $2a_1 = a_0 + a_2$. Set $\mu_M := \{(x, y, z) \in M^3 : 2y = x + z\}$ and let $\mathcal{M}_M := (M, \mu_M)$. In the case $M = \mathbb{N}$, we make the convention that $\mu_\mathbb{N}$ denotes the set of triples associated to the additive monoid on the non-negative integers.
In the case where $M = V$ is a vector space over $\mathbb{Q}$, which will be the main case of interest, then $\mu_V$ denotes the set of triples associated with the addition on $V$. Notice that in this case an arithmetic progression of length 3 is a sequence of three elements $a_0, a_1, a_2$ where $a_1$ is the mid-point of the segment joining $a_0$ and $a_2$. The ternary relational structure $\mathcal{M}_V = (V, \mu_V)$ is the midpoint structure associated with $V$. We shall show in particular that $\mathcal{M}_V$ is age indivisible but not weakly indivisible.

We first characterize the embeddings of such a structure $\mathcal{M}_V$ as simply the affine embedding of the underlying vector space.

**Lemma 15.** Let $V$ and $V'$ be two vector spaces over $\mathbb{Q}$. A map $\alpha : V \to V'$ is an embedding of $\mathcal{M}_V$ into $\mathcal{M}_{V'}$ if and only if it is an affine embedding of the underlying vector spaces.

**Proof.** First an affine embedding $\alpha$ of $V$ into $V'$ does satisfy $\alpha\left(\frac{a+b}{2}\right) = \frac{\alpha(a) + \alpha(b)}{2}$ for all $a, b \in V$, and is therefore an embedding of $\mathcal{M}_V$ into $\mathcal{M}_{V'}$.

Conversely this condition implies that $\alpha$ is an affine transformation, indeed it suffices to show that $\beta : V \to V'$ given by $\beta(a) = \alpha(a) - \alpha(0)$ for all $a \in V$ is a linear transformation.

Note that:

$$\frac{\alpha(a) + \alpha(b)}{2} = \alpha\left(\frac{a+b}{2}\right) = \alpha\left(\frac{a+b+0}{2}\right) = \alpha\left(\frac{a+b}{2}\right) + \alpha(0).$$

Hence $\alpha(a+b) = \alpha(a) + \alpha(b) - \alpha(0)$, and therefore $\beta(a+b) = \alpha(a+b) - \alpha(0) = \alpha(a) + \alpha(b) - 2\alpha(0) = \beta(a) + \beta(b)$. It follows immediately that $\beta(xa) = x\beta(a)$ for all rational $x$. Therefore $\beta$ is a linear transformation, and since moreover $\alpha$ is one to one, then it is an affine embedding as desired. 

We therefore immediately have the following Corollary.

**Corollary 16.** The group $\text{Aut}(\mathcal{M}_V)$ of automorphisms of $\mathcal{M}_V$ and the Inhomogeneous General Linear group $\text{IGL}(V)$ of $V$ coincide. Moreover, the closure of $\text{Aut}(\mathcal{M}_V)$ consists of exactly the affine embeddings of $V$ into itself, namely the set $\text{Emb}(V)$.

The age indivisibility of $\mathcal{M}_V$ will follow from the following Lemma.

**Lemma 17.** If $G$ is a torsion free abelian group then $\mathcal{M}_G$ and $\mathcal{M}_\mathbb{N}$ have the same age.

**Proof.** Since $\mathbb{N}$ can be identified with a subgroup of $G$, the age of $\mathcal{M}_\mathbb{N}$ is a subset of the age of $\mathcal{M}_G$. 

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Conversely let $F$ be a finite subset of $G$. We will show that there is a map $f : F \to \mathbb{N}$ which is an embedding of $\mathcal{M}_G \restriction F$ into $\mathcal{M}_\mathbb{N}$.

Since $G$ is torsion free, the subgroup generated by $F$ is isomorphic to a finite direct sum of the integers $\mathbb{Z}$. Without loss of generality, we may therefore suppose that $F \subseteq \mathbb{Z}_n^+$, and let $k \in \mathbb{N}$ be large enough such that $F \subseteq k^n$. Finally define $\sigma : k^n \to \mathbb{N}$ by

$$\sigma(<x_i : i \in k>) = \sum_i x_i k^i.$$ 

Then $\sigma \restriction F$ is easily seen to be the required embedding. \qed 

The following tool is key in showing the failure of weak indivisibility, and the proof is similar to that of Theorem 14.

**Lemma 18.** A countable commutative monoid $M$ in which equations of the form $a + x = b$ and $a + 2x = b$ have only finitely many solutions in $x$ for all $a, b \in M$ contains a subset having no three elements forming an arithmetic progression, but containing a point from every infinite arithmetic progression.

**Proof.** Note that the set of infinite arithmetic progressions is also countable. Also note that if $S \subseteq M$ is finite and $X$ is an infinite arithmetic progression with $X \cap S = \emptyset$ then there is an element $x \in X \setminus S$ which does not form a three element arithmetic progression with any two of the elements in $S$. This follows from the fact that, for every two elements $\{a, b\}$ in $S$, there are only finitely many equations of the above type in $M$ potentially producing a three element arithmetic progression with $\{a, b\}$, each with only finitely many solutions by hypothesis. Namely for any elements $x, y \in M$ with $a + x = b$ or $b + y = a$ respectively, then each of $a + 2x$ and $b + 2y$ will form an arithmetic progression with $\{a, b\}$; moreover for any such $x, y$ as above, then for any elements $x', y' \in M$ with $x' + x = a$ or $y' + y = b$ respectively, then $x'$ and $y'$ will also form an arithmetic progression with $\{a, b\}$; finally for any elements $x, y \in M$ with $a + 2x = b$ or $b + 2y = a$ respectively, then $a + x$ and $b + y$ will form an arithmetic progression with $\{a, b\}$.

Now enumerate the elements of $M$ into the $\omega$-sequence $x_0, \ldots, x_n, \ldots$ and the set of infinite arithmetic progressions into the $\omega$-sequence $X_0, \ldots, X_n, \ldots$. We construct the sequence $y_0, \ldots, y_n, \ldots$ such that for every integer $n$, the set $Y_n := \{y_i : i \in n\}$ contains no three elements forming an arithmetic progression, but meets $X_i$ for every $i \in n$. The element $y_0$ is an arbitrary element in $X_0$. If $Y_n$ is already constructed and $X_n \cap Y_n \neq \emptyset$, then let $y_n \in X_n \cap Y_n$. If on the other hand $X_n \cap Y_n = \emptyset$, then let $y_n \in X_n$ such
that it does not form a three element arithmetic progression with any pair
of elements in \( Y_n \). This completes the proof.

We are now ready to prove the main result of this section.

**Theorem 19.** Let \( V \) be a vector space of countable dimension over \( \mathbb{Q} \) and
\( M_V := (V, \mu_V) \) be the midpoint structure associated with the vector space
\( V \). Then:

1. \( M_V \) is age indivisible.
2. \( M_V \) is not weakly indivisible.
3. \( M_V \) is universal for its age: every countable \( R := (R, \mu') \) with the
   same age as \( M_V \) is embeddable into \( M_V \).

**Proof.** We prove each part separately.

Item 1. We already observed in Section 2 that two relational structures with
the same age either are both age indivisible or both age divisible.

According to Lemma 17, \( M_V \) and \( M_N \) have the same age. Thus it suffices
to prove that for each subset \( A \) of \( \mathbb{N} \) either \( M_N \upharpoonright A \) or \( M_N \upharpoonright (\mathbb{N} \setminus A) \) has the
same age as \( M_N \). This amounts to say that for each integer \( n \), \( M_N \upharpoonright [0, n] \) into a subset amounts the existence of an arithmetic progression
\( (a + ix)_{i \in \mathbb{N}} \) into that subset. Van der Waerden’s theorem on arithmetic
progressions [9] ensures the required conclusion.

Item 2. The additive structure on \( V \) is a torsion free abelian group and
hence satisfies the requirements of Lemma 18. Let \( A \) be given by Lemma 18
and let \( B := V \setminus A \). The age of \( M_V \upharpoonright A \) is a proper subset of the age of
\( M_V \), because \( A \) does not contain a three element arithmetic progression and
hence does not contain a triple in the relation \( \mu_V \). According to Lemma 15,
an embedding \( \alpha \) from \( M_V \) into itself is an affine map. There does not exist
such an embedding whose range is a subset of \( B \) because \( B \) contains no
affine line.

Item 3. Let \( V' := \mathbb{Q}[\mathbb{R}] \) be the set of functions \( h : \mathbb{R} \to \mathbb{Q} \) which are 0 almost
everywhere. Let \( \delta : \mathbb{R} \to V' \) be the map defined by \( \delta(x)(y) := 1 \) if \( x = y \)
and \( \delta(x)(y) := 0 \) otherwise. \( V' \) is a vector space over \( \mathbb{Q} \) under the natural
addition and scalar multiplication operations, and let \( W \) be the subspace
of \( V' \) generated by the vectors of the form \( \delta(x) + \delta(z) - 2\delta(y) \) such that
\( (x, y, z) \in \mu' \). Let \( V'/W \) be the quotient of \( V' \) by \( W \) and let \( \rho : V' \to V'/W \)
be the quotient map.
Claim 1. The map \( \rho' := \rho \circ \delta \) is an embedding of \( R \) into \( M_{V'/W} \).

Proof of Claim 1. We first verify that \((x, y, z) \in \mu'\) if and only if \((\rho'(x), \rho'(y), \rho'(z)) \in \mu_{V'/W}\). The “only if” part of this equivalence is immediate: by definition of \( W \), \((x, y, z) \in \mu'\) implies \( \delta(x) + \delta(z) - 2\delta(y) \in W \). This amounts to \( \rho(\delta(x) + \delta(z) - 2\delta(y)) = 0 \), that is \( \rho(\delta(x)) + \rho(\delta(z)) - 2\rho(\delta(y)) = 0 \) which rewrites \( \rho'(x) + \rho'(z) - 2\rho'(y) = 0 \), that is \( (\rho'(x), \rho'(y), \rho'(z)) \in \mu_{V'/W} \).

For the “if” part, it suffices to show that \( \delta(x) + \delta(z) - 2\delta(y) \in W \) implies \((x, y, z) \in \mu'\). So suppose that \( \delta(x) + \delta(z) - 2\delta(y) \) is a finite linear combination \( \sum_{i \in n} \lambda_i (\delta(x_i) + \delta(z_i) - 2\delta(y_i)) \) where \((x_i, y_i, z_i) \in \mu'\) and \( \lambda_i \in \mathbb{Q} \) for each \( i \in n \). Let \( F' := \{x, y, z, x_i, y_i, z_i : i \in n\} \), and by hypothesis let \( f \) be an isomorphism of \( R_{|F} \) into \( V \). As a map defined on a subset of \( R \), \( f \) extends to a linear map \( \mathcal{F} \) from \( V' \) to \( V \). As such it satisfies:

\[
\mathcal{F}(\delta(x)) + \mathcal{F}(\delta(z)) - 2\mathcal{F}(\delta(y)) = \sum_{i \in n} \lambda_i (\mathcal{F}(\delta(x_i)) + \mathcal{F}(\delta(z_i)) - 2\mathcal{F}(\delta(y_i))).
\]

Since \( f \) preserves \( \mu' \), \((f(x_i), f(y_i), f(z_i)) \in \mu_V \) for all \( i \in n \), hence \( \mathcal{F}(\delta(x_i)) + \mathcal{F}(\delta(z_i)) - 2\mathcal{F}(\delta(y_i)) = 0 \). This yields \( \mathcal{F}(\delta(x)) + \mathcal{F}(\delta(z)) - 2\mathcal{F}(\delta(y)) = 0 \), that is \((f(x), f(y), f(z)) \in \mu_V \) from which it follows that \((x, y, z) \in \mu'\).

To conclude, it suffices to prove that \( \rho' \) is one to one. Let \( a, a' \in R \) such that \( \rho'(a) = \rho'(a') \). This means that \( \delta(a) - \delta(a') \) is a finite linear combination \( \sum_{i \in n} \lambda_i (\delta(x_i) + \delta(z_i) - 2\delta(y_i)) \) where \((x_i, y_i, z_i) \in \mu'\) and \( \lambda_i \in \mathbb{Q} \) for each \( i \in n \). In order to prove that this linear combination is zero, we use the same technique that above. Let \( F' := \{a, a', x_i, y_i, z_i : i \in n\} \), and by hypothesis let \( f \) be an isomorphism of \( R_{|F} \) into \( V \). The map \( f \) extends to a linear map \( \mathcal{F} \) from \( V' \) to \( V \). It satisfies:

\[
\mathcal{F}(\delta(a)) - \mathcal{F}(\delta(a')) = \sum_{i \in n} \lambda_i (\mathcal{F}(\delta(x_i)) + \mathcal{F}(\delta(z_i)) - 2\mathcal{F}(\delta(y_i))).
\]

Since \( f \) preserves \( \mu' \), \((f(x_i), f(y_i), f(z_i)) \in \mu_V \) for all \( i \in n \), hence \( \mathcal{F}(\delta(x_i)) + \mathcal{F}(\delta(z_i)) - 2\mathcal{F}(\delta(y_i)) = 0 \). Hence \( \mathcal{F}(\delta(a)) = \mathcal{F}(\delta(a')) \) from which it follows that \( a = a' \). This proves our claim. \( \square \)

Since \( R \) is countable, \( V'/W \) is countable and hence it is embeddable into \( V \) by some linear map. It follows that \( M_{V'/W} \) is embeddable into \( M_V \).

Using Claim 1, it follows that \( R \) is embeddable into \( M_V \). This completes the proof of Theorem 19.
5 Conclusion

We have seen that a countable dimensional vector space over an infinite field is age indivisible, not weakly indivisible. Although homogeneous as a relational structure, it has infinitely many relations. The midpoint structure above is also age indivisible, not weakly indivisible. In this case, as a relational structure, it has a single ternary relation, but is not homogeneous. It can be made homogeneous by adding relations, but as above an infinite number is required.

It is natural to impose finiteness conditions in asking for a countable age indivisible, not weakly indivisible, homogeneous relational structure. Beside a finite number of relations, one may ask for an oligomorphic automorphism group (finite number or orbits on \(n\)-tuples for each \(n\), [2]), or an automorphism group of finite arity (types of \(n\)-tuples determined by their \(r\)-tuples for a fixed \(r\), see [3]). We therefore ask:

**Open Question 1.** *Can one impose “finiteness” conditions for an age indivisible, not weakly indivisible, countable homogeneous relational structure?*

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