THE A-POLYNOMIAL OF THE $(-2, 3, 3 + 2n)$ PRETZEL KNOTS

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Abstract. We show that the A-polynomial $A_n$ of the 1-parameter family of pretzel knots $K_n = (-2, 3, 3 + 2n)$ satisfies a linear recursion relation of order 4 with explicit constant coefficients and initial conditions. Our proof combines results of Tamura-Yokota and the second author. As a corollary, we show that the A-polynomial of $K_n$ and the mirror of $K_{-n}$ are related by an explicit $GL(2, \mathbb{Z})$ action. We leave open the question of whether or not this action lifts to the quantum level.

1. Introduction

1.1. The behavior of the A-polynomial under filling. In [CCG+94], the authors introduced the A-polynomial $A_W$ of a hyperbolic 3-manifold $W$ with one cusp. It is a 2-variable polynomial which describes the dependence of the eigenvalues of a meridian and longitude under any representation of $\pi_1(W)$ into $SL(2, \mathbb{C})$. The A-polynomial plays a key role in two problems:

- the deformation of the hyperbolic structure of $W$,
- the problem of exceptional (i.e., non-hyperbolic) fillings of $W$.

Knowledge of the A-polynomial (and often, of its Newton polygon) is translated directly into information about the above problems, and vice-versa. In particular, as demonstrated by Boyer and Zhang [BZ01], the Newton polygon is dual to the fundamental polygon of the Culler-Shalen seminorm [CGLS87] and, therefore, can be used to classify cyclic and finite exceptional surgeries.

In [Gar10], the first author observed a pattern in the behavior of the A-polynomial (and its Newton polygon) of a 1-parameter family of 3-manifolds obtained by fillings of a 2-cusped manifold. To state the pattern, we need to introduce some notation. Let $K = \mathbb{Q}(x_1, \ldots, x_r)$ denote the field of rational functions in $r$ variables $x_1, \ldots, x_r$.

Definition 1.1. We say that a sequence of rational functions $R_n \in K$ (defined for all integers $n$) is holonomic if it satisfies a linear recursion with constant coefficients. In other words, there exists a natural number $d$ and $c_k \in K$ for $k = 0, \ldots, d$ with $c_d \neq 0$ such that for all integers $n$ we have:

$$
\sum_{k=0}^{d} c_k R_{n+k} = 0
$$

Depending on the circumstances, one can restrict attention to sequences indexed by the natural numbers (rather than the integers).

Consider a hyperbolic manifold $W$ with two cusps $C_1$ and $C_2$. Let $(\mu_i, \lambda_i)$ for $i = 1, 2$ be pairs of meridian-longitude curves, and let $W_n$ denote the result of $-1/n$ filling on $C_2$. Let $A_n(M_1, L_1)$ denote the A-polynomial of $W_n$ with the meridian-longitude pair inherited from $W$.

Theorem 1.1. [Gar10] With the above conventions, there exists a holonomic sequence $R_n(M_1, L_1) \in \mathbb{Q}(M_1, L_1)$ such that for all but finitely many integers $n$, $A_n(M_1, L_1)$ divides the numerator of $R_n(M_1, L_1)$. In addition, a recursion for $R_n$ can be computed explicitly via elimination, from an ideal triangulation of $W$.
1.2. The Newton polytope of a holonomic sequence. Theorem 1.1 motivates us to study the Newton polytope of a holonomic sequence of Laurent polynomials. To state our result, we need some definitions. Recall that the Newton polytope of a Laurent polynomial in \( n \) variables \( x_1, \ldots, x_n \) is the convex hull of the points whose coordinates are the exponents of its monomials. Recall that a quasi-polynomial is a function \( p : \mathbb{N} \to \mathbb{Q} \) of the form \( p(n) = \sum_{k=0}^{d} c_k(n)n^k \) where \( c_k : \mathbb{N} \to \mathbb{Q} \) are periodic functions. When \( c_d \neq 0 \), we call \( d \) the degree of \( p(n) \). We will call quasi-polynomials of degree at most one (resp. two) quasi-linear (resp. quasi-quadratic). Quasi-polynomials appear in lattice point counting problems (see [Ehr62, CW10]), in the Slope Conjecture in quantum topology (see [Gar11b]), in enumerative combinatorics (see [Gar11a]) and also in the \( A \)-polynomial of filling families of 3-manifolds (see [Gar10]).

**Definition 1.2.** We say that a sequence \( N_n \) of polytopes is linear (resp. quasi-linear) if the coordinates of the vertices of \( N_n \) are polynomials (resp. quasi-polynomials) in \( n \) of degree at most one. Likewise, we say that a sequence \( N_n \) of polytopes is quadratic (resp. quasi-quadratic) if the coordinates of the vertices of \( N_n \) are polynomials (resp. quasi-polynomials) of degree at most two.

**Theorem 1.2.** [Gar10] Let \( N_n \) be the Newton polytope of a holonomic sequence \( R_n \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \). Then, for all but finitely many indices \( n \), \( N_n \) is quasi-linear.

1.3. Do favorable links exist? Theorems 1.1 and 1.2 are general, but in favorable circumstances more is true. Namely, consider a family of knot complements \( K_n \), obtained by \(-1/n\) filling on a cusp of a 2-component hyperbolic link \( J \). Let \( f \) denote the linking number of the two components of \( J \), and let \( A_n \) denote the \( A \)-polynomial of \( K_n \) with respect to its canonical meridian and longitude \((M, L)\). By definition, \( A_n \) contains all components of irreducible representations, but not the component \( L - 1 \) of abelian representations.

**Definition 1.3.** We say that \( J \), a 2-component link in 3-space, with linking number \( f \) is favorable if \( A_n(M, LM^{-f^2/n}) \in \mathbb{Q}[M^{\pm 1}, L^{\pm 1}] \) is holonomic.

The shift of coordinates, \( LM^{-f^2/n} \), above is due to the canonical meridian-longitude pair of \( K_n \) differing from the corresponding pair for the unfilled component of \( J \) as a result of the nonzero linking number. Theorem 1.2 combined with the above shift implies that, for a favorable link, the Newton polygon of \( K_n \) is quasi-quadratic.

Hoste-Shanahan studied the first examples of a favorable link, the Whitehead link and its half-twisted version (see Figure 1), and consequently gave an explicit recursion relation for the 1-parameter families of \( A \)-polynomials of twist knots \( K_{2,n} \) and \( K_{3,n} \) respectively; see [HS04].

![Figure 1. The Whitehead link on the left, the half-twisted Whitehead link in the middle and our seed link \( J \) at right.](image)

The goal of our paper is to give another example of a favorable link \( J \) (see Figure 1), whose 1-parameter filling gives rise to the family of \((-2, 3, 3 + 2n)\) pretzel knots. Our paper is a concrete illustration of the general Theorems 1.1 and 1.2 above. Aside from this, the 1-parameter family of knots \( K_n \), where \( K_n \) is the \((-2, 3, 3 + 2n)\) pretzel knot, is well-studied in hyperbolic geometry (where \( K_n \) and the mirror of \( K_{-n} \) are pairs of geometrically similar knots; see [BH96, MM08]), in exceptional Dehn surgery (where for instance
$K_2 = (-2, 3, 7)$ has three Lens space fillings $1/0, 18/1$ and $19/1$; see [CGLS87]) and in Quantum Topology (where $K_n$ and the mirror of $K_{-n}$ have different Kashaev invariant, equal volume, and different subleading corrections to the volume, see [GZ]).

The success of Theorems 1.3 and 1.4 below hinges on two independent results of Tamura-Yokota and the second author [TY04, Mat02], and an additional lucky coincidence. Tamura-Yokota compute an explicit recursion relation, as in Theorem 1.3, by elimination, using the gluing equations of the decomposition of the complement of $J$ into six ideal tetrahedra; see [TY04]. The second author computes the Newton polygon $N_n$ of the $A$-polynomial of the family $K_n$ of pretzel knots; see [Mat02]. This part is considerably more difficult, and requires:

(a) The set of boundary slopes of $K_n$, which are available by applying the Hatcher-Oertel algorithm [HO89, Dun01] to the 1-parameter family $K_n$ of Montesinos knots. The four slopes given by the algorithm are candidates for the slopes of the sides of $N_n$. Similarly, the fundamental polygon of the Culler-Shalen seminorm of $K_n$ has vertices in rays which are the multiples of the slopes of $N_n$. Taking advantage of the duality of the fundamental polygon and Newton polygon, in order to describe $N_n$ it is enough to determine the vertices of the Culler-Shalen polygon.

(b) Use of the exceptional $1/0$ filling and two fortunate exceptional Seifert fillings of $K_n$ with slopes $4n + 10$ and $4n + 11$ to determine exactly the vertices of the Culler-Shalen polygon and consequently $N_n$. In particular, the boundary slope $0$ is not a side of $N_n$ (unless $n = -3$) and the Newton polygon is a hexagon for all hyperbolic $K_n$.

Given the work of [TY04] and [Mat02], if one is lucky enough to match $N_n$ of [Mat02] with the Newton polygon of the solution of the recursion relation of [TY04] (and also match a leading coefficient), then Theorem 1.3 below follows; i.e., $J$ is a favorable link.

1.4. Our results for the pretzel knots $K_n$. Let $A_n(M, L)$ denote the $A$-polynomial of the pretzel knot $K_n$, using the canonical meridian-longitude coordinates. Consider the sequences of Laurent polynomials $P_n(M, L)$ and $Q_n(M, L)$ defined by:

$$P_n(M, L) = A_n(M, LM^{-4n})$$

for $n > 1$ and

$$Q_n(M, L) = A_n(M, LM^{-4n})M^{-4(3n^2+11n+4)}$$

for $n < -2$ and $Q_{-2}(M, L) = A_{-2}(M, LM^{-8})M^{-20}$. In the remaining cases $n = -1, 0, 1$, the knot $K_n$ is not hyperbolic (it is the torus knot $5_1$, $8_{19}$ and $10_{124}$ respectively), and one expects exceptional behavior. This is reflected in the fact that $P_n$ for $n = 0, 1$ and $Q_n$ for $n = -1, 0$ can be defined to be suitable rational functions (rather than polynomials) of $M, L$. Let $NP_n$ and $NQ_n$ denote the Newton polygons of $P_n$ and $Q_n$ respectively.

Theorem 1.3. (a) $P_n$ and $Q_n$ satisfy linear recursion relations

$$\sum_{k=0}^{4} c_k P_{n+k} = 0, \quad n \geq 0$$

and

$$\sum_{k=0}^{4} c_k Q_{n-k} = 0, \quad n \leq 0$$

where the coefficients $c_k$ and the initial conditions $P_n$ for $n = 0, \ldots, 3$ and $Q_n$ for $n = -3, \ldots, 0$ are given in Appendix A.

(b) In $(L, M)$ coordinates, $NP_n$ and $NQ_n$ are hexagons with vertices

$$(0, 0), (1, -4n + 16), (n - 1, 12n - 12), (2n + 1, 16n + 18), (3n - 1, 32n - 10), (3n, 28n + 6)$$

for $P_n$ with $n > 1$ and

$$(0, 4n + 28), (1, 38), (-n, -12n + 26), (-2n - 3, -16n - 4), (-3n - 4, -28n - 16), (-3n - 3, -32n - 6)$$

for $Q_n$. In particular, the boundary slope $0$ is not a side of $NP_n$ and $NQ_n$, respectively.
for \( Q_n \) with \( n < -1 \).

**Remark 1.4.** We can give a single recursion relation valid for \( n \in \mathbb{Z} \setminus \{-1, 0, 1\} \) as follows. Define

\[
R_n(M, L) = A_n(M, LM^{-4n})b^{|n|}c(M),
\]

where

\[
b = \frac{1}{LM^8(1-M^2)(1+LM^4)}
\]

\[
c = \frac{L^3M^{12}(1-M^2)^3}{(1+LM^4)^3}
\]

\[
\epsilon_n(M) = \begin{cases} 
1 & \text{if } n > 1 \\
LM^{-4(3+n)(2+3n)} & \text{if } n < -2 \\
cM^{-28} & \text{if } n = -2
\end{cases}
\]

Then, \( R_n \) satisfies the palindromic fourth order linear recursion

\[
\sum_{k=0}^{4} \gamma_k R_{n+k} = 0
\]

where the coefficients \( \gamma_k \) and the initial conditions \( R_n \) for \( n = 0, \ldots, 3 \) are given in Appendix B. Moreover, \( R_n \) is related to \( P_n \) and \( Q_n \) by:

\[
R_n = \begin{cases} 
P_n b^{|n|} & \text{if } n > 0 \\
Q_n b^{|n|}cM^{-8} & \text{if } n \leq 0
\end{cases}
\]

**Remark 1.5.** The computation of the Culler-Shalen seminorm of the pretzel knot \( K_n \) has an additional application, namely it determines the number of components (containing the character of an irreducible representation) of the \( \text{SL}(2, \mathbb{C}) \) character variety of the knot, and consequently the number of factors of its \( A \)-polynomial. In the case of \( K_n \), (after translating the results of [Mat02] for the pretzel knots \((-2, 3, n)\) to the pretzel knots \((-2, 3, 3 + 2n)\)) it was shown by the second author [Mat02, Theorem 1.6] that the character variety of \( K_n \) has one (resp. two) components when 3 does not divide \( n \) (resp. divides \( n \)). The non-geometric factor of \( A_n \) is given by

\[
\begin{cases} 
1 - LM^{4(n+3)} & n \geq 3 \\
L - M^{4(n+3)} & n \leq -3
\end{cases}
\]

for \( n \neq 0 \) a multiple of 3.

Since the \( A \)-polynomial has even powers of \( M \), we can define the \( B \)-polynomial by

\[
B(M^2, L) = A(M, L).
\]

Our next result relates the \( A \)-polynomials of the geometrically similar pair \((K_n, -K_{-n})\) by an explicit \( \text{GL}(2, \mathbb{Z}) \) transformation.
Theorem 1.4. For $n > 1$ we have:

\[(12)\quad B_{-n}(M, LM^{2n-5}) = (-L)^n M^{8(2n^2 - 7n + 7)} B_n(-L^{-1}, L^{2n+5} M^{-1}) \eta_n\]

where $\eta_n = 1$ (resp. $M^{22}$) when $n > 2$ (resp. $n = 2$).

2. Proofs

2.1. The equivalence of Theorem 1.3 and Remark 1.4. In this subsection we will show the equivalence of Theorem 1.3 and Remark 1.4. Let $\gamma_k = c_k/b^5$ for $k = 0, \ldots, 4$ where $b$ is given by (9). It is easy to see that the $\gamma_k$ are given explicitly by Appendix B, and moreover, they are palindromic. Since $R_n = P_n b^n$ for $n = 0, \ldots, 3$ it follows that $R_n$ and $P_n b^n$ satisfy the same recursion relation (10) for $n \geq 0$ with the same initial conditions. It follows that $R_n = P_n b^n$ for $n \geq 0$.

Solving (10) backwards, we can check by an explicit calculation that $R_n = Q_n b^{-n} c M^{-8}$ for $n = -3, \ldots, 0$ where $b$ and $c$ are given by (9). Moreover, $R_n$ and $Q_n b^{-n} c M^{-8}$ satisfy the same recursion relation (10) for $n < 0$. It follows that $R_n = Q_n b^{-n} c M^{-8}$ for $n < 0$. This concludes the proof of Equations (10) and (11).

2.2. Proof of Theorem 1.3. Let us consider first the case of $n \geq 0$, and denote by $P_n'$ for $n \geq 0$ the unique solution to the linear recursion relation (4) with the initial conditions as in Theorem 1.3. Let $R_n' = P_n' b^n$ be defined according to Equation (11) for $n \geq 0$.

Remark 1.4 implies that $R_n'$ satisfies the recursion relation of [TY04, Thm.1]. It follows by [TY04, Thm.1] that $A_n(M, LM^{-4n})$ divides $P_n'(M, L)$ when $n > 1$.

Next, we claim that the Newton polygon $NP_n'$ of $P_n'(M, L)$ is given by (6). This can be verified easily by induction on $n$.

Next, in [Mat02, p.1286], the second author computes the Newton polygon $N_n$ of the $A_n(M, L)$. It is a hexagon given in $(L, M)$ coordinates by:

\[
\begin{align*}
&\{\{0,0\}, \{1,16\}, \{n-1,4(n^2+2n-3)\}, \{2n+1,2(4n^2+10n+9)\}, \\
&\{3n-1,2(6n^2+14n-5)\}, \{3n,2(6n^2+14n+3)\}\}
\end{align*}
\]

when $n > 1$,

\[
\begin{align*}
&\{\{-3n-4,0\}, \{-3(1+n),10\}, \{-3-2n,4(3+4n+n^2)\}, \\
&\{-n,2(4n^2+16n+21)\}, \{0,4(3n^2+12n+11)\}, \{1,6(2n^2+8n+9)\}\}
\end{align*}
\]

when $n < -2$ and

\[
\begin{align*}
&\{\{0,0\}, \{1,0\}, \{2,4\}, \{1,10\}, \{2,14\}, \{3,14\}\}
\end{align*}
\]

when $n = -2$. Notice that the above 1-parameter families of Newton polygons are quadratic. It follows by explicit calculation that the Newton polygon of $A_n(M, LM^{-4n})$ is quadratic and exactly agrees with $NP_n'$ for all $n > 1$.

The above discussion implies that $P_n(M, L)$ is a rational multiple of $A_n(M, LM^{-4n})$. Since their leading coefficients (with respect to $L$) agree, they are equal. This proves Theorem 1.3 for $n > 1$. The case of $n < -1$ is similar.

2.3. Proof of Theorem 1.4. Using Equations (2) and (3), convert Equation (12) into

\[(13)\quad Q_{-n}(\sqrt{M}, L/M^5) = (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M).
\]

Note that, under the substitution $(M, L) \mapsto (i/\sqrt{L}, L^{2n+5}/M)$, $LM^{4n}$ becomes $L^5/M$. Similarly, $LM^{-4n}$ becomes $L/M^5$ under the substitution $(M, L) \mapsto (\sqrt{M}, LM^{2n-5})$.

It is straightforward to verify equation (13) for $n = 2, 3, 4, 5$. For $n \geq 6$, we use induction. Let $c_k$ denote the result of applying the substitutions $(M, L) \mapsto (\sqrt{M}, L/M^5)$ to the $c_k$ coefficients in the recursions (4) and (5). For example,

\[c_0 = \frac{L^4(1+L)^4(1-M)^4}{M^2}.\]
Similarly, define \( c_k^+ \) to be the result of the substitution \( (M, L) \mapsto (i/\sqrt{L}, L^5/M) \) to \( c_k \). It is easy to verify that for \( k = 0, 1, 2, 3 \),

\[
\frac{c_k^-}{c_4} (-LM)^{k-4} = \frac{c_k^+}{c_4}.
\]

Then,

\[
Q_n(\sqrt{M}, L/M^5) = -\frac{1}{c_4} \sum_{k=0}^{3} c_k^- Q_{n+4-k}(\sqrt{M}, L/M^5)
\]

\[
= -\frac{1}{c_4} \sum_{k=0}^{3} c_k^- (-L)^{n-4+k} M^{n-4+k+13} P_{n-4+k}(i\sqrt{L}, L^5/M)
\]

\[
= -(L)^{n} M^{n+13} \sum_{k=0}^{3} \frac{c_k^-}{c_4} (-LM)^{k-4} P_{n-4+k}(i\sqrt{L}, L^5/M)
\]

\[
= -(L)^{n} M^{n+13} \sum_{k=0}^{3} \frac{c_k^+}{c_4} P_{n-4+k}(i\sqrt{L}, L^5/M)
\]

\[
= -(L)^{n} M^{n+13} P_{n}(i\sqrt{L}, L^5/M).
\]

By induction, equation (13) holds for all \( n > 1 \) proving Theorem 1.4. \( \square \)

**Appendix A. The coefficients \( c_k \) and the initial conditions for \( P_n \) and \( Q_n \)**

\[
c_4 = M^4
\]

\[
c_3 = 1 + M^4 + 2LM^{12} + LM^{14} - LM^{16} + L^2 M^{20} - L^2 M^{22} - 2L^2 M^{24} - L^3 M^{32} - L^3 M^{36}
\]

\[
c_2 = (1 + LM^{12}) (-1 - 2LM^{10} - 3LM^{12} + 2LM^{14} - L^2 M^{16} + 2L^2 M^{18} - 4L^2 M^{20} - 2L^2 M^{22} + 3L^2 M^{24} - 3L^3 M^{28} + 2L^3 M^{30} + 4L^3 M^{32} - 2L^3 M^{34} + L^3 M^{36} - 2L^4 M^{38} + 3L^4 M^{40} + 2L^4 M^{42} + L^5 M^{52})
\]

\[
c_1 = -L^2 (-1 + M)^2 M^{16} (1 + M)^2 (1 + LM^{10})^2 (-1 - M^4 - 2LM^{12} - LM^{14} + LM^{16} - L^2 M^{20} + L^2 M^{22} + 2L^2 M^{24} + L^3 M^{28} + L^3 M^{32} + L^3 M^{36})
\]

\[
c_0 = L^4 (-1 + M)^4 M^{36} (1 + M)^4 (1 + LM^{10})^4
\]

\[
P_0 = \frac{(1 + LM^{12}) (1 + LM^{12})^2}{(1 + LM^{10})^2}
\]

\[
P_1 = \frac{(1 + LM^{12})^2 (1 + LM^{12})^2}{1 + LM^{10}}
\]

\[
P_2 = -1 + LM^8 - 2LM^{10} + LM^{12} + 2L^2 M^{20} + L^2 M^{22} - L^4 M^{40} - 2L^4 M^{42} - L^5 M^{50} + 2L^5 M^{52} - L^5 M^{54} + L^6 M^{62}
\]

\[
P_3 = (1 + LM^{12}) (-1 + LM^4 - LM^8 + 2LM^8 - 5LM^{10} + LM^{12} + 5L^2 M^{16} - 4L^2 M^{18} + L^3 M^{22} + L^3 M^{26} + 3L^3 M^{30} + 2L^3 M^{32} - 2L^3 M^{34} + 3L^3 M^{40} + 2L^3 M^{42} - 2L^3 M^{46} + 3L^3 M^{48} - L^5 M^{58} - L^6 M^{56} + 5L^6 M^{60} - L^7 M^{66} + 5L^7 M^{68} - 2L^7 M^{70} + L^7 M^{72} - L^7 M^{74} + L^8 M^{78})
\]
\[Q_0 = \frac{(-1 + LM^{12})(1 + LM^{12})^2}{L^4(-1 + M)^3M^4(1 + M)^3}\]
\[Q_{-1} = \frac{M^{12}(1 + LM^{12})^2}{L(-1 + M)(1 + M)}\]
\[Q_{-2} = M^{20}(1 - LM^8 + 2LM^{10} - LM^{12} - LM^{16} + LM^{18} + L^2M^{20} - L^2M^{22} + 2L^2M^{26} + 2L^2M^{28} - L^2M^{30} + L^2M^{38})\]
\[Q_{-3} = M^{16}(-1 + LM^{12})(1 + LM^{10} + 5LM^{12} - LM^{14} - 2LM^{16} + 2LM^{18} - LM^{20} + 2L^2M^{20} + LM^{22} + 4L^2M^{22} + 3L^2M^{26} - 3L^2M^{28} - L^3M^{28} + 5L^3M^{30} + 5L^2M^{32} - L^2M^{34} - 3L^3M^{34} + 3L^3M^{36} + 4L^3M^{40} + L^4M^{42} - L^4M^{42} + 2L^3M^{44} - 2L^4M^{46} - L^4M^{48} + 5L^4M^{50} + L^4M^{52} + L^5M^{62})\]

**Appendix B. The coefficients \(\gamma_k\) and the initial conditions for \(R_n\)**

\[\gamma_4 = L^4(-1 + M)^4M^{36}(1 + M)^4(1 + LM^{10})^4\]
\[\gamma_5 = L^3(-1 + M)^3M^{24}(1 + M)^3(1 + LM^{10})^3(-1 - M^4 - 2LM^{12} - LM^{14} + LM^{16} - L^2M^{20} + L^2M^{22} + 2L^2M^{24} + L^3M^{32} + L^3M^{36})\]
\[\gamma_7 = L^2(-1 + M)^2M^{16}(1 + M)^2(1 + LM^{12})^2(-1 - 2LM^{10} - 3LM^{12} + 2LM^{14} - L^2M^{16} + 2L^2M^{18} - 4L^2M^{20} - 2L^2M^{22} + 3L^2M^{24} - 3L^3M^{26} + 2L^3M^{30} + 4L^3M^{32} - 2L^3M^{34} - 2L^3M^{36} - 2L^4M^{38} + 3L^4M^{40} + 2L^4M^{42} + L^5M^{52})\]
\[\gamma_7 = \gamma_3\]
\[\gamma_0 = \gamma_4\]

Let \(P_n\) for \(n = 0, \ldots, 3\) be as in Appendix A. Then,

\[(14) \quad R_n = P_n b^n\]

for \(n = 0, \ldots, 3\) where \(b\) is given by Equation \((9)\).

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