Towards multi-scale dynamics on the baryonic branch of Klebanov-Strassler

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We construct explicitly a new class of backgrounds in type-IIB supergravity which generalize the baryonic branch of Klebanov-Strassler. We apply a solution-generating technique that, starting from a large class of solutions of the wrapped-D5 system, yields the new solutions, and then proceed to study in detail their properties, both in the IR and in the UV. We propose a simple intuitive field theory interpretation of the rotation procedure and of the meaning of our new solutions within the Papadopoulos-Tseytlin ansatz, in particular in relation to the duality cascade in the Klebanov-Strassler solution. The presence in the field theory of different VEVs for operators of dimensions 2, 3 and 6 suggests that this is an important step towards the construction of the string dual of a genuinely multi-scale (strongly coupled) dynamical model.

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I. INTRODUCTION

The modern formulation of gauge-string dualities offers a new computational tool allowing to study field theories in the strong coupling regime, which exhibit very non-trivial dynamical features, inaccessible to standard (perturbation-theory) methods. The most celebrated example of such a correspondence relates a superconformal four-dimensional theory ($N = 4$ super-Yang-Mills with $SU(N_c)$ gauge group) to type-IIB superstring theory, on a background with $AdS_5 \times S^5$ geometry ($AdS/CFT$). In particular, the regime of large ’t Hooft coupling and large $N_c$ of the field theory is related to the weakly-coupled, classical supergravity approximation of the ten-dimensional dual (see also [2] for a review).

Since the discovery of this correspondence, a large amount of effort has been put into looking for its generalizations to theories that have less supersymmetry, and that are not conformal, with the aim of applying some of the technology developed in the AdS/CFT context to situations closely related to phenomenologically relevant field theories. Among many, three main examples exist of regular type-IIB backgrounds that are dual to $\mathcal{N} = 1$ four-dimensional theories (see also the Klebanov-Tseytlin (KT) [3] background and the Baryonic Branch solution in [4]), in which the six-dimensional internal space is related to the conifold and its variations [5]. The Klebanov-Witten (KW) solution is dual to a (super-)conformal theory [6], having metric of the form $AdS_5 \times T^{1,1}$. The Klebanov-Strassler (KS) [4] and wrapped-D5 [5] ones are non-singular backgrounds yielding confinement in the IR. The confining field theories dual to these two models are quite non-trivial, characterized by one dynamically generated scale, that appears explicitly in many interesting physical quantities (such as the gaugino condensate, the string tension and the glueball spectrum).

The next order of complexity is to find the gravity dual of strongly-coupled field theories in which two or more (distinct and parametrically separated) scales are generated dynamically. Besides being an interesting field theory problem per se, this line of research has a possible field of application in the context of dynamical electro-weak symmetry breaking, or technicolor [8], in particular in what goes under the name of walking technicolor (WTC) [9] and of extended technicolor [10] (see [11] for reviews on the subject). These theories are strongly coupled, multi-scale theories, in which many operators develop condensates and large anomalous dimensions, and hence they are peculiarly difficult to study. Many phenomenological aspects of these models of electro-weak symmetry breaking are not well understood. For instance, it is an open problem whether they predict the existence of a light composite state (dilaton) in the spectrum [12, 13], which might have couplings very similar to those of the Higgs particle of the minimal version of the Standard Model [13], and hence very similar LHC signatures. It is hence useful to try to use the techniques of gauge-string dualities in order to study the non-perturbative aspects of multi-scale field theories.

A proposal in the direction of studying the dual of a supersymmetric field theory yielding the emergence of two dynamical scales is contained in [16, 17]. The starting point of this proposal is the type-IIB background generated by a stack of D5-branes wrapping a compact internal two-cycle. The background consists of a metric $g_{\mu\nu}$, dilaton $\Phi$ and flux for the RR three-form $F_3$. In the specific case of a solution in the form of [8], a suitable definition of the gauge coupling, in terms of the geometry, complemented by a specific radius-energy relation [19], yields a beta-function that is compatible with the NSVZ beta-function of SYM [20]. Besides the solution in [8], there exist several classes of solutions of the same equations for the wrapped-D5 system [21], for which the same definition of gauge coupling yields a beta-function that exhibits the features expected in a walking theory [16, 17]. As a function of the value of the radial direction $\rho$ at which the coupling is computed, three very different behaviors appear. The coupling runs towards small values going above some value $\rho_*$ of the radial direction, it is approximately constant over a finite range $\rho_1 < \rho < \rho_*$, and grows indefinitely below $\rho_1$, diverging for $\rho \to 0$. In one specific class of solutions, the behavior of the Wilson loops shows that the theory confines in the conventional sense of producing a linear quark-antiquark static potential [18], although a peculiar behavior similar to a phase transition appears. Interestingly, for the same class of backgrounds one finds that the spectrum of scalar excitations (glueballs) contains a parametrically light state, whose mass is suppressed as a function of $\rho_*$ [17].

Analyzing in detail solutions in the class of [16] is non-trivial. The geometry is very far from being AdS at all values of the radial direction. Furthermore, the background is singular: while the Ricci scalar and the square of the Ricci tensor are finite, the Kretschmann scalar diverges (in spite of this, the calculation of the Wilson loops and of the glueball spectrum yield physically sensible results). It is difficult to understand the dual field theory in detail and...
what its dynamical properties are, including the role of the scale $\rho_*$. In particular, it is not known what the precise nature of the light state found in [1] is. In this paper, we construct a more general class of type-IIB backgrounds which share the interesting features of the class in [10], but that are easier to analyze. This is to be understood as a further step towards the formulation of the (UV-complete and IR-smooth) gravity dual of a genuinely multi-scale field theory.

We will rely on many known results and build upon them, making extensive use of the vast amount of knowledge cumulated over the years about type-IIB backgrounds that are related to the conifold. We briefly summarize here the main elements that will be needed in the body of the paper. All the solutions [3, 7, 11, 18] are special cases of the Papadopoulos-Tseytlin (PT) ansatz [22] (see also [23]), and they can all be obtained by lifting to ten dimensions the solutions of a specific five-dimensional scalar sigma-model coupled to gravity containing eight dynamical scalars. The PT ansatz has recently been shown to yield a consistent truncation, a subsector of the more general consistent truncation on $T^{1,1}$ of ten-dimensional type-IIB supergravity, down to $\mathcal{N} = 4$ five-dimensional gauged-supergravity with non-compact gauge group $U(1) \times \text{Heis}_3$ [24, 25].

The generic solution of the BPS equations for the wrapped-D5 system can be obtained by solving a non-linear second-order equation for a generating function $P$ [21], all the other functions in the background being algebraically related to $P$. Furthermore, elaborating on [26], it was recently shown that given a solution of the wrapped-D5 system, subject to some restriction on its UV behavior, it is possible to algorithmically generate a whole class of more general solutions, still satisfying the PT ansatz, but in which a non-trivial flux for the RR five-form $F_5$ and the NS two-form $B_2$ are present [26-28]. We will refer to this algorithmic procedure as rotation. In particular, this allows to connect systematically the wrapped-D5 system, the baryonic branch discussed in [7] and the KS background. Finally, the relation between the five-dimensional scalars of the PT ansatz (near a KW fixed point) and the corresponding field theory operators is known and well understood [7, 29].

The paper is organized as follows. In Section II we review most of the material discussed above. We start from the wrapped-D5 system, rediscuss the class of solutions in [16] and apply to them the rotation of [26-28]. In doing so, we find it convenient to adopt the five-dimensional language of [23]. We also briefly summarize the five-dimensional perspective on the KW-KT-KS solutions. In Section III we study in detail the UV behavior of the rotated solutions, and compare them both to the original unrotated solution and to the KS solutions, in the language of the five-dimensional sigma-model, and in the light of the operator analysis of all the perturbations of the KW [3] fixed point, allowed within the PT ansatz [7, 24]. This allows us in particular to discuss the difference between these various cases in terms of the operators of the dual field theory. We present a detailed analysis of the dual field theory where striking coincidences between the perturbative behavior of the quiver field theory and the gravity solution emerge. In Section IV, we examine in detail the behavior of the solutions in the deep IR. We show that both the Ricci scalar and the square of the Ricci tensor are finite, while a singularity appears in the Kretschmann scalar (the invariant built as the square of the Riemann tensor). We also compute the expectation value of rectangular Wilson loops. The very mild nature of the IR singularity, and the comparatively nice behavior of the rotated backgrounds in the far UV, allow us to follow the prescription in [31], and extract from it the quark-antiquark static potential $E_{Q\bar{Q}}$. The results are very similar to those in [18]. Linear confinement appears at arbitrarily large quark separation $L_{Q\bar{Q}} \to +\infty$, accompanied by the non-standard feature of a first-order phase transition taking place at a finite value of $L_{Q\bar{Q}}$, for backgrounds where $\rho_*$ is large enough. The strength of the transition depends explicitly on $\rho_*$. We conclude in Section V by critically discussing our results and outlining a few possible directions for further development.

II. A MINI-REVIEW: A CLASS OF SOLUTIONS INTERPOLATING WITHIN THE PT ANSATZ

We start by identifying the class of solutions we are going to study, and by summarizing all the technology we need in the rest of the paper. In doing so we make extensive use of the results and language in [21] and [27], transcribed into the five-dimensional formulation of the PT ansatz [22], following closely the notation of [23]. We also briefly remind the reader what the KS [4], KT [6] and KW [3] solutions are.

A. Wrapped-D5 system

We start from the geometry produced by stacking on top of each other $N_c$ D5-branes that wrap an $S^2$ inside a CY3-fold and then taking the strongly coupled limit of the gauge theory on this stack, in the (type-IIB) supergravity approximation [3, 21]. We truncate type-IIB supergravity to include only gravity, dilaton $\Phi$ and RR three-form $F_3$, and define the $SU(2)$ left-invariant one-forms as

$$\tilde{\omega}_1 = \cos \psi d\tilde{\theta} + \sin \psi \sin \tilde{\theta} d\tilde{\phi} \ , \ \tilde{\omega}_2 = -\sin \psi d\tilde{\theta} + \cos \psi \sin \tilde{\theta} d\tilde{\phi} \ , \ \tilde{\omega}_3 = d\psi + \cos \tilde{\theta} d\tilde{\phi} \ .$$ (1)
We use an ansatz that assumes the functions appearing in the background depend only the radial coordinate $\rho$ (the range of the angles is $0 \leq \theta, \tilde{\theta} < \pi$, $0 \leq \phi, \tilde{\phi} < 2\pi$, $0 \leq \psi < 4\pi$). We write the background (in Einstein frame) as

\[
\begin{align*}
    ds^2 &= \alpha' g_s e^{\Phi(\rho)/2} \left[ (\alpha' g_s)^{-1} dx^2_{1,3} + ds^2_6 \right], \\
    ds^2_6 &= e^{2k(\rho)} d\rho^2 + e^{2h(\rho)} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{e^{2g(\rho)}}{4} \left( (\tilde{\omega}_1 + a(\rho) d\theta)^2 + (\tilde{\omega}_2 - a(\rho) \sin \theta d\phi)^2 \right), \\
    \quad F_3 &= \frac{\alpha' g_s N_c}{4} \left[ - (\tilde{\omega}_1 + b(\rho) d\theta) \wedge (\tilde{\omega}_2 - b(\rho) \sin \theta d\phi) \wedge (\tilde{\omega}_3 + \cos \theta d\phi) \right] + \partial_\rho b \ d\rho \wedge (-d\theta \wedge \tilde{\omega}_1 + \sin \theta d\phi \wedge \tilde{\omega}_2) + (1 - b(\rho)^2) \sin \theta d\theta \wedge d\phi \wedge \tilde{\omega}_3].
\end{align*}
\]

The full background is then determined by solving the equations of motion for the functions $(a, b, \Phi, g, h, k)$. Notice that from here on we set $\alpha' g_s = 1$.

The system of BPS equations derived using this ansatz can be rearranged in a convenient form, by rewriting the functions of the background in terms of a set of functions $P(\rho), Q(\rho), Y(\rho), \tau(\rho), \sigma(\rho)$ as [21]

\[
4 e^{2h} = \frac{P^2 - Q^2}{P \cosh \tau - Q}, \quad e^{2g} = P \cosh \tau - Q, \quad e^{2k} = 4Y, \quad a = \frac{P \sinh \tau}{P \cosh \tau - Q}, \quad N_c b = \sigma.
\]

Using these new variables, one can manipulate the BPS equations to obtain a single decoupled second order equation for $P(\rho)$, while all other functions are obtained from $P(\rho)$ as follows:

\[
\begin{align*}
    Q(\rho) &= (Q_0 + N_c) \cosh \tau + N_c (2\rho \cosh \tau - 1), \\
    \sinh \tau(\rho) &= \frac{1}{\sinh(2\rho - 2\rho_0)}, \quad \cosh \tau(\rho) = \coth(2\rho - 2\rho_0), \\
    Y(\rho) &= \frac{P'}{8}, \quad e^{4\Phi} = \frac{e^{4\Phi_0} \cosh(2\rho_0)^2}{(P^2 - Q^2) Y \sinh^2 \tau}, \\
    \sigma &= \tanh \tau(Q + N_c) = \frac{(2N_c \rho + Q_0 + N_c)}{\sinh(2\rho - 2\rho_0)}.
\end{align*}
\]

The second order equation mentioned above reads

\[
P'' + \left( \frac{P' + Q'}{P - Q} + \frac{P' - Q'}{P + Q} - 4 \coth(2\rho - 2\rho_0) \right) = 0.
\]

We will refer to Eq. (5) as the master equation: this is the only equation that needs solving in order to generate the large classes of solutions for the more general Papadopoulos-Tseytlin system we are interested in. In this paper we will always set $\rho_0 = 0$, which amounts to setting to 1 the dynamical scale in terms of which all other dimensionful parameters will be measured. Also, in order to avoid a nasty singularity (‘bad’ according to the criteria in [32]) in the IR we fine-tune $Q_0 = -N_c$.\footnote{As an example, the solution $P = 2N_c \rho$ gives the background of [3]. This solution will not be the focus of this paper.}

Finally, we revisit the definition of gauge coupling in the dual field theory. The six-dimensional theory on the D6-branes has a 't Hooft coupling given by the dimensionful $\lambda_6 = g_s \alpha' N_c$, and the supergravity limit is taken by keeping this fixed [33]. The branes wrap a small two-cycle $\Sigma_2$, so that at low energies an effectively four-dimensional theory emerges with gauge coupling $g_4$. Following [19], which considers a five-brane (in the probe approximation) extended along the Minkowski directions and the two-cycle defined by $\Sigma_2 = \{ \theta = \tilde{\theta}, \ \phi = 2\pi - \tilde{\phi}, \ \psi = \pi \}$, one arrives at [16]

\[
\frac{g_4^2 N_c}{8\pi^2} = \frac{N_c \coth(\rho)}{P}.
\]
B. Rotation: U-duality as a solution-generating technique

In the paper \cite{26} the authors proposed a U-duality that takes a particular solution to Eq. (5) — hence a background of the form of Eq. (2) — and maps it into another background where new fluxes are turned on. This U-duality can be seen to be equivalent to (a particular case of) a rescaling of the Kähler two-form and complex structure three-form characterizing the background (see \cite{27, 28} for details).

The effect of this solution-generating technique (that we call ‘rotation’) can be summarized by defining a basis (below we use the definition $\hat{h} \equiv 1 - k_2^2 e^{2\Phi}$, with the parameter $k_2$ restricted to $0 \leq k_2 \leq e^{-\Phi(\infty)}$)

\[
e^{i_1} = \hat{h}^{1/2} e^{\Phi/2} dx_i, \quad e^{\theta} = \hat{h}^{1/2} e^{\Phi/2 + k} dp, \quad e^3 = \hat{h}^{1/2} \frac{e^{\Phi/2 + k}}{2} (\omega_3 + \cos \theta d\varphi),
\]

\[
e^\theta = \hat{h}^{1/2} e^{\Phi/2 + k} d\theta, \quad e^\varphi = \hat{h}^{1/2} e^{\Phi/2 + k} \sin \theta d\varphi,
\]

\[
e^1 = \hat{h}^{1/2} \frac{e^{\Phi/2 + g}}{2} (\omega_1 + ad\theta), \quad e^2 = \hat{h}^{1/2} \frac{e^{\Phi/2 + g}}{2} (\omega_2 - a \sin \theta d\varphi),
\]

where $x_i$ are the four Minkowski directions. The (new) generated configuration is

\[
ds_E^2 = \sum_{i=1}^{10} (e^i)^2,
\]

\[
F_3 = \frac{e^{3\Phi}}{\hat{h}^{3/2}} \left[ f_1 e^{123} + f_2 e^{2\varphi} - f_3 (e^{31} + e^{23}) + f_4 (e^{31} + e^{23}) \right],
\]

\[
B_2 = k_2 \frac{e^{3\Phi/2}}{\hat{h}^{1/2}} \left[ e^{3\varphi} + \cos \mu (e^{\varphi} + e^{12}) + \sin \mu (e^{31} + e^{23}) \right],
\]

\[
H_3 = -k_2 \frac{e^{3\Phi}}{\hat{h}^{3/4}} \left[ -f_1 e^{\varphi \rho} - f_2 e^{2\varphi \rho} + f_3 (e^{\varphi \rho} + e^{3\varphi}) - f_4 (e^{2\varphi} - e^{3\varphi}) \right],
\]

\[
F_5 = k_2 \frac{d}{dp} \frac{e^{3\Phi}}{\hat{h}^{3/4}} \left[ -e^{23} + e^{31} + e^{23} \right],
\]

where $\cos \mu = -\frac{p - Q \coth(2\rho)}{p \coth(2\rho) - Q}$, the functions $f_i, i = 1, \ldots, 4$ are

\[
f_1 = -2N_c e^{-k - g}, \quad f_2 = \frac{N_c}{2} (a^2 - 2ab + 1) e^{-k - g},
\]

\[
f_3 = N_c (b - a) e^{-k - g}, \quad f_4 = \frac{N_c}{2} b' e^{-k - g},
\]

and we denoted

\[
e^{ijk \cdots l} = e^i \wedge e^j \wedge e^k \wedge \ldots \wedge e^l.
\]

A necessary condition to apply this solution-generating technique is that the quantity $e^\Phi$ is bounded from above (being an increasing function with $e^\Phi(\infty)$ its maximum value). This condition can be linked with the absence of D7-brane sources in the configuration of Eq. (8) (see \cite{27} for details). In most parts of this paper, we will choose $k_2 e^\Phi(\infty) = 1$.

This is basically keeping the sub-leading term at infinity in an expansion of the warp factor $\hat{h}(\rho)$. The rationale for this choice will be carefully discussed in the following sections.

For future reference we compare the background in Eq. (8) with the generic type-IIB background written in Eqs. (3.8)-(3.11) of the paper \cite{23} (a detailed comparison will be given in Appendix \textsection A). The functions
\( h_1, h_2, \chi, K, a, b, \Phi, p, x, \tilde{g} \) in Eq. (8) are given in terms of the functions \( h, g, k, a, b, \Phi \) in Eq. (8) as
\[
\begin{align*}
\frac{k_2}{4} f_a e^{2k+h+k+g}, & \quad h_2 - \frac{k_2}{4} e^{2k} \left[ \frac{3}{2} e^{k+h+g} - \frac{f_2}{4} a e^{k+2g} \right], \\
\chi' + h_1' &= -\frac{k_2}{4} e^{2k} \left[ f_1 e^{k+2h} + f_3 a e^{k+h+g} - \frac{f_2}{4} a^2 e^{k+2g} \right], \quad h_1' - \chi' = \frac{k_2}{4} e^{2k+k+2g}, \\
\mathcal{K} &= -\frac{k_2}{4} e^{2k+2h+2g} \Phi', \quad e^{2\tilde{g}} = 4 e^{2h-2g}, \\
e^{-6p} &= \frac{h}{8} e^{2k+h+g} e^{-2\tilde{g}}, \quad e^{2x} = \frac{h}{4} e^{2h+2g+\Phi}, \\
& \quad a \rightarrow a, \quad b \rightarrow b, \quad \Phi \rightarrow \Phi.
\end{align*}
\]

Let us now move on to describe the five-dimensional perspective for these new backgrounds, to be used later in this paper.

C. Five-dimensional language

Following the notation in [23], we describe the more general PT system using an effective five-dimensional action that reads, up to an overall normalization,
\[
S = \int d^5y \sqrt{-g} \left[ \frac{1}{4} R - \frac{1}{4} G_{ab} g^{MN} \partial_M \Phi^a \partial_N \Phi^b - V(\phi) \right],
\]

where \( \Phi^a = (\tilde{g}, x, p, a, b, \Phi, h_1, h_2, \chi, K) \) and \( y^M = (x^\mu, r) \). We impose two constraints
\[
\begin{align*}
\mathcal{K} &= M + 2N(h_1 + bh_2), \\
\partial_M \chi &= \frac{(e^{2\tilde{g}} + 2a^2 + e^{-2\tilde{g}} a^4 - e^{-2\tilde{g}}) \partial_M h_1 + 2a(1 - e^{-2\tilde{g}} + a^2 e^{-2\tilde{g}}) \partial_M h_2}{e^{2\tilde{g}} + (1 - a^2)^2 e^{-2\tilde{g}} + 2a^2},
\end{align*}
\]

where \( \mathcal{K} \) is the normalization of the \( F_5 \) form in ten dimensions, and \( \chi, h_1 \) and \( h_2 \) appear in the NS \( B_2 \) antisymmetric tensor of type IIB. The quantity \( N \) is (up to a proportionality constant) the normalization of the \( F_3 \) form, and essentially counts how many D5-branes are present, while \( M \) would count the number of D3-branes if \( N = 0 \).

The constraints allow to remove \( \chi \) and \( \mathcal{K} \) from the sigma-model, which is hence defined by
\[
G_{ab} \partial_M \Phi^a \partial_N \Phi^b = \frac{1}{2} \partial_M \tilde{g} \partial_N \tilde{g} + \partial_M x \partial_N x + 6 \partial_M p \partial_N p + \frac{1}{4} \partial_M \Phi \partial_N \Phi + \frac{1}{2} e^{-2\tilde{g}} \partial_M a \partial_N a \\
+ \frac{1}{2} N^2 e^{2\Phi} \partial_M b \partial_N b + \frac{1}{2} e^{2\tilde{g}} \partial_M h_1 \partial_N h_1 + \frac{1}{2} e^{2\tilde{g}} \partial_M h_2 \partial_N h_2 \\
+ \frac{1}{2} (e^{2\tilde{g}} + 2a^2 + e^{-2\tilde{g}}(1 + a^2) \partial_M h_2 \partial_N h_2 + 2a e^{-2\tilde{g}}(a^2 + 1)) \partial_M h_1 \partial_N h_2].
\]

The potential is
\[
V = -\frac{1}{2} e^{2\tilde{g}} (e^{\tilde{g}} + (1 + a^2) e^{-\tilde{g}}) + \frac{1}{8} e^{-4p-4x} (e^{2\tilde{g}} + (a^2 - 1)^2 e^{-2\tilde{g}}) + 2a^2 \\
+ \frac{1}{4} e^{-2\tilde{g}} + 2a^2 + \frac{1}{4} N^2 e^{2\Phi} \frac{1}{4} e^{2\tilde{g}} (a^2 - 2ab + 1)^2 + 2(a - b)^2 \\
+ \frac{1}{4} e^{-\Phi} + 2ab + 1)^2 \partial_M h_2 \partial_N h_2 + 2a e^{-2\tilde{g}}(a^2 + 1) \partial_M h_1 \partial_N h_2].
\]

The five-dimensional metric is written as (by convention the metric is mostly plus)
\[
dy^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2.
\]
The warp factor $A$ is determined by the Einstein equations.

In looking for solutions to the background, we assume that all the functions have a non-trivial dependence only on the radial direction $r$. For example, the system of wrapped-D5 in Eq. (2) is obtained, as discussed below, by setting $M = 0$ and $N = N_c/4$, in which case one can consistently set $h_1 = h_2 = \chi = K = 0$, reducing to six the number of scalar functions controlling the background. The radial directions in the ten and five-dimensional languages are connected by the change of variable $2e^{-\Phi}d\rho = dr$. Let us study this in more detail.

**D. The D5’s backgrounds: master equation and rotation**

Given a solution for $P$ is found, one can algebraically derive the background in Eq. (2) for all the active scalars of the five-dimensional model

$$a = \frac{P}{\sinh 2\rho (P \coth 2\rho - Q)}, \quad b = \frac{2\rho}{\sinh 2\rho},$$

$$\Phi = \frac{1}{4} \log \left( \frac{8e^{4\Phi_o} \sinh^2(2\rho)}{(P^2 - Q^2) P'} \right), \quad x = \frac{1}{8} \log \left( \frac{e^{4\Phi_o} \sinh^2(2\rho) (P^2 - Q^2)^3}{8192 P''} \right),$$

$$p = -\frac{1}{24} \log \left( \frac{e^{4\Phi_o} (P^2 - Q^2) \sinh^2(2\rho) (P')^3}{131072} \right), \quad \check{g} = \frac{1}{2} \log \left( \frac{P^2 - Q^2}{(Q - P \coth(2\rho))^2} \right),$$

$$h_1' = 0 = h_2.$$

As a consequence,

$$A = \frac{1}{6} \log \left( \frac{1}{256} e^{4\Phi_o} (P^2 - Q^2) \sinh^2(2\rho) \right), \quad K = 0 = \chi,$$

and the full type-IIB background in Eq. (2) is known. For future reference, we highlight an important subtlety: because $B_2 = 0 = F_5$ in the system of Eq. (2), one might think that there are six active scalars, and hence expect a general solution of the BPS equations to depend on six integration constants. This is not so: the BPS equations do not descend simply from a superpotential for the five-dimensional description of the wrapped-D5 system, but rather the supersymmetric backgrounds must satisfy a system of six first-order equations, supplemented by a Hamiltonian constraint. After repackaging the resulting system in terms of $P$ and $Q$, the general BPS solution depends on five integration constants: $Q_0$, $\rho_0$, $\Phi_o$, and the two integration constants of the general solution to the second-order equation for $P$. As anticipated above, we fine-tune $Q_0$, so the actual solution depends on four independent integration constants.

Provided $\Phi_{\infty} \equiv \lim_{\rho \rightarrow \infty} \Phi$ is finite (and the dilaton is a monotonically increasing function of $\rho$, which is always true for the solutions we consider in this paper), the rotation of (27) allows to algorithmically generate the full class of solutions, parameterized by $0 < k_2 < e^{-\Phi_{\infty}}$. Comparing with Eq. (11) we obtain that in the five-dimensional language, the rotation acts as (the superscript $(r)$ indicates a ‘rotated’ function)

$$a^{(r)} = a, \quad b^{(r)} = b, \quad \Phi^{(r)} = \Phi, \quad e^{2\check{g}^{(r)}} = e^{2\check{g}},$$

$$e^{2x^{(r)}} = (1 - k_2^2 e^{2\Phi}) e^{2x},$$

$$e^{-6p^{(r)}} = (1 - k_2^2 e^{2\Phi}) e^{-6p},$$

$$\partial_\rho h_1^{(r)} = \frac{k_2 N_c}{4} e^{2\Phi} [e^{2\check{g}} + 2a(a - b) + e^{-2\check{g}}(a^2 + 1)(a^2 - 2ab + 1)],$$

$$h_2^{(r)} = \frac{k_2 N_c}{8} e^{2\Phi} \partial_\rho b,$$

and hence

$$A^{(r)} = A + \frac{1}{6} \log (1 - k_2^2 e^{2\Phi}),$$

$$\check{K}^{(r)} = k_2 e^{\Phi + 2x} \partial_\rho \Phi,$$

$$\partial_\rho \chi^{(r)} = \frac{k_2 N_c}{4} e^{2\Phi} [e^{2\check{g}} + 2a(a - b) + e^{-2\check{g}}(a^2 + 1)(a^2 - 2ab + 1)].$$
Notice that the combination $x + 3p$ is unaffected by the rotation.

We now specify the type of solutions we will be mostly interested in in the remainder of this paper. We call them 'seed' solutions since from them, after the rotation procedure is applied, we construct the backgrounds that are the focus of this paper.

E. Seed solutions

The two-parameter family of solutions discussed in \[16\] is obtained by observing that if $P \gg Q$, the master equation is approximately solved by

$$ P_0 = c \left( \cos^3 \alpha + \sin^3 \alpha (\sinh 4\rho - 4\rho) \right)^{1/3}. $$

(23)

One can then construct the full solution for $P$ by expanding in powers of $N_c/c$, with $P = \sum_{n=0}^{\infty} P_n(\rho) \left( N_c \right)^{2n}$, and iteratively solving for each $P_n$ as a function of the parameters $c$ and $\alpha$. This procedure yields a smooth solution for $P$, provided $P > Q$ for all $\rho > 0$. Ultimately, this yields the constraint

$$ \cot \alpha \lesssim \exp \left[ \frac{2^{4/3}c}{3N_c} \right]. $$

(24)

If $\alpha$ is small, effectively the solution for $P$ is approximately constant for $\rho < \rho_*$, while for $\rho > \rho_*$ one sees that $P \simeq e^{4\rho}$. Much of this paper is devoted to analyzing the physical meaning of $\rho_*$. One finds that approximately $4\rho_* \simeq \log 2 \cot^3 \alpha$. By looking at the gauge coupling defined in Eq. (20), one sees that, provided $\rho_*$ is large, there exists an intermediate regime in the radial direction $\rho_I < \rho < \rho_*$ over which this effective four-dimensional gauge coupling is finite and approximately constant \[16\]. The scale $\rho_I \sim 1$ is the value of the radial coordinate below which the functions $a$ and $b$ (and hence the gaugino condensate) become non-trivial (it is the scale above which coth $2\rho \simeq 1$).

We plot in Figure 1 some examples of such backgrounds. Notice that we choose the integration constants in such a way as to make the value of the dilaton agree in the far UV and deep IR for all solutions. We will clarify later on the reason for this choice; for the time being, the figure has mainly illustrative purposes.

Finally, it is useful to remind the reader about the asymptotic expansions of solutions of this class \[21\], \[16\]. In the far UV, for $\rho \to \infty$:

$$ P = 3c_+ e^{4\rho/3} + \frac{4N_c^2}{3c_+} \left( \rho^2 - \rho + \frac{13}{16} \right) e^{-4\rho/3} + \left( -8c_+ \rho - \frac{c_-}{192c_+} \right) e^{-8\rho/3} + O(e^{-4\rho}), $$

(25)

where $c_{\pm}$ are the two constants characterizing all of these solutions. In the IR, for $\rho \to 0$ we have

$$ P = c_0 + k_3 c_0 \rho^3 + \frac{4}{5} k_3 c_0 \rho^5 - k_3^2 c_0 \rho^6 + \frac{16(2c_0^2 k_3^2 - 5k_3 N_c^2)}{105 c_0} \rho^7 + O(\rho^8), $$

(26)

where now $c_0$ and $k_3$ are the free parameters. One can hence write all of these solutions by specifying $N_c$ and any of the pairs $(c, \alpha)$, $(c_+, c_-)$ or $(c_0, k_3)$. The relation between these is not known in analytical form, and which parameterization to use is mostly a matter of convenience.

The subject of this paper is the class of solutions that are obtained from the seed solution discussed above by applying the rotation procedure. As it will be useful in following sections, we move on to describe in five-dimensional language the solutions discovered by Klebanov-Strassler \[4\], Klebanov-Witten \[3\], Klebanov-Tseytlin \[6\].

F. Summary of the Klebanov-Strassler, Klebanov-Tseytlin and Klebanov-Witten solutions

We briefly summarize in this subsection the relevant properties of the KS-KT-KW solutions. The Klebanov-Strassler system can be obtained from the PT one with the constraint

$$ a = \tanh y, \quad e^{-\tilde{g}} = \cosh y, $$

(27)

such that a superpotential exists:

$$ W = \frac{1}{4} e^{-2(p+x)} \left( e^{6p} \left( M - 2e^{2x}\cosh y + 2(h_1 + bh_2)N \right) - 2 \right). $$

(28)
This has to be understood in the sense that given a solution to the BPS equations,

$$\partial_r A = -\frac{2}{3} W, \tag{29}$$

$$\partial_r \Phi^a = G^{ab} \frac{\partial W}{\partial \Phi^b}, \tag{30}$$

the resulting $A$ and $\Phi^a$ satisfy automatically the classical equations derived from the sigma-model.

Starting from the first-order equations of the KS system, one finds that in the usual $\rho$ coordinate in which $dr = 2e^{-4\rho} d\rho$,

$$y = -\ln \tanh(\rho - \rho_0) = 2\arctanh e^{-2(\rho - \rho_0)}. \tag{31}$$

Setting $\rho_0 = 0$, as usual, yields a second-order equation for $b$ (obtained by combining with the equation for $h_2$) that is solved by

$$b = b_1 \cosh 2\rho + \frac{b_2 + (1 - b_1)2\rho}{\sinh 2\rho}. \tag{32}$$

Setting $b_1 = 0$ makes the function $b$ well behaved in the UV, while setting $b_2 = 0$ avoids the arising of a nasty singularity in the IR. With these three choices, one has six of the background scalars:

$$\Phi = \Phi_\infty, \quad a = \frac{1}{\cosh 2\rho},$$

$$e^{-\tilde{g}} = \coth 2\rho, \quad b = \frac{2\rho}{\sinh 2\rho},$$

$$h_2 = \frac{N}{2} e^{\Phi_\infty} \partial_\rho h = -N e^{\Phi_\infty} \frac{-1 + 2\rho \coth 2\rho}{\sinh 2\rho},$$

$$h_1 = \frac{N}{2} e^{\Phi_\infty} \frac{2\rho(1 + \cosh 4\rho) - \sinh 4\rho}{\sinh^2 2\rho} + \tilde{h}_1. \tag{33}$$

The integration constant $\tilde{h}_1$ just amounts to a rescaling $M \to \tilde{M} = M + 2N \tilde{h}_1$. The equations for $p$ and for $x$ are less friendly. By defining

$$f \equiv x + 3p + \frac{1}{2} \ln \frac{2}{3}, \tag{34}$$

one finds that

$$e^{2f} = \frac{\sinh 4\rho + f_0 - 4\rho}{\cosh 4\rho - 1}, \tag{35}$$

and again we set $f_0 = 0$ in order to avoid an IR singularity. The equation for $x$ reduces to

$$x' + Ke^{-2x} - \frac{4}{3} e^{-2f} = 0, \tag{36}$$
where the function \( K = M + 2N(h_1 + bh_2) \) is the normalization of the \( F_5 \) form, and is known in closed form from the previous functions. The regular KS solution is obtained by fine-tuning \( h_1 \) so that \( \hat{M} = 0 \). The equation for \( x \) can be solved only numerically. Finally, the solution for the warp factor \( A \) can be written in terms of \( x \) as

\[
A = A_0 + \frac{x + \log \sinh(2\rho)}{3},
\]

where \( A_0 \) is an integration constant that we put equal to zero (it can be reabsorbed into \( dx_{1,3}^2 \) and just sets an overall energy scale).

Because we are mostly interested in the UV expansion, some useful information can be obtained from the (singular) KT limit, obtained by retaining only \( x, p \) and \( h_1 \) as dynamical fields. In this case the solution is

\[
\begin{align*}
\Phi &= \Phi_\infty, \quad a = 0 = \tilde{g} = b = h_2, \\
\rho &= -\frac{4}{9}\rho - \frac{1}{6} \ln \left( -\frac{N^2e^{\Phi_\infty}}{4} + \frac{\hat{M}}{2} + 2N^2e^{\Phi_\infty}\rho \right) e^{-\frac{2}{9}\rho} + \tilde{p},
\end{align*}
\]

where \( \tilde{p} \) is the last integration constant in the system, while in \( x + 3p \) we set the integration constant \( f_0 = 0 \). Notice that setting \( \tilde{p} = 0 \) results in a softening of the divergence of \( p \) as a function of \( \rho \) (for large \( \rho \)).

For completeness, notice that for \( N = 0 \) there exists a constant solution

\[
\begin{align*}
\Phi &= \Phi_\infty, \quad a = 0 = \tilde{g} = b = h_2 = h_1, \\
x + 3p &= \frac{1}{2} \ln \frac{3}{2}, \quad p = -\frac{1}{6} \log \frac{M}{2}.
\end{align*}
\]

This is the KW solution that yields the \( AdS_5 \times T^{1,1} \) background geometry mentioned earlier on.

Summarizing, the constraint \( a^2 + e^{2g} - 1 = 0 \), see Eq. (27), allows to reduce the system to seven scalars, with a known superpotential. The solution of the resulting first-order equations for the scalars depends on seven integration constants \( (\Phi_\infty, \rho_0, b_1, b_2, h_1, f_0, \tilde{p}) \), besides \( N \) and \( M \). While \( \Phi_\infty \) has little to no physical effect on the resulting solution, and \( \rho_0 = 0 \) simply defines the (dynamical) scale of the theory, by setting the end-of-space in the radial direction, one must set \( b_1 = b_2 = f_0 = \tilde{p} = 0 \) together with \( \hat{h}_1 = -M/(2N) \) in order to avoid singular behaviors in the IR and in the UV. As a result, the general (regular) KS solution depends on the two harmless, independent integration constants \( \Phi_\infty \) and \( \rho_0 \).

### III. SHORT-DISTANCE PHYSICS: TOWARDS A SYSTEMATIC FIELD THEORY INTERPRETATION

This section is mostly devoted to the study of the UV asymptotic behavior of the rotated solutions. By doing so, we can interpret the integration constants in terms of the operators deforming the KW fixed points. By comparing the rotated solution with the unrotated solution and with the KS solution, we can precisely identify what is the difference between these three classes of backgrounds, in terms of couplings and VEVs of field theory operators.

#### A. General analysis

First, we summarize some general results that hold for all the backgrounds compatible with the PT ansatz. Because the background with \( AdS_5 \times T^{1,1} \) geometry is dual to a conformal theory, it is sensible to expand the potential around the KW fixed point(s). In doing so, one finds that the general solution differs from the conformal one(s) by the presence of terms that scale with power \( \Delta \), that is, for a generic field \( \varphi \sim z^\Delta \) as \( z \to 0 \), with \( z = e^{-\frac{2}{9}\rho} \), in the radial coordinate used in the previous section. This power is either the physical dimension of an operator of dimension \( \Delta \) that is developing a VEV, or the dimension of the coupling of an operator of dimension \( 4 - \Delta \) that is added to the dual theory. The allowed values of \( \Delta \) can be classified in full generality, by requiring that the background satisfies the PT ansatz. We start this analysis by recalling what are the allowed values of \( \Delta \), and what background fields they are associated with.

Perturbations driven by the scalar fields \( \Phi \) and \( h_1 \) correspond to the scaling dimensions \( \Delta = 0, 4 \) (which can be interpreted in terms of a marginal deformation and its conjugate VEV). Notice that while the dual of \( \Phi \) is exactly
marginal, the dual of $h_1$ is not, an observation that we will recall and use later on. With the field $a$ are associated scaling dimensions $\Delta = 1,3$, while to $\tilde{g}$ scaling dimension $\Delta = 2$. The system of $b_2$ and $\tilde{b}$ mixes, and the resulting scaling dimensions are $\Delta = -3,1,3,7$. Finally, the mixed system of $x$ and $p$ corresponds to scaling dimensions $\Delta = -4,-2,6,8$. Summarizing, the dual field theory can be described in terms of a conformal theory, perturbed by the presence of VEVs and couplings of a set of eight possible operators: we have operators of dimensions 2,6,7 and 8 (one operator for each dimension), two operators of dimension 3 and two of dimension 4.

At the microscopic level, the field theory dual of the KW background is based on an $\mathcal{N} = 1$ gauge theory with gauge group $SU(M) \times SU(M)$, containing chiral superfields $A_{1,2} \sim (M,M)$ and $B_{1,2} \sim (\bar{M},M)$. All the corresponding field theory operators of the dual field theory can be found in [29] and we summarize them schematically in Table I. Because we will always work with BPS equations, only at most half of the admissible scaling dimensions are going to appear in the UV expansions of the solutions. By inspection, it turns out that we expect at most the presence of four couplings: the two marginal couplings are related to the two gauge couplings, and the coupling of the dimension-7 and dimension-8 operators are allowed. Four possible VEVs are also present: for the two dimension-3 operators, for the dimension-2 operator and for the dimension-6 operator.

All the solutions we discuss differ by which of these couplings and VEVs are non-zero (and independent). In the KS case, as we discussed in Section [11] and summarized below Eq. (39), we are setting to zero five integration constants, plus imposing a constraint that reduces to seven the number of scalars. This means that the only allowed couplings are the two marginal ones, but with coefficients that are related to one another. Ultimately, the fact that one of the two is never exactly marginal, unless $N = 0$, is what makes KS not asymptotically AdS in the far UV. Also, the VEV of a combination of the dimension-3 operators is present (the gaugino condensate). This can be verified explicitly in the expansions in Appendix [13] where we retained for completeness $\bar{M} \neq 0$, and expanded for small $z$, with $\rho = -\frac{1}{2} \log z$.

Let us move on to study the new solutions. First of all, we notice that two important combinations of the scalars are unaffected by the rotation. One is

$$a^2 + e^{2\tilde{b}} - 1 = \frac{2Q}{P \coth 2\rho - Q}. \quad (40)$$

Setting this to zero would amount to imposing the constraint that defines the KS system. But this is not allowed, because $Q$ cannot vanish: all the rotated solutions belong on the baryonic branch of the KS system. This is also indicated by the fact that the dimension-2 VEV is turned on, as can be seen from the expansion of $\tilde{g}^{(r)}$ from Appendix [13] In the limit in which $P \gg Q$, equivalently $N_c/c \to 0$ — see Eq. (25) — the violation of the constraint becomes parametrically small, and this is the regime in which the approximation $P \simeq P_0$, see Eq. (23), becomes accurate.

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3 Notice that what we mean by exactly marginal here is only the fact that the leading-order expansion in small $z$ contains a constant, but not a logarithm.

4 In the literature, the dimension-3 and dimension-2 VEVs are associated with the deformation and the resolution of the conifold, respectively, while the dimension-6 VEV has been discussed for example in [30].
FIG. 2: Three exact BPS backgrounds in the PT system, obtained numerically. In blue (continuum line) a numerical solution for the wrapped-D5 system, obtained by solving the master equation. In red (long-dashed line) the result of rotating the solution in blue, and fine-tuning $k_2$. In green (short-dashing) a regular KS solution, obtained by matching (where possible) the boundary conditions.
The second interesting invariant is \( x + 3p \), which in terms of the variables entering the master equation is given by

\[
e^{4x+12p} = \frac{4(P^2 - Q^2)}{(\partial_P P)^2}.
\]

In the KS case, the solution for \( x + 3p \) depends on the integration constant \( f_0 \), that is set to zero—see around Eq. (35). In the case of the seed solutions in Section [11][12] this is not the case: the constant \( c_- \) appears in the coefficient of corrections scaling as \( z^6 \) (see again Appendix [13]). This means that when allowing a non-vanishing value of \( c_- \) we are turning on the VEV of the dimension-6 operator, with arbitrary strength.

Another important invariant of the rotation is the dilaton \( \Phi \). As a consequence of the fact that the constraint yielding KS is violated, the dilaton has non-trivial dynamical equations, and hence a non-trivial profile. Again from the expansion in Appendix [13] one sees that indeed the corrections are proportional to \( N_c/c_+ \).

The fact that the dilaton is non-trivial, and goes to a finite value in the UV, means that it is possible to fine-tune \( k_2 = e^{-\Phi_\infty} \). This is of crucial importance, let us explain why. First of all, notice that whenever an irrelevant operator is inserted, it makes little sense to perform the expansion as in Appendix [13] One should first find a regime in which the background is at least in some sense close to conformal, and expand from there.

In order to do so, we consider the UV expansion of \( P \) (setting \( c_- = 0 \) for simplicity), replace in the expression for \( x(k_2=0) - p(k_2=0) \), and (formally) expand first for small \( c_+ \), and then for small \( z \). The result is trustable only at the leading order, which yields

\[
x(k_2=0) - p(k_2=0) = \frac{10c_+^2(30 \log(z)(3 \log(z) + 2) + 37)}{3N_c^2 z^4 (12 \log(z)(3 \log(z) + 2) + 13)^2}.
\]

The choice of \( x - p \) is just dictated by convenience, similar results holding for any combination of \( x \) and \( p \) (aside from \( x + 3p \)). The conclusion of this exercise is that in the seed solutions, as well as in their rotation with generic values of \( k_2 \), the constant \( c_+^2/N_c^2 \) controls the coupling of the dimension-8 operator, and is the analog of \( \tilde{p} \) in the KS system. This is ultimately what renders pathological the UV behavior of the wrapped-D5 backgrounds with \( P \simeq c_+ e^{\tilde{p} \nu} \), which would correspond to field theories that need a UV completion, because their UV dynamics is dominated by the higher-dimensional operator.

The fine-tuning of \( k_2 \) allows to adiabatically switch off this higher-order operator, as we explain now. We start with two minor remarks, which are important for technical reasons. Let us try to identify (at least at leading-order) the expansions of the KS solutions — see Eqs. (31)-(38) — with those of the rotated and fine-tuned solutions — see Eqs. (B11)-(B18). In order to do so, one sees that by choosing

\[
k_2 = e^{-\Phi_\infty} = (18c_+^3)^{1/4} e^{-\Phi_\infty},
\]

together with \( N = N_c/4, c_+ = 0 \) and \( N_c/c_+ = 0 \), one makes \( x + 3p \), \( a, b, \Phi, \tilde{g}, h_1 \) and \( h_2 \) agree with KS in the far UV (at leading-order). Interestingly, for \( x \) to actually agree one needs also \( \tilde{M} = M + 2N\tilde{h}_1 = 0 \). This last observation will help us understand the field theory interpretation of the rotation itself and we will make extensive use of it in the following subsections. The dilaton being an invariant and well behaved in the far UV, we can use the expansion for \( \Phi \) from Appendix [13] from which one sees that by fine-tuning \( k_2 \) one ensures that the rotation factor \( \tilde{h} = 1 - k_2^2 e^{2\Phi} \propto z^4 \), which cancels the \( 1/z^4 \) term, for example in the expansion of Eq. (12). In practice, this means that the coupling of the marginal operator in \( h_1 \) (related to \( k_2 \)) is fine-tuned against the marginal operator in \( \Phi \) (related to \( e^{\Phi_\infty} \)) in such a way as to switch off the dimension-8 operator, while preserving the dimension-2 VEV (related to the function \( \tilde{g} \)).

Finally, let us summarize what couplings and VEVs are present in each of the cases. The seed solutions depend, as we said, on four integration constants \( \rho_0, \Phi_o, c_+ \) and \( c_- \). The quantity \( \rho_0 \) corresponds to a VEV for the dimension-3 operator (gaugino condensate), in the same sense as in the KS solution. The normalization of the dilaton corresponds to a marginal coupling, while the absence of the \( h_1 \) fields is related to the fact that there is only one gauge group, and hence one gauge coupling in the dual theory. The constant \( c_+^2/N_c^2 \) corresponds to a deformation due to the dimension-8 operator, while the quantity \( c_-/c_+^3 \) to the VEV of the dimension-6 operator. The generic rotated solution differs by the fact that the second gauge group is now present, and hence a second quasi-marginal deformation proportional to \( h_1 \) (hence proportional to \( k_2 \)) is driving the flow. The fine-tuned rotated solution corresponds to a peculiar choice such that the coupling of the dimension-8 operator is switched off adiabatically (i.e. keeping the dimension-2 VEV fixed), while at the same time relating among each other in a specific way the couplings of the two marginal operators.

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5 Notice that this has to be understood as a limit procedure, in which one keeps \( N_c \) fixed, and dials \( c_+ \) to large values, hence producing backgrounds that, after rotating, approximate to the KS ones [27].
All of these solutions live on the baryonic branch, because the dimension-2 VEV is present (though its boundary value is fixed, as we said, in such a way as to avoid a nasty singularity in the IR). The whole analysis is summarized by Table I.

Concluding, the main difference between the KS solutions and the rotated and fine-tuned solutions is the insertion of two VEVs: a dimension-2 one, which brings the background on the baryonic branch,\textsuperscript{6} and a dimension-6 one. Hence in the UV the rotated and fine-tuned solutions are going to be almost indistinguishable from KS. Big differences will emerge for $\rho < \rho_*$, due to the VEVs. This is illustrated graphically in Fig. 2. What the figure shows is the background value of the eight scalars for three solutions. First of all, we plot the original seed solution [14], which belongs to the wrapped-D5 system, as shown by the vanishing of $h_{1,2}$. The UV of such a solution is bad, as shown by the divergence of $p$ and $x$, ultimately due to the presence of the dimension-8 operator. $\Phi$ not being constant, but approaching a constant in the UV, one can apply the rotation and fine-tune $\kappa_2$ so as to remove the dimension-8 operator from the dual field theory, hence smoothening the far-UV behavior of $x$ and $p$. At the same time, this induces non-trivial profiles for $h_{1,2}$. We then compare to the KS solution, chosen so as to match the rotated solution in the far UV (in particular, by setting $M = 0$). Above $\rho_* \simeq 9$ the two are almost indistinguishable. However, below $\rho_*$ the VEVs are playing an important role. The KS solution has very different $\tilde{g}$, $\Phi$, $x$ and $p$, while $\alpha, b, h_1$ and $h_2$ are qualitatively very similar.

A very final comment concerns the relation of these classes of solutions to the baryonic branch in [7]. Indeed, the constraint $a^2 + e^{2\phi} - 1 = 0$ is always violated and hence the solutions never really agree with KS. Far in the UV, they rather agree with the solutions in [7]. The main difference with respect to [7] is the presence of the dimension-6 VEV, which results in the background being very different in the deep IR, where a (mild) singularity appears, which is absent in [7]. If one were to evolve from the UV the rotated solutions with $c_2 = 0$ towards the IR, the singularity at the end-of-space in the IR would disappear and one would exactly describe the baryonic branch, in the same sense as in [27].

### B. The rotation and its field theory interpretation

In this subsection we propose a field-theory interpretation of the rotation procedure, that integrates and complements the discussions in [4, 5, 26, 27, 37]. We start by highlighting a set of seemingly puzzling facts about the backgrounds we built. Some of what we say here repeats previous results, but we find it convenient to collect together all the useful information we have.

In short, the unrotated solutions differ from the KS one in four respects. Two of these are well known and admit a clean field-theory interpretation. There are neither $B_2$ nor $F_5$ in the wrapped-D5 system, and as a consequence one of the (quasi-) marginal deformations, signaled by $h_1$, is absent. In the dual field theory language, this means that there is only one gauge group with adjoint matter in contrast with two gauge groups and bifundamental matter as in the quiver theories dual to the KW and KS backgrounds. There is a dimension-8 operator deforming the theory. As a result, the dual field theory is not UV complete, not even in the generalized sense of KS. These first two differences are affected by the rotation, which depending on the value of $\kappa_2$ amounts to switching back on the second marginal deformation (and hence the dual field theory is a quiver). In particular, fine-tuning $\kappa_2$ to its maximal value leaves us with a KS-like quiver, in which only one tunable parameter controls both marginal deformations, and at the same time, in this limit the dimension-8 deformation is exactly switched off. 

Aside from this, there are two differences between the wrapped-D5 background and the KS background which are not affected by the rotation, and that yield two puzzling results. First of all, the quantity $e^{2\phi} + a^2 - 1$ (that when vanishing yields the KS system) is non-vanishing, and hence we are always describing backgrounds that, after rotation, belong on the baryonic branch (they are hence more closely related to the backgrounds in [7] than to the ones in [3]). So, we find the first puzzle:

- **Puzzle:** the UV expansion of the coefficient of this VEV (or $\tilde{g}$) is proportional to $N_\chi/c_+$, while the coefficient of the dimension-8 deformation is proportional to $e^{2\phi}/N_\chi^2$. Such a precise relation between two superficially independent coefficients demands an explanation.

Remember that in the process of solving the master equation we restricted ourselves to a subset of the possible solutions, by fine-tuning $Q_0 = -N_\chi$ in order to avoid a nasty singularity — see the discussion below Eq. 5. This is the technical reason that makes the VEV and the coupling related. But when such a kind of fine-tuning is needed on

\textsuperscript{6} We did not check the existence of a massless normalizable glueball associated with the breaking of baryonic symmetry.
the gravity side of the correspondence in order to avoid a singularity, it is often the case that the fine-tuning has a clean explanation in terms of the dual field theory, and this is the first thing we would like to understand.

\( a^2 + e^{2\theta} - 1 \)

\( x + 3p \)

**FIG. 3:** The (rotation-invariant) combinations \( a^2 + e^{2\theta} - 1 \) and \( x + 3p \), as a function of the radial direction \( \rho \), for the same backgrounds as in Fig. 2, with the same color-coding.

The second puzzling fact has to do with the relation between the behavior of three independent background functions, all of which are unaffected by the rotation. We plot in Fig. 3 the two invariant quantities \( a^2 + e^{2\theta} - 1 \) and \( x + 3p \) for the same backgrounds as in Fig. 2. First of all, \( x + 3p \) agrees with KS in the far UV, but differs for \( \rho < \rho_\star \approx 9 \). This is simply the effect of the presence of the dimension-6 VEV, which changes the IR, but not the UV dynamics.

- **Puzzle:** the puzzle comes from the fact that on the solutions we are interested in, the invariant combination \( a^2 + e^{2\theta} - 1 \) assumes a non-trivial profile at the same scale \( \rho_\star \) at which the function \( x + 3p \) sensitive to the dimension-6 VEV is taking over the dynamics.

While this could probably be explained in terms of the very non-trivial behavior of the RG evolution in the field theory language (in the gravity language, the fact that the BPS equations for the background scalars are coupled), what is surprising is that below the scale \( \rho_\star \) this combination is suppressed, and vanishes (exactly) at the end-of-space in the IR.

The third rotation-invariant quantity we referred to is the dilaton. The coefficient of the \( z^4 \) term in the UV expansion of the dilaton depends only on \( c_+ / N_c \), and yet the dilaton profile changes significantly at the scale \( \rho_\star \) controlled by \( c_- \). Deep in the IR the dilaton becomes again practically constant (see Fig. 2). All of this in spite of the fact that \( c_- \) appears nowhere in the UV expansion of the dilaton itself. Again, this might just be the effect of operator mixing. And yet, it demands a more precise explanation.

Probably connected to the second puzzle, we make an observation that anticipates one of the results of the next section. In the presence of \( c_- \neq 0 \), the background is singular. This might suggest that what we are doing by turning on \( c_- \) is not allowed in the dual gauge theory: after all, we are tampering with a dimension-6 VEV, and hence the vacuum structure, without changing the couplings (dynamics), and it is hence not surprising that we run into troubles in the deep IR. But this is too simplistic an explanation: the singularity we obtain is surprisingly mild, yielding finite Ricci scalar \( R \) and \( R_{\mu\nu} R^{\mu\nu} \). Only the invariant \( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) shows the singularity. In the presence of various matter fields, ‘mild’ singularities like this are not typical. As a result of this, many physical low-energy quantities can (and will, in the next sections) be computed without obvious obstacles, contrary to what is expected in the presence of a singular background. This suggests that the singularity is resolvable, probably by relaxing the constraints yielding the PT system, and considering a more general truncation such as those in \([2, 25]\). We postpone the (non-trivial) question of how to resolve the IR singularity in the presence of \( c_- \) to a future study, in the hope that this would shed some light also on the other aspects of this puzzle, and go back to the first of the two puzzles, taking the attitude that these solutions, although singular, admit a sensible field-theory interpretation.

The rotated solution with \( k_2 \) fine-tuned to its maximal value \( k_2 e^{\Phi_\infty} = 1 \) automatically enforces the constraint \( \bar{M} = 0 \). We can summarize the second puzzle by saying that there is some non-trivial relation between the dimension-2 and dimension-6 VEV, that demands an explanation in the context of the field theory based on the \( SU(M) \times SU(M+N) \) quiver. We will devote the following subsection to making more precise all the elements of this puzzle, while postponing its resolution to a dedicated field theory study, in which the precise role of dimension-2 and dimension-6 VEVs will be studied in detail, in the context of more general classes of solutions to the master equation for \( P \) than those addressed in this paper.
1. Higher-order operators

We want to understand why the coefficients of the UV expansion of the dimension-2 VEV and of the coupling of the dimension-8 operator for the solutions to the wrapped-D5 system are not independent, and why by fine-tuning the parameter $k_2$ (which controls the extended gauge symmetry of the dual of the KS system with respect to the dual of the wrapped-D5 system) one ends up switching off the dimension-8 operator, without affecting the VEV. What kind of field theories do we know of, in which the coefficients of a set of higher-order operators are precisely related to the value of a VEV? Two examples are the chiral Lagrangian of QCD and the electro-weak effective action obtained by integrating out the heavy gauge bosons from the Standard Model. Let us digress and remind the reader about the basic properties of the latter.

Suppose that one wants to compute the amplitude of a given flavor-changing neutral current process involving hadrons (i.e. one or more of the five lightest quarks), within the Standard Model. In principle what one could do is simply to compute all the relevant Feynman diagrams at some order in perturbation theory. But this is not a good idea, for two reasons. First of all, because one needs to compute also the relevant matrix elements of hadrons, for currents built out of quarks, and this is a strongly-coupled problem that requires input from the lattice (or from some other non-perturbative tool). But even more importantly, because the diagrams would become far too difficult, due to the fact that even perturbative-QCD effects are large enough that they must be included, often at the next-to-next-to-leading-order level. In particular, perturbation theory does not like calculations that involve largely separated mass scales (such as the masses of the $W$-boson and of the $b$-quark), because of potentially large logarithms appearing from the brute-force evaluation of the loop diagrams, and this requires RG-improve the perturbative calculations.

A systematic and organized way of proceeding exists, and yields sensible results in many phenomenologically relevant applications (see for example [33] for a pedagogical review on the subject). The basic idea is to break down the calculation in three stages. First one uses perturbation-theory methods to compute the relevant amplitudes in the original theory (the Standard Model with gauge group $SU(3) \times SU(2) \times U(1)$), up to some loop order. This intrinsically assumes that all the couplings are small, and that it makes sense to compute in terms of quarks and gluons, which is the case provided this is done at the electro-weak scale. Then one uses these amplitudes to match (at the electro-weak scale) onto the coefficients of an effective theory, which is obtained by suppressing the heavy degrees of freedom ($W$ and $Z$ gauge bosons, top quark and possible Higgs fields) and writing an effective Lagrangian with the unbroken $SU(3) \times U(1)$, and which contains a complete basis of higher-order operators involving the light degrees of freedom (quarks, leptons, photon and gluons), in which the coefficients are chosen in such a way as to yield the same amplitudes. The second stage consists of using the RG equations of the effective theory in order to evolve the coefficients from the electro-weak scale down to some relevant physical scale (the mass of the $B$ meson, for example). Finally, one uses the input from the lattice, computing the matrix elements at the same low-energy scale (and within the same renormalization scheme), and finally obtain the phenomenologically relevant amplitudes to be compared to the data (which are scheme and scale independent).

So much for this digression. The point is that the effective Lagrangian used in the second stage of this procedure is a generalization and refinement of the Fermi theory, supplemented by the interactions of the unbroken gauge group. At leading-order, the coefficients of the higher-dimensional operators are precisely related to the VEV in the original theory $\rho_{W}$ (the Standard Model Higgs VEV responsible for electro-weak symmetry breaking) by the Fermi constant $G_F$ as in

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8M_W^2} = \frac{1}{2\rho_{W}},$$

where $M_W$ is the mass of the $W$ gauge bosons, and $g_{W}$ the $SU(2)$ gauge coupling. Deep in the IR, this theory is equivalent to the Standard Model, up to some finite order in the perturbative expansion. However, on the one hand adopting the effective field theory language makes it easier (in practice) to keep into account precisely the dynamics of the unbroken gauge group. On the other hand, the effective theory contains higher-order operators, and if one were to evolve its RG equations from the electro-weak scale towards the UV (rather than towards the IR, as one should) one would run into big troubles. While more manageable when dealing with phenomenological questions, this effective theory is not UV complete, it is valid (and useful) only up to a UV cutoff, set by the masses of the heavy states that have been integrated out in the process of constructing it.

Let us now close the digression and get back to our problem. The analogies should be evident. In the wrapped-D5 system we have a VEV and a higher-dimensional operator which are strongly correlated. The dual field theory has gauge group $SU(N_c)$, but is UV incomplete, because of the higher-order operator, which takes over the dynamics above the scale set by $\rho_{k}$. The process of rotating for a generic value of $k_2$ changes the dual gauge theory into some quiver theory, which has a larger gauge group, more degrees of freedom and more couplings. The presence of more gauge degrees of freedom translates into the fact that, because the deep IR is very similar, the coefficient of the higher-order operator is modified as a function of $k_2$. However, there exists a special value of $k_2$ (and hence a special
choice of dual quiver gauge theory) for which this process ends up switching off the higher-dimensional operator, making the theory healthier when extrapolated towards the far UV.

Hence our proposal for the interpretation of what is going on.

- The UV-complete dual field theory of the backgrounds obtained by rotating with a fine-tuned value of $k_2$ is a quiver, similar to the one of Klebanov-Strassler. This theory undergoes a chain of Seiberg dualities (the cascade), as in KS. However, the cascade does not proceed all the way down to its latest stages. Rather, a non-trivial VEV Higgses the gauge symmetry. The VEV interrupts the cascade at a stage that is controlled by $\rho_*$. In the process, most of the vector and chiral multiplets acquire a mass and decouple. The wrapped-D5 system provides the gravity dual of the effective field theory description valid after integrating out these degrees of freedom, below the scale fixed by $\rho_*$. Ultimately, the unbroken $SU(N_c)$ gauge group leads to confinement and to the formation of the gaugino condensate.

One can integrate out the heavy degrees of freedom from the quiver gauge theory, and in this way obtain a new gauge theory in which the field content is the one of the dual description of the wrapped-D5 system. However, the result is an effective theory which contains higher-order operators, with coefficients determined by the symmetry-breaking VEV. The gravity dual of this is the wrapped-D5 (unrotated) background. It yields (almost) the same physics in the deep IR as the original quiver, but it is now UV incomplete. Rotated backgrounds with generic values of $k_2$ correspond to partial UV completions, in which an incorrect number of degrees of freedom has been added (the gauge group is not large enough), and hence the dimension-8 operator cannot be completely removed. But notice that in doing so one keeps the dimension-2 and dimension-6 VEV unchanged, which is reflected in the fact that (in the five-dimensional language) $a^2 + e^{2g} - 1$ and $x + 3p$ are unaffected by the rotation.

In short, what we are suggesting is that the UV completion of the backgrounds obtained in the wrapped-D5 system (and having UV asymptotics with $P \sim e^{2g/3}$) can be constructed by rotating according to \cite{26, 27} and fine-tuning the rotation parameter $k_2$. This yields the dual of a quiver (in our particular case, the theory is in the Higgs phase). Hence the rotated and unrotated backgrounds are not dual to two different unrelated field theories, but to two theories one of which is the low-energy effective description of the other.

A final cautionary remark, mostly technical in nature. By close inspection of the five-dimensional scalars that enter in the background values of the fields $h_1$, $h_2$ that enter in the background values of the fields $B_2$ and $F_5$ are non-trivially suppressed below the scale $\rho_*$. We will come back to this later on, but we anticipate here that this is not a reason for concern, because it turns out that both $F_5$ and $H_3$ are strongly suppressed below $\rho_*$.  

2. Perturbative results: a summary

In this subsection we collect from the literature a set of perturbative results that are relevant in order to provide a complete field theory interpretation of the backgrounds we discussed. We start this discussion by reminding the reader of the so-called planar equivalences. Since 1998, it has been proposed that orbifold and orientifold projections of parent supersymmetric field theories (for example $\mathcal{N} = 4$ SYM, $\mathcal{N} = 1$ SYM, et cetera, with gauge group $SU(N_c)$) to less symmetric daughter theories shared the same planar diagrams in the large $N_c$ limit. This implies that the perturbative expansions are coincident. While at this level the equivalence is kinematical, the non-perturbative nature of the equivalence was suggested to be valid. This equivalence relates the connected correlation functions and VEVs of corresponding neutral operators\footnote{By neutral we refer to operators in the parent theory that are gauge invariant, single trace and invariant under the discrete symmetries that define the projection.} in both theories. In order for this planar equivalence to be valid it is needed that the discrete symmetries that define the projection do not undergo spontaneous symmetry breaking. For a summary of this line of research, see the papers \cite{34}.

In this paper we will only suggest a planar equivalence between the $\mathcal{N} = 1$ single node $SU(N_c)$ supersymmetric theory with an infinite tower of massive excitations (that arise when compactifying with the D5-branes), the $\mathcal{N} = 1^*$ SYM theory around a particular Higgs vacuum and the two-node KS quiver with bifundamentals.\footnote{Something like this planar equivalence may exist, but it must be slightly different from the ones studied in \cite{34} and references therein.} Nevertheless,
sometimes we may use the language developed in flows of $\mathcal{N} = 1^*$ SYM (with three chiral multiplets $\Phi_i$, $i = 1, 2, 3$) to refer to the KS quiver with bifundamentals $A_\alpha, B_\beta$, with $\alpha, \beta = 1, 2$. In the following, we will give some details on key results that highlight this connection.

In Section III B 1 of this paper, we interpreted the rotation procedure as a ‘conspiration’ between a quasi-marginal coupling and a dimension-2 operator getting a VEV, so that an irrelevant operator $O_8$ (that without this tuning would be present and driving the UV dynamics) is actually not present. This proposal was exemplified by what happens in the Standard Model and its low-energy effective field theory, the Fermi theory, both of them being weakly coupled. But we would like to emphasize that our proposal in section III B 1 is for some dynamics that takes place at strong coupling.

A weakly coupled version of the connection we proposed was presented by Maldacena and Martelli in [36]. Interestingly, their proposal goes from the KS quiver field theory into the one node with adjoints field theory, while ours (just like the rotation on the string/gravity side) goes from the one-node QFT (with the irrelevant inserted $O_8$) into the quiver, that acts as the correct UV completion and decouples this $O_8$.

The authors of [26] consider the KS quiver and study its perturbative dynamics (by this, we mean that the Kähler potential is trivial, hence no gravity background can be a good approximation to the dynamics). We need to solve the D-term equations that read (see also [36])

\[
\sum_\alpha A_\alpha A_\alpha^\dagger - \sum_\beta B_\beta B_\beta^\dagger = \frac{U}{M^1_M},
\]

\[
\sum_\alpha A_\alpha^\dagger A_\alpha - \sum_\beta B_\beta B_\beta^\dagger = \frac{U}{M + N_c}^1_{(N_c + M)},
\]

\[
U = Tr(\sum_\alpha A_\alpha A_\alpha^\dagger - \sum_\beta B_\beta B_\beta^\dagger),
\]

where we have used that the quiver is $SU(M) \times SU(M + N_c)$ and that $A_\alpha, B_\beta$ transform as bifundamentals in each group ($\alpha, \beta = 1, 2$). It was shown in [36] that there are two types of solutions to these equations. Those where both $A_\alpha, B_\beta$ are nonzero and that correspond to mesonic branches (where $M \sim AB$) and those in which either $A = 0$ or $B = 0$ that correspond to baryonic branches and that arise only if $M = q N_c$ (where $q$ is an integer). The authors of [36] complement their analysis with the non-perturbative induced superpotentials, solve also the F-term equations and point out how the moduli space changes from the classical solution (the conifold) into non-singular deformed conifolds.

Coming back to the perturbative analysis in the paper [26], the authors proceed by expanding around a particular (perturbative) baryonic solution—presented in section 4.2 of [36], and find that the gauge group is Higgsed from $SU(M + N_c) \times SU(M) \to SU(N_c)$ (we emphasize, with $M = q N_c$, otherwise such a perturbative baryonic solution does not exist). Also, they obtain that the perturbative mass spectrum of this quiver in this particular vacuum is nearly coincident with the perturbative mass spectrum found by the authors of [37], that we now revisit.

Andrews and Dorey [37] studied the KK (with twisting) decomposition of the six-dimensional field theory with sixteen supercharges and gauge group $SU(N_c)$, that is the theory living on $N_c$ D5-branes that wrap a holomorphic two-cycle in the resolved conifold. After a careful analysis, they obtained a spectrum for a four-supercharge $SU(N_c)$ field theory that consists of a massless vector multiplet and a tower of massive vector and chiral multiplets. Degeneracies and masses at each level are given in [37]. Again, we stress that this is a classical calculation. In the same work, the authors of [37] studied the F-flatness condition coming from the $\mathcal{N} = 1^*$ SYM superpotential

\[
\mathcal{W}(\Phi_i) = Tr[i\sqrt{2} \Phi_1 \Phi_2 \Phi_3 + \eta(\Phi_1^2 + \Phi_2^2 + \Phi_3^2)] \to \{\Phi_i, \Phi_j\} = i\sqrt{2} \delta_{ij} \Phi_k,
\]

that after a rescaling of the fields $\Phi_i$ leads to the $SU(2)$ algebra. The solutions to the equations are any representation of $SU(2)$. Expanding around the Higgs vacuum $\Phi_i = J_i^{(N_c)}$ that breaks $U(N_c) \to U(1)^{N_c}$, they find that the vacuum defines a fuzzy sphere (and, in the limit $N_c \to \infty$, a sphere). Carefully expanding the Lagrangian for $N = 1^*$ and keeping only quadratic terms around the vacua $\Phi_i = J_i^{(N_c)} + \delta \Phi_i$ they also find the mass spectrum, that matches (for finite value of $N_c$ with a truncated version of) the one of the compactified D5-brane theory discussed above. Also, they showed that at leading order in the fluctuations, the Lagrangians match. In other words, they are showing how the four-dimensional $\mathcal{N} = 1^*$ SYM theory deconstructs the six-dimensional theory on the five-branes.

For future reference, we note that if the choice of Higgs vacuum is

\[
\Phi_i = 1_{N_c} \times J_i^{(q + 1)}
\]

then the gauge symmetry is broken according to

\[
U((q + 1)N_c) \to U(N_c)
\]
and the mass spectrum of the weakly coupled $\mathcal{N} = 1^*$ SYM contains a tower of massive chirals and massive vectors (aside from the massless vector multiplet). The heaviest state is a vector multiplet with mass

$$M^2 = \eta^2 q(q + 1)$$

followed by a chiral of mass $M^2 = \eta^2 (q + 1)^2$. The masses and degeneracies for vector and chiral multiplets are

$$M_v^2 = \eta^2 k(k + 1), \quad \text{deg} = (2k + 1)N^2_c,$$

$$M_{ch}^2 = \eta^2 k^2, \quad \text{deg} = 4kN^2_c, \quad k : 1, \ldots, q.$$  

In summary, Higgsing the $\mathcal{N} = 1^*$ SYM theory around one of its classical vacua exactly reproduces the truncated perturbative mass spectrum of the compactified theory on the D5-branes. Higgsing the KS quiver around one of its perturbative baryonic solutions roughly reproduces the perturbative spectrum of the theory on the compactified D5-branes. The coincidences are notable. The three theories are linked and this suggests a relation between $\mathcal{N} = 1^*$ and the KS quiver, perhaps along the lines of [38] (it would be nice to realize this in string theory). We emphasize that both in our strongly coupled version and in Maldacena-Martelli [26] weakly coupled connection, the Higgsing plays a central role. It is indeed the way of connecting a quiver theory with a single node theory. It should be interesting to make more formal the idea of a possible planar equivalence between these three theories.

We would like to make a brief comment about the phenomenon of Higgsing in these backgrounds. In the paper [39], Aharony proposed that, aside from a sequence of Seiberg dualities, the Higgs mechanism could be the reason why the decrease in ranks of the KS cascade takes place. It was argued that at every position where one usually performs a Seiberg duality, there is a source that, once crossed, Higgses the gauge groups. This proposal found a clean realization in the solutions with sources (flavor branes) of [27], where one can see that the warp factor is the superposition of both phenomena (the cascade and the Higgsing). Here, we are proposing that even in the absence of sources, the Higgsing interpretation may be adequate. Indeed, the equivalence between the two pictures (Seiberg duality and Higgsing) was argued in more generality in [38].

3. About the vacuum structure of the dual theory

We conclude the field theory analysis by discussing the physics connected with the second puzzle we highlighted earlier on, in particular with the roles played by the dimension-2 and dimension-6 VEVs. Because we can think of the process of rotating and fine-tuning (the choice of $k_2$) as yielding the UV completion of the dual to the wrapped-D5 system, and hence as a way of describing in different terms the same long-distance physics, we will here concentrate on the rotated and fine-tuned solutions, the results extending to the whole class under consideration.

We start by listing some important properties of the backgrounds we studied in this paper.

- The solution is controlled by the coefficients $c_+$ and $c_-$ appearing in the UV expansion, which correspond to dimension-2 and dimension-6 VEVs in the field theory.
- The physical meaning of the freedom we have in choosing backgrounds with different $\rho_+$ is connected with the scale at which the cascade stops and the gauge group is Higgsed, and with the parameter $q$ controlling the breaking $SU(qN_c + N_c) \times SU(qN_c) \to SU(N_c)$.
- The rotated and fine-tuned solutions always have $\tilde{M} = 0$.
- There is a non-trivial correlation between the behavior of $\Phi$, $a^2 + e^{2\tilde{g}} - 1$ and $x + 3\rho$ at and below the scale $\rho_+$, but not above it (as suggested by the UV expansions). Also, the combination $a^2 + e^{2\tilde{g}} - 1$ vanishes at the end-of-space in the IR.
- There are physical, measurable differences between backgrounds in which both dimension-2 and dimension-6 VEVs are present (the seed solution and the rotated one in this paper), the backgrounds in which the dimension-6 VEV is absent (such as Butti et al. [21]), and the KS background, in which both VEVs vanish. We will explicitly show this fact later on, by computing the expectation value of the rectangular Wilson loop.
- The singularity in the IR does not seem to be a reason for major concern, as it is not preventing us from a consistent field theory interpretation, but rather the calculation of physical quantities seems to proceed unaffected by it. For this reason, in this paper we took the pragmatic approach of analyzing the background in field-theory terms, assuming that the singularity is resolvable, and that the possible resolution does not affect the observables we are interested in.
There seems to be an emerging general picture, in field theory terms, explaining what the relation is between the rotated backgrounds, their relatives within the PT ansatz, and the deformations of \( \mathcal{N} = 4 \) super-Yang-Mills. This picture is, for the time being, based on circumstantial evidence and striking analogies at the perturbative level, elements of which appear to manifest themselves also in the gravity dual at the non-perturbative level.

Ultimately, we would like to understand if there is a comprehensive field theory picture that explains all of the above. This requires conducting a more systematic study of the dual field theory, which we postpone to a future study. Such a study requires including also solutions of the master equation that we did not include in the present paper; such as those in which \( P \simeq 2N_c \rho \) (as in [8], at least for some range of \( \rho \) (as in [17, 18] for example). In doing so, we should be able to ask whether the physics associated with the dimension-3, dimension-2 and dimension-6 VEVs which control the non-trivial features of these backgrounds can be interpreted as a genuinely multi-scale dynamical model.

We conclude with a geometric observation, possibly connected with the roles of the dimension-2 VEV represented by \( (a^2 + e^{2\theta} - 1) \) and the dimension-6 VEV in \( (x + 3\rho) \). Let us take the (string-frame) metric of the PT system. Consider now a pair of three-cycles \( \Sigma_3 = [\theta, \varphi, \psi] \) and \( \Sigma_3 = [\tilde{\theta}, \tilde{\varphi}, \tilde{\psi}] \) in the internal manifold. The resulting induced metrics on each of the cycles \( \Sigma_3, \Sigma_3 \) are

\[
\Sigma_3 : \quad ds_3^2 = e^{\Phi/2+x+\tilde{g}} (a^2 + e^{2\tilde{g}}) \left( \frac{e^{-6p-2x+\tilde{g}}}{(a^2 + e^{2\tilde{g}})} (d\psi + d\phi \cos \theta)^2 + (d\theta^2 + d\phi^2 \sin^2 \theta) \right),
\]

\[
\Sigma_3 : \quad ds_3^2 = e^{\Phi/2+x+\tilde{g}} (e^{-6p-2x+\tilde{g}} (d\psi + d\phi \cos \tilde{\theta})^2 + (d\theta^2 + d\phi^2 \sin^2 \tilde{\theta})).
\]

Both are proportional to the metric on the squashed three-sphere, which can be written in terms of the three angles \( 0 \leq \theta < \pi, \ 0 \leq \phi < 2\pi \) and \( 0 \leq \psi < 4\pi \)

\[
ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \alpha (d\psi + \cos \theta d\phi)^2,
\]

and which reproduces the metric on the 3-sphere for \( \alpha = 1 \).

Notice that in the KS system, in which the VEVs of baryon and antibaryon operators coincide, one has \( a^2 + e^{2\theta} = 1 \), so that the two three-dimensional surfaces have the same geometry. This is an effect of the global \( \mathbb{Z}_2 \) symmetry of the KS system.

Let us focus now on the cycle \( \Sigma_3 \) or \( ds_3^2 \). Expressed in terms of \( P \) and \( Q \), the squashing factor becomes

\[
\alpha_2 = e^{-6p-2x+2\tilde{g}} = \frac{\partial_\rho P}{2(P \coth(2\rho - Q)},
\]

with \( Q = N_c(2\rho \coth(2\rho - 1)) \).

When \( P \sim e^{2\tilde{g}} \) (in the case of this paper, when \( \rho > \rho_1 \)), then \( \alpha_2 \sim \frac{1}{\tilde{g}} \), which is the familiar geometric factor characterizing \( T^{1,1} \). This is not a surprising result. More interesting is the fact that when \( P \simeq c_0 \)---as in our background in the deep IR--- then \( \alpha_2 \sim e^{A(\rho - \rho_1)} \) is exponentially suppressed, and one obtains that \( ds_3^2 \) is a three-sphere which becomes extremely squashed near the origin of the space. By comparison, in the linear-dilaton solution of [3] one has \( P = 2N_c \rho \), and hence \( \alpha_2 = 1 \), so that \( ds_3^2 \) is exactly a three-sphere.

The Wilson-loop expectation value was calculated in [18] focusing on backgrounds in which \( P \sim c_0 \) for \( \rho < \rho_1 \), and \( P \simeq 2N_c \rho \) for \( \rho > \rho_1 \). A first-order phase transition was shown to appear when \( \rho > \rho_1 \). As we will see, this behavior is present also in the case discussed in this paper, suggesting that the phase transition has to do with the squashing of the sphere, and ultimately with the dimension-6 VEV. Again, a dedicated study of the field theory, and a systematic comparison among all possible classes of regular solutions to the master equation is necessary, in order to elucidate this point.

We will now study a couple of quantities that will reinforce the field theory interpretation put forward above.

### C. Central charge, Maxwell charge and the scale \( \rho_* \)

In this subsection we perform a study of a set of non-trivial field-theory quantities that can be computed in the rotated background. The main purpose of this subsection is to illustrate the difference between the rotated backgrounds in the ranges \( \rho < \rho_* \) and \( \rho > \rho_* \). In doing so, we are going to test the proposal for the field-theory interpretation of the previous subsection. In particular, we will show that the dual theory below \( \rho_* \) has a smaller number of effective degrees of freedom, and a smaller gauge group, compared to what is expected in the KS case or along the baryonic branch of KS described in [7].
1. Central charge

The holographic central charge is given by [40]

\[ c \sim \frac{1}{(\partial_r A)^3}. \]  

(55)

Using Einstein’s equations, one can show that this quantity is a monotonically decreasing function as one flows towards the IR. In terms of the ten-dimensional variables

\[ c \sim \frac{(1 - k_2 e^{2\Phi})^2 e^{2\Phi + 2h + 2g + 4k}}{\left(\partial_{\rho} \left[4\Phi + 4h + 4g + 2k + \log(1 - k_2 e^{2\Phi})\right]\right)^3}. \]  

(56)

We plot this quantity for a few of the rotated solutions as well as deformations of Klebanov-Strassler in Figure 4 (by ‘deformations of KS’ we mean solutions where the integration constant \( f_0 \) in eq. (35) is non-zero).

![Figure 4: The left panel shows the central charge as a function of the radial coordinate \( \rho \) for a few of the rotated solutions. The right panel shows the same, but for deformations of Klebanov-Strassler given by different values of \( f_0 \) (the black line corresponds to \( f_0 = 0 \), i.e. the original solution of Klebanov-Strassler).](image)

The presence of the scale \( \rho_* \) has, both in our rotated solutions and in the case of the singular modification of KS, the effect of drastically reducing the central charge at and below the scale \( \rho_* \). In field-theory terms, this means that a large number of degrees of freedom freeze below this scale, their dynamics becoming trivial and decoupled.

Whilst the central charge is only well defined at fixed points, we believe this analysis points to the particular behavior we proposed. Let us now analyze another observable.

2. Maxwell charge

We define a Maxwell charge associated with the D3-branes as

\[ Q_{\text{Maxwell},D3} = \frac{1}{16\pi^2} \int_{\Sigma_5} F_5 = \frac{4}{\pi} K, \]  

(57)

with the manifold \( \Sigma_5 = [\theta, \varphi, \tilde{\theta}, \tilde{\varphi}, \psi] \). We will use this Maxwell charge as an indicator of the ‘number of degrees of freedom’ in the quiver field theory as originally suggested in [4]. Notice that in the five-dimensional language \( Q_{\text{Maxwell},D3} \) is nothing but the function \( K \) in Eq. (11). For the rotated solutions this becomes equal to

\[ Q_{\text{Maxwell},D3} = \frac{k_2 e^{2(\Phi + \sigma + h)}}{\pi} \partial_{\rho} \Phi = \frac{k_2 e^{2\Phi}}{2\pi} Q \left( N_c \cosh(2\rho) - \sigma \right), \]  

(58)

whereas for Klebanov-Strassler (and deformations corresponding to non-zero \( f_0 \), but \( \tilde{M} = 0 \)) it is equal to

\[ Q_{\text{Maxwell},D3} = -\frac{4}{\pi} e^{\Phi - N_c^2(2\rho \coth(2\rho) - 1)(4\rho - \sinh(4\rho))}. \]  

(59)
Note that the deformation parameter $f_0$ does not enter into this expression. In Figure $5$, $Q_{\text{Maxwell},D3}$ is plotted as a function of $\rho$ for a few of the rotated solutions, as well as for Klebanov-Strassler.

If one is to interpret this quantity in terms of diluted D3 in the background, or equivalently as giving a rough estimation for the rank of the gauge group of the dual quiver theory, what this shows is the expected behavior of the cascade for $\rho > \rho_*$. However, below $\rho_*$ this rank suddenly drops virtually to zero. This supports the suggestion that the formation of the condensates results in the Higgsing of the theory, in which the last steps of the duality cascade are replaced by the spontaneous breaking $SU(M) \times SU(M + N) \to SU(N)$. In the ten-dimensional language, the fact that $K$ is strongly suppressed below $\rho_*$ means in turn that $B_2$, $H_3 = dB_2$ and $F_5$ are also suppressed compared to $F_3$. In this sense, approximating them with zero, and looking and the wrapped-D5 system instead, is another way of thinking of the latter as an effective field theory. It contains many less degrees of freedom, and is hence much simpler and convenient, while at the same time this is just a leading-order approximation, which yields quite accurate results below the cutoff connected with $\rho_*$, but above which a completion is needed.

Concluding this short subsection: both the central charge and the Maxwell charge, computed for the rotated backgrounds behave in two different ways for $\rho > \rho_*$ and for $\rho < \rho_*$. In the former case, a slow evolution is compatible with the duality cascade. In the latter case, there is a sudden drop, with the rank of the gauge group of the dual theory and its number of effective degrees of freedom falling towards their minimal values. Both these two derived quantities seem to support our interpretation of the background, according to which $\rho_*$ is the cutoff scale below which the dual theory is in the Higgsed phase.

We now move on to study IR aspects of our new backgrounds.

IV. LONG-DISTANCE PHYSICS: IR ASYMPTOTICS AND WILSON LOOP

This section is devoted to the long-distance physics of the solutions. As we said, in the deep-IR region there is little difference between the backgrounds that belong to the class of the wrapped-D5 system and their rotated counter-parts. However, the presence of the dimension-6 VEV is going to result in rather important differences compared to the KS solution [4] and to the baryonic branch solutions in [7]. We focus the study on the rotated and fine-tuned solutions, for simplicity.

A. IR asymptotics and curvature singularity

The original ‘seed’ solutions discussed in Section II E have singularities in the IR. This is also true after the rotation has been applied to them. Various criteria have been given in the literature, as to when an IR singularity is good (and

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$^9$ The Page charge, as used in [31], gives the same result as in the KS background $Q_{\text{Page},D3} = 0$. 

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FIG. 5: The right panel shows the Maxwell charge $Q_{\text{Maxwell},D3}$ for Klebanov-Strassler (black line), and a few different rotated solutions, the $P$ of which is shown in the left panel.
it is believed that the supergravity background captures the relevant physics) or bad (in which case the supergravity description breaks down and a resolution of the singularity in supergravity or even the full string theory is needed). One such criterion is the one due to Gubser \[42\], which states that the five-dimensional potential $V$ — see Eq. (13) — evaluated on a particular solution to the equations of motion, should be bounded from above in order for the singularity to be good. It is argued that this is a necessary condition for the existence of near-extremal generalizations of the background, in which the singularity is hidden behind the horizon of a black hole. Another criterion, given in \[32\], states in its strong form that the $g_{tt}$ component of the metric as function of the radial coordinate should not increase as one approaches the IR singularity. This is motivated by the interpretation of the radial coordinate as corresponding to the energy scale of the dual field theory: excitations in the bulk that are closer to the IR singularity should correspond to excitations of lower and lower energy as seen from the boundary.

In order to better understand the nature of the IR singularities for the rotated solutions, we will now study three invariant objects related to the curvature. These are the Ricci scalar $R$, the Ricci tensor squared $R_{\mu\nu}R^{\mu\nu}$, and the Riemann tensor squared $R_{\mu\nu\tau\sigma}R^{\mu\nu\tau\sigma}$. Using that $P$ can be expanded in the IR as

$$P = c_0 + c_0 k_3 \rho^3 + \frac{4}{5} c_0 k_3 \rho^5 - c_0 k_3 \rho^6 + \frac{16 k_3}{105 c_0} (2 c_0^2 - 5 N_c^2) \rho^7 + O(\rho^8),$$

(60)

where $c_0$ and $k_3$ are integration constants, one finds that these three objects have IR expansions given by

$$R = \frac{16}{3} \left(\frac{2}{3}\right)^{5/8} e^{-2\Phi_0} N_c^2 \left(\sqrt{6} \sqrt{c_0^2 k_3^5 - 8 k_2^2 e^{2 \Phi_0}}\right)^{3/2} - \frac{128}{9} \left(\frac{2}{3}\right)^{5/8} e^{-2\Phi_0} N_c^2 \left(\sqrt{6} \sqrt{c_0^2 k_3^5 - 8 k_2^2 e^{2 \Phi_0}}\right)^{3/2} \rho^2 + O(\rho^4),$$

(61)

$$R_{\mu\nu} R^{\mu\nu} = -\frac{512}{81} \left(\frac{2}{3}\right)^{5/4} e^{-\Phi_0} N_c^4 \left(8 \sqrt{6} k_2 e^{2 \Phi_0} c_0^2 k_3 - 93 c_0^3 k_2^3 - 992 k_2^4 e^{4 \Phi_0}\right) + \frac{8192}{243} \left(\frac{2}{3}\right)^{5/4} e^{-\Phi_0} N_c^4 \left(8 \sqrt{6} k_2 e^{2 \Phi_0} c_0^2 k_3^5 - 93 c_0^3 k_2^3 - 992 k_2^4 e^{4 \Phi_0}\right) \rho^2 + O(\rho^4),$$

(62)

$$R_{\mu\nu\tau\sigma} R^{\mu\nu\tau\sigma} = -\frac{160}{3} \left(\frac{2}{3}\right)^{5/4} e^{-\Phi_0} \sqrt{\frac{2}{k_3}} \rho^8 + \frac{1024}{72} \sqrt{\frac{2}{k_3}} e^{2 \Phi_0} \rho^6 \rho^6 + O(\rho^{-5}).$$

(63)

As can be seen, $R$ and $R_{\mu\nu} R^{\mu\nu}$ stay finite in the IR (another simple invariant that is finite is $\sqrt{|g|}$), whereas $R_{\mu\nu\tau\sigma} R^{\mu\nu\tau\sigma}$ diverges as $\rho^{-8}$. This agrees with the numerically obtained plots shown in Figure 5.

It did not escape our attention that many other backgrounds (like the negative-mass Schwarzschild solution) follow the same pattern. What is characteristic of the present example is the presence of many matter fields accompanied by the mildness of the singularity.

For more detail about these invariants in related solutions, see Appendix C. We move now to the study of an important observable, the Wilson loop.

B. Wilson loops, confinement and phase transition

We compute here the expectation value of the Wilson loop, and hence the quark-antiquark potential, following the prescription in \[31\]. For convenience, and because in the IR all the solutions discussed here are very similar to those in \[18\], we follow the notation introduced there. We probe the background with an open string, the end-points of which are fixed on a D3-brane that extends in the Minkowski directions, and that is located at some very large value $\rho_2$ of the radial direction. The string configuration (given the distance in the Minkowski directions $L_{QQ}$ between the end-points) is computed by minimizing the classical Nambu-Goto action. The result is a classically stable configuration in which the string hangs in the radial direction down to a minimal value $\rho_o$ at its middle-point. In practice, one varies $\rho_o$, and for each possible value computes the string configuration, the separation $L_{QQ}$ between the end-points of the string on the UV brane, and the energy $E_{QQ}$ as the classical action evaluated on the solution. Solving for $\rho_o$ yields the relation $E_{QQ}(L_{QQ})$, and one can then verify that when $\rho_o$ approaches the end-of-space $\rho_0 = 0$, then $L_{QQ}$ diverges, and $E_{QQ} = \sigma L_{QQ} + O(1/L_{QQ})$, yielding confinement.
In order to do so, one has to write the ten-dimensional metric in string frame. We do not allow the string to explore the internal space, hence only the $g_{tt}$, $g_{xx}$ and $g_{\rho\rho}$ components of the metric are used. One then defines

\[ F^2 = g_{tt}g_{xx}, \]
\[ G^2 = g_{tt}g_{\rho\rho}, \]  
(64)
(65)
\[ V_{\text{eff}}^2(\rho, \hat{\rho}_o) = \frac{F^2(\rho)}{F^2(\hat{\rho}_o)G^2(\rho)} \left( F^2(\rho) - F^2(\hat{\rho}_o) \right), \]  
(66)
\[ L_{\text{QQ}}(\hat{\rho}_o) = 2 \int_{\hat{\rho}_o}^{\rho_2} \frac{1}{V_{\text{eff}}(\rho, \hat{\rho}_o)} d\hat{\rho}, \]  
(67)
\[ E_{\text{QQ}}(\hat{\rho}_o) = 2 \int_{\hat{\rho}_o}^{\rho_3} d\hat{\rho} \frac{F^2(\hat{\rho})G^2(\hat{\rho})}{F^2(\hat{\rho}) - F^2(\hat{\rho}_o)}. \]  
(68)

In order for this calculation to make sense, it is necessary that in the UV

\[ \lim_{\rho \to +\infty} V_{\text{eff}}(\rho, \hat{\rho}_o) = +\infty, \]  
(69)
which encodes the fact that appropriate boundary conditions must be satisfied by the string in order to end on the D3-brane. In order for $L_{\text{QQ}} \to +\infty$ when $\hat{\rho}_o \to 0$, then one must have

\[ \lim_{\hat{\rho}_o \to 0} V_{\text{eff}}(\rho, \hat{\rho}_o) \propto \rho^\gamma + \cdots, \]  
(70)
with $\gamma \geq 1$, where an expansion in small $\rho$ is understood, see the discussion around Eq. (25) in the paper [18].
If both these conditions are satisfied, then one can ask if the theory confines, and what is the value of the string tension. The linear potential is recovered for $\gamma = 1$. The result is

$$\sigma = \lim_{L_{QQ} \to +\infty} \frac{dE_{QQ}}{dL_{QQ}} = \lim_{\rho \to 0} F(\rho).$$  \hspace{1cm} (71)$$

Within the PT ansatz, the string-frame metric is

$$ds^2 = e^{\frac{\Phi}{4}} dS^2 = e^{2p-x} dy^2 + \cdots$$

$$= e^{2p-x+\frac{2}{N} + 2A} dz^2 + e^{2p-x+\frac{2}{N}} dr^2 + \cdots$$

$$= e^{2p-x+\frac{2}{N} + 2A} dz^2 + e^{-6p-x+\frac{2}{N} + \log 4} d\rho^2 + \cdots,$$  \hspace{1cm} (72)

where we omitted the internal part of the metric. As a result, all the necessary information is contained in

$$F^2 = e^{4p-2x+\Phi+4A},$$

$$G^2 = e^{-4p-2x+\Phi+2A+\log 4}.$$  \hspace{1cm} (73) \hspace{1cm} (74)

1. The rotated solutions

We start from the wrapped-D5 system, with no rotation in place, in which case the exact relation $A = \Phi + \frac{x}{2} - p$ holds. Hence in this case one has

$$F^2_0 = e^{2\Phi},$$

$$G^2_0 = e^{-6p-x+\frac{2}{N} + \log 4} = F^2_0 \frac{P'}{2},$$

where the prime refers to a derivative with respect to $\rho$.

In order to see whether these solutions satisfy the UV conditions allowing for the probe-string calculation to be done, we study the asymptotic behavior of $V_{\text{eff}}$.

The dilaton approaches a constant in the far UV, while $P' \propto e^{4p/3}$, hence

$$V_{\text{eff}}^{(k_2=0)} \propto e^{-\frac{4}{3} \rho} \to 0.$$  \hspace{1cm} (77)

This is not compatible with the boundary conditions of the open string on the D3-brane at infinity. We will not discuss these solutions any further.

The action of the rotation for generic $k_2$ is,

$$F^2_{k_2} = \left(1 - k_2^2 F^2_0\right)^{-1} F^2_0,$$

$$G^2_{k_2} = G^2_0,$$  \hspace{1cm} (78) \hspace{1cm} (79)

hence it does affect $F^2$ but not $G^2$. Most importantly, the whole calculation requires to know and specify only $\Phi$ and $P'$.

The effect of the rotation is, as we said, to change the asymptotic behavior of $F$, but not $G$. In practice, at asymptotically large values of $\rho$ and having fine-tuned $k_2$:

$$V_{\text{eff}}^2(r) \propto \left( N_c^2 c_+ e^{-\frac{2}{N} \rho} \right)^{-2} e^{-\frac{2}{N} \rho} \propto e^{4\rho}.$$  \hspace{1cm} (80)

In these expressions, we neglected the log $\rho$ dependence of the $O(\rho^4)$ terms in the expansion of $\Phi$. Hence, within this class of solutions, only those in which the fine-tuning of $k_2$ has been implemented can be probed with the string and the procedure in [31] carried out. Notice that because of the rotation we must include corrections of order $N_c^2 / c_+^2$, or else the effective potential is not well defined.
2. IR expansions

In order to study the effective potential in the deep IR region, we need to expand first. Remember that, as we saw, only in the fine-tuned case does this calculation make sense. Within the wrapped-D5 system, this expansion yields

\[ P = c_0 + k_3 c_0 \rho^3 + \frac{4}{5} k_3 c_0 \rho^5 - k_3^2 c_0 \rho^7 + \frac{16 (2c_0^2 k_3 - 5k_3 N_c^2)}{105 c_0} \rho^7 + \mathcal{O}(\rho^8), \]  
\[ Q = N_c \left( \frac{4}{3} \rho^2 - \frac{16}{45} \rho^4 + \frac{128}{945} \rho^6 + \mathcal{O}(\rho^8) \right), \]

and after some algebra:

\[ F_0^2 = e^{2\Phi} = 4 \sqrt{\frac{2}{3c_0^3 k_3}} \left( 1 + \frac{16 N_c^2}{9 c_0^2} \rho^4 \right)^{1/2}, \]
\[ G_0^2 = 2e^{2\Phi} \sqrt{\frac{6k_3}{c_0}} \rho^2. \]

This means that for \( \hat{\rho}_o \to 0 \) we find that \( V_{\text{eff}} \propto \rho \). We can interpret this as linear confinement, with the string tension given by

\[ \sigma(r) = \left( 1 - e^{-2\Phi_0} \sqrt{\frac{32 e^{4\Phi_0}}{3c_0^3 k_3}} \right)^{-1/2} \left( \frac{32 e^{4\Phi_0}}{3c_0^3 k_3} \right)^{1/4}. \]

In order to learn something, we need to connect the IR and UV expansions. To do so, we can make use of \( P_0 \), but we must keep in the expression leading-order corrections in \( N_c^2 / c_+^2 \). This was done in part in [16], where the IR expansion of the solution was modified to (at \( \mathcal{O}(N_c^2 / c_+^2) \))

\[ P = c \cos \alpha \left( 1 + \left( \frac{25}{32} \tan^3 \alpha + \frac{24 N_c^2 \sin^3 \alpha}{3^2 c_+^2 \cos^3 \alpha} \log^2 (2 \cot^3 \alpha) \right) \rho^3 \right) + \cdots, \]

which yields the identifications

\[ c_0 = c \cos \alpha, \]
\[ k_3 = \frac{2^6}{3^2} \left( 2 \cot^3 \alpha \right)^{1/3} \left( 1 + \frac{N_c^2 \log^2 (2 \cot^3 \alpha)}{2 c_0^2} \right). \]

The UV expansion of \( P \) yields the identifications

\[ c_+ = \frac{c \sin \alpha}{2^{1/3}}, \]
\[ -\frac{c_-}{192 c_+^2} = 2 \cot^3 \alpha. \]

By combining these results one then concludes that (at least for small values of \( N_c / c_+ \))

\[ \frac{3c_0^2 k_3}{32} e^{-4\Phi_0} = e^{-4\Phi_0} \left( 1 + \frac{N_c^2 \log^2 \left( -\frac{c_-}{192 c_+^2} \right)}{18 c_+^2 \left( -\frac{c_-}{192 c_+^2} \right)^{2/3}} \right), \]

and the string tension is

\[ \sigma(r) \simeq e^{\Phi_0} \sqrt{\frac{6c_+ \left( -\frac{c_-}{192 c_+^2} \right)^{1/3}}{N_c \log \left( -\frac{c_-}{192 c_+^2} \right)}}. \]

Going a step further, again from [16] one has \( \log \left( -\frac{c_-}{192 c_+^2} \right) = \log (2 \cot^3 \alpha) \sim 4 \rho_*, \) provided \( \rho_* \gg 1 \), and if we replace:

\[ \sigma(r) \simeq e^{\Phi_0} \left( \frac{3c_+}{2 N_c \rho_*} e^{4 \rho_*} \right). \]
Notice how this implies that for large values of $\rho_*$ the string tension would increase, if we were to keep the other parameters fixed.

As we recalled around Eq. (24), one has to require that $P > Q$ at the scale $\rho_*$ [16]. Hence one has to impose the approximate bound

$$\frac{N_c}{c_+} \lesssim \frac{3\pi^4 \rho_*}{2^{2/3} \rho_*},$$

(93)

which, replacing in the expression we gave for the string tension, yields

$$\sigma_{(r)} \gtrsim 2^{-1/3} e^{\Phi_{\infty}}.$$  

(94)

The actual numerical coefficient should not be trusted, aside from the fact that it is $O(1)$. However, this exercise shows that by tuning appropriately the parameters in our class of solutions (besides tuning $k_2$), one can get a whole family of solutions which confine and have the same string tension, while differing by the value of $\rho_*$, and that this value is controlled by the value of the dilaton at infinity.

3. Numerical study

The class of solutions we are looking at can be characterized in terms of six parameters. We study numerically the Wilson loops restricting to a one-parameter family of rotated solutions selected in the following way. The generating function $P$, which solves the master equation, depends in general on three parameters: $c_+$, $c_-$ and $N_c$. The overall scale is fixed by $\rho_0$, which we choose to vanish $\rho_0 = 0$. The rotated solution depends explicitly on $k_2$. The calculation of the Wilson loop depends also on the value of the dilaton $\Phi_\rho$. We keep $N_c = 4$ fixed (an arbitrary numerical choice, that does not affect any of the physical results), and vary $c_+$ and $\Phi_\rho$ in such a way that the dilaton in the far UV and deep IR is kept fixed, while fine-tuning $k_2 = e^{-\Phi_{\infty}}$, so that the calculation is sensible. In doing so, we generate a one-parameter family of solutions to the equations that differ only by the value of $\rho_*$ (or $c_-$), the value of the radial coordinate below which $P$ is approximately constant and above which it is dominated by terms proportional to $e^{4\rho_\pi/3}$. Furthermore, we choose $P$ so that $P(\rho_*) \approx Q(\rho_*)$, in such a way as to maximize the effects of the $N_c/c_+$ corrections which (after the rotation) take us away from the KS solutions. The resulting function $P$, the ’t Hooft coupling $g^2 N_c/4\pi^2$ and the dilaton $\Phi$ are shown in Fig. 1.

With all of this in place, we perform numerically the calculation of $L_{QQ}$ and $E_{QQ}$. Numerically, our solutions extend towards the UV only up to $\rho \lesssim 15$. We hence vary $0 < \rho_0 \lesssim 12$, keeping explicitly a UV cutoff $\rho_2 = 14$, and compute $L_{QQ}$ and $E_{QQ}$. We plot the result, for the numerically chosen backgrounds, in Fig. 7.

The result is that there is a first-order phase transition, as a function of $L_{QQ}$. In order to visualize the strength, we plot in Fig. 2 the derivative of the energy with respect to the quark separation, $dE_{QQ}/dL_{QQ}$, computed only on the minimum-$E_{QQ}$ configurations. The result illustrates two things. First of all, the solutions have the exact same value of the string tension ($dE_{QQ}/dL_{QQ}$ is a universal constant at large $L_{QQ}$), as a result of the tuning we did on the IR value of the dilaton. Second, the discontinuity in $dE_{QQ}/dL_{QQ}$ depends on $\rho_*$, becoming larger when $\rho_*$ is large.

Some comments are in order. Qualitatively, these results are hardly any different from those in [18], reflecting the fact that the IR of the geometry of these classes of solutions is very similar. In particular, the last panel of Fig. 7 shows an interesting fact: those configurations of the string that solve the classical equations but correspond to a maximum of the energy (the choices of $\rho_0$ for which $L_{QQ}$ is an increasing function of $\rho_*$) have a peculiar shape. The solutions are completely smooth, however they do show a fast turning at their tip, which is ultimately responsible for the fact that the configuration has a comparatively high energy, and is unstable.

Let us comment more about the phase transition we observe. The exact value of the critical $L_{QQ}$ at which the transition takes place appears to depend on the solution. This is a numerical artifact: both the energy $E_{QQ}$ and the quark separation $L_{QQ}$ have been computed while keeping the same value of the UV cutoff $\rho_2$, and hence there is some intrinsic uncertainty due to a possible overall shift in the two, which is purely UV dependent. One should not give any special meaning to it. More interestingly, we can define the following dimensionless quantity

$$s = \frac{\frac{dE_{QQ}}{dL_{QQ}} - \frac{dE_{QQ}}{dL_{QQ}}}{\lim_{L_{QQ} \to +\infty} \frac{dE_{QQ}}{dL_{QQ}}},$$

(95)

and use it to classify how strong the transition is. In the examples in the plots this quantity appears to be $O(1)$, pointing to the fact that already for $\rho_* \sim 3$ we are in the presence of a strong first-order transition. The larger $\rho_*$, the larger $s$ becomes.
An interesting observation: this phase transition exists only provided $\rho_*> \rho_1 \sim 1$. Below some finite value of $\rho_*$, $E_{QQ}(L_{QQ})$ is single valued, and the transition between Coulomb phase and confined phase is completely smooth. We should emphasize that this ‘double turn around’ phenomenon discussed in [18], is present in our solution and also in many other systems with two independent scales. See for example the papers in [42].

V. CONCLUSIONS

We conclude the paper by summarizing the results we obtained, and our interpretation. And finally, we summarize what are the open problems, and possible ways to test and extend our results and their interpretation.

We started our analysis from a rather general, four-parameter class of type-IIB backgrounds obtained by solving the master equation characterizing the wrapped-D5 system (only gravity, dilaton and $F_3$ form are non-trivial). The main features of such backgrounds are that: the theory confines, and the gaugino condensate appears, in the deep IR ($\rho \to 0$), but the dynamics above this scale is characterized by two very different behaviors, with a smooth transition at a value $\rho_* \gg 0$ of the radial direction. We applied to the backgrounds a solution-generating technique (rotation) that (as a function of a new parameter $k_2$) allows to algebraically construct backgrounds that are more general and fall within the PT ansatz. In this way, a flux for $F_3$, $B_2$ and $H_3 = dB_2$ is induced. We exhibited explicitly the action of the rotation both using the ten-dimensional language and the five-dimensional language obtained by consistent truncation of the KK decomposition.

We then studied the backgrounds obtained by the rotation, and compared them to the original ones, to the KS backgrounds and to the baryonic branch of KS. Performing the study of the UV asymptotic behavior of all of these solutions, we concluded that the most important differences with KS are: the presence of a dimension-2 VEV that brings all our solutions on the baryonic branch, the presence of a dimension-8 operator, whose dynamics makes the dual models UV incomplete (unless $k_2$ is fine-tuned against the asymptotic value of the dilaton, in which case the dimension-8 operator disappears), and the presence of what appears as a dimension-6 VEV.

We provided a simple field-theory interpretation of the rotation and of the backgrounds it relates. The unrotated background, with $F_3 = H_3 = 0$, provides a simple gravity dual to the low-energy effective field theory description of the system, which consists of a one-site $\mathcal{N} = 1$, $SU(N_c)$ gauge theory coupled to adjoint matter. This description is good up to the cutoff scale indicated by $\rho_*$, above which the dynamics is driven by the dimension-8 effective operator. The rotation allows to adiabatically switch off the higher-dimensional operator of the dual theory, while keeping the VEV(s) fixed and by fine-tuning $k_2$ one finds that the resulting background roughly interpolates between the KS solution for $\rho > \rho_*$ and the original ‘seed’ wrapped-D5 background for $\rho < \rho_*$. In particular, for $\rho > \rho_*$ the field-theory dual is essentially the cascading $SU(M) \times SU(M + N_c)$ quiver as in KS.

We observed that the rotated backgrounds automatically implement the constraint $\tilde{M} = 0$.

This, put together with the observation that at the perturbative level the $SU(M) \times SU(M + N_c)$ quiver field theory and the $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ SYM result in a spectrum that (at large $N$) deconstructs the (fuzzy) sphere $\mathcal{S}^{2N}$, leads us to suggest that at the scale $\rho_*$ the gauge group is undergoing the Higgsing $SU(qN_c) \times SU(qN_c + N_c) \to SU(N_c)$. The perturbative analog to the (non-perturbative) scale separation $\rho_* \gg 1$ comes from the fact that the spectrum of heavy vector multiplets contains a (finite) tower of states with lowest states with mass $M^2 \sim q^2 \eta^2$. We hence interpreted the freedom in the choice of $\rho_*$ in terms of the freedom of the choice of Higgs vacuum in the dual theory (i. e. in the choice of $q$). We collected quite a large set of elements supporting this interpretation by studying the properties of the supergravity background.

We then studied the long-distance properties of the theory, in order to test our interpretation. Most striking is the fact that in the presence of a substantial hierarchy $\rho_* \gg 1$, the calculation of the Wilson loops yields a non-trivial result for the static quark-antiquark potential, exhibiting the features of a strong first-order transition at intermediate distances, while at long distances the theory confines in the traditional sense. The former behavior disappears when the coefficient of the dimension-6 operator is tuned to very small values. This agrees with what was found in [18], in a class of solutions that are very similar to the seed solution studied here in the IR (for $\rho < \rho_*$), but in which $P$ is linear, rather than exponential, in the UV for ($\rho > \rho_*$). This indicates that the phase transition, if physical, has to do with the coefficient of the dimension-6 VEV.

Interestingly, the Wilson loop yields a perfectly healthy long-distance behavior, as expected in a confining theory well captured by a supergravity dual, in spite of the fact that the generic background with non-vanishing dimension-6 VEV is singular in the IR. This surprising behavior is probably connected with the fact that the singularity is unusually mild for systems with many matter fields like ours: both the Ricci scalar $R$ and the invariant $R_{\mu\nu} R^{\mu\nu}$ are finite, while the singularity is manifest in $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$.

We conclude by suggesting some future research programs. First of all, a coherent picture starts to emerge, that unifies all the best-known solutions belonging to the PT ansatz, in which many features resemble what is expected in the case of mass deformations of $\mathcal{N} = 4$ (see also [44][18]). Making a satisfactory connection with the linear-dilaton
FIG. 7: The result of the Wilson-loop numerical analysis. The color coding in the figure is such that the same color always corresponds to quantities computed on the same background, and agrees with Fig. 1. The top-left panel shows $\hat{\rho}_o(L_{QQ})$. Notice that the resulting function is invertible only for the solution with smallest $\rho_*$ (black). The fact that backgrounds with large values of $\rho_*$ do not extend to $L_{QQ} \to +\infty$ is due only to the limited numerical precision. The next three plots show a detail of $E_{QQ}(L_{QQ})$ for the three backgrounds with smallest values of $\rho_*$. Notice that the result is multi-valued for two of them, while in the last case the function is invertible and the transition between Coulomb and confined phase is smooth. The fifth plot shows the discontinuity of $\frac{dE_{QQ}}{dL_{QQ}}$ at the transition, which is absent for small values of $\rho_*$, together with the fact that the string tension is identical in all the cases considered. The next two plots show the shape of the probe string for three choices of $\hat{\rho}_o$ for two of the backgrounds (notice the color coding), chosen so that $L_{QQ}$ is the same, and coincides with the critical value at which the phase transition is taking place. Hence, $E_{QQ}$ is the same for the upper and lower curves, while the intermediate have a higher energy $E_{QQ}$. The last panel shows one unstable solution, highlighting the cuspy shape at its middle already observed and discussed in detail in [18], and which is connected with the fact that this is an unstable classical solution of the equations derived from the Nambu-Goto action.
solution of wrapped-D5 system \(\mathbb{3}\), in which \(P = \hat{P} = 2N_c\rho\), requires an additional step, in which a larger class of solutions \(P\) of the master equation must be analyzed in detail. This is due to the fact that the rotation procedure cannot be applied to this specific class of backgrounds, in which the dilaton diverges in the UV. We suggested an interesting idea, on a geometric basis, which could help provide a field theory explanation for the presence of the dimension-6 VEV, particularly in relation to the phase transition we found by computing the Wilson loop expectation value. A more systematic analysis of the (weakly coupled) dual field theory would also be useful along these lines.

An immediate test of all that we said would be to compute the spectrum of scalar glueballs of the dual (confined) field theory, by studying the fluctuations of the five-dimensional sigma-model in full generality. The technology for doing so exists and is well understood \(\cite{14, 23}\), although some subtlety connected with holographic renormalization does require a careful analysis. This study would allow to answer unambiguously two open questions. One is whether the scale separation between the heaviest and lightest massive vector multiplets survives also at strong coupling, or if it is only a perturbative result. Another is to explicitly verify whether the spectrum is also at strong coupling deconstructing the (non-fuzzy, at large \(N\)) sphere, and whether this is affected by the presence and magnitude of the dimension-6 VEV.

The second open question relates to the IR singularity and its resolution. It is not known whether this singularity has to be taken as signaling the fact that the background is intrinsically pathological, whether it admits a resolution in terms of stringy physics (signaling that the singularity is actually due to a physical effect), or whether it is just an unfortunate result of the restrictions imposed in the derivation of the background equations, which admits a resolution within supergravity. On the basis of the arguments summarized earlier on, it seems plausible that the backgrounds we studied, while singular, provide a sensible quantitative description, in supergravity terms, of the physics of the dual field theory, but this needs to be tested. A good place to start is with the spectrum, in order to see if there are pathologies due to the singularity itself. Aside from this, it should also be interesting to reanalyze the results of the papers \(\cite{49}\), in the more complete setup presented here.

A final comment about applications. One phenomenological application of this research is related to the finding of an isolated, anomalously light scalar in the spectrum of glueballs of a background that shares many similarities with those analyzed here. It is tempting to identify this scalar as a light four-dimensional pseudo-dilaton, and if so, the conditions under which it appears are directly relevant to the strongly coupled physics of electro-weak symmetry breaking. However, the properties of this scalar are not fully understood and it would hence be useful to see if its existence is a generic feature, and to find a robust physical explanation for it.

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Appendix A: Relation to PT ansatz and details about the rotation

Here we would like to make clear the relation between the backgrounds described in Section II and the PT ansatz, as well as in some more detail the effect of the rotation. The PT ansatz is given by\(^\text{10}\)

\[
\begin{align*}
    ds^2 &= e^{2\Delta} ds^2_0 + (e^{r+\tilde{g}} + a^2e^{-\tilde{g}})(e_1^2 + e_2^2) + e^{r-\tilde{g}} \left( e_3^2 + e_4^2 - 2a(e_1e_3 + e_2e_4) \right) + e^{-6p-x} e_5^2, \\
    ds^2_0 &= dr^2 + e^{2A} dx^2_{1,3}, \\
    F_3 &= N \left[ e_5 \wedge (e_4 \wedge e_3 + e_2 \wedge e_1 - b(e_4 \wedge e_1 - e_3 \wedge e_2)) + dr \wedge (\partial_x b(e_4 \wedge e_2 + e_3 \wedge e_1)) \right], \\
    H_3 &= -h_2 e_5 \wedge (e_4 \wedge e_2 + e_3 \wedge e_1) + dr \wedge \left[ \partial_x h_1(e_4 \wedge e_3 + e_2 \wedge e_1) - \partial_r h_2(e_4 \wedge e_1 - e_3 \wedge e_2) + \partial_r \chi(-e_4 \wedge e_3 + e_2 \wedge e_1) \right], \\
    F_5 &= F_5 + \ast F_5, \quad \tilde{F}_5 = -Ke_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5.
\end{align*}
\]

Here, \(\{p, x, g, a, b, h_1, h_2, \chi, K\}\) are functions of the radial coordinate \(r\), and we have defined the one-forms

\[
\begin{align*}
    e_1 &= -\sin \theta d\phi, \\
    e_2 &= d\theta, \\
    e_3 &= \cos \psi \sin \tilde{d}\tilde{\phi} - \sin \psi d\tilde{\phi}, \\
    e_4 &= \sin \psi \sin \tilde{d}\tilde{\phi} + \cos \psi d\tilde{\phi}, \\
    e_5 &= d\psi + \cos \tilde{d}\tilde{\phi} + \cos \theta d\phi.
\end{align*}
\]

Now, consider the wrapped-D5 system described in Section II A. The ansatz for the metric there falls into the more general case

\[
    ds^2 = e^{2\Delta} \left( dx^2_{1,3} + ds_0^2 \right),
\]

where \(ds_0^2\) is of the same form as before and \(\Delta\) is a function of \(\rho\). Comparing with the PT ansatz, we obtain the following one-to-one map between variables (\(N = N_c/4\))

\[
\begin{align*}
    \Delta &= A + p - \frac{x}{2}, \\
    g &= -A - \frac{\tilde{g}}{2} - p + x + \log 2, \\
    h &= -A + \frac{\tilde{g}}{2} - p + x, \\
    k &= -A - 4p + \log 2, \\
    dp &= \frac{1}{2} e^{4p} dr.
\end{align*}
\]

The solution-generating technique outlined in Section II B starts with a solution of the kind described in Section II A, i.e., a background describing the wrapped-D5 system, and then, applying a set of transformations, generates a new (rotated) solution which has the following form (the superscript \((r)\) refers to the rotated solution):

\[
\begin{align*}
    ds_0^{(r)} &= e^{\Phi/2} \left[ (1 - k_2^2 e^{2\Phi})^{-1/2} ds^2_{1,3} + (1 - k_2^2 e^{2\Phi})^{1/2} ds_0^2 \right], \\
    \Phi^{(r)} &= \Phi, \\
    F_3^{(r)} &= F_3, \\
    H_3^{(r)} &= -k_2 e^{2\Phi} \ast_6 F_3, \\
    F_5^{(r)} &= -k_2 (1 + \ast_{10}) \text{vol}(4) \wedge d (e^{-2\Phi} - k_2^2)^{-1}.
\end{align*}
\]

\(^{10}\) We are following the notation of \cite{23}. However, in order to reproduce the conventions used in Eq. \cite{23}, we make use of the fact that the ten-dimensional equations are symmetric under a simultaneous change of sign of all the RR forms \(F_1, F_3\) and \(F_5\). Furthermore, the equations for the functions determining the background are also symmetric under the simultaneous change of sign of \(a, b\) and \(h_2\). With respect to \cite{23}, we apply both these changes of sign to all the functions appearing in the background.
More explicitly, the transformations are given by
\[
\begin{align*}
a^{(r)} &= a, \quad b^{(r)} = b, \quad \Phi^{(r)} = \Phi, \\
e^{2A^{(r)}} &= (1 - k_2^2 e^{2\Phi})^{-1/2} e^{\Phi/2}, \quad e^{2\rho^{(r)}} = (1 - k_2^2 e^{2\Phi}) e^{2\rho}, \\
e^{2k^{(r)}} &= (1 - k_2^2 e^{2\Phi}) e^{2k}, \\
\partial_\rho \chi^{(r)} &= \frac{k_2 N_c}{16} e^{-2(g + h - \Phi)} \left( 8ae^{2(g + h)} (a-b) + (a^2 - 1) (a^2 - 2eb+1) e^{4\rho} + 16 e^{4h} \right), \\
\partial_\rho h_1^{(r)} &= \frac{k_2 N_c}{16} e^{-2(g + h - \Phi)} \left( 8ae^{2(g + h)} (a-b) + (a^2 + 1) (a^2 - 2eb+1) e^{4\rho} + 16 e^{4h} \right), \\
h_2^{(r)} &= \frac{k_2 N_c}{8} e^{2\Phi - 2p} \partial_\rho b = \frac{k_2 e^{2(g + \Phi)} (e^{2\rho} (a^2 - 1) + 4e^{2h}) a}{4 e^{4\rho} (a^2 - 1)^2 + 16e^{4h} + 8 (a^2 + 1) e^{2(g + h)}}. \\
\mathcal{K}^{(r)} &= \frac{k_2}{4} e^{2(g + h + \Phi)} \partial_\rho \Phi = \frac{k_2 N_c e^{2\Phi} (e^{2\rho} (a^2 - 1) + 4e^{2h}) (e^{2\rho} (a^2 - 2eb+1) + 4e^{2h})}{\sqrt{e^{4\rho} (a^2 - 1)^2 + 16e^{4h} + 8 (a^2 + 1) e^{2(g + h)}}}.
\end{align*}
\] (A15)

Using the one-to-one map given in Eq. \((A13)\), one easily shows that in terms of the variables that appear in the PT ansatz, the rotation takes the form (note that for the unrotated solution \(A = \Phi - p + \tilde{\rho}\))
\[
\begin{align*}
a^{(r)} &= a, \quad b^{(r)} = b, \quad \Phi^{(r)} = \Phi, \quad e^{2\tilde{\rho}^{(r)}} = e^{2\tilde{\rho}}, \\
e^{2A^{(r)}} &= (1 - k_2^2 e^{2\Phi})^{1/3} e^{2A}, \quad e^{-6\rho^{(r)}} = (1 - k_2^2 e^{2\Phi}) e^{-6\rho}, \quad e^{2x^{(r)}} = (1 - k_2^2 e^{2\Phi}) e^{2x}, \\
\partial_\rho \chi^{(r)} &= \frac{k_2 N_c}{8} \left( 1 - k_2^2 e^{2\Phi} \right)^{-2/3} e^{4p + 2\Phi - 2\tilde{\rho}} (a^2 + e^{2\tilde{\rho}} - 1) (a^2 - 2eb + e^{2\tilde{\rho}} + 1), \\
\partial_\rho h_1^{(r)} &= \frac{k_2 N_c}{8} \left( 1 - k_2^2 e^{2\Phi} \right)^{-2/3} e^{4p + 2\Phi - 2\tilde{\rho}} (2ae^{2\tilde{\rho}} (a-b) + e^{2\tilde{\rho}} + (a^2 + 1) (a^2 - 2eb + 1)), \\
h_2^{(r)} &= \frac{k_2 N_c}{4} e^{2\Phi - 4p} \partial_\rho b = \frac{k_2 e^{2\Phi - 3\tilde{\rho}} (a^2 + e^{2\tilde{\rho}} - 1) a}{\sqrt{a^4 + 2 (-1 + e^{2\tilde{\rho}}) a^2 + (1 + e^{2\tilde{\rho}})^2}}, \\
\mathcal{K}^{(r)} &= 2k_2 e^{-4p + 2x + \Phi} \partial_\rho \Phi = \frac{k_2 N_c e^{x + 2\Phi - \tilde{\rho}} (a^2 + e^{2\tilde{\rho}} - 1) (a^2 - 2eb + e^{2\tilde{\rho}} + 1)}{4 \sqrt{a^4 + 2 (-1 + e^{2\tilde{\rho}}) a^2 + (1 + e^{2\tilde{\rho}})^2}}.
\end{align*}
\] (A16)

where
\[
\begin{align*}
dr^{(r)} &= 2e^{-4p^{(r)}} dp = e^{4(p - r^{(r)})} dr = (1 - k_2^2 e^{2\Phi})^{2/3} dr.
\end{align*}
\] (A17)

Appendix B: UV asymptotic expansions

In this appendix we expand by brute force the expressions for the eight background scalars for the various classes of solutions discussed in the paper, by defining \(\rho = -\frac{3}{2} \log z\), and by expanding for small \(z\).
For the KS solutions (having fixed $b_1 = 0 = b_2 = f_0 = \tilde{p}$):

\begin{align*}
a^{(KS)} & = 2z^3 + \mathcal{O}(z^9), \\
b^{(KS)} & = -6z^3 \log z + \mathcal{O}(z^9), \\
\Phi^{(KS)} & = \Phi_\infty, \\
x^{(KS)} & = \frac{1}{2} \log \left( \frac{3}{8} \left( -12 \log(z) N^2 e^{\Phi_\infty} - N^2 e^{\Phi_\infty} + 2\tilde{M} \right) \right) \\
& + \frac{2}{125} \left( -3 \left( 7N^2 e^{\Phi_\infty} - 10\tilde{M} \right) \right) + 150 \log(z) - 5 \right) z^6 + \mathcal{O}(z^9), \\
p^{(KS)} & = -\frac{1}{6} \log \left( -3 \log(z) N^2 e^{\Phi_\infty} - \frac{N^2 e^{\Phi_\infty}}{4} + \frac{\tilde{M}}{2} \right) + \frac{z^6 \left( -31N^2 e^{\Phi_\infty} + 70\tilde{M} - 30 \log(z) \left( 60 \log(z) N^2 e^{\Phi_\infty} + 23N^2 e^{\Phi_\infty} - 10\tilde{M} \right) \right)}{125 \left( -12 \log(z) N^2 e^{\Phi_\infty} - N^2 e^{\Phi_\infty} + 2\tilde{M} \right)} + \mathcal{O}(z^9), \\
\tilde{g}^{(KS)} & = -2z^6 + \mathcal{O}(z^9), \\
\partial_\rho h_1^{(KS)} & = 2e^{\Phi_\infty} N + 16e^{\Phi_\infty} N z^6 (1 + 3 \log z) + \mathcal{O}(z^9), \\
h_2^{(KS)} & = 2Ne^{\Phi_\infty} z^3 (1 + 3 \log z) + \mathcal{O}(z^9).
\end{align*}

In particular

\begin{equation}
x^{(KS)} + 3p^{(KS)} = \frac{1}{2} \log \frac{3}{2} + z^6 (1 + 6 \log z) + \mathcal{O}(z^9).
\end{equation}

Turning back to the wrapped-D5 system, and making use of the UV asymptotic expansion for $P$, one finds that

\begin{equation}
\lim_{\rho \to +\infty} e^{-4\Phi} = 18c_+^3 e^{-4\Phi_\infty},
\end{equation}

and hence we fine-tune $k_2 = e^{-\Phi_\infty} = \left( 18c_+^3 \right)^{1/4} e^{-4\Phi_\infty}$. The resulting rotated and fine-tuned solution, expanded in the
same way as the KS solution, yields
\[ a(r) = 2z^3 \left( 1 - \frac{3N_c(3 \log(z) + 1)z^2}{9c^+_e} + \frac{N^2_c(3 \log(z) + 1)^2z^4}{9c^+_e^2} \right) + O(z^9), \quad (B11) \]
\[ b(r) = -6z^3 \log(z) + O(z^9), \quad (B12) \]
\[ \phi(r) = \frac{1}{4} \log \left( \frac{e^{4\Phi_e}N^8_e}{18c^+_e} \right) + \frac{N^2_e(12 \log(z) + 1)z^4}{48c^+_e^2} + \frac{N^4_e(72 \log(z)(3 \log(z)(4 \log(z) + 7) + 13) + 257)z^8}{3456c^+_e^4} + O(z^9), \quad (B13) \]
\[ \rho(r) = \frac{1}{8} \left( \log \left( \frac{9e^{4\Phi_e}N^8_e}{36670912c^+_e} \right) + 4 \log(-12 \log(z) - 1) \right) \]
\[ + \left( \frac{(14400c^3_e - 37c_+ + 12 \log(z)(3 \log(z)(2304 \log(z)c^3_e + 7872c^3_e - c_+) - 8(c_+ - 1392c^3_e)))z^6}{1728c^+_e(12 \log(z) + 1)} \right) \]
\[ + \frac{z^8}{497664c^+_e(12 \log(z)N_e + N_e)^2} \left\{ -1218038N^6_e + 27(c_+ - 576c^3_e)(c_+ - 192c^3_e) \right. \]
\[ + 12 \log(z) \left[ -326360N^6_e + 27(258048c^6_e - 1152c^3_e + c^3_+ + 2c^6_+) + 144 \log(z) \left( 5737N^6_e - 864c^3_e (c_+ - 480c^3_e) \right) \right. \]
\[ + 8 \log(z) \left( 124416c^6_e + 5143N^6_e + 3N^6_e \log(z)(72 \log(z)(\log(z)(12 \log(z) + 37) + 58) + 3667) \right) \right\} + O(z^9), \quad (B14) \]
\[ \varrho(r) = \frac{1}{24} \left( -\log \left( \frac{9c^3_e e^{4\Phi_e}}{8192} \right) - 4 \log \left( \frac{N^2_e(12 \log(z) + 1)}{24c^+_e} \right) \right) \]
\[ - \frac{z^8}{108c^+_e(12 \log(z) + 1)} \left\{ -1271030N^6_e + 27(c_+ - 576c^3_e)(c_+ - 192c^3_e) \right. \]
\[ + 12 \log(z) \left[ -457112N^6_e + 27(258048c^6_e - 1152c^3_+ + c^3_+ + 2c^6_+) + 144 \log(z) \left( -3131N^6_e - 864c^3_e (c_+ - 480c^3_e) \right) \right. \]
\[ + 4 \log(z) \left( 248832c^6_e + 2078N^6_e + 3N^6_e \log(z)(72 \log(z)(8 \log(z)(3 \log(z) + 7) + 65) + 2861) \right) \right\} + O(z^9), \quad (B15) \]
\[ \gamma(r) = \frac{z^8}{3c^+_e} \left\{ 1 + 3 \log(z) \right\} \frac{z^6}{324c^3_e} \left( -648c^3_+ + 35N^3_e + 9N^3_e \log(z)(24 \log(z)(\log(z) + 1) + 17) \right) \]
\[ + \frac{N^3_e \left( 1152c^3_+ - c_+ \right) + z^6 \left( 2304 \log(z)c^3_+ + 786c^3_+ - c_+ \right)}{1728c^+_e^4} \right\} + O(z^9), \quad (B16) \]
\[ \partial_r h^r_1 \left( r^4 \right) = \sqrt{\frac{18c^+_e}{N_c}} \left\{ \frac{1}{2} + \frac{144N^2_e(4 \log^2(z^2) + 4(3 \log(z) + 1)z^4}{20736c^2_e} \right. \]
\[ - \frac{N^4_e(24 \log(z)(36 \log(z)(2 \log(z)(12 \log(z) + 7) + 7) + 101) - 239)z^8}{20736c^4_e} \right\} + O(z^9), \quad (B17) \]
\[ h^r_2 \left( r^4 \right) = \sqrt{\frac{18c^+_e}{N_c}} \left( 3 \log(z) + 1 \right) \left( 1 + \frac{N^2_e}{c^+_e} z^4 \left( \frac{1}{24} + \frac{1}{2} \log(z) \right) \right) + O(z^9). \quad (B18) \]
It is useful to look more in detail at the two specific quantities

\[
x^{(r)} + 3\rho^{(r)} = \frac{1}{2} \log \frac{3}{2} + \frac{N_c^2 z^4 (3 \log(z) + 4) + 2}{4 c_+^2} + \frac{z^6}{216 c_+^4} \frac{N_c^4 z^8 (3 \log(z) + 1)(3 \log(z)(3 \log(z) + 7) + 20) + 23}{216 c_+^4} + \mathcal{O}(z^9),
\]

\[
e^{2g^{(r)}} + \rho^{(r)2} - 1 = -\frac{2 N_c z^2 (3 \log(z) + 1)}{3 c_+} + \frac{2 z^4 N_c^2 (3 \log(z) + 1)^2}{9 c_+^2} + \frac{N_c^3 z^6 (3 \log(z) + 1)}{6 c_+^3}
- \frac{N_c z^8}{2592 c_+^7} \left[ - 3456 c_+^3 + 352 N_c^3 + 3 c_- + 3 \log(z) \left( - 2304 c_+^3 + 832 N_c^3 \right)
+ 3 c_- + 288 \log(z) \left( - 24 c_+^3 + 7 N_c^3 + 2 N_c^3 \log(z)(3 \log(z) + 4) \right) \right] + \mathcal{O}(z^9),
\]

both of which are unaffected by the rotation.

**Appendix C: Curvature invariants for KS deformations**

In Section IV A, a few curvature invariants for the rotated solutions were studied in order to understand the nature of the singularity in the IR. For comparison, we perform here the same analysis for the singular deformations of Klebanov-Strassler obtained by taking the integration constant \( f_0 \) of Eq. (35) to be non-zero.

All functions in the background are determined analytically except \( \rho \), which satisfies the equation of motion Eq. (36), i.e. the IR expansion of which determines \( x \) as

\[
x = x_0 + \frac{32 \rho^3}{9 f_0} - \frac{8}{9} \left( e^{\Phi_\infty - 2 x_0 N^2} f_0 \right) \rho^4 + \frac{128 \rho^5}{45 f_0} + \frac{64}{135} \left( e^{\Phi_\infty - 2 x_0 N^2} f_0 \right) \rho^6 + \mathcal{O} \left( \rho^7 \right),
\]

where \( x_0 \) is an integration constant. This, in turn, implies that \( R, R_{\mu\nu}, R_{\mu\nu\tau\sigma} R^{\mu\nu\tau\sigma} \) have IR expansions given by

\[
R = \frac{1}{2} e^{\Phi_\infty - 3 x_0 N^2} f_0 - \frac{4}{3} \left( e^{\Phi_\infty - 3 x_0 N^2} f_0 \right) \rho^2 + \mathcal{O} \left( \rho^4 \right),
\]

\[
R_{\mu\nu} R^{\mu\nu} = \frac{5}{8} \left( e^{2 \Phi_\infty - 6 x_0 N^2} f_0^2 \right) \rho^2 - \frac{10}{3} \left( e^{2 \Phi_\infty - 6 x_0 N^2} f_0^2 \right) \rho^6 + \mathcal{O} \left( \rho^8 \right),
\]

\[
R_{\mu\nu\tau\sigma} R^{\mu\nu\tau\sigma} = \frac{45 e^{2 \Phi_\infty} f_0^2 \rho^2}{256} - \frac{3 (e^{2 \Phi_\infty} f_0^2)}{8} \rho^6 + \frac{5 (e^{2 \Phi_\infty} f_0^2)}{8} \rho^8 + \mathcal{O} \left( \rho^{10} \right).
\]

As with the rotated solutions, \( R \) and \( R_{\mu\nu} R^{\mu\nu} \) stay finite in the IR, while \( R_{\mu\nu\tau\sigma} R^{\mu\nu\tau\sigma} \) diverges as \( \rho^{-8} \).

For completeness, let us also give the IR expansions for the non-singular solution obtained by putting \( f_0 = 0 \), i.e. the original solution of Klebanov-Strassler. Now, we have that

\[
e^z = \tilde{x} \rho + \left( \frac{4 \tilde{x}}{15} - \frac{16 e^{3 \Phi_\infty N^2}}{9 \tilde{x}} \right) \rho^3 + \left( \frac{128 e^{2 \Phi_\infty N^4}}{81 \tilde{x}^3} + \frac{32 e^{3 \Phi_\infty N^2}}{45 \tilde{x}} + \frac{16 \tilde{x}}{525} \right) \rho^5 + \mathcal{O} \left( \rho^7 \right),
\]

where \( \tilde{x} \) is an integration constant. (Note that this expansion is radically different from the one in the case of non-zero \( f_0 \).) This leads to

\[
R = \frac{16 e^{\Phi_\infty} N^2}{32 \tilde{x}^3} - \frac{128 \left( e^{\Phi_\infty} N^2 \left( \tilde{x}^2 - 2 e^{\Phi_\infty N^2} \right) \right)}{9 \tilde{x}^5} \rho^2 + \mathcal{O} \left( \rho^4 \right),
\]

\[
R_{\mu\nu} R^{\mu\nu} = \frac{640 e^{2 \Phi_\infty} N^4}{9 \tilde{x}^6} + \frac{2048 e^{2 \Phi_\infty} N^4 \left( 122 e^{\Phi_\infty} N^2 - 45 \tilde{x}^2 \right)}{243 \tilde{x}^8} \rho^2 + \mathcal{O} \left( \rho^4 \right),
\]

\[
R_{\mu\nu\tau\sigma} R^{\mu\nu\tau\sigma} = \frac{640 e^{2 \Phi_\infty} N^4}{9 \tilde{x}^6} + \frac{2048 e^{2 \Phi_\infty} N^4 \left( 122 e^{\Phi_\infty} N^2 - 45 \tilde{x}^2 \right)}{243 \tilde{x}^8} \rho^2 + \mathcal{O} \left( \rho^4 \right),
\]
\[ R_{\mu\nu\tau\sigma}R^{\mu\nu\tau\sigma} = \frac{32}{2025\pi^8} \left( 440e^{2\phi}N^4 + 812\phi^4 \right) \]
\[ + \frac{135\pi^6}{512 \left( 6000e^{2\phi}N^6 - 2600e^{2\phi} \phi^2 N^4 + 270e^{2\phi} \phi^4 N^2 - 243\phi^6 \right)\rho^2 + O(\rho^3) \],
\]
which, as expected, all stay finite in the IR.

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[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] arXiv:hep-th/9711200.
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) arXiv:hep-th/9802109.
[3] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) arXiv:hep-th/9802150.
[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183 (2000) arXiv:hep-th/9905111.
[5] I. R. Klebanov and E. Witten, Nucl. Phys. B 536, 199 (1998) arXiv:hep-th/9807080.
[6] I. R. Klebanov and M. J. Strassler, JHEP 0008, 052 (2000) arXiv:hep-th/0007191.
[7] J. M. Maldacena and C. Nunez, Phys. Rev. Lett. 86, 588 (2001) arXiv:hep-th/0008001.
[8] See also A. H. Chamseddine and M. S. Volkov, Phys. Rev. Lett. 79, 3343 (1997) arXiv:hep-th/9707176.
[9] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B 578, 123 (2000) arXiv:hep-th/0002159.
[10] A. Butti, M. Grana, R. Minasian, M. Petrini and A. Zaffaroni, JHEP 0503, 069 (2005) arXiv:hep-th/0412187.
[11] F. Bigazzi, L. Griguolo, A. Zaffaroni, Nucl. Phys. B 536, 161 (2000) arXiv:hep-th/0112236.
[12] M. Bertolini and P. Merlatti, Phys. Lett. B 556, 80 (2003) arXiv:hep-th/0208099.
[13] V. A. Bardeen et al., Phys. Rev. Lett. 56, 1230 (1986); M. Bando et al., Phys. Lett. B 178, 308 (1986); Phys. Rev. Lett. 56, 1335 (1986); B. Holdom and J. Terning, Phys. Lett. B 187, 357 (1987); Phys. Lett. B 200, 338 (1988); D. D. Dietrich, F. Sannino and K. Tuominen, Phys. Rev. D 72, 055001 (2005) arXiv:hep-ph/0505059.
[14] T. Appelquist and Y. Bai, arXiv:1006.3575 [hep-ph]; K. Haba, S. Matsuzaki, K. Yamawaki, Phys. Rev. D 82, 055007 (2010). arXiv:1006.2526 [hep-ph]; L. Vecchi, arXiv:1007.5735 [hep-ph]; M. Hashimoto, K. Yamawaki, Phys. Rev. D 83, 015008 (2011). arXiv:1009.5482 [hep-ph].
[15] See also [14].
[16] J. M. Maldacena, Adv. High Energy Phys. 2010, 104 [arXiv:0906.0591 [hep-th]].
[17] S. J. Rey and J. T. Yee, Eur. Phys. J. C 22, 379 (2001) arXiv:hep-th/9803002.
