Effective actions of local composite operators — case of $\varphi^4$ theory, itinerant electron model, and QED

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Abstract

The effective action $\Gamma[\varphi]$ is defined from the generating function(al) $W[J]$ through Legendre transformation and plays the role of action functional in the zero-temperature field theory and of generalized thermodynamical function(al) in equilibrium statistical physics. A compact graph rule for the effective action $\Gamma[\varphi]$ of a local composite operator is given in this paper. This long-standing problem of obtaining $\Gamma[\varphi]$ in this case is solved directly without using the auxiliary field. The rule is first deduced with help of the inversion method, which is a technique for making the Legendre transformation perturbatively. It is then proved by using a topological relation and also by the sum-up rule. The latter is a technique for making the Legendre transformation in a graphical language. In the course of proof a special role is played by $J^{(0)}[\phi]$, which is a function(al) of the variable $\phi$ and defined through the lowest inversion formula. Here $J^{(0)}[\phi]$ has the meaning of the source $J$ for the noninteracting system expressed by $\phi$. Explicitly derived are the rules for the effective action of $\langle \varphi(x)^2 \rangle$ in the $\varphi^4$ theory, of the number density $\langle n_{r\sigma} \rangle$ in the itinerant electron model, and of the gauge invariant operator $\langle \bar{\psi} \gamma^\mu \psi \rangle$ in QED.
1 Introduction

The effective action $\Gamma[\phi]$ or thermodynamical function introduced by Legendre transformation is a very convenient tool in various fields of physics. Actually this fact has long been realized in the condensed matter physics as well as in particle physics[1].

In spite of its wide-spread use, the precise rule of constructing the effective action for a local composite field has not been derived up to now although the graphical rules for an elementary field and for non-local composite fields up to four-body operators are already known[2, 3, 4, 5, 6]. The study of the effective action for local composite operator amounts to rewrite the theory in terms of physical variables such as the expectation values of the number density operator, spin density operator, local gauge invariant operator etc. Thus the importance of the investigation can not be overestimated. In the following we deal with three examples — the effective action of the $\varphi(x)^2$ operator in the $\varphi^4$ theory, a generalized free energy as a function(al) the spin and number density in the itinerant electron model, and the effective action of the $\bar{\psi}(x)\gamma^\mu\psi(x)$ operator in QED where $\psi$ is the electron field.

In some cases the hard problem to obtain the effective action of local composite operators has been avoided by Hubbard-Stratonovich transformation[7] or by introducing an auxiliary field [8]. In such a formulation, the auxiliary field is not equal to the local composite operator if one deals with the off-shell quantities and extra work is needed to extract the physical on-shell quantities, which are directly related to the original local composite operator. In the following we take the local composite operator itself without using auxiliary field and explicitly derive the graphical rule for the effective action. Difficulty is solved by use of the inversion method [9, 10, 11, 12, 13].

For later discussion let us define the effective action $\Gamma[\phi]$ explicitly. For the zero-temperature case it is introduced through a generating functional $W[J]$ with a source $J$ coupled to some operator $\hat{O}$: $e^{iW[J]} = \langle 0 \vert e^{i\hat{O}J} \vert 0 \rangle$. Here $\vert 0 \rangle$ represents the ground state. Then a dynamical variable $\phi$ is defined as $\phi = \frac{\delta W}{\delta J} \equiv \langle \hat{O} \rangle^J$ and the effective action, which is a functional of $\phi$, is give by $\Gamma[\phi] = W[J] - \int J \phi$ with $-J = \frac{\delta W}{\delta \phi}$. Here $J$ is given by a functional of $\phi$ by inverting $\phi = \frac{\delta W}{\delta J}$. For simplicity we have considered the $x$-independent variables $J$ and $\phi$ since it is straightforward to extend to the local variables $J(x)$ and $\phi(x)$. We have called a function of $J$
or \( \phi \) functional as we will do in what follows so that we can recover the \( x \)-dependence freely. In equilibrium statistical physics \( W \) corresponds to the thermodynamical potential \( \Omega \) and \( \Gamma \) to the Helmholtz free energy \( F \). For instance, \( \hat{O} \) is chosen to be the total number operator \( N \) then \( J \) is the chemical potential \( \mu(N) \).

The essential step of Legendre transformation is to invert the relation \( \phi = \frac{\delta W}{\delta J} \). The inversion method enables us to write down the explicit form of \( J \) in terms of \( \phi \) by perturbative calculation. The lowest relation of the method defines the functional \( J^{(0)}[\phi] \), which is the source as a functional of \( \phi \) in noninteracting system. As will become clear, it is \( J^{(0)}[\phi] \) that plays a fundamental role in deriving \( \Gamma[\phi] \). In fact it turns out that by use of the inversion method \( \Gamma[\phi] \) in the case of a local composite field is obtained as a class of irreducible graph in certain sense (plus simple terms) as a functional of \( J^{(0)}[\phi] \) rather than \( \phi \) (for the \( \varphi^4 \) theory, see (2.69) with (2.46) or (2.90) below). In other words, all the functional dependence on \( \phi \) is through \( J^{(0)}[\phi] \). This point is in remarkable contrast to the rules for the effective action of an elementary field and non-local composite operators where the rule is based on \( \phi \) itself. This is the reason why the problem of local composite operator is difficult and has been unresolved. The use of \( J^{(0)}[\phi] \) naturally comes out in the formulation through the inversion method.

In order to explain the inversion method\(^9\) again for the simple case of the \( x \)-independent variable \( J \) and \( \phi \), we assume that the theory has a coupling constant \( \lambda \). Then the expectation value \( \phi = \langle \hat{O} \rangle^J \) is calculated in the presence of \( J \) through the Feynman rule (like (2.9) below) to get a series expansion

\[
\phi = \sum_{n=0}^{\infty} \phi^{(n)}[J]
\]  

(1.1)

where \( \phi^{(n)}[J] \) is the \( n \)-th order of \( \lambda \) by regarding \( J \) as independent of \( \lambda \). This relation can be inverted to give

\[
J = \sum_{n=0}^{\infty} J^{(n)}[\phi]
\]  

(1.2)

where \( J^{(n)}[\phi] \) is the \( n \)-th order of \( \lambda \). To obtain the explicit form of \( J^{(i)}[\phi] \) as a functional of \( \phi \) we first assume (1.2) and get

\[
\phi = \sum_{n=0}^{\infty} \phi^{(n)} \left[ \sum_{n=0}^{\infty} J^{(n)}[\phi] \right] = \phi^{(0)}[J^{(0)}[\phi] + J^{(1)}[\phi] + \cdots] + \phi^{(1)}[J^{(0)}[\phi] + J^{(1)}[\phi] + \cdots] + \cdots
\]  

(1.3)
or

\[ \phi = \phi^{(0)}[J^{(0)}[\phi]] + \phi^{(0)'}[J^{(0)}[\phi]] J^{(1)}[\phi] + \cdots + \phi^{(1)}[J^{(0)}[\phi]] + \cdots \]  

(1.5)

where \( \phi^{(0)'}[J] = \frac{\delta \phi^{(0)}[J]}{\delta J} \). The inversion is made by regarding \( \phi \) as independent of \( \lambda \), namely, as of order \( \lambda^0 = 1 \). Then an explicit form for \( J^{(n)}[\phi] \) is known successively up to the desired \( n \) by writing down the \( n \)-th order of (1.5); \( \phi = \phi^{(0)}[J^{(0)}[\phi]] \), \( J^{(1)} = -\phi^{(1)}[J^{(0)}[\phi]]/\phi^{(0)'}[J^{(0)}[\phi]] \), \ldots.

Regarding \( \phi \) as independent of \( \lambda \) just corresponds to making the Legendre transformation from \( J \) to \( \phi \) (see Appendix A). The extension of the above formula to the case of local variables \( J(x) \) and \( \phi(x) \) can be done merely by recovering the \( x \)-dependence and appropriate space-time integrals. We will see that the series expansion (1.5) in the graphical form is directly given by (2.10) below. An explicit form of \( J^{(0)}[\phi] \) may not be obtainable in the cases studied in this paper because \( J^{(0)}[\phi] \) is defined by the inverse of a known functional \( \phi^{(0)}[J] \) or \( J^{(0)}[\phi] = \phi^{(0)}[\cdot]^{-1} \). However, examples in which \( J^{(0)}[\phi] \) is explicitly obtained are dealt with in Appendix C. But this is not necessarily an obstacle or rather may be a merit in actual calculation in some cases. An explicit instance in this respect has been provided for the case of the itinerant electron model (see [13]). In other cases it is more convenient to change the dynamical variable; \( \phi \rightarrow J^{(0)}[\phi] \) as in [11].

In Sec. 2 the case of the \( \varphi^4 \) theory is discussed in detail as the simplest example and also as a prototype for the subsequent two models. First we try to deduce the rule and arrive at the propositions to be proved later. Explicit rules are given in the form of Proposition A2) with A1') or Proposition A3') below. In the second subsection we rigorously prove these propositions in two ways; by use of a topological relation and by the sum-up rule[14]. In Sec. 3 the case of the itinerant model is studied as an example of the free energy of the condensed matter physics. More model specific study of the case has been carried out [13] to give a systematic improvement of the Stoner theory and to obtain the results similar to the SCR theory by Moriya and Kawabata [15]. Last example of QED is given in Sec. 4 which may be a one-step toward a gauge-invariant study of the gauge field theory. Appendix A explains the reason why \( \phi \) is to be considered as independent of \( \lambda \) in the process of inversion in a way different from the one given in the literature. In Appendix B the Feynman rules which are necessary for our discussion are given in detail because the symmetry factors play an important...
role in the deduction of the rule (although they are less important in the proof of the rule).

Appendix C reproduces the known rules of the effective actions for an elementary field and non-local 2-body composite operators by the inversion method. In these cases $J^{(0)}[\phi]$ can be explicitly given as stated before. In Appendix D we review the path-integral technique for the fermion coherent state used in Sec. 3.

2 Case of $\varphi^4$ theory

As the simplest example we consider the effective action for the expectation value of a local composite operator $\varphi(x)^2$ in the $\varphi^4$ theory — we take $\Gamma[\phi]$ with the local variable $\phi(x) \propto \langle \varphi(x)^2 \rangle$.

Let us introduce the generating functional $W[J]$ in the path-integral representation as follows;

$$e^{iW[J]} = \int \mathcal{D}\varphi e^{iS[\varphi,J]},$$

$$S[\varphi,J] = -\frac{1}{2} \int d^4xd^4y \varphi(x)G^{-1}(x,y)\varphi(y) - \frac{\lambda}{4!} \int d^4x \varphi(x)^4 + \frac{1}{2} \int d^4xJ(x)\varphi(x)^2,$$

$$G^{-1}(x,y) = (\Box + m^2)\delta^4(x - y),$$

where $\int \mathcal{D}\varphi$ denotes the functional path-integration by the field $\varphi$. Note here that an $x$-dependent local external source $J(x)$ is coupled to the local composite field operator $\varphi(x)^2$. Hereafter we frequently use the notation in which the space-time indices and their integrations are omitted if it causes no ambiguity. For example $S[\varphi,J]$ in (2.2) is denoted as

$$-\frac{1}{2}\varphi G^{-1}\varphi - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}J\varphi$$

in this symbolic notation.

It is straightforward to get the graphical rule for $W[J]$. We note here that different rules are obtained depending on how much part of $J$ is absorbed in the propagator. In this paper both the following two diagrammatic rules (2.4) and (2.5) are used;

$$W[J] - W_0 = -\frac{1}{2i} \text{Tr} \ln G^{-1} + \frac{1}{i} \langle e^{-\frac{1}{i}J\varphi^4} \rangle_{G,J},$$

(2.4)
that is, the sum of all the connected vacuum graphs built with the 4-point vertex $-\lambda$ and the propagator $G_J$, and

$$W[J] - W_0 = -\frac{1}{2}\text{Tr} \ln[G^{(0)}]^{-1} + \frac{1}{i} \left< e^{-\frac{\lambda}{4!}\varphi^4 + \frac{1}{2}(J^{(1)} + J^{(2)} + \ldots)\varphi^2} \right>_{G^{(0)}},$$

(2.5)

that is, the sum of all the connected vacuum graphs constructed out of the 4-point vertex $-\lambda$, the 2-point vertex $J^{(i)}$ with $i \geq 1$, and the propagator $G^{(0)}$. Here the propagators are defined as (with obvious symbolic notation)

$$G_J^{-1} = \Box + m^2 - J \quad \text{and} \quad [G^{(0)}]^{-1} = \Box + m^2 - J^{(0)}.$$  

(2.6)

$W_0$ is the trivial $J$-independent part of $W$ and $\text{Tr}$ represents the functional trace. The first term on the right-hand side of (2.4) or (2.5) (Tr ln term), is usually denoted by a circle in graphical representation and, in this paper, is called a trivial skeleton (the definition of the skeleton itself is given below). Furthermore the notation of the form $\langle O[\varphi] \rangle_A$ means the summation of all the possible connected Wick contraction of the operators contained in $O[\varphi]$ by using $A$ as propagators, that is,

$$\langle O[\varphi] \rangle_A = \int \mathcal{D}\varphi e^{iS_0} O[\varphi] \bigg|_{\text{conn.}} \quad \text{with} \quad S_0 = -\frac{1}{2}\varphi A^{-1}\varphi.$$  

(2.7)

Throughout this paper we employ this notation frequently from which the weights of graphs are explicitly obtained. Remember that the original notation $\langle \varphi(x)^2 \rangle$ and $\langle \varphi(x)^2 \rangle^J$ implies, however, the full order expectation value of course. The rule (2.3) contains 2-point vertices of $J^{(i)}\varphi^2$ ($i \geq 1$) because the absorption of $J$ into the propagator is not complete.

Now the expectation value of the local composite field will be called $\phi(x)$, specifically:

$$\phi(x) = \frac{\delta W}{\delta J(x)} \equiv \frac{1}{2}\langle \varphi(x)^2 \rangle^J.$$  

(2.8)

With the notation (2.7) the graphical rules corresponding to (2.4) and (2.3) are summarized as

$$\phi = \left< \frac{1}{2}\varphi^2 e^{-\frac{\lambda}{4!}\varphi^4} \right>_{G_J},$$

(2.9)

that is, the sum of all the connected graphs with one external point (where two propagators meet) built with the 4-point vertex $-\lambda$ and the propagator $G_J$, and

$$\phi = \left< \frac{1}{2}\varphi^2 e^{-\frac{\lambda}{4!}\varphi^4 + \frac{1}{2}(J^{(1)} + J^{(2)} + \ldots)\varphi^2} \right>_{G^{(0)}},$$

(2.10)
that is, the sum of all the connected graphs with one external point (where two propagators meet) built with the 4-point vertex $-\lambda$, the 2-point vertex $J^{(i)}$ ($i \geq 1$), and the propagator $G^{(0)}$.

To rewrite the theory in terms of this dynamical variable $\phi$ instead of $J$ we introduce as usual the effective action of $\phi$ through Legendre transformation:

$$\Gamma[\phi] = W[J] - \int d^4x J(x) \phi(x) \equiv W[J] - J\phi$$

(2.11)

with an identity

$$-J(x) = \frac{\delta \Gamma[\phi]}{\delta \phi(x)}.$$  

(2.12)

It is convenient to introduce $\Gamma^{(n)}$, which is the $n$-th order in $\lambda$, or

$$\Gamma = \sum_{n=0}^{\infty} \Gamma^{(n)}.$$  

(2.13)

Then we see in Sec. 2.1.2 that $\Gamma^{(0)}$ and $\Gamma^{(1)}$ are explicitly given by , suppressing the space-time integration;

$$\Gamma^{(0)} = -J^{(0)}[\phi]\phi - \frac{1}{2i} \text{Tr} \ln[G^{(0)}]^{-1},$$

(2.14)

$$\Gamma^{(1)} = -\frac{1}{2} \lambda \phi^2.$$  

(2.15)

In this case of the $\varphi^4$ theory, $J^{(0)}[\phi]$ is defined through

$$\phi(x) = \frac{1}{2i} G^{(0)}(x,x) = \frac{1}{2i} \left( \frac{1}{\Box + m^2 - J^{(0)}[\phi]} \right)_{xx},$$

(2.16)

which is to be proved in Sec. 2.1.1. We emphasize here that although the right-hand side is denoted by a single graph of (2.31) below, $\phi$ on the left-hand side is a full-order quantity, suggesting that $J^{(0)}[\phi]$ has a full-order information. The central part of our study is that for the remaining part of $\Gamma$, which is called $\Delta\Gamma$,

$$\Delta\Gamma = \sum_{i=2}^{\infty} \Gamma^{(i)}[\phi].$$

(2.17)
2.1 Perturbative derivation of the graphical rule for $\Gamma[\phi]$ through inversion method

An explicit calculation up to the fourth order of $\lambda$ is sketched and based on the result the general rule for full order is deduced. Full justification is given in Sec. 2.2. In Sec. 2.1.1 the rule for $J^{(n)}$ is inferred by use of the inversion method. We see that $J^{(n)}$ is successively given as a functional of $J^{(0)}[\phi]$. Then in Sec. 2.1.2 we obtain $\Gamma^{(n)}$ based on $J^{(n)}$ vertex in two ways; by integrating the diagrams of $J^{(n)}$ or by starting from a closed formula for $\Delta \Gamma$. Since $J^{(n)}$ has already been given as a functional of $J^{(0)}[\phi]$ in Sec. 2.1.1, the effective action $\Delta \Gamma$ is obtained as a functional of $J^{(0)}[\phi]$. Explicit rules for $\Delta J$ and $\Delta \Gamma$ are given in Sec. 2.1.3 in which their dependence on $J^{(0)}[\phi]$ is transparent. For this purpose an artificial bosonic field $\sigma$ whose propagator is a functional of $J^{(0)}[\phi]$ is introduced.

2.1.1 Rule for $J^{(n)}$

The original series of $\phi$ is first calculated as

$$\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \phi^{(3)} + \phi^{(4)} + \cdots$$  \hspace{1cm} (2.18)

by (2.9) regarding $J$ as order unity with graphical representation as follows;

$$\phi^{(0)}[J] = \quad \text{a diagram} \quad ,$$  \hspace{1cm} (2.19)

$$\phi^{(1)}[J] = \quad \text{a diagram} \quad ,$$  \hspace{1cm} (2.20)

$$\phi^{(2)}[J] = \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad ,$$  \hspace{1cm} (2.21)

$$\phi^{(3)}[J] = \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad$$  \hspace{1cm} \hspace{1cm} (2.22)

$$\phi^{(4)}[J] = \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad + \quad \text{a diagram} \quad$$  \hspace{1cm} \hspace{1cm} (2.23)
Here the black dot \( \bullet \) corresponds to an external point where two propagators meet and to the insertion of the operator \( \varphi(x)^2 \) which is effected by the derivative with respect to \( J(x) \).

Note here the relation \( \frac{\partial G(y,z)}{\partial J(x)} = G(y,x)G(x,z) \). The propagator \( G_J(x,y) \) and the factor \(-\lambda\) are associated with a line and a 4-point vertex respectively. (No factor is associated with a black dot. For detailed rule including the symmetry factor, see Appendix B.)

We mention here that the diagrams of \( \phi^{(n)} \) is obtained by attaching a black dot, in all possible ways, to one of the lines in the graphs of the \( n \)-th order of \( W \). For example, the 31 diagrams of \( \phi^{(4)} \) is obtained through the fourth order of \( W \);

\[
W^{(4)}[J] = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram1} \\
\includegraphics[width=0.5\textwidth]{diagram2} \\
\includegraphics[width=0.5\textwidth]{diagram3} \\
\includegraphics[width=0.5\textwidth]{diagram4} \\
\includegraphics[width=0.5\textwidth]{diagram5} \\
\includegraphics[width=0.5\textwidth]{diagram6} \\
\includegraphics[width=0.5\textwidth]{diagram7} \\
\includegraphics[width=0.5\textwidth]{diagram8}
\end{array}.
\] (2.24)

Since the above diagrams of \( \phi^{(i)} \) are all functional of \( J(x) \), which is contained in the propagator \( G_J(x,y) \), we get \( \phi(x) \) as a functional of \( J \); \( \phi = \phi[J] \). Assume that the relation \( \phi = \phi[J] \) is inverted to give the relation \( J = J[\phi] \) and this inversion is done perturbatively as in \((2.2)\) regarding \( \phi \) as an quantity independent of \( \lambda \) or the order \( \lambda^0 = 1 \). Then as in Introduction we get the following formulae of the inversion method;

\[
\phi = \phi^{(0)}[J^{(0)}],
\] (2.25)

\[
\phi^{(0)}J^{(1)} + \phi^{(1)} = 0,
\] (2.26)

\[
\phi^{(0)}J^{(2)} + \frac{1}{2}\phi^{(0)}J^{(1)} + \phi^{(2)} = 0,
\] (2.27)

\[
\phi^{(0)}J^{(3)} + \phi^{(0)}J^{(1)}J^{(1)} + \frac{1}{3!}\phi^{(0)}J^{(1)}J^{(1)} + \phi^{(3)} = 0,
\] (2.28)

\[
\phi^{(0)}J^{(4)} + \frac{1}{2}\phi^{(0)}J^{(2)} + \frac{1}{2}\phi^{(0)}J^{(2)} + \frac{1}{4}\phi^{(0)}J^{(1)}J^{(1)} + \phi^{(4)} = 0.
\] (2.29)
Here we have employed a concise notation. If we explicitly write (2.27), for example, it has the form;

\[
\int d^4x \frac{\delta \phi^{(0)}[J^{(0)}]}{\delta J^{(0)}(x)} J^{(2)}(x) + \frac{1}{2} \int d^4xd^4y \frac{\delta \phi^{(0)}[J^{(0)}]}{\delta J^{(0)}(x)\delta J^{(0)}(y)} J^{(1)}(x)J^{(1)}(y) \\
+ \int d^4x \frac{\delta \phi^{(1)}[J^{(0)}]}{\delta J^{(0)}(x)} J^{(1)}(x) + \phi^{(2)}[J^{(0)}] = 0.
\]

We emphasize here that all \( \phi^{(i)} (i = 0, 1, 2, \ldots) \) and their derivatives in (2.25) to (2.29) are evaluated at \( J = J^{(0)}[\phi] \) defined implicitly by (2.25). So equations (2.26) to (2.29) successively give the functional dependence of \( J^{(1)} \) to \( J^{(4)} \) on \( \phi \) through \( J^{(0)}[\phi] \).

Let us discuss the graphical expressions of (2.25) to (2.29). Note here that the propagator in the following graphs is \( G^{(0)} = \frac{1}{\Box + m^2 - J^{(0)}[\phi]} \) instead of \( G_J \). Then, from (2.19), eq. (2.23) is expressed as

\[
\phi = \frac{1}{\Box + m^2 - J^{(0)}[\phi]}.
\]

Here and hereafter the black dot represents derivative not by \( J \) but by \( J^{(0)} \). Notice also that the meanings of the graphs on the right-hand side of (2.19) and (2.31) are different because the line or the propagator in them is not the same: \( G_J \) for (2.19) and \( G^{(0)} \) for (2.31). Thus (2.31) reduces to (2.16). It is stressed here that \( J^{(0)}[\phi] \) is defined through (2.16) or (2.31) although its dependence on \( \phi \) is only implicit. By using (2.19) and (2.20), eq. (2.26) is also expressed as follows.

\[
\triangle J^{(1)} + \bigtriangleup J^{(1)} = 0.
\]

Here we have used the relation

\[
\phi^{(0)\prime} = \frac{1}{\Box + m^2 - J^{(0)}[\phi]}.
\]

Noting that a 4-point vertex makes a contribution \( -\lambda \) so that

\[
\triangle = \triangle (\lambda) \bigtriangleup,
\]

we get from (2.32)

\[
J^{(1)} = \lambda \bigtriangleup.
\]
\[ J^{(1)} = \lambda \phi = \frac{1}{2i} \text{Tr} \left( -\lambda \phi + m^2 - J^{(0)}[\phi] \right). \] 

(2.36)

Thus \( J^{(1)} \) is given by \( J^{(0)}[\phi] \). Consider next the graphical expression of \( (2.27) \) obtained through \( (2.19) \) to \( (2.21) \).

![Graphical expression](image)

(2.37)

We see that the second, fourth, and sixth graphs on the left-hand side are summed up to zero after replacing \( J^{(1)} \) by the right-hand side of \( (2.35) \) by explicitly taking symmetry factors into account of course – see Appendix B. A similar cancellation of the third and fifth graphs on the left-hand side of \( (2.37) \) occurs ending up with

\[ -\text{graph} = \text{zero}. \]

(2.38)

The graphs of \( (2.28) \) and \( (2.29) \) are also obtained through \( (2.19) \) to \( (2.24) \). These expressions originally consist of many terms but due to similar cancellation mechanism, they reduce to

\[ -\text{graph} = \text{zero}. \]

(2.39)

\[ -\text{graph} = \text{zero}. \]

(2.40)

These simple results lead us to the following proposition to be justified later. Before presenting the proposition it is convenient to introduce the terms 1VI and 1VR. The 1VI (1-vertex-irreducible) graph is a connected graph in which removal of any one of the 4-point vertices does not lead to two separate graphs. The 1VR (1-vertex-reducible) vertex is defined as a 4-point vertex in a connected diagram deletion of which results in a separation of the graph.
The 1VI graph can also be defined as the connected graph without any 1VR vertex while the 1VR graph is a graph in which at least one 1VR vertex is present. By definition a graph which does not have any 4-point vertex is not 1VR but 1VI although the trivial skeleton (Tr ln term) is not 1VR nor 1VI. Namely all the graphs are classified into three categories; 1VI, 1VR, and the trivial skeleton. For later convenience we introduce the skeleton. Both the 1VI graph and the trivial skeleton is called the skeleton. In other words the whole class of the skeleton is all the 1VI graphs plus the trivial skeleton. With this terminology we see that all the 1VR graph in (2.37) disappear to result in (2.38) after all the $J^{(1)}$’s are replaced by the right-hand side of (2.35). Thus we can deduce the following proposition.

**Proposition A1**) After replacing $J^{(1)}$ by its graphical expression of the right-hand side of (2.33), all the 1VR graphs originally appearing in the inversion formula of the $n$-th order with $n \geq 2$ (2.27) to (2.29) and higher relations) cancel out. In other words, only the 1VI graphs with correct weight remain in the inversion formulae.

Note here that 1VI can not be replaced by 2PI (2-particle-irreducible) as in the case of the effective action for the *non-local operator* $\varphi(x)\varphi(y)$\[2, 4, 6\]. This is clear from the second and third (1VI) graphs from the last on the left-hand side of (2.40), which are 2PR (2-particle-reducible).

We also note a very convenient way to express the *original* graphs of the inversion formulae (2.25) to (2.29) and higher order relations such as (2.31), (2.32), (2.37) in which graphs $J^{(1)}$’s still remain (without replacing them by the right-hand side of (2.33)). Let us turn our attention to (2.10) where graphs are built with propagators $G^{(0)}$ and 4-point vertices $-\lambda$ and *pseudo-vertices* of order $\lambda^i$ with $i \geq 1$, which is denoted as $J^{(i)}$. We have called the 2-point vertex originating from $J^{(n)}\varphi^2$ a *pseudo-vertex* since it has nothing to do with the definition of 1VI. The term 1VI is defined as 1-vertex-irreducible with respect to 4-point vertex. Then the graphs of the inversion formula are obtained as follows. *If one writes down the n-th order of (2.10) considering $\phi$ and $G^{(0)}$ (namely, $J^{(0)}$) as of order $\lambda^0 = 1$ one obtains the inversion formula of order $n$ in the graphical form.* For example, the 0-th order of (2.10) is

$$\phi = \left\langle \frac{1}{2} \varphi^2 \right\rangle_{G^{(0)}},$$

(2.41)
which is equivalent to (2.31) and the first order is

\[ 0 = \left\langle \frac{1}{2} \varphi^2 \left( -\frac{i}{4!} \varphi^4 + \frac{i}{2} J^{(1)} \varphi^2 \right) \right\rangle_{G^{(0)}}, \tag{2.42} \]

which is (2.32). Furthermore the second order of (2.10) reduces to (2.37). Here it is convenient to introduce the self-contraction of the pseudo-vertex (Fig. 1) and the 4-point vertex (Fig. 2). Since the relation \( \frac{\delta G^{(0)}}{\delta J^{(0)}} = G^{(0)} G^{(0)} \) holds the quantity

\[ - \left\langle \frac{1}{2} \varphi^2 e^{-\frac{i}{4!} \varphi^4 + \frac{i}{2} (J^{(1)} + J^{(2)} + \ldots) \varphi^2} \right\rangle_{G^{(0)}} \]

for \( n \geq 1 \) we get the following formula.

\[ - \quad \left\langle \frac{1}{2} \varphi^2 e^{-\frac{i}{4!} \varphi^4 + \frac{i}{2} (J^{(1)} + J^{(2)} + \ldots) \varphi^2} \right\rangle_{G^{(0)}} ^{\text{ndself.}} \tag{2.44} \]

where ndself. (no derivative of the self-contraction) implies that the derivative of the self-contraction is moved on the left-hand side. Since (2.45) is the original inversion formulae of the n-th order Proposition A1) implies that in (2.43) all the contribution from the \( J^{(1)} \) vertices should be eliminated if only 1VI graphs are kept. Thus Proposition A1') follows:

**Proposition A1’** \( J^{(n)}[\phi] \ (n \geq 2) \) is successively obtained as a functional of \( J^{(0)}[\phi] \) through

\[ - \quad \left\langle \frac{1}{2} \varphi^2 e^{-\frac{i}{4!} \varphi^4 + \frac{i}{2} (J^{(1)} + J^{(2)} + \ldots) \varphi^2} \right\rangle_{G^{(0)}} ^{\text{1VI/ndself.}} \tag{2.46} \]

that is, the sum of all the connected 1VI/ndself. diagrams with one external point built with the 4-point vertex of \(-\lambda\), the 2-point pseudo-vertex \( J^{(i)} \ (i \geq 2) \) and the propagator \( G^{(0)} \).

The restriction 1VI/ndself. implies that the derivative by \( J^{(0)} \) of the self-contracted diagram are excluded and, at the same time, only the 1VI graphs should be kept. This proposition is directly proved in the next subsection by using the sum-up rule[14].
We can directly get (2.38) to (2.40) from Proposition A1′ due to the 1VI/ndself. restriction (through the procedure similar to the one given in (2.41) or (2.42) etc.). We notice that the right-hand side of (2.46) contains $J^{(i)}'$s with $i < n$ (strictly speaking $i \leq n - 2$). Hence we successively obtain $J^{(i)}$ ($i \geq 2$) as functional of $J^{(0)}$. For example, $J^{(4)}$ is expressed by $J^{(0)}$ if one insert $J^{(2)}$ vertex given in (2.38) into (2.40). In this way $J^{(i)}$ can be successively given as a functional of $J^{(0)}[\phi]$ (without using $J^{(j)}$ ($j < i, i \neq 0$)).

2.1.2 Rule for $\Delta \Gamma$ by use of the pseudo-vertex $J^{(n)}$ with $n \geq 2$

Now we turn our attention to the effective action $\Gamma[\phi]$ itself. We notice that $\Gamma^{(n)}[\phi]$ satisfies

$$\frac{\delta \Gamma^{(n)}[\phi]}{\delta \phi(x)} = -J^{(n)}(x).$$

(2.47)

Then for $n = 0$ we get (2.14) because by differentiating the right-hand side of (2.14) with respect to $\phi$ one obtains $-J^{(0)}$ through (2.16). $\Gamma^{(1)}$ is also easily obtained by integration of (2.36) so that we have (2.13). To derive $\Gamma^{(n)}$ for higher $n$, it is convenient to note the fact that

$$-J^{(n)} = \frac{\delta \Gamma^{(n)}[\phi]}{\delta \phi} = \frac{\delta J^{(0)}[\phi]}{\delta \phi} \frac{\delta \Gamma^{(n)}[\phi]}{\delta J^{(0)}}$$

(2.48)

and that

$$\frac{\delta \phi}{\delta J^{(0)}} = -D^{-1} = \begin{array}{c} \circ \end{array}.$$ 

(2.49)

The quantity of the last equation is a kind of propagator for the composite operator $\varphi(x)^2$ and is called composite propagator ($\langle \varphi(x)^2 \varphi(y)^2 \rangle$). $D$ is called the inverse composite propagator in what follows. Thus we get from (2.48)

$$-\begin{array}{c} \circ \end{array} J^{(n)} = \frac{\delta \Gamma^{(n)}[\phi]}{\delta J^{(0)}}.$$ 

(2.50)

Therefore the right-hand side of (2.46) is just $\frac{\delta \Gamma^{(n)}[\phi]}{\delta J^{(0)}}$. Keeping (2.50) in mind and by integrating (2.38) to (2.40) we arrive at

$$\Gamma^{(2)} = \begin{array}{c} \circ \end{array},$$ 

(2.51)

$$\Gamma^{(3)} = \begin{array}{c} \circ \circ \end{array},$$ 

(2.52)

$$\Gamma^{(4)} = J^{(2)} \begin{array}{c} \circ \circ \end{array} J^{(2)} + \begin{array}{c} \circ \circ \end{array} J^{(2)} + \begin{array}{c} \circ \circ \circ \end{array} + \begin{array}{c} \circ \circ \circ \circ \end{array} + \begin{array}{c} \circ \circ \circ \circ \circ \circ \circ \end{array}.$$ 

(2.53)
with

\[ J^{(2)} = - \left( \begin{array}{c} \circ \hline \end{array} \right)^{-1} \left( \begin{array}{c} \circ \hline \end{array} \right). \] (2.54)

Notice that the symmetry factors play an important role in the integration (see Appendix B). Note also that the first factor on the right-hand side of (2.54) corresponds to the amputation of the composite propagator (2.49).

In fact we can derive (2.54) to (2.53) and higher order relations more easily. To this end we introduce a closed form of functional representation of \( \Delta \Gamma[\phi] \). We first write down the following equation, which is clear from (2.41), (2.42), and (2.41):

\[ e^{i\Gamma[\phi]} = \int \mathcal{D}\varphi e^{i(-\frac{1}{2}\varphi(\square+m^2)\varphi-\frac{1}{4}\lambda \varphi^4 + \frac{i}{2}J \varphi^2 - J \varphi)} \] (2.55)

where \( J \) is expressed by \( \phi \). Noting that

\[ J = J^{(0)}[\phi] + \lambda \phi + \Delta J[\phi] \] (2.56)

with

\[ \Delta J = J^{(2)} + J^{(3)} + \cdots = -\frac{\delta \Delta \Gamma}{\delta \phi} \] (2.57)

and that, apart from irrelevant constant factor,

\[ \int \mathcal{D}\varphi e^{-i\frac{1}{2}\varphi(\square+m^2-J^{(0)})\varphi} = e^{-\frac{1}{2}\text{Tr} \ln(\square+m^2-J^{(0)})}, \] (2.58)

we get

\[ e^{i\Gamma[\phi]} = e^{i[-J^{(0)}\phi-\frac{1}{4}\text{Tr} \ln(\square+m^2-J^{(0)})-\frac{i}{2}\varphi^2] \int \mathcal{D}\varphi e^{i(-\frac{1}{2}\varphi(\square+m^2-J^{(0)})\varphi-\frac{1}{4}\lambda \varphi^4 + \frac{i}{2}J(J^{(0)}) \varphi^2-(J^{(0)}-\frac{1}{2}J) \varphi)} \int \mathcal{D}\varphi e^{-i\frac{1}{2}\varphi(\square+m^2-J^{(0)})\varphi}.} \] (2.59)

In this way a closed formula for \( \Delta \Gamma \) is obtained;

\[ e^{i\Delta \Gamma[\phi]} = \int \mathcal{D}\varphi e^{i\left(-\frac{1}{2}\varphi(\square+m^2-J^{(0)})\varphi + \left(-\frac{1}{4}\lambda \varphi^4 + \frac{i}{2}J(J^{(0)}) \varphi^2-(J^{(0)}-\frac{1}{2}J) \varphi \right) - \frac{\delta \Delta \Gamma}{\delta \phi}(\frac{\varphi^2}{2} - \phi) \right) \int \mathcal{D}\varphi e^{-i\frac{1}{2}\varphi(\square+m^2-J^{(0)})\varphi}.} \] (2.60)

This equation indicates that \( \Delta \Gamma \) can be calculated perturbatively by using \( G^{(0)} = (\square + m^2 - J^{(0)})^{-1} \) as propagators. The role of the additional vertices \( \frac{\lambda}{2} \varphi^2 - \frac{\lambda}{2} \varphi^2 \) and \( \frac{\delta \Delta \Gamma}{\delta \phi} \phi \) are merely to suppress the self-contractions. In other words the graphs having the structure of Fig. 1 and Fig. 2 disappear. To see this let us take one specific \(-\frac{\lambda}{4!} \varphi^4\) vertex. Each one of 4 \( \varphi \)'s of the vertex is to be contracted with the other \( \varphi \). There are three possible ways of such contractions.

\[ -\frac{\lambda}{4!} \varphi^4 \Rightarrow -\frac{\lambda}{4!} \varphi^4; \quad -\frac{\lambda}{4!} \left( \frac{4}{2!} \varphi \varphi \varphi \right) \varphi^2; \quad -\frac{\lambda}{4!} \left( \frac{4}{2!} \varphi \varphi \varphi \right) \varphi \] (2.61)
where the normal ordering \( :\varphi^n:\) means that each one of the \( n \) \( \varphi \)'s is to be contracted with \( \varphi \) contained in a vertex different from the one we are taking. Note here the contraction within a single vertex (self-contraction) is given by

\[
\hat{\varphi} = \frac{1}{i} \left( \frac{1}{\Box + m^2 - J^{(0)}} \right)_{xx} = 2\phi. \tag{2.62}
\]

In a similar manner we can write

\[
\frac{\lambda}{2} \phi \varphi^2 \Rightarrow \frac{\lambda}{2} \phi :\varphi^2: + \frac{\lambda}{2} \phi \hat{\varphi}. \tag{2.63}
\]

Then the contractions of the set appearing in (2.60) becomes

\[
-\frac{\lambda}{4!} \varphi^4 + \frac{\lambda}{2} \phi \varphi^2 - \frac{\lambda}{2} \phi^2 \Rightarrow -\frac{\lambda}{4!} :\varphi^4:, \tag{2.64}
\]

which is clear from (2.61) to (2.63). In the same way \( \frac{\delta \Delta \Gamma}{\delta \phi} \left( \frac{\varphi^2}{2} \right) \) reduces to

\[
\frac{\delta \Delta \Gamma}{\delta \phi} \left( \frac{1}{2} \left( :\varphi^2: + \hat{\varphi} \varphi \right) - \phi \right) = \frac{\delta \Delta \Gamma}{\delta \phi} \frac{1}{2} :\varphi^2:. \tag{2.65}
\]

In this way we get a simple formula for \( \Delta \Gamma[\phi] \)

\[
\Delta \Gamma = \frac{1}{i} \left\langle e^{-\frac{i\lambda}{4!} \varphi^4 - \frac{i}{2} \frac{\delta \Delta \Gamma}{\delta \phi} \varphi^2} \right\rangle_{G^{(0)}_{nself}}^{\text{nself}} = \frac{1}{i} \left\langle e^{-\frac{i\lambda}{4!} \varphi^4 - \frac{i}{2} J^{(2)} \varphi^2} \right\rangle_{G^{(0)}} \tag{2.66}
\]

where the superscript \( \text{nself} \) implies that we have to keep all possible connected Wick contraction using the propagator \( G^{(0)} = (\Box + m^2 - J^{(0)})^{-1} \) excluding self-contractions of both the 4-point vertex and the pseudo-vertices.

From this formula we can successively derive \( \Gamma^{(n)} \) \( (n \geq 2) \) more easily than in the previous method in which we started from algebraic inversion formula to obtain \( J^{(n)} \) first and then get \( \Gamma^{(n)} \) through integration. This is seen as follows. First notice that (2.60) actually starts from \( \lambda^2 \) because the first order of the right-hand side of (2.66), which is \( \frac{1}{i} \left\langle e^{\frac{i\lambda}{4!} \varphi^4} \right\rangle_{G^{(0)}} \) becomes zero due to the \( \text{nself} \) restriction. Since \( \Gamma^{(2)} \) is of order \( \lambda^2 \), we get

\[
\Gamma^{(2)} = \frac{1}{i} \left\langle \frac{1}{2} \left( \frac{-i\lambda}{4!} \varphi^4 \right)^2 - \frac{i}{2} J^{(2)} \varphi^2 \right\rangle_{G^{(0)}}^{\text{nself}}. \tag{2.67}
\]

The second term on the right-hand side makes no contribution to \( \Gamma^{(2)} \), again due to the \( \text{nself} \) condition, thus leading to (2.51). In the same way \( \Gamma^{(3)} \) is calculated from the expression

\[
\Gamma^{(3)} = \frac{1}{i} \left\langle \frac{1}{3!} \left( \frac{-i\lambda}{4!} \varphi^4 \right)^3 - \frac{i\lambda}{4!} \varphi^4 \left( \frac{-i}{2} J^{(2)} \varphi^2 \right) - \frac{i}{2} J^{(3)} \varphi^2 \right\rangle_{G^{(0)}}^{\text{nself}}. \tag{2.68}
\]
and we get (2.52).

This course of study can be continued (up to the desired order) to get (2.53) and so on. In (2.66) we do not yet have the 1VI restriction explicitly, but we can see that, due to the additional vertex $-\frac{i}{2} \frac{\delta \Delta \Gamma}{\delta \phi} \varphi^2 = -\frac{i}{2} \Delta J \varphi^2$, all the 1VR structures in the diagrammatic expression of (2.66) exactly cancel out. For example the 1VR graph of Fig. 3 (a) appearing in (2.66) for $n = 5$ is canceled by those of Fig. 3 (b) and (c), which are supplied by the pseudo-vertex $-\frac{i}{2} \Delta J \varphi^2$. Thereby a practical formula for $\Delta \Gamma[\phi]$ is obtained;

**Proposition A2)** $\Delta \Gamma$ is given by the following rule;

$$\Delta \Gamma = \frac{1}{i} \left< e^{-\frac{i}{4!} \varphi^4 + \frac{1}{2} (J^{(2)} + J^{(3)} + \ldots) \varphi^2} \right>_{G^{(0)}}^{1VI/nself.},$$  \hspace{1cm} (2.69)

that is, the sum of all the connected 1VI/nself. vacuum diagram built with 4-point vertices of $-\lambda$, 2-point pseudo-vertices of $J^{(i)} (i \geq 2)$ and propagators $G^{(0)}$.

The condition 1VI/nself. implies that only the connected Wick contraction corresponding to the 1VI graph need to be considered and, at the same time, that the self-contractions of the pseudo-vertex of Fig. 1 are excluded. The self-contractions of 4-point vertex of Fig. 2 are automatically excluded by the restriction of 1VI. Corresponding to the relation (2.50) or

$$- \bigcirc \bigcirc \bigcirc \Delta J = \frac{\delta \Delta \Gamma}{\delta J^{(0)}},$$  \hspace{1cm} (2.70)

the ndself. restriction in (2.46) is changed to nself. in (2.69). Proposition A1') is the derivative form of Proposition A2). Proposition A2') is clearly equivalent to the following Proposition A2') and is justified rigorously in the next subsection.

**Proposition A2')** $\Gamma^{(n)} (n \geq 2)$ is the sum of all possible $n$-th order 1VI/nself. diagram constructed out of the 4-point vertex of order $\lambda$ and the *pseudo-vertices* of order $\lambda^i (2 \leq i < n-2)$, which is denoted by $\overbrace{J^{(i)}}$ and the propagator $G^{(0)} = (\Box + m^2 - J^{(0)})^{-1}$.

We put stress on the fact that Proposition A2) or A2') makes it possible to write down $\Gamma^{(n)} (n \geq 2)$ successively with its $\phi$ dependence coming only through $J^{(0)}[\phi]$, although the rule contains $J^{(2)}, J^{(3)}, J^{(4)}, \ldots$. This is because the graphs of $\Gamma^{(n)}$ contain $J^{(i)}$ with $i \leq n - 2$ while
the graphical rule for these \( J^{(i)} \) in terms of \( G^{(0)} \) propagators are already known in (2.46) or through \( \Gamma^{(i)} \) by the relation (2.48);

\[
J^{(i)} = - \left( \begin{array}{c}
\Gamma^{(i)} \\
\end{array} \right) = \frac{\delta \Gamma^{(i)}}{\delta J^{(0)}}.
\tag{2.71}
\]

Combined with the fact that \( \Gamma^{(0)} \) and \( \Gamma^{(1)} \) are also given only through \( J^{(0)} \), which is clear from (2.14) and (2.15) with (2.16), \( \Gamma \) itself is given by \( J^{(0)} \).

From Proposition A2) or A2′) we can directly obtain (2.51) to (2.53) and

\[
\Gamma^{(5)} = \bullet J^{(3)} + \circ J^{(2)} + \text{\ldots} + \frac{\lambda \phi}{16} \Gamma^{(1)} + \text{\ldots}
\tag{2.72}
\]

and so on. The directness comes from the 1VI restriction.

Now it is convenient to introduce the whole class of the 1VI vacuum graph \( \mathcal{K}[A] \);

\[
\mathcal{K}[A] = \langle e^{-\frac{\lambda \phi}{4!} \varphi^4} \rangle_A^{1VI}
\tag{2.73}
\]

where the propagator used in the diagram is \( A \). Note that the trivial skeleton \(-\frac{1}{2i} \text{Tr} \ln A^{-1}\) is not contained in \( \mathcal{K}[A] \) by the definition (2.7). Thus this quantity is described as the whole class of vacuum skeleton minus the trivial skeleton. The whole class of the vacuum skeleton is given by

\[
\bar{\mathcal{K}}[A] = \mathcal{K}[A] - \frac{1}{2i} \text{Tr} \ln A^{-1} = \int \mathcal{D} \varphi \ e^{-\frac{\lambda \phi}{4!} \varphi^4} \big|_{\text{excl. 1VR}}
\tag{2.74}
\]

where excl. 1VR implies that the 1VR graphs are excluded or that only the 1VI graph and the trivial skeleton are kept.

In (2.53), (2.72) and in the graphs of \( \Gamma^{(n)} \) with higher \( n \) obtained by Proposition A2′) we see that \( \Gamma^{(n)} \) is the sum of all the 1VI vacuum diagrams built with the 4-point vertex and the decorated \( G^{(0)} \) propagator. The decoration is done by \( J^{(n)} \) (\( n \geq 2 \)) pseudo-vertices which are inserted into the \( G^{(0)} \) propagators in all possible ways. We see also that \(-\frac{1}{2i} \text{Tr} \ln[G^{(0)}]^{-1}\) and the self-contractions of the pseudo-vertex \( J^{(i)} \) with \( i \geq 2 \) are not included in \( \Delta \Gamma \). Thereby we arrive at Proposition A2′′) below.

**Proposition A2′′)** \( \Delta \mathcal{K}[\phi] \) is given by \( \mathcal{K}[^{\bar{G}}] - \frac{1}{2i} \text{Tr} \ln[^{\bar{G}}]^{-1} - \Delta \mathcal{K}_{\text{tr}} = \bar{\mathcal{K}}[^{\bar{G}}] - \Delta \mathcal{K}_{\text{tr}} \) where

\[
\bar{G} = \left( \Box + m^2 - J^{(0)} - J^{(2)} - J^{(3)} - \ldots \right)^{-1}
= \left( \Box + m^2 + \lambda \phi - J[\phi] \right)^{-1}
\tag{2.75/2.76}
\]
or, with the line representing the propagator $G^{(0)}$,

$$
\bar{G} = \underbrace{J^{(2)}}_{\text{term}} + \underbrace{J^{(3)}}_{\text{term}} + \cdots + \underbrace{J^{(2)} J^{(3)}}_{\text{term}} + \cdots
$$

(2.77)

and

$$
\Delta K_{\text{tr}} = -\frac{1}{2i} \text{Tr} \ln[G^{(0)}]^{-1} + \phi \left( J - J^{(0)} - J^{(1)} \right).
$$

(2.78)

In other words, $\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Delta \Gamma$ is given by,

$$
\Gamma[\phi] = -\phi J[\phi] + \frac{\lambda}{2} \phi^2 - \frac{1}{2i} \text{Tr} \ln \left( \Box + m^2 - J[\phi] + \lambda \phi \right) + K[^{\text{tr}}{G}]
$$

$$
= -\phi J[\phi] + \frac{\lambda}{2} \phi^2 + \bar{K}[\bar{G}].
$$

(2.79)

The quantity $-\frac{1}{2i} \text{Tr} \ln[\bar{G}]^{-1} - \Delta K_{\text{tr}}$ is a $\text{Tr} \ln$ of a decorated propagator specified as follows. The decoration is made by $J^{(0)}, J^{(2)}, J^{(3)}, \ldots$ ($J^{(1)} = \lambda \phi$ is not included) but the decoration only by $J^{(0)}$'s (the first term on the right-hand side of (2.78)) and the decoration by one single $J^{(2)}, J^{(3)}, \ldots$ (the last term in (2.78), which is a summation of the self-contracted diagrams of Fig. 1 with $i \geq 2$) are excluded. Proposition A2$''$ will be justified in the next subsection precisely.

Although the appearance of the term $-J \phi$ in (2.79) seems to be somewhat curious it is not actually so. Differentiating (2.79) with respect to $\phi$ by noting (2.76) and (2.12) we get

$$
\left( \phi + \frac{\delta \bar{K}}{\delta G^{-1}} \right) \left( -\frac{\delta J}{\delta \phi} + \lambda \right) = 0.
$$

(2.80)

The second term is not zero because $\frac{\delta J}{\delta \phi}$ contains various orders of $\lambda$. Thus we get

$$
\phi = -\frac{\delta \bar{K}}{\delta G^{-1}},
$$

(2.81)

which is consistent with (2.8) or $\phi = \frac{\delta W}{\delta J}$ in the following sense. If one uses the relation

$$
W = \Gamma + J \phi = \frac{\lambda}{2} \phi^2 + \bar{K}[\bar{G}]
$$

(2.82)

(obtained from (2.79)) in the right-hand side of $\phi = \frac{\delta W}{\delta J}$ and then uses (2.81) one gets the left-hand side of this equation, that is, $\phi$. 
2.1.3 Rule for $\Delta \Gamma$ in terms of $J^{(0)}[\phi]$

From Proposition A2) and A1') we can deduce another graphical rule for $\Delta \Gamma$ and $\Delta J$ in which all the $\phi$ dependence is explicitly through $J^{(0)}[\phi]$. We arrive at the new rule by using (2.74) and by noting that $\frac{\delta \Gamma^{(0)}}{\delta J^{(0)}}$ is given by the right-hand side of (2.46) (in addition to the facts stated just above Proposition A2'')). To state the new rule we introduce $i$-vertex ($i = 0, 1, 2, \ldots$), which is defined as

$$v_i(x_1, \ldots, x_i) \equiv \frac{1}{i!} \frac{\delta^i \tilde{K}[G^{(0)}]}{\delta J^{(0)}(x_1) \cdots \delta J^{(0)}(x_i)} - \delta_{i,1} \begin{array}{c} \bigcirc \\ \bigcirc \end{array} - \delta_{i,2} \begin{array}{c} \bigcirc \\ \bigcirc \end{array}.$$ (2.83)

where $\delta_{i,j}$ is the Kronecker delta.

Now the final rule is given by the following statement where the graphs are built with the inverse composite propagator $D(x, y)$ and the vertices $v_i(x_1, \ldots, x_i)$ ($i = 0, 1, 2, \ldots$).

**Proposition A3)** $\Delta \Gamma$ and $\Delta J$ is given by the following rule;

$$\Delta \Gamma = \sum \left[ \text{ all the connected tree diagrams with all the pairs of the argument of } v_i \text{'s connected by } D \text{ propagators (vacuum graph).} \right]$$ (2.84)

$$\Delta J(x) = \int d^4 y D_{xy} \times \sum \left[ \text{ all the connected tree diagrams with one of the argument of one of the } v_i \text{'s being the point } y \text{ (graph with an external point).} \right]$$ (2.85)

The tree graph in terms of $D$ propagator is the graph in which all the $D$ propagators are articulate. Here the $D$ propagator in a connected graph is called articulate if removal of it leads to a separation of the graph. Note that $D(x, y)$ lines never make a loop because $D(x, y)$ comes from $J^{(i)}$ with $i \geq 2$ (see (2.71)). Proposition A3) is understood by examples. For instance, $\Gamma^{(4)}$ in (2.53) or

$$\Gamma^{(4)} = - \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right)^{-1} \begin{array}{c} \bigcirc \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \begin{array}{c} \bigcirc \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \\ \bigcirc \end{array}$$ (2.86)

can be written as

$$\Gamma^{(4)} = \text{the fourth order of } \left( \frac{1}{2} v_1 D v_1 + v_0 \right).$$ (2.87)
In (2.86) $J^{(0)}[\phi]$ dependence is evident because there is no $J^{(i)}$ pseudo-vertex $(i \geq 2)$. All the $\phi$-dependence is through $J^{(0)}$ contained in $G^{(0)}$ (and $D$). The sum of the first two terms of (2.53) exactly coincide with the first term of (2.86) with correct weight after substitution of (2.38) or (2.54).

Proposition A3) can be expressed as follows. For this purpose we introduce $\sigma$-field which has the propagator $D$. (The $\sigma$-field looks like the auxiliary field but has nothing to do with it.) Then $\Delta \Gamma$ is given by

$$\Delta \Gamma = \frac{1}{i} \left[ \frac{\mathcal{D}\varphi \mathcal{D}\sigma e^{iS_0} e^{iS_{\text{int.}}}}{\int \mathcal{D}\varphi \mathcal{D}\sigma e^{iS_0}} \right]_{\text{conn./tree/1VI/excl.}}$$

with

$$S_0 = -\frac{1}{2} \varphi \left[ G^{(0)} \right]^{-1} \varphi - \frac{1}{2} \sigma D^{-1} \sigma$$

$$S_{\text{int.}} = -\frac{\lambda}{4!} \varphi^4 + \frac{1}{2} \sigma \varphi^2$$

(2.88)

where

$$D = -\frac{\delta J^{(0)}}{\delta \varphi} = -\left( \begin{array}{c} \varphi \\ \end{array} \right)^{-1}$$

(2.89)

with the subscript conn./tree/1VI/excl. implies that only connected graphs which is tree in terms of $D$ propagator and also 1VI in terms of the 4-point vertex have to be considered and, at the same time, that the sub-structure of $\begin{array}{c} \varphi \\ \end{array}$ and $\begin{array}{c} \varphi \\ \end{array}$ has to be excluded. Hence with the compact and self-evident notation, Proposition A3) is rewritten in the following form:

**Proposition A3’** $\Delta \Gamma$ and $\Delta J$ is given by the following formula;

$$\Delta \Gamma = \frac{1}{i} \left\langle e^{-\frac{i}{2} \varphi^4 + \frac{1}{2} \sigma \varphi^2} \right\rangle_{\text{tree/1VI/excl.}}^{G^{(0)},D}$$

(2.90)

$$-\begin{array}{c} \varphi \\ \end{array} \Delta J = \left\langle \frac{1}{2} \varphi^2 e^{-\frac{i}{2} \varphi^4 + \frac{1}{2} \sigma \varphi^2} \right\rangle_{\text{tree/1VI/excl.}}^{G^{(0)},D}$$

(2.91)

where the connected graphs with tree/1VI/excl. restriction are constructed by 3-point $(\sigma \varphi^2)$ and 4-point $(\lambda \varphi^4)$ vertices and propagators $G^{(0)}$ of the $\varphi$ field and $D$ of the $\sigma$ field.

Recall that both $G^{(0)}$ and $D$ are functionals of $J^{(0)}$. Proposition A3’) is easily understood from the rule (2.69) with (2.46) but a rigorous proof is presented in Sec. 2.2. Notice that this rule does not contain the $J^{(i)}$ pseudo-vertex unlike the previous rules but instead $D$ is represented...
by the propagator of the artificially introduced \( \sigma \) field. From (2.90), the quantity \( \Gamma^{(4)} \), for example, can be directly obtained as (2.86) above.

Finally we note that a certain infinite series of the graphs appearing in \( \Delta \Gamma[\phi] \) can be conveniently summed up. The series \( \Gamma_{ch.} \) is the sum of all the possible closed chains constructed out of the unit element \( \bigcirc \) or

\[
\Gamma_{ch.} = \bigcirc \bigcirc + \bigcirc \triangleleft + \bigcirc \bigcirc + \cdots.
\]

This series is summed up to give

\[
\Gamma_{ch.} = -\frac{1}{2} \left[ \text{Tr} \ln \left( 1 - \lambda \bigcirc \bigcirc \right) + \lambda \text{Tr} \bigcirc \bigcirc + (3! - 2) \bigcirc \bigcirc \right].
\] (2.92)

2.2 Formal justification of Propositions

In this subsection we directly prove Proposition A2', A1', and A3'), leading to the full proof of all the propositions in the previous subsection.

Proposition A2') or (2.79) is proved easily by analyzing the graphic expression of \( W[J] \) rather than that of \( \Gamma[\phi] \). It is based on a similar topological proof given in [2]. If one writes the graph rule of \( W[J] \) using \( G_J = (\Box + m^2 - J)^{-1} \) as the propagator (the rule (2.4)), the whole dependence of \( W[J] \) on \( J \) is through the propagator \( G_J = (\Box + m^2 - J)^{-1} \). The contribution of all the 1VI graphs appearing in \( W[J] \) can be written as \( \mathcal{K}[G_J] \), the vacuum skeleton minus the trivial skeleton \( -\text{Tr} \ln G_J^{-1} \) (see (2.73)). Then all the graphs of \( W[J] \) seem to be generated by replacing \( G_J \) with \( [G_J^{-1} - (-\lambda \phi)]^{-1} \), that is, \( W[J] \) seems to be given by

\[
-\frac{1}{2i} \text{Tr} \ln \left( G_J^{-1} + \lambda \phi \right) + \mathcal{K}[(G_J^{-1} + \lambda \phi)^{-1}] = \mathcal{K}[\tilde{G}].
\] (2.94)

Note here that \( \phi \) is the sum of the all distinct connected diagrams with one external point where two propagators meet. (We here use the rule (2.4) and (2.8) in which the propagator \( G_J = (\Box + m^2 - J)^{-1} \) is used so that there are only the 4-point vertices \(-\lambda \) and the pseudo-vertex does not exist in the graphs of \( \phi \).) But the above statement is not exactly true because each element of the graphs of (2.94) is incorrectly weighted. To examine this point the number of the skeletons \( N(\tilde{K}) \) is defined as follows. Removal of all 1VR vertices in a graph leads to
separated graphs which no longer have any lines connecting them. Then all the resulting separated graphs are skeletons and the number of them is \( N(\bar{K}) \). Note here that the skeleton and \( v_j \) vertex are slightly different, that is, \( v_j \) does not contain the second and the last term in (2.83) and the trivial skeleton while the skeleton does. An example of the graph of \( N(\bar{K}) = 4 \) is given in Fig. 4. Now we see that each graph of \( W[J] \) is contained in (2.94) \( N(\bar{K}) \) times.

On the other hand, if we turn our attention to 1VR vertices the graphs of \( W[J] \) seem to be generated by
\[
\frac{1}{2} \phi(-\lambda) \phi = -\frac{\lambda}{2} \phi^2
\]
(2.95)
because \( \phi \) is all the distinct connected graphs with one external point (given by the rule (2.9)). Again this is not true however because each element of \( W[J] \) appears \( N(1VR) \) times where \( N(1VR) \) is the number of 1VR vertices in the graph.

Thus the above two ways to construct the graphs of \( W[J] \) is not satisfactory. But fortunately we have a simple topological relation
\[
N(\bar{K}) - N(1VR) = 1.
\]
(2.96)
This can be proved by noting that the addition of one skeleton having one external point necessarily increases the number of the 1VR vertex by one. Thus if we take the sum \( \bar{K}[\bar{G}] + \lambda \phi^2 \), each graph of \( W[J] \) is contained exactly once or with correct weight. Hence we have
\[
W[J] = \bar{K}[\bar{G}] + \frac{\lambda}{2} \phi^2.
\]
(2.97)
This proves (2.79) or Propositions A2′″), leading to the proof of Propositions A2) and A2′).

We show below that Proposition A2′″) can also be proved by use of the sum-up rule, which is established by the author[14, 1]. Indeed we see that eq. (2.81) is directly obtained by the sum-up rule in the following. If eq. (2.81) holds, by assuming the form
\[
\Gamma = -J[\phi] \phi + \frac{\lambda}{2} \phi^2 + \Delta[\bar{G}],
\]
(2.98)
we immediately know by differentiation with respect to \( \phi \) that \( \Delta[\bar{G}] \) is equal to \( \bar{K}[\bar{G}] \), leading to (2.79).

In order to prove (2.81) first we note that \( \phi \) is all the distinct graphs with one external point (representing the insertion of \( \varphi(x)^2 \)) which are built with the propagators \( G_J = (\Box + m^2 - J)^{-1} \)
and the 4-point vertices $-\lambda$ (the rule (2.9)). A 1-part is a subdiagram connected to the rest by one 4-point vertex. When cut out, the 1-part itself is one element of the graphs of $\phi$ (see Fig. 5). The sum-up rule is best explained by an example. In short it guarantees that we can sum up the graphs on the left-hand side of the following example to the single graph on the right-hand side \emph{with correct weight.}

\[
\begin{array}{c}
p \circ p + p \circ p + \cdots + p \circ p + \cdots = p \circ (\lambda \phi). \\
\end{array}
\]

(2.99)

In other words \emph{all the 1-parts directly attached to the skeleton through an external point are summed up to $\phi$.} The statement is proved rigorously as follows.

In the graphs of $\phi$, we can easily show that \emph{if two different 1-parts have a common part, one completely contains the other.} (Note here that in a vacuum graph this is not true so that the following arguments can not be applied to the graphs with no external point.) Thus one can \emph{unambiguously proceed to a larger 1-part starting from one of the 1-parts (which is smaller) in the graph and finally reach the second largest 1-part.} See Fig. 6 as an example. (The largest 1-part is the whole graph itself.) This procedure can be repeated to reach the second largest 1-part starting from another 1-part which is not contained in the former second largest 1-parts. We continue this until there are no 1-parts other than the second largest ones. Thereby we find the second largest 1-part structure of the graph. This operation to find the 1-part structure is done for all the graphs of $\phi$. After the operation we sum up all the graphs having the same structure. We thus know \emph{all the propagators in the graphs are modified to $\tilde{G} = (G_J^{-1} + \lambda \phi)^{-1}$ while 1VR graphs disappear} because all the second largest 1-parts are summed up to $\phi$ with correct weight.

Hence we know that $\phi$ is all the distinct 1VI graph (including a derivative of trivial skeleton) with one external point where propagator $G_J = (\Box + m^2 - J)^{-1}$ is replaced by $\tilde{G}$ or

\[
\phi = \left. \frac{\delta \mathcal{K}[G]}{\delta J} \right|_{J \to J - \lambda \phi},
\]

which is equivalent to (2.81). Thus (2.79) or Proposition A2") is justified.

Having shown that Propositions A2), A2') and A2") are true we can take it for granted that Propositions A1) and A1') also hold because Proposition A1') can be regarded as the
derivative form of Proposition A2). But Proposition A1) or A1′) can be directly proved by using the sum-up rule again. From the rule (2.10) or (2.45) we know

\[- \bigcirc \Delta J = \left\langle \frac{1}{2} \varphi^2 e^{-\frac{4}{\hbar} \varphi^4 + \frac{1}{2} (J^{(1)} + J^{(2)} + \ldots) \varphi^2} \right\rangle_{G^{(0)}}^{\text{excl.}}\]  

where the superscript \text{excl.} means that the contributions of the 0-th order and the first order in \(\lambda\) and the derivative of the self-contractions of \(J^{(i)}\) with \(i \geq 2\) are \textit{excluded} from the expression. The derivative of the self-contractions have been moved on the left-hand side. Keeping the graphical meaning of (2.10) in mind we apply the sum-up rule again to obtain

\[- \bigcirc \Delta J = \text{all distinct } 1VI \text{ graphs with one external point} \]

which are built with the propagator

\[\left( \left[ G^{(0)} \right]^{-1} + \lambda \phi \right)^{-1}\]

and the 4-point vertex \(\lambda\) and the pseudo-vertex \(J^{(i)}\) with \(i \geq 1\).

\[\Delta J = \begin{cases} \left( \left[ G^{(0)} \right]^{-1} + \lambda \phi \right)^{-1} \end{cases}\]  

In the above, all the corrections by the pseudo-vertex \(J^{(1)} \varphi^2\) change the propagator \(\left( \left[ G^{(0)} \right]^{-1} + \lambda \phi \right)^{-1}\) back to \(G^{(0)}\) hence we get

\[- \bigcirc \Delta J = \left\langle \frac{1}{2} \varphi^2 e^{-\frac{4}{\hbar} \varphi^4 + \frac{1}{2} (J^{(2)} + J^{(3)} + \ldots) \varphi^2} \right\rangle_{G^{(0)}}^{1VI/\text{ndself.}} = \left\langle \frac{1}{2} \varphi^2 e^{-\frac{4}{\hbar} \varphi^4 + \frac{1}{2} \Delta J \varphi^2} \right\rangle_{G^{(0)}}^{1VI/\text{ndself.}}.\]  

This equation is, of course, equivalent to Proposition A1′).

The remaining work is to prove Proposition A3′). First the rule (2.91) for \(\Delta J\) is easily proved by mathematical induction; we assume the rule is true up to \(J^{(n)}\) or the \(n\)-th order of \(\Delta J\) and then we can convince ourselves that the statement for \(J^{(n+1)}\) or the \((n+1)\)-th order of \(\Delta J\) is also true from Proposition A1′). For this purpose we have only to note that the graphs of \(J^{(n)}\) contain \(J^{(i)}\) \((i \leq n - 2)\) and have one external point so that the sum-up rule can be applied.

The last task is to prove the rule (2.90) for \(\Delta \Gamma\). It is clear from Propositions A2) and A1′) that all the graphs appearing in \(\Delta \Gamma\) are exhausted in the rule (2.90). Thus it is enough if we confirm that the graphs of \(\Delta \Gamma\) in Proposition A2′′) appear with the same weight as in the

\[1\] The author got the idea of the proof presented below from S. Yokojima, to whom he is very thankful.
rule (2.90). In other words we justify (2.90) on the basis of Proposition A2’). To this end, we expand $-\frac{1}{2i} \text{Tr} \ln[\bar{G}]^{-1}$ in terms of $\Delta J$ ($= J^{(2)} + J^{(3)} + \cdots$) and get

$$-\frac{1}{2i} \text{Tr} \ln[\bar{G}]^{-1} - \Delta \mathcal{K}_{tr} = \sum_{n=2}^{\infty} \frac{1}{2in} \text{Tr} (G^{(0)} \Delta J)^n.$$  (2.104)

$\Delta \mathcal{K}_{tr}$ is canceled by the 0-th and first order of the expansion. Therefore we get, from the expression $\Delta \Gamma = \mathcal{K}[\bar{G}] - \frac{1}{2i} \text{Tr} \ln[\bar{G}]^{-1} - \Delta \mathcal{K}_{tr}$,

$$\Delta \Gamma = \mathcal{K}[\bar{G}] + \sum_{n=3}^{\infty} \frac{1}{2in} \text{Tr} (G^{(0)} \Delta J)^n + \Delta J \bigcirc \Delta J$$

$$\begin{align*} 
&= \mathcal{K} \left[ ([G^{(0)}]^{-1} - \Delta J)^{-1} \right] + \sum_{n=3}^{\infty} \left( \bigcirc \Delta J \right)^n

&= \mathcal{K} \left[ ([G^{(0)}]^{-1} - \Delta J)^{-1} \right] + \sum_{n=3}^{\infty} \left( \bigcirc \Delta J \right)^n

&- \frac{1}{2} \left( -\Delta J \bigcirc \Delta J \right) D \left( - \bigcirc \Delta J \right). \quad (2.105)
\end{align*}$$

By this relation the rule for $\Delta \Gamma$ is also proved by mathematical induction. We assume that the rule is true up to the $n$-th order of $\Delta \Gamma$ or $\Gamma^{(n)}$. We notice here that the first two terms on the right-hand side of (2.105) contain each graph $N(v_j)$ times and the last term $N(D)$ times (see the graphical rule (2.91) for $\Delta J$). Here $N(v_j)$ and $N(D)$ are the number of $v_j$ vertices ($j = 1, 2, \ldots$) and that of the $D$ propagators respectively. Due to the topological relation

$$N(v_j) - N(D) = 1 \quad (2.106)$$

we confirm that $\Gamma^{(n+1)}$ is given correctly by the final rule (2.90).

### 3 Case of itinerant electron model

In the previous section we have taken the $\varphi^4$ theory which is simple and convenient to develop a general framework. In this section we take a physically more interesting system as another example — the itinerant electron model including the Hubbard model. We couple an external source to the local composite operator corresponding to the spin operator (and to the number density operator). Writing down the effective action for such a system is equivalent to rewrite the theory in terms of the expectation value of the spin operator or the magnetization instead of the external source or the magnetic field. Such a formulation is of course convenient for the study of magnetic phase of the system — problem of the spontaneous symmetry breaking of SU(2), which is inherent in the model.
The generating functional for this system (written as $\Omega$ in this section instead of $W$) is a generalization of the thermodynamical potential to the case where an external source, which depends on imaginary time $\tau$, is present. This is particularly useful for our purpose and is defined by

$$e^{-\Omega[J]} = \text{Tr}T_\tau e^{-\int_0^\beta d\tau \mathcal{H}[J]}$$  \hspace{1cm} (3.1)

$$\mathcal{H}[J] = \mathcal{H}_0 + \mathcal{H}_J$$  \hspace{1cm} (3.2)

$$\mathcal{H}_0 = \sum_{rr'} \sum_\sigma t_{rr'} a^\dagger_{r\sigma} a_{r'\sigma} + U \sum_r n_{r\uparrow} n_{r\downarrow}$$  \hspace{1cm} (3.3)

$$\mathcal{H}_J = -\sum_{r\sigma} J_\sigma (r\tau) n_{r\sigma}$$
$$= -\sum_r h(r\tau) \hat{S}_z(r) - \mu N$$  \hspace{1cm} (3.4)

where $\beta^{-1}$ is the temperature of the system and $T_\tau$ is the $\tau$-ordering operator. The creation and annihilation operators for the electron of spin $\sigma$ and $\sigma'$ at the lattice site $r$ and $r'$ satisfy

$$\{a_{r\sigma}, a^\dagger_{r'\sigma'}\} = \delta_{rr'} \delta_{\sigma\sigma'}.$$  \hspace{1cm} (3.6)

Furthermore $t_{rr'}$ represents the hoping term and $U$ the on-site Coulomb interaction. We have also introduced

$$n_{r\sigma} = a^\dagger_{r\sigma} a_{r\sigma},$$  \hspace{1cm} (3.7)

$$\hat{S}_z(r) = \frac{1}{2}(n_{r\uparrow} - n_{r\downarrow}),$$  \hspace{1cm} (3.8)

$$N = \sum_r (n_{r\uparrow} + n_{r\downarrow}),$$  \hspace{1cm} (3.9)

$$J_\sigma (r\tau) = \frac{\sigma}{2} h(r\tau) + \mu.$$  \hspace{1cm} (3.10)

We regard below both the chemical potential and the $\tau$-dependent magnetic field $h(r\tau)$ as external sources for convenience. They are combined to $J_\sigma (r\tau)$ as in (3.10). Note here that if we want to rewrite the theory in terms of the expectation value of the number density operator without taking the spin operator as another dynamical variable we have only to set $J_\uparrow = J_\downarrow$ in the following formulae. The spin index $\sigma$ is defined to take the value $(+1, -1)$ for $(\uparrow, \downarrow)$.

The path-integral representation in terms of Grassmann variables $z$ and $z^*$ (corresponding to the operators $a$ and $a^\dagger$ respectively) is given by (see Appendix D)

$$e^{-\Omega} = \int \mathcal{D}z^* \mathcal{D}z e^{S[z^*,z,J]}$$  \hspace{1cm} (3.11)
\[ S[z^*, z, J] = - \sum_{x\sigma} z_{x\sigma}^* [G_{\sigma}]_{xx\sigma}^{-1} z_{x\sigma} - U \sum_x z_{x\uparrow}^* z_{x\downarrow}^* z_{x\uparrow} z_{x\downarrow} + \sum_{x\sigma} J_{x\sigma} z_{x\sigma}^* z_{x\sigma} \] (3.12)

\[ \equiv - \sum_{\sigma} z_{\sigma}^* G_{\sigma}^{-1} z_{\sigma} - U z_{\uparrow}^* z_{\downarrow}^* z_{\uparrow} z_{\downarrow} \] (3.13)

\[ G_{xx'}^{-1} = \delta_{\tau\tau'} \left( \delta_{\tau\tau'} \frac{\partial}{\partial \tau'} + t_{\tau\tau'} \right) \quad \text{and} \quad \left[ G_{\sigma}^{-1} \right]_{xx'} = G_{xx'}^{-1} - \delta_{\tau\tau'} \delta_{\tau\tau'} J_{x\sigma} \] (3.14)

where \( x \) and \( x' \) denote the sets \((r\tau)\) and \((r'\tau')\) respectively. From this expression it is straightforward to get the Feynman diagram expansion for \( \Omega \) in powers of \( U \). The expectation value of the local number operator \( n_{r\sigma} \) is defined as

\[ \phi_{x\sigma} = - \frac{\delta \Omega}{\delta J_{x\sigma}} = \left\langle a_{x\sigma}^+ a_{x\sigma} \right\rangle_{\tau} = \left( \frac{a_{x\downarrow}^+ a_{x\uparrow} + a_{x\uparrow}^+ a_{x\downarrow}}{2} + \sigma \frac{a_{x\uparrow}^+ a_{x\uparrow} - a_{x\downarrow}^+ a_{x\downarrow}}{2} \right)_{\tau} = \frac{n_x}{2} - \sigma m_x \] (3.15)

where \( x \) again denotes the set \((r\tau)\) while \( n_x \) and \( -m_x \) are the expectation value of the local number operator and the \( z \)-component of the local spin operator respectively.

The effective action or a generalization of the free energy to the case of \( \tau \)-dependent dynamical variables is defined by

\[ F = \Omega + \int_0^\beta d\tau \sum_{r\sigma} J_{\sigma}(r\tau) \phi_{\sigma}(r\tau) \equiv \Omega + \sum_{x\sigma} J_{x\sigma} \phi_{x\sigma} \] (3.16)

with an identity

\[ J_{x\sigma} = \frac{\delta F}{\delta \phi_{x\sigma}}. \] (3.17)

\( F \) corresponds to \( \Gamma \) of the previous section. The rule for \( \phi \) corresponding to the rule (2.10) in this case is

\[ - \phi_{\sigma} = \left\langle z_{\sigma} z_{\sigma}^* e^{-U z_{\uparrow}^* z_{\uparrow} z_{\downarrow}^* z_{\downarrow} + \sum_{\sigma'} (J_{\sigma'}^{(1)} + J_{\sigma'}^{(2)} + \cdots) z_{\sigma'}^* z_{\sigma'}} \right\rangle_{G^{(0)}}, \] (3.18)

that is, the sum of all the connected graphs built with 4-point vertices \( U \), pseudo-vertices \( J_{\sigma}^{(i)} \) \((i \geq 1)\), and propagators \( G_{\sigma}^{(0)} \) with the notation similar to (2.7). Here \( G^{(0)} \) is defined as

\[ \left[ G_{\sigma}^{(0)} \right]_{xy}^{-1} = G_{xy}^{-1} - \delta_{xy} J_{x\sigma}^{(0)}. \] (3.19)

The extra minus sign in (3.18) originates from the sign in \( \phi = - \frac{\delta \Omega}{\delta J} \). Then as mentioned before (below Proposition A1)) the inversion formula of the \( n \)-th order in \( U \) is given by the \( n \)-th order of (3.18) regarding both \( \phi_{\sigma} \) and \( G_{\sigma}^{(0)} \) as order unity. Thus we obtain
and so on. Here a solid (dashed) line to which an arrow is attached (per loop of lines) represents the propagator of spin-up (spin-down) electron and it is $G^{(0)}_\uparrow (G^{(0)}_\downarrow)$. The black dot denote the place where two propagators meet (corresponding to a derivative with respect to $J^{(0)}$ – note $\delta G^{(0)}_\sigma = \delta G^{(0)}_\sigma G^{(0)}_\sigma$). The factor $U$ is associated with a 4-point vertex at which spin-up and spin-down propagators come in and out, while no factor is associated with the black dot (see Appendix B). Hence from (3.21) we get

\begin{equation}
(3.22)
\end{equation}

or

\begin{equation}
J^{(1)}_\sigma = U \phi_\sigma = -U \text{Tr} G_\sigma.
\end{equation}

(3.23)

The second order formula of the inversion method is also obtained as that order of (3.18);

\begin{equation}
(3.24)
\end{equation}

which reduces to, as eq. (2.37) do to (2.38),

\begin{equation}
(3.25)
\end{equation}

Further, it is easy to find that, corresponding to (2.39),

\begin{equation}
(3.26)
\end{equation}

The left-hand side of (3.25) or (3.26) can be written as $\frac{\delta \phi_\uparrow}{\delta J^{(0)}_\uparrow} J^{(i)}_\downarrow$ with $i = 2$ or 3. Following the procedure presented in the previous section we get

\begin{equation}
F = F^{(0)} + F^{(1)} + F^{(2)} + F^{(3)} + \cdots,
\end{equation}

\begin{equation}
F^{(0)} = \sum_{x\sigma} J^{(0)}_{x\sigma} \phi_{x\sigma} - \sum_\sigma \text{Tr} \ln \left[ G^{(0)}_\sigma \right]^{-1},
\end{equation}

\begin{equation}
F^{(1)} = U \sum_x \phi_{x\uparrow} \phi_{x\downarrow} = \frac{U}{2} \sum_{x\sigma} \phi_{x\sigma} \phi_{x-\sigma},
\end{equation}

(3.27)
where $F^{(n)}$ satisfies
\[
J^{(n)}_{\sigma} = \frac{\delta F^{(n)}}{\delta \phi_{\sigma}}. 
\]  

Note that $J^{(0)}$ contained in $G^{(0)}$ is a functional of $\phi$ defined by the solution of (3.20) or
\[
\phi_{x\sigma} = -G^{(0)}_{xx\sigma} = - \left( \frac{1}{G^{-1} - J^{(0)}_{\sigma} [\phi]} \right)_{xx}. 
\]

The free energy of the Stoner theory is recreated by $F^{(0)} + F^{(1)}$. Now it is clear that all the propositions given in Sec. 2 hold for the present model with minor and self-evident modifications. Here we repeat them for later convenience.

**Proposition B1**  The graphical rule for $\Delta F$ is given by the following equation;
\[
\Delta F = - \left\langle e^{-Uz^*_\up z^*_\up z^*_\down z^*_\down + \sum_{\sigma} (J^{(2)}_{\sigma} + J^{(3)}_{\sigma} + \cdots) z^*_\sigma z^*_\sigma} \right\rangle_{G^{(0)}}^{1\text{VI/nself}}, 
\]

that is, the sum of all the connected 1VI/nself. diagram constructed out of 4-point vertices, 2-point pseudo-vertices and propagators $G^{(0)}_{\sigma}$.

Here 1VI/nself. condition implies that only the 1VI graphs are kept and graphs corresponding to the self-contractions of the vertices are excluded.

**Proposition B2**  $J^{(n)}$ is successively given as a functional of $J^{(0)}_{\sigma}$ by the following formula.
\[
J^{(n)}_{\sigma} = n\text{-th order of } D_{\sigma} \times \left\langle z^*_{\sigma} z^*_{\sigma} e^{-Uz^*_\up z^*_\up z^*_\down z^*_\down + \sum_{\sigma'} (J^{(2)}_{\sigma'} + J^{(3)}_{\sigma'} + \cdots) z^*_\sigma z^*_\sigma'} \right\rangle_{G^{(0)}}^{1\text{VI/ndself}}. 
\]

where
\[
D^{-1}_{\sigma} = \frac{\delta \phi_{\sigma}}{\delta J^{(0)}_{\sigma}} = - \frac{1}{G^{-1} - J^{(0)}_{\sigma}} \frac{1}{G^{-1} - J^{(0)}_{\sigma}}. 
\]
**Proposition B3)** The graphical rule for $\Delta F$ is given by the following formula

$$
\Delta F = -\left\langle e^{-Uz_1^*z_1^*z_1^*z_1} + \sum_\sigma z_\sigma^* \phi_\sigma - \phi_\sigma \right\rangle_{G^{(0)},D},
$$

(3.37)

or, in a more detailed expression,

$$
\Delta F = -\left[ \int Dz^* Dz D\varphi e^{S_0 + S_{int.}} \right]_{\text{conn./tree/1VI/excl.}}
$$

$$
S_0 = -\sum_\sigma z_\sigma^*[G^{(0)}]^{-1} z_\sigma + \frac{1}{2} \sum_\sigma \varphi_\sigma D_\sigma^{-1} \varphi_\sigma
$$

$$
S_{int.} = -Uz_1^*z_1^*z_1^*z_1 + \sum_\sigma z_\sigma^* \varphi_\sigma - \varphi_\sigma
$$

(3.38)

where the subscript conn./tree/1VI/excl. implies that we should take only connected graphs which are tree with respect to $D_\sigma$ propagator of the bosonic field $\varphi_\sigma$ and also 1VI with respect to the 4-point vertex and the sub-structures of the graphs corresponding to $\frac{\delta}{\delta J^{(0)}} \text{Tr} \ln G^{(0)}$ and $\frac{\delta^2}{\delta J^{(0)} \delta J^{(0)}} \text{Tr} \ln G^{(0)}$ are excluded.

Note that Proposition B1) can be deduced from the formula

$$
\Delta F = -\left\langle e^{-Uz_1^*z_1^*z_1^*z_1 + \sum_\sigma (J_\sigma^{(2)} + J_\sigma^{(3)} + \cdots) z_\sigma^* z_\sigma} \right\rangle_{G^{(0)}},
$$

(3.39)

which is clear from the functional representation;

$$
e^{-F} = e^{\sum_\sigma (-J_\sigma^{(0)} + \text{Tr} \ln [G^{(0)}]^{-1}) - U\phi_\sigma}\frac{\int Dz^* Dz e^{-\sum_\sigma z_\sigma^*[G^{(0)}]^{-1} z_\sigma - Uz_1^*z_1^*z_1^*z_1 + \sum_\sigma ((J_\sigma^{(0)} - J_\sigma^{(0)}) z_\sigma^* z_\sigma - (J_\sigma^{(0)} - J_\sigma^{(0)}) \phi_\sigma) + U\phi_\sigma}}{\int Dz^* Dz e^{-\sum_\sigma z_\sigma^*[G^{(0)}]^{-1} z_\sigma}}
$$

(3.40)

or

$$
e^{-\Delta F} = \frac{\int Dz^* Dz e^{-\sum_\sigma z_\sigma^*[G^{(0)}]^{-1} z_\sigma - Uz_1^*z_1^*z_1^*z_1 + \sum_\sigma z_\sigma^* z_\sigma - U\phi_\sigma + \phi_\sigma} - \sum_\sigma \frac{\delta}{\delta \phi_\sigma} (z_\sigma^* z_\sigma - \phi_\sigma)}{\int Dz^* Dz e^{-\sum_\sigma z_\sigma^*[G^{(0)}]^{-1} z_\sigma}}.
$$

(3.41)

There is another way to state the graph rule. For this purpose $\mathcal{K}[A]$ is defined as follows:

$$
\mathcal{K}[A] = \left\langle e^{-Uz_1^*z_1^*z_1^*z_1} \right\rangle_{A}^{1VI}
$$

(3.42)

where $A$ is the propagator used in the graphical expression. Then the rule is summarized in the following proposition.
**Proposition B1′″** $\Delta F = F - (F^{(0)} + F^{(1)})$ is given by $\mathcal{K}[\bar{G}] - \bar{\Delta F}$ where

$$\bar{G}_\sigma = \left(G^{-1} - J^{(0)}_\sigma - J^{(2)}_\sigma - J^{(3)}_\sigma - \cdots\right)^{-1} = \left(G^{-1} - J_\sigma + U\phi_{-\sigma}\right)^{-1} \quad \text{(3.43)}$$

and

$$\bar{\Delta F} = \sum_\sigma \text{Tr} \ln\left(G^{-1} - J^{(0)}_\sigma - J^{(2)}_\sigma - J^{(3)}_\sigma - \cdots\right) - \sum_\sigma \text{Tr} \ln\left(G^{-1} - J^{(0)}_\sigma\right) - \sum_\sigma \phi_\sigma \left(J_\sigma - J^{(0)}_\sigma - J^{(1)}_\sigma\right) \quad \text{(3.44)}$$

In other words

$$F = \sum_\sigma \phi_\sigma J_\sigma - U\phi_\uparrow \phi_\downarrow - \sum_\sigma \text{Tr} \ln \bar{G}_\sigma^{-1} + \mathcal{K}[\bar{G}]$$

$$\equiv \sum_\sigma \phi_\sigma J_\sigma - U\phi_\uparrow \phi_\downarrow + \mathcal{K}[\bar{G}] \quad \text{(3.45)}$$

### 4 Case of QED

Final example is the effective action for the expectation value of gauge invariant local composite field $\phi^\mu(x) = \langle \bar{\psi}(x)\gamma^\mu\psi(x)\rangle$ in QED. The practical use of $\Gamma[\phi_\mu]$ in QED is as follows. Although $\langle \bar{\psi}\gamma^\mu\psi \rangle = 0$ for the vacuum, the lowest relation of the on-shell condition \[16\] (with the space-time integration over $y$ and the summation over $\nu$ suppressed),

$$\Gamma^{(2)}_{\mu x,\nu y} \Delta \phi^\nu(y) = 0, \quad \text{(4.1)}$$

where

$$\Gamma^{(2)}_{\mu x,\nu y} = \frac{\delta^2 \Gamma[\phi]}{\delta \phi_\mu(x)\delta \phi_\nu(y)} \bigg|_{\phi=0}, \quad \text{(4.2)}$$

determines the bound state in the channel specified by $\bar{\psi}\gamma^\mu\psi$. This allows us a *gauge invariant study of* $^3S_1$ of positronium state. The following work may also be a starting point for the study of the order parameter for the chiral symmetry breaking $\phi = \langle \bar{\psi}\psi \rangle$ in the massless QED and that of $\langle \bar{q}a_q \rangle$ or $\langle A_\mu^a A_\mu^a \rangle$ in QCD. Here $q$ and $A_\mu^a$ are operators for quarks and gluons respectively. All these are believed to be non-vanishing objects in contrast to $\bar{\psi}\gamma^\mu\psi$. The lowest order discussion of $\langle \bar{\psi}\psi \rangle$ has been given in \[10\].
The generating functional in this case is given by (with the space-time integration and the summation over the Greek index suppressed),

$$e^{iW[J,K]} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \mathcal{D}A e^{iS[\bar{\psi}\gamma^\mu\psi]}$$

(4.3)

$$S[\bar{\psi}, \psi, A, J] = -\bar{\psi}G^{-1}\psi - \frac{1}{2}A^\mu D^{-1}_\mu A^\nu + e\bar{\psi}\gamma_\mu\psi A^\mu + J_\mu \bar{\psi}\gamma^\mu\psi$$

$$= -\bar{\psi}G^{-1}_J\psi - \frac{1}{2}A^\mu D^{-1}_\mu A^\nu + e\bar{j}_\mu A^\mu$$

(4.4)

where

$$G^{-1} = -i\gamma_\mu \partial^\mu + m,$$

(4.5)

$$G^{-1}_J = G^{-1} - J_\mu \gamma^\mu,$$

(4.6)

$$D^{-1}_\mu = -\Box g_{\mu\nu} + \left(1 - \frac{1}{\lambda}\right) \partial_\mu \partial_\nu,$$

(4.7)

$$j_\mu = \bar{\psi}\gamma_\mu\psi.$$  

(4.8)

Here the parameter $\lambda$ specifies the gauge. Then we get Feynman graphs for $\phi_\mu = \frac{\delta W}{\delta J_\mu} = \langle j_\mu \rangle = \langle \bar{\psi}\gamma_\mu\psi \rangle$:

$$\phi_\mu = \left\langle \bar{\psi}\gamma_\mu\psi e^{\bar{\psi}\gamma_\mu\psi A^\mu + \left(J^{(1)}_\mu + J^{(2)}_\mu + \ldots \right) \bar{\psi}\gamma^\mu\psi} \right\rangle_{G^{(0)}, D},$$

(4.9)

that is, the sum of all the connected graphs built with 3-point vertices ($\bar{\psi}\gamma^\mu\psi A^\mu$), 2-point pseudo-vertices ($J^{(i)}_\mu \bar{\psi}\gamma^\mu\psi$), electron propagators $G^{(0)}$ and photon propagators $D$. Here $J^{(i)}_\mu$ is the $i$-th order (in $e^2$) of $J_\mu$ and the propagator $G^{(0)}$ is $G_J$ but $J$ replaced by $J^{(0)}$. The quantity $J^{(0)}$ is defined by

$$\phi_\mu(x) = i\gamma_\mu G^{(0)}_{\mu\mu}(x, x),$$

(4.10)

which is equivalent to (4.11) below. By writing down the $i$-th order of (4.9) one gets the inversion formula of that order. For example,
In the above graphs we associate the electron propagator \( G^{(0)} \) with each solid line and the photon propagator \( D \) with each dashed line. In addition a factor \( e \gamma^\mu \) and \( \gamma^\mu \) are assigned to a vertex and to a black dot (\( \bullet \)) respectively (see Appendix B).

If we define \( J_A^{(1)} \) and \( J_B^{(1)} \) from (4.12) by

\[
\tag{4.14}
\]
we see that all the \( J_A^{(1)} \)'s exactly cancel out the 1PR structure appearing in the \( i \)-th order of (4.9) with \( i \geq 2 \). Here 1PR means 1-particle-reducible with respect to the photon propagator. Hereafter 1PI graph is defined as the graph which is not 1PR in photon channel. Indeed all the 1PR graph in (4.13) disappear after substitution of the last equation due to \( J_A^{(1)} \) while \( J_B^{(1)} \) remains;

\[
\tag{4.15}
\]

The effective action in this case is defined by \( \Gamma = W - J_{\mu} \phi^\mu \) (with the space-time integration suppressed) as usual with an identity \(-J^\mu = \frac{\delta \Gamma}{\delta \phi^\mu}\). Thus integrating (4.13) one can obtain \( \Gamma^{(2)} \) (and higher order by using (4.9)). Here we can take another course instead. For this purpose let us first examine the path-integral representation of \( \Gamma \). Integrating out the photon field we get

\[
\tag{4.16}
\]

Since \( \Gamma^{(i)} \) is defined by \(-J_{\mu}^{(i)} = \frac{\delta \Gamma^{(i)}}{\delta \phi^\mu}\) the quantities \( \Gamma_A^{(1)} \) and \( \Gamma_B^{(1)} \) are defined as

\[
\tag{4.17}
\]
in accordance with (4.14). The quantity \( \Delta J \) and \( \Delta \Gamma \) in this case are expanded as

\[
\Delta J = J_B^{(1)} + J^{(2)} + J^{(3)} + \cdots, \tag{4.18}
\]

\[
\Delta \Gamma = \Gamma_B^{(1)} + \Gamma^{(2)} + \Gamma^{(3)} + \cdots. \tag{4.19}
\]
Noting that \( \Gamma_A^{(1)} = \frac{e^2}{2} \phi^\mu D_{\mu\nu} \phi^\nu \) and \( \Gamma^{(0)} = -J^{(0)}_{\mu} \phi^\mu - i \text{Tr} \ln[G^{(0)}]^{-1} \), we get

\[
e^{i\Gamma} = e^{i(-J^{(0)}_{\mu} \phi^\mu - i \text{Tr} \ln[G^{(0)}]^{-1}) + \frac{i e^2}{2} \phi^\mu D_{\mu\nu} \phi^\nu} \]

\[
e^{i\Delta \Gamma} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i([-\bar{\psi}[G^{(0)}]^{-1}\psi + e^2(\frac{1}{2} j^\mu D_{\mu\nu} j^\nu - \phi^\mu D_{\mu\nu} j^\nu) + \frac{1}{2} \phi^\mu D_{\mu\nu} \phi^\nu \frac{\delta \Delta \Gamma}{\delta \phi^\mu} (j^\mu - \phi^\mu)])} \]

or

\[
e^{i\Delta \Gamma} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i([-\bar{\psi}[G^{(0)}]^{-1}\psi + e^2(\frac{1}{2} j^\mu D_{\mu\nu} j^\nu + \frac{\delta \Delta \Gamma}{\delta \phi^\mu} (j^\mu - \phi^\mu)])}} \]

We write (4.21) as

\[
\Delta \Gamma[\phi] = \frac{1}{i} \left< e^{i \frac{e^2}{2} j^\mu D_{\mu\nu} j^\nu - \frac{\delta \Delta \Gamma}{\delta \phi^\mu} (j^\mu - \phi^\mu)} \right>_{G^{(0)}}^{\text{nsel}}. \]  

(4.22)

The meaning of nself. is that we have to exclude the self-contraction of the electron propagators. By using (4.22) and noting the cancellation similar to that in (4.13) we get

\[
\Delta \Gamma[\phi] = \frac{1}{i} \left< e^{i \frac{e^2}{2} j^\mu D_{\mu\nu} j^\nu - \frac{\delta \Delta \Gamma}{\delta \phi^\mu} (j^\mu - \phi^\mu)} \right>_{G^{(0)}}^{\text{nsel}}. \]  

(4.23)

Combined with the arguments similar to those of previous subsections, we arrive at the following proposition (with a similar statement for a graph of \( J^{(n)} \)):

**Proposition C)** \( \Gamma^{(n)} (n \geq 2) \) is the sum of all possible \( n \)-th order (in \( e^2 \)) 1PI diagram constructed out of 4-point vertex of order \( e^2 \) ( ), vertex of order \( e^2 \) ( ) and vertices of order \( e^{2i} \) ( ) (\( 2 \leq i < n \)). Here the propagator is \( G^{(0)} \). In other words

\[
\Delta \Gamma[\phi] = \frac{1}{i} \left< e^{i \frac{e^2}{2} j^\mu D_{\mu\nu} j^\nu + i(J^{(1)}_{\mu} + J^{(2)}_{\mu} + J^{(3)}_{\mu} + ...)) j^\mu} \right>_{G^{(0)}}^{\text{1PI}} \]  

(4.24)

where \( \langle \cdot \cdot \cdot \rangle_{\text{1PI}} \) means 1PI (in terms of the photon lines) connected Wick contraction using the propagators \( G^{(0)} \) that is a functional of \( J^{(0)} \).

Of course, there are various equivalent modifications of Proposition C).
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Appendix A — Legendre transformation and the inversion method

In this appendix we look at more carefully the reason why we should assume that $\phi$ is of order $\lambda^0 = 1$ or independent of $\lambda$ in our inversion process. This point has been exemplified in terms of diagrams, which is not necessarily familiar to everyone. Here we present a clear explanation in purely mathematical language. Although the following discussion is trivial it is worth while in order to understand the foundation of the inversion method. For brevity the case of $x$-independent variables $J$ and $\phi$ are considered.

Consider the quantity $W[J, \lambda] - J\phi[J, \lambda]$ in which $\phi[J, \lambda] \equiv \frac{\delta W[J, \lambda]}{\delta J}$. Here we have emphasized the $\lambda$-dependence. If we take a small variation of this quantity assuming that $J$ and $\lambda$ are independent variables it becomes

$$\delta W[J, \lambda] \delta J dJ + \delta W[J, \lambda] \delta \lambda d\lambda - dJ \phi[J, \lambda] = \frac{\delta W[J, \lambda]}{\delta \lambda} d\lambda - J d\phi[J, \lambda]. \quad (A.1)$$

Hence we see that the quantity can be regarded as a function(al) of two independent variables $\phi$ and $\lambda$. We thus write the quantity $W - J\phi$ as $\Gamma[\phi, \lambda]$. What is implied here is as follows: if we solve the relation $\phi = \frac{\delta W[J, \lambda]}{\delta J}$ in favor of $J$ assuming that the two quantities $\phi$ and $\lambda$ are mutually independent to get $J = J[\phi, \lambda]$ and then insert this expression of $J$ into all $J$ appearing in $W[J] - J\phi$, then $W[J] - J\phi$ is automatically written by only two independent variables $\phi$ and $\lambda$. In other words, the inversion process of Legendre transformation is carried out regarding $\phi$ as independent of $\lambda$. Hence the process in the inversion method exactly coincides with the inversion process of Legendre transformation. Note that once the inversion or Legendre transformation is performed and after the sources are set to the desired values, zero for example, the resultant $\phi$ depends on $\lambda$ of course.
Appendix B — Feynman rules

$\varphi^4$ theory

Although well known, we summarize for clarity the rule (Rule A) to get algebraic expressions from the corresponding graphs for the $\varphi^4$ theory.

Rule A1) In one specific way (as one likes), assign $n$ labels $x_1, \ldots, x_n$ (internal points) to all the 4-point vertices and the pseudo-vertices where $n$ is the total number of vertices (including pseudo-vertex).

Rule A2) Associate a propagator $G_J$ (for the rule (2.4) and (2.9)) or $G^{(0)}$ (for the rule (2.3) and (2.10)) with each line. A factor $-\lambda$, and $J^{(i)}$ are assigned to the 4-point vertex and the pseudo-vertex of the $i$-th order respectively. No factor is assigned to the black dot which corresponds external point.

Rule A3) Associate a factor $i^{-L}$ for a diagram where $L$ is the number of independent momenta of the graph.

Rule A4) Associate a symmetry factor $S$ for a diagram.

Rule A5) Sum (Integrate) the product of all factors in A2) to A4) over the space time index $x_1, \ldots, x_n$.

The symmetry factor $S$ for each graph is given by the line symmetry number $S_L$ and the vertex symmetry number $S_V$ as $S = \frac{1}{S_L} \cdot \frac{1}{S_V}$. As is well known, $S_L$ and $S_V$ are obtained through the following rule.

Rule $S_L1)$ If there is a line which starts from a vertex (including the black dot • and pseudo-vertex) and come back directly to the starting vertex, associate the factor 2.

Rule $S_L2)$ If there are $m$ lines ($m = 2, 3, 4$) directly connecting two common vertices (including pseudo-vertex), associate the factor $m!$.

Rule $S_L3)$ All the product of the factors in $S_L1)$ and $S_L2)$ is $S_L$. 
**Rule S_V**) Assign $n$ labels $1, \ldots, n$ to $n$ vertices (including pseudo-vertex) in an arbitrary way.

Count the number of all possible other ways of assigning $n$ labels that give the same topological structure as the first specific way. The number thus obtained plus 1 is $S_V$.

For definiteness we give some examples; the graph appearing in (2.19) has $(S_L, S_V)=(2, 1)$: three graphs of (2.21) have $(2!^2, 2, 1)$, $(2!, 2)$, and $(3!, 2)$ respectively.

As another example we consider the reduction of (2.37) to (2.38). Since the symmetry factors of the second, fourth, and sixth graphs on the left-hand side of (2.37) are $(S_L, S_V)=(1, 2)$, $(2, 1)$, and $(2^2, 2)$, the contribution of the three graphs becomes zero. This is because, after replacing $J^{(1)}$ by use of (2.35) (whose symmetry factor is 2), new symmetry factors of these graphs becomes $1 \cdot 2 \cdot 2^2$, $2 \cdot 1 \cdot 2$, and $2^2 \cdot 2$ respectively. By a similar argument we find the cancellation of the third and fifth graphs on the left-hand side of (2.37). Thus we get (2.38) from (2.37).

**Itinerant electron model**

The rules for the itinerant electron model are given as follows (Rule B). Rule B1), B4) and B5) are the same as Rule A1), A4) and A5) respectively. Rule B3) is Rule A3) with $i^{-L}$ replaced by $(-1)^L(-1)^L_f$ where $L_f$ is the number of Fermion loops. Rule A2) is changed into

**Rule B2**) Associate $\bullet_y \longrightarrow \bullet_x$ and $\bullet_y \longrightarrow \bullet_x$ with $[G_t^{(0)}]_{xy}$ and $[G_t^{(0)}]_{xy}$ respectively and the factor $U$ is assigned to the 4-point vertex. The factor $J^{(i)}_\sigma$ is also associated with the pseudo-vertex of the $i$-th order. No factor is assigned to the external point.

As for the symmetry factor $S(=\frac{1}{S_L} \cdot \frac{1}{S_V})$ rules for $S_L$ and $S_V$ is essentially the same as those of the $\varphi^4$ theory except for the fact that we have to distinguish the spin-up and spin-down propagators and their directions of the arrows when we consider the topological equivalence. Thus the factor $S_L$ is always 1 in this model.

**QED**

Finally the rules for QED are presented as follows.

**Rule C1**) Assign $n$ labels, in one specific way as one likes, $(x_1, \mu_1), \ldots (x_n, \mu_n)$ to its vertices (including pseudo-vertices).
Rule C2) Associate an electron propagator $G^{(0)}$ with each solid line and a photon propagator $D$ with each dashed line.

Rule C3) Associate a factor $e\gamma_\mu$ and $J^{(i)}\gamma^\mu$ with the 3-point vertex and the pseudo-vertex respectively. $\gamma^\mu$ is assigned to the black dot (●) which corresponds the external point.

Rule C4) Associate a factor $i^{−L}(-1)^{L_f}$ where $L$ is the number of loop momenta of the graph and $L_f$ is the number of the fermion loops.

Rule C5) Associate a symmetry factor $S$ for a diagram.

Rule C6) Sum the product of all factors in C2) to C5) over the $x_1 \cdots x_n$ and $\mu_1 \cdots \mu_n$.

The symmetry factors are calculated as before. Note that $S_L$ is always 1 in QED.

Appendix C — Inversion method for $\langle \varphi(x) \rangle$ and $\langle \varphi(x)\varphi(y) \rangle$

We show below how the inversion method works to reproduce well-known results for the effective action of $\langle \varphi(x) \rangle$ and $\langle \varphi(x)\varphi(y) \rangle$. For simplicity we consider the $\varphi^4$ theory and several lower orders of the known rule are explicitly studied rather than giving formal proof.

Case of $\langle \varphi(x) \rangle$

In order to study the effective action of elementary field $\phi(x)$, the generating functional $W[J]$ is defined as in (2.1) with $S[\phi,J]$ replaced by

$$S[\varphi,J] = -\frac{1}{2} \int d^4x\varphi(x)(\Box + m^2)\varphi(x) - \frac{\lambda}{4!} \int d^4x\varphi(x)^4 + \int d^4xJ(x)\varphi(x).$$

(C.1)

The dynamical variable $\phi$ for the effective action is

$$\phi(x) = \frac{\delta W}{\delta J(x)} \equiv \langle \varphi(x) \rangle^J$$

(C.2)

by use of which $\Gamma[\phi]$ is defined by (2.11) and eq. (2.12) holds as an identity.

Now the original series expansion in $\lambda$ is given by, suppressing the $x$-dependence,

$$\phi^{(0)} = \quad ,$$

(C.3)

$$\phi^{(1)} = \quad + \quad ,$$

(C.4)
Here a black dot denotes the external source $J$ and a line the propagator $\frac{1}{\Box + m^2}$. Thus (2.23), the right-hand side of which is (C.3) with $J$ replaced by $J^{(0)}$, becomes

\[ \phi^{(2)} = \quad + \quad + \quad + \quad \]

\[ + \quad + \quad + \quad . \]  

(C.5)

from which $J^{(0)}$ is obtained explicitly as opposed to the case of the local composite operators;

\[ J^{(0)} = \left( \Box + m^2 \right)_{xy} \phi(y). \]

(C.8)

Hereafter, all the black dots in the graphs denote $J^{(0)}$ instead of $J$ as in (C.6). We immediately know

\[ \Gamma^{(0)} = -\frac{1}{2} \phi \left( \Box + m^2 \right) \phi \]

(C.9)

by integrating $J^{(0)} = -\frac{\delta \Gamma^{(0)}}{\delta \phi}$. From (C.3) and (C.4), the inversion formula of order $\lambda$, (2.26), becomes

\[ \qquad = 0 \]  

(C.10)

by noting that $\phi^{(0)} = (\Box + m^2)^{-1}$, which is denoted by a line. The integration of $J^{(1)} = -\frac{\delta \Gamma^{(1)}}{\delta \phi}$ leads to

\[ \Gamma^{(1)} = \quad + \quad . \]  

(C.11)

By (C.6) we confirm that $\Gamma^{(1)}$ is a functional of $\phi$ indeed. Equation (C.9) and the first term in (C.11) constitute the usual tree part of the 1PI (1-particle-irreducible) effective action. From (C.3) to (C.4), the second order formula (2.27) is written as

\[ \quad + \quad + \quad \]

(C.12)
The second term in (2.27) disappears because $\phi(0)''[J(0)] = 0$. Using (C.10) we see that 1-particle-reducible (1PR) graphs in (C.12) exactly cancel out each other to get

$$\Gamma^{(2)} + \text{graphs as in (C.12)} = 0$$

from which we obtain

$$\Gamma^{(2)} = \text{graphs as in (C.14)}.$$  

This course of study can be continued up to desired order to give the well-known result;

$$\Gamma = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{\lambda}{4!}\phi^4 + K_{\text{1PI}}[\phi]$$  

where $K_{\text{1PI}}[\phi]$ is 1PI vacuum graph $K_{\text{1PI}}[(\Box + m^2)^{-1}J]$ (written in terms of original $J$-representation) but with $(\Box + m^2)^{-1}J$ replaced by $\phi$ or

$$K_{\text{1PI}}[\phi] = \text{graphs as in (C.16)}.$$ 

with the notation (C.6). We note here that without using (C.3) to (C.5) we can directly obtain (C.10), (C.12) and higher orders if we note the equation corresponding to (2.10). This point is taken in the following case of $\langle \varphi(x)\varphi(y) \rangle$. It is easy to convince oneself that if one uses $(\Box + m^2 + \lambda\phi^2/2)^{-1}$ instead of $(\Box + m^2)^{-1}$ then the result of (B) is obtained.

**Case of $\langle \varphi(x)\varphi(y) \rangle$**

Now we consider the effective action of the bilocal composite operator. The generating functional $W[J]$ in this case is defined as in (2.1) with $S[\varphi, J]$ replaced by

$$S[\varphi, J] = -\frac{1}{2}\int d^4 x \varphi(x)(\Box + m^2)\varphi(x) - \frac{\lambda}{4!}\int d^4 x \varphi(x)^4 + \frac{1}{2}\int d^4 x d^4 y J(x, y) \varphi(x)\varphi(y)$$

$$\equiv -\frac{1}{2}\varphi G^{-1}_J \varphi - \frac{\lambda}{4!}\varphi^4,$$  

$$G^{-1}_J \equiv G^{-1}_J(x, y) = (\Box + m^2)\delta^4(x - y) - J(x, y).$$  

(C.17)  

(C.18)
Note here that $J(x, y)$ has been absorbed in the propagator $G_J$. We define $\phi(x, y)$ and $\Gamma[\phi]$ by

$$\phi(x, y) = \frac{\delta W}{\delta J(x, y)} \equiv \frac{1}{2} \langle \varphi(x)\varphi(y) \rangle,$$

(C.19)

$$\Gamma[\phi] = W[J] - \int d^4x d^4y J(x, y) \phi(x, y).$$

(C.20)

Then the 0-th order inversion formula (2.25), which is directly obtained as the 0-th order of the relation corresponding to (2.10), gives (with the space-time index omitted)

$$\phi = 0.$$  

(C.21)

The line denotes the propagator $G$ evaluated at $J = J^{(0)}$, namely,

$$\phi = \frac{1}{i} \frac{1}{\Box + m^2 - J^{(0)}} \equiv \frac{1}{i} G^{(0)}.$$  

(C.22)

Unlike the local case the key point is that this relation can be explicitly inverted to give $J^{(0)}$ (compare with (2.16)), that is,

$$J^{(0)} = \Box + m^2 + i\phi^{-1},$$

(C.23)

which gives, by integration,

$$\Gamma^{(0)} = \text{Tr} \left( \Box + m^2 \right) \phi + i\text{Tr} \ln \phi.$$  

(C.24)

Eq. (2.26) or the inversion formula of order $\lambda$, which is obtained by the first order of the equation like (2.10), gives

$$\underline{J^{(1)}} \underline{\Box} = 0$$

(C.25)

or, through integration,

$$\Gamma^{(1)} = \Box \Box \Box.$$  

(C.26)

We have used the notation in which $\underline{J^{(1)}}$ stands for $G_{x}^{(0)} J_{x}^{(1)} G_{w}^{(0)}$ where $\frac{1}{i} G^{(0)} = \phi$ (see (C.22)). From (C.26) we make sure that $\Gamma^{(1)}$ is actually a functional of the bilocal variable
\( \phi \) because lines in the graphs represent \( \phi \). The second order formula (2.27) given by the equation like (2.10) is written as,

\[
J^{(2)} + J^{(1)} J^{(1)} + J^{(1)} J^{(1)} + J^{(1)} = 0.
\]  
(C.27)

Using (C.25) we see that the 1 or 2-particle-reducible (2PR) graphs in (C.27) exactly cancel out to give

\[
J^{(2)}(x, y) = x \bigotimes y
\]  
(C.28)

or

\[
\Gamma^{(2)}(x, y) = \bigotimes .
\]  
(C.29)

As in the case of \( \langle \varphi(x) \rangle \) we can continue the process and get the well-known result;

\[
\Gamma = \text{Tr} \left( \square + m^2 \right) \phi + i \text{Tr} \ln \phi + \mathcal{K}_{2PI}[\phi]
\]  
(C.30)

where \( \mathcal{K}_{2PI}[\phi] \) is the original 2PI graph \( \mathcal{K}_{2PI}[\frac{1}{\square + m^2}] \) with \( \frac{1}{\square + m^2} \) replaced by \( \phi \) or

\[
\mathcal{K}_{2PI}[\phi] = \bigotimes + \bigotimes + \cdots.
\]  
(C.31)

**Appendix D — Path-integral formula for the fermion coherent state**

In this appendix we derive (3.11) from (3.1). In order to clarify the notations, we first enumerate some formulae for the fermionic coherent state in the case of a single mode. The generalization to multi-mode is straightforward. For the anti-commuting operator \( a, a^\dagger \) like (3.6), the coherent state is defined as

\[
a|z\rangle = z|z\rangle, \quad \langle z|a^\dagger = \langle z|z^*
\]  
(D.1)

where \( z \) and \( z^* \) are Grassmann numbers. Then inner product of the two states becomes

\[
\langle z|z'\rangle = e^{z^* z'}
\]  
(D.2)
which means that the coherent state is neither normalized nor orthogonalized. The matrix element in the coherent state is
\[
\langle z | O(a\dagger, a) | z' \rangle = O(z^*, z') e^{z^* z'}
\]  
where O is a normal-ordered operator. The over-completeness is expressed as
\[
\int dz^* dz e^{-z^* z} |z\rangle \langle z| = 1.
\]  
The trace of a normal-ordered operator becomes
\[
\text{Tr} O(a\dagger, a) = \int dz^* dz e^{-z^* z} \langle -z | O(a\dagger, a) | z \rangle.
\]  
In order to derive (3.11), we first estimate
\[
\langle z_F | T_\tau e^{-\int d\tau (t_{\alpha\beta} a_\alpha^\dagger a_\beta + \mathcal{V}(a_\gamma^\dagger a_\gamma))} | z_I \rangle e^{-z_F^* z_F}\text{.}
\]  
Here $\mathcal{V}$ is the on-site Coulomb term and the source term appearing in (3.3) to (3.5) and $\mathcal{V}(a_\gamma^\dagger a_\gamma)$, $z_I$ and $z_F$ are abbreviations of $\mathcal{V}(\{a_\alpha^\dagger\}, \{a_\gamma\})$, $\{z_{I\gamma}\}$ and $\{z_{F\gamma}\}$ respectively. As usual we divide the exponential into $N + 1$ pieces and insert $N$ multi-mode complete sets like (D.4). We get
\[
\left( \prod_{i=1}^N \prod_{\alpha} dz_{i\alpha}^* dz_{i\alpha} \right) e^{-\epsilon \sum_{i=1}^{N+1} z_{i\alpha}^* z_{i\alpha} e^{-\epsilon \sum_{i=1}^{N+1} z_{i\alpha}^* z_{i-1,\alpha} e^{-\epsilon \sum_{i=1}^{N+1} \{t_{\alpha\beta} z_{i\alpha}^* z_{i-1,\beta} + \mathcal{V}(z_{i\gamma}^* z_{i-1,\gamma})\}}}
\]  
where $\epsilon = \beta / (N + 1)$, $z_{0\alpha} = z_{I\alpha}$, $z_{N+1,\alpha} = z_{F\alpha}$ and we have assumed $\mathcal{V}$ is normal-ordered. The first two exponential can be formally written as
\[
e^{-\epsilon \sum_{i=1}^{N+1} z_{i\alpha}^* (z_{i\alpha} - z_{i-1,\alpha}) / \epsilon} \to e^{-\int_0^\beta d\tau z_\alpha^* (\tau) z_\alpha (\tau)}\text{.}
\]  
In this way, through the trace formula (D.5), we obtain the path-integral representation of (3.1), arriving at (3.11).

References

[1] See for example, J. Iliopoulos, C. Itzykson and A. Martin, Rev. Mod. Phys. 47 (1975) 165;
R. W. Haymaker, Rivista del Nuovo Cimento 14, serie 3, No. 8 (1991);
J. Negele and H. Orland, Quantum Many-Particle Systems (Addison-Wesley, New York,
G. Parisi, Statistical Field Theory (Addison-Wesley, New York, 1988).
For a general reference, see R. Fukuda et al., Novel use of Legendre transformation in quantum field theory and many particle systems, Prog. Theor. Phys. Suppl. (to be published) and references cited therein.

[2] C. De Dominicis and P. C. Martin, J. Math. Phys. 5 (1964) 14, 31

[3] G. Jona-Lasinio, Nuovo Cimento 34 (1964) 1790

[4] A. N. Vasil'ev and A. K. Kazanskii, Theor. Math. Phys. 12 (1972) 875; 14 (1973) 215;
   Yu. M. Pis'mak, Theor. Math. Phys. 18 (1974) 211;
   A. N. Vasil'ev, A. K. Kazanskii, and Yu. M. Pis'mak, Theor. Math. Phys. 19 (1974) 443;
   20 (1974) 754

[5] R. Jackiw, Phys. Rev. D9 (1974) 1686

[6] John M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D10 (1974) 2428;
   R. Fukuda and E. Kyriakopoulos, Nucl. Phys. B85 (1975) 354

[7] R. L. Stratonovich, Doklady Akad. Nauk S.S.S.R. 115 (1957) 1097 [translation: Soviet
   Phys. Doklady 2 (1958) 416];
   J. Hubbard, Phys. Rev. Lett. 3 (1959) 77

[8] D. J. Gross and A. Noveu, Phys. Rev. D10 (1974) 3235

[9] R. Fukuda, Phys. Rev. Lett. 61 (1988) 1549

[10] M. Ukita, M. Komachiya, and R. Fukuda, Int. J. Mod. Phys. A5 (1990) 1789

[11] T. Inagaki and R. Fukuda, Phys. Rev. B46 (1992) 10931

[12] M. Ukita, doctoral thesis (Keio University).

[13] K. Okumura, in preparation.

[14] K. Okumura, Prog. Theor. Phys. 87 (1992) 703
[15] T. Moriya and A. Kawabata, J. Phys. Soc. Jpn. 34 (1973) 639; 35 (1973) 669

[16] R. Fukuda, Prog. Theor. Phys. 78 (1987) 1487;
    R. Fukuda, M. Komachiya, and M. Ukita, Phys. Rev. D38 (1988) 3747;
    M. Komachiya, M. Ukita, and R. Fukuda, Phys. Rev. D42 (1990) 2792
See also [1] and references therein.
Fig.1: The self-contraction of the pseudo-vertex.

Fig.2: The self-contraction of the 4-point-vertex.

Fig.3: An example of 1VR graph that is canceled in $\Gamma^{(5)}$.

Fig.4: An example of the graph of $N(\tilde{K}) = 4$.

Fig.5: The 1-part (encircled by the dashed line) in a graph of $\phi$.

Fig.6: The procedure to reach the second largest structure.

(a) proceed to a larger 1-part.
(b) reach the second largest 1-part.
(c) reach the second largest 1-parts.
(d) reach the second largest (1-part) structure.