Partition Function and Open/Closed String Duality in Type IIA String Theory on a PP-wave

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Abstract

We discuss partition functions of $\mathcal{N} = (4,4)$ type IIA string theory on the pp-wave background. This theory is shown to be modular invariant. The boundary states are constructed and possible D-brane instantons are classified. Then we calculate cylinder amplitudes in both closed and open string descriptions and check the open/closed string duality. Furthermore we consider general properties of modular invariant partition functions in the case of pp-waves.

Keywords: pp-wave, modular invariance, boundary state, D-brane, open/closed string duality
1 Introduction

Recently, superstring theories on pp-waves have been very focused upon. The maximally supersymmetric pp-wave type solution in eleven dimensions [1] has been known for a long time while the maximally supersymmetric type IIB pp-wave solution was found in [2] in the recent progress. It was also pointed out in [3] that this pp-wave background is related to the $AdS$-geometry via the Penrose limit [4, 5]. Then, the type IIB superstring theory on the maximally supersymmetric pp-wave background was constructed [6, 7]. By using this pp-wave superstring theory, the study of $AdS$/CFT correspondence has greatly proceeded [8]. In particular, the $AdS$/CFT correspondence has been studied at the stringy level beyond the supergravity analysis.

In the study of pp-wave backgrounds, the matrix model on the pp-wave, which was proposed by Berenstein-Maldacena-Nastase [8], has been much studied. This matrix model is closely related to a supermembrane theory on the pp-wave background [9, 10]. We have discussed the supermembrane theory and matrix model on the pp-wave from the several aspects [10–14]. In particular, we showed the correspondence of brane charges in the supermembrane theory [10] and matrix model [11] in the pp-wave case as well as in flat space [15].

A supermembrane in eleven dimensions is related to a string in ten dimensions via the double dimensional reduction. We constructed the type IIA pp-wave background with 24 supersymmetries and string theory on this pp-wave background [16, 17], which is called $\mathcal{N} = (4,4)$ type IIA string theory on the pp-wave. We discussed the classification of the allowed D-branes [16, 18] by following the work of Dabholkar and Parvizi [19]. After these works, the covariant classification of D-branes was done in [20] where D0-branes could be studied. In addition, the spectrum of this type IIA string theory was compared to fluctuations of the linearized type IIA supergravity around the pp-wave background [21]. The thermodynamics of this type IIA string theory was recently studied in [22] *. On the other hand, as an important and interesting subject, the modular invariance of string theories on pp-waves has been studied by several authors [22–28]. It has been turned out that these theories are modular invariant in spite of mass terms in the action. In this paper, motivated by the previous works, we will be interested in the $\mathcal{N} = (4,4)$ type IIA string theory on the pp-wave background obtained in [16, 17] and study its partition function and open/closed string duality. The pp-wave background we will consider is not maximally

*Section 2 has some overlap with the work [22]. Thermal partition function is discussed in the cases of other pp-wave strings [23, 24].
supersymmetric but has 24 supersymmetries. Hence it is interesting to study whether the modular invariance and consistency condition between open and closed strings are satisfied or not in such less supersymmetric case. Moreover, since the number of preserved supersymmetries is nontrivial even in the case of supersymmetric D-branes, it is also interesting to construct boundary states in our model†.

In this paper we discuss partition functions in the $\mathcal{N} = (4,4)$ type IIA string theory on the pp-wave. We first show that our theory is modular invariant. The boundary states is constructed and then we classify the allowed D-brane instantons. They are 1/2 BPS states (preserving 12 supersymmetries) at the origin and 1/3 BPS ones (preserving 8 supersymmetries) away from the origin. This result is consistent with the classification of D-branes. Then we calculate the cylinder amplitude in the closed and open string descriptions and study the open/closed string channel duality in our theory. Finally, we discuss general properties of modular invariant partition function and classify several models.

This paper is organized as follows: In section 2 we prove the modular invariance of $\mathcal{N} = (4,4)$ type IIA string theory on the pp-wave background. The Witten index of this theory is shown to be one. Section 3 is devoted to a brief review about the supersymmetries of our theory. In section 4 we construct boundary states and classify D-brane instantons. In section 5 we calculate the cylinder amplitude in the closed string description. In section 6 the amplitude is derived in terms of open string and the open/closed string channel duality in our theory is proven. From the channel duality, the normalization factor of the boundary state is determined. In section 7, based on the result of section 2, we discuss several properties of modular invariant partition function in some general setup. Finally, section 8 is devoted to conclusions and discussions.

2 Modular Invariance of Type IIA String Theory

In this section we will discuss the modular invariance of type IIA string theory in the closed string description.

†We note that, contrary to the present type IIA case, the boundary states in the type IIB string theory on the pp-wave background have been relatively much studied [25, 28–30].
The action of our type IIA string in the light-cone gauge is given by

\[
S_{\text{closed}} = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi} d\sigma \left[ \sum_{i=1}^8 \partial_+ x^i \partial_- x^i - \left( \frac{\mu}{3} \right)^2 \sum_{a=1}^4 (x^a)^2 - \left( \frac{\mu}{6} \right)^2 \sum_{b=5}^8 (x^b)^2 \right] + \frac{i}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \left[ \Psi^{1+} \partial_- \Psi^{1+} + \Psi^{1-} \partial_- \Psi^{1-} + \Psi^{2+} \partial_+ \Psi^{2+} + \Psi^{2-} \partial_+ \Psi^{2-} - \frac{\mu}{3} \Psi^{1-} \Pi^+ \Psi^{2+} + \frac{\mu}{3} \Psi^{2+} \Pi^+ \Psi^{1-} - \frac{\mu}{6} \Psi^{1+} \Pi^+ \Psi^{2-} + \frac{\mu}{6} \Psi^{2-} \Pi^+ \Psi^{1+} \right],
\]

where \(\alpha'\) is a string tension and we have set \(p^+ = 1\). The \(\gamma^r\)'s are \(16 \times 16 SO(9)\) gamma matrices and we defined \(\Pi \equiv \gamma^{1234}\) (\(\Pi^+ \equiv \gamma^{321}\)). Each of the spinors \(\Psi^{i\pm}\) \((i = 1, 2)\) has four independent components and the superscript \(\pm\) represents the chirality measured by \(\gamma^{1234}\): \(\gamma^{1234} \Psi^{i\pm} = \pm 1 \cdot \Psi^{i\pm}\). This theory has 24 supersymmetries (8 dynamical supersymmetries and 16 kinematical supersymmetries). The equations of motion are described by

\[
\partial_+ \partial_- x^a + \frac{\mu^2}{9} x^a = 0 \quad (a = 1, 2, 3, 4),
\]

\[
\partial_+ \partial_- x^b + \frac{\mu^2}{36} x^b = 0 \quad (b = 5, 6, 7, 8),
\]

\[
\partial_+ \Psi^{2+} + \frac{\mu}{3} \Pi^+ \Psi^{1-} = 0, \quad \partial_- \Psi^{1-} - \frac{\mu}{3} \Pi^+ \Psi^{2+} = 0,
\]

\[
\partial_+ \Psi^{2-} + \frac{\mu}{6} \Pi^+ \Psi^{1+} = 0, \quad \partial_- \Psi^{1+} - \frac{\mu}{6} \Pi^+ \Psi^{2-} = 0.
\]

By solving the equations of motion (2.2) and (2.3), we can obtain the mode-expansions of bosonic variables represented by

\[
x^a(\tau, \sigma) = x^a_0 \cos \left( \frac{\mu}{3} \tau \right) + \left( \frac{3}{\mu} \right) \alpha' p^a_0 \sin \left( \frac{\mu}{3} \right) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{\omega_n} \left[ \alpha^a_n \phi_n + \bar{\alpha}^a_n \tilde{\phi}_n \right],
\]

\[
x^b(\tau, \sigma) = x^b_0 \cos \left( \frac{\mu}{6} \tau \right) + \left( \frac{6}{\mu} \right) \alpha' p^b_0 \sin \left( \frac{\mu}{6} \right) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{\omega'_n} \left[ \alpha^b_n \phi'_n + \bar{\alpha}^b_n \tilde{\phi}'_n \right].
\]

From the equations of motion (2.4) and (2.5), the mode-expansions of fermionic variables are
represented by

\[
\Psi^-(\tau, \sigma) = \Pi^T \Psi_0 \sin \left( \frac{\mu}{3} \tau \right) - \Pi^T \tilde{\Psi}_0 \cos \left( \frac{\mu}{3} \tau \right) + \sum_{n \neq 0} c_n \left( \frac{3}{\mu} i (\omega_n - n) \Pi^T \Psi_n \phi_n + \tilde{\Psi}_n \phi_n \right),
\]

\[\text{(2.8)}\]

\[
\Psi^+(\tau, \sigma) = \Psi_0 \cos \left( \frac{\mu}{3} \tau \right) + \tilde{\Psi}_0 \sin \left( \frac{\mu}{3} \tau \right) + \sum_{n \neq 0} c_n \left[ \Psi_n \phi_n - \frac{3}{\mu} i (\omega_n - n) \Pi \tilde{\Psi}_n \phi_n \right],
\]

\[\text{(2.9)}\]

\[
\Psi^0(\tau, \sigma) = \Pi^T \Psi_0' \sin \left( \frac{\mu}{6} \tau \right) - \Pi^T \tilde{\Psi}_0' \cos \left( \frac{\mu}{6} \tau \right)
\]

\[\text{(2.10)}\]

\[
\Psi^0(\tau, \sigma) = \Psi_0' \cos \left( \frac{\mu}{6} \tau \right) \tilde{\Psi}_0' \sin \left( \frac{\mu}{6} \tau \right)
\]

\[\text{(2.11)}\]

\[
\sum_{n \neq 0} c_n' \left[ \Psi_n' \phi_n' - \frac{6}{\mu} i (\omega_n' - n) \Pi \tilde{\Psi}_n \phi_n \right].
\]

Here we have introduced several notations:

\[
\omega_n = \text{sgn}(n) \sqrt{n^2 + \left( \frac{\mu}{3} \right)^2}, \quad \omega_n' = \text{sgn}(n) \sqrt{n^2 + \left( \frac{\mu}{6} \right)^2},
\]

\[
\phi_n = \exp \left( -i (\omega_n \tau - n \sigma) \right), \quad \tilde{\phi}_n = \exp \left( -i (\omega_n \tau + n \sigma) \right),
\]

\[
\phi_n' = \exp \left( -i (\omega_n' \tau - n \sigma) \right), \quad \tilde{\phi}_n' = \exp \left( -i (\omega_n' \tau + n \sigma) \right),
\]

\[
c_n = \left( 1 + \left( \frac{3}{\mu} \right)^2 (\omega_n - n)^2 \right)^{-1/2}, \quad c_n' = \left( 1 + \left( \frac{6}{\mu} \right)^2 (\omega_n' - n)^2 \right)^{-1/2}.
\]

Now we shall quantize the theory by imposing (anti)commutation relations. The commutation relations for bosonic modes are given by

\[
[x^i_0, p^j_0] = i \delta^{ij}, \quad [\tilde{\alpha}^i_m, \alpha^j_n] = [\alpha^i_m, \tilde{\alpha}^j_n] = 0 \quad (i, j = 1, \ldots, 8),
\]

\[
[\alpha^a_m, \alpha^b_n] = \left[ \tilde{\alpha}^a_m, \tilde{\alpha}^b_n \right] = \omega_m \delta_{m+n,0} \delta^{aa'} \quad (a, a' = 1, 2, 3, 4),
\]

\[
[\alpha^a_m, \alpha^b_n] = \left[ \tilde{\alpha}^b_m, \tilde{\alpha}^b_n \right] = \omega'_m \delta_{m+n,0} \delta^{bb'} \quad (b, b' = 5, 6, 7, 8),
\]

and the anticommutation relations for fermionic modes are written as

\[
\{(\Psi_m)_\alpha, (\tilde{\Psi}_n)_\beta\} = \{(\tilde{\Psi}_m)_\alpha, (\Psi_n)_\beta\} = 0,
\]

\[
\{(\Psi_m)_\alpha, (\Psi_n)_\beta\} = \{(\tilde{\Psi}_m)_\alpha, (\tilde{\Psi}_n)_\beta\} = \frac{1}{2} \delta_{m+n,0} \delta_{\alpha\beta},
\]

\[
\{(\Psi'_m)_\alpha, (\tilde{\Psi}'_n)_\beta\} = \{(\tilde{\Psi}'_m)_\alpha, (\Psi'_n)_\beta\} = 0,
\]

\[
\{(\Psi'_m)_\alpha, (\Psi'_n)_\beta\} = \{(\tilde{\Psi}'_m)_\alpha, (\tilde{\Psi}'_n)_\beta\} = \frac{1}{2} \delta_{m+n,0} \delta_{\alpha\beta}.
\]
Now let us introduce the annihilation and creation operators:

\[
a_0^a \equiv \sqrt{\frac{\alpha'}{2}} \sqrt{\frac{3}{\mu}} \left( p_0^a - i \frac{\mu}{3\alpha'} x_0^a \right), \quad a_0^{\dagger a} \equiv \sqrt{\frac{\alpha'}{2}} \sqrt{\frac{3}{\mu}} \left( p_0^{\dagger a} + i \frac{\mu}{3\alpha'} x_0^{\dagger a} \right) \quad (a = 1, 2, 3, 4),
\]

\[
a_n^a \equiv \frac{1}{\sqrt{\omega_n}} a_n^a, \quad a_n^{\dagger a} \equiv \frac{1}{\sqrt{\omega_n}} a_n^{\dagger a}, \quad \bar{a}_n^a \equiv \frac{1}{\sqrt{\omega_n}} \bar{a}_n^a, \quad \bar{a}_n^{\dagger a} \equiv \frac{1}{\sqrt{\omega_n}} \bar{a}_n^{\dagger a} \quad (n > 0),
\]

for the sector with mass \( \mu/3 \) and

\[
a_0^b \equiv \sqrt{\frac{\alpha'}{2}} \sqrt{\frac{6}{\mu}} \left( p_0^b - i \frac{\mu}{6\alpha'} x_0^b \right), \quad a_0^{\dagger b} \equiv \sqrt{\frac{\alpha'}{2}} \sqrt{\frac{6}{\mu}} \left( p_0^{\dagger b} + i \frac{\mu}{6\alpha'} x_0^{\dagger b} \right) \quad (b = 5, 6, 7, 8),
\]

\[
a_n^b \equiv \frac{1}{\sqrt{\omega_n}} a_n^b, \quad a_n^{\dagger b} \equiv \frac{1}{\sqrt{\omega_n}} a_n^{\dagger b}, \quad \bar{a}_n^b \equiv \frac{1}{\sqrt{\omega_n}} \bar{a}_n^b, \quad \bar{a}_n^{\dagger b} \equiv \frac{1}{\sqrt{\omega_n}} \bar{a}_n^{\dagger b} \quad (n > 0),
\]

for the sector with mass \( \mu/6 \), and those for fermionic variables:

\[
S_0 = \Psi_0 + i \bar{\Psi}_0, \quad S_0^\dagger = \Psi_0 - i \bar{\Psi}_0,
\]

\[
S_n = \sqrt{2}\Psi_n, \quad S_n^\dagger = \sqrt{2}\bar{\Psi}_n, \quad \bar{S}_n = \sqrt{2}\bar{\Psi}_n, \quad \bar{S}_n^\dagger = \sqrt{2}\Psi_n \quad (n > 0),
\]

Then the commutation relations are rewritten as

\[
[a_n^a, a_{n'}^{\dagger a}] = \delta^{aa'} \delta_{m,n}, \quad [\bar{a}_n^a, \bar{a}_{n'}^{\dagger a}] = \delta^{aa'} \delta_{m,n} \quad (a, a' = 1, 2, 3, 4),
\]

\[
[a_n^b, a_{n'}^{\dagger b}] = \delta^{bb'} \delta_{m,n}, \quad [\bar{a}_n^b, \bar{a}_{n'}^{\dagger b}] = \delta^{bb'} \delta_{m,n} \quad (b, b' = 5, 6, 7, 8),
\]

and the anticommutation relations are given by

\[
\{(S_m)_\alpha, (S_n^\dagger)_\beta\} = \delta_{\alpha\beta} \delta_{m,n}, \quad \{(\bar{S}_m)_\alpha, (\bar{S}_n^\dagger)_\beta\} = \delta_{\alpha\beta} \delta_{m,n},
\]

\[
\{(S'_m)_\alpha, (S'_n^\dagger)_\beta\} = \delta_{\alpha\beta} \delta_{m,n}, \quad \{(\bar{S}'_m)_\alpha, (\bar{S}'_n^\dagger)_\beta\} = \delta_{\alpha\beta} \delta_{m,n}.
\]

By using the above (anti)commutation relations, we can represent the Hamiltonian and momentum in terms of creation and annihilation operators as follows:

\[
H = \sum_{n=-\infty}^{\infty} (\omega_n N_n + \omega'_n N'_n), \quad P = \sum_{n=-\infty}^{\infty} (n N_n + n N'_n). \quad (2.12)
\]

where \( N_n \) and \( N'_n \) are defined by

\[
N_0 \equiv \sum_{a=1}^{4} a_0^{\dagger a} a_0^a + S_0^\dagger S_0, \quad N'_0 \equiv \sum_{b=5}^{8} a_0^{\dagger b} a_0^b + S'_0^\dagger S'_0,
\]

\[
N_n \equiv \sum_{a=1}^{4} a_n^{\dagger a} a_n^a + S_n^\dagger S_n, \quad N'_n \equiv \sum_{b=5}^{8} a_n^{\dagger b} a_n^b + S'_n^\dagger S'_n \quad (n > 0),
\]

\[
N_{-n} \equiv \sum_{a=1}^{4} \bar{a}_n^{\dagger a} \bar{a}_n^a + \bar{S}_n^\dagger \bar{S}_n, \quad N'_{-n} \equiv \sum_{b=5}^{8} \bar{a}_n^{\dagger b} \bar{a}_n^b + \bar{S}'_n^\dagger \bar{S}'_n \quad (n > 0).
\]
Now we shall introduce the Casimir Energy defined by
\[ \Delta(\nu; a) \equiv \frac{1}{2} \sum_{n \in \mathbb{Z}} \sqrt{\nu^2 + (n + a)^2} - \frac{1}{2} \int_{-\infty}^{+\infty} dk \sqrt{\nu^2 + k^2} \]
\[ = -\frac{1}{2\pi^2} \sum_{\ell \geq 1} \cos(2\pi a \ell) \int_0^\infty ds e^{-\nu^2 + \frac{\ell^2}{s}} = -\frac{1}{2} \int_0^\infty ds e^{-\nu^2} (\theta_3(i\pi s, a) - 1). \tag{2.13} \]

Then we can express the vacuum energy for eight bosons as
\[ E_0^B = \sum_{a=1}^{4} \left( \frac{1}{2} \sum_{n \in \mathbb{Z}} \omega_n \right) + \sum_{b=5}^{8} \left( \frac{1}{2} \sum_{n \in \mathbb{Z}} \omega'_n \right) \cong 4(\Delta(\mu/3; 0) + \Delta(\mu/6; 0)), \tag{2.14} \]
and that for fermions as
\[ E_0^F = -4 \left( \frac{1}{2} \sum_{n \in \mathbb{Z}} \omega_n + \frac{1}{2} \sum_{n \in \mathbb{Z}} \omega'_n \right) \cong -4 [\Delta(\mu/3; 0) + \Delta(\mu/6; 0)], \tag{2.15} \]
where the symbol \( \cong \) means the equality after the regularization of zero-point energies and the factor 4 appears since each fermion considered here has four independent components.

Here let us evaluate the toroidal partition function:
\[ Z = \text{Tr} \left[ (-1)^F e^{-2\pi \tau_2 H + 2\pi i \tau_1 P} \right], \tag{2.16} \]
where \( \tau_1 \) and \( \tau_2 \) are modular parameters and \( F \) is the fermion number operator.

First we will consider a partition function for one boson with mass \( \nu \). The number operator, Hamiltonian and momentum are given by
\[ N_0 = a_0^\dagger a_0 , \quad N_n = a_n^\dagger a_n , \quad N_{-n} = a_n^\dagger a_0 \ (n > 0) , \]
\[ H = \omega_0 N_0 + \sum_{n=1}^{\infty} (\omega_n N_n + \omega_n N_{-n}) , \quad P = \sum_{n=1}^{\infty} (n N_n - n N_{-n}) , \]
and so we can obtain the partition function:
\[ Z = \left[ \Theta_{(0,0)}(\tau, \bar{\tau}; \nu) \right]^{-1/2} , \tag{2.17} \]
where we have introduced the ‘massive’ theta function defined by
\[ \Theta_{(a,b)}(\tau, \bar{\tau}; \nu) \equiv e^{4\pi \tau_2 \Delta(\nu; a)} \prod_{n \in \mathbb{Z}} \left| 1 - e^{-2\pi \tau_2 \sqrt{\nu^2 + (n+a)^2} + 2\pi \tau_1 (n+a) + 2\pi ib} \right|^2. \tag{2.18} \]

Next we consider a partition function for single component fermion. The number operator, Hamiltonian and momentum are given by
\[ N_0 = S_0^\dagger S_0 , \quad N_n = S_n^\dagger S_n , \quad N_{-n} = S_n^\dagger S_0 \ (n > 0) , \]
\[ H = \omega_0 N_0 + \sum_{n>0} (\omega_n N_n + \omega_n N_{-n}) , \quad P = \sum_{n>0} (n N_n - n N_{-n}) , \]
and hence we can obtain the partition function:

\[ Z = \left[ \Theta_{(0,0)}(\tau, \bar{\tau}; \nu) \right]^{+1/2}. \] (2.19)

If we recall the field contents of our model:

\[
\begin{aligned}
\text{bosons} & \quad \left\{ \begin{array}{ll}
x^a & (a = 1, 2, 3, 4) \\
x^b & (b = 5, 6, 7, 8)
\end{array} \right. \\
\text{fermions} & \quad \left\{ \begin{array}{ll}
(S, \bar{S}) & \text{(mass } \mu/3 \text{ sector)} \\
(S', \bar{S'}) & \text{(mass } \mu/6 \text{ sector)}
\end{array} \right.
\end{aligned}
\] (2.20)

the bosonic and fermionic partition functions \( Z_B \) and \( Z_F \) are given by

\[
\begin{aligned}
Z_B & = \left[ \Theta_{(0,0)}(\tau, \bar{\tau}; \mu/3) \right]^{-2} \left[ \Theta_{(0,0)}(\tau, \bar{\tau}; \mu/6) \right]^{-2}, \\
Z_F & = \left[ \Theta_{(0,0)}(\tau, \bar{\tau}; \mu/3) \right]^{+2} \left[ \Theta_{(0,0)}(\tau, \bar{\tau}; \mu/6) \right]^{+2},
\end{aligned}
\]

and hence the total partition function is given by

\[ Z = Z_B \cdot Z_F = 1. \] (2.21)

Thus we have shown that our theory is modular invariant at the one-loop level since the total partition function \( Z \) of (2.21) is independent of the modular parameters \( \tau \) and \( \bar{\tau} \). As will be shown in more detail with some generality in the section 7, this result implies that the Witten index is one as in the cases of other string theories on pp-waves. It should be remarked that our type IIA string theory is modular invariant in the sector with mass \( \mu/3 \) and in that with \( \mu/6 \), respectively.

3 Supersymmetries of Type IIA String Theory

In this section we will briefly review about \( \mathcal{N} = (4, 4) \) supersymmetries of the type IIA string theory on the pp-wave background \([17, 18]\), according to which the world-sheet variables are arranged into two supermultiplets, \((x^a, \Psi^{1-}, \bar{\Psi}^{2+})\) and \((x^b, \Psi^{1+}, \bar{\Psi}^{2-})\). Then we will rewrite the supercharges in terms of modes in order to construct the boundary states in section 4.

This type IIA string theory on the pp-wave background has 24 supersymmetries, among which 8 are dynamical and 16 are kinematical. The dynamical supersymmetry transformation laws for the multiplet \((x^a, \Psi^{1-}, \bar{\Psi}^{2+})\) with mass \( \mu/3 \) are given by

\[
\begin{aligned}
\delta x^a & = 2i\alpha' \left( \Psi^{1-} \gamma^a \epsilon^{1+} + \bar{\Psi}^{2+} \gamma^a \epsilon^{2-} \right), \\
\delta \Psi^{1-} & = \partial_+ x^a \gamma^a \epsilon^{1+} + \frac{\mu}{3} x^a \gamma^4 \epsilon^{2-}, \\
\delta \bar{\Psi}^{2+} & = \partial_- x^a \gamma^a \epsilon^{2-} - \frac{\mu}{3} x^a \gamma^4 \gamma^a \epsilon^{1+},
\end{aligned}
\] (3.1)
and those for the multiplet \((x^b, \Psi^1, \Psi^2)\) with mass \(\mu/6\), are written as
\[
\delta x^b = 2i\alpha' (\Psi^{1+T} \gamma^b \epsilon^{2-} + \Psi^{2-T} \gamma^b \epsilon^{1+}) ,
\]
\[
\delta \Psi^{1+} = \partial_+ x^b \gamma^b \epsilon^{2-} + \frac{\mu}{6} x^b \gamma^4 \gamma^b \epsilon^{1+} , \quad \delta \Psi^{2-} = \partial_+ x^b \gamma^b \epsilon^{1+} - \frac{\mu}{6} x^b \gamma^4 \gamma^b \epsilon^{2-} ,
\]
where the constant spinors \(\epsilon^{1+}\) and \(\epsilon^{2-}\) satisfy the following chirality conditions:
\[
\gamma^9 \epsilon^{1+} = +\epsilon^{1+} , \quad \gamma^{1234} \epsilon^{1+} = +\epsilon^{1+} , \quad \gamma^9 \epsilon^{2-} = -\epsilon^{2-} , \quad \gamma^{1234} \epsilon^{2-} = -\epsilon^{2-} ,
\]
respectively. As for the kinematical supersymmetry, the transformation laws are given by \(\tilde{\delta} x^a = \tilde{\delta} x^b = 0\) and
\[
\begin{align*}
\tilde{\delta} \Psi^{-1} &= \cos \left( \frac{\mu}{3} \right) \epsilon^{-1} - \sin \left( \frac{\mu}{3} \right) \gamma^{123} \epsilon^{2+} , \\
\tilde{\delta} \Psi^{2+} &= \cos \left( \frac{\mu}{3} \right) \epsilon^{2+} - \sin \left( \frac{\mu}{3} \right) \gamma^{123} \epsilon^{-1} , \\
\tilde{\delta} \Psi^{1+} &= \cos \left( \frac{\mu}{6} \right) \epsilon^{1+} - \sin \left( \frac{\mu}{6} \right) \gamma^{123} \epsilon^{2-} , \\
\tilde{\delta} \Psi^{2-} &= \cos \left( \frac{\mu}{6} \right) \epsilon^{2-} - \sin \left( \frac{\mu}{6} \right) \gamma^{123} \epsilon^{1+} ,
\end{align*}
\]
where the constant spinors \(\epsilon^{1+}\), \(\epsilon^{-1}\), \(\epsilon^{2+}\) and \(\epsilon^{2-}\) satisfy the chirality conditions in terms of \(\gamma^9\) and \(\gamma^{1234}\) in the same way as the dynamical supersymmetry case.

By the use of Noether’s theorem, we can construct the associated supercharges. Firstly the dynamical supercharges are obtained as
\[
\epsilon^T Q_{(\mu/3)} = \epsilon^{1+T} Q^{1+} + \epsilon^{2-T} Q^{2-} , \quad \epsilon^{i'T} Q_{(\mu/6)} = \epsilon^{2-T} Q'^{2-} + \epsilon^{1+T} Q'^{1+} .
\]
Here the \(Q^{1+}\) and \(Q^{2-}\) are the quantities defined as
\[
Q^{1+} \equiv -\frac{i}{2\pi} \int_0^{2\pi} d\sigma \left[ \partial_+ x^a \gamma^a \Psi^{-1} - \frac{\mu}{3} x^a \gamma^a \gamma^4 \Psi^{2+} \right] , \quad (3.7)
\]
\[
Q^{2-} \equiv -\frac{i}{2\pi} \int_0^{2\pi} d\sigma \left[ \partial_- x^a \gamma^a \Psi^{2+} + \frac{\mu}{3} x^a \gamma^a \gamma^4 \Psi^{-1} \right] , \quad (3.8)
\]
and, for the \(Q'^{2-}\) and \(Q'^{1+}\),
\[
Q'^{2-} \equiv -\frac{i}{2\pi} \int_0^{2\pi} d\sigma \left[ \partial_+ x^b \gamma^b \Psi^{1+} - \frac{\mu}{6} x^b \gamma^b \gamma^4 \Psi^{2-} \right] , \quad (3.9)
\]
\[
Q'^{1+} \equiv -\frac{i}{2\pi} \int_0^{2\pi} d\sigma \left[ \partial_- x^b \gamma^b \Psi^{2-} + \frac{\mu}{6} x^b \gamma^b \gamma^4 \Psi^{1+} \right] . \quad (3.10)
\]
Secondly, the kinematical supercharges are obtained as
\[
\tilde{Q}_{(\mu/3)} \equiv \epsilon^{1-T} \tilde{Q}^{-1} + \epsilon^{2+T} \tilde{Q}^{2+} , \quad \tilde{Q}_{(\mu/6)} \equiv \epsilon^{1+T} \tilde{Q}^{1+} + \epsilon^{2-T} \tilde{Q}^{2-} , \quad (3.11)
\]
where the $\tilde{Q}^1^-$ and $\tilde{Q}^2^+$ are defined by
\begin{align}
\tilde{Q}^1^- & \equiv i \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left[ \cos \left( \frac{\mu}{6} \tau \right) \Psi^1^- + \sin \left( \frac{\mu}{6} \tau \right) \gamma^{123} \Psi^{2^+} \right], \\
\tilde{Q}^2^+ & \equiv i \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left[ \cos \left( \frac{\mu}{6} \tau \right) \Psi^{2^+} + \sin \left( \frac{\mu}{6} \tau \right) \gamma^{123} \Psi^1^- \right],
\end{align}
and, for the $\tilde{Q}^1^-$ and $\tilde{Q}^2^+$,
\begin{align}
\tilde{Q}^1^+ & \equiv i \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left[ \cos \left( \frac{\mu}{6} \tau \right) \Psi^{1^+} + \sin \left( \frac{\mu}{6} \tau \right) \gamma^{123} \Psi^{2^-} \right], \\
\tilde{Q}^2^- & \equiv i \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left[ \cos \left( \frac{\mu}{6} \tau \right) \Psi^{2^-} + \sin \left( \frac{\mu}{6} \tau \right) \gamma^{123} \Psi^{1^+} \right].
\end{align}

Now we can rewrite the above supercharges in terms of creation and annihilation operators by inserting the mode-expansions of bosonic and fermionic degrees of freedom.
\begin{align}
\sqrt{\alpha'}^{-1} Q^{1^+} & = c_0 \sqrt{\frac{\mu}{3}} \left( a_0^\dagger \gamma^a \Pi^T S_0 - a_0^b \gamma^b \Pi^T S_0^\dagger \right) - i \sum_{n=1}^\infty c_n \sqrt{\omega_n} \left( \tilde{a}_n^\dagger \gamma^a \tilde{S}_n - \tilde{a}_n \gamma^a \tilde{S}_n^\dagger \right) \\
& \hspace{1cm} + \frac{1}{2} \cdot \frac{\mu}{3} \sum_{n=1}^\infty \frac{1}{c_n \sqrt{\omega_n}} \left( a_n^\dagger \gamma^a \Pi^T S_n - a_n \gamma^a \Pi^T S_n^\dagger \right), \\
\sqrt{\alpha'}^{-1} Q^{2^-} & = -i c_0 \sqrt{\frac{\mu}{3}} \left( a_0^\dagger \gamma^a S_0 + a_0^b \gamma^b S_0^\dagger \right) - i \sum_{n=1}^\infty c_n \sqrt{\omega_n} \left( \tilde{a}_n^\dagger \gamma^a \tilde{S}_n + \tilde{a}_n \gamma^a \tilde{S}_n^\dagger \right) \\
& \hspace{1cm} + \frac{1}{2} \cdot \frac{\mu}{3} \sum_{n=1}^\infty \frac{1}{c_n \sqrt{\omega_n}} \left( a_n^\dagger \gamma^a \Pi^T S_n - a_n \gamma^a \Pi^T S_n^\dagger \right),
\end{align}
and
\begin{align}
\sqrt{\alpha'}^{-1} Q^{2^+} & = c'_0 \sqrt{\frac{\mu}{6}} \left( a_0^b \gamma^b \Pi^T S_0' - a_0^\dagger \gamma^a \Pi^T S_0'^\dagger \right) - i \sum_{n=1}^\infty c'_n \sqrt{\omega_n} \left( \tilde{a}_n^{b\dagger} \gamma^b \tilde{S}_n' - \tilde{a}_n \gamma^a \tilde{S}_n'^\dagger \right) \\
& \hspace{1cm} + \frac{1}{2} \cdot \frac{\mu}{6} \sum_{n=1}^\infty \frac{1}{c'_n \sqrt{\omega_n}} \left( a_n^{b\dagger} \gamma^b \Pi^T S_n' - a_n \gamma^a \Pi^T S_n'^\dagger \right), \\
\sqrt{\alpha'}^{-1} Q^{1^-} & = -i c'_0 \sqrt{\frac{\mu}{6}} \left( a_0^\dagger \gamma^a S_0' + a_0^b \gamma^b S_0'^\dagger \right) - i \sum_{n=1}^\infty c'_n \sqrt{\omega_n} \left( \tilde{a}_n^\dagger \gamma^a \tilde{S}_n' + \tilde{a}_n \gamma^b \tilde{S}_n'^\dagger \right) \\
& \hspace{1cm} + \frac{1}{2} \cdot \frac{\mu}{6} \sum_{n=1}^\infty \frac{1}{c'_n \sqrt{\omega_n}} \left( a_n^\dagger \gamma^a \Pi^T S_n' - a_n \gamma^b \Pi^T S_n'^\dagger \right).
\end{align}

On the other hand, the kinematical supersymmetries are rewritten as
\begin{align}
\tilde{Q}^1^- & = i \Pi \tilde{\Psi}_0 = \frac{\Pi}{2} \left( S_0 - S_0^\dagger \right), \quad \tilde{Q}^{2^+} = i \tilde{\Psi}_0 = \frac{i}{2} \left( S_0 + S_0^\dagger \right), \\
\tilde{Q}^1^+ & = i \Pi \tilde{\Psi}_0 = \frac{\Pi}{2} \left( S_0 - S_0^\dagger \right), \quad \tilde{Q}^{2^-} = i \tilde{\Psi}_0 = \frac{i}{2} \left( S_0 + S_0^\dagger \right).
\end{align}

In the next section, the above expressions of supercharges will be used for constructing the fermionic boundary states.
4 Boundary States of Type IIA String Theory

In this section we will construct the boundary states of type IIA string theory on the pp-wave background with 24 supersymmetries. To begin with, the bosonic boundary states will be constructed. Next, we construct the fermionic boundary states and classify the allowed D-brane instantons in our theory. The resulting boundary states will be used for the calculation of amplitude in the closed string description.

4.1 Bosonic Boundary States

Here we will consider the bosonic part of boundary states in the Type IIA string theory. The bosonic coordinates \((x^a, x^b)\) \((a = 1, \ldots, 4, b = 5, \ldots, 8)\) are classified into \((x^a, x^b)\) (for the Neumann condition) and \((x^a, x^b)\) (for the Dirichlet condition).

The definition of bosonic boundary state \(|B\rangle\) is given by the following boundary conditions:

\[
\partial_{\tau} x^a \big|_{\tau=0} |B\rangle = 0, \quad \partial_{\tau} x^b \big|_{\tau=0} |B\rangle = 0 \quad \text{(Neumann)},
\]

\[
(x_0^a - q_0^a) \big|_{\tau=0} |B\rangle = 0, \quad (x_0^b - q_0^b) \big|_{\tau=0} |B\rangle = 0 \quad \text{(Dirichlet)}. \tag{4.2}
\]

The conditions (4.1) can be rewritten as

\[
(a_0^a + a_0^{\dagger a}) |B\rangle = 0, \quad (a_n^a + a_n^{\dagger a}) |B\rangle = 0 \quad (n > 0), \tag{4.3}
\]

\[
(a_0^b + a_0^{\dagger b}) |B\rangle = 0, \quad (a_n^b + a_n^{\dagger b}) |B\rangle = 0 \quad (n > 0), \tag{4.4}
\]

and lead to the bosonic boundary state \(|B\rangle\) for Neumann directions described by

\[
|B\rangle = \exp \left( -\frac{1}{2} \sum_a a_0^{\dagger a} a_0^a - \sum_b b_0^{\dagger b} b_0^b - \sum_{n=1}^{\infty} \left\{ \sum_a a_n^{\dagger a} a_n^a + \sum_b b_n^{\dagger b} b_n^b \right\} \right) |0\rangle, \tag{4.5}
\]

where \(|0\rangle\) is the bosonic Fock vacuum state annihilated by the operators, \(a_n^{a,b}\) and \(a_n^{a,b}\).

On the other hand, the second conditions (4.2) can be rewritten as

\[
\left( a_0^a - a_0^{\dagger a} + i \left( \frac{2\mu}{3\alpha'} \right)^{1/2} q_0^a \right) |B\rangle = 0, \quad \left( a_n^a - a_n^{\dagger a} \right) |B\rangle = 0, \tag{4.6}
\]

\[
\left( a_0^b - a_0^{\dagger b} + i \left( \frac{\mu}{3\alpha'} \right)^{1/2} q_0^b \right) |B\rangle = 0, \quad \left( a_n^b - a_n^{\dagger b} \right) |B\rangle = 0. \tag{4.7}
\]

With these boundary conditions, we can construct the boundary states for Dirichlet directions described by

\[
|B\rangle = e^{+\frac{1}{2} \sum_a \left\{ a_0^{\dagger a} - \left( \frac{2\mu}{3\alpha'} \right)^{1/2} q_0^a \right\}^2 + \frac{1}{2} \sum_b \left\{ a_0^{\dagger b} - \left( \frac{\mu}{3\alpha'} \right)^{1/2} q_0^b \right\}^2} e^{+\sum_{n=1}^{\infty} \left\{ \sum_a a_n^{\dagger a} a_n^a + \sum_b b_n^{\dagger b} b_n^b \right\} |0\rangle. \tag{4.8}
\]
Here we shall introduce a diagonal matrix $M_{ij} = \text{diag}(\pm 1, \cdots, \pm 1)$ with eight components where $+1$ is assigned for Dirichlet directions and $-1$ is assigned for Neumann ones. If we will set as $q^a_0 = q^b_0 = 0$, the bosonic boundary state can be rewritten as

$$|B\rangle = |B\rangle_{\mu/3} \otimes |B\rangle_{\mu/6},$$

$$|B\rangle_{\mu/3} \equiv e^{+\frac{i}{2}M_{\alpha\beta}a_0^a a_0^a}. e^{\sum_{n=1}^{\infty} M_{\alpha\beta}a_n^a a_n^a}|0\rangle,$$

$$|B\rangle_{\mu/6} \equiv e^{+\frac{i}{2}M_{\alpha\beta}b_0^a b_0^a}. e^{\sum_{n=1}^{\infty} M_{\alpha\beta}b_n^a b_n^a}|0\rangle.$$ 

Thus the bosonic boundary state is the product of two sectors with mass $\mu/3$ and that with $\mu/6$, and it has the $SO(4) \times SO(4)$ symmetry. We will study the fermionic boundary states in the next subsection.

### 4.2 Fermionic Boundary States

We will now consider the fermionic part of boundary states in our case. The fermionic boundary states are defined by

$$\left(Q_\alpha^{2-} - i\eta M^{(\mu/3)}_{\alpha\beta} Q_\beta^{1+}\right) |B\rangle = 0, \quad \left(Q_\alpha^{2-} - i\eta M^{(\mu/6)}_{\alpha\beta} Q_\beta^{1+}\right) |B\rangle = 0,$$

$$\left(\tilde{Q}_\alpha^{2-} + i\eta \tilde{M}^{(\mu/3)}_{\alpha\beta} \tilde{Q}_\beta^{1+}\right) |B\rangle = 0, \quad \left(\tilde{Q}_\alpha^{2+} + i\eta \tilde{M}^{(\mu/6)}_{\alpha\beta} \tilde{Q}_\beta^{1-}\right) |B\rangle = 0,$$

where matrices $M^{(\mu/3)}_{\alpha\beta}, M^{(\mu/6)}_{\alpha\beta}, \tilde{M}^{(\mu/3)}_{\alpha\beta}$ and $\tilde{M}^{(\mu/6)}_{\alpha\beta}$ satisfy the following relations:

$$M_{\alpha\beta}(M^T)_{\beta\gamma} = \delta_{\alpha\gamma}, \quad (M^T)_{\alpha\beta}M_{\beta\gamma} = \delta_{\alpha\gamma},$$

$$\tilde{M}_{\alpha\beta}(\tilde{M}^T)_{\beta\gamma} = \delta_{\alpha\gamma}, \quad (\tilde{M}^T)_{\alpha\beta}\tilde{M}_{\beta\gamma} = \delta_{\alpha\gamma}. $$

The definition of the fermionic boundary states (4.11) leads to the conditions written in terms of the zero-modes$^\dagger$:

$$\left((\Psi_0)_{\alpha} + i\eta \tilde{M}^{(\mu/3)}_{\alpha\beta}(\Pi \tilde{\Psi}_0)_{\beta}\right) |B\rangle = 0, \quad \left((\Psi_0')_{\alpha} + i\eta \tilde{M}^{(\mu/6)}_{\alpha\beta}(\Pi \tilde{\Psi}_0')_{\beta}\right) |B\rangle = 0.$$

These conditions suggest us to take the following ansatz:

$$\left((S_n)_{\alpha} + i\eta \tilde{M}^{(\mu/3)}_{\alpha\beta}(S_n^\dagger)_{\beta}\right) |B\rangle = 0, \quad \left((S_n')_{\alpha} + i\eta \tilde{M}^{(\mu/6)}_{\alpha\beta}(S_n'^\dagger)_{\beta}\right) |B\rangle = 0.$$

$^\dagger$If we redefine the $\tilde{\Psi}_0$ as $\Pi \tilde{\Psi}_0 \rightarrow \tilde{\Psi}_0$, then we obtain the usual expressions for zero-mode conditions. The effect from the redefinition of $\tilde{\Psi}_0$ are absorbed into the definition of the creation and annihilation operators without the modification of anticommutation relations, and so we have no trouble for our discussion.
The above equations can be easily solved and the boundary state is given by

$|B\rangle = \sum_{n=1}^{\infty} \{ -i \eta \hat{M}_{\alpha\beta}^{(\mu/3)} (S_{\alpha}^n)_\alpha (S_{\beta}^n)_\beta - i \eta \hat{M}_{\alpha\beta}^{(\mu/6)} (S_{\alpha}^n)_\alpha (S_{\beta}^n)_\beta \} |B\rangle_0 , \tag{4.16}$

where $|B\rangle_0$ is the fermionic vacuum state yet to be determined. By the way, this boundary state is by definition the state satisfying (4.10) and (4.11). If we now act the conditions (4.15) on (4.10), we have three types of conditions that lead us to determine the structure of the matrices $M^{(\mu/3)}$, $M^{(\mu/6)}$, $\hat{M}^{(\mu/3)}$ and $\hat{M}^{(\mu/6)}$. Firstly, we obtain the conditions

$M^{(\mu/3)} a_{\alpha'} \gamma^a = -M^{(\mu/3)} \gamma^a \hat{M}^{(\mu/3)} r , \quad M^{(\mu/6)} b_{\gamma'} \gamma^b = -M^{(\mu/6)} \gamma^b \hat{M}^{(\mu/6)} r , \tag{4.17}$

which are similar to those arising in flat space [29]. The second type of conditions, which appears only in the pp-wave case, is

$M^{a_{\alpha'}} \gamma^a \Pi = -M^{(\mu/3)} \gamma^a \Pi \hat{M}^{(\mu/3)} , \quad M^{b_{\gamma'}} \gamma^b \Pi = -M^{(\mu/6)} \gamma^b \Pi \hat{M}^{(\mu/6)} . \tag{4.18}$

Finally, the third type of conditions we get comes from the zero-mode parts:

$$\begin{align*}
\left\{ a^a_0 \dagger (\gamma^a + \eta M^{(\mu/3)} \gamma^a \Pi) S_0 + M^{a_{\alpha'}} a^a_{\alpha'} (\gamma^a - \eta M^{(\mu/3)} \gamma^a \Pi) S^\dagger_0 \right\} |B\rangle & = 0 , \tag{4.19} \\
\left\{ b^b_0 \dagger (\gamma^b + \eta M^{(\mu/6)} \gamma^b \Pi) S_0 + M^{b_{\gamma'}} b^b_{\gamma'} (\gamma^b - \eta M^{(\mu/6)} \gamma^b \Pi) S^\dagger_0 \right\} |B\rangle & = 0 . \tag{4.20}
\end{align*}$$

By the use of the definition of fermionic vacuum $S_0 |B\rangle_0 = 0$ and the identities $a^a_0 = M^{a_{\alpha'}} a^a_{\alpha'}$ and $b^b_0 = M^{b_{\gamma'}} b^b_{\gamma'}$, we can rewrite (4.19) and (4.20) as

$$\begin{align*}
\left\{ \phi^a_0 \left( \delta^{aa'} - M^{aa'} \right) + \frac{i \mu}{3 \alpha'} \bar{x}^a_0 \left( \delta^{aa'} + M^{aa'} \right) \right\} (\gamma^a - \eta M^{(\mu/3)} \gamma^a \Pi) S^\dagger_0 |B\rangle_0 & = 0 , \tag{4.21} \\
\left\{ \phi^b_0 \left( \delta^{bb'} - M^{bb'} \right) + \frac{i \mu}{6 \alpha'} \bar{x}^b_0 \left( \delta^{bb'} + M^{bb'} \right) \right\} (\gamma^b - \eta M^{(\mu/6)} \gamma^b \Pi) S^\dagger_0 |B\rangle_0 & = 0 . \tag{4.22}
\end{align*}$$

Using the conditions (4.21) and (4.22), we can read off supersymmetries preserved by D-brane instantons in terms of their positions. In the case that all position coordinates of a D-brane instanton $q^r$ for the Dirichlet directions equals zero, the D-brane instanton has 12 (4 dynamical + 8 kinematical) supersymmetries (i.e., $1/2(-12/24)$ BPS D-brane instanton). If the position coordinates for the Dirichlet directions are not at the origin, then the D-brane instanton has 8 (0+8) supersymmetries (i.e., $1/3(-8/24)$ BPS D-brane instanton). Thus all of the dynamical supersymmetries are broken. However the D-brane instantons apart from the origin are supersymmetric solutions since they have 8 kinematical supersymmetries.

As a final remark regarding the structure of the matrices, $M^{(\mu/3)}$, $M^{(\mu/6)}$, $\hat{M}^{(\mu/3)}$ and $\hat{M}^{(\mu/6)}$, we now consider the chirality condition while the above three types of conditions are almost
same as in the type IIB string case [30]. First we can easily find that both matrices \( M^{(\mu/3)} \) and \( M^{(\mu/6)} \) contain the odd number of gamma matrices in order to preserve the \( SO(8) \) chirality measured by \( \gamma^9 \). Moreover, if we consider the chirality in terms of the matrix \( R = \gamma^{1234} \), we have the following conditions basically from (4.10) and (4.11):

\[
\{ R, M^{(\mu/3)} \} = 0, \quad \{ R, M^{(\mu/6)} \} = 0, \quad \{ R, \hat{M}^{(\mu/3)} \} = 0, \quad \{ R, \hat{M}^{(\mu/6)} \} = 0.
\] (4.23)

Then we obtain the following complete boundary state:

\[
|B\rangle = e^{\sum_{n=1}^{\infty} \left\{ M_{\alpha\beta} a_n^\dagger a_{\alpha'}^\dagger + M_{\beta\alpha} a_n a_{\alpha'} - i\eta \hat{M}^{(\mu/3)}(S_n^\dagger)_{\alpha}(S_n^\dagger)_{\beta} - i\eta \hat{M}^{(\mu/6)}(S_n'_{\alpha})_{(S_n'_{\beta})} \right\}} |B\rangle_0
\]

\[
|B\rangle_0 = (M_{IJ} |I\rangle |J\rangle - i\eta M_{\alpha\beta} |\alpha\rangle |\beta\rangle) e^{+\frac{1}{2}M_{\alpha\beta} a_0^\dagger a_0^\dagger + \frac{1}{2}M_{\alpha\beta} a_0 a_0^\dagger} |0\rangle,
\] (4.24)

where the state \( |B\rangle_0 \) is the product of the bosonic vacuum state, which is given by picking up the zero-mode parts in Eq. (4.9), and the fermionic one which is the solution of (4.14).

The remaining task is to determine the matrices \( M \) and \( \hat{M} \) from the conditions (4.17), (4.18), (4.21), (4.22) and (4.23). The determined structure of the matrices leads to the classification of possible D-brane instantons. This will be done in the next subsection.

4.3 Classification of D-brane Instantons

We will classify the allowed D-brane instantons by determining the matrices \( M \) and \( \hat{M} \). Now let us analyze each of Dp-brane instantons.

D0: D0-brane instantons are expressed by the following matrices:

\[
M^{(\mu/3)} = M^{(\mu/6)} = \hat{M}^{(\mu/3)} = \hat{M}^{(\mu/6)} = \gamma^I \quad (I = 1, 2, 3).
\] (4.25)

The \( x^I \)-direction satisfies the Neumann boundary condition and other directions satisfy the Dirichlet boundary condition. When we consider \( M = \gamma^1 \) as an example, \( x^1 \) is a Neumann direction and other directions are Dirichlet ones. If we consider the D0-brane instanton at the origin \( q^{2,3,4,5,6,7,8} = 0 \), then it is a 1/2 BPS object. If we consider the D0-brane instanton apart from the origin, then it becomes a 1/3 BPS object.

D2: D2-brane instantons are given by

\[
M^{(\mu/3)} = M^{(\mu/6)} = \hat{M}^{(\mu/3)} = \hat{M}^{(\mu/6)} = \gamma^{134},
\] (4.26)

or

\[
M^{(\mu/3)} = M^{(\mu/6)} = \hat{M}^{(\mu/3)} = \hat{M}^{(\mu/6)} = \gamma^4 \gamma^{bb'},
\]

where \( \gamma^4 \) is a 4-gamma matrix in the 8-dimensional Euclidean space.
where $I$ and $J$ take values in 1, 2, 3, and $b$ and $b'$ run from 5 to 8. For example, if we take $M = \gamma^{124}$ then $x^{1,2,4}$-directions satisfy the Neumann condition and others are Dirichlet directions. When the D2-brane instanton sits at the origin $q^{3,5,6,7,8} = 0$, it is a 1/2 BPS object. Once it goes away from the origin, it becomes 1/3 BPS.

**D4**: D4-brane instantons are described by

\[
M^{(\mu/3)} = M^{(\mu/6)} = \tilde{M}^{(\mu/3)} = \tilde{M}^{(\mu/6)} = \gamma^{123}\gamma^{bb'},
\]

or

\[
M^{(\mu/3)} = M^{(\mu/6)} = \tilde{M}^{(\mu/3)} = \tilde{M}^{(\mu/6)} = \gamma^I\gamma^{5678}.
\]

When we take $M = \gamma^{12356}$, the $x^{1,2,3,5,6}$-directions satisfy the Neumann boundary condition and others are the Dirichlet directions. When the D4-brane instanton is at the origin $q^{4,7,8} = 0$, it is a 1/2 BPS object. If it is apart from the origin, it becomes 1/3 BPS.

**D6**: D6-brane instantons are given by

\[
M^{(\mu/3)} = M^{(\mu/6)} = \tilde{M}^{(\mu/3)} = \tilde{M}^{(\mu/6)} = \gamma^{134}\gamma^{5678}.
\]

For example, the case $M = \gamma^{1245678}$ leads to a D6-brane instanton that preserves 12 supersymmetries (i.e., 1/2 BPS) for $q^3 = 0$ and 8 supersymmetries (1/3 BPS) for $q^3 \neq 0$.

We should remark that the above classification of D-brane instantons is consistent with that of D-branes in the open string description [18, 20]. The Neumann (Dirichlet) boundary condition in the closed string description is simply related by the Dirichlet (Neumann) one in the open string description, and hence this identification should hold as a matter of course. For comparison with the classification of D-branes [18, 20], we shall summarize our result in **Tab. 1**.

| $N_N$ | $M^{(\mu/3)} = M^{(\mu/6)} = \tilde{M}^{(\mu/3)} = \tilde{M}^{(\mu/6)}$ |
|-------|---------------------------------------------------------------------|
| 1     | $\gamma^I$                                                          |
| 3     | $\gamma^{IJ4}$, $\gamma^A\gamma^{bb'}$                             |
| 5     | $\gamma^{123}\gamma^{bb'}$, $\gamma^I\gamma^{5678}$               |
| 7     | $\gamma^{IJ4}\gamma^{5678}$                                        |

**Tab. 1**: List of possible D-brane instantons in our Type IIA string theory. The $N_N$ is the number of Neumann directions.
5 Cylinder Amplitude in the Closed String Description

In this section we will calculate the tree amplitude in the closed string description. The interaction energy between a pair of D-branes comes from the exchange of a closed string between two boundary states (i.e., a cylinder diagram).

The expression of the cylinder diagram (tree diagram) in the light-cone formulation can be expressed as

\[
A_{Dp_1: Dp_2}(x^+, x^-, q^i_1, q^j_2) = \frac{1}{2\pi i} \int dp^+ dp^- e^{ip^+x^- + ip^-x^+} \langle Dp_1, -p^-, -p^+, q^i_1 | \left( \frac{1}{p^+(p^- + H)} \right) | Dp_2, p^-, p^+, q^j_2 \rangle 
\]

where \(H\) is the light-cone Hamiltonian of a closed string and the \(|Dp, p^+, q^i\rangle\) represents a boundary state of a Dp-brane instanton located at the transverse position \(q^i\) with the longitudinal momentum \(p^+\). We note that the prescription given in [25] has been used for obtaining the last line in the above equation. If we define the variable \(t\) by

\[x^+ = \pi \tau = -i\pi t\]

by performing the customary Wick rotation, the amplitude is rewritten as

\[
A_{Dp_1: Dp_2}(x^+, x^-, q^i_1, q^j_2) = \int_{-\infty}^{+\infty} dt e^{-x^+x^-} \tilde{A}_{Dp_1: Dp_2}(t, q^i_1, q^j_2),
\]

where \(\tilde{A}_{Dp_1: Dp_2}(t, q_1, q_2)\) is the expectation value:

\[
\tilde{A}_{Dp_1: Dp_2}(t, q^i_1, q^j_2) \equiv \langle Dp_1, -p^+, q^i_1 | e^{-2\pi t(H/2)} | Dp_2, p^+, q^j_2 \rangle.
\]

Now the tree amplitude (5.3) will be calculated by using the boundary states constructed before. We restrict ourselves to the case of identical Dp-brane instantons for simplicity. The calculus consists of three parts: 1) vacuum energies, 2) nonzero-modes, and 3) zero-modes.

Let us concentrate only on the sector with mass \(\nu \equiv \mu / 3\). The other sector with mass \(\nu' \equiv \mu / 6\) results in the same final expression only with the difference in the value of mass parameter. When we consider the contribution of vacuum energies to the amplitude, the evaluation of vacuum energies is identical to that of the partition function in the closed string case:

\[
e^{2\Delta(\nu; 0)} \quad \text{(for bosons),} \quad e^{-2\Delta(\nu; 0)} \quad \text{(for fermions).}
\]
The contribution of nonzero-modes to the amplitude is also the same as that of partition function of a closed string, and so it is readily written as

$$\prod_{n=1}^{\infty} (1 - q^{\omega_n})^{-4}, \quad \prod_{n=1}^{\infty} (1 - q^{\omega_n})^4 \quad (\text{fermions}), \quad q \equiv e^{-2\pi t}. \quad (5.5)$$

The contribution of bosonic zero-modes can be evaluated by using the formula:

$$\langle 0 \mid e^{\pm \frac{1}{2} a_0 a_0} q^{\pm \frac{1}{2} m a_0 a_0} e^{\pm \frac{1}{2} a_0^\dagger a_0^\dagger} | 0 \rangle = (1 - q^m)^{-1/2}. \quad (5.6)$$

As a result, the factor \((1 - q^\nu)^{-2}\) is obtained from bosonic zero-modes. The fermionic zero-modes can be evaluated by adopting the prescription given in the appendix of [25], and the resulting contribution is \((1 - q^\nu)^2\). Consequently, the contribution of zero-modes is summarized as follows:

$$\begin{align*}
(1 - q^\nu)^{-2} & \quad \text{(for bosons)}, \\
(1 - q^\nu)^2 & \quad \text{(for fermions)}.
\end{align*} \quad (5.7)$$

After taking account of the sector with mass \(\mu/6\), the total partition function is then represented by

$$\bar{A}_{Dp:Dp} = \bar{A}_{Dp:Dp}^B \cdot \bar{A}_{Dp:Dp}^F = \mathcal{N}_{Dp}^2, \quad (5.8)$$

where \(\mathcal{N}_{Dp}\) is the normalization factor of boundary states, which is not determined yet. This factor can be fixed by calculating the cylinder diagram in the open string channel. This task will be done in the next section.

In the work of [25], the “conformal field theory condition”

$$\bar{A}_{Dp_1:Dp_2}(t, q_1, q_2) = \tilde{Z}_{Dp_1:Dp_2}(\tilde{t}, q_1, q_2), \quad (5.9)$$

was analyzed in the case of the pp-wave background. This condition ensures the consistency between closed and open string channels. We now turn to the calculation of the partition function with D-branes (i.e., the cylinder diagram) in order to show that the condition (5.9) also holds in our theory.

### 6 Partition Function in the Open String Description

In this section we will discuss the one-loop amplitude of open string, and confirm the consistency condition between open and closed string channels.
We start from the light-cone action of open string defined by
\[
S_{\text{open}} = \frac{1}{4\pi \alpha'} \int d\tau \int_0^\pi d\sigma \left[ \sum_{i=1}^8 \left( (\partial_+ x^i)^2 - (\partial_\sigma x^i)^2 \right) - \left( \frac{\mu}{3} \right)^2 \sum_{a=1}^4 (x^a)^2 - \left( \frac{\mu}{5} \right)^2 \sum_{b=5}^8 (x^b)^2 \right] + \frac{i}{2\pi} \int d\tau \int_0^\pi d\sigma \left[ \Psi^{1+} \partial_- \Psi^{1+} + \Psi^{1-} \partial_- \Psi^{1-} + \Psi^{2+} \partial_+ \Psi^{2+} + \Psi^{2-} \partial_+ \Psi^{2-} - \frac{\mu}{3} \Psi^{1-} \Pi^+ \Psi^{2+} - \frac{\mu}{6} \Psi^{1+} \Pi^+ \Psi^{2-} + \frac{\mu}{6} \Psi^{2-} \Pi^+ \Psi^{1+} + \frac{\mu}{3} \Psi^{2+} \Pi^+ \Psi^{1-} \right]. \tag{6.1}
\]

To begin with, we shall present the mode-expansion in the case of open string. The mode-expansion of D-D string is expressed as
\[
x^a(\tau, \sigma) = q^a_0 \cdot \frac{\sinh \nu (\pi - \sigma)}{\sinh(\pi \nu)} + q^a_1 \cdot \frac{\sinh(\nu \sigma)}{\sinh(\pi \nu)} - \frac{2\alpha'}{\omega_n} \sum_{\nu \neq 0} \frac{1}{\omega_n} \alpha^a_n e^{-i\omega_n \tau} \sin(n\sigma), \tag{6.2}
\]
\[
x^b(\tau, \sigma) = q^b_0 \cdot \frac{\sinh \nu'(\pi - \sigma)}{\sinh(\pi \nu')} + q^b_1 \cdot \frac{\sinh(\nu' \sigma)}{\sinh(\pi \nu')} - \frac{2\alpha'}{\omega_n} \sum_{\nu' \neq 0} \frac{1}{\omega_n} \alpha^b_n e^{-i\omega_n \tau} \sin(n\sigma), \tag{6.3}
\]
where the endpoints satisfy the Dirichlet conditions: \( x^i(\sigma = 0) = q^i_0 \), \( x^i(\sigma = \pi) = q^i_1 \).

The mode expansion of N-N string, whose endpoints satisfy the Neumann conditions, is written as
\[
x^a(\tau, \sigma) = x^a_0 \cos(\nu \tau) + \frac{1}{\nu} \cdot 2\alpha' p^a_0 \sin(\nu \tau) + i\sqrt{2\alpha'} \sum_{\nu \neq 0} \frac{1}{\omega_n} \alpha^a_n e^{-i\omega_n \tau} \cos(n\sigma), \tag{6.4}
\]
\[
x^b(\tau, \sigma) = x^b_0 \cos(\nu' \tau) + \frac{1}{\nu'} \cdot 2\alpha' p^b_0 \sin(\nu' \tau) + i\sqrt{2\alpha'} \sum_{\nu' \neq 0} \frac{1}{\omega_n} \alpha^b_n e^{-i\omega_n \tau} \cos(n\sigma). \tag{6.5}
\]

The mode-expansion of fermions are the same with that in the case of closed string, but we have to take account of boundary conditions at \( \sigma = 0 \) and \( \pi \) described by
\[
\Psi^{1-} = \Omega \Psi^{2+}, \quad \Psi^{2+} = \Omega^* \Psi^{1-}, \quad \Psi^{1+} = \Omega \Psi^{2-}, \quad \Psi^{2-} = \Omega^* \Psi^{1+},
\]
\[
\tilde{\Psi}_n = \Omega \Psi_n, \quad \tilde{\Psi}_0 = -\Pi \Omega \Psi_0, \quad \Psi_0 = \Pi \Omega \tilde{\Psi}_0, \quad \Pi \Omega \Pi \Omega = -1,
\]
where \( \Omega \) is the gluing matrix for fermionic modes on the boundaries.

We now introduce the creation and annihilation operators given by
\[
a^a_n = \frac{1}{\sqrt{\omega_n}} \alpha^a_n, \quad a^a_n^\dagger = \frac{1}{\sqrt{\omega_n}} \alpha^a_n, \quad [a^a_m, a^b_n^\dagger] = \delta^{a,a'} \delta_{m,n} \quad (m, n > 0),
\]
\[
a^b_n = \frac{1}{\sqrt{\omega_n}} \alpha^b_n, \quad a^b_n^\dagger = \frac{1}{\sqrt{\omega_n}} \alpha^b_n, \quad [a^b_m, a^{b'}_n^\dagger] = \delta^{b,b'} \delta_{m,n} \quad (m, n > 0),
\]
\[
a^a_0 = \sqrt{\frac{\alpha'}{\nu}} \left( p^a_0 - i\nu \frac{1}{2\alpha'} x^a_0 \right), \quad a^a_0^\dagger = \sqrt{\frac{\alpha'}{\nu}} \left( p^a_0 + i\nu \frac{1}{2\alpha'} x^a_0 \right), \quad [a^a_0, a^a_0^\dagger] = \delta^{a,a'},
\]
\[
a^b_0 = \sqrt{\frac{\alpha'}{\nu}} \left( p^b_0 - i\nu' \frac{1}{2\alpha'} x^b_0 \right), \quad a^b_0^\dagger = \sqrt{\frac{\alpha'}{\nu'}} \left( p^b_0 + i\nu' \frac{1}{2\alpha'} x^b_0 \right), \quad [a^b_0, a^{b'}_0^\dagger] = \delta^{b,b'},
\]
then the Hamiltonian $H_B$ for the Dirichlet directions is expressed by

\[
H_B = \frac{1}{4\pi\alpha'} \cdot \frac{\nu}{\sinh(\pi\nu)} \left[ \{(q_0^a)^2 + (q_1^a)^2\} \cosh(\pi\nu) - 2q_0^aq_1^a\right]
\]

\[+\frac{1}{4\pi\alpha'} \cdot \frac{\nu'}{\sinh(\pi\nu')} \left[ \{(q_0^b)^2 + (q_1^b)^2\} \cosh(\pi\nu') - 2q_0^bq_1^b\right]
\]

\[+\frac{1}{2} \sum_{n=1}^{\infty} \omega_n (a_n^{\dagger a}a_n^a + a_n^{a\dagger a}a_n^a) + \frac{1}{2} \sum_{n=1}^{\infty} \omega'_n (a_n^{b\dagger b}a_n^b + a_n^{b\dagger b}a_n^b), \quad (6.6)
\]

and that for the Neumann directions is represented by

\[
H_B = \frac{1}{2} \omega_0 (a_0^{\dagger a}a_0^a + a_0^{a\dagger a}a_0^a) + \frac{1}{2} \sum_{n \geq 1} \omega_n (a_n^{\dagger a}a_n^a + a_n^{a\dagger a}a_n^a)
\]

\[+\frac{1}{2} \omega'_0 (a_0^{b\dagger b}a_0^b + a_0^{b\dagger b}a_0^b) + \frac{1}{2} \sum_{n \geq 1} \omega'_n (a_n^{b\dagger b}a_n^b + a_n^{b\dagger b}a_n^b). \quad (6.7)
\]

The Hamiltonian of fermions $H_F$ is rewritten as

\[
H_F = \sum_{n=1}^{\infty} (\omega_n S_n^\dagger S_n + \omega'_n S_n^\dagger S_n') - \frac{\mu}{3} \tilde{\Psi}_0^\dagger \Omega \tilde{\Psi}_0 - \frac{\mu'}{6} \tilde{\Psi}_0^\dagger \Omega' \tilde{\Psi}_0'. \quad (6.8)
\]

Now we shall evaluate the Casimir energy given as follows:

\[
\sum_{n=1}^{\infty} \frac{1}{2} \omega_n \approx \frac{1}{2} \left( \sum_{n=1}^{\infty} \sqrt{n^2 + \nu^2} - \int_0^\infty dk \sqrt{k^2 + \nu^2} \right) = \frac{1}{2} \left( -\frac{1}{2} \nu + \Delta(\nu; 0) \right),
\]

\[
\sum_{n=1}^{\infty} \frac{1}{2} \omega'_n \approx \frac{1}{2} \left( \sum_{n=1}^{\infty} \sqrt{n^2 + \nu'^2} - \int_0^\infty dk \sqrt{k^2 + \nu'^2} \right) = \frac{1}{2} \left( -\frac{1}{2} \nu' + \Delta(\nu'; 0) \right),
\]

in terms of zero-point energy.

The zero-point energies of a single boson with the Dirichlet condition are given by

\[(D): \frac{1}{2} \sum_{n \geq 1} \omega_n \approx \frac{1}{2} \left( -\frac{1}{2} \nu + \Delta(\nu; 0) \right), \quad \frac{1}{2} \sum_{n \geq 1} \omega'_n \approx \frac{1}{2} \left( -\frac{1}{2} \nu' + \Delta(\nu'; 0) \right), \]

and those with the Neumann condition are expressed as

\[(N): \frac{1}{2} \sum_{n \geq 0} \omega_n \approx \frac{1}{2} \left( +\frac{1}{2} \nu + \Delta(\nu; 0) \right), \quad \frac{1}{2} \sum_{n \geq 0} \omega'_n \approx \frac{1}{2} \left( +\frac{1}{2} \nu' + \Delta(\nu'; 0) \right).
\]

As for the zero-point energies for a fermion, we have

\[E^0 = -4 \cdot \frac{1}{2} \sum_{n \geq 1} \omega_n \approx \nu - 2\Delta(\nu; 0), \quad E'^0 = -4 \cdot \frac{1}{2} \sum_{n \geq 1} \omega'_n \approx \nu' - 2\Delta(\nu'; 0), \]

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where the factor 4 in front of the summation arises since each of fermions considered here has four independent components. The contributions of zero-point energies to the partition function $Z_F$ are then

\[ e^{-2\pi t(\nu - 2\Delta(\nu';0))} = e^{-2\pi t\nu} e^{4\pi t\Delta(\nu;0)} \quad \text{and} \quad e^{-2\pi t(\nu' - 2\Delta(\nu';0))} = e^{-2\pi t\nu'} e^{4\pi t\Delta(\nu';0)}. \]

We note that we have to treat carefully the zero-mode part of the Hamiltonian

\[ H_F^0 = -i\nu \Psi_0^\dagger \Omega \Psi_0 - i\nu' \Psi_0'^\dagger \Omega \Psi'_0. \]

The $(\Psi_0)_\alpha$ and $(\Psi_0')_\alpha$ have four non-vanishing components, and hence four sets of creation and annihilation operators $S_{1}^\pm, S_{2}^\pm$ and $S_{1}'^\pm, S_{2}'^\pm$ can be constructed. Here $S_{1,2}^+, (S_{1}'^+, S_{2}'^+)$ are creation operators and $S_{1,2}^-, (S_{1}'^-, S_{2}'^-)$ are annihilation ones. Thus the Hamiltonian can be rewritten as

\[ H_F^0 = \frac{\nu}{2} (S_1^+ S_1^- - S_1^- S_1^+) + \frac{\nu}{2} (S_2^+ S_2^- - S_2^- S_2^+) + \frac{\nu'}{2} (S_1'^+ S_1'^- - S_1'^- S_1'^+) + \frac{\nu'}{2} (S_2'^+ S_2'^- - S_2'^- S_2'^+), \]

and the associated energies are $\pm \frac{\nu}{2}$ and $\pm \frac{\nu'}{2}$. Consequently, the contribution from these zero-mode parts is evaluated as

\[ e^{2\pi \nu t}(1 - e^{-2\pi \nu t})^2 + e^{2\pi \nu' t}(1 - e^{-2\pi \nu' t})^2. \]

We now turn to the evaluation of the total partition function of the open string connecting two identical $D_p$-branes. Let us first consider the partition function for bosons:

\[ Z_B = \text{Tr} e^{-2\pi t H_B}, \quad q = e^{-2\pi t}. \]

We will consider the sector with the mass $\nu = \mu/3$. The bosonic partition function for the $(4 - p)$ Dirichlet conditions (i.e., $4 - p = \#(D-D \text{ strings})$) is given by

\[ Z_B^{(\nu)} = \prod_{n \geq 1} \frac{1}{(1 - q^{\omega_n})^{4-p}} \cdot q^{(4-p)\frac{1}{2}(-\frac{1}{2}\nu + \Delta(\nu;0))} \cdot f(q_0^a, q_1^a), \]

where the function $f(q_0^a, q_1^a)$ is defined by

\[ f(q_0^a, q_1^a) \equiv \exp \left[ -\frac{t}{2\alpha'} \frac{\nu}{\text{sinh}(\pi \nu)} \left\{ (q_0^a)^2 + (q_1^a)^2 \cosh(\pi \nu) - 2q_0^a q_1^a \right\} \right]. \]

The partition function for the $p$ Neumann directions (i.e., $p = \#(N-N \text{ strings})$) is written as

\[ Z_B^{(\nu)} = \prod_{n \geq 0} \frac{1}{(1 - q^{\omega_n})^p} \cdot q^{p\frac{1}{2}(+\frac{1}{2}\nu + \Delta(\nu;0))}. \]
Thus, the bosonic partition function on the sector with mass $\mu/3$ is represented by

$$Z_B^{(\nu)} = (1 - q^{\nu})^{-p} \cdot \prod_{n \geq 1} (1 - q^{\omega_n})^4 \cdot f(q^a_0, q^a_1) \cdot q^{\nu(-1 + \frac{1}{2}p) + 2\Delta(\nu;0)}.$$

where we have utilized the theta-like function: $\theta_{(a,b)}(t; \nu) = \sqrt{\Theta_{(a,b)}(it; -it; \nu)}$.

Next we shall evaluate the partition function for fermions with mass $\mu/3$. The fermionic partition function is given by

$$Z_F = \text{Tr}(-1)^F e^{-2\pi t H_F}.$$

After the similar calculation to bosonic case, we obtain the fermionic partition function:

$$Z_F^{(\nu)} = \left(e^{\pi \nu t} - e^{-\pi \nu t}\right)^2 \cdot e^{2\pi \nu t} e^{4\pi t \Delta(\nu;0)} \prod_{n=1}^{\infty} (1 - e^{-2\pi \omega_n})^4 \cdot f(q^a_0, q^a_1).$$

(6.9)

It is an easy task to include the sector with mass $\mu/6$, and thus the total partition function is described by

$$Z_{\text{tot}} = Z_B^{(\nu)} Z_F^{(\nu)} Z_B^{(\nu')} Z_F^{(\nu')}$$

$$= (2 \sinh(\pi \nu t))^2 \prod_{a \in D} f(q^a_0, q^a_1) \cdot (2 \sinh(\pi \nu t'))^2 \prod_{b \in D} f(q^b_0, q^b_1).$$

(6.10)

Here the numbers $p_1$ and $p_2$ are Neumann directions in the coordinates $x^a$'s and $x^b$'s, respectively. The net number of Neumann directions is represented by $\sharp(\text{Neumann}) = p_1 + p_2$. The $\prod_{i \in D}$ means the product in terms of Dirichlet directions $x^i$'s.

By comparing the cylinder amplitude (5.8) obtained in the last section with the resulting partition function (6.10) in the case of $q_0 = q_1 = 0$, we can determine the normalization factor of boundary states. The modular S-transformation of ‘massive’ theta function (2.18) relates the parameter $\mu (\equiv \mu_{cl})$ in the closed string to that $\mu (\equiv \mu_{op})$ in the open string through the corresponding law: $\mu_{op} t = \mu_{cl}$ [25]. Hence the normalization factor of boundary states for the D$p$-brane instanton is given by

$$N_{Dp} = \left(2 \sinh(\pi \mu/3)\right)^{(2-p_1)/2} \cdot \left(2 \sinh(\pi \mu/6)\right)^{(2-p_2)/2} \quad (p = p_1 + p_2).$$

(6.11)

Thus, we have shown the open/closed string duality in the $\mathcal{N} = (4,4)$ type IIA string theory at the origin. It was already shown in [25] that this duality holds in the case of the type IIB
string theory on the \textit{maximally} supersymmetric pp-wave background. Although we are in a situation of less supersymmetric case, the duality still holds at the origin.

It should be noted that the open/closed string duality holds at the origin. That is, the open/closed string duality requires that there is no dependence on transverse coordinates since the supersymmetry conditions require that both D-brane instantons should be at the origin, as discussed in the paper [25]. The cylinder amplitude does not have sensible behavior once the branes are located away from the origin.

7 General Properties of Partition Functions of Closed String

We have discussed the partition function and modular invariance of the type IIA string theory on the pp-wave background above. In this consideration there are two sectors with masses $\mu/3$ and $\mu/6$, and we have found that the modular properties hold in each sector. In this section, motivated by this fact, we will discuss general properties of partition functions of closed string apart from the type IIA string theory considered above. We suggest that some characteristics of string theories on pp-waves should be fixed from the requirement of modular invariance.

In the pp-wave case, theta-like function $\Theta_{(a,b)}(\tau, \bar{\tau}, \nu)$ should appear in the closed string partition function. It contains a mass parameter $\nu$ and has peculiar properties under modular transformations

$$\Theta_{(a,b)}(\tau + 1, \bar{\tau} + 1; \nu) = \Theta_{(a,b+a)}(\tau, \bar{\tau}; \nu), \quad \Theta_{(a,b)}\left(\frac{1}{\tau}, \frac{1}{\bar{\tau}}; |\tau|\nu\right) = \Theta_{(b,-a)}(\tau, \bar{\tau}; \nu).$$

Notably, the mass parameter $\nu$ changes into $|\tau|\nu$ under $S$-transformation $\tau \to -1/\tau$, and it gives us severe constraints in constructing modular invariant partition functions. In other words, we can make modular invariant partition functions with this clue to go upon. Now we will study a certain class of modular-invariant partition functions on the pp-wave background by using the modular properties of $\Theta_{(a,b)}(\tau, \bar{\tau}, \nu)$.

To simplify the problem, we put the following ansatz:

(1) There are several kinds of mass parameters $\nu$’s.

(2) For each $\nu$, the partition function $Z_B(\tau, \bar{\tau}, \nu)$ of the boson and that of fermion $Z_F(\tau, \bar{\tau}, \nu)$ cancel. That is to say, $Z_B(\tau, \bar{\tau}, \nu) \cdot Z_F(\tau, \bar{\tau}, \nu) = 1$.

We impose the ansatz (2) because the modular $S$-transformation changes the mass parameter $\nu$ into another one $|\tau|\nu$ and it is generally difficult to construct modular invariant combinations of $Z_B$’s and $Z_F$’s. In order to avoid this complicated problem, we take the simplest ansatz
\( Z_B(τ, \bar{τ}, ν) \cdot Z_F(τ, \bar{τ}, ν) = 1 \) here. But we should emphasize that there might be other modular invariant combinations without our ansatz and we cannot say there are no other possibilities. Here we will investigate such restricted cases only and compare our results with the models proposed earlier.

Now we will classify possible models by the use of the above ansatz. In order to realize the condition (2), the degrees of freedom of bosons must be identical with those of fermions. When we consider a transverse \( D \) dimensional space, the degrees of freedom of bosons are \( D \) (i.e., \( \sharp(\text{boson}) = D \)). On the other hand, the degrees of freedom of spinors in \( D \) dimensions are evaluated as

\[
\sharp(\text{fermion}) = 2^{\left\lfloor \frac{D}{2} \right\rfloor + \epsilon}, \quad \epsilon = \begin{cases} 
0 & \text{Majorana or Weyl} \\
-1 & \text{Majorana and Weyl} \\
+1 & \text{otherwise}
\end{cases}
\]

The value of \( \epsilon \) depends on what kinds of spinors we consider. From the consideration of dimensionality, we can understand that the matching of degrees of freedom between bosons and fermions happens for \( D = 1, 2, 4 \) and 8 only. For each \( D = 1, 2, 4, 8 \), the corresponding degree of freedom is 1, 2, 4, 8. That is, we have to consider four kinds of sets containing one, two, four, and eight bosons. Different sets are distinguished from mass parameters \( ν \)'s. Due to these mass terms, Lorentz symmetry is broken into smaller one. Next let us classify bosonic parts based on the Lorentz symmetry.

We study superstring theories and hence the dimension of transverse space should be eight. For massless cases, the associated Lorentz symmetry is \( SO(8) \) and there are eight massless bosons. However bosons have mass terms in our massive case. We set \( N_a \) as the number of sets with \( a(=1, 2, 4, 8) \) bosons with the same mass parameter. Let \( ν_{a,i} \) \((i = 1, 2, \cdots, N_a)\) be mass parameters for bosons and fermions in the same set. Due to mass terms, Lorentz symmetry is broken down to smaller one

\[
SO(1)^{N_1} \otimes SO(2)^{N_2} \otimes SO(4)^{N_4} \otimes SO(8)^{N_8} \subset SO(8),
\]

with \( N_1 + 2N_2 + 4N_4 + 8N_8 = 8 \), \( N_1, N_2, N_4, N_8 \in \mathbb{Z}_{\geq 0} \).
From this constraint, we can classify possible combinations:

|   |   |   |   |   |                   |
|---|---|---|---|---|-------------------|
| 0 | 0 | 0 | 1 |   | \(SO(8)\)        |
| 0 | 0 | 2 | 0 |   | \(SO(4) \times SO(4)\) |
| 0 | 2 | 1 | 0 |   | \(SO(2) \times SO(2) \times SO(4)\) |
| 2 | 1 | 1 | 0 |   | \(SO(1)^{\otimes 2} \times SO(2) \times SO(4)\) |
| 8 - 2\(\ell\) | \(\ell\) | 0 | 0 |   | \(SO(1)^{\otimes (8 - 2\ell)} \times SO(2)^{\otimes \ell}\) |

with \(\ell = 0, 1, 2, 3, 4\). Our type IIA model corresponds to the second case \((N_1, N_2, N_4, N_8) = (0, 0, 2, 0)\) in the list and symmetry is \(SO(4) \times SO(4)\). The type IIB string theory on the maximally supersymmetric background corresponds to \((N_1, N_2, N_4, N_8) = (0, 0, 0, 1)\). All of type IIB pp-wave backgrounds with the above-mentioned bosonic isometry are found (for example see [31]) and we can construct superstring theories on these backgrounds.

Here we turn to partition functions based on our ansatz. Each set is labelled by the mass parameter \(\nu_{a,i} (i = 1, 2, \cdots, N_a; a = 1, 2, 4, 8)\). The associated partition function of boson \(Z_B(\tau, \bar{\tau}, \nu_{a,i})\) and that of fermion \(Z_F(\tau, \bar{\tau}, \nu_{a,i})\) are written in the massive closed string case

\[
Z_B(\tau, \bar{\tau}, \nu_{a,i}) = \Theta_{(0,0)}(\tau, \bar{\tau}, \nu_{a,i}), \quad Z_F(\tau, \bar{\tau}, \nu_{a,i}) = \Theta_{(0,0)}(\tau, \bar{\tau}, \nu_{a,i})^{-1}.
\]

Then we can evaluate the total partition function \(Z\) as

\[
Z = \prod_{a=1,2,4,8} \prod_{i=1}^{N_a} Z_B(\tau, \bar{\tau}, \nu_{a,i}) \cdot Z_F(\tau, \bar{\tau}, \nu_{a,i}) = 1.
\]

It is actually modular invariant and many models proposed earlier are included in our results.

Last we explain the result \(Z = 1\) from the point of view of energy matching. Let \(\varepsilon\) be any energy level of states in our string system. We also introduce \(n_B(\varepsilon), n_F(\varepsilon)\) as the number of bosonic states and that of fermionic states at each energy level \(\varepsilon\) respectively. Then the associated partition function \(Z\) is defined as

\[
Z = \text{Tr}(-1)^F e^{-2\pi \tau_2 H} = \sum_{\varepsilon} (n_B(\varepsilon) - n_F(\varepsilon)) e^{-2\pi \tau_2 \varepsilon}.
\]

Here \(F\) is the fermion number operator and we also take \(\tau_1 = 0\) for simplicity. By comparing our result \(Z = 1\), we understand following relations

\[
1 = Z = (n_B(\varepsilon = 0) - n_F(\varepsilon = 0)) e^{-2\pi \tau_2 0} + \sum_{\varepsilon > 0} (n_B(\varepsilon) - n_F(\varepsilon)) e^{-2\pi \tau_2 \varepsilon},
\]

\[
n_B(\varepsilon = 0) - n_F(\varepsilon = 0) = 1, \quad n_B(\varepsilon) = n_F(\varepsilon) \quad (\varepsilon > 0).
\]
It shows that number of bosonic states and that of fermionic states match at each energy level $\varepsilon > 0$. Then total energy of bosonic states $E_B = n_B(\varepsilon) \cdot \varepsilon$ is equal to that of fermionic states $E_F = n_F(\varepsilon) \cdot \varepsilon$ at each energy level $\varepsilon > 0$. On the other hand, there is unbalance in number between bosonic states and fermionic states for the $\varepsilon = 0$ part. But the associated total energy of bosonic states $E_B^0 = n_B(\varepsilon = 0) \cdot 0 = 0$ equals to that of fermionic states $E_F^0 = n_F(\varepsilon = 0) \cdot 0 = 0$ in this $\varepsilon = 0$ sector. So the partition function $Z$ is nothing but the Witten index $Z = \text{Tr}(-1)^F$ in our massive case. This fact is already known in previous papers. Collecting these considerations, we conclude that our ansatz (2) is equivalent to a condition (Witten index) = 1. When we impose this condition (2) on $Z$, the resulting partition function is one and does not vanish. It also ensures that total energies of states at each energy level $\varepsilon$ match between bosonic states and fermionic states.

Our ansatz satisfies sufficient conditions to construct modular invariant partition functions. But we do not know necessary condition for this problem. We think it is important to find some further extra constraints in order to construct consistent string backgrounds and classify possible strings for massive cases.

8 Conclusions and Discussions

We have discussed the partition function of type IIA string theory obtained from the eleven-dimensional theory through the $S^1$-compactification of a transverse direction.

The modular invariance of our type IIA string theory has been proven. This type IIA string theory is less supersymmetric but it is modular invariant by virtue of the cancellation between bosonic and fermionic degrees of freedom.

We have constructed the boundary states and classified the D-brane instantons in our theory. The resulting list of the allowed D-brane instantons is consistent with that of the allowed D-branes obtained previously in different frameworks. In addition, we have calculated the amplitude between D-branes in the closed and open string descriptions, and checked the channel duality in our theory. Furthermore, we have briefly discussed general modular properties. There are many non-maximally supersymmetric pp-wave backgrounds, but not all of them would give ‘modular invariant’ superstring theories. Thus, we believe that the modular invariance is an available clue to classify the ‘physical’ string theories on pp-waves.
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References

[1] J. Kowalski-Glikman, “Vacuum States In Supersymmetric Kaluza-Klein Theory,” Phys. Lett. B 134 (1984) 194.

[2] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory,” JHEP 0201 (2002) 047 [arXiv:hep-th/0110242].

[3] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “Penrose limits and maximal supersymmetry,” Class. Quant. Grav. 19 (2002) L87 [arXiv:hep-th/0201081].

[4] R. Penrose, “Any spacetime has a plane wave as a limit,” Differential geometry and relativity, Reidel, Dordrecht, 1976, pp. 271-275.

[5] R. Gueven, “Plane wave limits and T-duality,” Phys. Lett. B 482 (2000) 255 [arXiv:hep-th/0005061].

[6] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background,” Nucl. Phys. B 625 (2002) 70 [arXiv:hep-th/0112044].

[7] R. R. Metsaev and A. A. Tseytlin, “Exactly solvable model of superstring in plane wave Ramond-Ramond background,” Phys. Rev. D 65 (2002) 126004 [arXiv:hep-th/0202109].

[8] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from N = 4 super Yang Mills,” JHEP 0204 (2002) 013 [arXiv:hep-th/0202021].

[9] K. Dasgupta, M. M. Sheikh-Jabbari and M. Van Raamsdonk, “Matrix perturbation theory for M-theory on a PP-wave,” JHEP 0205 (2002) 056 [arXiv:hep-th/0205185].
[10] K. Sugiyama and K. Yoshida, “Supermembrane on the pp-wave background,” Nucl. Phys. B 644 (2002) 113 [arXiv:hep-th/0206070].

[11] S. Hyun and H. Shin, “Branes from matrix theory in pp-wave background,” Phys. Lett. B 543 (2002) 115 [arXiv:hep-th/0206090].

[12] K. Sugiyama and K. Yoshida, “BPS conditions of supermembrane on the pp-wave,” Phys. Lett. B 546 (2002) 143 [arXiv:hep-th/0206132].

[13] K. Sugiyama and K. Yoshida, “Giant graviton and quantum stability in matrix model on PP-wave background,” Phys. Rev. D 66 (2002) 085022 [arXiv:hep-th/0207190].

[14] N. Nakayama, K. Sugiyama and K. Yoshida, “Ground state of the supermembrane on a pp-wave,” arXiv:hep-th/0209081, to appear in Phys. Rev. D.

[15] T. Banks, N. Seiberg and S. H. Shenker, Nucl. Phys. B 490 (1997) 91 [arXiv:hep-th/9612157].

[16] K. Sugiyama and K. Yoshida, “Type IIA string and matrix string on pp-wave,” Nucl. Phys. B 644 (2002) 128 [arXiv:hep-th/0208029].

[17] S. Hyun and H. Shin, “N = (4,4) type IIA string theory on pp-wave background,” JHEP 0210 (2002) 070 [arXiv:hep-th/0208074].

[18] S. Hyun and H. Shin, “Solvable N = (4,4) type IIA string theory in plane-wave background and D-branes,” Nucl. Phys. B 654 (2003) 114 [arXiv:hep-th/0210158].

[19] A. Dabholkar and S. Parvizi, “Dp branes in pp-wave background,” Nucl. Phys. B 641 (2002) 223 [arXiv:hep-th/0203231].

[20] S. Hyun, J. Park and H. Shin, “Covariant description of D-branes in IIA plane-wave background,” Phys. Lett. B 559 (2003) 80 [arXiv:hep-th/0212343].

[21] O. K. Kwon and H. Shin, “Type IIA supergravity excitations in plane-wave background,” arXiv:hep-th/0303153.

[22] S. Hyun, J. D. Park and S. H. Yi, “Thermodynamic behavior of IIA string theory on a pp-wave,” arXiv:hep-th/0304239.
[23] Y. Sugawara, “Thermal amplitudes in DLCQ superstrings on pp-waves,” Nucl. Phys. B \textbf{650} (2003) 75 [arXiv:hep-th/0209145].

[24] Y. Sugawara, “Thermal partition function of superstring on compactified pp-wave,” arXiv:hep-th/0301035.

[25] O. Bergman, M. R. Gaberdiel and M. B. Green, “D-brane interactions in type IIB plane-wave background,” JHEP \textbf{0303} (2003) 002 [arXiv:hep-th/0205183].

[26] M. R. Gaberdiel and M. B. Green, “The D-instanton and other supersymmetric D-branes in IIB plane-wave string theory,” arXiv:hep-th/0211122.

[27] T. Takayanagi, “Modular invariance of strings on pp-waves with RR-flux,” JHEP \textbf{0212} (2002) 022 [arXiv:hep-th/0206010].

[28] M. R. Gaberdiel, M. R. Green, S. S.-Nameki and A. Sinha, “Oblique and curved D-branes in IIB plane-wave string theory,” arXiv:hep-th/0306056.

[29] M. B. Green and M. Gutperle, “Light-cone supersymmetry and D-branes,” Nucl. Phys. B \textbf{476} (1996) 484 [arXiv:hep-th/9604091].

[30] M. Billo and I. Pesando, “Boundary states for GS superstrings in an Hpp wave background,” Phys. Lett. B \textbf{536} (2002) 121 [arXiv:hep-th/0203028].

[31] M. Sakaguchi, “IIB pp-waves with extra supersymmetries,” arXiv:hep-th/0306009.