TIME SINGULARITIES FOR 3D NAVIER-STOKES EQUATIONS
IN HOMOGENEOUS LEBESGUE SPACES

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Abstract. We prove a local-in-time regularity criterion for 3D Navier-Stokes equations. In particular from the criterion we obtain a new partial regularity result on the dimension of possible singular times. It is shown that the Hausdorff dimension of possible singular times for weak solutions $u \in L^s([0, T] \times \mathbb{R}^3)$ with $4 \leq s \leq 5$ is at most $\frac{5}{2} - \frac{s}{2}$ improving the previous bound $\frac{1}{2}$.

1. Introduction

Consider the 3D Navier-Stokes equations in the whole space

$$\frac{du}{dt} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p$$
$$\nabla \cdot u = 0.$$ \hfill (1.1)

where $u$ is the unknown vector field that describes velocity of the flow, $p$ is the scalar function that stands for the pressure of the fluid and $\nu$ is the positive coefficient of viscosity. The problem is supplemented by divergence free data $u_0 \in L^2$ ($\mathbb{R})^3$.

Even though weak solution has been constructed via various methods, the global regularity of (1.1) remains open. Extensive studies of global regularity had been initiated but only conditional or partial results are available. For example, if $u \in L^s_t L^p_x$ for some $\frac{2}{p} + \frac{3}{s} \leq 1$ then the solution is regular [1]. There is a long history of improvements of this conditional regularity result. The limit case above $s = 3$ is solved by Escauriaza, Seregin and Sverak in [4]. Note that their result is actually local and we will talk about this below.

Since it is very difficult to prove the regularity of weak solutions, the theory of partial regularity of the solutions of (1.1) arises, which focuses on estimating the size of the singular set in space and time. There is locality nature in this matter and instead of proving regularity in the whole $(0, T) \times \mathbb{R}^3$ local regularity results or criteria are considered.

Definition 1.1. Given a weak solution $u$ of (1.1), the singular set $\mathcal{S}(u) \subset \mathbb{R}^+ \times \mathbb{R}^3$ is the set in which $u(x, t)$ is not locally bounded.

Due to the parabolic nature of (1.1), it is natural to consider local regularity on parabolic cylinder $Q_r(x, t) = B_r(x) \times [t - r^2, t]$. At first glance $L^\infty$ does not seem to be very regular, but this definition makes sense due to the classical result of Serrin[5] in which he proved that if $u \in L^s_t L^p_x(Q_r)$ for $\frac{2}{p} + \frac{3}{s} < 1$ then $\partial_x^k u(x, t) \in C^\alpha(Q_r)$ for some $\alpha > 0$ and any $k > 0$. Later this result was improved by Struwe [6] requiring only $\frac{2}{p} + \frac{3}{s} = 1$ for $s < \infty$ and extended to $s = \infty$ in [4].

The following partial regularity results are known. The set of singular times, the projection of $\mathcal{S}(u)$ have zero $\frac{1}{2}$-Hausdorff measure $\mathcal{H}^{\frac{1}{2}}(\Pi_t(\mathcal{S}(u)))$ for weak solutions satisfying energy inequality including Leray-Hopf weak solution, see for example.

Date: June 16, 2018.
The energy inequality
\[ \|u(t)\|_2^2 + 2 \int_{t_0}^t \|\nabla u(s)\|_2^2 ds \leq \|u(0)\|_2^2 \]
for a.e. \( t_0 \) and all \( t > t_0 \) is crucial here because it guarantees the uniqueness of strong solution in the class of Leray-Hopf weak solution. This result dates back to Leray but it was implicit there. The latest attention in this field was brought to us by Scheffer, which leads to the well-known theorem of Caffarelli, Kohn and Nirenberg [3], the best partial regularity result so far. After introducing the notion of suitable weak solution that satisfy local energy inequality, they prove that the 1-dimensional parabolic Hausdorff measure of \( S(u) \) is zero. The solutions constructed by Leray are suitable but the suitability Leray-Hopf weak solutions constructed by Galerkin approximation is unknown. Without the help of local energy inequality it is extremely difficult to establish any local space-time regularity result.

Before diving into the discussion of main results in this paper, let us briefly explain the issue of supercriticality. The 3D Navier-Stokes equations are known to be supercritical, which means available \emph{a priori} bounds are not strong enough to control the solution. In order to guarantee regularity, one usually needs to impose some kind of condition that is subcritical or critical with respect to (1.1). Current techniques are not very effective in dealing with supercritical equations. The best possible result so far can only beat criticality by a logarithmic amount. See for instance [8].

Since there is little hope to overcome the supercriticality, we try to bridge the two ends of conditional regularity and partial regularity together. More precisely, we examine the following question: if we assume \( u \in L^s_t L^r_x \) for some \( \frac{3}{2} > \frac{s}{r} + \frac{3}{s} > 1 \) can we get better bounds in partial regularity dimensions? Notice \( \frac{s}{2} + \frac{3}{s} = \frac{3}{2} \) is satisfied for any weak solution by interpolation. In this paper we answer this question by showing that if \( u \in L^s([0, T] \times \mathbb{R}^3) \) for \( 4 \leq s \leq 5 \) then the time singular set has Hausdorff dimension less than \( \frac{1}{2} \) improving the previous \( \frac{1}{2} \) for Leray-Hopf weak solution. This can be viewed as an interpolation between the Ladyzenskaya-Prodi-Serrin regularity criteria and Scheffer’s result on Hausdorff dimension of the time singular set.

Our main results are as follows.

**Theorem 1.2.** Let \( u(t) \) be weak solution of (1.1). Suppose \( u \in L^s((0, T) \times \mathbb{R}^3) \) for some \( 4 \leq s \leq 5 \), then for possible singular times we have
\[ \mathcal{H}^{\frac{2s}{r+s}}(S_T) = 0. \]

The exact definition of singular set \( S_T \) will be introduced later. It is clear that when the parameter \( s \) ranges from 4 to 5 the dimension of singular times decreases accordingly. The dimension is sharp with respect to the scaling leaving it interesting to consider the regime \( \frac{10}{3} \leq s < 4 \). Since when \( s \geq 4 \) any weak solution in this function class become Onsager critical meaning that it automatically satisfies energy equality. Theorem 1.2 is an application of the following local-in-time regularity criterion in terms of Besov norm. To the author’s best knowledge, it is the first result of such type.

**Theorem 1.3.** There exist universal constants \( \delta > 0 \) and \( C_2 > 0 \) with the following property: if for a Leray-Hopf weak solution \( u \in L^s([0, T]; B^0_{s,s}) \) with \( 3 \leq s \leq 5 \) we have
\[ \limsup_{p \to \infty} \lambda_p^{s-s} \int_{t_0}^t \sum_{q \geq p-2} \|u_q\|_{s}^s dt \leq \delta^s, \]
then there exist a integer $p > 0$ and an interval $[\tau_p, t_0] \subset [t_0 - \lambda_p^{-2}, t_0]$ such that
$$
\sum_{q \geq p^{-2}} \| u_q \|_s^s \leq C_2 \lambda_p^{s-3}
$$
on $[\tau_p, t_0]$.

Remark 1.4. It’s worth noting that we actually obtain $\mathcal{H}^{5/3} (\mathcal{S}_T) = 0$ for $u \in L^s (B_{0, s}^0)$. One may push the argument to obtain better results in larger spaces. We will address this issue in future studies.

When $s = 5$ we recover a known borderline case of the famous Ladyzhenskaya-Prodi-Serrin regularity criteria.

Theorem 1.5. Weak solutions in $L^5 ([0, T]; B_{0, 5}^0)$ are regular.

When $s = \frac{10}{3}$ the space $L^{\frac{10}{3}} ([0, T] \times \mathbb{R}^3)$ is an interpolation of the energy spaces, for which we obtain a new criterion at the first time of blowup for smooth solutions.

Theorem 1.6. Suppose $u$ is a Leray-Hopf weak solution on $[0, T)$ with smooth initial data where $T$ is first time of possible blowup. Then $u$ is regular on $[0, T]$ if $u$ satisfies
$$
\limsup_{p \to \infty} \lambda_p^{\frac{5}{3}} \int_{T-\lambda_p^{-2}}^T \| u \|_{\frac{10}{3}}^{\frac{10}{3}} dt \leq \delta^*,
$$
where $\delta^*$ is a universal constant.

Remark 1.7. Note that any weak solution $u$ verifies $\int_{T-\lambda_p^{-2}}^T \| u \|_{\frac{10}{3}}^{\frac{10}{3}} dt \to 0$ as $p \to \infty$.

On the other hand our condition requires $\int_{T-\lambda_p^{-2}}^T \| u \|_{\frac{10}{3}}^{\frac{10}{3}} dt \leq O(\lambda_p^{-\frac{5}{3}})$.

The paper is organized as follows. In Section 2 we state some preliminaries on properties of weak solutions and Littlewood-Paley theory. Section 3 is devoted for two main propositions. We formulate our estimate there using Besov space for optimal results although our argument does not rely on the theory of Besov spaces. Finally with all ingredients in hand we prove main theorems in Section 4.

2. Preliminaries

2.1. Notations. We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some absolute constant $C$, and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with some absolute constants $C_1, C_2$. We write $\| \cdot \| = \| \cdot \|_{L^p}$ for Lebesgue norms. The symbol $(\cdot, \cdot)$ stands for the $L^2$-inner product. For any $p \in \mathbb{N}$ and $t > 0$ we let $\lambda_p = 2^p$ be the standard dyadic number and $I_p(t) = [t - \lambda_p^{-2}, t]$ be the dyadic time interval.

2.2. Weak solution.

Definition 2.1. A weak solution to (1.1) on $[0, T]$ (or $(0, \infty)$) with divergence-free initial data $u_0 \in L^2 (\mathbb{R}^3)$ is a function $u \in C_w (0, T; L^2 (\mathbb{R}^3)) \cap L^2 (0, T; \mathcal{H}^1 (\mathbb{R}^3))$ satisfying
$$
(u(t), \phi(t)) - (u_0, \phi(0)) = \int_0^t (u(s), \partial_s \phi(s)) + (\nabla u(s), \nabla \phi(s)) + (u(s) \cdot \nabla u(s), \phi(s)) ds,
$$
(2.1)
$\nabla u(t) = 0$ in the sense of distribution for all $t \in [0, T]$ and all divergence-free test functions $\phi \in C_0^\infty ([0, T] \times \mathbb{R}^3)$. 


A weak solution that satisfies energy inequality
\[ \| u(t) \|^2_2 + 2 \int_{t_0}^t \| \nabla u \|^2_2 \leq \| u(t_0) \|^2_2, \]  
(2.2)
for almost all \( t_0 \in (0, T) \) and all \( t \in (t_0, T] \) is called a Leray-Hopf solution. A major difference between general weak solution and Leray-Hopf solution is the weak-strong uniqueness, namely strong solution is unique in the class of Leray-Hopf solution. With this property we only need to consider blowup from the left.

**Theorem 2.2 (Leray).** Let \( u \) be a Leray-Hopf solution of (1.1). If \( u \) is regular on \([\alpha, \beta)\) and
\[ \limsup_{t \to \beta^-} \| \nabla u(t) \|_2 < \infty \]  
(2.3)
then \( u \) is regular on \([\alpha, \beta + \epsilon]\) for some small \( \epsilon \).

Note that by the classical result of Lions any weak solution \( u \in L^s(0, T; \mathbb{R}^3) \) for some \( s \geq 4 \) satisfies energy equality which means it possesses all the properties of Leray-Hopf weak solutions.

### 2.3. Littlewood-Paley decomposition

We introduce a standard Littlewood-Paley decomposition. Let \( \chi: \mathbb{R}^+ \to \mathbb{R} \) be a smooth function so that \( \chi(\xi) = 1 \) for \( \xi \leq \frac{3}{4} \), and \( \chi(\xi) = 0 \) for \( \xi \geq 1 \). We further define \( \varphi(\xi) = \chi(\lambda_1^{-1}\xi) - \varphi(\xi) \) and \( \varphi_q(\xi) = \varphi(\lambda_q^{-1}\xi) \). For a tempered distribution vector field \( u \) let us denote
\[ u_q = F^{-1}(\varphi_q) * u \quad \text{for } q > -1 \quad u_{-1} = u_q = F^{-1}(\chi) * u, \]
where \( F \) is the Fourier transform. With this we have \( u = \sum_{q \geq -1} u_q \) in the sense of distribution.

Also let us finally note that the Besov space \( B^0_{s,a} \) is the space consisting of all tempered distributions \( u \) satisfying
\[ \| u \|_{B^0_{s,a}} := \sum_{q \geq -1} \| u_q \|_a^s < \infty. \]

### 3. Regularity away from \( S_T \)

#### 3.1. Definition of singular points

For any \( p \in \mathbb{N} \) we let \( I_p(t) = [t - \lambda_p^{-2}, t] \) as the dyadic time interval.

**Definition 3.1.** Let \( u \) be a weak solution to (1.1) on \([0, T] \). For any \( p \in \mathbb{N} \) a point \( t_0 \subset (0, T] \) is said to be a bad point to \( u \) if
\[ \limsup_{p \to \infty} \lambda_p^{5-s} \int_{I_p(t_0)} \sum_{r \geq p^{-2}} \| u_r(t) \|_a^s dt \geq \delta^s, \]  
(3.1)
where the implied constant shall be determined later in the second step argument.

Denote \( S_T \) the union of bad points. A simple covering argument shows the following fractal bound.

**Lemma 3.2.** Let \( u \) be a weak solution with \( u \in L^s((0, T) \times \mathbb{R}^3) \). We have \( \mathcal{H}^{5-s}(S_T) = 0 \).

**Proof.** We observe that thanks to Vitali lemma for each \( p \in \mathbb{N} \), \( S_T \) can be covered by finitely many \( 5I_{p_i}(t_i) \) with \( I_{p_i}(t_i) \) being disjoint and \( p_i \geq p \) such that
\[ \lambda_{p_i}^{5-s} \int_{I_{p_i}(t_i)} \sum_{r \geq p_i^{-2}} \| u_r(t) \|_a^s dt \geq \delta. \]
Consider the sum
\[ \sum_{p} \lambda_p^{s-5} \lesssim \sum_{p} \int_{I_p(t)} \sum_{r \geq p-2} \|u_r(t)\|_s^2 dt \leq \int_{U_p} \|u\|_s^2 dt \]
where \(U_p\) is the union of \(I_p\)'s.

By using the fact \(u \in L^4((0,T) \times \mathbb{R}^3)\) and the absolute continuity of Lebesgue integral we obtain \(\sum_{p} \lambda_p^{s-5}\) goes to 0 as \(p \to \infty\). \(\square\)

The next proposition is a standard application of Littewood-Paley and paraproduct theory.

**Proposition 3.3.** Let \(u\) be a weak solution to (1.1) and \(s \geq 2\). Then \(\|u_q(t)\|_s^{s-1}\) is absolute continuous and for a.e. \(t \in [0,T]\)
\[
\frac{d}{dt}\|u_q(t)\|_s + \lambda_q^{s/2} \|u_q(t)\|_s \lesssim \sum_{\substack{p \leq q \\cap \\mathbb{N}}} \lambda_p^{s/2} \|u_p\|_s \sum_{|p-q| \leq 2} \lambda_p \|u_p\|_s + \lambda_q^{s/2+1} \sum_{p \geq q-2} \|u_p\|_s^2.
\]

**Remark 3.4.** The proof is just a standard application of Littewood-Paley and paraproduct theory. For the sake of completeness we sketch one in the appendix.

### 3.2. Step 1: critical regularity.

**Proposition 3.5.** There exists a constant \(C_1\) such that for any \(\delta > 0\) if a Leray-Hopf weak solution \(u \in L^4(0,T; \mathcal{B}_{s_k})\) with \(s \geq 2\) verifies the bound
\[
\limsup_{p \to \infty} \lambda_p^{s-5} \int_{I_p(t_0)} \sum_{r \geq p-2} \|u_r(t)\|_s^2 dt \leq \delta^s,
\]
then
\[
\limsup_{q \to \infty} \lambda_q^{s-1} \sup_{I_q(t_0)} \|u_q\|_s \leq C_1 \delta.
\]

**Proof.** By the definition of \(\limsup\) there exist \(p_0\) such that for any \(p \geq p_0\)
\[
\int_{I_p(t_0)} \sum_{r \geq p-2} \|u_r(t)\|_s^2 dt \leq 2\delta \lambda_p^{s-5}. \quad (3.2)
\]

Furthermore there exists \(p_1(p_0, \|u_0\|) > p_0\) such that \(\lambda_{p_0}^{s-3} \|u_0\|_2^3 \lesssim \lambda_{p_1}\). Since \(u \in L^4((0,T) \times \mathbb{R}^3)\) for \(p \geq p_1\) there exist \(t_p \in I_p(t_0)\) such that
\[
\sum_{r \geq p-2} \|u_r(t_p)\|_s^2 \lesssim \delta \lambda_p^{s-3}. \quad (3.3)
\]

Since \(\|u_p(t)\|_s\) is continuous let \(t_p^*\) be such that \(\|u_p(t_p^*)\|_s = \sup_{I_p(t_0)} \|u_p\|_s\). By the last Proposition we integrate from \(t_p\) to \(t_p^*\) for \(\frac{d}{dt}\|u_q(s)\|_s^2\) to find that
\[
\sup_{I_p(t_0)} \|u_p\|_s^2 - \|u_p(t_p)\|_s^2 + \int_{t_p}^{t_p^*} \lambda_p^{3/2} \|u_p\|_s^2 dt \lesssim \int_{t_p}^{t_p^*} \frac{\lambda_p^{3/2} \|u_p\|_s^2}{\sum_{|p-p'| \leq 2} \lambda_{p'} \|u_{p'}\|_s} \sum_{|p'-p| \leq 2} \lambda_{p'} \|u_{p'}\|_s \|u_p\|_s^{s-1} dt
\]
\[+ \int_{t_p}^{t_p^*} \lambda_p^{3/2+1} \sum_{r \geq p-2} \|u_r\|_s^2 \|u_p\|_s^{-1} dt.\]

We use triangle inequality to find that
\[
\sup_{I_p(t_0)} \|u_p\|_s^2 - \|u_p(t_p)\|_s^2 \lesssim + \int_{I_p(t_0)} \lambda_p^{3/2} \|u_p\|_s^2 dt + \int_{I_p(t_0)} \frac{\lambda_p^{3/2} \|u_p\|_s^2}{\sum_{|p-p'| \leq 2} \lambda_{p'} \|u_{p'}\|_s} \sum_{|p'-p| \leq 2} \lambda_{p'} \|u_{p'}\|_s \|u_p\|_s^{s-1} dt
\]
\[+ \int_{I_p(t_0)} \lambda_p^{3/2+1} \sum_{r \geq p-2} \|u_r\|_s^2 \|u_p\|_s^{-1} dt := A + B + C. \quad (3.4)
\]
For the first term on the right we have by the assumption that
\[ A \sim \int_{I_p(t_0)} \lambda_p^a \|u_p\|_s^a dt \lesssim \delta \lambda_p^{a-3}. \quad (3.5) \]

For the second term on the right by Hölder inequality with exponents \((s, s, \frac{s}{2}, \infty)\) we need to estimate
\[
\left[ \int_{I_p} \left[ \sum_{p' \leq p} \lambda_p^a \|u_p\|_s^a \right] ds \right]^\frac{1}{a} \left[ \int_{I_p} \lambda_p^a \sum_{|p'| - p| \leq 2} \|u_{p'}\|_s^a ds \right]^\frac{1}{a} \left[ \int_{I_p} \|u_p(s)\|_s^a ds \right] \sup_{I_p} \|u_p(s)\|_s.
\]

By (3.2) the above is bounded by
\[
\delta^{a-1} \left[ \int_{I_p} \left[ \sum_{p' \leq p} \lambda_p^a \|u_p\|_s^a \right] ds \right]^\frac{1}{a} \sup_{I_p} \|u_p\|_s \lambda_p^{\frac{a-3}{2a}}.
\]

To bound the first part above, we split the low-modes sum accordingly
\[
\int_{I_p} \left[ \sum_{p' \leq p} \lambda_p^a \|u_p\|_s^a \right] ds \leq \lambda_p^a \int_{I_p} \sum_{p' \leq p} \|u_p\|_s^a ds \leq \lambda_p^a \sum_{p' \leq p} \int_{I_p} \|u_p\|_s^a ds
\]
\[
\leq \lambda_p^a \sum_{p_0 \leq p' \leq p} \int_{I_{p'}} \|u_p\|_s^a ds + \lambda_p^a \sum_{p' \leq p_0} \int_{I_{p'}} \|u_p\|_s^a ds.
\]

Using (3.3) and Bernstein inequality we obtain
\[
\sum_{p \leq p_0} \int_{I_{p'}} \|u_p\|_s^a ds \lesssim \lambda_p^{\frac{a-3}{2}} \|a_0\|_2^2 \leq \lambda_p^{\frac{a-3}{2}}.
\]

By (3.2) we know that for any \(p \geq p_0\) the bound \(\int_{I_{p'}} \|u_p\|_s^a ds \lesssim \delta \lambda_p^{a-2}\) holds. And thus \(\lambda_p^a \sum_{p_0 \leq p' \leq p} \int_{I_{p'}} \|u_p\|_s^a ds \lesssim \delta \lambda_p^{a-2}\).

Putting together we have
\[
\int_{I_p} \left[ \sum_{p' \leq p} \lambda_p^a \|u_p\|_s^a \right] ds \lesssim \delta^{a} \lambda_p^{a-2}.
\]

So term \(B\) verifies the estimate:
\[
B \lesssim \delta^{a} \lambda_p^{\frac{(a-1)(a-3)}{2}} \sup_{I_p} \|u_p(s)\|_s.
\]

Next we apply Hölder inequality for term \(C\) to obtain
\[
\int_{I_p} \sum_{p' \geq p-2} \|u_{p'}\|_s^2 \|u_p\|_s^{a-1} dt \lesssim \sup_{I_p} \|u_p(s)\|_s \int_{I_p} \sum_{p' \geq p-2} \|u_{p'}\|_s^a dt,
\]
which can further be bounded by
\[
\delta^{a} \lambda_p^{\frac{(a-1)(a-3)}{2}} \sup_{I_p} \|u_p(s)\|_s.
\]

Putting together the estimates for \(A, B\) and \(C\) we have that
\[
\sup_{I_p} \|u_p(s)\|_s^a \lesssim \delta^{a} \lambda_p^{a-3} + \delta^{a} \lambda_p^{\frac{(a-1)(a-3)}{2}} \sup_{I_p} \|u_p\|_s.
\]

Using for example Young’s inequality finishes the proof. \qed
3.3. Step 2: Local-in-time regularity. The regularity in Proposition 3.5 is not enough to obtain smoothness of \(u\). We will close this gap by a continuity argument in the following theorem.

**Theorem 3.6.** There exist universal constants \(\delta > 0\) and \(C_2 > 0\) with the following property: if for a weak solution \(u \in L^s(0, T; B^0_{s,\alpha})\) with \(3 \leq s \leq 5\) we have

\[
\limsup_{p \to \infty} \lambda_p^{5-s} \int_{I_p(t_0)} \sum_{q \geq p-2} \|u_q\|_s^s dt \leq \delta^s
\]

then there exist a integer \(p > 0\) and an interval \([\tau_p, t_0]\) \(\subset I_p(t_0)\) such that

\[
\sum_{q \geq p-2} \|u_q\|_s^s \leq C_2 \lambda_p^{-s-3}
\]
on \([\tau_p, t_0]\).

**Proof.** The exact value of \(\delta\) will be choosen in the end while \(C_2\) will be a constant from applying various inequalities.

First of all by Proposition 3.5 there exists \(p_0\) such that for any \(p \geq p_0\) the following 2 conditions hold

\[
\lambda_p^{5-s} \int_{I_p(t_0)} \sum_{q \geq p-2} \|u_q\|_s^s dt \leq 2 \delta^s, \quad \lambda_p^{s-1} \sup_{I_p(t_0)} \|u_p\|_s \leq C_1 \delta.
\]

To handle the low modes errors in the later estimates, we introduce the lower bound \(p_1 = p_1(p_0, \delta, \|u_0\|_2) \geq p_0\) so that

\[
\lambda_{p_0}^{4} \|u_0\|_2^{2-\delta} \leq \delta^{s-1} \lambda_{p_1}^{2-\delta}. \tag{3.9}
\]

The goal is to to show that for any \(p \geq p_1\) the bound \(\sum_{q \geq p-2} \|u_q\|_s^s \leq \delta^s \lambda_p^{-s-3}\) holds on some interval.

By the first condition in (3.7) there exists \(\tau_p \in [t \leftarrow \lambda_p^{-2}, t_0]\) such that

\[
\sum_{q \geq p-2} \|u_q(\tau_p)\|_s^s dt \leq 2 \delta^s \lambda_p^{-s-3}, \tag{3.10}
\]
i.e. the desired bound is satisfied.

By local existence and uniqueness theory for Leray-Hopf weak solutions, there exists an nonempty interval \([\tau_p, t_p]\) on which

\[
\sum_{q \geq p-2} \|u_q\|_s^s \leq 4 \delta^s \lambda_p^{-s-3}. \tag{3.11}
\]

Next we will use a continuity argument to show that if the above inequality holds on the interval \([\tau_p, t_p]\), then \(\sum_{q \geq p-2} \|u_q(t_p)\|_s^s \leq 2 \delta^s \lambda_p^{-s-3}\).

Consider the equation for \(u_q\) on \([\tau_p, t_p]\) for every \(q \geq p-2\) in the following form:

\[
\frac{d}{dt}\|u_q\|_s^{s-1} + \lambda_q^2 \|u_q\|_s^{s-1} \preceq \sum_{p' \leq q} \lambda_{p'}^{2} \|u_{p'}\|_s^{s-1} \lambda_q \sum_{|p' - q| \leq 2} \|u_{p'}\|_s + \lambda_q \sum_{r \geq q-2} \|u_r\|_s. \tag{3.12}
\]

We will bound the terms on the right hand side of (3.12) on \([\tau_p, t_p]\), namely \(\sum_{p \leq q} \lambda_{p'}^{2} \|u_{p'}\|_s^{s-1}, \sum_{|p' - q| \leq 2} \|u_{p'}\|_s, \) and \(\sum_{r \geq q-2} \|u_r\|_s\).

Firstly we consider the split

\[
\sum_{p' \leq q} \lambda_{p'}^{2} \|u_{p'}\|_s^{s-1} \leq \sum_{p \leq p' \leq q} \lambda_{p'}^{2} \|u_{p'}\|_s^{s-1} + \sum_{p \leq p' \leq p-2} \lambda_{p'}^{3} \|u_{p'}\|_s^{s-1} + \sum_{p' \leq p_0} \lambda_{p'}^{3} \|u_{p'}\|_s^{s-1}.
\]
The idea is to bound modes below \( p_0 \) by energy, modes between \( p_0 \) and \( p - 2 \) by critical regularity and modes above \( p - 2 \) by our hypothesis. By Jensen inequality we have

\[
\sum_{p' = 2}^{p_0} \lambda_{p'}^2 \| u_{p'} \|^s_{s-1} \leq \lambda_{p_0}^2 \left[ \sum_{p = 2}^{p_0} \| u_p \|^s_{s} \right]^{s-1} \leq \lambda_{p_0}^2 \lambda_p^{(s-1)(p_0-1)} \delta^s \lambda_{p_0}^{-3}
\]

for the first part. Recall from (3.7) that for any \( r \geq p_0 \) we have \( \| u_r \|^s_s \lesssim \delta^s \lambda_r^{-3} \) and on the other hand \( t_p \in [t_{p}, t_0) \subset I_p(t_0) \). It follows that for any \( t \in [t_{p}, t_p] \)

\[
\sum_{p' = 2}^{p - 2} \lambda_{p'}^2 \| u_{p'} \|^s_{s-1} \lesssim \lambda_{p_0}^2 \lambda_p^{2-s+2-s} \sum_{p' \leq p_0} \lambda_{p'}^{2-s} \| u_{p'} \|^s_s \lesssim \delta^s \lambda_{p_0}^{-3} \lambda_p^{2-s+2-s} \sum_{p' \leq p_0} \lambda_{p'}\| u_{p'} \|^s_s \lesssim \delta^s \lambda_{p_0}^{-3} \lambda_p^{2-s+2-s} \sum_{p' \leq p_0} \lambda_{p'}\| u_{p'} \|^s_s.
\]

For the last part Bernstein inequality and energy inequality imply that

\[
\sum_{p' \leq p_0} \lambda_{p'}^2 \| u_{p'} \|^s_s \lesssim \lambda_{p_0}^4 \| u_0 \|^s_{s-1} \lesssim \delta^s \lambda_{p_0}^{-3} \lambda_p^{2-s+2-s} \sum_{p' \leq p_0} \lambda_{p'}\| u_{p'} \|^s_s \lesssim \delta^s \lambda_{p_0}^{-3} \lambda_p^{2-s+2-s} \sum_{p' \leq p_0} \lambda_{p'}\| u_{p'} \|^s_s.
\]

For the term \( \sum_{p' = q}^{q} \| u_{p'} \|_s \) we have \( \sum_{p' = q}^{q} \| u_{p'} \|_s \lesssim \delta \lambda_{p_0}^{q-\frac{2}{s}} \) by (3.7) and (3.11).

Combining these two the first part of the nonlinear term verifies

\[
\sum_{p' \leq q} \lambda_{p'}^2 \| u_{p'} \|^s_s \lesssim \delta \lambda_{p_0}^{q-\frac{2}{s}} \lambda_q^{q-3-s}.
\]

(3.13)

For the last term on the right we once again use (3.7) and (3.7) in the summation to obtain

\[
\lambda_{q}^{q} \sum_{p' \geq q} \| u_{p'} \|^s_s \lesssim \delta q \lambda_{q}^{q+1} \lambda_p^{-3-s}.
\]

(3.14)

Putting (3.13) and (3.14) together we obtain the growth from the right hand side is bounded by \( \delta \lambda_{p_0}^{q+2} \lambda_p^{-3} \).

So we obtain on \( [t_{p}, t_p] \) the differential inequality:

\[
\frac{d}{dt} \| u_q \|^s_s \lesssim \delta \lambda_{q}^{q+1} \lambda_p^{-3-s}.
\]

By Gronwall inequality we have for any \( t \in [t_{p}, t_p] \) that

\[
\| u_q(t) \|^s_s \leq \| u_q(t_{p}) \|^s_s e^{-\delta \lambda_{q}^{q+1} (t-t_{p})} + c \left[ 1 - e^{-\delta \lambda_{q}^{q+1} (t-t_{p})} \right] \delta \lambda_{q}^{q+1} \lambda_p^{-3-s},
\]

(3.15)

for every \( q \geq p - 2 \), where \( c > 0 \) is a universal constant.

We choose the value of \( \delta \leq \frac{1}{84c} \) and define the index set \( I_p \subset \mathbb{Z} \) in the following manner:

\[
I_p := \{ q : q \geq p - 2 \text{ and } \| u_q(t_p) \|^s_s \geq \frac{1}{8} \delta \lambda_{q}^{q+1} \lambda_p^{-3-s} \}.
\]

(3.16)

From this we have the following decomposition:

\[
\sum_{q \geq p-2} \| u_q(t_p) \|^s_s = \sum_{q \in I_p} \| u_q(t_p) \|^s_s + \sum_{q \notin I_p} \| u_q(t_p) \|^s_s.
\]

On the one hand for \( q \in I_p \) by (3.15) and (3.16) we obtain

\[
\| u_q(t_p) \|^s_s \leq \| u_q(t_{p}) \|^s_s e^{-\delta \lambda_{q}^{q+1} (t_{p}-t_{p})} + \frac{1}{8} \left[ 1 - e^{-\delta \lambda_{q}^{q+1} (t_{p}-t_{p})} \right] \| u_q(t_{p}) \|^s_s \leq \frac{9}{8} \| u_q(t_{p}) \|^s_s.
\]

Taking a summation in \( I_p \) yields

\[
\sum_{q \in I_p} \| u_q(t_p) \|^s_s \leq \frac{9}{8} \sum_{q \in I_p} \| u_q(t_p) \|^s_s \leq \frac{9}{8} \delta \lambda_{p}^{s-3-s}.
\]

where we have used the fact that \( \sum_q \| u_q(t_p) \|^s_s \leq \delta \lambda_{p}^{s-3} \).
On the other hand for \( q \notin \mathcal{I}_p \) once again by (3.15) and (3.16) we obtain
\[
\|u_q(t_p)\|_s^{s-1} < \frac{1}{8} \delta^{s-1} \lambda_q^{\frac{s}{p}} \lambda_p^{s-3} e^{-c \lambda_q^2 (t_p - \tau_p)} + \frac{1}{64} [1 - e^{-c \lambda_q^2 (t_p - \tau_p)}] \delta^{s-1} \lambda_q^{\frac{s}{p}} \lambda_p^{s-3} 
\leq \frac{9}{64} \delta^{s-1} \lambda_q^{\frac{s}{p}} \lambda_p^{s-3}.
\]
Since \( 4 \leq s \leq 5 \) taking a summation in \( \mathcal{I}_p \) yields
\[
\sum_{q \in \mathcal{I}_p^C} \|u_q(t_p)\|_s^{s} \leq \frac{9}{64} \sum_{q \in \mathcal{I}_p^C} (\delta^{s-1} \lambda_q^{\frac{s}{p}} \lambda_p^{s-3}) \leq \frac{45}{64} \delta^s \lambda_p^{s-3}.
\]
Combining the decomposition it follows that
\[
\sum_{q \geq p-2} \|u_q(t_p)\|_s^{s} < 2\delta^s \lambda_p^{s-3}.
\]
And hence an iteration of applying local regularity result and the above continuity argument yields the desire bound: \( \sup_{[\tau_p, t_0]} \sum_{q \geq p-2} \|u_q(t_p)\|_s^{s} \leq 4\delta^s \lambda_p^{s-3} \). □

4. PROOF OF MAIN RESULTS

Thanks to the above two theorems, we can prove our results stated in the introduction.

Proof of Theorem 1.2. By Proposition 3.5 and Theorem 3.6, we know that if \( t_0 \notin \mathcal{S}_T \) there exists a small \( \epsilon > 0 \) such that \( u \in L^\infty ([t_0 - \epsilon, t_0]; L^s) \).

Since \( 4 \leq s \leq 5 \), The space \( L^\infty ([t_0 - \epsilon, t_0]; L^s) \) is subcritical to the Navier-Stokes scaling. We can use for instance classical Serrin’s regularity result to bootstrap arbitrary regularity and obtain \( u \in C^\infty ((t_0 - \epsilon', t_0 + \epsilon') \times \mathbb{R}^3) \).

Weak solutions in \( L^s ([0, T] \times \mathbb{R}^3) \) are in the class of Leray-Hopf weak solutions, and therefore by local regularity result for Leray-Hopf weak solutions we can assert \( u \in C^\infty ((t_0 - \epsilon', t_0 + \epsilon') \times \mathbb{R}^3) \) for some small \( \epsilon' > 0 \). □

Proof of Theorem 1.5 and 1.6. To prove Theorem 1.5 we need to show that if \( u \in L^s ([0, T]; B_{\alpha, \delta}^{0, 3} (\mathbb{R}^3)) \), the limsup condition is satisfied for all points. Indeed, for any \( t_0 \in [0, T] \) we know that \( \limsup_{p \to \infty} \int_{t_0}^{\infty} \sum_{q \geq p-2} \|u_q\|_s^2 dt = 0 \).

Theorem 1.6 is a direct consequence of the embedding \( L^{\frac{4s}{4s-3}} (\mathbb{R}^3) \subset B_{\alpha, \delta}^{0, 3} (\mathbb{R}^3) \). □

APPENDIX: PROOF OF PROPOSITION 3.3

We only prove the estimates for strong solutions. To prove the validity for general weak solutions one can use (2.1) in the class of divergence-free Schwartz functions.

Let \( P \) be the Leray projection. Multiplying (1.1) by \( sP \Delta_q (u_q |u_q|^{s-2}) \) and integrating in space yields
\[
\frac{d}{dt} \|u_q\|_s^{s} + s \int \Delta_q u_q u_q |u_q|^{s-2} dx = -s \int P \Delta_q (u \cdot \nabla u) u_q |u_q|^{s-2} dx.
\]
(4.1)

Note that we have used the fact that \( Pu_q = u_q \).

It is known that \( \int \Delta_q u_q u_q |u_q|^{s-2} dx \sim \lambda_q^2 \|u_q\|_s^s \). We also use the following version of paraproduct decomposition:
\[
\Delta_q (u \cdot v) = \sum_{p: |p| \leq q, |q| \leq 2} \Delta_q \langle u_{p \leq q}, v_q \rangle + \sum_{p: |p| \leq q, |q| \leq 2} \Delta_q \langle u_q, v_{p \leq q} \rangle + \sum_{p \geq q-2} \Delta_q \langle u_q, v_q \rangle.
\]
From the above two facts it follows that
\[
\frac{d}{dt} \|u_q\|_s^{s} + \lambda_q^2 \|u_q\|_s^{s} \leq I_1 + I_2 + I_3.
\]
where
\[ I_1 \sim \left| \int \sum_{p:|p-q| \leq 2} \mathbb{P} \Delta_q (u_{\leq p-2} \cdot \nabla u_p) u_q |u_q|^{s-2} \, dx \right|, \]
\[ I_2 \sim \left| \int \sum_{p:|p-q| \leq 2} \mathbb{P} \Delta_q (u_p \cdot \nabla u_{\leq p-2}) u_q |u_q|^{s-2} \, dx \right|, \]
and
\[ I_3 \sim \left| \int \sum_{p: p \geq q-2} \mathbb{P} \Delta_q (\tilde{u}_p \cdot \nabla u_p) u_q |u_q|^{s-2} \, dx \right|. \]

By Hölder inequality and the boundedness of the operator \( \mathbb{P} \Delta_q \) we find:
\[ I_1 \lesssim \sum_{p:|p-q| \leq 2} \| u_{\leq p-2} \cdot \nabla u_p \|_s \| u_q \|_s^{s-1}. \]

It can be further bounded by
\[ \lesssim \sum_{p' \leq q} \| u_{p'} \|_\infty \sum_{p:|p-q| \leq 2} \| \nabla u_p \|_s \| u_q \|_s^{s-1}. \]

Thus Bernstein inequality gives:
\[ I_1 \lesssim \sum_{p' \leq q} \lambda_p \| u_{p'} \|_s \sum_{p:|p-q| \leq 2} \lambda_p \| u_p \|_s \| u_q \|_s^{s-1}. \quad (4.2) \]

For the second term \( I_2 \) Hölder inequality yields
\[ I_2 \lesssim \sum_{p:|p-q| \leq 2} \| u_p \|_s \| \nabla u_{\leq p-2} \|_\infty \| u_q \|_s^{s-1}. \]

Bernstein inequality now gives:
\[ I_2 \lesssim \sum_{p:|p-q| \leq 2} \| u_p \|_s \sum_{p' \leq q} \lambda_{p'}^{\frac{2}{2}} \| u_{p'} \|_s \| u_q \|_s^{s-1}. \quad (4.3) \]

Finally for the last term \( I_3 \) we integrate by parts to obtain:
\[ I_3 \lesssim \left| \int \sum_{p: p \geq q-2} \mathbb{P} \Delta_q (\tilde{u}_p \cdot u_p) \nabla (u_q |u_q|^{s-2}) \, dx \right|. \]

Direct computations and Hölder inequality yield:
\[ I_3 \lesssim \sum_{p: p \geq q-2} \| \tilde{u}_p \otimes u_p \|^{s-2}_s \| \nabla u_q \|_\infty \| u_q \|_s^{s-2}. \]

So we can obtain the desire bound:
\[ I_3 \lesssim \lambda_q^{1+\frac{2}{2}} \sum_{p: p \geq q-3} \| u_p \|_s^{2} \| u_q \|_s^{s-1}. \quad (4.4) \]

Putting the bounds for \( I_1, I_2 \) and \( I_3 \) together and dividing a common factor \( \| u_q \|_s^{s-1} \) we have
\[ \frac{d}{dt} \| u_q (t) \|_s + \lambda_q^2 \| u_q (t) \|_s \lesssim \sum_{p \leq q} \lambda_p^2 \| u_p \|_s \sum_{|p-q| \leq 2} \lambda_p \| u_p \|_s + \lambda_q^{2+1} \sum_{p \geq q-2} \| u_p \|_s^2. \]

Acknowledgement

The author would like to express sincere gratitude to his advisor Professor Alexey Cheskidov for proofreading early drafts and giving many suggestions for improvement.
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