Quasirotational disturbances
of the relativistic string with massive ends
and higher radial excitations of hadrons

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Abstract

For the relativistic string with massive ends the small disturbances of its rotational
motion (quasirotational states) are investigated. They are presented in the form of
Fourier series with the two types of oscillatory modes. They have the form of stationary
waves correspondently in the rotational plane and in the orthogonal direction. The cal-
culated values of the energy and angular momentum of these states give us possibility to
describe higher radial excitations for hadrons with the help of the planar quasirotational
oscillatory modes of the string with massive ends.

Introduction

The relativistic string with massive ends \[1, 2\] includes two massive points connected by
the string that simulates the QCD confinement mechanism. This system may be considered
as the string model of the meson \(q-\bar{q}\) or the quark-diquark model \(q-gq\) of the baryon \[3\].

This model in the natural way describes the (quasi)linear Regge trajectories for the orbital
excitations of mesons and baryons. For this purpose different authors use the rotational
motions (planar uniform rotations of the rectilinear string) \[3, 4, 5, 6\]. But for describing
the higher radial excitations and other hadron excited states we are to use more complicated
string motions than rotational ones. This problem of widening the application area of the string
model onto various types of hadron excited states has not been solved yet in the framework
of the relativistic string with massive ends.

From this point of view the small disturbances of rotational motions (quasirotational states)
of this system obtained in Ref. \[7\] are very interesting. In this paper after the brief review the
classical dynamics for the relativistic string with massive ends in Sect. 1 we consider in Sect. 2
these quasirotational states. In Sect. 3 possibilities of application these states for describing
the higher radial excitations of hadrons are investigated.

1. Dynamics and rotational motions of the string with massive ends

The relativistic string with the pointlike masses \(m_1, m_2\) at the ends of the string is described
by the action \[1\]

\[
S = -\int d\tau \left\{ \gamma \int \frac{\sigma_2(\tau)}{\sigma_1(\tau)} \left[ (\dot{X}, X')^2 - \dot{X}^2 X'^2 \right]^{1/2} d\sigma + \sum_{i=1}^{3} m_i \sqrt{\dot{x}_i^2(\tau)} \right\}.
\]  

(1)
Here $X^\mu(\tau, \sigma)$ are coordinates of a point of the string in $3 + 1$-dimensional Minkowski space $R^{1,3}$ with signature $(+, -, -, \ldots)$ and the (pseudo)scalar product $(a, b) = a^\mu b_\mu$, $\gamma$ is the string tension, the speed of light $c = 1$, $\dot{X}^\mu = \partial_\tau X^\mu$, $X^\mu = \partial_\sigma X^\mu$, $\dot{x}_i^\mu(\tau) = \frac{d}{d\tau} X^\mu(\tau, \sigma_i(\tau))$; $\sigma_i(\tau)$ are inner coordinates of endpoints’ world lines. These massive points simulate the quark-antiquark pair for the meson or the quark and diquark for the baryon model.

The equations of motion and the boundary conditions at the ends result from the action and take the following simplest form

\[ \dot{X}^\mu - X^\nu\gamma^\mu_\nu = 0, \tag{2} \]

\[ m_i \frac{d}{d\tau} U_i^\mu(\tau) + (-1)^i \gamma X^\nu(\tau, \sigma_i) = 0, \quad U_i^\mu(\tau) = \frac{\dot{x}_i^\mu(\tau)}{|X(\tau, \sigma_i)|}, \tag{3} \]

under the conditions $\sigma_1 = 0$, $\sigma_2 = \pi$ for the ends and the orthonormality conditions on the world surface

\[ \dot{X}^2 + X''^2 = 0, \quad (\dot{X}, X') = 0, \tag{4} \]

which always may be stated without loss of generality. Under conditions the equations of motion become linear but the boundary conditions for the massive points remain essentially nonlinear. They make the model much more realistic but they bring additional nonlinearity and (hence) a lot of problems with quantization of this model.

For the relativistic string with massive ends the classical solution describing the rotational motion (planar uniform rotation of the rectilinear string segment) is well known and may be represented in the form

\[ X^\mu(\tau, \sigma) = X^\mu_{rot}(\tau, \sigma) = \Omega^{-1}[\theta \tau e_0^\mu + \cos(\theta \sigma + \phi_1) \cdot e^\mu(\tau)]. \tag{5} \]

Here $\Omega$ is the angular velocity, $e_0^\mu$ is the unit time-like velocity vector of c.m.,

\[ e^\mu(\tau) = e_1^\mu \cos \theta \tau + e_2^\mu \sin \theta \tau \tag{6} \]

is the unit ($e^2 = -1$) space-like rotating vector directed along the string, $\sigma \in [0, \pi]$. The parameter $\theta$ (dimensionless frequency) is connected with the constant speeds $v_1$, $v_2$ of the ends through the relations

\[ v_1 = \cos \phi_1, \quad v_2 = -\cos(\pi \theta + \phi_1), \quad \frac{m_i \Omega}{\gamma} = \frac{1}{v_i} - \frac{v_1^2}{v_i}. \tag{7} \]

The implicit expression (5) under restrictions (6) is the exact solution of the classic equations of motion (2) and satisfies the orthonormality and boundary conditions (4). The classic expressions for the energy $E$ and the angular momentum $J$ (its projection onto $Oz$ or $e_3^\mu$-direction) of the states (5) are

\[ E_{rot} = \sum_{i=1}^{2} \left[ \frac{\gamma \arcsin v_i}{\Omega} + \frac{m_i}{\sqrt{1 - v_i^2}} \right], \quad J_{rot} = \frac{1}{2\Omega} \left\{ \sum_{i=1}^{2} \left[ \frac{\gamma \arcsin v_i}{\Omega} + \frac{m_i v_i^2}{\sqrt{1 - v_i^2}} \right] \right\}. \tag{8} \]

The implicit expression (8) with different form of taking into account quark spins and the spin-orbit correction describes quasilinear Regge trajectories $J = J(E^2)$ with the ultrarelativistic behavior $J \sim \alpha' E^2 - \alpha_1 E^{1/2}$, $E \to \infty$, where the slope has the Nambu value.

\[ \begin{array}{c}
\text{1This approach is applicable for an arbitrary dimensionality of } R^{1,D-1}.
\end{array} \]
\[ \alpha' = (2\pi\gamma)^{-1}. \]

So the rotational motions of the string models \( q-f \) and \( q-qq \) are widely used for modeling the orbitally excited hadron states on the leading (parent) Regge trajectories \( 3, 4, 5, 6 \).

But other excited hadron states, for example, the radial excitations on the daughter Regge trajectories \( 6 \) are so far beyond the field of application for the string model \( 1 \). There were many attempts to specify the motions of the relativistic string \( 1 \), which may be interpreted as the radial excitations of hadrons. In particular, for this purpose small disturbances of the rotational motion \( 5 \) were searched in Refs. \( 8, 9 \). But these attempts were not fruitful. It was shown in Ref. \( 10 \) that these disturbances do not satisfy the rectilinearity ansatz \( 3 \) in the string dynamics. The approach in Refs. \( 9 \) involves the complicated nonlinear form of string motion equations beyond the conditions \( 4 \) and includes some oversimplified assumptions, in particular, neglecting the boundary condition \( 3 \) at the moving end (the string with one fixed end was considered). So the solutions \( 9 \) were not correct (details are in Ref. \( 7 \)).

In Ref. \( 7 \) another approach for obtaining small disturbances of the rotational motion \( 5 \) was suggested for the relativistic string with massive and fixed ends \( 0 < m_1 < \infty, m_2 \to \infty \). In the next section this approach is generalized for the case of two finite masses \( m_1, m_2 \) at the ends of the string.

2. Quasirotational states for the meson string model

The quasirotational states or small disturbances of the rotational motion \( 3 \) are interesting due to the following three reasons: (a) we are to search the motions describing the radially excited hadron states, in other words, we are to describe the daughter Regge trajectories; (b) the quasirotational motions are necessary for solving the problem of stability of rotational states \( 3 \);

(c) the quasirotational states are the basis for quantization of these nonlinear problems in the linear vicinity of the solutions \( 3 \) (if they are stable).

For obtaining the quasirotational solutions of slightly curved string \( 1 \) the following approach was suggested in Ref. \( 7 \). Under the orthonormality conditions \( 4 \) we use the linear equations of motion \( 2 \) and substitute their general solution

\[
X^\mu(\tau, \sigma) = \frac{1}{2} \left[ \Psi^\mu_+(\tau + \sigma) + \Psi^\mu_-(\tau - \sigma) \right]
\]  

(9)

into the boundary conditions \( 3 \). So the problem is reduced to the system of ordinary differential equations with shifted arguments. The unknown function may be \( \Psi^\mu_+(\tau) \), \( \Psi^\mu_-(\tau) \), or unit velocity vectors \( 3 \) of the endpoints \( U^\mu_1(\tau) \) or \( U^\mu_2(\tau) \) — this is equivalent due to the relations

\[
\Psi^\mu_+(\tau \pm \sigma_i) = m_i \gamma^{-1} \sqrt{-U_i^\mu(\tau) U_i^\mu(\tau) \mp (-1)^i U_i^\mu(\tau)}. \]

(10)

Remind that \( \sigma_1 = 0, \sigma_2 = \pi \).

Taking relations \( 4 \) and \( 3 \) into account we transform the boundary conditions \( 3 \) into the systems \( 11 \)

\[
\frac{dU^\mu_i}{d\tau} = \mp (-1)^i \gamma \frac{\delta^\mu_\nu}{m_1} \left[ \delta^\mu_\nu - U^\mu_i(\tau) U_\nu(\tau) \right] \Psi^\nu_\pm(\tau \pm \sigma_i),
\]

where \( \delta^\mu_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases} \).

\[2\]We use below the term “quasirotational states” for these disturbances.
Substituting Eqs. (10) into these relations we obtain for the case $0 < m_i < \infty$ the system

$$U^\mu_1(\tau) = m_2 m_1^{-1} [\delta^\mu_1 - U^\mu_1(\tau) U_{1\nu}(\tau)] \sqrt{U^\eta_2(\tau - \pi) U^\eta_2(\tau - \pi) - U^\nu_{2\nu}(\tau - \pi)},$$

$$U^\mu_2(\tau) = m_1 m_2^{-1} [\delta^\mu_2 - U^\mu_2(\tau) U_{2\nu}(\tau)] \sqrt{U^\eta_1(\tau - \pi) U^\eta_1(\tau - \pi) - U^\nu_{1\nu}(\tau - \pi)},$$

(11)

that plays the role of the above mentioned system of ordinary differential equations with shifted arguments with respect to known vector functions $U^\mu_i(\tau)$. Note that the initial data (keeping all information about the string’s motion) may be given here as the function $U^\mu_i(\tau)$ or $U^\nu_{ij}(\tau)$ in the segment $I = [\tau_0, \tau_0 + 2\pi]$ with the parameters $m_i/\gamma, U^\mu_i(\tau_0 + \pi)$. Integration of the system (11) with this initial data yields the values $U^\mu_i(\tau)$ and $U^\nu_{ij}(\tau)$ for all $\tau$ and then we can obtain the world surface $X^\mu(\tau, \sigma)$ with the help of Eqs. (10) and (3).

For the rotational motion (3) the unit velocity vectors $U^\mu_i$ are

$$U^\mu_i(\tau) = U^\mu_{i(rot)}(\tau) = \Gamma_i [e^\mu_0 - (-1)^i v_i e^\mu(\tau)], \quad \Gamma_i = (1 - v_i^2)^{-1/2},$$

(12)

where $e^\mu(\tau) = -e^\mu_1 \sin \theta \tau + e^\mu_2 \cos \theta \tau = \theta^{-1} d \xi e^\mu(\tau)$ is the unit rotating vector, connected with the vector $e^\mu$ (5). Expressions (12) are solutions of the system (11) if the parameters $v_i, m_i, \theta$ related by Eqs. (4).

To study the small disturbances of the rotational motion (3) we consider arbitrary small disturbances of this motion or of the vectors (12) in the form

$$U^\mu_i(\tau) = U^\mu_{i(rot)}(\tau) + u^\mu_i(\tau), \quad |u^\mu_i| \ll 1.$$  

(13)

For the exhaustive description of this quasirotational state the disturbance $u^\mu_i(\tau)$ may be given in the initial segment $I = [\tau_0, \tau_0 + 2\pi]$. It is small so we neglect in the linear approximation the second order terms. The equality $U^\mu_i(\tau) = 1$ for both vectors $U^\mu_i$ and $U^\mu_{i(rot)}$ leads in the linear approximation to the condition

$$U^\mu_{i(rot)}(\tau) u_{i\mu}(\tau) = 0.$$  

(14)

When we substitute the expressions (13) into the system (11) and omit the second order terms we obtain the linearized system of equations describing the evolution of small arbitrary disturbances $u^\mu_i$. Considering projections of these two vector equations onto the basic vectors $e_0, e, \dot{e}, e_3$, we reduce it to the following system of equations with respect to projections of $u^\mu_i$:

$$u^i_{10}(\tau) + Q_1 u_{10}(\tau) - \Gamma_1 Q_1 u_{1e}(\tau) = M_0 [u^i_{20} - Q_2 u_{20} + \Gamma_2 Q_2 u_{2e}],$$

$$u^i_{1e}(\tau) + Q_1 u_{1e}(\tau) + \theta v_i^{-1} u_{10}(\tau) = M_1^{-1} [- u^i_{2e} - Q_1 u_{2e} + N^*_1 u^i_{20} + N^*_1 u_{20}],$$

$$u^i_{20} + Q_2 u_{20} + \Gamma_2 Q_2 u_{2e} = M_0^{-1} [u^i_{10}(-) - Q_1 u_{10}(-) - \Gamma_1 Q_1 u_{1e}(-)],$$

$$u^i_{2e} + Q_2 u_{2e} - \theta v_i^{-1} u_{20} = M_1^{-1} [ - u^i_{1e}(-) - Q_2 u_{1e}(-) + N^*_1 u^i_{10}(-) + N^*_1 u_{10}(-)],$$

$$u^i_{1\sigma}(\tau) + Q_1 u_{1\sigma}(\tau) = (m_2/m_1) [ - u^i_{2\sigma} + Q_2 u_{2\sigma}],$$

$$u^i_{2\sigma} + Q_2 u_{2\sigma} = (m_2/m_1) [ - u^i_{1\sigma}(-) + Q_1 u_{1\sigma}(-)].$$  

(15)

Here $Q_i = \Gamma_i \theta v_i = \text{const}, (-) \equiv (\tau - 2\pi)$, the functions

$$u^i_{0\sigma}(\tau) = (e_0, u_i) = e^0_i u_{i\mu}, \quad u^i_{e\sigma}(\tau) = (e, u_i), \quad u^i_{z\sigma}(\tau) = (e_3, u_i)$$  

(16)

are the projections of the vectors $u^\mu_i(\tau)$ onto the mentioned basis. The projections of $u^\mu_i$ onto $e^\mu$ may be expressed through $u^i_{0\sigma}$: $(\dot{e}, u_i) = (-1)^i v_i^{-1} u_{i0}$ due to the equality (14). Arguments $(\tau - \pi)$ of the functions $u_{20}, u_{2e}, u_{2\sigma}$ are omitted. The constants in Eqs. (15) are

$$M_0 = m_2 Q_1/(m_1 Q_2), \quad M_1 = m_1 \Gamma_1/(m_2 \Gamma_2),$$

$$N^*_i = -(-1)^i (1 + Q_{3-i}/Q_i)/\Gamma_i, \quad N_i = (-1)^i (Q_{3-i} + Q_i \kappa_i)/\Gamma_i, \quad \kappa_i = 1 + v_i^{-2}.$$
We shall search solutions of the linearized system (13) in the form
\[ u_i^\mu = A_i^\mu e^{-i\omega \tau}. \]  (17)

For the last two equations (15) (they form the closed subsystem) solutions in the form (17) exist only if the dimensionless frequency \( \omega \) satisfies the transcendental equation
\[ \frac{\omega^2 - Q_1 Q_2}{(Q_1 + Q_2) \omega} = \cot \pi \omega. \]  (18)

Equation (18) has the countable set of real roots \( \omega_n, n - 1 < \omega_n < n \), the minimal positive root \( \omega_1 \) is equal to the parameter \( \theta \) in Eq. (5). These pure harmonic \( z \)-disturbances corresponding to various \( \omega_n \) result in the following correction to the motion (5) [due to Eqs. (10), (9) there is only \( z \) or \( e^\mu_3 \) component of the correction]:
\[ X^\mu(\tau, \sigma) = X^\mu_{rot}(\tau, \sigma) + e^\mu_3 A \cos(\omega_n \sigma + \phi_n) \cdot \cos(\omega_n \tau + \varphi_0), \]  (19)

Here the amplitude \( A \) is to be small in comparison with \( \Omega^{-1} \).

Expression (19) describes oscillating string in the form of orthogonal (with respect to the rotational plane) stationary waves with \( n \) nodes in the interval \( 0 < \sigma < \pi \). Note that the string ends are not in nodes, they oscillate along \( z \)-axis at the frequency \( \Omega_n = \Omega \omega_n / \theta \). The shape \( F = A \cos(\omega_n \sigma + \phi_n) \) of the \( z \)-oscillation (13) is not pure sinusoidal with respect to the distance \( s = \Omega^{-1} \cos(\theta \sigma + \phi_1) \) from the center to a point “\( \sigma \)”: If \( n = 1 \) this dependence is linear. In this trivial case the motion is pure rotational (5) with a small tilt of the rotational plane. But the motions (19) with excited higher harmonics \( n = 2, 3, \ldots \) are non-trivial.

The transcendental equation (18) was studied in Ref. [11] where we proved that its roots \( \omega_n \) (with \( \omega_0 = 0 \)) and the functions in Eqs. (19)
\[ y_n(\sigma) = \cos(\omega_n \sigma + \phi_n), \quad \phi_n = \arctan(\omega_n / Q_1), \quad n = 0, 1, 2, \ldots \]
are correspondingly the eigen-values and eigen-functions of the boundary-value problem
\[ y''(\sigma) + \omega^2 y = 0, \quad \omega^2 y(0) + Q_1 y'(0) = 0, \quad \omega^2 y(\pi) - Q_2 y'(\pi) = 0. \]  (20)

It was proved in Ref. [11] that the functions \( y_n(\sigma), n = 0, 1, 2, \ldots \) form the complete system in the class \( C([0, \pi]) \), and the system \( \exp(-i \omega_n \tau), n \in \mathbb{Z} \) (with \( \omega_{-n} = -\omega_n \)) is complete in the class of function \( C(I) \), where \( I = [\tau_0, \tau_0 + 2\pi] \). So any continuous function \( f(\tau) \) given in the segment \( I \) may be expanded in the Fourier series
\[ f(\tau) = \sum_{n=-\infty}^{+\infty} f_n \exp(-i \omega_n \tau), \quad \tau \in I = [\tau_0, \tau_0 + 2\pi]. \]  (21)

As was mentioned above, all information about any motion of this system is contained in the function \( U^\mu(\tau) \) given in the segment \( I \). If we expand any small disturbance \( u_{1z}(\tau) \) or \( u_{2z}(\tau) \) in the segment \( I \) into the Fourier series (21), this series will describe the evolution of the given disturbance for all \( \tau \in \mathbb{R} \), because any term in (21) satisfies the evolution equations (13). So any small disturbance of the rotational motion (5) in \( e^\mu_3 \) direction may be expanded into the Fourier series with the oscillatory modes (19).
This also concerns the quasirotational motions in the rotational plane \(e_1, e_2\). They are determined by the first 4 equations \([5]\). If we substitute \(u_{i0} = A_i \exp(-i\tilde{\omega}\tau), u_{ie} = B_i \exp(-i\tilde{\omega}\tau)\) into this system we obtain the following condition for existence of its non-trivial solutions:

\[
\begin{vmatrix}
Q_1 - i\tilde{\omega} & -Q_1 & Q_1 + i\tilde{\omega}q_{12} & -Q_1 \\
Q_1/v_1^2 & Q_1 - i\tilde{\omega} & -Q_2 - i\tilde{\omega}q_{12} & Q_1 - i\tilde{\omega} \\
Q_1 + i\tilde{\omega}q_{12} & Q_1 & (Q_1 - i\tilde{\omega}q_{12}) e^{-2\pi i\tilde{\omega}} & Q_1 e^{-2\pi i\tilde{\omega}} \\
Q_1 + i\tilde{\omega}q_{12} & Q_2 - i\tilde{\omega} & -Q_2 v_2^2 e^{-2\pi i\tilde{\omega}} & (Q_2 - i\tilde{\omega}) e^{-2\pi i\tilde{\omega}}
\end{vmatrix} = 0. \tag{22}
\]

Here \(q_{12} = Q_1/Q_2, \tilde{Q}_j = Q_j\kappa_j + Q_{3-j} + i\tilde{\omega}\).

There are eigen-frequencies \(\tilde{\omega} = \tilde{\omega}_n = n, n \in \mathbb{Z}\) satisfying this equation. But the correspondent functions \(u_i^\mu(\tau) = B_i^\mu \exp(-in\tau)\) after substitution into Eqs. \([3], \[4], \[8]\) result in quasirotational excitations of the motion \([3]\), which may be obtained from Eq. \([3]\) through the following reparametrization \([11]\)

\[
\tilde{\tau} \pm \tilde{\sigma} = f(\tau \pm \sigma) : f(\xi + 2\pi) = f(\xi) + 2\pi, \quad f'(\xi) > 0, \quad \xi \in \mathbb{R}.
\]

on the same world surface. Eqs. \([2] – [11]\) and the conditions \(\sigma_1 = 0, \sigma_2 = \pi\) are invariant with respect to this reparametrization so the oscillations with \(u_i^\mu(\tau) = B_i^\mu \exp(-in\tau)\) have no physical manifestations. They may be interpreted as “longitudinal oscillations” inside the string or, oscillations of the grid chart on the world surface.

If we exclude these non-physical roots \(\tilde{\omega}_n = n\), the equation \([22]\) will reduce to following one:

\[
\frac{(\tilde{\omega}_n^2 - Q_1^2\kappa_1)(\tilde{\omega}_n^2 - Q_2^2\kappa_2) - 4Q_1 Q_2 \tilde{\omega}_n^2}{2\tilde{\omega}[Q_1(\tilde{\omega}_n^2 - Q_2^2\kappa_2) + Q_2(\tilde{\omega}_n^2 - Q_1^2\kappa_1)]} = \cot \pi \tilde{\omega}_n. \tag{23}
\]

One can numerate the roots \(\tilde{\omega} = \tilde{\omega}_n\) of Eq. \([23]\) in order of increasing so that \(\tilde{\omega}_0 = 0, n-1 < \tilde{\omega}_n < n\) for \(n \geq 1\) (and \(\tilde{\omega}_{-n} = -\tilde{\omega}_n\)). The value \(\tilde{\omega} = \theta\) is also the root of Eq. \([23]\) but it corresponds to the trivial quasirotational states, connected with a small shift of the rotational center with respect to the coordinate origin.

In the following table some first roots \(\omega_n\) of Eq. \([18]\) and \(\tilde{\omega}_n\) of Eq. \([23]\) are represented for the example of the rotational motion \([3]\) with \(m_1 = m_2, v_1 = v_2 = 1/\sqrt{2}, \theta = 0.5 = Q_1 = Q_2:\)

| \(n\) | 1    | 2    | 3    | 4    | 5    |
|------|------|------|------|------|------|
| \(\omega_n\) | 0.5  | 1.2434 | 2.1457 | 3.102 | 4.078 |
| \(\tilde{\omega}_n\) | 0.9262 | 1.5 | 2.299 | 3.206 | 4.157 |

The roots \(\tilde{\omega}_n\) of Eq. \([23]\) are eigen-values of the boundary-value problem

\[
y''(\sigma) + \omega^2 y = 0, \quad (\omega^2 - Q_1^2\kappa_1) y(0) + 2Q_1 y'(0) = 0, \quad (\omega^2 - Q_2^2\kappa_2) y(\pi) - 2Q_2 y'(\pi) = 0,
\]

similar to the problem \([20]\). So the corresponding eigen-functions and the system \(\exp(-i\tilde{\omega}_n\tau), n \in \mathbb{Z}\) is complete in the class of function \(C(I), I = [\tau_0, \tau_0 + 2\pi]\) \([11]\).

Hence, an arbitrary quasirotational disturbance \(u_i^\mu(\tau)\) may be expanded in the Fourier series similar to Eq. \([21]\). Using this expansion for the disturbance \([13]\) \(u_i^\mu\) of the velocity vectors \([12]\) we obtain with the help of Eqs. \([3], [4], [8]\) the following expression for an arbitrary quasirotational motion of the string with massive ends \([12]\):

\[
X^\mu(\tau, \sigma) = X_{rot}^\mu(\tau, \sigma) + \sum_{n=-\infty}^{\infty} \left\{ e^{\mu}_n \alpha_n \cos(\omega_n\sigma + \phi_n) \exp(-i\omega_n\tau) + \beta_n [e^{\mu}_n f_0(\sigma) + e^{\mu}_n f_+ (\sigma) + ie^{\mu}_n f_0(\sigma)] \exp(-i\tilde{\omega}_n\tau) \right\}. \tag{24}
\]
Here the first term $X_{\text{rot}}^\mu$ describes the pure rotation and

$$f_0(\sigma) = \frac{1}{2}(Q_1 \kappa_1 \tilde{\omega}_n^{-1} - Q_1^{-1} \tilde{\omega}_n) \cos \tilde{\omega}_n \sigma - \sin \tilde{\omega}_n \sigma,$$

$$f_\perp(\sigma) = \Gamma_1(\Theta_n \tilde{\omega}_n - h_n v_1) C_\theta C_\omega - v_1^{-1} C_\theta S_\omega + \Gamma_1 \theta \Theta_n S_\theta S_\omega + h_n S_\theta C_\omega,$$

$$f_\parallel(\sigma) = \Gamma_1(\Theta_n \tilde{\omega}_n - h_n v_1) S_\theta S_\omega + v_1^{-1} S_\theta C_\omega + \Gamma_1 \theta \Theta_n C_\theta C_\omega - h_n C_\theta S_\omega;$$

$$\kappa_i = 1 + v_i^{-2}, \quad \Theta_n = \frac{2\theta}{\tilde{\omega}_n^2 - \theta^2}, \quad h_n = \frac{1 - \theta}{\tilde{\omega}_n} \left( \frac{1}{v_1} + v_1 \right) + \tilde{\omega}_n \left( \frac{1}{v_1^2} - v_1 \right), \quad C_\theta(\sigma) = \cos \theta \sigma,$$

$$S_\theta(\sigma) = \sin \theta \sigma, \quad C_\omega(\sigma) = \cos \tilde{\omega}_n \sigma, \quad S_\omega(\sigma) = \sin \tilde{\omega}_n \sigma, \quad \alpha_n = \alpha_n^*, \quad \beta_n = -\beta_n^*.$$  

The quasirotational disturbances \(\tilde{\omega}_n\) with \(\beta_n \neq 0\) and \(\alpha_n = 0\) are small (if \(\beta_n \ll \Omega^{-1}\)) harmonic planar oscillations or stationary waves in the rotational plane. The shape of these stationary waves in the co-rotating frame of reference (where the axes \(x\) and \(y\) are directed along \(e^\mu\) and \(\hat{e}^\mu\)) is approximately described by the function \(\beta_n[f_\perp(\sigma) - f_0(\sigma) \cos(\theta \sigma + \phi_1)]\) if the deflection is maximal. For each \(n\) this rotating curved string oscillates at the frequency \(\Omega_n = \Omega \tilde{\omega}_n/\theta\), it has \(n\) nodes in \((0, \pi)\) (which are not strictly fixed because \(f_0\) and \(f_\parallel\) are non-zero) and the moving quarks are not in nodes. Note that Eq. \(\text{(24)}\) describes both the deflection of the endpoints \(\beta_n e^\mu f_\perp(\sigma_1)\) sin \(\omega_n \tau\) and their radial motion \(\beta_n e^\mu f_\parallel(\sigma_1)\) cos \(\omega_n \tau\).

The frequencies \(\omega_n\) and \(\tilde{\omega}_n\) from Eqs. \((18)\) and \((23)\) are real numbers, so the rotations \((3)\) of the string with massive ends are stable in the linear approximation. Hence, one may consider the Fourier series \((24)\) for an arbitrary quasirotational motion as the basis for quantization of some class of motions (quasirotational states) of the string with massive ends in the linear vicinity of the solution \((5)\). Note that for the string baryon models “three-string” (Y) and the linear configuration (g-q-q) their rotational motions are unstable, the analogs of Eq. \((24)\) for these models contain complex frequencies \((12, 13)\).

The quasirotational states \((24)\) of the relativistic string with massive ends may be used for describing radial excitations of hadrons. In this connection the most interesting states \((24)\) are oscillations in the rotational plane (the planar modes) with \(\alpha_n = 0, \beta_n \neq 0\), and among them — the main planar mode with \(n = 1\). Let us consider them in detail and calculate the energy \(E\) and angular momentum \(J\) for these excitations.

3. **Energy \(E\) and angular momentum \(J\) of quasirotational states**

The main planar quasirotational mode \((24)\) with \(n = 1\) is non-trivial\(^3\). If only this mode is excited the motion is quasiperiodical, the string at the frequency \(\Omega_1 = \Omega \tilde{\omega}_1/\theta\ (1.5 \Omega < \Omega_1 < 2 \Omega)\) slightly deflections from pure uniform rotation keeping almost rectilinear shape. The length of the string (distance between quarks) varies in accordance with this deflection at the frequency \(\Omega_1\). The endpoints draw curves close to ellipses both in the co-rotating frame of reference (the pure uniform rotational position of an end is in the center of this ellipse) and in the frame of reference \(Oxy\). In the latter case this ellipse (close to a circle) rotates in the main rotational direction because the frequencies \(\Omega_1\) and \(2 \Omega\) are incommensurable numbers: \(\Omega_1 < 2 \Omega\). The more this disturbance (its amplitude \(\beta_1\)) the more those rotating ellipses differ from circles.

For the second oscillatory mode \((n = 2)\) the middle part of the rotating string oscillates at the frequency \(\Omega_2\), but deviations of the endpoints are small in comparison with this amplitude in the middle part. This motion is shown in details in Ref. \([7]\).

Possible applications of these string states in hadron spectroscopy depend on their physical characteristics. Let us calculate the most important among them: the energy \(E\) and angular

\(^3\)Unlike the \(n = 1\) mode of orthogonal oscillations \((4)\).
momentum $J$ of the quasirotational state \( \text{(24)} \). For an arbitrary classic state of the relativistic string with massive ends they are determined by the following integrals (Noether currents) \( \text{(2)} \):

$$
P^\mu = \int_{\sigma_1}^{\sigma_2} p^\mu(\tau, \sigma) \, d\sigma + p_1^\mu(\tau) + p_2^\mu(\tau), \quad p^\mu(\tau, \sigma) = \gamma \frac{(\dot{X}, X') X^\mu - X'^2 \dot{X}^\mu}{[(X, X')^2 - \dot{X}^2 X'^2]^{1/2}}, \quad (25)$$

$$
J^{\mu\nu} = \int_{\sigma_1}^{\sigma_2} \left[ X^{\mu}(\tau, \sigma) p^{\nu}(\tau, \sigma) - X^{\nu}(\tau, \sigma) p^{\mu}(\tau, \sigma) \right] \, d\sigma + \sum_{i=1}^{2} \left[ p_i^\mu(\tau) p_i^{\nu}(\tau) - x_i^\nu(\tau) p_i^\mu(\tau) \right], \quad (26)
$$

where \( p_i^\mu(\tau) = m_i U_i^\mu(\tau), \) \( s_1 = 0, s_2 = \pi \). In the orthonormal gauge \( \text{(2)} \) \( p^\mu(\tau, \sigma) = \gamma X^\mu(\tau, \sigma) \).

The square of energy $E^2$ equals the scalar square of the conserved $R^{1,3}$-vector of momentum \( \text{(2)} \): $P_i^2 = P_\mu P_i^\mu = E^2$. In the center of mass reference frame $E = P_0$, the latter case takes place for the expression \( \text{(24)} \).

If we substitute the Fourier series \( \text{(24)} \) into expression \( \text{(25)} \) we’ll obtain the following equality for the 4-momentum:

$$
P^\mu = P_{\text{rot}}^\mu = c_0^\mu \sum_{i=1}^{2} [\gamma \Omega^{-1} \arcsin v_i + m_i \Gamma_i]. \quad (27)
$$

In other words, the energy of an arbitrary quasirotational state \( \text{(24)} \) equals the energy $E_{\text{rot}}$ \( \text{(8)} \) of the pure rotational motion \( \text{(8)} \), all quasirotational modes with amplitudes $\alpha_n, \beta_n$ yield zero contributions. This result is obtained in the linear approximation with respect to $\alpha_n, \beta_n$ in the expressions for the endpoints’ momenta $p_i^\mu = m_i U_i^\mu$, for example

$$
U_1^\mu(\tau) \approx U_{\text{1(rot)}}^\mu(\tau) + \Omega v_1 \Gamma_1^2 \sum_{n \neq 0} \left\{ \frac{c_0^\mu \alpha_n}{\sqrt{\omega_n^2 + \Omega_1^2}} e^{-i\omega_n \tau} - i \beta_n [c_0^\mu + v_1^{-1} \dot{\epsilon}^\mu(\tau) + i h_n e^\mu(\tau)] e^{-i\omega_n \tau} \right\}.
$$

Here $U_{\text{1(rot)}}^\mu(\tau)$ is the expression \( \text{(12)} \). In this approximation the contribution of the ends in Eq. \( \text{(25)} \) exactly compensates the string contribution $\int p^\mu(\tau, \sigma) \, d\sigma$ for each oscillatory mode.

This property is similar to vanishing contributions of high oscillatory modes in the Fourier series for the massless open string \( \text{(2)} \). In the latter case energy of these oscillation may be found from the orthonormality conditions \( \text{(4)} \) (the Virasoro conditions).

In the case of quasirotational states \( \text{(24)} \) we have no the Virasoro conditions, because the orthonormality conditions \( \text{(4)} \) were previously solved and taken into account in expressions \( \text{(9)}, \text{(10)}, \text{(11)}, \text{(13)} \) in the linear approximation with respect to $u_i^\mu$.

The equality \( \text{(27)} \) of the energies of the quasirotational $E$ and pure rotational motion $E_{\text{rot}}$ (compare with the similar result in Ref. \( \text{(4)} \)) looks questionable: we always can add a disturbance with nonzero energy $\Delta E$ to the pure rotation \( \text{(3)} \) $X_{\text{rot}}^\mu(\tau, \sigma)$. But there is no contradiction: the resulting motion will be the quasirotational state \( \text{(24)} \) with respect to the rotation \( \text{(3)} \) with the energy $E_{\text{rot}} + \Delta E$.

The classical angular momentum of the quasirotational motion \( \text{(24)} \) is determined by Eq. \( \text{(26)} \) and in the case\( ^4 \) $\alpha_1 = 0$ it takes the form

$$
J^{\mu\nu} = j_3^{\mu\nu}\left\{ J_{\text{rot}} - \gamma \sum_{n=1}^{\infty} |\beta_n|^2 \frac{\Gamma_1^2 \Theta Z_{1,n}}{\omega_n^2 (\omega_n^2 - \theta^2)} \left[ \pi \theta (3\omega_n^2 + \theta^2) + \sum_{i=1}^{2} \frac{1}{v_i \Gamma_i} \left( \frac{4\omega_n^2 - v_i^2 (\omega_n^2 - \theta^2)}{v_i^2} \right) \right] \right\} \quad (28)
$$

\( ^4 \)Remind that inequality $\alpha_1 \neq 0$ results in a trivial tilt of the rotational plane.
Here $j_3^{\mu\nu} = e_1^\mu e_2^\nu - e_1^\nu e_2^\mu = e^\mu \epsilon^\nu - e^\nu \epsilon^\mu$, $Z_{i,n} = 1 + \frac{(\bar{\omega}_n^2 - \theta^2)^2}{4v_i^2\Gamma_i^2\theta^4}$.

Note that the 2-nd order contribution $\Delta J$ (proportional to $|\beta_n|^2$) in Eq. (28) to the momentum $J_{\text{rot}}$ of the pure rotational motion is always negative. This is natural, the rotational motion (5) has the maximal angular momentum among the motions with given energy [2]. And we consider the quasirotational states (24) with fixed energy (27) $E = E_{\text{rot}}$.

This result shows the possibility to apply these states (especially the state with $\beta_1 \neq 0$) for describing high radial excitations of mesons and baryons. If for a planar excited mode the correction $\Delta J = -1$ (in the natural units $\hbar$) we may interpret this state as the radial excitation lying on the first daughter Regge trajectory. It has the same energy $E_{\text{rot}}$, so it may be only the excitation of the state with lower energy on this daughter trajectory.

Hence in this approach the slope of radial trajectories is equal to that for the orbital Regge trajectories because the increase of energy is the same for enlarging the momentum $J$ or the radial quantum number $n$. After developing the quantization procedure on the basis of the Fourier series (24) this method may be applied for describing higher radial excitations in hadron spectroscopy.

Conclusion

For the obtained class of quasirotational motions for the relativistic string with massive ends the possible applications are investigated. It is shown that the quasirotational states (24) may be used for describing radial excitations of mesons and baryons (in the frameworks of the model $qqqq$). This possibility may be brought about after developing the quantization procedure for the quasirotational states (24). This procedure is the object for further study. It will include the usual interpretation of the amplitudes $\alpha_n$, $\beta_n$ in this Fourier series as quantum operators with defining their commutability relations.

Note that the expressions (27) and (28) for the energy and angular momentum are obtained in the linear approximation with respect to small disturbances $u_i^\mu(\tau)$ in Eq. (18). If these disturbances are not small the second order corrections may change Eqs. (27), (28). But these second order corrections can not be determined uniquely from the given linear disturbances $u_i^\mu(\tau)$ or (this is equivalent) $\alpha_n$, $\beta_n$. These corrections $\bar{u}_i^\mu(\tau)$ have to satisfy only one condition — the equality $U_i^\mu(\tau) = 1$, where $U_i^\mu(\tau) = U_i^\mu(\text{rot})(\tau) + u_i^\mu(\tau) + \bar{u}_i^\mu(\tau)$, and other degrees of freedom in a choice of $\bar{u}_i^\mu(\tau)$ are not fixed. So one can choose the correction $\bar{u}_i^\mu$ resulting in the same value of energy (27).

This approach was developed for the string with massive ends. We are to emphasize that it is not applicable for more complicated string baryon models $q-q-q$ and $Y$ (“three-string”) because the rotational motions for them are unstable and we can’t quantize states in their linear vicinity [4, 12, 13].

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