On the arithmetic difference of middle Cantor sets

M. Pourbarat
Dep. Math., Shahid Beheshti University, Tehran, Iran.

Abstract: We introduce a dense subset $\mathcal{L}$ of $\mathcal{C} \times \mathcal{C}$, where $\mathcal{C}$ is the space of all middle Cantor sets. Then we transfer the renormalization operators on the space of limit geometries corresponding to pair $(C_\alpha, C_\beta) \in \mathcal{L}$ to a family of the iterated function systems whose attractors are $C_\alpha - \lambda C_\beta$. This could be useful in the investigation of having stable intersection of $C_\alpha$ and $C_\beta$.

Another application is to provide conditions on the functions $f$, $g$ and the pair $(C_\alpha, C_\beta) \in \mathcal{L}$ which $f(C_\alpha) - g(C_\beta)$ contains an interval. This leads us to denote another type of stability in the intersection of two Cantor sets. We prove the existence of this stability for regular Cantor sets that have stable intersection and its absence for those which the sum of their Hausdorff dimension is less than one.

In sequel, special middle Cantor sets $C_\alpha$ and $C_\beta$ are introduced. Then the iterated function system corresponding to the attractor $C_\alpha - \frac{2\alpha}{\beta} C_\beta$ is characterized. Some specifications of the attractor has been presented that keep our example as an exception. At the end, we show that $\sqrt{C_\alpha} - \sqrt{C_\beta}$ contains at least one interval.

Keywords: Arithmetic difference, Hausdorff dimension, middle Cantor sets, Palis conjecture.
AMS Classification: 28A78, 58F14.

1 Introduction

Regular Cantor sets play a fundamental role in dynamical systems and number theory. Intersections of hyperbolic sets with stable and unstable manifolds of its points are often regular Cantor sets. Also, related to diophantine approximations, many Cantor sets given by combinatorial conditions on the continued fraction of real numbers are regular. In studying the homoclinic bifurcations in dynamical systems, also the classical Markov and Lagrange spectra related to diophantine approximations in number theory, we deal with arithmetic difference of regular Cantor sets [2, 3, 12]. Many papers have been written about metrical and topological properties of sum or difference of regular Cantor sets [6, 13, 15]. Before stating
the results of this paper, we establish some notations.

A Cantor set $K$ is regular or dynamically defined if:

i) there are disjoint compact intervals $K_1, K_2, \cdots, K_r$ such that $K \subseteq K_1 \cup K_2 \cup \cdots \cup K_r$ and the boundary of each $K_i$ is contained in $K$,

ii) there is a $C^{1+\varepsilon}$ expanding map $\psi$ defined in a neighborhood of the set $K_1 \cup K_2 \cup \cdots \cup K_r$ such that $\psi(K_i)$ is the convex hull of a finite union of some intervals $K_j$ satisfying:

iii.1 For each $1 \leq i \leq r$ and $n$ sufficiently large, $\psi^n(K \cap K_i) = K$,

iii.2 $K = \bigcap_{n=0}^{\infty} \psi^{-n}(K_1 \cup K_2 \cup \cdots \cup K_r)$.

The set $\{K_1, K_2, \cdots, K_r\}$ is, by definition, a Markov partition of $K$, and the set $D := \bigcup_{i=1}^{r} K_i$ is the corresponding Markov domain of $K$.

The Cantor set $K$ is close on the topology $C^{1+\varepsilon}$ to a Cantor set $\tilde{K}$ with the Markov partition $\{\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_r\}$ defined by expanding map $\tilde{\psi}$ if and only if the extremes of each $K_i$ are near the corresponding extremes of $\tilde{K}_i$ and supposing $\psi \in C^{1+\varepsilon}$ with Holder constant $C$, we must have $\tilde{\psi} \in C^{1+\varepsilon}$ with Holder constant $\tilde{C}$ such that $(\tilde{C}, \tilde{\varepsilon})$ is near $(C, \varepsilon)$ and $\tilde{\psi}$ is close to $\psi$ in the $C^1$ topology. Regular Cantor sets $K$ and $K'$ have stable intersection if for any pair of regular Cantor sets $(\tilde{K}, \tilde{K}')$ near $(K, K')$, we have $\tilde{K} \cap \tilde{K}' \neq \emptyset$.

The concept of stable intersection has been introduced by Moreira in [6] for the first time. The Cantor sets $K$ and $K'$ that have stable intersection are useful in two points of view. First, in dynamical systems theory, when stable and unstable Cantor sets associated to a homoclinic bifurcation have a stable intersection, they present open sets in the parameter line with positive density at the initial bifurcating value, for which the corresponding diffeomorphisms are not hyperbolic. Second, in number theory, they guarantee the existence of an open set $U$ including $(K, K')$ such that for each $(\tilde{K}, \tilde{K}') \in U$, the set $\tilde{K} - \tilde{K}'$ contains an interval.

On the other hand, topological and metrical structure of the $K - \lambda K'$ plays a key role in investigation of having stable intersection of regular Cantor sets $K$ and $K'$ [7]. Hence, we can concentrate on the arithmetic difference of regular Cantor sets of this form. Herein, there are several classical results:

I) If $\tau(K) \cdot \tau(K') > 1$, then $K - \lambda K'$ contains an interval [12],

II) If $HD(K) + HD(K') > 1$, then $K - \lambda K'$ generically contains an interval [7],

III) If $HD(K) + HD(K') > 1$, then $|K - \lambda K'| > 0$ for almost every $\lambda \in \mathbb{R}^*$ [12],

IV) There exist regular Cantor sets $K$ and $K'$ such that $K - K'$ has positive Lebesgue measure, but does not contain any interval [14],

V) If $HD(K) + HD(K') < 1$, then $|K - \lambda K'| = 0$ [12].
A regular Cantor set $K$ is affine if $D\psi$ is constant on every interval $K_i$. Meanwhile, the following conjecture due to Palis is still open:

**Conjecture.** The arithmetic difference of two affine Cantor sets generically, if not always, contains an interval or has zero Lebesgue measure.

Many studies have been done on this conjecture [6, 7, 8]. This conjecture can be written for middle Cantor sets too; Cantor set $C_\alpha$ is a middle-$(1-2\alpha)$ or in simple words, middle Cantor set, if the convex hull of $C_\alpha$ is $[0,1]$ and the Markov partition of $C_\alpha$ has exactly two members with $D\psi = \frac{1}{\alpha}$ on their intervals. Regard to above discussion, the morphology of the arithmetic difference $C_\alpha - \lambda C_\beta$ on the mysterious region

$$\Omega := \{(C_\alpha, C_\beta) \mid HD(C_\alpha) + HD(C_\beta) > 1, \ \tau(C_\alpha) \cdot \tau(C_\beta) < 1\}$$

is unclear. This is our motivation in writing the present paper that is organized as follows:

In Section 2, we will prove a theorem that introduces the iterated function systems with their attractors $C_\alpha - \lambda C_\beta$, where $(C_\alpha, C_\beta) \in \mathcal{L}$. We obtain this theorem by the transferred renormalization operators corresponding to a pair $(C_\alpha, C_\beta)$ on the space $\mathbb{R}^* \times \mathbb{R}$ that explained in [4]. A reason of proposing the theorem is that, tracing and controlling points in $\mathbb{R}$ under suitable compositions of functions which constitute the iterated function system are easier than tracing and controlling points in $\mathbb{R}^* \times \mathbb{R}$ under suitable compositions of transferred operators. Although the number of functions which constitute the iterated function system could be so many, the methods and techniques of the theory of iterated function systems could be profitable. Moreira and Yoccoz in [7] introduced that, a way of having stable intersection of Cantor sets $C_\alpha$ and $C_\beta$ is to construct a recurrent compact set of relative configurations corresponding to the renormalization operators. To deal with Palis conjecture, these facts may be a step forward. In this direction, we have found an element of $\Omega$ that have stable intersection [10], (see Proposition 4.3 too). The other applications of the theorem will occur throughout the paper.

In Section 3, the first aim is to establish conditions on the function $f$, $g$ and the pair $(C_\alpha, C_\beta) \in \mathcal{L}$ to ensure the existence of an interval in the set $f(C_\alpha) - g(C_\beta)$. For instance, it will be applied to guarantee the existence of an interval in the sets $C^2 + C^2$ and $\sin C + \cos C$, where $C$ is the middle-$\frac{1}{3}$ Cantor set. This allows us to introduce the concept of “weak stable intersection” that could have a pair of arbitrary Cantor sets embedded in the real line; the pair $(K, K')$ has weak (or geometric) $C^r$–stable intersection, if for all $f$ and $g$ in a $C^r$ – neighborhood of the identity, we have $f(K) \cap g(K') \neq \emptyset$. Note that, the diffeomorphisms $f$ and $g$ do not change the Hausdorff dimensions of $K$ and $K'$. To continue, we compare our definition with Moreira’s definition in category of regular Cantor sets and their similar attitude on the classic known regions. Indeed, these two definitions are equivalent on an open and dense subset and it seems that this equivalence happens everywhere.
In Section 4, we introduce $\lambda \in \mathbb{R}^*$ and $(C_{\alpha}, C_{\beta}) \in \Omega$ such that $C_{\alpha} - \lambda C_{\beta}$ is not an affine Cantor set and $|C_{\alpha} - \lambda C_{\beta}| = 0$. Regarding to Theorem 2.1, we observe that the set $C_{\alpha} - \lambda C_{\beta}$ is the attractor of the iterated function system namely $S := \{S_i | S_i(t) = p^{-2}t + b_i\}_{i=1}^{21}$. It is not clear at this point that the Hausdorff dimension of the attractor is smaller than one since we have twenty one different affine maps with slopes $p^2 = 17, 94... < 21$. Nevertheless, we calculate its exact value in a different manner and then we present some other results about these Cantor sets. In the context of regular Cantor sets $K$ and $K'$ that $K - K'$ is not an affine Cantor set and $HD(K - K') < 1 < HD(K) + HD(K')$, there exist some properties that prove our example as an exceptional one among others:

- The set $C_{\alpha} - \lambda C_{\beta}$ forms the attractor of an iterated function system that is of finite type.
- The similarity dimension of $S$ is bigger than one and also it is not obvious to determine $HD(C_{\alpha} - \lambda C_{\beta}) < 1$ on the lower steps of the construction $C_{\alpha} \times C_{\beta}$. In fact, if we do twelve steps in the construction of $C_{\alpha}$ and eighteen steps in the construction of $C_{\beta}$, then we can select an iterated function system on $I \times I \subset \mathbb{R}^2$ of Hausdorff dimension smaller than one, such that $C_{\alpha} - \lambda C_{\beta}$ becomes the projection of its attractor under angle $\cot^{-1} \lambda$. While, this method does not apply for the lower steps of the construction $C_{\alpha} \times C_{\beta}$, (see Corollary 2.1 and the Remark 2 of [15]). Another purpose in presenting this method is to find an upper bound of Hausdorff dimension and it may be useful for the situations that the iterated function system is not of finite type.
- $\dim_H(C_{\alpha} - \lambda C_{\beta}) = \dim_B(C_{\alpha} - \lambda C_{\beta})$, that is computable and $C_{\alpha} - \lambda C_{\beta}$ is a s–set.
- There exists a dense subgroup $G \subset \mathbb{R}$ such that for each $g \in G$, the set $C_{\alpha} - gC_{\beta}$ contains an interval or has zero Lebesgue measure.
- We can not put them in a non constant continuous curve from the pair of regular Cantor sets that Hausdorff dimension of their arithmetic difference is less than one.

2 Iterated function system

Let us start with introducing the set $\mathcal{L}$. An element $(C_{\alpha}, C_{\beta})$ belongs to $\mathcal{L}$ if and only if $\frac{\log \alpha}{\log \beta} \in \mathbb{Q}$. Obviously, $\mathcal{L}$ is a dense subset in the space $C \times C$. Let $p, q > 2$, it has been shown in [4] that the transferred renormalization operators corresponding to pair $(C_{\alpha}, C_{\beta}) := (C_{\frac{1}{p}}, C_{\frac{1}{q}})$ are

$$
\begin{align*}
(s, t) &\xrightarrow{T_0} (ps, pt) \\
(s, t) &\xrightarrow{T_1} (ps, pt - p + 1) \\
(s, t) &\xrightarrow{T_0'} (\frac{s}{q}, t) \\
(s, t) &\xrightarrow{T_1'} (\frac{s}{q}, t + \frac{q - 1}{q} s)
\end{align*}
$$

(1)
If \( \frac{\log P}{\log q} =: \frac{n_0}{m_0} \in \mathbb{Q} \) with \((m_0, n_0) = 1\), then every vertical line \( s = \lambda =: \cot \theta \) passes over itself with suitable compositions of the operators (1). Hence, we can transfer the operators (1) on these lines by

**Theorem 2.1.** Let \( \lambda \in \mathbb{R}^* \) and \( \{a_k\}_{k=0}^{m_0-1}, \{b_k\}_{k=0}^{n_0-1} \) be two finite sequences of numbers 0 and 1. Then the maps

\[
T_\lambda(t) := p^{m_0}t + a_\lambda, \tag{2}
\]

\[
a_\lambda := -(p-1)p^{m_0-1} \left( \sum_{k=0}^{m_0-1} \frac{a_k}{p^k} - \frac{p(q-1)}{q(p-1)} \lambda \sum_{k=0}^{n_0-1} \frac{b_k}{q^k} \right)
\]

are return maps to the vertical line \( s = \lambda \) and the attractor of iterated function system \( \{T_\lambda^{-1}\} \) is \( C_\alpha = \lambda C_\beta \).

**Proof.** Suppose that \( \{b_k\}_{k=0}^{\infty} \) and \( \{a_k\}_{k=0}^{\infty} \) are two arbitrary sequences of numbers 0 and 1. For every \( a_k \) and \( b_k \), we can write the operators (1) in form

\[
T_{a_k}(s, t) := (ps, pt - (p-1)a_k), \quad T_{b_k}'(s, t) := (\frac{s}{q}, t + (\frac{q-1}{q})b_k s).
\]

Let \( m, n \in \mathbb{N} \), then we claim that

i) \( T_{a_{m-1}} \circ \ldots \circ T_{a_0}(s, t) = \left( p^{m}s, p^{m}t - (p-1)\sum_{k=0}^{m-1} a_k p^{m-k} \right) \),

ii) \( T_{b_{n-1}}' \circ \ldots \circ T_{b_0}'(s, t) = \left( \frac{s}{q^{n}}, t + \frac{s}{q^{n}}(q-1)\sum_{k=0}^{n-1} b_k q^{n-k} \right) \).

To prove the claim, we use induction. The case \( m = n = 1 \) is true. Assume that formulas are valid for the cases \( m = i \) and \( n = j \), then we have

\[
T_{a_i} \circ T_{a_{i-1}} \circ \ldots \circ T_{a_0}(s, t) = \left( p^{i+1}s, p^{i+1}t - (p-1)\sum_{k=0}^{i-1} a_k p^{i-k} \right)
\]

\[
= \left( p^{i+1}s, p^{i+1}t - (p-1)\sum_{k=0}^{i} a_k p^{i-k} \right),
\]

\[
T_{b_j} \circ T_{b_{j-1}} \circ \ldots \circ T_{b_0}'(s, t) = \left( \frac{s}{q^{j+1}}, t + \frac{s}{q^{j+1}}(q-1)\sum_{k=0}^{j} b_k q^{j-k} \right)
\]

and we see the validity of the relations (i) and (ii) for the cases \( m = i + 1 \) and \( n = j + 1 \).

Put \( s = \lambda, m = m_0 \) and \( n = n_0 \) in the relations (i) and (ii). Then we obtain the maps (2), since

\[
T_{b_{n_0-1}}' \circ \ldots \circ T_{b_0}' \circ T_{a_{m_0-1}} \circ \ldots \circ T_{a_0}(\lambda, t) = \left( \frac{p^{m_0}}{q^{n_0}} \lambda, p^{m_0}t - (p-1)\sum_{k=0}^{m_0-1} a_k p^{m_0-1-k} \right)
\]

\[
+ \frac{p^{m_0}}{q^{n_0}} (q-1)\sum_{k=0}^{n_0-1} b_k q^{n_0-1-k} \right)
\]

\[
= \left( \lambda, p^{m_0}t - (p-1)p^{m_0-1} \sum_{k=0}^{m_0-1} \frac{a_k}{p^k} - \frac{p(q-1)}{q(p-1)} \lambda \sum_{k=0}^{n_0-1} \frac{b_k}{q^k} \right).
\]

Note that, the operators \( T_{b_{n_0-1}}' \circ \ldots \circ T_{b_0}' \circ T_{a_{m_0-1}} \circ \ldots \circ T_{a_0} \) give all of the return maps, since the operators \( T_{a_i} \) and \( T_{b_j}' \) commute together. Moreover, the maps (2) establish an iterated function system with contractions \( T_{\lambda}^{-1}(t) = p^{-m_0}t + b_\lambda := p^{-m_0}t - \frac{a_\lambda}{p^{m_0}} \) whose attractor is \( C_\alpha = \lambda C_\beta \), see page 51 of [7]. This completes the proof. □
For a given \( \lambda \in \mathbb{R}^* \), rename the maps \((2)\) to \(T^i_\lambda\) with \(1 \leq i \leq 2^{m_0+n_0}\), (sometimes they are less than this number) and let \(S^i_\lambda := (T^i_\lambda)^{-1}\). We call \(S_\lambda := \{S^i_\lambda\}_{i=1}^{2^{m_0+n_0}}\) the iterated function systems corresponding to the pair \((C_\alpha, \ C_\beta) \in L\), (or the attractors \(C_\alpha - \lambda C_\beta\)). Indeed, the set \(C_\alpha - \lambda C_\beta\) forms a uniformly contracting self–similar set that obeys from the formula

\[
C_\alpha - \lambda C_\beta = \bigcap_{i \in \mathbb{N}} S^i_\lambda([-\lambda, 1]).
\]

If we do \(m_0\) steps in the construction of \(C_\alpha\) and \(n_0\) steps in the construction of \(C_\beta\), then the squares that obtain from their Cartesian product are called the first step of the construction of \(C_\alpha \times C_\beta\). The number of the squares are \(2^{m_0+n_0}\) and each of them has length \(p^{-m_0}\). Let \(\Pi_\theta := \text{Proj}_\theta\) be the projection onto the line \(\mathbb{R} \times \{0\}\) and let \(C\) be one of these squares. Therefore the affine map that sends the interval \([-\lambda, 1]\) to the interval \(\Pi_\theta(C)\) is one of the maps \((2)\). Basically, the calculation of the maps \((2)\) is easier in this way.

**Corollary 2.1.** Under the above notations,

\[\]

I) If \(\lambda = \frac{q(p-1)}{p(q-1)}\) and \((m_0 + n_0) \log 2 + \log \frac{3}{4} < m_0 \log p\), then \(|C_\alpha - \lambda C_\beta| = 0\),

II) If \(\lambda_1 = \frac{p^i}{q^j}\lambda_2\), then \(HD(C_\alpha - \lambda_1 C_\beta) = HD(C_\alpha - \lambda_2 C_\beta)\). Moreover, \(|C_\alpha - \lambda_1 C_\beta| = 0\) if and only if \(|C_\alpha - \lambda_2 C_\beta| = 0\).

**Proof.** I) In this case, \(\tan \theta = \frac{p(q-1)}{q(p-1)}\) and it is easy to check that \(\Pi_\theta((1, 1 - \frac{1}{q})) = (\frac{1}{p}, 0)\). Thus, the number of intervals emerged from the projection of squares in the first step of the construction of \(C_\alpha \times C_\beta\) is at most \(\frac{3}{4}2^{m_0+n_0}\). As above, the iterated function system corresponding to \(\lambda = \frac{q(p-1)}{p(q-1)}\) consists of at most \(\frac{3}{4}2^{m_0+n_0}\) maps, that ensures its similarity dimension is smaller than \(\log_{p^{m_0}} \frac{3}{4}2^{m_0+n_0}\). Therefore, \(HD(C_\alpha - \lambda C_\beta) < 1\).

This completes the proof of (I).

II) The assertion is obtained since operators \((1)\) are affine and the points on the vertical lines \(s = \lambda_1\) pass over the vertical lines \(s = \lambda_2\) with suitable compositions of them. \(\square\)

### 3 Weak stable intersection

In this section, we apply Theorem 2.1 to provide conditions that guarantee the existence of an interval in \(f(K) - g(K')\) and then we bring some results in this direction. First we state a definition.

**Definition 3.1.** The elements of the iterated function systems \(S_\lambda := \{S^i_\lambda\}\) corresponding to the pair \((C_\alpha, \ C_\beta)\) are regularly linked on \(\lambda \in (m_1, m_2) \subset \mathbb{R}^*\) if

i) the set \(\bigcup S^i_\lambda((-\lambda, 1))\) is connected, for each \(\lambda \in (m_1, m_2)\),

ii) the set \(\bigcup S^i_\lambda((\lambda, 1))\) is connected, for each \(\lambda \in (m_1, m_2)\),

iii) the set \(\bigcup S^i_\lambda((0, -\lambda))\) is connected, for each \(\lambda \in (m_1, m_2)\),

iv) the set \(\bigcup S^i_\lambda((0, \lambda))\) is connected, for each \(\lambda \in (m_1, m_2)\),

v) the set \(\bigcup S^i_\lambda((\lambda, \lambda + 1))\) is connected, for each \(\lambda \in (m_1, m_2)\),

vi) the set \(\bigcup S^i_\lambda((-\lambda, \lambda))\) is connected, for each \(\lambda \in (m_1, m_2)\).
ii) for each $1 \leq i, j \leq 2^{m_0+n_0}$, we have $S^i_\lambda((-\lambda, 1)) \cap S^j_\lambda((-\lambda, 1)) = \emptyset$. Otherwise there exists $d \in \mathbb{R}$ such that $d \in S^i_\lambda((-\lambda, 1)) \cap S^j_\lambda((-\lambda, 1))$, for every $\lambda \in (m_1, m_2)$.

**Proposition 3.1.** Suppose that the elements of the iterated function systems $S_\lambda$ corresponding to the pair $(C_\alpha, C_\beta) \in L$ are regularly linked on $(m_1, m_2)$. Furthermore, suppose that $f, g : [0, 1] \to \mathbb{R}$ are defined in a way that for one point $(x_0, y_0) \in C_\alpha \times C_\beta$, functions $f$ in a neighborhood $x_0$ and $g$ in a neighborhood $y_0$ are of the class $C^1$ and $m_1 < \frac{g'(y_0)}{f'(x_0)} < m_2$. Then $f(C_\alpha) - g(C_\beta)$ contains an interval.

**Proof.** As just mentioned in Section 2, the set $C_\alpha - \lambda C_\beta$ consists of all points that appear from the projection of $C_\alpha \times C_\beta$ under the angle $\cot^{-1} \lambda$. This fact together with the part (i) of Definition 3.1 give $C_\alpha - \lambda C_\beta = [-\lambda, 1]$, for each $\lambda \in (m_1, m_2)$. The second condition presented in the assumption of the proposition guarantees the existence of a square named $C$ situated in $(x_0, y_0)$, in further steps of the construction $C_\alpha \times C_\beta$, such that for the family of curves $f(x) - g(y) = c$ which stay in $C$ satisfying $m_2^{-1} < y' = \frac{f'(x)}{g'(y)} < m_1^{-1}$. This implies that if we take the projection map in the direction of the curves $f(x) - g(y) = c$ onto the line $y = y_0$, then the projection of all the sub squares of further construction situated in the square $C$ overlap each other. We draw this situation in Figure 1 for $m_1 > 0$.

![Figure 1](image)

Figure 1: $\Pi_1$ and $\Pi_2$ project two sub squares of $C$ under the angles $\cot^{-1} m_1$ and $\cot^{-1} m_2$, respectively. We observe that how the projection of these squares in the direction of curves $f(x) - g(y) = c$ are determined on the line $y = y_0$.

Note that, a curve from the family $f(x) - g(y) = c$ which passes through the point $(a, b)$ is forced to lie among the lines passing through the point $(a, b)$ with slopes $m_1^{-1}$ and $m_2^{-1}$. This process happens for all sub squares $C$. Putting these facts together, we conclude the existence of an interval in the set $f(C_\alpha) - g(C_\beta)$. □

Example 1. Suppose that $C$ is a middle-$\frac{1}{3}$ Cantor set. Then $C^2 + C^2$ and $\sin C + \cos C$ contain an interval.

The projection of all the squares in the first step of the construction $C \times C$ cover each other, when $-1 < \lambda = \cot \theta < \frac{-1}{3}$. We can select numbers $m_1$ and $m_2$ such that $\frac{-1}{2} \in (m_1, m_2)$ and the elements of
the iterated function systems corresponding to the pair \((C, C)\) are regularly linked on \(\lambda \in (m_1, m_2)\). Let \(f(x) = -g(x) = x^2\), hence the family of curves \(x^2 + y^2 = c\) satisfying the differential equation \(y' = \frac{x}{-y}\) and for the point \((\frac{2}{3}, \frac{1}{3})\) \(\in C \times C\) we have \(m_1 < \frac{1}{3} < m_2\). Considering Proposition 3.1, the set \(C^2 + C^2\) contains an interval.

For the second one, when \(\frac{1}{3} < \cot \theta < 1\), the projection of all the squares in the first step of the construction \(C \times C\) cover each other. Moreover, the family of the curves \(\sin x + \cos y = c\) satisfies \(y' = \frac{\cos x}{\sin y}\) and for the point \((\frac{1}{3}, \frac{1}{3})\) \(\in C \times C\) we have \(\frac{1}{3} < \frac{\sin \frac{1}{3}}{\cos \frac{1}{3}} < 1\). The assertion is obtained by using Proposition 3.1, where \(m_1\) and \(m_2\) are enough close to \(\frac{\sin \frac{1}{3}}{\cos \frac{1}{3}}\) and \(\frac{1}{3} < m_1 < \frac{\sin \frac{1}{3}}{\cos \frac{1}{3}} < m_2 < 1\).

Although the first condition in Proposition 3.1 seems weak, this defect disappears in the second condition as we have seen in Example 1. We will also see this below and in Example 2 at the end of the Section 4.

**Definition 3.2.** Suppose that \(K\) and \(K'\) are two Cantor sets of the real numbers. We say that the pair \((K, K')\) has weak stable intersection in the sense of topology \(C^r\) with \(r \geq 1\), if \(f(K) \cap g(K') \neq \emptyset\), for all diffeomorphisms of \(f\) and \(g\) in the \(C^r\)-neighborhood of the identity.

Thus, when \(K\) and \(K'\) have weak stable intersection, the set \(f(K) - g(K')\) contains an interval, for \(f\) and \(g\) selected in a \(C^r\)-neighborhood of the identity. Now, we obtain two important results.

i) If the pair \((K, K')\) has stable intersection, then we can take neighborhood \(U\) of the identity, such that for the \(f, g \in U\), the sets \(f(K)\) and \(g(K')\) are regular Cantor sets, and \((f(K), g(K'))\) has stable intersection. Hence, the pair \((K, K')\) has weak stable intersection too. Thus, an appropriate way to show that the Cantor sets \(K\) and \(K'\) does not have stable intersection is to introduce a sequence of diffeomorphisms \(\{h_n\}\) near \(h(x) = x\), such that the Lebesgue measure of \(K - h_n(K')\) is zero.

ii) If \(HD(K) + HD(K') < 1\) and \(K\) and \(K'\) are the regular Cantor sets, then \(f(K) - g(K')\) does not contain any interval for each \(f, g \in C^1\), since

\[
HD(f(K) - g(K')) \leq HD(f(K) \times g(K'))
\]

\[
= HD((f, g)(K \times K'))
\]

\[
= HD(K \times K')
\]

\[
= HD(K) + HD(K') < 1.
\]

Thus, the pair \((K, K')\) does not have weak stable intersection.

Note that if we take arbitrary Cantor sets \(K\) and \(K'\) with \(\dim_H K = \dim_B K\) instead of regular Cantor sets \(K\) and \(K'\) in (ii), then we obtain the same assertion. Before we state a result about the existence of weak stable intersection of Cantor sets \(C_\alpha\) and \(\lambda C_\beta\), we state following open problem:

**Open Problem 1.** Are there any \((C_\alpha, C_\beta) \in \Omega\) that has weak stable intersection while does not have
stable intersection? what about regular Cantor sets \((K, K')\)?

The pair \((C, C)\) with \(\frac{1}{3}\)-middle Cantor set does not have weak stable intersection too. In fact, natural variations of Sannami’s example [14], which follows from the results of [1], shows that there are central Cantor sets \(K\) which are diffeomorphic to \(C\) by diffeomorphisms \(C^\infty\) very close to the identity such that \(K - K\) has empty interior with positive Lebesgue measure.

Note that \(C - C = [0, 1]\) and that \(\bigcup S^i_1((-1, 1)) = (-1, -\frac{1}{3}) \cup (-\frac{1}{3}, \frac{1}{3}) \cup (\frac{1}{3}, 1)\). But for the iterated function systems \(S_\lambda = \{S^i_\lambda\}\) corresponding to the pair \((C_\alpha, C_\beta)\) that satisfies \(\bigcup S^i_\alpha((-1, 1)) = (-1, 1)\), we can select the interval \((m^{-1}, m)\) such that for every \(1 \leq i, j \leq 2^{m_0+n_0}\) there exists \(d \in \mathbb{R}\) if \(d \in S^i_\alpha((-1, 1)) \cap S^j_\alpha((-1, 1))\), then \(d \in S^i_\alpha((-\lambda, 1)) \cap S^j_\alpha((-\lambda, 1))\), for every \(\lambda \in (m^{-1}, m)\). Let \(U\) be a \(C^r\)-neighborhood of the identity map that \(m^{-1} < f^{(x)}_g < m\) for every \(x, y \in [0, 1]\) and \(f, g \in U\).

By planning the arguments similar to what employed in the proof of Proposition 3.1, we observe that \(f(C_\alpha) - g(C_\beta)\) contains an interval. Indeed, it is proved that the pair \((C_\alpha, C_\beta) \in \mathcal{L}\) has weak stable intersection. The below corollary can be a generalization of this result.

**Corollary 3.1.** The pair \((C_\alpha, \lambda C_\beta)\) with condition \(\bigcup S^i_1((-\lambda, 1)) = (-\lambda, 1)\) has weak stable intersection.

## 4 Hausdorff dimension

The results of this section begin by observing the complication of calculating Hausdorff dimension \(C_\alpha - \lambda C_\beta\) with the simplest non trivial choice of the middle Cantor sets \(C_\alpha\) and \(C_\beta\); indeed, \(\frac{\log \alpha}{\log \beta} = \frac{3}{2} \in \mathbb{Q}\), together with the special number \(\lambda\). Take \(\alpha := \frac{1}{p} := \frac{1}{\gamma}\) and \(\beta := \frac{1}{q} := \frac{1}{\gamma}\), where \(\gamma\) is the golden number \(\frac{\sqrt{5} + 1}{2}\).

Also, let \(C_\alpha\) and \(C_\beta\) be two middle Cantor sets with expanding maps \(\phi_\alpha\) and \(\phi_\beta\), respectively, as follows:

\[
C_\alpha : \begin{array}{c|c}
\frac{1}{p} & \frac{1}{p} \\
px & px - p + 1
\end{array} \quad C_\beta : \begin{array}{c|c}
\frac{1}{q} & \frac{1}{q} \\
qx & qx - q + 1
\end{array}
\]

\[
\phi_\alpha(x) := \begin{cases}
px & x \in [0, \frac{1}{p}], \\
px - p + 1 & x \in [1 - \frac{1}{p}, 1]
\end{cases}, \quad \phi_\beta(x) := \begin{cases}
qx & x \in [0, \frac{1}{q}], \\
qx - q + 1 & x \in [1 - \frac{1}{q}, 1]
\end{cases}.
\]

The pair \((C_\alpha, C_\beta) \in \Omega\), since \(HD(C_\alpha) + HD(C_\beta) = \frac{5}{6} \log_\gamma 2 \cong 1.2003\) and \(\tau(C_\alpha) \cdot \tau(C_\beta) = \frac{1}{3-\gamma} \cong 0.7236\).

**Proposition 4.1.** The iterated function system corresponding to \(C_\alpha - \frac{2}{\gamma} C_\beta\) is of finite type and Hausdorff dimension of its attractor is smaller than one.

**Proof.** Firstly, it is easy to check that

\[
\begin{align*}
\gamma^2 &= \gamma + 1, & \gamma^3 &= 2\gamma + 1, & \gamma^4 &= 3\gamma + 2, & \gamma^5 &= 5\gamma + 3, & \gamma^6 &= 8\gamma + 5, \ldots \\
\frac{1}{\gamma} &= \gamma - 1, & \frac{1}{\gamma^2} &= 2 - \gamma, & \frac{1}{\gamma^3} &= 2\gamma - 3, & \frac{1}{\gamma^4} &= 5 - 3\gamma, \ldots
\end{align*}
\]

\]
Consider \( \Pi \) with \( \cot \theta := \frac{q(p-1)}{p(q-1)} = \frac{2}{7} = \sqrt{2} - 1 \), (see part (I) of Corollary 2.1). Letting \( m_0 = 2 \), \( n_0 = 3 \) and \( \lambda = \frac{2}{7} \) in Theorem 2.1, then we obtain 21 maps \( T = \{ T_i | T_i(t) = \gamma^\theta + a_j \}_{i=1}^{21} \) on \( \mathbb{R} \), that are return maps to the vertical line \( s = \frac{2}{7} \). As we mentioned in Section 2, we can easily find \( a_i \)'s as below:

\[
\left(0, \left(1 - \frac{1}{q}\right)(1 + \frac{1}{q} + \frac{1}{q^2})\right) \rightsquigarrow a_1 = -(6\gamma + 4)(4\gamma - 8) = 8\gamma + 8,
\]

\[
\left(0, \left(1 - \frac{1}{q}\right)(1 + \frac{1}{q})\right) \rightsquigarrow a_2 = -(6\gamma + 4)(\gamma - 3) = 8\gamma + 6,
\]

\[
\left(1 - \frac{1}{p^2}, \left(1 - \frac{1}{q}\right)(1 + \frac{1}{q} + \frac{1}{q^2})\right) \rightsquigarrow a_3 = -(6\gamma + 4)(6\gamma - 11) = 6\gamma + 8,
\]

\[
\left(1 - \frac{1}{p^2}, \left(1 - \frac{1}{q}\right)(1 + \frac{1}{q})\right) \rightsquigarrow a_4 = -(6\gamma + 4)(3\gamma - 6) = 6\gamma + 6,
\]

\[
\left(0, \left(1 - \frac{1}{q}\right)\right) a_5 = -(6\gamma + 4)(0 - 1) = 6\gamma + 4,
\]

\[
\left(1 - \frac{1}{p^2}, \left(1 - \frac{1}{q}\right)(1 + \frac{1}{q^2})\right) \rightsquigarrow a_6 = -(6\gamma + 4)(5\gamma - 9) = 4\gamma + 6,
\]

\[
\left(1 - \frac{1}{p^2}, \left(1 - \frac{1}{q}\right)\right) \rightsquigarrow a_7 = -(6\gamma + 4)(2\gamma - 4) = 4\gamma + 4.
\]

Note that, the numbers in the right side of the notion \( \rightsquigarrow \) are the \( a_i \)'s which are related to the square of the first step of the construction \( C_{a} \times C_{2} \) appeared in the point presented in the left side. Of course, we can find other \( a_i \)'s as below:

For every \( 7 < i \leq 14 \), we use the relation \( a_i = a_{i-7} - p(p-1) = a_{i-7} - (6\gamma + 4) \) and we get

\[
a_8 = 2\gamma + 4, \quad a_9 = 2\gamma + 2, \quad a_{10} = 4, \quad a_{11} = 2,
\]

\[
a_{12} = 0, \quad a_{13} = -2\gamma + 2, \quad a_{14} = -2\gamma.
\]

Also, for every \( 14 < i \leq 21 \), we use the relation \( a_i = a_{i-7} - p(p-1) = a_{i-7} - (6\gamma + 4) \) and we get

\[
a_{15} = -4\gamma, \quad a_{16} = -4\gamma - 2, \quad a_{17} = -6\gamma, \quad a_{18} = -6\gamma - 2,
\]

\[
a_{19} = -6\gamma - 4, \quad a_{20} = -8\gamma - 2, \quad a_{21} = -8\gamma - 4.
\]

The first assertion is obtained by using Theorem 2.9 of [9], since \( b_i = -\frac{a_i}{p^2} \in \mathbb{Z}[\gamma] \). For the second, Theorem 2.1 implies that \( C_{a} - \frac{2}{7} C_{2} = \bigcap_{i \in \mathbb{N}} T^{-i}\left(\left(\frac{2}{7}, 1\right)\right) \). Now we are going to describe a scheme to estimate a suitable upper bound of its Hausdorff dimension. To do this, we split the interval \( \left[\frac{2}{7}, 1\right] \) by using the return maps \( \{ T_j^{-1}\}_{j=1}^{21} \) as follows:

\[
G_1 := [T_2^{-1}\left(\frac{2}{7}\right), T_1^{-1}(1)] = \left[\frac{-10\gamma - 4}{p^2}, \frac{-8\gamma - 7}{p^2}\right], \quad G_2 := [T_4^{-1}\left(\frac{2}{7}\right), T_3^{-1}(1)] = \left[\frac{-8\gamma - 4}{p^2}, \frac{-6\gamma - 7}{p^2}\right],
\]

\[
G_3 := [T_5^{-1}\left(\frac{2}{7}\right), T_4^{-1}(1)] = \left[\frac{-8\gamma - 2}{p^2}, \frac{-6\gamma - 5}{p^2}\right], \quad G_4 := [T_7^{-1}\left(\frac{2}{7}\right), T_6^{-1}(1)] = \left[\frac{-6\gamma - 2}{p^2}, \frac{-4\gamma - 5}{p^2}\right],
\]

\[
R_1 := [T_1^{-1}\left(\frac{2}{7}\right), T_2^{-1}(1)] = \left[\frac{-8\gamma - 6}{p^2}, \frac{-6\gamma - 5}{p^2}\right], \quad R_2 := [T_6^{-1}\left(\frac{2}{7}\right), T_5^{-1}(1)] = \left[\frac{-6\gamma - 4}{p^2}, \frac{-4\gamma - 3}{p^2}\right],
\]

\[
G_i := G_{i-4} + \frac{p+1}{p}, \quad 4 < i \leq 8, \quad G_i := G_{i-4} + \frac{p+1}{p}, \quad 8 < i \leq 12,
\]

\[
R_i := R_{i-2} + \frac{p+1}{p}, \quad 2 < i \leq 4, \quad R_i := R_{i-2} + \frac{p+1}{p}, \quad 4 < i \leq 6,
\]

\[
H_1 := (T_1^{-1}(1), T_8^{-1}\left(\frac{2}{7}\right)) = \left(\frac{-3\gamma - 3}{p^2}, \frac{-3\gamma - 4}{p^2}\right), \quad H_2 := (T_{14}^{-1}(1), T_{15}^{-1}\left(\frac{2}{7}\right)) = \left(\frac{2\gamma + 1}{p^2}, \frac{2\gamma + 2}{p^2}\right),
\]

\[
Z_1 := \text{ch}(G_2 \cup G_3) \setminus G_2 \cup G_3, \quad Z_2 := \text{ch}(G_6 \cup G_7) \setminus G_6 \cup G_7, \quad Z_3 := \text{ch}(G_{10} \cup G_{11}) \setminus G_{10} \cup G_{11},
\]

where \( \text{ch}(A) \) is the shorthand of the convex hull \( A \subseteq \mathbb{R} \). Let \( \{Y_i\}_{i=1}^{12} \) be the intervals that situated between two subsequent \( G_i \) and \( R_i \), respectively, and \( \{X_i\}_{i=1}^{6} \) are connected components of the complement of the above sets on the interval \( \left[\frac{2}{7}, 1\right] \), respectively, see Figure 2.
Regardless of the indexes, we have
\[ |H| = \frac{|X|}{2} = |Y| = |R| = \frac{1}{\gamma^6}, \quad |G| = \frac{2\gamma - 3}{\gamma^6}, \quad |Z| = |X| - |G|. \tag{3} \]

Also, it is straightforward to show that
\begin{align*}
X_1 &= T_1^{-1}\left( (ch\{R_6, \ 1\})^c \right), \quad X_2 = T_7^{-1}\left( (ch\{\frac{2}{\gamma}, \ R_1\})^c \right), \\
Y_1 &= T_2^{-1}\left( (ch\{\frac{2}{\gamma}, \ R_1\} \cup ch\{R_4, \ 1\})^c \right), \quad Y_2 = T_3^{-1}\left( (ch\{\frac{2}{\gamma}, \ R_3\} \cup ch\{R_6, \ 1\})^c \right), \\
Y_3 &= T_5^{-1}\left( (ch\{\frac{2}{\gamma}, \ R_1\} \cup ch\{R_4, \ 1\})^c \right), \quad Y_4 = T_6^{-1}\left( (ch\{\frac{2}{\gamma}, \ R_3\} \cup ch\{R_6, \ 1\})^c \right), \\
Z_1 &= T_4^{-1}\left( (ch\{\frac{2}{\gamma}, \ R_1\} \cup ch\{R_6, \ 1\})^c \right),
\end{align*}

where \( c \) is the complement on interval \( \left[ \frac{2}{\gamma}, \ 1 \right] \). Other sets have similar relations.

The intervals \( S^i[-\lambda, \ 1] \) overlap each other just on \( G_i \) or \( R_i \). It is important to know that in each \( G_i \) or \( R_i \), the inverse of \( G \)s (also \( R \)s) under the maps \( T \) is either the same, or there is not any intersection between them and they situate symmetrical as Figure 3.

\[ G : - - - \quad R : - - - - - - - - - - \]

Figure 3: Projection of the emerged squares in the second step of the construction on \( G, R \), respectively.

Henceforth, we show that in each stage of the construction \( C_\alpha \times C_\beta \), there exists an attractor namely \( F \), with the minimum number of the contractions, that satisfies
\[ \Pi_\theta(F) = \Pi_\theta(C_\alpha \times C_\beta) = C_\alpha - \frac{2}{\gamma}C_\beta \tag{5} \]

and then we see that \( HD(F) < 1 \) for the 6th step.

In the first step, we have the iterated function system which consists of 32 contractions
\[ S^1 := \left\{ S \mid S(x, y) = \frac{1}{p^2} (x, y) + \left( 1 - \frac{1}{p} \right) \sum_{k=0}^{1} a_k p^k, \ (1 - \frac{1}{q}) \sum_{k=0}^{2} b_k q^k \right\} \]
on the square \( I \times I \). An equivalence relation on \( S^1 \) defines as follows:
\[ S_1 \sim S_2 \quad \equiv \quad \Pi_\theta \circ S_1 = \Pi_\theta \circ S_2 \quad \forall \ S_1, \ S_2 \in S^1. \tag{6} \]

On our selection of number \( \gamma \) and angle \( \theta \), we can take \( \mathcal{F} := \{ S_1, \ldots, S_{21} \} \subset S^1 \), in this condition that \( S_i \sim S_j \) for each \( 1 \leq i \neq j \leq 21 \). Let \( F \) be the attractor of the family \( \mathcal{F} \) on \( I \times I \). Then \( \mathcal{F} \) satisfies the relation (5) and we have \( HD(F) = \log_{\gamma^6}^{21} \).
On the n–step, let $S^n := \left\{ S = S_{i_1} \circ S_{i_2} \circ \ldots \circ S_{i_n} | S_{i_j} \in S^1, 1 \leq j \leq n \right\}$. Again, for elements of $S^n$, we use the equivalence relation (6). If $1 \leq i \leq 6$, then we define $\mathcal{X}_i := \left\{ S \mid S(I \times I) \cap \Pi^{-1}_\theta(X_i) \neq \emptyset \right\}$. For $1 \leq i \leq 12$ the set $\mathcal{Y}_i$, and for $1 \leq i \leq 3$ the set $\mathcal{Z}_i$ have been defined similarly. Moreover, if $1 \leq i \leq 12$, then we define $\mathcal{G}_i := \left\{ S \mid S(I \times I) \subset \Pi^{-1}_\gamma(G_i) \right\}$. For $1 \leq i \leq 6$ the set $\mathcal{R}_i$ has been defined similarly. Regardless of the index $i$, we take $x_n := |\mathcal{X}_i|$. Numbers $y_n$, $z_n$, $g_n$ and $r_n$ are defined similarly. Now we claim that

$$\begin{bmatrix} x_n \\ y_n \\ z_n \\ g_n \\ r_n \end{bmatrix} = \begin{bmatrix} 5 & 11 & 3 & 11 & 5 \\ 4 & 10 & 3 & 10 & 4 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 8 & 2 & 6 & 5 \end{bmatrix}^{n-2} \begin{bmatrix} 19 \\ 17 \\ 2 \\ 10 \end{bmatrix} =: A^{n-2} \begin{bmatrix} 19 \\ 17 \\ 2 \\ 10 \end{bmatrix}, \quad n \geq 2.$$

The assertion holds for $n = 2$. Indeed, we know that the elements of $S^2$ are as follows:

$$S(x, y) := \frac{1}{p^4}(x, y) + \left(1 - \frac{1}{p}\right) \sum_{k=0}^3 \frac{a_k}{p^k} + \left(1 - \frac{1}{q}\right) \sum_{k=0}^5 \frac{b_k}{q^k}, \quad a_k, b_k = 0 \text{ or } 1.$$  

Consider contractions $S_i(x, y) = \frac{1}{p^4}(x, y) + a_i$, $1 \leq i \leq 4$ of $S^2$, where

$$a_1 := \left(1 - \frac{1}{p}\right)(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}), \quad \left(1 - \frac{1}{q}\right)(\frac{1}{q^4} + \frac{1}{q^5}),$$

$$a_2 := \left(1 - \frac{1}{p}\right)(1 + \frac{1}{p} + \frac{1}{p^3} + \frac{1}{p^4}), \quad \left(1 - \frac{1}{q}\right)(\frac{1}{q^4} + \frac{1}{q^5} + \frac{1}{q^6} + \frac{1}{q^7}),$$

$$a_3 := \left(1 - \frac{1}{p}\right)(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}), \quad \left(1 - \frac{1}{q}\right)(\frac{1}{q^4} + \frac{1}{q^5} + \frac{1}{q^6} + \frac{1}{q^7}),$$

$$a_4 := \left(1 - \frac{1}{p}\right)(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}), \quad \left(1 - \frac{1}{q}\right)(\frac{1}{q^4} + \frac{1}{q^5} + \frac{1}{q^6} + \frac{1}{q^7}).$$

Because of the characteristics of $\gamma$, we have $\Pi_\theta \circ S_1 = \Pi_\theta \circ S_2$ and $\Pi_\theta \circ S_3 = \Pi_\theta \circ S_4$. Regarding this fact and the relation (3), it is not hard to see that $g_2 = 2$ and $r_2 = 10$, see Figures (3) and (4).

![Figure 4](image-url)

Figure 4: The left (right) figure demonstrates all squares in the second step of the construction $C_\alpha \times C_\beta$ that entirely lie in the strip $\Pi^{-1}_\theta(G)$ ($\Pi^{-1}_\theta(R)$). We filled the squares which should be counted.

For $1 \leq i \neq j \leq 21$, we have $S^i[-\lambda, 1] \cap S^j[-\lambda, 1] \cap X_k = \emptyset$ and just the projection of two squares in the first step of the construction $C_\alpha \times C_\beta$ entirely stay in $\text{ch}\{R_6, 1\}$ so $x_2 = 19$, see the relation (4) and Figure (2). Similarly $y_2 = 10$ and $z_2 = 17$.

The general case is obtained by using induction method and the relation

$$[x_n, y_n, z_n, g_n, r_n]^T = A[x_{n-1}, y_{n-1}, z_{n-1}, g_{n-1}, r_{n-1}]^T.$$
Now we select the elements of \( F \subset S^n \), that are just in a class and \( F \) is its attractor on the square \( I \times I \). Therefore, the relation (5) is valid. To calculate \( HD(F) \) we define

\[
k_n := |F| = < (6, 12, 3, 12, 6), \ (x_n, y_n, z_n, g_n, r_n) >,
\]

that gives \( HD(F) = \log_{10}^{k_n} |F| = \log_{10}^{3/4} 6 \). By using Maple program, we have

\[
\begin{align*}
  n = 2 & \Rightarrow HD(F) = \log_{10}^{19.2093\ldots} > 1, \\
  n = 3 & \Rightarrow HD(F) = \log_{10}^{18.5246\ldots} > 1, \\
  n = 4 & \Rightarrow HD(F) = \log_{10}^{18.1817\ldots} > 1, \\
  n = 5 & \Rightarrow HD(F) = \log_{10}^{17.9782\ldots} > 1, \\
  n = 6 & \Rightarrow HD(F) = \log_{10}^{17.8437\ldots} < 1,
\end{align*}
\]

Thus, on the 6th step of the construction \( C_\alpha \times C_\beta \), we see that \( HD(F) < 0.9982 \) since \( \gamma^6 \approx 17.9442 \). This completes the proof of the proposition. \( \sqcup \sqcap \)

Proposition 4.1 not only yields that Lebesgue measure of the set \( C_\alpha - \frac{2}{\gamma} C_\beta \) is zero, but also says that the above Cantor sets with \( \lambda = \frac{2}{\gamma} \) are good candidates for the following case:

\[
HD(C_\alpha - \lambda C_\beta) < \min \{1, \ HD(C_\alpha) + HD(C_\beta) \}
\]  

(7)

In general, when \( \lambda = 1 \), it has been showed that the equality holds in (7), where \( \frac{\log_\alpha}{\log_\beta} \notin \mathbb{Q} \). In our example equality holds since \( C_\alpha - C_\beta = [-1, 1] \). Also, it is obvious that if \( HD(C_\alpha) + HD(C_\beta) < 1 \) and \( \frac{\log_\alpha}{\log_\beta} \in \mathbb{Q} \), then (7) is always valid [11]. Moreover, \( H^{HD(C_\alpha)+HD(C_\beta)}(C_\alpha + C_\beta) = 0 \) [5].

Although, the iterated function system \( \{S_i\}_{i=1}^{21} \) corresponding to the attractor \( C_\alpha - \frac{2}{\gamma} C_\beta \) is of finite type and \( HD(C_\alpha - \lambda C_\beta) \) could be calculated by characterizing the incidence matrix corresponding to this 21 maps, (we can not do it). But we find an easier way to do this by using the fact that the attractor \( F \) of the iterated function system of finite type satisfies \( 0 < H^s(F) < 1 \), where \( s=\dim_H(F) \) [9]. Take \( A \) as Proposition 4.1, hence

**Proposition 4.2.** The Hausdorff dimension of \( C_\alpha - \frac{2}{\gamma} C_\beta \) is \( \log_{p2} \lambda \), where \( \lambda \) is the largest eigenvalue of the matrix \( A \). Moreover, this number is the Box dimension of \( C_\alpha - \frac{2}{\gamma} C_\beta \).

**Proof.** By the same notations used in Proposition 4.1 and \( s := HD(C_\alpha - \frac{2}{\gamma} C_\beta) \). Noting to the scaling property of the s–dimensional Hausdorff measure \( H^s \), we obtain

\[
\begin{pmatrix}
  p^{2s} H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap X) \\
  p^{2s} H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap Y) \\
  p^{2s} H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap Z) \\
  p^{2s} H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap G) \\
  p^{2s} H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap R)
\end{pmatrix} =
\begin{pmatrix}
  5 & 1 & 1 & 3 & 1 \\
  2 & 6 & 2 & 6 & 2 \\
  4 & 10 & 3 & 10 & 4 \\
  0 & 2 & 0 & 1 & 2 \\
  0 & 8 & 2 & 6 & 5
\end{pmatrix}
\begin{pmatrix}
  H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap X) \\
  H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap Y) \\
  H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap Z) \\
  H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap G) \\
  H^s ((C_\alpha - \frac{2}{\gamma} C_\beta) \cap R)
\end{pmatrix},
\]

13
which is equivalent to

\[(p^{2s}I - A) \begin{bmatrix}
\nu^c((c_\gamma - \frac{g}{2}C_\beta) \cap X) \\
\nu^c((c_\gamma - \frac{g}{2}C_\beta) \cap Y) \\
\nu^c((c_\gamma - \frac{g}{2}C_\beta) \cap Z) \\
\nu^c((c_\gamma - \frac{g}{2}C_\beta) \cap R)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

As \( C_\alpha - \frac{g}{2}C_\beta \) is a \( s \)-set, \( \det(p^{2s}I - A) = 0 \). On the other hand, roots of the characteristic polynomial

\[x^5 - 20x^4 + 50x^3 - 28x^2 - 3x = x(x - 1)(x^3 - 19x^2 + 31x + 3)\]
corresponding to the matrix \( A \) are 0, 1, −0.0943..., 1.8434..., 17.2508.... Due to these facts \( s \) has to be \( \log_{p^2} 17.2508 = 0.9863.... \) The second assertion is a direct result of Theorem 1.1 of [9]. □

Regarding to our discussion, we convince that \( C_\alpha \times C_\beta \) has a better structure among other members of \( \mathcal{L} \). We finish this paper by posing several characteristics of the pair of Cantor sets \( C_\alpha \times C_\beta \). Take the dense subgroup \( G := \{c_\gamma + d|\ c, d \in \mathbb{Q}\} \) from the real numbers. Hence,

**Proposition 4.3.** For all \( g \in G \), the set \( C_\alpha - gC_\beta \) contains an interval or has zero Lebesgue measure.

**Proof.** Fix \( g \in G \). Then we can find the rational number \( r \in \mathbb{Q} \) such that all \( a_g \) appeared in Theorem 2.1 satisfy \( b_g = -\frac{a_g}{p^2} \in r\mathbb{Z}[\gamma] \), since \( G = \mathbb{Q}[\gamma] \) is a field (note that \( \frac{1}{c_\gamma + d} = \frac{c}{c^2 - cd - d^2} \cdot \gamma - \frac{c + d}{c^2 - cd - d^2} \)). When \( HD(C_\alpha - gC_\beta) = 1 \), then by using Theorems 2.9 and 1.3 of [9], we see that \( (C_\alpha - gC_\beta)^\circ \neq \emptyset \). Thus, the set \( C_\alpha - gC_\beta \) contains an interval which proves the corollary. □

Also, it is easy to find \( \lambda \)'s that \( C_\alpha - \lambda C_\beta \) contains an interval. This can be useful in providing the assumptions of Proposition 3.1, for given \( f \) and \( g \). By taking \( f(x) = g(x) = \sqrt{x} \), we pose the below example.

**Example 2.** The set \( \sqrt{C_\alpha} - \sqrt{C_\beta} \) contains an interval.

On the first step of the structure of \( C_\alpha \times C_\beta \), the projection under the angle \( \theta \) on all squares overlap each other, when

\[1 = \frac{(1 - \frac{1}{2}) - \frac{1}{2}}{\frac{\q}{p}} < \tan \theta < \frac{\q}{\frac{p(1 - \frac{1}{2})}{2}} = \frac{q}{p - 2} . \]

We can select a basic square in the next structures of \( C_\alpha \times C_\beta \) situated in \((x_0, y_0)\) between lines \( y = x \) and \( y = (\frac{q}{p - 2})^2x \) (there are plenty of squares close to the point \((1, 1)\)) and so \( 1 < \frac{y_0}{x_0} < (\frac{q}{p - 2})^2 \). On the other hand, we see that the family of curves \( \sqrt{x} - \sqrt{y} = c \) satisfies \( y' = \frac{\sqrt{x}}{2} \). Now we can select \( \frac{p - 2}{q} < m_1 < \frac{\sqrt{x}}{y_0} < m_2 < 1 \) such that the elements of the iterated function systems corresponding to \((C_\alpha, C_\beta)\) be regularly linked on \((m_1, m_2)\). Regarding to the Proposition 3.1, the set \( \sqrt{C_\alpha} - \sqrt{C_\beta} \) contains an interval.

Our example is also different from Solomyak’s example in this point that we can not put it in a continuous curve

\[r : [\alpha_1, \alpha_2] \mapsto \Omega \times \mathbb{R}^* \]
\[\alpha \rightarrow (C_\alpha, C_\beta(\alpha), \lambda(\alpha))\]
with condition $HD(C_\alpha - \lambda(\alpha)C_\beta(\alpha)) < 1$, for each $\alpha \in [\alpha_1, \alpha_2]$. Note that, Solomyak’s curves that can be written in the form $r_{ma,na}(\alpha) = (C_\alpha, C_\alpha \frac{ma}{na} - 1)$ stay close to $\{(C_\alpha, C_\beta, -1) | HD(C_\alpha) + HD(C_\beta) = 1\}$, of course the case $m_0 = n_0$ is an exception, (see Remark and Figure 2 of [15]). Another example of this kind is the family of the curves $r_{ma,na}(\alpha) = (C_\alpha, C_\alpha \frac{ma}{na}, \frac{1-\alpha}{1-\alpha \frac{ma}{na}})$ that can be obtained from the part (I) of Corollary 2.1. They stay close to the boundary $\Omega \times \mathbb{R}^*$ too.

In contrast, it seems that there exists a sequence $\{\lambda_i\}$ of real numbers with $\lambda_i > 1$ convergent to one which $C_\alpha - \lambda_i C_\beta$ have zero Lebesgue measure, (recall that $|C_\alpha - \frac{2}{3}C_\beta| = 0$ and $C_\alpha - 1C_\beta = [-1, 1]$). A positive answer to this not only rejects the below problem but also gives $\{\lambda | HD(C_\alpha - \lambda C_\beta) < 1 \}' \neq \{0\}$.

**Open Problem 2.** Does the pair $(C_\alpha, C_\beta)$ have stable intersection, what about weak?

**Acknowledgment 1.** The author deeply thanks C. G. Moreira, without his comments and suggestions this paper would not have been possible.

**References**

[1] R. Bamon, C.G. Moreira, S. Plaza and J.Vera, *Differentiable structures of central Cantor sets*, Ergod. TH. and Dynam. Sys. (1997), 17, 1027-1042

[2] G. A. Freiman, *Diophantine approximation and the geometry of numbers (Markov’s Problem)*, Kalinin. Gosudarstv. Univ., Kalink, 1975.

[3] M. Hall, *On the sum and product of continued fractions*, Ann. of Math. 48 (1947), 966-993.

[4] B. Honary, C. G. Moreira and M. Pourbarat, *Stable intersections of affine Cantor sets*, Bull. Braz. Math. Soc. 36 (2005), no. 3, 363-378.

[5] K. Ilgar Eroglu, *On the arithmetic sums of Cantor sets*, Nonlinearity, Volume 20, Issue 5, pp. 1145-1161 (2007).

[6] C. G. Moreira, *Stable intersections of Cantor sets and homoclinic bifurcations*, Ann. Inst. H. Poincare Anal. Non Lineaire 13 (1996), no. 6, 741-781.

[7] C. G. Moreira and J.-C. Yoccoz, *Stable intersections of regular Cantor sets with large Hausdorff dimension*, Ann. of Math. 154 (2001), no. 1, 45-96.

[8] P. Mendes and F. Oliveira, *On the topological structure of the arithmetic sum of two Cantor sets*, Nonlinearity. 7 (1994), no. 2, 329-343.
[9] S. Ngai and Y. Wang, *Hausdorff dimension of self-similar sets with overlaps*, J. London Soc. (2) 63 (2001) 655-672.

[10] M. Pourbarat, *Stable intersection of middle-α Cantor sets*, Submitted.

[11] Y. Peres and P. Shmerkin, *Resonance between Cantor sets*, available at arXiv:0705.2628v2 [math.CA] 28 Mar 2008.

[12] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Univ. Press, Cambridge, 1993.

[13] J. Palis and J.-C. Yoccoz, *On the arithmetic sum of regular Cantor sets*, Ann. Inst. Henri Poincare. 14 (1997), no. 4, 439-456.

[14] A. Sannami, *An example of a regular Cantor set whose difference set is a Cantor set with positive measure*, Hokkaido Math. J. 21 (1992), no. 1, 7-24.

[15] B. Solomyak, *On the arithmetic sums of Cantor sets*, Indag. Mathem. 1997, 133-141.