Extra large type Artin groups are CAT(0) and acylindrically hyperbolic

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June 10, 2019

Abstract. We describe a simple locally CAT(0) classifying space for extra large type Artin groups. Furthermore, when the Artin group is not dihedral, we describe a rank 1 periodic geodesic, thus proving that extra large type Artin groups are acylindrically hyperbolic. Together with Property RD proved by Ciobanu, Holt and Rees, the CAT(0) property implies the Baum-Connes conjecture for all extra large type Artin groups.

Introduction

Artin-Tits groups are natural combinatorial generalizations of Artin’s braid groups. For every finite simple graph $\Gamma$ with vertex set $S$ and with edges labeled by some integer in $\{2,3,\ldots\}$, one associates the Artin-Tits group $A(\Gamma)$ with the following presentation:

$$A(\Gamma) = \langle S \mid \forall \{s,t\} \in \Gamma^{(1)}, w_m(s,t) = w_m(t,s) \text{ if the edge } \{s,t\} \text{ is labeled } m \rangle,$$

where $w_m(s,t)$ is the word $stst\ldots$ of length $m$. Note that when $m = 2$, then $s$ and $t$ commute, and when $m = 3$, then $s$ and $t$ satisfy the classical braid relation $sts = tst$.

Also note that when adding the relation $s^2 = 1$ for every $s \in S$, one obtains the Coxeter group $W(\Gamma)$ associated to $\Gamma$. Most results about Artin-Tits groups only concern particular classes, which we recall now. The Artin group $A(\Gamma)$ is called:

- **of large type** if all labels are greater or equal to 3,
- **of extra large type** if all labels are greater or equal to 4,
- **right-angled** if all labels are equal to 2,
- **spherical** if $W(\Gamma)$ is finite, and
- **of type FC** if every complete subgraph of $\Gamma$ spans a spherical Artin subgroup.

The **rank** of an Artin-Tits group $A(\Gamma)$ is the number of vertices of $\Gamma$. The **dimension** of an Artin-Tits group $A(\Gamma)$ is the largest rank of a spherical Artin subgroup. In particular, every large type Artin group has dimension at most 2.

Many geometric questions are still open for general Artin groups (see [Cha] and [McC17]). In particular, Charney asks the following question, to which we believe the answer is positive:

**Conjecture A.** Every Artin-Tits group is CAT(0), i.e. acts properly and cocompactly on a CAT(0) metric space.

Keywords: Artin-Tits groups, CAT(0) space, acylindrical hyperbolicity, Baum-Connes conjecture.

AMS codes: 20F36, 20F65, 20F67
This conjecture has been proved for the following classes of Artin groups:

1. Right-angled Artin groups (see [CD95]).
2. Some classes of 2-dimensional Artin groups (see [BC02, BM00]).
3. Artin groups of finite type with three generators (see [Bra00]).
4. 3-dimensional Artin groups of type FC (see [Bel05]).
5. The n-strand braid group for \( n \leq 6 \) (see [BM10, HKS16]).
6. The spherical Artin group of type \( B_4 \) (see [BM10]).

Since the classes of 2-dimensional Artin groups studied by Brady and McCammond in [BM00] and the extra large type Artin groups we are studying in this article have a large intersection, we will state their results more precisely.

**Theorem** (Brady and McCammond [BM00]). Let \( A(\Gamma) \) be an Artin group such that one of the following holds:

- \(|S| = 3 \) and all labels are greater or equal to 3.
- \( \Gamma \) contains no triangles.
- All labels are greater or equal to 3, and there is a way of orienting the edges of \( \Gamma \) so that neither of the graphs in Figure 1 appear as subgraphs.

Then \( A(\Gamma) \) is CAT(0).

On the other hand, concerning the cubical world, very few Artin groups have proper and cocompact actions on CAT(0) cube complexes (see [Hae15]). Nevertheless, the question whether all Artin groups act properly on a CAT(0) cube complex, or more generally have the Haagerup property, is still open.

Concerning variations on the notion of nonpositive curvature, Bestvina defined a geometric action of Artin groups of spherical Artin on a simplicial complex with some nonpositive curvature features (see [Bes99]). More recently, Huang and Osajda proved (see [HO17]) that every Artin group of almost large type (a class including all Artin groups of large type) act properly and cocompactly on systolic complexes, which are a combinatorial variation of nonpositive curvature. They also proved (see [HO19]) that every Artin group of type FC acts geometrically on a Helly graph, which give rise to classifying spaces with convex geodesic bicombings.

Another variation on the notion of nonnegative curvature is the acylindrical hyperbolicity (see [Osi17] for a survey). A group \( G \) is called acylindrically hyperbolic if it admits an acylindrical action on some hyperbolic space \( X \) (and is not virtually cyclic), i.e. for every \( \varepsilon > 0 \), there exist \( N, R \geq 0 \) such that, for every \( x, y \in X \) at distance at least \( R \), we have

\[ |\{g \in G, d(x, g \cdot x) \leq \varepsilon \text{ and } d(y, g \cdot y) \leq \varepsilon \}| \leq N.\]
In most cases, it is much easier to find an action on some hyperbolic space with one element satisfying the WPD condition (see [BF02]), and then according to Osin (see [Osi16]) there exists an acylindrical action on some other hyperbolic space. Concerning Artin-Tits groups, Charney and Morris-Wright (see [CMW18]) ask the following question, to which we believe the answer is positive:

**Conjecture B.** For every Artin-Tits group $A$, the central quotient $A/Z(A)$ is acylindrically hyperbolic.

This conjecture has been proved for the following classes of Artin groups:

1. Right-angled Artin groups (see [CS11]).
2. Braid groups, seen as mapping class groups (see [MM99] and [Bow08]).
3. Artin-Tits groups of spherical type (see [CW16]).
4. Artin-Tits groups of type FC such that $\Gamma$ has diameter at least 3 (see [CM16]).
5. Artin-Tits groups such that $\Gamma$ does not decompose as a join of two subgraphs (see [CMW18]).

The purpose of this article is to define a new geometric model for extra large type Artin groups.

**Theorem C.** Every extra large type Artin group is the fundamental group of a compact locally CAT(0) 3-dimensional piecewise Euclidean complex. Furthermore, if the rank of the Artin group is at least 3, then some element acts as a rank 1 isometry.

And in the case where all labels are even, the complex is only 2-dimensional. An isometry of a CAT(0) space is called rank 1 if some axis does not bound a flat half-plane. An interesting consequence, due to Sisto (see [Sis18]), is that if a group $G$ acts properly on a proper CAT(0) space such that some element has rank 1, then $G$ is either virtually cyclic or acylindrically hyperbolic.

Also note that if $A$ has rank 2, then $A$ is virtually a direct product of $\mathbb{Z}$ and of a free group, so $A$ is not acylindrically hyperbolic, but its geometry is well understood. In particular, the central quotient $A/Z(A)$ is virtually free and thus acylindrically hyperbolic.

We can deduce the following consequence, regarding the two main conjectures.

**Corollary D.** Every extra large type Artin group of rank at least 3 is acylindrically hyperbolic. In particular, Conjecture A and Conjecture B hold for all extra large type Artin groups.

Note that the class of extra large type Artin groups is not contained in the classes studied by Brady and McCammond in ([BM00]), by Martin and Chatterji ([CM16]) or by Charney and Morris-Wright ([CMW18]). For instance, if $\Gamma$ is a complete graph on at least 4 vertices, with labels at least 4, then none of the previous results apply.

Many consequences of being CAT(0) are already consequences of being systolic, and as such are consequences of Huang and Osajda’s result (see [HO17]). For instance, the Novikov conjecture, the fact that centralizers virtually split, the quadratic Dehn function. Let us list a few general consequences of being CAT(0) and acylindrically hyperbolic, which are new for extra large type Artin groups.

**Corollary E.** Let $A$ be an extra large type Artin group.

- $A$ satisfies the $K$-theoretic and $L$-theoretic Farrell-Jones conjectures (see [BL13] and [Weg13]).
• A is SQ-universal, i.e. every countable group embeds in a quotient of A (see [Osi16]).
• If \( V = \mathbb{R} \) or \( V = t^p(A) \), for \( p \in [1, \infty) \), then \( H_2^\mathbb{R}(A,V) \) is infinite-dimensional (see [Osi16]).
• A has a free normal subgroup (see [Osi16]).
• A has Property \( P_{\text{naive}} \): for any finite subset \( F \subset A \backslash \{1\} \), there exists \( g \in A \) such that for all \( f \in F \), the group \( \langle f, g \rangle \) is freely generated by \( \{f, g\} \) (see [ADJ9]).
• A is not inner amenable (see [DGOT11]).
• The reduced \( C^* \)-algebra of \( A \) is simple (see [DGOT11]).

Ciobanu, Holt and Rees proved (see [CHR16]) that every extra large type Artin group satisfies the Rapid Decay Property. According to Lafforgue (see [Laf02]), the property RD together with the CAT(0) property imply the Baum-Connes conjecture, so we can state the following.

**Corollary F.** Every extra large type Artin group satisfies the Baum-Connes conjecture.

**Acknowledgments:** The author would like to thank warmly Chris Cashen for discussions and an invitation to the University of Vienna, where part of this work was initiated. The author would also like to thank Anthony Genevois and Damian Osajda for many insightful comments.

# 1 The case of dihedral Artin groups

We start by describing a very simple nonpositively curved metric model for dihedral Artin groups, which we will use in the sequel as building blocks. If \( m \geq 2 \), let us denote the dihedral Artin group by \( I_2(m) = \langle a, b \mid w_m(a, b) = w_m(b, a) \rangle \).

**Lemma 1.1.** For every \( m \geq 4 \), there exists a compact, locally CAT(0), 3-dimensional piecewise Euclidean complex \( X_m \) and \( x_0 \in X_m \) with \( \pi_1(X_m, x_0) \simeq I_2(m) = \langle a, b \mid w_m(a, b) = w_m(b, a) \rangle \). There exist locally geodesic oriented loops \( X_m^a, X_m^b \) of length \( 1 \) through \( x_0 \) such that \( \pi_1(X_m^a, x_0) = \langle a \rangle \) and \( \pi_1(X_m^b, x_0) = \langle b \rangle \). Let \( a^+, a^- \in \text{lk}_{x_0}(X_m^a) \) denote the images in the link of \( x_0 \) of the positive and negative sides of the loop \( X_m^a \), and similarly \( b^+, b^- \in \text{lk}_{x_0}(X_m^b) \) for \( X_m^b \). We have furthermore:

- \( X_m^a \cap X_m^b = \{x_0\} \)
- \( \angle_{x_0}(a^+, b^+) = \angle_{x_0}(a^-, b^-) > \frac{4\pi}{3} \)
- \( \angle_{x_0}(a^+, b^-) = \angle_{x_0}(a^-, b^+) > \frac{3\pi}{2} \)

**Proof.** Fix \( \alpha \in (0, \tan(\frac{\pi}{10})) \).

Assume first that \( m = 2p \) is even, then according to Brady and McCammond (see [BM00]), there is an interesting presentation of \( I_2(m) \) given by \( I_2(m) = \langle a, b \mid w_m(a, b) = w_m(b, a) \rangle = \langle a, t \mid at^p = t^pa \rangle \), where \( t = ab \), so the central quotient \( G \) of \( I_2(m) \) is isomorphic to \( \langle a, t \mid at^p = t^pa \rangle / \langle t^p \rangle \simeq \mathbb{Z} / p\mathbb{Z} \ast \mathbb{Z} \). Consider the action of \( G \) on the \( 2p \)-regular tree \( T_m = \text{Cay}(\mathbb{F}_p, \{a_1^\pm, \ldots, a_p^\pm\}) \) by \( a \cdot x = a_1 \cdot x \) and \( t \) fixes the basepoint \( e \in T_m \) and acts on \( T_m \) as the automorphism of \( \mathbb{F}_p \) permuting cyclically \( \{a_1, \ldots, a_p\} \) (see Figure 2).

Consider the action of \( I_2(m) \) on \( \mathbb{R} \) by \( a \cdot x = b \cdot x = x + \alpha \). We can endow \( \mathbb{R} \) with the piecewise Euclidean simplicial structure where the vertex set is \( a\mathbb{Z} \).

Let \( Y_m = T_m \times \mathbb{R} \), endowed with the diagonal action of \( I_2(m) \), with basepoint \( y_0 = (e, 0) \). The stabilizers of the points of \( T_m \) are conjugated to the cyclic subgroup spanned by \( t \), and
this subgroup acts freely properly by translations on $\mathbb{R}$, we deduce that the action of $I_2(m)$ on $Y_m$ is free. More precisely, since $I_2(m)$ acts transitively on vertices of $T_m$ and since the stabilizer of each vertex has 2 orbits of vertices in $\mathbb{R}$, we deduce that $Y_m$ has exactly two orbits of vertices. Furthermore, since $Y_m$ is locally finite, we deduce that $X_m = I_2(m) \setminus Y_m$ is a compact locally CAT(0) space such that $\pi_1(X_m, x_0)$ is isomorphic to $I_2(m)$, where $x_0 = I_2(m) \cdot y_0$.

Let $Y_m^a$, $Y_m^b$ denote the axes of $a$ and $b$ through $y_0$, their images in $X_m$ define locally geodesic oriented loops $X_m^a$ and $X_m^b$, such that the angles at $x_0$ between $a^+$, $b^+$ and between $a^-$, $b^-$ are both equal to

$$\pi - 2 \arctan(\alpha) > \frac{4\pi}{5}.$$ 

On the other hand, the angles at $x_0$ between $a^+$, $b^-$ and between $a^-$, $b^+$ are both equal to $\pi$.

Up to rescaling $X_m$, since $X_m^a$ and $X_m^b$ have the same length, we can assume that they both have length 1. We can also assume that, up to refining the piecewise Euclidean structure of $X_m$, the axes $X_m^a$ and $X_m^b$ lie in the 1-skeleton.

Assume now that $m$ is odd, then according to Brady and McCammond (see [BM03]), there is an interesting presentation of $I_2(m)$ given by $I_2(m) = \langle a, b \mid w_m(a, b) = w_m(b, a) = (t, u | t^m = u^2) \rangle$, where $t = ab$ and $u = w_m(a, b)$, so the central quotient $G$ of $I_2(m)$ is isomorphic to $\langle t, u | t^m = u^2 \rangle / \langle t^m = u^2 \rangle \simeq \mathbb{Z}/m\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$. Consider the action of $G$ on the Bass-Serre $(m, 2)$-biregular tree $T$, and consider the regular $m$-gonal complex $T_m$ obtained from $T$ by replacing the star of each vertex with valency $m$ by a regular $m$-gon with side length 1, where $t$ acts on the base $m$-gon $P$ by a rotation of angle $\frac{4\pi}{m}$. Note that $a = t^{-p}u$ and $b = ut^{-p}$, and $t^p$ acts on the base $m$-gon by a rotation of angle $\frac{4\pi}{m} = \frac{2\pi}{m}$. This way, the axes of $a$ and $b$ acting on $T_m$ intersect the boundary of the $m$-gon $P$ in consecutive sides. Let $e \in T_m$ denote the intersection of the axes of $a$ and $b$, it is also the unique vertex fixed by $u = w_m(a, b)$ (see Figure 4).

Consider the action of $I_2(m)$ on $\mathbb{R}$ by $a \cdot x = b \cdot x = x + \alpha$. We can endow $\mathbb{R}$ with the piecewise Euclidean simplicial structure where the vertex set is $\alpha \mathbb{Z}$.

Let $Y_m = T_m \times \mathbb{R}$, endowed with the diagonal action of $I_2(m)$, with basepoint $y_0 = (e, 0)$. The stabilizers of the points of $T_m$ are conjugated to either the cyclic subgroup spanned by $u$ or by $t$, and these subgroups act freely properly by translations on $\mathbb{R}$, we deduce that the action of $I_2(m)$ on $Y_m$ is free, with compact quotient. More precisely, since $I_2(m)$ acts transitively on vertices of $T_m$ and since the stabilizer of each vertex has $m$ orbits of vertices in $\mathbb{R}$, we deduce that $Y_m$ has exactly $m$ orbits of vertices. Furthermore, since $Y_m$ is locally finite, we deduce that $X_m = I_2(m) \setminus Y_m$ is a compact locally CAT(0) space such that $\pi_1(X_m, x_0)$ is isomorphic to $I_2(m)$, where $x_0 = I_2(m) \cdot y_0$ (see Figure 4).

Let $Y_m^a$, $Y_m^b$ denote the axes of $a$ and $b$ through $y_0$, their image in $X_m$ define locally geodesic oriented loops $X_m^a$ and $X_m^b$, such that the angle at $x_0$ between $a^+$ and $b^+$ (and

![Figure 2 – A part of the tree $T_4$, with the axes of $a$ and $b$.](image-url)
similarly between \(a^-\) and \(b^-\) is equal to \(\pi - 2 \arctan(\alpha) > \frac{4\pi}{5}\).

And the angle between \(a^+\) and \(b^-\) (and similarly between \(a^-\) and \(b^+\)) is equal to

\[
\arccos \left( \cos(2 \arctan(\alpha)) \cos \left( \frac{(m-2)\pi}{m} \right) \right) > \frac{(m-2)\pi}{m} \geq \frac{3\pi}{5}.
\]

Up to rescaling \(X_m\), since \(X^a_m\) and \(X^b_m\) have the same length, we can assume that they both have length 1.

\(\square\)

2 The general case of extra large Artin groups

We now describe a metric model for extra large type Artin groups, obtained by gluing the complexes obtained by dihedral Artin groups. This is the first part of Theorem C.

Theorem 2.1. For every extra large type Artin group \(A\), there exists a compact locally CAT(0) 3-dimensional piecewise Euclidean complex \(X_A\) and \(x_0 \in X_A\) such that \(\pi_1(X_A, x_0) \simeq A\).

Proof. For each \(s \in S\), let \(X_s\) denote a circle with length 1 and basepoint \(x_0 \in X_s\), such that \(\pi_1(X_s, x_0)\) will be identified with \(\langle s \rangle\). Let \(E\) denote the set of all edges of \(\Gamma\). Consider
the following space

\[ X_A = \left( \bigcup_{I \in E} X_I \cup \bigcup_{s \in S} X_s \right) / \sim, \]

where the identifications are given, for all \( s \in S \) and \( I = \{s, t\} \in S_2 \), by \( X_s \sim X_{s,t} \).

According to the Van Kampen Theorem, the fundamental group \( \pi_1(X_A, x_0) \) is isomorphic to \( A \). Up to refining the cell structure, we can assume that \( X_A \) is a piecewise Euclidean cell complex \( X_A \). In order to prove that \( X_A \) is locally CAT(0), according to Gromov’s link condition, it is sufficient to prove that the link of every vertex is CAT(1). For every edge \( e \) of \( X_A \), the link of \( e \) in \( X_A \) is the disjoint union of links of \( e \) in all \( X_I \)’s that contain \( e \). Since each \( X_I \) is CAT(0), the link of \( e \) in \( X_A \) is CAT(1).

In other words, it is enough to prove that the link of every vertex of \( X_A \) is large, i.e. every closed locally geodesic loop has length at least \( 2\pi \). Fix a vertex \( x \in X_A \), and assume that \( \ell \) is a locally geodesic loop in the link of \( x \). We will prove that \( \ell \) has length at least \( 2\pi \).

Assume first that \( \ell \) is contained in a unique \( X_{ab} \). Since \( X_{ab} \) is CAT(0), the link of \( x \) in \( X_A \) is large, so \( \ell \) has length at least \( 2\pi \).

Assume now that \( \ell \) is contained in \( X_{ab} \cup X_{be} \). Since \( X_b \) is convex in both \( X_{ab} \) and \( X_{be} \), we know that \( X_{ab} \cup X_{be} \) is CAT(0), hence \( \ell \) has length at least \( 2\pi \).

Assume now that \( \ell \) is contained in \( X_{ab} \cup X_{be} \cup X_{ac} \), but not less than three. Then the length of \( \ell \) is at least

\[
\angle_{x_0}(a'^+, b^-) + \angle_{x_0}(b^-, c^+) + \angle_{x_0}(c^+, a'^+) = 2 \times \frac{3\pi}{5} + \frac{4\pi}{5} = 2\pi \text{ or }
\angle_{x_0}(a'^+, b^+) + \angle_{x_0}(b^+, c^+) + \angle_{x_0}(c^+, a'^+) = 3 \times \frac{4\pi}{5} > 2\pi.
\]

Assume now that \( \ell \) is contained in no fewer than four \( X_I \)’s. Then its length is at least \( 4 \times \frac{4\pi}{5} > 2\pi \).

In conclusion, every locally geodesic loop in the link of \( x \) has length at least \( 2\pi \). So the link of \( x \) is CAT(1), and \( X_A \) is locally CAT(0).

\[ \square \]

Note that this construction is not sharp, meaning that we could also build this way a locally CAT(0) model for some Artin groups which are not of extra large type. However, the precise combinatorial conditions would not be very elegant to write down. Furthermore, such a construction cannot be adapted to take into account the \((3,3,3)\) triangle Artin group for instance, which is known by Brady and McCammond (see [BM00]) to be CAT(0) using another complex.

## 3 A rank one geodesic

We will now prove that the locally CAT(0) complex we built for extra large type Artin groups has rank 1, meaning that there exists a periodic geodesic in the universal cover which does not bound any flat half-plane. Fix \( \alpha \in (0, \tan(\pi/10)) \). We start by looking at a specific loop for the complex for dihedral Artin groups, first in the odd case.

**Lemma 3.1.** For every odd \( m \geq 5 \), there exists a locally geodesic oriented simple loop \( \ell \) in \( X_m \) based at \( x_0 \) such that, if we denote \( \ell^+, \ell^- \in \text{lk}_{x_0}(X_m) \) the images in the link of \( x_0 \) of the positive and negative sides of the loop \( \ell \), we have:

- \( X_m^a \cap \ell = X_m^b \cap \ell = \{x_0\} \),
For every even $m$-gons $P, P'$ adjacent to the base vertex $e$. Consider the vertex $x \in P$ such that $e$ and $x$ are "almost opposite" in $P$, i.e. form an angle of $\frac{2\pi}{m}$ from the center of $P$, where $m = 2p + 1$. Consider the unique vertex $x' \in P'$ such that $(x, 0)$ and $(x', 0)$ are in the same $I_2(m)$-orbit in $Y_m$, then $x'$ and $e$ form an angle of $\frac{2\pi}{m}$ from the center of $P'$ (see Figure 5).

The piecewise Euclidean path from $x'$ to $x$ consisting of the two segments from $x'$ to $e$ and from $e$ to $x$ projects to a locally geodesic oriented simple loop $\ell$ in $I_2(m) \setminus \{0\} \subset X_m$. By construction, we have $X_m^a \cap \ell = X_m^b \cap \ell = \{x_0\}$. Furthermore, we have

- $\angle_{x_0}(a^+, \ell^+) = \angle_{x_0}(a^-, \ell^-) > \frac{2\pi}{m} \geq \frac{2\pi}{5}$ and
- $\angle_{x_0}(b^-, \ell^-) > \frac{(p-1)\pi}{m} \geq \frac{\pi}{5}.$

Furthermore, since the angle at $e$ inside $T_m$ between $a^-$ and $\ell^+$ is infinite, we deduce that, in the whole space $Y_m = T_m \times \mathbb{R}$ we have $\angle_{x_0}(a^-, \ell^+) = \pi - \arctan(\alpha) > \frac{9\pi}{10} > \frac{4\pi}{5}.$ Similarly

$$\angle_{x_0}(a^-, \ell^+) = \angle_{x_0}(a^+, \ell^-) = \angle_{x_0}(b^+, \ell^+) = \angle_{x_0}(b^-, \ell^-) > \frac{4\pi}{5}.$$ 

We now turn to the even dihedral Artin groups.

**Lemma 3.2.** For every even $m \geq 4$, there exists a locally geodesic oriented simple loop $\ell$ in $X_m$ based at $x_0$ such that, if we denote $\ell^+, \ell^- \in \text{lk}_{x_0}(X_m)$ the images in the link of $x_0$ of the positive and negative sides of the loop $\ell$, we have:

- $X_m^a \cap \ell = X_m^b \cap \ell = \{x_0\},$
- $\angle_{x_0}(a^+, \ell^+), \angle_{x_0}(a^-, \ell^-) > 0.$
- $\angle_{x_0}(b^-, \ell^-), \angle_{x_0}(b^+, \ell^+), \angle_{x_0}(a^-, \ell^+), \angle_{x_0}(a^+, \ell^-), \angle_{x_0}(b^+, \ell^+), \angle_{x_0}(b^-, \ell^-) > \frac{\pi}{5}.$

**Proof.** Consider the lift in $T_m \times \{0\}$ of the axis of $a$ in $T_m$, and consider the projection $\ell$ in $(T_m \times \{0\})/I_2(m) \subset X_m$. Then $\ell$ is a simple locally geodesic closed loop consisting of two vertices and two edges. By construction, we have $X_m^a \cap \ell = X_m^b \cap \ell = \{x_0\}$. We also have $\angle_{x_0}(a^+, \ell^+) = \angle_{x_0}(a^-, \ell^-) = \arctan(\alpha) > 0.$

Furthermore, since the angle at $e$ inside $T_m$ between $a^-$ and $\ell^+$ is infinite, we deduce that, in the whole space $Y_m = T_m \times \mathbb{R}$ we have $\angle_{x_0}(a^-, \ell^+) = \pi - \arctan(\alpha) > \frac{9\pi}{10} > \frac{\pi}{5}.$ Similarly

$$\angle_{x_0}(b^-, \ell^+) = \angle_{x_0}(b^+, \ell^-) = \angle_{x_0}(a^-, \ell^+) = \angle_{x_0}(a^+, \ell^-) = \angle_{x_0}(b^+, \ell^+) = \angle_{x_0}(b^-, \ell^-) > \frac{\pi}{5}.$$ 

\[\square\]
We can now prove that the complex $X_A$ has rank one, thus proving the second part of Theorem 3.3.

**Theorem 3.3.** Assume that $A(\Gamma)$ is an extra large Artin group with at least three generators. Then there exists a locally geodesic loop in $X_A$ whose lifts in $\tilde{X}_A$ have rank 1, i.e. do not bound flat half-planes.

**Proof.**

- If $\Gamma$ has no edge, then $A(\Gamma)$ is the free group on $S$ and $X_A$ is a wedge of $|S|$ circles. So every geodesic in the tree $\tilde{X}_A$ has rank 1.

- Assume now that $\Gamma$ has at least one edge labeled by some odd number $m \geq 5$, between $a$ and $b$. Fix $c \in S\setminus\{a, b\}$. Let $\ell_{ab} \subset X_{ab} \subset X_A$ denote the loop given by Lemma 3.1 and consider the oriented loop $X_c \subset X_A$. Then consider the concatenation $\ell = \ell_{ab} \cdot X_c$. We will prove that the angle at $x_0$ between the incoming loop $c^-$ and the outgoing loop $\ell_{ab}^+$ is bigger than $\pi$.

By construction, in the link of $x_0$, every path from $\ell_{ab}^+$ to $c^-$ must pass through one of $\{a^+, a^-, b^+, b^\}$. Let us compute the four quantities:

\[
\begin{align*}
\angle_{x_0}(\ell_{ab}^+, a^+) + \angle_{x_0}(a^+, c^-) &> \frac{2\pi}{5} + \frac{3\pi}{5} = \pi, \\
\angle_{x_0}(\ell_{ab}^+, a^-) + \angle_{x_0}(a^-, c^-) &> \frac{4\pi}{5} + \frac{4\pi}{5} > \pi, \\
\angle_{x_0}(\ell_{ab}^+, b^+) + \angle_{x_0}(b^+, c^-) &> \frac{4\pi}{5} + \frac{3\pi}{5} > \pi, \\
\angle_{x_0}(\ell_{ab}^+, b^-) + \angle_{x_0}(b^-, c^-) &> \frac{\pi}{5} + \frac{4\pi}{5} = \pi.
\end{align*}
\]

We deduce that the distance in the link of $x_0$ between $\ell_{ab}^+$ and $c^-$ is bigger than $\pi$. Similarly, the distance in the link of $x_0$ between $\ell_{ab}^-$ and $c^+$ is also bigger than $\pi$.

- Assume now that $\Gamma$ has at least one edge, and that all edges are labeled by even numbers. Consider some edge between $a$ and $b$, labeled by some even $m \geq 4$. Fix $c \in S\setminus\{a, b\}$. Let $\ell_{ab} \subset X_{ab} \subset X_A$ denote the loop given by Lemma 3.2 and consider the oriented loop $X_c \subset X_A$. Then consider the concatenation $\ell = \ell_{ab} \cdot X_c$. We will prove that the angle at $x_0$ between the incoming loop $c^-$ and the outgoing loop $\ell_{ab}^+$ is bigger than $\pi$.

Note that since all labels are even, we know that $\angle_{x_0}(a^+, c^-), \angle_{x_0}(b^+, c^-) \geq \pi$ and that $\angle_{x_0}(a^-, c^-), \angle_{x_0}(b^-, c^-) \geq \pi - 2 \arctan(\alpha) > \frac{4\pi}{5}$.

By construction, in the link of $x_0$, every path from $\ell_{ab}^+$ to $c^-$ must pass through one of $\{a^+, a^-, b^+, b^\}$. Let us compute the four quantities:

\[
\begin{align*}
\angle_{x_0}(\ell_{ab}^+, a^+) + \angle_{x_0}(a^+, c^-) &> 0 + \pi = \pi, \\
\angle_{x_0}(\ell_{ab}^+, a^-) + \angle_{x_0}(a^-, c^-) &> \frac{\pi}{5} + \frac{4\pi}{5} = \pi, \\
\angle_{x_0}(\ell_{ab}^+, b^+) + \angle_{x_0}(b^+, c^-) &> \frac{\pi}{5} + \frac{4\pi}{5} = \pi, \\
\angle_{x_0}(\ell_{ab}^+, b^-) + \angle_{x_0}(b^-, c^-) &> \frac{\pi}{5} + \frac{4\pi}{5} = \pi.
\end{align*}
\]

We deduce that the distance in the link of $x_0$ between $\ell_{ab}^+$ and $c^-$ is bigger than $\pi$. Similarly, the distance in the link of $x_0$ between $\ell_{ab}^-$ and $c^+$ is also bigger than $\pi$.

In conclusion, in each of the last two cases, $\ell$ is a locally geodesic loop in $X_A$ such that the angle at (each of the two passings at) $x_0$ is bigger than $\pi$. In particular, any lift of $\ell$ in $\tilde{X}_A$ does not bound a flat half-plane, so it has rank 1. 

\[\square\]
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