Analytical representations based on $su(3)$ coherent states and Robertson intelligent states

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Abstract

Robertson intelligent states which minimize the Schrödinger-Robertson uncertainty relation are constructed as eigenstates of a linear combination of Weyl generators of the $su(3)$ algebra. The construction is based on the analytic representations of $su(3)$ coherent states. New classes of coherent and squeezed states are explicitly derived.
1 Introduction

The coherent states introduced as displacement of the ground state of harmonic oscillator by Schrödinger [1], have found revived interest when it was realized that they are eigenfunctions of the annihilation operator and minimize the Heisenberg uncertainty relation. The generalization of the usual coherent states from the Weyl-Heisenberg to the other Lie algebras and from harmonic oscillator to other potentials, followed these three approaches, namely, (i) eigenstates of lowering group generator for Lie algebras or annihilation operator of exactly solvable system, (ii) as orbits of the extremal weight state or (iii) as states minimizing the uncertainty relation. These different approaches lead to distinct sets of coherent states and coincide only in the special case of the harmonic oscillator (see [2-4] for review). Concerning the optimization of the uncertainty principle, it was observed that a relation more accurate than Heisenberg one may be used to construct generalized coherent states and squeezed states. Indeed this relation known as Schrödinger-Robertson uncertainty inequality [5] can be minimized and gives rise to new sets of coherent and squeezed states ( see the pioneering works [6-8]). The states resulting from this minimization have different names in the literature such as correlated states [6-8] or Robertson intelligent states [9].

More recently, there has been much interest in such states for Lie algebras [9-13] as well as for quantum systems evolving in various potentials [14-17]. Robertson intelligent states for the quadrature components of Weyl generators of the algebras $su(1,1)$ and $su(2)$ were constructed [9-13]. They were also defined for exactly solvable quantum systems as the eigenstates of complex combination of creation and annihilation operators [14-17].

The purpose of this paper is to further extend the classes of Robertson intelligent states for higher symmetries. In this sense, the main idea of this note is to construct the intelligent states for the quadrature components of Weyl operators of the algebra $su(3)$. For this end, it may be useful to start by giving the explicit computation of the associated coherent states and their analytic representations. Hence, one can introduce the differential realizations for the $su(3)$ generators. As we will see, the analytic realization enables us to convert the eigenvalue equations arising from the minimization of Schrödinger-Robertson inequality into quasi linear differential equations which provide the Robertson intelligent states.

The paper is organized as follows. In section 2 we review the derivation of $su(3)$ coherent states. We compute explicitly the action of unitary displacement operator on the highest weight vector of the finite dimension representation space of the algebra $su(3)$. We give the analytic representations of the $su(3)$ coherent states. We construct also the differential operators corresponding to the actions of the generators of $su(3)$ on the Fock-Bargmann space. In section 3, we show how the analytic representation based on the coherent states provides us with the tool to solve the eigenvalue equations resulting from the minimization of Schrödinger-Robertson relation and to obtain intelligent states for the quadrature components of $su(3)$ Weyl generators. We conclude in section 4 after pointed out a number of interesting open problems.
2 Analytic representations of $su(3)$-coherent states

2.1 Reconstructing $su(3)$-coherent states

We shall begin by reexamining the construction of the coherent states associated with a quantum system of dynamical symmetry $su(3)$. Although this subject has been considered previously in the works [18-19], we thought that it is always interesting to give another method based on the explicit computation of the action of displacement operator on the highest weight state (fundamental vector) of the finite dimensional representation space of the algebra $su(3)$. The explicit forms of such states are needed to perform their analytic representations and to give the realization of the generators of the algebra under consideration.

The algebra $su(3)$ is defined by the generators $e_i, f_i, h_i$ ($i = 1, 2$) and the relations

\[ [e_i, f_i] = \delta_{ij} h_j \]  \hspace{1cm} (1)
\[ [h_i, e_j] = a_{ij} e_j \quad [h_i, f_j] = -a_{ij} f_j \]  \hspace{1cm} (2)
\[ [e_i, e_j] = 0 \quad \text{for} \quad |i - j| > 1 \]  \hspace{1cm} (3)
\[ e_i^2 e_{i+1} - 2 e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0 \]  \hspace{1cm} (4)
\[ f_i^2 f_{i+1} - 2 f_i f_{i+1} f_i + f_{i+1} f_i^2 = 0 \]  \hspace{1cm} (5)

where $(a_{ij})_{i,j=1,2}$ is the Cartan matrix of $su(3)$, i.e. $a_{ii} = 2, a_{i,i\pm 1} = -1$ and $a_{ij} = 0$ for $|i - j| > 1$. Many aspects of Lie algebras are best considered after choosing a special type of the representation basis. Since one would write down the $su(3)$ coherent states, the most convenient choice, in this case, is the bosonic realization. Indeed, an adapted basis is given in term of three bosonic pairs of creation and annihilation operators; they satisfy the commutation relations

\[ [a_k^-, a_l^+] = \delta_{kl} \]  \hspace{1cm} (6)

where $k, l = 1, 2, 3$. The operators number are $N_k = a_k^+ a_k^-$. The Fock space is generated by the eigenstates $|n_1, n_2, n_3\rangle$ of number operators, namely,

\[ |n_1, n_2, n_3\rangle = \frac{(a_1^+)^{n_1} (a_2^+)^{n_2} (a_3^+)^{n_3}}{\sqrt{n_1!} \sqrt{n_2!} \sqrt{n_3!}} |0, 0, 0\rangle \]  \hspace{1cm} (7)

In this bosonic representation, we define the generators of $su(3)$ as

\[ e_i = a_i^+ a_{i+1}^- \quad f_i = a_i^- a_{i+1}^+ \quad h_i = N_i - N_{i+1} \]  \hspace{1cm} (8)

The generators $e_i$, $f_i$ are called step, ladder or Weyl operators. The Cartan sub-algebra is generated by the elements $h_i$. They act on the representation space of dimension $\frac{1}{2} (j_1 + 1)(j_2 + 1)$ that is obtained from the Fock space of three harmonic oscillators by restricting the total number of quanta to $j_1 = n_1 + n_2 + n_3$. In the present representation the state of highest weight is $|j_1, 0, 0\rangle$. The generators of $su(3)$ having a nontrivial action (non-vanishing and non-diagonal) on the fundamental
vector $|j_1,0,0\rangle$ are $f_1 = a_1^-a_3^+$ and $f_3 = [f_2,f_1] = a_1^-a_3^+$. At this stage, one can define the coherent state as

$$|z_1,z_2\rangle = D(z_1,z_2)|j_1,0,0\rangle = \exp(z_1 f_1 + z_2 f_3 - \bar{z}_1 e_1 - \bar{z}_2 e_3)|j_1,0,0\rangle \quad (9)$$

where $e_3 = [e_1,e_2] = a_1^+a_3^-$. Expanding the displacement operator $D(z_1,z_2)$ and using the action of creation and annihilation operators on the restricted Fock space $\mathcal{F} = \{|n_1,n_2,n_3\}; n_1 + n_2 + n_3 = j_1\}$, one get

$$|z_1,z_2\rangle = \sum_{j_1} \sum_{j_2} z_1^{j_1} z_2^{j_2} I_{j_1}^{j_2}(|z_1\rangle I_{j_2}^{j_2}(|z_2\rangle)|j_1 - j_2, j_2 - j_3, \bar{j}_3\rangle. \quad (10)$$

where

$$I_{j_{s+1}}^{j_s}(|z_s\rangle) = \sum_{k=0}^{\infty} (-)^k(2j_k)^k P(j_{s+1} + 1,k), \quad (11)$$

for $s = 1, 2$. The quantities $P$ occurring in (11) are given by

$$P(j_{s+1} + 1,k) = P(j_{s+1} + 1,0) \sum_{l_1=1}^{j_{s+1}+1} E_s(l_1) \sum_{l_2=1}^{l_1+1} E_s(l_2) \cdots \sum_{l_{k-1}+1}^{l_{k-1}+1} E_s(l_k) \quad (12)$$

with $P(j_{s+1} + 1,0) = \frac{\partial^{j_{s+1}+1}}{\partial j_{s+1}+1!}$ and $E_s(l) = (j_s - l + 1)l$. They satisfy the following recursion relation

$$P(j_{s+1} + 1,k) = \sqrt{E_s(j_{s+1})}P(j_{s+1},k) + \sqrt{E_s(j_{s+1} + 1)}P(j_{s+1} + 2,k - 1). \quad (13)$$

Setting

$$J_{j_{s+1}}^{j_s}(|z_s\rangle) = |z_s\rangle P(j_{s+1} + 1,0)I_{j_{s+1}}^{j_s}(|z_s\rangle), \quad (14)$$

we get the first order differential equation

$$\frac{dJ_{j_{s+1}}^{j_s}(|z_s\rangle)}{d|z_s\rangle} = J_{j_{s+1}-1}^{j_s}(|z_s\rangle) - (E_s(j_{s+1} + 1))^2 J_{j_{s+1}+1}^{j_s}(|z_s\rangle). \quad (15)$$

The solution of this equation takes the simple form

$$J_{j_{s+1}}^{j_s}(|z_s\rangle) = \frac{1}{j_{s+1}!}(\cos(|z_s\rangle)J_{j_{s+1}+1}^{j_s}-1)(tg(|z_s\rangle))^{j_{s+1}}, \quad (16)$$

and the $su(3)$ coherent states rewrite as

$$|\zeta_1,\zeta_2\rangle = (1 + |\zeta_1|^2 + |\zeta_1|^2|\zeta_2|^2)^{-1/4} \times \sum_{j_2=0}^{j_1} \sqrt{\frac{j_1!}{j_2!(j_1-j_2)!}} c_1^{j_2} \sum_{j_3=0}^{j_2} \sqrt{\frac{j_2!}{j_3!(j_2-j_3)!}} c_2^{j_3} |j_1 - j_2, j_2 - j_3, j_3\rangle \quad (17)$$

where $\zeta_s = \frac{i}{|z_s|}tg(|z_s|)\cos(|z_{s+1}|)^2-s$ for $s = 1, 2$. They have the property of strong continuity in the label space and overcompleteness in the sense that there exists a
positive measure such that they solve the resolution to identity. The appropriate
form of this resolution is

\[ \int d\mu(\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2) |\zeta_1, \zeta_2| \langle \zeta_1, \zeta_2 | \zeta_1, \zeta_2 \rangle = \sum_{j_1} \sum_{j_2} \hat{h}(|\zeta_1|^2) |\zeta_1| |d| \langle \zeta_1, \zeta_2 | \zeta_1, \zeta_2 \rangle. \]  

(18)

Assuming the isotropy of the measure \( d\mu(\zeta, \bar{\zeta}) \), we set

\[ d\mu(\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2) = \pi^2 (1 + |\zeta_1|^2 + |\zeta_2|^2)^2 h(|\zeta_1|^2) |d| \langle \zeta_1, \zeta_2 | \zeta_1, \zeta_2 \rangle. \]  

(19)

with \( \zeta_s = |\zeta_s| e^{i\theta_s} \). Substituting (19) in Eq.(18), we obtain the following sum

\[ \int_0^\infty x^{j_s+1} h(x_s) dx_s = \frac{j_s+1!(j_s-j_{s+1})!}{j_s!}. \]  

(20)

which should be satisfied by the function \( h(x_s = |\zeta_s|^2) \). One get

\[ h(x_s) = \frac{j_s+1}{(1 + x_s^2)^{j_s+2}}. \]  

(21)

This result can be obtained by using the definition of Meijer’s G-function and the Mellin inversion theorem [20]. The resolution to identity is necessary to build up the Fock-Bargamann space based on the set of \( su(3) \) coherent states.

2.2 Differential realization of the \( su(3) \) generators

It is well established that the use of the Fock-Bargmann representation is a powerful method for obtaining closed analytic expressions for various properties of coherent states. Calculation for some quantum exception values and solutions for some eigenvalue equations are simplified by exploiting the theory of analytical entire functions. Here, we give the Fock-Bargamnn representation of a \( su(3) \) quantum mechanical system. We define the Fock-bargamnn space as a space of functions which are holomorphic. The scalar product is written with an integral of the form

\[ \langle f | g \rangle = \int \bar{f}(\zeta_1, \zeta_2) g(\zeta_1, \zeta_2) d\mu(\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2) \]  

(22)

where the measure is defined above (see Eq.(19)). Due to overcomletion of the coherent states, it is induced by the scalar product in \( \mathcal{F} \). Let

\[ |\psi\rangle = \sum_{n_1,n_2,n_3} a_{n_1,n_2,n_3} |n_1, n_2, n_3\rangle \]  

(23)

an arbitrary quantum state of \( \mathcal{F} \), it can be represented as a function of the complex variables \( \zeta_1, \zeta_2 \) as

\[ \psi(\zeta_1, \zeta_2) = (1 + |\zeta_1|^2 + |\zeta_2|^2)^2 \hat{h}(\zeta_1, \bar{\zeta}_2) \langle \zeta_1, \zeta_2 | \psi \rangle \]  

(24)
In particular, the analytic functions associated to elements of the basis of $\mathcal{F}$ are defined as
\[ \psi_{j_1,j_2,j_3}(\zeta_1, \zeta_2) = (1 + |\zeta_1|^2 + |\zeta_2|^2)^{\frac{j_3}{2}} \langle \bar{\zeta}_1, \bar{\zeta}_2 | j_1 - j_2, j_2 - j_3, j_3 \rangle. \] (25)

We now investigate the form of the action of the operators $e_i$, $f_i$ and $h_i$ on Fock-Bargmann space. Indeed, any operator $O$ of the algebra $su(3)$ is represented in the space of entire analytical functions by some differential operator $\hat{O}$, defined by
\[ \langle \bar{\zeta}_1, \bar{\zeta}_2 | \hat{O} | \psi \rangle = \hat{O} \psi(\zeta_1, \zeta_2) \] (26)
for any state $|\psi\rangle$ of $\mathcal{F}$.

According this definition, we obtain
\[ e_1 = \frac{\partial}{\partial \zeta_1}, \quad e_3 = \frac{\partial}{\partial \zeta_2} \] (27)
\[ f_1 = j_1 \zeta_1 - \zeta_1^2 \frac{\partial}{\partial \zeta_1} - \zeta_1 \zeta_2 \frac{\partial}{\partial \zeta_2} \] (28)
\[ f_3 = j_1 \zeta_2 - \zeta_2^2 \frac{\partial}{\partial \zeta_2} - \zeta_1 \zeta_2 \frac{\partial}{\partial \zeta_1} \] (29)
\[ e_2 = \zeta_1 \frac{\partial}{\partial \zeta_2}, \quad f_2 = \zeta_2 \frac{\partial}{\partial \zeta_1} \] (30)
\[ h_1 = j_1 - 2 \zeta_1 \frac{\partial}{\partial \zeta_1} - \zeta_2 \frac{\partial}{\partial \zeta_2} \] (31)
\[ h_2 = \zeta_1 \frac{\partial}{\partial \zeta_1} - \zeta_2 \frac{\partial}{\partial \zeta_2} \] (32)

To obtain the above differential realization:
(i) we remark that the coherent states (17) can be also written as
\[ |\zeta_1, \zeta_2\rangle = (1 + |\zeta_1|^2 + |\zeta_2|^2)^{-\frac{j_3}{2}} D(\zeta_1, \zeta_2)|j_1, 0, 0\rangle \] (33)
where $D(\zeta_1, \zeta_2) = \exp(\zeta_1 f_1 + \zeta_2 f_3)$,
(ii) we observe that
\[ \frac{\partial}{\partial \zeta_1} D(\zeta_1, \zeta_2) = f_1 D(\zeta_1, \zeta_2), \quad \frac{\partial}{\partial \zeta_2} D(\zeta_1, \zeta_2) = f_3 D(\zeta_1, \zeta_2), \] (34)
(iii) we use the Hausdorff formula
\[ e^{-B} A e^B = \sum_{n \geq 0} \frac{1}{n!} (-adB)^n A \] (35)
where $(adB)A = [B, A]$,
(iv) we use also the actions of the elements of $su(3)$ on the basis of Fock space $\mathcal{F}$, in particular the fiducial vector $|j_1, 0, 0\rangle$, and the structure relations (1-5) of the
algebra $su(3)$. From the previous considerations, it follows that the $su(3)$ generators act as first-order holomorphic differential operators on the space of the analytic functions generated by the elements (25). One can verify that the commutation relations (1-5) are preserved. This result combined with eigenvalue equations ensuring the minimization of Schrödinger-Robertson inequality provides the intelligent states as that will be explained in the next section.

3 $su(3)$ Robertson states

As we have already mentioned, in this section, we will study the fluctuations of the quadrature components of Weyl generators which represent creation and annihilation of states for a quantum mechanical system of $su(3)$ symmetry. In this order, to construct the intelligent states of any pair of ladder operators $e_i, f_i$ ($i = 1, 2, 3$), it is natural to introduce the quantum observables

$$\sqrt{2} p_i = e_i + f_i \text{ and } i \sqrt{2} q_i = e_i - f_i$$

where $i^2 = -1$. These observables obey

$$[p_i, q_i] = i h_i. \quad (36)$$

We known that $p_i$ and $q_i$ satisfy, in a given state, the Robertson-Shrödinger uncertainty relation

$$(\Delta p_i)^2 (\Delta q_i)^2 \geq \frac{1}{4} (\langle h_i \rangle^2 + \langle c_i \rangle^2), \quad (37)$$

where $\Delta p_i$ and $\Delta q_i$ are the dispersions and the hermitian operator $c_i = \{p_i - \langle p_i \rangle, q_i - \langle q_i \rangle\}$ gives the covariance (correlation) of the observables $p_i$ and $q_i$. The symbol $\{, \}$ stands for the standard definition of the anticommutator. A state $|\Phi \rangle$ providing the equality in (37) is the so-called Robertson intelligent state. It was proven that such state satisfy the following eigenvalue equation

$$((1 + \alpha) e_i + (1 - \alpha) f_i) |\Phi \rangle = \lambda |\Phi \rangle \quad (38)$$

where $\alpha \neq 0$ and $\lambda = (1 + \alpha) \langle e_i \rangle + (1 - \alpha) \langle f_i \rangle$ are complex parameters. Furthermore, the variances and covariance, in the intelligent state $|\Phi \rangle$, are related by

$$(\Delta p_i)^2 = |\alpha| \Delta_i \quad (\Delta q_i)^2 = \frac{1}{|\alpha|} \Delta_i \quad (39)$$

where $\Delta_i = \frac{1}{2} \sqrt{\langle h_i \rangle^2 + \langle c_i \rangle^2}$. Remark that they can be also expressed as

$$(\Delta p_i)^2 = \frac{|\alpha|^2}{u} \langle h_i \rangle \quad (\Delta q_i)^2 = \frac{1}{u} \langle h_i \rangle \quad \langle c_i \rangle = \frac{v}{u} \langle h_i \rangle \quad (40)$$

where the real parameters $u$ and $v$ are such that $u^2 + v^2 = 4 |\alpha|^2$ ( As example, one can take $u = 2Re \alpha$ and $v = 2Im \alpha$ ). It is clear that the dispersions and the correlation can be obtained from the mean value of the observable $h_i$. The state $|\Phi \rangle$ satisfying (38) with $|\alpha| = 1$ are coherent because they satisfy $(\Delta p_i)^2 = (\Delta q_i)^2 = \Delta_i$. 

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The fluctuations are equal and minimized in the sense of Schrödinger-Robertson uncertainty relation. The state satisfying (38) with $|\alpha| \neq 1$ are squeezed because if $|\alpha| > 1$, we have $(\Delta p_i)^2 < \Delta_i < (\Delta q_i)^2$ and if $|\alpha| < 1$, we have $(\Delta q_i)^2 < \Delta_i < (\Delta p_i)^2$.

To solve the eigenvalues equation (38), we will use the analytic representations of coherent states as well as the differential realizations of the generators $e_i$ and $f_i$ given by Eqs.(27-30). So, let us start by deriving the eigenfunctions of Eq.(38) for the first pair $e_1$, $f_1$. By introducing the analytic function

$$ \Phi_1 \equiv \Phi_1(\zeta_1, \zeta_2, \alpha, \lambda, j_1) = (1 + |\zeta_1|^2 + |\zeta_2|^2)^{\frac{j_1}{2}} \langle \zeta_1, \zeta_2 | \Phi_1 \rangle, \quad (41) $$

it can be easily checked that the eigenvalue equation (38) can be converted in the following first order differential equation

$$ (j_1 \eta_1 - \lambda') \Phi_1 + (1 - \eta_1^2) \frac{\partial \Phi_1}{\partial \eta_1} - \eta_1 \eta_2 \frac{\partial \Phi_1}{\partial \eta_2} = 0, \quad (42) $$

where $\eta_1 = \sqrt{1 - \alpha} \zeta_1$, $\eta_2 = \zeta_2$ and $\lambda' = \frac{\lambda}{\sqrt{1 - \alpha^2}}$ for $\alpha \neq \pm 1$. The function $\Phi_1(\zeta_1, \zeta_2, \alpha, \lambda, j_1)$ can be expanded as

$$ \Phi_1 = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} a_{j_1,j_2,j_3} \eta_1^{j_2} \eta_2^{j_3}. \quad (43) $$

Substitution of (43) in (42) yields the recursion formula

$$ (j_1 - j_2 - j_3 + 1) a_{j_1,j_2-1,j_3} - \lambda' a_{j_1,j_2,j_3} + (j_2 + 1) a_{j_1,j_2+1,j_3} = 0 \quad (44) $$

which can be solved by the Laplace method. Indeed, we set

$$ a_{j_1,j_2,j_3} = \int_{-1}^{+1} x^{j_2} f(x) dx \quad (45) $$

that we introduce in (44) to obtain, after partial integration, the simple first order differential equation satisfied by the function $f(x)$

$$ (x - x^3) \frac{df}{dx} + (j_1 - j_3 + 1 - \lambda' x - x^2) = 0. \quad (46) $$

The last equation is easily solvable. Replacing in (45), one get

$$ a_{j_1,j_2,j_3} = \int_{-1}^{+1} x^{j_2-j_1+j_3-1} (1 - x)^{\lambda' + j_3 - j_1 - 1} (1 + x)^{\lambda' + j_3 - j_1 - 1} dx, \quad (47) $$

or

$$ a_{j_1,j_2,j_3} = (-)^{j_2} \frac{\Gamma(\lambda' + j_3 - j_1) \Gamma(-\lambda' + j_3 + 1)}{\Gamma(j_1 - j_3 + 2)} \times _2 F_1(j_1 - j_3 - j_2 + 1, \frac{\lambda' + j_1 - j_3}{2} + 1, j_1 - j_3 + 2, 2) \quad (48) $$
using the integral representation for the hypergeometric function \(_2F_1\) [20]. Comparing
the expansion (43) with the general formula (41), we have the decomposition of
Robertson intelligent states over the basis of Fock space \(\mathcal{F}\)

\[
|\Phi_1\rangle = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} a_{j_1,j_2,j_3} \left(\frac{1-\alpha}{1+\alpha}\right)^{j_2} \sqrt{\frac{(j_1-j_2)!j_3!(j_2-j_3)!}{j_1!}} |j_1-j_2,j_2-j_3,j_3\rangle
\]

(49)

where the coefficients \(a_{j_1,j_2,j_3}\) are given by Eq.(48).

Now we consider the construction of intelligent states for the second pair \(e_2,f_2\).
The eigenvalues equation (38) gives, in this case, the following quasi linear differential

\[
\xi_1 \frac{\partial \Phi_2}{\partial \xi_2} + \xi_2 \frac{\partial \Phi_2}{\partial \xi_1} - \lambda' \Phi_2 = 0
\]

(50)

equation

where \(\xi_1 = \sqrt{\frac{1+\alpha}{1-\alpha}} \xi, \xi_2 = \xi_2\) and \(\lambda' = \frac{\lambda}{\sqrt{1-\alpha}}\).

Here also, we expand the eigenfunction \(\Phi_2 \equiv \Phi_2(\xi_1, \xi_2, \alpha, \lambda, j_1)\) as

\[
\Phi_2 = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} b_{j_1,j_2,j_3} \xi_1^{j_2} \xi_2^{j_3}
\]

(51)

that we insert in the equation (50) to obtain the recursion relation linking the
coefficients \(b's\)

\[
(j_3+1)b_{j_1,j_2-1,j_3+1} - \lambda' b_{j_1,j_2,j_3} + (j_2+1)b_{j_1,j_2+1,j_3-1} = 0.
\]

(52)

Setting \(b_{j_1,j_2,j_3} \equiv b_{j_1,j_2-j,j}\) where \(2j = j_2 + j_3\), the previous relation can be transformed to

\[
(j_2+1)b_{j_1,j_2-j+1,j} - \lambda' b_{j_1,j_2-j,j} + (j_3+1)b_{j_1,j_2-j-1,j} = 0,
\]

(53)

sovable in a similar manner that one given the solution of recursion formula (44),
and one has

\[
b_{j_1,j_2,j_3} = (-)^{j_2} \frac{\Gamma(\lambda' j_2 + j_3 + 1) \Gamma(\lambda' j_2 + j_3 + 1)}{\Gamma(j_2 + j_3 + 2)} \times 2F_1(j_3 + 1, \frac{\lambda' j_2 + j_3}{2}; 1, j_2 + j_3 + 2, 2)
\]

(54)

Finally, one obtain

\[
|\Phi_2\rangle = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} b_{j_1,j_2,j_3} \left(\frac{1+\alpha}{1-\alpha}\right)^{j_2} \sqrt{\frac{(j_1-j_2)!j_3!(j_2-j_3)!}{j_1!}} |j_1-j_2,j_2-j_3,j_3\rangle
\]

(55)

It remains to determine the \(e_3,f_3\) intelligent states. In this case, the Robertson
states should satisfy the following equation

\[
(j_1 \partial_2 - \lambda') \Phi_3 + (1 - \partial_2^2) \frac{\partial \Phi_3}{\partial \partial_2} - \partial_1 \partial_2 \frac{\partial \Phi_3}{\partial \partial_1} = 0
\]

(56)
where \( \vartheta_1 = \zeta_1 \), \( \vartheta_2 = \sqrt{\frac{1-\alpha}{1+\alpha}} \zeta_2 \) and \( \lambda' \) is defined above. In a similar way that one presented above, one obtain the intelligent states

\[
\Phi_3 = \sum_{j_1} \sum_{j_2=0} \sum_{j_3=0} c_{j_1,j_2,j_3} \vartheta_1^{j_1} \vartheta_2^{j_2} \vartheta_3^{j_3}
\]

where \( \Phi_3 \equiv \Phi_3(\zeta_1, \zeta_2, \alpha, \lambda, j_1) \) and the constants \( c \)'s are given by

\[
c_{j_1,j_2,j_3} = (-)^{j_3} \frac{\Gamma(\frac{\lambda' + j_1 - j_2}{2} + 1) \Gamma(\frac{-\lambda' + j_1 - j_2}{2} + 1)}{\Gamma(j_1 - j_2 + 2)} \times {}_2F_1(j_1 - j_2 - j_3 + 1, \frac{\lambda' + j_1 - j_2}{2} + 1, j_1 - j_2 + 2, 2).
\]

Analogously to the above cases, the intelligent states \( \Phi_3 \) can be converted as follows

\[
|\Phi_3\rangle = \sum_{j_2=0} \sum_{j_3=0} c_{j_1,j_2,j_3} \left( \frac{1+\alpha}{1-\alpha} \right)^{\frac{j_3}{2}} \sqrt{(j_1 - j_2)!(j_2 - j_3)!(j_2 - j_3)!} |j_1 - j_2, j_2 - j_3, j_3\rangle.
\]

To close this section, let us note that it is clear that the Fock-Bargmann representation of the coherent states provide a simplification and a "minimization" in the problem of finding intelligent states of \( su(3) \) Weyl generators. It is evident that the procedure described here can be relevant in the derivation of intelligent states for other quadrature components of type, for instance, \( e_i, e_j \) and \( f_i, f_j \) \((i \neq j)\).

### 4 Discussion and outlook

In conclusion, we have developed a method for finding the Robertson intelligent states for linear combination of the Weyl operators \( e_i \) and \( f_i \) for \( i = 1, 2, 3 \), corresponding to the Lie algebra \( su(3) \). The use of the analytic representation enables us to write the eigenvalue equations, satisfied by states minimizing the Schrödinger-Robertson uncertainty relation, as quasi linear first order differential equation. Interestingly, new types of coherent states for \( su(3) \) emerge for \( |\alpha| = 1 \). Also when \( |\alpha| \neq 1 \), the solutions give squeezed states. As it is noted in the end of the previous section, the approach used through this work can be applied to derive Robertson intelligent states associated to the other quadratures of the \( su(3) \) generators. In an unified scheme, they can be obtained by considering the eigenvalue problem for an operator which is a complex linear combination of all elements of \( su(3) \)

\[
\sum_{i=1,2} (\alpha_i^+ e_i + \alpha_i^- f_i + \alpha_i^0 h_i) |\Phi\rangle = \lambda |\Phi\rangle
\]

The solutions of such general problem give the so-called in the literature algebra eigenstates or algebraic coherent states ([13] and references therein). Taking specific constraints on the complex parameters occurring in this general eigenvalue equation, one can get various kind of coherent and squeezed states, in particular ones not discussed in this paper. This constitutes the first possible prolongation of our results. Also, as continuation, it would be interesting to apply the approach presented here to other Lie algebras like \( su(n) \) or \( su(p,q) \).
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