The genetic code, algebra of projection operators and problems of inherited biological ensembles

Sergey V. Petoukhov

Head of Laboratory of Biomechanical System, Mechanical Engineering Research Institute of the Russian Academy of Sciences, Moscow

spetoukhov@gmail.com, petoukhov@imash.ru, http://petoukhov.com/

Comment: Some materials of this article were presented in the international “Symmetry Festival-2013” (Delft, Netherlands, August 2-7, 2013, http://symmetry.hu/festival2013.html), the international conference “Theoretical approaches to bioinformation systems - TABIS 2013” (Belgrad, Serbia, September 17-22, 2013, http://www.tabis2013.ipb.ac.rs/)

Summary. This article is devoted to applications of projection operators to simulate phenomenological properties of the molecular-genetic code system. Oblique projection operators are under consideration, which are connected with matrix representations of the genetic coding system in forms of the Rademacher and Hadamard matrices. Evidences are shown that sums of such projectors give abilities for adequate simulations of ensembles of inherited biological phenomena including ensembles of biological cycles, morphogenetic ensembles of phyllotaxis patterns, mirror-symmetric patterns, etc. For such modeling, the author proposes multidimensional vector spaces, whose subspaces are under a selective control (or coding) by means of a set of matrix operators on base of genetic projectors. Development of genetic biomechanics is discussed. The author proposes and describes special systems of multidimensional numbers under names “tensocomplex numbers”, “tensorhyperbolic numbers”, etc. Described results can be used for developing algebraic biology, biotechnical applications and some other fields of science and technology.

Content
1. About the partnership of the genetic code and mathematics
2. Genetic Rademacher matrices as sums of projectors
3. Genetic Hadamard matrices as sums of projectors
4. Inherited biocycles and a selective control of cyclic changes of vectors in a multidimensional space. Problems of genetic biomechanics.
5. About a direction of rotation of vectors under influence of the cyclic groups of the operators
6. Hamilton’s quaternions, Cockle’s split-quaternions, their extensions and projector operators
7. Genetic matrices as sums of tensor products of oblique (2*2)-projectors. Extensions of the genetic matrices into (2^n*2^h)-matrices
8. An application of oblique projectors to simulate ensembles of phyllotaxis patterns in living bodies
9. Hyperbolic numbers, genetic projectors and the Weber-Fechner law of psychophysics
10. Reflection operators and genetic projectors.
11. The symbolic matrices of genetic duplets and triplets
12. Genetic projectors and evolutionary changes of dialects of the genetic code
13. About «tensocomplex» numbers
14. About «tensorhyperbolic» numbers
Some concluding remarks
Appendix 1. Complex numbers, cyclic groups and sums of genetic projectors
Appendix 2. Hyperbolic numbers and sums of genetic projectors
Appendix 3. Another tensor family of genetic Hadamard matrices
Appendix 4. About some applications in robotics

1. ABOUT THE PARTNERSHIP OF THE GENETIC CODE AND MATHEMATICS

Science has led to a new understanding of life itself: “Life is a partnership between genes and mathematics” [Stewart, 1999]. But what kind of mathematics can be a partner for the genetic coding system? This article shows some evidences that algebra of projectors can be one of main parts of such mathematics. Till now the notion of projection operators (or briefly, projectors) was one of important in many fields of non-biological science: physics including quantum mechanics; mathematics; computer science and informatics including theory of digital codes; chemistry; mathematical logic, etc. On basis of materials of this article, the author thinks that projectors can become one of the main notions and effective mathematical tools in mathematical biology. Moreover they will help not only to a development of algebraic biology and a new understanding of living matter but also to a mutual enrichment of different branches of science.

Projectors are expressed by means of square matrices (http://mathworld.wolfram.com/ProjectionMatrix.html, https://en.wikipedia.org/wiki/Projection_(linear_algebra)). A necessary and sufficient condition that a matrix $P$ is a projection operator is the fulfillment of the following condition: $P^2 = P$. A set of projectors is separated into two sub-sets:

- orthogonal projectors, which are expressed by symmetric matrices and theory of which is well developed and has a lot of applications;
- oblique projectors, which are expressed by non-symmetric matrices; their theory and its applications are developed much weaker as the author can judge. Namely oblique projectors will be the main objects of attention in this article.

This article is a continuation and an essential development of the author's article about relations between the genetic system and projection operators [Petoukhov, 2010].

In accordance with Mendel's laws of independent inheritance of traits, information from the micro-world of genetic molecules dictates constructions in the macro-world of living organisms under strong noise and interference. This dictation is realized by means of unknown algorithms of multi-channel noise-immunity coding. For example, in human organism, his skin color, eye color and hair color are inherited genetically independently of each other. It is possible if appropriate kinds of information are conducted via independent informational channels and if a general "phase space" of living organism contains sub-spaces with a possibility of a selective control or a selective coding of processes in them. So, any living organism is an algorithmic machine of multi-channel noise-immunity coding with ability to a selective control and coding of different sub-spaces of its phase space (a model approach to phase spaces with a selective control of their sub-spaces is presented in this article). This machine works in conditions of ontogenetic development of the organism when a multi-dimensionality of its phase space is increased step by step.

To understand such genetic machine, it is appropriate to use the theory of noise-immunity coding and transmission of digital information, taking into account the discrete
nature of the genetic code. In this theory, mathematical matrices have the basic importance. The use of matrix representations and analysis in the study of phenomenological features of molecular-genetic ensembles has led to the development of a special scientific direction under a name "Matrix Genetics" [Petoukhov, 2008; Petoukhov, He, 2009]. Namely researches of the "matrix genetics" gave results that are represented in this article.

Concerning the theme of projectors in inherited biological phenomena, one can note that our genetically inherited visual system works on the principle of projection of external objects at the retina. This projection is modeled using projection operators. The author believes that the value of projectors for bioinformatics is not limited to this single fact of biological significance of projection operators, but that the whole system of genetic and sensory informatics is based on their active use. This ubiquitous use of projection operators reflects and ensures (in some degree) the unity of any organism and interrelations of its parts.

The set of projection operators, which are associated with the matrix representation of the genetic code, provides new opportunities for modeling ensembles of inherited cycles; ensembles of phyllotaxis structures; a numeric specificity of reproduction of genetic information in acts of mitosis and meiosis of biological cells, etc. In the frame of the "projector conception" arised here in genetic informatics, some features of evolutionary transformations of variants (or dialects) of the genetic code are clarified.

The main mathematical objects of the article are four matrices $R_4$, $R_8$, $H_4$ and $H_8$ shown on Figure 1. Why these numeric matrices are chosen from infinite set of matrices? The reason is that they are connected with phenomenology of the genetic code system in matrix forms of its representation as it was shown in works [Petoukhov, 2008b, 2011a,b, 2012a,b] and as it will be additionally demonstrated in the end of this article. The matrices $R_4$ and $R_8$ are conditionally termed “Rademacher matrices” because each of their columns represents one of known Rademacher functions. The matrices $H_4$ and $H_8$ belong to a great set of Hadamard matrices, which are widely used for noise-immunity coding in technologies of signals processing and which are connected with complete orthogonal systems of Walsh functions.

![Matrices R, H](image-url)
Every of these matrices can be decomposed into sum of sparse matrices, each of which contains only one non-zero column. Such decomposition can be conditionally termed a «column decomposition». Every of such matrices can be also decomposed into a sum of sparse matrices, each of which contains only one non-zero row. Such decomposition can be conditionally termed a «row decomposition».

2. GENETIC RADEMACHER MATRICES AS SUMS OF PROJECTORS

Let us begin with the column decomposition $R_4 = c_0 + c_1 + c_2 + c_3$ and the row decomposition $R_4 = r_0 + r_1 + r_2 + r_3$ of the Rademacher (4*4)-matrix $R_4$ (Figure 2).

$$
R_4 = c_0 + c_1 + c_2 + c_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
R_4 = r_0 + r_1 + r_2 + r_3 = \begin{bmatrix}
1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
-1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 2. The «column decomposition» (upper layer) and the «row decomposition» (bottom layer) of the Rademacher matrix $R_4$ from Figure 1

Each of these sparse matrices $c_0$, $c_1$, $c_2$, $c_3$ and $r_0$, $r_1$, $r_2$, $r_3$ is a projection operator because it satisfies the criterion of projectors $P^2 = P$ (for example, $c_0 = c_0$, etc.). Every of these projectors is an oblique (non-orthogonal) projector because it is expressed by means of a non-symmetrical matrix. Every of sets ($c_0$, $c_1$, $c_2$, $c_3$) and ($r_0$, $r_1$, $r_2$, $r_3$) consists of non-commutative projectors. We will conditionally name projectors $c_0$, $c_1$, $c_2$, $c_3$ as «column projectors» and projectors $r_0$, $r_1$, $r_2$, $r_3$ as «row projectors».

Let us examine all possible variants of sums of pairs of the different column projectors $c_0$, $c_1$, $c_2$ and $c_3$: ($c_0 + c_1$), ($c_0 + c_2$), ($c_0 + c_3$), ($c_1 + c_2$), ($c_1 + c_3$), ($c_2 + c_3$). The result of this examination is the following: matrices ($c_0 + c_1$) and ($c_2 + c_3$) with a weight coefficient $2^{-0.5}$ lead to cyclic groups with their period 8 in cases of their exponentiation: $(2^{-0.5}*(c_0+c_1))^n = (2^{-0.5}*(c_0+c_1))^{n+8}$, $(2^{-0.5}*(c_2+c_3))^n = (2^{-0.5}*(c_2+c_3))^{n+8}$, where $n = 1, 2, 3, \ldots$ (Figure 3).

\[
(2^{-0.5}*(c_0+c_1))^1 = \begin{bmatrix}
2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\
-2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\
2^{-0.5}, & -2^{-0.5}, & 0, & 0 \\
-2^{-0.5}, & -2^{-0.5}, & 0, & 0
\end{bmatrix} \quad \text{and} \quad (2^{-0.5}*(c_2+c_3))^1 = \begin{bmatrix}
0, & 0, & 2^{-0.5}, & -2^{-0.5} \\
0, & 0, & -2^{-0.5}, & -2^{-0.5} \\
0, & 0, & 2^{-0.5}, & 2^{-0.5} \\
0, & 0, & -2^{-0.5}, & 2^{-0.5}
\end{bmatrix}
\]
\[(2^{-0.5}(c_0+c_1))^2 = \begin{bmatrix} 0, & 1, & 0, & 0 \\ -1, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^2 = \begin{bmatrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & -1 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & -1, & 0 \end{bmatrix}\]

\[(2^{-0.5}(c_0+c_1))^3 = \begin{bmatrix} -2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\ -2^{-0.5}, & -2^{-0.5}, & 0, & 0 \\ 2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\ 2^{-0.5}, & -2^{-0.5}, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^3 = \begin{bmatrix} 0, & 0, & 2^{-0.5}, & 2^{-0.5} \\ 0, & 0, & -2^{-0.5}, & 2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & -2^{-0.5}, & -2^{-0.5} \end{bmatrix}\]

\[(2^{-0.5}(c_0+c_1))^4 = \begin{bmatrix} -1, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 1, & 0, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^4 = \begin{bmatrix} 0, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & -1 \end{bmatrix}\]

\[(2^{-0.5}(c_0+c_1))^5 = \begin{bmatrix} -2^{-0.5}, & -2^{-0.5}, & 0, & 0 \\ 2^{-0.5}, & -2^{-0.5}, & 0, & 0 \\ -2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\ 2^{-0.5}, & 2^{-0.5}, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^5 = \begin{bmatrix} 0, & 0, & -2^{-0.5}, & 2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \end{bmatrix}\]

\[(2^{-0.5}(c_0+c_1))^6 = \begin{bmatrix} 0, & -1, & 0, & 0 \\ 1, & 0, & 0, & 0 \\ -1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^6 = \begin{bmatrix} 0, & 0, & -1, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & -1 \\ 0, & 0, & 1, & 0 \end{bmatrix}\]

\[(2^{-0.5}(c_0+c_1))^7 = \begin{bmatrix} 2^{-0.5}, & -2^{-0.5}, & 0, & 0 \\ 2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\ -2^{-0.5}, & -2^{-0.5}, & 0, & 0 \\ -2^{-0.5}, & 2^{-0.5}, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^7 = \begin{bmatrix} 0, & 0, & -2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & -2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \end{bmatrix}\]

\[(2^{-0.5}(c_0+c_1))^8 = \begin{bmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & -1, & 0, & 0 \\ -1, & 0, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^8 = \begin{bmatrix} 0, & 0, & 0, & -1 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{bmatrix}\]

\[(2^{-0.5}(c_0+c_1))^9 = \begin{bmatrix} 2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\ -2^{-0.5}, & 2^{-0.5}, & 0, & 0 \\ 2^{-0.5}, & -2^{-0.5}, & 0, & 0 \\ -2^{-0.5}, & -2^{-0.5}, & 0, & 0 \end{bmatrix}; \quad (2^{-0.5}(c_2+c_3))^9 = \begin{bmatrix} 0, & 0, & 2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \\ 0, & 0, & 2^{-0.5}, & -2^{-0.5} \end{bmatrix}\]

Figure 3. The illustration of cyclic groups of operators on the basis of sums of projection operators \((c_0+c_1)\) and \((c_2+c_3)\) from Figure 2 in cases of their exponentiation.

These two sums of column projectors \((c_0+c_1)\) and \((c_2+c_3)\) are marked by green colour in the left table on Figure 4, where every of cells represents a sum of those projectors, which denote its column and row.

Two other examined sums of column projectors \((c_0+c_2)\) and \((c_1+c_3)\) is doubled when squaring: \((c_0+c_2)^n = 2^{n-1}*(c_0+c_2), (c_1+c_3)^n = 2^{n-1}*(c_1+c_3)\), where \(n = 1, 2, 3, \ldots\) (these sums are marked by red colours in the left table on Figure 4). This doubling reminds dichotomic
dividing of biological cells in a result of mitosis when doubling of genetic information occurs.

The last examined sums of the column projectors \((c_0+c_1)\) and \((c_1+c_2)\) possess the following feature. Matrices of their second power is quadrupled in a result of exponentiation in integer powers: \(((c_0+c_3)^{2})^n = 4^{n-1}*(c_0+c_3)^2\), \(((c_1+c_2)^{2})^n = 4^{n-1}*(c_1+c_2)^2\), where \(n = 1, 2, 3\)… (this feature can be used to simulate a genetic phenomenon of tetra-reproduction of gametes and genetic information in a course of meiosis). The cells with these sums \((c_0+c_3)\) and \((c_1+c_2)\) are marked by yellow colour in the left table on Figure 4.

If we examine the row projectors \(r_0, r_1, r_2\) and \(r_3\) from Figure 2, we receive the same tabular structure with a small change: red and yellow cells are swapped (Figure 4, right). Matrices \(2^{-0.5}*(r_0+r_1)\) and \(2^{-0.5}*(r_2+r_3)\) are bases for cyclic groups with a period 8 in relation to their exponentiation (green cells in the right table on Figure 4). Matrices \((r_0+r_2)\) and \((r_1+r_3)\) is doubled when squaring: \((r_0+r_2)^2 = 2^{n-1}*(r_0+r_2), (r_1+r_3)^2 = 2^{n-1}*(r_1+r_3)\), where \(n = 1, 2, 3,\ldots\) (red cells in the right table on Figure 4). Matrices \((r_0+r_3)\) and \((r_1+r_2)\) possess the «quadruplet» property: \(((r_0+r_3)^{2})^n = 4^{n-1}*(r_0+r_3)^2\), \(((r_1+r_2)^{2})^n = 4^{n-1}*(r_1+r_2)^2\), where \(n = 1, 2, 3\ldots\) (yellow cells in the right table on Figure 4). The cells on the main diagonal correspond to sum of two projectors themselves: \((c_0+c_1)^{2} = 2^n*c_i, (r_0+r_1)^{2} = 2^n*r_i\), where \(i = 0, 1, 2, 3\).

![Figure 4. Tables of some features of sums of pairs of different «column projectors» \(c_0, c_1, c_2, c_3\) and of «row projectors» \(r_0, r_1, r_2, r_3\) (from the Rademacher matrix \(R_4\) on Figure 2) in relation to their exponentiation. Explanations in text.](http://en.wikipedia.org/wiki/Split-complex_number)

It should be noted that cyclic features of \((4*4)\)-matrices \(2^{-0.5}*(c_0+c_1)\), \(2^{-0.5}*(c_2+c_3)\), \(2^{-0.5}*(r_0+r_1)\) and \(2^{-0.5}*(r_2+r_3)\) in cases of their exponentiation exist due to their connections with matrix representations of 2-parametric complex numbers in 4-dimensional space; these connections are shown by means of a new decomposition of each of these matrices into sum of new sparse matrices \(e_0\) and \(e_1\) (Figures 5, 6, two upper levels, where the multiplication table for \(e_0\) and \(e_1\) in the right column is identical to the multiplication table of complex numbers). The dichotomic features of \((4*4)\)-matrices \((c_0+c_2)\), \((c_1+c_3)\), \((r_0+r_3)\), \((r_1+r_2)\) and tetra-reproduction features of \((4*4)\)-matrices \(((c_0+c_3)^{2})^n = 4^{n-1}*(c_0+c_3)^2\), \(((c_1+c_2)^{2})^n = 4^{n-1}*(c_1+c_2)^2\), \(((r_0+r_2)^{2})^n = 4^{n-1}*(r_0+r_2)^2\), \(((r_1+r_3)^{2})^n = 4^{n-1}*(r_1+r_3)^2\) exist due to their connections with matrix representations of 2-parametric hyperbolic numbers in 4-dimensional space (Figures 5, 6, four bottom levels, where the multiplication table for \(e_0\) and \(e_1\) in the right column is identical to the multiplication table of hyperbolic numbers). Synonyms of hyperbolic numbers are Lorentz numbers, split-complex numbers, double numbers, perplex numbers, etc. [http://en.wikipedia.org/wiki/Split-complex_number](http://en.wikipedia.org/wiki/Split-complex_number). Hyperbolic numbers are a two-dimensional commutative algebra over the real numbers. Additional details about such
(4*4)-matrix representations of complex numbers and hyperbolic numbers see in [Petoukhov, 2012b].

| $c_0+c_1$ | 1 1 0 0 | 1 0 0 0 | 0 1 0 0 | $e_0$ $e_1$ |
|-----------|---------|---------|---------|-------------|
|           | -1 1 0 0 | 0 1 0 0 | -1 0 0 0 | $e_0$ $e_0$ $e_1$ |
|           | -1 -1 0 0 | 0 0 1 0 | 1 0 0 0 | $e_1$ $e_1$ $-e_0$ |
|           | -1 -1 0 0 | 0 0 0 1 | 0 -1 0 0 |             |

| $c_2+c_3$ | 0 0 1 -1 | 0 0 0 -1 | 0 0 1 0 | $e_0$ $e_1$ |
|-----------|---------|---------|---------|-------------|
|           | 0 0 -1 -1 | 0 0 -1 0 | 0 0 0 1 | $e_0$ $e_0$ $e_1$ |
|           | 0 0 1 1 | 0 1 0 0 | 0 0 -1 0 | $e_1$ $e_1$ $-e_0$ |
|           | 0 0 -1 1 | 0 0 0 1 |             |             |

| $c_0+c_2$ | 1 0 1 0 | 1 0 0 0 | 0 0 1 0 | $e_0$ $e_1$ |
|-----------|---------|---------|---------|-------------|
|           | -1 0 -1 0 | -1 0 0 0 | 0 0 -1 0 | $e_0$ $e_0$ $e_1$ |
|           | -1 0 1 0 | 0 0 1 0 | 1 0 0 0 | $e_1$ $e_1$ $e_0$ |
|           | -1 0 -1 0 | 0 0 0 1 | -1 0 0 0 |             |

| $c_1+c_3$ | 0 1 0 -1 | 0 1 0 0 | 0 0 0 -1 | $e_0$ $e_1$ |
|-----------|---------|---------|---------|-------------|
|           | 0 1 0 -1 | 0 1 0 0 | 0 0 0 1 | $e_0$ $e_0$ $e_1$ |
|           | 0 -1 0 1 | 0 0 1 0 | -1 0 0 0 | $e_1$ $e_1$ $e_0$ |
|           | 0 -1 0 1 | 0 0 0 1 |             |             |

| $0.5^2(c_0+c_3)^2$ | 1 0 0 -1 | 1 0 0 0 | 0 0 0 -1 | $e_0$ $e_1$ |
|--------------------|---------|---------|---------|-------------|
|                   | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | $e_0$ $e_0$ $e_1$ |
|                   | -1 0 0 1 | 0 0 0 1 | 0 0 0 0 | $e_1$ $e_1$ $e_0$ |
|                   |             |         |         |             |

| $0.5^2(c_1+c_2)^2$ | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | $e_0$ $e_1$ |
|--------------------|---------|---------|---------|-------------|
|                   | 0 1 -1 0 | 0 1 -1 0 | 0 0 0 0 | $e_0$ $e_0$ $e_1$ |
|                   | 0 -1 1 0 | 0 -1 1 0 | 0 0 0 0 | $e_1$ $e_1$ $e_0$ |
|                   | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |             |

Figure 5. The table represents special decompositions of (4*4)-matrices $(c_0+c_1)$, $(c_2+c_3)$, $(c_0+c_2)$, $(c_1+c_3)$, $0.5^2(c_0+c_3)^2$ and $0.5^2(c_1+c_2)^2$ into sum of two matrices $e_0+e_1$ (see also Figures 2 and 4). The table shows direct relations of these matrices with matrix representations of 2-parametric complex numbers and hyperbolic numbers. Here $c_0$, $c_1$, $c_2$ and $c_3$ are column projectors from Figure 2. For each set of matrices $c_0$ and $e_1$ at every tabular level, the right column of this table contains its multiplication table: for two upper levels it is a known multiplication table of complex numbers; for other four levels it is a known multiplication table of hyperbolic numbers.
Figure 6. The table represents special decompositions of (4*4)-matrices \( (r_0+r_1), (r_2+r_3), 0.5*(r_0+r_2)^2, 0.5*(r_1+r_3)^2, (r_0+r_3) \) and \( (r_1+r_2) \) into sum of two matrices \( e_0+e_1 \). The table shows direct relations of these matrices with matrix representations of 2-parametric complex numbers and hyperbolic numbers. Here \( r_0, r_1, r_2 \) and \( r_3 \) are row projectors from Figure 2. For each set of matrices \( e_0 \) and \( e_1 \) at every tabular level, the right column of this table contains its multiplication table: for two upper levels it is a known multiplication table of complex numbers; for other four levels it is a known multiplication table of hyperbolic numbers.

Now let us turn to the Rademacher (8*8)-matrix \( R_8 \) (Figure 1) to analyze its column decomposition \( R_8 = s_0+s_1+s_2+s_3+s_4+s_5+s_6+s_7 \) (Fugure 7) and its row decomposition \( R_8 = v_0+v_1+v_2+v_3+v_4+v_5+v_6+v_7 \) (Figure 8).

\[
\begin{array}{ccc}
10000000 & 01000000 \\
01000000 & 01000000 \\
-10000000 & 0-10000000 \\
-10000000 & 0-10000000 \\
00000000 & 00000000 \\
00000000 & 00000000 \\
00000000 & 00000000 \\
00000000 & 00000000 \\
\end{array}
\]
Figure 7. The «column decomposition» $R_8 = s_0 + s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7$ of the Rademacher (8*8)-matrix $R_8$ (Figure 1) where every of matrices $s_0$, $s_1$, $s_2$, $s_3$, $s_4$, $s_5$, $s_6$, $s_7$ is a projection operator.
The right table contains the correspond to the exponentiations of which generate 8 cyclic groups. The left table contains comparison with the cases on Figure 4.

Figure 8. The «row decomposition» $R_8 = v_0+v_1+v_2+v_3+v_4+v_5+v_6+v_7$ of the Rademacher (8*8)-matrix $R_8$ (Figure 1) where every of matrices $v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7$ is a projection operator.

By analogy with the described case of the projection (4*4)-operators $c_0, c_1, c_2, c_3$ and $r_0, r_1, r_2, r_3$ (Figures 3-6), one can analyse features of sums of pairs of the column projection (8*8)-operators $s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7$ (Figure 7) and of the row projection (8*8)-operators $v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7$ in relation to their exponentiation. In other words, one can analyse features of matrices $(s_0+s_1)^n, (s_0+s_3)^n, \ldots$ and $(v_0+v_1)^n, (v_0+v_2)^n, \ldots$ where $n = 1, 2, 3, \ldots$.

Such analysis leads to resulting tables on Figure 9.

Every of cells in these tables on Figure 9 represents a sum of those projectors which denote its column and row by analogy with Figure 4. Again we have three types of such sums which are marked by green, red and yellow and which possess the similar properties in comparison with the cases on Figure 4.

In each of tables on Figure 9, green cells correspond to those matrices, exponentiations of which generate 8 cyclic groups. The left table contains 16 green cells that correspond to the following cyclic groups with a period 8: $(2^{-0.5}*(s_0+s_2))^n, (2^{-0.5}*(s_0+s_3))^n, (2^{-0.5}*(s_1+s_2))^n, (2^{-0.5}*(s_1+s_3))^n, (2^{-0.5}*(s_4+s_6))^n, (2^{-0.5}*(s_4+s_7))^n, (2^{-0.5}*(s_5+s_6))^n, (2^{-0.5}*(s_5+s_7))^n$. The right table contains 16 green cells with the same tabular location that correspond to the

![Figure 9. Tables of some features of sums of pairs of different column projectors $s_0, s_1, \ldots, s_7$ (from Figure 7) and of row projectors $v_0, v_1, \ldots, v_7$ (from the Rademacher matrix $R_8$ on Figure 8) in relation to their exponentiation. Explanations in text.](image-url)
following cyclic groups with a period 8: \((2^{0.5}(v_0+v_2))^n, (2^{0.5}(v_0+v_3))^n, (2^{0.5}(v_1+v_2))^n, (2^{0.5}(v_1+v_3))^n, (2^{0.5}(v_4+v_5))^n, (2^{0.5}(v_4+v_7))^n, (2^{0.5}(v_5+v_6))^n, (2^{0.5}(v_5+v_7))^n\).

Red cells in tables on Figure 9 contain those matrices which possess a doubling property in relation to their exponentiation. The left table contains 24 red cells, matrices of which satisfy the following feature: \((s_0+s_1)^n = 2^{n-1}(s_0+s_1), (s_0+s_4)^n = 2^{n-1}(s_0+s_4), \) etc. The right table also contains 24 red cells with the same feature but with another tabular location: \((v_0+v_1)^n = 2^{n-1}(v_0+v_1), (v_0+v_4)^n = 2^{n-1}(v_0+v_4), \) etc.

16 yellow cells in each of tables on Figure 9 contain those matrices, which have «quadruplet property»: \(((s_0+s_6)^2)^n = 4^{n-1}(s_0+s_6)^2, ((v_0+v_4)^2)^n = 4^{n-1}(v_0+v_4)^2, \) etc. The cells on the main diagonal correspond to sum of two projectors themselves: \((s_i+s_i)^n = 2^n s_i, \) \((v_i+v_i)^n = 2^n v_i, \) where \(i = 0, 1, 2, ..., 7.\)

3. GENETIC HADAMARD MATRICES AS SUMS OF PROJECTORS

The genetic Hadamard matrix \(H_4\) from Figure 1 can be also decomposed into sum of 4 sparse matrices \(H_4 = h_0+h_1+h_2+h_3\) where each of sparse matrices contains only one non-zero column (in a case of the «column decomposition») or only one non-zero row (in a case of the «row decomposition») (Figure 10).

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{array}
\]

\[
H_4 = h_0+h_1+h_2+h_3 =
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & -1 & -1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
H_4 = g_0+g_1+g_2+g_3 =
\begin{array}{cccc}
1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
\end{array}
\]

Figure 10. The «column decomposition» (upper layer) and the «row decomposition» (bottom layer) of the Hadamard matrix \(H_4\) from Figure 1

Each of these sparse matrices \(h_0, h_1, h_2, h_3\) and \(g_0, g_1, g_2, g_3\) on Figure 10 is a projector. We will conditionally name projectors \(h_0, h_1, h_2, h_3\) again as «column projectors» and projectors \(g_0, g_1, g_2, g_3\) as «row projectors».

By analogy with the previous section about the Rademacher matrix \(R_4\), one can analyse features of sums of pairs of these column projectors and row projectors in relation to their exponentiation. In other words, one can analyze features of matrices \((h_0+h_1)^n, (h_0+h_2)^n, \ldots, (g_0+g_1)^n, (g_0+g_2)^n, \ldots\) where \(n = 1, 2, 3, \ldots\). Such analysis leads to resulting tables on Figure 11.
Figure 11. Tables of some features of sums of pairs of the different «column projectors» h₀, h₁, h₂, h₃ (in the left table) and of the «row projectors» g₀, g₁, g₂, g₃ (from the Hadamard matrix H₄ on Figure 10) in relation to their exponentiation. Explanations in text.

Both tables on Figure 11 are identical in their mosaics. Every of their cells correspond to a sum of those column projectors (or row projectors), which denote its column and row. 12 green cells correspond to matrices, exponentiations of which lead to cyclic groups with a period 8: (2⁻⁰.⁵*(h₀+h₁))ⁿ, (2⁻⁰.⁵*(h₀+h₂))ⁿ, … , (2⁻⁰.⁵*(g₀+g₁))ⁿ, (2⁻⁰.⁵*(g₀+g₁))ⁿ, … , where n = 1, 2, 3,… . The cells on the main diagonal correspond to sum of two projectors themselves: (hᵢ+hᵢ)ⁿ = 2ᵃ*hᵢ, (gᵢ+gᵢ)ⁿ = 2ᵃ*gᵢ, where i = 0, 1, 2, 3.

Cyclic properties of these (4*4)-matrix operators exist due to a connection of these operators with complex numbers. Figures 12 and 13 show existence of a special decomposition of every of these (4*4)-matrices into such set of two sparse matrices e₀ and e₁, which is closed relative to multiplication and which defines their multiplication table that coincides with the known multiplication table of complex numbers.

| h₀+h₁ = 1 1 0 0 | 1 0 0 0 | 0 1 0 0 | 0 1 0 0 | e₀ e₁ |
| -1 1 0 0 = e₀+e₁ = 0 1 0 0 | 0 1 0 0 | -1 0 0 0 | e₀ e₁ |
| 1 -1 0 0 | -1 0 0 0 | 1 0 0 0 | e₁ e₀ |
| -1 -1 0 0 | -1 0 0 0 | 1 0 0 0 | e₁ e₀ |

| h₀+h₂ = 1 0 -1 0 | 1 0 0 0 | 0 0 -1 0 | 0 0 -1 0 | e₀ e₁ |
| -1 0 1 0 = e₀+e₁ = -1 0 0 0 | 0 0 1 0 | 0 0 1 0 | e₀ e₁ |
| 1 0 1 0 | 0 0 1 0 | 1 0 0 0 | e₁ e₀ |
| -1 0 -1 0 | 0 0 -1 0 | -1 0 0 0 | e₁ e₀ |

| h₀+h₃ = 1 0 0 1 | 1 0 0 0 | 0 0 0 1 | 0 0 0 1 | e₀ e₁ |
| -1 0 0 1 = e₀+e₁ = 0 0 0 1 | 0 0 0 1 | -1 0 0 0 | e₀ e₁ |
| 1 0 0 1 | 0 0 0 1 | 0 0 0 1 | e₁ e₀ |
| -1 0 0 1 | 0 0 0 1 | -1 0 0 0 | e₁ e₀ |

| h₁+h₂ = 0 1 -1 0 | 0 0 -1 0 | 0 1 0 0 | 0 1 0 0 | e₀ e₁ |
| 0 1 1 0 = e₀+e₁ = 0 1 0 0 | 0 1 0 0 | 0 0 1 0 | e₀ e₁ |
| 0 -1 1 0 | 0 0 1 0 | 0 1 0 0 | e₁ e₀ |
| 0 -1 -1 0 | 0 -1 0 0 | 0 0 -1 0 | e₁ e₀ |

| h₁+h₃ = 0 1 0 1 | 0 1 0 0 | 0 0 0 1 | 0 0 0 1 | e₀ e₁ |
| 0 1 0 1 = e₀+e₁ = 0 1 0 0 | 0 1 0 0 | 0 0 0 1 | e₀ e₁ |
| 0 -1 0 1 | 0 0 0 1 | 0 1 0 0 | e₁ e₀ |
| 0 -1 0 1 | 0 0 0 1 | 0 0 0 1 | e₁ e₀ |
The table represents special decompositions of $(4*4)$-matrices $(h_0+h_1)$, $(h_0+h_2)$, $(h_0+h_3)$, $(h_1+h_2)$, $(h_1+h_3)$, $(h_2+h_3)$ into sum of two matrices $e_0+e_1$. The table shows direct relations of these matrices with matrix representations of 2-parametric complex numbers. Here $h_0$, $h_1$, $h_2$ and $h_3$ are column projectors of the Hadamard matrix $H_4$ from Figure 10. For each set of matrices $e_0$ and $e_1$ at every tabular level, the right column of the table contains its multiplication table, which coincides with the multiplication table of complex numbers.

\[
\begin{array}{c|cc|cc|ccc|c}
 & h_2+h_3 & = & e_0+e_1 & = & 0 0 1 0 & + & 0 0 1 0 & ; \\
\hline
0 0 1 1 & 0 0 1 1 & 0 0 1 1 & 0 0 1 1 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 \\
0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 & 0 0 0 0 \\
\end{array}
\]

Figure 12. The table contains its multiplication table, which coincides with the multiplication table of complex numbers.
Figure 13. The table represents special decompositions of (4*4)-matrices \((g_0+g_1), (g_0+g_2), (g_0+g_3), (g_1+g_2), (g_1+g_3), (g_2+g_3)\) into sum of two matrices \(e_0+e_1\). The table shows direct relations of these matrices with matrix representations of 2-parametric complex numbers. Here \(g_0, g_1, g_2\) and \(g_3\) are row projectors from Figure 10. For each set of matrices \(e_0\) and \(e_1\) at every tabular level, the right column of the table contains its multiplication table, which coincides with the multiplication table of complex numbers.

Now let us turn to the genetic Hadamard matrix \(H_8\) from Figure 1. It can be also decomposed into sum of 8 sparse matrices \(H_8=u_0+u_1+u_2+u_3+u_4+u_5+u_6+u_7\) where each of sparse matrices contains only one non-zero column (in a case of the «column decomposition») or only one non-zero row (in a case of the «row decomposition») (Figure 14).

\[
H_8 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] +

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\] +

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] +

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] +

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] +

Figure 14. The «column decomposition» \(H_8=u_0+u_1+u_2+u_3+u_4+u_5+u_6+u_7\) of the Hadamard \((8*8)\)-matrix \(H_8\) (Figure 1) where every of sparse matrices \(u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\) is a projection operator.
Other words, one can analyse features of these column projectors and row projectors in relation to their exponentiation. In other words, one can analyse features of matrices \((u_0 + u_1)^n\), \((u_0 + u_3)^n\), and \((d_0 + d_1)^n\), \((d_0 + d_2)^n\) where \(n = 1, 2, 3, \ldots\). Such analysis leads to resulting tables on Figure 16.

**Figure 15.** The «row decomposition» \(H_6 = d_0 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7\) of the Hadamard \((8*8)\)-matrix \(H_8\) (Figure 1) where every of sparse matrices \(d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7\) is a projection operator.

Every of these sparse matrices \(u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\) and \(d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7\) on Figures 14, 15 is a projector. We will conditionally name projectors \(u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\) again as «column projectors» and projectors \(d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7\) as «row projectors».

By analogy with the previous sections, one can analyse features of sums of pairs of these column projectors and row projectors in relation to their exponentiation.
Figure 16. Tables of some features of sums of pairs of the different column projectors \( u_0, u_1, \ldots, u_7 \) (from Figure 14) and of the row projectors \( d_0, d_1, \ldots, d_7 \) (from Figure 15) in relation to their exponentiation. This is the case of the Hadamard matrix \( H_8 \) from Figure 1. Explanations in text.

Both tables on Figure 16 have the identical mosaic with 32 green cells and 24 yellow cells. The green cells in these tables correspond to those matrices, exponentiations of which generate cyclic groups with a period 8:

- \((2^{-0.5}(u_0+u_1))^n, (2^{-0.5}(u_0+u_2))^n, (2^{-0.5}(u_0+u_4))^n, (2^{-0.5}(u_1+u_3))^n, (2^{-0.5}(u_1+u_5))^n, (2^{-0.5}(u_2+u_3))^n, (2^{-0.5}(u_2+u_4))^n, (2^{-0.5}(u_2+u_6))^n, (2^{-0.5}(u_3+u_5))^n, (2^{-0.5}(u_3+u_7))^n, (2^{-0.5}(u_4+u_5))^n, (2^{-0.5}(u_4+u_6))^n, (2^{-0.5}(u_5+u_7))^n, (2^{-0.5}(u_6+u_7))^n)\) (in the left table);
- \((2^{-0.5}(d_0+d_1))^n, (2^{-0.5}(d_0+d_2))^n, (2^{-0.5}(d_0+d_4))^n, (2^{-0.5}(d_1+d_3))^n, (2^{-0.5}(d_1+d_5))^n, (2^{-0.5}(d_2+d_3))^n, (2^{-0.5}(d_2+d_4))^n, (2^{-0.5}(d_2+d_6))^n, (2^{-0.5}(d_3+d_5))^n, (2^{-0.5}(d_3+d_7))^n, (2^{-0.5}(d_4+d_5))^n, (2^{-0.5}(d_4+d_6))^n, (2^{-0.5}(d_5+d_7))^n)\) (in the right table).

Cyclic properties of these \((8\times8)\)-matrix operators exist due to a connection of these operators with complex numbers. Figure 17 shows some examples of decompositions of the \((8\times8)\)-matrices from green cells on Figure 16 into corresponding sets of two sparse matrices, each of which is closed in relation to multiplication and each of which defines the multiplication table of complex numbers (see some additional details about representations of complex numbers by means of \((2^n\times2^n)\)-matrices in [Petoukhov, 2012b]). It should be noted here that our study in the field of matrix genetics has revealed methods of extension of these \((8\times8)\)-genetic matrices \(R_8, R_{8s}, H_4, H_8\) (Figure 1) into \((2^n\times2^n)\)-matrices which are also sums of "column projectors" and "row projectors" and which give by analogy as much cyclic groups as needed to model big ensembles of cyclic processes.
inherited biological phenomena including cooperative ensembles of processes. Forms of are connected with phenomenological properties of molecular inherited physiological systems and molecular features. We develop a "genetic biomechanics", which subsystems of the body should be agreed with the structural organization of genetic coding synchronization biorhythmology, various diseases of the body are associated with disturbances and they are then collected into a new protein. According to chronomedicine and "birth and death," because after a certain time it breaks down into its constituents. 

**GENETIC BIOMECHANICS**

Changes of

\[
\begin{align*}
\mathbf{u}_2 + \mathbf{u}_6 &= \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}_3 + \mathbf{u}_7 &= \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{e}_2 + \mathbf{e}_6 &= \begin{pmatrix}
e_2 & e_6 \\
e_2 & e_6 \\
e_6 & e_6 \\
e_6 & e_6 \\
e_6 & e_6 \\
e_6 & e_6 \\
e_6 & e_6 \\
e_6 & e_6
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{e}_3 + \mathbf{e}_7 &= \begin{pmatrix}
e_3 & e_7 \\
e_3 & e_7 \\
e_7 & e_7 \\
e_7 & e_7 \\
e_7 & e_7 \\
e_7 & e_7 \\
e_7 & e_7 \\
e_7 & e_7
\end{pmatrix}
\end{align*}
\]

Figure 17. The decomposition of the (8*8)-matrices \( \mathbf{u}_0 + \mathbf{u}_4, \mathbf{u}_1 + \mathbf{u}_5, \mathbf{u}_2 + \mathbf{u}_6, \mathbf{u}_3 + \mathbf{u}_7 \), which are examples of (8*8)-matrices from green cells on Figure 16, into corresponding sets of two sparse matrices \( \mathbf{e}_0 \) and \( \mathbf{e}_1 \), \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \) and \( \mathbf{e}_6 \) and \( \mathbf{e}_7 \), each of which is closed in relation to multiplication and each of which defines the multiplication table of complex numbers (on the right).

Figure 17 testifies that the Hadamard (8*8)-matrix \( \mathbf{H}_8 = (\mathbf{u}_0 + \mathbf{u}_4) + (\mathbf{u}_1 + \mathbf{u}_5) + (\mathbf{u}_2 + \mathbf{u}_6) + (\mathbf{u}_3 + \mathbf{u}_7) \) (Fig. 1) is a sum of 4 complex numbers in 8-dimensional space.

Yellow cells in tables on Figure 16 correspond to matrices with the following property: \( ((\mathbf{u}_i + \mathbf{u}_j)^z)^n = 2^{n-1}((\mathbf{u}_i + \mathbf{u}_j)^z) \) and \( ((\mathbf{d}_i + \mathbf{d}_j)^z)^n = 2^{n-1}((\mathbf{d}_i + \mathbf{d}_j)^z) \) where \( i \neq j \), \( i, j = 0, 1, 2, \ldots, 7 \), \( n = 1, 2, 3, \ldots \). Cells on the main diagonal correspond to matrices \( (\mathbf{u}_i + \mathbf{u}_i)^n = 2^{n*} \mathbf{u}_i \).

**4. INHERITED BIOCYCLES AND A SELECTIVE CONTROL OF CYCLIC CHANGES OF VECTORS IN A MULTIDIMENSIONAL SPACE. PROBLEMS OF GENETIC BIOMECHANICS**

Any living organism is an object with a huge ensemble of inherited cyclic processes, which form a hierarchy at different levels. Even every protein is involved in a cycle of the "birth and death," because after a certain time it breaks down into its constituent amino acids and they are then collected into a new protein. According to chronomedicine and biorhythmology, various diseases of the body are associated with disturbances (dys-synchronization) in these cooperative ensembles of biocycles. All inherited physiological subsystems of the body should be agreed with the structural organization of genetic coding for their coding and transmission to descendants; in other words, they bear the stamp of its features. We develop a "genetic biomechanics", which studies deep coherence between inherited physiological systems and molecular-genetic structures.

Our discovery of the described cyclic groups (on basis of genetic projectors), which are connected with phenomenological properties of molecular-genetic systems in their matrix forms of representation, gives a mathematical approach to simulate ensembles of cyclic processes. In this approach an idea of multi-dimensional vector space is used to simulate inherited biological phenomena including cooperative ensembles of cyclic processes.
Multidimensional vectors of this bioinformation space can be changed under influence of those matrix operators on the basis of genetic projectors that were described in previous section. Due to special properties of these operators a useful possibility exists to provide a selective control (or a selective coding) of cyclic changes (and some other changes) of separate coordinates of multidimensional vectors in this space.

Let us explain this by one example. Let us take, for instance, the cyclic group of operators \( Y^n = (2^{0.5} (s_0 + s_2))^n \) (see Figures 7 and 9, on the left) and an arbitrary 8-dimensional vector \( X = [x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] \). Then let us analyze an expression \( X*Y^n = [x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] \) that leads to a new vector \( Z_n = [z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_7] \) (here \( n = 1, 2, 3, \ldots \)). Figure 18 shows a cyclic transformation of coordinates of vectors \( Z_n \); vectors \( X*Y^1 \) and \( X*Y^5 \) are identical because the period of this cyclic group \( Y^n = (2^{0.5} (s_0 + s_2))^n \) is equal to 8.

\[
\begin{align*}
X*Y^1 &= Z_1 = 2^{0.5} [(x_0 + x_1 - x_2 + x_3 + x_4 + x_5 - x_6 - x_7), 0, (x_0 + x_1 - x_2 + x_3 - x_4 - x_5 + x_6 + x_7), 0, 0, 0, 0, 0] \\
X*Y^2 &= Z_2 = [(x_0 - x_1 + x_2 + x_3), 0, (x_0 - x_1 + x_2), 0, 0, 0, 0, 0] \\
X*Y^3 &= Z_3 = 2^{0.5} [(x_0 - x_1 + x_2 - x_3 + x_4 - x_5 + x_6 - x_7), 0, (x_0 - x_1 + x_2 - x_3 + x_4 + x_5 - x_6 + x_7), 0, 0, 0, 0, 0] \\
X*Y^4 &= Z_4 = [(x_0 - x_1 - x_2 + x_3), 0, (x_0 - x_1 - x_2), 0, 0, 0, 0, 0] \\
X*Y^5 &= Z_5 = 2^{0.5} [(x_0 - x_1 + x_2 + x_3 - x_4 + x_5 - x_6 + x_7), 0, (x_0 - x_1 + x_2 + x_3 + x_4 - x_5 + x_6 - x_7), 0, 0, 0, 0] \\
X*Y^6 &= Z_6 = [(x_2 + x_3 + x_4 + x_5), 0, (x_2 - x_3 - x_4), 0, 0, 0, 0, 0] \\
X*Y^7 &= Z_7 = 2^{0.5} [(x_0 + x_1 + x_2 + x_3 - x_4 - x_5 + x_6 + x_7), 0, (x_2 - x_1 - x_2 + x_3 - x_4 + x_5 + x_6 - x_7), 0, 0, 0, 0] \\
X*Y^8 &= Z_8 = [(x_0 + x_1 - x_2 - x_3), 0, (x_2 + x_3 - x_4), 0, 0, 0, 0, 0] \\
X*Y^9 &= Z_9 = 2^{0.5} [(x_0 + x_1 - x_2 - x_3 + x_4 + x_5 - x_6 - x_7), 0, (x_0 + x_1 + x_2 - x_3 + x_4 - x_5 - x_6 - x_7), 0, 0, 0, 0, 0]
\end{align*}
\]

Figure 18. The illustration of a selective control (or selective coding) of cyclic changes of coordinates of a vector \( X*Y^n = Z_n \) where \( Y^n \) is the cyclic group with its period 8.

One can see from Figure 18 that only coordinates \( z_0 \) and \( z_2 \) have cyclic changes in this set of new vectors \( Z_n \), all other coordinates are equal to zero. In other words, all cycles are realized on a 2-dimensional plane \( (z_0, z_2) \) inside the 8-dimensional space. If one uses another cyclic group of operators, for example, \( (2^{0.5} (s_1 + s_3))^n \) \( (s_1 \) and \( s_2 \) are from Figure 7) then the same initial vector \( X = [x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] \) will be transformed into a cyclic set of vectors in another 2-dimensional plane \( (z_1, z_3) \) of the same 8-dimensional space. One should conclude that, in this model approach, the same initial information in a form of a multidimensional vector \( X \) could generate a few cyclic processes in different planes of appropriate multidimensional space by means of using cyclic operators of the described type. In other words, we have here a multi-purpose using of vector information due to such operators (for instance, this informational vector can represent a fragment of a nucleotide sequence that can be used to organize many cyclic processes in different planes or subspace of a phase space of genetic phenomena).

In the proposed model approach, one more benefit is that different cyclic processes of such cooperative ensemble can be easy coordinated and synchronized including an
One technical remark is needed here. If we use a cyclic operator on the basis of the “column projectors”, then a vector X should be multiplied by the matrix on the right in accordance with the sample: \([x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] \ast (2^{-0.5 \ast (s_0 + s_2)})^n\). But if we use a cyclic operator on the basis of the “row projectors” (for instance, \(v_0\) and \(v_2\) from Figure 8) then a vector X should be multiplied by the matrix on the left in accordance with the following sample: \((2^{-0.5 \ast (v_0 + v_2)})^n \ast [x_0; x_1; x_2; x_3; x_4; x_5; x_6; x_7]\).

Below we will describe extensions of the genetic (4*)-matrices \(R_4\) and \(H_4\) (Figure 1) into \(2^n \ast 2^n\)-matrices every of which consists of \(2^n\) “column projectors” (or \(2^n\) “row projectors”); summation of projectors from this expanded set leads to new cyclic groups, etc. by analogy with the described cases (see Figures 4, 9, 11, 16). It gives a great number of cyclic groups of operators with similar properties of the selective control (or selective coding) of cyclic changes of coordinates of \(2^n\)-dimensional vectors. These numerous cyclic groups are useful to simulate big cooperative ensembles of cyclic processes, for instance, an ensemble of cyclic motions of legs, hands and separate muscles during different gaits (walking, running, etc.) simultaneously with heartbeats, breathing cycles, metabolic cycles, etc. Such models and their practical applications are created in the author's laboratory. The problem of inherited ensembles of biological cycles are closely linked to the fundamental problems of the biological clock and time, aging, etc. Taking into account results, which were obtained in "matrix genetics", the author puts forward "a biological concept of projectors", which interprets the living body as a colony of projection operators.

It should be noted that in a case of a cyclic group of vector transformations with a period 8 (for example, in the case of the cyclic group \((2^{-0.5 \ast (s_0 + s_2)})^n\) that has only 8 discrete stages inside one cycle, one can enlarge a quantity of stages in “k” times by changing of the power in a form \(n/k\): the cyclic group \((2^{-0.5 \ast (s_0 + s_2)})^{n/k}\) has \(k\ast 8\) stages inside one cycle (here "n" and "k" - integer positive numbers). The more value of "k", the less discretization of the cycle and the more smooth (uninterrupted) type of this cyclic process.

It can be added that many gaits (which are based on cyclic movements of limbs and corresponding muscle actuators) have genetically inherited character. So, newborn turtles and crocodiles, when they hatched from eggs, crawl with quite coordinated movements to water without any training from anybody; a newborn foal, after a bit time, begins to walk and run; centipedes crawl by means of coordinated movements of a great number of their legs (this number sometimes reaches up to 750) on the basis of inherited algorithms of control of legs. One should emphasize that, in the previous history, gaits and locomotion algorithms were studied in biomechanics of movements without any connection with the structures of genetic coding and with inheritance of unified control algorithms. The projection operations are associated with many kinds of movements and planned actions of our body to achieve the goal by the shortest path: for example, sending a billiard ball in the goal, we use a projection operation; directing a finger to the button of computer or piano, we make a projection action, etc. In other words, the concept of projection operators can be additionally used to simulate a broad class of such biomechanical actions.

Subject of genetically inherited ability of coordinating movements of body parts is connected with fundamental problems of congenital knowledge about surrounding space and of physiological foundations of geometry. Various researches have long put forward ideas about the importance of kinematic organization of body and its movements in the genesis of spatial representations of the individual. For example, H. Poincare has put these ideas into
the foundation of his teachings about the physiological foundations of geometry and about the origin of spatial representations in individuum.

According to Poincare, the concept of space and geometry arises from an individual on the basis of kinematic organization of his body with using characterizations of positions and movements of body parts relative to each other, i.e., in the kinematic organization of the body is something that precedes the concept of space [Poincare, 1913]. Evolutionary development of the whole apparatus of kinematic activity of our body has provided a coherence of this apparatus with realities of the physical world. Because of this, each newborn organism receives adequate spatial representations not only through personal contact during ontogeny with the objects of the surrounding world, but also at the expense of achievements of previous generations enshrined in the apparatus of body movements in the phylogenesis. According to Poincare, for organism, which is absolutely immobile, spatial and geometric concepts are excluded. «To localize an object simply means to represent to oneself the movements that would be necessary to reach it. I will explain myself. It is not a question of representing the movements themselves in space, but solely of representing to oneself the muscular sensations which accompany these movements and which do not presuppose the preexistence of the notion of space. [Poincare, 1913, p. 247]. «I have just said that it is to our own body that we naturally refer exterior objects; that we carry about everywhere with us a system of axes to which we refer all the points of space and that this system of axes seems to be invariably bound to our body. It should be noticed that rigorously we could not speak of axes invariably bound to the body unless the different parts of this body were themselves invariably bound to one another. As this is not the case, we ought, before referring exterior objects to these fictitious axes, to suppose our body brought back to the initial attitude” [Poincare, 1913, p. 247]. «We should therefore not have been able to construct space if we had not had an instrument to measure it; well, this instrument to which we relate everything, which we use instinctively, it is our own body. It is in relation to our body that we place exterior objects, and the only spatial relations of these objects that we can represent are their relations to our body. It is our body which serves us, so to speak, as system of axes of coordinates» [Poincare, 1913, p. 418]. In times of Poincare science did not know about the genetic code, but from the modern point of view these thoughts by Poincare testify in favor the importance of the structural organization of the genetic system for physiological foundations of geometry and innate notions of space, which are connected with inherited apparatus and algorithms of body movements. And they are in tune with the results of matrix genetics, which are presented in our paper.

Modern physiology makes a significant addition to the teachings of the Poincare about an innate relationship of body and spatial representations, claiming an existence of a priori notions about our body shell. This statement is due to the study of the so-called phantom sensations in disabled: a special sense of the presence of natural parts of the body, which are absent in reality. It was found [Vetter, Weinstein, 1967; Weinstein, Sersen, 1961] that phantom sensations occur not only in cases of disabled with amputees, but also in people with congenital absence of limbs. Hence, the notion of the individual scheme of our body is not conditioned by our experiences, but has an innate character.

Additional materials relating to innate spatial representations, including the concept of B. Russell [Russel, 1956] about an innate character of ideas of projective geometry for each
person, as well as an overview of works E. Schroedinger and other researchers about the geometry of spaces of visual perception, can be found in the book [Petoukhov, 1981].

We note here that although the concept of space is the primary concept for most physical theories, one can develop a meaningful theory in theoretical physics, in which it serves as only one of secondary notions, which are deduced from primary bases of a numeric system of a discrete character. We mean the "binary geometrophysics" [Vladimirov, 2008], ideas of which generate some associations with the ability of animal organisms (initially endowed with discrete molecular genetic information) to receive spatial representations and to create spatial movements on the basis of this primary information of discrete character.

5. ABOUT A DIRECTION OF ROTATION OF VECTORS UNDER INFLUENCE OF THE CYCLIC GROUPS OF THE OPERATORS

In configurations and functions of biological objects frequently one direction of rotation is preferable (it concerns the famous problem of biological dissymmetry). Taking this into account, it is interesting what one can say about directions of rotation of 4-dimensional and 8-dimensional vectors under influence of the cyclic groups described in previous sections. Figure 19 gives answer and shows directions of cyclic rotation of vectors $[x_0, x_1, x_2, x_3]$ and $[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ under enlarging "n" (i≠j; all these cyclic operators correspond to green cells in tables on Figure 19, they are based on summation of pairs of the projectors of the Rademacher matrices $R_4$ and $R_8$ from Figure 1).

Figure 19. In addition to Figures 4 and 9, the tables show directions of rotations of 4-dimensional and 8-dimensional vectors under influence of the cyclic groups of operators, which correspond to green cells and which are based on summation of pairs of the "column projectors" (on the left, see Figures 2, 4, 7, 9) and of the "row projectors" (on the right, see Figures 2, 4, 8, 9) of the Rademacher matrices $R_4$ and $R_8$ (Figure 1). The symbol $\bigcirc$ means counter-clockwise rotation, the symbol $\bigodot$ means clockwise rotation.

Figure 20 shows directions of cyclic rotation of vectors $[x_0, x_1, x_2, x_3]$, $[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ under enlarging "n" (i≠j; all these cyclic operators correspond to green cells in tables on Figure 19, they are based on summation of pairs of the projectors of the Rademacher matrices $R_4$ and $R_8$ from Figure 1).
[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] under enlarging “n” (i≠j; all these operators correspond to green cells in tables on Figure 19, they are based on summation of pairs of projectors of the Rademacher matrices R_4 and R_8 from Figure 1).

Figure 20. In addition to Figures 11 and 16, the tables show directions of rotations of 4-dimensional and 8-dimensional vectors under influence of the cyclic groups of operators, which correspond to green cells and which are based on summation of pairs of the “column projectors” (on the left, see Figures 10, 11, 14, 16) and of the “row projectors” (on the right, see Figures 10, 11, 15, 16) of the Hadamard matrices H_4 and H_8 (Figure 1). The symbol $\varnothing$ means counter-clockwise rotation, the symbol $\circ$ means clockwise rotation.

Each of tables on Figures 19 and 20 contains completely different (asymmetrical) number of rotations in the clockwise and counterclockwise. These facts give evidences in favor of an idea that living matter at its basic level of genetic information has certain informational reasons to provide dissymmetry of inherited biological structures and processes. Taking this into account, the author thinks about a possibility of informational reasons for biological dis-symmetry. Here one can remind for a comparison that usually scientists look for reasons of biological dis-symmetry in physical or chemical sciences but not in informatic science.

6. HAMILTON’S QUATERNIONS, COCKLE’S SPLIT-QUATERNIONS, THEIR EXTENSIONS AND PROJECTION OPERATORS

In previous sections we described cases of summation of pairs of the oblique projectors. Now let us consider cases of summation of 4 of these projectors and cases of summation of 8 of these projectors.

The matrix H_4 (Figure 1) is sum of the four “column projectors” or the four “row projector” (Figure 10). But H_4 has also another decomposition in a form of four sparse matrices H_{40}, H_{41}, H_{42} and H_{43} (Figure 21). This set is closed in relation to multiplication and it defines their multiplication table (Figure 21, bottom level) that is identical to the known multiplication table of quaternions by Hamilton. From this point of view, the matrix H_4 is the
quaternion by Hamilton with unit coordinates. (Such type of decompositions is termed a dyadic-shift decomposition because it corresponds to structures of matrices of dyadic shifts, well known in technology of signals processing [Ahmed, Rao, 1975]).

$$H_4 = H_{40} + H_{41} + H_{42} + H_{43} =$$  
\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} + 
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 \\
\end{array} + 
\begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array} + 
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Figure 21. The dyadic-shift decomposition of the $(4*4)$-matrix $H_4$ (from Figure 1) gives the set of 4 sparse matrices $H_{40}$, $H_{41}$, $H_{42}$ and $H_{43}$, which corresponds to the multiplication table of quaternions by Hamilton (bottom row). The matrix $H_{40}$ is identity matrix.

Here one can mention that Hamilton quaternions are closely related to the Pauli matrices, the theory of the electromagnetic field (Maxwell wrote his equation on the language of quaternions Hamilton), the special theory of relativity, the theory of spins, quantum theory of chemical valency, etc. In the twentieth century thousands of works were devoted to quaternions in physics [http://arxiv.org/abs/math-ph/0511092]. Now Hamilton quaternions are manifested in the genetic code system. Our scientific direction - "matrix genetics" - has led to the discovery of an important bridge among physics, biology and computer science for their mutual enrichment. In our studies, we have received a new example of the effectiveness of mathematics: abstract mathematical structures, which have been derived by mathematicians at the tip of the pen 160 years ago, are embodied long ago in the information basis of living matter - the system of genetic coding. The mathematical structures, which are discovered by mathematicians in a result of painful reflections (like Hamilton, who has wasted 10 years of continuous thought to reveal his quaternions), are already represented in the genetic coding system.

Let us turn now to the $(8*8)$-matrix $H_8$ (Figure 1) that can be represented as sum of two matrices $HL_8 = u_0 + u_2 + u_4 + u_6$ and $HR_8 = u_1 + u_3 + u_5 + u_7$ (Figure 22). Here $u_0$, $u_1$, ..., $u_7$ are the «column projectors» from Figure 14.

$$H_8 = HL_8 + HR_8 =$$  
\[
\begin{array}{cccc}
1 & 0 & 1 & 0 \\
1 & 0 & 1 & -1 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 \\
\end{array} + 
\begin{array}{cccc}
0 & -1 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & -1 & 0 & 1 \\
0 & -1 & 0 & 1 \\
\end{array} + 
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 \\
\end{array} + 
\begin{array}{cccc}
0 & -1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
\end{array}
\]

Figure 22. The representation of the Hadamard matrix $H_8$ (from Figure 1) as sum of two
sparse matrices $H_{L_8}$ and $H_{R_8}$

Figure 23 shows a decomposition of the matrix $H_{L_8}$ (from Figure 22) as a sum of 4 matrices: $H_{L_8} = H_{L_80} + H_{L_81} + H_{L_82} + H_{L_83}$. The set of matrices $H_{L_80}$, $H_{L_81}$, $H_{L_82}$ and $H_{L_83}$ is closed in relation to multiplication and it defines the multiplication table that is identical to the multiplication table of quaternions by Hamilton. General expression for quaternions in this case can be written as $Q_{L} = a_0^*H_{L_80} + a_1^*H_{L_81} + a_2^*H_{L_82} + a_3^*H_{L_83}$, where $a_0$, $a_1$, $a_2$, $a_3$ are real numbers. From this point of view, the $(8*8)$-genomatrix $H_{L_8}$ is the 4-parametric quaternion by Hamilton with unit coordinates.

\[
H_{L_8} = H_{L_80} + H_{L_81} + H_{L_82} + H_{L_83} = \\
\begin{array}{cccc}
1 & 0 & 1 & 0 \\
1 & 0 & 1 & -1 \\
-1 & 1 & 0 & 1 \\
-1 & 1 & -1 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{array}
\]

\[
+ \\
\begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
+ \\
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
H_{L_80} \quad H_{L_81} \quad H_{L_82} \quad H_{L_83}
\]

Figure 23. Upper rows: the decomposition of the matrix $H_{L_8}$ (from Figure 22) as sum of 4 matrices: $H_{L_8} = H_{L_80} + H_{L_81} + H_{L_82} + H_{L_83}$. Bottom row: the multiplication table of these 4 matrices $H_{L_80}$, $H_{L_81}$, $H_{L_82}$ and $H_{L_83}$, which is identical to the multiplication table of quaternions by Hamilton. The matrix $H_{L_80}$ represents the real unit for this matrix set.

The similar situation holds true for the matrix $H_{R_8}$ (from Figure 22). Figure 24 shows a decomposition of the matrix $H_{R_8}$ as a sum of 4 matrices: $H_{R_8} = H_{R_80} + H_{R_81} + H_{R_82} + H_{R_83}$. The set of matrices $H_{R_80}$, $H_{R_81}$, $H_{R_82}$ and $H_{R_83}$ is closed in relation to multiplication and it defines the multiplication table that is identical to the same multiplication table of quaternions by Hamilton. General expression of quaternions in this case can be written as $Q_{R} = a_0^*H_{R_80} + a_1^*H_{R_81} + a_2^*H_{R_82} + a_3^*H_{R_83}$, where $a_0$, $a_1$, $a_2$, $a_3$ are real numbers. From this point of view, the $(8*8)$-genomatrix $H_{R_8}$ is the quaternion by Hamilton with unit coordinates.
\[ H_{R8} = H_{R80} + H_{R81} + H_{R82} + H_{R83} = \]

\[
\begin{array}{cccc}
0 & -1 & 0 & 10 & 1 \ 0 & 0 & 0 & 0 & 1 \ 0 & 1 & 0 & 10 & 1 \ 0 & 0 & 1 & 0 & 0 \ 0 & 1 & 0 & 10 & 10 \ 0 & 0 & 1 & 0 & 0 \ 0 & 1 & 0 & 10 & 10 \ 0 & 0 & 1 & 0 & 0
\end{array}
\]

\[
\begin{array}{cccc}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
\]

\[
+ \begin{array}{cccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
\]

\[
+ \begin{array}{cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 0 & 0 & -1 \ 0 & 0 & 0 & 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
\]

| H_{R80} | H_{R81} | H_{R82} | H_{R83} |
|---------|---------|---------|---------|
| H_{R80} | H_{R80} | H_{R81} | H_{R82} | H_{R83} |
| H_{R81} | H_{R81} | -H_{R80} | H_{R83} | -H_{R82} |
| H_{R82} | H_{R82} | -H_{R83} | -H_{R80} | H_{R81} |
| H_{R83} | H_{R83} | H_{R82} | -H_{R81} | -H_{R80} |

Figure 24. upper rows: the decomposition of the matrix H_{R8} (from Figure 22) as sum of 4 matrices: H_{R8R} = H_{R80} + H_{R81} + H_{R82} + H_{R83}. Bottom row: the multiplication table of these 4 matrices H_{R80}, H_{R81}, H_{R82} and H_{R83}, which is identical to the multiplication table of quaternions by Hamilton. H_{R80} represents the real unit for this matrix set.

The initial (8*8)-matrix H_8 (Figure 1) can be also decomposed in another way on the base of dyadic-shift decomposition. Figure 25 shows such dyadic-shift decomposition H_8 = H_{80}H_{81}H_{82}H_{83}H_{84}H_{85}H_{86}H_{87}, when 8 sparse matrices H_{80}, H_{81}, H_{82}, H_{83}, H_{84}, H_{85}, H_{86}, H_{87} arise (H_{80} is identity matrix). The set H_{80}, H_{81}, H_{82}, H_{83}, H_{84}, H_{85}, H_{86}, H_{87} is closed in relation to multiplication and it defines the multiplication table on Figure 25. This multiplication table is identical to the multiplication table of 8-dimensional hypercomplex numbers that are termed as biquaternions by Hamilton (or Hamiltons’ quaternions over the field of complex numbers). General expression for biquaternions in this case can be written as Q_8 = a_0H_{80} + a_1H_{81} + a_2H_{82} + a_3H_{83} + a_4H_{84} + a_5H_{85} + a_6H_{86} + a_7H_{87}, where a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7 are real numbers. From this point of view, the (8*8)-genomatrix H_8 is Hamilton’s biquaternion with unit coordinates.
\[ H_8 = H_{80} + H_{81} + H_{82} + H_{83} + H_{84} + H_{85} + H_{86} + H_{87} = \]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} +
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}
\]

Figure 25. Upper rows: the decomposition of the matrix \( H_8 \) (from Figure 1) as sum of 8 matrices: \( H_8 = H_{80} + H_{81} + H_{82} + H_{83} + H_{84} + H_{85} + H_{86} + H_{87} \). Bottom row: the multiplication table of these 8 matrices \( H_{80}, H_{81}, H_{82}, H_{83}, H_{84}, H_{85}, H_{86}, H_{87} \), which is identical to the multiplication table of biquaternions by Hamilton (or Hamiltons’ quaternions over the field of complex numbers). \( H_{80} \) is identity matrix.

One can analyze the Rademacher genomatrices \( R_4 \) and \( R_8 \) (From Figure 1) by a similar way [Petoukhov, 2012b]). In particular, in this case the following results arise:

- The Rademacher (4*4)-matrix \( R_4 \) represents split-quaternion by J.Cockle with unit coordinates (http://en.wikipedia.org/wiki/Split-quaternion) in the case of its dyadic-shift decomposition;
- The Rademacher (8*8)-matrix \( R_8 \) represents bispit-quaternion by J.Cockle with unit coordinates in the case of its dyadic-shift decomposition;
- If the Rademacher (4*4)-matrix \( R_4 \) is represented as sum of two sparse matrices \((c_0+c_2) + (c_1+c_3)\) (here \( c_0, c_1, c_2, c_3 \) are the column projectors from Figure 2), then the matrix \( R_4 \) is sum of two hyperbolic numbers with unit coordinates because each of summands \((c_0+c_2)\) and \((c_1+c_3)\) is hyperbolic number with unit coordinates. A similar is true for the case of the “row projectors” \( r_0, r_1, r_2, r_3 \) from Figure 2.

Now let us pay a special attention to the Rademacher (8*8)-matrix \( R_8 \) as a sum of the following two sparse matrices \( RL_8 \) and \( RR_8 \), the first of which is a sum of 4 projectors with even indexes \( S_0, S_2, S_4, S_6 \) and the second of which is a sum of 4 projectors with odd indexes \( S_1, S_3, S_5, S_7 \): \( R_8 = (s_0+s_2+s_4+s_6) + (s_1+s_3+s_5+s_7) = RL_8 + RR_8 \) (here \( s_0, s_1, ..., s_7 \) are the column projectors from Figure 7). Below this decomposition will be useful for analysis of a
correspondence between 64 triplets and 20 amino acids with stop-codons.

\[
R_8 = RL_8 + RR_8 = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\
-1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\
-1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 \\
-1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Figure 26. The decomposition of the Rademacher (8*8)-genomatrix R_8 from Figure 1

Each of these sparse matrices RL_8 and RR_8 can be decomposed into a set of 4 sparse matrices: RL_8 = RL_{80} + RL_{81} + RL_{82} + RL_{83} and RR_8 = RR_{80} + RR_{81} + RR_{82} + RR_{83} (Figures 27 and 28). The first set of matrices RL_{80}, RL_{81}, RL_{82}, RL_{83} is closed relative to multiplication and it defines a known multiplication table of split-quaternions by J. Cockle (http://en.wikipedia.org/wiki/Split-quaternion) on Figure 27. The second set of matrices RR_{80}, RR_{81}, RR_{82}, RR_{83} is also closed relative to multiplication and it defines the same multiplication table of split-quaternions by J. Cockle (Figure 28). Consequently, each of matrices RL_8 and RR_8 is split-quatnion by Cockle with unit coordinates.

\[
RL_8 = RL_{80} + RL_{81} + RL_{82} + RL_{83} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
RR_8 = RR_{80} + RR_{81} + RR_{82} + RR_{83} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Figure 27. The decomposition of the matrix RL_8 from Figure 26 into the set of 4 matrices RL_{80}, RL_{81}, RL_{82}, RL_{83}, which defines the known multiplication table of split-quaternions by J. Cockle (http://en.wikipedia.org/wiki/Split-quaternion)
Figure 28. The decomposition of the matrix $RR_8$ from Figure 26 into the set of 4 matrices $RR_80$, $RR_81$, $RR_82$, $RR_83$, which defines the same multiplication table of split-quaternions by J. Cockle (http://en.wikipedia.org/wiki/Split-quaternion)
7. GENETIC MATRICES AS SUMS OF TENSOR PRODUCTS OF OBLIQUE (2*2)-PROJECTORS. EXTENSIONS OF GENETIC MATRICES INTO (2^N*2^N)-MATRICES

The Rademacher matrices \( R_4 \) and \( R_8 \) and also Hadamard matrices \( H_4 \) and \( H_8 \) (Figure 1) are interconnected by means of the following expressions:

\[
R_4 \otimes [1 \ 1; 1 \ 1] = R_8, \quad H_4 \otimes [1 -1; 1 \ 1] = H_8
\]

where \( \otimes \) means tensor multiplication; the matrix \([1 \ 1; 1 \ 1]\) is a traditional \((2*2)\)-matrix representation of hyperbolic number with unit coordinates; the matrix \([1 -1; 1 \ 1]\) is a traditional \((2*2)\)-matrix representation of complex number with unit coordinates.

The following extensions of the expressions (1) lead to \((2^n*2^n)\)-matrices \( R_K \) and \( H_K \) (where \( K=2^n, n = 4, 5, 6, \ldots; (n-2) \) means a tensor power):

\[
R_4 \otimes [1 \ 1; 1 \ 1]^{(n-2)} = R_K, \quad H_4 \otimes [1 -1; 1 \ 1]^{(n-2)} = H_K
\]

In this algorithmic way we get a great set of \((2^n*2^n)\)-matrices \( R_K \) and \( H_K \), each of which can be represented as a sum of \( 2^n \) «column projectors» (or \( 2^n \) «row projectors») by analogy with cases described above. Summations of these new «column projectors» (and also «row projectors») in different combinations (in pairs, in fours, in eights, etc.) give many new operators, exponentiation of which generates a great number of cyclic groups and other kind of operators. They also give many new representations of complex numbers, hyperbolic numbers, Hamilton’s quaternions, split-quaternions and their extensions in a form of \((2^n*2^n)\)-matrices that correspond to \( 2^n \)-dimensional spaces. These new operators possess many similar properties, including a selective control (or coding) of different subspaces in \( 2^n \)-dimensional space, in analogy with operators described in previous sections.

Why one can declare that each of matrices in the «column decomposition» (or in the «row decomposition») of any of matrices \( R_K \) and \( H_K \) in the expressions (2) is a projection operator? It can be declared on basis of the following simple theorem, taking into account that main diagonals of all matrices \( R_K \) and \( H_K \) contain only entries +1.

**Theorem:** any sparse square matrix \( P \), which contains only a single non-zero column or a single non-zero row and which has its entry +1 on the main diagonal, is a projection operator (it satisfies the criterion \( P^2=P \)).
**Proof.** When multiplying two matrices $|A_{ik}|$ and $|B_{kj}|$, the elements of the rows in the first matrix are multiplied with corresponding columns in the second matrix to receive the resulting matrix $(AB)_{ij} = \sum_{k=1}^{m} A_{ik} * B_{kj}$ (http://en.wikipedia.org/wiki/Matrix_multiplication). Let us consider a case of a square matrix $|P_{ij}|$ (here $i,j = 1, 2, \ldots, m$) with only a single non-zero column $P_{is} \neq 0$, which is numerated by an index “s” and which contains +1 in its cell on the main diagonal of this matrix: $P_{ss} = 1$. It means that all entries $P_{ik} = 0$ if $k \neq s$. The second degree of this matrix gives a square matrix:

$$\sum_{k=1}^{m} P_{ik} * P_{kj} = \sum_{k=1}^{m} P_{ik} * P_{ks} \quad (3)$$

But among all $P_{ik}$ only one column differs from zero: $P_{is} \neq 0$. In the equation (3), $P_{is}$ corresponds to the second factor $P_{ss} = 1$. By these reasons we have $P_{is} * P_{ss} = P_{is}$ for the equation (3). So the sparse square matrix $|P_{ij}|^2$ contains only the same single non-zero column $P_{is}$ like as the matrix $|P_{ij}|$. Consequently $|P_{ij}|^2 = |P_{ij}|$: in other words, this matrix $|P_{ij}|$ is a projection operator, Q.E.D. The case of similar representations of such $2^n*2^n$-matrices on basis of a sum of “row projectors” has its proof by analogy.

This theorem allows making the following conclusion about any variant of matrix presentations of complex numbers, hyperbolic numbers and their extensions into $2^n$-dimensional numerical systems (including Hamilton's quaternions and biquaternions, split-quaternions and bisplit-quaternions by Cockle, etc.): if the real part of such $2^n$-dimensional number is equal to +1, then its matrix presentation is a sum of $2^n$ «column projectors» (and «row projectors»). It is provided by the fact that real parts of such multidimensional numerical systems are represented by matrix diagonal that contains only entries +1. Figure 30 shows an example of one of matrix presentations of Hamilton quaternions in a case when their real parts are equal to +1.

| 1 | b | c | d |
|---|---|---|---|
| -b | 1 | -d | c |
| -c | d | 1 | -b |
| -d | -c | b | 1 |

| 1 0 0 0 | 0 b 0 0 | 0 0 c 0 | 0 0 0 d |
|---|---|---|---|
| -b 0 0 0 | + 0 1 0 0 | + 0 0 -d 0 | + 0 0 0 c |
| -c 0 0 0 | 0 d 0 0 | 0 0 1 0 | 0 0 0 -b |
| -d 0 0 0 | 0 -c 0 0 | 0 0 b 0 | 0 0 0 1 |

Figure 30. The "column decomposition" of the classical (4*4)-matrix representation of Hamilton's quaternions (http://en.wikipedia.org/wiki/Quaternion#Matrix_representations) in cases when their real parts are equal to +1. Each of 4 matrices (on the right) is a projection operator in accordance with the described theorem. Here b, c, d are real numbers.

So, many kinds of hypercomplex numbers are based on sums of projectors. In this sense the notion "projectors" can be considered as more fundemental than the notion "hypercomplex numbers" of the mentioned types. Many of these hypercomplex numbers are applied widely in different fields of science: physics, chemistry, informatics, etc. Awareness of the fact that these systems of hypercomplex numbers are based on sums of projectors may help in a rethinking of existing theories and in developing new theories in the field of mathematical natural science. In particularly, it concerns Hamilton's quaternions. For example, Maxwell has used them in creation of his equations of electro-magnetic field. Could one develop an alternative description and development of the theory of electro-magnetic field on basis of sums of appropriate projectors? It is one of many open questions in theoretical applications of projectors.
Now we show that each of genetic Rademacher and Hadamard matrices (including $R_2$, $R_8$, $H_4$, $H_8$ from Figure 1 and their extensions into $2^n * 2^n$-matrices $R_K$ and $H_K$ in expressions (2)) can be expressed as sums and tensor multiplications of four $(2*2)$-matrices of «column projectors» (or of analogical «row projectors»). Figure 31 shows these 4 basic $(2*2)$-projectors, which are marked by 4 different colours for visibility, and some examples of expressions of a few Rademacher and Hadamard matrices by means of their using.

\[
\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
R_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
H_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

Figure 31. Examples of using the 4 basic $(2*2)$-projectors (upper level) to express $(2*2)$-matrix representations of hyperbolic number and of complex number with unit coordinates (the second level) and to express the Rademacher $(4*4)$-matrix $R_4$ and the Hadamard $(4*4)$-matrix $H_4$ from Figure 1 (two lower levels)

One can also note that every of the genetic “column $(2^n * 2^n)$-projectors” and the “row $(2^n * 2^n)$-projectors” can be expressed by means of tensor multiplications of appropriate $(2*2)$-projectors from their basic set of the 4 projectors (Figure 31, upper level). It means that cases of 2-dimensional spaces can be considered as basic in this model approach. It is interesting because of the known fact that namely 2-dimensional sub-spaces play a fundamental role in morphological organization and development of living bodies (see for example about a fundamental role of primary tissue layers or primary germ layers in http://en.wikipedia.org/wiki/Germ_layer; in accordance with germ layer theory, for example, all different organs of human bodies develop from one of the 3 germ layers).

8. AN APPLICATION OF OBLIQUE PROJECTORS TO SIMULATE ENSEMBLES OF PHYLLOTAXIS PATTERNS IN LIVING BODIES

In the field of mathematical biology, phyllotaxis phenomena are one of the most known [Adler, Barabe, Jean, 1997; Jean, 1995; http://www.goldenmuseum.com/0604Phyllotaxis_engl.html]. Usually phyllotaxis laws are described as those inherited spiral-like dislocations of leaves and some other parts of plants, which are connected with Fibonacci numbers. But the similar phyllotaxis laws dictate also inherited configurations of some biological molecules, parts of animal bodies, etc. (see, for example, a review in [Jean, 1995]). In other words, phyllotaxis laws appear in inherited morphological structures at very different levels and branches of biological evolution. Figure 32 shows a few examples of phyllotaxis spirals.
Figure 32. Examples of phyllotaxis patterns: 1) sunflower; 2) alloe (from http://radiolus.com/index.php/nepoznamnoe/255-paradoksy-posledovatelnosti-chisel-fibonachchi); 3) a pine cone (from http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html); 4) a seashell (from http://www.eb.tuebingen.mpg.de/?id=476); 5) a fractal vegetable Romanesco Broccoli (Brassica oleracea) (from http://egregores.blogspot.ru/2010/12/extremely-cool-natural-fractals.html); 6) a spruce with cones (from http://foto.rambler.ru/users/nadezhda-rodnaja/albums/53780408/photo/4e9c302c-1a1a-3e9d-c61e-db70c258d714/).

On Figure 32, images 5 and 6 illustrate that a whole organism can contain many parts with similar phyllotaxis patterns in each (like as a spruce with many phyllotaxis cones). For physicist or mathematician is natural to think that such organism can be modelled as a multidimensional phase space (or a configuration space) with an appropriate number of similar subspaces, each of which receives the same phyllotaxis pattern due to a selective control of subspaces or their selective genetic coding. Such selective control or coding in this phase space should be based on an appropriate system of operators (about a phase space see http://en.wikipedia.org/wiki/Phase_space).

Our model approach allows such modelling due to a discovered system of operators based on described sums of oblique projectors (including $2^n \mathbb{R}^n$-dimensional matrices $R_k$ and $H_k$ in expressions (2)), which have such properties of a selective control (or coding) of subspaces of $2^n$-dimensional space. In other words, we propose an approach to model ensembles of phyllotaxis patterns (or other patterns and processes) inside a multidimensional phase space that represents a whole organism. The described system of operators for a
selective control or coding can be conditionally named briefly as a «genetic system of operators» (or more briefly, «G-system of operators). As we can judge, in the field of phyllotaxis study, other authors didn’t simulate such ensembles of phyllotaxis patterns though many different models of separate patterns (without their ensembles in a joint phase space) exist. In addition, known models of inherited phyllotaxis patterns don’t associate them with structural properties of the genetic coding system in contrast to our genetic approach. Let us explain our model approach to phyllotaxis phenomena in more details.

It is known that classical phyllotaxis patterns arise in the result of iterative rotations of initial object approximately on an angle $137^\circ$ with a simultaneous increase of its distance from the center of the phyllotaxis pattern. On a complex plane such iterative operations can be simulated by means of iterative multiplication of an initial vector (or a point) with appropriate complex number $z = x + iy$, which provides such angle of rotation (due to its appropriate argument) and increase (due to its appropriate modulus) in accordance with known properties of complex numbers. The described G-system of operators, which contains many variants of sparse $2^n \times 2^n$-matrix presentations of complex numbers on basis of sums of some genetic projectors (from "column decompositions" of $2^n \times 2^n$-matrices $R_K$ and $H_K$ in expressions (2)), allows generating many phyllotaxis patterns, each of which belong to its own 2-dimensional plane inside a whole phase space. In these cases, each of phyllotaxis patterns in a separate phase plane can have its own degree of maturation (or development) and its own type of a phyllotaxis picture; it depends on a kind of complex number, which is chosen for its iterative generating.

This model approach does not pretend to a new explanation for existence of Fibonacci numbers in phyllotaxis patterns. But this model approach gives abilities to simulate bunches of phyllotaxis patterns in separate organisms. Concerning to Fibonacci or Luca numbers in phyllotaxis laws, one should remind here that "the phyllotaxis rules ... cannot be taken as applying to all circumstances, like a law of nature. Rather, in the words of the famous Canadian mathematician Coxeter, they are 'only a fascinatingly prevalent tendency" (http://goldenratiomyth.weebly.com/phyllotaxis-the-fibonacci-sequence-in-nature.html). One can think that the role of iterative operations in living nature is much more important than particular realizations of Fibonacci or Luca numbers.

To simulate an ensemble of phyllotaxis 3d-patterns (an ensemble of many cones of a spruce, etc.), each of which belongs to a separate subspace of a whole $2^n \times 2^n$-dimensional phase space, an iterative application of Hamilton's quaternions can be used in their $2^n \times 2^n$-matrix forms of presentation in the described G-system of operators.

In addition the author reminds here about cyclic groups on basis of Hamilton’s quaternion and biquaternion with unit coordinates: these cyclic groups allow simulating some heritable biological phenomena including color perception, properties of which correspond to the Newton’s color circle (see [Petoukhov, 2011b] and Section 17 in [Petoukhov, 2012a]). Using Hamilton’s quaternions and biquaternions as $2^\times 2^n$-operators from the described G-system allows simulating some inherited ensembles of biological patterns including some inherited ensembles of color patterns and color changes of biological bodies.

9. HYPERBOLIC NUMBERS, GENETIC PROJECTORS AND THE WEBER-FECHNER LAW OF PSYCHOPHYSICS

This Section is devoted to hyperbolic numbers in separate planes of $2^N$-dimensional spaces, their connections with some genetic projectors (from Figures 2, 4-6, 8 and 9) and
with inherited psychophysical phenomena. The name «hyperbolic» for this kind of 2-dimensional numbers is traditionally used because of their connection with hyperbolas and hyperbolic rotations (http://en.wikipedia.org/wiki/Split-complex_number, http://www.physicsinsights.org/hyperbolic_rotations.html). Hyperbolic numbers have properties corresponding to Lorentz group of two-dimensional Special Relativity (www.springer.com/.../9783642179761-c1.pdf?, http://garretstar.com/secciones/publications/docs/HYP2.PDF).

$$G_{xy} = x^*1+y^*i = \begin{bmatrix} x, y \\ y, x \end{bmatrix} = x^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$J = x^*1+(x^2-a)^{0.5}i = \begin{bmatrix} x, (x^2-a)^{0.5} \\ (x^2-a)^{0.5}, x \end{bmatrix} = x^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (x^2-a)^{0.5} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Figure 33. Upper level: the matrix representation of hyperbolic numbers, where two sparse (2*2)-matrices represent real and imaginary units (1 and i) of these numbers. The multiplication table of these basic elements 1 and i is shown on the right side. Bottom level: the special case of the set of hyperbolic numbers J (with the fixed value “a”) describes a hyperbola $x^2+y^2=a$, where “x” is a variable.

Hyperbolic numbers $G_{xy} = x^*1+y^*i$ (where «i» is the imaginary unit of hyperbolic numbers with its property $i^2=-1$) have a known (2*2)-matrix form of their representation shown on Figure 33. A special case of hyperbolic numbers

$$J=[x, (x^2-a)^{0.5}; (x^2-a)^{0.5}, x]$$

(4)

(4)

(4)

(where “x” is a variable and the parameter “a” is fixed) describes a hyperbola, which corresponds to the equation $x^2+y^2=a$. This equation describes the hyperbola in the coordinate system (x,y), axes of which coincide with axes of symmetry of the hyperbole. In another coordinate system, axes of which coincide with asymptotes of the hyperbole, the same hyperbola is described by the equation $y=a/x$ (Figure 34). Any point of any hyperbola can be transformed into a new point of the same hyperbola by means of so called hyperbolic rotation, which is described by the same matrix representation of hyperbolic numbers $[x, y; y, x]$ if its determinant is equal to 1.

It is known that hyperbolic numbers and hyperbolic operators are closely connected with natural logarithms, which can be defined on the base of hyperbolic rotations because of their relations with values of area under hyperbolas (http://mathworld.wolfram.com/NaturalLogarithm.html). By this reason natural logarithm «was formerly called hyperbolic logarithm as it corresponds to the area under a hyperbola» (http://en.wikipedia.org/wiki/Natural_logarithm). Area s of a curvilinear trapezoid inside boundaries, which are created by the hyperbola $x*y=a$, the x-axis and the lines $x=x_0$ and $x=x_1$, is equal to

$$s = a*\ln(x_1/x_0) = a*\{\ln(x_1)-\ln(x_0)\}$$,
where ln – natural (or hyperbolic) logarithm (Figure 34). History of hyperbolic logarithms is described for example in the book [Klein, 2009].

Figure 34. The function \( a \ln(x_1/x_0) \) is illustrated as the area under the hyperbola \( y = a/x \) from \( x_0 \) to \( x_1 \).

But a wide class of genetically inherited physiological phenomena is organized by the nature by means of the same logarithmic law (5) and it can be described mathematically on the base of hyperbolic numbers. We mean here the main psychophysiologic law by Weber-Fechner (http://en.wikipedia.org/wiki/Weber–Fechner_law): the intensity of the perception is proportional to the logarithm of stimulus intensity; it is expressed by the equation

\[
p = k \ln(V/V_0) = k \{ \ln(V) - \ln(V_0) \},
\]

where \( p \) - the intensity of perception, \( V \) - stimulus, \( V_0 \) - threshold stimulus, \( \ln \) - natural logarithm. Proportionality factor \( k \) in the expression (6) is different for different channels of sensory perception (vision, hearing, etc.); this difference of values \( "k" \) is associated with different values "\( a \)" in the equation of the hyperbola: \( x*y = a \).

The identity of expressions (5) and (6) allows to propose a geometric model of the Weber-Fechner law on the base of the described matrix representation of hyperbolic numbers in connection with a phenomenology of the molecular-genetic system. In this model the threshold stimulus \( V_0 \) in the expression (6) is interpreted as the value \( x_0 \) from the expression (5); the proportionality factor “\( k \)" in (6) is interpreted as the value “\( a \)” from (5); the stimulus \( V(t) \), which is varied in time, is interpreted as the variable \( x(t) \) from (5) and the perception “\( p \)” in (6) is interpreted as the area “\( s \)” from (5). Taking into account the type of hyperbolic numbers \( J \) from the expression (4), it is obvious that the hyperbolic number \( J_0 = [x_0, (x_0^2-a)^{0.5}, (x_0^2-a)^{0.5}, x_0] \) corresponds to the fixed area \( s_0 = a*\ln(x_0) \) of the curvilinear trapezoid, points of which have their \( x \)-coordinates inside the interval from 1 till \( x_0 \); another hyperbolic number \( J = [x, (x^2-a)^{0.5}, (x^2-a)^{0.5}, x_0] \) corresponds to the area \( s_x = a*\ln(x) \) of the curvilinear trapezoid with its \( x \)-coordinates inside the bigger interval from 1 till \( x \). The total area \( s \) of the third curvilinear trapezoid, points of which have their \( x \)-coordinates inside the interval from \( x_0 \) till \( x \), is equal to difference \( s_x - s_0 \), that is

\[
s = s_x - s_0 = a*\ln(x) - a*\ln(x_0) = a*\ln(x_1/x_0)
\]

This curvilinear trapezoid with its area \( s \) corresponds to the hyperbolic number, which is equal to the following:

\[
J - J_0 = [(x-x_0), ((x-x_0)^2-a)^{0.5}, ((x-x_0)^2-a)^{0.5}, x_0]
\]

We have described the geometric model for the case of one sensory channel. Let us
generalize now this geometric model for a case of a multi-dimensional space with different 2-dimensional planes inside it, each of which contains the described hyperbolic model of the Weber-Fechner law for one of many sensory channels of an organism. It is known that different types of inherited sensory perception are subordinated to this law: sight, hearing, smell, touch, taste, etc. One can suppose that the Weber–Fechner law is the law specially for nervous system. But it is not true since its meaning is much more wider because it is realized in many kinds of lower organisms without a nervous system in them: “this law is applicable to chemo-tropical, helio-tropical and geo-tropical movements of bacteria, fungi and antherozoids of ferns, mosses and phanerogams .... The Weber-Fechner law, therefore, is not the law of the nervous system and its centers, but the law of protoplasm in general and its ability to respond to stimuli” [Shultz, 1916, p. 126]. So the whole system of perception of an organism demonstrates itself as a multi-parametric system with many sub-systems of perception, which are subordinated to the logarithmic Weber-Fechner law. A generalized model of multi-dimensional space with appropriate sub-spaces is needed to describe this inherited multi-channel organization of the logarithmic perception in the case of the whole living organism. Let us show that our approach on the base of sums of genetic projectors allows to create such generalized model.

Hyperbolic numbers $G_{xy}=x^1+y^i$ have the multiplication table of their real and imaginary units shown on Figure 33. But the same multiplication table of hyperbolic numbers we have already received above:

- on Figures 5 and 6 for sums $(c_0+c_2), (c_1+c_3), (r_0+r_3), (r_1+r_2)$ of genetic (4*4)-projectors $c_0, c_1, c_2, c_3, r_0, r_1, r_2, r_3$ from Figure 2;
- on Figure 29 for sums $(s_0+s_4), (s_1+s_5), (s_2+s_6), (s_3+s_7)$ of genetic (8*8)-projectors $s_0, s_1, ..., s_7$ from Figure 7.

It means that these sums of genetic (4*4) and (8*8)-projectors are matrix representations of hyperbolic numbers with unit coordinates in corresponding 2-dimensional planes of 4-dimensional and 8-dimensional spaces. On the basis of each of these sums, one can construct a general representation of hyperbolic numbers (with an arbitrary values of their coordinates) in an appropriate hyperbolic plane of such multi-dimensional space. For example, if $e_0$ and $e_4$ are (8*8)-matrices taken from the decomposition of the (8*8)-matrix $s_0+s_4$ on Figure 29, then the expression $G_{4y}=e_0^*e_0+e_4^*e_4$ represents the hyperbolic number with real coordinates «$a_0$» and «$b_3$» in the plane $(x_0, x_4)$ inside an 8-dimensional space with its coordinate system $(x_0, x_1, ..., x_7)$. Similar situations holds true for expressions $G_{15}=a_1^*e_1+b_5^*e_5, G_{26}=a_2^*e_2+b_6^*e_6, G_{37}=a_3^*e_3+b_7^*e_7$, each of which represents hyperbolic number in an appropriate plane inside (8*8)-dimensional space, if $e_1, e_2, e_3, e_5, e_6, e_7$ are (8*8)-matrices taken from decompositions on Figure 29. The (8*8)-matrix $G_{04}+G_{15}+G_{26}+G_{37}$ represents an operator of an 8-dimensional space with a set of 4 hyperbolic planes $(x_0, x_4), (x_1, x_5), (x_2, x_6), (x_3, x_7)$ inside it.

For each of these four 2-dimensional planes $(x_0, x_4), (x_1, x_5), (x_2, x_6), (x_3, x_7)$ of an 8-dimensional space one can choose an individual hyperbola $x^2-y^2=a$ and an individual threshold stimulus $x_0$ to create the hyperbolic model of the Weber-Fechner law in the united set of four perception channels of the organism. In this case we have the following expression of the model for each of the planes, if we take (8*8)-matrices $e_0, e_1, ..., e_7$ from Figure 29 to express real and imaginary units of hyperbolic numbers (here $V_0, V_1, V_2, V_3$ mean stimulus variables on $x_0$-, $x_1$-, $x_2$-, $x_3$-axis correspondingly; $V_{00}, V_{01}, V_{02}, V_{03}$ mean threshold stimulus on the same $x_0$-, $x_1$-, $x_2$-, $x_3$-axis correspondingly; parameters $a_0, a_1, a_2, a_3$ for
corresponding planes mean the parameter “a” from the general equation \(x^2 - y^2 = a\) of hyperbola; \(S_{04}, S_{15}, S_{26}, S_{37}\) mean area of corresponding trapezoids defined by the hyperbolic numbers in these four planes; parameters \(k_0, k_1, k_2, k_3\) for corresponding planes mean the parameter \(k\) from (6):

1. for the plane \((x_0, x_4)\): \(G_{04}=-(V_0 - V_{00})e_0 + \{(V_0 - V_{00})^2 - a_0\}^{0.5}e_4\); this hyperbolic number \(G_{04}\) corresponds to the area \(S_{04}= k_0\ln(V_0/V_{00})\), which simulates the Weber-Fechner law in the plane \((x_0, x_4)\);
2. for the plane \((x_1, x_5)\): \(G_{15}=(V_1 - V_{01})e_1 + \{(V_1 - V_{01})^2 - a_1\}^{0.5}e_5\); this hyperbolic number \(G_{15}\) corresponds to the area \(S_{15}= k_1\ln(V_1/V_{01})\), which simulates the Weber-Fechner law in the plane \((x_1, x_5)\);
3. for the plane \((x_2, x_6)\): \(G_{26}=(V_2 - V_{02})e_2 + \{(V_2 - V_{02})^2 - a_2\}^{0.5}e_6\); this hyperbolic number \(G_{26}\) corresponds to the area \(S_{26}= k_2\ln(V_2/V_{02})\), which simulates the Weber-Fechner law in the plane \((x_2, x_6)\);
4. for the plane \((x_3, x_7)\): \(G_{37}=(V_3 - V_{03})e_3 + \{(V_3 - V_{03})^2 - a_3\}^{0.5}e_7\); this hyperbolic number \(G_{37}\) corresponds to the area \(S_{37}= k_3\ln(V_3/V_{03})\), which simulates the Weber-Fechner law in the plane \((x_3, x_7)\).

Figure 35 shows the \((8*8)\)-matrix \(W\), which represents the sum of the \((8*8)\)-matrix representations of these hyperbolic numbers: \(W=G_{04}+G_{15}+G_{26}+G_{37}\).

| \((V_0 - V_{00})\) | \((V_1 - V_{01})\) | \((V_2 - V_{02})\) | \((V_3 - V_{03})\) | \(\{(V_0 - V_{00})^2 - a_0\}^{0.5}\) | \(\{(V_1 - V_{01})^2 - a_1\}^{0.5}\) | \(\{(V_2 - V_{02})^2 - a_2\}^{0.5}\) | \(\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) |
|---|---|---|---|---|---|---|---|
| \((V_0 - V_{00})\) | \((V_1 - V_{01})\) | \((V_2 - V_{02})\) | \((V_3 - V_{03})\) | \(\{(V_0 - V_{00})^2 - a_0\}^{0.5}\) | \(\{(V_1 - V_{01})^2 - a_1\}^{0.5}\) | \(\{(V_2 - V_{02})^2 - a_2\}^{0.5}\) | \(\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) |
| \(-\{(V_0 - V_{00})^2 - a_0\}^{0.5}\) | \(-\{(V_1 - V_{01})^2 - a_1\}^{0.5}\) | \(-\{(V_2 - V_{02})^2 - a_2\}^{0.5}\) | \(-\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) |
| \(-\{(V_1 - V_{01})^2 - a_1\}^{0.5}\) | \(\{(V_1 - V_{01})^2 - a_1\}^{0.5}\) | \(-\{(V_2 - V_{02})^2 - a_2\}^{0.5}\) | \(-\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) |
| \(-\{(V_2 - V_{02})^2 - a_2\}^{0.5}\) | \(-\{(V_2 - V_{02})^2 - a_2\}^{0.5}\) | \(\{(V_2 - V_{02})^2 - a_2\}^{0.5}\) | \(-\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) |
| \(-\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) | \(-\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) | \(-\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) | \(\{(V_3 - V_{03})^2 - a_3\}^{0.5}\) |

Figure 35. The matrix representation of the sum \(W=G_{04}+G_{15}+G_{26}+G_{37}\) of the hyperbolic numbers \(G_{04}, G_{15}, G_{26}, G_{37}\), which belong to four planes \((x_0, x_4), (x_1, x_5), (x_2, x_6), (x_3, x_7)\) inside the 8-dimensional space (explanation in the text).

One can see that the \((8*8)\)-matrix \(W\) on Figure 35 has a special structure: its both \((4*4)\)-quadrants along each diagonals are identical to each other. This fact allows expressing the matrix \(W\) in the following form:

\[
W = [1 \ 0 \ 0 \ 1] \otimes M_0 + [0 \ 1 \ 1 \ 0] \otimes M_1,
\]
where \([1 \ 0; \ 0 \ 1]\) and \([0 \ 1; \ 1 \ 0]\) are matrix representations of real and imaginary units of hyperbolic numbers; \(M_0\) is the \((4*4)\)-matrix, which reproduces each of \((4*4)\)-quadrants along the main diagonal; \(M_1\) is the \((4*4)\)-matrix, which reproduces each of \((4*4)\)-quadrants along the second diagonal. The expression (9) means that the matrix \(W\) belongs to so called «tensornumbers» (more precisely, to a category of «tensorhyperbolic numbers»), which will be introduced below in a special Section.

Our approach, which was described above on the base of genetic matrices, allows natural modeling such kind of \(2^N\)-parametric systems with its \(2^N\)-parametric hyperbolic sub-systems by means of the described type of a \((2^N*2^N)\)-matrix operator of a \(2^N\)-dimensional space with appropriate quantity of hyperbolic planes inside it (in this case each of hyperbolic planes corresponds to an individual channel of perception with its own coefficient \(k\) and the threshold value \(V_0\) in the expression (6)). This \((2^N*2^N)\)-matrix operator also belongs to the category of tensorhyperbolic numbers, because it can be expressed by means of the expression (9), where \(M_0\) and \(M_1\) are \((2^{N-1}*2^{N-1})\)-matrices.

So we have two important facts:

- the logarithmic Weber-Fechner law has a total meaning for different sub-systems of perception inside the whole perception system of organism;
- this unity of all subsystems of perception inside the whole organism, which are subordinated to the Weber-Fechner law, can be expressed by means of \(2^N\)-dimensional tensorhyperbolic numbers by analogy with the expression (9).

These facts allow the author to put forward the following statement (or hypothesis): a living organism perceives an external world as a multi-parametric system, which belongs to a tensorhyperbolic category. In other words, for the whole perception system of a living organism, the external world is a life of tensorhyperbolic numbers in time. Correspondingly, interrelations of a living organism with the external world are realized on the base of processing of perceived tensorhyperbolic numbers, which are systematically changed over time in accordance with changes of external stimuli (in addition, the author believes that a living organism can be regarded as a life of tensornumbers over time; in this approach tensorcomplex numbers and their extensions deserve special attention). This mathematical approach to phenomenology of perception of the world is closely connected with the multi-parametric system of the genetic coding in its matrix form of representation; such connection allows a genetical transfer of this general biological property along a chain of generations.

10. REFLECTION OPERATORS AND GENETIC PROJECTORS.

By definition, a linear operator \(L\) is the reflection operator (or briefly, a "reflection") if and only if it satisfies the following criterion: \(L^2 = E\), where \(E\) is the identity operator (it is also denoted as \(\langle 1 \rangle\)), that is the real unit (see for example [Vinberg, 2003, Chapter 6]). The imaginary unit \(\langle i \rangle\) of hyperbolic numbers satisfies this criterion and consequently it is the reflection operator: \(i^2 = +1\) (see Figure 33). Hyperbolic number with unit coordinates \((1+i)\) is sum of the identity operator \(\langle 1 \rangle\) and the reflection operator \(\langle i \rangle\). The well-known \((2*2)\)-matrix represenation of the imaginary unit is the following: \([0 1; 1 0]\) (Figure 33). The acion of this reflection operator on a voluntary \(2\)-dimensional vector generates a new vector, which is a mirror-symmetrical analogue of the initial vector relative to the bisector of the angle between the x-axis and y-axis of the coordinate system \((x, y)\). For example, \([3, 5]*[0 1; 1 0] = [5, 3]\). A reflection is an involution: when applied twice in succession, every point returns to its original location, and every geometrical object is restored to its original state.
But we have shown above that many of sums of genetic projectors are the \((2^N \times 2^N)\)-matrix representation of hyperbolic numbers, which have their own real and imaginary units in respective planes inside \(2^N\)-dimensional space (see for example Figures 5, 6 and 29). Correspondingly, imaginary units of these hyperbolic numbers are \((2^N \times 2^N)\)-operators of reflections in these planes inside \(2^N\)-dimensional space. It gives evidences in favor of that the system of genetic coding actively uses also reflection operators. It is interesting because mirror reflections exist in many genetically inherited biological structures, including left and right halves of human and animal bodies. In the author’s laboratory, the genetic \((2^N \times 2^N)\)-matrices of reflection operators are used to analyze mirror symmetries in molecular-genetic systems including long genetic sequences, genetic palindromes, chromosomal inversion, etc.

11. THE SYMBOLIC MATRICES OF GENETIC DUPLETS AND TRIPLETS

In Section 1 the author promised to explain a relation of numeric matrices \(R_4, R_8, H_4, H_8\) (Figure 1), which were the initial matrices in this article, with a phenomenology of the genetic coding system in matrix forms of its representation. This Section is devoted to the explanation. Theory of noise-immunity coding is based on matrix methods. For example, matrix methods allow transferring high-quality photos of Mars’s surface via millions of kilometers of strong interference. In particularly, tensor families of Hadamard matrices are used for this aim. Tensor multiplication of matrices is the well-known operation in fields of signals processing technology, theoretical physics, etc. It is used for transition from spaces with a smaller dimension to associated spaces of higher dimension.

By analogy with theory of noise-immunity coding, the 4-letter alphabet of RNA (adenine A, cytosine C, guanine G and uracil U) can be represented in a form of the symbolic \((2\times2)\)-matrix \([C \ U; \ A \ G]\) (Figure 36) as a kernel of the tensor family of symbolic matrices \([C \ U; \ A \ G]^{(n)}\), where \((n)\) means a tensor power (Figure 36). Inside this family, this 4-letter alphabet of monoplets is connected with the alphabet of 16 duplets and 64 triplets by means of the second and third tensor powers of the kernel matrix: \([C \ U; \ A \ G]^{(2)}\) and \([C \ U; \ A \ G]^{(3)}\), where all duplet and triplets are disposed in a strict order (Figure 36). We begin with the alphabet A, C, G, U of RNA here because of mRNA-sequences of triplets define protein sequences of amino acids in a course of its reading in ribosomes.

Figure 36 contains not only 64 triplets but also amino acids and stop-codons encoded by the triplets in the case of the Vertebrate mitochondrial genetic code that is the most symmetrical among known variants of the genetic code (http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi). Let us explain the black-and-white mosaics of \([C \ U; \ A \ G]^{(2)}\) and \([C \ U; \ A \ G]^{(3)}\) (Figure 36) which reflect important features of the genetic code. These features are connected with a specificity of reading of mRNA-sequences in ribosomes to define protein sequences of amino acids (this is the reason, why we use the alphabet A, C, G, U of RNA in matrices on Figure 36; below we will consider the case of DNA-sequences separately).
Figure 36. The first three representatives of the tensor family of RNA-alphabetic matrices \([C\ U; A\ G]\). Black color marks 8 strong duplets in the matrix \([C\ U; A\ G]\) (at the top) and 32 triplets with strong roots in the matrix \([C\ U; A\ G]\) (bottom). 20 amino acids and stop-codons, which correspond to the triplets, are also shown in the matrix \([C\ U; A\ G]\) for the case of the Vertebrate mitochondrial genetic code.

A combination of letters on the two first positions of each triplet is usually termed as a “root” of this triplet [Konopelchenko, Rumer, 1975a,b; Rumer, 1968]. Modern science recognizes many variants (or dialects) of the genetic code, data about which are shown on the NCBI’s website http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi. 19 variants (or dialects) of the genetic code exist that differ one from another by some details of correspondences between triplets and objects encoded by them (these dialects are known at July 10, 2013, but perhaps later their list be increased). Most of these dialects (including the so called Standard Code and the Vertebrate Mitochondrial Code) have the symmetric general scheme of these correspondences, where 32 “black” triplets with “strong roots” and 32 “white” triplets with “weak” roots exist (the next Section shows all of these 19 dialects in details). In this basic scheme, the set of 64 triplets contains 16 subfamilies of triplets, every one of which contains 4 triplets with the same two letters on the first positions (an example of such subsets is the case of four triplets CAC, CAA, CAT, CAG with the same two letters CA on their first positions). In the described basic scheme, the set of these 16 subfamilies of NN-triplets is divided into two equal subsets. The first subset contains 8 subfamilies of so called “two-position” NN-triplets, a coding value of which is independent on a letter on their third position: (CAC, CCA, CCG), (CTC, CTT, CTA, CTG), (CGC, CGT, CGA, CGG), (TCC, TCT, TCA, TCG), (ACC, ACT, ACA, ACG), (GCC, GCT, GCA, GCG), (GTC, GTT, GTA, GTG), (GGC, GGT, GGA, GGG). An example of such subfamilies is the four triplets CAC, CAA, CAT, CAC, two of which (CAC, CAT) encode the amino acid His and the other two (CAA, CAG) encode another amino acid.
Gln. The 32 triplets of the second subset are termed as “triplets with weak roots” [Konopelchenko, Rumer, 1975a,b; Rumer, 1968]. The following duplets are appropriate 8 weak roots for them: CA, AA, AT, AG, TA, TT, TG, GA (weak duplets). All members of these 32 NN-triplets and 8 weak duplets are marked by white color in the matrices [C U; A G]^{(3)} and [C U; A G]^{(2)} on Figure 36.

From the point of view of its black-and-white mosaic, each of columns of genetic matrices [C U; A G]^{(2)} and [C U; A G]^{(3)} has a meander-like character and coincides with one of Rademacher functions that form orthogonal systems and well known in discrete signals processing. These functions contain elements “+1” and “-1” only. Due to this fact, one can construct Rademacher representations of the symbolic genomatrices [C U; A G]^{(2)} and [C U; A G]^{(3)} (Figure 36) by means of the following operation: each of black duplets and of black triplets is replaced by number “+1” and each of white duplets and white triplets is replaced by number “-1”. This operation leads immediately to the matrices R_{8} and R_{8} from Figure 1, that are the Rademacher representations of the phenomenological genomatrices [C U; A G]^{(2)} and [C U; A G]^{(3)}.

If columns of the matrix [C U; A G]^{(3)} on Figure 36 are numerated from left to right by indexes 0, 1, 2, …, 7, one can see that 4 columns with even indexes 0, 2, 4, 6 contain 32 triplets, each of which has nitrogenous bases C or A on its third position, that is in its suffix (these C and A are usually termed “amino bases”). Other 4 columns with odd indexes 1, 3, 5, 7 contain other 32 triplets, each of which has nitrogenous bases T or G on its third position (these T and G are usually termed “keto bases”). The following important phenomenon is connected with this separation of the matrix [C U; A G]^{(3)} in columns with even and odd indexes: adjacent columns with indexes “0 and 1”, “2 and 3”, “4 and 5” and “6 and 7” contain identical list of amino acids and stop-codons (these adjacent columns are twins from this point of view). Consequently the symbolic matrix [C U; A G]^{(3)} can be represented as a sum of two sparse matrices with identical lists of amino acids and stop-codons: the first of these two matrices coincides with the matrix [C U; A G]^{(3)} in columns with even indexes and has zero columns with odd indexes; the second one coincides with the matrix [C U; A G]^{(3)} in columns with odd indexes and has zero columns with even indexes.

By analogy the Rademacher representation R_{8} of this symbolic matrix [C U; A G]^{(3)} can be also decomposed into two sparse matrices RL_{8} and RR_{8} (Figure 26), the first of which has all non-zero columns with even indexes and the second one has all non-zero columns with odd indexes. As it was shown above, each of these numeric (8*8)-matrices RL_{8} and RR_{8} represents split-quaternion by Cockle, whose coordinates are equal to 1, in an 8-dimensional space. It means that the system of correspondences between the set of 64 triplets (with their internal separation into subsets of triplets with strong and weak roots) and the set of 20 amino acids and stop-codon is created by the nature in accordance with the layout of these two split-quaternions RL_{8} and RR_{8} in an 8-dimensional space. In some extend this double numeric construction resembles double helix of DNA.

But each of these (8*8)-matrices RL_{8} and RR_{8} consists of two hyperbolic numbers (Figure 29): \( RL_{8} = (e_0+e_3)+(e_2+e_6) \) and \( RR_{8} = (e_1+e_2)+(e_3+e_7) \). In other words the system of correspondences between the set of triplets and the set of amino acids and stop-codons is based on the mentioned 4 hyperbolic numbers in an 8-dimensional space. One can mention that here we are meeting again with a set of 4 elements in some analogy with the sets of 4 elements in the genetic alphabets of nitrogenous bases in DNA and RNA - A, C, G, T/U (and also with the Ancient set of Pythagorean Tetraktys - http://en.wikipedia.org/wiki/Tetractys).
Whether any symmetry exists between structures of these 4 hyperbolic numbers and the phenomenological disposition of amino acids and stop-codons in the genomatrix [C U; A G] on Figure 36? To receive an answer on this question, let us compare a content of corresponding cells of the symbolic genomatrix [C U; A G] (Figure 36) with non-zero cells of matrices \((e_0+e_4)\), \((e_2+e_6)\), \((e_1+e_5)\) and \((e_3+e_7)\), which represent these 4 hyperbolic numbers (Figure 29). Figure 37 shows results of such comparison for the hyperbolic numbers \((e_0+e_4)\) and \((e_2+e_6)\) with even indexes of their column projectors; results of such comparison for the hyperbolic numbers \((e_1+e_5)\) and \((e_3+e_7)\) with odd indexes are identical because in the matrix [C U; A G] (Figure 36) adjacent columns with indexes “0 and 1”, “2 and 3”, “4 and 5” and “6 and 7” contain identical lists of amino acids and stop-codons. Those amino acids, which belong to matrix cells with triplets of strong roots, are marked by bold letters on Figure 37. One can see here the following symmetrical feature: each of real and imagine parts of these hyperbolic numbers \((e_0+e_4)\) and \((e_2+e_6)\) contains an equal quantity of amino acids marked by bold letters and also an equal quantity of amino acids of another type. By such way we receive a special separation of the set of 20 amino acids into a few groups, which belong to real parts or to imagine parts of these hyperbolic numbers and which should be analized in future more attentively.

\[
\begin{array}{cccc}
1 0 0 0 & 0 0 0 0 & 0 0 0 0 1 0 0 0 & 0 0 0 0 1 0 0 0 \\
1 0 0 0 & 0 0 0 0 & 0 0 0 0 1 0 0 0 & 0 0 0 0 1 0 0 0 \\
-1 0 0 0 & 0 0 0 0 & 0 0 0 0 1 0 0 0 & 0 0 0 0 1 0 0 0 \\
-1 0 0 0 & 0 0 0 0 & 0 0 0 0 1 0 0 0 & 0 0 0 0 1 0 0 0 \\
0 0 0 0 & 1 0 0 0 & 1 0 0 0 0 0 0 0 & 0 0 0 0 1 0 0 0 \\
0 0 0 0 & 1 0 0 0 & 1 0 0 0 0 0 0 0 & 0 0 0 0 1 0 0 0 \\
0 0 0 0 & -1 0 0 0 & -1 0 0 0 0 0 0 0 & 0 0 0 0 1 0 0 0 \\
0 0 0 0 & -1 0 0 0 & -1 0 0 0 0 0 0 0 & 0 0 0 0 1 0 0 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
1 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
1 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 & 0 0 0 0 0 0 0 0 \\
\end{array}
\]

Figure 37. The separation of 20 amino acids and stop-codons in accordance with real and imagine parts of hyperbolic numbers \(e_0+e_4\) and \(e_2+e_6\). Those amino acids, which belong to matrix cells with triplets of strong roots, are marked by bold letters.

Now let us pay attention to Figure 38, where beginnings of appropriate tensor family of matrices \([C T; A G]^{(n)}\) for the case of the DNA alphabet (adenine A, cytosine C, guanine G and thymine T) are shown. What kind of black-and-white mosaics (or a disposition of elements “+1” and “-1” in numeric representations of these symbolic matrices) can be appropriate in the case of the DNA alphabet for the basic matrix \([C T; A G]\) and \([C T; A G]^{(2)}\)? The important phenomenological fact is that the thymine T is a single nitrogenous base in DNA which is replaced in RNA by another nitrogenous base U (uracil) for unknown reason (this is one of the mysteries of the genetic system). In other words, in this system the letter T is the opposition in relation to the letter U, and so the letter T can be symbolized by number “-1” (instead of number “+1” for U). Taking this into account, a simple algorithm
exists, which transforms the black-and-white mosaics of matrices $[C \hspace{1mm} U; A \hspace{1mm} G]^{(2)}$ and $[C \hspace{1mm} U; A \hspace{1mm} G]^{(3)}$ into other mosaics of matrices $[C \hspace{1mm} T; A \hspace{1mm} G]^{(2)}$ and $[C \hspace{1mm} T; A \hspace{1mm} G]^{(3)}$ that are shown on Figure 38. Concerning to their mosaics, the matrices $[C \hspace{1mm} T; A \hspace{1mm} G]^{(2)}$ and $[C \hspace{1mm} T; A \hspace{1mm} G]^{(3)}$ coincide with mosaics of the Hadamard matrices $H_4$ and $H_8$ (Figure 1), which are their Hadamard representations (here one should remind that Hadamard matrices contain only entries +1 and -1). The mentioned algorithm was described in a few author's works (see for example [Petoukhov, 2012a,b]). The Appendix 3 describes another way to construct Hadamard $(4*4)$- and $(8*8)$-matrices on the base of the unique status of the letter T in the genetic alphabet A, C, G and T in DNA.

Figure 38. The first three representatives $[C \hspace{1mm} T; A \hspace{1mm} G]$, $[C \hspace{1mm} T; A \hspace{1mm} G]^{(2)}$ and $[C \hspace{1mm} T; A \hspace{1mm} G]^{(3)}$ of the tensor family of DNA-alphabetic matrices $[C \hspace{1mm} T; A \hspace{1mm} G]^{(n)}$. These symbolic matrices $[C \hspace{1mm} T; A \hspace{1mm} G]^{(3)}$ and $[C \hspace{1mm} T; A \hspace{1mm} G]^{(3)}$ have mosaics, which coincide with the mosaics of their Hadamard representations $H_4$ and $H_8$ on Figure 1. All amino acids and stop-codons are shown for the case of the Vertebrate mitochondrial genetic code by analogy with Figure 36.

### 12. GENETIC PROJECTORS AND THE EXCLUSION PRINCIPLE FOR EVOLUTIONARY CHANGES OF DIALECTS OF THE GENETIC CODE

This Section describes an exclusion principle of evolution of dialects of the genetic code. This principle, which was discovered by the author, shows that evolutionary changes of dialects of the genetic code are related with the genetic projectors. One should note that discovering of exclusion principles of nature is a significant task of mathematical natural science (the exclusion principle by Pauli in quantum mechanics is one of examples).

The following list contains all known 19 dialects of the genetic code presented at July 10, 2013 on the NCBI’s website [http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi](http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi):
| No. | Code Description                                         |
|-----|---------------------------------------------------------|
| 1   | The Standard Code                                       |
| 2   | The Vertebrate Mitochondrial Code                       |
| 3   | The Yeast Mitochondrial Code                            |
| 4   | The Mold, Protozoan, and Coelenterate Mitochondrial Code and the Mycoplasma/Spiroplasma Code |
| 5   | The Invertebrate Mitochondrial Code                     |
| 6   | The Ciliate, Dasycladacean and Hexamita Nuclear Code    |
| 7   | The Echinoderm and Flatworm Mitochondrial Code          |
| 8   | The Euplotid Nuclear Code                               |
| 9   | The Bacterial, Archaeal and Plant Plastid Code          |
| 10  | The Alternative Yeast Nuclear Code                      |
| 11  | The Ascidian Mitochondrial Code                         |
| 12  | The Alternative Flatworm Mitochondrial Code             |
| 13  | Blepharisma Nuclear Code                                |
| 14  | Chlorophycean Mitochondrial Code                        |
| 15  | Trematode Mitochondrial Code                            |
| 16  | Scenedesmus Obliquus Mitochondrial Code                 |
| 17  | Thraustochytrium Mitochondrial Code                     |
| 18  | Pterobranchia Mitochondrial Code                        |
| 19  | Candidate Division SR1 and Gracilibacteria Code          |

Figure 39 shows these dialects in typical forms of black-and-white genetic matrices $[C\ U;\ A\ G]^3$, where black cells correspond to triplets with strong roots and white cells correspond to triplets with weak roots in cases of each individual dialect. One can see that the vast majority of dialects (13 dialect from the set of 19 dialects) possesses the identical black-and-white mosaics though some triplets have different code meanings in different dialects (they encode different amino acid or stop signal in different dialects) in comparison with their meanings in the vertebrate mitochondria genetic code (the author takes the case of the vertebrate mitochondria genetic code as the basic case because this dialect is the most symmetrical). All such triplets, which change their code meaning, are marked by red letters on Figure 39. From the list of 19 dialects only the following 6 dialects have their matrix $[C\ U;\ A\ G]^3$ with atypical black-and-white mosaics (see Figure 39): 5) The Invertebrate Mitochondrial Code; 7) The Echinoderm and Flatworm Mitochondrial Code; 10) The Alternative Yeast Nuclear Code; 12) The Alternative Flatworm Mitochondrial Code; 15) Trematode Mitochondrial Code; 16) Scenedesmus Obliquus Mitochondrial Code.

By analogy with the Rademacher presentation $R_8$ (see above Figures 1, 36, and Section 10), one can again replace black triplets by elements $\langle +1 \rangle$ and white triplets by elements $\langle -1 \rangle$ to receive numeric representations of these genetic matrices $[C\ U;\ A\ G]^3$ of all dialects. Such numeric representations of genetic matrices can be conditionally called as $\pm 1$-representations. The result is the following: this numeric $\pm 1$-representation of matrices $[C\ U;\ A\ G]^3$ of every of 19 dialects is decomposed into a sum of 8 sparse (8*8)-matrices of $\langle$column projectors$\rangle$ or $\langle$row projectors$\rangle$ (see Figure 39). It is connected with the fact that all cells on main diagonals of these numeric matrices contain only $\langle +1 \rangle$ (see the theorem in Section 7). This general feature of all dialects is a consequence of the following phenomenologic fact: biological evolution never changes code meaning of 16 black triplets, which occupies (2*2)-sub-quadrants along the main diagonal of these matrices (CCC, CCU, CCA, CCG, CGC, CGU, CGA, CGG, GCC, GCU, GCA, GCG, GGC, GGU, GGA, GGG).

From the point of view of algebra of projection operators, the described facts mean that biologic evolution of dialects of the genetic code is connected with a condition of conservation of the numeric $\pm 1$-representation of the genetic matrix $[C\ U;\ A\ G]^3$ as a sum of
8 column projectors (or 8 row projectors). In other words, algebra of projectors shows an existence of an algebraic invariant of biological evolution.

One can formulate here the phenomenologic exclusion principle for evolutionary changes of dialects of the genetic code: it is forbidden for biological evolution to violate a separation of the set of 64 triplets into two subsets of triplets with strong and weak roots (black and white triplets) in a such way that a black-and-white mosaic of the genetic matrix \([\text{C U}; \text{A G}]^{(3)}\) in its \(\pm 1\)-representation ceases to be a sum of 8 column projectors (or 8 row projectors).

The Vertebrate Mitochondrial Code:

|     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| C   | C   | C   | U   | C   | U   | C   | U   | C   |
| C   | C   | C   | G   | C   | G   | C   | G   | C   |
| A   | A   | A   | A   | A   | A   | A   | A   | A   |
| G   | G   | G   | G   | G   | G   | G   | G   | G   |

The Standard Code:

|     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| C   | C   | C   | U   | C   | U   | C   | U   | C   |
| C   | C   | C   | G   | C   | G   | C   | G   | C   |
| A   | A   | A   | A   | A   | A   | A   | A   | A   |
| G   | G   | G   | G   | G   | G   | G   | G   | G   |
The Yeast Mitochondrial Code:

| Codon | Thr | Cus | Cua | Ucc | Ucu | Uuc | Phe | Uuu |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | Pro | Thr | Thr | Thr | Ser | Ser |
| CCA   | Pro | Pro | CCG | Pro | Thr | Cua | Ucc | Ser |
| CAC   | His | His | CGC | Arg | CGU | Arg | CUA | Tyr |
| CAA   | Gln | Gln | CGA | Arg | CGG | Arg | UCA | Ser |
| ACC   | Thr | Thr | ACU | UUC | Pro | CUC | Leu | Leu |
| ACA   | Thr | Thr | ACG | Met | AUA | Met | CUA | Leu |
| AAC   | Asn | Asn | AAG | Arg | CGA | Arg | UCA | Arg |
| AAA   | Lys | Lys | AGA | Arg | CGA | Arg | UAA | Stop |

The Mold, Protozoan, and Coelenterate Mitochondrial Code and the Mycoplasma/Spiroplasma Code:

| Codon | Thr | Cus | Cua | Ucc | Ucu | Uuc | Phe | Uuu |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| ACC   | Thr | Thr | ACU | UUC | Pro | CUC | Leu | Leu |
| ACA   | Thr | Thr | ACG | Met | AUA | Met | CUA | Leu |
| AAC   | Asn | Asn | AAG | Arg | CGA | Arg | UCA | Arg |
| AAA   | Lys | Lys | AGA | Arg | CGA | Arg | UAA | Stop |

The Invertebrate Mitochondrial Code:

| Codon | Thr | Cus | Cua | Ucc | Ucu | Uuc | Phe | Uuu |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| ACC   | Thr | Thr | ACU | UUC | Pro | CUC | Leu | Leu |
| ACA   | Thr | Thr | ACG | Met | AUA | Met | CUA | Leu |
| AAC   | Asn | Asn | AAG | Arg | CGA | Arg | UCA | Arg |
| AAA   | Lys | Lys | AGA | Arg | CGA | Arg | UAA | Stop |
## The Ciliate, Dasycladacean and Hexamita Nuclear Code:

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | CUU | UCC | UCU | UUC | UUU |
| CCA   | Pro | CCG | CUA | CUG | UCA | UCG | UUA | UUG |
| CAC   | His | CAU | CGC | CGU | UAC | UAU | UGC | UGU |
| CAA   | Gln | CAG | CGA | CGG | UAA | UAG | UGA | UGG |

## The Echinoderm and Flatworm Mitochondrial Code:

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | CUU | UCC | UCU | UUC | UUU |
| CCA   | Pro | CCG | CUA | CUG | UCA | UCG | UUA | UUG |
| CAC   | His | CAU | CGC | CGU | UAC | UAU | UGC | UGU |
| CAA   | Gln | CAG | CGA | CGG | UAA | UAG | UGA | UGG |

## The Euplotid Nuclear Code:

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | CUU | UCC | UCU | UUC | UUU |
| CCA   | Pro | CCG | CUA | CUG | UCA | UCG | UUA | UUG |
| CAC   | His | CAU | CGC | CGU | UAC | UAU | UGC | UGU |
| CAA   | Gln | CAG | CGA | CGG | UAA | UAG | UGA | UGG |
| ACC   | Thr | ACU | AUC | AUA | GCC | GCC | GCU | GGU |
| ACA   | Thr | ACG | AUA | AUG | GCA | GCG | GAU | GGU |
| AAC   | Asn | AAG | AGA | AGG | GAC | GAU | GCC | GGU |
| AAA   | Lys | AAG | Arg | Arg | GAA | GAG | GGA | GGG |

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | CUU | UCC | UCU | UUC | UUU |
| CCA   | Pro | CCG | CUA | CUG | UCA | UCG | UUA | UUG |
| CAC   | His | CAU | CGC | CGU | UAC | UAU | UGC | UGU |
| CAA   | Gln | CAG | CGA | CGG | UAA | UAG | UGA | UGG |
| ACC   | Thr | ACU | AUC | AUA | GCC | GCC | GCU | GGU |
| ACA   | Thr | ACG | AUA | AUG | GCA | GCG | GAU | GGU |
| AAC   | Asn | AAG | AGA | AGG | GAC | GAU | GCC | GGU |
| AAA   | Lys | AAG | Arg | Arg | GAA | GAG | GGA | GGG |

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | CUU | UCC | UCU | UUC | UUU |
| CCA   | Pro | CCG | CUA | CUG | UCA | UCG | UUA | UUG |
| CAC   | His | CAU | CGC | CGU | UAC | UAU | UGC | UGU |
| CAA   | Gln | CAG | CGA | CGG | UAA | UAG | UGA | UGG |
| ACC   | Thr | ACU | AUC | AUA | GCC | GCC | GCU | GGU |
| ACA   | Thr | ACG | AUA | AUG | GCA | GCG | GAU | GGU |
| AAC   | Asn | AAG | AGA | AGG | GAC | GAU | GCC | GGU |
| AAA   | Lys | AAG | Arg | Arg | GAA | GAG | GGA | GGG |
The Bacterial, Archaeal and Plant Plastid Code:

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | Leu | UCC | Ser | UCU | Ser |
| CCA   | Pro | Pro | CUA | Leu | UCA | Ser | UCG | Leu |
| CAC   | His | His | CGC | Arg | UAC | Tyr | UAU | Tyr |
| CAA   | Gln | Gln | CGA | Arg | UAA | Stop| UAG | Stop|
| ACC   | Thr | Thr | AUU | Ile | GCC | Ala | GCU | Val |
| ACA   | Thr | Thr | AUG | Met | GCA | Ala | GCG | Val |
| AAC   | Asn | Asn | AGU | Ser | GAC | Asp | GAU | Gly |
| AAA   | Lys | Lys | AGA | Arg | GAA | Glu | GAG | Gly |
| AAA   | Lys | Lys | AGG | Arg | GAA | Glu | GAG | Gly |

The Alternative Yeast Nuclear Code:

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | Leu | UCC | Ser | UCU | Ser |
| CCA   | Pro | Pro | CUA | Leu | UCA | Ser | UCG | Leu |
| CAC   | His | His | CGC | Arg | UAC | Tyr | UAU | Tyr |
| CAA   | Gln | Gln | CGA | Arg | UAA | Stop| UAG | Stop|
| ACC   | Thr | Thr | AUU | Ile | GCC | Ala | GCU | Val |
| ACA   | Thr | Thr | AUG | Met | GCA | Ala | GCG | Val |
| AAC   | Asn | Asn | AGU | Ser | GAC | Asp | GAU | Gly |
| AAA   | Lys | Lys | AGA | Arg | GAA | Glu | GAG | Gly |
| AAA   | Lys | Lys | AGG | Arg | GAA | Glu | GAG | Gly |

The Ascidian Mitochondrial Code:

| Codon | Pro | Pro | Leu | Leu | Ser | Ser | Phe | Phe |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| CCC   | Pro | Pro | CUC | Leu | UCC | Ser | UCU | Ser |
| CCA   | Pro | Pro | CUA | Leu | UCA | Ser | UCG | Leu |
| CAC   | His | His | CGC | Arg | UAC | Tyr | UAU | Tyr |
| CAA   | Gln | Gln | CGA | Arg | UAA | Stop| UAG | Stop|
| ACC   | Thr | Thr | AUU | Ile | GCC | Ala | GCU | Val |
| ACA   | Thr | Thr | AUG | Met | GCA | Ala | GCG | Val |
| AAC   | Asn | Asn | AGU | Ser | GAC | Asp | GAU | Gly |
| AAA   | Lys | Lys | AGA | Arg | GAA | Glu | GAG | Gly |
| AAA   | Lys | Lys | AGG | Arg | GAA | Glu | GAG | Gly |
### The Alternative Flatworm Mitochondrial Code:

| Codon | Alanine | Arginine | Asparagine | Aspartic acid | Cysteine | Glutamic acid | Glutamine | Histidine | Isoleucine | Leucine | Lysine | Methionine | Phenylalanine | Threonine | Tryptophan | Valine |
|-------|---------|----------|------------|--------------|----------|---------------|------------|-----------|------------|---------|----------|------------|----------------|-----------|-----------|--------|
| CCC   | Pro     | Pro      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CCA   | Pro     | Pro      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CAC   | CAU     | CAU      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CAA   | CAG     | CAG      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| ACC   | Thr     | AU U      | UCU        |              |          | GCC            | GCU        | GUC       | GGU       |         |         |             |                |           |          |        |
| ACA   | Thr     | AU A      | ACG        |              |          | GCC            | GCU        | GUA       | GUG       |         |         |             |                |           |          |        |
| AAC   | Asn     | AGC       | AGU        | UAC          | UAU      | UAG            | UGA        | UGG       |             |         |         |             |                |           |          |        |
| AAA   | Lys     | AGA       | AGG        | UAA          | Stop UAG  | UGA            | Stop UGG   |           |             |         |         |             |                |           |          |        |

### Blepharisma Nuclear Code:

| Codon | Alanine | Arginine | Asparagine | Aspartic acid | Cysteine | Glutamic acid | Glutamine | Histidine | Isoleucine | Leucine | Lysine | Methionine | Phenylalanine | Threonine | Tryptophan | Valine |
|-------|---------|----------|------------|--------------|----------|---------------|------------|-----------|------------|---------|----------|------------|----------------|-----------|-----------|--------|
| CCC   | Pro     | Pro      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CCA   | Pro     | Pro      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CAC   | CAU     | CAU      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CAA   | CAG     | CAG      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| ACC   | Thr     | AU U      | UCU        |              |          | GCC            | GCU        | GUC       | GGU       |         |         |             |                |           |          |        |
| ACA   | Thr     | AU A      | ACG        |              |          | GCC            | GCU        | GUA       | GUG       |         |         |             |                |           |          |        |
| AAC   | Asn     | AGC       | AGU        | UAC          | UAU      | UAG            | UGA        | UGG       |             |         |         |             |                |           |          |        |
| AAA   | Lys     | AGA       | AGG        | UAA          | Stop UAG  | UGA            | Stop UGG   |           |             |         |         |             |                |           |          |        |

### Chlorophycean Mitochondrial Code:

| Codon | Alanine | Arginine | Asparagine | Aspartic acid | Cysteine | Glutamic acid | Glutamine | Histidine | Isoleucine | Leucine | Lysine | Methionine | Phenylalanine | Threonine | Tryptophan | Valine |
|-------|---------|----------|------------|--------------|----------|---------------|------------|-----------|------------|---------|----------|------------|----------------|-----------|-----------|--------|
| CCC   | Pro     | Pro      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CCA   | Pro     | Pro      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CAC   | CAU     | CAU      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| CAA   | CAG     | CAG      |            |              |          |                |            |           |            |         |         |             |                |           |          |        |
| ACC   | Thr     | AU U      | UCU        |              |          | GCC            | GCU        | GUC       | GGU       |         |         |             |                |           |          |        |
| ACA   | Thr     | AU A      | ACG        |              |          | GCC            | GCU        | GUA       | GUG       |         |         |             |                |           |          |        |
| AAC   | Asn     | AGC       | AGU        | UAC          | UAU      | UAG            | UGA        | UGG       |             |         |         |             |                |           |          |        |
| AAA   | Lys     | AGA       | AGG        | UAA          | Stop UAG  | UGA            | Stop UGG   |           |             |         |         |             |                |           |          |        |
### Trematode Mitochondrial Code:

| Codon | Amino Acid |
|-------|------------|
| CCC   | Pro        |
| CCA   | Pro        |
| CAC   | His        |
| CAA   | Gln        |
| ACC   | Thr        |
| ACA   | Thr        |
| AAC   | Asn        |
| AAA   | Asn        |
| CUC   | Leu        |
| CUG   | Leu        |
| CGC   | Arg        |
| CGA   | Arg        |
| AGC   | Ser        |
| AGG   | Arg        |
| CUU   | Leu        |
| CUG   | Leu        |
| CGU   | Arg        |
| CGG   | Arg        |
| UCU   | Ser        |
| UCG   | Ser        |
| UCA   | Ser        |
| UAG   | Stop       |
| UAA   | Stop       |
| UAC   | Tyr        |
| UAU   | Tyr        |
| UGU   | Cys        |
| UGA   | Trp        |
| UUG   | Gly        |
| GCU   | Ala        |
| GCA   | Ala        |
| GAA   | Glu        |
| GAG   | Glu        |
| GGU   | Gly        |
| GCC   | Ala        |
| GGA   | Gly        |
| GUC   | Val        |
| UUC   | Ure        |
| UAC   | Tyr        |
| UAG   | Trp        |
| UAA   | Stop       |
| UAC   | Tyr        |
| UAU   | Tyr        |
| UGG   | Ure        |
| AAA   | Asn        |
| AAC   | Asn        |
| AGC   | Ser        |
| AGG   | Arg        |
| AAA   | Asn        |

### Scenedesmus Obliquus Mitochondrial Code:

| Codon | Amino Acid |
|-------|------------|
| CCC   | Pro        |
| CCA   | Pro        |
| CAC   | His        |
| CAA   | Gln        |
| ACC   | Thr        |
| ACA   | Thr        |
| AAC   | Asn        |
| AAA   | Asn        |
| CUC   | Leu        |
| CUG   | Leu        |
| CGC   | Arg        |
| CGA   | Arg        |
| AGC   | Ser        |
| AGG   | Arg        |
| CUU   | Leu        |
| CUG   | Leu        |
| CGU   | Arg        |
| CGG   | Arg        |
| UCU   | Ser        |
| UCG   | Ure        |
| UCA   | Ser        |
| UAG   | Stop       |
| UAA   | Stop       |
| UAC   | Tyr        |
| UAU   | Tyr        |
| UGU   | Cys        |
| UGA   | Trp        |
| UUG   | Gly        |
| GCU   | Ala        |
| GCA   | Ala        |
| GAA   | Glu        |
| GAG   | Glu        |
| GGU   | Gly        |
| GUC   | Val        |
| UUC   | Ure        |
| UAC   | Tyr        |
| UAG   | Trp        |
| UAA   | Stop       |
| UAC   | Tyr        |
| UAU   | Tyr        |
| UGG   | Ure        |
| AAA   | Asn        |
| AAC   | Asn        |
| AGC   | Ser        |
| AGG   | Arg        |
| AAA   | Asn        |

### Thraustochytrium Mitochondrial Code:

| Codon | Amino Acid |
|-------|------------|
| CCC   | Pro        |
| CCA   | Pro        |
| CAC   | His        |
| CAA   | Gln        |
| ACC   | Thr        |
| ACA   | Thr        |
| AAC   | Asn        |
| AAA   | Asn        |
| CUC   | Leu        |
| CUG   | Leu        |
| CGC   | Arg        |
| CGA   | Arg        |
| AGC   | Ser        |
| AGG   | Arg        |
| CUU   | Leu        |
| CUG   | Leu        |
| CGU   | Arg        |
| CGG   | Arg        |
| UCU   | Ser        |
| UCG   | Ure        |
| UCA   | Ser        |
| UAG   | Stop       |
| UAA   | Stop       |
| UAC   | Tyr        |
| UAU   | Tyr        |
| UGU   | Cys        |
| UGA   | Trp        |
| UUG   | Gly        |
| GCU   | Ala        |
| GCA   | Ala        |
| GAA   | Glu        |
| GAG   | Glu        |
| GGU   | Gly        |
| GUC   | Val        |
| UUC   | Ure        |
| UAC   | Tyr        |
| UAG   | Trp        |
| UAA   | Stop       |
| UAC   | Tyr        |
| UAU   | Tyr        |
| UGG   | Ure        |
| AAA   | Asn        |
| AAC   | Asn        |
| AGC   | Ser        |
| AGG   | Arg        |
| AAA   | Asn        |
Pterobranchia Mitochondrial Code:

|     | CCC  | CCU  | CUC  | CUU  | UCC  | UCU  | UUC  | UGU  | UUG  | UAA  | UAG  | UGA  | UGG  |
|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|
| Pro | Pro  | Leu  | Leu  | Leu  | Ser  | Ser  | Ser  | Leu  | Leu  | Ser  | Ser  | Ser  | Ser  |
| CCA | Pro  | Pro  | CUA  | CUG  | UCA  | UCG  | UUA  | UUG  | UUA  | UUG  | UUA  | UUG  | UUA  |
| CAC | His  | His  | CGC  | CGU  | UAC  | UAU  | UGC  | UGU  | UGC  | UGU  | UGC  | UGU  | UGC  |
| CAA | Gln  | Gln  | CGA  | CGG  | UAA  | UAG  | UGA  | UGG  | UGA  | UGG  | UGA  | UGG  | UGA  |
| ACC | Thr  | ACU  | AUC  | AUU  | GCC  | GCU  | GUC  | GGU  | GGU  | GGU  | GGU  | GGU  | GGU  |
| ACA | Thr  | ACG  | AUA  | AUG  | GCA  | GCG  | GUA  | GUG  | GUA  | GUG  | GUA  | GUG  | GUA  |
| AAC | Asn  | Asn  | AGC  | AGU  | GAC  | GAU  | GCC  | GGU  | GCC  | GGU  | GCC  | GGU  | GCC  |
| AAA | Lys  | Lys  | AGA  | AGG  | GAA  | GAG  | GAA  | GGG  | GAA  | GGG  | GAA  | GGG  | GAA  |

Candidate Division SR1 and Gracilibacteria Code:

|     | CCC  | CCU  | CUC  | CUU  | UCC  | UCU  | UUC  | UGU  | UUG  | UAA  | UAG  | UGA  | UGG  |
|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|
| Pro | Pro  | Leu  | Leu  | Leu  | Ser  | Ser  | Ser  | Leu  | Leu  | Ser  | Ser  | Ser  | Ser  |
| CCA | Pro  | Pro  | CUA  | CUG  | UCA  | UCG  | UUA  | UUG  | UUA  | UUG  | UUA  | UUG  | UUA  |
| CAC | His  | His  | CGC  | CGU  | UAC  | UAU  | UGC  | UGU  | UGC  | UGU  | UGC  | UGU  | UGC  |
| CAA | Gln  | Gln  | CGA  | CGG  | UAA  | UAG  | UGA  | UGG  | UGA  | UGG  | UGA  | UGG  | UGA  |
| ACC | Thr  | ACU  | AUC  | AUU  | GCC  | GCU  | GUC  | GGU  | GGU  | GGU  | GGU  | GGU  | GGU  |
| ACA | Thr  | ACG  | AUA  | AUG  | GCA  | GCG  | GUA  | GUG  | GUA  | GUG  | GUA  | GUG  | GUA  |
| AAC | Asn  | Asn  | AGC  | AGU  | GAC  | GAU  | GCC  | GGU  | GCC  | GGU  | GCC  | GGU  | GCC  |
| AAA | Lys  | Lys  | AGA  | AGG  | GAA  | GAG  | GAA  | GGG  | GAA  | GGG  | GAA  | GGG  | GAA  |

Figure 39. The matrices [C U; A G]^{19} show 19 known dialects of the genetic code. Black (white) cells contain triplets with strong (weak) roots. Red color shows triplets, which have different code meanings in a considered dialect in comparison with their code meanings in the Vertebrate Mitochondrial Code, which is the most symmetrical among all dialects.

13 ABOUT «TENSORCOMPLEX» NUMBERS

This Section describes a system of multidimensional numbers, which seem to be a new one for mathematical natural sciences and which are constructed on the basis of sums of genetic projectors described above. Here the author will take some data from his work [Petoukhov, 2012b].

First of all, let us return to two sums of genetic projectors h_{0} + h_{2} and h_{1} + h_{3} (from Figure 10), which were received from the Hadamard matrix H_{4} (Figure 1). The first sum h_{0} + h_{2} is decomposed into two basic matrices e_{0} = [1 0 0 0; -1 0 0 0; 0 0 1 0; 0 0 -1 0] and e_{2} = [0 0 -1 0; 0 0 1 0; 1 0 0 0; -1 0 0 0], the set of which is closed relative to multiplication and defines the multiplication table of complex numbers. The second sum h_{1} + h_{3} is decomposed into other basic matrices e_{1} = [0 1 0 0; 0 1 0 0; 0 0 0 1; 0 0 0 1] and e_{3} = [0 0 0 1; 0 0 0 1;
0 -1 0 0; 0 -1 0 0], the set of which is closed relative to multiplication and also defines the multiplication table of complex numbers (see Figure 10). Now one can consider linear compositions $C_L$ and $C_R$ on basis of these basic matrices (Figure 37, upper level). The work [Petoukhov, 2012b] shows that each of $C_L$ and $C_R$ is a ($4\times4$)-matrix representation of 2-parametric complex numbers over field of real numbers but these representations concern different 2-dimensional planes ($x_0, x_2$) and ($x_1, x_3$) of a 4-dimensional vector space.

Many people know that the sum of two complex numbers gives a new complex number and that the product of two complex numbers is commutative. This is true when these complex numbers belong to the same complex plane. But the sum of these ($4\times4$)-representations of two complex numbers $C_L$ and $C_R$, which belong to different planes of 4-dimensional space, is not equal to a new complex number, and their product is not commutative: $C_L*C_R \neq C_R*C_L$. Each of these products $C_L*C_R$ and $C_R*C_L$ gives a new complex number. Figure 40 (two middle levels) show expressions $C_L*C_R$ and $C_R*C_L$ with their decompositions into sets of two matrices, which correspond to the multiplication table of complex numbers. Figure 40 (bottom level) also shows an expression of a corresponding commutator. One should note here that the expression of the commutator $C_L*C_R-C_R*C_L$ on Figure 40 belongs to so called “tensorcomplex numbers”, which will be introduced below (Figures 41 and 42) with marks of their quadrants by means of yellow and green colors to emphasise a special cross-like structure of this type of numbers.

$$C_L = a_0e_0 + a_2e_2 = \begin{pmatrix} a_0 & 0 & -a_2 & 0 \\ -a_0 & 0 & a_2 & 0 \\ a_2 & 0 & a_0 & 0 \\ -a_2 & 0 & -a_0 & 0 \end{pmatrix} ; \quad C_R = a_1e_1 + a_3e_3 = \begin{pmatrix} 0 & a_1 & 0 & a_3 \\ 0 & a_3 & 0 & a_1 \\ 0 & -a_3 & 0 & a_1 \\ 0 & -a_1 & 0 & a_3 \end{pmatrix}$$

$$C_L*C_R = \begin{pmatrix} a_0a_1+a_2a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_R*C_L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_L*C_R-C_R*C_L = \begin{pmatrix} a_0^2+a_1^2 & a_0a_2+a_3 & a_0a_1+a_3 & a_3-a_1 \\ a_1a_2-a_0a_3 & a_1a_2-a_0a_3 & a_1a_2-a_0a_3 & a_1a_2-a_0a_3 \\ a_0a_1-a_3a_3 & a_0a_1-a_3a_3 & a_0a_1-a_3a_3 & a_0a_1-a_3a_3 \\ a_0a_1-a_3a_3 & a_0a_1-a_3a_3 & a_0a_1-a_3a_3 & a_0a_1-a_3a_3 \end{pmatrix}$$

Figure 40. Upper level: $C_L$ is a ($4\times4$)-matrix representation of complex numbers $z = a_0 + a_2^*i$ on a 2-dimensional plane ($x_0, x_2$) in a 4-dimensional space ($x_0, x_1, x_2, x_3$); here $i$ is imaginary unit of complex numbers ($i^2 = -1$). $C_R$ is a ($4\times4$)-matrix representation of complex numbers $z = a_1 + a_3^*i$ on another 2-dimensional plane ($x_1, x_3$) in the same 4-dimensional space ($x_0, x_1, x_2, x_3$). Here $a_0$, $a_1$, $a_2$ and $a_3$ are real numbers. Two middle levels: expressions of products $C_L*C_R$ and $C_R*C_L$; in both cases two basic matrices define the multiplication table of complex numbers. Bottom level: the expression of the commutator $C_L*C_R-C_R*C_L$. 
Now let us take a sum \( V = C_L + C_R \) of these (4*4)-matrix representations of complex numbers, which are related with different planes of the 4-dimensional space (Figure 41).

\[
V = C_L + C_R = \begin{bmatrix} a_0 & a_1 & -a_2 & a_3 \\ -a_0 & a_1 & -a_2 & a_3 \\ a_2 & -a_3 & a_0 & a_1 \\ -a_2 & -a_3 & -a_0 & a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes M + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes P
\]

\[
V^{-1} = (2*(a_0^2 + a_2^2))^{-1} \begin{bmatrix} a_0 & a_2 & -a_0 & -a_2 \\ -a_2 & a_0 & -a_0 & -a_2 \\ a_0 & -a_2 & a_0 & -a_2 \\ -a_2 & -a_0 & -a_2 & a_0 \end{bmatrix} + (2*(a_1^2 + a_3^2))^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_3 & a_3 & a_1 & a_1 \end{bmatrix}
\]

Figure 41. Upper level: the sum \( V = C_L + C_R \) (see Figure 40). Here \( M=\begin{bmatrix} a_0 & a_1 \\ -a_0 & a_1 \end{bmatrix} \) and \( P=\begin{bmatrix} -a_2 & a_3 \\ a_2 & a_3 \end{bmatrix} \); \( \otimes \) is a symbol of tensor (or Kronecker) multiplication. Bottom level: the expression of the inverse matrix \( V^{-1} \).

Figure 41 shows that \( V=C_L+C_R \) is a (4*4)-matrix of a special type, where both quadrants (marked by yellow color) along the main diagonal are identical each other, and two other quadrants (marked by green color) differ each from other only by inversion of sign in their entries. This matrix can be written in a form \( V = [1 \ 0 \ 0 \ 0] \otimes M + [0 \ 1 \ -1 \ 0] \otimes P \), where \( M=\begin{bmatrix} a_0 & a_1 \\ -a_0 & a_1 \end{bmatrix} \) and \( P=\begin{bmatrix} -a_2 & a_3 \\ a_2 & a_3 \end{bmatrix} \); \( \otimes \) is a symbol of tensor multiplication; \( [1 \ 0 \ 0 \ 0] \) – a matrix representation of real unit; \( [0 \ 1 \ -1 \ 0] \) – a matrix representation of imaginary unit; \( a_0, a_1, a_2 \) and \( a_3 \) are real numbers. Such form of notation resembles a well-known matrix representation of usual complex numbers: \( z = [1 \ 0 \ 0 \ 1]a + [0 \ 1 \ -1 \ 0]b \), where \( a \) and \( b \) are real numbers. But it differs in the following aspects:

1) the expression \( V = [1 \ 0 \ 0 \ 0] \otimes M + [0 \ 1 \ -1 \ 0] \otimes P \) includes tensor multiplication \( \otimes \) instead of usual multiplication in the matrix representation of complex numbers;
2) in the case of \( V \), multipliers of the basic elements are square matrices \( M \) and \( P \) instead of real numbers \( \langle \text{a} \rangle \) and \( \langle \text{b} \rangle \) in the case of complex numbers.
3) The order of the factors inside \( V \) is essential since tensor multiplication is not commutative.

Let us consider a set of matrices, which includes all matrices of this kind \( V \) together with their inverse matrices \( V^{-1} \) and together with all products of matrices of this kind. This set of matrices has properties of multi-dimensional numeric system as it is described below.

What one can say about algebraic properties of matrices of such type \( V \) (Figure 41)? Matrices of this type can be added and subtracted. The matrix \( V \) has its inverse matrix \( V^{-1} \) (Figure 42), which is defined on the basis of the condition \( V*V^{-1}=V^{-1}*V=E_4 \), where \( E_4 \) is identity matrix \( [1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \). Product of two different matrices of this type (for example \( G \) and \( S \) on Figure 42) generates a new (4*4)-matrix \( W=G*S \) (Figure 42), where both (2*2)-quadrants (marked by yellow color) along the main diagonal are identical each other, and two other (2*2)-quadrants (marked by green color) differ each from other only by inversion of sign in their entries. This matrix can be written in a form \( W = [1 \ 0 \ 0 \ 0] \otimes Q + [0 \ 1 \ -1 \ 0] \otimes K \), where \( Q=\begin{bmatrix} a^k*c-k*b*n-d*n, a^m*c^m+b^p-d*p, b*n-c^k-a^k-d*n, c^m-a^m-b^p-d*p \end{bmatrix} \) and \( K=\begin{bmatrix} c*n-a*n-d*k-b*k, a*p-b*m+a*d*m+c*p, b*k+a*n-d*k+c*n, b^k+a*n-d*k+c*n \end{bmatrix} \).
b\*m-a\*p+d\*m+c\*p]; \odot \text{ is a symbol of tensor multiplication; } [1 0; 0 1] – a matrix representation of real unit; [0 1; -1 0] – a matrix representation of imaginary unit; a, b, c, d, k, m, n and p are real numbers. Such form of denotation resembles the known matrix representation of usual complex numbers: z = [1 0; 0 1]\*a + [0 1; -1 0]\*b, where a and b are real numbers. But it differs again in the following aspects:

1) the expression W = [1 0; 0 1]⊗Q + [0 1; -1 0]⊗K includes tensor multiplication \odot instead of usual multiplication in the matrix representation of complex numbers;

2) in the case of W, multipliers of the basic elements are square matrices Q and K instead of real numbers «a» and «b» in the case of complex numbers.

3) The order of the factors inside W is essential since tensor multiplication is not commutative.

Taking into account a significant role of tensor multiplication \odot in W, the author names algebraic constructions in a form W as “tensorcomplex numbers” because such matrices W have the following algebraic properties in relation to usual operations of addition, subtraction, multiplication and division:

- Addition and subtraction of two different matrices of this type W create a new matrix of the same type. Multiplication of different matrices of this type with each other is noncommutative and it gives a new matrix of the same type (Figure 42).
- Each non-zero matrix W=G*S has an inverse matrix W−1 = G−1*S−1 (expressions for G−1 and S−1 were shown on Figure 41). It allows a definition of operation of division of two matrices of this type as a multiplication with an inverse matrix.

Such properties of tensorcomplex numbers resemble algebraic properties of quaternions by Hamilton, which represent noncommutative division algebra (http://en.wikipedia.org/wiki/Quaternion). Here one should emphasize that tensorcomplex numbers cardinaly differ from hypercomplex numbers \(x_0 + x_1*i + \ldots + x_n*i_n\), where \(x_0, x_1, \ldots, x_n\) are real numbers, because, in the case of tensorcomplex numbers, multipliers of the basic elements are square matrices but not real numbers. By this reason, the famous Frobenius theorem (http://en.wikipedia.org/wiki/Frobenius_theorem_(real_division_algebras)) for hypercomplex numbers is not related to tensorcomplex numbers. This theorem says that any finite-dimensional associative division algebra is isomorphic to one of the following algebras: the real numbers, the complex numbers, the quaternions by Hamilton.

\[
G = \begin{bmatrix} a & c & -b & d \\ -a & b & c & d \\ b & -d & a & c \\ -b & -d & -a & c \end{bmatrix}; \quad S = \begin{bmatrix} k & m & -n & p \\ -k & m & n & p \\ n & -p & k & m \\ -n & -p & -k & m \end{bmatrix}
\]

\[
G*S = \begin{bmatrix} a*k-c*k-b*n-d*n \\ b*n-c*k-a*k-d*n \\ b*k+a*n+d*k-c*n \\ d*k-a*n-b*k-c*n \end{bmatrix} \begin{bmatrix} a*m+c*m+b*p-d*p \\ c*m-a*m-b*p-d*p \\ b*k+a*n+d*k+c*n \\ c*n-a*n-d*k-b*k \end{bmatrix} + \begin{bmatrix} a*p-b*m+d*m+c*p \\ b*m-a*p+d*m+c*p \\ a*m+c*m+b*p-d*p \\ c*m-a*m-b*p-d*p \end{bmatrix}
\]

Figure 42. Multiplication of two matrices G and S gives a new matrix W = G*S, which belong to so called «tensorcomplex numbers». Here a, b, c, d, k, m, n and p are real numbers.
This set of tensorcomplex numbers is one of examples of numeric systems, where real numbers exist only inside matrices in a form of whole ensembles but not as individual multipliers (or as individual personages) inside such numeric systems. It is one of differences of tensorcomplex numbers from hypercomplex systems. One can mention that the commutator $C_r*C_l-C_l*C_r$ (Figure 40) belongs to tensorcomplex numbers.

Till now in this Section we considered the case of tensorcomplex numbers, which were represented by means of the expression $W = [1 0; 0 1] \otimes Q + [0 1; -1 0] \otimes K$, where $Q$ and $K$ are $(2*2)$-matrices. But our work [Petoukhov, 2012b, ...,] describes that complex numbers can be also represented by means of sparse $(2^N*2^N)$-matrices on the basis of sums of projectors (here $N = 2, 3, 4, ...$). One can take sum of two complex numbers, which belong to different planes of the same $2^N$-dimensional space and which are represented by means of appropriate $(2^N*2^N)$-matrices. In this case new types of tensorcomplex numbers arise. Theory and expressions for such tensorcomplex numbers in spaces of higher dimensions are developed now for a publication in the nearest future.

The author hopes that tensorcomplex numbers, which seem to be a new type of multidimensional numbers for mathematical natural sciences, will be useful not only in bioinformatics, but also in physics, theory of communication, logic and other fields.

14 ABOUT «TENSORHYPERBOLIC» NUMBERS

Now let us return to the Rademacher $(4*4)$-matrix $R_4$, which is a sum of 4 column projectors $c_0$, $c_1$, $c_2$, $c_3$: $R_4 = c_0+c_1+c_2+c_3$ (Figures 1 and 2). Our work [Petoukhov, 2012b, Figures 11-13] shows that sum $c_0+c_2$ is decomposed into basic matrices $c_0 = [1 0 0 0; -1 0 0 0; 0 0 1 0; 0 0 -1 0]$ and $c_2 = [0 0 1 0; 0 0 -1 0; 1 0 0 0; -1 0 0 0]$, the set of which is closed relative to multiplication and defines the multiplication table of hyperbolic numbers; their known synonyms are “split-complex numbers”, “double numbers” or “Lorentz numbers” (http://en.wikipedia.org/wiki/Split-complex_number). In this paper we will prefer using the name «hyperbolic numbers» for such type of 2-dimensional numbers. Another sum $c_1+c_3$ is decomposed into basic matrices $c_1 = [0 1 0 0; 0 1 0 0; 0 0 0 1; 0 0 0 1]$ and $c_3 = [0 0 0 -1; 0 0 1 0; 0 -1 0 0; 0 -1 0 0]$. Now one can consider linear compositions $D_L$ and $D_R$ on basis of these basic matrices (Figure 43). The work [Petoukhov, 2012b, Figures 12, 13] shows that each of $D_L$ and $D_R$ is a $(4*4)$-matrix representation of 2-parametric hyperbolic numbers over field of real numbers but these representations concern different 2-dimensional planes $(x_0, x_2)$ and $(x_1, x_3)$ of a 4-dimensional vector space.

$$D_L=a_0*e_0+a_2*e_2=\begin{vmatrix} a_0 & 0 & a_2 & 0 \\ -a_0 & 0 & -a_2 & 0 \\ a_2 & 0 & a_0 & 0 \\ -a_2 & 0 & -a_0 & 0 \end{vmatrix} \quad D_R=a_1*e_1+a_3*e_3=\begin{vmatrix} a_1 & 0 & -a_3 \\ a_3 & 0 & a_1 \\ -a_3 & 0 & a_1 \\ -a_1 & 0 & -a_3 \end{vmatrix}$$

Figure 43. Left side: $D_L$ is a $(4*4)$-matrix representation of hyperbolic (or split-complex) numbers $z = a_0 + a_2 * j$ on a 2-dimensional plane $(x_0, x_3)$ in a 4-dimensional space $(x_0, x_1, x_2, x_3)$; here $j$ is imaginary unit of hyperbolic numbers ($j^2 = 1$). Right side: $D_R$ is a $(4*4)$-matrix representation of hyperbolic numbers $z = a_1 + a_3 * j$ on another 2-dimensional plane $(x_1, x_3)$ in the same 4-dimensional space $(x_0, x_1, x_2, x_3)$. Here $a_0, a_1, a_2$ and $a_3$ are real numbers.

Now let us take a sum $DL+DR$ of these $(4*4)$-matrix representations of hyperbolic numbers, which are related with different planes of the 4-dimensional space (Figure 44).
Figure 4 shows that $D = D_L + D_R$ is a (4*4)-matrix of a special type, where both quadrants along each of diagonals are identical (they are marked by yellow and blue colors). This matrix can be written in a form $D = [1\; 0\; 0\; 1] \otimes M + [0\; 1\; 1\; 0] \otimes K$, where $M = [a_0\; a_1; -a_0\; -a_1]$ and $K = [a_2\; -a_3; -a_2\; -a_3]$, $\otimes$ is a symbol of tensor multiplication. Middle level: the example of zero divisors in this type of matrices. Bottom level: the inverse matrix $D^{-1}$.

The set of matrix D has zero divisors, examples of which are shown on Figure 4. Figure 44 also shows a general expression of the inverse matrix $D^{-1}$ for the matrix D.

Multiplication of two matrices $D_0$ and $D_1$ of this type gives a new matrix $L$ (Figure 45), where both quadrants along each of diagonals are identical (they are marked by yellow and blue colors). This matrix can be represented in the following form: $L = [1\; 0\; 0\; 1] \otimes Q + [0\; 1\; 1\; 0] \otimes K$, where $Q = [a_k\; c_k; b_k\; 0\; +d_k\; n; +b_k\; m; +c_k\; p; +d_k\; m; +b_k\; p; +d_k\; p], K = [a\; m;\; +c_k; b\; n; +d_k; k; b\; m; a\; p; c; p; d; m; b\; k; c\; n; +d_k; m; b\; k; c\; n; a\; n; a\; p; b\; m; c; p; d; m].$

Taking into account a significant role of tensor multiplication $\otimes$ in $L$, the author names algebraic constructions in a form L (Figure 45) as “tensorhyperbolic numbers” because such matrices L have the following algebraic properties in relation to usual operations of addition, subtraction, multiplication and division:
• Addition, subtraction and multiplication of two different matrices of this type create a new matrix of the same type. Multiplication of different matrices of this type with each other is noncommutative. The set of matrices L has zero divisors.

• Each non-zero matrix \( L = D_0 * D_1 \), if it is not a zero divisor, has an inverse matrix \( L^{-1} = D_0^{-1} * D_1^{-1} \) (expressions for \( D^{-1} \) was shown on Figure 44). It allows a definition of operation of division of two matrices of this type as a multiplication with an inverse matrix.

\[
\begin{bmatrix}
  a & c & b & -d \\
  -a & c & -b & -d \\
  b & -d & a & c \\
  -b & -d & -a & -c \\
\end{bmatrix} \quad \begin{bmatrix}
  k & m & n & -p \\
  -k & m & -n & -p \\
  n & -p & k & m \\
  -n & -p & -k & m \\
\end{bmatrix} \quad \begin{bmatrix}
  a^*k + b^n + d^*n, & a^*m + c^n + b^*p + d^*p \\
  d^*n + c^n + b^*a + k, & c^n + a^*m + b^*p + d^*p \\
  a^*n + b^n + d^*n, & b^*m - a^n + c^*p - d^*m \\
  d^*k - b^n + c^n - k, & c^n + a^*m + b^*p + d^*m \\
\end{bmatrix}
\]

L = \( D_0 * D_1 \)

Figure 45. Multiplication of two matrices \( D_0 \) and \( D_1 \) (of the type D from Figure 44) gives a new matrix \( L = D_0 * D_1 \), which belongs to so called «tensorhyperbolic numbers». Here a, b, c, d, k, m, n and p are real numbers.

This Section has described the case of tensorhyperbolic numbers in the form of \((4 * 4)\)-matrices for 4-dimensional spaces. But tensorhyperbolic numbers and their generalization can be expressed in forms of \((2^N * 2^N)\)-matrices for \(2^N\)-dimensional spaces. These materials will be published later together with data about «tensordual» numbers, «tensorquaternions», etc.

Different types of such multidimensional numbers can be combined under a brief name «tensornumbers». Tensornumbers \([1] \otimes M_0 + [i_1] \otimes M_1 + \ldots + [i_n] \otimes M_n \) (here \( M_n \) are square matrices) are a generalization of hypercomplex numbers in the case when the following changes are made in the usual denotation of hypercomplex numbers \( 1^*x_0 + i_1^*x_1 + \ldots + i_n^*x_n \):

• Real multipliers \( x_0, x_1, \ldots, x_n \) are replaced by square matrices \( M_0, M_1, \ldots, M_n \);
• Usual multiplication is replaced by tensor multiplication.

It is obvious that hypercomplex numbers \( 1^*x_0 + i_1^*x_1 + \ldots + i_n^*x_n \) are a degenerate case of tensornumbers when their matrices \( M_n \) have the first order: 1) \((1 * 1)\)-matrices \( M_n = [x_n] \) are real numbers \( x_n \); 2) tensor multiplication \([i_n] \otimes [x_n]\) of \((1 * 1)\)-matrix \([x_n]\) with the matrix representation \([i_n]\) of any of the basic elements is commutative and it coincides with usual multiplication. By these reasons the following equation is true: \( [1] \otimes [x_0] + [i_1] \otimes [x_1] + \ldots + [i_n] \otimes [x_n] = [1]^*x_0 + [i_1]^*x_1 + \ldots + [i_n]^*x_n \).

What one can say at this initial stage about a future of tensornumbers in mathematical natural sciences and technologies? Two extreme points of view are possible here: 1) tensorcomplex numbers will have no applications; 2) tensorcomplex numbers will have a great significance for mathematical natural sciences including a creation of new theories of physical fields and their generalization, new laws of conservation, generalization of many physical and other rules and knowledge, new approaches in engineering and biological informatics, mathematical logics, etc. The author believes that the second point of view will coincide with a real future in a higher extent.
SOME CONCLUDING REMARKS

As it was noted in the beginning of the article, living organism is a machine for coding and processing of information. For example, visual information about external objects is transmitted through the nerves from the eye retina to the brain already in a logarithmically encoded form. This article shows some evidences that oblique projection operators and their combinations, which are connected with matrix representations of the genetic coding system, can be a basis of adequate approaches to simulate ensembles of inherited biological phenomena. Only some of such phenomena were considered in this article. Some other phenomena will be described from the proposed point of view later. In addition, the author reminds about fractal genetic nets (FGN), which can be represented as a construction on a base of orthogonal projectors and which lead to new genetic rules in structures of long nucleotide sequences [Petoukhov, 2012; Petoukhov, Svirin, 2012].

The revealed genetic system of operators connected with oblique projectors allows modeling multi-dimensional phase spaces with many subspaces, processes in which can be selectively determined and controled. Speaking about importance of projectors in the nature, the following points should be noted (it seems that the nature «likes» projectors):

• Most people are familiar with the idea of projectors due to sun rays (and light rays in general) that are distributed in a straight line and provide shade from the subjects (from the ancient time sundials were constructed on the use of this). Light rays have the projection property;
• Electromagnetic vectors are the sum of their projections in the form of their electric and magnetic vectors;
• Evolution of living organisms is associated with the consumption of solar energy that is projected by means of sun rays to surfaces of living bodies (photosynthesis, which is for living matter one of its basic mechanisms, is the conversion of sunlight energy into biochemical energy for activity of organisms; circadian biorhythms are connected with external light cycles "day-night");
• Projection phenomena of birefringence in biological tissues and crystals exist;
• A great variety of living organisms has polarization eyesight;
• Our vision is based on the projection of images on the retina;
• Religious people may ask whether there is any indication in the Bible on this subject? Especially for them, it may be recalled the following. According to the Bible, God's creation of the world began with the creation of the light: "Let there be light." Many thinkers suggested previously for various reasons that, figuratively speaking, the body is woven from the light. This has some associations with our hypothesis that the body is woven from the projectors.

Now projectors and their combinations become interesting instruments to study and simulate genetic phenomena and inherited structures and processes in living matter. A new conceptual notion with appropriate mathematical formalisms are proposed about a multi-dimensional control space (or coding space) with subspaces of a selective control in each on basis of a participation of projection operators in such control. Here one can remember the statement: “Profound study of nature is the most fertile source of mathematical discoveries” (Fourier, 2006, Chapter 1, p. 7).

Using this ideology of projection operators, one can get many unexpected results and approaches. In author's opinion, one of many promising applications of projectors in
mathematical biology and bioinformatics is the study of connections between genetic projectors and Boolean algebra. It is known that every family of commutative projectors generates a Boolean algebra of projectors. The Boolean algebra plays a great role in the modern science because of its connections with many scientific branches: mathematical logic, the problem of artificial intelligence, computer technologies, bases of theory of probability, etc. G. Boole was creating such algebra of logics (or logical operators), which would reflect inherited laws of human thought. One should note here that some of genetic projectors (which are not described in this article) form commutative pairs; this fact provokes thoughts about Boolean algebras in genetics and bioinformatics and also about genetic basis of logics of human thought.

Genetic molecules are subordinated to laws of quantum mechanics, which has begun from matrix mechanics by W. Heisenberg. Till this pioneer work by Heisenberg, matrices were not used in physics where specialists operated only with numbers to study physical systems. It was very unexpected for scientific community that applying whole ensembles of numbers in a form of matrices can be useful and appropriate to describe natural phenomena and systems. That matrices were not used in physics till a creation of the matrix mechanics by W. Heisenberg (modern quantum mechanics was begun. Previously physicists operated with numbers in physical tasks. It was very unexpected that using whole ensembles of numbers in a form of matrices can be useful and appropriate to describe natural phenomena and systems. Contemporary science uses matrices widely in many fields, and our work uses matrices to study molecular-genetic systems.

Materials of this article reinforce the author’s point of view that living matter in its informational fundamentals is an algebraic essence. The author believes that a development of algebraic biology, elements of which are contained in this and other author’s articles, is possible. By analogy with the known fact that molecular foundations of molecular genetics turned up unexpectedly very simple, perhaps algebraic foundations of living matter are also relative simple. In the infinite set of matrices, we find a small subset, which simulates the world of molecular genetic coding with many of its phenomenologic features; this discovery was possible due to studying the family of alphabets in the molecular-genetic system.

Concerning the new theme of tensorsnumbers, which were revealed in our study of genetic projectors, one can remind that the idea of multi-dimensional numbers and multi-dimensional spaces works intensively for a long time in the theoretical physics and other fields of science for modeling the phenomena of our physical world. Our results add mathematical formalisms, first of all, into the fields of molecular genetics and bioinformatics. After the discovery of non-Euclidean geometries and of Hamilton quaternions, it is known that different natural systems can possess their own geometry and their own algebra (see about this [Kline, 1980]). The genetic code is connected with its own multi-dimensional numerical systems or the multi-dimensional algebras. These algebras allow revealing hidden peculiarities of the structure of the genetic code and its evolution. It seems that many difficulties of modern bioinformatics and mathematical biology are connected with utilizing for their natural structures inadequate algebras, which were developed for completely other natural systems. Hamilton had similar difficulties in his attempts to describe 3D-space transformations by means of 3-dimensional numbers while this description needs quaternions. The author hopes that proposed tensorsnumbers will help to make progress not only in bioinformatics and mathematical biology, where algebraization of biology seems to be possible, but also in many fields of sciences and technologies.
“Complexity of a civilization is reflected in complexity of numbers used by this civilization” [Davis, 1967]. Whether modern civilization will use tensor numbers or not? It is the open question. Pythagoras has formulated the idea: “all things in the world are numbers” or “number rules the world”. B. Russell noted that that he did not know other person who would exert such influence on thinking of people as Pythagoras. From this viewpoint, there is no more fundamental scientific idea in the world, than this idea about a basic meaning of numbers. Our researches of oblique projectors in the field of matrix genetics have led to new systems of multidimensional numbers and have given new materials to the great idea by Pythagoras in its modernized formulating: “All things are multi-dimensional numbers”.

This article proposes a new mathematical approach to study “a partnership between genes and mathematics” (see Section 1 above). In the author’s opinion, this kind of mathematics is beautiful and it can be used for further developing of algebraic biology and theoretical physics in accordance with the famous statement by P. Dirac, who taught that a creation of a physical theory must begin with the beautiful mathematical theory: “If this theory is really beautiful, then it necessarily will appear as a fine model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathematical theory and to interpret them as predictions of new laws of physics” (this quotation is taken from [Arnold, 2007]). According to Dirac, all new physics, including relativistic and quantum, are developing in this way. One can suppose that this statement is also true for mathematical biology.

APPENDIX 1. COMPLEX NUMBERS, CYCLIC GROUPS AND SUMS OF GENETIC PROJECTORS

This Appendix shows a connection between complex numbers and cyclic groups on the base of sums of (8*8)-projectors $s_0 + s_2$, $s_1 + s_3$, $s_4 + s_6$, $s_5 + s_7$ from Fig. 7 on the base of the Rademacher (8*8)-matrix $R_8$ (on Fig. 9 these sums were marked by green color and they corresponded to cyclic groups, if the weight coefficient $2^{-0.5}$ was used for them). Each of these 8 sums can be decomposed into two matrices $e_{2k}$ and $e_{2k+1}$ ($k=0, 1, ...7$), a set of whose is closed relative to multiplication and has a multiplication table, which coincides with the multiplication table of basic elements of complex numbers (Fig. 46). It means that these matrices $e_{2k}$ and $e_{2k+1}$ represent basic elements of complex numbers in corresponding 2-dimensional planes of a 8-dimensional vector space. Sets of matrices $a_{2k} e_{2k} + a_{2k+1} e_{2k+1}$ (here $a_{2k}$ and $a_{2k+1}$ are real numbers; each of matrices $e_{2k}$ plays a role of unitary matrix inside the appropriate set $a_{2k} e_{2k} + a_{2k+1} e_{2k+1}$) represent complex numbers inside these 2-dimensional planes of the 8-dimensional space.

$$s_0 + s_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = e_0 + e_1 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
e_0 e_1 \\
e_0 e_1 \\
e_1 e_0 \end{array}
\end{array}
\end{array}
\]
Fig. 46. The decomposition of each of (8*8)-matrices $s_0+s_2$, $s_0+s_3$, $s_1+s_2$, $s_1+s_3$, $s_4+s_6$, $s_4+s_7$, $s_5+s_6$, $s_5+s_7$ from Fig. 7 into a set of two matrices $e_{2k}$ and $e_{2k+1}$ ($k=0, 1, \ldots, 7$), a set of whose is closed relative to multiplication and gives the multiplication table of complex numbers (on the right)

In a general case, the described approach allows constructing selective operators of a 2^n-dimensional vector space with a set of different 2-dimensional planes, each of which can contain a function of complex numbers (parameters of these functions in different planes can be independent or interrelated). Such selective operator allows simulating a combinatorial behaviour of a multi-parametric system, which contains different 2-parametric subsystems, whose independent or interrelated behaviours can be simulated by means of functions of complex numbers. If these functions of complex numbers are cyclic, such selective operators describes a behaviour of a multi-parametric system, which contains an appropriate ensemble of 2-parametric subsystems with cyclic behaviours.

**APPENDIX 2. HYPERBOLIC NUMBERS AND SUMS OF GENETIC PROJECTORS**

Let us show now a connection between hyperbolic numbers and sums of (8*8)-projectors $s_0+s_4$, $s_0+s_5$, $s_1+s_4$, $s_1+s_5$, $s_2+s_4$, $s_2+s_5$, $s_3+s_4$, $s_3+s_5$, $s_4+s_6$, $s_4+s_7$, $s_5+s_6$, $s_5+s_7$ from Fig. 7 on the base of the Rademacher (8*8)-matrix $R_4$ (on Fig. 9 these sums were marked by red color). Each of these 12 sums can be decomposed into two matrices $j_{2k}$ and $j_{2k+1}$ ($k=0, 1, 2, \ldots, 11$), a set of whose is closed relative to multiplication and has a multiplication table, which coincides with the multiplication table of basic elements of hyperbolic numbers (Fig. 47). It means that these matrices $j_{2k}$ and $j_{2k+1}$ represent basic elements of hyperbolic numbers in corresponding 2-dimensional planes of a 8-dimensional vector space. Sets of matrices $a_{2k}j_{2k} + a_{2k+1}j_{2k+1}$ (here $a_{2k}$ and $a_{2k+1}$ are real numbers; each of matrices $j_{2k}$ plays a role of unitary matrix inside the appropriate set $a_{2k}j_{2k} + a_{2k+1}j_{2k+1}$) represent hyperbolic numbers inside these 2-dimensional planes of the 8-dimensional space.
\[
\begin{align*}
\mathbf{s_0+s_1} & = j_0 + j_1 = \\
\left[
\begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
\end{array}
\right]
& = \\
\left[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{array}
\right] + \\
\left[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{array}
\right]
\end{align*}
\]

\[
\begin{align*}
\mathbf{s_0+s_4} & = j_2 + j_3 = \\
\left[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 \\
\end{array}
\right] & = \\
\left[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\right] + \\
\left[
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\right]
\end{align*}
\]

\[
\begin{align*}
\mathbf{s_0+s_5} & = j_4 + j_5 = \\
\left[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 \\
\end{array}
\right] & = \\
\left[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\right] + \\
\left[
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\right]
\end{align*}
\]

\[
\begin{align*}
\mathbf{s_1+s_4} & = j_6 + j_7 = \\
\left[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\right] & = \\
\left[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\right] + \\
\left[
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\right]
\end{align*}
\]

\[
\begin{align*}
\mathbf{s_1+s_5} & = j_8 + j_9 = \\
\left[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\right] & = \\
\left[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
\end{array}
\right] + \\
\left[
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\right]
\end{align*}
\]
\[
\begin{array}{c|c|c}
\text{s}_2 + \text{s}_3 & = j_{10} + j_{11} & + \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
j_{10} \\
j_{11} \\
\end{bmatrix} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{s}_2 + \text{s}_6 & = j_{12} + j_{13} & + \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} & \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
j_{12} \\
j_{13} \\
\end{bmatrix} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{s}_2 + \text{s}_7 & = j_{14} + j_{15} & + \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} & \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
j_{14} \\
j_{15} \\
\end{bmatrix} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{s}_3 + \text{s}_6 & = j_{16} + j_{17} & + \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix} & \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
j_{16} \\
j_{17} \\
\end{bmatrix} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{s}_3 + \text{s}_7 & = j_{18} + j_{19} & + \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix} & \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
j_{18} \\
j_{19} \\
\end{bmatrix} \\
\end{array}
\]
**APPENDIX 3. ANOTHER TENSOR FAMILY OF GENETIC HADAMARD MATRICES**

Hadamard matrices are well-known in noise-immunity coding, quantum mechanics, etc. Their rows are Walsh functions, which are widely used in radiocommunication for a code division in systems with many channels, etc., for example, in cellular standards such as IS-95, CDMA2000 or UMTS. Walsh functions and corresponding series and transforms find various applications in physics and engineering, in particular, in digital signal processing. They are used in speech recognition, in medical and biological image processing, in digital holography, and other areas.

Above we have described the variant of the relation of the molecular-genetic system with the special tensor family of Hadamard matrices $H_4$, $H_8$, etc. (Fig. 1 and 38). This variant uses the phenomenological fact of existence of triplets with strong and weak roots; in other words this variant is based on a specificity of the degeneracy of the genetic code. This Appendix shows second variant of a relation of the genetic alphabets with another tensor family of Hadamard matrices. This new family of genetic Hadamard matrices is based only on properties of the genetic alphabet A, C, G, T and doesn't depend on the degeneracy of the genetic code. On the author's opinion, this new variant is more interesting and fundamental for further using in future genetic researches.

Fig. 48 shows the beginning of this new tensor family of Hadamard matrices $P^{(n)} = [1 \ 1; \ -1 \ 1]^{(n)}$ together with genetic matrices $[C \ T; \ A \ G]^{(m)}$ ($m=1, 2, 3$) of monoplets, duplets and triplets with their black-and-white mosaics which coincide with mosaics of the Hadamard matrices.

---

**Fig. 47.** The decomposition of each of ($8*8$)-matrices $s_0+s_1$, $s_0+s_4$, $s_0+s_5$, $s_1+s_4$, $s_1+s_5$, $s_2+s_3$, $s_2+s_6$, $s_2+s_7$, $s_3+s_6$, $s_3+s_7$, $s_4+s_5$, $s_6+s_7$ from Fig. 7 into a set of two matrices $j_{2k}$ and $j_{2k+1}$ ($k=0, 1, ..., 11$), a set of whose is closed relative to multiplication and gives the multiplication table of hyperbolic numbers (on the right)

\[
s_4+s_5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0
\end{bmatrix} = j_{20}+j_{21} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
s_6+s_7 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix} = j_{22}+j_{23} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Fig. 48. Upper level: the beginning of the tensor family of Hadamard matrices \( P^{(n)} = [1 1; \ -1 \ -1] \), where \( P_4 = [1 1; \ -1 \ -1] \) and \( P_8 = [1 1; \ -1 \ -1] \). Bottom level: genetic matrices \([C \ T; \ A \ G]\), \([C \ T; \ A \ G]^{(2)}\) and \([C \ T; \ A \ G]^{(3)}\) of monoplets, duplets and triplets with their black-and-white mosaics, which coincide with mosaics of the Hadamard matrices \( P_2, P_4, P_8 \).

On Fig. 48 the black-and-white matrices \( P_2, P_4, P_8 \) are Hadamard matrices because they satisfy the criterium \( H_n * H_n^T = n * E \) [Ahmed, Rao, 1975]. The genetic matrices \([C \ T; \ A \ G]\), \([C \ T; \ A \ G]^{(2)}\) and \([C \ T; \ A \ G]^{(3)}\) algorithmically have the same black-and-white mosaics on the base of fundamental properties of the DNA alphabet (adenine A, cytosine C, guanine G and thymine T). These properties contrapose the letter T against three other letters of the DNA-alphabet by the following phenomenological facts:

- the thymine T is a single nitrogenous base in DNA which is replaced in RNA by another nitrogenous base U (uracil) for unknown reason;
- the thymine T is a single nitrogenous base in DNA without the amino-group \( \text{NH}_2 \) (Fig. 49), which plays an important role in molecular genetics. For instance, the amino-group in amino acids provides a function of recognition of the amino acids by ferments [Chapeville, Haenni, 1974]. A detachment of amino-groups in nitrogenous bases A and C in RNA under action of nitrous acid \( \text{HNO}_2 \) determines a property of amino-mutating of these bases, which was used to divide the set of 64 triplets into eight natural subsets with 8 triplets in each [Wittmann, 1961].
Fig. 49. The complementary pairs of the four nitrogenous bases in DNA. A-T (adenine and thymine), C-G (cytosine and guanine). Amino-groups NH2 are marked by big circles. Black circles are atoms of carbon; small white circles are atoms of hydrogen; squares with the letter N are atoms of nitrogen; triangles with the letter O are atoms of oxygen.

From the point of view of these two facts, the letters A, C, G are identical to each other and the letter T is opposite to them. Correspondingly this binary-oppositional division inside the DNA-alphabet can be reflected by the symbol “+1” for each of the letters A, C, G and by the opposite symbol “−1” for the letter T. Concerning the genetic matrices [C T; A G] this approach leads to a simple algorithm for assigning a sign “+1” or “−1” to each of the letters A, C, G and by the opposite symbol “−1” for the letter T. For example, the triplet CTG has the sign “−1” because 1*(-1)*1=-1; the triplet CTT has the sign “+1” because 1*(-1)*(-1)=+1, etc.

Inside of the genetic matrices [C T; A G] on Fig. 48, all multiplets with the symbol “+1” are denoted by black color and all multiplets with the symbol “−1” are denoted by white color. In the result we have the connection of the genetic matrices [C T; A G] on the base of fundamental molecular-genetic properties, which can be used in genetic computers of living organisms.

The Hadamard matrices P4 and P8 (Fig. 48) can be correspondingly decomposed into 4 and 8 sparse matrices by analogy with the «column» decompositions of the Hadamard matrices H4 and H8 on Fig. 10 and 14. Results of such decompositions are shown on Fig. 50 and 51.

\[
P_4 = K_0 + K_1 + K_2 + K_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & -1 & 0 \\
al-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Fig. 50. The «column» decomposition of the Hadamard matrix P4 from Fig. 48 into 4 sparse matrices K0, K1, K2 and K3, each of which is a projector.

\[
P_8 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Fig. 51. The «column» decomposition of the Hadamard matrix P8 from Fig. 48 into 8 sparse matrices J0, J1, J2, J3, J4, J5, J6 and J7.
Tabular forms on Fig. 52 in analogy with one can analyze into sparse matrices

|   | 0 0 -1 0 0 0 0 0 | 0 0 1 0 0 0 0 0 | 0 0 0 0 -1 0 0 0 |
|---|---|---|---|
|   | 0 0 -1 0 0 0 0 0 | 0 0 1 0 0 0 0 0 | 0 0 0 0 -1 0 0 0 |
|   | 0 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 0 | 0 0 0 0 -1 0 0 0 |
|   | 0 0 -1 0 0 0 0 0 | 0 0 1 0 0 0 0 0 | 0 0 0 0 -1 0 0 0 |
|   | 0 0 -1 0 0 0 0 0 | 0 0 1 0 0 0 0 0 | 0 0 0 0 -1 0 0 0 |
|   | 0 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 0 | 0 0 0 0 -1 0 0 0 |
|   | 0 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 0 | 0 0 0 0 -1 0 0 0 |

Fig. 51. The «column» decomposition of the Hadamard matrix \(P_8\) from Fig. 48 into 8 sparse matrices \(J_0, J_1, J_2, J_3, J_4, J_5, J_6\) and \(J_7\), each of which is a projector.

Each of these sparse matrices \(K_0, K_1, K_2, K_3\) and \(J_0, J_1, ..., J_7\) is a projector. Now one can analyze properties of sums of pairs of these new projectors in relation to their exponentiation: \((K_0+K_1)^n\), \((K_0+K_2)^n\), ..., \((K_2+K_3)^n\), \((J_0+J_1)^n\), \((J_0+J_2)^n\), ..., \((J_6+J_7)^n\). By analogy with the similar analysis of sums of projectors from the first variant of the Hadamard matrices \(H_4\) and \(H_8\) (Fig. 1, 10-14, 16 and 38), we have got results of such analysis shown in tabular forms on Fig. 52 in the case of the Hadamard matrices \(P_4\) and \(P_8\).

![Fig. 52](image-url)

Fig. 52. Bi-symmetrical tables of features of sums of pairs of projectors \(K_0, K_1, K_2, K_3\) and \(J_0, J_1, ..., J_7\) (shown on Fig. 50 and 51) from the Hadamard matrices \(P_4\) and \(P_8\) in relation to their exponentiation: \((K_0+K_1)^n\), \((K_0+K_2)^n\), ..., \((K_2+K_3)^n\), \((J_0+J_1)^n\), \((J_0+J_2)^n\), ..., \((J_6+J_7)^n\).

By analogy with Fig. 4, 9, 11, 16 and 19, tabular cells with green color on Fig. 52 correspond to those matrices, exponentiations of which generate cyclic groups with a period 8 in the case of using the normalizing factor \(2^{-0.5}\). For example, \((2^{-0.5}(K_0+K_1))^n = (2^{-0.5}(K_0+K_1))^{n+8}\). One can show that the matrices in cells with green color represent
complex numbers with unit coordinates inside appropriate 2-dimensional planes of 4-dimensional or 8-dimensional spaces correspondingly. Tabular cells with yellow color on Fig. 52 correspond to matrices with the «quadruplet property»: for example, \((K_0+K_3)^2 = 4^{n-1}*(K_0+K_3)^2\) and \((J_0+J_3)^2 = 4^{n-1}*(J_0+J_3)^2\), where \(n = 1, 2, 3\ldots\). Similar tabular results can be also obtained for the case of «row» decompositions of Hadamard matrices \(P_4\) and \(P_8\).

By analogy with the equation (2) the following equation holds true to receive Hadamard \((2^n * 2^n)\)-matrices \(P_K\) (where \(K=2^n\), \(n = 4, 5, 6, \ldots\)): \(P_4 \otimes [1 -1; 1 \ 1]^{4n-2} = P_K\).

**APPENDIX 4. ABOUT SOME APPLICATIONS IN ROBOTICS**

Turning once more to tensornumbers (first of all, to tensorcomplex numbers), which were described above, one can note their possible application in some tasks of robotics when movement control of a group of robots is needed. In this case a special class of multi-parametric system of a tensornumber organization is under consideration. Control and encoding in these systems can be organized so that each of the representative set of subsystems may be selectively controlled and coded independently from other subsystems. An example of such a multi-parameter system is a group of robots, each of which moves along a certain trajectory plane in accordance with the program from a matrix operator, whose components are functions of time; the entire set of these individual operators incorporated into a single matrix operator, whose multiplication with another matrix operator of a similar structure generates a new matrix operator, endowed with the same property independent motion control of each robot in a new regime (This is the problem of collective motion control of a set of robots, each of which can move quite independently of the others due to the fact that the management of its movement is carried by its "personal" sub-operator from a general matrix operator, allowing collective restructuring of all sub-operators by means of simple multiplication of the general operator with a matrix operator of a similar structure).

Consider an example of collective management of an 8-parametric system, which has a tensorcomplex type of its organization and which consists of four 2-parametric subsystems (four robots), the status of each of which may change over time regardless of the status of the other three subsystems. This management can be carried out using an \((8 \times 8)\)-matrix \(M(t)\), which is an operator of the tensorcomplex type; all its 8 components \(a_0, a_1, \ldots, a_7\) are functions of time \(t\) (Figure 53). A state of the whole system inside its configuration space during time can be characterized by an 8-dimensional vector \([x_0, x_1, x_2, \ldots, x_7]\), which is determined by the operator \(M(t)\).

\[
M(t) = \begin{pmatrix}
-a_0(t) & a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t) \\
a_0(t) & -a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t) \\
-a_0(t) & a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t) \\
-a_0(t) & a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t) \\
a_0(t) & -a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t) \\
a_0(t) & a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t) \\
-a_0(t) & a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t) \\
-a_0(t) & a_1(t) & a_2(t) & a_3(t) & a_4(t) & a_5(t) & a_6(t) & a_7(t)
\end{pmatrix}
\]

Fig. 53. The example of \((8*8)\)-matrix as a general operator for movement control of 4 robots in a case when they form an 8-parametric system (explanation in text).
In this example, behavior of each of the four 2-parametric subsystems is graphically depicted by means of movement of a point inside one of four coordinate planes \((x_0, x_4), (x_1, x_5), (x_2, x_6), (x_3, x_7)\) of the configuration space of the system; this behavior can determined entirely independently from behavior of the other three subsystems. To specify the path and pace of movement of each point along its individual trajectory, which is defined parametrically, it is only necessary to set the functions \(a_n(t)\) of the operator \(M(t)\).

For instance, independent movements of the points in the planes \((x_0, x_4), (x_1, x_5), (x_2, x_6), (x_3, x_7)\) occur along known types of curves - "cardoid", "petal clover", "5-petal rose" and "logarithmic spiral" (Figure 54) - in case, when the functions \(a_n(t)\) are the following:

\[
\begin{align*}
a_0(t) &= \cos(t) \cdot (1 + \cos(t)); \\
a_1(t) &= \cos(t) \cdot \cos(3t) + \cos(t) \cdot \sin^2(3t); \\
a_2(t) &= 2 \cdot \cos(t) \cdot \sin(5/3t); \\
a_3(t) &= \cos(t) \cdot (13/12)^t; \\
a_4(t) &= \sin(t) \cdot \sin(5/3t); \\
a_5(t) &= \sin(t) \cdot \sin(5/3t); \\
a_6(t) &= \sin(t) \cdot \sin(5/3t); \\
a_7(t) &= \sin(t) \cdot (13/12)^t.
\end{align*}
\]

In the plane \((x_0, x_4)\), motion occurs along a cardoid:

In the plane \((x_1, x_5)\), motion occurs along a "petal clover":

In the plane \((x_2, x_6)\), motion occurs along a "5-petal rose":

In the plane \((x_3, x_7)\), motion occurs along a logarithmic spiral:

Fig. 54. The example of motion trajectories for 4 sub-systems of the 8-parametric system (explanation in text).

REFERENCES

Adler I., Barabe D., Jean R. V. (1997) A History of the Study of Phyllotaxis. *Annals of Botany* 80: 231–244, http://aob.oxfordjournals.org/content/80/3/231.full.pdf

Ahmed N., Rao K. (1975) Orthogonal transforms for digital signal processing. N-Y, Springer-Verlag Inc.

Arnold, V. (2007) A complexity of the finite sequences of zeros and units and geometry of the finite functional spaces. *Lecture at the session of the Moscow Mathematical Society*, May 13, http://elementy.ru/lib/430178/430281.

Chapeville F., Haenni A.-L. (1974). *Biosythese des proteins*. Paris, Hermann.

Cook, T.A. (1914). *The curves of life*. London: Constable and Company Ltd, 524 p.

Davis P.J. (1964). Arithmetics. – In: “Mathematics in the modern world”, N. Y., Scientific American, p. 29-45.

Eigen, M. (1979) *The hypercycle - a principle of natural self-organization*. Berlin: Springer-Verlag.
Fourier J. (1878). *The analytical theory of heat*. – Cambridge: Deighton, Bell and Co. https://archive.org/details/analyticaltheory00fourrich

Halmos P.R. (1974) *Finite-dimensional vector spaces*. London: Springer, 212 p.

He M., Petoukhov S.V. (2011) Mathematics of bioinformatics: theory, practice, and applications. USA: John Wiley & Sons, Inc., 295 p.

Jean, R.V. (1995) *Phyllotaxis: A Systemic Study in Plant Morphogenesis*. Cambridge University Press, 1995. (Russian edition, 2005, Scientific Editor – S. Petoukhov)

Kline, M. (1980) *Mathematics. The loss of certainty*. New York: Oxford University Press.

Klein F. (2009). *Elementary Mathematics from an Advanced Standpoint*. - Cosimo, 284 p.

Konopelchenko, B. G., Rumer, Yu. B. (1975a) Classification of the codons of the genetic code. I & II. Preprints 75-11 and 75-12 of the Institute of Nuclear Physics of the Siberian department of the USSR Academy of Sciences. Novosibirsk: Institute of Nuclear Physics.

Konopelchenko, B. G., Rumer, Yu. B. (1975b). Classification of the codons in the genetic code. *Doklady Akademii Nauk SSSR*, 223(2), 145-153 (in Russian).

Messiah A. (1999) *Quantum mechanics*. New York: Dover Publications, 1152 p.

Meyer C. D. (2000), *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, 2000. ISBN 978-0-89871-454-8.

Petoukhov S.V. (1981). *Biomechanics, bionics and symmetry*. – Moscow, Nauka, 239 p. (in Russian).

Petoukhov S.V. (2001a) Genetic codes I: binary sub-alphabets, bi-symmetric matrices and the golden section; Genetic codes II: numeric rules of degeneracy and the chronocyclic theory. *Symmetry: Culture and Science*, vol. 12, #3-4, p. 255-306.

Petoukhov S.V. (2005) The rules of degeneracy and segregations in genetic codes. The chronocyclic conception and parallels with Mendel’s laws. - In: *Advances in Bioinformatics and its Applications, Series in Mathematical Biology and Medicine*, 8, p. 512-532, World Scientific.

Petoukhov S.V. (2006a) Bioinformatics: matrix genetics, algebras of the genetic code and biological harmony. *Symmetry: Culture and Science*, 17, #1-4, p. 251-290.

Petoukhov S.V. (2008a) The degeneracy of the genetic code and Hadamard matrices. arXiv:0802.3366 [q-bio.QM].

Petoukhov S.V. (2008b) *Matrix genetics, algebras of the genetic code, noise immunity*. M., RCD, 316 p. (in Russian).

Petoukhov S.V. (2010) Matrix genetics, part 5: genetic projection operators and direct sums. http://arxiv.org/abs/1005.5101.

Petoukhov S.V. (2011a). Matrix genetics and algebraic properties of the multi-level system of genetic alphabets. - Neuroquantology, 2011, Vol 9, No 4, p. 60-81, http://www.neuroquantology.com/index.php/journal/article/view/501

Petoukhov S.V. (2011b). Hypercomplex numbers and the algebraic system of genetic alphabets. Elements of algebraic biology. – *Hypercomplex numbers in geometry and physics*, v.8, No2(16), p. 118-139 (in Russian).

Petoukhov S.V. (2012a) The genetic code, 8-dimensional hypercomplex numbers and dyadic shifts. (7th version from January, 30, 2012), http://arxiv.org/abs/1102.3596

Petoukhov S.V. (2012b) Symmetries of the genetic code, hypercomplex numbers and genetic matrices with internal complementarities. - “Symmetries in genetic information and algebraic biology”, special issue of journal “Symmetry: Culture and Science” (Guest editor: S. Petoukhov), vol. 23, № 3-4, p. 275-301. http://symmetry.hu/scs_online/SCS_23_3-4.pdf

Petoukhov S.V., He M. (2009) *Symmetrical Analysis Techniques for Genetic Systems and Bioinformatics: Advanced Patterns and Applications*. Hershey, USA: IGI Global. 271 p.

Poincare H. (1913). The foundations of science. Science and hypothesis. The value of science. Science and method. – N. Y., The Science Press New York and Garrison.
Rumer Yu. B. (1968). Systematization of the codons of the genetic code. - *Doklady Akademii Nauk SSSR*, vol. 183(1), p. 225-226 (in Russian).

Russel B.A.W. (1956). An essay on the foundations of geometry. N-Y: Dover, 201 p.

Shults E. (1916), "The organism as creativity". - In the book "Questions of theory and psychology of creativity", Russia, Kharkov, vol.7, p. 108-190 (in Russian, "Voprosy teorii i psikhologii tvorchestva").

Stewart I. (1999). *Life's other secret: The new mathematics of the living world*. New-York: Penguin.

Vetter R.J., Weinstein S. (1967). The history of the phantom in congenitally absent limbs. – *Neuropsychologia*, № 5, p.335-338

Vinberg E.B. (2003). *A Course in Algebra*. – USA: American Mathematical Society, 511 p.

Vladimirov Y.S. (2008). *Foundations of Physics*. – Moscow, Binom, 456 p. (in Russian)

Weinstein S., Sersen E.A. (1961). Phantoms in cases of congenital absence of limbs. – *Neurology*, №11, p.905-911.

Wittmann H.G. (1961) Ansätze zur Entschlüsselung des genetischen Codes. – Die Naturwissenschaften, B.48, 24, S. 55