Extended lattice Gelfand–Dickey hierarchy

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Abstract

The lattice Gelfand–Dickey (GD) hierarchy is a lattice version of the GD hierarchy. A special case is the lattice KdV hierarchy. Inspired by recent work of Buryak and Rossi, we propose an extension of the lattice GD hierarchy. The extended system has an infinite number of logarithmic flows alongside the usual flows. We present the Lax, Sato and Hirota equations and a factorization problem of difference operators that captures the whole set of solutions. The construction of this system resembles the extended 1D and bigraded Toda hierarchy, but exhibits several novel features as well.

Keywords: cohomological field theory, Gelfand–Dickey hierarchy, KdV hierarchy, KP hierarchy, Toda hierarchy, logarithmic flow

1. Introduction

Some years ago, Buryak and Rossi [1] pointed out that an extension of the lattice KdV hierarchy emerges in a variant of cohomological field theory of Witten’s r-spin class. The lattice KdV hierarchy (the ‘discrete’ KdV hierarchy in the terminology of Buryak and Rossi) is a system of Lax equations for a difference operator of second order. The extended part comprises evolution equations with an extra set of time variables and is constructed in a rather abstract way. Finding a more explicit construction of these extra flows, as well as a higher spin generalization on the basis of the Gelfand–Dickey (GD) hierarchy, is an open problem raised therein.

In this paper, we present an extension of the lattice Gelfand–Dickey (GD for short) hierarchy. Although evidence is fragmentary, we believe that this is the integrable structure sought for by Buryak and Rossi. The lattice GD hierarchy is a reduction of the lattice KP hierarchy (aka the discrete KP hierarchy, the modified KP hierarchy, etc) [2]. Our extended lattice GD
hierarchy is obtained by adding an infinite number of ‘logarithmic flows’ to the lattice GD hierarchy.

Logarithmic flows have been known for the 1D Toda hierarchy and its bigraded generalizations. The name reflects the fact that the Lax equations contain the logarithm of the Lax operator as a building block. These exotic flows were first discovered in the large-$N$ limit of a matrix model of topological string theory on $\mathbb{CP}^1$ [3, 4]. The Lax equations were accordingly formulated in the dispersionless limit of the 1D Toda hierarchy. A dispersive version was constructed later [5] and further generalized to the bigraded Toda hierarchy [6]. The meaning of the logarithm of the relevant Lax operator, which is a difference operator, was also clarified therein. Moreover, Hirota equations for the tau function in the presence of logarithmic flows were also subsequently obtained [7–9]. These extended 1D and bigraded Toda hierarchies are used to capture the all-genus Gromov–Witten partition functions of $\mathbb{CP}^1$ [10, 11] and its two-point orbifolds [12, 13].

The construction of logarithmic flows in the lattice GD hierarchy is mostly parallel to our previous work [9] on the extended 1D Toda hierarchy. The somewhat unusual Lax formalism proposed therein turns out to be applicable to the lattice GD hierarchy as well. Let us stress that the construction of logarithmic flows by Carlet et al [5, 6] is not very suited for this purpose. The bigraded Toda hierarchy is labeled by two positive integers $N$ and $\bar{N}$. The 1D Toda hierarchy is the case of $N = \bar{N} = 1$. The lattice GD hierarchy amounts to the case of $\bar{N} = 0$, which is exceptional in the perspectives of the bigraded Toda hierarchy.

This paper is organized as follows. Section 2 is a review of the notion of the lattice KP hierarchy and its reduction to the lattice GD hierarchy. Section 3 presents the construction of the extended lattice GD hierarchy in the language of the Lax and Sato equations. Section 4 deals with the Hirota equations for the tau function. Section 5 is devoted to the factorization problem.

2. Lattice KP and GD hierarchies

The lattice KP hierarchy may be thought of as a subsystem of the 2D Toda hierarchy with the second set $\tilde{t} = (\tilde{t}_k)_{k=0}^\infty$ of the time variables $t, \tilde{t}$ being turned off to $\tilde{t} = 0$ (see the recent review [14] for generalities of the 2D Toda hierarchy). We now consider an $h$-dependent version [15] of these hierarchies. Let $s$ be the spatial coordinate therein, which is now understood to be a continuous variable, and $\Lambda$ be the shift operator

$$\Lambda = e^{h\partial}, \quad \partial_s = \partial / \partial s,$$

that acts on a function $f(s)$ of $s$ as $\Lambda f(s) = f(s + h)$.

The lattice KP hierarchy comprises the Lax equations

$$h \frac{\partial L}{\partial \tilde{t}_k} = [B_k, L] = [L, B_k], \quad k = 1, 2, \ldots ,$$

for the difference (or pseudo-difference) Lax operator

$$L = \Lambda + \sum_{n=1}^\infty u_n \Lambda^{1-n}.$$ 

The coefficients $u_n$ depend on $h$ as well as $s$ and the time variables $t = (t_k)_{k=1}^\infty$. The generators $B_k$ and $\bar{B}_k$ of the flows are defined by the Lax operator as

$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (L^{\bar{k}})_{< 0}.$$
where \((\ )_{\geq 0}\) and \((\ )_{< 0}\) denote the projection
\[
\left( \sum_{n=-\infty}^{\infty} a_n \Lambda^n \right)_{\geq 0} = \sum_{n\geq 0} a_n \Lambda^n, \quad \left( \sum_{n=-\infty}^{\infty} a_n \Lambda^n \right)_{< 0} = \sum_{n<0} a_n \Lambda^n.
\]
to the non-negative and negative powers of \(\Lambda\).

The Lax equations can be converted to the Sato equations
\[
\hbar \frac{\partial W}{\partial t_k} = (W \Lambda^k W^{-1})_{\geq 0} W - W \Lambda^k = -(W \Lambda^k W^{-1})_{< 0} W
\]
for the dressing operator
\[
W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}.
\]
\(L\) and \(B_k\) are thereby expressed as
\[
L = W \Lambda W^{-1}, \quad B_k = (W \Lambda^k W^{-1})_{\geq 0}.
\]
Thus the Sato equation (2) imply the Lax equation (1). Conversely, given a Lax operator \(L\) satisfying (1), one can find a solution \(W\) of (2).

The wave function
\[
\Psi = \left( 1 + \sum_{n=1}^{\infty} w_n z^{-n} \right) e^{\hbar \xi(t,z)/\hbar}, \quad \xi(t,z) = \sum_{k=1}^{\infty} b_k z^k,
\]
satisfies the auxiliary linear equations
\[
\hbar \frac{\partial \Psi}{\partial t_k} = B_k \Psi, \quad L \Psi = z \Psi.
\]
The amplitude part of \(\Psi\) can be expressed by the tau function \(\tau = \tau(h,s,t)\) as
\[
1 + \sum_{n=1}^{\infty} w_n z^{-n} = \frac{\tau(h,s,t - \hbar[z^{-1}])}{\tau(h,s,t)} \left( \frac{z^k}{k} \right)_{k=1}^{\infty}, \quad [z] = \left( \frac{z^k}{k} \right)_{k=1}^{\infty}.
\]
Reduction to the lattice GD hierarchy [2] is induced by the reduction condition
\[
(L^N)_{< 0} = 0.
\]
The reduced Lax operator
\[
\mathcal{L} = L^N = B_N = \Lambda^N + b_1 \Lambda^{N-1} + \cdots + b_N
\]
satisfies the Lax equations
\[
\hbar \frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}] = [\mathcal{L}, B_k^-], \quad k = 1, 2, \ldots
\]
of the lattice GD hierarchy. \(B_k\) and \(B_k^-\) can be rewritten in terms of fractional powers of \(\mathcal{L}\) as
\[
B_k = (\mathcal{L}^{k/N})_{\geq 0}, \quad B_k^- = (\mathcal{L}^{k/N})_{< 0}.
\]
The $N = 2$ case is the lattice KdV hierarchy. The existence of two expressions (6) of the Lax equations ensures that the reduced form (5) of $\mathcal{L}$ is preserved by the flows. The flows with respect to $t_{kN}$’s become trivial, i.e.,

$$\frac{\partial \mathcal{L}}{\partial t_{kN}} = 0, \quad k = 1, 2, \ldots.$$  \hfill (7)

The Sato equations persist to take the same form

$$\hbar \frac{\partial W}{\partial t_{kN}} = B_k W - W \Lambda^k = -B_k^{-1} W$$  \hfill (8)

in the reduction. The associated auxiliary linear equations read

$$\hbar \frac{\partial \Psi}{\partial t_{kN}} = B_k \Psi, \quad \mathcal{L} \Psi = \zeta^N \Psi.$$  \hfill (9)

**Remark 1.** The usual KP and GD hierarchies are hidden in the lattice KP and GD hierarchies. The first time variable $t_1$ plays the role of spatial coordinate therein. Note that the lowest auxiliary linear equation

$$\hbar \frac{\partial \Psi}{\partial t_1} = B_1 \Psi, \quad B_1 = \Lambda + b,$$

implies the relation

$$\Lambda \Psi = (\hbar \partial - b) \Psi, \quad \partial = \partial / \partial t_1.$$  \hfill (10)

One can thereby rewrite the difference operators in the auxiliary linear equations (3) and (9) into differential or pseudo-differential operators with respect to $t_1$. In particular, the linear difference equation $\mathcal{L} \Psi = \zeta^N \Psi$ can be thus converted to a linear differential equation of the form

$$Q \Psi = \zeta^N \Psi, \quad Q = (\hbar \partial)^N + a_2 (\hbar \partial)^{N-2} + \cdots + a_N,$$  \hfill (11)

which may be thought of as the eigenvalue problem of the GD hierarchy. The Lax equation (6) turn into Lax equations of the form

$$\hbar \frac{\partial Q}{\partial t_{kN}} = [A_k, Q], \quad k = 1, 2, \ldots.$$  \hfill (12)

$A_k$’s are differential operators obtained from $B_k$’s just as $Q$ is converted from $\mathcal{L}$. This is exactly the usual GD hierarchy, in which the variable $s$ is left as a parameter. Shifting $s$ as $s \rightarrow s + \hbar$ amounts to Bäcklund–Darboux transformations of the GD hierarchy as the relation (10) suggests.

### 3. Extended lattice GD hierarchy

We now extend the lattice GD hierarchy by an infinite number of logarithmic flows. Let $x = (x_k)_{k=1}^\infty$ be the set of time variables of these flows.

The Lax equations of the extended flows take seemingly the same form

$$\hbar \frac{\partial \mathcal{L}}{\partial x_k} = \mathcal{L} \hbar \frac{\partial \mathcal{L}}{\partial s} + [P_k, \mathcal{L}] = [\mathcal{L} \hbar \partial + P_k, \mathcal{L}], \quad k = 1, 2, \ldots.$$  \hfill (13)
as those of the extended 1D/bigraded Toda hierarchy [9]. The structure of $P_k$’s, however, is different from that case. In the present setup, $P_k$’s are difference operators of the form

$$P_k = - \left( q^k h \frac{\partial W}{\partial s} W^{-1} \right) \geq 0,$$

thus having no negative powers of $\Lambda$. These extended Lax equations and the foregoing Lax equation (6) constitute the extended lattice GD hierarchy.

These Lax equations can be converted to the Sato equations

$$\bar{h} \frac{\partial W}{\partial x_k} = L_k \bar{h} \frac{\partial W}{\partial s} + P_k W = (q^k \bar{h} \partial_s + P_k) W - W \Lambda^N \bar{h} \partial_t,$$

for the dressing operator $W$. The associated wave function is redefined as

$$\Psi = \left( 1 + \sum_{n=1}^{\infty} w_n z^{-n} \right) z^{i/\bar{h}} \exp \left( \bar{h}^{-1}(t, z) + \bar{h}^{-1}(x, z^N) \log z \right).$$

The undressed wave function

$$\Psi_0 = z^{i/\bar{h}} \exp \left( \bar{h}^{-1}(t, z) + \bar{h}^{-1}(x, z^N) \log z \right)$$

satisfies the linear equations

$$\bar{h} \frac{\partial \Psi_0}{\partial t_k} = \Lambda_k \Psi_0, \quad \bar{h} \frac{\partial \Psi_0}{\partial x_k} = \Lambda^N \bar{h} \partial_t \Psi_0.$$

These equations are transformed, via the dressing relation $\Psi = W \Psi_0$, to the linear equations

$$\bar{h} \frac{\partial \Psi}{\partial t_k} = B_k \Psi, \quad \bar{h} \frac{\partial \Psi}{\partial x_k} = (q^k \bar{h} \partial_s + P_k) \Psi$$

for the dressed wave function $\Psi$. These equations and the eigenvalue problem $L \Psi = z^N \Psi$ give the auxiliary linear equations of the extended lattice GD hierarchy.

The presence of the logarithmic factors $\log z$ and $\log \Lambda = \bar{h} \partial_t$ in the wave functions and the auxiliary linear equations explains the name ‘logarithmic flows’. Moreover the logarithm

$$\log L = W \log \Lambda W^{-1} = W \bar{h} \partial_t W^{-1}$$

of the Lax operator $L$, too, plays a role as follows.

One can rewrite the product $L^{\text{KN}} \log L$ of $L^{\text{KN}}$ and $\log L$ as

$$L^{\text{KN}} \log L = W \Lambda^N \bar{h} \partial_t W^{-1} = q^k \bar{h} \partial_s - q^k \bar{h} \frac{\partial W}{\partial s} W^{-1},$$

which splits into two pieces as

$$L^{\text{KN}} \log L = (q^k \bar{h} \partial_s + P_k) + P_k^-,$$

where

$$P_k^- = - \left( q^k \bar{h} \frac{\partial W}{\partial s} W^{-1} \right)_{<0}.$$

The first piece $q^k \bar{h} \partial_s + P_k$ is the generator of the $x_k$-flow in the Lax equation (13) and the Sato
equation (14). Since $L^{kN}\log L$ commute with $\mathcal{L}$, one can rewrite the Lax equations as
\[ \hbar \frac{\partial \mathcal{L}}{\partial x_k} = [\mathcal{L}, P_k]. \quad (19) \]
In the same sense, the Sato equation turns out to have another expression as
\[ \hbar \frac{\partial W}{\partial x_k} = -P_k W. \quad (20) \]

The operators $\mathcal{L}k\hbar \partial_s + P_k$ and $P_k$ may be thought of as the $()_{\geq 0}$ and $()_{< 0}$ parts of $\mathcal{L}k\log L$ if these projectors are extended to operators of the form $A \partial_s + B$, where $A$ and $B$ are genuine difference operators, as
\[ (A \partial_s + B)_{\geq 0} = (A)_{\geq 0} \partial_s + (B)_{\geq 0}, \quad (A \partial_s + B)_{< 0} = (A)_{< 0} \partial_s + (B)_{< 0}. \]
Although being logically problematical, such an interpretation seems to be useful for the extended 1D/bigraded Toda hierarchies as well.

**Remark 2.** The Lax operator of the bigraded Toda hierarchy of type $(N, \bar{N})$ is a difference operator of the form
\[ \mathcal{L} = \Lambda^N + b_1 \Lambda^{N-1} + \cdots + b_{N+\bar{N}} \Lambda^{-\bar{N}}, \quad b_{N+\bar{N}} \neq 0, \]
where $N$ and $\bar{N}$ are positive integers. The 1D Toda hierarchy is the case of $N = \bar{N} = 1$. On the other hand, the Lax operator (5) of the lattice GD hierarchy amounts to the case of $\bar{N} = 0$. Carlet’s construction of logarithmic flows [6] is based on the averaged (or subtracted) logarithm
\[ \log \mathcal{L} = \frac{1}{2N} \log + \mathcal{L} + \frac{1}{2N} \log - \mathcal{L} = \frac{1}{2} \bar{W} \log \Lambda W^{-1} - \frac{1}{2} \bar{W} \log \Lambda \bar{W}^{-1} \]
of $\mathcal{L}$, where $\bar{W}$ is the second dressing operator of the 2D Toda hierarchy, and does not work in the $N = 0$ case. To overcome this difficulty, we have adopted our previous construction of logarithmic flows [9] with suitable modifications. Let us note that our construction for the $N > 0$ case is equivalent to Carlet’s construction.

**Remark 3.** The building blocks of the Lax, Sato and auxiliary linear equations of the $x$-flows have the common factor $\hbar$. Although this factor could be removed from these equations, we dare to keep it for symmetry with the equations of the $t$-flows.

**Remark 4.** The dual expressions (13) and (19) of the Lax equations imply that the reduced form (5) of the Lax operator is preserved by the $x$-flows as well as the $t$-flows. Moreover one can show, in much the same way as in the case of the extended bigraded Toda hierarchy [6], that the Lax equations imply the zero-curvature equations
\[ \hbar \frac{\partial B_k}{\partial t_l} - \hbar \frac{\partial B_l}{\partial t_k} + [B_k, B_l] = 0, \quad (21) \]
\[ \hbar \frac{\partial B_k}{\partial x_l} - \hbar \frac{\partial C_l}{\partial x_k} + [B_k, C_l] = 0, \quad (22) \]
\[ \hbar \frac{\partial C_k}{\partial x_l} - \hbar \frac{\partial C_l}{\partial x_k} + [C_k, C_l] = 0, \quad k, l = 1, 2, \ldots, \quad (23) \]
for $B_k$ and $C_k = \mathcal{L}k\hbar \partial_s + P_k$. These zero-curvature equations ensure the consistency of the Lax equations.
Remark 5. Equating the coefficients of powers of $\Lambda$ in both hand sides of the Lax equations (6) and (13), one can derive a system of evolution equations for the coefficients $b_1, \ldots, b_N$ of $L$. In particular, the $\Lambda^k$ terms give the evolution equations

$$\frac{\partial b_N}{\partial t_k} = 0, \quad \frac{\partial b_N}{\partial x_k} = b_N^k \frac{\partial b_N}{\partial s}, \quad k = 1, 2, \ldots,$$

(24)

for $b_N$. These equations in the case of $N = 2$ agree, up to rescaling of the time variables, with Buryak and Rossi’s result [1, theorem 4.2]. This is a main piece of evidence suggesting that our extended lattice KdV hierarchy will be the integrable structure sought for therein.

4. Hirota equations

As in the case of the extended 1D/bigraded Toda hierarchy [8, 9], Hirota equations for the tau function can be derived from bilinear equations for the wave functions. Let us first consider those bilinear equations for the wave functions of the extended lattice GD hierarchy.

It is convenient to start from the bilinear equations

$$\oint \frac{dz}{2\pi i} \Psi(s, t, x, z)\Psi^\ast(s', t', x, z) = 0$$

(25)

of the lattice KP hierarchy that holds under the condition

$$\frac{s - s'}{\bar{h}} \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$$

(26)

and for two independent copies $t = (t_k)_{k=1}^\infty$ and $t' = (t'_k)_{k=1}^\infty$ of the $t$-variables. $\oint dz/2\pi i$ denotes the residue

$$\oint \frac{dz}{2\pi i} \sum_{n=-\infty}^{\infty} f_n z^n = f_{-1}$$

of formal Laurent series of $z$. $\Psi^\ast = \Psi^\ast(s, t, x, z)$ is the dual wave function

$$\Psi^\ast = \left( 1 + \sum_{n=1}^{\infty} \Lambda^n w_n \right) z^{-s/\bar{h}} \exp(-\bar{h}^{-1} \xi(t, z) - \bar{h}^{-1} \xi(x, z^N) \log z)$$

defined in the same way as the dual wave function of the 2D Toda hierarchy. Namely, the amplitude part is determined by the formal adjoint

$$W^\ast = 1 + \sum_{n=1}^{\infty} \Lambda^n \cdot w_n = 1 + \sum_{n=1}^{\infty} w_n |s = a_n h \Lambda^n$$

of $W$ as

$$1 + \sum_{n=1}^{\infty} w_n \Lambda^n = \Lambda^{-1}(W^\ast)^{-1} \Lambda.$$

(25) is actually the bilinear equations of the 2D Toda hierarchy restricted to the range (26) of $s$ and $s'$. 7
In the reduction to the lattice GD hierarchy, the auxiliary linear equations for \(t_{kN}\), \(k = 1, 2, \ldots\), trivialize as
\[
\hbar \frac{\partial \Psi}{\partial t_{kN}} = \Lambda^k \Psi = z^N \Psi.
\]
Accordingly, one can insert arbitrary positive powers of \(z^N\) into the bilinear equation (25) as
\[
\oint \frac{dz}{2\pi i} z^{mN} \Psi(s, t, x, z) \Psi^*(s', t', x, z) = 0.
\]
(27)
These are bilinear equations that characterize the lattice GD hierarchy.

We can now use the technique developed for the extended 1D/bigraded Toda hierarchy [9] to derive the following bilinear equation for the wave functions of the extended lattice GD hierarchy.

**Proposition 1.** The bilinear equation
\[
\oint \frac{dz}{2\pi i} z^{mN} \Psi(s - \xi(\alpha, z^N), t, x + \alpha, z) \Psi^*(s' - \xi(\beta, z^N), t', x + \beta, z) = 0
\]
holds for \(m, (s - s')/\hbar \in \mathbb{Z}\) and two sets \(\alpha = (\alpha_k)_{k=1}^\infty, \beta = (\beta_k)_{k=1}^\infty\) of arbitrary constants.

**Proof.** One can rewrite the auxiliary linear equations with respect to \(x_k\) in (16) as
\[
\hbar \left( \frac{\partial}{\partial x_k} - z^N \frac{\partial}{\partial s} \right) \Psi = Q_k \Psi,
\]
where
\[
Q_k = P_k - \hbar \frac{\partial \Lambda^k}{\partial s}.
\]
These equations can be extended to higher orders as
\[
\prod_{k \geq 1} \left( \frac{\partial}{\partial x_k} - z^N \frac{\partial}{\partial s} \right)^{m_k} \Psi = Q_{m_1, m_2, \ldots} \Psi, \quad m_1, m_2, \ldots \in \mathbb{Z}_{\geq 0},
\]
where \(Q_{m_1, m_2, \ldots}\)’s are difference operators without negative powers of \(\Lambda\). Now let \(Q_{m_1, m_2, \ldots}\) act on both sides of the bilinear equation (27). Since this difference operator consists of only non-negative powers of \(\Lambda\), the outcome is a linear combination of (27) with \(m\) being shifted to the positive direction. Consequently, the bilinear equation persists to hold under the action of \(Q_{m_1, m_2, \ldots}\) as
\[
\oint \frac{dz}{2\pi i} z^{mN} Q_{m_1, m_2, \ldots} \Psi(s, t, x, z) \cdot \Psi^*(s', t', x, z) = 0,
\]
equivalently,
\[
\oint \frac{dz}{2\pi i} z^{mN} \prod_{k \geq 1} \left( \frac{\partial}{\partial x_k} - z^N \frac{\partial}{\partial s} \right)^{m_k} \Psi(s, t, x, z) \cdot \Psi^*(s', t', x, z) = 0.
\]

8
These equations can be packed into a generating functional form with the auxiliary variables \(\alpha_1, \alpha_2, \ldots\) as
\[
\oint \frac{dz}{2\pi i} z^{m_0} e \exp \left( \sum_{k=1}^{\infty} \alpha_k \left( \frac{\partial}{\partial x_k} - z^{[k]} \frac{\partial}{\partial s} \right) \right) \Psi(s, t, x, z) \cdot \Psi^*(s', t', x, z) = 0,
\]
i.e.,
\[
\oint \frac{dz}{2\pi i} z^{m_0} \Psi(s - \xi(\alpha, z^N), t, x + \alpha, z) \Psi^*(s', t', x, z) = 0.
\]
Note that the last equation, too, holds for \(m, (s - s')/h \in \mathbb{Z}_{\geq 0}\). Therefore one can apply the shift operator
\[
\exp \left( - \sum_{k=1}^{\infty} \beta_k z^{[k]} (\partial_s + \partial_t) \right)
\]
to the integrand to obtain the further deformed bilinear equation
\[
\oint \frac{dz}{2\pi i} z^{m_0} \Psi(s - \xi(\alpha + \beta, z^N), t, x + \alpha, z) \Psi^*(s' - \xi(\beta, z^N), t', x, z) = 0.
\]
This equation turns into (28) by substituting \(\alpha \to \alpha - \beta\) and \(x \to x + \beta\). □

Lastly, we rewrite the bilinear equation (28) in the language of the tau function. As in the case of the extended 1D/bigraded Toda hierarchy [8, 9], the wave functions are expressed by the tau function \(\tau = \tau(h, s, t, x)\) as
\[
\Psi = \frac{\tau(h, s, t - h[z^{-1}], x)}{\tau(h, s, t, x)} e^{\frac{1}{h} \log z} \exp \left( h^{-1} \xi(t, z) + h^{-1} \xi(x, z^N) \log z \right),
\]
\[
\Psi^* = \frac{\tau(h, s, t + h[z^{-1}], x)}{\tau(h, s, t, x)} e^{-\frac{1}{h} \log z} \exp \left( -h^{-1} \xi(t, z) - h^{-1} \xi(x, z^N) \log z \right).
\]
Plugging these expressions into (28), we obtain the bilinear equation
\[
\oint \frac{dz}{2\pi i} z^{m_0 + (s - s')/h} e^{\frac{1}{h} \log z} \frac{1}{\tau(h, s - \xi(\alpha, z^N), t - h[z^{-1}], x + \alpha)} \times \tau(h, s' - \xi(\beta, z^N), t' + \frac{h[z^{-1}], x + \beta}{h}) = 0 \tag{29}
\]
that holds for \(m, (s - s')/h \in \mathbb{Z}_{\geq 0}\). Note that the factors \(\tau(h, s - \xi(\alpha, z^N), t, x + \alpha)\) and \(\tau(h, s' - \xi(\beta, z^N), t', x + \beta)\) arising in the denominator has been removed by a simple trick [9, section 2.4]. The bilinear equation (29) is a generating functional expression of an infinite number of Hirota equations.

5. Factorization problem

We now turn to an analogue of the factorization problem proposed for solving the extended 1D/bigraded Toda hierarchy [16]. Given an invertible difference operator \(U\) of infinite order
that depends on the spatial coordinate and the time variable in a particular manner, the problem is to factorize $U$ into two difference operators of the form

$$W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^n$$

as

$$U = W^{-1} \bar{W}. \quad (30)$$

In the case of the extended lattice GD hierarchy, $U$ is assumed to satisfy the differential equations

$$\bar{h} \frac{\partial U}{\partial t_k} = \Lambda^k U, \quad \frac{\partial U}{\partial x_k} = \Lambda^k \frac{\partial U}{\partial s} \quad (31)$$

and the algebraic relation

$$\Lambda^N U = U \varphi(\Lambda), \quad (32)$$

where $\varphi(\Lambda)$ is a difference operator of the form

$$\varphi(\Lambda) = \sum_{n=0}^{\infty} \varphi_n \Lambda^n.$$

**Proposition 2.** If $W$ and $\bar{W}$ solve the factorization problem (30), $W$ is the dressing operator of a solution of the extended lattice GD hierarchy. Namely, $\mathcal{L} = W \Lambda^N W^{-1}$ takes the reduced form (5), and $W$ satisfies the Sato equations (8) and (14). Moreover, $\bar{W}$ satisfies the Sato equations

$$\bar{h} \frac{\partial \bar{W}}{\partial t_k} = B_k \bar{W}, \quad \bar{h} \frac{\partial \bar{W}}{\partial x_k} = \mathcal{L}^k \bar{h} \frac{\partial \bar{W}}{\partial s} + P_k \bar{W} \quad (33)$$

of a slightly different form.

**Proof.** Substituting $U = W^{-1} \bar{W}$ in (32) yields the algebraic relation

$$\mathcal{L} = W \Lambda^N W^{-1} = W \varphi(\Lambda) W^{-1}. \quad (34)$$

Since the right-hand side does not contain negative powers of $\Lambda$, $\mathcal{L}$ turns out to take the reduced form (5). In the same way, (31) yields the equations

$$\bar{h} \frac{\partial \bar{W}}{\partial t_k} W^{-1} + W \Lambda^k W^{-1} \frac{\partial W}{\partial t_k} = \frac{\partial W}{\partial t_k} W^{-1}$$

and

$$\bar{h} \frac{\partial \bar{W}}{\partial x_k} W^{-1} - W \Lambda^k W^{-1} \frac{\partial W}{\partial x_k} W^{-1} + \frac{\partial \bar{W}}{\partial x_k} W^{-1} = \frac{\partial \bar{W}}{\partial x_k} W^{-1} - W \Lambda^k W^{-1} \frac{\partial \bar{W}}{\partial s} W^{-1},$$

which one can rewrite as

$$\bar{h} \frac{\partial \bar{W}}{\partial t_k} W^{-1} + \mathcal{L}^k W^{-1} = \frac{\partial \bar{W}}{\partial t_k} W^{-1}.$$
and
\[ h \frac{\partial W}{\partial x_k} W^{-1} - \xi^k h \frac{\partial W}{\partial s} W^{-1} = h \frac{\partial W}{\partial x_k} W^{-1} - \xi^k h \frac{\partial W}{\partial s} W^{-1}. \]

Let \( B_k \) and \( P_k \) be the difference operators defined by both hand sides of these equations. Since the right-hand side have no \((\cdots)_{<0}\) part, \( B_k \) and \( P_k \) are equal to the \((\cdots)_{\geq 0}\) part of the left-hand side. One can thus identify these operators as

\[ B_k = \left( h \frac{\partial W}{\partial x_k} W^{-1} + \xi^k \right)_{\geq 0} = \left( \xi^k / N \right)_{\geq 0} \]

and

\[ P_k = \left( h \frac{\partial W}{\partial x_k} W^{-1} - \xi^k h \frac{\partial W}{\partial s} W^{-1} \right)_{\geq 0} = -\left( \xi^k h \frac{\partial W}{\partial s} W^{-1} \right)_{\geq 0} \]

and obtain (8) and (14).

Although looking arbitrarily given, the operator \( \varphi(\Lambda) \) in the algebraic relation (32) has to satisfy a set of differential equations. This is a consequence of the differential equation (31) for \( U \).

**Proposition 3.**

\[
\frac{\partial \varphi(\Lambda)}{\partial t_k} = 0, \quad \frac{\partial \varphi(\Lambda)}{\partial x_k} = \varphi(\Lambda) \frac{\partial \varphi(\Lambda)}{\partial s}, \quad k = 1, 2, \ldots \quad (35)
\]

**Proof.** Differentiating both hand sides of (32) by \( t_k \) gives

\[ N h \frac{\partial U}{\partial t_k} = h \frac{\partial U}{\partial t_k} \varphi(\Lambda) + U h \frac{\partial \varphi(\Lambda)}{\partial t_k}. \]

By (31), this equation turns into

\[ N U \varphi(\Lambda) = N U \varphi(\Lambda) + U \frac{\partial \varphi(\Lambda)}{\partial t_k}, \]

which implies the first equation of (35). One can derive the second equation of (35) by a similar reasoning as follows. Differentiating both hand sides of (32) by \( x_k \) and \( s \) yields the two equations

\[ N \frac{\partial U}{\partial x_k} = \frac{\partial U}{\partial x_k} \varphi(\Lambda) + U \frac{\partial \varphi(\Lambda)}{\partial x_k}, \quad N \frac{\partial U}{\partial s} = \frac{\partial U}{\partial s} \varphi(\Lambda) + U \frac{\partial \varphi(\Lambda)}{\partial s}. \]

By (31), the first equation can be rewritten as

\[ N^{n+1} \frac{\partial U}{\partial x_k} = N^{n+1} \frac{\partial U}{\partial x_k} \varphi(\Lambda) + U \frac{\partial \varphi(\Lambda)}{\partial x_k}. \]

The second equation, multiplied with \( N^{n+1} \), becomes

\[ N^{n+1} \frac{\partial U}{\partial s} = N^{n+1} \frac{\partial U}{\partial s} \varphi(\Lambda) + N^{n+1} N U \frac{\partial \varphi(\Lambda)}{\partial s}. \]

Subtracting this equation from the preceding equation and using (32), one obtains the second equation of (35). \( \square \)
Let us stress that the process from the factorization problem (30) to the solution of the extended lattice GD hierarchy is reversible. Namely, given a solution of the extended lattice GD hierarchy, one can reconstruct the operators \( U \) and \( \varphi(\Lambda) \) as
\[
U = W^{-1} \bar{W}, \quad \varphi(\Lambda) = \bar{W}^{-1} \Sigma \bar{W},
\]
which turn out to satisfy the differential equation (31) and the algebraic condition (32). Thus the factorization problem can capture all solutions of the extended lattice GD hierarchy.

**Remark 6.** The \( \Lambda^0 \) part of (35) gives the equations
\[
\frac{\partial \varphi_0}{\partial t_k} = 0, \quad \frac{\partial \varphi_0}{\partial x_k} = \varphi_0^k \frac{\partial \varphi_0}{\partial s}, \quad k = 1, 2, \ldots
\]
(36)
This is in accord with the equation (24) for \( b_N \), because \( b_N \) is equal to \( \varphi_0 \) as (34) implies.

**Remark 7.** If the coefficients \( \varphi_n \) of \( \varphi(\Lambda) \) are quasi-constants (i.e., \( h \)-periodic) with respect to \( s \), \( \varphi(\Lambda) \) commutes with \( \Lambda \), and the differential equation (31) can be reduced to the differential equations
\[
\frac{\partial \varphi(z)}{\partial t_k} = 0, \quad \frac{\partial \varphi(z)}{\partial x_k} = \varphi(z)^k \frac{\partial \varphi(z)}{\partial s}, \quad k = 1, 2, \ldots
\]
(37)
for the generating function
\[
\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n.
\]
Moreover, the algebraic relation (34) becomes the relation
\[
\mathcal{L} = L^N = \varphi(\bar{L}) = \sum_{n=0}^{\infty} \varphi_n L^n
\]
(38)
connecting \( L \) to the second Lax operator
\[
L = W \Lambda W^{-1}.
\]
of the 2D Toda hierarchy hidden behind.

**Remark 8.** Algebraic relations of the form (38) are commonly used as a reduction condition in the 2D Toda hierarchy. The coefficients of such a reduction condition, however, are constants. For instance, the bigraded Toda hierarchy of type \((N, \bar{N})\) is induced by the reduction condition
\[
L^N = L^{-\bar{N}}.
\]
In contrast, the coefficients \( \varphi_n \) of (38), as well as those of (32) and (34), are allowed to depend on the time variables of the logarithmic flows. This is a remarkable feature of the logarithmic flows in the extended lattice GD hierarchy.
6. Conclusion

We have constructed an extension of the lattice GD hierarchy with an infinite number of logarithmic flows. This system is not a special case of the extended bigraded Toda hierarchy [6] (even if the second set of the time variable $(t, \bar{t}, x)$ therein is turned off to $t = 0$). It is meaningless to let $\bar{N} = 0$ naively in the construction of the extended bigraded Toda hierarchy of type $(N, \bar{N})$. Therefore we return to the slightly different Lax formalism proposed in our previous work [9], and modify the generators $P_k$ of logarithmic flows to fit into the $\bar{N} = 0$ case. This leads to a consistent system of Lax, Sato and Hirota equations as well as a factorization problem that captures the whole set of solutions of this system.

The extended lattice GD hierarchy exhibits novel features. Firstly, the last term $b_N$ of the Lax operator $L$ satisfies the simple evolution equation (24) that are closed within themselves. These equations will correspond to the extended sector of cohomological field theory studied by Buryak and Rossi [1]. Secondly, the operator $U$ in the factorization problem (30) is required to satisfy the algebraic condition (32) of an unusual form. In particular, the coefficients $\phi_\alpha$ of the operator $\varphi(\lambda)$ on the right-hand side are allowed to depend on the time variables of the logarithmic flows. Last but not least, the structure of the Lax and Hirota equations is different from the extended bigraded Toda hierarchy.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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