A Quantum Calculus Analogue of Dynamic Leontief Production Model with Linear Objective Function

Rasheed Al-Salih*, Ali Habeeb, and Watheq Laith
University of Sumer, Iraq
*Corresponding Author: rbahhd@mst.edu

Abstract. In this paper, we present a new formulation for the Leontief production model using quantum calculus analogue. This formulation unifies discrete and continuous Leontief production models. Also, the classical Leontief production model is obtained by choosing q = 1. In addition, briefly give an introduction to quantum calculus. We present the formulation for continuous Leontief production models as well as quantum calculus models. Moreover, we establish the weak duality theorem and the strong duality theorem for quantum calculus analogue. Furthermore, using the objective functions for the primal and the dual quantum calculus models, we can easily obtain upper and lower bounds for the value of production at any production plan. Finally, examples are provided in order to illustrate the given results.

1. Introduction

The theory of continuous-time linear programming problems plays an important role in modelling various real world applications such as in operation research, economics, finance, production planning, and transportation problem. For more details we refer to [1-10]. On the other hand, quantum calculus has been recently used to model many applications in number theory and physics such as conformal quantum mechanics, nuclear and high energy physics, and internal energy and specific heat. See for e.g. [11 -21]. In this work we present quantum calculus formulation for Leontief production model with linear objective function. The paper is organized as follows. In Section 2, some basic concepts of the theory of quantum calculus are given. In Section 3, the formulation of continuous model is presented. Quantum calculus formulation is presented in Section 4. Section 5 states and proves the weak and the strong duality theorems on quantum calculus. Examples are presented in Section 6 in order to show our formulation. Some conclusions are given in Section 7.

2. Quantum Calculus

In this section, we briefly give some basic concepts of the theory of quantum calculus. The material in this section can be found in monographs [11,12,17,18,21], in which comprehensive details are given.

**Definition 2.1.** The q-derivation of a function \( f : q^N \to \mathbb{R}^n \) is defined as

\[
D_q f(t) = \frac{f(q^t) - f(t)}{(q - 1)t}
\]

The q-derivative is also called Jackson derivative. See [21].

**Theorem 2.2.** If \( f,g : q^N \to \mathbb{R}^n \) are \( q \)-differentiable, then we have the following:

1) \( D_q(af(t) + bg(t)) = aD_qf(t) + bD_qg(t), \ t \in q^N \).
2) \( D_q(f(t)g(t)) = f(qt)D_q g(t) + g(t)D_q f(t), \ t \in q^N \).

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Published under licence by IOP Publishing Ltd
3) \( D_{q} \left( \frac{f(t)}{a(t)} \right) = \frac{q(t)D_{q}f(t)+f(t)D_{q}a(t)}{a(t)g(qt)} \), \( t \in q^{|N_0}. \)

**Definition 2.3.** Assume \( f : q^{|N_0} \rightarrow \mathbb{R}^n \) and \( a, b \in q^{|N_0} \) with \( a < b \). The definite integral of the function \( f \) is given by

\[
\int f(t)d_{q}(t) = (q - 1) \sum_{(e\in[a,b])\cap q^{|N_0}} tf(t)
\]

**Definition 2.4.** If \( f : q^{|N_0} \rightarrow \mathbb{R}^n \) with \( q > 1 \), \( m, n \in \mathbb{N}_0 \) and \( m < n \), then

\[
\int_{q^{|m}} f(t)d_{q}(t) = \sum_{k=m}^{n-1} (q - 1) q^k f(q^k).
\]

### 3. Continuous-time Leontief production model

In this section, we formulate the continuous-time Leontief production model. This model is a closed dynamic economy in which the production of goods with capital goods is limited at any time by the current accumulation of capital goods. See [4, 6] for more details. Consider the following: \( B : \) be an \( n \times n \) matrix whose \( ij \)th entries represent the amount of the \( i \)th product consumed by the \( j \)th activity in producing one unit of \( j \)th product. \( S : \) be an \( nxn \) matrix whose \( ij \)th entries represent the amount of the \( i \)th product required as capital stock in order to produce the \( j \)th product at unit rate.

The function \( z : [0, T] \rightarrow \mathbb{R}^n \) be a bounded measurable function represents the activity levels. Now, assume \( z(t) \) is positive, then the net production at time \( t \) is \((I - B)z(t)\) which is also summed positive. If there is an injection of \( c(t) \) non negative into the system up to time \( t \), where \( c : [0, T] \rightarrow \mathbb{R}^n \) is a bounded measurable function, then we have the following inequality:

\[
Sz(t) \leq c(t) + \int_0^t (I - B)z(s)ds, t \in [0, T].
\]

Assume the value of the unit goods vector at time \( t \) is a bounded measurable function \( u : [0, T] \rightarrow \mathbb{R}^n \) then the objective function is formulated as

\[
\text{Max } W(z) = \int_0^T \dot{u}(I - B)z(t)dt.
\]

Since \((I - B)z\) represents the positive production of goods which can be achieved by some nonnegative \( z \) if \((I - B)^{-1}\) has a nonnegative inverse, see Gale [4]. By \( U_n \), we denote the space of bounded measurable function from \([0, T]\) into \( \mathbb{R}^n \). The primal continuous-time Leontief production model (PLPM) is formulated as

\[
\text{(PLPM)} \left\{ \begin{array}{l}
\text{Max } w(x) = \int_0^T \dot{a}(t)x(t)dt \\
\text{S. t. } A(t)x(t) \leq c(t) + \int_0^T Mx(s)ds, \ t \in [0, T] \\
\text{and } x \in E_n, \ x(t) \geq 0, \ t \in [0, T].
\end{array} \right.
\]

Where \( x(t) = (I - B)^{-1}Mz(t), \ M \) is an arbitrary positive \( n \times n \) matrix, \( A = S(I - B)^{-1}, M, \dot{a}(t) = \dot{u}(t) \times M, a \in E_n, c \in E_m \) and \( A \) and \( M \) are constants matrices of size \( m \times n \). The dual continuous-time Leontief production model is formulated as

\[
\text{(DLP) } \left\{ \begin{array}{l}
\text{Min } G(y) = \int_0^T C(t)y(t)dt \\
\text{S. t. } \dot{y}(t) \geq a(t) + \int_0^T \dot{M}y(s)ds, \ t \in [0, T] \\
\text{and } y \in E_m, \ y(t) \geq 0, \ t \in [0, T].
\end{array} \right.
\]


4. Leontief models in quantum calculus

Throughout this paper, we use $J$ to denote the quantum calculus interval

$$J = [1, T] \cap q^{N_0},$$

and by $U_k$, we denote the space of all rd-continuous functions from $J$ into $\mathbb{R}^n$. The primal quantum Leontief production model (PQLPM) is given as

$$\begin{align*}
\text{Max} \quad & w(x) = \int_0^{q^{N+1}} \hat{a}(t)x(t)d_q(t) \\
\text{s.t.} \quad & A(t)x(t) \leq c(t) + \int_0^{q^n} Mx(s)d_q(s), \quad q^n \in J \\
\text{and} \quad & x \in E_n, \quad x(t) \geq 0, \quad t \in J.
\end{align*}$$

Where $a \in U_n, c \in U_m$, and $A$ and $M$ are matrices of size $m \times n$. The dual quantum Leontief production model (DQLPM) is given as

$$\begin{align*}
\text{Min} \quad & G(x) = \int_0^{q^{N+1}} c(t)y(t)d_q(t) \\
\text{s.t.} \quad & \hat{A}(t)y(t) \geq a(t) + \int_0^{q^{N+1}} \hat{M}y(s)d_q(s), \quad q^n \in J \\
\text{and} \quad & y \in U_m, \quad y(t) \geq 0, \quad t \in J.
\end{align*}$$

5. Duality theorems in quantum calculus

In this section, we establish some of the duality theorems for quantum Leontief production models.

**Theorem 5.1** (Weak Duality Theorem). If $x$ and $y$ are arbitrary feasible solutions of the primal quantum Leontief production model (PQLPM) and the dual quantum Leontief production model (DQLPM), respectively, then $W(x) \leq G(y)$.

**Theorem 5.2** (Strong Duality Theorem). If the primal quantum Leontief production model (PQLPM) has an optimal solution $x^*$, then the dual quantum Leontief production model (DQLPM) has an optimal solution $y^*$ such that $W(x) = G(y)$.

Remark 5.3. Using Definition 2.4, the proof of the weak duality theorem and the proof strong duality theorem are immediate from the proof of standard duality theorems. See [22].

6. Examples

In this section, two examples are given in order to illustrate our formulation.

**Example 6.1.** $T = q^{N_0}$ and $J = \{1, 2, 4\}$ with $q = 2$. Using the integral given in Definition 2.4, the primal Leontief production model in quantum calculus is described as

$$\begin{align*}
\text{Max} \quad & w(x) = \int_2^3 100tx(t)d_q(t) = 100 \sum_{k=0}^{2} 4^k x(2^k) \\
\text{s.t.} \quad & 700x(t) \leq 500t + 250 \int_1^t x(s)d_q(s) = 500t + 250 \sum_{k=0}^{\log_2^t-1} 2^k x(2^k) \quad t \in J \\
\text{and} \quad & x(t) \geq 0, \quad t \in J.
\end{align*}$$

Using MATLAB, we get
\( x^*(1) = 0.714286, \quad x^*(2) = 1.683674, \)
\( x^*(4) = 4.314869, \quad W(x^*) = 7648.688. \)

On the other hand, using again Definition 2.4, the dual Leontief production model in quantum calculus is described as
\[
\begin{aligned}
\min \quad & G(y) = 500 \int_1^3 ty(t) d_q(t) = 500 \sum_{k=0}^{2} 4^k y(2^k) \\
\text{s.t.} \quad & 700y(t) \geq 100t + 250 \int_{2t}^3 y(s)d_q(s) = 100t + 250 \sum_{k=0}^{2} 2^k y(2^k) \quad t \in J \\
& \quad \text{and} \quad y(t) \geq 0, \quad t \in J.
\end{aligned}
\]

Using MATLAB, we get
\( y^*(1) = 1.746356, \quad y^*(2) = 1.102041, \)
\( y^*(2) = 0.571429, \quad G(y^*) = 7648.688; \)
\( \text{thus } W(x^*) = G(y^*). \)

**Example 6.2.**\( T = q_{N_0}^N \text{ and } J = \{1, 3, 9\} \) with \( q = 3. \) Using the integral given in Definition 2.4, the primal Leontief production model in quantum calculus is described as
\[
\begin{aligned}
\max \quad & w(x) = \int_1^3 100tx(t)d_q(t) = 200 \sum_{k=0}^{2} 3^k x(3^k) \\
\text{s.t.} \quad & 700x(t) \leq 500t + 250 \int_1^t x(s)d_q(s) = 500t + 500 \sum_{k=0}^{lo g_3^{-1}} 3^k x(3^k) \quad t \in J \\
& \quad \text{and} \quad x(t) \geq 0, \quad t \in J.
\end{aligned}
\]

Using MATLAB, we get
\( x^*(1) = 0.714286, \quad x^*(2) = 2.653061, \)
\( x^*(4) = 12.623907, \quad W(x^*) = 209425.7. \)

On the other hand, using again Definition 2.4, the dual Leontief production model in quantum calculus is described as
\[
\begin{aligned}
\min \quad & G(y) = 500 \int_1^3 ty(t)d_q(t) = 1000 \sum_{k=0}^{2} 9^k y(3^k) \\
\text{s.t.} \quad & 700y(t) \geq 100t + 250 \int_{2t}^3 y(s)d_q(s) = 100t + 500 \sum_{k=0}^{2} 3^k y(3^k) \quad t \in J \\
& \quad \text{and} \quad y(t) \geq 0, \quad t \in J.
\end{aligned}
\]

Using MATLAB, we get
\( y^*(1) = 27.037901, \quad y^*(2) = 8.693877, \)
\( y^*(2) = 1.285714, \quad G(y^*) = 209425.7, \)
\( \text{thus } W(x^*) = G(y^*). \)

**7. Conclusions**

In this paper, quantum calculus analogue of Leontief production model have been presented. We formulated the primal and the dual Leontief production models. Furthermore, the weak duality theorem and the strong duality theorem were given for arbitrary quantum set. The new formulation provides an exact optimal solution for the production models by solving either the primal model or the dual model, which reduced the large computation effort. Moreover, using the objective functions for the primal and
the dual quantum calculus models, we can easily obtain upper and lower bounds for the value of production at any production plan.

References

[1] Bellman R 2010, Dynamic programming. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ., Reprint of the 1957 edition, With a new introduction by Stuart Dreyfus.

[2] Bellman R and Dreyfus S 1962, Applied dynamic programming. Princeton University Press, Princeton, N.J.

[3] Buie R and Abrahm J 1973, Numerical solutions to continuous linear programming problems. Z. Operations Res. Ser. A-B, 17(3):A107-A117.

[4] Gale D 1960, The Theory of Linear Economic Model. McGraw-Hill, New York.

[5] Grinold R. 1969, Continuous programming. I. Linear objectives. J. Math. Anal. Appl., 28:32-51.

[6] Hanson M. 1967, A continuous leontief production model with quadratic objective function. Econometrica, 35(4):530-536.

[7] Levinson N 1966, A class of continuous linear programming problems. J. Math. Anal. Appl. 16:73-83.

[8] Tyndall W 1965, A duality theorem for a class of continuous linear programming problems. J. Soc. Indust. Appl. Math., 13:644-666.

[9] Tyndall W 1967, An extended duality theorem for continuous linear programming problems. SIAM J. Appl. Math., 15:1294-1298.

[10] Wen C, Lur Y and Lai H 2012, Approximate solutions and error bounds for a class of continuous-time linear programming problems. Optimization, 61(2):163-185.

[11] Advan M and Bohner M 2006, Spectral analysis of q-difference equations with spectral singularities. Mathematical and Computer Modelling, 43:695-703.

[12] Advan M and Koyuncuoglu H 2016, Floquet theory based on new periodicity concept for hybrid systems involving q-difference equations. Applied Mathematics and Computation, 273:1208-1233.

[13] Bohner M and Chieochan R 2013, The beverton-holt q-difference equations. J. Biol. Dyn., 7(1):8695.

[14] Bohner M and Chieochan R2013, Positive periodic solutions for higher order functional q-difference equations. J. Appl. Funct. Anal., 8(1):1422.

[15] Bohner M, Fan M and Zhang J 2007, Periodicity of scalar dynamic equations and applications to population models. J. Math. Anal. Appl., 330(1):1-9.

[16] Bohner M, Heim J and Liu A 2013, Solow models on time scales. Cubo, 15(1):13-31.

[17] Bohner M and Jaqueline G2016, Mesquita. Periodic averaging principle in quantum calculus. J. Math. Anal. Appl., 435:11461159.

[18] Bohner M and Peterson A 2001, Dynamic equations on time scales. Birkh auser Boston, Inc., Boston, MA, An introduction with applications.

[19] Bohner M and Peterson A 2003, Advances in dynamic equations on time scales. Birkh• auser Boston, Inc., Boston, MA.

[20] Bohner M and Wintz N 2013, The Kalman filter for linear systems on time scales. J. Math. Anal. Appl., 406(2):419-436.

[21] Kac V and Cheung P. 2002, Quantum calculus. Universitext. Springer-Verlag, New York.

[22] Chvatal V 1983, Linear programming. A Series of Books in the Mathematical Sciences. W. H. Freeman and Company, New York.