FUNCTORIALITY OF CUNTZ-PIMSNER CORRESPONDENCE MAPS

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Abstract. We show that the passage from a $C^*$-correspondence to its Cuntz-Pimsner $C^*$-algebra gives a functor on a category of $C^*$-correspondences with appropriately defined morphisms. Applications involving topological graph $C^*$-algebras are discussed, and an application to crossed-product correspondences is presented in detail.

1. Introduction

Cuntz-Pimsner algebras were first introduced by Pimsner in [Pim97] as $C^*$-algebras associated to $C^*$-correspondences with injective left actions; Katsura extended the definition in [Kat04b] to include $C^*$-correspondences with non-injective left actions. The class of Cuntz-Pimsner algebras is very rich, containing all Cuntz-Krieger algebras, crossed products by $\mathbb{Z}$, and topological graph algebras. Accordingly, there is a pressing need to understand how constructions at the level of $C^*$-correspondences carry over to the Cuntz-Pimsner algebras.

Now, many $C^*$-correspondence constructions naturally and necessarily involve multiplier correspondences $(M(Y), M(B))$. For example, if a group $G$ acts on a correspondence $(X, A)$, then $(X, A)$ embeds in the multipliers $(M(X \rtimes G), M(A \rtimes G))$ of the crossed-product correspondence, but not in general in $(X \rtimes G, A \rtimes G)$ itself. Our main result (Corollary 3.6) is that a $C^*$-correspondence homomorphism $(X, A) \to (M(Y), M(B))$ that is covariant in an appropriate sense (Definition 3.1) induces a $C^*$-algebra homomorphism $O_X \to M(O_Y)$ of the corresponding Cuntz-Pimsner algebras.

Given that the natural embedding of a $C^*$-correspondence in its Cuntz-Pimsner algebra is often degenerate, there is no reason to expect that a correspondence homomorphism $(X, A) \to (M(Y), M(B))$ will automatically extend to $(M(X), M(A))$; this makes composing two homomorphisms problematic. To deal with this, our notion of covariance...
incorporates the assumption that the image of $X$ lies in the so-called restricted multipliers $M_B(Y)$ that were introduced in [DKQ, Appendix A]. We then can show (Theorem 3.5) that the $C^*$-homomorphisms $\mathcal{O}_X \to M(\mathcal{O}_Y)$ are obtained in a functorial way on an appropriately-defined category of $C^*$-correspondences. Some earlier work has been done on functoriality of Cuntz-Pimsner algebras [Rob, RS], but our approach is the first to incorporate multipliers.

In Section 4 we give three applications of our techniques: to topological graph actions, topological graph coactions, and to crossed-product $C^*$-correspondences. The first two are really just brief indications of applications, the first to a recent result [DKQ] concerning free and proper actions of groups on topological graphs, where the argument was essentially a “bare-hands” special case of our Corollary 3.6 and the second is a foreshadowing of how we plan to use the technique to define coactions on Cuntz-Pimsner algebras from suitable coactions on the correspondences. For the third application, we bring our methods to bear on crossed-product $C^*$-correspondences $(X \rtimes_G A \ltimes_A G)$. We show that the canonical embedding $(X, A) \to (M(X \rtimes_G A), M(A \rtimes_A G))$ induces via functoriality a covariant homomorphism $(\mathcal{O}_X, G) \to M(\mathcal{O}_{X \rtimes_G A})$. As a result we find a surjective homomorphism $\mathcal{O}_X \rtimes_G G \to \mathcal{O}_{X \rtimes_G A}$, which serves as an alternative approach to the result of Hao and Ng [HN08, Theorem 2.10] that when the group $G$ is amenable there is an isomorphism $\mathcal{O}_{X \rtimes_G A} \cong \mathcal{O}_X \rtimes_G G$. By showing there is a maximal dual coaction on $\mathcal{O}_{X \rtimes_G A}$, we can show that our surjection is actually an isomorphism when the group $G$ is amenable, though this requires background on $C^*$-correspondence coactions and is saved for a forthcoming paper [KQRb].

2. Preliminaries

Given $C^*$-algebras $A$ and $B$, an $A - B$ correspondence (or a $C^*$-correspondence over $A$ and $B$) is a Hilbert $B$-module $X$ equipped with a left $A$-module action which is implemented by a homomorphism of $A$ into the $C^*$-algebra $\mathcal{L}(X)$ of adjointable operators on $X$. (We refer to [EKQR06, Kat04a, RW98] for background on Hilbert modules and further details on $C^*$-correspondences.) The homomorphism is generically called $\varphi_A$, but we usually suppress this notation and just write $a \cdot \xi$ for $\varphi_A(a)(\xi)$, where $a \in A$ and $\xi \in X$. We say $X$ is full if $\text{span}(X, X) \subset B$ is dense. If in addition, the left action is an isomorphism $\varphi_A : A \to \mathcal{K}(X)$ we will call $X$ an $A - B$ imprimitivity bimodule. We write $(A, X, B)$ to denote an $A - B$ correspondence $X$; when $A = B$ we just write $(X, A)$, and call $X$ an $A$-correspondence (or
a $C^*$-correspondence over $A$). In this paper we will make the standing assumption that all $C^*$-correspondences are nondegenerate in the sense that $A \cdot X = X$.

Given a $C^*$-correspondence $(A, X, B)$, the set $\mathcal{K}(X) = \overline{\text{span}}\{\theta_{\xi, \eta} \mid \xi, \eta \in X\}$ is a (closed) two-sided ideal in $\mathcal{L}(X)$ called the compact operators on $X$, where by definition $\theta_{\xi, \eta}(\zeta) = \xi \cdot \langle \eta, \zeta \rangle$ for $\zeta \in X$. The Banach space $\mathcal{L}(B, X)$ of adjointable operators from $B$ to $X$ (where $B$ is viewed as a Hilbert $B$-module in the natural way) becomes an $M(A) - M(B)$ correspondence when equipped with the natural left action of $M(A)$, right action of $M(B)$, and $M(B)$-valued inner product $\langle m, n \rangle = m^*n$ all given by composition of operators. This is called the multiplier correspondence of $X$, and is denoted by $M(X)$. There is an embedding $\xi \mapsto T_\xi : X \to M(X)$ given by $T_\xi(b) = \xi \cdot b$ for $b \in B$. The strict topology on $M(X)$ is generated by the seminorms

$$m \mapsto \|T \cdot m\| \quad \text{and} \quad m \mapsto \|m \cdot b\| \quad \text{for} \quad T \in \mathcal{K}(X), b \in B.$$ 

A correspondence homomorphism between two $C^*$-correspondences $(A, X, B)$ and $(C, Y, D)$ is a triple $(\pi, \psi, \rho)$ consisting of $C^*$-homomorphisms $\pi : A \to M(C)$ and $\rho : B \to M(D)$ and a linear map $\psi : X \to M(Y)$ satisfying

(i) $\psi(a \cdot \xi) = \pi(a) \cdot \psi(\xi),$ 
(ii) $\psi(\xi \cdot b) = \psi(\xi) \cdot \rho(b),$ and 
(iii) $\rho(\langle \xi, \eta \rangle) = \langle \psi(\xi), \psi(\eta) \rangle.$

(Note that condition (ii) follows from (iii).)

A correspondence homomorphism $(\pi, \psi, \rho)$ is nondegenerate when $\pi$ and $\rho$ are nondegenerate $C^*$-homomorphisms and $\psi(X) \cdot B = Y$. In this case, by [EKQR06, Theorem 1.30] there is a unique strictly continuous extension $\overline{\psi} : M(X) \to M(Y)$ such that

$$(\overline{\pi}, \overline{\psi}, \overline{\rho}) : (M(A), M(X), M(B)) \to (M(C), M(Y), M(D))$$ 

is a correspondence homomorphism, where $\overline{\pi}$ and $\overline{\rho}$ are the usual extensions of nondegenerate $C^*$-homomorphisms to multiplier algebras.

If $(X, A)$ is an $A$-correspondence and $B$ is a $C^*$-algebra, a correspondence homomorphism of the form $(\pi, \psi, \pi) : (A, X, A) \to (B, B, B)$ (where $B$ is viewed as a $B$-correspondence in the natural way) is called a Toeplitz representation of $(X, A)$ in $B$, and is denoted $(\psi, \pi)$. It is a critical observation, usually attributed to Pimsner [Pim97, Lemma 3.2] (see also [KPR98, Lemma 2.2]), that a Toeplitz representation of a $C^*$-correspondence determines a homomorphism of the algebra of compact operators. Here we derive this fact (Corollary 2.2 below) from the more fundamental Proposition 2.1 to emphasize that it is really a property of the underlying Hilbert module structure on $X$ and $B$. 
Proposition 2.1. Let $X$ and $Y$ be Hilbert modules over $C^*$-algebras $A$ and $B$, respectively, and suppose $(\psi, \rho): (X, A) \to (M(Y), M(B))$ is a correspondence homomorphism. Then there is a unique homomorphism $\psi^{(1)}: \mathcal{K}(X) \to M(\mathcal{K}(Y))$ such that
\[
\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^* \quad \text{for} \quad \xi, \eta \in X.
\]
If $\psi(X) \subseteq Y$, then $\psi^{(1)}(\mathcal{K}(X)) \subseteq \mathcal{K}(Y)$, with $\psi^{(1)}(\theta_{\xi, \eta}) = \theta_{\psi(\xi), \psi(\eta)}$.

Proof. Obviously there can be at most one such $\psi^{(1)}$. For existence, first suppose $\psi(X) \subseteq Y$. Without loss of generality, we may assume that $X$ is full (otherwise, replace $A$ by the closed span of $(X, X)$); similarly, we may assume that $Y$ is full, and that $\psi(X) = Y$ and $\rho(A) = B$.

Next, let $C = \mathcal{K}(X)$, so that $X$ is a $C - A$ imprimitivity bimodule. Let $I = \ker \rho$. The Rieffel correspondence (see Definition 1.7) induces an ideal $J = X - \text{Ind } I := \{c \in C : cX \subseteq XI\}$ so that $XI$ and $X/XI$ are $J - I$ and $C/J - A/I$ imprimitivity bimodules respectively. Then the quotient maps $(q_C, q_X, q_A)$ comprise a surjective imprimitivity bimodule homomorphism of $(C, X, A)$ onto $(C/J, X/XI, A/I)$, and there is an imprimitivity bimodule isomorphism $(\pi, \tilde{\psi}, \tilde{\rho})$ of $(C/J, X/XI, A/I)$ onto $(\mathcal{K}(Y), Y, B)$ such that $\tilde{\psi}(\xi + J) = \psi(\xi)$ for $\xi \in X$. Taking $\psi^{(1)} = \pi \circ q_C$, for $\xi, \eta \in X$ we have
\[
\psi^{(1)}(\theta_{\xi, \eta}) = \pi(q_C(\langle \xi, \eta \rangle)) = \pi(c(\langle \xi, \eta \rangle + J)) = \pi(c_{C/J}(\xi + J, \eta + J)) = \kappa_{(Y)}(\langle \tilde{\psi}(\xi + J), \tilde{\psi}(\eta + J) \rangle) = \kappa_{(Y)}(\langle \psi(\xi), \psi(\eta) \rangle) = \theta_{\psi(\xi), \psi(\eta)}.
\]

For the general case, apply the above to the Hilbert $M(B)$-module $M(Y)$, and note that by Remark A.10 we have $\mathcal{K}(M(Y)) \subseteq M(\mathcal{K}(Y))$, with $\kappa_{(M(Y))}(m, n) = mn^*$ for $m, n \in M(Y)$.

Corollary 2.2 ([Pim97, KPW98]). Let $(X, A)$ be a $C^*$-correspondence, and let $(\psi, \pi)$ be a Toeplitz representation of $(X, A)$ in a $C^*$-algebra $B$. Then there is a unique homomorphism $\psi^{(1)}: \mathcal{K}(X) \to B$ such that
\[
\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^*.
\]

For a $C^*$-correspondence $(X, A)$, we follow Katsura’s convention [Kat04a] and define an ideal $J_X$ of $A$ by
\[
J_X = \{a \in A \mid \varphi_A(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker(\varphi_A)\}.
\]
A Toeplitz representation $(\psi, \pi)$ of $(X, A)$ in $B$ is Cuntz-Pimsner covariant if
\[
\psi^{(1)}(\varphi_A(a)) = \pi(a) \quad \text{for } a \in J_X.
\]
We denote the universal Cuntz-Pimsner covariant Toeplitz representation by $(k_X, k_A)$, and the Cuntz-Pimsner algebra by $\mathcal{O}_X$. Note that
k_A: A → O_X will always be nondegenerate as a C*-homomorphism because of our assumption that all correspondences (X, A) are nondegenerate. Again, we refer the reader to [Kat04b] for details.

Finally, we will need the theory of “relative multipliers” from [DKQ, Appendix A], which is useful for extending degenerate homomorphisms. If (X, A) is a nondegenerate correspondence and κ: C → M(A) is a nondegenerate homomorphism, the C-multipliers of X are by definition

\[ M_C(X) = \{ m \in M(X) | \kappa(C) \cdot m \cup m \cdot \kappa(C) \subset X \} \]

The C-strict topology on MC(X) is generated by the seminorms

\[ m \mapsto \| \kappa(c) \cdot m \| \quad \text{and} \quad m \mapsto \| m \cdot \kappa(c) \| \quad \text{for} \ c \in C. \]

When A is viewed as an A-correspondence over itself in the usual way, MC(A) is a C*-subalgebra of M(A).

Relative multipliers possess the following elementary properties:

(i) The C-strict topology is stronger than the relative strict topology on MC(X).

(ii) MC(X) is an MC(A)-correspondence with respect to the restrictions of the operations of the M(A)-correspondence M(X), and the operations are separately C-strictly continuous.

(iii) If X = A, then MC(A) is a C*-subalgebra of M(A), and the multiplication and involution on MC(A) are separately C-strictly continuous.

(iv) K(MC(X)) ⊂ MC(K(X)).

(v) MC(X) is the C-strict completion of X.

(vi) MC(X) is an M(C)-sub-bimodule of M(X).

The main purpose of relative multipliers is the following extension theorem [DKQ, Proposition A.11]: Suppose (X, A) and (Y, B) are (nondegenerate) C*-correspondences and κ: C → M(A) and σ: D → M(B) are nondegenerate homomorphisms. For any correspondence homomorphism (ψ, π): (X, A) → (MD(Y), MD(B)), if there is a nondegenerate homomorphism λ: C → M(σ(D)) such that

\[ \pi(\kappa(c)a) = \lambda(c)\pi(a) \quad \text{for all} \ c \in C \text{ and } a \in A, \]

then there is a unique C-strict to D-strictly continuous correspondence homomorphism (ψ, π): (MC(X), MC(A)) → (MD(Y), MD(B)) that extends (ψ, π).

A closely related concept is the following, due to Baaj and Skandalis [BS89].
Definition 2.3. For an ideal $I$ of a $C^*$-algebra $A$, let

$$M(A; I) = \{ m \in M(A) \mid mA \cup Am \subset I \}.$$ 

Lemma 2.4. If $I$ is an ideal of a $C^*$-algebra $A$, then:

(i) $M(A; I)$ is the strict closure of $I$ in $M(A)$;

(ii) if $\pi: A \to M(B)$ is a nondegenerate homomorphism such that $\pi(I) \subset B$, then $\pi(M(A; I)) \subset M_A(B)$, and $\pi|: M(A; I) \to M_A(B)$ is strict to $A$-strictly continuous.

Proof. This is elementary, and the techniques are similar to those of [DKQ, Appendix A]. For (i), we first show that $I$ is strictly dense in $M(A; I)$. Let $m \in M(A; I)$, and let $\{ e_i \}$ be an approximate identity for $A$. Then $me_i \in I$ for all $i$, and $me_i \to m$ strictly in $M(A)$. To see that $M(A; I)$ is strictly closed in $M(A)$, let $\{ m_i \}$ be a net in $M(A; I)$ converging strictly to $m$ in $M(A)$. We must show that $m \in M(A; I)$. For $a \in A$ we have

$$\|m_i a - ma\| \to 0 \quad \text{and} \quad \|am_i - am\| \to 0,$$

so $ma, am \in I$ because $m_i a, am_i \in I$ for all $i$.

For (ii), note that by (i) it suffices to show that $\pi|: I \to B$ is continuous for the relative strict topology of $I$ in $M(A)$ and the relative $A$-strict topology of $B \subset M_A(B)$. Let $\{ c_i \}$ be a net in $I$ converging strictly to 0 in $M(A)$, and let $a \in A$. Then $\pi(c_i)\pi(a) = \pi(c_ia)$ and $\pi(a)\pi(c_i) = \pi(ac_i)$ converge to 0 in norm, so $\pi(c_i) \to 0 A$-strictly in $B$. □

Remark 2.5. Let $I$ be an ideal of a $C^*$-algebra $A$, and let $\rho: A \to M(I)$ be the canonical homomorphism. Then by Lemma 2.4, the canonical extension $\overline{\rho}: M(A) \to M(I)$ maps $M(A; I)$ into $M_A(I)$. In fact, the restriction $\overline{\rho}|_{M(A; I)}$ is injective, although we will not need this here.

3. Functoriality

Definition 3.1. Let $(X, A)$ and $(Y, B)$ be nondegenerate $C^*$-correspondences. A correspondence homomorphism $(\psi, \pi): (X, A) \to (M(Y), M(B))$ is Cuntz-Pimsner covariant if

(i) $\psi(X) \subset M_B(Y)$,

(ii) $\pi: A \to M(B)$ is nondegenerate,

(iii) $\pi(J_X) \subset M(B; J_Y)$, and
(iv) the diagram

\[
\begin{array}{ccc}
J_X & \xrightarrow{\pi} & M(B; J_Y) \\
\varphi_A | & & | \varphi_B |
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{K}(X) & \xrightarrow{\psi(1)} & M_B(\mathcal{K}(Y))
\end{array}
\]

commutes, where \(\psi(1)\) is the homomorphism provided by Proposition 2.1.

The above definition simplifies when the correspondence homomorphism is nondegenerate:

**Lemma 3.2.** A nondegenerate correspondence homomorphism \((\psi, \pi): (X, A) \to (M(Y), M(B))\) is Cuntz-Pimsner covariant if and only if items (i) and (iii) hold in Definition 3.1.

The lemma follows immediately from the following elementary result. Lemma 3.3 is presumably well-known, but we could not find a reference in the literature.

**Lemma 3.3.** For any nondegenerate correspondence homomorphism \((\pi, \psi, \rho): (A, X, B) \to (M(C), M(Y), M(D))\), the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & M(C) \\
\varphi_A | & & | \varphi_C |
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{L}(X) & \xrightarrow{\psi(1)} & \mathcal{L}(Y)
\end{array}
\]

commutes.

**Proof.** Fix \(a \in A\); we must show that \(\overline{\psi(1)}(\varphi_A(a))\eta = \overline{\varphi_C}(\pi(a))\eta\) for all \(\eta \in Y\). By nondegeneracy it suffices to consider elements of the form \(\eta = \psi(\xi)d\) with \(\xi \in X\) and \(d \in D\), in that case we have

\[
\begin{align*}
\overline{\psi(1)}(\varphi_A(a))\eta &= \overline{\psi(1)}(\varphi_A(a))\psi(\xi)d = \psi(\varphi_A(a)\xi)d = \psi(a \cdot \xi)d \\
&= \pi(a) \cdot \psi(\xi)d = \overline{\varphi_C}(\pi(a))\psi(\xi)d = \overline{\varphi_C}(\pi(a))\eta.
\end{align*}
\]

The following lemma addresses the overlap between Cuntz-Pimsner covariant correspondence homomorphisms and Cuntz-Pimsner covariant Toeplitz representations:

**Lemma 3.4.** Let \((X, A)\) be a nondegenerate \(C^*\)-correspondence, and let \((\psi, \pi): (X, A) \to M(B)\) be a Toeplitz representation of \((X, A)\) in a \(C^*\)-algebra \(B\). Then \((\psi, \pi)\) is Cuntz-Pimsner covariant as a correspondence homomorphism into \((B, B)\) (as in Definition 3.1) if and only if \(\pi: A \to
$M(B)$ is nondegenerate and $(\psi, \pi)$ is Cuntz-Pimsner covariant as a Toeplitz representation. In particular, $(k_X, k_A): (X, A) \to (\mathcal{O}_X, \mathcal{O}_X)$ is Cuntz-Pimsner covariant in the sense of Definition 3.1.

**Proof.** This follows from the identifications $J_B = \mathcal{K}(B) = B$ and $M_B(B) = M(B; B) = M_B(K(B)) = M(B)$, and the observation that $\varphi_B$ is the inclusion $B \hookrightarrow M(B)$.

The Cuntz-Pimsner covariant homomorphisms between correspondences are the morphisms in a suitable category, which we now define.

**Theorem 3.5.** There is a category $\text{CPCorres}$ that has:

- nondegenerate $C^*$-correspondences as objects, and
- Cuntz-Pimsner covariant homomorphisms $(\psi, \pi): (X, A) \to (M(Y), M(B))$ (as in Definition 3.1) as morphisms from $(X, A)$ to $(Y, B)$;
- and in which the composition of $(\psi, \pi): (X, A) \to (Y, B)$ and $(\sigma, \tau): (Y, B) \to (Z, C)$ is $(\sigma \circ \psi, \tau \circ \pi)$.

**Proof.** First of all, to see that composition is well-defined, note that since $\tau: B \to M(C)$ is nondegenerate by definition, it follows from [DKQ, Proposition A.11] that $\sigma: Y \to M_C(Z)$ extends uniquely to a $B$-strict to $C$-strictly continuous homomorphism $(\sigma, \tau): (M_B(Y), M(B)) \to (M_C(Z), M(C))$.

Thus we get a correspondence homomorphism $(\sigma \circ \psi, \tau \circ \pi): (X, A) \to (M_C(Z), M(C)),$

which we must check is Cuntz-Pimsner covariant in the sense of Definition 3.1. Certainly $\tau \circ \pi: A \to M(C)$ is nondegenerate, so it remains to verify items (iii)–(iv) in Definition 3.1.

For (iii), since $\tau$ is nondegenerate we have

$$(\tau \circ \pi)(J_X)C = \tau(\pi(J_X))\tau(B)C \subset \tau(M(B; J_Y))\tau(B)C$$

$$= \tau(M(B; J_Y)B)C \subset \tau(J_Y)C \subset J_Z,$$

and similarly $C(\tau \circ \pi)(J_X) \subset J_Z$. Therefore $\tau \circ \pi(J_X) \subset M(C; J_Z)$.

For (iv), let $a \in J_X$. We must show that

$$\overline{\varphi_C \circ (\tau \circ \pi)}(a) = (\overline{\sigma \circ \psi})^{(1)} \circ \varphi_A(a)$$

in $\mathcal{L}(Z)$, and by nondegeneracy it suffices to show equality after multiplying on the right by $\varphi_C(c)$ for an arbitrary $c \in C$. Again by nondegeneracy we can factor $c = \tau(b)c'$ for some $b \in B$ and $c' \in C$, and then

$$\overline{\varphi_C \circ (\tau \circ \pi)}(a)\varphi_C(c) = \overline{\varphi_C(\tau(a))}\varphi_C(\tau(b)c')$$

in $\mathcal{L}(Z)$.
$= \varphi_C \circ \tau(\pi(a)b) \varphi_C(c')$

$= \sigma(1) \circ \varphi_B(\pi(a)b) \varphi_C(c')$

$= \sigma(1) \left( \varphi_B \circ \pi(a) \varphi_B(b) \right) \varphi_C(c')$

$= \sigma(1) \left( \psi(1) \circ \varphi_A(a) \varphi_B(b) \right) \varphi_C(c')$

$= \sigma(1) \circ \psi(1) \circ \varphi_A(a) \sigma(1) \circ \varphi_B(b) \varphi_C(c')$

$= (\sigma \circ \psi)(1) \circ \varphi_A(a) \varphi_C(\tau(b)c')$

$= (\sigma \circ \psi)(1) \circ \varphi_A(a) \varphi_C(c')$

$= (\sigma \circ \psi)(1) \circ \varphi_A(a) \varphi_C(c').$

We have thus verified that composition is well-defined in \textit{CPCorres}.

To see that composition is associative is a routine exercise in the definitions and the properties of “barring” (see, e.g., [aQRW11, Appendix A]): if also \((\zeta, \rho) : (Z, C) \to (W, D)\) in \textit{CPCorres} then

\[
(\zeta, \rho) \circ ((\sigma, \tau) \circ (\psi, \pi)) = (\zeta, \rho) \circ (\sigma \circ \psi, \tau \circ \pi)
\]

\[
= (\zeta \circ (\sigma \circ \psi), \tau \circ (\sigma \circ \pi))
\]

\[
= ((\zeta \circ \sigma) \circ \psi, (\tau \circ \sigma) \circ \pi)
\]

\[
= (\zeta \circ \sigma \circ \psi, \tau \circ \sigma \circ \pi)
\]

\[
= (\zeta \circ \sigma \circ \psi) \circ (\psi, \pi)
\]

\[
= (\zeta, \rho) \circ ((\sigma, \tau) \circ (\psi, \pi)).
\]

It is now clear that \((\text{id}_X, \text{id}_A)\) is an identity morphism on each object \((X, A)\), and therefore \textit{CPCorres} is a category.

\section*{Corollary 3.6.}

Let \((X, A)\) and \((Y, B)\) be nondegenerate \textit{C*}-correspondences, and let \((\psi, \pi) : (X, A) \to (M(Y), M(B))\) be a Cuntz-Pimsner covariant correspondence homomorphism. Then there is a unique homomorphism \(O_{\psi, \pi}\) making the diagram

\[
\begin{array}{ccc}
(X, A) & \xrightarrow{(\psi, \pi)} & (M_B(Y), M(B)) \\
\downarrow{\left( k_X, k_A \right)} & & \downarrow{\left( k_Y, k_B \right)} \\
O_X & \xrightarrow{O_{\psi, \pi}} & M_B(O_Y)
\end{array}
\]

commute. Moreover, \(O_{\psi, \pi}\) is nondegenerate, and is injective if \(\pi\) is.

\textbf{Proof.} Applying Theorem 3.5 with \((Z, C)\) being the \textit{C*-algebra} \(O_Y\) viewed as a correspondence over itself in the canonical way, we see that \((\overline{k_Y \circ \psi}, \overline{k_B \circ \pi})\) is a Cuntz-Pimsner covariant Toeplitz representation
$(\sigma, \nu): X \to M_B(O_Y)$. Then the universal property of $O_X$ gives the unique homomorphism

$$O_{\psi, \pi} = (k_Y \circ \psi) \times (k_B \circ \pi).$$

Nondegeneracy of $O_{\psi, \pi}$ follows from nondegeneracy of $\pi$ and $k_B$, since

$$O_Y = k_B \circ \pi(A)O_Y = O_{\psi, \pi} \circ k_A(A)O_Y = O_{\psi, \pi}(k_A(A))O_Y \subseteq O_{\psi, \pi}(O_X)O_Y \subseteq O_Y$$

implies equality throughout.

If $\pi$ is injective, then so is $k_B \circ \pi$, so to show $O_{\psi, \pi}$ is injective we can apply the Gauge-Invariant Uniqueness Theorem \cite[Theorem 6.4]{Kat04b}: let $\gamma: T \to \text{Aut} O_Y$ be the gauge action. It suffices to observe that for all $z \in T$, $\xi \in X$, and $a \in A$ we have

$$\gamma_z \circ k_Y \circ \psi(\xi) = z k_Y \circ \psi(\xi)$$
$$\gamma_z \circ k_B \circ \pi(a) = k_B \circ \pi(a).$$

Recall from, e.g., \cite{aQRW11}, that there is a category $C_{\text{nd}}$ that has:

- $C^*$-algebras as objects, and
- nondegenerate homomorphisms $\pi: A \to M(B)$ as morphisms from $A$ to $B$;
- and in which the composition of $\pi: A \to B$ and $\tau: B \to C$ is $\tau \circ \pi$.

**Theorem 3.7.** The assignments $X \mapsto O_X$ and $(\psi, \pi) \mapsto O_{\psi, \pi}$ define a functor from $\text{CPCorres}$ to $C_{\text{nd}}$.

**Proof.** First of all, it follows from Corollary 3.6 that if $(\psi, \pi): (X, A) \to (Y, B)$ in $\text{CPCorres}$ then $O_{\psi, \pi}$ is a morphism from $O_X$ to $O_Y$ in $C_{\text{nd}}$. Moreover, $O_{\text{id}_X, \text{id}_A} = \text{id}_{O_X}$ by uniqueness.

To see that compositions are preserved, let $(\psi, \pi): (X, A) \to (Y, B)$ and $(\sigma, \tau): (Y, B) \to (Z, C)$ in $\text{CPCorres}$. We have

$$O_{(\sigma, \tau) \circ (\psi, \pi)} \circ k_X = k_Z \circ (\sigma \circ \psi) = k_Z \circ (\sigma \circ \psi) = \overline{O_{\sigma, \tau} \circ k_Y \circ \psi} = \overline{O_{\sigma, \tau} \circ O_{\psi, \pi} \circ k_X},$$

and similarly

$$O_{(\sigma, \tau) \circ (\psi, \pi)} \circ k_A = \overline{O_{\sigma, \tau} \circ O_{\psi, \pi} \circ k_A},$$

so that $O_{(\sigma, \tau) \circ (\psi, \pi)} = O_{\sigma, \tau} \circ O_{\psi, \pi}$ in $C_{\text{nd}}$.

**4. Applications**

We give three applications of Corollary 3.6.
Topological graph actions. Our first application is historical; we show that in [DKQ] the germ of the idea of Corollary 3.6 was introduced in an ad-hoc way, in a very special case.

Theorem 5.6 of [DKQ] shows that if a locally compact group \( G \) acts freely and properly on a topological graph \( E \), then the quotient \( E/G \) is also a topological graph, and \( C^*(E) \rtimes_r G \) and \( C^*(E/G) \) are Morita equivalent. The strategy for proving Morita equivalence in [DKQ] is to construct an isomorphism of \( C^*(E/G) \) with Rieffel’s generalized fixed-point algebra \( C^*(E)^G \), after showing that the action of \( G \) on \( C^*(E) \) is saturated and proper in the sense of [Rie90], and then appealing to the imprimitivity theorem \( C^*(E) \rtimes_r G \sim M C^*(E)^G \) of [Rie90].

By definition, \( C^*(E)^G \) is a \( C^* \)-subalgebra of \( M(C^*(E)) \), and the isomorphism of \( C^*(E/G) \) onto \( C^*(E)^G \) is constructed from a Cuntz-Pimsner covariant Toeplitz representation \((\tau, \pi)\) of the topological-graph correspondence \((X(E/G), C_0((E/G)^0))\) in \( M(C^*(E)) \), which in turn is constructed via a correspondence homomorphism \((\mu, \nu)\) from \((X(E/G), C_0((E/G)^0))\) to \((M(X(E)), M(C_0(E^0)))\). The proof in [DKQ] that \((\tau, \pi)\) is Cuntz-Pimsner covariant essentially uses a special case of the concept of Cuntz-Pimsner covariant correspondence homomorphisms defined in Definition 3.1.

We will now explain this in more detail. First, recall that the \( C_0(E^0)\)-correspondence \( X(E) \) is a completion of \( C_c(E^1) \), the Katsura ideal \( J_{X(E)} \) can be identified with \( C_0(E_{rg}^0) \), where \( E_{rg}^0 \) is a certain open subset of \( E^0 \), and the topological-graph algebra \( C^*(E) \) is the Cuntz-Pimsner algebra \( \mathcal{O}_{X(E)} \). It will help the exposition to introduce the following temporary notation:

- \( A = C_0((E/G)^0) \)
- \( X = X((E/G)) \)
- \( A_{rg} = J_X \)
- \( B = C_0(E^0) \)
- \( Y = X(E) \)
- \( B_{rg} = J_Y \)
- \( q : E \to E/G \) is the quotient map (both for edges \( E^1 \) and vertices \( E^0 \)).

(Warning: the roles of \( X, A \) and \( Y, B \) between [DKQ] and here are switched, to allow more convenient reference to the methods of the current paper.) [DKQ] constructed the correspondence homomorphism

\[(\mu, \nu) : (X, A) \to (M_B(Y), M(B))\]

starting with

- \( \nu(f) = f \circ q \)
• \((\mu(\xi) \cdot g)(e) = \xi(q(e))g(e)\) for \(\xi \in C_c((E/G)^1)\), \(g \in C_c(E^0)\), and \(e \in E^1\).

Then the pair \((X, A)\) was mapped into \(M(O_Y)\) in [DKQ] by the commutative diagram

\[
\begin{array}{ccc}
(X, A) & \xrightarrow{(\mu, \nu)} & (M_B(Y), M(B)) \\
\downarrow{(\tau, \pi)} & & \downarrow{(k_Y, k_B)} \\
M_B(O_Y) & & M_B(O_Y)
\end{array}
\]

In [DKQ] it was then recognized that Cuntz-Pimsner covariance of \((\tau, \pi)\) is expressed by commutativity of the left-hand triangle of the following diagram, which is a version of [DKQ, page 1547, diagram (5)] in which some of the notation has been modified to be consistent with the current paper:

(4.1)

\[
\begin{array}{ccc}
J_X & \xrightarrow{\nu'} & M_B(J_Y) \\
\downarrow{\varphi_A} & & \downarrow{\mu^{(1)}} \\
\varphi_B & & M_B(K(Y)) \\
\downarrow{\tau^{(1)}} & & \downarrow{\varphi_B} \\
K(X) & \xrightarrow{\varphi_B} & M_B(K(Y)),
\end{array}
\]

and the strategy was to verify that the other parts of the diagram commute. Here the homomorphism \(\nu'\) is constructed from the commutative diagram

\[
\begin{array}{ccc}
J_X & \xrightarrow{\nu} & M(B; J_Y) \\
\downarrow{\nu'} & & \downarrow{\nu'} \\
M_B(J_Y) & & M_B(J_Y),
\end{array}
\]

where the inclusion \(M(B : J_Y) \hookrightarrow M_B(J_Y)\) is given by restriction

\(g \mapsto g|_{J_Y^{\text{ng}}}\).

In [DKQ] it was not recognized that in fact it would be better to do away with the map \(\nu'\) altogether, so that the outer square of (4.1) is
replaced by

\[
\begin{array}{ccc}
J_X & \xrightarrow{\nu} & M(B; J_Y) \\
\varphi_A & & \varphi_B \\
\downarrow & & \downarrow \\
\mathcal{K}(X) & \xrightarrow{\mu} & M_B(\mathcal{K}(Y)),
\end{array}
\]

whose commutativity is precisely our definition Definition 3.1 of Cuntz-Pimsner covariance of the correspondence homomorphism \((\mu, \nu)\). The computations in [DKQ] were much more painstakingly “bare-hands” than in the current paper, because, again, the techniques were entirely ad-hoc, whereas here we take a more conceptual and systematic approach, developing appropriate machinery along the way.

**Topological graph coactions.** In a forthcoming paper [KQ], a continuous map (“cocycle”) \(\kappa\) on a topological graph \(E\) with values in a locally compact group \(G\) is used to construct a coaction \(\delta\) of \(G\) on \(C^*(E)\), with an eye toward proving that the crossed product \(C^*(E) \rtimes G\) is isomorphic to the \(C^*\)-algebra of the skew-product topological graph \(E \rtimes \kappa G\), thereby generalizing [KQRa, Theorem 2.4] from the discrete case.

To describe this application, let

- \(A = C_0(E^0)\)
- \(X = X(E)\)

and recall, e.g., from the discussion at the beginning of this section, that \(C^*(E)\) is the Cuntz-Pimsner algebra of the \(C^*\)-correspondence \((X, A)\).

To be a coaction, \(\delta\) must in particular be a homomorphism from \(C^*(E)\) to \(M_{1 \otimes C^*(G)}(C^*(E) \otimes C^*(G))\). As usual, \(\delta\) is constructed from a Cuntz-Pimsner covariant Toeplitz representation \((\delta_X, \delta_A)\) of \((X, A)\) in the \(C^*\)-algebra \(M(C^*(E) \otimes C^*(G))\), which in turn is constructed via a correspondence homomorphism \((\sigma, \text{id}_A \otimes 1)\) from \((X, A)\) to \((M(X \otimes C^*(G)), M(A \otimes C^*(G)))\). Techniques based upon Corollary 3.6 will be used to show that the correspondence homomorphism \((\sigma, \text{id}_A \otimes 1)\) gives rise to a coaction \(\delta\) on \(C^*(E)\). Interestingly, however, we will need a slight strengthening of the Cuntz-Pimsner covariance condition of Definition 3.1. The problem is that, due to nonexactness of minimal \(C^*\)-tensor products, we have no reason to believe that the minimal tensor product \(\mathcal{O}_X \otimes C^*(G)\) coincides with the Cuntz-Pimsner algebra \(\mathcal{O}_{X \otimes C^*(G)}\) of the external-tensor-product correspondence (where \(C^*(G)\) is regarded as a correspondence over itself in the standard way). In fact,
the basic theory of coactions on Cuntz-Pimsner algebras will require a significant amount of work, which we will do in \cite{KQRb}.

Anyway, once we have the machinery necessary to construct coactions on Cuntz-Pimsner algebras, our application in \cite{KQ} will go roughly as follows: first of all, a $G$-valued cocycle on a topological graph $E$ is just a continuous map $\kappa : E \to G$. Since the $A$-correspondence $X$ is a completion of $C_c(E^1)$, and the group $G$ embeds as unitary multipliers of $C^*(G)$, we are led to regard the cocycle $\kappa$ as an adjointable operator $v$ on $X \otimes C^*(G)$, and then we are able to define a coaction on $X$ by $\sigma(\xi) = v(\xi \otimes 1)$, where $\xi \otimes 1$ is regarded as a multiplier of the correspondence $X \otimes C^*(G)$. This is a continuous version of the coaction $\chi(e) \mapsto \chi(e) \otimes \kappa(e)$ of \cite{KQRa}. We emphasize that the justification that this actually gives a coaction will depend upon the preparation to come in \cite{KQRb}.

\textbf{$C^*$-correspondence action crossed products.} Let $(\gamma, \alpha)$ be an action of a locally compact group $G$ on a nondegenerate correspondence $(X, A)$. The \textit{crossed product} is the completion $(X \rtimes \gamma G, A \rtimes_\alpha G)$ of the $K$-module action \cite{EKQR00, HN08, EKQR06, Chapters 2 and 3, Kas88} for the elementary theory of actions and crossed products for correspondences.

Since $(\gamma_s, \alpha_s) : (X, A) \to (X, A)$ is a correspondence homomorphism for each $s \in G$, Proposition \ref{prop:crossed_product_homomorphism} provides homomorphisms $\gamma^{(1)}_s : K(X) \to K(X)$, which by uniqueness give rise to an action $(\gamma^{(1)}_s, \gamma, \alpha)$ of $G$ on the correspondence $(K(X), X, A)$, and such that $\varphi_A : A \to M(K(X))$ is $\alpha - \gamma^{(1)}$ equivariant.

There is an isomorphism (see, e.g., \cite[3.11]{Kas88})

$$
\tau : K(X \rtimes_\gamma G) \xrightarrow{\cong} K(X) \rtimes_{\gamma^{(1)}} G
$$

satisfying

$$
\tau(\theta_{\xi, \eta})(s) = \int_G \theta_{\xi(t), \gamma_s(\eta(s^{-1}t))} \Delta(s^{-1}t) dt.
$$
where $\xi, \eta \in C_c(G, X), s \in G$ and $\Delta$ is the modular function of $G$, and moreover the diagram
\[
\begin{array}{ccc}
A \rtimes_\alpha G & \overset{\varphi_{A \rtimes_\alpha G}}{\longrightarrow} & M(K(X \rtimes_\gamma G)) \\
\downarrow \varphi_{A \rtimes G} & & \downarrow \varphi \\
M(K(X) \rtimes_{\gamma(1)} G) & & \\
\end{array}
\]
commutes.

Let $(i_A, i_G): (A, G) \to M(A \rtimes_\alpha G)$ be the canonical covariant homomorphism.

**Proposition 4.1.** There is a Cuntz-Pimsner covariant correspondence homomorphism
\[
(i_X, i_A): (X, A) \to (M(X \rtimes_\gamma G), M(A \rtimes_\alpha G))
\]
such that for $x \in X, f \in C_c(G, A), s \in G$ we have
\[(4.2)\]
\[
(i_X(x) \cdot f)(s) = x \cdot f(s).
\]

**Proof.** We first claim that for fixed $x \in X$, $i_X(x)$ uniquely determines an operator $i_X(x): A \rtimes_\alpha G \to X \rtimes_\gamma G$ with adjoint given by
\[
(i_X(x)^*\xi)(s) = \langle x, \xi(s) \rangle
\]
for $\xi \in C_c(G, X)$ and $s \in G$. Indeed, (4.2) certainly defines a right $C_c(G, A)$-module map $C_c(G, A) \to C_c(G, X)$, and we can check the adjoint property on generators: for any $\xi \in C_c(G, X), f \in C_c(G, A)$, and $s \in G$ we simply calculate
\[
\langle i_X(x)f, \xi \rangle(s) = \int_G \alpha_{t^{-1}}(\langle (i_X(x)f)(t), \xi(ts) \rangle) \, dt
\]
\[
= \int_G \alpha_{t^{-1}}(f(t)^*\langle x, \xi(ts) \rangle) \, dt
\]
\[
= \int_G \alpha_t(f(t^{-1})\Delta(t^{-1})\langle x, \xi(t^{-1}s) \rangle) \, dt
\]
\[
= \int_G f^*(t)\alpha_t(\langle x, \xi(t^{-1}s) \rangle) \, dt,
\]
proving the claim.

It is now straightforward to verify that the pair $(i_X, i_A)$ is a correspondence homomorphism. For example, for $x, y \in X, f \in C_c(G, A)$ and $s \in G$ we calculate
\[
(\langle i_X(x), i_X(y) \rangle f)(s) = (i_X(x)^*i_X(y) \cdot f)(s) = \langle x, (i_X(y) \cdot f)(s) \rangle
\]
\[
= \langle x, y \cdot f(s) \rangle = \langle x, y \rangle f(s) = (i_A(\langle x, y \rangle) f)(s)
\]
as required.

To show that \((i_X, i_A)\) is Cuntz-Pimsner covariant, by Lemma 3.2 it suffices to show that \((i_X, i_A)\) is nondegenerate and satisfies items (i) and (iii) in Definition 3.1.

We already know that the coefficient map \(i_A\) is nondegenerate. Then for \(x \in X, a \in A, g \in C_c(G)\) we have

\[
i_X(x) \cdot (i_A(a)i_G(g)) = i_X(x \cdot a)i_G(g),
\]

so nondegeneracy of \((i_X, i_A)\) follows since \(X \cdot A = X\) and

\[
X \rtimes_\gamma G = i_X(X) \cdot i_G(C_c(G)).
\]

Item (i), that \(i_X(X) \subset M_{A \rtimes_\alpha G}(X \rtimes_\gamma G)\), is clear from the definition. To verify item (iii), that \(i_A(J_X)(A \rtimes_\alpha G) \subset J_X \rtimes_\alpha G \subset J_X \rtimes_\gamma G\), fix \(a \in J_X\). Firstly, [HN08, Lemma 2.6(a)] says that \(J_X\) is \(\alpha\)-invariant, and we know from [HN08, Proposition 2.7] that \(J_X \rtimes_\alpha G \subset J_X \rtimes_\gamma G\). For any \(f \in C_c(G, A)\) and \(s \in G\) we have

\[
(i_A(a)f)(s) = af(s) \in J_X
\]

so \(i_A(a)f \in C_c(G, J_X)\). Hence we get

\[
i_A(J_X)(A \rtimes_\alpha G) \subset J_X \rtimes_\alpha G \subset J_X \rtimes_\gamma G
\]
as required. \(\square\)

**Proposition 4.2.** With \(i_X\) as in Proposition 4.1, let

\[O_{i_X, i_A} : O_X \to M(O_{X \rtimes_\gamma G})\]

be the nondegenerate homomorphism vouchsafed by Corollary 3.6. Also define a strictly continuous unitary homomorphism \(u : G \to M(O_{X \rtimes_\gamma G})\) by

\[u = k_{A \rtimes_\alpha G} \circ i_G.\]

Then the pair \((O_{i_X, i_A}, u)\) defines a covariant homomorphism of the C*-dynamical system \((O_X, G, \beta)\) in the Cuntz-Pimsner algebra \(O_{X \rtimes_\gamma G}\).

**Proof.** We only need to verify the covariance condition, namely that for each \(t \in G\) we have

\[
\text{Ad } u(t) \circ O_{i_X, i_A} = O_{i_X, i_A} \circ \beta_t,
\]

and it is enough to check this on the generators from \(X\) and \(A\). For \(x \in X\) we have

\[
u(t)O_{i_X, i_A} \circ k_X(x) = k_{A \rtimes_\alpha G}(i_G(t))k_{X \rtimes_\gamma G}(i_X(x))
= k_{X \rtimes_\gamma G}(i_G(t) \cdot i_X(x))
= k_{X \rtimes_\gamma G}(i_X(\gamma_t(x)) \cdot i_G(t))
\]
\[
\begin{align*}
&= k_{X \rtimes_s G}(i_X(\gamma_t(x))) k_{A \rtimes_a G}(i_G(t)) \\
&= O_{iX,i_A} \circ k_X \circ \gamma_t(u(t)) \\
&= O_{iX,i_A} \circ \beta_t \circ k_X(x(u(t)),
\end{align*}
\]
and the calculation for generators from \(A\) is similar.

**Proposition 4.3.** The integrated form
\[
O_{iX,i_A} \times u : O_X \rtimes G \to O_{X \rtimes_s G}
\]
of the covariant pair \((O_{iX,i_A}, u)\) is surjective.

**Proof.** Let \(f \in C_c(G, A)\). Then \(k_A \circ f \in C_c(G, O_X)\), and since \((i_A, i_G)\) is a covariant pair we have
\[
(O_{iX,i_A} \times u)(k_A \circ f) = \int_G O_{iX,i_A}(k_A(f(t)))u(t) \, dt = k_{A \rtimes_a G} \left( \int_G i_A(f(t))i_G(t) \, dt \right) = k_{A \rtimes a G}((i_A \times i_G)(f)) = k_{A \rtimes a G}(f).
\]
Therefore the image of \(O_{iX,i_A} \times u\) contains \(k_{A \rtimes a G}(A \rtimes_a G)\).

Now let \(\xi \in C_c(G, X)\). A similar calculation shows that
\[
(O_{iX,i_A} \times u)(k_X \circ \xi) = k_{X \rtimes_s G} \left( \int_G i_X(\xi(t)) \cdot i_G(t) \, dt \right).
\]
Now, for any \(f \in C_c(G, A)\), \(s \in G\) we can calculate
\[
\left( \int_G i_X(\xi(t)) \cdot i_G(t) \, dt \right)(s) = \left( \int_G i_X(\xi(t)) \cdot i_G(t) \, dt \right)(s) = \int_G (i_X(\xi(t)) \cdot i_G(t)f) \, dt = \int_G \xi(t) \cdot (i_G(t)f) \, dt = \int_G \xi(t) \cdot \alpha_t(f(t^{-1}s)) \, dt = (\xi \cdot f)(s),
\]
and so we have
\[
\int_G i_A(\xi(t)) \cdot i_G(t) \, dt = \xi.
\]
Thus the image of \(O_{iX,i_A} \times u\) also contains \(k_{X \rtimes_s G}(X \rtimes \gamma G)\), and it now follows that \(O_{iX,i_A}\) is surjective. 

\(\square\)
Remark 4.4. Ideally we would like the map in Proposition 4.3 to be injective, which would give $O_X \rtimes_G \beta G \cong O_{X \times \mathbb{Z}^2}$ for an arbitrary locally compact group $G$. In a forthcoming paper [KQRb] we will use our techniques to prove this when the group $G$ is amenable, thereby recovering the result previously obtained by Hao and Ng [HN08, Theorem 2.10] using the theory of correspondence coactions.

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