Harmonic functions with finite $p$-energy on lamplighter graphs are constant.

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Abstract

The aim of this note is to show that lamplighter graphs where the space graph is infinite and at most two-ended and the lamp graph is at most two-ended do not admit harmonic functions with gradients in $\ell^p$ (i.e. finite $p$-energy) for any $p \in [1, \infty]$ except constants (and, equivalently, that their reduced $\ell^p$ cohomology is trivial in degree one). This answers a question of Georgakopoulos [3] on functions with finite energy in lamplighter graphs. The proof relies on a theorem of Thomassen on spanning lines in squares of graphs.

1 Introduction

Given two graphs $H = (X, E)$ (henceforth the "space" graph) and $L = (Y, F)$ (henceforth the "lamp" graph), the lamplighter graph $G := L \wr H$ is the graph constructed as follows. Fix some root vertex $o \in Y$ and let $(\oplus_X Y)$ be the set of "finitely supported" functions from $X \to Y$ (i.e. only finitely many elements of $X$ are not sent to $o \in Y$). Its vertices are elements of $X \times (\oplus_X Y)$. Two vertices $(x, f)$ and $(x', f')$ are adjacent if

- either $x \sim x'$ in $H$ and $f = f'$,
- or $x = x'$, $f(y) = f'(y)$ for all $y \neq x$ and $f(x) \sim f'(x)$ in $L$.

It is easy to see that $L \wr H$ is connected exactly when both $H$ and $L$ are. In fact, in this note, all graphs will be assumed to be connected (this is not important) and the graphs are locally finite.

The ends of a graph are the infinite components of a group which cannot be separated by a finite set. More precisely, an end $\xi$ is a function from finite sets to infinite connected components of their complement so that $\xi(F) \cap \xi(F') \neq \emptyset$ (for any $F$ and $F'$).

Given a graph $G$, a real-valued function $f$ on its vertices $V$ is said to be harmonic if it satisfies the mean value property

$$\forall v \in V, \ f(v) = \frac{1}{\deg(v)} \sum_{w \sim v} f(w).$$

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where $v$ is the degree (or valency) of $v$. The gradient of $f$ is the function on the edges $(v, w)$ defined by $\nabla f(v, w) = f(w) - f(v)$. The square of the $\ell^2$-norm of the gradient is often referred to as the energy of the function.

The main result here is the following corollary:

**Corollary 1.** Assume $H$ is infinite and has at most two ends, $L$ has at least one edge and at most two ends and that both $L$ and $H$ are locally finite, then there are no non-constant harmonic functions with gradient in $\ell^p$ in $L \wr H$ for any $p \in ]1, \infty[$.

This result is in contrast with the fact that lamplighter graphs have bounded harmonic functions as soon as $H$ is not recurrent. Indeed, a bounded function has necessarily its gradient in $\ell^\infty$.

In fact, this result uses (and, when the graphs have bounded valency, is equivalent to) the vanishing of the reduced $\ell^p$ cohomology in degree one, see [4] for definitions. The proof of Corollary 1 is essentially a particular case of [4, Question 1.6]. This corollary answers partially questions which may be found (in different guises) in Georgakopoulos [3, Problem 3.1] and Gromov [6, §8.1.2, (A2), p.226]. Regarding [3], this answer actually more than asked: the question there concerns harmonic functions with finite energy, i.e. with gradient in $\ell^2$.

As for [6], the question there concerns other types of graphs; for lamplighter graphs of Cayley graphs the answer to this question is essentially complete. Indeed, a wreath product (i.e. lamplighter group) is amenable exactly when the lamp and space groups are amenable. Since amenable groups have at most 2 ends, corollary 1 shows the reduced $\ell^p$-cohomology of any amenable wreath product is trivial. Note that Martin & Valette [8, Theorem.(iv)] show this is still true when $L$ is not amenable.

Corollary 1 extends probably to graphs with finitely many ends. To do this one would need to answer the following question. Assume $G$ is the set of graphs obtained by taking a cycle and attaching to it finitely many (half-infinite) rays. Is the lamplighter graph $L \wr H$ with $L, H \in G$ Liouville? This seems to follow from classical consideration of Furstenberg (coupling), since both $H$ and $L$ are recurrent.

## 2 Proof

Let $D^p(G)$ be the space of functions on the vertices of the graph $G$ with gradient in $\ell^p$ and $H D^p(G)$ be the subset of $D^p(G)$ consisting of functions which are furthermore harmonic. The notation $H D^p(G) \simeq \mathbb{R}$ means that the only functions in $H D^p(G)$ are constants.

For $F \subset X$ a subset of the vertices, let $\partial F$ be the edges between $F$ and $F^c$. Let $d \in \mathbb{R}_{\geq 1}$. Then, a graph $G = (X, E)$ has

\[
\text{IS}_d \text{ if there is a } \kappa > 0 \text{ such that for all finite } F \subset X, \ |F|^{(d-1)/d} \leq \kappa |\partial F|.
\]

Quasi-homogeneous graphs with a certain (uniformly bounded below) volume growth in $n^d$ will satisfy these isoperimetric profiles, see Woess’ book [12, (4.18) Theorem]. For example, the Cayley graph of a group $G$ satisfies $\text{IS}_d$ for all $d$ if and only if $G$ is not virtually nilpotent.
Let $G_0 = L \wr H$ the lamplighter graph where $L$ is either finite or a Cayley graph of $\mathbb{Z}$ and $H$ is a Cayley graph of $\mathbb{Z}$. For our current purpose it will suffice to note that $G_0$ has IS$_d$ for any $d \geq 1$, see Erschler [2]. A second important ingredient is that, using Kaimanovich [7, Theorem 3.3], $G_0$ is Liouville, i.e. a bounded harmonic function is constant.

The proof will be split in a few steps for convenience.

**Step 1** - assume that $H$ and $L$ have bounded valency. Two results from [4] can then be invoked. Using [4, Theorem 1.2], if the graph under consideration has IS$_d$ for any $d$, then $\mathcal{H}D^p \simeq \mathbb{R}$ for any $p < \infty$ is equivalent to vanishing of the reduced $\ell^p$-cohomology in degree one (for short, $\ell^pH^1 = \{0\}$) for any $q < \infty$. By [4, Corollary 4.2.1], if a graph $G$ has a spanning subgraph which is Liouville and has IS$_d$ for all $d$, then $\ell^qH^1(G) = \{0\}$ for any $q < \infty$.

Note that if a spanning subgraph of $G$ has IS$_d$, it implies that $G$ has IS$_d$. Summing up, if a graph $G$ admits $G_0$ as a subgraph then $\ell^qH^1 = \{0\}$ for any $q < \infty$ (and, equivalently $\mathcal{H}D^p(G) \simeq \mathbb{R}$ for any $p < \infty$).

It is also possible to work only up to quasi-isometry: if two graphs of bounded valency $\Gamma$ and $\Gamma'$ are quasi-isometric, then they have the same $\ell^p$-cohomology (in all degrees, reduced or not), see Ëlek [1, §3] or Pansu [9].

Recall that the $k$-fuzz of a graph $G$, is the graph $G^{[k]}$ with the same vertices as $G$ but now two vertices are neighbours in $G^{[k]}$ if their distance in $G$ is $\leq k$. $G^{[2]}$ is often called the square of $G$.

Lastly, using either Thomassen [11] or Seward [10, Theorem 1.6], the graphs $L$ and $H$ in Corollary 1 are bi-Lipschitz equivalent to graphs containing a spanning line (or cycle) whenever it is finite. In fact, this bi-Lipschitz equivalence is given by taking the $k$-fuzz of these graphs. An interested reader could probably show that $k = 4$ is sufficient. This means that $L \wr H$ is bi-Lipschitz equivalent (and so quasi-isometric) to a graph containing $G_0$. This finishes the proof of Corollary 1 when $H$ and $L$ both have bounded valency.

**Step 2** - Assume from now on that both $H$ and $L$ have connected spanning subgraphs of bounded valency, say $H'$ and $L'$ respectively. If there is a non-constant $f \in \mathcal{H}D^p(G)$ (where $G = L \wr H$). Then $f$ is not “constant at infinity”: if $B_n$ denotes a ball of radius $n$ around some fixed vertex $o$, then $f(B_n^o)$ does not converge to a single value. Indeed, the maximum principle would then imply $f$ is constant.

But $f$ is also a function on the vertices of $G' = L' \wr H'$ and it is also in $\mathcal{D}^p(G')$ (because deleting edges only reduces the $\ell^p$ norm of the gradient). On the other hand $G'$ contains $G_0$ up to quasi-isometry and hence $\ell^pH^1(G') = \{0\}$. However, by [4, Corollary 4.2.1], $\ell^pH^1(G') = \{0\}$ implies that all functions in $\mathcal{D}^p(G')$ are constant at infinity (see [4, Corollary 3.2.4]).

**Step 3** - Now assume $H$ and $L$ are only locally finite. The result of Thomassen [11] still implies that (for some $k$) the $k$-fuzz of $H$ and $L$ have a spanning line. However, given a function $f \in \mathcal{D}^p(G)$, it may no longer be in $\mathcal{D}^p(G^{[k]})$ if $k > 1$ and $G$ does not have bounded valency. To circumvent this problem, construct a graph $H$ by adding (when necessary) the edges of the spanning line in $H^{[k]}$. Construct $L'$ similarly.

Given $f \in \mathcal{D}^p(G)$ where $G = L \wr H$, one has that $f \in \mathcal{D}^p(G')$ with $G' = L' \wr H'$. Indeed, in passing from $G$ to $G'$ at most four edges are added to each vertex and the gradient along these edge is expressed as a sum of $k$ values of the gradient of $f$ on $G$. The triangle
inequality ensures that the $\ell^p$-norm of $\nabla f$ (on $G'$) is at most $(4k + 1)$ times the $\ell^p$-norm of the gradient of $f$ on $G$.

This last reduction yields the conclusion. Indeed, if there is an $f \in \mathcal{H}\mathcal{D}^p(G)$ which is not constant, then there is an $f \in \mathcal{D}^p(G')$ which takes different values at infinity. This is however excluded by step 2.

References

[1] G. Élek, Coarse cohomology and $\ell_p$-cohomology, K-Theory, 13:1–22, 1998.

[2] A. Erschler, On isoperimetric profiles of finitely generated groups, Geom. Dedic. 100(1):157–171, 2003.

[3] A. Georgakopoulos, Lamplighter graphs do not admit harmonic functions of finite energy, Proc. Amer. Math. Soc. 138(9):3057–3061, 2010.

[4] A. Gournay, Boundary values of random walks and $\ell^p$-cohomology in degree one, arXiv:1303.4091

[5] A. Gournay, Absence of harmonic functions with $\ell^p$ gradient in some semi-direct products, arXiv:1402.3126

[6] M. Gromov, Asymptotic invariants of groups, in Geometric group theory (Vol. 2), London Mathematical Society Lecture Note Series, Vol. 182, Cambridge University Press, 1993, viii+295.

[7] V. A. Kaimanovich, Poisson boundaries of random walks on discrete solvable groups, Probability measures on groups, X, (Oberwolfach, 1990):205–238, Plenum, New York, 1991.

[8] F. Martin and A. Valette, On the first $L^p$ cohomology of discrete groups. Groups Geom. Dyn., 1:81–100, 2007.

[9] P. Pansu, Cohomologie $\ell^p$: invariance sous quasi-isométrie. Unpublished, but available on P. Pansu’s webpage http://www.math.u-psud.fr/~pansu/liste-prepub.html, 1995 (updated in 2004).

[10] B. Seward, Burnside’s Problem, spanning trees, and tilings. arXiv:1104.1231 (v2), 2011.

[11] C. Thomassen, Hamiltonian paths in squares of infinite locally finite blocks, Annals of Discrete Mathematics 3:269–277, 1978.

[12] W. Woess, Random Walks on Infinite Graphs and Groups, Cambridge tracts in mathematics, 138. Cambridge University Press, 2000.