LOCALIZATION FOR SCHRÖDINGER OPERATORS
WITH RANDOM VECTOR POTENTIALS

F. GHIRIBI, P. D. HISLOP, AND F. KLOPP

ABSTRACT. We prove Anderson localization at the internal band-edges for periodic magnetic Schrödinger operators perturbed by random vector potentials of Anderson-type. This is achieved by combining new results on the Lifshitz tails behavior of the integrated density of states for random magnetic Schrödinger operators, thereby providing the initial length-scale estimate, and a Wegner estimate, for such models.

1. INTRODUCTION

Random magnetic Schrödinger operators have attracted much recent interest. These operators are technically challenging because the randomness enters the coefficients of first-order differential operators unlike the zero-order case of Schrödinger operators with random electrostatic potentials. This means that the variation of the eigenvalues of the finite-volume operators is not monotone with respect to the random variables. In this paper, we treat some new models and prove localization near the internal band-edges due to a random perturbation of the vector potential only. We consider random vector potentials $A_\omega(x)$ of Anderson-type having the form

$$A_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j),$$

where $\{\omega_j \mid j \in \mathbb{Z}^d\}$ is a family of independent, identically distributed (iid) random variables, and the single-site vector potential $u$ is a real, vector-valued function of compact support. Such a random vector potential generates a random magnetic field $B_\omega = dA_\omega$. Precise hypotheses are formulated below. We consider families of magnetic Schrödinger operators on $L^2(\mathbb{R}^d)$ obtained by random perturbations of periodic magnetic Schrödinger operators by a random vector potential $\{\omega\}$. Such random Schrödinger operators have the form

$$H_\omega(\lambda) \equiv (i\nabla + A_0 + \lambda A_\omega)^2 + V_0,$$

where $V_0$ is a real-valued, bounded, periodic electrostatic potential and $A_0$ is a real, bounded, periodic vector potential. The unperturbed operator is $H_0 \equiv (i\nabla + A_0)^2 + V_0$. For the disorder $\lambda > 0$ sufficiently small, the deterministic spectrum $\Sigma(\lambda)$ has

---

PDH partially supported by NSF grant DMS 0503784. FK partially supported by Institut Universitaire de France.
a band structure. We prove that under certain hypotheses, and for the disorder \( \lambda \) sufficiently small, there is a neighborhood of any internal band edge in the deterministic spectrum \( \Sigma(\lambda) \) that is purely pure point spectrum with probability one, with exponentially decaying eigenfunctions.

There are few results on localization for random magnetic Schrödinger operators. Nakamura \cite{Nakamura1, Nakamura2} considered the Lifshitz tails behavior of the integrated density of states (IDS) \( N(E) \) at the bottom of the spectrum for a general family of random vector potentials associated with random magnetic fields on the lattice \( \mathbb{Z}^2 \) and on the continuum \( \mathbb{R}^d \), respectively. For models \( H_\omega = (i\nabla + A_\omega)^2 \) on \( L^2(\mathbb{R}^d) \), for \( d \geq 2 \), Nakamura \cite{Nakamura2} proves an upper bound on \( N(E) \) as \( E \to 0^+ \). Since zero is the bottom of the deterministic spectrum provided zero is in the support of the distribution of the random variables, this upper bound on the decay of the IDS provides an initial length scale estimate for the multiscale analysis for an interval of energies near zero. In order to prove localization in this interval, it is also necessary to have a Wegner estimate near zero energy. However, since zero is not a fluctuation boundary, no such Wegner estimate is currently known.

The situation for lattice models is better. Nakamura \cite{Nakamura1} proved a Lifshitz tail behavior at zero energy for a discrete model in two-dimensions. The magnetic Hamiltonian is defined using a vector potential \( A((x, y)) \), defined on the edges \( (x, y) \), if \( |x - y| = 1 \), and has the form

\[
(H\psi)(x) = \sum_{y : |x-y| = 1} (\psi(x) - e^{iA((x, y))}\psi(y)),
\]

for \( u \in \ell^2(\mathbb{Z}^2) \). The vector potential satisfies \( A((y, x)) = -A((x, y)) \). The spectrum of this operator is \([0, 8]\). The magnetic field \( B \) is defined for each unit square \( F \) by \( B(F) = \sum_{e \in \partial F} A(e) \). Nakamura assumes that the collection \( B(F) \) is a family of independent, identically distributed random variables. He proves that the IDS exhibits Lifshitz tails as \( E \to 0 \). This calculation was extended by Klopp, Nakamura, Nakano, and Nomura \cite{Klopp_Nakamura_Nakano_Nomura} who considered a specific model for which \( E_0 = \inf \Sigma \) can be computed. They prove that the IDS has Lifshitz tails as \( E \to E_0 \). These authors also prove a Wegner estimate for this model. Using these two results, Klopp, Nakamura, Nakano, and Nomura \cite{Klopp_Nakamura_Nakano_Nomura} proved exponential localization near the bottom of the spectrum for this family of lattice models in two-dimensions. In a preprint, Nakano \cite{Nakano} proved localization for lattice models of the form

\[
(H\psi)(x) = \sum_{y : |x-y| = 1} e^{iA_\omega((x, y))}\psi(y),
\]

where \( \{A_\omega\} \in [-\pi, \pi) \) is a family of independent and uniformly distributed random variables. The deterministic spectrum is \( \Sigma = [-2d, 2d] \). Nakano proves that if the dimension \( d \geq 11 \), then there is a \( \delta > 0 \) so that the model exhibits Anderson localization on \([-2d, -2d + \delta] \cup [2d - \delta, 2d] \) with exponentially decaying eigenfunctions.

Ueki \cite{Ueki1, Ueki2} also consider random magnetic Schrödinger operators of the form \( (2) \). In \cite{Ueki1}, he studied Lifshitz tails at zero energy for models on \( L^2(\mathbb{R}^2) \) for which
the vector potential is constructed from a Gaussian random field. Ueki studied localization for a similar random vector potential model on $L^2(\mathbb{R}^d)$ in [26], but there is an additional a random electrostatic potential of Anderson-type. There, localization at band-edges is due to the random electrostatic potential, rather than the random vector potential. In this paper, we prove that a random vector potential alone can create pure point spectrum almost surely in neighborhoods of the band edges.

In order to state our main theorem, we list the necessary hypotheses on our model.

(H1) The bounded, real-valued function $V_0$ and the bounded, vector-valued function $A_0$ are $\mathbb{Z}^d$-periodic. The self-adjoint operator $H_0 = (i\nabla + A_0)^2 + V_0$ is $\mathbb{Z}^d$-periodic and essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. The semibounded operator $H_0$ has an open internal spectral gap. That is, there exist constants $-\infty < M_0 < C_0 \leq E_- < E_+ < C_1 \leq \infty$ so that $\sigma(H_0) \subset [M_0, \infty)$, and

$$\sigma(H_0) \cap (C_0, C_1) = (C_0, E_-] \cup [E_+, C_1).$$

(H2) Each component of the single-site vector potential $u_k$ is continuously differentiable and compactly supported, i.e. $u_k \in C_0^\infty(\mathbb{R}^d)$. For each $k = 1, \ldots, d$, there exists a nonempty open set $B_k$ containing so that the single-site potential $u_k \neq 0$ on $B_k$.

(H3) The probability distribution of $\omega_0$ is absolutely continuous with respect to Lebesgue measure. The density $h_0$ has compact support contained, say, in $[-1, 1]$ and the infimum of its support is negative and the supremum is positive. The density $h_0$ is assumed to be locally absolutely continuous.

The periodicity of $H_0$ and the construction of the random vector potential $V$ satisfying (H2)–(H3) imply that $H_\omega(\lambda)$ is a $\mathbb{Z}^d$-ergodic of the family of operators. As a consequence, there is closed subset $\Sigma(\lambda) \subset \mathbb{R}$, the deterministic spectrum of $H_\omega(\lambda)$, such that $\sigma(H_\omega(\lambda)) = \Sigma(\lambda)$ with probability one. Furthermore, there are closed subsets $\Sigma_X(\lambda) \subset \Sigma(\lambda)$, for $X = pp, ac, sc$ that are the pure point, absolutely continuous, and singular continuous components of the spectrum with probability one. Finally, it is known that the deterministic spectrum $\Sigma(\lambda)$ has an open spectral gap $G(\lambda) \equiv (E_-(\lambda), E_+ (\lambda)) \subset G = (E_-, E_+)$ for $\lambda$ sufficiently small (cf. [3]). We note that there are examples [14, 20] of magnetic periodic Schrödinger operators ($V_0 = 0$) with open spectral gaps. We need $V_0 \neq 0$ in order to construct examples for which condition (Gh) (see section 2) is satisfied.

**Theorem 1.1.** Suppose that $H_0$ is a periodic magnetic Schrödinger operator satisfying the condition (H1), and that $H_\omega(\lambda)$ is defined as in [2], with Anderson-type random vector potential given in [1] satisfying (H2)–(H3). Furthermore, we suppose that hypotheses (H4)–(H5) and condition (Gh) at both gap edges (see section 2) are satisfied. There is a $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0]$, there is an $\eta(\lambda) > 0$, so that $E_+(\lambda) + \eta(\lambda) < E_+$ and $E_- < E_-(\lambda) - \eta(\lambda)$, and the deterministic spectrum in $I(\lambda) \equiv [E_-(\lambda) - \eta(\lambda), E_-(\lambda)] \cup [E_+(\lambda), E_+(\lambda) + \eta(\lambda)]$ is purely
pure point with exponentially decaying eigenfunctions. That is, we have that
\( \Sigma(\lambda) \cap I(\lambda) = \Sigma_{pp}(\lambda) \cap I(\lambda), \)
and there is no absolutely continuous or singular continuous spectrum in \( I(\lambda) \):
\( \Sigma_X(\lambda) \cap I(\lambda) = \emptyset, \quad X = ac, \ X = sc. \)

It is clear that we can replace \( \mathbb{Z}^d \) by a nondegenerate lattice \( \Gamma \subset \mathbb{Z}^d \), but we will only consider \( \mathbb{Z}^d \) here.

To our knowledge, this is the first result on localization due to only a random vector potential for continuum models. We will prove Theorem 1.1 by combining the recent results of F. Ghribi [9, 10] on Lifshitz tails for the IDS of \( H_\omega(\lambda) \) at internal band edges and the result of [12] on the Wegner estimate.

There are now many results on localization for random Schrödinger operators on \( L^2(\mathbb{R}^d) \). Papers using multiscale analysis include [2, 4, 7, 14, 13], and recently, the fractional moment method was extended to these models [1].

2. Internal Lifshitz Tails for the IDS

We now discuss the main points of Ghribi [9, 10] on the Lifshitz tails behavior of the IDS \( N(E) \) at the inner band-edges of the deterministic spectrum \( \Sigma(\lambda) \) of \( H_\omega(\lambda) \). We recall from section 1 that we have a \( \mathbb{Z}^d \)-periodic operator \( H_0 = (i\nabla + A_0)^2 + V_0 \), on \( L^2(\mathbb{R}^d) \), where \( V_0 \) is a real, bounded, \( \mathbb{Z}^d \)-periodic, electrostatic potential, and \( A_0 \) is a real, bounded, vector-valued \( \mathbb{Z}^d \)-periodic potential. We denote by \( C_0 \subset \mathbb{R}^d \) the unit cell and by \( C^*_0 \) the dual cell. Since \( H_0 \) is periodic, it admits a Floquet decomposition. We denote the Floquet eigenvalues by \( E_n(\theta) \), for \( \theta \in C^*_0 \), and the corresponding normalized eigenfunctions are denoted by \( \phi_{0,n}(\theta, x) \). We make some additional hypotheses on \( H_0 \).

(H4) The edge of the spectrum \( E_+ \) is simple meaning that it is attained by a single Floquet eigenvalue \( E_{n_0}(\theta), \) for \( \theta \in C^*_0 \).

(H5) The IDS \( N_0(E) \) of \( H_0 \) at \( E_+ \) is nondegenerate which means that:
\[
\lim_{\delta \to 0^+} \frac{\log(N_0(E_+ + \delta) - N_0(E_+))}{\delta} = \frac{d}{2}.
\]
Concerning (H4), we let \( E_{n_0}(\theta) \) be the unique Floquet eigenvalue that attains the band edge \( E_+ \). Then, as proved in [16], there is a finite set of points \( \theta_k \in C^*_0 \) so that \( E_{n_0}(\theta_k) = E_+, \) for \( k = 1, \ldots, m \).

Let us suppose that \( H_0 \) and \( H_\omega(\lambda) \) satisfy hypotheses (H1)–(H5). There is an important hypothesis on \( A_0 \) and \( u \), necessary for the proof of Lifshitz tails, that we call condition (Gh).

(Gh) The matrix \( M \equiv (M_{kk'})_{1 \leq k, k' \leq m} \), with matrix elements given by
\[
M_{kk'} = \int_{C_0} ((u \cdot i\nabla + i\nabla \cdot u + 2u \cdot A_0)\phi_{0,n_0}(\theta_k, x)\overline{\phi_{0,n_0}(\theta_{k'}, x)} \, dx,
\]
is positive or negative definite.

The following theorem follows from the main result of [9].
Theorem 2.1. Assume that $H_0$ and $H_{\omega}(\lambda)$ satisfy hypotheses (H1)–(H5), and that condition (Gh) is satisfied. Then, there exists $\lambda_0 > 0$ and $\nu > 0$ so that for all $\lambda \in [0, \lambda_0]$, one has $E_+ - \nu \lambda \leq E_+ (\lambda) \leq E_+$, and the IDS $N_\lambda(E)$ satisfies

$$\lim_{E \to E_+ (\lambda)^+} \frac{\log |\log (N_\lambda(E) - N_\lambda(E_+ (\lambda)))|}{\log (E - E_+ (\lambda))} = -\frac{d}{2}.$$  

A similar statement holds at the lower band edge $E_- (\lambda)$.

We prove in section 6 that given a nontrivial $\mathbb{Z}^d$-periodic potential $V_0$, we can construct a $\mathbb{Z}^d$-periodic vector potential $A_0$, so that, with $\epsilon > 0$ sufficiently small, there are vector-valued functions $u$, and hence random Anderson-type vector potentials (1), so that condition (Gh) is satisfied with $A_0$ replaced by $\epsilon A_0$. We comment further on condition (Gh) in section 6.

Concerning hypothesis (H4), Klopp and Ralston [18] proved that generically the band edge of a periodic Schrödinger operator is obtained by a single Floquet eigenvalue. One expects that the same result should hold for a dense, open set of pairs $(A_0, V_0)$ in $L^\infty(C_0)$. It is known that hypothesis (H5) is true in the one-dimensional case, and not necessary for two-dimensional periodic Schrödinger operators [19]. That is, the band edge is always nondegenerate. Furthermore, hypothesis (H5) is known to be satisfied for many families of periodic Schrödinger operators, see the discussion in [19].

3. Initial Estimate on the Resolvent of the Localized Hamiltonian

We follow the method of using the Lifshitz tails behavior of the IDS in order to obtain an initial decay estimate on the localized resolvent of the local Hamiltonian obtained from the Hamiltonian by restricting to finite-volumes with periodic boundary conditions. This method has been used, for example, by Klopp [14], in order to prove localization at the bottom of the deterministic spectrum, and in Klopp-Wolff [19] in order to localization near internal band edges. The methods of [2] or [13] do not work for the models of random vector potentials considered here because the variation of the eigenvalues with respect to the random variables is not monotonic. We localize the Hamiltonian $H_0$ to cube $\Lambda \subset \mathbb{R}^d$, with integer length sides, and obtain a self-adjoint operator $H_0^{\Lambda}$ by imposing periodic boundary conditions. It follows from Floquet theory that the spectrum of $H_0^{\Lambda}$ still has a spectral gap containing $G$. We localize the random perturbation to such a region by writing

$$A_\Lambda(x) = \sum_{j \in \overset{-}{\Lambda}} \omega_j u(x - j).$$

We assume that $A_\Lambda(x)$ is supported in $\Lambda$ without loss of generality, which amounts to the hypothesis that supp $u \subset C_0$, the unit cube. We write $H_\Lambda(\lambda) = [(i\nabla + A_0 + \lambda A_\Lambda)^2 + V_0]|\Lambda$, with periodic boundary conditions. The IDS for the local
Hamiltonian $H_\Lambda(\lambda)$ is defined as

$$N_\Lambda(E) = \frac{1}{|\Lambda|} \# \{ \text{eigenvalues of } H_\Lambda(\lambda) \leq E \}. \tag{11}$$

For $\lambda$ sufficiently small, this operator $H_\Lambda(\lambda)$ also has an open spectral gap that contains $(E_-(\lambda), E_+(\lambda))$. We write $N_\Lambda(E)$ for the IDS for the operator $H_\Lambda(\lambda)$.

Theorem \[24\] states that the density of states in a small interval around the band-edge is small. For example, it follows from Theorem \[2.1\] that for any $n \geq 1$, we have

$$\lim_{E \to E_+(\lambda)^+} (E - E_+(\lambda))^{-n} [N(E) - N(E_+(\lambda))] = 0. \tag{12}$$

The problem is to obtain information about the finite volume IDS from this infinite volume result. For this, we use a result of Klopp and Wolff \[19\].

**Theorem 3.1.** For any $E_0 \in \mathbb{R}$, if

$$N(E_0 + \epsilon) - N(E_0 - \epsilon) = O(\epsilon^\infty), \tag{13}$$

as $\epsilon \to 0^+$, then for any $k \in \mathbb{Z}^+$, $k \geq 2$, any $\nu > 0$, and for any $L \in \mathbb{Z}^+$ sufficiently large,

$$\mathbb{E} \left( N_{\Lambda_k}(E_0 + L^{-1}) - N_{\Lambda_k}(E_0 - L^{-1}) \right) \leq L^{-\nu}. \tag{14}$$

We apply this vanishing result as follows. We work with the upper band edge $E_+(\lambda)$, a similar argument holds for the lower band edge. Let $\chi_B(\cdot)$ denote the characteristic function for the subset $B \subset \mathbb{R}$. We fix $\lambda > 0$, and consider an interval $I_\lambda(L) = [E_+(\lambda), E_+(\lambda) + L^{-1/2}]$ near the upper band edge $E_+(\lambda)$, for integer $L^{1/2}$ sufficiently large, depending on $\lambda$, so that $E_+(\lambda) + L^{-1/2} < E_+$. We apply Theorem \[3.1\] taking $k = 2$, and $\nu > 0$ arbitrary. By the Chebychev inequality, we have

$$\mathbb{P} \{ \sigma(H_{\Lambda_k}) \cap I_\lambda(L) \neq \emptyset \} \leq \mathbb{P} \{ \text{Tr}_{\Lambda_k}(\chi_{I_\lambda(L)}(H_{\Lambda_k})) \geq 1 \} \leq \mathbb{E} \{ N_{\Lambda_k}(I_\lambda(L)) \} \leq \mathbb{E} \left( N_{\Lambda_k}(E_+(\lambda) + L^{-1/2}) - N_{\Lambda_k}(E_+(\lambda) - L^{-1/2}) \right) \leq L^{-\nu}. \tag{15}$$

So we are assured that the interval $[E_+(\lambda), E_+(\lambda) + (4L)^{-1/2}]$ is separated from the spectrum of $H_{\Lambda_k}$ by an amount $(4L)^{-1/2}$ with a probability greater than $1 - L^{-\nu}$ according to the right side of \[15\]. We use this as input into the Combes-Thomas estimate \[2\] \[8\] \[9\] on the exponential decay of the localized resolvent of $H_{\Lambda_k}(\lambda)$. For a cube $\Lambda \subset \mathbb{R}^d$, we let $g_\Lambda$ be a smooth characteristic function of $\Lambda$ such that $g_\Lambda = 1$ except near a fixed neighborhood of the boundary $\partial \Lambda$. We will often write $g_L$ when $\Lambda$ is a cube of side length $L \in \mathbb{Z}^+$. We denote by $W(g_L)$ the commutator $W(g_L) = [H_{\Lambda_k}, g_L]$ that is a first-order, relatively $H_0$-bounded operator localized near $\partial \Lambda$. 
Theorem 3.2. Fix $\lambda > 0$. There is an $L_0 >> 0$ and a $\gamma_0 > 0$, depending on $\lambda > 0$, with $\gamma_0 L_0 \sim O(L_0^{1/2}) >> 1$, so that for any energy $E \in [E_+(-\lambda), E_+(-\lambda) + (4L_0)^{-1/2}]$, the local Hamiltonian $H_{L_0}(-\lambda)$ satisfies the following bound

$$\|W(g_{L_0})R_{L_0}(E)^{\chi_{L_0}}\| \leq e^{-\gamma_0 L_0},$$

with a probability greater than $1 - L_0^{-\xi}$, for any $\xi > 2d$.

4. The Wegner Estimate for Random Magnetic Schrödinger Operators

A Wegner estimate for random magnetic Schrödinger operators of the form (2) with vector potentials of Anderson-type (1) was proven in [12] for energies in the spectral gaps of $H_0$. The proof of the Wegner estimate differs from the usual proofs because the variation in the eigenvalues with respect to the random variables is not monotonic as it is in the random potential case. We recall the main argument here. As in section 3, we localize our magnetic Schrödinger operators (2) to integer side length cubes $\Lambda^{\lambda}$ with periodic boundary conditions. Expanding the form of the operator $H_{\Lambda}^{\lambda}$, it will be convenient to write the local operator as

$$H_{\Lambda}^{\lambda} = H_0^\Lambda + \lambda H_1^\Lambda + \lambda^2 H_2^\Lambda,$$

where $H_0^\Lambda = [(i \nabla + A_0)^2 + V_0]_{\Lambda}$, with periodic boundary conditions, is the fixed background operator with an open spectral gap containing the open spectral gap $G$ of $H_0$. As the perturbation is $\lambda A_{\Lambda}$, we see that

$$H_1^\Lambda = [(i \nabla + A_0) \cdot A_{\Lambda} + A_{\Lambda} \cdot (i \nabla + A_0)]_{\Lambda}, \text{ and } H_2^\Lambda = A_{\Lambda} \cdot A_{\Lambda},$$

with periodic boundary conditions.

Theorem 4.1. Suppose that the deterministic background operator $H_0$ satisfies hypothesis (H1), and that the random operators $H_{\Lambda}^{\lambda}$, $j = 1, 2$, defined in (15), are constructed with single-site vector potential $u$ and iid random variables satisfying (H2)–(H3). Suppose $G = (E_-, E_+)$ is an open gap in the spectrum of $H_0$. Then, there exists a constant $\lambda_0 > 0$, and, for any $q > 1$, a finite constant $C_W > 0$, independent of $\lambda$, such that for all $|\lambda| < \lambda_0$, $E_0 \in G$ and $\eta > 0$ such that

$$\frac{\lambda^2}{\text{dist}(E_0, \sigma(H_0))} \leq \lambda_0, \quad [E_0 - 2\eta, E_0 + 2\eta] \subset G,$$

we have

$$\mathbb{P}\{ \text{dist}(\sigma(H_{\Lambda}^{\lambda})(E_0) \leq \eta} \} \leq \frac{C_W}{\text{dist}(E_0, \sigma(H_0))} \eta^{1/q} |\Lambda|.$$

Remark. As we will show in section 4, $\text{dist}(E_0, \sigma(H_0)) \sim O(\lambda)$, so that the ratio in theorem 4.1 is effectively $|\lambda| < \lambda_0$. 
Proof. 1. We recall the basic ideas from [12]. Let \( R_\Lambda(z) \equiv (H_\Lambda(z) - z)^{-1} \) be the resolvent of the local operator \( H_\Lambda(z) \) on the Hilbert space \( L^2(\Lambda) \). We begin with the observation

\[
\|P \{ \text{dist}(\sigma(H_\Lambda(z)), E_0) < \eta \} = \|R_\Lambda(E_0)\| > 1/\eta \}
\]

Because we are working at the band-edges of an internal gap, we use the Feshbach projection method to express the resolvent \( R_\Lambda(E_0) \) in terms of various positive operators by reducing to the spectral subspace of \( H_0^\Lambda \) above \( E_+ \) and below \( E_- \) (recall that the spectrum of \( H_0^\Lambda \) always maintains the spectral gap). Let \( P_\pm \) be the spectral projectors for \( H_0^\Lambda \) corresponding to the spectral subspaces \([E_+, \infty) \) and \((\infty, E_-) \), respectively. We consider the case of the upper band edge so that \( E_0 \in G \) and \( E_0 > (E_+ + E_-)/2 \). The argument for the lower band edge is similar.

We will suppress \( \Lambda \) for notational simplicity, and write \( H_0^\Lambda = P_+ H_0^\Lambda \), and denote by \( H_\pm(z) = H_0^\Lambda + \lambda \mathcal{P}_\pm(\lambda H_0^\Lambda + \lambda^2 H_1^\Lambda) P_\pm \). We will need the various projections of operators \( A \) between the subspaces \( P_\pm L^2(\mathbb{R}^d) \), and we denote them by \( A_\pm \equiv P_\pm A P_\pm \), and \( A_{++} \equiv P_+ A P_+ \), with \( A_{+-} = A_{-+} = A_{-} - A_{+} P_+ \). Let \( z \in \mathbb{C} \), with \( \Im z \neq 0 \). We can write the resolvent \( R_\Lambda(z) \) in terms of the resolvents of the projected operators \( H_\pm(z) \) as follows. In order to write a formula valid for either \( P_+ \) or for \( P_- \), we let \( P = P_\pm, Q = 1 - P_\pm \), and write \( R_P(z) = (PH_0^\Lambda + P(\lambda H_1^\Lambda + \lambda^2 H_2^\Lambda) P - zP)^{-1} \). We write the effective perturbation as \( \mathcal{V}(\lambda) \equiv \lambda H_1^\Lambda + \lambda^2 H_2^\Lambda \).

We then have

\[
(21) \quad R_\Lambda(z) = PR_P(z)P + \{Q - PR_P(z)P \mathcal{V}(\lambda)Q\}G(z)\{Q - Q \mathcal{V}(\lambda)PR_P(z)P\},
\]

where the operator \( G(z) \) is given by

\[
(22) \quad G(z) = \{QH_0 + Q \mathcal{V}(\lambda)Q - zQ - Q \mathcal{V}(\lambda)PR_P(z)P \mathcal{V}(\lambda)Q\}^{-1}.
\]

2. Our first goal is to reduce the estimate on the resolvent on the right in (20) to an estimate on the operator \( G(E_0) \). Since we are working close to \( E_+ \), we take \( P = P_+ \) and \( Q = P_- \). We let \( \delta_\pm(E_0) = \text{dist } (E_0, E_\pm) = \text{dist } (\sigma(H_\pm^\Lambda), E_0) \). To this end, we note that the resulting formula for the first term on the right in (21), \( PR_P(E_0)P \equiv R_-(E_0) \), is

\[
(23) \quad R_-(E_0) = R_0^-(E_0)^{1/2}\{1 + R_0^-(E_0)^{1/2}P_-(\lambda H_1^\Lambda + \lambda^2 H_2^\Lambda) P_-(E_0)^{1/2}\}^{-1}R_0^-(E_0)^{1/2},
\]

provided the inverse exists. The first factor on the right in (23) exists provided \(|\lambda| < \lambda_0(1)\), where \( \lambda_0(1) \) is fixed by the requirement that

\[
(24) \quad \lambda_0(1) \delta_-^{-1/2}\{\|H_1^\Lambda R_0^-(E_0)^{1/2}\| + \lambda_0(1)\|H_2^\Lambda R_0^-(E_0)^{1/2}\|\} < 1.
\]

We note that \( \|R_0^-(E_0)\| \) depends only on the distance from \( E_0 \) to \( E_- \) which is independent of \( \lambda \). Consequently, condition (24) requires that \( \lambda_0(1) < C_0 \delta_- \). Similarly, the operator \( \{P_+ - R_-(E_0) \mathcal{V}(\lambda) P_+\} \) is bounded for \(|\lambda| < \lambda_0(1)\). Consequently, it follows from (21) that the norm on the right in (20) is large if the norm of \( G(E_0) \)
is large. To analyze this operator, note that \( \mathcal{G}(E_0) \) can be written as
\[
\mathcal{G}(E_0) = R_0^+(E_0)^{1/2} \left\{ 1 + \tilde{\Gamma}_+(E_0) \right\}^{-1} R_0^+(E_0)^{1/2}. 
\]
The compact, self-adjoint operator \( \tilde{\Gamma}_+(E_0) \) has an expansion in \( \lambda \) given by
\[
\tilde{\Gamma}_+(E_0) = \sum_{j=1}^{\lambda} \lambda^j M_j(E_0), 
\]
where the coefficients are given by
\[
\begin{align*}
M_1(E_0) &= R_0^+(E_0)^{1/2} P_+ H_1^\Lambda P_+ R_0^+(E_0)^{1/2}, \\
M_2(E_0) &= R_0^+(E_0)^{1/2} \{ P_+ H_2^\Lambda P_+ - P_+ H_1^\Lambda P_+ (E_0) P_+ H_1^\Lambda P_+ \} R_0^+(E_0)^{1/2}, \\
M_3(E_0) &= -R_0^+(E_0)^{1/2} \{ P_+ H_2^\Lambda P_+ R_-(E_0) P_+ H_1^\Lambda P_+ \} R_0^+(E_0)^{1/2}, \\
M_4(E_0) &= -R_0^+(E_0)^{1/2} \{ P_+ H_2^\Lambda P_+ R_-(E_0) P_+ H_2^\Lambda P_+ \} R_0^+(E_0)^{1/2}. 
\end{align*}
\]
(27)

3. The probability estimate in (20) is now reduced to
\[
\mathbb{P}\{ \text{dist}(\sigma(H_\Lambda(\lambda)), E_0) < \eta \} = \mathbb{P}\{ \| R_\Lambda(E_0) \| > 1/\eta \} 
\leq \mathbb{P}\{ \| \mathcal{G}(E_0) \| > 1/(8\eta) \} 
\leq \mathbb{P}\{ \|(1 + \tilde{\Gamma}_+(E_0))^{-1} > \delta_+(E_0)/(8\eta) \}
= \mathbb{P}\{ \text{dist}(\sigma(\tilde{\Gamma}_+(E_0)), -1) < 8\eta/\delta_+(E_0) \}.
\]
(28)

To estimate the probability on the last line of (28), we analyze the spectrum of the operator \( \tilde{\Gamma}_+(E_0) \). Let \( \kappa = 8\eta/\delta_+(E_0) \) and let \( E_\Lambda(\cdot) \) be the spectral projectors for \( \tilde{\Gamma}_+(E_0) \). Chebychev’s inequality implies that
\[
\mathbb{P}\{ \text{dist}(\sigma(\tilde{\Gamma}_+(E_0)), -1) < \kappa \} = \mathbb{P}\{ Tr(E_\Lambda(I_\kappa)) \geq 1 \} 
\leq \mathbb{E}(Tr(E_\Lambda(I_\kappa))).
\]
(29)

Let \( \rho \) be a nonnegative, smooth, monotone decreasing function such that \( \rho(x) = 1 \), for \( x < -\kappa/2 \), and \( \rho(x) = 0 \), for \( x \geq \kappa/2 \). We can assume that \( \rho \) has compact support since \( \tilde{\Gamma}_+(E_0) \) is lower semibounded, independent of \( \Lambda \). As in [5, 12], we have
\[
\mathbb{E}_\Lambda(Tr(E_\Lambda(I_\kappa))) \leq \mathbb{E}_\Lambda\left\{ Tr\left[ \rho(\tilde{\Gamma}_+(E_0) + 1 - 3\kappa/2 - \rho(\tilde{\Gamma}_+(E_0) + 1 + 3\kappa/2) \right] \right\} 
\leq \mathbb{E}_\Lambda\left\{ Tr\left[ \int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt} \rho(\tilde{\Gamma}_+(E_0) + 1 - t) \ dt \right] \right\}. 
\]
(30)
4. In order to evaluate the $\rho'$ term, we compute the action of the vector field $A_\Lambda$, defined by

$$A_\Lambda = \sum_{j \in \Lambda} \omega_j \frac{\partial}{\partial \omega_j},$$

on the operator $\tilde{\Gamma}_+(E_0)$ defined in \((30)\)–\((31)\). This calculation is carried out in \[12\], and we obtain

$$A_\Lambda \tilde{\Gamma}_+(E_0) = \tilde{\Gamma}_+(E_0) + \sum_{j=2}^6 \lambda^j K_j(E_0).$$

The remainder terms $K_j(E_0)$ are given by

$$K_2(E_0) = M_2(E_0),$$
$$K_3(E_0) = 2M_3(E_0) + R_0^+(E_0)^{1/2} \{ P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+ \} R_0^+(E_0)^{1/2},$$
$$K_4(E_0) = 3M_4(E_0) + R_0^+(E_0)^{1/2} \{ 2P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+ + P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+ \} R_0^+(E_0)^{1/2},$$
$$K_5(E_0) = R_0^+(E_0)^{1/2} \{ 2P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+ + P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+ \} R_0^+(E_0)^{1/2},$$
$$K_6(E_0) = R_0^+(E_0)^{1/2} \{ 2P_+ H_{2,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda P_+ \} R_0^+(E_0)^{1/2}. $$

\((33)\)

We need to compute $\|A_\Lambda \tilde{\Gamma}_+(E_0)\rho'(\tilde{\Gamma}_+(E_0) - t + 1)\|$. This requires that we choose $|\lambda|$ sufficiently small so that

$$\sum_{j=2}^6 \lambda^j \|K_j(E_0)\| < (1 - 2\kappa)/2.$$

\((34)\)

Now, by \((33)\) and as $E_0 \in G$ such that $2E_0 > E_+ + E_-$, one has

$$\sum_{j=2}^6 \lambda^j \|K_j(E_0)\| \leq \frac{C\lambda^2}{\delta_+(E_0)}$$

\((35)\)

where $C$ depends on the gap size and on the relative $H_0$-bounds of $H_{j,\omega}^\Lambda$ (but not on $\lambda$ is a compact interval). Let $\lambda^{(2)} > 0$ be chosen so that $|\lambda| < \lambda^{(2)}$ guarantees that \((34)\) holds; it clearly suffices that $\lambda^2/\delta_+(E_0)$ be sufficiently small. We now choose $\lambda_0 = \min (\lambda^{(1)}, \lambda^{(2)})$.

5. With this choice of $\lambda_0$, we obtain the following crucial lower bound

$$Tr \{ \rho'(\tilde{\Gamma}_+(E_0) + 1 - t)A_\Lambda \tilde{\Gamma}_+(E_0) \} \geq -C_1 Tr \{ \rho'(\tilde{\Gamma}_+(E_0) + 1 - t) \},$$

\((36)\)
for a finite constant $C_1 > 0$. Given this positivity condition, we can finish the proof as in [5, 12]. □

We need the explicit formulas in order to check the dependence of various constants in $\lambda$ in section 5. We mention that this proof implies the Hölder continuity of the IDS outside of the spectrum of $H_0$.

**Corollary 4.2.** Let $H_{\omega}(\lambda) = H_0 + \lambda H_{1,\omega} + \lambda^2 H_{2,\omega}$ be a random family of operators satisfying hypotheses (H1)–(H3). Then, for any closed interval $I \subset \mathbb{R} \setminus \sigma(H_0)$, there exists a constant $0 < \lambda_0(I)$ such that for any $|\lambda| < \lambda_0(I)$, the integrated density of states for $H_{\omega}(\lambda)$ on $I$ is Hölder continuous of order $1/q$, for any $q > 1$.

5. **The Proof of Localization**

We now combine the main Theorems 1.1 and 4.1 in order to prove localization near the internal band edges. We make the observation that for $\lambda > 0$ small, the variation of the eigenvalues $E_j(\lambda)$ of $H_{\Lambda}(\lambda)$ are $O(\lambda)$. This follows by the Feynman-Hellman Theorem since

$$ \frac{dE_j(\lambda)}{d\lambda} = \langle \phi_j, \frac{dH_\Lambda(\lambda)}{d\lambda} \phi_j \rangle $$

(37)

The second term on the last line of (37) is bounded above by

$$ \langle \phi_j, [H_1^\Lambda + 2\lambda H_2^\Lambda] \phi_j \rangle. $$

(38)

As for the first term, we have

$$ |\langle \phi_j, H_1^\Lambda \phi_j \rangle| \leq \|H_1^\Lambda \phi_j\|, $$

(39)

so that using the bound $E_j < E_+$ and the fact that $H_1^\Lambda$ is relatively $H_0^\Lambda$-bounded, we get

$$ \|H_1^\Lambda \phi_j\| \leq \|H_1^\Lambda(H_0^\Lambda + 1)^{-1}||E_+ + \lambda^2\|u\|_\infty^2 + \lambda\|H_1^\Lambda \phi_j\|, $$

(40)

from which it follows that for $\lambda$ small, the norm $\|H_1^\Lambda \phi_j\|$ is uniformly bounded in $\lambda$. This result and (38)–(40) imply that $\frac{dE_j(\lambda)}{d\lambda}$ is bounded by a constant. It follows from the fact that the deterministic spectrum is the union of the spectra of the periodic approximations and from Floquet theory that the band-edges $E_+(\lambda)$ scale at most linearly in $\lambda$ as $\lambda \to 0$. That the band edge scales at least linearly comes from condition (Gh) (see Theorem 2.1).

5.1. **Wegner Estimate.** We fix $0 < \lambda < \lambda_0$ and consider the Wegner estimate for energies in the interval $I_\lambda = [E_+(\lambda), E_+]$ given by Theorem 4.1. We note that, if we pick $E_0 \in G$ such that, $\inf(E_+ - E_0, E_0 - E_-) \geq \nu \lambda$ and $\lambda$ sufficiently small (depending on $\nu$), we obtain a Wegner estimate, as in Theorem 4.1 holds true where the constant in front of $\eta^{1/q}|\Lambda|$ in the right side of (19) is replaced with $C_W \lambda^{-1}$. 
5.2. Initial Length Scale Estimate. We fix $0 < \lambda < \lambda_0$ of Theorem 1.2 so a Wegner estimate holds for intervals $[E_+(\lambda), E_0]$, for any $E_0 < E_+$. We choose $L_1$, depending on $\lambda$, so that $[E_+(\lambda), E_+(\lambda) + L_1^{-1/2}] \subset [E_+(\lambda), E_0]$. Note that $L_1 = O(\lambda^{-2})$. We now apply Theorem 3.2. For our fixed $\lambda$, there exist $(\tilde{L}_0, \gamma_0)$ so that the conclusions of the theorem hold. Consequently, taking $L_0 \equiv \sup(\tilde{L}_0, L_1)$, we have a Wegner estimate and an initial decay estimate (16) for all energies in the interval $[E_+(\lambda), E_+(\lambda) + (4L_0)^{-1/2}]$.

5.3. Multiscale Analysis. The multiscale analysis for the fixed energy interval $[E_+(\lambda), E_+(\lambda) + (4L_0)^{-1/2}]$ can now be performed as described, for example, in [14].

6. Examples of Magnetic Schrödinger Operators satisfying the Hypotheses

In this section, we show that the hypotheses on the unperturbed magnetic Schrödinger operator are satisfied by many examples. Let $\Gamma \subset \mathbb{R}^d$ be a nondegenerate lattice with a fundamental cell $C$. Consider $V_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ a smooth, $\Gamma$-periodic potential. We define the unperturbed periodic Schrödinger operator $H_0$ as

$$H_0 = -\Delta + V_0.$$

Suppose $(E_-, E_+)$ is an open gap for $H_0$ and assume that there is a unique Floquet eigenvalue $E(\theta)$ taking the value $E_+$ (the simplicity hypothesis (H4)). We further assume, for clarity of presentation, that this value is taken at a single Floquet parameter, say $\theta_0$, so that $E(\theta_0) = E_+$. Finally, we assume that the extremum of the Floquet eigenvalue at $\theta_0$ is quadratic nondegenerate as a function of $\theta$. This is equivalent to hypothesis (H5), see [16]. Such potentials can be constructed as sums of one dimensional operators (i.e. with separate variables) or using semiclassical constructions (see [15] and references therein). Let $(x, \theta) \mapsto \psi(x, \theta)$ be the smooth Floquet eigenfunction associated to the Floquet eigenvalue $\theta \mapsto E(\theta)$ assuming the value $E_+$ so that $E(\theta_0) = E_+$. The eigenfunction $\psi$ satisfies

$$\begin{align*}
(H_0 - E(\theta))\psi_0(x, \theta) &= 0 \\
\forall \gamma \in \Gamma, \forall x \in \mathbb{R}^d, \psi_0(x + \gamma, \theta) &= e^{i\gamma \theta} \psi_0(x, \theta).
\end{align*}$$

We now consider a perturbation of $H_0$ by a $\Gamma$-periodic vector potential $A_0 \in C^\infty(C)$ with a coupling constant $\varepsilon > 0$, and define $H_\varepsilon$ by

$$H_\varepsilon = (i\nabla + \varepsilon A_0)^2 + V_0.$$

For $\varepsilon$ small, we will construct the Floquet eigenvalues and eigenfunctions of $H_\varepsilon$ near the energy $E_+$. By the boundedness of $A_0$ and $\nabla A_0$, the mapping $\varepsilon \mapsto H_\varepsilon$ is norm resolvent continuous. Consequently, we know that, for $\varepsilon$ small, there exists some $E_+(\varepsilon)$ in the spectrum of $H_\varepsilon$ and such that $(E_+ + (E_+ - E_-)/2, E_+(\varepsilon))$ is in a gap of $H_\varepsilon$. Moreover, for $|\theta - \theta_0| > \delta$, the Floquet spectrum of $H_\varepsilon$ for Floquet parameter $\theta$ is either above $E_+ + \delta/2$ or below $E_+ - \delta/2$; and, for $|\theta - \theta_0| \leq \delta$, there exists a unique Floquet eigenvalue $E_\varepsilon(\theta)$ of $H_\varepsilon$ in $[E_- + \delta/2, E_+ + \delta/2]$. Let
c be a small circle around $E_+$. for $\varepsilon$ small, the spectral projector on this Floquet eigenvalue can be expressed as

$$
\pi_\varepsilon = \frac{1}{2\pi i} \oint_c (z - H_\varepsilon)^{-1} dz,
$$

and the Floquet eigenvalue and a normalized Floquet eigenvector as

$$
\psi_\varepsilon(\theta) = \frac{1}{\|\pi_\varepsilon \psi_0(\theta)\|} \pi_\varepsilon \psi_0(\theta) \quad \text{and} \quad E_\varepsilon(\theta) = \langle H_\varepsilon \psi_\varepsilon(\theta), \psi_\varepsilon(\theta) \rangle.
$$

These functions are jointly real analytic in $\varepsilon$ and $\theta$ for $\varepsilon$ near zero and $\theta$ near $\theta_0$.

We now want to consider a perturbation of $H_\varepsilon$ by a vector potential $A : \mathbb{R}^d \to \mathbb{R}^d$ supported in $C$, the fundamental cell of $\Gamma$. We recall that given $(V_0, A_0)$, Ghibi’s criterion (see [9]) requires that a certain quadratic form be sign-definite:

$$
\int_C \left( (A \cdot i\nabla + i\nabla \cdot A) + 2 \varepsilon A \cdot A_0 \right) \psi_\varepsilon(x) \overline{\psi_\varepsilon(x)} dx
$$

$$
= 2 \int_C A \cdot \left( \text{Re}(i\nabla \psi_\varepsilon(x) \overline{\psi_\varepsilon(x)}) + \varepsilon A_0(x) |\psi_\varepsilon(x)|^2 \right) dx \neq 0.
$$

So if $(V_0, A_0, \varepsilon)$ are such that

$$
\text{Re}(i\nabla \psi_0(x) \overline{\psi_0(x)}) + \varepsilon A_0(x) |\psi_\varepsilon(x)|^2 \neq 0,
$$

we can certainly choose vector-valued functions $A$, supported on $C$, so that (46) is satisfied. In the applications to random magnetic Schrödinger operators, the function $A$ is the single-site vector-valued function $u$ appearing in (41).

We now construct $A_0$ such that condition (47) is satisfied for $\varepsilon$ small and nonzero. Notice that if $\text{Re}(i\nabla \psi_0(x) \overline{\psi_0(x)}) \neq 0$, we just need to take $\varepsilon = 0$ and we can proceed with the unperturbed periodic operator $H_0$. So we can assume that

$$
\text{Re}(i\nabla \psi_0(x) \overline{\psi_0(x)}) \equiv 0
$$

We note that if $\psi_0$ is real, then this condition (48) holds. This is the case in the standard examples of periodic Schrödinger operators having the band edge behavior that we require. Let us analyze this condition. We prove

**Lemma 6.1.** Let $\psi_0$ satisfy $(H_0 - E_+)\psi_0 = 0$, condition (42), and be such that (48) holds. Then, $\psi_0$ is collinear to a real-valued function. that is, there exists a real function $f$ and a complex number $\eta$ with $|\eta| = 1$ so that $\psi_0 = \eta f$.

This in particular implies that when (48) holds, $\theta_0$ has to be in $\frac{1}{2}\Gamma^*/\Gamma^*$.

**Proof of Lemma 6.1** We write $\psi_0(x) = e^{i\alpha(x)}r(x)$, where $\alpha$ and $r$ are real valued. Clearly $\alpha$ is well defined as soon as $r$ does not vanish. At points where $r$ is nonzero, $r$ and $\alpha$ enjoy the same regularity as $\psi_0$. Moreover, (48) gives

$$
\nabla \alpha(x) = 0 \text{ if } r(x) \neq 0.
$$

Hence, $\alpha$ is constant on the connected components of the complement of the nodal set of $\psi_0$. Fix such a component, say $C_0$ and let $\alpha_0$ be the value taken by $\alpha$ on $C_0$. Clearly $e^{-i\alpha_0} \psi_0 = r$ is real positive on this component. Let $\varphi = \text{Im}(e^{-i\alpha_0} \psi_0)$. As
the differential equation in (12) has real coefficients, we see that \( \varphi \) is a solution to \((H_0 - E_+)\varphi = 0\). Moreover \( \varphi \) vanishes on some open set. The operator \( H_0 - E_+ \) satisfies a unique continuation principle. Hence, \( \varphi \) vanishes everywhere which proves that \( e^{-i\omega_0} \psi_0 \) is real.

We now return to constructing \( A_0 \) so that (17) holds assuming (18). Differentiating in \( \varepsilon \) the eigenvalue equation \( H_\varepsilon \psi_\varepsilon = E_+(\varepsilon)\psi_\varepsilon \) (note that differentiating the boundary condition \( \psi_\varepsilon(x + \gamma, \theta) = e^{i\tau \theta} \psi_\varepsilon(x, \theta) \) does not change it), one computes the Taylor expansion in \( \varepsilon \) of \( \psi_\varepsilon \) and gets

\[
\psi_\varepsilon = \psi_0 + \varepsilon \psi_0' + O(\varepsilon^2)
\]

where

\[
\psi_0' = -(H_0 - E_+)^{-1}(1 - \pi_0)(A_0 \cdot i\nabla + i\nabla \cdot A_0)\psi_0.
\]

The Taylor expansion (49) can be differentiated in \( x \) and \( \theta \) (for \( \theta \) close to \( \theta_0 \)). Substituting (49) into (46), and taking (48) into account, we see that we need to find \( A_0 \) such that

\[
\text{Re}(i\nabla \psi_0'(x)\psi_0(x)) + \text{Re}(i\nabla \psi_0(x)\psi_0'(x)) + A_0(x)|\psi_0(x)|^2 \neq 0.
\]

We will now construct \( A_0 \) not identically vanishing such that \( \psi_0' \) vanishes identically. Hence, (51) is satisfied. Consider \( \nabla \perp \) a constant coefficient vector field such that

\[
\nabla \cdot \nabla \perp = 0
\]

where, as above, \( \cdot \) denotes the scalar product in \( \mathbb{R}^d \). For example, in dimension \( d = 2 \), we choose \( \nabla \perp = (-\partial_2, \partial_1) \). In general, the \( i^{th} \) component of \( \nabla \perp \) has the form \( M_{ij}\partial_j \) for a constant real skew symmetric matrix \( M_{ij} = -M_{ji} \). Notice that for any differentiable function \( \psi : \mathbb{R}^d \rightarrow \mathbb{C} \), one also has

\[
\nabla(\psi) \cdot \nabla \perp(\psi) = -\nabla \perp(\psi) \cdot \nabla(\psi) = \frac{1}{2} \nabla \cdot \nabla \perp(\psi^2) - \psi \nabla \cdot \nabla \perp(\psi) = 0.
\]

As \( \psi_0 \) is not a constant, we can choose \( \nabla \perp \) so that \( \nabla \perp \psi_0 \) does not vanish identically. Then, we set \( A_0 = \nabla \perp(\psi_0^2) \). As \( V_0 \) is infinitely differentiable, the eigenvector \( \psi_0 \) is too, and this implies that \( A_0 \) is infinitely differentiable. Moreover, as \( \theta_0 \in \frac{1}{2} \Gamma^* / \Gamma^* \), one has

\[
A_0(x + \gamma) = \nabla \perp(\psi_0^2(x + \gamma)) = e^{2i\gamma\theta_0} \nabla \perp(\psi_0^2(x)) = \nabla \perp(\psi_0^2(x)) = A_0(x).
\]

Finally, using (52), we compute

\[
(A_0 \cdot i\nabla + i\nabla \cdot A_0)\psi_0 = 4i\nabla(\psi_0) \cdot \nabla \perp(\psi_0)\psi_0 + 2i\nabla \cdot \nabla \perp(\psi_0)\psi_0^2 = 0.
\]

Hence, \( \psi_0' = 0 \). This completes the construction of the example. To summarize: Given a smooth periodic electrostatic potential \( V_0 \), we can compute a smooth periodic vector potential \( A_0 \) and a single-site vector potential \( u = A \) satisfying (H2) and (10) so that the random Schrödinger operator \( H_\omega(\lambda, \epsilon) = (-i\nabla - \epsilon A_0 - \lambda A_0)^2 + V_0 \), for \( \epsilon > 0 \) and small, satisfies the condition of Ghribi (10). Consequently, the IDS exhibits Lifshitz tails behavior at the inner band-edges.
Let us notice here that, by Lemma 6.1, if we hope to construct an example of a periodic Schrödinger operator without a periodic magnetic field satisfying (46), it is necessary and sufficient to find a periodic Schrödinger operator that the energy \( E(\theta) \) reaches a band-edge for a Floquet parameter not belonging to \( \frac{1}{2} \Gamma^*/\Gamma^* \) and for which the band edge is simple. To our knowledge, no such example is known.

In dimension \( d = 1 \), it is known that this can not happen. In larger dimensions, the difficulty in producing such an example can be understood as a consequence of the fact that the points in \( \frac{1}{2} \Gamma^*/\Gamma^* \) are always critical points for simple Floquet eigenvalue. This clearly makes the construction of such an example by perturbation theory difficult.

References

[1] M. Aizenman, A. Elgart, S. Naboko, G. Stolz, J. Schenker: Moment Analysis for Localization in Random Schrödinger Operators, Inventiones Mathematicae 163 (2006), 343–413.
[2] J.-M. Barbaroux, J. M. Combes, P. D. Hislop, Localization near band edges for random Schrödinger operators, Helv. Phys. Acta 70 (1997), 16–43.
[3] Ph. Briet, H. D. Cornean, Locating the spectrum for magnetic Schrödinger and Dirac operators, Comm. Partial Differential Equations 27 (2002),1079–1101.
[4] J. M. Combes, P. D. Hislop: Localization for some continuous, random Hamiltonians in d-dimensions, J. Funct. Anal. 124 (1994) 149 - 180.
[5] J. M. Combes, P. D. Hislop, S. Nakamura, The \( L^p \)-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators, Commun. Math. Phys. 218 (2001), 113–130.
[6] J. M. Combes, L. Thomas, Asymptotic behavior of eigenfunctions for multiparticle Schrödinger operators, Commun. Math. Phys. 34 (1973), 251–276.
[7] F. Germinet, A. Klein, Bootstrap multiscale analysis and localization in random media, Commun. Math. Phys. 222 (2001), no. 2, 415–448.
[8] F. Germinet, A. Klein, Operator kernel estimates for functions of generalized Schrödinger operators, Proc. Amer. Math. Soc. 131 (2003), no. 3, 911–920.
[9] F. Ghribi, Asymptotique de Lifshitz pour les opérateurs de Schrödinger magnétiques aléatoires, thèse doctorale, Université Paris 13, 2005.
[10] F. Ghribi, Internal Lifshitz tails for random magnetic Schrödinger operators, J. Funct. Anal. 248, Issue 2 (2007), 387–427.
[11] R. Hempel, I. Herbst, Strong magnetic fields, Dirichlet boundaries, and spectral gaps, Commun. Math. Phys. 169 (1995), 237–259.
[12] P. D. Hislop, F. Klopp, The integrated density of states for some random operators with nonsign definite potentials, J. Funct. Anal. 195 (2002), 12–47.
[13] W. Kirsch, P. Stollmann, G. Stolz, Localization for random perturbations of periodic Schrödinger operators, Random Oper. Stochastic Equations 6 (1998), no. 3, 241–268.
[14] F. Klopp, Localization for some continuous, random Schrödinger operators, Commun. Math. Phys. 167(1995), 553–569.
[15] F. Klopp, Étude semi-classique d’une perturbation d’un opérateur de Schrödinger périodique. Ann. Inst. H. Poincaré Phys. Théor., 55(1):459–509, 1991.
[16] F. Klopp, Internal Lifshits tails for random perturbations of periodic Schrödinger operators, Duke Math. J. 98, 335–396 (1999); Erratum, Duke Math. J. 109 (2001), no. 2, 411–412.
[17] F. Klopp, S. Nakamura, F. Nakano, Y. Nomura, Anderson localization for 2D discrete Schrödinger operators with random magnetic fields, Ann. Henri Poincaré 4 (2003), 795–811.
[18] F. Klopp, J. Ralston, Endpoints of the spectrum of periodic operators are generically simple, Methods Appl. Anal. 7 (2000), no. 3, 459–463.
[19] F. Klopp, T. Wolff, Lifshitz tails for 2-dimensional random Schrödinger operators, *J. Anal. Math.* 88 (2002), 63–147.
[20] S. Nakamura, Band spectrum for Schrödinger operators with strong magnetic fields, in “Operator Theory: Advances and Applications” (M. Demuth and B.-W. Schulze, Eds.), Vol. 78, Birkhäuser Verlag, 1995.
[21] S. Nakamura, Lifshitz tail for 2D discrete Schrödinger operator with random magnetic field, *Ann. Henri Poincaré* 1 (2000), 823–835.
[22] S. Nakamura, Lifshitz tail for Schrödinger operator with random magnetic field, *Commun. Math. Phys.* 214 (2000), 565–572.
[23] F. Nakano, Anderson localization for a random vector potential model at high dimensions, preprint.
[24] B. Simon, *Trace Ideals and their Applications*, London Mathematical Society Lecture Note Series 35, Cambridge University Press, 1979; second edition, Mathematical Surveys and Monographs vol. 120, Amer. Math. Soc., 2005.
[25] N. Ueki, Simple examples of Lifshitz tails in Gaussian random magnetic fields, *Ann. Henri Poincaré* 1 (2000), no. 3, 473–498.
[26] N. Ueki, Wegner estimates and localization for Gaussian random potentials, *Publ. RIMS. Kyoto Univ.* 40 (2004), 29–90.