ON FUNCTIONAL EQUATION $T(x)T(y) = T^2(xy)$ FOR
SYMMETRIC LINEAR MAPS ON C* ALGEBRAS

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Abstract. Let $A$ be a C* algebra and $T : A \rightarrow A$ be a symmetric linear map
which satisfies the functional equation $T(x)T(y) = T^2(xy)$. We prove that
under each of the following conditions, $T$ must be the trivial map $T(x) = \lambda x$
for some $\lambda \in \mathbb{R}$:

i) $A$ is a simple C*-algebra.
ii) $A$ is unital with trivial center and has a faithful trace such that each zero-
trace element lies in the closure of the span of commutator elements.
iii) $A = B(H)$ where $H$ is a separable Hilbert space.

The Hyers- Ulam - Rassias Stability of this functional equation is discussed.

1. INTRODUCTION*

Let $C = (C[x], \Delta, \varepsilon)$ be the coalgebra of polynomials with complex coefficients
where the coproduct $\Delta$ is $\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \otimes x^i$ and counit $\varepsilon$ is defined by
$\varepsilon(1) = 1, \varepsilon(x^n) = 0$ for $n > 0$. Assume that $T$ is the operator of differentiation on
$C[x]$. It can be easily shown that $T$ satisfies

(1) $(T \otimes T) \circ \Delta = \Delta \circ T^2$

Then we have the following commutative diagram:

\[
\begin{array}{ccc}
C \otimes C & \xleftarrow{\Delta} & C \\
T \otimes T & \downarrow & T^2 \\
C \otimes C & \xleftarrow{\Delta} & C
\end{array}
\]

Now assume that $A$ is a (not necessarily unital) algebra with multiplication
$m : A \otimes A \rightarrow A$ and $T$ is a linear map on $A$. By reversing the direction of arrows
in the above diagram and replacing the coproduct $\Delta$ of $C$ by product $m$ of $A$, we
find the following commutative diagram:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{m} & A \\
T \otimes T & \downarrow & T^2 \\
A \otimes A & \xrightarrow{m} & A
\end{array}
\]

This means that $T$ is a linear map on algebra $A$ which satisfies:

(2) $T(x)T(y) = T^2(xy)$

In fact, motivating by the classical operator of differentiation, we constructed \(^1\) as a coalgebraic functional equation on an arbitrary coalgebra. This equation
naturally gives us the equation \(^2\), as an algebraic functional equation for linear maps
on a complex algebra $A$. If we wish to consider \(^2\) on a C* algebra $A$, it is natural

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to add the symmetric condition $T(x^*) = (T(x))^*$. 

In this paper we are mainly interested in functional equation

$$T(x)T(y) = T^2(xy)$$

$$T(x^*) = (T(x))^*$$

where $T$ is a (not necessarily continuous) linear map on a $C^*$ algebra $A$ and $T^2 = T \circ T$. An operator which satisfies (3) is called a partial multiplier.

We observe that a partial multiplier is automatically a continuous operator. Despite of its pure algebraic nature, we will see that for certain $C^*$ algebras, this functional equation has a geometric interpretation in term of inner product preserving maps, see proposition 2. Our reason that we choose the name "partial multiplier" for such operators is that an injective partial multiplier on an algebra $A$ can be considered as an element of multiplier algebra of $A$, see (2) proposition 1.

Although a partial multiplier is not necessarily a multiplicative operator, we show that its kernel is always a closed two sided ideal.

Obviously for every $\lambda \in \mathbb{R}$, the linear map $T(x) = \lambda x$ is a partial multiplier which we call it a trivial map. In this paper we are interested in conditions on a $C^*$ algebra $A$, under which every partial multiplier is necessarily a trivial map. Our main result is the following:

**Main Theorem**. Every partial multiplier on a $C^*$ algebra $A$ is trivial if $A$ satisfies each of the following conditions:

(I) $A$ is a simple $C^*$-algebra.

(II) $A$ is unital, with trivial center, and has a faithful trace such that each zero-trace element lies in the closure of the span of commutator elements.

(III) $A = B(H)$ where $H$ is a separable Hilbert space.

2. Preliminaries

In this section we give some definitions and notations. For a $C^*$ algebra $A$, a positive linear map on $A$ is a linear map $T$ with the property $T(x) \geq 0$ if $x \geq 0$. $T$ is completely positive if $T \otimes I_n$ is a positive map on $M_n(A) \approx A \otimes M_n(\mathbb{C})$ for all $n \in \mathbb{N}$. A faithful (positive) trace on $A$ define $\text{tr} : A \to \mathbb{C}$ such that $\text{tr}(xy) = \text{tr}(yx)$ and $\text{tr}(x) > 0$ for $x > 0$ ($\text{tr}(x) \geq 0$ for $x \geq 0$). A zero trace element is an element $x \in A$ with $\text{tr}(x) = 0$. An element $xy - yx$ is called a commutator. For a $C^*$ algebra $A$ with a faithful trace $\text{tr}$ we define an inner product $\langle \cdot, \cdot \rangle_{\text{tr}}$ on $A$ with $\langle a, b \rangle_{\text{tr}} = \text{tr}(ab^*)$.

The dual and bidual of $A$ is denoted by $A^*$ and $A^{**}$, respectively. A bounded linear map $T$ on $A$ induces natural linear maps $T^*$ and $T^{**}$ on $A^*$ and $A^{**}$, respectively. The space $A^{**}$ is a $C^*$ algebra with the Arens products and a natural involution. The Arens products is defined in three stage as follows, for more information about Arens product see [3]:

- For $f \in A^*$ and $x \in A$ define $\langle f, x \rangle_{\text{tr}} \in A^*$ with $\langle f, x \rangle (y) = f(xy)$ for $y \in A$.
- For $F \in A^{**}$ and $g \in A^*$ define $[F, g] \in A^*$ with $[F, g](x) = F(\langle g, x \rangle)$ for $x \in A$.
- For $F, G \in A^{**}$ the Arens product $F.G \in A^{**}$ is defined with $F.G(f) = F([G, f])$ for $f \in A^*$.
The involution on $A^{**}$ is defined as follows: $F^*(\phi) = T(\phi^*)$, $\phi^*(a) = \phi(a^*)$, where $F \in A^{**}$, $\phi \in A^*$, $a \in A$. Then $A^{**}$ is a unital $C^*$ algebra which contains $A$ as a $C^*$ subalgebra, via the natural imbedding of $A$ into $A^{**}$. The multiplier algebra of $A$, denoted by $\mathcal{M}(A)$, is the idealizer of $A$ in $A^{**}$, that is the algebra $\{ z \in A^{**} \mid zA \subseteq A \text{ & } A^z \subseteq A \}$.

A pair $(L, R)$ of linear maps on $A$ is called a double centralizer if $R(x)y = xL(y)$ for $x, y \in A$. The space of double centralizers on $A$ is a $C^*$ algebra with a natural operations and is isomorphic to the multiplier algebra $\mathcal{M}(A)$. In fact for every double centralizer $(L, R)$ on $A$ there is a unique element $a \in \mathcal{M}(A)$ such that $L(x) = ax$, $R(x) = xa$ for $x \in A$.

A linear map $T$ on an algebra $A$ preserves zero product (resp. commutativity) if $xy = 0$ implies $T(x)T(y) = 0$ (resp. $xy = yx$ implies $T(x)T(y) = T(y)T(x)$). A linear map $T$ on an inner product space is called orthogonality preserving if $x \perp y$ implies that $T(x) \perp T(y)$.

A complex coalgebra is a complex vector space $C$ with linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{C}$ such that

\begin{align*}
(Id \otimes \Delta) \circ \Delta &= (\Delta \otimes Id) \circ \Delta \\
(Id \otimes \varepsilon) \circ \Delta &= Id = (\varepsilon \otimes Id) \circ \Delta
\end{align*}

The n-th power tensor product $\underbrace{C \otimes C \otimes \ldots \otimes C}_n$ is denoted by $\bigotimes^n C$. Similarly, for a linear map $T$ on $C$, $(T \otimes T, \ldots, \otimes T)$ is denoted by $\bigotimes^n T$. The n-th power coproduct $\Delta^n : C \to \bigotimes^n C$ is defined inductively by $\Delta^0 = (\Delta^{n-1} \otimes Id) \circ \Delta$, $\Delta^2 = \Delta$. A linear map $T$ on $C$ is called a partial comultiplier if it satisfies $(T \otimes T) \circ \Delta = \Delta \circ T^2$.

3. Partial Multipliers

Let $A$ be a $C^*$ algebra. A linear map $T : A \to A$ is called a partial multiplier if $T$ satisfies [3]. Some algebraic properties of partial multipliers are included in the following proposition:

**Proposition 1.** Let $T$ be a partial multiplier on a $C^*$ algebra $A$. Then

(a) $T$ preserves zero product and commutativity.

(b) $T$ is a bounded operator, $T^2$ is a completely positive map and $T^{**}$ is a partial multiplier on $A^{**}$.

(c) $\ker(T)$ is a closed two sided ideal in $A$.

(d) $\prod_{i=1}^n T(x_i) = T^n(\prod_{i=1}^n x_i)$ where $x_i \in A$ for $i = 1, 2, \ldots, n$.

(e) If $T$ is an injective operator then $(T, T)$ is a double centralizer on $A$.

**Proof.** The proof of (a) is obvious. We prove (b). Assume that $T$ is a partial multiplier on $A$. Then $T^2(xx^*) = T(x)(T(x))^*$ so $T^2$ is a positive map. Moreover $T \otimes id_n$ is a partial multiplier on $M_n(A)$. Therefore $(T \otimes id_n)^2 = T^2 \otimes id_n$ is a positive map, hence $T^2$ is a completely positive map. Since $T^2$ is a positive operator on a $C^*$ algebra, it is a bounded operator, see[4] page260]. This implies that $T$ is a
bounded operator too because
\[ \| T(x) \|^2 = \| T(x)(T(x)^*) \| = \| T^2(x^*) \| = \| x \|^2. \]

To prove the last part of (4), assume that \( T^{**} \) is the induced map on \( A^{**} \) and \( F, G \in A^{**} \). We have to prove
\[ T^{**}(F)T^{**}(G) = T^{**}(T^{**}(FG)) \]
and
\[ T^{**}(F^*) = (T^{**}(F))^*. \]

The proof of (4) is a mimic of the proof of theorem 6.1 in [3] for multiplicative operators, as follows: Assume that \( h \in A^* \) and \( x, y \in A \) then
\[ T^{**}(F).T^{**}(G)(h) = T^{**}(F)(\{T^{**}(G), h]) = F(T^*(\{T^{**}(G), h]) \]
and
\[ T^{**}(T^{**}(FG))(h) = FG(T^{**2}(h)) = F([G, T^{**2}(h)]) \]

So to complete the proof of (4), we show that
\[ T^*((\{T^{**}(G), h]) = [G, T^{**2}(h)] \]

For \( x \in A \) we have
\[ T^*((\{T^{**}(G), h]))(x) = [T^{**}(G), h](T(x) = T^{**}(G)(< h, T(x) >) = G(T^*(< h, T(x))) \]
and similarly
\[ [G, T^{**2}(h)](x) = G(< T^{**2}(h), x >) \]

Then in order to prove (5) it is sufficient to prove:
\[ T^*(< h, T(x) >)(y) = < T^{**2}(h), x > (y) \quad \text{for all } y \in A \]
The left side is \( h(T(x)T(y)) \) and the right side is \( h(T^2(xy)) \). These are equal since \( T \) is a partial multiplier. This completes the proof of (5) hence (4) is proved.

Now we prove (c). Since \( T \) is a symmetric operator we have \( T^*(h^*) = (T^*(h))^* \) because
\[ T^*(h^*)(x) = h^*(T(x)) = \overline{h(T(x)^*)} = \overline{h(T(x))} = (T^*(h))^*(x) \]

Assume that \( F \in A^{**} \), \( h \in A^* \), then
\[ T^{**}(F^*)(h) = F^*(T^*(h)) = \overline{F(T^*(h^*))} = \overline{h(T(x)^*)} = (T^{**}(F))^*(h) \]

Then \( T^{**}(F^*) = (T^{**}(F))^* \), the equation (5). So the proof of (4) is completed.

To prove the remaining parts of the proposition, without lose of generality, we assume that \( A \) is a unital algebra. Otherwise we consider the extension of \( T \) on \( A \) to \( T^{**} \) on \( A^{**} \).

Now we prove (c). Note that \( \ker T \) is invariant under the left and right action of unitary group. Because for a unitary element \( u \in A \) and \( x \in \ker T \) we have:
\[ T(xu)(T(xu))^* = T(xu)T(u^*x^*) = T^2(x^*) = T(x)x^* = 0 \]
Then $xu \in \ker T$ and a similar argument shows that $ux \in \ker T$. On the other hand every $C^*$ algebra is the span of its unitary elements, see [4, page 25, Theorem 1.8.4]. This shows that $\ker T$ is a closed two sided ideal.

To prove (1) we first note that $T(1)^k T(x) = T^{k+1}(x)$ for all $k \in \mathbb{N}$. Now we prove (1) by induction on $n$. Assume that the statement is true for all $k \leq n - 1$. Then

$$
\prod_{i=1}^n T(x_i) = T^{n-1} \left( \prod_{i=1}^{n-1} x_i \right) T(x_n) =
$$

$$
T(1)^{n-2} T(\prod_{i=1}^{n-1} x_i) T(x_n) = T(1)^{n-2} T^2(\prod_{i=1}^n x_i) =
$$

$$
T(1)^{n-2} T(\prod_{i=1}^n x_i) = T^{n-1}(\prod_{i=1}^n x_i) = T^n(\prod_{i=1}^n x_i).
$$

To prove (2) we note that for all $x, y \in A$

$$
T(x)T(y) = T^2(x)T(y)
$$

since each side of the equality is equal to $T(x)T(1)T(y)$. Now (1) implies that $T^2(xT(y)) = T^2(T(x)y)$. Since $T$ is injective we conclude that $xT(y) = T(x)y$. Then $(T, T)$ is a double centralizer on $A$.

This completes the proof of proposition 1.

In the next proposition we give a geometric interpretation for partial multipliers.

To prove this proposition we need to the following lemma which is proved in [2, Theorem 1]:

**Lemma 1.** Assume that $T$ is a linear map on a complex inner product space $V$ which preserves orthogonality. Then there is a real number $k$ such that $<T(x), T(y)> = k <x, y>$ for all $x, y \in V$.

In the following proposition by $<x, y>_{tr}$ we mean $tr(xy^*)$, the inner product which induces from a faithful trace.

**Proposition 2.** Assume that a $C^*$ algebra $A$ has a faithful trace such that every zero trace element lies in the closure of span of commutator elements. Let $T$ be a partial multiplier on $A$. Then there is a $\lambda \in \mathbb{C}$ such that $<T(x), T(y)>_{tr} = \lambda <x, y>_{tr}$, for all $x, y \in A$.

**Proof.** According to the above lemma, it is sufficient to prove that $T$ preserve the orthogonality with respect to the inner product $<\cdot, \cdot>_{tr}$. Note that for every $a, b \in A$, $tr(T^2(ab - ba)) = tr(T(a)T^2(b) - T(b)T^2(a)) = 0$. Then the functional $tr(T^2)$ vanishes on the closure of the span of commutator elements. Then $tr(xy^*) = 0$ implies that $tr(T^2(xy^*)) = 0$. Hence $tr(T(x)T(y)^*) = 0$. So $T$ preserve the orthogonality with respect to $<\cdot, \cdot>_{tr}$. This completes the proof of the proposition.

**Proof of the main theorem.** To prove (1), assume that $T$ is a non zero partial multiplier on a simple $C^*$ algebra $A$. By (2) the kernel of $T$ is an ideal in $A$ so $\ker T = \{0\}$, that is $T$ is injective. By (1), $(T, T)$ is a double centralizer on $A$. This means that there is an element $z \in \mathcal{M}(A)$ such that $zx = xz$ for all $x \in A$. By the following argument we conclude that $z$ is a multiple of the identity element of $\mathcal{M}(A)$. We thank professor J. Rosenberg for this argument:

Every $C^*$ algebra has an irreducible representation on a Hilbert space $H$, see [4].
Corollari I.9.11]. Since $A$ is simple this representation is injective. So $A$ is an irreducible subalgebra of $B(H)$. From an equivalent definition of the multiplier algebra which is mentioned in [12] proposition 2.2.11], we have that $\mathcal{M}(A)$ is the idealizer of $A$ in $B(H)$. Moreover irreducibility of $A$ in $B(H)$ implies that the centralizer of $A$ in $B(H)$ reduces to one dimensional scalars $\mathbb{C}$. [1, Lemma I.9.1]. This obviously shows that $x$ is a multiple of the identity. Then the partial multiplier $T$ is in the form $T(x) = \lambda x$ for some $\lambda \in \mathbb{C}$. Since $T$ is a symmetric operator, $\lambda$ is a real number. This complete the proof of (I).

Assume that $T$ is a partial multiplier on a $C^*$ algebra $A$ which satisfies the hypothesis of (I). Then $T$ is injective by proposition 2 and $\mathcal{M}(A) = A$ since $A$ is unital. Then, similar to the above situation, $(T,T')$ is a double centralizer for $A$. Since $\mathcal{M}(A) = A$ we have $T(x) = zx = xz$ for a central element $z \in A$. Since $A$ has trivial center we conclude that the symmetric linear map $T$ is in the form $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$. This proves (II).

The same argument as above, also shows that an injective partial multiplier on $B(H)$ is a trivial map. Then to prove (III) we assume that $\ker T$ is non trivial, then we will obtain a contradiction. $\mathcal{K}(H)$, the space of compact operators on $H$, is the unique closed two sided ideal in $B(H)$. So we assume that $\ker T = \mathcal{K}(H)$. Then $T$ induces the quotient operators $\tilde{T} : B(H)/\mathcal{K}(H) \to B(H)$ and $\hat{T} : B(H)/\mathcal{K}(H) \to B(H)/\mathcal{K}(H)$. The Calkin algebra $B(H)/\mathcal{K}(H)$ is a simple algebra and $\tilde{T}$ is a partial multiplier on $B(H)/\mathcal{K}(H)$. Then (I) implies that $\tilde{T}(a) \equiv \lambda a$ for some $\lambda \in \mathbb{R}$. If $\lambda = 0$ then $T(x)$ is a compact operator, for all $x \in B(H)$. Thus $T^2 = 0$ since $\ker T = \mathcal{K}(H)$. So $T(x)T(x)^* = T^2(xx^*) = 0$, then $T(x) = 0$ for all $x \in B(H)$. Now assume that $\lambda \neq 0$. After a resealing $T := T/\lambda$ we can assume that $\tilde{T}$ is the identity operator. This means that $T(x) - x \in \mathcal{K}(H) = \ker T$ for all $x \in B(H)$. Then $T^2 = T$ so $T$ is a $C^*$ morphism on $B(H)$, since $T$ satisfies (3). Let $\pi : B(H) \to B(H)/\mathcal{K}(H)$ be the canonical map. We have $\pi \circ \tilde{T} = \text{Id}$, the identity operator on the Calkin algebra. This is a contradiction by each of the following arguments:

- It is well known that the Calkin algebra can not be embedded in $B(H)$, see [5] page 41).
- The equality $\pi \circ \tilde{T} = \text{Id}$ implies that the following short exact sequence is splitting:

$$0 \to \mathcal{K}(H) \to B(H) \to B(H)/\mathcal{K}(H) \to 0$$

On the other hand an splitting short exact sequence of $C^*$ algebras, gives us an splitting exact sequence of their $K$-theory, see [12] Corollary 8.2.2]. In particular for $i = 0,1$ we would obtain that

$$K_i(B(H)) \cong K_i(\mathcal{K}(H)) \bigoplus K_i((B(H)/\mathcal{K}(H))$$

This is a contradiction, see the catalogue of $K$-groups in [12] pages 123]. This completes the proof of the main theorem.

4. Remarks and Questions

**Remark 1.** Recall that a partial comultiplier on a coalgebra $C$ with comultiplication $\Delta$ is a linear map on $C$ which satisfies $(T \otimes T) \circ \Delta = \Delta \circ T^2$. The reader may easily checks that each of the following statements is true:
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- Assume that $D$ is a derivation on $\mathbb{C}[x]$ which is also a partial comultiplier. Then $D$ is a scalar multiple of the standard differentiation.

- If $T$ is a partial comultiplier on a coalgebra, then

$$(T^2 \otimes T) \circ \Delta = (T \otimes T^2) \circ \Delta$$

This formula is the coalgebraic version of (9).

- For a partial comultiplier $T$, we have

$$(10) \bigotimes T \circ \Delta^n = \Delta^n \circ T^n$$

This is a coalgebraic version of (10) in proposition 1.

**Question 1.** As we stated in the first line of the paper, the operator of differentiation satisfies (10) for $n = 2$. What is an example of a classical operator which satisfies (10) for some $n > 2$ but does not satisfies for $n = 2$?

**Remark 2.** The following example shows that the assumption "faithful trace" in (11) in the main theorem can not be weakened to "positive trace": Let $\mathcal{K}$ be the algebra of compact operators on an infinite dimensional separable Hilbert space. Assume that $A = \mathcal{K} \oplus \mathbb{C}$ is the unitization of $\mathcal{K}$. Obviously $A$ is a unital algebra with trivial center. Then define $tr : A \to \mathbb{C}$ with $tr(x, \lambda) = \lambda$. This is a positive but not faithful trace on $A$. Every zero trace element is in the form $(T, 0)$ which is a sum of three commutator element. Because every compact operator on an infinite dimensional separable Hilbert space is a sum of three commutators, see [6]. The operator $T(x, \lambda) = (0, \lambda)$ is a nontrivial partial multiplier on $A$. This shows that we can not replace the assumption "faithful trace" in (II) by a weaker assumption "positive trace".

**Remark 3.** As a consequence of part (II) of the main theorem we conclude that existence of a nontrivial idempotent $C^*$ morphism on a unital $C^*$ algebra $A$ with trivial center, is an obstruction for $A$ to have a faithful trace with the property that each zero trace element is a sum of commutator elements.

**Remark 4.** We observed in proposition 1 that every partial multiplier on a $C^*$ algebra is automatically continuous. The following example shows that the symmetric condition $T(x^*) = (T(x))^*$ is a necessary condition. Let $H$ be an infinite dimensional Hilbert space and $A = B(H \oplus H)$. Then each element of $A$ is in the form $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ where $X, Y, Z$ and $W$ are elements of $B(H)$. Assume that $\phi$ is an unbounded functional on $B(H)$. Then $T(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}) = \begin{pmatrix} 0 & \phi(X)d \\ 0 & 0 \end{pmatrix}$ is an unbounded operator on $A$ which satisfies $T(x)T(y) = T^2(xy)$. So without symmetric condition on $T$ we may loose the automatic continuity.

**Remark 5.** The following example shows that if we remove the symmetric condition $T(x^*) = (T(x))^*$ from the definition of partial multiplier in (3), the main theorem is not necessarily valid: Put $A = M_2(\mathbb{C})$. $A$ satisfies all 3 hypothesis of the main theorem. Define $T$ on $A$ with $T(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Then $T$ satisfies (3) without symmetric condition but is not an scalar multiple of the identity operator.

**Remark 6.** As we stated in proposition 1, every partial multiplier obviously preserves commutativity and zero product. On the other hand there are some
classification of such operators on $C^*$ algebras, see for example [2], [10], [13]. But most of such classification assume that the operator which preserve commutativity or zero product is surjective or even bijective. So we think that our main theorem can not be (easily) deduced from such classifications, because we do not include the assumption of surjectivity in the definition of partial multipliers.

**The Hyers-Ulam-Rassias stability of partial multipliers.** The Hyers-Ulam-Rassias stability of functional equations is a theory about the perturbation of functional equations, see [11]. According to this theory, we ask the following question:

Assume that $\epsilon > 0$ is given. Is there a positive $\delta$ such that for every symmetric linear map $T$ with $\| T(x)T(y) - T^2(xy) \| \leq \delta \| x \| \cdot \| y \|$ there necessarily exists a symmetric operator $S$ which satisfies (3) and $\| T - S \| \leq \epsilon$.

Let’s restrict this question to the matrix algebra, that is $T$ is a linear map on $M_n(C)$. Assume that $V$ is the real vector space of all symmetric $C$-linear maps on $M_n(C)$ and $W = B^2(M_n(C)) = \{ \text{The linear space of all 2-linear maps from } M_n(C) \times M_n(C) \to M_n(C) \}$. Similar to [8], we define a mapping $P : V \to W$ with (11) $P(T)(x, y) = T(x)T(y) - T^2(xy)$

$V$ and $W$, as two real vector spaces, can be identified with two Euclidean vector spaces. In this new real Euclidean coordinates, $P$ is a homogenous polynomial of degree 2 in multi variable $T$ which has only two isolated zeros on the unit sphere of $V$. This is a consequence of our main theorem. Since all norms on finite dimensional space are equivalent, the above question is equivalents to ask : Is the homogenous polynomial $P$, an Ulam stable polynomial?

Ulam stability of a polynomial has the following natural definition: $P$ is stable if for every $\epsilon$ there exist a $\delta$ such that $\| F(x) \| \leq \delta$ implies that $d(x, Z) \leq \epsilon$ where $d$ is the standard Euclidean distance and $Z$ is the zero locus of $P$. Note that there are homogenous polynomials which are not stable. We thank professor A. Eremenko who pointed out to the following counter example, see [9]: In fact it can be shown that the homogenous polynomial $y^{2m} + (z^{m-1}y - x^{n})^2$ is not an stable polynomial. However the reader may easily shows that:

- Every homogenous polynomial in 2 variable is stable. This is a consequence of factorization to homogenous polynomials of degree at most 2 and using the normal form of degree 2 polynomials.
- Every polynomial between real Euclidean spaces which last homogenous part has only trivial zero, is an stable polynomial.

The later statement can be used to prove the following stability result:

For every $\epsilon$ there is a $\delta$ such that if $T \in B(M_n(C))$ is a symmetric operator which satisfies $\| T(xy) - T(x)T(y) \| \leq \delta \| x \| \cdot \| y \|$, then there is a symmetric multiplicative operator which is $\epsilon$-close to $T$.

This is a consequence of the fact that the mapping $T \mapsto T(x)T(y) - T(xy)$ is a polynomial map on $V$ which last homogenous part $T(x)T(y)$, has only trivial zero, since $T$ is symmetric.

This stability result is a very particular case of a more general result which is deduced from [8, proposition 1.3, Theorem 7.3]. In this paper, the author gives a question about global stability of multiplicative functional equation on matrix algebra, without symmetric assumption, see [8] page 295, the second paragraph. That question is equivalent to the following question:
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Is the polynomial mapping $Q: B(M_n(\mathbb{C})) \rightarrow B^2(M_n(\mathbb{C}))$ with

$$Q(T)(x, y) = T(x)T(y) - T(xy)$$

an Ulam stable polynomial?

As a conclusion of this final remark, we see that the stability of certain functional equations for linear maps on finite dimensional algebras, can be interpreted as a problem in real or complex algebraic geometry. To what extent, the computation of degree of affine complex algebraic variety defined by $Q = 0$ in (12) or projective variety defined by $P = 0$ in (11), is straightforward? Is this degree useful from the functional equations viewpoint? For definition of degree of a variety see [7, page 171].

We end this remark with the following question: The homogenous polynomial $P$ in (11), which naturally arises from the functional equation (3), has a unique isolated zero on projectivization of $V$, as a real projective space. Now after complexification of the real projective space, is this real isolated zero, a complex isolated zero, too?

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