A ROBIN FORMULA FOR THE FEKETE-LEJA TRANSFINITE DIAMETER

ROBERT RUMELY

The purpose of this note is to give a formula for the Fekete-Leja transfinite diameter on \( \mathbb{C}^N \), generalizing the classical Robin formula

\[
d_{\infty}(E) = e^{-V(E)}
\]

for the usual transfinite diameter.

We will disengage this from a formula for the sectional capacity proved in arithmetic intersection theory ([5], Theorem 1.1, p.233).

First recall the definition of the Fekete-Leja transfinite diameter for a compact set \( E \subset \mathbb{C}^N \) (see [1], [16]). Consider the set of monomials \( z^k = z_1^{k_1} \cdots z_N^{k_N} \) in the polynomial ring \( \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_N] \). Let \( \Gamma(n) \subset \mathbb{C}[z] \) be the space of polynomials of total degree at most \( n \), and let \( \mathcal{K}(n) = \{ k \in \mathbb{Z}^N : k_i \geq 0, k_1 + \cdots + k_N \leq n \} \) be the index set for the monomial basis of \( \Gamma(n) \). Put \( q_n = \#(\mathcal{K}(n)) = \binom{n+N}{n} \).

Fixing \( n \), take \( q_n \) independent vector variables \( z_i = (z_{i1}, \ldots, z_{iN}) \in \mathbb{C}^N, i = 1, \ldots, q_n \). Let \( k_1, \ldots, k_{q_n} \) be the indices in \( \mathcal{K}(n) \). The Vandermonde determinant

\[
Q_n(z_1, \ldots, z_{q_n}) := \det(z_i^{k_j})_{i,j=1}^{q_n}\]

is a homogeneous polynomial in the \( z_{ij} \) of total degree \( T_n = N \cdot \binom{n+N}{N+1} \). For each \( n \), put

\[
d_n(E) = \max_{z_1, \ldots, z_{q_n} \in E} |Q_n(z_1, \ldots, z_{q_n})|^{1/T_n}.
\]

The Fekete-Leja transfinite diameter is defined by

\[
d_{\infty}(E) = \lim_{n \to \infty} d_n(E).
\]

The existence of the limit is due to Zaharjuta ([16]).

Henceforth, we will assume that \( d_{\infty}(E) > 0 \). This is equivalent to \( E \) being non-pluripolar ([10]).

For \( f \in \mathbb{C}[z_1, \ldots, z_N] \) write \( \|f\|_E = \sup_{z \in E} |f(z)| \). The Green’s function \( G^*(z, E) \) is the upper semicontinuous regularization of the Siciak extremal function

\[
G(z, E) := \lim_{n \to \infty} \max_{f \in \Gamma(n)} \frac{1}{n} \log(\|f(z)\|).
\]

Date: September 21, 2005.

2000 Mathematics Subject Classification. Primary: 32U35, 32U20, 14G40; Secondary: 31B15, 31C10.

Key words and phrases. Fekete-Leja transfinite diameter, Robin formula.

Work supported in part by NSF grant DMS-0300784.
Since $E$ is not pluripolar, $G^*(z, E)$ is finite for all $z \in \mathbb{C}^N$ and is plurisubharmonic ([9], [10], [17]). Write \( \dd c = \frac{1}{2\pi i} \dd \theta \) and let

\[
\omega = \dd c(2G^*(z, E))
\]

be the associated positive \((1,1)\)-current. (The factor 2 is needed for compatibility with the Poincaré-Lelong formula).

Let \( Z_0, \ldots, Z_N \) be homogeneous coordinates on \( \mathbb{P}^N(\mathbb{C}) \). For \( k = 0, \ldots, N \) write \( H_k \) for the hyperplane \( \{ Z_k = 0 \} \) and let \( A_k^N = \mathbb{P}^N(\mathbb{C}) \setminus H_k \) be the corresponding affine patch. Identify \( \mathbb{C}^N \) with \( A_0^N \subset \mathbb{P}^N(\mathbb{C}) \) via the embedding \( (z_1, \ldots, z_N) \mapsto (1 : z_1 : \cdots : z_N) \). If \( \|z\| = (|z_1|^2 + \cdots + |z_N|^2)^{1/2} \), then \( |G^*(z, E) - \max(0, \log(\|z\|))| \) is uniformly bounded on \( \mathbb{C}^N \) (for boundedness from above, see [9], Corollary 5.2.2, p.193; for boundedness from below, note that since \( E \) is compact, it is contained in a ball \( B(0, R) = \{ z \in \mathbb{C}^N : \|z\| \leq R \} \) for some large \( R \), so \( G^*(z, E) \geq G^*(z, B(0, R)) = \max(0, \log(\|z\|/R)) \)). It follows that for each \( 1 \leq k \leq N \) the function \( g_k(z, E) \) defined on \( A_k^N \setminus H_0 \) by

\[
g_k(z, E) = G^*(z, E) - \log(|z_k|)
\]

extends uniquely to a plurisubharmonic function on \( A_k^N \). If \( k = 0 \), write \( g_0(z, E) = G^*(z, E) \). Since \( \log(|Z_i/Z_j|) \) is plurisubharmonic on \( A_i^N \cap A_j^N \), the currents \( \omega_k := \dd c(2g_k(z, E)) \) cohere to give a positive \((1,1)\)-current on \( \mathbb{P}^N(\mathbb{C}) \) extending \( \omega \). We will denote it by \( \omega \) as well.

Let \( \omega^k = \omega \wedge \cdots \wedge \omega \) be the \( k \)-fold exterior product of \( \omega \) with itself, and put \( \omega^0 = 1 \). Write \( Y_k \) for the space \( H_0 \cap \cdots \cap H_{k-1} \) and let \( U_k = Y_k \setminus H_k \), so \( U_k(\mathbb{C}) \cong \mathbb{C}^{N-k} \). Define the \textit{iterated Robin constant} by

\[
\tilde{V}(E) = \frac{1}{N} \sum_{k=1}^{N} \int_{U_k(\mathbb{C})} g_k(z, E) \omega^{N-k}
\]

\[
= \frac{1}{N} \left( \int_{Z_0=0, Z \neq 0} g_1(z, E) \omega^{N-1} + \int_{Z_0=Z_1=0, Z \neq 0} g_2(z, E) \omega^{N-2} + \cdots + g_N((0 : \cdots : 0 : 1), E) \right)
\]

Note that when \( N = 1 \), the sum consists of a single term and reduces to the usual Robin constant, since \( \infty = (0 : 1) \in \mathbb{P}^1(\mathbb{C}) \), and \( g_1(\infty, E) = \lim_{z \to \infty} G^*(z, E) - \log(|z|) = V(E) \).

\textbf{Theorem 0.1.} If \( E \subset \mathbb{C}^N \) is compact and not pluripolar, then

\[
d_\infty(E) = e^{-\tilde{V}(E)}
\]

Before giving the proof we will need some facts from arithmetic intersection theory. Let \( K \) be a number field, and let \( \mathcal{O}_K \) be the ring of integers of \( K \). Let \( X/K \) be a smooth, connected projective variety of dimension \( N \), and write \( K(X) \) for the field of \( K \)-rational functions on \( X \).

The sectional capacity \( S_\gamma(E, D) \) is a measure of size for an adelic set \( \mathbb{E} \) on \( X \) relative to an ample divisor \( D \). It was first proposed by Chinburg ([4]), and its existence was shown in ([14]), using methods from ([16]). The name ‘sectional capacity’ refers to asymptotics of volumes of spaces of sections of \( \mathcal{O}_X(nD) \) as \( n \to \infty \).
Let $D$ be an effective, ample $K$-rational Cartier divisor on $X$.

For each place $v$ of $K$, let $K_v$ be the completion of $K$ at $v$, and let $\mathbb{C}_v$ be the minimal complete algebraically closed field containing $K_v$. Let $|x|_v$ be the absolute value on $\mathbb{C}_v$ extending the canonical absolute value on $K_v$ given by the modulus of additive Haar measure. Without loss, we can assume that $X$ is embedded in $\mathbb{P}^M$ for a suitable $M$. There is a natural distance function $d_v(x, y)$ on $\mathbb{P}^M(\mathbb{C}_v)$: the chordal metric associated to the Fubini-Study metric, if $v$ is archimedean; the $v$-adic spherical metric, if $v$ is nonarchimedean (see [12], §1.1).

For each $v$, let $E_v \subset X(\mathbb{C}_v)$ be a nonempty set, and write $E = \prod_v E_v$; we will call $E$ an adelic set. We will assume that the $E_v$ and $D$ satisfy the following ‘Standard Hypotheses’:

1. Each $E_v$ is bounded away from $\operatorname{supp}(D)(\mathbb{C}_v)$ under $d_v(x, y)$ and is stable under the group of continuous automorphisms $\operatorname{Gal}(\mathbb{C}_v/K_v)$.

2. For all but finitely many $v$, $E_v$ and $\operatorname{supp}(D)(\mathbb{C}_v)$ specialize to disjoint sets (mod $v$); equivalently, for all but finitely many $v$, $d_v(x, y) = 1$ for all $x \in E_v$ and all $y \in \operatorname{supp}(D)(\mathbb{C}_v)$.

Note that if $v$ is archimedean and $K_v \cong \mathbb{R}$, then $\operatorname{Gal}(\mathbb{C}_v/K_v) = \{1, \tau\}$ where $\tau$ is complex conjugation; if $K_v \cong \mathbb{C}$, then $\operatorname{Gal}(\mathbb{C}_v/K_v)$ is trivial.

For each integer $n \geq 0$, put $\Gamma(nD) = \mathcal{H}^0(X, \mathcal{O}_X(nD))$, a finite dimensional vector space over $K$. Dehomogenizing at $D$, identify $\Gamma(nD)$ with $\{f \in K(X) : \operatorname{div}(f) + nD \geq 0\}$. For each $v$, consider the space of $K_v$-rational functions on $X$ with polar divisor at most $nD$,

$$\Gamma_v(nD) = K_v \otimes_K \Gamma(nD) = \{f \in K_v(X) : \operatorname{div}(f) + nD \geq 0\}.$$ For $f \in K_v(X)$, write $|f|_{E_v} = \sup_{x \in E_v} |f(x)|_v$ and let

$$\mathcal{F}(E_v, nD) = \{f \in \Gamma_v(nD) : |f|_{E_v} \leq 1\}.$$ Let $K_A = \{(x_v) \in \prod_v K_v : |x_v|_v \leq 1\}$ for all but finitely many $v$ be the adele ring of $K$. Identifying $K_A \otimes_K \Gamma(nD)$ with a subset of $\bigotimes_v (K_v \otimes_K \Gamma(nD))$, introduce the ‘adelic unit ball’

$$\mathcal{F}(E, nD) = (K_A \otimes_K \Gamma(nD)) \bigcap \left( \bigotimes_v \mathcal{F}(E_v, nD) \right).$$

Fix an additive Haar measure $\operatorname{vol}_A$ on $K_A$. By transport of structure using a $K$-basis for $\Gamma(nD)$, $\operatorname{vol}_A$ induces a Haar measure on $K_A \otimes_K \Gamma(nD)$. The Product Formula shows this measure is independent of the choice of basis; it too will be denoted $\operatorname{vol}_A$. The sectional capacity $S_\gamma(\mathbb{E}, D)$ is defined by

$$-\log(S_\gamma(\mathbb{E}, D)) = \lim_{n \to \infty} \frac{(N + 1)!}{n^{N+1}} \operatorname{vol}_A(\mathcal{F}(E, nD))$$

(see [14], pp.23-24). The limit is independent of the choice of Haar measure on $K_A$. Its existence is part of ([14], Theorem C, p.8).

We will now discuss Arakelov theory. Let $\mathfrak{X}$ be a model of $X$: an integral projective scheme of Krull dimension $N + 1$, flat and proper over $\operatorname{Spec}(\mathcal{O}_K)$, whose generic fibre is $K$-isomorphic to $X$.

Put $X_{\mathbb{C}} = X \times_{\mathfrak{X}} \operatorname{Spec}(\mathbb{C})$; then $X_{\mathbb{C}}(\mathbb{C}) = \bigsqcup \sigma X_{\sigma}(\mathbb{C})$ is a complex manifold with a component corresponding to each embedding $\sigma : K \hookrightarrow \mathbb{C}$. Write $\tau$ for complex conjugation on $\mathbb{C}/\mathbb{R}$; there is a natural operation of $\tau$ on $X_{\mathbb{C}}(\mathbb{C})$ coming from its action on the base. If $\sigma$ is a real embedding, then $\tau$ acts as complex conjugation on $X_{\sigma}(\mathbb{C})$; if $\{\sigma, \overline{\sigma}\}$ is a pair of conjugate complex embeddings, then $\tau$ interchanges the components $X_{\sigma}$ and $X_{\overline{\sigma}}$, mapping $P \in X_{\sigma}(\mathbb{C})$
to \( P \in X_\tau(\mathbb{C}) \), where \( P \) is the point whose coordinates are the complex conjugates of those of \( P \).

Let \( L \) be a line bundle on \( X \). A metrized line bundle \( \overline{L} = (\mathcal{L}, [\cdot]) \) on \( X \), extending \( L \), is a pair consisting of a locally free sheaf \( \mathcal{L} \) of rank 1 on \( X \) which induces \( L \) on \( X \), and a smooth positive Hermitian metric \([\cdot]\) on the fibres of \( L \) over \( X_\tau(\mathbb{C}) \), which is invariant under the action of \( \tau \). Let \( c_1(\overline{L}) \) be the first Chern class of \( \overline{L} \), the smooth (1, 1)-form on \( X_\tau(\mathbb{C}) \) defined in a neighborhood of any point \( z_0 \in X_\tau(\mathbb{C}) \) by \( dd^c(-2 \log([s(z)])) \), where \( s \) is a meromorphic section of \( L \) which is defined and does not vanish at \( z_0 \).

For each cycle \( \mathcal{Y} \) on \( X \), Bost, Gillet and Soulé ([2]) define the “height” \( h_\mathcal{Z}(\mathcal{Y}) \) of \( \mathcal{Y} \) relative to \( \mathcal{Z} \) to be the self-intersection number \( \deg(\hat{\mathcal{C}}(\overline{L})^{\dim(\mathcal{Y})}|_{\mathcal{Z}}) \) in the arithmetic intersection theory Gillet and Soulé developed in ([3], [7], [8]). The height is additive in \( \mathcal{Y} \). There is a recursive formula for \( h_\mathcal{Z}(\mathcal{Y}) \) (see [2], Proposition 3.2.1(iv), p.949, and the remarks after it dealing with the non-regular case): when \( \mathcal{Y} \) is irreducible, let \( s \neq 0 \) be a section of \( \mathcal{L}|_{\mathcal{Y}} \), and let \( \text{div}\mathcal{Y}(s) \) be its divisor. Write \( \mathcal{Y} \) for the generic fibre \( \mathcal{Y}_k \). Then

\[
(5) \quad h_\mathcal{Z}(\mathcal{Y}) = \int_{\mathcal{Y}_k(\mathbb{C})} -\log([s(z)]) c_1(\overline{L})^{\dim(\mathcal{Y})} + h_\mathcal{Z}(\text{div}\mathcal{Y}(s)).
\]

Eventually the recursion abuts at a 0-cycle \( \mathcal{Z} = \sum p_n \cdot p \), a finite sum of closed points, and then \( h_\mathcal{Z}(\mathcal{Z}) = \sum n_p \log(Np) \) where \( Np \) is the order of the residue field at \( p \) ([2], formula 3.1.4, p.946).

It is customary to write \( \overline{L}^{N+1} \) for \( h_\mathcal{Z}(\mathcal{Z}) \). If \( \overline{L} \) is replaced by \( \overline{L}^{\otimes m} \), then \( (\overline{L}^{\otimes m})^{N+1} = m^{N+1} \cdot \overline{L}^{N+1} \) ([2], Proposition 3.2.1(i)). This leads to the notion of a fractional metrized line bundle: if \( n > 0 \) is an integer, and \( \overline{L} \) is a metrized line bundle inducing \( L^{\otimes n} \) on \( X \), then we call the formal object \( \frac{1}{n} \overline{L} \) a fractional metrized line bundle, and define \( (\frac{1}{n} \overline{L})^{N+1} = n^{-(N+1)} \overline{L}^{N+1} \).

Chinburg, Lau and Rumely ([5]) expressed the sectional capacity as a limit of self-intersection numbers of fractional metrized line bundles. Given an adelic set \( \mathbb{E} \) and an effective ample divisor \( D \) on \( X \) satisfying the Standard Hypotheses, they constructed a sequence of models \( \mathbb{X}_n \), and metrized line bundles \( \overline{L}_n \) on \( \mathbb{X}_n \) extending \( L^{\otimes n} = \mathcal{O}_X(nD) \) on \( X \), such that

\[
(6) \quad -\log(S_1(\mathbb{E}, D)) = \lim_{n \to \infty} \left( \frac{1}{n} \overline{L}_n \right)^{N+1};
\]

see ([5], Theorem 1.1, p.233).

The models \( \mathbb{X}_n \) are defined using the nonarchimedean part of \( \mathbb{E} \). For each \( n \), put \( S_n = \{ f \in \Gamma(nD) : \|f\|_{E_v} \leq 1 \text{ for all nonarchimedean } v \} \), and let \( \mathcal{O}_K[S_n] \) be the graded \( \mathcal{O}_K \)-algebra generated in degree 1 by \( S_n \). Then \( \mathbb{X}_n = \text{Proj}(\mathcal{O}_K[S_n]) \). Let \( \mathcal{L}_n = \mathcal{O}_{\mathbb{X}_n}(1) \); then \( \mathcal{L}_n \) induces \( L^{\otimes n} \) on \( X \).

The metrics \([\cdot]\) are constructed using the archimedean part of \( \mathbb{E} \). For each archimedean place \( v \) of \( K \), fix an isomorphism \( \mathbb{C}_v \cong \mathbb{C} \), and choose a sequence of sets \( E_{v,1} \supseteq E_{v,2} \supseteq \cdots \) containing \( E_v \) for which the extremal functions

\[
G(z, E_{v,n}, D) := \lim_{m \to \infty} \sup_{f \in \Gamma(mD)} \frac{1}{m} \log(\|f(z)\|)
\]

are continuous and increase monotonically to \( G(z, E_v, D) \). (The existence of such sets \( E_{v,n} \) follows from the proof of ([5], Lemma 1.2, p.234).)
Since \( G(z, E_{v,n}, D) \) is continuous, it is plurisubharmonic. By a theorem of Richburg (\[1\], Satz 4.7) there is a smooth plurisubharmonic function \( G_{v,n}(z) \) such that
\[
G(z, E_{v,n}, D) - \frac{1}{n} \leq G_{v,n}(z) \leq G(z, E_{v,n}, D) - \frac{1}{n + 1}
\]
for all \( z \in X(\mathbb{C}_v) \setminus \text{supp}(D)(\mathbb{C}_v) \). Hence
\[
G(z, E_v, D) = \lim_{n \to \infty} G_{v,n}(z)
\]
as an increasing limit.

By (\[5\], Theorem 2.13, p.253), the smoothings \( G_{v,n}(z) \) can be chosen so that for each \( z_0 \in \text{supp}(D)(\mathbb{C}_v) \), if \( s \) is a local equation for \( D \) at \( z_0 \), then \( G_{v,n}(z) + \log(|s(z)|) \) extends to a smooth plurisubharmonic function in a neighborhood of \( z_0 \). If \( K_v \cong \mathbb{R} \), then since \( E_v \) has been assumed to be stable under complex conjugation, the \( G_{v,n}(z) \) can be chosen to be invariant under complex conjugation as well.

Each embedding \( \sigma : K \hookrightarrow \mathbb{C} \) determines a place \( v \) of \( K \). If \( \sigma \) is a real embedding, put \( G_{\sigma,n}(z) = G_{v,n}(z) \). If \( \sigma \) is a complex embedding, then precisely one of \( \sigma \) and its complex conjugate \( \overline{\sigma} \) induces the chosen isomorphism \( \mathbb{C}_v \cong \mathbb{C} \); if it is \( \sigma \), put \( G_{\sigma,n}(z) = G_{v,n}(z) \), if not, put \( G_{\sigma,n}(z) = G_{v,n}(\overline{\sigma}) \).

The metric \([1]_n \) is defined by requiring that for the tautological section 1 of \( L^\otimes n \), if \( z \in X_{\sigma}(\mathbb{C}) \subset X_{\mathbb{C}}(\mathbb{C}) \) then
\[
[1(z)]_n = \exp(-nG_{\sigma,n}(z)).
\]
By construction, \([1]_n \) is invariant under \( \tau \).

Proof of Theorem 0.1 Let \( E \subset \mathbb{C}^N \) be a compact, non-pluripolar set. We will embed it as the archimedean component of an adelic set, and apply the machinery above.

Take \( K = \mathbb{Q}(\sqrt{-1}) \) as the ground field, so \( \mathcal{O}_K \) is the ring of Gaussian integers. Write \( \mathbb{P}_K^N \) for \( \mathbb{P}^N / \text{Spec}(\mathcal{O}_K) \), and write \( \mathbb{P}_K^N \) for its generic fibre. Let \( X/K \) be \( \mathbb{P}_K^N \), and let \( D \) be defined by \( \{Z_0 = 0\} \).

Put \( \mathbb{A}_K^N = \mathbb{P}_K^N \setminus H_0 \), and define \( \mathbb{E} = \prod_v E_v \subset \prod_v \mathbb{A}_K^N(\mathbb{C}_v) \) as follows. For each place \( v \), identify \( \mathbb{A}_K^N(\mathbb{C}_v) \) with \( \mathbb{C}_v^N \). There is one archimedean place \( v_{\infty} \) for \( K \); fix an isomorphism \( K_{v_{\infty}} \cong \mathbb{C} \) and put \( E_{v_{\infty}} = E \subset \mathbb{C}^N \). Condition (1) of the Standard Hypotheses holds trivially.

For each nonarchimedean \( v \), put \( E_v = B(0, 1) \) if \( \max(|z_i|_{\infty} \leq 1) \). Since \( \text{Gal}(\mathbb{C}_v/K_v) \) preserves \( |x|_v \), again condition (1) of the Standard Hypotheses holds. The sets \( E_v \) and \( H_0(\mathbb{C}_v) \) specialize to disjoint sets (mod \( v \)) for all nonarchimedean \( v \), so condition (2) of the Standard Hypotheses holds as well.

By (\[13\], Theorem 3.1, p.551) the sectional capacity \( S_{\sigma}(\mathbb{E}, D) \) can be decomposed as a product of ‘local sectional capacities’
\[
S_{\gamma}(\mathbb{E}, D) = \prod_v S_{\gamma}(E_v, D).
\]
Here the \( S_{\gamma}(E_v, D) \) depend on the choice of an ordered basis for the graded ring \( \oplus_{n=0}^{\infty} \Gamma(nD) \): we take this to be the monomial basis, equipped with the lexicographic order graded by the degree. In this situation, \( S_{\gamma}(E_{v_{\infty}}, D) = d_{v_{\infty}}(E)^{2N} \). This follows from (\[13\], Theorems 2.3 and 2.6), combined with the discussion on (\[13\], p.557). (In (\[13\], p.557), the formula \( S_{\gamma}(E_{\infty}, D) = d_{\infty}(E)^N \) is given when \( \mathbb{R} \) is the ground field. Here, since \( K_{v_{\infty}} \cong \mathbb{C} \), the normalized absolute value \( |x|_{v_{\infty}} = |x|^2 \) used in computing \( S_{\gamma}(E_{v_{\infty}}, D) \) is the square of the
usual absolute value; this is the source of the 2 in the exponent.) Furthermore, \( S_\gamma(E_v, D) = 1 \) for each nonarchimedean \( v \) by ([13], Example 4.1, p.555). Hence

\[
S_\gamma(E, D) = d_\infty(E)^{2N}.
\]

Let \( \{ \frac{1}{n} \mathcal{T}_n \} \) be the sequence of fractional metrized line bundles constructed in ([5], Theorem 1.1) as described above. Since \( E_v \) is the ‘trivial set’ \( B(0, 1) \) for each nonarchimedean \( v \), the Maximum Modulus Principle of nonarchimedean analysis shows that

\[
S_n = \oplus_{k_0 + \ldots + k_N = n} \mathcal{O}_K \cdot Z_{k_0} \ldots Z_{k_N}.
\]

It follows that \( \mathcal{X}_n \cong \mathbb{P}^N_{\mathcal{O}_K} \), and \( \mathcal{L}_n \cong \mathcal{O}_{\mathbb{P}^N_{\mathcal{O}_K}}(n) \). Let \( \mathcal{T}_n \) be \( \mathcal{O}_{\mathbb{P}^N_{\mathcal{O}_K}}(1) \), equipped with the metric \( \left[ \cdot \right]'_n \) defined by \( \left[ 1(z) \right]'_n = \exp(-G_n(z)) \), where \( G_n(z) \) is metric constructed as above by smoothing \( G(z, E, D) \). Then \( \mathcal{T}_n \cong (\mathcal{T}_n')^\otimes_n \), so \( (\frac{1}{n} \mathcal{T}_n)^{N+1} = (\mathcal{T}_n')^{N+1} \) and without loss we can replace \( \frac{1}{n} \mathcal{T}_n \) by \( \mathcal{T}_n \).

We will now compute the intersection product \( (\mathcal{T}_n')^{N+1} \) using Bost-Gillet-Soulé’s recursive formula. There are two embeddings of \( K \) into \( \mathbb{C} \), both of which correspond to the place \( \nu_\infty \), so \( (\mathbb{P}^N_K)_C(\mathbb{C}) \) has two components which are interchanged by \( \tau \). Write \( \mathbb{P}^N(\mathbb{C}) \) for the one corresponding to our chosen isomorphism \( K_{v_\infty} \cong \mathbb{C} \), and \( (\mathbb{P}^N)_C(\mathbb{C}) \) for the other. Since they are isomorphic, and the metrics \( \left[ \cdot \right]'_n \) are \( \tau \)-invariant, the integrals over the two components in the archimedean part of the intersection product are the same. Therefore, in what follows, we will compute the integrals over \( \mathbb{P}^N(\mathbb{C}) \) and double the answer. For any \( K \)-rational subvariety \( Y \subset \mathbb{P}^N_K \), write \( Y(\mathbb{C}) \) for the part of \( Y_C(\mathbb{C}) \) in the chosen component \( \mathbb{P}^N(\mathbb{C}) \).

Write \( \mathbb{A}^N_K \) for \( \mathbb{P}^N_K \setminus H_k \), where \( H_k = \{ Z_k = 0 \} \). Let \( \mathcal{P}_k \) be the Zariski closure of \( H_k \) in \( \mathbb{P}^N_{\mathcal{O}_K} \). Then \( \mathcal{P}_0, \ldots, \mathcal{P}_N \) meet transversely on \( \mathbb{P}^N_{\mathcal{O}_K} \). Write \( g_{n,0}(x) = G_n(x) \) on \( (\mathbb{A}^N_0)(\mathbb{C}) \) and for each \( k = 1, \ldots, N \) let \( g_{n,k}(x) \) be the natural extension of \( G_n(x) - \log(|z_k(x)|) \) to a plurisubharmonic function on \( \mathbb{A}^N_k(\mathbb{C}) \); here \( z_k = Z_k/Z_0 \). As before, the \( (1,1) \)-forms \( dd^c(2g_{n,k}) \) glue to give a well-defined \( (1,1) \)-form \( \omega_n \) on \( \mathbb{P}^N(\mathbb{C}) \).

Since \( L = \mathcal{O}_{\mathbb{P}^N_K}(D) \) where \( D = \{ Z_0 = 0 \} \), the canonical section ‘1’ of \( L \) is \( Z_0 \). This means that for each \( x \in \mathbb{A}^N_0(\mathbb{C}) \)

\[
- \log(\left[ Z_0(x) \right]'_n) = G_n(x) = g_{n,0}(x).
\]

Similarly, on \( \mathbb{A}^N_K(\mathbb{C}) \),

\[
- \log(\left[ Z_k(x) \right]'_n) = - \log(\left[ (Z_k/Z_0)(x) \cdot Z_0(x) \right]'_n) = - \log(|z_k(x)|) + G_n(x) = g_{n,k}(x).
\]

Put \( \mathcal{Y}_0 = \mathcal{X}_n = \mathbb{P}^N_{\mathbb{Z}} \) and take \( s = Z_0 \). Let \( \mathcal{Y}_0 = \mathbb{P}^N_{\mathbb{Q}} \) be the generic fibre of \( \mathcal{Y}_0 \) and put \( \mathcal{Y}_1 = \text{div}(Z_0) = \mathcal{H}_0 \). By Bost-Gillet-Soulé’s formula ([5]) and the remarks above,

\[
(\mathcal{T}_n')^{N+1} = h_{\mathcal{T}_n}(\mathcal{X}_n) = 2 \int_{\mathcal{Y}_0(\mathbb{C})} g_{n,0}(z) \omega^N + h_{\mathcal{T}_n}(\mathcal{Y}_1).
\]

Inductively apply this formula to the sections \( Z_1, \ldots, Z_N \), putting \( \mathcal{Y}_{k+1} = \text{div}\mathcal{Y}_k(Z_k \vert \mathcal{Y}_k) = \mathcal{H}_0 \cdot \ldots \cdot \mathcal{H}_k \). Note that \( \mathcal{H}_0 \cdot \ldots \cdot \mathcal{H}_N = 0 \), so the final abutment term vanishes. It follows that

\[
(\mathcal{T}_n')^{N+1} = 2 \sum_{k=0}^{N} \int_{\mathcal{Y}_k(\mathbb{C})} g_{n,k}(z) \omega^{N-k}.
\]
Since \( \omega_n \) is smooth, the \( (N - k - 1) \)-dimensional subspace \( Y_{k+1}(\mathbb{C}) \) of \( Y_k(\mathbb{C}) \) has measure 0 under \( \omega^{N-k}_n \). Put \( U_k = Y_k \setminus Y_{k+1} \). Then for each \( k \)

\[
\int_{Y_k(\mathbb{C})} g_{n,k}(z) \omega^{N-k}_n = \int_{U_k(\mathbb{C})} g_{n,k}(z) \omega^{N-k}_n.
\]

Now let \( n \to \infty \). For each \( k \) the \( g_{n,k} \) increase monotonically on \( U_k(\mathbb{C}) \) to \( G(z,E) - \log(|z_k|) \), whose upper semicontinuous regularization is \( g_k(z,E) \). Noting that \( U_k(\mathbb{C}) \cong \mathbb{C}^{N-k} \), it follows from (3.4, Theorem 7.4) that

\[
\lim_{n \to \infty} \int_{U_k(\mathbb{C})} g_{n,k}(z) \omega^{N-k}_n = \int_{U_k(\mathbb{C})} g_k(z,E) \omega^{N-k}_n.
\]

Combining (6), (8), (9), and (10) gives

\[
- \log(S_{\gamma}(\mathbb{E}, D)) = 2 \sum_{k=0}^{N} \int_{U_k(\mathbb{C})} g_k(z,E) \omega^{N-k}_n.
\]

When \( k = 0 \), \( g_0(z,E) \) is the extremal plurisubharmonic function \( G^*(z,E) \) on \( U_0(\mathbb{C}) \cong \mathbb{C}^N \), and \( \omega^N \) is the Bedford-Taylor measure, which is supported on \( E \). The set \( \{ z \in E : G^*(z,E) > 0 \} \) is a negligible set, so it has measure 0 under \( \omega^N \). Hence

\[
\int_{U_0(\mathbb{C})} g_{0}(z,E) \omega^{N}_n = 0.
\]

Using (7), (11), and (12) we obtain

\[
-2N \cdot \log(d_{\infty}(E)) = 2 \sum_{k=1}^{N} \int_{U_k(\mathbb{C})} g_k(z,E) \omega^{N-k}_n
\]

which is equivalent to (11). \( \square \)

**Generalizations.** It is tempting to assert that the formula in Theorem 0.1 gives a new definition of the capacity. However, we do not do so because the stated formula is only one of a class of formulas with the same property.

In particular, the order in which the hyperplanes \( H_k \) are intersected in the definition of \( \tilde{V}(E) \) is immaterial. More generally, if \( A = (a_{kj}) \in \text{GL}_N(\mathbb{C}) \), and if \( Z'_k = \sum a_{kj}Z_j \) for \( k = 1, \ldots, N \) then the constant \( \tilde{V}(E) \) can equally well be defined by

\[
\tilde{V}(E) = \frac{1}{N} \sum_{k=1}^{N} \int_{Z_0=Z'_{1}=\ldots=Z'_{k-1}=0, Z'_k \neq 0} g'_k(z,E) \omega^{N-k}_n + \frac{1}{N} \log(|\det(A)|)
\]

where \( g'_k(z,E) = G^*(z,E) - \log(|\sum a_{kj}z_j|) \) on \( \mathbb{C}^N \).

Indeed, if we put \( z' = t(z'_1, \ldots, z'_{N}) = A(z) \), where \( z'_k = \sum a_{kj}z_j \) for \( k = 1, \ldots, N \), this follows from Sheĭnov’s formula (see [15], or [11], p.287)

\[
d_{\infty}(A(E)) = |\det(A)|^{1/N} \cdot d_{\infty}(E),
\]

by applying Theorem 0.1 to the set \( A(E) \), and noting that \( G(z', A(E)) = G(z, E) \), as follows easily from the definitions.
ROBERT RUMELY

References

[1] T. Bloom and J.P. Calvi, On the multivariate transfinite diameter, Annales Polonici Mathematici 72 (1999), 285-305.
[2] J.-B. Bost, H. Gillet and C. Soulé, Heights of projective varieties and positive Green’s forms, Journal of the American Mathematical Society 7 (1994), 903-1027.
[3] E. Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Mathematica 149 (1982), 1-40.
[4] T. Chinburg, Capacity theory on varieties, Compositio Math 80 (1991), 71-84.
[5] T. Chinburg, C. F. Lau and R. Rumely, Capacity theory and arithmetic intersection theory, Duke Mathematical Journal 117 (2003), 229-285.
[6] H. Gillet and C. Soulé, Arithmetic intersection theory, IHES Publ Math. 72 (1990), 94-174.
[7] H. Gillet and C. Soulé, Characteristic classes for algebraic vector bundles with Hermitian metric, I, II, Ann. of Math. (2) 131 (1990), 13-203, 205-238.
[8] H. Gillet and C. Soulé, An arithmetic Riemann-Roch Theorem, Invent. Math. 110 (1992), 473-543.
[9] M. Klimek, Pluripotential theory, Oxford, 1991.
[10] N. Levenberg and B. A. Taylor, Comparison of capacities in $\mathbb{C}^N$, in: Complex Analysis (Toulouse, 1983), 162-172, SLNM 1094, Springer-Verlag, Berlin, 1984.
[11] R. Richburg, Stetige streng pseudokonvexe Funktionen, Math. Ann 175 (1968), 257-286.
[12] R. Rumely, Capacity theory on algebraic curves, Lecture Notes in Mathematics 1378, Springer-Verlag, Berlin-Heidelberg-New York, 1989.
[13] R. Rumely and C. F. Lau, Arithmetic capacities on $\mathbb{P}^n$, Math. Zeit. 215 (1994) 533-560.
[14] R. Rumely, C. F. Lau, and R. Varley, Existence of the Sectional Capacity, AMS Memoires 690 (vol. 145), American Mathematical Society, Providence, R.I., 2000.
[15] V. P. Sheinov, Invariant form of Pólya’s inequalities, Siberian Math. J. 14 (1973), 138-145.
[16] V. Zaharjuta, Transfinite diameter, Chebyshev constants, and capacity for compacta in $\mathbb{C}^N$, Math. USSR Sbornik vol 25 (1975), 350-364.
[17] V. Zaharjuta, Extremal plurisubharmonic functions, orthogonal polynomials and Bernstein-Walsh theorem for analytic functions of several variables, Annales Polonici Mathematici 33 (1976), 137-148.

ROBERT RUMELY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30602, USA

E-mail address: rr@math.uga.edu