Inferring entropy production from short experiments

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We provide a strategy for the exact inference of the average as well as the fluctuations of the entropy production in non-equilibrium systems in the steady state, from the measurements of arbitrary current fluctuations. Our results are built upon the finite-time generalization of the thermodynamic uncertainty relation, and require only very short time series data from experiments. We illustrate our results with exact and numerical solutions for two colloidal heat engines.

A fundamental property of non-equilibrium systems is the existence of currents which are fueled by a non-vanishing average rate of total entropy production \( \sigma := \langle \Delta S_{\text{tot}} \rangle / \tau \), where \( \tau \) is the time interval over which we observe the system. An estimate of \( \sigma \) quantifies how much heat is dissipated to the environment on average. More information is available from fluctuations of \( \Delta S_{\text{tot}} \). These are governed by the fluctuation theorems \[1,2\] and can be used for the estimation of free energy differences \[13,14\] or studying the binding energies in single-molecule \[15,16\] or multi-molecular experiments \[17\]. An accurate quantification of the statistics of \( \Delta S_{\text{tot}} \) could also help improve our understanding of the non-equilibrium physics of active matter systems \[18\], biological systems \[19,20\] and nanoscale devices \[21–24\] such as colloidal heat engines \[25,26\].

The main challenge in the thermodynamic characterization of microscopic systems continues to be however, the lack of a general scheme for the measurement and characterization of \( \Delta S_{\text{tot}} \). For systems such as colloidal particles, for which the full dynamical equations are known, \textit{stochastic thermodynamics} provides a framework to quantify \( \Delta S_{\text{tot}} \) from individual trajectories \[11,27,28\]. For more complex systems where not all relevant mesostates are accessible, these direct strategies fail \[29,30\]. The only options are either to perform local calorimetric measurements to directly measure the heat emitted to the bath \[31\] or to come up with a new scheme for inferring \( \sigma \) indirectly.

Recently, for non-equilibrium systems in a steady state, such a scheme for identifying \( \sigma \) has been proposed \[29,32\] using the thermodynamic uncertainty relation \[33,34\]. Using this scheme, a lower bound \( \sigma_L \) for \( \sigma \) can be obtained from the measurement of \( \text{any} \) fluctuating current \( J \), in terms of its mean \( \langle J \rangle \) and variance \( \text{Var}(J) \) as:

\[
\sigma \geq \sigma_L = \frac{2k_B \langle J \rangle^2}{\tau \text{Var}(J)}.
\]

Here, \( k_B \) is the Boltzmann constant. Eq. (1) holds for arbitrary \( \tau \) for non-equilibrium systems in a steady state, and the proof follows from a \( \sigma \)-dependent parabolic bound on the large deviation function \[35\] of \( J \) \[36,37\].

This inference scheme for \( \sigma \) has been shown \[32\] to perform better than more direct methods that use spatial or temporal averages. However, since the uncertainty relation is an inequality, Eq. (1) still only gives a bound for \( \sigma \) even when \( J = \Delta S_{\text{tot}} \). How tight this bound is depends in general on model details and the \( J \) chosen.

As a result, there has been much interest recently on how to choose \( J \) such that the bound value is the tightest \[32,35\]. For \( \tau \to \infty \), it is known that the current \( J \) that gives the best bound is \( J = \Delta S_{\text{tot}} \).

Eq. (1) could be used to predict \( \sigma \) exactly if the equality was to hold. One case when this is known to happen is the equilibrium limit \[33,36,39\] with \( J \) chosen so that \( J = \Delta S_{\text{tot}} \). This means, that for systems working in the close-to-equilibrium/linear response regimes, there is a possibility to estimate \( \sigma \) arbitrarily close to the exact value by using Eq. (1). The equality in Eq. (1) is also met for arbitrary non-equilibrium conditions if along with \( J = \Delta S_{\text{tot}} \), certain conditions are met by the steady state current and probability distributions \[40\]. However it is difficult to tailor models so as to satisfy such conditions. Hence, there is no general scheme available so far for inferring sigma exactly under arbitrary non-equilibrium conditions. In addition no scheme exists, to our knowledge, for inferring fluctuations in \( \Delta S_{\text{tot}} \).

We address precisely these issues in this Letter. Our first central contribution is to provide a new strategy which, in principle, can estimate \( \sigma \) exactly at arbitrary non-equilibrium conditions, by using Eq. (1) in the \( \tau \to 0 \) limit. In this limit, for the current \( J = \Delta S_{\text{tot}} \), it can be shown that the equality condition holds, just as for the equilibrium limit. Using this feature, we show that we can infer \( \sigma \) arbitrarily close to the exact value, by evaluating \( \sigma_L \) for a variety of \( J \) calculated over very short time durations, and then choosing the largest value of \( \sigma_L \) that results. A very important point for this inference scheme to work, is how to define \( J \). To get a value of \( \sigma \) as close as possible to the exact steady-state value, we demonstrate that the time-intensive, boundary contributions to \( J \) play a crucial role. Another point, appealing for experimental studies, is that, because we need to only evaluate the RHS of Eq. (1) over very short trajectories, a single long time-series should give a very good estimate for both \( J \) and \( \text{Var}(J) \). In addition, for very short trajectories, we expect that \( \sigma_L \) will depend quite sensitively on the choice of \( J \). This is advantageous when searching through the space of currents \( J \) to find the highest value for the RHS of Eq. (1). Note, that the value of \( \sigma \) so inferred is then
valid for any time since the system is in a steady state.

Our second contribution is to demonstrate that, by combining the value of $\sigma$ inferred from the previous step and the structure of the large deviation function of arbitrary currents \[\{36, 37, 41\}, we can also infer the distribution of $\Delta S_{tot}$, and as a result all the cumulants, arbitrarily close to their exact values. There by, we also extend the thermodynamic inference problem to inferring the fluctuations of $\Delta S_{tot}$. We illustrate all our findings using exact and numerical solutions for two models of colloidal engines, namely the Brownian gyrator \[22, 42\] as well as the isothermal work-to-work converter engine \[44\].

We begin by considering the uncertainty relation for $J = \Delta S_{tot}$, which reads (setting $k_B = 1$),

$$\frac{\text{Var}(\Delta S_{tot})}{\langle \Delta S_{tot} \rangle} \geq 2. \tag{2}$$

To motivate that this inequality saturates at $\tau \to 0$, we consider the arbitrary time, scaled cumulant generating function (SCGF) $\phi_{\Delta S_{tot}}(\lambda, \tau) \equiv \frac{1}{\tau} \log \langle e^{-\lambda \Delta S_{tot}} \rangle_\tau$. $\phi$ is a convex function by definition \[43\]. For short time durations, when $|\Delta S_{tot}| \ll 1$, we can express $\phi_{\Delta S_{tot}}(\lambda, \tau)$ as a series expansion in terms of the cumulants of $\Delta S_{tot}$.

Then, to the leading order that respects convexity, we get (see the supplemental material \[44\] for more details),

$$\phi_{\Delta S_{tot}}(1, \tau) = 0,$$

and

$$\phi_{\Delta S_{tot}}(1, \tau) \sim -\frac{\lambda}{\tau} \frac{\Delta S_{tot}}{\tau} + \frac{\lambda^2 \text{Var}(\Delta S_{tot})}{2 \tau}.$$ \[3\]

Now applying the integral fluctuation theorem \[12\]:

$$\frac{\text{Var}(\Delta S_{tot})}{\langle \Delta S_{tot} \rangle} \to 2 \quad \text{as} \quad \tau \to 0. \tag{4}$$

A more rigorous proof is provided in \[45\] to the effect that the bound in Eq. \[2\] is always satisfied when $\Delta S_{tot} \to 0$. This happens for the equilibrium limit but $\Delta S_{tot} \to 0$ also when $\tau \to 0$ and hence the same result \[45\] applies. A model which can be solved exactly for the LHS of Eq. \[2\] has also been shown \[39\] to display this behaviour as $\tau \to 0$.

We now demonstrate the usefulness of Eq. \[1\] for inferring $\sigma$ for two non-trivial models of colloidal engines, the Brownian gyrorator model \[22, 42\] and the work-to-work converter engine \[22, 43\], in both of which the working substance is a single colloidal particle. In the first case, the particle is in contact with external reservoirs at hot ($T_1$) and cold ($T_2$) temperatures and in the second case, the particle is subjected to two white-noise forces, interpreted as a load and drive force. The individual time-extensive and -intensive contributions to $\Delta S_{tot}$ can then be written as,

$$\Delta S_{tot} = -\frac{Q_1}{T_1} - \frac{Q_2}{T_2} + \Delta S_{sys},$$

$$= \eta_C \frac{T_2}{T_1} Q_1 + \frac{1}{T_2} W + \Delta S_{int}. \tag{5}$$

FIG. 1. An illustration of the exact estimation of the entropy production rate $\sigma$, using the $\tau \to 0$ limit of Eq. \[1\] for two colloidal engine models in non-equilibrium steady states. \textit{Left}: $\sigma$ inferred as a function of time, for the Brownian gyrorator model, with analytic solutions for $\sigma_L$ in Eq. \[1\]. The black horizontal line corresponds to the actual entropy production rate. The red solid and blue dotted lines correspond to arbitrary currents with and without boundary contributions (see the main text). The results show that the best inference of $\sigma$ is given by $\Delta S_{tot}$ (green dashed line) itself, and that several of the currents having boundary contributions (red solid lines) infer $\sigma$ arbitrarily close to the actual value, in the $\tau \to 0$ limit. \textit{Right}: Inferring $\sigma$, as the maximum of the measured $\sigma_L$'s of $N$ arbitrary currents (Eq. \[3\]), for the isothermal work-to-work converter engine, from numerical simulations. Here the black dashed line corresponds to the actual $\sigma$, obtainable from a large-time computation \[43\]. $\sigma_N$ corresponds to the maximum $\sigma_L$ inferred by the $N$ currents, at $\tau \to 0$. We see that as $N$ increases, $\sigma_N$ saturates to the known value of $\sigma$. The inset shows that the inference procedure makes a large error if the boundary terms are not included.

Here $\eta_C = 1 - \frac{T_2}{T_1}$, is the Carnot efficiency. The second equation in Eq. \[5\] is valid also for the work-to-work converter if $Q_1$ and $W$ are interpreted as work done arising from the driving and loading terms respectively and $\eta_C = 1$ \[44\].

The term $\Delta S_{int} = -\frac{T_2}{T_1} \Delta E + \Delta S_{sys}$ collects the time-intensive contributions to the total entropy production that depend only on the initial and final states of the system. $\Delta E$ denotes the change in internal energy, which is, according to the First Law, $\Delta E = W + Q_1 + Q_2$. Although $\Delta S_{int}$ is a time intensive contribution to $\Delta S_{tot}$, it can significantly fluctuate for infinite state space systems as discussed recently in \[22\], and cannot be neglected. We define an arbitrary current $J$ in the system as the linear combination $J = c_1 \frac{T_2}{T_1} Q_1 + c_2 \frac{1}{T_2} W + c_3 \Delta S_{int}$, where $c_1$, $c_2$ and $c_3$ are random real numbers, taken uniformly from the interval $[-1, 1]$. In particular, when $c_1 = c_2 = c_3 = 1$, we get $J = \Delta S_{tot}$ \[46\]. It is important to note that, for a generic non-equilibrium system, the decomposition of $\Delta S_{tot}$ as given in Eq. \[6\] is usually not straightforward. In such cases, one can generate random currents $J$ from the phase space trajectories of the system \[32, 47\], as discussed later. The results we present here could then be applied to such currents.
temperature, we relegate their detailed description to the supplementary material [44]. The Brownian gyrator can be solved exactly [22] for the full SCGF $\Phi(\lambda, \lambda_w, \lambda_S, \tau) \equiv \frac{1}{2} \log \langle e^{-\frac{\lambda w Q_1 - \lambda W + \lambda S \Delta S_{\text{tot}}}{\tau}} \rangle$, at arbitrary times and hence provides us with the means to check the inference procedure analytically. The second model of the work-to-work converter can only be solved for large times [43]. We hence use it to test our inference scheme in a situation where we can only rely on numerics.

In the left panel of Fig. 1 we compute $\sigma_L$ for the Brownian gyrator, using our analytical solutions (see [44] for more details) for arbitrary currents $J = c_1 \frac{1}{\tau_2} Q_1 + c_2 \frac{1}{\tau_2} W + c_3 \Delta S_{\text{tot}}$ at any time $\tau$. The exact value of $\sigma$ is marked by the black horizontal line. At any time $\tau$, the current which infers $\sigma$ the best is $J = \Delta S_{\text{tot}}$, which is the green dashed curve in the figure. In particular, in the $\tau \to 0$ limit, $\Delta S_{\text{tot}}$ infers $\sigma$ exactly. More interestingly, notice that there are currents which are not necessarily $\Delta S_{\text{tot}}$ which perform almost as good as $\Delta S_{\text{tot}}$, and infer $\sigma$ arbitrarily close to the actual value, in the $\tau \to 0$ limit. The red solid lines correspond to a value of $\sigma_L$ computed from currents for which $c_3 \neq 0$. The blue dotted lines correspond to $\sigma_L$ calculated from currents for which $c_3 = 0$ and hence which are only linear combinations of $Q_1$ and $W$, the time-extensive contributions to $\Delta S_{\text{tot}}$. We find that inference with currents for which $c_3 \neq 0$ gives better results in many cases, particularly at short times. The best inference strategy is therefore to measure the mean and variance of an ensemble of randomly generated currents with boundary contributions at arbitrary short times. Since, the bound in Eq. (1) saturates for $\tau \to 0$, we are guaranteed to obtain a value for $\sigma_L$ arbitrarily close to the actual $\sigma$ as,

$$\sigma = \max_j \left\{ \lim_{\tau \to 0} \sigma_L \right\}. \tag{6}$$

Note that for large $\tau$ all currents, including $J = \Delta S_{\text{tot}}$ give a similar estimate, which is considerably less than the actual value. Hence, the small-time saturation of Eq. (1) as well as its sensitivity to the $J$ chosen, both work in favour of getting a better estimate for $\sigma$ than at large $\tau$. In practice, the $\tau \to 0$ limit may be achieved in experiments by choosing trajectory lengths corresponding to the minimal temporal resolution accessible to the experiment [48, 49].

In the right panel of Fig. 1 we numerically compute $\sigma_L$ for the second model of the work-to-work converter, by computing the mean and variance of different randomly chosen $J$ for very short trajectories and using Eq. (6) to estimate $\sigma$ as the maximum value obtained over all the chosen currents. Since this model can be solved in the steady state [43], we know the exact value of $\sigma$. As can be seen, the inferred value is in very good agreement with the exact value after the inference procedure has been applied to the order of about 20 currents. As in the inset shows, if $c_3 = 0$, no current is able to obtain the exact value of $\sigma$.

To further analyze the inference of $\sigma$ by the finite time inference scheme, Eq. (6), we have identified the optimal currents that infer $\sigma$ the best. These currents have been referred to as hyper-accurate currents recently [35]. In Fig. 2, we illustrate this in the $\tau \to 0$ limit for the Brownian gyrator. When $c_3 \neq 0$ the best inference is given by the $\Delta S_{\text{tot}}$ current itself, as expected. However, for $c_3 = 0$, the best inference is given by some other direction in the $(c_1, c_2)$ plane. When $\tau \neq 0$ also, we can show (see supplemental material), that the optimal current is in general different from $\Delta S_{\text{tot}}$ as recently suggested in [38], and becomes equal to $\Delta S_{\text{tot}}$ only for large $\tau$ [39].

So far, we have shown that the finite time thermodynamic uncertainty relation can be used at very short observational times to infer $\sigma$ arbitrarily close to the actual value, arbitrarily far from equilibrium. It is then natural to ask, if there can be similar inference strategies for the fluctuations of $\Delta S_{\text{tot}}$ as well. Our second result is to show that the steady state distribution of $\Delta S_{\text{tot}}$ can also be obtained to great accuracy, if we have access to an exact estimate of $\sigma$.

We begin by considering the structure of the cumulant generating function of an arbitrary current in the steady state, $\phi_J(\lambda, \tau) \equiv \frac{1}{2} \log \langle e^{-\lambda \frac{Q_1 + W + \lambda S \Delta S_{\text{tot}}}{\tau}} \rangle$ at large $\tau$. Using large deviation techniques, it has been shown recently that $\phi_J$ obeys the bound [36, 37, 41],

$$-\sigma \lambda (1 - \lambda) \leq \phi_{\Delta S_{\text{tot}}}(\lambda, \tau) \leq \phi_J(\lambda, \tau). \tag{7}$$

The uncertainty relation in Eq. (1) can be directly proved from this result [46, 47]. Interestingly, Eq. (7) constrains the fluctuations of $\Delta S_{\text{tot}}$ strongly, by providing both a lower bound and an upper bound for $\phi_{\Delta S_{\text{tot}}}$. In particular, one can saturate the bound $\phi_{\Delta S_{\text{tot}}}(\lambda, \tau) \leq \phi_J(\lambda, \tau)$ if $J = \Delta S_{\text{tot}}$. We therefore get a natural scheme for
We illustrate Eq. (8) and Eq. (9) in Fig. 3 for the isothermal work-to-work converter engine. We have first obtained an analytic expression for the joint SCGF
\[ \Phi(\lambda_Q, \lambda_W, \lambda_S, \tau) = \frac{1}{\tau} \log \left( e^{\lambda_Q n_Q Q_\tau \tau^2} - \lambda_W W \lambda_S S \Delta S_{\text{tot}} \right), \]
which is exact at large but finite times (see the supplemental material [44]). The geometry of \( \Phi \) was recently conjectured and discussed in some detail in [22]. Due to the fluctuation theorem, \( \Phi \) is a reflection symmetric object around the point \( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \), and typically has a limited domain of convergence (cut-offs) that depend on \( \lambda_S \). We illustrate this for a fixed, large value of \( \tau \) in Fig. 3. The SCGF of an arbitrary current \( J \) can be obtained from \( \Phi \) by evaluating it along a straight line passing through the origin and the point \( (c_1, c_2, c_3) \), where \( c_1, c_2, c_3 \) are random numbers. In particular, \( \phi_{\Delta S_{\text{tot}}} \) is \( \Phi \) evaluated along the \((1,1,1)\) direction marked by the red solid line in Fig. 3.

In Fig. 3a and Fig. 3b, we illustrate the inference of \( \phi_{\Delta S_{\text{tot}}} \) using Eq. (8). The blue solid and red dashed curves in Fig. 3a correspond to \( \phi_{\tau J} \) with and without contributions from \( \Delta S_{\text{int}} \). Since \( \phi_{\tau J} \) of currents with \( c_3 = 0 \), can have restricted domains of convergences (see \( \lambda_S = 0 \) plane of Fig. 3a), they will end up inferring a limited domain of \( \phi_{\Delta S_{\text{tot}}} \), as shown in Fig. 3a, with the blue circles. On the other hand, if we apply Eq. (8) to the same currents, with arbitrary boundary contributions \( (c_3 \neq 0) \), we see significant improvement in the estimate of \( \phi_{\Delta S_{\text{tot}}} \), as shown in Fig. 3b, with the red squares. In Fig. 3d, we illustrate the inference of \( \text{Var}(\Delta S_{\text{tot}}) \) using Eq. (9), numerically.

Finally, we would like to discuss how the results presented here can be generalized to non-equilibrium systems, where the decomposition of \( \Delta S_{\text{tot}} \) as given by Eq. (1) is not straightforward [29, 32]. The first step is to identify the relevant dynamical degrees of freedom \( x(t) \) of the system. Arbitrary currents \( J_d \) can be constructed using random vectors \( d(x) \) using the formalism recently elaborated in [32] as,
\[ J_d = \int_{x(0)}^{x(T)} d(x) \ d(x) \ j(x), \]
where, \( j(x) = \frac{1}{T} \int_0^T \delta(x - x(t)) \ dx(t) \), an estimate for the steady state current. The inference scheme we have presented here can then be straightforwardly implemented by using Eq. (6), Eqs. (8) and (9) with \( J = J_d \).

In summary, we have presented here an indirect scheme to exactly infer the average entropy production rate \( \sigma \) as well as the distribution of entropy production \( P(\Delta S_{\text{tot}}) \) in non-equilibrium steady state systems. The scheme for identifying \( \sigma \) is built upon the finite time thermodynamic uncertainty relation [37, 50] and it’s saturation in the very short time limit. The inference of \( P(\Delta S_{\text{tot}}) \) is then built upon an exactly estimated value of \( \sigma \) and the dissipation bounded structure of steady state current fluctuations [36, 41]. We have found that, for this inference scheme to return an accurate estimate, the time intensive contributions to \( \Delta S_{\text{tot}} \), play a crucial role.
It will be very interesting to test this inference scheme in biological systems or active matter systems in the steady state, to infer $\Delta S_{\text{tot}}$, or to indirectly test fluctuation theorems. Generalizations to time symmetrically driven systems could also be interesting.

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Supplemental Material for “Inferring entropy production from short experiments”

SATURATION OF THE THERMODYNAMIC UNCERTAINTY RELATION FOR $\Delta S_{\text{tot}}$ IN THE LIMIT $\tau \to 0$

In this section, we discuss the saturation of the thermodynamic uncertainty relation for $\Delta S_{\text{tot}}$ in the $\tau \to 0$ limit. We begin with considering a generic non-equilibrium system in contact with thermal reservoirs at temperature $T_i$. The total entropy production $\Delta S_{\text{tot}}$ is given by,

$$\Delta S_{\text{tot}} = \sum_i \frac{Q_i}{T_i} + \Delta S_{\text{sys}},$$

where $Q_i$ is the heat dissipated in the $i$-th bath at temperature $T_i$, and $\Delta S_{\text{sys}} = -\log P_{ss}(x) + \log P_{ss}(x_0)$ is the system entropy production [12]. The first term in the above equation is a time-extensive quantity and vanishes in the $\tau \to 0$ limit whereas the last term $\Delta S_{\text{sys}}$, the change in system entropy, is a time-intensive quantity which however also vanishes in the $\tau \to 0$ limit. Thus, the total entropy production $\Delta S_{\text{tot}} \to 0$ for each realization in the limit $\tau \to 0$.

Consider the scaled cumulant generating function $\phi_{\Delta S_{\text{tot}}}(\lambda, \tau) \equiv \frac{1}{\lambda} \log \braket{e^{-\lambda \Delta S_{\text{tot}}}}_{\tau}$. In the limit $\tau \to 0$, we write the series expansion in $\Delta S_{\text{tot}}$ as

$$\phi_{\Delta S_{\text{tot}}}(\lambda, \tau) = -\lambda \frac{\braket{\Delta S_{\text{tot}}}}{\tau} + \lambda^2 \frac{\braket{\Delta S_{\text{tot}}^2}}{2 \tau} - \lambda^3 \frac{\braket{\Delta S_{\text{tot}}^3}}{3! \tau} + \lambda^4 \frac{\braket{\Delta S_{\text{tot}}^4}}{4! \tau} + \ldots,$$

where $\braket{\Delta S_{\text{tot}}^n}$ is $n$-th cumulant of $\Delta S_{\text{tot}}$.

To the leading order,

$$\phi_{\Delta S_{\text{tot}}}(\lambda, \tau) \to -\lambda \frac{\braket{\Delta S_{\text{tot}}}}{\tau} + \lambda^2 \frac{\braket{\Delta S_{\text{tot}}^2}}{2 \tau} \quad \text{as} \quad \tau \to 0.$$  \hspace{1cm} (S2)

Notice that the above approximation preserves the convex nature of $\phi_{\Delta S_{\text{tot}}}(\lambda, \tau)$ [23]. Substituting $\lambda = 1$ in the above equation and invoking the integral fluctuation theorem $\phi_{\Delta S_{\text{tot}}}(1, \tau) = 0$ yields

$$\frac{\text{Var}(\Delta S_{\text{tot}})}{\braket{\Delta S_{\text{tot}}}} \to 2 \quad \text{as} \quad \tau \to 0.$$ \hspace{1cm} (S4)

The above equation gives the saturation of the thermodynamic uncertainty relation Eq. (1) in the $\tau \to 0$ limit for $J = \Delta S_{\text{tot}}$. Similarly, when one considers the next higher order terms in the series [S2], the saturation involving the higher order cumulants translates to the following condition for the moments:

$$\frac{\braket{\Delta S_{\text{tot}}^4}}{4 \braket{\Delta S_{\text{tot}}^3} - 12 \braket{\Delta S_{\text{tot}}^2} + 24 \braket{\Delta S_{\text{tot}}}} \to 1 \quad \text{as} \quad \tau \to 0.$$ \hspace{1cm} (S5)

PROOF OF THE INEQUALITY $M_{\Delta S_{\text{tot}}}^{(n)} \leq M_{J}^{(n)}$

From Eq.(7), we see that the scaled cumulant generating function for the entropy production is bounded by the scaled cumulant generating function of the normalized currents (defined in main text) as

$$\phi_{\Delta S_{\text{tot}}}(\lambda, \tau) \leq \phi_{J}^{\gamma}(\lambda, \tau).$$ \hspace{1cm} (S6)

We expand both sides, and obtain

$$(-\lambda)M_{\Delta S_{\text{tot}}}^{(1)} + (-\lambda)^2 \frac{M_{\Delta S_{\text{tot}}}^{(2)}}{2!} + (-\lambda)^3 \frac{M_{\Delta S_{\text{tot}}}^{(3)}}{3!} + \ldots \leq (-\lambda)M_{J}^{(1)} + (-\lambda)^2 \frac{M_{J}^{(2)}}{2!} + (-\lambda)^3 \frac{M_{J}^{(3)}}{3!} + \ldots$$ \hspace{1cm} (S7)

Notice that the above equation [S7] holds for an arbitrary $\lambda$. Hence, comparing the coefficients of $(-\lambda)^n$, we obtain the bounds on the cumulants of the total entropy production as given by

$$M_{\Delta S_{\text{tot}}}^{(n)} \leq M_{J}^{(n)}$$ \hspace{1cm} (S8)
BROWNIAN GYRATOR MODEL

In this model, we consider a Brownian particle in two dimensions. The particle is coupled to two thermal reservoirs at different temperatures $T_1 > T_2$ acting in the $x_1$ and $x_2$ directions, respectively. Moreover, the particle is confined in a parabolic potential $U(x)$ with stiffnesses $u_1$ and $u_2$ along its principal axes tilted by an angle $\alpha$ with respect to the coordinate axes [22, 42, 51].

$$U(x) = \frac{1}{2} x^T R_\alpha^T u R_\alpha x,$$  \hspace{1cm} (S9)

$$R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},$$  \hspace{1cm} (S10)

$$u = \begin{pmatrix} u_1 \\ 0 \\ 0 \\ u_2 \end{pmatrix}.$$  \hspace{1cm} (S11)

In the above equation, $x = (x_1, x_2)^T$ is the position of the particle at time $t$, and $R_\alpha$ is the rotation matrix. Due to an asymmetry in the thermal and restoring forces (for e.g., $T_1 \neq T_2$, $u_1 \neq u_2$, and $\alpha \neq \pi n/4$, $n \in \mathbb{Z}$), the particle reaches a non-equilibrium stationary state and gyrates about the origin on average [42]. This systematic motion and torque exerted on the medium can be used to extract thermodynamic work from this system by introducing an additional external force [21, 22],

$$f_{ext}(x) = -f_{ext} \epsilon x,$$  \hspace{1cm} (S12)

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In the overdamped limit, the dynamics of the Brownian Gyrator is described by the following equations of motion:

$$\dot{x}(t) = -A x(t) + B \eta(t),$$  \hspace{1cm} (S13)

where

$$A = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$  \hspace{1cm} (S14)

$$B = \begin{pmatrix} \sqrt{2k_B T_1} / \gamma_1 \\ 0 & \sqrt{2k_B T_2} / \gamma_2 \end{pmatrix}.$$  \hspace{1cm} (S15)

where $K = R_\alpha^T u R_\alpha + f_{ext} \epsilon$. Here $\eta_t(t)$ is an independent Gaussian white-noise with $\langle \eta_t(t) \rangle = 0$ and $\langle \eta_t(t) \eta_{t'}(t') \rangle = \delta_{jj} \delta(t - t')$. For a particular range of parameters where the matrix $A$ is positive definite, the system reaches a non-equilibrium steady state with the following probability distribution [51]

$$P_{ss}(x) = \frac{1}{\sqrt{(2\pi)^2 \det \Sigma(\infty)}} \exp \left( -\frac{1}{2} x^T \Sigma^{-1}(\infty) x \right),$$  \hspace{1cm} (S16)

where $\Sigma(\infty)$ is obtained from the solution of

$$A \Sigma(\infty) + \Sigma(\infty) A^T = 2D,$$  \hspace{1cm} (S17)

and the matrix $D = \frac{1}{2} BB^T$.

The work done by the external load force $f_{ext}$ and the heat taken from the hot reservoir in time duration $\tau$ are given by [22]

$$W = \sum_{i,j} \int_0^\tau Y_{ij}^W x_j dx_i,$$  \hspace{1cm} (S18)

$$Q_1 = \sum_{i,j} \int_0^\tau Y_{ij}^Q x_j dx_i$$

where

$$Y^W = -f_{ext} \epsilon,$$  \hspace{1cm} and  \hspace{1cm} $$Y^Q = \begin{pmatrix} K_{11} & K_{12} \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (S19)
In the following, we are interested in the total entropy production \( \Delta S_{\text{tot}} \) in the steady state given as
\[
\Delta S_{\text{tot}} = \frac{\eta_C}{T_2} Q_1 + \frac{W}{T_2} + \Delta S_{\text{int}},
\]  
(S20)
where the time-intensive contribution to the total entropy production \( \Delta S_{\text{int}} \) is given by,
\[
\Delta S_{\text{int}} = -x_0^T Y^0 x_0 + x_0^T Y^0 x_\tau,
\]
\[
Y^0 = Y^\tau = \frac{R_a^T u R_a}{2TC} - \frac{\Sigma^{-1}(\infty)}{2}.
\]  
(S21)
Using the path integral formalism [52], the moment generating function (MGF) for \( Q_1, W \) and \( \Delta S_{\text{int}} \) at any arbitrary time can be obtained as
\[
\Psi(\lambda_Q, \lambda_W, \lambda_S, \tau) = \left\langle e^{-\lambda_Q Q_1 - \lambda_W W - \lambda_S \Delta S_{\text{int}}} \right\rangle_{\tau} 
= \int dx_0 P_{\text{ex}}(x_0) \int dx_\tau \int_{x_0}^{x_\tau} Dx[\cdot] P[x[\cdot]] e^{-\lambda_Q Q_1 - \lambda_W W - \lambda_S \Delta S_{\text{int}}},
\]  
(S22)
where
\[
P[x[\cdot]] \propto \exp \left( -\int_0^\tau dt \left[ \dot{x}(t) + A x(t) \right] \right) \frac{1}{2D} \left[ \dot{x}(t) + A x(t) \right]
\]  
(S23)
is the Onsager-Machlup weight of the path [53, 54]. Since all terms in the exponent of the RHS of Eq. (S22) are quadratic in \( x_1, x_2 \) and in their derivatives, we rewrite Eq. (S22) as
\[
\left\langle e^{-\lambda_Q Q_1 - \lambda_W W - \lambda_S \Delta S_{\text{int}}} \right\rangle_{\tau} = \int dx_0 \int dx_\tau \int_{x_0}^{x_\tau} Dx[\cdot] \exp \left( -\int_0^\tau x(t) \hat{O}_{\lambda_Q, \lambda_W, \lambda_S} x(t) + \text{Boundary terms} \right)
\]  
(S24)

\[
= \sqrt{\frac{\det \hat{O}_{0,0,0}}{\det \hat{O}_{\lambda_Q, \lambda_W, \lambda_S}}},
\]  
(S25)
Here the operator \( \hat{O} \) is a matrix whose elements are differential operators [56] and functional determinants. In our case, it can be shown that the matrix \( \hat{O} \) has the following form
\[
\hat{O} = \begin{bmatrix}
a & b e^{\frac{d}{dt}} + d e^{\frac{d^2}{dt^2}} + f e^{\frac{d^3}{dt^3}} \\
b e^{\frac{d}{dt}} + d e^{\frac{d^2}{dt^2}} + f e^{\frac{d^3}{dt^3}} & c e^{\frac{d}{dt}} + d e^{\frac{d^2}{dt^2}} + f e^{\frac{d^3}{dt^3}}
\end{bmatrix},
\]  
(S26)
where
\[
a = \frac{1}{4D_{11}},
\]
b = \[
\frac{1}{2} \left( \frac{A_{11}^2}{2D_{11}} + \frac{A_{12}^2}{2D_{22}} \right),
\]
c = \[
\frac{1}{2} \left( -\frac{A_{12}}{2D_{11}} + \frac{A_{21}}{2D_{22}} \right) - \lambda_Q \frac{A_{12}}{2} + \lambda_W f_{\text{ext}},
\]
d = \[
\frac{1}{2} \left( \frac{A_{11}A_{12}}{2D_{11}} + \frac{A_{21}A_{22}}{2D_{22}} \right),
\]
e = \[
\frac{1}{4D_{22}},
\]
f = \[
\frac{1}{2} \left( \frac{A_{12}^2}{2D_{11}} + \frac{A_{22}^2}{2D_{22}} \right).
\]  
(S27)
The ratio of determinants in Eq. (S25) can be computed using a technique based on spectral-\( \xi \) functions of Sturm-Liouville type operators as described in [56] and can be obtained in terms of a characteristic polynomial function \( F \) as
\[
\left\langle e^{-\lambda_Q Q_1 - \lambda_W W - \lambda_S \Delta S_{\text{int}}} \right\rangle_{\tau} = \sqrt{\frac{F_{0,0,0}(0)}{F_{\lambda_Q, \lambda_W, \lambda_S}(0)}},
\]  
(S28)
where \( F_{\lambda_1, \lambda_2, \lambda_3} = \text{det} \{ M + NH(\tau) \} \) in which \( H \) is a matrix of independent and suitably normalized fundamental solutions \( x^1(t), \ldots, x^4(t) \) of the homogeneous equation \( \hat{\Omega} x = 0 \):

\[
H(t) = \begin{bmatrix}
    x^1_1(t) & x^1_2(t) & x^1_3(t) & x^1_4(t) \\
    \dot{x}^1_1(t) & \dot{x}^1_2(t) & \dot{x}^1_3(t) & \dot{x}^1_4(t) \\
    x^2_1(t) & x^2_2(t) & x^2_3(t) & x^2_4(t) \\
    \dot{x}^2_1(t) & \dot{x}^2_2(t) & \dot{x}^2_3(t) & \dot{x}^2_4(t)
\end{bmatrix}, \quad \text{and} \quad H(0) = I_4,
\]

(S29)

and \( M \) and \( N \) contain the information about the boundary conditions from Eq. \( (S24) \), and they satisfy

\[
M \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = 0 \quad \text{and} \quad N \begin{bmatrix} x(\tau) \\ \dot{x}(\tau) \end{bmatrix} = 0.
\]

(S30)

We stress that the expression given in Eq. \( (S28) \) is valid within the domain \( C_{\lambda_1, \lambda_2, \lambda_3} \) for which the operator \( \hat{\Omega} \) doesn’t have negative eigenvalues. The MGF is not convergent outside this domain.

In our problem, we obtain the four independent solutions of \( \hat{\Omega} x = 0 \) as

\[
x^1_1(t) = \exp \left( \pm \frac{\sqrt{t}}{\sqrt{2}} \right) \sqrt{\frac{\pm \sqrt{\alpha^2 f^2 - 2abc f + 2ad^2 e + b^2 e^2 - 2bc^2 e + c^4}}{ae + b^2 - \frac{c^2}{ae} + \frac{f}{e}}} \right)
\]

(S31)

\[
x^2_1(t) = \frac{x^1_1(t) \left( (c^2 d - a d f) + c(a f - c^2) x^1_1'(t) - a c e x^1_1''(t) - a d e x^1_1'''(t)(t) \right)}{bc^2 - ad^2 - 2}
\]

(S32)

The matrices \( M \) and \( N \) are given by

\[
M = \begin{pmatrix}
    -2D_1\lambda_1A_1 + A_{11} - 2D_1\lambda_2A_2 - 2D_1\lambda_3A_3 + \lambda_1 - 2\Sigma_{3j}^1 T_2 u_4 \cos(2\Omega_2) u_4 + u_4 \cos(2\Omega_2) - u_4 \cos(2\Omega_2) + u_4 \\
    -2D_1\lambda_1A_1 + A_{11} - 2D_1\lambda_2A_2 - 2D_1\lambda_3A_3 + \lambda_1 - 2\Sigma_{3j}^1 T_2 u_4 \cos(2\Omega_2) u_4 + u_4 \cos(2\Omega_2) - u_4 \cos(2\Omega_2) + u_4 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

(S33)

\[
N = \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

(S34)

Now the scaled cumulant generating function \( \phi(\lambda_1, \lambda_2, \lambda_3, \tau) \equiv \frac{1}{\tau} \left( e^{-\lambda_1 Q_1 - \lambda_2 W - \lambda_3 S_{\text{tot}}} \right) \) can be computed for arbitrary values of \( \tau \), using Eq. \( (S28) \). For explicit parameter choices, the first few cumulants of arbitrary currents can be straightforwardly computed. For the parameter choice: \( f_{\text{ext}} = -1, u_1 = 4, u_2 = 2, \gamma_1 = 1, \gamma_2 = 1, T_1 = 2, T_2 = 6, \alpha = \frac{\pi}{4}, k_B = 1 \), we get,

\[
\sigma = 2,
\]

\[
\sigma^2 \tau(\Delta S_{\text{tot}}) = \frac{2\pi e^\tau \sinh(2\pi e^\tau)}{2\pi e^\tau + 1}
\]

(S35)

It is this expression that is plotted as red solid lines (\( (c_1, c_2, c_3) \in [-1,1] \)), blue dotted lines (\( c_1 = c_2 = c_3 = 1 \)) and green dashed line (\( c_1 = c_2 = c_3 = 0 \)) in Fig. 1a. It is also this expression, in the \( \tau \to 0 \) limit, that is plotted in Fig. 2, as contour plots for fixed values of \( c_3 \). Similarly, we show the contour plots for \( \lim_{\tau \to 0} \sigma^2 \tau \) in the absence of boundary contribution (i.e., \( c_3 \to 0 \)) in Fig. 3. We see that the currents along the diagonal direction are not the best current to infer \( \sigma \).

**Optimal currents**

Using \( \sigma^2 \tau(\Delta S_{\text{tot}}) \) given in \( (S35) \), it is possible to identify the coefficients \( \{ c_1^*, c_2^*, c_3^* \} = \{ 1, 0, 0 \} \) (see left panel in Fig. \( (S2) \) that maximize it for a given time \( \tau \). For \( \tau \gg 1 \) as well as \( \tau \to 0 \), \( \Delta S_{\text{tot}} \) becomes the optimal current since \( c_1^* = c_2^* = c_3^* = 1 \). In the right panel of Fig. \( (S2) \), we plot \( \sigma^2 \tau(\Delta S_{\text{tot}}) \) as a function of \( \tau \) and compare the inference of \( \sigma \) using \( \sigma^2 \tau(\Delta S_{\text{tot}}) \) with \( c_1^*, c_2^*, c_3^* \) (blue dashed line) with that of \( c_1 = c_2 = c_3 = 1 \) (\( \Delta S_{\text{tot}} \)). It is clear that there exist certain values of \( \Delta S_{\text{tot}} \) for which one can infer \( \sigma \) more accurately using currents other than \( \Delta S_{\text{tot}} \).
FIG. S1. Contour plot for $\sigma_L$ in the limit $\tau \to 0$ at $c_3 = 0$ (absence of time-intensive contributions). The black dashed line is the diagonal direction. It is clear that the optimal currents with coefficients $c_1$ and $c_2$ along the diagonal direction are not the best currents to infer the actual entropy production $\sigma$.

FIG. S2. Left: $c_1^*, c_2^*$ and $c_3^*$ which maximize $\sigma_L(\tau)$ as a function of $\tau$. We see that $c_1 = c_2 = c_3 = 1$ at $\tau \to 0$ as well as $\tau \gg 1$ implying that the optimal current is equal to $\Delta S_{\text{tot}}$ (see main text). Right: $\sigma_L$ with coefficients $c_1^*, c_2^*$ and $c_3^*$ (blue dashed line) is more accurate to infer the $\sigma$ than when $c_1 = c_2 = c_3 = 1$ (red solid line) at intermediate times $0 < \tau \sim 1$.

**ISOTHERMAL WORK-TO-WORK CONVERTER ENGINE**

We consider a stochastic engine composed of a single Brownian particle coupled to a heat bath at temperature $T$. The particle is driven out of equilibrium using two stochastic external Gaussian noises $f_1$ (load force) and $f_2$ (drive force). The system evolves according to the following underdamped dynamics

$$m\dot{v} = \gamma v + \eta(t) + f_1(t) + f_2(t),$$

(S37)

where $m$ is the mass of the particle and $\gamma$ the dissipation constant. The thermal $\eta(t)$ and external noises $f_i(t)$ have mean zero and variances $\langle \eta(t)\eta(t') \rangle = 2\gamma k_B T \delta(t-t')$, $\langle f_i(t)f_j(t') \rangle = 2A_i \delta(t-t')$, where $A_1 = \theta \gamma k_B T$, and $A_2 = \alpha^2 A_1$. Moreover, these noises are independent of each other. For convenience, we set the Boltzmann’s constant $k_B = 1$.

The observable we are interested in is the total entropy production in the non-equilibrium steady state:

$$\Delta S_{\text{tot}} = \Delta S_{\text{sys}} + \Delta S_{\text{med}},$$

(S38)

where $\Delta S_{\text{sys}}$ and $\Delta S_{\text{med}}$, respectively, are the system and medium entropy productions observed over a time $\tau$.

Identifying $\Delta S_{\text{sys}}$ and $\Delta S_{\text{med}}$, we write the total entropy production as

$$\Delta S_{\text{tot}} = W_1 + W_2 + \Delta S_{\text{int}},$$

(S39)

where $W_i = 1/T \int_0^T dt f_i(t) v(t)$ is the (dimensionless) work done, and $\Delta S_{\text{int}} = - \log P_{ss}(v) + \log P_{ss}(v_0) - \Delta E/T$ is the time-intensive contribution to the total entropy production.
One can write the joint characteristic function for $W_1$, $W_2$, and $\Delta S_{int}$

$$Z(\lambda_1, \lambda_2, \lambda_3) = \langle e^{-\lambda_1 W_1 - \lambda_2 W_2 - \lambda_3 \Delta S_{int}} \rangle_\tau.$$  \hfill (S40)

Computation of the above characteristic function in the large time limit ($\tau \gg \tau_\gamma$) yields

$$Z(\lambda_1, \lambda_2, \lambda_3) \approx g(\lambda_1, \lambda_2, \lambda_3) e^{(\tau/\tau_\gamma) \mu(\lambda_1, \lambda_2)},$$  \hfill (S41)

where

$$\mu(\lambda_1, \lambda_2) = \frac{1}{2} [1 - \nu(\lambda_1, \lambda_2)],$$  \hfill (S42)

$$g(\lambda_1, \lambda_2, \lambda_3) = \frac{2 \sqrt{\nu(\lambda_1, \lambda_2)}}{\sqrt{\nu(\lambda_1, \lambda_2) + 2\theta (\alpha^2 \lambda_2 - (\alpha^2 + 1) \lambda_3 + \lambda_1) + 1 \sqrt{\nu(\lambda_1, \lambda_2) + 2\theta (\alpha^2 (\lambda_3 - \lambda_2) - \lambda_1 + \lambda_3) + 1}}}.  \hfill (S43)$$

In the above equations, $\nu(\lambda_1, \lambda_2)$ is given by

$$\nu(\lambda_1, \lambda_2) = \sqrt{1 + 4\theta [\lambda_1 (1 - \lambda_1)] + \alpha^2 \lambda_2 (1 - \lambda_2) - \alpha^2 \theta (\lambda_1 - \lambda_2)^2}$$ \hfill (S44)\hfill (S44)

Therefore, the scaled cumulant generating function is

$$\Phi(\lambda_1, \lambda_2, \lambda_3, \tau) \equiv \frac{1}{\tau} \log Z(\lambda_1, \lambda_2, \lambda_3).$$  \hfill (S45)

**MODEL PARAMETERS**

*Brownian Gyrator*

Fig. 1a, Fig. 2: $f_{ext} = -1$, $u_1 = 4$, $u_2 = 2$, $\gamma_1 = 1$, $\gamma_2 = 1$, $T_1 = 2$, $T_2 = 6$, $\alpha = \frac{\pi}{4}$, $k_B = 1$.

*Isothermal work-to-work converter*

Fig 1b: $\theta = 0.5$, $\alpha = 1$, $\tau = 0.01$, $\tau_\gamma = 1$.
Fig 3a,b,c: $\theta = 10$, $\alpha = \frac{1}{4}$, $\tau = 1000$, $\tau_\gamma = 1$.
Fig 3d: $\theta = 0.5$, $\alpha = 1$, $\tau = 1$, $\tau_\gamma = 1$. 