MAXIMAL MOMENTS AND UNIFORM MODULUS OF CONTINUITY FOR STABLE RANDOM FIELDS

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Dedicated to the memory of Professor Wenbo Li

Abstract. Based on the seminal works of Rosiński (1995, 2000) and Samorodnitsky (2004a), we solve an open problem mentioned in a paper of Xiao (2010) and provide sharp results on the rate of growth of maximal moments for many stationary symmetric stable random fields. We also investigate the relationship between this rate of growth and the path regularity properties of self-similar stable random fields with stationary increments, and establish uniform modulus of continuity of such fields.

1. Introduction

A real-valued stochastic process \( \{X(t) : t \in \mathbb{T}^d\} \) (\( \mathbb{T} = \mathbb{Z} \) or \( [0,1] \) or \( \mathbb{R} \)) is called a symmetric \( \alpha \)-stable (S\( \alpha \)S) random field if each of its finite linear combination follows a S\( \alpha \)S distribution. In general, the parameter \( \alpha \) satisfies \( 0 < \alpha \leq 2 \), although in this paper, we assume our random fields to be non-Gaussian and therefore \( 0 < \alpha < 2 \). See, for example, Samorodnitsky and Taqqu (1994) for detailed discussions on non-Gaussian stable distributions and processes.

Sample path continuity and Hölder regularity of stochastic processes and random fields have been studied for many years. The main tool behind such investigation has been a powerful chaining argument mainly applicable to Gaussian and other light-tailed processes; see Adler and Taylor (2007), Khoshnevisan (2002), Marcus and Rosen (2006), Talagrand (2006). Recently, there has been a significant interest in establishing uniform modulus of continuity of sample paths for stable and other non-Gaussian infinitely divisible processes; see, for instance, Ayache et al. (2009), Bierné and Lacaux (2009, 2015), Xiao (2010).

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Motivated by Kôno and Maejima (1991), Xiao (2010) modified the existing chaining arguments and made it amenable to heavy-tailed random fields. This technique uses estimates of the lower order moments of the maximum increments over the two consecutive steps of the chain to obtain a uniform modulus of continuity for stable and other heavy-tailed random fields. In this context, it was stated in Xiao (2010) that for a stationary $\alpha$-stable sequence $\{\xi_k : k \geq 1\}$, it is an open problem to give sharp upper and lower bounds for the maximal moment sequence $E(\max_{1 \leq k \leq n} |\xi_k|^\gamma)$ for $\gamma \in (0, \alpha)$.

Borrowing techniques from Samorodnitsky (2004a, b), we have solved the aforementioned open problem for stationary $\alpha$S discrete-parameter random fields having various generic dependence structures based on ergodic and group theoretic properties of the underlying nonsingular group action. Our work easily extends to the continuous parameter case provided the random field is measurable and stationary. Solution to this open problem in the discrete-parameter case allows us to prove results on uniform modulus of continuity for a large class of self-similar $\alpha$S random fields with stationary increments.

Based on the ergodic theoretic properties of the underlying nonsingular group action, Samorodnitsky (2004a) obtained a phase transition boundary for the partial maxima of stable processes. It was also conjectured in this work that many other important phase transitions for stable processes should occur at this boundary. While this conjecture has indeed been established for extremal point processes (see Resnick and Samorodnitsky (2004), Roy (2010a)), ruin probabilities (see Mikosch and Samorodnitsky (2000)), large deviations issues (see Fasen and Roy (2016)), etc., the effects of this transition boundary on path properties have not yet been explored. In this work, we bridge this gap and establish that the uniform modulus of continuity does change significantly at this boundary (see Section 4) and also at the group theoretic boundaries obtained by Roy and Samorodnitsky (2008).

This paper is organized as follows. In Section 2, we recall the result in Xiao (2010), explain how it naturally leads to an open problem in extreme value theory, and describe the ergodic-theoretic and group-theoretic connections to this problem. Section 3 deals with the asymptotic behavior of the maximal moments of stationary $\alpha$S random fields as the index parameter runs over $d$-dimensional hypercubes of increasing edge-length. In Section 4, we establish results on uniform modulus of continuity for self-similar $\alpha$S random fields whose first order increments are stationary. Finally, two examples of fractional stable processes are discussed in Section 5.

Throughout this paper, we will use $K$ to denote a positive and finite constant which may differ in each occurrence. Some specific constants will be denoted by $K_1, K_2, \ldots$. For two sequences of real numbers $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ the notation $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$. For $u, v \in \mathbb{R}^d$, $u = (u_1, u_2, \ldots, u_d) \leq v = (v_1, v_2, \ldots, v_d)$ means $u_i \leq v_i$ for all $i = 1, 2, \ldots, d$. The vectors $0 = (0, 0, \ldots, 0)$, $1 = (1, 1, \ldots, 1)$ are elements of $\mathbb{Z}^d$. We shall abuse the notation and use $[u, v]$ to denote the set $\{t \in \mathbb{Z}^d :$
u \leq t \leq v\} or the set \{t \in \mathbb{R}^d : u \leq t \leq v\} depending on the context. For some standard Borel space \((S, \mathcal{S})\) with a \(\sigma\)-finite measure \(\mu\) on it, we define the space \(L^\alpha(S, \mu) := \{f : S \to \mathbb{R} \text{ measurable} : \|f\|_\alpha < \infty\}\), where \(\|f\|_\alpha := \left(\int_S |f(s)|^\alpha \mu(ds)\right)^{1/\alpha}\). For two random variables \(Y, Z\), we write \(Y \overset{\mathcal{L}}{=} Z\) if \(Y\) and \(Z\) are identically distributed. For two stochastic processes \(\{Y_t\}_{t \in T}\) and \(\{Z_t\}_{t \in T}\), the notation \(\{Y_t\} \overset{\mathcal{L}}{=} \{Z_t\}\) (or simply \(Y_t \overset{\mathcal{L}}{=} Z_t\)) means that they have same finite-dimensional distributions.

2. Preliminaries

We start with a brief description of the main result in Xiao (2010), which is built upon a modification of the chaining arguments used in the proofs of Kolmogorov’s continuity theorem, Dudley’s entropy theorem and other results on path regularity properties in the light-tailed situations; see Adler and Taylor (2007), Khoshnevisan (2002), Marcus and Rosen (2006), Talagrand (2006). To this end, let \(\{X(t) : t \in T\}\) be a random field indexed by a compact metric space \((T, \rho)\), and let \(\{D_n : n \geq 1\}\) be a sequence (which is called a chaining sequence) of finite subsets of \(T\) satisfying the following conditions:

1. There exists a positive integer \(\kappa_0\) depending only on \((T, \rho)\) such that for every \(\tau_n \in D_n\), the set
   \[
   O_{n-1}(\tau_n) := \{\tau'_{n-1} \in D_{n-1} : \rho(\tau_n, \tau'_{n-1}) \leq 2^{-n}\}
   \]
   has at most \(\kappa_0\) many elements.
2. (The Chaining Property) For every \(s, t \in T\) with \(\rho(s, t) \leq 2^{-n}\), there exist two sequences \(\{\tau_p(s) : p \geq n\}\) and \(\{\tau_p(t) : p \geq n\}\) such that \(\tau_n(s) = \tau_n(t)\) and, for every \(p \geq n\), \(\tau_p(s), \tau_p(t) \in D_p, \rho(\tau_p(s), s) \leq 2^{-p}, \rho(\tau_p(t), t) \leq 2^{-p}\), and \(\tau_p(s) \in O_p(\tau_{p+1}(s)), \tau_p(t) \in O_p(\tau_{p+1}(t))\).

   If \(s \in D := \bigcup_{k=1}^\infty D_k\) (if \(t \in D\)), then there exists an integer \(q \geq n\) such that \(\tau_p(s) = s\) (\(\tau_p(t) = t\), resp.) for all \(p \geq q\).

Note that Condition (2) yields immediately that for each \(n\), \(T\) can be covered by open balls with radius \(2^{-n}\) and centers in \(D_n\), and the set \(\cup_{n \geq 1} D_n\) is dense in \(T\). The following result from Xiao (2010) provides an upper bound for the uniform modulus of continuity for a class of random fields, including those with heavy-tailed distributions.

**Proposition 2.1.** Let \(X = \{X(t) : t \in T\}\) be a real-valued random field indexed by a compact metric space \((T, \rho)\) and let \(\{D_n : n \geq 1\}\) be a chaining sequence satisfying Conditions (1) and (2) above. Suppose \(\sigma : \mathbb{R}_+ \to \mathbb{R}_+\) is a nondecreasing continuous function which is regularly varying at the origin with index \(\delta > 0\) (i.e., \(\lim_{h \to 0+} \sigma(ch)/\sigma(h) = c^\delta\) for all \(c > 0\)). If there are constants \(\gamma > 0\), and \(K > 0\) such that

\[
(2.1) \quad \mathbb{E} \left( \max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in D_{n-1}(\tau_n)} |X(\tau_n) - X(\tau'_{n-1})|^\gamma \right) \leq K (\sigma(2^{-n}))^\gamma
\]
for all integers $n \geq 1$, then for all $\epsilon > 0,$

\begin{equation}
\lim_{h \to 0+} \frac{\sup_{t \in T} \sup_{\rho(s,t) \leq h} |X(t) - X(s)|}{\sigma(h)(\log 1/h)^{1+\epsilon}/\gamma} = 0
\end{equation}

almost surely.

In this paper we will focus on studying the maximal moments of SoS random fields indexed by $\mathbb{Z}^d$ or $\mathbb{R}^d$ so that we can apply Proposition 2.1 to self-similar SoS random fields with stationary increments. Recall that a random field $\{X(t)\}_{t \in \mathbb{R}^d}$ is called $H$-self-similar ($H > 0$) if $\{X(ct)\}_{t \in \mathbb{R}^d} \overset{\mathcal{L}}{=} \{c^H X(t)\}_{t \in \mathbb{R}^d}$ for all $c > 0$. $\{X(t)\}_{t \in \mathbb{R}^d}$ is said to have stationary increments if, $\{X(t + u) - X(u)\}_{t \in \mathbb{R}^d} \overset{\mathcal{L}}{=} \{X(t) - X(0)\}_{t \in \mathbb{R}^d}$, for each $u \in \mathbb{R}^d$.

Now we take $T = [0, 1]^d$ with $\rho(s, t) = \max_{1 \leq i \leq d} |s_i - t_i|$, $D_n = \{2^{-n} u : u \in [1, 2^n 1] \cap \mathbb{Z}^d\}$ and apply Proposition 2.1 above to a self-similar SoS random field $\{X(t)\}_{t \in \mathbb{R}^d}$ whose first order increments are stationary. Using the self-similarity of $\{X(t)\}_{t \in \mathbb{R}^d}$ it follows (see the proof of Theorem 4.1 below) that for all $\gamma \in (0, \alpha \land 1)$ and for all $n \geq 1$,

\begin{equation}
\mathbb{E} \left( \max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau'_{n-1})|^\gamma \right) \\
\leq 2^{-nH\gamma} \sum_{v \in V} \mathbb{E} \left( \max_{t \in [1, 2^n 1] \cap \mathbb{Z}^d} |Y^{(v)}(t)|^\gamma \right),
\end{equation}

where $Y^{(v)} = \{Y^{(v)}(t)\}_{t \in \mathbb{Z}^d}$ is the discrete-parameter increment field defined by

$Y^{(v)}(t) = X(t + v) - X(t), \ t \in \mathbb{Z}^d$

in the direction $v \in V := \{-1, 0, 1\}^d \setminus \{0\}$.

The crucial observation is that due to the stationarity of the increments, each discrete-parameter field $Y^{(v)}$ is stationary. Therefore, in order to estimate the quantity in (2.3), it suffices to establish sharp upper bounds on

\begin{equation}
\mathbb{E} \left( \max_{t \in [0, (2^n - 1)1] \cap \mathbb{Z}^d} |Y(t)|^\gamma \right),
\end{equation}

where $Y = \{Y(t)\}_{t \in \mathbb{Z}^d}$ is a stationary SoS random field, $n \geq 1$ and $\gamma \in (0, \alpha \land 1)$. This translates an investigation of sample path regularity properties into an extreme value theoretic question. Along this direction, some partial results were obtained in Xiao (2010) which are applicable to stable random fields with certain specific dependence structures. In this work, we have improved upon these results and computed the exact rate of growth of the maximal moment sequence (2.4) for a large class of stationary SoS random fields, thus solving an open problem in Xiao (2010) (see pages 173-174 therein). The main tools used in our solution are ergodic-theoretic and algebraic in nature as described below.
It was established by Rosiński (1995, 2000) that every stationary SoS random field $Y = \{Y_t\}_{t \in \mathbb{Z}^d}$ has an integral representation of the form

$$
Y_t \overset{d}{=} \int_S c_t(s) \left( \frac{d\mu \circ \phi_t}{d\mu} (s) \right)^{1/\alpha} f \circ \phi_t(s) M(ds), \quad t \in \mathbb{Z}^d,
$$

where $M$ is a SoS random measure on some standard Borel space $(S, \mathcal{S})$ with $\sigma$-finite control measure $\mu$, $f \in L^\alpha(S, \mu)$, $\{\phi_t\}_{t \in \mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on $(S, \mathcal{S}, \mu)$ (i.e., each $\phi_t : S \to S$ is a measurable map, $\phi_0$ is the identity map on $S$, $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in \mathbb{Z}^d$ and each $\mu \circ \phi_t$ is equivalent to $\mu$), and $\{c_t\}_{t \in \mathbb{Z}^d}$ is a measurable cocycle for $\{\phi_t\}$ (i.e., each $c_t$ is a $\{\pm 1\}$-valued measurable map defined on $S$ satisfying $c_{u+v}(s) = c_u(\phi_v(s))c_v(s)$ for all $u, v \in \mathbb{Z}^d$ and for all $s \in S$). See, for example, Aaronson (1997), Krengel (1985), Varadarajan (1970) and Zimmer (1984) for discussions on nonsingular (also known as quasi-invariant) group actions.

The Rosiński Representation (2.5) is very useful in determining various probabilistic properties of $Y$; see, for example, Mikosch and Samorodnitsky (2000), Samorodnitsky (2004a,b), Resnick and Samorodnitsky (2004), Samorodnitsky (2005), Roy and Samorodnitsky (2008), Roy (2010a,b), Wang et al. (2013), Chakrabarty and Roy (2013), Fasen and Roy (2016). In this work, we shall focus on estimating the maximal moment in (2.4) and its connection to uniform modulus of continuity of SoS random fields. We say that a stationary SoS random field $\{Y_t\}_{t \in \mathbb{Z}^d}$ is generated by a nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}$ on $(S, \mu)$ if it has an integral representation of the form (2.5) satisfying the full support condition $\bigcup_{t \in \mathbb{Z}^d} \text{Support}(f \circ \phi_t) = S$, which will be assumed without loss of generality.

A measurable set $W \subseteq S$ is called a wandering set for the nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}_{t \in \mathbb{Z}^d}$ if $\{\phi_t(W) : t \in \mathbb{Z}^d\}$ is a pairwise disjoint collection. The set $S$ can be decomposed into two disjoint and invariant parts as follows: $S = C \cup D$, where $D = \bigcup_{t \in \mathbb{Z}^d} \phi_t(W^*)$ for some wandering set $W^* \subseteq S$, and $C$ has no wandering subset of positive $\mu$-measure; see Aaronson (1997) and Krengel (1985). This decomposition is called the Hopf decomposition, and the sets $C$ and $D$ are called conservative and dissipative parts (of $\{\phi_t\}_{t \in \mathbb{Z}^d}$), respectively. The action is called conservative if $S = C$ and dissipative if $S = D$.

Denote by $f_t(s)$ the family of functions on $S$ in the representation (2.5):

$$
f_t(s) = c_t(s) \left( \frac{d\mu \circ \phi_t}{d\mu} (s) \right)^{1/\alpha} f \circ \phi_t(s), \quad t \in \mathbb{Z}^d.
$$

The Hopf decomposition of $\{\phi_t\}_{t \in \mathbb{Z}^d}$ induces the following unique (in law) decomposition of the random field $Y$

$$
Y_t \overset{d}{=} \int_C f_t(s) M(ds) + \int_D f_t(s) M(ds) := Y_t^C + Y_t^D, \quad t \in \mathbb{Z}^d,
$$

where the two random fields $Y^C$ and $Y^D$ are independent and are generated by conservative and dissipative $\mathbb{Z}^d$-actions, respectively; see Rosiński (1995,
2000), and Roy and Samorodnitsky (2008). This decomposition reduces the study of stationary S\(\alpha\)S random fields to that of the ones generated by conservative and dissipative actions.

It was argued by Samorodnitsky (2004a) (see also Roy and Samorodnitsky (2008)) that stationary S\(\alpha\)S random fields generated by conservative actions have longer memory than those generated by dissipative actions and therefore, the following dichotomy were observed:

\[
\max_{\|t\|_\infty \leq n} |Y_t| \Rightarrow \begin{cases} 
  c_Y Z_\alpha, & \text{if } Y \text{ is generated by a dissipative action}, \\
  0, & \text{if } Y \text{ is generated by a conservative action}
\end{cases}
\]

as \(n \to \infty\). Here \(Z_\alpha\) is a standard Frechet type extreme value random variable with distribution function

\[
P(Z_\alpha \leq x) = e^{-x^{-\alpha}}, \quad x > 0,
\]

and \(c_Y\) is a positive constant depending on the random field \(Y\). In fact, this is closely tied with the limit of the deterministic sequence

\[
\{b_n\}_{n \geq 1} = \left\{ \left( \int_0^\infty \max_{0 \leq t \leq (n-1)} |f_t(s)|^\alpha \mu(ds) \right)^{1/\alpha} \right\}_{n \geq 1},
\]

which has been proved by Samorodnitsky (2004a), Roy and Samorodnitsky (2008) to satisfy

\[
n^{-d/\alpha} b_n \to \begin{cases} 
  \tilde{c}_Y & \text{if action is dissipative}, \\
  0 & \text{if action is conservative},
\end{cases}
\]

where \(\tilde{c}_Y\) is a positive constant. For conservative actions, the actual rate of growth of the partial maxima sequence \(M_n\) depends on further properties of the action as investigated in Roy and Samorodnitsky (2008), The work above hinges on some group theoretic preliminaries, as discussed briefly below. Let

\[
A = \{ \phi_t : t \in \mathbb{Z}^d \}
\]

be a subgroup of the group of invertible nonsingular transformations on \((S, \mu)\) and define a group homomorphism, \(\Phi : \mathbb{Z}^d \to A\) by \(\Phi(t) = \phi_t\) for all \(t \in \mathbb{Z}^d\). Let,

\[
K = \text{Ker}(\Phi) = \{ t \in \mathbb{Z}^d : \phi_t = 1_S \},
\]

where \(1_S\) denotes the identity map on \(S\). Then \(K\) is a free abelian group and by the first isomorphism theorem of groups, we have

\[
A \cong \mathbb{Z}^d / K.
\]

Now, by the structure theorem of finitely generated abelian groups (see, for example, Theorem 8.5 in Chapter I of Lang (2002)), we get,

\[
A = F \oplus \bar{N},
\]

where \(F\) is a free abelian group and \(\bar{N}\) is a finite group. Assume \(\text{rank}(\bar{F}) = p \geq 1\) and \(|\bar{N}| = l\). Since, \(\bar{F}\) is free abelian, there exists an injective group
homomorphism,
\[ \Psi : \bar{F} \rightarrow \mathbb{Z}^d, \]
such that \( \Phi \circ \Psi = 1_{\bar{F}} \). Then \( F = \Psi(\bar{F}) \) is a free subgroup of \( \mathbb{Z}^d \) of rank \( p \). The subgroup \( F \) can be regarded as the effective index set and its rank \( p \) is the effective dimension of the random field, giving more precise information on the rate of growth of the partial maximum than the nominal dimension \( d \).

The deterministic sequence \( b_n \) controlling the rate of partial maxima shows the following asymptotic behavior:
\[ n^{-p/\alpha} b_n \rightarrow \begin{cases} c & \text{if action restricted to } F \text{ is dissipative,} \\ 0 & \text{if action restricted to } F \text{ is conservative,} \end{cases} \]
where \( c \) is a positive and finite constant and \( p \) is the effective dimension of the field.

The theorem in Roy and Samorodnitsky (2008) thereby sharpens the description of the asymptotic behavior of the partial maxima of a random field when the action is conservative by observing the behavior of the action when restricted to the free subgroup \( F \) of \( \mathbb{Z}^d \), leading to the conclusion that \( \max_{\|t\|_\infty \leq n} |Y_t| = O(n^{p/\alpha}) \) when the effective \( F \)-action is dissipative, and is \( o(n^{p/\alpha}) \) in the conservative scenario. That is,
\[ n^{-p/\alpha} \max_{\|t\|_\infty \leq n} |Y_t| \Rightarrow \begin{cases} c Y Z_{\alpha} & \text{if } Y \text{ is generated by a dissipative } F \text{-action,} \\ 0 & \text{if } Y \text{ is generated by a conservative } F \text{-action.} \end{cases} \]

Similar rates of growth are extended to continuous random fields in Samorodnitsky (2004b).

This work provides the rates of growth of the \( \beta \)-th moments of the partial maxima sequence denoted as
\[ M_n = \max_{0 \leq t \leq (n-1)1} |Y_t| \]
for \( 0 < \beta < \alpha \) for a stationary \( SaS \) process \( Y = \{Y_t\}_{t \in \mathbb{Z}^d} \) with an integral representation given by (2.5). Theorem 3.1 in Section 3 shows that the \( \beta \)-th moments of maxima of such discrete random fields are \( O(n^{d\beta/\alpha}) \) for a dissipative action and \( o(n^{d\beta/\alpha}) \) for a conservative one. We sharpen the above asymptotics in the case of a conservative action by looking at properties of the underlying action restricted to the free subgroup \( F \) with effective dimension \( p \). Again, we achieve similar rates of growth with the effective dimension of the action \( p \), as stated in Theorem 3.4.

Finally, we use the rates of growth of the partial maxima sequence for stationary random fields \( Y \) to derive path properties of a real valued \( H \)-self-similar \( SaS \) random field \( X \) with stationary increments. Our main result is Theorem 4.1 which establishes uniform modulus of continuity for a large class of such random fields. As a consequence (see Corollary 4.3), we prove that the paths of \( X \) are uniformly \( H \) Hölder continuous of all orders \( < H - \frac{2}{\pi} \) when the corresponding increment processes \( Y^{(v)} \) are generated by actions.
with effective dimension $p$. The short memory case when the effective dimension $p = d$ is considered in Corollary 4.2. These results show that in presence of stronger dependence, $p < d$ and the sample paths of $X$ become smoother because stronger dependence prevent erratic jumps. Therefore, Hölder continuity of $\alpha S\alpha$ random fields also changes at the boundary between short and long memory, which is consistent with the conjecture in (Samorodnitsky, 2004a, p.1440).

3. Maximal Moments of Stationary $\alpha S\alpha$ Random Fields

The following is our main result on the asymptotic behaviour of the maximal moments of stationary $\alpha S\alpha$ random fields indexed by $\mathbb{Z}^d$.

**Theorem 3.1.** Let $Y = \{Y_t\}_{t \in \mathbb{Z}^d}$ be a stationary $\alpha S\alpha$ random field with $0 < \alpha < 2$ and having integral representation as

\[ Y_t \overset{d}{=} \int_S f_t(s) M(ds) \]

where $M$ is an $\alpha S\alpha$ random measure on $(S, S)$ with a control measure $\mu$ as in (2.5).

(1) If $Y$ is generated by a dissipative action or equivalently\(^1\), $Y$ has a mixed moving average representation given by

\[ Y_t \overset{d}{=} \left\{ \int_{W \times \mathbb{Z}} f(v, t + s) M(dv, ds) \right\}_{t \in \mathbb{Z}^d}, \]

then, for $0 < \beta < \alpha$,

\[ n^{-d\beta/\alpha} \mathbb{E}\left[ M_n^{\beta} \right] \to C \quad \text{as } n \to \infty, \]

where $C = \tilde{c}_Y C_{\alpha}^{\beta/\alpha} \mathbb{E}\left[ Z_{\alpha/\beta} \right]$, with $Z_{\alpha/\beta}$ denoting a Frechét random variable with shape parameter $\alpha/\beta$, $\tilde{c}_Y$ is the constant in (2.9) and

\[ C_{\alpha} = \begin{cases} \frac{1 - \alpha}{\Gamma(2 - \alpha) \cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1, \\ 2/\pi, & \text{if } \alpha = 1. \end{cases} \]

(2) If $Y$ is generated by a conservative action, then for $0 < \beta < \alpha$,

\[ n^{-d\beta/\alpha} \mathbb{E}\left[ M_n^{\beta} \right] \to 0 \quad \text{as } n \to \infty. \]

**Proof.** Our proof is built upon the arguments in Samorodnitsky (2004a) and Roy and Samorodnitsky (2008). In the following, $\{\Gamma_n\}_{n \geq 1}$ denotes a sequence of arrival times of a unit rate Poisson process on $(0, \infty)$, $\{\xi_n\}_{n \geq 1}$

\(^1\text{See Theorem 3.3 of Roy and Samorodnitsky (2008)}\)
are i.i.d. Rademacher random variables, and $\{U_\ell^{(n)}\}_{n \geq 1}$ ($\ell = 1, 2$) are i.i.d. $S$-valued random variables with common law $\eta_n$ whose density is given by
\[
\frac{d\eta_n}{dt} = b_n^{-\alpha} \max_{0 \leq t \leq (n-1)} |f_t(s)|^\alpha, \quad s \in S.
\]
All four sequences are independent. We will make use of the following series representation for $\{Y_k, 0 \leq k \leq (n-1)\}$:
\[
Y_k \overset{d}{=} b_n C_1^{1/\alpha} \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{m \in \{0, (n-1)\}} |f_m(U_j^{(n)})|}.
\]
See Section 3.10 in Samorodnitsky and Taqqu (1994).

We first consider case (1) when $Y$ is generated by a dissipative action, it follows from (2.9) that the deterministic sequence $\{b_n\}_{n \geq 1}$ satisfies
\[
\lim_{n \to \infty} n^{-d/\alpha} b_n = \tilde{c}_Y,
\]
where $\tilde{c}_Y > 0$ is a constant. This implies that $b_n$ satisfies condition (4.6) in Samorodnitsky (2004a) with $\theta = d/\alpha$ and, additionally, its condition (4.8) also holds; thanks to Remark 4.2 in Samorodnitsky (2004a) (or Remark 4.4 in Roy and Samorodnitsky (2008)). Further, (3.5) implies that for any $p > \alpha$, there is a finite constant $A$ such that
\[
n^{d/p} b_n^{-p} < n^{d} b_n^{-\alpha} \leq A.
\]
Let $K = d$, $\epsilon$ and $\delta$ be chosen such that
\[
0 < \epsilon < \frac{\delta}{K}.
\]
Then we obtain from (4.21) in Samorodnitsky (2004a) the following upper bound on the tail distribution of $b_n^{-1} M_n$:
\[
\mathbb{P}(b_n^{-1} M_n > \lambda) \leq \mathbb{P}(C_1^{1/\alpha} \Gamma_1^{-1/\alpha} > \lambda(1 - \delta)) + \phi_n(\epsilon, \lambda) + \psi_n(\epsilon, \delta, \lambda).
\]
Here, similarly to (4.22) in Samorodnitsky (2004a),
\[
\phi_n(\epsilon, \lambda) = \mathbb{P}\left( \exists k \in [0, (n-1)1], \frac{\Gamma_j^{-1/\alpha} |f_k(U_j^{(n)})|}{\max_{m \in \{0, (n-1)\}} |f_m(U_j^{(n)})|} > \frac{\epsilon \lambda}{C_1^{1/\alpha}} \right.
\]
for at least 2 different $j$
\[

\leq n^d \mathbb{P}\left( \Gamma_j^{-1/\alpha} > \frac{b_n \epsilon \lambda}{C_1^{1/\alpha} \|f\|_\alpha} \right. \text{ for at least 2 different } j \right).
\]
In deriving the last equality, we have applied the fact that for every $k \in [0, (n-1)1]$, the points
\[
b_n \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{0 \leq s \leq (n-1)1} |f_s(U_j^{(n)})|}, \quad j = 1, 2, \ldots
\]
have the same joint distribution as the points
\[ \xi_j \|f\|_\alpha \Gamma_j^{-1/\alpha}, \quad j = 1, 2, \ldots \]
which represent a symmetric Poisson random measure on \( \mathbb{R} \) with mean measure
\[ (3.8) \quad \Lambda((x, \infty)) = x^{-\alpha}\|f\|_\alpha^\alpha/2, \quad \text{for } x > 0. \]

In the above, \( \|f\|_\alpha = (\int_S |f(s)|^\alpha \mu(ds))^{1/\alpha} \). Similarly, we have
\[
\psi_n(\epsilon, \delta, \lambda) = \mathbb{P}\left( \max_{k \in [0,(n-1)\|f\|_\alpha]} \left| \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} |f_k(U_j^{(n)})| \right| > \frac{\lambda}{C_\alpha^{1/\alpha}\|f\|_\alpha}, \right.
\]
\[ \Gamma_1^{-1/\alpha} \leq \frac{b_n \lambda (1 - \delta)}{C_\alpha^{1/\alpha}\|f\|_\alpha}, \quad \text{and } \Gamma_2^{-1/\alpha} \leq \frac{b_n \lambda \epsilon}{C_\alpha^{1/\alpha}\|f\|_\alpha}. \]
\[ \leq n^d \mathbb{P}\left( \left| \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \right| > \frac{b_n \lambda}{C_\alpha^{1/\alpha}\|f\|_\alpha}, \Gamma_1^{-1/\alpha} \leq \frac{b_n \lambda (1 - \delta)}{C_\alpha^{1/\alpha}\|f\|_\alpha}, \right.
\]
\[ \quad \text{and } \Gamma_2^{-1/\alpha} \leq \frac{b_n \lambda \epsilon}{C_\alpha^{1/\alpha}\|f\|_\alpha} \right) \]

For any \( 0 < \beta < \alpha \), using the tail bound in (3.6) we have
\[ \mathbb{E}[b_n^{-\beta} M_n^\beta] = \int_0^\infty \mathbb{P}(b_n^{-1} M_n > \tau^{1/\beta}) d\tau \]
\[ \leq \int_0^\infty \mathbb{P}(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \tau^{1/\beta} (1 - \delta)) d\tau \]
\[ + \int_0^\infty \phi_n(\epsilon, \tau^{1/\beta}) d\tau + \int_0^\infty \psi_n(\epsilon, \delta, \tau^{1/\beta}) d\tau \]
\[ := T_1(\delta) + T_2^{(n)}(\epsilon) + T_3^{(n)}(\epsilon, \delta). \]

It is shown in Samorodnitsky (2004a) that for every \( \tau > 0 \),
\[ \phi_n(\epsilon, \tau^{1/\beta}), \quad \text{and } \psi_n(\epsilon, \delta, \tau^{1/\beta}) \]
\[ \text{converge to } 0, \quad \text{as } n \to \infty \text{ for choices of } \epsilon \text{ adequately smaller in comparison to } \delta. \]

Next we present non-trivial integrable bounds on \((1, \infty)\) for integrands \( \phi_n(\epsilon, \tau^{1/\beta}) \) and \( \psi_n(\epsilon, \delta, \tau^{1/\beta}) \) in \( T_2^{(n)}(\epsilon) \) and \( T_3^{(n)}(\epsilon, \delta) \) in (3.9) respectively, and use the trivial bound of 1 on \((0, 1)\). Finally, we apply DCT to show that the integrals in \( T_2^{(n)}(\epsilon) \) and \( T_3^{(n)}(\epsilon, \delta) \) converge to 0 as \( n \to \infty \).
We begin by providing an integrable upper bound for \( \phi_n(\epsilon, \tau^{1/\beta}) \) on \((1, \infty)\). It follows from (3.7) that

\[
\phi_n(\epsilon, \tau^{1/\beta}) \leq n^d \mathbb{P} \left( \sum_{j=1}^{\infty} 1_{B_n} \right) \leq n^d \mathbb{P} \left( \sum_{j=1}^{\infty} \xi_j \|f\|_\alpha \gamma_j^{-1/\alpha} \right)
\]

\[
\left( -\infty, -C_{\alpha}^{-1/\alpha} b_n \epsilon \tau^{1/\beta} \right) \cup \left( C_{\alpha}^{-1/\alpha} b_n \epsilon \tau^{1/\beta}, \infty \right) \geq 2
\]

(3.10)

\[
= n^d \mathbb{P}(\text{Poi}(\Lambda(B_n)) \geq 2),
\]

where

\[
B_n = \left( -\infty, -C_{\alpha}^{-1/\alpha} b_n \epsilon \tau^{1/\beta} \right) \cup \left( C_{\alpha}^{-1/\alpha} b_n \epsilon \tau^{1/\beta}, \infty \right)
\]

and we have used the fact that

\[
\sum_{j=1}^{\infty} 1_{B_n} \sim \text{Poi}(\Lambda(B_n)).
\]

Thus, the Markov inequality and definition (3.8) of the mean measure \( \Lambda \) imply

\[
\phi_n(\epsilon, \tau^{1/\beta}) \leq n^d \mathbb{E}(\text{Poi}(\Lambda(B_n))) = n^d \Lambda(B_n)/2
\]

(3.11)

\[
= n^d b_n^{-\alpha} C_{\alpha}^{-1} \epsilon^{-\alpha} \frac{1}{\tau^{\alpha/\beta}}
\]

\[
\leq A_n C_{\alpha}^{-1} \epsilon^{-\alpha} \frac{1}{\tau^{\alpha/\beta}}.
\]

The last term in (3.11) is clearly integrable in \( \tau \) on \((1, \infty)\). We apply DCT to \( T_2^{(n)}(\epsilon) \) as

\[
T_2^{(n)}(\epsilon) = \int_0^1 \phi_n(\epsilon, \tau^{1/\beta}) d\tau + \int_{1}^{\infty} \phi_n(\epsilon, \tau^{1/\beta}) d\tau
\]

by using the trivial bound of 1 on \((0,1)\) and the bound derived in (3.11) on \((1, \infty)\) to conclude

\[
T_2^{(n)}(\epsilon) \to 0 \quad \text{as} \quad n \to \infty.
\]
We next derive an upper bound for \( \psi_n(\epsilon, \delta, \tau^{1/\beta}) \). It follows from (3.9) that \( \psi_n(\epsilon, \delta, \tau^{1/\beta}) \) is bounded from above by

\[
\begin{align*}
&t_{n} \mathbb{P} \left( |C_{\alpha}^{1/\alpha} \sum_{j=1}^{\infty} \xi_{j} \Gamma_{j}^{-1/\alpha}| \leq \frac{b_{n} \tau^{1/\beta}}{\|f\|_{\alpha}}, \ C_{\alpha}^{1/\alpha} \Gamma_{1}^{-1/\alpha} \leq \frac{b_{n} \tau^{1/\beta}(1 - \delta)}{\|f\|_{\alpha}} \right), \\
&\text{and } C_{\alpha}^{1/\alpha} \Gamma_{j}^{-1/\alpha} \leq \frac{b_{n} \tau^{1/\beta} \epsilon}{\|f\|_{\alpha}} \text{ for all } j \geq 2.
\end{align*}
\]

(3.12)

\[
\begin{align*}
&\leq n \mathbb{P} \left( C_{\alpha}^{1/\alpha} \left| \sum_{j=K+1}^{\infty} \xi_{j} \Gamma_{j}^{-1/\alpha} \right| > \frac{b_{n} \tau^{1/\beta}(\delta - \epsilon(K - 1))}{\|f\|_{\alpha}} \right), \\
&\leq n \mathbb{P} \left( C_{\alpha}^{1/\alpha} \left| \sum_{j=K+1}^{\infty} \xi_{j} \Gamma_{j}^{-1/\alpha} \right| > \frac{b_{n} \tau^{1/\beta}}{\|f\|_{\alpha}} \right), \\
&\leq n \mathbb{P} \left( \frac{\|f\|_{p}^{p} \mathbb{E} \left| C_{\alpha}^{1/\alpha} \sum_{j=K+1}^{\infty} \xi_{j} \Gamma_{j}^{-1/\alpha} \right|^{p}}{\tau^{p/\beta} e^{p}} \right), \\
&\leq A \mathbb{P} \left( \frac{\|f\|_{p}^{p} \mathbb{E} \left| C_{\alpha}^{1/\alpha} \sum_{j=K+1}^{\infty} \xi_{j} \Gamma_{j}^{-1/\alpha} \right|^{p}}{\tau^{p/\beta} e^{p}} \right).
\end{align*}
\]

For choice of \( p \) such that \( \alpha < p < \alpha(K + 1) \) in the Markov inequality in the third step of (3.12)

\[
\mathbb{E} \left| C_{\alpha}^{1/\alpha} \sum_{j=K+1}^{\infty} \xi_{j} \Gamma_{j}^{-1/\alpha} \right|^{p} < \infty
\]

and also,

\[
n^{d} b_{n}^{-p} \leq A, \text{ a constant,}
\]

which gives an integrable upper bound for \( \psi_n(\epsilon, \delta, \tau^{1/\beta}) \) on \((1, \infty)\). By a similar DCT argument, we have

\[
T_{3}^{(n)}(\epsilon) \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Using (3.9), we complete the proof by noting

\[
\limsup_{n \rightarrow \infty} \mathbb{E} \left[ b_{n}^{-\beta} M_{n}^{\beta} \right] \leq \int_{0}^{\infty} \mathbb{P} \left( \Gamma_{1}^{-1/\alpha} > C_{\alpha}^{-1/\alpha} \tau^{1/\beta}(1 - \delta) \right) d\tau
\]

\[= \int_{0}^{\infty} \left( 1 - \exp \left( - C_{\alpha}^{-\alpha/\beta}(1 - \delta)^{-\alpha} \right) \right) d\tau.
\]

By letting \( \delta \rightarrow 0^{+} \), and applying DCT again and using (3.5), we have

\[
\limsup_{n \rightarrow \infty} \mathbb{E} \left[ n^{-d\beta/\alpha} M_{n}^{\beta} \right] \leq \bar{c}_{Y} \gamma_{\alpha}^{\beta/\alpha} \mathbb{E} \left[ Z_{\alpha/\beta} \right].
\]

The argument for establishing corresponding lower bound is similar. We start with the following lower bound for the tail distribution of \( b_{n}^{-1} M_{n} \) from
\( Z \) field generated by a conservative integral representation of \( Y \).

By applying DCT with the integrable bounds derived in (3.7) and \( \tilde{\psi}_n(\epsilon, \delta, \lambda) \) is defined by

\[
\tilde{\psi}_n(\epsilon, \delta, \lambda) = \mathbb{P}\left( \max_{k \in \{0, (n-1)1\}} \left| \sum_{j=1}^{\infty} \frac{\xi_j \Gamma_j^{-1/\alpha}|f_k(U_j^{(n)})|}{\max_{m \in \{0, (n-1)1\}} |f_m(U_j^{(n)})|} \right| \leq \frac{\lambda}{C_{\alpha}^{1/\alpha} \|f\|_\alpha}, \quad \Gamma_j^{-1/\alpha} \\
\leq \frac{\frac{b_n \lambda(1+\delta)}{C_{\alpha}^{1/\alpha} \|f\|_\alpha}}, \quad \Gamma_2^{-1/\alpha} \leq \frac{\frac{b_n \lambda}{C_{\alpha}^{1/\alpha} \|f\|_\alpha}}{1 + \delta}.
\]

By a similar argument leading to (3.9), we obtain

\[
\mathbb{E}[b_n^{-\beta} M_n^\beta] \geq \int_0^\infty \mathbb{P}(C_{\alpha}^{1/\alpha} \Gamma_1^{-1/\alpha} > P_{1/\beta}(1+\delta)) d\tau \\
- \int_0^\infty \bar{\phi}_n(\epsilon, \tau^{1/\beta}) d\tau - \int_0^\infty \tilde{\psi}_n(\epsilon, \delta, \tau^{1/\beta}) d\tau \\
:= \bar{T}_1(\delta) - T_2(\alpha)(\epsilon) - \tilde{T}_3(\alpha)(\epsilon, \delta).
\]

By applying DCT with the integrable bounds derived in (3.10) and (3.12), we derive

\[
\liminf_{n \to \infty} \mathbb{E}[n^{-d/\alpha} M_n^\beta] \geq \frac{C_{\alpha}^{1/\alpha} \mathbb{E}[Z_\alpha/\beta]}{\tilde{\psi}_n(\epsilon, \delta, \tau^{1/\beta})}.
\]

Combining the above inequalities, we prove (3.2), that is

\[
n^{-d/\alpha} \mathbb{E}[M_n^\beta] \to C \text{ as } n \to \infty.
\]

In the case of a conservative action, let \( W \) be a stationary SoS random field independent of \( Y \), having a similar integral representation with SoS measure \( M' \) on space \( S' \) with control measure \( \mu' \), independent of \( M \) in the integral representation of \( Y \). That is,

\[
W_t = \int_{S'} c'_t(s) \left( \frac{d\mu' \circ \phi'_t(s)}{d\mu'(s)} \right)^{1/\alpha} g \circ \phi'_t(s) M'(ds), \quad t \in \mathbb{Z}^d.
\]

Denote the above integrand by \( g_t(s) \). We further assume that the sequence

\[
b_n^W = \left( \int_{S'} \max_{0 \leq t \leq (n-1)1} |g_t(s)|^{\alpha} \mu'(ds) \right)^{1/\alpha}, \quad n \geq 1,
\]

satisfies equation (4.6) in Samorodnitsky (2004a) for some \( \theta > 0 \). Define \( Z = W + Y \). Then \( Z \) inherits its natural integral representation on \( S \cup S' \) and the naturally defined action on that space is a stationary SoS random field generated by a conservative \( \mathbb{Z}^d \)-action. The deterministic maximal
sequence \( b_n^Z \) corresponding to conservative \( Z \) satisfies (4.6) in Samorodnitsky (2004a) as

\[
b_n^Z \geq b_n^W \quad \text{for all } n.
\]

Using symmetry, we have

\[
(3.15) \quad P(M_n^Z > x) \geq \frac{1}{2} P(M_n > x)
\]

and

\[
E[n^{-d/\alpha} M_n^\beta] = \int_0^\infty P\left(n^{-d/\alpha} M_n > \tau^{1/\beta}\right) d\tau
\]

\[
= 2 \int_0^1 P\left((b_n^Z)^{-1} M_n^Z > C\tau^{1/\beta}\right) d\tau + 2 \int_1^\infty P\left((b_n^Z)^{-1} M_n^Z > C\tau^{1/\beta}\right) d\tau
\]

\[
= S_n^{(1)} + S_n^{(2)}
\]

with the second step following from (3.15) and that \( n^{-d/\alpha} b_n^Z \) converges to 0 and hence is bounded by a constant \( 1/C \) say. We use the fact from Samorodnitsky (2004a) that

\[
n^{-d/\alpha} M_n \to 0 \quad \text{as } n \to \infty,
\]

and conclude (3.4) via a DCT argument by using the trivial bound on \( P\left((b_n^Z)^{-1} M_n^Z > C\tau^{1/\beta}\right) \) on \((0,1)\) and obtaining a non-trivial integrable bound for the same on \((1,\infty)\). Again with a similar choice of \( \epsilon \) as in the dissipative case we have

\[
P\left(M_n^Z > C b_n^Z \tau^{1/\beta}\right) \leq P\left(\Gamma_1^{-1/\alpha} > C\tau^{1/\beta} \epsilon\right)
\]

\[
+ P\left(M_n^Z > C b_n^Z \tau^{1/\beta}, \Gamma_1^{-1/\alpha} \leq C\tau^{1/\beta} \epsilon\right),
\]

where \( M_n^Z \) is the maxima, and \( b_n^Z \) is the corresponding deterministic maximal sequence for \( Z \). Let \( Z \) have a series representation in terms of arrival times of a unit Poisson process, \( \Gamma_j \) and Rademacher variables \( \xi_j \). Now choose \( K \) large enough so that \( \alpha(K+1) > d/\theta \). For \( p \) satisfying

\[
\frac{d}{\theta} < p < \alpha(K+1),
\]
using a technique similar to (3.12) by an application of Markov’s inequality, we derive an integrable bound on \((1, \infty)\) as

\[
P\left( M_n^Z > Cb_n^{\tau^{1/\beta}}, \; \Gamma_1^{-1/\alpha} \leq C\tau^{1/\beta} \right)
\]

\[
\leq n^d P\left( |C_{1/\alpha}^{\Gamma_j^{-1/\alpha}}| > \frac{Cb_n^{\tau^{1/\beta}}}{\|f^Z\|_\alpha}, \; C_{1/\alpha}\Gamma_1^{-1/\alpha} \leq \frac{Cb_n^{\tau^{1/\beta}}}{\|f^Z\|_\alpha} \right.
\]

\[
\leq n^d P\left( |C_{1/\alpha}^{\Gamma_j^{-1/\alpha}}| > \frac{Cb_n^{\tau^{1/\beta}}}{\|f^Z\|_\alpha}, \; C_{1/\alpha}\Gamma_1^{-1/\alpha} \leq \frac{Cb_n^{\tau^{1/\beta}}}{\|f^Z\|_\alpha} \right.$

\[
\leq n^d \left[ C_{1/\alpha}^{\Gamma_j^{-1/\alpha}} \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right]^{p} \frac{\|f^Z\|_\alpha}{\tau^{p/\beta p}}
\]

\[
\leq AC_p \|f^Z\|_\alpha \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{\|f^Z\|_\alpha}{\tau^{p/\beta p}}.
\]

Observing that

\[
\int_0^\infty P\left( \Gamma_1^{-1/\alpha} > \epsilon\tau^{1/\beta} \right) d\tau = e^{-\beta} E[Z_\alpha^{1/\beta}] = e^{-\beta} \Gamma(1 - \beta/\alpha) < \infty,
\]

and using integrable bound for

\[
P\left( M_n^Z > \tau^{1/\beta} b_n^Z, \; \Gamma_1^{-1/\alpha} \leq \epsilon\tau^{1/\beta} \right)
\]

as derived in (3.16), we obtain a nontrivial bound for \(S_n^{(2)}\). Equipped to apply DCT with the trivial bound 1 for \(S_n^{(1)}\) and an integrable bound for \(S_n^{(2)}\), we conclude (3.4).

The above result solves an open problem mentioned (right after the proof of Lemma 3.5) in Xiao (2010) when the underlying group action is dissipative. Note that as long as the action is not conservative, the same asymptotics will hold for the maximal moment sequence. In the next result, we present a solution to the problem in a more general situation.

**Theorem 3.2.** Consider a stationary SoS random field with \(0 < \alpha < 2\), \(Y = \{Y_t\}_{t \in \mathbb{Z}^d}\) with integral representation as (3.1). If there are constants \(c_1, c_2 > 0\), \(0 < \theta_1 < d/\alpha\), and \(\theta_2 > 0\) such that

\[
c_1 n^{\theta_1} \leq b_n \leq c_2 n^{\theta_2}\quad \text{for all sufficiently large } n,
\]

(3.17)
then for all \( n \geq 1 \),

\[
(3.18) \quad n^{-\beta_2} E[M_n^\beta] \leq K',
\]

where \( K' \) is a finite constant.

**Proof.** The proof again follows by noting that

\[
E[b_n^{-\beta} M_n^\beta] = \int_0^\infty \mathbb{P}(b_n^{-1} M_n > \tau^{1/\beta}) d\tau
\]

\[
\leq \int_0^\infty \{ \mathbb{P}(\Gamma_1^{-1/\alpha} > \tau^{1/\beta})
+ \mathbb{P}(M_n > \tau^{1/\beta} b_n, \Gamma_1^{-1/\alpha} \leq \tau^{1/\beta} \epsilon) \} d\tau
\]

\[
= \epsilon^{-\beta} \Gamma(1 - \beta/\alpha) + \int_0^1 \mathbb{P}(M_n > \tau^{1/\beta} b_n, \Gamma_1^{-1/\alpha} \leq \tau^{1/\beta} \epsilon) d\tau
\]

\[
+ \int_1^\infty \mathbb{P}(M_n > \tau^{1/\beta} b_n, \Gamma_1^{-1/\alpha} \leq \tau^{1/\beta} \epsilon) d\tau.
\]

The integral over \([0, 1]\) is bounded by 1. To bound the integral over \((1, \infty)\), we choose \( K \) large enough so that \( \alpha(K + 1) > \frac{d}{\theta_1} \). Fix \( \epsilon \) satisfying \( 0 < \epsilon < \frac{1}{K} \) and \( p \) satisfying

\[
\frac{d}{\theta_1} < p < \alpha(K + 1).
\]

The same argument as in (3.16), together with the lower bound in (3.17), gives

\[
\mathbb{P}(M_n > \tau^{1/\beta} b_n, \Gamma_1^{-1/\alpha} \leq \tau^{1/\beta} \epsilon) \leq \frac{B}{\tau^{p/\beta} \epsilon^p},
\]

where

\[
B = A\|f\|_\alpha^p C_1^{1/\alpha} \sum_{j=K+1}^\infty \epsilon_j \Gamma_1^{-1/\alpha} |p|.
\]

It follows from above that

\[
E[b_n^{-\beta} M_n^\beta] \leq \epsilon^{-\beta} \Gamma(1 - \beta/\alpha) + 1 + \int_1^\infty \frac{B}{\tau^{p/\beta} \epsilon^p} d\tau
\]

\[
= K_1 < \infty.
\]

Hence

\[
E[M_n^\beta] \leq K_1 \cdot b_n^\beta \leq K_1 c_2 \cdot n^{\beta_2}
\]

for all sufficiently large \( n \), say \( n \geq n_0 \). Taking \( K' = \max\{c_2 K_1; E[M_k^\beta], k \leq n_0\} \) yields (3.18).

**Remark 3.3.** By Theorem 2.1 of Marcus (1984) (see also (3.4) in Samorodnitsky (2004a)), as long as \( \alpha \in (0, 1) \),

\[
E(M_n^\beta) \leq b_n^\beta
\]

always holds for all \( \beta \in (0, \alpha) \) and for all \( n \geq 1 \). Therefore, the lower bound in (3.17) is not required when \( 0 < \alpha < 1 \).
Now we consider the case when the underlying group action is conservative and establish refined results in terms of the effective dimension $p$ of $Y$.

**Theorem 3.4.** Let $Y = \{Y_t\}_{t \in \mathbb{Z}^d}$ be a stationary $S\alpha S$ random field with $0 < \alpha < 2$, with integral representation written in terms of functions $\{f_t\}$ as in (3.1).

1. If the underlying action $\{\phi_t\}_{t \in F}$ is dissipative when restricted to free subgroup $F$ with rank $p$, then

$$n^{-p_\beta/\alpha}E[M_n^\beta] \to C \quad \text{as} \quad n \to \infty,$$

where constant $C = c_\beta c_\alpha^{-\beta/\alpha}E[Z_\alpha/\beta]$, with $Z_\alpha/\beta$ denoting a Frechet random variable with shape parameter $\alpha/\beta$ and constant $c = \lim_{n \to \infty} n^{-p/\alpha}b_n$.

2. If the underlying action $\{\phi_t\}_{t \in F}$ is conservative when restricted to free subgroup $F$ with rank $p$, then

$$n^{-p_\beta/\alpha}E[M_n^\beta] \to 0 \quad \text{as} \quad n \to \infty.$$

**Proof.** (1). When the action $\{\phi_t\}_{t \in F}$ restricted to the free group $F$ is dissipative, then by Proposition 5.1 of Roy and Samorodnitsky (2008), the sequence $\{b_n\}_{n \geq 0}$ satisfies (4.6) in Samorodnitsky (2004a) with $\theta = p/\alpha$. Also, (4.17) of Roy and Samorodnitsky (2008) holds; see the proof of Theorem 5.4 in Roy and Samorodnitsky (2008).

Now we choose $K$ large enough so that $\alpha (K + 1) > da/p$, use the same tail bound as in (3.6) and apply DCT using integrable bounds on

$$\phi_n(\epsilon, \tau^{1/\beta}) \leq n^d b_{n}^{-\alpha} C_\alpha^{-1} \epsilon^{-\alpha} / \tau^{\alpha/\beta},$$

$$\psi_n(\epsilon, \delta, \tau) \leq n^d b_{n}^{-p} \| f \|_{p'} C_\alpha^{1/\alpha} \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_{j-1/\alpha}^{-1/\alpha} / \tau^{p'/\beta} \epsilon_{p'},$$

for $p'$ satisfying

$$\frac{d\alpha}{p} \leq p' \leq \alpha (K + 1).$$

Then as in the proof of (3.2), (3.19) follows.

(2). When the action $\{\phi_t\}_{t \in F}$ is conservative, we can obtain a stationary $S\alpha S$ random field $Z$ generated by a conservative $\mathbb{Z}^d$-action such that $b_n^Z$ satisfies (4.6) in Samorodnitsky (2004a) for some $\theta > 0$ and

$$n^{-p_\beta/\alpha}b_n^Z \to 0 \quad \text{as} \quad n \to \infty.$$
Again by the exact argument used to prove (3.4), we obtain (3.20).

\textbf{Remark 3.5.} The asymptotic properties of maximal moments can easily be extended to stationary measurable symmetric $\alpha$-stable random fields indexed by $\mathbb{R}^d$. This can be done based on the works of Samorodnitsky (2004b), Roy (2010b) and Chakrabarty and Roy (2013). Since the results (and the proofs) are similar to those presented in this section, we omit them in this paper.

\section{4. Uniform Modulus of Continuity}

Now we combine the moment estimates in Section 3 with Proposition 2.1 to establish uniform moduli of continuity of self-similar $\alpha$S random fields with stationary increments. In the following theorem, we take $T = [0, 1]^d$ for simplicity.

\textbf{Theorem 4.1.} Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a real-valued $H$-self-similar $\alpha$S random field with stationary increments and with the following integral representation

\begin{equation}
X_t = \int_E f_i(s)M(ds), \quad t \in \mathbb{R}^d,
\end{equation}

where $M$ is a $\alpha$S random measure on a measurable space $(E, \mathcal{E})$ with a $\sigma$-finite control measure $m$, while $f_i \in L^\alpha(m, \mathcal{E})$ for all $t \in \mathbb{R}^d$.

Let $V = \{(v_1, \cdots, v_d) : v_i \in \{-1, 0, 1\}\} \setminus \{(0, \cdots, 0)\}$ be the set of vertices of unit cubes in $[-1, 1]^d$, excluding the origin $0$. Define for each $v \in V$, the random field $Y = \{Y^{(v)}(t), t \in \mathbb{R}^d\}$ by $Y^{(v)}(t) = X(t + v) - X(t)$, with the integral representation given by

\begin{equation}
Y^{(v)}_t = \int_E f_i^{(v)}(x)M(dx),
\end{equation}

where $f_i^{(v)} = f_{v+i} - f_i$ for all $t \in \mathbb{R}^d$. If either

1. $X$ is a $\alpha$S process with $0 < \alpha < 1$ and for all $v \in V$ and for some constants $0 < \theta_2 < H$ and $c > 0$,

\begin{equation}
b_n^{(v)} := \left( \int_E \max_{0 \leq t \leq (n-1)1} |f_i^{(v)}(x)|^\alpha m(dx) \right)^{1/\alpha} \leq cn^{\theta_2}
\end{equation}

for all sufficiently large $n$; or

2. $X$ is a $\alpha$S process with $1 \leq \alpha < 2$ and for all $v \in V$ and for some constants $0 < \theta_1, 0 < \theta_2 < H$ and $c_1 > 0, c_2 > 0$,

\begin{equation}
c_1n^{\theta_1} \leq b_n^{(v)} \leq c_2n^{\theta_2}
\end{equation}

for all sufficiently large $n$.

Then for any $0 < \gamma < \alpha$,

\begin{equation}
\lim_{h \to 0+} \sup_{h \leq s-t \leq h} \frac{\sup_{h \leq s-t \leq h} |X(t) - X(s)|}{h^{(H-\theta_2)(\log 1/h)^{1/\gamma}}} = 0 \quad \text{a.s.,}
\end{equation}

where $|s-t|_\infty = \max_{1 \leq j \leq d} |s_j - t_j|$ is the $\ell^\infty$ metric on $\mathbb{R}^d$. 


Proof. We first give the proof under condition (2) (i.e., when $1 \leq \alpha < 2$). In this case, define the sequence \( \{D_n, n \geq 0\} \) as,

\[
D_n = \left\{ \left( \frac{k_1}{2^n}, \frac{k_2}{2^n}, \cdots, \frac{k_d}{2^n} \right) : 0 \leq k_j \leq 2^n - 1, 1 \leq j \leq d \right\}.
\]

Then the sequence \( \{D_n, n \geq 0\} \) satisfies the assumptions 1) and 2) for a chaining sequence in Section 2.

Observe that for any $0 < \gamma < \alpha$,

\[
\begin{align*}
\mathbb{E}\left( \max_{\tau_n \in D_n, \tau_{n-1} \in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau_{n-1})|^{\gamma} \right) & \\
\leq \sum_{v \in V} \mathbb{E}\left( \max_{0 \leq k_j \leq 2^n - 1, \forall j = 1, \ldots, d} \left| X\left( \frac{k_1}{2^n}, \frac{k_2}{2^n}, \cdots, \frac{k_d}{2^n} + \frac{v}{2^n} \right) - X\left( \frac{k_1}{2^n}, \frac{k_2}{2^n}, \cdots, \frac{k_d}{2^n} \right)^\gamma \right) \\
= 2^{-n\gamma}H \sum_{v \in V} \mathbb{E}\left( \max_{0 \leq k_j \leq 2^n - 1, \forall j = 1, \ldots, d} |Y^{(v)}((k_1, \cdots, k_d))|^{\gamma} \right) \\
= 2^{-n\gamma}H \sum_{v \in V} \mathbb{E}\left( (M_{2n}^{(v)})^\gamma \right),
\end{align*}
\]

where \( M^{(v)} \) is the partial maxima sequence of the stationary SoS random field \( Y^{(v)} \), and where the first equality follows from the self-similarity of \( X \). Under the assumption of Theorem 4.1 we have that for some constants $\theta_2 > 0$ and $c > 0$,

\[ c_1 n^{\theta_1} \leq b_n^{(v)} \leq c_2 n^{\theta_2}. \]

It follows from Theorem 3.2 that the sequence \( \mathbb{E}\left( (b_n^{(v)})^{-1} M_n^{(v)} \right)^\gamma \) is bounded above by a constant $K'$ > 0. Hence

\[ \mathbb{E}\left( (M_n^{(v)})^\gamma \right) \leq K'(b_n^{(v)})^\gamma \leq K n^{\theta_2 \gamma} \]

for a finite constant $K > 0$. Noting that the cardinality of \( V \) is $|V| = 3^d - 1$, we have

\[ \mathbb{E}\left( \max_{\tau_n \in D_n, \tau_{n-1} \in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau_{n-1})|^{\gamma} \right) \leq C 2^{-n\gamma(H-\theta_2)}, \]

where $C = (3^d - 1)K$. It follows immediately from Proposition 2.1 that for any $\epsilon > 0$ and $\gamma \in (0, \alpha)$,

\[
\limsup_{h \to 0+} \frac{\sup_{t \in \mathbb{T}} \sup_{|s-t| \leq h} |X(t) - X(s)|}{h^{(H-\theta_2)} (\log 1/h)^{1+\epsilon}/\gamma} = 0 \quad a.s.
\]

Since $\epsilon > 0$ and $\gamma$ are arbitrary, (4.2) follows.

Under condition (1), the same proof will go through because when $0 < \alpha < 1$, the lower bound on $b_n^{(v)}$ is not needed for establishing $\mathbb{E}\left( (M_n^{(v)})^\gamma \right) \leq \ldots$. 
Kn^{\theta \gamma} \text{ for some } K > 0 \text{ (see Remark 3.3 above). This completes the proof of Theorem 4.1.} \qed

Corollary 4.2. Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a real-valued $H$-self-similar SoS random field with stationary increments and with the integral representation (4.1). If, for every vertex $v \in V$, the increment process $Y^{(v)}$ defined as in Theorem 4.1 is generated by a dissipative action and $\alpha > \frac{d}{H}$, then for any $0 < \gamma < \alpha$,

$$
\limsup_{h \to 0+} \frac{\sup_{t \in T} \sup_{|s-t|_\infty \leq h} |X(t) - X(s)|}{h^{(H-d/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \text{ a.s.}
$$

Proof. Considering the same chaining sequence as in the proof of Theorem 4.1 with the $\ell^\infty$ metric, we may proceed similarly as in (4.3) to derive that for any $0 < \gamma < \alpha$,

$$
E\left(\max_{\tau_n \in D_n} \max_{\tau_{n-1} \in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau_{n-1})|^\gamma\right) \leq 2^{-n\gamma H} \sum_{v \in V} E\left( (M^{(v)}_{2^n_{n-1}})^\gamma \right),
$$

where $M^{(v)}$ is the partial maxima sequence of the stationary SoS random field $Y^{(v)}$. From Theorem 3.1 in Section 3, when $Y^{(v)}$ is generated by a dissipative action, we have

$$
\lim_{n \to \infty} E\left( (2^n - 1)^{-\gamma d/\alpha} (M^{(v)}_{2^n_{n-1}})^\gamma \right) = c,
$$

where $c > 0$ is a finite constant. Hence, there exists a finite constant $K$ such that

$$
E\left(\max_{\tau_n \in D_n} \max_{\tau_{n-1} \in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau_{n-1})|^\gamma\right) \leq K 2^{-n\gamma (H-d/\alpha)},
$$

for all sufficiently large $n$. It is now clear that (4.4) follows from Proposition 2.1. \qed

Corollary 4.3. Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a real-valued $H$-self-similar random field with stationary increments as in Theorem 4.1. If, for every vertex $v \in V$, the increment process $Y^{(v)}$ has effective dimension $p \leq d$ and $\alpha > \frac{p}{H}$, then for any $0 < \gamma < \alpha$,

$$
\limsup_{h \to 0+} \frac{\sup_{t \in T} \sup_{|s-t|_\infty \leq h} |X(t) - X(s)|}{h^{(H-p/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \text{ a.s.}
$$

Proof. The proof follows similarly along the lines of Corollary 4.2 by using the bound on moments in terms of the effective dimension in Theorem 3.4. \qed
5. Examples

The theorems in Sections 5 can be applied to various classes of self-similar random fields with stationary increments. In the following, we mention two examples of them: linear fractional stable motion and harmonizable fractional stable motion. We refer to Samorodnitsky and Taqqu (1994) for more information on these two classes of important self-similar stable processes. For further examples of self-similar processes with stationary increments, see Pipiras and Taqqu (2002).

5.1. Linear fractional stable motion. For any given constants $0 < \alpha < 2$ and $H \in (0, 1)$, we define a SoS process $Z^H = \{Z^H(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}$ by

\[
Z^H(t) = \kappa \int_{\mathbb{R}} \left\{ (t-s)^{H-1/\alpha} - (-s)^{H-1/\alpha} \right\} \alpha(ds),
\]

where $\kappa > 0$ is a normalizing constant, $t_+ = \max\{t, 0\}$ and $\alpha$ is a SoS random measure with Lebesgue control measure.

Using (5.1) one can verify that the stable process $Z^H$ is $H$-self-similar and has stationary increments. It is a stable analogue of fractional Brownian motion, and it called a linear fractional stable motion (LFSM).

Many sample path properties of $Z^H$ are different from those of fractional Brownian motion. For example, Maejima (1983) showed that, if $H\alpha < 1$, then $Z^H$ has a.s. unbounded sample functions on all intervals. Takahashi (1989) showed that, if $H\alpha > 1$, then the index of uniform Hölder continuity of $Z^H$ is $H - \frac{1}{\alpha}$.

In order to apply the results in Section 5, we consider for every $v \in \{-1, 1\}$ the increment process

\[
Y^{(v)}(t) = \int_{\mathbb{R}} \left\{ (t+v-s)^{H-1/\alpha} - (t-s)^{H-1/\alpha} \right\} \alpha(ds).
\]

Then for any $n \geq 1$,

\[
b_n^{(v)}(\alpha) = \left( \int_{\mathbb{R}} \max_{0 \leq k \leq n-1} \left| (k+v-s)^{H-1/\alpha} - (k-s)^{H-1/\alpha} \right|^{\alpha} ds \right)^{1/\alpha}.
\]

For simplicity, we only consider the case of $v = 1$ and write $b_n^{(1)}$ as $b_n$. The case of $v = -1$ can be treated the same way. For integers $k \in \{0, 1, \ldots, n-1\}$, let $g_k(s) = (k+1-s)^{H-1/\alpha} - (k-s)^{H-1/\alpha}$. It is easy to see that for each fixed $s \leq k$, the sequence $g_k(s)$ is non-negative and non-increasing in $k$. We write $b_n^\alpha$ as

\[
b_n^\alpha = \int_{-\infty}^0 \max_{0 \leq k \leq n-1} g_k(s)^\alpha ds + \sum_{\ell=0}^{n-1} \int_{\ell}^{\ell+1} \max_{0 \leq k \leq n-1} g_k(s)^\alpha ds
\]

\[
= \int_{-\infty}^0 g_k(s)^\alpha ds + \sum_{\ell=0}^{n-1} \int_{\ell}^{\ell+1} \max \left\{ g_\ell(s), g_{\ell+1}(s) \right\}^\alpha ds,
\]
where \( g_n \equiv 0 \). Now it is elementary to verify that each of the \((n+1)\) integrals in the right hand side of (5.3) is a positive and finite constant depending only on \( \alpha \) and \( H \). Except the first and the last integrals, all the other integrals are equal. Consequently, there is a positive and finite constant \( K \) such that

\[
\lim_{n \to \infty} n^{-1/\alpha} b_n = K.
\]

Hence, we have proved that, for \( v \in \{1, -1\} \), the stationary \( \text{S\alphaS} \) process \( \{Y(v)(n), n \in \mathbb{Z}\} \) is not generated by a conservative action. Moreover, Condition (2) of Theorem 4.1 is satisfied with \( \theta_1 = \theta_2 = 1/\alpha \). It follows that, if \( H > 1/\alpha \) then for any \( 0 < \gamma < \alpha \),

\[
\limsup_{h \to 0^+} \sup_{t \in [0,1]} \sup_{|s-t| \leq h} \frac{|Z(t) - Z(s)|}{h^{(H-1/\alpha)}(\log 1/h)^{1/\gamma}} = 0 \quad \text{a.s.}
\]

This result improves Theorem 2 in Kôno and Maejima (1991). We mention that, by using more delicate analysis, Takahima (1989) established the exact uniform and local moduli of continuity of linear fractional stable motion \( Z_H \) with \( H > 1/\alpha \).

5.2. Harmonizable fractional stable motion. For any given \( \alpha \in (0, 2) \) and \( H \in (0, 1) \), let \( \tilde{Z}^H = \{\tilde{Z}^H(t), t \in \mathbb{R}\} \) be the real-valued harmonizable fractional \( \text{S\alphaS} \) process (HF\( \alpha \)SF or HFSF, for brevity) with Hurst index \( H \), defined by:

\[
\tilde{Z}^H(t) := \kappa \Re \int_{\mathbb{R}} e^{itx} - 1 \, \tilde{M}_\alpha(dx),
\]

where \( \kappa \) is the positive normalizing constant given by

\[
\kappa = 2^{1/2} \left( \int_{\mathbb{R}} \left( 1 - \cos x \right)^{\alpha/2} \frac{dx}{|x|^\alpha H+1} \right)^{-1/\alpha},
\]

\( \Re \) denotes the real-part, and \( \tilde{M}_\alpha \) a complex-valued rotationally invariant \( \alpha \)-stable random measure with Lebesgue control measure.

For every \( v \in \{-1, 1\} \) consider the increment process

\[
\tilde{Y}^{(v)}(t) = \kappa \Re \int_{\mathbb{R}} e^{i(t+v)x} - e^{ix} \, \tilde{M}_\alpha(dx).
\]

Then for any integer \( n \geq 1 \),

\[
h_n^{(v)} = \kappa \left( \int_{\mathbb{R}} \max_{0 \leq k \leq n-1} \left| e^{i(k+v)x} - e^{ix} \right|^{\alpha} \frac{dx}{|x|^\alpha H} \right)^{1/\alpha},
\]

which is independent of \( n \). Hence the results in Section 2 are not applicable for determining the magnitude of the maximal moments \( \mathbb{E} \left[ \max_{0 \leq k \leq n-1} |Y^{(v)}(k)|^\gamma \right] \) for \( \gamma \in (0, \alpha) \). By appealing to the fact that \( \tilde{Y}^{(v)}(t) \) is conditionally Gaussian, see Biémé and Lacaux (2009, 2015), or Kôno and Maejima (1991), we
can modify the proof of Proposition 4.3 in Xiao (2010) to derive the following upper and lower bounds

\begin{equation}
K \leq \mathbb{E} \left[ \max_{0 \leq k \leq n-1} |Y^{(v)}(k)^\gamma| \right] \leq K' \left( \log n \right)^{\gamma/2},
\end{equation}

where $K$ and $K'$ are positive and finite constants. We omit a detailed verification of (5.8) here because it is lengthy and does not produce the optimal bounds.

It follows from (5.8), (4.3) and Proposition 2.1 with $\sigma(h) = h^H \log 1/h^{1/2}$ that for any $\epsilon > 0$,

\begin{equation}
\limsup_{h \to 0^+} \frac{\sup_{t \in [0,1]} \sup_{|t-s| \leq h} |\tilde{Z}^H(t) - \tilde{Z}^H(s)|}{h^H (\log 1/h)^{\frac{1}{2} + \frac{1}{\alpha} + \epsilon}} = 0 \text{ a.s.}
\end{equation}

This recovers Theorem 1 in Kôno and Maejima (1991). However, it is an open problem in determining the exact uniform modulus of continuity for HFSM $\tilde{Z}^H$.

Even though LFSM $Z^H$ and HFSM $\tilde{Z}^H$ are both $H$-self-similar with stationary increments, their properties are very different. By the exact modulus of continuity in Takahshima (1989) and (5.9), it is clear that the laws of $Z^H$ and $\tilde{Z}^H$ are singular with respect to each other.

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