THE SMALL INDEX PROPERTY OF AUTOMORPHISM GROUPS OF AB-INITIO GENERIC STRUCTURES

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Abstract. Suppose $M$ is a countable ab-initio (uncollapsed) generic structure which is obtained from a pre-dimension function with rational coefficients. We show that if $H$ is a subgroup of $\text{Aut}(M)$ with $[\text{Aut}(M):H] < 2^{\aleph_0}$, then there exists a finite set $A \subseteq M$ such that $\text{Aut}_A(M) \subseteq H$. This shows that $\text{Aut}(M)$ has the small index property.

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1. Introduction

1.1. Background. It is well-known that the automorphism group of a countable structure, with the point-wise convergence topology, is a closed subgroup of the symmetric group of its underlying set. Conversely, one can associate a first-order structure to every closed subgroup of the symmetric group of a countable set in such way that the automorphism group of the associated structure is exactly the group that one started with.

Suppose $M$ is a first-order countable structure and let $G := \text{Aut}(M)$. A subgroup $H$ of $G$ is said to have small index in $G$ if $[G:H] < 2^{\aleph_0}$. One can easily see that open subgroups of $G$ has small index in $G$. 

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We say Aut(M) has the \textit{small index property}, denoted by SIP, if every subgroup of Aut(M) of small index is open. The small index property for Aut(M) indicates a condition under which the topology on Aut(M) can be recovered from its abstract group structure.

When the structure M is $\omega$-categorical (or equivalently when Aut(M) is \textit{oligomorphic}) from the small index property of Aut(M) one can ‘reconstruct’ M from its automorphism group; namely the topology determines the structure M up to \textit{bi-interpretability} (see Section 5. in [18], for more details). The small index property has been proved for the automorphism groups of various first-order $\omega$-categorical structures: the countable infinite set without structure; the countable dense linear ordering $(\mathbb{Q}, <)$; a vector space of dimension $\omega$ over a finite or countable division ring; the random graph; countable $\omega$-stable $\omega$-categorical structures (see [16] for references and more details).

Outside the $\omega$-categorical context there are few known results. The small index property has also been proved for some countable structures which are not saturated: the free groups of countable rank ([2]); arithmetically saturated models of arithmetic ([14]). It is worth noting that, in [17], Lascar and Shelah proved that the automorphism group of every uncountable saturated structures has SIP.

There are few known methods for proving the small index property (cf. [18], for an overview). In this paper, we adopt the method in [11]. One key combinatorial property to prove Aut(M) has SIP is to show the class of all finite substructures of M, up to isomorphism, has the \textit{extension property} (see Definition 4.2). The extension property has been originally proved in [12] for the class of all finite graphs and later generalized by Herwig in [8, 9]. The extension property is used to prove Aut(M) has \textit{ample homogeneous generic automorphisms} (see Definition 4.1). It is shown (in [11], Theorem 5.3) that if M is $\omega$-categorical with ample homogeneous generic automorphisms then Aut(M) has SIP.

Moreover, Lascar shows the following interesting theorem (Théorème 1 in [15]): Suppose M is a countable saturated structure with a $\emptyset$-definable strongly minimal subset D such that M is in the algebraic closure of D. If H is a subgroup of Aut(M) of countable index there there is a finite set A of M such that every A-strong automorphism is in H. We refer to this as \textit{almost} SIP.

1.2. Setting. The Hrushovski construction which originated in [13] admits many variations and can be presented at various levels of generality. Here, we consider the following basic case and comment on
Let \( \mathcal{L} = \{ \mathcal{R} \} \) be a first-order language where \( \mathcal{R} \) is a binary relation that is irreflexive and symmetric. Let \( \mathcal{K} \) be the class of all finite \( \mathcal{L} \)-structures (i.e., \( \mathcal{K} \) is the class of all finite graphs). Suppose \( M, N \subseteq P \) and \( M, N, P \) are \( \mathcal{L} \)-structures, we will often write \( MN \) for the \( \mathcal{L} \)-substructure of \( P \) with domain \( M \cup N \). We write \( M \subseteq \text{fin} \, P \) when \( M \) is a finite substructure of \( P \). We write \( \mathcal{R}(M) \) for the set of all edges of \( M \) and, write \( \mathcal{R}(M; N) = \{ \{ m, n \} : m \in M, n \in N \text{ and } \mathcal{R}(M; N)(m, n) \} \).

Suppose \( m \geq 2 \) is a fixed integer. If \( A \in \mathcal{K} \) consider the pre-dimension \( \delta : \mathcal{K} \rightarrow \mathbb{Z} \) such that \( \delta(A) = m \cdot |A| - |\mathcal{R}(A)| \). Let \( A, B \in \mathcal{K} \), we say \( A \) is \( \preceq \)-closed or self-sufficient in \( B \) and write \( A \preceq B \), if and only if \( \delta(A') \geq \delta(A) \) for all \( A \subseteq A' \subseteq B \). If \( N \) is infinite and \( A \subseteq N \), we write \( A \preceq N \) when \( A \preceq B \) for every finite substructure \( B \) of \( N \) that contains \( A \). The following is standard (cf. [13]).

**Lemma 1.1.** Suppose \( A, B \in \mathcal{K} \). Then:

1. \( \delta(AB) \leq \delta(A) + \delta(B) - \delta(A \cap B) \);
2. If \( A \preceq B \) and \( X \subseteq B \), then \( A \cap X \preceq X \);
3. If \( A \preceq B \preceq C \), then \( A \preceq C \).

Let \( \mathcal{K}_0 \subset \mathcal{K} \) be the set of all \( A \in \mathcal{K} \) such that \( \delta(A') \geq 0 \) for all \( A' \subseteq A \). The class \( (\mathcal{K}_0, \preceq) \) is called an ab-initio class that is obtained from \( \delta \). Let \( A, B, C \in \mathcal{K} \) with \( A \subseteq B, C \). Then the free-amalgam of \( B \) and \( C \) over \( A \), denoted by \( B \otimes_A C \), consists of the disjoint union of \( B \) and \( C \) over \( A \) whose only relations are those from \( B \) and \( C \).

**Theorem 1.2.** The class \( (\mathcal{K}_0, \preceq) \) has the free-amalgamation property; namely, if \( A, B, C \in \mathcal{K}_0 \) and \( A \subseteq B, C \) such that \( A \preceq B \) and \( A \preceq C \), then \( B \otimes_A C \in \mathcal{K}_0 \).

Using Fact 1.2 and a standard Fraïssé-style construction, we obtain the following well-known result (cf. [19, 6] for more details).

**Theorem 1.3.** There is a unique countable structure \( M \) such that:

- \( M \) is the union of a chain of finite \( \preceq \)-closed sets;
- every isomorphisms between finite \( \preceq \)-closed subsets of \( M \) extend to an automorphism of \( M \);
- every element of \( \mathcal{K}_0 \) is isomorphic to a \( \preceq \)-closed subset of \( M \).

The structure \( M \) that is obtained from Theorem 1.3 is referred to as the \( (\mathcal{K}_0, \preceq) \)-generic structure and sometimes as an ab-initio case of the Hrushovski constructions. Throughout the paper \( \mathcal{K}_0 \) and the \( (\mathcal{K}_0, \preceq) \)-generic structure \( M \) is fixed.

The following is well-known (cf. [11, 6]).
Theorem 1.4. The structure $\mathbf{M}$ is saturated and $\text{Th}(\mathbf{M})$ is $\omega$-stable of infinite Morley rank.

Suppose $A \subseteq_{\text{fin}} \mathbf{M}$. Define $d(A) := \delta(\text{cl}(A))$, referred to as dimension of $A$, where $\text{cl}(A)$ is the smallest $\leq$-closed finite subset of $\mathbf{M}$ that contains $A$. The uniqueness of $\text{cl}(A)$ can be proved using Lemma 1.1(2). The following is also standard (cf. [19]).

Fact 1.5. Suppose $A, B$ are finite subset of $\mathbf{M}$. Then

1. $\text{cl}(A) = \text{acl}(A)$;
2. $\delta(\text{cl}(A)) \leq \delta(A)$, and $\delta(B) \geq d(A)$ for all $B \supseteq \text{cl}(A)$;
3. $d(AB) + d(A \cap B) \leq d(A) + d(B)$.

When $X$ is an infinite subset of $\mathbf{M}$, define $d(X) := \max\{d(A) : A \subseteq_{\text{fin}} X\}$ (cf. [19] for more details). It is clear that $d(X) \leq d(Y)$ when $X \subseteq Y \subseteq \mathbf{M}$.

The following geometric closure operator appears naturally: Let $X \subseteq M$. Define $\text{gcl}(X) := \{m \in \mathbf{M} : d((m/A)) = 0, \text{ for some } A \subseteq_{\text{fin}} X\}$, where we write $d(m/A)$ for $d(mA) - d(A)$. Note that $\text{gcl}(A)$ is an infinite set when $A$ is a finite subset of $\mathbf{M}$; unlike $\text{cl}(A)$ which is finite. Moreover $d(\text{gcl}(A)) = d(A) = \delta(\text{cl}(A))$.

Finally we need the following definition (cf. [13] page 150).

Definition 1.6. Suppose $A, B \subseteq_{\text{fin}} \mathbf{M}$ such that $A \cap B = \emptyset$. We say $B$ is $0$-algebraic over $A$ if $\delta(BA) - \delta(A) = 0$ and $\delta(B'A) - \delta(A) > 0$ for all proper subsets $\emptyset \neq B' \subsetneq B$. The set $B$ is called $0$-minimally algebraic over $A$ if there is no proper subset $A'$ of $A$ such that $B$ is $0$-algebraic over $A'$.

Fix the following notation for the automorphism groups: Suppose $\mathbf{M}$ is a countable first order structure and put $G := \text{Aut}(\mathbf{M})$. Let $S_\omega := \text{Sym}(\Omega)$, where $\Omega$ is the countable underlying set of $\mathbf{M}$. Suppose $X \subseteq M$ and $g \in G$. We write $g[X]$ for the image of $X$ under $g$. Then $G_X := \{g \in G : g(x) = x, \forall x \in X\}$ and $G_{\{x\}} := \{g \in G : g[X] = X\}$. It is well-known that $G$ with the point-wise convergence topology is a closed subgroup of $S_\omega$. Suppose $N_0 \subseteq N_1$ are two $\mathfrak{L}$-structures and $g_0 \in \text{Aut}(N_0)$ and $g_1 \in \text{Aut}(N_1)$, we write $g_0 \leq g_1$ when $g_1$ is an extension of $g_0$ i.e. $g_1 \upharpoonright N_0 = g_0$.

1.3. Main results. In Section 2 using the same technique as Lascar in [13], we prove Theorem 1.7 that has been suggested in [5]. This is what we call almost SIP. In Theorem 5.1.6 and Corollary 5.1.7 in [6], similar results have been shown for the automorphism groups of almost strongly minimal structures, and the automorphism groups of generic almost strongly minimal structures.
Fix $G := \text{Aut}(M)$. In Section 2 we prove the following

**Theorem 1.7.** Let $H$ be a subgroup of $G$ with $[G : H] < 2^{\aleph_0}$. Then there exists $X \subseteq M$ such that $G_X \leq H$ where $X = \text{gcl} (A)$ for some $A \subseteq_{\text{fin}} M$.

Then in Section 3 we show:

**Lemma 1.8.** Let $X = \text{gcl} (A)$ where $A \subseteq_{\text{fin}} M$. Then:

1. $G \{X\}$ is a clopen subgroup of $G$, hence it is a Polish group.
2. $G \{X\}$ has small index in $G$.
3. If $G \{X\}$ has SIP, then $G$ has SIP.
4. Let $\pi_X : G \{X\} \to \text{Aut}(X)$ be the projection map with $h \mapsto h \upharpoonright X$, then $\pi_X$ is a homomorphism which is continuous, surjective and open.

As mentioned before we adopt the method in [11] however here we do not show directly that the structure $M$ has ample homogeneous generic automorphisms; the definition of ample homogeneous generic automorphisms is technical and hence only given in Chapter 4 (see Definition 4.1). For our case of $M$, it would have been enough to show that $(K_0, \leq)$ has the extension property (cf. Definition 4.2). As proved in Corollary 5.1.15 in [6], the class $(K_0, \leq)$ does not have the extension property (for more details see Remark 4.3). However, we prove the following theorem in Section 4.

**Theorem 1.9.** Let $\hat{M} := \text{gcl}(\emptyset)$ and $C = \{ A \in K_0 : \delta(A) = 0 \}$. The class $(C, \leq)$ has the extension property. Therefore, $\hat{M}$ has ample homogeneous generic automorphisms and $\text{Aut}(\hat{M})$ has SIP.

Moreover, using a similar technique one can show the following theorem.

**Theorem 1.10.** Suppose $A \subseteq_{\text{fin}} M$ and let $X = \text{gcl}(A)$. Then $\text{Aut}_A(X)$ has SIP and hence $\text{Aut}(X)$ has SIP.

Now by combining Theorem 1.7, Theorem 1.10 and Lemma 1.8 we conclude the following:

**Theorem 1.11.** Suppose $M$ is an ab-initio generic structure that is obtained from a pre-dimension function with rational coefficients. Suppose $H$ is a subgroup of $G = \text{Aut}(M)$ of small index. Then $H$ is an open subgroup of $G$.

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1 Theorem 1.10 was suggested by David M. Evans after pointing out some problems in earlier versions of the proof of Theorem 1.11.
Proof. By Theorem 1.7 there is a finite subset $A$ of $M$ such that $G_X \leq H$ where $X = \text{gcl}(A)$. Let $H' := H \cap G_{\{X\}}$. It is clear that $G_X \leq H'$ and $[G_{\{X\}} : H'] \leq \aleph_0$. Now let $\pi_X : G_{\{X\}} \to \text{Aut}(X)$ be the projection map that has been defined in Lemma 1.8. By Lemma 1.8 (4), the projection map $\pi_X$ is surjective. Therefore, $\pi_X(H')$ is a small index subgroup of $\text{Aut}(X)$. From Theorem 1.10 we know that $\text{Aut}(X)$ has SIP. Therefore $\pi_X(H')$ is open in $\text{Aut}(X)$. Since $\pi_X$ is continuous, $\pi_X^{-1}(\pi_X(H'))$ is open in $G_{\{X\}}$. Note that $\pi_X^{-1}(\pi_X(H')) = H' \ker(\pi_X)$. By our assumption $\ker(\pi_X) = G_X \leq H'$ and hence $\pi_X^{-1}(\pi_X(H')) = H'$. Therefore $H'$ is open in $G_{\{X\}}$ and then open in $G$. □

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2. The almost small index property

In this section we first prove the following

Lemma 2.1. Let $M$ be the $(K_0, \leq)$-generic structure. There exists a countable subset $B \subseteq M$ such that:

1. $\text{gcl}(B) = M$;
2. $B_0 \leq M$ for all $B_0 \subseteq B$;
3. Every permutation of $B$ extends to an automorphism of $M$.

Proof. Fix an enumeration $\langle m_i : i \in \omega \rangle$ of elements of $M$. We start finding elements $b_i$ in $M$ for $i \in \omega$ inductively such that:

1. $b_0 \cdots b_i$ is $\leq$-closed in $M$ for $i \in \omega$;
2. $m_i \in \text{gcl}(b_0 \cdots b_{i+1})$ for $i \in \omega$.

Choose $b_0$ to be a $\leq$-closed element in $M$. Assume $b_i$'s is chosen for $i \leq s$ and they satisfy the conditions above. If $m_s \in \text{gcl}(b_0 \cdots b_s)$ then let $b_{s+1}$ be an element that $d(b_{s+1}/b_0 \cdots b_s) = m$ (i.e. $b_0 \cdots b_s \leq b_0 \cdots b_s b_{s+1} \leq M$ and $R^M(b_{s+1} ; b_0 \cdots b_s) = \emptyset$). Otherwise $1 \leq d(m_s/b_0 \cdots b_s) \leq m$. If $d(m_s/b_0 \cdots b_s) = m$, then $b_0 \cdots b_s m_s$ is a $\leq$-closed set and in this case let $b_{s+1} = m_s$. Suppose now $d(m_s/b_0 \cdots b_s) = l$ where $1 \leq l < m$. Let $B_s := \text{cl}(m_s b_0 \cdots b_s)$ and then let $C = B_s \cup \{c_1, \cdots, c_{m-l}\}$ be an $\mathcal{L}$-structure such that $\delta(c_1/B_s) = \delta(c_i/B_{s+i-1} \cdots c_1) = 1$ for $1 < i \leq m - l$. Then $\delta(C/B_s) = m - l$, $C \in K_0$ and $B_s \leq C$. By Theorem 1.3 we can strongly embed $C$ over $B_s$ inside $M$. With
it follows that \( C \gamma \) and forth argument in the following manner. We build finite partial isomorphisms \( g \) such that:

\[
\gamma = \text{gcl} (b_0 \cdots b_s) = m. \quad \text{Choose } b_{s+1} \quad \text{be } c_{m-t}. \quad \text{In all the above cases } d (m_s/b_0 \cdots b_{s+1}) = 0 \quad \text{and hence } m_s \in \text{gcl} (b_0 \cdots b_{s+1}).
\]

Suppose \( \langle b_i : i < \omega \rangle \) satisfies Conditions (1) and (2). Let \( B := \{ b_i : i < \omega \} \) and suppose \( \gamma \) is a permutation of \( B \). We want to show that \( \gamma \) extends to an automorphism of \( M \). This is feasible by a back and forth argument in the following manner. We build finite partial isomorphisms \( g_0 \leq g_1 \leq \cdots \) between \( \leq \)-closed subsets of \( M \) and, then \( \tilde{\gamma} := \bigcup_{i < \omega} g_i \) will be the desired automorphism of \( M \) that extends \( \gamma \). We only explain how to define \( g_0 \) and the forth step of extending \( g_0 \) to \( g_1 \). The back step can be done with a similar argument.

Suppose \( m_i \) is the first element in the enumeration of \( M \) that \( m_i \notin B \). Let \( j \) be the smallest index in the sequence \( \langle b_i : i < \omega \rangle \) that \( m_i \in \text{gcl} (b_0 \cdots b_j) \). Write \( B' \) for the set \( \{ b_i : i \leq j \} \) and let \( g_0 := \gamma \upharpoonright B' \) and \( B_i := \text{cl} (B'm_i) \). Let \( D \) be an isomorphic copy of \( B_i \) over \( g_0 [B'] \) such that \( (D \setminus g_0[B']) \cap (B \setminus B') = \emptyset \). Extend \( g_0 \) to \( g_1 \) such that \( g_1 [B_i] = D \). Notice that \( B^i \subseteq B_i \) and \( B_i C \subseteq M \) for all \( B^i \subseteq C \subseteq \text{fin } B \) and therefore \( g_1 \cup \gamma \) is a partial isomorphism. We can continue building partial isomorphism \( g_i \)'s for \( i \in \omega \) and, then \( \tilde{\gamma} \) will be the desired automorphism of \( M \) that extends \( \gamma \) and hence, the sequence \( \langle b_i : i \in \omega \rangle \) satisfies Condition (1-3).

**Proof of Theorem 1.7.** Suppose \( B \) is a countable set that is obtained from Lemma 2.1. Our aim is to enrich the language \( \mathcal{L} \) to \( \mathcal{L}^* := \mathcal{L} \cup \mathcal{F} \cup \{ \mathcal{I} \} \) where \( \mathcal{F} \) is a countable set of functions and \( \mathcal{I} \) is a unary predicate such that:

1. \( \langle B_0 \rangle_{\mathcal{F}} = \text{gcl} (B_0) \) for all \( B_0 \subseteq B \);
2. \( \mathcal{F} \) is compatible with permutations of \( B \): For each permutation \( \beta \) of \( B \), there is a unique \( \tilde{\beta} \in \text{Aut}(M) \) such that \( \beta \leq \tilde{\beta} \), and \( \tilde{\beta} [\langle B_0 \rangle_{\mathcal{F}}] = \langle \beta [B_0] \rangle_{\mathcal{F}} \) for all \( B_0 \subseteq_{\text{fin }} B \);
3. \( \mathcal{I}(M) = B \).

First suppose such an enrichment \( \mathcal{L}^* \) of \( \mathcal{L} \) exists. Let \( M^* \) be the structure \( M \) in the expanded language \( \mathcal{L}^* \). It is clear that \( \text{Aut}(M^*) \) is a closed subgroup of \( \text{Aut}(M) \). Assume \( H \) is a subgroup of small index in \( \text{Aut}(M) \). Then \( H \cap \text{Aut}(M^*) \) has small index in \( \text{Aut}(M^*) \). Note that by Condition (2) the family \( \mathcal{F} \) is compatible with the permutations of \( B \), and the unary predicate \( \mathcal{I} \) guarantees that every automorphism of \( M^* \) preserves \( B \) set-wise. Therefore, \( \text{Aut}(M^*) \) and the group of permutations of \( B \) are isomorphic (the restriction map from \( \text{Aut}(M^*) \) to \( \text{Aut}(B) \) is an isomorphism). By the result of Dixon, Neumann
and Thomas in [3] the group of permutations of $B$ which is isomorphic to $S_\omega$, has the small index property. Hence, there is $B_0 \subseteq_{fin} B$ such that $\text{Aut}_{B_0}(M^*) \leq H \cap \text{Aut}(M^*)$. Now we want to show that $\text{Aut}_{B_0}(M) \leq H$ where $\hat{B}_0 = \text{gcl}(B_0)$.

Similar to [5], let $X = \{ \text{gcl}(A) : A \subseteq_{fin} M \}$ and $\mathcal{F}$ consist of all maps $f : X \to Y$ with $X, Y \in \mathcal{X}$ which extend to automorphisms. By Lemma 4.3. and Corollary 4.8 in [5], the independence notion that is derived from $\text{gcl}(-)$, is a stationary independence (cf. Definition 2.2 in [5]) that is compatible with the class $\mathcal{X}$. Suppose $S \subseteq \mathcal{F}$ and let $\hat{G} (S) = \{ g \in G : g \restriction X \in S \text{ for all } X \in \mathcal{X} \}$. By Lemma 2.3 in [5] if $S_0 \subseteq \mathcal{F}$ is a countable subset, then there exists a countable $\mathcal{S}$ with $S_0 \subseteq \mathcal{S}$ such that $\hat{G} (S)$ is a Polish group: when we topologise $G$ by taking the basic open sets those of the form $O(f) = \{ g \in G : f \leq g \}$ where $f \in \mathcal{F}$.

Suppose now $h \in \text{Aut}_{\hat{B}_0}(M)$. We want to show that $h \in H$. Let $\mathcal{X}_{\hat{B}_0} := \{ X \in \mathcal{X} : B_0 \subseteq X \}$ and $\mathcal{P} \subseteq \mathcal{F}$ be a countable set that contains the identity maps, is closed under inverses, restrictions and compositions, and:

1. If $\epsilon \in \mathcal{P}$, then $\text{id}_{\hat{B}_0} \leq \epsilon$;
2. $h \restriction X \in \mathcal{P}$ for all $X \in \mathcal{X}_{\hat{B}_0}$;
3. For all finite subset $B_1 \subseteq B$ that contains $B_0$, and $u$ a partial isomorphism of $B_1$ into a subset of $B$ which is identity on $B_0$, there is a unique $\mathfrak{L}^*$-extension of $u$ to $\text{gcl}(B_1)$ in $M^*$ which belongs to $\mathcal{P}$.
4. If $\epsilon, \nu \in \mathcal{P}$ such that $\epsilon, \nu \leq \sigma$ for some $\sigma \in \text{Aut}(M)$, then there is $\hat{\sigma} \in \mathcal{P}$ such that $\epsilon, \nu \leq \hat{\sigma}$;
5. If $\epsilon \in \mathcal{P}$, $\epsilon : X \to Y$ and $Z \in \mathcal{X}_{\hat{B}_0}$, $X \cup Y \subseteq Z$, then there exists $\lambda \in \mathcal{P}$ such that $\epsilon \leq \lambda$ and $\lambda : Z \to Z$.

Let $\hat{G} := G (\mathcal{P})$ and $K := \text{Aut}_{\hat{B}_0}(M^*)$. It is clear that $h \in \hat{G}$. From (3) follows that $K \subseteq \hat{G}$, and we know $K \subseteq H$.

See Lascar’s proof of Propositions 7 and 8 in [15] for the following claim:

Claim. The followings hold:

1. The set of all $\mathcal{P}$-generic automorphisms (see Definition 4.1) of $\hat{G}$ is $G_\delta$, and comeager in $\hat{G}$;
2. Suppose $g$ and $g'$ are two $\mathcal{P}$-generic automorphisms, then there exists $\alpha \in K$ such that $g = \alpha \circ g' \circ \alpha^{-1}$.

Now, we want to show that $H$ contains $\hat{G}$. Note that $H \cap \hat{G}$ has small index in $\hat{G}$. The group $\hat{G}$ is a Polish group. Hence $H \cap \hat{G}$ is not meager; meager subgroups has large index in $\hat{G}$. Then by the above
Lemma, \( H \cap \hat{G} \) contains an \( \mathcal{P} \)-generic element. Since \( K \subseteq H \), \( \hat{G} \) again by the above Lemma the group \( H \cap \hat{G} \) contains the set of all \( \mathcal{P} \)-generic automorphisms. Therefore, \( H \cap \hat{G} \) is a comeager subgroup of \( G \). Hence \( H \cap \hat{G} = \hat{G} \) and then \( h \in H \).

Now, we show the enrichment that was claimed exists. Suppose \( E \subseteq M \) and define the following operation

\[
\mathcal{H}(E) := \bigcup \{A \subseteq M : A \text{ is 0-algebraic over a finite subset of } E\}.
\]

Let \( M_0 := \mathcal{H}(B) \), \( M_i := \mathcal{H}(M_{i-1}) \) for \( i > 0 \) and finally \( M_\omega := \bigcup_{i \in \omega} M_i \). Note that \( M_\omega = M \). We define a family \( \mathfrak{g}^i \) of functions for each \( i \in \omega \). Let \( M_{-1} := B \) and fix \( s \in \omega \). Assume \( A \in M_s \) and assume \( C \) is the finite subset of \( M_{s-1} \) that \( A \) is 0-minimally algebraic over \( C \).

Let \( \{A_i : i \in \omega\} \) be an enumeration of all the isomorphic copies of \( A \) that are 0-minimally algebraic over \( C \); without repetition. Fix \( \bar{c} \) an enumeration of \( C \) and assume \( |\bar{c}| = n \). For each \( i \in \omega \), let \( \bar{a}_i \) to be enumeration of \( A_i \) such that \( \text{tp}(\bar{a}_i/\bar{c}) \equiv \text{tp}(\bar{a}_j/\bar{c}) \) when \( i, j \in \omega \). For \( i \in \omega \) let \( f^s_{i,A} \) be a map that \( f^s_{i,A}(\bar{c}) = \bar{a}_i \). Extend the domain of \( f^s_{i,A} \) to \( M^n \) as follows: \( f^s_{i,A}(\bar{x}) = (x_0, \ldots, x_0) \) if \( \bar{x} \in M^n \) and \( \bar{x} \neq \bar{c} \). Let \( \mathfrak{g}_A^i := \{f^s_{i,A} : i \in \omega\} \). We assume for elements \( A_1, A_2 \) in \( M_s \) if \( A_1, A_2 \) are 0-minimally algebraic over a the same finite set \( C \subseteq M_{s-1} \) and \( A_1 \cong_C A_2 \), then \( \mathfrak{g}_A^i = \mathfrak{g}_A^{i'} \). Now define \( \mathfrak{g}^i := \bigcup_{A \in M_s} \mathfrak{g}_A^i \).

Finally, let \( \mathfrak{g} := \bigcup_{i \in \omega} \mathfrak{g}^i \). Notice that elements of \( \mathfrak{g} \) are not necessarily functions from some power of \( M \) to \( M \) as to be considered in \( \mathfrak{L} \). But this can be fixed by considering all the projections of them to each single arity. It is clear that \( \langle B_0 \rangle_{\mathfrak{g}} = \text{gcl}(B_0) \) for all \( B_0 \subseteq B \). One can extend any permutation of \( B \) step by step to \( M_i \) in a unique way similar to the proof of Lemma 2.1 for each \( i \in \omega \).

3. Proof of Lemma 1.8

First we prove the following lemma whose proof is very similar to the proof of Lemma 3.2.19 in [6].

**Lemma 3.1.** Let \( X = \text{gcl}(A) \) where \( A \subseteq_{\text{fin}} M \) and suppose \( g \in \text{Aut}(X) \). Then, there is \( \gamma \in \text{Aut}(M) \) that extends \( g \).

**Proof.** Without loss of generality we can assume \( A \) is \( \leq \)-closed. Fix \( \langle B_i : i < \omega \rangle \) a chain of finite \( \leq \)-closed subsets of \( M \), such that \( B_0 := A \) and \( M = \bigcup_{i < \omega} B_i \). Similarly, fix \( \langle C_i : i < \omega \rangle \) a chain of finite \( \leq \)-closed subsets of \( M \), such that \( C_0 := g[A_0] \) and \( M = \bigcup_{i < \omega} C_i \). Let \( g_0 := g \mid B_0 \). Using a back and forth construction in the following, we build finite partial isomorphisms \( g_0 \leq g_1 \leq \cdots \) between \( \leq \)-closed subsets of \( M \) and, then \( \gamma := \bigcup_{i < \omega} g_i \) will be the desired automorphism of \( M \).
that extends $g$. When $i = 2k$ we make sure that $C_k$ is in the range of $g_i$ and agrees with $g \upharpoonright (C_k \cap X)$, and when $i = 2k + 1$ we make sure $B_{k+1}$ is in the domain of $g_i$ and agrees with $g \upharpoonright (B_{k+1} \cap X)$. As $\bigcup_{i<\omega} B_i = \bigcup_{i<\omega} C_i = M$, then $\gamma$ will be an automorphism of $M$.

Assume $g_i$ is defined for $i = 2k$ and we want to construct $g_{i+1}$. Let $D_i = \text{dom}(g_i)$. If $B_{k+1} \subseteq D_i$ then let $g_{i+1} = g_i$. Suppose now $B_{k+1} \setminus D_i \neq \emptyset$ and let $\check{B}_{k+1} := \text{cl}(B_{k+1} \setminus D_i)$ and $\check{B}_k := \check{B}_{k+1} \setminus (D_i \cup X)$. It is clear that $D_i \cup (X \cap \check{B}_{k+1}) = (D_i \cup X) \cap \check{B}_{k+1} \subseteq \check{B}_{k+1}$. By the $\leq$-genericity of $M$, we can find $E$, an isomorphic copy of $\check{B}_k$, over $g_i[D_i] \cup g[\check{B}_{k+1} \cap X]$ such that $g_i[D_i] \cup g[\check{B}_{k+1} \cap X] \cup E$ is $\leq$-closed in $M$. Now let $g_{i+1}$ be the partial isomorphism that extends $g_i[D_i] \cup g[\check{B}_{k+1} \cap X]$ and sends $\check{B}_k$ to $E$ (note that $g_i[D_i] \cup g[\check{B}_{k+1} \cap X]$ is already a partial isomorphism of $\leq$-closed sets). Similarly we can extend $g_i$ for $i = 2k + 1$ such that $C_k \subseteq \text{rang}(g_i)$. 

**Proof of Lemma 7.8** (1) Let $A' := \text{cl}(A)$. It is clear that $G_{A'} \leq G_{\{X\}}$ and therefore $G_{\{X\}}$ is open.

(2) Follows immediately from (1).

(3) (Special case of Theorem 5.1.5 in [6]) Let $H \leq G$ with $[G : H] \leq \aleph_0$. Then $H' := H \cap G_{\{X\}}$ has small index in $G_{\{X\}}$. If $G_{\{X\}}$ has SIP, then $H'$ is open in $G_{\{X\}}$. Therefore from (1) follows that $H'$ is open in $G$, thus $H$ is open in $G$.

(4) It is clear that $\pi_X$ is a group homomorphism. Surjectivity follows from Lemma 3.1. Let $K := \text{Aut}_{X_0}(X)$ be a basic open neighbourhood of the identity in $\text{Aut}(X)$ where $X_0 \subseteq_{\text{fin}} X$. Then $\pi^{-1}_X(K) = G_{X_0} \cap G_{\{X\}}$ which is a basic open neighbourhood of the identity in $G_{\{X\}}$. Also it is clear that $\pi_X(G_{X_0} \cap G_{\{X\}}) = \text{Aut}_{X_0}(X)$. 

□

4. THE SMALL INDEX PROPERTY OF $\text{Aut}(M)$

In this section we prove Theorem 1.9 and Theorem 1.10.

Fix the following notation: Suppose $A, B, C$ are $\mathcal{L}$-structures and $A, B \subseteq C$ such that $A \cap B = \emptyset$, $A \leq C$ and $AB \leq C$. Write $\binom{C}{A}$ for the set of all $\leq$-embeddings of $A$ in $C$. Let $\alpha \in \binom{C}{A}$, and write $\mu_C(B, \alpha)$ for the set $\{\alpha' \in \binom{C}{AB} : \alpha' \upharpoonright A = \alpha\}$.

Recall the following definitions from [11]:

**Definition 4.1.** Suppose $M$ is a countable first order structure, and let $G := \text{Aut}(M)$.
A countable class of substructures $\mathcal{B}$ of $M$ is called a base for $M$ if:

(a) $G_A$ is open in $G$ for all $A \in \mathcal{B}$;
(b) If $A \in \mathcal{B}$ and $g \in G$, then $g[A] \in \mathcal{B}$.

Let $\mathcal{B}$ be a base for $M$ and $n \in \omega$ a nonzero integer. Let $\gamma = (g_1, \ldots, g_n)$ be a sequence of elements of $G$. We say $\gamma$ is $\mathcal{B}$-generic if the following two conditions hold:

(a) If $A \in \mathcal{B}$, then $\{G_B : A \subseteq B \in \mathcal{B}, g[B] = B \text{ for } i \leq n\}$ is a base of open neighbourhoods of $1$ in $G$.
(b) Whenever $A \in \mathcal{B}$ is such that $\gamma \upharpoonright A$ is a sequence of automorphisms of $A$ and $A_1 \in \mathcal{B}$ and $\theta = (t_i : i \leq n)$ is a sequence of automorphisms of $A_1$ extending $\gamma \upharpoonright A$ i.e. $g_i \upharpoonright A \leq t_i$ for all $i \leq n$, then there exists $\alpha \in G_A$ such that $\gamma$ extends $\alpha \circ \theta \circ \alpha^{-1}$ (or $\gamma^\alpha := (g_i^\alpha : i \leq n)$ extends $\theta$).

Suppose $\mathcal{B}$ is a base for $M$. Then the structure $M$ has ample $\mathcal{B}$-generic automorphisms if for all non-zero $n \in \omega$, the set of $\mathcal{B}$-generic elements of $G^n$ is comeager in $G^n$, in the product topology.

We say $M$ has ample homogeneous generic automorphisms if there exists a base $\mathcal{B}$ for $M$ such that $M$ has $\mathcal{B}$-generic automorphism.

Suppose $\mathcal{B}$ is a base for $M$. We say $\mathcal{B}$ is an amalgamation base if

(a) $\mathcal{B}$ is countable.
(b) If $e_1, \ldots, e_n$ are finite elementary maps from $M$ to $M$ and $A \in \mathcal{B}$. Then there is $B \in \mathcal{B}$ containing $A$ and $f_i \in \text{Aut}(B)$ such that $e_i \leq f_i$ for $0 \leq i \leq n$.
(c) Let $A, B, C \in \mathcal{B}$ with $A \subseteq B, C$. Then there is $\alpha \in G_A$ such that whenever $g \in \text{Aut}(\alpha[B]), h \in \text{Aut}(C)$ satisfy $g \upharpoonright A = h \upharpoonright A \in \text{Aut}(A)$, then $g \cup h$ is an elementary map that can be extended to an automorphism of $M$.

In order to show $\text{Aut}(\hat{M})$ has SIP, we prove $\hat{M}$ has ample homogeneous generic automorphisms and for that we need to show the existence of an amalgamation base for $\hat{M}$. We now introduce the following key combinatorial definition of the extension property:

**Definition 4.2.** Suppose $\mathcal{E}$ is a subclass of $K_0$. We say $\mathcal{E}$ has the extension property (EP) if for every $A \in \mathcal{E}$ and every finite set $e_0, \ldots, e_n$ of elementary maps of $\leq$-closed subsets of $A$, which are extendable to
automorphisms of $M$, there exist $B \in E$ and $f_i \in \text{Aut}(B)$ such that $A \subseteq B$ and $e_i \leq f_i$ for $0 \leq i \leq n$.

To prove a certain class of substructures is an amalgamation base, its extension property appears as a technical part.

Remark 4.3. As mentioned before, in [6] it has been shown that $(K_0, \leq)$ does not have the extension property; EP does not hold for some elements of $K_0$ with even with only one partial $\leq$-closed map. Similarly one can to show that $(C_A, \leq)$ does not have EP where $A \subseteq \text{fin} M$ with $d(A) > 0$ and $C_A := \{ B \in K_0 : A \leq B \}$. It is interesting to comment that for the classes that are obtained from pre-dimensions with irrational coefficients (or simple $\omega$-categorical generic structures with rational coefficients see [4, 5]) one can still show EP does not hold with a slightly different argument, however one needs to consider at least two partial $\leq$-closed maps. More recently, in [7] a connection between having a tree-pair and EP has been observed. Moreover, David M. Evans in an email correspondence has also noted that using a different proof, he can show EP does not hold for either of the classes that are obtained from pre-dimensions with rational and irrational coefficients.

The main technical lemma in this section is the following (the proof is given later).

Lemma 4.4. Let $C := \{ A \in K_0 : \delta(A) = 0 \}$. Then the class $(C, \leq)$ has the extension property.

Then, we conclude

Corollary 4.5. The class $C := \{ A \subseteq \hat{M} : \delta(A) = 0 \}$ is an amalgamation base for $\hat{M}$.

Proof. Condition in 5(a) in Definition 4.1 is obvious. Condition 5(b) follows from Lemma 4.4. For 5(c) let $B'$ be an isomorphic copy of $B$ such that $B' \downarrow_A C$ and $B' \cap C = \emptyset$ (this follows from $M$ being stable or the $(K_0, \leq)$-genericity of $M$ plus the free-amalgamation property). It is also clear that there is $\alpha \in G_A$ such that $\alpha[B] = B'$ and the result follows. \qed

In Theorem 2.9 in [11] it is shown if $M$ is a countable $\omega$-categorical structure and $\mathcal{B}$ an amalgamation base, then $M$ has ample $\mathcal{B}$-generic automorphisms. Moreover, in Theorem 5.3 in [11] it is shown if $M$ is a countable structure with ample homogeneous generics, then $M$ has SIP. Now using Corollary 4.5 we can finish the proof Theorem 1.9.
Proof of Theorem 1.9. We follow a similar method in [11]. In Corollary 4.5 we proved \( \mathcal{C} := \left\{ A \subseteq \hat{M} : \delta(A) = 0 \right\} \) is an amalgamation base for \( \hat{M} \). Notice that in our case \( \hat{M} \) is not \( \omega \)-categorical, however we can prove, following the proof of Theorem 2.9 in [11] using Lemma 3.1 instead of Corollary 2.5 in [11], that the structure \( \hat{M} \) has ample \( \mathcal{C} \)-generic automorphisms. Then, from Theorem 5.3 in [11] we conclude \( \hat{M} \) has SIP. \[ \square \]

Before starting the proof of Lemma 4.4, we need to consider the following definitions.

**Definition 4.6.**

1. Suppose \( A \in \mathcal{C} \) and \( E \subseteq A \). We say \( E \) is **minimally closed (m.c.)** in \( A \) if \( E \preceq A \) and \( \delta(E') > \delta(E) \) for all \( E' \preceq E \) (or equivalently \( \text{cl}(E') = E \) for all \( E' \subseteq E \)). Define \( \mathcal{E}(A) := \{ E \subseteq A : E \text{ is m.c. in } A \} \).
2. Suppose \( A \in \mathcal{C} \) and \( C \subseteq A \). We say \( C \) is a **connected zero-set (c.z.)** of \( A \) if \( C \preceq A \) and \( C \) cannot be partitioned into nonempty disjoint \( \preceq \)-closed subsets. We say \( C \) is a **maximal connected zero-set (m.c.z.)** if there is no connected zero-set \( C' \subseteq A \) that contains \( C \) and \( C' \neq C \). Write \( \mathcal{F}(A) \) for the set \( \{ C \subseteq A : C \text{ is m.c.z.} \} \).
3. To each \( C \) in \( \mathcal{F}(A) \), we assign a number \( l_C \) which is the minimum natural number such that \( C = \bigcup_{i \leq l_C} C_i \) where:
   - \( a) \ C_0 := \bigcup \mathcal{E}(C) \);
   - \( b) \ C_{i+1} := C_i \cup \bigcup \{ D \subseteq C : D \text{ is 0-algebraic over } C_i \} \) and \( C_{i+1} \neq C_i \) for \( 0 < i < l_C \), and \( C_{l_C} = C \).
   - We call \( l_C \) the **level of complexity** of \( C \).

**Remark 4.7.**

1. Suppose \( A \in \mathcal{C} \). Then elements of \( \mathcal{E}(A) \) are disjoint (see Lemma 3.1.5. in [6]). Moreover, if \( E_1, E_2 \in \mathcal{E}(A) \) are distinct, then \( \mathcal{R}^A(E_1; E_2) = \emptyset \).
2. Suppose \( C \in \mathcal{F}(A) \). It is easy to see that there is \( E \in \mathcal{E}(A) \) such that \( E \subseteq C \), and \( \mathcal{E}(C) \subseteq \mathcal{E}(A) \). Similar to (1) elements of \( \mathcal{F}(A) \) are disjoint, and for any two distinct \( C_i, C_j \in \mathcal{F}(A) \) we have \( \mathcal{R}^A(C_i; C_j) = \emptyset \).

We use the following definitions in the proof EP for \((\mathcal{C}, \preceq)\).

**Definition 4.8.** Suppose \( A \in \mathcal{C} \) and let \( i \in \omega \) be a nonzero integer.

1. We call \( B \subseteq A \) an **i-base** subset of \( A \) if there is \( C \in \mathcal{F}(A) \) such that:
   - \( a) \ l_C \geq i \) and \( B \subseteq C_{i-1} \);
Proof of Lemma 4.4. In the following, we are going to construct by induction partial maps to an automorphism in a bigger structure. When the level of the complexity of every m.c.z of the given element is zero as follows: As the c.z. sets are the smallest \( \leq \)-closed sets, the partial maps for each c.z. set are either defined for the whole set or not defined at all. So we can see them as colored single points (color determines the isomorphism type). Then it is easy to see how we can extend the partial maps to an automorphism in a bigger structure. When the level of complexity is higher it becomes more complicated but still doable by induction.

For each \( 0 \leq i \leq n \) write \( D_i := \text{dom} (e_i) \) and \( R_i := \text{ran} (e_i) \). Notice that \( e_i \)'s are isomorphisms of \( \leq \)-closed sets and elements of \( \mathcal{E}(A) \) are the smallest \( \leq \)-closed subsets of \( A \). Therefore for an element \( E \in \mathcal{E}(A) \) either \( E \subseteq D_i \cup R_i \) or \( E \cap (D_i \cup R_i) = \emptyset \).

For each \( 0 \leq i \leq n \) and \( d \in D_i \), let \( \alpha_{i,d} \) be the smallest natural number that \( e_i^{(\alpha_{i,d})} (d) = e_i \circ \cdots \circ e_i (d) = d \) if exists; otherwise let \( \alpha_{i,d} = 1 \). Put \( \alpha_i := \{ \alpha_{i,d} : d \in D_i \} \) and let \( \alpha := \prod_{0 \leq i \leq n} \alpha_i \).

Fix an enumeration \( \{ E_1, \cdots, E_k \} \) of the elements of \( \mathcal{E}(A) \) and put \( \mu_j := \left| \left( \begin{array}{c} A \\ E_j \end{array} \right) \right| \) for \( 1 \leq j \leq k \). Note that \( \mu_i = \mu_j \) for \( 1 \leq i, j \leq k \) when \( E_i \cong E_j \). Put \( \tilde{E} = \bigcup_{0 \leq j \leq k} E_j \) and let \( B_0 \) be the \( \mathfrak{L} \)-structure that is the disjoint union of isomorphic copies of \( E_i \)'s for each \( 1 \leq i \leq k \) such that \( \left| \left( \begin{array}{c} B_0 \\ E_i \end{array} \right) \right| = \alpha \cdot \mu_i \), and they are connected by an edge. Then \( \delta (B_0) = 0 \), \( \tilde{E} \leq B_0 \) and \( B_0 \in \mathcal{K}_0 \).
In the following, for each $0 \leq i \leq n$, we introduce $f_{i,0}$, an automorphism of $B_0$, that it extends $e_i \mid \bar{E}$. Fix $0 \leq i \leq n$, and let $D_{i,0} := D_i \cap \bar{E}$ and $R_{i,0} := R_i \cap \bar{E}$.

**Case 1.** Suppose $o_i = 1$. Define $f_{i,0}$ be as follows:

1. $f_{i,0} \upharpoonright D_{i,0} = e_i \upharpoonright D_{i,0}$;
2. For each $r \in R_{i,0} \setminus D_{i,0}$, define $f_{i,0}(r) = d$ where $d \in D_{i,0}$ such that there exists $s \geq 1$ with $e_i^{(s)}(d) = e_i \circ \cdots \circ e_i(d) = r$, and $d \notin R_i$. Notice that in this case $f_{i,0}^{(s+1)}(d) = d$.
3. $f_{i,0}$ fixes all elements of $B_0 \setminus (R_{i,0} \cup D_{i,0})$.

For $b_1, b_2 \in B_0$ one can check that $R^{B_0}(b_1, b_2)$ if and only if $R^{B_0}(f_{i,0}(b_1), f_{i,0}(b_2))$.

**Case 2.** Suppose $o_i \neq 1$. Assume $E_j \subseteq D_{i,0}$ and $E_j$ has the smallest index in $\{E_1, \ldots, E_k\}$.

**Case i.** If $e_i^{(s)}[E_j] = E_j$ for some $s \leq o_i$, then define $f_{i,0}$ be the same as $e_i$ for $E_j$.

**Case ii.** Otherwise, define $f_{i,0}$ as follows: First let $s$ be such that $e_i^{(s)}[E_j] \in R_{i,0}$ but $e_i^{(s+1)}[E_j]$ is not defined. By our assumption $s \leq o_j$ and moreover $\left|\left(D_{i,0} \setminus E_j\right)\right| \leq o_j$. $B_0$ has $(o \cdot o_j)$-many distinct isomorphic copies of $E_j$. Pick $o_i \cdot (s - 1)$-many distinct elements $\{E_j^r : 1 \leq r \leq o_i \cdot (s - 1)\}$ of the isomorphic copies of $E_j$ in $B_0 \setminus (D_{i,0} \cup R_{i,0})$ and extend $e_i$ to $f_{i,0}$ in such way that:

1. $f_{i,0}
[e_i^{(s)}[E_j]] = E_j^1$;
2. $f_{i,0}
[E_j^r] = E_j^{r+1}$ for $1 \leq r < o_i \cdot (s - 1)$;
3. $f_{i,0}
[E_j^{o_i \cdot (s - 1)}] = E_j$.

Notice that in this case $f_{i,0}^{(o_i \cdot s)}[E_j] = E_j$. We continue this procedure similarly and define $f_{i,0}$ for each element of $\mathcal{E}(A)$ in the domain of $e_i$ and in each stage we make sure that we pick those isomorphic copies that is not chosen in previous steps. There are enough isomorphic copies of each element of $\mathcal{E}(A)$ in $B_0$ to allow us to extend $e_i$ to $f_{i,0}$ as we desire. Finally let $f_{i,0}$ to allow us to extend $e_i$ to $f_{i,0}$ as we desire. One can check that $f_{i,0}$ is an automorphism of $B_0$.

If $\bigcup \mathcal{E}(A) = A$, then we are finished in this first step and, $B_0$ and $f_{i,0}$‘s for $0 \leq i \leq n$ are our solution.
Suppose now $\bigcup \mathcal{E}(A) \neq A$. Let $I := \max \{ I_C : C \in \mathcal{F}(A) \}$ and note that in this case $I > 0$. Our aim is to construct $B_j \in \mathcal{C}$ for $1 \leq j \leq I$ by induction such that:

1. $B_0 \leq B_1 \leq \cdots \leq B_i$, and $A \subseteq B_i$;
2. $B_q$ contains all subsets of $A$ with the level of complexity $\leq q$, for $0 < q \leq I$;
3. $B_q$ has $q$-uniform algebraicity, for $0 < q \leq I$.

And then for each $0 \leq i \leq n$ we explain how to extend $f_{i,q}$ to $f_{i,q+1}$, an automorphism of $B_q$, that extends $e_i \upharpoonright (B_q \cap A)$ for $0 \leq q < I$. Our final solution for EP is $B_I$ and automorphisms $f_i := f_{i,1}$ for $0 \leq i \leq n$.

We only explain how to construct $B_1$ from $B_0$, and for a fixed $0 \leq i \leq n$ how to extend $f_{i,0}$ to an automorphism $f_{i,1}$ of $B_1$; the rest can be done inductively in a similar way. Suppose $S$ is a 1-base subset of $A$.

Let $\mathcal{G}_A(S) := \{ Z \subseteq A : Z \text{ is z.m. set over } \alpha[S] \text{ for } \alpha \in \binom{A}{S} \}$. For an element $Z \in \mathcal{G}_A(S)$ put $\nu = \max \{ |\mu_{B_0}(Z, \alpha)| : \alpha \in \binom{B_0}{S} \}$.

Suppose $|\mu_{B_0}(Z, \alpha)| < \nu$ for $\alpha \in \binom{B_0}{S}$. Let $I = \{ 1, \ldots, \nu - |\mu_{B_0}(Z, \alpha)| \}$, and let $B^\alpha$ be an $\mathcal{L}$-structure that contains $B_0$ such that

1. For each $i \in I$ there is $Z^i_\alpha$ such that $\text{tp}(Z^i_\alpha/\alpha[D]) \equiv \text{tp}(\beta[Z]/\alpha[D])$ for any $\beta \in \mu_{B_0}(Z, \alpha)$ and $Z^i_\alpha \cap B_0 = \emptyset$; and,
2. $Z^i_\alpha \cap Z^j_\alpha = \emptyset$ and $\mathcal{R}^{B^\alpha}(Z^i_\alpha, Z^j_\alpha) = \emptyset$ for $i \neq j \in I$.

It is clear that $\delta(B^\alpha) = 0$ and $B^\alpha \in \mathcal{K}_0$. Then, there is an isomorphic copy of $B^\alpha$ in $C$ over $B_0$ which with abuse of notation we assume $B^\alpha \in \mathcal{C}$. Now let $B^Z$ be the free-amalgam of $B^\alpha$’s over $B_0$ for all $\alpha \in \binom{B_0}{S}$ with $|\mu_{B_0}(Z, \alpha)| < \nu$. Since $\mathcal{K}_0$ has the free-amalgamation property then $B^Z \in \mathcal{K}_0$ and with abuse of notation we assume $B^Z \in \mathcal{C}$. It is easy to check that $B^Z$ has 1-uniform algebraicity for isomorphic copies of $Z$. Using the free-amalgamation property we construct $B^Z$ for each element $Z \in \mathcal{G}_A(S)$ and then let $B^S$ be the free-amalgam of all $B^Z$’s over $B_0$ for $Z \in \mathcal{G}_A(S)$. If $S$ and $S'$ are isomorphic and both 1-base subset of $A$ we let $B^S = B^{S'}$.

Repeat the same procedure and construct $B^S$ for every isomorphism type of 1-base subset $S$ of $A$. Now let $B_1$ be the free-amalgam of all $B^S$’s over $B_0$ where $S$ is a 1-base subset of $A$. One can check that $B_1$ has 1-uniform algebraicity and $B_1$ contains all subsets of $A$ with level of complexity $\leq 1$.

We now explain how one can extend $f_{i,0}$ and $e_i \upharpoonright (B_1 \cap A)$, simultaneously, to an automorphism $f_{i,1}$ of $B_1$. Suppose $S$ is a 1-base subset
of $A$ and $Z \subseteq A$ is a zero-minimal set over $S$. Let $\mathfrak{o}_S$ be the smallest number that $f_{i,0}^{(\mathfrak{o}_S)}[S] = S$. Note that $S \subseteq B_0$ and $\mathfrak{o}_S$ exists as $f_{i,0}$ is an automorphism of $B_0$. First suppose $Z \cap (D_i \cup R_i) = \emptyset$. Then pick $(\mathfrak{o}_S - 1)$-many distinct copies $Z^j$ of $Z$ such that $Z^j \cap (D_i \cup R_i) = \emptyset$ and $\text{tp} \left( Z^j / f_{i,0}^{(j)}[S] \right) \equiv \text{tp}(Z/S)$ for $1 \leq j \leq \mathfrak{o}_S - 1$. Extend $f_{i,0}$ to $f_{i,1}$ such that

\begin{enumerate}
\item $f_{i,1}[Z] = Z^1$;
\item $f_{i,1}[Z^j] = Z^{j+1}$ for $1 \leq j < \mathfrak{o}_S - 1$ and $f_{i,1}[Z^{\mathfrak{o}_S - 1}] = Z$.
\end{enumerate}

Now suppose $Z \subseteq (D_i \cup R_i)$.

**Case 1.** If $e_i^{(\mathfrak{o}_S)}(z) = z$ for all $z \in Z$, then let $f_{i,1}$ be an extension of $f_{i,0}$ and $e_i \upharpoonright Z$. Note that since distinct copies of $Z$ are disjoint such extension of $f_{i,0}$ exists.

**Case 2.** Otherwise, suppose $e_i^{(r)}[Z] \in R_i$ but $e_i^{(r+1)}[Z]$ is not defined. Without loss of generality we also assume $Z \not\subseteq R_i$. Note that $r < \mathfrak{o}_S$. Then pick $(\mathfrak{o}_S - r)$-many distinct elements $\{Z^q : 1 \leq q \leq (\mathfrak{o}_S - r)\}$ of copies of $Z$ such that $\text{tp} \left( Z^q / f_{i,0}^{(q)}[S] \right) \equiv \text{tp}(Z/S)$ for all $1 \leq q \leq (\mathfrak{o}_S - r)$. Then extend $f_{i,0}$ to $f_{i,1}$ such that

\begin{enumerate}
\item $f_{i,1}[e_i^{(\mathfrak{o}_S)}[Z]] = Z^1$;
\item $f_{i,1}[Z^q] = Z^{q+1}$ for $1 \leq q < (\mathfrak{o}_S - r)$;
\item $f_{i,1}[Z^{(\mathfrak{o}_S - r)}] = Z$.
\end{enumerate}

Note that this is guaranteed by 1-uniform algebraicity of $B_1$ and it is clear that $f_{i,1}^{(\mathfrak{o}_S)}[Z] = Z$. We continue this procedure and define $f_{i,1}$ be an extension of $f_{i,0}$ and $e_i \upharpoonright (A \cap B_1)$ for all 1-based subsets of $A$ in the domain of $e_i$. Finally let $f_{i,1}$ fixes the rest of the elements of $B_1$ that has not been already in the domain or range of $f_{i,1}$. One can check that $f_{i,1}$ is an automorphism of $B_1$. \hfill $\Box$

And finally

**Proof of Lemma 1.11.** It is clear that if $\text{Aut}_A(X)$ has SIP, then $\text{Aut}(X)$ has SIP (for example it follows from Theorem 5.1.5 in [6]). Let $\mathcal{C}_A := \{ B \subseteq X : A \subseteq B \}$. It is easy to show that $\mathcal{C}_A$ is an amalgamation base: With a similar argument for proving EP for $(\mathcal{C}, \subseteq)$ in Section 4, one can show if $f_0, \ldots, f_n$ are partial isomorphisms of $\subseteq$-closed subsets of $D \in \mathcal{C}_A$ that are extendable to automorphisms of $\hat{M}_A$, then there is $D' \in \mathcal{C}_A$ such that $D \subseteq D'$ and $f_i$’s extend to automorphisms of $D'$. \hfill $\Box$
5. Remaining cases

It is known that if the automorphism group of an \( \omega \)-categorical structure \( M \) has the strong small index property then \( \text{Th}(M) \) admits weak elimination of imaginaries (see [10] p.161 for definition and reference). Furthermore, \( \text{Th}(M) \) has weak elimination of imaginaries (cf. [1], Proposition 5.3) and it is interesting to determine whether or not \( \text{Aut}(M) \) has the strong small index property.

When the coefficient of the pre-dimension \( \delta \) is rational, using a finite-to-one function \( \mu \) over the 0-minimally algebraic elements, one can restrict the ab-initio class \( K_0 \) to \( K_0^\mu \) such that \( (K_0^\mu, \leq) \) has the amalgamation property (see [13] for details). Let \( M^\mu \) be the \( (K_0^\mu, \leq) \)-generic structure. Note that \( M^\mu \) is the original ‘collapsed’ version of the construction from [13] which produces structures of finite Morley rank. In Chapter 5 in [6], some results have been given about the small index subgroups of the automorphism group of some collapsed ab-initio generic structures (see for example Theorem 5.1.6 in [6]). Using similar arguments as of the uncollapsed case one can show the following:

**Theorem 5.1.** Suppose \( M^\mu \) is a countable collapsed Hrushovski ab-initio generic structure and \( H \) is a subgroup of small index in \( G := \text{Aut}(M^\mu) \). Then \( H \) is an open subgroup of \( G \).

However, the small index property and almost SIP for the automorphism groups of the following generic structures remain unanswered in this paper: ab-initio generic structures which are obtained from pre-dimension functions with irrational coefficients, and simple \( \omega \)-categorical generic structures (see [4] [5]).

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