Guided waves as superposition of body waves

David R. Dalton*, Michael A. Slawinski†, Theodore Stanoev‡

Abstract

We illustrate properties of guided waves in terms of a superposition of body waves. In particular, we consider the Love and \( SH \) waves. Body-wave propagation at postcritical angles—required for a total reflection—results in the speed of the Love wave being between the speeds of the \( SH \) waves in the layer and in the halfspace. A finite wavelength of the \( SH \) waves—required for constructive interference—results in a limited number of modes of the Love wave. Each mode exhibits a discrete frequency and propagation speed; the fundamental mode has the lowest frequency and the highest speed.

1 Introduction

Let us consider Love wave and the \( SH \) waves to examine the concept of a guided wave within a layer as an interference of body waves therein. In the \( x_1x_3 \)-plane, the nonzero component of the displacement vector of the Love wave is (e.g., Slawinski, 2018, Section 6.3)

\[
 u^\ell_2(x_1, x_3, t) = C_1 \exp \left( -i \kappa s^\ell x_3 \right) \exp \left[ i \left( \kappa x_1 - \omega t \right) \right] + C_2 \exp \left( i \kappa s^\ell x_3 \right) \exp \left[ i \left( \kappa x_1 - \omega t \right) \right],
\]

where \( s^\ell := \sqrt{(v/\beta^\ell)^2 - 1} \), with \( v \) being the speed of the Love wave and \( \beta^\ell \) the speed of the \( SH \) wave; \( \omega \) and \( \kappa \) are the temporal and spatial frequencies, related by \( \kappa = \omega/v \). The \( SH \) waves travel obliquely in the \( x_1x_3 \)-plane; different signs in front of \( x_3 \) mean that one wave travels upwards and the other downwards. Their wave vectors are \( k^\pm := (\kappa, 0, \pm \kappa s^\ell) \). Considering their magnitudes,

\[
 \|k^\pm\| = \sqrt{\kappa^2 + (\kappa s^\ell)^2},
\]

we have

\[
 \|k^\pm\| = \kappa \sqrt{1 + (s^\ell)^2} = \kappa \frac{v}{\beta^\ell};
\]

from which it follows that

\[
 \frac{\beta^\ell}{v} = \frac{\kappa}{\|k^\pm\|} = \sin \theta,
\]

where \( \theta \) is the angle between \( k^\pm \) and the \( x_3 \)-axis. Thus, \( \theta \) is the angle between the \( x_3 \)-axis and a wavefront normal, which means that—exhibiting opposite signs—it is the propagation direction of upward and downward wavefronts.

2 Total reflection

A necessary condition for the existence of a guided wave is a total reflection on either side of the layer; the energy must remain within a layer. For the Love waves, this is tantamount to no transmission of the \( SH \)

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*Department of Earth Sciences, Memorial University of Newfoundland, Canada; dalton.nfld@gmail.com
†Department of Earth Sciences, Memorial University of Newfoundland, Canada; mslawins@mac.com
‡Department of Earth Sciences, Memorial University of Newfoundland, Canada; theodore.stanoev@gmail.com
waves through the surface or the interface. The former is ensured by the assumption of vacuum above the surface; hence, total reflection occurs for all propagation angles, \(\theta\). The latter requires \(\beta_\ell < \beta_h\), where \(\beta_h\) is the speed of the \(SH\) wave within the halfspace. This inequality results in the existence of a critical angle, \(\theta_c = \arcsin(\beta_\ell/\beta_h)\), which is required for a propagation at postcritical angles, \(\theta > \theta_c\).

In view of expression (1), the lower limit of \(v\) is \(\beta_\ell\), for which \(\sin \theta = \beta_\ell/\beta_h = 1\); hence, \(\theta = \pi/2\). It corresponds to the \(SH\) waves that propagate parallel to the \(x_1\)-axis, and can be viewed as the Love wave.

The upper limit, \(v = \beta_h\), is a consequence of the critical angle, for which \(\sin \theta_c = \beta_\ell/\beta_h\). If \(\beta_h \to \infty\) — which corresponds to a rigid halfspace—\(\theta_c \to 0\); hence, the \(SH\) waves within the layer can propagate nearly perpendicularly to the interface and still exhibit a total reflection. This means that \(v \to \infty\), as can be also inferred from Figure 2.

These limits, \(\beta_\ell < v < \beta_h\), are a consequence of total reflection. Also, the upper limit needs to be introduced to ensure an exponential amplitude decay in the halfspace (e.g., Slawinski, 2018, Section 6.3.2).

3 Constructive interference

Guided waves—as superpositions of body waves—require a constructive interference of body waves. A necessary condition of such an interference is the same phase among the wavefronts of parallel rays. In Figure 1, this condition means that the difference between \(\|AB\|\) and \(\|AB'\|\) must be equal to a positive-integer multiple of the wavelength, \(\lambda\), taking into account the phase shift due to reflection. A reflection at the surface results in no phase shift (Udías, 1999, Sections 5.4 and 10.3.1), and the \(SH\)-wave postcritical phase shift at the elastic halfspace is presented by Udías (1999, equation (5.74)).

To illustrate the constructive interference—without discussing the phase shift as a function of the incidence angle—let us consider an elastic layer above a rigid halfspace, on which a transverse wave undergoes a phase change of \(\pi\) radians for any angle. In such a case, the propagation angle is (e.g., Saleh and Teich, 1991, Section 7.1)

\[
\theta_n = \arcsin \left( \frac{n\lambda}{Z} \right), \quad n = 1, 2, \ldots
\]  

(2)

where \(\lambda\) is the wavelength of the \(SH\) wave, \(Z\) is the layer thickness and \(n\) is a mode of the guided wave; \(n = 1\) is the fundamental mode.

Thus—as a consequence of constructive interference—for a given \(SH\) wavelength and layer thickness, the propagation angles, \(\theta_n\), form a set of discrete values; each \(n\) corresponds to a mode of the guided wave. As illustrated in Figure 2, each mode has its propagation speed, which—in accordance with expression (1)—is

\[
v_n = \frac{\beta_\ell}{\sin \theta_n},
\]

(3)
Figure 2: Constructive interference for Love wave: The upgoing and downgoing $SH$ wavefronts at two instants; their speed, $\|\beta_r\|$, remains constant—regardless of the wavefront orientation—but the Love-wave speed, $\|v_n\|$, whose direction, $v_n$, remains constant, increases as $\theta_n$ decreases.

where, as a consequence of total reflection, $\theta_n \in (\theta_1, \pi/2)$, where $\theta_1 > \theta_c$. The specific value of $\theta_1$ depends on $Z$ and $\lambda$; it corresponds to the first postcritical value for which $\|AB\| - \|AB'\| = 2 \|AB\| \cos^2 \theta = 2Z \cos \theta$ is a multiple integer of $\lambda$.

Examining Figure 2, we distinguish the upgoing and downgoing wavefronts, which compose the guided wave. Its longest permissible wavelength is twice the layer thickness, $\lambda_1 = 2Z$, which corresponds to the fundamental mode; $\lambda_2 = Z$, $\lambda_3 = 2Z/3$, and, in general, $\lambda_n = 2Z/n$.

4 Frequencies of body and guided waves

$\lambda$, referred to in the caption of Figure 1 and used in expression (2), corresponds to the $SH$ wave; $\lambda_n$, where $n = 1, 2, \ldots$, corresponds to the guided wave. They are related by the propagation angle, $\theta_n$, and by the layer thickness, $Z$.

The radial frequency of a monochromatic $SH$ wave is constant, $\omega = 2\pi \beta_r / \lambda$. The radial frequencies of the Love wave are distinct for distinct modes, $\omega_n = 2\pi v_n / \lambda_n$. For a given model, $\beta_r$, $\beta_h$ and $Z$, the relations between $\omega$ and $\omega_n$, as well as among $\omega_n$, where $n = 1, 2, \ldots$, are functions of $n$ and $\theta_n$; explicitly, $\omega_n = n \pi \beta_r / (Z \sin \theta_n)$, and, in general, its behaviour as a function of $n$ cannot be examined analytically. However, in an elastic layer above a rigid halfspace, in accordance with expression (2),

$$\omega_n = n \pi \frac{\beta_r}{Z \sin \theta_n} = \pi \frac{\beta_r}{\lambda} = \frac{\omega}{2},$$

which is constant for all modes, and depends only on the radial frequency of the $SH$ wave.

The constructive interference, illustrated in Figure 1, requires that

$$\|AB\| - \|AB'\| = a \lambda - b \lambda = (a - b) \lambda,$$

where—in contrast to $a = \|AB\| / \lambda$ and $b = \|AB'\| / \lambda$—$a - b$ is a positive integer; $\lambda$ is the $SH$ wavelength. Following trigonometric relations, we write

$$\|AB\| - \|AB'\| = \|AB\| - \|AB\| \cos(\pi - 2\theta) = \|AB\| (1 - \cos(\pi - 2\theta)) = 2 \|AB\| \cos^2 \theta.$$

Since $\|AB\| = Z / \cos \theta$, where $\theta$ is the $SH$-wave propagation angle, it follows that $2 \|AB\| \cos^2 \theta = 2Z \cos \theta$, and the constructive interference requires that $2Z \cos \theta = (a - b) \lambda$, where $(a - b) \in \mathbb{N}$; in other words,

$$\cos \theta = \frac{a - b}{2Z} \lambda,$$
where $\theta \geq \theta_c$, to ensure the total reflection, and $(a - b) \lambda \leq 2Z$, for $\theta \in \mathbb{R}$.

Using this result and the inverse trigonometric function, we write the first equality of expression (4) as

$$\omega_n = n\pi - \frac{\beta_\ell}{Z\sqrt{1 - \left(\frac{a_n - b_n}{2Z}\right)^2}\lambda^2},$$  

(5)

which corresponds only to a given value of $n$ and, hence, of $\theta_n$, since $a_n - b_n$ changes with the propagation angle, and needs to be restricted to integer values for each $n$.

Following expression (5), we obtain

$$\frac{\omega_n}{\omega_{n+1}} = \frac{n}{n+1} \sqrt{1 - \left(\frac{a_{n+1} - b_{n+1}}{2Z}\right)^2\lambda^2}.$$  

(6)

Since $\theta_{n+1} > \theta_n$, examining Figure 1 and considering given values of $\lambda$ and $Z$, we see that—as $\theta$ increases—$\|\text{AB}\| - \|\text{AB'}\|$ decreases. Hence, $(a_n - b_n) > (a_{n+1} - b_{n+1})$, and the root in the numerator is greater than in the denominator. Consequently, the ratio of roots is greater than unity. However, $n/(n+1) < 1$.

We cannot, in general, determine analytically if the radial frequency of the $n$th mode is higher or lower than the frequency of the $n+1$ mode. To determine it, we need not only to specify $Z$ and the model parameters, which result in $\theta_c$, but also $\lambda$ and $n$, to obtain $\theta_n$ and $\theta_{n+1}$, with integer values of $\|\text{AB}\| - \|\text{AB'}\|$.

5 Numerical example

To obtain specific values, we let $Z = 1000$, $\beta_\ell = 2000$, $\beta_h = 3000$ and $\lambda = 50$, which means that $\theta_c \approx 0.73$, in radians, and $\omega = 2\pi\beta_\ell/\lambda \approx 251$. For the guided wave, in accordance with Figure 1, we obtain—numerically—$\theta_1 \approx 0.76$, which corresponds to $(a_1 - b_1) = 29$. To include higher modes, using expression (6), we obtain $\omega_1/\omega_2 \approx 0.52$, $\omega_2/\omega_3 \approx 0.69$ and $\omega_3/\omega_4 \approx 0.77$, which corresponds to, respectively, $(a_{n+1} - b_{n+1}) = 29 - n = 28$, 27 and 26, and to $\omega_2 = 17.60$, $\omega_3 = 25.55$ and $\omega_4 = 33.07$.

We might infer that the Love-wave fundamental mode, $n = 1$, exhibits the lowest radial frequency—which, following expression (5), is 9.123—and that the frequency increases monotonically with $n$. The highest allowable mode corresponds to $n = 29$, since, for that value, $(a_n - b_n) = 1$. For this mode, $\omega_{29} \approx 182.27$; also, $\omega_{28}/\omega_{29} \approx 0.966$. Frequencies of distinct modes are shown in the left-hand plot of Figure 3.

Examining expression (6), in view of these results, we conclude that—as $n$ increases—both $n/(n+1)$ and the ratio of roots tend to unity; the former from below, the latter from above. The ratio of successive frequencies approaches the ratio of successive overtones for a vibrating string, $\frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots, \frac{29}{27}$.

Furthermore, using expression (3) and the computed values of $\theta_n$, we can obtain the corresponding propagation speeds of the Love-wave modes. For the fundamental mode,

$$v_1 = \frac{\beta_\ell}{\sin\theta_1} = \frac{2000}{\sin(0.76)} = 2903.09,$$

which is the highest speed of this Love wave; it is smaller than $\beta_h = 3000$, as required. The lowest speed corresponds to $\theta_{29} = 1.55$, which is nearly $\pi/2$; hence, the $SH$ waves propagate almost parallel to the layer. The speed of the resulting Love wave is $v_{29} = \beta_\ell/\sin\theta_{29} = 2000.63$, which is greater than $\beta_\ell = 2000$, as required. Speeds of distinct modes are shown in right-hand plot of Figure 3.
6 Conclusions

Superposition of body waves allows us to examine several properties of guided waves. Body-wave propagation at postcritical angles—required for a total reflection—results in the speed of the Love wave being between the speeds of the $SH$ waves in the layer and in the halfspace. A finite wavelength of the $SH$ waves—required for constructive interference—results in a limited number of modes of the Love wave. Each mode exhibits a discrete frequency and propagation speed; the first mode has the lowest frequency and the highest speed.

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