A CERTAIN VECTOR-VALUED FUNCTION SPACE AND ITS APPLICATIONS TO MULTILINEAR OPERATORS

BAE JUN PARK

Abstract. In this paper we present several (quasi-)norm equivalences involving $L^p(l^q)$ norm of a certain vector-valued functions and extend the equivalences to $p = \infty$ and $0 < q < \infty$ in the scale of Triebel-Lizorkin spaces, motivated by Fraizer, Jawerth [13]. We also study a Fourier multiplier theorem for $L^p_{\mathcal{A}}(l^q)$. By applying the results, we will improve the multilinear Hörmander multiplier theorems in Tomita [27], Grafakos, Si [18], and the boundedness results for bilinear pseudo-differential operators in Koezuka, Tomita [20].

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1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be the collections of all natural numbers and all integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will work on $d$-dimensional Euclidean space $\mathbb{R}^d$. We denote by $S = S(\mathbb{R}^d)$ the space of all Schwartz functions on $\mathbb{R}^d$ and by $S'$ the set of all tempered distributions. Let $\mathcal{D}$ denote the set of all dyadic cubes in $\mathbb{R}^d$, and for each $k \in \mathbb{Z}$ let $\mathcal{D}_k$ be the subset of $\mathcal{D}$ consisting of the cubes with side length $2^{-k}$. For each $Q \in \mathcal{D}$ we denote the side length of $Q$ by $l(Q)$.

The symbol $X \lesssim Y$ means that there exists a positive constant $C$, possibly different at each occurrence, such that $X \leq CY$, and $X \approx Y$ signifies $C^{-1}Y \leq X \leq CY$ for a positive unspecified constant $C$. For $f \in S$ the Fourier transform is defined by the formula

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx$$

while $f^\vee(\xi) = \hat{f}(-\xi)$ denotes the inverse Fourier transform.

For $r > 0$ let $\mathcal{E}(r)$ denote the space of all distributions whose Fourier transforms are supported in $\{\xi \in \mathbb{R}^d : |\xi| \leq 2r\}$. Let $A > 0$. For $0 < p < \infty$ and $0 < q \leq \infty$ or for $p = q = \infty$ we define

$$L^p_{\mathcal{A}}(l^q) := \{f = \{f_k\}_{k \in \mathbb{Z}} \subset S' : f_k \in \mathcal{E}(A2^k), \|f\|_{L^p(l^q)} < \infty\}.$$

Then it is known in [25] that $L^p_{\mathcal{A}}(l^q)$ is a quasi-Banach space (Banach space if $p, q \geq 1$) with a (quasi-)norm $\|f\|_{L^p_{\mathcal{A}}(l^q)}$. We will study some (quasi-)norm equivalences on $L^p_{\mathcal{A}}(l^q)$ and one of main results is an extension of the norm equivalences to the case $p = \infty$ and $0 < q < \infty$. 
in the scale of Triebel-Lizorkin space. Suppose $0 < q < \infty$ and $q \leq t \leq \infty$, and $\sigma > d/q$. Then for each $Q \in \mathcal{D}$ and $f_k \in \mathcal{E}(A2^k)$ there exists a proper measurable subset $S_Q$ of $Q$, depending on $\{f_k\}_{k \in \mathbb{Z}}$, $\sigma$, and $t$, such that

\[
\sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^{\infty} |f_k(x)|^q \, dx \right)^{1/q} \approx \left\| \left\{ \sum_{y \in Q} \inf_{y \in Q} \left( \mathcal{M}_{\sigma, 2^k} f_k(y) \right) \chi_{S_Q} \right\}_{k \in \mathbb{N}} \right\|_{L^\infty(I^t)}.
\]

Here $\mathcal{M}_{\sigma, 2^k}$ means a variant of Peetre’s maximal operator, defined in Section 2. This can be compared with the estimate that for $0 < p < \infty$ or $p = \infty$\n
\[
\|f\|_{L^p(I^t)} \approx \left\| \left\{ \sum_{y \in Q} \inf_{y \in Q} \left( \mathcal{M}_{\sigma, 2^k} f_k(y) \right) \chi_{S_Q} \right\}_{k \in \mathbb{N}} \right\|_{L^p(I^t)}, \quad f \in L^p_{\text{loc}}(I^t),
\]

if $\min(p, q) \leq t \leq \infty$ and $\sigma > d/\min(p, q)$. Note that for $1 < p < \infty$, according to Littlewood-Paley theory,\n
\[
\|f\|_{L^p} \approx \left\| \left\{ \phi_k \ast f \right\}_{k \in \mathbb{N}} \right\|_{L^p(I^t)}
\]

and, using (1.2), this is also equivalent to\n
\[
\left\| \left\{ \sum_{Q \in \mathcal{D}_h} \inf_{y \in Q} \left( \mathcal{M}_{\sigma, 2^k} (\phi_k \ast f)(y) \right) \chi_{S_Q} \right\}_{k \in \mathbb{N}} \right\|_{L^p(I^t)}
\]

where $\{\phi_k\}_{k \in \mathbb{N}}$ is a homogeneous Littlewood-Paley partition of unity, defined in Section 2. On the other hand, using a deep connection between BMO and Carleson measure,\n
\[
\|f\|_{BMO} \approx \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^{\infty} \left| \phi_k \ast f(x) \right|^2 \, dx \right)^{1/2}.
\]

The main value of (1.1) is that $\| \cdot \|_{BMO}$ can be expressed in the form $\| \cdot \|_{L^\infty(I^t)}$ as an extension of (1.3) to $p = \infty$. To be specific,\n
\[
\|f\|_{BMO} \approx \left\| \left\{ \sum_{Q \in \mathcal{D}_h} \inf_{y \in Q} \left( \mathcal{M}_{\sigma, 2^k} (\phi_k \ast f)(y) \right) \chi_{S_Q} \right\}_{k \in \mathbb{N}} \right\|_{L^\infty(I^t)}.
\]

The results (1.1) and (1.2) are stated in Theorem 3.4 and Lemma 3.1 (1), respectively.\n
Other (quasi-)norm equivalences will be provided by using a method of $\varphi$-transform and by using Fefferman-Stein’s sharp maximal function in Section 4 and 5. Based on those results we will improve the $L^p_{\text{loc}}(I^t)$-multiplier theorem of Triebel [28, 1.6.3, 2.4.9]. Suppose $f_k \in \mathcal{E}(A2^k)$. Then for $s > d/\min(1, p, q) - d/2$ one has \n
\[
\|\{ (m_k \widehat{f_k})^\vee \}_{k \in \mathbb{N}} \|_{L^p(I^t)} \lesssim \sup_{l \in \mathbb{N}} \|m_l(2^l \cdot)\|_{L^p(I^t)} \|f\|_{L^p(I^t)}, \quad p < \infty \text{ or } p = \infty = \infty,
\]

and \n
\[
\sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^{\infty} \left| (m_k \widehat{f_k})^\vee (x) \right|^q \, dx \right)^{1/q} \lesssim \sup_{l \in \mathbb{N}} \|m_l(2^l \cdot)\|_{L^2} \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^{\infty} \left| f_k(x) \right|^q \, dx \right)^{1/q}.
\]

Note that $m_k$ is not necessarily compactly supported. The main technique of Triebel to prove the multiplier theorem is a complex interpolation theorem for analytic families of operators, but the interpolation method cannot be applied to the endpoint case $q = \infty$ or $p = \infty$. In our proof we adopt the idea in the author [24], which provides a totally different and elementary proof. The multiplier theorem is stated in Theorem 6.1.
One of problems in bilinear operator theory is $L^p \times L^q \to L^r$ boundedness estimates for $1/r = 1/p + 1/q$, and Hölder’s inequality is primarily required. Note that the $BMO$-norm equivalence (1.3) enables us to still utilize Hölder’s inequality to obtain some boundedness results involving $BMO$-type function spaces. In this context the reader will readily notice that (1.4) may be very suitable to endpoint estimates for bilinear or multilinear operators. Indeed, in Theorem 7.1 and 7.2 we will extend and improve the multilinear version of Hörmander’s multiplier theorems of Tomita [27] and Grafakos, Si [18] using (1.1). We will also improve the boundedness result of multilinear pseudo-differential operators of Koezuka and Tomita [20] by using the (quasi-)norm equivalence results and the multiplier theorem in Theorem 6.1 and furthermore, this immediately establishes $bmo$-endpoint Kato-Ponce inequality. See Section 8 for details.

The paper is organized as follows. In Section 2 we present definitions of several function spaces and some maximal inequalities, which will be basic ingredients of our results. Section 3.5 contain some (quasi-)norm equivalences on the space $L^p_A(L^q)$ and its extensions to $p = \infty$ and $0 < q < \infty$. In Section 6 we state and prove a generalized $L^p_A(L^q)$-multiplier theorem. Section 7 and 8 are devoted to applications of our results to multilinear multiplier operators and multilinear pseudo-differential operators of type $(1,1)$.

2. Function spaces and some maximal inequalities

2.1. Function spaces. Let $\Phi_0 \in S$ satisfy $\text{Supp}(\Phi_0) \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq 1 \}$ and $\widehat{\Phi_0}(\xi) = 1$ for $|\xi| \leq 1/2$. Define $\phi := \Phi_0 - 2^{-d}\Phi(2^{-1} \cdot)$ and $\phi_k := 2^{kd}\phi(2^{k} \cdot)$. Then $\{\Phi_0\} \cup \{\phi_k\}_{k \in \mathbb{N}}$ and $\{\phi_k\}_{k \in \mathbb{Z}}$ form inhomogeneous and homogeneous Littlewood-Paley partition of unity, respectively. Note that $\text{Supp}(\phi_k) \subset \{ \xi \in \mathbb{R}^d : 2^{k-2} \leq |\xi| \leq 2^k \}$ and

$$\widehat{\Phi_0}(\xi) + \sum_{k \in \mathbb{N}} \widehat{\phi_k}(\xi) = 1, \quad \text{(inhomogeneous)}$$

$$\sum_{k \in \mathbb{Z}} \widehat{\phi_k}(\xi) = 1, \quad \xi \neq 0. \quad \text{(homogeneous)}$$

For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, inhomogeneous Triebel-Lizorkin space $F^s_{p,q}$ is the collection of all $f \in S'$ such that

$$\|f\|_{F^s_{p,q}} := \|\Phi_0 \ast f\|_{L^p} + \left\| \{2^{sk}\phi_k \ast f\}_{k \in \mathbb{N}} \right\|_{L^p(\mathbb{R}^d)} < \infty, \quad 0 < p < \infty \text{ or } p = q = \infty,$$

$$\|f\|_{F^s_{\infty,q}} := \|\Phi_0 \ast f\|_{L^\infty} + \sup_{P \in \mathbb{D}, l(P) < 1} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^{\infty} 2^{skq} |\phi_k \ast f(x)|^q dx \right)^{1/q}, \quad 0 < q < \infty$$

where the supremum is taken over all dyadic cubes whose side length is less than 1. Similarly, homogeneous Triebel-Lizorkin space $F^s_{p,q}$ is defined to be the collection of all $f \in S'/\mathcal{P}$ (tempered distribution modulo polynomials) such that

$$\|f\|_{F^s_{p,q}} := \left\| \{2^{sk}\phi_k \ast f\}_{k \in \mathbb{Z}} \right\|_{L^p(\mathbb{R}^d)} < \infty, \quad 0 < p < \infty \text{ or } p = q = \infty,$$

$$\|f\|_{F^s_{\infty,q}} := \sup_{P \in \mathbb{D}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^{\infty} 2^{skq} |\phi_k \ast f(x)|^q dx \right)^{1/q}, \quad 0 < q < \infty.$$
Then these spaces provide a general framework that unifies classical function spaces.

- **$L^p$ space**
  \[ \hat{F}^{0,2}_p = F^{0,2}_p = L^p \quad 1 < p < \infty \]

- **Hardy space**
  \[ \hat{F}^{0,2}_p = H^p, \quad F^{0,2}_p = h^p \quad 0 < p \leq 1 \]

- **Fractional Sobolev space**
  \[ \hat{F}^{s,2}_p = \dot{L}^p_s, \quad F^{s,2}_p = L^p_s \quad 1 < p < \infty \]

- **Hardy-Sobolev space**
  \[ \hat{F}^{s,2}_p = H^p_s, \quad F^{s,2}_p = h^p_s \quad 0 < p \leq 1 \]

- **$BMO, bmo$**
  \[ \hat{F}^{0,2}_\infty = BMO, \quad F^{0,2}_\infty = bmo \]

- **Sobolev-BMO**
  \[ \hat{F}^{s,2}_\infty = BMO_s, \quad F^{s,2}_\infty = bmo_s. \]

The Hardy space $h^p$, $0 < p \leq \infty$, consists of all $f \in S'$ such that

\[ (2.1) \quad \|f\|_{h^p} := \left\| \sup_{0 < t \leq 1} |\Phi_0^t \ast f| \right\|_{L^p} < \infty, \]

where $\Phi_0^t := t^{-d}\Phi_0(t^{-1} \cdot)$, and the space $bmo$ is a localized version of $BMO$ defined as the set of locally integrable functions $f$ satisfying

\[ \|f\|_{bmo} := \sup_{l(Q) \leq 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx + \sup_{l(Q) > 1} \frac{1}{|Q|} \int_Q |f(x)| \, dx < \infty \]

where $f_Q$ is the average of $f$ over a cube $Q$. It is known that $(h^1)^s = bmo$ and $h^p = L^p$ for $1 < p \leq \infty$. For $s \in \mathbb{R}$ the spaces $h^p_s$ and $bmo_s$ are defined similarly, using

\[ \|f\|_{h^p_s} := \|J^s f\|_{h^p}, \quad \|f\|_{bmo_s} := \|J^s f\|_{bmo} \]

where $J^s := (1 - \Delta)^{s/2}$ is the fractional Laplacian operator. See [13] [15] [23] for further details.

### 2.2. Maximal inequalities

Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator, defined by

\[ \mathcal{M} f(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy \]

where the supremum is taken over all cubes containing $x$, and for $0 < r < \infty$ let $\mathcal{M}_r f := (\mathcal{M}(|f|)^r)^{1/r}$. Then Fefferman-Stein’s vector-valued maximal inequality in [9] says that for $0 < p < \infty$, $0 < q \leq \infty$, and $0 < r \leq \min(p, q)$ one has

\[ (2.2) \quad \left\| \{\mathcal{M}_r f_k\}_{k \in \mathbb{Z}} \right\|_{L^p(\mathbb{N})} \lesssim \left\| \{f_k\}_{k \in \mathbb{Z}} \right\|_{L^p(\mathbb{N})}. \]

Clearly, (2.2) also holds when $p = q = \infty$.

We now introduce a variant of Hardy-Littlewood maximal function. For $\varepsilon \geq 0$, $r > 0$, and $k \in \mathbb{Z}$, let $\mathcal{M}_r^{k,\varepsilon}$ be defined by

\[ \mathcal{M}_r^{k,\varepsilon} f(x) := \sup_{x \in Q, 2^k l(Q) \leq 1} \left( \frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right)^{1/r} \]

\[ + \sup_{x \in Q, 2^k l(Q) > 1} (2^k l(Q))^{-\varepsilon} \left( \frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right)^{1/r}. \]

Note that $\mathcal{M}_r^{k,\varepsilon} f(x)$ is decreasing function of $\varepsilon$, and $\mathcal{M}_r^{k,0} f(x) \approx \mathcal{M}_r f(x)$. Then the following maximal inequality holds.

**Lemma 2.1.** [23] Let $0 < r < q < \infty$, $\varepsilon > 0$, and $\mu \in \mathbb{Z}$. For $k \in \mathbb{Z}$ let $f_k \in \mathcal{E}(A2^k)$ for some $A > 0$. Then one has

\[ \sup_{P \in \mathcal{D}_\mu} \left( \frac{1}{|P|} \int_P \sum_{k=\mu}^{\infty} (\mathcal{M}_r^{k,\varepsilon} f_k(x))^q \, dx \right)^{1/q} \lesssim \sup_{R \in \mathcal{D}_\mu} \left( \frac{1}{|R|} \int_R \sum_{k=\mu}^{\infty} |f_k(x)|^q \, dx \right)^{1/q}. \]
Here, the implicit constant of the inequality is independent of $\mu$.

For $k \in \mathbb{Z}$ and $\sigma > 0$ we define the Peetre’s maximal operator $\mathcal{M}_{\sigma,2^k}$ by

$$\mathcal{M}_{\sigma,2^k} f(x) := \sup_{y \in \mathbb{R}^d} \frac{|f(x - y)|}{(1 + 2^k|y|)^\sigma}.$$  

It is proved in [23] that if $\sigma > d/r$ and $f \in \mathcal{E}(A2^k)$ for some $A > 0$ then

$$\mathcal{M}_{\sigma,2^k} f(x) \lesssim \mathcal{M}_{\sigma}^{k,d/r} f(x)$$  

uniformly in $k$

and this allow us to replace $\mathcal{M}_{\sigma}$ and $\mathcal{M}_{\sigma}^{k,e}$ by $\mathcal{M}_{\sigma,2^k}$ in (2.2) and Lemma 2.1.

Now we generalize the Peetre’s maximal operator. For $k \in \mathbb{Z}$, $\sigma > 0$, and $0 < t \leq \infty$ let

$$\mathcal{M}_{\sigma,2^k}^t f(x) := 2^{kd/t} \left\| \frac{f_k(x - \cdot)}{(1 + 2^k|\cdot|)^\sigma} \right\|_{L^t},$$

which is an extension of $\mathcal{M}_{\sigma,2^k}^\infty f(x) = \mathcal{M}_{\sigma,2^k} f(x)$.

**Lemma 2.2.** Let $\sigma > 0$ and $k \in \mathbb{Z}$. Suppose $0 < t \leq s \leq \infty$ and $f \in \mathcal{E}(A2^k)$ for some $A > 0$. Then

$$\mathcal{M}_{\sigma,2^k}^s f(x) \lesssim \mathcal{M}_{\sigma,2^k}^t f(x).$$

**Proof.** Since the case $t = s$ is trivial, we only consider the case $t < s$. Let $\Psi_0 \in S$ satisfy

$$\text{Supp}(\hat{\Psi}_0) \subset \{ \xi : |\xi| \leq 2A \} \quad \text{and} \quad \hat{\Psi}_0(\xi) = 1 \quad \text{for} \quad |\xi| \leq 2A.$$  

Then we note that $f = \Psi_k * f$.

We first assume $s = \infty$ and $0 < t < \infty$. If $1 < t < \infty$, then it follows from Hölder’s inequality that

$$\frac{|f(x - y)|}{(1 + 2^k|y|)^\sigma} \leq \int_{\mathbb{R}^d} \frac{|f(x - z)|}{(1 + 2^k|z|)^\sigma} |\Psi_k(z - y)|(1 + 2^k|z - y|)^\sigma dz \leq \mathcal{M}_{\sigma,2^k}^t f(x) 2^{-kd/t} \left( \int_{\mathbb{R}^d} (|\Psi_k(z)|(1 + 2^k|z|)^\sigma)^t dz \right)^{1/t} \lesssim \mathcal{M}_{\sigma,2^k}^t f(x).$$

If $0 < t \leq 1$ then we apply Nikolskii’s inequality to obtain

$$|f(x - y)| \lesssim 2^{kd(1/t - 1)} \left( \int_{\mathbb{R}^d} |f(x - z)|^t |\Psi_k(z - y)|^t dz \right)^{1/t}$$

and thus

$$\frac{|f(x - y)|}{(1 + 2^k|y|)^\sigma} \lesssim 2^{kd(1/t - 1)} \left( \int_{\mathbb{R}^d} |f(x - z)|^t |\Psi_k(z - y)|^t (1 + 2^k|z - y|)^\sigma dz \right)^{1/t} \lesssim \mathcal{M}_{\sigma,2^k}^t f(x).$$

This proves

$$\mathcal{M}_{\sigma,2^k} f(x) \lesssim \mathcal{M}_{\sigma,2^k}^t f(x).$$
Now assume $0 < t < s < \infty$. Then one has
\[
\mathcal{M}_{\sigma,2k}^s f(x) = 2^{kd/s} \left( \int_{\mathbb{R}^d} \left( \frac{|f(x-y)|}{(1+2^k|y|)^\sigma} \right)^s dy \right)^{1/s} \\
\leq (\mathcal{M}_{\sigma,2k}^t f(x))^{1-t/s} 2^{kd/t} \left( \int_{\mathbb{R}^d} \left( \frac{|f(x-y)|}{(1+2^k|y|)^\sigma} \right)^t dy \right)^{1/s} \\
\lesssim (\mathcal{M}_{\sigma,2k}^t f(x))^{1-t/s} (\mathcal{M}_{\sigma,2k}^t f(x))^{t/s} = \mathcal{M}_{\sigma,2k}^t f(x).
\]
by applying (2.4). \qed

**Lemma 2.3.** Let $\sigma > 0$ and $k \in \mathbb{Z}$. Suppose $0 < t \leq s \leq \infty$. Then
\[
\mathcal{M}_{\sigma,2k}^s \mathcal{M}_{\sigma,2k}^t f(x) \approx \mathcal{M}_{\sigma,2k}^t f(x).
\]

**Proof.** One direction is clear because
\[
\mathcal{M}_{\sigma,2k}^t f(x) \leq \mathcal{M}_{\sigma,2k}^t \mathcal{M}_{\sigma,2k}^s f(x) \leq \mathcal{M}_{\sigma,2k}^s \mathcal{M}_{\sigma,2k}^t f(x),
\]
which follows from Lemma 2.2.

For the opposite direction we only concern ourselves with the case $0 < t < s < \infty$ since the other cases follow in a similar and simpler way. By applying Minkowski’s inequality with $s/t > 1$, one has
\[
\mathcal{M}_{\sigma,2k}^s \mathcal{M}_{\sigma,2k}^t f(x) = 2^{kd/s} 2^{kd/t} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{|f(x-z)|^q}{(1+2^k|y-z|)^{\sigma q}(1+2^k|y|)^{\sigma q}} dz \right)^{s/t} dy \right)^{1/s} \\
\leq 2^{kd/s} 2^{kd/t} \left( \int_{\mathbb{R}^d} |f(x-z)|^t \left( \int_{\mathbb{R}^d} \frac{1}{(1+2^k|y-z|)^{\sigma s}(1+2^k|y|)^{\sigma s}} dy \right)^{t/s} dz \right)^{1/t}.
\]
and a standard computation (see [16, Appendix K]) yields that
\[
\int_{\mathbb{R}^d} \frac{1}{(1+2^k|y-z|)^{\sigma s}(1+2^k|y|)^{\sigma s}} dy \lesssim 2^{-kd}.
\]
Therefore,
\[
\mathcal{M}_{\sigma,2k}^s \mathcal{M}_{\sigma,2k}^t f(x) \lesssim \mathcal{M}_{\sigma,2k}^t f(x).
\]

Elementary considerations reveal that for $\sigma > 0$ and $Q \in \mathcal{D}_k$
\[
(2.5) \quad \sup_{y \in Q} |f(y)| \lesssim \inf_{y \in Q} \mathcal{M}_{\sigma,2k} f(y)
\]
and then it follows from Lemma 2.3 that for $0 < t \leq \infty$
\[
(2.6) \quad \sup_{y \in Q} \mathcal{M}_{\sigma,2k}^t f(y) \lesssim \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t \mathcal{M}_{\sigma,2k}^s f(y) \approx \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f(y)
\]
if $f \in \mathcal{E}(A2^k)$ for some $A > 0$.

**Lemma 2.4.** Let $0 < t \leq s \leq \infty$, $0 < t < \infty$, $k \in \mathbb{Z}$, $\sigma > d/t$, and $0 < \epsilon < \sigma - d/t$. Suppose $f_k \in \mathcal{E}(A2^k)$ for some $A > 0$. Then
\[
\mathcal{M}_{\sigma,2k}^s f(x) \lesssim \mathcal{M}_{t}^{k,\epsilon} f(x).
\]

**Proof.** Due to Lemma 2.2 it suffices to show
\[
\mathcal{M}_{\sigma,2k}^s f(x) \lesssim \mathcal{M}_{t}^{k,\epsilon} f(x)
\]
Let
\[
E_0 := \{ y \in \mathbb{R}^d : |y| \leq 2^{-k} \} \quad \text{and} \quad E_j := \{ y \in \mathbb{R}^d : 2^{-k+j-1} < |y| \leq 2^{-k+j} \}, \; j \geq 1.
\]
Then one has
\[
\int_{\mathbb{R}^d} \frac{|f(x-y)|^t}{(1 + 2^k|y|)^{\sigma t}} dy \lesssim \sum_{j=0}^{\infty} 2^{-j\sigma t} \int_{E_j} |f(x-y)|^t dy \leq 2^{-kd} \left( M^{t,k}_{\sigma} f(x) \right)^t \sum_{j=0}^{\infty} 2^{-j(\sigma - d/t - \epsilon)},
\]
which concludes the proof since \( \sigma - d/t - \epsilon > 0 \). \( \square \)

Note that, from (2.3), Lemma 2.4 also holds for \( s = \infty \) and \( \epsilon = \sigma - d/t \).

As a immediate consequence of (2.2) and lemma 2.4, one has the following inequality.

**Lemma 2.5.** Let \( 0 < p, q \leq \infty \), \( \min(p, q) \leq t \leq \infty \), and \( \sigma > d/\min(p, q) \). Suppose \( A > 0 \) and \( f_k \in \mathcal{E}(A2^k) \) for each \( k \in \mathbb{Z} \).

1. For \( 0 < p < \infty \) or \( p = q = \infty \)
\[
\left\| \left\{ M_{\sigma,2k}^t f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\mu)} \lesssim \left\| \left\{ f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\mu)}.
\]
2. For \( p = \infty \), \( 0 < q < \infty \), and \( \mu \in \mathbb{Z} \)
\[
\sup_{P \in \mathcal{D}_{\mu}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} \left( M_{\sigma,2k}^t f_k(x) \right)^q dx \right)^{1/q} \lesssim \sup_{P \in \mathcal{D}_{\mu}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q}
\]
where the constant in the inequality is independent of \( \mu \).

As an application of Lemma 2.5 (2), for \( \mu \in \mathbb{Z} \), \( q_1 < q_2 < \infty \), and \( f_k \in \mathcal{E}(A2^k) \) for some \( A > 0 \), one has
\[
\left\| \left\{ f_k \right\}_{k \geq \mu} \right\|_{L^\infty(\mathbb{R}^n)} \lesssim \sup_{P \in \mathcal{D}_{\mu}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^{q_2} dx \right)^{1/q_2} \lesssim \sup_{P \in \mathcal{D}_{\mu}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^{q_1} dx \right)^{1/q_1}.
\]

We omit the proof and refer to [23].

3. Equivalence of (quasi-)norms by using \( M_{\sigma,2k}^t \)

Let \( f_k \in \mathcal{E}(A2^k) \) for some \( A > 0 \) and \( f := \{ f_k \}_{k \in \mathbb{Z}} \). For convenience in notation we will occasionally write
\[
M_{\sigma}(f) := \{ M_{\sigma,2k}^t f_k \}_{k \in \mathbb{Z}}, \quad M_{\sigma}^t(f) := \{ M_{\sigma,2k}^t f_k \}_{k \in \mathbb{Z}}.
\]
If \( 0 < p < \infty \) or \( p = q = \infty \), then one has the (quasi-)norm equivalence
\[
\| f \|_{L^p(\mu)} \approx \| M_{\sigma}(f) \|_{L^p(\mu)} \approx \| M_{\sigma}^t(f) \|_{L^p(\mu)}.
\]
for \( t \geq \min(p, q) \) and \( \sigma > d/\min(p, q) \) due to Lemma 2.5. Moreover, it follows from (2.5) and (2.6) that
\[
|f_k(x)| = \sum_{Q \in \mathcal{D}_k} |f_k(x)\chi_Q(x)| \lesssim \sum_{Q \in \mathcal{D}_k} \left( \inf_{y \in Q} M_{\sigma,2k}^t f_k(y) \right) \chi_Q(x) \leq M_{\sigma,2k}^t f_k(x)
\]
and thus Lemma 2.5 (1) implies
\[
\| f \|_{L^p(\mu)} \approx \left\| \left\{ \sum_{Q \in \mathcal{D}_k} \left( \inf_{y \in Q} M_{\sigma,2k}^t f_k(y) \right) \chi_Q \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\mu)}.
\]
Similarly, if $0 < q < \infty$ and $\mu \in \mathbb{Z}$, then
\[
\sup_{P \in \mathcal{D}, |x| \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty |f_k(x)|^q dx \right)^{1/q} \approx \sup_{P \in \mathcal{D}, |x| \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty |f_k(x)|^q dx \right)^{1/q}.
\]
(3.2)
Similarly, if $0 < q < \infty$, then
\[
\sup_{P \in \mathcal{D}, |x| \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty |f_k(x)|^q dx \right)^{1/q} \leq \sup_{P \in \mathcal{D}, |x| \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty \left( \inf_{y \in Q} M_{\sigma,2k} f_k(y) \right)^q \chi_Q(x) |dx| \right)^{1/q}.
\]
(3.3)

**Lemma 3.1.** Let $0 < p, q \leq \infty$, $\min(p,q) \leq t < \infty$, $A > 0$, and $\mu \in \mathbb{Z}$. For each $Q \in \mathcal{D}$ let $S_Q$ be a measurable subset of $Q$ with $|S_Q| > \gamma |Q|$ for some $0 < \gamma < 1$. Suppose $\sigma > d/\min(p,q)$ and $f_k \in \mathcal{E}(A2^k)$ for each $k \in \mathbb{Z}$.

1. For $0 < p < \infty$ or $p = q = \infty$
\[
\|f\|_{L^p(Q)} \approx \left\| \left\{ \sum_{Q \in \mathcal{D}_k} \left( \inf_{y \in Q} M_{\sigma,2k} f_k(y) \right) \chi_{S_Q} \right\}_{k \in \mathbb{Z}} \right\|_{L^p(Q)}.
\]
(3.1)

2. For $p = \infty$ and $0 < q < \infty$
\[
\sup_{P \in \mathcal{D}, |x| \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty |f_k(x)|^q dx \right)^{1/q} \approx \sup_{P \in \mathcal{D}, |x| \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty \left( \inf_{y \in Q} M_{\sigma,2k} f_k(y) \right)^q \chi_{S_Q}(x) |dx| \right)^{1/q}.
\]
(3.3)

Note that the constants in the estimates are independent of $S_Q$ as long as $S_Q$ satisfies $|S_Q| > \gamma |Q|$.

**Proof of Lemma 3.1.** The second assertion follows immediately from (3.3) and the condition $|S_Q| > \gamma |Q|$. Thus we only consider the first one. Assume $0 < p < \infty$ or $p = q = \infty$. Since $\chi_Q \geq \chi_{S_Q}$, one direction is obvious due to (3.1). We will base the converse on the pointwise estimate that for $0 < r < \infty$
\[
\chi_Q(x) \leq M_r(\chi_{S_Q})(x) \chi_Q(x),
\]
(3.4)
which is due to the observation that for $x \in Q$
\[
1 < \frac{1}{\gamma_{1/r}} \frac{|S_Q|^{1/r}}{|Q|^{1/r}} = \frac{1}{\gamma_{1/r}} \left( \frac{1}{|Q|} \int_Q \chi_{S_Q}(y) dy \right)^{1/r} \leq \gamma_{-1/r} M_r(\chi_{S_Q})(x).
\]
Choose $r < p, q$ and then apply (3.1) and (3.4) to obtain
\[
\|f\|_{L^p(Q)} \lesssim \left\| \left\{ \sum_{Q \in \mathcal{D}_k} \left( \inf_{y \in Q} M_{\sigma,2k} f_k(y) \right) M_r(\chi_{S_Q}) \chi_Q \right\}_{k \in \mathbb{Z}} \right\|_{L^p(Q)}
\]
\[
\lesssim \left\{ \left\{ \inf_{y \in Q} M_{\sigma,2k} f_k(y) \right\} M_r(\chi_{S_Q}) \right\}_{Q \in \mathcal{D}} \left\| \left\{ \left\{ \inf_{y \in Q} M_{\sigma,2k} f_k(y) \right\} \chi_{S_Q} \right\}_{k \in \mathbb{Z}} \right\|_{L^p(Q)}
\]
where the maximal inequality (2.2) is applied in the third inequality (with a different countable index set $\mathcal{D}$).
We are mainly interested in the following result.

**Theorem 3.2.** Let \(0 < q < \infty, q \leq t \leq \infty, \mu \in \mathbb{Z},\) and \(\sigma > d/q.\) Suppose \(f_k \in \mathcal{E}(A2^k)\) for some \(A > 0.\) For \(0 < \gamma < 1\) and \(Q \in \mathcal{D}\) there exists a proper measurable subset \(S_Q\) of \(Q,\) depending on \(\gamma, q, \sigma, t, f,\) such that \(|S_Q| > (1 - \gamma)|Q|\) and

\[
\sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^{\infty} |f_k(x)|^q \, dx \right)^{1/q} \\
\approx \left\{ \sum_{Q \in D_k} \left( \inf_{y \in Q} |\mathcal{M}_{\sigma,2^k} (\phi_k \ast f)(y)| \chi_{S_Q} \right) \right\}_{k \geq \mu} \|\|_{L^\infty(l^2)}, \text{ uniformly in } \mu.
\]

We note that the constant in the equivalence is independent of \(f,\) just depending on \(\gamma.\) Moreover, by taking the supremum over \(\mu \in \mathbb{Z}\) we may replace \(\sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \) by \(\sup_{P \in \mathcal{D}}\) and \(\|\{ \cdot \cdot \cdot \}_k \|_{L^\infty(l^2)}\) by \(\|\{ \cdot \cdot \cdot \}_k \|_{L^\infty(l^2)}\). Then as a corollary we have the following \(BMO\) characterization.

**Corollary 3.3.** Let \(2 \leq t \leq \infty\) and \(\sigma > d/2.\) Suppose \(f \in BMO.\) For \(0 < \gamma < 1\) and \(Q \in \mathcal{D}\) there exists a proper measurable subset \(S_Q\) of \(Q,\) depending on \(\gamma, \sigma, t, f,\) such that \(|S_Q| > (1 - \gamma)|Q|\) and

\[
\|f\|_{BMO} \approx \left\{ \sum_{Q \in D_k} \inf_{y \in Q} |\mathcal{M}_{\sigma,2^k} (\phi_k \ast f)(y) \chi_{S_Q} \right\}_{k \in \mathbb{Z}} \|\|_{L^\infty(l^2)}.
\]

A simple application of Corollary 3.3 is the estimate

\[
(f, g) \lesssim \|f\|_{BMO} \|g\|_{H^1}.
\]

This provides one direction of the duality between \(H^1\) and \(BMO,\) which was first announced in [8] and the proof appeared in [4, 9]. By using Corollary 3.3 and Hölder’s inequality (3.5) can be also proved. To be specific,

\[
(f, g) = \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \phi_k * f(x) \bar{\phi}_k * g(x) \, dx \leq \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} |\phi_k * f(x)||\bar{\phi}_k * g(x)| \, dx
\]

\[
= \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \sum_{Q \in D_k} |\phi_k * f(x)||\bar{\phi}_k * g(x)| \chi_Q(x) \, dx
\]

\[
\lesssim \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \sum_{Q \in D_k} \inf_{y \in Q} (\mathcal{M}_{\sigma,2^k} (\phi_k \ast f)(y)) \inf_{y \in Q} (\mathcal{M}_{\sigma,2^k} (\bar{\phi}_k \ast g)(y)) \chi_Q(x) \, dx
\]

\[
\lesssim \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \sum_{Q \in D_k} \inf_{y \in Q} (\mathcal{M}_{\sigma,2^k} (\phi_k \ast f)(y)) \inf_{y \in Q} (\mathcal{M}_{\sigma,2^k} (\bar{\phi}_k \ast g)(y)) \chi_{S_Q}(x) \, dx
\]

\[
\leq \left\{ \sum_{Q \in D_k} \inf_{y \in Q} (\mathcal{M}_{\sigma,2^k} (\phi_k \ast f)(y)) \chi_{S_Q} \right\}_{k \in \mathbb{Z}} \|\|_{L^\infty(l^2)} \times \left\{ \sum_{Q \in D_k} \inf_{y \in Q} (\mathcal{M}_{\sigma,2^k} (\bar{\phi}_k \ast g)(y)) \chi_{S_Q} \right\}_{k \in \mathbb{Z}} \|\|_{L^1(l^2)}
\]

\[
\approx \|f\|_{BMO} \|g\|_{H^1}
\]

where the third inequality follows from (3.4) and Peetre’s maximal inequality and \(S_Q\) is the subset of \(Q\) for \(BMO\) norm equivalence of \(f\) as in Corollary 3.3. Here \(\{\bar{\phi}_k\}\) is a sequence of Schwartz functions such that \(\text{Supp}(\bar{\phi}_k) \subset \{\xi \in \mathbb{R}^d : |\xi| \approx 2^k\}\) and \(\sum_{k \in \mathbb{Z}} \bar{\phi}_k(\xi) \phi_k(\xi) = 1\) for \(\xi \neq 0.\)
3.1. **Proof of Theorem 3.2**  One direction is clear, for any subset \( S_Q \) of \( Q \) with \(|S_Q| > (1 - \gamma)|Q|\), due to Lemma 3.1. Therefore, we will prove that there exists a measurable subset \( S_Q \) such that \(|S_Q| > (1 - \gamma)|Q|\) and

\[
\left\| \left\{ \sum_{Q \in D_k} \left( \inf_{y \in Q} |\mathfrak{N}_{\sigma_k} f_k(y)| \chi_{S_Q} \right) \right\}_{k \geq \mu} \right\|_{L^\infty(\nu)} \lesssim \sup_{P \in D, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q}.
\]

(3.6)

To choose such a subset \( S_Q \) we set up notation and terminology. For \( 0 < q \leq \infty \) and \( P \in \mathcal{D} \) we define

\[
G^q_P(f)(x) := \left\| \left\{ \sum_{Q \in D_k, Q \subset P} \left( \inf_{y \in Q} |f_k(y)| \right) \chi_Q(x) \right\}_{k \geq -\log_2 l(P)} \right\|_{L^q(\nu)}.
\]

Recall that the nonincreasing rearrangement \( f^* \) of a non-negative measurable function \( f \) is given by

\[
f^*(\gamma) := \inf \left\{ \lambda > 0 : \left| \left\{ x \in \mathbb{R}^d : f(x) > \lambda \right\} \right| \leq \gamma \}
\]

and satisfies

\[
\left| \left\{ x \in \mathbb{R}^d : f(x) > f^*(\gamma) \right\} \right| \leq \gamma, \quad \gamma > 0.
\]

(3.7)

For \( P \in \mathcal{D} \), \( 0 < \gamma < 1 \), and a non-negative measurable function \( f \), the “\( \gamma \)-median of \( f \) over \( P \)” is defined as

\[
m^\gamma_P(f) := \inf \left\{ \lambda > 0 : \left| \left\{ x \in P : f(x) > \lambda \right\} \right| \leq \gamma |P| \}.
\]

We consider the \( \gamma \)-median of \( G^q_P(f) \) over \( P \) and the supremums of the quantity over \( P \in \mathcal{D} \), \( l(P) \leq 2^{-\mu} \). That is,

\[
m^{\gamma,q}_P(f) := m^\gamma_P(G^q_P(f)) = \inf \left\{ \lambda > 0 : \left| \left\{ x \in P : G^q_P(f)(x) > \lambda \right\} \right| \leq \gamma |P| \},
\]

\[
m^{\gamma,q,\mu}_P(f)(x) := \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} m^{\gamma,q}_P(f) \chi_P(x),
\]

Observe that

\[
m^{\gamma,q}_P(f) = (G^q_P(f) \chi_P)^*(\gamma |P|)
\]

and by (3.7) one has

\[
\left| \left\{ x \in P : G^q_P(f)(x) > \mathfrak{m}^{\gamma,q,-\log_2 l(P)}(f)(x) \right\} \right|
\]

\[
\leq \left| \left\{ x \in P : G^q_P(f)(x) > \mathfrak{m}^{\gamma,q}_P(f) \right\} \right| \leq \gamma |P|.
\]

(3.8)

Moreover,

\[
m^{\gamma,q,\mu_1}_P(f)(x) \leq m^{\gamma,q,\mu_2}_P(f)(x) \quad \text{for} \quad \mu_1 \geq \mu_2.
\]

(3.9)

Now for each \( P \in \mathcal{D} \) we define

\[
S^{\gamma,q}_P(f) := \left\{ x \in P : G^q_P(f)(x) \leq \mathfrak{m}^{\gamma,q,-\log_2 l(P)}(f)(x) \right\}.
\]

Then (3.8) yields that

\[
\left| S^{\gamma,q}_P(f) \right| \geq (1 - \gamma) |P|.
\]

(3.10)

Then (3.6) can be established by the following proposition.
Proposition 3.4. Let $0 < q < \infty$, $q \leq t \leq \infty$, $\mu \in \mathbb{Z}$, and $\sigma > d/q$. Suppose $0 < \gamma < 1$ and $f_k \in \mathcal{E}(\Lambda 2^k)$ for $k \in \mathbb{Z}$. Then

$$
\left\{ \sum_{Q \in D_k} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)} \right\}_{k \geq \mu} \right\}_{L^\infty(\nu)} \\
\lesssim \sup_{P \in D, l(P)} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} |f_k(x)|^q \, dx \right)^{1/q}
$$

uniformly in $\mu$.

**Remark.** For $0 < q < \infty$

$$
\|f\|_{L^\infty(\nu)} \approx \left\{ \sum_{Q \in D_k} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)} \right\}_{k \in \mathbb{Z}} \right\}_{L^\infty(\nu)}
$$

while for $0 < p < \infty$ or $p = q = \infty$

$$
\|f\|_{L^p(\nu)} \approx \left\{ \sum_{Q \in D_k} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)} \right\}_{k \in \mathbb{Z}} \right\}_{L^p(\nu)},
$$

which is due to Lemma 3.1 (1).

**Proof of Proposition 3.4.** Assume $0 < q < \infty$, $q \leq t \leq \infty$, $\mu \in \mathbb{Z}$, and $\sigma > d/q$. Our claim is

$$
\left\{ \sum_{Q \in D_k} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)} \right\}_{k \geq \mu} \right\}_{L^\infty(\nu)} \\
= \sup_{P \in D, l(P) \leq 2^{2-\mu}} \left\{ \sum_{Q \in D_{k}, Q \subset P} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)} \right\}_{k \geq \mu} \right\}_{L^\infty(\nu)}
$$

(Claim 1)

$$
\lesssim \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} |f_k(x)|^q \, dx \right)^{1/q}
$$

(3.11)

To verify (Claim 1) let $\nu \geq \mu$ and fix $P \in D_\nu$ and $x \in P$. Then it suffices to show that

$$
\left\{ \sum_{Q \in D_{k}, Q \subset P} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)}(x) \right\}_{k \geq \mu} \right\}_{L^q(\nu)} \\
\leq m^{\gamma,q,\mu}(\mathcal{M}_\sigma^t(f))(x)
$$

due to (3.9). Suppose that the left hand side of (3.11) is a nonzero number. Then there exists the “maximal” dyadic cube $P_0(x) \subset P$ such that $x \in S_{P_0(x)}^{\gamma,q}(\mathcal{M}_\sigma^t(f))$, and thus

$$
G^q_{P_0}(x)(\mathcal{M}_\sigma^t(f))(x) \leq m^{\gamma,q,\mu, \log_2 l(P_0(x))}(\mathcal{M}_\sigma^t(f))(x) \leq m^{\gamma,q,\mu}(\mathcal{M}_\sigma^t(f))(x)
$$

where the second inequality holds due to (3.9). The maximality of $P_0(x)$ yields that the left hand side of (3.11) is

$$
\left\{ \sum_{Q \in D_{k}, Q \subset P_0(x)} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)}(x) \right\}_{k \geq \mu} \right\}_{L^q(\nu)} \\
\leq \left\{ \sum_{Q \in D_{k}, Q \subset P_0(x)} \left( \inf_{y \in Q} \mathcal{M}_{\sigma,2k}^t f_k(y) \right) \chi_{S_{Q}^{\gamma,q}(\mathcal{M}_\sigma^t)(f)}(x) \right\}_{k \geq \mu} \right\}_{L^q(\nu)} \\
= G^q_{P_0}(x)(\mathcal{M}_\sigma^t(f))(x) \leq m^{\gamma,q,\mu, \log_2 l(P_0(x))}(\mathcal{M}_\sigma^t(f))(x),
$$

where the last one follows from (3.12). This proves (3.11).
To achieve [Claim 2] fix $\nu \geq \mu$ and let us assume

$$\epsilon > \gamma^{-1/q} \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\nu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q}. \tag{3.13}$$

Then, using Chebyshev’s inequality, (3.2), and (3.13), there exists a constant $C_{A,q,t,\sigma} > 0$ such that for $R \in \mathcal{D}_\nu$,

$$\left| \{ x \in R : G^q_R (M^t_\sigma (f)) (x) > \epsilon \} \right| \leq \frac{1}{\epsilon^q} \| G^q_R (M^t_\sigma (f)) \|_{L^q}^q = \frac{1}{\epsilon^q} \int_{R} \sum_{k=-\log_2 l(R)}^{\infty} \left( \sum_{Q \in \mathcal{D}_k} \left( \inf_{y \in Q} M^t_{\sigma,2k} f_k(y) \right) \chi_Q(x) \right)^q dx \leq C_{A,q,t,\sigma} \frac{|R|}{\epsilon^q} \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\nu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q} \leq C_{A,q,t,\sigma,\gamma} |R|.$$  

This yields that

$$\mathbf{m}^C_{A,q,t,\sigma,\gamma,q} (M^t_\sigma (f)) \leq \epsilon < 2\epsilon.$$  

So far, we have proved that for any $R \in \mathcal{D}_\nu$,

$$\mathbf{m}^C_{A,q,t,\sigma,\gamma,q} (M^t_\sigma (f)) \leq 2\gamma^{-1/q} \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\nu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q},$$

which is equivalent to

$$\mathbf{m}^{2,q}_{R} (M^t_\sigma (f)) \leq 2C_{A,q,t,\sigma} \gamma^{-1/q} \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\nu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q}.$$  

We complete the proof by taking the supremum over $R \in \mathcal{D}, l(R) = 2^{-\nu} \leq 2^{-\mu}$.  

4. Equivalence of (quasi-)norms by using the method of $\varphi$-transform

For a sequence of complex numbers $b := \{b_Q\}_{Q \in \mathcal{D}}$ we define

$$\| b \|_{p,q} := \| g^q (b) \|_{L^p}, \quad 0 < p < \infty \quad \text{or} \quad p = q = \infty,$$

$$\| b \|_{\infty,q} := \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_{P} \sum_{Q \subseteq P} (|b_Q| \|Q|^{-1/2} \chi_Q(x))^q dx \right)^{1/q}, \quad 0 < q < \infty$$

where

$$g^q (b) (x) := \| \{ |b_Q| \|Q|^{-1/2} \chi_Q(x) \}_{Q \in \mathcal{D}} \|_{L^q}.$$  

Furthermore, for $C > 0$ let $\varphi, \tilde{\varphi} \in S$ satisfy

$$\text{Supp}(\tilde{\varphi}), \text{Supp}(\tilde{\varphi}) \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \},$$

$$|\tilde{\varphi}(\xi)|, |\tilde{\varphi}(\xi)| \geq C > 0 \text{ for } 3/4 \leq |\xi| \leq 5/3,$$

and

$$\sum_{k \in \mathbb{Z}} \tilde{\varphi}_k (\xi) \tilde{\varphi}_k (\xi) = 1, \quad \xi \neq 0.$$
where \( \varphi_k(x) := 2^{kd} \varphi(2^k x) \) and \( \bar{\varphi}_k(x) := 2^{kd} \bar{\varphi}(2^k x) \) for \( k \in \mathbb{Z} \). Then the norms in \( \dot{F}^{0,q}_{p,q} \) can be characterized by the discrete \( \dot{f}^{0,q}_{p,q} \) norms. For each \( Q \in \mathcal{D} \) let \( x_Q \) denote the lower left corner of \( Q \). Every \( f \in \dot{F}^{0,q}_{p,q} \) can be decomposed as

\[
(4.2) \quad f(x) = \sum_{Q \in \mathcal{D}} b_Q \varphi^Q(x)
\]

where \( \varphi^Q(x) := |Q|^{1/2} \varphi_k(x - x_Q) \) for \( Q \in \mathcal{D}_k \) and \( b_Q := \langle f, \varphi^Q \rangle \). Moreover, in this case, one has

\[
(4.3) \quad \|b\|_{\dot{f}^{0,q}_{p,q}} \lesssim \|f\|_{\dot{F}^{0,q}_{p,q}}.
\]

The converse estimate also holds. For any sequence \( b = \{b_Q\}_{Q \in \mathcal{D}} \) of complex numbers satisfying \( \|b\|_{\dot{F}^{0,q}_{p,q}} < \infty \),

\[
(4.4) \quad f(x) = \sum_{Q \in \mathcal{D}} b_Q \varphi^Q(x)
\]

belongs to \( \dot{F}^{0,q}_{p,q} \) and

\[
\|f\|_{\dot{F}^{0,q}_{p,q}} \lesssim \|b\|_{\dot{F}^{0,q}_{p,q}}.
\]

See \cite{11,12} for more details.

In this section we will give an analogous properties of \( \{f_k\}_{k \in \mathbb{Z}} \) with \( f_k \in \mathcal{E}(A2^k) \) for some \( A > 0 \), like \( (4.2), (4.3), \) and \( (4.4) \). From now on we fix \( A > 0 \) and suppose \( f_k \in \mathcal{E}(A2^k) \). Let \( \Psi_0 \in \mathcal{S} \) satisfy

\[
\text{Supp}(\hat{\Psi}_0) \subset \{ \xi : |\xi| \leq 2^2 A \} \quad \text{and} \quad \hat{\Psi}_0(\xi) = 1 \quad \text{for} \quad |\xi| \leq 2A.
\]

For each \( k \in \mathbb{Z} \) and \( Q \in \mathcal{D}_k \) let \( \Psi_k := 2^{kd} \Psi_0(2^k \cdot) \) and

\[
\Psi^Q(x) := |Q|^{1/2} \Psi_k(x - x_Q).
\]

**Lemma 4.1.** Let \( 0 < p < \infty \) or \( p = q = \infty \).

1. Assume \( f_k \in \mathcal{E}(A2^k) \) for each \( k \in \mathbb{Z} \). Then there exists a sequence of complex numbers \( b := \{b_Q\}_{Q \in \mathcal{D}} \) such that

\[
f_k(x) = \sum_{Q \in \mathcal{D}_k} b_Q \Psi^Q(x)
\]

and

\[
\|b\|_{\dot{F}^{0,q}_{p,q}} \lesssim \|f\|_{L^p(\mu)}.
\]

2. For any sequence \( b = \{b_Q\}_{Q \in \mathcal{D}} \) of complex numbers satisfying \( \|b\|_{\dot{F}^{0,q}_{p,q}} < \infty \),

\[
f_k(x) = \sum_{Q \in \mathcal{D}_k} b_Q \Psi^Q(x)
\]

satisfies

\[
(4.5) \quad \|f\|_{L^p(\mu)} \lesssim \|b\|_{\dot{F}^{0,q}_{p,q}}.
\]

For the case \( p = \infty \) and \( 0 < q < \infty \) we fix \( \mu \in \mathbb{Z} \) and let

\[
\|b\|_{\dot{F}^{0,q}_{p,q}(\mu)} := \sup_{P \in \mathcal{D}, \mu(P) \geq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{Q \subset P} (|b_Q||Q|^{-1/2} \chi_Q(x))^q dx \right)^{1/q}.
\]

**Lemma 4.2.** Let \( 0 < q < \infty \) and \( \mu \in \mathbb{Z} \).
(1) Assume $f_k \in \mathcal{E}(A2^k)$ for each $k \geq \mu$. Then there exists a sequence of complex numbers $b := \{b_Q\}_{Q \in \mathcal{D}, l(Q) \leq 2^{-\mu}}$ such that

$$f_k(x) = \sum_{Q \in \mathcal{D}_k} b_Q \psi_Q(x)$$

and

$$||b||_{p,q,\mu} \lesssim \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \left( \sum_{k = -\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q} \right).$$

(2) For any sequence $b = \{b_Q\}_{Q \in \mathcal{D}, l(Q) \leq 2^{-\mu}}$ of complex numbers satisfying $\|b\|_{p,q,\mu} < \infty$,

$$f_k(x) := \sum_{Q \in \mathcal{D}_k} b_Q \psi_Q(x)$$

satisfies

$$\sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \left( \sum_{k = -\log_2 l(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q} \right) \lesssim \|b\|_{p,q,\mu}.$$

For the proof we assume $A = 2^{-2}$, for otherwise the conclusions follow from a standard dyadic dilation method (with an inessential change of constant).

**Proof of Lemma 4.1.** Let $0 < p < \infty$ or $p = q = \infty$.

(1) Since $\text{Supp}(f_k(2^k \cdot)) \subset \{ |\xi| \leq 1/2 \}$, using the Fourier series representation of $\hat{f}_k(2^k \cdot)$, one has

$$\hat{f}_k(\xi) = 2^{-kd} \sum_{l \in \mathbb{Z}^d} f_k(2^{-k} l) e^{-2\pi i \langle l/2^k, \xi \rangle}.$$

Then

$$f_k(x) = (\hat{f}_k \hat{\psi}_k)^\vee(x) = 2^{-kd} \sum_{l \in \mathbb{Z}^d} f_k(2^{-k} l) \psi_k(x - 2^{-k} l)$$

$$= \sum_{l \in \mathbb{Z}^d} 2^{-kd/2} f_k(2^{-k} l) 2^{-kd/2} \psi_k(x - 2^{-k} l).$$

For any $Q \in \mathcal{D}_k$ we write

$$Q = Q_{k,l} := \{ x \in \mathbb{R}^d : 2^{-k} l_i \leq x_i \leq 2^{-k} (l_i + 1), \ i = 1, \ldots , d \}$$

where $l = (l_1, \ldots , l_d) \in \mathbb{Z}^d$. That is, $Q_{k,l}$ is the dyadic cube in $\mathcal{D}_k$ whose lower left corner is $2^{-k} l$. Now let

$$b_{Q_{k,l}} := 2^{-kd/2} f_k(2^{-k} l) = |Q_{k,l}|^{1/2} f_k(x_{Q_{k,l}})$$

$$\psi_{Q_{k,l}}(x) := 2^{-kd/2} \psi_k(x - 2^{-k} l) = |Q_{k,l}|^{1/2} \psi_k(x - x_{Q_{k,l}})$$

and then one can write

$$f_k(x) = \sum_{Q \in \mathcal{D}_k} b_{Q} \psi_{Q}(x).$$

Furthermore, for a.e. $x \in \mathbb{R}^d$ there exists $Q_0 \in \mathcal{D}_k$ whose interior contains $x$. Therefore, for any $\sigma > 0$ one has

$$\sum_{Q \in \mathcal{D}_k} |b_Q| |Q|^{-1/2} \chi_Q(x) = |b_{Q_0}| |Q_0|^{-1/2} = |f_k(x_{Q_0})| \lesssim M_{\sigma,2^k} f_k(x) \ \text{a.e.} \ x.$$
Now we select $\sigma > d/\min (p, q)$ and then
\[
\|b\|_{p,q} = \left\| \{ |b_Q| |Q|^{-1/2} \chi_Q \} \right\|_{L^p(D)} = \left\| \left\{ \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \chi_Q \right\}_{k \in \mathbb{Z}} \right\|_{L^p(D)}
\]
\[
\lesssim \left\| \{ \mathcal{M}_{\sigma,2k} f_k \} \right\|_{L^p(D)} \lesssim \|f\|_{L^p(D)}
\]
where (1.1) and Lemma 2.5 (1) are applied.

(2) For a given $b = \{ b_Q \}_{Q \in D}$ and $k \in \mathbb{Z}$ let
\[
f_k(x) := \sum_{Q \in D_k} b_Q \psi^Q(x).
\]
For each $k \in \mathbb{Z}$ and $x \in \mathbb{R}^d$ let
\[
E_k^k(x) := \{ Q \in D_k : |x - x_Q| < 2^{-k} \}
\]
\[
E_j^k(x) := \{ Q \in D_k : 2^{-k+j+1} \leq |x - x_Q| < 2^{-k+j}, \ j \in \mathbb{N}.
\]
Choose $0 < \epsilon < \min (1, p, q)$ and $M > d/\epsilon$. Since $|\psi^Q(x)| \lesssim 2^{-jM} |Q|^{-1/2}$ on $E_j^k$, by decomposing $\sum_{Q \in D_k} = \sum_{j=0}^\infty \sum_{Q \in E_j^k(x)}$ and using $t^\epsilon \rightarrow t^1$ one has
\[
|f_k(x)| \lesssim \sum_{j=0}^\infty 2^{-jM} \left( \sum_{Q \in E_j^k(x)} |b_Q| |Q|^{-1/2} \right)^{1/\epsilon}
\]
\[
\approx \sum_{j=0}^\infty 2^{-j(M-d/\epsilon)} \left( \frac{1}{2^{-kd/2d}} \int_{\mathbb{R}^d} \sum_{Q \in E_j^k(x)} (|b_Q| |Q|^{-1/2} \chi_Q(y))^\epsilon dy \right)^{1/\epsilon}
\]
\[
\lesssim \mathcal{M}_\epsilon \left( \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \chi_Q \right)(x).
\]
Then, using the estimate (4.8) and the maximal inequality (2.2), one has
\[
\|f\|_{L^p(D)} \lesssim \left\{ \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \chi_Q \right\}_{k \in \mathbb{Z}} \|b\|_{p,q},
\]
as required. \qed

Proof of Lemma 4.2. Assume $0 < q < \infty$ and $\mu \in \mathbb{Z}$. 
(1) We apply (4.6), (4.7) and Lemma 2.5 (2), choosing $\sigma > d/q$, to obtain
\[
\|b\|_{p,q} = \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^\infty \left( \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \chi_Q(x) \right)^q dx \right)^{1/q}
\]
\[
\lesssim \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^\infty \left( \mathcal{M}_{\sigma,2k} f_k(x) \right)^q dx \right)^{1/q}
\]
\[
\lesssim \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^\infty |f_k(x)|^q dx \right)^{1/q}.
\]
(2) We first observe that
\[
|b|_{p,q}^{p,q} = \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \sum_{Q \in P} (|b_Q| |Q|^{-1/2} q |Q|)^1 \right)^{1/q}.
\]
Let
\[
f_k(x) := \sum_{Q \in D_k} b_Q \psi^Q(x)
\]
and choose $M > d/\min(1, q)$. Using Hölder’s inequality if $q > 1$ or the embedding $l^q \hookrightarrow l^1$ if $q \leq 1$, one has

$$|f_k(x)| \lesssim \sum_{Q \in \mathcal{D}_k} |b_Q||Q|^{-1/2} \frac{1}{(1 + 2^k|x - x_Q|)^{2M}}$$

$$\lesssim \left( \sum_{Q \in \mathcal{D}_k} (|b_Q||Q|^{-1/2})^q \frac{1}{(1 + 2^k|x - x_Q|)^{Mq}} \right)^{1/q}$$

and thus

$$\sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 l(P)}^\infty |f_k(x)|^q dx \right)^{1/q}$$

$$\lesssim \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \sum_{k = -\log_2 l(P)}^\infty \sum_{Q \in \mathcal{D}_k} (|b_Q||Q|^{-1/2})^q \frac{1}{|P|} \int_P \frac{1}{(1 + 2^k|x - x_Q|)^{Mq}} dx \right)^{1/q}.$$
Now we apply triangle inequality if $q \geq 1$ or $l^q \hookrightarrow l^1$ if $q < 1$ to obtain that

\[
\left( \sum_{k=-\log_2 l(P)}^{\infty} \mathcal{J}^P_{k,M} \right)^{\min(1,q)/q} \leq \sum_{m \in \mathbb{Z}^d} \frac{1}{|m|^\min(1,q)} \left( \sum_{k=-\log_2 l(P)}^{\infty} \frac{1}{2^{kMq} |l(P)|^M} \sum_{Q \subset P + m l(P)} (|b_Q||Q|^{-1/2})^q \right)^{\min(1,q)/q} \]

because $M \min(1,q) > d$ and $2^k l(P) \geq 1$.

Combining these estimates, one has

\[
\sup_{P \in D, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{k=-\log_2 l(P)}^\infty \sum_{k=-\log_2 l(P)}^\infty |f_k(x)|^q dx \right)^{1/q} \leq \sup_{R \in D, l(R) \leq 2^{-\mu}} \left( \frac{1}{|R|} \sum_{k=-\log_2 l(R)}^{\infty} \sum_{Q \subset P + m l(P)} (|b_Q||Q|^{-1/2})^q |Q| \right)^{1/q} \leq \|b\|_{f^0, q(\mu)}. \quad \square
\]

5. Equivalence of (quasi-)norms by using sharp maximal functions

Given a locally integrable function $f$ on $\mathbb{R}^d$ the Fefferman-Stein sharp maximal function $f^*$ is defined by

\[
f^*(x) = \sup_{P : x \in P} \frac{1}{|P|} \int_P |f(y) - f_P| dy
\]

where $f_P := \frac{1}{|P|} \int_P f(z) dz$ and the supremum is taken over all cubes $P$ containing $x$. Then a fundamental inequality of Fefferman and Stein [10] says that for $1 < p < \infty$, $1 \leq p_0 \leq p$ if $f \in L^{p_0}(\mathbb{R}^d)$ then we have

\[
\|Mf\|_{L^p} \lesssim \|f^*\|_{L^p}. \quad (5.1)
\]

Now one has the following characterization of $f^0_{p,q}$ by the sharp maximal functions. For $0 < q < p < \infty$,

\[
\|f\|_{f^0_{p,q}} \approx \left\| \sup_{x \in P} \left( \frac{1}{|P|} \int_{k=-\log_2 l(P)}^\infty |\phi_k * f(y)|^q dy \right)^{1/q} \right\|_{L^p(x)}
\]

where the supremum is taken over all cubes containing $x$ (not necessarily dyadic cubes).

Observe that one can actually replace the maximal functions by dyadic maximal ones in (5.1). That is, for locally integrable function $f$ we define the dyadic maximal function

\[
\mathcal{M}^{(d)}f(x) := \sup_{P \in D, x \in P} \frac{1}{|P|} \int_P |f(y)| dy,
\]

and the dyadic sharp maximal function

\[
\mathcal{M}^2f(x) := \sup_{P \in D, x \in P} \frac{1}{|P|} \int_P |f(y) - f_P| dy
\]
where the supremums are taken over all dyadic cubes $Q$ containing $x$. Then for $1 < p < \infty$, $1 \leq p_0 \leq p$, and $f \in L^{p_0}$ one has

\[(5.3) \quad \|M^{(d)} f\|_{L^p} \lesssim \|M^2 f\|_{L^p}.
\]

The proof of $(5.2)$ is based on $(5.1)$, and by applying $(5.3)$ instead of $(5.1)$ one may replace “sup” in $(5.2)$ by “sup”, which means the supremum over all dyadic cubes containing $x$.

We refer the reader to [25, 26, Proposition 6.1 and 6.2] for details.

We characterize $L^p_A(\ell^q)$, $0 < q < p < \infty$, by using analogous sharp maximal functions.

**Lemma 5.1.** Let $0 < q < p < \infty$ and $A > 0$. Suppose $f_k \in \mathcal{E}(A2^k)$ for each $k \in \mathbb{Z}$. Then

\[
\|f\|_{L^p(\ell^q)} \approx \sup_{x \in P \in D} \left( \frac{1}{|P|} \int_{\mathbb{R}^n} \left| \sum_{k=-\log_2 l(P)}^{\infty} f_k(y)|dy| \right|^q \right)^{1/q} \|L^p(x)\|
\]

where the supremum is taken over all dyadic cubes containing $x$.

The proof of Lemma 5.1 is essentially the same as the proof of $(5.2)$, which is given in [24], replacing $(5.1)$ by $(5.3)$. We omit the proof and refer to [25, 26].

### 6. A MULTIPLIER THEOREM FOR A VECTOR-VALUED FUNCTION SPACE

In this section we will study Fourier multipliers for $L^p_A(\ell^q)$ for $0 < p < \infty$ or $p = q = \infty$, and a proper extension to the case $p = \infty$ and $0 < q < \infty$. We continue to use the notation $f := \{f_k\}_{k \in \mathbb{Z}}$.

**Theorem A.** ([28, 1.6.3, 2.4.9]) Let $0 < p < \infty$, $0 < q \leq \infty$, and $A > 0$. Suppose $f_k \in \mathcal{E}(A2^k)$ for each $k \in \mathbb{Z}$, and $\{m_k\}_{k \in \mathbb{Z}}$ satisfies

\[(6.1) \quad \sup_{l \in \mathbb{N}} \|m_l(2^l \cdot)\|_{L^2} < \infty
\]

for

\[
s > \begin{cases} 
  d/\min(1, p, q) - d/2 & \text{if } q < \infty \\
  d/p + d/2 & \text{if } q = \infty
\end{cases}
\]

Then

\[(6.2) \quad \|\{ (m_k\hat{f}_k)^\vee \}_{k \in \mathbb{N}}\|_{L^p(\ell^q)} \lesssim \sup_{l \in \mathbb{N}} \|m_l(2^l \cdot)\|_{L^2} \|f\|_{L^p(\ell^q)}.
\]

It was first proved that for $1 < p, q < \infty$ if $(6.1)$ holds for $s > d/2$, then $(6.2)$ works by using the classical Hörmander-Mikhlin multiplier theorem. Moreover, for $0 < p < \infty$ and $0 < q \leq \infty$ it is easy to obtain that $(6.2)$ is true under the assumption $(6.1)$ with $s > d/2 + d/\min(p, q)$. Then a complex interpolation method is applied to derive $s > d/\min(1, p, q) - d/2$ when $0 < p, q < \infty$. However, this method cannot be applied to the endpoint case $p = \infty$ or $q = \infty$ and thus one does not have any conclusion when $p = \infty$, and the assumption $s > d/p + d/2$ is required when $q = \infty$, which is stronger than seemingly “natural” condition $s > d/\min(1, p) - d/2$. In [24] the author proved the $\tilde{F}^{0, q}_{p, q}$ multiplier theory for the full range $0 < p, q \leq \infty$ with the condition $s > d/\min(1, p, q) - d/2$ in a different method. By using some techniques in [24] we will improve Theorem A.

**Theorem 6.1.** Let $0 < p, q \leq \infty$, $j \geq 0$, and $\mu \in \mathbb{Z}$. Suppose $f_k \in \mathcal{E}(A2^k)$ for each $k \in \mathbb{Z}$, and $\{m_k\}_{k \in \mathbb{Z}}$ satisfies

\[
\sup_{l \in \mathbb{Z}} \|m_l(2^{l+j} \cdot)\|_{L^2} < \infty \quad \text{for } s > d/\min(1, p, q) - d/2.
\]
(1) For $0 < p < \infty$ or $p = q = \infty$,
\[ \left\| \left\{ (m_k \hat{f}_{k+j})^\vee \right\}_{k \in \mathbb{Z}} \right\|_{L^p(l^q)} \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l(2^{l+j} \cdot) \right\|_{L^2} \left\| f \right\|_{L^p(l^q)} \]
uniformly in $j$.

(2) For $p = \infty$ and $0 < q < \infty$,
\[ \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} \left| (m_k \hat{f}_{k+j})^\vee (x) \right|^q \, dx \right)^{1/q} \]
\[ \lesssim \sup_{l \geq \mu} \left\| m_l(2^{l+j} \cdot) \right\|_{L^2} \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} \left| f_{k+j}(x) \right|^q \, dx \right)^{1/q} \]
uniformly in $\mu$ and $j$.

Note that the condition $s > d/\min(1, p, q) - d/2$ is sharp except the case $p = \infty$ and $q < 1$, and counter examples for the sharpness can be found in [23].

Since the inequality in Theorem 6.1 (2) holds uniformly in $\mu$, by taking supremum over $\mu \in \mathbb{Z}$, one has
\[ \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} \left| (m_k \hat{f}_{k+j})^\vee (x) \right|^q \, dx \right)^{1/q} \]
\[ \lesssim \sup_{l \geq \mu} \left\| m_l(2^{l+j} \cdot) \right\|_{L^2} \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} \left| f_{k+j}(x) \right|^q \, dx \right)^{1/q} . \]

Observe that for $j \geq 0$
\[ (m_k \hat{f}_{k+j})^\vee (x) = (m_k(2^{j} \cdot \left( f_{k+j}(-/2^{j}) \right)^\wedge)^\vee (2^{j} x) \]
and by using a change of variables, one may assume $j = 0$ in the proof of Theorem 6.1.

Indeed, once the theorem is established for $j = 0$, then for $0 < p < \infty$ or $p = q = \infty$
\[ \left\| \left\{ (m_k \hat{f}_{k+j})^\vee \right\}_{k \in \mathbb{Z}} \right\|_{L^p(l^q)} = 2^{-j/d/p} \left\| \left\{ (m_k(2^{j} \cdot \left( f_{k+j}(-/2^{j}) \right)^\wedge)^\vee \right\}_{k \in \mathbb{Z}} \right\|_{L^p(l^q)} \]
\[ \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l(2^{l+j} \cdot) \right\|_{L^2} 2^{-j/d/p} \left\| \left\{ f_{k+j}(-/2^{j}) \right\}_{k \in \mathbb{Z}} \right\|_{L^p(l^q)} \]
\[ = \sup_{l \in \mathbb{Z}} \left\| m_l(2^{l+j} \cdot) \right\|_{L^2} \left\| f \right\|_{L^p(l^q)} \]
uniformly in $j$, since $f_{k+j}(2^{j} \cdot) \in \mathcal{E}(A2^k)$. When $p = \infty$ and $0 < q < \infty$, one has
\[ \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} \left| (m_k \hat{f}_{k+j})^\vee (x) \right|^q \, dx \right)^{1/q} \]
\[ = \sup_{R \in \mathcal{D}, l(R) \leq 2^{-\mu+j}} \left( \frac{1}{|R|} \int_{R} \sum_{k = -\log_2 l(R)+j}^{\infty} \left| (m_k(2^{j} \cdot \left( f_{k+j}(-/2^{j}) \right)^\wedge)^\vee (x) \right|^q \, dx \right)^{1/q} \]
\[ \lesssim \sup_{l \geq \mu} \left\| m_l(2^{l+j} \cdot) \right\|_{L^2} \sup_{R \in \mathcal{D}, l(R) \leq 2^{-\mu+j}} \left( \frac{1}{|R|} \int_{R} \sum_{k = -\log_2 l(R)}^{\infty} \left| f_{k+j}(x/2^{j}) \right|^q \, dx \right)^{1/q} \]
\[ = \sup_{l \geq \mu} \left\| m_l(2^{l+j} \cdot) \right\|_{L^2} \sup_{P \in \mathcal{D}, l(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 l(P)}^{\infty} \left| f_{k+j}(x) \right|^q \, dx \right)^{1/q} \]
uniformly in $j$. 

As mentioned above, the proof of Theorem A in [28] relies on the classical Mikhlin-Hörmander multiplier theorem. In order to prove Theorem 6.1 we will, instead, make use of the following lemma.

**Lemma 6.2.** Suppose $0 < p \leq \infty$, $k \in \mathbb{Z}$, and $s > 0$. Suppose $f_k \in \mathcal{E}(A^2^k)$ and $\{m_k\}_{k \in \mathbb{Z}}$ satisfies
\[
\|m_k(2^k \cdot)\|_{L^2^s} < \infty \quad \text{for } s > d/\min(1, p) - d/2.
\]
Then
\[
\|(m_k \hat{f}_k)^\vee\|_{L^p} \lesssim \|m_k(2^k \cdot)\|_{L^2} \|f_k\|_{L^p} \quad \text{uniformly in } k.
\]

**Proof.** Let $\Psi_k \in \mathcal{S}$ be defined as before. That is, $\text{Supp}(\hat{\Psi}_0) \subset \{ ||\xi|| \leq 2^A \}$, $\hat{\Psi}_0(\xi) = 1$ for $||\xi|| \leq 2A$, and $\Psi_k := 2^{kd}\hat{\Psi}_0(2^k \cdot)$. Then $f_k = \Psi_k \ast f_k$ and $(m_k \hat{f}_k)^\vee = (m_k \hat{\Psi}_k)^\vee \ast f_k$.

Our claim is that
\[
\|(m_k \hat{\Psi}_k)^\vee \ast f_k\|_{L^p} \lesssim \|m_k(2^k \cdot)\|_{L^2} \|f_k\|_{L^p}.
\]
Then (6.3) follows from the observation that
\[
\|m_k(2^k \cdot)\|_{L^2^s} \leq \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s/2} \|m_k(2^k \cdot)\|_{L^2} \leq \int_{\mathbb{R}^d} (1 + |\eta|^2)^{s/2} \|m_k(2^k \cdot)\|_{L^2} \lesssim \|m_k(2^k \cdot)\|_{L^2},
\]
where Minkowski’s inequality is applied with $1 + |\xi|^2 \leq 2(1 + |\eta|^2)(1 + |\xi - \eta|^2)$.

In order to prove (6.4) we first assume $1 \leq p \leq \infty$ and $s > d/2$. Using Young’s inequality,
\[
\|\hat{(m_k \Psi_k)^\vee} \ast f_k\|_{L^p} \leq \|\hat{(m_k \Psi_k)^\vee}\|_{L^1} \|f_k\|_{L^p}
\]
and one has
\[
\|\hat{(m_k \Psi_k)^\vee}\|_{L^1} = \|2^{-kd}(m_k \hat{\Psi}_k)^\vee(\cdot / 2^k)\|_{L^1} \leq \|2^{-kd}(m_k \hat{\Psi}_k)^\vee(\cdot / 2^k)\hat{\phi}_0\|_{L^1} + \sum_{l=1}^{\infty} \|2^{-kd}(m_k \hat{\Psi}_k)^\vee(\cdot / 2^k)\hat{\phi}_l\|_{L^1} \lesssim \|\hat{\phi}_0 \ast (m_k(2^k \cdot)\hat{\Psi}_0)\|_{L^2} + \sum_{l=1}^{\infty} 2^{ld/2} \|\hat{\phi}_l \ast (m_k(2^k \cdot)\hat{\Psi}_0)\|_{L^2} \lesssim \|m_k(2^k \cdot)\hat{\Psi}_0\|_{L^2} \lesssim \|f_k\|_{L^p}.
\]
where the second inequality follows from Schwarz inequality and Plancherel’s theorem, and the third one from Schwarz inequality with $s > d/2$. This proves (6.4) for $1 \leq p \leq \infty$.

For $0 < p < 1$ assume $s > d/p - d/2$ and apply Nikolski’s inequality to obtain
\[
\|\hat{(m_k \Psi_k)^\vee} \ast f_k\|_{L^p} \lesssim 2^{kd(1/p - 1)} \|\hat{(m_k \Psi_k)^\vee}\|_{L^1} \|f_k\|_{L^p}.
\]
Now we observe that
\[ 2^{kd(1/p - 1)} \| (m_k \hat{\Psi}_k)^\vee \|_{L^p} = \| 2^{-kd}(m_k \hat{\Psi}_k)^\vee (-2^k \cdot) \|_{L^p} \]
\[ = \left( \| 2^{-kd}(m_k \hat{\Psi}_k)^\vee (-2^k \cdot) \Phi_0 \|_{L^p}^p + \sum_{l=1}^{\infty} \| 2^{-kd}(m_k \hat{\Psi}_k)^\vee (-2^k \cdot) \hat{\phi}_l \|_{L^p}^p \right)^{1/p} \]
\[ \lesssim \| \Phi_0 \ast (m_k(2^k \cdot) \hat{\Psi}_0) \|_{L^2} + \left( \sum_{l=1}^{\infty} 2^{lp(d/p - d/2)} \| \hat{\phi}_l \ast (m_k(2^k \cdot) \hat{\Psi}_0) \|_{L^2}^p \right)^{1/p} \]
\[ \lesssim \| m_k(2^k \cdot) \hat{\Psi}_0 \|_{L^2} \approx \| m_k(2^k \cdot) \hat{\Psi}_0 \|_{L^2}^p \]
where H"older’s inequality and Plancherel’s theorem are applied in the first inequality. This completes the proof of (6.4) for $0 < p < 1$. \( \blacksquare \)

We now proceed with the proof of Theorem 6.1. We assume $j = 0$ as mentioned above. Since the constant $A$ plays a minor role and affects the result only up to a constant, we fix $A = 2^{-2}$ in the proof to avoid unnecessary complications. Moreover, if $f_k \in \mathcal{E}(2^k \cdot - 2)$, then $(m_k \hat{f}_k)^\vee = (m_k \hat{\Psi}_k)^\vee * f_k$ and due to (6.5), one has
\[ \| (m_k \hat{\Psi}_k)(2^k \cdot) \|_{L^p_2} = \| m_k(2^k \cdot) \hat{\Psi}_0 \|_{L^2} \lesssim \| m_k(2^k \cdot) \|_{L^2} \].
This enables us to assume
\[ (6.6) \quad \text{Supp}(m_k) \subset \{ |\xi| \leq 2^k \} \]
in the proof. With this assumption, $(m_k \hat{f}_k)^\vee = m_k^\vee \ast f_k$.

We first deal with the case $p = \infty$ and $0 < q < \infty$.

**Proof of Theorem 6.1 (2).** Suppose $\nu \geq \mu$ and $P \in \mathcal{D}_\nu$. Let $P^* = 9P$ denote the dilate of $P$ by a factor of $9$ with the same center. Then $P^*$ is a union of some dyadic cubes near $P$. Then we decompose
\[ \left( \frac{1}{|P|} \int_P \sum_{k=\nu}^{\infty} |m_k^\vee \ast f_k(x)|^q \, dx \right)^{1/q} \lesssim \left( \frac{1}{|P|} \int_P \sum_{k=\nu}^{\infty} |m_k^\vee \ast (\chi_{P^*} f_k)(x)|^q \, dx \right)^{1/q} \]
\[ + \left( \frac{1}{|P|} \int_P \sum_{k=\nu}^{\infty} |m_k^\vee \ast (\chi_{(P^*)^c} f_k)(x)|^q \, dx \right)^{1/q} \]
\[ =: \mathcal{U}_P + \mathcal{V}_P \]

Note that
\[ (6.7) \quad m_k^\vee \ast (\chi_{P^*} f_k) = m_k^\vee \ast \Psi_{k+1} \ast (\chi_{P^*} f_k). \]
Therefore, using Lemma 6.2 for $s > d/\min(1, q) - d/2$ one has
\[ \mathcal{U}_P \lesssim \left( \frac{1}{|P|} \sum_{k=\nu}^{\infty} \| m_k^\vee \ast \Psi_{k+1} \ast (\chi_{P^*} f_k) \|_{L^q} \right)^{1/q} \]
\[ \lesssim \sup_{l \geq \nu} \left\| m_l(2^l \cdot) \right\|_{L^2} \left( \frac{1}{|P|} \sum_{k=\nu}^{\infty} \| \Psi_{k+1} \ast (\chi_{P^*} f_k) \|_{L^q} \right)^{1/q}. \]
Then for any $\sigma > 0$
\[ (6.8) \quad \| \Psi_{k+1} \ast (\chi_{P^*} f_k) \|_{L^q} \lesssim \left( \int_{P^*} (m_{\sigma, 2^k} f_k(y))^q \, dy \right)^{1/q}. \]
This follows immediately from Young’s inequality if $q \geq 1$. If $q < 1$ we apply Hölder’s inequality with $1/q > 1$ and (2.5) to obtain
\[
\|\Psi_{k+1} \ast (\chi_{P^*} f_k)\|_{L^2}^q = \sum_{Q \in D_k} \|\Psi_{k+1} \ast (\chi_Q f_k)\|_{L^3}^q
\]
\[
\leq \sum_{Q \in D_k} \int_Q \left( \int |\Psi_{k+1}(x-y)| dy \right)^q dx \|f_k\|_{L^\infty(Q)} \lesssim \sum_{Q \in D_k} 2^{-kd} \inf_{y \in Q} \|\mathcal{M}_{\sigma,2^k} f_k(y)\|^q
\]
\[
\leq \sum_{Q \in D_k} \int_Q (\mathcal{M}_{\sigma,2^k} f_k(y))^q dy = \int_{P^*} (\mathcal{M}_{\sigma,2^k} f_k(y))^q dy
\]
and this proves (6.8). Therefore
\[
U_P \lesssim \sup_{l \geq \mu} \|m_l(2^l \cdot)\|_{L^2} \left( \frac{1}{|P|} \int_{P^*} \sum_{k=\nu}^\infty (\mathcal{M}_{\sigma,2^k} f_k(y))^q dy \right)^{1/q}
\]
\[
\lesssim \sup_{l \geq \mu} \|m_l(2^l \cdot)\|_{L^2} \sup_{R \in D_l} \left( \frac{1}{|R|} \int_R \sum_{k=\nu}^\infty (\mathcal{M}_{\sigma,2^k} f_k(y))^q dy \right)^{1/q}.
\]
Choosing $\sigma > d/q$ and applying Lemma 2.5 (2), one obtains
\[
U_P \lesssim \sup_{l \geq \mu} \|m_l(2^l \cdot)\|_{L^2} \sup_{R \in D_l} \left( \frac{1}{|R|} \int_R \sum_{k=\nu}^\infty |f_k(x)|^q dx \right)^{1/q}.
\]
To estimate $\mathcal{V}_P$ choose $\epsilon > 0$ such that $s > d/\min(1, q) - d/2 + \epsilon \geq d/2 + \epsilon$. Then for $x \in P$ and for some $C > 0$
\[
|m_l^\gamma \ast (\chi_{P^*} f_k)(x)| \leq \int_{|z| \geq l(P)} |m_l^\gamma(z)| f_k(x-z) dz
\]
\[
\leq \mathcal{M}_{\epsilon,2^k} f_k(x) \int_{|z| \geq l(P)} (1 + 2^k |z|)^\epsilon |m_l^\gamma(z)| dz
\]
\[
\leq \mathcal{M}_{\epsilon,2^k} f_k(x) \sum_{l=k-\nu+C}^\infty \int_{|z| \geq 2^{-l-k}} (1 + 2^k |z|)^\epsilon |m_l^\gamma(z)| \phi_l(2^k z) dz
\]
\[
\lesssim \mathcal{M}_{\epsilon,2^k} f_k(x) \sum_{l=k-\nu}^\infty 2^{(\epsilon+\mu)q} \|\phi_l(2^k \cdot)|_{L^2}
\]
\[
\lesssim \mathcal{M}_{\epsilon,2^k} f_k(x) 2^{-(k-\nu)(s-\epsilon-d/2)} \|m_k(2^l \cdot)\|_{L^2},
\]
where the penultimate inequality follows from the Schwarz inequality and Plancheral’s theorem, and the last one from Schwarz inequality and the fact $F_2^{-2} = L_2^2$.

Now choose $t > \max(d/\epsilon, q)$ and then
\[
\mathcal{V}_P \lesssim \sup_{l \geq \mu} \|m_l(2^l \cdot)\|_{L^2} \left( \frac{1}{|P|} \int_P \sum_{k=\nu}^\infty 2^{-q(k-\nu)(s-\epsilon-d/2)} (\mathcal{M}_{\epsilon,2^k} f_k(x))^q dx \right)^{1/q}
\]
\[
\lesssim \sup_{l \geq \mu} \|m_l(2^l \cdot)\|_{L^2} \left( \frac{1}{|P|} \int_P \sum_{k=\nu}^\infty (\mathcal{M}_{\epsilon,2^k} f_k(x))^t dx \right)^{1/t}
\]
\[
\lesssim \sup_{l \geq \mu} \|m_l(2^l \cdot)\|_{L^2} \sup_{R \in D_l} \left( \frac{1}{|R|} \int_R \sum_{k=\nu}^\infty |f_k(x)|^t dx \right)^{1/t}
\]
\[
\lesssim \sup_{l \geq \mu} \|m_l(2^l \cdot)\|_{L^2} \sup_{R \in D_l, l(R) \leq 2^{-\mu}} \left( \frac{1}{|R|} \int_R \sum_{k=\nu}^\infty |f_k(x)|^q dx \right)^{1/q}.
\]
where the second inequality follows from Hörder’s inequality, the third and fourth ones from Lemma 2.5 and (2.7), respectively.

By taking the supremum of $U_P$ and $V_P$ over all dyadic cubes $P$ whose side length is less or equal to $\mu$, the proof of Theorem 6.1(2) ends.

**Proof of Theorem 6.1 (1).** The proof of the case $0 < p = q \leq \infty$ is a straightforward application of Lemma 6.2 and therefore we work with only the case $p \neq q$ and $0 < p < \infty$.

**The case** $0 < p \leq 1$ and $p < q \leq \infty$. Assume $s > d/p - d/2$. The proof is based on “$\infty$-atoms” for $\dot{f}_{p,q}^0$. We recall in [14] that a sequence of complex numbers $r = \{r_Q\}_{Q \in D}$ is called an $\infty$-atom for $\dot{f}_{p,q}^0$ if there exists a dyadic cube $Q_0$ such that

$$r_Q = 0 \text{ if } Q \not\subset Q_0$$

and

$$(6.11) \quad \|g^q(r)\|_{L^\infty} \leq |Q_0|^{-1/p.}$$

where $g^q(r)$ is defined as in (4.1). Then one has the following atomic decomposition of $\dot{f}_{p,q}^0$, which is analogous to the $H^p$ atomic decomposition for $0 < p \leq 1$.

**Lemma B.** (14) Suppose $0 < p \leq 1$, $p \leq q \leq \infty$, and $b = \{b_Q\}_{Q \in D} \in \dot{f}_{p,q}^0$. Then there exist $C_{d,p,q} > 0$, a sequence of scalars $\{\lambda_j\}$, and a sequence of $\infty$-atoms $r_j = \{r_{j,Q}\}_{Q \in D}$ for $\dot{f}_{p,q}^0$ such that

$$b = \{b_Q\} = \sum_{j=1}^{\infty} \lambda_j \{r_{j,Q}\} = \sum_{j=1}^{\infty} \lambda_j r_j$$

and

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \leq C_{d,p,q} \|b\|_{\dot{f}_{p,q}^0}.$$

Moreover,

$$\|b\|_{\dot{f}_{p,q}^0} \approx \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} : b = \sum_{j=1}^{\infty} \lambda_j r_j, \ r_j \text{ is an } \infty \text{-atom for } \dot{f}_{p,q}^0 \right\}.$$

According to Lemma 4.1 and Lemma B if $\text{Supp}(\hat{f}_k) \subset \{\xi : |\xi| \leq 2^{k-1}\}$ for each $k \in \mathbb{Z}$, then there exist $\{b_Q\}_{Q \in D} \in \dot{f}_{p,q}^0$, a sequence of scalars $\{\lambda_j\}$, and a sequence of $\infty$-atoms $\{r_{j,Q}\}$ for $\dot{f}_{p,q}^0$ such that

$$f_k(x) = \sum_{Q \in D_k} b_Q \Psi^Q(x) = \sum_{j=1}^{\infty} \lambda_j \sum_{Q \in D_k} r_{j,Q} \Psi^Q(x), \ k \in \mathbb{Z},$$

and

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \lesssim \|b\|_{\dot{f}_{p,q}^0} \lesssim \|\{f_k\}_{k \in \mathbb{Z}}\|_{L^p(\mathbb{Z})}.$$
Therefore, it suffices to show that the supremum in the last expression is dominated by a constant times \( \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2} \), which is equivalent to

\[
\| \{ m_k^\nu \ast A_{Q_0,k} \}_{k \in \mathbb{Z}} \|_{L^p(\nu)} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2}
\]

uniformly in \( Q_0 \)

where \( \{ r_Q \} \) is an \( \infty \)-atom for \( f_p^0 \) associated with \( Q_0 \in \mathcal{D} \) and

\[
A_{Q_0,k}(x) := \sum_{Q \in \mathcal{D}_k, Q \subset Q_0} r_Q \Psi^Q(x).
\]

Suppose \( Q_0 \in \mathcal{D}_\nu \) for some \( \nu \in \mathbb{Z} \). Then the condition \( Q \subset Q_0 \) ensures that \( A_{Q_0,k} \) vanishes unless \( \nu \leq k \), and thus we actually need to prove

\[
\| \{ m_k^\nu \ast A_{Q_0,k} \}_{k \geq \nu} \|_{L^p(\nu)} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2}
\]

uniformly in \( \nu \) and \( Q_0 \).

We observe that for \( x \in \mathbb{R}^d \)

\[
(6.12) \quad \| \{ |r_Q||Q|^{-1/2} \chi_Q(x) \}_{Q \subset Q_0} \|_{L^\nu} \leq |Q_0|^{-1/p}
\]

and for \( 0 < r < \infty \)

\[
(6.13) \quad \| A_{Q_0,k} \|_{L^r} \lesssim \| \sum_{Q \in \mathcal{D}_k, Q \subset Q_0} |r_Q||Q|^{-1/2} \chi_Q \|_{L^r} \leq |Q_0|^{-1/p+1/r}
\]

by using the argument in (6.5) and the estimate (6.11). Moreover,

\[
\text{Supp}(\widehat{A_{Q_0,k}}) = \text{Supp}(\widehat{\Psi_k}) \subset \{ \xi : |\xi| \leq 2^k \}.
\]

Let \( Q^*_0 := 9Q_0 \) be the dilate of \( Q \), concentric with \( Q \), with side length \( 9l(Q_0) \), and \( Q^{**}_0 := 9Q^*_0 (= 81Q_0) \). Then

\[
(6.14) \quad \| \{ m_k^\nu \ast A_{Q_0,k} \}_{k \geq \nu} \|_{L^p(\nu)} \lesssim \left( \int_{Q^{**}_0} \| \{ m_k^\nu \ast A_{Q_0,k} \}_{k \geq \nu} \|_{L^p(\nu)} \right)^{1/p}
\]

and the first one is dominated by

\[
\| Q^{**}_0 \|^{1/p-1/q} \| \{ m_k^\nu \ast A_{Q_0,k} \}_{k \geq \nu} \|_{L^p(\nu)} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2} |Q_0|^{1/p-1/q} \| A_{Q_0,k} \|_{L^8(\nu)}
\]

\[
\lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2} |Q_0|^{1/p-1/q} \| r_Q \|_{Q \subset Q_0} \| r_Q \|_{\mathcal{F}_p^0} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2}
\]

where the first inequality follows from Lemma 6.2, the second one from (4.5), and the last one from (6.12).

To handle the term (6.14) we apply the embedding \( L^p \rightarrow L^q \) and then obtain

\[
(6.14) \quad \lesssim \left( \sum_{k=\nu}^{\infty} \| m_k^\nu \ast A_{Q_0,k} \|_{L^p((Q^{**}_0)^c)}^p \right)^{1/p}.
\]

Writing

\[
\| m_k^\nu \ast A_{Q_0,k} \|_{L^p((Q^{**}_0)^c)}^p \leq \| m_k^\nu \ast (A_{Q_0,k} \chi_{Q_0}) \|_{L^p((Q^{**}_0)^c)}^p + \| m_k^\nu \ast (A_{Q_0,k} \chi_{Q^{**}_0}) \|_{L^p((Q^{**}_0)^c)}^p,
\]

the proof will be finished once we establish the estimates that for some \( \epsilon > 0 \)

\[
(6.15) \quad \| m_k^\nu \ast (A_{Q_0,k} \chi_{Q_0}) \|_{L^p((Q^{**}_0)^c)} \lesssim 2^{-\epsilon(k-\nu)} \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2}
\]

and

\[
(6.16) \quad \| m_k^\nu \ast (A_{Q_0,k} \chi_{Q^{**}_0}) \|_{L^p((Q^{**}_0)^c)} \lesssim 2^{-\epsilon(k-\nu)} \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^2}.
\]
By applying the embedding $l^p \hookrightarrow l^1$

$$\left\| m_k^\vee \ast (A_{Q_0,k} \chi_{Q_0^c}) \right\|_{L^p(Q_0^{**c})} \leq \left( \sum_{Q \in D_k, Q \subseteq Q_0^c} \int_{(Q_0^{**c})^c} \left| m_k^\vee \ast (A_{Q_0,k} \chi_{Q}) (x) \right|^p dx \right)^{1/p}$$

$$\leq \left( \sum_{Q \in D_k, Q \subseteq Q_0^c} \| A_{Q_0,k} \|_{L^\infty(Q)} \int_{(Q_0^{**c})^c} \left( \int_Q \left| m_k^\vee (x-y) \right| dy \right)^p dx \right)^{1/p} \cdot$$

Since $s > d/p - d/2$ there exists $M > d(1 - p)$ such that $s > M/p + d/2 > d/p - d/2$. We choose $\epsilon > 0$ such that $s > M/2 + d/2 + \epsilon$. Recall that $x_Q$ denotes the left lower corner of $Q \in D$ and observe that for $Q \subset Q_0^c$

$$\int_{(Q_0^{**c})^c} \left( \int_Q \left| m_k^\vee (x-y) \right| dy \right)^p dx \lesssim 2^{-kM} l(Q_0)^{-M+d(1-p)} \left( \int_Q \int_{(Q_0^{**c})^c} (1 + 2^k |x-x_Q|)^{M/p} \left| m_k^\vee (x-y) \right| dx dy \right)^p$$

$$\lesssim 2^{-k(M+pd)} l(Q_0)^{-M+d(1-p)} \left( \int_{\mathbb{R}^d} (1 + 2^k |y|)^{M/p} \left| m_k^\vee (y) \right| dy \right)^p$$

$$\lesssim 2^{-k(M+pd)} l(Q_0)^{-M+d(1-p)} 2^{-kdp} \left( \int_{\mathbb{R}^d} (1 + |\cdot|)^{M/p+d/2+\epsilon} m_k^\vee(\cdot/2^k) \right)^p \lesssim 2^{-k(M+pd)} l(Q_0)^{-M+d(1-p)} \sup_{l \in \mathbb{Z}} \left\| m_l(2^l \cdot) \right\|_{L^p_2}^p$$

where the first one follows from Hölder’s inequality if $0 < p < 1$ (it is trivial if $p = 1$), the second one from the fact that $|x - x_Q| \lesssim |x - y|$ for $x \in (Q_0^{**c})^c$ and $y \in Q \subset Q_0^c$, and the third one from Schwarz inequality. By applying (2.5) one obtains that for any $\sigma > 0$

$$\left\| m_k^\vee \ast (A_{Q_0,k} \chi_{Q_0^c}) \right\|_{L^p(Q_0^{**c})} \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l(2^l \cdot) \right\|_{L^p_2} 2^{-k(M/p+d/p-d)} \left( \sum_{Q \in D_k, Q \subseteq Q_0^c} \left( \inf_{y \in Q} m_{\sigma,2^k} A_{Q_0,k}(y) \right) \right)^p \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l(2^l \cdot) \right\|_{L^p_2} 2^{-k(M/p-d(1/p-1))} \left( \int_{Q_0^c} \left( \inf_{y \in Q} m_{\sigma,2^k} A_{Q_0,k}(x) \right)^p dx \right)^{1/p}$$

and then Lemma 2.5 (1) with $\sigma > d/p$ and (6.13) prove (6.15) with $\epsilon = M/p-d(1/p-1) > 0$. To verify (6.16) we observe that, similar to (6.7) under the assumption (6.6),

$$m_k^\vee \ast \left( A_{Q_0,k} \chi_{Q_0^c} \right) = m_k^\vee \ast \Psi_{k+1} \ast \left( A_{Q_0,k} \chi_{Q_0^c} \right)$$

and it follows from Lemma 6.2 that

$$\left\| m_k^\vee \ast \left( A_{Q_0,k} \chi_{Q_0^c} \right) \right\|_{L^p} \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l(2^l \cdot) \right\|_{L^p_2} \left\| \Psi_{k+1} \ast \left( A_{Q_0,k} \chi_{Q_0^c} \right) \right\|_{L^p}.$$

Moreover,

$$\left\| \Psi_{k+1} \ast \left( A_{Q_0,k} \chi_{Q_0^c} \right) \right\|_{L^p} \lesssim \left( \int_{\mathbb{R}^d} \left( \sum_{Q \in D_k, Q \subseteq Q_0^c} |r_Q||Q|^{-1/2} \int_{(Q_0^c)^c} \left| \Psi_{k+1}(x-y) \right| \frac{1}{(1 + 2^k |y-x_Q|)^2} dx \right)^p dy \right)^{1/p} \lesssim 2^{-kL} \left( \sum_{Q \in D_k, Q \subseteq Q_0^c} |r_Q||Q|^{-1/2} \left( \int_{\mathbb{R}^d} \left( \int_{(Q_0^c)^c} \frac{\left| \Psi_{k+1}(x-y) \right|}{|y-x_Q|} dx \right)^p dy \right)^{1/p} \right).$$
because \( |y - x_Q| \gtrsim l(Q_0) \) and
\[
\frac{1}{(1 + 2^k |y - x_Q|)^{2L}} \lesssim (2^k l(Q_0))^{-L} \frac{1 + 2^k |x_Q - x_Q_0|}{(1 + 2^k |y - x_Q|)^{2L}} \lesssim \frac{1}{(2^k |y - x_Q|)^{L}}
\]
for \( y \in (Q_0)^c \) and \( Q \subset Q_0 \). Notice that due to (6.12)
\[
\sum_{Q \in \mathcal{D}_k, Q \subset Q_0} |r_Q| |Q|^{-1/2} \leq 2^{kd(1/p-1)} 2^{kd}
\]
and, using Hölder’s inequality (if \( p < 1 \)), one obtains
\[
\left( \int_{\mathbb{R}^d} \left( \int_{(Q_0)^c} \frac{|\Psi_{k+1}(x - y)|}{|y - x_Q|^{L}} dy \right)^p dx \right)^{1/p} \lesssim 2^{-kd(1/p-1)} \int_{(Q_0)^c} \frac{1}{|y - x_Q|^{L}} \int_{\mathbb{R}^d} (1 + 2^k |x - x_Q_0|)^{N/p} |\Psi_{k+1}(x - y)| dxdy
\]
\[
\lesssim 2^{-kd(1/p-1)} \int_{(Q_0)^c} \frac{(1 + 2^k |y - x_Q_0|)^{N/p}}{|y - x_Q|^{L}} dy \lesssim 2^{-kd(1/p-1)} 2^{kN/p} \int_{(Q_0)^c} \frac{1}{|y - x_Q|^{L-N/p}} dxdy
\]
\[
\lesssim 2^{-kd(1/p-1)} 2^{kN/p} 2^d(L-N/p-d)
\]
for \( N > d(1 - p) \) and \( L - N/p > d \).

In conclusion, one has
\[
\| \Psi_{k+1} \ast (A_{Q_0,k} \chi_{(Q_0)^c}) \|_{L^p} \lesssim 2^{-(k-\nu)(L-N/p+d/p-2d)}
\]
and this proves (6.10) with \( \epsilon = L - N/p + d/p - 2d > 0 \).

**The case** \( 1 < p < \infty \) and \( p < q \leq \infty \). Assume \( s > 0 \) and interpolate two estimates
\[
\| \{ m_k^\nu \ast f_k \}_{k \in \mathbb{Z}} \|_{L^1(\nu)} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^1_l} \| f \|_{L^1_l}
\]
and
\[
\| \{ m_k^\nu \ast f_k \}_{k \in \mathbb{Z}} \|_{L^q(\nu)} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^1_l} \| f \|_{L^q_l},
\]
which have been already proved.

**The case** \( 0 < q < p < \infty \). Assume \( s > d/\min(1,q) - d/2 \), and choose \( \epsilon > 0 \) and \( t > 0 \) such that \( s > d/\min(1,q) - d/2 + \epsilon \geq d/2 + \epsilon \) and \( t > \max(d/\epsilon,q) \).

We first consider the case \( t < p < \infty \). In this case we apply Lemma 5.1 to obtain
\[
\| \{ m_k^\nu \ast f_k \}_{k \in \mathbb{Z}} \|_{L^p(\nu)} \lesssim \sup_{x \in \mathcal{P} \subset \mathbb{D}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty |m_k^\nu \ast f_k(y)|^q dy \right)^{1/q} \| m_l(2^l \cdot) \|_{L^1_l} \| f \|_{L^p(\nu)}.
\]
Now let \( x \in P \in \mathcal{D}_\nu \) for some \( \nu \in \mathbb{Z} \) and define \( P^* = 9P \) as before. Then, using (6.9),
\[
\left( \frac{1}{|P|} \int_{P} \sum_{k=\nu}^\infty |m_k^\nu \ast (\chi_{P^*} f_k)(x)|^q dy \right)^{1/q} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^1_l} \left( \frac{1}{|P|} \int_{P^*} \sum_{k=\nu}^\infty (\mathcal{M}_{\sigma,2^k} f_k(y))^q dy \right)^{1/q}
\]
\[
\lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^1_l} \left[ \mathcal{M} \left( \sum_{k=\nu}^\infty (\mathcal{M}_{\sigma,2^k} f_k(y))^q \right)(x) \right]^{1/q}
\]
for \( \sigma > d/q \). By the \( L^{p/q} \) boundedness of \( \mathcal{M} \) and Lemma 2.3 (1)
\[
\| \sup_{x \in \mathcal{P} \subset \mathbb{D}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 l(P)}^\infty |m_k^\nu \ast (\chi_{P^*} f_k)(y)|^q dy \right)^{1/q} \|_{L^p(\nu)} \lesssim \sup_{l \in \mathbb{Z}} \| m_l(2^l \cdot) \|_{L^1_l} \| f \|_{L^p(\nu)},
\]
Furthermore, one obtains, from (6.10), that
\[
\left( \frac{1}{|P|} \int_P \sum_{k=\nu}^{\infty} |m_k^\vee \ast (\chi_{(P')^c} f_k)(x)|^q dy \right)^{1/q} \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l (2^l \cdot) \right\|_{L^2} \left( \frac{1}{|P|} \int_P \sum_{k=\nu}^{\infty} (\mathcal{M}_{\epsilon,2} f_k(y))' dy \right)^{1/q} \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l (2^l \cdot) \right\|_{L^2} \left[ \mathcal{M} \left( \sum_{k \in \mathbb{Z}} (\mathcal{M}_{\epsilon,2} f_k)'(x) \right) \right]^{1/l}.
\]

Then by the $L^{p/t}$ boundedness of $\mathcal{M}$ (since $p/t > 1$ from our assumption), Lemma 2.5 (1) with $\epsilon > d/t$, and the embedding $l^q \hookrightarrow l^t$, one has
\[
\left\| \sup_{x \in P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} |m_k^\vee \ast (\chi_{(P')^c} f_k)(y)|^q dy \right)^{1/q} \right\|_{L^{p}(x)} \lesssim L_s^2(\{m_k\}_{k \in \mathbb{Z}}) \left\| f \right\|_{L^p(l^q)}.
\]

This proves that for $s > d/\min(1, q) - d/2$
\[
(6.17) \quad \left\| \{m_k^\vee \ast f_k\}_{k \in \mathbb{Z}} \right\|_{L^p(l^q)} \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l (2^l \cdot) \right\|_{L^2} \left\| f \right\|_{L^p(l^q)}, \quad q < t < p.
\]

The general case $q < p$ follows from interpolation between (6.17) and $L^q(l^q)$ estimate with the same value $s > d/\min(1, q) - d/2$. Here $l^q, 0 < q \leq \infty$, is a quasi-Banach space and Peetre’s real interpolation method, so called $K$-method, works. See [13, Chapter 6] for more details about the interpolation method.

7. HÖRMANDER MULTIPLIER THEOREM FOR MULTILINEAR OPERATORS

For notational convenience we will occasionally write $\tilde{f} := (f_1, \ldots, f_n), \tilde{\xi} := (\xi_1, \ldots, \xi_n), k := (k_1, \ldots, k_n), \tilde{v} := (v_1, \ldots, v_n), d\tilde{\xi} := d\xi_1 \cdots d\xi_n$, and $d\tilde{v} := dv_1 \cdots dv_n$.

For $m \in L^\infty(\mathbb{R}^d)^n$ the n-multilinear multiplier operator $T_m$ is defined by
\[
T_m \tilde{f}(x) := \int_{\mathbb{R}^d} m(\tilde{\xi}) \left( \prod_{j=1}^n \hat{f}_j(\xi_j) \right) e^{2\pi i (x, \sum_{j=1}^n \xi_j)} d\tilde{\xi}
\]
for $f_j \in S(\mathbb{R}^d)$. Let $\vartheta^{(n)} \in S(\mathbb{R}^d)^n$ satisfy the properties that $0 \leq \vartheta^{(n)} \leq 1$, $\vartheta^{(n)} = 1$ for $2^{-1} \leq |\tilde{\xi}| \leq 2$, and $\text{Supp}(\vartheta^{(n)}) \subset \{ \tilde{\xi} \in \mathbb{R}^d : 2^{-2} \leq |\tilde{\xi}| \leq 2^2 \}$. Define
\[
L_s^{r, \vartheta^{(n)}}[m] := \sup_{l \in \mathbb{Z}} \left\| m(2^l \cdot, \ldots, 2^l \cdot) \vartheta^{(n)} \right\|_{L^2(\mathbb{R}^dn)}.
\]

A multilinear version of Hörmander’s multiplier theorem was established by Tomita [27].

Theorem C. Suppose $1 < p, p_1, \ldots, p_n < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_n$. If $m \in L^\infty(\mathbb{R}^d)^n$ satisfies $L_s^{2, \vartheta^{(n)}}[m] < \infty$ for $s > nd/2$, then there exists a constant $C > 0$ so that
\[
\left\| T_m \tilde{f} \right\|_{L^p} \leq C L_s^{2, \vartheta^{(n)}}[m] \prod_{j=1}^n \left\| f_j \right\|_{L^{p_j}}.
\]

Another boundedness result was obtained by Grafakos, Si [18].

Theorem D. Let $0 < p < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_n$. Suppose $1 < r \leq 2$ and $m$ satisfies $L_s^{r, \vartheta^{(n)}}[m] < \infty$ for $s > nd/r$. Then there exists a number $\delta > 0$ satisfying $0 < \delta \leq r - 1$, such that
\[
\left\| T_m \tilde{f} \right\|_{L^p} \lesssim L_s^{r, \vartheta^{(n)}}[m] \prod_{j=1}^n \left\| f_j \right\|_{L^{p_j}}.
\]

whenever $r - \delta < p_j < \infty$ for $1 \leq j \leq n$. 
In this section we will generalize Theorem [C] and [D]. Let
\[ X^p := \begin{cases} H^p & \text{if } p < \infty, \\ BMO & \text{if } p = \infty. \end{cases} \]

**Theorem 7.1.** Let \( 1 < p < \infty \) and \( 1 < p_{i,j} \leq \infty \), \( 1 \leq i, j \leq n \), satisfies
\[
\frac{1}{p} = \frac{1}{p_{i,1}} + \cdots + \frac{1}{p_{i,n}} \quad \text{for } 1 \leq i \leq n. 
\]

Suppose \( m \) satisfies \( L_{s,\widetilde{\vartheta}^{(n)}}^2[m] < \infty \) for \( s > nd/2 \). Then
\[
\| T_m \tilde{f} \|_{L^p} \lesssim L_{s,\widetilde{\vartheta}^{(n)}}^2[m] \sum_{i=1}^n \left( \| f_i \|_{X^{p_{i,i}}} \prod_{1 \leq j \leq n, j \neq i} \| f_j \|_{L^{p_{i,j}}} \right).
\]

**Theorem 7.2.** Let \( 1 < p < \infty \) and \( 1 < p_{i,j} \leq \infty \), \( 1 \leq i, j \leq n \), satisfies
\[
\frac{1}{p} = \frac{1}{p_{i,1}} + \cdots + \frac{1}{p_{i,n}} \quad \text{for } 1 \leq i \leq n. 
\]

Suppose \( 1 < u \leq 2, 0 < r \leq 2, \) and \( m \) satisfies \( L_{s,\widetilde{\vartheta}^{(n)}}^u[m] < \infty \) for \( s > nd/r \). Then there exists a number \( \delta > 0 \) such that
\[
\| T_m \tilde{f} \|_{L^p} \lesssim L_{s,\widetilde{\vartheta}^{(n)}}^u[m] \sum_{i=1}^n \left( \| f_i \|_{X^{p_{i,i}}} \prod_{1 \leq j \leq n, j \neq i} \| f_j \|_{H^{p_{i,j}}} \right)
\]
whenever \( r - \delta < p_{i,j} \leq \infty \) for \( 1 \leq i, j \leq n \).

Under the same hypothesis \( s > nd/r \), the condition \( L_{s,\widetilde{\vartheta}^{(n)}}^r[m] < \infty \) in Theorem [D] is improved by \( L_{s,\widetilde{\vartheta}^{(n)}}^{u_1,\widetilde{\vartheta}^{(n)}}[m] < \infty \) for any \( 1 < u \leq 2 \) in Theorem 7.2. In turn, due to the independence of \( u \) in \( L_{s,\widetilde{\vartheta}^{(n)}}^{u_1,\widetilde{\vartheta}^{(n)}}[m] < \infty \), one has better freedom in the range \( 0 < r \leq 2 \) and \( r - \delta < p_{i,j} \leq \infty \). Note that \( L_{s,\widetilde{\vartheta}^{(n)}}^{u_1,\widetilde{\vartheta}^{(n)}}[m] \lesssim L_{s,\widetilde{\vartheta}^{(n)}}^{u_2,\widetilde{\vartheta}^{(n)}}[m] \) if \( 1 < u_1 < u_2 \leq 2 \).

We will first prove Theorem 7.2 and then turn to the proof of Theorem 7.1.

### 7.1. Proof of Theorem 7.2

Choose \( 0 < t < r \) such that \( s > nd/t > nd/r \) and let \( \delta = r - t > 0 \). Suppose \( p_1, \ldots, p_n > t = r - \delta \).

Let \( \vartheta^{(n)} \) be a cutoff function on \( (\mathbb{R}^d)^n \) such that \( 0 \leq \vartheta^{(n)} \leq 1, \vartheta^{(n)}(\xi) = 1 \) for \( 2^{-1}n^{-1/2} \leq |\xi| \leq 2n^{1/2} \), and \( \text{Supp}(\vartheta^{(n)}) \subset \{ \xi \in (\mathbb{R}^d)^n : 2^{-2}n^{-1/2} \leq |\xi| \leq 2^2n^{1/2} \} \). Then using Calderón reproducing formula, Littlewood-Paley partition of unity \( \{ \varphi_k \}_{k \in \mathbb{Z}} \), and triangle inequality, we first see that
\[
L_{s,\vartheta^{(n)}}^u[m] \lesssim L_{s,\widetilde{\vartheta}^{(n)}}^u[m].
\]

Thus it suffices to prove the estimate that
\[
(7.1) \quad \| T_m \tilde{f} \|_{L^p} \lesssim L_{s,\vartheta^{(n)}}^u[m] \sum_{i=1}^n \left( \| f_i \|_{X^{p_{i,i}}} \prod_{1 \leq j \leq n, j \neq i} \| f_j \|_{H^{p_{i,j}}} \right).
\]

We use a notation \( L_u^u[m] := L_u,\vartheta^{(n)}[m] \).
By using Littlewood-Paley partition of unity \( \{ \phi_k \}_{k \in \mathbb{Z}} \), \( m(\vec{\xi}) \) can be decomposed as
\[
m(\vec{\xi}) = \sum_{k \in \mathbb{Z}^d} m(\vec{\xi})\hat{\phi}_k(\xi_1) \cdot \cdot \cdot \hat{\phi}_n(\xi_n) = \left( \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \leq k_1} \cdots \right) + \left( \sum_{k_2 \in \mathbb{Z}} \sum_{k_1 < k_2} \cdots \right) + \cdots + \left( \sum_{k_n \in \mathbb{Z}} \sum_{k_1 \leq k_2 \leq k_3 \leq \cdots \leq k_n} \cdots \right)
\]
\[
=: m^{(1)}(\vec{\xi}) + m^{(2)}(\vec{\xi}) + \cdots + m^{(n)}(\vec{\xi}).
\]
Then (7.2) is a consequence of the following estimates that for \( s > nd/r \)
\[
\| T_{m^{(s)}} \vec{f} \|_{L^p} \lesssim \mathcal{L}^u_s[m] \| f_1 \|_{X^{p,1}} \prod_{1 \leq j \leq n, j \neq i} \| f_j \|_{H^{p,1}}
\]
for each \( 1 \leq i \leq n \).

We only concern ourselves with the case \( i = 1 \) and use symmetry for other cases by setting \( p_j := p_{1,j} \) for \( 1 \leq j \leq n \).

We write
\[
m^{(1)}(\vec{\xi}) = \sum_{k \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \leq k} m(\vec{\xi})\hat{\phi}_k(\xi_1)\hat{\phi}_{k_2}(\xi_2) \cdot \cdot \cdot \hat{\phi}_{k_n}(\xi_n)
\]
\[
= \sum_{k \in \mathbb{Z}} m(\vec{\xi})\hat{\vartheta}^{(n)}(\vec{\xi}/2^k)\hat{\phi}_k(\xi_1) \sum_{k_2 \leq \ldots \leq k_n \leq k} \hat{\phi}_{k_2}(\xi_2) \cdot \cdot \cdot \hat{\phi}_{k_n}(\xi_n)
\]
since \( \hat{\vartheta}^{(n)}(\vec{\xi}/2^k) = 1 \) for \( 2^{k-1} \leq |\xi_1| \leq 2^{k+1} \) and \( |\xi_j| \leq 2^{k+1} \) for \( 2 \leq j \leq n \). Let
\[
m_k(\vec{\xi}) := m(\vec{\xi})\hat{\vartheta}^{(n)}(\vec{\xi}/2^k)
\]
and then we note that
\[
(7.2) \quad \| m_k(2^k \cdot) \|_{L^p((\mathbb{R}^d)^n)} \leq \mathcal{L}^u_s[m]
\]
and
\[
m^{(1)}(\vec{\xi}) = \sum_{k \in \mathbb{Z}} m_k(\vec{\xi})\hat{\phi}_k(\xi_1) \sum_{k_2, \ldots, k_n \leq k} \hat{\phi}_{k_2}(\xi_2) \cdot \cdot \cdot \hat{\phi}_{k_n}(\xi_n).
\]

We further decompose \( m^{(1)} \) as
\[
m^{(1)}(\vec{\xi}) = m^{(1)}_{\text{low}}(\vec{\xi}) + m^{(1)}_{\text{high}}(\vec{\xi})
\]
where
\[
m^{(1)}_{\text{low}}(\vec{\xi}) := \sum_{k \in \mathbb{Z}} m_k(\vec{\xi})\hat{\phi}_k(\xi_1) \sum_{k_2, \ldots, k_n \leq k} \hat{\phi}_{k_2}(\xi_2) \cdot \cdot \cdot \hat{\phi}_{k_n}(\xi_n), \quad \max_{2 \leq j \leq n} (k_j) \geq k - 3 - \lfloor \log_2 n \rfloor
\]
\[
m^{(1)}_{\text{high}}(\vec{\xi}) := \sum_{k \in \mathbb{Z}} m_k(\vec{\xi})\hat{\phi}_k(\xi_1) \sum_{k_2, \ldots, k_n \leq k} \hat{\phi}_{k_2}(\xi_2) \cdot \cdot \cdot \hat{\phi}_{k_n}(\xi_n), \quad \max_{2 \leq j \leq n} (k_j) \geq k - 4 - \lfloor \log_2 n \rfloor
\]
We refer to \( T_{m^{(1)}_{\text{low}}} \) as the low frequency part, and \( T_{m^{(1)}_{\text{high}}} \) as the high frequency part of \( T_{m^{(1)}} \) (due to the Fourier supports of \( T_{m^{(1)}_{\text{low}}} \vec{f} \) and \( T_{m^{(1)}_{\text{high}}} \vec{f} \)).
7.1.1. Low frequency part. To obtain the estimates for the operator $T_{m_{low}}^{(1)}$, we observe that

$$T_{m_{low}}^{(1)} \hat{f}(x) = \sum_{k \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \leq k} T_m((f_1)_k, (f_2)_k, \ldots, (f_n)_k)(x)$$

where $(g)_l := \phi_l * g$ for $g \in S$ and $l \in \mathbb{Z}$. It suffices to consider only the sum over $k_3, \ldots, k_n \leq k_2$ and $k - 3 - \lfloor \log_2 n \rfloor \leq k_2 \leq k$, and we will actually prove that

$$\left\| \sum_{k \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \leq k} T_m((f_1)_k, (f_2)_k, \ldots, (f_n)_k) \right\|_{L^p} \lesssim \mathcal{L}_s^{u}[m] \| f_1 \|_{X^p_1} \prod_{j=2}^n \| f_j \|_{H^{p_j}}.$$  

We define $\Phi_l := 2^d \Phi_0(2^l \cdot)$ for $l \in \mathbb{Z}$ and then observe that for any $g \in S$

$$\sum_{m \leq l} \phi_m * g = \Phi_l * g$$

and

$$(7.3) \quad \left\| \sup_{k \in \mathbb{Z}} |\Phi_k * f| \right\|_{L^p} \lesssim \| f \|_{H^p}, \quad 0 < p \leq \infty.$$  

We see that

$$\sum_{k \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \leq k} \sum_{k_3, \ldots, k_n \leq k_2} T_m((f_1)_k, (f_2)_k, (f_3)_k, \ldots, (f_n)_k)(x)$$

$$= \sum_{k \in \mathbb{Z}} \sum_{k_2, \ldots, k_n \leq k_2} T_m((f_1)_k, (f_2)_k, (f_3)_k, \ldots, (f_n)_k)(x)$$

where $(f_j)_k := \Phi_k * f_j$. Since the second sum is a finite sum over $k_2$ near $k$, we may only consider the case $k_2 = k$ and thus our claim is

$$(7.4) \quad \left\| \sum_{k \in \mathbb{Z}} T_m((f_1)_k, (f_2)_k, (f_3)_k, \ldots, (f_n)_k) \right\|_{L^p} \lesssim \mathcal{L}_s^{u}[m] \| f_1 \|_{X^p_1} \prod_{j=2}^n \| f_j \|_{H^{p_j}}.$$  

To prove (7.4) let $0 < \epsilon < \min(1, t)$ such that $1/\epsilon = 1 - 1/u + 1/t$, which implies $u' = \frac{1}{1-1/\epsilon}$ where $1/u + 1/u' = 1$. Then using Nikol'ski’s inequality and Hölder’s inequality with $u'/\epsilon > 1$ one has

$$|T_m((f_1)_k, (f_2)_k, (f_3)_k, \ldots, (f_n)_k)(x)|$$

$$= \left| \int_{(\mathbb{R}^d)^n} m_k^\vee(\tilde{v})(f_1)_k(x - v_1)(f_2)_k(x - v_2) \prod_{j=3}^n (f_j)_k(x - v_j) d\tilde{v} \right|$$

$$\lesssim 2^{nkd(1/\epsilon - 1)} \left( \int_{(\mathbb{R}^d)^n} |m_k^\vee(\tilde{v})|^\epsilon |(f_1)_k(x - v_1)|^\epsilon |(f_2)_k(x - v_2)|^\epsilon \prod_{j=3}^n |(f_j)_k(x - v_j)|^\epsilon d\tilde{v} \right)^{1/\epsilon}$$

$$\leq 2^{nkd(1/\epsilon - 1)} \left( \int_{(\mathbb{R}^d)^n} \left( 1 + 2^k |v_1| + \cdots + 2^k |v_n| \right)^{su'} |m_k^\vee(\tilde{v})|^{\epsilon' d\tilde{v}} \right)^{1/u'}$$

$$\times \left( \int_{(\mathbb{R}^d)^n} \left( 1 + 2^k |v_1| + \cdots + 2^k |v_n| \right)^{su} \prod_{j=3}^n |(f_j)_k(x - v_j)|^\epsilon d\tilde{v} \right)^{1/\epsilon}.$$  

We observe that by using Hausdorff Young’s inequality with \( u' \geq 2 \) and (2.6),
\[
\left( \int_{\mathbb{R}^d} (1 + 2^k|v_1| + \cdots + 2^k|v_n|)^{su'} \left| m_k^v(\mathbf{v}) \right|^{u'} d\mathbf{v} \right)^{1/u'} \lesssim 2^{nk/u} \left\| m_k(2^k \cdot) \right\|_{L^u_x} \lesssim 2^{nk/u} L^u_s [m],
\]
and
\[
\left( \int_{\mathbb{R}^d} \frac{|(f_1)_k(x - v_1)|^t |(f_2)_k(x - v_2)|^t}{(1 + 2^k|v_1| + \cdots + 2^k|v_n|)^{st/n}} \prod_{j=3}^n |(f_j)_k(x - v_n)|^t d\mathbf{v} \right)^{1/t} \leq \left( \int_{\mathbb{R}^d} \left( (1 + 2^k|v_1|)^{st/n} dv_1 \right)^{1/t} \left( \int_{\mathbb{R}^d} \left( (1 + 2^k|v_2|)^{st/n} dv_2 \right)^{1/t} \prod_{j=3}^n \left( (1 + 2^k|v_j|)^{st/n} dv_j \right)^{1/t} \right) \lesssim 2^{-nk/t} \mathcal{M}^{s/n,2k}_s (f_1)_k(x) \mathcal{M}^{s/n,2k}_s (f_2)_k(x) \prod_{j=3}^n \mathcal{M}^{s/n,2k}_s (f_j)_k(x).
\]

Therefore
\[
|T_{mk}((f_1)_k, (f_2)_k, (f_3)_k, \ldots, (f_n)_k)(x)| \lesssim L^u_s [m] \mathcal{M}^{s/n,2k}_s (f_1)_k(x) \mathcal{M}^{s/n,2k}_s (f_2)_k(x) \prod_{j=3}^n \mathcal{M}^{s/n,2k}_s (f_j)_k(x)
\]
(7.5)
because \( 1/\epsilon - 1 + 1/u - 1/t = 0 \). Now fix \( 0 < \gamma < 1 \) and for each \( Q \in \mathcal{D}_k \) let \( S_Q := S_Q^{\gamma,2} (\mathcal{M}^{s/n,2k}_s (f_1)_k) \). Then it follows from (3.10) and (3.4) that
\[
|S_Q| \geq (1 - \gamma)|Q|, \quad \chi_Q(x) \lesssim \mathcal{M}_r (\chi_{S_Q})(x) \chi_Q(x) \quad \text{for } 0 < r < \infty.
\]

Now we choose \( \tau < \min (1, p) \) and apply (2.6) to obtain
\[
\left\| \sum_{k \in \mathbb{Z}} T_{mk}((f_1)_k, (f_2)_k, (f_3)_k, \ldots, (f_n)_k) \right\|_{L^p} \leq \mathcal{L}^u_s [m] \left\| \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_k} \mathcal{M}^{s/n,2k}_s (f_1)_k \mathcal{M}^{s/n,2k}_s (f_2)_k \chi_Q \right\|_{L^{p_1/p_2 + p_2}} \prod_{j=3}^n \left\| \{ \mathcal{M}^{s/n,2k}_s (f_j)_k \} \right\|_{L^{p_2}/(p_2)}
\]
\[
\lesssim \mathcal{L}^u_s [m] \left\| \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_k} \left( \inf_{y \in Q} \mathcal{M}^{s/n,2k}_s (f_1)_k(y) \right) \left( \inf_{y \in Q} \mathcal{M}^{s/n,2k}_s (f_2)_k(y) \right) \mathcal{M}_r (\chi_{S_Q}) \right\|_{L^{p_1/p_2 + p_2}} \prod_{j=3}^n \left\| f_j \right\|_{H^{p_2}}
\]
\[
\lesssim \mathcal{L}^u_s [m] \left\| \left\{ \sum_{Q \in \mathcal{D}_k} \left( \inf_{y \in Q} \mathcal{M}^{s/n,2k}_s (f_1)_k(y) \right) \chi_{S_Q} \right\} \right\|_{L^{p_1/(2p_2)}} \prod_{j=2}^n \left\| f_j \right\|_{H^{p_2}}
\]
\[
\lesssim \mathcal{L}^u_s [m] \left\| f_1 \right\|_{X^{p_1}} \prod_{j=2}^n \left\| f_j \right\|_{H^{p_2}}
\]
where the first inequality follows from Hölder’s inequality, the second one from Lemma 2.5 (1), the third one from (7.3), the fourth one from Hölder’s inequality, and the last one from Lemma 3.1 (1) if \( p_1 < \infty \) or Theorem 3.3 if \( p_1 = \infty \).
7.1.2. High frequency part. The proof for the high frequency part is based on the property that if \( \hat{g}_k \) is supported on \( \{ \xi : A^{-1}2^k \leq |\xi| \leq A2^k \} \) for \( A > 1 \) then

\[
(7.6) \quad \left\| \{ \phi_k * \left( \sum_{l=k-h}^{k+h} g_l \right) \}_{k \in \mathbb{Z}} \right\|_{L^p(l_0)} \lesssim \left\| \{ g_k \}_{k \in \mathbb{Z}} \right\|_{L^p(l_0)}
\]

for \( h \in \mathbb{N} \). The proof of \((7.6)\) is elementary and standard, so we omit it. Just use the estimate \( |\phi_k * g_l(x)| \lesssim m_{s/2}^t g_l(x) \) for \( k - h \leq l \leq k + h \) and apply Lemma 2.5 \((1)\).

We note that

\[
T_{m_{\text{high}}} \hat{f}(x) = \sum_{k \in \mathbb{Z}} T_{m_k} ((f_1)_k, (f_2)_k, \ldots, (f_n)_k)(x)
\]

where \((f_j)_{k,n} := \Phi_{k-4-\lfloor \log_2 n \rfloor} * f_j\).

Observe that the Fourier transform of \( T_{m_k} ((f_1)_k, (f_2)_k, \ldots, (f_n)_k) \) is supported in \( \{ \xi \in \mathbb{R}^d : 2^k - 2 \leq |\xi| \leq 2^k + 2 \} \) and thus \((7.3)\) yields that

\[
\left\| T_{m_{\text{high}}} \hat{f} \right\|_{L^p} \lesssim \left\| T_{m_{\text{high}}} \hat{f} \right\|_{H^p} \approx \left\| T_{m_{\text{high}}} \hat{f} \right\|_{F^0,p,2} \lesssim \left\| \{ T_{m_k} ((f_1)_k, (f_2)_k, \ldots, (f_n)_k) \}_{k \in \mathbb{Z}} \right\|_{L^p(l_0)}.
\]

Using the argument that led to \((7.3)\), one has

\[
|T_{m_k} ((f_1)_k, (f_2)_k, \ldots, (f_n)_k)(x)| \lesssim L^u_s[m] \mathcal{M}_s^t_{s/2}((f_1)_k(x)) \prod_{j=2}^n \mathcal{M}_{s/2}^t(f_j)_k(x).
\]

Fix \( 0 < \gamma < 1 \) and for \( Q \in D_k \) let \( S_Q := S^{\gamma,2}_Q \mathcal{M}_{s/2}^t((f_1)_k) \) as before and proceed the similar arguments to obtain that

\[
\left\| T_{m_{\text{high}}} \hat{f} \right\|_{L^p} \lesssim L^u_s[m] \left\| \prod_{j=2}^n \mathcal{M}_{s/2}^t(f_j)_k(x) \right\|_{L^p(l_0)} \lesssim L^u_s[m] \left\| \prod_{j=2}^n \mathcal{M}_{s/2}^t(f_j)_k(x) \right\|_{L^p(l_0)} \lesssim L^u_s[m] \left\| f_1 \right\|_{X^{p_1}} \prod_{j=2}^n \left\| f_j \right\|_{H^{p_j}}
\]

for \( p_1, \ldots, p_n > t \).

7.2. Proof of Theorem 7.1. As in the proof of Theorem 7.2 it suffices to deal with \( T_{m(1)} \).

Suppose \( 1 < p < \infty \) and \( 1 < p_j \leq \infty \) for \( 1 \leq j \leq n \). Then we will prove

\[
(7.7) \quad \left\| T_{m(1)} \hat{f} \right\|_{L^p} \lesssim L^2_s[m] \left\| f_1 \right\|_{X^{p_1}} \prod_{j=2}^n \left\| f_j \right\|_{L^{p_j}}, \quad s > nd/2
\]

for each \( 1 \leq i \leq n \). First of all, it follows, from Theorem 7.2 with \( r = u = 2 \), that \((7.1)\) holds for \( 2 \leq p_j \leq \infty \).

Now assume \( 1 < p \leq \min (p_1, \ldots, p_n) < 2 \). Observe that only one of \( p_j's \) could be less than 2 because \( 1/p = 1/p_1 + \cdots + 1/p_n < 1 \), and we will actually look at two cases
1 < p_1 < 2 ≤ p_2, . . . , p_n and 1 < p_2 < 2 ≤ p_1, p_3, . . . , p_n. Let \( T_{m(1)}^{j} \) be the jth transpose of \( T_{m(1)} \), defined by the unique operator satisfying
\[
\langle T_{m(1)}^{j}(f_1, . . . , f_n), h \rangle := \langle T(f_1, . . . , f_j-1, h, f_j+1, . . . , f_n), f_j \rangle
\]
for \( f_1, . . . , f_n, h \in S \). Then it is known in [27] that \( T_{m(1)}^{j} = T_{(m(1))^{*}j} \) where
\[
(m(1))^{*j}(ξ_1, . . . , ξ_n) = m(1)(ξ_1, . . . , ξ_j-1, -(ξ_1 + . . . + ξ_n), ξ_j+1, . . . , ξ_n),
\]
and then
\[
(7.8) \quad \mathcal{L}^{2}_{s}[(m(1))^{*j}] \lesssim \mathcal{L}^{2}_{s}[m(1)] \lesssim \mathcal{L}^{2}_{s}[m].
\]

7.2.1. The case \( 1 < p < p_1 < 2 \). Let \( 2 < p', p'_1 < \infty \) be the conjugates of \( p, p_1 \), respectively. That is, \( 1/p + 1/p' = 1/p_1 + 1/p'_1 = 1 \). Then \( X^{p_1} = L^{p_1} \) and \( 1/p'_1 = 1/p'_1 + 1/p_2 + . . . + 1/p_n \). Therefore
\[
\| T_{m(1)}(f_1, . . . , f_n) \|_{L^p} = \sup_{\|h\|_{L^{p'_1}} = 1} \| \langle T_{m(1)}(h, f_2, . . . , f_n), f_1 \rangle \|
\]
\[
\leq \| f_1 \|_{L^{p_1}} \sup_{\|h\|_{L^{p'_1}} = 1} \| T_{m(1)}(h, f_2, . . . , f_n) \|_{L^{p'_1}}
\]
\[
\lesssim \mathcal{L}^{2}_{s}[(m(1))^{*1}] \prod_{j=1}^{n} \| f_j \|_{L^{p_j}} \lesssim \mathcal{L}^{2}_{s}[m] \prod_{j=1}^{n} \| f_j \|_{L^{p_j}}
\]
where the second inequality follows from Theorem 7.2 and the last one from (7.8).

7.2.2. The case \( 1 < p < p_2 < 2 \). Similarly, let \( 2 < p', p'_2 < \infty \) be the conjugates of \( p, p_2 \) and then
\[
\| T_{m(1)}(f_1, . . . , f_n) \|_{L^p} = \sup_{\|h\|_{L^{p'_2}} = 1} \| \langle T_{m(1)}(h, f_3, . . . , f_n), f_2 \rangle \|
\]
\[
\leq \| f_2 \|_{L^{p_2}} \sup_{\|h\|_{L^{p'_2}} = 1} \| T_{m(1)}(h, f_3, . . . , f_n) \|_{L^{p'_2}}
\]
\[
\lesssim \mathcal{L}^{2}_{s}[(m(1))^{*2}] \| f_1 \|_{X^{p_1}} \prod_{j=2}^{n} \| f_j \|_{L^{p_j}} \lesssim \mathcal{L}^{2}_{s}[m] \| f_1 \|_{X^{p_1}} \prod_{j=2}^{n} \| f_j \|_{L^{p_j}}.
\]

8. Multilinear pseudo-differential operators of type (1, 1)

We use notations \( \tilde{f} := (f_1, . . . , f_n), \tilde{ξ} := (ξ_1, . . . , ξ_n), \tilde{l} := (l_1, . . . , l_n), \tilde{d}ξ := dξ_1 . . . dξ_n, \tilde{d}η := dη_1 . . . dη_m, \tilde{α} := (α_1, . . . , α_n), |\tilde{α}| := |α_1| + . . . + |α_n|, \tilde{∂}^α := \tilde{∂}^{α_1} . . . \tilde{∂}^{α_n}, \) and \( \tilde{d}\tilde{ξ} := \tilde{d}ξ_1 . . . \tilde{d}ξ_n \).

Multilinear pseudo-differential operators were studied by Coifman and Meyer [5, 6, 7] and there have been a large number of variants of their results. In this section, we will study boundedness of n-linear pseudo-differential operators associated with forbidden symbols. The n-linear Hörmander symbol class \( M^{n}_{\alpha, \beta} \) consists of all \( a \in C^\infty((\mathbb{R}^d)^{n+1}) \) having the property that for all multi-indices \( α_1, . . . , α_n, β \) there exists a constant \( C = C_{α, β} \) such that
\[
|\tilde{∂}^{α}_ξ β_a(x, \tilde{ξ})| \leq C\left(1 + \sum_{j=1}^{n} |ξ_j|\right)^{m-|α|+|β|},
\]
and the corresponding n-linear pseudo-differential operator \( T_{[α]} \) is defined by
\[
T_{[α]}\tilde{f}(x) := \int_{(\mathbb{R}^d)^{n}} a(x, \tilde{ξ}) \prod_{j=1}^{n} \tilde{f}_j(ξ_j)e^{2πi(x, Σ_{j=1}^{n} ξ_j)} d\tilde{ξ}
\]
for \( f_1, \ldots, f_n \in S(\mathbb{R}^d) \). Denote by \( \text{OpM}_n S_{1,1}^m \) the class of \( n \)-linear pseudo-differential operators with symbols in \( M_{n} S_{1,1}^m \).

Bilinear pseudo-differential operators (\( n=2 \)) in \( \text{OpM}_2 S_{1,1}^0 \) have bilinear Calderón-Zygmund kernels, but in general they are not bilinear Calderón-Zygmund operators. In particular, they do not always give rise to a mapping \( L^{p_1} \times L^{p_2} \to L^p \) for \( 1 < p, p_1, p_2 \leq \infty \) with \( 1/p = 1/p_1 + 1/p_2 \).

The boundedness properties of operators in \( \text{OpM}_2 S_{1,1}^0 \) have been studied by Bényi and Torres \([2]\), and Bényi, Nahmod, and Torres \([1]\) in the scale of Lebesgue-Sobolev spaces. To be specific, Bényi and Torres \([2]\) proved that if \( a \in M_{2} S_{1,1}^1 \), then

\[
\| T[a](f_1, f_2) \|_{L^p} \lesssim \| f_1 \|_{L^{p_1}} \| f_2 \|_{L^{p_2}} \| f_1 \|_{L^{p_1}} \| f_2 \|_{L^{p_2}}
\]

for \( 1 < p, p_2, p \leq \infty \), \( 1/p_1 + 1/p_2 = 1/p \), and \( s > 0 \). Moreover, this result was generalized to \( a \in M_{2} S_{1,1}^m \), \( m \in \mathbb{R} \), by Bényi-Nahmod-Torres \([1]\). Naibo \([22]\) investigated bilinear pseudo-differential operators on Triebel-Lizorkin spaces and Koezuka and Tomita \([20]\) slightly developed the result of Naibo. These results can be readily extended to a multilinear operators. For \( a \in M_{n} S_{1,1}^m \) and \( N \in \mathbb{N}_0 \) we define

\[
\| a \|_{M_{n} S_{1,1}^m} := \max \left[ \sup \left( 1 + \sum_{j=1}^{n} |\xi_j| \right)^{-m+|\alpha|-|\beta|} \left| \partial_{\xi}^\alpha \partial_{\bar{\xi}}^\beta a(x, \bar{\xi}) \right| \right]
\]

where the supremum is taken over \( (x, \bar{\xi}) \in (\mathbb{R}^d)^{n+1} \) and the maximum is taken over \( |\alpha_1|, \ldots, |\alpha_n|, |\beta| \leq N \). For \( 0 < p, q \leq \infty \) let

\[
\tau_p := d/\min(1, p) - d, \quad \tau_{p, q} := d/\min(1, p, q) - d.
\]

**Theorem E.** \([20, 22]\) Let \( 0 < p < \infty \), \( 0 < q \leq \infty \), \( m \in \mathbb{R} \), and \( a \in M_{n} S_{1,1}^m \). Let \( \{p_{i,j}\}_{1 \leq i, j \leq n} \) satisfy \( 1 < p_{i,j} < \infty \) and

\[
\frac{1}{p} = \frac{1}{p_{i,1}} + \cdots + \frac{1}{p_{i,n}} \quad \text{for} \quad 1 \leq i \leq n.
\]

If

\[
(8.1)\quad s > \begin{cases} \tau_{p, q} & \text{if} \quad q < \infty \\ \tau_{p, \infty} + d & \text{if} \quad q = \infty \end{cases}
\]

then there exists a positive integer \( N \) such that

\[
\| T[a] f \|_{F^{p, q}} \lesssim \| a \|_{M_{n} S_{1,1}^m} \sum_{i=1}^{n} \left( \| f_i \|_{F^{p_{i,1}, q}} \prod_{1 \leq j \leq n, j \neq i} \| f_j \|_{h^{p_{i,j}}_n} \right)
\]

for \( f_1, \ldots, f_n \in S(\mathbb{R}^d) \). Moreover, the inequality also holds for \( p_{i,j} = \infty \), \( i \neq j \).

We recall that \( h^p = L^p \) for \( 1 < p \leq \infty \) and \( F^{p, 2} = h^p \) for \( 0 < p < \infty \).

In this section we extend Theorem E to the full range \( 0 < p, p_{i,j} \leq \infty \) with the weaker condition \( s > \tau_{p, q} \), instead of \((8.2)\), using Theorem 3.4 and Theorem 6.1.

**Theorem 8.1.** Suppose \( 0 < p, q \leq \infty \), \( m \in \mathbb{R} \), and \( a \in M_{n} S_{1,1}^m \). Let \( \{p_{i,j}\}_{1 \leq i, j \leq n} \) satisfy \( 0 < p_{i,j} \leq \infty \) and \((8.7)\). If \( s > \tau_{p, q} \), then there exists a positive integer \( N \) such that

\[
\| T[a] f \|_{F^{p, q}} \lesssim \| a \|_{M_{n} S_{1,1}^m} \sum_{i=1}^{n} \left( \| f_i \|_{F^{p_{i,1}, q}} \prod_{1 \leq j \leq n, j \neq i} \| f_j \|_{h^{p_{i,j}}_n} \right)
\]

for \( f_1, \ldots, f_n \in S(\mathbb{R}^d) \).
As a corollary, from \( h_s^p = F_s^{p,2} \) and \( bmo_s = F_s^{p,2} \), the following estimates hold. Let

\[
Y_s^p := \begin{cases} h_s^p & \text{if } p < \infty \\ bmo_s & \text{if } p = \infty \end{cases}
\]

**Corollary 8.2.** Suppose \( 0 < p \leq \infty \), \( m \in \mathbb{R} \), and \( a \in M_n S_{1,1}^m \). Let \( \{p_{i,j}\}_{1 \leq i,j \leq n} \) satisfy \( 0 < p_{i,j} \leq \infty \) and (8.1). If \( s > \tau_p \), then there exist positive integers \( N > 0 \) such that

\[
\|T_{[a]} \tilde{f}\|_{Y_s^p} \lesssim \|a\| \|M_n S_{1,1}^m\| \sum_{i=1}^n \left( \|f_i\|_{Y_{p_{i,j}}} \prod_{1 \leq j \leq n, j \neq i} \|f_j\|_{h_{p_{i,j}}} \right)
\]

for \( f_1, \ldots, f_n \in S(\mathbb{R}^d) \).

**Generalization of Kato-Ponce inequality.** The classical Kato-Ponce commutator estimate [19] plays a key role in the wellposedness theory of Navier-Stokes and Euler equations in Sobolev spaces. The commutator estimate has been recast later on into the following fractional Leibniz rule, so called Kato-Ponce inequality. Recall that \( J^s := (1 - \Delta)^{s/2} \) be the fractional Laplacian operators. Then

\[
\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^p_1} \|g\|_{L^{p_2}} + \|f\|_{L^{p_2}} \|J^s g\|_{L^{p_1}}
\]

where \( 1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2 \), \( 1 < p < \infty \), and \( 1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty \). Grafakos, Oh [17] and Muscalu, Schlag [21] extended the inequality (8.3) to the wider range \( 1/2 < p < \infty \) under the assumption that \( s > \tau_p \) or \( s \in 2\mathbb{N} \). The case \( p = \infty \) was settled by Bourgain and Li [3].

**Theorem F.** Let \( 1/2 < p \leq \infty \) and \( 1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty \) satisfy \( 1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2 \). Suppose \( s > \tau_p \) or \( s \in 2\mathbb{N} \). Then for \( f, g \in S(\mathbb{R}^d) \) one has

\[
\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_2}} \|J^s g\|_{L^{p_1}}
\]

As a consequence of Corollary 8.2 in the case \( a \equiv 1 \), one obtains the following extension of Kato-Ponce inequality, which includes an endpoint case of bmo type.

**Corollary 8.3.** Let \( 0 < p, p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty \) satisfy \( 1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2 \). Suppose \( s > \tau_p \). Then for \( f, g \in S(\mathbb{R}^d) \) one has

\[
\|J^s(fg)\|_{Y_s^p} \lesssim \|J^s f\|_{Y_{p_1}} \|g\|_{h^{p_2}} + \|f\|_{h^{\tilde{p}_1}} \|J^s g\|_{h^{\tilde{p}_2}}
\]

where

\[
Y_s^p := \begin{cases} h^p & \text{if } p < \infty \\ bmo & \text{if } p = \infty \end{cases}
\]

### 8.1. **Proof of Theorem 8.1**

The proof is based on the decomposition technique by Bényi and Torres [2]. By using Littlewood-Paley partition of unity \( a \in M_n S_{1,1}^m \) can be written as

\[
a(x, \xi) = \sum_{k_1, \ldots, k_n \in \mathbb{N}_0} a(x, \xi) \hat{\phi}_{k_1}(\xi_1) \cdots \hat{\phi}_{k_n}(\xi_n)
\]

\[
= \left( \sum_{k_2, \ldots, k_n \leq k_1} \cdots + \left( \sum_{k_3, \ldots, k_n \leq k_2} \cdots + \left( \sum_{k_1, \ldots, k_n-1 < k_n} \cdots \right) \right) \right)
\]

\[
= a^{(1)}(x, \xi) + a^{(2)}(x, \xi) + \cdots + a^{(n)}(x, \xi)
\]

where we use \( \Phi_0 \), instead of \( \phi_0 \). Then, due to the symmetry, it is enough to work only with \( a^{(1)} \) and our actual goal is to show that if \( s > \tau_{p,q} \) then

\[
\|T_{[a^{(1)}]} \tilde{f}\|_{L^p_{\tau_{p,q}}} \lesssim \|f_1\|_{L^{p_1}} \prod_{j=2}^n \|f_j\|_{h^{p_j}}
\]
where $1/p = 1/p_1 + \cdots + 1/p_n$. 

Observe that

$$a^{(1)}(x, \xi) = \sum_{k=0}^{\infty} a(x, \xi) \hat{\phi}_k(\xi_1) \hat{\Phi}_k(\xi_2) \cdots \hat{\Phi}_k(\xi_n) =: \sum_{k=0}^{\infty} a_k(x, \xi).$$

Of course, we regard $\phi_0$ as $\Phi_0$ when $k = 0$. Then each $a_k$ belongs to $M_n S^m_{1,1}$ and for each $N \in \mathbb{N}_0$

$$\tag{8.5} \|a_k\|_{M_n S^m_{1,1}} \lesssim \|a\|_{M_n S^m_{1,1,N}} \quad \text{uniformly in } k.
$$

Let $\widetilde{\phi}_0 := \Phi_1$ and $\widetilde{\phi}_k := \phi_k - \phi_k + \phi_{k+1}$ for $k \geq 1$. By using Fourier series expansion and the fact that $\widetilde{\phi}_k = 1$ on $\text{Supp}(\phi_k)$ and $\widetilde{\Phi}_{k+1} = 1$ on $\text{Supp}(\Phi_k)$, one can write

$$a_k(x, \xi) = \sum_{\vec{l} \in (\mathbb{Z}^d)^n} \vec{c}_k(x) \varphi^{l_1}_k(\xi_1) \varphi^{l_2}_k(\xi_2) \cdots \varphi^{l_n}_k(\xi_n)$$

where

$$\vec{c}_k(x) := \int_{(\mathbb{R}^d)^n} a_k(x, 2^k \eta_1, \ldots, 2^k \eta_n) e^{-2\pi i (\eta_1, t_1)} \ldots e^{-2\pi i (\eta_n, t_n)} d\vec{\eta}$$

$$\varphi^{l_1}_k(\xi_1) := e^{2\pi i (l_1, 2^{-k}\xi_1)} \hat{\phi}_k(\xi_1), \quad \varphi^{l_j}_k(\xi_j) := e^{2\pi i (l_j, 2^{-k}\xi_j)} \hat{\Phi}_{k+1}(\xi_j), \quad 2 \leq j \leq n.$$ It can be verified that for $l \in \mathbb{Z}^d$ and multi-index $\alpha$ one has 

- $\text{Supp}(\varphi^l_0) \subset \{ |\xi| \leq 2 \}$, 
- $\text{Supp}(\varphi^l_k) \subset \{ 2^{k-3} \leq |\xi| \leq 2^{k+1} \}$ for $k \geq 1$
- $\text{Supp}(\varphi^l_k) \subset \{ |\xi| \leq 2^{k+1} \}$ for $k \geq 0$
- $|\partial_\xi^\alpha \varphi^l_k(\xi)|, |\partial_\xi^\alpha \varphi^l_k(\xi)| \lesssim 2^{-k|\alpha|}$ for $k \geq 0$.

Now for $N > 0$ let 

$$m_{\vec{l},N}(x) := \left( \prod_{j=1}^{n} (1 + |l_j|)^N \right) c_{\vec{l}}(x)$$

and we claim that for $|\beta| \leq N$

$$\tag{8.6} \|\partial_\vec{x}^\beta m_{\vec{l},N}\|_{L^\infty} \lesssim \|a\|_{M_n S^m_{1,1,N}} 2^{k(m+|\beta|)} \quad \text{uniformly in } \vec{l}.$$ 

Indeed,

$$|\partial_\vec{x}^\beta m_{\vec{l},N}(x)| = \left( \prod_{j=1}^{n} (1 + |l_j|)^N \right) \left| \int_{(\mathbb{R}^d)^n} \partial_\vec{x}^\beta a_k(x, 2^k \eta_1, \ldots, 2^k \eta_n) e^{-2\pi i \sum_{j=1}^{n} (\eta_j, l_j)} d\vec{\eta} \right|$$

$$\lesssim \sum_{|\alpha_1|, \ldots, |\alpha_n| \leq N} \left| \int_{(\mathbb{R}^d)^n} \partial_\vec{\eta}^\alpha \partial_\vec{x}^\beta a_k(x, 2^k \eta_1, \ldots, 2^k \eta_n) 2^{k|\alpha|} d\vec{\eta} \right|$$

$$\lesssim \|a\|_{M_n S^m_{1,1,N}} \sum_{|\alpha_1|, \ldots, |\alpha_n| \leq N} 2^{k|\alpha|} 2^{k(m - |\alpha| + |\beta|)} \lesssim \|a\|_{M_n S^m_{1,1,M}} 2^{k(m+|\beta|)}$$

where the first inequality follows from integration by parts and the second inequality is due to (8.5) and the fact that the domain of the integral is actually $\{ |\eta_1| \leq 2 \} \times \cdots \times \{ |\eta_n| \leq 2 \}$.

Moreover, we decompose $m_{\vec{l},N}$ as

$$m_{\vec{l},N} = \sum_{j=0}^{\infty} m_{\vec{l},N}^{k,j}$$

where

$$m_{\vec{l},0}^{k,j} := \phi_k \ast m_{\vec{l},N}^{k,j} \quad \text{(low frequency part)}$$
\begin{align*}
m_{k,j}^{\tilde{L},N} & := \phi_{k+j} \ast m_{k}^{\tilde{L},N}, \quad j \geq 1 \text{ (high frequency part)}
\end{align*}
and then (8.6) yields that for \( j \in \mathbb{N}_0 \)
\begin{equation}
(8.7) \quad |m_{k,j}^{\tilde{L},N}(x)| \lesssim \|a\|_{M_{n}S_{1,1,N}^m} 2^{-jN} 2^{kn} \quad \text{uniformly in } \tilde{I}.
\end{equation}
To be specific,
\begin{align*}
|m_{k,0}^{\tilde{L},N}(x)| & \lesssim \|m_{k}^{\tilde{L},N}\|_{L_{\infty}} \lesssim \|a\|_{M_{n}S_{1,1,N}^m} 2^{kn}
\end{align*}
and for \( j \geq 1 \), using vanishing moment condition of \( \phi_{k+j} \) and Taylor expansion,
\begin{align*}
|m_{k,j}^{\tilde{L},N}(x)| & = \left| \int_{\mathbb{R}^d} \phi_{k+j}(x-y)m_{k}^{\tilde{L},N}(y)dy \right|
\leq \sum_{|\beta|=N} \|\partial_{x}^{\beta}m_{k}^{\tilde{L},N}\|_{L_{\infty}} \int_{\mathbb{R}^d} |\phi_{k+j}(x-y)||x-y|^{N}dy
\lesssim \|a\|_{M_{n}S_{1,1,N}^m} 2^{kn} 2^{-jN}.
\end{align*}
We also observe that due to the Fourier support of \( \Phi_{k} \) and \( \phi_{k+j} \)
\begin{align*}
\text{Supp}(m_{k,j}^{\tilde{L},N}) & \subset \{ |\xi| \leq 2^{k+j} \}.
\end{align*}
Then
\begin{align*}
d^{(1)}(x,\xi) & = \sum_{k=0}^{\infty} \sum_{\tilde{I} \in (\mathbb{Z}^d)^n} \left( \prod_{j=1}^{n} \frac{1}{(1 + |\xi_j|)^{N}} \right) m_{k}^{\tilde{L},N}(x) \varphi_{k}^{l_{1}}(\xi_{1}) \varphi_{k}^{l_{2}}(\xi_{2}) \cdots \varphi_{k}^{l_{n}}(\xi_{n})
= \sum_{\tilde{I} \in (\mathbb{Z}^d)^n} \left( \prod_{j=1}^{n} \frac{1}{(1 + |\xi_j|)^{N}} \right) \sum_{k,j \in \mathbb{N}_0} m_{k,j}^{\tilde{L},N}(x) \varphi_{k}^{l_{1}}(\xi_{1}) \varphi_{k}^{l_{2}}(\xi_{2}) \cdots \varphi_{k}^{l_{n}}(\xi_{n}).
\end{align*}
Setting
\begin{align*}
A_{k,j}^{\tilde{L},N}(x,\xi) & := m_{k,j}^{\tilde{L},N}(x) \varphi_{k}^{l_{1}}(\xi_{1}) \varphi_{k}^{l_{2}}(\xi_{2}) \cdots \varphi_{k}^{l_{n}}(\xi_{n}),
\end{align*}
we write
\begin{align*}
T_{[a^{(1)}]} \tilde{f} & = \sum_{\tilde{I} \in (\mathbb{Z}^d)^n} \left( \prod_{j=1}^{n} \frac{1}{(1 + |\xi_j|)^{N}} \right) \sum_{k,j \in \mathbb{N}_0} A_{k,j}^{\tilde{L},N} \tilde{f}.
\end{align*}
Now one has
\begin{align*}
\|T_{[a^{(1)}]} \tilde{f}\|_{F_{p,q}^{s,q}}^{\min(1,p,q)} & \leq \sum_{\tilde{I} \in (\mathbb{Z}^d)^n} \left( \prod_{j=1}^{n} \frac{1}{(1 + |\xi_j|)^{N \min(1,p,q)}} \right) \sum_{k,j \in \mathbb{N}_0} A_{k,j}^{\tilde{L},N} \tilde{f}\|_{F_{p,q}^{s,q}}^{\min(1,p,q)}.
\end{align*}
Choose \( N > 0 \) sufficiently large so that \( N > s \) and \( N > d/ \min(1,p,q) + d/ \min(p_1, \ldots, p_n, q) \).
Then the proof of (8.4) can be deduced from the following estimate that
\begin{equation}
(8.8) \quad \left\| \sum_{k,j \in \mathbb{N}_0} T_{[A_{k,j}^{\tilde{L},N}]} \tilde{f}\right\|_{F_{p,q}^{s,q}} \lesssim \|a\|_{M_{n}S_{1,1,N}^m} \left( \prod_{j=1}^{n} (1 + |\xi_j|)^{\sigma} \right) \|f_1\|_{F_{p_1}^{s,q}} \left( \prod_{j=2}^{n} \|f_j\|_{H_{p_j}} \right)
\end{equation}
for \( d/ \min(p_1, \ldots, p_n, q) < \sigma < N - d/ \min(1,p,q) \).
From now on we shall prove (8.8). Let \( \phi_{k}^{0} := \Phi_{2} \) and \( \phi_{k}^{*} \) be Schwartz functions such that
\begin{align*}
\text{Supp}(\widehat{\phi_{k}^{0}}) & \subset \{ \xi : 2^{-k-2} \leq |\xi| \leq 2^{k+2} \},
\text{Supp}(\widehat{\phi_{k}^{*}}) & = 1 \text{ on } \text{Supp}(\widehat{\phi_{k}}), \quad \text{for } k \geq 1.
\end{align*}
Choose $\sigma$ satisfying $\sigma > d / \min (p_1, \ldots, p_n)$ and let
\[
(f_1)_k := \phi_k^* f_1 \quad \text{and} \quad (f_j)^k := \Phi_{k+2} * f_j, \quad 2 \leq j \leq n.
\]
Observe that
\[
|\varphi^l_k(D) f_1(x)| = |\varphi^l_k(D)(f_1)_k(x)| \leq \int_{\mathbb{R}^d} |(\varphi^l_k)^1(y)(f_1)_k(x - y)| dy
\]
\[
\leq \mathcal{M}_{\sigma,2^k}(f_1)_k(x) \int_{\mathbb{R}^d} (1 + 2^k |y|)^\sigma |\varphi^l_k(y + 2^{-k} l_1)| dy
\]
\[
|\varphi^l_k(D) f_j(x)| \lesssim (1 + |l_1|)^\sigma \mathcal{M}_{\sigma,2^k}(f_j)^k(x).
\]
Moreover, from (2.1),
\[
\sum_{j \in \mathbb{N}} |(f_j)^k| \lesssim \| f_j \|_{L^p}, \quad 0 < p \leq \infty.
\]
Now (8.7), (8.9), and (8.10) establish the pointwise estimate that
\[
|T_{[A^L_{k,j}]} \tilde{f}(x)| = |m_{k,j}^L (x) \varphi^l_k(D) f_1(x) \prod_{j=2}^n \varphi^l_k(D) f_j(x)|
\]
\[
\lesssim \| a \|_{L^p} S_{n_1, n_2}^{\alpha n} 2^{km} 2^{-jn} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \mathcal{M}_{\sigma,2^k}(f_1)_k(x) \left( \prod_{j=2}^n \mathcal{M}_{\sigma,2^k}(f_j)^k(x) \right) \right).
\]
We observe that
\[
\text{Supp}(T_{[A^L_{k,j}]} \tilde{f}) \subset \{ |\xi| \leq 2^{k+j} + n 2^k \}
\]
and this yields, with the support condition of $\tilde{\varphi}_h$, that for $h \in \mathbb{N}_0$
\[
\phi_h * \left( \sum_{k,j \in \mathbb{N}_0} T_{[A^L_{k,j}]} \tilde{f} \right) = \sum_{k,j \in \mathbb{N}_0} \phi_h * T_{[A^L_{k,j}]} \tilde{f} = \sum_{j,k \in \mathbb{N}_0} \phi_h * T_{[A^L_{k,j}]} \tilde{f}.
\]
By assuming $A^L_{k,j} = 0$ for $k < 0$ and applying a change of variables the last expression is
\[
\sum_{j=0}^{\infty} \phi_h * T_{[A^L_{k,j}]} \tilde{f}
\]
\[
= \sum_{j,u \in \mathbb{N}_0} \phi_h * T_{[A^L_{u+h-j-3-[\log_2 n]}]} \tilde{f} = \phi_h \left( \sum_{j,u \in \mathbb{N}_0} T_{[A^L_{u+h-j-3-[\log_2 n],j}]} \tilde{f} \right).
\]
That is, for $h \in \mathbb{N}_0$
\[
\phi_h * \left( \sum_{k,j \in \mathbb{N}_0} T_{[A^L_{k,j}]} \tilde{f} \right) = \phi_h \left( \sum_{j,u \in \mathbb{N}_0} T_{[A^L_{u+h-j-3-[\log_2 n],j}]} \tilde{f} \right).
\]
8.1.1. The case $0 < p < \infty$ or $p = \infty$. From (8.14) one has

$$\left\| \sum_{k,j \in \mathbb{N}_0} T_{A_{k,j}} \vec{f} \right\|_{F^p}^{\min(1,p,q)} \leq \sum_{j \in \mathbb{N}_0} \left\| \left\{ 2^{sh} \hat{\phi}_h \ast T_{A_{u+h-j-3-[\log_2 N],j}} \vec{f} \right\}_{h \in \mathbb{N}_0} \right\|_{L^p(I^q)}^{\min(1,p,q)}.$$ 

It follows from the observation (8.13) that the Fourier transform of $T_{A_{u+h-j-3-[\log_2 N],j}} \vec{f}$ is supported on $\{ |\xi| \leq 2^{u+h} \}$. We choose $t > 0$ such that $s > t - d/2 > r_{p,q}$ and apply Theorem 6.1 (1) to obtain

$$\left\| \left\{ 2^{sh} \hat{\phi}_h \ast T_{A_{u+h-j-3-[\log_2 N],j}} \vec{f} \right\}_{h \in \mathbb{N}_0} \right\|_{L^p(I^q)} \leq \sup_{l \in \mathbb{N}_0} \left\| \hat{\phi}(2^{u+l}) \right\|_{L^2} \left\| \left\{ 2^{sh} T_{A_{u+h-j-3-[\log_2 N],j}} \vec{f} \right\}_{h \in \mathbb{N}_0} \right\|_{L^p(I^q)} \approx 2^{u(t-d/2)} \left\| \left\{ 2^{sk} T_{A_{k,j}} \vec{f} \right\}_{k \in \mathbb{N}_0} \right\|_{L^p(I^q)} \leq 2^{-u(s-t+d/2)} 2^{sj} \left\| \left\{ 2^{sk} T_{A_{k,j}} \vec{f} \right\}_{k \in \mathbb{N}_0} \right\|_{L^p(I^q)} (8.15)$$

where we applied a change of variables in the last inequality. Then the estimate (8.12) proves

$$\left(8.15\right) \lesssim \|a\|_{M_n S^m_{1,1,N}} 2^{-u(s-t+d/2)} 2^{-j(N-s)} \left( \prod_{j=1}^n \left( 1 + |j| \right) \right)^{\sigma}$$

and one has

$$\| \sum_{k,j \in \mathbb{N}_0} T_{A_{k,j}} \vec{f} \|_{F^p}^{q} \lesssim \|a\|_{M_n S^m_{1,1,N}} \left( \prod_{j=1}^n \left( 1 + |j| \right) \right)^{\sigma} \left\| \left\{ 2^{s+m} k \mu_{\sigma,2k}(f_1)_k \prod_{j=2}^n \mu_{\sigma,2k}(f_j)_k \right\}_{k \in \mathbb{N}_0} \right\|_{L^p(I^q)}.$$ 

because $N > s > t - d/2$. Moreover, using (2.6),

$$\left\| \left\{ 2^{s+m} k \mu_{\sigma,2k}(f_1)_k \prod_{j=2}^n \mu_{\sigma,2k}(f_j)_k \right\}_{k \in \mathbb{N}_0} \right\|_{L^p(I^q)} = \left\| \left\{ \sum_{Q \in D_k} 2^{s+m} k \mu_{\sigma,2k}(f_1)_k \left( \prod_{j=2}^n \mu_{\sigma,2k}(f_j)_k \right) \chi_Q \right\}_{k \in \mathbb{N}_0} \right\|_{L^p(I^q)}$$

$$\lesssim \left\| \left\{ \sum_{Q \in D_k} \inf_{y \in Q} \mu_{\sigma,2k}(f_1)_k \left( \prod_{j=2}^n \inf_{y \in Q} \mu_{\sigma,2k}(f_j)_k \right) \chi_Q \right\}_{k \in \mathbb{N}_0} \right\|_{L^p(I^q)}.$$
Now let $S_Q := S^r_{Q}(\{\mathbf{M}_{\sigma, 2^k}(f_1)k\}_{k \in \mathbb{N}_0})$ and apply (3.1) and (2.2) for $0 < r < \min(1, p)$ to show that the last expression is

$$\lesssim \|\{ \sum_{Q \in \mathcal{D}_k} 2^{(s+m)k} \inf_{y \in Q} \mathbf{M}_{\sigma, 2^k}(f_1)(y) \left( \prod_{j=2}^n \inf_{y \in Q} \mathbf{M}_{\sigma, 2^k}(f_j)(y) \right) \chi_S Q \}_{k \in \mathbb{N}_0}\|_{L^p(Q)}$$

$$\leq \|\{ \sum_{Q \in \mathcal{D}_k} 2^{(s+m)k} \inf_{y \in Q} \mathbf{M}_{\sigma, 2^k}(f_1)(y) \chi_S Q \}_{k \in \mathbb{N}_0}\|_{L^p(Q)} \prod_{j=2}^n \|\{ \mathbf{M}_{\sigma, 2^k}(f_j) \}_{k \in \mathbb{N}_0}\|_{L^{p_j}(Q)}$$

$$\lesssim \|f_1\|_{F^{a+m,q}_{p_1}} \prod_{j=2}^n \|f_j\|_{h^{p_j}}$$

where Hölder’s inequality, Lemma 3.1 (1) (for $p_1 < \infty$), Theorem 3.4 (with $\mu = 0$ for $p_1 = \infty$), and (3.11) are applied.

Combining all together the proof of (8.8) ends for $0 < p < \infty$ or $p = q = \infty$.

### 8.1.2. The case $p = \infty$ and $0 < q < \infty$.

Suppose $\sigma > d/q$. First of all, by using (8.8) for $p = q = \infty$ and the embedding $F^a_{\infty,m,q} \hookrightarrow F^a_{\infty,m,\infty}$ one has

$$\|\Phi_0 \ast \left( \sum_{k,j \in \mathbb{N}_0} T_{[A_{k,j}^N]} f \right)\|_{L^\infty} \leq \sum_{k,j \in \mathbb{N}_0} T_{[A_{k,j}^N]} f \right)\|_{F^a_{\infty,m,\infty}}$$

$$\lesssim \|a\|_{M^{\infty}_m} \|f_1\|_{F_{\infty,m,q}^{a+m,q}} \left( \prod_{j=2}^n \|f_j\|_{h^{p_j}} \right) \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right).$$

Now we fix a dyadic cube $P \in \mathcal{D}$ with $l(P) < 1$. Then it follows from (8.14) that

$$\left( \frac{1}{|P|} \int_{h = - \log_2 l(P)}^\infty 2^{shq} \left| \phi_h \ast \left( \sum_{k,j \in \mathbb{N}_0} T_{[A_{k,j}^N]} f \right) \right|^q dx \right)^{\min(1,q)/q} \leq \sum_{j,u \in \mathbb{N}_0} \left( \frac{1}{|P|} \int_{h = - \log_2 l(P)}^\infty 2^{shq} \left| \phi_h \ast T_{[A_{u+h,j-3-[\log_2 n],j}]} f \right|^q dx \right)^{\min(1,q)/q}.$$

We choose $t > 0$ such that $s > t - d/2 > \tau_q$ and apply Theorem 6.1 (2) with $\mu = 1$. Then

$$\left( \frac{1}{|P|} \int_{h = - \log_2 l(P)}^\infty 2^{shq} \left| \phi_h \ast T_{[A_{u+h,j-3-[\log_2 n],j}]} f \right|^q dx \right)^{1/q} \lesssim \sup_{l \in \mathbb{N}_0} \left| \phi_l \ast (2^{u+l}) \right|_{L^2} \sup_{R \in \mathcal{D},l(R) < 1} \left( \frac{1}{|R|} \int_{h = - \log_2 l(R)}^\infty 2^{skq} \left| T_{[A_{u+h,j-3-[\log_2 n],j}]} f \right|^q dx \right)^{1/q}$$

$$\lesssim 2^{-u(s-t+d/2)/2} \sup_{R \in \mathcal{D},l(R) < 1} \left( \frac{1}{|R|} \int_{k = u-j-3-[\log_2 n] - \log_2 l(R)}^\infty \sum_{k = u-j-3-[\log_2 n]}^\infty 2^{skq} \left| T_{[A_{k,j}^N]} f \right|^q dx \right)^{1/q}.$$

We only concern ourselves with the case $u - j - 3 - [\log_2 n] \leq -1$ since the other case follows in a similar and simpler way. The supremum in the last expression is less than a constant times the sum of

$$\sup_{R \in \mathcal{D},l(R) < 1} \left( \frac{1}{|R|} \int_{k = u-j-3-[\log_2 n] - \log_2 l(R)}^\infty \sum_{k = u-j-3-[\log_2 n] - \log_2 l(R)}^\infty 2^{skq} \left| T_{[A_{k,j}^N]} f \right|^q dx \right)^{1/q}$$

and

$$\sup_{R \in \mathcal{D},l(R) < 1} \left( \frac{1}{|R|} \int_{k = u-j-3-[\log_2 n] - \log_2 l(R)}^\infty \sum_{k = u-j-3-[\log_2 n] - \log_2 l(R)}^\infty 2^{skq} \left| T_{[A_{k,j}^N]} f \right|^q dx \right)^{1/q}.$$
By using (8.12) with the estimate
\[ \mathcal{M}_{\sigma,2^k}(f_j)^k(x) \leq \| (f_j)^k \|_{L^\infty} \lesssim \| f_j \|_{L^\infty}, \]
(8.18) is less than a constant multiple of
\[ \left\| \left\{ g^{sk} T_{[x^h]} f_j \right\}_{k \in \mathbb{N}_0} \right\|_{L^\infty(L^\infty)} \]
\[ \lesssim \| a \|_{M_n S^m_{1,1,N}} 2^{-jN} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right) \right\|_{M_n S^m_{1,1,N}} 2^{-jN} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right) \right\|_{M_n S^m_{1,1,N}} \]
\[ \lesssim \| a \|_{M_n S^m_{1,1,N}} 2^{-jN} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right) \right\|_{M_n S^m_{1,1,N}} \]
by using Lemma 2.5 (1) and the embedding \( F^{s+m,q}_\infty \hookrightarrow F^{s+m,q}_\infty \). This, with \( N > s \), proves that the term corresponding to (8.18) in (8.16) is dominated by a constant times
\[ \| a \|_{M_n S^m_{1,1,N}} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right) \right\|_{M_n S^m_{1,1,N}} \]
\[ \lesssim \| a \|_{M_n S^m_{1,1,N}} 2^{-jN} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right) \right\|_{M_n S^m_{1,1,N}} \]
which proves (8.8) for the part corresponding to (8.18).

Similarly, (8.12) yields that for \( N > s \)
\[ (8.17) \]
\[ \lesssim \| a \|_{M_n S^m_{1,1,N}} 2^{-jN} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right) \right\|_{M_n S^m_{1,1,N}} \]
\[ \times \sup_{P \in \mathcal{D}, l(P) < 1} \left( \frac{1}{|P|} \right) \int_{P} \left( \sum_{k=-\log_2 l(P)}^\infty 2^{(s+m)kq} \mathcal{M}_{\sigma,2^k}(f_1(k) x)^q dx \right)^{1/q} \]
\[ \lesssim \| a \|_{M_n S^m_{1,1,N}} 2^{-jN} \left( \prod_{j=1}^n (1 + |l_j|)^\sigma \right) \right\|_{M_n S^m_{1,1,N}} \]
where Lemma 2.5 (2) is applied in the last inequality. This implies that the term corresponding to (8.17) in (8.16) is bounded by a constant times
\[ \| a \|_{M_n S^m_{1,1,N}} \left[ \prod_{j=1}^n \| f_j \|_{L^\infty} \right] \right\|_{M_n S^m_{1,1,N}} \]
\[ \lesssim \| a \|_{M_n S^m_{1,1,N}} \left[ \prod_{j=1}^n \| f_j \|_{L^\infty} \right] \right\|_{M_n S^m_{1,1,N}} \]
which completes the proof of (8.8).

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School of Mathematics, Korea Institute for Advanced Study, Seoul, Republic of Korea
E-mail address: qkrqowns@kias.re.kr