SIMPLE PHYSICAL APPROACH TO THERMAL CUTTING RULES

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Abstract

A first-principles derivation is given of the imaginary part of a Green’s function in real-time thermal field theory. The analysis and its conclusions are simpler than in the usual circled-vertex formalism. The relationship to Cutkosky-like cutting rules is explained.

In a recent paper, Bedaque, Das and Naik\cite{1} have discussed some cancellations among diagrams that contribute to the imaginary parts of finite-temperature amplitudes. A purpose of this note is to point out that any such cancellations occur because the usual treatment of the imaginary parts, which starts in $x$ space and introduces circled vertices\cite{2}, is unnecessarily complicated. A momentum-space treatment, which starts from first principles\cite{3}, and stays close to the physics leads to the answer much more simply and directly.

Suppose, for definiteness, that we want to calculate\cite{4} dilepton production from a plasma at temperature $1/\beta$. The plasma is defined to be composed of quarks and gluons only, with any leptons or photons that may be produced immediately escaping from it. The initial density matrix is

$$\rho_0 = Z^{-1} \mathcal{P} \exp(-\beta H) \quad (1a)$$

where $H$ is the hadronic part of the Heisenberg-picture Hamiltonian and $\mathcal{P}$ is a projection operator which removes any states that do not contain just quarks and gluons. We may choose to express $\mathcal{P}$ in terms of a complete orthonormal set of plasma states $|i\text{ in}\rangle$, so that

$$\rho_0 = Z^{-1} \sum_i |i\text{ in}\rangle\langle i\text{ in}| e^{-\beta H} \quad (1b)$$

The partition function $Z$ is defined in the usual way, so as to make $\text{tr} \rho_0 = 1$. The probability of emission of a lepton pair of total momentum $q$ is then calculated from

$$W_{\mu\nu}(q) = \int d^4x e^{iq.x} \sum_f \langle f\text{ out}| J_\mu(x) \rho_0 J_\nu(0) |f\text{ out}\rangle \quad (2)$$

where $J(x)$ is the hadronic part of the Heisenberg-picture electromagnetic current and $|f\text{ out}\rangle$ are a complete set of final plasma states.

The usual zero-temperature perturbation theory applies to the matrix elements that arise in the sum (2). Introduce first an interaction picture that coincides with the Heisenberg picture at time $t_0$:

$$J(x) = U^{t_0}(t_0, x^0) J^{t_0}(x) U^{t_0}(x^0, t_0) \quad (3)$$

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where

\[ U^{t_0}(t_2, t_1) = \Lambda(t_2)\Lambda^{-1}(t_1) = T \exp \left( -i \int_{t_1}^{t_2} dt \int d^3x \, H_{\text{INT}}^{t_0}(t, x) \right) \]

\[ \Lambda(t) = e^{i(t-t_0)H_0} e^{i(t-t_0)H} \]  

(4)

Here \( T \) denotes ordinary time-ordering and \( H_{\text{INT}}^{t_0} \) is the interaction Hamiltonian density written as a functional of interaction-picture fields and canonical momenta. We have

\[ e^{-\beta H} = e^{-\beta H_0^{t_0}} U^{t_0}(t_0 - i\beta, t_0) \]  

(5)

We take the limit \( t_0 \to -\infty \), so that the interaction-picture fields become the in fields. We use also

\[ |f \text{ out} \rangle = U^{t_0}(-\infty, \infty) |f \text{ in} \rangle \]  

(6)

Then

\[ W_{\mu\nu}(q) = Z^{-1} \int d^4x \, e^{iq.x} \sum_i \langle f \text{ in}|U^{t_0}(\infty, x^0) J^{\text{in}}_{\mu}(x) U^{t_0}(x^0, -\infty)|i \text{ in} \rangle \]

\[ \langle i \text{ in}|e^{-\beta H_0^{t_0}} U^{t_0}(-\infty - i\beta, 0) J^{\text{in}}_{\nu}(0) U^{t_0}(0, \infty)|f \text{ in} \rangle \]  

(7)

By using completeness for the states \( f \) we obtain:

\[ Z^{-1} \int d^4x \, e^{iq.x} \sum_i \]

\[ \langle i \text{ in}|e^{-\beta H_0^{t_0}} U^{t_0}(-\infty - i\beta, 0) J^{\text{in}}_{\nu}(0) U^{t_0}(0, \infty)|f \text{ in} \rangle \]  

(8a)

We recognise this\[5\] as the standard perturbation-theory expression for

\[ G^{12}_{\mu\nu}(q) = \int d^4x \, e^{iq.x} \text{tr}(\rho_0 J_{\nu}(0) J_{\mu}(x)) \]  

(9a)

with the Keldysh contour\[6\], which in the complex \( t \) plane runs along the real axis from \(-\infty\) to \(+\infty\), back along the real axis to \(-\infty\), and then down to \(-\infty - i\beta\). We may obtain an alternative expression by using instead completeness of the states \( i \) in (6b), and the fact that \( H_0^{t_0} \) is the time-translation operator for the in fields. We find, instead of (9a),

\[ e^{-\beta q^0} G^{21}_{\mu\nu}(q) = e^{-\beta q^0} \int d^4x \, e^{iq.x} \text{tr}(\rho_0 J_{\mu}(x) J_{\nu}(0)) \]  

(9b)

For the conserved hermitean electromagnetic current, \( G^{21}_{\mu\nu}(q) \) is symmetric in \( \mu, \nu \) and is real. Of course, the equality of (9a) and (9b) is nothing but the standard relation\[5\]

\[ G^{12}(t, x) = G^{21}(t - i\beta, x) \]  

(9c)

and one of its consequences is that

\[ G^{11}_{\mu\nu}(q) = \int d^4x \, e^{iq.x} \text{tr}(\rho_0 T J_{\mu}(x) J_{\nu}(0)) \]  

(10a)

satisfies

\[ \text{Im } G^{11}_{\mu\nu}(q) = \frac{1}{2}(e^{\beta q^0} + 1)G^{12}_{\mu\nu}(q) \]  

(10b)
We shall find that there are simple generalised Feynman rules to calculate $G^{12}_{\mu\nu}(q)$, and that they may be recast into a formalism resembling the zero-temperature Cutkosky formula\cite{7}, though with important differences\cite{8}.

To derive these, we start with the expression (8a) for $G^{12}_{\mu\nu}(q)$. It is well-known\cite{9} that one may replace the argument $(-\infty - i\beta)$ in the first $U$ with $-\infty$, so long as one does the same in the calculation of $Z$. That is, in the original definition (1a) of $\rho_0$ we may replace $H$ with $H_0$. Of course, the full Hamiltonian is still used to describe how the system evolves away from its initial density matrix $\rho_0$.

The physical reason that allows this modification of the initial density matrix is as follows. The photon emission rate which we wish to calculate is constant in time and depends only on the density matrix at the given time. If we change the density matrix in the remote past, by multiplying the interaction Hamiltonian is still used to describe how the system evolves away from its initial density matrix $\rho_0$.

There is a similar matrix propagator for the spin-$\frac{1}{2}$ case. For gauge fields, we use the formalism\cite{10} in which only the two physical degrees of freedom of the gauge field have a thermal term; the other two, and the Faddeev-Popov ghost field, have only the first matrix in (12a). However, for simplicity pretend first that the fields of the quarks and gluons that make up the plasma are scalar.

To derive the diagrammatic expansion of (11), expand the $U$’s as in (4) and apply the zero-temperature Wick theorem to each matrix element in the double sum. Each product of operators $UJU$ in the second matrix element is written as a sum of products of zero-temperature Feynman propagators $(0|T\phi\phi|0)$ and normal-ordered creation and annihilation operators. For the first matrix element we instead have $(0|T\phi\phi|0)$ and therefore complex conjugates of propagators. In order to evaluate the matrix elements of the normal products, we must introduce the finite volume $V$ of the plasma, and a discrete spectrum for the fields:

$$\phi(x) = \sum_r \frac{1}{\sqrt{2\omega_r V}} a_r \exp(-i\omega_r x^0 + ik_r \cdot x) + h.c. \tag{13a}$$

with

$$[a_r, a^+_s] = \delta_{rs} \tag{13b}$$

The states are then labelled by the occupation numbers $n_r$ of the various single-particle modes $r$. In the $V \to \infty$ continuum limit

$$\sum_r \to \frac{V}{(2\pi)^3} \int d^3k$$
\[ \phi(x) \rightarrow \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} a(k) e^{-ik.x} + \text{h.c} \]

\[ [a(k), a^\dagger(k')] = (2\pi)^3 2k^0 \delta^{(3)}(k - k') \quad (14) \]

but we do not take this limit until we have used the standard relations

\[ a_r|n_1, n_2 \ldots \rangle = \sqrt{n_r}|n_1, n_2 \ldots n_r - 1 \ldots \rangle, \quad a^\dagger_r|n_1, n_2 \ldots \rangle = \sqrt{n_r + 1}|n_1, n_2 \ldots n_r + 1 \ldots \rangle \quad (15) \]

In order to give a non-zero contribution, the indices \( r \) on the operators \( a \) and \( a^\dagger \) in (11) must be equal in pairs. The possible pairings are

(i) the index on one of the operators \( a \) in the normal product in the second matrix element matches that on one of the operators \( a^\dagger \) in the anti-normal product in the first matrix element

(ii) the index on one of the operators \( a^\dagger \) in the second matrix element matches that on one of the operators \( a \) in the first matrix element

(iii) the index on one of the operators \( a \) in the first matrix element matches that on one of the operators \( a^\dagger \) in the same matrix element

(iv) the index on one of the operators \( a \) in the second matrix element matches that on one of the operators \( a^\dagger \) in the same matrix element

We need not consider the case where three or more indices are equal to each other; this would reduce the number of indices left to be summed and so\([11]\), according to (14), give fewer powers of \( V \) when we go to the continuum limit. (For a system in thermal equilibrium these nonleading powers actually cancel\([12]\), though this is not true more generally.) For each pair of equal indices \( r \), we use (15) to replace the corresponding two operators with \( n_r \) in the cases (i),(iii),(iv), and \( n_r + 1 \) in case (ii). We then sum over \( n_r \) using

\[ \frac{\left( \sum_{n=0}^{\infty} ne^{-\beta n\omega} \right)}{\left( \sum_{n=0}^{\infty} e^{-\beta n\omega} \right)} = n(\omega) \quad (16) \]

where \( n(\omega) \) is again the Bose distribution.

Figure 1: (a) a matrix element that contributes to (11), with (b) its square summed over the momenta. The lines at the top of the diagrams are the spectator particles in the heat bath.
We then take the continuum limit and represent the results diagrammatically. Consider first the contribution from cases (i) and (ii) only. In (11) we have a matrix element, shown in figure 1a, which has to be squared and integrations applied to the momenta on the incoming and outgoing lines. The lines at the top of the diagram are the spectator particles of the heat bath. In the squared matrix element the incoming lines, those on the left in figure 1a, correspond to case (i); with each of them is associated $2\pi\delta^+(k^2 - m^2)n(k^0)$. The outgoing lines on the right correspond to case (ii); with them is associated $2\pi\delta^+(k^2 - m^2)(n(k^0) + 1)$. The result of squaring figure 1a and performing the momentum integrations may be depicted as in figure 1b, where 2 labels the first matrix element in (11) and 1 the second. To agree with what has been said, the lines joining them correspond to the $\{21\}$ element of the thermal propagator matrix $iD$ of (12a); the direction of the arrow denotes whether the energy flow from 2 to 1 is positive or negative. The matrix element 1 is an ordinary zero-temperature matrix element; its internal lines are zero-temperature Feynman propagators $iD_F$. The matrix element 2 is its complex conjugate, so its internal lines are rather $(iD_F)^*$. 

In figure 2a we expose one such internal line $k$ in the matrix element 1. Figures 2b and 2c show diagrams corresponding to case (iii). The internal line $k$ is replaced with two external lines of the same momentum. In the matrix element 1, this corresponds to a heat-bath particle $k$ undergoing forward scattering and returning to the heat bath with the same momentum. This matrix element

Figure 2: (a) The diagram of figure 1b with one of the internal lines $k$ shown explicitly; in (b) and (c) this line instead is a heat-bath particle.
Figure 3: (a) an “uncuttable graph”, with an example (b) of a pair of interfering physical processes to which it corresponds

is multiplied by a corresponding matrix element 2 in which the heat-bath particle \( k \) is a spectator. Figure 2b may be obtained from figure 2a by replacing \( iD_F(k) \) with \( 2\pi \delta^+(k^2 - m^2)n(k^0) \), and for figure 2c we instead need \( 2\pi \delta^-(k^2 - m^2)n(k^0) \). If we add the three figures together, \( iD_F(k) \) becomes just the \( \{11\} \) element of the thermal propagator matrix \( iD \). Exactly similar arguments apply to case \( (iv) \) and the matrix element 2, where instead we arrive at the \( \{22\} \) element of \( iD \). So we may omit graphs where there is interference with spectator particles in the heat bath, and instead use thermal propagators on internal lines of the matrix elements 1 and 2. Then we need no longer explicitly draw the heat-bath spectators in our diagrams.

Notice that the matrix elements 1 and 2 need not both be connected. This can lead to the “uncuttable” diagrams of Kobes and Semenoff\cite{8}. An example of such an uncuttable graph is figure 3a. One of the physical processes to which it corresponds is shown in figure 3b, where a connected graph on the left interferes with a disconnected one on the right.

The prescription, then, is that the incoming-current vertex is of type 1 and the outgoing one of type 2. For the internal vertices we sum over either possibility. Of course, some of the resulting graphs vanish for kinematic reasons. An example is figure 4: one of its internal vertices is a type-1 vertex connected just to three type-2 vertices. Because the \( \{21\} \) element of \( iD \) puts the corresponding line on-shell, and – at least in the case of nonzero mass – it is kinematically impossible to have a three-line vertex where all three lines are on shell, the graph vanishes.

The graphs of figure 5 contain “self-energy” insertions. Figures 5a and 5b are associated with forward scattering on a heat-bath particle, as has been explained above, while figure 5c represents either the absorption of a particle from the heat bath or the emission of an extra one into it. The graph of figure 5d vanishes for the kinematic reasons explained above. These self-energy diagrams are delicate, in that two singular propagators having the same argument are multiplied together, but by switching off the interaction in the remote past and future one may show\cite{13} that one should handle them in a
way similar to zero-temperature self-energy insertions. The thermal propagator \( iD(k) \) of (12a) may be written as[14]

\[
iD(k) = M(k)(i\hat{D}(k))M(k)
\]

\[
M(k) = \sqrt{n(k_0)} \begin{pmatrix} e^{\beta k_0^2/2} & e^{-\beta k_0^2/2} \\ e^{-\beta k_0^2/2} & e^{\beta k_0^2/2} \end{pmatrix}
\]

\[
i\hat{D}(k) = \begin{pmatrix} iD_F(k) & 0 \\ 0 & -iD_F^*(k) \end{pmatrix}
\]

(17a)

where \( D_F \) is, as usual, the zero-temperature Feynman propagator. The matrix self energy \( \Pi(k) \) has the form

\[
\Pi(k) = M^{-1}(k)\hat{\Pi}(k)M^{-1}(k)
\]

\[
\hat{\Pi}(k) = \begin{pmatrix} \Pi(k) & 0 \\ 0 & -\Pi^*(k) \end{pmatrix}
\]

(17b)

In consequence, the Dyson sum of repeated self-energy insertions gives the dressed thermal propagator

\[
iD'(k) = M(k)(i\hat{D}'(k))M(k)
\]

\[
i\hat{D}'(k) = \begin{pmatrix} i(k^2 - m^2 - \Pi(k) + i\epsilon)^{-1} & 0 \\ 0 & -i(k^2 - m^2 - \Pi^*(k) - i\epsilon)^{-1} \end{pmatrix}
\]

(17c)

To calculate the sum of graphs in figure 5 we need for the upper line with its insertions

\[
i[D'(k)]^{21} = i \left[ \frac{1}{k^2 - m^2 - \Pi(k) + i\epsilon} - \frac{1}{k^2 - m^2 - \Pi^*(k) - i\epsilon} \right] n(k_0) \left( e^{\beta k_0^0} \theta(k_0^0) + \theta(-k_0^0) \right)
\]

(18a)

For values of \( k \) for which \( \Pi(k) \) is real, this is

\[
i[D'(k)]^{21} = 2\pi \delta(k^2 - m^2 - \Pi(k)) \left( n(k_0^0) + \theta(k_0^0) \right)
\]

\[
= 2\pi Z_\beta \delta(k^2 - m_\beta^2(k_0^0)) \left( n(k_0^0) + \theta(k_0^0) \right)
\]
\[ Z_{\beta}^{-1}(k^0) = 1 - \frac{\partial \Pi(k)}{\partial k_0^2} \bigg|_{k^2 = m^2_{\beta}(k^0)} \]  

where \( m^2_{\beta}(k^0) \) is the value of \( k^2 \) for which \( (k^2 - m^2 - \Pi(k)) \) vanishes. The zero-temperature values of \( m_\beta \) and \( Z_\beta \) are just the renormalised mass and the contribution from the self energy to the charge renormalisation constant. These zero-temperature values arise just from figures 5a and 5b, not 5c. At nonzero temperature the additional contributions to \( m_\beta \) and \( Z_\beta \) may be thought of as providing extra shifts to the mass and the charge. Usually these are \( k^0 \)-dependent; they still arise only from figures 5a and 5b because, as can be checked from (17b), \( \Pi^{21} \) vanishes when \( \Pi \) is real. When \( \Pi(k) \) is complex, all three diagrams of figure 5 contribute.

Similar considerations apply to self-energy insertions in the 11 and 22 elements of the thermal propagator \( D \), which occur within the 1 and 2 matrix elements in figure 1b.

The extension of this treatment to include spin-\( \frac{1}{2} \) particles is obvious. In the case of gauge particles, note that the sum over states in (1b) should extend only over physical states \( i \), since the statistical mechanics from which it is derived deals only with the possible physical states of the ensemble. The summation (2) over states is again a summation over probabilities of achieving any physical state \( j \). However, here we may include also unphysical states, so as to make up a complete set. This is because a process that begins from a physical state has total probability zero to end up in an unphysical state: in the canonical quantisation the Faddeev-Popov ghosts are introduced expressly to achieve this\[15].

Hence we again arrive at the expression (9a), though with the trace understood to be restricted to a summation over the physical states \( i \). In a similar way, if we restrict the states \( j \) to be physical, we may extend the set of states \( i \) to make up a complete set, because the ghosts also ensure that a process that finishes in a physical state has zero total probability that it began in an unphysical one. So again we may derive (9b). So all our diagrammatic analysis is still valid, provided that any external gauge particles in the graphs are restricted to their physical degrees of freedom, with no external ghost lines, and for internal gauge particles only the physical degrees of freedom are thermalised, with their unphysical modes and the ghosts having just the zero-temperature Feynman propagators\[14\]. Of course, when zero-mass gauge particles are present there are various infrared divergences. These must cancel in physical quantities, and the gauge invariance together with unitarity can also cause other cancellations\[3,13\].

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