On the Chern-Simons Gauge Field

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Abstract

We show the relationship between a fluid of particles having charge and magnetic moment in 2 + 1 dimensional electromagnetism and the Chern-Simons statistical field. The matter current which is minimally coupled to the electromagnetic field has two parts: the global electromagnetic current, and the corresponding topological current. The topological current is associated to the induced electromagnetic current, via a simple constitutive relation between charges and magnetic moments. We also study the edge states, when the region that the currents occupy is bounded.

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I. INTRODUCTION

Over the last years, theories of interacting particles in $2 + 1$ dimensions have received considerable attention. Perhaps the most relevant application of these theories is the Quantum Hall Effect \[1\]. In this context new phenomena arise connected with the possibility of statistical transmutation \[2\], or in general with the possibility of associating charges and fluxes.

Within this approach, theories involving gauge fields with Chern-Simons dynamics became very popular \[3\]. For instance, the generally accepted explanation of the Fractional Quantum Hall Effect (FQHE) is based on the idea that the ground state is basically the ground state of the Integer Quantum Hall Effect (IQHE) for composite objects having charge and flux \[4\]. The more accurate ground state known up to now is the $N$-particle Laughlin wave function \[5\]

$$\psi(z_1 \ldots z_n) = \prod_{i<j} (z_i - z_j)^m \prod_i e^{-\frac{1}{4l^2}|z_i|^2}$$

where $l$ is the magnetic length. Eq. (1) can be associated with a system of $N$ particles having an elementary charge $e$ and $m - 1$ elementary fluxes, where $m$ must be an odd integer in order for the wave function to be antisymmetric.

The implementation of this idea in Quantum Field Theory is performed through the Chern-Simons theory \[6\]. By coupling the matter current to a “statistical” gauge field $a_\mu$ with Chern-Simons dynamics

$$S = \int d^3x \, J^\mu a_\mu - \frac{\theta}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho,$$

the Euler-Lagrange equations lead to the constraints

$$\theta \epsilon^{\mu\nu\rho} \partial_\nu a_\rho = J^\mu$$

Note that these equations completely determine the field $a_\mu$ from the matter configuration. It is clear that these constraints associate every particle with a “statistical” magnetic field
and a “statistical” electric field perpendicular to the current, both fields being localized at the position of the particle.

The parameter $\theta$ has [charge/flux] units and can be expressed as $\theta = \alpha \frac{\phi}{\phi_0}$, where $e$ is the electronic charge and $\phi_0$ is the flux quantum. In other words, $1/\theta$ is the number of quantum statistical fluxes attached per unit charge.

The complete action including a Maxwell gauge field is

$$S = \int d^3x \mathcal{L}_M - \frac{1}{4} F^\mu\nu F_{\mu\nu} + J^\mu A_\mu + J^\mu a_\mu - \frac{\theta}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho,$$  \hspace{1cm} (4)

where $\mathcal{L}_M$ is the matter lagrangian. This action is quadratic in the gauge fields; then, changing to euclidean space, and by conveniently fixing the gauge we can integrate out the fields $A_\mu$ and $a_\mu$ to obtain the effective action for the matter fields

$$S_{\text{eff}} = \int d^3x \mathcal{L}_M - \int d^3x \int d^3y \left( \frac{1}{2} J^\mu \Box_\mu J_\mu + \frac{1}{2\theta} \epsilon^{\alpha\beta} J_\mu \partial_\alpha \frac{1}{\Box} J_\beta \right)$$  \hspace{1cm} (5)

the third term corresponds to an electromagnetic interaction between the currents $J$, while the last term comes from the Chern-Simons sector of the original action and is responsible for the statistical transmutation of the matter fields. This mechanism of associating charge and flux is widely accepted in the construction of Landau-Ginzburg theories for the FQHE \[7\]. Nevertheless, the interpretation of the “statistical field” is not clear.

Recently, it was proposed \[8\] that the Chern-Simons term could be obtained by projecting a topological term from $3 + 1$ to $2 + 1$ dimensions; in this way the statistical field is associated with the gauge field of a “modified” $(3 + 1)$ QED. We present in this paper a model which is appropriate for the description of a fluid of vortices that interact electromagnetically in $2 + 1$ dimensions. This description leads to statistical transmutation and has a very clear interpretation that could allow future studies of the real dynamics of the charge-flux association.
II. THE MODEL

If we are interested in describing the long distance behavior of particles having a given micro-structure, as a first approximation, we can consider the first moments of the internal charge and current distributions. Let us suppose that, in the frame where the micro-structure is globally at rest, the first non-zero moments are the total charge and the magnetic dipole moment. Then, we can assume that we are working with a fluid of "particle-like" vortices. In this approximation we can think of each constituent as having a charge, and an intrinsic magnetic moment arising from microscopic currents moving on "circuits" or "loops". These loops can be taken as a way of representing the intrinsic currents of each micro-structure, in order to facilitate the understanding of the origin of the induced currents.

If we consider a neutral loop at rest in a plane, it is easy to see that there is a magnetic field (in (2 + 1) electromagnetism) which is only non-zero inside the loop. This is to be considered as an approximation, when we look at the local effects, to the real situation in (3+1) electromagnetism where the magnetic field is not localized at the loop. Now, if we consider a neutral loop with constant velocity $\vec{v}$, it is a simple task to evaluate the corresponding electromagnetic fields (by Lorentz transforming). The result is that the electromagnetic field is localized in the inner part of the loop. The electric field is perpendicular to $\vec{v}$ and comes from the appearance of a non-zero charge density (having total charge equal to zero) due to the movement of the loop. Note also that the velocity $\vec{v}$ is proportional to the "global" current associated to the motion of the loop.

Now, let us consider a "fluid" of particle-like vortices. In order to describe it, we give the global distribution of charge by a density $J^0_{global}$ and a current density $\vec{J}_{global}$, due to the global motion of the charged vortices. We also give a constitutive relation which simply says that the internal state is the same for all the particles of the fluid. That is, all the particles have the same total charge $Q$ and, when considered at rest, they display the same magnetic dipole moment $\mu \hat{z}$ ($\hat{z}$ is a versor perpendicular to the plane in consideration). Then, in the case where the charge density $J^0_{global}$ is stationary and $\vec{J}_{global} = 0$, in every place where we
have a charge $Q$ we also have a magnetic dipole moment $\mu \hat{z}$ and therefore the magnetic dipole moment density is given by (from now on, we will simply write $J_\rho$ for the global charge and current densities)

$$\vec{m}(\vec{x}) = \frac{\mu}{Q} J_0(\vec{x}) \hat{z}$$

(6)

This density implies an induced electric current (due to the possible lack of compensation of currents coming from different loops) given by

$$\vec{J}_{ind} = \vec{\nabla} \times \vec{m} = g \vec{\nabla} \times (J_0 \hat{z}) = g \hat{z} \times \vec{\nabla} J_0$$

(7)

where we have called $g = \mu/Q$. Written in two-dimensional notation this current takes the form

$$J_{ind}^i = g e^{0ik} \partial_k J_0$$

(8)

The electromagnetic interaction lagrangian is obtained by minimally coupling the total current to the electromagnetic field,

$$\mathcal{L}_{int} = A_\mu J^\mu_{total}$$

(9)

Using $J^\mu_{total} = J^\mu + J^\mu_{ind}$ and recalling that $\vec{J} = 0$, $J^0_{ind} = 0$ we get

$$\mathcal{L}_{int} = A_0 J^0 + g A_k e^{0ik} \partial_k J_0$$

(10)

The Lorentz-invariant generalization of this interaction lagrangian is

$$\mathcal{L}_{int} = A_\mu (J^\mu + ge^{\mu\nu\rho} \partial_\nu J_\rho)$$

$$= A_\mu (J^\mu + gG^\mu) , \quad G^\mu = e^{\mu\nu\rho} \partial_\nu J_\rho$$

(11)

$G^\mu$ is the topological current ($\partial_\mu G^\mu = 0$) associated to the global current $J_\rho$, and gives (up to the constant $g$) the induced charge density and induced currents in a general situation.

The interpretation of an interaction term proportional to $J_\mu \tilde{F}^\mu$, $\tilde{F}^\mu = e^{\mu\nu\rho} \partial_\nu A_\rho$ as associated to particles having magnetic moment has also been given in Refs. [9] and [10]. This
term coincides with that given in (11) up to boundary terms, which are relevant when the region that the currents occupy is bounded (see section IV). In Ref. [9] it is also shown the relationship between this term and a Chern-Simons term, when the matter is given by a scalar field that undergoes spontaneous symmetry breaking (SSB). In the next section we will see that the relation between (11) and the Chern-Simons theory is more general and does not depend on SSB.

III. RELATIONSHIP WITH THE CHERN-SIMONS THEORY

According to (11), the lagrangian for a fluid of particle-like vortices whose total current (global and induced) is minimally coupled to the electromagnetic field is

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu(J^\mu + gG^\mu) + \mathcal{L}_M \]  

(12)

The Euler-Lagrange equation for the electromagnetic field is (in the gauge \( \partial_\mu A^\mu = 0 \))

\[ \Box A^\mu = -(J^\mu + gG^\mu) \]

(13)

Given a matter distribution we can write

\[ A^\mu = A^\mu + b^\mu \]  

(14)

where \( A^\mu \) is the electromagnetic field due to the global currents, and \( b^\mu \) is the field having sources on the induced currents:

\[ \Box A^\mu = -J^\mu, \quad \Box b^\mu = -gG^\mu = -g\epsilon^{\mu\alpha\beta} \partial_\alpha J_\beta \]  

(15)

Multiplying the second equation by \( \epsilon_{\mu\nu\rho} \partial^\rho \), we obtain

\[ \Box \epsilon_{\mu\nu\rho} \partial^\rho b^\mu = -g \partial_\nu (\partial \cdot J) + g \Box J_\nu \]  

(16)

using that \( J \) is a conserved current, we get the equation \( \Box (\epsilon_{\mu\nu\rho} \partial^\rho b^\mu - gJ_\nu) = 0 \), and up to a solution to the homogeneous equation of the form \( \epsilon_{\mu\nu\rho} \partial^\rho c^\mu, \Box c^\mu = 0 \), which can be absorbed as a radiation field in \( A^\mu \) (cf. [14]): \( A^\mu \rightarrow A^\mu + c^\mu \), we get
\[ J^\mu = \frac{1}{g} \epsilon^{\mu\nu\rho} \partial_\nu b_\rho \] (17)

or, by fixing the gauge \( \partial_\mu b^\mu = 0 \),

\[ b_\mu(x) = -g\epsilon_{\mu\nu\rho} \partial_\nu \int dx' J^\rho(x') G(x - x') \] (18)

where \( G(x - x') \) is a Green function for the operator \( \Box \) with a given boundary condition. For instance, the retarded function is \( G_R(x) = 1/(2\pi)Q_+^{-1/2}\theta(t) \) where \( Q_+^{-1/2} = (t^2 - \vec{x}^2)^{-1/2} \), if \( t^2 > \vec{x}^2 \) and \( Q_+^{-1/2} = 0 \) otherwise.

From Eq. (17) we can see that the field \( b_\mu \) and the global current \( J^\mu \) satisfy a Chern-Simons relation. Note that in this case \( b_\mu \) is a true electromagnetic field and not the statistical field used in (3). The physical interpretation for the expression (17) can be traced back from the discussion at the beginning of section II: the magnetic and electric fields having sources in the intrinsic currents of the moving loops are localized at these loops, the electric field being perpendicular to the global current.

In Eq. (7), we have introduced the coupling \( g = \mu/Q \). From Eq. (14) and (17), we find that \( g \) also represents the number of (intrinsic) real flux quanta per unit charge: \( g = \Phi/Q \). Note that if we take a static vortex at \( \vec{x}_0 \), the magnetic field is equal to the magnetic dipole moment density: \( \vec{B} = \vec{m} = \mu \delta(\vec{x} - \vec{x}_0)\hat{z} \), and integrating over an area containing \( \vec{x}_0 \) we obtain \( \Phi = \mu \), in accordance with our previous definition \( g = \mu/Q \).

Now, we will study the possibility of considering \( b \) as a background field. For instance, if we take an external classical source \( J \) (with zero divergence), we can obtain, from the partition function \( Z \), the probability \( P = ZZ^* \) to remain in the vacuum state (vacuum persistence). From (12), path integrating over \( A \), and using Feynman’s prescription to invert \( \Box \) we get

\[
Z = \exp \left( \frac{i}{2} \int dx dy H_\mu(x) \int \frac{dk}{(2\pi)^3} e^{ik(x-y)} \frac{\eta^{\mu\nu}}{k^2 + i\epsilon} H_\nu(y) \right)
\]

\[
= \exp \left( \frac{i}{2} \int \frac{dk}{(2\pi)^3} \frac{\tilde{H}_\mu(k)\tilde{H}_\mu(-k)}{k^2 + i\epsilon} \right) \]

where \( H_\mu = J_\mu + g\epsilon_{\mu\nu\rho} \partial_\nu J^\rho \) and \( \tilde{H}_\mu \) is its Fourier transform:
\[ \tilde{H}_\mu(k) = \tilde{J}_\mu(k) + g\epsilon_{\mu\nu\rho}k^\nu\tilde{J}^\rho(k) \]
\[ \tilde{H}_\mu(-k) = \tilde{J}_\mu^*(k) - g\epsilon_{\mu\nu\rho}k^\nu\tilde{J}^{*\rho}(k) \]  \hspace{1cm} (20)

\[(J(x) \text{ is real and therefore we have } \tilde{J}(-k) = \tilde{J}^*(k)). \]

Now, the probability to remain in the vacuum comes from the real part \((R)\) of the exponent in \(Z\):

\[ R = \frac{\pi}{2} \int \frac{dk}{(2\pi)^3} \delta(k^2) \tilde{H}_\mu(k)\tilde{H}^\mu(-k) \]
\[ P = e^{2R} \]  \hspace{1cm} (21)

and using (20) in (21), we get

\[ R = \frac{\pi}{2} \int \frac{dk}{(2\pi)^3} \delta(k^2) \left( \tilde{J}_\mu(k)\tilde{J}^{\mu*}(k) + g\epsilon_{\mu\nu\rho}k^\nu\tilde{J}^\rho(k)\tilde{J}^{*\rho}(k) \right. \]
\[ - g\tilde{J}^\mu(k)\epsilon_{\mu\nu'\rho'}k^{\nu'}\tilde{J}^{*\rho'}(k) - g^2(k^2\tilde{J}(k)\tilde{J}^*(k) - k.\tilde{J}(k)k.\tilde{J}^*(k)) \]
\[ = \frac{\pi}{2} \int \frac{dk}{(2\pi)^3} \delta(k^2) \tilde{J}_\mu(k)\tilde{J}^{\mu*}(k) \]  \hspace{1cm} (22)

where we have used that \(J\) is conserved (note that \(k.\tilde{J} = 0, k^2 = 0\) imply \(R < 0\)). Then, the property of having a non-zero \(g\) does not modify the vacuum survival probability.

The canonically quantized field \(A\) can be written as \(A(x) = \mathcal{A}(x) + b(x), \quad \mathcal{A}(x) = \hat{A}(x) - \int dx'G(x-x')J(x')\), where \(\mathcal{A}\) contains the fluctuating part \(\hat{A}, \square \hat{A} = 0\). Now, suppose that the field \(A\) were only coupled to the induced current: \(Q\) tends to zero while keeping \(\mu = gQ\) finite, that is, \(J^\mu \to 0\) keeping \(gG^\mu\) finite. In this case, from (22), the probability to remain in the vacuum is equal to 1; then, from unitarity, the probability to go from the vacuum to any other state \(|n\rangle \langle 0| = 0\) is equal to zero. Therefore, in this case, a situation where the field is just the classical part \(b\) and the free fluctuating part \(\hat{A}\) is always in the vacuum state is self-consistent at the quantum level. This differs from the case were the \(A\) field is only coupled to \(J\). In this case, according to (22), the classical current excites particles from the vacuum state and the field \(\hat{A}\) can not be left in the vacuum.

Then, we could expect, in the general case, to be able to replace the full electromagnetic field by a background field \(b\) and a fluctuating field \(\mathcal{A}\) having sources on the global current \(J\). In order to look for this possibility we will suppose that the current \(J\) comes from a (quantum) matter sector represented by a field \(\psi\) and we will consider the corresponding partition function, obtained from the path integral.
\begin{align*}
Z[H, K] &= \int [\mathcal{D} \psi][\mathcal{D} A] \exp i \int dx (-\frac{1}{4} F^2 +
+ A(J(\psi) + gG(\psi)) + \mathcal{L}_M(\psi) + \mathcal{L}_{GF}(A) + HA + K\psi) 
\tag{23}
\end{align*}

where $\mathcal{L}_{GF} = -1/2(\partial A)^2$ is the (Lorentz) gauge fixing term. In the path integral over $A$, we can make the change of variables (having jacobian equal to 1):

$$A^\mu = \mathcal{A}^\mu + b^\mu(\psi)$$

(24)

where $b^\mu(\psi) = -g\epsilon_{\mu\nu\rho} \partial^\nu \int dx' J^\rho(\psi(x')) G(x-x')$. If we call $\mathcal{F}$ and $f$ the field-strength tensors for $\mathcal{A}$ and $b$, respectively, the exponent in (23) now reads

$$-\frac{1}{4} F^2 - \frac{1}{2} \mathcal{F} f - \frac{1}{4} f^2 + (\mathcal{A} + b)J + g\mathcal{A}G + gbG + \mathcal{L}_M + \mathcal{L}_{GF}(\mathcal{A}) + H(\mathcal{A} + b) + K\psi$$

(25)

Here, $\mathcal{L}_{GF}(\mathcal{A}) = -1/2(\partial \mathcal{A})^2$ ($\partial b = 0$). From (17) we have $f_{\mu\nu} = g\epsilon_{\mu\nu\rho} J^\rho$, then (up to divergences) we obtain

$$-\frac{1}{2} \mathcal{F} f = -g\mathcal{A}G \quad , \quad -\frac{1}{4} f^2 = -\frac{g^2}{2} J^2 \quad , \quad gbG = g^2 J^2$$

(26)

and results

$$Z[H, K] = \int [\mathcal{D} \psi][\mathcal{D} \mathcal{A}] \exp i \int dx (-\frac{1}{4} F^2 + (\mathcal{A} + b)J +
+ \frac{g^2}{2} J^2 + \mathcal{L}_M + \mathcal{L}_{GF}(\mathcal{A}) + H(\mathcal{A} + b) + K\psi)$$

(27)

Then we find, by taking functional derivatives with respect to $H$ and $K$ that we can compute $n$-point Green functions for the fields $\mathcal{A}$ and $\psi$ using for this fields the dynamics (12) or, equivalently, we can replace $A$ by $\mathcal{A} + b$ and compute the corresponding Green functions using the dynamics given by

$$\tilde{\mathcal{L}} = -\frac{1}{4} F^2 + \mathcal{A}J + \mathcal{L}_M + bJ + \frac{g^2}{2} J^2$$

(28)

Here, we see that the $b$ field appears as a background and the $\mathcal{A}$ field has sources only on $J$; the matter lagrangian is also modified by the presence of a local current-current term. The $bJ$ term leads to a non-local interaction.
\[ bJ = -g \int d^3x \, d^3y \, \epsilon^{\mu\alpha\beta} J_\mu(x) \partial_\alpha \frac{1}{\Box} J_\beta(y) \, , \quad J = J(\psi) \quad (29) \]

which can be made local by introducing a Chern-Simons field \( a \), with \( \theta = 1/(2g) \) (cf. (1) and (3)):

\[ \tilde{\mathcal{L}} \rightarrow -\frac{1}{4} F^2 + A J + a J - \frac{1}{4g} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \mathcal{L}_M + \frac{g^2}{2} J^2 \quad (30) \]

Then, we can compute a probability amplitude between two \( A, \psi \) configurations, using \( \mathcal{L} \) (Eq. (12)), or we can compute the corresponding expression for the \( A, \psi \) fields, using \( \tilde{\mathcal{L}} \) (Eq. (28) or (30)). Clearly, this result also holds when the current comes from the motion of a given number of point particles, replacing in the path integral \( \psi \) by the particle's coordinates.

The additional terms in the matter lagrangian can also be understood if we consider the effective dynamics for the vortices that interact via \((2+1)\) electromagnetism, obtained by averaging over the gauge field. From (12), integrating out the field \( A_\mu \), we obtain the effective action

\[ S_{\text{eff}} = \int d^3x \, \mathcal{L}_M - \frac{1}{2} \int d^3x \, d^3y \, (J_\mu(x) + gG_\mu(x)) \frac{1}{\Box} (J_\mu(y) + gG_\mu(y)) \]

\[ = \int d^3x \, \mathcal{L}_M + \frac{g^2}{2} J_\mu J^\mu - \int d^3x \, d^3y \, \left( \frac{1}{2} J_\mu \frac{1}{\Box} J_\mu + g\epsilon^{\mu\alpha\beta} J_\mu \partial_\alpha \frac{1}{\Box} J_\beta \right) \quad (31) \]

where \( 1/\Box \) stands for \( G_F(x-y) \), the Feynman’s Green function: \( G_F(x) = 1/(2\pi)(Q-i\epsilon)^{-1/2} \), \( Q = i^2 - \vec{x}^2 \). We have just seen in (22) that the \( g \)-dependent part of (31) is real: the second current-current interaction term in (31) comes from the interaction of the induced currents via \( 1/\Box \), while the last term comes from the interaction of the induced and global currents and is known as the Hopf term, which changes the particle’s statistics (for a review, see Ref. [11]). This effective action can also be obtained by averaging over \( A \) in (28), that is, integrating the \( A \) field in (27) and setting to zero the external currents.

The current-current term is a contact term, when the matter comes from the motion of a given number of point particles. In this case, if we look at the long distance properties of the theory, we can drop out the current-current interaction in (30), obtaining the Maxwell and Chern-Simons model of Eq. (3) (making the identification \( \theta = 1/(2g) \)).
The short distance behavior differs by the presence of a current-current term. This term distinguishes the interaction of particles having attached a real electromagnetic flux from the interaction of particles having attached a statistical flux. This type of “velocity dependent contact interaction” has been considered in Ref. [10], in the context of classical Maxwell-Chern-Simons theory.

IV. EDGE STATES

It is well known that the Chern-Simons action is gauge invariant up to boundary terms. So if we work in a bounded region, in order to make the theory gauge invariant, it is necessary to add a 1 + 1 dimensional action to the 2 + 1 dimensional action (4). In the case of the Quantum Hall Effect this will produce edge currents that satisfy a Kac-Moody algebra [12]. These edge currents appear to be fundamental when explaining the universal aspects of the QHE [13] [14].

In our case, to maintain the gauge invariance of the theory it is sufficient to require the local conservation of the total electric charge. The induced part of the total current is automatically divergenceless, as it corresponds to a topological current. This can be related to the fact that the induced current comes from the part of the vortex charge distribution that is a magnetic moment in the vortex rest-frame, and gives no net charge transportation. Then, the charge is locally conserved when the global current is divergenceless everywhere. Note that when the currents vanish outside a finite region $D$, we can write

$$J_\rho(t, \vec{x}) = j_\rho(t, \vec{x}) \theta(D)$$

where $\theta(D)$ is a function whose value is one when $\vec{x}$ is inside $D$ and zero otherwise. If the boundary is static we have

$$\partial^\rho J_\rho(t, \vec{x}) = \partial^\rho j_\rho(t, \vec{x}) \theta(D) + j_i(t, \vec{x}) \partial^\rho \theta(D)$$

then, in this case, the charge will be locally conserved (even at the edge) if $j_\rho$ is divergenceless
in the bulk, and the spatial part of \( j_\rho \) is tangential to the border of the bounded region. Under these circumstances, the theory will be gauge invariant.

From (32) we see that the induced current has two terms (cf. (11))

\[
gG^\mu = \theta(D) \epsilon^{\nu\rho} \partial_\nu j_\rho + g \epsilon^{\mu\nu\rho} j_\rho \partial_\nu \theta(D)
\]

the first term in (33) has support at \( D \) and gives the induced currents in the bulk: \( gG_{\text{bulk}} \), while the second term gives the induced currents at \( \partial D \) (the border of \( D \)): \( gG_{\partial D} \). If the border is static, \( \theta(D) \) is time-independent and we obtain for the second term

\[
- g \epsilon^{\kappa\rho} j_\rho n_k \delta(\partial D)
\]

where \( n_k \) is the \( k \)-th component of the versor contained in the plane, which is normal to \( \partial D \) (and external to \( D \)), and \( \delta(\partial D) \) is a delta-function with support at \( \partial D \). Then, the charge and current density at \( \partial D \) are

\[
gG^0_{\partial D} = -g j_j n_k \delta(\partial D) = -g \hat{j} \hat{t} \delta(\partial D) \quad , \quad gG^1_{\partial D} = g j_0 \hat{t} \delta(\partial D)
\]

where \( \hat{t} = \hat{z} \times \hat{n} \) is the tangent versor to \( \partial D \). The physical interpretation of the charge and current densities at the edge is clear: the charge density is due to the fact that the moving loops imply a distribution of electric dipole moment, perpendicular to the current, and the associated charge distribution is not compensated at the edge. Similarly, the induced current comes from the lack of compensation of the intrinsic currents at the edge.

If a part of \( D \) is given by the points below \( y = 0 \), then the edge densities at \( y = 0 \) are

\[
gG^0_{\partial D} = +g j_j \delta(y) \quad , \quad gG^1_{\partial D} = -g j_0 \delta(y)
\]

this can be written using two-dimensional notation as

\[
gG^\mu_{\partial D} = g \epsilon^{\mu\nu} j_\nu \delta(y) \quad , \quad \mu = 0, 1
\]

If the border of \( D \) changes with time, as it occurs for a droplet of fluid, we should consider in (33) the time derivative of \( \theta(D) \); note however that the expression for the charge density at the border does not depend on this derivative and is still given by
\[ gG^0_{\partial D} = -g \vec{j} \cdot \delta(\partial D) \quad (38) \]

where \( \partial D \) is the border of the droplet, which can change with time.

Recalling that the time-component of the topological current represents the induced charge density coming from the (intrinsic) magnetic moment part of each micro-structure, we have that the total induced charge (in the bulk and at the edge) must be zero. This can be verified as the induced charge in the bulk is (cf. [33])

\[ g \int_D d^2x \, \epsilon^{ik} \partial_i j_k = g \oint_{\partial D} \vec{j} \cdot d\vec{l} = g \oint_{\partial D} \vec{j} \cdot \hat{t} \, dl \quad (39) \]

which cancels against the induced charge at \( \partial D \) (cf. [33]):

\[ Q_{\partial D} = -g \oint_{\partial D} \vec{j} \cdot \hat{t} \, dl \quad (40) \]

We can see that the 1 + 1 quiral currents that are necessary to restore the gauge invariance of the Chern-Simons theory are the analog of the induced charge and current densities, at the edge of a fluid of particle-like vortices. Now, we may ask about the conditions that lead to the conservation of the induced charge at \( \partial D \). The quantum states carrying this charge can be constructed from an operator \( L \) that satisfies

\[ Q_{\partial D} L |0\rangle = qL |0\rangle \quad (41) \]

In other words, the equation

\[ [Q_{\partial D}, L] = qL \quad (42) \]

must be solved.

In the case of a linear infinite edge \((y = 0)\), the currents are given by \([36]\) and the topological charge can be written as

\[ Q_{\partial D} = g \int_{-\infty}^{\infty} j(x) \, dx \quad (43) \]

where \( j = j_1(x) \). It is well known \([12] [13] [14]\) that if the dynamics of the model leads to an incompressible fluid state, then the edge currents will satisfy a Kac-Moody algebra
\[ [j(x), j(x')] = i \frac{d}{dx} \delta(x - x') \] (44)

In this case, the operators that obey (42) can be characterized by:

\[ L_\alpha = \exp \left\{ -i \int_{-\infty}^{\infty} \alpha(x) j(x) \, dx \right\} \] (45)

where \( \alpha(x) \) is an arbitrary function. Using this expression and (44), we immediately obtain the induced charge at the border:

\[ q = g [\alpha(+) - \alpha(-)] \] (46)

Similarly, in the case of a droplet of incompressible fluid, the states are characterized by a function \( \alpha(\theta) \) (where \( \theta \) parametrizes the border) and the induced charge is \( g[\alpha(2\pi) - \alpha(0)] \).

That is, the topological charge \( q \) that a state \( |\alpha\rangle = L_\alpha |0\rangle \) supports only depends on the boundary conditions on \( \alpha \).

These excitations \( |\alpha\rangle \) that occur at the border of an incompressible quantum fluid are the so called “ripplons” and have been introduced in Ref. [14]. These “ripplons” were shown to correspond to deformations of the droplet surface that move along the surface with a fixed velocity and their shape unchanged. For instance, for a linear edge, the ripplon evolves according to \( \alpha(x) \rightarrow \alpha(x - vt) \) and the charge \( q \), given by (46), does not change. Then, in general, we see that in the case of an incompressible fluid of particle-like vortices, the induced electric charge at the border is conserved.

**V. DISCUSSION**

In this work we have considered a fluid of particle-like vortices interacting electromagnetically in \( 2+1 \) dimensions. The matter current has two parts: a global current associated to the motion of the total charge of each vortex, and an induced current arising from the lack of compensation of the intrinsic currents coming from different vortices. By considering a simple constitutive relation (which says that the internal state of every vortex is the same), we can write the induced current as the topological current associated to the global current.
Then, given a matter distribution we obtain a Chern-Simons relation between the global current and the electromagnetic field having sources on the induced currents.

At the quantum level, we have found that the n-point Green functions for a system where the electromagnetic field is minimally coupled to the global current $J$ and the induced (topological) current $gG$ (Eq. (12)) can be computed from a theory containing a Maxwell field and a Chern-Simons field (with $\theta = 1/(2g)$) minimally coupled to $J$, plus a local current-current term that modifies the matter lagrangian (Eq. (30)). In particular, after averaging over the gauge fields, the effective dynamics for both models is the same, except for the current-current term coming from the interaction of the intrinsic electromagnetic fields of a given vortex with the intrinsic currents of the other. If the matter current comes from the motion of point particles, the current-current term is a contact term which is unimportant, when we consider the long distance behavior of the model. In this case we make contact with Maxwell’s theory with a Chern-Simons field.

Our model is constructed just in terms of elecromagnetic fields; then, unlike the Chern-Simons construction, it is gauge invariant by itself even when the region that the currents occupy is bounded. Note that there are induced currents localized at the edge which are the analog of the quiral currents needed to restore the gauge symmetry in the Chern-Simons theory. These currents can be used to characterize the edge states, as the corresponding induced charge is conserved for an incompressible quantum droplet.

In this article we have not been concerned with the possible mechanisms that can lead to the dynamical formation of structures having charge and flux. It may be possible that in non-relativistic systems the competition between the electron-electron interaction and the interaction with a strong external magnetic field could produce effective objects that correspond to the quiral structures we have considered here. In this framework, the understanding of the FQHE would be related to the understanding of the origin of micro-structures with a given relation of [flux/charge] per structure. Given this particle-like vortices, their effective dynamics could be described by the lagrangian (30), and it would be interesting to explore the system’s response to external electromagnetic fields in order to
compute physical parameters as the conductivity.

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