AFFINE ALGEBRAIC VARIETIES

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Abstract. In this paper, we give new criteria for affineness of a variety defined over \( \mathbb{C} \). Our main result is that an irreducible algebraic variety \( Y \) (may be singular) of dimension \( d \geq 1 \) defined over \( \mathbb{C} \) is an affine variety if and only if \( Y \) contains no complete curves, \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \) and the boundary \( X - Y \) is support of a big divisor, where \( X \) is a projective variety containing \( Y \). We construct three examples to show that a variety is not affine if it only satisfies two conditions among these three conditions. We also give examples to demonstrate the difference between the behavior of the boundary divisor \( D \) and the affineness of \( Y \).

If \( Y \) is an affine variety, then the ring \( \Gamma(Y, \mathcal{O}_Y) \) is noetherian. However, to prove that \( Y \) is an affine variety, we do not start from this ring. We explain why we do not need to check the noetherian property of the ring \( \Gamma(Y, \mathcal{O}_Y) \) directly but use the techniques of sheaf and cohomology.

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1. Introduction

We work over complex number field \( \mathbb{C} \).

Affine varieties are important in algebraic geometry. In 1957, J.-P. Serre introduced sheaf and cohomology techniques to algebraic geometry and discovered his well-known cohomology criterion ([26], [8], Chapter 2, Theorem 1.1): a variety (or a noetherian scheme) \( Y \) is affine if and only if for all coherent sheaves \( F \) on \( Y \) and all \( i > 0 \), \( H^i(Y, F) = 0 \). The point of Serre’s criterion is that instead of looking at the noetherian property of the ring \( \Gamma(Y, \mathcal{O}_Y) \), to check the affineness of \( Y \), we examine the cohomology of the coherent sheaves on \( Y \). In 1962, Narasimhan solved the Levi problem for complex spaces ([21], [22]). The corresponding analytic variety in complex geometry, which holds the similar important position as an affine variety in algebraic geometry, is a Stein variety. We know that to verify the Steinness of an analytic variety \( Y \), we do not look at the ring \( \Gamma(Y, \mathcal{O}_Y) \), but consider the holomorphic functions and check whether \( Y \) is holomorphically separable and holomorphically convex ([6], Page 143). Inspired by Serre’s criterion and the analytic method of Narasimhan, in 1969, Goodman and Hartshorne proved that \( Y \) is an affine variety if and only if \( Y \) contains no complete curves and the dimension \( \dim(Y, F) \) of the linear space \( H^1(Y, F) \) is bounded for all coherent sheaves \( F \) on \( Y \) ([4]).

Let \( X \) be the completion of \( Y \). In 1969, Goodman also proved that \( Y \) is affine if and only if after suitable blowing up the closed subvariety on the boundary \( X - Y \), the new boundary \( X' - Y \) is support of an ample divisor, where \( X' \to X \).
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is the blowing up with center in $X - Y$ ([3]; [8], Chapter 2, Theorem 6.1). For any quasi-projective variety $Y$, we may assume that the boundary $X - Y$ is the support of an effective divisor $D$ with simple normal crossings by blowing up the closed subvariety in $X - Y$. $Y$ is affine if $D$ is ample. So if we can show the ampleness of $D$, $Y$ is affine. There are two important criteria for ampleness due to Nakai-Moishezon and Kleiman ([12]; [15], Chapter 1, Section 1.5). Another sufficient condition is that if $Y$ contains no complete curves and the linear system $|nD|$ is base point free, then $Y$ is affine ([8], Chapter 2, Page 64). Therefore we can apply base point free theorem if we know the numerical condition of $D$ ([24], [15], Chapter 3, Page 75, Theorem 3.3). In 1988, Neeman proved that if $Y$ can be embedded in an affine scheme Spec$A$, then $Y$ is affine if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ [23]. The significance of Neeman’s Theorem is that it is not assumed that the ring $A$ is noetherian.

In higher dimension (at least, in our problem), it is very hard to check the ampleness of a big (even big and nef) divisor $D$ and the base point freeness of the linear system $|nD|$. We need to search for a different approach.

Iitaka’s $D$-dimension theory is widely used in classification of algebraic varieties ([10], [11] [19], [30]). Recall the definition of notation $\kappa(D, X)$.

**Definition 1.1.** If for all $m > 0$, $H^0(X, \mathcal{O}_X(mD)) = 0$, then we define the $D$-dimension (or Iitaka dimension) $\kappa(D, X) = -\infty$. Otherwise,

$$\kappa(D, X) = \text{tr.deg}_\mathbb{C} \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)) - 1.$$ 

In particular, the Kodaira dimension $\kappa(X)$ of $X$ is defined to be $\kappa(K_X, X)$, where $K_X$ is the canonical divisor of $X$.

A quasi-affine variety is a Zariski open subset of an affine variety. Throughout this paper, we assume that $Y$ is an irreducible algebraic variety of dimension $d$ defined over $\mathbb{C}$ such that it is the complement of an effective divisor $D$ in a projective variety $X$. We may assume that $D$ has simple normal crossings.

**Theorem 1.2.** An irreducible algebraic surface $Y$ is an affine surface if and only if $Y$ contains no complete curves, the boundary $X - Y$ is connected and $\kappa(D, X) = 2$.

Theorem 1.2 is not true for higher dimensional varieties (see the example in Section 3).

**Theorem 1.3.** An irreducible algebraic variety $Y$ of dimension $d$ ($d \geq 1$) is a quasi-affine variety if and only if $Y$ contains no complete curves and $\kappa(D, X) = d$.

**Theorem 1.4.** An irreducible algebraic variety $Y$ of dimension $d$ ($d \geq 1$) is an affine variety if and only if $Y$ contains no complete curves, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ and $\kappa(D, X) = d$.

If we have a surjective morphism from a variety $Y$ to an affine variety such that every fibre is affine, then $Y$ may not be affine (see Example 3.11 in Section 3). But we have the following theorem.
Theorem 1.5. Let \( f : Y \to W \) be a surjective morphism from an irreducible quasi-projective variety \( Y \) to an irreducible affine variety \( W \). If every fiber is an affine subvariety of \( Y \) and \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \), then \( Y \) is an affine variety.

Corollary 1.6. Let \( f : Y \to W \) be a surjective morphism from an irreducible quasi-projective variety \( Y \) to an irreducible affine variety \( W \). If \( Y \) has no complete curves, a general fiber is an affine subvariety of \( Y \) and \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \), then \( Y \) is an affine variety.

In Section 3, we will give three examples to show that a variety is not affine if it only satisfies two conditions in Theorem 1.4.

If \( Y \) is an affine variety, then the ring \( \Gamma(Y, \mathcal{O}_Y) \) is noetherian. However, in our proof of Theorem 1.4, we do not directly check the noetherian property of this ring, which I do not know whether it is possible or not, because our condition is rather geometric. In [36], we proved Mohan Kumar’s affineness conjecture for an algebraic manifold and gave a partial answer to J.-P. Serre’s Steinness question [27]. After we carefully examined the conditions and where we used them, we found that we can get general theorems for singular varieties by changing the assumption and modifying the proof. The vanishing Hodge cohomology \( H^i(Y, \Omega_Y^j) = 0 \) for all \( i > 0 \) and \( j \geq 0 \) are replaced by two conditions: \( Y \) has no complete curves and \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \). The advantage of these two conditions is that they are well-defined for singular varieties. This makes the generalization possible.

The question of a quasi-projective variety \( Y \) to be affine is different from the behavior of the boundary divisor \( D \), in particular, the numerical condition of \( D \) like nefness and finitely generated property of the graded ring

\[
\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)).
\]

The reason is that

\[
\Gamma(Y, \mathcal{O}_Y) \neq \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)).
\]

We will give two examples to demonstrate this difference in Section 3. One example (due to Zariski) gives a surface \( Y = X - D \), which is affine but the the corresponding graded ring \( \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)) \) is not finitely generated for an effective divisor \( D \). The other example ([10; 8], Page 232) is a surface \( Y = X - D \) such that

\[
H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X(nD)) = \mathbb{C}.
\]

A necessary condition for the affineness of \( Y \) with dimension \( d \) is that \( Y \) must have plenty of nonconstant regular functions. More precisely, \( Y \) must have \( d = \dim Y \) algebraically independent nonconstant regular functions. This means that the corresponding effective boundary divisor \( D \) must be big, i.e.,

\[
h^0(X, \mathcal{O}_X(nD)) \geq an^d
\]

for some positive number \( a \) and \( n \gg 0 \). So the above surface \( Y \) is not affine but \( \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)) \) is finitely generated.

We will prove the results in Section 2 and give examples in Section 3.
2. Proof of the Theorems

Recall our notation: \( Y \) is an open subset of a projective variety \( X \) with dimension \( d \geq 1 \) and \( D \) is the effective boundary divisor with support \( X - Y \). We may assume that the boundary divisor \( D \) has simple normal crossings by further blowing up suitable closed subvariety of \( X - Y \).

**Proof of Theorem 1.2.** The proof is the same as the proof of Theorem 1.1 \[35\]. The idea is to show that the boundary \( X - Y \) is support of an ample divisor.

Q.E.D.

**Proof of Theorem 1.3.** One direction is trivial. If \( Y \) is an irreducible quasi-affine variety, then it has no complete curves since it is a closed subset of affine space \( \mathbb{C}^N \). Let \( U \) be an affine variety such that \( Y \subset U \subset X \), then the boundary \( X - U \) is of pure codimension 1 \([8], \text{Chapter II, Proposition 3.1}\). By Goodman’s theorem \([8], \text{Chapter II, Theorem 6.1}\), since \( U \) is affine we may assume that \( X - U \) is support of an ample divisor \( A \) on \( X \). Therefore \( \kappa(A, X) = d \) \([15], \text{Proposition 2.61}\).

Suppose now that \( Y \) satisfies the condition in Theorem 1.3. We will prove that it is quasi-affine. We may assume that both \( X \) and \( Y \) are normal. In fact, let \( Y' \) be the normalization of \( Y \), then we have finite morphism from \( Y' \) to \( Y \). Thus \( Y' \) satisfies two conditions in Theorem 1.3 and \( Y \) is affine if and only if \( Y' \) is affine.

**Lemma 2.1.** Theorem 1.3 holds for curves and surfaces.

**Proof.** If a curve is not complete then it is affine \([8], \text{Chapter II, Proposition 4.1}\). Surface case is true by Theorem 1.1. Q.E.D.

We may assume that Theorem 1.3 holds for \((d - 1)\)-dimensional algebraic varieties. Suppose \( \dim Y = d \).

**Lemma 2.2.** Under the condition of Theorem 1.3, if \( Y \) is smooth, then for every point \( y \in Y \), there is a smooth prime principal divisor \( Z \) passing through \( y \).

**Proof.** Since \( X \) is projective and smooth, there is a hypersurface \( H_1 \) defined by a homogeneous polynomial \( h_1 \) of degree at least 2 passing through \( y_1 \) and \( Z = H_1 \cap X \) is a prime principal divisor on \( X \).

Let \( H_2 \) be a distinct hypersurface defined by \( h_2 \) with the same degree such that \( h_2(y_1) \neq 0 \) and \( H_2 \cap X \) is a prime principal divisor on \( X \). Let \( h = \frac{h_1}{h_2} \), then \( h \) is a rational function on \( Y \) and regular on an open subset of \( Y \) containing \( y_1 \). By the following claim, there are two regular functions \( f \) and \( g \) on \( Y \) such that \( h = \frac{f}{g} \). So \( h_1 = 0 \) if and only if \( f = 0 \). Therefore the subvariety \( Z \) in \( Y \) is defined by \( f \).

If \( h^0(X, \mathcal{O}_X(mD)) > 0 \) for some \( m \in \mathbb{Z} \) and \( X \) is normal, choose a basis \( \{ f_0, f_1, \ldots, f_n \} \) of the linear space \( H^0(X, \mathcal{O}_X(mD)) \), it defines a rational map \( \Phi_{|mD|} \) from \( X \) to the projective space \( \mathbb{P}^N \) by sending a point \( x \) on \( X \) to \( (f_0(x), f_1(x), \ldots, f_N(x)) \) in \( \mathbb{P}^N \). By definition of \( D \)-dimension \([30], \text{Definition 5.1}\),

\[
\kappa(D, X) = \max_m \{ \dim(\Phi_{|mD|}(X)) \}.
\]
Let $\mathbb{C}(X)$ be the function field of $X$. Let

$$R(X, D) = \bigoplus_{\gamma=0}^{\infty} H^0(X, \mathcal{O}_X(\gamma D))$$

be the graded $\mathbb{C}$-domain and $R^* \subset R$ the multiplicative subset of all nonzero homogeneous elements. Then the quotient ring $R^{*-1}R$ is a graded $\mathbb{C}$-domain and its degree 0 part $(R^{*-1}R)_0$ is a field denoted by $Q((X, D))$, i.e.,

$$Q((X, D)) = (R^{*-1}R)_0.$$  

Let $X$ be normal proper over $\mathbb{C}$, then we have [19].

1. If there is an $m_0 > 0$ such that for all $m > m_0$, $h^0(X, \mathcal{O}_X(mD)) > 0$, then

$$\mathbb{C}(\Phi_{|mD|}(X)) = Q((X, D)).$$

2. If $\kappa(D, X) = \dim X$, then $\Phi_{|mD|}$ is birational for all $m \gg 0$. In particular, $\mathbb{C}(X) = Q((X, D))$.

So if $D$ is a big divisor, then any rational function on $X$ can be written as a quotient of two elements in $H^0(X, \mathcal{O}_X(mD))$ for sufficiently large $m$. These two elements are regular on $Y$.

In the proof, we used the generalized version of Bertini’s Theorem: if $X$ is a projective algebraic manifold of dimension at least 2, then for any point $x_0$ on $X$, there is an irreducible smooth hypersurface $H$ of degree at least 2 passing through $x_0$ such that $H$ intersects $X$ with an irreducible smooth codimension 1 subvariety of $X$. We give a proof here for completeness.

Let $X$ be a closed subset of $\mathbb{P}^n$, $n \geq 3$. We may assume that $X$ is not contained in any hyperplane and after coordinate transformation, the homogeneous coordinate of $x_0$ is $(1, 0, ..., 0)$.

Let $H$ be a hypersurface defined by a homogeneous polynomial $h$ of degree 2 passing through $x_0$, then

$$h = \sum_{j=1}^{n} a_{0j}x_0x_j + \sum_{i=1}^{n} \sum_{j \geq i} a_{ij}x_ix_j.$$  

$H$ is nonsingular at $x_0$ if at least one $a_{0j} \neq 0$. Let $V$ be the linear space of these hypersurfaces, then the dimension of $V$ is

$$\dim_{\mathbb{C}} V = \frac{(n+2)(n+1)}{2} - 1 = \frac{n^2 + 3n}{2}.$$  

By Euler’s formula, the hypersurface is singular at a point $x = (x_0, x_1, ..., x_n)$ if and only if

$$\frac{\partial h}{\partial x_0} = \frac{\partial h}{\partial x_1} = ... = \frac{\partial h}{\partial x_n} = 0.$$  

It is a system of linear equations

\begin{align*}
    a_{01}x_1 + a_{02}x_2 + \cdots + a_{0n}x_n &= 0 \\
    a_{01}x_0 + 2a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
    \cdots \cdots \cdots \\
    a_{0n}x_0 + a_{1n}x_1 + a_{2n}x_2 + \cdots + 2a_{nn}x_n &= 0
\end{align*}
The above system has a solution in $\mathbb{P}^n$ if and only if the determinant of the following symmetric matrix $A$ is zero,

$$
\begin{pmatrix}
0 & a_{01} & a_{02} & \cdots & a_{0n} \\
a_{01} & 2a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{0n} & a_{1n} & a_{2n} & \cdots & 2a_{nn}
\end{pmatrix}.
$$

Considering $(a_{01}, a_{02}, \ldots, a_{(n-1)n})$ as a point in the projective space $\mathbb{P}^{\frac{n^2+n}{2}-1}$, the system has solution only on the hypersurface defined by $\det A = 0$. So the degree 2 hypersurface $H$ in $V$ is nonsingular on an open subset of $\mathbb{P}^{\frac{n^2+n}{2}-1}$, i.e., a general member $H$ of $V$ is smooth.

Let $x$ be a closed point of $X$ and define $S_x$ to be the set of smooth hypersurfaces $H$ (defined by $h$) of degree 2 such that $x$ is a singular point of $H \cap X$. $H$ is ample and $X$ is not contained in $H$ by our assumption so $X \cap H$ is not empty. Fix a smooth irreducible hypersurface $H_0$ of degree 2 such that $x$ is not a point of $H_0$. Let $h_0$ be the defining homogeneous polynomial of $H_0$, then $h/h_0$ gives a regular function on $\mathbb{P}^n - H_0$. When restricted on $X$, it is a regular function on $X - X \cap H_0$.

Define a map $\xi_x$ from the linear space $V$ to $\mathcal{O}_{x,X}/\mathcal{M}_x^2$ as follows: for every element $h$ in $V$ (a homogeneous polynomial of degree 2 such that the corresponding hypersurface $H$ is smooth and passes through $x_0$), the image $\xi_x(h)$ is the image of $h/h_0$ in the local ring $\mathcal{O}_{x,X}$ modulo $\mathcal{M}_x^2$. It is easy to see that $x$ is a point of $H \cap X$ if and only if the image $\xi_x(h)$ of the defining polynomial $h$ of $H$ is contained in $\mathcal{M}_x$. And $x$ is singular on $H \cap X$ if and only if the image $\xi_x(h)$ is contained in $\mathcal{M}_x^2$ because the local ring $\mathcal{O}_{x}/\xi_x(h)$ will not be regular. So there is the following one-to-one correspondence

$$H \in S_x \iff h \in \ker \xi_x.$$

Since $x$ is a closed point and the ground field is $\mathbb{C}$, the maximal ideal $\mathcal{M}_x$ is generated by linear forms in the coordinates. The map $\xi_x$ is surjective if $x \neq x_0$. The map $\xi_{x_0}$ is not surjective to $\mathcal{O}_{x,X}/\mathcal{M}_x^2$ but surjective to $\mathcal{M}_{x_0}/\mathcal{M}_{x_0}^2$.

Let $d$ be the dimension of $X$, then the linear space $\mathcal{O}_{x,X}/\mathcal{M}_x^2$ has dimension $d+1$ over $\mathbb{C}$. Considering the map

$$\xi_x : V \longrightarrow \mathcal{O}_{x,X}/\mathcal{M}_x^2,$$

if $x \neq x_0$, the kernel has dimension

$$\dim \ker \xi_x = \frac{n(n+3)}{2} - d - 1.$$

Therefore the linear space $S_x$ is a linear system of hypersurfaces with dimension $\frac{n(n+3)}{2} - d - 2$ if $x \neq x_0$. If $x = x_0$, then the projective dimension of $S_{x_0}$ is $\frac{n(n+3)}{2} - d - 1$.

Considering the linear system $V$ as a projective space, then $X \times V$ is a projective variety. The subset $S \subset X \times V$ consists of all pairs $<x, H>$ such that $x \in X$ is a closed point and $H \in S_x$. 
$S$ is the set of closed points of a closed subset of $X \times V$ and we give a reduced induced scheme structure to $S$. The first projection $p_1 : S \to X$ is surjective. If $x \neq x_0$, the fiber $p^{-1}_1(x)$ is a projective space with dimension $\frac{n(n+3)}{2} - d - 2$. The special fiber $p^{-1}_1(x_0)$ is a projective space with dimension $\frac{n(n+3)}{2} - d - 1$. Hence $S$ has dimension

$$\left[ \frac{n(n+3)}{2} - d - 2 \right] + d = \frac{n(n+3)}{2} - 2.$$

Let $S = \cup_{i=0}^m S_i$ be an irreducible decomposition. Then every $p_1(S_i)$ is closed and there is an $i$, such that $p_1(S_i) = X$. For every $S_i$ with $p_1(S_i) = X$, there is an open subset $U_i \subset S_i$ such that for every $x \in U_i$, the fiber $p^{-1}_1(x)$ has constant dimension $n_i$. Let $x \in \cap U_i$, since the fiber $p^{-1}_1(x)$ is irreducible, it is contained in some $S_i$. Suppose $p^{-1}_1(x) \subseteq S_i$. Let $f_1$ be the restriction of $p_1$ on $S_1$, i.e., $p_1|_{S_1} = f_1$, then $p^{-1}_1(x) \subseteq f^{-1}_1(x)$ since $p^{-1}_1(x)$ is irreducible. The opposite inclusion is obvious, so $p^{-1}_1(x) = f^{-1}_1(x)$ for $x \in \cap U_i$ and $n_1 = \frac{n(n+3)}{2} - d - 2$.

Since $f_1$ is surjective and $S_1$ is one irreducible component of $S$, for every $x \in X$, the fiber $f^{-1}_1(x)$ is not empty and contained in $p^{-1}_1(x)$. But the dimension of $f^{-1}_1(x)$ is at least $\frac{n(n+3)}{2} - d - 2$, so for every $x \in X$, $p^{-1}_1(x) = f^{-1}_1(x)$. Thus $S_1 = S$ and $S$ is irreducible.

Looking at the second projection (a proper morphism) $p_2 : S \to V$. The dimension of the image

$$\dim p_2(S) \leq \dim S = \frac{n(n+3)}{2} - 2.$$

Since $S$ is closed in $X \times V$ and the dimension of $V$ (as a projective space) is $\frac{n(n+3)}{2} - 1$, $V - p_2(S)$ is an open subset of $V$. This implies that a general member $H$ of $V$ intersects $X$ with a smooth variety $X \cap H$.

Next we will prove that $X \cap H$ is irreducible.

From the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-H) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_H \longrightarrow 0,$$

since $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-H)) = 0$ ([7], Page 225, Theorem 5.1), we have a surjective map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \mathbb{C} \longrightarrow H^0(H, \mathcal{O}_H).$$

So $H^0(H, \mathcal{O}_H) = \mathbb{C}$ and the hypersurface $H$ is connected.

Since $X$ is closed in $\mathbb{P}^n$, $H|_X$ is ample on $X$. By Kodaira Vanishing Theorem, $H^1(X, \mathcal{O}_X(-H)) = 0$ ([15], Page 62). Applying the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{H \cap X} \longrightarrow 0,$$

we get

$$H^0(H \cap X, \mathcal{O}_{H \cap X}) = H^0(X, \mathcal{O}_X) = \mathbb{C}.$$

Thus the intersection $H \cap X$ is connected. Therefore for a general hypersurface $H$ of degree 2, $H \cap X$ is smooth and irreducible.

We have proved that a general smooth hypersurface of degree 2 passing through $x_0$ intersects $X$ with an irreducible smooth subvariety of codimension 1.
Lemma 2.3. Under the condition of Theorem 1.3, any prime principal divisor $Z = \{ f = 0, f \in H^0(Y, O_Y) \}$ satisfies the same condition, i.e., $Z$ contains no complete curves, $H^i(Z, O_Z) = 0$ for all $i > 0$ and $\kappa(D|_Z, \bar{Z}) = d - 1$, where $\bar{Z} \subset X$ is a closed subvariety of $X$ containing $Z$.

Proof. It is obvious that $Z$ contains no complete curves since $Z$ is a subvariety of $Y$.

From the short exact sequence

$$0 \rightarrow O_Y \rightarrow O_Y \rightarrow O_Z \rightarrow 0,$$

where the first map is defined by $f$, we have $H^i(Z, O_Z) = 0$ for all $i > 0$ by the corresponding long exact sequence and the assumption $H^i(Y, O_Y) = 0$ for all $i > 0$.

The regular function $f$ gives a rational map from $X$ to $\mathbb{P}^1$. It is a morphism when restricted to $Y$. Let $C' = f(Y)$, the image of $Y$ under the map $f$. By Hironaka’s elimination of indeterminacy, there is a smooth projective variety $X'$ such that the morphism $\sigma : X' \rightarrow X$ is a composite of finitely many monoidal transformations. $\sigma$ is an isomorphism when restricted to $Y$. So $Y$ is fixed and $g = f \circ \sigma : X' \rightarrow \mathbb{P}^1$ is proper, surjective and we have a commutative diagram

$$\begin{array}{ccc}
Y & \rightarrow^g & X' \\
\downarrow & & \downarrow \\
C' & \rightarrow & \mathbb{P}^1.
\end{array}$$

Notice that the $D$-dimension does not change under blowing up or blowing down: Let $\xi : V \rightarrow W$ be a surjective morphism of two varieties and let $D$ be a Cartier divisor on $W$, then we have (\cite{30}, Chapter 2, Theorem 5.13)

$$\kappa(\xi^*D, V) = \kappa(D, W).$$

In particular, in the above blowing up $\sigma : X' \rightarrow X$, let $E$ be an effective divisor on $X'$ such that $\text{codim} \sigma(E) \geq 2$, then

$$\kappa(\sigma^{-1}(D) + E, X') = \kappa(D, X),$$

where $\sigma^{-1}(D)$ is the reduced transform of $D$, defined to be $\sigma^{-1}(D) = \sum D_i$, $D_i$’s are the irreducible components of $D$.

The $D$-dimension also does not depend on the choice of coefficients if $D$ is an effective divisor with simple normal crossings. Let $D_1, D_2, \ldots, D_n$ be any divisor on $X$ such that for every $i, 1 \leq i \leq n$, $\kappa(D_i, X) \geq 0$, then for integers $p_1 > 0, \ldots, p_n > 0$, we have (\cite{10})

$$\kappa(D_1 + \cdots + D_n, X) = \kappa(p_1D_1 + \cdots + p_nD_n, X).$$

In particular, if $D_i$’s are irreducible components of $D$ and $D$ is effective, then we can change its coefficients to different positive integers but do not change the $D$-dimension.

A fiber space is defined to be a proper surjective morphism $f : V \rightarrow W$ between two varieties $V$ and $W$ such that the general fiber is connected. We cannot use
the above morphism in the commutative diagram from $X'$ to $\mathbb{P}^1$ to compute $D$-dimension because we do not know whether a general fiber is connected.

By Stein factorization, we can factor the map $g$ through

$$X' \xrightarrow{h} \bar{C} \xrightarrow{\pi} \mathbb{P}^1$$

where $g = \pi \circ h$, $\pi$ is a finite morphism, $h$ is proper, surjective and every fiber of $h$ in $X'$ is connected. Let $C = h(Y)$, then we have the following new commutative diagram

$$
\begin{array}{ccc}
Y & \hookrightarrow & X' \\
\downarrow h|Y & & \downarrow h \\
C & \hookrightarrow & \bar{C}.
\end{array}
$$

Now we have a fiber space such that every fiber in $X'$ is connected. Consider the image of $D$ under $h$. If $h(D)$ has dimension 0, then $Y$ contains complete curves, so $h(D) = \bar{C}$.

Since $Z = \{ f = 0, f \in H^0(Y, \mathcal{O}_Y) \}$ is irreducible and $g|_Y = f$, $\bar{Z}$ is an irreducible component of $g^{-1}(0)$ and $g^{-1}(0) \cap Y = \bar{Z} \cap Y = Z$ is irreducible.

By a theorem of Iitaka ([30], Chapter II, Theorem 5.11), for a general point $t$ on $C$, we have

$$d = \kappa(D, X') \leq \kappa(D_t, X'_t) + 1 \leq d,$$

where $X'_t = h^{-1}(t)$ is the irreducible fiber in $X'$ and $D_t$ is the restriction of $D$ on $X'_t$. Thus $\kappa(D_t, X'_t) = d - 1$. Let $t_0 \in C$ such that $\pi(t_0) = 0$ and $\bar{Z}$ is an irreducible component of $h^{-1}(t_0) = X_0$. Then $h^{-1}(t_0)$ is a connected component of $g^{-1}(0)$. By upper semi-continuity theorem, $\kappa(D_0, X_0') = d - 1$, where $D_0 = D|_{X_0'}$. By the properties of $D$-dimension, $\kappa(D|_Z, \bar{Z}) = \kappa(D_0, X_0') = d - 1$.

Q.E.D.

**Lemma 2.4.** Under the condition of Theorem 1.3, if $Y$ is smooth, then the regular functions on $Y$ separate points on $Y$.

*Proof.* For two distinct points $y_1$ and $y_2$ on $Y$, we need to find a regular function $R$ on $Y$ such that $R(y_1) \neq R(y_2)$. We will use induction on the dimension of $Y$. When $Y$ is a curve, the claim is true by Lemma 2.3. We may assume that the claim is true for $(d - 1)$-dimensional varieties. Consider the $d$-dimensional variety $Y$.

By Lemma 2.2, there is a smooth prime principle divisor $Z$ passing through $y_1$. Let $f \in H^0(Y, \mathcal{O}_Y)$ be the defining function of $Z$. If $f(y_2) \neq 0$, we are done. Assume $f(y_2) = 0$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where the first map is defined by $f$, we have a surjective map from $H^0(Y, \mathcal{O}_Y)$ to $H^0(Z, \mathcal{O}_Z)$ since $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

By Lemma 2.3 and the inductive assumption, $Z$ is affine. Therefore there is a regular function $r$ on $Z$ such that $r(y_1) \neq r(y_2)$. Lift this function from $Z$ to $Y$, we get a global regular function $R$ on $Y$ such that it separates $y_1$ and $y_2$. 
Lemma 2.4 only holds for smooth varieties. The reason is that if \( y_0 \) is a singular point on \( Y \), we cannot expect that there exists a prime principal divisor passing through \( y_0 \). For example, let \( A = \mathbb{C}[x, y, z]/(xy - z^2) \) and \( Y = \text{Spec} A \). However, to prove that \( Y \) is an affine variety, the following lemma is sufficient.

Lemma 2.5. Under the condition of Theorem 1.3, if \( Y \) is not smooth but normal, then for any irreducible curve \( F \) on \( Y \), there is a regular function \( f \) on \( Y \) such that the restriction \( f|_F \) is not a constant.

Proof. Let \( \bar{F} \) be an irreducible complete curve on \( X \) such that \( F \) is an open subset of \( \bar{F} \). Then the complement \( F^c = \bar{F} - F \) is a set of finitely many points on the boundary \( X - Y \). An old theorem of Seidenberg says that a general hyperplane section \( H \) of \( X \) is normal and irreducible \(^{25}\). We may choose \( H \) such that all points of \( F^c \) are not contained in \( H \). Since \( H \) is ample and \( \bar{F} \) is complete, \( H \cap \bar{F} = H \cap F \neq \emptyset \). If \( F \) is contained in \( H \), then \( \bar{F} \) is a curve on \( H \) since \( \bar{F} \) is connected. So \( F \) is not contained in \( H \), i.e., there is a point \( p \) in \( F \) such that \( p \) is not a point of \( H \).

Let \( h \) be the defining linear form of \( H \). Let \( h' \) be a different linear form (linearly independent with \( h \)) such that the hyperplane \( H' \) defined by \( h' \) does not contain the point \( p \). Then \( h/h' \) gives a rational function on \( X \) and defines \( H \cap X \).

Since \( D \) is a big divisor, by \(^{19}\), there are two regular functions \( f \) and \( g \) on \( Y \) such that \( h/h' = f/g \). So the irreducible open subvariety \( Z = H \cap Y \) is defined by \( f \). Since \( F \) is not a curve on \( Z \), the restriction function \( f|_F \) is not a constant.

Q.E.D.

Lemma 2.6. \( Y \) is a quasi-affine variety under the assumption of Theorem 1.3.

Proof. By \(^{4}\), there is a proper morphism \( \xi : Y \to U \) to a quasi-affine variety \( U \) since for any irreducible curve \( F \) on \( Y \), there is a regular function \( f \) on \( Y \) such that the restriction \( f|_Y \) is not a constant. We know that \( Y \) has no complete curves, so the fiber of the map \( \xi \) is of 0 dimensional and finite. Therefore \( \xi \) is a quasi-finite morphism. Zariski’s Main Theorem \(^{20}\), Chapter III, Section 9) says that if \( \xi : Y \to U \) is a morphism of varieties with finite fibers, then the map \( \xi \) can be factored through an embedding \( i \) from \( Y \) to a variety \( Y' \) followed by a finite morphism \( \psi : Y' \to U \). Hence \( Y \) is a quasi-affine variety since \( Y' \) is quasi-affine.

Q.E.D.

Proof of Theorem 1.4. In 1988, Neeman proved a very nice local criterion for affineness: Let \( V = \text{Spec} A \) be a scheme and \( U \subset V \) a quasi-compact Zariski open subset. Here we don’t assume that \( A \) is noetherian \(^{23}\). Then \( U \) is affine if and only if \( H^i(U, \mathcal{O}_U) = 0 \) for all \( i \geq 1 \).

Combining with Theorem 1.3, we immediately see that \( Y \) is affine.

Q.E.D.

Proof of Theorem 1.5. Let \( X \) be an irreducible projective variety containing \( Y \). Let \( V \) be an irreducible projective variety containing \( W \). Then we have a
rational map \( g \) from \( X \) to \( V \) such that \( g|_Y = f \). By Hironaka’s elimination of indeterminancy, we may assume that \( g \) is a proper surjective morphism. Let \( \pi_V : V' \to V \) be the blow up of closed subset of \( V \) such that \( V' \) is smooth. Let \( \pi_X : X' \to X \) be the resolution of the singularities of \( X \) such that we have the following commutative diagram

\[
\begin{array}{ccc}
Y & \hookrightarrow & X \\
\downarrow g|_Y = f & \downarrow g & \downarrow h \\
W & \hookrightarrow & V \\
\end{array}
\]

where \( f \) is a surjective morphism, \( g \) and \( h \) are surjective proper morphisms.

Suppose that the dimension of \( Y \) is \( d \) and the dimension of \( W \) is \( m \). Choose suitable projective variety \( V \) such that the boundary \( V - W \) is support of an ample divisor \( A \). Let \( D \) be an effective divisor on \( X \) with support \( X - Y \). Then we have (\cite{fulton}, Chapter II, Theorem 5.13)

\[
\kappa(\pi^*_V A, V') = \kappa(A, V) = m
\]

and

\[
\kappa(\pi^*_X D, X') = \kappa(D, X).
\]

If we can prove that \( Y \) has no complete curves and \( \kappa(D, X) = d \), then \( Y \) is affine by Theorem 1.4. The first property is obvious since every fiber of \( f \) in \( Y \) is affine.

To compute the \( D \)-dimension of \( X \), we need a theorem of Fujita: Let \( M \) and \( S \) be two projective manifolds. Let \( \pi : M \to S \) be a fiber space and let \( L \) and \( H \) be line bundles on \( M \) and \( S \) respectively. Suppose that \( \kappa(H, S) = \dim S \) and that \( \kappa(aL - b\pi^*(H)) \geq 0 \) for certain positive integers \( a, b \). Then

\[
\kappa(L, M) = \kappa(L|_F, F) + \kappa(H, S)
\]

for a general fiber \( F \) of \( \pi \). Here if \( L \) is a line bundle on a projective manifold \( M \), it determines a Cartier divisor \( D \). We define

\[
\kappa(L, M) = \kappa(D, M).
\]

We don’t know whether the fiber of \( h \) is connected. Let

\[
X' \xrightarrow{j} V'' \xrightarrow{\alpha} V'
\]

be the Stein factorization, then \( h = \alpha \circ j \), \( \alpha \) is a finite morphism and \( j \) has connected fibers. And a general fiber of \( j \) is smooth and irreducible.

Since the support of \( D \) is \( X - Y \), the image \( \pi_V \circ h(\pi_X^* D) = V - W \) is support of \( A \). So the support of pull back divisor \( h^*(\pi_V^*(A)) \) is contained in the support of \( \pi_X^*(D) \).

Since the dimension of \( V'' \) is \( m \), we have (\cite{fulton}, Chapter II, Theorem 5.13)

\[
\kappa(\alpha^*(\pi_V^*(A)), V'') = \kappa(A, V) = m.
\]

Let \( H = \alpha^*(\pi_V^*(A)) \), then \( \kappa(H, V'') = m = \dim V'' \). Let \( L = \pi_X^* D \), then for sufficiently large \( n \), we have

\[
\kappa(nL - j^*H) \geq 0.
\]
This is because the exceptional divisors do not change the Iitaka dimension and the support of $g^*(A)$ is contained in $X - Y$, the support of $D$. By Fujita’s formula,
\[
\kappa(L, X') = \kappa(nL, X') = \kappa(nL|_F, F) + m,
\]
where $F$ is a general fiber of $j$. A general fiber of $g$ has dimension $d - m$. Since every fiber of $f$ is affine and $\pi_X(F)$ is a fiber of $g$, we have
\[
\kappa(nL|_F, F) = \kappa(nD|_{\pi_X(F)}, \pi_X(F)) = d - m.
\]
Therefore
\[
\kappa(D, X) = \kappa(L, X') = \kappa(nL|_F, F) + m = d.
\]
Hence $Y$ is affine by Theorem 1.4.

Q.E.D.

**Proof of Corollary 1.6.** By the same calculation as in the proof of Theorem 1.5, we have $\kappa(D, X) = d$, the dimension of $Y$. The conclusion is obvious by Theorem 1.4.

Q.E.D.

3. **Examples**

Again $Y$ is a variety contained in a projective variety $X$ such that $Y = X - D$, where $D$ is an effective boundary divisor with support $X - Y$.

**Example 3.1.** There is an affine surface $Y$ such that the graded ring $\oplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD))$ is not finitely generated. This example is due to Zariski ([31], page 562-564).

Let $C$ be a smooth curve of degree 3 in $\mathbb{P}^2$. Let $\Lambda$ be a divisor class cut out on $C$ by a curve of degree 4 in $\mathbb{P}^2$. There exist 12 distinct points $p_1, p_2, \cdots, p_{12}$ on $C$ such that
\[
m(p_1 + p_2 + \cdots + p_{12}) \notin m\Lambda
\]
for all positive integers $m$. Let $X$ be a surface obtained by blowing up $\mathbb{P}^2$ at these 12 points $p_1, p_2, \cdots, p_{12}$. Let $\bar{C}$ be the strict transform of $C$ (i.e., the closure of the inverse image of $C - \{p_1, p_2, \cdots, p_{12}\}$ in $X$). Let $L$ be a line not passing through any point $p_i$ in these 12 points. Let $\bar{L}$ be the strict transform of $L$. Then the complete linear system
\[
|m(\bar{C} + \bar{L})|
\]
has a fixed locus $C$ for all $m \geq 1$ and
\[
|m\bar{C} + (m - 1)\bar{L}|
\]
has no fixed components and is base point free. By Nakai-Moishezon’s ampleness criterion ([7], Chapter V, Section 1), the divisor
\[
m\bar{C} + (m - 1)\bar{L}
\]
is ample. Hence the complement $Y = X - (m\bar{C} + (m - 1)\bar{L})$ is affine but the graded ring
\[
R = \oplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(\bar{C} + \bar{L})))
\]
is not finitely generated.
Example 3.2. There is a nonaffine surface $Y$ such that the graded ring
\[ \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)) \]
is finitely generated [16].

Let $C$ be an elliptic curve and $E$ the unique nonsplit extension of $\mathcal{O}_C$ by itself. Let $X = \mathbb{P}_C(E)$ and $D$ be the canonical section, then $Y = X - D$ is not affine and $H^0(X, \mathcal{O}_X(nD)) = \mathbb{C}$ [16]. So
\[ \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)) \]
is finitely generated.

The above two examples demonstrate that the affineness of $Y$ and the finitely generated property of the graded ring
\[ \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)) \]
are different in nature. The reason is
\[ \Gamma(Y, \mathcal{O}_Y) \neq \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)). \]

In fact, we have [4]

Lemma 3.3. [Goodman, Hartshorne] Let $V$ be a scheme and $D$ be an effective Cartier divisor on $V$. Let $U = V - \text{Supp}D$ and $F$ be any coherent sheaf on $V$, then for every $i \geq 0$,
\[ \lim_n H^i(V, F \otimes \mathcal{O}(nD)) \cong H^i(U, F|_U). \]

So we have
\[ \Gamma(Y, \mathcal{O}_Y) \cong \lim_n H^0(X, \mathcal{O}_X(nD)). \]

The direct limit is the quotient of the direct sum and its subring, so it is much “smaller” than direct sum ([17], Chapter II, Section 10). And even though $Y$ is affine, the boundary divisor can be very bad. For example, $D$ may not be nef. It is easy to see this by blowing up $\mathbb{P}^2$ at a point. Let $L$ be a line in $\mathbb{P}^2$, let $O$ be a point on $L$. Let $\pi : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $O$. Let $E$ be the exceptional divisor and $D = \pi^{-1}(L) + mE$, where $\pi^{-1}(L)$ is the strict transform of $L$ and $m$ is a large positive integer. Then $D \cdot E = 1 - m < 0$ ([7], Chapter V, Corollary 3.7). Therefore $D$ is not nef.

Example 3.4. There exists a threefold $Y$ such that $Y$ contains no complete curves, $H^i(Y, \Omega_Y^j) = 0$ for all $i > 0$ and $j \geq 0$ but is not affine [34]. Let $C_t$ be a smooth projective elliptic curve defined by $y^2 = x(x - 1)(x - t)$, $t \neq 0, 1$. Let $Z$ be the elliptic surface defined by the same equation, then we have surjective morphism from $Z$ to $C = \mathbb{C} - \{0, 1\}$ such that for every $t \in C$, the fiber $f^{-1}(t) = C_t$. In [34], we proved that there is a rank 2 vector bundle $E$ on $Z$ such that when restricted to $C_t$, $E|_{C_t} = E_t$ is the unique nonsplit extension of $\mathcal{O}_{C_t}$ by $\mathcal{O}_{C_t}$, where $f$ is the morphism from $Z$ to $C$. We also proved that there is a divisor $D$ on $X = \mathbb{P}_Z(E)$ such that when restricted to $X_t = \mathbb{P}_{C_t}(E_t)$, $D|_{X_t} = D_t$ is the canonical section of $X_t$. Let $Y = X - D$, we have $H^i(Y, \Omega_Y^j) = 0$ for all $i > 0$ and
\[ j \geq 0 \quad [31] \]. We know that this threefold \( Y \) contains no complete curves \[33\] and \( \kappa(D, X) = 1 \). So \( Y \) is not affine.

**Example 3.5.** There is a surface \( Y \) without complete curves such that \( \kappa(D, X) = 2 \) but is not affine.

Remove a line \( L \) from \( \mathbb{P}^2 \), we have \( \mathbb{C}^2 = \mathbb{P}^2 - L \). Remove the origin \( O \) from the complex plane \( \mathbb{C}^2 \), let \( Y = \mathbb{C}^2 - \{O\} \). Then \( Y \) is not affine since the boundary is not connected \((8, \text{Chapter II, Section 3 and Section 6})\). Blow up \( \mathbb{P}^2 \) with center \( O \), let \( E \) be the exceptional divisor and \( \pi : X \to \mathbb{P}^2 \) be the blowup. Let \( D = \pi^{-1}(L) + E \), where \( \pi^{-1}(L) \) is the strict transform of \( L \). Then by Iitaka’s result, on \( X \), \( \kappa(D, X) = 2 \) and \( X - D \cong Y \) has no complete curves. But \( Y \) is not affine.

**Lemma 3.6.** \([\text{Kollár}]\) Let \( \pi : X \to Z \) be a surjective map between projective varieties, \( X \) smooth, \( Z \) normal. Let \( F \) be the geometric generic fiber of \( \pi \) and assume that \( F \) is connected. The following two statements are equivalent:

(i) \( R^i \pi_\ast \mathcal{O}_X = 0 \) for all \( i > 0 \);

(ii) \( Z \) has rational singularities and \( H^i(F, \mathcal{O}_F) = 0 \) for all \( i > 0 \).

**Example 3.7.** There is a smooth variety \( Y \) of dimension \( d \geq 1 \) with \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \) and \( \kappa(D, X) = d \) but is not affine.

Let \( X \) be the smooth projective variety obtained by blowing up a point \( O \) in \( \mathbb{P}^d \). Let \( \pi : X \to \mathbb{P}^d \) be the blowup. Let \( H \) be a hyperplane not passing through \( O \). Let \( D = \pi^{-1}(H) \), the strict transform of \( H \) and \( Y = X - D \). Then \( \kappa(D, X) = d \) \((30, \text{Chapter 2, Theorem 5.13})\). Let \( E \) be the exceptional divisor on \( X \), then \( E \cong \mathbb{P}^{d-1} \). So \( H^i(E, \mathcal{O}_E) = 0 \) for all \( i > 0 \). By Lemma 3.6, we have \( R^i \pi_\ast \mathcal{O}_X = 0 \) for all \( i > 0 \).

We can get the vanishing of direct images by Grauert’s upper semi-continuity theorem. For every point \( p \in \mathbb{P}^d \), \( p \neq O \), then \( \pi^{-1}(p) = q \) is a point in \( X \). So for every \( i > 0 \),

\[ h^i(q, \mathcal{O}_q) = 0. \]

We already saw

\[ h^i(\pi^{-1}(O), \mathcal{O}_{\pi^{-1}(O)}) = 0. \]

So \( R^i \pi_\ast \mathcal{O}_X = 0 \) for all \( i > 0 \) \((30, \text{Chapter 1, Theorem 1.4})\).

Let \( U = \mathbb{P}^d - H \), then \( U \cong \mathbb{C}^d \). For the global sections on affine space \( U \), we have \((7, \text{Chapter III, Proposition 8.1, 8.5 and Chapter II, Proposition 5.1(d)})\)

\[ 0 = R^i \pi_\ast \mathcal{O}_X(U) = H^i(Y, \mathcal{O}_Y) \]

for all \( i > 0 \).

It is obvious that \( Y \) is not affine since it contains a projective space \( \mathbb{P}^{d-1} \).

**Remark 3.8.** Examples 3.5-3.7 show that \( Y \) is not affine if we drop any condition from three conditions in Theorem 1.4.

**Example 3.9.** There is a threefold \( Y \) such that it satisfies the followings three conditions but is not affine:

1. \( Y \) contains no complete curves;
2. the boundary \( X - Y \) is connected;
3. \( \kappa(D, X) = 3 \).
Let $H$ be a hyperplane in $\mathbb{P}^3$. Let $L$ be a line not contained in $H$. Blow up $\mathbb{P}^3$ along $L$, let $\pi: X \to \mathbb{P}^3$ be the blowup. Define a divisor $D$ on $X$ such that $D = \pi^{-1}(H) + E$, where $E$ is the exceptional divisor. Let $Y = X - D$, then $Y = \mathbb{P}^3 - H - L$.

It is easy to see that the above three conditions are satisfied but $Y$ is not affine.

Remark 3.10. Theorem 1.2 is false for higher dimensional varieties by the above example.

Example 3.11. Let $(x, y)$ be the coordinates of $\mathbb{C}^2$. Define a projection
\[
\pi: \mathbb{C}^2 - \{(0, 0)\} \to \mathbb{C}
\]
\[
(x, y) \mapsto x.
\]
Then $\pi$ is a surjective morphism and every fiber is affine. Let $Y = \mathbb{C}^2 - \{(0, 0)\}$, then the boundary of $Y$ in $\mathbb{P}^2$ is not connected. Therefore $\mathbb{C}^2 - \{0\}$ is not affine (\cite{8}, Page 67). By Neeman’s theorem \cite{23}, we know $H^i(Y, \mathcal{O}_Y) \neq 0$. Thus if we drop the cohomology assumption, then Theorem 1.5 does not hold.

Example 3.12. There is a quasi-projective variety $Y$ with a surjective morphism $f: Y \to U$ such that $U$ is affine, a general fiber is affine and $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ but $Y$ is not affine.

Let $Y, U$ be the varieties defined in Example 3.7. Then the fiber space $\pi: Y \to U$ satisfied the above requirements. $Y$ is not affine because it has a projective space $\mathbb{P}^{d-1}$.

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