LIPS AND SWALLOW-TAILS OF SINGULARITIES OF PRODUCT MAPS

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Abstract. Lips and swallow-tails are generic local moves of singularities of a smooth map to a 2-manifold. We prove that these moves of singularities of the product map of two functions on a 3-manifold can be realized by isotopies of the functions.

1. Introduction

We consider the relationship between a pair of maps and the product map. Let $M, P, Q$ be smooth manifolds and let $C^\infty(M, *)$ denote the space of smooth maps from $M$ to a smooth manifold $*$ endowed with the Whitney $C^\infty$ topology. Two smooth maps $F \in C^\infty(M, P)$ and $G \in C^\infty(M, Q)$ determine the product map $(F, G) \in C^\infty(M, P \times Q)$ by $(F, G)(p) = (F(p), G(p))$. Conversely, a smooth map $\varphi \in C^\infty(M, P \times Q)$ can be decomposed into $\pi_P \circ \varphi \in C^\infty(M, P)$ and $\pi_Q \circ \varphi \in C^\infty(M, Q)$, where $\pi_P : P \times Q \to P$ and $\pi_Q : P \times Q \to Q$ are the projections. By [1, Chapter II, Proposition 3.6], this correspondence gives the homeomorphism $C^\infty(M, P) \times C^\infty(M, Q) \cong C^\infty(M, P \times Q)$.

The homeomorphism however does not mean the singularity theoretic equivalence. More specifically, isotopies of $F$ and $G$ do not always induce an isotopy of $(F, G)$, and an isotopy of $\varphi$ does not always induce isotopies of $\pi_P \circ \varphi$ and $\pi_Q \circ \varphi$. While one of the general interests in singularity theory is to understand the partition of a mapping space by isotopy classes of smooth maps, few things are known about the relation between the partitions of the both sides of the homeomorphism.

We focus on the case where $M$ is closed and 3-dimensional, and $P, Q$ are 1-dimensional. Suppose $F : M \to P$ and $G : M \to Q$ are smooth functions such that $\varphi = (F, G)$ is stable. A singular point of $\varphi$ is then either a fold point or a cusp point. By Levine’s [5] theorem, we can eliminate the cusp points by a homotopy of $\varphi$. It implies that we can eliminate the cusp points by homotopies of $F$ and $G$. Note that we cannot reduce the number of cusp points by an isotopy of $\varphi$. We propose the following question.

**Question 1.** Can we eliminate the cusp points of $\varphi$ by (quasi-)isotopies of $F$ and $G$?

Our strategy to attack Question 1 is to deform $\varphi$ by a sequence of global isotopies and local homotopies which can be realized by (quasi-)isotopies of $F, G$. Johnson [3, Section 6] showed what kind of global isotopy of $\varphi$ can be realized by (quasi-)isotopies of $F, G$ (see Corollary 7). In this paper, we prove the following.

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Theorem 2. The local moves of the discriminant set of \( \varphi \) as in Figure 1 can be realized by isotopies of \( F \) and \( G \).

Figure 1. Local moves which reduce the number of cusp points of \( \varphi \). Here, \((f, g)\) is the coordinate system of \( P \times Q \) with respect to which \( \varphi = (F, G) \) is defined. We require that the local moves do not involve tangent lines of the discriminant set parallel to the axises.

Forgetting the axises, these moves are ones of generic local moves of singularities of \( \varphi \) known as a “lip” and a “swallow-tail”, respectively. See [11] for the classification of generic local moves. We expect that we can use our method (Proposition 11) to work out other local moves, and we hope that we can use them to approach a global theory.

We would like to mention here the relation of this work to Heegaard theory of 3-manifolds. Rubinstein–Scharlemann [13] introduced the graphic for comparing two Heegaard splittings. Kobayashi–Saeki [4] interpreted the graphic as the discriminant set of the product map of two functions representing the splittings. Johnson [3] gave an upper bound for the Reidemeister–Singer distance between two Heegaard splittings in terms of the graphic. The author [14] developed Johnson’s idea to show that the Reidemeister–Singer distance is at most the sum of the genera of the splittings plus the number of cusp points of the product map. If Question 1 is answered positively, it ensures that the Reidemeister–Singer distance is at most the sum of the genera, which is the best possible bound by Hass–Thompson–Thurston’s example.

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2. Morse functions and stable maps

In this section, we briefly review standard definitions and facts on singularities of smooth maps. We refer the reader to [10] for basic notions in Morse theory, and to [1] for detailed description of stable maps.
A Morse function on a compact smooth manifold \( M \) possibly with boundary is a smooth function \( F \) from \( M \) to a smooth 1-dimensional manifold \( P \), namely \( \mathbb{R} \) or \( S^1 \), satisfying the following:

- All the critical points of \( F \) are non-degenerate and belong to \( \text{int} M \).
- The function \( F \) is constant on each component of \( \partial M \).

Note that each component of \( \partial M \) is either locally minimal or locally maximal with respect to \( F \).

A quasi-isotopy is a smooth homotopy consisting of Morse functions. In the case where \( M = \Sigma \) is a connected surface and \( P = \mathbb{R} \), Maksymenko [2] showed the following.

**Theorem 3** (Maksymenko). Two Morse functions on \( \Sigma \) to \( \mathbb{R} \) are quasi-isotopic if and only if they have the same number of critical points at each index, and the same sets of locally minimal and locally maximal components of \( \partial \Sigma \).

Note that a quasi-isotopy \( \{ \alpha_r : \Sigma \to \mathbb{R} \}_{r \in [0,1]} \) has the local form \( \alpha_\lambda(\xi, \eta) = \xi^2 \pm \eta^2 \) at each critical point of \( \alpha_r \), where \( \{ (\xi, \eta) \mapsto (\xi\lambda, \eta\lambda) \}_\lambda \) is a smooth family of coordinate transformations. A birth (resp. death) is a singularity of a homotopy with the local form \( \alpha_\lambda(\xi, \eta) = \lambda\xi^2 - \xi^4 + \eta^2 \), where the direction of the local coordinate \( \lambda \) agrees (resp. disagrees) with that of \( r \). A homotopy \( \{ \alpha_r : \Sigma \to \mathbb{R} \}_{r \in [0,1]} \) is said to be generic if it consists of Morse functions whose critical points have pairwise distinct values except that at each \( r \) in a finite subset of \( [0,1] \), \( \{ \alpha_r \}_{r \in [0,1]} \) has a single birth, a single death or a single transverse passing of critical values.

A stable map between smooth manifolds \( M, N \) is a smooth map \( \varphi : M \to N \) such that there exists an open neighborhood \( U \) of \( \varphi \) in \( C^\infty (M, N) \) such that every map in \( U \) can be obtained by an isotopy of \( \varphi \). An isotopy of \( \varphi = \varphi_0 \) is a homotopy \( \{ \varphi_r \}_{r \in [0,1]} \) which is decomposed as \( \varphi_r = H^N_r \circ \varphi_0 \circ H^M_r \), where \( \{ H^M_r \}_{r \in [0,1]} \}, \{ H^N_r \}_{r \in [0,1]} \) are smooth ambient isotopies of \( M, N \), respectively. In the case where \( M \) is closed and \( N \) is 1-dimensional, the smooth map \( \varphi \) is stable if and only if \( \varphi \) is a Morse function whose critical points have pairwise distinct values.

Consider the case where \( M \) is a closed smooth 3-manifold and \( N \) is a smooth surface. Recall that \( p \in M \) is a regular point of \( \varphi \) if the differential \( (d\varphi)_p : T_p M \to T_{\varphi(p)} N \) is surjective, and otherwise a singular point. The set \( S_\varphi \) of singular points of \( \varphi \) is called the singular set and its image \( \varphi(S_\varphi) \) is called the discriminant set of \( \varphi \). At a regular point \( p \in M \setminus S_\varphi \), the map \( \varphi \) has the standard form \( \varphi(u, x, y) = (u, x) \) for some coordinate neighborhoods of \( p \) and \( \varphi(p) \). Standard forms are also known for generic types of singular points as follows.

A fold point is a singular point \( p \) where \( \varphi \) has the form \( \varphi(u, x, y) = (u, x^2 \pm y^2) \) for a coordinate neighborhood \( U \) of \( p = (0, 0, 0) \) and a coordinate neighborhood of \( \varphi(p) = (0, 0) \). The Jacobian matrix of \( \varphi(u, x, y) = (u, x^2 \pm y^2) \) says that the singular set \( S_\varphi \cap U \) is the arc \( \{(u, 0, 0)\} \). It follows that each singular point on \( \{(u, 0, 0)\} \) is also a fold point by a translation of the local coordinates. The arc \( \{(u, 0, 0)\} \) is embedded to the arc \( \{(u, 0)\} \subset N \) by \( \varphi \).

A cusp point is a singular point \( p \) where \( \varphi \) has the local form \( \varphi(u, x, y) = (u, ux - x^3 + y^2) \). One can check that the singular set \( S_\varphi \cap U \) is the arc \( \{(3x^2, x, 0)\} \), and consists of fold points except for the cusp point \( p = (0, 0, 0) \). Note that the arc \( \{(3x^2, x, 0)\} \) is a regular curve but its image \( \{(3x^2, 2x^3)\} \subset N \) has a cusp at \( \varphi(p) = (0, 0) \).
Assume that the singular set $S_\varphi$ consists only of fold points and cusp points. By the above local observations and the compactness of $M$, we can see the outline of $S_\varphi$. It is a 1-dimensional submanifold of $M$, namely a collection of smooth circles. There are finitely many cusp points and the restriction $\varphi|_{S_\varphi}$ is an immersion except that each cusp point maps to a cusp. The next characterization of stable maps follows from Mather’s theorems [8, Theorem A, Proposition 1.8] and [9, Theorem 4.1].

**Theorem 4** (Mather). A smooth map $\varphi$ from a closed smooth 3-manifold to a smooth surface is stable if and only if:

- The singular set $S_\varphi$ consists only of fold points and cusp points.
- The restriction $\varphi|_{S_\varphi}$ has no double points at cusps, and the immersion $\varphi|_{S_\varphi \setminus \text{cusp points}}$ has normal crossings.

The Stein factorization $W_\varphi$ of a general smooth map $\varphi : M \to N$ is the quotient space $M/\sim$, where $p_1 \sim p_2$ if $p_1, p_2$ belong to the same connected component of a level set of $\varphi$. Let $q_\varphi$ denote the quotient map from $M$ to $W_\varphi$. We can see that there is also a unique continuous map $\bar{\varphi} : W_\varphi \to N$ such that $\varphi = \bar{\varphi} \circ q_\varphi$. The Stein factorization of a stable map from a closed smooth 3-manifold to a smooth surface is, in fact, a 2-dimensional cell complex. See [6] for example.

3. TWO FUNCTIONS AND THE PRODUCT MAP

In this section, we review a local theory of singularities of two functions and the product map.

We use the following notation. Suppose $M$ is a closed smooth 3-manifold and $P, Q$ are either $\mathbb{R}$ or $S^1$. Let $F : M \to P$ and $G : M \to Q$ be smooth functions, and $\varphi$ denote the product map $(F, G)$. While we do not assume $F, G$ to be Morse, we assume $\varphi$ to be stable in this section.

The singular set $S_\varphi$ includes the critical points of $F$ and $G$, which can be seen as follows. Note that there is a global coordinate system $(f, g)$ of $P \times Q$ with respect to which $(F, G)$ is defined as $(F, G)(p) = (F(p), G(p))$. The Jacobian matrix of $\varphi$ with respect to the coordinate system is composed of the gradients of $F$ and $G$. If $p$ is a critical point of $F$ or $G$, the Jacobian matrix has rank at most one, namely $p$ is a singular point of $\varphi$.

We read information about $F, G$ from the discriminant set of $\varphi$. Note that $\varphi|_{S_\varphi}$ is an immersion of circles with finitely many cusps. We can define the slope of the discriminant set $\varphi(S_\varphi)$ at $\varphi(p)$ for each $p \in S_\varphi$ with respect to the coordinate system $(f, g)$. In particular, a point on the discriminant set with slope zero (resp. infinity) is called a horizontal (resp. vertical) point. We can also define the second derivative of $\varphi(S_\varphi)$ outside of vertical points and cusps. In particular, a point with second derivative zero is called an inflection point. Since zero or non-zero of the second derivative is preserved by rotating the coordinate system, inflection can be defined also for vertical points.

**Lemma 5.** A point $p \in M$ is a critical point of $F$ if and only if $\varphi(p)$ is a vertical point of the image of a small neighborhood of $p$ in $S_\varphi$. The same holds for $G$ by replacing “vertical” to “horizontal”.

**Lemma 6.** A critical point $p$ of $F$ degenerates if and only if $p$ is a fold point of $\varphi$ and $\varphi(p)$ is a vertical inflection point of the image of a small neighborhood of $p$ in $S_\varphi$. The same holds for $G$ by replacing “vertical” to “horizontal”.
We include the proofs of these lemmas for completeness though those in the case of $P = Q = \mathbb{R}$ have been given in [3, Lemmas 10 and 11] and [14, Lemmas 11 and 12], and the proofs are independent of whether $\mathbb{R}$ or $S^1$.

**Proof of Lemma 3** There are a coordinate system $(u, x, y)$ of a small neighborhood $U$ of $p = (0, 0, 0)$ and a local coordinate system $(s, t)$ at $\varphi(p) = (0, 0)$ such that either $(s, t) = (u, x^2 + y^2)$ if $p$ is a fold point, or $(s, t) = (u, y^2 + ux - x^3)$ if $p$ is a cusp point. On the other hand, the global coordinate system $(f, g)$ of $P \times Q$ satisfies $(f, g) = (F(u, x, y), G(u, x, y))$. There exists a smooth regular coordinate transformation from $(s, t)$ to $(f, g)$.

**Fold Case.** If $p$ is a fold point of $\varphi$, the form $(s, t) = (u, x^2 + y^2)$ and the chain rule give

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial s} \quad \frac{\partial F}{\partial x} = 2x \frac{\partial f}{\partial t} \quad \frac{\partial F}{\partial y} = \pm 2y \frac{\partial f}{\partial t}.$$  

Substituting $(u, x, y) = (0, 0, 0)$, the gradient vector of $F$ at $p$ is $((\frac{\partial f}{\partial s})_{\varphi(p)}, 0, 0)$. The point $p$ is a critical point of $F$ if and only if the coordinate transformation satisfies $(\frac{\partial f}{\partial s})_{\varphi(p)} = 0$. It means that the $s$-axis is parallel to the $g$-axis at $\varphi(p)$. Recall that the singular set $S_\varphi \cap U$ is embedded to the $s$-axis by $\varphi$. The image $\varphi(S_\varphi \cap U)$ is therefore vertical at $\varphi(p)$.

**Cusp Case.** If $p$ is a cusp point of $\varphi$, the form $(s, t) = (u, y^2 + ux - x^3)$ and the chain rule give

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial s} + x \frac{\partial f}{\partial t} \quad \frac{\partial F}{\partial x} = (u - 3x^2) \frac{\partial f}{\partial t} \quad \frac{\partial F}{\partial y} = 2y \frac{\partial f}{\partial t}.$$  

The gradient vector at $p$ is $((\frac{\partial f}{\partial s})_{\varphi(p)}, 0, 0)$. The point $p$ is a critical point if and only if $(\frac{\partial f}{\partial s})_{\varphi(p)} = 0$. It means that the $s$-axis is parallel to the $g$-axis at $\varphi(p)$. Note that the $s$-axis is the tangent line of $\varphi(S_\varphi \cap U) = \{(s, t) = (3x^2, 2x^3)\}$ at the cusp $\varphi(p)$. The image $\varphi(S_\varphi \cap U)$ is therefore vertical at $\varphi(p)$.

□

**Proof of Lemma 4** We continue with the notation in the previous proof. Suppose $p$ is a critical point of $F$, and hence $(\frac{\partial f}{\partial s})_{\varphi(p)} = 0$. The regularity of the coordinate transformation requires $(\frac{\partial f}{\partial s})_{\varphi(p)} \neq 0$ and $(\frac{\partial f}{\partial s})_{\varphi(p)} \neq 0$.

**Cusp Case.** If $p$ is a cusp point of $\varphi$, continuing partial differentiations by the chain rule,

$$\frac{\partial^2 F}{\partial u \partial t} = 2x \frac{\partial^2 f}{\partial s \partial t} + x^2 \frac{\partial^2 f}{\partial t^2} \quad \frac{\partial^2 F}{\partial u \partial x} = \frac{\partial f}{\partial t} + (u - 3x^2) \frac{\partial^2 f}{\partial s \partial t} + x(u - 3x^2) \frac{\partial^2 f}{\partial t^2} \quad \frac{\partial^2 F}{\partial u \partial y} = 2y \frac{\partial^2 f}{\partial s \partial t} + 2xy \frac{\partial^2 f}{\partial t^2} \quad \frac{\partial^2 F}{\partial x ^2} = -6x \frac{\partial f}{\partial t} + (u - 3x^2) \frac{\partial^2 f}{\partial t^2} \quad \frac{\partial^2 F}{\partial x \partial y} = 2y(u - 3x^2) \frac{\partial^2 f}{\partial t^2} \quad \frac{\partial^2 F}{\partial y ^2} = 2 \frac{\partial f}{\partial t} + 4y^2 \frac{\partial^2 f}{\partial t^2}.$$
Substituting \((u, x, y) = (0, 0, 0)\), the Hessian matrix of \(F\) at \(p\) is
\[
\begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} (\varphi(p)) & \frac{\partial f}{\partial t} (\varphi(p)) & 0 \\
\frac{\partial f}{\partial t} (\varphi(p)) & 0 & 0 \\
0 & 0 & 2 \left( \frac{\partial f}{\partial t} \right) (\varphi(p))
\end{pmatrix}
\]
and its determinant is \(-2\left( \frac{\partial f}{\partial t} \right)^3 (\varphi(p)) \neq 0\). The critical point \(p\) is thus non-degenerate.

**Fold Case.** If \(p\) is a fold point,
\[
\begin{align*}
\frac{\partial^2 F}{\partial u^2} &= \frac{\partial^2 f}{\partial s^2} \\
\frac{\partial^2 F}{\partial u \partial y} &= \pm 2y \frac{\partial^2 f}{\partial s \partial t} \\
\frac{\partial^2 F}{\partial x^2} &= \pm 4xy \frac{\partial^2 f}{\partial t^2} \\
\frac{\partial^2 F}{\partial x \partial y} &= \pm 4xy \frac{\partial^2 f}{\partial t^2}
\end{align*}
\]
The Hessian matrix of \(F\) at \(p\) is
\[
\begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} (\varphi(p)) & 0 & 0 \\
0 & 2 \left( \frac{\partial f}{\partial t} \right) (\varphi(p)) & 0 \\
0 & 0 & \pm 2 \left( \frac{\partial f}{\partial t} \right) (\varphi(p))
\end{pmatrix}
\]
and its determinant is \(\pm 4 \left( \frac{\partial^2 f}{\partial x^2} \right) \varphi(p) \left( \frac{\partial f}{\partial t} \right)^2 \varphi(p)\). The critical point \(p\) degenerates if and only if \(\left( \frac{\partial^2 f}{\partial x^2} \right) \varphi(p) = 0\).

The discriminant set \(\varphi(S \cap U) = \{(s, t) = (u, 0)\}\) is regarded as the graph of a function \(f = \theta(g)\) near the vertical point \(\varphi(p)\). The first and second derivatives of \(\theta\) are
\[
\frac{d\varphi}{dg} = \frac{d}{dg} f(u, 0) = \frac{d}{du} f(u, 0) + \frac{d}{du} (0) \frac{\partial f}{\partial t} (u, 0) = \frac{\partial f}{\partial s} (u, 0),
\]
\[
\frac{d^2 \varphi}{dg^2} = \frac{d}{dg} \frac{d\varphi}{ds} (u, 0)
\]
\[
= \left\{ \frac{d}{dg} \left( \frac{\partial f}{\partial s} (u, 0) \right) \frac{\partial g}{\partial s} (u, 0) - \frac{\partial f}{\partial s} (u, 0) \frac{d}{dg} \left( \frac{\partial g}{\partial s} (u, 0) \right) \right\} / \left( \frac{\partial g}{\partial s} (u, 0) \right)^2
\]
\[
= \left\{ \frac{\partial^2 f}{\partial x^2} (u, 0) \frac{\partial g}{\partial s} (u, 0) - \frac{\partial f}{\partial s} (u, 0) \frac{\partial^2 g}{\partial s^2} (u, 0) \right\} / \left( \frac{\partial g}{\partial s} (u, 0) \right)^2
\]
Noting that \(\left( \frac{\partial f}{\partial s} \right) \varphi(p) = 0\) and \(\left( \frac{\partial^2 f}{\partial x^2} \varphi(p) \right) \neq 0\), the second derivative of \(\theta\) at \(\varphi(p)\) is \(\left( \frac{\partial^2 f}{\partial x^2} \right) \varphi(p) / \left( \frac{\partial^2 f}{\partial x^2} \varphi(p) \right)^2\). The horizontal point \(\varphi(p)\) is thus an inflection point if and only if \(\left( \frac{\partial^2 f}{\partial x^2} \right) \varphi(p) = 0\).
By the above lemmas, the function $F$ is Morse if the discriminant set $\varphi(S_\varphi)$ does not have vertical inflection points. Note that the $f$-coordinate of each vertical point of $\varphi(S_\varphi)$ corresponds to the critical value of $F$. It follows that the Morse function $F$ is stable if $\varphi(S_\varphi)$ does not have vertical double tangent lines. The same holds for $G$ by replacing “vertical” to “horizontal”.

**Corollary 7.** A deformation of $\varphi(S_\varphi)$ by an ambient isotopy of $P \times Q$ can be realized by isotopies of $F,G$ if it keeps $\varphi(S_\varphi)$ without horizontal or vertical, inflection points and double tangent lines.

**Proof.** Let $\{H_r\}_{r \in [0,1]}$ be a smooth ambient isotopy of $P \times Q$ such that $H_0 = id_{P \times Q}$. By the definitions, $\{H_r \circ \varphi\}_{r \in [0,1]}$ is an isotopy of $\varphi$ and consists of stable maps. It induces homotopies $\{F_r = \pi_P \circ H_r \circ \varphi\}_{r \in [0,1]}$ of $F$ and $\{G_r = \pi_Q \circ H_r \circ \varphi\}_{r \in [0,1]}$ of $G$. The isotoped discriminant set $H_r(\varphi(S_\varphi))$ is the discriminant set of $H_r \circ \varphi = (F_r,G_r)$ for each $r \in [0,1]$. Since $H_r(\varphi(S_\varphi))$ does not have horizontal or vertical, inflection points and double tangent lines, $F_r$ and $G_r$ are stable for each $r \in [0,1]$. The homotopies $\{F_r\}_{r \in [0,1]}$ and $\{G_r\}_{r \in [0,1]}$ are therefore isotopies. □

4. **Restrictions to level surfaces**

In this section, we consider the relation between a product map restricted to an appropriate domain and the family of the restrictions of one function to level surfaces of the other function.

We use the following notation again. Suppose $M$ is a closed smooth 3-manifold and $P, Q$ are either $\mathbb{R}$ or $S^1$. Let $F : M \to P$ and $G : M \to Q$ be smooth functions, and $\varphi$ denote the product map $(F,G)$. We do not assume $F,G$ to be Morse nor $\varphi$ to be stable at this stage.

We consider the restriction of $F$ to a level surface of $G$. Suppose $p \in M$ is a regular point of $G$, and hence the level set $G^{-1}(G(p))$ is a regular surface near the point $p$. The point $p$ is a critical point of $F|_{G^{-1}(G(p))}$ if and only if $p$ is a singular point of $\varphi$, which can be seen as follows. The gradient of the restriction $F|_{G^{-1}(G(p))}$ is the projection of the gradient of $F$ to the orthogonal compliment of the gradient of $G$. It is zero if and only if the gradients of $F$ and $G$ are linearly dependent. They are linearly dependent if and only if the differential $(d \varphi)_p$ is not surjective.

**Lemma 8.** A critical point $p$ of $F|_{G^{-1}(G(p))}$ is non-degenerate if and only if $p$ is a fold point of $\varphi$.

**Proof.** There is a local coordinate system $(\xi, \eta, r)$ of $M$ at $p = (0,0,0)$ such that $G(\xi, \eta, r) = r$ and $(\xi, \eta)$ is a local coordinate system of $G^{-1}(G(p))$ at $p$. The map $\varphi$ then has the local form $\varphi(\xi, \eta, r) = (F(\xi, \eta, r), r)$, and $F|_{G^{-1}(G(p))}(\xi, \eta) = F(\xi, \eta, 0)$. By Morin’s [12] Lemme 1 characterization, the point $p$ is a fold point of $\varphi$ if and only if the critical point $p$ of $F(\xi, \eta, 0)$ is non-degenerate. □

We consider the restrictions of $F$ to the level surfaces of $G$ in a domain $V \subset M$ which is defined as follows. Note that there is a canonical covering $\mathbb{R}^2 \to P \times Q$ by identifying $S^1$ with $\mathbb{R}/\mathbb{Z}$. Let $\tilde{R} \subset \mathbb{R}^2$ be the region $\{(f,g) \in \mathbb{R}^2 \mid h_-(g) \leq f \leq h_+(g), \ g_- \leq g \leq g_+\}$, where $g_-, g_+ \in \mathbb{R}$ are constants such that $g_- < g_+$ and $h_-, h_+ : \mathbb{R} \to \mathbb{R}$ are smooth functions such that $h_-(g) < h_+(g)$ for every $g \in [g_-, g_+]$. We assume that $\tilde{R}$ is embedded to $R \subset P \times Q$ by the covering map. Let $V$ be a connected component of the preimage $\varphi^{-1}(\tilde{R})$. From now on, we consider $\varphi, F,G$ restricted to $V$, which allows us to assume that $P = Q = \mathbb{R}$. We
assume that $G$ does not have critical points in $V$, and that the discriminant set of $φ$ does not intersect with the two edges $\{f = h_-(g), h_+(g), g_- \leq g \leq g_+\}$ of $R$. Each level set $G^{-1}(g) \cap V$ is then a regular surface whose boundaries are regular level curves of $F|_{G^{-1}(g)}$. The space $V$ is therefore a $Σ$-bundle over $[g_-, g_+]$, namely the direct product $Σ \times [g_-, g_+]$. Here $Σ$ is a compact connected surface and each $Σ \times \{g\}$ is the level surface $G^{-1}(g) \cap V$.

The restrictions of $F$ to the level surfaces of $G$ determine a homotopy $\{α_r : Σ \to \mathbb{R}\}_{r \in [g_-, g_+]}$. That is to say, $α_r(σ) = F(σ, r)$ for each point $(σ, r)$ in $Σ \times [g_-, g_+] = V$. The range of each $α_r$ is contained in $[h_-(r), h_+(r)]$, and each component of $∂Σ$ is either at the minimal level $h_-(r)$ or at the maximal level $h_+(r)$. In particular, $\{α_r\}_{r \in [g_-, g_+]}$ preserves the sets of locally minimal and locally maximal components of $∂Σ$. By Lemma 8 and the definition of a quasi-isotopy, we have the following.

**Corollary 9.** The homotopy $\{α_r\}_{r \in [g_-, g_+]}$ is a quasi-isotopy if and only if the singular set $S_φ \cap V$ consists only of fold points.

The “only if” direction of this corollary extends to the following.

**Lemma 10.** If the homotopy $\{α_r\}_{r \in [g_-, g_+]}$ is generic, the map $φ$ is stable in $V$.

**Proof.** At each birth or death on $\{α_r\}_{r \in [g_-, g_+]}$, it has the local form $α_λ(ξ, η) = λξ + λ^2 + η^2, \text{ where } \{(ξ, η) \to (ξ, η)}_{λ}$ is a smooth family of coordinate transformations. Choosing a local coordinate system $(u, x, y)$ of $M$ as $u = λ, x = ξ, y = η$, the map $φ$ has the local form $φ(u, x, y) = (u, ux - x^3 + fy^2)$, which is of a cusp point. Taking this together with the “only if” direction of Corollary 9, the singular set $S_φ \cap V$ consists only of fold points and cusp points. The conditions of a generic homotopy about the critical values implies the second condition in Theorem 2. $\square$

The discriminant set $φ(S_φ \cap V)$ is the so-called Cerf graphic of $\{α_r\}_{r \in [g_-, g_+]}$. That is to say, the intersection of $φ(S_φ \cap V)$ with each line $l_r = \{(f, g) \in \mathbb{R}^2 | g = r\}$ corresponds to the critical values of $α_r$, and we can read from $φ(S_φ \cap V)$ how the critical values of $α_r$ moves as $r$.

We can read more about the behavior of $\{α_r\}_{r \in [g_-, g_+]}$ from the Stein factorization $q_φ(V)$. For a general homotopy $\{β_r : Σ \to \mathbb{R}\}_{r \in [g_-, g_+]}$, we call the Stein factorization of the map $(σ, r) \to (β_r(σ), r)$ from $Σ \times [g_-, g_+]$ to $\mathbb{R}^2$ the Cerf complex of $\{β_r\}_{r \in [g_-, g_+]}. Note that for each $r \in [g_-, g_+]$, the intersection of the Cerf complex $q_φ(V)$ of $\{α_r\}_{r \in [g_-, g_+]}$ with the preimage $φ^{-1}(l_r)$ is the Stein factorization $W_α_r$ of $α_r$, and that the composition $π_ρ φ$ is the map $α_r : W_α_r \to \mathbb{R}$. Suppose $φ$ is stable in $V$ and $l_r$ is disjoint from cusps and crossing points of the discriminant set $φ(S_φ \cap V)$. The function $α_r$ is Morse by Lemma 8 and the critical points have pairwise distinct values. The Stein factorization $q_φ(V) \cap φ^{-1}(l_r)$ is then a trivalent graph. We can see that a valence 3 vertex of $q_φ(V) \cap φ^{-1}(l_r)$ corresponds to an index 1 critical point of $α_r$. Regarding $π_ρ φ$ as a height function, a locally minimal (resp. locally maximal) valence 1 vertex corresponds to an index 0 (resp. 2) critical point of $α_r$ except that those at the minimal level $h_-(r)$ (resp. the maximal level $h_+(r)$) correspond to minimal (resp. maximal) components of $∂Σ$.

For example, consider the situation of the bottom left of Figure 4. We can choose a parallelogram $R$ as in the top of Figure 2 after an appropriate isotopy of $φ(S_φ)$. There exists a component $V$ of the preimage $φ^{-1}(R)$ containing the two cusp points. The bottom left of Figure 2 illustrates one of possible structures of $q_φ(V)$, and the bottom right illustrates $q_φ(V) \cap φ^{-1}(l_r)$ for $r = g_-, g^-, g^+, g_+$. 
5. PROOF OF THE THEOREM

We use the following notation. Suppose $M$ is a closed smooth 3-manifold and $P, Q$ are either $\mathbb{R}$ or $S^1$. Let $F : M \to P$ and $G : M \to Q$ be smooth functions, and $\varphi$ denote the product map $(F, G)$. Let $\mathcal{R} \subset P \times Q$, $\mathcal{V} \subset M$ and $\{ \alpha_r : \Sigma \to \mathbb{R} \}_{r \in [g_- g_+]}$ be as described in Section 4. We consider $\varphi, F, G$ restricted to $\mathcal{V}$ and we can assume that $P = Q = \mathbb{R}$. We assume that $\varphi$ is stable in $\mathcal{V}$, and the following:

1. The region $\mathcal{R}$ is a parallelogram $\{(f, g) \in \mathbb{R}^2 \mid f_- + a(g - g_-) \leq f \leq f_+ + a(g - g_-), g_- \leq g \leq g_+\}$, where $f_- < f_+, g_- < g_+, a \in \mathbb{R}$ and $f_+ < f_- + a(g_+ - g_-)$.
2. The discriminant set $\varphi(\mathcal{S}_\varphi \cap \mathcal{V})$ does not have tangent lines parallel to the $f$-axis nor the $g$-axis.
3. The Stein factorization $q_{\varphi}(\mathcal{V})$ has at least one edge which maps to one of the edges $\{f = f_\pm + a(g - g_-), g_- \leq g \leq g_+\}$ of $\mathcal{R}$.

We can then regard a modification of the homotopy $\{\alpha_r\}_{r \in [g_- g_+]}$ as an isotopy of the function $F$ in the sense of the following proposition.

**Proposition 11.** Let $\{\beta_r : \Sigma \to \mathbb{R}\}_{r \in [g_- g_+]}$ be a generic homotopy from $\beta_{g_-} = \alpha_{g_-}$ to $\beta_{g_+} = \alpha_{g_+}$. We can then isotope $F$ in $\mathcal{V}$ to $\tilde{F}$ such that $\tilde{\varphi} = (\tilde{F}, G)$ is stable, and the Stein factorization $q_{\tilde{\varphi}}(\mathcal{V})$ is homeomorphic to the Cerf complex of $\{\beta_r\}_{r \in [g_- g_+]}$.

**Proof.** We can assume that $a = 1, f_- = 0, f_+ = \frac{1}{3}, g_- = 0$ and $g_+ = 1$ after an isotopy of $\varphi(\mathcal{S}_\varphi)$ by the condition (1) and Corollary 5. Let $\tilde{\beta}_r(\sigma) = \beta_r(\sigma) - r$ for $r \in [0, 1], \sigma \in \Sigma$, and let $c = \inf\{\frac{\partial}{\partial r} \tilde{\beta}_r(\sigma) \mid r \in [0, 1], \sigma \in \Sigma\}$. We define a continuous homotopy $\{\hat{\beta}_r\}_{r \in [0, 1]}$ by

\[
\hat{\beta}_r(\sigma) = \begin{cases} 
\left(1 - \frac{6r + 2}{2c + 1}\right) \tilde{\beta}_0(\sigma) + \frac{4}{3}r & (r \in [0, \frac{1}{3}]) \\
\frac{1}{6r + 3} \tilde{\beta}_{3r - 1}(\sigma) + r + \frac{1}{3} & (r \in [\frac{1}{3}, \frac{2}{3}]) \\
\left(\frac{6c + 2}{2c + 1} - \frac{4c + 1}{2c + 1}r\right) \tilde{\beta}_1(\sigma) + \frac{2}{3}r + \frac{1}{3} & (r \in [\frac{2}{3}, 1]) 
\end{cases}
\]
In the first interval $[0, \frac{1}{3}]$, the derivative $\frac{\partial}{\partial r} \hat{\beta}_r(\sigma)$ is positive by $\frac{\partial}{\partial r} \hat{\beta}_r(\sigma) = -\frac{6c+3}{2c+1} \hat{\beta}_0(\sigma) + \frac{3}{4} = -\frac{6c+3}{2c+1} a_0(\sigma) + \frac{3}{4}$ and $0 < a_0(\sigma) \leq \frac{1}{3}$. It is positive also in the last interval $[\frac{2}{3}, 1]$ similarly. In the middle interval, by $\frac{\partial}{\partial r} \hat{\beta}_r(\sigma) = \frac{1}{6c+3} \frac{\partial}{\partial r} \hat{\beta}_{3r-1}(\sigma) + 1$ and $-3c \leq -\frac{3}{6c+3} \hat{\beta}_{3r-1}(\sigma) \leq 3c$, we have $\frac{1}{3} < -\frac{3c}{6c+3} + 1 \leq \frac{\partial}{\partial r} \hat{\beta}_r(\sigma) \leq \frac{3c}{6c+3} + 1 < \frac{5}{3}$. The derivative $\frac{\partial}{\partial r} \hat{\beta}_r(\sigma)$ is thus positive for $r \in [0, 1]$ except that the right and left derivatives may disagree at $r = \frac{1}{3}$, $\frac{2}{3}$.

Note that $\beta_0(\sigma) = \hat{\beta}_0(\sigma) = \bar{\beta}_0(\sigma) = a_0(\sigma) \in [0, \frac{1}{3}]$ and $\hat{\beta}_1(\sigma) = \bar{\beta}_1(\sigma) + 1 = \beta_1(\sigma) = \alpha_1(\sigma) \in [1, \frac{3}{4}]$. Note also that $\hat{\beta}_\frac{1}{4}(\sigma) = \frac{1}{6c+3} \bar{\beta}_0(\sigma) + \frac{3}{4} \in \left[\frac{1}{6c+3}, \frac{1}{4}\right]$, and $\hat{\beta}_\frac{1}{2}(\sigma) = \frac{1}{6c+3} \bar{\beta}_1(\sigma) + \frac{3}{4} \in \left[\frac{1}{6c+3} + \frac{3}{4}, \frac{1}{2}\right]$. Since $\{\beta_r\}_{r \in [0, \frac{1}{3}]}$ connects $\beta_0$ and $\beta_1$ linearly, we can see that $\hat{\beta}_r(\sigma) \in [r, \frac{1}{3} + r]$ for $r \in [0, \frac{1}{3}]$. The same holds for $r \in \left[\frac{1}{4}, \frac{1}{2}\right]$. We can see that the same holds also for $r \in \left[\frac{1}{2}, \frac{3}{4}\right]$ by $\hat{\beta}_\frac{3}{4}(\sigma) \in \left[\frac{1}{6c+3}, \frac{1}{4}\right]$, $\hat{\beta}_\frac{1}{2}(\sigma) \in \left[\frac{1}{6c+3} + \frac{3}{4}, \frac{1}{2}\right]$ and $\frac{1}{3} < \frac{\partial}{\partial r} \hat{\beta}_r(\sigma) < \frac{5}{6}$. The range of $\hat{\beta}_r$ is thus contained in $[r, \frac{1}{3} + r]$ for $r \in [0, 1]$ as well as that of $\alpha_r$.

Note that the differential $(d\hat{\beta}_r)_\sigma$ is a scalar multiplication of $(d\hat{\beta}_r)_r$, where $r' = 0$ if $r \in [0, \frac{1}{3}]$, $r' = 3r - 1$ if $r \in \left[\frac{1}{3}, \frac{2}{3}\right]$ or $r' = 1$ if $r \in \left[\frac{2}{3}, 1\right]$. The homotopy $\{\beta_r\}_{r \in [0, 1]}$ therefore keeps $\partial \Sigma$ without critical points as well as $\{\beta_r\}_{r \in [0, 1]}$. In particular, $\{\beta_r\}_{r \in [0, 1]}$ preserves the sets of locally minimal and locally maximal components of $\Sigma$. By deforming $\{\beta_r\}_{r \in [0, 1]}$ in a collar neighborhood of $\partial \Sigma$, we can obtain a homotopy $\{\hat{\beta}_r\}_{r \in [0, 1]}$ such that the locally minimal (resp. the locally maximal) components of $\partial \Sigma$ are at the level $r$ (resp. $\frac{1}{3} + r$) for each $r \in [0, 1]$.

The homotopy $\{\hat{\beta}_r\}_{r \in [0, 1]}$ then determines a continuous function $\hat{F} : M \to \partial \Sigma$. That is to say, $\hat{F}(\sigma, r) = \hat{\beta}_r(\sigma)$ for each point $(\sigma, r)$ in $V = \Sigma \times [0, 1]$ and $\hat{F}$ is equal to $F$ outside of $V$. By arbitrarily small deformation in $V$, we can make $\{\hat{\beta}_r\}_{r \in [0, 1]}$ generic and $\hat{F}$ smooth keeping the differential $\frac{\partial}{\partial r} \hat{F}(\sigma, r) = \frac{\partial}{\partial r} \hat{\beta}_r(\sigma)$ positive. The map $\hat{\varphi} = (\hat{F}, G)$ is then stable by Lemma 10. The Stein factorization $\varphi_\Sigma(V)$ is homeomorphic to the Cerf complex of $\{\beta_r\}_{r \in [g_-, g_+]}$ by the constructions of $\{\hat{\beta}_r\}_{r \in [0, 1]}$, $\{\beta_r\}_{r \in [0, 1]}$ and $\{\hat{\beta}_r\}_{r \in [0, 1]}$.

The condition (3) implies that $\Sigma$ has non-empty boundary, and hence $V = \Sigma \times [0, 1]$ is a handlebody. By the condition (2) and Lemma 5 the original function $F$ has no critical points in the handlebody $V$. By $\frac{\partial}{\partial r} \hat{F}(\sigma, r) > 0$, the new function $\hat{F}$ also has no critical points in $V$. The topologies of the level sets $F^{-1}(f) \cap V$ and $\hat{F}^{-1}(f) \cap V$ change as $f$ according to singularities of $F_{|\partial V}$ and $\hat{F}_{|\partial V}$, respectively. Since $F_{|\partial V}$ and $\hat{F}_{|\partial V}$ are coincident, there is a homeomorphism of $V$ which takes each $F^{-1}(f) \cap V$ to $\hat{F}^{-1}(f) \cap V$. It is known that the canonical homomorphism from the mapping class group of a handlebody to the mapping class group of the boundary surface is injective. It follows that the homeomorphism is isotopic to the identity, and so $\hat{F}$ is isotopic to $F$.

**Corollary 12.** Assume the following in addition to the above. We can then isotope $F$ in $V$ to $\hat{F}$ such that $\hat{\varphi} = (\hat{F}, G)$ is a stable map without cusp points in $V$.

(4) The discriminant set $\varphi(S_g \cap V)$ has no cusps and no crossing points on the two edges $\{f_- + a(g - g_-) \leq f \leq f_+ + a(g - g_-), \ g = g_-, g_+\}$ of $R$. 
The intersections of the Stein factorization $q_\varphi(V)$ with the preimages $\varphi^{-1}(l_{g_-}), \varphi^{-1}(l_{g_+})$ have the same numbers of locally minimal valence 1 vertices, valence 3 vertices and locally maximal valence 1 vertices.

Proof. By the condition (4) and Lemma 8, $\alpha_{g_-}, \alpha_{g_+}$ are Morse functions whose critical points have pairwise distinct values. By the condition (5) and the arguments in Section 4, $\alpha_{g_-}, \alpha_{g_+}$ have the same number of critical points at each index. By Theorem 3, there exists a quasi-isotopy $\{\beta_r\}_{r \in [g_-,g_+]}$ from $\beta_{g_-} = \alpha_{g_-}$ to $\beta_{g_+} = \alpha_{g_+}$. Corollary 12 follows from the proposition and Corollary 9.

The local moves in Theorem 2 are the simplest ones to which we can apply the above. We can see that the choice of $R$ in Figure 2 satisfies the conditions (1), (2) and (4). We can also see that $q_\varphi(V)$ in Figure 2 satisfies the conditions (3) and (5). Finitely many other structures are possible for $q_\varphi(V)$, and the reader can check that they all satisfy the requirements. By Corollary 12, we can cancel the pair of cusp points, and the result is uniquely as in the bottom right of Figure 1. Similarly, we can obtain the local move of the top of Figure 1.

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