The two-dimensional OLCT of angularly periodic functions in polar coordinates

Hui Zhao\textsuperscript{a,b}, Bing-Zhao Li\textsuperscript{a,b}\textsuperscript{*}

\textsuperscript{a}School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China
\textsuperscript{b}Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081, China

Abstract

The two-dimensional (2D) offset linear canonical transform (OLCT) in polar coordinates plays an important role in many fields of optics and signal processing. This paper studies the 2D OLCT in polar coordinates. Firstly, we extend the 2D OLCT to the polar coordinate system, and obtain the offset linear canonical Hankel transform (OLCHT) formula. Secondly, through the angular periodic function with a period of $2\pi$, the relationship between the 2D OLCT and the OLCHT is revealed. Finally, the spatial shift and convolution theorems for the 2D OLCT are proposed by using this relationship.

Keywords: Polar coordinates, Offset linear canonical transform, Offset linear canonical Hankel transform, Spatial shift theorem, Convolution theorem

1. Introduction

In recent years, the linear canonical transform (LCT) \cite{1-6} has become a new and more extensive signal analysis tool in the fields of applied mathematics, signal processing and optical system analysis \cite{3-5}. Compared with the LCT, the offset linear canonical transform (OLCT) \cite{7-9} adds two additional parameters $\tau$ and $\eta$ (corresponding to time offset and frequency modulation, respectively) on
its basis, which is more versatile and flexible than the original LCT. The OLCT, also known as special affine Fourier transform [10] or non-homogeneous regular transform [7], is a linear integral transform with six parameters \((a, b, c, d, \tau, \eta)\). It is a time-shifted and frequency-modulated version of LCT [11–13]. Many linear transforms widely used in practical applications, such as Fourier transform (FT), offset FT [7, 9], fractional Fourier transform (FRFT) [7, 9, 14], offset FRFT [12, 15], Fresnel transform (FRST) [2], LCT, etc., are all special cases of the OLCT. Therefore, studying the OLCT and developing related theories of the OLCT may help to gain more insights into its special situation and transfer the knowledge gained from one discipline to other disciplines.

As an extension of many other linear transform, the OLCT has a wide range of applications in optics and signal processing. For the OLCT domain signal, people have carried out a lot of promotion, and got a lot of research results on the OLCT convolution, sampling, etc., which can be found in [16–20]. All these results are signals for dealing with one-dimensional (1D) problems. As we all know, in addition to the usual Cartesian coordinates, polar coordinates can also be used to represent signals. This situation is particularly suitable for the transformation of functions that are naturally described by polar coordinates, such as photoacoustics [21], computed tomography [22], and magnetic resonance imaging [23]. The situation of two-dimensional (2D) OLCT in polar coordinates is still unknown. Therefore, it is very important to understand whether the results of existing one-dimensional case can be extended to the two-dimensional polar coordinate case.

The two-dimensional polar coordinate transform means that the general function in the Cartesian coordinate system produces a special function, which forms an angular period with a period of \(2\pi\) in the polar coordinate system, and vice versa. Therefore, the purpose of this article is to obtain a two-dimensional correlation study of the specific function class in the polar coordinate system. First, a new representation of the 2D OLCT in polar coordinates is studied, and the expression of offset linear canonical Hankel transform (OLCHT) is derived in detail. Second, using an extension of the celebrated exponent expansion for-
mula (24), the relationship between the 2D OLCT and the OLCHT is obtained. Third, based on this relationship, the spatial shift theorem and convolution theorem of the 2D OLCT are studied. The results of this paper not only study some useful properties of the 2D OLCT in polar coordinates, but also provide a theoretical basis for the practical applications of optics and signal processing.

The paper is organized as follows. Section 2 provides some basic knowledge of the 2D OLCT in Cartesian coordinate system. In Section 3, the definition of the 2D OLCT and the OLCHT in polar coordinates is given. In Section 4, the relationship between the 2D OLCT and the OLCHT is discussed. Section 5 proves the space shift theorem and convolution theorem of the 2D OLCT. Section 6 concludes the article.

2. Preliminaries

In this section, we give some necessary background and nation on the 2D OLCT.

Let \( t = (t_1, t_2) \), \( u = (u_1, u_2) \) and \( t \cdot u = t_1 u_1 + t_2 u_2 \). The 2D OLCT with real parameters of \( A = (a, b, c, d, \tau, \eta) \) of a signal \( f(t) \) is defined by

\[
F^A(u) = O^A_L[f(t)](u) = \begin{cases} 
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t) h_A(t, u) dt, & b \neq 0 \\
\sqrt{d} e^{i [\frac{1}{2} (u^2 - \tau^2) + \frac{d}{2}]} f[d(u - \tau)], & b = 0 
\end{cases}
\]

where the kernel \( h_A(t, u) \) is

\[
h_A(t, u) = K_A e^{i \left[ \frac{1}{2} |t|^2 + \frac{1}{4} (\tau - u)^2 - \frac{1}{2} (\tau - u)(d \tau - b \eta) + \frac{d}{2} |u|^2 \right]},
\]

\[
K_A = \frac{1}{2\pi b} e^{\frac{i}{2b} \tau^2},
\]

where \( ad - bc = 1 \), \( O^A_L \) denotes the OLCT operator, \( |t|^2 = t_1^2 + t_2^2 \), \( |u|^2 = u_1^2 + u_2^2 \) and \( dt = dt_1 dt_2 \).

The inverse of the 2D OLCT with real parameters \( A = (a, b, c, d, \tau, \eta) \) is given by the 2D OLCT with parameters \( A^{-1} = (d, -b, -c, a, b \eta - d \tau, c \tau - a \eta) \).
Table 1: Some of the specific cases of the 2D OLCT

| Transform | Parameters $A$ |
|-----------|----------------|
| $A = (a, b, c, d, \tau, \eta)$ | Offset linear canonical transform (OLCT) |
| $A = (a, b, c, d, 0, 0)$ | Linear canonical transform (LCT) |
| $A = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta, 0, 0)$ | Fractional Fourier transform (FRFT) |
| $A = (0, 1, -1, 0, 0, 0)$ | Fourier transform (FT) |
| $A = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta, \tau, \eta)$ | Offset fractional Fourier transform (OFRFT) |
| $A = (1, b, 0, 1, 0, 0)$ | Fresnel transform (FRST) |
| $A = (1, 0, 0, 1, 0, \eta)$ | Frequency modulation |
| $A = (d^{-1}, 0, 0, d, 0, 0)$ | Time scaling |
| $A = (1, 0, 0, 1, \tau, 0)$ | Time shifting |

The exact inverse OLCT expression is given by [7–9, 18]

$$f(t) = O_B^{-1} \left[ F^A(u) \right](t) = C \int_{-\infty}^{+\infty} F^A(u) h_{A^{-1}}(u, t) du,$$

where $C = e^{i \frac{1}{2} \left[ c d \tau^2 - 2a d^2 \eta + a b \eta^2 \right]}$.

The definition for case $b = 0$ is the limit of the integral in (1) for the case $b \neq 0$ as $|b| \to 0$, the OLCT is simply a time scaled version off multiplied by a linear chirp. Therefore, from now on we shall confine our attention to the 2D OLCT for $b \neq 0$. And without loss of generality, we assume $b > 0$ in the following sections. Some of the special cases of the OLCT are listed in Table 1.

3. The 2D offset linear canonical transform and offset linear canonical Hankel transform in polar coodinates

3.1. 2D Offset linear canonical transform

Here we make the following symbolic regulations: the function $f(r, \theta)$, its 2D FT $F(\rho, \phi)$ and the 2D OLCT $F^A(\rho, \phi)$ are angularly periodic with period $2\pi$ as they are essentially in the form

$$f(r \cos \theta, r \sin \theta) \stackrel{\Delta}{=} f(r, \theta),$$
F(ρcosφ, ρsinθ) \triangleq F(ρ, φ),
F^M(ρcosφ, ρsinθ) \triangleq F^M(ρ, φ).

By changing Cartesian coordinates to polar coordinates related to general functions \( f(t) \), its 2D FT \( F(u) \) and 2D OLCT \( F^A(u) \), respectively.

According to the 2D OLCT, we can obtain the definition of the 2D OLCT in polar coordinates as follows:

**Definition 1 (2D OLCT).** Let polar coordinates \( t_1 = r\cosθ, t_2 = r\sinθ, u_1 = ρ\cosφ, u_2 = ρ\sinφ. \) Assume the function \( f(r, θ) \) is angularly periodic in \( 2\pi \), then its 2D OLCT with real parameters of \( A = (a, b, c, d, τ, η) \) in polar coordinates is defined by

\[
F^A(ρ, φ) = O_A^A[f](ρ, φ) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, θ)P_A(r, θ; ρ, φ)rdθdτ,
\]

where the \( P_A(r, θ; ρ, φ) \) denotes the 2D OLCT kernel in polar coordinates and is given by

\[
P_A(r, θ; ρ, φ) = \frac{ℓ_A}{2πb} e^{−dΔ[2(\sqrt{a} + \frac{bτ}{\sqrt{c}}) + 2τ(τ - u) - 2u(dτ - bη) + d|u|^2]} \]

where \( b \neq 0 \) and \( ℓ_A = e^{−dΔ[2(\sqrt{a} + \frac{bτ}{\sqrt{c}}) + 2τ(τ - u) - 2u(dτ - bη) + d|u|^2]} \).

**Proof.** Let \( t = (t_1, t_2), u = (u_1, u_2), \) and the 2D OLCT of \( f(t) \) in (1), we have

\[
F^A(u) = O_A^A[f(t)](u)
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)K_A e^{−dΔ[2u|t|^2 + 2τ(τ - u) - 2u(τ - bη) + d|u|^2]}dt
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)K_A e^{−dΔ[2τ(τ - u) - 2u(τ - bη) + d|u|^2]}dt
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)K_A e^{−dΔ[2(τ - u) + (τ - bη) - 2u(τ - bη)]}dt
\times e^{−dΔ[−2(τ - bη)(u_1 + u_2) + \frac{(τ - bη)^2}{d} - 2(t_1 u_1 + t_2 u_2)]}dt_1dt_2,
\]

where \( K_A(t, u) \) is given by (3).

Using polar coordinates \( t_1 = r\cosθ, t_2 = r\sinθ, u_1 = ρ\cosφ, u_2 = ρ\sinφ, \) we
obtain

\[ F^A (\rho, \phi) = O^A_L [f](\rho, \phi) = F \left( \frac{d_2 b \rho^2}{2 \pi} \right) e^{i \frac{d_2 b (d - b)n}{2 \pi}} \]

\[ = \int_0^{+\infty} \int_0^{2\pi} f(r, \theta) \frac{1}{2\pi} e^{i \frac{d_2 b \rho^2}{2 \pi} \left( \frac{x_2^2}{\frac{2\pi}{x_2^2}} \right)} r dr d\theta \]

\[ = F \left( \frac{d_2 b \rho^2}{2 \pi} \right) e^{i \frac{d_2 b (d - b)n}{2 \pi}} \]

(8)

Hence

\[ F^A (\rho, \phi) = O^A_L [f](\rho, \phi) = \int_0^{+\infty} \int_0^{2\pi} f(r, \theta) P_A (r, \theta; \rho, \phi) r dr d\theta, \]

(9)

where

\[ P_A (r, \theta; \rho, \phi) = e^{i \frac{d_2 b \rho^2}{2 \pi} \left( \frac{x_2^2}{\frac{2\pi}{x_2^2}} \right)} \]

(10)

which completes the proof.

The inversion formula of the 2D OLCT in polar coordinates takes

\[ f(r, \theta) = O^{-1}_L [F^A](r, \theta), \]

(11)

where \( A^{-1} = (d, -b, -c, a, b\eta - d\tau, c\tau - a\eta) \) and \( b \neq 0 \).

As it is seen, when \( A = (0, 1, -1, 0, 0, 0) \), the 2D OLCT can be reduced to the 2D FT

\[ F(\rho, \phi) = \mathcal{F}[f](\rho, \phi) \]

\[ = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} f(r, \theta) e^{-i r \rho \cos(\theta - \phi)} r dr d\theta. \]

(12)

It follows that there is a relation between the 2D OLCT and the 2D FT

\[ F^A (\rho, \phi) = \frac{\ell_A}{b} e^{i \frac{d_2 b \rho^2}{2 \pi} \left( \frac{x_2^2}{\frac{2\pi}{x_2^2}} \right)} F[f \left( \frac{b}{\ell_A}, \phi \right)], \]

(13)

where \( b \neq 0 \), \( \ell_A \) is given by (6) and \( f(r, \theta) = e^{i \frac{d_2 b \rho^2}{2 \pi} \left( \frac{x_2^2}{\frac{2\pi}{x_2^2}} \right)} f(r, \theta). \)
3.2. Offset linear canonical Hankel transform

In this subsection, we discuss definition of the 2D OLCHT in polar coordinates. Inspired by the literature [25–27], under the premise that the transformation function has rotational symmetry, the OLCHT is obtained from the OLCT.

**Definition 2 (OLCHT).** The \( n \)-th-order OLCHT of the real parameters matrix of \( A = (a, b, c, d, \tau, \eta) \) is defined by
\[
H_n^A[f](\rho) = i^n \frac{w_A}{b} e^{i\frac{\pi}{2} \rho^2} \int_0^{+\infty} w_2 e^{i\frac{\pi}{2} \tau^2} J_n \left( \frac{\rho}{b} \right) f(r) r dr,
\]
(14)
where the \( J_n \) is the \( n \)-th-order Bessel function of the first kind and order \( n \geq -\frac{1}{2} \), \( b \neq 0 \), \( \ell_A \) is given by (6), and
\[
w_1 = \sum_{m=-\infty}^{+\infty} J_m \left( \frac{\sqrt{2} \rho (d \tau - b \eta)}{b} \right), \quad w_2 = \sum_{m=-\infty}^{+\infty} J_m \left( \frac{\sqrt{2} \tau}{b} \right).
\]
(15)

**Proof.** From (5), it follows that
\[
F^A(\rho, \phi) = \frac{\ell_A}{2\pi b} \int_0^{+\infty} f(r, \theta) e^{i \left[ \frac{\pi}{4} r^2 + \frac{\sqrt{2} \tau r}{b} \sin(\theta + \frac{\pi}{4}) \right]} e^{i \left[ -\frac{\sqrt{2} \rho (d \tau - b \eta) b \sin(\phi + \frac{\pi}{4})}{2} \cos(\theta - \phi) + \frac{\pi}{4} \rho^2 \right]} r dr d\theta,
\]
(16)
where \( \ell_A \) is given by (6).

In view of the relation \( 28 \)
\[
e^{-it \sin \theta} = \sum_{m=-\infty}^{+\infty} J_m(t) e^{-im \theta}. \quad (17)
\]
So we can obtain
\[
e^{i \frac{\sqrt{2} \tau r}{b} \sin(\theta + \frac{\pi}{4})} = \sum_{m=-\infty}^{+\infty} J_m \left( \frac{\sqrt{2} \tau}{b} \right) e^{im(\theta + \frac{\pi}{4})}, \quad (18)
\]
\[
e^{-i \frac{\sqrt{2} \rho (d \tau - b \eta) b \sin(\phi + \frac{\pi}{4})}{2}} = \sum_{m=-\infty}^{+\infty} J_m \left( \frac{\sqrt{2} \rho (d \tau - b \eta)}{b} \right) e^{-im(\phi + \frac{\pi}{4})}. \quad (19)
\]
When \( f(r, \theta) \) is circularly symmetric, there is
\[
f(r, \theta) = f(r) e^{i k \theta}. \quad (20)
\]
Let

\[ B = \int_0^{2\pi} f(r, \theta) e^{i \left[ \sqrt{2} \rho \sin(\theta + \phi) \right. - \sqrt{2} \rho (d \tau - b \eta) \sin(\phi + \phi) - \sqrt{2} \rho \cos(\theta - \phi) \right] d\theta. \]  

(21)

Using (18), (19) and (20), we can derive the following result

\[ B = \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} J_m \left( \sqrt{2} \rho \right) J_m \left( \sqrt{2} \rho (d \tau - b \eta) \right) \times e^{im\theta - im\phi - \frac{\sqrt{2} \rho}{b} \cos(\theta - \phi)} e^{ik\theta} f(r) d\theta. \]  

(22)

According to the famous formula \[24, 28\]

\[ J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n\theta - x \sin \theta)} d\theta. \]  

(23)

We have

\[ \int_0^{2\pi} e^{im\theta - im\phi - \frac{\sqrt{2} \rho}{b} \cos(\theta - \phi)} e^{ik\theta} d\theta \]

\[ = \int_0^{2\pi} e^{im\theta - im\phi - \frac{\sqrt{2} \rho}{b} \sin(\theta + \phi)} e^{ik\theta} d\theta \]

\[ = e^{-in(\pi/2 + \theta - \phi) + im\theta - im\phi + ik\theta} \int_0^{2\pi} e^{i[n(\pi/2 + \theta - \phi) - \frac{\sqrt{2} \rho}{b} \sin(\theta + \phi)]} d\theta \]

\[ = 2\pi J_n \left( \frac{\sqrt{2} \rho}{b} \right) e^{i(m-k)\theta + (n-m)\phi - \frac{i\pi}{2} n}. \]  

(24)

By making the change of \( k = n-m \) in the above expression, we obtain

\[ \int_0^{2\pi} e^{im\theta - im\phi - \frac{\sqrt{2} \rho}{b} \cos(\theta - \phi)} e^{ik\theta} d\theta = 2\pi J_n \left( \frac{\sqrt{2} \rho}{b} \right) e^{i\phi - i\frac{\pi}{2} n}. \]  

(25)

Hence

\[ F^A(\rho, \phi) = i^n \frac{\ell_A}{2\pi b} \sum_{m=-\infty}^{+\infty} J_m \left( \sqrt{2} \rho (d \tau - b \eta) \right) e^{i\frac{\pi}{2} \rho \phi^2} \]

\[ \times \int_0^{+\infty} \sum_{m=-\infty}^{+\infty} J_m \left( \sqrt{2} \rho \right) e^{i\frac{\pi}{2} \rho \phi^2} 2\pi J_n \left( \frac{\rho}{b} \right) e^{ik\phi} d\tau. \]  

(26)

The output function also has circular symmetry, we have

\[ F^A(\rho, \phi) = F^A(\rho) e^{ik\phi}. \]  

(27)
So

\[ H_n^A[f](\rho) = F^A(\rho) = i^n \frac{\ell_A}{b} \sum_{m=-\infty}^{+\infty} J_m \left( \frac{\sqrt{2} \rho (\tau - b\eta)}{b} \right) e^{i\frac{\pi}{4} \rho^2} \]

\[ \times \int_0^{+\infty} \sum_{m=-\infty}^{+\infty} J_m \left( \frac{\sqrt{2} \tau r}{b} \right) e^{i\frac{\pi}{4} r^2} J_n \left( \frac{r \rho}{b} \right) r dr. \]  

(28)

The proof is completed.

The inversion formula of nth-order OLCHT takes

\[ f(r) = H_n^{-A^{-1}} [H_n^A[f]](r). \]  

(29)

It is evident that the nth-order OLCHT and its inverse with \( A = (0, 1, -1, 0, 0, 0) \) reduces to the conventional nth-order Hankel transform (HT) 29.

\[ H_n[f](\rho) = \int_0^{+\infty} f(r) J_n(\rho r) r dr, \]

and the corresponding inversion formula

\[ f(r) = H_n[H_n[f]](r), \]

respectively.

4. Relationship between the 2D OLCT and the OLCHT

Under the condition of rotational symmetry, the OLCHT can be deduced from the OLCT, and there is a close relationship between them.

4.1. Relationship between the 2D OLCHT and the HT

The purpose of this section is to extend the above relationship to preparing for the OLCT domain. Let the function \( f(r, \theta) \) and its 2D FT \( F(\rho, \phi) \) satisfy the Dirichlet conditions. Since they can angularly periodic with period \( 2\pi \), then their Fourier series are well-defined as

\[ f(r, \theta) = \sum_{n=-\infty}^{+\infty} f_n(r) e^{in\theta}, \]

(32)
\[ F(\rho, \phi) = \sum_{n=-\infty}^{+\infty} F_n(r) e^{in\phi}. \] (33)

As we all know, the \( n \)th term in Fourier series of primitive functions and their 2D FT versions \( f_n(r) \) and \( F_n(\rho) \), forming an \( n \)th-order HT pair [29–31]

\[ F_n(\rho) = i^{-n} H_n[f_n](\rho), \] (34)
\[ f_n(r) = i^n H_n[F_n](r). \] (35)

From the above, (14) and (30), we can obtain the relationship between the OLCHT and the HT, it follows that

\[ H_{n}^{A}[f](\rho) = i^{n} \frac{w_1}{b} e^{i \frac{n}{2} \rho^2} H_{n}[\tilde{f}] \left( \frac{\rho}{b} \right), \] (36)

where \( w_1, w_2 \) is given by (15), and \( \ell_A \) is given by (6)

\[ \tilde{f}(r) = w_2 e^{i \frac{\pi}{2} r^2} f(r). \] (37)

Remark 1. If it is assumed that \( f \) is radially symmetric, then it can be written as a function of \( r \) only and can thus be taken out of the integration over the angular coordinate.

Remark 2. When the function \( f(r, \theta) \) is not radially symmetric and is a function of both \( r \) and \( \theta \), the preceding result can be generalized. Since \( f(r, \theta) \) depends on the angle \( \theta \), it can be expanded into a Fourier series (32) and (33).

4.2. Relationship between the 2D OLCT and the OLCHT

Next, we will show that the relation between the 2D OLCT and the OLCHT.

Theorem 1. Let the function \( f(r, \theta) \) satisfy Dirichlet conditions, be angularly periodic in \( 2\pi \), and have a Fourier expansion

\[ f(r, \theta) = \sum_{n=-\infty}^{+\infty} f_n(r) e^{in\theta}, \] (38)

then the Fourier series expansion of the 2D OLCT of \( f(r, \theta) \) has a form

\[ F^A(\rho, \phi) = \sum_{n=-\infty}^{+\infty} H_{n}^{A}[f_n](\rho) e^{in\phi}. \] (39)
Proof. The 2D OLCT of \( f(r, \theta) \) as expanded in (38), takes

\[
F_A(\rho, \phi) = \sum_{n = -\infty}^{+\infty} \int_0^{+\infty} \int_0^{2\pi} e^{in\theta} f_n(r) P_A(r, \theta; \rho, \phi) r dr d\theta
\]

\[
= \sum_{n = -\infty}^{+\infty} \int_0^{+\infty} e^{i \left[ \sqrt{2\tau r} \sin(\theta + \frac{\pi}{2}) - \frac{\sqrt{2\tau (r^2 - b^2)}}{2b} \sin(\phi + \frac{\pi}{4}) - \frac{\sqrt{2} \rho b}{4} \cos(\theta - \phi) \right]} e^{in\theta} D(r, \theta; \rho, \phi)
\]

\[
\times \frac{\ell_A}{2\pi b} e^{i \left( \frac{2\pi r^2}{b} + \frac{2\pi \rho^2}{b} \right)} \int_0^{2\pi} f_n(r) r dr d\theta.
\]

(40)

According to the celebrated exponent expansion formula \[28\]

\[
e^{i \frac{2\pi}{b} \cos(\theta - \phi)} = e^{i \frac{2\pi}{b} \cos(\theta - \phi)} = \sum_{v = -\infty}^{+\infty} i^v J_v \left( \frac{\tau b}{r} \right) e^{-iv(\phi - \theta)}.
\]

(41)

Using (18), (19) and (41), we get

\[
D(r, \theta; \rho, \phi) = \sum_{v = -\infty}^{+\infty} i^v J_v \left( \frac{\tau b}{r} \right) w_1 w_2 e^{im\theta - im\phi - iv(\theta - \phi) + i\theta}
\]

(42)

where \( w_1, w_2 \) is given by (15).

For \( n = v - m \), (42) can be written as

\[
D(r, \theta; \rho, \phi) = \sum_{v = -\infty}^{+\infty} i^v J_v \left( \frac{\tau b}{r} \right) w_1 w_2 e^{i\phi}.
\]

(43)

Hence

\[
F_A(\rho, \phi) = \sum_{n = -\infty}^{+\infty} \int_0^{+\infty} \frac{\ell_A}{2\pi b} e^{i \left( \frac{2\pi r^2}{b} + \frac{2\pi \rho^2}{b} \right)} \sum_{v = -\infty}^{+\infty} i^v J_v \left( \frac{\tau b}{r} \right) w_1 w_2 e^{i\phi} f_n(r) r dr \int_0^{2\pi} d\theta
\]

\[
= \sum_{n = -\infty}^{+\infty} \sum_{v = -\infty}^{+\infty} i^v w_1 \ell_A \left( \frac{\tau b}{r} \right) e^{i \frac{2\pi r^2}{b}} \int_0^{+\infty} w_2 e^{i \frac{2\pi \rho^2}{b}} J_v \left( \frac{\tau b}{r} \right) f_n(r) e^{i\phi} r dr
\]

\[
= \sum_{n = -\infty}^{+\infty} i^v w_1 \ell_A \left( \frac{\tau b}{r} \right) e^{i \frac{2\pi r^2}{b}} \int_0^{+\infty} w_2 e^{i \frac{2\pi \rho^2}{b}} J_n \left( \frac{\tau b}{r} \right) f_n(r) e^{i\phi} r dr
\]

\[
= \sum_{n = -\infty}^{+\infty} H_n^A \left[ f_n \right] (\rho) e^{i\phi}.
\]

(44)
The proof is completed. □

Through the above series of discussions, the Fourier series expansion of the 2D OLCT $F^A(\rho, \phi)$ has a form

$$F^A(\rho, \phi) = \sum_{n=-\infty}^{+\infty} F^A_n(r) e^{in\phi}.$$  \hspace{1cm} (45)

According to Theorem 1 and (45), we get

$$F^A_n(\rho) = H^A_n[f_n](\rho).$$  \hspace{1cm} (46)

5. The properties of 2D OLCT in polar coordinates

The 1D OLCT has many important properties, such as spatial shift and convolution. Similarly, the 2D OLCT also has some important nature. In this section, we use two theorems to understand the properties of the 2D OLCT in polar coordinates, namely spatial shift theorem and convolution theorem, respectively.

5.1. Spatial shift theorem

See [18] for the spatial shift properties of the 1D OLCT. Below we give the spatial shift theorem of the 2D OLCT in polar coordinates.

**Theorem 2** (Spatial shift theorem). Let $\mathbf{t}_0 = (t_3, t_4)$, $t_3 = r_0 \cos \theta_0$, $t_4 = r_0 \sin \theta_0$, using polar coordinates. Then we have

$$f(t - \mathbf{t}_0) = (-i)^{\frac{3}{2}} 2\pi w_3 \gamma(r, \theta, r_0, \theta_0) \sum_{n=-\infty}^{+\infty} e^{in(\theta-\theta_0)} H^A_n[f_n](\rho) e^{-\frac{1}{2\pi} (\rho^2)}$$

$$\times J_n \left( \frac{r \rho}{b} \right) J_n \left( \frac{r_0 \rho}{b} \right) J_n \left( \frac{\sqrt{2} \rho (b \eta - d \tau)}{b} \right) \rho d \rho,$$  \hspace{1cm} (47)

where

$$\gamma(r, \theta, r_0, \theta_0) = e^{i \left[-\frac{1}{2}(r^2 + r_0^2) + \frac{\pi r_0}{2} \cos(\theta - \theta_0) + \frac{\pi r_0 (b_0 - a_0)}{2} \sin(\theta + \frac{\pi}{4}) - \frac{\pi r (b_0 - a_0)}{2} \sin(\theta_0 + \frac{\pi}{4}) \right]},$$

$$w_3 = e^{-i \frac{(b_0 - a_0)^2}{2a b}}.$$  \hspace{1cm} (48)
Proof. According to (11), it follows that

\[
f(t - t_0) = C \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F^A(u) K_A^{-1} e^{i \frac{1}{t-a} |u|^2 + \frac{1}{t-a} u(t-t_0)} du.
\]

Using polar coordinates \( t_1 = r\cos \theta, \ t_2 = r\sin \theta, \ t_3 = r_0\cos \theta_0, \ t_4 = r_0\sin \theta_0, \)

\( u_1 = r\cos \phi, \ u_2 = r\sin \phi, \) we obtain

\[
E \triangleq e^{-\frac{1}{t-a} [d(u^2) + 2(b\eta - d\tau)u + a(t-t_0)^2 - 2r(t-t_0)(bc-ad) - 2ut+2ut_0]} \]

\[
e^{-\frac{1}{t-a} \left[ \left( \sqrt{a^2 + b^2} \right)^2 - 2a \right] e^{-\frac{1}{t-a} \left[ \left( \sqrt{a^2 + b^2} \right)^2 - 2a \right]}} e^{\frac{1}{t-a} \left[ \left( \sqrt{a^2 + b^2} \right)^2 - 2a \right]}
\]

\[
= w_4 \left[ -2\sqrt{p(b\eta - d\tau)} \sin(\phi + \frac{\pi}{2}) + 2r_0\cos(\theta - \phi) - 2r_0\cos(\theta_0 - \phi) \right]
\]

\[
\times e^{\frac{1}{t-a} \left[ \left( \sqrt{a^2 + b^2} \right)^2 - 2a \right]}
\]

According to the celebrated exponent expansion formula (17), split each item of \( w_4, \) we have

\[
e^{i\rho_0\cos(\theta - \phi)} = \sum_{m=-\infty}^{+\infty} i^m J_m \left( \frac{r\rho}{b} \right) e^{im(\theta - \phi)},
\]

\[
e^{-i\rho_0\cos(\theta_0 - \phi)} = \sum_{m=-\infty}^{+\infty} i^{-m} J_m \left( \frac{r_0\rho}{b} \right) e^{-im(\theta_0 - \phi)},
\]

and

\[
e^{-i\sqrt{\rho_0(b\eta - d\tau)} \sin(\phi + \frac{\pi}{2})} = \sum_{n=-\infty}^{+\infty} J_n \left( \frac{\sqrt{\rho_0(b\eta - d\tau)}}{b} \right) e^{-in(\phi + \frac{\pi}{2})}.
\]
Using (39), (49), (50), (51) and (52), we obtain

\[
\begin{aligned}
\quad f(t - t_0) &= C w_3 K_{A-1} \int_0^\infty \sum_{m = -\infty}^{+\infty} J_m \left( \frac{r \rho}{b} \right) J_m \left( \frac{r_0 \rho}{b} \right) \sum_{n = -\infty}^{+\infty} J_n \left( \frac{\sqrt{2} \rho (b \eta - d \tau)}{b} \right) \\
&\quad \times \sum_{n = -\infty}^{+\infty} H_n^A [f_n] (\rho) e^{in(\theta - \theta_0)} - i \int_0^{2\pi} d\phi d\rho \\
&\quad \times e^{-\frac{\pi}{4} \rho^2 \left( r^2 + r_0^2 \right) + \frac{\pi r_0}{b} \cos(\theta - \theta_0) + \frac{\sqrt{2} \pi (bc - ad)}{b} \sin(\theta + \frac{\pi}{4}) - \frac{\sqrt{2} \pi (bc - ad)}{b} \sin(\theta_0 + \frac{\pi}{4})} \\
&\quad \times H_n^A [f_n] (\rho) e^{in(\theta - \theta_0)} d\rho d\rho.
\end{aligned}
\]

(53)

Let \( n = m \), we get

\[
\begin{aligned}
\quad f(t - t_0) &= (-i)^{\frac{n}{2}} 2\pi C w_3 \int_0^\infty \sum_{n = -\infty}^{+\infty} J_n \left( \frac{r \rho}{b} \right) J_n \left( \frac{r_0 \rho}{b} \right) J_n \left( \frac{\sqrt{2} \rho (b \eta - d \tau)}{b} \right) \\
&\quad \times e^{-\frac{\pi}{4} \rho^2 \left( r^2 + r_0^2 \right) + \frac{\pi r_0}{b} \cos(\theta - \theta_0) + \frac{\sqrt{2} \pi (bc - ad)}{b} \sin(\theta + \frac{\pi}{4}) - \frac{\sqrt{2} \pi (bc - ad)}{b} \sin(\theta_0 + \frac{\pi}{4})} \\
&\quad \times H_n^A [f_n] (\rho) e^{in(\theta - \theta_0)} d\rho.
\end{aligned}
\]

(54)

Change the integral and summation signs, we have

\[
\begin{aligned}
\quad f(t - t_0) &= (-i)^{\frac{n}{2}} 2\pi C w_3 \sum_{n = -\infty}^{+\infty} e^{in(\theta - \theta_0)} \\
&\quad \times e^{-\frac{\pi}{4} \rho^2 \left( r^2 + r_0^2 \right) + \frac{\pi r_0}{b} \cos(\theta - \theta_0) + \frac{\sqrt{2} \pi (bc - ad)}{b} \sin(\theta + \frac{\pi}{4}) - \frac{\sqrt{2} \pi (bc - ad)}{b} \sin(\theta_0 + \frac{\pi}{4})} \\
&\quad \times \int_0^{+\infty} H_n^A [f_n] (\rho) J_n \left( \frac{r \rho}{b} \right) J_n \left( \frac{r_0 \rho}{b} \right) J_n \left( \frac{\sqrt{2} \rho (b \eta - d \tau)}{b} \right) d\rho.
\end{aligned}
\]

(55)

The proof is completed. \( \square \)

5.2. Convolution theorem

Let us give the convolution concept of the 2D OLCT in polar coordinates as follows:
Definition 3. The convolution operation $*^A$ of the 2D OLCT in polar coordinates for two functions $f(r, \theta)$ and $g(r, \theta)$ is defined by

$$
(f *^A g)(r, \theta) = \frac{e^{-i\left(\frac{\pi}{2}r^2 + \frac{\sqrt{2}\pi}{2}r b \sin(\theta + \frac{\pi}{4})\right)}}{2\pi} \int_0^{+\infty} \int_0^{2\pi} e^{i\left(\frac{\pi}{2}r^2 + \frac{\sqrt{2}\pi}{2}r b \sin(\theta_0 + \frac{\pi}{4})\right)} 
\times F^{-1}\left(e^{-ipr_0 \cos(\phi - \theta_0)} F^A(b\rho, \phi)\right)(r, \theta) r_0 dr_0 d\theta_0.
$$

(56)

In the following, we obtain the convolution theorem of the 2D OLCT in polar coordinates.

Theorem 3 (Convolution theorem). Let $z(r, \theta) = (f *^A g)(r, \theta)$. Then, there is a relation

$$
Z^A(\rho, \phi) = F^A(\rho, \phi) G^A(\rho, \phi),
$$

(57)

where $Z^A(\rho, \phi), F^A(\rho, \phi)$ and $G^A(\rho, \phi)$ denote the 2D OLCT in polar coordinates of the functions $z(r, \theta), f(r, \theta)$ and $g(r, \theta)$, respectively.

Proof. In view of (39), we get

$$
e^{-ipr_0 \cos(\phi - \theta_0)} = \sum_{m=-\infty}^{+\infty} i^{-m} J_m(\rho r_0) e^{-im\theta_0} e^{im\phi}.
$$

(58)

By (12) and (58), it follows that

$$
F^{-1}\left[e^{-ipr_0 \cos(\phi - \theta_0)} F^A(b\rho, \phi)\right](r, \theta) = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} i^{-m} J_m(\rho r_0) e^{-im\theta_0} e^{im\phi} 
\times \sum_{n=-\infty}^{+\infty} H^A_n[f_n](b\rho) e^{in\phi} \sum_{k=-\infty}^{+\infty} i^k J_k(r\rho) e^{ik\theta} e^{-ik\phi} d\rho d\phi.
$$

(59)

Since

$$
\int_0^{2\pi} e^{im\phi + in\phi - ik\phi} d\phi = 2\pi \delta_{m+n+k} = \begin{cases} 
2\pi, & m + n - k = 0 \\
0, & \text{otherwise}
\end{cases},
$$

(60)
where $\delta_{\cdot\cdot}$ denotes the kronecker delta operator.

Hence, (59) can be written as

$$i^n \int_0^{+\infty} \sum_{m=-\infty}^{+\infty} i^{-m} J_m (\rho r_0) e^{-im\theta_0} \sum_{k=-\infty}^{+\infty} H^A_{k-m} [f_{k-m}] (b\rho) J_k (\rho) e^{ik\theta} \rho d\rho. \tag{61}$$

We make the following symbols

$$\tilde{z} (r, \theta) = e^{i \left[ a_2 r^2 + \sqrt{2} r \sin (\theta + \pi) \right]} z (r, \theta), \tag{62}$$

$$\tilde{g} (r_0, \theta_0) = e^{i \left[ a_2 r_0^2 + \sqrt{2} r_0 \sin (\theta_0 + \pi) \right]} g (r_0, \theta_0), \tag{63}$$

and the Fourier series expansion of the 2D OLCT has a form

$$\tilde{g} (r_0, \theta_0) = \sum_{n=-\infty}^{+\infty} \tilde{g}_n (r_0) e^{in\theta_0}, \tag{64}$$

where $\tilde{g}_n (r_0) = w_2 e^{i \left[ a_2 r_0^2 + \sqrt{2} r_0 \sin (\theta_0 + \pi) \right]} g (r_0, \theta_0)$, $w_2$ is given by (15) and $g_n (r_0)$ is the nth term in the Fourier series for the function $g (r_0, \theta_0)$.

Using (56), (61) and (64) that

$$\tilde{z} (r, \theta) = i^n \frac{1}{2\pi} \int_0^{+\infty} \sum_{n=-\infty}^{+\infty} \tilde{g}_n (r_0) \sum_{m=-\infty}^{+\infty} i^{-m} J_m (\rho r_0)$$

$$\times \sum_{k=-\infty}^{+\infty} H^A_{k-m} [f_{k-m}] (b\rho) J_k (\rho) e^{ik\theta} \int_0^{2\pi} e^{im\theta_0 - im\theta_0} r_0 d\theta_0 d\rho. \tag{65}$$

Let $m = n$ and by (30), we get

$$\tilde{z} (r, \theta) = \int_0^{+\infty} \sum_{m=-\infty}^{+\infty} \tilde{g}_m (r_0) J_m (\rho r_0)$$

$$\times \sum_{k=-\infty}^{+\infty} H^A_{k-m} [f_{k-m}] (b\rho) J_k (\rho) e^{ik\theta} \rho d\rho$$

$$= \sum_{k=-\infty}^{+\infty} \int_0^{+\infty} \sum_{m=-\infty}^{+\infty} H_m [\tilde{g}_m (r_0)] (\rho)$$

$$\times H^A_{k-m} [f_{k-m}] (b\rho) J_k (\rho) e^{ik\theta} \rho d\rho. \tag{66}$$
The (66) is a Fourier series expansion, then the $k$th term is

$$\tilde{z}_k(r) = \int_0^{+\infty} \sum_{m=-\infty}^{+\infty} H_m[\tilde{g}_m(r_0)](\rho) H_{k-m}^A[f_{k-m}](b\rho) J_k(r\rho) \rho d\rho,$$

(67)

where $\tilde{z}_k(r) = w_2 e^{i2\pi r_0^2} g_k(r_0)$, $w_2$ is given by (15) and $z_k(r)$ is the $k$th term in the Fourier series for the function $z(r, \theta)$.

Apply the inverse HT formula, (67) becomes

$$H_k[\tilde{z}_k](\rho) = \sum_{m=-\infty}^{+\infty} H_m[\tilde{g}_m](\rho) H_{k-m}^A[f_{k-m}](b\rho).$$

(68)

According to the relationship between the OLCHT and the HT, we obtain

$$H_k^A[\tilde{z}_k](b\rho) = i^k \frac{w_1 \ell_A}{b} e^{i2\pi^2 \rho^2} H_k[\tilde{z}_k](\rho),$$

$$H_m^A[g_m](b\rho) = i^m \frac{w_1 \ell_A}{b} e^{i2\pi^2 \rho^2} H_m[g_m](\rho),$$

(69)

where $w_1$ is given by (15) and $\ell_A$ is given by (6). Hence

$$i^{-k} H_k^A[\tilde{z}_k](\rho) = i^{-m} \sum_{m=-\infty}^{+\infty} H_m^A[\tilde{g}_m](\rho) H_{k-m}^A[f_{k-m}](\rho).$$

(70)

By (60), we have

$$H_k^A[\tilde{z}_k](\rho) = \sum_{m=-\infty}^{+\infty} H_m^A[\tilde{g}_m](\rho) H_{k-m}^A[f_{k-m}](\rho),$$

(71)

that is, a convolution relation

$$Z_k^A(\rho) = \sum_{m=-\infty}^{+\infty} C_m^A(\rho) F_{k-m}^A(\rho).$$

(72)

According to the formula (46), as we all know, the convolution of two sets of the Fourier coefficients is equivalent to the multiplication of functions [32]. Therefore, it can be seen from (72) that there is a multiplicative relationship

$$Z^A(\rho, \phi) = F^A(\rho, \phi) G^A(\rho, \phi),$$

(73)

where $Z^A(\rho, \phi)$, $F^A(\rho, \phi)$ and $G^A(\rho, \phi)$ denote the 2D OLCT in polar coordinates of the functions $z(r, \theta)$, $f(r, \theta)$ and $g(r, \theta)$, respectively.
6. Conclusions

According to the definition of the OLCT in the two-dimensional Cartesian coordinate system, this paper obtains its new expression in polar coordinates, and derives the mathematical formula of the OLCHT. Then, the relationship between the 2D OLCT and the OLCHT is derived. Finally, using this relationship, the spatial shift theorem and convolution theorem of the 2D OLCT are proved. These contents are new achievements in the field of the 2D OLCT polar coordinates. In further work, we will consider the sampling theorem of the 2D OLCT in optics and signal processing.

References

[1] R. Tao, B. Z. Li, Y. Wang, G. K. Aggrey, On sampling of bandlimited signals associated with the linear canonical transform, IEEE Trans. Signal Process. 56 (11) (Aug. 2008) 5454-5464.

[2] D. F. V. James, G. S. Agarwal, The generalized Fresnel transform and its application to optics, Opt. Commun. 126 (4-6) (May. 1996) 207-212.

[3] M. F. Erden, M. A. Kutay, H. M. Ozaktas, Repeated filtering in consecutive fractional fourier domains and its application to signal restoration, IEEE Trans. Signal Process. 47 (5) (May. 1999) 1458-1462.

[4] Y. X. Fu, L. Q. Li, Generalized analytic signal associated with linear canonical transform, Opt.Commun. 281 (6) (Mar. 2008) 1468-1472.

[5] J. J. Healy, M. A. Kutay, H. M. Ozaktas, J. T. Sheridan, Linear Canonical Transforms: Theory and Applications, Springer, NewYork, 2016.

[6] R. F. Bai, B. Z. Li, Q. Y. Cheng, Wigner-Ville distribution associated with the linear canonical transform, J. Appl. Math. 2012 (2012) 1-14.

[7] S. C. Pei, J. J. Ding, Eigenfunctions of the offset Fourier, fractional Fourier and linear canonical transforms, J. Opt. Soc. Am. A. 20 (3) (Mar. 2003) 522–532.
[8] Stern, A. Sampling of compact signals in offset linear canonical domains, Signal Image Video Process. 1 (4) (Jul. 2007) 359–367.

[9] S. C. Pei, J. J. Ding, Eigenfunctions of Fourier and fractional Fourier transforms with complex offsets and parameters, IEEE Trans. Circuits Syst. I. 54 (7) (Jul. 2007) 1599–1611.

[10] S. Abe, J. T. Sheridan, Optical operations on wave-functions as the Abelian subgroups of the special affine Fourier transformation, Opt. Lett. 19 (22) (Nov, 1994) 1801–1803.

[11] M. Moshinsky, C. Quesne, Linear canonical transformations and their unitary representations, J. Math. Phys. 12 (8) (1971) 1772–1783.

[12] S. C. Pei, J. J. Ding, Relations between fractional operations and time-frequency distributions, and their applications, IEEE Trans. Signal Process. 49 (8) (Aug. 2001) 1638–1655.

[13] K. K. Sharma, S. D. Joshi, Signal separation using linear canonical and fractional Fourier transforms, Opt. Commun. 265 (2) (Sep. 2006) 454–460.

[14] Z. W. Li, W. B. Gao, B. Z. Li. The solvability of a class of convolution equations associated with 2D FRFT, Mathematics. 8 (11) (Nov. 2020) 1-12.

[15] L. B. Almeida, The fractional Fourier-transform and time-frequency representations, IEEE Trans. Signal Process. 42 (11) (Nov. 1994) 3084–3091.

[16] D. Y. Wei, Y. M. Li, Convolution and multichannel sampling for the offset linear canonical transform and their applications, IEEE Trans. Signal Process. 67 (23) (Dec. 2019) 6009-6024.

[17] X. Y. Zhi, D. Y. Wei, W. Zhang, A generalized convolution theorem for the special affine Fourier transform and its application to filtering, Optik. 127 (5) (2016) 2613-2616.
[18] Q. Xiang, K. Y. Qin. Convolution, correlation, and sampling theorems for the offset linear canonical transform, Signal Image Video Process. 8 (3) (Mar. 2014) 433-442.

[19] D. Urynbassarova, A. Urynbassarova, E. Al-Hussam, The Wigner-Ville distribution based on the offset linear canonical transform domain, in: International Conference on Modelling Simulation and Applied Mathematics (MSAM), March 26, 2017, pp. 139–142.

[20] J. Shi, X. P. Liu, N. T. Zhang, Generalized convolution and product theorems associated with linear canonical transform, Signal, Signal Image Video Process. 8 (5) (Jul. 2014) 967-974.

[21] Y. Xu, M. H. Xu, L. H. V. Wang, Exact frequency-domain reconstruction for thermoacoustic tomography-II: Cylindrical geometry, IEEE Trans. Med. Imag. 21 (7) (Jul. 2002) 829–833.

[22] B. Rigaud et al. , Statistical shape model to generate a planning library for cervical adaptive radiotherapy, IEEE Trans. Med. Imag. 38 (2) (Feb. 2019) 406–416.

[23] J. D. Suever et al. , Right ventricular strain, torsion, and dyssynchrony in healthy subjects using 3D spiral cine DENSE magnetic resonance imaging, IEEE Trans. Med. Imag. 36 (5) (May. 2017) 1076–1085.

[24] G. Chirikjian, A. Kyatkin, Engineering applications of noncommutative harmonic analysis: with emphasis on rotation and motion groups. New York, NY, USA: Academic, 2001.

[25] Z. C. Zhang. Convolution theorem for two-dimensional LCT of angularly periodic functions in polar coordinates, IEEE Signal Process. Lett. 26 (8) (Aug. 2019) 1142-1146.

[26] A. I. Zayed, Sampling of signals bandlimited to a Disc in the linear canonical transform, IEEE Signal Process. Lett. 25 (12) (Dec. 2018) 1765-1769.
[27] Z. C. Zhang et al., Sampling theorems for bandlimited function in the two-dimensional LCT and the LCHT domains. Digital Signal Process. 114 (1) (Apr. 2021) 103053.

[28] I. Gradshteyn, I. Ryzhik, Tables of integrals, series, and products. New York, NY, USA: Academic, 1965.

[29] J. V. Cornacchio, R. P. Soni, On a relation between two-dimensional Fourier integrals and series of Hankel transforms, J. Res. Nat. Bureau. Standard. B, Math. Math. Phys. 69B (3) (Jul. 1965) 173-174.

[30] N. Baddour, Two-dimensional Fourier transforms in polar coordinates, Adv. Imag. Electron. Phys. 165 (2011) 1–45.

[31] N. Baddour, Operational and convolution properties of two-dimensional Fourier transforms in polar coordinates, J. Opt. Soc. Amer. A. 26 (8) (Aug. 2009) 1767–1777.

[32] A. Oppenheim, R. Schafer, Discrete-time signal processing. Englewood Cliffs, NJ, USA: Prentice-Hall, 1989.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Acknowledgments**

This work was supported by the National Natural Science Foundation of China [No. 61671063].