BPS States of the Non–Abelian Born–Infeld Action

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Abstract

We argue that the trace structure of the non–abelian Born–Infeld action can be fixed by demanding that the action be linearised by certain energy–minimising BPS–like configurations. It is shown how instantons in D4-branes, $SU(2)$ monopoles and dyons in D3-branes, and vortices in D2-branes are all BPS states of the action recently proposed by Tseytlin. All such configurations can be dealt with exactly within the context of non–abelian Born–Infeld theory since, given the relevant BPS–like condition, the action reduces to that of Yang–Mills theory. It would seem, moreover, that such an analysis holds for the symmetrised trace structure of Tseytlin’s proposal only.
1 Introduction

The Dirac–Born–Infeld (DBI) action has some remarkable properties. Not least is the fact that it “knows” about energy–minimising BPS states, or worldvolume “solitons”, in the sense that it admits a supersymmetric extension. For such states, the action is linearised, reducing to the simpler Maxwell theory. The worldvolume theories of D_\(p\)-branes with specific values of \(p\) admit the following such solitons: abelian instantons in the D4-brane [1]; abelian monopoles and dyons in the D3-brane [2]; abelian vortices in the D2-brane [3, 2]; and kinks in the D-string [3]. For all these cases, it was shown in [1] that, given the relevant BPS–like condition, the energy is minimised; in fact, all of the above configurations follow from the D4-brane case by successive applications of T–duality.

In this latter work it was, furthermore, claimed that the same ideas should hold for the non–abelian generalisation of the DBI action, relevant to the description of multiple D-branes; that, indeed, such properties could be viewed as criteria for fixing the form of this, the non–abelian Born–Infeld (NBI) action. This is the view taken in this letter, in which we consider the non–abelian generalisation of the results of [1] concentrating, where necessary, on the SU(2) case for definiteness. Although there has been some work on the question as to what is the correct generalisation of the DBI action to the non–abelian case, the issue still seems to be somewhat ambiguous. We will show here that the action recently proposed by Tseytlin [5, and verified in [6], is singled out by demanding such BPS properties; the arguments being really very simple. This would suggest the existence of some supersymmetric extension of this action, as opposed to any other.

2 General Considerations

SU(\(N\)) Yang–Mills theory should provide a good description of the relevant dynamics of \(N\) coincident D-branes [7]. It would seem, however, that an NBI action, of which Yang–Mills theory is just a “non–relativistic” approximation, should be used. The natural such action would be a generalisation of the DBI action, in which the field strength is replaced by its non–abelian counterpart, and in which the worldvolume metric is multiplied by a unit matrix in the group space. Then, since the action must be a group scalar, we should trace over it, e.g. [8]:

\[ L_p = T_p \ Tr \left[ I - \sqrt{-\det(\eta_{ab}I + F_{ab})} \right], \]  

(1)
where $I$ is the unit $SU(N)$ matrix, $F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b] = F^A_{ab} t^i$ is the non–abelian field strength, $\{t^i, i = 1, \ldots, N^2 - 1\}$ are an hermitian basis of the $SU(N)$ algebra, $[t^i, t^j] = i \varepsilon^{ijk} t^k$, and the trace is over the fundamental representation. We work throughout in units such that $2\pi \alpha' = 1$; the tension of the branes is then $T_p = g_s^{-1}(2\pi)^{(1-p)/2}$, which includes a factor of the string coupling constant $g_s = e^{\Phi}$. We have, for the time being, ignored the (matrix–valued) transverse coordinates. As explained in [6], other trace structures, such as \( \sqrt{\text{Tr}(-\det(\eta_{ab} I + F_{ab}))} \), which is used implicitly in [1], can be ruled out immediately.

We can, however, consider different group trace operations, and this is where some ambiguity over the form of the action appears. Tseytlin has argued [5] that the NBI action should take the form

\[
L_p = T_p \text{STr} \left[ I - \sqrt{-\det(\eta_{ab} I + F_{ab})} \right],
\]

where STr is a symmetrised trace, given by $\text{STr}(M_1, \ldots, M_n) = \frac{1}{n!} \sum_{\pi} \text{Tr}(M_{\pi(1)} \ldots M_{\pi(n)})$. By including all possible permutations of the matrices, the STr operation resolves the matrix–ordering ambiguities involved in taking the determinant of a matrix–valued function. We can also consider an antisymmetrised trace $\text{ATr}(M_1, \ldots, M_n) = \frac{1}{n!} \sum_{\pi} (-1)^\pi \text{Tr}(M_{\pi(1)} \ldots M_{\pi(n)})$ and make use of the combination $\text{STr} + i\text{ATr}$, e.g. [3], the factor of $i$ being necessary since the basis of the group algebra is hermitian. These would seem to be the only possibilities.

We will argue that, of these three trace structures, Tseytlin’s proposal is the only one which allows for the BPS properties with which this paper is concerned.

To this end, then, we will first consider D4-branes, with no scalar fields excited; that is, we set $X^\mu = \text{constant}$. The expansion of the spacetime determinant in (2) gives

\[
- \det(\eta_{ab} I + F_{ab}) = I + \frac{1}{2} F^2 + \frac{1}{3} F^3 - \frac{1}{4} \left[ F^4 - \frac{1}{2} (F^2)^2 \right] + \frac{1}{5} F^5 + \frac{1}{12} (F^2 F^3 + F^3 F^2),
\]

where $F^2 = F_{ab} F^{ab}$, $F^3 = F_{ab} F^{bc} F^d c^a$, $F^4 = F_{ab} F^{bc} F_{cd} F^{de} F^f c^a$ and $F^5 = F_{ab} F^{bc} F_{cd} F^{de} F^f c^a$. In the abelian case, all the odd powers of $F_{ab}$ vanish identically, but this is not so for the case at hand. A binomial expansion of this expression results in a infinite series, which is where the difficulties in the STr prescription occur, since we must expand the binomial series before the trace can be taken\[^3\]. At any rate, it should be clear that the resulting expansion is just a sum of both even and odd powers of $F_{ab}$.

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\[^2\]A note on indices: $a, b = 0, 1, \ldots, p$ denote worldvolume directions; $\alpha, \beta = 1, \ldots, p$ denote worldspace directions; $\mu, \nu = p + 1, \ldots, 9$ denote directions transverse to the brane; and $i, j$ will be group indices.

\[^3\]Interesting progress has recently been made [10], in which it has been shown that the action describing two D0-branes can be written in a closed form, after having taken the symmetrised trace.
The important properties of the STr and ATr operations is that they pick out the even and odd powers of $F_{ab}$ respectively. Moreover, it is clear that, at least for the $SU(2)$ case, and to the first few orders, the same will apply to the cross terms generated by the binomial expansion; that is, e.g., $\text{STr}(F^2 F^3) = \text{STr}(F^2 F^5) = 0$. So we can state, quite generally, that the lagrangian (4) can be written as a sum of even powers of $F_{ab}$ alone, whereas if we were to use either Tr or ATr, odd powers would also be included. Indeed, this was the motivation behind Tseytlin’s proposal. Since odd powers of $F_{ab}$ can be written in terms of derivatives of $F_{ab}$, e.g. $F^3 \sim [F,F]F \sim [D,D]F$, and in analogy with the abelian case, in which the DBI action does not include derivatives of the field strength, Tseytlin was led simply to define the NBI action to depend on even powers of $F_{ab}$ alone.

It must be noted that this discussion should be viewed from within the context of a general analysis of possible NBI actions. That is, the action relevant to the description of multiple D-branes has its origin in open superstring theory coupled to a non–abelian gauge field; and it is known that the effective action of this latter does not contain a term of the form $F^3$. From the point of view of string theory, then, it would seem that the STr prescription alone is acceptable. Given that we want the NBI action to have the BPS properties discussed, the arguments presented here should then be taken as evidence for the STr prescription, string theory aside.

## 3 Instantons in D4-Branes

For static configurations of D4-branes, $F_{a0} = E_a = 0$, and we have $-\det(\eta_{ab}I + F_{ab}) = \det(\delta_{\alpha\beta}I + F_{\alpha\beta})$. Then

\[
\mathcal{L}_4 = T_4 \text{Str} \left[ I - \sqrt{I + \frac{1}{4} F^2 + \frac{1}{4} \tilde{F}^2 + \frac{1}{16} (F \cdot \tilde{F})^2} \right] = T_4 \text{Str} \left[ I - \sqrt{(I \pm \frac{1}{4} F \cdot \tilde{F})^2 - \frac{1}{4} \text{tr} |F \mp \tilde{F}|^2} \right],
\]  

(4)

where, since we are dealing with static configurations, $\tilde{F}_{\alpha\beta}$ is the Hodge dual of $F_{\alpha\beta}$, with respect to the worldspace indices only, and $F \cdot \tilde{F} = F_{\alpha\beta} \tilde{F}_{\alpha\beta}$. It is important to note that in deriving this equation, and those appearing below, use has been made of the symmetry properties of the STr operation. That is, the matrices can be treated as if they were abelian until the last, at which point the non–commuting group generators can be re–inserted.
This seems somewhat odd, but is in fact not at all since, under STr, we can assume $AB = BA$. Any matrices can be freely interchanged under this operation; and the procedure in [13, 14] is then fully justified. Indeed, such a procedure automatically removes all odd powers of $F_{ab}$ explicitly since, for the abelian case, all such powers vanish identically.

It is easy to see, then, that for the (anti–)self–dual configuration, for which $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$, the determinant can be written as a complete square, as has already been noted in [14, 3]. The action is then linearised, becoming that of Yang–Mills theory. Since we are dealing with static configurations, the energy density is just $T_{400}^{00} = -L_4$, this being minimised if and only if $F_{\alpha\beta} = \pm \tilde{F}_{\alpha\beta}$. Thus, the (anti–)self–duality condition at once linearises the action and minimises the energy; hence its BPS interpretation. We will see such properties for all the static configurations considered below. The (anti–)self–duality condition is, as usual, solved by multi–(anti–) instanton configurations, D0-branes from the worldvolume point of view.

How is this story changed if we were to use a different group trace operation? That is, how do the above considerations single out the STr operation alone? A very simple argument shows that, if we were to use either Tr or STr + $i$ATr, the above analysis would no longer follow through. In either of these cases, odd powers of $F_{\alpha\beta}$ would be introduced into the NBI action (4). In the generic case, we would have to consider the $O(F^5)$ terms in (3) although, for the static case at hand, the only additional term is that of $F^3 = F_{\alpha\beta} F_{\beta\gamma} F_{\gamma\alpha}$. Under either Tr or ATr, this term can be rewritten in terms of the dual field strength as $F^3 = \tilde{F}_{\alpha\beta} \tilde{F}_{\beta\gamma} F_{\gamma\alpha}$. It is unclear, however, as to how such a term could be included in the structure of (4) viz., a sum of squares. Under the standard Tr operation, the picture becomes more complicated still, since we can no longer treat the matrices as if abelian. At any rate, it would seem that the determinant simply cannot be written as a sum of squares if we are to include odd powers of $F_{\alpha\beta}$. If we cannot write the determinant as a sum of squares, it certainly will not reduce to the linear Yang–Mills action given the (anti–)self–duality condition. It would be difficult to claim that the energy is minimised by such configurations in this case, since the NBI action cannot be linearised in any simple fashion. If we are to demand both that the action is linearised, and that the energy is minimised by such (anti–)self–dual configurations, we simply cannot include odd powers of $F_{\alpha\beta}$. And the only way of ensuring this is to use the STr operation and no other. This same argument holds for all the configurations considered below.
4 Monopoles and Dyons in D3-Branes

Since D0-branes in D4-branes are T–dual to D-strings ending on D3-branes, the above instanton configurations will give rise to monopoles and dyons within D3-branes [1]. In [14], it was shown how the standard $SU(2)$ BPS monopole solution of Yang–Mills theory can be interpreted as a D-string joining two D3-branes; and, via S–duality, a fundamental string joining the two branes. This work follows through for precisely the reason that the BPS condition linearises the NBI action, and so a mapping between the latter and the Yang–Mills action is possible. We will consider the dyonic generalisation of these results here.

Including the transverse scalars, we have [5, 6]

$$L_p = T_p \text{STr} \left[ I - \sqrt{\det(\delta_{\mu\nu}I - i[X_\mu, X_\nu])} \right] \times \sqrt{\det(\eta_{ab}I + D_aX_\mu(\delta_{\mu\nu}I - i[X_\mu, X_\nu])^{-1}D_bX_\nu + F_{ab})}.$$  

(5)

Since we will excite a single scalar only, the commutators vanish identically here; such a simplification cannot be made, however, in sections to follow. We excite both the electric and magnetic worldvolume fields, and impose the “static” condition $D_0X = 0$, in which case the lagrangian is given by

$$L_3 = T_3 \text{STr} \left[ I - \sqrt{I + |\vec{D}X|^2 - |\vec{E}|^2 + |\vec{B}|^2 - |\vec{E} \cdot \vec{B}|^2 - |\vec{E} \times \vec{D}X|^2 + (\vec{B} \cdot \vec{D}X)^2} \right].$$  

(6)

The energy density is no longer simply the negative of the lagrangian density, however; and this is where our results depart somewhat from those of [1]. That is, the latter work seems to make use of an energy functional different to that of [3, 2], for reasons which are unclear to the author.

We claim here that the energy density can be given by a similar Legendre transform as in the abelian case [15]. Since taking the variation of the determinant in the NBI action is an operation which commutes with STr, $\delta(\text{STr}(F^n)) = \text{STr}(\delta(F^n))$, we can, as usual, define the energy–momentum tensor by $T^{ab}_p \sim \frac{\delta L_p}{\delta g_{ab}}$. The result is just what we find via the prescription

$$T^{00}_p = \vec{D}^i \cdot \vec{E}^i - L_p,$$  

(7)

where $\vec{D}^i = \frac{\partial L_p}{\partial \vec{E}^i}$. Using (6), we then have

$$T^{00}_3 = T_3 \text{STr} \left[ \frac{(I + |\vec{D}X|^2)(I + |\vec{B}|^2)}{\sqrt{I + |\vec{D}X|^2 - |\vec{E}|^2 + |\vec{B}|^2 - |\vec{E} \times \vec{D}X|^2 - |\vec{E} \cdot \vec{B}|^2 + (\vec{B} \cdot \vec{D}X)^2}} - I \right].$$  

(8)
where the denominator must be viewed formally as the inverse of some matrix-valued (binomial) function. There is no inconsistency here since the fields can be taken to be abelian, as explained above. At any rate, we see that the energy is bounded from below by configurations for which \( \vec{E} = \sin \theta \vec{D} \Phi \) and \( \vec{B} = \cos \theta \vec{D} \Phi \), for arbitrary angle \( \theta \) \( ^{[1]} \). In this case, the lagrangian is again linearised, and we have

\[
\mathcal{L}_3 = -\frac{T_3(2\pi \alpha' g)^2}{2} \text{Tr} \left[ |\vec{B}|^2 - |\vec{E}|^2 + (2\pi \alpha' g)^{-2} |\vec{D} \Phi|^2 \right],
\]

(9)

\[
T_3^{00} = \frac{T_3(2\pi \alpha' g)^2}{2} \text{Tr} \left[ |\vec{B}|^2 + |\vec{E}|^2 + (2\pi \alpha' g)^{-2} |\vec{D} \Phi|^2 \right],
\]

(10)

where we have reinserted the relevant factors of \( 2\pi \alpha' \) and the coupling constant \( g \). Comparing with the usual Yang–Mills lagrangian, \( \mathcal{L}_{YM} = -\text{Tr} \left[ |\vec{B}|^2 - |\vec{E}|^2 + |\vec{D} \Phi|^2 \right] \), we see that if we take \( \frac{1}{2} T_3(2\pi \alpha')^2 = g^{-2} \) and \( (2\pi \alpha' g)^{-1} X_i = \Phi^i \) solutions of the Yang–Mills theory will be solutions of the linearised NBI theory \( ^{[14]} \).

The Prasad and Sommerfield solution \( ^{[16]} \) is given in terms of the ansatz

\[
\begin{align*}
A^i_{\alpha} &= \varepsilon_{i\alpha\beta} \hat{x}_\beta (1 - K(r))/gr, \\
A^0_i &= \hat{x}_i J(r)/gr, \\
\Phi^i &= \hat{x}_i H(r)/gr,
\end{align*}
\]

(11)

where \( r \) is a radial coordinate and \( \{\hat{x}_i\} \) are unit vectors. The dyonic solution, satisfying \( \vec{E} = \sin \theta \vec{D} \Phi \) and \( \vec{B} = \cos \theta \vec{D} \Phi \), has

\[
\begin{align*}
K(r) &= Cr/\sinh(Cr), \\
J(r) &= \frac{\sin \theta}{\cos \theta} [Cr \coth(Cr) - 1], \\
H(r) &= \frac{1}{\cos \theta} [Cr \coth(Cr) - 1],
\end{align*}
\]

(12)

where the constant \( C = vg \), \( v \) being the expectation value of the Higgs field. Standard analysis then gives \( T_{YM}^{00} = v \sqrt{q_E^2 + q_M^2} \) where \( q_M = 4\pi/g \) and \( q_E = \tan \theta q_M \). Asymptotically, the Higgs field can always be diagonalised by performing a gauge transformation, interpreting the diagonal entries as the asymptotic positions of the two branes: \( X = X^3 \sigma^3/2 = X(r) \). Then

\[
X(r) = \pm \frac{1}{2} \frac{2\pi \alpha'}{\cos \theta} \left[ C \coth(Cr) - \frac{1}{r} \right],
\]

(13)

in which \( \theta = 0 \) corresponds to the results of \( ^{[14]} \), the purely magnetic case. Taking \( C = (2\pi \alpha')^{-1} \Delta X \), with \( \Delta X \) the separation of the branes gives, as \( r \to \infty \), \( X(r) \to \)
\( \pm (1/\cos \theta) \Delta X/2 \). Turning on the electric field (increasing \( \theta \)) increases the effective separation of the branes. \( \theta = \pi/2 \), the purely electric case, gives an infinite separation, corresponding to the semi–infinite string solution of [3, 4]. Quantum mechanically, we can take \( q_E = ng, n \) an integer. Then, upon making the relevant substitutions, we have

\[
T_{YM}^{00} = (2\pi \alpha')^{-1} \Delta X \sqrt{n^2 + \frac{1}{g_s^2}}
\]  

which is precisely the energy of an \((n, 1)\) string of length \( \Delta X \), a dyon from the worldvolume point of view [1].

5 Vortices in D2-Branes and Hitchin’s Equations

By dimensionally reducing the (anti–)self–duality condition of section three a second time, we obtain Hitchin’s equations [17]: with \( X \) and \( Y \) the two relevant transverse coordinates of the branes

\[
\begin{align*}
\bar{D}X &= \mp \star \bar{D}Y, \\
\star F &= \mp i[X, Y],
\end{align*}
\]  

which should describe non–abelian vortices in the worldvolume of D2-branes [1]. As above, the Hodge dual is taken with respect to the spatial directions only: \( \bar{D} = (D_1, D_2), \star \bar{D} = (D_2, -D_1) \) and \( \star F = F_{12} \).

In the generic case, the two scalars will not be simultaneously diagonalisable, the commutator terms in (5) will be non–vanishing, and no natural interpretation in terms of classical coordinates will be possible. Then

\[
\det(\delta_{\mu\nu} I - i[X_\mu, X_\nu]) = I - [X, Y]^2,
\]  

and

\[
D_a X_\mu (\delta_{\mu\nu} I - i[X_\mu, X_\nu])^{-1} D_b X_\nu = (I - [X, Y]^2)^{-1} (D_a X_\mu D_b X_\mu + i D_a X_\mu [X_\mu, X_\nu] D_b X_\nu). \]  

Evaluating the determinant over the \( a, b \) indices gives [1]

\[
\mathcal{L}_2 = T_2 \text{STr} \left[ I - \left\{ I + |\bar{D}X|^2 + |\star \bar{D}Y|^2 - |X, Y|^2 + |\star F|^2 \\
+ |\bar{D}X \cdot \star \bar{D}Y|^2 - |\star F|^2 |X, Y|^2 + 2i \star F (\bar{D}X \cdot \star \bar{D}Y) [X, Y] \right\}^{1/2} \right] = T_2 \text{STr} \left[ I - \sqrt{\left( I \mp (\star F i[X, Y] + \bar{D}X \cdot \star \bar{D}Y) \right)^2 + |\bar{D}X \pm \star \bar{D}Y|^2 + |\star F \pm i[X, Y]|^2} \right].
\]
Thus, the energy density \( T_{00}^2 = -L \) is

\[
T_{00}^2 \geq \mp T_2 \Tr \left[ \star F_i [X, Y] + \bar{D} X \cdot \star \bar{D} Y \right] = T_2 \Tr \left[ \frac{1}{2} |\bar{D} X|^2 + \frac{1}{2} |\bar{D} Y|^2 - [X, Y]^2 \right],
\]

the last step being valid for the energy–minimising configurations for which Hitchin’s equations (15) are obeyed. In this case, the lagrangian (18) is once again linearised. The, superficially complicated, lagrangian (5), involving the product of two determinants taken over different indices reduces to the simple dimensionally reduced Yang–Mills lagrangian, when the BPS–like conditions hold.

6 D-strings and Nahm’s equations

Nahm’s equations [18], the dimensionally reduced version of Hitchin’s equations, reduce the \( SU(2) \) monopole problem in three (spatial) dimensions to a one–dimensional problem. That is, the dyon of section four can be described from within either the D3-branes’ worldvolume, or from within that of the string [1]. Indeed, it was shown in [19] from the point of view of Yang–Mills theory, that Nahm’s equations do in fact follow from the worldvolume theory of the D-string. Here we will show this to be true for the full–blown NBI action.

To this end, we consider static configurations, in which case \( F_{ab} \rightarrow F_{10} = E_1 = 0 \). Then the relevant lagrangian is the non–abelian generalisation of the Dirac lagrangian, in which three scalars are excited:

\[
L_1 = T_1 \text{STr} \left[ I - \sqrt{\det(\delta_{\mu \nu} - i[X_\mu, X_\nu])} \det(I + DX_\mu(\delta_{\mu \nu} I - i[X_\mu, X_\nu])^{-1} DX_\nu) \right],
\]

(20)

where \( D_\alpha X_\mu \rightarrow D_1 X_\mu \equiv DX_\mu \) and \( \mu, \nu \) run over the three spatial directions of the D3-brane. Then

\[
\det(\delta_{\mu \nu} I - i[X_\mu, X_\nu]) = I - \frac{1}{2} [X_\mu, X_\nu]^2,
\]

(21)

and

\[
DX_\mu(\delta_{\mu \nu} I - i[X_\mu, X_\nu])^{-1} DX_\nu = \left( I - \frac{1}{2} [X_\mu, X_\nu]^2 \right)^{-1} \left[ DX_\mu DX_\nu - \left( \frac{1}{2} \varepsilon_{\mu \nu \rho} X_\mu [X_\nu, X_\rho] \right)^2 \right].
\]

(22)

Substituting these latter into (20) gives the lagrangian

\[
L_1 = T_1 \text{STr} \left[ I - \sqrt{I + DX_\mu DX_\mu - \frac{1}{2} [X_\mu, X_\nu]^2 - \left( \frac{1}{2} \varepsilon_{\mu \nu \rho} DX_\mu [X_\nu, X_\rho] \right)^2} \right]
\]

\[
= T_1 \text{STr} \left[ I - \sqrt{I + \frac{i}{2} \varepsilon_{\mu \nu \rho} DX_\mu [X_\nu, X_\rho]^2} + \text{tr} \left| DX_\mu \mp \frac{i}{2} \varepsilon_{\mu \nu \rho} [X_\nu, X_\rho] \right|^2 \right].
\]

(23)
The energy density, $T_{00}^1 = -\mathcal{L}_1$, and so

$$T_{00}^1 \geq \pm T_1 \frac{i}{2} \text{Tr} [\varepsilon_{\mu\nu\rho} DX_\mu [X_\nu, X_\rho]] = T_1 \frac{1}{2} \text{Tr} \left[ DX_\mu DX_\mu - \frac{1}{2} [X_\mu, X_\nu]^2 \right],$$  \hspace{1cm} (24)$$

the last step being valid if Nahm’s equations,

$$DX_\mu = \pm \frac{i}{2} \varepsilon_{\mu\nu\rho} [X_\nu, X_\rho],$$  \hspace{1cm} (25)$$

are obeyed. In this case, as expected, the action is linearised, and the energy minimised.

A final T–duality takes us back to the configuration of section three, but now from the point of view of the D0-branes lying within the D4-branes. We excite the four relevant scalars, those corresponding to the four spatial directions of the D4-brane, and consider static configurations, in which case the determinant over the $a, b$ indices in (5) drops out entirely. As should be expected, the lagrangian is formally identical to (4), with $F_{\alpha\beta} \rightarrow -i[X_\mu, X_\nu]$. The energy $T_{00}^0 = -\mathcal{L}_0$ is minimised, and the lagrangian is linearised if

$$[X_\mu, X_\nu] = \pm \varepsilon_{\mu\nu\rho\sigma} X_\rho X_\sigma,$$  \hspace{1cm} (26)$$

which is just the (anti–)self–duality condition of section three from the D0-branes’ point of view. When this condition holds \[1\]

$$T_{00}^0 = \pm T_0 \frac{1}{4} \text{Tr} [\varepsilon_{\mu\nu\rho\sigma} X_\mu X_\nu X_\rho X_\sigma],$$  \hspace{1cm} (27)$$

which vanishes identically for finite dimensionally matrices. In the M(atrix) theory $N \rightarrow \infty$ limit, however, the energy density (27) in the case of D0-branes on $T^4$ corresponds to a single unit of D4-brane charge \[20\]. In this limit, then, the instanton configurations of section three can be described from the point of view of the D0-branes’.

### 7 Discussion

The moral of the story is that we can use the facts that BPS–like conditions should, firstly, linearise the NBI action and, secondly, should minimise the NBI energy, as criteria to fix the trace structure of this action. That is, from the three possible trace structures which such an action could have — Tr, STr, and STr + iATr — Tseytlin’s STr prescription is singled out. In this case, we have shown how certain worldvolume “solitons” are BPS states of the NBI action. All such configurations minimise the worldvolume energy, this being due to the fact that, given the relevant BPS–like condition, the determinant in the NBI action can be
written as a complete square. The relevant action for all these configurations is just that of (dimensionally reduced) Yang–Mills theory. The reason that STr is singled out by this analysis is simple: only in this case do no odd powers of the field strength, or of the world-volume scalars, appear in the NBI action and so, only in this case, can the determinants be written as complete squares. Only for this prescription, then, can such exact statements concerning the form of the NBI action be made. Moreover, as the usual BPS equation follows ultimately from supersymmetry considerations, it would be natural to postulate that the STr prescription, and no other, allows a supersymmetric extension.

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