Quantum discontinuity fixed point and renormalization group flow of the SYK model

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We determine the global renormalization group (RG) flow of the Sachdev-Ye-Kitaev (SYK) model. This flow allows for an understanding of the surprising role of critical slowing down at a quantum first-order transition in strongly-correlated electronic systems. From a simple truncation of the infinite hierarchy of the exact functional RG flow equations we identify several fixed points: Apart from a stable fixed point, associated with the celebrated non-Fermi liquid state of the model, we find another stable fixed point related to an integer-valence state. These stable fixed points are separated by a discontinuity fixed point with one relevant direction, describing a quantum first-order transition. Most notably, the fermionic spectrum continues to be quantum critical even at the discontinuity fixed point. This rules out a description of this quantum first-order transition in terms of a local effective Ising variable that is established for classical transitions. It reveals that quantum phase coexistence can be a genuine critical state of matter.

The liquid-vapor transition of real gases is the prime example of a first-order phase transition that preserves the symmetry at a critical end-point [1]. Such transitions are of importance in systems as diverse as hot and dense nuclear matter [2], polymer-gel fluid mixtures [3], and correlated-electron systems. The famous Mott transition between states of localized and delocalized electrons [4–17], the Kondo volume-collapse transition [18–22], or valence transitions in inter-metallic compounds [23] are prominent correlated-electron problems of this kind. Significant experimental and theoretical insights have been gained with regards to the behavior of the classical critical end-point of these electronic states; it is described in terms of an effective Ising model and governed by critical elasticity, a vanishing bulk modulus, and rich crossover behavior due to the finite shear modulus of crystalline solids [5, 6, 24–28].

Much less is known about the associated quantum first-order transition. It is an open question whether the description in terms of an effective Ising model, successful for classical transitions, continues to be appropriate. This is particularly ambiguous if the transition is from a non-Fermi liquid, strange-metal state to a fully localized incompressible state of matter. Exotic behavior was found near several quantum first-order transitions [29–32]. A symmetry-preserving first-order transition that terminates at a critical end-point was recently identified [33–38] in the Sachdev-Ye-Kitaev (SYK) model [39, 40] and generalizations thereof. The SYK model proved important for our understanding of the intriguing properties of interacting quantum matter without quasiparticles [35, 41–49]. The ability to perform controlled calculations in the strong-coupling regime of the SYK model makes it a promising platform to elucidate the role of electronic dynamics at a quantum first-order transition.

In this paper, using a functional renormalization group (FRG) approach, we analyze the scaling behavior at a first-order quantum phase transition in the SYK model. Performing a large-\(N\) truncation of the formally exact FRG flow equations [50–54], we derive the global renormalization group (RG) flow of the SYK model at zero temperature. As shown in Fig.1, a flow profile emerges that exhibits four different fixed points: Apart from a
trivial vacuum fixed point $V$ we find two stable fixed points $S_+$ and $S_-$ separated by a discontinuity fixed point $D$. The latter is driven by changing the chemical potential $\mu$ and, as we will show, describes the first-order transition between a critical non-Fermi liquid state ($S_+$) and an integer-valence state ($S_-$). The equation of state of this transition was analyzed in Refs. [34, 37]; similar transitions were discussed in a number of closely related models [33, 35, 36], reflecting for example a drastic change from fast to slow quantum information scrambling [33]. Here we determine the relevant exponent of the discontinuity fixed point in accordance with the scaling theory of first-order transitions [55, 56]. In addition, we find that fermions at the transition are governed by quantum-critical dynamics with an anomalous dimension, despite the discontinuous change of thermodynamic variables. Our results are a consequence of the subtle interplay between high-energy and low-energy dynamics that can be captured by the FRG approach developed here. Our results may also shed light onto the recent observation of powerlaw behavior of the dielectric function at a first-order Mott transition [16, 17].

SYK model: The SYK model describes $N$ species of fermions interacting with random two-body interactions,

$$\mathcal{H} = -\mu \sum_i c_i^\dagger c_i + \sum_{i<j,k<l} J_{ij,kl} c_i^\dagger c_j^\dagger c_k c_l. \quad (1)$$

Here $c_i$ and $c_i^\dagger$ are fermionic annihilation and creation operators, $i,j\cdots$ label $N$ different orbitals or lattice sites, and $\mu$ is the chemical potential. The $J_{ij,kl}$ are random variables with Gaussian distribution of zero mean and variance $\langle J_{ij,kl}^2 \rangle = 2J^2/N^3$. The model is exactly solvable in the limit $N \to \infty$ where the single-particle properties are determined by the self energy $\Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$ together with the Dyson equation $G^{-1}(\omega) = i\omega + \mu - \Sigma(\omega)$. At temperature $T = 0$ one can construct two solutions of this set of equations. On the one hand, there is the critical, non-Fermi liquid (NFL) solution which takes for $|\omega| \ll J$ the form

$$\Sigma_{\text{NFL}}(\omega) = \mu - \frac{J A(\theta, \omega)}{\pi^{1/4}} \frac{|\omega|^\eta}{J}, \quad (2)$$

with anomalous dimension $\eta = 1/2$ [57]. The coefficient $A = \cos^{1/4}(2\theta) e^{-i\text{sign}(\omega)(\theta - \pi/2)}$ depends on the particle number $n = 1/2 + \theta/\pi + \sin(2\theta)/4$ through the angle $\theta \in [-\pi/2, \pi/2]$ [41]. The solution, Eq.(2), with power-law propagator $G(\tau) \propto \tau^{-1/2}$ yields the much discussed finite ground state entropy, displays analogies to the physics of black holes with regards to its information scrambling, led to theories of the transport properties of strange metals, and allowed for the analysis of superconductivity in non-Fermi liquids [35, 41-49]. However, at $T = 0$ and finite $\mu$ one can easily show that

$$\Sigma_{\text{IV}}(\omega) = 0 \quad (3)$$

is another solution of the large-$N$ equations describing an integer-valence (IV) state. Depending on the sign of $\mu$ we have $n = 1$ or $0$, i.e. a completely filled or empty system. As shown in Ref. [34] a first-order transition occurs between both solutions at $\mu = \pm \mu_*(T)$. At $T = 0$ and $\mu = \mu_*(0) \approx 0.212 J$ the density jumps from $n \approx 0.76$ to 1. For $\mu = -\mu_*(0)$ it jumps from 0.24 to 0. At $T > 0$, $\mu_*$ terminates at a critical end-point. In Fig. 1 we reproduce the phase diagram already discussed in Ref. [34].
Analyzing Eq. (5) we find the fixed points shown in Fig. 1: i) the unstable vacuum fixed point $V$ at $r = u = g = 0$; ii) the discontinuity fixed point $D$ with one relevant direction; iii) the stable NFL fixed point $S_+$ with $\Sigma(0) = \mu$, preserving scaling. Here, the renormalized two-body interaction diverges to $+\infty$ and the coupling $g$ approaches $\pi^2/3$; and iv) the stable integer-valence fixed point $S_-$ with $\Sigma(0) \neq \mu$, corresponding to a gapped state yet with a divergent attractive two-body interaction $u \to -\infty$. Notice, the large-$N$ self-consistency equations for the self-energy $\Sigma(\omega)$ can be recovered from our FRG approach. This is achieved by substituting our result for the four-point vertex into the flow equation for the self-energy and integrating over the flow parameter $\Lambda$.

In order to characterize the fermionic dynamics we determine the scale-dependent anomalous dimension via

$$\eta = \frac{\Lambda Z_\Lambda}{\beta} \sum_{\omega_2} G_\Lambda(\omega_2) \left. \frac{\partial \Gamma^{(4)}_\Lambda(\omega_1, \omega_2; \omega_2, \omega_1)}{\partial (\omega_1)} \right|_{\omega_1 = 0},$$

with single-scale propagator $G_\Lambda(\omega)$ [52, 53]. The four-point vertex $\Gamma^{(4)}_\Lambda(\omega)$ is expressed in terms of scale dependent particle-particle and particle-hole susceptibilities [59]. The behavior of $\eta$ during the flow from the vicinity of the vacuum fixed point with $\eta = 0$ is shown
In Fig. 4 for slightly different initial conditions. As expected, the NFL fixed point $S_\lambda$ has $\eta = 1/2$ in agreement with Eq. (2), while we find $\eta = 0$ for the integer-valence fixed point $S_\mu$, consistent with Eq. (3). The most remarkable finding of our analysis is however that $\eta = 1/2$ at the discontinuity fixed point, see lower part of Fig. 4. While this fixed point describes a discontinuous change of the particle number, the fermionic dynamics is quantum critical. The first order transition is characterized by critical slowing down of the fermions. A direct implication would be the scaling behavior of the longitudinal dielectric constant $\varepsilon_l(\omega) \propto \omega^{2\nu - 2}$ right at phase coexistence. We followed Ref. [44] to determine the electromagnetic response. Power law behavior of $\varepsilon_l(\omega)$ that would imply $\eta \approx 0.75$ was recently observed in Refs. [16, 17] at a Mott transition. While there is no reason to expect that the SYK model yields the value of the measured exponent, its observation supports the concept of phase coexistence as a genuine critical state of matter.

Discontinuity fixed point: At the discontinuity fixed point $D$ all rescaled couplings approach finite limits. This enables us to analyze the linearized flow in its vicinity. Using $\eta = 1/2$ we can determine the numerical values of our couplings by demanding that all scale derivatives in Eq. (3) vanish. We obtain

$$r_s = \frac{1}{3}, \quad u_* = \frac{2\pi}{3}, \quad g_* = \frac{1}{2c} \approx 8.52.$$  \hfill (7)

The result for $g_*$ follows from an analysis of Eq. (6) near $D$ which yields $\eta_\lambda = c_ga_\lambda$ with numerical coefficient $c = 3(\log 2 - \frac{1}{2})/\pi^2 \approx 0.0587$. The precise value of $c$ may weakly depend on details of the regularization scheme. From Fig. 1 we see that the discontinuity fixed point has an unstable and a stable direction. To calculate the corresponding scaling variables, we linearize the flow in its vicinity. Setting $r_1 = r_s + \delta r_1$, and similar for $u_1$ and $g_1$, we obtain the linearized flow equations. Within our truncation $\delta g_1$ decouples and we can focus on the $\delta r - \delta u$-plane. With $\delta u_t = \frac{3}{4}\sqrt{\frac{3}{c}}\delta y_t$ we obtain

$$\partial_t \left( \begin{array}{c} \delta r_1 \\ \delta y_t \end{array} \right) = \left( \begin{array}{cc} \frac{1}{4} & -a \\ -a & 0 \end{array} \right) \left( \begin{array}{c} \delta r_1 \\ \delta y_t \end{array} \right),$$  \hfill (8)

with numerical constant $a = \frac{3}{8\pi}\sqrt{3/c}$. The eigenvalues of the matrix are $\lambda_+ \approx 0.987$ and $\lambda_- = 1/4 - \lambda_+$. The corresponding eigen-directions in the $r-u$-plane follow easily. Near the transition, the relevant eigenvector is proportional to $\mu - \mu_*$. Hence, near the transition, the grand canonical potential per fermion species should - in addition to a regular piece - be characterized by a singular contribution with following scaling behavior,

$$\Omega(\mu) = \Omega_{\text{reg}}(\mu) + e^{-1}\Omega_{\text{sing}}(e^{\lambda_+}(\mu - \mu_*)).$$  \hfill (9)

This yields for the particle density $n = -\partial\Omega/\partial\mu = n_{\text{reg}} \pm B_\pm [\mu - \mu_*]^{-1 + 1/\lambda_+}$. Keeping in mind that our determination of $c$ is approximate, the above result for $\lambda_+$ is consistent with $\lambda_+ = 1$, yielding indeed a discontinuity of $n$ at $\mu_*$. According to scaling arguments near first order transitions [55, 56] the relevant eigenvalue is the space dimension, i.e. $\lambda_+ = d$. For our zero-dimensional quantum system with one time direction this yields indeed $\lambda_+ = 1$.

The linear relation between $\eta_\lambda$ and $g_\lambda$ near the discontinuity fixed point can also be used to determine the crossover energy to the regime with $\eta \approx \frac{1}{2}$. Expressing $g_\lambda$ and $\eta_\lambda$ in terms of $Z_\lambda$, we obtain the closed flow equation of the wave-function renormalization near $D$,

$$\partial_\lambda Z_\lambda = cJ^2\Lambda^{-3}Z_\lambda^2.$$  \hfill (10)

From the exact solution of this equation with initial condition $Z_{\lambda_0} = 1$ we find that critical behavior with $\eta \rightarrow 1/2$ sets in at the crossover scale $\Lambda_c \approx 0.343J$, see the lower part of Fig. 4.

Integer-valence fixed point: Finally, we elucidate our finding that the rescaled two-body interaction $u_1$ at the integer valence fixed point $S_\mu$ shown in Fig. 1 flows to $-\infty$, which corresponds to a finite attractive two-body interaction. Of course, the single-particle properties for occupations $n = 0$ or 1 are trivial. A single electron or hole added to the system has no particle to interact with. This does not apply to higher-particle correlations. Following the classical work of Galitskii [60] the two-particle vertex of a such a dilute system can be obtained by summing up all particle-particle ladder diagrams. We perform this analysis prior to the disorder averaging,

$$\Gamma_{ijkl}(\omega) = J_{ijkl} - \sum_{m<n} J_{ijmn}\Gamma_{mnkl}(\omega)\chi(\omega),$$  \hfill (11)

with particle-particle bubble $\chi(\tau) = G^2(\tau)$. Summing up the series and averaging term-by-term over the $J_{ijkl}$ yields at large $N$ for $i < j$ and $k < l$ and after analytic continuation to retarded functions,

$$\Gamma_{ijkl}^{\text{ret}}(\epsilon - 2\mu) = -\delta_{ik}\delta_{jl}\epsilon \left( 1 - 2 \frac{1}{1 + \sqrt{\frac{c^2 - 4J^2}{\epsilon^2}}} \right).$$  \hfill (12)

This result implies that a spectrum of bonding and anti-bonding two-particle scattering states of bandwidth $4J/\sqrt{N}$ emerges through $\text{Im} \Gamma_{ijkl}^{\text{ret}} \neq 0$ below and above the trivial two-particle energy $-2\mu$. While the single-particle physics of the integer-valence phase is trivial, pairs of correlated electrons or holes propagate coherently, a behavior that is signaled by $u \rightarrow -\infty$ in Fig. 1.

In summary, we formulated the functional RG of the SYK model and determined its global RG flow. Our approach reproduces known results of the two stable phases, the non-Fermi liquid state and the integer-valence state, see Fig. 2. For the integer-valence phase it also revealed interesting two-particle correlations. Most importantly, the FRG allows for insights into the discontinuity fixed point that separates these two phases. The relevant scaling dimension of this fixed point is consistent with the
behavior near first-order transitions [55, 56]. In addition, fermionic excitations at phase coexistence behave quantum critical. Such a behavior cannot be captured in terms of a local Ising variable. Thus, quantum first-order transitions that do not break a symmetry are shown to be dramatically altered by the presence of non-Fermi liquid electronic excitations.

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