Charm Effects in the $\overline{\text{MS}}$ Bottom Quark Mass from $\Upsilon$ Mesons

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Abstract

We study the shift in the $\Upsilon$ mass due to a non-zero charm quark mass. This shift affects the value of the $\overline{\text{MS}}$ $b$-quark mass extracted from the $\Upsilon$ system by about $-20$ MeV, due to an incomplete cancellation of terms that are non-analytic in the charm quark mass. The precise size of the shift depends on unknown higher order corrections, and might have a considerable uncertainty if they are large.
The bottom quark mass is an important parameter for the theoretical description of $B$ meson decays and $b$ jet production cross sections in collider experiments. In continuum QCD, the most precise determinations of the bottom quark mass parameter have been obtained from data on the spectrum and the electronic partial widths of the $\Upsilon$ mesons. Recently, a number of $\overline{\text{MS}}$ bottom quark determinations have been carried out, which were based on $\Upsilon$ meson sum rules at next-to-next-to-leading order (NNLO) in the non-relativistic expansion, and which consistently eliminated all linear sensitivity to small momenta [1–3]. The latter is mandatory to reduce the systematic uncertainty in the bottom quark mass below the typical hadronization scale $\Lambda_{\text{QCD}}$ [4, 5]. The analyses mentioned above, however, treated all quarks other than the $b$ quark as massless. This treatment is justified for light quarks that have masses much smaller than the inverse Bohr radius $1/\langle r \rangle$ of the non-relativistic bottom–antibottom system i.e. for up, down and strange quarks, because in this case the theoretical expressions describing the bottom–antibottom dynamics and the conversion to the $\overline{\text{MS}}$ bottom mass definition can be expanded in the light quark masses. Like the contributions that are linearly sensitive to small momenta, the terms linear (and non-analytic) in these light quark masses cancel out in the analysis. At NLO in the non-relativistic expansion this can be seen explicitly by considering the effects of a light virtual quark to the static energy of a bottom–antibottom quark pair with spatial distance $r$,

$$E_{\text{stat}} = 2M_b + V_{\text{stat}}(r),$$

where $M_b$ is the bottom quark pole mass and $V_{\text{stat}}$ the potential energy of the non-relativistic bottom–antibottom quark system. At order $\alpha_s^2$ the correction coming from the finite mass of a light quark $q$ to the pole mass contribution reads [3]

$$\delta M_b^q = \frac{4}{3} \frac{\alpha_s^2}{\pi^2} M_b \Delta \left( \frac{m_q}{M_b} \right),$$

$$\Delta(r) = \frac{\pi^2}{8} r - \frac{3}{4} r^2 + \frac{\pi^2}{8} r^3 - \left( \frac{1}{4} \ln^2 r - \frac{13}{24} \ln r + \frac{\pi^2}{24} + \frac{151}{288} \right) r^4$$

$$- \sum_{n=3}^{\infty} \left( 2F(n) \ln r + F'(n) \right) r^{2n},$$

$$F(n) \equiv \frac{3(n-1)}{4n(n-2)(2n-1)(2n-3)},$$

where $\delta M_b^q$ is the shift in the $b$-quark pole mass keeping the $b$-quark $\overline{\text{MS}}$ mass fixed, $m_q$ is the mass of the light quark, and $F'(n) \equiv \frac{\partial}{\partial n} F(n)$. In the limit $m_q/M_b \to 0$, Eq. (2) reduces to

$$\delta M_b^q = \frac{1}{6} \alpha_s^2 m_c.$$
The shift $\delta M_b^q$ is non-analytic in the quark masses, and should be regarded as being of the form

$$\delta M_b^q = \alpha_s^2 \sqrt{m_c^2/M_b^2} (M_b/6),$$

so that it is explicitly proportional to $M_b$, which breaks the $b$-quark chiral symmetry. The limiting value Eq. (5) can be easily computed using heavy quark effective theory. At the scale $\mu = m_b$, one matches to a theory in which the $b$-quark is treated as a static field, with the residual mass term equal to zero, so that the propagator is $i/(k \cdot v)$. At the lower scale $\mu = m_c$, one integrates out the charm quark. The matching condition at this scale induces a residual mass term for the $b$-quark, which is given by computing the graph in Fig. 1.

The shift in the bottom–antibottom quark potential energy due to a light quark $q$ is

$$\delta V_{\text{stat}}^q(r) = -\frac{4}{3} \alpha_s \left(\frac{\alpha_s}{6 \pi}\right) \left\{ \ln \left(\frac{m_q^2}{\mu^2}\right) + \int_{4m_q^2}^\infty \frac{d\lambda^2}{\lambda^2} R_{q\bar{q}}(m_q, \lambda) \exp(-\lambda r) \right\},$$

and the contribution linear in $m_q$ cancels in the total static energy, Eq. (1) between $M_b$ and $V(r)$. The dominant light quark correction is of order $(\alpha_s/\pi)^2 m_q^2/M_b$ and can be neglected for all practical purposes.

On the other hand, for the case that a light quark mass is comparable in size to the inverse Bohr radius, i.e. for the charm quark, the full $m_q$-dependence of $\delta V_{\text{stat}}^q$ has to be taken into account. In this case the cancellation of the linear charm mass term between the pole mass and potential contributions to the static energy, Eq. (2), is incomplete. In this letter we show that this feature can lead to sizeable effects in the $\overline{\text{MS}}$ bottom quark mass determination, compared to when the charm quark is treated as massless.

There are two reasons why the incomplete cancellation of the linear charm mass contribution can lead to a sizeable effect. First, the linear charm mass term is not suppressed by a factor $\pi^2$, as one might expect from a contribution coming from a loop integration...
This is because the linear charm mass contribution represents a correction generated by a non-analytic linear dependence on infrared momenta. A similar effect is known in chiral perturbation theory where non-analytic $m_{\pi}^3$ corrections from loop diagrams are also not suppressed by powers of $\pi$. Second, the scale of the strong coupling of the linear charm mass term in Eq. (2) is of order the charm mass rather than the bottom quark mass. This can be understood from the fact that the charm mass serves as an infrared cutoff for the non-analytic linear dependence on infrared momenta just mentioned before. Thus, the linear charm mass term is generated at momenta of order $m_c$. The effective field theory computation of Fig. 1 also indicates that one should use $\alpha_s(m_c)$. These arguments are supported by an explicit calculation of the order $\alpha^3_s$ BLM corrections to the linear charm mass term in Eq.(2). For this calculation it is sufficient to consider the static limit, i.e. we only need to determine the linear charm mass corrections to the bottom quark self-energy due to the chromostatic Coulomb field. The order $\alpha^3_s$ BLM linear charm mass contributions in the bottom quark pole mass can be obtained from the formula

$$\delta M_{c,b,\text{stat}} = \frac{\alpha_s}{6\pi} \int_{4m_c^2}^{\infty} \frac{d\lambda^2}{\lambda^2} R_{qq}(m_c, \lambda) \int \frac{d^2p}{(2\pi)^2} \frac{2}{3} \frac{4\pi \alpha_s}{p^2 + \lambda^2} \left[ 1 + 2 \frac{\alpha_s}{4\pi} \beta_0 \left( \ln \left( \frac{\mu^2}{p^2} \right) + \frac{5}{3} \right) \right],$$

(9)

where we assumed that $m_c$ is the charm pole mass, and $\beta_0 = 11 - \frac{2}{3} n_l$ for $n_l = 3$ light quark flavors. The factor of two in front of the $\beta_0$ term arises because the charm quark loop can be inserted on both sides of the massless fermion loops. We have also included the chromostatic self-energy contribution at order $\alpha^2_s$. The term linear in $m_c$ contained in Eq. (9) reads

$$\delta M_{c,b,\text{linear}} = \frac{\alpha^2_s(\mu)}{6} m_c \left[ 1 + 2 \frac{\alpha_s}{4\pi} \beta_0 \left( \ln \left( \frac{\mu^2}{m_c^2} \right) - 4 \ln 2 + \frac{14}{3} \right) \right],$$

(10)

which corresponds to the BLM scale $\mu_{\text{BLM}} = 0.388 m_c$. Obviously, the BLM calculation indicates a very low renormalization scale for the strong coupling governing the linear charm quark mass contribution.

To illustrate the size of the corrections caused by the incomplete cancellation of the linear charm quark terms let us examine the difference in the N\overline{S} bottom quark mass $m_{b}(\overline{m}_{b})$, determined from the mass of the $\Upsilon(1S)$ meson, $M_{\Upsilon(1S)} = 9.460$ GeV, for the two cases that that the finite charm mass effects are either taken into account or neglected. For simplicity, we only consider an extraction of the N\overline{S} bottom quark mass at NLO in the non-relativistic expansion. A more thorough analysis using full NNLO expressions and including also a sum rule analysis based on data for all known $\Upsilon$ mesons will be carried out elsewhere.

Including the effects of the charm quark mass from Eq. (7) properly in first order time-independent perturbation theory, the $\Upsilon(1S)$ meson mass at NLO in the non-relativistic expansion reads

$$M_{\Upsilon(1S)} = 2 M_b \left\{ 1 - \frac{C_F a_s^2}{8} \left[ \beta_0 \left( L + 1 \right) + \frac{a_1}{2} + \frac{2}{3} \left( \ln \left( \frac{m_c}{\mu} \right) + h\left( \frac{2m_c}{M_b C_F a_s} \right) \right) \right] \right\},$$

(11)

where
\[ h(x) \equiv -\frac{11}{6} - 2x^2 + \frac{3x\pi}{4} + x^3\pi + \begin{cases} \frac{(2-x^2-4x^4)}{2\sqrt{x^2-1}} \tan^{-1} \left( \frac{\sqrt{x^2-1}}{x-1} \right) & : x > 1 \\ \frac{(2-x^2-4x^4)}{2} & : x = 1 \\ \frac{(2-x^2-4x^4)}{4\sqrt{1-x^2}} \ln \left( \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \right) & : x < 1 \end{cases} \]  

\[ L = \ln \left( \frac{\mu}{C_F a_s M_b} \right), \]  

(13)

and \( a_1 = \frac{31}{3} - \frac{10}{9} n_l \) for \( n_l = 3 \) massless quark flavors \[ 11, 12 \]. For \( x \to 0 \) the function \( h \) has the limiting behavior \( h(x \to 0) = -\frac{11}{6} + \ln(2/x) + \frac{3\pi}{4} + O(x^2) \). Using the upsilon expansion up to order \( \epsilon^2 \) \[ 13 \] and the expressions given in Eqs. (2) and (11) we arrive at the following formula for the shift in the \( \overline{\text{MS}} \) bottom quark mass \((C_F = 4/3)\):

\[ \Delta \overline{m}_b = \frac{M_{\Upsilon(1S)}}{2} \left\{ \frac{C_F^2 \alpha_s^2}{8} \left( \frac{\alpha_s}{\pi} \right) \frac{2}{3} \left[ h \left( \frac{4m_c}{M_{\Upsilon(1S)} C_F \alpha_s} \right) + \frac{11}{6} - \ln \left( \frac{M_{\Upsilon(1S)} C_F \alpha_s}{2m_c} \right) \right] \right\} \]  

(14)

where we have taken into account only the term linear in the charm quark mass from Eq. (2). The scale of the strong coupling contained in the first line of Eq. (14) is of order the inverse Bohr radius. We identify this scale with the one of the linear charm mass term. In deriving Eq. (14), the terms \(-\frac{11}{6} + \ln(2/x)\) in \( h(x \to 0) \) have been absorbed into Eq. (14), with the replacement \( n_l \to n_l + 1 = 4 \). This gives the usual relation between the 1S and \( \overline{\text{MS}} \) masses neglecting the charm quark mass, so these terms in \( h(x) \) do not contribute to the shift \( \Delta \overline{m}_b \).

In Fig. 2 we have displayed \(-\Delta \overline{m}_b\) as a function of \( \alpha_s \). \( \alpha_s \) has to be evaluated at a scale of order the charm mass as discussed above. The solid, dashed, dash-dotted and dotted lines correspond to the choices \( m_c = 1.7, 1.5, 1.3 \) and 1.1 GeV, respectively, for the charm quark mass. For \( \alpha_s \approx 0.4 \), which corresponds to a choice of the renormalization scale equal to the charm quark mass, we find that the shift is between \(-10 \) and \(-20 \) MeV. If a smaller renormalization scale is assumed, the shift can amount to more than \(-50 \) MeV for larger choices of the charm quark mass. The spread in the curves displayed in Fig. 2 shows that the shift in the \( \overline{\text{MS}} \) bottom quark mass can be of order several tens of MeV. The exact value, however, contains a considerable uncertainty, which is amplified by the large value of the strong coupling governing the \( \Delta \overline{m}_b \). A complete NNLO analysis for the \( \overline{\text{MS}} \) bottom quark extraction from \( \Upsilon \) sum rules will be indispensable to accurately determine the effect of a nonzero charm mass. However, if the scale governing the strong coupling constant in \( \Delta \overline{m}_b \) is indeed as low as the BLM scale estimate carried out above indicates, a considerable uncertainty might persist. For a moderate choice of the strong coupling in \( \Delta \overline{m}_b \) between 0.4 and 0.6 the charm mass shift is smaller than the uncertainties of 60–80 MeV in the

\[ It \ is \ sufficient \ to \ only \ include \ the \ term \ linear \ in \ m_c \ in \ Eq. (2), \ since \ the \ higher \ order \ terms \ are \ suppressed \ by \ powers \ of \ m_c/M_b \ which \ is \ small. \ Higher \ order \ terms \ in \ Eq. (14) \ depend \ on \ powers \ of \ 2m_c/(M_b C_F a_s), \ which \ is \ not \ small, \ and \ it \ is \ necessary \ to \ include \ the \ full \ functional \ dependence \ in \ h(x). \]
FIG. 2. The function $-\Delta m_b$ as a function of $\alpha_s$. The solid, dashed, dash-dotted and dotted lines correspond to the choices $m_c = 1.7, 1.5, 1.3$ and 1.1 GeV, respectively, for the charm quark mass.

value of $\overline{m}_b(\overline{m}_b)$ obtained in recent NNLO analyses of the Υ sum rules [1–3], where charm mass effects have been neglected. However, the inclusion of the charm mass effects will be essential for a future extraction of $\overline{m}_b(\overline{m}_b)$ at NNNLO.

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REFERENCES

[1] K. Melnikov and A. Yelkhovky, *Phys. Rev. D* **59** (1999) 114009.
[2] A. H. Hoang, to appear in *Phys. Rev. D*, hep-ph/9905550.
[3] M. Beneke and A. Signer, hep-ph/9906473.
[4] A. H. Hoang, M. C. Smith, T. Stelzer and S. S. Willenbrock, *Phys. Rev. D* **59** (1999) 114014.
[5] M. Beneke, *Phys. Lett. B* **434** (1998) 115.
[6] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, *Z. Phys. C* **48** (1990) 673.
[7] A.V. Manohar and M.B. Wise, unpublished.
[8] A. H. Hoang and M. Melles, in preparation.
[9] S. Titard and F. J. Yndurain, *Phys. Rev. D* **49** (1994) 6007.
[10] D. Eiras and J. Soto, hep-ph/9905543.
[11] W. Fischler, *Nucl. Phys. B* **129** (1977) 157.
[12] A. Billoire, *Phys. Lett. B* **92** (1980) 343.
[13] A. H. Hoang, Z. Ligeti and A. V. Manohar, *Phys. Rev. Lett.* **82** (1999) 277, *Phys. Rev. D* **59** (1999) 074017.