FINDING TOPOLOGY IN A FACTORY: CONFIGURATION SPACES

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It is perhaps not universally acknowledged that an outstanding place to find interesting topological objects is within the walls of an automated warehouse or factory.

The examples of topological spaces constructed in this exposition arose simultaneously from two seemingly disparate fields: the first author, in his thesis [1], discovered these spaces after working with the group of H. Landau, Z. Landau, J. Pommersheim, and E. Zaslow [2] on problems about multiple random walks on graphs. The second author [8, 7] discovered these same spaces while collaborating with D. Koditschek in the Artificial Intelligence Lab at the University of Michigan. The net result makes evident the abundance of topological objects within the physical world.

Topology seeks to describe, as one author puts it, the “shape of space” [15], with “shape” being interpreted as appropriate for the context at hand. We will begin with thinking about spaces up to homeomorphism (continuous maps with continuous inverse), but will quickly need to abandon this class in favor of a looser form of equivalence: homotopy type.

Although few topological prerequisites are necessary for fully appreciating the examples discussed here, the class of spaces we consider gives an earthly incarnation of several intricate ideas from topology, such as \( K(\pi, 1) \) spaces (a.k.a. Eilenberg-MacLane spaces of type \( K(\pi, 1) \)) and NPC (or non-positively curved) spaces.

1. Configurations and Braids

Our story begins with a classical construction: that of a configuration space of points. We consider first the configuration space of \( N \) distinct labelled points

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in the plane $\mathbb{R}^2$. That is, we consider all ordered $N$-tuples of points in $\mathbb{R}^2$ with the property that no two of the points coincide. To each such $N$-tuple, visualized as a collection of labeled tokens on a tabletop, we assign a point in the configuration space. Two points in the configuration space are close if the $N$-tuples are close as measured in $\mathbb{R}^{2N}$. One can write this mathematically as

$$C^N(\mathbb{R}^2) := (\mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2) - \Delta,$$

where $\Delta$ denotes the “pairwise diagonal”

$$\Delta := \{(x_1, x_2, \ldots, x_N) \in (\mathbb{R}^2)^N : x_i = x_j \text{ for some } i \neq j\}.$$

This configuration space is not easy to visualize, in no small part because the dimension of the space is $2N$.

**Exercise:** Show that $C^2(\mathbb{R}^2)$ is homeomorphic to $\mathbb{R}^2 \times (\mathbb{R}^2 - (0, 0))$ or equivalently $\mathbb{R}^3 \times S^1$, where $S^1$ denotes the unit circle in $\mathbb{R}^2$. Hint: think about placing tokens on the table one at a time. Does your method of proof give a simple presentation for $C^3(\mathbb{R}^2)$?

The configuration space of points on a plane, like so many other mathematical objects, finds multiple applications to physical problems. For example, any loop (closed curve) in $C^N(\mathbb{R}^2)$ is a *pure braid on $N$ strands*, so called because of the space-time graph of the loop in $C^N(\mathbb{R}^2)$: see Figure 1. (The word *pure* indicates that each token returns to its original location, rather than the tokens getting permuted in some other way. Other permutations give rise to general *braids*, which are loops in the space of *unlabeled* configurations of points in the plane.) Braids are ubiquitous in the study of knots and links [3], and thus find their way into applications ranging from quantum field theory to dynamical systems to DNA modeling.

### 2. Robotics and Topological Motion Planning

The space $C^N(\mathbb{R}^2)$ itself appears naturally in robotics.

Consider an automated factory equipped with a cadre of Automated Guided Vehicles (AGVs), or mobile robots, capable of transporting items from place to place. A common goal is to place several, say $N$, of these AGVs in motion simultaneously. It is necessary to enact a control algorithm which perhaps moves the AGVs from their initial positions to their goal positions (in a warehousing application), or which executes a cyclic pattern (as arises in manufacturing applications). As might be expected, these robots are somewhat costly and
cannot tolerate collisions without a loss of performance. Thus, modeling the factory floor as $\mathbb{R}^2$ and the AGVs as points, one often wishes to find paths or cycles in $C^N(\mathbb{R}^2)$ to enact specific behaviors. Of course, since the robots are not truly points, and since no control algorithm implementation is of infinite precision, we require that the control path reside outside of a neighborhood of the diagonal $\Delta$ in $(\mathbb{R}^2)^N$.

Fortunately, this problem is not impossible to solve. The work of Koditschek and Rimon [12] provides one example of a concrete solution: they write out explicit vector fields on these configuration spaces which can be used to flow from initial to goal positions in the presence of certain types of obstacles. By arranging these vector fields so that they strongly push away from the vestiges of the diagonal $\Delta$ on the boundary of $C^N(\mathbb{R}^2)$, the control scheme is provably safe from collisions (as opposed to being statistically safe via computer simulations): no path can ever intersect the diagonal. Furthermore, since a neighborhood of the diagonal is repelling, the control scheme is stable with respect to perturbations to the system. This is quite important as mechanical systems have an annoying tendency to occasionally malfunction. Drawing the appropriate vector field on a configuration space yields an excellent method of self-correction.

Obstacles can easily be incorporated into these models — a vast literature on this subject exists [13]. Executing cyclic motions is more complex but can be at first approximated by composed point-to-point motions. There are several assumptions required for the simple construction we present, and various kinematic issues (e.g., steering geometry) must in general be addressed.
This is a clean, direct application of topological and dynamical ideas to a matter of great practical relevance which is currently used in various industrial settings.

One reason why this problem has a nice solution is evident to anyone who regularly shops at a large supermarket: numerous shopping carts trace out paths in a workspace that consists of wide aisles. If two carts are headed toward each other, one needs merely steer out of the way a little bit (assuming the other does not move in the same direction!) to avoid a collision. The resolution of collisions on \( \mathbb{R}^2 \) is a local phenomenon.

3. Graphs

The robotics community, largely independently of the topology community, has enjoyed great success at identifying and manipulating configuration spaces to their advantage in control problems. There is, however, a class of simple, physically relevant scenarios whose configuration spaces have been untapped: the configuration spaces of points on a graph, or a network of edges and vertices.

Consider the situation where the AGVs must move about on a collection of tracks embedded in the floor, or via a path of electrified guide-wires from the ceiling (see [6] for examples). Such a restricted network is quite common, mainly because it is more cost-effective than a full two-degree-of-freedom steering system for AGVs. In this setting, the state of the system at any instant of time is a point in the configuration space of the graph \( \Gamma \):

\[
\mathcal{C}^N(\Gamma) := (\Gamma \times \cdots \times \Gamma) - \Delta.
\]

The same principles previously mentioned still apply. To navigate safely on a graph, one must construct appropriate paths which remain strictly within \( \mathcal{C}^N(\Gamma) \) and are repulsed by any boundaries near \( \Delta \). Several problems have arisen which heretofore have prevented an analogous solution.

1. What do these spaces look like?
2. How does one resolve an impending collision?

Notice the differences between this problem and the problem of \( \mathcal{C}^N(\mathbb{R}^2) \). First of all, in relation to (1) above, \( \mathcal{C}^N(\Gamma) \) is not a manifold: that is, you cannot hope that every point has a neighborhood which is locally homeomorphic to a Euclidean space. Indeed, if we ignore the trivial graphs which are homeomorphic to a line segment or a circle, then the graph itself is not locally
Euclidean and products of the graph will share this feature. Second, concerning (2) above, collisions within the interior of an edge are no longer locally resolvable. Imagine that the aisles of the grocery store are only as wide as the shopping carts, so that passing another person is impossible. A store full of shoppers (using carts) would pose a difficult coordinated control problem. How can carts avoid a collision in the interior of an aisle? Clearly at least one of the participants must make a large-scale change in plans and back up to the end of the aisle. *The resolution of a collision on a graph is a non-local phenomenon.*

4. **Examples: Two Robots**

Since a graph is a one-dimensional object, the configuration space $C^N(\Gamma)$ is $N$-dimensional. Thus configuration spaces of two robots are two-dimensional objects, which (at least in simple cases) one should be able to visualize. We therefore give numerous examples of configuration spaces of two robots.

**Example 1: $C^2(\mathcal{T})$**

Let $\mathcal{T}$ denote the graph of three edges attached at a central vertex. The cellular structure of $C^2(\mathcal{T})$ is simple to procure. There are six (two robots times three-choose-two edges) squares or “2-cells”, each with one corner punctured, corresponding to those configurations where the two robots are on distinct edges of $\mathcal{T}$. Since there are three edges in $\mathcal{T}$, those remaining configurations in which both robots are on the same edge yield three square cells, each of which is divided by the diagonal $\Delta$ into a pair of triangular cells. Thus there are six triangular 2-cells corresponding to those configurations where both robots are on the same edge, but distinct. By enumerating the behaviors of each of these 2-cells, one can make the identifications to arrive at the space given in Figure 2 (or, one can use a slightly more sophisticated argument as in [8, 9]).

**Exercise:** Place two coins on a piece of paper with a large $\mathcal{T}$ drawn on it so that both coins are on the same edge of the graph. Using two fingers, execute a path which exchanges the coins’ positions without collisions. Draw this motion as a path on Figure 2.

**Example 2: $C^2(\mathcal{O}-)$**

Let $\mathcal{O}-$ denote the graph with three edges obtained from $\mathcal{T}$ by gluing two boundary vertices together. One method of constructing $C^2(\mathcal{O}-)$ would be to
first remove the configurations where both robots are on the vertices to be glued. Then identify those portions of the boundary of $C^2(\mathcal{T})$ which have a robot at the vertices to be glued in $\mathcal{O}$, and glue these portions of $C^2(\mathcal{T})$ together. The result, although a very simple configuration space, is already somewhat difficult to visualize: we illustrate the space, embedded in $\mathbb{R}^3$, in Figure 3 [left]. There are three “punctures” at which both robots collide at one of the three vertices. The six dotted edges are the images of the diagonal curves from Figure 2.

Example 3: $C^2(\times)$
Increasing the incidence number of the central vertex complicates the configuration space. Consider \( \times \) a radial tree of four edges emanating from a central vertex. The visualization of \( \mathcal{C}^2(\times) \) is a bit more involved and requires some work to obtain. For the purpose of stimulating curiosity, we include this configuration space as Figure 3 [right].

5. Simplification: discretization

To visualize these configuration spaces, it is clear that some simplification into a more manageable form is necessary. There are two principal methods we use: the first is a way of removing the “unsafe” cells of the space near the diagonal \( \Delta \). The second, deformation retraction, is a more drastic crushing of the space down to a lower dimensional “skeleton,” as discussed in the subsequent section.

Any graph \( \Gamma \) comes equipped with a cellular structure: 0-cells (vertices) and 1-cells (edges). The \( N \)-fold cross product of \( \Gamma \) with itself inherits a cell structure, each cell being a product of \( N \) (not necessarily distinct) cells in \( \Gamma \): cf. Example 1. However, the configuration space does not quite have a natural cell structure, since the diagonal \( \Delta \) slices through all product cells with repeated factors. Notice, however, that in several of the previous examples, these partial cells dangle “inessentially” and could be collapsed onto a more “essential” skeleton of the configuration space.

Such an operation can be interpreted as follows [4]. Consider the *discretized configuration space* of \( \Gamma \), denoted \( D^N(\Gamma) \), defined as \( (\Gamma \times \cdots \times \Gamma) - \tilde{\Delta} \), where \( \tilde{\Delta} \) denotes the set of all product cells in \( \Gamma \times \cdots \times \Gamma \) whose closures intersect the diagonal \( \Delta \). An equivalent description of \( D^N(\Gamma) \) is that it is the set of configurations for which, given any two robots on \( \Gamma \) and any path in \( \Gamma \) connecting them, the path contains at least one entire edge. Thus, instead of restricting robots to be at least some intrinsic distance \( \epsilon \) apart (i.e., removing an \( \epsilon \) neighborhood of \( \Delta \)), one now restricts robots on \( \Gamma \) to be “at least one full edge apart.” This is a natural kind of configuration space in the context of random walks on graphs [2]. Note that \( D^N(\Gamma) \) is a subcomplex of \( \mathcal{C}^N(\Gamma) \) (it does not contain “partial cells” which arise when cutting along the diagonal), and is, in fact, the largest subcomplex of \( \mathcal{C}^N(\Gamma) \) which does not intersect \( \Delta \).

With this natural cell structure, one can think of the vertices (0-cells) of \( D^N(\Gamma) \) as “discretized” configurations — arrangements of labeled tokens at the vertices of the graph. The edges of \( D^N(\Gamma) \), or 1-cells, tell you which discrete configurations can be connected by moving one token along an edge of \( \Gamma \).
Each 2-cell in $D^N(\Gamma)$ represents two independent (or “commuting”) edges: one can move a pair of tokens independently along disjoint edges. A $k$-cell in $D^N(\Gamma)$ likewise represents the ability to move $k$ tokens along $k$ disjoint edges in $\Gamma$.

Returning to Figure 2, the discretization of $C^2(T)$ removes much of the space. For example, the triangular two-dimensional cells represent configurations where both robots are on the interior of the same edge. Since these are not “one full edge apart,” these cells are deleted. The same is true of all the other two-dimensional cells which represent robots in the interior of separate edges. Which configurations of two robots on $T$ are separated by a full edge?

**Exercise:** Show that discretizing the configuration spaces of Examples 1 through 3 yields the configuration spaces of Figure 4. How good of an “approximation” are these spaces?

![Figure 4. The discretizations of the configuration spaces in Examples 1-3 (left to right).](image)

One could compute the discretization of Example 1 in a less direct manner which generalizes to some lovely examples to follow. Recall that the discretized configuration spaces inherit a cell structure from $\Gamma$: all the product cells of $\Gamma^N$ not entirely in $C^N(\Gamma)$ are removed by the discretization. Thus, in the case of Example 1, simple counting reveals that the space $D^2(T)$ possesses twelve 0-cells (where both robots are at distinct vertices of $T$), twelve 1-cells (where one robot is at a vertex and the other is on an edge whose closure does not contain said vertex), and zero 2-cells (since every pair of edges intersect along their boundaries). With a little thought, one can see that $D^2(T)$ is a connected manifold: each zero-cell connects to exactly two 1-cells, and all of the 1-cells are joined end-to-end cyclically. Thus, $D^2(T)$ is a topological
circle, precisely as obtained by deleting all the near-diagonal cells from $C^2(\mathcal{T})$ in Figure 2. The discretization operation yields a subcomplex of $C^2(\mathcal{T})$ which appears to contain all the “essential” topology (more specifically, the spaces $C^N(\mathcal{T})$ and $D^N(\mathcal{T})$ are of the same homotopy type — see the next section for definitions); however, this is certainly not the case for the discretization of $C^2(\mathcal{O})$, which becomes disconnected! In the next section, we will state the criteria under which discretization is topologically faithful.

The counting arguments used above often can determine the discretized configuration space, even when the full configuration space is unknown. The following are some surprising examples of interesting spaces arising as the discretized configuration space of non-planar graphs [1].

![Figure 5. The non-planar graphs $K_5$ (left) and $K_{3,3}$ (right). This notation comes from graph theory, where these are fundamental examples of non-planar graphs.](image)

**Example 4: $D^2(K_5)$**

Consider the complete bipartite graph $K_5$ pictured in Figure 2 (left). The discretized configuration space of two robots on this graph is a two-dimensional complex. A simple counting argument reveals the cell-structure:

**0-cells** Each 0-cell corresponds to a configuration in which the two robots are at distinct vertices. Since $K_5$ has five vertices, there are exactly $(5)(5 - 1) = 20$ such 0-cells. (Remember, there is no vertex where two edges cross in the picture; there are only vertices at the corners of the pentagon.)

**1-cells** Each 1-cell corresponds to a configuration in which one robot is at a vertex and the other is on an edge whose endpoints do not include the vertex already occupied. From the diagram of $K_5$ one counts that there are $(2)(5)(6) = 60$ such 1-cells, as in Figure 2 (left). The factor of two comes from the fact that we label the two robots on $K_5$.  

FIGURE 6. [left] For every vertex in the space $K_5$ there are six disjoint edges. Likewise [middle] for each edge there are three totally disjoint edges. In $D^2(K_5)$, these cells fit together to form a locally Euclidean two-dimensional complex [right].

2-cells Each 2-cell corresponds to a configuration in which the two robots occupy edges whose closures are disjoint. Again, from the diagram (and Figure [middle]) one counts that there are $(10)(3) = 30$ such 2-cells in the complex.

One then demonstrates that each edge borders a pair of 2-cells preserving an orientation and that each vertex is incident to six edges, as in Figure [right]. Also, the space $D^2(K_5)$ is connected: it is not hard to see that you can move from any configuration to any other. Thus $D^2(K_5)$ is a connected orientable surface, and the Classification Theorem for surfaces implies that the space is determined uniquely up to homeomorphism by the Euler characteristic,

\[
\chi(D^2(K_5)) := \text{#faces} - \text{#edges} + \text{#vertices} = 30 - 60 + 20 = -10.
\]

Thus, $D^2(K_5)$ is a closed orientable surface of genus $g := 1 - \frac{1}{2}\chi = 6$. It is not at all obvious that the motion of two robots on this graph should produce a genus six surface. Obtaining a manifold is surprising enough, but a manifold with genus larger than one really goes against the notion that all of the interesting topology in these spaces is “localized” in configurations about a vertex.

Example 5: $D^2(K_{3,3})$
Figure 7. The space $D^2(K_5)$ is homeomorphic to a closed orientable surface of genus six.

A near-identical analysis on the graph $K_{3,3}$ of Figure 5 reveals that $D^2(K_{3,3})$ is also a connected closed orientable surface. The natural cell structure on $D^2(K_{3,3})$ possesses exactly 36 faces, 72 edges, and 30 vertices. Thus,

$$\chi(D^2(K_5)) = 36 - 72 + 30 = -6,$$

and we conclude that the genus of this surface is four.

In order to understand discretizations of higher-dimensional configuration spaces, one can employ an appropriate version of duality: instead of tracking how distinct robots on $\Gamma$ move about, one considers configurations of “holes” — regions on $\Gamma$ which have no robots on them. As the holes have no natural labeling, the duality applies directly to the unlabeled configuration spaces. It nevertheless implies a relationship between $D^N(\Gamma)$ and $D^{V-N}(\Gamma)$, where $V$ is the number of vertices of $\Gamma$. Using this duality, one obtains the following examples.

**Example 6:** The space $D^3(K_5)$ is homeomorphic to a connected closed orientable surface of genus 16.

**Example 7:** Likewise, $D^4(K_{3,3})$ is homeomorphic to a connected closed orientable surface of genus 37.

6. **Simplification: Deformation**

We are naturally led to the question of how good an approximation the discretized configuration space is. Heuristically, the spaces $C^N$ and $D^N$ should be “similar,” since the latter is a subset of the former obtained by collapsing out those cells which border the remains of the diagonal $\Delta$. That the discretization is not always faithful is evidenced by the fact that $C^2(\text{--})$ is connected while $D^2(\text{--})$ is not. Indeed, by definition, $D^N(\Gamma)$ is the empty set whenever $N$ is greater than the number of vertices of $\Gamma$: i.e., when the discretization of $\Gamma$ is too “coarse.”
The notion of “sameness” appropriate here is that of deformation retract. A subspace $A$ of a space $X$ is a (strong) deformation retract of $X$ if there exists a continuous family of continuous maps $f_t : X \to X$ such that $f_0$ is the identity map on $X$ and $f_1$ is a map which sends $X$ onto $A$, and such that $f_t$ fixes $A$ pointwise for all $t$. The image of each $f_t$ can be seen as frames in a movie which exhibits a continuous shrinking of $X$ onto $A$. Deformation retracts are an excellent way to simplify a space without changing any essential topological properties.

The most fundamental notion of topological equivalence, homotopy type, can be defined in terms of deformation retracts: two spaces $X$ and $Y$ are of the same homotopy type if and only if they are both deformation retracts of a “larger” space $Z$.

The key result is that $D^N(\Gamma)$ is a deformation retract of $C^N(\Gamma)$ as long as the discretization of the graph $\Gamma$ is not too coarse. More specifically,

**Theorem 1:** [Abrams [1]] For any $N > 1$ and any graph $\Gamma$ with at least $N$ vertices, $C^N(\Gamma)$ deformation retracts to $D^N(\Gamma)$ if and only if

1. Each path between distinct vertices of valence not equal to two passes through at least $N - 1$ edges; and
2. Each loop from a vertex to itself which cannot be shrunk to a point in $\Gamma$ passes through at least $N + 1$ edges.

(The valence of a vertex is the number of incident edges.)

It thus follows that the spaces $C^2(K_5)$ and $C^2(K_{3,3})$ deformation retract to the discretized configuration spaces computed in Examples 4 and 5. This is extremely useful information: trying to compute the configuration space $C^2(K_5)$ directly would appear hopelessly complex. Note that the second condition fails for the discretization of $C^2(\bigcirc\bigcirc)$, but adding one more vertex would yield a faithful discretized configuration space. The discretizations of Examples 6 and 7 are not fine enough to give an equivalence.

The dimension of the smallest subcomplex to which a configuration space deformation retracts is an important quantity in practice, since a large dimension greatly increases the complexity of the computational work needed to control the system. The following theorem reveals that the essential dimension of the configuration space is governed by properties of the graph, independent of the number of robots on the graph.

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2Except perhaps dimension, but for the applications at hand, reducing the dimension of the configuration space is a boon.
**Theorem 2**: [Ghrist 7] Given a graph \( \Gamma \) having \( V \) vertices of valence greater than two, the space \( C^N(\Gamma) \) deformation retracts to a subcomplex of dimension at most \( V \).

**Example 8**: \( C^N(\Upsilon_k) \)

Consider the radial \( k \)-prong tree \( \Upsilon_k \) having \( k > 2 \) edges and \( k + 1 \) vertices, all edges being attached at a single central vertex. For example, \( \Upsilon_3 = \bigcirc \) and \( \Upsilon_4 = \times \). From the above theorem, we see that \( C^N(\Upsilon_k) \) must deformation retract to a 1-dimensional subcomplex — that is, a graph. Since the essential topological features of a graph are determined by its Euler characteristic, one need merely compute the number of vertices and edges to classify these spaces. Using a double-induction argument on \( N \) and \( k \) [7], one derives a two-variable recursion relation for the Euler characteristic. By solving this equation, one can prove that \( C^N(\Upsilon_k) \) has the homotopy type of a graph which is a “bouquet” of \( P \) distinct loops joined together like petals on a daisy, where

\[
P = 1 + (Nk - 2N - k + 1)\frac{(N + k - 2)!}{(k - 1)!}.
\]

Note, for example, that \( C^2(\Upsilon_3) \) has exactly one generating loop, as Figures 2 and 4 [left] confirm. Figure 4 [right] provides another confirmation of this equation for \( k = 4 \), as the reader should verify. The factorial growth of \( P \) in \( N \) is due to the fact that we label the \( N \) robots on \( \Upsilon_k \). If one considers the unlabeled configuration spaces, then the second term in the expression above is reduced by a factor of \( N! \).

It is worth emphasizing that while the control problem of robots on a graph is rather intuitive for two robots, it quickly builds in complexity. Since the dimension alone makes the configuration spaces in general nearly impossible to visualize, Theorem 2 is quite helpful — the “essential” dimension of the configuration space is independent of the number of robots on the graph. For the graph \( \Upsilon_k \), Theorem 2 implies that there is a one-dimensional roadmap which gives a perfect representation of the configuration space: no topological data is lost. Since the proof of Theorem 2 is constructive, one can use standard algorithms for determining shortest paths on a graph in order to develop efficient path planning for multiple robots on \( \Upsilon_k \) via the roadmap.
7. Conclusions

Applications of configuration spaces to robotics are by no means novel: ideas by mathematicians, computer scientists, and engineers have been steadily developing since the 1960’s (see, e.g., [3, 10] for an introduction). In addition, various kinds of configuration spaces arise in topology and physics rather often, as in the study of braids [3], abstract linkages [14, 11], or invariants of manifolds [4].

In the particular application involving multiple independent robots, the global aspect of the control problem is the principal difficulty when the robots are constrained to a network. It is this global nature which hints at the efficacy of a topological viewpoint. Indeed, the determination and simplification of the configuration spaces of graphs can be used to construct practical control schemes (see [8, 9] for some simple examples).

Reversing the perspective, it is remarkable that this class of topologically rich configuration spaces were virtually untouched until motivated by problems from other fields. There are many deeper properties of these spaces which one can prove [7, 1]. We list a few below:

1. For any graph $\Gamma$, $C^N(\Gamma)$ is an Eilenberg-MacLane space of type $K(\pi, 1)$. That is, the image of any continuous map from a $k$-dimensional sphere $S^k$, $k > 1$, into $C^N(\Gamma)$ can be shrunk to a point in $C^N(\Gamma)$. Such a space is sometimes descriptively called “aspherical.”

2. The discretized configuration space has a natural structure of a cube-complex, since all the cells are products of intervals. This allows one to use recent fast algorithms from computational homology to determine homology groups and generators in practical settings.

3. From the cube complex structure of $D^N(\Gamma)$, one can show that $C^N(\Gamma)$ is an NPC (non-positively curved) space: there exists a metric whose curvature (defined appropriately at the non-manifold points) is never positive.

4. The fundamental groups of these spaces are all torsion-free. In other words, if a loop in $C^N(\Gamma)$ cannot be shrunk to a point, then neither can any multiple of the loop. This property is also true for configuration spaces of $\mathbb{R}^2$, but not for $C^N(S^2)$ — robots on a two-dimensional sphere.

5. The fundamental groups of these spaces have solvable word problem, which means that there is an algorithm which can be used to decide whether any given loop in $C^N(\Gamma)$ can be shrunk to a point in $C^N(\Gamma)$. 
6. The fundamental group $\pi_1(C^N(\Gamma))$ can be realized as a graph of groups — that is, as a collection of groups (abstractly thought of as vertices in some graph) which are pairwise glued together or “amalgamated” along a network of subgroups (each gluing represented by an edge between vertices).

For those not familiar with these more subtle features of topological spaces and their fundamental groups, we would offer the examples presented here as an excellent concrete manifestation of these properties. One can easily explain what a configuration space of robots is to a high-school class\(^3\), and, upon demonstrating that, e.g., two robots on $K_5$ yields a genus six surface, one has an excellent demonstration of the sublime nature of configuration spaces and of topology in general.

References

[1] A. Abrams. Configuration Spaces and Braid Groups of Graphs. PhD thesis, UC Berkeley, 2000.
[2] A. Abrams, H. Landau, Z. Landau, J. Pommersheim, and E. Zaslow. Evasive random walks. Preprint, submitted for publication, 2000.
[3] J. Birman. Braids, Links, and Mapping Class Groups. Princeton University Press, Princeton, N.J., 1974.
[4] R. Bott and C. Taubes. On the self-linking of knots: topology and physics. J. Math. Phys. 35(10), 5247–5287, 1994.
[5] J. Canny. The Complexity of Robot Motion Planning. MIT Press, Cambridge, MA, 1988.
[6] G. Castleberry. The AGV Handbook. Braun-Brumfield, Ann Arbor, MI, 1991.
[7] R. Ghrist. Configuration spaces of graphs in robotics. In Braids, Links, and Mapping Class Groups: the Proceedings of Joan Birman’s 70th Birthday, AMS/IP Studies in Mathematics vol. 19, 31–41, 2000. ArXiv preprint math.GT/9905023.
[8] R. Ghrist and D. Koditschek. Safe cooperative robot dynamics via dynamics on graphs. In Y. Nakayama, editor, Proceedings of the Eighth International Symposium on Robotics Research, pages 81–92. Springer-Verlag, 1998.
[9] R. Ghrist and D. Koditschek. Safe, cooperative robot dynamics on graphs. Preprint, submitted for publication, 2000. ArXiv preprint cs.RO/0002014.
[10] D. Gottlieb. Topology and the robot arm. Acta Appl. Math., 11(2): 117–121, 1988.
[11] M. Kapovich and J. J. Millson. The symplectic geometry of polygons in Euclidean space. J. Differential Geom., 44(3):479–513, 1996.
[12] D. Koditschek and E. Rimon. Robot navigation functions on manifolds with boundary. Adv. in Appl. Math., 11(4):412–442, 1990.
[13] J.-C. Latombe. Robot Motion Planning. Kluwer Academic Press, Boston, MA, 1991.

\(^3\)Both authors have done this on several occasions with positive results.
[14] K. Walker. *Configuration Spaces of Linkages*. Undergraduate thesis, Princeton University, 1985.
[15] J. Weeks. *The Shape of Space*. Marcel Dekker Inc., New York, 1985.

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