Suppressing and restoring constants in physical equations

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Abstract. A simple procedure for restoring the constants in physical equations is introduced by a consistent consideration of the correspondence between the symbols representing physical quantities and pairs formed by numerical values and unit symbols. This procedure is applied to the very used unit systems stressing also the direct relations between the atomic, natural systems and SI.

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A frequent procedure for obtaining simple calculations consists in setting some fundamental constants equal to unity. Sometimes, from physical interpretation reasons, it is necessary to restore these constants in the physical equations and in their solutions. This is just the point less treated in the textbooks (see for example [1, 2]). There are some general theoretical expositions in the literature on this problem [3] but here we adopt a practical point of view trying to give some simple notation and prescriptions for restoring the constants in any expression. Illustrations for the very used unit systems as the atomic and natural ones are presented. Certainly, such notation and prescriptions are almost everyday discovered and rediscovered by professors and students teaching or learning physics. The present note, not claimed as an original one, is intended as a didactic teaching aid for the beginners in physics.

Let us an unit system labeled by $\alpha$. A physical quantity $X$ is specified in the given system by the correspondence $X \rightarrow \{X_\alpha, \hat{X}_\alpha\}$ where the first element of the pair is a number and the second one symbolizes the unit used for measuring $X$. By the same rule we can write $\hat{X}_\alpha \rightarrow \{X_{\alpha\beta}, \hat{X}_\beta\}$ relating the units of $X$ from two systems $\alpha$ and $\beta$ such that

$$X_\beta = X_\alpha X_{\alpha\beta}. \quad (1)$$

By writing the successive correspondences

$$X \rightarrow \{X_\alpha, \hat{X}_\alpha\} \rightarrow \{X_\alpha X_{\alpha\beta}, \hat{X}_\beta\} \rightarrow \{X_\alpha X_{\alpha\beta} X_{\beta\alpha}, \hat{X}_\alpha\}$$

one obtains the simple and natural relation $X_{\beta\alpha}X_{\alpha\beta} = 1$.

Let a physical equation written in the $\alpha$-system as

$$F^{(\alpha)}(X^{(1)}_\alpha, \ldots, X^{(n)}_\alpha) = 0.$$
Suppressing and restoring constants in physical equations

For passing to the $\beta$-system, we write

$$F^{(\alpha)}(X^{(1)}_{\beta} X^{(1)}_{\beta a}, \ldots, X^{(n)}_{\beta} X^{(n)}_{\beta a}) = 0$$

such that the rule for obtaining the equation written in the $\beta$-system is given by

$$F^{(\alpha)}(X^{(1)}_{\alpha}, \ldots, X^{(n)}_{\alpha}) = 0 \quad X_{\alpha} \xrightarrow{\text{gauss}} F^{(\beta)}(X^{(1)}_{\beta}, \ldots, X^{(n)}_{\beta}) = 0. \quad (2)$$

The following examples are the most encountered in the physics literature.

1) The atomic unit system ($\alpha = a$) is introduced by the following values of electron’s mass and charge and of the reduced Planck’s constant:

$$(m_e)_a = 1, \quad (e)_a = 1, \quad (\hbar)_a = 1. \quad (3)$$

Considering the relation between this system and the Gauss unit system (cgsG: $\alpha = G$), these equations introduce the atomic units of mass, action and electric charge which, expressed in cgsG units, are represented by $M_{aG} = (m_e)_G, S_{aG} = (\hbar)_G, Q_{aG} = (e)_G$. From the equations (3) we can write

$$(m_e)_G M_{aG} = 1, \quad (e)_G^2 M_{aG} L_{aG}^3 T_{aG}^{-2} = 1, \quad (\hbar)_G M_{aG} L_{aG}^2 T_{aG}^{-1} = 1$$

resulting the well known cgsG expressions of the atomic units as

$$M_{aG} = \frac{1}{M_{aG}} = (m_e)_G, \quad L_{aG} = \frac{1}{L_{aG}} = \left( \frac{\hbar^2}{m_e e^2} \right)_G, \quad T_{aG} = \frac{1}{T_{aG}} = \left( \frac{\hbar^3}{m_e e^4} \right)_G \quad (4)$$

$L_{aG}$ being the Bohr radius expressed in cgs units. The unit of the velocity $v$, the value of the vacuum light speed and the energy unit are given by

$$v_{aG} = \frac{1}{v_{Ga}} = \frac{L_{aG}}{T_{aG}} = \left( \frac{e^2}{\hbar} \right)_G, \quad (e)_a = (e)_G v_{Ga} = \left( \frac{c \hbar}{e^2} \right)_G = \frac{1}{\alpha} \approx 137$$

$$W_{aG} = \frac{1}{W_{Ga}} = M_{aG} L_{aG}^2 T_{aG}^{-2} = \left( \frac{m_e e^4}{\hbar^2} \right)_G = 2 \left( \text{Rydberg}_G \right).$$

For passing to Heaviside-Lorentz unit system ($\alpha = H$), it suffices to give the relation between the electric charge units $Q_H$ and $Q_G$: from Coulomb’s law written for the same charges and distances in the two unit systems we have $F = \frac{Q_H^2}{r^2} = \frac{Q_G^2}{4 \pi r^2}$ and, using equation (7), $Q_G = Q_H Q_{HG}$, one obtains $Q_{HG} = \frac{1}{\sqrt{4 \pi}}$.

We may pass from the atomic system to the SI one ($\alpha = I$) via the cgsG system by well known relations. Alternatively, one may define the atomic unit system starting from SI by writing equations (3) with the SI numerical values for $m_e, e, \hbar$ and with the substitutions $G \rightarrow I$. Because there are four basic units in SI $\frac{1}{4}$ we can impose still another condition. From Coulomb’s law, $F = \frac{Q^2}{4 \pi \varepsilon_0 r^2}$, one sees that a reasonable condition is

$$(4 \pi \varepsilon_0)_a = 1 : \quad 4 \pi (\varepsilon_0)_I M_{Ia}^{-1} L_{Ia}^{-3} T_{Ia}^2 Q_{Ia}^2 = 1. \quad (5)$$

The SI version of equations (3) furnishes the atomic units for mass and electric charge

$$M_{aI} = 1/M_{Ia} = (m_e)_I, \quad Q_{aI} = 1/Q_{Ia} = (e)_I. \quad (6)$$

Considering also equation (7), one obtains the atomic units for length and time:

$$L_{aI} = \frac{1}{L_{Ia}} = \left( \frac{\hbar^2}{m_e c^2} \right)_I = (\text{Bohr’s radius})_I, \quad T_{aI} = \frac{1}{T_{Ia}} = \left( \frac{\hbar^3}{m_e c^4} \right)_I \quad (7)$$

$\frac{1}{4}$ We consider only the mechanical and electromagnetic units
Be chosen arbitrary and, usually, this is the length unit chosen as the Compton length.

Concerning the fundamental units of mass, length and time, one of these units may be chosen as the Coulomb’s law in cgsG and natural systems. We can write, using equation (2),

\[ Q \text{ (10)} \text{ in cgsG system, one obtains} \]
\[ \rho_{Ia} = \left(\frac{\hbar^6}{\sqrt{4\pi\varepsilon_0}m^3e^7} \right)_I, \quad j_{Ia} = \left(\frac{\hbar^7}{\sqrt{4\pi\varepsilon_0}m^4e^8} \right)_I, \quad E_{Ia} = \left(\frac{\sqrt{4\pi\varepsilon_0}\hbar^4}{m^2e^7} \right)_I, \]
\[ B_{Ia} = \left(\frac{e_0^2}{\hbar} \right)_I E_{Ia}, \quad \Phi_{Ia} = \left(\frac{\sqrt{4\pi\varepsilon_0}\hbar^2}{me_0} \right)_I, \quad A_{Ia} = \left(\frac{\sqrt{4\pi\varepsilon_0}\hbar}{me_0} \right)_I \]

(9)
given as an exercise for the reader. It is also interesting to see how the equations and are used for restoring the constants in Maxwell’s equations and obtaining the corresponding SI equations.

Let a simple example for the restoration of constants in the expression of the electromagnetic energy density. The energy density associated with equations (4) is given as an exercise for the reader. It is also interesting to see how the equations and are used for restoring the constants in Maxwell’s equations and obtaining the corresponding SI equations.

Let a simple example for the restoration of constants in the expression of the electromagnetic energy density. The energy density associated with equations (5) is given as an exercise for the reader. It is also interesting to see how the equations and are used for restoring the constants in Maxwell’s equations and obtaining the corresponding SI equations.

2) The natural system of units \((\alpha = n)\) is defined by the following values of the reduced Planck’s constant and of the vacuum light speed:

\[ (\hbar)_n = 1, \quad (c)_n = 1. \]

(10)

These equations introduce directly the natural units of action and velocity expressed, for example, in cgsG units, as \(S_{nG} = (\hbar)_G, \quad v_{nG} = (c)_G. \)

Expressing the equations in cgsG system, one obtains

\[ (\hbar)_G M_{Gn} L_{Gn}^2 T_{Gn}^{-1} = 1, \quad (c)_G L_{Gn} T_{Gn}^{-1} = 1. \]

(11)

Concerning the fundamental units of mass, length and time, one of these units may be chosen arbitrary and, usually, this is the length unit chosen as the Compton length of the electron: \(L_{nG} = (\hbar/m_e)_G. \)

From these definitions one obtains

\[ L_{nG} = (\hbar/m_e)_G, \quad T_{nG} = (\hbar/m_e^2)_G, \quad M_{nG} = (m_e)_G. \]

(12)

The natural unit of electric charge is obtained requiring the same formulation of Coulomb’s law in cgsG and natural systems. We can write, using equation (2),

\[ F_G = Q_G^2 R_{G}^{-2} \longrightarrow F_n F_{nG} = Q_n^2 R_n^{-2} Q_{nG}^2 L_{nG}^{-2} \]

and, because \(F_n = Q_n^2/R_n^2, \)

\[ Q_{nG} = \sqrt{(\hbar c)_G, \quad (e)_n = \left(e/\sqrt{nG} \right)_G \approx 1/\sqrt{137}.} \]

(13)

§ Equivalently we may choose the natural unit of mass as the electron mass.
We can write also the following useful relations:

\[ \rho_{nG} = \left( \sqrt{\hbar cm^3 c^3 / \hbar^3} \right)_G, \quad j_{nG} = (c)_G \rho_{nG}, \quad E_{nG} = \left( \sqrt{\hbar cm^2 c^2 / \hbar^2} \right)_G, \]
\[ B_{nG} = E_{nG}, \quad \Phi_{nG} = (\hbar/m_e c) G E_{nG}, \quad A_{nG} = \Phi_{nG}. \]  

(14)

If equations (10) are written in SI units one obtains the natural values \( M_{nI}, L_{nI}, T_{nI} \) by the same procedure as in the case of cgsG:

\[ L_{nI} = (\hbar/m_e c)_I, \quad T_{nI} = (\hbar/m_e c^2)_I, \quad M_{nI} = (m_e)_I. \]

As in the atomic unit system case, we require \((4\pi \varepsilon_0)_n = 1\) resulting \((\mu_0)_n = 4\pi\).

Considering Coulomb’s law, required to be expressed in natural units as \( F = Q^2/r^2 \), one obtains \( Q^2_{nI} = (4\pi \varepsilon_0 \hbar c)_I \), and for the electron charge \((e)_n = (e_0/\sqrt{\hbar c})_I\) as in equation (13).

We can also derive the SI values of the units for the electric charge and current densities, electric and magnetic fields and electromagnetic potentials:

\[ \rho_{nI} = \left( \sqrt{4\pi \varepsilon_0 \sqrt{\hbar cm^3 c^3 / \hbar^3}} \right)_I, \quad j_{nI} = (c)_I \rho_{nI}, \quad E_{nI} = \left( \sqrt{\hbar cm^2 c^2 / 4\pi \varepsilon_0 \hbar^2} \right)_I, \]
\[ B_{nI} = (1/c)_I E_{nI}, \quad \Phi_{nI} = (\hbar/m_e c)_I E_{nI}, \quad A_{nI} = (1/c)_I \Phi_{nI} \]

(15)

Let us, as an example, the Dirac equation for a charged particle in an external electromagnetic field written in natural units

\[ (i\hat{\partial} - q \hat{A} - m)\psi(x) = 0, \quad (A^\mu) = (\Phi, \mathbf{A}). \]

Passing to the cgsG or SI units, \( \alpha = G \) or \( \alpha = I \), we write

\[ \left[ (i\hat{\partial} - q \hat{A} - m)\right]_{\alpha} \psi = 0 \rightarrow \]
\[ \left\{ \frac{i}{L_{\alpha n}} \hat{\partial} - Q_{\alpha n} q \left[ \gamma^0 \Phi_{\alpha n} - \gamma \cdot \mathbf{A}_{\alpha n} \right] - m M_{\alpha n} \right\}_{\alpha} \right) \psi = 0. \]

For \( \alpha = G \), using equations (14) one obtains the Dirac equation in cgsG units

\[ (i\hbar \hat{\partial} - q \hat{A} - m c)\psi = 0, \quad (A^\mu) = (\Phi, \mathbf{A}) \]

and, for \( \alpha = I \), the same equation in SI units

\[ (i\hbar \hat{\partial} - q \hat{A} - m c)\psi = 0, \quad (A^\mu) = (\Phi/c, \mathbf{A}). \]

References

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