Self-Dual SU(3) Chern-Simons Higgs Systems†

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ABSTRACT

We explore self-dual Chern-Simons Higgs systems with the local \( SU(3) \) and global \( U(1) \) symmetries where the matter field lies in the adjoint representation. We show that there are three degenerate vacua of different symmetries and study the unbroken symmetry and particle spectrum in each vacuum. We classify the self-dual configurations into three types and study their properties.

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1. Introduction

There have been some recent activities related to the various self-dual Chern-Simons Higgs systems. A bound on the energy functional of these systems is saturated by the solitonic configurations of the fractional spin. When the gauge symmetry is abelian, the structure of these configurations has been studied quite well.\textsuperscript{1,2} While the self-dual systems with an arbitrary gauge group and matter are shown to exist in the theories with a global $U(1)$ symmetry, the self-dual configurations in these systems have been studied only in the cases where either the gauge symmetry or the matter is simple.\textsuperscript{3} The nonrelativistic limit of these system with matter in the adjoint representation has been also studied extensively, where the classification of finite energy soliton solutions has been found.\textsuperscript{4}

There are some studies of the simpler cases where the self-dual solitons is equivalent to the abelian self-dual solitons.\textsuperscript{3} However, the soliton structure with nonabelian symmetries turns out in general very intricate and interesting. In this paper we study the theory with $SU(3)$ gauge group with matter made of a complex scalar field in the adjoint representation. This model is one of the simple models with a nontrivial nonabelian feature and exhibits rich vacuum and soliton structures. Solitons in this theory would carry fractional spins and nonabelian charges. We hope our work would shed some light on the general structures of the self-dual systems.

First, we investigate the general consequences of the self-dual equations for the configurations saturating the energy bound. Then, we show that there are three degenerate vacua of various unbroken symmetries and topologies, and analyze the particles spectrum at each vacuum. After that we study the characteristics of the self-dual configurations and classify them into three types. In general we ex-
pect these self-dual configurations describe topological and nontopological solitons dwelling on each phase. We study the topology of these solutions. Our analysis here provides a significant but not complete understanding of these classical solutions of the self-dual equations.

There is usually an underlying $N = 2$ supersymmetry behind every self-dual model. There have been some studies of the underlying $N = 2$ supersymmetry in (Maxwell) Chern-Simons Higgs systems. In addition, it is obvious that the maximum possible symmetry for three dimensions is $N = 3$ because a maximum vector multiplet can have spin $1, 1/2, 0, -1/2$ up to sign. All $N = 3$ supersymmetric theories have been constructed recently.

In Sec. 2, we review briefly the self-dual model with $SU(3)$ gauge group and a complex scalar field in the adjoint representation. We then investigate in some detail the restrictions on the field configurations imposed by the self-dual equations. In Sec. 3 we study the ground states, their symmetric properties and elementary excitations. In Sec. 4, we classify the self-dual configurations to three types and study their properties. In Sec. 5 we conclude with some remarks.
2. Model

Let us consider a Chern-Simons-Higgs theory with local $SU(3)$ and global $U(1)$ symmetries. The generators of $SU(3)$ in the fundamental representation are made of $3 \times 3$ Hermitian matrices, $T^a$, with $a = 1, 2, \ldots, 8$ and satisfy the commutation relations $[T^a, T^b] = i f^{abc} T^c$ with $f^{abc}$ as the structure constants of $SU(3)$. The normalization is such that $\text{tr} T^a T^b = \delta^{ab} / 2$. The scalar matter field $\phi = (\phi_R^a + i \phi_I^a) T^a$ is made of a pair $\phi^a_R, \phi^a_I$ in the adjoint representation of $SU(3)$.

The Lagrangian density for the theory is given by

$$
\mathcal{L} = \kappa \epsilon^{\mu
u\rho} \text{tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + 2 \text{tr}|D_\mu \phi|^2 - \frac{2}{\kappa^2} \text{tr}[[\phi, [\phi^\dagger, \phi]] - v^2 \phi]^2
$$

(2.1)

where $D_\mu \phi = \partial_\mu \phi - i [A_\mu, \phi]$ with $A_\mu = A^{a}_\mu T^a$. The gauge field strength is given by $F_{\mu\nu} \equiv F^a_{\mu\nu} T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]$. The theory is renormalizable because in $2 + 1$ dimensions the sixth order term in the potential energy has a dimensionless coupling constant. For the consistent quantum mechanics, the coefficient $\kappa$ should be quantized so that $\kappa = k/4\pi$ with a nonzero integer $k$.

Gauss’s law constraint obtained from the variation of $A^a_0$ is

$$
\mathcal{G} \equiv -\kappa F_{12} - i([D_0 \phi^\dagger, \phi] - [\phi^\dagger, D_0 \phi]) = 0
$$

(2.2)

The local gauge transformation generators are made of $\text{tr}(\mathcal{G} T^a)$. The Lagrangian (2.1) is invariant under the global phase rotation of the scalar field. The charge density of the corresponding global $U(1)$ symmetry is

$$
\rho_Q = 2i \text{tr}(D_0 \phi^\dagger \phi - \phi^\dagger D_0 \phi)
$$

(2.3)

and the charge is $Q = \int d^2 r \rho_Q$. 


We are interested in finding bound on the energy functional. The energy functional for the Lagrangian (2.1) is

\[ E = \int d^2r \, 2 \left\{ \text{tr} |D_0 \phi|^2 + \text{tr} |D_i \phi|^2 + \frac{1}{\kappa^2} \text{tr} \left[ [\phi, [\phi^\dagger, \phi]] - v^2 \phi \right] \right\} \] (2.4)

With Gauss’s law (2.2), the second term in the bracket becomes

\[ \text{tr} |D_i \phi|^2 = \text{tr} |(D_1 \pm iD_2)\phi|^2 \]

\[ \pm \frac{i}{\kappa} \left\{ \text{tr} \left[ D_0 \phi^\dagger([\phi, [\phi^\dagger, \phi]] - v^2 \phi) \right] - \text{h.c.} \right\} \pm \frac{v^2}{2\kappa} Q \] (2.5)

up to a total derivative. This allows us to put $|D_0 \phi|^2$ and the potential energy density into a total square. After integrating by parts, the energy functional can be written as

\[ E = \int d^2r \, 2 \left\{ \text{tr} \left| D_0 \phi \pm \frac{i}{\kappa} ([\phi, [\phi^\dagger, \phi]] - v^2 \phi) \right|^2 + \text{tr} |D_1 \phi \pm iD_2 \phi|^2 \right\} \pm \frac{v^2}{\kappa} Q \] (2.6)

Since the integrand in Eq. (2.6) is nonnegative, there is a bound on the energy functional

\[ E \geq m |Q| \] (2.7)

where $m \equiv v^2/\kappa$ is the mass of elementary particles in the symmetric phase. The bound is given by a total global charge, which is not a priori related to any topological quantity. The bound (2.7) is saturated by the configurations satisfying Gauss’s law and the self-dual equations

\[ D_0 \phi \pm \frac{i}{\kappa} ([\phi, [\phi^\dagger, \phi]] - v^2 \phi) = 0 \] (2.8)

\[ D_1 \phi \pm iD_2 \phi = 0 \] (2.9)

where the upper (lower) sign corresponds to the positive (negative) value of $Q$. 

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Gauss’s law and Eq.(2.8) can be combined to

\[ F_{12} = \pm \frac{2}{\kappa^2} ([\phi^\dagger, [\phi, [\phi^\dagger, \phi]]] - v^2 [\phi^\dagger, \phi]) \]  

(2.10)

When Eq.(2.8) is satisfied the \( U(1) \) charge density (2.3) becomes

\[ \rho_Q = \mp \frac{4}{\kappa} (\text{tr}[\phi^\dagger, \phi])^2 - v^2 \text{tr}[\phi^\dagger, \phi] \]

(2.11)

For the configurations satisfying Eqs.(2.8) and (2.9), the total angular momentum becomes

\[ J = - \int d^2r \ 2 \epsilon_{ij} r_i \ \text{tr}(D_0 \phi^\dagger D_j \phi + D_j \phi^\dagger D_0 \phi) \]

\[ = \int d^2r \ r_i \partial_i \ \text{tr}([(\phi^\dagger, \phi])^2 - 2v^2 \phi^\dagger \phi) \]

\[ = 2 \int d^2r \left\{ v^4 C_{\text{phase}} - \text{tr} \left( [(\phi^\dagger, \phi]^2 - 2v^2 \phi^\dagger \phi) \right) \right\} \]

(2.12)

where nonnegative \( C_{\text{phase}} \) is the spatial asymptotic value of \( \text{tr}([(\phi^\dagger, \phi]^2 - 2v^2 \phi^\dagger \phi)/v^4. \)

\( C_{\text{phase}} \) depends on the phase or vacuum the system resides and will be calculated Sec.3 for each phase. As the potential energy density is nonnegative, the ground states are characterized by the zeros of the potential energy density, satisfying

\[ [\phi, [\phi^\dagger, \phi]] - v^2 \phi = 0 \]

(2.13)

which implies \( \text{tr} \phi^n = 0 \) for any natural number \( n. \)

Let us now explore some aspects of the self-duality equations Eqs.(2.9) and (2.10). One can easily see that the solutions of these equations satisfy \( \partial_0 \text{tr} \phi^n \mp \frac{2}{\kappa} v^2 \text{tr} \phi^n = 0 \) and \( (\partial_1 \pm i \partial_2) \text{tr} \phi^n = 0 \) for any natural number \( n, \) which imply that \( \text{tr} \phi^n \) is a (anti)holonomic function. As the field configuration approaches one
of vacua at spatial infinity where \( \text{tr}\phi^n = 0 \) because of Eq.(2.13), the holonomic function should vanish everywhere, i.e., \( \text{tr}\phi^n(x) = 0 \). After triangularization with a similar transformation, the trace conditions imply that the diagonal elements of the triangularized matrix vanish, leading to \( \phi^n = 0 \) for \( n \geq 3 \).

The relation \( \phi^3(x) = 0 \) everywhere is an important property of the self-dual configurations. If \( \phi^3 = 0 \) and \( \phi^2 \neq 0 \), there is a three dimensional complex vector \( \vec{u} \) such that \( \phi^2 \vec{u} \neq 0 \). Three vectors \( \vec{u}, \phi \vec{u}, \phi^2 \vec{u} \) are linearly independent and form a basis of a three dimensional complex vector space. Starting from \( \phi^2 \vec{u} \), we can form an orthonormal basis by the standard procedure in linear algebra. In this orthonormal basis, \( \phi \) becomes

\[
\phi = v \begin{pmatrix} 0 & f & h \\ 0 & 0 & g \\ 0 & 0 & 0 \end{pmatrix}
\]

where dimensionless \( f, g, h \) are in general complex. If \( \phi^2 = 0 \) and \( \phi \neq 0 \) then, one can show easily that \( \phi \) is again given by Eq.(2.14) with either \( f \) or \( g \) to be zero. As there is an orthonormal basis where \( \phi \) is given by the above triangular matrix, one can see that there is always a special unitary transformation from any \( \phi \) satisfying \( \phi^3 = 0 \) to this triangular matrix. We can use a local gauge transformation to put the \( \phi \) field in the above form at each spacetime point. Here we will not consider the possibility of configurations, e.g., magnetic monopole instantons, for which there may be a topological obstruction to choose such a gauge globally.
3. Ground States and Spectra

Let us now consider the ground states of the model and explore the unbroken symmetries and particle spectra. As the potential energy in the Lagrangian (2.1) is nonnegative, the ground state configurations of zero energy satisfy Eq.(2.13). By identifying $J_z = [\phi^\dagger, \phi]/v^2$, $J_+ = \sqrt{2}\phi^\dagger/v$, $J_- = \sqrt{2}\phi/v$, one can see that $J_x = (J_+ + J_-)/2$, $J_y = (J_+ - J_-)/2i$ and $J_z$ satisfy the angular momentum commutation relation. As $\phi$ is a $3 \times 3$ triangular matrix, ‘the total angular momentum’ can be zero, one half and one, having one, two and three dimensional representations, respectively.

Alternatively, we notice that the vacuum configurations are the solutions of the self-dual equations. Thus a vacuum configuration can be chosen to the triangular form (2.14). We solve Eq.(2.13) with the triangular scalar field (2.14) to find the vacuum configurations. Since the vacuum energies of three phases are degenerate, there will be topological domain walls interpolating two different vacua.

1) Phase I

Let us consider first the one-dimensional representation. The vacuum expectation value of $\phi$ becomes $<\phi> = 0$. This is the symmetric phase or Phase I where the global $U(1)$ and local $SU(3)$ symmetries are preserved. There is no propagating mode for the gauge field. The scalar field $\phi$ carries unit global charge and forms the adjoint representation of $SU(3)$. The mass of the scalar field is $m \equiv v^2/\kappa$. $C_{\text{phase}}$ of Eq.(2.12) vanishes.

2) Phase II

For the two-dimensional representation the vacuum expectation value of $\phi$ can
be chosen to be

\[
\langle \phi \rangle_v = \frac{v}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (3.1)

In this phase \(C_{\text{phase}}\) of Eq.(2.12) becomes 1/2. In the unitary gauge, the scalar field becomes

\[
\phi = \frac{1}{\sqrt{2}} \begin{pmatrix}
\alpha/\sqrt{6} & 0 & v + \delta/\sqrt{2} \\
-2\alpha/\sqrt{6} & \beta_1 & 0 \\
\gamma & \beta_2 & \alpha/\sqrt{6}
\end{pmatrix}
\] (3.2)

All components except \(\delta\) are complex. The fields are normalized to have the standard kinetic term. The masses of the fields are \(m_\alpha = m\), \(m_{\beta_1} = m_{\beta_2} = 3m/2\), and \(m_\gamma = 2m\), and \(m_\delta = 2m\). The gauge field in the unitary gauge becomes

\[
A_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix}
a_\mu/\sqrt{6} + b_\mu/\sqrt{2} & c_{1\mu} & d_\mu \\
c_{1\mu} & -2a_\mu/\sqrt{6} & e_{2\mu} \\
d_\mu & e_{2\mu} & a_\mu/\sqrt{6} - b_\mu/\sqrt{2}
\end{pmatrix}
\] (3.3)

with masses \(m_b = 2m\), \(m_{c_1} = m_{c_2} = m/2\), \(m_d = m\). The field \(a_\mu\) is the gauge field for the leftover abelian local gauge symmetry and there is no corresponding propagating degrees of freedom.

By examining the symmetry generators which leave Eq.(3.1) invariant, we can find the unbroken generators. The \(SU(3) \times U(1)\) group is spontaneously broken to the global \(U(1)_R\) and local \(U(1)_S\) symmetry group. With the definition, \(\tilde{T}^3 \equiv \text{diag}(1,0,-1)\) and \(\tilde{T}^8 \equiv \text{diag}(1,-2,1)/\sqrt{3}\), the generators of the unbroken symmetries are given as \(R = \int d^2r (\rho_Q - \text{tr}\tilde{T}^3G)\) and \(S = \int d^2r \text{tr}\tilde{T}^8G\).

Since the gauge group acting on the adjoint representation is really \(SU(3)/Z_3\) where \(Z_3\) is the center of \(SU(3)\), the vacuum manifold of Phase II would be \([SU(3)/Z_3 \times U(1)_Q]/[U(1)_R \times U(1)_S]\) and 7 dimensional. We argue in the next section that the first fundamental homotopy group of this vacuum manifold is \(Z_2\).
We can write the lagrangian in terms of the fields (3.2) and (3.3), which would be invariant under these unbroken symmetries. One can calculate the charge density for these generators. For the global $U(1)_R$ symmetry, the charge density is given by

$$\rho_Q - \text{tr} T^3 = i \left\{ (\alpha \pi_\alpha - \bar{\alpha} \pi_{\bar{\alpha}}) + \frac{3}{2} (\beta_1 \pi_{\beta_1} - \bar{\beta}_1 \pi_{\bar{\beta}_1}) + 2 (\gamma \pi_\gamma - \bar{\gamma} \pi_{\bar{\gamma}}) \right\}$$

$$+ \frac{\kappa}{2} \left\{ 2 \vec{\nabla} \times \vec{b} - i \vec{c}_1 \times \vec{\bar{c}}_1 - i \vec{c}_2 \times \vec{\bar{c}}_2 - 2 i \vec{d} \times \vec{\bar{d}} \right\}$$

(3.4)

For the local $U(1)_S$ symmetry, the charge density is

$$\text{tr} T^8 = 3 \left\{ - (\beta_1 \pi_{\beta_1} - \bar{\beta}_1 \pi_{\bar{\beta}_1}) + (\beta_2 \pi_{\beta_2} - \bar{\beta}_2 \pi_{\bar{\beta}_2}) \right\}$$

$$+ \kappa \left\{ \vec{\nabla} \times \vec{a} + 3 i \vec{c}_1 \times \vec{\bar{c}}_1 - 3 i \vec{c}_2 \times \vec{\bar{c}}_2 \right\}$$

(3.5)

In Phase II, the energy bound (2.7) becomes

$$\mathcal{E} \geq \pm m \int d^2r (\rho_Q - \text{tr} T^3 G)$$

(3.6)

because of Gauss’s law on the physical configurations. Note that all particles of global $U(1)_R$ charge saturate the above bound.

Here we should note that there could be magnetic monopole instantons in this phase, leading to the violation of the charge (3.5) for the local gauge symmetry.\(^9\) The gauge charge would be conserved modulo a integer which depends on the coefficient of the Chern-Simons term and the minimum monopole magnetic flux. Further investigation is necessary to settle this interesting possibility.

3) Phase III
The three dimensional representation would lead to the ground configuration
\[
\langle \phi \rangle_v = v \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\] (3.7)

In this phase \( C_{\text{phase}} \) in Eq.(2.12) becomes 2. In the unitary gauge we can choose the scalar field to be
\[
\phi = \begin{pmatrix} \chi/\sqrt{30} & v + (\zeta + \xi)/2\sqrt{2} & -3\bar{\chi}/\sqrt{30} \\ \eta/2 & -2\chi/\sqrt{30} & v + (\zeta - \xi)/2\sqrt{2} \\ \psi/\sqrt{2} & \eta/2 & \chi/\sqrt{30} \end{pmatrix}
\] (3.8)

The mass spectrum is given by \( m_\chi = 5m, m_\eta = 2m, m_\psi = 3m, m_\zeta = 2m \) and \( m_\xi = 6m \). The gauge field in the unitary gauge is given by
\[
A_\mu = \begin{pmatrix} p_\mu/2 + q_\mu/2\sqrt{3} & (r_\mu + s_\mu)/2 & t_\mu/\sqrt{2} \\ (\bar{r}_\mu + \bar{s}_\mu)/2 & -q_\mu/\sqrt{3} & (r_\mu - s_\mu)/2 \\ \bar{t}_\mu/\sqrt{2} & (\bar{r}_\mu - \bar{s}_\mu)/2 & -p_\mu/2 + q_\mu/2\sqrt{3} \end{pmatrix}
\] (3.9)

with masses \( m_p = 2m, m_q = 6m, m_r = m, m_s = 5m, \) and \( m_t = 2m \).

The original symmetry is then spontaneously broken to a global \( U(1)_U \) symmetry. The generator of this symmetry is \( U = \int d^2r (\rho_Q - 2 \text{tr} \bar{T}^3 G) \), where
\[
\rho_Q - 2 \text{tr} \bar{T}^3 G = i \left\{ \left( \chi \pi - \bar{\chi} \pi \bar{\chi} \right) + 2(\eta \pi - \bar{\eta} \pi \bar{\eta}) + 3(\psi \pi \psi - \bar{\psi} \pi \bar{\psi}) \right\} + \kappa \left\{ 2\bar{\nabla} \times \vec{p} + i\vec{r} \times \vec{\bar{r}} + i\vec{s} \times \vec{\bar{s}} + 2i\vec{t} \times \vec{\bar{t}} \right\}
\] (3.10)

In Phase III, the energy bound (2.7) becomes
\[
E \geq \pm m \int d^2r (\rho_Q - 2 \text{tr} \bar{T}^3)
\] (3.11)
due to Gauss’s law. The masses of all charged field except those of \( \chi \) and \( s_\mu \) saturate the energy bound. Usually the masses of charged particles in self-dual models...
saturate the energy bound. This seems to be the first example where the bound is not saturated by charged particles. If our theory is a part of $N = 2$ supersymmetric theory, $\chi$ and $s_\mu$ would be the bosonic part of a vector supermultiplet.

The vacuum manifold of Phase III would be given by a 8-dimensional space $[SU(3)/Z_3 \times U(1)_Q]/U(1)_U$. We argue in the next section that the first homotopy group of this manifold is $Z_3$. 
4. Self-Dual Configurations

In this section, we study some properties of the self-dual configurations which satisfy Eqs.(2.9) and (2.10). Let us first try to classify the possible configurations. There is always a gauge where the scalar field is given by Eq.(2.14) as argued before. For convenience, we introduce three dimensionless quantities $F, G, H$ such that

$$F = |f|^2, \ G = |g|^2, \ H = |h|^2$$

(4.1)

where $f, g, h$ are given in Eq.(2.14).

From gauge invariant combinations $\text{tr}\phi^\dagger\phi$, $\text{tr}\phi^\dagger\phi^2$, $\text{tr}(\phi^\dagger\phi)^2$, and $\text{det}[\phi^\dagger, \phi]$ of the scalar field, one can obtain some dimensionless gauge invariant quantities,

$$K = F + G + H$$
$$L = FG$$
$$M = fg\bar{h}$$
$$N = (F - G)L$$

(4.2)

We classify the nontrivial ($K \neq 0$) solutions of the self-dual equations into three types: $\text{Type A}$ with $M = L = 0$, $\text{Type B}$ with $M = 0, L \neq 0$ and $\text{Type C}$ with $M \neq 0$. As we will see, $\text{Type A}$ is the simplest and $\text{Type C}$ is the most complicated and interesting. For each phase studied in the previous section, the above three types of self-dual solutions might exist. Some of them would be topological and others would be nontopological.

In terms of the above gauge invariant quantities, the global charge density
\[ \rho_Q = \pm 4mv^2[K(1 - 2K) + 6L] \]  

(4.3)

The total angular momentum (2.12) becomes

\[ J = 2v^4 \int d^2r \left\{ C_{\text{phase}} + 2(K - K^2 + 3L) \right\} \]  

(4.4)

To understand further implications of the self-dual equations, we define \( \partial = \partial_1 + i\partial_2 \), \( \bar{\partial} = \partial_1 - i\partial_2 \), \( A = A_1 + iA_2 \) and \( \bar{A} = A_1 - iA_2 \). The magnetic field becomes \( F_{12} = (\partial A - \partial \bar{A} - i[A, \bar{A}])/2i \). From now on we will be only interested in positive \( Q \) configurations. Eq.(2.9) can be written as

\[ \partial \phi - i[A, \phi] = 0 \]  

(4.5)

With \( \phi \) given in Eq.(2.14), the above equation implies that \( A \) should be an traceless triangular matrix,

\[ A = \begin{pmatrix} a_1 & b_1 & c \\ 0 & -(a_1 + a_2) & b_2 \\ 0 & 0 & a_2 \end{pmatrix} \]  

(4.6)

Furthermore Eq.(4.5) in components becomes

\[ \partial f - i(2a_1 + a_2)f = 0 \]
\[ \partial g + i(a_1 + 2a_2)g = 0 \]  

(4.7)

\[ \partial h - i(a_1 - a_2)h + ib_2f - ib_1g = 0 \]

With the gauge field (4.6), the off-diagonal components of the self-dual equation
(2.10) become

$$\begin{align*}
\partial \bar{b}_1 + i(2a_1 + a_2)\bar{b}_1 - ib_2 \bar{c} - 4im^2(2K - 1)g\bar{h} &= 0 \\
\partial \bar{b}_2 - i(a_1 + 2a_2)\bar{b}_2 + ib_1 \bar{c} + 4im^2(2K - 1)f\bar{h} &= 0 \\
\partial \bar{c} + i(a_1 - a_2)\bar{c} &= 0
\end{align*}$$

(4.8)

The diagonal components of Eq.(2.10) become

$$\begin{align*}
\bar{\partial}a_1 - \partial \bar{a}_1 + i|b_1|^2 + i|c|^2 + 4im^2[(2K - 1)(K - G) - 3L] &= 0 \\
\bar{\partial}a_2 - \partial \bar{a}_2 - i|b_2|^2 - i|c|^2 - 4im^2[(2K - 1)(K - F) - 3L] &= 0
\end{align*}$$

(4.9)

The self-dual equations in components are then given by Eqs.(4.7), (4.8) and (4.9).

Let us now examine more closely what the self-dual equations imply for each type of solutions.

1) Type A Solutions

For Type A solutions we can see easily that there is a local gauge transformation where \( f \neq 0 \) and \( g = h = 0 \) everywhere. (This is gauge equivalent to the case only \( h \) is not vanishing.) Thus, this type of configuration can exist only in Phases I, II. As there is no contribution to the energy from \( a_2, b_2, c \), we can regard \( a_2, b_2, c \) to be zero. (In the lagrangian equation, the field strength is zero and so the vector potential can be chosen to be zero.) Then, Eqs.(4.7) and (4.8) lead to

$$\partial(f\bar{b}_1) = 0$$

(4.10)

As \( f\bar{b}_1 \) is holonomic function and the gauge field goes to zero at the spatial infinity, \( \bar{b}_1 \) should be zero everywhere.
Thus Eqs.(4.7) and (4.8) become identical to the self-dual equations studied before Ref.[1,2] with different numerical factors:

\[ \partial f - 2ia_1 f = 0 \]
\[ \frac{1}{2i} (\partial a_1 - \partial \bar{a}_1) + 2m^2 F(2F - 1) = 0 \]  

(4.11)

With \( \ln f \equiv \frac{1}{2} \ln F + i \sum_{\alpha} \text{Arg}(\vec{r} - \vec{q}_\alpha) \) with vortices at \( \vec{q}_\alpha \), we can combine the above two equations to

\[ \vec{\nabla}^2 \ln F + 8m^2 F(1 - 2F) = 4\pi \sum_\alpha \delta(\vec{r} - \vec{q}_\alpha) \]

(4.12)

where \( \epsilon_{ij} \partial_i \partial_j \text{Arg}(\vec{r} - \vec{q}) = 2\pi \delta(\vec{r} - \vec{q}) \) is used. The solutions of this equation are made of Q-balls in the symmetric Phase I and vortices in the asymmetric Phase II.

The topology of vortices in the asymmetric phase is interesting. In the \( SU(2) \) case, vortices are shown to have the \( Z_2 \) topology. In our case, the \( f \) field of an elementary vortex in Phase II would be given as \( f \approx e^{i\varphi}/\sqrt{2} \) in large distance, which is equivalent to applying a gauge transformation \( \exp[i\varphi(\lambda_3/2 \pm \lambda_8/\sqrt{3})] \) on the vacuum expectation value \( f = 1/\sqrt{2} \). Since both of these mappings lead to the same vortex and are nontrivial elements of the first homotopy group \( \pi_1(SU(3)/Z_3) = Z_3 \), the topology of elementary vortices in Phase II should be \( Z_2 \). This can be confirmed by noticing that the asymptotic \( e^{i2\varphi}/\sqrt{2} \) of the \( f \) field for vorticity 2 is represented by the gauge transformation \( \exp[i\varphi\lambda_3] \) which is a trivial element of \( \pi_1(SU(3)/Z_3) \).

There is a simplification of the global charge and the total angular momentum.
Eqs.(4.3) and (4.11) leads to the charge density as a total derivative

\[ \rho_Q = 2\kappa \vec{\nabla} \times \vec{a}_1 \]  

The total global charge would then get a contribution only from spatial infinity. To simplify the angular momentum (2.12), we introduce a transverse vector \( \vec{a}_1 = \vec{a}_1 - (1/2) \sum_{\alpha} \vec{\nabla} \text{Arg}(\vec{r} - \vec{q}_\alpha) \), which is not well defined at the vortex position. Since the angular momentum density is finite everywhere, there is no finite contribution from the vortex positions to the angular momentum (2.12) and the integration region may be reduced from \( R^2 \) to \( R^2_* = R^2 - \{ \vec{q}_\alpha \} \). The angular momentum for Type A can then be written as

\[
J = -8\kappa m^2 \int_{R^2_*} d^2r \vec{r} \times \vec{a}_1 (2F - 1) F \\
= 4\kappa \int_{R^2_*} d^2r \vec{r} \times \vec{\nabla} \times \vec{a}_1 \\
= 4\kappa \int_{R^2} d^2r \vec{\nabla} \cdot \left\{ \frac{1}{2} \vec{r} \vec{a}_1^2 - \vec{a}_1 \vec{r} \cdot \vec{a}_1 \right\}
\]

(4.14)

For a given Type A self-dual configuration, the angular momentum can be evaluated as the sum of the boundary contributions from the vortex positions and spatial infinity. Eq.(4.14) was used extensively in the second paper of Ref.[2] to study the vortex dynamics. Especially, the statistical phase of vortices is argued to be originated from both the Aharonov and Bohm phase and the quantum Magnus phase. Similar arguments would apply to our case under the study.

2) Type B Solutions
Let us here start by considering Types B, C in general terms. For Type B, C solutions, \( fg \neq 0 \) and from Eqs.(4.7) and (4.8) we get

\[
\partial(f \bar{c}) = 0
\]
\[
\partial(f \bar{b}_1 + g \bar{b}_2) - i(f b_2 - g b_1) \bar{c} = 0
\]

(4.15)

As gauge fields vanish at the spatial infinity, the first part of Eq.(4.15) implies that \( fg \bar{c} = 0 \) everywhere, which in turn implies \( \bar{c} = 0 \) everywhere. The second part of the above equation implies \( f \bar{b}_1 + g \bar{b}_2 = 0 \), which can be satisfied by introducing a new variable \( u \) such that

\[ b_1 = -i\bar{g}u, \ b_2 = i\bar{f}u \]

(4.16)

Eqs.(4.7) and (4.16) lead to an equation for the field \( h \),

\[
\partial h - i(a_1 - a_2) h - (F + G) u = 0
\]

(4.17)

For Type B solutions where \( h = 0 \), the off-diagonal elements \( b_i \)'s of the gauge field vanish everywhere as we can see from Eqs.(4.16) and (4.17). Eqs.(4.7) and (4.9) become

\[
\partial f - i(2a_1 + a_2) f = 0
\]
\[
\partial g + i(a_1 + 2a_2) g = 0
\]
\[
\bar{\partial}a_1 - \bar{\partial}a_1 + 4im^2(2F - G - 1) F = 0
\]
\[
\bar{\partial}a_2 - \bar{\partial}a_2 - 4im^2(2G - F - 1) G = 0
\]

(4.18)

Eq.(4.18) is invariant under two U(1) gauge symmetries. We choose the gauge so
that

\[
\ln f = \frac{1}{2} \ln F + i \sum_{\alpha} \text{Arg}(\vec{r} - \vec{q}_{f\alpha}) \\
\ln g = \frac{1}{2} \ln G + i \sum_{\beta} \text{Arg}(\vec{r} - \vec{q}_{g\beta})
\]

where \(\vec{q}_{f\alpha}, \vec{q}_{g\beta}\) are positions of \(f, g\) vortices. Then, Eq.(4.18) can be written as

\[
\vec{\nabla}^2 \ln F - 4m^2(4F^2 - 2G^2 - FG - 2F + G) = 4\pi \sum_{\alpha} \delta(\vec{r} - \vec{q}_{f\alpha}) \\
\vec{\nabla}^2 \ln G - 4m^2(-2F^2 + 4G^2 - FG + F - 2G) = 4\pi \sum_{\beta} \delta(\vec{r} - \vec{q}_{g\beta})
\]  

We expect Type B solutions in all three phases. In Phase II, one of \(f\) or \(g\) would take the vacuum expectation value \(1/\sqrt{2}\) at spatial infinity. By similar argument for vortices of Type A in Phase II, elementary vortices of Type B in Phase II would have a topology \(Z_2\). In Phase III, vortices of Type B would have the \(Z_3\) topology. To see this, we assume that at spatial infinity \(f \approx e^{ik\phi}\) and \(g \approx e^{il\phi}\) with integers \(k, l\). This is equivalent to a gauge transformation \(\exp[i\phi\text{diag}(2k + l, -k + l, -k - 2l)/3]\) of \(\langle \phi \rangle\), which is a \(Z_3\) element of \(SU(3)\).

From Eq.(4.18), the global charge density (4.3) for Type B becomes

\[
\rho_Q = 2\kappa \vec{\nabla} \times (\vec{a}_1 - \vec{a}_2)
\]  

Since the gauge fields should be smooth functions, the total charge will get a contribution only from spatial infinity. To understand the angular momentum better, let us define

\[
\bar{a}_1 = a_1 - [2\partial\text{Arg}(f) + \partial\text{Arg}(g)]/3 \\
\bar{a}_2 = a_2 + [\partial\text{Arg}(f) + 2\partial\text{Arg}(g)]/3
\]  

From Eq.(4.18), one can see they are transverse vector fields. Similar to Type A, we subtract the vortex positions from the integration domain, \(R^2_* = R^2 - \{\vec{q}_{f\alpha}, \vec{q}_{g\beta}\}\).
without any change of the angular momentum. With Eqs. (2.12) and (4.16) the angular momentum becomes

\[
J = 2\kappa \int_{\mathbb{R}^2} \left\{ \nabla \times \tilde{a}_1 (2\vec{r} \times \tilde{a}_1 + \vec{r} \times \tilde{a}_2) + \nabla \times \tilde{a}_2 (2\vec{r} \times \tilde{a}_2 + \vec{r} \times \tilde{a}_1) \right\}
\]

Thus, the total angular momentum would get contributions from the vortex positions and spatial infinity. For a given self-dual configurations, we can write down the total angular momentum as a function of vortex positions in principle. As discussed for Type A, Eq.(4.23) would lead to a considerable understanding of the dynamics of the slowly moving vortices.

3) Type C Solutions

From Eqs. (4.8) and (4.16), we get the equation for the \( u \) field,

\[
\partial \bar{u} + i(a_1 - a_2) \bar{u} - 4m^2(2K - 1)\bar{h} = 0
\]  

(4.24)

From Eqs.(4.7), (4.17) and (4.24), we get

\[
\partial (fg \bar{u}) - 4m^2(2K - 1)fg\bar{h} = 0
\]

\[
\partial (h \bar{u}) - (F + G)|u|^2 - 4m^2(2K - 1)H = 0
\]

(4.25)

which implies that \( \partial (h \bar{u}) \) is a real field.

Since \( M = fg\bar{h} \) is a gauge invariant quantity, the vorticities of the \( f \) and \( g \) field should be closely related to that of the \( h \) field. However, it is not easy to see what kind of solutions will exist because the self-dual equations Eqs.(4.7), (4.9), (4.17) and (4.25) are rather complicated. In principle, Type C could exist in all
phases of the theory. The topology of Type C vortices in the broken phases would be identical to that of Type B vortices because Type C solutions become Type A or Type B solutions at spatial infinity.

Note that the charge density is given as a total derivative,

\[
\rho_Q = \frac{\kappa}{2i} \left( \bar{\partial}(a_2 - a_1) - \partial(\bar{a}_2 - \bar{a}_1) \right) + \kappa \partial(h\bar{u}) \quad (4.26)
\]

However, we have not been successful to express the angular momentum as a boundary contributions as in Eq.(4.23). The self-dual equations satisfied by Type C is rather complicated and needs further consideration.

5. Conclusion

We have studied the self-dual Chern-Simons Higgs systems with SU(3) gauge symmetry and U(1) global symmetry. The matter field is made of a complex scalar field in the adjoint representation. Our work is a first step towards understanding the self-dual Chern-Simons Higgs systems where the nonabelian symmetry plays a crucial role. We have analyzed the vacuum structure, particle spectrum and unbroken symmetries. In addition, we classified the self-dual configurations into three types of increasing complexity. We have shown that vortices in Phase II would have the $\mathbb{Z}_2$ topology and vortices in Phase III would have the $\mathbb{Z}_3$ topology. We have seen the global charge of the self-dual configurations is given as a boundary contribution from spatial infinity, making topological the total energy of those
configurations. In addition, the self-dual configurations are characterized by the total angular momentum, which we have shown to take a rather simple form for at least Types A, B.

Ideally, we want to understand the nature of self-dual solitons completely and there are many directions to take to reach that goal. Here are some ideas to be explored: the rotationally symmetric solutions, the topological domain walls interpolating degenerate vacua, the self-dual solutions of Type C, the classical dynamics of slowly moving solitons, the relation between relativistic and nonrelativistic solutions, and the possible magnetic monopole instantons in Phase II. We would like to understand the quantum aspects of these solitons. One novel possibility might be the “nonabelian Magnus force and phase” between vortices in the asymmetric phase. We note that some of understandings gained here could be easily generalized to the cases with more complicated gauge groups and matter fields.

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