On the adjoint action of the group of symplectic diffeomorphisms

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Abstract

We study the action of Hamiltonian diffeomorphisms of a compact symplectic manifold \((X, \omega)\) on \(C^\infty(X)\) and on functions \(C^\infty(X) \to \mathbb{R}\). We describe various properties of invariant convex functions on \(C^\infty(X)\). Among other things we show that continuous convex functions \(C^\infty(X) \to \mathbb{R}\) that are invariant under the action are automatically invariant under so called strict rearrangements and they are continuous in the sup norm topology of \(C^\infty(X)\); but this is not generally true if the convexity condition is dropped.

1 Introduction

Consider a connected, compact, symplectic manifold \((X, \omega)\), without boundary, of dimension \(2n\). According to Omori \cite{O}, symplectic self-diffeomorphisms of \(X\) form a Fréchet–Lie group \(\text{Symp}(\omega)\), with Lie algebra the space \(\mathfrak{v}(\omega)\) of smooth vector fields on \(X\) that are locally Hamiltonian. In this paper we will be interested in the action of \(\text{Symp}(\omega)\), by pull back, on the Fréchet space \(C^\infty(X)\) of smooth real functions

\[
\text{Symp}(\omega) \times C^\infty(X) \ni (g, \xi) \mapsto \xi \circ g^{-1} \in C^\infty(X),
\]

and on functions on \(C^\infty(X)\) that (1.1) leaves invariant. This action is no adjoint action, but it is close to one. The adjoint action \(\text{Ad}_g\) of \(g \in \text{Symp}(\omega)\) is, rather, push forward by \(g^{-1}\) of vector fields in \(\mathfrak{v}(\omega)\). The subspace \(\text{ham}(\omega) \subset \mathfrak{v}(\omega)\) of globally Hamiltonian vector fields, those that are symplectic gradients \(\text{sgrad} \xi\) of some \(\xi \in C^\infty(X)\), is invariant under \(\text{Ad}_g\), and (1.1) induces via the projection \(\xi \mapsto \text{sgrad} \xi\) the restriction of the adjoint action to \(\text{ham}(\omega)\).

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Other diffeomorphism groups of $X$ also act on $C^\infty(X)$ by pull back. Our focus will be on the subgroup $\Ham(\omega) \subset \Symp(\omega)$ of Hamiltonian diffeomorphisms. Hamiltonian diffeomorphisms are the time 1 maps of time dependent Hamiltonian vector fields $\text{sgrad} \xi_t, \xi_t \in C^\infty(X)$. Continuous norms—and also seminorms—on the Fréchet space $C^\infty(X)$, invariant under $\Ham(\omega)$, are of potential interest in symplectic geometry because they give rise to bi-invariant metrics on $\Ham(\omega)$, and have been investigated in the past. An obvious norm is $\|\xi\|_\infty = \max_X |\xi|$. That it gives rise to a genuine metric on $\Ham(\omega)$ was proved first by Hofer in $\mathbb{R}^{2n}$, and in general by Lalonde and McDuff; see also Polterovich’s book, [Ho, LM, P]. Work by Ostrover–Wagner, Han, and Buhovsky–Ostrover [BO, Ha, OW] gave the following. Let $(X, \mu)$ and $(Y, \nu)$ be measure spaces. We say that measurable functions $\xi : X \to \mathbb{R}$, $\eta : Y \to \mathbb{R}$ are equidistributed. Theorem 1.1 is known (for example along the lines of the proof of Theorem 4.1 below). We obtain the simplification by restructuring the proof. First we prove that any $\lambda \in \Ham(\omega)$, $\lambda$ is invariant under the action of $\Ham(\omega)$, and proves by an involved argument that $\|\| \leq c\|$ is invariant under volume preserving diffeomorphisms. Han in [Ha] subsequently strengthened this to invariance under strict rearrangements. All this is obtained as a consequence of a lemma of Katok [K] Section 3. The final step is in [BO], that takes an arbitrary continuous $\Ham(\omega)$ invariant seminorm $\|\|$ on $C^\infty(X)$, and proves by an involved argument that $\|\| \leq c\|$.

We obtain the simplification by restructuring the proof. First we prove that $p$ in Theorem 1.2 is a limit point of the set of $\Ham(\omega)$ invariant functions $q$ that are continuous in the $L^1$ topology on $C^\infty(X)$. This depends on studying linear forms on $C^\infty(X)$, i.e., distributions, and regularizing them using the action of $\Ham(\omega)$. Katok’s

When $\mu(X), \nu(Y) < \infty$, this is equivalent to $\mu\{x \in X : \xi(x) > t\} = \nu\{y \in Y : \eta(y) > t\}$ for all $t \in \mathbb{R}$. We have to use the qualifier ‘strict’, since the notion of rearrangement in harmonic analysis and Banach space theory typically refers to the relation $\mu\{x \in X : |\xi(x)| > t\} = \nu\{y \in Y : |\eta(y)| > t\}$. Back to our symplectic manifold $(X, \omega)$, we write $\mu$ for the measure on $X$ defined by $\omega^n$; the action (1.1) clearly sends functions on $(X, \mu)$ to their strict rearrangements.

**Theorem 1.1 ([BO, H, OW]).** If $\|\|$ is a $\Ham(\omega)$ invariant continuous seminorm on the Fréchet space $C^\infty(X)$, then $\|\xi\| = \|\eta\|$ whenever $\xi, \eta \in C^\infty(X)$ are equidistributed. These seminorms satisfy $\|\| \leq c\|\|_\infty$ with some $c \in (0, \infty)$. Unless $\|\|$ and $\|\|_\infty$ are equivalent, the pseudodistance on $\Ham(\omega)$ induced by $\|\|$ is identically 0.

One of our goals in this paper is to offer a simpler proof to the first two statements, in fact in a slightly greater generality:

**Theorem 1.2.** Suppose $p : C^\infty(X) \to \mathbb{R}$ is a continuous, convex function that is invariant under the action of $\Ham(\omega)$. Then $p$ is continuous in the topology of $C^\infty(X)$ induced by $\|\|_\infty$, and is invariant under strict rearrangements: $p(\xi) = p(\eta)$ whenever $\xi, \eta$ are equidistributed.

The point is not the modest gain in generality, which can easily be achieved once Theorem 1.1 is known (for example along the lines of the proof of Theorem 4.1 below). Rather, it is the simplification of the proof. This is how the two proofs compare. [OW] first proved that any $\Ham(\omega)$ invariant seminorm $\|\| \leq c\|\|_\infty$ is invariant under volume preserving diffeomorphisms. Han in [Ha] subsequently strengthened this to invariance under strict rearrangements. All this is obtained as a consequence of a lemma of Katok [K] Section 3. The final step is in [BO], that takes an arbitrary continuous $\Ham(\omega)$ invariant seminorm $\|\|$ on $C^\infty(X)$, and proves by an involved argument that $\|\| \leq c\|\|_\infty$.
lemma now gives that the functions \( q \) are invariant under strict rearrangements, whence so must be their limit point \( p \). Another application of Katok’s lemma, combined with real analysis type arguments then gives the continuity of \( p \) with respect to \( \| \cdot \|_\infty \).

Continuity of \( p \) with respect to \( \| \cdot \|_\infty \) in Theorem 1.2 is essentially an upper estimate of \( p \). We will also prove a lower estimate:

**Theorem 1.3.** Let \( p : C^\infty(X) \to \mathbb{R} \) be Ham(\( \omega \)) invariant, convex, and continuous. Then either

(i) \( p(\xi) = p_1(\int_X \xi \omega^n) \), where \( p_1 : \mathbb{R} \to \mathbb{R} \) is convex; or
(ii) there are \( a \in \mathbb{R}, b \in (0, \infty) \) such that

\[
p(\xi) \geq a + b \int_X |\xi| \omega^n \quad \text{if} \quad \begin{cases} \int_X \xi \omega^n = 0, & \text{or} \\ \int_X \xi \omega^n \geq 0 & \text{and} \lim_{\lambda \to \infty} \mathbb{R} \ni \lambda \to \infty p(\lambda) = \infty, & \text{or} \\ \int_X \xi \omega^n \leq 0 & \text{and} \lim_{\lambda \to -\infty} \mathbb{R} \ni \lambda \to -\infty p(\lambda) = \infty. \end{cases}
\]

If \( p \) is positively homogeneous (\( p(c\xi) = cp(\xi) \) for positive constants \( c \)), then \( a = 0 \).

In particular, if \( p \) is a norm, then it dominates \( L^1 \) norm, something that [OW] also found (cf. Proposition 6.1 there and its proof).

Above we have insisted on the difference between rearrangements and strict rearrangements. Nevertheless, Theorem 1.3 implies that in our setting the difference between the two is minimal. The notion of rearrangement invariant Banach spaces in the next theorem is defined in [BS], see also section 6.

**Theorem 1.4.** Given a Ham(\( \omega \)) invariant continuous norm \( p \) on \( C^\infty(X) \), there is a rearrangement invariant Banach function space on \( X \) whose norm, restricted to \( C^\infty(X) \), is equivalent to \( p \).

A natural question is whether Theorem 1.2 holds for all continuous Ham(\( \omega \)) invariant functions \( p \), independently of convexity. It does not:

**Theorem 1.5.** If \( \dim X \geq 4 \), there is a smooth Ham(\( \omega \)) invariant function \( p : C^\infty(X) \to \mathbb{R} \) that is not invariant under volume preserving diffeomorphisms \( X \to X \).

The last statement of Theorem 1.1 suggests that, after all, the only invariant norm on \( C^\infty(X) \) that is of interest for symplectic geometry, is Hofer’s norm \( \| \cdot \|_\infty \). However, all invariant norms are of interest for Kähler geometry. The groups \( \text{Symp}(\omega) \) and Ham(\( \omega \)) can be regarded as symmetric spaces. When \( (X, \omega) \) is Kähler, Donaldson, Mabuchi, and Semmes proposed that the infinite dimensional manifold \( \mathcal{H}_\omega \) of relative Kähler potentials, endowed with a natural connection on its tangent bundle, should be viewed as the dual symmetric space, at least in a formal sense; see [Do, M, S1, S2]. Ham(\( \omega \)) invariant norms on \( C^\infty(X) \) induce Finsler metrics on \( \mathcal{H}_\omega \) that are invariant under parallel transport, and, perhaps surprisingly, all these Finsler metrics induce genuine metrics on \( \mathcal{H}_\omega \). Mabuchi was the first to study such a metric, associated with \( L^2 \) norm \( \| \xi \| = (\int_X |\xi|^2 \omega^n)^{1/2} \); more recently, Darvas in [Da] introduced various Orlicz norms on \( C^\infty(X) \) and the induced metrics on \( \mathcal{H}_\omega \). Generalizing Darvas’s norms and metrics, in [L] we study general Ham(\( \omega \)) invariant Lagrangians and the associated action on \( \mathcal{H}_\omega \), and most results here are motivated by the needs of that paper.
2 Reduction to linear forms

In this section \((X, \omega)\) can be any \(2n\) dimensional symplectic manifold, not necessarily compact. The space of compactly supported smooth functions on \(X\) will be denoted \(\mathcal{D}(X)\), with its usual locally convex inductive limit topology. Its dual is \(\mathcal{D}'(X)\), the space of distributions. The group \(\text{Ham}_0(\omega)\), time 1 maps of compactly supported Hamiltonian flows, acts on \(\mathcal{D}(X)\) by pull back and on \(\mathcal{D}'(X)\) by push forward. We denote the pairing between \(\mathcal{D}'(X)\) and \(\mathcal{D}(X)\) by \(\langle \cdot, \cdot \rangle\). The locally convex topology of \(\mathcal{D}'(X)\) is generated by the seminorms \(\|f\|_\xi = |\langle f, \xi \rangle|\) with \(\xi \in \mathcal{D}(X)\). Integration against any smooth \(2n\)-form defines a distribution. Such distributions will be called smooth. If \(h \in \mathcal{D}'(X)\), we denote by \(\text{conv}(h)\) the closed convex hull of the \(\text{Ham}_0(\omega)\) orbit of \(h\).

The main result of this section is

\[\text{Lemma 2.1.}\] Suppose \(p : \mathcal{D}(X) \to \mathbb{R}\) is a \(\text{Ham}_0(\omega)\) invariant, continuous, convex function. There is a family \(A \subset \mathbb{R} \times C^\infty(X)\) such that

\[p(\xi) = \sup \left\{ a + \int_X f \xi a^n : (a, f) \in A \right\}, \quad \text{for all } \xi \in \mathcal{D}(X).\]

If \(p\) is positively homogeneous as well (\(p(c\xi) = cp(\xi)\) for \(0 < c < \infty\)), then \(A\) can be chosen in \((0) \times C^\infty(X)\).

For the proof we need certain regularization maps \(\mathcal{D}'(X) \to \mathcal{D}'(X)\). Let \(U \subset X\) be open, and assume that on a neighborhood of \(\overline{U}\) there are local coordinates \(x_\nu\) in which \(\omega\) takes the form \(\sum_1^n dx_\nu \wedge dx_{n+\nu}\). Let \(C \subset X \setminus \overline{U}\) be compact. Fix \(\varphi_\nu \in \mathcal{D}(X)\), \(\nu = 1, \ldots, 2n\), vanishing on a neighborhood of \(C\), such that \(\varphi_\nu = x_\nu\) in a neighborhood of \(\overline{U}\). Let \(g^t_\nu, \tau \in \mathbb{R}\), denote the Hamiltonian flow of \(\varphi_\nu\) for \(\nu \leq n\) and of \(-\varphi_\nu\) for \(\nu > n\); i.e., the flow of the vector fields \(\pm \text{grad} \varphi_\nu\). If \(t = (t_1, \ldots, t_{2n}) \in \mathbb{R}^{2n}\), put

\[g^t = g_{1}^{t_1} \circ g_{2}^{t_2} \circ \cdots \circ g_{2n}^{t_{2n}}.\]

Near \(C\) we have \(g^t = \text{id}\); on \(U\), for small \(t\), \(g^t(x) = x - t\). Let furthermore \(\chi \in \mathcal{D}(\mathbb{R}^{2n})\) be nonnegative, \(\int_{\mathbb{R}^{2n}} \chi(t) dt_1 \ldots dt_{2n} = 1\). For \(\lambda \in (0, \infty)\) define operators \(R_\lambda : \mathcal{D}'(X) \to \mathcal{D}'(X)\) by

\[R_\lambda h = \lambda^{2n} \int_{\mathbb{R}^{2n}} \chi(\lambda t)(g^t_\ast h) dt_1 \ldots dt_{2n} \in \text{conv}(h), \quad h \in \mathcal{D}'(X).\]

Standard properties of convolutions imply

\[\text{Lemma 2.2.}\] \(\lim_{\lambda \to \infty} R_\lambda h = h\) for \(h \in \mathcal{D}'(X)\). If the support of \(\chi\) is sufficiently close to 0, then \(R_\lambda h \in \text{conv}(h)\) is smooth on \(U\) and \(R_\lambda h = h\) on a neighborhood of \(C\). Furthermore, if \(V \subset W \subset X\) are open, and \(h\) is smooth on \(W\), then \(R_\lambda h\) is smooth on \(V\) for sufficiently large \(\lambda\).

\[\text{Lemma 2.3.}\] For any \(h \in \mathcal{D}'(X)\), smooth distributions are dense in \(\text{conv}(h)\).
Proof. It will suffice to prove that given a finite $\Xi \subset \mathcal{D}(X)$ and $\varepsilon > 0$, there is a smooth $h' \in \text{conv}(h)$ such that $|\langle h' - h, \xi \rangle| \leq \varepsilon$ for all $\xi \in \Xi$. To show this latter, for each $z \in X$ construct an open neighborhood $V(z) \subset X$ so that in a neighborhood of $\overline{V(z)}$ we can write $\omega = \sum dx_\nu \wedge dx_{n+\nu}$ in suitable local coordinates. Select a locally finite cover $V(z_1), V(z_2), \ldots$ of $X$. Thus the $V(z_j)$ form a finite or infinite cover depending on whether $X$ is compact or not. For each $j$ we can find $U_j \supset V(z_j)$ such that $\{U_j\}_j$ is still locally finite, and $\omega = \sum dx_\nu \wedge dx_{n+\nu}$ is still valid in some neighborhood of $U_j$.

Fix furthermore open sets $V_j^i$, $i \in \mathbb{N}$, such that

$$U_j = V_j^1 \supset V_j^2 \supset \cdots \supset V(z_j),$$

and compact sets $C_j \subset X \setminus \bigcup_{k>j} U_k$, $C_0 = \emptyset$, such that $C_{j-1} \subset \text{int} C_j$ and $\bigcup_j C_j = X$.

We let $h_0 = h$ and construct $h_j \in \text{conv}(h)$ so that for $j \geq 1$

$$|\langle h_j - h, \xi \rangle| < \varepsilon \quad \text{if} \quad \xi \in \Xi;$$

$$h_j|V_j^1 \cup \cdots \cup V_j^j \quad \text{is smooth ;}$$

$$h_j = h_{j-1} \quad \text{on int} C_{j-1}.$$

Assuming we already have $h_{j-1}$, we apply Lemma 2.2 with $U = U_j$, $C = C_j$, $V = V_j^1 \cup \cdots \cup V_j^j$, and $W = V_j^{j-1} \cup \cdots \cup V_{j-1}^1$. If $\lambda$ is sufficiently large, then $h_j = R_\lambda h_{j-1}$ will do as the next function. Note that $h_j$ is smooth over $V \cup U_j \supset V_j^1 \cup \cdots \cup V_j^j \cup V_{j-1}^1$.

Thus $h_j = h_{j+1} = \ldots$ on int $C_j$ and $h_j|V(z_1) \cup \cdots \cup V(z_j)$ is smooth. If $X$ is compact, we take $h'$ to be the last $h_j$; otherwise we take $h' = \lim_{j \to \infty} h_j$.

Proof of Lemma 2.1. By an affine function we mean a function $\mathcal{D}(X) \to \mathbb{R}$ of the form $\text{const} + \text{linear}$. Clearly, if an affine function is bounded above on a symmetric neighborhood of $0 \in \mathcal{D}(X)$, it is bounded below as well, hence continuous.

Let $\mathcal{B}$ denote a collection of affine functions $\beta : \mathcal{D}(X) \to \mathbb{R}$ such that $\beta \leq p$. Thus $\beta \in \mathcal{B}$ can be written

$$\beta(\xi) = a + \langle h, \xi \rangle, \quad \text{with} \quad a \in \mathbb{R}, \quad h \in \mathcal{D}'(X).$$

The Banach–Hahn separation theorem gives that $p = \sup_{\beta \in \mathcal{B}} \beta$ with a suitable choice of $\mathcal{B}$. If $p$ is positively homogeneous, another version of the Banach–Hahn theorem, see e.g. [Sc, p.317-319], gives that $\mathcal{B}$ can be taken to consists of linear forms, i.e. all $a$ will be 0.

By the invariance of $p$, if $\beta$ in (2.2) is in $\mathcal{B}$, then for any $g \in \text{Ham}(\omega)$

$$(g_* \beta)(\xi) = a + \langle g_* h, \xi \rangle = a + \langle h, g^* \xi \rangle \leq p(\xi).$$

This means that all $g_* \beta$ can be adjoined to $\mathcal{B}$, and in fact we can arrange that all $a + \langle h', \xi \rangle$ are in $\mathcal{B}$ for any $h \in \mathcal{B}$ and $h' \in \text{conv}(h)$. Therefore if we take all $\beta \in \mathcal{B}$ of form (2.2) with smooth $h$ and write $h$ as $f\omega^n$, the family $\mathcal{A}$ of pairs $(a,f)$ thus obtained will do according to Lemma 2.3.
3 Proof of the second part of Theorem 1.2

The second part was:

**Theorem 3.1.** Let \((X, \omega)\) be a connected, compact, symplectic manifold. Any continuous, convex, and \(\text{Ham}(\omega)\) invariant function \(p : C^\infty(X) \rightarrow \mathbb{R}\) is strict rearrangement invariant: \(p(\xi) = p(\eta)\) if \(\xi, \eta\) are equidistributed.

As before, \(\mu\) denotes the Borel measure on \(X\) that the form \(\omega^n\) determines. In our integrals below we will often omit \(d\mu\) and write \(\int_E f\) for \(\int_E f\,d\mu\); and when \(E = X\), we will even omit \(X\) and write \(\int f\) for \(\int_X f\,d\mu\). In the same spirit, we write \(L^q(X, \mu)\) for \(L^q(X, \mu)\).

We need the following result, an equivalent of Katok’s Basic Lemma, valid for non-compact (but connected) \(X\) as well:

**Lemma 3.2.** If \(\xi, \eta \in L^1(X)\) are equidistributed, then there is a sequence of \(g_k \in \text{Ham}_0(\omega)\) such that
\[
\lim_{k \to \infty} \int_X |\xi - \eta \circ g_k|\,d\mu = 0.
\]

**Proof.** (Essentially as in [OW], [Ha, Proposition 1.12].) Given \(\varepsilon > 0\), we will find \(g \in \text{Ham}_0(\omega)\) such that \(\int |\xi - \eta \circ g| < 5\varepsilon\). Assume first \(\mu(X) < \infty\).

The measure \(|\xi|d\mu\) is absolutely continuous with respect to \(d\mu\), hence there is a \(\delta > 0\) such that \(\int_E |\xi| < \varepsilon\) if \(\mu(E) < \delta\). Construct disjoint intervals \(J_1, \ldots, J_N \subset \mathbb{R}\) of length \(< \varepsilon/\mu(X)\) so that \(\mu(X \setminus \bigcup_i \xi^{-1}J_i) < \delta/2\), and choose compact sets \(K_i \subset \xi^{-1}J_i\) so that also
\[
(3.1) \quad \mu(X \setminus \bigcup_i K_i) < \delta/2.
\]

By equidistribution \(\mu(\eta^{-1}J_i) = \mu(\xi^{-1}J_i)\), hence there are compact \(L_i \subset \eta^{-1}J_i\) such that \(\mu(L_i) = \mu(K_i)\). The \(K_i\) are disjoint among themselves and so are the \(L_i\). In this situation Katok’s Basic Lemma [K, Section 3] provides a \(g \in \text{Ham}_0(\omega)\) such that
\[
(3.2) \quad \mu(K_i \setminus g^{-1}L_i) < \delta/2N, \quad i = 1, \ldots, N.
\]

If \(x \in K_i \cap g^{-1}L_i\) then \(\xi(x), \eta(gx) \in J_i\) and so \(|\xi(x) - \eta(gx)| < \varepsilon/\mu(X)\). Conversely, \(|\xi(x) - \eta(gx)| \geq \varepsilon/\mu(X)\) can happen only if
\[
x \in E, \quad \text{where } E = (X \setminus \bigcup_i K_i) \cup \bigcup_i (K_i \setminus g^{-1}L_i).
\]

By (3.1), (3.2) \(\mu(E) < \delta\), whence \(\mu(gE) < \delta\) and
\[
\int |\xi - \eta \circ g| = \int_{X \setminus E} |\xi - \eta \circ g| + \int_E |\xi - \eta \circ g| < \varepsilon + \int_E |\xi| + \int_{gE} |\eta| < 3\varepsilon.
\]

This takes care of \(X\) of finite measure. In general, choose an \(a > 0\) so that the level sets \(Y_1 = \{|\xi| \geq a\}\) and \(Y_2 = \{|\eta| \geq a\}\) satisfy \(\int_{X \setminus Y_1} |\xi| = \int_{X \setminus Y_2} |\eta| < \varepsilon\). Then \(\mu(Y_1) = \mu(Y_2) < \infty\). The functions
\[
\xi' = \begin{cases} \xi & \text{on } Y_1 \\ 0 & \text{on } X \setminus Y_1 \end{cases} \quad \text{and} \quad \eta' = \begin{cases} \eta & \text{on } Y_2 \\ 0 & \text{on } X \setminus Y_2 \end{cases}
\]
are also equidistributed. Construct a connected open $X' \subset X$ of finite measure containing $Y_1 \cup Y_2$. By what we have proved so far, there is a $g \in \text{Ham}_0(\omega|X')$ such that $\int_{X'} |\xi' - \eta' \circ g| < 3\varepsilon$. Extend $g$ to all of $X$ by identity on $X \setminus X'$. Denoting this extension also by $g$, we have

$$
\int |\xi - \eta \circ g| \leq \int |\xi' - \eta' \circ g| + \int |\xi - \xi'| + \int |\eta - \eta'| < 3\varepsilon + \varepsilon + \varepsilon = 5\varepsilon.
$$

To finish the proof, we let $\varepsilon = 1/k$ and $g = g_k, k \in \mathbb{N}$, and obtain the sequence sought.

**Proof of Theorem 3.1.** Consider $A \subset \mathbb{R} \times C^\infty(X)$ of Lemma 2.1:

$$
p(\xi) = \sup \left\{ a + \int f\xi : (a, f) \in A \right\}.
$$

Suppose $\xi, \eta \in C^\infty(X)$ are equidistributed, and let $g_k$ be as in Lemma 3.2. With any $(a, f) \in A$

$$
p(\eta) = p(\eta \circ g_k) \geq a + \int (\eta \circ g_k)f \to a + \int f\xi \quad \text{as } k \to \infty.
$$

Taking sup over $(a, f) \in A, p(\eta) \geq p(\xi)$ follows, and in fact $p(\xi) = p(\eta)$ by symmetry.

### 4 Proof of the first part of Theorem 1.2

This is what the first part says:

**Theorem 4.1.** If $(X, \omega)$ is a connected compact symplectic manifold, any continuous, convex, $\text{Ham}(\omega)$ invariant function $p : C^\infty(X) \to \mathbb{R}$ is continuous in the sup norm topology on $C^\infty(X)$.

We will use the following standard fact:

**Lemma 4.2.** Let $V$ be a locally convex topological vector space over $\mathbb{R}$. If $p : V \to \mathbb{R}$ is convex and bounded above on some open $U \subset V$, then it is continuous on $U$.

**Proof.** We can assume $U$ is convex. Say, we want to prove continuity at $0 \in U$. Let $s = \sup_U p < \infty$. With $0 < \lambda < 1$ and $v \in (\lambda U) \cap (-\lambda U)$ convexity implies

$$
\begin{align*}
p(v) - p(0) &\leq \lambda(p(v/\lambda) - p(0)) \leq \lambda(s - p(0)) \\
p(0) - p(v) &\leq \lambda(p(-v/\lambda) - p(0)) \leq \lambda(s - p(0))
\end{align*}
$$

then $0 \to 0$, as needed.

The key to the proof of Theorem 4.1 is the following.

**Lemma 4.3.** Let $F \subset L^1(X)$ be a $\text{Ham}(\omega)$ invariant family of functions. If for every $\xi \in C^\infty(X)$

$$
\sup_{f \in F} \int_X f\xi d\mu < \infty,
$$

then $\sup_{f \in F} \int_X |f| d\mu < \infty$. 

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This is not hard to show and will suffice to prove Theorem 4.1; but later we will need a more precise statement, whose proof is just a little more involved. Let \( \xi^+ = \max(\xi, 0) \) and \( \xi^- = \max(-\xi, 0) \) denote the positive and negative parts of functions \( \xi: X \to \mathbb{R} \). If \( E \subset X \) is measurable, write \( \int_E \xi \) for the average \( \int_X \xi / \mu(E) \) of an integrable function. If \( \mu(E) = 0 \), we let \( \int_E \xi = 0 \).

**Lemma 4.4.** Let \( f \in L^1(X) \), \( \xi \in L^\infty(X) \), and \( S, T \subset X \) be of equal measure. If \( \xi \geq 0 \) on \( T \) and \( \xi \leq 0 \) on \( X \setminus T \), then

\[
\sup \{ \int_X (f \circ g) \xi : g \in \text{Ham}(\omega) \} \geq \int_S f \int \xi^+ - \int_{X \setminus S} f \int \xi^-.
\]  

First we show how this implies Lemma 4.3.

**Proof of Lemma 4.3.** We can assume \( \mu(X) = 1 \). Let \( M(\xi) \) denote the left hand side of (4.1). Fix a nonnegative \( \xi \in C^\infty(X) \) that is not identically 0, but \( T' = \{ \xi > 0 \} \) has measure \( \leq 1/2 \). Let \( f \in F \). Suppose first that \( S = \{ f \geq 0 \} \) has measure \( \geq 1/2 \), and choose \( T \supset T' \) so that \( \mu(S) = \mu(T) \). By Lemma 4.4 \( M(\xi) \geq \int_S f \int \xi \), hence

\[
\int f^+ \leq 2M(\xi) / \int \xi.
\]

If, instead of \( S \), \( \{ f \leq 0 \} \) has measure \( \geq 1/2 \), Lemma 4.4 implies in the same way that

\[
\int f^- \leq 2M(-\xi) / \int \xi.
\]

Since \( \int |f^+ - f^-| = \int |f| \leq M(1) + M(-1) \), in both cases we obtain a bound for \( \int |f| = \int f^+ + \int f^- \), as claimed.

Given \( f \in L^1(X) \), we will write \( \text{conv}_1(f) \) for the closure, in the \( L^1(X) \) topology, of the convex hull of the orbit of \( f \) under \( \text{Ham}(\omega) \). In light of Lemma 3.2 this is the same as the closed convex hull of all strict rearrangements of \( f \). To prove Lemma 4.4 we need the following.

**Lemma 4.5.** If \( f \in L^1(X) \) and \( E \subset X \) has positive measure, then the function

\[
f' = \begin{cases} 
\int_E f & \text{on } E \\
f & \text{on } X \setminus E
\end{cases}
\]

is in \( \text{conv}_1(f) \).

**Proof.** If two functions \( f, h \in L^1(X) \) are at \( L^1 \) distance \( \leq \varepsilon \), then their \( \text{Ham}(\omega) \) orbits are at Hausdorff distance \( \leq \varepsilon \), and so are therefore \( \text{conv}_1(f) \) and \( \text{conv}_1(h) \). Hence, given \( E \), if the lemma holds for a sequence \( f = f_k, k = 1, 2, \ldots, \) and \( f_k \to f_0 \) in \( L^1 \), then the lemma will hold for \( f_0 \) as well.

Now suppose that \( E \) is the disjoint union of \( E_j, j = 1, \ldots, m \), of equal measure, and \( f = c_j \) is constant on each \( E_j \). If \( \sigma \) is a permutation of \( 1, \ldots, m \), define \( f_\sigma \in L^1(X) \) by

\[
f_\sigma = c_{\sigma(j)} \quad \text{on } E_j, \quad f_\sigma = f \quad \text{on } X \setminus E.
\]

As a strict rearrangement of \( f \), by Lemma 3.2 \( f_\sigma \) is in the closure of the \( \text{Ham}(\omega) \) orbit of \( f \). Therefore

\[
f' = \sum_\sigma f_\sigma / m!
\]
is indeed in \( \text{conv}_1(f) \). Since any \( f \in L^1(X) \) is the limit of functions of the above type, the claim follows.

**Proof of Lemma 4.4.** Write \( \chi_A \) for the characteristic function of a set \( A \). By Lemma 3.2 there is a sequence \( g_k \in \text{Ham}(\omega) \) such that \( \chi_S \circ g_k \to \chi_T \) in \( L^1 \). Two applications of Lemma 4.5 give that

\[
\begin{align*}
    f' &= \begin{cases} 
        f_S f & \text{on } S \\
        f_{X \setminus S} f & \text{on } X \setminus S 
    \end{cases} \\
    f'' &= \lim_k f' \circ g_k = \begin{cases} 
        f_S f & \text{on } T \\
        f_{X \setminus S} f & \text{on } X \setminus T 
    \end{cases}
\end{align*}
\]

are in \( \text{conv}_1(f) \). Lemma 4.4. follows, since the left hand side in (4.2) is

\[
\geq \int f'' \xi = \int_S f \int_T \xi + \int_{X \setminus S} f \int_{X \setminus T} \xi = \int_S f \int \xi^+ - \int_{X \setminus S} f \int \xi^-.
\]

**Proof of Theorem 4.1.** If a function is continuous in the sup norm topology, we will say it is \( \| \|_{\infty} \)-continuous, and use similar terminology for other topological notions. First assume that \( p \) of the theorem is positively homogeneous as well. By Lemma 2.1 there is a family \( F \subset L^1(X) \) such that

\[
(4.3) \quad p(\xi) = \sup \left\{ \int f \xi : f \in F \right\}.
\]

If we replace \( F \) by its \( \text{Ham}(\omega) \) orbit, the supremum in (4.3) will not change, for

\[
\int (f \circ g) \xi = \int (\xi \circ g^{-1}) f \leq p(\xi \circ g^{-1}) = p(\xi) \quad \text{if } f \in F, g \in \text{Ham}(\omega).
\]

Therefore we may assume that the family \( F \) in (4.3) is already invariant under \( \text{Ham}(\omega) \). Hence Lemma 4.3 gives \( \sup_F \int |f| < \infty \). This implies \( p \) is bounded on \( \| \|_{\infty} \)-bounded subsets of \( C^\infty(X) \), and by Lemma 4.2 it is \( \| \|_{\infty} \)-continuous.

For general \( p \), pick a number \( c > p(0) \) and consider the Minkowski functional \( q \) of the convex set \( \{ p < c \} \) (see e.g. 

\[
q(\xi) = \inf \{ \lambda \in (0, \infty) : p(\xi/\lambda) < c \} \in [0, \infty).
\]

This is a convex, positively homogeneous, strict rearrangement invariant function, that is continuous—because locally bounded—in the topology of \( C^\infty(X) \). By what we have already proved, it is \( \| \|_{\infty} \)-continuous. In particular, the set \( U_c = \{ q < 1 \} \supset \{ p < c \} \) is \( \| \|_{\infty} \)-open. If \( \xi \in U_c \) then \( p(\xi/\lambda) < c \) with some \( \lambda < 1 \). Also \( p(0) < c \). As \( \xi \) is a point on the segment connecting 0, \( \xi/\lambda \), convexity implies \( p(\xi) < c \). Thus \( p \) is bounded above on the \( \| \|_{\infty} \)-open set \( U_c \), and by Lemma 4.2 it is continuous there. The theorem follows since \( \bigcup U_c = C^\infty(X) \).

5 **Extending convex functions**

The above ideas can be developed to prove that \( p \) can be extended to \( C(X) \) and, under an additional assumption, to the Banach space \( B(X) \) of bounded Borel functions, with the supremum norm. (Thus \( L^\infty(X) \) is a quotient of \( B(X) \), but \( B(X) \) is more natural to use in our setting.)
**Definition 5.1.** If $V \subset B(X)$ is a vector subspace, we say that a function $p: V \to \mathbb{R}$ is strongly continuous if $p(\xi_k)$ is convergent whenever $\xi_k \in V$ is a pointwise convergent sequence of uniformly bounded functions.

This is stronger than continuity in the topology inherited from $B(X)$. The limit $\lim p(\xi_k)$ depends only on $\lim \xi_k = \xi$, since two such sequences can be combined into one sequence, converging to $\xi$.

**Theorem 5.2.** Any continuous, convex, Ham($\omega$) invariant $p: C^\infty(X) \to \mathbb{R}$ has a unique continuous extension to $C(X)$; this extension is convex and Ham($\omega$) (hence strict rearrangement) invariant. If $p$ is strongly continuous, then it has a unique strongly continuous, strict rearrangement invariant extension $q: B(X) \to \mathbb{R}$. This extension is convex, and satisfies $\lim_k q(\xi_k) = q(\xi)$ whenever uniformly bounded $\xi_k \in B(X)$ converge almost everywhere to $\xi$.

Since $C^\infty(X)$ is dense in $C(X)$, and $p$ is known to be continuous in supremum norm, for the first part of Theorem 5.2 one only needs to prove that a continuous extension exists. This is a special case of the following:

**Lemma 5.3.** Let $W$ be a locally convex topological vector space over $\mathbb{R}$, $V \subset W$ a dense subspace. Any continuous, convex $p: V \to \mathbb{R}$ can be extended to a continuous $q: W \to \mathbb{R}$.

**Proof.** First we show that any $w \in W$ has a convex neighborhood $U$ such that $p$ is bounded on $V \cap U$. By continuity, there certainly is a symmetric, convex neighborhood $U_0 \subset W$ of $0$ such that $p$ is bounded on $V \cap 4U_0$. Now $w + 2U_0$ is a neighborhood of $w$, and if $v_1 \in V$ is sufficiently close to $w$, then $U = v_1 + 2U_0$ is also. For any $v \in V \cap U$ convexity implies

$$2p(v) \leq p(2v_1) + p(2(v-v_1)).$$

Since $v - v_1 \in 2U_0$, the right hand side is bounded as $v$ varies in $V \cap U$. Thus $p$ is bounded above on $V \cap U$. But then $p(v) + p(2v_1 - v) \geq 2p(v_1)$ gives that $p$ is also bounded below. Set $s = \sup_U |p|$.

We let $U' = v_1 + U_0$ and show that $p$ is uniformly continuous on $V \cap U'$. For suppose $\lambda \in (0, \infty)$. If $u,v \in V \cap U'$ and $v-u \in U_0/\lambda$, then $v + \lambda(v-u) \in v_1 + U_0 + U_0 = U$, hence by convexity

$$p(v) - p(u) \leq \frac{p(v + \lambda(v-u)) - p(u)}{1+\lambda} \leq \frac{2s}{1+\lambda}.$$  

Since the roles of $u,v$ are symmetric, this indeed proves locally uniform continuity; which in turn implies continuous extension.

The proof of the second part of Theorem 5.2 requires some preparation.

**Lemma 5.4.** There is a continuous $\theta: X \to [0, \mu(X)]$ that is smooth away from the preimage of finitely many $t \in [0,\mu(X)]$, and that preserves measure (the target is endowed with Lebesgue measure).
Proof. If $\zeta \in C^\infty(X)$ is a Morse function, its reverse distribution function
\[
\lambda(t) = \mu(\zeta < t), \quad t \in [\min \zeta, \max \zeta],
\]
is continuous, strictly increasing, and smooth away from the set $C$ of critical values of $\zeta$. It is a homeomorphism $[\min \zeta, \max \zeta] \to [0, \mu(X)]$, and a diffeomorphism away from $C$. The function $\theta = \lambda \circ \zeta$ will therefore do, as
\[
\mu(\theta < s) = \mu(\zeta < \lambda^{-1}(s)) = \lambda(\lambda^{-1}(s)) = s, \quad s \in [0, \mu(X)].
\]

We will need the notion of decreasing rearrangement of a measurable $\xi : X \to \mathbb{R}$. It is the decreasing, say, upper semicontinuous function $\xi^* : [0, \mu(X)] \to \mathbb{R}$ that is equidistributed with $\xi$. Thus $\mu(s \leq \xi \leq t)$ is equal to the length of the maximal interval on which $s \leq \xi^* \leq t$. In particular,
\[
(5.1) \quad \mu(\xi \geq \xi^*(s)) = s.
\]
The upper semicontinuity requirement translates to left continuity of the decreasing function $\xi^*$, which differs from the more usual convention of right continuity, but the difference is inconsequential. Obviously, with $\theta$ of Lemma 5.3 $\xi$ and $\xi^* \circ \theta$ are equidistributed.

**Lemma 5.5.** If $\xi \in C(X)$, then $\xi^*$ is continuous.

**Proof.** Since $\xi^*$ is always u.s.c., i.e., left continuous, all we need to show is that if $s_j \in [0, \mu(X)]$ decreases to $s$, then $\lim_j \xi^*(s_j)$ cannot be $> \xi^*(s)$. Suppose it were, and let $\xi^*(s) < \alpha < \beta < \lim_j \xi^*(s_j)$. Then $\xi^{-1}(\alpha, \beta) \subset X$ would be a nonempty open subset, of positive measure, contradicting (cf.(5.1))
\[
\mu(\xi \geq \xi^*(s_j)) = s_j \to s = \mu(\xi \geq \xi^*(s)).
\]

The continuity property in Definition 5.1 implies a stronger property:

**Lemma 5.6.** Suppose $p : C^\infty(X) \to \mathbb{R}$ is Ham($\omega$) invariant, continuous, and convex. If $\xi_k \in C^\infty(X)$ is a uniformly bounded sequence that converges almost everywhere, then $p(\xi_k)$ is also convergent.

**Proof.** By the first part of Theorem 5.2, already proved, $p$ has a continuous invariant extension to $C(X)$, still denoted $p$. Suppose uniformly bounded $\xi_k \in C^\infty(X)$ converge a.e. to $\xi \in B(X)$. This implies that the rearrangements $\xi_k^*$ converge everywhere to $\xi^*$, see [BS Proposition 1.7, p.41]. Immediately that Proposition only gives a $\xi^* \leq \liminf_k \xi_k^*$, when $\xi_k \geq 0$; but applying it with $c + \xi_k$, $c - \xi_k$ and a suitable constant $c$ we obtain what we need. If $\theta : X \to [0, \mu(X)]$ is as in Lemma 5.4, $\xi_k$ and $\eta_k = \xi_k^* \circ \theta$ are equidistributed, and $\eta_k \to \eta = \xi^* \circ \theta$. For each $k$ we can uniformly approximate $\xi_k^*$ by smooth functions on $[0, \mu(X)]$, and $\eta_k = \xi_k^* \circ \theta$ by their pullbacks along $\theta$. Since $p$ (extended to $C(X)$) is continuous, there are $u_k \in C^\infty[0, \mu(X)]$ such that
\[
\max |u_k - \xi_k^*| < 1/k \quad \text{and} \quad |p(u_k \circ \theta) - p(\eta_k)| < 1/k.
\]
We can arrange that $u_k$ is constant in a neighborhood of the critical values of $\theta$. This implies that $u_k \circ \theta \in C^\infty(X)$, and $\lim_k u_k \circ \theta = \lim_k \eta_k = \eta$ pointwise. Hence $p(u_k \circ \theta)$ converges, and so does $p(\xi_k) = p(\eta_k)$.
Proof of Theorem 5.2. We have already seen that the first half of the theorem follows from Lemma 5.3. As to the extension to \( B(X) \), note that with \( \theta \) of Lemma 5.4 for any \( \xi \in B(X) \) its strict rearrangement \( \xi^* \circ \theta \) is u.s.c. Thus it is the pointwise limit of a uniformly bounded sequence of continuous, hence also of smooth functions \( \xi_k \). Therefore at \( \xi \) the extension \( q \) of \( p \) must take the value \( q(\xi^* \circ \theta) = \lim_k p(\xi_k) \), so it is unique. What remains is to construct the required extension \( q \).

If \( \xi \in B(X) \), predictably we let \( q(\xi) = \lim_k p(\xi_k) \), where the uniformly bounded sequence \( \xi_k \in C^\infty(X) \) converges to \( \xi \) a.e. By Lemma 5.6 the limit exists and, as we saw, it is independent of the choice of the sequence \( \xi_k \). Clearly \( p = q \) on \( C^\infty(X) \). If uniformly bounded \( \eta_k \in C^\infty(X) \) converge to \( \eta \in B(X) \) a.e., and \( \lambda \in [0,1] \), then

\[
q(\lambda \xi + (1 - \lambda) \eta) = \lim_k p(\lambda \xi_k + (1 - \lambda) \eta_k) \\
\leq \lim_k \lambda p(\xi_k) + (1 - \lambda) p(\eta_k) = \lambda q(\xi) + (1 - \lambda) q(\eta),
\]

i.e., \( q \) is convex. It is also strongly continuous, and in fact if uniformly bounded \( \xi_k \in B(X) \) a.e. converge to \( \xi \in B(X) \), then \( q(\xi_k) \to q(\xi) \). It suffices to show that this latter convergence holds along some subsequence.

By dominated convergence,

\[
(5.2) \quad \lim_k \int |\xi_k - \xi| = 0.
\]

Let each \( \xi_k \) be the a.e. limit of a uniformly bounded sequence \( \xi^i_k \in C^\infty(X) \), as \( i \to \infty \). We can arrange that the double sequence \( \xi^i_k \) is also uniformly bounded. Thus \( \lim_{i \to \infty} p(\xi^i_k) = q(\xi_k) \). For each \( k \) choose \( i = i_k \) so that \( \eta_k = \xi^i_k \) satisfies

\[
(5.3) \quad |p(\eta_k) - q(\xi_k)| < 1/k, \quad \int |\eta_k - \xi_k| < 1/k.
\]

In view of (5.2) \( \lim_k \int |\eta_k - \xi| = 0 \), so a subsequence \( \eta_{kj} \) converges to \( \xi \) a.e. Hence, by (5.3)

\[
q(\xi) = \lim_j p(\eta_{kj}) = \lim_j q(\xi_{kj}),
\]

as needed.

Finally, to show that \( q \) is invariant under strict rearrangements, consider equidistributed \( \xi, \eta \in B(X) \). By Lemma 3.2 there are \( g_k \in \text{Ham}(\omega) \) such that \( \int |\eta - \xi \circ g_k| \to 0 \) as \( k \to \infty \). Choose uniformly bounded \( \xi_k \in C^\infty(X) \) converging to \( \xi \) a.e. In particular, \( \lim_k \int |\xi_k - \xi| = 0 \). Then

\[
\lim_k \int |\xi_k \circ g_k - \eta| \leq \limsup_k \int |(\xi_k - \xi) \circ g_k| + \limsup_k \int |\xi \circ g_k - \eta| = 0.
\]

Again, this means that a subsequence of \( \xi_k \circ g_k \) converges a.e. to \( \eta \), whence

\[
q(\xi) = \lim_k p(\xi_k) = \lim_k p(\xi_k \circ g_k) = q(\eta),
\]

which proves that \( q \) is indeed invariant under strict rearrangements.
Here is the last result in this section.

**Theorem 5.7.** If a strict rearrangement invariant convex $p : B(X) \to \mathbb{R}$ is strongly continuous, then it is Lipschitz continuous on bounded sets.

**Proof.** Let $\theta$ be as in Lemma 5.4. We start by showing that $p$ is bounded on bounded sets. Otherwise there would be a bounded sequence $\xi_k \in B(X)$ such that $|p(\xi_k)| \to \infty$. The decreasing rearrangements $\xi_k^\ast$ are uniformly bounded, hence by Helly’s theorem contain a pointwise convergent subsequence. But along that subsequence $\xi_k^\ast \circ \theta$ converges pointwise and therefore by strong continuity

$$p(\xi_k) = p(\xi_k^\ast \circ \theta)$$

also converges, a contradiction.

Now boundedness on bounded sets implies Lipschitz continuity on bounded sets. For suppose $\xi \neq \eta$ have norm $\leq R$, and let $\rho$ be the unit vector in the direction of $\xi - \eta$. With $M = \sup_{||\xi||_\infty \leq R+1} |p(\xi)|$, by convexity

$$\frac{p(\xi) - p(\eta)}{||\xi - \eta||_\infty} \leq \frac{p(\xi + \rho) - p(\eta)}{||\xi + \rho - \eta||_\infty} \leq 2M.$$

The roles of $\xi, \eta$ being symmetric, we obtain Lipschitz continuity.

### 6 Proof of Theorem 1.3

To simplify notation, we will assume $\mu(X) = 1$. By Lemma 2.1 a Ham($\omega$) invariant convex, continuous, $p : C^\infty(X) \to \mathbb{R}$ can be written

$$(6.1) \quad p(\xi) = \sup \left\{ a + \int f\xi : (a, f) \in A \right\}$$

with a family $A \subset \mathbb{R} \times C^\infty(X)$, that can be chosen convex and invariant under Ham($\omega$). The possible behaviors of $p$ described in Theorem 1.3 are determined by whether all functions $f$ that occur in $A$ are constant or not.

If in $A$ only constant functions occur, then (6.1) gives $p(\xi) = p(\int f \xi)$. Henceforward we will assume $A$ contains a pair $(a, f)$ with a nonconstant function $f$. According to (ii) of Theorem 1.3, we must estimate $p(\xi)$ from below with the $L^1$ norm of $\xi \in C^\infty(X)$. We do this do in a somewhat greater generality, that we will need in the next section.

**Lemma 6.1.** Suppose $A \subset \mathbb{R} \times L^1(X)$ is convex and invariant under Ham($\omega$). For $\xi \in L^\infty(X)$ let $q(\xi) = \sup_{(a, f) \in A} a + \int f\xi$. If $A$ contains a pair $(a, f)$ with $f$ nonconstant, then there are $a_0 \in \mathbb{R}$ and $b \in (0, \infty)$ such that

$$q(\xi) \geq a_0 + b \int |\xi| \quad \text{if} \quad \begin{cases} \int \xi = 0, & \text{or} \\ \int \xi \geq 0 & \text{and} \quad \lim_{\lambda \to -\infty} q(\lambda) > q(0), \quad \text{or} \\ \int \xi \leq 0 & \text{and} \quad \lim_{\lambda \to -\infty} q(\lambda) > q(0). \end{cases}$$

If $A \subset \{0\} \times L^1(X)$, then $a_0$ can be chosen 0.
Lemma 4.4 implies
\[(6.4)\]
is equivalent to \(\mu\) since \(\alpha\) trivially holds at \(\alpha\) (according to Lemma 4.5, and \((f, \mu)\) fixed with respect to \(\alpha\)) even if \(\alpha\) and let \(s_0 = \text{ess sup} f\), \(i_0 = \text{ess inf} f\). For every \(\alpha > 0\) there is an \(S = S_\alpha \subset X\) of measure \(\alpha\) for which \(\int_S f = s_\alpha\). Indeed, consider
\[u = \inf\{t \in \mathbb{R}: \mu\{f > t\} \leq \alpha\}.
\]
Since \(\mu\{f > u\} \leq \alpha \leq \mu\{f \geq u\}\), any set \(S\) of measure \(\alpha\) sandwiched between \(\{f > u\}\) and \(\{f \geq u\}\) will provide the sup in (6.2). Similarly, \(S' = X \setminus S\), of measure \(1 - \alpha\), satisfies \(i_{1-\alpha} = f_{S'} f\). This implies that \(s_\alpha > i_{1-\alpha}\). From the absolute continuity of \(fd\mu\) with respect to \(d\mu\) we deduce that \(s_\alpha, i_\alpha\) are continuous functions of \(\alpha > 0\); continuity trivially holds at \(\alpha = 0\) as well. Hence
\[(6.3)\]
\[2c = 2c(f) = \min_{0 \leq \alpha \leq 1} (s_\alpha - i_{1-\alpha}) > 0, \quad 2m = 2m(f) = \max_{0 \leq \alpha \leq 1} |s_\alpha| + |i_{1-\alpha}| < \infty.
\]
Consider a \(\xi \in L^\infty(X)\) and let \(T = \{\xi \geq 0\}\). With \(\alpha = \mu(T)\) and \(S = S_\alpha\) as above, Lemma 4.4 implies
\[(6.4)\]
\[q(\xi) \geq a + s_\alpha \int \xi^+ - i_{1-\alpha} \int \xi^- = a + s_\alpha = \frac{s_\alpha - i_{1-\alpha}}{2} \int |\xi| + \frac{s_\alpha + i_{1-\alpha}}{2} \int \xi
\]
(even if \(\alpha = 0\)). When \(\int \xi = 0\), by (6.3) we obtain \(q(\xi) \geq a + c \int |\xi|\).

Next suppose that \(\lim_{\lambda \to \infty} q(\lambda) > q(0)\). There are \(\lambda > 0\) and \((a_1, f_1) \in \mathcal{A}\) with \(a_1 + \int f_1 \lambda > q(0) \geq a_1\); hence \(\int f_1 > 0\). Because \(\mathcal{A}\) is convex, we can arrange that our fixed \((a, f) \in \mathcal{A}\) already satisfies \(\int f > 0\). Let \(b = s_1 c / (s_1 + m)\). We will show that if \(\int \xi \geq 0\), then \(q(\xi) \geq a + b \int |\xi|\). Note that the constant function \(f' = \int f\) is in \(\mathcal{C}(\mathbb{R})\) according to Lemma 4.5, and \((a, f')\) is in \(\mathcal{A}\). Hence \(q(\xi) \geq a + \int f' \xi = a + s_1 \int \xi\). By (6.4) \(q(\xi) \geq a + c \int |\xi| - m \int \xi\). Combining these two we can eliminate \(\int \xi\) and obtain
\[mq(\xi) + s_1 q(\xi) \geq (m + s_1)a + s_1 c \int |\xi|,
\]
as needed. Finally, if \(\lim_{\lambda \to -\infty} q(\lambda) > q(0)\), we choose \((a, f) \in \mathcal{A}\) such that \(f\) is nonconstant and \(\int f < 0\). Letting \(b = c(f) |s_1(f)| / (|s_1(f)| + m(f))\) we can similarly prove \(q(\xi) \geq a + b \int |\xi|\) whenever \(\int \xi \leq 0\). This completes the proof of the lemma, and also of the theorem.

7 Proof of Theorem 1.4

This was the theorem:

Theorem 7.1. Given a Ham(\(\omega\)) invariant continuous norm \(p\) on \(C^\infty(X)\), there is a rearrangement invariant Banach function space on \(X\) whose norm, restricted to \(C^\infty(X)\), is equivalent to \(p\).
We will show that each term on the right is determined by \( \varphi )\) (\( \psi (x) - \psi (y) \)) \( \geq 0 \) for all \( x, y \in X \).

Put it differently, \( \phi (x) > \psi (y) \) should imply \( \psi (x) \geq \psi (y) \). In spite of what the language may suggest, this is not an equivalence relation (all functions are similarly ordered as a constant). However, it is true that if \( \phi \) and \( \psi \) are similarly ordered, and \( U : \mathbb{R} \to \mathbb{R} \) is increasing, then \( \phi \) and \( U \circ \psi \) are also similarly ordered.

We will write \( \phi \sim \psi \) for measurable functions \( X \to \mathbb{R} \) if they are equidistributed.  

The following lemma in one form or another is known and, like Lemmas 7.3, 7.4, 7.5, holds in any finite measure space \( (X, \mu) \) without atoms.

**Lemma 7.2.** Let \( \phi_0 \in L^1(X) \) be bounded below and \( \psi_0 \in L^\infty(X) \).

(a) \( \sup_{\phi \sim \phi_0} \int \phi \psi_0 = \sup_{\psi \sim \psi_0} \int \phi_0 \psi \).

(b) The suprema in (a) are attained, by \( \phi \) and \( \psi \) that are similarly ordered as \( \psi_0 \) and \( \phi_0 \).

(c) \( \int \phi \psi \) is independent of the choice of \( \phi \sim \phi_0, \psi \sim \psi_0 \), as long as \( \phi, \psi \) are similarly ordered.

**Proof.** (b) That the suprema are attained, at least when \( \phi_0, \psi_0 \geq 0 \), is proved in [BS, Chapter 2, Theorems 2.2 and 2.6]. The general result follows upon adding a constant to the functions. The proof in [BS, pp. 49-50], say, for the first supremum in (a), proceeds by first considering simple \( \phi_0 \) and representing the maximizing \( \phi \) by an explicit formula, then passing to a limit. The formula shows that \( \phi \) and \( \psi \) are similarly ordered when \( \phi_0 \) is simple; but similar ordering is preserved under pointwise limits, and must hold in general.

(c) Again, first assume that \( \psi_0 \) is simple, and takes values \( a_1 < a_2 < \cdots < a_k \). Let \( A_j = \{ x : \psi(x) = a_j \} \). If necessary, we can change the values of \( \phi, \psi \) on a set of zero measure to arrange that each \( \mu(A_j) > 0 \). Let

\[
m_j = \inf_{A_j} \phi, \quad M_j = \sup_{A_j} \phi.
\]

If \( x \in A_j \) and \( y \in A_{j+1} \), then \( \psi(x) < \psi(y) \) and \( \phi(x) \leq \phi(y) \), hence

\[
(7.1) \quad \cdots \leq m_j \leq M_j \leq m_{j+1} \leq \cdots
\]

It follows that the set \( B_j = \{ x : m_j < \phi(x) < M_j \} \) is included in \( A_j \). With \( C_j = \{ x \in A_j : \phi(x) = m_j \} \) and \( D_j = \{ x \in A_j : \phi(x) = M_j \} \) therefore

\[
\int \phi \psi = \sum_j a_j \int_{A_j} \phi = \sum_j a_j \left( \int_{B_j} \phi + m_j \mu(C_j) + M_j \mu(D_j) \right) = \int_{B_j} \phi + m_j \mu(A_j) + M_j \mu(A_j)
\]

We will show that each term on the right is determined by \( \phi_0, \psi_0 \).

To start,

\[
(7.2) \quad m_j = \sup \left\{ m : \mu(\phi \geq m) \geq \mu \left( \bigcup_{i=j}^{k} A_i \right) \right\},
\]

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because by (7.1)
\[
\phi \leq m_j \text{ on } \bigcup_{i=1}^{j-1} A_i, \quad \phi \geq m_j \text{ on } \bigcup_{i=j}^{k} A_i.
\]

Since \( \mu(\phi \geq m) = \mu(\phi_0 \geq m) \) and \( \mu(\bigcup_i^k A_i) = \mu(\psi_0 \geq a_j) \), (7.2) shows that the \( m_j \) are determined by \( \phi_0, \psi_0 \); and so are the \( M_j \). It follows that
\[
\mu(B_j) = \mu(m_j < \phi_0 < M_j) \quad \text{and} \quad \int_{B_j} \phi = \int_{\{m_j < \phi_0 < M_j\}} \phi_0
\]
are also determined by \( \phi_0, \psi_0 \). Next, \( \mu(A_j) = \mu(\psi_0 = a_j) \). Finally, if \( j \) is such that \( m_j < M_j \), then in light of (7.1) \( C_j = (\phi \leq m_j) \setminus \bigcup_{i=1}^{j-1} A_i \),
\[
\mu(C_j) = \mu(\phi \leq m_j) - \mu(\bigcup_{i=1}^{j-1} A_i) = \mu(\phi_0 \leq m_j) - \mu(\psi_0 < a_j) \quad \text{and}
\]
\[
\mu(D_j) = \mu(A_j) - \mu(B_j) - \mu(C_j).
\]

This proves (c) for a simple \( \psi_0 \). To finish the proof, consider a general \( \psi_0 \). Let \( \lfloor \cdot \rfloor \) denote integer part and for \( k \in \mathbb{N}, t \in \mathbb{R} \) let \( U_k(t) = \lfloor kt \rfloor / k \), an increasing function of \( t \). By what we have proved \( \int (U_k \circ \psi) \phi \) is determined by \( \phi_0, \psi_0 \), hence so is (by the dominated convergence theorem)
\[
\int \phi \psi = \lim_{k \to \infty} \int (U_k \circ \psi) \phi.
\]

(a) now follows from (b) and (c).

**Lemma 7.3.** If \( \phi \in L^1(X) \) and \( \psi \in L^\infty(X) \) are similarly ordered, then \( \int \phi \psi \geq \int \phi \int \psi \).

**Proof.** This is Chebyshev’s integral inequality. See for the discrete version of the inequality—from which the lemma follows—p. 43 in [HLP], and also p. 168.

**Lemma 7.4.** If \( \phi_0, \psi \in L^\infty(X) \), then
\[
(7.3) \quad \sup_{\phi \sim \phi_0} \int |\phi| \psi \leq \sup_{\phi \sim \phi_0} \int \phi \psi + \sup_{\phi \sim \phi_0} \int (-\phi) \psi + \int |\phi_0| \int \psi.
\]

**Proof.** First we estimate \( \int \phi^+ \psi \). By Lemma 7.2 we can choose \( \phi_1 \sim \phi_0 \), similarly ordered as \( \psi \), that realizes \( \sup_{\phi \sim \phi_0} \int \phi \psi \). It follows that \( \phi_1^+ \), a composition of \( \phi_1 \) with an increasing function, is also similarly ordered as \( \psi \). Using Lemma 7.2 once more we obtain
\[
\sup_{\phi \sim \phi_0} \int \phi^+ \psi = \int \phi_1^+ \psi = \int \phi_1 \psi + \int \phi_1^- \psi.
\]
As \( -\phi_1^- \) and \( \psi \) are similarly ordered, Lemma 7.3 gives \( -\int \phi_1^- \psi \geq -\int \phi_1^- \int \psi \), and so
\[
(7.4) \quad \sup_{\phi \sim \phi_0} \int \phi^+ \psi \leq \int \phi_1 \psi + \int \phi_1^- \int \psi = \sup_{\phi \sim \phi_0} \int \phi \psi + \int \phi_0^- \int \psi.
\]
Replacing \( \phi_0 \) with \(-\phi_0\),

\[
\sup_{\phi \sim \phi_0} \int \phi^{-} \psi \leq \sup_{\phi \sim \phi_0} \int (-\phi) \psi + \int \phi^+ \int \psi,
\]

and (7.3) follows by adding (7.4) and (7.5).

**Lemma 7.5.** If \( f_0, \xi \in L^\infty(X) \) then \( \sup_{f \sim f_0} \int |f| \leq 4 \sup_{f \sim f_0} \int f \xi \leq 4 \int |f_0| \int |\xi| \).

*Proof.* Let us start with a simple \( \xi \). Lemma 7.4, with \( \phi_0 = f_0, \psi = |\xi| \) gives

\[
\sup_{f \sim f_0} \int |f| \leq 2 \sup_{f \sim f_0} \int |f| \xi| + \int |f_0| \int |\xi|.
\]

By Lemma 7.2

\[
\sup_{f \sim f_0} \int \xi| = \sup_{\xi \sim |\xi|} \int f_0 \xi.
\]

Any \( \zeta \sim |\xi| \) can be written as \( \zeta = |\eta| \) with \( \eta \sim \xi \). Indeed, suppose \( \xi \) takes distinct values \( a_1, \ldots, a_k \). If for some \( i \) there is no \( j \) with \( a_i = -a_j \), we let \( \eta = a_i \) on the set \( \{ \zeta = |a_i| \} \). If for some \( i \) there is a (necessarily unique) \( j \) with \( a_i = -a_j \), for each such pair we divide the set \( \{ \zeta = |a_i| = |a_j| \} \) in two parts, of measures \( \mu(\xi = a_i), \mu(\xi = a_j) \), and define \( \eta \equiv a_i \) on the former, \( \eta \equiv a_j \) on the latter.

Hence, applying Lemma 7.4 again, this time with \( \phi_0 = \xi, \psi = f_0 \), we obtain

\[
\sup_{\zeta \sim |\xi|} \int f_0 \xi = \sup_{\eta \sim \xi} \int f_0 |\eta| \leq 2 \sup_{\eta \sim \xi} \int f_0 |\eta| + \int f_0 \int |\xi|.
\]

In light of (7.7) and Lemma 7.2 therefore

\[
\sup_{f \sim f_0} \int f |\xi| \leq 2 \sup_{f \sim f_0} \int f |\xi| + \int f_0 \int |\xi|.
\]

Substituting this, and its counterpart with \( f_0 \) replaced by \(-f_0\), into (7.6) gives the lemma, when \( \xi \) is simple. A general \( \xi \) can be uniformly approximated by simple functions \( \xi_m \), and knowing the estimate for each \( \xi_m \) gives the estimate for \( \xi \) in the limit.

**Proof of Theorem 7.1.** By Lemma 2.1 \( p(\xi) = \sup\{ \int f \xi : f \in \mathcal{F} \} \) with a family \( \mathcal{F} \subset L^\infty(X) \), that we can choose to be invariant under \( \text{Ham}(\omega) \). Because of Lemma 3.2 we can even choose it to be invariant under strict rearrangements. For any measurable \( \zeta : X \to [-\infty, \infty] \) define

\[
q(\zeta) = \sup \left\{ \int |f \xi| : f \in \mathcal{F} \right\} \in [0, \infty],
\]

and let \( B = \{ \zeta : q(\zeta) < \infty \} \), \( \| \| = q|B| \). Some obvious properties of \( q \) are: it is positively homogeneous, \( q(\eta + \zeta) \leq q(\eta) + q(\zeta) \), and \( |\eta| \leq |\zeta| \) a.e. implies \( q(\eta) \leq q(\zeta) \). If \( q(\zeta) = 0 \) then \( \zeta = 0 \) a.e. on any set where some \( f \in \mathcal{F} \) is nonzero; since \( \mathcal{F} \) is invariant under strict rearrangements, this simply means \( \zeta = 0 \) a.e. By Lemma 4.3 \( \sup_{f \in \mathcal{F}} \int |f| < \infty \),
hence $L^\infty(X) \subset B$. Furthermore, $q$ is invariant under all rearrangements, strict or not; this also implies by Lemma 6.1, with a suitable $b > 0$,

$$q(\zeta) \geq b \int |\zeta|$$

if $\zeta \in L^\infty(X)$.

Following Bennett–Sharpley’s definition [BS, pp. 2, 59], $(B, \| \|)$ is a rearrangement invariant Banach space if, in addition to the properties above, (7.8) holds for all measurable $\zeta$, and

$$\lim_{k \to \infty} q(\zeta_k) = q(\zeta)$$

for every increasing sequence $\zeta_k \geq 0$ converging to $\zeta$. We start with the latter. On the one hand, since $q$ is monotone, the limit in (7.9) exists, and is $\leq q(\zeta)$. On the other, the monotone convergence theorem implies that with any $f \in F$

$$\int |f\zeta| = \lim_{k \to \infty} \int |f\zeta_k| \leq \lim_{k \to \infty} q(\zeta_k).$$

Taking the sup over all $f \in F$ we obtain $q(\zeta) \leq \lim_k q(\zeta_k)$, which proves (7.9). That (7.8) holds for all measurable $\zeta$ now follows because $|\zeta|$ is the limit of an increasing sequence of functions in $L^\infty(X)$.

It remains to verify that $p$ and $\| \|$ are equivalent on $C^\infty(X)$. Clearly $p \leq \| \|$. By Lemma 7.5

$$\|\xi\| = \sup_{f \in F} \int |f\xi| \leq 4 \sup_{f \in F} \left| \int f\xi \right| + 3 \sup_{f \in F} \left| \int |f| \int |\xi| \right|, \quad \xi \in C^\infty(X).$$

Equivalence follows, because the first supremum on the right is $p(\xi)$ and the last term is $\leq Cp(\xi)$ by Lemma 4.3 and Theorem 1.3.

### 8 Proof of Theorem 1.5

The construction of a smooth, Ham$(\omega)$ invariant function $p : C^\infty(X) \to \mathbb{R}$ that is not invariant under volume preserving diffeomorphisms is based on symplectic rigidity; but linear rigidity, the easy kind, suffices. Let $V$ be a $2n \geq 4$ dimensional symplectic vector space over $\mathbb{R}$, and $\Omega$ the vector space of quadratic forms $Q : V \to \mathbb{R}$. Linear maps of $V$ act on $\Omega$ by composition. It is easy to construct a smooth function $t : \Omega \to \mathbb{R}$ that is invariant under the symplectic group Sp$(V)$, but not under SL$(V)$. For Poisson bracket $\{ , \}$ turns $\Omega$ into a Lie algebra, and induces the adjoint action $\text{ad}_Q : \Omega \to \Omega$,

$$\text{ad}_Q(R) = \{Q, R\} = (\text{grad} Q)R, \quad Q, R \in \Omega.$$ 

We let $t(\Omega) = \text{tr} \text{ad}_\Omega^2$. Thus $t$ is a polynomial on $\Omega$. If $V \to V'$ is an isomorphism of symplectic vector spaces under which quadratic forms $Q, Q'$ correspond, then $t(Q) = t(Q')$. 

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For example, suppose that $V$ is $\mathbb{R}^{2n}$ with coordinates $x_\nu, y_\nu$ and symplectic form $\sum_1^n dx_\nu \wedge dy_\nu$. Consider

$$Q(x, y) = \sum q_\nu x_\nu y_\nu, \quad q_\nu \in \mathbb{R}. $$

As $\text{sgrad} \ Q = \sum_\nu q_\nu (x_\nu \partial x_\nu - y_\nu \partial y_\nu)$, monomials $x_\lambda x_\mu, x_\lambda y_\mu, \text{and } y_\lambda y_\mu$ form an eigenbasis of $\text{ad}_Q$, with eigenvalues $q_\lambda + q_\mu, \text{resp. } q_\lambda - q_\mu, \text{resp. } -q_\lambda - q_\mu$. Hence

$$t(Q) = \sum_{\lambda \leq \mu} (q_\lambda + q_\mu)^2 + \sum_{\lambda, \mu} (q_\lambda - q_\mu)^2 + \sum_{\lambda \geq \mu} (q_\lambda + q_\mu)^2 $$

$$= \sum_{\lambda} (2q_\lambda)^2 + \sum_{\lambda, \mu} (q_\lambda + q_\mu)^2 + (q_\lambda - q_\mu)^2 $$

$$= 4 \sum_\lambda q_\lambda^2 + 2 \sum_{\lambda, \mu} (q_\lambda^2 + q_\mu^2) = (4n + 4) \sum_\lambda q_\lambda^2. $$

(8.1)

Note that $Q$ and $R = \sum r_\nu x_\nu y_\nu$ are on the same $\text{SL}(V)$ orbit whenever $\prod q_\nu = \prod r_\nu$. We conclude $t$ is not $\text{SL}(V)$ invariant.

We need to introduce one more player. If a general quadratic form $Q : V \to \mathbb{R}$ is written in a symplectic basis $z_\nu, \nu = 1, \ldots, 2n$, as $Q(z) = \sum a_{\nu \lambda} z_\nu z_\lambda$, with $a_{\nu \lambda} = a_{\lambda \nu}$, we let

$$\text{Det} \ Q = \text{det}(a_{\nu \lambda}). $$

Thus $\text{Det} \ Q$ is independent of the choice of basis, and is even $\text{SL}(V)$ invariant.

Fix a smooth function $\varphi: \mathbb{R} \to \mathbb{R}$ such that $\varphi(t) = 0$ for $|t| \leq 1/2$ and $\varphi(t) = t$ for $|t| \geq 1$. If $\xi \in C^\infty(X)$ and $x$ is a critical point of $\xi$, let $Q_x = Q_{\xi,x}$ stand for the quadratic Taylor polynomial of $\xi - \xi(x)$ at $x$, a quadratic form on the symplectic vector space $T_xX$ (the Hessian). Given $\varepsilon > 0$, critical points $x$ of $\xi$ for which $|\text{Det} \ Q_x| \geq \varepsilon$ form a discrete and compact, hence finite set. In particular $\xi$ has countably many nondegenerate critical points, that we denote $x_i$. Define $p: C^\infty(X) \to \mathbb{R}$ by letting

$$p(\xi) = \sum_i \varphi(\text{Det} \ Q_{x_i}) t(Q_{x_i}); $$

we are summing over all nondegenerate critical points $x_i$ of $\xi$, or only over those for which $|\text{Det} \ Q_{x_i}| > 1/2$. We claim that $p$ is smooth.

Indeed, given $\eta \in C^\infty(X)$, let $C$ consist of its critical points $y$ for which $|\text{Det} \ Q_y| \leq 1/4$, a compact subset of $X$, and let $y_i, 1 \leq i \leq k$ denote the rest of its critical points. It is possible that $k = 0$, and even that $\eta$ has no nondegenerate critical point at all. About each $y_i$ construct a neighborhood $U_i$ so that the only critical point within $\overline{U_i}$ is $y_i$. About each $y \in C$ construct a neighborhood $V \subset X$ with local coordinates $z_1, \ldots, z_{2n}$ so that $\omega|V = \sum_\nu dz_\nu \wedge dz_{n+\nu}$. Let $U \subset \subset V$ be a neighborhood of $y$ consisting of $x$ such that the quadratic form $Q(z) = \sum \partial_\nu \partial_\eta(x) z_\nu z_\nu$ has determinant $|\text{Det} \ Q| < 1/3$. Choose a finite cover $\{U_{k+1}, \ldots, U_l\}$ of $C$ by such neighborhoods $U$. If $\xi \in C^\infty(X)$ is in a sufficiently small neighborhood of $\eta$,

- in each $\overline{U_j}, j \leq k$, $\xi$ has a single critical point, which depends smoothly on $\xi$;
- all critical points $x$ of $\xi$ in $\bigcup_{j>k} \overline{U_j}$ satisfy $|\text{Det} \ Q_{\xi,x}| < 1/2$; and
\( \xi \) has no critical points outside \( \bigcup_j U_j \).

Therefore \( p \) in (8.2) is a smooth function in this neighborhood of \( \eta \), hence everywhere.

Invariance of \( \text{Det} \) and \( t \) implies that \( p \) is \( \text{Ham}(\omega) \) invariant. It is, however, not invariant under general volume preserving diffeomorphisms for the following reason. Fix a coordinate system \( x_\nu, y_\nu \) on an open \( W \subset X \), centered at some \( o \in W \), such that \( \omega|W = \sum dx_\nu \wedge dy_\nu \). Let \( \xi \in C^\infty(X) \) be given by \( \xi = 2 \sum x_\nu y_\nu \) on \( W \).

The local flow of a vector field \( v = \sum a_\nu(x, y)\partial x_\nu + b_\nu(x, y)\partial y_\nu \) preserves \( \omega^n \) if and only if \( \text{div} \; v = 0 \); that is, if the \((2n-1)\)-form

\[
\alpha = \sum_\nu (a_\nu dx_\nu - b_\nu dy_\nu) \wedge \bigwedge_{\lambda \neq \nu} dx_\lambda \wedge dy_\lambda
\]

is closed, or if locally \( \alpha = d\beta \). This shows that the germ of any volume preserving flow at \( o \) can be continued to a volume preserving flow on all of \( X \), that will be supported in our coordinate neighborhood. With \( c_\nu \in \mathbb{R} \) consider the germ of a diffeomorphism at \( o \)

\[ (x, y) \mapsto (e^{c_\nu x_\nu}, e^{c_\nu y_\nu})_{1 \leq \nu \leq n} . \]

This is the time 1 map of a volume preserving flow if \( \sum c_\nu = 0 \). If so, there is a volume preserving diffeomorphism \( g : X \to X \), supported in \( W \), whose germ at \( o \) is (8.3). Now \( \xi \) and \( \eta = \xi \circ g \) have the same critical points, and even their germs agree at all critical points except possibly at \( o \). Hence the contributions to \( p(\xi) \) and \( p(\eta) \) of critical points different from \( o \) are the same. At \( o \)

\[
Q_{\xi, o} = \xi = 2 \sum x_\nu y_\nu, \quad Q_{\eta, o} = \eta = 2 \sum e^{2c_\nu x_\nu y_\nu} .
\]

This means that \( \text{Det} \; Q_{\xi, o} = \text{Det} \; Q_{\eta, o} = \pm 1 \), while in general, in view of (8.1)

\[
t(Q_{\xi, o}) = 4(4n + 4)n \neq 4(4n + 4) \sum e^{4c_\nu} = t(Q_{\eta, o}) .
\]

Therefore \( p(\xi) \neq p(\eta) \), as claimed.

Note also that \( p \) is discontinuous in the sup norm topology, since arbitrarily \( \| \cdot \|_\infty \) close to \( 0 \in C^\infty(X) \) there are \( \xi \) with a unique nondegenerate critical point \( x \), where the Hessian \( Q_{\xi, x} \) can be arbitrarily prescribed.

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