A blue sky catastrophe in double-diffusive convection

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A global bifurcation of the blue sky catastrophe type has been found in a small Prandtl number binary mixture contained in a laterally heated cavity. The system has been studied numerically applying the tools of bifurcation theory. The catastrophe corresponds to the destruction of an orbit which, for a large range of Rayleigh numbers, is the only stable solution. This orbit is born in a global saddle-loop bifurcation and becomes chaotic in a period doubling cascade just before its disappearance at the blue sky catastrophe.

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Bifurcation theory has long been a very helpful tool in the analysis of complex dynamics of nonlinear systems [1, 2]. Whereas different devised scenarios have been found in theoretical models with a few variables, there is a growing interest both in relating real systems with that kind of models (e.g., projecting their dynamics to some relevant degrees of freedom [3]) and in directly analyzing the behavior of these systems in terms of dynamical systems theory (by studying them either experimentally or by realistic models). In this context a great deal of work has been devoted to convection in fluids. Qualitative changes in the dynamics of fluxes maintained out of equilibrium by imposed thermal gradients have provided examples of most of the known bifurcations, and have become a main subject in the area of nonlinear dynamics.

In this letter we will show the occurrence of a blue sky catastrophe [BSC] in double diffusive convection. The BSC is a codimension-1 bifurcation that consists in the destruction of a stable periodic orbit as its length and period tend to infinity, while the cycle remains bounded and located at a finite distance from all the equilibrium solutions [1, 2]. This destruction is caused by the collision with a non-hyperbolic cycle that appears at the bifurcation point. While approaching the bifurcation the orbit increasingly coils in the zone where the new cycle will appear, which originates the divergence in both period and length. In that point the original cycle becomes an orbit homoclinic to the new cycle. This type of bifurcation is relatively exotic, but can easily be found in slow-fast (i.e., singularly perturbed) systems with at least two fast variables [3].

We are interested in double-diffusive fluxes that occur when convection is driven by simultaneous thermal and concentration gradients in a binary mixture [4]. Double-diffusive convection in cavities with imposed vertical gradients exhibits very rich dynamics, and has been used as a system to study pattern formation [5] and transition to chaos [6]. The case of horizontal gradients, which arises naturally in applications such as crystal growth [7] or oceanography [8], has received less attention. In this work we numerically study this latter configuration for a small Prandtl number binary mixture. We consider the case when thermal and solutal buoyancy forces exactly compensate each other, which allows the existence of a quiescent (conductive) state [9, 10, 11, 12, 13]. We have found that in this system there exists a large range of Rayleigh numbers in which the only stable solution is an orbit that features a low-frequency spiking behavior. This orbit appears associated to a global bifurcation and loses stability when a period doubling cascade takes place originating a chaotic attractor. However, the most remarkable feature of this chaotic attractor is its sudden disappearance in a BSC of the chaotic type. As far as we know this is the first example of such bifurcation in an extended system.

We have considered a binary mixture in a 2-D rectangular cavity of aspect ratio $\Gamma = d/h = 2$, where $d$ is the length and $h$ is the height of the cavity. A difference of temperature $\Delta T$ is maintained between both vertical boundaries. Dimensionless equations in Boussinesq approximation explicitly read

$$
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + \sigma \nabla^2 \mathbf{u} + \sigma \text{Ra} (1 + S) (-0.5 + x/\Gamma) + \theta + SC \hat{z},$

$$
\partial_t \theta + (\mathbf{u} \cdot \nabla) \theta = -v_x/\Gamma + \nabla^2 \theta,$$

$$
\partial_t C + (\mathbf{u} \cdot \nabla) C = -v_x/\Gamma - \tau \nabla^2 (\theta - C),$$

$$
\nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u} = (v_x, v_z)$ is the velocity field in $(x, z)$ coordinates, $P$ is the pressure over the density, $\theta$ denotes the departure of the temperature from a linear horizontal profile. $C$ is the scaled deviation of the concentration of the heavier component relative to the linear horizontal profile which equilibrates that of the temperature in the expression of the mass flux. Lengths and times are scaled with $h$ and $t_\kappa = h^2/\kappa$, respectively, being $\kappa$ the thermal diffusivity. The dimensionless parameters are the Prandtl number $\sigma = \nu/\kappa$, the Rayleigh number $\text{Ra} = agh^3 \Delta T/\nu \kappa$ and the Lewis number $\tau = D/\kappa$, where $\nu$ denotes the kinematic viscosity, $g$ the gravity level, $\alpha$ the thermal ex-
pansion coefficient, and $D$ is the mass diffusivity. The separation ratio $S = C_0(1 - C_0) \frac{\beta}{\alpha} S_T$ will be taken $S = -1$. Here, $S_T$ is the Soret coefficient, $C_0$ is the concentration of the heavier component in the homogeneous mixture, and $\beta$ is the mass expansion coefficient ($\beta > 0$ for the heavier component).

The boundaries are taken to be no-slip and with no mass flux. Lateral walls are maintained at constant temperatures and at the horizontal plates a linear profile of temperature between the two prescribed temperatures is imposed. Thus, boundary conditions are written as

$$u = \theta = n \cdot \nabla (C - \theta) = 0, \quad \text{at } \partial \Omega. \quad (5)$$

Notice that these boundary conditions prevent one to absorb the Soret terms into the equations like in Refs. [10, 11, 12, 13]. On the other hand this system is $Z_2$-equivariant. Eqs. (1-4), together with boundary conditions (5), are invariant under a transformation $\pi$, a central symmetry around the point ($\Gamma/2, 1/2$), i.e. $\pi : (v_x, v_z, \theta, C) \rightarrow (-v_x, -v_z, -\theta, -C), (x, z) \rightarrow (\Gamma - x, 1 - z)$. Hence any solution of these equations either is $\pi$-invariant (for now on we will call it symmetric) or its image under $\pi$ is also a solution (constituting a pair of asymmetric solutions). This has important consequences on the nature of its possible bifurcations [1].

We have obtained time-dependent solutions of equations (1-4) and boundary conditions (5) by using a second order time-splitting algorithm, proposed in Ref. [14], and a pseudo-spectral Chebyshev method for the space discretization. Furthermore we have calculated (both stable and unstable) steady solutions and analyzed their stability by adapting a pseudospectral first-order time-stepping formulation, as described in Ref. [15, 16, 17]. The values of the parameters have been $\sigma = 0.00715$ and $\tau = 0.03$, close to that characteristic of molten doped germanium [18, 19]. Spatial discretization has typically been between $60 \times 30$ and $90 \times 60$ mesh grid points.

The scenario provided by the analysis of the steady solutions is shown in the bifurcations diagram of Fig. 1. In this figure the Nusselt number ($Nu$), defined as the quotient of heat flux through the hot wall with that of the corresponding conductive solution, is represented for the steady states as a function of the Rayleigh number ($Ra$). For the sake of clarity only one asymmetric solution of each pair has been shown. For small $Ra$ the conductive solution (allowed here by the choice $S = -1$) is stable, but loses stability, maintaining the symmetry, through a transcritical bifurcation at $Ra = 541.9$. The supercritical branch of the bifurcating solution is only stable up to a pitchfork bifurcation at $Ra = 542.4$, following a scenario similar to that described in Ref. [12].

The interesting behavior in this system originates from the subcritical branch. This branch gains stability via a saddle node bifurcation at $Ra = 90$ ($SN_1$), and loses it again at $Ra = 245$ in a Pitchfork bifurcation ($P$) where a couple of stable asymmetric branches appear. In Fig. 2 we represent the concentration for a symmetric (left) and an asymmetric (right) steady states. We can see that concentration is roughly homogeneous inside rolls, displacing concentration gradients to the lateral boundaries.

The asymmetrical steady state is stable until $Ra = 1209$, where it loses stability at a saddle-node bifurcation ($SN_2$). The full branch of asymmetrical steady states is depicted in Fig. 1 where we can see that it changes again the direction at a turning point at $Ra = 865.6$, but without gaining stability. Increasing the Rayleigh number Hopf bifurcations of the symmetric and asymmetric branches take place at $H_1$ ($Ra = 2137$) and $H_2$ ($Ra = 2218$) respectively. The branch of symmetric periodic orbits emanating from $H_1$ will play an essential role in the subsequent evolution of the system.

In the range from $Ra = 1209$ until $Ra = 2253$ we have found no stable solution connected with the above branches by local bifurcations. Integrating the evolution equations we have obtained a branch of asymmetric periodic solutions that dominates the dynamics of the system in this range of parameters. In Fig. 3 we represent time series and phase space plots of the orbits of this branch for two different values of the Rayleigh number. The oscillations first appear in the form of spikes of very large period (see Fig. 3b), according to the proximity to a global saddle-loop bifurcation that occurs at

![FIG. 1: Nusselt number of steady solutions versus Ra, and its corresponding bifurcations. Continuous lines: stable states. Dashed lines: unstable states.](image1)

![FIG. 2: Concentration levels of the steady solutions of the symmetric branch ($Ra = 1252$) and the non symmetric branch ($Ra = 888$).](image2)
decreases toward (mic divergence of the period when the Rayleigh number
this global bifurcation can be inferred from the logarith-
$Ra$-connection. Right: Square-root fit of the periods for the BSC.

FIG. 4: Left: logarithmic fit of the periods for the SL con-
Asymmetric orbit at $\lambda = 0.079$ results to be quite close to the unstable
value of the saddle stationary point, as obtained by the stability calculation. Near that
global bifurcation the time evolution of the velocity of a represen-
tative point is shown in Fig. 3 (a). The value for the
saddle asymmetric state is also represented. We can see
how the solution spends a long time near it. The spike
corresponds to a rapid and large excursion by the phase
frequency of the unstable symmetric orbit that appears
in $H_1$. In fact we have been able to calculate this unsta-
ble branch by temporal evolution forcing the symmetry
of the system, and its frequency coincides with that of
the windings of the attractor on all the branch. If we in-
crease further $Ra$, the asymmetric orbit follows a period-
doubling cascade, becoming chaotic. This is revealed in
the phase of the winding of the trajectory, as can be seen
in Fig. 5 where a detail of the orbit during the cascade
is shown. This cascade seems to move to slightly higher
$Ra$ values as spatial resolution is increased, but we have
not been able to obtain the precise values due to the
extremely large duration of the orbits in this regime. In
Fig. 4(c,d) the attractor thus generated is represented at
$Ra = 2255$. For this value of $Ra$ the symmetric orbit has
already become stable at a Pitchfork bifurcation ($P_{so}$, at
$Ra = 2253$), and both coexist. Very shortly afterwards,
the whole attractor disappears at $Ra = 2257.5$.

This destruction of the attractor exhibits characteristics
that permit to identify it as the chaotic counterpart
of the scenario for BSC bifurcation described in Refs.
[1, 20]. Indeed in all the process the attractor remains
bounded and at a finite distance of any steady solution,
as required [1]. The average length and time between
spikes (which are reproducible with variations smaller
than 1 over 1000) diverges as the windings start to accu-
rate, which occurs at a specific location in the attrac-
tor. That indicates that the solution is colliding there
with a new cycle that appears at the bifurcation point,
and to which it becomes homoclinic. Furthermore this
divergence, shown in Fig. 4(right), is very well fitted by
a square-root law:

$$T \sim \frac{A}{\sqrt{Ra - Ra_c}} + B.$$  \hspace{1cm} (7)

This law of divergence particularly corresponds to that
scenario, since it demonstrates that the new cycle to
which the attractor is connecting is the saddle-node of
two orbits ($SNO_1$). In principle there are several possi-
bilities for the topology of the attractor [2]. In our case
the successive windings are braided by the tip of the at-
tractor into an almost one dimensional tube or filament
(Fig. 4). This filament reintroduces the orbit into the
vicinity of the saddle-node orbit, and it starts winding
again accumulating curls near it. Therefore, in the limit,
the attractor has the topology of a French Horn. This
feature is also shared with Ref. [20].

After the BSC, one would expect the system to reach
the stable member of the pair of asymmetric solutions
born at $SNO_1$. On the contrary, simulations show that
the system evolves through a extremely long transient,
during which the trajectory accumulates curls near the
saddle-node before being rejected to the symmetric orbit
that became stable at $P_{so}$. That could mean either that
the stability range of the asymmetric orbit is very small
(which would require a much finer exploration in $Ra$ to

$Ra = 1183.67$ (SL) where the orbit connects with the
unstable branch of $SN_2$ (see Fig. 1). The character of
this global bifurcation can be inferred from the logarith-
mic divergence of the period when the Rayleigh number
decreases toward (SL). We have fitted that period to

$$T \sim -\frac{1}{\lambda} \log (Ra - Ra_{SL}) + A.$$  \hspace{1cm} (6)

We can see the fit in Fig. 3 (left). The resulting value
$\lambda_{fit} = 0.079$ results to be quite close to the unstable
eigenvalue $\lambda = 0.074$ of the saddle stationary point, as
obtained by the stability calculation. Near that global
bifurcation the time evolution of the velocity of a represen-
tative point is shown in Fig. 3 (a). The value for the
saddle asymmetric state is also represented. We can see
how the solution spends a long time near it. The spike
corresponds to a rapid and large excursion by the phase
space, as seen in Fig. 3 (b), during which the roll alter-
nately switches between a confined and a more centered
positions (analogous to the patterns shown in Fig. 2).

Increasing $Ra$, at $Ra = 2137$ the orbit starts to curl,
showing ripples in the time dependence, reflecting the

FIG. 3: Velocity components of a representative point. Top:
Asymmetric orbit at $Ra = 1183.68$. a) time series, with the
value of the saddle stationary solution marked. b) orbit in
the phase space. Bottom: Attractor at $Ra = 2255$. c) time
series. d) attractor in the phase space with the stable sym-
metric orbit. The unstable stationary symmetric solution is
also shown.

FIG. 4: Left: logarithmic fit of the periods for the SL con-
nector. Right: Square-root fit of the periods for the BSC.
would slow down the dynamics for recurrence of this BSC. For example, the attractor could be as it is actually observed. making the transient to the symmetric solution very long, \(\omega\) of the windings of the attractor, SNO and with such connection. A small distance between SNO the additional saddle node orbit of the asymmetrical solution. Finally, the presence of the separation ratio \(S\). In particular we have obtained a similar BSC in simulations performed with \(S = -0.99\). That means that the additional symmetry introduced in the system by the special value \(S = -1\) is not an essential ingredient of the phenomena described here.

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![FIG. 5: Period-doubling cascade (zoom of the tip of the attractor). a) period 1 (Ra = 2220). b) period 2 (Ra = 2232). c) period 4 (Ra = 2235). d) chaotic solution (Ra = 2240).](image)

![FIG. 6: Diagram of the conjectured unstable asymmetric orbit (thick line) and its connections to other branches](image)

find it, a formidable task in this slow regime), or that its basin of attraction is very reduced (and the nearby symmetric orbit attracted all the calculated orbits).

We propose that the SNO\(_1\) is located in the branch of unstable asymmetric orbits created at the pitchfork bifurcation where the new orbit becomes stable (\(P_{SO}\)). This hypothetical scenario is shown in Fig. 6 and is the simplest one in which the attractor presents at the BSC an homoclinic connection to a branch coming from known solutions. This conjecture requires the unstable asymmetric branch to gain stability in a first saddle node bifurcation SNO\(_2\) and to lose it again at the SNO\(_1\), as can be seen in Fig. 6. The coincidence of the frequency value of the symmetric orbit at \(P_{SO}, \omega_{SO} = 7.01\), to that of the windings of the attractor, \(\omega = 7.01\), is consistent with such connection. A small distance between SNO\(_1\) and SNO\(_2\) would explain the reduced stability domain of the asymmetrical solution. Finally, the presence of the additional saddle node orbit SNO\(_2\) in the proximity would slow down the dynamics for \(Ra\) slightly above, making the transient to the symmetric solution very long, as it is actually observed.

One could devise more complex scenarios for the occurrence of this BSC. For example, the attractor could be destroyed on a boundary crisis associated to global connections of the unstable orbits coming from the period doubling bifurcations.

Finally, it is worth remarking than the BSC displayed by this system is robust against small changes in the value of the Rayleigh number, \(Ra\).

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