Abstract.
We apply algebraic and vertex operator techniques to solve two dimensional reduced vacuum Einstein’s equations. This leads to explicit expressions for the coefficients of metrics solutions of the vacuum equations as ratios of determinants. No quadratures are left. These formulas rely on the identification of dual pairs of vertex operators corresponding to dual metrics related by the Kramer-Neugebauer symmetry.

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1 Introduction.

Surprisingly, four dimensional gravity admits an integrable sector. It corresponds to ansatz metrics which admit two surface orthogonal Killing vectors. They describe stationary axis symmetric situations or colliding gravitational waves depending on the nature of the Killing vectors. For these ansatz, the vacuum Einstein’s equations reduce to the Ernst’s equations. An infinite dimensional solution generating group for them was constructed by Geroch \cite{2} and later identified by Julia \cite{3} as the affine $SL(2, R)$ Kac-Moody group. The integrable character of Ernst’s equations was deciphered by Belinskii-Zakharov in ref.\cite{4}. See also ref.\cite{5} for a discussion relating these two aspects. Since then, various methods have been applied to solve 2d reduced Einstein’s equations: either using Backlund or solution generating transformations or the Belinskii-Zakharov method, cf eg. \cite{6, 7, 8, 9, 10, 11} for a sample of references and \cite{12, 13} and references therein, or using analytical finite-gap techniques \cite{14}. However, the by-now standard vertex operator approach to integrable models developed by the Kyoto school \cite{15} was never applied to this problem. This is probably due to the fact that the Belinskii-Zakharov method involves the so-called moving poles which forbids a direct application of the vertex operator technique. This problem was partially overcome in ref.\cite{16}. The aim of this paper is to fill the gap left open in this work and to describe how vertex operators may be used to solve Einstein’s equations. It leads to determinant formula for the metrics which are described in the following section. This vertex operator approach, which is based on an algebraic formulation of the dressing group, may also be useful for deciphering the Lie-Poisson properties of the solution generating groups of Einstein’s equations. See refs.\cite{17, 18} for a discussion of the Lie-Poisson properties of the dressing transformations and ref.\cite{19} for a discussion of these properties for the Geroch group. In supergravity context these groups are called duality groups and they are important for quantization.

Besides providing explicit formulas for exact solutions of Ernst’s equations, (but whose possible physical applications are not discussed), one of the aim of this paper is to decipher the algebraic structures underlying the solvability of Ernst’s equation. Contrary to the impression that one may get from the Belinskii-Zakharov approach which uses space-time dependent spectral parameters, we shall show that Ernst’s equations belong to the usual class of integrable hierarchies, such as the KP or sine-Gordon equations, and that they may be solved using usual algebraic technical tools. Vertex operators is one of those techniques which provide tools for algebraically solving matrix Riemann-Hilbert problems. Deciphering these algebraic structures leads us to identify dual pairs of elements of an affine Kac-Moody group corresponding to solutions paired via the Kramers-Neugebauer duality. This could be interesting in view of the importance of the duality group in supergravity analogues of Ernst’s equations. But the long term motivation for this algebraic detours is quantization. 2D reduced gravity provides a toy model in application to quantization technique to gravity. The dressing method, on which our approach is based, is particularly adapted to quantization since the dressing group usually acts on the phase space of integrable hierarchies by a Lie-Poisson action. So it is promoted to a quantum group symmetry after quantization. The next step in that direction would thus be to find a concise description of the symplectic struture of this system and of the Lie-Poisson property of the dressing group.

We have tried to write the paper such that it may be read in two different ways depending if one is only interested in explicit formulas for the solutions or if one is willing to learn the algebraic structures underlying the derivation of these formulas. Readers who are just interested in concrete formulas to obtain solutions may restrict their attention to Section 2, beginning of Section 3 and Section 6. Algebraically oriented readers may look at Section 4 and Section 5. The main new trick which allows us to complete the approach initiated in \cite{16} is a construction of dual pairs of elements in the $SL(2, R)$ Kac-Moody group associated to dual pairs of solutions exchangeable by Kramers-Neugebauer involution. This leads us to solve a problem of factorization that was left opened in \cite{16} and to give determinant formulas for the metrics.

The vertex operator and the determinant formula of the metrics are described in the following Section 2. The rest of the paper is devoted to the proof of these formulas and it is organized as follows. In Section 3, we recall basic facts concerning the 2d reduced Einstein’s equations and present them in a way convenient for the following. In particular we introduce the appropriate tau functions. Section 4 is devoted to the construction of dual pairs of vertex operators which allow us to compute the dual pairs of tau functions. Section 5 describe how to get the coefficients of the metric given the dual pair of vertex operators. Finally, a few explicit examples and comparisons with the previous result are described in Section 6. Appendix A presents a rapid survey of the method described in \cite{16}, and we have gathered a few useful formulas in Appendices B and C.
2 Determinant formula for the metrics.

As is well known, solutions of the reduced Einstein’s equations come in pairs which are related via the Kramer-Neugebauer duality. The two dual metrics, that we shall denote by $ds^2$ and $ds^*_2$, can be parametrized in terms of Weyl coordinates $z$ and $\rho$ as:

$$
\begin{align*}
    ds^2 &= \rho^{-\frac{1}{2}} e^{2\widehat{\sigma}} (dz^2 - d\rho^2) + G_{ab} dx^a dx^b \\
    ds^*_2 &= \rho^{-\frac{1}{2}} e^{2\widehat{\sigma}^*} (dz^2 - d\rho^2) + G^*_{ab} dx^a dx^b
\end{align*}
$$

(1) All fields only depend on the two coordinates $z, \rho$. The indices $a, b$ run from one to two and $\rho^2 = \det G_{ab} = \det G^*_{ab}$. The prefactor $\widehat{\sigma}$, or $\widehat{\sigma}^*$, is usually called the conformal factor. The precise form of the duality relation mapping $ds^2$ into $ds^*_2$ is recalled in the following section as well as an alternative parametrization of the metrics.

The components of the metrics would be parametrized in terms of expectation values of certain vertex operators as follows:

$$
\begin{align*}
    e^{2\widehat{\sigma}} &= |\langle g \rangle_{z,\rho}|^2 \\
    e^{2\widehat{\sigma}^*} &= |\langle g^* \rangle_{z,\rho}|^2 \\
    e^{2\widehat{\sigma}} G_{22} &= \sqrt{p} |\langle g \rangle_{z,\rho}|^2 \\
    e^{2\widehat{\sigma}^*} G^*_{22} &= \sqrt{p} |\langle g^* \rangle_{z,\rho}|^2 \\
    e^{2\widehat{\sigma}} G_{12} &= \sqrt{p} \Im \left[ \langle g^* \rangle_{z,\rho} \overline{\langle W g^* \rangle_{z,\rho}} \right] \\
    e^{2\widehat{\sigma}^*} G^*_{12} &= \sqrt{p} \Im \left[ \langle g \rangle_{z,\rho} \overline{\langle W g \rangle_{z,\rho}} \right] \\
    e^{2\widehat{\sigma}} G_{11} &= \sqrt{p} |\langle W g \rangle_{z,\rho}|^2 \\
    e^{2\widehat{\sigma}^*} G^*_{11} &= \sqrt{p} |\langle W g^* \rangle_{z,\rho}|^2
\end{align*}
$$

(3) Here $g$, $W g$ and $g^*$, $W g^*$ denote the vertex operators associated to the two dual solutions. See eqs. (12), (43) and eqs. (58), (59) below for their definitions. The overbar denotes complex conjugation. The indices $z, \rho$ are here to recall that these expectation values depend on the Weyl coordinates. Notice the interplay between $g$ and $g^*$ in the above formula: the element $g$ enters in the conformal factor $\widehat{\sigma}$ while the dual element $g^*$ enters in the metric $G_{ab}$.

The solutions we shall describe depend on two sets of real parameters with total number $2(m+n)$: the first set is made of $2m$ parameters $(z_p, u_p)$, $p = 1, \ldots, m$, while the second set is made of $2n$ parameters denoted $(z_j, y_j)$, $j = 1, \ldots, n$. All parameters are real but $|z_j| > 1$ and $|z_p| > 1$. It is convenient to introduce the notations $X_j, \mu_j$ and $X_p, \mu_p$ such that:

$$
X_j = \rho \frac{z_j^2 - 1}{(z_j - z)^2 + \rho^2} \quad \text{and} \quad \mu_j = \frac{(z_j - z) + \rho}{(z_j - z) - \rho}
$$

and similarly for $X_p$ and $\mu_p$. The functions $\mu_j$ are related in a simple way to the usual moving poles in the Belinskii-Zakharov approach. The validity of the metric is restricted to $(z, \rho)$ domains such that $z \pm \rho < z_j$ for $z_j > 1$ and $z \pm \rho > z_j$ for $z_j < -1$, and similarly for $z_p$.

The explicit determinant formula for the metrics come from the following expressions for the vertex operator expectation values which we shall derive in the following sections:

$$
\begin{align*}
    \langle g \rangle_{z,\rho} &= \Omega \cdot \tau(Y_j | \mu_j) \\
    \langle g^* \rangle_{z,\rho} &= \rho^\frac{1}{2} \Omega^* \cdot \tau(Y_j B_j | \mu_j)
\end{align*}
$$

(8) The tau functions $\tau(Y | \mu)$ could be written as $n \times n$ determinants:

$$
\tau(Y | \mu) = \det_{n \times n} \left[ 1 + iV \right] \quad \text{with} \quad V_{ij} = \frac{2\mu_i Y_j}{\mu_i + \mu_j}
$$

(10) The parameters are the following:

$$
Y_j = y_j X_j \left( \prod_{p=1}^{m} B^u_{j_p} \right) \quad \text{with} \quad B_{jp} = \frac{\mu_j - \mu_p}{\mu_j + \mu_p} \quad \text{and} \quad B_j = \frac{1 - \mu_j}{1 + \mu_j}
$$

(11) Finally the prefactors $\Omega$ and $\Omega^*$ are given by:

$$
\begin{align*}
    \Omega &= \left( \prod_{p=1}^{m} X^u_{p} \right) \left( \prod_{p < q} B^u_{pq} \right) \Omega \\
    \Omega^* &= \left( \prod_{p=1}^{m} B^u_{p} \right) \Omega
\end{align*}
$$

(12)
of the metrics are thus the ratio of these determinants. Notice that the prefactor $\Omega$ cancels when computing these ratios.

Stationary axis symmetric solutions may be obtained by analytic continuation, see Section 6.3 below. However the reality conditions are then more involved and not all of these solutions are physical. Notice also that for an infinite number of parameters $(z_j, y_j)$ the above solutions may be described in terms of Fredholm determinants.

Of course many solutions of Ernst equations have already been described in the literature, see Refs. [4, 5, 6] for a sample of references and Refs. [7, 8] and references therein. The formula closest to the ones we are describing here are those obtained by P. Letelier, cf. Ref. [9]. These later formulas also present the standard in integrable models, may be applied to the Ernst equations.

To be solved. But, again our principal aim was to show explicitly how vertex operator methods, which are standard in integrable models, may be applied to the Ernst equations.

3 Equations of motion and dualisation.

This section is devoted to recall a few well known facts on the Ernst equations: its duality property and the triangular gauge condition.

Following Papapetrou, let us parametrize the metric (1) as:

$$ds^2 = \rho^{-\frac{1}{2}} e^{2\tilde{\sigma}} (dz^2 - d\rho^2) + \rho \Delta^{-1} dx^2 + \rho \Delta(dy - N dx)^2$$ (14)

This means that $G_{22} = \rho \Delta$ and $G_{12} = -\rho \Delta N$. The fields $(\tilde{\sigma}, \Delta, N)$ only depends on the two coordinates $z$ and $\rho$. This ansatz can be made more covariant by introducing two light-cone coordinates $u$ and $v$ such that

$$\rho = a(u) + b(v) \quad , \quad z = a(u) - b(v)$$ (15)

with $a$ and $b$ any functions. The metric is then:

$$ds^2 = -4\rho^{-\frac{1}{2}} e^{2\tilde{\sigma}} dudv + \rho \Delta^{-1} dx^2 + \rho \Delta(dy - N dx)^2$$

Consistency of this parametrization follows from the fact that Einstein equations imply that $\rho$ defined as the square root of $\text{det} G_{ab}$ is harmonic: $\partial_u \partial_v \rho = 0$. In the following we shall denote by $\partial_+$ and $\partial_-$ the derivatives with respect to $u$ and $v$. 

Here, $\Psi_0$ was derived in Ref. [16] using the algebraic method of dressing transformations:

$$\frac{\rho}{G_{22}} + \frac{G_{12}}{G_{22}} = \frac{\langle \Psi_0 \cdot (g^{-1} W_2(1) g_+ \cdot \Psi_0^{-1}) \rangle}{\langle \Psi_0 \cdot (g^{-1} g_+ \cdot \Psi_0^{-1}) \rangle} = \frac{\langle W g \rangle_{z, \rho}}{\langle g \rangle_{z, \rho}}$$ (13)

Here, $\langle \cdot \rangle$ is an explicit operator defined in eq. (32) below which carries all $(z, \rho)$ dependence.

$W$ and $g_{\pm}$ are acting operators on an auxiliary Fock space and $(\cdot \cdot \cdot)$ denotes the vacuum expectation value in that Fock space. $\Psi_0$ is an explicit operator defined in eq. (32) below which carries all $(z, \rho)$ dependence. $W_2(1)$ is a specific vertex operator associated to the Kramer-Neugebauer duality and $g = g^{-1} g_+$ are elements of an SL(2, R) affine Kac-Moody group.

However, the formula (13) is not the final answer since it requires factorizing the element $g$ in two pieces as $g = g^{-1} g_+$. This factorization problem is specified by Einstein's equations up to a residual $SO(2)$ gauge freedom which reflects the gauge symmetry of these equations. Eq. (13) is only true in a specific gauge: the triangular gauge. The problem of fixing this gauge in the factorization of the group element as $g = g^{-1} g_+$ remained unsolved in Ref. [14]. The main new technical point of this paper is to solve algebraically this gauge condition. This will be done in three steps: (i) by relating the triangular gauge condition to the dual pair of solution of Ernst equations, cf Proposition 1, (ii) by establishing the relation between pairs of group elements $g$ and $g^*$ corresponding to dual pairs of solutions, cf Proposition 2, and (iii) by using these two results to factorize the element $g = g^{-1} g_+$ and to compute $g^{-1} W_2(1) g_+$, cf Proposition 3. This then leads to the determinant formula (4). The key algebraic result is the duality $(g, g^*)$, see eqs. (42,43), or eqs. (62,63) for its interpretation in the dressing group.

Although this paper is necessarily quite technical, we have tried to minimize the technical aspects. In particular the proofs could probably be omitted when reading it. Since this paper is a continuation of Ref. [16] and has for aim to fill the gap left open there, we shall freely use results from that reference.
As is well known, given any solution of the reduced Einstein’s equations one gets using the Kramer-Neugebauer symmetry a dual solution by introducing the metric $ds^2_\ast$ parametrized as above but with the original fields $(\hat{\sigma}, \Delta, N)$ replaced by the dual fields $(\check{\sigma}, \check{\Delta}, \check{N})$ with:

$$\Delta^\ast = 1/(\rho \Delta) , \quad \Delta^\ast \partial_\nu N^\ast = \epsilon_{\nu\alpha} \Delta \partial_\alpha N , \quad \Delta^\ast e^{i\check{\sigma}} = e^{i\sigma}$$  \hspace{1cm} (16)

Hence, the dual metric $ds^2_\ast$ can be written as:

$$ds^2_\ast = \Delta e^{2\check{\sigma}} (dz^2 - dp^2) + \rho^2 \Delta dx^2 + \Delta^{-1}(dy - N^* dx)^2$$ \hspace{1cm} (17)

### 3.1 Reduced vacuum Einstein’s equations.

The vacuum Einstein’s equations, which codes for the Ricci flatness of the metric \([1]\), reduce to the Ernst equations. They are those of a non linear sigma model on the coset space $SL(2,R)/SO(2)$. They can be written using a first order formalism. This amounts to introduce a connection taking values in the $\text{sl}_2(2)$ subalgebra. We denote by $Q$ and $P$ its components with $Q$ antisymmetric and $P$ traceless symmetric $2 \times 2$ matrices. The equations for $Q$ and $P$ are then, cf eg. ref.\([2]\) and references therein:

$$[\partial_+ + Q_+ + P_+, \partial_- + Q_- + P_-] = 0$$ \hspace{1cm} (18)

$$D_-(\rho P_+) + D_+(\rho P_-) = 0$$ \hspace{1cm} (19)

$$2(\partial_+ \sigma)(\rho^{-1} \partial_+ \rho) = tr(P_\pm P_\pm)$$ \hspace{1cm} (20)

where $D_\pm = \partial_\pm + [Q_\pm, \cdot ]$ is the covariant derivative. This system is gauge covariant with gauge group $SO(2)$. It acts as $Q_\pm \rightarrow Q_\pm + \Lambda^{-1} \partial_\pm \Lambda$ and $P_\pm \rightarrow \Lambda^{-1} P_\Lambda \Lambda$ for $\Lambda \in SO(2)$ while $\hat{\sigma}$ is gauge invariant.

The first equation \([13]\) means that $(Q_\pm + P_\pm)$ is flat; this ensures that there exists a $2 \times 2$ matrix $\mathcal{V}$, with $\mathcal{V} \in SL(2,R)$ such that $\mathcal{V} \partial_+ \mathcal{V}^{-1} = Q_\pm + P_\pm$. The matrix $\mathcal{V}$ is the matrix of the zwei-beins such that the metric coefficients $G_{ab}$ are such that:

$$G_{ab} = \rho \ (\mathcal{V} \mathcal{V})_{ab}$$

Note that $\det G_{ab} = \rho^2$ since $\det \mathcal{V} = 1$. It is often convenient to choose a gauge in which $\mathcal{V}$ is triangular:

$$\mathcal{V} = \begin{pmatrix} \Delta^{-\frac{i}{2}} & 0 \\ -N \Delta \frac{i}{2} & \Delta^\frac{i}{2} \end{pmatrix} = \begin{pmatrix} \Delta^{-\frac{i}{2}} & 0 \\ 0 & \Delta^\frac{i}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$$

It reproduces the parametrization \([14]\) for the metric. In this gauge the connection is:

$$Q_\pm + P_\pm = \frac{1}{2}(\Delta^{-1} \partial_\pm \Delta) \cdot \sigma^\pm + \Delta(\partial_\pm \nu) \cdot \sigma^\pm$$ \hspace{1cm} (21)

Plugging this parametrization into eqs.\([18]\) gives the Ernst equations.

### 3.2 Alternative parametrization of Ernst equations.

To make contact with the vertex operator approach described in the following we need to introduce an alternative parametrization of the connection. Since $Q$ is antisymmetric and $P$ traceless symmetric, we may parametrize them as follows:

$$Q_\pm = \frac{1}{2} (\partial_\pm \phi_\pm) \ [\sigma^+ - \sigma^-]$$  \hspace{1cm} (22)

$$P_\pm = \frac{1}{4}(\rho^{-1} \partial_\pm \rho) (Z_\pm e^{i\phi_\pm}) \ [\sigma^+ - i(\sigma^+ + \sigma^-)] + \text{c.c.}$$

with $\sigma^\pm, \sigma^\pm$ Pauli matrices. The dynamical fields are thus the two real fields $\phi_\pm$ and the two complex ones $Z_\pm$. Gauge transformations act by translations as $\phi_\pm \rightarrow \phi_\pm + \lambda$ with $\lambda$ a real function. In particular $Z_\pm$ as well as $\phi = \phi_+ - \phi_-$ are gauge invariant.
In this parametrization, the Ernst equations become:

\[ \partial_{\pm} Z_\pm = \frac{1}{2}(\rho^{-1}\partial_{\pm}\rho) \left( Z_\pm - e^{i\phi_\pm} Z_{\mp} \right) \]  
\[ \partial_\phi \partial_\phi = -\frac{i}{2}(\rho^{-1}\partial_\phi \rho)(\rho^{-1}\partial_\phi \rho) \left( Z_+ Z_- e^{i\phi} - Z_+ Z_- e^{-i\phi} \right) \]  
\[ 4\partial_\phi \sigma = (\rho^{-1}\partial_\phi \rho) \left( Z_+ Z_- \partial_\phi \right) \]  

These equations are gauge invariant since \( \phi \) and \( Z_\pm \) are gauge invariant. The equivalence between eqs.\((23)\) and the more usual Ernst equations follows by plugging the parametrization \((24)\) in the first order formalism eqs.\([19,19]\). Note the similarity with the sine-Gordon equation.

Let us also mention that eqs.\((23,24,25)\) may be rewritten in a bilinear form similar to Hirota’s equations. Namely, let \( \tau_0 \) and \( \tau_{\pm} \) be tau-functions defined by:

\[ \tau_0 = \exp(\sigma - \frac{i}{4}\phi), \quad \tau_+ = Z_+ \tau_0, \quad \tau_- = Z_- \tau_0 \]

then, eqs.\((23,24,25)\) are equivalent to the following bilinear equations:

\[ \tau_0(\partial_+ \tau_+) - \tau_+(\partial_0 \tau_0) = \frac{1}{2}(\rho^{-1}\partial_0 \rho) (\tau_0 \tau_+ - \tau_0 \tau_+) \]  
\[ \tau_0(\partial_+ \tau_-) - (\partial_0 \tau_-) (\partial_0 \tau_+) = -\frac{1}{4}(\rho^{-1}\partial_0 \rho)(\rho^{-1}\partial_0 \rho) \tau_+ \tau_- \]

This is proved by direct substitution. Eqs.\((26)\) are Hirota’s form of Ernst’s equations.

### 3.3 Dualisation and triangular gauge.

Let us now explain the interplay between the condition for choosing the triangular gauge and the existence of two dual solutions.

**Proposition 1.** Let \((\phi_\pm, Z_\pm)\) and \((\phi^*_\pm, Z^*_\pm)\) be the components of the two dual solutions assuming that both connections are in the triangular gauge. The duality relation is then equivalent to:

— a duality relation between the fields \( \phi_\pm \) and \( \phi^*_\pm \), which is valid only in the triangular gauge:

\[ \phi_\pm = \frac{1}{2}(\phi^* \pm \phi) = \pm \phi^*_\pm, \]  

— a duality relation between \( Z_\pm \) and \( Z^*_\pm \), which is gauge invariant:

\[ Z^*_+ + Z_+ = -e^{i(\phi + \phi^*)/2} \]  
\[ Z^*_- + Z_- = -e^{-i(\phi - \phi^*)/2} \]

**Proof.** Indeed, let \((\phi_\pm, Z_\pm)\) be a solution of the Ernst equations \((23,24)\). Imposing the triangular gauge as in eq.\((22)\) demands that the coefficient of \( Q_\pm + P_\pm \) along \( \sigma^+ \) be zero. In the parametrization \((22)\) this amounts to impose:

\[ 2\partial_\pm \phi_\pm = i(\rho^{-1}\partial_\pm \rho) \left( Z_\mp e^{i\phi_\mp} - Z_\mp e^{-i\phi_\mp} \right) \]

Let us now define the dual solution \( \phi^* \) and \( Z^*_\pm \) by the formulas \((27)\) and \((28)\). Notice that when parametrized in terms of \( \phi \) and \( \phi^* \), the condition \((24)\) becomes:

\[ \partial_\pm (\phi^* \pm \phi) = i(\rho^{-1}\partial_\pm \rho) \left[ Z_\mp e^{i(\phi^* + \phi)} - Z_\mp e^{-i(\phi^* + \phi)} \right] \]

One has to prove that \((\phi^*, Z^*_\pm)\) are solutions of eqs.\((23,24)\). Consider eq.\((23)\) for \( Z^*_+ \). Using the definition of \( Z^*_+ \) and eq.\((23)\) satisfied by \( Z_+ \) and eq.\((29)\) for \( \phi + \phi^* \), we get:

\[ \partial_+ Z^*_+ = -\frac{1}{2}(\rho^{-1}\partial_+ \rho) \left[ Z_+ - e^{-i\phi} Z_- + \left( Z_- e^{i\phi^*} - Z_- e^{-i\phi^*} \right) e^{-i(\phi^* + \phi)} \right] \]

\[ = \frac{1}{2}(\rho^{-1}\partial_+ \rho) \left[ Z^*_+ - Z^*_+ e^{-i\phi^*} \right] \]
This shows that $Z_\ast^\pm$ is the solution. The equation for $Z_\ast^\pm$ is proved in a similar way. The equation for the dual field $\phi^\ast$ is proved by taking derivatives of eq.(29). Finally, since the transformation (28) is an involution the triangular gauge condition for $\phi^\ast$ follows from that of $\phi$, eq.(29).

Remark that the duality relation (27) and (28) are purely algebraic and that eq.(27) is similar to the usual abelian T-duality of string theory. This proposition also shows that solving for the triangular gauge or for the dual solution are equivalent problems.

Using the defining relation between $\phi, Z_\pm^\pm$ and the connection $Q_\pm^+ + P_\pm^+$ in the triangular gauge, one may translate the duality (27,28) on the connection as:

$$Q_\pm^+ + P_\pm^+ = \frac{1}{2} (\Delta^{-1} \partial_{\pm} \Delta) \cdot \sigma^z + \Delta(\partial_{\pm} N) \cdot \sigma^-$$

$$\rightarrow \quad Q_\pm^* + P_\pm^* = -\frac{1}{2} (\rho^{-1} \partial_{\pm} \rho) \cdot \sigma^z - \frac{1}{2} (\Delta^{-1} \partial_{\pm} \Delta) \cdot \sigma^z \pm \Delta(\partial_{\pm} N) \cdot \sigma^-$$

This is clearly equivalent to the Kramer-Neugebauer duality (14).

4 Integrability and vertex operators.

The aim of this section is to recall the use of vertex operators for solving the Ernst equations following the method initiated in ref.[16]. This method was based on an application of the dressing method [17]. We have written this section in a down-to-earth way by just presenting propositions which give the rules for computing solutions of the Ernst equation using vertex operators. These rules are analogous of the famous Kyoto formula [13] for the tau function of the KP hierarchy in terms of fermions or vertex operators. The logic which allows us to go from the dressing method to these vertex operator formulas is recalled in Appendix A. We shall also describe the dual pairs of vertex operators corresponding to dual pairs of solutions

4.1 Vertex operators and tau functions.

Vertex operators are exponentials of a free bosonic field acting on an auxiliary Fock space. They may be used to find solutions of the Ernst equations in a way similar to their use for solving the KP hierarchy as described the Kyoto school [13]. Let us denote by $\hat{X}(w)$ the bosonic field:

$$\hat{X}(w) = -i \sum \left. \frac{w^n}{n} \right|_{n \, \text{odd}}$$

The operators $p_n$ generate a Fock space, we denote by $|0\rangle$ its vacuum: $p_n|0\rangle = 0$ for $n > 0$. For any number $u$, let $W_u(w)$ be the vertex operators:

$$W_u(w) = \exp(-iu\hat{X}(w)) :$$

The double dots refer to the normal ordering which amounts to move to the right the oscillators $p_n$ with $n$ positive. The parameter $w$ is called the spectral parameter. The Virasoro algebra acts on the Fock space generated by the $p_n$. The Virasoro generators $L_n$ are represented by

$$\sum_n (L_n - \frac{1}{16} \delta_{n,0}) w^{-2n-2} = -\frac{1}{4} : (\partial_w \hat{X})^2 :$$

The algebraic dressing method applied to the Ernst equations leads to the following result:

Proposition [16]. Let $\Psi_0$ be defined as:

$$\Psi_0 = \left( \rho + z + \frac{1}{2} \rho \right)^{L_0 - L_1} \left( \rho + z + \frac{1}{2} \rho \right)^{L_0 - L_{-1}}$$

Let $g$ be any product of vertex operators of the following form:

$$g = \text{const.} \prod_{p=1}^{m} W_{u_p}(w_p) \cdot \prod_{j=1}^{n} (1 + iy_j W_2(w_j))$$

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where \((y_j, w_j)\) and \((u_p, w_p)\) are \(2(m + n)\) real parameters, then the fields \((\bar{\sigma}, \phi, Z_\pm)\), defined by the following expectation values,

\[
\exp\left(\bar{\sigma} - \frac{i}{4} \phi\right) = \langle \Psi_0 g \Psi_{0}^{-1} \rangle = \langle g \rangle_{z,\rho}
\]

are solutions of equations \(24,24,24\).

The above equations serve as the definition of \(\langle g \rangle_{z,\rho}, \langle gp_{-1} \rangle_{z,\rho}\) and \(\langle p_{1} g \rangle_{z,\rho}\). This is the notation used in Section 2, eqs.\(\text{(34)}\). These expectation values are the tau-functions of the model.

To make sense the vertex operators in eq.\(\text{(34)}\) have to be ordered in decreasing order of the \(|w_j|\)'s. Modification of this order may be done by analytic continuation in the spectral parameters.

To compute these expectation values one needs to know how to conjugate vertex operators \(W_u(w)\) with \(\Psi_0\). One has \(\text{(37)}\):

\[
\Psi_0 \cdot W_u(w_j) \cdot \Psi_0^{-1} = X_j^{u^2/4} \cdot W_u(\mu_j)
\]

with \(X_j\) and \(\mu_j\) defined in eq.\(\text{(35)}\) with the parameter \(z_j = \frac{w_j^2 + 1}{w_j^2 - 1}\). Notice in particular that \(\Psi_0 \cdot W_u(1) \cdot \Psi_0^{-1} = \rho^{\frac{u^2}{2}} W_u(1)\). With this result in hand, the computation of the expectation values \(\text{(34,35,36)}\) is reduced to the computation of expectation values of vertex operators. As recalled in Appendix B, this is done using the usual Wick’s theorem. For example:

\[
\langle g \rangle_{z,\rho} = \left(\prod_{p=1}^{m} X_p^{u_p^2/4}\right) \left(\prod_{p=1}^{m} W_{u_p}(\mu_p) \cdot \prod_{j=1}^{n} \left(1 + iy_j W_2(\mu_j)\right)\right) = \Omega \cdot \tau(Y|\mu_j)
\]

with \(\Omega\) and \(\tau(Y|\mu)\) defined in eqs.\(\text{(11,12)}\) above. The expectation values \(\langle gp_{-1} \rangle_{z,\rho}\) and \(\langle p_{1} g \rangle_{z,\rho}\) are computed similarly using the formula recalled in Appendix B. They may be obtained from the previous expression for \(\langle g \rangle_{z,\rho}\) by replacing each monome \(X_{k_1} \cdots X_{k_p}\) by,

\[
X_{k_1} \cdots X_{k_p} \rightarrow X_{k_1} \cdots X_{k_p} \left(\sum_m u_{k_m} \mu_{k_m}^{-1}\right) \text{ for } \langle gp_{-1} \rangle_{z,\rho}
\]

\[
X_{k_1} \cdots X_{k_p} \rightarrow - X_{k_1} \cdots X_{k_p} \left(\sum_m u_{k_m} \mu_{k_m}\right) \text{ for } \langle p_{1} g \rangle_{z,\rho}
\]

in the formula \(\text{(38)}\).

### 4.2 Vertex operators and duality.

Two dual solutions correspond to two dual products of vertex operators that we shall denote by \(g\) and \(g^*\). The duality relations \(\text{(27)}\) and \(\text{(28)}\) can be translated into quadratic relations for the expectation values of these operators similar to the Hirota equations. Namely:

\[
\langle g \rangle_{z,\rho} \langle g^* \rangle_{z,\rho} + \langle g^* \rangle_{z,\rho} \langle gp_{-1} \rangle_{z,\rho} = -\langle g \rangle_{z,\rho} \langle g \rangle_{z,\rho} \langle gp_{-1} \rangle_{z,\rho}
\]

\[
\langle g \rangle_{z,\rho} \langle p_{1} g \rangle_{z,\rho} + \langle g^* \rangle_{z,\rho} \langle g^* \rangle_{z,\rho} \langle p_{1} g \rangle_{z,\rho} = +\langle g \rangle_{z,\rho} \langle g \rangle_{z,\rho} \langle gp_{-1} \rangle_{z,\rho}
\]

**Proposition 2.** Pairs of solutions of the above duality relations, eqs.\(\text{(40,41)}\), are provided by the following pairs of vertex operators:

\[
g = \text{const.} \prod_{p=1}^{m} W_{u_p}(w_p) \cdot \prod_{j=1}^{n} \left(1 + iy_j W_2(w_j)\right)
\]

\[
g^* = \text{const.} \cdot W_{-1}(1) \prod_{p=1}^{m} W_{-u_p}(w_p) \cdot \prod_{j=1}^{n} \left(1 + iy_j W_{-2}(w_j)\right)
\]
Up the multiplication by $W_{-1}(1)$ the dual vertex operator is obtained by the charge $u$ into $-u$. The constant prefactors in eqs. (12, 13) are irrelevant.

**Proof.** It relies on an identity for the tau-functions proved in ref. [22]. Let us sketch the proof of eq. (11) for $m = 0$. Recall eq. (38) for $\langle g \rangle_{z,\rho}$ and eq. (47) below for $\langle g^* \rangle_{z,\rho}$:

$$
\langle g \rangle_{z,\rho} = \tau(Y_j|\mu_j) \quad \text{and} \quad \langle g^* \rangle_{z,\rho} = \rho^{\dagger} \tau(B_j Y_j|\mu_j)
$$

(44)

From eqs. (38) one infers that the expectation values $\langle gp_{-1} \rangle_{z,\rho}$ and $\langle g^* p_{-1} \rangle_{z,\rho}$ may be written in terms of derivatives of tau-functions. Namely:

$$
\langle gp_{-1} \rangle_{z,\rho} = \frac{\partial}{\partial \mu_0} \tau(B_{0j} Y_j|\mu_j)|_{\mu_0=0}
$$

$$
\rho^{\dagger} \langle g^* p_{-1} \rangle_{z,\rho} = -\tau(B_j Y_j|\mu_j) - \frac{\partial}{\partial \mu_0} \tau(B_{0j} B_j Y_j|\mu_j)|_{\mu_0=0}
$$

with $B_{0j} = \frac{\mu_0 + \mu_j}{\mu_0 - \mu_j}$. Eq. (11) may then be written as a bilinear identity for the tau-functions. The later follows by adding and expending in power of $\mu_0$ the following two relations proved in [22]:

$$
\tau(B_{0k} B_{1k} Y_k)\tau(Y_k) + \tau(B_{0k} B_{1k} Y_k)\tau(Y_k) = \tau(B_{0k} Y_k)\tau(B_{1k} Y_k) + \tau(B_{0k} Y_k)\tau(B_{1k} Y_k)
$$

(45)

$$
B_{10} \left[ \tau(B_{0k} B_{1k} Y_k)\tau(Y_k) - \tau(B_{0k} B_{1k} Y_k)\tau(Y_k) \right] = \tau(B_{0k} Y_k)\tau(B_{1k} Y_k) - \tau(B_{0k} Y_k)\tau(B_{1k} Y_k)
$$

(46)

with $B_{1k} = B_k$ and $B_{10} = \frac{1-\mu_0}{1+\mu_0}$. The proofs of the general case $m \neq 0$ as well as eq. (11) are similar. □

In eq. (11) the order of the operators matters: the operator $W_{-1}(1)$ has to be on the left. When changing the order of the operators one has to take into account their commutation relations, i.e. $W_{-1}(1)$ anticommutes with $W_{-2}(w)$.

Expectation values of the dual vertex operators may be evaluated using the conjugation formula (37) and Wick’s theorem as explained for $\langle g \rangle_{z,\rho}$. One gets:

$$
\langle g^* \rangle_{z,\rho} = \langle \Psi_0 g^* \Psi_0^{-1} \rangle = \rho^{\dagger} \Omega^* \cdot \tau(B_j Y_j|\mu_j)
$$

(47)

as in eq. (5).

### 5 Algebraic computation of the metric coefficients.

The previous section made precise the relation between the two dual vertex operators $g$ and $g^*$. It allows us to compute algorithmically the gauge invariant fields $(\tilde{\sigma}, \phi, Z_\pm)$ and their duals $(\tilde{\sigma}^*, \phi^*, Z^*_\pm)$. This is not quite the final answer for the metric since to go from $(\phi, Z_\pm)$ to the metric coefficients $G_{ab}$, or $\Delta$ and $N$, one needs to impose the triangular gauge and then integrate the connection $Q_\pm + P_\pm$ to obtain the zwei-bein $\mathcal{V}$.

Imposing the triangular gauge requires solving eq. (29). This is a non-linear problem which was actually not solved in [11]. We shall now solve it using our knowledge on the dual pairs of vertex operators, eqs. (42-43), and on the link between the duality and the triangular gauge, eqs. (27, 28). We will then be able to use formula (43) to compute the metric coefficients $G_{ab}$.

#### 5.1 Vertex operator representation and factorization.

In order to be able to apply formula (43) we need to make a small detour into group theory in order to explain the relation between factorization problem in affine Kac-Moody group and vertex operators.

This relation arises because the vertex operators (31) may be used to represent the $sl(2, R)$ affine Kac-Moody algebra on the Fock space. The commutation relations of the $sl(2, R)$ affine Kac-Moody algebra are:

$$
[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \frac{n}{2} tr(xy) \delta_{n+m,0}
$$

(48)

The affine Kac-Moody $sl(2, R)$ algebra is twisted in the sense that its elements $x \otimes t^n$ are such that $x \in so(2)$ if $n$ is even while $x$ is an $2 \times 2$ traceless symmetric matrix if $n$ is odd. We are actually considering the semi
The crossed Lie bracket is: \([L_n, x \otimes t^m] = -\frac{m}{2} x \otimes t^{m+2n}\).

The representation of the affine algebra on the Fock space is specified by the following relations \[\text{Eq.(51)}\] is only valid in the triangular gauge. Let us explain in more detail how this gauge choice fixes the factorization of \(g\) as \(g^{-1}g_+\). Since \(g_{\pm} \in B_{\pm}\), one has:

\[g_{\pm} = \exp \left( \pm \frac{\varphi}{2} \right) \exp \left( -\frac{\varphi_{\pm}}{2} (\sigma^+ - \sigma^-) \right) \times [\text{degree} \geq 1]\]

For a given group element \(g = g^{-1}g_+\) only the difference \(\varphi_+ - \varphi_-\) is fixed. To translate \(\varphi_{\pm}\) keeping this difference fixed amounts to multiply \(g_{\pm} \rightarrow hg_{\pm}\) by elements \(h \in SO(2)\). The link between this freedom and the \(SO(2)\) gauge freedom of the Ernst equation relies on the fact \[\varphi_\pm\] coincide with the fields \(\varphi_\pm\) at \(z = 0, \rho = 1\). Thus in the triangular gauge,

\[\varphi_\pm = \frac{1}{2}(\varphi^* \pm \varphi)\]

with \(\varphi\) and \(\varphi^*\) equal to \(\Phi\) and \(\Phi^*\) at \(z = 0, \rho = 1\). This follows from eq.\[\text{Eq.(27)}\]. Since \(\Psi_0 = 1\) at \(z = 0, \rho = 1\), they may be evaluated using eq.\[\text{Eq.(34)}\]:

\[\exp \left( i\frac{\varphi}{2} \right) = \frac{\langle g \rangle}{\langle g \rangle}, \quad \exp \left( i\frac{\varphi^*}{2} \right) = \frac{\langle g^* \rangle}{\langle g^* \rangle}\]

Here, the expectation values are the vacuum expectation values, without insertions of \(\Psi_0\).
5.2 Factorization and dualisation.

The factorization problem in the affine Kac-Moody group may be understood as a kind of normal ordering. So when considering the vertex operator representation one has to face two different normal orderings: the one associated to bosonic oscillators $p_n$ and the group theoretical one. We shall now explain the link between them for the vertex operators $\tilde{W}_g$.

First, consider vertex operators $\tilde{W}_g$. Since $(\sigma^z \otimes t^n)$ is represented by $p_n$, we may consider them as elements of the affine group. Namely:

$$ W_u(w) = \exp \left( -\frac{u}{2} \sigma^z \log \left( \frac{1 + w/t}{1 - w/t} \right) \right) \cdot \exp \left( \frac{u}{2} \sigma^z \log \left( \frac{1 + t/w}{1 - t/w} \right) \right) $$

(54)

The last equation serves as definition of $W_u(w)_{\pm}$, elements of the Borel subgroups $B_{\pm}$. Thus, the two normal orderings coincide for these group elements.

Consider now the product of vertex operators of the form $\prod_j (1 + iy_j W_2(w_j))$. Since $W_2(w)$, which are generating functions representing elements of the affine algebra, are nilpotent inside any correlation functions, ie. $W_2(w)W_2(w) = 0$, these products are representations of elements of the Kac-Moody group. As shown in ref. [22], these products may be factorized in the affine Kac-Moody group. More precisely, let $g_{\pm}(j)$ be the elements of the Borel subgroups $B_{\pm}$ defined by

$$ g_{\pm}(j) = \exp \left( \pm \frac{r_j}{2} k \right) \cdot \exp \left( \frac{v_j}{2} (\sigma^+ - \sigma^-) \right) \cdot \exp \left( \frac{s_j}{2} \tilde{E}_{\pm}(w_j) \right) $$

(55)

with

$$ \tilde{E}_{\pm}(w) = \pm \left[ (\sigma^+ - \sigma^-) \otimes \left( \frac{1 + (t/w)^{\pm 2}}{1 - (t/w)^{\pm 2}} - (\sigma^+ + \sigma^-) \otimes \left( \frac{2(t/w)^{\pm 1}}{1 - (t/w)^{\pm 2}} \right) \right) \right] $$

(56)

Then, for $k = 1, \ldots, n$ one has [22],

$$ g_{\pm}(1) \cdots g_{\pm}(k) = \prod_{j=1}^k (1 + iy_j W_2(w_j)) $$

(57)

Eq. (57) is valid in the Fock space representation. The relation between the parameters $(s_j, r_j, v_j)$ and $(y_j, w_j)$ is explained in the following proposition.

We can then solve for the factorization problem:

**Proposition 3.** For $g$ and $g^*$ the dual vertex operators [42] and [43], then:

$$ W g = g_{\pm}^{-1} W_{2}(1) g_+ = (-ia + b W_{2}(1)) \cdot \prod_{p=1}^m W_{u_p}(w_p) \cdot \prod_{j=1}^n (1 + iy_j W_2(w_j)) $$

(58)

$$ W g^* = g_{\pm}^{-1} W_{2}(1) g^*_+ = (-ia^* + b^* W_{1}(1)) \cdot \prod_{p=1}^m W_{-u_p}(w_p) \cdot \prod_{j=1}^n (1 + iy_j W_{-2}(w_j)) $$

(59)

in the triangular gauge. Since the Ernst potential is defined up to a multiplicative constant and a constant translation on $N$, $a$ and $b$ are irrelevant when computing the metrics.

**Proof.** This relies on the relation [22] between the parameters $(y_j, w_j)$ involved in the vertex operators and the parameters $(s_j, r_j, v_j)$ in the group elements $g_{\pm}(j)$ such that:

$$ g_{\pm}(1) \cdots g_{\pm}(n) = \prod_{j=1}^n (1 + iy_j W_2(w_j)) $$

(60)

Let us introduce the obvious notation $\varphi_j$ and $\varphi^*_j$ for $j \leq n$ by

$$ \varphi_j = -2 \sum_{k=1}^j s_k \quad , \quad \varphi^*_j = -2 \sum_{k=1}^j v_k $$

11
such that \( s_j = -\frac{1}{2}(\varphi_j - \varphi_{j-1}) \) and similarly for \( v_j \). Of course \( \varphi_{\pm} = \frac{1}{2}(\varphi_n^* \pm \varphi_n) \). First \((s_j, r_j)\) are recursively determined as functions of \((y_k, w_k)\) with \( k \leq j \) by computing the expectation values of eq.\((67)\):

\[
\exp \left( \sum_{k=1}^{j} (r_k + i \frac{s_k}{2}) \right) = \left( \prod_{k=1}^{j} (1 + i y_k W_2(w_k)) \right)
\]

These equations determine \( \varphi_j(y_k) \) as functions of \( y_k, k \leq j \). The \( v_j \)'s are then given by \((22)\):

\[
\varphi_j^*(y_k) = \varphi_j(\beta_{j+1,k} y_k) \quad \text{with} \quad \beta_{j+1,k} = \frac{w_{j+1} - w_k}{w_{j+1} + w_k}
\]

This leaves \( v_n \), which actually cancels in eq.\((20)\), underdetermined. However, \( v_n \), or equivalently \( \varphi_n^* \), is fixed once we impose the triangular gauge. Indeed the triangular gauge condition written as in eq.\((58)\) and formulas \((12,13)\) for the dual vertex operators leads to:

\[
\varphi_n^*(y_j) = \varphi_n(\beta_j y_j) \quad \text{with} \quad \beta_j = \frac{1 - w_j}{1 + w_j}
\]

(61)

This allows us to go one step further in the recursion relation \((57)\) by inserting one extra vertex operator with spectral parameter \( w \) equals to 1. Let \( g_\pm(n+1) \) be such that \( g_-^{-1}(n+1) g_+(n+1) = \hat{a} + i \hat{b} W_2(1) \) with \( \hat{a}, \hat{b} \) numbers. The triangular gauge condition \((1)\) then implies:

\[
g_-^{-1}(1) \cdots g_-^{-1}(n) \cdot (\hat{a} + i \hat{b} W_2(1)) \cdot g_+(n) \cdots g_+(1) = (a + i b W_2(1)) \cdot \prod_{j=1}^{n} (1 + i y_j W_2(w_j))
\]

with \( a, b \) functions of \( \hat{a}, \hat{b} \). Taking \( \hat{a} = 0 \) and \( \hat{b} = 1 \), this proves eq.\((18)\). The dual equation \((59)\) is proved similarly.

Once the operators \( g_-^{-1}(1) g_+ \) and \( g_-^{-1}(1) g^*_+ \) are expressed in terms of the product of vertex operators, it easy to conjugate them with \( \Psi_0 \) using eq.\((17)\). One may then evaluate \( \langle W g \rangle_{z,\rho} \) and \( \langle W g^* \rangle_{z,\rho} \) using Wick’s theorem, cf. Appendix B. Of course one gets eq.\((3)\). This ends the algebraic proof of the determinant formula for the metrics.

Remark that eq.\((13)\) for the tau-functions with \( \mu_0 = 1 \) implies:

\[
\text{Re} \left( \langle g \rangle_{z,\rho} \langle W g \rangle_{z,\rho} \right) = \sqrt{\rho} \left| \langle g^* \rangle_{z,\rho} \right|^2
\]

This shows that \( G_{ab}^* \) defined in eqs.\((13,14)\) satisfies \( \det G_{ab}^* = \rho^2 \). It provides a non-trivial check of the construction.

### 5.3 Dualisation in the dressing group.

The dressing group is the group whose elements are the pairs \((g_-, g_+)\) factorizing \( g \) as \( g = g_- g_+ \). It is different from the affine \( SL(2, \mathbb{R}) \) Kac-Moody group since their multiplication laws do not coincide, \((17)\). In the dressing group the product is given by \( (g_-, g_+) (h_-, h_+) = (g_- h_-, g_+ h_+) \).

The solutions we have obtained should actually be labeled by elements of the dressing group since this is the solution generating group. As a consequence, the duality between the vertex operators \((22)\) and \((33)\) should be thought of as a duality in the dressing group. Writing the vertex operator \((22)\) in the dressing group amounts factorizing them according to the rules explained in the previous section:

\[
g_-^{-1} g_+ = \prod_{p=1}^{m} W_{u_p}(w_p)_-^{-1} \cdot \tilde{g}_-^{-1}(1) \cdots \tilde{g}_-^{-1}(n) \cdot \tilde{g}_+(n) \cdots \tilde{g}_+(1) \cdot \prod_{p=1}^{m} W_{u_p}(w_p)_+
\]

(62)

where the middle term corresponds the factorization of \( \prod_{j=1}^{n} (1 + i \tilde{y}_j W_2(w_j)) \) according to eq.\((17)\),

\[
\tilde{g}_-^{-1}(1) \cdots \tilde{g}_-^{-1}(n) \cdot \tilde{g}_+(n) \cdots \tilde{g}_+(1) = \prod_{j=1}^{n} (1 + i \tilde{y}_j W_2(w_j))
\]
Here \( \hat{y}_j = y_j \prod_{p} \beta_j^{\nu_p} \). Similarly the dual vertex operator is factorized as:

\[
g_{-}^{*} g_{+} = W_{-1}(1)^{-1} \cdot \prod_{p=1}^{m} W_{-u_p}(w_p)^{-1} \cdot \hat{g}_{-}^{*}(1) \cdots \hat{g}_{-}^{*}(n) \cdot \hat{g}_{+}^{*}(n) \cdots \hat{g}_{+}^{*}(1) \cdot \prod_{p=1}^{m} W_{-u_p}(w_p)_+ \cdot W_{-1}(1)_+ \tag{63}
\]

where

\[
\hat{g}_{-}^{*}(1) \cdots \hat{g}_{-}^{*}(n) \cdot \hat{g}_{+}^{*}(n) \cdots \hat{g}_{+}^{*}(1) = \prod_{j=1}^{n} (1 + i \hat{y}_j W_{-2}(w_j))
\]

with \( \hat{y}_j = \beta_j \hat{y}_j \).

In eqs. (62,63) the dual elements \((g_-, g_+)\) and \((g_{-}^{*}, g_{+}^{*})\) are written as elements of the dressing group (and not in a particular representation). It is then clear that the map from \((g_-, g_+)\) to \((g_{-}^{*}, g_{+}^{*})\) is an involution. However it is not a group automorphism. It is unfortunate and frustrating that we do not know how to write this involution in more group theoretical way without relying on these particular elements. A better group theoretical understanding of the dualisation, ie. of the relation between \(g\) and \(g^*\), will provide a way to decipher how general the duality property and the involution trick are and whether they apply to other integrable systems.

6 A few examples.

Since our aim was to describe the use of vertex operators for solving Ernst equations and not to describe the physical properties of the solutions, we shall only discuss a few examples (which are actually already known in the literature, cf eg. [6, 7, 8, 9] and [12] and references therein).

6.1 Diagonal solutions.

Diagonal solutions correspond to solutions for which \(G_{ab}\) is diagonal, ie. \(N = 0\). They are obtained by imposing \(y_j = 0\) in the parametrization of Section 2. They correspond to vertex operators of the form

\[
g_{-}^{*} g_{+} = \prod_{p=1}^{m} W_{u_p}(w_p) \tag{64}
\]

They depend on the \(2m\) parameters \((z_p, u_p)\); recall that \(z_p = \frac{w_p^2 + 1}{w_p} \). Here and in the following examples, we drop insignificant multiplicative constants in front of the vertex operators.

The simplest of such solutions is the well known Khan-Penrose metric [6]. It describes the interaction region of two plane impulsive gravitational waves having their polarization vectors aligned. For more details on this subject, see eg ref. [12]. It corresponds to the following set of parameters, \(\{(z_p, u_p)\} = \{(1, 1), (-1, 1)\}\), or equivalently to the following vertex operator:

\[
W_1(\infty) W_1(0)
\]

Notice that this choice of singular values for \(z_p\) leads to null values for \(X_p\). However, we may reabsorb these singular constant factors in the normalisation since Einstein’s equations determine the conformal factor only up to an additive constant. In order to have the formula closed to those which may be found in the literature, we introduce two new positive variables \(u\) and \(v\) defined by:

\[
\rho = 1 - u^2 - v^2 \quad \text{and} \quad z = v^2 - u^2
\]

\(^1\)Note however that the relation between \(g^*\) and \(g\) may be written as \(g^* = W_{-1}(1)T(g)\) with \(T\) the automorphism of \(sl(2, R)\) fixing \(so(2)\) and multiplying \(2 \times 2\) traceless symmetric matrices by minus one.
The conformal factor is determined by computing the expectation value
\[ \langle g \rangle_{z,\rho} \]
with Einstein’s equations allow to adjust metrics elements. In particular, we can add a constant to the imaginary identities may be found in Appendix C.

\[ \rho \]
\[ G_{22} \]
\[ G_{12} \]
\[ \langle W g^* \rangle_{z,\rho} \]
\[ \langle g^* \rangle_{z,\rho} \]
\[ \rho \]
\[ (1 - p\xi)^2 + q^2 w^2 - i q (-2\xi(1 - w^2) + 2p(\xi^2 - w^2) + \frac{1}{1 - p^2}(p^2 w^2 - \xi^2) + (1 - w^2)) \]
The conformal factor is determined by computing the expectation value \( \langle g \rangle_{z,\rho} \). With
\[ X = (1 - p\xi)^2 + q^2 w^2 \]
\[ Y = 1 - p^2 \xi^2 - q^2 w^2 \]
the line element can be written as:
\[ ds^2 = -\frac{X}{2} \left( \frac{d\xi^2}{1 - \xi^2} - \frac{dw^2}{1 - w^2} \right) + \frac{Y}{X} \left( dx - \frac{2q}{p(p + 1)} dy \right)^2 + \frac{4q(1 - w^2)(1 - p\xi)}{pX} \left( dx - \frac{2q}{p(p + 1)} dy \right) dy + \frac{(1 - w^2)}{p^2 X} (1 - p\xi)^2 + q^2 + p^2 q^2 (1 - \xi^2)(1 - w^2) dy^2 \]
This is the Chandrasekhar-Xanthopoulos solution written in the same form as in \([12]\).
To illustrate the duality formulas, we compute the dual of this metric. It is the Nuktu-Halil solution \([5]\). The dual vertex operators are:

\[
\langle g \rangle_{z,\rho} = \langle g^* \rangle_{z,\rho} = 1 + p\xi + iqw
\]

from which the metric can be computed.

These solutions may be seen as members of a larger class of solutions, the so-called Ernst family of solutions \([6]\). These correspond to the following sets of parameters:

\[
\{(1, -1), (+\infty, -n), (-1, -1)\} \quad \text{and} \quad \left\{ \left( 1, -\frac{q - q'}{4(p + p')} \right), \left( -1, \frac{q + q'}{4(p + p')} \right) \right\}
\]

with \(p^2 + q^2 = 1\) and \(p'^2 + q'^2 = 1\). Introducing \(p, q\) and \(p', q'\) is just a convenient way of parametrizing \(y_1\) and \(y_{-1}\). The vertex operators are:

\[
g = W_{-n}(\infty) W_{-1}(1) W_{-1}(0) \left( 1 + i \frac{q - q'}{4(p + p')} W_2(\infty) \right) \left( 1 + i \frac{q + q'}{4(p + p')} W_2(0) \right)
\]

The expression of the metric is a little lengthy. For the conformal factor we have

\[
e^{\tilde{z}^2} (d\rho^2 - dz^2) = \langle g \rangle_{z,\rho}^2 (d\rho^2 - dz^2)
\]

\[
= \frac{1}{4\rho^2} \left( \frac{d\xi^2}{1 - \xi^2} - \frac{dw^2}{1 - w^2} \right) \left[ (1 - \xi^2)((p + p')^2(1 - \xi) + (p' - p)^2(1 + \xi)^n) + (1 - w)^2((q + q')^2(1 - w) + (q' - q)^2(1 + w)^n) \right] + 2(q^2 - q'^2)(\xi^2 - w^2)
\]

For the Ernst potential we have

\[
\rho G_{22} + iG_{12} = \frac{\langle Wg^* \rangle_{z,\rho}}{\langle g^* \rangle_{z,\rho}} = \rho^n \frac{A}{B}
\]

with

\[
A = (1 - \xi^2)^{\frac{1}{2}} \left[ (p + p') \left( \frac{1 - \xi}{1 + \xi} \right)^{n+1} + (p' - p) \left( \frac{1 + \xi}{1 - \xi} \right)^{n+1} \right]
\]

\[
+ i(1 - w^2)^{\frac{1}{2}} \left[ (q + q') \left( \frac{1 - w}{1 + w} \right)^{n+1} + (q' - q) \left( \frac{1 + w}{1 - w} \right)^{n+1} \right]
\]

and

\[
B = (1 - \xi^2)^{\frac{1}{2}} \left[ (p + p') \left( \frac{1 - \xi}{1 + \xi} \right)^{n+1} + (p' - p) \left( \frac{1 + \xi}{1 - \xi} \right)^{n+1} \right]
\]

\[
+ i(1 - w^2)^{\frac{1}{2}} \left[ (q + q') \left( \frac{1 - w}{1 + w} \right)^{n+1} + (q' - q) \left( \frac{1 + w}{1 - w} \right)^{n+1} \right]
\]

These are the solutions found in \([5]\). The dual metrics can be obtained by changing \(n\) into \(1 - n\) and exchanging \((p, q)\) with \((p', q')\). This may be checked by comparing the dual vertex operators.

### 6.3 Analytic continuation.

Stationary axis symmetric solutions of the vacuum Einstein equations may formally be obtained by analytic continuation:

\[
\rho \to i\rho, \quad x \to i\varphi, \quad z \to z, \quad y \to i\tau
\]
The (dual) metric then reads:

\[
ds_2^2 = -\Delta^{-1}(d\tau + \omega d\varphi)^2 + \Delta e^{z} (dz^2 + d\rho^2) + \rho^2 \Delta d\varphi^2
\]

with \( \omega = N^* \). However the reality conditions for axis symmetric solutions are more involved.

By analytic continuation, the Khan-Penrose solution is mapped into the Schwarzschild solution. Let’s see what happen in the case of the Chandrasekhar-Xanthopoulos solution. Using the standard parametrisation recalled in Appendix C, cf eg. [12], one obtains the line element of the Kerr solution:

\[
2M^2 ds^2 = -\left(1 - \frac{2Mr}{R^2}\right) d\tau^2 + \frac{4aMr}{R^2} \sin^2 \theta d\phi d\tau
\]

\[
+ \left(r^2 + a^2 - \frac{2a^2 Mr}{R^2}\right) \sin^2 \theta d\phi^2 + R^2 \left(\frac{1}{D} dr^2 + d\theta^2\right)
\]

with \( R^2 = r^2 + a^2 \cos^2 \theta \) and \( D = r^2 - 2Mr + a^2 \). Here the domain of validity of the metric is such that \(-\Delta a^2 < D \leq 0\), which corresponds to the region inside the ergo-sphere. However, using the analytic continuation described above, eq.(68), the variables \( \xi \) and \( w \) become

\[
\xi = \frac{1}{2} \left(\sqrt{(1-z)^2 + \rho^2} + \sqrt{(1+z)^2 + \rho^2}\right)
\]

\[
w = \frac{1}{2} \left(\sqrt{(1-z)^2 + \rho^2} - \sqrt{(1+z)^2 + \rho^2}\right)
\]

The domain is now such that \( D \geq 0 \). The solution thus describes the asymptotically flat exterior Kerr solution, which is stationary axis symmetric.

Other physically realistic axis symmetric solutions would probably require infinite sets of parameters \((z_p, u_p)\) and \((z_j, y_j)\), i.e. infinite products of vertex operators. In such a case the metric coefficients could be expressed in terms of Fredholm determinants. However the analysis of such cases is beyond the scope of this paper.

### 6.4 Belinskii-Zakharov approach : one soliton case.

To make contact with the Belinskii-Zakharov approach, we describe in more detail the vertex operator construction of the one soliton solution found in [4]. The Belinskii-Zakharov approach [4] starts from Kasner’s metric as a seed solution. Since these Kasner’s solutions are obtained from the vertex operator with parameter \(\{(z_p, u_p)\} = \{(+\infty, u)\}\), we insert vertex operators whose sets of parameters are now

\[
\{(z_p, u_p)\} = \{(+\infty, u), (-1, -1)\} \quad \text{and} \quad \{(z_j, y_j)\} = \{(-1, y)\}
\]

They correspond to

\[
g = W_u(1)W_{-1}(0) (1 + iyW_2(0))
\]

Given these values of the parameters \( z_j \) one has to compute the values of \( \mu_j \) using eq.(4). To compare with ref[4] let us introduce the same notation as this reference:

\[
e^\pm \equiv \frac{1}{1 + \mu_{-1}} = \frac{(z + 1)}{\rho} - \sqrt{\left(\frac{(z + 1)^2}{\rho^2} - 1\right)}
\]

Using this parametrization and appropriate normalization, we find using formulas of Section 1 the metric:

\[
ds^2 = \frac{\rho^2 \cosh(qr + C)}{\sqrt{(z + 1)^2 + \rho^2}} \left((-\rho d^2 + dz^2) + \cosh \left(\frac{1}{2} + q\right) r + C\right) \rho^{1 + 2q} dr^2
\]

\[
- \frac{2 \sinh \left(\frac{1}{2}\right)}{\cosh(qr + C)} \rho d\tau d\phi + \cosh \left(\frac{1}{2} - q\right) r - C\right) \rho^{1 - 2q} d\tau d\phi
\]

with \( e^C = 4y \) (choosing \( y \) positive) and \( u = 2q \). This coincides with the one soliton solution of Belinskii and Zakharov.
So the formulas we have found can be used in the same way as those found following the Belinskii-Zakharov method. One may first consider a seed solution and then insert more vertex operators to generate multiple solitons solutions. As pointed out in Section 2, this leads to solutions similar to those found by P. Letelier but expressed and parametrized in different ways and with no quadrature left.

The parameters \( \mu_j \) defined in eq.\[(\ref{5.1})\] correspond to the moving poles and the parameter \( y \) to the integration constants of Belinskii and Zakharov. The relation between the parameters of the vertex operators and the moving poles can be made more explicit. Namely, let \( z_j \) the positions of the vertex operators and \( \mu_j \) defined as in eq.\[(\ref{5.1})\] by \( \mu_j^2 = \frac{z_j - z + \rho}{z_j - z - \rho} \), then the functions

\[
\lambda_j \equiv \rho \left( \frac{1 - \mu_j}{1 + \mu_j} \right) = (z - z_j) - \sqrt{(z - z_j)^2 - \rho^2}
\] (70)

are the moving poles of the Belinskii-Zakharov method. However the main difference between the vertex operator and the Belinskii-Zakharov approaches is the fact that no quadrature is needed to obtain the metric in the vertex operator approach.

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7 Appendix A: A survey of the general method.

To make the paper more self-contained, we recall here some of the results obtained in [16]. They could be helpful to understand the general method based on vertex operators. There are a few steps which we now explain. We will use notations and formulas introduced in Section 3 and Section 5. In particular we need to use the Kac-Moody and Virasoro algebras defined in Section 5.1, eq.\[(\ref{5.1})\] and below.

Let us first introduce the Lax connection:

\[
A_{\pm} = \pm d_{\pm}(L_0 - L_{\pm1}) + Q_{\pm} + P_{\pm} \otimes t_{\pm1} \mp (\partial_{\pm} \hat{\sigma}) \frac{k}{2}
\]

It is such that its zero curvature condition \([\partial_+ + A_+, \partial_- + A_-] = 0\) is equivalent to \( d_\pm = \rho^{-1} \partial_{\pm} \rho \) with \( \partial_+ \partial_- \rho = 0 \) and the reduced Einstein equations. As usual the zero curvature condition is the compatibility condition for an auxiliary linear system:

\[
(\partial_\pm + A_\pm)\Psi = 0
\] (71)

The solution \( \Psi \) of that system is called the wave function.

The simplest solution to Einstein’s equations corresponds to \( Q_\pm = P_\pm = \hat{\sigma} = 0 \). We call it the vacuum solution. It is easy to realise that it is associated to Minkowski’s flat solution (using the dual metric and analytic continuation). For that solution the Lax connection is simply \( A_{\pm} = \pm d_{\pm}(L_0 - L_{\pm1}) \). Its wave function is:

\[
\Psi_0(u, v) = \left( \frac{b(v) + c_1}{\rho} \right)^{L_0 - L_1} \left( \frac{b(v) + c_1}{c_2} \right)^{L_0 - L_{-1}} = \left( \frac{\rho}{a(u) + c_3(c_1, c_2)} \right)^{L_0 - L_1} \left( \frac{c_4(c_1, c_2)}{a(u) + c_3(c_1, c_2)} \right)^{L_0 - L_{-1}}
\]

with \( \rho = a(u) + b(v) \) and \( z = a(u) - b(v) \) as in eq.\[(\ref{5.1})\] and where \( c_1, c_2 \) are constants depending on the initial conditions. In this paper, we take \( c_1 = c_3 = \frac{1}{2} \) and \( c_2 = c_4 = 1 \) (choosing other values amounts rescaling and translating \( z \) and \( \rho \)).

The algebraic vertex operator method is based on manipulating the wave function to generate new solutions from old ones. These manipulations are dressing transformations [17]. They were applied to 2D reduced Einstein equations in ref.\.[16]. Dressing symmetries are associated to the factorization problem in the Kac-Moody group recalled in Section 5.1, eq.\[(\ref{5.1})\]. The point is that given the vacuum wave function \( \Psi_0 \) and
any element \( g = g_+^{-1}g_- \) of the affine \( SL(2, R) \) Kac-Moody group with the factorisation \( g_\pm \in \exp (\mathcal{B}_\pm \oplus \mathfrak{C}^k) \), then the wave function
\[
\Psi = (\Psi_0 g_0^{-1})_+ \Psi_0 g_0^{-1} = (\Psi_0 g_0^{-1})_- \Psi_0 g_0^{-1}
\]
is a solution of a compatible auxiliary linear system (71).
Given a wave function \( \Psi \), the original fields \( Q_\pm, P_\pm \) and \( \phi, \hat{\sigma} \), solutions of the Einstein equations, are then reconstructed by evaluating matrix elements of the wave function. This is done with the help of vertex operators. Choosing as in ref. [16] highest weight representations of the Kac-Moody group in which the group elements \( g \) can be written as products of vertex operators one recovers formulas (eq.(34, 35, 36)) for the Lax connection, and more interestingly formulas for the elements of the metric (eq.(51). (More details may be found in ref.[16].) We mark that in formula (51) all the coordinates’ dependence is contained in the vacuum wave function \( \Psi_0 \).

Note that in this approach the Lax connection does contain any space-time dependance spectral parameter. There is no moving spectral parameter. In particular the poles, which are the \( w \)-argument of the vertex operators, are fixed in contrast with other methods based on the Belinskii-Zakharov approach. The space-time dependence comes back when we conjugate the vertex operators with \( \Psi_0 \), cf eq.(37). This is manifest in the definition (70) of the moving pole which may be rewritten as:
\[
\lambda = \rho \Psi_0 \left( \frac{1 - w}{1 + w} \right) \Psi_0^{-1}
\]
So the Virasoro algebra appears as a way to encode the coordinates’ dependence of the moving poles.

8 Appendix B: Vertex operator expectation values.

Here we have gathered a few formulas for the vertex operator expectation values. These are computed using Wick’s theorem:
\[
\langle \prod_p W_{u_p}(\mu_p) \rangle = \prod_{p<q} \left( \frac{\mu_p - \mu_q}{\mu_p + \mu_q} \right)^{u_p u_q / 2}
\]
(72)
To evaluate the tau-function (34) or its dual one needs to compute expectation values such as
\[
\langle \prod_{p=1}^m W_{u_p}(\mu_p) \cdot \prod_{j=1}^n (1 + i Y_j W_2(\mu_j)) \rangle
\]
This is done using Wick’s theorem (72). For example
\[
\langle \prod_{j=1}^n (1 + i Y_j W_2(\mu_j)) \rangle = \sum_{p=0}^n i^p \sum_{k_1 < \ldots < k_p} Y_{k_1} \cdots Y_{k_p} \prod_{k_i < k_j} \left( \frac{\mu_{k_i} - \mu_{k_j}}{\mu_{k_i} + \mu_{k_j}} \right)^2
\]
The determinant formula for the tau-functions then follows from the Cauchy determinant formula
\[
\det \left( \frac{2\mu_i}{\mu_i + \mu_j} \right) = \prod_{i<j} \left( \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^2
\]
To evaluate the tau-functions (35, 36) one needs to know
\[
\langle \prod_j W_{u_j}(\mu_j) \rangle = \prod_{i<j} \left( \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^{u_i u_j / 2} \cdot \sum_j u_j \mu_j^{-1}
\]
\[
\langle p_1 \cdot \prod_j W_{u_j}(\mu_j) \rangle = - \prod_{i<j} \left( \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^{u_i u_j / 2} \cdot \sum_j u_j \mu_j
\]
from which equations (39) follow.
9 Appendix C: Some details about the examples

Here we have gathered a few formulas and identities that can be helpful in the computations used in the examples. First, let introduce the two variables $\xi$ and $w$ by

$$
\xi = \frac{1}{2} \left( \sqrt{(1-z+\rho)(1-z-\rho)} + \sqrt{(1+z+\rho)(1+z-\rho)} \right)
$$

$$
w = \frac{1}{2} \left( \sqrt{(1-z+\rho)(1-z-\rho)} - \sqrt{(1+z+\rho)(1+z-\rho)} \right)
$$

Reciprocally

$$
\rho = \sqrt{(1-\xi^2)(1-w^2)}
$$

$$
z = -w\xi
$$

The domain of definition of $\rho$ and $z$ implies that $|w| \leq \xi \leq 1$.

We use the notation $\mu_z$ and $X_z$ for the variables $\mu$ and $X$ associated to $z$ as defined in eq.(8). The singular values of $z$ we have chosen, eg. $z = \pm 1$ or $z = \infty$, lead to simple expressions for the factor using $\mu_z$. For example:

$$
\lim_{z \to -1} X_z \left( \frac{\mu_z - \mu_{-1}}{\mu_z + \mu_{-1}} \right)^{-1} = \lim_{z \to 1} X_z \left( \frac{\mu_1 - \mu_z}{\mu_1 + \mu_z} \right)^{-1} = 4
$$

and

$$
\left( \frac{\mu_z - 1}{\mu_z + 1} \right) \sim \frac{\rho}{2z} \quad \text{for} \quad z \to +\infty
$$

The variables $\mu_z$ for $z = \pm 1$ are related to the variables $\xi$ and $w$ by

$$
\left( \frac{\mu_1 - 1}{\mu_1 + 1} \right) \left( \frac{1 - \mu_{-1}}{1 + \mu_{-1}} \right) = \frac{1 - \xi}{1 + \xi}
$$

$$
\left( \frac{\mu_1 - 1}{\mu_1 + 1} \right) \left( \frac{1 + \mu_{-1}}{1 - \mu_{-1}} \right) = \frac{1 - w}{1 + w}
$$

$$
\left( \frac{\mu_1 - \mu_{-1}}{\mu_1 + \mu_{-1}} \right) = \left( \frac{1 - \xi^2}{1 - w^2} \right)^{1/2}
$$

The variables $X_z$ for $z = \pm 1$ are related to the variables $\xi$ and $w$ by

$$
\frac{(X_1 X_{-1})^{1/2}}{\rho} \left( d\rho^2 - dz^2 \right) = \text{const.} \left( \frac{d\xi^2}{1 - \xi^2} - \frac{dw^2}{1 - w^2} \right)
$$

Finally we give the parametrisation used to deduce the Kerr solution from the Chandrasekahar-Xanthopoulos one. The relation between the parameters $(p, q)$ and the mass and the angular momentum is

$$
p = -\frac{\sqrt{M^2 - a^2}}{M}; \quad q = \frac{a}{M}
$$

The relations between the coordinates are

$$
\xi = \frac{r - M}{\sqrt{M^2 - a^2}} \quad ; \quad w = \cos \theta
$$

$$
\tau = -\sqrt{2}M \left( x - \frac{2q}{p(1+p)} y \right) \quad ; \quad \phi = \frac{\sqrt{2}M}{\sqrt{M^2 - a^2}} y
$$

Recall that $R^2 = r^2 + a^2 \cos^2 \theta$ and $D = r^2 - 2Mr + a^2$. 

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References

[1] F. Ernst, Phys. Rev. 167 (1968) 1175.
[2] R. Geroch, J. Math. Phys. 13 (1972) 394.
[3] B. Julia, Infinite dimensional algebras in physics, in "John Hopkins workshop on current problems in particle physics: Unified theories and beyond", Johns Hopkins University, Baltimore (1981).
[4] V.A. Belinskii and V.E. Zakharov, Sov. Phys. JETP 48 (1978) 985.
[5] P. Breitenlohner and D. Maison, Ann. Inst. H. Poincare 46 (1987) 215.
[6] K.A. Khan and R. Penrose, Nature 229 (1971) 185; See also P. Szekeres Nature 19 (1970) 1183.
[7] Y. Nuktu and M. Halil, Phys. Rev. Lett. 39 (1977) 1379.
[8] S. Chandrasekhar and B. Xanthopoulos, Proc. R. Soc. London Ser. A 408 (1986) 175
[9] F. Ernst, A. Garcia and I. Hauser, J. Math. Phys. 28 (1987) 2555; J. Math. Phys. 28 (1987) 2951; J. Math. Phys. 29 (1988) 681.
[10] P. Letelier, J. Math. Phys. 25 (1984) 2675 and J. Math. Phys. 26 (1985) 467; See also: P. Letelier and S. Oliveira, J. Math. Phys. 28 (1987) 165; P. Letelier and S. Oliveira, Class. Quant. Grav. 15 (1998) 421.
[11] V. Ferrari, J. Ibanez and M. Bruni, Phys. Lett A (1987) 49; J. Ibanez and E. Verdaguer, Phys. Rev. Lett. 51 (1983) 1313; V. Ferrari and J. Ibanez, Proc. R. Soc. Lond. A417 (1988) 417.
[12] J.B. Griffiths, Colliding Plan Waves in General Relativity., Oxford University Press 1990.
[13] D. Kramer, H. Stephani, M.A.H. MacCallum and E. Herlt, Exact solutions of Einstein's field equations., Cambridge University Press (1980).
[14] D. Korotkin, Commun. Math. Phys. 137 (1991) 383 and Phys. Lett A 229 (1997) 195; G. Neugebauer and R. Meinel, Phys. Rev. Lett. 73 (1994) 2166 and Phys. rev. Lett. 75 (1995) 3046; R. Meinel and G. Neugebauer, Phys. Lett A 210 (1996) 160; C. Klein and O. Richter, Phys. Rev. Lett. 79 (1997) 565 and gr-qc/9806051.
[15] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Proc. Japan Acad. A 57 (1981) 3806; Physica D4 (1982) 343; Publ. RIMS Kyoto Univ. 18 (1982) 1077; M. Jimbo and T. Miwa, Publ. RIMS Kyoto Univ. 19 (1983) 943.
[16] D. Bernard and B. Julia, hep-th/9712254, to appear in Nucl. Phys. B.
[17] M. Semenov-Tian-Shansky, Publ. RIMS 21 (1985) 1237.
[18] O. Babelon and D. Bernard, Commun. Math. Phys. 149 (1992) 279.
[19] D. Korotkin and H. Samtleben, Nucl. Phys. B527 (1998) 657.
[20] cf. eg. H. Nicolai, D. Korotkin and H. Samtleben, Integrable classical and quantum gravity, hep-th/9612065.
[21] B. Julia and H. Nicolai, Nucl. Phys. B482 (1996) 431.
[22] O. Babelon and D. Bernard, Int. J. Mod. Phys. A8 (1993) 507.
[23] J. Lepowsky and R. Wilson, Commun. Math. Phys. 62 (1978) 43.