EXISTENCE AND A BLOW-UP CRITERION OF SOLUTION TO THE 3D COMPRESSIBLE NAVIER-STOKES-POISSON EQUATIONS WITH FINITE ENERGY

ANTHONY SUEN*

Department of Mathematics and Information Technology
The Education University of Hong Kong
10 Lo Ping Road, Tai Po, New Territories, Hong Kong, China

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Abstract. We study the low-energy solutions to the 3D compressible Navier-Stokes-Poisson equations. We first obtain the existence of smooth solutions with small $L^2$-norm and essentially bounded densities. No smallness assumption is imposed on the $H^4$-norm of the initial data. Using a compactness argument, we further obtain the existence of weak solutions which may have discontinuities across some hypersurfaces in $\mathbb{R}^3$. We also provide a blow-up criterion of solutions in terms of the $L^\infty$-norm of density.

1. Introduction. In this present paper, we consider the following isentropic compressible Navier-Stokes-Poisson (NSP) equations in the whole space $\mathbb{R}^3$ ($j = 1, 2, 3$):

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u_j)_t + \text{div}(\rho u_j u) + (P)_{x_j} &= \mu \Delta u_j + \lambda (\text{div} u) u_j + \rho \phi_{x_j}, \\
\Delta \phi &= \rho - \tilde{\rho},
\end{align*}
$$

Here $x \in \mathbb{R}^3$ is the spatial coordinate and $t \geq 0$ stands for the time. The unknown functions $\rho = \rho(x, t)$, $u = (u^1, u^2, u^3)(x, t)$ and $\phi = \phi(x, t)$ represent the electron density, electron velocity and the electrostatic potential respectively. $P = P(\rho)$ is a function in $\rho$ which denotes the pressure and $\tilde{\rho} > 0$ is a fixed constant, and $\mu$, $\lambda$ are positive viscosity constants. The system (1) is equipped with initial condition

$$
(\rho(\cdot, 0) - \tilde{\rho}, u(\cdot, 0), \phi(\cdot, 0)) = (\rho_0 - \tilde{\rho}, u_0, \phi_0)
$$

with the compatibility condition on $\rho_0$, namely

$$
\int_{\mathbb{R}^3} (\rho_0 - \tilde{\rho}) = 0.
$$

The NSP system (1) was used for describing the dynamics of a compressible fluid of electron in which the fluid interacts with its own electric field under the influence of a charged ion background at a given temperature. Equations (1)$_1$ and (1)$_2$ give the conservation of charge and conservation of momentum respectively, while equation (1)$_3$ is the self-consistent Poisson equation which relates the electron density and electrostatic potential. We refer to [1], [2], [3], [4] for more detailed discussions.

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* Corresponding author: Anthony Suen.
The system (1)-(2) has been studied by various mathematicians and we first recall some known results from the literature. On the one hand, Li-Matsumura-Zhang [9] obtained the global existence of small-smooth type solutions using the method by Matsumura-Nishida [13]-[14] with smallness assumptions on the initial data. The authors in [9] further proved that the density \( \rho \) converges to its equilibrium state in \( L^2 \) and \( L^\infty \)-norm with optimal rates of convergence. On the other hand, the global existence of large-weak type solutions of (1)-(2) with large initial data was proved in Donatelli [3] and Zhang-Tan [22] using the theory of P. L. Lions [12].

In this present work, we try to deepen our understanding on the compressible NSP system by addressing the solutions to (1) from another new perspective, in the sense that the initial data (2) is assumed to be small in some weaker norms (\( L^2 \)) with nonnegative and essentially bounded initial densities, and no further smallness assumption is imposed on the higher-regularity norms of the initial data. Such idea was first initiated by Hoff [6]-[8] in studying compressible Navier-Stokes system which was later extended by Suen [17]-[18] for compressible Naiver-Stokes system with potential forces as well as by Suen-Hoff [20] and Suen [19] for compressible magnetoohydrodynamics (MHD). The weak solutions obtained in the present work are known as the intermediate weak solutions which enjoy the following properties:

- The density \( \rho \) and velocity gradient \( \nabla u \) may exhibit discontinuities across some hypersurfaces in \( \mathbb{R}^3 \), such phenomenon cannot be observed by those small-smooth type classical solutions.
- These intermediate weak solutions would have more regularity than the large-weak type solutions developed by Lions [12], so that the uniqueness and continuous dependence of solutions may still be obtained (see [8] for the case without external force).

Furthermore, as a by-product of our analysis, we provide a blow-up criterion of smooth solution to (1)-(2) in terms of density. Such result is parallel to those obtained in Navier-Stokes system [21] as well as for compressible MHD system [16]. In the present work, we allow vacuum in the initial density \( \rho_0 \) and there is no smallness assumption imposed on the initial data in obtaining such blow-up criterion.

The main novelties of this current work can be summarised as follows:

1. We generalise the results obtained in [9] in the way that we obtain the existence of classical solution to (1)-(2) without smallness assumption on the \( H^4 \) of the initial data.
2. We prove the existence of intermediate weak solutions to (1)-(2) which can be viewed as an extensions from those in [6]-[8] for compressible Navier-Stokes.
3. We obtain a blow-up criterion for (1)-(2) in terms of the \( L^\infty \)-norm on the density. Such result is parallel to the one for the compressible Navier-Stokes system as given in [21].

We provide a brief outline on the analysis and idea behind our work. We introduce the following auxiliary variable associated with the system (1) which is known as the effective viscous flux \( F \). It is given by

\[
F = (\mu + \lambda)\text{div}(u) - (P(\rho) - P(\tilde{\rho})).
\]  

The variable \( F \) has been studied extensively by Hoff in [6]-[8], and we refer to those references for more detailed discussions on \( F \). Upon rearranging terms, the
momentum equation (1)_2 can be rewritten in terms of $F$:
\[ \rho \ddot{u}^j = F_{x_j} + \mu \omega_{x_k}^{j,k} - \rho \phi_{x_j}. \]  
Differentiate (5) with respect to $x_j$ and sum over $j$, we obtain the following equation for $F$:
\[ \Delta(F) = \text{div}(\rho \dot{u} - \rho \nabla \phi). \]  
Equations (5)-(6) will be crucial in obtaining \textit{a priori} bounds on the solutions, and the importance can be explained heuristically as follows:

1. In estimating higher-regularity norms (for example $\int_{0}^{t} |\nabla u|^4 \, dx \, dt$) of the weak solutions, one cannot merely applying the embedding $W^{2,2} \hookrightarrow W^{1,4}$ as $\nabla u$ may be discontinuous across hypersurfaces of $\mathbb{R}^3$ (see for example [6] for more details). With the help of $F$ and $\omega$, we can observe the following decomposition of $\Delta u$:
\[ u_{x_k x_k}^j = \omega_{x_j}^{j,k} + [(\mu + \lambda)^{-1} F]_{x_j} + [(\mu + \lambda)^{-1} (P - P(\bar{\rho}))]_{x_j}. \]

If we anticipate that $F(\cdot,t), \omega(\cdot,t) \in H^1$ and $P(\cdot,t) - P(\bar{\rho}) \in L^2 \cap L^\infty$, the term $u_{x_k x_k}^j$ should then be in $W^{-1,4}$, and hence the desired bound for $\nabla u$ in $L^4$ follows by a Fourier-type multiplier theorem.

2. Another important application of the equations (5)-(6) can be revealed in studying the pointwise bounds on the density. With the help of the effective viscous flux $F$, we can rewrite equation (1)_1 as follows:
\[ (\mu + \lambda) \frac{d}{dt} [\log(\rho(x(t),t))] + P(\rho(x(t),t)) - P(\bar{\rho}) = -\dot{\rho} F(x(t),t), \]
where $(x(t),t)$ is a particle path governed by $u$. Upon integrating with respect to time, we observe that the oscillation in density can be controlled by the time integral of $-\dot{\rho} F(x(t),t)$. By utilizing the Poisson’s equation (6) and the claimed \textit{a priori} bounds on $F$, we are able to show that such time integral is bounded by the initial energy of the system which is taken to be small by our assumption. Hence the density remains bounded above in $L^\infty$ as compare to itself initially.

We now give a precise formulation of our results. We first define the system parameters $P,$ $\mu,$ $\lambda$ as follows. For the pressure function $P = P(\rho),$ we assume that
\[ P(\rho) \in C^2((0,\infty)) \quad \text{with} \quad P(0) = 0; \quad P'(\rho) > 0; \quad P'(\rho) > 0 \quad \text{for} \quad \rho > 0; \quad (7) \]
For the diffusion coefficients $\mu$ and $\lambda,$ we assume that
\[ \mu > 0 \quad \text{and} \quad 0 < \lambda < \frac{5q}{4}. \quad (8) \]
It follows that
\[ \frac{\mu}{\lambda} > \frac{(q - 2)^2}{4(q - 1)} \quad (9) \]
for $q = 6$ and consequently for some $q > 6$, which we now fix. We also remark that the above conditions (7)-(9) as imposed on $P,$ $\mu$ and $\lambda$ are consistent with those used by Li-Matsumura [10] and Hoff [7] for compressible Navier-Stokes system. Condition (7) is considered to be more general than those used in Hoff [6] which includes the special case $P(\rho) = K \rho^\gamma$ for $\gamma \geq 1$ and $K > 0$. The assumptions (8)-(9) are required for technical reasons in obtaining \textit{a priori} estimates and will be particularly used in proving Lemma 2.5 (notice that (8)-(9) are consistent with those given in [6]).

Next we state the assumptions on the initial data $(\rho_0, u_0, \phi_0).$ We assume there is a positive number $N,$ which may be arbitrarily large such that
\[ \|u_0\|_{L^q} \leq N \quad (10) \]
where \( q \) is defined in (9). From now on, for \( \rho_0 - \bar{\rho}, u_0, \nabla \phi_0 \in L^2(\mathbb{R}^3) \), we also write

\[
C_0 = \|\rho_0 - \bar{\rho}\|^2_{L^2} + \|u_0\|^2_{L^2} + \|\nabla \phi_0\|^2_{L^2}
\]  

(11)

for the sake of convenience without further referring.

Weak solutions to the system (1)-(2) can be defined as follows. Given \( T > 0 \), we say that \((\rho, \bar{\rho}, u, \phi)\) is a weak solution of (1)-(2) if

- \((\rho - \bar{\rho}, pu, \nabla \phi) \in C([0, T]; H^{-1}(\mathbb{R}^3))\);
- \((\rho - \bar{\rho}, u, \phi)|_{t=0} = (\rho_0, u_0, \phi_0)\);
- \(\nabla u \in L^2(\mathbb{R}^3 \times (0, T))\);

and the following integral identities (12)-(14) hold for all \( t_1, t_2 \in [0, T] \) and \( C^1 \) test functions \( \varphi \) having uniformly bounded support in \( x \) for \( t \in [t_1, t_2] \):

\[
\int_{\mathbb{R}^3} \rho(x, \cdot) \varphi(x, \cdot) dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho \varphi_t + pu \cdot \nabla \varphi) dx dt; 
\]  

(12)

\[
\int_{\mathbb{R}^3} (pu)(x, \cdot) \varphi(x, \cdot) dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} [p \rho \varphi_t + pu \cdot \nabla \varphi + P(\rho) \varphi_x] dx dt
- \int_{t_1}^{t_2} \int_{\mathbb{R}^3} [\mu \nabla u \cdot \nabla \varphi + (\mu - \xi)(\text{div} u) \varphi_x] dx dt
+ \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \rho \phi_x \varphi dx dt. 
\]  

(13)

\[
- \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \nabla \phi \cdot \nabla \varphi dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho - \bar{\rho}) \varphi dx dt. 
\]  

(14)

We adopt the following usual notations for Hölder seminorms: for \( v : \mathbb{R}^3 \to \mathbb{R}^m \) and \( \alpha \in (0, 1) \),

\[
\langle v \rangle^\alpha = \sup_{x_1, x_2 \in \mathbb{R}^3, x_1 \neq x_2} \frac{|v(x_2) - v(x_1)|}{|x_2 - x_1|^\alpha};
\]

and for \( v : Q \subseteq \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^m \) and \( \alpha_1, \alpha_2 \in (0, 1) \),

\[
\langle v \rangle_{Q, \alpha_1, \alpha_2}^\alpha = \sup_{(x_1, t_1), (x_2, t_2) \in Q, (x_1, t_1) \neq (x_2, t_2)} \frac{|v(x_2, t_2) - v(x_1, t_1)|}{|x_2 - x_1|^\alpha_1 + |t_2 - t_1|^\alpha_2}.
\]

We denote the material derivative of a given function \( v \) by \( \dot{v} = v_t + \nabla v \cdot u \). Finally, if \( I \subset [0, \infty) \) is an interval, \( C^1(I; X) \) will be the elements \( v \in C(I; X) \) such that the distribution derivative \( v_t \in \mathcal{D}'(\mathbb{R}^3 \times \text{int} I) \) is an element of \( C(I; X) \).

We make use of the following standard facts (refer to Ziemer [23] for details). First, given \( r \in [2, 6] \) there is a constant \( C(r) \) such that for \( w \in H^1(\mathbb{R}^3) \),

\[
\|w\|_{L^r} \leq C(r) \left( \|w\|_{L^2}^{(6-r)/2r} \|w\|_{L^2}^{(3r-6)/2r} \right). 
\]  

(15)

Next, for any \( r \in (3, \infty) \) there is a constant \( C(r) \) such that for \( w \in W^{1,r}(\mathbb{R}^3) \),

\[
\|w\|_{L^\infty} \leq C(r) \|w\|_{W^{1,r}} 
\]  

(16)

and

\[
\|w\|_{L^3}^\alpha \leq C(r) \|\nabla w\|_{L^r},
\]  

(17)

where \( \alpha = 1 - 3/r \). If \( \Gamma \) is the fundamental solution of the Laplace operator on \( \mathbb{R}^3 \), then there is a constant \( C \) such that for any \( g \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \),

\[
\|\Gamma_{x_j} * g\|_{L^\infty} \leq C \left( \|g\|_{L^2} + \|g\|_{L^4} \right).
\]  

(18)
Theorem 1.1. Let the system parameters \( P, \mu, \lambda \) be given and satisfy the conditions (7)-(9). Given \( N, d, \tilde{C}, \tilde{\rho} > 0 \) and \( q > 6 \), for each \( T > 0 \), there exists \( \delta_T > 0 \) such that if the initial data \((\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0) \in H^4 \) satisfies (3) and (10)-(11) and
\[
0 \leq \text{ess inf} \rho_0 \leq \text{ess sup} \rho_0 < \tilde{\rho} - d, \tag{19}
\]
then the classical solution \((\rho - \tilde{\rho}, u, \phi)\) of (1)-(2) exists on \( [0, T] \).

Next in Theorem 1.2, we show that for a given \( L^2 \) initial data, under the smallness assumption (19) on \( C_0 \), there exists a weak solution \((\rho - \tilde{\rho}, u, \phi)\) of (1)-(2) defined on \( \mathbb{R}^3 \times [0, T] \) for each \( T > 0 \).

Theorem 1.2. Let the system parameters \( P, \mu, \lambda \) be given and satisfy the conditions (7)-(9). Given \( N, d, \tilde{C}, \tilde{\rho} > 0 \) and \( q > 6 \), for each \( T > 0 \), there are constants \( C_T, \theta_T, \delta_T > 0 \) such that if the initial data \((\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0) \in L^2 \) satisfies (3) and (10)-(11) and
\[
0 \leq \text{ess inf} \rho_0 \leq \text{ess sup} \rho_0 < \tilde{\rho} - d, \tag{20}
\]
then a weak solution \((\rho - \tilde{\rho}, u, \phi)\) of (1)-(2) in the sense of (12)-(14) exists on \( \mathbb{R}^3 \times [0, T] \). In particular, \((\rho - \tilde{\rho}, u, \phi)\) satisfies the following:
\[
\rho - \tilde{\rho}, \nabla \phi, \rho u \in C([0, T]; H^{-1}(\mathbb{R}^3)), \tag{23}
\]
\[
\nabla u \in L^2(\mathbb{R}^3 \times [0, T]), \tag{24}
\]
\[
u(t) \in H^1(\mathbb{R}^3), \quad t \in (0, T], \tag{25}
\]
\[
\omega(t), F(t) \in H^1(\mathbb{R}^3), \quad t \in [0, T], \tag{26}
\]
\[
\langle \omega \rangle^{\frac{1}{2}} \leq C(\tau)C_0^{\theta}, \quad \tau \in (0, T], \tag{27}
\]
where \( C(\tau) \) may depend additionally on a positive lower bound for \( \tau \),
\[
0 \leq \rho(x, t) \leq \tilde{\rho} \text{ a.e. on } \mathbb{R}^3 \times [0, T], \tag{28}
\]
and
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} [\rho^2 + \rho |u|^2 + |\nabla \phi|^2 + \sigma |\nabla u|^2 + \sigma^2 (F^2 + |\nabla \omega|^2)] dx ds \leq C_T C_0^{\theta_T} \tag{29}
\]
where \( \sigma(t) = \min\{1, t\} \).

Finally in Theorem 1.3, we obtain a blow-up criterion for the classical solutions to (1)-(2) for the isothermal case without any smallness assumption on the initial data.
Theorem 1.3. Let the system parameters $P$, $\mu$, $\lambda$ be given and satisfy the conditions (8)-(9) and

$$P(\rho) = K\rho,$$

where $K > 0$ is a given constant. Given $\tilde{\rho} > 0$ and $(\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0) \in H^4(\mathbb{R}^3)$, assume that $(\rho - \tilde{\rho}, u, \phi)$ is the smooth classical solution on $\mathbb{R}^3 \times [0, T]$. Let $T^* \geq T$ be the maximal existence time of the solution. If $T^* < \infty$, then we have

$$\lim_{t \to T^*} \|\rho\|_{L^\infty(\mathbb{R}^3 \times (0, t))} = \infty. \quad (31)$$

The rest of the paper is organised as follows. In Section 2, we derive a priori bounds for smooth, local-in-time solutions to (1)-(2) under the assumption that densities are non-negative and bounded. We first recall the following local-in-time existence theorem for the system (1)-(2) (see for example [9] and the references therein):

Theorem 2.1. Given $\tilde{\rho} > 0$ and initial data $(\rho_0 - \tilde{\rho}, u_0, \phi_0) \in H^4(\mathbb{R}^3)$, we can find $T > 0$ such that the classical solution $(\rho - \tilde{\rho}, u, \phi)$ to (1)-(2) exists on $\mathbb{R}^3 \times [0, T]$. Moreover, $(\rho - \tilde{\rho}, u, \phi)$ satisfies

$$\rho - \tilde{\rho} \in C^0([0, T]; H^4(\mathbb{R}^3)) \cap C^4([0, T]; H^3(\mathbb{R}^3)), \quad (32)$$

$$u \in C^0([0, T]; H^4(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3)), \quad (33)$$

and

$$\nabla \phi \in C^0([0, T]; H^5(\mathbb{R}^3)). \quad (34)$$

Given $T > 0$, we now fix $\tilde{\rho}, \rho$ as described in Section 1 and a smooth classical solution $(\rho - \tilde{\rho}, u, \phi)$ of (1)-(2) on $\mathbb{R}^3 \times [0, T]$ satisfying (32)-(34) with initial data $(\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0)$. With respect to $(\rho - \tilde{\rho}, u, \phi)$, we then define functionals $\Phi(t)$ and $H(t)$ for a given such solution by

$$\Phi(t) = \sup_{0 \leq s \leq t} \left[\|\partial_t \tilde{\rho}(\cdot, s)\|_{L^2}^2 + \|(\rho - \tilde{\rho})(\cdot, s)\|_{L^2}^2 + \|\nabla \phi(\cdot, s)\|_{L^2}^2\right]$$

$$+ \sup_{0 \leq s \leq t} \left[\sigma(s)\|\nabla u(\cdot, s)\|_{L^2}^2 + \sigma^3(s)\|\partial_t \tilde{\rho}(\cdot, s)\|_{L^2}^2\right]$$

$$+ \int_0^t \left[\|\nabla u(\cdot, s)\|_{L^2}^2 + \sigma(s)\|\partial_t \tilde{\rho}(\cdot, s)\|_{L^2}^2 + \sigma^3(s)\|\nabla \phi(\cdot, s)\|_{L^2}^2\right] ds. \quad (35)$$

$$H(t) = \int_0^t \sigma^2(s)\|\nabla u(\cdot, s)\|_{L^2}^2 ds + \int_0^t \sigma^3(s)\|\nabla u(\cdot, s)\|_{L^4}^4 ds$$

$$+ \sum_{1 \leq k, j, m \leq 3} \int_{\mathbb{R}^3} \sigma u_{x_k}^j u_{x_k}^j u_{x_k}^j dx ds, \quad (36)$$

where $\sigma(s)$ is a given constant.
where $\sigma(t) \equiv \min\{1, t\}$. We obtain a priori bounds for $\Phi(t)$ and $H(t)$ under the assumptions that

- the initial energy $C_0$ in (11) is small;
- the density $\rho$ remains bounded above and non-negative.

The results can be summarised in the following theorem:

**Theorem 2.2.** Let $N$, $d$, $\bar{\rho}$, $\bar{\rho} > 0$ be given. Assume that the system parameters in (1) satisfy the conditions in (7)-(9). For $T > 0$, if $(\rho - \bar{\rho}, u, \phi)$ is the classical solution of (1)-(2) defined on $\mathbb{R}^3 \times [0, T]$ with smooth initial data $(\rho_0 - \bar{\rho}, u_0, \nabla \phi_0) \in H^4(\mathbb{R}^3)$ satisfying (3) and (10)-(11), and if

$$0 \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T],$$

then we can find positive constants $\delta_T$, $M_T$ and $\theta_T$ such that: if $(\rho_0 - \bar{\rho}, u_0, \nabla \phi_0)$ further satisfies

$$C_0 \leq \delta_T$$

with

$$0 \leq \text{ess inf } \rho_0 \leq \text{ess sup } \rho_0 < \bar{\rho} - d,$$

then the following bound holds

$$\Phi(t) + H(t) \leq M_T \rho_0^{\delta_T} \text{ on } \mathbb{R}^3 \times [0, T].$$

(37)

Unless otherwise specified, $M$ will denote a generic positive constant which depends on $T$ and the same quantities as the constant $C_T$ in the statement of Theorem 1.2 but independent the regularity of initial data. And for simplicity, we write $P = P(\rho)$ and $\bar{P} = P(\bar{\rho})$, etc., without further referring.

We begin with the following $L^2$-estimate on $(\rho - \bar{\rho}, u, \phi)$ which is valid for all $t \in [0, T]$.

**Lemma 2.3.** For all $t \in [0, T]$, we have

$$\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \left( \rho |u|^2 + |\rho - \bar{\rho}|^2 + |\nabla \phi|^2 \right) + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \leq M C_0.$$  

(40)

**Proof.** By direct computation, we readily have

$$\frac{1}{2} \int_{\mathbb{R}^3} \rho |u|^2 \bigg|_0^t + \int_0^t \int_{\mathbb{R}^3} \left( \mu |\nabla u|^2 + \lambda (\text{div}(u))^2 \right) + \int_0^t \int_{\mathbb{R}^3} G(\rho) \bigg|_0^t = \int_0^t \int_{\mathbb{R}^3} \rho \nabla \phi \cdot u,$$

where

$$\int_{\mathbb{R}^4} G(\rho) = \int_{\mathbb{R}^4} \rho \int_{\rho}^\infty s^{-2} (P(s) - P(\bar{\rho})) ds$$

is comparable to the $L^2(\mathbb{R}^3)$ norm of $(\rho - \bar{\rho})$ (see [6] for related discussion). On the other hand, using the equations $(1)_1$ and $(1)_3$, we have

$$\Delta \phi = \rho_t = -\text{div}(\rho u),$$

and hence

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi(x, t)|^2 \bigg|_0^t = -\int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \phi.$$ 

Therefore we obtain the following energy balance equation:

$$\frac{1}{2} \int_{\mathbb{R}^3} \rho |u|^2 \bigg|_0^t + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 \bigg|_0^t + \int_0^t \int_{\mathbb{R}^3} \left( \mu |\nabla u|^2 + \lambda (\text{div}(u))^2 \right) + \int_{\mathbb{R}^3} G(\rho) \bigg|_0^t = 0,$$

(41)

and the result (40) follows.
Next we prove the following lemma which gives some auxiliary bounds on \( \phi \). These bounds will be useful in later analysis.

**Lemma 2.4.** For all \( t \in [0, T] \), we have

\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla \phi_t(x,t)|^2 \, dx \leq M \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \rho|u|^2(x,s) \, dx, \tag{42}
\]

\[
\sup_{0 \leq s \leq t} \|\nabla \phi(\cdot, s)\|_{L^\infty} \leq C(r) \left( C_0 + C_0^\frac{r}{2} \right), \tag{43}
\]

\[
\sup_{0 \leq s \leq t} \|\Delta \phi(\cdot, s)\|_{L^\infty} \leq M \sup_{0 \leq s \leq t} \|\rho - \bar{\rho}(\cdot, s)\|_{L^\infty}, \tag{44}
\]

where \( r > 3 \) and \( C(r) > 0 \) depends on \( r \).

**Proof.** Since \( \Delta \phi_t = \rho_t = -\text{div}(\rho u) \), so we have

\[
\int_{\mathbb{R}^3} \Delta \phi_t \cdot \phi_t(x,t) \, dx = \int_{\mathbb{R}^3} \rho u \cdot \nabla \phi_t(x,t) \, dx,
\]

which implies

\[
\int |\nabla \phi_t(x,t)|^2 \, dx \leq C \left( \int_{\mathbb{R}^3} \rho|u|^2(x,t) \, dx \right)^{\frac{1}{2}} \left( \int |\nabla \phi_t(x,t)|^2 \, dx \right)^{\frac{1}{2}},
\]

and (42) follows by (40). To prove (43), using (15) and (40), for \( r > 3 \), there exists \( \alpha_r \in (0, 1) \) and \( C(r) > 0 \) such that

\[
\|\nabla \phi(\cdot, t)\|_{L^\infty} \leq C(r) \left( \|\nabla \phi(\cdot, t)\|_{L^r} + \|\Delta \phi(\cdot, t)\|_{L^r} \right)
\]

\[
\leq C(r) \left( \|\nabla \phi(\cdot, t)\|_{L^\infty}^{\frac{\alpha_r}{2}} \|\Delta \phi(\cdot, t)\|_{L^\infty}^{\frac{1-\alpha_r}{2}} + M \|\Delta \phi(\cdot, t)\|_{L^2} \right)
\]

\[
\leq C(r) \left( C_0 + C_0^\frac{1}{2} \right).
\]

Finally, (44) follows immediately from the equation (1). \( \square \)

We prove the following auxiliary \( L^q \) estimates on the velocity \( u \) which will be used for controlling the \( \hat{L} \) norm of \( \nabla u \).

**Lemma 2.5.** For all \( t \in [0, T] \), we have

\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |u(t)|^q + \int_{0}^{t} \int_{\mathbb{R}^3} |u|^q \, |\nabla u|^2 \leq M \left[ C_0 + \int_{\mathbb{R}^3} |u_0|^q \right] \leq M \left[ C_0 + N \right]. \tag{45}
\]

**Proof.** The proof is similar to the one given in [7] page 323–324. From the momentum equation, we have

\[
\rho \left[ (|u|^q)_t + (\nabla |u|^q) \cdot u \right] + q |u|^{q-2} u \cdot \nabla (P - \bar{P}) + \mu q |u|^{q-2} |\nabla u|^2 + \lambda q |u|^{q-2} (\text{div} u)^2 + |u|^q \rho \nabla \phi
\]

\[
= q |u|^{q-2} \left[ \frac{1}{2} \mu \Delta |u|^2 + \lambda \text{div}((\text{div} u) u) \right].
\]
Adding the equation $|u|^q(\rho_t + \text{div}(\rho u)) = 0$ and integrating, we then obtain, for $0 \leq t \leq T$,

\[
\int_{\mathbb{R}^3} \rho|u|^q \mathbb{I}^t_0 + \int_0^t \int_{\mathbb{R}^3} q|u|^{q-2} \left[ \mu|\nabla u|^2 + \lambda(\text{div} u)^2 + \mu(q-2)|\nabla|u|^2 \right] \\
+ \int_0^t \int_{\mathbb{R}^3} q\lambda(\nabla|u|^{q-2}) \cdot u \text{div} u + \int_0^t \int_{\mathbb{R}^3} |u|^q \rho \nabla \phi \\
= \int_0^t \int_{\mathbb{R}^3} \left[ q\text{div}(|u|^{q-2}u)(P - \tilde{P}) \right]
\]  

(46)

Using the estimate (43), for $r = 4$ the term $\int_0^t \int_{\mathbb{R}^3} |u|^q \rho \nabla \phi$ can be bounded by

\[
\sup_{0 \leq s \leq t} \|\rho \nabla \phi(\cdot, t)\|_{L^\infty} \int_{\mathbb{R}^3} |u(\cdot, t)|^q \leq M(C_0 + C_0^2) \left( \int_{\mathbb{R}^3} |u(\cdot, t)|^q \right),
\]

and using the hypothesis (9) on $\mu$ and $\lambda$, we can bound the integrand in the double integral on the left side of (46) from below as follows.

\[
q|u|^{q-2} \left[ \mu|\nabla u|^2 + \lambda(\text{div} u)^2 + \mu(q-2)|\nabla|u|^2 - \lambda(q-2)|\nabla|u||\text{div} u| \right] \\
\geq q|u|^{q-2} \left[ \mu(q-1) - \frac{1}{4}(q-2)^2 \right]|\nabla u|^2 \\
\geq M^{-1}|u|^{q-2}|\nabla u|^2.
\]

On the other hand, there exists some $M > 0$ such that for each $\varepsilon > 0$, the right side of (46) is bounded by

\[
M \left[ \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 + \frac{1}{2\varepsilon} \left( \int_0^t \int_{\mathbb{R}^3} |P - \tilde{P}|^q \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |u|^q \right)^{\frac{1}{2}} \right] \\
\leq M \left[ \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 + \frac{1}{2\varepsilon} \left( \int_0^t \int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^q \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |u|^q \right)^{\frac{1}{2}} \right].
\]

Hence we conclude that for all $\varepsilon > 0$,

\[
\int_{\mathbb{R}^3} |u|^q \mathbb{I}^t_0 + \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 \\
\leq M \left[ \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 + \frac{1}{2\varepsilon} \left( \int_0^t \int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^q \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |u|^q \right)^{\frac{1}{2}} \right] \\
+ M(C_0 + C_0^2) \left( \int_{\mathbb{R}^3} |u(\cdot, t)|^q \right).
\]

By choosing $\varepsilon$ sufficiently small and applying Gronwall’s inequality, (45) follows for all $t \in [0, T]$.

We recall the following bounds for $u, \omega$ and $F$ in $W^{1,r}$ which are required for the derivation of estimates for the auxiliary functionals $H(t)$ and $\Phi(t)$. The proof can be found on page 505 in [10].

**Proposition 2.6.** For $r_1, r_2 \in (1, \infty)$ and $t \in [0, T]$, we have the following estimates:

\[
||\nabla u(\cdot, t)||_{L^{r_1}} \leq C(r_1) \left[ ||F(\cdot, t)||_{L^{r_2}} + ||\omega(\cdot, t)||_{L^{r_2}} + ||\rho - \tilde{\rho}(\cdot, t)||_{L^{r_2}} \right]  
\]  

(47)
and

\[ \| \nabla F(\cdot,t) \|_{L^{r_2}} + \| \nabla w(\cdot,t) \|_{L^{r_2}} \leq C(r_2) \left[ \| \rho^{\frac{2}{3}} \dot{u}(\cdot,t) \|_{L^{r_2}} + \| \nabla u(\cdot,t) \|_{L^{r_2}} + \| (\rho - \bar{\rho})^2(\cdot,t) \|_{L^{r_2}} + \| \rho^{\frac{2}{3}} \nabla \phi(\cdot,t) \|_{L^{r_2}} \right]. \]

(48)

The constants \( C(r_1), C(r_2) \) in (47)-(48) may depend additionally on \( r_1 \) and \( r_2 \) respectively.

We are ready to prove some higher order estimates on \( u \) and \( \dot{u} \) which are crucial in bounding the functional \( \Phi(t) \) in terms of \( H(t) \) for all \( t \in [0,T] \).

**Lemma 2.7.** For all \( t \in [0,T] \), we have

\[
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \sigma \left( \begin{array}{c} \nabla u^2 \\sigma u \dot{u} \end{array} \right) \leq M \left( C_0 + \sum_{1 \leq k,j,m \leq 3} \int_{0}^{t} \int_{\mathbb{R}^3} \sigma u_{x_{k_1}}^{j_1} u_{x_{k_2}}^{j_2} u_{x_{k_3}}^{j_3} dx \right),
\]

(49)

and

\[
\sup_{0 \leq s \leq t} \sigma^3 \left( \begin{array}{c} \int_{\mathbb{R}^3} \sigma \dot{u}^2 \\int_{\mathbb{R}^3} \sigma \nabla u^2 \end{array} \right) \leq M \left( C_0 + \sum_{1 \leq k,j,m \leq 3} \int_{0}^{t} \int_{\mathbb{R}^3} \sigma u_{x_{k_1}}^{j_1} u_{x_{k_2}}^{j_2} u_{x_{k_3}}^{j_3} dx \right) + \| \sigma \|_{L^\infty} \| \nabla u \|_{L^2}^2.
\]

(50)

for some \( \theta > 0 \).

**Proof.** The proof is similar to the one given in [6]. To prove (49), we first multiply the equation (1) by \( \sigma \dot{u} \) and integrate to get

\[
\sup_{0 \leq s \leq t} \left( \sigma \int_{\mathbb{R}^3} \nabla u^2 \right) + \int_{0}^{t} \int_{\mathbb{R}^3} \sigma \dot{u}^2 \leq M \left( C_0 + \sum_{1 \leq k,j,m \leq 3} \int_{0}^{t} \int_{\mathbb{R}^3} \sigma u_{x_{k_1}}^{j_1} u_{x_{k_2}}^{j_2} u_{x_{k_3}}^{j_3} dx \right).
\]

For the term involving \( \phi \), using Hölder’s inequality and the estimate (40),

\[
\left| \int_{0}^{t} \int_{\mathbb{R}^3} \sigma \dot{u} \cdot \nabla \phi \right| \leq M \left( \int_{0}^{t} \int_{\mathbb{R}^3} \sigma \dot{u}^2 \right)^{\frac{1}{2}} \left( \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla \phi|^2 \right)^{\frac{1}{2}} \leq M \left( \int_{0}^{t} \int_{\mathbb{R}^3} \sigma \dot{u}^2 \right)^{\frac{1}{2}} \left( MC_0 \right)^{\frac{1}{2}},
\]

hence the term can be absorbed by the left side of (49), and hence (49) follows.

Next, we apply the differential operator \( \partial_t + u \cdot \nabla \) on the equation (1) and make use of the transport theorem to obtain

\[
\sup_{0 \leq s \leq t} \sigma^3 \left( \begin{array}{c} \int_{\mathbb{R}^3} \rho \dot{u}^2 \\int_{\mathbb{R}^3} \sigma \nabla u^2 \end{array} \right) \leq M \left( \int_{0}^{t} \int_{\mathbb{R}^3} \sigma \dot{u}^2 \right)^{\frac{1}{2}} \left( \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla \phi|^2 \right)^{\frac{1}{2}} \leq M \left( \int_{0}^{t} \int_{\mathbb{R}^3} \sigma \dot{u}^2 \right)^{\frac{1}{2}} \left( MC_0 \right)^{\frac{1}{2}},
\]

(51)
To estimate the term \( M \left| \int_0^t \int_{\mathbb{R}^3} \sigma^3 \dot{u} \cdot \frac{\partial}{\partial t} (\rho \nabla \phi) \right| \), using equation (1)_1, we have

\[
\left| \int_0^t \int_{\mathbb{R}^3} \sigma^3 \dot{u} \cdot \frac{\partial}{\partial t} (\rho \nabla \phi) \right| = \left| \int_0^t \int_{\mathbb{R}^3} \sigma^3 \dot{u} \cdot (-\text{div}(\rho u) \nabla \phi + \rho \nabla \dot{\phi}) \right| \\
\leq M \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho |\nabla \dot{u}| ||\nabla \phi| + M \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho ||\dot{u}|| \Delta \phi| + M \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho ||\dot{u}|| \nabla \phi_t|.
\]

Using the estimates (42), (43) and (44), we have

\[
\int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho |\nabla \dot{u}| ||\nabla \phi| \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} \rho |u|^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq s \leq t} \|\nabla \phi(\cdot, t)\|_{L^\infty} \right) \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2 \right)^{\frac{1}{2}} C_0^\frac{1}{2} \left( C_0 + C_0^\frac{1}{2} \right),
\]

\[
\int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho ||\dot{u}|| ||\Delta \phi| \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma \rho ||\dot{u}||^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} \rho |u|^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq s \leq t} ||\Delta \phi(\cdot, t)||_{L^\infty} \right) \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2 \right)^{\frac{1}{2}} C_0^\frac{1}{2},
\]

and

\[
\int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho ||\dot{u}|| ||\nabla \phi_t|| \leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma \rho ||\dot{u}||^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla \phi_t|^2 \right)^{\frac{1}{2}} \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma |\dot{u}||^2 \right)^{\frac{1}{2}} C_0^\frac{1}{2},
\]

hence \( \int_0^t \int_{\mathbb{R}^3} \sigma^3 \dot{u} \cdot \frac{\partial}{\partial t} (\rho \nabla \phi) \) can be readily bounded in terms of \( \Phi \). For the term \( M \left| \int_0^t \int_{\mathbb{R}^3} \sigma^3 \dot{u} \cdot (u \cdot \nabla (\rho \nabla \phi)) \right| \), we can bound it as follows.

\[
\left| \int_0^t \int_{\mathbb{R}^3} \sigma^3 \dot{u} \cdot (u \cdot \nabla (\rho \nabla \phi)) \right| \\
\leq M \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho ||u|| ||\nabla \phi| + M \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho ||\nabla \dot{u}| |u|| \nabla \phi| \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho ||u||^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq s \leq t} ||\nabla \phi(\cdot, t)||_{L^\infty} \right) \\
+ M \left( \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2 \right)^{\frac{1}{2}} C_0^\frac{1}{2} \left( C_0 + C_0^\frac{1}{2} \right) \\
\leq M \Phi^{\frac{1}{2}} C_0^\frac{1}{2} \left( C_0 + C_0^\frac{1}{2} \right) + M \left( \int_0^t \int_{\mathbb{R}^3} \sigma^3 |\nabla \dot{u}|^2 \right)^{\frac{1}{2}} C_0^\frac{1}{2} \left( C_0 + C_0^\frac{1}{2} \right).
\]

Therefore the result (50) follows. \( \square \)
By (49) and (50) as obtained in Lemma 2.7, we have the following bound on $\Phi(t)$ in terms of $H(t)$:

$$\Phi(t) \leq M \left[ C_0^{\theta_1} + C_0 + H(t) \right].$$

(52)

In Lemma 2.8 listed below, we obtain the bound on $H(t)$ in terms of $\Phi(t)$.

**Lemma 2.8.** Assume that the hypotheses and notations of Theorem 2.1 are in force. Then there are positive numbers $\theta_1 > 0$ and $\theta_2 > 1$ such that for all $t \in [0, T]$, we have

$$H(t) \leq M \left[ C_0^{\theta_1} + \Phi(t)^\theta_2 \right].$$

(53)

**Proof.** First we bound the term $\int_0^t \sigma^3 \| \nabla u(\cdot, s) \|_{L^4}^4 ds$ as appeared in the definition of $H(t)$. With the help of (47),

$$\int_0^t \sigma^3 \| \nabla u(\cdot, s) \|_{L^4}^4 ds \leq M \int_0^t \sigma^3 (\| F(\cdot, s) \|_{L^4}^2 + \| \omega(\cdot, s) \|_{L^4}^2 + \| (\rho - \tilde{\rho})(\cdot, s) \|_{L^4}^2) ds.$$  

(54)

We estimate $\int_0^t \sigma^3 \| F(\cdot, s) \|_{L^4}^4 ds$ using (15), (48) and the definition of $\Phi$, it can be estimated by

$$\int_0^t \sigma^3 \| F(\cdot, s) \|_{L^4}^4 ds \leq M \int_0^t \sigma^3 (\| F(\cdot, s) \|_{L^2}^2 \| \nabla F(\cdot, s) \|_{L^2}^2 + \| \omega \|_{L^2} \| \nabla \omega(\cdot, s) \|_{L^2}^2) ds + MC_0.$$  

(55)

Next we estimate the term $\int_0^t \sigma^3 \| \nabla u(\cdot, s) \|_{L^4}^4 ds$ as follows.

$$\int_0^t \sigma^3 \| \nabla u(\cdot, s) \|_{L^4}^4 ds \leq M (\Phi(t)^2 + C_0^2 + C_0).$$

(55)
The first term on the right can be estimated by
\[
\int_0^t \sigma \frac{d}{ds} M_F(s) ds \leq M \left( \sup_{0 \leq s \leq t} \sigma \frac{d}{ds} M_F(s) \right) \left( \int_0^t \sigma \frac{d}{ds} M_F(s) ds \right) \leq M(\Phi(t) + C_0) \left( \int_0^t \sigma \frac{d}{ds} M_F(s) ds + \sup_{0 \leq s \leq t} \|\rho - \bar{\rho}(s)\|_2^2 \right) \leq M(\Phi(t) + C_0)^{\frac{3}{2}}.
\]

Proceeding in the same way, the second term involving \(\omega\) is also bounded by
\[
\int_0^t \sigma \frac{d}{ds} M_\omega(s) ds \leq M(\Phi(t) + C_0)^{\frac{3}{2}}.
\]

Hence we obtain
\[
\int_0^t \sigma \frac{d}{ds} M_\nabla u(s) ds \leq M(\Phi(t) + C_0)^{\frac{3}{2}}.
\]

It remains to estimate the summation term
\[
\sum_{1 \leq k_1, k_2, k_3 \leq 3} \left| \int_0^t \sigma u_{x_1} u_{x_2} u_{x_3} \right|.
\]

The proof is similar to the one given in [6] page 239–241 and we just sketch here. We make use of the decomposition
\[
u_{x_kx_k} = (u_{x_k} - u_{x_j})_{x_k} + (u_{x_j})_{x_k} = \omega_{x_j} + (\text{div} u)_{x_j}
\]
and write \(u\) as \(u = z + w\) so that
\[
z_{x_kx_k} = \omega_{x_j} + [(\mu + \lambda)^{-1} \tilde{P}]_{x_j} \quad \text{and} \quad w_{x_kx_k} = [(\mu + \lambda)^{-1} (P - \tilde{P})]_{x_j}.
\]

Hence for each \(r \in (1, \infty)\), there is a constant \(M(r)\) such that for \(t > 0\),
\[
\|\nabla z(s, t)\|_{L^r} \leq M(r) \|F(s, t)\|_{L^r} + \|\omega(s, t)\|_{L^r},
\]
and
\[
\|\nabla w(s, t)\|_{L^r} \leq M(r) \|(P - \tilde{P})(s, t)\|_{L^r}.
\]

Given \(j_1, j_2, j_3, k_1, k_2, k_3 \in \{1, 2, 3\}\), we have
\[
\int_0^t \sigma u_{x_{k_1}x_{k_1}} u_{x_{k_2}x_{k_2}} u_{x_{k_3}x_{k_3}} = A_1 + A_2 + A_3,
\]
where
\[ |A_1| \leq \int_0^t \int_{\mathbb{R}^3} \sigma |\nabla u||\nabla z||\nabla w|, \]
\[ |A_2| \leq \int_0^t \int_{\mathbb{R}^3} \sigma |\nabla u||\nabla w|^2, \]
\[ A_3 = \int_0^t \int_{\mathbb{R}^3} \sigma u^{j_1}_{x_{k_1}} z^{j_2}_{x_{k_2}} z^{j_3}_{x_{k_3}}. \]

We readily have
\[ \int_0^t \int_{\mathbb{R}^3} \sigma |\nabla u||\nabla z||\nabla w| \leq M \left( \int_0^t \sigma \frac{2}{3} \|\nabla u(\cdot, t)\|^3_{L^3} \right)^{\frac{3}{4}} \left( \int_0^t \|\nabla w(\cdot, t)\|^3_{L^3} \right)^{\frac{1}{4}}, \]
\[ \leq M \left[ \Phi(t) + C_0^2 \right] + C_0^2 \left( C_0^2 + C_0 \right)^{\frac{3}{4}} C_0^{\frac{3}{2}}, \]
and
\[ \int_0^t \int_{\mathbb{R}^3} \sigma |\nabla u||\nabla w|^2 \leq MC_0. \]

For \( A_3 \), using (40) and (2.5), we can estimate it as follows.
\[ \int_0^t \int_{\mathbb{R}^3} \sigma |\nabla u||\nabla z||\nabla w|^2 \leq \int_0^t \int_{\mathbb{R}^3} \left( |u|^2 |\nabla u|^2 + |u|^2 |\nabla w|^2 \right) \]
\[ \leq \left( \int_0^t \|\nabla u(\cdot, s)\|^2_{L^2} \right)^{\frac{3}{5}} \left( \int_0^t \|\nabla u(\cdot, s)\|^2 \right)^{\frac{2}{5}} \]
\[ + \int_0^t \|\nabla u(\cdot, s)\|^2_{L^6} \|\nabla w(\cdot, s)\|^2_{L^3} ds \]
\[ \leq C_0^{\frac{3}{10}} (C_0 + N)^{\frac{3}{5}} + MC_0 C_0^{\frac{3}{2}}. \]

Substituting the above estimates back into (54), we obtain the bound (53) as required.

**Proof of Theorem 2.2.** Theorem 2.2 follows immediately from the bounds (52) and (53), and the fact that those functionals \( \Phi(t) \) and \( H(t) \) are all continuous in time. \( \square \)

### 3. Pointwise bounds for the density and Proof of Theorem 1.1

In this section we derive pointwise bounds for the density \( \rho \), bounds which are independent both of time and of initial smoothness. This will then close the estimates of Theorem 2.2 to give an uncontingent estimate for the functionals \( \Phi(t) \) and \( H(t) \) defined in (35)-(36). The result is formulated as Theorem 3.1. Theorem 1.1 will then be proved using the a priori estimates derived from Theorem 3.1.

**Theorem 3.1.** Let \( N, d, \bar{\rho}, \bar{\rho} > 0 \) be given. Assume that the system parameters in (1) satisfy the conditions in (7)-(9). For \( T > 0 \), if \( (\rho - \bar{\rho}, u, \phi) \) is the classical solution of (1)-(2) defined on \( \mathbb{R}^3 \times [0, T] \) with initial data \( (\rho_0 - \bar{\rho}, u_0, \nabla \phi_0) \in H^4(\mathbb{R}^3) \) satisfying (3) and (10)-(11), then we can find positive constants \( \delta_T, M_T \) and \( \theta_T \) such that: if \( (\rho_0 - \bar{\rho}, u_0, \nabla \phi_0) \) further satisfies
\[ C_0 \leq \delta_T \]
(57)
with
\[ 0 \leq \text{ess inf} \rho_0 \leq \text{ess sup} \rho_0 < \bar{\rho} - d, \] (58)
then in fact
\[ \Phi(t) + H(t) \leq M_T C_0^{\theta'} \text{ on } \mathbb{R}^3 \times [0, T], \]
as well as
\[ 0 \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T]. \]

Proof. The proof consists of a maximum-principle argument applied along particle trajectories of \( u \) which is similar to the one given in [20] pg. 48–50 for the corresponding magnetohydrodynamics system as well as those in [17]-[18] with slight modifications. We choose a positive number \( b' \) satisfying
\[ \bar{\rho} - d < b' < \bar{\rho}. \]
Recall that \( \rho_0 \) takes values in \([0, \bar{\rho} - d] \), so that \( \rho \in [0, \bar{\rho}] \) on \( \mathbb{R}^3 \times [0, \tau] \) for some positive \( \tau \). It then follows from Theorem 2.2 that
\[ \Phi(\tau) + H(\tau) \leq M C_0^\theta, \] (59)
where \( M \) is now fixed. We shall show that if \( C_0 \) is further restricted, then in fact
\[ 0 \leq \rho < b' \text{ on } \mathbb{R}^3 \times [0, \tau], \]
and so by an open-closed argument that \( 0 \leq \rho < b' \) on all of \( \mathbb{R}^3 \times [0, T] \), we have \( \Phi(t) + H(t) \leq M C_0^\theta \) as well. We will only give the proof for the upper bound, the proof of the lower bound is just similar.

Fix \( y \in \mathbb{R}^3 \) and define the corresponding particle path \( x(t) \) by
\[ \begin{cases} 
\dot{x}(t) = u(x(t), t) \\
x(0) = y.
\end{cases} \]
Suppose that there is a time \( t_1 \leq \tau \) such that \( \rho(x(t_1), t_1) = b' \). We may take \( t_1 \) minimal and then choose \( t_0 < t_1 \) maximal such that \( \rho(x(t_0), t_0) = \bar{\rho} \). Thus \( \rho(x(t), t) \in [\bar{\rho}, b'] \) for \( t \in [t_0, t_1] \).

We have from the definition (4) of \( F \) and the mass equation that
\[ (\mu + \lambda) \frac{d}{dt} [\log \rho(x(t), t)] + P(\rho(x(t), t)) - P(\bar{\rho}) = -F(x(t), t). \]
Integrating from \( t_0 \) to \( t_1 \) and abbreviating \( \rho(x(t), t) \) by \( \rho(t) \), etc., we then obtain
\[ (\mu + \lambda) [\log \rho(s) - \log(\bar{\rho})]_{t_0}^{t_1} + \int_{t_0}^{t_1} [P(s) - \bar{P}] ds = -\int_{t_0}^{t_1} F(s) ds. \] (60)
We shall show that
\[ -\int_{t_0}^{t_1} \bar{\rho}(s) F(s) ds \leq \bar{M} C_0^{b'} \] (61)
for a constant \( \bar{M} \) which depends on the same quantities as the \( M_T \) from Theorem 3.1 and \( \theta' > 0 \). If so, then from (60),
\[ (\mu + \lambda) [\log \rho(s) - \log(\bar{\rho})]_{t_0}^{t_1} \leq \bar{M} C_0^{b'}, \] (62)
where the last inequality holds because \( P \) is increasing and \( P(s) - \bar{P}(s) \) is nonnegative on \([t_0, t_1] \). But (62) cannot hold if \( C_0 \) is small depending on \( \bar{M}, b' \), and \( \bar{\rho} \). Stipulating this smallness condition, we therefore conclude that there is no time \( t_1 \) such that \( \rho(t_1) = \rho(x(t_1), t_1) = b' \). Since \( y \in \mathbb{R}^3 \) was arbitrary, it follows that \( \rho < b' \) on \( \mathbb{R}^3 \times [0, \tau] \), as claimed.
To prove (61), we rewrite the right hand side of (60) as a space-time integral. Let \( \Gamma \) be the fundamental solution of the Laplace operator in \( \mathbb{R}^3 \), then we apply (6) to obtain

\[
- \int_{t_0}^{t_1} F(s) ds = - \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(s) - y) \rho \dot{u}^j(y, s) dy ds \\
+ \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(s) - y) \left[ \rho \phi_{x_j}(y, s) \right] dy ds.
\]

(63)

Using (18), the first integral on the right of (63) is bounded in the same way as in Lemma 4.2 of Hoff [6]

\[
\int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(t) - y) \rho \dot{u}^j(y, t) dy ds \\
\leq |\Gamma_{x_j} (\rho u^j)(\cdot, t_1)|_{L^\infty} + |\Gamma_{x_j} (\rho u^j)(\cdot, t_0)|_{L^\infty} \\
+ \left| \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_jx_k}(x(s) - y) \left[ u^k(x(s), s) - u^k(y, s) \right] (\rho u^j)(y, s) dy ds \right|
\leq \tilde{M} C_0^\theta.
\]

The second integral on the right side of (63), we observe that, for all \( s \in [0, \tau] \),

\[
\int_{\mathbb{R}^3} \Gamma_{x_j}(x(s) - y) \left[ \rho \phi_{x_j}(y, s) \right] dy \\
\leq \tilde{M} \int_{\mathbb{R}^3} |y|^{-2} |\nabla \phi(y, s)| dy \\
\leq \tilde{M} \left( \int_{|y| \leq 1} |y|^{-2} dy \right)^{\frac{3}{4}} \left( \int_{|y| \leq 1} |\nabla \phi(y, s)|^2 dy \right)^{\frac{1}{4}} \\
+ \tilde{M} \left( \int_{|y| > 1} |y|^{-2} dy \right)^{\frac{3}{4}} \left( \int_{|y| > 1} |\nabla \phi(y, s)|^2 dy \right)^{\frac{1}{4}} \\
\leq \tilde{M} \left( \int_{\mathbb{R}^3} |\nabla \phi(y, s)| dy \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} |\Delta \phi(y, s)| dy \right)^{\frac{1}{3}} + \tilde{M} C_0^\frac{3}{4} \\
\leq \tilde{M} C_0^\frac{1}{2}
\]

where we have used (15) for \( r = 5 \). Therefore we finish the proof.

**Proof of Theorem 1.1.** Let \( T > 0 \) be given. By Theorem 2.1, for a given initial data \((\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0) \in H^4(\mathbb{R}^3)\), we can find \( \bar{t} > 0 \) such that the classical solution \((\rho - \tilde{\rho}, u, \phi)\) to (1)-(2) exists on \( \mathbb{R}^3 \times [0, \bar{t}] \). Furthermore, by Theorem 3.1 just proved, there exists \( \delta_1, M_1, \theta_1 > 0 \) such that if \((\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0)\) satisfies the bounds (3) and (10)-(11) and (58) and the smallness assumption

\[
C_0 < \delta_1,
\]

then \((\rho - \tilde{\rho}, u, \phi)\) satisfies

\[
0 \leq \rho(x, t) \leq \tilde{\rho},
\]

and

\[
\Phi(t) + H(t) \leq M_1 C_0^{\theta_1}.
\]
for all $(x,t) \in \mathbb{R}^3 \times [0,T]$. In particular, we have
\[ \rho(x,t) \in [0,\bar{\rho}], \]
and $(\rho - \bar{\rho}, u, \nabla \phi)(\cdot,t) \in H^4(\mathbb{R}^3)$. Since the system (1) is autonomous, we can therefore reapply Theorem 2.1 at the new initial time $\bar{t}$ to extend the solution $(\rho - \bar{\rho}, u, \phi)$ eventually to all $[0,T]$ in finitely many steps by replacing $\delta_\tau$ with some smaller number $\delta_T$ in (64). In other words, there exists $\delta_T > 0$ depending on $T$ such that if the initial data $(\rho_0 - \bar{\rho}, u_0, \nabla \phi_0) \in H^4(\mathbb{R}^3)$ is given satisfying (3) and (10)-(11) with $C_0 < \delta_T$, then $(\rho - \bar{\rho}, u, \phi)$ can be extended to the whole time interval $[0,T]$. It finishes the proof of Theorem 1.1.

4. Existence of weak solutions and Proof of Theorem 1.2. In this section, we prove Theorem 1.2 by obtaining weak solutions to the system (1)-(2) on $\mathbb{R}^3 \times [0,T]$ when $T > 0$ is any given time. To begin with, let initial data $(\rho_0, u_0, \phi_0)$ be given satisfying the hypotheses (3) and (10)-(11) of Theorem 3.1, and we fix those constants $\delta_T$, $M_T$ and $\theta_T$ defined in Theorems 3.1.

Upon choosing $(\rho_0', u_0', \phi_0')$ as a smooth approximation of $(\rho_0, u_0, \phi_0)$ which can be obtained by convolving $(\rho_0, u_0, \phi_0)$ with the standard mollifying kernel of width $\eta > 0$, we can apply Theorem 1.1 to show that there is a smooth solution $(\rho^\eta, u^\eta, \phi^\eta)$ of (1)-(2) with initial data $(\rho_0', u_0', \phi_0')$ defined on the time interval $[0,T]$. The \textit{a priori} estimates of Theorem 3.1 then apply to show that
\[ \Phi(t) + H(t) \leq M_T C_0^{\theta_T} \quad \text{on} \quad \mathbb{R}^3 \times [0,T], \] (65)
as well as
\[ 0 \leq \rho(x,t) \leq \bar{\rho} \quad \text{on} \quad \mathbb{R}^3 \times [0,T], \] (66)
where $\Phi(t)$ and $H(t)$ are defined by (35)-(36) but with $(\rho, u, \phi)$ replaced by $(\rho^\eta, u^\eta, \phi^\eta)$.

We prove that the approximate solution $(\rho^\eta, u^\eta, \phi^\eta)$ satisfies the following uniform Hölder continuity estimates. It will be used for obtaining weak solutions via compactness arguments.

\textbf{Lemma 4.1.} Given $\tau \in (0,T]$, there is a constant $C = C(\tau)$ such that for all $\alpha \in (0, \frac{1}{2}]$ and $\eta > 0$,
\[ \langle u^\eta(\cdot,t) \rangle^\alpha \leq M \left[ \left( C_0 + \| \nabla u^\eta(\cdot,t) \|_{L^2}^2 \right)^\frac{1-2\alpha}{2} \left( \| (\rho^\eta)^{\frac{1}{2}} \dot{u}^\eta(\cdot,t) \|_{L^2}^2 + \| \rho^\frac{1}{2} \nabla \phi^\eta(\cdot,t) \|_{L^2}^2 \right)^\frac{1+2\alpha}{2} \right. \]
\[ + \| \nabla u^\eta(\cdot,t) \|_{L^2}^{1-2\alpha} \| \nabla \omega^\eta(\cdot,t) \|_{L^2}^{\frac{1+2\alpha}{2}} + C_0^{\frac{1+2\alpha}{2}} \right], \]
\[ \leq C(\tau) C_0^{\theta_T}, \] (67)
\[ \langle u^\eta(\cdot,t) \rangle \leq C(\tau) C_0^{\theta_T}. \] (68)

\textit{Proof:} First we prove (67). Let $\alpha \in (0, \frac{1}{2}]$ and define $r \in (3,6]$ by $r = 3/(1-\alpha)$. Then by (17) and (47),
\[ \langle u^\eta(\cdot,t) \rangle^\alpha \leq M \| u^\eta(\cdot,t) \|_{L^r} \]
\[ \leq M \left( \| F^\eta(\cdot,t) \|_{L^r} + \| \omega^\eta(\cdot,t) \|_{L^r} + \| (\rho^\eta - \bar{\rho})(\cdot,t) \|_{L^r} \right). \] (69)
But by (15), we also have
\[ \|F^n(\cdot,t)\|_{L^r} \leq M \left( \|F^n(\cdot,t)\|_{L^2}^{(6-r)/2r} \|\nabla F^n(\cdot,t)\|_{L^2}^{(3r-6)/2r} \right) \]
\[ \leq M \left( \|((\rho^n - \rho)(\cdot,t))\|_{L^2}^2 + \|\nabla u^n(\cdot,t)\|_{L^2}^2 \right)^{1-2\alpha} \times \left( \|((\rho^n)^{\frac{1}{2}}u^n(\cdot,t))\|_{L^2}^2 + \|((\rho^n)^{\frac{1}{2}}\nabla \phi^n(\cdot,t))\|_{L^2}^2 \right)^{1+2\alpha} \]
and
\[ \|\omega^n(\cdot,t)\|_{L^r} \leq M \left( \|\omega^n(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \|\nabla \omega^n(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r} \right) \]
\[ \leq M \left( \|\nabla u^n(\cdot,t)\|_{L^2}^{1-2\alpha} \|\nabla \omega^n(\cdot,t)\|_{L^2}^{1+2\alpha} \right), \]
where \( F^n \) is defined in (4) with \( u, \rho \) replaced by \( u^n, \rho^n \) respectively. The result (67) then follows by applying these bounds in (69) as well as the uniform estimates (65).

Next we consider (68). First, the Hölder-\( \frac{1}{2} \) continuity of \( u^n \) in space was just proved in (67) by taking \( \alpha = \frac{1}{2} \), and to infer Hölder continuity in time we fix \( x \) and \( \tau \leq t_1 \leq t_2 \leq T \) and compute
\[
\left| u^n(x,t_2) - u^n(x,t_1) \right| \leq \frac{1}{|B_R(x)|} \int_{B_R(x)} \left| u^n(z,t_2) - u^n(z,t_1) \right| dz + C(\tau) C_0^{\theta^p} R^2 \]
\[ \leq R^{-\frac{1}{2}}|t_2 - t_1| \sup_{\tau \leq t \leq T} \left( \int |u^n_\tau|^2 dx \right)^{\frac{1}{2}} + C(\tau) C_0^{\theta^p} R^4 \]
\[ \leq C(\tau) C_0^{\theta^p} \left[ R^{-\frac{1}{2}}|t_2 - t_1| + R^2 \right] \]
by the bounds in (65). Taking \( R = |t_2 - t_1|^{\frac{1}{2}} \) we then obtain the estimate in (68) for \( u^n \).

In view of Lemma 4.1, the desired weak solution \( (\rho, u, \phi) \) will then be obtained by taking \( \eta \rightarrow 0 \), with the use of the compactness provided by those bounds in (65)-(66) and (68). It can be summarised in the following lemma:

**Lemma 4.2.** There is a sequence \( \eta_k \rightarrow 0 \) and functions \( u, \rho \) and \( \phi \) such that as \( k \rightarrow \infty \),
\[ u^{n_k} \rightarrow u \] uniformly on compact sets in \( \mathbb{R}^3 \times (0,T) \);  
\[ \nabla u^{n_k}(\cdot,t), \nabla \omega^{n_k}(\cdot,t) \rightharpoonup \nabla u(\cdot,t), \nabla \omega(\cdot,t) \]
weakly in \( L^2(\mathbb{R}^3) \) for all \( t \in (0,T) \);  
\[ \sigma^{\frac{1}{2}} \dot{u}^{n_k}, \sigma^{\frac{1}{2}} \nabla \dot{u}^{n_k} \rightharpoonup \sigma^{\frac{1}{2}} \dot{u}, \sigma^{\frac{1}{2}} \nabla \dot{u} \]
weakly in \( L^2(\mathbb{R}^3 \times [0,T]) \); and  
\[ \rho^{n_k}(\cdot,t), \Delta \phi^{n_k} \rightharpoonup \rho(\cdot,t), \Delta \phi(\cdot,t) \]
strongly in \( L^2_{loc}(\mathbb{R}^3) \) for every \( t \in [0,T] \).

**Proof.** The uniform convergence (71) follows from Lemma 4.1 via a diagonal process, thus fixing the sequence \{ \( \eta_k \) \}. The weak-convergence statements in (72) and (73) then follow from the bound (65) and considerations based on the equality of weak-L\(^2\) derivatives and distribution derivatives. The convergence of approximate densities in (74) for a further subsequence can be achieved by applying the argument given...
Proof of Theorem 1.2. In view of the bounds (65) and (68) and the convergence results obtained from Lemma 4.2, it is clear that the limiting functions \((\rho, u, \phi)\) of Lemma 4.2 inherit the bounds in (24)-(29). It is also clear from the modes of convergence described in Lemma 4.2 that \((\rho, u, \phi)\) satisfies the weak forms (12)-(14) of the system (1) as well as the initial condition (2). The continuity statement (23) then follows easily from these weak forms together with the bounds in (29). This completes the proof of Theorem 1.2.

5. Blow-up criteria and Proof of Theorem 1.3. In this section, we study the blow-up criterion for classical solutions to the system (1)-(2) and prove Theorem 1.3. First we define the so-called maximal time of existence of smooth solutions to (1)-(2):

**Definition 5.1.** We call \(T^* \in (0, \infty)\) to be the maximal time of existence of a smooth solution \((\rho - \tilde{\rho}, u, B)\) to (1)-(2) if for any \(0 < T < T^*\), \((\rho - \tilde{\rho}, u, \phi)\) solves (1)-(2) in \([0, T]\times \mathbb{R}^3\) and satisfies

\[
\rho - \tilde{\rho} \in C^0([0, T]; H^4(\mathbb{R}^3)) \cap C^1([0, T]; H^3(\mathbb{R}^3)),
\]
(75)

\[
u \in C^0([0, T]; H^4(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3)),
\]
(76)

\[
\nabla \phi \in C^0([0, T]; H^5(\mathbb{R}^3)).
\]
(77)

Moreover, the conditions (75)-(77) fail to hold when \(T = T^*\).

We will prove Theorem 1.3 using a contradiction argument. Specifically, for the sake of contradiction, we assume that

\[
\|\rho\|_{L^\infty((0, T^*) \times \mathbb{R}^3)} \leq \bar{C}.
\]
(78)

for some constant \(\bar{C} > 0\). Based on the assumption (78), we derive \(a\ priori\) estimates for the local smooth solution \((\rho - \tilde{\rho}, u, \phi)\) on \([0, T]\) with \(T \leq T^*\). Those estimates are different from we did in Section 2 and Section 3 in the sense that there is no smallness assumption imposed on the initial data, hence a more delicate analysis is required in bounding the solution \((\rho - \tilde{\rho}, u, \phi)\).

To facilitate the proof, we introduce the following auxiliary functionals:

\[
\Phi_1(t) = \sup_{0 \leq s \leq t} \|\rho^{\frac{1}{2}} \dot{u}(. , s)\|_{L^2}^2 + \int_0^t \|\rho^{\frac{1}{2}} \dot{u}(. , s)\|_{L^2}^2 ds,
\]
(79)

\[
\Phi_2(t) = \sup_{0 \leq s \leq t} \|\rho^{\frac{1}{2}} \dot{u}(. , s)\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}(. , s)\|_{L^2}^2 ds
\]
(80)

\[
\Phi_3(t) = \int_0^t \int_{\mathbb{R}^3} |\nabla u|^4 dx ds
\]
(81)

We first recall the following lemma which gives estimates on the solutions of the Lamé operator \(\mu \Delta + (\mu + \lambda) \nabla \text{div}\). More detailed discussions can also be found in Sun-Wang-Zhang [21].

**Lemma 5.2.** Consider the following equation:

\[
\mu \Delta v + (\mu + \lambda) \nabla \text{div}(v) = J,
\]
(82)

where \(v = (v^1, v^2, v^3)(x)\), \(J = (J^1, J^2, J^3)(x)\) with \(x \in \mathbb{R}^3\) and \(\mu, \lambda > 0\). Then for \(r \in (1, \infty)\), we have:
1. if $J \in W^{2,r}(\mathbb{R}^3)$, then $\|\Delta v\|_{L^r} \leq \tilde{C}\|J\|_{L^r}$;  
2. if $J = \nabla \varphi$ with $\varphi \in W^{2,r}(\mathbb{R}^3)$, then $\|\nabla v\|_{L^r} \leq \tilde{C}\|\varphi\|_{L^r}$;  
3. if $J = \nabla \text{div}(\varphi)$ with $\varphi \in W^{2,r}(\mathbb{R}^3)$, then $\|v\|_{L^r} \leq \tilde{C}\|\varphi\|_{L^r}$.

Here $\tilde{C}$ is a positive constant which depends only on $\mu$, $\lambda$ and $r$.

\textbf{Proof.} A proof can be found in [21] pg. 39 and we omit the details here. \hfill $\Box$

Given $\tilde{\rho} > 0$ and initial data $(\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0) \in H^4(\mathbb{R}^3)$, we define

$$S_0 = \|\rho_0 - \tilde{\rho}, u_0, \nabla \phi_0\|_{H^4}^2. \tag{83}$$

We begin to estimate the functionals $\Phi_1$, $\Phi_2$ and $\Phi_3$ under the assumption (78). Similar to the previous cases, $M$ will denote a generic positive constant which further depends on $\bar{C}, \bar{C}, T^*$ and $S_0$.

We first have the following bounds based on the results we obtained in Section 2, namely for any $0 \leq t \leq T \leq T^*$ and $r > 3$,

$$\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|\rho|u^2 + |\rho - \tilde{\rho}|^2 + |\nabla \varphi|^2) + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \leq M, \tag{84}$$

$$\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|u|^4 + |B|^4) \leq M, \tag{85}$$

$$\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla \phi(x, t)|^2 dx \leq M \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\rho u|^2(x, s)dx, \tag{86}$$

$$\sup_{0 \leq s \leq t} \|\nabla \phi(\cdot, s)\|_{L^\infty} \leq C(r) \left( S_0 + S_0^\frac{4}{3} \right), \tag{87}$$

$$\sup_{0 \leq s \leq t} \|\Delta \phi(\cdot, s)\|_{L^\infty} \leq M \sup_{0 \leq s \leq t} \|(\rho - \tilde{\rho})(\cdot, s)\|_{L^\infty}. \tag{88}$$

Next we are going to estimate $\Phi_1$ which is given in the following lemma:

\textbf{Lemma 5.3.} For any $0 \leq t \leq T \leq T^*$,

$$\Phi_1(t) \leq M[1 + \Phi_3(t)]. \tag{89}$$

\textbf{Proof.} Following the proof of Lemma 2.7, we have

$$\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^t \int_{\mathbb{R}^3} |\rho \dot{u}|^2 \leq \left| \int_0^t \int_{\mathbb{R}^3} \rho \dot{u} \cdot \nabla \phi \right| + M \left( S_0 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 \right). \tag{90}$$

Using (78) and (84), the first term on the right side of (90) can be estimated as follows.

$$\left| \int_0^t \int_{\mathbb{R}^3} \rho \dot{u} \cdot \nabla \phi \right| \leq \tilde{C} \left( \int_0^t \int_{\mathbb{R}^3} |\rho u|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla \varphi|^2 \right)^\frac{1}{2} \leq \tilde{C} \Phi_3^\frac{1}{2} T^\frac{1}{2} (MS_0)^\frac{1}{2} \leq M \Phi_3^\frac{1}{2},$$

and the second term can be bounded by

$$\int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 \leq \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^4 \right)^\frac{1}{2} \leq M \Phi_3^\frac{1}{2}.$$

Applying the above bounds on (90), the result follows. \hfill $\Box$
Before we estimate $\Phi_2$, we introduce the following decomposition on $u$. We write
\[ u = u_p + u_s, \]  
where $u_p$ and $u_s$ satisfy
\[
\begin{cases}
\mu \Delta (u_p) + (\mu + \lambda) \nabla \text{div}(u_p) = \nabla (P - \bar{P}), \\
\rho(u_s)_t - \mu \Delta u_s - (\mu + \lambda) \nabla \text{div}(u_s) = -\rho u \cdot \nabla u - \rho (u_p)_t + \rho \nabla \phi.
\end{cases}
\]  
(92)

Using Lemma 5.2, the term $u_p$ can be bounded by
\[
\int_{\mathbb{R}^3} |\nabla u_p|^r \leq M \int_{\mathbb{R}^3} |P - \bar{P}|^r \leq M \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^r.
\]  
(93)

We give the estimates for $u_s$ as follows.

**Lemma 5.4.** For any $0 \leq t \leq T \leq T^*$, we have
\[
\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla u_s|^2 + \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 + \int_0^t \int_{\mathbb{R}^3} |\Delta u_s|^2 \leq M.
\]  
(94)

**Proof.** We multiply (92) by $\partial_t (u_s)$ and integrate to obtain
\[
\int_{\mathbb{R}^3} \mu |\nabla u_s|^2_t |_0^t + \int_0^t \int_{\mathbb{R}^3} (\mu + \lambda) |\text{div} u_s|^2 + \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2
\]
\[
= - \int_0^t \int_{\mathbb{R}^3} (\rho u \cdot \nabla u) \cdot \partial_t (u_s) - \int_0^t \int_{\mathbb{R}^3} (\rho \partial_t (u_p)) \cdot \partial_t (u_s) + \int_0^t \int_{\mathbb{R}^3} \rho \nabla \phi \cdot \partial_t (u_s).
\]
(95)

We estimate the right side of (95) term by term. Using (85) and (93), the first integral can be bounded by
\[
\left( \int_0^t \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \right)^\frac{1}{2}
\]
\[
\leq M \left[ \int_0^t \left( \int_{\mathbb{R}^3} |u|^4 \right)^\frac{1}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^4 \right)^\frac{1}{2} \right] \left[ \int_0^t \left( \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \right)^\frac{1}{2} \right]
\]
\[
\leq M \left[ \int_0^t \left( \int_{\mathbb{R}^3} |\nabla u_s|^2 \right)^\frac{1}{2} \left( \int_{\mathbb{R}^3} |\Delta u_s|^2 \right)^\frac{1}{2} + \int_0^t \left( \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^4 \right)^\frac{1}{2} \right] \frac{1}{2}
\]
\[
\times \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \right)^\frac{1}{2}
\]
\[
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \right)^\frac{1}{2} \left[ \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u_s|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{R}^3} |\Delta u_s|^2 \right)^\frac{1}{2} + 1 \right].
\]

Next to estimate $-\int_0^t \int_{\mathbb{R}^3} (\rho \partial_t (u_p)) \cdot \partial_t (u_s)$, we differentiate (92) with respect to $t$ and obtain
\[
\mu \Delta \partial_t (u_p) + (\mu + \lambda) \nabla \text{div} \partial_t (u_p) = \nabla \text{div} (-P \cdot u).
\]

Using Lemma 5.2 and (84), we have
\[
\int_0^t \int_{\mathbb{R}^3} |\partial_t (u_p)|^2 \leq M \int_0^t \int_{\mathbb{R}^3} |P \cdot u|^2 \leq M.
\]  
(96)

Therefore
\[
-\int_0^t \int_{\mathbb{R}^3} (\rho \partial_t (u_p)) \cdot \partial_t (u_s) \leq \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_p)|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{R}^3} |\partial_t (u_p)|^2 \right)^\frac{1}{2}
\]
\[
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \right)^\frac{1}{2}.
\]
To estimate $\int_0^t \int_{\mathbb{R}^3} \rho \nabla \phi \cdot \partial_t (u_s)$, using (84), we readily have
\[
\int_0^t \int_{\mathbb{R}^3} \rho \nabla \phi \cdot \partial_t (u_s) \leq \left( \int_0^t \int_{\mathbb{R}^3} \rho |\nabla \phi|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \right)^{\frac{1}{2}} \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \right)^{\frac{1}{2}}.
\]
Combining the above, we have from (95) that
\[
\int_{\mathbb{R}^3} |\nabla u_s|^2 (x, t) dx + \int_0^t \int_{\mathbb{R}^3} |\text{div}(u_s)|^2 + \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u_s|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\Delta u_s|^2 \right)^{\frac{1}{2}} + M. \tag{97}
\]
It remains to estimate the term $\int_0^t \int_{\mathbb{R}^3} |\Delta u_s|^2$. Rearranging the terms in (92), we have that
\[
\mu \Delta u_s + (\mu + \lambda) \nabla \text{div}(u_s) = \rho \partial_t (u_s) + \rho u \cdot \nabla u + \rho \partial_t (u_p) - \rho \nabla \phi.
\]
Therefore, we can apply Lemma 5.2 to get
\[
\int_0^t \int_{\mathbb{R}^3} |\Delta u_s|^2 \\
\leq M \left[ \int_0^t \int_{\mathbb{R}^3} \left( |\rho \partial_t (u_s)|^2 + |\rho u \cdot \nabla u|^2 + |\rho \partial_t (u_p)|^2 + |\rho \nabla \phi|^2 \right) \right].
\]
Using (84) and (96), we obtain
\[
\int_0^t \int_{\mathbb{R}^3} \left( |\rho \partial_t (u_s)|^2 + |\rho u \cdot \nabla u|^2 + |\rho \partial_t (u_p)|^2 + |\rho \nabla \phi|^2 \right) \leq M \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 + 1 \right),
\]
and hence
\[
\int_0^t \int_{\mathbb{R}^3} |\Delta u_s|^2 \leq M \left( \int_0^t \int_{\mathbb{R}^3} \rho |\partial_t (u_s)|^2 + 1 \right). \tag{98}
\]
Applying the estimate (98) on (97) and using Gröwall’s inequality, we conclude that for $0 \leq t \leq T^*$,
\[
\int_{\mathbb{R}^3} |\nabla u_s|^2 (x, t) dx \leq M,
\]
and the result (94) follows.

We are now ready to estimate $\Phi_2$ as defined in (80). The result is given in the following lemma.

**Lemma 5.5.** For any $0 \leq t \leq T \leq T^*$,
\[
\Phi_2(t) \leq M [\Phi_1(t) + \Phi_3(t) + 1]. \tag{99}
\]

**Proof.** Following the proof of Lemma 2.7, we have
\[
\int_{\mathbb{R}^3} |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \\
\leq M \left[ S_0 + \left| \int_0^t \int_{\mathbb{R}^3} \dot{u} \cdot \frac{\partial}{\partial t} (\rho \nabla \phi) \right| + \left| \int_0^t \int_{\mathbb{R}^3} \dot{u} \cdot (u \nabla (\rho \nabla \phi)) \right| + \Phi_1 + \Phi_3 \right]. \tag{100}
\]
Using the estimates (84)-(88), we readily have the bound
\[
\left| \int_0^t \int_{\mathbb{R}^3} \dot{u} \cdot \frac{\partial}{\partial t} (\rho \nabla \phi) \right| \\
\leq M \left( \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho |\nabla \dot{u}| |u| |\nabla \phi| + M \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho |\dot{u}| |\nabla \phi| + M \int_0^t \int_{\mathbb{R}^3} \sigma^3 \rho |\nabla \phi| \right) \\
\leq M \left[ \left( \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \right)^{\frac{1}{2}} + \int_0^t \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right]^{\frac{1}{2}} \leq M [\Phi_1^2 + \Phi_2^2],
\]
as well as
\[
\left| \int_0^t \int_{\mathbb{R}^3} \dot{u} \cdot (u \nabla (\rho \nabla \phi)) \right| \\
\leq M \left[ \int_0^t \int_{\mathbb{R}^3} \rho |\dot{u}| |\nabla u| |\nabla \phi| + \int_0^t \int_{\mathbb{R}^3} \rho |\dot{u}| |u| |\nabla \phi| \right] \\
\leq M \left[ \left( \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \right)^{\frac{1}{2}} + \int_0^t \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \right]^{\frac{1}{2}} \leq M [\Phi_1^2 + \Phi_2^2].
\]
Therefore by Cauchy’s inequality, the result (99) follows. \qed

We finally obtain the bound on \( \Phi_3 \) in terms of \( \Phi_2 \).

**Lemma 5.6.** For any \( 0 \leq t \leq T \leq T^* \),
\[
\Phi_3(t) \leq M [\Phi_1(t)^{\frac{1}{2}} + 1], \tag{101}
\]

**Proof.** Using the decomposition (91) on \( u \) and the estimates (93) and (94), we have
\[
\Phi_3 \leq \int_0^t \int_{\mathbb{R}^3} |\nabla u_s|^4 + \int_0^t \int_{\mathbb{R}^3} |\nabla \rho|^4 \\
\leq M \int_0^t \left( \int_{\mathbb{R}^3} |\nabla u_s|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\Delta u_s|^2 \right)^{\frac{1}{2}} + \int_0^t \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^4 \\
\leq M \left[ \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\Delta u_s|^2 \right)^{\frac{1}{2}} + 1 \right].
\]

To estimate \( \int_{\mathbb{R}^3} |\Delta u_s|^2 \), we rearrange the terms in (92)_2 to obtain
\[
\mu \Delta u_s + (\mu + \lambda) \nabla \text{div}(u_s) = \rho \dot{u} - \rho \nabla \phi.
\]
Therefore Lemma 5.2 implies that
\[
\int_{\mathbb{R}^3} |\Delta u_s|^2 \leq M \left[ \int_{\mathbb{R}^3} (|\rho \dot{u}|^2 + |\rho \nabla \phi|^2) \right] \leq M (\Phi_2 + 1),
\]
and the result follows. \qed

**Proof of Theorem 1.3.** In view of the bounds (89), (99) and (101), we can conclude that for \( 0 \leq t \leq T \leq T^* \),
\[
\Phi_1(t) + \Phi_2(t) + \Phi_3(t) \leq M, \tag{102}
\]
Together with the pointwise boundedness assumption (78) on \( \rho \) and apply the similar argument given in [16], we can show that for \( 0 \leq t \leq T \leq T^* \),
\[
\sup_{0 \leq s \leq t} ||(\rho - \bar{\rho}, u, \nabla \phi)(\cdot, s)||_{H^4} + \int_0^t ||u(\cdot, s)||_{H^1}^2 ds \leq M'', \tag{103}
\]
for some \( M'' \) which depends on \( C_0, \bar{C}, T^* \) and the system parameters \( P, \mu, \lambda \) and \( K \). An open-and-closed argument on the time interval can then be applied.
which shows that the local solution $(\rho - \tilde{\rho}, u, \phi)$ can be extended beyond $T^\ast$, which contradicts the maximality of $T^\ast$. Therefore the assumption (78) does not hold and this completes the proof of Theorem 1.3.

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E-mail address: acksuen@edu.hk