Sally modules of m-primary ideals in local rings

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Abstract: Given a local Noetherian ring \((R, m)\) of dimension \(d > 0\) and infinite residue field, we study the invariants (dimension and multiplicity) of the Sally module \(S_J(I)\) of any \(m\)-primary ideal \(I\) with respect to a minimal reduction \(J\). As a by-product we obtain an estimate for the Hilbert coefficients of \(m\) that generalizes a bound established by J. Elias and G. Valla in a local Cohen-Macaulay setting. We also find sharp estimates for the multiplicity of the special fiber ring \(F(I)\), which recover previous bounds established by C. Polini, W.V. Vasconcelos and the author in the local Cohen-Macaulay case. Great attention is also paid to Sally modules in local Buchsbaum rings.

1. Introduction

Let \((R, m)\) be a local Noetherian ring of dimension \(d > 0\) and with infinite residue field and let \(I\) be an \(m\)-primary ideal. The Rees algebra \(R(I)\) (often denoted \(R[It]\), where \(t\) is an indeterminate over \(R\)), the associated graded ring \(G(I)\) (often denoted \(gr_J(R)\)), and the special fiber ring \(F(I)\) of \(I\)

\[
R(I) = \bigoplus_{n=0}^{\infty} I^n t^n, \quad G(I) = R \otimes R/I, \quad F(I) = R \otimes R/m,
\]

collectively referred to as blowup algebras of \(I\), play an important role in the process of blowing up the variety \(\text{Spec}(R)\) along the subvariety \(V(I)\). In particular, their depth properties have been under much scrutiny in the past two decades. Moreover, these algebras are also extensively used as the means to examine diverse properties of the ideal \(I\).

\[1\]

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A successful approach to the study of blowup algebras, initiated by S. Goto, S. Huckaba, C. Huneke, D. Johnston, D. Katz, K. Nishida, N.V. Trung and others, uses minimal reductions of the ideal. This notion was first introduced and exploited by D.G. Northcott and D. Rees fifty years ago for its effectiveness in studying multiplicities in Noetherian local rings. We recall that, in our setting, a minimal reduction $J$ of $I$ is a $d$-generated subideal of $I$ such that $I^{r+1} = JI^r$ for some non-negative integer $r$. Phrased otherwise, we say that any such $J$ is a minimal reduction of $I$ if the inclusion of Rees algebras $R(J) \hookrightarrow R(I)$ is module finite and $r$ is the bound of the degrees required to generate $R(I)$ as a module over $R(J)$. The underlying philosophy is that it is reasonable to expect to recover some of the properties of $R(I)$ from the amenable structure of $R(J)$, especially whenever $r$ is sufficiently small.

An earlier approach to the depth properties of the blowup algebras dates back to J. Sally and involves a detailed analysis of numerical information encoded in the Hilbert-Samuel function of $I$, that is the function that measures the growth of the length $\lambda(R/I^n)$ of $R/I^n$ for all $n \geq 1$. It is well known that for $n \gg 0$ the function $\lambda(R/I^n)$ is a polynomial in $n$ of degree $d$, say

$$\lambda(R/I^n) = e_0(I)\left(\frac{n + d - 1}{d}\right) - e_1(I)\left(\frac{n + d - 2}{d - 1}\right) + \cdots + (-1)^d e_d(I),$$

where $e_0(I), e_1(I), \ldots, e_d(I)$ are called the Hilbert coefficients of $I$.

From a more recent vintage is a remarkable novelty which bridges the previous two approaches. More precisely, in [15] W.V. Vasconcelos enlarged the list of blowup algebras by introducing the Sally module $S_J(I)$ of $I$ with respect to a minimal reduction $J$. This is the graded $R(J)$-module defined in terms of the short exact sequence

$$0 \to IR(J) \hookrightarrow IR(I) \twoheadrightarrow S_J(I) = \bigoplus_{n=2}^{\infty} I^n/I^nJ^{n-1} \to 0.$$

When $R$ is a local Cohen-Macaulay ring, he shows that the Sally module $S_J(I)$ is a $d$-dimensional graded $R(J)$-module with a unique associated prime, namely $\mathfrak{m}R(J)$, provided $S_J(I)$ is not the trivial module. He also finds precise relations among the Hilbert coefficients of $I$ and $S_J(I)$, which in turn enable him to recover the bound $e_1(I) - e_0(I) + \lambda(R/I) \geq 0$, originally due to D.G. Northcott [10]. At the same time, he obtains the result of C. Huneke [5] and A. Ooishi [12], which says that equality holds in Northcott’s estimate if and only if $I^2 = JI$ for some (equivalently, any) minimal reduction $J$ of $I$. In particular, it follows that $G(I)$ is Cohen-Macaulay when equality holds. Later, the Sally module has been further studied by M. Vaz Pinto [18], H.-J. Wang [19, 20, 21], the author, C. Polini and M. Vaz Pinto [2], L. Doering and M. Vaz Pinto [3], and C. Polini [13]. More recently, in a joint work with C. Polini and W.V. Vasconcelos [11, 1], we have used the Sally module as the means to obtain information on the multiplicity of the special fiber ring $F(I)$, on the unmixedness (or even Cohen-Macaulayness) of $F(I)$, and, ultimately, on the reduction number $r$ of $I$.

Our investigation has been prompted by the lack of knowledge of the properties of Sally modules in non Cohen-Macaulay settings. In this paper we study the invariants (dimension and multiplicity) of the Sally module $S_J(\mathfrak{m})$ in an arbitrary local Noetherian ring $R$ of dimension $d > 0$ and infinite residue field. In particular, in Proposition 2.1 we show that the
Sally module $S_J(I)$ has dimension $d$ if and only if $e_1(m) - e_0(m) - e_1(J) + 1$ is strictly positive. However, the dimension of the Sally module $S_J(m)$ may assume intermediate values, as shown in Example 2.2. Interestingly enough, the situation becomes rather extremal in the case of a local Buchsbaum ring, as we show in Proposition 2.6 that in this setting the Sally module $S_J(m)$ has either dimension $d$ or 0. As a consequence of Proposition 2.6 we prove in Theorem 2.3 that in an arbitrary local Noetherian ring $R$ the estimate $2e_0(m) - e_1(m) + e_1(J) \leq \mu(m) - d + 2$ always holds. This bound was first obtained by J. Elias and G. Valla in the case of a local Cohen-Macaulay ring [1].

As far as the case of an arbitrary $m$-primary ideal is concerned, we do not find a closed formula for the multiplicity of its Sally module with respect to a minimal reduction, but we give an upper bound. In Proposition 3.1, we show that the multiplicity of a $d$-dimensional Sally module $S_J(I)$ is at most $e_1(I) - e_0(I) - e_1(J) + \lambda(R/I)$, with equality if and only if $I$ contains the zero-th local cohomology $H^0_\mathfrak{m}(R)$ of $R$ with support in $m$. Nevertheless, this estimate allows us to find sharp bounds for the multiplicity $f_0(I)$ of the special fiber ring $F(I)$. To be more specific, in Theorem 3.3 we show that $f_0(I) \leq e_1(I) - e_0(I) - e_1(J) + \lambda(R/I) + \mu(I) - d + 1$, thus generalizing a previous result obtained jointly with C. Polini and W.V. Vasconcelos [11]. If in addition $R$ is a local Buchsbaum ring, in Theorem 3.5 we show that $f_0(I) \leq e_1(I) + I(R) - e_1(J) + 1$, where $I(R)$ is the Buchsbaum invariant of $R$ introduced by J. Stückrad and W. Vogel [14]. This result generalizes a previous estimate due to W.V. Vasconcelos [17]. We also show that if equality holds in the latter estimate then the ideal $I$ has minimal multiplicity in the sense of S. Goto (see [11] for a similar statement).

Still in a Buchsbaum setting, we show in Proposition 3.5 that the Sally module of an ideal $I$ containing $H^0_\mathfrak{m}(R)$ has either dimension $d$ or 0.

2. The case of the maximal ideal and applications to Hilbert coefficients

We first do some calculations in order to compute the dimension and the multiplicity of the Sally module $S_J(m)$ in an arbitrary local Noetherian ring $R$ of dimension $d > 0$ with infinite residue field. We show in Proposition 2.1 that the dimension is exactly $d$ if and only if its multiplicity has a precise value, namely $e_1(m) - e_0(m) - e_1(J) + 1$. This allows us to obtain a general estimate relating the first two Hilbert coefficients of $m$. However, the situation becomes more interesting in the case of a local Buchsbaum ring, as we show in Proposition 2.6 that in this setting the Sally module has either dimension $d$ or 0.

**Proposition 2.1.** Let $(R, m)$ be a local Noetherian ring of dimension $d > 0$ with infinite residue field and let $J$ be a minimal reduction of $m$. Then the Sally module $S_J(m)$ of $m$ with respect to $J$ has dimension $d$ if and only if $e_1(m) - e_0(m) - e_1(J) + 1$ is strictly positive. In this event, the multiplicity of $S_J(m)$ is exactly $e_1(m) - e_0(m) - e_1(J) + 1$.

**Proof:** We compute the Hilbert function/polynomial of the Sally module $S_J(m)$. Chasing lengths in the short exact sequences

$$0 \to mJ^{n-1}/J^n \to m^n/J^n \to \boxed{m^n/mJ^{n-1}} \to 0,$$

$$0 \to mJ^{n-1}/J^n \to J^{n-1}/J^n \to J^{n-1}/mJ^{n-1} \to 0$$
leads to the following equality that provides a formula for the length of the component of degree \( n - 1 \) of the Sally module \( S_J(\mathfrak{m}) \)

\[
\lambda(\mathfrak{m}^n/\mathfrak{m}J^{n-1}) = \lambda(R/J^{n-1}) - \lambda(R/\mathfrak{m}^n) + \lambda(J^{n-1}/\mathfrak{m}J^{n-1}).
\]

Now observe that, for \( n \gg 0 \), both \( \lambda(R/\mathfrak{m}^n) \) and \( \lambda(R/J^{n-1}) \) can be replaced with their respective Hilbert-Samuel polynomials

\[
\lambda(R/\mathfrak{m}^n) = e_0(\mathfrak{m}) \binom{n+d-1}{d} - e_1(\mathfrak{m}) \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathfrak{m}),
\]

\[
\lambda(R/J^{n-1}) = e_0(J) \binom{n+d-2}{d} - e_1(J) \binom{n+d-3}{d-1} + \cdots + (-1)^d e_d(J).
\]

On the other hand \( J^{n-1}/\mathfrak{m}J^{n-1} \) is the component of degree \( n - 1 \) of the special fiber ring \( \mathcal{F}(J) \) of \( J \), which is a polynomial ring in \( d \) variables with coefficients over the residue field. In particular we have that

\[
\lambda(J^{n-1}/\mathfrak{m}J^{n-1}) = \binom{n+d-2}{d-1}.
\]

Finally, using the fact that \( e_0(\mathfrak{m}) = e_0(J) \) as shown in [11, Section 1, Theorem 1], since \( J \) is a reduction of \( \mathfrak{m} \), and the combinatorial identity \( \binom{p}{q} + \binom{p}{q+1} = \binom{p+1}{q+1} \) for non-negative integers \( p \) and \( q \), we have

\[
\lambda(\mathfrak{m}^n/\mathfrak{m}J^{n-1}) = s_0 \binom{n+d-2}{d-1} - s_1 \binom{n+d-3}{d-2} + \cdots + (-1)^{d-1} s_{d-1},
\]

where \( s_0 = e_1(\mathfrak{m}) - e_0(\mathfrak{m}) - e_1(J) + 1 \) and \( s_i = e_{i+1}(\mathfrak{m}) - e_i(J) - e_{i+1}(J) \) for \( i = 1, \ldots, d-1 \). This proves that the dimension of \( S_J(\mathfrak{m}) \) is \( d \) if and only if \( e_1(\mathfrak{m}) - e_0(\mathfrak{m}) - e_1(J) + 1 \) is strictly positive.

In a local Cohen-Macaulay setting the Sally module has the same dimension as the ambient ring, unless it is the trivial module. This is no longer the case in an arbitrary local Noetherian ring of positive dimension, as the next example taken from [5, 4.2] shows.

**Example 2.2.** Let \( k \) be a field and let \( S = k[X,Y,Z,W] \) be the polynomial ring in 4 variables over \( k \). Define \( T = S/(X^2,Y) \cap (Z,W) \), \( M = T_+ \) and \( R = T_M \) and \( \mathfrak{m} = MR \). Let \( x, y, z \) and \( w \) denote the images in \( R \) of \( X, Y, Z \) and \( W \), respectively. The ring \( R \) is a two-dimensional local ring such that the Sally module \( S_J(\mathfrak{m}) \) of \( \mathfrak{m} \) with respect to \( J = (x-z, y-w) \) has dimension one. In fact \( e_1(\mathfrak{m}) - e_0(\mathfrak{m}) - e_1(J) + 1 = 0 \) whereas \( -s_1 = -e_2(\mathfrak{m}) + e_1(J) + e_2(J) = 1 \).

In Theorem 2.3 below we use the techniques of [11] to generalize to a local Noetherian setting an estimate of J. Elias and G. Valla [4, Section 2], which involves the Hilbert coefficients of \( \mathfrak{m} \) and the embedding codimension of \( R \). Also, in Example 2.4 we use a well-known example, even studied by F.S. Macaulay as early as 1916, to provide an instance that illustrates when equality in Theorem 2.3 is attained.

**Theorem 2.3.** Let \( (R, \mathfrak{m}) \) be a local Noetherian ring of dimension \( d > 0 \) with infinite residue field. Then \( 2e_0(\mathfrak{m}) - e_1(\mathfrak{m}) + e_1(J) \leq \mu(\mathfrak{m}) - d + 2 \), where \( J \) is any minimal reduction of \( \mathfrak{m} \).
Proof: Let \( J \) be a minimal reduction of \( \mathfrak{m} \) and write \( \mathfrak{m} = (J, a_1, \ldots, a_{m-d}) \), where \( m \) denotes the minimal number of generators of \( \mathfrak{m} \). We now consider the Sally module \( S_J(\mathfrak{m}) \) of \( I \) with respect to \( J \) defined by means of the exact sequence introduced in [1] proof of 2.1

\[
\mathcal{R}(J) \oplus \mathcal{R}(J)^{m-d}[-1] \xrightarrow{\varphi} \mathcal{R}(\mathfrak{m}) \rightarrow S_J(\mathfrak{m})[-1] \rightarrow 0,
\]

where \( \varphi \) is the map defined by \( \varphi(r_0, r_1, \ldots, r_{m-d}) = r_0 + r_1a_1 t + \cdots + r_{m-d}a_{m-d} t \), for any element \( (r_0, r_1, \ldots, r_{m-d}) \in \mathcal{R}(J) \oplus \mathcal{R}(J)^{m-d}[-1] \). Tensoring the above exact sequence with \( R/\mathfrak{m} \) yields the bottom row in the diagram

\[
\begin{align*}
S_J(\mathfrak{m})[-1] & \\
\downarrow & \\
\mathcal{F}(J) \oplus \mathcal{F}(J)^{m-d}[-1] & \rightarrow \mathcal{G}(\mathfrak{m}) \rightarrow S_J(\mathfrak{m})[-1] \otimes R/\mathfrak{m} \rightarrow 0.
\end{align*}
\]

Hence we obtain the following multiplicity (degree) estimate

\[
e_0(\mathfrak{m}) \leq e_1(\mathfrak{m}) - e_0(\mathfrak{m}) - e_1(J) + 1 + \deg(\mathcal{F}(J) \oplus \mathcal{F}(J)^{m-d}[-1]).
\]

Notice that the contribution involving \( S_J(\mathfrak{m}) \) only occurs if its dimension is \( d \), in which case we use the result of Proposition 2.1. On the other hand, \( \mathcal{F}(J) \oplus \mathcal{F}(J)^{m-d}[-1] \) is a free \( \mathcal{F}(J) \)-module of rank \( m - d + 1 \). Thus, its multiplicity is \( m - d + 1 \), since \( \mathcal{F}(J) \) is isomorphic to a polynomial ring. The result now easily follows. \( \square \)

In the rest of the section we restrict our attention to the case of a local Buchsbaum ring. We briefly review some basic notions, whereas we refer the reader to the monograph of J. Stückrad and W. Vogel for a comprehensive treatment of the subject [14]. In short, the theory of Buchsbaum rings is a natural generalization of the concept of a Cohen-Macaulay ring and started in a remarkable series of papers by J. Stückrad and W. Vogel to answer negatively a problem of D.A. Buchsbaum. A local Noetherian ring \((R, \mathfrak{m})\) of positive dimension \( d \) is said to be a \textit{Buchsbaum ring} if and only if there exists a non-negative integer \( I(R) \) such that \( \lambda(R/J) = I(R) \) for every system of parameters \( J = (x_1, \ldots, x_d) \) of \( R \). In this setting any such \( J \) is no longer generated by a regular sequence (as in the Cohen-Macaulay case, where \( I(R) = 0 \)), but by a \( d \)-sequence. The number \( I(R) \) is the so-called \textit{Buchsbaum invariant} of \( R \) and has an explicit description either in terms of the lengths of the local cohomology modules \( H^i_{\mathfrak{m}}(R) \), for \( i = 0, \ldots, d - 1 \) (as they are annihilated by the maximal ideal \( \mathfrak{m} \)), or in terms of the higher Hilbert coefficients of any system of parameters \( J \) of \( R \). Namely, we have that

\[
I(R) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(H^i_{\mathfrak{m}}(R)) \quad \text{or} \quad I(R) = \sum_{i=1}^{d} (-1)^i e_i(J),
\]

for every parameter ideal \( J \) of \( R \). Even more surprisingly, in a local Buchsbaum ring \( R \) one has that the Hilbert coefficients \( e_i(J) \), for \( i = 1, \ldots, d \), do not depend on the system of parameters \( J \) but only on the ring \( R \). (Note that, in a local Cohen-Macaulay ring, \( e_i(J) = 0 \) for \( i > 0 \).) In particular, we observe that if \( R \) is a Buchsbaum ring then the formula in
Theorem 2.3 does not depend on the reduction $J$ of $\mathfrak{m}$, but solely on the ring $R$. Indeed, for any parameter ideal $J$ of a local Buchsbaum ring $R$ one has that

$$-e_1(J) = \sum_{i=0}^{d-1} \binom{d-2}{i-1}\lambda(H^0_m(R)),$$

where $\binom{p}{i}$ is either 0 when $p \neq -1$, or 1 when $p = -1$ (see [14] 2.7(ii)).

**Example 2.4.** The curve $X$ in $\mathbb{P}^3$ given parametrically by $\{s^4, s^3u, su^3, u^4\}$ is such that the local ring of the affine cone over $X$ at the vertex is a Buchsbaum ring of dimension 2 with invariant $I(R) = 1 = -e_1(J)$ for every system of parameters $J$ of $R$. One can verify that equality holds in the bound established in Corollary 2.3 as $e_0(\mathfrak{m}) = 4$ and $e_1(\mathfrak{m}) = 3$. Moreover, the maximal ideal $\mathfrak{m}$ has reduction number 2, the associated graded ring $G(\mathfrak{m}) \cong R$ is Buchsbaum, the Sally module $S_J(\mathfrak{m})$ of $\mathfrak{m}$ with respect to a minimal reduction $J$ is two-dimensional with multiplicity 1 and $mS_J(\mathfrak{m}) = 0$.

The above example and other similar ones led us to ask the following question, which is motivated by an analogous result due to J. Elias and G. Valla in a local Cohen-Macaulay setting [41 2.1].

**Question 2.5.** Let $(R, \mathfrak{m})$ be a local Buchsbaum ring and suppose that $2e_0(\mathfrak{m}) - e_1(\mathfrak{m}) + e_1(J) = \mu(\mathfrak{m}) - d + 2$, where $J$ is any minimal reduction of $\mathfrak{m}$. Is the associated graded ring $G(\mathfrak{m})$ always Buchsbaum?

We end this section by showing that the dimension of the Sally module is rather extremal (either $d$ or 0) in the case in which the ambient ring is Buchsbaum. Our proof uses a remarkable generalization due to S. Goto and K. Nishida [5 1.1] of the result by C. Huneke [8 2.1] and A. Ooishi [12 3.2 and 3.3] quoted in the introduction.

**Proposition 2.6.** Let $(R, \mathfrak{m})$ be a local Buchsbaum ring of dimension $d > 0$ with infinite residue field and let $J$ be a minimal reduction of $\mathfrak{m}$. Then the Sally module $S_J(\mathfrak{m})$ of $\mathfrak{m}$ with respect to $J$ has either dimension $d$ or 0.

**Proof:** As shown in the proof of Proposition 2.4.1 if $e_1(\mathfrak{m}) - e_0(\mathfrak{m}) - e_1(J) + 1 > 0$ then the Sally module $S_J(\mathfrak{m})$ has dimension $d$. Suppose now that $e_1(\mathfrak{m}) - e_0(\mathfrak{m}) - e_1(J) + 1 = 0$. Let $H$ denote the zero-th local cohomology module $H^0_m(R)$ of $R$ with support in $\mathfrak{m}$ and let $\overline{}$ denote images in the ring $\overline{R} = R/H$. The two Sally modules $S_J(\mathfrak{m})$ and $S_J(\overline{\mathfrak{m}})$ are related by the short exact sequence

$$0 \rightarrow K = \bigoplus_{n \geq 2} \frac{H \cap \mathfrak{m}^n + mJ^{n-1}}{mJ^{n-1}} \rightarrow S_J(\mathfrak{m}) \rightarrow S_J(\overline{\mathfrak{m}}) \rightarrow 0.$$

Since $H$ is Artinian and therefore $H \cap \mathfrak{m}^n = 0$ for $n$ sufficiently large, it follows that the kernel $K$ has only finitely many components, hence it is Artinian. On the other hand, by [5 1.1] we have that $\mathfrak{m}^2 \subset J\mathfrak{m} + H$. Hence $S_J(\overline{\mathfrak{m}}) = 0$ so that $S_J(\mathfrak{m}) = K$ is Artinian.

We also observe that in the case of a zero-dimensional Sally module all the Hilbert coefficients of $\mathfrak{m}$ only depend on the multiplicity of $\mathfrak{m}$ and the local Buchsbaum ring $R$. 
Corollary 2.7. Let \((R, \mathfrak{m})\) be a local Buchsbaum ring of dimension \(d > 0\) with infinite residue field and let \(I\) be a minimal reduction of \(\mathfrak{m}\). If \(e_1(\mathfrak{m}) = e_0(\mathfrak{m}) + e_1(I) - 1\) then

\[ e_i(\mathfrak{m}) = e_{i-1}(I) + e_i(J), \]

for \(i = 2, \ldots, d\).

3. Sally modules of \(\mathfrak{m}\)-primary ideals and special fiber rings

It is natural to ask what happens in the case of an arbitrary \(\mathfrak{m}\)-primary ideal \(I\) of a local Noetherian ring \(R\). We address this issue next. In this case we can only give an upper bound for the multiplicity of the Sally module \(S_J(I)\) of \(I\) with respect to a minimal reduction \(J\), unless the ideal \(I\) contains the zero-th local cohomology module \(H^0_{\mathfrak{m}}(R)\) of \(R\) with support in \(\mathfrak{m}\). Nevertheless, this enables us to obtain general multiplicity estimates for the special fiber ring \(\mathcal{F}(I)\): see Theorem \[\text{[5.4] and Theorem [5.5]}\]

In a Cohen-Macaulay setting, the estimates had been previously obtained in a paper by V.W. Vasconcelos [17 2.4] and in a joint work with C. Polini, V.W. Vasconcelos [1 2.1, 2.2].

Proposition 3.1. Let \((R, \mathfrak{m})\) be a local Noetherian ring of dimension \(d > 0\) with infinite residue field and let \(I\) be an \(\mathfrak{m}\)-primary ideal. Then a \(d\)-dimensional Sally module \(S_J(I)\) of \(I\) with respect to a minimal reduction \(J\) has multiplicity at most \(e_1(I) - e_0(I) - e_1(J) + \lambda(R/I)\), with equality if and only if \(I\) contains \(H^0_{\mathfrak{m}}(R)\).

Proof: In a similar fashion as in the proof of Proposition \[\text{2.1}\] we obtain a formula for the length of the component of degree \(n - 1\) of the Sally module \(S_J(I)\)

\[ \lambda(I^n/IJ^{n-1}) = \left( e_1(I) - e_0(I) - e_1(J) + \hat{e}_0(J, I) \right) \left( \frac{n + d - 2}{d - 1} \right) + \text{lower terms}, \]

where \(\hat{e}_0(J, I)\) is the multiplicity of graded module \(G(J) \otimes R/I\). Observe that the statement follows once we show that \(\hat{e}_0(J, I) \leq \lambda(R/I)\), with equality if and only if \(I\) contains \(H^0_{\mathfrak{m}}(R)\). We prove this claim by induction on the dimension \(d\) of \(R\). If \(d = 1\), let \(H\) denote the zero-th local cohomology module \(H^0_{\mathfrak{m}}(R)\) of \(R\) with support in \(\mathfrak{m}\) and let \(\mathfrak{m}\) denote images in the one-dimensional local Cohen-Macaulay ring \(\overline{R} = R/H\). Since \(H \cap J^n\) is eventually the zero ideal as \(H\) is Artinian, it is not difficult to verify that \(J^n/IJ^n \cong \overline{J}^n/\overline{J}^{n+1}\) for \(n \gg 0\). Thus \(\hat{e}_0(J, I) = \hat{e}_0(J', I') = \lambda(\overline{R}/I') \leq \lambda(R/I)\), with equality if and only if \(I\) contains \(H\).

Suppose now \(d > 1\). We can find an element \(x\) which belongs to \(J \setminus \mathfrak{m}J\) and whose image is superficial in \(G(J) \otimes R/I\). Let \(I'\) denote images in the ring \(R' = R/(x)\), which has dimension \(d - 1\). Observe that \(\hat{e}_0(J, I) = \hat{e}_0(J', I')\), as the image of \(x\) is superficial in \(G(J) \otimes R/I\) (see \[\text{[11 22.6]}\]). Thus, by inductive hypothesis we conclude that \(\hat{e}_0(J, I) = \hat{e}_0(J', I') \leq \lambda(R'/I') = \lambda(R/I)\). On the other hand, by induction we have that \(\hat{e}_0(J', I') = \lambda(R'/I')\) if and only if \(H^0_{\mathfrak{m}}(R') \subseteq I'\). Our assertion now follows as \(H^0_{\mathfrak{m}}(R)R' \subseteq H^0_{\mathfrak{m}}(R')\).
Remark 3.2. If in addition to the assumptions in Proposition 3.1 all the local cohomology modules $H^i_{m}(R)$ have finite length for $i < d$, S. Goto and K. Nishida have shown that

$$-e_1(J) \leq \sum_{i=0}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{m}(R)).$$

(see [5, 2.4]). This yields another bound on the multiplicity of a $d$-dimensional Sally module which depends solely on the ideal and still reduces to the classical bound of W.V. Vasconcelos in the Cohen-Macaulay case.

The next result is similar to the one in Proposition 2.6.

Proposition 3.3. Let $(R, m)$ be a local Buchsbaum ring of dimension $d > 0$ with infinite residue field and let $I$ be an $m$-primary ideal containing $H^0_{m}(R)$. Then the Sally module $S_J(I)$ of $I$ with respect to a minimal reduction $J$ has either dimension $d$ or 0.

Proof: The proof is similar to the one of Proposition 2.6. In fact, if $S_J(I)$ has not dimension $d$ then then $e_1(I) - e_0(I) - e_1(J) + \lambda(R/I) = 0$ by Proposition 3.1 and hence $I^2 \subset JI + H^0_{m}(R)$ by [5, 1.1].

We now use the previous results to obtain estimates on the multiplicity of the special fiber ring of any $m$-primary ideal.

Theorem 3.4. Let $(R, m)$ be a local Noetherian ring of dimension $d > 0$ with infinite residue field and let $I$ be an $m$-primary ideal. Then the multiplicity $f_0(I)$ of the special fiber ring $F(I)$ of $I$ is at most $e_1(I) - e_0(I) - e_1(J) + \lambda(R/I) + \mu(I) - d + 1$, where $J$ is any minimal reduction of $I$.

Proof: Let $J$ be any minimal reduction of $I$ and let $m$ denote the minimal number of generators of $I$. We proceed in a similar fashion as in the proof of Theorem 2.5. Namely, tensoring with $R/m$ the defining sequence of the Sally module $S_J(I)$ of $I$ with respect to $J$

$$R(J) \oplus R(J)^{m-d}[-1] \xrightarrow{\varphi} R(I) \longrightarrow S_J(I)[-1] \rightarrow 0$$

yields the following exact sequence

$$F(J) \oplus F(J)^{m-d}[-1] \rightarrow F(I) \rightarrow S_J(I)[-1] \otimes R/m \rightarrow 0.$$

Our assertion now follows after taking into account the estimate of Proposition 3.1.

Theorem 3.5. Let $(R, m)$ be a local Buchsbaum ring of dimension $d > 0$ with infinite residue field and let $I$ be an $m$-primary ideal. Then the multiplicity $f_0(I)$ of the special fiber ring $F(I)$ of $I$ is at most $e_1(I) + I(R) - e_1(J) + 1$, where $J$ is any minimal reduction of $I$. Furthermore, if the bound is attained then the ideal $I$ has minimal multiplicity in the sense of S. Goto, that is $mI = mJ$ for any minimal reduction $J$ of $I$.

Proof: Our assertions follow from Theorem 3.4 and the equality due to K. Yamagishi $e_0(I) = \mu(I) - d + \lambda(R/I) - I(R) + \lambda(mI/mJ)$ [22, 2.5, 2.6], which generalizes the classical result of S. Abhyankar.

The next example shows that the ideal $I$ may have minimal multiplicity even if the inequality in Theorem 3.5 is strict.
Example 3.6. Let $k$ be a field and let $S = k[X_1, X_2, X_3, X_4, V, A_1, A_2, A_3]$ be the polynomial ring in 8 variables over $k$. and put
\[ a = (X_1, X_2, X_3)^2 + (X_2^2) + (X_1 V, X_2 V, X_3 V, X_4 V) + (V^2 - A_1 X_1 - A_2 X_2 - A_3 X_3). \]
Define $T = S/a$, $M = T_+$ and $R = T_M$ and $m = MR$. Let $x_i$, $v$ and $a_j$ denote the images of $X_i$, $V$ and $A_j$ modulo $a$, respectively. It follows from \[ \text{Section 4} \] that $R$ is a local Buchsbaum ring of dimension 3 with $I(R) = 1$. Moreover, the ideal $I = J$: $m$, where $J = (a_1, a_2, a_3) R$, is such that $f_0(I) = 5$, $e_1(I) = 3$ and $e_1(J) = -1$. Thus we have a strict inequality in the bound of Theorem 3.7. However, the ideal $I$ satisfies $I^3 = JI^2$ and has minimal multiplicity, that is $mI = mJ$.

We conclude with a generalization of \[ \text{[11] 2.9} \] to a non Cohen-Macaulay setting.

Proposition 3.7. Let $(R, m)$ be a local Noetherian ring of dimension $d > 0$ with infinite residue field and let $I$ be an $m$-primary ideal. Then
\[ 2e_0(I) - e_1(I) + e_1(J) \leq \lambda(R/I) (\mu(I) - d + 2), \]
where $J$ is any minimal reduction of $I$.

Proof: Repeat the proof of Theorem 2.3 using the estimate given in Proposition 3.1.

References

[1] A. Corso, C. Polini and W.V. Vasconcelos, Multiplicity of the special fiber of blowups, preprint 2003 in arXiv.math.AC/0307037.
[2] A. Corso, C. Polini and M. Vaz Pinto, Sally modules and associated graded rings, Comm. Algebra 26 (1998), 2689-2708.
[3] L.R. Doering and M. Vaz Pinto, On the monotonicity of the Hilbert function of the Sally module, Comm. Algebra 28 (2000), 1861-1866.
[4] J. Elias and G. Valla, Rigid Hilbert functions, J. Pure and Appl. Algebra 71 (1991), 19-41.
[5] S. Goto and K. Nishida, Hilbert coefficients and Buchsbaumness of associated graded rings, J. Pure and Appl. Algebra 181 (2003), 61-74.
[6] S. Goto and H. Sakurai, The equality $I^2 = Qi$ in Buchsbaum rings, to appear in Rend. Sem. Mat. Univ. Padova 110 (2003).
[7] ________, The equality $I^2 = Qi$ in Buchsbaum rings with multiplicity two, preprint 2003 in arXiv.math:03060146.
[8] C. Huneke, Hilbert functions and symbolic powers, Michigan Math. J. 34 (1987), 293-318.
[9] M. Nagata, Local rings, Interscience, New York, 1962.
[10] D.G. Northcott, A note on the coefficients of the abstract Hilbert function, J. London Math. Soc. 35 (1960), 209-214.
[11] D.G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Camb. Phil. Soc. 50 (1954), 145-158.
[12] A. Ooishi, $\Delta$-genera and sectional genera of commutative rings, Hiroshima Math. J. 17 (1987), 361-372.
[13] C. Polini, A filtration of the Sally module and the associated graded ring of an ideal, Comm. Algebra 28 (2000), 1335-1341.
[14] J. St"uckrad and W. Vogel, Buchsbaum rings and applications, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.
[15] W.V. Vasconcelos, Hilbert functions, analytic spread and Koszul homology, in Commutative algebra: syzygies, multiplicities, and birational algebra, W. Heinzer, C. Huneke and J. Sally Eds., Contemporary Mathematics 159, American Mathematical Society, Providence, 1994, 401-422.
[16] ________, Cohomological degrees of graded modules, in Six lectures on commutative algebra, J. Elias, J.M. Giral, R.M. Miró-Roig and S. Zarzuela Eds., Progress in Mathematics 166, Birkhäuser, Basel, 1998, 345-392.
\[ \text{17} \] Multiplicities and reduction numbers, to appear in Compositio Math.

\[ \text{18} \] M. Vaz Pinto, Hilbert functions and Sally modules, J. Algebra 192 (1997), 504-523.

\[ \text{19} \] H.-J. Wang, An interpretation of $\text{depth}(G(I))$ and $e_1(I)$ via the Sally module, Comm. Algebra 25 (1997), 303-309.

\[ \text{20} \] On Cohen-Macaulay local rings with embedding dimension $e + d - 2$, J. Algebra 190 (1997), 226-240.

\[ \text{21} \] Hilbert coefficients and the associated graded rings, Proc. Amer. Math. Soc. 128 (2000), 963-973.

\[ \text{22} \] K. Yamagishi, Buchsbaumness in Rees modules associated to ideals of minimal multiplicity in the equi-$\ell$-invariant case, J. Algebra 251 (2002), 213-255.