A differential topological invariant on spin manifolds from supersymmetric path integrals

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November 25, 2021

Abstract

We show that the $N = 1/2$ supersymmetric path integral on a closed even dimensional Riemannian spin manifold, realized via Chen forms and recent results from noncommutative geometry, induces a differential topological invariant (which does not depend on the Riemannian metric).

1 Motivation

Let $X$ be a compact even dimensional topological spin manifold[1]. The fixed topological spin structure induces an orientation (cf. Corollary E in [10]) on the Fréchet manifold $LX$ of smooth loops $\gamma : \mathbb{T} \to X$, whose tangent space $T_\gamma LX$ at a fixed loop $\gamma \in LX$ is given by the space of vector fields on $X$ along $\gamma$, that is, smooth maps $A : \mathbb{T} \to TX$ with $\dot{\gamma}(s) \in T_{\gamma(s)}X$ for all $s \in \mathbb{T}$. Given a Riemannian metric $g$ on $X$ let $E^g \in C^\infty(LX)$ and $\omega^g \in \Omega^2(LX)$ denote the energy functional and, respectively, the presymplectic form

$$E^g_\gamma := \int_\mathbb{T} g(\dot{\gamma}, \dot{\gamma}), \quad \omega^g(A, B) := \int_\mathbb{T} g(\nabla_{\dot{\gamma}} A, B),$$

where we will occasionally identify $\mathbb{T} = [0, 1)/\sim$. The following $N = 1/2$ supersymmetric path integral plays a crucial role in the context of Duistermaat-Heckman localization on $LX$: with

$$\widehat{\Omega}(LX) := \prod_{j=0}^\infty \Omega^j(LX)$$

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1 We work exclusively in the category of smooth manifolds without boundary.
the space smooth differential forms of $LX$, one formally sets

$$ \mathcal{J}^g : \hat{\Omega}(LX) \longrightarrow \mathbb{C}, \quad \mathcal{J}^g[\alpha] := \int_{LX} e^{-E^g + \omega^g} \wedge \alpha. \quad (1.1) $$

Note that even though $LX$ is oriented, as it stands, the definition of $\mathcal{J}^g$ does not make sense for (at least) the following reasons:

- there exists no infinite dimensional Lebesgue measure;
- the integral of an inhomogeneous differential form (which are the ones of interest) should by definition be the integral of its top degree part, however, $LX$ is infinite dimensional;
- $LX$ is noncompact, so even if one finds a natural way to integrate differential forms on $LX$, some care has to be taken concerning the question of finding a class of 'integrable' (smooth) differential forms.

As we are going to explain in a moment, the mathematical solution of these problems is tied together and manifests itself in a construction of $\mathcal{J}^g$ via Chen integrals and the differential graded Chern character on $(M,g)$. However, in order to motivate our main results, let us continue with our heuristic observations for the moment.

With $\iota$ the contraction by the vector field $A$ on $LX$ given by $\gamma \mapsto \dot{\gamma}$, which generates the natural $T$-action on $LX$ given by rotating loops, and

$$ \hat{\Omega}_T(LX) := \{ \alpha \in \hat{\Omega}(LX) : \mathcal{L}_A \alpha = 0 \} $$

the space of $T$-invariant differential forms, there is a supercomplex

$$ \cdots \xrightarrow{d + \iota} \hat{\Omega}^+_T(LX) \xrightarrow{d + \iota} \hat{\Omega}_T(LX) \xrightarrow{d + \iota} \hat{\Omega}^-_T(LX) \xrightarrow{d + \iota} \cdots, \quad (1.2) $$

and (with a slight abuse of notation) the dual supercomplex

$$ \cdots \xrightarrow{d + \iota} \hat{\Omega}^+_T(LX) \xrightarrow{d + \iota} \hat{\Omega}_T(LX) \xrightarrow{d + \iota} \hat{\Omega}^-_T(LX) \xrightarrow{d + \iota} \cdots. \quad (1.3) $$

Note that these complexes are actually well-defined within the differential calculus of Fréchet manifolds. Now, supersymmetry takes the form

$$ \mathcal{J}^g[(d + \iota)\alpha] = 0 \quad \text{for all } \alpha \in \hat{\Omega}(LX). $$

Moreover, $\mathcal{J}^g$ is an even current, as $LX$ is formally even-dimensional, so that $\mathcal{J}^g$ determines an even homology class in the homology of $(1.3)$. Finally, one can derive the following infinite dimensional analogue of the Duistermaat-Heckman localization formula,

$$ \mathcal{J}^g[\alpha] = \int_X \hat{A}(M,g) \wedge \alpha|_X \quad \text{for all } \alpha \in \hat{\Omega}(LX) \text{ with } (d + \iota)\alpha = 0, $$
which leads to a simple and differential geometric 'proof' of the Atiyah-Singer index theorem \[3, 2, 1\], and which was in fact, the main motivation that lead to the discovery of \(J^g\).

The aim of this paper is to examine the dependence of \(J^g\) on \(g\). To this end, let \(g_\bullet = (g_t)_{t \in [0,1]}\) be a smooth family of Riemannian metrics on \(X\) and define for every fixed \(t \in [0,1]\) a differential form

\[
\beta_t^{g_\bullet} \in \Omega^1(LX), \quad \beta_t^{g_\bullet}(A) := -\frac{1}{2} \int_T g_t(\dot{\gamma}, A),
\]

and the induced odd current

\[
\mathcal{C}_t^{g_\bullet} : \widehat{\Omega}(LX) \rightarrow \mathbb{C}, \quad \mathcal{C}_t^{g_\bullet}(\alpha) := J^{g_t}(\beta_t^{g_\bullet} \wedge \alpha).
\]

In the appendix, we are going to derive the formula

\[
\frac{d}{dt}J^{g_t} = (d + \iota)\mathcal{C}_t^{g_\bullet} \quad \text{for all } t \in [0,1]. \tag{1.4}
\]

This equality has an important consequence: defining the (odd) Chern-Simons current \(\mathcal{C}^{g_\bullet}\) by

\[
\mathcal{C}^{g_\bullet} := \int_0^1 \mathcal{C}_t^{g_\bullet} dt : \widehat{\Omega}(LX) \rightarrow \mathbb{C},
\]

one gets the transgression formula

\[
J^{g_1} - J^{g_0} = (d + \iota)\mathcal{C}^{g_\bullet},
\]

so that the homology class induced by \(J^g\) in the homology of \(\text{eqs}\) does not depend on a particular choice of a Riemannian metric \(g\) on \(X\). These heuristic considerations show that any mathematically rigorous definition of the supersymmetric current \(J^g\) should lead to a differential topologic invariant of \(X\).

2 Main results

Let us explain now how these heuristic considerations can be verified in a mathematically rigorous way. To this end, we first explain the natural class of (smooth) integrable differential forms on \(LX\): we turn \(\widehat{\Omega}(LX)\) into a complete locally convex Hausdorff space by equipping \(\Omega_j(LX)\) with the family of seminorms \(\nu_f(\alpha) := \nu(f^* \alpha)\), where \(f\) is a smooth map from a finite dimensional manifold \(Y\) to \(LX\), and \(\nu\) is a continuous seminorm on the Fréchet space \(\Omega_j(Y)\), and by equipping \(\widehat{\Omega}(LX)\) with the product topology. Given \(\alpha \in \Omega(X)\) and \(t \in T\) one defines \(\alpha(t) \in \Omega(LX)\) to be the pullback of \(\alpha\) with respect to the evaluation \(\gamma \mapsto \gamma(t)\).

Consider the Fréchet space of \(T\)-invariant differential forms \(\Omega_T(X \times T)\) on \(X \times T\), with \(T\) acting on the second slot. With \(\vartheta_T \in \Omega(T)\) the volume form, any \(\theta \in \Omega_T(X \times T)\) can be uniquely written in the form \(\theta = \theta' + \vartheta_T \wedge \theta''\) with \(\theta', \theta'' \in \Omega(X)\).
Associated to this construction, there is the space of entire chains $C^\epsilon_T(X)$ which is defined as the completion of

$$C^\epsilon_T(X) := \bigoplus_{N=0}^\infty \Omega_T(X \times T) \otimes \Omega_T(X \times T)^{\otimes N},$$

with

$$\Omega_T(X \times T)^{\otimes N} := \Omega_T(X \times T)^{\otimes N}/(C \cdot 1)$$

and where $C_T(X)$ is equipped with the following family of seminorms: given any continuous seminorm $\nu$ on $\Omega_T(X \times T)$, one gets the induced projective tensor norm

$$\pi_{\nu,N} \quad \text{on} \quad \Omega_T(X \times T) \otimes \Omega_T(X \times T)^{\otimes N},$$

and then a seminorm $\epsilon_{\nu}$ on $C_T(X)$ by setting

$$\epsilon_{\nu}(c) := \sum_{N=0}^\infty \pi_{\nu,N}(c_N) \cdot \left\lfloor \frac{N}{2} \right\rfloor !,$$

if

$$c = \sum_{N=0}^\infty c_N \in C_T(X), \quad \text{with} \quad c_N \in \Omega_T(X \times T) \otimes \Omega_T(X \times T)^{\otimes N} \quad \text{for all} \quad N.$$

The required family of seminorms is now given by $\epsilon_{\nu}$, where $\nu$ is a continuous seminorm on $\Omega_T(X \times T)$.

There exists a uniquely determined continuous map $\Psi$, the equivariant Chen iterated integral map,

$$\Psi : C^\epsilon_T(X) \longrightarrow \hat{\Omega}(LX),$$

such that for all $N \in \mathbb{N}_\geq 0, \theta_0, \ldots, \theta_N \in \theta \in \Omega_T(X \times T)$, one has

$$\Psi(\theta_0 \otimes \cdots \otimes \theta_N) = \int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta_0(0) \wedge (i\theta'_{1}(t_1) - \theta''_{1}(t_1)) \wedge \cdots \wedge (i\theta'_{N}(t_N) - \theta''_{N}(t_N)) \, dt_1 \cdots dt_N. \quad (2.3)$$

**Definition 2.1.** The space of integrable Chen forms $\tilde{\Omega}(LX) \subset \hat{\Omega}(LX)$ is defined as the image of $\Psi$.

Set

$$\tilde{\Omega}_T(LX) := \tilde{\Omega}(LX) \cap \hat{\Omega}_T(LX).$$

The following result follows essentially from calculations made in [6]. A detailed proof will be given in Section 3.

**Proposition 2.2.** There is a well-defined supercomplex

$$\cdots \longrightarrow \tilde{\Omega}_T^+(LX) \xrightarrow{d+i} \tilde{\Omega}_T^-(LX) \xrightarrow{d+i} \tilde{\Omega}_T^+(LX) \xrightarrow{d+i} \cdots. \quad (2.4)$$
The associated dual supercomplex will be denoted with
\[ \cdots \to \widetilde{\Omega}_+^d(LX) \to \tilde{\Omega}_+^d(LX) \to \widetilde{\Omega}_+^d(LX) \to \cdots. \tag{2.5} \]
Let us now give the formula for \( \mathcal{J}^g \). Recall that we have fixed a topologic spin structure on \( X \). Consider the spinor bundle \( \Sigma_g \to X \) induced by \( g \), with its (essentially self-adjoint) Dirac operator \( D_g \) on the super-Hilbert-space of \( L^2 \)-spinors \( \Gamma_{L^2}(X,\Sigma_g) \), and the (natural extension to differential forms of all degrees of the) Clifford multiplication
\[ c_g : \Omega(X) \to \Gamma_C(S^\infty(X,End(\Sigma_g))). \]
Given any \( N \in \mathbb{N}_{\geq 1} \) and any tupel \( (\theta_1, \ldots, \theta_N) \) of elements of \( \Omega_T(X \times \mathbb{T}) \), define a differential operator \( F_g(\theta_1, \ldots, \theta_N) \) in \( \Sigma_g \to X \) as follows,
\[ F_g(\theta) = c_g(d\theta') - [D_g, c_g(\theta')] - c_g(\theta'') \]
\[ F_g(\theta_1, \theta_2) = (-1)^{|[\theta_1]|}(c_g(\theta_1'\theta_2') - c_g(\theta_1')c_g(\theta_2')) \]
\[ F_g(\theta_1, \ldots, \theta_N) = 0, \quad \text{if} \quad N \geq 3, \]
where here and in the sequel all commutators are super-commutators. For \( M \leq N \) denote with \( P_{M,N} \) all tuples \( I = (I_1, \ldots, I_M) \) of subsets of \( \{1, \ldots, N\} \) with \( I_1 \cup \cdots \cup I_M = \{1, \ldots, N\} \) and with each element of \( I_a \) smaller than each element of \( I_b \) whenever \( a < b \). Given
\[ \theta_1, \ldots, \theta_N \in \Omega_T(X \times \mathbb{T}), \quad I = (I_1, \ldots, I_M) \in P_{M,N}, \quad 1 \leq a \leq M, \]
set
\[ \theta_{I_a} := (\theta_{i+1}, \ldots, \theta_{i+m}), \quad \text{if} \quad I_a = \{j \mid i < j \leq i + m\} \text{ for some } i, m. \]
With these preparations, the following is the main result of [7]:

**Theorem 2.3.** There exists a uniquely determined current \( \mathcal{J}^g : \tilde{\Omega}(LX) \to \mathbb{C} \) such that for all \( N \in \mathbb{N}_{\geq 0} \), \( \theta_0, \ldots, \theta_N \in \Omega_T(X \times \mathbb{T}) \) one has
\[ \mathcal{J}^g \left[ \int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta_0(0) \wedge (i\theta_1'(t_1) - \theta_1''(t_1)) \wedge \cdots \wedge (i\theta_N'(t_N) - \theta_N''(t_N)) \, dt_1 \cdots dt_N \right] \]
\[ = \sum_{M=1}^N (-1)^M \sum_{I \in P_{M,N}} \int_{\{0 \leq t_1 \leq \cdots \leq t_M \leq 1\}} \text{Str}_g \left( c_g(\theta_0)e^{(t_1-t_2)D_g^2}F_g(\theta_{I_1}) \times \right. \]
\[ \left. \times e^{-(t_2-t_1)D_g^2}F_g(\theta_{I_2}) \cdots e^{-(t_{M-1}-t_M)D_g^2}F_g(\theta_{I_M})e^{-(t_{M-1}-t_M)D_g^2} \right) \, dt_1 \cdots dt_M, \]
where \( \text{Str}_g \) denotes the supertrace in \( \Gamma_{L^2}(X,\Sigma_g) \). Moreover, \( \mathcal{J}^g \) is even and \( (d+\imath)\mathcal{J}^g = 0 \), so that \( \mathcal{J}^g \) defines an even homology class in the homology of (2.7), and one has the localization formula
\[ \mathcal{J}^g[\alpha] = \int_X \widehat{A}(M,g) \wedge \alpha|_X \quad \text{for all} \ \alpha \in \widetilde{\Omega}(LX) \text{ with } (d+\imath)\alpha = 0. \]
That this definition of $J^g$ is natural, in the sense that it really serves as an implementation of the right hand side of (1.1), has been indicated in [9] using the Pfaffian line bundle. A probabilistic representation of $J^g$ has been derived in [10], generalizing the earlier result from [5] for $N = 1$ to all orders.

Here comes the main result of this note:

**Theorem 2.4.** Assume $g_\bullet = (g_t)_{t \in [0,1]}$ is a smooth family of Riemannian metrics on $X$. Then there exists a canonically given odd current $C^g_\bullet : \tilde{\Omega}(LX) \to \mathbb{C}$ with $J^g_1 - J^g_0 = (d + \iota) C^g_\bullet$; in particular, the homology class induced by $J^g$ in the homology of (2.5) does not depend on a particular choice of a Riemannian metric $g$ on $X$.

Our main result yields a new differential topological invariant:

**Corollary 2.5.** Let $M$ and $N$ be compact even-dimensional, oriented spin manifolds with fixed topological spin-structures. Assume there exists a diffeomorphism $f : M \to N$ preserving orientations and topological spin-structures. Then, for any choice of Riemannian metrics $g$ and $h$ on $M$ resp. on $N$, the homology class induced by $J^g_M$ in the homology of (2.5) equals the homology class of $f^* J^h_N$.

**Proof.** Setting $g_1 := f^* h$, the diffeomorphism $f$ becomes an orientation and metric spin-structure preserving isometry $f : (M,g_1) \to (N,h)$ furnishing unitary equivalences between Clifford multiplications and Dirac operators on $(M,g_1)$ and $(N,h)$. Formula (2.6) shows that $J^g_M$ and $f^* J^h_N$ are equal, and Theorem 2.4 establishes the claim. □

### 3 Proof of Proposition 2.2

We have to show that $d + \iota$ maps

$$\tilde{\Omega}_T(LX) = \tilde{\Omega}(LX) \cap \tilde{\Omega}_T(LX)$$

to itself. We give $\Omega_T(X \times \mathbb{T})$ the $\mathbb{Z}$-grading

$$\theta' + \theta_T \wedge \theta'' \in \Omega_T(X \times \mathbb{T})^j \iff \theta' \in \Omega^j(X), \theta'' \in \Omega^{j+1}(X)$$

and turn it into a locally convex DGA using the differential $d + \iota \partial_T$ with $\partial_T$ the canonical vector field on $\mathbb{T}$. Then $C_T(X)$ inherits the $\mathbb{Z}$-grading induced by

$$C_T(X) = \bigoplus_{N=0}^{\infty} \Omega_T(X \times \mathbb{T}) \otimes \Omega_T(X \times \mathbb{T})[1]^{\otimes N},$$

where $\Omega_T(X \times \mathbb{T})[1]$ denotes $\Omega_T(X \times \mathbb{T})$ as a set with the shifted grading

$$\Omega_T(X \times \mathbb{T})[1]^j := \Omega_T(X \times \mathbb{T})^{j+1}.$$

With $b$ the Hochschild differential and $B$ the Connes differential in the $\mathbb{Z}$-graded category, the space $C_T(X)$ becomes a supercomplex with the differential $d + \iota \partial_T + b + B$. By continuity,
the same holds true for $C^*_T(X)$.

Let

$$A : \tilde{\Omega}(LX) \longrightarrow \tilde{\Omega}(LX)$$

be the idempotent linear operator obtained by averaging the $T$-action on $LX$. Then as shown in [6] one has the formulae

$$\Psi(d + \iota \partial_t + b) = d\Psi, \quad \Psi B + A \Psi \iota \partial_t = A \iota \Psi,$$

noting that in the notation of [6] one has $\rho = A \Psi$. Note also that $A$ commutes with $d$ and $\iota$, so that

$$A \Psi(d + \iota \partial_t + b + B) = (d + \iota) A \Psi.$$ 

Assume that $\alpha \in \tilde{\Omega}(LX)$ is $T$-invariant. This means that $\alpha = \Psi(\theta)$ for some $\theta \in C^*_T(X)$ and that $A \Psi(\theta) = \Psi(\theta)$. Clearly, $(d + \iota)\alpha = (d + \iota)\Psi(\theta)$, so that $(d + \iota)\alpha$ is in $\tilde{\Omega}(LX)$, furthermore,

$$(d + \iota)\alpha = (d + \iota)A \Psi(\theta) = A \Psi((d + \iota \partial_t + b + B)\theta),$$

which shows that $(d + \iota)\alpha$ is $T$-invariant. This completes the proof.

4 Proof of Theorem 2.4

We briefly recall the Bourguignon-Gauduchon machinery for metric changes of the Dirac operator [4]. For any $t \in [0, 1]$, define a section $A_t$ of $\text{End}(TX)$ by

$$g_0(u,v) = g_t(A_t u, v) \quad \text{for all} \quad x \in X, u,v \in T_x X.$$ 

Then $A_t$ is strictly positive w.r.t. $g_t$ and $g_0$ and $A^{-1/2}_t$ is a pointwise isometry $(TX,g_t) \rightarrow (TX,g_0)$. It therefore lifts canonically to an $SO(n)$-equivariant bundle map

$$b_t : \text{SO}(X,g_t) \longrightarrow \text{SO}(X,g_0),$$

where $\text{SO}(X,g_t)$ denotes the bundle of oriented orthonormal frames of $X$ w.r.t. the Riemannian metric $g_t$.

Now recall that we have fixed a topological spin structure. This implies that every Riemannian metric $g_t$ canonically induces a Riemannian spin structure on $X$, i.e., a Spin$(n)$-principal fibre bundle $P_t$ over $X$ together with a $\xi$-equivariant map $\pi_t : P_t \rightarrow \text{SO}(X,g_t)$ such that $(P_t,\pi_t)$ is a $\xi$-reduction of $\text{SO}(X,g_t)$. Here, $\xi : \text{Spin}(n) \rightarrow \text{SO}(n)$ is the canonically given double cover. Furthermore, $(P_t,\pi_t)$ being associated with a fixed topological spin structure, the map $b_t$ lifts to an equivariant bundle map $\tilde{b}_t : P_t \rightarrow P_0$ and through the associated vector bundle construction, we obtain a fibrewise isometric vector bundle isomorphism

$$\beta_t : \Sigma_{g_t} \longrightarrow \Sigma_{g_0},$$

which moreover satisfies

$$\beta_t(c_{g_t}(\theta)(\varphi)) = c_{g_0}(A^{1/2}_t(\theta))(\beta_t(\varphi)) \quad \text{for all} \quad x \in X, \theta \in T^*_x X, \varphi \in (\Sigma_{g_t})_x,$$
where $A' \in \text{End}(T^*X)$ is the transpose of $A$. With

$$0 < \rho_t = d\mu_{g_0}/d\mu_{g_t} \in C^\infty(X)$$

the Radon-Nikodym density of $\mu_{g_0}$ w.r.t. $\mu_{g_t}$, we obtain the canonically given unitary operator

$$U_t : \Gamma_{L^2}(X,\Sigma_{g_t}) \longrightarrow \Gamma_{L^2}(X,\Sigma_{g_0})$$

$$U_t \phi(x) = \rho_t^{-1/2} \beta_t(\phi(x)),$$

which we use to define a family of $\vartheta$-summable Fredholm modules $M^{g\bullet}$ over $\Omega(X)$ in the sense of Definition 2.1 in [7], by

$$M^{g\bullet}_t : (\Gamma_{L^2}(X,\Sigma_{g_0}),c^t, Q_t) := (\Gamma_{L^2}(X,\Sigma_{g_0}), U_t c_{g_t} U_t^* ; U_t D_{g_t} U_t^*),$$

where $D_{g_t}$ is the Dirac operator acting on $L^2$-sections of $\Sigma_{g_t}$. Consider the Chern character

$$\text{Ch}_{g_t} : C^\epsilon_T(X) \longrightarrow \mathbb{C},$$

whose value at

$$\theta_0 \otimes \cdots \otimes \theta_N \in C^\epsilon_T(X)$$

is given by the RHS of (2.6) for $g = g_t$. Then $\text{Ch}_{g_t}$ vanishes on the kernel of $\Psi$ and this defines $\mathcal{J}^{g_t}$. If we can show that $M^{\vartheta\bullet}$ satisfies the axioms of Definition 6.1 in [7], then (using that Chern characters are invariant under unitary transformations) it follows that the (odd) Chern-Simons form

$$\text{CS}(M^{\vartheta\bullet}_T) : C^\epsilon_T(X) \longrightarrow \mathbb{C}$$

constructed on page 31 in [7] satisfies

$$\text{Ch}_{g_1} - \text{Ch}_{g_0} = (d + i\partial_T + b + B)\text{CS}(M^{\vartheta\bullet}_T)$$

and vanishes on the kernel of $\Psi$, too. It follows that

$$\mathcal{E}^{g_0}(\Psi(\theta)) := \text{CS}(M^{\vartheta\bullet}_T)(\theta), \quad \theta \in C^\epsilon_T(X),$$

is well-defined and, being invariant under $A$ (which follows from its very construction), has the desired properties, in view of

$$A \Psi(d + i\partial_T + b + B) = (d + i) A \Psi.$$

It remains to show (H1) and (H2) from Definition 6.1 in [7], where (H1) is the condition

$$\sup_{t \in [0,1]} \text{tr} \left( e^{-Q_t^2} \right) < \infty,$$

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and (H2) is the condition
\[
\sup_{t \in [0,1]} \left\| \dot{Q}_t (Q_t^2 + 1)^{-1/2} \right\| + \sup_{t \in [0,1]} \left\| (Q_t^2 + 1)^{-1/2} \dot{Q}_t \right\| < \infty.
\]

Here, (H1) can be seen as follows: one can appeal to the Lichnerowicz formula for \( D_t^2 \) and semigroup domination (cf. Theorem 3.1 in [11]) to get
\[
\text{tr} \left( e^{-Q_t^2} \right) \leq \text{rank}(\Sigma_0) e^{-\min_{x \in X} (1/4) \text{scal}_{gt}(x)} \text{tr} \left( e^{-\Delta_{gt}} \right),
\]
which entails (H1), as \( t \mapsto \min_{x \in X} (1/4) \text{scal}_{gt}(x) \) is clearly continuous, and \( t \mapsto \text{tr} \left( e^{-\Delta_{gt}} \right) \) is smooth by Proposition 6.1 from [13]. To see (H2) note that from elliptic regularity, each \( Q_t := U_{gt} D_{gt} U_{gt}^* \) has the same domain of definition \( W^{1,2}(X) \). Furthermore, \( \dot{Q}_t := (d/dt)Q_t \) is a first order differential operator, which we consider as acting on smooth spinors. The proof of (H2) is based on the following lemma, which is a modification of Lemma 4.17 in [8]:

**Lemma 4.1.** Let \( S \) be a densely defined, closed linear operator from a Hilbert space \( \mathcal{H}_1 \) to a Hilbert space \( \mathcal{H}_2 \), and let \( T \) be a self-adjoint bounded linear operator in \( \mathcal{H}_1 \) with \( T \geq -\lambda \) for some \( \lambda \geq 0 \). Assume that
\[
\| S(S^*S + T + 1)^{-1/2} \| \leq \sqrt{\lambda + 1}.
\]

**Proof.** By assumption we have
\[
S^*S + 1 \leq S^*S + T + \lambda + 1,
\]
which means
\[
\|(S^*S + 1)^{1/2} f \| \leq \|(S^*S + T + \lambda + 1)^{1/2} f \| \quad \text{for all } \quad f \in \text{dom}(S^*S)^{1/2}.
\]

From this we obtain
\[
\|(S^*S + 1)^{1/2} (S^*S + T + 1)^{-1/2} h \| \leq \|(S^*S + T + \lambda + 1)^{1/2} (S^*S + T + 1)^{-1/2} h \|
\]
for all \( h \in \mathcal{H}_1 \). Using the functional calculus associated with the operator \( S^*S + T \), we calculate the norm of the operator appearing on the right hand side to be
\[
\|(S^*S + T + \lambda + 1)^{1/2} (S^*S + T + 1)^{-1/2} \| \leq \sup_{t \geq 0} \sqrt{\frac{t + \lambda + 1}{t + 1}} = \sqrt{\lambda + 1},
\]
which implies
\[
\|(S^*S + 1)^{1/2} (S^*S + T + 1)^{-1/2} \| \leq \sqrt{\lambda + 1}.
\]
Now we can estimate
\[
\|S(S^*S + T + 1)^{-1/2}\| = \|S(S^*S + 1)^{-1/2}(S^*S + 1)^{1/2}(S^*S + T + 1)^{-1/2}\|
\leq \sqrt{\lambda + 1}\|S(S^*S + 1)^{-1/2}\|
\leq \sqrt{\lambda + 1}\|(S^*S)^{1/2}(S^*S + 1)^{-1/2}\|
\leq \sqrt{\lambda + 1}\sup_{t \geq 0} \sqrt{\frac{t}{t + 1}}
\leq \sqrt{\lambda + 1},
\]
where we have used the polar decomposition \(S = U(S^*S)^{1/2}\) with a partial isometry \(U\) on the third line and the functional calculus associated with the operator \(S^*S\) on the fourth line.

Using this lemma, we are going to prove that one has (H2): first of all, note that \(Q_t\) acting on \(\Gamma_{C^\infty}(X,\Sigma_{g_t})\) is a first order differential operator whose coefficients depend smoothly on \(t \in [0,1]\). Since \(X\) is compact, it follows that
\[
\langle \dot{Q}_t\varphi,\psi \rangle = (d/dt) \langle Q_t\varphi,\psi \rangle = (d/dt) \langle \varphi,Q_t\psi \rangle = \langle \varphi,\dot{Q}_t\psi \rangle
\]
for all \(\varphi,\psi \in \Gamma_{C^\infty}(X,\Sigma_{g_t})\), i.e., \(\dot{Q}_t\) is symmetric.

Secondly, the operator \((Q_t^2 + 1)^{-1/2}\) being elliptic, it follows from a classical result of Seeley \cite{14} that \((Q_t^2 + 1)^{-1/2}\) is a pseudo-differential operator. In particular, it maps \(\Gamma_{C^\infty}(X,\Sigma_{g_t})\) to itself.

Turning to operator norms, note that \(\dot{Q}_t(Q_t^2 + 1)^{-1/2}\) is bounded if and only if
\[
\sup \left\{ \left\| \dot{Q}_t(Q_t^2 + 1)^{-1/2}\varphi \right\| : \varphi \in \Gamma_{C^\infty}(X,\Sigma_{g_t}) \right\} < \infty.
\]

The operators \(\dot{Q}_t\) and \((Q_t^2 + 1)^{-1/2}\) being symmetric this, in turn, is equivalent to \((Q_t^2 + 1)^{-1/2}\dot{Q}_t\) being bounded. Hence, it suffices to show that
\[
\sup_{t \in [0,1]} \left\| \dot{Q}_t(Q_t^2 + 1)^{-1/2}\right\| < \infty. \tag{4.1}
\]

To this end, we first use the unitary invariance of the functional calculus to compute
\[
\left\| \dot{Q}_t(Q_t^2 + 1)^{-1/2}\right\| = \left\| \dot{Q}_t((U_tD_{g_t}U_t^*)^2 + 1)^{-1/2}\right\| = \left\| \dot{Q}_tU_t(D_{g_t}^2 + 1)^{-1/2}U_t^*\right\|
\leq \left\| U_t^*\dot{Q}_tU_t(D_{g_t}^2 + 1)^{-1/2}\right\|.
\]

Next, we decompose
\[
U_t^*\dot{Q}_tU_t = \sigma_t \circ \nabla_t + \tau_t,
\]
with \(\nabla_t\) the spinor connection of \(\Sigma_{g_t}\), and
\[
\sigma_t \in \Gamma_{C^\infty}(X,\text{Hom}(T^*X \otimes \Sigma_{g_t},\Sigma_{g_t})), \quad \tau_t \in \Gamma_{C^\infty}(X,\text{End}(\Sigma_{g_t})),
\]

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so that by the Lichnerowicz formula we have

\[ U_t^* \dot{Q} U_t (D_{g_t}^2 + 1)^{-1/2} = \sigma_t \nabla (\nabla^* \nabla + \frac{1}{4} \text{scal}_{g_t} + 1)^{-1/2} + \tau_t (D_{g_t}^2 + 1)^{-1/2} \]  

(4.2)

Because \(||(D_{g_t}^2 + 1)^{-1/2}\| \leq 1\), the operator norm of the second term on the right hand side is bounded by \(||\tau_t||\), which is continuous in \(t\). Hence,

\[ \sup_{t \in [0,1]} ||\tau_t (D_{g_t}^2 + 1)^{-1/2}|| < \infty. \]

Regarding the first term on the right hand side of (4.2), we appeal to the above lemma with

\[ S = \nabla, \quad T = (1/4) \text{scal}_{g_t}, \quad \lambda_t := (1/4) \max_{x \in X} |\text{scal}_{g_t}(x)|, \]

to see that

\[ ||\sigma_t \nabla (\nabla^* \nabla + \frac{1}{4} \text{scal}_{g_t} + 1)^{-1/2}|| \leq ||\sigma_t|| \sqrt{\lambda_t + 1}, \]

which is also continuous in \(t\), thereby completing the proof of (4.1) and hence also of Theorem 2.4.

**Appendix: formal proof of formula (1.4)**

We start by calculating the derivative of \(\mathcal{I}^{g_t}\) w.r.t. \(t\),

\[ (d/dt) \mathcal{I}^{g_t} \alpha = \int_{LX} (d/dt) e^{-E^{g_t} + \omega^{g_t}} \land \alpha = \int_{LX} e^{-E^{g_t} + \omega^{g_t}} \land (d/dt) (-E^{g_t} + \omega^{g_t}) \land \alpha. \]

Let \(\nabla(t)\) denote the Levi-Civita connection for \(g_t\), and let \(\gamma \in LX, X,Y \in T \gamma LX\). The \(t\)-derivative appearing in the integrand on the right-hand side is

\[ (d/dt) (-E^{g_t}_\gamma + \omega^{g_t}_\gamma)(Y,Z) = -\frac{1}{2} \int_T g'_t(\dot{\gamma},\dot{\gamma}) + \int_T g'_t(Y,\nabla(t)\gamma Z) + \int_T g_t(Y,\nabla(t)\gamma Z), \]

(4.3)

where we have used primes to denote derivatives w.r.t. \(t\) and dots to denote derivatives w.r.t. the loop parameter.

Using that the covariant derivative commutes with every contraction, the second integral in (4.3) is equal to

\[ \frac{1}{2} \int_T g'_t(Y,\nabla(t)\gamma Z) + \frac{1}{2} \int_T \{\dot{\gamma} g'_t(Y,Z) - \nabla(t)\gamma (g'_t(Y,\cdot))(Z)\} = \frac{1}{2} \int_T g'_t(Y,\nabla(t)\gamma Z) - \frac{1}{2} \int_T \nabla(t)\gamma (g'_t(Y,\cdot))(Z) = \frac{1}{2} \int_T \{g'_t(Y,\nabla(t)\gamma Z) - g'_t(Z,\nabla(t)\gamma Y)\} - \frac{1}{2} \int_T (\nabla(t)\gamma g'_t)(Y,Z). \]
For the third term on the right-hand side of (4.3), we use the well-known formula (see, e.g., [15, Proposition 2.3.1]) for the time derivative of the Levi-Civita connection,

\[
\int_T g_t(Y, \nabla(t) \dot{\gamma}; Z) = \frac{1}{2} \int_T \{ (\nabla(t) Z g'(t))(Y, \dot{\gamma}) + (\nabla(t) g'_t)(Y, Z) - (\nabla(t) Y g'_t)(Z, \dot{\gamma}) \}. 
\]

Putting the above together, we obtain

\[
\frac{d}{dt} \left( -E^g_t + \omega^g_t \right)(Y, Z) = \frac{1}{2} \int_T g_t'(\gamma, \dot{\gamma}) + \frac{1}{2} \int_T \{ g_t'(Y, \nabla(t) \dot{\gamma} Z) - g'_t(Z, \nabla(t) \gamma Y) \}
- \frac{1}{2} \int_T \{ (\nabla(t) Y g'_t)(\dot{\gamma}, Z) - (\nabla(t) Z g'_t)(\dot{\gamma}, Y) \}. 
\] (4.4)

On the other hand, defining the 1-form \( \sigma_t \) on \( LX \) by

\[
(\sigma_t)_\gamma(Y) = -\frac{1}{2} \int_T g'_t(\dot{\gamma}, Y),
\]
its exterior derivative \( d\sigma_t \) is defined by the Cartan formula,

\[
d(\sigma_t)_\gamma(Y, Z) = Y\sigma_t(\tilde{Z}) - Z\sigma_t(\tilde{Y}) - \sigma_t([\tilde{Y}, \tilde{Z}]),
\]

where \( \tilde{Y} \) and \( \tilde{Z} \) are local extensions of \( Y, Z \), i.e., vector fields defined on a neighborhood of \( \gamma \in LX \) with \( \tilde{Y}_\gamma = Y \) and \( \tilde{Z}_\gamma = Z \) (this definition is independent of the extensions \( \tilde{Y}, \tilde{Z} \)). Using 1- and 2-parameter variations of \( \gamma \) with variation vector fields \( X \) and \( Y \) respectively and formula (4.4), one easily computes

\[
d(\sigma_t)_\gamma(Y, Z) = (d/dt) \left( -E^g_t + \omega^g_t \right)(Y, Z) - \iota_{\sigma_t}.
\]

Hence, for any differential form \( \alpha \) on \( LX \) we have

\[
(d/dt) \mathcal{I}^g_t[\alpha] = \int_{LX} e^{-E^g_t + \omega^g_t} \wedge (d + \iota)\sigma_t \wedge \alpha = \int_{LX} e^{-E^g_t + \omega^g_t} \wedge \sigma_t \wedge (d + \iota)\alpha,
\]
where the last equality follows from

\[
(d + \iota) \mathcal{I}^g_t[\alpha] = \mathcal{I}^g_t[(d + \iota)\alpha] = 0.
\]

Defining

\[
\mathcal{C}^g_t(\alpha) := \int_{LX} e^{-E^g_t + \omega^g_t} \wedge \sigma_t \wedge \alpha,
\]
we end up with

\[
(d/dt) \mathcal{I}^g_t = (d + \iota)\mathcal{C}^g_t,
\]
formally proving (1.4).

**Acknowledgements:** The authors would like to thank Matthias Ludewig and Konrad Waldorf for very helpful discussions!
References

[1] L. Álvarez-Gaumé: Supersymmetry and the Atiyah-Singer index theorem. Comm. Math. Phys. 90 (1983), no. 2, 161–173.

[2] M. Atiyah: Circular symmetry and stationary-phase approximation. Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983). Astérisque No. 131 (1985), 43–59.

[3] J.-M. Bismut: Index theorem and equivariant cohomology on the loop space. Comm. Math. Phys. 98 (1985), no. 2, 213–237.

[4] J.-P. Bourguignon & P. Gauduchon: Spineurs, opérateurs de Dirac et variations de métriques. Comm. Math. Phys. 144 (1992), no. 3, 581–599.

[5] S. Boldt & B. Güneysu: Feynman-Kac formula for perturbations of order $\leq 1$ and noncommutative geometry. Preprint, 2021, arXiv:2012.15551.

[6] S. Cacciatori & B. Güneysu: Odd characteristic classes in entire cyclic homology and equivariant loop space homology. To appear in Journal of Noncommutative Geometry.

[7] B. Güneysu & M. Ludewig: The Chern Character of $\theta$-summable Fredholm Modules over dg Algebras and Localization on Loop Space. Preprint, 2020, arXiv:1901.04721.

[8] B. Güneysu & S. Pigola: The Calderón-Zygmund inequality and Sobolev spaces on noncompact Riemannian manifolds. Adv. Math. 281 (2015), 353–393.

[9] F. Hanisch & M. Ludewig: The Fermionic integral on loop space and the Pfaffian line bundle, preprint 2021, arXiv:1709.10028.

[10] F. Hanisch & M. Ludewig: A Rigorous Construction of the Supersymmetric Path Integral Associated to a Compact Spin Manifold, preprint, 2021, arXiv:1709.10027.

[11] H. Hess & R. Schrader & D.A. Uhlenbrock: Kato’s inequality and the spectral distribution of Laplacians on compact Riemannian manifolds. J. Differential Geometry 15 (1980), no. 1, 27–37 (1981).

[12] Free loop spaces in geometry and topology. Including the monograph Symplectic cohomology and Viterbo’s theorem by Mohammed Abouzaid. Edited by Janko Latschev and Alexandru Oancea. IRMA Lectures in Mathematics and Theoretical Physics, 24. European Mathematical Society (EMS), Zürich, 2015.

[13] D. B. Ray & I. M. Singer: $R$-torsion and the Laplacian on Riemannian manifolds. Advances in Math. 7 (1971), 145–210.

[14] R. T. Seeley: Complex powers of an elliptic operator. 1967 Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966) pp. 288–307 Amer. Math. Soc., Providence, R.I.
[15] P. Topping: Lectures on the Ricci flow. London Mathematical Society Lecture Note Series, 325. Cambridge University Press, Cambridge, 2006.

[16] K. Waldorf: Transgression to loop spaces and its inverse, II: Gerbes and fusion bundles with connection. Asian J. Math. 20 (2016), no. 1, 59–115.