Admissible $\hat{sl}(2|1; \mathbb{C})_k$ Characters and Parafermions.

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Abstract

The branching functions of the affine superalgebra $\hat{sl}(2|1; \mathbb{C})_k$ characters into characters of the subalgebra $\hat{sl}(2; \mathbb{C})_k$ are calculated for fractional levels $k = \frac{1}{u} - 1$, $u \in \mathbb{N}$. They involve rational torus $A_{u(u-1)}$ and $Z_{u-1}$ parafermion characters.

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1 Introduction.

It has long been stressed that non-critical strings might well be described in terms of a topological $G/G$ Wess-Zumino-Novikov-Witten model \[1, 2, 3\], where $G$ is a Lie supergroup or a Lie group, depending on whether the string theory considered is supersymmetric or not. The exact correspondence between the traditional approach to non-critical strings and the latter is yet to be proven. However, a crucial ingredient in the description of the spectrum in the $G/G$ picture is the representation theory of the corresponding affine Lie (super)algebra, $\hat{g}$, at fractional level $k = p/u - h$, $p \in \mathbb{Z} \setminus \{0\}$, $u \in \mathbb{N}$, gcd($p, u$) = 1, with $h$ the dual Coxeter number of $\hat{g}$.

In this paper we investigate further the characters of the complex affine superalgebra $\hat{sl}(2|1; \mathbb{C})_k$ because of its potential relevance to the description of $N = 2$ non-critical strings. Indeed, when the matter, which is coupled to supergravity in the $N = 2$ non-critical string, is minimal, i.e. taken in an $N = 2$ super Coulomb gas representation with central charge,

$$c_{\text{matter}} = 3 \left(1 - \frac{2p}{u}\right), \quad p, u \in \mathbb{N}, \quad \text{gcd}(p, u) = 1,$$

the level of the ‘matter’ affine superalgebra $\hat{sl}(2|1; \mathbb{C})$ appearing in the $\text{SL}(2|1; \mathbb{R})/\text{SL}(2|1; \mathbb{R})$ model is of the form

$$k = \frac{p}{u} - 1.$$  \hspace{1cm} (1.2)

That is, the level precisely takes values for which admissible representations of $\hat{sl}(2|1; \mathbb{C})_k$ do exist \[4\]. The character formulae obtained in \[3\] as functions of three variables $\tau, \sigma$ and $\nu$ are not directly suitable for the analysis of their behaviour under the modular group. In order to make this analysis straightforward, we provide here a decomposition of the $\hat{sl}(2|1; \mathbb{C})_k$ characters into characters of the subalgebra $\hat{sl}(2; \mathbb{C})_k$. It turns out that for a level of the form,

$$k = \frac{1}{u} - 1, \quad u = 2, 3, ...,$$  \hspace{1cm} (1.3)

i.e. of the form \((1.2)\) with $p = 1$, the branching functions involve characters of a rational torus model $A_{u(u-1)}$ as well as $\mathbb{Z}_{u-1}$ parafermion characters. The identification of the branching functions relies on the residue analysis at the simple pole developed, in the limit where $\sigma$ tends to zero, by a subset of $\hat{sl}(2|1; \mathbb{C})_k$ characters. One particular feature of $\hat{sl}(2|1; \mathbb{C})_k$ is that the central charge of the associated Virasoro algebra is zero (a consequence of equal number of bosonic and fermionic generators in $sl(2|1)$),

$$c_{sl(2|1)} = 0,$$  \hspace{1cm} (1.4)

so that the coset $\hat{sl}(2|1; \mathbb{C})_k/\hat{sl}(2; \mathbb{C})_k$ has central charge $c_{\text{coset}} = c_{sl(2|1)} - c_{sl(2)} = -\frac{3k}{k+2}$.

Since we restrict ourselves here to a family of conjectured rational theories at levels of the form \((1.3)\), the coset central charge is positive,

$$c_{\text{coset}} = \frac{3(u-1)}{u+1},$$  \hspace{1cm} (1.5)
and its value is one when \( u = 2 \). The branching functions in this case are just the characters of the rational torus algebra \( A_2 \), as already noticed in [3].

The paper is organised as follows. In section 2, the \( \hat{sl}(2|1; \mathbb{C})_k \) characters given in [3] are rewritten in terms of infinite products and theta functions. The latter formulation leads in section 3 to the decomposition of \( \hat{sl}(2|1; \mathbb{C})_k \) characters into \( \hat{sl}(2; \mathbb{C})_k \) characters. Section 4 uses the residues of singular \( \hat{sl}(2|1; \mathbb{C})_k \) characters at the pole \( \sigma = 0 \) to rewrite the branching functions in such a way that the \( S \) transform of the \( \hat{sl}(2|1; \mathbb{C})_k \) characters can be easily worked out. The decomposition formulae (4.4) and (4.5) are conjectured for higher values of the parameter \( u \) on the basis of a detailed analysis of the cases \( u = 3, 4 \) sketched in the appendix. We offer some comments on the structure of the coset \( \hat{sl}(2|1; \mathbb{C})_k \hat{sl}(2; \mathbb{C})_k \) in our conclusions.

2 New formulae for \( \hat{sl}(2|1; \mathbb{C})_k \) admissible characters.

In [3], we derived formulae for the characters of irreducible representations of \( \hat{sl}(2|1; \mathbb{C})_k \) at fractional level \( k = \frac{p}{u} - 1 \), where \( p \) and \( u \) are positive coprime integers. Those characters are obtained in a standard way from the knowledge of the quantum numbers and embedding diagrams of singular vectors within a given Verma module [3]. By definition, \( \hat{sl}(2|1; \mathbb{C})_k \) characters are given by,

\[
\chi_{h_-,h_+}^{\hat{sl}(2|1;\mathbb{C})_k}(\tau,\sigma,\nu) \overset{d}{=} \text{tr} \exp(2\pi i(\tau L_0 + \sigma J^3_0 + \nu U_0))
\]

(2.1)

where \( J^3_0 \) and \( U_0 \) are the zero modes of the \( \hat{sl}(2|1; \mathbb{C})_k \) Cartan generators, and the isospin \( \frac{1}{2}h_- \) and charge \( \frac{1}{2}h_+ \) are their eigenvalues when they act on the highest weight state \( |\Omega\rangle \) of the representation,

\[
J^3_0|\Omega\rangle = \frac{1}{2}h_-|\Omega\rangle, \quad U_0|\Omega\rangle = \frac{1}{2}h_+|\Omega\rangle.
\]

(2.2)

The variables \( q, z \) and \( \zeta \) are defined by,

\[
q \overset{d}{=} \exp(2\pi i \tau), \quad \tau \in \mathbb{C} \quad \text{Im}(\tau) > 0 \Rightarrow |q| < 1,
\]

\[
z \overset{d}{=} \exp(2\pi i \sigma), \quad \sigma \in \mathbb{C},
\]

\[
\zeta \overset{d}{=} \exp(2\pi i \nu), \quad \nu \in \mathbb{C}.
\]

(2.3)

For instance, the class IV Ramond characters read,

\[
\chi^{R,IV,\hat{sl}(2|1;\mathbb{C})_k}_{h_-h_+}(\tau,\sigma,\nu) = q^{h_-^R}z^{\frac{1}{2}h_-^R}\zeta^{\frac{1}{2}h_+^R}F^R(\tau,\sigma,\nu) \times \sum_{a \in \mathbb{Z}} q^{a^2pu+amz-a} \frac{1-q^{2apz^{-1}}}{1+q^{apu+am'}z^{-\frac{1}{2}\zeta^{-\frac{1}{2}}}(1+q^{apu+am'-m'}z^{-\frac{1}{2}\zeta^{\frac{1}{2}}})},
\]

(2.4)

where \( m, m' \in \mathbb{Z}_+, \, 0 \leq m' \leq m \leq u - 1 \), and

\[
h_-^R = -m(k+1),
\]

\[
h_+^R = (2m' - m)(k+1).
\]

(2.5)
By substituting $M + M' + 2$ for $m$, $M + 1$ for $m'$ and $z^{-1}$ for $z$ in the above, one obtains an expression for the class V Ramond characters, up to an overall sign. The range of the parameters $M$ and $M'$ in the Ramond sector class V is $M, M' \in \mathbb{Z}_+, \ 0 \leq M + M' \leq u - 2$, and the quantum numbers associated to these parameters are,

$$h_R^R = (M + M' + 2)(k + 1),$$
$$h_R^+ = (M - M')(k + 1). \tag{2.6}$$

In both classes, the conformal weight is given by

$$h^R = \frac{1}{4(k + 1)}((h_-^R)^2 - (h_+^R)^2). \tag{2.7}$$

On the other hand, the class IV (resp. class V) NS characters are easily obtained from class IV (resp. class V) R characters by spectral flow, namely,

$$\chi_{\hat{sl}(2|1; \mathbb{C})_k}^{\text{NS, IV}}(\tau, \sigma, \nu) = q^{k/4} z^{k/2} \chi_{h_R^R, h_R^+}^{\text{NS, IV}}(\tau, -\sigma - \tau, \nu), \tag{2.8}$$

where

$$h_{\text{NS}} = h_R^R - \frac{1}{2} h_+^R + \frac{1}{4} k,$$
$$\frac{1}{2} h_-^R = -\frac{1}{2} h_+^R + \frac{1}{2} k,$$
$$\frac{1}{2} h_+^R = \frac{1}{2} h_+^R. \tag{2.9}$$

The above formulae are not suited to the discussion of the modular properties of characters. Our ultimate goal is to identify how the above $\hat{sl}(2|1; \mathbb{C})_k$ characters branch into $\hat{sl}(2; \mathbb{C})_k$ characters, in order to make their modular transforms straightforward to obtain. We restrict ourselves to levels of the form $k = \frac{1}{u} - 1$, and we mainly concentrate on the Neveu-Schwarz sector of the theory.

We first use standard techniques to produce expressions for the characters in terms of infinite products. Namely, we analyse the pole structure of the series appearing in the NS version of (2.4) and manufacture an infinite product with the same singularities and same residues at the poles. It is not too difficult to show that,

$$\chi_{h_{\text{NS}}, h_{\text{NS}}^+}^{\text{NS, IV, } \hat{sl}(2|1; \mathbb{C})_k}(\tau, \sigma, \nu) = q^{h_{\text{NS}}^R} z^{h_{\text{NS}}^R} \zeta^{h_{\text{NS}}^R \frac{1}{2}} F^{\text{NS}}(\tau, \sigma, \nu) \times$$

$$\prod_{n=1}^{\infty} (1 - q^n)^2(1 - zq^{u - m})(1 - z^{-1} q^{u - m+1}) h_n^{-1}(m, m'; \tau, \sigma, \nu) \tag{2.10}$$

where,

$$F^{\text{NS}}(\tau, \sigma, \nu) =$$

$$\prod_{n=1}^{\infty} \frac{(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{-\frac{1}{2}})}{(1 - q^n)^2(1 - zq^n)(1 - z^{-1} q^{n-1})}, \tag{2.11}$$
and,

\[
{\hat h}_n(m, m'; \tau, \sigma, \nu) = (1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{u(n-1)+m-m'+\frac{1}{2}})(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{uu(n-1)+m'+\frac{1}{2}}) \\
(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{un-m'-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{un+m'-m-\frac{1}{2}}).
\] (2.12)

In (2.11), one has,

\[
h^NS_{\pm} = k + m(k + 1) \quad \text{and} \quad h^NS_{\pm} = (2m' - m)(k + 1).
\] (2.13)

An easy way to obtain the class V NS characters is to substitute \( M + M' \) for \( m, M \) for \( m' \) and \( z \to z^{-1} \) in the above expression, and flip the overall sign.

Note that, at fixed level \( k = \frac{1}{u} - 1 \), there exist \( u^2 \) NS characters in classes IV and V, \( u \) of which are regular at \( \sigma = 0 \). The others develop a simple pole at this value of \( \sigma \) and the residue at the pole is given by,

\[
\lim_{\sigma \to 0} 2\pi i\sigma \chi^NS,\hat{sl}(2|1;\mathbb{C})^{k}_{h^NS,h^NS_{\pm}}(\tau, \sigma, \nu) = \frac{\vartheta_{0,2}(\tau, \frac{1}{2} \nu) + \vartheta_{2,2}(\tau, \frac{1}{2} \nu)}{\eta^3(\tau)} \chi^NS,N=2_{r,s}(\tau, \zeta^{\frac{1}{2}}),
\] (2.14)

where the \( N = 2 \) superconformal characters appear in the infinite product form first derived by Matsuo [7]. In the above, the \( N = 2 \) central charge is \( c = 3(1 - \frac{2}{u}) \) and \( m = r + s - 1, m' = r - \frac{1}{2} \) in class IV, while \( M = r - \frac{1}{2}, M' = s - \frac{1}{2} \) in class V.

Similar formulae exist for singular Ramond characters,

\[
\lim_{\sigma \to 0} 2\pi i\sigma \chi^R,\hat{sl}(2|1;\mathbb{C})^{k}_{h^R_R,h^R_R_{\pm}}(\tau, \sigma, \nu) = \frac{\vartheta_{1,2}(\tau, \frac{1}{2} \nu) + \vartheta_{-1,2}(\tau, \frac{1}{2} \nu)}{\eta^3(\tau)} \chi^NS,N=2_{r,s}(\tau, \zeta^{\frac{1}{2}}).
\] (2.15)

Although interesting, the infinite product formula (2.10) has a denominator whose behaviour under the modular group is nontrivial. Using the Jacobi triple identity repetitively, as well as the standard properties of theta functions, a tedious calculation leads from (2.10) to the following elegant expression,

\[
\chi^NS,IV,\hat{sl}(2|1;\mathbb{C})^{k}_{h^NS,h^NS_{\pm}}(\tau, \sigma, \nu) = \frac{\vartheta_{-u+2(m+1),2u}(\tau, \frac{u}{2}) - \vartheta_{u+2(m+1),2u}(\tau, \frac{u}{2})}{\vartheta_{1,2}(\tau, \sigma) - \vartheta_{-1,2}(\tau, \sigma)} \eta(\tau)^{-1} \eta^3(2u)(u\tau) \times \\
\prod_{r=1}^{u-1} \sum_{s=0}^{1} \vartheta_{u+s+m-2m'u}(\tau, \frac{u}{2}) \vartheta_{u(u+1)+m+1+2r,u}(\tau, \frac{u}{2}).
\] (2.16)

The last obstacle to easy modular transformations is the presence of the function \( \eta^3(2u)(u\tau) \).

Our strategy to eliminate this function consists of two steps. The first is to rewrite the expression (2.16) in terms of \( \hat{sl}(2|1;\mathbb{C})_{k} \) characters, as described in the next section. The branching coefficients are functions of \( \nu \) and \( \tau \) and still involve the function \( \eta^3(2u)(u\tau) \). The second step, described in section 4, eliminates this function from the branching coefficients by calculating the residue at the pole \( \sigma = 0 \) of each singular \( \hat{sl}(2|1;\mathbb{C})_{k} \) character when decomposed in \( \hat{sl}(2|1;\mathbb{C})_{k} \) characters (formulae (4.4),(4.5)), and comparing the result obtained with the expressions (2.14) and (2.13).
3 Branching $\hat{sl}(2|1; \mathbb{C})_k$ into $\hat{sl}(2; \mathbb{C})_k$.

The $\hat{sl}(2; \mathbb{C})_k$ characters can be written as, see [4, 8]

$$\chi_{n,n'}^{\hat{sl}(2;\mathbb{C})_k}(\tau, \sigma) = \frac{\vartheta_{b,n,a}(\tau, \frac{\sigma}{u}) - \vartheta_{b-2,n,a}(\tau, \frac{\sigma}{u})}{\vartheta_{1,2}(\tau, \sigma) - \vartheta_{-1,2}(\tau, \sigma)}, \quad (3.1)$$

where the level is parametrized as,

$$k = \frac{t}{u}, \quad \gcd(t, u) = 1, \quad u \in \mathbb{N}, \quad t \in \mathbb{Z}, \quad (3.2)$$

with $0 \leq n \leq 2u + t - 2$ and $0 \leq n' \leq u - 1$ and,

$$b_{\pm} \equiv u(\pm(n + 1) - n'(k + 2)), \quad a \equiv u^2(k + 2). \quad (3.3)$$

In order to identify which $\hat{sl}(2; \mathbb{C})_k$ characters enter in the decomposition of the class IV, NS $\hat{sl}(2|1; \mathbb{C})$ characters, we first rewrite (2.16) as,

$$\chi_{h_{NS}^{IV, \hat{sl}(2|1; \mathbb{C})}}^{\hat{sl}(2|1; \mathbb{C})_k}(\tau, \sigma, \nu) = \frac{\vartheta_{u+2(m+1),2u}(\tau, \frac{\nu}{u}) - \vartheta_{u+2(m+1),2u}(\tau, \frac{\sigma}{u})}{\vartheta_{1,2}(\tau, \sigma) - \vartheta_{-1,2}(\tau, \sigma)} \eta(\tau)^{-1}\eta^{3-2u}(u\tau) \times$$

$$\sum_{n=1}^{u} \left\{ \theta_{m-2m', u}(\tau, \frac{\nu}{u})^{u-n}\theta_{m-2m'+u, u}(\tau, \frac{\nu}{u})^{n-1} \times \right.$$  

$$\sum \left\{ \prod_{\{p_i\}_{i=1}^{u-n} \subset S} \prod_{i=1}^{u-n} \theta_{m+1+u+2p_i, u}(\tau, \frac{\sigma}{u}) \prod_{j=1}^{n-1} \theta_{m+1+u+2p_{u-n+j}, u}(\tau, \frac{\sigma}{u}) \right\} \right\}, \quad (3.4)$$

where the sum $\sum_{\{p_i\}_{i=1}^{u-n} \subset S}$ is over the $(u-n)!/(u-n)!$ possible subsets $S_{(n)}$ of $(u-n)$ distinct integers $p_i$ included in the set $S = \{1, ..., u-1\}$. For each choice of subset $S_{(n)}$, the variables $p_{u-n+1}, ..., p_{u-1}$ take the distinct values in $S \setminus S_{(n)}$.

The following expression is central in our discussion of branching functions. It reads,

$$\chi_{h_{NS}^{IV, \hat{sl}(2|1; \mathbb{C})}}^{\hat{sl}(2|1; \mathbb{C})_k}(\tau, \sigma, \nu) = \eta(\tau)^{-1}\eta^{3-2u}(u\tau) \times \sum_{n=1}^{u} \left\{ \mathcal{F}(n, \tau, \frac{\nu}{u}) \times \right.$$  

$$\sum \left\{ \prod_{\{p_i\}_{i=1}^{u-n} \subset S} \prod_{r=0}^{u-2} \left\{ \sum_{D(\mu_1, ..., \mu_{u-n-1}; \nu_1, ..., \nu_{u-2}; r)} \left\{ \mathcal{G}(p_1, ..., p_{u-1}; \bar{\mu}; \bar{\nu}) \times \right.$$  

$$\sum_{\ell=0}^{u-2} \left\{ \theta_{u(u-n)(n-1)(2\ell+2(n-1)(\bar{\mu}+u) \bar{\mu}+u)} - 2u(n)(\bar{\mu}+u)(u-n)(n-1)(\tau) \right.$$  

$$\sum_{\lambda=0}^{u} \theta_{u(u-n)(4\lambda+3)(2u-1)(2\ell-4r, 2u-1)}(\tau)(-1)^{\epsilon \chi_{\hat{sl}(2; \mathbb{C})}}_{h_{NS}^{IV, \hat{sl}(2|1; \mathbb{C})}}(\tau, \sigma)} \right\} \right\} \right\} \right\} \right\} \right\} \right\}. \quad (3.5)$$

In the above formula, the quantum numbers of the representation considered are,

$$h_{NS}^- = \frac{1}{u}(u - m - 1), \quad h_{NS}^+ = \frac{1}{u}(2m' - m), \quad 0 \leq m' \leq m \leq u - 1. \quad (3.6)$$
Given a set of \( n \) integers \( \alpha_i, i = 1, \ldots, n \), one also introduces,
\[
\tilde{\alpha}_i = \sum_{j=i}^{n} (n + 1 - j)\alpha_j \tag{3.7}
\]
and the domain,
\[
D(\alpha_1, \ldots, \alpha_n; r) = \{ \alpha_j : 0 \leq \alpha_j \leq n + 1 - j, \ j = 1, \ldots, n : \tilde{\alpha}_1 = k'(n + 1) + r, k' \in \mathbb{N} \}, \tag{3.8}
\]
with
\[
0 \leq \tilde{\alpha}_1 \leq \frac{1}{6}n(n + 1)(2n + 1). \tag{3.9}
\]
In particular, one has,
\[
D(\mu_1, \ldots, \mu_{u-n-1}; \nu_1, \ldots, \nu_{n-2}; r) = \{ (\mu_j; \nu_j') : 0 \leq \mu_j \leq u - n - j, \ j = 1, \ldots, u - n - 1; \\
0 \leq \nu_j' \leq n - 1 - j', \ j' = 1, \ldots, n - 2 : \tilde{\mu}_1 + \tilde{\nu}_1 = k'(u - 1) + r, k' \in \mathbb{N} \}, \tag{3.10}
\]
with
\[
0 \leq \tilde{\mu}_1 \leq \frac{1}{6}(u - n - 1)(u - n)(2(u - n - 1) + 1), \\
0 \leq \tilde{\nu}_1 \leq \frac{1}{6}(n - 2)(n - 1)(2(n - 2) + 1). \tag{3.11}
\]

The function \( G \) is given by the following product,
\[
G(p_1, \ldots, p_{u-1}; \tilde{\mu}; \tilde{\nu}) = \prod_{i=1}^{u-n-1} \theta_{2P(0; u-n-i)-2u\tilde{\mu}_i,(u-n-i)(u-n-i+1)u(\tau)} \\
\times \prod_{j=1}^{n-2} \theta_{2P(u-n;n-1-j)-2u\tilde{\nu}_j,(n-1-j)(n-j)u(\tau)}, \tag{3.12}
\]
where one defines,
\[
P(\alpha; \beta) = \bar{p}_{\alpha, \beta} - \beta p_{\alpha+\beta+1}, \tag{3.13}
\]
with
\[
\bar{p}_{j,n} = \sum_{k=1}^{n} p_{j+k}, \quad \bar{p}_{0,n} \equiv \bar{p}_n. \tag{3.14}
\]

The function \( \mathcal{F} \) reads,
\[
\mathcal{F}(n; \tau, \frac{\nu}{u}) = \sum_{s=0}^{u-n-1} \left\{ \sum_{t=0}^{n-2} \left\{ \sum_{D(\rho_1, \ldots, \rho_{u-n-1}; s)} \sum_{D(\sigma_1, \ldots, \sigma_{n-2}; t)} \{ G(0; \bar{\rho}; \bar{\sigma}) \\
\theta_{u[-(u-n)(n-1)(2\lambda'+1)-2(n-1)s+2(u-n)t],u(u-1)(u-n)(n-1)\lambda'+(u-1)(n-1)(n-1)}(\tau) \\
\theta_{u[-2(u-n)\lambda'-2s+2(u-1)]+(u-1)(m-2m'),u(u-1)\lambda'+(u-1)\lambda'}(\tau, \frac{\nu}{u}) \right\} \right\} \right\}. \tag{3.15}
\]
Finally, the label \( n_\epsilon \) in the \( \hat{sl}(2; C)_k \) characters entering the formula (3.3) is the residue modulo \( 2(u+1) \) of \( n_\epsilon \) defined by, \( (\epsilon = 0, 1) \)

\[
n_\epsilon = -1 + (1 - 2\epsilon) \left( (u-1)(2\lambda + 1) + u - 2r + (u-n)(1-2\ell) \right).
\] (3.16)

For each choice of variables \( \lambda, r, n, \ell \), either \( [n_0] \) or \( [n_1] \) is in the set \( S = \{1, \ldots, u-1\} \), or else, \( [n_0] = [n_1] \). In the latter case, there is no contribution proportional to \( \chi_{[n_0],u-m-1}^{\hat{sl}(2; C)} \) in (3.3), while in the former case, one gets a contribution \( \chi_{[n_1],u-m-1}^{\hat{sl}(2; C)} \) with \( \epsilon = 0 \) (resp. 1) according to whether \( [n_0] \) (resp. \( [n_1] \)) is in the set \( S \).

In obtaining (3.3) from (3.4), the use of the theta function identity,

\[
\theta_{m,k}(\tau, \sigma)\theta_{m',k'}(\tau, \sigma) = \sum_{\ell=1}^{k+k'} \theta_{mk'-m'k+2\ell kk',kk'(k+k')}(\tau)\theta_{m+m'+2\ell k,k+k'}(\tau, \sigma)
\] (3.17)

is crucial. In particular, it allows to write,

\[
\prod_{i=1}^{n} \theta_{p_i,u}(\tau, \sigma) = \sum_{r=0}^{n-1} \left\{ \sum_{D(n_1, \ldots, n_{n-1}; r)} \prod_{i=1}^{n-1} \theta_{P(0;n-i)-2u\tilde{n}_{i},(n-i)(n-i+1)u}(\tau) \right\} \theta_{p_n-2ur,nu}(\tau, \sigma). \] (3.18)

Note that the invariance of the left hand side of (3.18) under permutations of \( \{p_1, \ldots, p_n\} \) provides identities between sums of products of theta functions at fixed values of \( r \). This type of identity has been used in deriving (3.5). Let us end this section by noting that the corresponding decomposition in \( \hat{sl}(2; C)_k \) characters for NS class V \( \hat{sl}(2;1; C) \) characters is readily obtained by making the substitution \( M+M' \) for \( m \), \( M \) for \( m' \), \( z \to z^{-1} \) and an overall flip of sign in (3.5).

### 4 Parafermionic characters as branching functions.

As stressed in the introduction, the expression (3.3) neatly isolates the \( \hat{sl}(2; C)_k \) character dependence, but the branching functions are still written in a way which obscures their modular properties. However, it is easy to calculate the residue at the simple pole \( \sigma = 0 \) in the NS singular characters, both in class IV and class V, when they are decomposed in \( \hat{sl}(2; C)_k \) characters. Indeed, as discussed in [3], when the level \( k \) is of the form \( 1,3 \), the residue at the pole \( \sigma = 0 \) of singular \( \hat{sl}(2; C)_k \) characters are unitary minimal Virasoro characters at level \( u \), multiplied by \( \eta^{-2}(\tau) \).

On the other hand, we have the residue calculated as in (2.14), which can be rewritten using \( \hat{su}(2) \) string functions at level \( u-2 \) [4, 10].
result is that the branching functions involve a rational torus
central charge value (1.5), since parafermion characters \[12, 13\]. With hindsight, this structure can be derived from the coset
transformations can be carried out easily. We have worked out in detail the branching
provides enough information to express the branching functions in a form where the modular
where \( Z \)
parafermions and the branching functions are just the
where
\[
\varepsilon = 1 = \lim_{\tilde{m} \to \infty} \sum_{\tilde{m}} \Theta_{\tilde{m}, \tilde{m}'}(\tau) \Xi_{\tilde{m}, \tilde{m}'}^{\mp}(\tau, \nu)
\]
and

\[
\hat{c} = m = r + s - 1 \quad \text{and} \quad \hat{m} = 2m' - m
\]
in class IV.

The above residues are thus expanded in a basis of theta functions at level \( u(u - 1) \) with arguments \( \tau \) and \( \frac{\nu}{\mu} \), as are the residues calculated from (3.3) (see (3.15)). A comparison between these two methods of obtaining the residues of singular NS \( \hat{sl}(2|1; \mathbb{C}) \) characters provides enough information to express the branching functions in a form where the modular transformations can be carried out easily. We have worked out in detail the branching functions for the cases \( u = 3 \) and \( u = 4 \) (see the appendix), and the truly remarkable result is that the branching functions involve a rational torus \( A_{u(u - 1)} \) \[11\] as well as \( Z_{u-1} \) parafermion characters \[12, 13\]. With hindsight, this structure can be derived from the coset central charge value \( (1,5) \), since

\[
c_{\text{cosp}} = \frac{3(u - 1)}{u + 1} + \frac{2(u - 2)}{u + 1}
\]

where the first term is the torus central charge while the second term is precisely the central charge of the parafermionic algebra based on \( Z_{u-1} \) parafermions. When \( u = 2 \), there are no parafermions and the branching functions are just the \( A_2 \) torus characters, as discovered in \[14\]. Based on the analysis for low values of the parameter \( u \), we conjecture the following \( \hat{sl}(2|1; \mathbb{C})_k \) character decomposition formulae. The class IV characters can be written,

\[
\chi_{h^{NS}, h^{NS}}^{NS, \hat{sl}(2|1; \mathbb{C})_k}(\tau, \sigma, \nu) = \sum_{i=0}^{u-1} \hat{sl}(2; \mathbb{C})_k \chi_{i, u - m - 1}(\tau, \sigma) \times \left\{ \sum_{s=0}^{u-2} C_{i, 3i + 2us}(\tau) \theta(u - 1)(m - 2m') + u(u - 1)(i + 1) + 2iu(\frac{\nu}{\mu} - \frac{\nu}{\mu}) - 2us, u(u - 1)(\tau, \nu) \right\}
\]

where \( h^{NS}_- \) and \( h^{NS}_+ \) are given by (3.6), while the class V NS characters are decomposed as,

\[
\chi_{h^{NS}, h^{NS}}^{NS, \hat{sl}(2|1; \mathbb{C})_k}(\tau, \sigma, \nu) = \sum_{i=0}^{u-1} \hat{sl}(2; \mathbb{C})_k \chi_{i, M + M' + 1}(\tau, \sigma) \times \left\{ \sum_{s=0}^{u-2} C_{i, 3i + 2us}(\tau) \theta(u - 1)(M' - M) + u(u - 1)i + 2iu(\frac{\nu}{\mu} - \frac{\nu}{\mu}) - 2us, u(u - 1)(\tau, \nu) \right\}
\]
where
\[ h_{NS}^{-} = -\frac{1}{u} (M + M' + 1 + u), \quad \text{and} \quad h_{NS}^{+} = \frac{1}{u} (M - M'). \] (4.6)

The symbol \( \lfloor \frac{u}{2} \rfloor \) is the integer part of \( \frac{u}{2} \). In the above expressions, one interprets \( \eta(\tau)c_{\ell,m}^{(u-1)}(\tau) \) as the partition function for the \( \mathbb{Z}_{u-1} \) parafermionic theory where the lowest dimensional field is \( \Phi_{-3i-2us}^{i} \) in the notations of \([20]\). Recall that the \( \hat{su}(2) \) string functions at level \( u-1 \) have the following symmetries \([9]\),
\[ c_{\ell,m}^{(u-1)}(\tau) = c_{\ell,-m}^{(u-1)}(\tau) = c_{\ell,m+2(u-1)\mathbb{Z}}^{(u-1)}(\tau) = c_{u-1-\ell,u-1-m}^{(u-1)}(\tau), \]
\[ c_{\ell,m}^{(u-1)}(\tau) = 0 \text{ for } \ell - m \not\equiv 0 \mod 2. \] (4.7)

The \( A_{u(u-1)} \) torus characters are given by the theta functions at level \( u(u-1) \) multiplied by \( \eta^{-1}(\tau) \). The decomposition (4.4) encodes all information needed to obtain a parafermionic realisation of \( \hat{sl}(2|1; \mathbb{C})_{k} \) at fractional level \([12]\).

5 Conclusions

Admissible representations of affine Lie algebras and superalgebras have been known to exist for fractional levels since the pioneering work of Kac and Wakimoto \([4]\). It is argued there that the corresponding characters can be expressed in terms of modular forms and provide a finite representation of the modular group.

Over the last two years, we have calculated explicit character formulae for the affine superalgebra \( \hat{sl}(2|1; \mathbb{C})_{k} \) using the Kac-Kazhdan determinant formula \([15, 16]\) and the embedding diagrams for singular vectors one can derive from it \([4, 5]\). A necessary condition for admissibility is that the level of the algebra \( \hat{sl}(2|1; \mathbb{C})_{k} \) is of the form \((1.2)\). In this paper, we restrict ourselves to levels of the form \( k = \frac{1}{u} - 1 \), \( u \in \mathbb{N} \), and to a certain class of highest weight representations labeled class IV and class V in \([5]\), where the highest weight state quantum numbers are given by (3.6) and (1.6). The corresponding characters, first derived in \([4]\), are decomposed here into characters of the bosonic subalgebra \( \hat{sl}(2; \mathbb{C})_{k} \), which has been extensively studied. The branching functions involve characters of the rational torus \( A_{u(u-1)} \) and partition functions of the parafermionic algebra \( \mathbb{Z}_{u-1} \), given by \( \hat{su}(2) \) string functions at level \( (u-1) \). The decomposition (4.4) and (1.3) are conjectured for arbitrary value of \( u \in \mathbb{N} \) on the basis of a detailed analysis of the cases \( u = 2 \) \([5]\) and \( u = 3, 4 \) (see Appendix). They reflect the structure of the central charge of the coset \( \hat{sl}(2|1; \mathbb{C})_{k}/\hat{sl}(2; \mathbb{C})_{k} \), given by (1.3) as the sum of a torus and a \( \mathbb{Z}_{u-1} \) parafermionic central charge.

There are three immediate byproducts of these decomposition formulae. First of all, the S modular transform of the \( \hat{sl}(2|1; \mathbb{C})_{k} \) characters can easily be obtained from (4.4) and (1.3) since the modular transformations laws of theta functions, \( \hat{su}(2) \) string functions and \( \hat{sl}(2; \mathbb{C})_{k} \) characters are standard. We have explicitly checked that in the cases \( u = 2, 3 \), the irreducible \( \hat{sl}(2|1; \mathbb{C})_{k} \) characters provide a finite representation of the modular group, indicating the existence of an underlying rational theory. Second, as pointed out in the appendix, the more technical steps taken in obtaining the branching functions produce mathematical identities reminiscent of those discovered and discussed in \([17, 18]\), and may well put the latter in a new perspective. Third, the decomposition formulae encode all
necessary information to obtain a new representation of the $\hat{sl}(2|1; \mathbb{C})_k$ currents in terms of a primary parafermionic conformal field, as described in [14]. In particular, in the case $u = 3$ (resp. $u = 4$), the primary field corresponds to the Ising model (resp. 3-states Potts model) spin field $\sigma$ of conformal weight $1/16$ (resp. $1/15$). The deep rôle of such a representation has yet to be understood, particularly in connection with our interest in the family of levels studied here, which stems from the non trivial rôle $\hat{sl}(2|1; \mathbb{C})_k$ plays in the description of non-critical unitary minimal $N = 2$ strings, when the matter central charge is given by $c_{\text{matter}} = 3 \left(1 - \frac{2}{u}\right)$. The most straightforward generalisation of this work would be to relax the condition $p = 1$ and investigate the theory of nonunitary minimal $N = 2$ strings.

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Appendix

In this appendix, we look in some detail at the derivation of branching functions for the cases $u = 3$ and $u = 4$. Incidentally, the corresponding parafermionic theories are the Ising model and the 3-states Potts model, a fact which will be highlighted below by the natural occurrence of unitary Virasoro characters at level 3 and 5 in connection with $\hat{su}(2)$ string functions at level 2 and 3 respectively. As explained in section 2, at fixed values of $u$, there exist $u^2$ characters corresponding to irreducible representations, $u$ of which are regular when the variable $\sigma$ tends to zero. We will view the latter as characters obtained by spectral flow from the other (Ramond or Neveu-Schwarz) sector, where they are singular. To fix the ideas, let us start in the NS sector of the $u = 3$ theory, where six characters are singular in the limit described above. We will study in detail the class IV character labeled by $h_{\text{NS}}^\text{NS} = -2/3$, $h_+^\text{NS} = 0$, which is obtained by spectral flow from the (regular) vacuum character in the Ramond sector, corresponding to $(m, m') = (0, 0)$ in (2.3).

Using the decomposition formula (3.3), one gets,

$$
\chi_{-2/3,0}^{IV,\text{NS},\hat{sl}(2|1;\mathbb{C})} - 2/3 (\tau, \sigma, \nu) = \eta^{-1}(\tau) \times \left\{ \chi_{\nu,0}^{\hat{sl}(2|1;\mathbb{C})} (\tau, \sigma) \left[ C_1 (\tau) \theta_{0,6} (\tau, \frac{3}{\nu}) + C_2 (\tau) \theta_{6,6} (\tau, \frac{3}{\nu}) \right] + \chi_{2,\nu}^{\hat{sl}(2|1;\mathbb{C})} (\tau, \sigma) \left[ C_2 (\tau) \theta_{0,6} (\tau, \frac{3}{\nu}) + C_1 (\tau) \theta_{6,6} (\tau, \frac{3}{\nu}) \right] + \chi_{1,2}^{\hat{sl}(2|1;\mathbb{C})} (\tau, \sigma) C_3 (\tau) \left[ \theta_{3,6} (\tau, \frac{\nu}{3}) + \theta_{9,6} (\tau, \frac{\nu}{3}) \right] \right\},
$$

(A.1)
where
\[
\begin{align*}
C_1(\tau) &= \eta^{-3}(3\tau)[\theta_{15,36}(\tau) - \theta_{33,36}(\tau)]\theta_{2,6}(\tau) + [\theta_{3,36}(\tau) - \theta_{21,36}(\tau)]\theta_{4,6}(\tau), \\
C_2(\tau) &= \eta^{-3}(3\tau)[\theta_{15,36}(\tau) - \theta_{33,36}(\tau)]\theta_{4,6}(\tau) + [\theta_{3,36}(\tau) - \theta_{21,36}(\tau)]\theta_{4,6}(\tau), \\
C_3(\tau) &= \eta^{-3}(3\tau)[\theta_{1,6}(\tau) + \theta_{5,6}(\tau)][\theta_{6,36}(\tau) - \theta_{30,36}(\tau)].
\end{align*}
\] (A.2)

The residue of the above character when \( \sigma \to 0 \) can be calculated in two different ways. From the decomposition \([A.2]\), it is readily obtained by noticing that the three \( sl(2; \mathbb{C}) \) characters have a simple pole singularity when \( \sigma \to 0 \), and that their residue at the pole are Virasoro characters at level 3 multiplied by the function \( \eta^{-2}(\tau) \) \([8]\). On the other hand, formula \([4.1]\) gives the residue in terms of the unique level one string function \( c_{0,0}^{(1)}(\tau) = \eta^{-1}(\tau) \) (see e.g.\([1]\)), i.e.
\[
\lim_{\sigma \to 0} 2\pi i \sigma \chi_{-2/3,0}^{IV,NS,\hat{sl}(2;\mathbb{C})}(\tau, \sigma, \nu) = \eta^{-3}(\tau)c_{0,0}^{(1)}(\tau) \times \left\{ \theta_{0,2}(\tau)\theta_{0,6}(\tau, \frac{\nu}{3}) + \theta_{2,2}(\tau)\theta_{0,6}(\tau, \frac{\nu}{3}) + \theta_{1,2}(\tau)\theta_{3,6}(\tau, \frac{\nu}{3}) + \theta_{9,6}(\tau, \frac{\nu}{3}) \right\}. \] (A.3)

Comparison of these two residue calculations yields the following expressions for the coefficients \( C_i(\tau) \),
\[
\begin{align*}
[x_{1,1}^{Vir(3)}(\tau)^2 - x_{2,1}^{Vir(3)}(\tau)^2]C_1(\tau) &= c_{0,0}^{(1)}x_{1,1}^{Vir(3)}(\tau)\theta_{2,2}(\tau) - x_{2,1}^{Vir(3)}(\tau)\theta_{0,2}(\tau), \\
[x_{1,1}^{Vir(3)}(\tau)^2 - x_{2,1}^{Vir(3)}(\tau)^2]C_2(\tau) &= c_{0,0}^{(1)}x_{1,1}^{Vir(3)}(\tau)\theta_{0,2}(\tau) - x_{2,1}^{Vir(3)}(\tau)\theta_{2,2}(\tau), \\
x_{2,2}^{Vir(3)}(\tau)[x_{1,1}^{Vir(3)}(\tau) + x_{2,1}^{Vir(3)}(\tau)]C_3(\tau) &= c_{0,0}^{(1)}\theta_{1,2}(\tau)[x_{1,1}^{Vir(3)}(\tau) + x_{2,1}^{Vir(3)}(\tau)],
\end{align*}
\] (A.4)

or again, using well-known identities relating the Ising model characters to square roots of theta functions at level 2 \([14]\),
\[
C_1(\tau) = x_{2,1}^{Vir(3)}(\tau), \quad C_2(\tau) = x_{1,1}^{Vir(3)}(\tau), \quad C_3(\tau) = x_{2,2}^{Vir(3)}(\tau). \] (A.5)

It is also worthwhile noticing that the three unitary Virasoro characters at level 3 coincide with the three partition functions of the \( \mathbb{Z}_2 \) parafermionic theory whose lowest dimensional fields are \( \Phi_{2,0}^2, \Phi_{0}^0 \) and \( \Phi_{1}^1 \) \([20]\). The latter is actually a primary conformal field which is identified as the spin field of the Ising model. These partitions are given in terms of \( \hat{su}(2) \) string functions at level 2. Namely,
\[
\begin{align*}
\chi_{1,1}^{Vir(3)}(\tau) = \eta(\tau)c_{2,2}^{(2)}(\tau), \quad \chi_{2,1}^{Vir(3)}(\tau) = \eta(\tau)c_{0,2}^{(2)}(\tau), \quad \chi_{2,2}^{Vir(3)}(\tau) = \eta(\tau)c_{1,1}^{(2)}(\tau),
\end{align*}
\] (A.6)

and therefore, the above expressions can also be viewed as relating \( \hat{su}(2) \) string functions at level one and two.

The same analysis can be applied to the other five singular NS \( \hat{sl}(2;1;\mathbb{C}) \) characters at level \( k = -2/3 \), and to the three singular R characters which flow to the three regular NS characters at the same level. They all involve the three functions \([A.3]\). One then obtains expressions which can be read off \([14],[13]\) for the nine NS \( sl(2;1;\mathbb{C}) \) characters at this level. The modular S transform of these nine characters is encoded in a \( 9 \times 9 \) matrix which is unitary and whose fourth power is the identity \([21]\).
A similar analysis can be performed for the case \( u = 4 \). For instance, using again the decomposition formula (3.4), one obtains,

\[
\chi_{-3/4,0}^{IV,NS}(\tau, \sigma, \nu) = \eta^{-1}(\tau) \times \left\{ \chi_{0,3}^{sl(2;C)}(\tau, \sigma)[D_1(\tau)\theta_{12,12}(\tau, \frac{\nu}{4}) + D_2(\tau)\{\theta_{4,12}(\tau, \frac{\nu}{4}) + \theta_{20,12}(\tau, \frac{\nu}{4})\}] + \chi_{3,3}^{sl(2;C)}(\tau, \sigma)[D_1(\tau)\theta_{0,12}(\tau, \frac{\nu}{4}) + D_2(\tau)\{\theta_{8,12}(\tau, \frac{\nu}{4}) + \theta_{16,12}(\tau, \frac{\nu}{4})\}] + \chi_{1,3}^{sl(2;C)}(\tau, \sigma)[D_3(\tau)\theta_{0,12}(\tau, \frac{\nu}{4}) + D_4(\tau)\{\theta_{8,12}(\tau, \frac{\nu}{4}) + \theta_{16,12}(\tau, \frac{\nu}{4})\}] + \chi_{2,3}^{sl(2;C)}(\tau, \sigma)[D_3(\tau)\theta_{12,12}(\tau, \frac{\nu}{4}) + D_4(\tau)\{\theta_{4,12}(\tau, \frac{\nu}{4}) + \theta_{20,12}(\tau, \frac{\nu}{4})\}] \right\}, \tag{A.7}
\]

where the four functions \( D_i(\tau), i = 1, ..., 4 \) are all sums of quintic expressions in theta functions (two at level 8, two at level 24 and one at level 120) times the function \( \eta^{-5}(4\tau) \). The important remark is that all sixteen NS \( sl(2|1;C) \) characters at level \( k = -3/4 \) have the same structure as the one above, with the same four \( D_i \) functions. Here again, the residues can be calculated in two ways. The decomposition (A.7) yields Virasoro characters at level 4 multiplied by the function \( \eta^{-2}(\tau) \), while formula (4.1) expresses the residue in terms of \( s\bar{u}(2) \) string functions at level 2. By comparison of these two calculations, one can write,

\[
\chi_{2,2}^{Vir(3)}(\tau)D_1(\tau) = c_{2,0}^{(2)}(\tau)[\chi_{3,3}^{Vir(4)}(\tau)\theta_{6,6}(\tau) - \chi_{3,2}^{Vir(4)}(\tau)\theta_{0,6}(\tau)] + c_{2,2}^{(2)}(\tau)[\chi_{3,3}^{Vir(4)}(\tau)\theta_{0,6}(\tau) - \chi_{3,2}^{Vir(4)}(\tau)\theta_{6,6}(\tau)],
\]

\[
\chi_{2,2}^{Vir(3)}(\tau)D_2(\tau) = c_{2,0}^{(2)}(\tau)[\chi_{3,3}^{Vir(4)}(\tau)\theta_{2,6}(\tau) - \chi_{3,2}^{Vir(4)}(\tau)\theta_{4,6}(\tau)] + c_{2,2}^{(2)}(\tau)[\chi_{3,3}^{Vir(4)}(\tau)\theta_{4,6}(\tau) - \chi_{3,2}^{Vir(4)}(\tau)\theta_{2,6}(\tau)],
\]

\[
\chi_{2,2}^{Vir(3)}(\tau)D_3(\tau) = c_{2,0}^{(2)}(\tau)[\chi_{1,1}^{Vir(4)}(\tau)\theta_{0,6}(\tau) - \chi_{3,1}^{Vir(4)}(\tau)\theta_{6,6}(\tau)] + c_{2,2}^{(2)}(\tau)[\chi_{1,1}^{Vir(4)}(\tau)\theta_{6,6}(\tau) - \chi_{3,1}^{Vir(4)}(\tau)\theta_{0,6}(\tau)],
\]

\[
\chi_{2,2}^{Vir(3)}(\tau)D_4(\tau) = c_{2,0}^{(2)}(\tau)[\chi_{1,1}^{Vir(4)}(\tau)\theta_{4,6}(\tau) - \chi_{3,1}^{Vir(4)}(\tau)\theta_{2,6}(\tau)] + c_{2,2}^{(2)}(\tau)[\chi_{1,1}^{Vir(4)}(\tau)\theta_{2,6}(\tau) - \chi_{3,1}^{Vir(4)}(\tau)\theta_{4,6}(\tau)], \tag{A.8}
\]

where we have used the identity [17],

\[
\chi_{1,1}^{Vir(4)}(\tau)\chi_{3,3}^{Vir(4)}(\tau) - \chi_{3,1}^{Vir(4)}(\tau)\chi_{3,2}^{Vir(4)}(\tau) = \chi_{2,2}^{Vir(3)}(\tau). \tag{A.9}
\]

It turns out that the functions \( D_i(\tau) \) enter the partition function of the 3-states Potts model, and can therefore be interpreted as the partition functions for the parafermionic algebra \( \mathbb{Z}_3 \). One has,

\[
D_1(\tau) = \chi_{1,1}^{Vir(5)}(\tau) + \chi_{4,1}^{Vir(5)}(\tau) = \eta(\tau)c_{0,0}^{(3)}(\tau), \quad D_2(\tau) = \chi_{4,3}^{Vir(5)}(\tau) = \eta(\tau)c_{3,1}^{(3)}(\tau),
\]

\[
D_3(\tau) = \chi_{2,1}^{Vir(5)}(\tau) + \chi_{3,1}^{Vir(5)}(\tau) = \eta(\tau)c_{2,0}^{(3)}(\tau), \quad D_4(\tau) = \chi_{3,3}^{Vir(5)}(\tau) = \eta(\tau)c_{1,1}^{(3)}(\tau). \tag{A.10}
\]
The relations (A.8) can therefore be viewed as relations between $\hat{\mathfrak{su}}(2)$ string functions at level two and level three. It should also be stressed that the relations (A.8), with the functions $D_i$ expressed in terms of Virasoro characters at level five as in (A.10), provide new identities very similar to the ones obtained in [17].

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