Box complexes, neighborhood complexes, and the chromatic number

PÉTER CSORBA
Institute of Theoretical Computer Science
ETH Zürich, 8092 Zürich, Switzerland
E-mail: pcsorba@inf.ethz.ch

CARSTEN LANGE
Institute of Mathematics, MA 6-2
TU Berlin D-10623 Berlin, Germany
E-mail: lange@math.tu-berlin.de

INGO SCHURR
Institute of Theoretical Computer Science
ETH Zürich, 8092 Zürich, Switzerland
E-mail: schurr@inf.ethz.ch

ARNOLD WASSMER
Institute of Mathematics, MA 6-2
TU Berlin D-10623 Berlin, Germany
E-mail: wassmer@math.tu-berlin.de

October 21, 2003

Abstract

Lovász’s striking proof of Kneser’s conjecture from 1978 using the Borsuk–Ulam theorem provides a lower bound on the chromatic number \( \chi(G) \) of a graph \( G \). We introduce the shore subdivision of simplicial complexes and use it to show an upper bound to this topological lower bound and to construct a strong \( \mathbb{Z}_2 \)-deformation retraction from the box complex (in the version introduced by Matoušek and Ziegler) to the Lovász complex. In the process, we analyze and clarify the combinatorics of the complexes involved and link their structure via several “intermediate” complexes.

1 Introduction

The topological method in graph theory was introduced by Lovász [L78] to prove Kneser’s conjecture [K55]. The pattern to obtain a lower bound of the chromatic number \( \chi(G) \) of a graph \( G \) is to associate a topological space and bound the chromatic number by a topological invariant of this space, e.g. connectivity or \( \mathbb{Z}_2 \)-index. In this note we present a subdivision technique that shows that the complex \( L(G) \) which Lovász used (and which we call Lovász complex for that reason) is a \( \mathbb{Z}_2 \)-deformation retract of the box complex \( B(G) \) described by

Supported by the joint Berlin/Zürich graduate program “Combinatorics, Geometry, and Computation (CGC),” financed by ETH Zürich and the Deutsche Forschungsgemeinschaft (DFG grant GRK 588/1).

Supported by the DFG Sonderforschungsbereich 288 “Differentialgeometrie und Quantenphysik” in Berlin.
Matoušek and Ziegler [MZ03]. The advantage of the box complex is that for any graph homomorphism $f : G \rightarrow H$ one obtains an induced simplicial $\mathbb{Z}_2$-map $B(f) : B(G) \rightarrow B(H)$. This functorial property gives elegant conceptual proofs which was not the case for the Lovász complex. Walker [W83] constructed a $\mathbb{Z}_2$-map $\varphi : \|L(G)\| \rightarrow \|L(H)\|$. Such a map could also be constructed using $B(f)$ and the $\mathbb{Z}_2$-deformation retraction constructed below.

The box complex of a graph yields a lower bound for its chromatic number: \(\text{ind}(B(G)) + 2 \leq \chi(G)\). It is known that this topological bound can get arbitrarily bad: Walker [W83] shows that if a graph $G$ does not contain a $K_{2,2}$ then the associated invariant yields 3 as largest possible lower bound for the chromatic number $\chi(G)$. In section 4 we generalize this result to the following statement: If $G$ does not contain a complete bipartite graph $K_{\ell, m}$ then the index of the box complex $B(G)$ is bounded by $\ell + m - 3$ and this bound is sharp.

Finally, we show in section 5 that $L(G)$ is $\mathbb{Z}_2$-isomorphic to a subcomplex of the shore subdivision of the box complex $B(G)$ (which is introduced in section 3) and that this copy of $L(G)$ is a strong $\mathbb{Z}_2$-deformation retract of $B(G)$.

2 Preliminaries

In this section we recall some basic facts of graphs and simplicial complexes to fix notation. The interested reader is referred to [M03] or [B93] for details.

**Graphs:** Any graph $G$ considered will be assumed to be finite, simple, connected, and undirected, i.e. $G$ is given by a finite set $V(G)$ of nodes (we use vertices for associated complexes) and a set of edges $E(G) \subseteq \binom{V(G)}{2}$. A proper graph coloring with $n$ colors is a homomorphism $c : G \rightarrow K_n$, where $K_n$ is the complete graph on $n$ nodes and the chromatic number $\chi(G)$ of $G$ is the smallest $n$ such that there exists a proper graph coloring of $G$ with $n$ colors. The neighborhood $N(u)$ of $u \in V(G)$ is the set of all nodes adjacent to $u$. For a set of nodes $A \subseteq V(G)$ a node $v$ is in the common neighborhood $CN(A)$ of $A$, if $v$ is adjacent to all $a \in A$: we define $CN(\emptyset) := V(G)$. For $A \subseteq B \subseteq V(G)$ the common neighborhood relation satisfies (a) $A \cap CN(A) = \emptyset$, (b) $CN(B) \subseteq CN(A)$, (c) $A \subseteq CN^2(A)$, and (d) $CN(A) = CN^3(A)$. For two disjoint sets of nodes $A, B \subseteq V(G)$ we define $G[A; B]$ as the (not necessarily induced) subgraph of $G$ with node set $V(G[A; B]) = A \cup B$ and all edges $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. In this notation $CN(A)$ is the inclusion-maximal set $B$ such that $G[A; B]$ is complete bipartite.

**Simplicial Complexes:** An abstract simplicial complex $K$ is a finite hereditary set system. We denote its vertex set by $V(K)$ and its barycentric subdivision by $sd(K)$. For sets $A, B$ define $A \cup B := \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}$. An important construction in the category of simplicial complexes is the join operation. For two simplicial complexes $K$ and $L$ the join $K \ast L$ is defined as $\{F \cup G \mid F \in K$ and $G \in L\}$. Any abstract simplicial complex $K$ can be realized as a topological space $\|K\|$ in $\mathbb{R}^d$ for some $d$.

**$\mathbb{Z}_2$-spaces:** A $\mathbb{Z}_2$-space is a topological space $X$ together with a homeomorphism $\nu : X \rightarrow X$ that is self-inverse and free, i.e. has no fixed points. The
map \( \nu \) is called free \( \mathbb{Z}_2 \)-action. The fundamental example for a \( \mathbb{Z}_2 \)-space is the \( d \)-sphere \( S^d \) together with the antipodal map \( \nu(x) = -x \). A continuous map \( f \) between \( \mathbb{Z}_2 \)-spaces \((X, \nu)\) and \((Y, \mu)\) is \( \mathbb{Z}_2 \)-equivariant (or a \( \mathbb{Z}_2 \)-map for simplicity) if \( f \) commutes with the \( \mathbb{Z}_2 \)-actions, i.e. \( f \circ \nu = \mu \circ f \). A simplicial complex \((K, \nu)\) is a simplicial \( \mathbb{Z}_2 \)-space if \( \nu : K \to K \) is a simplicial map such that \( \|\nu\| \) is a free \( \mathbb{Z}_2 \)-action on \( |K| \). A simplicial \( \mathbb{Z}_2 \)-equivariant map \( f \) is a simplicial map between two simplicial \( \mathbb{Z}_2 \)-spaces that commutes with the simplicial \( \mathbb{Z}_2 \)-actions.

The index of a \( \mathbb{Z}_2 \)-space \((X, \nu)\) is the smallest \( d \) such that there is a \( \mathbb{Z}_2 \)-map \( f : X \to S^d \), i.e. \( f \circ \nu = -f \). The Borsuk–Ulam theorem provides the index for spheres: \( \text{ind}(S^d) = d \). Since the \( \mathbb{Z}_2 \)-actions are usually clear, we tend to refer to a \( \mathbb{Z}_2 \)-space \( K \) without explicit reference to \( \nu \).

**Chain Notation:** We denote by \( \mathcal{A} \) a chain \( A_1 \subset \ldots \subset A_p \) of subsets of \( V(G) \). A chain \( \mathcal{A} \) will be of length \( p \) and a chain \( \mathcal{B} \) of length \( q \). For \( 1 \leq t \leq p \) we denote by \( \mathcal{A}_{\leq t} \) the chain \( A_1 \subset \ldots \subset A_t \). A similar convention is used for \( \mathcal{A}_{>t} \). For chains \( \mathcal{A}, \mathcal{B} \) satisfying \( A_p \subseteq B_1 \) the chain \( A_1 \subset \ldots \subset A_p \subseteq B_1 \subset \ldots \subseteq B_q \) will be denoted by \( \mathcal{A} \sqsubseteq \mathcal{B} \), where we omit \( A_p \) or \( B_1 \) in case \( A_p = B_1 \). If a map \( f \) preserves (resp. reverses) orders, we write \( f(\mathcal{A}) \) instead of \( f(A_1) \subset \ldots \subset f(A_p) \) (resp. \( f(A_p) \subseteq \ldots \subseteq f(A_1) \)).

**Neighborhood Complex:** The neighborhood complex \( \mathcal{N}(G) \) of a graph \( G \) has \( V(G) \) as vertices and the sets \( A \subseteq V(G) \) with \( \text{CN}(A) \not= \emptyset \) as simplices.

**Lovász Complex:** In general \( \mathcal{N}(G) \) is not a \( \mathbb{Z}_2 \)-space. However, the neighborhood complex can be retracted to a \( \mathbb{Z}_2 \)-subspace, the Lovász complex. This complex \( \mathcal{L}(G) \) is the subcomplex of \( \text{sd}(\mathcal{N}(G)) \) induced by the vertices that are fixed points of \( \text{CN}^2 \). The Lovász complex is

\[
\mathcal{L}(G) = \{ \mathcal{A} \mid \mathcal{A} \text{ a chain of node sets of } G \text{ with } \mathcal{A} = \text{CN}^2(\mathcal{A}) \}
\]

which is a \( \mathbb{Z}_2 \)-space with \( \mathbb{Z}_2 \)-action \( \text{CN} \).

**Box Complex:** Different versions of a box complex are described by Alon, Frankl, and Lovász [AL80], Sarkaria [S90], Kríž [K92], and Matoušek and Ziegler [MZ03]. The box complex \( \mathcal{B}(G) \) of a graph which we are interested is the one introduced by Matoušek and Ziegler and is defined by

\[
\mathcal{B}(G) := \{ A \sqcup B \mid A, B \in \mathcal{N}(G) \text{ and } G[A; B] \text{ is complete bipartite} \} = \{ A \sqcup B \mid A, B \in \mathcal{N}(G), A \subseteq \text{CN}(B), \text{ and } B \subseteq \text{CN}(A) \}.
\]

The vertices of the box complex are \( V_1 := \{ v \} \sqcup \emptyset \) and \( V_2 := \emptyset \sqcup \{ v \} \) for all vertices of \( G \). The subcomplexes of \( \mathcal{B}(G) \) induced by \( V_1 \) and \( V_2 \) are disjoint subcomplexes of \( \mathcal{B}(G) \) that are both isomorphic to the neighborhood complex \( \mathcal{N}(G) \). We refer to these two copies as **shores** of the box complex. The box complex is endowed with a \( \mathbb{Z}_2 \)-action \( \nu \) which interchanges the shores.

### 3 Shore Subdivision and Useful Subcomplexes

**Shore Subdivision:** More general, for a simplicial complex \( K \) and any partition \( V_1 \sqcup V_2 \) of its vertex set, we call the simplicial subcomplexes \( K_1 \) and \( K_2 \) induced
by the vertex sets \( V_1 \) and \( V_2 \) its shores. The *shore subdivision* of \( K \) is

\[
\text{ssd}(K) := \{ \text{sd}(\sigma \cap K_1) + \text{sd}(\sigma \cap K_2) \mid \sigma \in K \}.
\]

The shores of the box complex define a partition of the vertex set which allows us to define the shore subdivision \( \text{ssd}(B(G)) \) of the box complex \( B(G) \). The vertices of \( \text{ssd}(B(G)) \) are of type \( A \cup \emptyset \) and \( \emptyset \cup A \) where \( \emptyset \neq A \subset V(G) \) with \( \text{CN}(A) \neq \emptyset \). A simplex of \( \text{ssd}(B(G)) \) is denoted by \( A \cup B \) (the simplex spanned by the vertices \( A \cup \emptyset \) and \( \emptyset \cup B \) where \( A \in A, B \in B \)).

**Doubled Lovász Complex:** The map \( c^{n^2} : \text{ssd}(B(G)) \rightarrow \text{ssd}(B(G)) \) defined on the vertices by \( A \cup \emptyset \mapsto \text{CN}^2(A) \cup \emptyset \) and \( \emptyset \cup A \mapsto \emptyset \cup \text{CN}^2(A) \) is simplicial and \( \mathbb{Z}_2 \)-equivariant. We refer to its image \( \text{Im} c^{n^2} \) as *doubled Lovász complex* \( DL(G) \).

It is

\[
DL(G) = \left\{ A \cup B \mid G[A; B] \text{ is complete bipartite for all } A \in A, B \in B \right\}.
\]

A copy of the Lovász complex can be found on each shore of \( DL(G) \subset \text{ssd}(B(G)) \), but these copies do not respect the induced \( \mathbb{Z}_2 \)-action.

**Halved Doubled Lovász Complex:** We partition the vertex set of the doubled Lovász complex \( DL(G) \) into pairs of type \( \{ A \cup \emptyset, \emptyset \cup \text{CN}(A) \} \) to define a simplicial \( \mathbb{Z}_2 \)-map \( j : DL(G) \rightarrow DL(G) \). Our aim is to specify one vertex for every pair and map both vertices of a pair to this chosen “smaller” vertex. To do this we refine the partial order by cardinality to a linear order “\( \prec \)” on the vertices of the original Lovász complex \( L(G) \) using the lexicographic order:

\[
A \prec B \iff \begin{cases} |A| < |B| \text{ or} \\ |A| = |B| \text{ and } A \prec_{\text{lex}} B. \end{cases}
\]

In fact any refinement would work in the following. A partial order on the vertices of the doubled Lovász complex \( DL(G) \) is now obtained:

\[
A \cup \emptyset \prec \emptyset \cup \text{CN}(A) \iff A \prec \text{CN}(A).
\]

We define the map \( j \) using this partial order by \( j(A \cup \emptyset) := \min_{\prec} \{ A \cup \emptyset, \emptyset \cup \text{CN}(A) \} \) and \( j(\emptyset \cup B) := \min_{\prec} \{ \emptyset \cup B, \text{CN}(B) \cup \emptyset \} \). Since the image \( \text{Im} j \) has half as many vertices as \( DL(G) \), we refer to \( \text{Im} j \) as *halved doubled Lovász complex* \( HDL(G) \).

**An example:** The neighborhood complex \( N(C_5) \) of the 5-cycle \( C_5 \) is the 5-cycle; its Lovász complex \( L(C_5) \) is the 10-cycle \( C_{10} \). The box complex \( B(C_5) \) consists of two copies of \( N(C_5) \) (the two shores) such that simplices of different shores are joined iff their vertex sets are common neighbors of each other. The shore subdivision \( \text{ssd}(B(C_5)) \) is a subdivision of the box complex induced from a barycentric subdivision of the shores. The map \( c^{n^2} \) maps a vertex of \( \text{ssd}(B(C_5)) \) to the common neighborhood of its common neighborhood. In our example, every vertex is mapped to itself, hence \( \text{ssd}(B(C_5)) = DL(C_5) \). The partitioning of the vertex set of \( DL(C_5) \) into pairs of type \( \{ A \cup \emptyset, \emptyset \cup \text{CN}(A) \} \) can be visualized.
by edges of $\mathbb{D}L(C_5)$ that connect singletons from one shore with two-element sets from the other. The smaller vertex of each such pair is actually a vertex of the original box complex $B(C_5)$. Hence the map $j$ collapses all edges of type $(A \uplus \emptyset, \emptyset \uplus \text{CN}(A))$, which yields the halved doubled Lovász complex $\mathbb{H}DL(G)$. The maps $f_i$ introduced in section 5 are these collapses and they are used to show that $L(G)$ is a $\mathbb{Z}_2$-deformation retract of $\text{ssd}(B(G))$. All these complexes are illustrated in Figure 1.

4 The $K_{1,m}$-Theorem

**Theorem 1** If a graph $G$ does not contain a complete bipartite subgraph $K_{\ell,m}$ then the index of its box complex is bounded by

$$\text{ind}(B(G)) \leq \ell + m - 3.$$ 

Since $\text{ind}(B(K_{\ell,m-1})) = \ell + m - 3$, the statement of the theorem is best possible. On the other hand, we obtain $\text{ind}(B(K_{k,k})) \leq k - 1$, but it can be shown that $\text{ind}(B(K_{k,k})) = 0$. So the gap in the inequality can arbitrarily large.

We give two proofs for this theorem. The first one uses the shore subdivision and the halved doubled Lovász complex, the other is a direct argument on $L(G)$ along the lines of Walker [W83].

**Proof (using Shore Subdivision)** Let $\Phi : \text{ssd}(B(G)) \to \text{ssd}(B(G))$ be the simplicial $\mathbb{Z}_2$-map defined by $j \circ \text{cn}^2$. Using that the index is dominated by dimension, it suffices to show the last inequality of

$$\text{ind}(B(G)) = \text{ind}(\text{ssd}(B(G))) \leq \text{ind}(\text{Im } \Phi) \leq \dim(\text{Im } \Phi) \leq \ell + m - 3.$$ 

To estimate the dimension of $\text{Im } \Phi = \mathbb{H}DL(G)$, we use that the graph $G$ does not contain a $K_{\ell,m}$ as a subgraph and assume without loss of generality that $\ell \leq m$. A vertex of $\mathbb{H}DL(G)$ or $\mathbb{D}L(G)$ of the form $A \uplus \emptyset$ or $\emptyset \uplus A$ is called small if $|A| < \ell$, medium if $\ell \leq |A| < m$, and large if $m \leq |A|$. For $\ell = m$ there are no medium vertices. Let $\sigma = A \uplus B$ be a simplex of $\mathbb{H}DL(G)$ and consider the set of vertices

$$M_\sigma := j^{-1}(\sigma) = \bigcup_{A \in A} \{A \uplus \emptyset, \emptyset \uplus \text{CN}(A)\} \cup \bigcup_{B \in B} \{\text{CN}(B) \uplus \emptyset, \emptyset \uplus B\}.$$ 

Clearly, $|M_\sigma|$ is at most twice $|V(\sigma)|$. If $\sigma$ has a large vertex $A \uplus \emptyset$, then the vertex $\emptyset \uplus \text{CN}(A)$ must be small, otherwise $G$ would contain a $K_{\ell,m}$. Hence
there are at most $2 \cdot 2(\ell - 1)$ many vertices in $M_\sigma$ that are large or small. Since the number of medium vertices is at most $2(m - \ell)$, we have

$$|M_\sigma| \leq 2 \cdot 2(\ell - 1) + 2(m - \ell) = 2(\ell + m - 2).$$

Hence $|V(\sigma)| \leq \ell + m - 2$ for all $\sigma$, and therefore $\dim(\text{HDL}(G))$ is at most $\ell + m - 3$. \hfill \square

Proof: (using Lovász Complex) It suffices to prove $\dim(\text{L}(G)) \leq \ell + m - 3$ since $\text{ind}(\text{B}(G)) = \text{ind}(\text{L}(G)) \leq \dim(\text{L}(G))$, \textsc{MZ03} or section 4. Without loss of generality let $\ell \leq m$ and consider a simplex $\sigma = A_1 \subseteq \ldots \subseteq A_p$ of $\text{L}(G)$ of maximal dimension $p - 1$. If $p < \ell$ we are done. Suppose therefore that $p \geq \ell$. Then $G[A; \text{CN}(A)]$ is a bipartite subgraph of $G$ and we have $|A| \geq \ell$ as well as $|\text{CN}(A)| \geq p - \ell + 1$. The assumption that $G$ does not contain a $K_{\ell,m}$ implies that $m > p - \ell + 1$, i.e. $\dim(\sigma) \leq \ell + m - 3$. \hfill \square

5 \hspace{1cm} \text{L}(G) as a $\mathbb{Z}_2$-Deformation Retract of $\text{B}(G)$

Theorem 2 The Lovász complex $\text{L}(G)$ and the halved doubled Lovász complex $\text{HDL}(G)$ are $\mathbb{Z}_2$-isomorphic.

Proof. First we have $|V(\text{L}(G))| = |V(\text{HDL}(G))|$ since each shore of $\text{DL}(G)$ is isomorphic (but not $\mathbb{Z}_2$-isomorphic) to $\text{L}(G)$. To define a simplicial $\mathbb{Z}_2$-map $f : \text{L}(G) \to \text{HDL}(G)$, we partition $V(\text{L}(G))$ into

$$S := \left\{ A \mid A \in V(\text{L}(G)) \text{ and } j(A \cup \emptyset) = A \cup \emptyset \right\} \text{ and } J := \left\{ A \mid A \in V(\text{L}(G)) \text{ and } j(A \cup \emptyset) = \emptyset \cup \text{CN}(A) \right\},$$

(where “$S$” and “$J$” denote the vertices that stay fixed or jump to their neighbor), and set

$$f(A) := \begin{cases} A \cup \emptyset & \text{if } A \in S \\ \emptyset \cup \text{CN}(A) & \text{if } A \in J. \end{cases}$$

This map is a bijection between the vertex sets, surjective, simplicial, and $\mathbb{Z}_2$-equivariant. For simpliciality, consider a simplex $\mathcal{A}$ in $\text{L}(G)$. Let $t$ denote the largest index $i$ such that $A_i$ is mapped onto the first shore. The image of $\mathcal{A}$ under $f$ is $A_{\leq t} \cup \text{CN}(A_{t+1})$. This is a simplex since $G[A; \text{CN}(A_{t+1})]$ is complete bipartite. For surjectivity consider a simplex $\mathcal{A} \cup \mathcal{B}$ of $\text{HDL}(G)$, i.e. $G[A_p; B_q]$ is complete bipartite. This is the image of the simplex $\mathcal{A} \subseteq \text{CN}(\mathcal{B})$ of $\text{L}(G)$. \hfill \square

Theorem 3 The halved doubled Lovász complex $\text{HDL}(G)$ is a strong $\mathbb{Z}_2$-deformation retract of the box complex $\text{B}(G)$.

Proof. First we observe that $\|\text{DL}(G)\|$ is a strong $\mathbb{Z}_2$-deformation retract of $\|\text{ssd}(\text{B}(G))\|$. This follows from the fact that a closure operator induces a strong deformation retraction from its domain to its image \textsc{B95}, \textsc{M03}. Explicitly,
this map is obtained by sending each point \( p \in \| \text{ssd}(B(G)) \| \) towards \( \| \text{CN}^2 \| (p) \) with uniform speed, which is \( \mathbb{Z}_2 \)-equivariant at any time of the deformation.

To show that \( \| \text{HDL}(G) \| \) is a strong \( \mathbb{Z}_2 \)-deformation retract of \( \| \text{DL}(G) \| \), we define simplicial complexes and simplicial \( \mathbb{Z}_2 \)-maps

\[
\text{DL}(G) =: S_0 \xrightarrow{f_0} S_1 \xrightarrow{f_1} \ldots \xrightarrow{f_N} S_{N+1} := \text{HDL}(G)
\]

such that \( S_{i+1} \) is a \( \mathbb{Z}_2 \)-subcomplex of \( S_i \) and \( S_{i+1} \) is a strong \( \mathbb{Z}_2 \)-deformation retract of \( S_i \). The composition of the \( f_i \) yields the earlier defined map \( j \), i.e. \( j = f_N \circ \cdots \circ f_1 \circ f_0 \). To construct \( S_{i+1} \) inductively from \( S_i \), we consider \( X := \max \{ Y \in J \mid Y \cup \emptyset \in S_i \} \) and obtain \( S_{i+1} \) from \( S_i \) by deleting each simplex of \( S_i \) that contains \( X \cup \emptyset \) or its \( \mathbb{Z}_2 \)-pair \( \emptyset \cup X \), i.e.

\[
S_{i+1} := \{ \sigma \mid \sigma \in S_i \text{ and } X \cup \emptyset \not\in \sigma \text{ and } \emptyset \cup X \not\in \sigma \}.
\]

The maximality of \( X \) implies that a maximal simplex which contains \( X \cup \emptyset \) (resp. \( \emptyset \cup X \)) does also contain \( \emptyset \cup \text{CN}(X) \) (resp. \( \text{CN}(X) \cup \emptyset \)). Hence the map \( f_i \) defined on the vertices \( v \in V(S_i) \) via

\[
f_i(v) := \begin{cases} 
\emptyset \cup \text{CN}(X) & \text{if } v = X \cup \emptyset \\
\text{CN}(X) \cup \emptyset & \text{if } v = \emptyset \cup X \\
v & \text{otherwise}
\end{cases}
\]

is simplicial and \( \mathbb{Z}_2 \)-equivariant.

Thus \( F: \| S_i \| \times [0,1] \to \| S_i \| \) given by \( F(x,t) := t \cdot x + (1-t) \cdot \| f_i \| (x) \) is a well-defined \( \mathbb{Z}_2 \)-homotopy from \( \| f_i \| \) to \( \text{Id}_{\| S_i \|} \) that fixes \( \| S_{i+1} \| \).

Acknowledgments

The authors thank Günter M. Ziegler for bringing the problems studied in this paper to their attention, and Tibor Szabó for numerous discussions.

References

[AFL86] N. Alon, P. Frankl, L. Lovász: The chromatic number of Kneser hypergraphs, Transactions Amer. Math. Soc., 298, pp. 359-370, 1986.

[B95] A. Björner: Topological methods. In R. Graham, M. Grötschel, and L. Lovász, editors, Handbook of Combinatorics Vol. II, Chapter 34, pp. 1819-1872. North-Holland, Amsterdam, 1995.

[K55] M. Kneser: Aufgabe 360, Jahresbericht der Deutschen Mathematiker-Vereinigung, 58, 2. Abteilung, p. 27, 1955.

[K92] I. Kriz: Equivariant cohomology and lower bounds for chromatic numbers, Transactions Amer. Math. Soc., 333, pp. 567-577, 1992.
[L78] L. Lovász: *Kneser’s conjecture, chromatic number and homotopy*, J. Comb. Theory, Ser. A, 25, pp. 319-324, 1978.

[M03] J. Matoušek: *Using the Borsuk–Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer Verlag, Heidelberg, 2003.

[MZ03] J. Matoušek, G. M. Ziegler: *Topological lower bounds for the chromatic number: A hierarchy*, Preprint, arXiv:math.CO/0208072, 2002.

[S90] K. S. Sarkaria: *A generalized Kneser conjecture*, J. Comb. Theory, Ser. B, 49, pp. 236-240, 1990.

[W83] J. W. Walker: *From graphs to ortholattices and equivariant maps*, J. Comb. Theory, Ser. B, 35, pp. 171-192, 1983.