STRUCTURE OF THE UNRAMIFIED $L$-PACKET

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Abstract. Let $G$ be an unramified connected reductive group defined over a non-archimedean local field $k$ and let $T$ be a maximal torus in $G$. Let $\lambda$ be an unramified character of $T$. Then the conjugacy classes of hyperspecial subgroups of $G(k)$ is a principal homogenous space for a certain finite abelian group $\hat{\Omega}$. Also, the $L$-packet $\Pi(\varphi_\lambda)$ associated to $\lambda$ is parametrized by an abelian group $\hat{R}$. We show that $\hat{R}$ is naturally a homogenous space for $\hat{\Omega}$. Further, let $\pi_\rho \in \Pi(\varphi_\lambda)$, where $\rho \in \hat{R}$ and let $[K]$ denote the conjugacy class of hyperspecial subgroup $K$. Then we show that $\pi^K_\rho \neq 0$ if and only if $\pi^{K_\omega}_{\rho\omega} \neq 0$ where $\omega \in \hat{\Omega}$ and $K_\omega$ is any hyperspecial subgroup in the conjugacy class $\omega \cdot [K]$.

Introduction

Let $G$ be a connected reductive group defined over a local field $k$. The local Langlands conjectures predict that the irreducible admissible representations of $G(k)$ can be partitioned into finite sets, known as $L$-packets, in a certain natural way. Each packet is expected to be associated to what is known as a Langlands parameter for $G$.

Now assume $G$ is unramified, i.e., it admits hyperspecial subgroups. A representation of $G$ is called unramified if it has a non-zero vector fixed under some hyperspecial subgroup of $G(k)$. Unramified representations are of central importance in the Langlands program, as almost all local components of global representations are unramified. They were the first representations to be grouped into packets and associated with Langlands parameters. In this paper, we answer the question of how $L$-indistinguishability is related to the different choices of hyperspecial subgroups.

Let $T$ be a maximal torus in $G$ contained in a Borel subgroup $B$, both defined over $k$. A character of $T(k)$ is called unramified if it is trivial on the maximal compact subgroup $T(k)_0$ of $T(k)$. An unramified character $\lambda$ of $T(k)$ corresponds to a Langlands parameter $\varphi_\lambda$ of $G$ via the local Langlands correspondence for tori. The local Langlands conjectures stipulate what the $L$-packet $\Pi(\varphi_\lambda)$ associated to such a Langlands parameter $\varphi_\lambda$ has to be. Namely, (see [Bor79, 10.4]) $\Pi(\varphi_\lambda)$ should

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consist precisely of those subquotients of the induced representation $\text{Ind}_{B(k)}^{G(k)} \lambda$, that are unramified. For a hyperspecial subgroup $K$ of $G(k)$, denote by $\tau_{K,\lambda}$ the unique subquotient of $\text{Ind}_{B(k)}^{G(k)} \lambda$ having a $K$-fixed vector. It is well known that the choice of $K$ determines a bijection between the group $\hat{\mathbf{R}}_{\phi} \lambda$ of characters of a certain finite abelian group $R_{\phi} \lambda$, and the elements of $\Pi(\varphi_{\lambda})$, with the trivial character of $R_{\phi} \lambda$ corresponding to $\tau_{K,\lambda}$. However, this does not tell us which character of $R_{\phi} \lambda$ corresponds to which representation in $\Pi(\varphi_{\lambda})$. This hitherto unexplored question on the finer internal structure of the $L$-packet is answered by Theorem 1 (see Section 2) which, given a character $\rho$ of $R_{\phi} \lambda$, specifies the various hyperspecial subgroups $K'$ for which $\tau_{K',\lambda}$ corresponds to $\tau_{K,\lambda}$.

Further, it has long been known that the various $\tau_{K,\lambda}$ as above are all the unramified representations of $G$. This gets sharpened into a parametrization of the unramified representations of $G$ when combined with the Corollary 2 to Theorem 1 (see Section 2), which spells out what the relation between pairs $(K, \lambda)$ and $(K', \lambda')$ as above has to be, for $\tau_{K,\lambda}$ to be isomorphic to $\tau_{K',\lambda'}$.

To put this result in perspective, let us briefly review some history of previous work related to this problem. When $G$ is (in addition to being unramified) simply connected and almost simple and when $\lambda$ is unitary and unramified, D. Keys showed that all irreducible constituents of the principal series representation $\text{Ind}_{B(k)}^{G(k)} \lambda$ are unramified [Key82a, sec 4]. Without these extra conditions on $G$, D. Keys and F. Shahidi specified a non-zero Whittaker functional afforded by $\tau_{K,\lambda}$ [KS88, Theorem 4.1].

A key idea required in the proof of Theorem 1 is that the Knapp-Stein $R$-group can be realized as the stabilizer, in a certain subgroup of the affine Weyl group, of a point $x_{\lambda}$ in the Bruhat-Tits building of $G$. This idea develops over the work of M. Reeder [Ree10]. Note that we are allowing $\lambda$ to be non-unitary, so the principal series representation $\text{Ind}_{B(k)}^{G(k)} \lambda$ does not, in general decompose as a direct sum of irreducible representations. The construction of the $L$-packet as described in [Sha11] is required.

1. Notations

Let $G$ be an unramified, connected reductive linear algebraic group defined over a $p$-adic local field $k$. We denote the $k$-point of $G$ by $G$ and likewise for all its subgroups. Let $A$ denote a maximal split $k$-torus in $G$ and let $T = Z_G A$, the centralizer of $A$ in $T$. Then $T$ is a maximal torus in $G$, since $G$ is quasi-split. Let $B$ be a Borel subgroup containing $T$. Let $\Psi = (X, \Phi, \Delta, \check{X}, \check{\Phi}, \check{\Delta})$ be the based root datum of $(G, B, T)$. Let $W_k$ denote the Weil group of $k$. Let $I = I_k$ be the inertia subgroup of $W_k$ and let $\sigma = \sigma_k$ be the Frobenius element in $W_k/I$. The Frobenius element $\sigma$ induces an automorphism of $\Psi$ which we again denote by $\sigma$. If $K$ is a
compact subgroup of \( G \), we denote the set of conjugates of \( K \) in \( G \) by \([K]\). If \( \pi \) is a representation of \( G \), by the expression \( \pi^{[K]} \neq 0 \), we mean that there is a non-zero \( K \)-fixed vector in the space realizing \( \pi \) (consequently, for any conjugate \( K' \in [K] \), there is a non-zero \( K' \)-fixed vector).

A character of \( T \) is called unramified, if it is trivial on the maximal compact subgroup \( T_0 \) of \( T \). Let \( \lambda \) be such a character of \( T \). To this character, one can associate a Langlands parameter \( \varphi = \varphi_\lambda \). Denote by \( \Pi(\varphi_\lambda) \) the \( L \)-packet associated to \( \varphi_\lambda \). Let \( \hat{G} \) be the complex dual of \( G \). Let \( S_{\varphi} = Z_G \text{Im}(\varphi) \), the centralizer in \( \hat{G} \) of the image of \( \varphi \) and let \( S_{\varphi}^0 \) be its connected component containing the identity.

Let \( \hat{\mathcal{Z}} = Z(\hat{G}) \), the center of the dual group. Let \( R_\varphi = R_\varphi(G) := S_{\varphi}/S_{\varphi}^0 \hat{\mathcal{Z}}^{\sigma} \) be the \( R \) group defined by Langlands. This group turns out to be abelian. The elements of the \( L \)-packet \( \Pi(\varphi) \) are parametrized by \( \check{R}_\varphi \), the group of characters of \( R_\varphi \). This parametrization is not canonical, however. It depends on the choice of a nilpotent orbit. We fix a parametrization and denote the representation corresponding to a character \( \rho \in \check{R}_\varphi \) under this parametrization by \( \pi_\rho \).

2. Statement of the theorems

Let \( G_{\text{ad}} \) be the adjoint group of \( G \) and let \( T_{\text{ad}} \) be the maximal torus of \( G_{\text{ad}} \) which is the image of \( T \) in \( G_{\text{ad}} \). The set of conjugacy classes of hyperspecial subgroups of \( G \) is a principal homogenous space for the finite abelian group \( \hat{\Omega} := \text{coker}(\hat{X}^{\sigma} \to \hat{X}_{\text{ad}}^{\sigma}) \), where \( \hat{X}_{\text{ad}} := \hat{X}(T_{\text{ad}}) \). The action is defined as follows: the natural map \( \hat{X}(T_{\text{ad}})^{\sigma} \otimes \mathbb{R} \to \hat{X}(\mathcal{A}) \otimes \mathbb{R} \) gives an action of \( \hat{X}_{\text{ad}}^{\sigma} \) on the apartment \( \mathcal{A}(\mathcal{A}) \). Let \( \mathcal{H} = \{ x \in \mathcal{A}(\mathcal{A}) : x \text{ is hyperspecial} \} \). Sine \( G \) is unramified, we may and do assume that the hyperspecial subgroups of \( G \) are the stabilizers \( G_x \) of hyperspecial points \( x \).

For \( t \in \hat{X}_{\text{ad}}^{\sigma} \) and \( x \in \mathcal{H} \), define \( t \cdot [G_x] := [G_{t \cdot x}] \). Then this action is well defined and it makes the conjugacy classes of hyperspecial subgroups, a principal homogenous space for \( \hat{\Omega} \).

**Theorem 1.** There exists a canonical surjection \( \zeta : \hat{\Omega} \to \check{R}_\varphi \) and

\[
\omega \cdot [K] \neq 0 \iff \pi_\rho^{[K]} \neq 0,
\]

for all \( \omega \in \hat{\Omega}, \rho \in \check{R}_\varphi \), where \( K \) is a hyperspecial subgroup of \( G \). Here \( \omega \cdot \rho := \rho + \zeta(\omega) \).

For a hyperspecial subgroup \( K \), a representation of \( G \) is called \( K \)-spherical if it is smooth, irreducible, admissible and contains a non-zero vector invariant under \( K \). Write \( B = TU \), where \( U \) is the unipotent radical of \( B \). The \( K \)-spherical representations have the following explicit description:

Define the complex valued function \( \Lambda_{K,\lambda} : G \to \mathbb{C} \)

\[
\Lambda_{K,\lambda}(tuk) = \lambda(t)\delta^\frac{s}{2}(t),
\]
for \( t \in T, u \in U \) and \( k \in K \). Here \( \delta \) is the modulus function. Put
\[
\Gamma_{K,\lambda}(g) = \int_{K} A_{K,\lambda}(kg)dk,
\]
for \( g \in G \). Denote by \( V_{[K],\lambda} \) the space of functions \( f \) on \( G \) of the form
\[
f(g) = \sum_{i=1}^{n} c_i \Gamma_{K,\lambda}(gg_i)
\]
for \( c_1, \ldots, c_n \) in \( \mathbb{C} \) and \( g_1, \ldots, g_n \) in \( G \). We let \( G \) operate on \( V_{[K],\lambda} \) by right translations, namely
\[
(\pi_{[K],\lambda}(g)f)(g_1) = f(g_1g)
\]
for \( f \) in \( V_{[K],\lambda} \) and \( g,g_1 \) in \( G \). Then it is well known that the representation \((\pi_{[K],\lambda}, V_{[K],\lambda})\) is \( K \)-spherical and any \( K \)-spherical representation is isomorphic to \((\pi_{[K],\lambda}, V_{[K],\lambda})\) for some unramified character \( \lambda \) (see [Car79] sec. 4.4).

For different choices of \( K \), the following corollary of theorem \( \Pi \) describes the condition for the representations \((\pi_{[K],\lambda}, V_{[K],\lambda})\) to be isomorphic.

**Corollary 2.** \((\pi_{\omega_1,[K],\lambda}, V_{\omega_1,[K],\lambda}) \cong (\pi_{\omega_2,[K],\lambda}, V_{\omega_2,[K],\lambda})\) if and only if \( \zeta(\omega_1) = \zeta(\omega_2) \).

Let \( \hat{\rho} \) denote any element of \( \zeta^{-1}(\rho) \) for \( \rho \in \hat{R}_\varphi \). Theorem \( \Pi \) and corollary 3 immediately imply the following explicit description of the elements of the \( L \)-packet \( \Pi(\varphi,\lambda) \).

**Corollary 3.** The elements of the \( L \)-packet \( \Pi(\varphi,\lambda) \) are \( \{ (\pi_{\hat{\rho}[K],\lambda}, V_{\hat{\rho}[K],\lambda}) : \rho \in \hat{R}_\varphi \} \).

3. The group \( \Omega \)

Let \( \Psi = (X, \Phi, \Delta, \hat{X}, \hat{\Phi}, \hat{\Delta}) \) be a based root datum as defined in [Spr79 1.9]. Let \( W = W(\Psi) \) be the Weyl group. Let \( \{ \alpha_1, \ldots, \alpha_l \} \) be the set of simple roots \( \Delta \). Let \( \Phi_{\text{aff}} = \{ a + n : a \in \Phi, n \text{ is an integer if } \frac{1}{2}a \notin \Phi \text{ (resp. any odd integer if } \frac{1}{2}a \in \Phi \} \). Then \( \Phi_{\text{aff}} \) is an affine root system in the real vector space \( V = V(\Psi) := \mathbb{R} \otimes X \) (see [Mac03 Theorem 1.2,1]). The Weyl group \( W \) acts on \( V \) as the group of reflections generated by \( r_1, \ldots, r_l \), where \( r_i \) fixes the hyperplace \( \{ x \in V : \alpha_i(x) = 0 \} \). The basis \( \Delta \) determines a particular alcove \( C \) in \( V \) as follows: let \( \hat{\alpha}_0 = \sum_{i=1}^{l} a_i \alpha_i \) be the highest root with respect to \( \Delta \). Let \( \alpha_0 \) be the affine linear function \( 1 - \hat{\alpha}_0 \) on \( V \) and set \( a_0 = 1 \), so that
\[
\sum_{i=0}^{l} a_i \alpha_i \equiv 1.
\]
Then the alcove determined by \( \Delta \) is the intersection of the half spaces:
\[
C = \{ x \in V : \alpha_i(x) > 0 \text{ for } 0 \leq i \leq l \}.
\]
The group $\tilde{W} = W \ltimes \tilde{X}$ is called the extended affine Weyl group. It contains the affine Weyl group $\tilde{W}^\circ = W \ltimes \mathbb{Z}\tilde{\Phi}$ as a normal subgroup. $\tilde{W}$ acts transitively on the set of alcoves in $V$. Let
\[
\Omega = \Omega(\Psi) := \left\{ \rho \in \tilde{W} : \rho \cdot C = C \right\}.
\]
Then, $\tilde{W} = \Omega \ltimes \tilde{W}^\circ$ and therefore $\Omega \cong \tilde{W}/\tilde{W}^\circ \cong \tilde{X}/\mathbb{Z}\tilde{\Phi}$.

4. R-groups and intertwining operators

Let $U$ denote the unipotent radical of $B$. Then $B = TU$. We will abbreviate by $I(\lambda)$, the principal series representation $\text{Ind}_B^G \lambda$ of $G$. Let $N := NA$ and let $W$ be the relative Weyl group of $G$. Then $W \cong N/T$. Denote by $\bar{w}$, a representative of an element $w$ of $W$. Let $\bar{B} = T\bar{U}$ be the Borel subgroup opposite to $B$. Define the usual intertwining operators:
\[
A(w, \lambda): I(\lambda) \to I(w\lambda),
\]
\[
(A(\bar{w}, \lambda)f)(g) = \int_{U \cap wU_{\bar{w}}} f(gu\bar{w})du.
\]
These operators converge in an appropriate domain and they may be analytically continued so that they are defined for all unitary $\lambda$ [Kac82b Sec. 2]. Further, they satisfy the co-cycle relation
\[
A(\bar{w}_1\bar{w}_2, \lambda) = A(\bar{w}_1, \bar{w}_2\lambda)A(\bar{w}_2, \lambda),
\]
provided
\[
\text{length}(w_1w_2) = \text{length}(w_1) + \text{length}(w_2).
\]
Define the normalized intertwining operators
\[
\mathcal{A}(\bar{w}, \lambda) := \frac{1}{c_w(\lambda)}A(\bar{w}, \lambda),
\]
where $c_w(\lambda)$ is the Harish-Chandra $c$-function. Then with appropriate choices of the coset representatives $\bar{w}$ of the Weyl group elements, the co-cycle relation
\[
\mathcal{A}(\bar{w}_1\bar{w}_2, \lambda) = \mathcal{A}(\bar{w}_1, \bar{w}_2\lambda)\mathcal{A}(\bar{w}_2, \lambda),
\]
holds without any condition on the lengths of $w_1$ and $w_2$.

Let $\Phi_{rel}$ be the relative roots. Then $W$ is the Weyl group associated to $\Phi_{rel}$. For $\alpha \in \Phi_{rel}$, let $s_\alpha$ denote the reflection associated to $\alpha$. Define
\[
\Delta' = \{ \alpha \in \Phi_{rel} : A(w, \lambda) \text{ is scalar} \},
\]
and let $W'$ be the reflection group $<s_\alpha : \alpha \in \Delta'>$. Let

$$W_\lambda = \{ w \in W : w\lambda = \lambda \}.$$ 

Define

$$R = R(G, \lambda) = \{ w \in W_\lambda : \alpha \in \Delta' \text{ and } \alpha > 0 \text{ imply } w\alpha > 0 \}. \tag{4.5}$$

Then

$$W_\lambda = W' \rtimes R. \tag{4.6}$$

For a unitary, unramified character, the Knapp-Stein $R$-group is abelian. This is shown in [Key82b] for simply connected, almost simple, semi-simple groups. In Corollary 10 we have proved it for any connected reductive group. It then follows from [Key87, Thm. 2.4],

**Theorem 4.** For an unramified unitary character $\lambda$ of $T$,

1. The commuting algebra $\text{End}(I(\lambda))$ of $I(\lambda)$ is isomorphic to the group algebra $\mathbb{C}[R]$.
2. $I(\lambda)$ decomposes with multiplicity one.
3. The irreducible components of $I(\lambda)$ are parametrized by the characters of $R$.

Let $I(\lambda) \cong \bigoplus_{\rho \in \hat{R}} \pi_\rho$ be a decomposition of the principal series, where $\hat{R}$ is the group of characters of $R$. This parametrization is not unique. It depends on the choice of normalizations of intertwining operators giving $\text{End}(I(\lambda)) \cong \mathbb{C}[R]$. But any two normalizations must differ by a one-dimensional character $\rho'$ of $R$, i.e.,

$$\mathcal{A}'(r, \lambda) = \rho'(r) \mathcal{A}(r, \lambda).$$

Then if $\pi$ corresponds to the character $\rho$ in the first normalization, then it corresponds to the character $\rho + \rho'$ in the second normalization.

5. **Image of an Unramified Character under LLC**

In this section, we review the description of the image of an unramified character, under the local Langlands correspondence for tori.

Let $T$ be a torus defined over $k$. Assume that $T$ splits over an unramified extension $k'$ of $k$. As before, we denote by $T_o$, the maximal compact subgroup of $T$. Given $t \in T(k')$, let $\nu(t) \in \text{Hom}(X^*(T), \mathbb{Z})$ be defined by $\nu(t)(m) = \text{ord}(m(t))$. 

Then the map $t \mapsto \nu(t)$ induces an isomorphism (see [Bor79, Sec 9.5]):

$$T/T_\sigma \cong X_*(T)^\sigma = X^*(\hat{T})^\sigma \cong X^*(\hat{T}_\sigma).$$

Here $\hat{T}_\sigma$ denotes the co-invariant of $\hat{T}$ with respect to $\sigma$. From this we get

$$\text{Hom}(T/T_\sigma, \mathbb{C}^\times) \cong \text{Hom}(X^*(\hat{T}_\sigma), \mathbb{C}^\times) \cong \hat{T}_\sigma. \quad (5.1)$$

Thus to an unramified character $\chi$ of $T$, we can associate the $\sigma$-conjugacy class of a semisimple element $s_\chi$ of $\hat{T}$. In fact, this is the image of $\chi$ under the Local Langlands correspondence for tori,

$$\text{Hom}(T/T_\sigma, \mathbb{C}^\times) \xrightarrow{\text{LLC}} H^1(W_k, \hat{T}) \xrightarrow{\text{infl}} H^1(W_k/I, \hat{T})$$

6. Coefficient of the spherical vector

Assume $\lambda$ to be unitary. Let $K$ be a hyperspecial subgroup of $G$. Let $f_K \in I(\lambda)^K$ be such that $f_K(1) = 1$. Let $\rho_{\lambda,[K]}$ be the character of $R$ defined by $\mathcal{A}(r, \lambda) f_K = \rho_{\lambda,[K]}(r) f_K$. Then,

**Lemma 5.** $\pi^K_p \neq 0 \iff \rho_{\lambda,[K]} = \rho$.

**Proof.** The operators

$$P_\rho = \frac{1}{|R|} \sum_{r \in R} \rho(r)^{-1} \mathcal{A}(r, \lambda)$$

are $|\hat{R}|$ orthogonal projections onto the invariant subspaces of $I(\lambda)$. Let $U_\rho = P_\rho U$, where $U$ is the space realizing $I(\lambda)$. Then

$$U = \bigoplus_{\rho \in \hat{R}} U_\rho.$$

Since

$$\mathcal{A}(r, \lambda) f_K = \rho_{\lambda,[K]}(r) f_K,$$}

Therefore
\[ P_{\rho f_K} = \frac{1}{|R|} \left( \sum_{r \in R} \rho(r)^{-1} \rho_{\lambda,[K]}(r) \right) f_K \]

\[ = \begin{cases} 
0 & \rho_{\lambda,[K]} \neq \rho \\
 f_K & \rho_{\lambda,[K]} = \rho,
\end{cases} \]

i.e., \( U^K_{\rho} \neq 0 \iff \rho_{\lambda,[K]} = \rho. \)

Thus, to understand which representation in the \( L \)-packet \( \Pi(\varphi_\lambda) \) is spherical for which hyperspecial conjugacy class, we need to understand how the characters \( \rho_{\lambda,[K]} \) relate to each other for different choices of conjugacy classes \([K]\).

7. Proof in a special case

To give the main ideas of the proof quickly, we first prove our theorem for the simplest case when \( G \) is semisimple, almost simple, simply connected and split and when \( \lambda \) is unitary. We will use the notations introduced in section \([3]\).

The local Langlands correspondence for tori induces an isomorphism,

\[ \text{Hom}(T/T_\circ, \mathbb{C}^\times) \cong \hat{T}. \]

The image of the unramified unitary characters under this isomorphism is the subtorus \( \hat{U} \cong X \otimes \mathbb{S}^1 \subset \hat{T} \cong X \otimes \mathbb{C}^\times. \) Now \( \hat{U} \cong X \otimes (\mathbb{R}/\mathbb{Z}) \cong X \otimes \mathbb{R}/X. \) Therefore \( \hat{U}/W \cong X \otimes \mathbb{R}/W \ltimes X = V(\hat{\psi})/\hat{W}. \) Thus, the LLC induces a natural isomorphism \( \text{Hom}(T/T_\circ, \mathbb{S}^1)/W \cong V(\hat{\psi})/\hat{W}. \) Here \( \mathbb{S}^1 \) denotes the unit circle in \( \mathbb{C}. \)

Thus, to a unitary unramified character \( \lambda \), we can naturally associate a semisimple element \( s = s_\lambda \in \hat{T} \) which corresponds to a point \( x = x_\lambda \) in the closure of the alcove \( C(\hat{\psi}). \) We'll denote \( \Omega(\hat{\psi}) \) by \( \Omega \) whenever there is no ambiguity. Let \( \Omega_x \) denote the stabilizer of \( x \) in \( \Omega \).

**Proposition 6.** There is a natural isomorphism \( \Omega_x \cong \mathbb{R}. \)

**Proof.** By \([\text{Ree10}, \text{Prop.} 2.1]\), \( \Omega_x \cong \pi_0(\hat{Z}_{\hat{G}}s) \), the connected component group of the centralizer of the semi-simple element \( s \). By \([\text{Key87}, \text{Proposition} 2.6]\), \( \pi_0(\hat{Z}_{\hat{G}}s) \cong \mathbb{R}. \) Thus, \( \Omega_x \cong \mathbb{R}. \) In fact, it follows from the proofs of \([\text{Ree10}, \text{Prop.} 2.1]\) & \([\text{Key87}, \text{Prop.} 2.6]\) that isomorphism is induced by the natural projection \( \hat{W} \rightarrow W. \) We defer these details until proposition \([9]\) where we prove our statement for more general groups. \( \square \)

Let \( \hat{P} \) be the co-weight lattice of \( G. \) Then the set of conjugacy classes of hyperspecial points of \( G \) form a principal homogenous space for the group \( \hat{P}/\hat{X}. \) This group is in duality with the group \( \Omega \cong X/\mathbb{Z}\Phi. \) Denote by \( (,): (\hat{P}/\hat{X}) \times \Omega \rightarrow \mathbb{Q}/\mathbb{Z} \)
the pairing between them. Using the last proposition, we can realize $R$ as a subgroup of $\Omega$ by the natural embedding $R \cong \Omega_x \subset \Omega$. For $r \in R$, and $\omega \in \hat{\mathcal{P}}/\hat{\mathcal{X}}$, let $\rho_\omega(r) = e^{-2\pi i (\omega, r)}$. Then we have a natural surjection $\hat{\Omega} \twoheadrightarrow \hat{R}$ given by $\omega \mapsto \rho_\omega$.

Let $K_0$ be a hyperspecial subgroup satisfying $\pi [K_0] \neq 0$. Pick $K_\omega \in \omega \cdot [K_0]$ and let $f_\omega \in I(\lambda)^{K_\omega}$ be the spherical vector such that $f_\omega(1) = 1$.

**Proposition 7.** $\mathcal{A}(r, \lambda)f_\omega = \rho_\omega(r)f_\omega$.

**Proof.** By [Key82b, Section 4 lemma], one can do a case by case computation to show this. The calculations are shown in the section 11. □

From the surjection $\hat{\Omega} \twoheadrightarrow \hat{R}$, we get a natural action of $\hat{\Omega}$ on $\hat{R}$, namely for $\rho \in \hat{R}$ and $\omega \in \hat{\Omega}$, we let $\omega \cdot \rho = \rho + \rho_\omega$. Let $K$ be any hyperspecial subgroup. Let $\omega' \in \hat{\Omega}$ be such that $\omega' \cdot [K_0] = [K]$. Then from lemma 5 it follows:

$$\pi_{\omega'}^{[K]} \neq 0 \iff \omega' \cdot \rho = \rho_{\omega' + \omega}$$

$$\iff \rho = \rho_{\omega'}$$

$$\iff \pi_{\rho}^{[K_0]} \neq 0$$

$$\iff \pi_{\rho}^{[K]} \neq 0.$$  

8. CONSTRUCTION OF THE RELATIVE BASED ROOT DATUM

Let $\Psi = (X, \Phi, \Delta, \hat{X}, \hat{\Phi}, \hat{\Delta})$ be the based root datum of $(G, B, T)$, where $G$ is an unramified, connected reductive group, $B$ is a borel subgroup of $G$ containing a maximal torus $T$. The following construction is given in [Yu].

Define a new 6-touple:

$$Y = (X \sigma)/\text{torsion},$$

$$\hat{Y} = (\hat{X})^e,$$

$$S = \{\underline{a} : a \in \Phi\}, \text{ where } \underline{a} = a|_{\hat{Y}},$$

$$\hat{S} = \{\hat{a} : a \in S\},$$

$$E = \{a : a \in \Delta\},$$

$$\hat{E} = \{\hat{a} : a \in E\}.$$  

(8.1)

The explanation for the defining formulas is as follows. We first note that $Y$ and $\hat{Y}$ are free abelian groups, dual to each other under the canonical pairing $(\underline{x}, v) \mapsto <x, v>$, for $\underline{x} \in Y$, $v \in \hat{Y}$, where $x$ is any preimage of $\underline{x}$ in $X$. Define $\hat{a}$
for $\alpha \in S$ as follows:

\[
\bar{\alpha} = \begin{cases} 
\sum_{a \in \Phi; a | Y = \alpha} \bar{a}, & \text{if } 2\alpha \notin S \\
2 \sum_{a \in \Phi; a | Y = \alpha} \bar{a}, & \text{if } 2\alpha \in S
\end{cases}
\] (8.2)

**Theorem 8.** [Yn] The 6-touple $\Psi = (Y, S, E, \tilde{Y}, \tilde{S}, \tilde{E})$, with the canonical pairing between $Y$ and $\tilde{Y}$ and the correspondence $S \rightarrow \tilde{S}$, $\alpha \mapsto \bar{\alpha}$, is a based root datum, which is not necessarily reduced. Moreover,

(1) The homomorphism $W(\Psi)^\sigma \rightarrow GL(\tilde{Y})$, $w \mapsto w|_Y$ is injective and the image is $W(\Psi)$.

(2) Let $a, b \in \Phi$. Then $a = b$ if and only if $a, b$ are in the same $\sigma$-orbit.

9. **Geometric description of the $R$-group**

Let $\Psi$ denote the dual of the based root datum $\tilde{\Psi}$ defined in section 8. Let $V = V(\Psi) = Y \otimes \mathbb{R}$ and let $\tilde{W} = W \ltimes Y$, where $W = W(\Psi)$ is the relative Weyl group. By equation 5.1, we have $\text{Hom}(T/T_\sigma, \mathbb{C}^\times) \cong T_\sigma$. Under this isomorphism, the unramified unitary characters will correspond to the $\sigma$-conjugacy classes of the compact subtorus $\hat{U} \cong X \otimes \mathbb{S}^1$ of $\hat{T} \cong X \otimes \mathbb{C}^\times$. We have,

$\hat{U} \cong X \otimes \mathbb{R}/\mathbb{Z}$

Then,

$\hat{U}/W \cong X \otimes \mathbb{R}/W \cong Y/\tilde{W}$.

By equation 5.1 it follows that the class of unitary unramified characters in $\text{Hom}(T/T_\sigma, \mathbb{C}^\times)/W$ are in one to one correspondence with the points of $Y/\tilde{W}$. Assume that the unramified character $\lambda$ of $T$ is unitary. To the $W$ orbit of $\lambda$ we can therefore associate a point $x = x_\lambda$ in the closure of the alcove $\Sigma := C(\Psi)$ of $V$.

Let $\Omega := \Omega(\Psi)$. We will also denote $\Omega(\Psi)$ by $\Omega_G$ when the based root-datum is clear from the context.

We have $\Omega \cong Y/\mathbb{Z}E$. Let $\Omega_x$ be the stabilizer of $x$ in $\Omega$.

**Proposition 9.** The natural projection $\tilde{W} \twoheadrightarrow W$ induces an isomorphism $\Omega_x \cong R$.

**Proof.** Let $\tilde{W}_x = \{ w \in \tilde{W} : w \cdot x = x \}$ be the stabilizer of $x$ in $\tilde{W}$. This group is finite and its normal subgroup $\tilde{W}_x^\sigma$, generated by reflections about the hyperplanes through $x$, acts simply transitively on the set of alcoves containing $x$ in their closure.
It follows that
\[ \tilde{W}_x = \Omega_x \ltimes \tilde{W}^o_x \]
and therefore,
\[ (9.1) \quad \Omega_x \cong \tilde{W}_x / \tilde{W}^o_x. \]
Let \( \pi: \tilde{W} \to W \) be the projection map. Let \( \delta = s \times \sigma \). Let \( \tilde{N} = N_G \tilde{T} \) be the normalizer of \( \tilde{T} \) in \( \tilde{G} \). Let \( S_\delta = Z_{\tilde{G}} \delta \). Let \( \tilde{Z} = Z(\tilde{G}) \) be the center of \( \tilde{G} \). Then the proof in [Ree10, lemma 3.9] shows:
\[ (9.2) \quad \tilde{W}_x \cong \tilde{N} \cap S_\delta / \tilde{T}^o, \]
\[ (9.3) \quad \tilde{W}^o_x \cong \tilde{N} \cap S_\delta^o / (\tilde{T}^o)^o. \]
We will include the proofs of 9.2 and 9.3 here for the sake of completeness. Let \( W_\delta = \tilde{N} \cap S_\delta / \tilde{T}^o \). Let \( R_\lambda = S_\delta / S_\delta^o \tilde{Z}^o \). Then Proposition 2.6 in [Key87] states that there is a short exact sequence
\[ (9.5) \quad 1 \to \tilde{N} \cap S_\delta^o / (\tilde{T}^o)^o \to \tilde{N} \cap S_\delta / \tilde{T}^o \to R_\lambda \to 1 \]
and \( R \cong R_\lambda \). Therefore,
\[
R \cong (\hat{N} \cap S_\delta /\hat{T}^\sigma) / (\hat{N} \cap S_\delta^c / (\hat{T}^\sigma)^c)
\cong \hat{W}_x / \hat{W}_x^o
\cong \Omega_x.
\]
The last isomorphism follows from equation (9.1). This completes the proof of the proposition.

This result immediately gives the classification of \( R \)-groups. The same classification is obtained in [Key82a, Sec. 3, Theorem], in a case by case manner. In particular, we have shown:

**Corollary 10.** The Knapp-Stein \( R \)-group for an unramified unitary character is abelian.

**Lemma 11.** \( \hat{\Omega}^{tor} = \ker(\hat{X}^\sigma \to \hat{X}^\sigma_{ad}) \).

*Proof.* Let \( L \) denote the lattice \( ZE \). We have the short exact sequence
\[
0 \to L \to Y \to Y/L \to 0.
\]
Applying the contravariant functor \( \text{Hom}(\cdot, Z) \) to this, we get
\[
\hat{Y} \to \hat{L} \to \text{Ext}^1(Y/L, Z) \to 0.
\]
This implies
\[
(9.6) \quad \text{Ext}^1(Y/L, Z) \cong \hat{L}/\text{im}(\hat{Y})
\cong \hat{X}^\sigma_{ad}/\text{im}(\hat{X}^\sigma).
\]
Also, we have the short exact sequence
\[
0 \to Z \to \mathbb{Q} \to \mathbb{Q}/Z \to 0.
\]
Let \( A \) be an abelian group. Applying the functor \( \text{Hom}(A, \cdot) \), we get,
\[
\text{Hom}(A, \mathbb{Q}/Z)/\text{im}(\text{Hom}(A, \mathbb{Q})) \cong \text{Ext}^1(A, Z).
\]
When \( A \) is finitely generated, we also have that
\[
0 \to \text{im}(\text{Hom}(A, \mathbb{Q})) \to \text{Hom}(A, \mathbb{Q}/Z) \to \text{Hom}(A^{tor}, \mathbb{Q}/Z) \to 0.
\]
Here \( A^{tor} \) denotes the torsion part of \( A \). From the last sequence, we get that,
\[
(9.8) \quad \hat{A}^{tor} \cong \text{Ext}^1(A, Z).
\]
Putting $A = Y/L$ and using equations 9.7 and 9.8 we get,

\begin{equation}
\check{(Y/L)}^{tor} \cong \text{Ext}^{1}(Y/L, \mathbb{Z})
\end{equation}
\begin{equation}
\cong \check{X}_{ad}/\text{im}(\check{X})
\end{equation}

This completes the proof of the lemma. \hfill \square

10. PROOF OF THE MAIN THEOREM

We assume $\lambda$ to be unitary till the end of section 10.2.

10.1. $G$ semisimple, simply connected and unramified. Assume $G$ to be unramified, semisimple and simply connected. Then we can write $G$ as the direct product

$G = H_1 \times \ldots \times H_n$,

where $H_i$ are semisimple, almost simple, simply connected and unramified. Thus, it suffices to prove the result when $G$ is unramified, semisimple, almost simple, simply connected.

Let $\check{\mathcal{P}}$ be the co-weight lattice of $G$. Then the set of conjugacy classes of hyperspecial points of $G$ form a principal homogeneous space for the group $\check{\mathcal{P}}/\check{X}$. By lemma 11, this group is dual to the group $\Omega \cong Y/\mathbb{Z}$ (notations as in equation 8.1). Denote by $(\cdot, \cdot) : (\check{\mathcal{P}}/\check{X}) \times \Omega \rightarrow \mathbb{Q}/\mathbb{Z}$ the pairing between them. By proposition 9, $R \cong \Omega$. So we can realize $R$ as a subgroup of $\Omega$. Thus we have a pairing $(\cdot, \cdot) : (\check{P}/\check{X}) \times R \rightarrow \mathbb{Q}/\mathbb{Z}$. For $r \in R$ and $\omega \in \check{P}/\check{X}$, let $\rho(\omega)(r) = e^{-2\pi i (\omega, r)}$. Then we have a natural surjection $\Omega \rightarrow \check{R}$ given by $\omega \mapsto \rho$. Denote by $[K_0]$ the conjugacy class of a hyperspecial subgroup $K_0$ satisfying $\pi^{[K_0]}_0 \neq 0$. Pick $K_\omega \in \omega \cdot [K_0]$ and let $f_\omega \in I(\lambda)^{K_\omega}$ be the spherical vector such that $f_\omega(1) = 1$.

**Proposition 12.** $\omega(r, \lambda)f_\omega = \rho(\omega(r))f_\omega$.

**Proof.** The proof again is computation like in the split case. It is worked out in section 11. \hfill \square

Then from the lemma 11 we get that $\pi^{[K_0]}_0 \neq 0$ iff $\rho = \rho_\omega$. The rest of the proof of the theorem in this case is now identical to the split case.

10.2. $G$ reductive and unramified. Let $\tilde{G} = Z^{c} \times (G^{\text{der}})^{sc}$ where $G^{\text{der}}$ is the derived group of $G$ and $(G^{\text{der}})^{sc}$ is its simply connected cover. There is an isogeny $\zeta : \tilde{G} \rightarrow G$. Let $\tilde{T} = Z^{c} \times (T^{\text{der}})^{sc}$. Let $\tilde{\lambda}$ be the pull back of the character $\lambda$ of $T$ to $\tilde{T}$. If $f \in I(\lambda)$, then $f \circ (\zeta|_{\tilde{G}})$ is a map $\tilde{G} \rightarrow \mathbb{C}$ and it satisfies the principal series condition. Thus, $\zeta$ induces

$\rho : I(\lambda) \rightarrow I(\tilde{\lambda})$

a $\tilde{G}$ map of the principal series representations.
Lemma 13. \(\varrho\) is surjective.

Proof. Let \(\bar{f}\) (resp. \(\bar{g}\)) denote the image of an element \(f \in I(\tilde{\lambda})\) (resp \(g \in \tilde{G}\)) under \(\varrho\) (resp \(\varsigma\)). Let \(W\) be a representation of \(\tilde{G}\) such that

\[I(\tilde{\lambda}) = I(\lambda) \oplus W.\]

If \(W\) is non-trivial, then there exists a hyperspecial subgroup \(\tilde{K}\) of \(\tilde{G}\) such that \(W_{\tilde{K}} \neq 0\) [Key82b, Sec. 4, Theorem]. Let \(K\) be a hyperspecial subgroup of \(G\) such that \(K \supset \tilde{K}\), where \(\tilde{K} = \varrho(\tilde{K})\). Then \(I(\lambda)^K \neq 0\) implies \(I(\lambda)_{\tilde{K}} \neq 0\). But \(\dim I(\tilde{\lambda})_{\tilde{K}} = 1\), a contradiction. This implies \(W = 0\) and therefore the claim. \(\square\)

Lemma 14. There is a natural inclusion

\[\Omega^\text{tor}_G \hookrightarrow \Omega^\text{tor}_{\tilde{G}}.\]

Proof. Let the notations be as in equation 8.1. Let \(X_o\) (respectively \(Y_o\)) be the subgroup of \(X\) (respectively \(Y\)) orthogonal to \(\Phi\) (respectively \(S\)). Then \(Y/Y_o \cong (X/X_o)_o/torsion\). We have,

\[\Omega^\text{tor}_G \cong (Y/ZE)^\text{tor}\]

and

\[\Omega^\text{tor}_{Gder} \cong ((Y/Y_o)/ZE)^\text{tor} \cong Y/(Y_o + ZE).\]

[Spr79, 2.15 (a)]. The surjection \(Y/ZE \rightarrow (Y/Y_o)/ZE\) induces a map \(\Omega^\text{tor}_G \rightarrow \Omega^\text{tor}_{Gder}\). Let \(\chi \in (Y_o + ZE)\) such that \(n\chi \in ZE\). Write \(\chi = \chi_o + q\) where \(\chi_o \in Y_o\) and \(q \in ZE\). Then \(n\chi \in ZE\) implies \(n\chi_o \in Y_o \cap ZE = 0\) [Spr79, lemma 1.2]. This shows that the map \(\Omega^\text{tor}_G \rightarrow \Omega^\text{tor}_{Gder}\) is injective. Since \(\Omega^\text{tor}_{Gder} \hookrightarrow \Omega^\text{tor}_{(Gder)_sc}\), it follows that there is a natural inclusion

\[\Omega^\text{tor}_G \hookrightarrow \Omega^\text{tor}_{\tilde{G}}.\]

\(\square\)

We then have,

(10.1)

\[
\begin{array}{c}
\Omega^\text{tor}_G \\
\Omega^\text{tor}_{G,x}
\end{array} \quad \begin{array}{c}
\hookrightarrow \\
\hookrightarrow
\end{array} \quad \begin{array}{c}
\Omega^\text{tor}_{\tilde{G}} \\
\Omega^\text{tor}_{\tilde{G},x}
\end{array}
\]

It follows that
(10.2) $$\hat{\Omega}^\text{tor}_G \rightarrow \hat{\Omega}^\text{tor}_G$$

$$\hat{\Omega}^\text{tor}_{G,x} \rightarrow \hat{\Omega}^\text{tor}_{G,x}.$$  

By lemma [11], the conjugacy classes of hyperspecial subgroups of $G$ (resp $\tilde{G}$) is a principal homogenous space for $\hat{\Omega}^\text{tor}_G$ (resp $\hat{\Omega}^\text{tor}_{\tilde{G}}$). Let $\hat{\mathcal{R}} = R(\tilde{G}, \lambda)$. Then by proposition [9] $\hat{\Omega}^\text{tor}_{G,x} \cong \hat{\mathcal{R}}$ and $\hat{\Omega}^\text{tor}_{\tilde{G},x} \cong \mathcal{R}$. (note that $\hat{\Omega}^\text{tor}_{G,x} = \Omega_{G,x}$ and $\hat{\Omega}^\text{tor}_{\tilde{G},x} = \Omega_{G,x}$)

Let $\mu \mapsto \rho$ under $\hat{\mathcal{R}} \rightarrow \hat{\mathcal{R}}$ and $\omega \mapsto \omega$ under $\hat{\Omega}^\text{tor}_G \rightarrow \hat{\Omega}^\text{tor}_{\tilde{G}}$. Then from the diagram (10.2), it follows that $\omega \cdot \mu \mapsto \omega \cdot \rho$. Using lemma [13] we get the diagram

(10.3) $$I(\lambda) \rightarrow I(\tilde{\lambda})$$

$$\pi_\rho \rightarrow \pi_\rho \oplus \tau_\mu.$$

Let $\pi_\rho$ be an irreducible component of $I(\lambda)$. Then it follows from [Key82b, Sec 4, Theorem], that there exits a hyperspecial vertex $h$ such that $\bar{\pi}_{\tilde{G}_h} \neq 0$. Choose $0 \neq \tilde{f} \in I(\lambda)^{\tilde{G}_h}$ such that $f(1) = 1$. Then $0 \neq \tilde{f} \in \bar{\pi}_{\tilde{G}_h}$. This implies $f \in \pi_\rho$ and therefore $\pi_{\tilde{G}_h} \neq 0$. Thus, $\bar{\pi}_{\rho}^{[G_h]} \neq 0$ iff $\bar{\pi}_{\rho}^{[\tilde{G}_h]} \neq 0$. Equivalently,

(10.4) $$\bar{\pi}_{\omega \cdot \rho}^{[G_h]} \neq 0 \text{ iff } \bar{\pi}_{\omega \cdot \rho}^{[\tilde{G}_h]} \neq 0.$$  

Also, from the statement of our theorem proved for the semisimple, simply connected and unramified groups in section [10.1] it follows that

(10.5) $$\bar{\pi}_{\omega \cdot \rho}^{[\tilde{G}_h]} \neq 0 \text{ iff } \bar{\pi}_{\rho}^{[\tilde{G}_h]} \neq 0.$$  

From equations [10.4] and [10.5] it follows that for any hyperspecial subgroup $K$ of $G$, $\pi_{\rho}^{[K]} \neq 0$ iff $\pi_{\omega \cdot \rho}^{[K]} \neq 0$.

10.3. Non-unitary case.

10.3.1. Construction of the L-packet. The following construction is given in [Sha11].

Let $\varphi = \varphi_{\lambda}$ be the Langlands parameter attached to the character $\lambda$. The parameter $\varphi$ determines a commuting pair $\varphi_\circ$ and $\varphi_+$ such that $\varphi_\circ$ is tempered and

$$\varphi(w) = \varphi_\circ(w) \varphi_+(w)$$

for all $w \in W_k'$ where $W_k'$ is the Weil-Deligne group.
Let $\mu : T \to \mathbb{C}^\times$ be the character attached to $\varphi_0$ by Langlands. Similarly let $\nu : T \to \mathbb{C}^\times$ be the character attached to $\varphi_+$. Let $\varpi$ be a uniformizer in $k$. The simple roots (notations as in equation 8.1)

$$E' = \{ \alpha \in E : |\nu(\alpha(\varpi))| = 1 \}$$

generate a Levi subgroup $M$ of $G$ whose dual group $\hat{M}$ contains $\text{Im}(\varphi)$ in $\hat{G}$. Also, $R_{\varphi_+}(M) \cong R_{\varphi}(G)$.

Let $P = MN$ be the parabolic subgroup of $G$ with $M$ as a Levi subgroup and $N \subset U$ where $U$ is the unipotent radical of $B$. Let $U_M = U \cap M$. Then

$$\tau = \text{Ind}^M_{U_M} \mu$$

is a tempered representation of $M$ which may not be irreducible. Write

$$\tau = \bigoplus_{i=1}^n \tau_i,$$

where $\tau_i$ are irreducible. By replacing $\nu$, $\tau_i$ and $M$ with a $W(G, A)$ conjugate, we may assume that

$$I(\nu, \tau_i) = \text{Ind}_G^P \tau_i \otimes \nu$$

is in the Langlands setting. The elements of the $L$-packet are then the unique Langlands quotients $J(\nu, \tau_i)$ of each $I(\nu, \tau_i)$ $1 \leq i \leq n$, i.e.,

$$\Pi(\varphi_\lambda) = \{ J(\nu, \tau_i) : 1 \leq i \leq n \}.$$

**Lemma 15.** There is a natural embedding $\hat{\Omega}_G^{\text{tor}} \hookrightarrow \hat{\Omega}_M^{\text{tor}}$.

**Proof.** We have $\Omega_M \cong Y/ZE'$ and $\Omega_G \cong Y/ZE$. So we have a natural surjection $\text{Pr} : \Omega_M \twoheadrightarrow \Omega_G$. Let $\tilde{\chi} \in \hat{\Omega}_M^{\text{tor}}$ such that $\text{Pr}(\tilde{\chi}) = 0$. Then we can choose a representative $\chi \in ZE$ of $\tilde{\chi}$ such that $n \chi \in ZE'$ for some integer $n$. Write

$$\chi = \sum_{i \in I} a_i \alpha_i,$$

where $a_i \in \mathbb{Z}$, $\alpha_i \in E$ and $I$ is an indexing set for $E$. Write

$$n \chi = \sum_{j \in J} b_j \alpha_j,$$

where $J \subset I$ is an indexing set for $E'$. Then

$$n \sum_{i \in I \setminus J} a_i \alpha_i + \sum_{j \in J} (na_j - b_j) \alpha_j = 0.$$

This implies $a_i = 0$ for $i \in I \setminus J$ and therefore $\chi \in ZE'$, i.e., $\tilde{\chi} = 0$. Thus, the projection $\text{Pr}$ induces an embedding $\hat{\Omega}_M^{\text{tor}} \hookrightarrow \hat{\Omega}_G^{\text{tor}}$. \qed

By the lemma, we have a surjection $\mathcal{L} : \hat{\Omega}_G^{\text{tor}} \twoheadrightarrow \hat{\Omega}_M^{\text{tor}}$. By proposition 9 there is a natural surjection $\hat{\Omega}_M^{\text{tor}} \twoheadrightarrow R_{\varphi_+}(M)$. Therefore, we have a natural surjection
Let $K_M$ be a hyperspecial subgroup of $M$ such that $\tau_{\rho}^{K_M} \neq 0$, $\rho \in \hat{R}_\rho$. Then there is a hyperspecial subgroup $K_G$ of $G$ containing some conjugate of $K_M$. Without loss of generality, $K_G \supseteq K_M$.

Claim 16. $I(\nu, \tau_{\rho})^{K_G} \neq 0$.

Proof. Denote by $U_{\rho}$ the space realizing $\tau_{\rho}$ and let $u$ be a $K_M$ fixed vector in $U_{\rho}$. Let $f: G \to U_{\rho}$ be the function defined by

$$f(mnk) = \delta^{1/2}(m)\tau_{\rho}(m)u,$$

where $m \in M$, $n \in N$ and $k \in K_G$. Then $f$ is well defined and is fixed by $K_G$. □

Lemma 17. $J(\nu, \tau_{\rho})^{K_G} \neq 0$.

Proof. The Langlands quotient $J(\nu, \tau_{\rho})$ is the image of an intertwining operator

$$A(\nu, \tau_{\rho}, w): I(\nu, \tau_{\rho}) \to I(w\nu, w\tau_{\rho}),$$

for some $w \in W$. Then $J(\nu, \tau_{\rho})^{K_G} \neq 0$ follows from the fact that the image of a $K_G$ spherical vector in $I(\lambda)$ under

$$A(\lambda, w): I(\lambda) \to I(w\lambda)$$

is non-zero. □

Conversely, if $J(\nu, \tau_{\rho})^{K_G} \neq 0$, then some conjugate of $K_G$ contains a hyperspecial $K_M$ of $M$. Without loss of generality, $K_G \supseteq K_M$. If $0 \neq g \in I(\nu, \tau_{\rho})^{K_G}$, then $0 \neq g(1) \in \tau_{\rho}^{K_M}$.

In the notations as above, let $\pi_{\rho} = J(\nu, \tau_{\rho})$. Let $\omega \in \hat{\Omega}_G^{tor}$ and $\bar{\omega} \in \hat{\Omega}_M^{tor}$ be its image under $\mathcal{L}: \omega \mapsto \bar{\omega}$. Then it follows from lemma 17 and the diagram (10.6)

$$\pi_{\omega, \rho}^{[K_G]} \neq 0 \iff \bar{\omega}_{\rho}^{[K_M]} \neq 0 \iff \tau_{\rho}^{[K_M]} \neq 0 \iff \pi_{\rho}^{[K_G]} \neq 0.$$

This completes the proof of the main theorem.
11. Proof of Proposition [7] and [12]

11.1. Affine Root Structure. Let $G$ be semisimple, almost simple, simply connected and unramified. Let $N = N_G A$ be the normalizer of $A$ and let $W$ be the relative Weyl group of $G$. Let $S$ denote the set of relative roots. Bruhat and Tits define affine roots $\Sigma$ associated to $G$. In the split case or in the case that $G$ is unramified and of type $2A_{2n-1}(n \geq 3)$ or $2D_{n+1}(n \geq 2)$,

$$\Sigma = \{a + n : a \in S, n \in \mathbb{Z}\}.
$$

Affine roots can be regarded as affine linear functions $x \mapsto a(x) + n$ on the vector space spanned by $S$. Let $h_\alpha$ be the hyperplane on which $\alpha \in \Sigma$ vanishes. Let $s_\alpha$ be the reflection in the hyperplane $h_\alpha$. The affine Weyl group $\tilde{W}$ is generated by the reflections $s_\alpha$, $\alpha \in \Sigma$. For $\alpha \in \Sigma$, let $t_\alpha$ be the translation $s_\alpha s_{\alpha+1}$. There exists a canonical surjection $\nu : N \to \tilde{W}$. Let $H$ be the kernel of $\nu$.

There exists a collection of compact unipotent subgroups $U_\alpha$ of $G$, $\alpha \in \Sigma$, satisfying the following properties (see [Mac71, page 27]):

1. $xU_\alpha x^{-1} = U_{\nu(x)\alpha}$ for $x \in N$, $\alpha \in \Sigma$.
2. $U_\alpha \subseteq U_{\alpha-1}$, $q_\alpha = (U_{\alpha-1} : U_\alpha)$ is finite and $\bigcap_{k \in \mathbb{Z}} U_{\alpha+k} = \{1\}$.
3. If $\beta = -\alpha + r$, $r \geq 0$, then $<U_\alpha, U_{\beta}, H> = U_\alpha H U_{\beta}$.
4. $<U_\alpha, U_{\beta}, H> = U_\alpha \nu^{-1}(s_\alpha) U_\alpha \cup U_\beta U_{\alpha+1} U_\beta$.
5. If $h_\alpha$ and $h_\beta$ are not parallel, i.e., if $\beta \neq \pm \alpha + r$ for any $r \in \mathbb{Z}$, then the commutator group $[U_\alpha, U_{\beta}]$ is contained in $<U_\alpha U_{\alpha+\beta}|r\alpha + s\beta \in \Sigma, r, s \geq 0>$.
6. Let $S^+$ and $S^-$ denote the positive and negative roots respectively in $S$.
   For each $\alpha \in S$, let $U_{(\alpha)} = \bigcup_{r \in \mathbb{Z}} U_{\alpha+r}$. Let $U^+ = <U_{(\alpha)} : \alpha \in S^+>$ and $U^- = <U_{(\alpha)} : \alpha \in S^->$. Then $U^+ \cap MU^- = \{1\}$.
7. $G = <N, U_\alpha : \alpha \in \Sigma>$.

Lemma 18. [Key82a, Lemma] Let $\alpha$ be a positive simple root and let $f_{K,\lambda} \in I(\lambda)$ be the $K$-fixed vector with $f_{K,\lambda}(1) = 1$. Then $\mathcal{A}(s_\alpha, \lambda)f_{K,\lambda} = f_{K,\alpha s_{\lambda}}$ if $U_{(\alpha)} \cap K = U_\alpha$ and $\mathcal{A}(s_\alpha, \lambda)f_{K,\lambda} = \lambda(t_\alpha)^{-1} f_{K,\alpha s_{\lambda}}$ if $U_{(\alpha)} \cap K = U_{\alpha-1}$ and $q_{\alpha/2} = 1$.

11.2. More about the group $\Omega$. Let $G$ be semisimple, almost simple, simply connected and unramified. Let $\Psi = (Y, S, E, \tilde{Y}, S, E)$ be the based root datum defined in section [8]. $W$ is the relative Weyl group. Let $\tilde{E} = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ where $\tilde{\beta}_i$, $1 \leq i \leq n$, are the simple relative co-roots. Let $\tilde{\beta}_0 = \sum_{i=1}^n n_i \tilde{\beta}_i$ be the highest co-root. Put $\tilde{\beta}_0 = 1 - \tilde{\beta}_0$. Let $h = \sum_{i=1}^n n_i + 1$ be the Coxeter number and put $\delta_0 = \frac{1}{2} \sum_{s \in \tilde{S}} \beta$. Recall that $\tilde{C} = C(\tilde{\Psi})$ was the alcove in the apartment $A = Y \otimes \mathbb{R}$ defined by $\{x \in A : \beta_0(x) \geq 0, \ldots, \beta_n(x) \geq 0\}$. Put $n_0 = 1$ and give the weight $n_i$ to the $i^{th}$ vertex of $\tilde{C}$. Then $c_0 = h^{-1} \delta_0$ is the (weighted) barycenter of $\tilde{C}$. For any
Lemma 19. [AYY13] Lemma 6.2] For any $w \in W$, the following are equivalent:

1. $\bar{w} \in W \ltimes Y$.
2. $\bar{w} C = C$.
3. $\bar{w}(\hat{E}_0) = \hat{E}_0$ where $\hat{E}_0 = \hat{E} \cup \{\hat{\beta}_0\}$.
4. $w(\hat{E} \cup \{-\hat{\beta}_0\}) = \hat{E} \cup \{-\hat{\beta}_0\}$.

Then $\Omega = \Omega(\hat{Y})$ is precisely the set of those $w \in W$ satisfying these conditions and $w \mapsto \bar{w}(\hat{\beta}_0)$ is a bijection between $\Omega$ and $\{\hat{\beta}_i \in \hat{E}_0 : n_i = 1\}$.

There is an isomorphism $\iota : \Omega \to Y/Q$ defined by any of the following ways:

(a) $\iota(w) = (1 - w) h^{-1} \delta + Q$.
(b) $\iota(w)$ is the image of $\bar{w}$ under $W \ltimes Y \to (W \ltimes Y)/(W \ltimes Q) = Y/Q$.
(c) If $w \neq 0$ and $\bar{w}(\hat{\beta}_0) = \hat{\beta}_i$, $\iota(w)$ is the $i^{th}$ fundamental weight $\omega_i$ modulo $Q$.

For $w = 1$, $\iota(w) = 0$.

11.3. Calculations. Let the notations be as in section 8. Further, $G$ is semisimple, almost simple, simply connected and unramified. Let $K_0$ be a hyperspecial subgroup satisfying $\pi_0^{K_0} \neq 0$. Then $K_0$ determines the choice of an origin in the Bruhat-Tits building of $G$. For $K_\omega \in \omega \cdot [K_0]$, let $f_{\omega,\lambda} \in I(\lambda)^{K_\omega}$ be the spherical vector such that $f_{\omega,\lambda}(1) = 1$. For $w \in W$, let $c_\lambda(\omega, w)$ be the coefficient of $f_{\omega,\lambda}$ in the equation: $\mathcal{A}(w, \lambda)f_{\omega,\lambda} = c_\lambda(\omega, w)f_{\omega,\lambda}$. Then $c_\lambda(\omega, w)$ satisfies the co-cycle relation:

\[
(11.1) \quad c_\lambda(\omega, w_1 w_2) = c_{w_2,\lambda}(\omega, w_1) \cdot c_\lambda(\omega, w_2).
\]

Let $x_\lambda$ denote the point in $\mathbb{C}$ associated to $\lambda$. Define $\tilde{c}_\lambda(\omega, w) \in \mathbb{Q}/\mathbb{Z}$ by $c_\lambda(\omega, w) = e^{-2\pi i \tilde{c}_\lambda(\omega, w)}$. Then $\tilde{c}(\omega, w)$ satisfies the cocycle relation:

\[
(11.2) \quad \tilde{c}_\lambda(\omega, w_1 w_2) = \tilde{c}_{w_2,\lambda}(\omega, w_1) + \tilde{c}_\lambda(\omega, w_2).
\]

Let $\mathcal{C} = \{x : x$ is a hyperspecial vertex in the closure of $\mathbb{C}\}$. Then the representatives of the conjugacy classes of hyperspecial subgroups can be chosen to be $\{G_x : x \in \mathcal{C}\}$. Also, the action of $\hat{\Omega}$ on the conjugacy classes of hyperspecial subgroups gives a bijection $\hat{\Omega} \to \mathcal{C}$. We use this bijection to identify $\hat{\Omega}$ with $\mathcal{C}$ whenever there is no ambiguity. By lemma 13, we have,

\[
\mathcal{A}(s_\alpha, \lambda)f_{\omega,\lambda} = \begin{cases} 
\alpha(\omega) = 1 & \text{otherwise.} \\
(\lambda(t_\alpha)^{-1}f_{\omega,\lambda})_\alpha = 1 
\end{cases}
\]

Therefore,
\[ c_\lambda(\omega, s_\alpha) = \begin{cases} e^{-2\pi i (x_\lambda, \tilde{\alpha})} & \alpha(\omega) = 1 \\ 1 & \text{otherwise.} \end{cases} \]

Here \((x_\lambda, \tilde{\alpha})\) denotes \(\tilde{\alpha}(x_\lambda)\). Then,

\[(11.3)\quad \tilde{c}_\lambda(\omega, s_\alpha) = \begin{cases} (x_\lambda, \tilde{\alpha}) & \alpha(\omega) = 1 \\ 0 & \text{otherwise.} \end{cases} \]

Using proposition \(10\) we identify \(R_\lambda\) as a subgroup of \(\tilde{\Omega}\). For \(r \in R_\lambda\), we calculate \(\tilde{c}_\lambda(\omega, r)\) by using equations \(11.1\) and \(11.3\). Using lemma \(19\) we realize \(R_\lambda\) as a subgroup of \(Y/Q\). Finally, we calculate \((\omega, r)\) and show that \(\tilde{c}_\lambda(\omega, r) = (\omega, r) \mod Z\), implying \(c_\lambda(\omega, r) = \rho_\omega(r)\).

When \(G\) is split, \(\tilde{\Omega}\) is non-trivial for the cases \(A_n, B_n, C_n, D_n, E_6\) and \(E_7\). When \(G\) is unramified but not split, \(\tilde{\Omega}\) is non-trivial only for the cases \(2A_{2n-1}(n \geq 3)\) and \(2D_{n+1}(n \geq 2)\) [Ree10, table-1, page 29]. The method of calculation of \(c_\lambda(\omega, w)\) is the same in all cases. We illustrate the cases below:

11.3.1. Type \(A_n\). Let \(\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}\}\) be the standard orthonormal basis of \(\mathbb{R}^{n+1}\). We realize \(A_n\) as a root system in the vector space \(E\) where,

\[ E = \{ \sum_{i=1}^{n+1} c_i \epsilon_i : c_i \in \mathbb{R}, \sum c_i = 0 \} \]

Then \(\Phi = \{ \epsilon_i - \epsilon_j : i \neq j, 0 \leq i, j \leq n \}\) is the set of roots. Let \(\alpha_i = \epsilon_i - \epsilon_{i+1}\) for \(1 \leq \alpha \leq n\). The set \(\Delta = \{ \alpha_1, \ldots, \alpha_n \}\) is a fundamental system in \(\Phi\). Also, \(\tilde{\Phi} = \Phi\) and \(\tilde{\Delta} = \Delta\). The highest root is \(\beta = \alpha_1 + \ldots + \alpha_n = \epsilon_1 - \epsilon_{n+1}\). The fundamental weights are \(\{x_1, \ldots, x_n\}\), where

\[ x_k = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k - \frac{k}{n+1}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{n+1}). \]

Let \(\tilde{w}_0\) be the generator of the \(\Omega(\tilde{\Phi})\) and let \(w_0\) be the vector part. Then \(w_0 \cdot \alpha_i = \alpha_{i+1}\) for \(1 \leq i \leq n-1\) and \(w_0 \cdot \alpha_n = -\beta\), by lemma \(19\). Write \(s_i\) for the reflection about the hyperplane determined by the root \(\alpha_i\). As a product of reflections, \(w_0 = (12\ldots n+1) = s_1 s_2 \ldots s_n\). Write

\[ x_\lambda = \sum_{i=1}^{n} a_i x_i = (b_1, b_2, \ldots, b_{n+1}) \]

for some \(a_i \in \mathbb{Q}\) and where \(b_i = \frac{1}{(n+1)}(-a_1 - 2a_2 - \ldots - (i-1)a_{i-1} + (n+1-i)a_i + \ldots + (n+1-k)a_k + \ldots + na_n)\).
If \( r \in R_\lambda \), then \( r = w_0^d \) for some integer \( d \). Then by the cocycle relation, we get

\[
(11.4) \quad \tilde{c}_\lambda(\omega, w_0^d) = \tilde{c}_\lambda(\omega, w_0) + \ldots + \tilde{c}_{w_0^d-2\lambda}(\omega, w_0) + \tilde{c}_{w_0^d-\lambda}(\omega, w_0).
\]

Let \( \omega_i \) be the \( i \)-th fundamental co-weight, i.e., \( \alpha_j(\omega_i) = \delta_{ij} \). The set of hyperspecial vertices in the closure of the fundamental alcove is \( C = \{ 0, \omega_i \text{ for } 1 \leq i \leq n \} \). Using equations (11.2) and (11.3), we get,

\[
\tilde{c}_\lambda(\omega, w_0) = b_i - b_{n+1}.
\]

From this and the equation (11.4), we get

\[
(11.5) \quad \tilde{c}_\lambda(\omega_i, w_0^d) = (b_i - b_{n+1}) + (b_{i-1} - b_n) + \ldots.
\]

Since \( w_0^d \cdot x_\lambda = x_\lambda \), it follows that \( a_i = a_j \) iff \( i \equiv j \mod d \). From this and the last equation, it follow that

\[
\tilde{c}_\lambda(\omega_i, w_0^d) = (a_0 + \ldots + a_{d-1})(n + 1 - i) \equiv \frac{(n + 1 - i)d}{n + 1} \mod \mathbb{Z}.
\]

The last equality follows because \( \sum_{i=0}^n a_i = 1 \). Now \( w_0^d \cdot x_0 = x_d \), where \( x_0 \) is the origin in \( V(\Psi) \). Therefore,

\[
(\omega_i, r) = (\omega_i, x_d) = \frac{(n + 1 - i)d}{n + 1}.
\]

Thus,

\[
\tilde{c}_\lambda(\omega_i, r) = (\omega_i, r) \mod \mathbb{Z}.
\]

This proves the result for type \( A_n \).

11.3.2. Type \( B_n \). The dual root system of \( B_n \) is \( C_n \). Let \( \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \) be the standard orthonormal basis of \( \mathbb{R}^n \). Then \( B_n \) root system is given by \( \Phi = \{ \pm \epsilon_i \pm \epsilon_j \} \cup \{ \pm \epsilon_i \} \). The co-roots are \( \Phi = \{ \pm \epsilon_i \pm \epsilon_j \} \cup \{ \pm 2\epsilon_i \} \). Let \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) for \( 1 \leq i \leq n-1 \) and \( \alpha_n = \epsilon_n \). The set \( \Delta = \{ \alpha_1, \ldots, \alpha_n \} \) is a set of simple roots in \( \Phi \). The corresponding simple co-roots are \( \check{\Delta} = \{ \check{\alpha}_1, \ldots, \check{\alpha}_n \} \) where \( \check{\alpha}_i = \alpha_i \) for \( 1 \leq i \leq n-1 \) and \( \check{\alpha}_n = 2\alpha_n \). The highest co-root is \( \check{\beta} = 2(\check{\alpha}_1 + \cdots + \check{\alpha}_{n-1}) + \check{\alpha}_n \).

The fundamental weights are \( \{x_1, \ldots, x_n\} \) where,

\[
x_k = \epsilon_1 + \ldots + \epsilon_k \text{ for } 1 \leq k \leq n-1,
\]

\[
x_n = \frac{1}{2}(\epsilon_1 + \ldots + \epsilon_{n-1} + \epsilon_n).
\]

Let \( \check{w}_0 \) be the generator of the \( \Omega(\check{\Psi}) \) and let \( w_0 \) be the vector part. Then \( w_0 \cdot \check{\alpha}_i = \check{\alpha}_{n-i} \) for \( 1 \leq i \leq n-1 \) and \( w_0 \cdot \check{\alpha}_n = -\check{\beta} \). Thus for \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), \( w_0 \cdot y = \)
\((-y_n, -y_{n-1}, \ldots, -y_1)\). As a product of reflections, 
\(w_0 = \ldots q s_{n-2} s_{n-1} q s_{n-1} q q\), where \(q = s_n \ldots s_1\). Write 
\[
x_\lambda = \sum_{i=1}^{n} a_i x_i \\
= (b_1, b_2, \ldots, b_n),
\]
for some \(a_i \in \mathbb{Q}\) and where \(b_i = a_i + \ldots + a_{n-1} + \frac{1}{2} a_n\). Let \(\omega_1\) be the first fundamental co-weight. Then \(\omega_1 = \epsilon_1\) and \(C = \{0, \omega_1\}\). Using equations \(11.172\) and \(11.3\) to calculate \(\hat{c}(\omega_1, w_0)\) we get, 
\[
\hat{c}(\omega_1, w_0) = (h, \tilde{\alpha}_1)
\]
where \(h = x_\lambda + q \cdot x_\lambda + s_{n-1} q q \cdot x_\lambda + \ldots\). Thus 
\[
\hat{c}(\omega_1, w_0) = (h, \epsilon_1 - \epsilon_2) \\
= b_1 + b_n \\
= a_1 + \ldots + a_{n-1} + \frac{1}{2} a_n + \frac{1}{2} a_n \\
= a_1 + \ldots + a_n.
\]
(11.6)
Since \(\tilde{w}_0 \cdot x_\lambda = x_\lambda\), it follows that \(a_i = a_{n-i}\) for \(1 \leq i \leq n - 1\) and \(a_0 = a_n\). From this and the relation 
\[
a_0 + 2(a_1 + \ldots + a_{n-1}) + a_n = 1,
\]
it follows that \(\hat{c}(\omega_1, w_0) = \frac{1}{2}\). Also \(\omega_1, x_n = \frac{1}{2}\). This completes the case of \(B_n\).

11.3. Type \(C_n\). Then \(C_n\) root system is given by \(\Phi = \{\pm \epsilon_i \pm \epsilon_j\} \cup \{\pm 2 \epsilon_i\}\). Let \(\alpha_i = \epsilon_i - \epsilon_{i+1}\) for \(1 \leq i \leq n - 1\) and \(\alpha_n = 2 \epsilon_n\). The set \(\Delta = \{\alpha_1, \ldots, \alpha_n\}\) is a set of simple roots in \(\Phi\). The corresponding simple co-roots are \(\tilde{\Delta} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}\), where \(\tilde{\alpha}_i = \alpha_i\) for \(1 \leq i \leq n - 1\) and \(\tilde{\alpha}_n = \frac{1}{2} \alpha_n\). The highest co-root is \(\tilde{\beta} = \tilde{\alpha}_1 + 2(\tilde{\alpha}_2 + \cdots + \tilde{\alpha}_n) = \epsilon_1 + \epsilon_2\). The fundamental weights are \(\{x_1, \ldots, x_n\}\) where, 
\[
x_k = \epsilon_1 + \ldots + \epsilon_k\text{ for } 1 \leq k \leq n.
\]
Let \(\tilde{w}_0\) be the generator of the \(\Omega(\tilde{\Psi})\) and let \(w_0\) be the vector part. Then 
\(w_0 \cdot \tilde{\alpha}_i = \tilde{\alpha}_i\) for \(2 \leq i \leq n\) and \(w_0 \cdot \tilde{\alpha}_1 = -\tilde{\beta}\). Thus for \(y = (y_1, \ldots, y_n) \in \mathbb{R}^n\), 
\(w_0 \cdot y = (-y_1, y_2, \ldots, y_n)\). As a product of reflections, \(w_0 = s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_1\). Write 
\[
x_\lambda = \sum_{i=1}^{n} a_i x_i \\
= (b_1, b_2, \ldots, b_n),
\]
for some \( a_i \in \mathbb{Q} \) and where \( b_i = a_i + \ldots + a_{n-1} + a_n \). Let \( \omega_n \) be the \( n \)th fundamental co-weight. Then \( \omega_n = \frac{1}{2}(\epsilon_1 + \ldots + \epsilon_n + \epsilon_{n-1}) \) and \( C = \{0, \omega_n\} \). Using equations 11.2 and 11.3 to calculate \( \hat{c}(\omega_n, w_0) \), we get,

\[
\hat{c}(\omega_n, w_0) = (s_{n-1} \ldots s_1 \cdot x_\lambda, \tilde{\alpha}_n) = a_1 + \ldots + a_n.
\]

Since \( w_0 \cdot x_\lambda = x_\lambda \), it follows that \( a_0 = a_1 \). From this and the relation \( a_0 + a_1 + 2(a_2 + \ldots + a_n) = 1 \), it follow that \( \hat{c}(\omega_n, w_0) = \frac{1}{2} \). Also \( (\omega_n, x_1) = \frac{1}{2} \). This completes the case of \( C_n \).

11.3.4. Type \( D_n \).

Case: \( n \) is odd

If \( \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \) is the standard orthonormal basis of \( \mathbb{R}^n \), then the root system of \( D_n \) is given by \( \Phi = \{\pm \epsilon_i \pm \epsilon_j\} \). Let \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) for \( 1 \leq i \leq n-1 \) and \( \alpha_n = \epsilon_{n-1} + \epsilon_n \). The set \( \Delta = \{\alpha_1, \ldots, \alpha_n\} \) is a set of simple roots in \( \Phi \). We have \( \tilde{\Phi} = \Phi \) and \( \tilde{\Delta} = \Delta \). The highest root is \( \beta = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n = \epsilon_1 + \epsilon_2 \). The fundamental weights are

\[
x_k = \epsilon_1 + \ldots + \epsilon_k \text{ for } 1 \leq k \leq n - 2,
\]

\[
x_{n-1} = \frac{1}{2}(\epsilon_1 + \ldots + \epsilon_{n-1} - \epsilon_n),
\]

\[
x_n = \frac{1}{2}(\epsilon_1 + \ldots + \epsilon_{n-1} + \epsilon_n).
\]

Let \( \tilde{w}_0 \) be the generator of the \( \Omega(\tilde{\Psi}) \) and let \( w_0 \) be the vector part. Then \( w_0 \cdot \alpha_i = \alpha_{n-i} \) for \( 2 \leq i \leq n-2 \), \( w_0 \cdot \alpha_1 = \alpha_{n-1} \), \( w_0 \cdot \alpha_{n-1} = -\beta \), \( w_0 \cdot (-\beta) = \alpha_n \) and \( w_0 \cdot \alpha_n = \alpha_1 \). Thus for \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), \( w_0 \cdot y = (y_n, -y_{n-1}, -y_{n-2}, \ldots, -y_1) \). As a product of reflections,

\[
w_0 = \ldots s_4 \ldots s_{n-2} s_n s_3 \ldots s_{n-1} s_2 \ldots s_{n-2} s_n s_1 \ldots s_{n-1}.
\]

Write

\[
x_\lambda = \sum_{i=1}^{n} a_i x_i = (b_1, b_2, \ldots, b_n)
\]

for some \( a_i \in \mathbb{Q} \) and where \( b_i = a_i + \ldots + a_{n-2} + \frac{a_{n-1} + a_n}{2} \) for \( 1 \leq i \leq n-1 \) and \( b_n = -\frac{a_{n-1} + a_n}{2} \). Let \( \omega_i \) be the \( i \)th fundamental co-weight. Then \( \tilde{\mathcal{C}} = \{0, \omega_1, \omega_{n-1}, \omega_n\} \). The first fundamental co-weight \( \omega_1 = x_1 \). Using equations 11.2 and 11.3 to calculate
\( \tilde{c}(\omega_1, w_0) \), we get,

\[
\tilde{c}(\omega_1, w_0) = (s_2 \cdots s_{n-1} \cdot x_\lambda, \bar{a}_1)
\]

\[
= b_1 - b_n
\]

\[
= a_1 + \ldots + a_{n-1}.
\]

Since \( w_0 \cdot x_\lambda = x_\lambda \), it follows that \( a_i = a_{n-i} \) for \( 2 \leq i \leq n - 2 \) and \( a_0 = a_1 = a_{n-1} = a_n \). From this and the relation

\[
a_0 + a_1 + 2(a_2 + \ldots + a_{n-2}) + a_{n-1} + a_n = 1
\]

(11.7) it follow that \( \tilde{c}(\omega_1, w_0) = \frac{1}{2} \). Also \( (\omega_1, x_n) = \frac{1}{2} \).

To compute \( \tilde{c}(\omega_1, w_2^0) \) and \( \tilde{c}(\omega_1, w_3^0) \) we can use the co-cycle relation 11.2.

For \( r = w_2^0 \in R_\lambda \) we get

\[
\tilde{c}_\lambda(\omega_1, w_2^0) = \tilde{c}_\lambda(\omega_1, w_0) + \tilde{c}_{w_0^2 \lambda}(\omega_1, w_0)
\]

\[
= b_1 - b_n + b_n + b_1
\]

\[
= 2b_1.
\]

Now if \( w_0^2 \cdot x_\lambda = x_\lambda \), then it follows \( a_0 = a_1 \) and \( a_{n-1} = a_n \). From this and the relation 11.7 we get \( b_1 = \frac{1}{2} \) and therefore \( \tilde{c}_\lambda(\omega_1, w_0^2) = 1 \). Also \( (\omega_1, x_1) = 1 \).

For \( r = w_3^0 \in R_\lambda \) we get,

\[
\tilde{c}_\lambda(\omega_1, w_3^0) = \tilde{c}_\lambda(\omega_1, w_0^2) + \tilde{c}_{w_0^3 \lambda}(\omega_1, w_0)
\]

\[
= 2b_1 + b_n - b_1
\]

\[
= b_n + b_1.
\]

Now if \( w_0^3 \cdot x_\lambda = x_\lambda \), then it follows \( a_0 = a_{n-1} = a_n \) and \( a_i = a_{n-i} \) for \( 2 \leq i \leq n - 2 \). From this and the relation 11.7 we get \( b_1 + b_n = \frac{1}{2} \) and therefore \( \tilde{c}_\lambda(\omega_1, w_0^3) = \frac{1}{2} \). Also \( (\omega_{n-1}, x_1) = \frac{1}{2} \).

The calculation for other hyperspecial vertices follows by the same method. This completes the case of \( D_n \) when \( n \) is odd.

**Case:** \( n \) is even

In this case \( \Omega(\Psi) \) is not cyclic. The non-trivial automorphisms of the extended Dynkin diagram are:
• $w_1 : \{-\beta\} \cup \Delta \to \{-\beta\} \cup \Delta$
  
  $-\beta \longleftrightarrow \alpha_n,$
  
  $\alpha_1 \longleftrightarrow \alpha_{n-1},$
  
  $\alpha_i \longleftrightarrow \alpha_{n-i}.$

• $w_2 : \{-\beta\} \cup \Delta \to \{-\beta\} \cup \Delta$
  
  $-\beta \longleftrightarrow \alpha_1,$
  
  $\alpha_{n-1} \longleftrightarrow \alpha_n,$
  
  $\alpha_i \longleftrightarrow \alpha_i.$

• $w_3 : \{-\beta\} \cup \Delta \to \{-\beta\} \cup \Delta$
  
  $-\beta \longleftrightarrow \alpha_{n-1},$
  
  $\alpha_1 \longleftrightarrow \alpha_n,$
  
  $\alpha_i \longleftrightarrow \alpha_{n-i}.$

For the fundamental weight $\omega_1,$ we calculate $\tilde{c}_\lambda(\omega_1, w)$ for each $w \in \{w_1, w_2, w_3\}$ and verify our result.

\[ w_1 : \mathbb{R}^n \to \mathbb{R}^n \text{ is the map } (y_1, \ldots, y_n) \mapsto (-y_n, -y_{n-1}, \ldots, -y_1). \text{ As a product of reflections } w_1 = \cdots s_4 \cdots s_{n-1}s_3 \cdots s_{n-2}s_n s_2 \cdots s_{n-1}\cdots s_1 \cdots s_{n-2}s_n. \text{ Using equations 11.2 and 11.3 to calculate } \tilde{c}(\omega_1, w_1) \text{ we get,} \]

\[
\tilde{c}(\omega_1, w_1) = (s_2 \cdots s_{n-2}s_n \cdot x_\lambda, \tilde{\alpha}_1)
\]

\[
= ((b_1, -b_n, b_2, \ldots, b_{n-2}, -b_{n-1}), \epsilon_1 - \epsilon_2)
\]

\[
= b_1 + b_n
\]

\[
= a_1 + \cdots + a_{n-1}.
\]

Since $w_1 \cdot x_\lambda = x_\lambda,$ it follows $a_0 = a_n$ and $a_1 = a_{n-1}.$ From this and the relation 11.7 we get that $\tilde{c}(\omega_1, w_1) = \frac{1}{2}.$ Also, $(w_1, x_n) = \frac{1}{2}.$

\[ w_2 : \mathbb{R}^n \to \mathbb{R}^n \text{ is the map } (y_1, \ldots, y_n) \mapsto (-y_1, y_2, \ldots, y_{n-1}, -y_n). \text{ As a product of reflections } w_2 = s_1 \cdots s_{n-2}s_n s_{n-1} \cdots s_1. \text{ Using equations 11.2 and 11.3 to calculate } \tilde{c}(\omega_1, w_2) \text{ we get,} \]

\[
\tilde{c}(\omega_1, w_2) = (x_\lambda + s_2 \cdots s_{n-2}s_n s_{n-1} \cdots s_1 \cdot x_\lambda, \tilde{\alpha}_1)
\]

\[
= ((b_1, \ldots, b_n) + (b_2, -b_1, b_3, \ldots, b_{n-1}, -b_n), \epsilon_1 - \epsilon_2)
\]

\[
= 2b_1.
\]

$w_2 \cdot x_\lambda = x_\lambda$ implies $a_0 = a_1$ and $a_{n-1} = a_n.$ From this and the relation 11.7 we get that $\tilde{c}(\omega_1, w_2) = 1.$ Also, $(w_1, x_1) = 1.$
$w_3 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the map $(y_1, \ldots, y_n) \mapsto (y_n, -y_{n-1}, \ldots, y_1)$. As a product of reflections $w_3 = s_{n-1} \cdots s_1 s_n \cdots s_n$. Using equations 11.2 and 11.3 to calculate $c(\omega_1, w_3)$ we get,

\[
c(\omega_1, w_3) = (s_n \cdots s_1 s_n \cdots s_n - \omega_1 \cdot x_\lambda, \bar{a}_1) = ((b_1, b_n, -b_{n-1}, -b_{n-2}, \ldots, -b_2), \epsilon_1 - \epsilon_2) = b_1 - b_n = a_1 + \cdots + a_{n-1}.
\]

Since $w_3 \cdot x_\lambda = x_\lambda$ implies $a_0 = a_{n-1}$ and $a_1 = a_n$. From this and the relation 11.7 we get that $c(\omega_1, w_3) = \frac{1}{2}$. Also, $(w_1, x_{n-1}) = \frac{1}{2}$.

The calculation for other hyperspecial vertices follows by the same method. This completes the case of $D_n$ when $n$ is even.

11.3.5. Type $E_6$. $E_6$ can be realized as a subspace of $\mathbb{R}^8$. A set of simple roots would then be $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ where $\alpha_1 = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $\alpha_2 = (1, 1, 0, 0, 0, 0, 0, 0)$, $\alpha_3 = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $\alpha_4 = (0, -1, 1, 0, 0, 0, 0, 0)$, $\alpha_5 = (0, 0, -1, 1, 0, 0, 0, 0)$, $\alpha_6 = (0, 0, 0, -1, 1, 0, 0, 0)$. The fundamental weights are,

\[
x_1 = (0, 0, 0, 0, 0, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}), \quad x_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \quad x_3 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right),
\]

\[
x_4 = (0, 0, 1, 1, -1, -1, 1, 1), \quad x_5 = (0, 0, 0, 1, 1, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}), \quad x_6 = (0, 0, 0, 0, 1, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}).
\]

Let $\beta$ denote the highest root. Let $\bar{w}_0$ be the generator of the $\Omega$ and let $w_0$ be the vector part. Then under the action of $w_0$ we have $-\beta \mapsto \alpha_1 \mapsto \alpha_6 \mapsto -\beta$ and $\alpha_2 \mapsto \alpha_3 \mapsto \alpha_5 \mapsto \alpha_2$. As a product of reflections, $w_0 = s_{183} s_{3456} s_{2845} s_{3841} s_{3824} s_{456}$. Write

\[
x_\lambda = \sum_{i=1}^{6} a_i x_i = (b_1, b_2, \ldots, b_8),
\]

for some $a_i \in \mathbb{Q}$ and where $b_i$ denote the $i^{th}$ co-ordinate in the standard basis. Let $\omega_i$ be the $i^{th}$ fundamental co-weight. Then $\mathcal{C} = \{0, \omega_1, \omega_6\}$. Using equations 11.2
and [11.3] to calculate \( \tilde{c}(\omega_6, w_0) \) we get,

\[
\tilde{c}(\omega_6, w_0) = (x_\lambda + s_2 s_4 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_6 \cdot x_\lambda, \tilde{\alpha}_6)
\]

\[
= 2b_5
\]

\[
= a_2 + a_3 + 2(a_4 + a_5 + a_6).
\]

From the relation

(11.8) \[ a_0 + a_1 + a_6 + 2(a_2 + a_3 + a_5) + 3a_4 = 1 \]

and the relations

(11.9) \[ a_0 = a_1 = a_6, \]

(11.10) \[ a_2 = a_3 = a_5, \]

we get that \( \tilde{c}(\omega_6, w_0) = \frac{2}{3} \). Also \( (\omega_6, x_1) = \frac{2}{3} \).

The calculation for the other hyperspecial vertex is very similar.

11.3.6. Type \( E_7 \). \( E_7 \) can be realized as a subspace of \( \mathbb{R}^8 \). A set of simple roots would then be \( \Delta = \{\alpha_1, \ldots, \alpha_6\} \) where \( \alpha_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \), \( \alpha_2 = (1, 1, 0, 0, 0, 0, 0, 0, 0) \), \( \alpha_3 = (-1, 1, 0, 0, 0, 0, 0, 0, 0) \), \( \alpha_4 = (0, -1, 1, 0, 0, 0, 0, 0, 0) \), \( \alpha_5 = (0, 0, -1, 1, 0, 0, 0, 0, 0) \), \( \alpha_6 = (0, 0, 0, -1, 1, 0, 0, 0) \) and \( \alpha_7 = (0, 0, 0, 0, -1, 1, 0, 0) \). The fundamental weights are,

\[
x_1 = (0, 0, 0, 0, 0, 0, -1, 1),
\]

\[
x_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, 1),
\]

\[
x_3 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 3, 3),
\]

\[
x_4 = (0, 0, 1, 1, 1, 1, 1, -2, 2),
\]

\[
x_5 = (0, 0, 0, 1, 1, 1, 3, 3, 2, 2),
\]

\[
x_6 = (0, 0, 0, 0, 1, 1, -1, 1),
\]

\[
x_7 = (0, 0, 0, 0, 0, 1, -1, 2, 2).
\]

Let \( \beta \) denote the highest root. Let \( \tilde{w}_0 \) be the generator of the \( \Omega(\tilde{\Psi}) \) and let \( w_0 \) be the vector part. Then \( w_0 \) is of order two and it sends \( -\beta \mapsto \alpha_7, \alpha_1 \mapsto \alpha_6, \alpha_3 \mapsto \alpha_5 \) and it fixes \( \alpha_2 \) and \( \alpha_4 \). As a product of reflections \( w_0 = s_7 s_6 s_5 s_4 s_3 s_2 s_1 s_4 s_3 s_5 s_4 s_2 s_6 s_5 s_4 s_3 s_1 s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7 \). Write

\[
x_\lambda = \sum_{i=1}^{6} a_i x_i
\]

\[
= (b_1, b_2, \ldots, b_8),
\]
for some $a_i \in \mathbb{Q}$ and where $b_i$ denote the $i^{th}$ co-ordinate in the standard basis. Let $\omega_i$ be the $i^{th}$ fundamental co-weight. Then $\mathcal{C} = \{0, \omega_7\}$. Using equations (11.2) and (11.3) to calculate $\tilde{c}(\omega_7, w_0)$ we get,

$$\tilde{c}(\omega_7, w_0) = (x_\lambda + s_6 s_5 s_4 s_3 s_2 s_1 s_0 s_7 \cdot x_\lambda + s_6 s_5 s_4 s_3 s_2 s_1 s_0 s_7 s_6 s_5 s_4 s_3 s_2 s_1 s_0 s_7 \cdot x_\lambda, \tilde{a}_7)$$

$$= 2b_6 - b_7 + b_8.$$

From the relations

(11.11) $a_0 + 2a_1 + 2a_2 + 3a_3 + 4a_4 + 3a_5 + 2a_6 + a_7 = 1,$

(11.12) $a_0 = a_7; a_1 = a_6; a_3 = a_5,$

we get that $\tilde{c}(\omega_7, w_0) = \frac{3}{2}$. Also $(\omega_7, x_7) = \frac{3}{2}$. This completes the case of $E_7$.

11.3.7. Types $^2A_{2n-1}(n \geq 3)$ and $^2D_{n+1}(n \geq 2)$. The relative root system for

$^2A_{2n-1}(n \geq 3)$ is of type $B_n$ and that $^2D_{n+1}(n \geq 2)$ is of type $C_n$. So the

calculations in these cases are the same as in the unramified case.

This completes the proof of the Proposition 7 and 12.

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