A NOTE ON THE FOURIER COEFFICIENTS AND PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to investigate Paley type and Hardy-Littlewood type inequalities and strong convergence theorem of partial sums of Vilenkin-Fourier series.

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Let \( \mathbb{N}_+ \) denote the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \).

Let \( m := (m_0, m_1, \ldots) \) denote a sequence of the positive numbers, not less than 2.

Denote by
\[
Z_{m_k} := \{0, 1, \ldots, m_k - 1\}
\]
the additive group of integers modulo \( m_k \).

Define the group \( G_m \) as the complete direct product of the group \( Z_{m_j} \) with the product of the discrete topologies of \( Z_{m_j} \)’s.

The direct product \( \mu, \) of the measures
\[
\mu_k (\{j\}) := 1/m_k, \quad (j \in Z_{m_k})
\]
is the Haar measure on \( G_m \), with \( \mu (G_m) = 1 \).

If \( \sup_n m_n < \infty \), then we call \( G_m \) a bounded Vilenkin group. If the generating sequence \( m \) is not bounded then \( G_m \) is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.

The elements of \( G_m \) represented by sequences
\[
x := (x_0, x_1, \ldots, x_j, \ldots), \quad (x_k \in Z_{m_k}).
\]

It is easy to give a base for the neighborhood of \( G_m \) :
\[
I_0 (x) := G_m,
\]
\[
I_n (x) := \{ y \in G_m \mid y_0 = x_0, \ldots, y_{m-1} = x_{n-1} \}, \quad (x \in G_m, \ n \in \mathbb{N}).
\]

Denote \( I_n := I_n (0) \), for \( n \in \mathbb{N} \) and \( \bar{I}_n := G_m \setminus I_n \).

If we define the so-called generalized number system, based on \( m \) in the following way :
\[
M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),
\]
then every \( n \in \mathbb{N} \) can be uniquely expressed as 
\[
 n = \sum_{j=0}^{\infty} n_j M_j,
\]
where \( n_j \in \mathbb{Z} \) and only a finite number of \( n_j \)'s differ from zero.

Let \( |n| := \max \{ j \in \mathbb{N} : n_j \neq 0 \} \).

Denote by \( \mathbb{N}_{n_0} \) the subset of positive integers \( \mathbb{N}_+ \), for which \( |n| = n_0 = 1 \).

Then for every \( n \in \mathbb{N}_{n_0}, M_k < n < M_{k+1} \) can be written as 
\[
 n = M_0 + \sum_{j=1}^{k-1} n_j M_j + M_k = 1 + \sum_{j=1}^{k-1} n_j M_j + M_k, \quad \text{where} \quad n_j \in \{0, m_j - 1\}, \quad (j \in \mathbb{N}_+).
\]

By simple calculation we get
\[
(1) \quad \sum_{\{n : M_k \leq n \leq M_{k+1}, n \in \mathbb{N}_{n_0}\}} 1 = \frac{M_{k-1}}{m_0} \geq c M_k,
\]
where \( c \) is absolute constant.

Denote by \( L_1(G_m) \) the usual (one dimensional) Lebesgue space.

Next, we introduce on \( G_m \) an orthonormal system, which is called the Vilenkin system.

At first define the complex valued function \( r_k(x) : G_m \to \mathbb{C} \), the generalized Rademacher functions as 
\[
 r_k(x) := \exp (2\pi i x k / m_k), \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).
\]

Now define the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) as:
\[
\psi_n(x) := \prod_{k=0}^{\infty} r_k^n (x), \quad (n \in \mathbb{N}).
\]

Specifically, we call this system the Walsh-Paley one if \( m \equiv 2 \).

The Vilenkin system is orthonormal and complete in \( L_2(G_m) \).

Now we introduce analogues of the usual definitions in Fourier-analysis.

If \( f \in L_1(G_m) \) we can establish the the Fourier coefficients, the partial sums, the Dirichlet kernels, with respect to the Vilenkin system in the usual manner:
\[
(\hat{f})(k) := \int_{G_m} f \overline{\psi_k} d\mu \quad (k \in \mathbb{N}),
\]
\[
S_n f := \sum_{k=0}^{n-1} (\hat{f})(k) \psi_k \quad (n \in \mathbb{N}_+, \ S_0 f := 0),
\]
\[
D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+),
\]

Recall that
\[
(2) \quad D_{M_n}(x) = \begin{cases} 
 M_n & \text{if} \ x \in I_n \\
 0 & \text{if} \ x \notin I_n
\end{cases}
\]
and
\[
(3) \quad D_n = \psi_n \left( \sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j-n_j}^{m_j-1} r_j^u \right).
\]
The norm (or quasinorm) of the space $L^p(G_m)$ is defined by

$$
\|f\|_p := \left( \int_{G_m} |f|^p \, d\mu \right)^{1/p} \quad (0 < p < \infty).
$$

The space $L^{p, \infty}(G_m)$ consists of all measurable functions $f$ for which

$$
\|f\|_{L^{p, \infty}} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < +\infty.
$$

The $\sigma$-algebra, generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\mathcal{F}_n$ ($n \in \mathbb{N}$). The conditional expectation operators relative to $\mathcal{F}_n$ ($n \in \mathbb{N}$) are denoted by $E_n$.

Then

$$
E_n f(x) = S_{M_n} f(x) = \sum_{k=0}^{M_n-1} \hat{f}(k) w_k
$$

$$
= \frac{1}{|I_n(x)|} \int_{I_n(x)} f(x) \, d\mu(x),
$$

where $|I_n(x)| = M_n^{-1}$ denotes the length of $I_n(x)$.

A sequence $f = (f^{(n)}, n \in \mathbb{N})$ of functions $f_n \in L^1(G)$ is said to be a dyadic martingale if (for details see e.g. [14])

(i) $f^{(n)}$ is $\mathcal{F}_n$ measurable, for all $n \in \mathbb{N},$

(ii) $E_n f^{(m)} = f^{(n)}$, for all $n \leq m$.

The maximal function of a martingale $f$ is denoted by

$$
f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.
$$

In case $f \in L^1$, the maximal functions are also be given by

$$
f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \, d\mu(u) \right|.
$$

For $0 < p < \infty$, the Hardy martingale spaces $H^p(G_m)$ consist of all martingales, for which

$$
\|f\|_{H^p} := \|f^*\|_{L^p} < \infty.
$$

If $f \in L^1$, then it is easy to show that the sequence $(S_{M_n} f : n \in \mathbb{N})$ is a martingale. If $f = (f^{(n)}), n \in \mathbb{N}$ is martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$
\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \Psi_i(x) \, d\mu(x).
$$

The Vilenkin-Fourier coefficients of $f \in L^1(G_m)$ are the same as those of the martingale $(S_{M_n} f : n \in \mathbb{N})$ obtained from $f$. 
A bounded measurable function $a$ is $p$-atom, if there exist a dyadic interval $I$, such that

\[
\begin{align*}
(a) & \quad \int_I a \, d\mu = 0 \\
(b) & \quad \|a\|_\infty \leq \mu(I)^{-1/p} \\
(c) & \quad \text{supp}(a) \subset I.
\end{align*}
\]

The Hardy martingale spaces $H_p(G_m)$, for $0 < p \leq 1$ have an atomic characterization. Namely, the following theorem is true (see [15]):

**Theorem W.** A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p (0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers, such that for every $n \in \mathbb{N}$:

\[
\begin{align*}
\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k &= f^{(n)}, \\
\sum_{k=0}^{\infty} |\mu_k|^p &< \infty.
\end{align*}
\]

Moreover, $\|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}$, where the infimum is taken over all decomposition of $f$ of the form (5).

When $0 < p \leq 1$, the Hardy martingale space $H_p$ is proper subspace of Lebesque space $L_p$. It is well known that for $1 < p < \infty$ the space $H_p$ is nothing but $L_p$.

The classical inequality of Hardy type is well known in the trigonometric as well as in the Vilenkin-Fourier analysis. Namely,

\[
\sum_{k=1}^{\infty} \left| \frac{\hat{f}(k)}{k} \right| \leq c \|f\|_{H_1},
\]

where the function $f$ belongs to the Hardy space $H_1$ and $c$ is an absolute constant. This was proved in the trigonometric case by Hardy and Littlewood [6] (see also Coifman and Weiss [2]) and for Walsh system it can be found in [8].

Weisz [14, 16] generalized this result for Vilenkin system and proved:

**Theorem A.** Let $0 < p \leq 2$. Then there is an absolute constant $c_p$, depend only $p$, such that

\[
\sum_{k=1}^{\infty} \left| \frac{\hat{f}(k)}{k^{2-p}} \right|^p \leq c_p \|f\|_{H_p},
\]

for all $f \in H_p$.

Paley [7] proved that the Walsh- Fourier coefficients of a function $f \in L_p$ $(1 < p < 2)$ satisfy the condition

\[
\sum_{k=1}^{\infty} \left| \hat{f}(2^k) \right|^2 < \infty.
\]
This result fails to hold for \( p = 1 \). However, it can be verified for functions \( f \in L_1 \), such that \( f^* \) belongs \( L_1 \), i.e \( f \in H_1 \) (see e.g Coifman and Weiss [2]).

For the Vilenkin system the following theorem (see Weisz [17]) is proved:

**Theorem B.** Let \( 0 < p \leq 1 \). Then there is an absolute constant \( c_p \), depends only \( p \), such that

\[
\left( \sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 \right)^{1/2} \leq c_p \|f\|_{H_p},
\]

for all \( f \in H_p \).

It is well-known that Vilenkin system forms not basis in the space \( L_1 \). Moreover, there is a function in the dyadic Hardy space \( H_1 \), such that the partial sums of \( f \) are not bounded in \( L_1 \)-norm. However, in Simon [10] the following strong convergence result was obtained for all \( f \in H_1 \):

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,
\]

where \( S_k f \) denotes the \( k \)-th partial sum of the Walsh-Fourier series of \( f \). (For the trigonometric analogue see Smith [11], for the Vilenkin system by Gát [3]). For the Vilenkin system Simon [9] proved:

**Theorem C.** Let \( 0 < p < 1 \). Then there is an absolute constant \( c_p \), depends only \( p \), such that

\[
\sum_{k=1}^{\infty} \|S_k f\|^p_{L^p_{0,k}} \leq c_p \|f\|^p_{H_p},
\]

for all \( f \in H_p \).

Strong convergence theorems of two-dimensional partial sums was investigate by Weisz [18], Goginava [5], Gogoladze [4], Tephnadze [12].

The main aim of this paper is to prove that the following is true:

**Theorem 1.** Let \( \{\Phi_n\}_{n=1}^{\infty} \) is any nondecreasing sequence, satisfying the condition \( \lim_{n \to \infty} \Phi_n = +\infty \). Then there exists a martingale \( f \in H_p \), such that

\[
\sum_{k=1}^{\infty} \left| \hat{f}(k) \right|^p \Phi_k \left( \sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 \right) \leq \infty,
\]

for \( 0 < p \leq 2 \),

\[
\sum_{k=1}^{\infty} \Theta_k \left( \sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 \right) \leq \infty,
\]

for \( 0 < p \leq 1 \)

and

\[
\sum_{k=1}^{\infty} \left\| S_k f \right\|^p_{L^p_{0,k}} \Phi_k \left( \sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 \right) \leq \infty,
\]

for \( 0 < p < 1 \).
Proof of Theorem 1. Let $0 < p \leq 2$ and $\{\Phi_n\}_{n=1}^{\infty}$ is any nondecreasing, nonnegative sequence, satisfying condition

$$\lim_{n \to \infty} \Phi_n = \infty.$$  

For this function $\Phi(n)$, there exists an increasing sequence $\{\alpha_k \geq 2 : k \in \mathbb{N}_+\}$ of the positive integers such that:

$$(12) \quad \sum_{k=1}^{\infty} \frac{1}{\Phi_{M_{\alpha_k}}^{p/4}} < \infty.$$  

Let

$$f^{(A)}(x) := \sum_{\{k; \alpha_k < A\}} \lambda_k a_k(x),$$  

where

$$\lambda_k = \frac{1}{\Phi_{M_{\alpha_k}}^{1/4}}$$  

and

$$a_k(x) = \frac{M_{\alpha_k}^{1/p-1}}{M} \left( D_{M_{\alpha_k} + 1}(x) - D_{M_{\alpha_k}}(x) \right),$$  

where $M = \sup_{n \in \mathbb{N}} m_n$.

It is easy to show that the martingale $f = (f^{(1)}, f^{(2)}, ..., f^{(A)}, ...) \in H_p$.

Indeed, since

$$(13) \quad S_{M_A}(a_k(x)) = \begin{cases} a_k(x) & \alpha_k < A \\ 0 & \alpha_k \geq A, \end{cases}$$  

$$\text{supp}(a_k) = I_{\alpha_k},$$  

$$\int_{I_{\alpha_k}} a_k d\mu = 0,$$

and

$$\|a_k\|_\infty \leq \frac{M_{\alpha_k}^{1/p-1}}{M} M_{\alpha_k + 1} \leq M_{\alpha_k}^{1/p} = \mu(\text{supp } a_k)^{-1/p},$$

if we apply Theorem W and (12) we conclude that $f \in H_p$.

It is easy to show that
\((14) \hat{f}(j) = \begin{cases} \frac{1}{M} \frac{M_{\alpha k}^{1/p-1}}{\Phi_{M_{\alpha k}}} \frac{1}{M_{\alpha k}^{1/p-1}} \Phi_{M_{\alpha k}}^2, & \text{if } j \in \{M_{\alpha k}, \ldots, M_{\alpha k} + 1 - 1\}, \ k = 1, 2, \ldots \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M_{\alpha k}, \ldots, M_{\alpha k} + 1 - 1\}. \end{cases} \)

First we prove equality (9). Using (14) we can

\[
\sum_{l=1}^{M_{\alpha k} + 1 - 1} \left| \hat{f}(l) \right|^p \Phi_l = \sum_{n=1}^{k} \sum_{l=M_{\alpha n}}^{M_{\alpha k} + 1 - 1} \left| \hat{f}(l) \right|^p \Phi_l \geq \sum_{l=M_{\alpha k}}^{M_{\alpha k} + 1 - 1} \left| \hat{f}(l) \right|^p \Phi_l \geq c \Phi_{M_{\alpha k}} \sum_{l=M_{\alpha k}}^{M_{\alpha k} + 1 - 1} \frac{1}{l^{2-p}} \geq c \Phi_{M_{\alpha k}}^\frac{1}{2} M_{\alpha k}^1 - \frac{1}{2} M_{\alpha k}^1 - \frac{2}{p - 2} \frac{1}{M_{\alpha k}^{1/p}} \geq c \Phi_{M_{\alpha k}}^\frac{1}{2} \to \infty, \quad \text{when } k \to \infty.
\]

Next we prove equality (10). Let \(0 < p \leq 1\). Using (14) we get

\[
\sum_{i=1}^{k} \sum_{j=1}^{M_{\alpha i}^{2-2/p} \Phi_{M_{\alpha i}}} \left| \hat{f}(j M_{\alpha i}) \right|^2 \geq M_{\alpha k}^2 - \frac{2}{p} \Phi_{M_{\alpha k}} \sum_{j=1}^{M_{\alpha k}^{2-2/p} \Phi_{M_{\alpha k}}} \left| \hat{f}(j M_{\alpha k}) \right|^2 \geq c M_{\alpha k}^2 - \frac{2}{p} \Phi_{M_{\alpha k}} \sum_{j=1}^{M_{\alpha k}^{2-2/p} \Phi_{M_{\alpha k}}} \left| \hat{f}(j M_{\alpha k}) \right|^2 \geq c \Phi_{M_{\alpha k}}^\frac{1}{2} \to \infty, \quad \text{when } k \to \infty.
\]
Finally we prove equality (11). Let $0 < p < 1$ and $M_{\alpha k} \leq j < M_{\alpha k+1}$. From (11) we have

\[
S_j f(x) = \sum_{l=0}^{M_{\alpha k+1} - 1} \hat{f}(l) \psi_l(x) + \sum_{l=M_{\alpha k}}^{j-1} \hat{f}(l) \psi_l(x)
\]

\[
= \sum_{\eta=0}^{k-1 M_{\alpha q+1} - 1} \sum_{v=M_{\alpha q}}^{j-1} \frac{M_{\alpha q}^{1/p-1}}{M \Phi_{M_{\alpha q}}^{1/4}} \psi_v(x) + \sum_{v=M_{\alpha k}}^{j-1} \frac{M_{\alpha k}^{1/p-1}}{M \Phi_{M_{\alpha k}}^{1/4}} \psi_v(x)
\]

\[
= \sum_{\eta=0}^{k-1 M_{\alpha q+1} - 1} \frac{M_{\alpha q}^{1/p-1}}{M \Phi_{M_{\alpha q}}^{1/4}} \left(D_{M_{\alpha q}+1}(x) - D_{M_{\alpha q}}(x)\right) + \frac{M_{\alpha k}^{1/p-1}}{M \Phi_{M_{\alpha k}}^{1/4}} \left(D_j(x) - D_{M_{\alpha k}}(x)\right)
\]

\[
= I + II.
\]

Let $j \in \mathbb{N}_{n_0}$ and $x \in G_m \setminus J_1$. Since $j - M_{\alpha k} \in \mathbb{N}_{n_0}$ and

\[
D_{j+M_{\alpha k}}(x) = D_{M_{\alpha k}}(x) + \psi_{M_{\alpha k}}(x) D_j(x),
\]

when $j < M_{\alpha k}$, combining (2) and (3) we can write

\[
|II| = \frac{1}{M} \frac{M_{\alpha k}^{1/p-1}}{\Phi_{M_{\alpha k}}^{1/4}} \left|\psi_{M_{\alpha k}} D_{j-M_{\alpha k}}(x)\right|
\]

\[
= \frac{1}{M} \frac{M_{\alpha k}^{1/p-1}}{\Phi_{M_{\alpha k}}^{1/4}} \left|\psi_{M_{\alpha k}}(x) \psi_{j-M_{\alpha k}}(x) r_{M_{\alpha k}}^{m_{\alpha k}}(x) D_1(x)\right|
\]

\[
= \frac{1}{M} \frac{M_{\alpha k}^{1/p-1}}{\Phi_{M_{\alpha k}}^{1/4}}.
\]
Applying (2) and condition $\alpha_n \geq 2 \ (n \in \mathbb{N})$ for $I$ we have

\begin{equation}
I = 0, \quad \text{for } x \in G_m \setminus I_1.
\end{equation}

It follows that

$$|S_j f(x)| = |II| = \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}, \quad \text{for } x \in G_m \setminus I_1.$$ 

Hence

\begin{equation}
\|S_j (f(x))\|_{L_{p, \infty}} \geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left( x \in G_m : |S_j f(x)| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p}
\end{equation}

\begin{equation}
\geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left( x \in G_m \setminus I_1 : |S_j f(x)| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p}
\end{equation}

\begin{equation}
= \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} |G_m \setminus I_1|
\end{equation}

\begin{equation}
\geq \frac{cM_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}.
\end{equation}

Combining (1) and (17) we have

\begin{equation}
\sum_{j=1}^{M_{\alpha_k+1}-1} \frac{\|S_j (f(x))\|_{L_{p, \infty}}^{p}}{j^{2-p}} \Phi_j
\end{equation}

\begin{equation}
\geq \sum_{j=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{\|S_j (f(x))\|_{L_{p, \infty}}^{p}}{j^{2-p}} \Phi_j
\end{equation}

\begin{equation}
\geq \Phi_{M_{\alpha_k}} \sum_{\{n: M_k \leq n \leq M_{k+1}, \ n \in \mathbb{N}_0\}} \frac{\|S_j (f(x))\|_{L_{p, \infty}}^{p}}{j^{2-p}}
\end{equation}

\begin{equation}
\geq c \Phi_{M_{\alpha_k}} \frac{M_{\alpha_k}^{1-p}}{\Phi_{M_{\alpha_k}}^{p/4}} \sum_{\{n: M_k \leq n \leq M_{k+1}, \ n \in \mathbb{N}_0\}} \frac{1}{j^{2-p}}
\end{equation}

\begin{equation}
\geq c \Phi_{M_{\alpha_k}}^{3/4} \frac{M_{\alpha_k}^{1-p}}{M_{\alpha_k+1}^{2-p}} \sum_{\{n: M_k \leq n \leq M_{k+1}, \ n \in \mathbb{N}_0\}} \frac{1}{M_{\alpha_k+1}^{2-p}}
\end{equation}
\[ \begin{align*}
&\geq c \Phi^{3/4}_{M_{\alpha_k}} \\
&\sum_{\{n: M_k \leq n \leq M_{k+1}, n \in \mathbb{N}_{\infty}\}} 1 \\
&\geq c \Phi^{3/4}_{M_{\alpha_k}} \to \infty, \quad \text{when } k \to \infty.
\end{align*} \]

Theorem 1 is proved.

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