UNITARY EXTENSION PRINCIPLE FOR NONUNIFORM WAVELET FRAMES IN $L^2(\mathbb{R})$

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Abstract. Gabardo and Nashed introduced and studied nonuniform multiresolution analysis which is based on the theory of spectral pairs, where the related translation set is not necessary a group. Motivated by the work of Gabardo and Nashed for nonuniform wavelets, we study the construction of nonuniform tight wavelet frames for the Lebesgue space $L^2(\mathbb{R})$. The main purpose of this paper is to prove the unitary extension principle (UEP) and the oblique extension principle (OEP) for construction of multi-generated nonuniform tight wavelet frames for $L^2(\mathbb{R})$. Some examples are also give to illustrate our results.

1. Introduction

Wavelets have been extensively studied over the last few years and its role in both pure and applied mathematics is well known. It is not possible to give complete list of applications of wavelets, let us at least mention some [1, 2, 7, 8, 15, 16, 18, 23], also see many references therein. Wavelets in $L^2(\mathbb{R})$ are very efficient tools as it gives orthonormal basis for $L^2(\mathbb{R})$ in form of dilation and translation of finite numbers of function in $L^2(\mathbb{R})$ which is very simple and convenient form of basis for $L^2(\mathbb{R})$. Gabardo and Nashed [13] considered a generalization of Mallat’s classic multiresolution analysis (MRA), which is based on the theory of spectral pairs.

Definition 1.1. [13, Definition 3.1] Let $N \geq 1$ be a positive integer and $r$ be an odd integer relatively prime to $N$ such that $1 \leq r \leq 2N - 1$, an associated nonuniform multiresolution analysis (abbreviated NUMRA) is a collection $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
(ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,
(iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
(iv) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$,
(v) There exists a function $\phi \in V_0$, called the scaling function, such that the collection $\{\phi(x - \lambda)\}_{\lambda \in \Lambda}$, where $\Lambda = \{0, \frac{r}{N}\} + 2\mathbb{Z}$, is a complete orthonormal system for $V_0$.

Here, the translate set $\Lambda = \{0, \frac{r}{N}\} + 2\mathbb{Z}$ may not be a group. One may observe that the standard definition of a one-dimensional multiresolution analysis with dilation factor equal to 2 is a special case of NUMRA given in Definition 1.1. Related to the one-dimensional spectral pairs, Gabardo and Yu [14] considered sets of nonuniform wavelets in $L^2(\mathbb{R})$. For fundamental properties of nonuniform wavelets based on the spectral pair, we refer to [13, 14, 20].

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Ron and Shen [17] introduced the unitary extension principle which gives the construction of a multi-generated tight wavelet frame for $L^2(\mathbb{R}^d)$, based on a given refinable function. Tight wavelet frames gives more convenient way to represent a function in $L^2(\mathbb{R})$ in comparison of non-tight wavelet frames as in that case frame operator is constant multiple of identity operator in $L^2(\mathbb{R})$. Christensen and Goh in [6] generalized the unitary extension principle to locally compact abelian groups. They gave general constructions, based on B-splines on the group itself as well as on characteristic functions on the dual group. Motivated by the work of Gabardo and Nashed [13] for the construction of nonuniform wavelets, and application of frames in applied and pure mathematics, we study nonuniform wavelet frames for the Lebesgue space $L^2(\mathbb{R})$. Notable contribution in the paper is to introduce the unitary extension principle for the construction of multi-generated tight nonuniform wavelet frames of the form

$$\{\Psi_{j,\lambda, f}\}_{j \in \mathbb{Z}, \lambda \in \Lambda} = \left\{(2N)^j \psi_1 (2N)^j \gamma - \lambda \right\}_{j \in \mathbb{Z}, \lambda \in \Lambda} \bigcup \cdots \bigcup \left\{(2N)^j \psi_n (2N)^j \gamma - \lambda \right\}_{j \in \mathbb{Z}, \lambda \in \Lambda}$$

in $L^2(\mathbb{R})$.

1.1. Overview and main results. The paper is organized as follows. In Section 2, we give basic notations, definitions and properties of operators related with nonuniform wavelet frames in $L^2(\mathbb{R})$. The general setup for nonuniform wavelet frame system in $L^2(\mathbb{R})$ is also given in Section 2. Section 3 gives some auxiliary results needed in the rest of the paper. The main results are given in Section 4. Theorem 4.1 gives the unitary extension principle (UEP) for the construction of multi-generated tight nonuniform wavelet frames for $L^2(\mathbb{R})$. The extended version of UEP (or oblique extension principle) for nonuniform wavelet frames for $L^2(\mathbb{R})$ can be found in Theorem 4.2. Some examples are given in Section 5 to illustrate our results.

1.2. Relation to existing work and motivation. Duffin and Schaeffer [12] introduced the concept of a frame for separable Hilbert spaces, while addressing some difficult problems from the theory of nonharmonic analysis. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with inner product $\langle ., . \rangle$. The norm induced by the inner product $\langle ., . \rangle$ is given by $\|f\| = \sqrt{\langle f, f \rangle}$, $f \in \mathcal{H}$. A family $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ is called a frame for $\mathcal{H}$, if there exist positive scalars $A_o \leq B_o < \infty$ such that for all $f \in \mathcal{H}$,

$$A_o \|f\|^2 \leq \sum_{k=1}^\infty |\langle f, f_k \rangle|^2 \leq B_o \|f\|^2. \quad (1.1)$$

The scalars $A_o$ and $B_o$ are called lower frame bound and upper frame bound, respectively. If it is possible to choose $A_o = B_o$, then we say that $\{f_k\}_{k=1}^\infty$ is a $A_o$-Parseval frame (or $A_o$-tight frame); and Parseval frame if $A_o = B_o = 1$. If only upper inequality in (1.1) holds, then we say that $\{f_k\}_{k=1}^\infty$ is a Bessel sequence sequence with Bessel bound $B_o$. If $\{f_k\}_{k=1}^\infty$ is a frame for $\mathcal{H}$, then $S : \mathcal{H} \rightarrow \mathcal{H}$ given by $Sf = \sum_{k=1}^\infty \langle f, f_k \rangle f_k$ is a bounded, linear and invertible on $\mathcal{H}$, and is called the frame operator. This gives the reconstruction formula of each vector $f \in \mathcal{H}$,

$$f = SS^{-1} f = \sum_{k=1}^\infty \langle S^{-1} f, f_k \rangle f_k.$$ 

Thus, each vector has an explicit series expansion which need not be unique. For application of frames in both pure and applied mathematics, we refer to book of Casazza and Kutyniok [3], Christensen [5] and Han [15]. Nowadays the theory of iterated function systems, quantum mechanics
and wavelets is emerging in important applications in frame theory, see [11, 21, 22] and many references therein. Very recent work on discrete frames of translates and discrete wavelet frames, and their duals in finite dimensional spaces can be found in [9, 10]. Wavelet frames in $L^2(\mathbb{R})$ are also very powerful tool for representing functions in $L^2(\mathbb{R})$ as sum of series of functions which are dilation and translation of finite number of functions in $L^2(\mathbb{R})$. It provides us convenient tool to expansion of functions in $L^2(\mathbb{R})$ of similar type as one that arise in orthonormal basis, however, wavelet frame conditions are weaker that makes wavelet frame more flexible. Nonuniform wavelet frames could be used in signal processing, sampling theory, speech recognition and various other areas, where instead of integer shifts nonuniform shifts are needed. Some necessary and sufficient conditions for nonuniform wavelet frames for $L^2(\mathbb{R})$ can be found in [19].

Motivated by the work of Gabardo and Nashed [13] and Gabardo and Yu [14] in the study of nonuniform wavelets, we study frame properties of nonuniform wavelets in the Lebesgue space $L^2(\mathbb{R})$. We recall that the extension problems in frame theory has a long history. It is showed in [4] that the extension problem has a solution in the sense that “any Bessel sequence can be extended to a tight frame by adjoining a suitable family of vectors in the underlying space.” Ron and Shen introduced unitary extension principle for construction of tight wavelet frames in the Lebesgue space $L^2(\mathbb{R}^d)$. The unitary extension principle allows construction of tight wavelet frames with compact support, desired smoothness; and good approximation of functions. In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of this types of signals by a stable mathematical tool. Gabardo and Nashed [13] and Gabardo and Yu [14] filled this gap by the concept of nonuniform multiresolution analysis. In the direction of construction of Parseval frames from nonuniform multiwavelets systems, we develop a general setup and prove the unitary extension principle for construction of multi-generated nonuniform tight wavelet frames for $L^2(\mathbb{R})$. Ron and Shen [17] gave the unitary extension principle, where conditions for the construction of multi-generated tight wavelet frames for the Lebesgue space $L^2(\mathbb{R}^d)$ are based on a given refinable function. The conditions in our general set up and in the unitary extension principle are different than that given by Ron and Shen, see Remark 2.2 for details.

2. Preliminaries

As is standard, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all integers, positive integers and real numbers, respectively. Throughout the paper, $N \in \mathbb{N}$, $r$ be an odd integer relative prime to $N$ such that $1 \leq r \leq 2N - 1$ and $\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$. Notice that the discrete set $\Lambda$ is not always a group. The support of a function $\psi$ is denoted by Supp $\psi$, and defined as $\text{Supp} \psi = \text{clo}\left\{x : \psi(x) \neq 0\right\}$. Symbol $\overline{z}$ denote the complex conjugate of a complex number $z$. The conjugate transpose of a matrix $H$ is denoted by $H^*$. The characteristic function of a set $E$ is denoted by $\chi_E$. The spaces $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ denote the equivalence classes of square-integrable functions and essentially bounded functions on $\mathbb{R}$, respectively. Next, we recall the Parseval identity. Let $\{e_k\}_{k \in \mathbb{Z}}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Then,

$$\sum_{k \in \mathbb{Z}} |\langle f, e_k \rangle|^2 = \|f\|^2, \ f \in \mathcal{H} \quad \text{(Parseval identity)}.$$
For \( a, b \in \mathbb{R} \), we consider the following operators on \( L^2(\mathbb{R}) \).

\[
T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_a f(\gamma) = f(\gamma - a) \quad \text{(Translation by } a), \\
E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad E_b f(\gamma) = e^{2\pi ib\gamma} f(\gamma) \quad \text{(Modulation by } b), \\
L : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad L f(\gamma) = \sqrt{2N} f(2N\gamma) \quad \text{(N-Dilation operator)}.
\]

The \( j \) fold \( N \)-dilation, where \( j \in \mathbb{Z} \), is given by

\[
L^j f(\gamma) = (2N)^{j/2} f((2N)^j \gamma).
\]

**Definition 2.1.** Let \( \{\psi_1, \psi_2, \ldots, \psi_n\} \subset L^2(\mathbb{R}) \) be a finite set, where \( \psi_\ell \neq 0, 1 \leq \ell \leq n \). The family

\[
\{L^jT_\lambda \psi_\ell\}_{\lambda \in \mathbb{Z}, \ell = 1, 2, \ldots, n} = \left\{(2N)^{j/2} \psi_1(2N)^j \gamma - \lambda\right\}_{\lambda \in \mathbb{Z}, \ell = 1, 2, \ldots, n}
\]

is called a nonuniform wavelet frame for \( L^2(\mathbb{R}) \), if there exist finite positive constants \( A \) and \( B \) such that

\[
A\|f\|^2 \leq \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \sum_{\ell = 1}^n |(f, L^jT_\lambda \psi_\ell)|^2 \leq B\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}).
\]

The Fourier transform of a function \( f \) is denoted by \( \mathcal{F} f \) or \( \hat{f} \), and defined as

\[
\mathcal{F} f = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \gamma} dx.
\]

For \( N \in \mathbb{N}, j \in \mathbb{Z} \) and \( a \in \mathbb{R} \), we have the following properties.

(i) \( L^j : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is unitary map.

(ii) \( L^jT_a = T_{(2N)^{-j}a}L^j \).

(iii) \( \mathcal{F} L^j = L^{-j}\mathcal{F} \).

(iv) \( \mathcal{F}T_a = E_{-a}\mathcal{F} \).

In formulation of the unitary extension principle there is long list of assumption, instead of writing each assumption again and again, we state all assumptions at once and call it **general setup**.

**General Setup:** Let \( \psi_0 \in L^2(\mathbb{R}) \) be such that

(i) \( \hat{\psi}_0(2N\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma), \ H_0(\gamma) \in L^\infty(\mathbb{R}) \);

(ii) \( \text{Supp } \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{2N}] \);

(iii) \( \lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1 \).

Further, let \( H_1, H_2, \ldots, H_n \in L^\infty(\mathbb{R}) \), and define \( \psi_1, \psi_2, \ldots, \psi_n \in L^2(\mathbb{R}) \) such that

\[
\hat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \ \ell = 1, 2, \ldots, n.
\]

Let \( H(\gamma) \) be a \((n + 1) \times 1\) matrix given by

\[
H(\gamma) = \begin{bmatrix}
H_0(\gamma) \\
H_1(\gamma) \\
\vdots \\
H_n(\gamma)
\end{bmatrix}
\]

Then, the collection \( \{\psi_\ell, H_\ell\}_{\ell = 0}^n \) is called a general setup.
**Remark 2.2.** Note that the conditions in our general set up are different than that given by Ron and Shen [17] for unitary extension principle for standard wavelet frames. To precise, in [17], the functions $H_i$ are 1-periodic essential bounded functions on $\mathbb{R}$ (also see [5, p. 446]) for details, but in our general set up $H_i$ are merely essential bounded functions on $\mathbb{R}$. Further, conditions (i)-(iii) of our general set up are also different than the set up given in [17].

With this general setup, our aim is to find conditions on $H(\gamma)$ such that the nonuniform system \{\text{\hat{L}}_j^T \psi_0\}_{j \in \mathbb{Z}, \lambda \in \Lambda}$ constitutes a Parseval frame for $L^2(\mathbb{R})$.

All the inequalities and equations in the entire paper are assumed to hold almost everywhere (a.e.).

### 3. Some Auxiliary Results

In this section, we give some auxiliary results that will be used in the sequel.

**Lemma 3.1.** For any $f \in L^1(\mathbb{R})$, the function $S_f(\gamma) = \sum_{k \in \mathbb{Z}} f(\gamma + Nk)$ is well defined, $N$-periodic and belongs to $L^1(0, N)$.

**Proof.** It is clear that $S_f(\gamma) = \sum_{k \in \mathbb{Z}} f(\gamma + Nk)$ is $N$-periodic. For any $f \in L^1(\mathbb{R})$, we have
\[
\int_0^N \sum_{k \in \mathbb{Z}} |f(\gamma + Nk)|d\gamma = \int_\mathbb{R} |f(\gamma)|d\gamma < \infty.
\]
Thus, $S_f(\gamma)$ is well defined a.e. on $\mathbb{R}$, and also belongs to $L^1(0, N)$. \qed

**Lemma 3.2.** Assume that

(i) $\psi_0 \in L^2(\mathbb{R})$, $\lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1$ and $\text{Supp} \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{N}]$;

(ii) $f \in L^2(\mathbb{R})$ such that $\hat{f} \in C_c(\mathbb{R})$.

Then, for any $\epsilon > 0$ there exist $j \in \mathbb{Z}$ such that

\[(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \text{\hat{L}}_j^T \psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2 \text{ for all } j \geq J.
\]

**Proof.** For any $j \in \mathbb{Z}$, $(L^j \hat{f}) \hat{\psi}_0 \in L^1(\mathbb{R})$. Therefore, by Lemma 3.1, the function $S(L^j \hat{f}) \hat{\psi}_0$ is well defined. Further, for $\gamma \in [0, N]$, we have
\[
S(L^j \hat{f}) \hat{\psi}_0 = \sum_{k \in \mathbb{Z}} ((L^j \hat{f}) \hat{\psi}_0)(\gamma - Nk)
= \sum_{k \in \mathbb{Z}} (L^j \hat{f})(\gamma - Nk) \hat{\psi}_0(\gamma - Nk).
\]
Thus, $S(L^j \hat{f}) \hat{\psi}_0$ is bounded by finite linear combinations of translates of $\hat{\psi}_0$ and $S(L^j \hat{f}) \hat{\psi}_0 \in L^2[0, N]$.

Note that
\[
\langle f, \text{\hat{L}}_j^T \psi_0 \rangle = \langle \hat{f}, \text{\hat{L}}_j^T \hat{\psi}_0 \rangle = \langle \hat{f}, L^{-j}E_{-\lambda} \hat{\psi}_0 \rangle = \langle L^j \hat{f}, E_{-\lambda} \hat{\psi}_0 \rangle.
\]

Using $\text{Supp} \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{4N}]$ and $\frac{1}{4N} < \frac{1}{2}$, we compute
\[
\sum_{\lambda \in \Lambda} |\langle f, \text{\hat{L}}_j^T \psi_0 \rangle|^2
\]
Applying the Parseval identity on $L^2(0, \frac{1}{2})$ with respect to an orthonormal basis $\{\sqrt{2\pi}e^{2\pi i(2m)\gamma}\}$ in (3.1), we obtain

$$
\sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2 = \frac{1}{2} \int_0^{\frac{1}{2}} |\langle L^j \hat{f}, \tilde{\psi}_0 \rangle|^2 d\gamma + \frac{1}{2} \int_0^{\frac{1}{2}} |\langle L^j \hat{f}, \tilde{\psi}_0 \rangle|^2 d\gamma
$$

$$
= \frac{1}{2} \int_0^{\frac{1}{2}} |\langle L^j \hat{f}, \tilde{\psi}_0 \rangle|^2 d\gamma. \quad (3.2)
$$

Let $\epsilon > 0$ be given. Since $\hat{\psi}_0(\gamma) \to 1$ as $\gamma \to 0^+$, we can choose $b \in ]0, \frac{1}{2}]$ so that

$$
(1 - \epsilon) \leq |\hat{\psi}_0(\gamma)|^2 \leq (1 + \epsilon), \quad \text{where } 0 < \gamma < b. \quad (3.3)
$$

Choose $J \in \mathbb{Z}$ large enough, so that $\text{Supp } (L^j \hat{f}) \subseteq [-b, b]$ for all $j \geq J$. Then, by (3.2), we have

$$
\sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2 = \frac{1}{2} \int_0^{\frac{1}{2}} |\langle L^j \hat{f}, \tilde{\psi}_0 \rangle|^2 d\gamma \quad \text{for all } j \geq J. \quad (3.4)
$$

By (3.3), (3.4) and the fact that $L^j$ is unitary map, we have

$$
(1 - \epsilon)\|\hat{f}\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2 \leq (1 + \epsilon)\|\hat{f}\|^2 \quad \text{for all } j \geq J.
$$

Since the Fourier transform is unitary map, we get

$$
(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2 \quad \text{for all } j \geq J.
$$

This concludes the proof. \(\square\)

**Lemma 3.3.** Suppose that

(i) $\psi_0 \in L^2(\mathbb{R})$ satisfies $\text{Supp } \hat{\psi}_0 \subseteq [0, \frac{1}{4\pi}]$ and $\hat{\psi}_0(2N\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma)$, where $H_0(\gamma) \in L^\infty(\mathbb{R})$;

(ii) $f \in L^2(\mathbb{R})$ with $\hat{f} \in C_c(\mathbb{R})$, and $H_1, H_2, \ldots, H_n \in L^\infty(\mathbb{R})$ such that the $(n + 1) \times 1$ matrix

$$
H(\gamma) = \begin{bmatrix}
H_0(\gamma) \\
H_1(\gamma) \\
\vdots \\
H_n(\gamma)
\end{bmatrix}_{(n+1)\times 1}
$$
satisfies $H(\gamma)^* H(\gamma) = 1$ a.e.;

(iii) $\psi_1, \psi_2, \ldots, \psi_n \in L^2(\mathbb{R})$ such that $\hat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \ \ell = 1, 2, \ldots, n$. Then

$$\sum_{\ell=0}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_\lambda \psi_\ell \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^{j}T_\lambda \psi_0 \rangle|^2.$$ 

Proof. For any $j \in \mathbb{Z}$ and for any $\ell = 0, 1, \ldots, n$, we have

$$\langle f, L^{j-1}T_\lambda \psi_\ell \rangle = \langle L^{-j} f, L^{-1}T_\lambda \psi_\ell \rangle$$

$$= \langle L^{-j} f, T(2N)\lambda L^{-1} \psi_\ell \rangle$$

$$= \langle L^{j} \hat{f}, E_{-(2N)\lambda} \hat{L} \psi_\ell \rangle$$

$$= \int_{\mathbb{R}} \langle L^{j} \hat{f}(\gamma) \sqrt{2N} \hat{\psi}_\ell(2N\gamma)e^{2\pi i (2N\lambda)\gamma} d\gamma$$

$$= \sqrt{2N} \int_{\mathbb{R}} \langle L^{j} \hat{f}(\gamma) H_\ell(\gamma) \psi_\ell(\gamma)e^{2\pi i (2N\lambda)\gamma} d\gamma.$$ (3.5)

Using $\text{Supp} \ \hat{\psi}_0 \subseteq [0, \frac{1}{4N}]$, and Parseval identity on $L^2(0, \frac{1}{4N})$ with respect to orthonormal basis \{2 N e^{2\pi i (4Nm) \gamma} \}_{m \in \mathbb{Z}}$, we have

$$\sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_\lambda \psi_\ell \rangle|^2 = \sum_{\lambda \in \mathbb{Z}} |\langle f, L^{j-1}T_\lambda \psi_\ell \rangle|^2 + \sum_{\lambda \in \mathbb{Z} \cup \mathbb{Z}^c} |\langle f, L^{j-1}T_\lambda \psi_\ell \rangle|^2$$

$$= \sum_{m \in \mathbb{Z}} \left| \sqrt{2N} \int_{0}^{N} \mathcal{S}((L^{j} \hat{f})H_\ell \hat{\psi}_0(\gamma)e^{2\pi i (2N)(2m)\gamma} d\gamma \right|^2$$

$$+ \sum_{m \in \mathbb{Z}} \left| \sqrt{2N} \int_{0}^{N} \mathcal{S}((L^{j} \hat{f})H_\ell \hat{\psi}_0(\gamma)e^{2\pi i (2N)(\frac{1}{4N}+2m)\gamma} d\gamma \right|^2 \text{ (using (3.5))}$$

$$= \frac{1}{2} \sum_{m \in \mathbb{Z}} \left| \int_{0}^{\frac{1}{4N}} (L^{j} \hat{f})(\gamma) H_\ell(\gamma) \hat{\psi}_0(\gamma)e^{2\pi i (4Nm)\gamma} 2\sqrt{N} d\gamma \right|^2$$

$$+ \frac{1}{2} \sum_{m \in \mathbb{Z}} \left| \int_{0}^{\frac{1}{4N}} (L^{j} \hat{f})(\gamma) H_\ell(\gamma) \hat{\psi}_0(\gamma)e^{2\pi i (2(\frac{1}{4N})+2m)\gamma} 2\sqrt{N} d\gamma \right|^2$$

$$= \frac{1}{2} \int_{0}^{\frac{1}{4N}} |(L^{j} \hat{f})(\gamma) H_\ell(\gamma) \hat{\psi}_0(\gamma)|^2 d\gamma + \frac{1}{2} \int_{0}^{\frac{1}{4N}} |(L^{j} \hat{f})(\gamma) H_\ell(\gamma) \hat{\psi}_0(\gamma)|^2 d\gamma$$

$$= \int_{0}^{\frac{1}{4N}} |(L^{j} \hat{f})(\gamma) H_\ell(\gamma) \hat{\psi}_0(\gamma)|^2 d\gamma.$$ (3.6)

Since $H(\gamma)^* H(\gamma) = 1$ a.e., so $H(\gamma)$ could be consider as an isometry from $\mathbb{C}^1$ into $\mathbb{C}^{n+1}$. Using (3.6), we have

$$\sum_{\ell=0}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_\lambda \psi_\ell \rangle|^2 = \sum_{\ell=0}^{n} \int_{0}^{\frac{1}{4N}} |(L^{j} \hat{f})(\gamma) H_\ell(\gamma) \hat{\psi}_0(\gamma)|^2 d\gamma.$$
Lemma 3.4. Let \( \{\psi_\ell, H_\ell\}_{\ell=0}^\infty \) be a general setup, and let \( H(\gamma)^* H(\gamma) = 1 \). Then, the following holds.

(i) \( \{T_\lambda \psi_0\}_{\lambda \in \Lambda} \) is Bessel sequence with Bessel bound 1.

(ii) For any \( f \in L^2(\mathbb{R}) \),

\[
\lim_{j \to -\infty} \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_0 \rangle|^2 = 0.
\]
Proof. (i): Let \( f \in L^2(\mathbb{R}) \) be such that \( \hat{f} \in C_c(\mathbb{R}) \), and let \( \epsilon > 0 \) be given. Then, by Lemma 3.2, we can find an integer \( j > 0 \) such that
\[
\sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2 \leq (1 + \epsilon) \|f\|^2.
\] (3.10)
Also, by Lemma 3.3, we have
\[
\sum_{\lambda \in \Lambda} |\langle f, L^{j-1} T_{\lambda} \psi_1 \rangle|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2.
\] (3.11)
Applying (3.11) \( j \) times and using (3.10), we get
\[
\sum_{\lambda \in \Lambda} |\langle f, T_{\lambda} \psi_0 \rangle|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2 \leq (1 + \epsilon) \|f\|^2.
\]
Since \( \epsilon > 0 \) was arbitrary, we have
\[
\sum_{\lambda \in \Lambda} |\langle f, T_{\lambda} \psi_0 \rangle|^2 \leq \|f\|^2.
\]
Because this inequality holds on a dense subset of \( L^2(\mathbb{R}) \), it holds on \( L^2(\mathbb{R}) \). This proves (i).

(ii): Let \( f \in L^2(\mathbb{R}) \). Since \( L^j \) is unitary map for all \( j \in \mathbb{Z} \), by using (i), the family \( \{L^j T_{\lambda} \psi_0\}_{\lambda \in \Lambda} \) is Bessel sequence with Bessel bound 1. For any \( j \in \mathbb{Z} \) and for any bounded interval \( I \subset \mathbb{R} \), we have
\[
\sum_{\lambda \in \Lambda} |\langle f, L^j T_{\lambda} \psi_0 \rangle|^2 \leq 2 \sum_{\lambda \in \Lambda} |\langle f \chi_I, L^j T_{\lambda} \psi_0 \rangle|^2 + 2 \sum_{\lambda \in \Lambda} |\langle f (1 - \chi_I), L^j T_{\lambda} \psi_0 \rangle|^2
\]
\[
\leq 2 \sum_{\lambda \in \Lambda} |\langle f \chi_I, L^j T_{\lambda} \psi_0 \rangle|^2 + 2 \|f (1 - \chi_I)\|^2.
\]
Now, \( \|f (1 - \chi_I)\|^2 \to 0 \), if we choose \( I \) to be sufficiently large. Therefore, we only need to show
\[
\sum_{\lambda \in \Lambda} |\langle f \chi_I, L^j T_{\lambda} \psi_0 \rangle|^2 \to 0 \text{ as } j \to -\infty.
\]
Using the Cauchy-Schwarz’s inequality for integrals, we obtain
\[
\sum_{\lambda \in \Lambda} |\langle f \chi_I, L^j T_{\lambda} \psi_0 \rangle|^2 = (2N)^j \sum_{\lambda \in \Lambda} \left| \int_I f(\gamma) \bar{\psi}_0((2N)^j \gamma - \lambda) d\gamma \right|^2
\]
\[
\leq (2N)^j \|f\|^2 \sum_{\lambda \in \Lambda} \int_I |\psi_0((2N)^j \gamma - \lambda)|^2 d\gamma
\]
\[
= \|f\|^2 \sum_{\lambda \in \Lambda \cap (2N)^j I - \lambda} \int_I |\psi_0(\gamma)|^2 d\gamma.
\] (3.12)
Applying the Lebesgue dominated convergence theorem in (3.12), we have
\[
\sum_{\lambda \in \Lambda} |\langle f \chi_I, L^j T_{\lambda} \psi_0 \rangle|^2 \to 0 \text{ as } j \to -\infty.
\]
Hence (ii) is proved.

4. Unitary Extension Principle for Nonuniform Wavelet Frames

We start this section with the UEP for nonuniform wavelet frames for \( L^2(\mathbb{R}) \).

**Theorem 4.1.** Let \( \{\psi_t, H_t\}_{t=0}^n \) be a general setup and \( H(\gamma)^* H(\gamma) = 1 \). Then, the nonuniform multiwavelets system \( \{L^j T_{\lambda} \psi_t\}_{j \in \mathbb{Z}, \lambda \in \Lambda} \) constitutes a Parseval frame for \( L^2(\mathbb{R}) \).
Proof. Let $\epsilon > 0$ be given. Consider a function $f \in L^2(\mathbb{R})$ such that $\hat{f} \in C_c(\mathbb{R})$. By Lemma 3.2, we can choose $J > 0$ such that for all $j \geq J$,

$$(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^jT_{\lambda}\psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2.$$  \hfill (4.1)

Using Lemma 3.3, we have

$$\sum_{\lambda \in \Lambda} |\langle f, L^jT_{\lambda}\psi_0 \rangle|^2 = \sum_{\ell=0}^n \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda}\psi_0 \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda}\psi_0 \rangle|^2 + \sum_{\ell=1}^n \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda}\psi_\ell \rangle|^2.$$ \hfill (4.2)

Applying Lemma 3.3 on $\sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda}\psi_0 \rangle|^2$, we get

$$\sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda}\psi_0 \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^{j-2}T_{\lambda}\psi_0 \rangle|^2 + \sum_{\ell=1}^n \sum_{\lambda \in \Lambda} |\langle f, L^{j-2}T_{\lambda}\psi_\ell \rangle|^2.$$ \hfill (4.3)

By (4.2) and (4.3), we have

$$\sum_{\lambda \in \Lambda} |\langle f, L^jT_{\lambda}\psi_0 \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^{j-2}T_{\lambda}\psi_0 \rangle|^2 + \sum_{\ell=1}^n \sum_{\lambda \in \Lambda} \sum_{p=j-2}^{j-1} |\langle f, L^pT_{\lambda}\psi_\ell \rangle|^2.$$ \hfill (4.4)

Repeating the above arguments, for any $m < j$, we have

$$\sum_{\lambda \in \Lambda} |\langle f, L^jT_{\lambda}\psi_0 \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^mT_{\lambda}\psi_0 \rangle|^2 + \sum_{\ell=1}^n \sum_{\lambda \in \Lambda} \sum_{p=m}^{j-1} |\langle f, L^pT_{\lambda}\psi_\ell \rangle|^2.$$ \hfill (4.4)

It follows from (4.1) and (4.4) that for all $j \geq J$, $m < j$,

$$(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^mT_{\lambda}\psi_0 \rangle|^2 + \sum_{\ell=1}^n \sum_{\lambda \in \Lambda} \sum_{p=m}^{j-1} |\langle f, L^pT_{\lambda}\psi_\ell \rangle|^2 \leq (1 + \epsilon)\|f\|^2.$$ \hfill (4.5)

Letting $m \to -\infty$ in above and using (ii) of Lemma 3.4, we have

$$(1 - \epsilon)\|f\|^2 \leq \sum_{\ell=1}^n \sum_{\lambda \in \Lambda} \sum_{p=-\infty}^{j-1} |\langle f, L^pT_{\lambda}\psi_\ell \rangle|^2 \leq (1 + \epsilon)\|f\|^2.$$ \hfill (4.5)

Letting $j \to \infty$ in (4.5), we have

$$(1 - \epsilon)\|f\|^2 \leq \sum_{\ell=1}^n \sum_{\lambda \in \Lambda} \sum_{p=-\infty}^{\infty} |\langle f, L^pT_{\lambda}\psi_\ell \rangle|^2 \leq (1 + \epsilon)\|f\|^2.$$ \hfill (4.5)

Since $\epsilon > 0$ was arbitrary, we obtain

$$\sum_{\ell=1}^n \sum_{\lambda \in \Lambda} \sum_{p \in \mathbb{Z}} |\langle f, L^pT_{\lambda}\psi_\ell \rangle|^2 = \|f\|^2 \text{ for all } f \in L^2(\mathbb{R}),$$

as desired. \hfill \Box

The next theorem gives the generalized (or oblique) extension principle for nonuniform wavelet frames in $L^2(\mathbb{R})$. It gives the more flexible technique to construct nonuniform wavelet frames. For the applications and other technical details about the oblique extension principle for standard wavelet frames, we refer to [5, p. 460] and [8, Proposition 1.11].
Theorem 4.2. Let \( \{\psi_{\ell}, H_{\ell}\}_{\ell=0}^{n} \) be a general setup. Assume that there exist strictly positive function \( \theta \in L^\infty(\mathbb{R}) \) for which

\[
\lim_{\gamma \to 0^+} \theta(\gamma) = 1,
\]

and

\[
\theta(2N\gamma)|H_0(\gamma)|^2 + \sum_{\ell=1}^{n} |H_\ell(\gamma)|^2 = \theta(\gamma).
\]

Then, \( \{L^T \chi_{\ell} \psi_{\ell}\}_{\ell \in \Lambda_{\Sigma \Xi}} \) is a Parseval nonuniform wavelet frame for \( L^2(\mathbb{R}) \).

Proof. Define \( \widetilde{\psi}_0 \in L^2(\mathbb{R}) \) such that

\[
\mathcal{F}_{\gamma} \widetilde{\psi}_0(\gamma) = \sqrt{\theta(\gamma)} \mathcal{F}_{\gamma} \hat{\psi}_0(\gamma).
\]

(4.6)

Define functions \( \widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_n \) as follows

\[
\widetilde{H}_0(\gamma) = \sqrt{\frac{\theta(2N\gamma)}{\theta(\gamma)}} H_0(\gamma),
\]

\[
\widetilde{H}_\ell(\gamma) = \sqrt{\frac{1}{\theta(\gamma)}} H_\ell(\gamma), \quad \ell = 1, 2, \ldots, n.
\]

Then, we have

\[
\mathcal{F}_{\gamma} \widetilde{\psi}_0(2N\gamma) = \sqrt{\theta(2N\gamma)} \mathcal{F}_{\gamma} \hat{\psi}_0(2N\gamma)
\]

\[
= \sqrt{\theta(2N\gamma)} \{H_0(\gamma) \mathcal{F}_{\gamma} \hat{\psi}_0(\gamma)\}
\]

\[
= \sqrt{\theta(2N\gamma)} \left\{ H_0(\gamma) \sqrt{\frac{\theta(\gamma)}{\theta(\gamma)}} \right\}
\]

\[
= \sqrt{\theta(2N\gamma)} \frac{\theta(\gamma)}{\theta(\gamma)} H_0(\gamma) \mathcal{F}_{\gamma} \hat{\psi}_0(\gamma)
\]

\[
= \mathcal{F}_{\gamma} \widetilde{H}_0(\gamma) \mathcal{F}_{\gamma} \hat{\psi}_0(\gamma),
\]

(4.7)

and

\[
\lim_{\gamma \to 0^+} \mathcal{F}_{\gamma} \widetilde{\psi}_0(\gamma) = \lim_{\gamma \to 0^+} \sqrt{\theta(\gamma)} \mathcal{F}_{\gamma} \hat{\psi}_0(\gamma) = 1.
\]

(4.8)

Since \( \{\psi_{\ell}, H_{\ell}\}_{\ell=0}^{n} \) is a general setup, by (4.6), we have

\[
\text{Supp} \mathcal{F}_{\gamma} \widetilde{\psi}_0(\gamma) \subseteq \left[0, \frac{1}{4N}\right],
\]

(4.9)

and

\[
\sum_{\ell=0}^{n} |\mathcal{F}_{\gamma} \widetilde{H}_\ell(\gamma)|^2 = |\mathcal{F}_{\gamma} \widetilde{H}_0(\gamma)|^2 + \sum_{\ell=1}^{n} |\mathcal{F}_{\gamma} \widetilde{H}_\ell(\gamma)|^2
\]

\[
= \frac{\theta(2N\gamma)}{\theta(\gamma)} |H_0(\gamma)|^2 + \sum_{\ell=1}^{n} \frac{|H_\ell(\gamma)|^2}{\theta(\gamma)}
\]

\[
= \frac{1}{\theta(\gamma)} \theta(\gamma)
\]

\[
= 1.
\]

(4.10)
Thus

\[ \tilde{H}_\ell(\gamma) \in L^\infty(\mathbb{R}) \text{ for } \ell = 0, 1, \ldots, n. \]  

(4.11)

Let \( \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_n \in L^2(\mathbb{R}) \) be such that

\[ \tilde{\psi}_\ell(2N\gamma) = \tilde{H}_\ell(\gamma)\tilde{\psi}_0(\gamma), \ell = 1, \ldots, n. \]  

(4.12)

Define

\[
\tilde{H}(\gamma) = \begin{bmatrix}
H_0(\gamma) \\
H_1(\gamma) \\
\vdots \\
H_n(\gamma)
\end{bmatrix}_{(n+1) \times 1}.
\]

Then, by (4.7), (4.8), (4.9) and (4.11), the collection \( \{\tilde{\psi}_\ell, \tilde{H}_\ell\}_{\ell=0}^n \) is a general setup.

Using (4.10), we have

\[
\tilde{H}(\gamma)^* \tilde{H}(\gamma) = \sum_{\ell=0}^n |\tilde{H}_\ell(\gamma)|^2 = 1.
\]

Hence, by Theorem 4.1, \( \{L^1T_\lambda \tilde{\psi}_\ell\}_{j \in \mathbb{Z}, \lambda \in \Lambda} \) is a Parseval nonuniform wavelet frames for \( L^2(\mathbb{R}) \).

Next, we compute

\[
{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma) \\
= \left\{ \begin{array}{l}
\tilde{H}_\ell(\gamma)\sqrt{\theta(\gamma)} \\
\tilde{\psi}_0(\gamma)
\end{array} \right\} \\
= \tilde{H}_\ell(\gamma)\tilde{\psi}_0(\gamma) \\
= \tilde{\psi}_\ell(2N\gamma).
\]

This gives, \( \psi_\ell = \tilde{\psi}_\ell \). Hence, the system \( \{L^1T_\lambda \psi_\ell\}_{j \in \mathbb{Z}, \lambda \in \Lambda} \) is a Parseval nonuniform wavelet frames for \( L^2(\mathbb{R}) \).

\[ \square \]

**Remark 4.3.** It is worth noticing that, when \( \theta = 1 \), Theorem 4.1 can be obtained from Theorem 4.2.

**Construction of nonuniform wavelet frame with two generators:** Computation effort reduces if we have less number of generator or window functions, so we wish to have as minimum numbers of generators as is it possible. In this direction, we have the following result as an application of Theorem 4.2.

**Corollary 4.4.** Let \( \psi_0 \in L^2(\mathbb{R}) \) such that

(1) \( \hat{\psi}_0(2N\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma) \), where \( H_0(\gamma) \in L^\infty(\mathbb{R}) \);

(2) \( \text{Supp } \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{2N}] \); and

(3) \( \lim_{\gamma \to 0^+} \psi_0(\gamma) = 1 \).
If we choose \( H_1(\gamma) = \sqrt{\theta(2N\gamma)}H_0(\gamma)i, \) \( H_2(\gamma) = \sqrt{\theta(\gamma)}, \) and \( \psi_1, \psi_2 \in L^2(\mathbb{R}) \) such that
\[
\hat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \quad \ell = 1, 2.
\]
Then
\[
\theta(2N\gamma)|H_0(\gamma)|^2 + |H_1(\gamma)|^2 + |H_2(\gamma)|^2 = \theta(\gamma).
\]
Hence, by Theorem 4.2, \( \{L^\ell T_\chi \psi_\ell\}_{\ell \in 1, 2} \) form a Parseval nonuniform wavelet frame for \( L^2(\mathbb{R}) \).

5. Examples

This section gives some applicative examples of the UEP and its generalized version. The following example illustrates Theorem 4.1.

**Example 5.1.** Let \( \psi(\gamma) \in L^2(\mathbb{R}) \) such that \( \hat{\psi}_0(\gamma) = \chi_{[0, 1]}(\gamma) \).

Then
\[
(i) \text{ Supp } \hat{\psi}_0(\gamma) \subset [0, \frac{1}{4N}];
\]
\[
(ii) \lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1;
\]
\[
(iii) \hat{\psi}_0(2N\gamma) = \chi_{[0, \frac{1}{8N}]}(2N\gamma)
\]
\[
= \chi_{[0, \frac{1}{8N}]}(\gamma)
\]
\[
= \chi_{[0, \frac{1}{8N}]}(\gamma)\chi_{[0, \frac{1}{4N}]}(\gamma)
\]
\[
= H_0(\gamma)\hat{\psi}_0(\gamma),
\]
where \( H_0(\gamma) = \chi_{[0, \frac{1}{8N}]}(\gamma) \in L^\infty(\mathbb{R}). \)

Define
\[
H_1(\gamma) = \chi_{\mathbb{R}^{-}\frac{1}{8N}}(\gamma), \quad H_2(\gamma) = 1 \quad \text{and} \quad H_3(\gamma) = i.
\]

Choose \( H(\gamma) = \begin{bmatrix} H_0(\gamma) \\ H_1(\gamma) \\ H_2(\gamma) \\ H_3(\gamma) \end{bmatrix} \) and \( \psi_1, \psi_2, \psi_3 \in L^2(\mathbb{R}) \) such that \( \hat{\psi}(2N\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \quad \ell = 1, 2, 3. \)

Then, the collection \( \{\psi_\ell, H_\ell\}_{\ell \in 0}^3 \) is a general setup such that
\[
H(\gamma)^*H(\gamma) = |H_0(\gamma)|^2 + |H_1(\gamma)|^2 + |H_2(\gamma)|^2 + |H_3(\gamma)|^2 = 1.
\]
Hence, by Theorem 4.1, the family \( \{L^\ell T_\chi \psi_\ell\}_{\ell \in 1, 2, 3} \) is a nonuniform Parseval wavelet frame for \( L^2(\mathbb{R}) \).

The next example illustrates Theorem 4.1 in form of trigonometric nonuniform wavelet frames.

**Example 5.2.** Let \( N = 2, r = 3, \) and \( \psi_0 \in L^2(\mathbb{R}) \) be such that
\[
\hat{\psi}_0(\gamma) = \frac{\sin(\gamma)}{\gamma} \chi_{[0, 1]}(\gamma).
\]
Then
\[
(i) \lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1;
\]
Choose

\[ H(\gamma) = \begin{bmatrix} H_0(\gamma) \\ H_1(\gamma) \\ H_2(\gamma) \\ H_3(\gamma) \end{bmatrix}. \]

Then, \( \{\psi_\ell, H_\ell\}_{\ell=0}^3 \) is a general setup such that

\[ H(\gamma)^* H(\gamma) = |H_0(\gamma)|^2 + |H_1(\gamma)|^2 + |H_2(\gamma)|^2 + |H_3(\gamma)|^2 = 1. \]

Hence, by Theorem 4.1, \( \{L^jT_\lambda \psi_\ell\}_{\ell \in \mathbb{Z}, \lambda \in \{0, \frac{1}{32}\} + 2\mathbb{Z}} \) is a nonuniform Parseval wavelet frame \( L^2(\mathbb{R}) \).

To conclude the paper, we illustrate Theorem 4.2 with the following example.

**Example 5.3.** Let \( N = 2, r = 3 \) and \( \psi_0 \in L^2(\mathbb{R}) \) be such that

\[ \hat{\psi}_0(\gamma) = \cos(\gamma) \chi_{[0, \frac{1}{2}]}(\gamma). \]

Then

(i) \( \lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1; \)

(ii) \( \text{Supp} \ \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{8}]; \)

(iii) \( \psi_0(4\gamma) = \cos(4\gamma) \chi_{[0, \frac{1}{2}]}(4\gamma) \)

\[ = \cos(4\gamma) \chi_{[0, \frac{1}{2}]}(\gamma) \chi_{[0, \frac{1}{8}]}(\gamma) \]

\[ = \frac{\cos(4\gamma)}{\cos(\gamma)} \chi_{[0, \frac{1}{2}]}(\gamma) \left\{ \cos(\gamma) \chi_{[0, \frac{1}{32}]}(\gamma) \right\} \left( \text{note that } \cos(\gamma) \neq 0, \gamma \in \left[0, \frac{1}{32}\right] \right) \]

\[ = H_0(\gamma) \hat{\psi}_0(\gamma), \]

where \( H_0(\gamma) = \frac{\cos(4\gamma)}{\cos(\gamma)} \chi_{[0, \frac{1}{2}]}(\gamma) \in L^\infty(\mathbb{R}). \)

Let \( \theta(\gamma) = \sin^2(\gamma) + 1 \) and define \( H_i(\gamma) \) as follows

(i) \( H_1(\gamma) = i \left\{ \sqrt{\sin^2(4\gamma) + 1} \right\} \frac{\cos(4\gamma)}{\cos(\gamma)} \chi_{[0, \frac{1}{8}]}(\gamma). \)
(ii) $H_2(\gamma) = \sin(\gamma)$.

(iii) $H_3(\gamma) = 1$.

Then, the collection $\{\psi_\ell, H_\ell\}^3_{\ell=0}$ is a general setup such that

$$\theta(4\gamma)|H_0(\gamma)|^2 + |H_1(\gamma)|^2 + |H_2(\gamma)|^2 + |H_3(\gamma)|^2 = \theta(\gamma).$$

Hence, by Theorem 4.2, the nonuniform wavelet system $\{L^j T_\lambda \psi_\ell\}_{\ell=1,2,3}^j \in \mathbb{Z}$ is a Parseval frame for $L^2(\mathbb{R})$.

REFERENCES

[1] J. Benedetto and O. Treiber, Wavelet frames: multiresolution analysis and extension principles. In: "Wavelet transforms and time-frequency signal analysis", 1-36, Birkhäuser, Boston, 2001.

[2] A. Boggess and F. J. Narcowich, A First Course in Wavelets with Fourier Analysis, John Wiley & Sons, Inc., Hoboken, NJ, 2009.

[3] P. G. Casazza and G. Kutyniok, Finite frames: Theory and Applications, Birkhäuser, 2012.

[4] O. Christensen, H. O. Kim and R. Y. Kim, Extensions of Bessel sequences to dual pairs of frames, Appl. Comput. Harmon. Anal., 34 (2)(2013), 224–253.

[5] O. Christensen, An introduction to frames and Riesz bases, Second edition, Birkhäuser, 2016.

[6] O. Christensen and S. S. Goh, The unitary extension principle on locally compact abelian groups, Appl. Comput. Harmon. Anal., to appear. Available at http://dx.doi.org/10.1016/j.acha.2017.07.004.

[7] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[8] I. Daubechies, B. Han, A. Ron and Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal., 14 (1) (2003), 1–46.

[9] Deepshikha and L. K. Vashisht, A note on discrete frames of translates in $\mathbb{C}^N$, TWMS J. Appl. Eng. Math., 6 (1) (2016), 143–149.

[10] Deepshikha and L. K. Vashisht, Necessary and sufficient conditions for discrete wavelet frames in $\mathbb{C}^N$, J. Geom. Phys., 117 (2017), 134–143.

[11] Dao-Xin Ding, Generalized continuous frames constructed by using an iterated function system, J. Geom. Phys., 61 (2011) 1045–1050.

[12] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., 72 (1952), 341–366.

[13] J. P. Gabardo and M. Z. Nashed, Nonuniform multiresolution analysis and spectral pairs, J. Funct. Anal., 158 (1998), 209–241.

[14] J. P. Gabardo and X. Yu, Wavelets associated with nonuniform multiresolution analyses and one-dimensional spectral pairs, J. Math. Anal. Appl., 323 (2006), 798–817.

[15] B. Han, Framelets and Wavelets: Algorithms, Analysis, and Applications, Birkhäuser, 2017.

[16] E. Hernandez and G. Weiss, A First Course on Wavelets. CRC Press, Boca Raton, 1996.

[17] A. Ron and Z. Shen, Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator, J. Funct. Anal., 148 (1997), 408–447.

[18] M. B. Ruskai, G. Beylkin, R. Coifman, I. Daubechies, S. Mallat, Y. Meyer and L. Raphael, Wavelets and Their Applications, Jones and Bartlett Publishers, Boston, MA, 1992.

[19] V. Sharma and P. Manchanda, Nonuniform wavelet frames in $L^2(\mathbb{R})$, Asian-Eur. J. Math., 8 (2) (2015), 1550034, 15 pp.

[20] X. Yu and J. P. Gabardo, Nonuniform wavelets and wavelet sets related to one-dimensional spectral pairs, J. Approx. Theory, 145 (2007) (1), 133–139.

[21] L. K. Vashisht and Deepshikha, Weaving properties of generalized continuous frames generated by an iterated function system, J. Geom. Phys., 110 (2016), 282–295.

[22] R. A. Zalik, Riesz bases and multiresolution analyses, Appl. Comput. Harmon. Anal., 7 (3) (1999), 315–331.

[23] R. A. Zalik, Orthonormal wavelet systems and multiresolution analyses, J. Appl. Funct. Anal., 5 (1) (2010), 31–41.

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