Robust Ising Gates for Practical Quantum Computation

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I describe the use of techniques based on composite rotations to combat systematic errors in controlled phase gates, which form the basis of two qubit quantum logic gates. Although developed and described within the context of Nuclear Magnetic Resonance (NMR) quantum computing these sequences should be applicable to any implementation of quantum computation based on Ising couplings. In combination with existing single qubit gates this provides a universal set of robust quantum logic gates.

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Quantum computers use quantum mechanical effects to implement algorithms which are not accessible to classical computers, and thus are able to tackle otherwise intractable problems. They are extremely vulnerable to the effects of errors, and considerable effort has been expended on correcting random errors arising from decoherence processes. It is, however, also important to consider the effects of systematic errors, which arise from reproducible imperfections in the experimental apparatus used to implement quantum computations.

It makes sense to consider systematic errors as some of them can be tackled relatively easily. In the context of Nuclear Magnetic Resonance (NMR) quantum computation the concept of composite rotations (also called composite pulses) has been used to tackle both off-resonance effects and more recently pulse length errors. All these results, however, have been concerned with single qubit gates, and while these play a central role in quantum logic circuits it is also vital to consider two qubit controlled logic gates, such as the controlled-NOT gate.

The combination of a complete set of single qubit gates and the controlled-NOT gate is universal for quantum computation, that is any quantum logic circuit can be built entirely from single qubit gates and controlled-NOT gates. There are, however, other two qubit gates which can be used to form a universal set, most notably the controlled phase-shift (controlled-$\sigma_z$) gate. This performs the transformation

$$|1\rangle\langle 1| \rightarrow -|1\rangle\langle 1|$$

while leaving other eigenstates unchanged, and so is described by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
$$

(2)

Applying Hadamard gates to the target spin before and after the controlled-$\sigma_z$ gate converts it to a controlled-$\sigma_x$ gate, that is controlled-NOT.

The controlled phase-shift gate can itself be considered as a derivative of the Ising ($\sigma_z$) coupling gate

$$
\begin{pmatrix}
e^{i\phi/2} & 0 & 0 & 0 \\
0 & e^{-i\phi/2} & 0 & 0 \\
0 & 0 & e^{-i\phi/2} & 0 \\
0 & 0 & 0 & e^{i\phi/2}
\end{pmatrix}
$$

(3)

which corresponds to evolution under the Ising ($\sigma_1 \sigma_2$) Hamiltonian; Eqns. 2 and 3 are related by single qubit $z$ rotations and a global phase shift. The $z$ rotations can be adsorbed into abstract reference frames and the global phase shift has no physical significance, and thus the two equations are essentially equivalent.

The Ising coupling is of great experimental importance, as it provides the basic quantum logic gate for many proposed implementations of quantum computing. Its role is particularly clear in NMR systems, where the scalar coupling ($J$ coupling) takes the Ising form in the weak coupling limit, and from this point I will adopt the product operator notation widely used in NMR. However the underlying principles remain applicable in any system based on Ising couplings.

Consider a system of two spin-$\frac{1}{2}$ nuclei, $I$ and $S$. The Ising gate (Eq. 3) is implemented by free evolution under the $J$ coupling Hamiltonian

$$\mathcal{H}_{JS} = \pi J 2I_z S_z$$

(4)

for a time $\tau = \phi/\pi J$, where $J$ is the coupling strength and $\phi$ is the desired evolution angle. In order to implement accurate controlled phase-shift gates by this means it is clearly necessary to know $J$ with corresponding accuracy.

Obtaining accurate values of $J$ for individual spin–spin couplings is easy in small molecules, but is much more difficult in larger systems. In particular, consider an array of qubits coupled by Ising interactions, with $J$ couplings that are nominally identical but in fact differ from one another as a result of imperfections in the lattice. In such a system it is desirable to be able to perform some accurately known $\sigma_z$ evolution over a range of values of
While this might appear tricky, it is in fact easy to achieve using composite pulse techniques.

The problem of performing accurate $zz$ rotations is conceptually similar to that of correcting for pulse length errors in single qubit gates \[17\], and the solutions are closely related. I begin by describing existing techniques for tackling pulse length errors. While a wide variety of composite pulse sequences have been developed within the context of NMR experiments \[14\], most of these are not suitable for use in quantum computers as they make assumptions about the form of the initial state. Instead it is necessary to use Class A composite pulses \[14\] which are closely related. I begin by describing existing techniques for tackling pulse length errors. While a wide variety of composite pulse sequences have been developed within the context of NMR experiments \[14\], most of these are not suitable for use in quantum computers as they make assumptions about the form of the initial state. Instead it is necessary to use Class A composite pulses \[14\] which work well for any initial state. Two suitable families of composite pulses have been described: the SCROFULOUS pulse sequences of Cummins et al. \[17\], and the BB1 family, initially described by Wimperis \[15\], and subsequently developed by Cummins et al. \[17\]. As the performance of the BB1 sequence is much better than that of SCROFULOUS I will concentrate on the time symmetrized variant of BB1.

Consider a $\theta_0$ pulse, which implements a single qubit rotation around the $x$ axis on the Bloch sphere \[12\]. In an ideal system the propagator will be

$$U = \exp[-i\theta I_x]$$

but in the presence of a fractional error $\epsilon$ in the strength of the driving field the actual propagator will be

$$V = \exp[-i\theta(1+\epsilon)I_x].$$

The accuracy of this experimental implementation can be assessed using the propagator fidelity

$$F = \frac{|\text{Tr}(VU^\dagger)|}{\text{Tr}(UU^\dagger)}$$

(note that it is necessary to take the absolute value of the numerator as $U$ and $V$ could in principle differ by an irrelevant global phase shift). In this case

$$F = \cos\left(\frac{\epsilon\theta}{2}\right) \approx 1 - \frac{\epsilon^2\theta^2}{8}. \quad (8)$$

A better approach is to replace the naive implementation, Eq. 3, with the composite pulse sequence

$$(\theta/2)_0 \pi_{\phi_1}, 2\pi_{\phi_2}\pi_{\phi_1}(\theta/2)_0$$

with $\phi_2 = 3\phi_1$ and $\phi_1 = \arccos(-\theta/4\pi)$. This gives a fidelity expression

$$F \approx 1 - \frac{\epsilon^6(\theta^6 - 14\pi^2\theta^4 - 32\pi^4\theta^2)}{9216}$$

in which the second and fourth order error terms have been completely removed \[17\].

A very similar approach can be used to tackle systematic errors in Ising coupling gates; in effect Ising coupling corresponds to rotation about the $2I_zS_z$ axis, and errors in $J$ correspond to errors in a rotation angle about this axis. These can be parameterised by the fractional error in $J$:

$$\epsilon = \frac{J_{\text{real}}}{J_{\text{nominal}}} - 1. \quad (11)$$

Such errors can be overcome by rotating about a sequence of axes tilted from $2I_zS_z$ towards another axis, such as $2I_zS_y$. Defining

$$\theta_0 = \exp[-i\times(2I_zS_z\cos\phi + 2I_zS_x\sin\phi)]$$

allows the naive sequence $\theta_0$ to be replaced by Eq. 8 as before. The tilted evolutions can be achieved by sandwiching a $2I_zS_z$ rotation (that is, free evolution under the Ising Hamiltonian) between $\phi_{\pm y}$ pulses applied to spin $S_z$ \[13\]. For the case that $\theta = \pi/2$ (which forms the basis of the controlled-NOT gate) the final sequence takes the form shown in Fig. 1.

![FIG. 1: Pulse sequence for a robust Ising gate to implement a controlled-NOT gate. Boxes correspond to single qubit rotations with rotation angles of $\phi = \arccos(-1/8) \approx 97.2^\circ$ applied along the $\pm y$ axes as indicated; time periods correspond to free evolution under the Ising coupling, $\pi J 2I_zS_z$ for multiples of the time $t = 1/4J$. The naive Ising gate corresponds to free evolution for a time $2t$.](image)

Clearly it is vital that any robust implementation of a quantum gate must be built from components that can themselves be implemented robustly. The robust Ising gate uses only two components: single qubit rotations around the $\pm y$ axes, for which robust versions are already known \[17\], and periods of evolution under the Ising coupling. Note that the five time periods must have lengths in the integer ratios $1 : 4 : 8 : 4 : 1$, but it is not necessary to accurately control the absolute length of the periods, as errors in absolute lengths are equivalent to errors in the value of $J$.

The fidelity gain achieved for coupling gates by this approach is identical to that achieved for single qubit rotations. In particular for the case that $\theta = \pi/2$ the naive fidelity is

$$F \approx 1 - \frac{\epsilon^6\pi^6}{32} \quad (13)$$

while the BB1 approach gives

$$F \approx 1 - \frac{\epsilon^6\pi^6}{65536} \quad (14)$$

Fidelity plots are shown in Fig. 2. Clearly the BB1 approach compensates extremely well for small errors in
J values, especially within the range ±10%. Over this range the infidelity of the BB1 sequence is always less than one part in 10^6. To achieve comparable fidelity with a naive gate it is necessary to determine the value of J to better than 0.2%. Thus the robust gate can achieve an infidelity of 10^-6 over a range of J values more than 50 times wider than the naive gate. If even higher fidelities are desired the improvement provided by the robust gate is even greater. In combination with existing robust single qubit gates, the robust Ising gate provides a complete set of robust gates for quantum computation within the Ising model.

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[1] C. H. Bennett, and D. P. DiVincenzo, Nature (London) 404, 247 (2000).
[2] P. W. Shor, SIAM Rev. 41, 303 (1999).
[3] P. W. Shor, Phys. Rev. A 52, 2493 (1995).
[4] A. Steane, Phys. Rev. Lett. 77, 793 (1996).
[5] A. Steane Nature 399, 124 (1999).
[6] D. G. Cory, A. F. Fahmy and T. F. Havel, in Proceedings of PhysComp ‘96 (New England Complex Systems Institute, Cambridge MA, 1996).
[7] D. G. Cory, A. F. Fahmy and T. F. Havel, Proc. Nat. Acad. Sci. USA 94, 1634 (1997).
[8] N. A. Gershenfeld and I. L. Chuang, Science 275, 350 (1997).
[9] J. A. Jones and M. Mosca, J. Chem. Phys. 109, 1648.
[10] J. A. Jones, Prog. NMR. Spectrosc. 38, 325 (2001).
[11] L. M. K. Vandersypen, M. Steffen, G. Breyla, C. S. Yannoni, M. H. Sherwood, and I. L. Chuang, Nature 414, 883 (2001).
[12] R. R. Ernst, G. Bodenhausen, and A. Wokaun, Principles of Nuclear Magnetic Resonance in One and Two Dimensions (Oxford University Press, 1987).
[13] R. Freeman, Spin Choreography (Spektrum, Oxford, 1997).
[14] M. H. Levitt, Prog. NMR Spectrosc. 18, 61 (1986).
[15] S. Wimperis, J. Magn. Reson. A 109, 221 (1994).
[16] H. K. Cummins and J. A. Jones, New J. Phys. 2, 1 (2000).
[17] H. K. Cummins, G. Llewellyn and J. A. Jones, quant-ph/0208092, submitted to Phys. Rev. A.
[18] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).
[19] E. Knill, R. Laflamme, R. Martinez, and C.-H. Tseng, Nature 404, 368 (2000).
[20] S. Lloyd, Science 261, 1569 (1993).
[21] L. B. Ioffe, V. B. Geshkenbein, M. V. Feigel’man, A. L. Fauchère, and G. Blatter, Nature 398, 679 (1999).
[22] J. I. Cirac and P. Zoller, Nature 404, 579 (2000).
[23] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. 86, 910 (2001).
[24] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 6, 5188 (2001).
[25] O. W. Sørensen, G. W. Eich, M. H. Levitt, G. Bodenhausen and R. R. Ernst, Prog. NMR. Spectrosc. 16, 163 (1983).