We show that if $H$ is a hereditary finite dimensional algebra, $M$ is a finitely generated $H$-module and $B$ is a semisimple subalgebra of $\text{End}_H(M)^{op}$, then the representation dimension of $\Lambda = \begin{pmatrix} B & 0 \\ M & H \end{pmatrix}$ is less or equal to 3 whenever one of the following conditions hold: i) $H$ is of finite representation type; ii) $H$ is tame and $M$ is a direct sum of regular and preprojective modules; iii) $M$ has no self-extensions.

The representation dimension of an Artin algebra is the infimum of the global dimensions of the endomorphism algebras of generator-cogenerators of its category of finitely generated modules. It was introduced by Auslander (cf. [2]) and, his own words, it was aimed at being a measure of how far an (in this paper always Artin) algebra is from being of finite representation type. Indeed, in that same paper, Auslander proved that an algebra is of finite representation type if, and only if, its representation dimension is less or equal than 2. While the concept was essentially forgotten for almost thirty years, two breaking recent results have put it into the spotlight again. On one side, Iyama [10] proved that the representation dimension of an algebra is always finite, and, on the other, Rouquier [16] showed that all natural numbers can be attained. With these two results at hand, Artin algebras can be, at least in theory, classified numerically. In addition, representation dimension is invariant under stable
equivalences (cf. [7] and [5]) and, when restricted to self-injective algebras, invariant under derived equivalences (cf. [17]), facts that allow to construct classes of algebras of a given representation dimension from others having the same property.

It is a natural goal to discover classes of algebras of infinite representation type that, from the point of view of representation dimension, are the nearest to being of finite type, namely, those having representation dimension equal to 3. Examples of these algebras available in the literature include the hereditary [2], stably hereditary [17], special biserial [6], Schur algebras of tame representation type [8], local algebras of quaternion type [9], selfinjective algebras (socle equivalent to) weakly symmetric algebras of Euclidean type [4], tilted and laura algebras [1] and canonical algebras [12].

In this paper we consider generalizations of one-point extensions of hereditary algebras, namely, triangular algebras of the form \( \Lambda = \begin{pmatrix} B & 0 \\ M & H \end{pmatrix} \), where \( H \) is a hereditary algebra over an algebraically closed field \( K \), \( M \) is a (left) \( H \)-module and \( B \) is a semisimple subalgebra of \( \text{End}_H(M)^{op} \). We find sufficient conditions for those algebras to have representation dimension 3. Note that, due to recent results of Oppermann [13], every wild algebra admits one-point extensions of representation dimension \( \geq 4 \). So there are choices of \( H \) and \( M \) for which \( \text{rep.dim}(\Lambda) > 3 \).

The first main result of the paper, Proposition 0.1, states that \( \text{rep.dim}(\Lambda) \leq 3 \) whenever one of the following two conditions holds: i) \( H \) is of finite representation type; ii) \( H \) is tame and \( M \) is a direct sum of preprojective and regular modules. The second main result, Theorem 0.5, states that if \( H \) is of infinite representation type and \( M \) has no self-extensions, then \( \text{rep.dim}(\Lambda) = 3 \). The proof of this theorem is based on the construction of an Auslander generator \( \hat{G} \) of \( \Lambda - \text{mod} \) derived from the existence of an Auslander generator \( G \) of \( H - \text{mod} \) which contains \( M \) as a direct summand (see Proposition 0.2 and Proposition 0.4).

With notation as above, notice that if \( B = B_1 \times \ldots \times B_r \) is the decomposition of \( B \) into a direct product of simple algebras, then the central idempotents of \( B \) corresponding to that decomposition give a decomposition \( M = \oplus_{1 \leq i \leq r} M_i \), such that \( B_i \subseteq \text{End}_H(M_i) \), for every \( i = 1, \ldots, r \). Moreover, if \( B_i \cong M_{n_i}^i(K) \) then \( M_i \cong M_{n_i}^{op} \). It is clear that \( \Lambda \) is a basic algebra if, and only if, \( H \) is basic and \( n_i = 1 \) for \( i = 1, \ldots, r \). Without loss of generality, we can and shall assume in the sequel that these two conditions hold. Notice that, even with that restriction, the \( M_i \) need not be indecomposable. Notice also that if the chosen decomposition of \( M \) is the trivial one (i.e. \( r = 1 \) and \( M = M_1 = \hat{M}_1 \) above), then \( \Lambda \) is just the one-point extension \( \begin{pmatrix} K & 0 \\ M & H \end{pmatrix} \).

It is well-known (cf. [ARS]) that every \( \Lambda \)-module is then identified by a triple \((V, X, f)\) consisting of a \( B \)-module \( V \), an \( H \)-module \( X \) and a homomorphism of \( H \)-modules \( f : M \otimes_B V \rightarrow X \). Implicitly assuming \( f \), we shall write \( \Lambda \)-modules as 2-entry columns \((V, X)\) and the multiplications by elements of \( \Lambda \) will be just
left matrix multiplication. In that case the full subcategory of $\Lambda - \text{mod}$ formed by the objects of the form $\begin{pmatrix} 0 \\ X \end{pmatrix}$ is canonically identified with the category $H - \text{mod}$.

Note that if $f^t : V \rightarrow Hom_H(M, X)$ denotes the transpose of $f$, which is a homomorphism of $B$-modules, then due to the semisimplicity of $B$ we have a decomposition $\begin{pmatrix} V \\ X \end{pmatrix} \cong \begin{pmatrix} Ker(f^t) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Im(f^t) \\ X \end{pmatrix}$. That will allows us to reduce many arguments to the case in which $V \subseteq Hom_H(M, X)$ is a $B$-submodule and the map $M \otimes_B V \rightarrow X$ is the canonical one: $m \otimes v \mapsto v(m)$.

**Proposition 0.1.** Let $\Lambda = \begin{pmatrix} B & 0 \\ M & H \end{pmatrix}$ be as above. If $H$ is of finite representation type or if $H$ is tame and $M$ is a direct sum of regular and preprojective $H$-modules, then there only finitely many indecomposable torsionless $\Lambda$-modules up to isomorphism. In particular

$$\text{rep.dim}(\Lambda) \leq 3$$

**Proof.** The final assertion is a consequence of the first due to a recent result of Ringel ([15]). As for the first sentence, notice that the indecomposable projective $\Lambda$-modules are the projective $H$-modules plus the modules $\begin{pmatrix} Kp_i \\ M_i \end{pmatrix}$, where $p_i : M \rightarrow M_i$ is the $i$-th projection associated to the given decomposition of $M$.

Since the radical of $\begin{pmatrix} Kp_i \\ M_i \end{pmatrix}$ is $\begin{pmatrix} 0 \\ M_i \end{pmatrix}$ the indecomposable torsionless $\Lambda$-modules are the projective ones plus all the indecomposable $H$-modules in $\text{Sub}(M)$. Therefore the case in which $H$ is of finite representation type is obvious.

We assume in the sequel that $H$ is tame and $M$ admits a decomposicion $M = X \oplus R$ as a direct sum of a preprojective $H$-module $X$ and a regular $H$-module $R$. If $\begin{pmatrix} u \\ v \end{pmatrix} : Z \rightarrow X^n \oplus R^n = M^n$ is an monomorphism from the indecomposable $H$-module $Z$, then either $u \neq 0$, in which case $Z$ is a (preprojective) predecessor of some of the indecomposable summands of $X$ or, else, $v$ is a monomorphism so that $Z \in \text{Sub}(R)$. By the well-known structure of the subcategory of regular $H$-modules, the number of regular indecomposable $H$-modules in $\text{Sub}(R)$ is finite. So the problem is reduced to prove that if $R$ is any regular $H$-module, then $\text{Sub}(R)$ contains only finitely many preprojective indecomposable $H$-module. For that there is no loss of generality in assuming that $R$ is multiplicity-free and, by adding some regular indecomposable summands if necessary, also that $\tau_H R = R$. Notice that if $f : Z \rightarrow R^m$ is an (indecomposable) monomorphism, where $Z$ is a preprojective nonprojective indecomposable, then $\tau_H(f) : \tau_H Z \rightarrow \tau_H(R)^m \cong R^m$ is also an (indecomposable) monomorphism (cf. [Kerner, Lemma 2.2]). In particular, given any preprojective indecomposable $H$-module $Z$, the set of natural numbers $S_Z := \{n \geq 0 : \tau^{-n}Z \in \text{Sub}(R)\}$ is closed under predecessors (i.e. $n \in S_Z$ implies $n - 1 \in S_Z$). If there were infinitely many indecomposable preprojective modules $Z$ in $\text{Sub}(R)$
we would conclude that there is a projective indecomposable $H$-module $P$ such that $\tau^{-n}P \in \text{Sub}(R)$ for all $n \geq 0$.

Let us assume that such a $P$ exists. We then denote by $\varphi(n)$ the largest of the positive integers $r$ such that there is a monomorphism $\tau^{-n}P \rightarrow R^r$ which is an indecomposable map. The argument in the above paragraph shows that $\varphi(n-1) \geq \varphi(n)$. As a consequence the map $\varphi : \mathbf{N} \rightarrow \mathbf{N}$ is eventually constant, so that we have a natural number $q$ such that $\varphi(n) = q$ for $n \gg 0$. But then $\dim(\tau^{-n}P) \leq q \cdot \dim(R)$ for all $n \gg 0$. That implies that there are only finitely many dimension vectors of modules in the $\tau$-orbit of $P$. This is known to be false for preprojective indecomposable modules are identified by their dimension vectors (cf. [14]).

We now give an auxiliary result which is valid for every hereditary algebra $H$.

**Proposition 0.2.** Suppose that, in our situation, the $H$-module $M$ has no selfextensions and that we have found an Auslander generator of $H - \text{mod}$ containing $M$ as a direct summand. Then the $\Lambda$-module $\hat{G} = \bigoplus \bigoplus (B_{M} \oplus \text{DA})$ satisfies that

$$
\text{gl.dim}(\text{End}_\Lambda(\hat{G})) \leq 3.
$$

**Proof.** Since the simple modules $\left( \begin{smallmatrix} Kp_i \\ 0 \end{smallmatrix} \right)$ are injective, whence belong to $\text{Add}(\hat{G})$, without loss of generality, we can deal only with $\Lambda$-modules of the form $\left( \begin{smallmatrix} V \\ X \end{smallmatrix} \right)$, with $V$ a $B$-submodule of $\text{Hom}_H(M, X)$. In that case, we claim that if $\left( \begin{smallmatrix} V \\ X \end{smallmatrix} \right)$ is indecomposable and the canonical map $f : M \otimes_B V \rightarrow X$ is surjective, then $\left[ \left( \begin{smallmatrix} V \\ X \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 \\ G \end{smallmatrix} \right) \right] = 0$ and $\text{rad}\left( \left( \begin{smallmatrix} V \\ X \end{smallmatrix} \right), \left( \begin{smallmatrix} B \\ M \end{smallmatrix} \right) \right] = 0$, where $[-,-]$ denotes $\text{Hom}_\Lambda(-,-)$.

Indeed in the first case a morphism $\left( \begin{smallmatrix} V \\ X \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} 0 \\ G \end{smallmatrix} \right)$ is identified by a morphism $u : X \rightarrow G$ such that $u \circ v = 0$, for all $v \in V$. But then $u \circ f = 0$ and so $u = 0$. In the second case we consider the initial decomposition $M = M_1 \oplus \ldots \oplus M_r$ and the associated projections $p_i : M \rightarrow M_i$, so that $\left( \begin{smallmatrix} B \\ M \end{smallmatrix} \right) = \bigoplus_{1 \leq i \leq r} \left( \begin{smallmatrix} Kp_i \\ M_i \end{smallmatrix} \right)$ is the decomposition of $\left( \begin{smallmatrix} B \\ M \end{smallmatrix} \right)$ into a direct sum of (projective) indecomposable $\Lambda$-modules. By the above argument, a nonzero morphism $\psi : \left( \begin{smallmatrix} V \\ X \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} Kp_i \\ M_i \end{smallmatrix} \right)$ cannot have image contained in $\text{rad}\left( \left( \begin{smallmatrix} Kp_i \\ M_i \end{smallmatrix} \right) \right) = \left( \begin{smallmatrix} 0 \\ M_i \end{smallmatrix} \right)$ because $\left( \begin{smallmatrix} 0 \\ M_i \end{smallmatrix} \right)$ is a direct summand of $\left( \begin{smallmatrix} 0 \\ G \end{smallmatrix} \right)$. Therefore $\psi$ is a (split) epimorphism.
We need to prove that the projective dimension of \[\begin{pmatrix} V \\ X \end{pmatrix}, \hat{G}\] as an \(\text{End}_\Lambda(\hat{G})\)-module is \(\leq 1\), for all indecomposable \(\Lambda\)-modules \(\begin{pmatrix} V \\ X \end{pmatrix}\). By the above paragraph, if \(f : M \otimes_B V \rightarrow X\) is surjective and \(\begin{pmatrix} V \\ X \end{pmatrix} \notin \text{Add}(\begin{pmatrix} B \\ M \end{pmatrix})\), then the only indecomposable summands of \(\hat{G}\) on which \(\begin{pmatrix} V \\ X \end{pmatrix}, -\) does not vanish are the injectives.

Since \(X \in \text{Fac}(M)\) and \(M\) has no self-extensions, we get that \(\text{Ext}^1_H(M, X) = 0\) and then the minimal injective resolution of \(\begin{pmatrix} V \\ X \end{pmatrix}\) is of the form

\[
0 \rightarrow \begin{pmatrix} V \\ X \end{pmatrix} \rightarrow \left(\frac{\text{Hom}_H(M, E(X))}{E(X)}\right) \rightarrow \left(\frac{W}{0}\right) \oplus \left(\frac{\text{Hom}_H(M, \Omega^{-1}X)}{\Omega^{-1}X}\right) \rightarrow 0,
\]

for some \(B\)-module \(W\), and is kept exact by the functor \([- , \hat{G}]\). Then \(\text{pd}(\begin{pmatrix} V \\ X \end{pmatrix}, \hat{G}) \leq 1\) in this case.

We next consider the case in which \(V = 0\), i.e., \(\begin{pmatrix} V \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ X \end{pmatrix}\) is an \(H\)-module. Since \(G\) is an Auslander generator of \(H - \text{mod}\), we have an exact sequence \(0 \rightarrow X \rightarrow G_0 \rightarrow G_1 \rightarrow 0\) which is kept exact when applying \(\text{Hom}_H(-, \hat{G})\). Then we also get an exact sequence of \(\text{End}_\Lambda(\hat{G})\)-modules

\[
0 \rightarrow \left[\begin{pmatrix} 0 \\ G_1 \end{pmatrix}, \hat{G}\right] \rightarrow \left[\begin{pmatrix} 0 \\ G_0 \end{pmatrix}, \hat{G}\right] \rightarrow \left[\begin{pmatrix} 0 \\ X \end{pmatrix}, \hat{G}\right] \rightarrow 0,
\]

thus showing that \(\text{pd}(\begin{pmatrix} 0 \\ X \end{pmatrix}, \hat{G}) \leq 1\).

Finally, we consider an arbitrary indecomposable \(\Lambda\)-module \(\begin{pmatrix} V \\ X \end{pmatrix}\). Then we have an exact sequence

\[
0 \rightarrow \left(\frac{V}{\text{Im}(f)}\right) \rightarrow \begin{pmatrix} V \\ X \end{pmatrix} \xrightarrow{j} \begin{pmatrix} 0 \\ \text{Coker}(f) \end{pmatrix} \rightarrow 0.
\]

If now \(Z\) is any indecomposable summand of \(\hat{G}\), then, by the first paragraph of this proof, the map \([j, Z]\) is surjective except in case \(Z \cong \begin{pmatrix} Kp_i \\ M_i \end{pmatrix}\), for some \(i = 1, \ldots, r\). But that means that all composition factors of the \(\text{End}_\Lambda(\hat{G})\)-module \(\text{Coker}[j, \hat{G}]\) are of the form \(\Sigma_i =: \begin{pmatrix} Kp_i \\ M_i \end{pmatrix}/\text{rad}(\begin{pmatrix} Kp_i \\ M_i \end{pmatrix}, \hat{G})\), with \(i = 1, \ldots, r\). Suppose we prove that \(\text{pd}(\Sigma_i) \leq 2\) for all \(i = 1, \ldots, r\). Then we consider the exact sequence

\[
0 \rightarrow \text{Im}[j, \hat{G}] \rightarrow \left[\begin{pmatrix} V \\ \text{Im}(f) \end{pmatrix}, \hat{G}\right] \rightarrow \text{Coker}[j, \hat{G}] \rightarrow 0.
\]
By the above paragraphs of this proof, we know that its central term has projective dimension \( \leq 1 \) and, hence, we also have \( \text{pd}(\text{Im}[j, \hat{G}]) \leq 1 \). But then the outer nontrivial terms in the sequence

\[
0 \to \left[ \left( \begin{array}{c} 0 \\ Coker(f) \end{array} \right), \hat{G} \right] \to \left[ \left( \begin{array}{c} V \\ X \end{array} \right), \hat{G} \right] \to \text{Im}[j, \hat{G}] \to 0
\]

have projective dimension \( \leq 1 \), so that \( \text{pd}(\left[ \left( \begin{array}{c} V \\ X \end{array} \right), \hat{G} \right]) \leq 1 \) and the proof would be finished.

It remains to prove that \( \text{pd}(\Sigma_i) \leq 2 \) for all \( i = 1, ..., r \). Since the canonical map \( M \otimes_B Kp_i \to M_i \) is surjective, by the first paragraph of this proof, we know that if \( Z \) is an indecomposable summand of \( \hat{G} \) such that \( \text{rad}(\left[ \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right), Z \right]) \neq 0 \), then \( Z \) is injective. We then consider the injective envelope \( u : \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right) \hookrightarrow \left( \text{Hom}_H(M, E(M_i)) \right) / E(M_i) \) in \( H \text{-mod} \). Then the image of the map \( [u, \hat{G}] : \left[ \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right), \hat{G} \right] \to \left[ \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right), \hat{G} \right] \) is precisely \( \text{rad}(\left[ \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right), \hat{G} \right]) \), and therefore its cokernel is \( \Sigma_i \). Note that, due to the fact that \( \text{Ext}_H^1(M, M_i) = 0 \), the minimal injective resolution of \( \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right) \) is

\[
0 \to \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right) \hookrightarrow \left( \text{Hom}_H(M, E(M_i)) \right) / E(M_i) \to \left( W \right) \oplus \left( \text{Hom}_H(M, \Omega^{-1}M_i) \right) / \Omega^{-1}M_i \to 0,
\]

where \( W \) is a \( B \)-submodule of \( \text{Hom}_H(M, M_i) \) complementary of \( Kp_i \). We then get as projective resolution of \( \Sigma_i \):

\[
0 \to \left[ \left( \begin{array}{c} W \\ 0 \end{array} \right) \oplus \left( \text{Hom}_H(M, \Omega^{-1}M_i) \right) / \Omega^{-1}M_i, \hat{G} \right] \to \left[ \left( \text{Hom}_H(M, E(M_i)) \right) / E(M_i), \hat{G} \right] \to \left[ \left( \begin{array}{c} Kp_i \\ M_i \end{array} \right), \hat{G} \right] \to \Sigma_i \to 0,
\]

which shows that \( \text{pd}(\Sigma_i) \leq 2 \).

\[ \square \]

Lemma 0.3. Let \( M \) be an \( H \)-module such that \( \text{Ext}_H^1(M, M) = 0 \). The following assertions are equivalent for an indecomposable module \( U \):

1. \( U \) belongs to \( \text{Sub}(M) \cap \text{KerExt}_H^1(M, -) \)

2. \( U \) is either a direct summand of \( M \) or a direct summand of \( \text{Ker}(f) \), for some minimal right add(\( M \))-approximation \( f : M' \to X \).
Moreover, up to isomorphism, there only finitely many indecomposable modules in $\text{Sub}(M) \cap \text{KerExt}_H^1(M,-)$.

Proof. 1) $\implies$ 2) Suppose that $U$ is not a direct summand of $M$ and let $u : U \to M'$ be the minimal left $\text{add}(M)$-approximation, then the induced map $\text{Hom}_H(M,M') \to \text{Hom}_H(M,\text{Coker}(u))$ is surjective due to the fact that $\text{Ext}_H^1(M,U) = 0$. That means that the cokernel map $p : M' \to \text{Coker}(u)$ is a right $\text{add}(M)$-approximation. But $p$ is right minimal since $u$ is left minimal. Therefore $U$ is the kernel of a minimal right $\text{add}(M)$-approximation.

2) $\implies$ 1) If $U$ is a direct summand of $M$ there is nothing to prove, so we assume that $U$ is not so. Let $f : M' \to X$ be the minimal right $\text{add}(M)$-approximation of an indecomposable module $X$ such that $U$ is a direct summand of $\text{Ker}(f)$. Without loss of generality, we can assume that $X \in \text{Fac}(M) \setminus \text{add}(M)$. Since the map $f : \text{Hom}_H(M,M') \to \text{Hom}_H(M,X)$ is surjective and $\text{Ext}_H^1(M,M') = 0$, we conclude that $\text{Ext}_H^1(M,\text{Ker}(f)) = 0$ and hence $U \in \text{Sub}(M) \cap \text{KerExt}_H^1(M,-)$.

Let $\{U_1,\ldots,U_r\}$ be any finite set of nonisomorphic indecomposable modules in $\text{Sub}(M) \cap \text{KerExt}_H^1(M)$ including the direct summands of $M$. Then we put $U = \oplus_{i=1}^r U_i$ and claim that $\text{Ext}_H^1(U,U) = 0$. Indeed there exists a monomorphism $U \to M^\ast$, for some $s > 0$, which yields an epimorphism $0 \to \text{Ext}_H^1(M^\ast,U) \to \text{Ext}_H^1(U,U)$. As a consequence $U$ is a partial tilting module, and hence $r \leq n$, where $n$ is the number of simple $H$-modules.

Our last auxiliary proposition leads directly to the main result.

**Proposition 0.4.** Suppose that $\text{Ext}_H^1(M,M) = 0$. Then there exists an Auslander generator of $H-\text{mod}$ containing $M$ as a direct summand.

Proof. Our goal is to construct an Auslander generator $G$ of $H$-module containing $M$ as a direct summand. Let $\{V_1,\ldots,V_m\}$ be the finite set of indecomposable modules in $\text{Sub}(M) \cap \text{KerExt}_H^1(M,-)$ which are not in $\text{add}(M)$. We put $V = \oplus_{i=1}^m V_i$ and shall prove that $G := H \oplus V \oplus M \oplus DH$ is an Auslander generator of $H-\text{mod}$.

We need to show that if $X \notin \text{add}(G)$ is an indecomposable $H$-module, then there is a right $\text{add}(G)$-approximation $G_X \to X$ whose kernel is in $\text{add}(G)$. Since $X$ is not injective and $H$ is hereditary, every right $\text{add}(H \oplus V \oplus M)$-approximation is already an $\text{add}(G)$-approximation. We put $T =: \text{tr}_M(X) = \sum_{f \in \text{Hom}_H(M,X)} \text{Im}(f)$. Then the minimal right $\text{add}(M)$-approximation $p : M' \to X$ has $\text{Im}(p) = T$ and we get two induced exact sequences:

\[
0 \to U \to M' \xrightarrow{\tilde{p}} T \to 0
\]

\[
0 \to T \to X \xrightarrow{q} X/T \to 0.
\]

By Lemma 0.3, we know that $U \in \text{add}(M \oplus V)$. In the proof of that lemma we have shown that $M \oplus V$ is a partial tilting module, which implies also that $\text{Ext}_H^1(M \oplus V,-)$ vanishes over all modules in $\text{Fac}(M)$. In particular, we get
that $\text{Hom}_H(V, -)$ keeps exact both sequences (*). Keeping exact the first one means that every morphism $V \to T$ factors through $\tilde{p}$, while keeping exact the second one implies that we can choose a morphism $g : V' \to X$ such that the composition $V' \xrightarrow{g} X \xrightarrow{q} X/T$ is the minimal right $\text{add}(V)$-approximation of $X/T$.

We claim that $(g \ p) : V' \oplus M' \to X$ is a right $\text{add}(V \oplus M)$-approximation. Clearly, every morphism $M \to X$ factors through $(g \ p)$, so we only need to see that the same is true for every morphism $u : V \to X$. By definition of $g$, we have that $q \circ u$ factors through $q \circ g : V' \to X/T$ so that there is a $v : V \to V'$ such that $q \circ u = q \circ g \circ v$. Then $u - g \circ v$ factors through $\text{Ker}(q) = T$ and we have a morphism $w : V \to T$ such that $j \circ w = u - g \circ v$, where $j : T \to X$ is the inclusion. From the previous paragraph we get that $w$ factors through $\tilde{p}$ and so there is a morphism $h : V \to M'$ such that $w = \tilde{p} \circ h$ and then $u = p \circ h + g \circ v = (g \ p) \circ \begin{pmatrix} \tilde{v} \\ h \end{pmatrix}$. This settles our claim.

We next look at $Z =: \text{Ker}(g \ p)$. By explicit construction of the pullback of $g$ and $p$, we see that $Z$ fits into an exact sequence

$$0 \to U \to Z \to \text{Ker}(q \circ g) \to 0,$$

and we already know that $U \in \text{add}(M \oplus V)$. On the other hand, the exact sequence

$$0 \to \text{Hom}_H(M, T) \cong \text{Hom}_H(M, X) \to \text{Hom}_H(M, X/T) \to \text{Ext}_H^1(M, T) = 0$$

gives that $\text{Hom}_H(M, X/T) = 0$ and so $\text{Hom}_H(M, \text{Im}(q \circ g)) = 0$. We then get an exact sequence

$$0 = \text{Hom}_H(M, \text{Im}(q \circ g)) \to \text{Ext}_H^1(M, \text{Ker}(q \circ g)) \to \text{Ext}_H^1(M, V') = 0,$$

which shows that $\text{Ker}(q \circ f) \in \text{Sub}(M) \cap \text{Ker}\text{Ext}_H^1(M, -) = \text{add}(M \oplus V)$. But then the sequence (**) splits, because $\text{Ext}_H^1(M \oplus V, M \oplus V) = 0$. Therefore $Z = \text{Coker}(g \ p) \in \text{add}(M \oplus V)$.

In order to complete the desired right $\text{add}(G)$-approximation of $X$ we only need to consider a morphism $t : Q \to X$ such that the composition $Q \xrightarrow{t} X \xrightarrow{\tilde{p}} \text{Coker}(g \ p)$ is a projective cover. It is straightforward to see that the map

$$(g \ p \ t) : V' \oplus M' \oplus Q \to X$$

is a right $\text{add}(G)$-approximation and, by explicit construction of the pullback of $(g \ p)$ and $t$, we readily see that

$$\text{Ker}(g \ p \ t) \cong \text{Ker}(g \ p) \oplus \Omega^1(\text{Coker}(g \ p)),$$

which belongs to $\text{add}(G)$ because $\Omega^1(\text{Coker}(g \ p))$ is a projective $H$-module. \qed
As a straightforward consequence of the two propositions, we derive the main result of the paper.

**Theorem 0.5.** Let $H$ be a hereditary algebra of infinite representation type, $M$ be a left $H$-module such that $\text{Ext}_H^1(M, M) = 0$ and $B$ be a semisimple subalgebra of $\text{End}_H(M)^{\text{op}}$. Then $\Lambda = \begin{pmatrix} B & 0 \\ M & H \end{pmatrix}$ has representation dimension equal to 3.

**Proof.** As mentioned at the beginning of the paper, there is no loss of generality in assuming that $H$ is basic and $B \cong K \times \ldots \times K$ is the semisimple subalgebra of $\text{End}_H(M)^{\text{op}}$ associated to a fixed decomposition $M = \oplus _{1 \leq i \leq r} M_i$. Then from Proposition 0.4 we know that there is an Auslander generator $G$ of $H - \text{mod}$ containing $M$ as a direct summand. Finally Proposition 0.2 gives a generator-cogenerator $\hat{G}$ of $\Lambda - \text{mod}$ such that $\text{gl.dim(End}_\Lambda(\hat{G})) \leq 3$. Since $\Lambda$ is of infinite representation type, we conclude that $\text{rep.dim(}\Lambda) = 3$. $\square$

**References**

[1] ASSEM, I.; PLATZECK, M.I.; TREPODE, S.: On the representation dimension of tilted and laura algebras. J. Algebra 296 (2006), 426-439.

[2] AUSLANDER, M.: Representation dimension of Artin algebras. Queen Mary College Maths. Notes. London (1971).

[3] AUSLANDER, M.; REITEN, I.; SMALØ, S.O.: "Representation theory of Artin algebras". Cambridge Studies in Adv. Maths. 36. Cambridge Univ. Press (1995).

[4] BOCIAN, R.; HOLM, T.; SKOWRONSKI, A.: The representation dimension of domestic weakly symmetric algebras. Cent. Eur. J. Math. 2(1) (2004), 67-75.

[5] DUGAS, A.: Representation dimension as a relative homological invariant of stable equivalence. Algebras and Repres. Theory 10(3) (2007), 223-240.

[6] ERDMANN, K.; HOLM, T.; IYAMA, O.; SCHRÖER, J.: Radical embeddings and representation dimension. Adv. Maths. 185(1) (2004), 159-177.

[7] GUO, X.: Representation dimension: an invariant under stable equivalence. Trans. Amer. Math. Soc. 357(8), 3255-3263.

[8] HOLM, T.: The representation dimension of Schur algebras: the tame case. Quart. J. Math. 55 (2004), 477-490.

[9] HOLM, T.: Representation dimension of some tame blocks of finite groups. Algebra Colloq. 10 (2003), 275-284.
[10] IYAMA, O.: Finiteness of representation dimension. Proc. Amer. Math. Soc. 131(4) (2003), 1011-1014.

[11] KERNER, O.: Representations of wild quivers. On "Representation theory and related topics" (México 1994). CMS Conference Proceedings 19. Amer. Math. Soc. (1996).

[12] OPPELMANN, S.: Private communication.

[13] OPPELMANN, S.: Wild algebras have one-point extensions of representation dimension at least four. Preprint (available at http://www.math.ntnu.no~opperman/).

[14] RINGEL, C.M.: "Tame algebras and integral quadratic forms". Lect. Notes Maths. 1099. Springer-Verlag (1984).

[15] RINGEL, C.M.: An introduction to the representation dimension of Artin algebras. Oberwolfach Lecture (available at http://www.math.uni-bielefeld.de~ringel/lectures.html).

[16] ROUQUIER, R.: Representation dimension of exterior algebras. Invent. Math. 165(2) (2006), 357-367.

[17] XI, C.: Representation dimension and quasi-hereditary algebras. Adv. Maths. 168 (2002), 193-212.