CHARACTERIZATION OF SCHATTEN CLASS HANKEL OPERATORS ON WEIGHTED BERGMAN SPACES

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Abstract. We completely characterize the simultaneous membership in the Schatten ideals $S_p$, $0 < p < \infty$ of the Hankel operators $H_f$ and $H_{\overline{f}}$ on the Bergman space, in terms of the behaviour of a local mean oscillation function, proving a conjecture of Kehe Zhu from 1991.

1. Main results

Problem: Describe the simultaneous membership in the Schatten ideals $S_p$ of the Hankel operators $H_f$ and $H_{\overline{f}}$ acting on weighted Bergman spaces.

The answer given below is the main result of the paper.

Theorem 1. Let $f \in L^2(\mathbb{B}_n, dv_\alpha)$ and $0 < p < \infty$. The following are equivalent:

(a) $H_f$ and $H_{\overline{f}}$ are in $S_p(A_n^2, L^2(\mathbb{B}_n, dv_\alpha))$.
(b) $MO_r(f) \in L^p(\mathbb{B}_n, d\lambda_n)$ for some (any) $r > 0$.

Here

$$d\lambda_n(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}$$

is the Möbius invariant volume measure on $\mathbb{B}_n$, and $MO_r(f)$ is a certain type of local mean oscillation function to be defined next after we discuss briefly the history of the problem.

When $f$ is holomorphic on $\mathbb{B}_n$ one has $H_f = 0$, and the membership of $H_{\overline{f}}$ in $S_p$ is described by $f$ being in the analytic Besov space $B_p$ if $p > \gamma_n$, and $f$ constant if $0 < p \leq \gamma_n$ [1, 2, 5, 11, 15, 16]. The cut-off point is $\gamma_n = 1$ if $n = 1$, and $\gamma_n = 2n$ if $n \geq 2$. The equivalence between (a) and (b) was conjectured (at least for $p \geq 1$) in 1991 by K. Zhu in [16]. It was previously known that, if $p > \frac{2n}{n+1+\alpha}$, then (a) is equivalent to

(c) $MO_\alpha(f) \in L^p(\mathbb{B}_n, d\lambda_n)$,
where $MO_\alpha(f)$ is a “global” mean oscillation type function. The equivalence between (a) and (c) for $p > \frac{2n}{n+1+\alpha}$ was proved in several steps: K. Zhu [16] proved the case $p \geq 2$; J. Xia [12, 13] obtained the case $\max(1, \frac{2n}{n+1+\alpha}) < p \leq 2$, and the last case $\frac{2n}{n+1+\alpha} < p \leq 1$ has been proved recently by J. Isralowitz [4]. It is also well known that condition (c) can not characterize the membership on the Schatten ideals on the missing range $0 < p \leq 2n/(n+1+\alpha)$, since on this range, condition (c) implies $f$ is a constant (see [19, p.233]). Now we recall the concepts and definitions.

We denote by $\mathbb{B}_n$ the open unit ball of $\mathbb{C}^n$, and let $dv$ be the usual Lebesgue volume measure on $\mathbb{B}_n$, normalized so that the volume of $\mathbb{B}_n$ is one. We fix a real parameter $\alpha$ with $\alpha > -1$ and write $dv_\alpha(z) = c_\alpha (1-|z|^2)^\alpha dv(z)$, where $c_\alpha$ is a positive constant chosen so that $v_\alpha(\mathbb{B}_n) = 1$. The weighted Bergman space $A^2_\alpha := A^2_\alpha(\mathbb{B}_n)$ is the closed subspace of $L^2_\alpha := L^2(\mathbb{B}_n, dv_\alpha)$ consisting of holomorphic functions. It is a Hilbert space with inner product
\[
\langle f, g \rangle_\alpha = \int_{\mathbb{B}_n} f(z) \overline{g(z)} dv_\alpha(z).
\]

The corresponding norm is denoted by $\|f\|_\alpha$. The orthogonal (Bergman) projection $P_\alpha : L^2(\mathbb{B}_n, dv_\alpha) \to A^2_\alpha(\mathbb{B}_n)$ is an integral operator given by
\[
P_\alpha f(z) = \int_{\mathbb{B}_n} \frac{f(w) dv_\alpha(w)}{(1-\langle z, w \rangle)^{n+1+\alpha}}, \quad f \in L^2(\mathbb{B}_n, dv_\alpha).
\]

Given a function $f \in L^2(\mathbb{B}_n, dv_\alpha)$, the Hankel operator $H_f$ with symbol $f$ is
\[
H_f = (I - P_\alpha)M_f,
\]
where $M_f$ denotes the operator of multiplication by $f$. It is well known that the simultaneous study of the Hankel operators $H_f$ and $H_f$ is equivalent to the study of the commutator $[M_f, P_\alpha] := M_f P_\alpha - P_\alpha M_f$ acting on $L^2_\alpha$, by virtue of the identity
\[
[M_f, P_\alpha] = \widetilde{H_f} - (\widetilde{H_f})^*,
\]
where $\widetilde{H_f} := H_f P_\alpha$ acts now on $L^2_\alpha$.

Let $H$ and $K$ be separable Hilbert spaces, and let $0 < p < \infty$. A compact operator $T$ from $H$ to $K$ is said to belong to the Schatten class $S_p = S_p(H, K)$ if its sequence of singular numbers belongs to the sequence space $\ell^p$ (the singular numbers are the square roots of the eigenvalues of the positive operator $T^*T$, where $T^*$ is the Hilbert adjoint of $T$). For $p \geq 1$, the class $S_p$ is a Banach space with the norm $\|T\|_p = (\sum_n |\lambda_n|^p)^{1/p}$, while for $0 < p < 1$ one has [4] Theorem 2.8] the inequality $\|S + T\|_p^p \leq \|S\|_p^p + \|T\|_p^p$. Also, if $A$ is a bounded operator on $H$, $B$ a bounded operator on $K$, and $T$ is in $S_p$, then $BTA$ is in $S_p$. We refer to [19, Chapter 1] for a brief account on Schatten classes.

For $z \in \mathbb{B}_n$ and $r > 0$, the Bergman metric ball at $z$ is given by $D(z,r) = \{ w \in \mathbb{B}_n : \beta(z,w) < r \}$, where $\beta(z,w)$ denotes the hyperbolic distance between $z$ and $w$ induced by
It is well known \cite{16, 19} that the simultaneous boundedness and compactness of the Hankel operators $H_f$ and $H_g$ acting on the Bergman space $A^2_\alpha$ can be characterized in terms of the properties of the function $MO_\alpha(f)$. The Hankel operators $H_f$ and $H_g$ are both bounded if and only if $MO_\alpha(f) \in L^\infty(\mathbb{B}_n)$, and compact if and only if $MO_\alpha(f) \in C_0(\mathbb{B}_n)$. The same characterization holds using a more “global” oscillation function that we introduce next.

For any $f \in L^2(\mathbb{B}_n, dv_\alpha)$ and $z \in \mathbb{B}_n$, let

$$MO_\alpha(f)(z) = \left[ B_\alpha(\|f\|^2)(z) - |B_\alpha(f)(z)|^2 \right]^{1/2},$$

where $B_\alpha(g)$ denotes the Berezin transform of a function $g \in L^1(\mathbb{B}_n, dv_\alpha)$ defined as

$$B_\alpha(g)(z) = \langle g k_z, k_z \rangle_\alpha,$$

where $k_z$ are the normalized reproducing kernels of $A^2_\alpha$, that is, $k_z = K_z/\|K_z\|_\alpha$ with $K_z$ being the reproducing kernel of $A^2_\alpha$ at the point $z$, given by

$$K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1+\alpha}}, \quad w \in \mathbb{B}_n.$$

In order to prove Theorem \cite{1}, we must introduce a more general Berezin type transform $B_{\alpha,t}f$, and a more general “mean oscillation” function $MO_{\alpha,t}(f)$. For $\alpha > -1$ and $t \geq 0$, let

$$(1.1) \quad K_z^t(w) = R^{\alpha,t}K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1+\alpha+t}}.$$  

We also denote by $h_z^t$ to be its normalized function, that is, $h_z^t = K_z^t/\|K_z^t\|_\alpha$. Because $\|K_z^t\|_\alpha \approx (1 - |z|^2)^{-\frac{t}{2(n+1+\alpha+2t)}}$, we have

$$|h_z^t(w)| \approx \frac{(1 - |z|^2)^{\frac{t}{2(n+1+\alpha+2t)}}}{(1 - \langle w, z \rangle)^{n+1+\alpha+t}}.$$

If $g \in L^1(\mathbb{B}_n, dv_\alpha)$, the Berezin type transform $B_{\alpha,t}(g)$ is defined as

$$B_{\alpha,t}(g)(z) = \langle g h_z^t, h_z^t \rangle_\alpha.$$  

For $f \in L^2(\mathbb{B}_n, dv_\alpha)$, we also set

$$MO_{\alpha,t}(f)(z) = \left( B_{\alpha,t}(|f|^2)(z) - |B_{\alpha,t}(f)(z)|^2 \right)^{1/2}.$$  

It is easy to see that

$$MO_{\alpha,t}(f)(z) = \|fh_z^t - B_{\alpha,t}(f)(z)h_z^t\|_\alpha.$$

the Bergman metric. If $f$ is locally square integrable with respect to the volume measure on $\mathbb{B}_n$, the *mean oscillation* of $f$ at the point $z \in \mathbb{B}_n$ in the Bergman metric is

$$MO_r(f)(z) = \left[ \frac{1}{v_\alpha(D(z, r))} \int_{D(z, r)} |f(w) - \tilde{f}_r(z)|^2 dv_\alpha(w) \right]^{1/2},$$

where the averaging function $\tilde{f}_r$ is given by

$$\tilde{f}_r(z) = \frac{1}{v_\alpha(D(z, r))} \int_{D(z, r)} f(w) dv_\alpha(w).$$
and that one has also the following double integral expression

\[ MO_{\alpha, t}(f)(z)^2 = \int_{B_n} \int_{B_n} |f(u) - f(w)|^2 |h_z^t(u)|^2 |h_z^t(w)|^2 \, dv_\alpha(u) \, dv_\alpha(w). \]

The idea to use the function \( MO_{\alpha, t}(f) \) in the study of Hankel operators has been also suggested by other authors independently (see [4, 14] for example). We have the following result.

**Theorem 2.** Let \( \alpha > -1, \; r > 0, \; f \in L^2(B_n, dv_\alpha), \) and \( 0 < p < \infty. \) Then, for each \( t \geq 0 \) such that \( p > 2n/(n+1+\alpha+2t) \), we have

\[
\int_{B_n} MO_{\alpha, t}(f)(z)^p \, d\lambda_n(z) \leq C \int_{B_n} MO_r(f)(z)^p \, d\lambda_n(z).
\]

It is easy to check that, for any \( z \in B_n \) and \( r > 0, \) one has

\[
MO_r(f)(z) = \left[ \frac{1}{2 \nu_\alpha(D(z, r))^2} \int_{D(z, r)} \int_{D(z, r)} |f(u) - f(w)|^2 \, dv_\alpha(u) \, dv_\alpha(w) \right]^{1/2}.
\]

From this expression it follows that the behaviour of the local mean oscillation function \( MO_r(f) \) is independent of the parameter \( \alpha. \) Also, from this and the double integral expression of \( MO_{\alpha, t}(f), \) it is straightforward to see that \( MO_r(f)(z) \leq C MO_{\alpha, t}(f)(z). \) From this observation and Theorem 2, we see that Theorem 1 is equivalent to the following one.

**Theorem 3.** Let \( \alpha > -1, \; f \in L^2(B_n, dv_\alpha) \) and \( 0 < p < \infty. \) The following are equivalent:

1. \( H_f \) and \( H_{\overline{f}} \) are both in \( S_p(A_\alpha^2, L_\alpha^2) \)
2. For each (or some) \( t \geq 0 \) with \( p(n+1+\alpha+2t) > 2n, \) one has \( MO_{\alpha, t}(f) \in L^p(B_n, d\lambda_n). \)

From this, it can be seen that the conjecture stated at the end of [4] is also true.

The paper is organized as follows. After some preliminaries given in Section 2, we prove Theorem 2 in Section 3. All the implications in Theorem 1 are proved in Section 4 except the necessity in the case \( 0 < p < 2. \) This part is proved in Section 5 (the case \( 2n/(n+1+\alpha) < p < 2), \) and in Section 6 where we deduce the last case from the previous one in a tricky way.

We are not worried on the computation of the exact values of certain constants when are not depending on the important quantities involved, so that we use \( C \) to denote a positive constant like that, whose exact value may change at different occurrences, and sometimes we use the notation \( A \lesssim B \) to indicate that there is a positive constant \( C \) such that \( A \leq CB, \) and the notation \( A \asymp B \) means that both \( A \lesssim B \) and \( B \lesssim A \) hold.
2. Some known lemmas

We need a well-known result on decomposition of the unit ball \( \mathbb{B}_n \). A sequence \( \{a_k\} \) of points in \( \mathbb{B}_n \) is called a \emph{separated sequence} (in the Bergman metric) if there exists a positive constant \( \delta > 0 \) such that \( \beta(a_i, a_j) > \delta \) for any \( i \neq j \). By Theorem 2.23 in [18], there exists a positive integer \( N \) such that for any \( 0 < r < 1 \) we can find a sequence \( \{a_k\} \) in \( \mathbb{B}_n \) with the following properties:

(i) \( \mathbb{B}_n = \bigcup_k D(a_k, r) \).
(ii) The sets \( D(a_k, r/4) \) are mutually disjoint.
(iii) Each point \( z \in \mathbb{B}_n \) belongs to at most \( N \) of the sets \( D(a_k, 4r) \).

Any sequence \( \{a_k\} \) satisfying the above conditions is called an \emph{r-lattice} in the Bergman metric. Obviously any r-lattice is separated.

We need the following well known integral estimate that has become very useful in this area of analysis (see [18, Theorem 1.12] for example).

**Lemma A.** Let \( t > -1 \) and \( s > 0 \). There is a positive constant \( C \) such that

\[
\int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t \, dv(w)}{|1 - \langle z, w \rangle|^{n+1+t+s}} \leq C (1 - |z|^2)^{-s}
\]

for all \( z \in \mathbb{B}_n \).

We also need the following well known discrete version of the previous lemma.

**Lemma B.** Let \( \{z_k\} \) be a separated sequence in \( \mathbb{B}_n \), and let \( n < t < s \). Then

\[
\sum_k \frac{(1 - |z_k|^2)^t}{|1 - \langle z, z_k \rangle|^s} \leq C (1 - |z|^2)^{t-s}, \quad z \in \mathbb{B}_n.
\]

Lemma B can be deduced from Lemma A after noticing that, if a sequence \( \{z_k\} \) is separated, then there is a constant \( r > 0 \) such that the Bergman metric balls \( D(z_k, r) \) are pairwise disjoints.

We also need the following version of Lemma A with an extra (unbounded) factor \( \beta(z, w) \) in the integrand.

**Lemma 2.1.** Let \( t > -1 \) and \( s, c > 0 \). There is a positive constant \( C \) such that

\[
I := \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t \beta(z, w)^c \, dv(w)}{|1 - \langle z, w \rangle|^{n+1+t+s}} \leq C (1 - |z|^2)^{-s}
\]

for all \( z \in \mathbb{B}_n \).

**Proof.** Pick \( \varepsilon > 0 \) so that \( t - c \varepsilon > -1 \) and \( s - c \varepsilon > 0 \). Since \( \beta(z, w) \) grows logarithmically, we have

\[
\beta(z, w) = \beta(0, \varphi_z(w)) \leq C(1 - |\varphi_z(w)|^2)^{-\varepsilon}.
\]
Here $\varphi_z$ denotes the Möbius transformation sending $z$ to 0. It follows from the basic identity

\begin{equation}
1 - |\varphi_z(w)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \langle z, w \rangle|^2},
\end{equation}

that

\[ I \lesssim (1 - |z|^2)^{-c\varepsilon} \int_{B_n} \frac{(1 - |w|^2)^{t-c\varepsilon}}{|1 - \langle z, w \rangle|^{n+1+t+s-2c\varepsilon}} \, dv(w). \]

The desired result then follows from Lemma A. \hfill \Box

The corresponding discrete version is stated below.

**Lemma 2.2.** Let $\{z_k\}$ be a separated sequence in $\mathbb{B}_n$. Let $n < t < s$ and $c > 0$. Then

\[ \sum_k \frac{(1 - |z_k|^2)^t}{|1 - \langle z, z_k \rangle|^s} \beta(z, z_k)^c \leq C (1 - |z|^2)^{t-s}, \quad z \in \mathbb{B}_n. \]

We also need the following elementary result.

**Lemma 2.3.** For $r > 0$ let $\{a_k\}$ be an $r$-lattice on $\mathbb{B}_n$. Then

\[ \sum_k MO_r(f)(a_k)^p \leq C_1 \int_{B_n} MO_{2r}(f)(z)^p \, d\lambda_n(z) \leq C_2 \sum_k MO_{4r}(f)(a_k)^p \]

for all $0 < p < \infty$.

**Proof.** It follows from the double integral expression of the mean oscillation that

\[ MO_r(f)(a_k) \leq CMO_{2r}(f)(z), \quad z \in D(a_k, r). \]

From this, the result is easily deduced. \hfill \Box

### 3. Proof of Theorem 2

Let $\{a_k\}$ be an $(r/3)$-lattice on $\mathbb{B}_n$. Because $r > 0$ is arbitrary, due to Lemma 2.3 it is enough to prove

\begin{equation}
\int_{B_n} MO_{\alpha,t}(f)(z)^p \, d\lambda_n(z) \leq C \sum_k MO_{2r}(f)(a_k)^p.
\end{equation}

Let $D_k = D(a_k, r)$. Then, using the double integral expression of $MO_{\alpha,t}(f)$, we have

\[ MO_{\alpha,t}(f)(z)^2 \leq \sum_{j,k} \int_{D_j} \int_{D_k} |f(u) - f(w)|^2 |h^t_z(u)|^2 \, dv_\alpha(u) \, dv_\alpha(w). \]

Since $|h^t_z(u)| \leq |h^t_z(a_k)|$ for $u \in D_k$ (see estimate (2.20) in p.63 of [18]), we get

\[ MO_{\alpha,t}(f)(z)^2 \lesssim \sum_{j,k} |h^t_z(a_k)|^2 |h^t_z(a_j)|^2 \int_{D_j} \int_{D_k} |f(u) - f(w)|^2 \, dv_\alpha(u) \, dv_\alpha(w). \]
Due to the triangle inequality, we see that

$$MO_{\alpha, r}(f)(z)^2 \lesssim A_1(f, z) + A_2(f, z),$$

and because of the symmetry of the terms, in order to establish (3.1) it is enough to show that

$$\int_{\mathbb{B}_n} A_1(f, z)^{p/2} d\lambda_n(z) \lesssim \sum_j MO_{2r}(f)(a_j)^p$$

with

$$A_1(f, z) := \sum_{j,k} |h^l_z(a_k)|^2 |h^l_z(a_j)|^2 |D_k|_\alpha \int_{D_j} |f(u) - f(z)|^2 dv_\alpha(u).$$

Here we use the notation $|D_k|_\alpha = v_\alpha(D_k) \asymp (1 - |a_k|^2)^{n+1+\alpha}$. By Lemma 3.1 we get

$$A_1(f, z) \lesssim \sum_j |h^l_z(a_j)|^2 \int_{D_j} |f(u) - f(z)|^2 dv_\alpha(u).$$

Now we need the following technical lemma.

**Lemma 3.1.** Let $r > 0$ and $\{\xi_m\}$ be an $(r/3)$-lattice on $\mathbb{B}_n$. Let $0 < p < \infty$ and $d, \delta > 0$. For $a, z \in \mathbb{B}_n$, we have

$$|f(z) - f(a)| \lesssim h_\delta(a, z) N_p(f, a)^{1/p} |1 - \langle z, a \rangle|^d,$$

with

$$N_p(f, a) = \sum_m \frac{MO_{2r}(f)(\xi_m)^p (1 - |\xi_m|^2)^{dp}}{|1 - \langle \xi_m, a \rangle|^{pd}},$$

and

$$h_\delta(a, z) = (1 + \beta(a, z)) \left[ \min(1 - |z|, 1 - |a|) \right]^{-\delta}.$$

**Proof.** Let $\xi$ be a point in the lattice with $\beta(z, \xi) \leq r/3$. Since

$$|f(z) - \hat{f}_r(z)| \lesssim MO_{r}(f)(\xi) \frac{MO_{r}(f)(\xi)(1 - |\xi|)^d}{|1 - \langle \xi, a \rangle|^d |1 - \langle z, a \rangle|^d (1 - |z|)^{-\delta}},$$

and a similar estimate can be obtained for $|f(a) - \hat{f}_r(a)|$, it is enough to prove

$$|\hat{f}_r(z) - \hat{f}_r(a)| \lesssim h_\delta(a, z) N_p(f, a)^{1/p} |1 - \langle z, a \rangle|^d.$$

Denote by $\gamma(t)$, $0 \leq t \leq 1$, the geodesic in the Bergman metric going from $z$ to $a$. Let $N = \lceil \beta(z, a)/R \rceil + 1$ with $R = r/3$, and $t_m = m/N$, $0 \leq m \leq N$, where $\lceil x \rceil$ denotes the largest integer less than or equal to $x$. Set $z_m = \gamma(t_m)$, $0 \leq m \leq N$. Clearly

$$\beta(z_m, z_{m+1}) = \frac{\beta(z, a)}{N} \leq R = r/3.$$

By the triangle inequality, we have

$$|\hat{f}_r(z) - \hat{f}_r(a)| \leq \left( \sum_{m=1}^{N} |\hat{f}_r(z_{m-1}) - \hat{f}_r(z_{m})| \right).$$
For each $m$, take a point $\xi_m$ in the lattice with $\beta(z_m, \xi_m) < r/3$. It is not difficult to see that $|\widehat{f}_r(\xi) - \widehat{f}_r(w)| \lesssim MO_{2r}(f)(\xi)$ if $\beta(\xi, w) \leq r$ (see [19, p.211]). Then

$$|\widehat{f}_r(z_{m-1}) - \widehat{f}_r(z_m)| \leq |\widehat{f}_r(z_{m-1}) - \widehat{f}_r(\xi_m)| + |\widehat{f}_r(\xi_m) - \widehat{f}_r(z_m)| \lesssim MO_{2r}(f)(\xi_m).$$

This gives

$$|\widehat{f}_r(z) - \widehat{f}_r(a)| \lesssim \sum_{m=1}^N MO_{2r}(f)(\xi_m).$$

Because the Möbius transformation $\varphi_z$ sends the geodesic joining $z$ and $a$ to the geodesic joining $0$ and $\varphi_z(a)$, we have

$$|1 - \langle \varphi_z(a), \varphi_z(z_m) \rangle| = 1 - |\varphi_z(a)||\varphi_z(z_m)| \leq 1 - |\varphi_z(z_m)|^2.$$

Developing this inequality using the basic identity (2.1) together with its polarized analogue [18, Lemma 1.3]

$$1 - \langle \varphi_z(a), \varphi_z(b) \rangle = \frac{(1 - |z|^2)(1 - \langle a, b \rangle)}{(1 - \langle a, z \rangle)(1 - \langle z, b \rangle)},$$

we arrive at

$$\frac{|1 - \langle a, z_m \rangle|}{|1 - \langle a, z \rangle|} \leq \frac{(1 - |z_m|^2)}{|1 - \langle z, z_m \rangle|},$$

which gives

$$|1 - \langle a, z_m \rangle| \leq 2|1 - \langle a, z \rangle|.$$

Putting these inequality into (3.4), with the help of the estimate $|1 - \langle \xi_m, a \rangle| \asymp |1 - \langle z_m, a \rangle|$ (see [18, p.63]), we obtain

$$|\widehat{f}_r(z) - \widehat{f}_r(a)| \lesssim \sum_{m=1}^N \frac{MO_{2r}(f)(\xi_m)}{|1 - \langle \xi_m, a \rangle|^d} |1 - \langle z, a \rangle|^d.$$

From here, the result easily follows, since

$$\sum_{m=1}^N \frac{MO_{2r}(f)(\xi_m)}{|1 - \langle \xi_m, a \rangle|^d} \leq \left( \sum_{m=1}^N \frac{MO_{2r}(f)(\xi_m)^p}{|1 - \langle \xi_m, a \rangle|^{pd}} \right)^{1/p}, \quad 0 < p \leq 1,$$

and Hölder’s inequality yields

$$\sum_{m=1}^N \frac{MO_{2r}(f)(\xi_m)}{|1 - \langle \xi_m, a \rangle|^d} \lesssim N^{1/p'} \left( \sum_{m=1}^N \frac{MO_{2r}(f)(\xi_m)^p}{|1 - \langle \xi_m, a \rangle|^{pd}} \right)^{1/p}, \quad 1 < p < \infty.$$

Finally, since $N \lesssim (1 + \beta(a, z))$, the inequality

$$\min(1 - |a|, 1 - |z|) \leq (1 - |z_m|) \asymp (1 - |\xi_m|)$$

completes the proof of the lemma. □
Returning to the estimate for $A_1(f, z)$, putting the inequality of Lemma 3.1 into (3.3), with $d = \frac{1}{2}(n + 1 + \alpha + 2t) - \varepsilon$, where $\varepsilon > 0$ is taken so that $pd > n$, we see that $A_1(f, z)$ is less than constant times

$$(1 - |z|^2)^{n+1+\alpha+2t} N_p(f, z)^{2/p} \sum_j \frac{(1 - |a_j|^2)^{n+1+\alpha}}{|1 - \langle a_j, z \rangle|^{2(n+1+\alpha+t-d)}} h_\delta(a_j, z)^2,$$

with $\delta > 0$ taken so that $\alpha - 2\delta > -1$ and $pd - p\delta > n$. By Lemma 3.1 and Lemma 2.2, we have

$$A_1(f, z) \lesssim (1 - |z|^2)^{2d-2\delta} N_p(f, z)^{2/p}.$$

Then

$$\int_{B_n} A_1(f, z)^{p/2} d\lambda_n(z) \lesssim \int_{B_n} N_p(f, z) (1 - |z|^2)^{p(d-\delta)} d\lambda_n(z)$$

$$= \sum_m MO_{2r}(f)(\xi_m)^p (1 - |\xi_m|^2)^{\delta p} \int_{B_n} \frac{(1 - |z|^2)^{p(d-\delta)} d\lambda_n(z)}{|1 - \langle \xi_m, z \rangle|^{pd}}$$

$$\lesssim \sum_m MO_{2r}(f)(\xi_m)^p,$$

after an application of Lemma A. This proves (3.2) finishing the proof of the theorem.

4. Proof of Theorem 1: first steps

We first establish some auxiliary results that can be of independent interest. Recall that $h^t_z = K^t_z /\|K^t_z\|_\alpha$ with $K^t_z$ defined in (1.1). We begin with a tricky lemma.

**Lemma 4.1.** Let $\alpha > -1$, $t \geq 0$ and $f \in L^2(\mathbb{B}_n, dv_\alpha)$. Then

$$MO_{\alpha, t}(f)(z) \leq C \cdot (\|H f h^t_z\|_\alpha + \|H_\perp h^t_z\|_\alpha)$$

for each $z \in \mathbb{B}_n$.

**Proof.** An easy computation gives

$$\|(f - \lambda)h^t_z\|_\alpha^2 = B_{\alpha, t}(|f|^2)(z) - |B_{\alpha, t}(f)(z)|^2 + |B_{\alpha, t}(f)(z) - \lambda|^2.$$

Thus, we have

$$MO_{\alpha, t}(f)(z) = \left( B_{\alpha, t}(|f|^2)(z) - |B_{\alpha, t}(f)(z)|^2 \right)^{1/2}$$

$$\leq \|f h^t_z - g_z(z) h^t_z\|_\alpha,$$

$$\leq \|f h^t_z - P_\alpha(f h^t_z)\|_\alpha + \|P_\alpha(f h^t_z) - g_z(z) h^t_z\|_\alpha$$

$$= \|H f h^t_z\|_\alpha + \|P_\alpha(f h^t_z) - g_z(z) h^t_z\|_\alpha.$$
where \( g_z \) denotes the holomorphic function on \( B_n \) given by

\[
g_z(w) = \frac{P_\alpha(f h^t_z)(w)}{h^t_z(w)}, \quad w \in B_n.
\]

Now we use the identity

\[
(4.1) \quad g_z(z) h^t_z = P_{\alpha+t}(g_z h^t_z).
\]

To see this, since \( K^t_z(w) = K^t_w(z) \), by the reproducing formula

\[
g_z(z) h^t_z(w) = \| K^t_z \|^{-1}_\alpha g_z(z) K^t_w(z) = \| K^t_z \|^{-1}_\alpha \langle g_z K^t_w, K_z \rangle
\]

\[
= \| K^t_z \|^{-1}_\alpha \langle K_z, g_z K^t_w \rangle = \| K^t_z \|^{-1}_\alpha \langle K^t_z, g_z K^t_w \rangle_{\alpha+t}
\]

\[
= \langle \overline{g_z h^t_z}, K^t_w \rangle_{\alpha+t} = P_{\alpha+t}(\overline{g_z h^t_z})(w).
\]

Therefore, \((4.1)\) together with the boundedness of \( P_{\alpha+t} \) on \( L^2(B_n, dv_\alpha) \) yields

\[
\| P_\alpha(f h^t_z) - g_z(z) h^t_z \|_\alpha = \| P_\alpha(f h^t_z) - P_{\alpha+t}(\overline{g_z h^t_z}) \|_\alpha
\]

\[
(4.2) \quad = \| P_{\alpha+t}(P_\alpha(f h^t_z) - g_z h^t_z) \|_\alpha
\]

\[
\leq \| P_{\alpha+t} \| \cdot \| P_\alpha(f h^t_z) - g_z h^t_z \|_\alpha.
\]

Finally,

\[
\| P_\alpha(f h^t_z) - g_z h^t_z \|_\alpha \leq \| f h^t_z - P_\alpha(f h^t_z) \|_\alpha + \| f h^t_z - g_z h^t_z \|_\alpha
\]

\[
= \| H_f h^t_z \|_\alpha + \| f h^t_z - g_z h^t_z \|_\alpha
\]

\[
= \| H_f h^t_z \|_\alpha + \| f h^t_z - P_\alpha(\overline{f h^t_z}) \|_\alpha
\]

\[
= \| H_f h^t_z \|_\alpha + \| H_\overline{f} h^t_z \|_\alpha.
\]

This proves the result with constant \( C = (1 + \| P_{\alpha+t} \|) \). Observe that, when \( t = 0 \), since \( \| P_\alpha \| = 1 \), one gets \( C = 1 \) since in \( (4.2) \) one has the term \( \| P_\alpha(f k_z - g_z k_z) \|_\alpha \), and thus it is not necessary to use again the triangle inequality. \( \square \)

The case \( t = 0 \) of Lemma 4.1 appears in [3] and [16], with a proof that seems to be specific of the Hilbert space case. Observe that our proof is flexible enough to work when studying Hankel operators acting on \( A^p_\alpha \) (see [17], where some version of Lemma 4.1 for \( t = 0 \) in this
setting was proved with a different method).

The following inequality is also satisfied:

\[(4.3) \quad \|H_fh_z^t\| + \|Hfh_z^t\| \leq 2 \text{MO}_{\alpha,t}(f)(z).\]

Indeed, we have

\[
\|H_fh_z^t\|_\alpha^2 = \|(I - Pa)(fh_z^t)\|_\alpha^2 = \|fh_z^t\|_\alpha^2 - \|Pa(fh_z^t)\|_\alpha^2.
\]

Thus, by Cauchy-Schwarz,

\[
|B_{\alpha,t}(f)(z)| = |\langle fh_z^t, h_z^t \rangle_\alpha| = |\langle Pa(fh_z^t), h_z^t \rangle_\alpha| \leq \|Pa(fh_z^t)\|_\alpha.
\]

The following inequality is also satisfied:

\[
\|fh_z^t\|_\alpha^2 = B_{\alpha,t}(|f|^2)(z),
\]

and it follows that \(\|H_fh_z^t\|_\alpha \leq \text{MO}_{\alpha,t}(f)(z)\).

The following result can be found in [8, Lemma 2].

**Lemma C.** Let \(\alpha > -1\) and \(T : A_\alpha^2(\mathbb{B}_n) \to A_\alpha^2(\mathbb{B}_n)\) be a positive operator. For \(t \geq 0\) set

\[\widetilde{T}^t(z) = \langle Th_z^t, h_z^t \rangle_\alpha, \quad z \in \mathbb{B}_n.\]

(a) Let \(0 < p \leq 1\). If \(\widetilde{T}^t \in L^p(\mathbb{B}_n, d\lambda_n)\) then \(T\) is in \(S_p\).

(b) Let \(p \geq 1\). If \(T\) is in \(S_p\) then \(\widetilde{T}^t \in L^p(\mathbb{B}_n, d\lambda_n)\).

If we apply this lemma with the positive operator \(T = H_f^*H_f\), then due to (4.3) and Lemma 4.1 we obtain the necessity in Theorem 3 for \(p \geq 2\) and the sufficiency for \(p \leq 2\). This together with the inequality \(\text{MO}_r(f)(z) \lesssim \text{MO}_{\alpha,t}(f)(z)\) gives the implication (a) implies (b) in Theorem 4 for \(p \geq 2\), and if we use Theorem 2 we see that (b) implies (a) for \(p \leq 2\). Summarizing, the following proposition has been proved.

**Proposition 4.2.** Let \(\alpha > -1\) and \(f \in L_\alpha^2\). Then

(i) Let \(2 \leq p < \infty\). If the Hankel operators \(H_f\) and \(H_T\) are simultaneously in \(S_p(A_\alpha^2, L_\alpha^2)\), then \(\text{MO}_r(f)\) is in \(L^p(\mathbb{B}_n, d\lambda_n)\).

(ii) Let \(0 < p \leq 2\). If \(\text{MO}_r(f) \in L^p(\mathbb{B}_n, d\lambda_n)\) then \(H_f\) and \(H_T\) are both in \(S_p(A_\alpha^2, L_\alpha^2)\).

Next, we consider the Hankel operator \(H_f^\gamma\) defined by

\[
H_f^\gamma = (I - P_\gamma)M_f.
\]

With this notation, we have \(H_f = H_f^0\).

**Lemma 4.3.** Let \(\alpha > -1\), \(f \in L_\alpha^2\) and \(\gamma > \alpha\). If \(H_\alpha^\alpha\) is in \(S_\alpha(A_\alpha^2, L_\alpha^2)\) (or compact), then \(H_f^\gamma\) is also in \(S_\alpha(A_\alpha^2, L_\alpha^2)\) (or compact).

**Proof.** Since for \(\gamma > \alpha\), the projection \(P_\gamma\) is bounded on \(L_\alpha^2\) and \(P_\gamma P_\alpha = P_\alpha\), we have

\[
H_f^\gamma = H_f^\alpha + (P_\alpha - P_\gamma)M_f = H_f^\alpha - P_\gamma H_f^\alpha.
\]

Hence the result follows. \(\Box\)
Proposition 4.4. Let $\alpha > -1$, $f \in L^2_\alpha$ and $2 < p < \infty$. If $MO_r(f) \in L^p(\mathbb{R}, d\lambda_n)$ then $H_f$ and $H_f^\gamma$ are both in $S_p(A_\alpha^2, L^2_\alpha)$.

Proof. This follows from Theorem 2 with $t = 0$ and the well know fact that $MO_{\alpha,0}(f) \in L^p(\mathbb{R}, d\lambda_n)$ implies the conclusion of the Proposition. However, we will provide a self-contained proof based on Lemma 3.1. Note that the condition implies that both $H_f^\alpha$ and $H_f^\gamma$ are compact (just take a look at Lemma 2.3 which implies $MO_r(f)(z) \to 0$ as $|z| \to 1$); and in view of Lemma 4.3, the operators $H_f^\gamma$ and $H_f^\gamma$ are also compact for all $\gamma > \alpha$. Since $P_\gamma = P_\alpha P_\gamma$ on $L^2_\alpha$ and

$$H_f^\alpha - H_f^\gamma = (P_\gamma - P_\alpha)M_f = P_\alpha H_f^\gamma,$$

it is enough to show that $H_f^\gamma$ belongs to $S_p(A_\alpha^2, L^2_\alpha)$ for $\gamma$ big enough, say $\gamma = \alpha + 4t$ with $pt > n$. By [19, Theorem 1.33], it suffices to prove that

$$\sum_n \|H_f^\gamma e_n\|_\alpha^p \leq C$$

for any orthonormal set $\{e_n\}$ of $A_\alpha^2$, with a constant $C$ not depending on the choice of the orthonormal set. Let $\varepsilon > 0$ so that $\alpha - \varepsilon > -1$. By Cauchy-Schwarz and Lemma A we have

$$\|H_f^\gamma e_n\|_\alpha^2 \leq \int_{\mathbb{R}} \left( \frac{|f(z) - f(w)|}{|1 - \langle z, w \rangle|^{n+1+\gamma}} dv_\gamma(w) \right)^2 \, dv_\alpha(z)$$

$$\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{|f(z) - f(w)|^2}{|1 - \langle z, w \rangle|^{n+1+\gamma}} dv_\gamma(w) \right) \, dv_{\alpha-\varepsilon}(z).$$

Now, Fubini’s theorem, Hölder’s inequality with exponent $p/2 > 1$ and $\|e_n\|_\alpha = 1$ yield

$$\|H_f^\gamma e_n\|_\alpha^p \leq \int_{\mathbb{R}} |e_n(w)|^2 \left( \int_{\mathbb{R}} \frac{|f(z) - f(w)|^2}{|1 - \langle z, w \rangle|^{n+1+\gamma}} \right)^{p/2} \, dv_{\alpha+1/2(\gamma-\alpha+\varepsilon)}(w).$$

Because $\{e_n\}$ is an orthonormal set, we can use the inequality

$$\sum_n |e_n(w)|^2 \leq \|K_w\|_\alpha^2 = (1 - |w|^2)^{-(n+1+\alpha)}$$

to obtain

$$\sum_n \|H_f^\gamma e_n\|_\alpha^p \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{|f(z) - f(w)|^2}{|1 - \langle z, w \rangle|^{n+1+\gamma}} \right)^{p/2} (1 - |w|^2)^{\frac{p}{2}(\gamma-\alpha+\varepsilon)} \, d\lambda_n(w).$$

Set

$$I_p(f, w) := \left( \int_{\mathbb{R}} \frac{|f(z) - f(w)|^2}{|1 - \langle z, w \rangle|^{n+1+\gamma}} \right)^{p/2}.$$

Take a lattice $\{\xi_k\}$ and apply Lemma 3.1 with $d = t$ and $\delta > 0$ satisfying $pt - p\delta > n$ and $\alpha - \varepsilon - 2\delta > -1$, to obtain

$$I_p(f, w) \lesssim N_p(f, w) \left( \int_{\mathbb{R}} \frac{h(z, w) \, dv_{\alpha-\varepsilon}(z)}{|1 - \langle z, w \rangle|^{n+1+\gamma-2\delta}} \right)^{p/2}.$$
with
\[ N_p(f, w) = \sum_k \frac{MO_{2r}(f)(\xi_k)^p (1 - |\xi_k|^2)^{\delta p}}{|1 - \langle w, \xi_k \rangle|^{pd}} \]
and
\[ h(z, w) = [1 + \beta(z, w)]^2 \left( \min(1 - |z|, 1 - |w|) \right)^{-\delta}. \]
By Lemma A and Lemma 2.1, we have
\[ \int_{B_n} h(z, w) dv_{\alpha - \epsilon}(z) \lesssim (1 - |w|^2)^{2d - \gamma - \alpha - \epsilon - 2\delta}. \]
This, together with Lemma A gives
\[ \sum_n \| H_{\gamma}^\gamma e_n \|^p_\alpha \lesssim \int_{B_n} I_p(f, w) (1 - |w|^2)^{2(\gamma - \alpha + \epsilon)} d\lambda_n(w) \]
\[ \lesssim \int_{B_n} N_p(f, w) (1 - |w|^2)^{pd - \delta} d\lambda_n(w) \]
\[ = \sum_k MO_{2r}(f)(\xi_k)^p (1 - |\xi_k|^2)^{\delta p} \int_{B_n} (1 - |w|^2)^{pd - \delta} |1 - \langle w, \xi_k \rangle|^{pd} d\lambda_n(w) \]
\[ \lesssim \sum_k MO_{2r}(f)(\xi_k)^p. \]
This finishes the proof. \qed

Taking into account Propositions 4.2 and 4.4 in order to complete the proof of Theorem 1 it remains to show that (a) implies (b) for 0 < p < 2. This is done in the next two sections.

5. Necessity: the case \( \frac{2n}{n+1+\alpha} < p < 2 \)

This follows immediately from condition (c) and the fact that \( MO_r(f)(z) \lesssim MO_n(f)(z) \). However, we are going to give a direct proof of the implication (a) \( \Rightarrow \) (b) in that case. There are several reasons for doing that. First, to be self-contained, and also because the proof we give here has some independent interest and works equally well in the full range \( \frac{2n}{n+1+\alpha} < p < 2 \), while the proof of (c) given in [12, 13] worked only for \( p > 1 \), and the proof of (c) given in [4] was specific of the case \( \frac{2n}{n+1+\alpha} < p \leq 1 \). Finally, the proof given here can be of some help when looking for some extensions of Theorem 1 in some contexts where the equivalence with (c) is not available.

The proof follows the lines of [19, Theorem 7.16], a method that seems to have its roots on previous work of S. Semmes [10] and D. Luecking [6], but with much more complicated estimates. If \( H_f \) is in \( S_p \), then \( H_{\gamma}^\gamma \) is also in \( S_p \) for all \( \gamma > \alpha \), as a consequence of Lemma 4.3. Let \( \gamma = \alpha + t \), with \( t > 0 \) taken big enough so that all the applications of the lemmas
appearing in Section 2 are going to be correct (for example, satisfying \( pt > 10n \)). Let \( \Lambda = \{ \xi_k \} \) be a regular \((r/3)\)-lattice in the Bergman metric. Fix a sufficiently large positive radius \( R \), and partition the lattice \( \{ \xi_k \} \) into \( M \) subsequences so that the Bergman metric between any two points in each subsequence is at least \( R \). Let \( \{ a_k \} \) be such a subsequence.

Fix an orthonormal basis \( \{ e_k \} \) for \( A^2_\alpha(\mathbb{B}_n) \), and define an operator \( A \) on \( A^2_\alpha(\mathbb{B}_n) \) by

\[
Ae_k = h^t_k := h^t_{a_k}.
\]

The boundedness of \( A \) follows easily from the fact that \( \{ a_k \} \) is separated in the Bergman metric. Let

\[
T_1 = A^* (H^\gamma_T)^* H^\gamma_T A, \quad T_2 = A^* (H^\gamma_T)^* H^\gamma_T A.
\]

We split these operators as \( T_1 = D_1 + E_1 \), and \( T_2 = D_2 + E_2 \), where \( D_1 \) and \( D_2 \) are the diagonal operators defined by

\[
D_i = \sum_k \langle T_i e_k, e_k \rangle \langle f, e_k \rangle e_k, \quad f \in A^2_\alpha, \quad i = 1, 2.
\]

By the triangle inequality,

\[
\| T_i \|_{S_{p/2}}^{p/2} \geq \| D_i \|_{S_{p/2}}^{p/2} - \| E_i \|_{S_{p/2}}^{p/2}, \quad i = 1, 2.
\]

Since \( A \) is a bounded operator, we have

\[
\| T_i \|_{S_{p/2}}^{p/2} \lesssim \| (H^\gamma_T)^* H^\gamma_T \|_{S_{p/2}}^{p/2} = \| H^\gamma_T \|_{S_p}^p,
\]

and

\[
\| T_2 \|_{S_{p/2}}^{p/2} \lesssim \| H^\gamma_T \|_{S_p}^p.
\]

Therefore,

\[
\| H^\gamma_T \|_{S_p}^p + \| H^\gamma_T \|_{S_p}^p \gtrsim \| T_1 \|_{S_{p/2}}^{p/2} + \| T_2 \|_{S_{p/2}}^{p/2} \geq \| D_1 \|_{S_{p/2}}^{p/2} + \| D_2 \|_{S_{p/2}}^{p/2} - \left( \| E_1 \|_{S_{p/2}}^{p/2} + \| E_2 \|_{S_{p/2}}^{p/2} \right).
\]

(5.1)

Since \( D_1 \) and \( D_2 \) are positive diagonal operators, then

\[
\| D_1 \|_{S_{p/2}}^{p/2} = \sum_k \langle T_1 e_k, e_k \rangle^p = \sum_k \langle A^* (H^\gamma_T)^* H^\gamma_T A e_k, e_k \rangle^p = \sum_k \langle (H^\gamma_T)^* H^\gamma_T h^t_k, h^t_k \rangle^p = \sum_k \| H^\gamma_T h^t_k \|_{S_p}^p.
\]

In the same way, we also have

\[
\| D_2 \|_{S_{p/2}}^{p/2} = \sum_k \| H^\gamma_T h^t_k \|_{S_p}^p.
\]
It is very easy to see that \( \|H_f h_k\|_\alpha \lesssim \|H_f^\alpha h_k\|_\alpha \), and the same holds if we replace \( f \) by \( \overline{f} \). Therefore, an application of Lemma 4.1 gives
\[
\|D_1\|_{S_p/2}^{p/2} + \|D_2\|_{S_p/2}^{p/2} = \sum_k \left( \|H_f^\alpha h_k\|_\alpha + \|H_f^\alpha h_k\|_\alpha \right)^p \\
\gtrsim \sum_k \left( \|H_f^\alpha h_k\|_\alpha + \|H_f^\alpha h_k\|_\alpha \right)^p \\
\gtrsim \sum_k MO_{\alpha,t}(f)(a_k)^p.
\]
Because \( MO_{2r}(f)(a_k) \lesssim MO_{\alpha,t}(f)(a_k) \), we see that there exists a constant \( C_1 \) such that
\[
(5.2) \quad \|D_1\|_{S_p/2}^{p/2} + \|D_2\|_{S_p/2}^{p/2} \geq C_1 \sum_k MO_{2r}(f)(a_k)^p.
\]
Thus, if we are able to show that, given \( \varepsilon > 0 \) we can take \( R > 0 \) big enough such that
\[
(5.3) \quad \|E_1\|_{S_p/2}^{p/2} + \|E_2\|_{S_p/2}^{p/2} \leq C_2 \varepsilon \sum_k MO_{2r}(f)(a_k)^p,
\]
then, bearing in mind (5.1) and (5.2), we will obtain
\[
\|H_f^\alpha\|_{S_p}^p + \|H_f^\alpha\|_{S_p}^p \gtrsim (C_1 - C_2 \varepsilon) \sum_k MO_{2r}(f)(a_k)^p.
\]
Hence, choosing \( \varepsilon > 0 \) such that \( C_2 \varepsilon < C_1/2 \), we get
\[
\sum_k MO_{2r}(f)(a_k)^p \lesssim \|H_f^\alpha\|_{S_p}^p + \|H_f^\alpha\|_{S_p}^p.
\]
Since this holds for each one of the \( M \) subsequences of our lattice \( \{\xi_m\} \), we obtain
\[
\sum_m MO_{2r}(f)(\xi_m)^p \lesssim M \cdot (\|H_f^\alpha\|_{S_p}^p + \|H_f^\alpha\|_{S_p}^p).
\]
Finally, due to Lemma 2.3, it follows that \( MO_r(f) \in L^p(\mathbb{B}_n, d\lambda_n) \) finishing the proof. It remains to prove that (5.3) holds, and for that it is enough to show that
\[
(5.4) \quad \|E_1\|_{S_p/2}^{p/2} \leq C_3 \varepsilon \sum_k MO_{2r}(f)(a_k)^p.
\]
The proof of this inequality is the goal of the rest of the section.

**Lemma 5.1.** Let \( \{a_j\} \) be a separated sequence in \( \mathbb{B}_n \). Let \( n < a < b \), and \( c \geq 0 \). Given \( \varepsilon > 0 \), there is \( R > 0 \) such that
\[
\sum_{j: \beta(a_j,w) > R} \frac{(1 - |a_j|^2)^a (1 + \beta(a_j,w)^c)}{|1 - \langle a_j, w \rangle|^b} \leq C\varepsilon (1 - |w|^2)^{a-b}.
\]
Proof. As in the proof of Lemma 2.1 it is enough to show that
\[ \sum_{j, \beta(a_j, w) > R} \frac{(1 - |a_j|^2)^a}{|1 - \langle a_j, w \rangle|^b} \leq C \varepsilon (1 - |w|^2)^{a-b}, \]
for \( R \) large enough. Taking into account Lemma 13, the elementary proof is left to the interested reader. \( \square \)

We return to the proof of (5.4). Since \( 0 < p \leq 2 \), by [19] Proposition 1.29, we have
\[ \| E_1 \|^p_{L^p} \leq \sum_k \sum_j \| \langle E_1 e_j, e_k \rangle \|^p = \sum_{j, k, j \neq k} \| \langle T_1 e_j, e_k \rangle \|^p \]
(5.5)

Using the expression of \( H_f^\gamma h \) given by
\[ H_f^\gamma h(z) = \int_{B_n} \frac{(f(z) - f(u)) h(z)}{(1 - \langle z, u \rangle)^{n+1+\gamma}} dv_\gamma(u), \]
and Fubini’s theorem, we have
\[ \| \langle H_f^\gamma h_j^t, H_f^\gamma h_k^t \rangle \| \leq \int_{B_n} |H_f^\gamma h_j^t(z)| |H_f^\gamma h_k^t(z)| dv_\alpha(z) \]
\[ \leq \int_{B_n} \int_{B_n} |h_j^t(u)| |h_k^t(w)| \left( \int_{B_n} |f(z) - f(u)| |f(z) - f(w)| dv_\gamma(z) \right) dv_\gamma(u) dv_\gamma(w). \]

Then, with the notation \( D_m = D(\xi_m, r) \), we have
\[ \| \langle H_f^\gamma h_j^t, H_f^\gamma h_k^t \rangle \| \leq \sum_{\ell, m} |h_j^t(\xi_\ell)| |h_k^t(\xi_m)| J_{\ell, m}(f) \]
with
\[ J_{\ell, m}(f) = \int_{B_n} \left( \int_{D_\ell} \frac{|f(z) - f(u)|}{1 - \langle z, u \rangle} dv_\gamma(u) \right) \left( \int_{D_m} \frac{|f(z) - f(w)|}{1 - \langle z, w \rangle} dv_\gamma(w) \right) dv_\alpha(z). \]

As \( 0 < p < 2 \),
\[ \sum_{j, k, j \neq k} \| \langle H_f^\gamma h_j^t, H_f^\gamma h_k^t \rangle \|^p \leq \sum_{\ell, m} \left( \sum_{j, k, j \neq k} |h_j^t(\xi_\ell)|^p |h_k^t(\xi_m)|^p \right)^{p/2} J_{\ell, m}(f)^{p/2}. \]

For the pairs \( (\ell, m) \) for which, either \( \beta(a_j, \xi_\ell) > R/4 \) or \( \beta(a_k, \xi_m) > R/4 \), because of Lemma 5.1 we have
\[ \sum_{j, k, j \neq k} |h_j^t(\xi_\ell)|^p |h_k^t(\xi_m)|^p \leq \varepsilon \left[ (1 - |\xi_\ell|^2) (1 - |\xi_m|^2) \right]^{-\frac{p}{4}(n+1+\alpha)}. \]
For the other pairs \((\ell, m)\), we have the same inequality without the \(\varepsilon\) but, due to the triangle inequality, one has \(\beta(\xi_\ell, \xi_m) > R/4\). Thus, we assume that we are in this last case. Then

\[
\|E_1\|_{L^p}^p \lesssim \sum_{\ell, m} \left[ (1 - |\xi_\ell|^2)(1 - |\xi_m|^2) \right]^{-\frac{p}{2}(n+\alpha)} J_{\ell, m}(f)^{p/2}.
\]

We estimate \(J_{\ell, m}(f)\) with the help of Lemma 5.2. Let \(d = t/4\) and take \(\delta > 0\) satisfying \(\frac{p}{2}(n + 1 + \alpha - 3\delta) > n\), and also \(p(d - \delta) > n\). By Lemma 3.1 with the same meaning for \(N_p(f, u)\) and \(h_\delta(z, w)\) as in that lemma, and taking into account that \(N_p(f, u) \simeq N_p(f, \xi_\ell)\) if \(u \in D_\ell\), we have

\[
J_{\ell, m}(f) \lesssim \left[ (1 - |\xi_\ell|^2)(1 - |\xi_m|^2) \right]^{n+\gamma} N_p(f, \xi_\ell)^{1/p} N_p(f, \xi_m)^{1/p} J(\ell, m),
\]

with

\[
J(\ell, m) = \int_{\mathbb{S}_n} \frac{h_\delta(z, \xi_\ell) h_\delta(z, \xi_m)}{|1 - \langle z, \xi_\ell \rangle|^{n+\gamma-d}|1 - \langle z, \xi_m \rangle|^{n+\gamma-d}} dv_\alpha(z).
\]

**Lemma 5.2.** We have

\[
J(\ell, m) \lesssim \frac{h_\delta(\xi_\ell, \xi_m)}{|1 - \langle \xi_\ell, \xi_m \rangle|^{n+1+\gamma-d}} \left( (1 - |\xi_\ell|^2)^{d-t-\delta} + (1 - |\xi_m|^2)^{d-t-\delta} \right)
\]

*Proof.* Since \(d(z, w) = |1 - \langle z, w \rangle|^{1/2}\) satisfies the triangle inequality \([9, Proposition 5.1.2]\), we have

\[
\frac{1}{|1 - \langle z, \xi_\ell \rangle| |1 - \langle z, \xi_m \rangle|} \lesssim \frac{1}{|1 - \langle z, \xi_\ell \rangle| |1 - \langle \xi_\ell, \xi_m \rangle|} + \frac{1}{|1 - \langle \xi_m, \xi_\ell \rangle| |1 - \langle z, \xi_m \rangle|}.
\]

Then, bearing in mind that \(\gamma = \alpha + t\),

\[
h_\delta(z, w) = \left( 1 + \beta(z, w) \right) \left[ \min(1 - |z|, 1 - |w|) \right]^{-\delta},
\]

and the triangle inequality for \(\beta(z, w)\), the result follows easily from Lemma 3.1 and Lemma 2.1. \(\square\)

Putting the estimate of Lemma 5.2 into (5.7), and taking into account (5.6) and the symmetry of the two terms, it is enough to show

\[
A_p(f) \lesssim \varepsilon \sum_{\xi \in \Lambda} MO_{2r}(f)(\xi)^p
\]

where \(A_p(f)\) is given by the expression

\[
\sum_{\ell, m} \frac{(1 - |\xi_\ell|^2)^{\frac{p}{2}(n+1+\alpha+2d-2\delta)}(1 - |\xi_m|^2)^{\frac{p}{2}(n+1+\alpha+2t)}}{|1 - \langle \xi_\ell, \xi_m \rangle|^{\frac{p}{2}(n+1+\gamma-d)}} H_{\delta, p}(\xi_\ell, \xi_m)
\]

with

\[
H_{\delta, p}(\xi_\ell, \xi_m) = h_\delta(\xi_\ell, \xi_m)^{p/2} N_p(f, \xi_\ell)^{1/2} N_p(f, \xi_m)^{1/2}.
\]
Using the inequality $2AB \leq A^2 + B^2$, we have
\begin{equation}
A_p(f) \lesssim B_p(f) + C_p(f),
\end{equation}
\begin{align*}
B_p(f) &= \sum_{\ell,m} \frac{(1 - |\xi\ell|^2)^{pd} (1 - |\xi_m|^2)^{\frac{r}{2}(n+1+\alpha+t-\delta)}}{|1 - \langle \xi\ell, \xi_m \rangle|^{\frac{r}{2}(n+1+\alpha+t)}} h_\delta(\xi\ell, \xi_m)^{p/2} N_p(f, \xi\ell)
\end{align*}
and
\begin{align*}
C_p(f) &= \sum_{\ell,m} \frac{(1 - |\xi\ell|^2)^{\frac{r}{2}(n+1+\alpha-2d)} (1 - |\xi_m|^2)^{\frac{r}{2}(t+\delta)}}{|1 - \langle \xi\ell, \xi_m \rangle|^{\frac{r}{2}(n+1+\alpha+t-2d)}} h_\delta(\xi\ell, \xi_m)^{p/2} N_p(f, \xi_m)
\end{align*}
We begin with the estimate for $B_p(f)$. Because we are assuming that $\beta(\xi\ell, \xi_m) > R/4$, by Lemma 5.1 given $\varepsilon > 0$, we can take $R$ big enough so that
\begin{equation}
\sum_m \frac{(1 - |\xi_m|^2)^{\frac{r}{2}(n+1+\alpha+t-\delta)} h_\delta(\xi\ell, \xi_m)^{p/2}}{|1 - \langle \xi\ell, \xi_m \rangle|^{\frac{r}{2}(n+1+\alpha+t)}} \lesssim \varepsilon (1 - |\xi\ell|^2)^{-pd}. \tag{5.10}
\end{equation}
Therefore,
\begin{align*}
B_p(f) &\lesssim \varepsilon \sum_{\ell} (1 - |\xi\ell|^2)^{pd} N_p(f, \xi\ell)
\end{align*}
\begin{align*}
&= \varepsilon \sum_i MO_{2r}(f)(\xi_i)^p (1 - |\xi_i|^2)^{pd} \sum_{\ell} \frac{(1 - |\xi\ell|^2)^{pd} N_p(f, \xi\ell)}{|1 - \langle \xi\ell, \xi_i \rangle|^{pd}}
\end{align*}
\begin{align*}
&\lesssim \varepsilon \sum_i MO_{2r}(f)(\xi_i)^p.
\end{align*}
In the estimate for $C_p(f)$ is when we use our assumption $\frac{r}{2}(n + 1 + \alpha) > n$. Since $\delta > 0$ has been taken so that $\frac{r}{2}(n + 1 + \alpha - 3\delta) > n$, we can apply Lemma 5.1 to get
\begin{equation}
\sum_{\ell} \frac{(1 - |\xi\ell|^2)^{\frac{r}{2}(n+1+\alpha-2d)} h_\delta(\xi\ell, \xi_m)^{p/2}}{|1 - \langle \xi\ell, \xi_m \rangle|^{\frac{r}{2}(n+1+\alpha+t-2d)}} \lesssim \varepsilon (1 - |\xi_m|^2)^{-\frac{r}{2}(t+3\delta-2d)}. \tag{5.11}
\end{equation}
Then, proceeding as before, we obtain
\begin{equation}
C_p(f) \lesssim \varepsilon \sum_m (1 - |\xi_m|^2)^{pd} N_p(f, \xi_m) \lesssim \varepsilon \sum_i MO_{2r}(f)(\xi_i)^p.
\end{equation}
Joining (5.11), (5.10) and (5.9), we have proved that (5.8) holds. The proof is complete.

6. The last case: $0 < p \leq \frac{2n}{n+1+\alpha}$

In order to prove this case, we will fix a number $\beta > \alpha$ satisfying $p(n + 1 + \beta) > 2n$. We will show that condition (a) of Theorem 1 implies that both $H^\beta_f$ and $H^\beta_{\ell,1}$ are in $S_p(A^2_\beta, L^2_\beta)$. Then the case already proved will give $MO_{\ell}(f) \in L^p(\mathbb{R}_n, d\lambda_n)$. 
We will use that, under the pairing \( \langle \cdot , \cdot \rangle_\gamma \) with \( \gamma = (\alpha + \beta)/2 \), the dual of \( L^2_\alpha \) can be identified with \( L^2_\beta \). Thus, if \( T \) is an operator in \( L^2_\alpha \), we can consider its adjoint operator \( S \) respect to the pairing \( \langle \cdot , \cdot \rangle_\gamma \) (acting now on \( L^2_\beta \)) defined by the relation

\[
(Tu, v)_{\gamma} = \langle u, Sv \rangle_{\gamma}, \quad u \in L^2_\alpha, \quad v \in L^2_\beta.
\]

**Lemma 6.1.** Let \( T \in S_p(L^2_\alpha) \). Then the operator \( S \) defined by (6.1) is in \( S_p(L^2_\beta) \). Moreover \( \|T\|_{sp} \leq \|S\|_{sp} \).

**Proof.** Let

\[
Tu = \sum_n \lambda_n \langle u, e_n \rangle_\alpha \sigma_n, \quad u \in L^2_\alpha
\]

be the canonical decomposition of the operator \( T \), where \( \{e_n\} \) and \( \{\sigma_n\} \) are orthonormal sets of \( L^2_\alpha \), and \( \{\lambda_n\} \) are the singular values of \( T \). For each \( n \), consider the functions

\[
f_n(z) = e_n(z)(1 - |z|^2)^{\alpha - \gamma} \quad \text{and} \quad h_n(z) = \sigma_n(z)(1 - |z|^2)^{\alpha - \gamma}.
\]

Then \( \{f_n\} \) and \( \{h_n\} \) are orthogonal sets in \( L^2_\beta \), with \( \|f_n\|_\beta = \|h_n\|_\beta = \sqrt{c_\beta/c_\alpha} \), where \( c_\alpha \) is the normalizing constant appearing in the definition of \( dv_\alpha \). Also

\[
\langle u, e_n \rangle_\alpha = K_{\alpha, \gamma} \langle u, f_n \rangle_\gamma
\]

with \( K_{\alpha, \gamma} = c_\alpha/c_\gamma \). Then it follows that

\[
Sv = K_{\alpha, \gamma} \sum_n \lambda_n \langle v, \sigma_n \rangle_\gamma f_n, \quad v \in L^2_\beta.
\]

Since \( \langle \sigma_n, v \rangle_\gamma = (c_\gamma/c_\beta) \langle h_n, v \rangle_\beta \), normalizing the functions \( f_n \) and \( h_n \) in \( L^2_\beta \) we see that \( \{\lambda_n\} \) are the singular values of the operator \( S \) acting on \( L^2_\beta \). This gives the result. \( \square \)

**Lemma 6.2.** Suppose that \( H^\alpha_f \) and \( H^\alpha_T \) are both in \( S_p(A^2_\alpha, L^2_\alpha) \). Then the commutator \([M_f, P_\gamma]\) is in \( S_p(L^2_\alpha) \).

**Proof.** It is enough to show that \([M_f, P_\gamma] - [M_f, P_\alpha]\) is in \( S_p(L^2_\alpha) \). Some algebraic manipulations give

\[
[M_f, P_\gamma] - [M_f, P_\alpha] = M_f P_\gamma - P_\gamma M_f - M_f P_\alpha + P_\alpha M_f
\]

\[
= M_f (P_\gamma - P_\alpha) - \tilde{H}^\alpha_f - P_\gamma M_f P_\gamma + \tilde{H}^\alpha_f + P_\alpha M_f P_\alpha.
\]

Here \( \tilde{H}^\alpha_f = (I - P_\alpha)M_f P_\alpha \). We already know that \( \tilde{H}^\alpha_f \) is in \( S_p(L^2_\alpha) \), and by Lemma 4.3 we also have \( \tilde{H}^\alpha_f \in S_p(L^2_\alpha) \) because \( P_\gamma : L^2_\alpha \to A^2_\alpha \) is bounded. Thus, it is enough to see that the operator

\[
T := M_f (P_\gamma - P_\alpha) - P_\gamma M_f P_\gamma + P_\alpha M_f P_\alpha
\]

is in \( S_p(L^2_\alpha) \). Since \( P_\gamma = P_\alpha P_\gamma \) and \( P_\alpha = P_\gamma P_\alpha \) on \( L^2_\alpha \), we have

\[
T = (I - P_\alpha)M_f (P_\gamma - P_\alpha) + (P_\alpha - P_\gamma)M_f P_\gamma
\]

\[
= \tilde{H}^\alpha_f (P_\gamma - I) - P_\gamma \tilde{H}^\alpha_f P_\gamma.
\]
This shows that $T$ is in $S_p(L^2_\alpha)$ finishing the proof. \hfill \square

Now that we know that the commutator $T = [M_f, P_\gamma]$ is in $S_p(L^2_\alpha)$, an application of Lemma 6.1 gives that its adjoint $S$ respect to the pairing $\langle \cdot, \cdot \rangle_\gamma$ is in $S_p(L^2_\beta)$. A simple computation gives $S = -[M_{\overline{T}}, P_\gamma]$. Since $P_\gamma$ is bounded on $L^2_\beta$ (from Theorem 2.11), we have that $P_\gamma$ is bounded on $L^2_\beta$. Hence $H^\beta_T$ belongs to $S_p(A^2_\beta, L^2_\beta)$. To see this it is enough to prove that $H^\beta_T - H^\alpha_T$ is in $S_p(A^2_\beta, L^2_\beta)$, but using that $P_\gamma = P_\beta P_\gamma$, we have

$$H^\beta_T - H^\alpha_T = (P_\gamma - P_\beta)M_{\overline{T}} = -P_\beta H^\alpha_T,$$

and the result follows. In the same manner we also have $H^\beta_T$ in $S_p(A^2_\beta, L^2_\beta)$. This completes the proof of Theorem 1.

7. Further remarks

One can also consider the problem of describing the simultaneous membership of $H^\alpha_T$ and $H^\beta_T$ in $S_p(L^2_\beta, A^2_\alpha)$, that is, when the weights are not necessarily the same, in the lines of the results of Janson and Wallstén in the holomorphic case. The result that can be obtained following the proof given here is that $H^\alpha_T$ and $H^\beta_T$ are both in $S_p(L^2_\beta, A^2_\alpha)$ if and only if the function $(1 - |z|^2)^\gamma MO_{\alpha,t}(f)(z)$ is in $L^p(\mathbb{B}_n, d\lambda_n)$, with $\gamma = (\alpha - \beta)/2$. The general form of Theorem 2 that can be proved is: let $\gamma \in \mathbb{R}$ with $2\gamma < 1 + \alpha$ and $0 < p < \infty$. Then, for each $t \geq 0$ such that $p > 2n/(n + 1 + \alpha + \gamma + 2t)$, one has

$$\int_{\mathbb{B}_n} (1 - |z|^2)^\gamma p MO_{\alpha,t}(f)(z)^p d\lambda_n(z) \leq C \int_{\mathbb{B}_n} (1 - |z|^2)^\gamma p MO_{\alpha}(f)(z)^p d\lambda_n(z).$$

The proof, as well as the other analogues needed, is essentially the same but more technical in the sense that more parameters are involved.

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