Acoustic and Filtration Properties of Thermo-elastic porous medium: Biot’s Equations of Thermo-Poroelasticity.

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Abstract. A linear system of differential equations describing a joint motion of thermoelastic porous body and incompressible thermofluid occupying porous space is considered. Although the problem is linear, it is very hard to tackle due to the fact that its main differential equations involve non-smooth oscillatory coefficients, both big and small, under the differentiation operators. The rigorous justification is fulfilled for homogenization procedures as the dimensionless size of the pores tends to zero, while the porous body is geometrically periodic. As the results, we derive Biot’s like system of equations of thermo-poroelasticity, system of equations of thermo-viscoelasticity, or system of non-isotropic Lamé’s equations depending on ratios between physical parameters and geometry of porous space. The proofs are based on Nguetseng’s two-scale convergence method of homogenization in periodic structures.

Key words: Biot’s equations, Stokes equations, Lamé’s equations, two-scale convergence, homogenization of periodic structures, thermo-poroelasticity.

Introduction

In the present publication we consider a problem of a joint motion of thermoelastic deformable solid (thermoelastic skeleton), perforated by a system of channels (pores) and incompressible thermofluid occupying a porous space. We refer to this model as to model (NA). In dimensionless variables (without primes)

\[ x' = Lx, \quad t' = \tau t, \quad w' = Lw, \quad \theta' = \partial_t \frac{L}{\tau v_*} \theta \]

the differential equations of the model in a domain \( \Omega \subseteq \mathbb{R}^3 \) for the dimensionless displacement vector \( w \) of the continuum medium and the dimension-
less temperature $\theta$, have a form:

$$
\alpha_\tau \bar{\rho} \frac{\partial^2 w}{\partial t^2} = \text{div}_x \mathbb{P} + \bar{\rho} \mathbf{F},
$$

(0.1)

$$
\alpha_\tau \bar{c}_p \frac{\partial \theta}{\partial t} = \text{div}_x (\bar{\alpha}_\kappa \nabla_x \theta) - \bar{\alpha}_\theta \frac{\partial}{\partial t} \text{div}_x \mathbf{w} + \Psi,
$$

(0.2)

$$
\mathbb{P} = \bar{\chi} \alpha \mu D(x, \frac{\partial \mathbf{w}}{\partial t}) + (1 - \bar{\chi}) \alpha \lambda D(x, \mathbf{w}) - (q + \pi) \mathbb{I},
$$

(0.3)

$$
q = p + \frac{\alpha_\nu \frac{\partial p}{\partial t}}{\alpha_p} + \bar{\chi} a_{\nu f} \theta,
$$

(0.4)

$$
p + \bar{\chi} \alpha_p \text{div}_x \mathbf{w} = 0,
$$

(0.5)

$$
\pi + (1 - \bar{\chi}) (\alpha_\eta \text{div}_x \mathbf{w} - \alpha_\theta s \theta) = 0.
$$

(0.6)

Here and further we use notations

$$
D(x, u) = (1/2) \left( \nabla_x u + (\nabla_x u)^T \right),
$$

$$
\bar{\rho} = \bar{\chi} \rho_f + (1 - \bar{\chi}) \rho_s, \quad \bar{c}_p = \bar{\chi} c_{pf} + (1 - \bar{\chi}) c_{ps},
$$

$$
\bar{\alpha}_\kappa = \bar{\chi} \alpha_{\kappa f} + (1 - \bar{\chi}) \alpha_{\kappa s}, \quad \bar{\alpha}_\theta = \bar{\chi} \alpha_{\theta f} + (1 - \bar{\chi}) \alpha_{\theta s}.
$$

In this model the characteristic function of the porous space $\bar{\chi}(x)$ is a known function. For derivation of (0.1)–(0.6) and description of dimensionless constants (all these constants are positive) see [11].

We endow model (NA) with initial and boundary conditions

$$
\mathbf{w}|_{t=0} = \mathbf{w}_0, \quad \frac{\partial \mathbf{w}}{\partial t}|_{t=0} = \mathbf{v}_0, \quad \theta|_{t=0} = \theta_0, \quad \mathbf{x} \in \Omega
$$

(0.7)

$$
\mathbf{w} = 0, \quad \theta = 0, \quad \mathbf{x} \in S = \partial \Omega, \quad t \geq 0.
$$

(0.8)

From the purely mathematical point of view, the corresponding initial-boundary value problem for model (NA) is well-posed in the sense that it has a unique solution belonging to a suitable functional space on any finite temporal interval (see [11]). However, in view of possible applications this model is ineffective. Therefore arises a question of finding an effective approximate models. If the model involves the small parameter $\varepsilon$, the most natural approach to this problem is to derive models that would describe limiting regimes arising as $\varepsilon$ tends to zero. Such an approximation significantly simplifies the original problem and at the same time preserves all of its main features. In the model under consideration we define $\varepsilon$ as the characteristic size of pores $l$ divided by the characteristic size $L$ of the entire porous body:

$$
\varepsilon = \frac{l}{L}.
$$
But even this approach is too hard to work out, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, the most appropriate is to simplify the problem postulating that the porous structure is periodic. Further by model \(\text{(NB)}^\varepsilon\) we will call model NA supplemented by this periodicity condition. Thus, our main goal now is a derivation of all possible homogenized equations in the model \(\text{(NB)}^\varepsilon\).

We accept the following constraints

**Assumption 0.1.** domain \(\Omega = (0, 1)^3\) is a periodic repetition of an elementary cell \(Y^\varepsilon = \varepsilon Y\), where \(Y = (0, 1)^3\) and quantity \(1/\varepsilon\) is integer, so that \(\Omega\) always contains an integer number of elementary cells \(Y^\varepsilon\). Let \(Y_s\) be a ”solid part” of \(Y\), and the ”liquid part” \(Y_f\) – is its open complement. We denote as \(\gamma = \partial Y_f \cap \partial Y_s\) and \(\gamma\) is \(C^1\)-surface. A porous space \(\Omega_f^\varepsilon\) is the periodic repetition of the elementary cell \(\varepsilon Y_f\), and solid skeleton \(\Omega_s^\varepsilon\) is the periodic repetition of the elementary cell \(\varepsilon Y_s\). A boundary \(\Gamma^\varepsilon = \partial \Omega_f^\varepsilon \cap \partial \Omega_s^\varepsilon\) is the periodic repetition in \(\Omega\) of the boundary \(\varepsilon \gamma\). The ”solid skeleton” \(\Omega_s\) is a connected domain.

In these assumption

\[
\bar{\chi}(\mathbf{x}) = \chi^\varepsilon(\mathbf{x}) = \chi(\mathbf{x}/\varepsilon),
\]
\[
\bar{c}_p = c_p^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})c_{pf} + (1 - \chi^\varepsilon(\mathbf{x}))c_{ps},
\]
\[
\bar{\rho} = \rho^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\rho_f + (1 - \chi^\varepsilon(\mathbf{x}))\rho_s,
\]
\[
\bar{\alpha}_\kappa = \alpha^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\alpha_{\kappa f} + (1 - \chi^\varepsilon(\mathbf{x}))\alpha_{\kappa s},
\]
\[
\bar{\alpha}_\theta = \alpha^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\alpha_{\theta f} + (1 - \chi^\varepsilon(\mathbf{x}))\alpha_{\theta s},
\]

where \(\chi(y)\) is a characteristic function of \(Y_f\) in \(Y\).

We say that a porous space is disconnected (isolated pores) if \(\gamma \cap \partial Y = \emptyset\).

In the present work we suppose that all dimensionless parameters depend on the small parameter \(\varepsilon\) and there exist limits (finite or infinite)

\[
\lim_{\varepsilon \searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon \searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon \searrow 0} \alpha_\tau(\varepsilon) = \tau_0, \quad \lim_{\varepsilon \searrow 0} \alpha_p(\varepsilon) = p_*.\n\]

Moreover, we restrict ourself with the case when \(\tau_0 < \infty\) and

\[
\mu_0 = 0, \quad p_* = \infty, \quad 0 < \lambda_0 < \infty.
\]
If $\tau_0 = \infty$, then, re-normalizing the displacement vector and temperature by setting
\[ w \rightarrow \alpha_\tau w, \quad \theta \rightarrow \alpha_\tau \theta \]
we reduce the problem to the previous case. The condition $p_* = \infty$ means that the liquid in consideration is incompressible.

Using Nguetseng’s two-scale convergence method \cite{9, 12} we derive Biot’s like systems of thermo-poroelasticity or system of non-isotropic Lamé’s, depending on the ratios between dimensionless parameters and geometry of the porous space.

Different isothermic models have been considered in \cite{14}, \cite{2}, \cite{13}, \cite{4, 5, 6}, \cite{10}.

§1. Model $(NB)^\varepsilon$

§1. Formulation of the main results.

As usual, equations (0.1)-(0.6) are understood in the sense of distributions. They involve the equations (0.1)–(0.6) in a usual sense in the domains $\Omega^\varepsilon_f$ and $\Omega^\varepsilon_s$ and the boundary conditions
\begin{align}
[\vartheta] &= 0, \quad [w] = 0, \quad x_0 \in \Gamma^\varepsilon, \ t \geq 0, \tag{1.1} \\
[P] &= 0, \quad [\alpha_\varepsilon \nabla x \theta] = 0, \quad x_0 \in \Gamma^\varepsilon, \ t \geq 0 \tag{1.2}
\end{align}
on the boundary $\Gamma^\varepsilon$, where
\[ \varphi(x_0) = \varphi(s)(x_0) - \varphi(f)(x_0), \]
\[ \varphi(s)(x_0) = \lim_{x \to x_0} \varphi(x), \quad \varphi(f)(x_0) = \lim_{x \to x_0} \varphi(x). \]

There are various equivalent in the sense of distributions forms of representation of equations (0.1)-(0.2) and boundary conditions (1.1)-(1.2). In what follows, it is convenient to write them in the form of the integral equalities.

**Definition 1.1.** Five functions $(w^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ are called a generalized solution of model $(NB)^\varepsilon$ if they satisfy the regularity conditions in the domain $\Omega_T = \Omega \times (0, T)$
\begin{align}
\{w^\varepsilon, w^\varepsilon(x), \text{div}_x w^\varepsilon, q^\varepsilon, p^\varepsilon, \frac{\partial p^\varepsilon}{\partial t}, \pi^\varepsilon, \theta^\varepsilon, \nabla x \theta^\varepsilon \} &\in L^2(\Omega_T) \tag{1.3}
\end{align}
in the domain $\Omega_T = \Omega \times (0, T)$, boundary conditions (1.8), equations

\begin{align*}
q^\varepsilon &= p^\varepsilon + \frac{\alpha_\nu}{\alpha_p} \frac{\partial p^\varepsilon}{\partial t} + \chi^\varepsilon \alpha_{\theta f} \theta^\varepsilon, \quad (1.4) \\
p^\varepsilon + \chi^\varepsilon \alpha_p \text{div}_x w^\varepsilon &= 0, \quad (1.5) \\
\pi^\varepsilon + (1 - \chi^\varepsilon)(\alpha_\eta \text{div}_x w^\varepsilon - \alpha_{\theta s} \theta^\varepsilon) &= 0 \quad (1.6)
\end{align*}

a.e. in $\Omega_T$, and integral identities

\begin{align*}
\int_{\Omega_T} \left( \alpha \tau \rho \varepsilon \cdot \frac{\partial^2 \varphi}{\partial t^2} - \chi^\varepsilon \alpha_\mu \mathbb{D}(x, w^\varepsilon) : \mathbb{D}(x, \frac{\partial \varphi}{\partial t}) - \rho^\varepsilon \mathbf{F} \cdot \varphi + \{(1 - \chi^\varepsilon) \alpha \lambda \mathbb{D}(x, w^\varepsilon) - (q^\varepsilon + \pi^\varepsilon) \mathbb{I}\} : \mathbb{D}(x, \varphi) \right) dx dt + \\
\int_{\Omega} \alpha \tau \rho^\varepsilon \left( w^\varepsilon_0 \cdot \frac{\partial \varphi}{\partial t} \big|_{t=0} - w^\varepsilon_0 \cdot \varphi \big|_{t=0} \right) dx &= 0 \quad (1.7)
\end{align*}

for all smooth vector-functions $\varphi = \varphi(x, t)$ such that $\varphi|_{\partial \Omega} = \varphi|_{t=T} = \frac{\partial \varphi}{\partial t}|_{t=T} = 0$ and

\begin{align*}
\int_{\Omega_T} \left( (\alpha \tau c_\theta \theta^\varepsilon + \alpha_{\theta f} \text{div}_x w^\varepsilon) \frac{\partial \xi}{\partial t} - \alpha \chi \nabla_x \theta^\varepsilon \cdot \nabla_x \xi + \Psi \xi \right) dx dt + \\
\int_{\Omega} (\alpha \tau c_\theta \theta^\varepsilon + \alpha_{\theta f} \text{div}_x w^\varepsilon_0) \xi \big|_{t=0}) dx &= 0 \quad (1.8)
\end{align*}

for all smooth functions $\xi = \xi(x, t)$ such that $\xi|_{\partial \Omega} = \xi|_{t=T} = 0$.

In (1.4) by $A : B$ we denote the convolution (or, equivalently, the inner tensor product) of two second-rank tensors along the both indexes, i.e., $A : B = \text{tr} (B^* \circ A) = \sum_{i,j=1}^3 A_{ij} B_{ji}$.

Suppose additionally that there exist limits (finite or infinite)

\begin{align*}
\lim_{\varepsilon \to 0} \alpha_\nu(\varepsilon) &= \nu_0, \quad \lim_{\varepsilon \to 0} \alpha_\eta(\varepsilon) = \eta_0, \quad \lim_{\varepsilon \to 0} \alpha_{\theta s}(\varepsilon) = \kappa_0, \quad \lim_{\varepsilon \to 0} \alpha_{\theta f}(\varepsilon) = \beta_{0f}, \\
\lim_{\varepsilon \to 0} \alpha_{\theta s}(\varepsilon) &= \beta_{0s}, \quad \lim_{\varepsilon \to 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \quad \lim_{\varepsilon \to 0} \frac{\alpha_{\theta f}}{\alpha_\mu} = \kappa_f.
\end{align*}

In what follows we suppose to be held

**Assumption 1.1.** 1) Dimensionless parameters in the model $\{NB\}^\varepsilon$ satisfy to next restrictions

\begin{align*}
\mu_0 = 0; \quad 0 < r_0 + \mu_1, \quad \kappa_0, \quad \kappa_f, \quad \lambda_0, \quad \eta_0.
\end{align*}
\[ \tau_0, \quad \kappa, \quad \tau_0, \quad \nu_0, \quad \beta_{0f}, \quad \beta_{0s} \quad \lambda_0 < \infty. \]

2) Sequences \( \{\sqrt{\alpha}(1-\chi^\varepsilon)\nabla w_0^\varepsilon\}, \{\sqrt{\alpha_p}v_0^\varepsilon\}, \{\sqrt{\alpha_p}\alpha \nabla \cdot (1-\chi^\varepsilon)\nabla \vartheta_0^\varepsilon\}, \{\sqrt{\alpha_p}\alpha \nabla \cdot (1-\chi^\varepsilon)\nabla \varphi_0^\varepsilon\}, \{\sqrt{\alpha_p}\alpha \nabla \cdot (1-\chi^\varepsilon)\nabla \varphi_0^\varepsilon\}, \{\alpha_0^\varepsilon\}, \{\beta_0^\varepsilon\} \) are uniformly in \( \varepsilon \) bounded in \( L^2(\Omega) \) and \( |F|, |\partial F/\partial t|, \Psi, \partial \Psi/\partial t \in L^2(\Omega_T) \).

Here
\[
\begin{align*}
a_0^\varepsilon &= \text{div}_x \alpha \nabla \cdot w_0^\varepsilon + \tilde{\rho} F(x, 0), \\
c_p^\varepsilon b_0^\varepsilon &= \text{div}_x \alpha (\alpha \nabla \cdot \vartheta_0^\varepsilon) - \alpha \nabla v_0^\varepsilon + \Psi(x, 0), \\
\mathbb{P}_0^\varepsilon &= \chi \alpha \mu \mathbb{D}(x, v_0^\varepsilon) + (1-\chi^\varepsilon)\alpha \mu \mathbb{D}(x, w_0^\varepsilon) + \\
&\quad (\chi (\alpha_p \text{div}_x w_0^\varepsilon + \alpha \text{div}_x v_0^\varepsilon) + (1-\chi^\varepsilon)\alpha \text{div}_x w_0^\varepsilon) \mathbb{I}.
\end{align*}
\]

In what follows all parameters may take all permitted values. For example, if \( \tau_0 = 0 \) or \( \eta_0^{-1} = 0 \), then all terms in final equations containing these parameters disappear. The following Theorems 1.1, 1.2 are the main results of the paper.

**Theorem 1.1.** For all \( \varepsilon > 0 \) on the arbitrary time interval \( [0, T] \) there exists a unique generalized solution of model \( (NB)^\varepsilon \) and

\[
\max_{0 \leq t \leq T} \|w^\varepsilon(t)\|, \sqrt{\alpha_p} \|\nabla x w^\varepsilon(t)\|, (1-\chi^\varepsilon) \|\nabla_x w^\varepsilon(t)\||_{2, \Omega} \leq C_0, \tag{1.9}
\]

\[
\|\theta^\varepsilon\|_{2, \Omega_T} + \sqrt{\alpha_p} \|\chi \nabla \theta_0^\varepsilon\|_{2, \Omega_T} + \|(1-\chi^\varepsilon)\nabla \theta^\varepsilon\|_{2, \Omega_T} \leq C_0, \tag{1.10}
\]

\[
\|q^\varepsilon\|_{2, \Omega_T} + \|p^\varepsilon\|_{2, \Omega_T} + \frac{\alpha_p}{\alpha} \|\partial \varphi^\varepsilon\|_{2, \Omega_T} + \|\pi^\varepsilon\|_{2, \Omega_T} \leq C_0 \tag{1.11}
\]

where \( C_0 \) does not depend on the small parameter \( \varepsilon \).

**Theorem 1.2.** Functions \( w^\varepsilon \) and \( \theta^\varepsilon \) admit an extension \( u^\varepsilon \) and \( \vartheta^\varepsilon \) respectively from \( \Omega^\varepsilon_T = \Omega^\varepsilon \times (0, T) \) into \( \Omega_T \) such that the sequences \( \{u^\varepsilon\} \) and \( \{\vartheta^\varepsilon\} \) converge strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0, T); W^1_2(\Omega)) \) to the functions \( u \) and \( \vartheta \) respectively. At the same time, sequences \( \{w^\varepsilon\}, \{\theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\}, \) and \( \{\pi^\varepsilon\} \) converge weakly in \( L^2(\Omega_T) \) to \( w, \theta, p, q, \) and \( \pi \), respectively.

The following assertions for these limiting functions hold true:

1. If \( \mu_1 = \infty \) then \( w = u \), \( \theta = \vartheta \) and the weak limits \( u, \vartheta, p, q, \) and \( \pi \) satisfy in \( \Omega_T \) the initial-boundary value problem

\[
\tau_0 \partial^2 u + \nabla (q + \pi) - \tilde{\rho} F = \\
\text{div}_x \{ \lambda_0 A_0^\varepsilon : \mathbb{D}(x,u) + B_0^\varepsilon (\text{div}_x u - \frac{\beta_{0s}}{\beta_{0f}} \vartheta) + B_1^\varepsilon q \}. \tag{1.12}
\]
\begin{align}
(\tau_0 \hat{c}_p + \frac{\beta_0^2}{\eta_0}(1-m)\frac{\partial \hat{c}}{\partial t} - \frac{\beta_0 \partial \pi}{\eta_0} \frac{\partial q}{\partial t} + (a_1^s - \frac{1}{\eta_0})\frac{\partial q}{\partial t})_{\Omega} &= \text{div}_x (B^\theta \cdot \nabla \vartheta) + \Psi, \quad (1.13) \\
\frac{1}{\eta_0}(\pi + \langle q \rangle_{\Omega}) + C_0^s : \text{D}(x, u) + a_0^s(\text{div}_x u - \frac{\beta_0}{\eta_0}(\vartheta - \langle \vartheta \rangle_{\Omega})) + a_1^s(q - \langle q \rangle_{\Omega}) &= 0, \quad (1.14) \\
\frac{1}{\eta_0}(\pi + \langle q \rangle_{\Omega}) + \text{div}_x u + \frac{(1-m)\beta_0}{\eta_0}(\vartheta - \langle \vartheta \rangle_{\Omega}) &= 0, \quad (1.15) \\
q - \langle q \rangle_{\Omega} &= p + \beta_0 f(\theta - \langle \theta \rangle_{\Omega}), \quad (1.16)
\end{align}

where
\begin{align*}
\hat{\rho} &= m \rho_f + (1-m) \rho_s, \quad \hat{c}_p = m c_{pf} + (1-m) c_{ps}, \quad m = \int_Y \chi(y) dy.
\end{align*}

The symmetric strictly positively defined constant fourth-rank tensor $A_0^s$, constant matrices $C_0^s, B_0^s, B_1^s$, strictly positively defined constant matrix $B^\theta$ and constants $a_0^s, a_1^s$ and $a_2^s$ are defined below by Eqs. \((4.32)-\quad (4.34)\) and \((4.37)\).

Differential equations \((1.12)-\quad (1.16)\) are endowed with initial conditions at $t = 0$ and $x \in \Omega$
\begin{align}
(\tau_0 + \beta_0)(\vartheta - \vartheta_0) &= 0, \quad \tau_0(u - u_0) = \tau_0(\frac{\partial u}{\partial t} - v_0) = 0; \quad (1.17)
\end{align}

and boundary conditions
\begin{align}
\vartheta(x, t) &= 0, \quad u(x, t) = 0, \quad x \in S, \quad t > 0. \quad (1.18)
\end{align}

\textbf{(II)} If the porous space is disconnected, then $w = u$ and strong and weak limits $u, \vartheta, p, q, \pi$ together with a weak limit $\theta^f$ of the sequence $\{\chi^\varepsilon \theta^\varepsilon\}$ satisfy in $\Omega_T$ equations \((1.12), (1.14) - (1.15),\) the state equation
\begin{align}
q - \langle q \rangle_{\Omega} &= p + \beta_0 f(\theta^f - \langle \theta^f \rangle_{\Omega}), \quad (1.19)
\end{align}

and heat equation
\begin{align}
\tau_0 c_{pf} \frac{\partial \theta^f}{\partial t} + (\tau_0 c_{ps} + \frac{\beta_0^2}{\eta_0})(1-m)\frac{\partial \theta^f}{\partial t} - \frac{\beta_0 \partial \pi}{\eta_0} \frac{\partial q}{\partial t} + (a_1^s - \frac{1}{\eta_0}) \frac{\partial q}{\partial t}_{\Omega} = \\
\text{div}_x (B^\theta \cdot \nabla \vartheta) + \Psi. \quad (1.20)
\end{align}

Here $\theta^f$ is defined below by formulas \((4.39)-\quad (4.44)\).

The problem is endowed with initial and boundary conditions \((1.17) - (1.18)\).
If \( \mu_1 < \infty \) then strong and weak limits \( u, \vartheta, w^f, \theta^f, p, q \) and \( \pi \) of the sequences \( \{u^\varepsilon\}, \{\vartheta^\varepsilon\}, \{\chi^\varepsilon w^\varepsilon\}, \{\chi^\varepsilon \theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\} \) and \( \{\pi^\varepsilon\} \) satisfy the initial-boundary value problem in \( \Omega_T \), consisting of the balance of momentum equation

\[
\tau_0(\rho_f \frac{\partial^2 w^f}{\partial t^2} + \rho_s(1 - m) \frac{\partial^2 u}{\partial t^2}) + \nabla (q + \pi) - \hat{\rho} F = \left\{ \begin{array}{l}
\text{div}_x \{\lambda_0 A^s_0 : D(x, u) + B^s_0 \text{div}_x u + B^s_1 q\},
\end{array} \right.
\]

(1.21)

where \( A^s_0, B^s_0, B^s_1 \) are the same as in Eq. (1.12), continuity equation (1.14), continuity equation

\[
\frac{1}{\eta_0} (\pi + \langle q \rangle_\Omega) + \text{div}_x w^f + \frac{(1 - m) \beta_0}{\eta_0} (\vartheta - \langle \vartheta \rangle_\Omega) = (m - 1) \text{div}_x u,
\]

(1.22)

state equation (1.19), heat equation (1.20) and Darcy’s law in the form

\[
\frac{\partial w^f}{\partial t} = m \frac{\partial u}{\partial t} + \int_0^t B_1(\mu_1, t - \tau) \cdot (-\nabla_x q + \rho_f F - \tau_0 \rho_f \frac{\partial^2 u}{\partial \tau^2})(x, \tau)d\tau
\]

(1.23)

if \( \tau_0 > 0, \mu_1 > 0 \), Darcy’s law in the form

\[
\frac{\partial w^f}{\partial t} = \frac{\partial u}{\partial t} + B_2(\mu_1) \cdot (-\nabla_x q + \rho_f F)
\]

(1.24)

if \( \tau_0 = 0 \) and, finally, Darcy’s law in the form

\[
\frac{\partial w^f}{\partial t} = B_3 \cdot \frac{\partial u}{\partial t} + \frac{1}{\tau_0 \rho_f} (m I - B_3) \cdot \int_0^t (-\nabla_x q + \rho_f F)(x, \tau)d\tau
\]

(1.25)

if \( \mu_1 = 0 \).

The problem is supplemented by boundary and initial conditions (1.17)-(1.18) for the displacement \( u \) and temperature \( \vartheta \) of the rigid component and by the boundary condition

\[
w^f(x, t) \cdot n(x) = 0, \quad (x, t) \in \partial \Omega, \; t > 0
\]

(1.26)

for the displacement \( w^f \) of the liquid component. In Eqs. (1.23)-(1.26) \( n(x) \) is the unit normal vector to \( S \) at a point \( x \in S \), and matrices \( B_1(\mu_1, t), B_2(\mu_1) \), and \( B_3 \) are given below by formulas (4.50)-(4.55).

\[\S 2. \text{Preliminaries}\]
2.1. Two-scale convergence. Justification of Theorems 1.1–1.2 relies on systematic use of the method of two-scale convergence, which had been proposed by G. Nguetseng [12] and has been applied recently to a wide range of homogenization problems (see, for example, the survey [9]).

Definition 2.1. A sequence \( \{ \varphi^\varepsilon \} \subset L^2(\Omega_T) \) is said to be two-scale convergent to a limit \( \varphi \in L^2(\Omega_T \times Y) \) if and only if for any 1-periodic in \( y \) function \( \sigma = \sigma(x, t, y) \) the limiting relation

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \varphi^\varepsilon(x, t)\sigma(x, t, x/\varepsilon) \, dx \, dt = \int_{\Omega_T} \int_Y \varphi(x, t, y)\sigma(x, t, y) \, dy \, dx \, dt \tag{2.1}
\]

holds.

Existence and main properties of weakly convergent sequences are established by the following fundamental theorem [12, 9]:

Theorem 2.1. (Nguetseng’s theorem)

1. Any bounded in \( L^2(Q) \) sequence contains a subsequence, two-scale convergent to some limit \( \varphi \in L^2(\Omega_T \times Y) \).

2. Let sequences \( \{ \varphi^\varepsilon \} \) and \( \{ \varepsilon \nabla_x \varphi^\varepsilon \} \) be uniformly bounded in \( L^2(\Omega_T) \). Then there exist a 1-periodic in \( y \) function \( \varphi = \varphi(x, t, y) \) and a subsequence \( \{ \varphi^\varepsilon \} \) such that \( \varphi, \nabla_y \varphi \in L^2(\Omega_T \times Y) \), and \( \varphi^\varepsilon \) and \( \varepsilon \nabla_x \varphi^\varepsilon \) two-scale converge to \( \varphi \) and \( \nabla_y \varphi \), respectively.

3. Let sequences \( \{ \varphi^\varepsilon \} \) and \( \{ \nabla_x \varphi^\varepsilon \} \) be bounded in \( L^2(Q) \). Then there exist functions \( \varphi \in L^2(\Omega_T) \) and \( \psi \in L^2(\Omega_T \times Y) \) and a subsequence from \( \{ \varphi^\varepsilon \} \) such that \( \psi \) is 1-periodic in \( y \), \( \nabla_y \psi \in L^2(\Omega_T \times Y) \), and \( \varphi^\varepsilon \) and \( \nabla_x \varphi^\varepsilon \) two-scale converge to \( \varphi \) and \( \nabla_x \varphi(x, t) + \nabla_y \psi(x, t, y) \), respectively.

Corollary 2.1. Let \( \sigma \in L^2(Y) \) and \( \sigma^\varepsilon(x) := \sigma(x/\varepsilon) \). Assume that a sequence \( \{ \varphi^\varepsilon \} \subset L^2(\Omega_T) \) two-scale converges to \( \varphi \in L^2(\Omega_T \times Y) \). Then the sequence \( \sigma^\varepsilon \varphi^\varepsilon \) two-scale converges to \( \sigma \varphi \).

2.2. An extension lemma. The typical difficulty in homogenization problems while passing to a limit in Model (\( NB \)) as \( \varepsilon \to 0 \) arises because of the fact that the bounds on the gradient of displacement \( \nabla_x w^\varepsilon \) may be distinct in liquid and rigid phases. The classical approach in overcoming this difficulty consists of constructing of extension to the whole \( \Omega \) of the displacement field defined merely on \( \Omega_s \). The following lemma is valid due to the well-known results from [1, 7]. We formulate it in appropriate for us form:
Lemma 2.1. Suppose that assumptions of Sec. 1.2 on geometry of periodic structure hold, $\psi^\varepsilon \in W^1_2(\Omega^\varepsilon_g)$ and $\psi^\varepsilon = 0$ on $S^\varepsilon_\delta = \partial \Omega^\varepsilon_s \cap \partial \Omega$ in the trace sense. Then there exists a function $\sigma^\varepsilon \in W^1_2(\Omega)$ such that its restriction on the sub-domain $\Omega^\varepsilon_s$ coincide with $\psi^\varepsilon$, i.e.,

$$(1 - \chi^\varepsilon(x))(\sigma^\varepsilon(x) - \psi^\varepsilon(x)) = 0, \quad x \in \Omega,$$  

(2.2)

and, moreover, the estimate

$$\|\sigma^\varepsilon\|_{2,\Omega} \leq C\|\psi^\varepsilon\|_{2,\Omega^\varepsilon_s}, \quad \|\nabla_x \sigma^\varepsilon\|_{2,\Omega} \leq C\|\nabla_x \psi^\varepsilon\|_{2,\Omega^\varepsilon_s}$$  

(2.3)

hold true, where the constant $C$ depends only on geometry $Y$ and does not depend on $\varepsilon$.

2.3. Friedrichs–Poincaré’s inequality in periodic structure. The following lemma was proved by L. Tartar in [14, Appendix]. It specifies Friedrichs–Poincaré’s inequality for $\varepsilon$-periodic structure.

Lemma 2.2. Suppose that assumptions on the geometry of $\Omega^\varepsilon_f$ hold true. Then for any function $\varphi \in W^1_2(\Omega^\varepsilon_f)$ the inequality

$$\int_{\Omega^\varepsilon_f} |\varphi|^2 dx \leq C\varepsilon^2 \int_{\Omega^\varepsilon_f} |\nabla_x \varphi|^2 dx$$  

(2.4)

holds true with some constant $C$, independent of $\varepsilon$.

2.4. Some notation. Further we denote

1) $\langle \Phi \rangle_Y = \int_Y \Phi dy$, $\langle \Phi \rangle_{Y_f} = \int_{Y_f} \Phi dy$, $\langle \Phi \rangle_{Y_s} = \int_{Y_s} \Phi dy$.

2) If $a$ and $b$ are two vectors then the matrix $a \otimes b$ is defined by the formula

$$(a \otimes b) \cdot c = a(b \cdot c)$$

for any vector $c$.

3) If $B$ and $C$ are two matrices, then $B \otimes C$ is a forth-rank tensor such that its convolution with any matrix $A$ is defined by the formula

$$(B \otimes C) : A = B(C : A)$$

4) By $\mathbb{I}^{ij}$ we denote the $3 \times 3$-matrix with just one non-vanishing entry, which is equal to one and stands in the $i$-th row and the $j$-th column.

5) We also introduce

$$J^{ij} = \frac{1}{2}(\mathbb{I}^{ij} + \mathbb{I}^{ji}) = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i),$$

where $(e_1, e_2, e_3)$ are the standard Cartesian basis vectors.
§3. Proof of Theorem 1.1

Under restriction $\tau_0 > 0$ estimates (1.9)- (1.10) follow from

\[
\max_{0 \leq t < T} \left( \sqrt{\alpha_\eta} \| \text{div}_x \frac{\partial w^\varepsilon}{\partial t} (t) \|_{2, \Omega} + \sqrt{\alpha_p} \| \nabla_x \frac{\partial w^\varepsilon}{\partial t} (t) \|_{2, \Omega} \right)
+ \sqrt{\alpha_\tau} \left( \frac{\partial^2 w^\varepsilon}{\partial t^2} (t) \right) \|_{2, \Omega} + \sqrt{\alpha_{\kappa}} \left( \frac{\partial \theta^\varepsilon}{\partial t} (t) \right) \|_{2, \Omega} 
+ \sqrt{\alpha_\kappa} \| \chi^\varepsilon \nabla_x \frac{\partial^2 w^\varepsilon}{\partial t^2} \|_{2, \Omega} 
+ \sqrt{\alpha_\kappa} \| \chi^\varepsilon \nabla_x \frac{\partial \theta^\varepsilon}{\partial t} \|_{2, \Omega}
\]

\[
\leq C_0 \sqrt{\alpha_\tau}, \tag{3.1}
\]

where $C_0$ is independent of $\varepsilon$. Last estimate we obtain if we differentiate equations for $w^\varepsilon$ and $\theta^\varepsilon$ with respect to time, multiply first equation by $\frac{\partial^2 w^\varepsilon}{\partial t^2}$, second equation–by $\frac{\partial \theta^\varepsilon}{\partial t}$, integrate by parts and sum the result. The same estimate guaranties the existence and uniqueness of the generalized solution for the model $(NB)^\varepsilon$.

Estimate (1.11) for pressures follows from integral identity (1.7) and estimates (3.1) as an estimate of the corresponding functional, if we re-normalized pressures, such that

\[
\int_\Omega (q^\varepsilon(x, t) + \pi^\varepsilon(x, t)) \, dx = 0.
\]

Indeed, integral identity (1.7) and estimates (3.1) imply

\[
| \int_\Omega (q^\varepsilon + \pi^\varepsilon) \text{div}_x \psi \, dx | \leq C \| \nabla \psi \|_{2, \Omega}.
\]

Choosing now $\psi$ such that $(q^\varepsilon + \pi^\varepsilon) = \text{div}_x \psi$

we get the desired estimate for the sum of pressures $(q^\varepsilon + \pi^\varepsilon)$. Such a choice is always possible (see [3]), if we put

\[
\psi = \nabla \varphi + \psi_0, \quad \text{div}_x \psi_0 = 0, \quad \Delta \varphi = q^\varepsilon + \pi^\varepsilon, \quad \varphi \big|_{\partial \Omega} = 0, \quad (\nabla \varphi + \psi_0) \big|_{\partial \Omega} = 0.
\]

Note that the re-normalization of the pressures $(q^\varepsilon + \pi^\varepsilon)$ transforms continuity and state equations (1.4)-(1.6) for pressures into

\[
q^\varepsilon = p^\varepsilon + \frac{\alpha_\nu}{\alpha_p} \frac{\partial p^\varepsilon}{\partial t} + \chi^\varepsilon \left( \alpha_{\eta} \theta^\varepsilon + \gamma_f^\varepsilon \right), \tag{3.2}
\]

\[
\frac{1}{\alpha_p} \frac{\partial p^\varepsilon}{\partial t} + \chi^\varepsilon \text{div}_x w^\varepsilon = -\frac{1}{m} \beta^\varepsilon \chi^\varepsilon, \tag{3.3}
\]

\[
\frac{1}{\alpha_\eta} \pi^\varepsilon + (1 - \chi^\varepsilon) \left( \text{div}_x w^\varepsilon - \frac{\alpha_{\gamma s}}{\alpha_\eta} \theta^\varepsilon + \gamma_s^\varepsilon \right) = 0, \tag{3.4}
\]
where
\[
\beta^\varepsilon = (1 - \chi^\varepsilon) \text{div}_x w^\varepsilon)_\Omega, \quad m^\varepsilon_f = \langle q^\varepsilon \rangle_\Omega - \alpha_{\theta f} \langle \chi^\varepsilon \theta^\varepsilon \rangle_\Omega,
\]
\[
(1 - m) \gamma^\varepsilon_s = \frac{1}{\alpha_\eta} \langle q^\varepsilon \rangle_\Omega + \frac{\alpha_{\theta s}}{\alpha_\eta} \langle (1 - \chi^\varepsilon) \theta^\varepsilon \rangle_\Omega - \beta^\varepsilon.
\]

Note that the basic integral identity (1.7) permits to bound only the sum \((q^\varepsilon + \pi^\varepsilon)\). But thanks to the property that the product of these two functions is equal to zero, it is enough to get bounds for each of these functions. The pressure \(p^\varepsilon\) is bounded from the state equation (3.2), if we substitute the term \((\alpha_\nu / \alpha_\eta) \partial p^\varepsilon / \partial t\) from the continuity equation (3.3) and use estimate (3.1).

Estimation of \(w^\varepsilon\) and \(\theta^\varepsilon\) in the case \(\tau_0 = 0\) is not simple, and we outline it in more detail. As usual, we obtain the basic estimates if we multiply equations for \(w^\varepsilon\) by \(\partial w^\varepsilon / \partial t\), equation for \(\theta^\varepsilon\) by \(\theta^\varepsilon\), sum the result and then integrate by parts all obtained terms. The only two terms \(F \cdot \partial w^\varepsilon / \partial t\) and \(\Psi \cdot \theta^\varepsilon\) heed additional consideration here. First of all, on the strength of Lemma 2.1, we construct an extension \(u^\varepsilon\) of the function \(w^\varepsilon\) from \(\Omega^\varepsilon_s\) into \(\Omega^\varepsilon_f\) such that \(u^\varepsilon = w^\varepsilon\) in \(\Omega^\varepsilon_s\), \(u^\varepsilon \in W^1_2(\Omega)\) and
\[
\|u^\varepsilon\|_{2, \Omega} \leq C \|\nabla_x u^\varepsilon\|_{2, \Omega} \leq \frac{C}{\sqrt{\alpha_\lambda}} \|\sqrt{1 - \chi^\varepsilon} \nabla x \omega^\varepsilon\|_{2, \Omega}.
\]

After that we estimate \(\|w^\varepsilon\|_{2, \Omega}\) with the help of Friedrichs–Poincaré’s inequality in periodic structure (lemma 2.2) for the difference \((u^\varepsilon - w^\varepsilon)\):
\[
\|w^\varepsilon\|_{2, \Omega} \leq \|u^\varepsilon\|_{2, \Omega} + \|u^\varepsilon - w^\varepsilon\|_{2, \Omega} \leq \|u^\varepsilon\|_{2, \Omega} + C\varepsilon \|\chi^\varepsilon \nabla_x (u^\varepsilon - w^\varepsilon)\|_{2, \Omega}
\]
\[
\leq \|u^\varepsilon\|_{2, \Omega} + C\varepsilon \|\nabla_x u^\varepsilon\|_{2, \Omega} + C\varepsilon \|\nabla_x \omega^\varepsilon\|_{2, \Omega}
\]
\[
\leq \frac{C}{\sqrt{\alpha_\lambda}} \|\sqrt{1 - \chi^\varepsilon} \nabla x \omega^\varepsilon\|_{2, \Omega} + C\varepsilon \|\nabla_x \omega^\varepsilon\|_{2, \Omega}.
\]

The same method we apply for the temperature \(\theta^\varepsilon\): there is an extension \(\vartheta^\varepsilon\) of the function \(\theta^\varepsilon\) from \(\Omega^\varepsilon_s\) into \(\Omega^\varepsilon_f\) such that \(\vartheta^\varepsilon = \theta^\varepsilon\) in \(\Omega^\varepsilon_s\), \(\vartheta^\varepsilon \in W^1_2(\Omega)\) and
\[
\|\vartheta^\varepsilon\|_{2, \Omega} \leq C \|\nabla_x \vartheta^\varepsilon\|_{2, \Omega} \leq \frac{C}{\sqrt{\alpha_{\theta s}}} \|\sqrt{1 - \chi^\varepsilon} \nabla_x \vartheta^\varepsilon\|_{2, \Omega},
\]
\[
\|\theta^\varepsilon\|_{2, \Omega} \leq \frac{C}{\sqrt{\alpha_{\theta s}}} \|\sqrt{1 - \chi^\varepsilon} \nabla_x \theta^\varepsilon\|_{2, \Omega} + C\varepsilon \|\nabla_x \theta^\varepsilon\|_{2, \Omega}.
\]
Next we pass the derivative with respect to time from $\frac{\partial w^\varepsilon}{\partial t}$ to $\rho^\varepsilon F$ and bound all obtained new terms in a usual way with the help of Hölder and Grownwall’s inequalities.

The rest of the proof is the same as for the case $\tau_0 > 0$, if we use the consequence of (3.1):

$$\max_{0 < t < T} \alpha_t \| \frac{\partial^2 w^\varepsilon}{\partial t^2}(t) \|_{2,\Omega} \leq C_0.$$  

§4. Proof of Theorem 1.2

4.1. Weak and two-scale limits of sequences of displacement, temperatures and pressures. On the strength of Theorem 1.1, the sequences $\{\theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\}, \{\pi^\varepsilon\}$ and $\{w^\varepsilon\}$ are uniformly in $\varepsilon$ bounded in $L^2(\Omega_T)$. Hence there exist a subsequence of small parameters $\{\varepsilon > 0\}$ and functions $\theta, p, q, \pi$ and $w$ such that

$$\theta^\varepsilon \rightarrow \theta, \quad p^\varepsilon \rightarrow p, \quad q^\varepsilon \rightarrow q, \quad \pi^\varepsilon \rightarrow \pi, \quad w^\varepsilon \rightarrow w$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Due to Lemma 2.1 there is a function $u^\varepsilon \in L^\infty((0, T); W^1_2(\Omega))$ such that $u^\varepsilon = w^\varepsilon$ in $\Omega_s \times (0, T)$, and the family $\{u^\varepsilon\}$ is uniformly in $\varepsilon$ bounded in $L^\infty((0, T); W^1_2(\Omega))$. Therefore it is possible to extract a subsequence of $\{\varepsilon > 0\}$ such that

$$u^\varepsilon \rightarrow u \text{ weakly in } L^2((0, T); W^1_2(\Omega))$$

as $\varepsilon \searrow 0$.

Applying again the same lemma 2.1 we conclude that there is a function $\vartheta^\varepsilon \in L^\infty((0, T); W^1_2(\Omega))$ such that $\vartheta^\varepsilon = \theta^\varepsilon$ in $\Omega_s \times (0, T)$, and the family $\{\vartheta^\varepsilon\}$ is uniformly in $\varepsilon$ bounded in $L^2((0, T); W^1_2(\Omega))$. Therefore it is possible to extract a subsequence of $\{\varepsilon > 0\}$ such that

$$\vartheta^\varepsilon \rightarrow \vartheta \text{ weakly in } L^2((0, T); W^1_2(\Omega))$$

as $\varepsilon \searrow 0$.

Moreover,

$$\chi^\varepsilon \alpha_\mu \mathbb{D}(x, w^\varepsilon) \rightarrow 0, \quad \chi^\varepsilon \alpha_\kappa f \nabla \theta^\varepsilon \rightarrow 0 \quad (4.1)$$

as $\varepsilon \searrow 0$.

Relabelling if necessary, we assume that the sequences converge themselves.
On the strength of Nguetseng’s theorem, there exist 1-periodic in \( y \) functions \( \Theta(x, t, y), P(x, t, y), \Pi(x, t, y), Q(x, t, y), W(x, t, y), \Theta_s(x, t, y) \) and \( U(x, t, y) \) such that the sequences \( \{\theta^\varepsilon\}, \{p^\varepsilon\}, \{\pi^\varepsilon\}, \{q^\varepsilon\}, \{w^\varepsilon\}, \{\nabla_x \vartheta^\varepsilon\} \) and \( \{\nabla_x u^\varepsilon\} \) two-scale converge to \( \Theta(x, t, y), P(x, t, y), \Pi(x, t, y), Q(x, t, y), W(x, t, y), \nabla_x \vartheta + \nabla_y \Theta_s(x, t, y) \) and \( \nabla_x u + \nabla_y U(x, t, y) \), respectively.

Note that the sequence \( \{\text{div}_x w^\varepsilon\} \) weakly converges to \( \text{div}_x w \) and \( \vartheta, |u| \in L^2((0, T); W^1_2(\Omega)) \). Last assertion for disconnected porous space follows from inclusion \( \vartheta^\varepsilon, |u^\varepsilon| \in L^2((0, T); W^1_2(\Omega)) \) and for the connected porous space it follows from the Friedrichs–Poincaré’s inequality for \( u^\varepsilon \) and \( \vartheta^\varepsilon \) in the \( \varepsilon \)-layer of the boundary \( S \) and from convergence of sequences \( \{u^\varepsilon\} \) and \( \{\vartheta^\varepsilon\} \) to \( u \) and \( \vartheta \) respectively strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0, T); W^1_2(\Omega)) \).

4.2. Micro- and macroscopic equations I..

**Lemma 4.1.** For all \( x \in \Omega \) and \( y \in Y \) weak and two-scale limits of the sequences \( \{\theta^\varepsilon\}, \{p^\varepsilon\}, \{\pi^\varepsilon\}, \{q^\varepsilon\}, \{w^\varepsilon\}, \{\nabla_x \vartheta^\varepsilon\} \) and \( \{\nabla_x u^\varepsilon\} \) satisfy the relations

\[
\begin{align*}
Q &= \frac{1}{m} \chi q, \quad Q = P + \chi(\beta_0 f + \gamma f); \\
\frac{1}{\eta_0} \Pi + (1 - \chi)(\text{div}_x u + \text{div}_y U - \frac{\beta_0}{\eta_0} (\vartheta - \langle \vartheta \rangle_\Omega) + \gamma s) &= 0; \\
\text{div}_y W &= 0; \\
W &= \chi W + (1 - \chi) u; \\
\Theta &= \chi \Theta + (1 - \chi) \vartheta; \\
q &= p + \beta_0 f \theta f + m \gamma f; \\
\frac{1}{\eta_0} \pi + (1 - m)(\text{div}_x u - \frac{\beta_0}{\eta_0} (\vartheta - \langle \vartheta \rangle_\Omega) + \gamma s) + \langle \text{div}_y U \rangle_{Y_f} &= 0; \\
\frac{1}{\eta_0} \pi + \text{div}_x w - (1 - m)(\frac{\beta_0}{\eta_0} (\vartheta - \langle \vartheta \rangle_\Omega) - \gamma s) + \beta &= 0,
\end{align*}
\]

where

\[
\begin{align*}
\beta &= \langle (\text{div}_y U)_{Y_f} \rangle_\Omega, \quad \theta f = \langle \Theta \rangle_{Y_f}, \\
m \gamma f &= \langle q \rangle_\Omega - \beta_0 f \langle \theta f \rangle_\Omega, \quad (1 - m) \gamma s = \frac{1}{\eta_0} \langle q \rangle_\Omega - \beta.
\end{align*}
\]

**Proof.** In order to prove first equation in (4.2) into Eq.(1.7) insert a test function \( \psi^\varepsilon = \varepsilon \psi(x, t, x/\varepsilon) \), where \( \psi(x, t, y) \) is an arbitrary 1-periodic in \( y \) and finite on \( Y_f \) function. Passing to the limit as \( \varepsilon \to 0 \), we get

\[
\nabla_y Q(x, t, y) = 0, \quad y \in Y_f.
\]
The weak and two-scale limiting passage in Eq. (3.4) yield that Eq. (4.7) and the second equation in (4.2).

Next, fulfilling the two-scale limiting passage in the equalities

\[(1 - \chi^\epsilon)p^\epsilon = 0, \quad (1 - \chi^\epsilon)q^\epsilon = 0\]

we get

\[(1 - \chi)P = 0, \quad (1 - \chi)Q = 0,\]

which justify first equation in (4.2). Eqs. (4.3), (4.4), (4.8), and (4.9) appear as the results of two-scale limiting passages in Eqs. (3.2)–(3.4) with the proper test functions being involved. Thus, for example, Eq. (4.8) is just a subsequence of Eq. (4.3) and Eq. (4.9) is a result of two-scale convergence in the sum of Eq. (3.3) and Eq. (3.4) with the test functions independent of the “fast” variable \(y = x/\epsilon\). Eq. (4.4) is derived quite similarly if multiply the same sum of Eq. (3.3) and Eq. (3.4) by an arbitrary function \(\psi^\epsilon = \epsilon \psi(x, t, x/\epsilon)\) and pass to the limit as \(\epsilon \to 0\).

In order to prove Eqs. (4.5) and (4.6) it is sufficient to consider the two-scale limiting relations in

\[(1 - \chi^\epsilon)(w^\epsilon - u^\epsilon) = 0, \quad (1 - \chi^\epsilon)(\vartheta^\epsilon - \varphi^\epsilon) = 0.\]

\[\square\]

**Lemma 4.2.** For all \((x, t) \in \Omega_T\) and \(y \in Y\) the relation

\[\text{div}_{y}\{\lambda_0(1 - \chi)((\mathbb{D}(y, U) + \mathbb{D}(x, u)) - (\Pi + \frac{1}{m} q \chi) \cdot 1\} = 0. \quad (4.11)\]

holds true.

**Proof.** Substituting a test function of the form \(\psi^\epsilon = \epsilon \psi(x, t, x/\epsilon)\), where \(\psi(x, t, y)\) is an arbitrary 1-periodic in \(y\) function vanishing on the boundary \(\partial \Omega\), into Eq. (1.7) and passing to the limit as \(\epsilon \to 0\), we arrive at the desired microscopic relation on the cell \(Y\). \(\square\)

In the same way using additionally continuity equations (3.3) and (3.4) one gets from Eq. (1.8)

**Lemma 4.3.** For all \((x, t) \in \Omega_T\) the relations

\[\begin{align*}
\triangle_{y} \Theta^s &= 0, \quad y \in Y_s, \\
\frac{\partial \Theta^s}{\partial n} &= -\nabla_{x} \vartheta \cdot n, \quad y \in \gamma
\end{align*}\]

hold true.
Now we pass to the macroscopic equations for the solid displacements.

**Lemma 4.4.** Let \( \hat{\rho} = m\rho_f + (1-m)\rho_s \), \( \mathbf{w}^f = \langle \mathbf{W} \rangle_Y \). Then functions \( \mathbf{u}, \mathbf{w}^f, q, \pi, \theta^f, \vartheta \) satisfy in \( \Omega_T \) the system of macroscopic equations

\[
\begin{align*}
\tau_0 \rho_f \frac{\partial^2 \mathbf{w}^f}{\partial t^2} + \tau_0 \rho_s (1-m) \frac{\partial^2 \mathbf{u}}{\partial t^2} - \hat{\rho} \mathbf{F} = & \quad \text{(4.13)} \\
\text{div}_x \{ \lambda_0 ((1-m)\mathbb{D}(x, \mathbf{u}) + (\mathbb{D}(y, \mathbf{U}))_Y) - (q + \pi) \cdot \mathbb{I} \}, \\
\tau_0 c_{pf} \frac{\partial \theta^f}{\partial t} + (\tau_0 c_{ps} + \beta_0^2) (1-m) \frac{\partial \vartheta}{\partial t} - \frac{\beta_{0s}}{\eta_0} \frac{\partial \pi}{\partial t} - \beta_{0f} \frac{\partial \beta}{\partial t} = & \quad \text{(4.14)} \\
(1-m) \beta_{0s} \frac{\partial \vartheta}{\partial t} = & \quad \nu_{0s} \text{div}_x \{(1-m)\nabla \vartheta + \langle \nabla \Theta^s \rangle_Y \} + \Psi.
\end{align*}
\]

**Proof.** Eqs. (4.13) and (4.14) arise as the limit of Eqs. (1.7) and (1.8) with test functions being finite in \( \Omega_T \) and independent of \( \varepsilon \). In Eq. (1.8) we have used continuity equations (3.3) and (3.4). \( \square \)

### 4.3. Micro- and macroscopic equations II.

**Lemma 4.5.** If \( \mu_1 = \infty \), then \( \mathbf{u} = \mathbf{w} \) and \( \vartheta = \vartheta^f \).

**Proof.** In order to verify, it is sufficient to consider the differences \( (\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon) \) and \( (\vartheta^\varepsilon - \vartheta^f) \) and apply Friedrichs–Poincaré’s inequality, just like in the proof of Theorem 1.1. \( \square \)

**Lemma 4.6.** Let \( \mu_1 < \infty \) and \( \mathbf{V} = \chi \partial \mathbf{W} / \partial t \). Then

\[
\begin{align*}
\tau_0 \rho_f \frac{\partial \mathbf{V}}{\partial t} - \rho_f \mathbf{F} = & \quad \mu_1 \Delta_y \mathbf{V} - \nabla_y R - \nabla_x q, \quad y \in Y_f, \quad (4.15) \\
\tau_0 c_{pf} \frac{\partial \Theta}{\partial t} = & \quad \chi \mu_1 \Delta_y \Theta + \frac{\beta_{0f}}{m} \frac{\partial \beta}{\partial t} + \Psi, \quad y \in Y_f, \quad (4.16) \\
\mathbf{V} = & \quad \frac{\partial \mathbf{u}}{\partial t}, \quad \Theta = \vartheta, \quad y \in \gamma 
\end{align*}
\]

for \( \mu_1 > 0 \), and

\[
\begin{align*}
\tau_0 \rho_f \frac{\partial \mathbf{V}}{\partial t} = & \quad -\nabla_y R - \nabla_x q + \rho_f \mathbf{F}, \quad y \in Y_f, \quad (4.18) \\
\tau_0 c_{pf} \frac{\partial \Theta}{\partial t} = & \quad \frac{\beta_{0f}}{m} \frac{\partial \beta}{\partial t} + \Psi, \quad y \in Y_f, \quad (4.19) \\
(\chi \mathbf{W} - \mathbf{u}) \cdot \mathbf{n} = & \quad 0, \quad y \in \gamma 
\end{align*}
\]

for \( \mu_1 = 0 \).

In Eq. (4.20) \( \mathbf{n} \) is the unit normal to \( \gamma \).

16
Proof. Differential equations (4.15) and (4.18) follow as \( \varepsilon \to 0 \) from integral equality (1.7) with the test function \( \psi = \varphi(x\varepsilon^{-1}) \cdot h(x, t) \), where \( \varphi \) is solenoidal and finite in \( Y \) vector-function.

The same arguments apply for the Eq. (4.16) and Eq. (4.19). The only difference here is that we use the continuity equation (3.3) to exclude the term \( \chi \varepsilon \text{div}_x (\partial w^\varepsilon / \partial t) \).

First boundary condition in (4.17) is the consequence of the two-scale convergence of \( \{ \alpha_1^2 \nabla_x u^\varepsilon \} \) to the function \( \mu_1^2 \nabla_y W(x, t, y) \). On the strength of this convergence, the function \( \nabla_y W(x, t, y) \) is \( L^2 \)-integrable in \( Y \). As above we apply the same argument to the second boundary condition in (4.17). The boundary conditions (4.20) follow from Eqs. (4.4) and (4.5).

Lemma 4.7. If the porous space is disconnected, which is the case of isolated pores, then \( u = w \).

Proof. Indeed, in the case \( 0 \leq \mu_1 < \infty \) the systems of equations (4.4), (4.15) and (4.17), or (4.4), (4.18) and (4.20) have the unique solution \( V = \partial u / \partial t \).

4.4. Homogenized equations I.

Lemma 4.8. If \( \mu_1 = \infty \) then \( w = u \), \( \vartheta = \vartheta \) and the weak limits \( u \), \( \vartheta \), \( p \), \( q \), and \( \pi \) satisfy in \( \Omega_T \) the initial-boundary value problem

\[
\begin{align*}
\tau_0 \hat{c}_p \frac{\partial^2 u}{\partial t^2} + \nabla (q + \pi) - \rho \mathbf{F} = \\
\text{div}_x \{ \lambda_0 A_0^s : \nabla (x, u) + B_0^s (\text{div}_x u - \beta_0 \psi) + B_1 q \},
\end{align*}
\]

(4.21)

\[
(\tau_0 \hat{c}_p + \frac{\beta_0}{\eta_0} (1 - m) \frac{\partial \vartheta}{\partial t} - \frac{\beta_0 \beta}{\eta_0} \frac{\partial \pi}{\partial t} + (a_0^s - \frac{1}{\eta_0} \frac{\partial q}{\partial t}) \Omega = \text{div}_x (B^\theta \cdot \nabla \vartheta) + \Psi, (4.22)
\]

\[
\frac{1}{\eta_0} \pi + C_0^s : \nabla (x, u) + a_0^s (\text{div}_x u - \beta_0 \vartheta) + a_0^s q = \tilde{\gamma},
\]

(4.23)

\[
\frac{1}{\eta_0} \pi + \text{div}_x u + (1 - m) \beta_0 \vartheta = \tilde{\beta},
\]

(4.24)

\[
q = p + \beta_0 \rho \vartheta + m \gamma_f,
\]

(4.25)

where the symmetric strictly positively defined constant fourth-rank tensor \( A_0^s \), constant matrices \( C_0^s, B_0^s, B_1^s \), strictly positively defined constant matrix
\[ B^0 \] and constants \( a_0^0, a_1^1 \) and \( a_2^0 \) are defined below by formulas (4.32) - (4.34) and (4.37) and

\[
\tilde{\gamma} = (a_1^a - \frac{1}{\eta_0})\langle q \rangle_\Omega - a_0^a \frac{\beta_{0s}}{\eta_0} \langle \vartheta \rangle_\Omega, \quad -\tilde{\beta} = (1 - m) \frac{\beta_{0s}}{\eta_0} \langle \vartheta \rangle_\Omega + \frac{1}{\eta_0} \langle q \rangle_\Omega.
\]

Differential equations (4.21) and (4.22) are endowed with initial conditions at \( t = 0 \) and \( x \in \Omega \)

\[
(\tau_0 + \beta_{0s})(\vartheta - \vartheta_0) = 0, \quad \tau_0(u - u_0) = \tau_0\frac{\partial u}{\partial t} - v_0 = 0;
\]
and boundary conditions

\[
\vartheta(x, t) = 0, \quad u(x, t) = 0, \quad x \in S, \quad t > 0.
\]

**Proof.** In the first place let us notice that \( u = w \) and \( \theta = \vartheta \) due to Lemma 4.5.

The differential equations (4.21) follow from the macroscopic equations (4.13), after we insert in them the expression

\[
\langle D(y, U) \rangle_{Y_s} = \mathbb{A}_1 : \mathbb{D}(x, u) + B_0^s(\mathrm{div}_x u - \frac{\beta_{0s}}{\eta_0} (\vartheta - \langle \vartheta \rangle_\Omega)) + B_1^s(q - \langle q \rangle_\Omega).
\]

In turn, this expression follows by virtue of solutions of Eqs. (4.3) and (4.11) on the pattern cell \( Y_s \). Indeed, setting

\[
U = \sum_{i,j=1}^3 U_{ij}(y) D_{ij} + U_0(y)(\mathrm{div}_x u - \frac{\beta_{0s}}{\eta_0} (\vartheta - \langle \vartheta \rangle_\Omega)) + U_1(y)(q - \langle q \rangle_\Omega) + U_2(y)\langle q \rangle_\Omega,
\]

\[
\Pi = \lambda_0 \sum_{i,j=1}^3 \Pi_{ij}(y) D_{ij} + \Pi_0(y)(\mathrm{div}_x u - \frac{\beta_{0s}}{\eta_0} (\vartheta - \langle \vartheta \rangle_\Omega)) + \Pi_1(y)(q - \langle q \rangle_\Omega) + \Pi_2(y)\langle q \rangle_\Omega,
\]

where

\[
D_{ij} = \frac{1}{2} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),
\]
we arrive at the following periodic-boundary value problems in \( Y \):

\[
\begin{align*}
\mathrm{div}_y \left\{ (1 - \chi)(\mathbb{D}(y, U_{ij}) + J_{ij}) - \Pi_{ij} \cdot \mathbb{I} \right\} &= 0, \\
\frac{\lambda_0}{\eta_0} \Pi_{ij} + (1 - \chi)\mathrm{div}_y U_{ij} &= 0;
\end{align*}
\]
\[ \text{div}_y \{ \lambda_0 (1 - \chi) \mathcal{D}(y, U_0) - \Pi_0 \cdot \mathbb{I} \} = 0, \]
\[ \frac{1}{\eta_0} \Pi_0 + (1 - \chi) (\text{div} U_0 + 1) = 0; \]

\[ \text{div}_y \{ \lambda_0 (1 - \chi) \mathcal{D}(y, U_1) - (\Pi_1 + \frac{1}{m} \chi) \cdot \mathbb{I} \} = 0, \]
\[ \frac{1}{\eta_0} \Pi_1 + (1 - \chi) \text{div}_y U_1 = 0. \]

\[ \frac{1}{\eta_0} \Pi_2 + (1 - \chi) \text{div}_y U_2 - \frac{(1 - \chi)}{(1 - m)} (\langle \text{div}_y U_2 \rangle_{Y_s} + \frac{1}{\eta_0}) = 0. \]

Note, that

\[ \beta = \sum_{i,j=1}^{3} \langle \text{div}_y U_{ij} \rangle_{Y_s} \langle D_{ij} \rangle_{\Omega} + \langle \text{div}_y U_0 \rangle_{Y_s} \langle \text{div}_x u \rangle - \frac{\beta_{0s}}{\eta_0} (\bar{\vartheta} - \langle \bar{\vartheta} \rangle_{\Omega}) \rangle_{\Omega} + \\
\langle \text{div}_y U_1 \rangle_{Y_s} (q - \langle q \rangle_{\Omega}) + \langle \text{div}_y U_2 \rangle_{Y_s} (q) = \langle \text{div}_y U_2 \rangle_{Y_s} (q) \]
due to homogeneous boundary conditions for \( u(x, t) \).

On the strength of the assumptions on the geometry of the pattern “liquid” cell \( Y_s \), problems (4.29)–(4.31) have unique solution, up to an arbitrary constant vector. In order to discard the arbitrary constant vectors we demand

\[ \langle U_{ij} \rangle_{Y_s} = \langle U_0 \rangle_{Y_s} = \langle U_1 \rangle_{Y_s} = \langle U_2 \rangle_{Y_s} = 0. \]

Thus

\[ A_0^s = \sum_{i,j=1}^{3} J_{ij} \otimes J_{ij} + A_1^s, \quad A_1^s = \sum_{i,j=1}^{3} \langle (1 - \chi) D(y, U_{ij}) \rangle_{Y} \otimes J_{ij}. \] (4.32)

Symmetry and strict positiveness of the tensor \( A_0^s \) have been proved in [10].

Finally, Eqs.(4.23)–(4.25) for the pressures follow from Eqs. (4.7)–(4.9), after we insert in them the expression

\[ \langle \text{div}_y U \rangle_{Y_s} = C_0^s : \mathcal{D}(x, u) + \tilde{\alpha}_0^s (\text{div}_x u - \frac{\beta_{0s}}{\eta_0} (\bar{\vartheta} - \langle \bar{\vartheta} \rangle_{\Omega})) + \tilde{\alpha}_1^s (q - \langle q \rangle_{\Omega}) + \tilde{\alpha}_2^s (q) \]

where

\[ B_0^s = \langle \mathcal{D}(y, U_0) \rangle_{Y_s}, \quad B_1^s = \langle \mathcal{D}(y, U_1) \rangle_{Y_s}, \quad C_0^s = \sum_{i,j=1}^{3} \langle \text{div}_y U_{ij} \rangle_{Y_s} J_{ij}. \] (4.33)
\[
\tilde{a}_0^s = \langle \text{div}_y U_0 \rangle_{Y_s} = a_0^s - 1 + m, \quad a_1^s = \langle \text{div}_y U_1 \rangle_{Y_s}, \quad a_2^s = \langle \text{div}_y U_2 \rangle_{Y_s}.
\] (4.34)

Now for \(i = 1, 2, 3\) we consider the model problems

\[
\Delta_y \Theta^s_i = 0, \quad y \in Y_s,
\]
(4.35)

\[
\frac{\partial \Theta^s_i}{\partial n} = -e_i \cdot n, \quad y \in \gamma
\]
and put

\[
\Theta^s = \sum_{i=1}^{3} (\Theta^s_i \otimes e_i) \cdot \nabla_x \vartheta.
\]
(4.36)

Then \(\Theta^s\) solves the problem (4.12)–(4.14) and if we insert an expression \(\langle \nabla_y \Theta^s \rangle_{Y_s}\) into (4.14) we get

\[
B_{\theta}^s = \kappa_0^s ((1 - m) I + \sum_{i=1}^{3} (\langle \nabla_y \Theta^s_i \rangle_{Y_s} \otimes e_i)).
\]
(4.37)

All properties of the matrix \(B_{\theta}^s\) are well known (see [14], [7]).

**Lemma 4.9.** If the porous space is disconnected, then \(w = u\) and the weak limits \(\theta^f, u, \vartheta, p, q,\) and \(\pi\) satisfy in \(\Omega_T\) equations (4.21), (4.23), (4.24), (4.10), where \(A_0^s, C_0^s, B_0^s, B^s, a_0^s, a_1^s\) and \(a_2^s\) are the same as in Lemma 4.8, the state equation (4.7), and heat equation

\[
\begin{align*}
\tau_0 c_{pf} \frac{\partial \theta^f}{\partial t} + (\tau_0 c_{ps} + \frac{\beta_0^s}{\eta_0})(1 - m) \frac{\partial \vartheta}{\partial t} - \frac{\beta_{ox}}{\eta_0} \frac{\partial \pi}{\partial t} + (a_1^s - \frac{1}{\eta_0}) (\frac{\partial q}{\partial t})_\Omega &= \rightleftharpoons \nabla_x (B_{\theta}^s \cdot \nabla \vartheta) + \Psi,
\end{align*}
\]
(4.38)

where for \(\mu_1 > 0\) and \(\tau > 0\)

\[
\theta^f(x, t) = m \vartheta(x, t) + \int_0^t b_0^f(t - \tau) (\frac{1}{\tau_0 c_{pf}} (\beta_0 \frac{\partial \beta}{\partial t} + \Psi) - \frac{\partial \vartheta}{\partial t})(x, \tau) d\tau.
\]
(4.39)

If \(\mu_1 > 0\) and \(\tau = 0\), then

\[
\theta^f(x, t) = m \vartheta(x, t) - c_0^f (\frac{\beta_0 \frac{\partial \beta}{\partial t}}{m} (t) + \Psi(x, t)).
\]
(4.40)

Finally, if \(\mu_1 = 0\), then

\[
\theta^f(x, t) = m \vartheta_0(x) + \frac{m}{\tau_0 c_{pf}} \int_0^t (\frac{\beta_0 \frac{\partial \beta}{\partial t}}{m} (\tau) + \Psi(x, \tau)) d\tau.
\]
(4.41)

Here \(b_0^f(t)\) and \(c_0^f\) are defined below by formulas (4.42)–(4.44).

The problem is endowed with initial and boundary conditions (4.26) (4.27), (4.28)
Proof. The only one difference here with the previous lemma is the heat equation for \( \vartheta \) and the state equation for pressures, because \( \theta \neq \vartheta \). The function \( \theta^f = \langle \Theta \rangle_{\gamma_f} \) now is defined from microscopic equations (4.16) and (4.17), if \( \mu^1 > 0 \) and microscopic equations (4.19), if \( \mu^1 = 0 \).

Indeed, the solutions of above mentioned problems are given by formulas

\[
\Theta = \vartheta(x, t) + \int_0^t \Theta_1^f(y, t-\tau) h(x, \tau) d\tau,
\]

if \( \mu^1 > 0 \) and \( \tau > 0 \) and

\[
\Theta = \vartheta(x, t) - \Theta_0^f(y)(\frac{\beta_0 f \partial \beta}{m \partial t}(t) + \Psi(x, t)),
\]

if \( \mu^1 > 0 \) and \( \tau = 0 \), where

\[
h = \frac{1}{\tau_0 c_{pf}} \left( \frac{\beta_0 f \partial \beta}{m \partial t} + \Psi \right) - \frac{\partial \vartheta}{\partial t}
\]

and functions \( \Theta_1^f \) and \( \Theta_0^f \) are 1-periodic in \( y \) solutions of the problems

\[
\tau_0 c_{pf} \frac{\partial \Theta_1^f}{\partial t} = \kappa_1 \mu^1 \Delta_y \Theta_1^f, \quad y \in Y_f, \quad \Theta_1^f(y, 0) = 1, \quad y \in Y_f; \quad \Theta_1^f = 0, \quad y \in \gamma,
\]

(4.42)

and

\[
\kappa_1 \mu^1 \Delta_y \Theta_0^f = 1, \quad y \in Y_f; \quad \Theta_0^f = 0, \quad y \in \gamma.
\]

(4.43)

Then, in accordance with definition, the function \( \theta^f \) is given by (4.39) or (4.40), where

\[
b^f_j(t) = \langle \Theta_1^f \rangle_{\gamma_j}, \quad c^g_f = \langle \Theta_0^f \rangle_{\gamma_j}.
\]

(4.44)

If \( \mu^1 = 0 \), then \( \Theta \) is found by a simple integration in time.

\[ \square \]

4.5. Homogenized equations II.

Let \( \mu^1 < \infty \). In the same manner as above, we verify that the weak limit \( u \) of the sequence \( \{u^\varepsilon\} \) satisfies some initial-boundary value problem like problem (4.21) – (4.27) because, in general, the weak limit \( u \) of the sequence \( \{u^\varepsilon\} \) differs from \( u \). More precisely, the following statement is true.

Lemma 4.10. If \( \mu^1 < \infty \) then the weak limits \( u, w^f, \theta^f, \vartheta, p, q, \) and \( \pi \) of the sequences \( \{u^\varepsilon\}, \{x^\varepsilon u^\varepsilon\}, \{x^\varepsilon \theta^\varepsilon\}, \{\vartheta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\}, \) and \( \{\pi^\varepsilon\} \) satisfy the
initial-boundary value problem in $\Omega_T$, consisting of the balance of momentum equation

$$\tau_0(\rho_f \frac{\partial^2 w^f}{\partial t^2} + \rho_s(1 - m) \frac{\partial^2 u}{\partial t^2} + \nabla(q + \pi) - \dot{\rho} F = \right)$$

$$\text{div}_x\{\lambda_0 A^s_0 : \mathbb{D}(x, u) + B^s_0 \text{div}_x u + B^s_1 q\},$$

where $A^s_0$, $B^s_0$, and $B^s_1$ are the same as in (4.21), continuity equation (4.23),

continuity equation

$$\frac{1}{\eta}(\pi + \langle q \rangle_\Omega) + \text{div}_x w^f + \frac{(1 - m) \beta_0}{\eta_0}(\vartheta - \langle \vartheta \rangle_\Omega) = (m - 1) \text{div}_x u,$$  \hspace{1cm} (4.46)

state equation (4.7), heat equation (4.38) and Darcy’s law in the form

$$\frac{\partial w^f}{\partial t} = \frac{\partial u}{\partial t} + \int_0^t B_1(\mu_1, t - \tau) \cdot (-\nabla_x q + \rho_f F - \tau_0 \rho_f \frac{\partial^2 u}{\partial \tau^2})(x, \tau) d\tau \hspace{1cm} (4.47)$$

if $\tau_0 > 0$ and $\mu_1 > 0$, Darcy’s law in the form

$$\frac{\partial w^f}{\partial t} = \frac{\partial u}{\partial t} + B_2(\mu_1) \cdot (-\nabla_x q + \rho_f F) \hspace{1cm} (4.48)$$

if $\tau_0 = 0$ and, finally, Darcy’s law in the form

$$\frac{\partial w^f}{\partial t} = B_3 \cdot \frac{\partial u}{\partial t} + \frac{1}{\tau_0 \rho_f}(m \mathbb{1} - B_3) \cdot \int_0^t (-\nabla_x q + \rho_f F)(x, \tau) d\tau \hspace{1cm} (4.49)$$

if $\mu_1 = 0$. The problem is supplemented by boundary and initial conditions (4.26)–(4.27) for the displacement $u$ and temperature $\vartheta$ of the rigid component and by the boundary condition

$$w^f(x, t) \cdot n(x) = 0, \hspace{1cm} (x, t) \in S = \partial \Omega, \hspace{1cm} t > 0,$$  \hspace{1cm} (4.50)

for the displacement $w^f$ of the liquid component. In Eqs. (4.45)–(4.49) $n(x)$ is the unit normal vector to $S$ at a point $x \in S$, and matrices $B_1(\mu_1, t)$, $B_2(\mu_1)$, and $B_3$ are defined below by formulas (4.51)–(4.56).

Proof. Eqs. (4.45) and (4.46) derived in a usual way like Eqs. (4.21) and (4.24). For example, to get Eq. (4.46) we just expressed $\text{div}_x w$ in Eq. (4.49) using homogenization in Eq. (4.5): $w = w^f + (1 - m) u$. Therefore we omit the relevant proofs now and focus only on derivation of homogenized equations for the velocity $\mathbf{v}$ in the form of Darcy’s laws. The derivation of Eq. (4.50) is standard [14].
If \( \mu_1 > 0 \) and \( \tau_0 > 0 \), then the solution of the microscopic equations (4.4), (4.15) and (4.17) is given by formula

\[
V = \frac{\partial u}{\partial t} + \int_0^t B^f_1(y, t - \tau) \cdot (-\nabla q + \rho_f F - \tau_0 \rho_f \frac{\partial^2 u}{\partial \tau^2})(x, \tau) d\tau,
\]

where

\[
B^f_1(y, t) = \sum_{i=1}^{3} V^i(y, t) \otimes e_i,
\]

and functions \( V^i(y, t) \) are defined by virtue of the periodic initial-boundary value problem

\[
\begin{aligned}
\tau_0 \rho_f \frac{\partial V^i}{\partial t} - \mu_1 \Delta V^i + \nabla Q^i &= 0, & \text{div}_y V^i = 0, & \quad y \in Y_f, t > 0, \\
V^i &= 0, & y \in \gamma, t > 0; & \quad \tau_0 \rho_f V^i(y, 0) = e_i, & y \in Y_f.
\end{aligned}
\]

(4.51)

In (4.51) \( e_i \) is the standard Cartesian basis vector of the coordinate axis \( x_i \). Therefore

\[
B_1(\mu_1, t) = \langle B^f_1(y, t) \rangle_{Y_s}.
\]

(4.52)

b) If \( \tau_0 = 0 \) and \( \mu_1 > 0 \) then the solution of the stationary microscopic equations (4.4), (4.15) and (4.17) is given by formula

\[
V = \frac{\partial u}{\partial t} + B^f_2(y) \cdot (-\nabla q + \rho_f F),
\]

where

\[
B^f_2(y) = \sum_{i=1}^{3} U^i(y) \otimes e_i,
\]

and functions \( U^i(y) \) are defined from the periodic boundary value problem

\[
\begin{aligned}
-\mu_1 \Delta U^i + \nabla R^i &= e_i, & \text{div}_y U^i = 0, & \quad y \in Y_f, \\
U^i &= 0, & y \in \gamma.
\end{aligned}
\]

(4.53)

Thus

\[
B_2(\mu_1) = \langle B^f_2(y) \rangle_{Y_s}.
\]

(4.54)

Matrices \( B_1(\mu_1, t) \) and \( B_2(\mu_1) \) are symmetric and positively defined [14, Chap. 8].
c) Finally, if $\tau_0 > 0$ and $\mu_1 = 0$, then in the process of solving the system (4.14), (4.18) and (4.20) we firstly find the pressure $R(\mathbf{x}, t, \mathbf{y})$ by virtue of solving the Neumann problem for Laplace’s equation in $Y_f$. If

$$
\mathbf{h}(\mathbf{x}, t) = -\tau_0 \rho_f \frac{\partial^2 \mathbf{u}}{\partial t^2}(\mathbf{x}, t) - \nabla q(\mathbf{x}, t) + \rho_f \mathbf{F}(\mathbf{x}, t),
$$

then

$$
R(\mathbf{x}, t, \mathbf{y}) = \sum_{i=1}^{3} R_i(\mathbf{y}) \mathbf{e}_i \otimes \mathbf{h}(\mathbf{x}, t),
$$

where $R_i(\mathbf{y})$ is the solution of the problem

$$
\Delta R_i = 0, \quad \mathbf{y} \in Y_f; \quad \nabla R_i \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{e}_i, \quad \mathbf{y} \in \gamma. \quad (4.55)
$$

The formula (4.49) appears as the result of integration with respect to time in the homogenization of Eq.(4.18) and

$$
B_3 = \sum_{i=1}^{3} \langle \nabla R_i(\mathbf{y}) \rangle_{Y_i} \otimes \mathbf{e}_i, \quad (4.56)
$$

where the matrix $(mI - B_3)$ is symmetric and positively definite [14, Chap. 8].

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