On linearization problems in the plane
Cremona group

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Abstract

We study finite non-linearizable subgroups of the plane Cremona group which potentially could be stably linearizable.

1 Introduction

Let $k$ be an algebraically closed field of characteristic 0. Recall that by definition the Cremona group $\text{Cr}_n(k)$ is the group of birational transformations of the projective space $\mathbb{P}^n$ over $k$. This group is isomorphic to the $k$-linear automorphism group of the field $k(x_1, \ldots, x_n)$. Also note that for $n \leq N$ there is an embedding $\text{Cr}_n(k) \subset \text{Cr}_N(k)$ which is induced by a birational map $\mathbb{P}^N \dashrightarrow \mathbb{P}^n \times \mathbb{P}^{N-n}$. This leads us to the following definition.

**Definition 1.1.** Finite subgroups $G_1 \subset \text{Cr}_n(k)$ and $G_2 \subset \text{Cr}_m(k)$ are **stably conjugate** if there exists an integer $N \geq n, m$ such that $G_1$ and $G_2$ are conjugate in $\text{Cr}_N(k)$ under the embedding described above.

Recall that any embedding of a finite subgroup $G \subset \text{Cr}_n(k)$ is induced by a biregular action on a rational variety $X$.

**Definition 1.2.** The group $G$ is **linearizable** if the embedding $G \subset \text{Cr}_n(k)$ is induced by a linear action on $\mathbb{P}^n$.

**Definition 1.3.** The group $G$ is **stably linearizable** if $G$ is stably conjugate to a linear action on $\mathbb{P}^N$ for some $N$. Equivalently, there exist a linear action of $G$ on $\mathbb{P}^N$, integers $n, m$ and a $G$-equivariant birational map

$$X \times \mathbb{P}^n \dashrightarrow \mathbb{P}^N \times \mathbb{P}^m$$

such that $G$ acts trivially on $\mathbb{P}^n$ and $\mathbb{P}^m$. 
Remark 1.4. There are no generally accepted agreements about the definition of stable linearization and one can define it in several ways. We refer to [Pro15, Remark 2.3.3] for more definitions and details.

If $G$ is linearizable, then it is stably linearizable. However, the converse is not true, which shows the following example suggested by V. Popov.

Example 1.5. Let $G_1 \cong \mathfrak{S}_3 \times \mathfrak{C}_2$ act on a surface $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ which is defined by the equation

$$x_0y_0z_0 = x_1y_1z_1,$$

where $x_i$, $y_i$, $z_i$ are homogeneous coordinates of each $\mathbb{P}^1$ respectively. Let the subgroup $\mathfrak{S}_3$ permute $\mathbb{P}^1$’s and the subgroup $\mathfrak{C}_2$ permute homogeneous coordinates of each $\mathbb{P}^1$. On the other hand, let $G_2 \cong \mathfrak{S}_3 \times \mathfrak{C}_2 \subset \text{PGL}(3, \mathbb{C})$ act linearly on $Y = \mathbb{P}^2$. V. Iskovskikh showed [Isk08] that there are no equivariant birational maps between $X$ and $Y$, thus the group $\mathfrak{S}_3 \times \mathfrak{C}_2$ has a non-linearizable embedding into $\text{Cr}_2(\mathbb{C})$. However, in [LPR06] it is proved that the varieties $X \times \mathbb{P}^2$ and $Y \times \mathbb{P}^2$ are equivariantly birationally equivalent. The question remains open whether it is possible to multiply only by $\mathbb{P}^1$. Moreover, other examples are currently unknown.

Recall that a smooth projective algebraic surface $X$ is called a del Pezzo surface if its anticanonical sheaf is ample.

In this paper we classify up to conjugation non-linearizable finite subgroups $G$ of $\text{Cr}_2(\mathbb{k})$ which potentially could be stably linearizable assuming that the embedding $G \subset \text{Cr}_2(\mathbb{k})$ is induced by a biregular action on a del Pezzo surface $X$ such that the invariant Picard number $\rho(X)^G = 1$. This work also contributes to the open questions from [DI09, Section 9]. Without loss of generality, we assume that $\mathbb{k}$ is the field of complex numbers $\mathbb{C}$. The main result is the following.

Theorem. Let $X$ be a del Pezzo surface and $G$ be a finite group acting biregularly on $X$ such that $\rho(X)^G = 1$. Then the group $G$ is non-linearizable if and only if the pair $(X, G)$ satisfies one of the following conditions:

1. $X$ is a del Pezzo surface of degree $\leq 3$. ([Pro15, Theorem 1.2]);
2. $X$ is a del Pezzo surface of degree 4 and $G \cong \mathfrak{C}_3 \times \mathfrak{C}_4$ (Proposition 3.18);
3. $X$ is a del Pezzo surface of degree 5 and $G \cong \mathfrak{C}_5 \times \mathfrak{A}_5$, $\mathfrak{S}_5$ (Theorem 3.20).
(4) $X$ is a del Pezzo surface of degree 6. Then either $G \cap (\mathbb{C}^*)^2 \neq 0$ or $\psi(G) \cong \mathfrak{S}_3 \times \mathfrak{C}_2$, where $\psi$: Aut$(X) \rightarrow \mathfrak{S}_3 \times \mathfrak{C}_2$ is a projection homomorphism to the discrete part of Aut$(X)$ (Theorem 3.31);

(5) $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then the following sequence is exact:

$$1 \longrightarrow A \times_D A \longrightarrow G \longrightarrow \mathfrak{C}_2 \longrightarrow 1,$$

where $A \cong \text{G}_n, \text{A}_4, \text{A}_5$ and $D$ is some group (Theorem 3.25).

Remark 1.6. Actually if $X$ is the del Pezzo surface of degree 3, then $G$ is not even stably linearizable by [Pro15, Theorem 1.2].

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2 Preliminaries

Notation.

- $\rho(X) := \text{rk Pic}(X)$ is the Picard number of an algebraic variety $X$;
- Aut$(X)$ is the automorphism group of $X$;
- Bir$(X)$ is the group of birational transformations of $X$;
- $\mathfrak{C}_n$ is the cyclic group of order $n$;
- $\mathfrak{S}_n$ is the symmetric group on $n$ elements;
- $\mathfrak{A}_n$ is the alternating group on $n$ elements;
- $\mathfrak{D}_n$ is the dihedral group of order $2n$;
- $G \times_D H$ is the fiber product of groups $G$ and $H$ over a group $D$, i.e.

$$G \times_D H := \{ (g, h) \in G \times H \mid \phi(g) = \psi(h) \},$$

where $\phi: G \rightarrow D$ and $\psi: H \rightarrow D$ are epimorphisms.
2.1 \(G\)-varieties

**Definition 2.7.** Let \(X\) be an algebraic variety and \(G\) be a group. According to Yu. Manin [Man67] we will say that \(X\) is a \(G\)-variety if \(G\) is finite and its action on \(X\) is defined by the homomorphism \(\theta: G \to \text{Aut}(X)\).

Usually \(G\)-variety are denoted as \((X,\theta)\), \((X,G)\) or simply \(X\) when it does not cause confusion.

**Definition 2.8.** Let \((X,\theta)\), \((X',\theta')\) be \(G\)-varieties. A morphism \(f: X \to X'\) (resp. a rational map \(f: X \dashrightarrow X'\)) is a \(G\)-morphism (resp. a \(G\)-rational map) between the \(G\)-varieties if \(\theta'(g) \circ f = f \circ \theta(g)\) for all \(g \in G\).

**Definition 2.9.** We say that the \(G\)-varieties \((X,\theta)\) and \((X',\theta')\) are \(G\)-stably birationally equivalent if there exist integers \(n, m\) and a \(G\)-birational map \(X \times \mathbb{P}^n \to X' \times \mathbb{P}^m\), where \(G\) acts trivially on \(\mathbb{P}^n\) and \(\mathbb{P}^m\).

One can easily see that two subgroups \(G\) and \(G' \cong G\) of \(\text{Aut}(X)\) define isomorphic (resp. birationally equivalent) \(G\)-varieties, if and only if these subgroups are conjugate in \(\text{Aut}(X)\) (resp. in \(\text{Bir}(X)\)).

**Definition 2.10.** Let \(X\) and \(X_{\min}\) be \(G\)-varieties. We say that \(X_{\min}\) is a \(G\)-minimal model of \(X\) if there is a \(G\)-birational morphism \(X \to X_{\min}\) and any \(G\)-birational morphism \(X_{\min} \to Y\) is a \(G\)-isomorphism.

Let \(G \subset \text{Cr}_2(\mathbb{C})\) be a finite subgroup. After a \(G\)-equivariant resolution of indeterminacy we may assume that \(G\) acts biregularly on some rational smooth projective surface \(X\).

**Theorem 2.11** ([Isk79]). Let \(X\) be a rational smooth projective \(G\)-surface and let \(X_{\min}\) be its \(G\)-minimal model. Then \(X_{\min}\) is one of the following:

1. \(X_{\min}\) is a del Pezzo surface with \(\rho(X_{\min})^G = 1\);
2. \(X_{\min}\) admits a structure of \(G\)-conic bundle, i.e. there exists a surjective \(G\)-equivariant morphism \(f: X_{\min} \to \mathbb{P}^1\) such that \(f_*\mathcal{O}_{X_{\min}} = \mathcal{O}_{\mathbb{P}^1}\), the divisor \(-K_{X_{\min}}\) is \(f\)-ample and \(\rho(X_{\min})^G = 2\).

We will call the surfaces with the described conditions above as \(G\)-minimal or just minimal if the group is clear from the context.

As was mentioned above, the pair \((X,G)\) up to a \(G\)-birational equivalence corresponds to a conjugacy class of \(G\) in \(\text{Cr}_2(\mathbb{C})\). By Theorem 2.11 we may assume that \((X,G)\) is the \(G\)-minimal del Pezzo surface or the \(G\)-minimal conic bundle. In this paper we will study the case of \(G\)-minimal del Pezzo surfaces only, i.e. the case when \(\rho(X)^G = 1\).
2.2 $G$-Sarkisov links

From now on, let $(X, G)$ be a smooth projective $G$-variety and the action of the group $G$ is faithful. Our main tool in this paper will be $G$-Sarkisov links. Recall that the $G$-Sarkisov links or the $G$-links are elementary $G$-birational transformations of four types. Any $G$-link from the $G$-minimal surface $X$ is defined by its center which is a 0-dimensional $G$-orbit of length $d < K_X^2$. We refer to [Isk96] for more details. Also note that all the $G$-Sarkisov links are classified [Isk96, Theorem 2.6]. We will use the following important result.

**Theorem 2.12** ([Isk96, Theorem 2.5]). Any $G$-birational map between rational $G$-minimal projective surfaces can be factorized in a sequence of the $G$-Sarkisov links.

2.3 $G$-stable birational invariants

2.3.1 Amitsur subgroup

Let $\mathcal{L}$ be a line bundle on $X$ with total space $L$ and $\pi: L \to X$ be a structure morphism.

**Definition 2.13.** A $G$-linearization of $\mathcal{L}$ is an action of $G$ on $L$ such that:

1. the structure morphism $\pi$ is $G$-equivariant;
2. the action is linear on its fibers, i.e. for any $g \in G$ and $x \in X$ the map on the fibers $L_x \to L_{g\cdot x}$ is linear.

Denote by Pic $(X, G)$ the group of the $G$-linearized line bundles on $X$ up to isomorphism. Then one has the following exact sequence (see [BCDP18, Section 6] for more details):

$$1 \longrightarrow \text{Hom}(G, \mathbb{C}^*) \longrightarrow \text{Pic}(X, G) \longrightarrow \text{Pic}(X)^G \longrightarrow H^2(G, \mathbb{C}^*) \longrightarrow 0.$$  

According [BCDP18] we define the *Amitsur subgroup* of $X$ as follows

$$\text{Am}(X, G) := \text{im} \left( \delta: \text{Pic}(X)^G \to H^2(G, \mathbb{C}^*) \right).$$

The Amitsur subgroup is a $G$-birational invariant [BCDP18, Theorem 6.1]. Using this fact it is easy to prove that $\text{Am}(X, G)$ is actually a $G$-stable birational invariant.

**Theorem 2.14.** Let $X, Y$ be smooth projective $G$-stably birationally equivalent $G$-varieties. Then

$$\text{Am}(X, G) \cong \text{Am}(Y, G).$$
Proof. By assumption, there exist integers \( n, m \) and a \( G \)-birational map \( X \times \mathbb{P}^n \to Y \times \mathbb{P}^m \). So, \( \text{Am}(X \times \mathbb{P}^n, G) \cong \text{Am}(Y \times \mathbb{P}^m, G) \). Also,
\[
\text{Pic}(X \times \mathbb{P}^n)^G \cong \text{Pic}(X)^G \oplus \mathbb{Z}[\mathcal{O}_{\mathbb{P}^n}(1)]^G.
\]
Consequently, \( \text{Am}(X \times \mathbb{P}^n, G) \cong \text{Am}(Y \times \mathbb{P}^m, G) \). Since the action of \( G \) is trivial on \( \mathbb{P}^n \) by assumption, we have \( \delta([\mathcal{O}_{\mathbb{P}^n}(1)]) = 0 \). Therefore, \( \text{Am}(X,G \times \mathbb{P}^n) \cong \text{Am}(Y,G) \).

2.3.2 The \( H^1(G, \text{Pic}(X)) \) group

The action of \( G \) on \( X \) induces an action on \( \text{Pic}(X) \). Hence, one can consider the first cohomology group \( H^1(G, \text{Pic}(X)) \). As in the arithmetic case, it is an obstruction to the stable linearization.

**Proposition 2.15** ([BP13, Proposition 2.5]). Let \( X, Y \) be smooth projective \( G \)-stably birational \( G \)-varieties. Then there exist permutation \( G \)-modules \( \Pi_1, \Pi_2 \) such that the following isomorphism of \( G \)-modules holds
\[
\text{Pic}(X)^G \oplus \Pi_1 \cong \text{Pic}(Y)^G \oplus \Pi_2.
\]
Consequently,
\[
H^1(G, \text{Pic}(X)) \cong H^1(G, \text{Pic}(Y)).
\]

**Corollary 2.16** ([BP13, Corollary 2.5.2]). In the above notation if \( G \) is stably linearizable, then \( H^1(G', \text{Pic}(X)) = 0 \) for any subgroup \( G' \subseteq G \).

**Theorem 2.17** ([Pro15, Theorem 1.2]). Let \((X,G)\) be a minimal del Pezzo surface. Then the following are equivalent:

1. \( H^1(G', \text{Pic}(X)) = 0 \) for any subgroup \( G' \subseteq G \);
2. any nontrivial element of \( G \) does not fix a curve of positive genus;
3. either
   a. \( K_X^2 \geq 5 \), or
   b. \( X \subseteq \mathbb{P}^4 \) is a quartic del Pezzo surface given by
      \[
      x_0^2 + \zeta_3 x_1^2 + \zeta_3^2 x_2^2 + x_3^2 = x_0^2 + \zeta_3^2 x_1^2 + \zeta_3 x_2^2 + x_4^2 = 0,
      \]
      where \( \zeta_3 = \exp(2\pi i/3) \) and \( G \cong \mathbb{C}_3 \rtimes \mathbb{C}_4 \) is generated by
      \[
      \gamma: (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_1 : x_2 : x_0 : \zeta_3 x_3 : \zeta_3^2 x_4);
      \]
      \[
      \beta: (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 : x_2 : x_1 : -x_4 : x_3).
      \]
3 \textbf{G-minimal del Pezzo surfaces}

Let us start to prove the main result. The proof will be divided into four parts. We will discuss the \(G\)-minimal del Pezzo surfaces of each degree separately. Due to a proof’s structure the cases will be considered in the following order: degree 4, degree 5, degree 8 and degree 6.

As was noted, if the group \(G\) admits regularization on a del Pezzo surface of degree \(\leq 3\), then \(G\) is not stably linearizable by Theorem 2.17. Also, a del Pezzo surface of degree 7 and the blowup of a point on \(\mathbb{P}^2\) are never \(G\)-minimal.

3.1 \textbf{Del Pezzo surface of degree 4}

Let \(X \subset \mathbb{P}^4\) be the quartic del Pezzo surface from Theorem 2.17 and the group \(G \cong \mathfrak{C}_3 \rtimes \mathfrak{C}_4\).

\textbf{Proposition 3.18.} In the above notation \(G\) is non-linearizable.

\textit{Proof.} We will describe all \(G\)-birational models of \(X\) using the \(G\)-links. Firstly, we find the \(G\)-orbits of length \(< 4\).

There is a unique \(G\)-fixed point \((1 : 1 : 1 : 0 : 0)\), the blowup of which is a cubic surface with a \(G\)-minimal conic bundle structure. Then there exist only the inverse \(G\)-link to \(X\) and birational transformations of the conic bundle structure which preserve the degree of the surface by [Isk96, Theorem 2.6]. Consequently, the \(G\)-link from the \(G\)-fixed point does not yield a \(G\)-birational map from \(X\) to \(\mathbb{P}^2\).

There are no orbits of length 2. An orbit of length 3 does exist, it is unique and has the following form:

\[
\{(-1 : 1 : 1 : 0 : 0), \ (1 : 1 : -1 : 0 : 0) \ (1 : -1 : 1 : 0 : 0)\}.
\]

The \(G\)-link corresponding to this orbit is the Bertini involution, which brings us again to the del Pezzo surfaces of degree 4.

Thus, any sequence of \(G\)-links from \(X\) leads us either to the \(G\)-minimal del Pezzo surface of degree 4 or to the \(G\)-minimal conic bundle of degree 3. Hence, the group \(G\) is non-linearizable. \(\square\)

3.2 \textbf{Del Pezzo surface of degree 5}

Let \(X\) be a del Pezzo surface of degree 5. Recall that points \(x_1, \ldots, x_k\) are in a \textit{general position} on a del Pezzo surface \(S\) if the blowup of \(x_1, \ldots, x_k\) is also a del Pezzo surface.
Theorem 3.19 ([DI09, Theorem 6.4]). Let $(X, G)$ be minimal. Then
\[ G \cong C_5, D_5, C_5 \times C_4, A_5, S_5. \]

Theorem 3.20. Let $(X, G)$ be minimal. Then $G$ is non-linearizable if and only if $G \cong C_5 \times C_4, A_5, S_5$.

Proof. The groups $C_5 \times C_4, S_5$ do not act linearly on $\mathbb{P}^2$, consequently they are non-linearizable.

Let $G \cong A_5$. The action of $G$ has no fixed points on $X$, otherwise $A_5$ would act faithfully on the tangent space at the $G$-fixed point, which is impossible since $A_5$ does not have a 2-dimensional faithful representations. Also, there are no orbits of length $\leq 4$, since $A_5$ has no subgroups of index $\leq 4$. Thus, there are no $G$-Sarkisov links from $X$. Hence, $A_5$ is non-linearizable.

Let $G \cong D_5$. We will construct a $G$-birational map from $\mathbb{P}^2$ to $X$. Recall that $D_5 \cong C_5 \times C_2$. Let us consider such action of $G$ on $\mathbb{P}^2$ that a $C_5$-orbit of a $C_2$-fixed point is a set of five points in a general position. The blowup of these five points is a del Pezzo surface $X_4$ of degree 4 with $\rho(X_4)^G = 2$.

Notice that a proper transform of a conic passing through the $C_5$-orbit of the $C_2$-fixed point is a $G$-invariant $(-1)$-curve. So, it can be $G$-equivariantly contracted and as a result we get $X$ with $\rho(X)^G = 1$. Since all subgroups which are isomorphic to $D_5$ are conjugated in $\text{Aut}(X) \cong S_5$, we are done. Similar arguments show that $C_5$ is linearizable too. $\Box$

3.3 Del Pezzo surfaces of degree 8

Let us consider $\mathbb{P}^1 \times \mathbb{P}^1$ with an action of $G$ such that $\rho(\mathbb{P}^1 \times \mathbb{P}^1)^G = 1$. It is well known that
\[ \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \cong (\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})) \times C_2, \]
where two copies of PGL(2, $\mathbb{C}$) act on each $\mathbb{P}^1$ and $C_2$ swaps factors in $\mathbb{P}^1 \times \mathbb{P}^1$.

Denote by $\pi$ a projection homomorphism:
\[ \pi: \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \to C_2. \]

We will use the following result from the group theory.

Theorem 3.21 (Goursat’s Lemma). Let $A_1, A_2, B$ be finite groups such that $B \subseteq A_1 \times A_2$ and projection maps to each $A_i$ are surjective. Then
\[ B \cong A_1 \times_D A_2, \]
for some group $D$. 

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According to we prove the following.

**Theorem 3.22.** Let \((\mathbb{P}^1 \times \mathbb{P}^1, G)\) be minimal. Then the following sequence is exact:

\[1 \longrightarrow A \times_D A \longrightarrow G \longrightarrow \mathcal{C}_2 \longrightarrow \pi \longrightarrow 1,\]

where \(A \cong \mathcal{C}_n, \mathcal{D}_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5\) and \(D\) is some group.

**Proof.** Consider a subgroup

\[G_0 = G \cap (\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})).\]

It acts naturally on \(\mathbb{P}^1 \times \mathbb{P}^1\) and does not swap the factors. Let \(A_i\) be an image of \(G\) under the projection of \(\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})\) on the \(i\)-th factor. If the groups \(A_1\) and \(A_2\) are not isomorphic, then \(G = G_0\) and \(\rho(\mathbb{P}^1 \times \mathbb{P}^1)^G = 2\); a contradiction with minimality. So, \(A_1 \cong A_2 \cong A\), where \(A\) is a finite subgroup of \(\text{PGL}(2, \mathbb{C})\), i.e. \(A \cong \mathcal{C}_n, \mathcal{D}_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5\). Now we are done by Goursat’s Lemma.

One can easily see that \(\text{Am}(\mathbb{P}^2, G)\) is either \(\mathcal{C}_3\) or trivial [BCDP18, Proposition 6.7].

**Lemma 3.23.** In the above notation if \(A\) has no 2-dimensional faithful representations, then \(G\) is non-linearizable.

**Proof.** Notice that \(G\) has a subgroup

\[\{ (a, a) \in A \times_D A \mid a \in A \} \cong A.\]

Consider \(\mathbb{P}^1 \times \mathbb{P}^1\) with a diagonal action of \(A\). Since \(A\) has no 2-dimensional faithful representations, then \(\text{Am}(\mathbb{P}^1 \times \mathbb{P}^1, A) \cong \mathcal{C}_2\). Consequently, \(A\) is non-linearizable by Theorem 2.14, so \(G\) too.

Therefore, it remains to consider only cases when \(A\) is isomorphic either to \(\mathcal{C}_n\) or \(\mathcal{D}_n\). Let us first discuss which \(G\)-links exist from \(\mathbb{P}^1 \times \mathbb{P}^1\).

Any \(G\)-link with center at \(d \neq 1, 3, 5\) points is either a \(G\)-birational self-map of \(\mathbb{P}^1 \times \mathbb{P}^1\) or a transformation of \(\mathbb{P}^1 \times \mathbb{P}^1\) to a \(G\)-minimal conic bundle by [Isk96, Theorem 2.6]. If the \(G\)-link transforms \(\mathbb{P}^1 \times \mathbb{P}^1\) to the \(G\)-minimal conic bundle, then all the following \(G\)-links are either the transformations of the \(G\)-minimal conic bundle structure, i.e. do not change the degree of the surface, or bring us back to \(\mathbb{P}^1 \times \mathbb{P}^1\) with \(\rho(X)^G = 1\). Thus, the \(G\)-links with such centers do not lead us to a \(G\)-birational map to \(\mathbb{P}^2\).

The \(G\)-link with center at one point is a \(G\)-birational map to \(\mathbb{P}^2\). Indeed, it is a composition the blowup of the \(G\)-fixed point and a \(G\)-contraction of \((-1)\)-curves.
The $G$-link with center at three points determines a $G$-birational map to a del Pezzo surface $X_6$ of degree 6. More precisely, let $C \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|$ be a curve passing through the $G$-orbit of length 3. We blow up this $G$-orbit and $G$-contract a proper transform of $C$. Thereby we get a $G$-birational map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $X_6$. But the $G$-links from $X_6$ are either $G$-birational selfmaps or bring us back to $\mathbb{P}^1 \times \mathbb{P}^1$ by [Isk96, Theorem 2.6]. Thus, the $G$-link with center at three point does not lead us to a $G$-birational map to $\mathbb{P}^2$.

Finally, the $G$-link with center at five points is a $G$-birational map to the del Pezzo surface of degree 5. Thus, for linearizability of $G$ it is necessary either the $G$-fixed point or the orbit of length five.

**Lemma 3.24.** Let $(\mathbb{P}^1 \times \mathbb{P}^1, G)$ be minimal. Then $G$ is linearizable if and only if $A \cong \mathcal{C}_n$.

**Proof.** Consider two cases.

Let $A \cong \mathcal{C}_n$. Choose $\tau \in G \setminus (\mathcal{C}_n \times_D \mathcal{C}_n)$. By definition $\tau$ swaps projection morphisms

$$\pi_i : \mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)} \to \mathbb{P}^1_{(i)};$$

i.e.

$$\pi_2 \circ \tau = F \circ \pi_1,$$

where $F : \mathbb{P}^1_{(1)} \to \mathbb{P}^1_{(2)}$ is an isomorphism. Then

$$\pi_1 \circ \tau = F^{-1} \circ \pi_2 \circ \tau^2.$$

Notice that $\tau^2 \in \mathcal{C}_n \times_D \mathcal{C}_n$, i.e. $\tau^2(x, y) = (\xi_1(x), \xi_2(y))$, where $\xi_i \in \mathcal{C}_n$ is an automorphism of $\mathbb{P}^1_{(i)}$. Consequently,

$$\tau(x, y) = \left( F^{-1} \circ \xi_2(y), F(x) \right).$$

Thereby,

$$\tau^2(x, y) = \left( F^{-1} \circ \xi_2 \circ F(x), \xi_2(y) \right);$$

$$\xi_1 = F^{-1} \circ \xi_2 \circ F.$$

Identify $\mathbb{P}^1_{(1)}$ and $\mathbb{P}^1_{(2)}$ by $F$. Then $\xi_1 = \xi_2 = \xi$, so $\tau(x, y) = (\xi(y), x)$. By the Lefschetz fixed-point theorem there exist a $\mathcal{C}_n$-fixed point $\alpha \in \mathbb{P}^1$. Therefore, the point $(\alpha, \alpha) \in \mathbb{P}^1 \times \mathbb{P}^1$ is fixed by $\tau$. Consequently, $(\alpha, \alpha)$ is a $G$-fixed point and the $G$-link with center at this point yields the $G$-birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Thus, $G$ is linearizable.

Let $A \cong \mathcal{D}_n$. Since the action of $\mathcal{D}_n$ has no fixed points on $\mathbb{P}^1$, the action of $G$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has no too. Hence, for linearizability of $G$ it is necessary to have the orbit of length 5 which defines the $G$-link to the del Pezzo surface.
of degree 5. So, \( G \) is isomorphic either to \( C_5 \) or \( D_5 \) by Theorem 3.20. But the group \( G \) is non-abelian, consequently \( G \) is not isomorphic to \( C_5 \).

We claim that \( G \) is also not isomorphic to \( D_5 \). Indeed, by Theorem 3.22 the following sequence is exact:

\[
1 \longrightarrow D_n \times_D D_n \longrightarrow G \longrightarrow C_2 \longrightarrow 1.
\]

We see that \( D_n \times_D D_n \) must be isomorphic to \( C_5 \). However, it is impossible, since \( D_n \times_D D_n \) contains a diagonal subgroup \( D_n \).

Finally, combining Theorem 3.22, Lemma 3.23 and Lemma 3.24, we obtain the following theorem.

**Theorem 3.25.** Let \((\mathbb{P}^1 \times \mathbb{P}^1, G)\) be minimal. Then the following sequence is exact:

\[
1 \longrightarrow A \times_D A \longrightarrow G \longrightarrow C_2 \longrightarrow 1,
\]

where \( A \cong C_n, D_n, A_4, S_4, A_5 \) and \( D \) is some group. Moreover, \( G \) is non-linearizable if and only if \( A \cong D_n, A_4, S_4, A_5 \).

### 3.4 Del Pezzo surface of degree 6

Let \( X \) be a del Pezzo surface of degree 6. Without loss of generality, we regard \( X \) as the blowup of \( \mathbb{P}^2 \) in points

\[
p_1 = (1 : 0 : 0), \quad p_2 = (0 : 1 : 0), \quad p_3 = (0 : 0 : 1).
\]

Exceptional curves and proper transforms of lines passing through pairs of the blowup points combinatorially form a hexagon. These are exactly all the lines on \( X \). Recall that \( X \) can be defined in \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) by the equation

\[
x_0y_0z_0 = x_1y_1z_1,
\]

where \( x_i, y_i, z_i \) are homogeneous coordinates of each \( \mathbb{P}^1 \) respectively.

Let \( W \cong S_3 \times C_2 \) be a subgroup of \( \text{Aut} \ (X) \), where \( S_3 \) acts as a lift of permutations of coordinates on \( \mathbb{P}^2 \) and \( C_2 \) acts as a lift of the standard quadratic involution on \( \mathbb{P}^2 \) which is defined as follows

\[
[x_0 : x_1 : x_2] \rightarrow [x_1x_2 : x_0x_2 : x_0x_1].
\]

**Theorem 3.26** ([Dol12, Theorem 6.4.2]). In the above notation \( \text{Aut} \ (X) \cong (\mathbb{C}^*)^2 \rtimes W \).
Denote by $\psi$ a projection homomorphism:

$$
\psi: \text{Aut} (X) \rightarrow W.
$$

A straightforward computation yields the following.

**Proposition 3.27.** Let $(X, G)$ be minimal. Then

$$
\psi (G) \cong C_6, \ G_3^{tw}, G_3 \times C_2,
$$

where $G_3^{tw}$ is $G_3$ twisted by the standard quadratic involution in $W$.

**Lemma 3.28.** Let $G \subset \text{Aut} (X)$ be a subgroup such that $\psi (G) \cong C_6$ and $G \cap (\mathbb{C}^*)^2 = 0$. Then $G$ is linearizable.

**Proof.** We will construct a $G$-birational map between $X$ and $\mathbb{P}^2$ as a composition of two $G$-links. By the holomorphic Lefschetz fixed-point formula the action of $G$ has a fixed point on $X$. We claim that this fixed point is in a general position. Indeed, if not, then the $G$-fixed point must lie on an intersection of all $(-1)$-curves on $X$, since $C_6$ cyclically permutes the hexagon of the $(-1)$-curves. However, this intersection is empty. Contradiction. Then denote by $x$ an image of the $G$-fixed point on $\mathbb{P}^2$ under the blowup $X \rightarrow \mathbb{P}^2$.

The blowup of the point $x$ and a $G$-contraction of proper transforms of each line passing through $x$ and $p_i$ for $i = 1, 2, 3$ yield a $G$-link from $X$ to $\mathbb{P}^1 \times \mathbb{P}^1$.

Again by the holomorphic Lefschetz fixed-point formula we there exists a $G$-fixed point on $\mathbb{P}^1 \times \mathbb{P}^1$. However, the $G$-link from $\mathbb{P}^1 \times \mathbb{P}^1$ with center at the point gives a $G$-birational map to $\mathbb{P}^2$ as we discussed earlier. Thus, the composition of two built $G$-links gives the $G$-birational map from $X$ to $\mathbb{P}^2$. So, $G$ is linearizable. \qed

**Lemma 3.29.** Let $G \subset \text{Aut} (X)$ be a subgroup such that $\psi (G) \cong G_3^{tw}$ and $G \cap (\mathbb{C}^*)^2 = 0$. Then the following holds:

1) all such the subgroups are conjugate in $\text{Aut} (X)$;

2) $G$ is linearizable.

**Proof.** 1) It is sufficient to consider the action of $G$ on the torus $xyz = 1$. Denote by $c_2$ and $c_3$ generators of $G$ of orders 2 and 3 respectively. Firstly, conjugating $G$ we can consider that $c_3$ is as follows

$$
c_3: (x; y) \mapsto \left( y; \frac{1}{xy} \right)
$$
Since $c_3^{-1} = c_2 c_3 c_2^{-1}$ holds, $c_2$ is as follows
\[ c_2: (x; y) \mapsto \left( \frac{\zeta_3}{y}; \frac{\zeta_4}{x} \right), \]
where $\zeta_3$ is a cubic root of unity. Then we are done by conjugating $G$ by
\[ (x; y) \mapsto \left( \sqrt{\zeta_3 x}; \sqrt{\zeta_3 y} \right). \]

2) Now we can consider that up to conjugation the action of $G$ on $X$ is generated by the lift from $\mathbb{P}^2$ of the following birational transformations
\[
\begin{align*}
[x_0 : x_1 : x_2] & \to [x_0 x_2 : x_1 x_2 : x_0 x_1]; \\
[x_0 : x_1 : x_2] & \to [x_1 : x_2 : x_0].
\end{align*}
\]
Thus, a preimage of the point $[1 : 1 : 1]$ on $\mathbb{P}^2$ under the blowup $X \to \mathbb{P}^2$ is a $G$-fixed point on $X$. Thereby, the $G$-link started at this $G$-fixed point determines the $G$-birational map to $\mathbb{P}^1 \times \mathbb{P}^1$ as we saw in Lemma 3.28. Then one can easily see that by Theorem 3.22 the following sequence is exact:
\[
1 \longrightarrow C_3 \times \epsilon_3 C_3 \longrightarrow G \overset{\pi}{\longrightarrow} C_2 \longrightarrow 1.
\]
So, $G$ is linearizable by Theorem 3.25.

**Lemma 3.30.** Let $G \subset \text{Aut}(X)$ be a group such that $\psi(G) \cong W$ and $G \cap (\mathbb{C}^*)^2 = 0$. Then there exists a $G$-fixed point in a general position on $X$ if and only if $G$ is conjugate to $W$ in $\text{Aut}(X)$.

**Proof.** As above, we can assume that up to conjugation an action of $G$ on the torus is generated by
\[
\begin{align*}
c_2: & \quad (x; y) \mapsto (y; x); \\
c_3: & \quad (x; y) \mapsto \left( y; \frac{1}{xy} \right); \\
\tau: & \quad (x; y) \mapsto \left( \frac{\alpha}{x}; \frac{\beta}{y} \right),
\end{align*}
\]
where $\alpha, \beta \in (\mathbb{C}^*)^2$. Then a straightforward computation shows that the action of $G$ has the fixed point if and only if $\alpha = \beta = \zeta_3$. Thus again, we are done by conjugating $G$ by
\[ (x; y) \mapsto \left( \sqrt{\zeta_3 x}; \sqrt{\zeta_3 y} \right). \]
Now we are ready to prove the main theorem of this section. As was noted in Example 1.5 the group $W$ is non-linearizable.

**Theorem 3.31.** Let $(X, G)$ be minimal. Then $G$ is non-linearizable if and only if either $G \cap (\mathbb{C}^*)^2 \neq 0$ or $\psi(G) \cong S_3 \times \mathbb{C}_2$.

**Proof.** As we discussed earlier any $G$-link with center at more than one point on $X$ is the $G$-birational selfmap [Isk96 Theorem 2.6], but the $G$-link with center at one point gives the $G$-birational map to $\mathbb{P}^1 \times \mathbb{P}^1$ which was constructed in Lemma 3.28. Hence, for the linearizability of $G$ it is necessary to have a $G$-fixed point in a general position.

Let $G_0 = G \cap (\mathbb{C}^*)^2$. For each $G$ the following sequence is exact:

$$1 \longrightarrow G_0 \longrightarrow G \overset{\psi}{\longrightarrow} H \longrightarrow 1,$$

where $H \cong \mathbb{C}_6, S_3^{tw}, S_3 \times \mathbb{C}_2$. Any nontrivial element of $(\mathbb{C}^*)^2$ can have the fixed points only on the exceptional set. But such points are not in the general position, so we cannot start the $G$-links in such centers. Thus, $G_0 = 0$ and consequently $G \cong H$.

If $H \cong \mathbb{C}_6$, then $G$ is linearizable by Lemma 3.28. If $G \cong S_3^{tw}$, then $G$ is linearizable by Lemma 3.29. If $H \cong S_3 \times \mathbb{C}_2$, then $G$ is non-linearizable by Lemma 3.30 and Example 1.5.

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