Experiments on extreme wave generation using the Soliton on Finite Background

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Introduction

Freak waves are very large water waves whose heights exceed the significant wave height of a measured wave train by a factor of more than 2.2. However, this in itself is not a well established definition of a freak wave. The mechanism of freak wave generation in reality as well as modeling it in a wave basin has become an issue of great importance.

Recently one is aware of the generation of freak wave through the Benjamin–Feir type of instability or self-focussing. Consequently the Non-Linear–Schrödinger (NLS) equation forms a good basis for understanding the formation of freak waves. However, the complex generation of a freak wave in nature within a sea condition is still not well understood, when the non-linearity of the carrier wave is not small. In our study we will focus on the Soliton on Finite Background, an exact solution of the NLS equation, as a generating mechanism for extreme waves.

Apart from a numerical investigation into the evolution of a soliton on a finite background also extensive detailed model tests have been performed for validation purposes in the hydrodynamic laboratories of the Maritime Research Institute Netherlands (MARIN). Furthermore, a numerical wave tank [12] is used to model the complete non-linear non-breaking wave evolution in the basin.

Properties of the Soliton on Finite Background

The NLS equation is chosen as a mathematical model for the non-linear evolution of the envelope of surface wave packets. For spatial evolution problems, it is given in non-dimensional form and in a frame of reference moving with the group velocity by

\[ \partial_\xi \psi + i\beta \partial_{\tau}^2 \psi + i\gamma |\psi|^2 \psi = 0, \tag{1} \]

where \( \xi \) and \( \tau \) are the corresponding spatial and temporal variables, respectively; \( \beta \) and \( \gamma \) are the dispersion and non-linearity coefficients. This equation has many
families of exact solutions. One family of exact solutions is known as the Soliton on Finite Background (SFB) and this is a good candidate for describing extreme waves. This exact solution has been found by Akhmediev, Eleonski˘i & Kulagin [3], see also [2] and [1].

This SFB solution describes the dynamic evolution of an unstable modulation process, with dimensionless modulation frequency $\nu$. In the context of water waves, for infinitesimal modulational perturbations to a finite-amplitude plane wave, this process is known as Benjamin-Feir (BF) instability [5]. However, non-linearity will limit this exponential growth and the SFB is one (of many other) non-linear extension of the BF instability for larger amplitudes of the modulation. Extensive research on the NLS equation and the SFB solution, to obtain a better understanding of deterministic extreme-wave phenomena has been conducted in the past few years (see e.g. [10], [11], [4] and [9]).

An explicit expression for the SFB is given as the following complex-valued function

$$\psi(\xi, \tau; \tilde{\nu}, r_0) = A(\xi) \cdot \left\{ \frac{\tilde{\nu}^2 \cosh(\sigma \xi) - i \left[ \sigma / (\gamma r_0^2) \right] \sinh(\sigma \xi)}{\cosh(\sigma \xi) - \sqrt{1 - \frac{1}{2} \tilde{\nu}^2 \cos(\nu \tau)}} - 1 \right\} ,$$

where $A(\xi) = r_0 e^{-i \sigma \xi}$ is the plane-wave or the continuous wave solution of the NLS equation, $\sigma = \gamma r_0^2 \tilde{\nu} \sqrt{2 - \tilde{\nu}^2}$ is the growth rate corresponds to the Benjamin-Feir instability, $\nu = \tilde{\nu} R_0 \sqrt{\gamma / 2}$ is the modulation frequency, and $\tilde{\nu}$, $0 < \tilde{\nu} < \sqrt{2}$ is the normalized modulation frequency. This SFB reaches its maxima at $(\xi, \tau) = (0, 2n\pi)$, $n \in \mathbb{Z}$. It has a soliton-like form with a finite background in the spatial $\xi$-direction. The SFB is periodic along in the temporal $\tau$-direction, with period $2\pi$. For $|\xi| \to \infty$, the SFB turns into the continuous wave solution $A(\xi)$. It possesses two essential parameters: $r_0$ and $\tilde{\nu}$.

**Fig. 1.** Density plots of a physical wave packet profile according to an SFB envelope for $\tilde{\nu}_1 = 1$, showing the wave dislocation phenomenon.
The first-order part of the corresponding physical wave packet profile \( \eta(x,t) \) for a given complex-valued function \( \psi(\xi,\tau) \) is expressed as follows

\[
\eta(x,t) = \psi(\xi,\tau)e^{i(k_0x - \omega_0t)} + \text{c.c.,} \tag{3}
\]

where c.c. means the complex conjugate of the preceding term, the wave number \( k_0 \) and frequency \( \omega_0 \) satisfy the linear dispersion relation \( \omega = \Omega(k) \equiv \sqrt{k \tanh k} \).

The variables \((x,t)\) in the non-moving frame of reference are related to \((\xi,\tau)\) in the moving frame of reference by the transformation \( \xi = x \) and \( \tau = t - x/\Omega'(k_0) \).

The modulus of \( \psi \) represents the wave group envelope, enclosing the wave packet profile \( \eta(x,t) \). The dimensional laboratory quantities are related to the non-dimensional quantities by the following Froude scaling, using the gravitational acceleration \( g \) and the depth of the basin \( h \):

\[
x_{\text{lab}} = x \cdot h, \quad t_{\text{lab}} = t \cdot \sqrt{g/h}, \quad k_{\text{lab}} = k/h, \quad \omega_{\text{lab}} = \omega \cdot \sqrt{g/h}, \quad \text{and} \quad \eta_{\text{lab}} = \eta \cdot h.
\]

In principle, the wave profile including the higher-order terms represents a good approximation to the situation in real life. To accommodate this fact, we will include higher-order terms up to second order. We apply a perturbation-series expansion (Stokes’ expansion) to the physical wave-packet profile \( \eta(x,t) \) and the multiple-scale approach using the variables \( \xi \) and \( \tau \), where \( \xi = \epsilon^2 x, \quad \tau = \epsilon(t - x/\Omega'(k_0)) \), and \( \epsilon \) is a small positive non-linearity and modulation parameter. The corresponding physical wave packet profile, consisting of the superposition of the first-order harmonic term of \( O(\epsilon) \) and a second-order non-harmonic long wave as well as a second-order double-frequency harmonic term of \( O(\epsilon^2) \), is given by

\[
\eta(x,t) = \epsilon \left[ \psi^{(10)}(\xi,\tau)e^{i(k_0x - \omega_0t)} + \text{c.c.} \right] \\
+ \epsilon^2 \left\{ \psi^{(20)}(\xi,\tau) + \left[ \psi^{(22)}(\xi,\tau)e^{2i(k_0x - \omega_0t)} + \text{c.c.} \right] \right\}. \tag{4}
\]

We find from the multiple-scales perturbation-series approach that \( \psi^{(10)}(\xi,\tau) = \psi(\xi,\tau) \) satisfies the spatial NLS equation and

\[
\psi^{(20)}(\xi,\tau) = -\frac{1}{\Omega'(k_0)} \frac{4k_0\Omega''(k_0) - \Omega'(k_0)}{[\Omega'(0)]^2 - [\Omega'(k_0)]^2} |\psi^{(10)}(\xi,\tau)|^2 \tag{5}
\]

\[
\psi^{(22)}(\xi,\tau) = k_0 \frac{3 - \tanh^2 k_0}{2 \tanh^3 k_0} [\psi^{(10)}(\xi,\tau)]^2. \tag{6}
\]

A similar derivation for the temporal NLS equation resulting from the KdV equation can be found in [8]. By including this second-order term, the wave signal \( \eta(x,t) \) experiences the well-known Stokes’ effect: the crests become steeper and the troughs becomes shallower [6].

The coefficients \( \beta \) and \( \gamma \) of the spatial NLS equation are given, in nondimensional form, as:

\[
\beta = -\frac{1}{2} \frac{\Omega''(k_0)}{[\Omega'(k_0)]^3}, \tag{7}
\]

\[
\gamma = \gamma_1 + k_0\alpha_U + \lambda \alpha_c \frac{\Omega'(k_0)}{\Omega''(k_0)}, \tag{8}
\]
where
\[
\gamma_1 = k_0^2 \Omega'(k_0) \frac{9 \tanh^4 k_0 - 10 \tanh^2 k_0 + 9}{4 \tanh^4 k_0},
\]
(9)
\[
\lambda = \frac{1}{2} k_0 \frac{1 - \tanh^2 k_0}{\Omega(k_0)},
\]
(10)
\[
\alpha_\zeta = -\frac{1}{\Omega(k_0)} \frac{4 k_0 \Omega'(k_0) - \Omega(k_0)}{|\Omega'(k_0)|^2 - |\Omega'(0)|^2} \quad \text{and}
\]
(11)
\[
\alpha_U = \alpha_\zeta \Omega'(k_0) - \frac{2 k_0}{\Omega(k_0)}.
\]
(12)

These can be used to compute the SFB solution \( \psi(\xi, \tau) \) from Equation (2).

Phase singularity and wave dislocation

![Fig. 2. The evolution of the SFB for \( \tilde{\nu}_1 = 1 \) from a modulated continuous wave signal into the extreme position. From top to bottom, the signals are taken at \( x = -200 \), \( x = -100 \), \( x = -50 \), \( x = -30 \), and \( x = 0 \).](image)
By writing the complex-valued function $\psi$ in a polar (or phase-amplitude) representation, it is found that for modulation frequency $\tilde{\nu}$ in the range $0 < \tilde{\nu} < \sqrt{\frac{3}{2}}$ a phase singularity phenomenon occurs. It happens when the real-valued amplitude $|\psi|$ vanishes and therefore there is no way of ascribing a value to the real-valued phase when it occurs. The local wave number $k \equiv k_0 + \partial_x \theta$ and local frequency $\omega \equiv \omega_0 - \partial_t \theta$, with $\theta(\xi, \tau) \equiv \arg(\psi(\xi, \tau))$, become unbounded when this happens. The corresponding physical wave packet profile $\eta(x, t)$ confirms this by showing a wave dislocation phenomenon. When the real-valued amplitude $|\psi|$ vanishes at that specific position and time, waves merge or split. For $\sqrt{\frac{3}{2}} < \tilde{\nu} < \sqrt{2}$, the real-valued amplitude is always definite positive, and thus there is no wave dislocation. Furthermore, in one modulation period, there is a pair of wave dislocations. Before or after this dislocation, the real-valued amplitude reaches its maximum value. Figure 1 shows the density plot of a physical wave packet profile $\eta(x, t)$. The wave dislocation is also visible in this figure. Figure 2 shows the evolution of the SFB from a modulated wave signal until it reaches the extreme position. We can see also in this figure that the amplitude $|\psi|$ vanishes at some moments for the extremal position $x = 0$, causing phase singularity.

The phase singularity is a well known phenomenon in physical optics. In the context of water waves, similar observations can be made, and also wave dislocations occur. Trulsen [13] calls it as crest pairing and crest splitting and he explains this phenomenon as a consequence of linear dispersion.

**Maximum temporal amplitude**

The maximum temporal amplitude (MTA) is a useful concept to understand long-time behavior of wave elevation. For wave propagation in the laboratory, it also gives a direct view of the consequences of an initial wave signal on the corresponding extreme-wave signal. It is defined as

$$\mu(x) = \max_t \eta(x, t), \quad (13)$$

where $\eta(x, t)$ is the surface elevation as a function of space $x$ and time $t$. It describes the largest wave elevation that can appear at a certain position. For laboratory wave generation, it describes the boundary between the wet and dry parts of the wall of the basin after a long time of wave evolution.

Figure 3 shows the MTA plot of the SFB in the laboratory coordinates. In this example, the mean water depth is 3.55 m and the wavelength is approximately 6.2 m. The wave signal is generated at the left side, for example at $x_{\text{lab}} = -350$ m, and it propagates to the right and reaches its extremal condition at $x_{\text{lab}} = 0$. A slightly modulated wave train increases in amplitude as the SFB waves travels in the positive $x$-direction. Furthermore, in this example a SFB wave signal with initial amplitude around 0.19 m can reach an extreme amplitude of 0.45 m, an amplification factor of around 2.4. After reaching its maximum amplitude, the MTA decreases monotonically and returns to its initial value.
**Experimental Result**

For the validation of the proposed method we performed experiments in one of the wave basins of MARIN. The basin dimensions amounted to \(L \times B \times D\) as 200 m \(\times\) 4.0 m \(\times\) 3.55 m. In the basin an array of wave probes were mounted as indicated in the set-up in figure (4). The predefined wave board control signal was put onto the hydraulic wave generator. The stroke of the wave flap was measured. Main characteristics of the model test experiments:

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**Fig. 3.** The MTA plot of the SFB, the corresponding wave profiles at \(t = 0\), and its envelope.

**Fig. 4.** The set-up of the wave probe array in the wave basin

Carrier wave period is 1.685 sec. maximum wave height to be achieved (MTA) varies from 0.213m to 0.2485m
As a explanation the results for the tests with an MTA of 0.2485 will be shown, see figure 5 to figure 7.

![Figure 5](image1.png)

**Fig. 5.** Comparison Non-Linear Wave model HUBRIS with results from experiments and sNLS

Figure 8 shows the SFB signal based on the experiment at distance 150 m from the wave maker, where it is expected that the signal to be extreme. That figure also shows the phase singularity phenomenon when the local frequency becomes unbounded when the real-valued amplitude vanishes or almost vanishes. The experiment result shows asymmetric form of the extreme signal while the theoretical result of the SFB preserve the symmetry of the signal. It is suspected that if the modified NLS equation of Dysthe [7] is used as the governing equation for the wave signal evolution, then there are good comparisons with experimental measurements. The good comparisons are observed for the case of bi-chromatic waves, where the modified NLS equation predicts both the evolution of individual wave crests and the modulation of the envelope over longer fetch [14].

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Fig. 6. Comparison Non-Linear Wave model HUBRIS with results from experiments and sNLS

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Fig. 7. Comparison Non-Linear Wave model HUBRIS with results from experiments and sNLS

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Fig. 8. The SFB signal plot based on the experiment at 150 m (top) and the corresponding local frequency plot (bottom).