Non-stationary localized oscillations of an infinite Bernoulli-Euler beam lying on the Winkler foundation with a point elastic inhomogeneity of time-varying stiffness

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Abstract

We consider non-stationary localized oscillations of an infinite Bernoulli-Euler beam. The beam lies on the Winkler foundation with a point inhomogeneity (a concentrated spring with negative time-varying stiffness). In such a system with constant parameters (the spring stiffness), under certain conditions a trapped mode of oscillation exists and is unique. Therefore, applying a non-stationary external excitation to this system can lead to the emergence of the beam oscillations localized near the inhomogeneity. We provide an analytical description of non-stationary localized oscillations in the system with time-varying properties using the asymptotic procedure based on successive application of two asymptotic methods, namely the method of stationary phase and the method of multiple scales. The obtained analytical results were verified by independent numerical calculations. The applicability of the analytical formulas was demonstrated for various types of external excitation and laws governing the varying stiffness. In particular, we have shown that in the case when the trapped mode frequency approaches zero, localized low-frequency oscillations with increasing amplitude precede the localized beam buckling. The dependence of the ampli-

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tude of such oscillations on its frequency is more complicated in comparison with the case of a one degree of freedom system with time-varying stiffness.

**Keywords:** trapped modes, linear wave localization, non-stationary oscillation, systems with time-varying properties

1. Introduction

In this paper we consider a mechanical system with mixed spectrum of natural oscillations. Namely, we deal with an infinite Bernoulli-Euler beam lying on the Winkler foundation with a point inhomogeneity (a concentrated spring with negative slowly time-varying stiffness). In the case of a constant spring stiffness the discrete part of the spectrum for such a system may contain unique (positive) eigenvalue, which is less than the lowest frequency for the beam on the uniform foundation. This special natural frequency corresponds to a trapped mode of oscillation with eigenform localized near the spring. The phenomenon of trapped modes was discovered in the theory of surface water waves [1]. The examples of various mechanical systems, where trapped modes can exist, can be found in studies [2–22]. Note that very similar problems for structures of finite length were considered in studies [23–26].

It is known [2, 7, 16, 21, 22, 27, 28] that applying non-stationary external excitation to a system possessing trapped modes leads to the emergence of undamped oscillations localized near the inhomogeneity. The large time asymptotics for such oscillation can be found [7, 21, 22, 27] by means of the method of stationary phase [29, 30]. Gavrilov in [6, 7] suggested an asymptotic procedure based on successive application of two asymptotic methods, namely, the method of stationary phase [29, 30] and the method of multiple scales [30, 31] that allows us to investigate non-stationary processes in perturbed systems, with slowly time-varying parameters, possessing trapped modes. In studies [6, 7] the problem concerning non-uniform motion of a point mass along a taut string on the Winkler foundation was considered and solved. The asymptotic procedure suggested in [6, 7] was successfully applied to describe the evolution of the
amplitude of the trapped mode of oscillations in a taut string on the Winkler foundation with a point inertial inclusion with time-varying mass [21] and in a taut string on the Winkler foundation with a concentrated spring with negative time-varying stiffness [22]. In preprint [28] non-stationary localized oscillations of an infinite string, with time-varying tension, lying on the Winkler foundation with a point elastic inhomogeneity are considered.

All problems considered in previous papers [6, 7, 21, 22, 28] deals with a string on the Winkler foundation and are formulated for the Klein-Gordon-type PDE. The aim of this paper is to generalize the suggested approach to the case when the waveguide under consideration is a Bernoulli-Euler beam. The consideration of the problem for a compressed beam with time-varying compression has some geophysical motivation [14]. This problem is quite complicated and we need to begin with some more simple test problems to develop the suitable mathematical apparatus. The problem considered in this paper is a simplest problem of such kind formulated in the case of a beam.

The paper is organized as follows. In Section 2, we consider the formulation of the problem. In Section 3, an infinite Bernoulli-Euler beam lying on the Winkler foundation with a point elastic inhomogeneity of a constant stiffness is considered. To deal with such a system we use the corresponding Green function in the frequency domain (see Appendix A). In Section 3.1 we consider the system without an external excitation and solve the spectral problem. We demonstrate that the system under consideration can possess a mixed spectrum of natural frequencies and find the necessary and sufficient conditions for the existence of a trapped mode. In Section 3.2 we consider the system with an external excitation and use the method of stationary phase to find non-vanishing localized non-stationary oscillations with the trapped mode frequency. In Section 4 we consider the case of inhomogeneity (the concentrated spring) with a slowly time-varying stiffness. We apply the asymptotic procedure based on the method of multiple scales and obtain the asymptotic solution. To find the unknown constants we use the results obtained in Section 3.2. In Section 5, we compare the analytical results with numerical ones obtained by means of solving the
Volterra integral equation of second kind with its kernel expressed in terms of the corresponding Green function in the time domain (see Appendix B). In the conclusion (Section 6), we discuss the basic results of the paper.

2. Mathematical formulation

We consider transverse oscillation of an infinite Bernoulli-Euler beam on the Winkler elastic foundation. The elastic foundation has a point inhomogeneity in the form of a concentrated spring of a negative stiffness.

The dimensionless equations describing transverse oscillation of the beam can be written as follows:

\[ \ddot{w} + w^{(4)} + w = P(t)\delta(x), \]  
\[ P(t) = -K(t)w(0, t) + p(t). \]  

Here \( w(x, t) \) is the displacement of a point of the beam at the dimensionless position \( x \) and dimensionless time \( t \), \( K(t) \) is the dimensionless stiffness of the concentrated spring (a given function of time), \( P(t) \) is the unknown dimensionless force on the beam from the spring, \( p \) is the given dimensionless external force on the beam. The initial conditions for Eq. (1) can be formulated in the following form, which is conventional for distributions (or generalized functions) [32]:

\[ w|_{t<0} = 0. \]  

Integrating (1) over \( x = 0 \) results in the following condition

\[ [w^{(4)}] = -K(t)w(0, t) + p(t). \]  

Here, and in what follows, \([\mu] = \mu(x + 0) - \mu(x - 0)\) for any arbitrary quantity \( \mu \).
3. The unperturbed problem

In this section we consider the case of concentrated spring of a constant
stiffness \( K = \text{const} \).

3.1. Spectral problem

Put \( p = 0 \) and consider the steady-state problem concerning the natural
oscillations of the system described by Eqs. \((1)-(2)\). Take

\[
w = \hat{w}(x) \exp(-i\Omega t).
\]

Let us show that such a system possesses a mixed spectrum of natural frequen-
cies. There exists a continuous spectrum of frequencies \( |\Omega| \geq \Omega_* \), which lies
higher than the cut-off (or boundary) frequency \( \Omega_* = 1 \). The modes corre-
sponding to the frequencies from the continuous spectrum are harmonic waves.
Trapped modes correspond to the frequencies from the discrete part of the spec-
trum, which lies lower than the cut-off frequency: \( 0 < |\Omega| < \Omega_* \). We want to
demonstrate that for the problem under consideration the only one trapped
mode can exist. Trapped modes are modes with finite energy, therefore, we
require \( \hat{w} \to 0, \hat{w}' \to 0 \) as \( x \to \infty \), and

\[
\int_{-\infty}^{+\infty} \hat{w}^2 \, dx < \infty, \quad \int_{-\infty}^{+\infty} \hat{w}'^2 \, dx < \infty.
\]

Substituting Eq. \((5)\) into Eqs. \((1)-(2)\) and taking into account the expression for
the Green function in the frequency domain (see Appendix A, formulas \((A.3)\),
\((A.14)\)) results in

\[
\hat{w}(x) = -\frac{K\hat{w}(0)}{2} \frac{e^{-a|x|} \cos(a|x| - \pi/4)}{(1 - \Omega^2)^{3/4}}.
\]

Calculating the right-hand side of \((7)\) at \( x = 0 \) yields the equation for eigen-
frequencies of the discrete spectrum

\[
K + 2\sqrt{2} (1 - \Omega^2)^{3/4} = 0.
\]
It is clear that frequency equation (8) can have real solutions only if the stiffness of the inclusion is negative:

\[ K < 0, \]  

i.e. we deal with a destabilizing spring. The solution of Eq. (8) is

\[ \Omega = \Omega_0 \equiv \sqrt{1 - \frac{|K|^{4/3}}{4}}, \]  

where \( \Omega_0 \) is the trapped mode frequency. The corresponding localized form is proportional to the right-hand side of Eq. (A.14).

Thus, there exists the unique positive natural frequency less than the cut-off frequency \( \Omega_\ast = 1 \) if and only if inequality (9) is true and

\[ |K| < K_\ast \equiv 2\sqrt{2}. \]  

The critical value \( K_\ast \) corresponds to the possibility of the localized buckling of the beam.

3.2. Inhomogeneous non-stationary problem

Put now \( p \neq 0 \). Applying to Eq. (1)–(2) the Fourier transform in time \( t \) results in

\[ w'''_F + (1 - \Omega^2)w_F = (-K w_F(0, \Omega) + p_F(\Omega)) \delta(x), \]  

where \( w_F(x, \Omega), \ p_F(\Omega) \) are the Fourier transforms of \( w(x, t) \) and \( p(t) \), respectively. Using expressions (A.3), (A.14) for the corresponding Green function to resolve Eq. (12) with respect to \( w_F(0, \Omega) \) and applying the inverse transform yields

\[ w(0, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{p_F e^{-i\Omega t} d\Omega}{K + 2\sqrt{2}(1 - \Omega^2)^{3/4}}. \]  

Consider the case when \( p(t) \) is a vanishing as \( t \to \infty \) function such that its Fourier’s transform \( p_F(\Omega) \) does not have singular points on the real axis. Applying the residue theorem, Jordan’s lemma, and the method of stationary phase to asymptotic evaluation of the integral in the right-hand side of (13)
results in $[29, 33]$

$$\begin{align*}
w(0, t) &= -i \sum_{\tilde{\Omega} = \pm \Omega_0 - i0} p_f(\Omega) \text{Res} \left( \frac{1}{K + 2 \sqrt{2} (1 - \Omega^2)^{3/4}, \tilde{\Omega}} \right) \exp(-i\tilde{\Omega}t) + o(1), \\
t \to \infty. & \quad (14)
\end{align*}$$

Here symbol $\text{Res} (f(\Omega), \tilde{\Omega})$ means the residue of function $f(\Omega)$ at a pole $\Omega = \tilde{\Omega}$. The terms $-i0$ in the expression for the poles

$$\tilde{\Omega} = \pm \Omega_0 - i0 \quad (15)$$

are taken in accordance with the principle of limit absorption. The asymptotic order of the error term in formula (14) depends on the properties of $p_F$. One has

$$\text{Res} \left( \frac{1}{K + 2 \sqrt{2} (1 - \Omega^2)^{3/4}, \pm \Omega_0 - i0} \right) = \pm \frac{\sqrt{1 - \Omega_0^2}}{3 \sqrt{2} \Omega_0} \quad (16)$$

thus

$$w(0, t) = \sqrt{2} \frac{\sqrt{1 - \Omega_0^2} \left| p_F(\Omega_0) \right|}{3 \Omega_0} \sin \left( \Omega_0 t - \arg p_F(\Omega_0) \right) + o(1), \quad t \to \infty. \quad (17)$$

Taking into account (8), one can rewrite the last formula in the equivalent form:

$$w(0, t) = \frac{2 \sqrt{\sqrt{K} \left| p_F(\Omega_0) \right|}}{3 \sqrt{4 - \left| K \right|^{2/3}}} \sin \left( \Omega_0 t - \arg p_F(\Omega_0) \right) + o(1), \quad t \to \infty. \quad (18)$$

Hence, for the large times, the non-stationary response of the system under consideration is undamped oscillations with the trapped mode frequency $\Omega_0$.

4. The perturbed problem

Assume that the stiffness of the concentrated beam $K$ is slowly varying piecewise monotone functions of the dimensionless time $t$: $K = K(\epsilon t)$. Here $\epsilon$ is a formal small parameter. We use an approach $[6, 7, 21]$ based on the modification of the method of multiple scales $[31]$ (Section 7.1.6) for equations with
slowly varying coefficients. The corresponding rigorous proofs, which validates such asymptotic approach in the case of a one degree of freedom system, can be found in [34]. We look for the asymptotics for the solution under the following conditions:

- \( \epsilon = o(1) \),
- \( t = O(\epsilon^{-1}) \),
- \( K(\epsilon t) \) satisfies restriction (11) for all \( t \).

To construct the particular solution of (1), (2) which describes the evolution of the trapped mode of oscillation in the case of slowly varying \( K \), we require that in the perturbed system

- Frequency equation (8) for the trapped mode holds for all \( t \);
- Dispersion relation (A.2) at \( x = \pm 0 \) holds for all \( t \).

Accordingly, we use the following ansatz \((t > 0, x \ll 0)\):

\[
\begin{align*}
    w(x,t) &= W(X,T) \exp \varphi(x,t), \\
    T &= \epsilon t, \quad X = \epsilon x, \\
    \varphi' &= i\omega(X,T), \quad \dot{\varphi} = -i\Omega(X,T), \\
    W(X,T) &= \sum_{j=0}^{\infty} \epsilon^j W_j(X,T).
\end{align*}
\]  

Here the amplitude \( W(X,T) \), the wavenumber \( \omega(X,T) \), and the frequency \( \Omega(X,T) \) are the unknown functions to be defined in accordance with Eq. (1). The variables \( X, T, \varphi \) are assumed to be independent. Accordingly, we use the
following representations for the differential operators:

\[
\begin{align*}
\ddot{\varphi} &= -\Omega \partial_x + \epsilon \partial_T, \\
\dddot{\varphi} &= -\Omega^2 \partial^2_x - 2\epsilon\Omega \partial^2_x T - \epsilon \Omega' \partial_x + O(\epsilon^2), \\
\partial_x &= i\omega \partial_x + \epsilon \partial_X, \\
\partial^2_{xx} &= -\omega^2 \partial^2_x + \epsilon(2i\omega \partial^2_x X + i\omega' \partial_x) + O(\epsilon^2), \\
\partial^3_{xxx} &= -i\omega^3 \partial^3_x + \epsilon(-3\omega^2 \partial^3_x X - 3\omega \omega' \partial^2_x) + O(\epsilon^2), \\
\partial^4_{xxxx} &= \omega^4 \partial^4_x + \epsilon(-4i\omega^3 \partial^4_x X - 6i\omega^2 \omega' \partial^3_x) + O(\epsilon^2).
\end{align*}
\]

We require that \(\omega(X, T)\) and \(\Omega(X, T)\) satisfy dispersion relation (A.2) and equation

\[
\Omega' X + \omega' T = 0
\]

that follows from (21). Since in the case of a concentrated spring with constant stiffness the undamped oscillation can be described by Eq. (17), we assume that

\[
\Omega(\pm 0, T) = \Omega_0(T).
\]

Additionally, we require that

\[
[W] = 0, \quad [\varphi] = 0.
\]

In Eq. (25) the right-hand side is defined in accordance with the frequency equation (8), wherein \(K = K(T)\). The phase \(\varphi(x, t)\) should be defined by the formula

\[
\varphi = i \int (\omega \, dx - \Omega \, dt).
\]

For large times, integrating formally Eq. (1) with respect to \(x\) over the infinitesimal vicinity of \(x = 0\) taking into account (2), one gets (4), wherein \(p = 0\). Now we substitute ansatz (19)–(22) and representations (23) into Eq. (4) and equate coefficients of like powers \(\epsilon\). Taking into account frequency equation
(8), and Eq. (25), one obtains that to the first approximation

$$\left[\omega^2 W'_0 X \right] = -[\omega' X] W_0. \quad (28)$$

Here $\omega = \omega_{\pm}$ at $x = \pm 0$, and $\omega_{\pm}$ is defined by (A.5).

On the other hand, the quantity in the left-hand side of (28) can be defined by consideration of Eq. (1) at $x = \pm 0$. To do this, we substitute ansatz (19)–(22) and representations (23) into Eq. (1) and equate coefficients of like powers $\epsilon$. Taking into account dispersion relation (A.2) and Eq. (25), one obtains that to the first approximation

$$4\omega^3 W'_0 X + 6\omega^2 \omega' X W_0 + (2\Omega_0 W'_0 T + \Omega'_0 W_0) = 0 \quad (29)$$

at $x = \pm 0$. Equation (29) can be rewritten in the following form:

$$\omega^2 W'_0 X + \frac{\Omega_0}{2\omega} W'_0 T + \left(\frac{1}{4\omega} \Omega'_0 T + \frac{3\omega' X}{2}\right) W_0 = 0. \quad (30)$$

Calculating the jump results in

$$\left[\omega^2 W'_0 X \right] + \frac{1}{2} \left[\frac{1}{\omega}\right] \Omega_0 W'_0 T + \left(\frac{1}{4} \left[\frac{1}{\omega}\right] \Omega'_0 T + \frac{3\omega' X}{2}\right) W_0 = 0. \quad (31)$$

Now, taking into account (28), one gets

$$\bar{W}_0' T + \left(\frac{\Omega'_0 T}{2\Omega_0} + \frac{[\omega' X]}{\Omega_0[1/\omega]}\right) \bar{W}_0 = 0, \quad (32)$$

where $\bar{W}_0(T) \equiv W_0(0,T)$.

Due to (24) one has

$$\omega'_X = \omega'_\Omega \Omega'_X = -\omega'_\Omega \omega'_T = -\omega'_\Omega^2 \Omega'_T, \quad (33)$$

where according to dispersion relation

$$\omega'_\Omega = \frac{\Omega_0}{2\omega^3}. \quad (34)$$
Hence,
\[ \omega'_X = -\frac{\Omega_0^2}{4\omega^6} \Omega_0 T', \] (35)
and
\[ [\omega_0] = -\frac{1}{4} \Omega_0^2 \Omega_0 T \left[ \frac{1}{\omega^2} \right]. \] (36)

Taking into account the dispersion relation (A.2), Eq. (36) can be rewritten in the following form:
\[ [\omega_0] = \frac{1}{4} \frac{\Omega_0^2 \Omega_0 T}{1 - \Omega_0^2} \left[ \frac{1}{\omega} \right]. \] (37)

Now we substitute the equation obtained into Eq. (32), and finally obtain the first approximation equation
\[ \bar{W}_0 T + \left( \frac{\Omega_0 T}{2\Omega_0} + \frac{1}{4} \frac{\Omega_0 \Omega_0 T}{1 - \Omega_0^2} \right) \bar{W}_0 = 0. \] (38)

The general solution of the above equation is
\[ \bar{W}_0 = C_0 \exp \left( -\int \frac{\Omega_0 T}{2\Omega_0} dT \right) \exp \left( -\int \frac{1}{4} \frac{\Omega_0 \Omega_0 T}{1 - \Omega_0^2} dT \right), \] (39)
where \( C_0 \) is an arbitrary constant. One has
\[ \int \frac{\Omega_0 T}{2\Omega_0} dT = \frac{1}{2} \ln \Omega_0, \] (40)
and
\[ \int \frac{1}{4} \frac{\Omega_0 \Omega_0 T}{1 - \Omega_0^2} dT = \frac{1}{8} \int \frac{d\Omega_0^2}{1 - \Omega_0^2} = -\frac{1}{8} \ln(1 - \Omega_0^2). \] (41)

Thus, one obtains:
\[ \bar{W}_0 = C_0 \exp \left( -\frac{1}{2} \ln \Omega_0 \right) \exp \left( \frac{1}{8} \ln(1 - \Omega_0^2) \right) = C_0 \frac{\sqrt{1 - \Omega_0^2}}{\sqrt{\Omega_0}}. \] (42)

Taking into account (8), one can rewrite the last formula in the equivalent form:
\[ \bar{W}_0 = C_0 \frac{\sqrt{2}}{\sqrt{4 - |K|^2/3}}. \] (43)
One can see that $\Omega_0 \to 0$ if $K \to 2\sqrt{2}$. When $\Omega_0 \to 0$,

$$\bar{W}_0 = \frac{C_0}{\Omega_0^{1/2}} + o(1). \quad (44)$$

This result is analogous to the classical result for a one degree of freedom system

$$\ddot{y} + \dot{\Omega}^2(\epsilon \tau)y = 0, \quad (45)$$

where the following formula

$$Y \propto \frac{1}{\dot{\Omega}^{1/2}} \quad (46)$$

for the amplitude of a natural oscillations $Y$ is valid (the Liouville – Green approximation [31]). On the other hand, unlike one degree of freedom system (45), for the system under consideration, formula (44) is valid only in the limiting case $\Omega_0 \to +0$. For finite $\Omega_0$ the dependence for the amplitude is more complicated and is given by (42). The analogous results were obtained for other similar problems (for a taut string on the Winkler foundation with point inhomogeneity [22, 28]).

Combining the solution in the form of Eqs. (19)–(22) with its complex conjugate, we get the non-stationary solution as the following ansatz:

$$w(0, t) \sim \bar{W}_0(\Omega_0(T)) \sin \left( \int_0^T \Omega_0(T) \, dT - D_0 \right), \quad (47)$$

where $\bar{W}_0$ is defined by (42). The unknown constants $C_0$ and $D_0$ should be defined by equating the right-hand sides of (17) and (47) taken at $t = 0$. This yields

$$C_0 = \frac{2^{3/4} \sqrt{1 - \Omega_0^2(0)}}{3 \sqrt{\Omega_0(0)}} |p_F(\Omega_0(0))| = \frac{2^{3/4} \sqrt{|K(0)|}}{3 \sqrt{4 - |K(0)|^{4/3}}} |p_F(\Omega_0(0))|, \quad (48)$$

$$D_0 = \arg p_F(\Omega_0(0)). \quad (49)$$

In the particular case $p = \delta(t)$ that corresponds to the initial conditions in the
classical form

\[ w(0, 0) = 0, \quad \dot{w}(0, 0) = 1 \]  

one has

\[ C_0 = \frac{\sqrt{2} \sqrt{1 - \Omega_0^2(0)}}{3 \sqrt{\Omega_0(0)}}, \]  

\[ D_0 = 0. \]  

In the particular case

\[ p(t) = H(t) \exp(-\lambda t), \]  

where \( H(t) \) is the Heaviside function, and \( \lambda = \text{const} > 0 \), one has

\[ p_F(\Omega_0(0)) = \frac{1}{\lambda - i \Omega_0(0)}, \]  

\[ |p_F(\Omega_0(0))| = \frac{1}{\sqrt{\lambda^2 + \Omega_0^2(0)}}, \]  

\[ \arg (p_F(\Omega_0(0))) = \arctan \frac{\Omega_0(0)}{\lambda}. \]  

5. Numerics

The solution satisfying Eq. (1) and initial conditions (3) can be written in the following form

\[ w(x, t) = \int_0^t P(\tau) \mathcal{H}(t - \tau, x) \, d\tau, \]  

where \( \mathcal{H} \) is the fundamental solution (Appendix B, formula (B.3)). On the other hand, from Eq. (2) one obtains

\[ w(0, t) = -K^{-1}(P(t) - p(t)). \]  

Thus, it is easy to get the following Volterra integral equation of the second kind for \( P(t) \):

\[ P(t) = p(t) + K \int_0^t P(\tau) \mathcal{H}(t - \tau, 0) \, d\tau. \]
To solve integral equation (59) we approximate the integral in the right-hand side according to the trapezoidal rule [35] and reduce the problem to a system of linear equations with a triangular matrix. To perform the numerical calculations we use SciPY software. All numerical results below are obtained for the following choice of the time step

$$\Delta \tau = 0.025.$$  \hfill (60)

Calculating the numerical solutions, which corresponds to \( p = \delta(t) \), we approximate the Dirac delta-function as follows:

$$p = t_0^{-1}(H(t) - H(t - t_0)),$$  \hfill (61)

where \( H(t) \) is the Heaviside function.

A comparison between the analytical solution for \( P(t) \) (given by formulas (2), (43), (47), (48), (49)) and the numerical one is presented in Figures 1–3. In Figure 1 we compare the results obtained for the case of \( p = \delta(t) \) and monotonically increasing \( |K(\epsilon t)| \). The asymptotic solution approaches the numeric one very quickly. The localized buckling occurs at \( t \approx 242 \) that corresponds to the critical value (11).

In Figure 2 we compare the results obtained for the case of \( p = \delta(t) \) and monotonically decreasing (vanishing) \( |K(\epsilon t)| \). Again the asymptotic solution approaches the numeric one very quickly.

In Figure 3 we compare the results obtained for the case of \( p(t) = H(t) \exp(-\lambda t) \) and monotonically increasing \( |K(\epsilon t)| \). Since \( \lambda = 0.1 \) is taken small enough, the method of the stationary phase gives a reasonable result only after some time \( t \approx 20 \). After that time the analytical solution approaches the numerical one.
Figure 1: Comparing the analytical solution (2), (43), (47), (51), (52) obtained for $p = \delta(t)$ (the red dotted line) and the numerical solution obtained for $p(t) = t_0^{-1}(H(t) - H(t - t_0))$ (the blue solid line) in the case $K(\epsilon t) = -0.4 - \epsilon t$. Here $\epsilon = 0.01$, $t_0 = 0.1$. The localized buckling occurs at $t \approx 242$. 
Figure 2: Comparing the analytical solution (2), (43), (47), (51), (52) obtained for \( p = \delta(t) \) (the red dotted line) and the numerical solution obtained for \( p(t) = t_0^{-1}(H(t) - H(t - t_0)) \) (the blue solid line) in the case \( K(\epsilon t) = -2.0 \exp(-\epsilon t) \). Here \( \epsilon = 0.01 \), \( t_0 = 0.1 \).
Figure 3: Comparing the analytical solution (43), (47), (48), (49), (55), (56) (the red dashed line) and the numerical solution obtained for $p = H(t) \exp(-\lambda t)$ (the blue solid line) in the case $K(\epsilon t) = -0.4 - \epsilon t$. Here $\epsilon = 0.01$, $\lambda = 0.1$. The localized buckling occurs at $t \simeq 242$. 
6. Conclusion

In the paper we consider a non-stationary localized oscillations of an infinite Bernoulli-Euler beam lying on the Winkler foundation with a point inhomogeneity (a concentrated spring with negative slowly time-varying stiffness. In the case of the concentrated spring with a constant stiffness a trapped mode of oscillations exists and is unique if and only if conditions (9), (11) are satisfied. The existence of a trapped mode leads to the possibility of the wave localization near the inhomogeneity in the Winkler foundation. Applying a vanishing as $t \to \infty$ external excitation to the point of the beam under the inclusion leads to the emergence of undamped free oscillations localized near the spring. The most important result of the paper is analytical formulas (43), (47), (48), (49), which allow us to describe for large times such non-stationary localized oscillations in the case of slowly time-varying spring stiffness. The obtained analytical results were verified by independent numerical calculations based on the Volterra integral equation of the second kind (59), which is equivalent to the initial value problem (1)–(3). The applicability of the analytical formulas was demonstrated for various types of external excitation and laws governing the varying spring stiffness (see Figures 1–3).

We also have shown that localized low-frequency oscillations with increasing amplitude precede the localized beam buckling (see (44)). However, unlike the case (45) of a one degree of freedom system with time-varying stiffness, in the framework of the problem under consideration formula (44) is correct only in the limiting case, where the frequency of localized oscillations approaches zero. The analogous results were obtained for other similar problems (for a taut string on the Winkler foundation with point inhomogeneity [22, 28]).

Finally, it may be noted that the results of the paper can be used when constructing dynamic materials [36, 37].
Appendix A. The Green function, for a beam on the Winkler foundation, in the frequency domain

Consider equation

\[ \ddot{G} + G'''' + G = \exp(-i\Omega t)\delta(x), \]  

(A.1)

where \(|\Omega| < 1\). The dispersion relation for the operator in the left-hand side of (A.1) is

\[ \omega^4 - \Omega^2 + 1 = 0. \]  

(A.2)

We look for the solution of this equation in the form of

\[ G = 2 \text{Re} \left( G_0(x, \Omega) \right) \exp(-i\Omega t), \]  

(A.3)

\[ G_0(x, \Omega) = \begin{cases} 
  \frac{C}{2} \exp \left( i(\omega_+ x - \varphi/2) \right), & x > 0; \\
  \frac{C}{2} \exp \left( i(\omega_- x + \varphi/2) \right), & x < 0;
\end{cases} \]  

(A.4)

where \(C, \varphi\) are real constants, and \(\omega_\pm\) are the wavenumbers

\[ \omega_\pm = \pm a i + a, \]  

(A.5)

\[ a = \frac{\sqrt{2}}{2} \sqrt{1 - \Omega^2}, \]  

(A.6)

which satisfy dispersion relation (A.2). Calculating the real parts of jumps for the first, the second, and the third derivative of the right-hand side of (A.3) results in the following set of boundary conditions at \(x = 0\):

\[ \text{Re}[i\omega] = 0, \]  

(A.7)

\[ \text{Re}[(i\omega)^2] = 0, \]  

(A.8)

\[ 2C \text{Re}[(i\omega)^3] = 1. \]  

(A.9)

The left-hand side of Eq. (A.8) equals zero identically. Calculating the left-hand
sides of (A.7), (A.9) yields two equations to define unknown constants $\varphi$ and $C$:

\[
\sin \frac{\varphi}{2} - \cos \frac{\varphi}{2} = 0, \quad (\text{A.10})
\]

\[
a^3 \left( -\sin \frac{\varphi}{2} + 3 \sin \frac{\varphi}{2} + 3 \cos \frac{\varphi}{2} - \cos \frac{\varphi}{2} \right) = \frac{1}{2C}. \quad (\text{A.11})
\]

Resolving these equations yields:

\[
C = \frac{1}{2(1 - \Omega^2)^{3/4}}, \quad (\text{A.12})
\]

\[
\tan \frac{\varphi}{2} = 1. \quad (\text{A.13})
\]

Hence,

\[
2 \text{Re} G_0(x, \Omega) = \frac{1}{2} e^{-a|x|} \cos \left( a|x| - \pi/4 \right) \frac{1}{(1 - \Omega^2)^{3/4}}. \quad (\text{A.14})
\]

Appendix B. The Green function, for a beam on the Winkler foundation, in the time domain (the fundamental solution)

Consider equation

\[
\ddot{G} + G''' + G = \delta(t) \delta(x). \quad (\text{B.1})
\]

According to (A.1), (A.3), (B.1) the Laplace transform $G_L(0, p)$ of $G(x, t)|_{x=0}$ equals

\[
G_L(0, p) = 2 \text{Re} G_0(0, -pi) = \frac{1}{2^{5/4}(p^2 + 1)^{3/4}}. \quad (\text{B.2})
\]

Applying the inverse Laplace transform yields [38]

\[
G(0, t) = \frac{\sqrt{\pi}}{2^{7/4} \Gamma(3/4)} t^{1/4} J_{1/4}(t), \quad (\text{B.3})
\]

where $\Gamma(\cdot)$ is the Gamma function (the Euler integral of the second kind), $J_{1/4}(\cdot)$ is the Bessel function of the first kind of order $1/4$. 

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Acknowledgements

The authors are grateful to Prof. D.A. Indeitsev for useful and stimulating discussions.

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