RANK GRADIENT OF SEQUENCES OF SUBGROUPS IN A DIRECT PRODUCT

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Abstract. For a sequence \( \{U_n\}_{n=1}^{\infty} \) of finite index subgroups of a direct product \( G := A \times B \) of finitely generated groups, we show that

\[
\lim_{n \to \infty} \min \left\{ |X| : \langle X \rangle = U_n \right\} \left[ G : U_n \right] = 0
\]

once \( [A : A \cap U_n], [B : B \cap U_n] \to \infty \) as \( n \to \infty \). Our proof relies on the classification of finite simple groups.

1. Introduction

Motivated by the theory of 3-manifolds, Lackenby defined in [12] the rank gradient of a sequence \( \{U_n\}_{n=1}^{\infty} \) of finite index subgroups of a finitely generated group \( G \) to be the following combinatorial invariant

\[
\inf \frac{d(U_n) - 1}{[G : U_n]}
\]

where \( d(K) \) is the least cardinality of a generating set for the group \( K \). As can be seen from [1, 2, 3, 4, 5, 6, 9, 11, 14, 15, 18, 19, 20] the rank gradient has been extensively studied, calculated in various cases, and related to the notion of cost in ergodic theory. We show that the rank gradient of sequences in direct products vanishes.

**Theorem 1.1.** Let \( A, B \) be finitely generated groups, set \( G := A \times B \), and let \( \{U_n\}_{n=1}^{\infty} \) be a sequence of finite index subgroups of \( G \). Assume that \( [A : A \cap U_n], [B : B \cap U_n] \to \infty \) as \( n \to \infty \). Then

\[
\lim_{n \to \infty} \frac{d(U_n) - 1}{[G : U_n]} = 0.
\]

An open problem in measured group theory analogous to [Theorem 1.1] is to determine whether the product of every pair of infinite countable groups has fixed price. Indeed, the special case of [Theorem 1.1] considering Farber chains, is the focus of [17] that was motivated by the fixed price problem. For more on this fascinating topic see [10].

2. Preliminaries

Let \( G \) be a finitely generated group and let \( H \leq G \) be a finite index subgroup of \( G \). It follows from [16, Theorem 11.44] that

\[
d(H) \leq d(G)[G : H].
\]
Proposition 2.1. Let $A,B$ be groups, set $G := A \times B$ and let $K \leq G$ be a subgroup for which $BK = G$. Then $A \cap K \triangleleft A$.

Proof. Take $a \in A$, $x \in A \cap K$ and note that since $A \subseteq BK$, there exist $b \in B$, $k \in K$ such that $a = bk$. As $A \triangleleft G$ we see that $kxk^{-1} \in A \cap K$, and $[A,B] = \{1\}$ implies that $axa^{-1} = bkxk^{-1}b^{-1} = kxk^{-1} \in A \cap K$. \hfill \Box

Definition 2.2. Let $G$ be a group and let $S,X \subseteq G$ be subsets. We denote by $\langle S \rangle^X$ the subgroup of $G$ generated by the conjugates of $S$ elements of $X$. For a normal subgroup $N \triangleleft G$ we define $d_G(N)$ to be the least cardinality of a subset $S \subseteq N$ for which $\langle S \rangle^G = N$.

Definition 2.3. Let $K$ be a finite group, let $d \in \mathbb{N}$, and let $T = \{t_1, \ldots, t_d\}$ be a generating multiset of $K$. Take a free group $F$ on $X = \{x_1, \ldots, x_d\}$ and let $\varphi : F \to K$ be the unique surjection with $\varphi(x_i) = t_i$ for $1 \leq i \leq d$. We set $r(K,T) := d_F(\text{Ker}(\varphi))$ and think of this quantity as the least number of relations needed to present $K$ using the generating multiset $T$.

Proposition 2.4. Let $G$ be a group generated by a finite subset $T \subseteq G$, let $N \triangleleft G$ be a normal subgroup of finite index in $G$, and define a multiset $\overline{T} \subseteq G/N$ by $\overline{T} := \{tN \mid t \in T\}$. Then $d_G(N) \leq r(G/N, \overline{T})$.

Proof. Let $F$ be the free group on $T$, and let $\varphi : F \to G/N$ be the unique surjection for which $\varphi(t) = tN$ for all $t \in T$. By 2.3 there exists a subset $S \subseteq \text{Ker}(\varphi)$ of cardinality $r(G/N, \overline{T})$ such that $\langle S \rangle^F = \text{Ker}(\varphi)$. Define a surjection $\psi : F \to G$ by $\psi|_T = \text{Id}_T$, and observe that

$$N = \psi(\text{Ker}(\varphi)) = \psi(\langle S \rangle^F) = \langle \psi(S) \rangle^G$$

so that $d_G(N) \leq |\psi(S)| \leq |S| = r(G/N, \overline{T})$ as required. \hfill \Box

3. Presentations of finite groups

Let $K$ be a finite group generated by a multiset $T = \{t_1, \ldots, t_d\}$, let $N$ be a minimal normal subgroup of $K$, let $S$ be a finite simple group onto which $N$ surjects, and define the multiset $\overline{T} := \{t_1N, \ldots, t_dN\}$. The following bound comes from the argument appearing in the proof of Theorem 1 in [13] and in the discussion following the proof of Theorem 2 therein.

$$r(K,T) \leq r(K/N, \overline{T}) + 6d \log_2 |N| + r(S,W) \quad (3.1)$$

where $W$ is any pair of elements generating $S$. It follows from [7, Corollary A', Lemma 2.1, Theorem 4.34] and [13, Theorem 1] that

$$r(S,W) \leq 8|S|^{3/7} \quad (3.2)$$

Corollary 3.1. Let $K$ be a finite group generated by a multiset $T$. Then

$$r(K,T) \leq 128|T||K|^{3/7} \quad (3.3)$$
Proof. Let $N$ be a minimal normal subgroup of $K$, and let $S$ be a finite simple factor of $N$. By induction, (3.1), and (3.2), we have

\[ r(K, T) \leq r(K/N, T) + 6|T| \log_2 |N| + 8|S|^{3/7} \]
\[ \leq 128|T| \left( \frac{|K|}{|N|} \right)^{3/7} + 6|T| \log_2 |N| + 8|N|^{3/7} \]
\[ \leq 128|T| \left( \frac{|K|}{|N|} \right)^{3/7} + 24|T||N|^{3/7} + 8|T||N|^{3/7} \]
\[ = 128|T| \left( \frac{|K|}{|N|} \right)^{3/7} + |N|^{3/7} - \frac{3}{4} |N|^{3/7} \]
\[ \leq 128|T| \left( |K|^{3/7} + 1 - \frac{3}{4} |N|^{3/7} \right) \]
\[ \leq 128|T| \left( |K|^{3/7} + 1 - \frac{3}{4} 2^{3/7} \right) \leq 128|T||K|^{3/7}. \]

4. Upper bounds on the number of generators

We establish several bounds on the number of generators of a finite index subgroup of a direct product, and conclude that the rank gradient vanishes.

**Theorem 4.1.** Let $A, B$ be finitely generated groups, and let $H$ be a finite index subgroup of $G := A \times B$. Then $d(H)$ is bounded by:

1. $d(G)([G : AH] + [AH : H])$
2. $d(G)([G : BH] + [BH : H])$
3. $d(G)([G : AH] + 130[G : BH][G : H]^{3/7}).$

**Proof.** For (1) note that

\[ d(H) \leq d(H/H \cap A) + d(H \cap A) \leq d(AH/A) + d(A)[A : H \cap A] \]
\[ \leq d(AH) + d(A)[AH : H] \leq d(G)[G : AH] + d(G)[AH : H] \]

as required. To get (2) just replace $A$ with $B$ in (1).

For (3) let $\pi_A, \pi_B$ be the projections from $G$ onto $A, B$ and observe that $H \leq \pi_A(H) \times \pi_B(H)$ is a subgroup that complements $\pi_B(H)$ (that is, $\pi_B(H)H = \pi_A(H) \times \pi_B(H)$). By Proposition 2.1, $\pi_A(H) \cap H \trianglelefteq \pi_A(H)$.

We can thus take a subset $S \subseteq \pi_A(H) \cap H$ of least cardinality, with

\[ (S)_{\pi_A(H)} = \pi_A(H) \cap H. \]

Furthermore, take $R_A$ (respectively, $R_B$) to be a subset of $H$ mapped bijectively by $\pi_A$ (respectively, $\pi_B$) onto a generating set of $\pi_A(H)$ (respectively, $\pi_B(H)$) of least cardinality. Set $L := (S \cup R_A \cup R_B)$ and note that $L \leq H$. 
Similarly, $G(2)$ are tight up to a constant once $A$ subgroups with $B$.

On the other hand, $\pi_B(H) = \pi_B(\langle R_B \rangle) \leq \pi_B(L)$ so in conjunction with (4.3) we conclude that $L = H$. Thus

\begin{equation}
\label{eq:4.4}
d(H) = d(L) \leq |S| + |R_A| + |R_B|
\end{equation}

\begin{equation}
\leq d_{\pi_A(H)}(\pi_A(H) \cap R_A) + d(\pi_A(H)) + d(\pi_B(H))
\end{equation}

Moreover,

\begin{equation}
\label{eq:4.5}
d(\pi_A(H)) \leq d(A)[A : \pi_A(H)] = d(A)[G : BH] \leq d(G)[G : BH]
\end{equation}

and similarly, we have $d(\pi_B(H)) \leq d(G)[G : AH]$. Combining this inequality, (4.4), and (4.5) we obtain (3). \hfill \Box

Let us now deduce Theorem 1.1.

Proof. Suppose that our claim is false, so (after passing to a subsequence) we may assume that the limit in (1.2) is positive. By Theorem 4.1 (1),

\begin{equation}
\label{eq:4.6}
\frac{d(U_n)}{[G : U_n]} \leq d(G)\left(\frac{1}{[A_{U_n} : U_n]} + \frac{1}{[G : A_{U_n}]}\right)
\end{equation}

and the first summand on the right hand side tends to 0 in view of our assumption that $[A : A \cap U_n] \to \infty$ as $n \to \infty$. Since the left hand side of (4.6) tends to some $c > 0$, we conclude that $[G : A_{U_n}]$ is bounded as $n \to \infty$. Similarly, $[G : BU_n]$ is bounded. Finally, apply Theorem 4.1 (3) to $U_n$. \hfill \Box

5. Lower bounds on the number of generators

To which extent are the bounds in Theorem 4.1 tight? Suppose that $A$ and $B$ are isomorphic to a free group $F$ on two generators. Clearly, (1) and (2) are tight up to a constant once $H := A_0 \times B_0$ where $A_0, B_0 \leq A, B$ are subgroups with $[A : A_0] = [B : B_0]$.

It is conjectured ([13, Conjecture 2]) that every finite group has a presentation with a logarithmic number of relations. If this improvement of Corollary 3.1 holds, then the argument from the proof of Theorem 4.1 (3) gives a logarithmic bound on the number of generators as a function of the index of $H$ in $G$. Let us show that such a bound is tight (up to a constant). Fix a prime $p$, and let $\{P_n\}_{n=1}^{\infty}$ be the lower $p$-central series defined by

\begin{equation}
P_1 = F, \ P_{n+1} = P_n^p[F, P_n].
\end{equation}
Set
\[ p'^n := |P_n/P_{n+1}|, \quad b_n := \sum_{i=1}^{n} a_i. \]

By [8, Corollary 3.4], \( a_n = \sum_{i=1}^{n} r_i \) where \( r_i \) are the Witt numbers given by
\[ r_i = \frac{1}{l} \sum_{j|i} \mu \left( \frac{i}{j} \right) 2^j. \]

It is thus easy to see that \( r_n \sim 2^n/n \), and in particular the radius of convergence of \( \sum_{n=1}^{\infty} r_n x^n \) is \( 1/2 \). Multiplying by \( (\sum_{n=0}^{\infty} x^n)^2 \) we deduce that \( \sum_{n=1}^{\infty} b_n x^n \) has the same radius of convergence, and thus
\[ \limsup_{n \to \infty} \frac{b_n}{b_{n-1}} \geq 2. \]

Set \( U_n := \{(x,y) \in F \times F \mid xP_n = yP_n\} \). We have
\[ [F \times F : U_n] = p^{b_n-1} = [F \times \{1\} : F \times \{1\} \cap U_n]. \]

We claim that \( U_n \) maps onto the elementary abelian group \( P_n/P_{n+1} \). Let \( \Delta \leq (F/P_n) \times (F/P_{n+1}) \) be the image of \( U_n \) mod \( P_{n+1} \). That is
\[ \Delta := \{(xP_{n+1},yP_{n+1}) \in (F/P_{n+1}) \times (F/P_{n+1}) \mid xP_n = yP_n\}. \]

Let \( L := \{(xP_{n+1},xP_{n+1}) \mid x \in F\} \) be the diagonal subgroup of \( \Delta \). Clearly, \( L \) is a normal subgroup of \( \Delta \) that commutes with \( (P_n/P_{n+1}) \times (P_n/P_{n+1}) \). Moreover, \( L \cap ((P_n/P_{n+1}) \times (P_n/P_{n+1})) \) is the diagonal subgroup isomorphic to \( P_n/P_{n+1} \). It follows that
\[ \frac{\Delta}{L} \cong \frac{(P_n/P_{n+1}) \times (P_n/P_{n+1})}{L \cap ((P_n/P_{n+1}) \times (P_n/P_{n+1}))} \cong P_n/P_{n+1}. \]

Hence \( U_n \) has \( P_n/P_{n+1} \) as a homomorphic image and therefore \( d(U_n) \geq a_n \). At last, note that for any \( \epsilon > 0 \) we have
\[ \frac{d(U_n)}{\log_p[F \times F : U_n]} \geq \frac{a_n}{b_{n-1}} = \frac{b_n}{b_{n-1}} - 1 \geq 1 - \epsilon \]
where the last inequality holds for infinitely many values of \( n \) (that is, \( (5.8) \) holds for a subsequence of \( \{U_n\}_{n=1}^{\infty} \)).

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