Non-conventional Anderson localization in bilayered structures

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Abstract - We resolve the problem of non-conventional Anderson localization emerging in bilayered periodic-on-average structures with alternating layers of materials with positive and negative refraction indices. Recently, it was numerically discovered that in such structures with weak fluctuations of refractive indices, the localization length $L_{\text{loc}}$ can be enormously large for small wave frequencies $\omega$. Within a new approach allowing us to go beyond the second order of perturbation theory, we derive the expression for $L_{\text{loc}}$ valid for any $\omega$ and small variance of disorder, $\sigma^2 \ll 1$. In the limit $\omega \to 0$ one gets a quite specific dependence, $L_{\text{loc}} \propto \sigma^4 \omega^8$. Our approach allows one to establish the conditions under which this effect occurs.

Introduction. – The remarkable success in nano- and material science has led to a burst of interest in the study of wave propagation and electron transport in periodic one-dimensional (1D) systems (see, e.g., ref. [1] and references therein). Among others, the systems of particular interest are bilayered structures in optics [2] and electromagnetics [3], or semiconductor superlattices [4], as well as alternating quantum wells and barriers in electronics. The interest in this subject is due to the possibility to create devices with prescribed properties of transmission and/or reflection.

As is shown in recent studies [5–10], the new perspectives are related to specific optic properties of metamaterials that can be embedded in periodic structures. A particular system that has attracted much attention, represents an array of two alternating $a,b$ layers with equal widths, $d_a = d_b$, one of which is of right-handed (RH) material with positive refractive index, $n_a > 0$, and the other is of left-handed (LH) material with negative index, $n_b < 0$. In such periodic structure with equal $n_a = |n_b|$ the phase shift of the wave function gained in the $a$-layer is canceled by the subsequent shift in the next $b$-layer. Therefore, the total phase shift vanishes after passing $N$ unit $(a,b)$ cells. Together with a perfect transmission, this results in a kind of “non-visible” structure.

However, even a small amount of disorder can destroy the above ideal picture due to an emergence of localization [6]. To a great surprise, the numerical data [7] obtained for the model with weakly disordered refractive indices $n_a$ and $n_b$, have demonstrated very fast divergence of localization length, $L_{\text{loc}} \sim \omega^{-6}$, for $\omega \to 0$. More detailed numerical calculations [8] have shown that the power $\kappa$ in dependence $L_{\text{loc}} \propto \omega^{-\kappa}$ increases with the sample length, and even reaches the value $\kappa \approx 8.78$ for the total number of layers $N \sim 10^{12}$. The influence of polarization on this abnormal localization have been considered in ref. [9], and the role of dispersion and absorption was analyzed in refs. [10,11].

In ref. [12] another type of disorder was analyzed according to which both widths $d_a$ and $d_b$ are slightly fluctuating. Specifically, the general case of correlated disorder was analytically studied by assuming any kind of statistical correlations in widths of $a$ and $b$ layers, as well as the inter-correlations between the two widths. It was shown that for any ratio between $d_a$ and $d_b$ the localization length is governed by the unique expression, no matter whether the $a$, $b$ layers are both of normal materials (RH-RH array) or with alternating right-handed and left-handed materials (RH-LH array). The same statement stems from the analysis of RH-RH and RH-LH arrays with fluctuating refractive indices [13], however, only when $d_a \neq d_b$ in the latter case. These results indicate that the

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abnormal localization emerges in a very specific situation, and not only due to inclusion of LH arrays into the structure.

Later on, it was found [14] that for \( d_a = d_b \) the phase of wave, propagating through the RH-LH array with fluctuating refractive indices, is described by a highly non-uniform distribution. This effect is somewhat similar to that arising in standard tight-binding models close to band edges (see, e.g., review [15] and references therein). Further analysis of this fact has led to a remarkable result [14]: the Lyapunov exponent vanishes in the quadratic approximation in disorder parameter. Nevertheless, the problem has remained open since the evaluation of next-order terms for the Lyapunov exponent was confronted with severe technical difficulties.

In this letter we suggest the method allowing one to resolve the above problem of non-conventional Anderson localization. We derive the expression for the Lyapunov exponent within the fourth-order perturbation theory, which is valid for any value of frequency \( \omega \) and leads to the asymptotic expression \( L_{\text{loc}} \propto \sigma^{-4} \omega^{-8} \) for small \( \omega \). Our approach elucidates the mechanism that is responsible for such an abnormal localization, and allows one to specify the conditions under which it emerges.

**Model.** – The model describes the perpendicular propagation of electromagnetic waves of frequency \( \omega \) through an infinite array of two alternating \( a \) and \( b \) layers. Every kind of slabs is, respectively, specified by the dielectric permittivity \( \varepsilon_{a,b} \), magnetic permeability \( \mu_{a,b} \), corresponding refractive index \( n_{a,b} = \sqrt{\varepsilon_{a,b} \mu_{a,b}} \), impedance \( Z_{a,b} = \mu_{a,b}/n_{a,b} \) and wave number \( k_{a,b} = \omega n_{a,b}/c \). We consider two systems: the homogeneous stack when both \( a \) and \( b \) layers are made of right-handed optic materials, and mixed stack when \( a \)-layers contain RH-material while \( b \)-layers are of left-handed material. Following ref. [7], we admit that the compositional disorder is incorporated via dielectric constants \( \varepsilon_{a,b} \) only,

\[
n_a(n) = 1 + \eta_a(n), \quad n_b(n) = \pm [1 + \eta_b(n)],
\]

with \( \mu_a = 1 \) and \( \mu_b = \pm 1 \). Here \( n \) enumerates the \( n \)-th unit \((a,b)\) cell. The upper/lower sign stands for the RH/LH material respectively. Without disorder, \( \eta_{a,b}(n) = 0 \), the RH-RH structure is just the air without any stratification, while the RH-LH array represents the ideal mixed stack (\( \varepsilon_a = \mu_a = 1 \), \( \varepsilon_b = \mu_b = -1 \)) with perfectly matched slabs (\( Z_a = Z_b = 1 \)). All slabs have equal thicknesses, \( d_a = d_b = d/2 \), thus the size of any \((a,b)\) cell is \( d \).

We specify that delta-correlated sequences \( \eta_{a,b}(n) \) have zero average, \( \langle \eta_{a,b}(n) \rangle = 0 \), and variance \( \sigma^2 \),

\[
\langle \eta_a(n) \eta_b(n') \rangle = \sigma^2 \delta_{a,b} \delta_{n,n'}.
\]

Here \( \langle \ldots \rangle \) stands for the average along arrays, which is assumed to be equivalent to ensemble average. Numerically, to generate \( \eta_a(n) \) and \( \eta_b(n) \) we use flat distribution, however our expressions are given in general form.

**Basic relations.** – Within any of \( a \) or \( b \) layers the electric field of the wave, \( E(x) \exp(-i\omega t) \), obeys the 1D Helmholtz equation with two boundary conditions at the interfaces between neighboring slabs,

\[
\left( \frac{d^2}{dx^2} + k_{a,b}^2 \right) E_{a,b}(x) = 0,
\]

\[ E_a(x_i) = E_b(x_i), \quad \mu_a^{-1} E'_a(x_i) = \mu_b^{-1} E'_b(x_i). \]

The \( x \)-axis is directed along the array of bi-layers perpendicular to the stratification, with \( x = x_i \) standing for the interface coordinate.

The general solution of eq. (3a) within the \( n \)-th \((a,b)\) cell can be presented in the following form:

\[
E_a(x) = E_a(x_n) \cos \left[ k_a(x - x_n) \right] + k_a^{-1} E'_a(x_n) \sin \left[ k_a(x - x_n) \right],
\]

inside \( a \)-layer, where \( x_n \leq x \leq x_{bn} \);

\[
E_b(x) = E_b(x_{bn}) \cos \left[ k_b(x - x_{bn}) \right] + k_b^{-1} E'_b(x_{bn}) \sin \left[ k_b(x - x_{bn}) \right],
\]

inside \( b \)-layer, where \( x_{bn} \leq x \leq x_{a(n+1)} \).

The coordinates \( x_{an} \) and \( x_{bn} \) refer to the left-hand edges of successive \( a \) and \( b \) layers. Note that \( x_{bn} - x_{an} = d_a \) and \( x_{a(n+1)} - x_{bn} = d_b \). By combining eqs. (4) with boundary conditions (3b) at \( x = x_{bn} \) and \( x = x_{a(n+1)} \), one can rewrite the model in the form of area-preserving map,

\[
Q_{n+1} = A_n Q_n + B_n P_n, \quad P_{n+1} = -C_n Q_n + D_n P_n.
\]

Here the “coordinate” \( Q_n = E_a(x_n) \) and “momentum” \( P_n = c/\omega E'_a(x_n) \) refer to the electric field and its derivative, respectively, taken at left-hand edge of the \( n \)-th unit \((a,b)\) cell. The factors \( A_n, B_n, C_n, D_n \) read

\[
A_n = \cos \varphi_a \cos \varphi_b - Z_{a,b}^{-1} Z_b \sin \varphi_a \sin \varphi_b, \quad B_n = Z_a \sin \varphi_a \cos \varphi_b + Z_b \cos \varphi_a \sin \varphi_b, \quad C_n = Z_{a,b}^{-1} \sin \varphi_a \cos \varphi_b + Z_b^{-1} \cos \varphi_a \sin \varphi_b, \quad D_n = \cos \varphi_a \cos \varphi_b - Z_a Z_b^{-1} \sin \varphi_a \sin \varphi_b.
\]

For non-zero disorder the coefficients (6) depend on the cell number (discrete “time”) \( n \), due to randomized refractive indices (1) entering both the impedances \( Z_{a,b}(n) \) and phase shifts \( \varphi_{a,b}(n) \),

\[
\varphi_a(n) = k_a(n) d/2 = \varphi[1 + \eta_a(n)], \quad \varphi_b(n) = k_b(n) d/2 = \pm \varphi[1 + \eta_b(n)],
\]

with \( \varphi = \omega d/2c \).

It is useful to pass to polar coordinates \( R_n \) and \( \theta_n \) by the standard transformation,

\[
Q_n = R_n \cos \theta_n, \quad P_n = R_n \sin \theta_n.
\]
The map in the radius-angle presentation gets the form
\[
\left( \frac{R_{n+1}}{R_n} \right)^{-2} = \frac{d \theta_{n+1}}{d \theta_n},
\]
\[
\theta_{n+1} = \arctan \left[ \frac{-C_n + D_n \tan \theta_n}{A_n + B_n \tan \theta_n} \right].
\]  
(9a)
(9b)

The localization length \( L_{\text{loc}} \) can be derived via the Lyapunov exponent \( \lambda \) [14,15],
\[
\frac{d}{L_{\text{loc}}} \equiv \lambda = -\frac{1}{2} \left\langle \ln \frac{d \theta_{n+1}}{d \theta_n} \right\rangle.
\]  
(10)

Thus, the \( \theta \)-map (9b) is the unique equation to be treated. We assume that the disorder is weak,
\[ \sigma^2 \ll 1 \quad \text{and} \quad (\sigma \varphi)^2 \ll 1, \]
(11)

that allows us to develop a proper perturbation theory.

**Phase distribution.** – The phase distribution \( \rho(\theta) \) can be found by expanding the exact \( \theta \)-map (9b) up to the second order in perturbation and taking into account the uncorrelated nature of the disorder, see eq. (2), one can obtain
\[
\theta_{n+1} - \theta_n = -\gamma - \eta_a(n)U(\theta_n) + \eta_b(n)U(\theta_n - \gamma/2) - \sigma^2 W(\theta_n),
\]
(12)

where
\[
U(\theta) = \varphi + \sin \varphi \cos(2\theta - \varphi),
\]
\[
W(\theta) = \varphi[\cos(2\theta - 2\varphi) + \cos(2\theta - 2\gamma)] + \sin \varphi[\sin \theta \sin(\theta - \varphi) + \sin(\theta - \gamma/2) \sin(\theta - \varphi - \gamma/2)] + \sin^2 \varphi\cos(4\theta - 2\varphi - \gamma) \cos \gamma.
\]
(13)

The unperturbed Bloch phase shift \( \gamma \) over a unit \((a, b)\) cell is defined as
\[ \gamma = \varphi \pm \varphi = (1 \pm 1)\omega d/2c, \]
(14)

where “plus” stands for the RH-RH structure, and “minus” refers to the mixed RH-LH array. To proceed further, we pass from eq. (12) to stationary Fokker-Planck equation for the phase distribution \( \rho(\theta) \) [16],
\[
\frac{d^2}{d\theta^2} \left[ U^2(\theta) + U^2(\theta - \gamma/2) \right] \rho(\theta) + 2\frac{d}{d\theta} \left[ \frac{\gamma}{\sigma^2} + W(\theta) \right] \rho(\theta) = 0,
\]
(15)

which is complemented by the normalization condition and the condition of periodicity, \( \rho(\theta + \pi) = \rho(\theta) \). One can see that the form of solution to eq. (15) strongly depends on whether the phase shift \( \gamma \) is non-zero (RH-RH array) or vanishes (RH-LH array).

**Homogeneous RH-RH array.** – In such a structure the Bloch phase (14) is non-zero, \( \gamma = 2\varphi = \omega d/c \), and for weak disorder the term in eq. (15) containing \( \gamma/\sigma^2 \) prevails over the others for any, even arbitrarily small, value of phase shift \( \varphi \). Therefore, the phase distribution is uniform within the first order of perturbation theory,
\[ \rho(\theta) = 1/\pi. \]
(16)

The Lyapunov exponent can be derived by substitution of eq. (12) into definition (10), and expanding the logarithm up to quadratic terms in disorder. After, the subsequent averaging over \( \theta \) is trivial, and one gets
\[ d/L_{\text{loc}} \equiv \lambda = \sigma^2 \varphi^2, \quad \varphi = \omega d/2c. \]
(17)

If the phase shift \( \varphi \) is small, this result yields the asymptotics,
\[ d/L_{\text{loc}} \equiv \lambda \approx \sigma^2 \omega^2 d^2/4c^2 \quad \text{for} \quad \omega \ll 2c/d, \]
(18)

that gives rise to standard \( \omega \)-dependence, \( \lambda \propto \omega^2 \) when \( \omega \rightarrow 0 \).

**Mixed RH-LH array.** – The principally different situation emerges for the RH-LH array. In this case \( \gamma = 0 \) independently of the phase shift \( \varphi \). As a result, we have \( W(\theta) = -U(\theta)U'(\theta) \) in eq. (13), and eq. (15) leads to a highly non-uniform phase distribution,
\[ \rho(\theta) = \frac{1}{\pi} \sqrt{\varphi^2 - \sin^2 \varphi}/U(\theta). \]
(19)

Figure 1 displays a perfect agreement between analytical expressions (16), (19) and the data obtained by the iteration of the exact map (5).

The above result means that to calculate the Lyapunov exponent via eq. (10), one needs to perform an average with the distribution \( \rho(\theta) \) given by eq. (19). Surprisingly, the use of eqs. (10), (12) and (19) results in the vanishing value of \( \lambda \) [14]. Therefore, the Lyapunov exponent is determined by the next orders of the perturbation theory. However, the evaluation of high-order terms in \( \rho(\theta) \) is not possible with the above method [17].

In order to proceed further, we suggest another method. The numerical data in fig. 1(b) manifest that the trajectory has the form of fluctuating ellipse specified by angle with respect to axes, and by fixed aspect ratio. Therefore, one can expect that in new variables \( Q_n, P_n \) obtained by rotating and rescaling the axes \( Q, P \), the trajectory transforms into fluctuating circle. Thus, the distribution of a new phase \( \Theta_n \) in the considered approximation will be uniform.

To follow this recipe, we rotate the \( Q, P \) axes by the angle \( \tau \), with further rescaling the axes due to the free
parameter $\alpha$. In new coordinates the expressions (5) and (8)-(10) conserve their forms, however, with the factors

$$\tilde{A}_n = A_n \cos^2 \tau + (B_n - C_n) \sin \tau \cos \tau + D_n \sin^2 \tau,$$

$$\tilde{B}_n \alpha^{-1} = B_n \cos^2 \tau - (A_n - D_n) \sin \tau \cos \tau + C_n \sin^2 \tau,$$

$$\tilde{C}_n \alpha = C_n \cos^2 \tau + (A_n - D_n) \sin \tau \cos \tau + B_n \sin^2 \tau,$$

$$\tilde{D}_n = D_n \cos^2 \tau - (B_n - C_n) \sin \tau \cos \tau + A_n \sin^2 \tau. \tag{20}$$

Now the distribution $\rho(\Theta)$ for new phase $\Theta$ can be found starting from the quadratic expansion of eq. (9b) with new coefficients (20) and $\gamma = 0$,

$$\Theta_{n+1} - \Theta_n = [\eta_n(n) - \eta_0(n)]V(\Theta_n) + \sigma^2 V(\Theta_n)V'(\Theta_n). \tag{21}$$

Here the function $V(\Theta)$ is defined by

$$V(\Theta) = \sin \varphi \sin(2\tau - \varphi) \sin 2\Theta$$

$$+ \frac{\alpha}{2} (\varphi - \sin \varphi \cos(2\tau - \varphi)][\cos 2\Theta - 1]$$

$$- \frac{1}{2\alpha} [\varphi + \sin \varphi \cos(2\tau - \varphi)][\cos 2\Theta + 1]. \tag{22}$$

The stationary Fokker-Planck equation corresponding to the $\Theta$-map (21) reads

$$\frac{d}{d\Theta} \left[ V'(\Theta) \frac{d}{d\Theta} \rho(\Theta) + V(\Theta)V'(\Theta)\rho(\Theta) \right] = 0. \tag{23}$$

From this equation one gets that the phase distribution is uniform, $\rho(\Theta) = 1/\pi$, and the trajectory is, indeed, a fluctuating circle provided that

$$\frac{d}{d\Theta} V(\Theta)V'(\Theta) = 0. \tag{24}$$

With the use of eqs. (22) and (24), we now can obtain the desired expressions for the angle $\tau$, parameter $\alpha$ and function $V(\Theta)$ (which is actually no more $\Theta$-dependent),

$$\tau = \frac{\varphi}{2}, \quad \alpha^2 = \frac{\varphi + \sin \varphi}{\varphi - \sin \varphi}, \quad V(\Theta) = \sqrt{\varphi^2 - \sin^2 \varphi}. \tag{25}$$

The data presented in fig. 2 confirm the success of our approach: in new variables the trajectory is a fluctuating circle and the phase distribution is uniform.

Fig. 2: (Color online) (a) Phase space trajectory in new variables $(\tilde{Q}, \tilde{P})$; (b) distribution $\rho(\Theta)$ generated by the transformed map with eqs. (20) and (25), for $\gamma = 0$, $\varphi = 2\pi/5$, $\sigma^2 = 0.02$ and $N = 10^5$.

The Lyapunov exponent $\lambda$ can be now obtained via eq. (10) with the change $\theta_n \rightarrow \Theta_n$. Taking into account that $\lambda$ vanishes within quadratic approximation in disorder, we expand the $\Theta$-map of the form (9b) with the coefficients (20) up to the fourth order in perturbation. By substituting the resulting expression into eq. (10) and expanding the logarithm within the same approximation, after the averaging over $\Theta_n$ with uniform distribution, we arrive at the final expression

$$\frac{d}{d\log \bar{\lambda}} \equiv \lambda = \zeta + 2 \sigma^4 \left[ \frac{(2\varphi^2 - \sin^2 \varphi) \cos \varphi - \varphi \sin^2 \varphi}{\varphi^2 - \sin^2 \varphi} \right]. \tag{26}$$

Here $-2 < \zeta < \infty$ stands for the excess kurtosis, $\zeta = (\eta^4(n))/(\eta^2(n))^2 - 3$, the constant specified by the form of distribution of $\eta_{a,b}(n)$. For Gaussian and flat distributions we have $\zeta = 0, -6/5$, respectively.

Equation (26) determines the asymptotics for small $\varphi$,

$$\frac{d}{d\log \bar{\lambda}} \equiv \lambda \approx \frac{24}{35} \zeta + 2 \sigma^4 \zeta \frac{\bar{\lambda}}{1} \sigma^{-1} \equiv \lambda, \tag{27}$$

that results in a quite surprising frequency dependence of the Lyapunov exponent, $\lambda \propto \omega^{2/3}$. Thus, the dependence $\lambda \propto \omega^6$, numerically found for small $\omega$ in refs. [7,8] should be regarded as the intermediate one, apparently emerging due to not sufficiently large lengths $N$ over which the average of $\lambda$ is performed.

Figure 3 shows excellent agreement between the numerical data obtained for the localization length from exact eq. (10) and eqs. (26), (27) for RH-LH arrays, as well as eqs. (17), (18) for RH-RH arrays with $\zeta = -6/5$. For sequence lengths $N = 10^5, 10^7$ and $10^9$ the data are
obtained with ensemble averaging over 100 realizations of disorder, while at $N = 10^{12}$ only one realization is used.

**Concluding remarks.** – Our approach allows one to resolve the problem of non-conventional Anderson localization for bilayered RH-LH arrays with equal widths, $d_a = d_b$, and derive eq. (26) for the localization length. As we show, the peculiarity of this model is entirely due to zero value of the unperturbed Bloch phase, $\gamma = 0$. We derive the expression for the Lyapunov exponent, which appears to be defined by the fourth order of perturbation theory in disorder, and valid for any value of frequency $\omega$. Our results prove that for small $\omega$ the localization length is enormously large, $L_{\text{loc}} \propto \sigma^{-4} \omega^{-8}$.

According to the analysis given above, one can conclude that the non-conventional dependence $L_{\text{loc}} \propto \sigma^{-4} \omega^{-8}$ also emerges when $d_a \neq d_b$, in case of equal unperturbed optic lengths, $n_a d_b = n_b d_a$, provided that the unperturbed impedances are also equal, $Z_a = Z_b$. Another generalization is due to eq. (12), according to which the same dependence for $L_{\text{loc}}$ is expected to occur when the difference $\Delta = n_a d_b - |n_b d_a|$ is sufficiently small, $\Delta \ll \sigma^2$.

The obtained results show that the non-conventional Anderson localization observed numerically in refs. [8–10] is very fragile with respect to the choice of model parameters, and can be hardly observed experimentally. Our additional study [17] demonstrates that with an additional weak disorder in the widths $d_a$ and/or $d_b$ the standard dependence $L_{\text{loc}} \propto \sigma^2 \omega^2$ quickly recovers. This means that the discussed effect of anomalous localization has to be considered as a kind of correlated disorder, one part of which is embedded into the phases $\varphi_{a,b}(n)$, and the other is absorbed by the impedances $Z_{a,b}(n)$.

Finally, we would like to note that the effect of vanishing of the unperturbed Bloch phase shift $\gamma$ in eq. (12), is a typical effect occurring at band edges in discrete tight-binding models. Therefore, our approach may be considered as a powerful tool to analyze the localization length near band edges in various disordered models.

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