Construction of the equation of fractals structure based on the rvachev r-functions theories

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Abstract. The article is devoted to the construction of the equation of fractal complex structures based on the theories of Rvachev R-functions (RFM), the application of the recursion procedure. Using the equation of objects elementary geometries and constructive means of the method of R-functions R0: R-conjunctions, R-disjunctions and R-reflections, various fractal equations are constructed. Based on these equations, specifying the number of iteration and the angle of inclination various pre-fractals are generated.

1. Introduction
Fractal is a word discussed by a number of people currently, from physicists to high school students. It appears on the covers of many math textbooks, scientific journals, and computer software boxes. Color fractal images can be found everywhere today: from cards to T-shirts. Over the past two decades, the number of units produced per month associated with fractals has increased from a few dozen to many thousands. So, what are the colored shapes that we see everywhere? In simple terms, a fractal is a geometric shape, a certain part of which repeats over and over again, changing in size. Hence the principle of self-similarity. All fractals are similar to themselves, that is, they are similar at all levels. However, fractals are not just complex shapes generated by computers. Anything that seems random and wrong can be a fractal. Theoretically, it can be said that all that exists in the real world is a fractal, be it a cloud or a small oxygen molecule.

At the present time, fractals have found wide application in various fields of science and technology: computer graphics, physics, and other natural sciences, radio engineering, telecommunications, television, medicine, and etc.

The first and obvious application of fractal algorithms was the so-called fractal image compression. Fractal image compression is a lossy image compression algorithm based on applying systems of iterated functions to images. This algorithm is known for the fact that in some cases it allows to obtain very high compression ratios (the best examples are up to 1000
times with acceptable visual quality) for real photos of natural objects, which is inaccessible for other image compression algorithms in principle.

The basis of the fractal coding method is the detection of self-similar areas in the image. An opportunity of applying the theory of iterated function system (IFS) to the problem of image compression was investigated by Michael Barnsley and Alan Sloan for the first time in history.

The most well-known are two images obtained using the IFS: the Sierpinski triangle and the Barnsley fern. The first is given by three, and the second by five affine transformations. Each conversion is literally set by bytes, while an image built with their help can take several megabytes.

It becomes clear how the compression utility works, and why it takes so much time. In fact, fractal compression is the search for self-similar regions in the image and the determination of the parameters of affine transformations for them.

In the worst case, if an optimizing algorithm is not used, it will be necessary to search and compare all possible image fragments of different sizes. Even for small images, considering discreteness, we get an astronomical number of enumerated variants. Even a sharp narrowing of the classes of transformations, for example, due to scaling only a certain number of times, will not allow to achieve an acceptable time. In addition, the image quality is lost. The vast majority of research in the field of fractal compression is now aimed at reducing the archiving time needed to obtain a high-quality image.

In computer graphics, fractals are used to build realistic images of natural objects, such as sea surfaces, trees, bushes, mountain landscapes, etc. Therefore, fractal images can be applied in various fields, ranging from the creation of ordinary textures and background images to fantastic landscapes for computer games or book illustrations. And similar fractal masterpieces are created (as well as vector ones) by mathematical calculations, but unlike vector graphics, the basic element of fractal graphics is the mathematical formula itself - this means that no objects are stored in the computers memory and the image (no matter how intricate) is based solely on the basis of equations.

In radio engineering, the use of fractal geometry in the design of antenna devices was first used by the American engineer Nathan Cohen, who then lived in the center of Boston, where the installation of external antennas on buildings was prohibited. Nathan cut a figure in the shape of a Koch curve from aluminum foil and pasted it on a piece of paper, then attached it to the receiver. It turned out that such an antenna works as well as usual. And, although the physical principles of operation of such an antenna have not been studied so far, this did not prevent Cohen from starting his own company and establishing their serial production.

Fractal antennas are a relatively new class of electrically small antennas (ESA), fundamentally different in their geometry from known solutions. In fact, the traditional evolution of antennas was based on Euclidean geometry, operating with objects of integer dimension (line, circle, ellipse, paraboloid, etc.). Fractal antennas with a surprisingly compact design provide superior broadband performance in a small form factor. Compact enough to be installed or embedded in various locations, fractal antennas are used for marine, air vehicles, or personal devices.

In telecommunications in the field of network technologies, a lot of research has been conducted showing the self-similarity of traffic transmitted over various networks. Especially it concerns speech, audio and video services. Therefore, research and development of the possibility of fractal compression of the traffic transmitted over networks is being conducted with a view to more efficient transmission of information.

Decentralized networks. The Netsukuku IP Addressing System uses the principle of fractal data compression to compactly store information about network nodes. Each node of the Netsukuku network stores only 4 KB of information on the status of neighboring nodes, while any new node connects to a common network without the need for central regulation of distribution of IP addresses, which, for example, is typical of the Internet. Thus, the principle of fractal
information compression ensures a fully decentralized and, therefore, the most stable operation
of the entire network.

Many objects in nature (for example, the human body) consist of a set of fractals mixed with
each other, each fractal having its own dimension different from the others. For example, the
two-dimensional surface of the human vascular system bends, branches, twists and shrinks so
that its fractal dimension is 3.0. But if it were divided into separate parts, the fractal dimension
of the arteries would be only 2.7; whereas the bronchial passages in the lungs would have a
fractal dimension of 1.07 [1].

An analysis of the world's literary sources and Internet resources on fractals shows that at
the present time fractal geometries have been studied quite deeply.

In Uzbekistan, the theory of fractals and its applications is dealt with by D.Sc., academician
B.A. Bondarenko [5-6]. He studied classical and new arithmetic, geometric and combinatorial
properties of arithmetic triangles and pyramids, generalizing the Pascal triangle. The problems
of divisibility and distribution of elements of generalized triangles and Pascal pyramids modulo
p are studied. On the basis of this, flat and spatial classes of fractals and generalized arithmetic
graphs, which are discrete mathematical models of certain structures and processes of engineering
and science, are constructed and investigated. He developed combinatorial algorithms for
applying arithmetic triangles to construct non-orthogonal polynomials and use them to solve
problems of mathematical physics. They also considered examples of applications of arithmetic
triangles, their fractals and graphs to specific problems.

Writing equations of objects of fractal geometry was occupied by professor Sh.A. Nazirov and
his students. The results of these studies are presented in [7-17].

In the proposed article, on the basis of new constructive tools of the Rvachev theory of R-
functions (RFM) [2], equations of a number of objects of fractal geometry will be constructed
with the subsequent modernization of the color design of products.

But in order to stamp images of fractals on materials, it is necessary to write their equations,
i.e. construct the geometry of the domain of fractals, which can be implemented using the
Rvachev R-functions method (RFM) [2-4].

The construction of equations for the boundaries of the geometry of a region (GR) requires the
specification of both support functions and a logical formula, which allows, with an appropriate
choice of a system of R-functions, to obtain an equation in an analytical form. This requires
certain mathematical knowledge and skills, which makes the system difficult to access for
engineers and researchers who are not familiar with the R-function method, analytical and
differential geometry. The formation of the equations of composite geometric objects and their
support functions from standard (proposed to the user) primitives seems promising in this
direction.

The method of constructing equations of HE, including normalized ones, is a good
technological basis for automating the very process of drawing up these equations. In fact, it is
only necessary to automate the process of constructing predicate equations, since the transition
from these equations to ordinary elementary equations can be accomplished by formally replacing
the symbols of the functions of logic with the corresponding symbols of the R-functions, and the
symbols of the domains with the left-hand parts of the corresponding inequalities.

So, the input information for the algorithm should include: 1. Types of standard primitives
used: circle, ellipse, rectangle, triangle, convex polygon, etc. (depending on user requests,
the menu can be replenished), or their appearance (negation). 2. Geometric parameters that
determine the position and size of the standard primitive.

Using this information, the support functions are automatically generated, the normalized
equations of the primitives are called, and the predicate and analytic functions of the composite
GO are formed on the basis of inside – appearance.

This is quite a relevant work, since cotton is grown in large volumes in Uzbekistan and
cotton fibers are obtained from them and on the basis of which materials are created. We are interested in the material to be beautiful, the patterns stamped in them must be quite beautiful. In addition, the color of the patterns plays a special role in determining the cost of the material. The price of one color is quite expensive than another color. These circumstances can also be taken into account in the developed algorithmic - software development tools.

2. Construction of the equation and computational experiments.
Let a complex domain $\Omega \subset \mathbb{R}^2$ with a boundary $\Gamma$ be given as a combination of simple domains $\{\Omega_k\}_{k=1}^m$ with the help of set-theoretic operations of intersection, union and complement. If the implicit equations of the boundaries of these regions are known, $\{\omega_k(x, y) = 0\}_{k=1}^m$ such as $\omega_k > 0$ with $(x, y) \in \Omega_i$ and $\omega_k < 0$ with $(x, y) \notin \Omega_k = \Omega_k \cup \Gamma_k$ then using RFM it is possible to build the boundary equation $\gamma \omega(x, y) = 0$, and the function $\omega$ is positive inside, negative outside of it and is zero on $\Gamma$. The most common system of R-functions is the system 0, whose algebra logical operations are

$$f_1 \land f_2 \equiv f_1 + f_2 - \sqrt{f_1^2 + f_2^2}, \quad f_1 \lor f_2 \equiv -(f_1 \land f_2), \quad \overline{f} \equiv -f.$$ 

Consider constructing the equations of the Koch curve based on RFM (Fig. 1). Perform construction on the interval $-3a \leq x \leq 3a$. Then

$$\omega_0 = -y \geq 0; \quad \omega_{00} = \omega_0 \land (f_1 \land 0) f_2 \geq 0;$$

$$f_1 = \frac{1}{2} \left( x \sqrt{3} - y + a \sqrt{3} \right) \geq 0; \quad f_2 = \frac{1}{2} \left( -x \sqrt{3} - y + a \sqrt{3} \right) \geq 0;$$

In this case, $\omega_{00}$ there is an equation of an isosceles triangle. If we write together an $\omega_{00}$ equation of the form

$$\omega_1 = \omega_0 \lor (f_1 \land 0) f_2 \geq 0;$$

Then we get the equation corresponding to Fig. 1.b. This is the essence of the generating rule: newf: $= F-F++F-F$

So:

$$\omega_1 = \omega_0 \lor (f_1 \land 0) f_2 \geq 0;$$

$$\omega_{21} = \omega_1 \left( 3 \left( \frac{x + 2a}{2} + \left( y - \frac{a \sqrt{3}}{2} \right) \frac{\sqrt{3}}{2} \right) \right) \geq 0;$$

$$\omega_{22} = \omega_1 \left( 3 \left( \frac{x + a/2}{2} + \left( y - \frac{a \sqrt{3}}{2} \right) \frac{\sqrt{3}}{2} \right) \right) \geq 0;$$

$$\omega_2 = (\omega_{21}(x, y) \lor \omega_{22}(x, y)) \land (-x, y) \lor \omega_{22}(-x, y) \geq 0;$$

Figure 1. Curve Kochw.
\[ \omega_{kl} = \omega_{k-1}(3(x + 2a), 3y) \geq 0; \]

\[ \omega_{k2} = \omega_{k-1}\left(3\left(\frac{x + \alpha/2}{2} + \left(y - \frac{a\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2}\right), 3\left(-\left(x + a/2\right)\frac{\sqrt{3}}{2} + \left(y - \frac{a\sqrt{3}}{2}\right)\frac{1}{2}\right)\right) \geq 0; \]

\[ \omega_k = (\omega_{k1}(x, y) \lor \omega_{k2}(x, y)) \land 0\left(\omega_{k1}(-x, y) \lor \omega_{k2}(-x, y)\right) \geq 0 (k = 3, 4, \ldots). \]

In fig. The patterns of the level lines of the function

\[ \omega_k(x, y) \geq 0; \]

defining the Koch curve for various values of k are shown. Sierpinski carpet. The Sierpinski carpet is constructed as follows. The initial square is divided into 9 equal squares, the central of which is excluded (Fig. 2). Then each of the remaining squares undergo a similar procedure, etc. Letting this process go to infinity, we end up with a fractal object - the Sierpinski carpet. Its fractional dimension is \( D = \log 8 / \log 3 \approx 1.893 \). If we interrupt the infinite process at the \( k^{th} \) step, then we obtain a prefractal \( k^{th} \) level. According to [3-4], the boundary function of a pre-fractal Sierpinski has the form

\[ f_1 = a^2 - x^2 \geq 0, \quad f_2 = b^2 - y^2 \geq 0, \]

then prefractal zero. We pass auxiliary functions using the property of self-similarity. \( \omega_1(x, y) = \omega_0(3x, 3y) \geq 0, \quad \omega_k(x, y) = \omega_{k-1}(3\mu_{hx}, 3\mu_{hy}) \geq 0 \ldots (k = 2, 3, \ldots) \) where

\[ \mu_{hx} = \frac{h_x}{\pi} \arcsin \left(\frac{sin \frac{\pi y}{h_y}}{h_y}\right), \quad \mu_{hy} = \frac{h_y}{\pi} \arcsin \left(\frac{sin \frac{\pi y}{h_y}}{h_y}\right), \quad h_x = \frac{2a}{3}, \quad h_y = \frac{2b}{3}. \]

Then \( K\omega_k(x, y) = \omega_0(x, y) \lor \omega_1(x, y) \lor \omega_2(x, y) \lor \ldots \lor \omega_k(x, y) \geq 0 \) Figure 2 shows the pictures of the level lines of the functions \( K\omega_k(x, y) \geq 0 \) defining the Sierpinski carpet for various values of k. Napkin

![Figure 2. Building a square carpet Sierpinski.](image)
it, a triangle is cut out from the center of an equilateral triangle. Repeat the same procedure for the three triangles formed (with the exception of the central one), and so on to infinity. If we now take any of the resulting triangles and increase it, we get an exact copy of the whole. In this case, there is complete self-similarity. In this fractal, the initiator and generator, as in the previous case, are the same. At each iteration, a reduced copy of the initiator is added to each corner of the generator, and so on. If you create an infinite number of iterations while creating this fractal, it would occupy the entire plane. Therefore, its fractal dimension is \( \frac{\ln 9}{\ln 3} = 2 \).

Write the equation of a regular triangle in the form

\[ \omega_0(x, y) = \left( \frac{y}{\sqrt{3}} - \frac{a}{2} \right)^2 - 2 \geq 0 \land_0 \left( \frac{\sqrt{3}}{4} - (y - \frac{a\sqrt{3}}{4}) \right)^2 \geq 0. \]

When \( k = 1 \), the Sierpinski Napkin equation takes the following form (Fig. 1 b):

\[ \omega_0(2x + \frac{a}{2}, 2y) \lor_0 \omega_0(2x - \frac{a}{2}, 2y) \lor_0 \omega_0(2x, 2y - \frac{a\sqrt{3}}{2}). \]

Similarly, we can write formulas of the R-function for the Sepinsky Napkin for the remaining \( k \) (Fig. 3)

\[ \omega_k(x, y) = \omega_{k-1}(2x + \frac{a}{2}, 2y) \lor_0 \omega_{k-1}(2x - \frac{a}{2}, 2y) \lor_0 \omega_{k-1}(2x, 2y - \frac{a\sqrt{3}}{2}); k = 1, 2, 3, \ldots. \]

In fig. 3 pictures of the lines of the function level defining the Sierpinski cloth for various values of \( k \) are constructed.

![Picture of Sierpinski napkin at various values of k](image)

**Figure 3.** Sierpinski napkin at various values of \( k \).

Sierpinski curve [3]. First, we construct the base equation of Figure 4.

\[ x_1 = x \cos(\alpha_1) + y \sin(\alpha_1), \quad y_1 = -x \sin(\alpha_1) + y \cos(\alpha_1), \]
\[ x_2 = x \cos(\alpha_2) + y \sin(\alpha_2), \quad y_2 = -x \sin(\alpha_2) + y \cos(\alpha_2), \]
\[ f_1(x, y) = (a^2 - x_1^2) \land_0 (a^2 - y_1^2) \geq 0, \]
\[ f_2(x, y) = (a^2 - x_2^2) \land_0 (a^2 - y_2^2) \geq 0, \]
\[ f_3(x, y) = f_2(-x, y); \]
\[ \omega_1(x, y) = f_1(x, y) \lor_0 f_2(x, y) \lor_0 f_3(x, y) \]

Here we use the formulas for turning the axes, which are necessary in fig. below. Next, we construct an iterative process and as a result we get,

\[ \omega_n(x, y) = \omega_{n-1}(x, y) \lor_0 \omega_{n-1}(x - 2a, y - 2a) \lor_0 \omega_{n-1}(x + 2a, y - 2a) \lor_0 \]
\[ \lor_0 \omega_{n-1}(x + 2a, y + 2a) \lor_0 \omega_{n-1}(x - 2a, y + 2a), n = 2, 3, 4, 5, ... \]

In the calculation \( a_1 = \frac{3}{8}a, b_1 = \frac{7}{4}a \) are used. The calculation results for \( \alpha_1 = 0 \) and \( \alpha_2 = \frac{\pi}{4} \) and with different values in Fig. four.

![Figure 4. Sierpinski curve for different values of k and n](image)

3. Conclusion.
Using the equation of objects, elementary geometries and constructive means of the method of R-functions R0: R-conjunctions, R-disjunctions and R-reflections can construct various tree-like fractals. Based on these equations, setting the number of iterations and the angle of inclination can generate various pre-fractals, which can be used in creating computer landscapes, in various illustrations, the textile industry, etc.

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