THE GEOMETR Y OF MODIFIED RIEFFANNIAN EXTENSIONS

E. CALVIÑO-LOUZAO, E. GARCÍA-RÍO, P. GILKEY, AND R. VÁZQUEZ-LORENZO

Abstract. We show that every paracomplex space form is locally isometric to a modified Riemannian extension and give necessary and sufficient conditions so that a modified Riemannian extension is Einstein. We exhibit Riemannian extension Osserman manifolds of signature (3, 3) whose Jacobi operators have non-trivial Jordan normal form and which are not nilpotent. We present new four dimensional results in Osserman geometry.

1. Introduction

Walker metrics are indecomposable pseudo-Riemannian metrics which are not irreducible (i.e., they admit a null parallel distribution). They play a distinguished role in geometry and physics [2, 4, 16, 18, 27, 30, 31]. Lorentzian Walker metrics have been studied extensively in the physics literature since they constitute the background metric of the pp-wave models (see for example [19] and references therein). From a purely geometric point of view, Lorentzian Walker metrics naturally appear in the investigation of the non-uniqueness of the metric for the Levi Civita connection [32], a question which is meaningless in the positive definite case. Moreover, Walker metrics are the underlying structure of many geometrical structures like para-Kaehler and hypersymplectic structures (see [3, 10, 11, 26, 29]).

The simplest examples of non Lorentzian Walker metrics are provided by the so-called Riemannian extensions. This construction, which relates affine and pseudo-Riemannian geometries, associates a neutral signature metric on $T^* M$ to any torsion free connection $\nabla$ on the base manifold $M$. Riemannian extensions have been used both to understand questions in affine geometry and to solve curvature problems (see for example [1, 8, 21]). It is a remarkable fact that Walker metrics satisfying some natural curvature conditions are locally Riemannian extensions, thus leading the corresponding classification problem to a task in affine geometry as shown in [9, 11, 16].

In this paper we introduce a modification of the usual Riemannian extensions with special attention to the behaviour of their curvature. The geometry of modified Riemannian extensions is much less rigid than that of the Riemannian extensions, allowing the existence of many non Ricci flat Einstein metrics, which can be further specialized to be Osserman (i.e., the eigenvalues of the Jacobi operators are constant on the unit pseudo-sphere bundles) since their scalar curvature invariants do not vanish. In particular, we show that any paracomplex space form is locally a modified Riemannian extension where the corresponding torsion free connection is necessarily flat (cf. Theorem [2.2]). This description seems to be well suited to further investigations, specially for the consideration of the Lagrangian submanifolds of paracomplex space forms.

Modified Riemannian extensions turn out to be very useful in describing four dimensional Walker geometry. Indeed, we show in Theorem [7.1] that any self-dual Walker metric is a modified Riemannian extension. As a consequence, a description
of all four dimensional Osserman metrics whose Jacobi operator has a non-zero double root of its minimal polynomial is given in Theorem 7.3. This result, coupled with recent work of Derdzinski [13], completes the classification of four dimensional Osserman metrics. Finally, as an application of the four dimensional results, one obtains a procedure to construct new Osserman metrics in higher dimensions, a problem that has been a task in the field (cf. Theorem 2.3).

2. Summary of results

2.1. Affine geometry. Let $R^\nabla$ be the curvature operator of a torsion free connection $\nabla$ on the tangent bundle of a smooth manifold $M$ of dimension $n$; if $X$ and $Y$ are smooth vector fields on $M$, then

$$R^\nabla(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{\left[ X, Y \right]}.$$

The Ricci tensor $\rho^\nabla$ is defined by contracting indices:

$$\rho^\nabla(X, Y) := \text{Tr} \{ Z \to R^\nabla(Z, X)Y \}.$$

In contrast to the Riemannian setting, $\rho^\nabla$ need not be a symmetric 2-tensor field. We refer to [7] for further details concerning curvature decompositions. We denote the symmetric and anti-symmetric Ricci tensors by:

$$\rho^\nabla,s(X, Y) := \frac{1}{2} \{ \rho^\nabla(X, Y) + \rho^\nabla(Y, X) \},$$

$$\rho^\nabla,a(X, Y) := \frac{1}{2} \{ \rho^\nabla(X, Y) - \rho^\nabla(Y, X) \}.$$

2.2. The modified Riemannian extension. Let $\Phi \in C^\infty(S^2(T^*M))$ be a symmetric 2-tensor field and let $T, S \in C^\infty(\text{End}(TM))$ be $(1, 1)$-tensor fields on $M$. In Section 5 we will use these data to define (see Equation (5.13)) a neutral signature pseudo-Riemannian metric $g^\nabla,\Phi,T,S$ on the cotangent bundle $T^*M$ which is called the modified Riemannian extension which is a Walker metric. The case $T = \text{c} \text{id}$ and $S = \text{id}$ is of particular importance in our treatment and will be denoted by $g^\nabla,c$ if $\Phi = 0$ and by $g^\nabla,\Phi,c$ if $\Phi \neq 0$.

2.3. Einstein geometry. We have the following result

**Theorem 2.1.** The modified Riemannian extension $g^\nabla,\Phi,c$ on the cotangent bundle of an $n$ dimensional affine manifold is Einstein if and only if $\Phi = \frac{4}{c(n-1)} \rho^\nabla,s$.

2.4. Para-Kaehler geometry. Our fundamental result concerning such geometry is the following which illustrates the importance of the modified Riemannian extension:

**Theorem 2.2.** A para-Kaehler metric of non-zero constant para holomorphic sectional curvature $c$ is locally isometric to the cotangent bundle of an affine manifold which is equipped with the modified Riemannian extension $g^\nabla,c$ where $\nabla$ is a flat connection.

2.5. Osserman geometry. We can use the modified Riemannian extension to exhibit the following examples which extend previous results in signature $(2, 2)$ (see Theorem 2.1 of [17]) – it has long been a task in this field to build examples of Osserman manifolds which were not nilpotent and which exhibited non-trivial Jordan normal form.

**Theorem 2.3.** Let $M = \mathbb{R}^3$ and let $\nabla$ be the torsion free connection whose only non-zero Christoffel symbol is $\nabla_{\partial_1} \partial_1 = x_2 \partial_3$. Let $g := g^\nabla,1$ on $T^*M$. Then $g$ is an Osserman metric of signature $(3, 3)$ with eigenvalues $\{0, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \}$ which is neither spacelike Jordan Osserman nor timelike Jordan Osserman at any point. The Jacobi operator is neither diagonalizable nor nilpotent for a generic tangent vector.
2.6. Notational conventions. We shall let \( M := (M, \nabla) \) denote an affine manifold and \( R \) the associated curvature operator. Similarly, we shall let \( N := (N, g) \) denote a pseudo-Riemannian manifold, \( \nabla^g \) denote the associated Levi-Civita connection, and \( R^g \) the associated curvature operator. If \( \xi_i \in T^* N \), then the symmetric product is defined by
\[
\xi_1 \circ \xi_2 := \frac{1}{2} (\xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_1) \in S^2 (T^* N).
\]

2.7. Outline of the paper. In Section 3, we establish notation and recall some basic definitions in the geometry of the curvature operator. We shall also discuss Osserman, Szabó, and Ivanov–Petrova geometry. In Section 4, we present a brief introduction to Walker geometry and in Section 5, we define the modified Riemannian extension and establish Theorem 2.1. In Section 6, we discuss para-Kaehler geometry and prove Theorem 2.2. In Section 7, we present some results in four dimensional geometry with a description of four dimensional Osserman metrics whose Jacobi operators have a non-zero double root of its minimal polynomial, as a generalization of paracomplex space forms (cf. Theorem 7.3). We conclude in Section 8 by proving Theorem 2.3 and discussing some additional results for this six dimensional example that relate to Szabó and Ivanov–Petrova geometry. Throughout, we shall adopt the Einstein convention and sum over repeated indices. We shall suppress many of the technical details in the interests of brevity in giving various proofs in this paper – further details are available from the authors upon request.

3. The geometry of the curvature operator

3.1. Osserman geometry. Let \( J^g(X) : Y \to R^g(Y, X)X \) be the Jacobi operator on a pseudo-Riemannian manifold \( N \) of signature \((p, q)\). One says that \( N \) is timelike Osserman (resp. spacelike Osserman) if the eigenvalues of \( J^g \) are constant on the pseudo-sphere of unit timelike (resp. spacelike) vectors; these are equivalent concepts if \( p > 0 \) and \( q > 0 \) [5, 20, 21]. Similarly, we say that \( N \) is timelike Jordan Osserman or spacelike Jordan Osserman if the Jordan normal form of \( J^g \) is constant on the appropriate pseudo-sphere; these are in general not equivalent concepts.

3.2. Szabó geometry. A pseudo-Riemannian manifold \( N \) is said to be Szabó if the Szabó operator \( S_X : Y \to \nabla_X R^g(Y, X)X \) has constant eigenvalues on \( S^\pm (TN) \) [23]. Any Szabó manifold is locally symmetric in the Riemannian [34] and the Lorentzian [26] setting but the higher signature case supports examples with nilpotent Szabó operators (cf. [24] and the references therein).

3.3. Ivanov–Petrova geometry. For any oriented non-degenerate 2-plane \( \pi \), the skew-symmetric curvature operator \( R^g(\pi) \) of the Levi-Civita connection is defined by
\[
R^g(\pi) = \left| g(X, X)g(Y, Y) - g(X, Y)^2 \right|^{-1/2} R^g(X, Y); \]
\( R^g(\pi) \) is a skew-adjoint operator which is independent of the oriented basis \( \{X, Y\} \) of \( \pi \). \( N \) is said to be spacelike (respectively, timelike or mixed) Ivanov–Petrova if the eigenvalues of \( R^g(\pi) \) are constant on the appropriate Grassmannian (see [28, 37]). If \( p \geq 2 \) and \( q \geq 2 \), these are equivalent conditions [24] so one simply says the metric is Ivanov–Petrova in this setting.

4. Walker geometry

A Walker manifold is a triple \((N, g, D)\), where \( N \) is an \( n \) dimensional manifold, where \( g \) is a pseudo-Riemannian metric of signature \((p, q)\) on \( N \), and where \( D \) is an \( r \) dimensional parallel null distribution.
4.1. Geometrical contexts. Walker metrics appear as the underlying structure of several specific pseudo-Riemannian structures. For instance, indecomposable metrics which are not irreducible play a distinguished role in investigating the holonomy of indefinite metrics. Those metrics are naturally equipped with a Walker structure (see for example [4] and the references therein). Einstein hypersurfaces in indefinite real space forms with two-step nilpotent shape operators [31] are Walker. Similarly, locally conformally flat manifolds with nilpotent Ricci operator are Walker manifolds [27]. Also, non-trivial conformally symmetric manifolds (i.e., neither symmetric nor locally conformally flat) may only occur in the pseudo-Riemannian setting and they are Walker manifolds [14].

4.2. Neutral signature Walker manifolds. Of special interest are those manifolds admitting a field of null planes of maximum dimension \( r = \frac{p}{2} \). This is the case of para-Kaehler [29] and hyper-symplectic structures [26]. Note that, in opposition to the non-degenerate case, the complementary distribution to a parallel degenerate plane field is not necessarily parallel (even not integrable). Moreover, parallelizability of the complementary plane field is indeed equivalent to the existence of a para-Kaehler structure [4]. Note that any four dimensional Osserman manifold of neutral signature whose Jacobi operators have a non-zero double root of their minimal polynomial is necessarily Walker with a parallel field of planes of maximal dimensionality [6, 18].

4.3. Walker coordinates. Walker [34] (see also the discussion in [15]) constructed simplified local coordinates in this setting. We shall restrict our attention to neutral signature \((p,p)\). Let \((N^n, g, \mathcal{D})\) be a Walker manifold of signature \((p,p)\) where \(\dim\{\mathcal{D}\} = p\). There are local coordinates \((x_1, \ldots, x_p, x_{p'}, \ldots, x_{p'})\) so that
\[
\begin{align*}
g = 2\, dx^i \circ dx^{i'} + B_{ij} dx^i \circ dx^j.
\end{align*}
\]
Here \(B\) is a symmetric matrix and the parallel degenerate distribution is given by \(\mathcal{D} = \text{Span}\{\partial_1, \ldots, \partial_p\}\) where \(\partial_i := \frac{\partial}{\partial x^i}\) and \(\partial_{i'} := \frac{\partial}{\partial x^{i'}}\).

4.4. The Christoffel symbols of \(\nabla^g\). We sum over \(1 \leq s \leq p\):
\[
\begin{align*}
\Gamma^g_{ij} &= -\frac{1}{2} \partial_{i'} g_{ij}, \\
\Gamma^g_{ij'} &= \frac{1}{2} \partial_{i'} g_{jk}, \\
\Gamma^g_{i'j'} &= \frac{1}{2} \left(-\partial_k g_{ij} + \partial_j g_{ik} + \partial_i g_{jk} + g_{ks} \partial_{s'} g_{jk}\right).
\end{align*}
\]

4.5. The Riemann curvature tensor of \(\nabla^g\). We sum over \(1 \leq s \leq p, 1 \leq t \leq p\):
\[
\begin{align*}
R^g_{ijk} &= -\frac{1}{2} \left(\partial_k \partial_i g_{jk} - \partial_j \partial_i g_{jk} - \frac{1}{3} (\partial_{i'} g_{ik} \partial_t g_{jk} - \partial_{i'} g_{jk} \partial_t g_{ik})ight), \\
R^g_{ij'} &= -\frac{1}{2} \left(\partial_j \partial_k g_{ik} - \partial_j \partial_k g_{jk} + \partial_i \partial_k g_{jk} - \partial_i \partial_k g_{ik}\right), \\
R^g_{i'j'} &= -\frac{1}{2} \left(\partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk}\right), \\
R^g_{i'j'} &= -\frac{1}{2} \left(\partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk}\right), \\
R^g_{i'j'} &= -\frac{1}{2} \left(\partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk}\right), \\
R^g_{i'j'} &= -\frac{1}{2} \left(\partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk}\right), \\
R^g_{i'j'} &= -\frac{1}{2} \left(\partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk} - \partial_{i'} g_{jk}\right).
\end{align*}
\]
5. Modified Riemannian Extensions

5.1. The geometry of the cotangent bundle. We refer to [36] for further details concerning the material of this Section. Let $T^*M$ be the cotangent bundle of an $n$ dimensional manifold $M$ and let $\pi : T^*M \to M$ be the projection. Let $\tilde{p} = (p, \omega)$ where $p \in M$ and $\omega \in T^*_pM$ denote a point of $T^*M$. Local coordinates $(x_i)$ in a neighborhood $U$ of $M$ induce coordinates $(x_i, x^i)$ in $\pi^{-1}(U)$ where we decompose

$$\omega = \sum x^i \frac{\partial}{\partial x^i}.$$ 

For each vector field $X$ on $M$, define a function $\iota X : T^*M \to \mathbb{R}$ by

$$\iota X(p, \omega) = \omega(X_p).$$

We may expand $X = X^j \partial_j$ and express:

$$\iota X(x_i, x^i) = \sum x^j X_i^j.$$

Lemma 5.1. Let $\tilde{Y}, \tilde{Z} \in C^\infty(T(T^*M))$ be smooth vector fields on $T^*M$. Then $\tilde{Y} = \tilde{Z}$ if and only if $\tilde{Y}(\iota X) = \tilde{Z}(\iota X)$ for all smooth vector fields $X \in C^\infty(TM)$.

Let $X \in C^\infty(TM)$ be a vector field on $M$. The complete lift $X^C$ is, by Lemma 5.1 characterized by the identity

$$X^C(\iota Z) = [X, Z] \quad \text{for all } Z \in C^\infty(TM).$$

We then have $T_{(p, \omega)}(T^*M) = \{X^C_{p, \omega} : X \in C^\infty(TM)\}$ and consequently

Lemma 5.2. A $(0, s)$-tensor field on $T^*M$ is characterized by its evaluation on complete lifts of vector fields on $M$.

Let $T$ be a tensor field of type $(1, 1)$ on $M$, i.e. $T \in C^\infty(\text{End}(TM))$. We define a 1-form $\iota(T) \in C^\infty(T(T^*M))$ which is characterized by the identity

$$\iota(T)(X^C) = \iota(TX).$$

5.2. The Riemannian extension. Let $\nabla$ be a torsion free affine connection on $M$. The Riemannian extension $g_\nabla$ is the pseudo-Riemannian metric $g_\nabla$ on $N := T^*M$ of neutral signature $(n, n)$ characterized by the identity:

$$g_\nabla(X^C, Y^C) = -\iota(\nabla_X Y + \nabla_Y X).$$

Let $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$ give the Christoffel symbols of the connection $\nabla$. Then:

$$g_\nabla = 2 \, dx^i \circ dx^i - 2 x^k \Gamma^k_{ij} dx^j \circ dx^j.$$

Riemannian extensions were originally defined by Patterson and Walker [33] and further investigated in [1] thus relating pseudo-Riemannian properties of $T^*M$ with the affine structure of the base manifold $(M, \nabla)$. Moreover, Riemannian extensions were also considered in [21] in relation to Osserman manifolds (see also [14]).

5.3. The modified Riemannian extension. Let $\Phi \in C^\infty(S^2(T^*M))$ be a symmetric $(0, 2)$-tensor field on $M$ and let $T, S \in C^\infty(\text{End}(TM))$ be tensor fields of type $(1, 1)$ on $M$. The modified Riemannian extension is the neutral signature metric on $T^*M$ defined by

$$g_\nabla, \Phi : T^*M := \iota T \circ \iota S + g_\nabla + \pi^*\Phi.$$

In a system of local coordinates one has

$$(5.b) \ g_\nabla, \Phi : T^*M = 2 \, dx^i \circ dx^i + \{\frac{1}{2} x^r x^s (T^r_i S^s_j + T^r_j S^s_i) + \Phi_{ij}(x) - 2 x^k \Gamma^k_{ij}\} dx^i \circ dx^j.$$

The case when $T = c \text{id}$ and $S = \text{id}$ is important and plays a central role in our treatment. More precisely, if

$$g_\nabla, \Phi, c := c \cdot \iota \text{id} \circ \iota \text{id} + g_\nabla + \pi^*\Phi,$$
then one has in a system of local coordinates that
\[(5.c) \quad g_{\nabla,\Phi,c} = 2 \, dx^i \circ dx^j + \{ c \, x_i \, x_{j'} + \Phi_{ij}(x) - 2 \, x_k \, \Gamma_{ij}^k \} \, dx^i \circ dx^j. \]
These metrics are Walker metrics on $T^*M$ where the tensor $B_{ij}(x, x')$ of Equation \[(4.a) \quad \text{is a quadratic function of } x' \quad \text{(and affine if } c = 0). \]

The parallel degenerate distribution $D = \ker(\pi_\perp)$ and the scalar curvature is a suitable multiple (depending on the dimension) of the parameter $c$.

The modified Riemannian extensions $g_{\nabla,\Phi,0}$ have been used in [8] to construct Kaehler and para-Kaehler Osserman metrics with one-side bounded (para) holomorphic sectional curvature.

5.4. The proof of Theorem 2.1 Let $g = g_{\nabla,\Phi,c} = c \cdot \text{id} \circ \text{id} + g_{\nabla} + \pi^* \Phi$ and let $\tau^g$ be the scalar curvature. The trace free Ricci tensor $\rho_0^g = \rho^g - \frac{\pi^*}{\pi^*}g$ can then be determined to be
\[\rho_0^g = 2 \pi^* \rho_{\nabla^g} - \frac{1}{2}c(n - 1) \pi^* \Phi. \]

Theorem 2.1 now follows. \hfill \Box

6. PARA-KAHLER MANIFOLDS

A para-Kaehler manifold is a symplectic manifold $N$ admitting two transversal Lagrangian foliations (see [11][29]). Such a structure induces a decomposition of the tangent bundle $TN$ into the Whitney sum of Lagrangian subbundles $L$ and $L'$, that is, $TN = L \oplus L'$. By generalizing this definition, an almost para-Hermitian manifold is defined to be an almost symplectic manifold $(N, \Omega)$ whose tangent bundle splits into the Whitney sum of Lagrangian subbundles. This definition implies that the $(1,1)$-tensor field $J$ defined by $J = \pi_L - \pi_{L'}$ is an almost paracomplex structure, that is $J^2 = \text{id}$ on $N$, such that $\Omega(JX, JY) = -\Omega(X, Y)$ for all vector fields $X, Y$ on $N$, where $\pi_L$ and $\pi_{L'}$ are the projections of $TN$ onto $L$ and $L'$, respectively. The 2-form $\Omega$ induces a non-degenerate $(0,2)$-tensor field $g$ on $N$ defined by $g(X, Y) = \Omega(X, JY)$, where $X, Y$ are vector fields on $N$. Now the relation between the almost paracomplex and the almost symplectic structures on $N$ shows that $g$ defines a pseudo-Riemannian metric of signature $(n, n)$ on $N$ and moreover, one has that $g(JX, JY) + g(X, JY) = 0$, where $X, Y$ are vector fields on $N$. We refer to [11] for further details on paracomplex geometry.

The specific importance of the para-Kaehler condition is equivalently stated in terms of the parallelizability of the paracomplex structure with respect to the Levi-Civita connection of $g$, that is $\nabla^g J = 0$. The $\pm$ eigenspaces $\mathcal{D}_\pm$ of the paracomplex structure $J$ are null distributions. Moreover, since $J$ is parallel in the para-Kaehler setting, the distributions $\mathcal{D}_\pm$ are parallel. This shows that any para-Kaehler structure $(g, J)$ necessarily has an underlying Walker metric.

6.1. Proof of Theorem 2.2 Choose an affine manifold $(M, \nabla)$, with $\nabla$ a flat connection on $M$. We normalize the choice of coordinates so the associated Christoffel symbols vanish. Consider the modified Riemannian extension
\[g = g_{\nabla,\Phi,c} = 2 \, dx^i \circ dx^j + c \, x_i \, x_{j'} \circ dx^i \circ dx^j. \]
Let $\Omega = dx^i \wedge dx^j$ be the symplectic form. Now, the associated almost para-Hermitian structure $J$, defined by $\Omega(X, Y) = g(JX, Y)$, is given by (sum over $j$)
\[J\partial_i = \partial_i - c \, x_i \, x_{j'} \partial_j', \quad J\partial_{j'} = -\partial_{j'}, \]
for $i = 1, \ldots, n$, and a direct calculation shows that the Nijenhuis tensor
\[N_{j'}(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0. \]
As $J$ is integrable, and as $\Omega$ is closed, $(T^*M, g, J)$ is a para-Kaehler manifold.

We finish the proof showing that $(T^*M, g, J)$ has constant para holomorphic sectional curvature $c$. First, a straightforward calculation from the expressions
for the curvature given in Section 4.5 shows that the non-null components of the curvature tensor of the Levi-Civita connection are determined by (we do not sum over repeated indices)

\[ R^g_{ij\alpha} = \frac{\tau^g}{12} x_j x^i x^\alpha, \quad R^g_{ij} = -c, \quad R^g_{ijk} = \frac{\tau^g}{12}, \quad R^g_{ij\alpha} = \frac{\tau^g}{12}, \]

adding the symmetry condition \( R^g_{abc} = -R^g_{bac} \), and where \( i, j, k \) are different indexes in \( \{1, \ldots, n\} \). Now, for any vector field \( X = (\alpha_i \partial_i + \alpha_I \partial_I) \), a long but straightforward calculation from the above relations shows that (sum over \( i \) and \( j \))

\[
R^g \left( JX, X \right) X = -\varepsilon_X \left( \alpha_i R^g_{iij} \partial_i + 2\alpha_I R^g_{ijI} \partial_I - \alpha_i R^g_{ijI} \partial_I - \alpha_I R^g_{iij} \partial_I \right) = c \varepsilon_X \left( \alpha_i \partial_i - \alpha_I c x_i x^i \partial_I \right).
\]

Thus, \( R^g \left( JX, X \right) X = cg(X, X) JX \), showing that \( (T^* M, g, J) \) has constant para holomorphic sectional curvature \( c \) and finishing the proof.

7. Four Dimensional Geometry

7.1. Self-dual Walker metrics. In the particular case of \( n = 4 \), we choose suitable coordinates \( (x_1, x_2, x_1, x_2) \) where the Walker metric takes the form:

\[(7.a) \quad g = 2 dx^1 \circ dx^1' + 2 dx^2 \circ dx^2' + a dx^1 \circ dx^1 + b dx^2 \circ dx^2 + 2 c dx^1 \circ dx^2, \]

for some functions \( a, b \) and \( c \) depending on the coordinates \( (x_1, x_2, x_1, x_2) \).

Considering the Riemann curvature tensor as an endomorphism of \( \Lambda^2(N) \), we have the following \( O(2,2) \)-decomposition

\[
R^g = \frac{\tau^g}{12} \text{id}_{\Lambda^2} + \rho^g_0 + W^g : \Lambda^2 \to \Lambda^2,
\]

where \( W^g \) denotes the Weyl conformal curvature tensor, \( \tau^g \) the scalar curvature, and \( \rho^g_0 \) the traceless Ricci tensor,

\[
\rho^g_0(X, Y) = \rho^g(X, Y) - \frac{\tau^g}{4} g(X, Y).
\]

The Hodge star operator \( \star : \Lambda^2 \to \Lambda^2 \) associated with any metric of signature \( (2, 2) \) induces a further splitting \( \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_- \), where \( \Lambda^2_\pm \) denotes the \( \pm 1 \)-eigenspaces of the Hodge star operator, that is

\[
\Lambda^2_\pm = \{ \alpha \in \Lambda^2(N) : \star \alpha = \pm \alpha \}.
\]

Correspondingly, the curvature tensor decomposes as

\[
R^g = \frac{\tau^g}{12} \text{id}_{\Lambda^2} + \rho^g_0 + W^g_+ + W^g_- \quad \text{for} \quad W^g_\pm = \frac{1}{2} (W^g \pm \star W^g).
\]

Recall that a pseudo-Riemannian 4-manifold is called self-dual (resp., anti-self-dual) if \( W^g = 0 \) (resp., \( W^g_\pm = 0 \)).

Self-dual Walker metrics have been previously investigated in [18] (see also [12]), where a local description in Walker coordinates of such metrics was obtained. As a further application, self-dual Walker metrics can be completely described in terms of the modified Riemannian extension as follows.

Theorem 7.1. A four dimensional Walker metric is self-dual if and only if it is locally isometric to the cotangent bundle \( T^* \Sigma \) of an affine surface \( (\Sigma, \nabla) \), with metric tensor

\[
g = \iota X (\iota \text{id} \circ \iota) + \iota \text{id} \circ \iota T + g_{\nabla} + \pi^* \Phi
\]
where $X$, $T$, $\nabla$ and $\Phi$ are a vector field, a $(1,1)$-tensor field, a torsion free affine connection and a symmetric $(0,2)$-tensor field on $\Sigma$, respectively.

**Proof.** It follows from [18] that the metric of Equation (7.a) is self-dual if and only if the functions $a$, $b$, $c$ have the form

$$a(x_1, x_2, x_1', x_2') = x_1^2 A + x_2^2 B + x_1^2 x_2 C + x_1 x_2 D + x_1 P + x_2 Q + \xi,$$

$$b(x_1, x_2, x_1', x_2') = x_1^2 C + x_2^2 \xi + x_1 x_2 A + x_1 x_2 F + x_1 S + x_2 T + \eta,$$

$$c(x_1, x_2, x_1', x_2') = \frac{1}{2} x_1^2 F + \frac{1}{2} x_2^2 D + x_1 x_2 A + x_1 x_2 C + \frac{1}{2} x_1 x_2 (B + \xi)$$

$$+ x_1 U + x_2 V + \gamma,$$

where all capital, calligraphic and Greek letters stand for arbitrary smooth functions depending only on the coordinates $(x_1, x_2)$.

For a vector field $X = A(x_1, x_2) \partial_1 + C(x_1, x_2) \partial_2$ on $\Sigma$ we have

$$\iota X = x_1 A(x_1, x_2) + x_2 C(x_1, x_2),$$

and hence

$$(\iota X \cdot \iota \text{id} \circ \iota \text{id})_{11} = x_1^2 A(x_1, x_2) + x_2^2 x_1 x_2 C(x_1, x_2),$$

$$(\iota X \cdot \iota \text{id} \circ \iota \text{id})_{12} = x_1^2 x_2 A(x_1, x_2) + x_2^2 C(x_1, x_2),$$

$$(\iota X \cdot \iota \text{id} \circ \iota \text{id})_{22} = x_1 x_2^2 A(x_1, x_2) + x_2^3 C(x_1, x_2).$$

Next, let $T$ be a $(1,1)$-tensor field on $\Sigma$ with components

$$T^1_1 = B(x_1, x_2), \quad T^2_2 = D(x_1, x_2), \quad T^1_2 = F(x_1, x_2), \quad T^2_1 = \mathcal{E}(x_1, x_2).$$

It follows from the definition of $\iota T$ in Equation (4.4) that:

$$(\iota T)_1 = x_1 B(x_1, x_2) + x_2 D(x_1, x_2),$$

$$(\iota T)_2 = x_1 F(x_1, x_2) + x_2 \mathcal{E}(x_1, x_2)$$

and therefore

$$(\iota T \circ \iota \text{id})_{11} = x_1^2 B(x_1, x_2) + x_1 x_2 D(x_1, x_2),$$

$$(\iota T \circ \iota \text{id})_{12} = \frac{1}{2} (x_1^2 F(x_1, x_2) + x_2^2 D(x_1, x_2))$$

$$+ x_1 x_2 (B(x_1, x_2) + \mathcal{E}(x_1, x_2)),$$

$$(\iota T \circ \iota \text{id})_{22} = x_1 x_2 F(x_1, x_2) + x_2^3 \mathcal{E}(x_1, x_2).$$

Now the result follows from Equation (7.1). \hfill \Box

Note that, as a direct consequence of Equation (7.b), any modified Riemannian extension $g_{\nabla, \Phi, 0}$ is necessarily a self-dual Walker metric. Recently, the authors showed that Ivanov–Petrova self-dual Walker 4-metrics correspond to modified Riemannian extensions $g_{\nabla, \Phi, 0}$ [9], and the same was shown by Derdzinski for Ricci-flat self-dual Walker 4-metrics [14]. For sake of completeness, we recall the following [14, Thm. 6.1], which strengthen the main result in [21].

**Theorem 7.2.** A four dimensional Ricci flat self-dual Walker metric is locally isometric to the cotangent bundle $T^* \Sigma$ of an affine surface $(\Sigma, \nabla)$ equipped with the modified Riemannian extension $g_{\nabla, \Phi, 0} = g_{\nabla} + \pi^* \Phi$, where $\nabla$ is a torsion free connection with skew-symmetric Ricci tensor which expresses in adapted coordinates $(x_1, x_2)$ by

$$\Gamma^{\nabla}_{11} = -\partial_1 \varphi, \quad \Gamma^{\nabla}_{22} = \partial_2 \varphi$$

for an arbitrary function $\varphi$, and where $\Phi$ is an arbitrary symmetric $(0,2)$-tensor field on $\Sigma$. 
7.2. Osserman 4-metrics with non-diagonalizable Jacobi operators. Let \( \mathcal{N} \) be a pseudo-Riemannian manifold of signature \((2,2)\). Then, for each non-null vector \( X \), the induced metric on \( X^\perp \) is of Lorentzian signature and thus the Jacobi operator \( J^g(X) = R^g(\cdot, X)X \), viewed as an endomorphism of \( X^\perp \), corresponds to one of the following possibilities [6]:

\[
\begin{pmatrix}
\alpha & \beta \\
\beta & \gamma \\
\end{pmatrix}, \quad \begin{pmatrix}
\alpha & -\beta \\
\beta & \alpha \\
\end{pmatrix}, \quad \begin{pmatrix}
\alpha & \beta \\
1 & \beta \\
\end{pmatrix}, \quad \begin{pmatrix}
\alpha & \beta \\
1 & 1 \\
\end{pmatrix}.
\]

Type Ia Osserman metrics correspond to real, complex and paracomplex space forms, Type Ib Osserman metrics do not exist [6], and Types II and III Osserman metrics with non-nilpotent Jacobi operators have recently been classified in [18] and [13], respectively. Further, note that any Type II Osserman metric whose Jacobi operator has non-zero eigenvalues is necessarily a Walker metric. Moreover, since four dimensional Osserman metrics are locally Einstein and self-dual, they can be described by means of modified Riemannian extensions as follows.

**Theorem 7.3.** A four dimensional Type II Osserman metric whose Jacobi operator has non-zero eigenvalues is locally isometric to the cotangent bundle \( \mathcal{T}^*\Sigma \) of an affine surface \((\Sigma, \nabla)\), with metric tensor

\[
g = \frac{\tau}{6} \cdot \iota \cdot \text{id} \circ \text{id} + g_\nabla + \frac{24}{\tau} \pi^* \Phi,
\]

where \( \tau \neq 0 \) denotes the scalar curvature of \((\mathcal{T}^*\Sigma, g)\), \( \nabla \) is an arbitrary non-flat connection on \( \Sigma \) and \( \Phi \) is the symmetric part of the Ricci tensor of \( \nabla \).

**Proof.** It follows after some straightforward calculations as in Theorem 7.1 that any Type II Osserman metric of non-zero scalar curvature \( \tau \) is obtained by the above modified Riemannian extension of a torsion free connection \( \nabla \) given by

\[
\Gamma^\nabla_{ij} = \begin{cases}
\frac{1}{2} P(x_1, x_2), & \text{Type Ia}, \\
\frac{1}{2} Q(x_1, x_2), & \text{Type Ib}, \\
-\frac{1}{2} U(x_1, x_2), & \text{Type II}, \end{cases}
\]

just considering [18] Thm. 3.1. \( \square \)

**Remark 7.4.** The first examples of non Ricci flat Type II Osserman metrics were given in [17] as follows. Let \( \mathcal{N} := \mathbb{R}^4 \) with usual coordinates \((x_1, x_2, x_3, x_4)\) and define a metric by

\[
g = 2(dx^1 \circ dx^3 + dx^2 \circ dx^4) + (4kx_1^2 - \frac{1}{12} f(x_4)^2)dx^3 \circ dx^3 + 4kx_2^2dx^4 \circ dx^4 + 2(4kx_1x_2 + x_2 f(x_4) - \frac{1}{12} f'(x_4))dx^3 \circ dx^4,
\]

where \( k \) is a non-zero constant and \( f(x_4) \) is an arbitrary function.

Now, an easy calculation shows that Equation (7.3) is nothing but the modified Riemannian extension \( g = 4k \cdot \iota \cdot \text{id} \circ \text{id} + g_\nabla + \frac{1}{4} \pi^* \Phi \) of the torsion free connection \( \nabla \) given by \( \Gamma^\nabla_{ij} = -\frac{1}{2} f(x_2) \), whose Ricci tensor is given by

\[
\rho^\nabla = -\frac{1}{4} f(x_2)^2dx^3 \otimes dx^1 - \frac{1}{2} f'(x_2)dx^3 \otimes dx^2.
\]

This is neither symmetric nor skew-symmetric. We symmetrize to see

\[
\Phi = -\frac{1}{4} f(x_2)^2dx^1 \circ dx^1 - \frac{1}{2} f'(x_2)dx^1 \circ dx^2.
\]

**Remark 7.5.** Four dimensional Type II Osserman metrics have been studied intensively during the last years. The existence of many nilpotent examples suggested that the family of Osserman metrics with two-step nilpotent Jacobi operators was larger than the non-nilpotent one. However, this seems not to be true (see Theorems 7.2 and 7.3).
Let $p = 3$. We consider on $\mathbb{R}^3$ the torsion free connection $\nabla$ with the only non-zero Christoffel symbol given by $\nabla_{\partial_1} \partial_1 = x_2 \partial_3$. This connection is Ricci flat, but not flat. We set $\Phi = 0$ and take $c = 1$ in Equation (4.1) to define a metric $g = g_{\gamma \lambda}$ where the tensor $B$ of Equation (4.1) is given by:

$$B = \begin{pmatrix}
    x_1^2 - 2x_2x_3' & x_1'x_3' & x_1x_3' \\
x_1'x_2' & x_2^2 & x_2x_3' \\
x_1x_3' & x_2x_3' & x_3^2
\end{pmatrix}.$$ 

8. Six Dimensional geometry

The curvature tensor of the Levi-Civita connection is given by:

$$R^g_{1212} = \frac{1}{2} x_2 x_3 (x_2 x_2' - 4), \quad R^g_{1213} = \frac{1}{2} x_3^2 (x_2 x_2' - 2),$$

$$R^g_{1211'} = -R^g_{1222'}, \quad R^g_{1333'} = -R^g_{13'23} = \frac{1}{4} x_1 x_2'^2,$$

$$R^g_{1212'} = -R^g_{1313} = -\frac{1}{4} (x_1' - 2x_2 x_3'), \quad R^g_{1221'} = -R^g_{2333'} = \frac{1}{4} x_1 x_2'^2,$$

$$R^g_{1231} = -R^g_{1321'} = -R^g_{2322} = -R^g_{2333} = \frac{1}{4} x_1 x_3'^2,$$

$$R^g_{1313} = \frac{1}{4} x_2 x_3', \quad R^g_{1331'} = R^g_{2332} = \frac{1}{4} x_3'^2,$$

$$R^g_{11'22'} = R^g_{11'33'} = R^g_{12'21'} = R^g_{13'31'} = R^g_{22'33'} = R^g_{23'32'} = \frac{1}{2}.$$

The eigenvalues of the Jacobi operator of an Osserman metric change sign when passing from timelike to spacelike directions. Thus, for the purpose of studying the Osserman property, it is convenient to consider the operator given by setting $\mathcal{J}^g(v) = g(v, v)^{-1} \mathcal{J}^g(v)$ associated to each non-null vector $v$, whose eigenvalues must be constant if and only if the manifold is Osserman.

We now determine the Jacobi operator. Let $v = \sum_{i=1}^3 (\alpha_i \partial_i + \alpha_i' \partial_i$) be a non-null vector, where $\{\partial_i, \partial_i'\}$ denotes the coordinate basis. The associated Jacobi operator $\mathcal{J}^g(v) = R^g(\cdot, \cdot, v)$ can be expressed, with respect to the coordinate basis, as

$$\mathcal{J}^g(v) = \frac{1}{4} \begin{pmatrix}
    a_{11} & a_{12} & a_{13} & -4a_1^3 & -4a_1 a_2 & -4a_1 a_3 \\
    a_{21} & a_{22} & a_{23} & -4a_2 a_1 & -4a_2^3 & -4a_2 a_3 \\
    a_{31} & a_{32} & a_{33} & -4a_3 a_1 & -4a_3 a_2 & -4a_3^3 \\
    a_{11'} & a_{11'} & a_{11'} & a_{11'} & a_{11'} & a_{11'} \\
    a_{21'} & a_{22'} & a_{23'} & a_{21'} & a_{22'} & a_{23'} \\
    a_{31'} & a_{32'} & a_{33'} & a_{31'} & a_{32'} & a_{33'}
\end{pmatrix}$$

with

$$a_{11} = x_2^2 a_2^2 + 2x_2 x_3 x_2 a_3 + x_3^2 a_3^2 + x_1' (x_2' a_1 + x_3' a_3) + 2(20 a_1 a_1 + a_2 a_2 + a_3 a_3),$$

$$a_{1j} = -x_1 x_j a_1^2 - x_2 x_j' a_1 a_2 - x_j x_3 a_1 a_3 + 2a_1 a_j', \quad j = 2, 3,$$

$$a_{2j} = -x_1^2 a_1 a_2 + 2x_2 x_3 a_1 a_2 - x_1 (x_2 a_2^2 + x_3 a_3 a_3) + 2a_2 a_1'',$$

$$a_{22} = x_1 x_2 x_3 a_1 + x_2 x_3 a_2 + x_3 a_3 a_3 + 2a_2 a_2',$$

$$a_{23} = x_1 x_2 x_3 a_1 - x_2 x_3 (a_2^2 + a_3 a_3) + 2a_2 a_3,$$

$$a_{33} = -4a_1 a_2 - x_1 x_2 x_3 a_1 + 2x_2 x_3 a_3 - x_1 (x_2 a_3 + x_3 a_3^2) + 2a_3 a_1'',$$

$$a_{32} = 4a_1^2 - x_1 x_2 a_3 - x_2 a_2 a_2 + 2a_2 a_2,$$

$$a_{31} = -x_1 x_2 x_3 a_3 + x_2 x_3 a_3 + 2x_2 x_3 a_3 - x_1 (x_2 a_3 + x_3 a_3^2) + 2a_3 a_1'',$$

$$a_{11'} = 8x_2 x_3 a_2^2 + 8x_3 a_3 a_2 - 4x_2 a_2 a_3 + 8a_2 a_3.$$

This completes the proof of Theorem 2.3.
\[ a_{12} = -4x_3^2\alpha_1\alpha_3 - 2x_2^2\alpha_2\alpha_1 - x_2^2x_3^2(8\alpha_1\alpha_2 + 3\alpha_1\alpha_3) - 3x_1^2\alpha_1\alpha_2 \]
\[ + 6x_2x_3\alpha_1\alpha_2 - x_1(x_2(\alpha_1\alpha_1 + 3\alpha_2\alpha_2) + x_3(4\alpha_4^2 + 3\alpha_3\alpha_2)) \]
\[ - 4(\alpha_1\alpha_2 + \alpha_1\alpha_3), \]
\[ a_{13} = -2x_2x_3\alpha_2\alpha_1 - x_3^2(4\alpha_1\alpha_2 + 3\alpha_1\alpha_3) - 3x_1^2\alpha_1\alpha_3 - 6x_2x_3\alpha_1\alpha_3 \]
\[ - 4\alpha_1\alpha_3 - x_1(x_2(\alpha_2\alpha_2 + x_3(\alpha_1\alpha_1 + 3\alpha_3\alpha_3))), \]
\[ a_{1'1'} = 4\varepsilon_v - 3x_2^2\alpha_2^2 - 6x_2x_3'\alpha_2\alpha_3 - 3x_3^2\alpha_3^2 - x_1'(3x_2\alpha_1\alpha_2 + 3x_3\alpha_1\alpha_3) \]
\[ - 4\alpha_1\alpha_1 - 6\alpha_2\alpha_2 - 6\alpha_3\alpha_3, \]
\[ a_{1'2'} = 3x_1^2\alpha_1\alpha_2 - 6x_2x_3\alpha_1\alpha_2 + x_1'(3x_2\alpha_2^2 + 3x_3\alpha_2\alpha_3) + 2\alpha_2\alpha_1, \]
\[ a_{1'3'} = -4\alpha_1\alpha_2 + 3x_1^2\alpha_1\alpha_3 - 6x_2x_3\alpha_1\alpha_3 + x_1'(3x_2\alpha_2\alpha_3 + 3x_3\alpha_1^2) + 2\alpha_3\alpha_1, \]
\[ a_{2'1} = -4x_1^2\alpha_1\alpha_3 - 3x_2^2\alpha_2\alpha_1 - x_2'x_3'(4\alpha_1\alpha_2 + 3\alpha_3\alpha_1') + 2x_2x_3'\alpha_1\alpha_2' \]
\[ - x_1'(x_3\alpha_3\alpha_2 + x_2'(3\alpha_1\alpha_1 + \alpha_2\alpha_2')) - 4(\alpha_1\alpha_2 + \alpha_1\alpha_3), \]
\[ a_{2'2} = -4x_1x_2'\alpha_1\alpha_2 - 4x_1^2\alpha_2\alpha_2 - 4\alpha_2' + 4x_2'x_3'(\alpha_1^2 - \alpha_3\alpha_2'), \]
\[ a_{2'3} = x_3^2(4\alpha_1^2 - \alpha_3\alpha_2) - 3x_2^2\alpha_2\alpha_3' - 4\alpha_2\alpha_2' - x_1'(x_3\alpha_1\alpha_2 + 3x_2\alpha_1\alpha_3') \]
\[ - x_2x_3(\alpha_2\alpha_2' + 3\alpha_3\alpha_3'), \]
\[ a_{2'2'} = 3x_1x_2'\alpha_1\alpha_2 + 3x_2^2\alpha_2\alpha_2 + 3x_2'x_3\alpha_2\alpha_3 + 2\alpha_2\alpha_1', \]
\[ a_{2'2'} = \varepsilon_v + 3x_1x_2\alpha_1\alpha_2 + 3x_2^2\alpha_2^2 + 3x_2'x_3\alpha_2\alpha_3 + 2\alpha_2\alpha_1', \]
\[ a_{2'3'} = 4\alpha_1^2 + 3x_1x_2\alpha_1\alpha_3 + 3x_2'\alpha_2\alpha_3 + 3x_2x_3\alpha_3^2 + 2\alpha_3\alpha_2, \]
\[ a_{3'1} = -3x_2x_3'\alpha_2\alpha_1' - 3x_2\alpha_3^2\alpha_3' - x_1'(x_3\alpha_1\alpha_2 - 3x_2\alpha_1\alpha_3' - 4\alpha_1\alpha_3') \]
\[ - x_1'(x_2\alpha_2\alpha_3 + x_3(3\alpha_1\alpha_1' + 3\alpha_3\alpha_3')) \]
\[ a_{3'2} = -3x_2^2\alpha_3\alpha_3' - 2x_2^2\alpha_2\alpha_3' - 4\alpha_2\alpha_2' - x_1'(3x_2\alpha_1\alpha_2 + 3x_2\alpha_1\alpha_3') \]
\[ - x_2x_3'(3\alpha_2\alpha_2' + 3\alpha_3\alpha_3'), \]
\[ a_{3'3} = -4x_1x_3\alpha_1\alpha_3' - 4x_2'x_3\alpha_2\alpha_3' - 4x_3^2\alpha_3\alpha_3' - 4\alpha_3^2, \]
\[ a_{3'2} = 3x_1x_2\alpha_1\alpha_j + 3x_2'x_3\alpha_2\alpha_j + 3x_3^2\alpha_3\alpha_j + 2\alpha_3\alpha_j, \]
\[ a_{3'3} = \varepsilon_v + 3x_1x_2\alpha_1\alpha_3 + 3x_2'x_3\alpha_2\alpha_3 + 3x_3^2\alpha_3^2 + 2\alpha_3\alpha_3. \]

The characteristic polynomial of the Jacobi operator is now seen to be:
\[ p_{\lambda}(\mathcal{J}^g(v)) = \lambda(\lambda - 1)(\lambda - \frac{1}{2})^4, \]
and therefore g is Osserman with eigenvalues \( \{0, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \} \). Now, for any unit vector v set
\[ \mathcal{A}(v) = \mathcal{J}^g(v) \cdot (\mathcal{J}^g(v) - \varepsilon_v \text{id}) \cdot (\mathcal{J}^g(v) - \frac{\varepsilon}{4} \text{id}). \]
For the particular choice of the unit vectors
\[ v = \partial_3 + \frac{1}{2}(\varepsilon - x_3^2)\partial_3', \quad \hat{v} = \partial_1 - \frac{1}{2}(x_2^2 - 2x_2x_3 - \varepsilon)\partial_1', \]
we have \( g(v, v) = \varepsilon \) and \( g(\hat{v}, \hat{v}) = \varepsilon \), and a straightforward calculation shows that \( \mathcal{A}(v) = 0 \), while \( (\mathcal{A}(v))_{32} = -\frac{3}{50} \). Therefore, at any point, \( J^g(v) \) diagonalizes while \( J^g(\hat{v}) \) is not diagonalizable, and hence the metric is neither spacelike Jordan Osserman nor timelike Jordan Osserman at any point.

**Remark 8.1.** We make the following observations concerning the metric of Theorem 2.3.2:

1. The Jacobi operator \( J^g(v) \) associated to a unit vector v is either diagonalizable, has a single 2 \times 2 Jordan block, or has a 3 \times 3 Jordan block, or has two 2 \times 2 Jordan blocks depending on the point and the vector v considered; all possibilities can arise. More precisely, if \( v = \sum_{i=1}^{3}(\alpha_i\partial_i + \alpha_i'\partial_i') \), setting
\[ \xi_1 = x_2^2\varepsilon_v + x_1^2\varepsilon_v + \alpha_1\alpha_2 + \alpha_2\alpha_1 \]
\[ + 2x_3'x_1\alpha_1 + x_2'\alpha_2\alpha_3 + 2\alpha_3^2, \]
\[ \xi_2 = x_2^2\alpha_2\alpha_3 + x_2^2\alpha_3\varepsilon_v + 3\alpha_3\varepsilon_v + 2\alpha_3\alpha_2 + 2\alpha_3\alpha_3 \]
\[ + 2x_3'\alpha_2\alpha_3(\varepsilon_v + 2\alpha_3\varepsilon_v), \]
we have:
(a) If $\alpha_1$ and $\xi_1$ do not vanish, then $J^g(v)$ has a $3 \times 3$ Jordan block.
(b) If $\alpha_1 = 0$ and $\xi_2 \neq 0$, or otherwise $\xi_1 = 0$ and $\alpha_1 \neq 0$, then $J^g(v)$ has
a single $2 \times 2$ Jordan block or has two $2 \times 2$ Jordan blocks.
(c) If $\alpha_1 = 0$ and $\xi_2 = 0$, then $J^g(v)$ is diagonalizable.

Proof. Set

$$A^r(v) = J^g(v) \cdot (J^g(v) - \varepsilon_v \text{id}) \cdot (J^g(v) - \frac{\xi_v}{2} \text{id})^r, \quad r = 1, \ldots, 4,$$

for any unit vector $v$. First, a very long but direct computation shows that

(8.a)

$$A(v) = \begin{pmatrix} A & 0 \\ B & A' \end{pmatrix},$$

where

$$A = \frac{3}{16} \varepsilon_v \begin{pmatrix} -\eta \alpha_1^2 \alpha_2 & \eta \alpha_1^3 & 0 \\ -\frac{\eta \alpha_1^2 \alpha_2}{\varepsilon_v - \eta \alpha_3} & \eta \alpha_1^3 & 0 \\ \alpha_1 \alpha_2 (\varepsilon_v - \eta \alpha_3) & -\alpha_1^2 (\varepsilon_v - \eta \alpha_3) & 0 \end{pmatrix},$$

with $\eta = x_3'(x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3) + 2 \alpha_3$, and $B = \frac{1}{16} (b_{ij})$ is given by

$$b_{11} = 2 \alpha_2 \{3x_3' \alpha_2 (x_2' \alpha_1 \alpha_1' - x_1 \alpha_1 \alpha_2' + 2x_2 \alpha_3 \alpha_3')
+ x_2' (2 - 3x_2 x_2') \alpha_1^2 \alpha_2 + 3 \alpha_3 (\varepsilon_v - \alpha_2 \alpha_2' + \alpha_3 \alpha_3')
- 3(4 \varepsilon_v \xi_1 \alpha_1^2 \alpha_2 + x_1 x_2' \alpha_1 \alpha_2 \alpha_3' + \alpha_3 (\xi_1 - 2 \alpha_3')) \} - 6 \xi_2,$

$$b_{12} = \alpha_1 \{6 \xi_1 (4 \varepsilon_v \alpha_1^2 \alpha_2 + \alpha_3) + 3x_1 x_3' \alpha_1 (\alpha_1 \alpha_1' + 3 \alpha_2 \alpha_2')
- x_3' (4 \alpha_1 \alpha_2' - 3 \alpha_1 \alpha_3 \alpha_1' - 3 \alpha_3 (\varepsilon_v - 3 \alpha_2 \alpha_2' - 6x_2 \alpha_2^2 \alpha_3')
+ 3x_2 \alpha_2 (x_3' (2 \varepsilon_v + 2x_2 x_2' \alpha_1^2 - \alpha_1 \alpha_1' + \alpha_2 \alpha_2') + 2x_1 \alpha_1 \alpha_3')
- 3(\varepsilon_v - 2 \alpha_1 \alpha_1' - 6 \alpha_2 \alpha_2' \alpha_3') \},$$

$$b_{13} = 3 \alpha_1 \alpha_2 \{\varepsilon_v x_3' + x_3' (x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3) \alpha_3 + 2 \alpha_3^2 \},$$

$$b_{21} = \alpha_1 \{24 \varepsilon_v \xi_1 \alpha_1^2 \alpha_2 - 3x_1 x_3' \alpha_1 (\alpha_1 \alpha_1' - \alpha_2 \alpha_2')
- x_3' (4 \alpha_1 \alpha_2' - 3 \alpha_1 \alpha_3 \alpha_1' - 3 \alpha_3 (\varepsilon_v + \alpha_2 \alpha_2') + 6x_2 \alpha_2^2 \alpha_3'
+ 3x_2 \alpha_2 (x_3' (2 \varepsilon_v + 2x_2 x_2' \alpha_1^2 - 3 \alpha_1 \alpha_1' - 2 \alpha_2 \alpha_2') + 2x_1 \alpha_1 \alpha_3')
+ 3(\varepsilon_v - 2 \alpha_1 \alpha_1' + 2 \alpha_2 \alpha_2' \alpha_3') \},$$

$$b_{22} = -2 \alpha_1^2 \{3x_1 x_3' \alpha_3 \alpha_2' - x_3' (2 \alpha_1^2 - 3 \alpha_3 \alpha_2') + 3x_2 \alpha_2 \alpha_3
+ 3x_2 (\varepsilon_v + \alpha_1 (x_2 x_3' \alpha_3 - \alpha_1')) + x_1 \alpha_1 \alpha_3') + 6(2 \varepsilon_v \xi_1 \alpha_1^2 + 2 \alpha_2 \alpha_3') \},$$

$$b_{23} = -3 \alpha_1^2 \{\varepsilon_v x_3^2 + x_3' (x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3) \alpha_3 + 2 \alpha_3^2 \},$$

$$b_{31} = -3 \alpha_1 \alpha_2 \{\varepsilon_v x_3^2 - 2 \xi_1 + x_3' (x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3) \alpha_3 + 2 \alpha_3^2 \},$$

$$b_{32} = 3 \alpha_1^2 \{\varepsilon_v x_3^2 - 2 \xi_1 + x_3' (x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3) \alpha_3 + 2 \alpha_3^2 \},$$

$$b_{33} = 0.$$

Moreover,

(8.b)

$$A^2(v) = \frac{3}{8} \varepsilon_v \xi_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_1^2 \alpha_2 & \alpha_3^2 \alpha_2 & 0 & 0 & 0 & 0 \\ \alpha_1 \alpha_2 & -\alpha_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

It follows that $A^3(v) = 0$, and therefore the Jacobi operator $J^g(v)$ associated to a unit vector $v$ is either diagonalizable, has a single $2 \times 2$ Jordan block, or has a $3 \times 3$ Jordan block, or has two $2 \times 2$ Jordan blocks.
Note that (a) follows from (8.3). Now, assuming \( A^2(v) = 0 \), i.e., \( \alpha_1 = 0 \) or \( \xi_1 = 0 \), we analyze the vanishing of \( A(v) \) using (8.3). If \( \alpha_1 = 0 \), then the only non-vanishing element of \( A(v) \) is \( (A(v))_{11} = -\frac{4}{15}v_1\xi_2 \); if \( \xi_1 = 0 \), then \( (A(v))_{12} = \frac{4}{15}v_1^2\gamma_3^3 \) and \( (A(v))_{12} = -\frac{4}{15}v_1\gamma_3^3(v - \gamma_3^3) \), for \( \gamma = x_3(x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + 2\alpha_3) \), which implies that \( A(v) \) does not diagonalize if \( \alpha_1 \neq 0 \). Thus one easily gets (b) and (c).

Note that (a), (b) and (c) are possible at any point, and for spacelike or timelike unit vectors. Indeed, the following can be easily checked:

(i) For the special choice of \( v = \partial_1 - \frac{1}{2}(x_1^2 - 2x_2x_3 - \epsilon)\partial_2 + \lambda\partial_3 \), we have \( g(v, v) = \epsilon \), \( \alpha_1 = 1 \neq 0 \), and \( \xi_1 = \frac{1}{2}(4\lambda^2 + 4\lambda x_1x_3 + x_3^2(x_1^2 + \epsilon)) \).

(ii) For the special choice of \( v = \partial_2 - \frac{1}{2}(x_2^2 - \epsilon)\partial_2 + \lambda\partial_3 \), we have \( g(v, v) = \epsilon \), \( \alpha_1 = 0 \), and \( \xi_3 = (\lambda + x_2x_3)\epsilon \). Note that, for a fixed point, \( \lambda \) can be chosen so that \( \xi_1 \neq 0 \). So (a) holds.

(iii) For the special choice of \( v = \partial_3 + \frac{1}{2}(-x_3^2)\partial_3 \), we have \( g(v, v) = \epsilon \), \( \alpha_1 = 0 \), and \( \xi_2 = 0 \). So (b) holds.

(2) Observe that the eigenvalues of the Jacobi operator of a pseudo-Riemannian Osserman metric change sign from spacelike to timelike vectors, and thus they are all zero for null vectors (cf. [22, 23]), which shows that any Osserman metric is null Osserman. Hence, the metric of Theorem 2.3 is null Osserman; moreover, proceeding as above, one checks that the null Jacobi operators can be two-step nilpotent, or three-step nilpotent, or four-step nilpotent, changing even at a fixed point, and therefore the metric is not pointwise null Jordan Osserman. Indeed:

(a) For the special choice of the null vector \( v = \partial_1 \), \( J^g(v) \) is two-step nilpotent at any point.

(b) At points with \( x_3 \neq 0 \), let \( v = \partial_1 - \frac{2}{x_3^2}\partial_3 + x_2x_3\partial_2 \). Then \( J^g(v) \) is three-step nilpotent. Such a degree of nilpotency is not possible at points with \( x_3 = 0 \).

(c) For the special choice of the null vector \( v = \partial_1 - \frac{1}{2}(x_1^2 - 2x_2x_3)\partial_1 \), \( J^g(v) \) is four-step nilpotent at any point with \( x_1x_3 \neq 0 \). At points with \( x_1 = 0 \) take \( v = \partial_1 + x_2x_3\partial_2 + \partial_3 \), and at points with \( x_3 = 0 \) take \( v = \partial_1 - \frac{1}{2}x_1^2\partial_1 + \partial_2 + \partial_3 \), to get a four-step nilpotent null Jacobi operator.

(3) The metric of Theorem 2.3 is timelike and spacelike nilpotent Szabó, it is not symmetric, and it is not Jordan Szabó. To be more explicit. Given a non-null vector \( v = \sum_{i=1}^{3}(\alpha_i\partial_i + \alpha_i\partial_i) \), the associated Szabó operator, when expressed in the coordinate basis, takes the form

\[
\nabla_v J^g_v = \begin{pmatrix} A & 0 \\ B & \iota A \end{pmatrix},
\]

where

\[
A = \begin{pmatrix} 2x_3\alpha_1^2\alpha_2 & -2x_3\alpha_1^3 & 0 \\ 2x_3\alpha_1\alpha_2^2 & -2x_3\alpha_1^2\alpha_2 & 0 \\ -2\alpha_1\alpha_2(x_1\alpha_1 + x_2\alpha_2) & 2\alpha_1^2(x_1\alpha_1 + x_2\alpha_2) & 0 \end{pmatrix}.
\]

Hence the characteristic polynomial of the Szabó operators is

\[
p_A(\nabla_v J^g_v) = \lambda^6
\]

(independently of the \( 3 \times 3 \)-matrix \( B \)), so metric is Szabó of signature \((3, 3)\) with zero eigenvalues. A further analysis shows that \( \nabla_v J^g_v \) can be one-step nilpotent, or two-step nilpotent, or three-step nilpotent, changing even at a fixed point, and therefore the metric is not pointwise Jordan Szabó. Indeed:
(a) For the special choice of $v = \partial_3 + \frac{1}{2}(\varepsilon - x_3^2)\partial y$, we have $g(v, v) = \varepsilon$, and $\nabla_v J^g v$ is one-step nilpotent at any point.

(b) For the special choice of $v = \partial_2 + \frac{1}{2}(\varepsilon - x_2^2)\partial y + \lambda \partial y$, we have $g(v, v) = \varepsilon$, and choosing appropriate $\lambda \in \mathbb{R}$ we obtain that $\nabla_v J^g v$ is two-step nilpotent at any point.

(c) For the special choice of $v = \partial_1 + \frac{1}{2}(\varepsilon - (x_1^2 - 2x_2x_3))\partial_3 + \lambda \partial y$, we have $g(v, v) = \varepsilon$, and choosing appropriate $\lambda \in \mathbb{R}$ we obtain that $\nabla_v J^g v$ is three-step nilpotent at any point.

(4) A straightforward calculation shows that metric of Theorem 2.3 is not Ivanov–Petra's.

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C-L, G-R, V-L: Department of Geometry and Topology, Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain

E-mail address: estebcl@edu.xunta.es, Eduardo.Garcia.Rio@usc.es, ravazlor@edu.xunta.es

G: Mathematics Department, University of Oregon, Eugene, Oregon 97403, USA

E-mail address: gilkey@uoregon.edu