COINCIDENCE CLASSES IN NONORIENTABLE MANIFOLDS

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ABSTRACT. In this article we studied Nielsen coincidence theory for maps between manifolds of same dimension without hypotheses on orientation. We use the definition of semi-index of a class, we review the definition of defective classes and study the appearance of defective root classes. We proof a semi-index product formula type for lifting maps and we presented conditions such that defective coincidence classes are the only essential classes.

1. Introduction

Nielsen coincidence theory was extended ([Dobreńko & Jezierski] and [Jezierski 2]) to maps between nonorientable topological manifolds using the notion of semi-index (a non negative integer) for a coincidence set.

We consider maps \( f, g : M \to N \) between manifolds without boundary of the same dimension \( n \), we define \( h = (f, g) : M \to N \times N \), then using microbundles (see [Jezierski 2] for details) we can suppose that \( h \) is in a transverse position.

Let \( w \) be a path satisfying the Nielsen relation between \( x, y \in \text{Coin}(f, g) \). We choose a local orientation \( \gamma_0 \) of \( M \) in \( x \) and denote by \( \gamma_t \) the translation of \( \gamma_0 \) along \( w(t) \).

**Definition 1.1.** [Jezierski 2, 1.2] We will say that two points \( x, y \in \text{Coin}(f, g) \) are \( R \)-related (\( xRy \)) if and only if there is a path \( w \) establishing the Nielsen relation between them such that the translation of the orientation \( h_*\gamma_0 \) along a path in the diagonal \( \Delta(N) \subset N \times N \) homotopic to \( h \circ w \) in \( N \times N \) is opposite to \( h_*\gamma_1 \). In this case the path \( w \) is called graph-orientation-reversing.

Since \( (f, g) \) is transverse, \( \text{Coin}(f, g) \) is finite. Let \( A \subset \text{Coin}(f, g) \), then \( A \) can be represented as \( A = \{a_1, a_2, \ldots, a_s; b_1, c_1, \ldots, b_k, c_k\} \) where \( b_iRc_i \) for any \( i \) and \( a_iRa_j \) for no \( i \neq j \). The elements \( \{a_i\}_i \) of this decomposition are called free.

**Definition 1.2.** In the above situation the semi-index of the pair \( (f, g) \) in \( A = \{a_1, a_2, \ldots, a_s; b_1, c_1, \ldots, b_k, c_k\} \) (denoted \( \text{ind}(f, g; A) \))

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is the number of free elements of this decomposition of $A$.
\[ |\text{ind}|(f, g; A) = s. \]

In [Dobreńko & Jezierski] and [Jezierski 2] we can find the proof of the fact that this definition does not depend on the decomposition of $A$. Moreover, if $U \subset M$ is an open subset we can extend this definition to a semi-index of a pair on the subset $U$ ($|\text{ind}|(f, g; U)$).

**Definition 1.3.** A coincidence class $C$ of a transverse pair $(f, g)$ is essential if $|\text{ind}|(f, g; A) \neq 0$.

In [Jezierski 1] the authors studied when a coincidence point $x \in \text{Coin}(f, g)$ satisfy $xRx$. These points can only appear in the nonorientable case, they will be called self-reducing points. They represent a new situation (see [Jezierski 1, Example 2.4]) that never occurs in the orientable case or in the fixed point context.

**Definition 1.4.** (Jezierski 1, 2.1) Let $x \in \text{Coin}(f, g)$ and let $H \subset \pi_1(M)$, $H' \subset \pi_1(N)$ denote the subgroups of orientation-preserving elements. We define:
\[
\text{Coin}(f\# , g\# )_x = \{ \alpha \in \pi_1(M, x) \mid f\# (\alpha) = g\# (\alpha) \},
\]
\[
\text{Coin}^+(f\# , g\# )_x = \text{Coin}(f\# , g\# )_x \cap H.
\]

**Lemma 1.5.** (Jezierski 1, 2.2) Let $f, g : M \to N$ be transverse and $x \in \text{Coin}(f, g)$. Then $xRx$ if and only if $\text{Coin}^+(f\# , g\# )_x \neq \text{Coin}(f\# , g\# )_x \cap f\#^{-1}(H')$ (in other words, if there exists a loop $\alpha$ based at $x$ such that $f \circ \alpha \sim g \circ \alpha$ and exactly one of the loops $\alpha$ or $f \circ \alpha$ is orientation-preserving).

**Definition 1.6.** A coincidence class $A$ is called defective if $A$ contains a self-reducing point.

**Lemma 1.7.** (Jezierski 1, 2.3) If a Nielsen class $A$ contains a self-reducing point (i.e. $A$ is defective) then any two points in this class are $R$-related, and thus:
\[
|\text{ind}|(f, g; A) = \begin{cases} 
0 & \text{if } \#A \text{ is even;} \\
1 & \text{if } \#A \text{ is odd.}
\end{cases}
\]

2. **The root case**

In [Brown & Schirmer] we can find a different approach to extending the Nielsen root theory to the nonorientable case. Using the concept of orientation-true\(^1\) map they classified maps between manifolds of the same dimension in three types (see also [Olum] and [Skora]):

**Definition 2.1.** (Brown & Schirmer 2.1) Let $f : M \to N$ be a map of manifolds. Then three types of maps are defined as follows.

\(^1f\) is orientation-true if for each loop $\alpha \in \pi_1(M)$, $f(\alpha)$ is orientation-preserving if and only if $\alpha$ is orientation-preserving.
(1) Type I: \( f \) is orientation-true.
(2) Type II: \( f \) is not orientation-true but does not map an orientation-reversing loop in \( M \) to a contractible loop in \( N \).
(3) Type III: \( f \) maps an orientation-reversing loop in \( M \) to a contractible loop in \( N \).

Further, a map \( f \) is defined to be orientable if it is of Type I or II, and nonorientable otherwise.

For orientable maps they describe an Orientation Procedure ([Brown & Schirmer, 2.6]) for root classes, using local degree with coefficients in \( \mathbb{Z} \). For maps of Type III the same procedure is only possible with coefficients in \( \mathbb{Z}_2 \).

They then defined the multiplicity of a root class, that is an integer for orientable maps and an element of \( \mathbb{Z}_2 \) for maps of Type III.

Now if we consider the root classes for a map \( f \) to be the coincidence classes of the pair \((f, c)\) where \( c \) is the constant map, we have:

**Theorem 2.2.** Let \( f : M \to N \) be a map of manifolds of the same dimension.

(i) If \( f \) is orientable, no root class of \( f \) is defective.
(ii) If \( f \) is of Type III, then all root classes of \( f \) are defective.

**Proof:** By lemma 1.5, a coincidence class \( C \) of the pair \((f, c)\) is defective if and only if there exists a point \( x \in C \) and a loop \( \alpha \) in \( x \) such that \( f \circ \alpha \sim 1 \) and \( \alpha \) is orientation-reversing. This proves the first statement.

Now let \( f \) be a Type III map. Then there exists a loop \( \alpha \in \pi_1(M, x_0) \) such that \( \alpha \) is orientation-reversing and \( f \circ \alpha \sim 1 \). If \( x \) is a root of \( f \) choosing a path \( \beta \) from \( x \) to \( x_0 \) we have that \( \gamma = \beta^{-1} \alpha \beta \) is a loop in \( x \) such that \( \gamma \) is orientation-reversing and \( f \circ \gamma \sim 1 \). Then all roots of \( f \) are self-reducing points. \qed

In fact [Brown & Schirmer, Lemma 4.1] shows the equality between the multiplicity of a root class and its semi-index.

**Theorem 2.3.** Let \( M \) and \( N \) be manifolds of the same dimension such that \( M \) is nonorientable and \( N \) is orientable. If \( f : M \to N \) is a map then all essential root classes of \( f \) are defective.

**Proof:** There are no orientation-true maps from a nonorientable to an orientable manifold. If \( f \) is a Type II map then by [Brown & Schirmer, Lemma 3.10] \( \deg(f) = 0 \) and \( f \) has no essential root classes. The result follows by proposition 2.2. \qed

Using the ideas of Proposition 2.2 we can also state:

**Lemma 2.4.** Let \( f, g : M \to N \) be two maps between manifolds of the same dimension. If there exist a coincidence point \( x_0 \) and a graph-orientation-reverse loop \( \alpha \) in \( x_0 \) such that \( f(\alpha) \) is in the center of \( \pi_1(N) \), then all coincidence points of the pair \((f, g)\) are self-reducing points.
Proof: We can suppose that all coincidences of the pair \((f, g)\) have image on the same point in \(N\). If \(x_1 \in \text{Coin}(f, g)\) we choose a path \(\beta\) from \(x_1\) to \(x_0\) and take the loop \(\gamma = \beta^{-1} \circ \alpha \circ \beta\) at \(x_1\). Since \(\alpha\) belongs to the center of \(\pi_1(N)\), \(\gamma\) is a graph-orientation-reverse loop at \(x_1\). \(\square\)

Corollary 2.5. Let \(f, g : M \to N\) be two maps between manifolds of the same dimension such that \(f_\#(\pi_1(M))\) is contained in the center of \(\pi_1(N)\). If \((f, g)\) has a defective class then all classes of \((f, g)\) are defective.

3. Covering maps

Let \(M\) and \(N\) be compact, closed manifolds of the same dimension; \(f, g : M \to N\) be two maps such that \(\text{Coin}(f, g)\) is finite. and \(p : \tilde{M} \to M\) and \(q : \tilde{N} \to N\) be finite coverings such that there exist lifts \(\tilde{f}, \tilde{g} : \tilde{M} \to \tilde{N}\) of the pair \(f, g\).

![Diagram](image)

In this situation the groups of Deck transformations of the lifts \(\tilde{M}\) and \(\tilde{N}\) can be described by:

\[
\mathcal{D}(\tilde{M}) = \frac{\pi_1(M)}{p_\#(\pi_1(M))}; \quad \mathcal{D}(\tilde{N}) = \frac{\pi_1(N)}{q_\#(\pi_1(N))}
\]

Choosing a point \(x_0 \in \text{Coin}(f, g)\) we define \(\tilde{f}_{*, x_0}, \tilde{g}_{*, x_0} : \mathcal{D}(\tilde{M}) \to \mathcal{D}(\tilde{N})\) such that, for each \(\alpha \in \mathcal{D}(\tilde{M})\) we take \(\overline{\alpha} \in \pi_1(M, x_0)\) such that \(\mathcal{D}(\tilde{M}) \ni [\overline{\alpha}] = \alpha\) and:

\[
\tilde{f}_{*, x_0}(\alpha) = [f_\#(\overline{\alpha})] \in \mathcal{D}(\tilde{N})
\]

\[
\tilde{g}_{*, x_0}(\alpha) = [g_\#(\overline{\alpha})] \in \mathcal{D}(\tilde{N})
\]

We note that \(\tilde{f}_{*, x_0}, \tilde{g}_{*, x_0}\) may depend on \(x_0\). Choosing a \(x_0 \in \text{Coin}(f, g)\) we have. \(\forall \tilde{x} \in \tilde{M}\) and \(\forall \alpha \in \mathcal{D}(\tilde{M})\):

\[
\tilde{f}(\alpha(\tilde{x})) = \tilde{f}_{*, x_0}(\alpha) \circ \tilde{f}(\tilde{x})
\]

\[
\tilde{g}(\alpha(\tilde{x})) = \tilde{g}_{*, x_0}(\alpha) \circ \tilde{g}(\tilde{x})
\]

Lemma 3.1. Let \(\tilde{x}_0 \in \text{Coin}(\tilde{f}, \tilde{g})\) and \(\alpha \in \mathcal{D}(\tilde{M})\). Then \(\alpha(\tilde{x}_0) \in \text{Coin}(\tilde{f}, \tilde{g})\) if and only if \(\tilde{f}_{*, x_0}(\alpha) = \tilde{g}_{*, x_0}(\alpha)\) where \(x_0 = p(\tilde{x}_0)\).

Proof: By the above observation we have \(\tilde{f}(\alpha(\tilde{x}_0)) = \tilde{f}_{*, x_0}(\alpha) \circ \tilde{f}(\tilde{x}_0)\) and \(\tilde{g}(\alpha(\tilde{x}_0)) = \tilde{g}_{*, x_0}(\alpha) \circ \tilde{g}(\tilde{x}_0)\). This proves the lemma. \(\square\)
**Corollary 3.2.** Let \( \tilde{x}_0 \in \text{Coin}(\tilde{f}, \tilde{g}) \) and \( x_0 = p(\tilde{x}_0) \). Then \( p^{-1}(x_0) \cap \text{Coin}(\tilde{f}, \tilde{g}) \) have exactly \( \#\text{Coin}(\tilde{f}_{*x_0}, \tilde{g}_{*x_0}) \) elements. \( \square \)

**Lemma 3.3.** Let \( \tilde{x}_0 \) and \( \tilde{x}_0' \) be two coincidences of the pair \( (\tilde{f}, \tilde{g}) \) such that \( p(\tilde{x}_0) = p(\tilde{x}_0') = x_0 \), and let \( \gamma \) the unique element of \( \mathcal{D}(M) \) such that \( \gamma(\tilde{x}_0) = \tilde{x}_0' \). The points \( \tilde{x}_0 \) and \( \tilde{x}_0' \) are in the same coincidence class of \( (\tilde{f}, \tilde{g}) \) if and only if there exists \( \gamma \in \pi_1(M, x_0) \) such that:

- \( \mathcal{D}(M) \ni \gamma \)
- \( f_\#(\gamma) = g_\#(\gamma) \).

**Proof:** (\( \Rightarrow \)) If \( \tilde{x}_0 \) and \( \tilde{x}_0' \) are in the same coincidence class of \( (\tilde{f}, \tilde{g}) \), there exists a path \( \beta \) from \( \tilde{x}_0 \) to \( \tilde{x}_0' \) that realizes the Nielsen relation, (i.e. \( \tilde{f} \circ \beta \sim \tilde{g} \circ \beta \)).

Take \( \gamma = p(\beta) \in \pi_1(M, x_0) \). We can see that \( \gamma = \gamma \) and \( f \circ \gamma = g \circ \gamma \) showing that \( f_\#(\gamma) = g_\#(\gamma) \).

(\( \Leftarrow \)) Take the lift \( \tilde{\gamma} \) of \( \gamma \) starting at \( \tilde{x}_0 \). It is a path from \( \tilde{x}_0 \) to \( \tilde{x}_0' \) that realizes the Nielsen relation, (i.e. \( \tilde{f} \circ \tilde{\gamma} \sim \tilde{g} \circ \tilde{\gamma} \)). \( \square \)

**Corollary 3.4.** In lemma 3.3, if the points \( \tilde{x}_0 \) and \( \tilde{x}_0' \) are in the same coincidence class of \( (\tilde{f}, \tilde{g}) \) then \( \tilde{x}_0 R \tilde{x}_0' \) if and only if \( \text{sign}(\tilde{f}_{*x_0}(\gamma)) \cdot \text{sign}(\gamma) = -1 \). In this case, \( x_0 \) is a self-reducing coincidence point.

**Proof:** First we note that since \( f_\#(\gamma) = g_\#(\gamma) \) then \( f_{*x_0}(\gamma) = g_{*x_0}(\gamma) \) and we can see that \( \text{sign}(\tilde{f}_{*x_0}(\gamma)) \cdot \text{sign}(\gamma) = -1 \) if and only if the paths \( \gamma \) and \( \tilde{\gamma} \) in the proof of lemma 3.3 are both graph orientation-reversing. \( \square \)

Denoting the natural projection by \( j_{x_0} : \pi_1(M, x_0) \to \mathcal{D}(M) \) and the set \( \{ \alpha \in \pi_1(M, x_0) \mid f_\#(\alpha) = g_\#(\alpha) \} \) by \( \text{Coin}(f_\#, g_\#, x_0) \), we have:

**Corollary 3.5.** If \( x_0 \) is a coincidence of the pair \( (f, g) \) then the set \( p^{-1}(x_0) \cap \text{Coin}(\tilde{f}, \tilde{g}) \) can be partitioned in \( \frac{\#\text{Coin}(f_{*x_0}, g_{*x_0})}{\#j_{x_0}(\text{Coin}(f_\#, g_\#, x_0))} \) disjoint subsets, each of which contain Nielsen related (therefore contained in the same coincidence class of the pair \( (\tilde{f}, \tilde{g}) \)). Moreover, no two points of different subsets are Nielsen related. \( \square \)

**Lemma 3.6.** Let \( x_0, x_1 \) be coincidence points in the same coincidence class of the pair \( (f, g) \), \( \overline{\alpha} \) be a path from \( x_0 \) to \( x_1 \) that realizes the Nielsen relation, \( \tilde{x}_0, \tilde{x}_0' \) be coincidence points of the pair \( (\tilde{f}, \tilde{g}) \) such that \( p(\tilde{x}_0) = p(\tilde{x}_0') = x_0 \), and \( \gamma \) the unique element of \( \mathcal{D}(M) \) such that \( \gamma(\tilde{x}_0) = \tilde{x}_0' \). If \( \tilde{\alpha} \) and \( \tilde{\alpha}' \) are the two liftings of \( \overline{\alpha} \) starting at \( \tilde{x}_0 \) and \( \tilde{x}_0' \) respectively then:

- (i) \( \tilde{\alpha}(1) \) and \( \tilde{\alpha}'(1) \) are coincidence points of the pair \( (\tilde{f}, \tilde{g}) \);
- (ii) \( \tilde{\alpha}(1) = \tilde{\alpha}'(1) \) is in the same coincidence class as \( \tilde{x}_0 \) \( (\tilde{x}_0') \). \( \square \)
- (iii) \( p(\tilde{\alpha}(1)) = p(\tilde{\alpha}'(1)) = x_1 \);
(iv) $\gamma(\alpha(1)) = \alpha'(1)$.

(v) If $\alpha$ is a graph orientation-reversing-path (in this case $x_0 Rx_1$) then $\alpha(1)$ and $\alpha'(1)$ are graph orientation-reverse-paths (in this case $\tilde{x}_0 R\tilde{x}_1$ and $\tilde{x}'_0 R\tilde{x}'_1$).

**Proof:** (i), (ii) and (iii) are known (we can prove them using covering space theory). To prove (iv), we take $\gamma \in \pi_1(M, x_0)$ such that $\mathcal{D}(\tilde{M}) \ni [\gamma] = \gamma$ and we can see that projection of the path $\pi \circ \gamma \circ \pi^{-1} \in \pi_1(M, x_1)$ in $\mathcal{D}(\tilde{M})$ coincides with $\gamma$.

To prove (v), we use [Dobreńko & Jezierski, lemma 2.1, page 77].

**Theorem 3.7.** Let $M$ and $N$ be compact, closed manifolds of the same dimension, $f, g : M \to N$ be two maps such that $\text{Coin}(f, g)$ is finite, and $p : \tilde{M} \to M$ and $q : \tilde{N} \to N$ be finite coverings such that there exist lifts $\tilde{f}, \tilde{g} : \tilde{M} \to \tilde{N}$ of the pair $(f, g)$. If $\tilde{C}$ is a coincidence class of the pair $(\tilde{f}, \tilde{g})$ then $C = p(\tilde{C})$ is a coincidence class of the pair $(f, g)$ and

$$|\text{ind}|(\tilde{f}, \tilde{g}; \tilde{C}) = \begin{cases} s \cdot k \pmod{2} & \text{if } C \text{ is defective;} \\ s \cdot k & \text{otherwise,} \end{cases}$$

where $s = |\text{ind}|(f, g, C)$, $k = \#j(\text{Coin}(f_#, g_#)_{x_0})$ and $x_0 \in C$.

**Proof:** The fact that $C = p(\tilde{C})$ is a coincidence class of the pair $(f, g)$ is known. We choose a point $x_0 \in C$. Since $\text{Coin}(f, g)$ is finite we can suppose that a decomposition $C = \{x_1, x_2, \ldots, x_s; c_1, c_1', c_2, c_2', \ldots, c_n, c_n'\}$ is such that each $x_i$ is free, and for all pairs $c_j, c_j'$ we have $c_j R c_j'$.

Now we choose paths $\{\alpha_i\}_i$, $2 \leq i \leq s$; $\{\beta_j\}_j$ and $\{\gamma_j\}_j$, $1 \leq j \leq n$ (see figure 1) such that:

- $\alpha_i$ is a path in $M$ from $x_1$ to $x_i$ that realizes the Nielsen relation.
- $\beta_j$ is a path in $M$ from $x_1$ to $c_j$ that realizes the Nielsen relation.
- $\gamma_j$ is a graph-orientation-reversing path in $M$ from $c_j$ to $c_j'$.

![Figure 1](image.png)

**Figure 1.** The class $C$ and the chosen paths.

For each element $\{x^k_i\}$ of $p^{-1}(x_1) \cap \text{Coin}(\tilde{f}, \tilde{g})$ (by Corollary 3.2 there were $\#\text{Coin}(\tilde{f}_{s,x_1}, \tilde{g}_{s,x_1})$ such elements) we take lifts $\{\tilde{\alpha}^k_i\}_{i,k}$, $\{\tilde{\beta}^k_j\}_{j,k}$ and $\{\tilde{\gamma}^k_j\}_{j,k}$ of the paths $\{\alpha_i\}_i$, $\{\beta_j\}_j$ and $\{\gamma_j \circ \beta_j\}_j$ respectively, starting at $x^k_i$.

Using Corollary 3.5 and Lemma 3.6 at the point $x_1 \in C$ we obtain that the set $p^{-1}(C) \cap \text{Coin}(\tilde{f}, \tilde{g})$ is the union of $\#\text{Coin}(\tilde{f}_{s,x_1}, \tilde{g}_{s,x_1})$ copies.
of the class \( C \) and these copies can be divided in disjoint coincidence classes of the pair \((\tilde{f}, \tilde{g})\). Each one of these classes contains \( \#j_{x_1}(\text{Coin}(f, g)) \) copies of the class \( C \).

The result follows by Corollary 3.3.

4. Two folded orientable covering

Let \( M \) and \( N \) be manifolds of same dimension with \( M \) nonorientable and \( N \) orientable; \( f, g : M \to N \) be two maps such that \( \text{Coin}(f, g) \) is finite, and \( p : \tilde{M} \to M \) be the two-fold orientable covering of \( M \). We define \( \tilde{f}, \tilde{g} : \tilde{M} \to N \) by \( \tilde{f} = f \circ p \) and \( \tilde{g} = g \circ p \).

Lemma 4.1. Under the above conditions, if \( C \) is a coincidence class of the pair \((f, g)\) then \( p^{-1}(C) \subset \text{Coin}(\tilde{f}, \tilde{g}) \) is such that:

1. \( p^{-1}(C) \) can be divided in two disjoint sets \( \tilde{C} \) and \( \tilde{C}' \), of the same cardinality, such that \( p(\tilde{C}) = p(\tilde{C}') = C \).
2. If \( \tilde{x}_1, \tilde{x}_2 \in \tilde{C} \) (or \( \tilde{C}' \)) then \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are in the same coincidence class of \((\tilde{f}, \tilde{g})\)
3. \( \tilde{C} \) and \( \tilde{C}' \) are in the same coincidence class of the pair \((\tilde{f}, \tilde{g})\) if and only if \( C \) is defective.

Proof: We need only to take \( q : \tilde{N} \to N \) as the identity map in the Corollary 3.2, Corollary 3.4 and Lemma 3.6.

Corollary 4.2. Under the hypotheses of lemma 4.1 we have:

1. If \( C \) is not defective then \( \tilde{C} \) and \( \tilde{C}' \) are two coincidence classes of the pair \((\tilde{f}, \tilde{g})\) with \( \text{ind}((\tilde{f}, \tilde{g}), \tilde{C}) = -\text{ind}(\tilde{f}, \tilde{g}, \tilde{C}') \) and \( |\text{ind}((\tilde{f}, \tilde{g}), \tilde{C})| = |\text{ind}(f, g, C)| \).
2. If \( C \) is defective then \( \tilde{C} \cup \tilde{C}' \) is a unique coincidence class of the pair \((\tilde{f}, \tilde{g})\) with \( \text{ind}(\tilde{f}, \tilde{g}, \tilde{C} \cup \tilde{C}') = 0 \).

Proof: The first part is a consequence of lemma 4.1 and the second can be proved using the fact that \( \tilde{M} \) is the double orientable covering of \( M \). It is useful to remember that the pair \((\tilde{f}, \tilde{g})\) is a pair of maps between orientable manifolds.

Corollary 4.3. Under hypotheses of lemma 4.1 we have:

1. \( L(\tilde{f}, \tilde{g}) = 0 \);
2. \( N(\tilde{f}, \tilde{g}) \) is even;
3. \( N(f, g) \geq N(\tilde{f}, \tilde{g}) / 2 \).
(4) If $N(\tilde{f}, \tilde{g}) = 0$ then all coincidence classes with nonzero semi-index of the pair $(f, g)$ are defective.

**Proof:** We need only to see that $p(Coin(\tilde{f}, \tilde{g})) = Coin(f, g)$, and that in the pair $(\tilde{f}, \tilde{g})$ we "lose" the defective classes.  

5. APPLICATIONS

**Theorem 5.1.** Let $f, g : M \rightarrow N$ be two maps between compact closed manifolds of the same dimension such that $M$ is nonorientable and $N$ is orientable. Suppose that $N$ is such that for all orientable manifolds $M'$ of the same dimension of $N$ and all pairs of maps $f', g' : M' \rightarrow N$ we have $L(f', g') = 0 \Rightarrow N(f', g') = 0$. Then all coincidence classes with nonzero semi-index of the pair $(f, g)$ are defective.

**Proof:** The hypotheses on $N$ are enough to show, using the notation of the proof of theorem 4.1, that $N(\tilde{f}, \tilde{g}) = 0$. So by corollary 4.3 all coincidence classes with nonzero semi-index of the pair $(f, g)$ are defective.  

We note that the hypotheses on the manifold $N$ in theorem 5.1 in dimension greater then 2, are equivalent to the converse of Lefschetz theorem. In dimension 2 these hypotheses are not equivalent but necessary for the converse of Lefschetz theorem.

**Remark 5.2.** The following manifolds satisfy the hypotheses on the manifold $N$ in the theorem 5.1:

1. Jiang spaces ([Gonçalves & Wong 1, Corollary 1]).
2. Nilmanifolds ([Gonçalves & Wong 2, Theorem 5]).
3. Homogeneous spaces of a compact connected Lie group $G$ by a finite subgroup $K$ ([Gonçalves & Wong 1, Theorem 4]).
4. Suitable manifolds ([Wong, Theorem 3 e Theorem 4]).

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