A Collection of Results Relating the Geometry of Plane Domains and the Exit Time of Planar Brownian Motion

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Received: 3 September 2021 / Revised: 15 March 2022 / Accepted: 31 March 2022 / Published online: 3 June 2022

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Abstract

We prove a number of results relating exit times of planar Brownian motion with the geometric properties of the domains in question. Included are proofs of the conformal invariance of moduli of rectangles and annuli using Brownian motion; similarly probabilistic proofs of some recent results of Karafyllia on harmonic measure on starlike domains; examples of domains and their complements which are simultaneously large when measured by the moments of exit time of Brownian motion, and examples of domains and their complements which are simultaneously small; and proofs of several identities involving the Cauchy distribution using the optional stopping theorem.

Keywords Brownian motion · Analytic functions · Harmonic measure · Exit times

Mathematics Subject Classification 30A99 · 60J65

1 Introduction

In this paper, we prove a series of loosely connected results relating the geometry of domains in the plane with the distribution of the exit time of planar Brownian motion from those domains. Results of this type seem to have originated in such seminal papers as [7, 11], and continue to attract current researchers (see the references contained in [22] for a large list). We first informally list our results.
In Sect. 2, we show how Brownian motion can be used to prove a staple of complex analysis, which is the conformal invariance of the modulus of a rectangle, or of an annulus (Theorems 2.1, 2.3 below). In other words, conformal maps between rectangles which fix the vertices cannot change the ratio of the side lengths, and conformal maps between annuli cannot change the ratio of the radii of the annuli.

In Sect. 3, we show how Brownian motion can be used to prove some recent analytic results on harmonic measure on starlike domains due to Karafyllia [14] (Theorem 3.4). We also present a small generalisation (Theorem 3.5) and discuss related topics.

In Sect. 4, we discuss the relationship between moments of Brownian exit time and the Hardy norm of domains. We discuss how these quantities can indicate the size of a domain at infinity, but also discuss potentially surprising examples of domains and their complements which are simultaneously large when measured by the moments of exit time of Brownian motion, and examples where domains and their complements are simultaneously small (Theorem 4.10).

In the Sect. 5, we show how the optional stopping theorem can be used to prove a number of identities concerning the Cauchy distribution which are proved in the literature by other methods (Propositions 5.1, 5.4).

We now fix notation. $B_t$ will always denote a planar Brownian motion, and given a domain $U$ let $\tau(U)$ denote the exit time of planar Brownian motion from $U$; that is, $\tau(U) = \inf\{t \geq 0 : B_t \notin U\}$. The next theorem is the core of planar Brownian motion theory (see [23] for a proof).

**Theorem 1.1** (Lévy [19]) If $f$ is a non constant analytic function and $B_t$ is a planar Brownian motion starting at $a$ then $f(B_t)$ is a time-changed Brownian motion starting at $f(a)$, where the time change rate is governed by

$$\sigma(t) := \int_0^t |f'(B_s)|^2 ds.$$  

In other words, the process $f(B_{\sigma^{-1}(t)})$ is a planar Brownian motion.

Lévy’s Theorem is often referred to as the conformal invariance principle, although a more accurate name might be the analytic invariance principle, since injectivity plays no role in the theorem. This result will play the major role in everything that is to follow.

## 2 Exit Distributions of Brownian Motion from Rectangles and Annuli, and Applications to Conformal Maps

We will use the following notation for a rectangle (with $a, b > 0$).

$$R_{a,b} := (-a, a) \times (-b, b).$$

Fix a rectangle $R_{a,b}$, and label the vertices clockwise as $z_1, \ldots, z_4$, starting with $z_1 = a + bi$. Now suppose that we have a similarly defined rectangle $R_{a',b'}$ with
vertices \( z_1', \ldots, z_4' \), and that there is a conformal map \( f \) between them such that \( f(z_j) = z_j' \) for \( j = 1, \ldots, 4 \); note that in making this last statement we are implicitly using Carathéodory’s Theorem (see [18]), which implies that such a conformal map between Jordan domains extends to a homeomorphism between the boundaries. The following is a standard result in complex analysis.

**Theorem 2.1** Given the setup described above, we have \( a/b = a'/b' \), and furthermore \( f(z) = cz \), where \( c \) is a positive constant.

There are at least two known proofs of this. The standard one uses the concept of extremal length (see [1, 13, 17]). The second, which is not as widely known, is to repeatedly apply Schwarz’s reflection principle in order to extend \( f \) to a conformal self-map of \( \mathbb{C} \). We present here a third, using the conformal invariance of Brownian motion.

**Proof** Label the right side of \( R_{a,b} \) as \( S_1 \), and then label the remaining sides \( S_2, S_3, S_4 \) clockwise. Label the sides of \( R_{a',b'} \) similarly as \( S_1', \ldots, S_4' \). Let

\[
W = \{ z \in R_{a,b} : P_z(B_\tau(R_{a,b})) \in S_1 \} = P_z(B_\tau(R_{a,b})) \in S_3). \]

It is clear by symmetry that \( i\mathbb{R} \cap R_{a,b} \subseteq W \), where \( i\mathbb{R} \) denotes the imaginary axis. It is also not hard to see that \( W \subseteq i\mathbb{R} \cap R_{a,b} \), since a Brownian motion starting from some point not on \( i\mathbb{R} \cap R_{a,b} \) has a positive probability to exit on the nearest vertical side before hitting \( i\mathbb{R} \cap R_{a,b} \), but if it hits \( i\mathbb{R} \cap R_{a,b} \) before hitting the nearest vertical side it has equal probabilities of exiting on \( S_1 \) and \( S_3 \) after that by symmetry and the strong Markov property. We conclude that \( W = i\mathbb{R} \cap R_{a,b} \). By the conformal invariance of Brownian motion, for \( z \in W \) we must have

\[
P_{f(z)}(B_\tau(R_{a',b'})) \in S_1' = P_{f(z)}(B_\tau(R_{a',b'})) \in S_3', \]

and this implies that \( f(i\mathbb{R} \cap R_{a,b}) = i\mathbb{R} \cap R_{a',b'} \). The same argument, applied to the horizontal sides, shows that \( f(R_{a,b}) = R_{a',b'} \). In particular, we see that \( f(0) = 0 \). Furthermore, \( R_{a,b} \) and \( R_{a',b'} \) are divided into four smaller rectangles by the real and imaginary axes, and we have shown that \( f \) maps each of these rectangles in \( R_{a,b} \) onto the corresponding one in \( R_{a',b'} \). Applying the same argument to each of these smaller rectangles, and then iterating the argument shows that \( f(j2^{-n}a + k2^{-n}bi) = j2^{-n}a' + k2^{-n}b'i \), where \( n \) is any positive integer and \( j, k \) are any odd integers with \( |j|, |k| < 2^n \). These points form a dense set in \( R_{a,b} \) and \( f \) is continuous, so \( f \) must be the map \((x, y) \mapsto (a'/a)x, (b'/b)y)\). However, the Cauchy–Riemann equations now imply that \( a'/a = b'/b \), and the result follows.

\( \square \)

There is a well-known analogue for an annulus (again, see [1, 13], or [17]). This states that if there is a conformal map between two annuli then the ratios of their inner and outer radii must be the same, and that the map must be either a linear map or an inversion. Two standard proofs are well-known which parallel the ones for the rectangle, one using extremal length and the other using Schwarz reflection. We will offer here a third using Brownian motion, and will note that our proof yields a more general result which applies to analytic functions which are not necessarily univalent. To state this generalisation, we need the definition of a proper map.
Definition 2.2. We say that an analytic function \( f : U \to W \) is proper if for any compact set \( K, f^{-1}(K) \) is also compact.

Definition 2.2 extends to any continuous function between two topological spaces [27]. The interpretation of properness is the following: If \( (z_n)_n \) is a sequence of \( U \) that converges to \( \partial U \), i.e., the complement of any compact set of \( U \) contains all but a finite number of \( (z_n)_n \), then \( (f(z_n))_n \) converges necessarily to \( \partial W \). Often, we express that as \( f \) maps boundary to boundary. Note that conformal maps are automatically proper, since \( f^{-1} \) is itself conformal.

We will use the following notation for an annulus

\[
A_{r,R} := \{z \mid r < |z| < R\}.
\]

Our generalisation is as follows.

Theorem 2.3. Suppose \( f : A_{r,R} \to A_{r',R'} \) is a proper analytic function. Then there is a positive integer \( n \) such that \( R^n/r^n = R'/r' \), and \( f \) is of the form \( f(z) = \xi z^n \) or \( f(z) = \xi/z^n \) for some constant \( \xi \).

Proof. We will consider a Brownian motion \( B_t \) running in \( A_{r,R} \), and the image time-changed Brownian motion \( B'_t \) running in \( A_{r',R'} \). Since \( f \) is proper, one can see that

\[
\sigma(\tau(A_{r,R})) = \tau'(A_{r',R'}),
\]

where \( \tau'(A_{r',R'}) \) denotes the exit time of the Brownian motion \( B'_t = f(B_{\sigma^{-1}(t)}) \). Fix \( a \in A_{r,R} \) and choose \( r'', R'' \) such that \( r' < r'' < |f(a)| < R' < R' \). Since \( f \) is proper, \( f^{-1}(\partial(A_{r'',R''})) \) is compact (here \( \partial \) denotes the closure), and we may therefore find \( \tilde{r} > r \) and \( \tilde{R} < R \) such that \( f^{-1}(\partial(A_{r'',R''})) \subseteq A_{\tilde{r},\tilde{R}} \). The sets \( f(|z| = \tilde{r}) \) and \( f(|z| = \tilde{R}) \), must lie in the complement of \( A_{r'',R''} \), however their union must separate \( f(a) \) from the boundary of \( A_{r',R'} \); to see this, note that the set \( \{B_t : 0 \leq t \leq \tau(A_{r,R})\} \) must intersect \( \{z = \tilde{r}\} \cup \{z = \tilde{R}\} \), and thus \( \{B'_t : 0 \leq t \leq \tau'(A_{r',R'})\} \) must intersect \( f(|z| = \tilde{r}) \cup f(|z| = \tilde{R}) \).

Now, as continuous images of connected sets, each of \( f(|z| = \tilde{r}) \) and \( f(|z| = \tilde{R}) \) is connected. Since they do not intersect \( A_{r'',R''} \), each of them must therefore lie in just one of \( \{|z| < r''\} \) and \( \{|z| > R''\} \). We can then, if necessary, replace \( f \) by \( Rr/f \) and assume that \( f(|z| = \tilde{r}) \subseteq \{|z| < r''\} \) and \( f(|z| = \tilde{R}) \subseteq \{|z| > R''\} \). Now let \( \tilde{r} \setminus r \) and \( \tilde{R} \setminus R' \); as \( f \) is proper, the distances from the images \( f(|z| = \tilde{r}) \) and \( f(|z| = \tilde{R}) \) to the boundary of \( A_{r',R'} \) must go to 0, and we conclude the following equality of events:

\[
\{|B_{\tau(A_{r,R})}| = r\} = \{|B'_{\tau'(A_{r',R'})}| = r'\}
\]

\[
\{|B_{\tau(A_{r,R})}| = R\} = \{|B'_{\tau'(A_{r',R'})}| = R'\}
\]

Fix \( \theta \in (0,1) \), and let \( W = \{a : P_a(|B_{\tau(A_{r,R})}| = r) = \theta\} \). It is well known that \( W \) is the circle \(|z| = R - \theta \ln(R/r)| \); see for example [23, Thm. 3.17]. By conformal invariance, we must have \( f(W) \subseteq W' \), where \( W' = \{a : P_a(|B'_{\tau'(A_{r',R'})}| = r') = \theta\} = \{|z| = R' - \theta \ln(R'/r')| \}. Rearranging and dividing yields

\[
\frac{\ln |f(z)/R'|}{\ln |z/R|} = \frac{\ln |R'/r|}{\ln |R/r|}.
\]
Setting $\eta := \ln|R'/r|/\ln|R/r|$, we obtain

$$|f(z)| = \frac{R'}{R\eta}|z|^\eta.$$ 

We want to now consider $f(z)/z^\eta$ as an analytic function, however it cannot be defined to be analytic on the entire annulus unless $\eta$ is an integer. Since we do not, as of yet, know that it is an integer, we solve this problem by removing the slit $S = \{\text{Im}(z) = 0, -R < \text{Re}(z) < -r\}$, from the annulus, and choose a branch of the function $z \rightarrow z^\eta$ which is analytic on $A_{r, R} \setminus S$. On this domain, the function $f(z)/z^\eta$ is indeed analytic, and as it has constant absolute value it must be a constant by the open mapping theorem for analytic functions. Thus, $f(z) = \xi z^\eta$ for some constant $\xi$. However, $f$ is analytic on all of $A_{r, R}$, and the result now follows upon recalling again that $z \rightarrow z^\eta$ can only be analytic on the annulus if $\eta$ is an integer $n$. $\square$

**Remark** The ratio of the two dimensions of a rectangle (resp. inner and outer radii of an annulus) is referred to as the *modulus* of the rectangle (resp. annulus), and these results imply that it is a conformal invariant. Interestingly, a different probabilistic interpretation of modulus has recently been given in [2]. The approach taken there is quite different from ours, and makes use of discretization. We should also mention that Theorem 2.3 follows from the purely analytic arguments given in [26, p. 216]

### 3 Brownian Motion and Starlike Domains

The main purpose of this section is to examine a recent result of Karafyllia [14] on the harmonic measure of starlike domains. We will show how the result can be extended, and will give a proof using Brownian motion. First, we need some definitions.

**Definition 3.1** (*Harmonic measure*) For a simply connected domain $D$ and a subset $A \subset \partial D$, we define the harmonic measure as

$$\omega_D(a, A) := \mathbb{P}_a(B_{\tau(D)} \in A)$$

**Remark** The harmonic measure can also be defined analytically, i.e. without mentioning Brownian motion. This is the definition used by Karafyllia in [14], but the two definitions are equivalent.

**Definition 3.2** (*Starlike domain*) An open set $D$ on $\mathbb{C}$ is starlike (with respect to $z_0$) if there exists some $z_0 \in D$ such that $\{(1 - t)z_0 + tz : t \in [0, 1]\} \subset D$ for all $z \in D$.

In other words, the line segment between $z_0$ and any $z \in D$ lies entirely within $D$.

**Definition 3.3** (*Starlike at infinity domain*) A domain $U$ is starlike at infinity if, given any $z \in U$, the horizontal ray

$$\{w : \text{Im}(w) = \text{Im}(z), \text{Re}(w) \leq \text{Re}(z)\}$$

lies entirely in $U$. \[ Springer \]
Remark The reader may find this notion in other places in the literature with such names as horizontally convex.

The following result was proved analytically for starlike domains by Karafyllia [14], in order to answer a question posed by Betsakos in [4, 5].

**Theorem 3.4** Let $D$ be a domain in $\mathbb{C}$ which is starlike with respect to 0. Then for every $R > 0$,

$$
\hat{\nu}_D(0, R) \leq 2\nu_{0,D}(R),
$$

where $\nu_D(0, R) := \omega_D(0, \partial D \cap \{|z| > R\})$, $\hat{\nu}_D(0, r) := \omega_{D \cap \{|z| > r\}}(0, D \cap \{|z| = R\})$. The constant 2 is the best possible.

Now a generalisation is offered and proved using the reflection principle of Brownian motion.

**Theorem 3.5** Let $U$ be a starlike at infinity domain in $\mathbb{C}$. Then for every $r \in \mathbb{R}$, $a \in \{\text{Re}(z) < r\}$, we have

$$
\hat{\nu}_U(a, r) \leq 2\nu_U(a, r),
$$

where

$$
\nu_U(a, r) := \omega_U(a, \partial U \cap \{|\text{Re}(z)| > r\}),
\hat{\nu}_U(a, r) := \omega_{U \cap \{|\text{Re}(z)| < r\}}(a, U \cap \{|\text{Re}(z)| = r\}).
$$

The constant 2 is the best possible.

Note that Theorem 3.4 is an immediate consequence of Theorem 3.5, as follows.
Proof of Theorem 3.4 assuming Theorem 3.5: Assume $D$ is starlike with respect to 0. We can find a starlike at infinity $D'$ with periodic boundary of period $2\pi i$ such that $D \setminus \{0\}$ is the image of $D'$ through the complex exponential map. The image of a Brownian motion starting at $a \in D'$ will be a Brownian motion starting at $e^a \in D \setminus \{0\}$, and with probability 1 the exit time of this Brownian motion from $D$ is the same as from $D \setminus \{0\}$ as planar Brownian motion doesn’t hit points (see [22, Sect. 3.1]). The set $\{\text{Re}(z) = \ln R\} \cap D'$ is mapped to $\{|z| = R\} \cap D$ by the exponential map, and Theorem 3.4 then follows. To include the origin as starting point of the Brownian motion in $D$, simply take a point near the origin, conclude the theorem for that starting point, and then let the point approach the origin.

Remark Note that this proof shows that the harmonic measure can be taken with respect to any point in $a \in \{|z| < R\}$, rather than just 0. That is, given the conditions of Theorem 3.4, we have proved

$$\hat{\nu}_D(a, R) \leq 2\nu_{a, D}(R)$$

for any $a \in \{|z| < R\}$.

Proof of Theorem 3.5 It suffices to assume $r = 0$. Let $\mathbb{I} := \{z : \text{Re}(z) = 0\}$, be the imaginary axis, $\tau_0 := \inf\{t : B_t \notin \{\text{Re}(B_t) < 0\} \cap U\}$ the time that the Brownian motion exits either $U$ or the negative half plane, and as before $\tau(U)$ is the exit time of $B_t$ from $U$. Define

$$\hat{B}_t = \begin{cases} B_t & t \leq \tau_0 \\ -\bar{B}_t & t > \tau_0 \end{cases}.$$

By the reflection principle for planar Brownian motion [22], $\hat{B}_t$ is a Brownian motion. Let $\hat{\tau}(U)$ be the exit time of $\hat{B}_t$ from $U$. One can see, as a consequence of being within a starlike at infinity domain, if the Brownian motion exits to the left of $\mathbb{I}$ then its reflection must have exited to the right of $\mathbb{I}$, however the converse is not true. That is:

$$\{\text{Re}(\hat{B}_{\hat{\tau}(U)}) < 0, \tau(U) > \tau_0\} \subseteq \{\text{Re}(B_{\tau(U)}) > 0, \tau(U) > \tau_0\}.$$

Figure 2 captures this idea, that once the Brownian motion hits $\mathbb{I}$ it will be at least as likely to exit to the right of this line than to the left.

Note also that, up to a set of probability 0, $\{\tau(U) > \tau_0\}$ is the disjoint union of $\{\text{Re}(B_{\tau(U)}) < 0, \tau(U) > \tau_0\}$ and $\{\text{Re}(B_{\tau(U)}) > 0, \tau(U) > \tau_0\}$. Using this and the fact that since $\hat{B}$ is a Brownian motion,

$$P(\text{Re}(B_{\tau(U)}) < 0, \tau(U) > \tau_0) \leq P(\text{Re}(B_{\tau(U)}) > 0, \tau(U) > \tau_0),$$

we obtain

$$P(\text{Re}(B_{\tau(U)}) > 0, \tau(U) > \tau_0) \geq \frac{1}{2} P(\tau(U) > \tau_0).$$
The result now follows upon noting that

$$
\hat{v}_U(a, 0) = P(\tau(U) > \tau_0) \geq P(\text{Re}(B_{\tau(U)}) > 0, \tau(U) > \tau_0)
\geq P(\text{Re}(B_{\tau(U)}) > 0) = v_U(a, 0)
$$

Note that the second to last equality of the previous line follows from the continuity of the Brownian paths.

**Remark** We note that one may obtain Karafyllia’s result directly in a similar manner (avoiding Theorem 3.5) by reflecting the Brownian motion over the circle of radius $R$. Such a reflection does preserve Brownian motion but also introduces a time change, however that is not relevant for our problem since we are only interested in the distribution of the Brownian motion at time $\tau$, rather than the distribution of $\tau$ itself. The constant 2 in our result is optimal, and occurs when the starlike at infinity domain is a horizontal strip or half-plane with horizontal boundary. We note that the constant 2 is also optimal in Karafyllia’s result as well, but in that case the inequality is always strict; that is, the constant 2 is never obtained for a given domain. This is because the starlike domain can never actually equal its reflection over the circle, since 0 is in the starlike domain but $\infty$ is in its complement.
4 Domains with Constraints on the Exit Moments

4.1 Hardy Spaces and $p$-th Moments of Exit Times

The theory of Hardy spaces provides another point of intersection between the theory of Brownian motion and complex analysis. Hardy spaces provide an analytic way of putting a measure on the size of the domain as seen by Brownian motion through its exit time out of the domain. Moreover, Hardy spaces play a major role when it comes to the problem of finiteness of the moments of the exit times. To this end, let $f$ be an analytic function on the unit disc and for any $p > 0$ and $0 \leq r < 1$ set

$$N_{p,r}(f) := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$  

The quantity $N_{p,r}(f)$ can be interpreted as the $L^p$ norm\(^1\) of the function $\theta \mapsto f(re^{i\theta})$. It can be shown, using harmonic analysis techniques, that $N_{p,r}(f)$ is non-decreasing in terms of $r$ \cite{27}. Hence, we are ready now to define the $p$th-Hardy norm.

**Definition 4.1** For any analytic function $f$ on the unit disc, the $p$th-Hardy norm of $f$ is defined by

$$H_p(f) := \sup_{0 \leq r < 1} N_{p,r}(f) = \sup_{0 \leq r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$  

The set of analytic functions whose $p$th-Hardy norm is finite is denoted by $\mathcal{H}^p$ and called Hardy space (of index $p$). A crucial result about Hardy norms is that, if $H_p(f)$ is finite then $f$ has a radial extension to the boundary. More precisely $f^*(z) := \lim_{r \to 1} f(rz)$ exists a.e for all $z \in \partial \mathbb{D}$ and it belongs to $L^p$ as well. More details about the topic can be found in \cite{12}. A consequence of Hölder’s inequality is the inclusion $\mathcal{H}^q \subseteq \mathcal{H}^p$ whenever $0 < p \leq q$. This leads to the following definition.

**Definition 4.2** (Hardy number) Given a simply connected domain $W$ and conformal $f : \mathbb{D} \to W$, the Hardy number, $H(W)$, of $W$ is defined as

$$H(W) := \sup\{p > 0 | H_p(f) < \infty\}.$$  

Although the definition is in terms of $f$ it can be shown that the Hardy number does not depend on the particular choice of $f$ and so it is well defined. We will be interested in the Hardy numbers of a particular type of domain, namely Jordan* domains, and we now work towards the definition of these.

**Definition 4.3** (Jordan curve) A Jordan curve is a closed curve in $\mathbb{C}$ which is the homeomorphic image of a circle.

\(^1\) The word norm is an abuse of language as $N_{p,r}(f)$ is not a true norm when $p < 1.$
Definition 4.4 (Jordan domain) A domain whose boundary is a Jordan curve is called a Jordan domain.

Note that Jordan domains are automatically simply connected. The question of calculating the Hardy number of a Jordan domain is not interesting, because a Jordan domain is bounded and it follows easily that its Hardy number is infinite. The question is more interesting for Jordan* domains, which were previously considered in [20].

Definition 4.5 (Jordan* Domain) A Jordan* domain is a domain in \( \mathbb{C} \) which is the image of Jordan domain \( U \) under a Mobius transformation which takes a boundary point of \( U \) to \( \infty \).

Essentially we are modifying the class of Jordan domains to require \( \infty \) to be a boundary point, and the Hardy number of Jordan* domains is no longer trivial. Alternatively, Jordan* domains could be defined to be domains whose boundary contains \( \infty \) and is homeomorphic (as a set in the Riemann sphere) to a circle. An important paper which relates the theory of Hardy spaces to the exit time of Brownian motion is that of Burkholder [7] and we will implement from it without proof the following result.

Theorem 4.6 Suppose \( f \) is a conformal map taking the unit disc to a simply connected domain \( R \). Then for all \( x \in R \),

\[
f \in H^{2p} \iff \mathbb{E}_x[\tau(R)^p] < \infty \iff \mathbb{E}_x[|B_{\tau(R)}|^{2p}] < \infty.
\]

The \( p \)-Hardy norm can be seen as the expectation of the \( p \)th moment of \( f(B_T(\mathbb{D})) \) as \( B_T(\mathbb{D}) \) is uniformly distributed on the circle. Thus, the equivalence of the finiteness of the \( p \)th moment of the stopped Brownian motion and the \( 2p \)th Hardy norm of a conformal map can be seen as a consequence of the conformal invariance of Brownian motion. Another crucial fact about Theorem 4.6, proved by Burkholder in the same paper, is that the finiteness of \( \mathbb{E}_x[\tau(R)^p] \) and \( \mathbb{E}_x[|B_{\tau(R)}|^{2p}] \) does not depend on the starting point \( x \) [7]. The following is another important fact about the \( p \)th moment of the exit time of a Brownian motion exiting any simply connected domain, which is a consequence of the corresponding result for Hardy norms (this was initially proved in [25] but appears also in [12]).

Theorem 4.7 [7] If \( R \) is simply connected and not the whole complex plane then, for all \( z \in R \),

\[
\mathbb{E}_z[\tau(R)^p] < \infty \text{ for any } p < 1/4.
\]

In other words, the minimum Hardy number for any simply connected domain is \( 1/2 \). This is therefore considered an extremal case with regards to the size of a simply connected domain as seen by Brownian motion. The Koebe domain, \( K := \mathbb{C}\backslash(-\infty, -1/4] \) is considered in many ways extremal among simply connected domains, and it has infinite \( 1/4 \)th moment of \( \tau_K \) and thus a Hardy number of \( 1/2 \) (this is proved in [12], and can also be proved using Burkholder’s result and calculations similar to Example 4.8 below). There are examples of smaller domains which also have infinite \( 1/4 \)th moment. One of these will be important in what follows, so we take the time to describe it.
Example 4.8 Consider a half-strip, for example \{\Re(z) < 0, -1 < \Im(z) < 1\} and label the complement of this half strip \(V\). Now consider the complex plane split into separate domains by the parabola \(x = 1 - y^2/4\). We label the right of the two domains \(W\), that is \(W = \{x > 1 - y^2/4\}\), and notice \(W \subset V\). Also consider the conformal map from the unit disk to \(W\), \(f(z) = 4/(1 + z)^2\). We see

\[
H_p(f)^p = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^{2p} dt = \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{4}{(1 + e^{it})^2}\right|^{2p} dt.
\]

This integral is finite only when \(p < 1/4\), thus the \(2p\)-th-Hardy norm is infinite for \(p = 1/4\). By Theorem 4.6 we have \(f \notin H^{1/2}\), meaning \(E[\tau_W^{1/4}] = \infty\) and since \(W \subset V\), \(E[\tau_V^{1/4}] = \infty\) hence \(H(V) = 1/2\). We may translate the parabola as required such that any half strip may be fit inside, meaning any half strip will have infinite 1/4th moment of its exit time out of the domain and thus a Hardy number of 1/2.

As another application of Theorem 4.6 (applied to the conformal map \(f(z) = ((1 - z)/(1 + z))/\theta/\pi\)), let us consider the following result (which can also be deduced from the explicit formula for the distribution of \(\tau(W)\) given in [21]).

Proposition 4.9 [7] Let \(W = \{-\theta/2 < \Arg(z) < \theta/2\}\) be a wedge of aperture \(\theta \in (0, 2\pi]\). Then

\[E(\tau(W)^{p/2}) < +\infty \iff p < \frac{\pi}{\theta}.\]

Note that the Koebe domain can be considered as a wedge of size \(2\pi\) with \(\theta = 2\pi\) giving the required result of

\[E(\tau^{p/2}) < +\infty \iff p < \frac{1}{2},\]

resulting in a Hardy number of 1/2 as expected.

We will now give an intuitive discussion to motivate a natural conjecture that we have considered. Proposition 4.9 indicates that the aperture of a domain at \(\infty\) essentially decides whether the domain is viewed as “small” or “large” by Brownian motion; we will continue to use these terms in quotation marks to indicate their informal nature. Consider now a wedge \(V\) of size \(\theta \in (0, 2\pi)\) and its complement \(W\) of size \(\alpha = 2\pi - \theta\). Suppose that \(E(\tau_V^{p/2}) < +\infty\) for some \(p\). This means that \(V\) is “small”, so \(W\) must be “large”. To be precise, suppose \(1/p + 1/q \leq 2\); we will show that \(E(\tau_W^{q/2}) = +\infty\). We know \(p < \pi/\theta\), so

\[
2 > \frac{1}{q} + \frac{\theta}{\pi} = \frac{1}{q} + \frac{2\pi - \alpha}{\pi},
\]

hence \(q > \pi/\alpha\) and thus \(E(\tau_W^{q/2}) = +\infty\). On the other hand, if we suppose \(E(\tau_V^{p/2}) = +\infty\) (i.e. \(V\) is “large”) and \(1/p + 1/q > 2\) then we can show \(E(\tau_W^{q/2}) < +\infty\) (i.e. \(W\)
is “large”). To see this, note that $p \geq \pi/\theta$, so
\[
2 < \frac{1}{q} + \frac{\theta}{\pi} = \frac{1}{q} + \frac{2\pi - \alpha}{\pi},
\]
hence $q < \pi/\alpha$ and $\mathbb{E}(\tau_W^{q/2}) < +\infty$. These observations have led us to an attempt at
generalization, and the Jordan* domains appear to be ideally suited for this purpose,
since the Jordan Curve Theorem implies that the interior of the complement of a
Jordan* domain is again a Jordan* domain. The following conjecture is therefore
natural.

**Conjecture:** If $V$ and $W$ are complementary Jordan* domains (that is, they share the
same boundary), then they cannot both be "small" or both be "large" as viewed by
Brownian motion.

We have not attempted to make this statement more rigorous, since we have found
example domains that show that virtually any formulation of it will be false. We will
now take the time to describe them; in particular, we will prove the following theorem.

**Theorem 4.10**  (i) There exist complementary Jordan* domains $V$ and $W$ such that
$\mathbb{E}[\tau(V)^{1/4}] = \mathbb{E}[\tau(W)^{1/4}] = \infty$.

(ii) There exist complementary Jordan* domains $V$ and $W$ such that $\mathbb{E}[\tau(V)^p] < \infty$
and $\mathbb{E}[\tau(W)^p] < \infty$ for all $p > 0$.

Part 1 of the theorem provides an example of complementary Jordan* domains
which are both "large" as viewed by Brownian motion, and part 2 does the same with
both "small". We remind the reader that any Jordan* domain is simply connected and
therefore has Hardy number at least $1/2$. This implies that part 1 of the theorem cannot
be improved. The remainder of this subsection will provide the proof of part 1, while
part 2 will be proved in the next subsection.

**Proof** We will utilise the following result for domains, which is a straightforward
consequence of the monotone convergence theorem.

**Theorem 4.11** If $\Omega_n$ is an increasing sequence of domains (i.e. $\Omega_n \subseteq \Omega_{n+1}$) and
$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, then $\mathbb{E}[\tau(\Omega_n)^p] \to \mathbb{E}[\tau(\Omega)^p]$.

We will construct a sequence of domains $(W_n)_{n \in \mathbb{N}_0}$ and a sequence of each respec-
tive complement $(V_n)_{n \in \mathbb{N}_0}$ in which the limiting domains of each sequence will both
have infinite $1/4$th moment. In each $W_n$ we start a Brownian motion $B_t$ at $-1$ and in
each $V_n$ we start a Brownian motion $\hat{B}_t$ at $1$.

We begin with an extension of Example 4.8, in which we restrict the complement
of the half strip by the lines shown in Fig. 3
\[
\text{Re}(z) = b_1, \quad \text{Im}(z) = a_1 \quad \text{Im}(z) = -a_1,
\]
where $a_1 > a_0 := 1$. $V_1$ is a subset of the complement of a half strip and so, if
we take $a_1 \to \infty$ and $b_1 \to -\infty$, then $W_1$ becomes a half strip, so $V_1$ becomes the
complement of a half strip and by (4.11), $\mathbb{E}[\tau(V_1)^{1/4}] \to \infty$. Thus we may take $a_1, b_1$
large enough such that $E[\tau(V_1)] > 1$, and may also ensure that $a_1 \geq a_0 + 1$. Since the complement of a half strip can fit inside $W_1$, we have that $E[\tau(W_1)] = \infty$.

We can then restrict $W_1$ by the lines shown in Fig. 4

$$\text{Re}(z) = b_2, \quad \text{Im}(z) = a_2 \quad \text{Im}(z) = -a_2.$$ 

As $a_2 \to \infty$ and $b_2 \to \infty$, $W_2 \to W_1$ and we have that $W_2 \subset W_1$, meaning by Theorem 4.11,

$$E[\tau(W_2)] \to E[\tau(W_1)] = \infty.$$ 

We may then take these limits large enough such that $E[\tau(W_2)] > 2$ while again ensuring $a_2 \geq a_1 + 1$. Since the complement of a half strip can fit inside $V_2$, we also have that $E[\tau(V_2)] = \infty$.

We may continue iterating this process. In the odd iterations, for all $n \in \mathbb{N}_0$, as $a_{2n+1} \to \infty$, $b_{2n+1} \to -\infty$, then $V_{2n+1} \to V_{2n}$ and since $V_{2n+1} \subset V_{2n}$, by Theorem 4.11,

$$E[\tau(V_{2n+1})] \to E[\tau(V_{2n})] = \infty.$$ 

Thus, the limits may be taken large enough such that $E[\tau(V_{2n+1})] > 2n + 1$ while once again ensuring that $a_{2n+1} \geq a_{2n} + 1$. The complement of a half strip can fit inside $W_{2n+1}$ meaning $E[\tau(W_{2n+1})] = \infty$.

Following the same logic with the as the odd iterations we construct the even iterations such that $E[\tau(W_{2n+2})] > 2n + 2$ and $E[\tau(V_{2n})] = \infty$, with $a_{2n} \geq a_{2n-1} + 1$.

We show three more iterations in Fig. 5.
Notice \( V_{2n+1} \subset V_{2n+3} \) and \( W_{2n+2} \subset W_{2n+4} \) for all \( n \in \mathbb{N}_0 \). Thus we define the required domains

\[
V_\infty = \bigcup_{n=0}^{\infty} V_{2n+1} \quad W_\infty = \bigcup_{n=1} W_{2n+2}.
\]
By construction, the $a_n$’s go to $+\infty$, so $V_\infty$ and $W_\infty$ are both Jordan* domains. Since $(V_{2n+1})_{n=0}^\infty$ is an increasing sequence of domains, by Theorem 4.11 we have

$$\mathbb{E}[\tau(V_{2n+1})^{1/4}] \to \mathbb{E}[\tau(V_\infty)^{1/4}] = \infty.$$ 

The same is true of $W_\infty$ and thus in the limit we have a domain and its complement who both have infinite $1/4$th moment of Brownian exit time, as required. \qed

### 4.2 Hyperbolic Geometry and Hardy Numbers

We now construct a second counterexample to the conjecture in which the domain and its complement both have finite $p$th moment of the exit time for any $p > 0$. To construct this second counterexample we must recall theory from hyperbolic geometry [3, 16]. We first define hyperbolic distance on the unit disc, which is induced by the hyperbolic density.

**Definition 4.12 (Hyperbolic density)** The hyperbolic density on the unit disc $\mathbb{D}$ is defined as

$$\lambda_\mathbb{D}(z)|dz| = \frac{2|dz|}{1 - |z|^2}$$

The definition of hyperbolic density is motivated by the following isometric property

$$\lambda_\mathbb{D}(\phi(z))|\phi'(z)| = \lambda_\mathbb{D}(z)$$

for a conformal automorphism $\phi$. The hyperbolic density induces hyperbolic length and hyperbolic distance in the following way.

**Definition 4.13 (Hyperbolic length)** For any two points, $z, w \in \mathbb{D}$, we define the hyperbolic length $\ell_\mathbb{D}$ as

$$\ell_\mathbb{D}(\gamma) = \int_\gamma \lambda_\mathbb{D}(z)|dz|$$

As such, we define the hyperbolic distance as follows:

**Definition 4.14 (Hyperbolic distance)** The hyperbolic distance on the unit disc $\mathbb{D}$, for $z, w \in \mathbb{D}$ is defined by

$$d_\mathbb{D}(z, w) = \inf_\gamma \ell_\mathbb{D}(\gamma)$$

More generally, for any simply connected domain $U \neq \mathbb{C}$, any conformal $\phi : \mathbb{D} \to U$, and $z, w \in U$,

$$d_U(z, w) = d_\mathbb{D}(\phi^{-1}(z), \phi^{-1}(w))$$
and for any $E \subset U$,

$$d_U(z, E) = \inf \{d_U(z, w) : w \in E\}.$$  

We may also define a related quantity used to estimate the hyperbolic distance.

**Definition 4.15 (Quasi-hyperbolic distance)** For any two points, $z, w$, in a simply connected domain $U$, the quasi-hyperbolic distance is defined as

$$\delta_U(z, w) = \inf_{\gamma: z \to w} \int_{\gamma} \frac{|ds|}{d(s, \partial U)}.$$  

Where $d(s, \partial U)$ is the shortest distance between $s$ and $\partial U$ and the infimum is taken over all paths $\gamma$ which connect $z$ to $w$.

The quasi-hyperbolic distance estimates the hyperbolic distance by the following result, which is a consequence of the celebrated Kobe–1 theorem [8].

**Theorem 4.16** $\frac{1}{2} \delta_U \leq d_U \leq 2 \delta_U$.

We may apply this theory to the theory of Hardy domains to construct the second counterexample using the following theorem given in [15].

**Theorem 4.17** Let $U$ be a simply connected domain, $a \in U$, $W_R = \{|z - a| = R\}$, and $F_R = W_R \cap U$ for $R > 0$, then

$$H(U) = \liminf_{R \to \infty} \frac{d_U(a, F_R)}{\ln(R)}.$$  

**Remark 4.18** In [15], $F_R$ was defined as $R \partial \mathbb{D} \cap U$, but here we require the natural extension to circles centered at arbitrary values $a$ in $U$.

Finally we will implement the following lemma.

**Lemma 4.19** If $U$ is simply connected and $d(z, \partial U) < K$ for all $z \in U$ for some $K \in \mathbb{R}^+$, then $H(U) = \infty$.

**Proof** By definition and assumption we have

$$\delta_U(a, z) = \inf_{\gamma: a \to z} \int_{\gamma} \frac{|ds|}{d(s, \partial U)} > \inf_{\gamma: a \to z} \int_{\gamma} \frac{|ds|}{K} > \frac{|z - a|}{K}.$$  

From (4.16),

$$d_U(a, z) \geq \frac{1}{2} \delta_U(a, z) > \frac{|z - a|}{2K}.$$  

Hence

$$H(U) = \liminf_{R \to \infty} \frac{d_U(a, F_R)}{\ln(R)} \geq \liminf_{R \to \infty} \frac{R}{2K \ln(R)} = \infty.$$  

\[ \square \] Springer
Example 4.20  We will construct a domain and its complement which will both have finite $p$th moment for any $p$ for the exit time of a Brownian motion exiting the domains.

Let $\gamma_1 = \{ t \geq 0 : te^{it} \}$ shown in (6) in orange and $\gamma_2 = \{ t : te^{i(t-\pi)} \}$ shown in blue. Concatenating these curves we split the complex plane into $U$ and $\mathbb{C}\setminus U$. Any point $a$ in $U$ or $b$ in $\mathbb{C}\setminus U$ will be within $\pi$ of $\gamma_1$ or $\gamma_2$. Hence by (4.17), $H(U) = \infty$ and $H(\mathbb{C}\setminus U) = \infty$ meaning in turn both have exit times with finite $p$th moment for any $p$.

5 Characterisations of the Cauchy Distribution

In this short final section, we examine a method of deducing identities related to planar Brownian motion by applying the optional stopping theorem to complex-valued martingales. This method has been used previously in [9] to deduce the exit distributions of Brownian motion from a half-plane and from a strip, as well as the identity

$$\mathbb{E}[e^{i\lambda(2/\pi)\ln|C_1|}] = \frac{1}{\cosh \lambda},$$

with $C_1$ distributed Cauchy$(0, 1)$, which was proved in [6] by other methods (however, unbeknownst to the authors of [9] at the time, the arguments for the strip and half-plane were already present in Exercises 2.18 and 2.19 in [23]).

We will focus the method on two identities of the Cauchy distribution given in [24] and proved via the residue theorem. Here alternate probabilistic proofs are offered.
We recall Cauchy\((a, b)\) distribution has pdf
\[
b \frac{1}{\pi \left( (x-a)^2 + b^2 \right)}.
\]

**Proposition 5.1** Let \(C\) be distributed Cauchy\((a, b)\) and \(\gamma = a + bi \in \mathbb{C}\). If \(\alpha \in \mathbb{C}\) and \(\text{Im}(\alpha) > 0\) then
\[
\mathbb{E}\left[ \frac{(C - \alpha)}{(C - \overline{\alpha})} \right] = \frac{(\gamma - \alpha)}{(\gamma - \overline{\alpha})}.
\]

**Proof** Start a Brownian motion at \(\gamma\) in the upper half plane and stop it at \(T(\mathbb{H})\). As is well-known (see [9]), \(B_{T(\mathbb{H})} \sim\) Cauchy\((a, b)\) so \(B_{T(\mathbb{H})} \overset{d}{=} C\). We can transform this distribution by \(\phi(z) = \frac{(z - \alpha)}{(z - \overline{\alpha})}\) which maps \(\mathbb{H}\) to \(\mathbb{D}\) and \(\alpha\) to 0. Since \(\phi\) maps to \(\mathbb{D}\), it is bounded and since it is analytic, \(\phi(B_t)\) is a martingale. Thus we may utilise the optional stopping theorem.

\[
\mathbb{E}\left[ \frac{C - \alpha}{C - \overline{\alpha}} \right] = \mathbb{E}\left[ \frac{B_t - \alpha}{B_t - \overline{\alpha}} \right] = \mathbb{E}\left[ \frac{B_0 - \alpha}{B_0 - \overline{\alpha}} \right] = \frac{\gamma - \alpha}{\gamma - \overline{\alpha}}.
\]

\[\square\]

In proving the second Cauchy distribution identity, the following result from Burkholder [7] will be utilised.

**Theorem 5.2** If \(0 < p < \infty\), \(B_t\) is a planar Brownian motion and \(\tau\) is a stopping time such that \(\mathbb{E}[\log(1 + \tau)] < \infty\), then
\[
\mathbb{E}\left( \sup_{0 \leq t < \infty} |B_{t \wedge \tau}| \right)^p \leq c_p \mathbb{E}|B_{t}|^p.
\]

**Remark** Clearly, if \(\mathbb{E}[\tau^p] < \infty\) for some \(p > 0\), then \(\mathbb{E}[\ln(1 + \tau)] < \infty\) and Theorem 5.2 may be applied.

We will also use the following convergence theorem given in [10].

**Lemma 5.3** If \(X = (X_n, \mathcal{F}_n)\) is a discrete time submartingale and, for some \(p > 1\),
\[
\sup_n \mathbb{E}|X_n|^p < \infty
\]
then there is an integrable random variable \(X_\infty\) such that
\[
X_n \rightarrow X_\infty \quad a.s. \quad X_n \overset{L^1}{\rightarrow} X_\infty.
\]

We thus show a probabilistic proof for the following Cauchy distribution identity.

**Proposition 5.4** Let \(C\) be distributed distributed Cauchy\((a, b)\) and \(\gamma = a + bi \in \mathbb{C}\). Let \(\phi(z) = z^\alpha\) be defined for \(z \in \mathbb{C}\) and \(\alpha \in (0, 1)\), using a principal branch of the logarithm; that is, \(z^\alpha = e^{\alpha \text{Log}(z)}\), where \(\text{Log}(z) = \ln |z| + i\text{Arg}(z)\) and \(-\pi < \text{Arg}(z) \leq \pi\). Then
\[
\mathbb{E}[C^\alpha] = \gamma^\alpha.
\]
Proof} Start a Brownian motion at \( \gamma \in \mathbb{H} \) and stop it at \( T(H) \). \( \phi \) is defined to be analytic, thus \( B_{\alpha}^{t} = M_{t} \) is a martingale. Since \( \alpha \in (0, 1) \), \( \phi(H) \subset H \) and is a wedge of size \( \alpha \pi \). Define \( \tau = T(\phi(H)) \). Thus, using (4.7) for \( p < 1/\alpha \), \( \mathbb{E}(\tau^{p/2}) < +\infty \), and so \( \mathbb{E}[\text{Log}(1 + \tau)] < \infty \).

Thus, by (5.2), for any \( 0 < q < \infty \)

\[
\mathbb{E}_{x}[\left( \sup_{0 \leq t < \infty} |B_{\tau \wedge t}| \right)^{q}] \leq c_{q} \mathbb{E}|B_{\tau}|^{q}.
\]

We define \( 1 < \beta < 1/\alpha \) so that \( \alpha \beta < 1 \). Setting \( q = \alpha \beta \) we have, for all \( t \)

\[
\sup_{0 \leq t < \infty} \mathbb{E}_{x}[|M_{\tau \wedge t}|^{\beta}] \leq \mathbb{E}_{x}[\left( \sup_{0 \leq t < \infty} |M_{\tau \wedge t}| \right)^{\beta}] \\
\leq c_{p} \mathbb{E}|M_{\tau}|^{\beta} = D < \infty,
\]

where \( D = \mathbb{E}|C|^{{\alpha \beta}} \) is a finite constant since the Cauchy distribution has finite absolute fractional moments. By discretising we have \( \sup_{n} \mathbb{E}_{x}[|M_{\tau \wedge n}|^{\beta}] < \infty \) and so by (5.3), as \( n \to \infty \), \( M_{\tau \wedge n} \to M_{\tau} \) in \( L^1 \) and almost surely and thus in turn \( \mathbb{E}[M_{\tau \wedge n}] \to \mathbb{E}[M_{\tau}] \).

However since \( \tau \wedge n \) is bounded almost surely, we can apply the optional stopping theorem to the stopped martingale \( M_{\tau \wedge n} \) hence achieving the result

\[
\mathbb{E}[M_{\tau \wedge n}] = \mathbb{E}[M_{0}] = \mathbb{E}[M_{\tau}].
\]

Hence we arrive at the result

\[
\mathbb{E}[C^{\alpha}] = \mathbb{E}[B_{\tau}^{\alpha}] = \mathbb{E}[B_{0}^{\alpha}] = \gamma^{\alpha}.
\]

\[\square\]

Acknowledgements} The authors would like to thank an anonymous referee, whose helpful comments have significantly improved the paper.

Funding} Open Access funding enabled and organized by CAUL and its Member Institutions.

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References

1. Ahlfors, L.: Conformal Invariants: Topics in Geometric Function Theory, vol. 371. American Mathematical Society, Providence (2010)
2. Albin, N., Lind, J., Poggi-Corradi, P.: Convergence of the probabilistic interpretation of modulus. arXiv:2106.11418 (2021)
3. Beardon, A., Minda, C.: The hyperbolic metric and geometric function theory. 01 (2007)
4. Betsakos, D.: Harmonic measure on simply connected domains of fixed inradius. Ark. Mat. 36(2), 275–306 (1998)
5. Betsakos, D.: Geometric theorems and problems for harmonic measure. Rocky Mt. J. Math. 31(3), 773–795 (2001)
6. Bourgade, P., Fujita, T., Yor, M.: Euler’s formulae for \( \zeta(2n) \) and products of Cauchy variables. Electron. Commun. Probab. 12, 73–80 (2007)
7. Burkholder, D.: Exit times of Brownian motion, harmonic majorization, and Hardy spaces. Adv. Math. 26(2), 182–205 (1977)
8. Carleson, L., Gamelin, T.W.: Complex Dynamics. Springer, Berlin (2013)
9. Chin, W., Jung, P., Markowsky, G.: Some remarks on invariant maps of the Cauchy distribution. Stat. Prob. Lett. 158, 108668 (2020)
10. Chung, K.L.: Lectures from Markov Processes to Brownian Motion. Springer, Berlin (1982)
11. Davis, B.: Brownian motion and analytic functions. Ann. Probab. 7(6), 913–932 (1979)
12. Duren, P.: Theory of \( H^p \) Spaces. Courier Corporation, North Chelmsford (2000)
13. Gardiner, F., Lakic, N.: Quasiconformal Teichmüller Theory. American Mathematical Association, New York (2000)
14. Karafyllia, C.: On a property of harmonic measure on simply connected domains. Can. J. Math. 20, 1–21 (2019)
15. Karafyllia, C.: On the Hardy number of a domain in terms of harmonic measure and hyperbolic distance. Ark. Mat. 58(2), 307–331 (2020)
16. Karafyllia, C.: On the Hardy number of comb domains. arXiv:2101.10477 (2021)
17. Keen, L., Lakic, N.: Hyperbolic Geometry from a Local Viewpoint, vol. 68. Cambridge University Press, Cambridge (2007)
18. Krantz, S., Epstein, C.: Geometric Function Theory: Explorations in Complex Analysis. Springer, Berlin (2006)
19. Lévy, P.: Processus Stochastiques et Mouvement Brownien. Gauthier-Villars, Paris (1948)
20. Markowsky, G.: The exit time of planar Brownian motion and the Phragmén-Lindelöf principle. J. Math. Anal. Appl. 422(1), 638–645 (2015)
21. Markowsky, G.: On the distribution of planar Brownian motion at stopping times. Ann. Acad. Sci. Fenn. Math. 43, 597–616 (2018)
22. Markowsky, G.: Planar Brownian motion and complex analysis. arXiv:2012.08574 (2020)
23. Mörters, P., Peres, Y.: Brownian Motion, vol. 30. Cambridge University Press, Cambridge (2010)
24. Okamura, K.: Characterizations of the Cauchy distribution associated with integral transforms. Stud. Sci. Math. Hung. 57(3), 385–396 (2020)
25. Prawitz, H.: Über die Mittelwerte analytischer Funktionen. Arkiv Mat. Astr. Fys 20(6), 1–12 (1927)
26. Remmert, R.: Classical Topics in Complex Function Theory, vol. 172. Springer, Berlin (2013)
27. Rudin, W.: Function Theory in the Unit Ball of \( \mathbb{C}^n \), vol. 241. Springer, Berlin (2012)

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