An output-sensitive Algorithm to partition a Sequence of Integers into Subsets with equal Sums

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We present a polynomial time algorithm, which solves a nonstandard variation of the well-known PARTITION problem: Given positive integers \( n,k \) and \( t \) such that \( t \geq n \) and \( k \cdot t = \binom{n+1}{2} \), the algorithm partitions the elements of the set \( I_n = \{1, \ldots, n\} \) into \( k \) mutually disjoint subsets \( T_j \) such that \( \cup_{j=1}^k T_j = I_n \) and \( \sum_{x \in T_j} x = t \) for each \( j \in \{1,2,\ldots,k\} \). The algorithm needs \( \mathcal{O}(n \left( \frac{n}{t} + \log \frac{n(n+1)}{2t}\right)) \) steps to insert the \( n \) elements of \( I_n \) into the \( k \) sets \( T_j \).

Keywords: Set partition problem, Cutting sticks problem

1 Introduction

For \( n \in \mathbb{N} \) let \( I_n = \{1, \ldots, n\} \) be the set of integers from 1 to \( n \), and \( \Delta_n = \frac{n(n+1)}{2} \) the sum of these elements. In this paper we consider a variant of the PARTITION problem and present a solution for a class of special instances of this variant. The general version of our variant is given by \( n,k,t_1,\ldots,t_k \in \mathbb{N} \), and the question is whether there exists \( k \) pairwise disjoint subsets \( T_j \subseteq I_n \) such that the elements of \( T_j \) add up to \( t_j \), and the union of these sets equals \( I_n \). We call such a collection of sets \( T_j \) a \((t_1, t_2, \ldots, t_k)\)-partition of \( I_n \).

Fu and Hu (2015) show, that for \( k,l,t \in \mathbb{N} \) with \( 0 < l \leq \Delta_n \) and \((k-1)t + l + \Delta_{k-2} = \Delta_n \) a \((t, t+1, \ldots, t+k-2, l)\)-partition of \( I_n \) exists. Chen et al (2015) prove, that a \((t_1, \ldots, t_k)\)-partition of \( I_n \) exists, if \( \sum_{j=1}^k t_j = \Delta_n \) and \( t_j \geq t_{j+1} \) for \( 1 \leq j \leq k-1 \) and \( t_{k-1} \geq n \) hold. In Büchel et al (2016) we present a 0/1-linear program to solve partition problems.

In the special case, where \( t_j = t = \text{const} \) we call \( T_1, \ldots, T_k \) a \((k,t)\)-partition of \( I_n \). Given \( n,k,t \in \mathbb{N} \) with \( t \geq n \) and \( \Delta_n = k \cdot t \) the decision problem reduces to the question, whether a \((k,t)\) partition of \( I_n \) exists. Straight and Schillo (1975) show that for all \( k,t \) with \( \Delta_n = k \cdot t \) and \( t \geq n \) a partition of \( I_n \) exists. Ando et al (1990) withdraw the condition \( \Delta_n = k \cdot t \) and prove that for positive integers \( n,k \) and \( t \), the set \( I_n \) contains \( k \) disjoint subsets having a constant sum \( t \) if and only if \( k(2k-1) \leq k \cdot t \leq \Delta_n \).

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Where as the cited papers study for which $k$-tuples $(t_1, \ldots, t_k)$-partitions of $I_n$ exist, we are interested in efficient algorithms to determine partitions. In this paper we consider problem instances $\Pi(n, k, t)$ with $t \geq n$ and $\Delta_n = k \cdot t$. In Section 2 we introduce the recursive algorithm $\PiSolve$ which determines a partition for each instance $\Pi(n, k, t)$. Before, in Section 2 we present the so called meander algorithm which solves problem instances $\Pi(n, k, t)$, where $n$ is even and $2k$ is a divisor of $n$ or where $n$ is odd and $2k$ divides $n + 1$, respectively. The reason is, that $\PiSolve$ can be stopped, when one of these conditions is reached, and the remaining partition can be determined directly by means of the meander algorithm. In Section 4 we analyze the run time complexity of $\PiSolve$. Section 5 summarizes the paper and mentions some ideas to improve $\PiSolve$.

Inputs for the algorithms are $n$, $k$ and $t$, hence these have length $O(\log n)$. Since it is to be expected that the complexity to insert $n$ elements into $k$ sets is at least $O(n)$, we will consider the complexity of the algorithms not depending on the size of the inputs, but output-sensitive, i.e. depending on $n$ and $k$.

## 2 Meander Algorithm

For $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$ we denote $b|a$ if $b$ is a divisor of $a$. Given the problem instance $\Pi(n, k, t)$ the meander algorithm applies if $n$ is even and $2k|n$ or if $n$ is odd and $2k|n + 1$, respectively. The algorithm distributes the elements of the set $I_n$ into the subsets $T_j$ such that these sets build a $(k, t)$-partition of $I_n$, i.e. the sets $T_j$ fulfill the conditions

\begin{align*}
T_i \cap T_j &= \emptyset, \quad 1 \leq i, j \leq k, \quad i \neq j \quad (1) \\
\bigcup_{j=1}^{k} T_j &= I_n \quad (2) \\
\sum_{x \in T_j} x &= t, \quad 1 \leq j \leq k \quad (3)
\end{align*}

### 2.1 Case: $n$ even and $2k|n$

Figure 2 shows the part of the meander algorithm which solves problem instances $\Pi(n, k, t)$ when $n$ is even and $2k$ divides $n$. To prove that the algorithm determines a correct $(k, t)$-partition of $I_n$ we have to show that the partition fulfills the conditions above. Condition (1) is obviously fulfilled. We will verify (2) in Lemma 2.1 and (3) in Lemma 2.2.

Let

\begin{align*}
X_1(n, k) &= \left\{ 2ki - (j - 1) \mid 1 \leq i \leq \frac{n}{2k}, \quad 1 \leq j \leq k \right\} \quad (4) \\
X_2(n, k) &= \left\{ 2k(i - 1) + j \mid 1 \leq i \leq \frac{n}{2k}, \quad 1 \leq j \leq k \right\} \quad (5)
\end{align*}

be the sets of elements of $I_n$ which are distributed in assignment (I) or assignment (II), respectively.

**Lemma 2.1** Let $\Pi(n, k, t)$ be a problem instance such that $n$ even and $2k|n$, then $I_n = X_1(n, k) \cup X_2(n, k)$. 


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meandereven\((n,k,t)\);
input: \(n, k, t\) with \(n\) even, \(2k\mid n\), \(t \geq n\), and \(\Delta_n = k \cdot t\);
output: \((k,t)\)-partition \(T_j, 1 \leq j \leq k\), of \(I_n\);
\(T_j := \emptyset, 1 \leq j \leq k\);
\(\text{for } j := 1 \text{ to } k \text{ do}\)
\(\text{for } i := 1 \text{ to } \frac{n}{2k} \text{ do}\)
\(\text{I) } T_j := T_j \cup \{2ki - (j - 1)\};\)
\(\text{II) } T_j := T_j \cup \{2k(i - 1) + j\};\)
\(\text{endfor}\)
\(\text{endfor}\)
\(\text{end.}\)

Fig. 1: Meander Algorithm in case \(n\) even and \(2k\mid n\).

Proof: For each \(x \in I_n\) there exist unambiguously \(i, r\) such that
\[ x = 2k(i - 1) + r, 1 \leq i \leq \frac{n}{2k}, 1 \leq r \leq 2k \] (6)
We consider the two following sets of remainders \(r \in I_{2k}\): \(R_1 = \{2k - (j - 1) \mid 1 \leq j \leq k\}\) and \(R_2 = \{j \mid 1 \leq j \leq k\}\). Since \(r \in R_1\), if \(k + 1 \leq r \leq 2k\), it follows \(R_1 \cap R_2 = \emptyset\) and \(R_1 \cup R_2 = I_{2k}\). Thus with respect to (6) we get either
\[ x = 2k(i - 1) + 2k - (j - 1) = 2ki - (j - 1) \] (7)
or
\[ x = 2k(i - 1) + j \] (8)
It follows \(x \in X_1(n,k) \cup X_2(n,k)\). Hence we have shown \(I_n \subseteq X_1(n,k) \cup X_2(n,k)\).

If \(x \in X_1(n,k)\), then \(k + 1 \leq x \leq n\), and if \(x \in X_2(n,k)\) then \(1 \leq x \leq n - k\). Thus, if \(x \in X_1(n,k) \cup X_2(n,k)\), we have \(1 \leq x \leq n\), hence \(x \in I_n\) and thereby \(X_1(n,k) \cup X_2(n,k) \subseteq I_n\). \(\square\)

Lemma 2.2 Let \(\Pi(n,k,t)\) be a problem instance with \(n\) even and \(2k\mid n\), then the output \(T_j, 1 \leq j \leq k\), of meandereven\((n,k,t)\) fulfills condition (3).

Proof: For each \(j \in \{1, \ldots, k\}\) we have:
\[ \sum_{x \in T_j} x = \frac{k}{2} \sum_{i=1}^{\frac{n}{2k}} (2ki - (j - 1)) + \frac{k}{2} \sum_{i=1}^{\frac{n}{2k}} (2k(i - 1) + j) = 2k \sum_{i=1}^{\frac{n}{2k}} (2i - 1) + \frac{n}{2k} = 2k \frac{n^2}{4k^2} + \frac{n}{2k} = \frac{n(n+1)}{2k} = t \]
\(\square\)
Theorem 2.1  \textit{meandereven}(n, k, t) \\
\textbf{a}) determines a correct partition of } I_n \text{ for all problem instances } \Pi(n, k, t) \text{ with } n \text{ even and } 2k|n, \text{ and} \\
\textbf{b}) needs } O(n) \text{ steps to insert the } n \text{ elements of } I_n \text{ into the sets } T_j.

\textbf{Proof:} a) follows immediately from Lemmas 2.1 and 2.2, and b) is obvious. \hfill \Box

2.2 Case: \textit{n} odd and \textit{2k} \textit{|} \textit{n+1}

To solve problem instances \( \Pi(n, k, t) \) with \textit{n odd} and \textit{2k|n+1} we adapt slightly the \textit{meandereven}-algorithm (see Fig. 2). The correctness of the \textit{meanderodd}-algorithm can be shown analogously to the proof of the correctness of the \textit{meandereven}-algorithm. At this point we define the sets of elements assigned due to labels (I) and (II) in the \textit{meanderodd}-algorithm as

\begin{align*}
X_1'(n, k) &= \left\{ 2ki - j \mid 1 \leq i \leq \frac{n+1}{2k}, 1 \leq j \leq k \right\} \\
X_2'(n, k) &= \left\{ 2k(i - 1) + (j - 1) \mid 1 \leq i \leq \frac{n+1}{2k}, 1 \leq j \leq k \right\}
\end{align*}

\textit{meanderodd}(n, k, t); \\
\textbf{input: } n, k, t \text{ with } n \text{ odd, } 2k|n+1, \ t \geq n, \text{ and } \Delta_n = k \cdot t; \\
\textbf{output: } (k, t)-\text{partition } T_j, 1 \leq j \leq k, \text{ of } I_n; \\
& \quad T_j := \emptyset, \ 1 \leq j \leq k; \\
& \quad \text{for } j := 1 \text{ to } k \text{ do} \\
& \quad \quad \text{for } i := 1 \text{ to } \frac{n}{2k} \text{ do} \\
& \quad \quad \quad \text{(I) } T_j := T_j \cup \{2ki - j\}; \\
& \quad \quad \quad \text{(II) } T_j := T_j \cup \{2k(i - 1) + (j - 1)\}; \\
& \quad \quad \text{endfor}; \\
& \quad \endfor; \\
& \text{end.}

\textbf{Remark 2.1} \text{ In order to avoid a case distinction, we first assign the element } 0 \ (i = 1, \ j = 1) \text{ to set } T_1. \\
\text{For this reason, in the following we assume that } I_n \text{ contains the element } 0, \text{ too.} \hfill \Box

\textbf{Lemma 2.3} \text{ Let } \Pi(n, k, t) \text{ be a problem instance such that } n \text{ odd and } 2k|n+1, \text{ then } I_n = X_1'(n, k) \cup X_2'(n, k).
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Proof: For each $x \in I_n$, there exist unambiguously $i, r$ such that

$$x = 2k(i - 1) + r, \quad 1 \leq i \leq \frac{n+1}{2k}, \quad 0 \leq r \leq 2k - 1$$

We consider the sets of remainders $r \in I_{2k-1}$: $R'_1 = \{2k-j \mid 1 \leq j \leq k\}$ and $R'_2 = \{j-1 \mid 1 \leq j \leq k\}$ = $\{j \mid 0 \leq j \leq k-1\}$. Since $r \in R'_1$, if $k \leq r \leq 2k - 1$, it follows $R_1 \cap R_2 = \emptyset$ and $R_1 \cup R_2 = I_{2k-1}$. Thus with respect to (11) we get

$$x = 2k(i - 1) + 2k - j = 2ki - j$$

or

$$x = 2k(i - 1) + (j - 1)$$

respectively. It follows $x \in X'_1(n, k) \cup X'_2(n, k)$. Hence we have shown $I_n \subseteq X'_1(n, k) \cup X'_2(n, k)$.

If $x \in X'_1(n, k)$, then $k \leq x \leq n$, and if $x \in X'_2(n, k)$ then $0 \leq x \leq n - k$. Thus, if $x \in X'_1(n, k) \cup X'_2(n, k)$, we have $0 \leq x \leq n$, hence $x \in I_n$ and thereby $X'_1(n, k) \cup X'_2(n, k) \subseteq I_n$. □

Lemma 2.4 Let $\Pi(n, k, t)$ be a problem instance with $n$ odd and $2k|n + 1$, then the output $T_j$, $1 \leq j \leq k$, of $\text{meander}^{\text{odd}}(n, k, t)$ fullfills condition (3).

Proof: For each $j \in \{1, \ldots, k\}$ we have

$$\sum_{x \in T_j} x = \sum_{i=1}^{n+1} (2ki - j) + \sum_{i=1}^{n+1} (2k(i - 1) + (j - 1))$$

$$= 2k \sum_{i=1}^{n+1} (2i - 1) - \frac{n+1}{2k} = 2k \frac{(n+1)^2}{4k^2} - \frac{n+1}{2k}$$

$$= \frac{n(n+1)}{2k} = t$$

□

Theorem 2.2 $\text{meander}^{\text{odd}}(n, k, t)$

a) determines a correct partition of $I_n$ for all problem instances $\Pi(n, k, t)$ with $n$ odd and $2k|n + 1$, and

b) needs $O(n)$ steps to insert the $n$ elements of $I_n$ into the sets $T_j$.

Proof: a) follows from Lemmas 2.3 and 2.4, and b) is obvious. □

3 The Algorithm $\Pi$ Solve

In this section we present the different cases which the $\Pi$ Solve-algorithm distinguishes using ideas similar to those used in Straight and Schillo (1979). The input to the algorithm are the integers $n, k, t \in \mathbb{N}$ with $t \geq n$ and $\Delta_n = k \cdot t$. The output is a $(k, t)$-partition $T_j$, $1 \leq j \leq k$, of $I_n$, which fullfills condition (3). We prove that the algorithm works correctly in all cases.
3.1 Case: $2n > t$

In this case the algorithm makes a distinction between the cases $t$ even and $t$ odd.

3.1.1 Case: $t$ even

The algorithm starts with filling \( \frac{2n-t}{2} \) sets as follows:

\[
T_j = \{ t - n + (j-1), n - (j-1) \}, \quad 1 \leq j \leq \frac{2n-t}{2}
\]  

(14)

Obviously these sets are disjoint and fulfill condition (3). The union of these sets is the set \( \{ t-n, \ldots, \frac{t}{2} - 1, \frac{t}{2} + 1, \ldots, n \} \). Thus the elements of the set \( I_{t-n-1} \) and the element \( \frac{t}{2} \) remain, these have to be distributed into the empty \( k - \frac{2n-t}{2} \) sets. To do this, each of these sets is split into two subsets:

\[
T_j = T_{j,1} \cup T_{j,2}, \quad \frac{2n-t}{2} + 1 \leq j \leq k
\]  

(15)

The total number of these subsets is \( 2(k-n) + t \). The set \( T_{2n-t+1,1} \) is filled with the element \( \frac{t}{2} \):

\[
T_{2n-t+1,1} = \left\{ \frac{t}{2} \right\}
\]  

(16)

Thus it remains to distribute the elements of \( I_{t-n-1} \) into the \( 2(k-n) + t - 1 \) sets \( T_{2n-t+1,2} \) and \( T_{j,s} \), \( \frac{2n-t}{2} + 2 \leq j \leq k, \quad s \in \{ 1, 2 \} \), i.e. it remains to solve the problem instance \( \Pi(n', k', t') \) where

\[
n' = t - n - 1
\]  

(17)

\[
k' = 2(k-n) + t - 1
\]  

(18)

\[
t' = \frac{t}{2}
\]  

(19)

We have to verify that this instance fulfills the input conditions

\[
\Delta_{n'} = k' \cdot t'
\]  

(20)

and

\[
t' \geq n'
\]  

(21)

Using (17) - (19) we get on one side

\[
\Delta_{n'} = \frac{n'(n'+1)}{2} = \frac{(t-n-1)(t-n)}{2} = \Delta_n + \frac{t^2 - 2tn - t}{2}
\]  

(22)

and on the other side

\[
k' \cdot t' = (2(k-n) + t - 1) \cdot \frac{t}{2} = k \cdot t + \frac{t^2 - 2tn - t}{2}
\]  

(23)

Since for our initial problem \( \Pi(n, k, t) \) the condition \( \Delta_n = k \cdot t \) holds, the verification of (20) follows immediately from (22) and (23).

From \( 2n > t \) immediately follows \( \frac{t}{2} > t - n - 1 \). Using (17) and (19) condition (21) is verified, too.

Thus the algorithm can recursively continue to solve the initial problem by determining a solution for the instance \( \Pi(n', k', t') \).
3.1.2 Case: $t$ odd

In this case the algorithm initially fills \(\frac{2n-t+1}{2}\) sets as follows:

\[
T_j = \{t - n + (j - 1), n - (j - 1)\}, \quad 1 \leq j \leq \frac{2n-t+1}{2}
\]  

(24)

Obviously these sets are disjoint and fulfill condition (3). The union of these sets builds the set \(\{t - n, \ldots, n\}\). Thus the elements of the set \(I_{t-n-1}\) remain, these have to be distributed into the empty \(k - \frac{2n-t+1}{2}\) sets. Therefore, the instance \(\Pi(n', k', t')\) has to be solved, where

\[
n' = t - n - 1
\]  

(25)

\[
k' = k - \frac{2n-t+1}{2}
\]  

(26)

\[
t' = t
\]  

(27)

To proof that this instance is feasible we have to verify, that the input conditions (20) and (21) are fulfilled in this case as well.

Using (25) – (27) we get on one side

\[
\Delta n' = \frac{n'(n'+1)}{2} = \frac{(t-n-1)(t-n)}{2} = \Delta n + \frac{t^2 - 2tn - t}{2}
\]  

(28)

and on the other side

\[
k' \cdot t' = \left(k - \frac{2n-t+1}{2}\right) \cdot t = k \cdot t + \frac{t^2 - 2tn - t}{2}
\]  

(29)

Since \(\Delta n = k \cdot t\) the verification of (21) follows immediately from (28) and (29).

From \(2n > t\) it follows \(n > t - n - 1\). From this we get by means of the input condition \(t \geq n\) and the definitions (25) and (27): \(t' = t \geq n > t - n - 1 = n'\), i.e. condition (21) is fulfilled.

3.2 Case: \(2n \leq t\)

In this case each set \(T_j\) is split into two disjoint subsets: \(T_j = T_{j,1} \cup T_{j,2}, 1 \leq j \leq k\). The sets \(T_{j,1}\) will be filled as follows:

\[
T_{j,1} = \{n - 2k + j, n - (j - 1)\}
\]  

(30)

Hence the elements \(n - 2k + 1, \ldots, n\) are already distributed, and the two elements in each of these sets add up to

\[
n - (i - 1) + n - 2k + i = 2(n - k) + 1
\]  

(31)

It remains to partition the elements of \(I_{n-2k}\) into the sets \(T_{j,2}\) such that the sum of elements in each \(T_{j,2}\) equals \(t - (2(n - k) + 1)\). Thus it remains to solve the problem instance \(\Pi(n', k', t')\) with

\[
n' = n - 2k
\]  

(32)

\[
k' = k
\]  

(33)

\[
t' = t - 2(n - k) - 1
\]  

(34)
As well as in the former cases we have to assure, that the input conditions (20) and (21) are fulfilled. On the one side we have

\[ \Delta_{n'} = \frac{(n - 2k)(n - 2k + 1)}{2} = \Delta_n + 2k^2 - k - 2kn \]  

(35)

and on the other side

\[ k' \cdot t' = k \cdot (t - 2(n - k) - 1) = k \cdot t - 2kn + 2k^2 - k \]  

(36)

(20) follows immediately from (35) and (36).

From \( t \geq 2n \) it follows \( n + 1 \geq 4k \). By subtraction we get \( t - n - 1 \geq 2n - 4k \) and from this and definitions (32) and (34) \( t' = t - 2n + 2k - 1 \geq n - 2k = n' \), i.e. condition (21) is verified.

The considerations so far lead to the algorithm \( \Pi_{Solve} \) shown in Figure 3, and we proved that it works correctly in all cases.

4 Complexity

In this section we analyse the worst case run time complexity of the \( \Pi_{Solve} \)-Algorithm. The algorithm consists of four subalgorithms related to the cases we distinguish: (I) \( 2k|n \) or \( 2k|n + 1 \), (II) \( t \geq 2n \), (III) \( t < 2n \) and \( t \) even, (IV) \( t < 2n \) and \( t \) odd. We abbreviate these cases by \( m \) (meander), \( s \) (smaller), \( ge \) (greater even), and \( go \) (greater odd), respectively. Then the run \( \Pi_{Solve}(n, k, t) \) can be represented by a sequence \( \rho'(n, k, t) \in \{m, s, ge, go\}^\ast \).

**Example 4.1**  

a) Let \( n = 1337 \). The list of runs for all partitions of \( I_{1337} \) is:

\[ \rho'(1337, 3, 298151) = m \]
\[ \rho'(1337, 7, 127779) = s^{94} go m \]
\[ \rho'(1337, 21, 42593) = s^{30} go s ge m \]
\[ \rho'(1337, 191, 4683) = ss go m \]
\[ \rho'(1337, 223, 4011) = m \]
\[ \rho'(1337, 573, 1561) = go m \]
\[ \rho'(1337, 669, 1337) = m \]

b) Let \( n = 9999 \), then we have

\[ \rho'(9999, 4444, 11250) = ge s^3 ge^4 go m \]
\[ \rho'(9999, 4040, 12375) = go s^4 go s^4 go s ge m \]
\[ \rho'(9999, 3960, 12625) = go s^3 ge go s^8 go m \]
\[ \rho'(9999, 3333, 15000) = ge^3 go m \]
\[ \rho'(9999, 12, 4166250) = s^{415} go s ge^2 m \]
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\[ \Pi \text{Solve}(n, k, t); \]
input: \( n, k, t \) with \( t \geq n \), and \( \Delta_n = k \cdot t \);
output: \((k, t)\)-partition \( T_j, 1 \leq j \leq k \), of \( I_n \);

(I) \text{case } 2k \mid n \\
\text{then fill } \{ T_j \}_{1 \leq j \leq k} \text{ by } \text{meandereven}(n, k, t) \\
\text{case } 2k \mid n + 1 \\
\text{then fill } \{ T_j \}_{1 \leq j \leq k} \text{ by } \text{meanderodd}(n, k, t) \\

(II) \text{case } t \geq 2n \\
\text{then for } 1 \leq j \leq k \text{ do } T_{j,1} = \{ n - 2k + j, n - (j - 1) \} \text{ endfor; } \\
\text{fill } \{ T_{j,2} \}_{1 \leq j \leq k} \text{ by } \Pi \text{Solve}(n - 2k, k, t - 2(n - k) - 1)); \\
\text{for } 1 \leq j \leq k \text{ do } T_j = T_{j,1} \cup T_{j,2} \text{ endfor} \\

(III) \text{case } t < 2n \text{ and } t \text{ even} \\
\text{then for } 1 \leq j \leq \frac{2n - t}{2} \text{ do } T_j = \{ t - n + (j - 1), n - (j - 1) \} \text{ endfor; } \\
\text{T}_{\frac{2n - t}{2} + 1,1} = \{ \frac{t}{2} \}; \\
\text{fill } T_{\frac{2n - t}{2} + 1,2}, \{ T_{j,1} \}_{\frac{2n - t}{2} + 2 \leq j \leq k} \text{ and } \{ T_{j,2} \}_{\frac{2n - t}{2} + 2 \leq j \leq k} \\
\text{by } \Pi \text{Solve}(t - n - 1, 2(k - n) + t - 1, \frac{t}{2}); \\
\text{for } \frac{2n - t}{2} + 1 \leq j \leq k \text{ do } T_j = T_{\frac{2n - t}{2} + j,1} \cup T_{\frac{2n - t}{2} + j,2} \text{ endfor} \\

(IV) \text{case } t < 2n \text{ and } t \text{ odd} \\
\text{then for } 1 \leq j \leq \frac{2n - t + 1}{2} \text{ do } T_j = \{ t - n + (j - 1), n - (j - 1) \} \text{ endfor; } \\
\text{fill } \{ T_j \}_{\frac{2n - t + 1}{2} + 1 \leq j \leq k} \text{ by } \Pi \text{Solve}(t - n - 1, k - \frac{2n - t + 1}{2}, t) \\
end.

Fig. 3: Algorithm \( \Pi \text{Solve} \).
Let $\alpha$ be a non empty sequence over $\Omega' = \{m, s, ge, go\}$, then $\text{first}(\alpha)$ is the first and $\text{last}(\alpha)$ the last symbol of $\alpha \in \Omega'^+$, and $\text{head}(\alpha)$ is the sequence without the last symbol. $|w|_a$ is the number of occurrences of symbol $a \in \Omega'$ in the sequence $w \in \Omega'^*$.

Obviously we have

**Lemma 4.1** Let $\Pi(n, k, t)$ be a problem instance, then $\text{last}(\rho'(n, k, t)) = m$ and $m$ is not a member of $\text{head}(\rho'(n, k, t))$. $\square$

Thus, we may neglect the last symbol of $\rho'(n, k, t)$ and denote $\rho(n, k, t) = \text{head}(\rho'(n, k, t))$. As well we do not need the alphabet $\Omega'$, because $\rho(n, k, t) \in \{s, ge, go\}^*$. We denote this alphabet by $\Omega$.

Next we show, that the last call before the recursion stops with the $m$-case cannot be $s$.

**Lemma 4.2** Let $\Pi(n, k, t)$ be a problem instance. If $|\rho(n, k, t)| \geq 1$, then $\text{last}(\rho(n, k, t)) \neq s$.

**Proof:** We assume $\text{last}(\rho(n, k, t)) = s$. Let $\Pi(\nu, \kappa, \tau)$ be the problem instance before the last $s$-call. Then by (32) and (33) after the $s$-call we have $\nu' = \nu - 2\kappa$ and $\kappa' = \kappa$. Since the next call is $m$ it has to be $2\kappa'|\nu'$ or $2\kappa'|\nu' + 1$, thus we have $2\kappa|\nu - 2\kappa$ or $2\kappa|\nu - 2k + 1$. It follows $2\kappa|\nu$ or $2\kappa|\nu + 1$. Hence the instance $\Pi(\nu, \kappa, \tau)$ would have been solved by an $m$-call, a contradiction to our assumption $\text{last}(\rho(n, k, t)) = s$. $\square$

**Corollary 4.1** If $|\rho(n, k, t)| \geq 1$, then $\text{last}(\rho(n, k, t)) \in \{ge, go\}$.

4.1 Case: $2n > t$ and $t$ odd

From $2n > t$ we can conclude $t > 2(t - n - 1)$. Using (25) and (27) we get $t' > 2n'$. This leads to

**Lemma 4.3** Let $\Pi(n, k, t)$ be a problem instance with $2n > t$, $t$ odd and $\rho'(n, k, t) = \alpha\beta$, $\alpha \in \Omega^*$, $\beta \in \Omega'^+$, then

a) $\text{first}(\beta) = m$, if $|\beta| = 1$.

b) $\text{first}(\beta) = s$, if $|\beta| \geq 2$. $\square$

Thus, after the case $go$ the recursion ends by call of the meander algorithm or the recursion continues with the $s$ case either.

**Corollary 4.2** Let $\Pi(n, k, t)$ be a problem instance with $2n > t$ and $t$ odd, then

$$|\rho(n, k, t)|_s \geq |\rho(n, k, t)|_{go}. \quad (37)$$

4.2 Case: $2n > t$ and $t$ even

From (19) it follows immediately

$$|\rho(n, k, t)|_{ge} \leq \log t = \log \frac{n(n + 1)}{2k} \quad (38)$$
4.3 Case: $2n \leq t$

In this case if the algorithm performs the instance $\Pi(n', k, t')$ with $n' = n - 2k$ and $t' = t - 2(n - k) - 1$ (cf. Subsection 3.2, equations (32) and (34), respectively). By $n^{(\ell)}$ and $t^{(\ell)}$ we denote the value of $n$ and $t$ in the $\ell$th recursion call in the case $2n^{(\ell)} \leq t^{(\ell)}$. Thus we have $n^{(0)} = n$, $n^{(1)} = n' = n - 2k$ and $t^{(0)} = t$, $t^{(1)} = t' = t - 2(n - k) - 1$, for example. By induction we get

$$n^{(\ell)} = n - 2k \cdot \ell$$

$$t^{(\ell)} = t - 2n \cdot \ell + 2k \cdot \ell^2 - \ell$$

$$= t - (2(n - k \cdot \ell) + 1) \cdot \ell$$

Now we determine the order of the maximum value of $\ell$ guaranteeing the condition $2n^{(\ell)} \leq t^{(\ell)}$. Using (39) and (40) we get

$$0 \leq t^{(\ell)} - 2n^{(\ell)}$$

$$= t - (2(n - k \cdot \ell) + 1) \cdot \ell - 2(n - 2k \cdot \ell)$$

To determine $\ell$ we solve the quadratic equation

$$0 = \ell^2 + \frac{4k - 2n - 1}{2k} \cdot \ell + \frac{t - 2n}{2k}$$

which has the solutions

$$\ell_{1,2} = \frac{-4k - 2n - 1}{4k} \pm \sqrt{\left(\frac{4k - 2n - 1}{4k}\right)^2 - \frac{t - 2n}{2k}}$$

$$= \frac{-4k - 2n - 1}{4k} \pm \frac{4k - 1}{4k}$$

i.e.

$$\ell_1 = \frac{n}{2k}, \quad \ell_2 = \frac{n + 1}{2k} - 2$$

Finally we get

$$\ell \leq \frac{n}{2k}$$

Thus, we have just proven

**Lemma 4.4** Let $\Pi(n, k, t)$ be a problem instance. If $p(n, k, t) = s^x$ with $x \in \{ge, go\}$, then $\ell \leq \frac{n}{2k}$. $\square$
Corollary 4.2, inequality (38) and Lemma 4.4 lead to

**Theorem 4.1** Let
\[
\Pi(n,k,t)
\]
be a problem instance.

\textbf{a)} Then the recursion depth of \(\Pi\text{Solve}(n,k,t)\) is \(O\left(\frac{n}{2k} + \log \frac{n(n+1)}{2k}\right)\).

\textbf{b)} Since the complexity of operations the algorithm performs in each recursion call (assigning elements of \(I_n\) to some set \(T_j\), arithmetic comparisons and operations) is \(O(n)\) it follows that \(\Pi\text{Solve}\) needs
\[
O\left(n \cdot \left(\frac{n}{2k} + \log \frac{n(n+1)}{2k}\right)\right)
\]
steps to insert the \(n\) elements of \(I_n\) into the \(k\) sets \(T_j\).

\(\square\)

5 Conclusion

In Section 3 we present the recursive algorithm \(\Pi\text{Solve}\) which solves following special \textsc{Partition} problems \(\Pi(n,k,t)\): Given \(n,k,t \in \mathbb{N}\) with \(t \geq n\) and \(\Delta_n = k \cdot t\), then the algorithm partitions the set \(I_n = \{1, \ldots, n\}\) into \(k\) mutually disjoint sets such that the elements in each set add up to \(t\). The recursion can be stopped, if \(n\) is even and \(2k\) is a divisor of \(n\) or if \(n\) is odd and \(2k\) is a divisor of \(n+1\), respectively, because in these cases the meander algorithms presented in Section 2 can be applied, which directly determines a partition.

We prove that the algorithm works correctly and needs
\[
O\left(n \cdot \left(\frac{n}{2k} + \log \frac{n(n+1)}{2k}\right)\right)
\]
steps to assign the elements of \(I_n\) to the \(k\) subsets \(T_j\) for each problem instance \(\Pi(n,k,t)\). Taking into account that the algorithm for the inputs \(n\) and \(k\) determines an output consisting of \(k\) sets to which the elements of \(I_n\) are to be distributed so that all constraints are met, \(\Pi\text{Solve}\) is a polynomial output-sensitive time algorithm.

In Jagadish (2015) an approximation algorithm for the cutting sticks-problem is presented. Because the cutting sticks-problem can be transformed into an equivalent partitioning problem our algorithms can be applied to the corresponding cutting sticks-problems.

Further research may investigate whether ideas from the previous chapters and cited papers can be used to improve the efficiency of the \(\Pi\text{Solve}\)-algorithm. In Büchel et al. (2016), Büchel et al. (2017a) and Büchel et al. (2017b) we present efficient solutions for problem instances \(\Pi(n,k,t)\), where \(n = q \cdot k\), \(q,k\) odd; \(n = m^2 - 1\), \(m \geq 3\); \(n = p - 1\), \(n = p\), \(n = 2p\), \(p \in \mathbb{P}\), where \(\mathbb{P}\) is the set of prime numbers. Thus we may augment the \(\Pi\text{Solve}\)-algorithm by related conditions to stop further recursion calls.

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