Self-similar Scalings in Focusing Flows Driven by Capillary and Gravitational Forces

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We study converging fluid films driven by both surface tension and gravitational forcing. Our numerical study explores this complete range of flow regimes. Even in this intermediate flow regime, away from well-known limiting behavior, we find similarity forms. The intermediate regime spans a huge range of Bond numbers, and limits all realistic convergent flows to the intermediate scaling regime. We verify the validity of numerical methodology and results with various experiments.

Keywords: viscous fluids, gravity currents, focusing flows, similarity solutions

I. INTRODUCTION

A thin layer of fluid on a solid substrate displays surprisingly rich dynamics, due to the interplay of forces at many length scales. Gravity currents are fluid flows in which the main driving force is gravity. Viscous gravity currents occur on many scales: in liquids spreading on flat surfaces, inclined surfaces, dam breaks, mud slides and snow avalanches. They are well represented in a wide range of applications in for example geophysical and industrial (spin coating) contexts, where a good understanding of their dynamics is of considerable practical interest.

Surface tension may also play a role in the dynamics of gravity currents on smaller scales. The capillary length \( l_d = \sqrt{\frac{\gamma}{\rho g}} \), the ratio between surface tension forces, given by \( \gamma \), and gravity, given by \( \rho g \), gives the lengthscale on which surface tension effects are significant. For example, in spin coating application, a thin film of the order of a micron is spreading on a rotating substrate. Under such conditions, surface tension gradients can dominate the spreading dynamics of the flow. The role of surface tension forces in thin films with considerable gravitational driving has received some attention; this work is intended to map out the complete range of flows, from gravitational to capillary driven.

We study converging viscous gravity currents with surface tension through experiments and numerics. The experimental approach to study these convergent flows starts by first creating an axisymmetric reservoir of fluid in a rotating container. Fluid confinement inside the container leads trivially to the following heuristic picture: during rotation, centrifugal forces drive the fluid to the outer edge of the container. Cessation of the rotation then ‘releases’ the fluid and creates an axi-symmetric collapsing fluid flow.

There are many experimental advantages of this approach: it creates highly reproducible initial conditions in a mechanically simple and small setup. The dependence on experimental parameters can be easily tested and there is full optical access to the complete fluid film height profile, via various techniques. Determining the collapsing surface structure is critical to determining the collapse dynamics, especially the possibly self-similar structure of the fluid film close to the time of collapse.

Gravity currents of thin layers of very viscous fluids have been modeled using lubrication theory by a nonlinear diffusion equation for the film thickness. The equation is often called the porous medium equation. Theoretical and one experimental study have shown that under appropriate conditions, focusing flows that converge to fill-in an exposed circular dry-spot on a substrate take a self-similar form. While properties of similarity solutions (called first kind similarity solutions) for many problems obey power-law scalings with rational exponents, for this problem the radius of the dry spot was predicted to collapse like \( R_S(t) \propto (t_c - t)^{\delta} \) with \( \delta = 0.762... \) where \( t_c \) is the critical collapse moment. This is called a second-kind similarity solution, since the exponent is not derivable from dimensional scaling arguments and must be obtained numerically as part of the process of determining the similarity solution.

Extracting scaling exponents in power-law contexts is a challenge in many experimental and numerical data analysis. In our data, the canonical log-fitting method yields large uncertainties on the extracted exponents, even for high resolution numerical data. To resolve this, we develop two other methods to extract scaling exponents. One method is directly applicable to the surface structure data that we obtain experimentally and numerically. The second method is computationally simpler, but relies on surface gradient information, which is not well resolved experimentally. Both methods rely on the mathematical
properties of the scaling functions in our experiments, yet seem generically applicable in other contexts.

With these new methods, accurate exponents are extracted, and used to map out with, unprecedented detail, the transition from gravity to surface tension driven collapse. We verify numerical results, where possible, against experiments.

The article is arranged as follows: first, we discuss the theoretical background, introduce the relevant scaling parameters and the numerical methods. Then we describe the experimental approach in our studies, including the setting up of initial conditions for the collapse. The numerical results reveal the typical difficulties encountered with traditional log-fitting of power laws, both in experimental and numerical data. We describe two methods to circumvent the difficulties, using our numerical data as a guide. The effectiveness of the methods is then demonstrated, and used to explore the transition in dynamics from gravity driven, to surface tension driven flows.

II. BACKGROUND

We consider thin film dynamics in the geometry shown in Fig. 1h. In the low Reynolds number, creeping viscous flow limit, the time dependent film height \( h(r,t) \) in the rotating container is described with a time dependent axisymmetric lubrication equation that includes surface tension, centrifugal force and disjoining pressure:

\[
\frac{\partial h}{\partial t} + \frac{\nabla}{r} \left\{ -r h^3 \frac{\partial h}{\partial r} + \frac{r h^3}{Bo} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{h}{r} \right) \right] \right\} = 0 ,
\]

(3)

with

\[
\mathcal{V} = \frac{\rho g H_0^3}{3\eta R^2}.
\]

Note that the governing equation for thin films can be cast into a particular nonlinear diffusion equation form \[16\] sometimes referred to as the porous media equation. This equation is therefore also of considerably wider interest in diffusion, porous media and dynamics of non-Newtonian fluids \[17, 19\].

III. EXPERIMENTS

The experimental system consists of an initially \( \sim 1 \) mm thick layer of fluid in a shallow cylindrical container – see Fig. 1. The bucket is rotated using a stepper motor with closed loop controller (Parker Zeta Drive 6104). The closed loop stepper motor controller can be programmed to run any time dependent rotation speed profile \( \Omega(t) \) with a maximum of two rotations per second (rps). The bucket measures 13 cm in diameter and 2 cm in height. To fix the temperature dependent viscosity \( \eta \) and surface tension \( \gamma \) of the fluid, the bucket is uniformly heated to a temperature of 24°C unless otherwise noted by running water at a set temperature through the double-walled rotating axis – for details see \[11\]. On the bottom of the bucket, a silicon wafer (University Wafers) is placed; the wafer is fixed to the base through the deposition of a small (\( \lesssim 1 \) ml) amount of fluid between wafer and the bottom of the bucket. Suction force remains even after complete submersion of the wafer. The bucket is filled with a volume \( V \) of fluid which gives an initial filling height \( H_0 = V/\pi R^2 \) with \( R \) the radius of the bucket. We use polydimethylsiloxane (PDMS) for all experiments described in this work; this fluid completely wets the silicon wafer. The properties of PDMS, with density \( \rho = 965 \) kg/m\(^3\) and \( \gamma = 0.02 \) N/m together with the bucket radius \( R = 6.5 \) cm fixes \( Bo = 2 \times 10^3 \), with only a weak dependence on temperature through \( dr/dT \sim 0.6 \times 10^{-3} \) N/Km. The transparency of the PDMS and reflectivity of the silicon wafer allows for a laser-assisted alignment of the gravity-leveled fluid surface and the silicon wafer in the bucket, whose orientation can be tuned by set screws. Interferometry provides access to the spatial structure in the thin film dynamics – see Fig. 4. Bucket illumination is provided with a uniform sodium light via a beam splitter. The spatial structure of the interference pattern
FIG. 1: (a) Schematic drawing of the bucket and all the relevant parameters: $\Omega$ the rotation speed, $H_0$ the initial filling height, $R$ the radius of the bucket, $\eta, \gamma$ the viscosity and surface tension of the fluid respectively. (b) Sketch of the initial conditions for the collapse dynamics numerics (upper figure), and for the initial conditions used in the numerics(lower figure). (c) Schematic drawing of the interferometry setup. (d) Schematic of the surface structure scanning experiment. (e) A typical interferometry image from a dry spot. (f) CTF size versus rotation rate: experimental observations for $H_0 = 1.4, 1.9, 2.4, 3.2, 3.9$ mm, numerical results (solid lines, see text) and the mass conservation constraint [20] (dash-dot) for comparison.

Choosing $H_0, \eta$ — As shown below, the collapse time strongly decreases with the initial layer thickness $H_0$, and as such, it is favorable to choose thicker layers to keep experimental timescales manageable. However, for thicker layers the fluid reservoir has not completely drained at the moment of the collapse. One solution for this challenge is to speed up the collapse by choosing less viscous PDMS, but for fluid viscosities of less than $\sim 50$ mPa·s the Stokes flow approximation breaks down during the collapse [21]. We are hence limited in choosing $\eta \sim 100 - 1000$ mPa·s and $H_0 \sim 3$ mm.

A. Initial Conditions in Experiments

In a rotating bucket, a fluid will set up a parabolic surface profile to balance gravitational pressure and centrifugal forces. However, in a shallow bucket the presence of the bottom interferes with the parabolic solution above a critical rotation rate $\Omega > \Omega_c$ [20] and requires at least a piecewise continuous solution for the height profile of the fluid surface. The height profile remains parabolic for large enough $r$; for smaller $r$ there are two options, depending on the wetting properties of the fluid on the surface of the bottom. The parabolic solution can connect to a thin film covering the bottom in the center that we call the central thin film (CTF), or the parabolic solution can end in a contact line at some finite radius and contact angle leaving a completely dewetted surface area in the center. In this paper, we discuss only the case of completely wetting fluids, which sets up a height profile as sketched in Fig. 1b.

To find the initial conditions before the collapse created by the rotation, we solve the steady state $\frac{\partial h}{\partial t} = 0$ lubrication equation to obtain the complete surface profile $h(r, \Omega, H_0)$ for different rotation rates and initial volumes set by $H_0$; the CTF radii extracted from the numerics are in good agreement with the approximation from Linden [20] that assumes a piecewise continuous surface structure, consisting of only a flat layer and a parabolic surface. For such a solution, mass conservation implies that

$$R_s = R \left[1 + \frac{1}{R} \left(\frac{2\Omega^2 H_0}{g}\right)^{1/2}\right]^{1/2},$$

in which the CTF mass and surface tension at the boundary of the bucket [21] are neglected. The size of the truncated parabola $R_s$ depends on the square root of the rotation rate [20] and the total volume of fluid in the bucket. Fig. 1f shows that these two methods of determining the CTF size versus $\Omega$ are consistent with each other.

We verify Eq. 5 by using interferometry to characterize the thin fluid film in the center of the bucket. Fig. 1e shows a typical interferometric image of the CTF. There are several features in this image. When the bucket is properly leveled, the CTF is axisymmetric and flat with only modest height variations at best [22]. The fine structure of the edge of the CTF is not visible through interferometry, so we arbitrarily but consistently define the edge of the CTF by the faint ring indicated by the arrow in Fig. 1e, which is visible in all experiments. The obvious fine structure within the ring is related to the contact line between the thin film and the parabolic solution. This fine structure is nontrivial; it is left for future work. The radius $R_s$ of the CTF for various rotation rates and fluid volumes is shown in Fig. 1f. $R_s \sim \sqrt{\Omega}$ above $\Omega_c$ is as predicted by [20]. Theoretical
and numerical predictions are made with the same $R, H$, some minutes at $\Omega = 1 \text{ rps}$. Blue lines for reference are power in a static bucket, the rotation rate is ramped up to procedure: starting from an initially flat fluid surface indicated in Fig. 2a. We use the following experimental center of the bucket. An example of this behavior is cessation of the bucket rotation, is ‘released’ in a dam break fashion. This creates a fluid flux towards the edge of the bucket serves as a reservoir that, upon spinning for some minutes at $\Omega = 1 \text{ rps}$. Blue lines for reference are power laws of exponents 0.5 and 0.6.

During the process of establishing a parabolic profile, the CTF is also continuously draining fluid, making the CTF change in thickness. This drainage process is governed by a balance between centrifugal forces and viscous drag in the thinning layer. The efflux of fluid, radially outward from the CTF, slows progressively over time towards an equilibrium height profile, as described by the scaling laws first given by Emslie, Bonner and Peck (EBP) [23]. In order to observe the collapse dynamics, it is however important to make sure that the CTF is as thin as possible. Therefore, in the numerics, we set up initial conditions as shown in Fig. 1b. Specifically, the initial conditions are $h(r, t = 0) = \min(h_{00}, 20(r - 0.07747))$. This creates a fluid mass of surface one which can then be rescaled with the height factor $H_0$: we choose the CTF layer thickness $h_{00} = 10^{-4} H_0$ sufficiently small that it does not affect the results.

**B. Top View Imaging**

The fluid volume under the parabolic solution at the edge of the bucket serves as a reservoir that, upon cessation of the bucket rotation, is ‘released’ in a dam break fashion. This creates a fluid flux towards the center of the bucket. An example of this behavior is indicated in Fig. 2a. We use the following experimental procedure: starting from an initially flat fluid surface in a static bucket, the rotation rate is ramped up to $\Omega > \Omega_c$, which creates the partial parabolic profile. After rotating at constant $\Omega$ for a finite time, rotation is stopped virtually instantaneously. This removes the centrifugal force and initiates the collapse of the fluid reservoir at the boundary. Tracer particle tracking on the surface of the fluid showed that all experiments are at low enough Reynolds number $Re$ such that rotational flow never persists more than a small fraction of the initial collapse. The CTF spot in the center of the bucket thus disappears relatively slowly. This process is indicated in Fig. 2c: these experiments are similar as the one described in Ref. [16]. Note that the collapse dynamics of the front over a too-thick CTF layer ends with a standard diffusive $R_S(t) \sim t^{1/2}$ exponent for a fixed height threshold. This is due to the fact that with a too thick CTF layer, any cavity is simply a dip in the fluid surface, with an approximately parabolic shape. To avoid mixing this low exponent with nontrivial scaling dynamics it is essential to make the CTF as thin as possible just before the collapse. The disk is spun sufficiently long to ensure this is the case. Imaging suggests that the typical initial film thickness is about 100 micrometer or less.

From Eq. [3] it is evident that the speed of the nonlinear evolution of the collapsing front is proportional to $H_0^{-3}$. Experimental results are consistent with this. We measure the collapse timescale by direct imaging the CTF during collapse for a $\eta = 1000 \text{ mPas}$ and a range of $H = 1.7$ to 7.3 mm. Results are shown in Fig. 2b; in the accessible range of data, the experimental results are indeed consistent with a total collapse time scaling of $H_0^{-3}$.

At the end of the collapse, just before cavity closure, the spot size versus time is clearly nonlinear: from the data shown in Fig. 2b, the size of the shrinking CTF spot is tracked with an intensity threshold technique. The result is shown in Fig. 2c: the spot size varies as $R_S(t) \sim t^{0.55}$ with an uncertainty of about 0.05 in the exponent. However, as we will see below, log-fitting routine is unable to capture an accurate exponent for the collapse. To better characterize the collapse dynamics, we turn to surface structure imaging.

**C. Surface Structure Imaging**

To obtain full surface structure data in the experiments, we step away from interferometry and instead image a cross section of the (radially symmetric) height profile with a high speed camera (Photon) – see Fig. 3. Surface contrast is created by adding an oil-soluble fluorescent dye in the fluid (Pyromethene 567) and illuminating the dye with a 532nm laser line. There is sufficient dye concentration in the PDMS, that only the fluid close to the surface fluoresces (Fig. 3a).

The laser line scanning method is used to obtain the
due to centrifugal force, which is quantitatively in agreement with the prediction from [20]. The typical behavior of the fluid layer after cessation of rotation is shown in Fig. 3a. The parabolic surface profile first inverts its curvature to form the collapsing front shape as visible in Fig. 3b. The curvature inversion happens all while the contact line is essentially static – this is a typical waiting time solution [25, 26] of the governing equation, which creates reproducible initial conditions for the collapse experiments. Beyond this point a symmetric collapse of the central cavity is observed. In Fig. 3c, $h(t)$ for $r = \pm 0, \pm 10, \pm 25$ and $\pm 60$ mm radial positions along the diameter marked by the laser line in the 65 mm radius bucket is shown. Contour lines for both positive and negative $r$ are shown to indicate the radial symmetry in the surface shape. The figure shows that the collapse point is ambiguously defined due to the curvature in the fluid surface: even at $r = 0$ the surface profile $h(r,t)$ has a finite slope. Nevertheless we can extract the CTF collapse exponent by choosing a height threshold: we define the radius of the CTF with $R_S(t) = \max(r) h(r,t) < h_T$, the largest $r$ for which the experimentally determined height profile $h(r,t)$ is smaller than threshold $h_T$. $R_S(t)$ for two different collapse experiments and four different $h_T$ for each experiment is shown in Fig. 4a,b. For the experimental settings used, the data in Fig. 4b shows that $R_S(t) \sim t^{0.65}$ with an uncertainty in the exponent of about 0.05, but very similar to the exponent obtained with the top view experiments. These two front tracking methods therefore yield consistent results.

### IV. NUMERICAL RESULTS

We use a second-order-accurate implicit finite difference scheme to solve the time dependent axisymmetric lubrication equation (Eq. 3). Time stepping is dynamic to allow us to resolve the fluid surface motion just before collapse. Numerics give us access to a wide range of radii $R = 0.01 \cdots 5$ m (using $\cdots$ to indicate a range) and
surface tensions $\gamma = 2 \times 10^{-6} \cdots 2 \times 10^6$ N/m, which allow us to study a large range of $Bo = 10^{-2} \cdots 10^{10}$. The ratio of $\eta$ and the time step in the numerics is set to keep computation time small; we ensure mass conservation inside the film to within $10^{-6}H_0$ or better.

A typical $h(r,t)$ is shown in Fig. 5a. The initial conditions, set up to mimic the surface profile during rotation, clearly evolve towards a collapsing CTF upon cessation of the rotation. From these data, we extract several quantities. The total collapse time scaling is $\propto H_0^{-3}$, as is evident from Eq. 3 in which $h_c \propto h^3$. We extract this time scale by measuring the time it takes for the thin film to rise to $0.75 \gg h_{00}$ at $r = 0.6R$ after the cessation of rotation; scaling results are insensitive to the choice of these parameters. However, they induce an arbitrariness in the time scale through the threshold and via viscous slowing, so only scaling with $H_0$ is verified here. The numerical collapse dynamics are indeed consistent with $H_0^{-3}$, as shown in Fig. 5b.

Second, the size, $R_S(t)$, of the CTF or the ‘spot’ in the center of the container is shown in Fig. 6 where $R_S(t)$ is shown as a function of the time to collapse $t_c - t$ with $t_c$ the collapse moment, defined as the moment at which all film heights exceed a minimum threshold level, $h(r,t_c) > h_T$.

The extracted scaling exponents tend to the value predicted by Diez et. al. for large values of $Bo$, that is, in an infinite domain. However, the exponents extracted with log fitting show considerable scatter and no systematic convergence towards the 0.762 predicted by Diez et. al., indicated by the dash-dotted line in Fig. 6. We also find no trend of the extracted exponents with $H_0$ or the thresholds used for the extraction of $R_S(t)$ (not shown). This suggests that a better method to extract exponents is needed. We discuss here the results of two such methods; details can be found in the Appendix.

A. Triangulation Method

One alternative approach to extract scaling exponent starts by noting that the thin film equation Eq. [1] in the large $R$ limit allows for similarity solutions of the form

$$h(r,t) = (t_c - t)^\alpha F \left( \frac{r}{(t_c - t)^\beta} \right),$$

in which $\alpha, \beta$ are two scaling exponents, $F$ is a shape function and $t_c$ is the collapse moment. We can use this feature to find the $\alpha, \beta$ for a particular experimental or numerical solution of the $h(r,t)$ profile.

We apply the method from Sec. A.1A to a series of numerical simulations at a range of $R, \gamma$ as discussed above, we extract $\alpha, \beta$ for each of the simulations. Results are shown in Fig. 7a,b. Fig. 7a shows $\beta$ versus the dimensionless ratio $\rho g R^2 / \gamma$ (the color code indicates the value of $\alpha$). This panel shows that there are two regimes: for large $Bo$, the $\beta$ that best describe the shape of $h(r,t)$ asymptotes to the expected value of 0.762 indicated with the dashed line. For smaller $Bo$ there is a large crossover range to a regime in which $\beta \sim 0.5$. In Fig. 7b we show $\alpha(\beta)$. Scaling theory predicts a relationship between the exponents $\alpha = (2\beta - 1)/3$ as indicated by the dash dotted line; clearly the numerics for large $Bo$ converges to this solution. At small $Bo$, a different set of solutions emerges, clearly separated from the other self-similar solutions. Note that the underlying assumption in all these analyses is that in all regimes, a self-similar solution exist; this need not be the case for small $Bo$.
B. Least Squares method

The second test relies on knowledge of the spatial and temporal derivative $h_r, h_t$, respectively. For the scaling form of Eq. 6, some algebra provides an expression relating $\alpha$ and $\beta$ directly for each point on $h(r, t)$ within the scaling regime:

$$(t_c-t)h_t = -\alpha h + \beta rh_r , \quad (7)$$

Having access to the gradients of $h$, for example in the numerics, Eq. 7 provides a set of linearly independent equations, with only $\alpha$ and $\beta$ as fit parameters. We can thus solve this equation in a least squares sense. This method is computationally much more efficient than the triangulation method discussed above. Details are discussed in Sec. VI.B Fig. 8b, which shows that the LS method gives results consistent with the triangulation method, yet its $\alpha(\beta)$ relation satisfies the imposed constraint $\alpha = (2\beta - 1)/3$ much better. The transition region between large and small $Bo$ is more consistently captured with the least squares method, as shown in Fig. 8b.

The numerical results described in this section have shown that extracting power-law exponents on log-log scale is not accurate enough in the context of the exponents of similarity solutions. The two alternative, complementary methods described provide a better way to analyze similarity solutions. The results are consistent with experimental data.

V. CONCLUSIONS

We studied the effect of surface tension on viscous gravity currents in a novel experimental setup. We developed two complementary methods to extract scaling information from the self-similar structure of the converging surface profile. These methods allow us to analyze numerical data over the complete range of purely gravity driven, to purely surface tension dominated flows, and allow for a comparison the experimental data where that is available. For smaller Bond numbers, we find that there is a new set of solutions that deviates substantially from the solutions found by Diez et. al. [10]. The crossover regime spans a surprisingly large range of $Bo$ numbers, and makes the Diez et. al. solutions accessible only for fluids of extremely low surface tension, or in an infinite domain.

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VI. APPENDIX

We discuss here two method to extract scaling exponents from surface structure data as obtained in the numerics described in this article. We apply one method to the experimental surface structure data to provide a quantitative comparison between experiments and numerics.

A. Triangulation method

The first method works by extracting all $h(r, t)$ for which $t_c - t > 0$ and $|t_c - t| < 0.05t_c$, meaning all $t$ before the collapse, but sufficiently close to the collapse that the scaling assumptions should hold. We can plot the datapoints $h(r, t)$ from this set on rescaled Cartesian coordinate axis $\{ x, y \} = \{ r/(t_c - t)^\alpha, h(t_c - t)^\alpha \}$. An example of such a set is shown in Fig. 9a for reasonable choice of $\alpha, \beta$. In general, this plotting method will produce a scatter cloud of datapoints. The crux of this method is that for the $\alpha, \beta$, that best represents the scaling function, the scatter plot of all the surface points collapses to produce the function $F$ from Eq. 6. This optimal collapse, and thus the form of $F$, is shown in Fig. 9a.

To find the best $\alpha, \beta$, we triangulate the surface spanned by the aforementioned set and measure the statistical properties of the set of triangulated areas $\{ V \}$ found this way. Fig. 9b shows the distribution of areas found for the example from Fig. 9a. The best collapse minimizes the total triangulated surface area. This minimum is best captured by the median $\{ V \}_{med}$ of the set $\{ V \}$, in order to be less sensitive to outliers in the dataset. In Fig. 9c, we show $\{ V \}_{med}(\alpha, \beta)$. The minimum of the triangulated surface is indicated by the cross, whose collapse is shown in Fig. 9b – indeed the minimum $\{ V \}_{med}(\alpha, \beta)$ corresponds to a very good collapse of the data.
FIG. 9: (a) Scatter plot of rescaled h obtained for \( \gamma = 2 \times 10^{-6} \text{ N/m} \) and \( R = 1 \text{ m} \); \( Bo = 4.7 \times 10^6 \). Quantities for \( \alpha = 0.2, \beta = 0.6 \). The blue lines represent the triangulation of the datapoints. Color indicates \( \log h \). (b) Same as (a), without triangulations and with \( \alpha = 0.18, \beta = 0.78 \), representing the collapse properties indicated by the + in (d), the \( \alpha, \beta \) for which \( \{V\}_{med} \) is minimal. (c) The probability distribution function of the surface areas within the triangulation from (a). The dash-dotted line indicates the median. (d) \( \{V\}_{med}(\alpha, \beta) \) for this particular numerical run. Color scale indicates ranges from low (blue) to high (red).

FIG. 10: (a,b) \( \alpha, \beta \) as a function of band height, extracted with the least squares method described in the text. Color indicates least squares error is shown next to (c). (c) Band height for which the best \( \alpha, \beta \) solution was found, as a function of \( Bo \). Color scale is the same as in (a,b).

B. Least-Squares Scaling Form Test

Eq. \( 7 \) should be valid for all \( h(r, t) \); however, we can further maximize the resolving power of the exponent extraction method by applying this relationship to height bands within the surface. As before, we use the data for which \( t_c > t > 0 \) and limit ourselves to a time period close to the collapse, of five percent of the total collapse duration: \( |t_c - t| < 0.05t_c \); the band selection \( h_{Ti} < h < h_{Ti+1} \) selects a band of 0.01\( H_0 \), from \( h = 0 \) to \( h = 0.9H_0 \). For each numerical data set we compute for each band an \( \alpha, \beta \). The results for a typical run are shown in Fig. \( 10 \), b for \( \beta \) and \( \alpha \) respectively. The color coding indicates the least squares error \( LSE \equiv (h_i(t_c - t) + \alpha_i h(r, t) - \beta_i r_h)_r^2 \).

Both figures show similar trends: For low and high bands, the \( \alpha, \beta \) found with the least squares method give poor fits as indicated by the large error. In an intermediate regime the error has a clear minimum. This minimum is consistently of order \( 10^{-14} \), as shown in Fig. \( 10 \). Also this panel indicates at which band the best fit is found; the optimal band is around 0.5\( H_0 \) for large \( Bo \), moves up in the intermediate regime and is lower for small \( Bo \).

C. Applying the Triangulation Method to Experimental Data

The triangulation based \( h(r, t) \) rescaling technique introduced above, can now be used to extract scaling exponents in the experimental data. \( \{V\}_{med}(\alpha, \beta) \) for several time ranges \( |t_c - t| < 0.05 \cdots 0.015t_c \) are shown in Fig. \( 11 \) for an experiment run at 45 \( ^\circ \text{C} \); results are similar for experiments at other temperatures. There are two important observations: the overall structure of the experimentally determined \( \{V\}_{med}(\alpha, \beta) \) is very similar to the numerical one; for comparison, see Fig. \( 9 \). The location of the minimum in \( \{V\}_{med}(\alpha, \beta) \) is, however, somewhat dependent on the choice of thresholds; we attribute this to the intermediate \( Bo \) number at which this experiment was performed.

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FIG. 11: $\{V\}_{med}(\alpha, \beta)$ for a particular time cutoff of 0.4 seconds for the 45°C PDMS dataset shown in Fig. 4b. The stars indicate minima in the $\{V\}_{med}(\alpha, \beta)$ found for a range of time cutoffs and choices for $t_c$. Based on these arbitrary thresholds, the error bars on the determination of $\alpha, \beta$ are as indicated in Fig. 10e.

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