Abstract: We investigate the fluctuations of the free energy of the 2-spin spherical Sherrington-Kirkpatrick model at critical temperature $\beta_c = 1$. When $\beta > 1$ we find asymptotic Gaussian fluctuations with variance $\frac{1}{2N} \log(N)$, confirming in the spherical case a physics prediction for the SK model with Ising spins. We furthermore prove the existence of a critical window on the scale $\beta = 1 + \alpha \sqrt{\log(N)} N^{-1/\beta}$. For any $\alpha \in \mathbb{R}$ we show that the fluctuations are at most order $\sqrt{\log(N)} / N$, in the sense of tightness. If $\alpha \to \infty$ at any rate as $N \to \infty$ then, properly normalized, the fluctuations converge to the Tracy-Widom$_1$ distribution. If $\alpha \to 0$ at any rate as $N \to \infty$ or $\alpha < 0$ is fixed, the fluctuations are asymptotically Gaussian as in the $\alpha = 0$ case. In determining the fluctuations, we apply a recent result of Lambert and Paquette [23] on the behavior of the Gaussian-$\beta$-ensemble at the spectral edge.

1 Introduction and main results

The Hamiltonian of the 2-spin spherical Sherrington-Kirkpatrick (SSK) model is given by,

$$H_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i,j} \sigma_i g_{ij} \sigma_j \quad (1.1)$$

where the $\{g_{ij}\}_{i,j}$ are independent standard normal random variables and the vector of spins lies in the state space $\sigma \in S^{N-1} := \{ \sigma \in \mathbb{R}^N : ||\sigma||_2^2 = N \}$. The SSK model was introduced by Kosterlitz, Thouless and Jones [22] as a simplification of the usual SK model which has the same form except that the spins $\sigma$ are assumed to lie in the hypercube. The model with Ising spins was introduced in 1975 in order to explain strange phenomena of various alloys [38]. Since its introduction, this model and its generalizations have been the subject of much research in both the physics and mathematics communities. We refer the interested reader to, e.g., [27, 30, 39, 41].

One of the primary thermodynamic quantities of interest is the free energy $F_N(\beta)$ which is defined by,

$$F_N(\beta) := \frac{1}{N} \log(Z_N(\beta)), \quad Z_N(\beta) := \frac{1}{|S^{N-1}|} \int_{S^{N-1}} e^{-\beta H_N(\sigma)} d\omega_{N-1}(\sigma) \quad (1.2)$$

where $d\omega_{N-1}$ is the uniform surface measure on $S^{N-1}$. Here, $Z_N(\beta)$ is the partition function, the normalizing constant of the Gibbs measure on $S^{N-1}$ that has weight proportional to $e^{-\beta H_N(\sigma)}$. Both the SK and SSK models are known to exhibit a phase transition at the critical temperature $\beta_c := 1$. In the case of the SSK, the limit of the free energy was calculated by [14] to be,

$$\lim_{N \to \infty} F_N(\beta) = f(\beta) := \begin{cases} \beta^2 / 4, & \beta \leq 1, \\ \beta - \frac{\log(\beta)}{2} - \frac{3}{4} & \beta \geq 1. \end{cases} \quad (1.3)$$

This was later rigorously proven by Talagrand [40].

Baik and Lee [6] further described the fluctuations of the free energy around its limiting value for non-critical $\beta \neq 1$ as follows. In the high temperature regime $\beta < 1$ the fluctuations are $O(N^{-1})$ and are asymptotically Gaussian. In the low temperature regime $\beta > 1$, the fluctuations are $O(N^{-2/3})$ and converge to the Tracy-Widom$_1$ distribution (TW$_1$) of random matrix theory, the limiting distribution

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of the largest eigenvalue of a symmetric random matrix with Gaussian entries [42] (note that [6] uses a different scaling in which the transition between high and low temperature regimes is at $\frac{1}{\sqrt{N}}$). The high temperature central limit theorem for the SK model with Ising spins was proven earlier by Aizenman, Lebowitz and Ruelle [1].

Less is known about the size or nature of the fluctuations of either the SK or SSK models at or near the critical temperature $\beta_c = 1$. Talagrand examined the SK model on the scale $\frac{1}{\sqrt{N}}$ and found that the overlap undergoes a phase transition depending on whether or not $c$ diverges [39]. At $\beta_c$ he showed that the variance is at most $O(N^{-3/2})$. Chatterjee proved a general estimate that for fixed $\beta$ the variance of the free energy of the SK model is at most $O((N \log(N))^{-1})$ [12]. Chen and Lam [13] recently obtained the estimate for the SK model,

$$\text{Var}(F_N^{(SK)}(\beta_c)) \leq C \frac{\log(N)^2}{N^2},$$  \hspace{1cm} (1.4)

which is the tightest bound for the variance at critical temperature.

For the SK model, it is predicted that [2, 34]

$$\text{Var}(F_N^{(SK)}(\beta_c)) = \frac{\log(N)}{6N^2} + O(1).$$  \hspace{1cm} (1.5)

Our first main result addresses this prediction for the SSK model.

**Theorem 1.1.** Let $\beta = \beta_c = 1$ and $F_N(\beta)$ be the free energy of the SSK model, with $f(\beta)$ its limiting value as above. Then, the random variable

$$\frac{N(F_N(\beta_c) - f(\beta_c)) + \frac{1}{12} \log(N)}{\sqrt{\frac{1}{6} \log(N)}}$$  \hspace{1cm} (1.6)

converges in distribution to a standard normal random variable as $N \to \infty$.

Based on the form of the variance that they find in the high and low temperature regimes, Baik and Lee [6] predicted that there is a critical regime in the SSK model near the critical temperature of the form

$$\beta = 1 + \alpha \sqrt{\frac{\log(N)}{N}}$$  \hspace{1cm} (1.7)

for $\alpha \in \mathbb{R}$ of order 1, for which the size of the fluctuations are $O(N^{-1} \sqrt{\log(N)})$. Our next theorem confirms this prediction by showing that the fluctuations are at most of this order. Moreover, we find that in the cases $(\alpha)_{\alpha} \to 0$ and $\alpha \to \infty$, the free energy fluctuations converge to the Gaussian and Tracy-Widom distributions that characterize the high and low temperature regimes, respectively.

**Theorem 1.2.** Let $\beta = 1 + \alpha \sqrt{\log(N)}/N^{1/3}$. Then the random variable,

$$\frac{N(F_N(\beta) - f(\beta)) + \frac{1}{12} \log(N)}{\sqrt{\frac{1}{6} \log(N)}}$$  \hspace{1cm} (1.8)

is tight. If $\alpha \to 0$ or $\alpha < 0$ is fixed, then it converges to a standard normal random variable. Suppose that $\alpha \to \infty$ as $N \to \infty$ but slowly enough that $\beta$ remains bounded above. Then,

$$\frac{1}{N^{1/3}(\beta - 1)} (NF_N(\beta) - f(\beta)) + \log(N)/12$$  \hspace{1cm} (1.9)

converges to a Tracy-Widom random variable.

Note that Theorem 1.1 follows from the above result. The restriction that $\beta$ is bounded above is for convenience; it is likely that the result holds without this condition under minimal modification of our methods or those of Baik-Lee [6]. We are interested in the behavior of $\beta$ near 1 so we do not pursue this minor extension in our work.
In many respects, the behavior of the SSK model turns out to be quite different from the SK model with Ising spins or higher-order spin models with either continuous or discrete state spaces. This is primarily due to the fact that the combination of the quadratic nature of the SSK model and the continuous nature of the state space reduces the complexity of the energy landscape, compared to other “true” spin glasses. For more complex models the number of critical points of the Hamiltonian is typically exponential in $N$ whereas for the SSK it is linear - see, e.g., [3]. Moreover, the SSK model is usually studied using random matrix methods instead of approaches more commonly associated with the theory of spin glasses.

The main simplification arising in the SSK model is a contour integral formula for observables in terms of the eigenvalues of a matrix formed from the disorder random variables,

$$H_{ij} = -\frac{g_{ij} + g_{ji}}{\sqrt{2N}}.$$  \hspace{1cm} (1.10)

The matrix $H$ is an element of the Gaussian Orthogonal Ensemble (GOE), one of the main objects of random matrix theory, and accordingly, much is known about its spectral properties. The contour integral was used by Kosterlitz-Thouless-Jones [22] in their original paper to study the free energy. Baik and Lee’s work [6] further examined this formula and used it to expand the free energy in terms of the eigenvalues of $H$. In the high temperature regime, the leading order fluctuations of the free energy are given by a linear spectral statistic,

$$\sum_i \varphi(\lambda_i)$$  \hspace{1cm} (1.11)

for a certain test function $\varphi$. Such objects are well studied in random matrix theory, see e.g., [26, 37] where it is proven for sufficiently regular $\varphi$ that the limiting fluctuations are Gaussian. In the low temperature case, Baik and Lee showed that the free energy fluctuations are determined by the largest eigenvalue of $H$, whose convergence was established by Tracy and Widom [42]; the limiting distribution is now called the Tracy-Widom$_1$ distribution.

Baik-Lee and Baik-Lee-Wu have since extended their analysis of the free energy to other models related to the SSK [7–9], including a bipartite model and models incorporating a Curie-Weiss-type interaction. Other recent developments have studied the overlap between two samples (replicas) from the random Gibbs measure as well as the SSK model with a magnetic field. The overlap is an important quantity in the theory of spin glasses, begin linked to the free energy via the Parisi formula [31–33]. In the work [28], Nguyen and Sosoe extended the contour integral formula of [22] to prove a central limit theorem for the overlap in the high-temperature regime. Subsequently, the author with Sosoe [24] examined the overlap fluctuations in the low temperature regime and found that they are determined by an explicit function of the Airy$_1$ random point field, which is the joint limit of the largest eigenvalues of the GOE. In the work [25] we analyzed the fluctuations of the free energy and overlap for the SSK model with magnetic field in various scaling regimes, confirming some predictions of Fyodorov and Le Doussal [19]. This model was also investigated by Baik, Collins-Wildman, Le Doussal and Wu [5].

We now recall the notation $\beta = 1 + \alpha \sqrt{\log(N)/N^{1/3}}$. When $(\alpha)_+ \to 0$ we show that the fluctuations of the free energy are determined by the logarithm of the characteristic polynomial of the matrix $H$ evaluated at a point at the edge of the limiting spectral measure (in this case at $E = 2 + O(N^{-2/3})$). Formally, this has the form of (1.11) but with a singular function $\varphi$, for which the existing central limit theorems of random matrix theory do not apply. However, recently Lambert and Paquette [23] have analyzed precisely the quantity we find. In particular, they prove a central limit theorem for the log-characteristic polynomial, allowing us to obtain Gaussian fluctuations. When $\alpha > 0$ we find that the free energy fluctuations are determined by a sum of the log-characteristic polynomial and the largest eigenvalue of $H$, and so we obtain tightness of the free energy fluctuations.$^1$

The main technical element of the present work is then to relate the free energy to the eigenvalues of $H$ using the method of steepest descent. In particular, the method of steepest descent has not been

1. In private communication, Lambert and Paquette indicate that it follows from their methods [23] that the log-characteristic polynomial and the largest eigenvalues of $H$ are asymptotically independent. Conditional on this, the $\alpha > 0$ regime of our main theorem is restated as convergence to an independent sum of Tracy-Widom$_1$ and Gaussian random variables.
carried out for the SSK model at critical temperature. The main technical complication is that the saddle itself fluctuates on the same scale as the eigenvalues. In order to handle this we require as input estimates for the eigenvalue positions on their natural scale (estimates in particular stronger than the well-known rigidity estimates and local semicircle law of random matrix theory -see, e.g., [11]). The estimates we require were proven with the author with Sosoe in our work [24]. They are proven using a result of Gustavson [20] on the eigenvalues of the Gaussian Unitary Ensemble (GUE) and a coupling between the GOE and GUE due to Forrester-Rains [18].

**Applications to statistics.** Consider the measure $P_\lambda$ on $N \times N$ symmetric matrices induced by

$$H_\lambda := H + \lambda vv^T$$

where $H$ is a matrix distributed according to the GOE and $v$ is a vector uniformly distributed on the unit sphere. This is an example of a spiked random matrix, introduced by Johnstone [21] as a simple high-dimensional model of the form “signal+ noise.” After its introduction, further investigations have shown the existence of a spectral transition at the point $\lambda_c = 1$ where the largest eigenvalue of $H_\lambda$ separates from the spectral bulk $[-2, 2]$ for larger $\lambda$ (the celebrated Baik-Ben Arous-Péché transition [4] see also, e.g., [10, 17] for important work). Since Johnstone’s seminal work a large statistical and mathematical literature on spiked models has emerged; we refer to, e.g., [15, 35] and the references therein for further discussion.

In this context, the quantity

$$N(F_N(\beta) - \beta^2/4)$$

turns out to be the likelihood ratio $\frac{P_\lambda}{P_0}$ under the “null hypothesis” $\lambda = 0$. In the somewhat different setting of sample-covariance matrices, Onatski, Moreira and Hallin [29] proved a central limit theorem for the likelihood ratio below the spectral transition (which is at a different point for their model but the behavior is analogous to our setting), implying the mutual contiguity of the null and non-null measures. They moreover proved that above the spectral transition the likelihood ratio converges to 0 (in fact is exponentially small). As a corollary, our work extends this result that the likelihood ratio tends to 0 as $N \to \infty$ near the spectral transition $\lambda = 1$.

**Organization of paper.** In the next section we introduce some notation as well as collect the results of random matrix theory that we use in our work. In Section 3 we establish preliminary estimates on the saddle that we use in the method steepest descent as well as analyze the leading order term. In Section 4 we carry out the method of steepest descent. In Section 5 we prove our technical propositions which expand the free energy of the SSK in terms of random matrix quantities. Theorem 1.2 is proven in Section 5.3.

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### 2 Preliminaries

#### 2.1 Notation

We fix some notation. When we have a complex parameter $z$ we will denote its real and imaginary parts by

$$z = E + i\eta.$$  

(2.1)

The Hamiltonian of the SSK model is

$$H_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i,j} \sigma_i g_{ij} \sigma_j =: -\sigma^T H \sigma$$ 

(2.2)

where $g_{ij}$ are independent standard normal random variables and $H$ has matrix elements,

$$H_{ij} = \frac{g_{ij} + g_{ji}}{\sqrt{2N}}.$$ 

(2.3)
Then $H$ has a random matrix from the Gaussian Orthogonal Ensemble and the relation between the Hamiltonian and $H$ is,

$$H_N(\sigma) = -\frac{1}{2} \sigma^T H \sigma.$$  \hfill (2.4)

We will denote the eigenvalues of $H$ in decreasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Note that they are almost surely distinct. For convenient reference we recall the definitions of the free energy and the partition function,

$$F_N(\beta) = \frac{1}{N} \log(Z_N(\beta)), \quad Z_N(\beta) := \frac{1}{|S^{N-1}|} \int_{S^{N-1}} e^{-\beta H_N(\sigma)} d\omega_{N-1}(\sigma)$$ \hfill (2.5)

where the $S^{N-1}$ is the $N-1$ dimensional sphere of radius $\sqrt{N}$, $\omega_{N-1}$ is the uniform measure on $S^{N-1}$ and so

$$|S^{N-1}| = \int d\omega_{N-1}(\sigma) = 2\pi^{N/2} \Gamma\left(\frac{N}{2}\right) N^{N-1}. \hfill (2.6)$$

**Assumption.** In general we allow $\beta$ to depend on $N$. However for definiteness throughout the remainder of the paper we will assume that there is a $m_0 > 0$ so that,

$$\frac{1}{m_0} \leq \beta \leq m_0 \hfill (2.7)$$

While it is possible to consider the cases of $\beta \to 0$ or $\infty$ using minor modifications of either our methods or those of Baik-Lee [6] we refrain from doing so as we are mainly interested in the behavior near $\beta = \beta_c = 1$.

An important role will be played by the function,

$$G(z) := \beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i)$$ \hfill (2.8)

where the log is the principal branch of the logarithm. Throughout the paper we will denote the saddle $\gamma$ as the unique solution $\gamma > \lambda_1$ to,

$$G'(\gamma) = 0. \hfill (2.9)$$

The almost sure limit of the empirical eigenvalue measure of $H$ is given by Wigner’s semicircle distribution,

$$\rho_{sc}(E) = \frac{1}{2\pi} \sqrt{4 - E^2} 1_{\{|E| \leq 2\}}, \hfill (2.10)$$

which has Stieltjes transform,

$$m_{sc}(z) = \int \frac{1}{x-z} \rho_{sc}(x) dx = \frac{-z + \sqrt{z^2 - 4}}{2}. \hfill (2.11)$$

The empirical Stieltjes transform will be denoted by,

$$m_N(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_i - z}. \hfill (2.12)$$

We say that an event $A$ (really, a sequence of events $A = A_N$) holds with overwhelming probability if for every $D > 0$ there is a constant $C_D$ on which

$$\mathbb{P}[A] \geq 1 - C_D N^{-D}. \hfill (2.13)$$

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2.2 Estimates on eigenvalue positions

The \( N \)-quantiles or classical eigenvalue locations \( \{\gamma_i\}_{i=1}^N \) of the semicircle distribution are defined by,

\[
\frac{i}{N} = \int_{\gamma_i}^{\gamma_{i+1}} \rho_{sc}(E)\mathrm{d}E.
\]  
(2.14)

A straightforward calculation gives for \( i \leq N/2 \),

\[
2 - \gamma_i = \left( \frac{3\pi i}{2N} \right)^{2/3} + \mathcal{O}\left( \frac{i^{4/3}}{N^{4/3}} \right).
\]  
(2.15)

Note that the error term is less than the rigidity error appearing in the following result as long as \( i \leq N^{2/5} \).

The following result is from, e.g., [11].

**Theorem 2.1.** For any \( \xi > 0 \) define the event \( \mathcal{F}_\xi \) by

\[
\mathcal{F}_\xi := \bigcap_{1 \leq i \leq N} \left\{ \left| \lambda_i - \gamma_i \right| \leq \frac{N^\xi}{N^{2/3} \min\{i^{1/3}, (N + 1 - i)^{1/3}\}} \right\}.
\]  
(2.16)

Then \( \mathcal{F}_\xi \) holds with overwhelming probability.

The following is a result of Section 6.2 of [24] as well as the fact that \( N^{2/3}(\lambda_1 - 2) \) converges to a random variable.

**Lemma 2.2.** For \( K > 0 \) and \( A > 0 \) let us denote the event \( \mathcal{G}_{K,A} \), by

\[
\mathcal{G}_{K,A} := \left\{ \bigcap_{K \leq j \leq N^{2/5}} \left\{ \left| N^{2/3}(\lambda_j - 2) + \left( \frac{3\pi j}{2} \right)^{2/3} \right| \leq \sqrt{j^{2/3}} \frac{1}{10} \right\} \bigcap \{N^{2/3}(\lambda_1 - 2) \leq A\} \right\}.
\]  
(2.17)

Let \( \varepsilon > 0 \). Then there are \( K, A \) so that \( \mathbb{P}[\mathcal{G}_{K,A}] \geq 1 - \varepsilon \) for all \( N \) large enough.

Since \( \lambda_2 \) and \( \lambda_1 \) converge jointly to the largest particles of the Airy\(_1\) random point field which is almost surely simple (see, e.g., Proposition 3.5 of [36]) we have the following.

**Lemma 2.3.** Define for \( b > 0 \) the event

\[
\mathcal{S}_b := \{N^{2/3}(\lambda_2 - \lambda_1) > b\}.
\]  
(2.18)

Let \( \varepsilon > 0 \). There is a \( b > 0 \) so that \( \mathbb{P}[\mathcal{S}_b] \geq 1 - \varepsilon \) for all \( N \) large enough.

We have also the following, proven in Section 6.2 of [24].

**Lemma 2.4.** There are positive \( C, C_1 \) so that,

\[
\mathbb{E} \left[ 1_{\{N^{2/3}(\lambda_j - 2) \leq -C_1\}} \left| N^{2/3}(\lambda_j - 2) + \left( \frac{3\pi k}{2} \right)^{2/3} \right| \right] \leq \frac{C \log(j)^2}{j^{1/3}}, \quad j \leq N^{2/5}.
\]  
(2.19)

The following is a consequence of Theorem 6.1 of [24].

**Lemma 2.5.** For any \( D > 0 \), let \( \mathcal{J}_D \) be the event that

\[
N^{1/3} \left| 1 + \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - \lambda_1} \right| \leq D
\]  
(2.20)

For any \( \varepsilon > 0 \) there is a \( D > 0 \) so that \( \mathbb{P}[\mathcal{J}_D] \geq 1 - \varepsilon \) for all \( N \) large enough.

We now prove the following simple consequence of the above lemmas.
Lemma 2.6. Let $A, K > C$ for $C$ sufficiently large. There is a $C' > 0$ so that for $N^{2/3}(E - 2) \geq 2A$ we have for all small $1/100 \geq \xi > 0$,

$$
E \left[ 1_{\mathcal{G}_{K,A} \cap \mathcal{F}_t} N^{1/3} \left| \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j - E} - m_{sec}(E) \right|^2 \right] \leq \frac{C'(K + \log^2(N^{2/3}(E - 2))}{N^{2/3}(E - 2)}
$$

$$
+ C' \left( N^{-2/3 + \xi} + 1_{\{N^{2/3}(E - 2) \geq N^{1/3 - \xi}\}} \frac{N^\xi}{N^{2/3}(E - 2)} \right)
$$

(2.21)

Proof. We write,

$$
\frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j - E} - m_{sec}(E) = \sum_{j > K} \int_{\gamma_{i-1}}^{\gamma_i} \frac{(x - \gamma_i) + (\gamma_i - \lambda_j)}{(\lambda_j - E)(x - E)} \, dE 
$$

$$
+ \frac{1}{N} \sum_{i=1}^{K} \frac{1}{\lambda_i - E} + \int_{0}^{\gamma_{K}} \frac{\rho_{sec}(x)}{x - E} \, dx.
$$

(2.22)

As long as $N^{2/3}(E - 2) > 2A$, the denominators in the last line are bounded below by $N^{2/3}(E - A)/2$ on the event $\mathcal{G}_{K,A}$. Therefore,

$$
1_{\mathcal{G}_{K,A}} N^{1/3} \left| \frac{1}{N} \sum_{j=1}^{K} \frac{1}{\lambda_j - E} + \int_{0}^{\gamma_{K}} \frac{\rho_{sec}(x)}{x - E} \, dx \right| \leq \frac{CK}{N^{2/3}(E - A)}.
$$

(2.23)

As long as $K$ is sufficiently large, then $N^{2/3}(\lambda_j - 2) \geq c j^{2/3}$ on $\mathcal{G}_{K,A} \cap \mathcal{F}_t$, for all $j > K$. From this we see that,

$$
N^{1/3} 1_{\mathcal{G}_{K,A} \cap \mathcal{F}_t} \left| \sum_{j=1}^{K} \int_{\gamma_{i-1}}^{\gamma_i} \frac{(x - \gamma_i) + (\gamma_i - \lambda_j)}{(\lambda_j - E)(x - E)} \, dE \right| 
$$

$$
\leq C 1_{\mathcal{G}_{K,A}} \sum_{j=K}^{N^{1/10}} \left( j^{-1/3} + N^{2/3}(\lambda_j - 2) + \left( \frac{3\pi j}{2} \right)^{2/3} \right) \frac{1}{j^{4/3} + (N^{2/3}(E - 2))^2}
$$

$$
+ C \sum_{j > N^{1/10}} \frac{1}{j^{4/3} + (N^{2/3}(E - 2))^2} \min\{j^{1/3}, (N + 1 - j)^{1/3}\}
$$

(2.24)

Note we used (2.15). For the last line, the contributions for terms in the sum for $j > N / 2$ are $O(N^{-2/3 + \xi})$. The remaining terms contribute $O(N^{-15 + \xi})$ if $N^{2/3}(E - 2) \leq N^{1/15}$ which can be absorbed into the first line of (2.21) if $N^{2/3}(E - 2) \leq N^{1/15 - \xi}$ and the second otherwise. If $N^{2/3}(E - 2) \geq N^{1/15}$ then the sum can be further estimated by dividing into cases when $j^{2/3}$ is larger or less than $N^{2/3}$. Ultimately, this is bounded by the second line of (2.21).

Taking $K$ large enough so that on $\mathcal{G}_{K,A}$ we have that $N^{2/3}(\lambda_j - 2) \leq -C_1$ where $C_1$ is from Lemma 2.4, we see

$$
E \left[ 1_{\mathcal{G}_{K,A}} \sum_{j=K}^{N^{1/10}} \left( j^{-1/3} + N^{2/3}(\lambda_j - 2) + \left( \frac{3\pi j}{2} \right)^{2/3} \right) \right] \leq C \sum_{j=K}^{N^{1/10}} \frac{j^{-1/3} \log(j)^2}{j^{4/3} + (N^{2/3}(E - 2))^2}
$$

$$
\leq C \sum_{j^{2/3} < N^{2/3}(E - 2)} \frac{j^{-1/3} \log(j)^2}{j^{4/3} + (N^{2/3}(E - 2))^2} + C \sum_{j^{2/3} \geq N^{2/3}(E - 2)} \frac{j^{-1/3} \log(j)^2}{j^{4/3} + (N^{2/3}(E - 2))^2}
$$

$$
\leq C \frac{\log^2(N^{2/3}(E - 2))}{N^{2/3}(E - 2)}
$$

(2.25)

This concludes the claim.

We require also the following.
Lemma 2.7. For $10 \geq E \geq 0$ and $0 \leq \eta \leq 10$ we have,
\[ c\sqrt{|E-2|} + \eta \leq |1 + m_{sc}(z)| \leq C\sqrt{|E-2|} + \eta. \]  
\hspace{1cm} (2.26)

For $2 \leq E \leq 10$ we have,
\[ -m_{sc} \leq 1 - c\sqrt{E-2}. \]  
\hspace{1cm} (2.27)

**Proof.** The first follows from the estimate (4.2) of [16] if we can show that $c \leq |1 - m_{sc}(z)| \leq C$ for $z$ as specified above. This is easily verified using (2.12). The second follows from the first estimate together with the fact that $m_{sc}^2(z) = 1$ and that $m_{sc}^2(E) > 0$ for $E > 2$. \quad \square

2.3 Convergence result

For any $Q > 0$ define the random variable
\[ X_Q := \sum_{i=1}^{N} \log |2 + Q N^{-2/3} - \lambda_i| - \frac{N}{2} - N^{1/3}Q + \frac{1}{6} \log(N). \]  
\hspace{1cm} (2.28)

The following is Corollary 1.2 of [23]. Note that they use a different scaling, where the eigenvalues of the GOE are asymptotically in the interval $[-1, 1]$.

**Theorem 2.8.** For any $Q > 0$ we have that
\[ \frac{X_Q}{\sqrt{2 \log(N)^3}} \]  
converges to a standard normal random variable.

2.4 Integral representation

Due to [6] we have for any matrix $H$,
\[ \int \exp \left[ \frac{\beta}{2} \sigma^{T} H \sigma \right] d\omega_{N-1}(\sigma) = \frac{\beta N^{1/2}}{2\pi i} \left( \frac{2\pi}{\beta} \right)^{\frac{N}{2}} \int_{a-i\infty}^{a+i\infty} \exp \left[ \frac{N}{2} G(z) \right] dz \]  
\hspace{1cm} (2.30)

for any $a > \lambda_1(H)$. Hence we have,

**Lemma 2.9.** The following representation for the free energy holds.
\[ F_N(\beta) = \frac{1}{N} \log(Z_N(\beta)) = \frac{1}{N} \log \Gamma \left( \frac{N}{2} \right) - \frac{N-2}{2N} \log(\beta N) + \frac{N}{2} \log(2) \]
\[ + \log \left( \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \exp \left[ \frac{N}{2} G(z) \right] dz \right) \]  
\hspace{1cm} (2.31)

where $a > \lambda_1$.

Note that due to Stirling’s formula, for $|1 - \beta| \leq 1/2$,
\[ \log \Gamma(N/2) - \frac{N-2}{2} \log(\beta N) = \frac{1}{2} \log(N) - \frac{N}{2} \log(2) - \frac{N}{2} \log(\beta) - \frac{N}{2} + O(1). \]  
\hspace{1cm} (2.32)

3 Estimates on saddle and $G(\gamma)$

Throughout this section we will make use of the events $S_b, J_D, G_{K,A}$ and $F_\xi$ that were defined in Section 2.2. Recall that the saddle $\gamma$ the unique solution satisfying $\gamma > \lambda_1$ to the equation
\[ G'(\gamma) = 0. \]  
\hspace{1cm} (3.1)
Note that this is the solution to,

\[ \beta = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i}. \]  

(3.2)

Indeed, note that for \( \gamma > \lambda_1 \) the RHS is a monotonically decreasing function that goes to \( \infty \) as \( \gamma \to \lambda_1 \) and 0 as \( \gamma \to \infty \), guaranteeing a unique solution.

We first prove the following general estimate.

**Proposition 3.1.** Assume \( \beta > c > 0 \). Let \( \varepsilon > 0 \). There is a constant \( C_\varepsilon > 0 \) so that,

\[ \frac{1}{C_\varepsilon(1 + N^{1/3}(\beta - 1)_+)} \leq N^{2/3}(\gamma - \lambda_1) \leq C_\varepsilon(1 + (N^{1/3}(1 - \beta)_+)^2). \]

(3.3)

with probability at least \( 1 - \varepsilon \) for all \( N \) large enough.

**Lemma 3.2.** Let \( b, A, K, D > 0 \), with \( K \) sufficiently large. On the event \( \mathcal{G}_{K, A} \cap \mathcal{J}_D \cap \mathcal{S}_b \cap \mathcal{F}_\xi \) with \( \xi = 1/100 \) we have for \( E > \lambda_1 \),

\[ \frac{1}{N} \sum_{i=1}^{N} \frac{1}{E - \lambda_i} \geq 1 - DN^{-1/3} - C'N^{-1/3}(Kb^{-1} + N^{2/3}(E - \lambda_1)) + \frac{1}{N^{1/3}N^{2/3}(\lambda_1 - E)}. \]

(3.4)

**Proof.** We write,

\[ 1 + \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - E} = \left( 1 + \frac{1}{N} \sum_{i=2}^{N} \frac{1}{\lambda_i - \lambda_1} \right) \]

\[ + \left( \frac{1}{N} \sum_{i=2}^{N} \frac{1}{\lambda_i - E} - \frac{1}{\lambda_i - \lambda_1} \right) \]

\[ + \frac{1}{N^{1/3}N^{2/3}(E - \lambda_1)}. \]

(3.5)

Note that for \( K \) large enough we have on \( \mathcal{G}_{K, A} \cap \mathcal{F}_\xi \) (with \( \xi = 1/100 \)) that \( N^{2/3}(\lambda_j - \lambda_1) > cj^{2/3} \) for \( j \geq 10K \). Hence, the term on the second line is bounded by,

\[ \left| \frac{1}{N} \sum_{i=2}^{N} \frac{1}{\lambda_i - E} - \frac{1}{\lambda_i - \lambda_1} \right| \leq \frac{CK}{N^{1/3}} + N^{2/3}(E - \lambda_1) \sum_{j>1}^{N} \frac{1}{N^{1/3}j^{4/3}}. \]

(3.6)

This yields the claim. \( \square \)

**Proof of Proposition 3.1.** From Lemma 2.6 and Markov’s inequality, we see that for any \( \varepsilon > 0 \), there is a \( C_2 > 0 \) so that for any \( N^{2/3}(E - 2) \geq C_2 \) there is an event (depending on \( E \)) on which \( \lambda_1 \leq E \) and

\[ \frac{1}{N} \sum_{j=1}^{N} \frac{1}{E - \lambda_j} \leq N^{-1/3} - m_{\text{sec}}(E) \leq N^{-1/3} + \beta + (1 - \beta) - cN^{-1/3} \sqrt{N^{2/3}(E - 2)}, \]

(3.7)

where we used Lemma 2.7 in the second estimate. For \( N^{2/3}(E - 2) \geq C(1 + N^{2/3}(1 - \beta)_+) \) for some \( C > 0 \), the RHS is less than \( \beta \). This proves the upper bound we require. The lower bound follows immediately from Lemma 3.2. \( \square \)

The following lemma will be useful.

**Lemma 3.3.** On \( \mathcal{G}_{K, A} \cap \mathcal{F}_\xi \) for \( K \) sufficiently large and \( \xi < 1/100 \) we have for \( E > \max\{2, \lambda_1\} \) that,

\[ \frac{1}{N^{4/3}} \sum_{j} \frac{1}{(\lambda_j - E)^2} \leq \frac{K}{N^{4/3}(E - \lambda_1)^2} + \frac{C}{\sqrt{N^{2/3}(E - 2)}}. \]

(3.8)

If \( \mathcal{S}_b \) holds,

\[ \frac{1}{N^{4/3}} \sum_{j>1} \frac{1}{(\lambda_j - \lambda_1)^2} \leq C_{b,k} \]

(3.9)
Proof. For \( j > K \) it holds that \( E - \lambda_j \geq (E - 2) + (2 - \lambda_j) \geq (E - \lambda_1) + cN^{-2/3}j^{2/3} \) if \( K \) is sufficiently large. Hence,

\[
\frac{1}{N^{4/3}} \sum_j \frac{1}{(\lambda_j - E)^2} \leq \frac{K}{N^{4/3}(E - \lambda_1)^2} + \sum_{j > K} \frac{1}{j^{4/3} + N^{4/3}(E - 2)^2}.
\] (3.10)

The second sum is easily seen to be \( O((N^{2/3}(E - 2))^{-1/2}) \), by dividing it into the cases \( j^{2/3} < N^{2/3}(E - 2) \) and \( j^{2/3} \geq N^{2/3}(E - 2) \). The second estimate is similar.

We now proceed slightly differently in the high temperature and low temperature cases to find better estimates.

### 3.1 High temperature \( \beta \leq 1 \)

In the high temperature case we define \( \tilde{\gamma} \) by

\[
\beta + m_{sc}(\tilde{\gamma}) = 0.
\] (3.11)

Due to Lemma 2.7 we have in the case that \( \beta < 1 \) that,

\[
c(1 - \beta)^2 \leq \tilde{\gamma} - 2 \leq C(1 - \beta)^2.
\] (3.12)

We first need the following rough bound.

**Lemma 3.4.** Suppose that \( \beta \leq 1 \). For any \( \varepsilon > 0 \) there is a \( C_{\varepsilon} \) for with probability at least \( 1 - \varepsilon \) for all large \( N \) we have for

\[
(C_{\varepsilon})^{-1}(1 + N^{2/3}(1 - \beta)^2) \leq N^{2/3}(\gamma - 2) \leq C_{\varepsilon}(1 + N^{2/3}(1 - \beta)^2).
\] (3.13)

**Proof.** We only need to prove the lower bound. From Lemmas 2.6 and 2.7 we see that there is a \( C_{\varepsilon} > 0 \) so that for \( N^{2/3}(E - 2) \geq C_{\varepsilon} \) we have with probability at least \( 1 - \varepsilon \) that

\[
\frac{1}{N} \sum_{j=1}^{N} \frac{1}{E - \lambda_j} \geq -m_{sc} - N^{-1/3} \geq \beta + (1 - \beta) - C\sqrt{(E - 2)} - N^{-1/3}.
\] (3.14)

Hence if \( N^{1/3}(1 - \beta) \geq 10C(C_{\varepsilon})^2 + 10 \) we see that \( -m_{N}(E) > \beta \) for \( N^{2/3}(E - 2) \leq C_{\varepsilon}(\beta - 1)^2 \). This yields the claim, together with Proposition 3.1, which gives us the lower bound in the case that \( N^{1/3}(1 - \beta) \leq 10C(C_{\varepsilon})^2 + 10 \).

Now we need the following.

**Lemma 3.5.** Let \( \varepsilon > 0 \) and \( \xi > 0 \). Assume \( \beta \leq 1 \). There is a constant \( C_{\varepsilon} \) on which the following holds with probability at least \( 1 - \varepsilon \) that, for large enough \( N \),

\[
N^{2/3}|\tilde{\gamma} - \gamma| \leq C_{\varepsilon} \left\{ \frac{\log^2(1 + N^{1/3}(1 - \beta))}{1 + N^{1/3}(1 - \beta)} + N^{-2/3 + \xi}(1 + N^{1/3}(1 - \beta)) + 1_{(N^{2/3}(1 - \beta)^2 \geq N^{1/15 - \xi})} \frac{N^{\xi}}{1 + N^{1/3}(1 - \beta) + 1} \right\}
\] (3.15)

**Proof.** First we see that for \( N^{2/3}(\tilde{\gamma} - 2) \geq C_{\varepsilon} \) we have with probability at least \( 1 - \varepsilon \) that,

\[
N^{1/3}|m_{N}(\tilde{\gamma}) - m_{sc}(\tilde{\gamma})| \leq C_{\varepsilon} \left( \frac{\log^2(N^{2/3}(\tilde{\gamma} - 2))}{N^{2/3}(\tilde{\gamma} - 2)} + N^{-2/3 + \xi} + 1_{(N^{2/3}(\tilde{\gamma} - 2) \geq N^{1/15 - \xi})} \frac{N^{\xi}}{N^{2/3}(\tilde{\gamma} - 2)} \right)
\] (3.16)

for any \( \xi > 0 \) and \( N \) large enough, by Lemma 2.6. On the other hand, on \( \mathcal{G}_{K,A} \) and if \( N^{2/3}(\tilde{\gamma} - 2) \geq 2A \) we have,

\[
|m_{N}(\gamma) - m_{N}(\tilde{\gamma})| \geq |\gamma - \tilde{\gamma}| \frac{1}{N} \sum_{j=2}^{N} \frac{1}{(\lambda_j - \max\{\tilde{\gamma}, \gamma\})^2}.
\] (3.17)
There is an event of probability at least $1 - \varepsilon$ on which \(\max\{\tilde{\gamma}, \gamma\} - 2 \leq N^{-2/3}C_0(1 + N^{2/3}(1 - \beta)^2).\) When this estimate holds and on \(\mathcal{G}_{K,A} \cap \mathcal{F}_\xi\) we have that for \(j^{2/3} \leq N^{2/3}(1 + N^{2/3}(1 - \beta)^2)\) that

\[
N^{2/3}|\lambda_j - \max\{\tilde{\gamma}, \gamma\}| \leq K^{2/3} + C_\varepsilon(1 + N^{2/3}(1 - \beta)^2). \tag{3.18}
\]

Hence, on this event,

\[
\frac{1}{N^{4/3}} \sum_{j=2}^{N} \frac{1}{(\lambda_j - \max\{\tilde{\gamma}, \gamma\})^2} \geq \frac{c_{K,\varepsilon}}{N^{1/3}(1 - \beta) + 1}. \tag{3.19}
\]

This yields the claim after taking \(K, A\) large enough, from the equality

\[
m_N(\tilde{\gamma}) - m_{sc}(\tilde{\gamma}) = m_N(\tilde{\gamma}) - m_N(\gamma), \tag{3.20}
\]

which holds by definition of \(\gamma, \tilde{\gamma}\). Define now,

\[
\tilde{\gamma}_s = \max\{\tilde{\gamma}, 2 + sN^{-2/3}\}. \tag{3.21}
\]

We have the following.

**Lemma 3.6.** Let \(\varepsilon > 0\). Assume \(\beta \leq 1\). There are \(C_1\) and \(C_2\) depending on \(\varepsilon\) so that any \(s \geq C_1\) the following holds with probability at least \(1 - \varepsilon\).

\[
|G(\gamma) - G(\tilde{\gamma}_s)| \leq \frac{C_2(1 + s^2)}{N}. \tag{3.22}
\]

**Proof.** Assume \(\mathcal{G}_{K,A}\) holds and choose \(A, K\) large enough so that \(\mathbb{P}[\mathcal{G}_{K,A}] > 1 - \varepsilon\). Choose \(C_1 > 2A\). We may further assume that the estimates of the previous two lemmas hold.

By a second order Taylor expansion around \(\gamma\) (using that \(G'(\gamma) = 0\)) we have,

\[
|G(\gamma) - G(\tilde{\gamma}_s)| \leq C|\gamma - \tilde{\gamma}_s|^2 \max_{x \in [\gamma, \tilde{\gamma}_s]} |G''(x)|. \tag{3.23}
\]

On the event \(\mathcal{G}_{K,A} \cap \mathcal{F}_\xi\) for \(\xi < 1/100\), we have for \(x > \lambda_1\),

\[
\frac{1}{N^{4/3}} |G''(x)| \leq \frac{K}{N^{4/3}(x - \lambda_1)^2} + C \sum_{j > 10K} \frac{1}{j^{4/3} + N^{4/3}(x - \lambda_1)^2} \leq \frac{K}{N^{4/3}(x - \lambda_1)^2} + \frac{C}{\sqrt{N^{2/3}(x - \lambda_1)}}. \tag{3.24}
\]

This yields the claim after we note that we have,

\[
|\gamma - \tilde{\gamma}_s|^2 \leq C \frac{1 + s^2}{N^{4/3}}, \tag{3.25}
\]

on the event of Lemma 3.5. □

**Proposition 3.7.** Let \(\varepsilon > 0\). Let \(\beta \leq 1\) and assume \((1 - \beta)N^{1/3} \leq N^{1/10}\). There is a \(Q > 0\) and \(P > 0\) so that there is an event with probability at least \(1 - \varepsilon\) on which,

\[
\left|G(\gamma) - \left( R(2 + QN^{-2/3}) + \beta^2/2 - 1/2 + \log(\beta) \right) \right| \leq P \frac{1 + \log^5(N^{1/3}(1 - \beta))}{N} \tag{3.26}
\]

where \(R(z) := z - N^{-1} \sum_i \log |z - \lambda_i|\).

**Proof.** First assume that the estimate of the previous lemma holds for some \(\tilde{\gamma}_s\). We can assume also that \(\mathcal{G}_{K,A} \cap \mathcal{F}_\xi\) holds and \(s > 2A\). We take \(Q = s\). If \(\tilde{\gamma}_s = 2 + sN^{-2/3}\) we are finished, after noting that this implies that \(1 - \beta \leq CN^{-1/3}\), and that

\[
\beta(2 + sN^{-2/3}) - \beta^2/2 - 1/2 + \log(\beta) = 2 + sN^{-2/3} + \mathcal{O}(N^{-1}) \tag{3.27}
\]
in this case. Otherwise, we integrate,
\[
\frac{1}{N} \sum_{j=1}^{N} \log(\hat{\gamma} - \lambda_j) - \log(2 + sN^{-2/3} - \lambda_j) = - \int_{2 + sN^{-2/3}}^{\hat{\gamma}} m_N(t) dt
\]
\[
= - \int_{2 + sN^{-2/3}}^{\hat{\gamma}} m_{sc}(t) dt - \int_{2 + sN^{-2/3}}^{\hat{\gamma}} (m_N(t) - m_{sc}(t)) dt
\]
(3.28)

By an application of Lemma 2.6 we have,
\[
\mathbb{E} \left[ 1_{\mathcal{G}_{K,A} \cap \mathcal{F}_\xi} \int_{2 + sN^{-2/3}}^{\hat{\gamma}} (m_N(t) - m_{sc}(t)) dt \right] \leq C \frac{\log^3(1 + N^{1/3}(1 - \beta)) + N^\xi(1 - \beta)^2}{N}.
\]
(3.29)

We can assume \( \xi < 1/100 \), so that the second term in the numerator is \( \mathcal{O}(1) \). Hence by Markov’s inequality this is less than \( C \varepsilon \log^3(1 + N^{1/3}(1 - \beta)) \) with probability at least \( 1 - \varepsilon \). Moreover,
\[
\int_{2 + sN^{-2/3}}^{\hat{\gamma}} m_{sc}(t) dt = \int_{2}^{\hat{\gamma}} m_{sc}(t) dt + sN^{-2/3} + \mathcal{O}(sN^{-1}).
\]
(3.30)

By explicit calculation,
\[
\int_{2}^{\hat{\gamma}} m_{sc}(t) dt = \frac{1}{2} \left( 2 - \hat{\gamma}^2/2 + \frac{\hat{\gamma}}{2} \sqrt{\hat{\gamma}^2 - 4} - 2 \log((\sqrt{\hat{\gamma}^2 - 4} + \hat{\gamma})) + 2 \log(2) \right)
\]
(3.31)

Since \( \hat{\gamma} = (\beta^2 + 1)/\beta \) this simplifies to,
\[
\int_{2}^{\hat{\gamma}} m_{sc}(t) dt = \frac{1}{2}(1 - \beta^2) + \log(\beta),
\]
(3.32)

which yields the claim. \( \square \)

### 3.2 Low temperature case

We now turn to the low temperature case. The following is clear.

**Lemma 3.8.** On \( \mathcal{F}_D \) we have for \( E > \lambda_1 \),
\[
\frac{1}{N} \sum_{i=1}^{N} \frac{1}{E - \lambda_i} \leq 1 + DN^{-1/3} + \frac{1}{N^{1/3}} \frac{1}{N^{2/3}(E - \lambda_1)}.
\]
(3.33)

and so
\[
N^{2/3}(\gamma - \lambda_1) \leq \frac{1}{(N^{1/3}(\beta - 1) - D)_+}.
\]
(3.34)

In the low temperature regime we define \( \hat{\gamma} \) by
\[
\hat{\gamma} = \lambda_1 + \frac{1}{N(\beta - 1)}.
\]
(3.35)

We prove the following detailing the quality of approximation of \( \gamma \) by \( \hat{\gamma} \).

**Lemma 3.9.** Assume \( \beta \geq 1 \). On \( \mathcal{F}_D \cap \mathcal{S}_b \cap \mathcal{G}_{K,A} \cap \mathcal{F}_\xi \) for \( \xi = 1/100 \) we have,
\[
\frac{N^{2/3}|\gamma - \hat{\gamma}|}{N^{4/3}(\gamma - \lambda_1)(\gamma - \lambda_1)} \leq D + N^{2/3}(\gamma - \lambda_1)C(Kb^{-1} + 1).
\]
(3.36)

Therefore,
\[
N^{2/3}|\gamma - \hat{\gamma}| \leq \left( \frac{1}{(N^{1/3}(\beta - 1) - D)_+} \right)^2 \left( D + N^{2/3}(\gamma - \lambda_1)C(Kb^{-1} + 1) \right).
\]
(3.37)
Proof. We have,

\[
\frac{1}{N(\gamma - \lambda_1)} - \frac{1}{N(\hat{\gamma} - \lambda_1)} = 1 + \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - \lambda_1} + \frac{\gamma - \lambda_1}{N} \sum_{j=2}^{N} \frac{1}{(\lambda_j - \gamma)(\lambda_j - \lambda_1)}.
\] (3.38)

On \( S_b \cap G_{K,A} \cap F_\xi \) we have,

\[
\frac{1}{N^{4/3}} \sum_{j=2}^{N} \frac{1}{(\lambda_j - \gamma)(\lambda_j - \lambda_1)} \leq C \left( \frac{K}{b} + 1 \right).
\] (3.39)

This yields the claim. \( \square \)

Proposition 3.10. Suppose that \( J_D \cap S_b \cap G_{K,A} \cap F_\xi \) hold and that \( N^{1/3}(\beta - 1) \geq 2D \). Let \( Q > 2A \). Then,

\[
G(\gamma) = (\beta - 1)(\lambda_1 + N^{-1}(\beta - 1)^{-1}) + \frac{1}{N} \log(N^{1/3}(\beta - 1))
+ \left( 2 + N^{-2/3}Q - \frac{1}{N} \sum_{j=1}^{N} \log(2 + N^{-2/3}Q - \lambda_j) \right) + \mathcal{O}(N^{-1}).
\] (3.40)

Proof. Taylor expanding around \( \gamma \) and using \( G'(\gamma) = 0 \) we have,

\[
|G(\gamma) - G(\hat{\gamma})| \leq |\gamma - \hat{\gamma}|^2 \max_{x \in [\gamma, \hat{\gamma}]} |G''(x)|.
\] (3.41)

We see that on the events we are considering,

\[
|G''(x)| \leq \frac{CK}{1} \left( \frac{1}{(\lambda_1 - \gamma)^2} + \frac{1}{(\lambda_1 - \gamma_2)^2} \right) + CN^{1/3} \left( \frac{1}{\sqrt{N^{2/3}(\gamma - \lambda_1)}} + \frac{1}{\sqrt{N^{2/3}(\lambda_1 - \lambda_1)}} \right).
\] (3.42)

Hence by applying the previous two lemmas and the assumption that \( N^{1/3}(\beta - 1) \geq 2D \) we have,

\[
|G(\gamma) - G(\hat{\gamma})| \leq \frac{C}{N(N^{1/3}(\beta - 1))^2}.
\] (3.43)

We now write

\[
G(\hat{\gamma}) = (\beta - 1)\hat{\gamma} - \frac{1}{N} \log(\hat{\gamma} - \lambda_1) + \left( \hat{\gamma} - \frac{1}{N} \sum_{j=2}^{N} \log(\hat{\gamma} - \lambda_2) \right).
\] (3.44)

Fix now a \( Q > 2A \). We have, using the second estimate of Lemma 3.3 and that \( |\hat{\gamma} - \lambda_1| \leq CN^{-2/3} \) and the assumption that we are working on the event \( J_D \) that

\[
\left| 1 + \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j - \hat{\gamma}} \right| \leq CN^{-1/3},
\] (3.45)

and so by a second order Taylor expansion around \( \hat{\gamma} \) (applying the second estimate of Lemma 3.3) we find,

\[
\left| \hat{\gamma} - \frac{1}{N} \sum_{j=2}^{N} \log(\hat{\gamma} - \lambda_2) - (2 + N^{-2/3}Q) + \frac{1}{N} \sum_{j=2}^{N} \log((2 + N^{-2/3}Q) - \lambda_2) \right| \leq \frac{C}{N}(Q + A + 1)^2
\] (3.46)

Also,

\[
\frac{1}{N} \log(2 + QN^{-1/3} - \lambda_1) - \frac{1}{N} \log(\hat{\gamma} - \lambda_1) = \frac{1}{N} \log(N^{1/3}(\beta - 1)) + N^{-1}\mathcal{O}(\log(Q + A)).
\] (3.47)

This yields the claim. \( \square \)

We also have the following expansion which will be used for the case that \( N^{1/3}(\beta - 1) \) is bounded.
Proposition 3.11. Assume that $\beta \geq 1$. For every $\varepsilon > 0$ there is a $C_\varepsilon > 0$ so that the following holds with probability at least $1 - \varepsilon$ for $Q \geq C_\varepsilon$. We have for all $N$ large enough,

$$|G(\gamma) - G(2 + QN^{-2/3})| \leq \frac{C_\varepsilon}{N} \left(1 + N^{1/3}(\beta - 1)\right)^2$$

(3.48)

Proof. This follows immediately from a second order Taylor expansion of $G$ around $\gamma$, the lower bound of Proposition 3.1 and the first estimate of Lemma 3.3.

4 Estimates on contour integral

Recall that the saddle $\gamma$ is the unique solution $\gamma > \lambda_1$ to the equation

$$G'(\gamma) = 0.$$  (4.1)

We define the contour $\Gamma$ as the steepest descent curve, i.e., the curve in the complex plane passing through $\gamma$ and satisfying

$$\text{Im}[G(z)] = 0.$$  (4.2)

Note that since $G(\bar{z}) = G(\bar{z})$ this curve is symmetric in the upper and lower half-planes and we will often just discuss its behavior in the upper half-plane. Recall the following elementary consequences of the Cauchy-Riemann equations,

$$\partial_E \text{Re}[f] = \text{Re}[f'], \quad \partial_\eta \text{Re}[f'] = -\text{Im}[f'], \quad \partial_E \text{Im}[f] = \text{Im}[f'], \quad \partial_\eta \text{Im}[f] = \text{Re}[f'].$$  (4.3)

for analytic $f$. From the fact that $\partial_E \text{Im}[G(z)] = \text{Im}[m_N] > 0$ for $\eta > 0$ we see that $\text{Im}[G(z)]$ is a monotonic function of $E$, and so the curve maybe parameterized by $\eta$, and that once the curve leaves the real axis at $\gamma$ it never intersects the real axis again. From the fact that $G'(z) \neq 0$ off of the real-axis we see that $\Gamma$ is $C^1$ curve by the implicit function theorem. As $\eta \to \pi/\beta$, we see that $E \to -\infty$.

We first prove,

Lemma 4.1. If $N \geq 20$ then for any $a > \lambda_1$,

$$\int_{\alpha - i\infty}^{\alpha + i\infty} e^{NG(z)/2} dz = \int_{\Gamma} e^{NG(z)/2} dz.$$  (4.4)

Proof. Let $L = \sup_\gamma |\lambda_i|$. Let $C > 0$. If $\eta$ is sufficiently large we use that $\partial_\eta \text{Re}[G(z)] \leq -(2\eta)^{-1}$ for $|E| \leq 10L + 100(1 + \beta) + C(1 + |q|)$. For $R$ sufficiently large depending on $L$ and $E \leq 0$ and $|z| \geq R$,

$$\text{Re}[G'(z)] \leq \beta E - \log(|z|/2) \leq -\log(|z|/2).$$  (4.5)

Hence, if $N \geq 20$ we have that on the circle $|z| = R$ and $E \leq C$ for any $C > 0$ we have for $R$ large enough,

$$|e^{NG(z)/2}| \leq C'R^{-10}.$$  (4.6)

Hence the contribution from this arc goes to 0 as $N \to \infty$ and we can shift the contour to $\Gamma$.

Lemma 4.2. Suppose that $G_{K,A} \cap \mathcal{F}_\xi$ hold with $\xi = 1/100$ and that $(\gamma - \lambda_1) \leq C'$. Then, for $N$ sufficiently large,

$$\int_{-\infty}^{\infty} \left|e^{N(G(\gamma + it) - G(\gamma))/2}\right| dt \leq C(N^{-2/3}(K + A + 1) + (\gamma - 2)_+)$$  (4.7)

Proof. We write,

$$\text{Im}[m_N(z)] \geq \frac{1}{N} \sum_{j \leq N\eta/2} \frac{\eta}{\eta^2 + (\lambda_j - \gamma)^2}.$$  (4.8)

For $N$ sufficiently large we may assume that $K + N^{2/3}(\gamma - 2) + A \leq (2C' + 1)N^{2/3}$. If $(4C' + 2)N^{2/3} \geq N^{2/3}\eta \geq K + N^{2/3}(\gamma - 2) + A$, then on $G_{K,A} \cap \mathcal{F}_\xi$ the denominator in the sum above is smaller than

$$\eta^2 + (\lambda_j - \gamma)^2 \leq C(\eta^2).$$  (4.9)
Hence, since $\text{Re}[G(\gamma + it)]$ is decreasing for $t > 0$, we have
\[
\text{Re}[G(\gamma + it)] \leq -c \int_{N^{-2/3}(K + A) + (\gamma - 2) + }^{t} \eta^{1/2}d\eta \leq -c\eta^{3/2}
\] (4.10)
if $10C' + 10 > t > 2(N^{-2/3}(K + A) + (\gamma - 2) + )$. Hence,
\[
\int_{|t| \leq 1} \left| e^{N(G(\gamma + it) - G(\gamma))/2} \right| dt \leq C(N^{-2/3}(K + A + 1) + (\gamma - 2) + )
\] (4.11)
For the regime $10(C' + 1) \leq |t| \leq N^{10}$ we can just the fact that
\[
\text{Re}[G(\gamma + it)] - \text{Re}[(G\gamma)] \leq \text{Re}[G + i0(C' + 1)] - \text{Re}[(G\gamma)] \leq -c,
\] (4.12)
by our above calculation, so this part of the integral is exponentially small. Finally we use that on $\mathcal{F}_\xi$ for $\eta \geq 1$ we have $\text{Im}[m_N] \geq c/\eta$ and so,
\[
\text{Re}[G'(\gamma + it)] \leq -c\log(t)
\] (4.13)
for $|t| \geq 1$. This allows us to estimate the rest of the integral. 

We now move towards proving a lower bound.

**Lemma 4.3.** Suppose that the events $\mathcal{G}_{K,A} \cap \mathcal{F}_\xi \cap S_b$ hold with $\xi = 1/100$ and $K, A, b > 0$ and $K \geq A^{3/2}$ sufficiently large. Let $\delta = N^{2/3}(\gamma - \lambda_1)$. Then for $N^{2/3}|z - \gamma| \leq \delta/2$ we have for $k \geq 2$, for a universal $C_1 > 0$,
\[
\frac{N}{N^{2k/3}} |G^{(k)}(z)| \leq C_1^k (k - 1)! \left[ \frac{1}{\delta^k} + \frac{K}{(\delta + b)^k} + \frac{1}{\delta^{k-3/2}} \right],
\] (4.14)
Assume further that $\delta \leq C' N^{2/3}$ for some $C' > 0$. Then for $k = 2, 3, 4$,
\[
\frac{N}{N^{2k/3}} |G^{(k)}(\gamma)| \geq c \left[ \frac{1}{\delta^k} + 1_{(\delta/2 > K)} \frac{1}{\delta^{k-3/2}} \right],
\] (4.15)
where $c$ depends on $C'$.

**Proof.** For the upper bound we have,
\[
\frac{N}{N^{2k/3}} |G^{(k)}(z)| \leq C_1^k (k - 1)! \left[ \frac{1}{\delta^k} + \frac{K}{(\delta + b)^k} + \sum_{j > K} \frac{1}{N^{2/3}(\lambda_j - z)^k} \right].
\] (4.16)
Under our assumptions we have that for $K$ large enough and $j > K$ on $\mathcal{G}_{K,A} \cap \mathcal{F}_\xi$,
\[
N^{2/3}|\lambda_j - z| \leq C(\delta + j^{2/3}).
\] (4.17)
Hence the estimate follows from breaking up the sum into $j^{2/3} \leq \delta$ and $j^{2/3} \geq \delta$.

For the lower bound,
\[
\frac{N}{N^{2k/3}} |G^{(k)}(\gamma)| \geq c \frac{1}{\delta^k} + \sum_{j^{2/3} \leq \delta} \frac{1}{N^{2/3}(\lambda_j - \gamma)^k}.
\] (4.18)
As long as $\delta \geq K^{2/3}$ the factor in the denominator in the second term is larger than $c\delta$. Our assumed upper bound on $\delta$ ensures that the sum has at least $(\delta/C')^{3/2}$ terms. This yields the lower bound. 

With this, we now prove the following concerning the behavior of the contour $\Gamma$ in a small disc around the saddle $\gamma$.

**Proposition 4.4.** Suppose that the events $\mathcal{G}_{K,A} \cap \mathcal{F}_\xi$ hold with $\xi = 1/100$. Define $\delta = N^{2/3}(\gamma - \lambda_1)$ and assume that $\delta \leq C_2 N^{2/3}$. There are constants $c_1, c_2$ and $C_1$ depending on $K$ and $C_2$ so that for $N^{2/3}|z - \gamma| \leq c_1 \delta$ we have for $z \in \Gamma$,
\[
c_2 N^{2/3}|E - \gamma| \delta \leq N^{4/3}\eta^2 \leq C_1 N^{2/3}|E - \gamma| \delta.
\] (4.19)
Proof. Throughout the proof we will always assume $G_{K,A} \cap F_{\xi}$ holds without comment. Let us change variables to,

$$N^{2/3}(z - \gamma) = w = u + iv.$$  

By either a power series expansion (using estimates from the previous lemma) or Taylor’s theorem with explicit remainder we have for $|w| \leq c\delta$ that,

$$N(G(\gamma + z) - G(\gamma)) = w^2T_2 + w^3T_3 + w^4g(w)$$  

where,

$$c' \left( \frac{1}{\delta k} + 1_{\{\delta^{3/2} > K\}} \frac{1}{\delta^{k-3/2}} \right) \leq |T_k| \leq C \left[ \frac{K}{\delta k} + \frac{1}{\delta^{k-3/2}} \right],$$  

and,

$$|g(w)| \leq C \left[ \frac{K}{\delta^k} + \frac{1}{\delta^{5/2}} \right].$$

Furthermore, $T_2 > 0$ and $T_3 < 0$. Along the real axis $g(w)$ and its derivatives are real so,

$$|\text{Im}[g(w)]| \leq C|v| \left( \delta^{-5} + \delta^{-7/2} \right).$$

Taking the imaginary part of (4.21) and setting it equal to 0 we obtain

$$0 = 2uvT_2 + u^2v3T_3 - v^3T_3 + (4u^3v - 4uv^3)\text{Re}[g(w)] + (u^4 - 6u^2v^2 + v^4)\text{Im}[g(w)].$$

Dividing through by $v$ we find,

$$v^2T_3 = 2uT_2 + u^2v3T_3 + (4u^3v - 4uv^3)\text{Re}[g(w)] + (u^4 - 6u^2v^2 + v^4)(\text{Im}[g(w)]/v).$$

Note that we see that

$$\frac{K + \delta^{3/2}}{\delta^k} \frac{c}{K + 1} \leq |T_k| \leq C \frac{K + \delta^{3/2}}{\delta^k}.$$  

Hence, for constants $c_1, c_2$ and $C_1$ depending on $K$ we have that for $|w| \leq c_1\delta$ that

$$c_2|u|\delta \leq v^2 \leq C_1|u|\delta.$$  

This yields the claim after noting the signs of $T_2$ and $T_3$. \hfill \Box

**Proposition 4.5.** Suppose that $G_{K,A} \cap F_{\xi}$ with $\xi = 1/100$ and that $\delta = N^{2/3}(\gamma - \lambda_1)$. Then there are constants $c, C$ depending on $K$ so that,

$$-i \int_{\Gamma} e^{N(G(z) - G(\gamma))/2}dz \geq c\delta N^{-2/3}e^{-C\sqrt{\delta}}.$$  

**Proof.** The integral is real so for any parameterization of $\Gamma$ by $z = dx + idy$, only the $dy$ part contributes. Moreover, the integrand is positive and along $\Gamma$, the imaginary part of $z \in \Gamma$ is monotonically increasing. Hence for a lower bound we can take the portion of the curve intersecting a disc of radius $c_1N^{-2/3}\delta$ centered at $\gamma$, where $c_1$ is from Proposition 4.4. From Proposition 4.4, the curve exits the disc at a point with $\text{Im}[z] \geq c_2N^{-2/3}$. From the fact that $G'(\gamma) = 0$ and the upper bounds on $G''(z)$ of Lemma 4.3 we get

$$|\text{Re}[G(z)]| \leq C\delta^{1/2}$$  

for $z$ in this disc. This yields the claim. \hfill \Box
4.1 Better upper bound for contour integral in low temperature regime

In the low temperature regime we can obtain a better upper bound. Recall the definition of the contour $\Gamma$. For $\alpha > 0$ we define,

$$\Gamma_\alpha := \{ z \in \Gamma : E - \lambda_1 \geq -\alpha N^{-2/3} \}. \quad (4.31)$$

and we also introduce the shorthand,

$$b = \frac{1}{N^{1/3}(\beta - 1)}. \quad (4.32)$$

The next proposition and lemma control the behavior of the contour as well as the integrand along the contour.

**Proposition 4.6.** Let $\varepsilon > 0$. There is an $\alpha > 0$ depending on $\varepsilon$ and a $C_\varepsilon > 0$ so that the following holds with probability at least $1 - \varepsilon$. If $N^{1/3}(\beta - 1) \geq C_\varepsilon$ and $c_1 > 0$ is a constant we have for $z \in \Gamma_\alpha$ satisfying $E - \gamma \leq c_1 b N^{-2/3}$ that there is a $C > 0$ depending on $c_1$ so that,

$$\frac{1}{C C_\varepsilon} b \leq N^{2/3} \eta \leq b C C_\varepsilon,$$

with probability at least $1 - \varepsilon$. Moreover, for $z \in \Gamma_\alpha$, $\eta$ is a monotonic function of $E$ and

$$\text{Re}[G'(z)] \geq \frac{1}{2}(\beta - 1)1_{\{E \leq \lambda_1\}}. \quad (4.34)$$

**Proof.** Since $\partial_E \text{Im}[G(z)] = \text{Im}[G'(z)] > 0$ we see that the contour intersects each horizontal line the complex plane only once. Hence, the lower bound on $\eta$ follows from Propositions 3.1 and 4.4.

For the upper bound we will argue by finding a vertical line $\eta \rightarrow E_\eta + i\eta$ on which the sign of $\text{Im}[G(z)]$ changes at a point $\eta N^{2/3} \leq C b$.

Let us assume that $S_b$ holds, taking $b$ small enough so that $P[S_b] \geq 1 - \varepsilon$. By taking $b$ small we can assume that

$$N^{2/3}(\gamma - \lambda_1) \leq C_\varepsilon b \leq b/100 \quad (4.35)$$

with probability at least $1 - \varepsilon$ by applying the upper bound of Lemma 3.8. For $|z - \lambda_1| \leq b/10$ we have the following expansion,

$$G(z) = (\beta + \Xi)(z - \lambda_1) + \beta \lambda_1 - \frac{1}{N} \log(z - \lambda_1) + (z - \lambda_1)^2 g(z) \quad (4.36)$$

for a function $g(z)$ satisfying $|g(z)| \leq N^{1/3} c$ on $G_{K,A} \cap \mathcal{F}_\xi$ (and $\xi = 1/100$) where $C$ depends on $K, \varepsilon, b$ and we defined $\Xi$ as

$$\Xi := \frac{1}{N} \sum_{j > 1} \frac{1}{j - \lambda_1}. \quad (4.37)$$

For $E < \lambda_1$ we have $\text{Im}[G(E + i0)] < 0$ (as a boundary value). On $\mathcal{J}_D$ and the current event we are considering, we can derive from (4.36),

$$\text{Im}[G(z)] \geq (\beta - 1 - DN^{-1/3}) \eta - \frac{\pi}{N} - CN^{1/3}((E - \lambda_1)^2 + \eta^2). \quad (4.38)$$

We choose $E_\alpha < \lambda_1$ so that $|E_\alpha - \lambda_1| = \min\{cN^{-2/3}, b N^{-2/3}/100\}$. Assume that $(\beta - 1) \geq 2DN^{-1/3}$ so that with $y = N(\beta - 1)\eta$ we have,

$$N\text{Im}[G(E_\alpha + i\eta)] \geq \frac{y}{2} - 10 - Cy^2 b^2. \quad (4.39)$$

If $b$ is small enough this will be positive for $y = 100$, or for $\eta = 100N^{-2/3} b$. The fact that we used a Taylor expansion above means that we need to ensure that this choice of $y$ does not conflict with the constraint that $N^{2/3} \eta \leq b/100$. So we choose $b$ small enough so that this is the case. This means that for this chosen $E_\alpha < \lambda_1$ the equation $\text{Im}[G(E + i\eta)] = 0$ has a solution that is less than $CN^{-2/3} b$. This yields the upper bound we desire, with the $a$ for $\Gamma_\alpha$ coming from $E_\alpha$. 

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For monotonicity, by the equation
\[
\frac{d\eta}{dE} = -\frac{\text{Im}[G'(z)]}{\text{Re}[G'(z)]}
\]  
(4.40)

it suffices to check that \(\text{Re}[G'(z)] > 0\). This argument is slightly different depending on whether \(E\) is less than or greater than \(\lambda_1\). For \(\lambda_1 < E < \gamma\) consider first as a function of \(\eta\),
\[
\text{Im}[G(z)] = \beta \eta - \frac{1}{N} \sum_j \log(E + i\eta - \lambda_j),
\]  
(4.41)

and
\[
\partial_\eta \text{Im}[G(z)] = \beta + \frac{1}{N} \sum_{j=1}^{N} \frac{\lambda_j - E}{(\lambda_j - E)^2 + \eta^2}.
\]  
(4.42)

This is a convex function of \(\eta\) that is 0 at \(\eta = 0\) and \(\partial_\eta \text{Im}[G(z)]|_{\eta=0} < 0\) by definition of the saddle \(\gamma\). We also see that for large \(\eta\) that \(\text{Im}[G(z)] > 0\). The positive solution to \(\text{Im}[G(z)] = 0\) is the point on the contour \(\Gamma\) lying above \(E\) and we see, by the strict convexity, that \(\partial_\eta \text{Im}[G(z)] = \text{Re}[G'(z)] > 0\) at this point. This completes the case \(\lambda_1 < E < \gamma\).

For \(E < \lambda_1\) and \(z \in \Gamma_a\) we have, on \(\mathcal{J}_D \cap \mathcal{S}_b \cap \mathcal{G}_{K,A} \cap \mathcal{F}_\xi\), (choose all these parameters so that the events hold with probability at least \(1 - \varepsilon\))
\[
\text{Re}[G'(z)] \geq \beta - 1 + \frac{1}{N} \frac{\lambda_1 - E}{(\lambda_1 - E)^2 + \eta^2} - DN^{-1/3} - |z - \lambda_1|CN^{1/3}
\]  
(4.43)

using a similar expansion as before, as long as \(|z - \lambda_1| \leq b/100\). Here, \(C\) depends on \(\varepsilon\) but not on \(b\).

We can decrease \(a\) and assume that \(b\) is sufficiently small so that the portion of \(\Gamma_a\) for which \(E \leq \lambda_1\) lies in the disc \(|z - \lambda_1| \leq b/100\) with probability at least \(1 - \varepsilon\), using the upper bound just proven for \(\eta\). We can further let \(a\) and \(b\) be small enough so that the very last term in the above inequality is less than \(|z - \lambda_1|CN^{1/3} \leq N^{-1/3}\). Now, we see that if \((\beta - 1)N^{1/3}\) is large enough, depending on \(D\) this is positive and moreover exceeds \((\beta - 1)/2\).

**Lemma 4.7.** Under the conditions of the previous Proposition, we have that for \(z \in \Gamma_a\) that,
\[
\text{Re}[G(z) - G(\gamma)] \leq -c(\beta - 1)(E - \lambda_1)1_{\{E \leq \lambda_1\}}
\]  
(4.44)

and if \(E \leq \lambda_1\) that,
\[
\left| \frac{d\eta}{dE} \right| \leq C.
\]  
(4.45)

**Proof.** We have,
\[
\frac{d}{dE} \text{Re}[G(z)] = \text{Re}[G'(z)] + \frac{(\text{Im}[G'(z)])^2}{\text{Re}[G'(z)]} \geq \text{Re}[G'(z)].
\]  
(4.46)

The first claim is then an application of the lower bound for \(\text{Re}[G'(z)]\) we found in the previous proposition. For the second we see that for \(E \leq \lambda_1\)
\[
\text{Im}[G'(z)] \leq \frac{1}{N\eta} + C\eta N^{1/3} \leq C(\beta - 1),
\]  
(4.47)

on \(\mathcal{G}_{K,A} \cap \mathcal{F}_\xi\). This together with the equality
\[
\left| \frac{d\eta}{dE} \right| = \frac{|\text{Im}[G'(z)]|}{|\text{Re}[G'(z)]|}
\]  
(4.48)

and the lower bound for \(\text{Re}[G'(z)]\) yields the claim. 

With the two previous results we can now estimate the integral.
Proposition 4.8. Let \( \varepsilon > 0 \). There is a \( C_\varepsilon \) so that if \( N^{1/3}(\beta - 1) \geq C_\varepsilon \) then with probability at least \( 1 - \varepsilon \),
\[
\left| \frac{1}{2\pi i} \int_{\Gamma} e^{N(G(\gamma) - G(\gamma'))/2} d\gamma \right| \leq C_\varepsilon bN^{-2/3}. \tag{4.49}
\]

Proof. For any \( a \) we can deform the contour to be the union of \( \Gamma_a \) and the vertical lines extending from the end of \( \Gamma_a \) to \( \pm i\infty \), using the argument of Lemma 4.1. First we estimate the contribution of \( \Gamma_a \). First, for \( \lambda_1 \leq E \leq \gamma \) we just use that the length of this contour is \( \mathcal{O}(bN^{-2/3}) \) and that \( \text{Re}[G(\gamma)] \) is decreasing there, so that the integrand is bounded by 1. The estimate on the length of the contour follows from the fact that \( \eta \) and \( E \) are monotonically related, the upper bound on \( \eta \) from Proposition 4.6, and the upper bound on the saddle of Lemma 3.8.

For the rest of \( \Gamma_a \) we can assume that \( (d\eta)/(dE) \) is bounded due to the previous lemma and so the contribution here is bounded by,
\[
\int_0^\infty \exp \left[ -cN(\beta - 1)t \right] dt \leq C \frac{b}{N^{2/3}},
\]
where we also used the previous lemma to estimate the integrand.

The remaining integral consists of the vertical lines from the endpoints of \( \Gamma_a \) to \( \pm i\infty \). Their contribution is bounded by,
\[
C e^{-c(\beta - 1)N^{1/3}} \left( \int_{\eta > N^{-2/3}} \exp \left[ -N \int_{N^{-2/3}}^{\eta} \text{Im}[G'(E_a + it)]dt \right] d\eta + N^{-2/3} \right)
\]
where \( E_a = \lambda_1 - aN^{-2/3} \). Note that we estimated the contribution for \( bcn^{-2/3} \leq \eta \leq N^{-2/3} \) by \( N^{-2/3} \). The same argument from the start of the proof of Lemma 4.2 shows that \( \text{Im}[G'(\eta)] \geq c\eta^{1/2} \) for \( N^{-2/3} \leq \eta \leq N^{-2/3+1/10} \). Hence, the contribution in this range is \( \mathcal{O}(N^{-2/3}) \). At the point \( \eta = N^{-2/3+1/10} \), the integrand is exponentially small, so we can do away with the contribution for \( N^{-2/3+1/10} \leq \eta \leq NC \).

For \( \eta \geq 1 \) we have \( \text{Im}[G'(\eta)] \geq c/\eta \) so \( \text{Re}[G(\eta)] \leq -c\log(\eta) \) here. Hence, the contribution for \( \eta > NC \) is \( \mathcal{O}(N^{-10}) \) if \( C \) is large enough. This yields the claim. \( \square \)

4.2 Tighter estimates in high temperature regime

We first have the following.

Lemma 4.9. Suppose that \( G_{K,A} \cap F_\xi \) holds with \( \xi = 1/100 \). Assume that there is a constant \( C_1 \geq 0 \) so that \( \gamma - 2 \leq C_1 \) and furthermore that \( N^{2/3}(\gamma - 2) \geq 10(A + K) \). There is a \( C > 0 \) so that,
\[
\frac{1}{C(\gamma - 2)^{1/2}} \leq |G''(\gamma)| \leq \frac{C}{(\gamma - 2)^{1/2}}, \quad |G'''(\gamma)| \leq \frac{C}{(\gamma - 2)^{3/2}}
\]
and for \( 0 \leq \eta \leq 1 \),
\[
\text{Im}[m_N(\gamma + i\eta)] \geq \frac{1}{C \sqrt{\eta + (\gamma - 2)}}. \tag{4.53}
\]

Proof. Under the assumptions we see that,
\[
\frac{1}{C} (N^{2/3}(\gamma - 2) + j^{2/3}) \leq N^{2/3}(\gamma - \lambda_j) \leq C(N^{2/3}(\gamma - 2) + j^{2/3}),
\]
and so
\[
\frac{1}{C} \sum_{j=1}^N \frac{1}{(N^{2/3}(\gamma - 2) + j^{2/3})^k} \leq \sum_{j=1}^N \frac{1}{(N^{2/3}(\gamma - \lambda_j))^k} \leq C \sum_{j=1}^N \frac{1}{(N^{2/3}(\gamma - 2) + j^{2/3})^k}
\]
for \( k = 2, 3 \). The upper bound follows from consider separately the contributions from \( j^{2/3} \leq N^{2/3}(\gamma - 2) \) and \( j^{2/3} \geq N^{2/3}(\gamma - 2) \) separately, and estimating the factor in the denominator from below by \( N^{2/3}(\gamma - 2) \) and \( j^{2/3} \) in each case, respectively. The lower bound comes from the terms \( j^{2/3} \leq N^{2/3}(\gamma - 2) \) for which the factor in the denominator is less than \( 2N^{2/3}(\gamma - 2) \).
For the lower bound on $\text{Im}[m_N(\gamma + i\eta)]$ we have,

$$\text{Im}[m_N(E + i\eta)] \geq c \frac{1}{N} \sum_{j=1}^{N} \frac{\eta}{(\gamma - 2)^2 + N^{-4/3}j^{4/3} + \eta^2} \geq \frac{c}{N} \sum_{j^{2/3} \leq N^{2/3} \eta/ N^{2/3}} \frac{\eta}{(\gamma - 2)^2 + \eta^2} \quad (4.56)$$

There are at least $c(N^{2/3}(\gamma - 2) + N^{2/3}(\eta))^{3/2}$ terms in the final sum on the RHS due to our assumption $N^{2/3}(\gamma - 2) \leq C_1$. This yields the claim. \qed

**Proposition 4.10.** Let $\varepsilon > 0$ and $\beta \leq 1$. There is a $C_\varepsilon > 0$ so that if $N^{1/3}(1 - \beta) \geq C_\varepsilon$ then the following holds with probability at least $1 - \varepsilon$.

$$\frac{1}{C_\varepsilon} \left( \frac{(N^{1/3}(1 - \beta))^{1/2}}{N^{2/3}} \right) \leq \int_{\mathbb{R}} \exp \left[ \frac{N}{2} (G(\gamma + it) - G(\gamma)) \right] dt \leq C_\varepsilon \left( \frac{(N^{1/3}(1 - \beta))^{1/2}}{N^{2/3}} \right) \quad (4.57)$$

**Proof.** We assume that $G_{K,A} \cap F_\varepsilon$ hold with probability at least $1 - \varepsilon$. We furthermore assume that the event of Lemma 3.4 holds. We then assume that $(1 - \beta)N^{1/3}$ is sufficiently large so that the estimates of Lemma 3.4 imply that $\gamma - 2$ is sufficiently large that the previous lemma applies. Fixing a small $c_1 > 0$ we split the integral into a few regions

(i) $|t| \leq 2c_1(\gamma - 2)^{1/2} N^{-1/3}$

(ii) $2c_1(\gamma - 2)^{1/2} N^{-1/3} < |t| < (\gamma - 2)$

(iii) $(\gamma - 2) < |t| < 1$

(iv) $|t| > 1$

For region (i) we use, the second order Taylor expansion,

$$\frac{N}{2} (G(\gamma + it) - G(\gamma)) = -\frac{Nt^2}{4} G''(\gamma) + \mathcal{O} \left( |t|^3 N(\gamma - 2)^{-3/2} \right) \quad (4.58)$$

where we used the estimates of Lemma 4.9 and the fact that $|G''(\gamma + it)| \leq |G''(\gamma)|$. Hence, we can choose $c_1$ sufficiently small so that,

$$\exp \left[ \frac{N}{2} (G(\gamma + it) - G(\gamma)) \right] = \exp \left[ -\frac{Nt^2}{4} G''(\gamma) \right] (1 + f(t)) \quad (4.59)$$

for a function that satisfies $|f(t)| \leq \frac{1}{100}$ for $|t| \leq 100c_1(\gamma - 2)^{1/3} N^{-1/3}$. We see also that if $N^{1/3} (\beta - 1)$ is large enough, then

$$\text{Re}[G(\gamma + ic_1(\gamma - 2)^{1/2} N^{-1/3}) - G(\gamma)] \leq -cN^{-2/3}(\gamma - 2)^{1/2}. \quad (4.60)$$

Hence,

$$\int_{|t| \leq c_1(\gamma - 2)^{1/3} N^{2/3}} \exp \left[ \frac{N}{2} (G(\gamma + it) - G(\gamma)) \right] dt = \frac{N^{2/3}(\gamma - 2)^{1/4}}{\sqrt{2\pi G''(\gamma)(\gamma - 2)^{-1/2}}} \left( 1 + \mathcal{O} \left( e^{-cN^{1/3}(\gamma - 2)} \right) \right) (1 + X) \quad (4.61)$$

where $|X| \leq \frac{1}{100}$. For region (ii) we have, using that $\text{Re}[G(\gamma + it)]$ is decreasing for $t > 0$ that,

$$\text{Re}[G(\gamma + it) - G(\gamma)] \leq \text{Re}[G(\gamma + it) - G(\gamma + ic_1(\gamma - 2)^{1/2} N^{-1/3})] - cN^{-2/3}(\gamma - 2)^{1/2} \quad (4.62)$$

and then,

$$\text{Re}[G(\gamma + it) - G(\gamma + ic_1(\gamma - 2)^{1/2} N^{-1/3})] = -\int_{c_1(\gamma - 2)^{1/3} N^{-1/3}}^{t} \text{Im}[m_N(\gamma + i\eta)] d\eta \leq -c \int_{c_1(\gamma - 2)^{1/3} N^{-1/3}}^{t} \frac{\eta}{\sqrt{(\gamma - 2)}} d\eta \leq -c \frac{t^2}{\sqrt{(\gamma - 2)}} \quad (4.63)$$

where we used Lemma 4.9 in the second line and that $t > 2(c_1(\gamma - 2)^{1/3}N^{-1/3})$ in the third line. Hence,

$$
\int_{2c_1(\gamma - 2)^{1/3}N^{-1/3}(\gamma - 2)^{1/3}(\gamma - 2)^{1/3}} \exp \left[ -\frac{N}{2}(G(\gamma + it) - G(\gamma)) \right] dt 
\leq Ce^{-cN^{3/2}(\gamma - 2)^{1/3}} \int_{|t| < (\gamma - 2)} \exp \left[ -cN\epsilon^2(\gamma - 2)^{-1/2} \right] dt 
\leq Ce^{-cN^{3/2}(\gamma - 2)^{1/3}} \frac{(N^{2/3}(\gamma - 2))^{1/4}}{N^{2/3}}. 
(4.64)
$$

For region (iii) we use that

$$
\text{Re}[G(\gamma + it) - G(\gamma)] = \text{Re}[G(\gamma + it) - G(\gamma + i(\gamma - 2)/2)] + \text{Re}[G(\gamma + i(\gamma - 2)/2) - G(\gamma)] 
\leq \text{Re}[G(\gamma + it) - G(\gamma + i(\gamma - 2)/2)] - c(\gamma - 2)^{3/2}, 
(4.65)
$$

and then,

$$
\text{Re}[G(\gamma + it) - G(\gamma + i(\gamma - 2)/2)] = -\int_{(\gamma - 2)/2}^{\epsilon} \text{Im}[m_N(\gamma + \epsilon)] d\eta 
\leq -c \int_{(\gamma - 2)/2}^{\epsilon} \sqrt{\eta} d\eta 
\leq -c t^{3/2} 
(4.66)
$$

where we used Lemma 4.9 and the fact that $t > (\gamma - 2)$. Hence,

$$
\int_{(\gamma - 2)^{1/3}}^{(\gamma - 2)^{1/3}} \exp \left[ \frac{N}{2}(G(\gamma + it) - G(\gamma)) \right] dt 
\leq Ce^{-cN(\gamma - 2)^{3/2}} \int_{|t| \leq 1} \exp \left[ -cNt^{3/2} \right] dt 
\leq CN^{-2/3}e^{-cN(\gamma - 2)^{3/2}}. 
(4.67)
$$

For $|t| > 1$ we use,

$$
\text{Re}[G(\gamma + it) - G(\gamma)] \leq \text{Re}[G(\gamma + it) - G(\gamma + i)] - c, 
(4.68)
$$

and that for $t > 1$ we have $\text{Im}[m_N(\gamma + it)] \geq ct^{-1}$ and so,

$$
\text{Re}[G(\gamma + it) - G(\gamma)] \leq -c \log(t) - c. 
(4.69)
$$

Hence the contribution from region (iv) is $O(e^{-N})$. The leading order contribution from region (i) is positive and is,

$$
\frac{(N^{2/3}(\gamma - 2))^{1/4}}{\sqrt{2\pi G''(\gamma)(\gamma - 2)^{1/2}}} \frac{1}{N^{2/3}}. 
(4.70)
$$

By taking $N^{1/3}(\beta - 1)$ large enough we see that this ensures $(\gamma - 2)N^{2/3}$ large enough to ensure that all the other contributions from the integral are less than half of this quantity. This yields the claim.

5 Expansion of free energy

Define

$$
X_Q = \sum_j \log |2 + QN^{-2/3} - \lambda_j| - QN^{1/3} - N/2 + \log(N)/6. 
(5.1)
$$

5.1 High temperature regime

**Proposition 5.1.** Assume $\beta \leq 1$. For any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ so that with probability at least $1 - \varepsilon$ we have for a large $Q$ depending on $\varepsilon > 0$,

$$
|N(F_N - \beta^2/4) + \log(N)/12 + X_Q/2| \leq C_\varepsilon \left( 1 + N^{1/3}(\beta - 1) \right). 
(5.2)
$$
Proof. We first have by Lemma 2.9 and (2.32) that,

\[ N(F_N - \beta^2/4) = \frac{N}{2} G(\gamma) + \frac{1}{2} \log(N) - \frac{N}{2} \log(\beta) - \frac{N}{2} - \beta^2 N/4 + O(1) \]

\[ + \log \left( \frac{1}{2\pi i} \int \exp \left[ \frac{N}{2} (G(z) - G(\gamma)) \right] dz \right) \]  
(5.3)

By Lemma 4.2, Proposition 4.5 and Lemma 3.4 we see that there is a \( C_\varepsilon > 0 \) on which,

\[ \left| \log \left( \frac{1}{2\pi i} \int \exp \left[ \frac{N}{2} (G(z) - G(\gamma)) \right] dz \right) + \frac{2}{3} \log(N) \right| \leq C_\varepsilon \left( 1 + N^{1/3}(1 - \beta) + \log(1 + N^{1/3}(\beta - 1)) \right), \]

with probability at least \( 1 - \varepsilon \). We have,

\[ \frac{N}{2} G(\gamma) + \frac{1}{2} \log(N) - \frac{N}{2} \log(\beta) - \frac{N}{2} - \beta^2 N/4 \]

\[ = \frac{N}{2} \left( G(\gamma) - \beta^2/2 - \log(\beta) - 1 \right) + \frac{1}{2} \log(N) \]

\[ = \frac{N}{2} \left( G(\gamma) - \beta^2/2 - \log(\beta) - 1 + X_Q/N - \frac{1}{6} \log(N)/N \right) + \frac{7}{12} \log(N) - X_Q/2. \]

(5.5)

Now note that,

\[ G(\gamma) - \beta^2/2 - \log(\beta) - 1 + X_Q/N - \frac{1}{6} \log(N)/N = G(\gamma) - \beta^2/2 - R(2 + QN^{-2/3}) + 1/2 - \log(\beta), \]

where \( R(z) \) is as in Proposition 3.7. This proposition also gives us a bound for the RHS, and gives us the choice of \( Q \). This concludes the proof. 

In the above proof if we instead use Proposition 4.10 to estimate (5.4) we easily deduce the following.

Proposition 5.2. Let \( \varepsilon > 0 \) and \( \beta \leq 1 \). There is a \( C_\varepsilon > 0 \) and a \( Q > 0 \) so that if \( N^{1/3}(1 - \beta) \geq C_\varepsilon \) then

\[ \left| N(F_N - \beta^2/4) + \log(N)/12 + X_Q/2 \right| \leq C_\varepsilon \left( 1 + \log^4(N^{1/3}(1 - \beta)) \right) \]  
(5.7)

5.2 Low temperature regime

Proposition 5.3. For any \( \varepsilon > 0 \) there is a \( Q > 0 \) and \( C_\varepsilon > 0 \) so that with probability with at least \( 1 - \varepsilon \) we have that,

\[ \left| N(F_N - [\beta - 3/4 - \log(\beta)/2]) \right| \]

\[ - \left( -X_Q + N(\beta - 1)(\lambda_1 - 2) - \log(N)/12 \right) - \log(N^{1/3}(\beta - 1) + 1)/2 \]  
\( \leq C_\varepsilon \)  
(5.8)

Proof. We first have, by combining Propositions 3.10 and 3.11 that, for any \( \varepsilon > 0 \) there is a \( Q > 0 \) and \( C_\varepsilon > 0 \) so that,

\[ G(\gamma) - \left( 2\beta - \frac{1}{2} + (\beta - 1)(\lambda_1 - 2) + \frac{1}{N} \log(N^{1/3}(\beta - 1) + 1) + \frac{\log(N)}{6N} - X_Q/N \right) \leq \frac{C_\varepsilon}{N}, \]

with probability at least \( 1 - \varepsilon \). Furthermore, by Propositions 4.5, 3.1 and 4.8 we see that there is a \( C_\varepsilon > 0 \) so that with probability at least \( 1 - \varepsilon \),

\[ \left| \log \left( \frac{1}{2\pi i} \int \exp \left[ N(G(z) - G(\gamma))/2 \right] dz \right) + \log(N^{2/3}(\beta - 1) + 1) \right| \leq C_\varepsilon. \]  
(5.10)
Hence,
\[
NF_N = \frac{N}{2}G(\gamma) + \log \left(\frac{1}{2\pi i} \int_\Gamma \exp \left[ N(G(z) - G(\gamma))/2 \right] \, dz \right) + \frac{1}{2} \log(N) - \frac{N}{2} \log(\beta) - \frac{N}{2} + O(1)
\]
\[
= N \left( \beta - \frac{3}{4} - \frac{\log(\beta)}{2} \right) - \frac{X_Q}{2} + (\beta - 1)(\lambda_1 - 2) - \frac{1}{12} \log(N) - \frac{1}{2} \log(N^{1/3}(\beta - 1) + 1) + O(1).
\]  
(5.11)

This yields the claim.

\[
\square
\]

5.3 Proof of Theorem 1.2

The fact that the quantity,
\[
Y_N := \frac{N(F_N - f(\beta))}{\sqrt{\frac{1}{6} \log(N)}} + \frac{X}{\sqrt{\frac{1}{2} \log(N)}}
\]
(5.12)
is tight for \( \beta \) of the form \( \beta = 1 + \alpha \sqrt{\log(N)}N^{-1/3} \) with \( \alpha \) fixed follows immediately from Propositions 5.1 and 5.3 as well as the fact that the random variables \( X_Q/\sqrt{\log(N)} \) and \( N^{2/3}(\lambda_1 - 2) \) are themselves tight due to Theorem 2.8 and the convergence of the latter to the Tracy-Widom distribution.

For the Gaussian fluctuations in the case that \( \alpha \to 0 \), we see first that for any \( \varepsilon > 0 \), by Propositions 5.1 and 5.3 (and the tightness of the random variable \( N^{2/3}(\lambda_1 - 2) \)) that there is a \( Q > 0 \) and a \( C_\varepsilon > 0 \) so that,
\[
\left| Y_N + \frac{X_Q}{\sqrt{2 \log(N)/3}} \right| \leq C_\varepsilon \left( \frac{1 + \log(\log(N))}{\sqrt{\log(N)}} + |\alpha| \right)
\]
(5.13)
with probability at least \( 1 - \varepsilon \) for all \( N \) large enough. Hence, for Lipschitz \( O \), (with \( Y \) as above)
\[
\limsup_{N \to \infty} \left| \mathbb{E}[O(Y_N)] - \mathbb{E}[O(-X_Q/\sqrt{2 \log(N)/3})] \right| \leq ||O||_\infty \varepsilon.
\]
(5.14)

On the other hand,
\[
\lim_{N \to \infty} \mathbb{E}[O(-X_Q/\sqrt{2 \log(N)/3})] = \mathbb{E}[O(Z)]
\]
(5.15)
where \( Z \) is a standard normal random variable. Therefore, taking \( \varepsilon \to 0 \) yields the claim.

From Proposition 5.2 we see that if \( \beta = 1 + \alpha \sqrt{\log(N)}N^{-1/3} \) with \( \alpha < 0 \) fixed, then for any \( \varepsilon > 0 \) there is a \( C_\varepsilon > 0 \) and \( Q > 0 \) so that,
\[
\left| Y_N + \frac{X_Q}{\sqrt{2 \log(N)/3}} \right| \leq C_\varepsilon \left( \frac{1 + \log(\alpha \log(N))}{\sqrt{\log(N)}} \right).
\]
(5.16)
The Gaussian fluctuations for \( \alpha < 0 \) then follows from the same argument as in the \( \alpha \to 0 \) case.

Finally, we consider the case \( \alpha \to \infty \) as \( N \to \infty \). From Proposition 5.3 we see that for any \( \varepsilon > 0 \) there is a \( C_\varepsilon > 0 \) on which,
\[
\left| \frac{N(F_N(\beta) - f(\beta)) + \log(N)/12}{N^{1/3}(\beta - 1)} - N^{2/3}(\lambda_1 - 2) \right| \leq C_\varepsilon \left( \frac{1}{\sqrt{\log(N)}} + \frac{1 + \log(1 + \alpha)}{\alpha} \right)
\]
(5.17)
where we used that \( X_Q/\sqrt{\log(N)} \) is tight. Hence, if \( \alpha \to \infty \) as \( N \to \infty \), then for Lipschitz \( O \),
\[
\limsup_{N \to \infty} \left| \mathbb{E}[O(\sqrt{\frac{1}{6} \log(N)N^{-1/3}(\beta - 1)^{-1}Y_N})] - \mathbb{E}[O(N^{2/3}(\lambda_1 - 2))] \right| \leq \varepsilon ||O||_\infty.
\]
(5.18)
The claim follows by the convergence of \( N^{2/3}(\lambda_1 - 2) \) to the Tracy-Widom\(_1\) distribution. 
\[
\square
\]
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