REVERSING BELLMAN OPERATOR INEQUALITY

MOHAMMAD SABABHEH, HAMID REZA MORADI AND SHIGERU FURUICHI

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Abstract. The main aim of the present paper is to obtain several reverses of the operator Bellman inequality. To this end, we employ Mond-Pečarić method to achieve a general inequality treating the arithmetic mean and unital positive linear maps. In particular, we show that, for certain scalars $\alpha, \beta$,

$$\alpha(\Phi(I - A\nabla_v B))^{1/p} + \beta I \leq \Phi\left((I - A)^{1/p} \nabla_v (I - B)^{1/p}\right)$$

for the positive operators $A, B$, the normalized positive linear map $\Phi$ and $p > 1$. As a consequence, we get multiplicative and additive reverses of operator Bellman inequality. Further, we show some inequalities involving concave and convex functions. In the end, we present a simple proof of the scalar Bellman inequality and its reverses.

1. Introduction

Throughout this paper, $A$ and $B$ are positive operators on a Hilbert space $\mathcal{H}$, with identity $I$. For convenience, we write $A \geq 0$ (respectively, $A > 0$) if $A$ is a positive (respectively, positive invertible) operator. In the sequel, we use $m$ and $M$ for positive real numbers, and the order between operators is that in which $A \leq B$ means $B - A$ is positive. The notation $\nabla_v$ will be used for the arithmetic mean, defined for two positive operators $A$ and $B$ by $A \nabla_v B = (1 - v)A + vB$. A real valued function $f : J \to (0, \infty)$ is said to be operator concave if $f(A\nabla_v B) \geq f(A)\nabla_v f(B)$ for $0 \leq v \leq 1$ and all self adjoint operators $A, B$ whose spectra are contained in the real interval $J$. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is said to be a normalized positive linear map if $\Phi(A) \geq 0$ whenever $A \geq 0$ and $\Phi(I) = I$. For further details about the notations of this paper, we refer the reader to [4]. In this context, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators acting on $\mathcal{H}$.

The following inequality is well known in the literature as the operator Bellman inequality [6]

$$(\Phi(I - A\nabla_v B))^{1/p} \geq \Phi\left((I - A)^{1/p} \nabla_v (I - B)^{1/p}\right)$$  \hspace{1cm} (1.1)

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for \( 0 \leq v \leq 1, \ p > 1, \ 0 < A, B \leq I \) and a normalized positive linear map \( \Phi \). This inequality was proved in [6] as an operator version of the scalar Bellman inequality [2]

\[
\left( a^p - \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} + \left( b^p - \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}} \leq \left( (a + b)^p - \sum_{k=1}^{n} (a_k + b_k)^p \right)^{\frac{1}{p}}, \tag{1.2}
\]

for the positive numbers \( a, a_k, b, b_k \) satisfying \( \sum_{k=1}^{n} a_k^p \leq a^p \) and \( \sum_{k=1}^{n} b_k^p \leq b^p \), where \( p > 1 \).

The proof of (1.1) was based on the operator inequality [6]

\[
f(\Phi(A\nabla v B)) \geq \Phi(f(A)\nabla v f(B)), \tag{1.3}
\]

valid for the operator concave function \( f : J \subset (0, \infty) \to (0, \infty) \), the normalized positive linear map \( \Phi \) and the positive operators \( A, B \) whose spectra are contained in the interval \( J \).

We refer the reader to [5, 7] for further discussion of (1.1).

In this article, we prove a more elaborated reverse of (1.3), valid for concave functions (not necessarily operator concave). This reverse-type inequality will be used to find a reversed version of (1.1) and a reversed version of (1.2). Further, we present a simple approach that can be used to prove the scalar Bellman inequality and its reverse. The new approach will be useful in obtaining several refinements of these inequalities.

## 2. Main results

In the sequel, we present a general inequality by applying Mond-Pečarić method. We refer the reader to [4] as a comprehensive reference of this method.

The following notations will be used in Theorem 1, for the positive numbers \( m, M \) and the function \( f : [m, M] \to \mathbb{R} \).

\[
a_f = \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad b_f = \frac{Mf(m) - mf(M)}{M - m}.
\]

**Theorem 1.** Let \( \Phi \) be a normalized positive linear map on \( \mathcal{B}(\mathcal{H}) \), \( A, B \in \mathcal{B}(\mathcal{H}) \) be two positive operators such that \( mI \leq A, B \leq MI \) for some scalars \( 0 < m < M \). If \( f, g : [m, M] \to [0, \infty) \) are continuous functions such that \( f \) is concave, then for a given \( \alpha > 0 \),

\[
\alpha g(\Phi(A\nabla v B)) + \beta I \leq \Phi(f(A)\nabla v f(B)) \tag{2.1}
\]

where \( \beta = \min_{t \in [m, M]} \{ a_f t + b_f - \alpha g(t) \} \).

The reverse inequality of (2.1) holds when \( f \) is a convex function.

**Proof.** According to the assumptions, we have, for any \( t \in [m, M] \),

\[
f(t) \geq a_f t + b_f.
\]
A standard functional calculus argument implies
\[ f(A) \geq afA + bfI \quad \text{and} \quad f(B) \geq afB + bfI. \]

Consequently, we infer for any \( v \in [0, 1] \),
\[ (1 - v)f(A) \geq (1 - v)afA + (1 - v)bfI \quad \text{and} \quad vf(B) \geq vafB + vbfI, \]
and hence
\[ f(A)\nabla_v f(B) \geq af(A\nabla_v B) + bfI. \]

It follows from the linearity and the normality of \( \Phi \) that
\[ \Phi(f(A)\nabla_v f(B)) \geq af(\Phi(A\nabla_v B)) + bfI. \]

Whence
\[ \Phi(f(A)\nabla_v f(B)) - \alpha g(\Phi(A\nabla_v B)) \geq af(\Phi(A\nabla_v B)) + bfi - \alpha g(\Phi(A\nabla_v B)) \]
\[ \geq \min_{t \in [m, M]} \left\{ at + bf - \alpha g(t) \right\} I \]
which implies the desired inequality (2.1).

A reverse of the operator Bellman inequality (1.1) is obtained by taking \( f(t) = g(t) = (1 - t)^{1/p} \) on \((0, 1)\) with \( p > 1 \) in Theorem 1.

**COROLLARY 2.1.** (Reverse of operator Bellman inequality) Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be two positive invertible operators such that \( 0 < ml \leq A, B \leq MI < I \), and \( \Phi \) be a normalized positive linear map on \( \mathcal{B}(\mathcal{H}) \). Then for a given \( \alpha > 0 \),
\[ \alpha(\Phi(I - A\nabla_v B))^{1/p} + \beta I \leq \Phi\left( (I - A)^{1/p} \nabla_v (I - B)^{1/p} \right) \]
where \( p > 1, \ v \in [0, 1] \) and
\[ \beta = \min_{t \in [m, M]} \left\{ \frac{(1 - M)^{1/p} - (1 - m)^{1/p}}{M - m} t + \frac{M(1 - m)^{1/p} - m(1 - M)^{1/p}}{M - m} - \alpha(1 - t)^{1/p} \right\}. \]

We remark that a similar result as in Corollary 2.1 was shown in [1, Corollary 2.8]. However, the advantage of our result is that the inclusion of a free constant \( \alpha \). This allows obtaining a multiplicative reverse, by choosing appropriate \( \alpha \) and \( \beta \) in Corollary 2.1. This is our next result.

**COROLLARY 2.2.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be two positive invertible operators such that \( 0 < ml \leq A, B \leq MI < I \), and \( \Phi \) be a normalized positive linear map on \( \mathcal{B}(\mathcal{H}) \). Then
\[ \alpha(\Phi(I - A\nabla_v B))^{1/p} \leq \Phi\left( (I - A)^{1/p} \nabla_v (I - B)^{1/p} \right) \]
where \( p > 1, \nu \in [0, 1] \) and

\[
\alpha = \min_{t \in [m, M]} \left\{ \frac{1}{(1-t)^{1/p}} \left( \frac{(1-M)^{1/p} - (1-m)^{1/p}}{M-m} t + \frac{M(1-m)^{1/p} - m(1-M)^{1/p}}{M-m} \right) \right\}.
\]

Additionally,

\[
(\Phi (I-A\nabla_v B))^{1/p} + \beta I \leq \Phi \left( (I-A)^{1/p} \nabla_v (I-B)^{1/p} \right)
\]

where

\[
\beta = \min_{t \in [m, M]} \left\{ \frac{(1-M)^{1/p} - (1-m)^{1/p}}{M-m} t + \frac{M(1-m)^{1/p} - m(1-M)^{1/p}}{M-m} - (1-t)^{1/p} \right\}.
\]

**Remark 2.1.** Here we find the exact value of \( \alpha \) appearing in Corollary 2.2. This will help us better understand the operator Bellman inequality.

For simplicity, let

\[
a = \frac{(1-M)^{1/p} - (1-m)^{1/p}}{M-m}, \quad b = \frac{M(1-m)^{1/p} - m(1-M)^{1/p}}{M-m}, \quad r = \frac{1}{p},
\]

and let

\[
f(t) = \frac{at + b}{(1-t)^r}, \quad 0 < m \leq t \leq M < 1.
\]

To find \( \alpha \), we find \( \min_{m \leq t \leq M} f(t) \). Notice that

\[
f'(t) = \frac{a + br + a(r-1)t}{(1-t)^{r+1}}.
\]

Solving \( f'(t) = 0 \), we obtain \( t_0 = \frac{a + br}{a(1-r)} \). Noting that \( a, r-1 < 0 \), it is easily seen that \( f \) attains its minimum at \( t_0 \), provided that \( m \leq t_0 \leq M \); which we show in this remark.

We will prove that \( m \leq t_0 \) and leave the similar proof of \( t_0 \leq M \) to the reader. So, define \( g(m) = t_0 - m \). Simplifying this using the above \( a, b \), we obtain

\[
g(m) = \frac{1}{p-1} \left( p(1-m) + \frac{(1-m)^r(m-M)}{(1-m)^r - (1-M)^r} \right).
\]

Calculus computations show that

\[
g'(m) = \frac{h(M)}{p(p-1)(m-1)((1-M)^r - (1-M)^r)^2},
\]

where

\[
h(M) = (1-m)((1-m)^r - (1-M)^r)^2(p-1) + (1-M)^r((1-m)(1-M)^r
\]
\[+ (1-m)^r(-1 + m - mr + Mr)).]
Then

\[ h'(M) = -(1-M)^{r-1}H(M), \]

where

\[ H(M) = 2(1-m)(1-M)^r + (1-m)^r(-2+Mr(1+r)-m(-2+r+r^2)). \]

Further

\[ H'(M) = (1-m)r[(1+r)(1-m)^{r-1} - 2(1-M)^{r-1}]. \]

Noting that \( r < 1 \) and \( m < M \), it follows that \( H'(M) \leq 0 \). Since \( m \leq M \), it follows that \( H(M) \leq H(m) = 0 \), and hence \( h'(M) \geq 0 \). Consequently, \( h(M) \geq h(m) = 0 \) and \( g'(m) \leq 0 \). This implies \( g(m) \geq \lim_{m \to M^-} g(m) = 0 \), showing that \( g \geq 0 \) and hence \( t_0 \geq m \).

Following similar computations, one can show that \( t_0 \leq M \). We leave these computations to the reader.

Now having shown that \( m \leq t_0 \leq M \), it follows that \( f \) attains its minimum on \([m,M]\) at \( t_0 \). That is

\[ \alpha = f(t_0) = \frac{|a|}{r} \left( \frac{r(a+b)}{|a|(1-r)} \right)^{1-r} = p|a|^\frac{1}{p} \left( \frac{a+b}{p-1} \right)^{\frac{p-1}{p}}. \]

**Remark 2.2.** To find \( \beta \) appearing in Corollary 2.2, we set

\[ f(t) = at + b - (1-t)^r \]

for the same parameters as in Remark 2.1. Direct computations show that \( f \) attains its minimum on \([m,M]\) at

\[ t_0 = 1 - \left( \frac{r}{|a|} \right)^{\frac{1}{1-r}}, \]

provided that \( t_0 \in [m,M] \). In fact, tedious Calculus computations show that this is always the case. Consequently,

\[ \beta = f(t_0) = a + b - a(p|a|)^{\frac{1}{p}} - (p|a|)^{\frac{1}{p-1}} = a + b - \frac{p-1}{p} (p|a|)^{-\frac{1}{p-1}}. \]

As an application of Corollary 2.2, we have the following scalar Bellman-type inequality.

**Corollary 2.3.** For \( 1 \leq i \leq n \), let \( a_i, b_i \) be positive numbers satisfying \( 0 < m \leq a_i, b_i \leq M < 1 \) for some scalars \( m, M \). Then, for \( p > 1 \) and \( q \leq 1 \),

\[ 2^{1-\frac{q}{p}} \alpha \sum_{i=1}^{n} \left( 2^q - (a_i + b_i)^q \right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} \left\{ (1-a_i^q)^{\frac{1}{p}} + (1-b_i^q)^{\frac{1}{p}} \right\}, \]

where \( \alpha \) is as in Corollary 2.2.
Proof. For the given $a_i, b_i$, define the $n \times n$ matrices $A = \text{diag}(a_i^q)$ and $B = \text{diag}(b_i^q)$. Apply the first inequality of Corollary 2.2 with $v = \frac{1}{p}$ to get

$$\alpha(I - A\nabla B)^{\frac{1}{p}} \leq (I - A)^{\frac{1}{p}} \nabla (I - B)^{\frac{1}{p}},$$

where we have chosen $\Phi$ to be the identity mapping. In particular, it follows that

$$\alpha \| (I - A\nabla B)^{\frac{1}{p}} \| \leq \| (I - A)^{\frac{1}{p}} \nabla (I - B)^{\frac{1}{p}} \|,$$

for any unitarily invariant norm $\| \|$. Selecting the trace norm $\| \|_1$, we obtain

$$\alpha \sum_{i=1}^{n} s_i \left((I - A\nabla B)^{\frac{1}{p}}\right) \leq \sum_{i=1}^{n} s_i \left((I - A)^{\frac{1}{p}} \nabla (I - B)^{\frac{1}{p}}\right),$$

where $s_i$ is the $i^{th}$ singular value. This implies

$$\alpha \sum_{i=1}^{n} \left(1 - a_i^q \nabla b_i^q\right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} \left(1 - a_i^q \nabla (1 - b_i^q)^{\frac{1}{p}}\right).$$

That is, noting concavity of the mapping $t \mapsto t^q$,

$$\frac{1}{2} \sum_{i=1}^{n} \left\{ (1 - a_i^q)^{\frac{1}{p}} + (1 - b_i^q)^{\frac{1}{p}} \right\} \geq \alpha \sum_{i=1}^{n} \left(1 - \frac{a_i^q + b_i^q}{2}\right)^{\frac{1}{p}} \geq \alpha \sum_{i=1}^{n} \left(1 - \left(\frac{a_i + b_i}{2}\right)^q\right)^{\frac{1}{p}} = \frac{\alpha}{2q/p} \sum_{i=1}^{n} (2^q - (a_i + b_i)^q)^{\frac{1}{p}},$$

which completes the proof.

The main observation in [3, Lemma 3.2] can be stated as follows.

**Corollary 2.4.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators such that $mI \leq A, B \leq M I$ for some scalars $0 < m < M$. If $f : [m, M] \rightarrow [0, \infty)$ is a concave function and $v \in [0, 1]$, then the ratio inequality

$$\alpha f(A\nabla v, B) \leq f(A) \nabla_v f(B)$$

holds, where $\alpha = \min_{t \in [m, M]} \left\{ \frac{a f + b f}{f(t)} \right\}$. Additionally, the following difference inequality

$$f(A\nabla_v B) + \beta I \leq f(A) \nabla_v f(B)$$

holds, where $\beta = \min_{t \in [m, M]} \left\{ a f + b f - f(t) \right\}$.

The reverse inequalities in (2.2) and (2.3) hold when $f$ is a convex function.

We conclude this paper, by presenting the following simple proof of (1.2) and some reversed versions.
PROPOSITION 2.1. Let \( a_k, b_k \) be positive numbers such that \( \sum_{k=1}^{n} a_k^p \leq 1 \) and \( \sum_{k=1}^{n} b_k^p \leq 1 \), for \( p \in \mathbb{R} \). Then, for \( 0 \leq v \leq 1 \),

\[
\left( 1 - \frac{1}{p} \sum_{k=1}^{n} (a_k^p \nabla_v b_k^p) \right)^{\frac{1}{p}} \geq \left( 1 - \frac{1}{p} \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \nabla_v \left( 1 - \frac{1}{p} \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}}, \quad \text{if } p > 1
\]

and

\[
\left( 1 - \frac{1}{p} \sum_{k=1}^{n} (a_k^p \nabla_v b_k^p) \right)^{\frac{1}{p}} \leq \left( 1 - \frac{1}{p} \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \nabla_v \left( 1 - \frac{1}{p} \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}}, \quad \text{if } p < 1.
\]

**Proof.** For \( 0 \leq v \leq 1 \), let

\[
f(v) = \left( 1 - \frac{1}{p} \sum_{k=1}^{n} (a_k^p \nabla_v b_k^p) \right)^{\frac{1}{p}}.
\]

Since the summands are linear in \( v \), it is readily seen that \( f \) is concave if \( p > 1 \) and is convex if \( p < 1 \). Then both inequalities follow from concavity/convexity of \( f \).

Notice that when \( p > 1 \), the function \( x \mapsto x^p, x > 0 \) is convex. Therefore, \( a_k^p \nabla_v b_k^p \geq (a_k \nabla_v b_k)^p \). This observation together with (2.4) imply

\[
\left( 1 - \frac{1}{p} \sum_{k=1}^{n} (a_k^p \nabla_v b_k^p) \right)^{\frac{1}{p}} \geq \left( 1 - \frac{1}{p} \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \nabla_v \left( 1 - \frac{1}{p} \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}}, \quad \text{if } p > 1.
\]

An elaborated proof of this inequality was given in [6] as an application of (1.1). Further, in [6], it was shown that this last inequality is equivalent to (1.2).

Notice that convexity of the mapping \( x \mapsto x^p, p > 1 \) allowed the passage from \( a_k^p \nabla_v b_k^p \) to \( (a_k \nabla_v b_k)^p \). Unfortunately, the same logic does not apply for \( p < 1 \). However, the following is a more elaborated convexity result. The proof follows immediately upon finding the second derivative of the given function.

PROPOSITION 2.2. For the positive numbers \( a_k, b_k \) satisfying \( \sum_{k=1}^{n} a_k^p, \sum_{k=1}^{n} b_k^p \leq 1 \), where \( p \in \mathbb{R} \), define the function

\[
f(v) = \left( 1 - \frac{1}{p} \sum_{k=1}^{n} (a_k^p \nabla_v b_k^p) \right)^{\frac{1}{p}}, \quad 0 \leq v \leq 1.
\]

Then \( f \) is concave if \( p > 1 \), while it is convex if \( p < 0 \).

From this, we have

\[
\left( 1 - \frac{1}{p} \sum_{k=1}^{n} (a_k^p \nabla_v b_k^p) \right)^{\frac{1}{p}} \leq \left( 1 - \frac{1}{p} \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \nabla_v \left( 1 - \frac{1}{p} \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}}, \quad \text{if } p < 0.
\]
Using this inequality and following the proof of [6, Theorem 2.5] imply the following reverse of (1.2).

COROLLARY 2.5. Let $a, a_k, b, b_k$ be positive scalars satisfying $\sum_{k=1}^{n} a_k^p \leq a^p$ and $\sum_{k=1}^{n} b_k^p \leq b^p$, where $p < 0$. Then the following reverse of (1.2) holds

$$\left( a^p - \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} + \left( b^p - \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}} \geq \left( (a + b)^p - \sum_{k=1}^{n} (a_k + b_k)^p \right)^{\frac{1}{p}}.$$ 

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