THE WEAK MAXIMUM PRINCIPLE FOR SECOND-ORDER ELLIPTIC AND PARABOLIC CONORMAL DERIVATIVE PROBLEMS

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Abstract. We prove the weak maximum principle for second-order elliptic and parabolic equations in divergence form with the conormal derivative boundary conditions when the lower-order coefficients are unbounded and domains are beyond Lipschitz boundary regularity. In the elliptic case we consider John domains and lower-order coefficients in $L_n$ spaces ($a^i, b^i \in L_q, c \in L_{q/2}$, $q = n$ if $n \geq 3$ and $q > 2$ if $n = 2$). For the parabolic case, the lower-order coefficients $a^i, b^i$, and $c$ belong to $L_{q,r}$ spaces ($a^i, b^i, |c|^{1/2} \in L_{q,r}$ with $n/q + 2/r \leq 1$), $q \in (n, \infty]$, $r \in [2, \infty]$, $n \geq 2$. We also consider coefficients in $L_{n,\infty}$ with a smallness condition for parabolic equations.

1. Introduction. The classical properties of solutions to elliptic and parabolic differential equations, such as the maximum principle, Harnack’s inequality, Hölder estimates, and $L_p$ estimates, have been studied vastly over the last 70 years. Among them, the maximum principle plays an important role in the study of second-order linear and nonlinear elliptic and parabolic equations. For instance, it is an essential ingredient in proving a priori estimates and existence (see, for instance, [7, 8]). In this paper, we consider the weak maximum principle for second-order linear elliptic and parabolic equations in divergence form with the conormal derivative boundary conditions (Neumann boundary conditions) on bounded non-smooth domains.

We mainly study the conormal derivative problem for parabolic operators in divergence form

$$\mathcal{L}u = -u_t + D_i(a^{ij}(x,t)D_ju + a^i(x,t)u) + b^i(x,t)D_iu + c(x,t)u$$

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whose coefficients $a^{ij}$, $a^i$, $b^i$, $c$, $i,j = 1,2,\ldots,n$, are measurable functions on a bounded cylindrical domain in $\mathbb{R}^n \times \mathbb{R}$. The leading coefficients $a^{ij}(x,t)$ are assumed to be uniformly elliptic and bounded; that is, there exists $\delta \in (0,1)$ such that
\[
\delta |\xi|^2 \leq a^{ij}(x,t)\xi^j \xi^i \quad \text{and} \quad |a^{ij}(x,t)| \leq 1/\delta
\]
for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Throughout the paper, we adopt the summation convention over repeated indices.

We focus on two objectives: unbounded lower-order coefficients and non-smooth domains (beyond Lipschitz category). To get the weak maximum principle for equations with such coefficients and domains, we first suggest minimal integrability assumptions on the lower-order coefficients without any extra structural conditions on $b := (b^1, \ldots, b^n)$, such as $\text{div} b \leq 0$. We suppose that $a^i, b^i \in L_{q,r}$, $\frac{n}{q} + \frac{2}{r} \leq 1$. Notably, in the case $\frac{n}{q} + \frac{2}{r} = 1$, we establish the weak maximum principle for equations defined on non-smooth domains without smallness assumptions on $\|a^i\|_{L_{q,r}}$ or $\|b^i\|_{L_{q,r}}$, provided that $2 \leq n < q \leq \infty$. In the case $q = n \geq 3$ with $r = \infty$, we obtain the same result under a smallness assumption on $\|a^i\|_{L_{n,\infty}}$ and $\|b^i\|_{L_{n,\infty}}$. For the case $n = 1$, see Remark 2.5. In contrast, for the elliptic conormal derivative problem, if $n \geq 3$, we do not require the smallness assumption on $\|a^i\|_{L_n}$ or $\|b^i\|_{L_n}$. The choice of $a^i, b^i \in L_n$, $n \geq 3$, is optimal in the sense that, for $a^i, b^i \in L_{n-\varepsilon}$, weak solutions may not be well-defined unless they have sufficient smoothness. Concerning this for the parabolic case, see Remark 2.2. We note that there is a research activity in the case $1 \leq \frac{n}{q} + \frac{2}{r} < 2$ for parabolic problems. Indeed, a version of the strong maximum principle is proved for any Lipschitz solutions in [21]. In this case the additional assumption $\text{div} b \leq 0$ in the sense of distributions is imposed.

Secondly, regarding the maximum principle on non-smooth domains, we consider John domains. They can be defined in many equivalent ways and we refer the reader to [16,17] for various definitions of John domains.

**Definition 1.1.** A bounded domain $\Omega$ in $\mathbb{R}^n$ is a John domain with center $z_0 \in \Omega$ and constant $K \geq 1$ if for each $z \in \Omega \setminus \{z_0\}$, there is a rectifiable curve $\gamma(z,z_0) \subset \Omega$ connecting $z$ and $z_0$ such that
\[
|\gamma(z,x)| \leq K d(x), \quad \forall x \in \gamma(z,z_0),
\]
where $d(x) := \text{dist}(x, \partial \Omega)$, and $|\gamma(z,x)|$ denotes the length of the subcurve $\gamma(z,x) \subset \gamma(z,z_0)$ connecting $z$ and $x$.

Note that it follows from (1.1) with $x = z_0$ that $|\gamma(z,z_0)| \leq K d(z_0)(=: R_0)$ for all $z \in \Omega$. Since $|z-z_0| \leq \gamma(z,z_0)$, we have $\Omega \subset B_{R_0}(z_0)$, and
\[
\text{diam} \Omega \leq 2R_0 = 2K d(z_0) \leq K \cdot \text{diam} \Omega.
\]
If there is no confusion, hereafter, we shall simply say “John domain” instead of saying “John domain with center $z_0$ and constant $K$”. Every Lipschitz domain is a John domain. John domains are typical examples of non-smooth domains, and non-tangentially accessible (NTA) domains and uniform domains are examples of John domains. We have the following inclusions between different classes of domains:

\[
\text{Lipschitz} \subset \text{NTA} \subset \text{Uniform} \subset \text{John}.
\]

See [1,17]. In general, the Hausdorff dimension of the boundary of a John domain in $\mathbb{R}^n$ can be strictly larger than $n-1$. The boundary of a (planar) John domain may contain an interior cusp, while exterior cusps are ruled out. Moreover, John
domains satisfy an optimal geometric condition for the Sobolev-Poincaré inequality to hold. See Proposition 3.1 and Remark 3.2.

We point out that if all lower-order coefficients are bounded, then the weak maximum principle for elliptic and parabolic conormal derivative problems is established, for instance, on Lipschitz domains. We refer to [19, Chapter VI] for the parabolic case and [20, Chapter 5] for the elliptic case. In [3], the authors proved a generalized maximum principle for degenerate parabolic operators with discontinuous coefficients on a Lipschitz domain, when \( n \geq 2 \). Due to the degeneracy, the indices for the integrability of lower-order coefficients are implicitly given. In this paper, the case \( n = 1 \) is also mentioned and, for \( n \geq 3 \), the smallness of \( \|a^i\|_{L^{n,\infty}} \) and \( \|b^i\|_{L^{n,\infty}} \) required for the maximum principle is explicitly estimated. We believe that some of our results belong to the folklore and are more or less known to the experts. However, it has been very hard to find a specific reference and we anticipate that our results fill a gap in the literature.

Finally, a few remarks are in order regarding the proofs of our main results. To prove the critical case \( a^i, b^i \in L^{q,r} \) with \( \frac{n}{q} + \frac{2}{r} = 1 \), \( r \neq \infty \), of the main results, we use a decomposition on \( a^i \) and \( b^i \), that is, for instance, \( b^i = b^i_1 + b^i_2 \) such that \( b^i_1 \) with its small norm and \( b^i_2 \in L^{\infty} \). See the proof of Theorem 2.1 and Lemma 6.2. In addition, we prove a multiplicative version of the Gagliardo-Nirenberg-Sobolev inequality on John domains (Lemma 3.4), which is the main tool of the proof. Lemma 3.4 clearly holds on \( W^{1,1}_p \)-extension domains. According to Jones’ results [15], uniform domains (as mentioned earlier, a strict subclass of John domains) are \( W^{1,1}_p \)-extension domains and these two domains are equivalent only for bounded and finitely connected domains in the plane. We refer to [10, 12–14] for more details.

A brief outline of the paper is as follows. In Section 2, we state the main results (Theorems 2.1 and 2.6). Section 3 is devoted to some embeddings for parabolic function spaces based on multiplicative inequalities. We prove Theorem 2.1 for the parabolic problem and further discuss the critical case in Section 4. The proof of Theorem 2.1 for elliptic problems is in Section 5. Finally, in Section 6, we give proofs for subtle issues.

2. Main results. The parabolic function spaces considered here are following. Let \( \Omega_T = \Omega \times (0, T) \) be a parabolic cylinder. Let \( q, r \geq 1 \) and consider the Banach space

\[
L_{q,r}(\Omega_T) = L_r(0, T; L^q(\Omega))
\]

with the norm \( \|v\|_{L_{q,r}(\Omega_T)} := \|\|v(\cdot, t)\|_{L^q(\Omega)}\|_{L^r(0, T)} \). \( L_{q,q}(\Omega_T) \) will be denoted by \( L_q(\Omega_T) \). \( W^{1,0}_2(\Omega_T) \) is the Hilbert space with the scalar product

\[
(u, v)_{W^{1,0}_2(\Omega_T)} = \iint_{\Omega_T} (uv + D_i u D_i v) \, dx \, dt.
\]

\( W^{1,1}_2(\Omega_T) \) is the Hilbert space with the scalar product

\[
(u, v)_{W^{1,1}_2(\Omega_T)} = \iint_{\Omega_T} (uv + D_i u D_i v + u_t v_t) \, dx \, dt.
\]

Let \( p \geq 1 \) and consider the Banach space

\[
V_p(\Omega_T) := L_\infty(0, T; L^p(\Omega)) \cap L^p(0, T; W^{1,1}_2(\Omega))
\]
with the norm  
\[ \|v\|_{V_{p}(\Omega_T)} := \text{ess sup}_{0 < t < T} \|v(\cdot, t)\|_{L_p(\Omega)} + \|Dv\|_{L_p(\Omega_T)}. \]

\(V_{2}^{1,0}(\Omega_T)\) is the Banach space consisting of all elements of \(V_{2}(\Omega_T)\) that are continuous in \(t\) in the norm of \(L_2(\Omega)\) with the norm  
\[ \|v\|_{V_{2}^{1,0}(\Omega_T)} := \max_{0 \leq t \leq T} \|v(\cdot, t)\|_{L_2(\Omega)} + \|Dv\|_{L_2(\Omega_T)}. \]

The continuity in \(t\) of a function \(v(x, t)\) in the norm of \(L_2(\Omega)\) means that  
\[ \|v(\cdot, t + \Delta t) - v(\cdot, t)\|_{L_2(\Omega)} \to 0 \quad \text{as} \quad \Delta t \to 0. \]

The space \(V_{2}^{1,0}(\Omega_T)\) is obtained by completing the set \(W_{2}^{1,1}(\Omega_T)\) in the norm of \(V_{2}(\Omega_T)\).

Let \(S\Omega_T = \partial\Omega \times (0, T)\) be the lateral boundary of \(\Omega_T\) and \(B\Omega_T = \Omega \times \{0\}\) be the bottom of \(\Omega_T\). For \(v \in V_{2}^{1,0}(\Omega_T)\), \(sup_{B\Omega_T} v\) is defined by  
\[ \sup_{B\Omega_T} v = \inf \{k : v(x, 0) \leq k, \ x \in \Omega\}. \]

We now present the main results of this paper, the first of which is the weak maximum principle for parabolic equations with the conormal derivative boundary condition.

**Theorem 2.1.** Let \(\Omega\) be a John domain in \(\mathbb{R}^n, n \geq 2\). Suppose that \(a^i, b^i, a^{ij}\) and \(c^{ij}\) are in \(L_{q,r}(\Omega_T)\) with \(q \in (n, \infty)\) and \(r \in [2, \infty]\) satisfying  
\[ \frac{n}{q} + \frac{2}{r} \leq 1, \tag{2.1} \]

and  
\[ \iint_{\Omega_T} (-a^i D_i \varphi + c \varphi) \, dx \, dt \leq 0 \tag{2.2} \]

for all nonnegative \(\varphi\) defined on \(\Omega_T\) such that  
\[ \varphi \in L_{\frac{n}{q-1}, \frac{2}{r-1}}(\Omega_T), \quad D\varphi \in L_{\frac{n}{q-1}, \frac{2}{r-1}}(\Omega_T). \]

Assume that \(u \in V_{2}^{1,0}(\Omega_T)\) satisfies  
\[
\begin{cases}
-u_i + D_i (a^{ij}(x, t)D_j u + a^i(x, t)u) + b^i(x, t)D_i u + c(x, t)u \geq 0 & \text{in } \Omega_T \\
-(a^{ij} D_j u + a^i u)\nu^i \geq 0 & \text{on } S\Omega_T
\end{cases}
\]

in the weak sense, where \(\nu = (\nu^1, \ldots, \nu^n)\) is the outward normal direction of \(S\Omega_T\). In other words, \(u \in V_{2}^{1,0}(\Omega_T)\) satisfies  
\[ \iint_{\Omega_T} u \phi_i \, dx \, dt - \iint_{\Omega_T} \left\{ (a^{ij} D_j u + a^i u) D_i \phi - (b^i D_i u + c u) \phi \right\} \, dx \, dt \geq 0 \tag{2.3} \]

for all nonnegative \(\phi \in W_{2}^{1,1}(\Omega_T)\) with \(\phi(x, 0) = \phi(x, T) = 0\) for a.e. \(x\). Then  
\[ \sup_{\Omega_T} u \leq \sup_{B\Omega_T} u^+, \]

where \(u^+ := \max\{u, 0\}\).

**Remark 2.2.** The condition (2.1) is unavoidable in terms of well-posedness of the integrals. In fact, we have  
\[ \iint_{\Omega_T} a^i u D_i v \, dx \, dt \leq \frac{1}{2} \|a^i\|_{L_{q,r}(\Omega_T)}^2 \|u\|_{L_{\frac{n}{q-1}, \frac{2}{r-1}}(\Omega_T)}^2 + \frac{1}{2} \|D_i v\|_{L_2(\Omega_T)}^2, \]
provided that $2 < q \leq \infty$ and $2 \leq r \leq \infty$. The condition (2.1) guarantees $\|u\|_{L^\frac{2q}{q-2}, \frac{2r}{r-2}(\Omega_T)} < +\infty$ by Proposition 3.5.

Remark 2.3. As is seen in the proof of Theorem 2.1, the inequality (2.2) is required to hold only for $uv$ in place of $\varphi$, where $u, v \in V^{1,0}_2(\Omega_T)$. In this case, from the definition of $V^{1,0}_2(\Omega_T)$ and Proposition 3.5, we see that $u, v \in L^\frac{2q}{q-2}, \frac{2r}{r-2}(\Omega_T)$ and $Du, Dv \in L^2(\Omega_T)$. Then

$$uv \in L^\frac{q}{q-2}, \frac{r}{r-2}(\Omega_T), \quad D(uv) = uDv + vDu \in L^\frac{q}{q-2}, \frac{r}{r-2}(\Omega_T).$$

Also note that by Hölder’s inequality, (2.2) is well-defined.

Remark 2.4. In the case $q = n \geq 3$ with $r = \infty$, we obtain the same result under an additional smallness assumption on $||a^i| + |b^i||_{L^{q, r}(\Omega_T)}$. See Section 4.

Remark 2.5. Note that John domains are irrelevant when $n = 1$. In this case, Theorem 2.1 still holds, provided that $q \in [2, \infty]$ and $r \in [2, \infty]$ satisfy (2.1) and that $\Omega$ is a bounded open interval. The proof is the same as that of Theorem 2.1 for $n \geq 2$ using the embedding inequality (3.3) for $n = 1$. One can find a proof of the inequality (3.3) in [18, Chapter II] when $\Omega$ is a bounded open interval or has a piecewise smooth $\partial \Omega$ in the case $n \geq 2$.

We next state the weak maximum principle for second-order elliptic equations in divergence form with the conormal derivative boundary condition.

Theorem 2.6. Let $\Omega$ be a John domain in $\mathbb{R}^n$, $n \geq 2$. Suppose that $a^i, b^i$, and $|c|^{1/2}$ are in $L^q(\Omega)$ with $q \in [n, \infty]$ if $n \geq 3$ and $q \in (2, \infty]$ if $n = 2$, and

$$\int_{\Omega} (-a^i D_i \varphi + c \varphi) \, dx \leq 0$$

for all nonnegative $\varphi \in W^{1,1}_0(\Omega)$. Assume that $u \in W^{1}_{2}(\Omega)$ satisfies

$$\left\{ \begin{array}{ll} D_i(a^{ij}(x) D_j u + a^i(x) u) + b^i(x) D_i u + c(x) u & \geq 0 \quad \text{in } \Omega \\ -(a^{ij} D_j u + a^i u) \nu^i & \geq 0 \quad \text{on } \partial \Omega \end{array} \right.$$  

in the weak sense, where $\nu = (\nu^1, \ldots, \nu^n)$ is the outward normal direction of $\partial \Omega$. In other words, $u \in W^1_{2}(\Omega)$ satisfies

$$\int_{\Omega} \left\{ \left( a^{ij} D_j u + a^i u \right) D_i \varphi - \left( b^i D_i u + cu \right) \varphi \right\} \, dx \leq 0$$

for all nonnegative $\varphi \in W^1_{2}(\Omega)$. Then either $u$ is a constant or else $u \leq 0$ in $\Omega$.

As mentioned in the introduction, the result concerning elliptic conormal derivative problems does not require any smallness assumptions on the $L_n$ norms of lower-order coefficients in case $q = n \geq 3$. See Section 5. We finally remark that Theorem 2.6 is a natural extension of [20, Theorem 5.15] to both unbounded coefficients and non-smooth domains.

3. Multiplicative inequalities. In this section, we provide some multiplicative inequalities on John domains. To do this, we first introduce the Sobolev-Poincaré inequality on John domains.
Proposition 3.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a John domain with center $z_0 \in \Omega$ and constant $K \geq 1$. The Sobolev-Poincaré inequality
\[ \left( \int_{\Omega} |u - u_\Omega|^{\frac{n_p}{Np}} \, dx \right)^{\frac{n-1}{n}} \leq c_1 \left( \int_{\Omega} |Du|^p \, dx \right)^{\frac{1}{p}} \tag{3.1} \]
holds for $1 \leq p < n$ whenever $u \in W^1_p(\Omega)$ and $u_\Omega = \frac{1}{|\Omega|} \int_{\Omega} u \, dx$. Note that the constant $c_1$ depends on $n$, $p$, $K$, and $d(z_0)$. See [2, Section 6].

Remark 3.2. John domain satisfies an optimal geometric condition (in some sense) for the Sobolev-Poincaré inequality (3.1) to hold. If $\Omega$ is a simply connected plane domain of finite area, for instance, the necessary and sufficient condition for the validity of (3.1) is that $\Omega$ is a John domain. Similarly if $\Omega$ satisfies an additional assumption – the so-called separation property (see [4]), then John domain is optimal in the higher dimension case. We refer to [4,5,11,17] for more details.

Corollary 3.3. Let $\Omega$ be a John domain in $\mathbb{R}^n$, $n \geq 2$. Let $u \in W^1_p(\Omega)$ with average value zero on a set $E \subset \Omega$ of positive volume. For $N > p \geq 1$ and $N \geq n$, there is a constant $C$, depending only on $n$, $p$, $N$, $c_1$, $|\Omega|$, and $|\Omega|/|E|$, such that
\[ \|u\|_{L^{\frac{n_p}{Np}}(\Omega)} \leq C \|Du\|_{L^p(\Omega)}. \]

Proof. Find $p_1$ such that
\[ 1 < p_1 < n, \quad p_1 \leq p, \quad \text{and} \quad \frac{Np}{N-p} \leq \frac{np_1}{n-p_1}. \]
This is possible because $p_1 = p$ if $p \in [1,n)$. Otherwise, one can find $p_1$ such that $1 < p_1 < n \leq p$ because
\[ \frac{Nnp}{Np + Nn - np} < n. \]
Let $q = \frac{np_1}{n-p_1}$. Then $q > 1$ and
\[ \|u\|_{L^q(\Omega)} = \left( \int_{\Omega} \left| u(x) - \frac{1}{|E|} \int_{E} u(y) \, dy \right|^q \, dx \right)^{1/q} \leq |E|^{-1/q} \left( \int_{\Omega} \int_{\Omega} |u(x) - u(y)|^q \, dy \, dx \right)^{1/q} \leq 2 \left( \frac{|\Omega|}{|E|} \right)^{1/q} \left( \int_{\Omega} |u(x) - u_\Omega|^q \, dx \right)^{1/q}. \]
Hence, by Proposition 3.1, we have
\[ \|u\|_{L^{\frac{n_p}{Np}}(\Omega)} \leq C \|u\|_{L^{\frac{np_1}{nt}}(\Omega)} \leq C \|u - u_\Omega\|_{L^{\frac{np_1}{nt}}(\Omega)} \leq C \|Du\|_{L^{p_1}(\Omega)} \leq C \|Du\|_{L^p(\Omega)}, \]
where $C$ depends also on $|\Omega|/|E|$. We refer to [9, Theorem 3.16] and [20, Lemma 5.11] for similar results.

We have the following multiplicative inequality on John domains.

Lemma 3.4. Let $\Omega$ be a John domain in $\mathbb{R}^n$, $n \geq 2$. For $N > p \geq 1$ and $N \geq n$, there is a constant $C$, depending only on $n$, $p$, $N$, $c_1$, and $|\Omega|$, such that
\[ \left( \int_{\Omega} |u|^{\frac{n_p}{Np}} \, dx \right)^{\frac{n-1}{n}} \leq C \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} (|Du|^p + |u|^p) \, dx \right)^{\frac{n-1}{n}} \]
for any $u \in W^1_p(\Omega)$, where $c_1$ is the constant in (3.1).
Proof. The proof is a detailed version of [20, Theorem 5.8] on John domains. If $N = n > p$, then it follows directly from (3.1) that
$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \leq c_1 \|Du\|_{L^p(\Omega)} + |\Omega|^{-1/n} \|u\|_{L^p(\Omega)}.$$ 
So we assume that $N > n$. A simple calculation yields that
$$\left( \int_\Omega |u|^{\frac{np}{n-p}} dx \right)^{\frac{n-1}{n}} \leq C(n) \left( \int_\Omega |u - u_\Omega|^{\frac{np}{n-p}} dx + \int_\Omega |u_\Omega|^{\frac{np}{n-p}} dx \right)^{\frac{n-1}{n}}.$$ 
From this and the Sobolev-Poincaré inequality (3.1) with $p = 1$, we have that
$$\|u\|_{L^{\frac{n}{n-p}}(\Omega)} \leq C(n, c_1, |\Omega|) \left( \|Du\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \right). \tag{3.2}$$ 
Set $\alpha = \frac{p(n-1)}{N + np - n - p}$. Then $0 < \alpha < 1$ and $1 - \alpha = \frac{N - n}{N + np - n - p}$. Using Hölder’s inequality for $\alpha$ and $\frac{1}{\alpha}$, we see that
$$\int_\Omega |u|^{\frac{np}{n-p}} dx = \int_\Omega |u|^{\frac{np}{n-p}(\frac{N}{N+n(p-n-1)})} \cdot |u|^{\frac{(N-n)p}{N+n(p-n-1)}} dx \leq \left( \int_\Omega |u|^p dx \right)^\alpha \left( \int_\Omega |u|^p dx \right)^{1-\alpha},$$ 
where $r = \frac{np(N-1)}{(N-p)(n-1)}$. Due to (3.2),
$$\left( \int_\Omega |u|^p dx \right)^{\frac{n-1}{n}} = \left( \int_\Omega |u|^\frac{r(n-1)}{n} dx \right)^{\frac{n-1}{n}} \leq C(c_1, r) \int_\Omega \left( |u|^{\frac{r(n-1)}{n}} + |u|^{\frac{r(n-1)}{n}-1} |Du| \right) dx.$$ 
If $p > 1$, Hölder’s inequality shows that
$$\int_\Omega |u|^{\frac{r(n-1)}{n}-1} |Du| dx \leq \left( \int_\Omega |u|^{\frac{np}{n-p}} dx \right)^{\frac{p-1}{p}} \left( \int_\Omega |Du|^p dx \right)^{\frac{1}{p}}.$$ 
Indeed,
$$\frac{r(n-1)}{n} - 1 = \frac{np(N-1)}{(N-p)(n-1)} - 1 = \frac{Np - p - N + p}{N - p} = \frac{N(p-1)}{N - p}.$$ 
Similarly,
$$\int_\Omega |u|^{\frac{r(n-1)}{n}} dx = \int_\Omega |u|^{\frac{r(n-1)}{n}-1} |u| dx \leq \left( \int_\Omega |u|^{\frac{np}{n-p}} dx \right)^{\frac{p-1}{p}} \left( \int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$ 
While $p = 1$ implies $\frac{r(n-1)}{n} = 1$ in which case the inequality is immediate. Therefore, 
$$\int_\Omega |u|^{\frac{np}{n-p}} dx \leq \left( \int_\Omega |u|^p dx \right)^\alpha \left( \int_\Omega |u|^p dx \right)^{1-\alpha} \leq C \left( \int_\Omega \{ |u|^{\frac{r(n-1)}{n}} + |u|^{\frac{r(n-1)}{n}-1} |Du| \} dx \right)^{\frac{n-1}{n}} \left( \int_\Omega |u|^p dx \right)^{1-\alpha} \leq C \left( \int_\Omega |u|^{\frac{np}{n-p}} dx \right)^{\frac{n-1}{n}} \left( \|u\|_p + \|Du\|_p \right)^{\frac{n-1}{n}} \left( \int_\Omega |u|^p dx \right)^{1-\alpha}.$$
Since $\alpha = \frac{p(n-1)}{N + np - n - p}$,
\[
\frac{n\alpha}{n-1} = \frac{np}{N + np - n - p} \quad \text{and} \quad \frac{p-1}{p} \frac{n\alpha}{n-1} = \frac{np-n}{N + np - n - p}.
\]
Thus, using $1 - \alpha = \frac{N-n}{N + np - n - p}$,
\[
\left(\int_\Omega |u|^{\frac{Np}{N+n}} \, dx\right)^{N+n-p-n} \leq C \left(\int_\Omega \left(|u|^p + |Du|^p\right) \, dx\right)^{\frac{n}{n}} \left(\int_\Omega |u|^p \, dx\right)^{N-n}
\]
and consequently,
\[
\left(\int_\Omega |u|^{\frac{Np}{N+n}} \, dx\right)^{N-n} \leq C \left(\int_\Omega \left(|u|^p + |Du|^p\right) \, dx\right)^{\frac{n}{n}} \left(\int_\Omega |u|^p \, dx\right)^{N-n}.
\]
This completes the proof.

We next state embedding inequalities for the parabolic spaces $V_p(\Omega_T)$. One can prove Proposition 3.5 in the same way as in the proof of Proposition 3.4 in [6, Chapter 1] with Theorem 2.1 replaced by Lemma 3.4. For the reader’s convenience, we give a short proof.

**Proposition 3.5.** Let $\Omega$ be a John domain in $\mathbb{R}^n$, $n \geq 2$. Let $N$ and $p$ be constants such that $N > p \geq 1$ and $N \geq n$. There exists a constant $C$, depending only on $n$, $p$, $N$, $c_1$, $|\Omega|$, and $T$, such that, for every $v \in V_p(\Omega_T)$,
\[
\|v\|_{L_{\frac{Np}{N+n}}(\Omega_T)} \leq C \|v\|_{V_p(\Omega_T)}.
\]
In particular, we have that
\[
\|v\|_{L_{\frac{2q}{nq+2r}}(\Omega_T)} \leq C \|v\|_{V_2(\Omega_T)},
\]
where the constants $q \in (n, \infty]$ and $r \in [2, \infty)$ are linked by $\frac{2}{q} + \frac{2}{r} = 1$. Moreover, the inequality (3.3) holds for $q = n$ and $r = \infty$ if $n \geq 3$.

**Proof.** By Lemma 3.4, for a.e. $t \in [0, T]$,
\[
\|v(\cdot, t)\|_{L_{\frac{Np}{N+n}}(\Omega_T)} \leq C \|v(\cdot, t)\|_{L_p(\Omega)}^\frac{Np-n}{N} \left(\|Dv(\cdot, t)\|_{L_p(\Omega)}^p + \|v(\cdot, t)\|_{L_p(\Omega)}^p\right).
\]
Then
\[
\|v\|_{L_{\frac{Np}{N+n}}(\Omega_T)} \leq C T \frac{\pi}{2} \text{ess sup}_{0 < t < T} \|v(\cdot, t)\|_{L_p(\Omega)}
\]
\[
+ C \left(\text{ess sup}_{0 < t < T} \|v(\cdot, t)\|_{L_p(\Omega)}\right)^{1-\frac{n}{N}} \|Dv\|_{L_p(\Omega_T)}^n.
\]
This along with Young’s inequality implies the desired inequalities. In particular, the inequality (3.3) for $q = \infty$ and $r = 2$ follows directly from the definition of $V_2(\Omega_T)$.
WEAK MAXIMUM PRINCIPLE

4. Proof of Theorem 2.1. There is a well-recognized inconvenience when dealing with weak solutions of parabolic problems: weak solutions may not be differentiable in the time variable. Regarding this, we adopt, whenever needed, the so-called Steklov average of weak solutions. More precisely, due to the Steklov average and a proper smooth cut-off function depending only on the time variable, it may be possible to take a weak solution, even any function in $V^1_2(\Omega_T)$, as a test function. We refer to [18, Chapter III] or [19, Chapter VI] for more details about the Steklov averaging.

Proof of Theorem 2.1. Suppose that, to the contrary of Theorem 2.1, there exists $k$ such that $\sup_{\partial\Omega_T} u^+ \leq k < \sup_{\Omega_T} u$. (If no such $k$ exists, we are done.) Define $v = v_k = (u - k)^+$. Then $v \in V^1_2(\Omega_T)$ and take

$$Dv = \begin{cases} Du & \text{on } \{u > k\}, \\ 0 & \text{on } \{u \leq k\}. \end{cases}$$

We write

$$\Gamma_k = \{(x, t) \in \Omega_T : Dv \neq 0\}.$$

It follows from the definitions that $\Gamma_k \subset \{u > k\}$, $Du = Dv$ in $\Gamma_k$, and $Dv = 0$ on $\Omega_T \setminus \Gamma_k$. If $v$ can be taken as a test function, the inequality (2.3) can be written as

$$\iint_{\Omega_T} \left[ -uv_t + a^{ij}D_juD_iv - (a^i + b^i)vD_iu \right] dxdt \leq \iint_{\Omega_T} \left[ -a^iD_i(uv) + c(uv) \right] dxdt.$$

Since (2.2) is valid for $uv$, in place of $\varphi$, we have that

$$-\iint_{\Omega_T} uv_t dxdt + \iint_{\Omega_T} a^{ij}D_jD_iv dxdt \leq \iint_{\Gamma_k} (a^i + b^i)vD_i v dxdt.$$

From this inequality we derive

$$\frac{1}{2} \max_{0 \leq t \leq T} \int_{\Omega} v^2 dx + \int_{\Omega_T} |Dv|^2 dxdt \leq \int_{\Gamma_k} |(a^i + b^i)vD_i v| dxdt. \quad (4.1)$$

There are a few comments. The inequality (4.1) can be derived by a standard argument based on the Steklov average or a proper mollification technique. In fact, (4.1) is obtained by the method presented in [18, pages 142–143 and 182–183] for the problems subject to the Dirichlet boundary condition. Since we are dealing with the Neumann boundary condition, for the reader’s convenience we give a proof of (4.1) in Appendix. See Section 6.1.

We only deal with the case $\frac{n}{q} + \frac{2}{r} = 1$ with $q \in (n, \infty]$ and $r \in [2, \infty)$. Otherwise, that is, if $\frac{n}{q} + \frac{2}{r} < 1$, one can find $Q \in (n, q)$ and $R \in (2, r)$ such that $\frac{n}{Q} + \frac{2}{R} = 1$. Then $a^i$, $b^i$, $|c|^{1/2} \in L_{Q,R}(\Omega_T)$. Moreover, the sign condition (2.2) is satisfied for all nonnegative $\varphi$ on $\Omega_T$ such that

$$\varphi \in L_{\frac{n}{Q}, \frac{n}{R}}(\Omega_T), \quad D\varphi \in L_{\frac{n}{Q}, \frac{n}{R}}(\Omega_T)$$

because

$$L_{\frac{n}{Q}, \frac{n}{R}}(\Omega_T) \subset L_{\frac{n}{Q}, \frac{n}{R}}(\Omega_T), \quad L_{\frac{n}{Q}, \frac{n}{R}}(\Omega_T) \subset L_{\frac{n}{Q}, \frac{n}{R}}(\Omega_T).$$

Note that even if $n/q + 2/r < 1$ with $r = \infty$, we have $n/Q + 2/R = 1$ with $R < \infty$ because $Q$ is to be found so that $Q > n$. 


We decompose $a^i$ and $b^i$ as follows (for a detailed proof of the decompositions, see Lemma 6.2). For $\varepsilon > 0$

\[ a^i = a^i_1 + a^i_2, \quad ||a^i_1||_{L^q, r(\Omega_T)} \leq \varepsilon, \quad ||a^i_2||_{L^\infty(\Omega_T)} \leq \Lambda_1 \]

and

\[ b^i = b^i_1 + b^i_2, \quad ||b^i_1||_{L^q, r(\Omega_T)} \leq \varepsilon, \quad ||b^i_2||_{L^\infty(\Omega_T)} \leq \Lambda_2 \]

for some $\Lambda_1$ and $\Lambda_2$. Then it follows from (4.1) that

\[
\begin{align*}
\frac{1}{2} \max_{0 \leq t \leq T} \int_{\Omega} v^2 \, dx + \delta \int_{\Omega_T} |Dv|^2 \, dx \, dt \\
\leq \int_{\Gamma_k} |(a^i_1 + b^i_1)vD_iv| \, dx \, dt \quad (=: I_1) \\
+ \int_{\Gamma_k} |(a^i_2 + b^i_2)vD_iv| \, dx \, dt \quad (=: I_2). 
\end{align*}
\]

By Proposition 3.5 with $\frac{n}{q} + \frac{2}{r} = 1$, we have that

\[
||v||_{L^{\frac{n}{2}, \frac{n}{r}}(\Omega)} \leq C_* ||v||_{V^2(\Omega_T)}. 
\]

Hence, $I_1$ is estimated as follows:

\[
I_1 = \int_{\Gamma_k} |(a^i_1 + b^i_1)vD_iv| \, dx \, dt \\
\leq C(\delta)||a^i_1|| + ||b^i_1||_{L^q, r(\Omega_T)} ||v||_{L^{\frac{n}{2}, \frac{n}{r}}(\Omega_T)}^2 + \frac{\delta}{2} \int_{\Omega_T} |Dv|^2 \, dx \, dt \\
\leq C_* ||a^i_1||_V + ||b^i_1||_{L^q, r(\Omega_T)} ||v||_{V^2(\Omega_T)}^2 + \frac{\delta}{2} \int_{\Omega_T} |Dv|^2 \, dx \, dt \\
\leq 4C_* \varepsilon^2 ||v||_{V^2(\Omega_T)}^2 + \frac{\delta}{2} \int_{\Omega_T} |Dv|^2 \, dx \, dt. 
\]

By Young’s inequality, we see that for any $\gamma > 0$

\[
I_2 = \int_{\Gamma_k} |(a^i_2 + b^i_2)vD_iv| \, dx \, dt \leq (\Lambda_1 + \Lambda_2) \int_{\Gamma_k} ||v||Dv| \, dx \, dt \\
\leq 2\Lambda_* \int_{\Gamma_k} \left( \frac{1}{4\gamma} ||v|^2 + \gamma |Dv|^2 \right) \, dx \, dt, 
\]

where $\Lambda_* = \max\{\Lambda_1, \Lambda_2\}$. Now, we combine (4.2), (4.4), and (4.5) to get

\[
\frac{1}{2} \max_{0 \leq t \leq T} \int_{\Omega} v^2 \, dx + \frac{\delta}{2} \int_{\Omega_T} |Dv|^2 \, dx \, dt \\
\leq 4C_* \varepsilon^2 ||v||_{V^2(\Omega_T)}^2 + 2\Lambda_* \int_{\Gamma_k} \left( \frac{1}{4\gamma} ||v|^2 + \gamma |Dv|^2 \right) \, dx \, dt. 
\]

Now we select $\varepsilon$ and $\gamma$ satisfying

\[
\varepsilon^2 = \frac{\delta}{32C_*} \quad \text{and} \quad \gamma = \frac{\delta}{16\Lambda_*}. 
\]

Thus we have

\[
\left( \frac{1}{2} - \frac{\delta}{4} \right) \max_{0 \leq t \leq T} \int_{\Omega} v^2 \, dx + \frac{\delta}{8} \int_{\Omega_T} |Dv|^2 \, dx \, dt \leq \frac{8}{\delta} \Lambda_*^2 \int_{\Gamma_k} ||v||^2 \, dx \, dt, 
\]
that is,
\[
\max_{0 \leq t \leq T} \int_{\Omega} v^2 \, dx + \int_{\Gamma_T} |Dv|^2 \, dx \, dt \leq C(\delta) \Lambda^2 \int_{\Omega} |v|^2 \, dx \, dt. \tag{4.6}
\]
Combining (4.3) and (4.6) yields
\[
\left( \left\| v \right\|_{L^{\frac{2n}{n-2}, \frac{2n}{n-2}}(\Omega_T)} \right)^2 \leq C \Lambda^2 \int_{\Gamma_k} \left| v \right|^2 \, dx \, dt,
\]
where by Hölder’s inequality,
\[
\int_{\Gamma_k} \left| v \right|^2 \, dx \, dt \leq \left\| v \right\|_{L^{\frac{2n}{n-2}, \frac{2n}{n-2}}(\Omega_T)}^2 \left( \int_0^T \left( \int_{\Omega_k} \chi_k \, dx \right)^{\frac{2}{q}} \, dt \right)^{\frac{q}{2}} \leq \left\{ \begin{array}{ll}
\frac{1}{\Omega^{\frac{2(\alpha-q)}{\alpha}} \left( \left\| v \right\|_{L^{\frac{2n}{n-2}, \frac{2n}{n-2}}(\Omega_T)} \right)^2}, & \text{if } r \geq q, \\
T^{\frac{q}{2} - \frac{2}{q}} \left\| v \right\|_{L^{\frac{2n}{n-2}, \frac{2n}{n-2}}(\Omega_T)}^2 \left| \Gamma_k \right|^{\frac{q}{2}}, & \text{if } r < q.
\end{array} \right.
\]
Consequently, we have
\[
\left\| v \right\|_{L^{\frac{2n}{n-2}, \frac{2n}{n-2}}(\Omega_T)} \leq C \Lambda^2 \left\| v \right\|_{L^{\frac{2n}{n-2}, \frac{2n}{n-2}}(\Omega_T)} \left| \Gamma_k \right|^\alpha
\]
for some $C > 0$ and $\alpha > 0$, that is,
\[
0 < \frac{1}{C \Lambda^2} \leq \left| \Gamma_k \right|^\alpha.
\]
Therefore, it follows that $\left| \Gamma_k \right|$ is bounded from below by a positive constant, independent of $k$, so is $\{u > k\}$ because $\Gamma_k \subset \{u > k\}$. This shows that the supremum of $u$ on $\Omega_T$ is finite (otherwise, $u$ is not, for instance, in $L_2(\Omega_T)$). We now see that $\Gamma_k \subset \Gamma_i$ whenever $\sup_{\Omega_T} u^{+} \leq l < k < \sup_{\Omega_T} u$. Hence
\[
\Gamma = \bigcap_{\sup_{\Omega_T} u^{+} \leq l < \sup_{\Omega_T} u} \Gamma_k
\]
has a positive measure. Set
\[
\Gamma^* := \bigcap_{\sup_{\Omega_T} u^{+} \leq l < \sup_{\Omega_T} u} \{u > k\}.
\]
Since $u = \sup_{\Omega_T} u$ on $\Gamma^*$, $\Gamma^* \subset \Gamma^*$, and $D u \neq 0$ on $\Gamma^*$, we have that $u = \sup_{\Omega_T} u$ on $\Gamma^*$ as well as $D u \neq 0$ on $\Gamma^*$, which is a contradiction. Thus, $\sup_{\Omega_T} u \leq \sup_{\Omega_T} u^{+}$. The theorem is proved. \(\square\)

In the remaining part of this section, we briefly discuss the critical case $q = n \geq 3$ with $r = \infty$. As mentioned in Remark 2.4, we obtain the same result under a smallness assumption on $\left\| |a'| + |b'| \right\|_{L_{q,r}(\Omega_T)}$.

To show this, let us assume that there exists a constant $k$ satisfying $\sup_{\Omega_T} u^{+} \leq k < \sup_{\Omega_T} u$. We again derive (4.1), from which we get
\[
\frac{1}{2} \max_{0 \leq t \leq T} \int_{\Omega} v^2 \, dx + \frac{\delta}{2} \int_{\Omega_T} |Dv|^2 \, dx \, dt \leq \frac{1}{2\delta} \left\| |a'| + |b'| \right\|_{L_{q,r}(\Omega_T)}^2 \left\| v \right\|_{L^{\frac{2n}{n-2}, \frac{2n}{n-2}}(\Gamma_k)}^2. \tag{4.7}
\]
Then by Proposition 3.5 with $q = n \geq 3$ and $r = \infty$, and the definition of $\| \cdot \|_{V^2(\Omega_T)}$, we obtain from (4.7) that

$$
\frac{\delta}{2} \left( \max_{0 \leq t \leq T} \int_{\Omega} v^2 dx + \int_{\Omega_T} |Dv|^2 dx dt \right) \leq \frac{C^2}{\delta} \|a^i| + |b^i|\|_{L_{q,r}(\Omega_T)}^2 \times \left( \max_{0 \leq t \leq T} \int_{\Omega} v^2 dx + \int_{\Omega_T} |Dv|^2 dx dt \right),
$$

where we used the fact that $\delta < 1$. Now, we additionally assume that $\|a^i| + |b^i\|_{L_{q,r}(\Omega_T)} < \frac{\delta^2}{2C^2}$.

Then we have

$$
\max_{0 \leq t \leq T} \int_{\Omega} v^2 dx + \int_{\Omega_T} |Dv|^2 dx dt \leq 0,
$$

which means that $v = (u - k)^+ = 0$. Consequently we see $u(x, t) \leq k$ a.e. $(x, t)$, which is a contradiction to $\sup_{\Omega_T} u^+ \leq k < \sup_{\Omega_T} u$. Hence $\sup_{\Omega_T} u \leq \sup_{\Omega_T} u^+$.

5. Proof of Theorem 2.6. In this section, we prove the weak maximum principle for second-order elliptic equations in divergence form with the conormal derivative boundary condition.

**Proof of Theorem 2.6.** It suffices to prove that $u$ is constant if $\sup_{\Omega} u > 0$. Thus, to get a contradiction, assume that $\sup_{\Omega} u > 0$, but $u$ is not constant. In this case there has to be a constant $k_1$ such that $0 < k_1 < \sup_{\Omega} u$ and $|\{u < k_1\}| > 0$.

Now we write (2.5) as follows.

$$
\int_{\Omega_T} \left\{ a^{ij} D_j u D_i \phi - (a^i + b^i) \phi D_i u \right\} dx \leq \int_{\Omega_T} \left\{ c(u \phi) - a^i D_i(u \phi) \right\} dx,
$$

where $0 \leq \phi \in W^1_2(\Omega)$. Let $k$ be a positive constant such that $k_1 \leq k < \sup_{\Omega} u$. For each $k \in [k_1, \sup_{\Omega} u)$, define

$$
v = v_k = \begin{cases} u - k & \text{if } u > k, \\ 0 & \text{if } u \leq k. \end{cases}
$$

Also define

$$w = w_k = \begin{cases} D u & \text{if } u > k, \\ 0 & \text{if } u \leq k. \end{cases}
$$

Then we have that $Dv = w$ a.e. in $\Omega$. We write

$$\Gamma_k = \{x \in \Omega : w(x) \neq 0\}.
$$

It follows by the definitions that $\Gamma_k \subset \{u > k\}$, $Du = Dv$ a.e. in $\Gamma_k$, and $Dv = 0$ a.e. in $\Omega \setminus \Gamma_k$. Hence we have

$$
\int_{\Omega} a^{ij} D_j v D_i v dx \leq \int_{\Gamma_k} (a^i + b^i) v D_i v dx,
$$

because (2.4) holds true for $\varphi = uv$.

Note that it follows from Corollary 3.3 with $N = q \geq n$ if $n \geq 3$ and $N = q > 2$ if $n = 2$, and $p = 2$ that

$$
\|v\|_{L^2_q(\Omega)} \leq C_* \|Dv\|_{L^2(\Omega)},
$$

(5.1)
where \( v = 0 \) on a set of positive volume because of the choice of \( k \). Now, we decompose \( a^i \) and \( b^i \) as follows. For \( \varepsilon > 0 \)
\[
a^i = a^i_1 + a^i_2, \quad \|a^i_1\|_{L_q(\Omega)} \leq \varepsilon, \quad \|a^i_2\|_{L_\infty(\Omega)} \leq L_1
\] (5.2)
and
\[
b^i = b^i_1 + b^i_2, \quad \|b^i_1\|_{L_q(\Omega)} \leq \varepsilon, \quad \|b^i_2\|_{L_\infty(\Omega)} \leq L_2
\] (5.3)
for some \( L_1 \) and \( L_2 \). From these decompositions and the ellipticity of \( a^{ij} \), we have that
\[
\delta \int_{\Omega} |Dv|^2 \, dx \leq \int_{\Gamma_k} (a^i_1 + b^i_1)vD_i v \, dx + \int_{\Gamma_k} (a^i_2 + b^i_2)vD_i v \, dx.
\] (5.4)
Using Hölder’s inequality,
\[
\int_{\Gamma_k} (a^i_1 + b^i_1)vD_i v \, dx \leq \|a^i_1\|_{L_q(\Omega)} \|v\|_{L_{\frac{2q}{q-2}}(\Omega)} \|Dv\|_{L_2(\Omega)}
\]
and so
\[
\int_{\Gamma_k} (a^i_1 + b^i_1)vD_i v \, dx \leq 2\varepsilon C_* \int_{\Omega} |Dv|^2 \, dx,
\] (5.5)
by (5.1), (5.2), and (5.3). Young’s inequality with \( \gamma > 0 \) implies that
\[
\int_{\Gamma_k} (a^i_2 + b^i_2)vD_i v \, dx \leq (L_1 + L_2) \int_{\Gamma_k} \left\{ \frac{1}{4\gamma} |v|^2 + \gamma |Dv|^2 \right\} \, dx.
\] (5.6)
Combining (5.4), (5.5), and (5.6) yields
\[
\delta \int_{\Omega} |Dv|^2 \, dx \leq (2\varepsilon C_* + (L_1 + L_2)\gamma) \int_{\Omega} |Dv|^2 \, dx + \frac{L_1 + L_2}{4\gamma} \int_{\Gamma_k} |v|^2 \, dx
\]
for all \( \gamma > 0 \). Now, we take \( \varepsilon \) and \( \gamma \) such that
\[
0 < \varepsilon < \frac{\delta}{8C_*} \quad \text{and} \quad 0 < \gamma < \frac{\delta}{4(L_1 + L_2)}.
\]
Thus, we obtain that
\[
\int_{\Omega} |Dv|^2 \, dx \leq C \int_{\Gamma_k} |v|^2 \, dx.
\] (5.7)
Combining (5.1) and (5.7) yields
\[
\|v\|_{L_{\frac{2q}{q-2}}(\Omega)} \leq C \|v\|_{L_2(\Gamma_k)} \leq C |\Gamma_k|^{\frac{1}{q}} \|v\|_{L_{\frac{2q}{q-2}}(\Gamma_k)}.
\]
Hence we have that
\[
\|v\|_{L_{\frac{2q}{q-2}}(\Omega)} \leq C |\Gamma_k|^{\frac{1}{q}} \|v\|_{L_{\frac{2q}{q-2}}(\Omega)}.
\]
It follows that \( |\Gamma_k| \) is bounded below by a positive constant, independent of \( k \), so is \( \{u > k\} \) because \( \Gamma_k \subset \{u > k\} \). This shows that the supremum of \( u \) on \( \Omega \) is finite. Now we see that \( \Gamma_k \subset \Gamma_l \) whenever \( l < k \) and \( l, k \in [k_1, \sup_{\Omega} u) \). Hence
\[
\Gamma' = \bigcap_{k_1 \leq k < \sup_{\Omega} u} \Gamma_k
\]
has a positive measure. Set
\[
\Gamma^* := \bigcap_{k_1 \leq k < \sup_{\Omega} u} \{u > k\}.
\]
Since \( u = \sup_{\Omega} u \) on \( \Gamma^* \), \( \Gamma' \subset \Gamma^* \), and \( Du \neq 0 \) on \( \Gamma' \), we have that \( u = \sup_{\Omega} u \) on \( \Gamma' \) as well as \( Du \neq 0 \) on \( \Gamma' \), which is a contradiction. Therefore, \( u \) is constant. \( \square \)
6. Appendix.

6.1. Proof of (4.1). In this subsection, we drive (4.1), where as we recall \( v = (u - k)^+ \), \( \sup_{B^2 T} u^+ \leq k < \sup_{\Omega_T} u \), and \( \Gamma_k = \{ Dv \neq 0 \} \cap \Omega_T \).

To do this, we briefly review the definition of Steklov average. The Steklov average \([w]_h\) of a function \( w \) with a nonzero constant \( h \) is defined by

\[
[w]_h(x, t) = \frac{1}{h} \int_{t-h}^{t+h} w(x, s) \, ds.
\]

Note that

\[
[w]_{-h}(x, t) = \frac{1}{-h} \int_{t-h}^{t} w(x, s) \, ds = \frac{1}{h} \int_{t-h}^{t} w(x, s) \, ds.
\]

**Remark 6.1.** If \( w(x, t) \) belongs to class \( V^{1, 0}_2(\Omega_T) \), then the averagings \([w]_h(x, t)\) for \( h \leq \delta \) belong to class \( W^{2, 1}_2(\Omega_T - \delta) \), with \( \| [w]_h - w \|_{V^{2, 0}_2(\Omega_T - \delta)} \to 0 \) as \( h \to 0 \).

(See [18, Lemma 4.7, page 85].)

**Proof of (4.1).** We take as \( \phi(x, t) \) in (2.3) the function \([v]_{-h}(x, t)\), where \( v(x, t) \) is an arbitrary element of \( W^{2, 1}_2(\Omega \times (-h, T)) \) that is equal to zero for \( t \leq 0 \) and \( t \geq t_1 \), where \( t_1 \in (0, T - h) \), and we transform the first term in it as follows:

\[
- \int_{\Omega_T} u ([v]_{-h})_t \, dx \, dt = - \int_{\Omega_T} u [v]_{-h} \, dx \, dt
\]

\[
= - \int_{\Omega_1} [u]_h \, v_t \, dx \, dt = \int_{\Omega_1} ([u]_h)_t \, v \, dx \, dt.
\]

(See [19, Section 6.1] or [18, Chapter III-§2] for more details.) In all the other terms of (2.3) we also transfer the averaging \([\cdot]_{-h}\) from \( v \) to their coefficients, taking into account the permutability of this averaging with differentiation with respect to \( x \).

This gives that

\[
\int_{\Omega_1} \left\{ ([u]_h)_t v + [a^{ij} D_i u + a^i u]_h D_i v \right\} \, dx \, dt
\]

\[
\leq \int_{\Omega_1} [b^i D_i u + c u]_h v \, dx \, dt. \tag{6.1}
\]

Using standard cut-off functions with respect to \( t \), we see that this inequality is actually valid for any \( v(x, t) \in W^{1, 1}_2(\Omega \times (0, t_1)) \), where \( t_1 \in [0, T - h] \). (See [18, page 142] for more details.)

Now we take a test function of (6.1) as follows:

\[
\eta(x, t) = \max \{ [u]_h(x, t) - k, 0 \}, \quad k \geq \sup_{B^2 T} u^+.
\]

This is possible, since the function \([u]_h\) and consequently the function \([u]_h^k\) belong to \( W^{2, 1}_2(\Omega_{T-h}) \). (See [18, Lemma 4.9, page 87].) Since, for each \( t \in [0, t_1] \), where \( t_1 \in [0, T - h] \),

\[
\int_{\Omega} ([u]_h^k)_t(x, t) \, ([u]_h^k)(x, t) \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} ([u]_h^k(x, t))^2 \, dx,
\]
we have that
\[
\frac{1}{2} \int_\Omega (|u_h^{(k)}(x,t)|^2) \, dx \bigg|_{t=t_1}^{t=t_2} + \int_{\Omega} \left[ a^{ij} D_j u + a^i u \right] D_i |u_h^{(k)}| \, dx \, dt \leq \int_{\Omega} \left[ b^i D_i u + c u \right] |u_h^{(k)}| \, dx \, dt. \tag{6.2}
\]

We let \( h \) tend to zero in (6.2). Since \( u \in V_2^{1,0}(\Omega_T) \), it follows that \([u]_h\) converges to \( u \) in the norm of \( V_2^{1,0}(\Omega_T-\delta) \), \( \delta > 0 \), and therefore the “undercut” functions \([u]_h^{(k)}\) converge to \( u^{(k)} \) in the same norm. (See Remark 6.1 and [18, Lemma 4.5, page 83].)

Thus, we have that
\[
\frac{1}{2} \int_\Omega (u^{(k)}(x,t))^2 \, dx \bigg|_{t=t_1}^{t=t_2} + \int_{\Omega} \left( a^{ij} D_j u + a^i u \right) D_i u^{(k)} \, dx \, dt \leq \int_{\Omega} (b^i D_i u + c u) u^{(k)} \, dx \, dt,
\]

where
\[
u^{(k)}(x,t) := \max\{u(x,t) - k, 0\} = (u - k)^+(x,t) =: v(x,t).
\]

Since \( k \geq \sup_{\partial \Omega_T} u^+ \), it follows that \( \int_{\Omega} (u^{(k)}(x,0))^2 \, dx = 0 \) and consequently we obtain that
\[
\frac{1}{2} \int_\Omega \left[ u^{(k)}(x,t_1) \right]^2 \, dx + \int_{\Omega} \left[ a^{ij} D_j u D_i u^{(k)} - (a^i + b^i) u^{(k)} D_i u \right] \, dx \, dt \leq \int_{\Omega} \left[ -a^i D_i (uu^{(k)}) + c(uu^{(k)}) \right] \, dx \, dt.
\]

Note that \( u, u^{(k)} \in V_2^{1,0}(\Omega_T) \) and \( uu^{(k)} \geq 0 \). Thus, the sign condition (2.2) shows that
\[
\int_{\Omega} \left[ -a^i D_i (uu^{(k)}) + c(uu^{(k)}) \right] \, dx \, dt \leq 0.
\]

It then follows that
\[
\frac{1}{2} \int_\Omega \left[ u^{(k)}(x,t_1) \right]^2 \, dx + \int_{\Omega} a^{ij} D_j u D_i u^{(k)} \, dx \, dt \leq \int_{\Omega} (a^i + b^i) u^{(k)} D_i u \, dx \, dt. \tag{6.3}
\]

Note that \( \Gamma_k := \{(x,t) \in \Omega_T : Dv \neq 0\} \subset \{u > k\}, Du = Dv \text{ a.e. in } \Gamma_k, \text{ and } Dv = 0 \text{ a.e. in } \Omega_T \setminus \Gamma_k \). Hence, by the ellipticity
\[
\int_{\Omega} a^{ij} D_j u D_i u^{(k)} \, dx \, dt = \int_{\Omega \cap \Gamma_k} a^{ij} D_j u D_i v \, dx \, dt \geq \delta \int_{\Omega} |Dv|^2 \, dx \, dt = \delta \int_{\Omega} |Du|^2 \, dx \, dt.
\]
The right-hand side of (6.3) is estimated as follows:
\[
\iint_{\Omega_{1}} (a^{i} + b^{i}) D_{i} u \, dx \, dt = \iint_{\Omega_{1} \cap \{ u > k \}} (a^{i} + b^{i}) v D_{i} v \, dx \, dt
\]
\[
= \iint_{\Omega_{1} \cap \Gamma_{k}} (a^{i} + b^{i}) v D_{i} v \, dx \, dt
\]
\[
\leq \iint_{\Gamma_{k}} (a^{i} + b^{i}) v D_{i} v \, dx \, dt.
\]
Finally, we have that
\[
\frac{1}{2} \int_{\Omega} [v(x, t_{1})]^{2} \, dx + \delta \iint_{\Omega_{1}} |Dv|^{2} \, dx \, dt \leq \iint_{\Gamma_{k}} (a^{i} + b^{i}) v D_{i} v \, dx \, dt
\]
for all \( t_{1} \) in \((0, T)\) and thus
\[
\frac{1}{2} \max_{0 \leq t \leq T} \int_{\Omega} v^{2} \, dx + \delta \iint_{\Omega_{T}} |Dv|^{2} \, dx \, dt \leq \iint_{\Gamma_{k}} (a^{i} + b^{i}) v D_{i} v \, dx \, dt,
\]
(6.4)
because \( v \in V_{2}^{1,0}(\Omega_{T}) \). This completes the proof.

6.2. Decomposition of \( L_{q,r}(\Omega_{T})\)-functions.

Lemma 6.2. Let \( 1 \leq q, r < \infty \) and let \( v(x, t) \in L_{q,r}(\Omega_{T}) \). Then for any \( \varepsilon > 0 \) there exist \( v_{1} \in L_{q,r}(\Omega_{T}) \) and \( v_{2} \in L_{\infty}(\Omega_{T}) \) such that
\[
v = v_{1} + v_{2}, \quad \|v_{1}\|_{L_{q,r}(\Omega_{T})} \leq \varepsilon, \quad \text{and} \quad \|v_{2}\|_{L_{\infty}(\Omega_{T})} \leq \Lambda
\]
for some \( \Lambda > 0 \). We also have such a decomposition if \( q = \infty \) and \( 1 \leq r < \infty \).

Proof. For \( 1 \leq q, r < \infty \), set
\[
v_{k}(x, t) := \max\{-k, \min\{v(x, t), k\}\}.
\]
By the Lebesgue dominated convergence theorem
\[
\|v - v_{k}\|_{L_{q,r}(\Omega_{T})} \to 0 \quad \text{as} \quad k \to \infty.
\]
Then we take
\[
v_{1} = v - v_{k} \quad \text{and} \quad v_{2} = v_{k}
\]
for a sufficiently large \( k \).

When \( q = \infty \) and \( 1 \leq r < \infty \), we set \( \tilde{v}(t) = \|v(\cdot, t)\|_{L_{\infty}(\Omega)} \) and define
\[
v_{1}(x, t) = \begin{cases} 
0, & (x, t) \in \Omega \times \{ t \in (0, T) : \tilde{v}(t) \leq k \}, \\
v(x, t), & (x, t) \in \Omega \times \{ t \in (0, T) : \tilde{v}(t) > k \},
\end{cases}
\]
and \( v_{2}(x, t) = v(x, t) - v_{1}(x, t) \). Since \( \tilde{v}(t) \in L_{r}(0, T) \), it follows that
\[
\int_{0}^{T} \|v_{1}(\cdot, t)\|_{L_{\infty}(\Omega)}^{r} \, dt = \int_{\{t \in (0, T) : \tilde{v}(t) > k\}} \|v(\cdot, t)\|_{L_{\infty}(\Omega)}^{r} \, dt
\]
\[
= \int_{\{t \in (0, T) : \tilde{v}(t) > k\}} |\tilde{v}(t)|^{r} \, dt \to 0
\]
as \( k \to \infty \). For each \( t \in (0, T) \), we see that
\[
\|v_{2}(\cdot, t)\|_{L_{\infty}(\Omega)} = \begin{cases} 
0, & t \in \{ t \in (0, T) : \tilde{v}(t) > k \}, \\
\tilde{v}(t), & t \in \{ t \in (0, T) : \tilde{v}(t) \leq k \}.
\end{cases}
\]
Thus, \( \|v_{2}\|_{L_{\infty}(\Omega_{T})} \leq k \). To finish the proof, we take a sufficiently large \( k \).
Remark 6.3. The same conclusion does not hold true for $L_{q,\infty}(\Omega_T)$ with $1 \leq q < \infty$. Indeed, set $\Omega = (0,1) \subset \mathbb{R}$ and

$$v(x,t) = t^{-1/q} \chi_{(0,t)}(x)$$

in $(x,t) \in \Omega \times (0,1)$. Note that $v$ is unbounded, but

$$\|v(t, \cdot)\|_{L_q(Q)}^q = \int_0^1 t^{-1} \chi_{(0,t)}(x) \, dx = \int_0^t t^{-1} \, dx = 1$$

for all $t \in (0,1)$. Hence,

$$\|v\|_{L_{q,\infty}(\Omega \times (0,1))} < \infty.$$

Suppose that, for a given $\varepsilon > 0$, there is a bounded function $v_2$ such that $|v_2| \leq K$, $K > 1$, and

$$\|v - v_2\|_{L_{q,\infty}(\Omega \times (0,1))} \leq \varepsilon.$$

This means that, for a.e. $t \in (0,1)$, we have

$$\int_0^1 |v(x,t) - v_2(x,t)|^q \, dx \leq \varepsilon^q.$$

However, for $t \in (0, 1/K^q)$,

$$\int_0^1 |v(x,t) - v_2(x,t)|^q \, dx \geq \int_0^t |t^{-1/q} \chi_{(0,t)}(x) - v_2(t,x)|^q \, dx \geq \int_0^t (t^{-1/q} - K)^q \, dx = (t^{-1/q} - K)^q t,$$

which is bigger than $\varepsilon^q > 0$ if $t$ is sufficiently small.

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