I. INTRODUCTION

In a relativistic quantum theory, amplitudes can be calculated using Feynman rules derived from a Lagrangian. The alternative is to directly construct on-shell amplitudes using the basic principles of Poincaré symmetry, locality, and unitarity, without using the crutches of fields and Lagrangians. This approach is based on the fact that, on kinematic poles, residues of an N-point amplitude factorize into products of lower-point amplitudes. For example, a tree-level amplitude with 4 massless particles on the external legs can be represented as

$$A(1234) = \frac{A(12\tilde{p}_s)A(34p_s)}{s} - \frac{A(13\tilde{p}_t)A(24p_t)}{t} + \frac{A(14\tilde{p}_u)A(23p_u)}{u},$$

where 1...4 label incoming particles, $$p_s = p_1 + p_2$$, $$p_t = p_3 + p_4$$, $$p_u = p_1 + p_4$$, the Mandelstam invariants are $$x \equiv p_x^2$$ for $$x = s, t, u$$, and the hats denote outgoing particles. The last piece stands for contact terms, which are regular functions of $$s, t, u$$ without poles or other singularities, therefore they are not connected to lower-point amplitudes by unitarity. All in all, an on-shell 4-point amplitude can be bootstrapped from 3-point ones, up to contact terms. The latter are the focus of this paper.

One can fix the contact terms through physically motivated assumptions about the high-energy or analytic behavior of the amplitudes. This path is relevant for certain important theories, such as QCD or general relativity (GR), which are completely fixed by their 3-point amplitudes and do not admit any free parameters entering at $$N > 3$$. The EFT framework takes exactly the opposite path. There, the contact terms are assumed to have the most general form allowed by the symmetries, and are organized as an expansion in inverse powers of a high mass scale $$\Lambda$$:

$$C(1234) = \sum_{D=4}^{\infty} \sum_k \frac{c_{D,k} O_{D,k}(1234)}{\Lambda^{D-4}},$$

where $$O_{D,k}$$ are the basis elements spanning the space of all possible Lorentz-invariant contact terms. Each $$O_{D,k}$$ is a regular function of Mandelstam variables with mass dimension [mass]^{D-4}, while $$c_{D,k}$$ are free numerical parameters called the Wilson coefficients. Formally, contact terms depend on an infinite number of Wilson coefficients. However, for processes with a characteristic energy scale $$E \ll \Lambda$$ only $$O_{D,k}$$ with low enough $$D$$ are numerically important, and the amplitude can be well approximated using a finite number of $$c_{D,k}$$ and $$O_{D,k}$$. The number of basis elements at a given order in the EFT expansion determines the number of free parameters entering at that order.

There remains the highly non-trivial issue of writing down all possible $$O_{D,k}$$ given the quantum numbers (in particular spin and helicity) of the external particles. In the Lagrangian language, the parallel problem is constructing all independent Lorentz-invariant local operators of canonical dimension $$D$$ from the fields corresponding to the external particles. That task can be systematically organized thanks to the Hilbert series techniques. Alternatively, for massless theories one can bypass Lagrangians and work at the amplitude level using the spinor helicity variables (helicity spinors, in short). This method may be simpler, especially for higher spins, given that the spinors have simple transformation properties under the little group, and therefore they encode helicity information in a transparent way. Recently, Ref. [9] proposed an algorithm for constructing the basis $$O_{D,k}$$ as harmonic modes of the physical manifold of helicity spinors parametrizing $$N$$-point amplitudes.

In this paper I propose another way of constructing a basis of contact terms in massless EFTs. The departure point is the parameterization of N-point amplitudes using the momentum twistor variables. Namely, the kinematic data can be encoded in N spinor pairs $$Z_i = (\lambda_i, \tilde{\mu}_i)$$. From these, the four-momenta $$p_i$$ of all external particles can be reconstructed, and they automatically satisfy the on-shell condition ($$p_i^2 = 0$$) and momentum conservation ($$\sum_i p_i = 0$$). The basic facts about momentum twistors are summarized in Section III. These are very convenient and natural variables on the space of N-body kinematics. In particular, they greatly simplify...
the task of constructing independent Lorentz invariants from the kinematic data. Section III shows how to generate all independent rational functions of Lorentz invariant \((\lambda_i\lambda_j)\) and \((\bar{\mu}_i\bar{\mu}_j)\) spinor contractions at a given order in the EFT expansion and with a given little group transformation properties. Given that set, only a finite subset has a regular behavior as a function of \(s,t,u\) to make a valid contact term. In this paper I focus on 4-point amplitudes, however the method can be generalized to higher \(N\) in a straightforward way. For \(N = 4\), the candidate basis elements are characterized by a compact formula in Eq. (13) and labeled by one integer bounded to a finite range, thus they can be enumerated by order in order in the EFT expansion. Section IV illustrates this method with a number of simple examples rooted in physically relevant theories. A slightly more involved example of constructing contact terms and scattering amplitudes in the EFT extension of GR is relegated to Appendix A.

I work in 4 spacetime dimensions with the mostly minus metric: \(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). The Lorentz algebra decomposes into \(SU(2) \times SU(2)\), with holomorphic and anti-holomorphic spinors \(\lambda_\alpha\) and \(\tilde{\lambda}_\lambda\) transforming under the respective \(SU(2)\) factors. For the spinors I adopt the conventions of Ref. [9]. Spinor indices are raised with the antisymmetric tensor \(\epsilon^{\alpha\beta}\) and lowered with \(\epsilon_{\alpha\beta}\): \(\lambda^{\alpha} = \epsilon^{\alpha\beta}\lambda_\beta\), \(\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta\), and idem for \(\tilde{\lambda}\), with the convention \(\epsilon^{12} = -\epsilon_{12} = 1\). The Lorentz invariant spinor contractions are \(\lambda_\alpha\lambda_\beta\equiv (\lambda_\lambda) \equiv (ij)\), \(\lambda_\alpha\tilde{\lambda}_\beta\equiv (\tilde{\lambda}_\lambda) \equiv (ij)\). Vector and spinor Lorentz indices can be traded with the help of the sigma matrices \(\sigma^{\alpha\beta}\).

II. FLASH REVIEW OF MOMENTUM TWISTORS

I start by reviewing the momentum twistor description of massless \(N\)-body kinematics in four spacetime dimensions, following closely the presentation in Ref. [9]. Consider a set of four-momenta \(p_i\), \(i = 1 \cdots N\), subject to the on-shell conditions \(p_i^2 = 0\) and the momentum conservation \(\sum_{i=1}^N p_i = 0\). The restrictions on \(p_i\) may be inconvenient to work with, and for many applications it is beneficial to introduce different variables that trivialize the constraints. The on-shell conditions are dealt with by introducing the helicity spinors, that is \(N\) holomorphic and anti-holomorphic \(2\)-component spinors \(\lambda_i\), \(\tilde{\lambda}_i\) related to the four-momenta by \(p_i\cdot\sigma = \lambda_i\tilde{\lambda}_i\). This trivializes the on-shell constraints, in the sense that an arbitrary pair \((\lambda_i, \tilde{\lambda}_i)\) defines a \(N\)-body kinematics with \(p_i^2 = 0\). As a bonus, the transformation \(\lambda_i \to t_i^{-1}\lambda_i\), \(\tilde{\lambda}_i \to t_i\tilde{\lambda}_i\) does not change \(p_i\), therefore it represents the little group action on the particle \(i\). However, momentum conservation is not automatic in these variables, and implies one non-trivial constraint on the \(N\) spinors \(\lambda_i\) and \(\tilde{\lambda}_i\). The idea behind the momentum twistors is to trade the helicity spinors into a different set of variables so as to trivialize the momentum conservation as well. To this end, one defines the dual coordinates \(y_i\) via the relation

\[
p_i = y_i - y_{i-1} \Rightarrow y_i = y_N + \sum_{j=1}^i p_j,
\]

with the cyclic identification \(y_0 = y_N\). One can think of \(y_i\) as vertices of a polygon in the dual coordinate space, and of \(p_i\) as the sides of that polygon. Note that \(y_i\) have units of \([\text{mass}]^3\), unlike the spacetime coordinates. The four-momenta \(p_i\) do not depend on \(y_i\): they do not change when the polygon is moved around in the dual spacetime. This is just the translation invariance in the dual coordinate space. We can always gauge-fix it, e.g. by setting \(y_N = 0\).

Introducing \(y_i\) via Eq. (3) trivializes the momentum conservation. However the goal is to construct variables that trivialize both the momentum conservation and the on-shell conditions. This is achieved by introducing the set of \(N\) anti-holomorphic spinors \(\bar{\mu}_i\) defined as

\[
\bar{\mu}_i = \lambda_i\sigma \cdot \bar{y}_i = \lambda_i\sigma \cdot y_{i-1}.
\]

Given that \(\lambda_i\) have dimension \([\text{mass}]^{1/2}\), \(\bar{\mu}_i\) have dimension \([\text{mass}]^{3/2}\).

The spinor pairs \(Z_i = (\lambda_i, \bar{\mu}_i)\) are called the momentum twistors. Any set of \(N\) such pairs automatically defines an \(N\)-body kinematics with \(p_i^2 = 0\) and \(\sum_{i=1}^N p_i = 0\). In order to see this, note that Eq. (4) can be solved for the dual coordinate:

\[
y_i \cdot \sigma = \frac{\lambda_{i+1}\bar{\mu}_i - \lambda_i\bar{\mu}_{i+1}}{(\lambda_i\lambda_{i+1})}.
\]

Therefore, starting from \(Z_i\) one can reconstruct all \(y_i\), and thus \(p_i\) via Eq. (3). Furthermore, one can show that these four-momenta can be decomposed as \(p_i \cdot \sigma = \lambda_i\tilde{\lambda}_i\), where

\[
\tilde{\lambda}_i = -\bar{\mu}_{i-1}(\lambda_i\lambda_{i+1}) + \bar{\mu}_i(\lambda_i\lambda_{i+1}) + \bar{\mu}_{i+1}(\lambda_{i+1}\lambda_{i-1})
\]

From Eq. (5) and Eq. (6) it follows that \(\lambda_i\) and \(\bar{\mu}_i\) transform in the same way under the little group: \(Z_i \to t_i^{-1}Z_i\). In other words, \(Z_i\) are defined projectively: an independent rescaling of each \(Z_i\) does not change \(p_i\). In the literature, momentum twistors are most often used in superconformal frameworks (see e.g. Ref. [10]), in which case \(Z_i\) are unbreakable building blocks of the amplitudes. In this paper however I am interested in more down-to-earth theories, and I will treat \(\lambda_i\) and \(\bar{\mu}_i\) as separate building blocks.

Let us compare the number of degrees of freedom on the four-momentum and momentum twistor sides. In complex kinematics, each \(p_i\) has 6 real degrees of freedom after imposing the on-shell condition. Momentum conservation fixes 8 degrees of freedom, thus \(N\)-body kinematics is defined by \(6N - 8\) free parameters. On the twistor
side, fixing the translation invariance by setting \( y_N = 0 \) corresponds via Eq. \( \text{[4]} \) to setting \( \mu_1 = \mu_N = 0 \). We thus have \( N \) spinors \( \lambda_i \) and \( N-2 \) spinors \( \hat{\mu}_i \). Each spinor has 4 real degrees of freedom, however 2 degrees of freedom in each \( \lambda_i \) can be removed by little group transformations \( Z_i \to t_i^{-1} Z_i \). This leaves \( 2N + 4(N-2) = 6N - 8 \) degrees of freedom, in agreement with the number on the momentum side. This shows that the projective space spanned by \( N \) momentum twistors \( Z_i \) is in 1-to-1 correspondence (after gauge-fixing the translations) with the space of all independent massless \( N \)-body kinematics.

Instead of \( Z_i = (\lambda_i, \hat{\mu}_i) \), one could represent the kinematic data by momentum anti-twistors \( \hat{Z}_i \equiv (\mu_i, \lambda_i) \) related to the dual coproducts by \( \mu_i = y_i \cdot \sigma \lambda_i = y_{i-1} \cdot \sigma \lambda_i \). In the spinor helicity variables, parity exchanges \( \lambda_i \leftrightarrow \hat{\lambda}_i \). The binary choice of either \( Z_i \) or \( \hat{Z}_i \) to represent the kinematic data is not parity invariant, therefore parity is not manifest in momentum twistor variables.

### III. CONTACT TERMS VIA MOMENTUM TWISTORS

Contact terms are functions of the helicity spinors that can be written in the form

\[
O = \Pi_{i<j} (\lambda_i \lambda_j)^{n_{ij}} \Pi_{k<l} (\hat{\lambda}_k \hat{\lambda}_l)^{\hat{n}_{ij}}, \quad n_{ij} \geq 0, \hat{n}_{ij} \geq 0, \tag{7}
\]

that is without any spinor contractions in the denominator. In this section I lay out an algorithm to construct a basis of contact terms for \( N \)-point amplitudes. To this end, I parametrize the \( N \)-body kinematics by momentum twistors \( (\lambda_i, \hat{\mu}_i) \) subject to the gauge fixing condition \( \mu_1 = \hat{\mu}_N = 0 \). From these building blocks one can construct a set of rational functions of the holomorphic \( (\lambda_i \lambda_j) \) and anti-holomorphic \( (\hat{\mu}_i \hat{\mu}_j) \) spinor contractions. These functions can be classified according to their little group transformation properties, corresponding to different helicity configurations of the amplitude. For a given configuration, a basis of \( N \)-point contact terms consists of the rational functions that can be cast in the form of Eq. \( \text{[7]} \). Of course, such a basis is infinite, however its elements can be organized according to their mass dimensions, which directly translates into the EFT expansion in powers of \( 1/\Lambda \). The point is that, at a given EFT order, the number of basis elements is finite and can be described by a compact algebraic formula depending on discrete parameters.

Below I perform this construction for \( N = 4 \), and then comment on extending it to higher \( N \).

#### A. 4-point

For \( N = 4 \) the building blocks are 4 holomorphic spinors \( \lambda_{1,2,3,4} \) and 2 anti-holomorphic spinors \( \hat{\mu}_{2,3} \). Thus, there is only one possible anti-holomorphic contraction \( (\hat{\mu}_2 \hat{\mu}_3) \), which greatly simplifies the discussion.

The most general Lorentz-invariant rational function of the momentum twistors is a linear combination of

\[
(\hat{\mu}_2 \hat{\mu}_3)^n \langle 12 \rangle^k \langle 13 \rangle^{n_{13}} \langle 23 \rangle^{n_{23}} - n \langle 24 \rangle^{n_{24}} \langle 34 \rangle^{n_{34}}, \tag{8}
\]

where \( n, k \), and \( n_{ij} \) are integers. The spinor contraction \( \langle 14 \rangle \) does not appear above because it has been eliminated via the Schouten identity \( \langle 12 \rangle \langle 34 \rangle - \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle = 0 \). For helicities \( h_{1,2,3,4} \) of the external particles, one can solve for the integers \( n_{ij} \) so as to arrive at the correct little group transformations:

\[
O_{n,k}^{h_1 h_2 h_3 h_4} = \left( (\hat{\mu}_2 \hat{\mu}_3)^n \right)^k \left( \langle 12 \rangle^{\langle 34 \rangle} \langle 23 \rangle^{\langle 13 \rangle} \langle 14 \rangle \right)^{h_1-h_2-h_3+h_4} \times \langle 13 \rangle^{2h_1} \langle 24 \rangle^{h_1+h_2-h_3-h_4} \langle 34 \rangle^{h_1+h_2-h_3-h_4}.
\tag{9}

For a given helicity configuration, the candidate basis elements are parametrized by two integers: \( n \) and \( k \). The former controls the canonical dimension, which is related to \( n \) by

\[
D = 2n + 4 - \sum_i h_i. \tag{10}
\]

The latter counts the number of basis elements for each canonical dimension, that is at each EFT order. The important point is that, for \( n \) fixed, \( k \) is constrained to a finite range for \( O_{n,k}^{h_1 h_2 h_3 h_4} \) to possibly be a contact term. This can be seen by looking at the scaling: \( (\hat{\mu}_2 \hat{\mu}_3) \sim s \sqrt{u} \), \( \langle 23 \rangle \sim \sqrt{u} \), \( \langle 12 \rangle \sim \sqrt{s} \), \( \langle 34 \rangle \sim \sqrt{s} \), \( \langle 13 \rangle \sim \sqrt{t} \), \( \langle 24 \rangle \sim \sqrt{t} \). A contact term must be non-singular when \( s \) or \( t \) go to zero, which leads to the necessary condition:

\[
- n - \frac{h_1 + h_2 - h_3 - h_4}{2} \leq k \leq - \frac{3h_1 + h_2 - h_3 + h_4}{2}.
\tag{11}
\]

It selects a finite (or empty) set of \( n - h_1 - h_4 + 1 \) possible choices of \( k \). I stress that this is only a necessary condition. The sufficient condition for \( O_{n,k}^{h_1 h_2 h_3 h_4} \) to be a local contact term is that it can be written in the form of Eq. \( \text{[9]} \). That is often more restrictive than the necessary condition, thus the allowed range of \( k \) can be smaller (but never larger) than suggested by Eq. \( \text{[11]} \). For the allowed range of \( k \) to be non-empty, \( n \) is bounded by

\[
n \geq h_1 + h_4. \tag{12}
\]

In the EFT approach one is usually interested in lowest dimension terms of the \( 1/\Lambda \) expansion. Then it suffices to inspect \( O_{n,k}^{h_1 h_2 h_3 h_4} \) for a few values of \( n \) close to \( h_1 + h_4 \) and for the corresponding range of \( k \).

The formulas in Eqs. \( \text{[9]} \), \( \text{[12]} \) do not treat all incoming particles in the same way, e.g. \( h_1 \) enters in Eq. \( \text{[11]} \) with a different coefficient than the other helicities. This is because arbitrary choices that break the interchange symmetry have been made along the way: gauge-fixing \( \mu_1 \) and \( \mu_4 \), and eliminating \( \langle 14 \rangle \) via the Schouten identity. One could of course alter these choices to arrive...
at a different, equivalent representation of the basis candidates $O_{n,k}^{h_1h_2h_3h_4}$. Note that Eq. (9) is not manifestly parity-invariant because the momentum twistor formalism is not. Furthermore, the scaling $(23) \sim \sqrt{u}$ implies that Eq. (8) has a singularity in the $u$-variable for $h_1 - h_2 - h_3 + h_4 < 0$. For such helicity configurations a local term may be a linear combination of $O_{n,k}^{h_1h_2h_3h_4}$ with different $k$. This may be cumbersome in practice, therefore it is easier to work with the configurations satisfying $h_1 - h_2 - h_3 + h_4 \geq 0$, and obtain the basis elements for $h_1 - h_2 - h_3 + h_4 < 0$ via the parity operation $P$ acting as $h_i \to -h_i$, $\lambda_i \to \lambda_i$. The final comment is that Eq. (9) is not automatically symmetric under permutations of external particles $i, j$, even when $h_i = h_j$. For identical particles, symmetrization or anti-symmetrization of the basis elements has to be performed in addition.

To summarize the algorithm, the candidate basis elements span the contact terms of massless 4-point amplitudes are given in Eq. (3). They are parametrized by two integers: $n$ and $k$. The former is constrained by the inequality in Eq. (12), and controls the EFT expansion, with increasing $n$ corresponding to increasing canonical dimensions. For a fixed $n$, the latter is constrained to a finite range by Eq. (11). The fact that momentum twistors are unconstrained variables on the manifold of $N$-body kinematics ensures the independence of $O_{n,k}^{h_1h_2h_3h_4}$ for different $n$ and $k$. The elements of this set that are local, meaning they can be written as in Eq. (7), form a basis of contact terms for a given helicity configuration $h_1, h_2, h_3, h_4$ and canonical dimension $D = 2n - \sum h_i + 4$. This prescription is a purely algebraic algorithm to write down a basis of 4-point contact terms in any massless theory.

For practical applications it is more convenient to trade momentum twistors for the standard helicity spinors using the identity $(\mu_2 \mu_3) = -s(23) = (23)^2(23)$, which follows directly from Eq. (8). Furthermore, one can simplify spinor expressions using $(13)(34) = -s(13)(24)$. Dropping the irrelevant $(-1)^{n+k}$ factor, Eq. (9) can be recast as

$$O_{n,k}^{h_1h_2h_3h_4} = s^{n+k}k^{n-k}(13)^{2h_1}h_2^{h_2}h_3^{h_3}h_4^{h_4} \times (24)^{-h_1-h_2+h_3-h_4}(34)^{h_1+h_2-h_3-h_4}. \tag{13}$$

This is the central result of this paper.

B. Higher point

A similar algorithm as in Section III A can be worked out for $N$-point functions with $N > 4$. In such a case the building blocks are $N$ holomorphic spinors $\lambda_1 \ldots \lambda_N$ and $N - 2$ anti-holomorphic spinors $\mu_2 \ldots \mu_{N-1}$. From those one can construct the general rational function of Lorentz-invariant spinor contractions

$$\Pi_{k < l = 2}(\mu_k \mu_l)\tilde{n}_{kl} = \Pi_{k < l = 1}(\lambda_k \lambda_l)\tilde{n}_{kl}. \tag{14}$$

Subsequently, one should eliminate the dependent contractions using the Schouten identities, and relate the exponents $n_{ij}$, $\tilde{n}_{ij}$ to the external helicities. Finally, among these candidates one identifies the contact terms that can be written in the form of Eq. (4). The procedure is relatively straightforward for small enough $N$, however selection of independent Schouten identities, testing for locality, and eventual symmetrization for identical particles becomes more pesky with increasing $N$. I leave for future publications the details of this procedure and results for concrete physical theories.

IV. EXAMPLES

This section provides some applications of the master formula Eq. (13) to construct bases of 4-point contact terms in concrete physical theories.

A. Scalar

I start with a trivial example of scattering of 4 distinct scalars, $h_i = 0$. Eq. (13) reduces to

$$O_{n,k}^{s} = s^{n+k}k^{n-k}, \tag{15}$$

where $n \geq 0$ from Eq. (12), and $k$ is constrained in the range $-n \leq k \leq 0$ following Eq. (11). In this case $O_{n,k}$ simply generates all independent kinematic invariants at a given order in the EFT expansion. For $n = 0$ ($D=4$, or $O(\Lambda^0)$) the only option is $k = 0$, that is a constant contact term $O_{0,0} = 1$. For $n = 1$ ($D=6$ or $O(\Lambda^{-2})$) we have 2 options: $k = -1$, and $k = 0$, which correspond to the 2 independent Mandelstam invariants $O_{1,0} = s$ and $O_{1,-1} = t$. For $n = 2$ ($O(\Lambda^{-4})$) there are 3 independent invariants $O_{2,0} = s^2$, $O_{2,-1} = st$, and $O_{2,-2} = t^2$. And so on... For 4 identical scalars one needs to construct linear combinations of the basis elements that are invariant under the $S_4$ permutation symmetry. $O_{0,0}$ is trivially invariant. For $n = 1$ no invariant combination exists. For $n = 2$ the unique permutation-invariant combination is $O_{2,0} + O_{2,-1} + O_{2,-2} = (s^2 + t^2 + u^2)/2$. And so on...

B. Scalar and Spin-h

A less trivial example is determination of 4-point contact terms of 2 scalars and 2 identical spin-h particles at the lowest order in the EFT expansion. We take $h_1, h_4 = 0$ and $h_{2,3} = \pm h$, in which case Eq. (12) implies $n \geq 0$. Increasing $n$ corresponds to increasing canonical dimension, thus we are interested in contact terms with the lowest possible $n$. For the both-minus helicity configuration we try Eq. (13) with $n = 0$ and $k = 0$:

$$O^{--} = O_{0,0}^{--} = (23)^{2h}. \tag{16}$$

This is manifestly local, therefore the lowest order contact term in this helicity sector is unique and is $O(\Lambda^{-2h})$. 
The lowest order contact term in the both-plus sector can be immediately obtained via the parity transformation:

\[ O^{++} \equiv P \cdot O^{-+} = [23]^{2h}. \] (17)

Obtaining the same result directly from Eq. (13) is more tricky, as \( O^{++} \) is a linear combination \( \sum_k a_k O_{2h,k}^{++} \), with the coefficients \( a_k \) chosen such that the singularities in the \( u \) variable cancel. This illustrates the fact that the method is not manifestly parity invariant, due to working with the momentum twistor variables. Note that, for any integer or half-integer spin \( h \), \( O^{-+} \) and \( O^{++} \) automatically have the correct symmetry properties under the exchange \( n \leftrightarrow 3 \).

In the opposite helicity sector, the necessary condition Eq. (12) may suggest that the lowest order contact term corresponds to \( n = 0 \). However, upon inspection of Eq. (13) for \( h_2 = -h_3 = -h \):

\[ O_{n,k}^{2h} = (-)^k s^n \left( \frac{\langle 12 \rangle^{34}}{\langle 13 \rangle^{24}} \right)^k \langle 24 \rangle^{2h} \langle 34 \rangle^{-2h}, \] (18)

it is clear that at least \( n = 2h \) is needed to cancel the singularity in \( \langle 34 \rangle \) while keeping \( k = 0 \) to avoid singularity in \( \langle 13 \rangle \). Thus, the lowest order contact term is \( O(\Lambda^{-4h}) \):

\[ O^{-+} \equiv O_{2h,0}^{++} = \left( \frac{s^{(24)}}{\langle 34 \rangle} \right)^{2h} = (\lambda_2 p_4 \sigma \lambda_3)^{2h}. \] (19)

This is higher order in the EFT expansion than \( O^{-+} \) and \( O^{++} \) for any \( h > 0 \). The parity mirror is \( O^{-+} = P \cdot O^{--} = (\lambda_3 p_4 \sigma \lambda_2)^{2h} \).

In summary, at the leading order in the EFT expansion, the basis of contact terms in 2 scalar and 2 spin-h particles is 2-dimensional, consisting of \( O^{-+} \) and \( O^{++} \). These correspond to operators of canonical dimension \( D = 2h + 4 \). In the language of Lagrangians, the case \( h = 1/2 \) translates to dimension-5 operators \( \phi^2(\bar{c} \psi \psi + \bar{\psi} \psi) \), \( h = 1 \) to the dimension-6 operators \( \phi^2(c F_{\mu \nu} F^{\mu \nu} + c F_{\mu \nu} F^{\mu \nu}) \), and \( h = 2 \) to the dimension-8 operators \( \phi^2(c C_{\mu \nu \omega \beta}^{\alpha \beta} C^{\mu \nu \omega \beta} + \bar{c} C_{\mu \nu \omega \beta}^{\alpha \beta} C^{\mu \nu \omega \beta}) \) where \( C_{\mu \nu \omega \beta}^{\alpha \beta} \) is the Weyl tensor. The expressions in Eqs. (16)-(19) can also be used for \( h > 2 \). Of course, massless particles with spin higher than two cannot be consistently coupled to gravity, thus for strictly massless particles this only makes sense as an academic exercise in the limit \( M_{p_1} \to \infty \). Nevertheless, the structure of the contact terms in the massless limit may give some guidance for constructing a basis of contact terms for massive higher-spin particles, for which consistent theories exist as EFTs.

### C. Euler-Heisenberg

The final example is derivation of the leading order contact terms with spin-1 particles, \( h_i = \pm 1 \). I first discuss an academic theory of 4 distinguishable massless spin-1 particles, and then restrict to 4 identical particles, aka photons. The latter case corresponds to the Euler-Heisenberg Lagrangian.

Starting with all-minus helicities, the necessary condition Eq. (12) is \( n > -2 \), however local terms first arise at \( n = 0 \), corresponding to \( O(\Lambda^{-4}) \). In this case, Eq. (11) allows for 3 discrete options: \( k = 0, 1, 2 \), which yield the contact terms

\[ O_1^{-+++} = \langle 13 \rangle^2 \langle 24 \rangle^2, \]
\[ O_2^{-+++} = \frac{s^2}{t^2} \langle 13 \rangle^2 \langle 24 \rangle^2 = -\langle 12 \rangle \langle 13 \rangle \langle 24 \rangle \langle 34 \rangle, \]
\[ O_3^{-+++} = \frac{s^2}{t^2} \langle 13 \rangle^2 \langle 24 \rangle^2 = \langle 12 \rangle^2 \langle 34 \rangle^2. \] (20)

where \( O_i^{-+++} = O_{0,i-1}^{-} \). These are all manifestly local, thus \( O_i^{-+++} \) spans the basis of contact terms in the all-minus sector. Exactly the same basis would be obtained via the harmonics method of Ref. [10]. The basis in the all-plus sector is most easily obtained by the parity operation, \( O_i^{-+++} = P \cdot O_i^{--} \):

\[ O_1^{+++} = \langle 13 \rangle^2 \langle 24 \rangle^2, \quad O_2^{+++} = -\langle 12 \rangle \langle 13 \rangle \langle 24 \rangle \langle 34 \rangle, \]
\[ O_3^{+++} = \langle 12 \rangle^2 \langle 34 \rangle^2. \] (21)

Equivalently, these elements could be obtained from Eq. (13) for \( n = 4 \) and \( k = -4, -3, -2 \), respectively.

Next, consider the configuration \( h_{2,3} = -1, h_{1,4} = +1 \), which implies \( n \geq 2, -n \leq k \leq -2 \). The leading contact terms corresponds to \( n = 2, k = -2 \):

\[ O^{-+-+} = t^2 \left( \frac{\langle 23 \rangle^4}{\langle 13 \rangle^2 \langle 24 \rangle^2} \right) = \langle 23 \rangle^2 \langle 14 \rangle^2, \] (22)

and is also \( O(\Lambda^{-4}) \). For other 2-plus-2-minus configurations the leading contact terms can be trivially obtained by permutations: \( O^{-++-} = \langle 12 \rangle^2 \langle 34 \rangle^2, O^{-+-+} = \langle 13 \rangle^2 \langle 24 \rangle^2 \), etc.

For configurations with a single minus or a single plus helicity Eq. (13) does not generate any \( O(\Lambda^{-4}) \) contact terms (local terms appear at \( O(\Lambda^{-6}) \)). All in all, for amplitudes with 4 distinct spin-1 particles, all possible contact terms are spanned by the following 12 basis elements:

\[ O_{1,2,3}^{-}, O_{1,2,3}^{++}, O^{-+} + \text{permutations}. \] (23)

In the Lagrangian parlance they correspond to the basis of independent dimension-8 operators: \( F_{\mu \nu}^4 F_{\rho \sigma}^4 F_{\alpha \beta}^4 F_{\delta \gamma}^4 \)
\( F_{\mu \nu}^4 F_{\rho \sigma}^4 F_{\alpha \beta}^4 F_{\delta \gamma}^4 \)
\( F_{\mu \nu}^4 F_{\rho \sigma}^4 F_{\alpha \beta}^4 F_{\delta \gamma}^4 \)
\( F_{\mu \nu}^4 F_{\rho \sigma}^4 F_{\alpha \beta}^4 F_{\delta \gamma}^4 \)
\( 9 \) analogous one with one one or two field strengths replaced by the dual field strength: \( F \to ̂F \). One could easily continue this exercise into higher orders in the EFT expansion, simply by incrementing \( n \) in the master formula Eq. (13). Then Eq. (11) suggests that, for every helicity configuration, the number of basis elements increases by one at each consecutive EFT order.

For photons, the basis of contact terms consists of linear combinations of the elements in Eq. (23) that are
invariant under permutations of identical particles. In the all-minus and all-plus sectors these are
\[
O^- = O_1^{---} + O_2^{---} + O_3^{---},
O^+ = O_1^{+++} + O_2^{+++} + O_3^{+++}.
\]
(24)
The four-photon helicity amplitudes are given by
\[
A^{+++} = C_4 O^+ + O(\Lambda^{-6}), \quad A^{---} = C_0 O^- + O(\Lambda^{-6}),
A^{--+} = C_6 O^{--} + O(\Lambda^{-6}).
\]
At the leading order in the EFT they are characterized by 3-independent parameters: \(C_-, \ C_+, \ \text{and} \ C_0). These correspond to the 3 independent dimension-8 four-photon operators:
\[
\mathcal{L} = \frac{1}{\Lambda^4} \left[ a F_{\mu \nu}^2 F_{\alpha \beta}^2 + b F_{\mu \nu} \tilde{F}_{\mu \nu} F_{\alpha \beta} \tilde{F}_{\alpha \beta} + c F_{\mu \nu}^2 F_{\alpha \beta} \tilde{F}_{\alpha \beta} \right],
\]
where the map is \(C_- = 8(a - b + ic), C_+ = 8(a - b - ic), C_0 = 8(a + b). If parity is conserved (as in QED) then \(C_- = C_+ \), or equivalently \(c = 0\), reducing the number of parameters to two.

V. SUMMARY AND DISCUSSION

This paper brings you an algorithm for constructing a basis of contact terms of 4-point amplitudes in massless EFTs. The master formula is Eq. (13). It gives a compact expression for candidate basis elements for any helicity configuration \(h_{1,2,3,4}\) of the external particles. The candidates \(O_{n,k}\) are labeled by two integers \(n\) and \(k\). The former fixes the canonical dimension of the contact term, thus the order in the EFT expansion. At any order there is a finite number of candidates counted by the integer \(k\). All \(O_{n,k}\) in Eq. (13) are independent, but they are not guaranteed to be regular function of Mandelstam variables. Selecting from this set local terms, that is those that can be written in the form of Eq. (7) without any spinor contractions in the denominator, yields a basis of 4-point contact terms.

It is worth noting that this method directly constructs the basis elements, although it does not a priori count them. In this sense it is orthogonal to the Hilbert series method. Indeed, the two can be used together, with the Hilbert series method providing a useful guidance about the EFT order where local contact terms appear and the number of them. I verified that for all examples studied in this paper the number of local contact terms generated by Eq. (13) agrees with that predicted by the Hilbert series method.

It is straightforward to generalize this method to higher-point contact terms, but I leave the details to a future publication. Less straightforward is to imagine generalization to EFTs in different spacetime dimensions, or with massive particles. Indeed, momentum twistors are only defined for massless particles in four dimensions. New clever variables encoding the kinematics data seem necessary to perform a similar program beyond 4-dimensional massless EFTs.

The final comment is that Eq. (13) contains more information than just contact terms. In particular, it also generates all independent terms with a single pole in the Mandelstam variables. This may provide a shortcut to constructing full tree-level amplitudes (rather than just contact terms), which may be useful especially for higher-point amplitudes.

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Appendix A: GR EFT

In this appendix I consider the EFT of a massless spin-2 particle, which I call GR EFT. At the lowest order in the EFT expansion the theory is the same as the ordinary GR described by the Einstein-Hilbert Lagrangian: \(\mathcal{L}_{EH} = \frac{M_{Pl}^2}{2} \sqrt{-g} R\). The higher orders correspond to other coordinate invariant operators added to the Einstein-Hilbert Lagrangian. Much as in the rest of this paper, I work at the amplitude level, and only mention Lagrangians for the sake of reference to earlier literature. I will start by writing down the most general 3-point amplitudes for spin-2 particles. Then I will use Eq. (13) to derive the leading contact terms in the 4-point amplitudes for different helicity configurations. Finally I will derive the pole terms in the 4-point amplitude required by unitarity, and compare their contributions with that of the contact terms.

Before we begin, note that the power counting here will be somewhat different than in the previous examples. Two scales are introduced: the Planck scale \(M_{Pl} = (8\pi G)^{-1/2}\) and the EFT scale \(\Lambda\) which is unknown. The 3-graviton amplitudes are \(O(1/M_{Pl})\), the 4-graviton amplitudes are \(O(1/M_{Pl}^2)\), and so on. Additional mass scales needed to arrive at a correct dimension of the amplitude will be filled by \(\Lambda\). This way the EFT deformation of GR is more transparent, and the GR limit is recovered for \(\Lambda \to \infty\). Of course, one cannot identify \(\Lambda = M_{Pl}\) a posteriori, so as to simplify the power counting.

We start with the 3-point amplitudes. The minimal self-coupling of 3 gravitons is described by
\[
A^{---} = -\frac{\langle 12 \rangle^6}{M_{Pl} (13)^2 (23)^2}, \quad A^{++-} = -\frac{\langle 12 \rangle^6}{M_{Pl} (13)^2 (23)^2}.
\]
(11)
This corresponds to the usual 2-derivative graviton cubic interaction in the Einstein-Hilbert Lagrangian. In addition, Poincaré symmetry allows for maximally helicity
violating amplitudes:
\[ A^{---} = c_\epsilon \frac{(12)^2(23)^2(31)^2}{M^4 \Lambda^4}, \quad A^{+++} = c_\epsilon \frac{[12][23][31]^2}{M^4 \Lambda^4}. \] (A2)

They map to the cubic Weyl tensor terms in the effective Lagrangian of Ref. [11].

We move to discussing 4-point contact terms, which first appear at \( O(M^{-2} \Lambda^{-6}) \) in the EFT expansion. The derivation is similar to the spin-1 case in Section IV.C. In the all-minus sector, for distinguishable spin-2 particles we would find 5 local contact terms at the leading order:
\[ O_k^{---} = (-1)^k \langle 12 \rangle^k \langle 34 \rangle^k \langle 13 \rangle^{4-k} \langle 24 \rangle^{4-k}, \] (A3)
corresponding to \( n = 0 \) and \( k = 0 \ldots 4 \) in Eq. [13]. The Bose symmetric combination is
\[ O^- = 2O_0^{---} + 2O_4^{---} + 4O_1^{---} + 4O_3^{---} + 6O_2^{---}, \] (A4)
which can be simplified as
\[ O^- = (12)^4(34)^4 + (13)^4(24)^4 + (14)^4(23)^4. \] (A5)

The parity mirror of Eq. [A3], \( O^+ = P\cdot O^- \), corresponds to replacing \( \langle \cdot \rangle \) with \([\cdot]\). Moving to the zero total helicity sector, we have
\[ O^0 \equiv O^{-----} = (12)^4(34)^4. \] (A6)

up to \( O(\Lambda^{-8}) \) corrections. The EFT corrections interfere with GR in Eq. [A8] but not in Eq. [A9]. Therefore the leading EFT effect on graviton scattering observables (proportional to the helicity amplitudes squared) is determined by the contact term \( O^{---} \) and parametrized by a single Wilson coefficient \( d_0 \).

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