Holomorphic bundles on diagonal Hopf manifolds

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Abstract
Let $A \in GL(n, \mathbb{C})$ be a diagonal linear operator, with all eigenvalues satisfying $|\alpha_i| < 1$, and $M = (\mathbb{C}^n \setminus 0)/\langle A \rangle$ the corresponding Hopf manifold. We show that any stable holomorphic bundle on $M$ can be lifted to a $\tilde{G}_F$-equivariant coherent sheaf on $\mathbb{C}^n$, where $\tilde{G}_F \cong (\mathbb{C}^*)^l$ is a commutative Lie group acting on $\mathbb{C}^n$ and containing $A$. This is used to show that all stable bundles on $M$ are filtrable, that is, admit a filtration by a sequence $F_i$ of coherent sheaves, with all subquotients $F_i/F_{i-1}$ of rank 1.

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1 Introduction

In this paper we study the Hopf manifolds of form $M = (\mathbb{C}^n \setminus 0) / \langle A \rangle$, where $A \in GL(n, \mathbb{C})$ is a linear operator with all eigenvalues satisfying $|\alpha_i| < 1$ (such an operator is called a linear contraction). Deforming $A$ to an operator $\lambda \cdot Id$, $0 < |\lambda| < 1$, we find that $M$ is diffeomorphic to $S^{2n-1} \times (\mathbb{R} / \mathbb{Z}) \cong S^{2n-1} \times S^1$. The odd Betti numbers of $M$ are odd, hence $M$ is not Kähler. This is the first example of non-Kähler manifold known in algebraic geometry.

When $A$ is diagonal and has form $A = \tau \cdot Id$, $M$ is elliptically fibered over $\mathbb{C}P^{n-1}$, with all fibers isomorphic to an elliptic curve $C_{\tau} = \mathbb{C}^* / \langle \tau \rangle$. In this (so-called “classical”) case, the algebraic dimension is maximal possible. For arbitrary $A$, the algebraic dimension of $M$ can reach any value from 0 to $n - 1$.

Algebraic geometry of Hopf manifolds, especially Hopf surfaces, is well studied ([Ka1], [Ka2], [BM2], [BM1], [M1]). For $\dim M = 2$, one has a good understanding of the geometry of holomorphic vector bundles on $M$ ([M2]). A typical stable vector bundle in this situation is non-filtrable, and actually contains no proper holomorphic subsheaves.

For $\dim M > 2$, geometry of holomorphic vector bundles is drastically different. In [Ve2], it was shown that any bundle (and any coherent sheaf) on a classical Hopf manifold

$$(\mathbb{C}^n \setminus 0) / \langle \lambda \cdot Id \rangle, \quad n > 2, \quad 0 < |\lambda| < 1$$

is filtrable. In the present paper, we generalize this theorem to an arbitrary diagonal Hopf manifold.

**Theorem 1.1:** Let $A \in GL(n, \mathbb{C})$ be a diagonal linear operator, with all eigenvalues satisfying $|\alpha_i| < 1$, and $M = (\mathbb{C}^n \setminus 0) / \langle A \rangle$ the corresponding Hopf manifold. Then any coherent sheaf $F \in \text{Coh}(M)$ is filtrable, that is, admits a filtration

$$0 = F_0 \subset F_1 \subset ... \subset F_m = F$$

with $\text{rk} F_i / F_{i-1} \leq 1$.

**Proof:** Using induction, we can always assume that any sheaf $F'$ with $\text{rk} F' < \text{rk} F$ is filtrable. Then $F$ is filtrable unless $F$ has no proper coherent subshesaf. In the latter case, $F$ is stable. Therefore, **Theorem 1.1** is implied by the following theorem, which is proven in Section 6 by the means of gauge theory.

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Theorem 1.2: Let $A \in GL(n, \mathbb{C})$ be a diagonal linear operator, with all eigenvalues satisfying $|\alpha_i| < 1$, and $M = (\mathbb{C}^n \setminus 0)/\langle A \rangle$ the corresponding Hopf manifold. We choose a locally conformally Kähler Hermitian structure on $M$ as in Subsection 2.2. Let $F$ be a holomorphic bundle (or a reflexive coherent sheaf) which is stable with respect to this Hermitian structure. Then $F$ is filtrable.

Proof: See Remark 6.4.

The proof of Theorem 1.2 goes as follows. Using the Kobayashi-Hitchin correspondence on complex Hermitian manifolds (Section 3), we show that any stable bundle on a diagonal Hopf manifold is equivariant with respect to a certain holomorphic flow (Corollary 4.4). Taking a completion of this flow in $GL(n, \mathbb{C})$, we obtain an abelian Lie group, which is isomorphic to $(\mathbb{C}^\times)^l$ (Proposition 6.1). This allows us to treat stable holomorphic bundles (or reflexive sheaves) on $M$ as objects in a category $(\mathbb{C}^\times)^l$-equivariant coherent sheaves on $\mathbb{C}^n \setminus 0$ (Remark 6.4). Then we show that all objects in this category are filtrable (Theorem 6.5).

2 Diagonal Hopf manifolds in Vaisman geometry

2.1 An introduction to Vaisman geometry

Definition 2.1: Let $M$ be a complex manifold, $\dim_{\mathbb{C}} M > 1$, and $\tilde{M}$ its covering. Assume that $\tilde{M}$ is equipped with a Kähler form $\omega_K$, in such a way that the deck transform of $\tilde{M}/M$ acts on $(\tilde{M}, \omega_K)$ by homotheties. The form $\omega_K$ defines on $M$ a conformal class by $[\omega_K]$. The pair $(M, [\omega_K])$ is called locally conformally Kähler (LCK). A Hermitian form $\omega_H$ on $M$ is called an LCK-form if it belongs to the conformal class $[\omega_K]$.

Definition 2.2: Consider an LCK-manifold $M$ with an LCK-form $\omega_H$. A pullback of $\omega_H$ to $\tilde{M}$ is written as $f^* \omega_K$, where $f$ is a function and $\omega_K$ is the Kähler form on $\tilde{M}$. Therefore, $d\omega_H = \omega_H \wedge \theta$, where $\theta = df$ is a 1-form on $M$. Clearly, $\theta$ is defined uniquely. Since $\theta = d \log f$, $\theta$ is also closed. This form is called the Lee form of $(M, \omega_H)$.

Remark 2.3: For a general Hermitian complex manifold $(M, \omega_H)$, the Lee form

\footnote{For a definition of stability on Hermitian manifolds, see Section 3.}
form is defined as $d^c \ast \omega_H$, where $d^c = I \circ d \circ I^{-1}$ is the twisted de Rham differential, and $d^c \ast$ its Hermitian adjoint. It is not difficult to check that this definition is compatible with the one we used above.

**Definition 2.4:** Let $(M, \omega_H)$ be a Hermitian complex manifold, $\dim \mathbb{C} M = n$. Then $\omega_H$ is called a **Gauduchon metric** if $d^c \ast d^c \ast \omega_H = 0$, or, equivalently, $dd^c(\omega_H^{n-1}) = 0$.

**Remark 2.5:** In [Ga], P. Gauduchon proved that such a metric on $M$ exists and is unique, up to a constant multiplier, in any conformal class, provided that the manifold $M$ is compact.

**Remark 2.6:** On a compact LCK-manifold, this result translates into an existence of a unique metric with a harmonic Lee form $\theta$. Indeed, $d^c \ast \omega_H = \theta$ is always closed, hence the Gauduchon condition $d^c \ast d^c \ast \omega_H = 0$ is equivalent to $d^c \theta = 0$.

Further on, we shall always fix a choice of a Hermitian metric on an LCK-manifold by choosing a Gauduchon metric.

**Definition 2.7:** Let $M$ be an LCK-manifold equipped with a Gauduchon metric $\omega_H$, $\theta$ its Lee form and $\nabla$ the Levi-Civita connection associated with $\omega_H$. Assume that $\theta$ is parallel: $\nabla \theta = 0$. Then $M$ is called a **Vaisman manifold**.

**Remark 2.8:** According to Kamishima-Ornea ([KO]), a compact LCK-manifold $M$ is Vaisman if and only if it admits a holomorphic vector field acting on $M$ conformally, in such a way that its lifting to $\tilde{M}$ is not an isometry of $(\tilde{M}, \omega_K)$.

**Remark 2.9:** It is easy to see ([DO]) that the condition $\nabla \theta = 0$ implies that the dual to $\theta$ vector field $\theta^\sharp$ (called the **Lee field**) is a holomorphic isometry of $M$ and acts on $\tilde{M}$ by non-isometric conformal automorphisms. This gives the “only if” part of Kamishima-Ornea theorem.

For further results, details and calculations in Vaisman geometry, the reader is referred to [DO], [GO], [OV1], [OV2], [OV3].

Further on, we shall use the following lemma, which is proven in [Ve1] (see also [OV2]).
**Lemma 2.10:** Let $M$ be a Vaisman manifold, $\theta^\sharp$ its Lee field, and $\Sigma$ the complex holomorphic foliation generated by $\theta^\sharp$. Denote by $\omega_0 := d^c \theta$ the real $(1,1)$-form obtained as $d^c = I \circ d \circ I^{-1}$-differential of the Lee field $\theta$. Then $\omega_0 \geq 0$, and the null direction of $\omega_0$ is precisely $\Sigma$.

**Remark 2.11:** Let $L_\mathbb{R}$ be a real flat line bundle on $M$ with the same automorphy factors as the Kähler form $\omega_K$ (in conformal geometry, it is known as the weight bundle). Any non-degenerate positive section of $L_\mathbb{R}$ corresponds uniquely to a metric on $M$ conformally equivalent to $\omega_K$, and the converse is also true. The Gauduchon metric gives a rise to a section $\mu_G$ of $L_\mathbb{R}$. Consider $L := L_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ as a holomorphic Hermitian line bundle, with a holomorphic structure induced from the flat connection on $L = L_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, and Hermitian structure defined by $|\mu_G| = const$. Denote by $\nabla_C$ the corresponding Chern connection. Then $\omega_0$ is the curvature of $\nabla_C$ ($[Ve1]$, $[OV2]$).

### 2.2 LCK structure on diagonal Hopf manifolds

The main examples of LCK and Vaisman geometries are provided by the theory of Hopf manifolds.

**Definition 2.12:** Let $A \in GL(n)$ be a linear transform, acting on $\mathbb{C}^n$ with all eigenvalues satisfying $|\alpha_i| < 1$. Denote by $\langle A \rangle \subset \text{GL}(n, \mathbb{C})$ the cyclic group generated by $A$. The quotient $(\mathbb{C}^n \setminus 0)/\langle A \rangle$ is called a **linear Hopf manifold**. If $A$ is diagonalizable, $(\mathbb{C}^n \setminus 0)/\langle A \rangle$ is called a **diagonal Hopf manifold**.

**Remark 2.13:** If one takes an arbitrary holomorphic contraction $A$ instead of a linear contraction, one obtains the general definition of a Hopf manifold (see e.g. $[Ka1]$, $[Ka2]$ for details).

**Remark 2.14:** Izu Vaisman, who introduced the subject and studied the Vaisman manifolds at great length (see $[Va1]$, $[Va2]$), called them the generalized Hopf manifolds. This name is not suitable because many Hopf manifolds are not Vaisman. For linear Hopf manifolds, $(\mathbb{C}^n \setminus 0)/\langle A \rangle$ is Vaisman if and only if $A$ is diagonalizable (see $[OV3]$).
Let \( A \in GL(n, \mathbb{C}) \) be a diagonal linear transform:
\[
\begin{bmatrix}
\alpha_1 & 0 & \cdots & 0 \\
0 & \alpha_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_n
\end{bmatrix}, |\alpha_i| < 1
\]
Consider the Kähler metric \( \omega_K := -\sqrt{-1} \partial \bar{\partial} \varphi \) on \( \mathbb{C}^n \setminus 0 \), defined using the Kähler potential \( \varphi : \mathbb{C}^n \setminus 0 \rightarrow \mathbb{R} \). The \( \varphi \) is defined via the formula
\[
\varphi(t_1, \ldots, t_n) = \sum |t_i|^{\beta_i}, \tag{2.1}
\]
where \( \beta_i := \log |\alpha_i|^{-1} C \) are positive real numbers which satisfy \( |\alpha_i|^{-\beta_i} = C \) for some fixed real constant \( C > 1 \), chosen in such a way that all \( \beta_i \) satisfy \( |\beta_i| \geq 2 \), and \( t_i \) are complex coordinates. By construction, \( A^* \varphi = C^{-1} \varphi \).

Indeed,
\[
A^* \varphi(t_1, \ldots, t_n) = \varphi(A(t_1, \ldots, t_n)) = \sum |\alpha_i t_i|^{\beta_i} = C^{-1} \varphi(t_1, \ldots, t_n).
\]
Therefore, \( \omega_K := -\sqrt{-1} \partial \bar{\partial} \varphi \) is a Kähler form which satisfies \( A^* \omega_K = C^{-1} \omega_K \). This implies that the diagonal Hopf manifold \( (\mathbb{C}^n \setminus 0)/\langle A \rangle \) is LCK.

To see that it is Vaisman, we notice that the holomorphic vector field \( \log A \) acts on \( (\mathbb{C}^n \setminus 0, \omega_K) \) conformally and apply the Kamishima-Ornea theorem \( \text{[Remark 2.8]} \).

We proceed with computing the Lee field for the Gauduchon metric on \( (\mathbb{C}^n \setminus 0)/\langle A \rangle \), equipped with a conformal structure defined by the Kähler form described above.

Consider the action of the complex Lie group \( V(t) = e^{Cv} \) generated by the holomorphic vector field \( v := \sum_i -t_i \log |\alpha_i| \frac{d}{dt_i} \). By construction, \( V(\lambda) \) is a linear operator which can be written as \( \sum e^{i |\alpha_i| \lambda t_i} \). For \( \lambda \) real, this operator multiplies \( \varphi \) by a constant \( C^\lambda \) (this is proven in the same way as one proves that \( A(\varphi) = C^{-1} \varphi \)), and for \( \lambda \) purely imaginary, \( V(\lambda) \) preserves \( \varphi \) (this is clear). Therefore, \( v^c := I(v) \) acts on \( (\mathbb{C}^n \setminus 0, \omega_K) \) by holomorphic isometries.

The corresponding moment map \( \mu : \mathbb{C}^n \setminus 0 \rightarrow \mathbb{R} \) is given by \( d\mu = \omega_K(v^c, \cdot) \). The latter differential form is written as
\[
(dd^c \varphi) \cdot v^c = \text{Lie}_{v^c} d^c \varphi - d(d^c \varphi \cdot v^c). \tag{2.2}
\]
The first term of the right hand side of (2.2) vanishes because \( v^c \) acts on \( (\mathbb{C}^n \setminus 0) \) preserving \( \varphi \) and a complex structure. This gives
\[
\omega_K(v^c, \cdot) = (dd^c \varphi) \cdot v^c = -d(d^c \varphi \cdot v^c) = d(d\varphi \cdot v) = \log C \cdot d\varphi
\]
(the last equation holds because \( d\varphi \cdot v = \text{Lie}_v \varphi = \log C \cdot \varphi \)). We obtained that \( \log(C)\varphi \) is the moment map for \( V(t) \) acting on \((\mathbb{C}^n \setminus 0, \omega_K)\).

We obtained the following claim, which is well known in many similar situations.

**Claim 2.15:** Let \( A \subset GL(n) \) be a diagonal contraction of \( \mathbb{C}^n \), with all eigenvalues \( \alpha_i \) satisfying \( |\alpha_i| < 1 \). Consider a Kähler metric \( \omega_K := -\sqrt{-1} \partial \bar{\partial} \varphi \) on \( \mathbb{C}^n \setminus 0 \), where the Kähler potential \( \varphi \) is defined by the formula (2.1), and let \( V(t) = e^{\mathbb{C}v} \) be the holomorphic flow generated by \( v := \sum \alpha_i \text{log} |\alpha_i| \frac{d}{dt_i} \). Let \( v^c := I(v) \) be the complex adjoint of \( v \). Then \( e^{\mathbb{R}v^c} \subset V(t) \), preserves the Kähler structure on \( \mathbb{C}^n \setminus 0 \), and the corresponding moment map is \( \log(C)\varphi \):

\[
d(\log(C)\varphi) = \omega_K(v^c, \cdot).
\]

Consider the Hermitian form \( \omega_H = \frac{\omega_K}{\varphi} \) on \( M = (\mathbb{C}^n \setminus 0)/\langle A \rangle \). The corresponding Lee form \( \theta \) is obtained via

\[
d\omega_H = -\frac{\omega_K}{\varphi^2} = -\omega_H \wedge \log d\varphi,
\]

hence \( \theta = \frac{d\varphi}{\varphi} \). The dual under \( \omega_H \) vector field (Lee field) is given by \( \theta^d = v \), where \( v = \log(C) \sum \alpha_i \text{log} |\alpha_i| \frac{d}{dt_i} \). This is clear because \( v \) is dual to \( \log(C)d\varphi \) with respect to \( \omega_K \) as Claim 2.15 implies, and \( \omega_H = \frac{\omega_K}{\varphi} \).

This gives the following Proposition.

**Proposition 2.16:** In assumptions of Claim 2.15 consider the Hermitian form \( \omega_H = \frac{\omega_K}{\varphi} \) on \( M = (\mathbb{C}^n \setminus 0)/\langle A \rangle \). Then the corresponding Lee field is given as

\[
\theta^d = \log(C) \sum \alpha_i \text{log} |\alpha_i| \frac{d}{dt_i}.
\]

Moreover, \( \omega_H \) is Gauduchon.

**Proof:** The equation (2.3) is proven above. To see that \( \omega_H \) is Gauduchon, it suffices to see that \( |\theta^d|_{\omega_H} \) is constant. Indeed, from the definition of \( d^* \) it follows easily that

\[
d^* \theta = \nabla_{\theta^d} \theta^d.
\]

However, \( \theta^d \) is Killing, because \( \text{Lie}_{\theta^d} \varphi = C \varphi \), \( \text{Lie}_{\theta^d} \omega_K = C \omega_K \), and therefore

\[
\text{Lie}_{\theta^d} \omega_H = C \omega_H - C \omega_H = 0.
\]
By another definition of Killing fields, this means that

\[(\nabla_X \theta^z, Y)_{\omega_H} = -(\nabla_Y \theta^z, X)_{\omega_H}\]

for all vector fields \(X, Y\). Taking \(Y = \theta^z\), and applying \(\text{Lie}_X (\theta^z, \theta^z)_{\omega_H} = 0\), we obtain

\[0 = (\nabla_X \theta^z, \theta^z)_{\omega_H} = -(\nabla_{\theta^z} \theta^z, X)_{\omega_H}.\]

As \(X\) is arbitrary, this implies \(\nabla_{\theta^z} \theta^z = 0\). Therefore, Proposition 2.16 is implied by the equation \(\omega_H (\theta^z, \bar{\theta}^z) = \text{const}\), or, equivalently,

\[\omega_K (\theta^z, \bar{\theta}^z) = \text{const} \cdot \varphi. \quad (2.4)\]

Writing \(\omega_K\) as

\[\omega_K = -\sqrt{-1} \partial \bar{\partial} \varphi = \sum_i dt_i \wedge d\bar{t}_i |t_i|^\beta_i - \beta_i^2 |t_i|^2 / 4,\]

and using \(\theta^z = \log(C) \sum_i -t_i \log |\alpha_i| \frac{dt_i}{|t_i|}\), we obtain

\[\omega_K (\theta^z, \bar{\theta}^z) = \log(C)^2 \sum_i (\log |\alpha_i|)^2 |t_i|^\beta_i \frac{\beta_i^2}{4}. \quad (2.5)\]

By definition, \(e^{-\log |\alpha_i| \beta_i} = C\), in other words, \(\beta_i = -\frac{\log C}{\log |\alpha_i|}\). Plugging this into (2.5), we obtain

\[\omega_K (\theta^z, \bar{\theta}^z) = \sum_i |t_i|^\beta_i \frac{(\log C)^4}{4} = \frac{(\log C)^4}{4} \varphi.\]

This proves Proposition 2.16.

3 Stable bundles on Hermitian manifolds

3.1 Gauduchon metrics and stability

**Definition 3.1:** Let \(M\) be a compact complex Hermitian manifold. Choose a Gauduchon metric in the same conformal class.\(^1\) Consider a torsion-free coherent sheaf \(F\) on \(M\). Denote by \(\text{det} F\) its determinant bundle. Pick a

\(^1\)A Hermitian metric on a complex manifold of dimension \(n\) is called **Gauduchon** if \(\partial \bar{\partial} (\omega^{n-1}) = 0\), where \(\omega\) is its Hermitian form (Definition 2.4). On a compact manifold, a Gauduchon metric exists in any conformal class, and is unique up to a constant multiplier, see [Ga].
Hermitian metric $\nu$ on $\det F$, and let $\Theta$ be the curvature of the associated Chern connection. We define the degree of $F$ as follows:

$$\deg F := \int_M \Theta \wedge \omega^\dim C M^{-1},$$

where $\omega \in \Lambda^{1,1}(M)$ is the Hermitian form of the Gauduchon metric. This notion is independent from the choice of the Hermitian structure $\nu$ in $F$. Indeed, if $\nu' = e^\psi \nu$, $\psi \in C^\infty(M)$, then the associated curvature form is written as $\Theta' = \Theta + \bar{\partial} \partial \psi$, and

$$\int_M \bar{\partial} \partial \psi \wedge \omega^\dim C M^{-1} = 0$$

because $\omega$ is Gauduchon.

If $F$ is a Hermitian vector bundle, $\Theta_F$ its curvature, and the metric $\nu$ is induced from $F$, then $\Theta = \text{Tr}_F \Theta_F$. In Kähler case this allows one to relate the degree of a bundle with the first Chern class. However, in non-Kähler case, the degree is not a topological invariant — it depends fundamentally on the holomorphic geometry of $F$. Moreover, the degree is not discrete, as in the Kähler situation, but takes values in continuum.

Further on, we shall see that one can in some cases construct a holomorphic structure of any given degree $\lambda \in \mathbb{R}$ on a fixed $C^\infty$-bundle. In our examples, such holomorphic structures are constructed on a topologically trivial line bundle over a Vaisman manifold (Remark 4.3).

**Definition 3.2:** Let $F$ be a non-zero torsion-free coherent sheaf on $M$. Then slope$(F)$ is defined as

$$\text{slope}(F) := \frac{\deg F}{\text{rk} F}.$$ 

The sheaf $F$ is called

- **stable** if for all subsheaves $F' \subset F$, we have slope$(F') < $ slope$(F)$
- **semistable** if for all subsheaves $F' \subset F$, we have slope$(F') \leq $ slope$(F)$
- **polystable** if $F$ can be represented as a direct sum of stable coherent sheaves with the same slope.

**Remark 3.3:** This definition is stability is “good” as most standard properties of stable and semistable bundles hold in this situation as well. In particular, all line bundles are stable; all stable sheaves are simple; the Jordan-Hölder and Harder-Narasimhan filtrations are well defined and behave in the same way as they do in the usual Kähler situation (Lüt, Br).
However, not all bundles are filtrable, that is, are obtained as successive extensions by coherent sheaves of rank 1. There are non-filtrable holomorphic vector bundles on most non-algebraic K3 surfaces.

### 3.2 Kobayashi-Hitchin correspondence

The statement of Kobayashi-Hitchin correspondence (Donaldson-Uhlenbeck-Yau theorem) is translated to the Hermitian situation verbatim, following Li and Yau ([LY]).

**Definition 3.4:** Let $B$ be a holomorphic Hermitian vector bundle on a Hermitian manifold $M$, and $\Theta \in \Lambda^{1,1}(M) \otimes \text{End}(B)$ the curvature of its Chern connection $\nabla$. Consider the operator $\Lambda : \Lambda^{1,1}(M) \otimes \text{End}(B) \to \text{End}(B)$ which is a Hermitian adjoint to $b \to \omega \otimes b$, $\omega$ being the Hermitian form on $M$. The connection $\nabla$ is called **Hermitian-Einstein** (or **Yang-Mills**) if $\Lambda \Theta = \text{const} \cdot \text{Id}_B$.

**Theorem 3.5:** (Kobayashi-Hitchin correspondence) Let $B$ be a holomorphic vector bundle on a compact complex manifold equipped with a Gauduchon metric. Then $B$ admits a Hermitian-Einstein connection $\nabla$ if and only if $B$ is polystable. Moreover, the Hermitian-Einstein connection is unique.

**Proof:** See [LY], [LT1], [LT2].

### 4 Stable bundles on Vaisman manifolds

Existence of the positive exact $(1,1)$-form $\omega_0$, defined in [Lemma 2.10] brings many consequences for algebraic geometry of the Vaisman manifolds (see e.g. [Ve1] and [OV2]). One of these is the structure theorem for Hermitian-Einstein bundles of degree 0.

The following result was stated and proven as Theorem 4.3, [Ve2] for positive principal elliptic fibrations, which admit a similar structure. These manifolds are not always Vaisman (e.g. Calabi-Eckmann manifolds are not Vaisman). However, the proof of this theorem can be repeated almost verbatim in the Vaisman situation.

**Theorem 4.1:** Let $M$ be a compact Vaisman manifold, $\dim \mathbb{C} M > 2$, and $B$ a stable bundle of degree 0 on $M$. Denote by $\Sigma$ the 1-dimensional complex holomorphic foliation generated by the Lee field $\theta^2$. Then $\Theta(v, \cdot) = 0$ for any $v \in \Sigma$. In particular, $B$ is equivariant with respect to the complex Lie group
$V(t)$ generated by $\theta^t$, and this equivariant structure is compatible with the connection.

**Proof:** Consider the map

$$\Lambda : \Lambda^{1,1}(M, \text{End}(B)) \to \text{End}(B)$$

defined in Subsection 3.2. By definition, $\Theta$ is **primitive**, that is, satisfies $\Lambda \Theta = 0$. Then Theorem 4.1 is implied by the following proposition.

**Proposition 4.2:** Let $M$ be a compact Vaisman manifold, $\dim \mathbb{C} M > 2$, $B$ a Hermitian bundle with connection, and $\Theta \in \Lambda^{1,1}(M, \mathbb{R}) \otimes \mathfrak{u}(B)$ a closed skew-Hermitian real $(1,1)$-form. Assume that $\Theta$ is primitive, that is, $\Lambda \Theta = 0$. Then $\Theta(v, \cdot) = 0$ for any $v \in \Sigma$.

**Proof:** Rescaling the metric, we normalize the Lee form $\theta$ so that $|\theta| = 1$. Let $\theta, \theta_1, ..., \theta_{n-1}$ be an orthonormal basis in $\Lambda^{1,0}(M)$, with $\theta \in \Sigma, \theta_i \in \Sigma^\perp$. Consider the form $\omega_0$ (Lemma 2.10). This form is exact, positive, and has $n - 1$ strictly positive eigenvalues. Using the basis described above, we can write

$$\omega_H = -\sqrt{-1} \left( \theta \wedge \bar{\theta} + \sum_i \theta_i \wedge \bar{\theta}_i \right), \quad \omega_0 = -\sqrt{-1} \left( \sum_i \theta_i \wedge \bar{\theta}_i \right) \quad (4.1)$$

where $\omega_H$ is the Hermitian form of $M$ (see [Ve1], Proposition 6.1).

In this basis, we can write $\Theta$ as

$$\Theta = \sum_{i \neq j} (\theta_i \wedge \bar{\theta}_j + \bar{\theta}_i \wedge \theta_j) \otimes b_{ij} + \sum_i (\theta_i \wedge \bar{\theta}_i) \otimes a_i \quad (4.2)$$

$$+ \sum_i (\theta \wedge \bar{\theta}_i + \bar{\theta} \wedge \theta_i) \otimes b_i + \theta \wedge \bar{\theta} \otimes a, \quad (4.3)$$

with $b_{ij}, b_i, a_i, a \in \mathfrak{u}(B)$ being skew-Hermitian endomorphisms of $B$.

Let $\Xi := \text{Tr}(\Theta \wedge \Theta)$. This is a closed $(2,2)$-form on $M$. Then [12] implies

$$(\sqrt{-1})^n \Xi \wedge \omega_0^{n-2} = \text{Tr} \left( -\sum b_i^2 + a \left( \sum a_i \right) \right)$$

On the other hand, $\sum a_i + a = \Lambda \Theta = 0$, hence

$$(\sqrt{-1})^n \Xi \wedge \omega_0^{n-2} = \text{Tr} \left( -\sum b_i^2 - a^2 \right).$$
Since \( u \rightarrow \text{Tr}(-u^2) \) is a positive definite form on \( u(B) \), the integral
\[
\int_M (\sqrt{-1})^n \Xi \wedge \omega_0^{n-2}
\]
is non-negative, and positive unless \( b_i \) and \( a \) both vanish everywhere. Using \( n > 2 \), we find that (4.4) vanishes, because \( \omega_0 \) is exact and \( \Xi \) is closed. Therefore, \( b_i \) and \( a \) are identically zero, which is exactly the claim of Proposition 4.2. We proved Theorem 4.1.

**Remark 4.3:** The results of Theorem 4.1 can be applied to arbitrary stable bundle on \( M \) using the following trick. Consider the line bundle \( L \) (Remark 2.11). Write the Chern connection on \( L \) as
\[
\nabla_C = \nabla_{\text{triv}} - \sqrt{-1} \theta^c,
\]
where \( \theta^c = I(\theta) \) is the complex conjugate of \( \theta \) (see [Ve1], (6.11)), and \( \nabla_{\text{triv}} \) is a trivial connection associated to the trivialization of \( L \) constructed in Remark 2.11. Since \( d\theta^c = \omega_0 \), \( L \) has a degree \( \delta := \int \omega_0 \wedge \omega_H^{n-1} \) which is clearly positive (see (4.1)). Given \( \lambda \in \mathbb{R} \), denote by \( L_\lambda \) a holomorphic Hermitian bundle with the connection \( \nabla_{\text{triv}} - \sqrt{-1} \frac{\delta}{2} \theta^c \). Then \( L_\lambda \) has degree \( \lambda \). We obtain that a Vaisman manifold admits a line bundle \( L_\lambda \) of arbitrary degree \( \lambda \). Moreover, \( L_\lambda \) is by construction \( V(t) \)-equivariant (the form \( \theta^c \) is \( V(t) \)-invariant, as \( V(t) \) acts on \( M \) preserving the metric and the holomorphic structure). This brings the following corollary.

**Corollary 4.4:** Let \( M \) be a compact Vaisman manifold, and \( B \) a stable bundle. Consider a complex holomorphic flow \( V(t) = e^{it\theta} \) generated by the Lee field \( \theta^\sharp \). Then \( B \) admits a natural \( V(t) \)-equivariant structure.

**Proof:** Tensoring \( B \) by \( L_\lambda \) for appropriate choice of \( \lambda \in \mathbb{R} \), we obtain a stable bundle of degree 0. Then Theorem 4.1 implies Corollary 4.4.

5 Stable bundles on Hopf manifolds and coherent sheaves on \( \mathbb{C}^n \)

5.1 Admissible Hermitian structures on reflexive sheaves

**Definition 5.1:** Let \( X \) be a complex manifold, and \( F \) a coherent sheaf on \( X \). Consider the sheaf \( F^\vee := \text{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X) \). There is a natural functorial map
$\rho_F : F \rightarrow F^{**}$. The sheaf $F^{**}$ is called a reflexive hull, or reflexization, of $F$. The sheaf $F$ is called reflexive if the map $\rho_F : F \rightarrow F^{**}$ is an isomorphism.

**Remark 5.2:** For all coherent sheaves $F$, the map $\rho_{F^*} : F^* \rightarrow F^{***}$ is an isomorphism ([OSS], Ch. II, the proof of Lemma 1.1.12). Therefore, a reflexive hull of a sheaf is always reflexive.

Reflexive hull can be obtained by restricting to an open subset and taking the pushforward.

**Lemma 5.3:** Let $X$ be a complex manifold, $F$ a coherent sheaf on $X$, $Z$ a closed analytic subvariety, $\text{codim } Z \geq 2$, and $j : (X \setminus Z) \hookrightarrow X$ the natural embedding. Assume that the pullback $j^*F$ is reflexive on $(X \setminus Z)$. Then the pushforward $j_*j^*F$ is also reflexive.

**Proof:** This is [OSS], Ch. II, Lemma 1.1.12. □

**Remark 5.4:** From Lemma 5.3 it is apparent that one could obtain a reflexization of a non-singular in codimension 1 coherent sheaf $F$ by taking $j_*j^*F$, where $j : (X \setminus Z) \hookrightarrow X$ the natural open embedding, and $Z$ the singular locus of $F$.

Using the results of [BS], we are able to apply the Kobayashi-Hitchin correspondence to reflexive sheaves.

**Definition 5.5:** [BS] Let $F$ be a coherent sheaf on $M$ and $\nabla$ a Hermitian connection on $F$ defined outside of its singularities. Denote by $\Theta$ the curvature of $\nabla$. Then $\nabla$ is called admissible if the following holds

(i) $\Lambda \Theta \in \text{End}(F)$ is uniformly bounded

(ii) $|\Theta|^2$ is integrable on $M$.

**Theorem 5.6:** [BS] Any torsion-free coherent sheaf admits an admissible connection. An admissible connection can be extended over the place where $F$ is smooth. Moreover, if a bundle $B$ on $M \setminus Z$, $\text{codim}_C Z \geq 2$ is equipped with an admissible connection, then $B$ can be extended to a coherent sheaf on $M$. □
A version of Donaldson-Uhlenbeck-Yau theorem exists for coherent sheaves (Theorem 5.7); given a torsion-free coherent sheaf $F$, $F$ admits an admissible Hermitian-Einstein connection $\nabla$ if and only if $F$ is polystable.

**Theorem 5.7:** Let $M$ be a compact Kähler manifold, and $F$ a coherent sheaf without torsion. Then $F$ admits an admissible Hermitian-Einstein metric if and only if $F$ is polystable. Moreover, if $F$ is stable, then this metric is unique, up to a constant multiplier.

**Proof:** [BS], Theorem 3. [QED]

This proof can be adapted for Hermitian complex manifolds with Gauduchon metric.

### 5.2 Hermitian-Einstein bundles on Hopf manifolds and admissibility

**Theorem 5.8:** Let $M = (\mathbb{C}^n \setminus 0)/\langle A \rangle$ be a diagonal Hopf manifold, $n \geq 3$, and $B$ a stable holomorphic bundle on $M$ of degree 0. Denote by $\tilde{B}$ the pullback of $B$ to $\mathbb{C}^n \setminus 0$. Then $\tilde{B}$ can be extended to a reflexive coherent sheaf $F$ on $\mathbb{C}^n$. Moreover, $F$ is $V(t)$-equivariant, where $V(t)$ is the complex holomorphic flow on $\mathbb{C}^n$ generated by the Lee field $\theta^t = \log(C) \sum_i t_i \log |\alpha_i| \frac{d}{dt_i}$.

**Proof:** Consider a Hermitian-Einstein metric on $B$, and lift it to $\tilde{B}$. Denote by $\tilde{\Theta}$ the curvature of $\tilde{B}$. To extend $\tilde{B}$ to $\mathbb{C}^n$, we apply the Bando-Siu theorem (Theorem 5.6). We need to show that $\tilde{B}$ is admissible, in the sense of Definition 5.5. The Kähler metric $\omega_K$ on $\mathbb{C}^n$ is conformally equivalent to that lifted from $M$, hence $\Lambda \tilde{\Theta} = 0$ (this condition means that $\tilde{\Theta}$ is orthogonal to the Hermitian form pointwise, and therefore it is conformally invariant). To prove that $\tilde{B}$ is admissible, it remains to show that $\tilde{\Theta}$ is square-integrable. The function $|\tilde{\Theta}|^2$ can be expressed, using the Hodge-Riemann relations, as follows.

**Lemma 5.9:** Let $B_1$ be a Hermitian bundle on a Hermitian almost complex manifold $M_1$, of dimension $n$, and

$$\nu \in \Lambda^{1,1}(M_1, \mathfrak{su}(B_1))$$

a $\mathfrak{su}(B_1)$-valued (1,1)-form satisfying $\Lambda(\nu) = 0$. Then

$$|\nu|^2 = -\sqrt{-1} \frac{n-1}{2n} \Tr(\Lambda^2(\nu \wedge \nu)), \quad (5.1)$$
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\[ \Lambda : \Lambda^{p,q}(M_1, \mathfrak{su}(B_1)) \rightarrow \Lambda^{p-1,q-1}(M_1, \mathfrak{su}(B_1)) \]

is the standard Hodge operator on differential forms.

**Proof:** An elementary calculation, and essentially the same as one which proves the Hodge-Riemann bilinear relations (see e.g. [BS]).

**Remark 5.10:** The equation (5.1) can be stated as

\[ |\nu|^2 \text{Vol}(M_1) = -\sqrt{-1} \frac{n-1}{2n \cdot 2^{n-1} \cdot n!} \text{Tr}(\nu \wedge \nu) \wedge \omega_1^{n-2}, \]  

(5.2)

where \( \omega \) is the Hermitian form on \( M_1 \), and \( \text{Vol}(M_1) \) the Riemannian volume. This is clear from the definition of \( \Lambda \) and the relation \( \text{Vol}(M_1) = \frac{1}{2^{n-1} \cdot n!} \omega_1^n \).

Using (5.2), we obtain that \( L^2 \)-integrability of \( \tilde{\Theta} \) is equivalent to integrability of the form

\[ \text{Tr}(\tilde{\Theta} \wedge \tilde{\Theta}) \wedge \omega_K^{n-2}. \]  

(5.3)

The form \( \tilde{\Theta} \) is by construction \( A \)-invariant, and \( \omega_K \) satisfies \( A^*(\omega_K) = c\omega_K \) because \( M \) is LCK. Therefore, the form (5.3) is homogeneous with respect to the action of \( A \):

\[ A^* \left( \text{Tr}(\tilde{\Theta} \wedge \tilde{\Theta}) \wedge \omega_K^{n-2} \right) = c^{n-2} \text{Tr}(\tilde{\Theta} \wedge \tilde{\Theta}) \wedge \omega_K^{n-2}, c < 1. \]  

(5.4)

Denote by \( D \) the fundamental domain for \( \langle A \rangle \),

\[ D := \{ x \in \mathbb{C}^n \setminus \{0\} \mid 1 \leq \rho(x) < C \} \]

Then \( \mathbb{C}^n \setminus \{0\} = \cap_{i \in \mathbb{Z}} A^i(D) \). To check that \( \tilde{\Theta} \) is \( L^2 \)-integrable in a neighbourhood of 0, we need to show that the series

\[ \sum_{i=0}^{\infty} \int_{A^i(D)} |\tilde{\Theta}|^2 \text{Vol} = -\sqrt{-1} \frac{n-1}{2n \cdot 2^{n-1} \cdot n!} \sum_{i=0}^{\infty} \int_{A^i(D)} \text{Tr}(\tilde{\Theta} \wedge \tilde{\Theta}) \wedge \omega_K^{n-2} \]

converges. However, by homogeneity, the latter integral is power series, and (5.4) implies that it converges whenever \( n > 2 \). We have shown that \( \tilde{B} \) is admissible. Now, Bando-Siu theorem (Theorem 5.6) implies the first assertion of Theorem 5.8. The second assertion is implied immediately by Lemma 5.3. Indeed, let \( (\mathbb{C}^n \setminus \{0\}) \mapsto \mathbb{C}^n \) be the standard embedding. Then
\( F = j_! \tilde{B} \) (Lemma 5.3). By Corollary 4.4, \( \tilde{B} \) is \( V(t) \)-equivariant. Then \( j_! \tilde{B} \) is also \( V(t) \)-equivariant. ■

Remark 5.11: Using the Bando-Siu version of Donaldson-Uhlenbeck-Yau theorem, we can extend Theorem 5.8 verbatim to reflexive coherent sheaves.

6 Equivariant sheaves on \( \mathbb{C}^n \)

6.1 Extending \( V(t) \)-equivariance to \( (\mathbb{C}^*)^l \)-equivariance

Let \( M = (\mathbb{C}^n \setminus 0)/\langle A \rangle \) be a diagonal Hopf manifold, and \( V(t) = e^{C\theta t}, t \in \mathbb{C} \) the holomorphic flow generated by the Lee field \( \theta \) as above. Then \( V(t) \) acts on \( M \) by holomorphic isometries ([KO]). Consider the closure \( G \) of \( V(t) \), \( t \in \mathbb{C} \), within the group \( \text{Iso}(M) \) of isometries of \( M \). Denote by \( \tilde{G} \) the the lifting of \( G \) to \( \text{Aut}(\tilde{M}) \) ([OV1], [OV2]). By construction, \( \tilde{G} \) is the smallest closed Lie subgroup of \( GL(n, \mathbb{C}) \) containing \( V(t) \) and \( A \). It is easy to check that \( \tilde{G} \) is a reductive complex commutative Lie group. A similar result is true for all Vaisman manifolds.

Proposition 6.1: For any Vaisman manifold \( M \), let \( \theta \) be its Lee field, \( G \) the closure of the corresponding complex holomorphic flow within \( \text{Iso}(M) \), and \( \tilde{G} \) its lift to \( \text{Aut}(\tilde{M}) \). Then \( \tilde{G} \cong (\mathbb{C}^*)^k \), and the deck transform map \( \gamma \in \text{Aut}(\tilde{M}, M) \) lies in \( \tilde{G} \).

Proof: This is [OV2], Proposition 4.3 ■

Proposition 6.2: In assumptions of [Theorem 5.8], consider the action of the group \( V(t) \) on \( \Gamma(\mathbb{C}^n, F) \). Consider the adic topology on \( \mathcal{O}_{\mathbb{C}^n} \) and \( \Gamma(\mathbb{C}^n, F) \), with \( \lim f_i \to 0 \) as \( [f_i]_0 \to \infty \), where \( [f_i]_0 \) denotes the order of zeroes of \( f_i \) in \( 0 \in \mathbb{C}^n \). Clearly, \( V(t) \) is continuous in adic topology. Let \( \tilde{G}_F \) be the closure of \( V(t) \)-action on \( \Gamma(\mathbb{C}^n, F) \times \mathcal{O}_{\mathbb{C}^n} \) in adic topology. Then

(i) The natural map \( \tilde{G}_F \longrightarrow GL(F/mF) \times GL(m/m^2) \) is injective, where \( m \) is the maximal ideal of \( 0 \) in \( \mathcal{O}_{\mathbb{C}^n} \).

(ii) \( \tilde{G}_F \) is a closure of \( V(t) \) under the natural map \( V(t) \longrightarrow GL(F/mF) \times GL(m/m^2) \).

(iii) Consider the natural projection \( \tilde{G}_F \longrightarrow \tilde{G} \) induced by \( GL(F/mF) \times GL(m/m^2) \longrightarrow GL(m/m^2) \).
Then \( \pi \) satisfies \( g(af) = \pi(g)(a)g(f) \), for any \( f \in \Gamma(\mathbb{C}^n, F) \), \( a \in \mathcal{O}_{\mathbb{C}^n} \), \( g \in \tilde{G}_F \). This gives a \( \tilde{G}_F \)-equivariant structure on \( F \).

(iv) The group \( \tilde{G}_F \) is isomorphic to \( (\mathbb{C}^*)^l \).

**Proof:** Proposition 6.2 (i) is clear from Nakayama’s lemma. Proposition 6.2 (ii) is immediately implied by Proposition 6.2 (i). Proposition 6.2 (iii) follows from Proposition 6.2 (ii) and \( V(t) \)-equivariance of \( F \).

To prove Proposition 6.2 (iv), we use Proposition 6.2 (ii), and notice that \( \tilde{G}_F \) is commutative as a closure of a 1-parametric group within a Lie group \( GL(F/mF) \times GL(m/m^2) \). To show that \( \tilde{G}_F \cong (\mathbb{C}^*)^l \), we need to prove that it is reductive, that is, to show that \( V(t) \) acts diagonally on \( (F/mF) \times (m/m^2) \).

The group \( V(t) \) acts on \( M \) holomorphically and conformally. Since the Hermitian-Einstein metric on \( B \) is unique, up to a constant multiplier, the group \( V(t) \) acts on \( B \) also conformally. Then, \( V(t) \) acts conformally on the Hermitian space \( \Gamma(B_{\mathbb{C}^n}, F) \) of holomorphic sections of \( F \) on an open ball \( B_{\mathbb{C}^n} \subset \mathbb{C}^n \) and on \( \Gamma(B_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n}) \). Since orthogonal matrices in finite dimension are diagonalizable, \( V(t) \) acts diagonally on any finite-dimensional subspace in \( \Gamma(B_{\mathbb{C}^n}, F) \times \Gamma(B_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n}) \) preserved by \( V(t) \). Using the same classical Poincare-Dulac argument as used in the proof of Theorem 3.3 in \[OV3\], we find that \( \Gamma(B_{\mathbb{C}^n}, F) \times \Gamma(B_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n}) \) contains a dense (in appropriate, e.g. \( m \)-adic topology) subspace which is generated by finite-dimensional \( V(t) \)-invariant subspaces. Then \( V(t) \)-action on the space \( \Gamma(B_{\mathbb{C}^n}, F) \times \Gamma(B_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n}) \) is diagonal in a dense subspace. Therefore, this action is diagonal on its quotient \( (F/mF) \times (m/m^2) \). We proved Proposition 6.2 (iv). \( \blacksquare \)

**Remark 6.3:** Using the Bando-Siu version of Donaldson-Uhlenbeck-Yau theorem (see Remark 5.11), we can extend Proposition 6.2 verbatim to reflexive coherent sheaves.

**Remark 6.4:** Denote by \( \mathbb{C}^n_{\times} \) the complex manifold \( \mathbb{C}^n \setminus \{0\} \). Given a \( \tilde{G}_F \)-equivariant coherent sheaf on \( \mathbb{C}^n_{\times} \), we can obtain a coherent sheaf on \( \mathbb{C}^n_{\times}/\langle A \rangle \). Indeed, coherent sheaves on \( \mathbb{C}^n_{\times}/\langle A \rangle \) are the same as \( \langle A \rangle \)-equivariant sheaves on \( \mathbb{C}^n_{\times} \), and \( \langle A \rangle \) lies in \( \tilde{G} \) as Proposition 6.1 implies. Therefore, to prove the filtrability of a stable bundle \( B \) on \( M = (\mathbb{C}^n_{\times})/\langle A \rangle \), it suffices to show that the corresponding \( \tilde{G}_F \)-equivariant coherent sheaf \( F \) is filtrable on \( \mathbb{C}^n_{\times} \) in the category \( \text{Coh}_{\tilde{G}_F}(\mathbb{C}^n_{\times}) \) of \( \tilde{G}_F \)-equivariant coherent sheaves. Then, the following theorem proves Theorem 1.2.

**Theorem 6.5:** Let \( \tilde{G}_F \cong (\mathbb{C}^*)^l \) be a commutative Lie group, acting on \( \mathbb{C}^n_{\times} \) via a homomorphism \( \tilde{G}_F \rightarrow GL(\mathbb{C}, n) \), and \( \text{Coh}_{\tilde{G}_F}(\mathbb{C}^n_{\times}) \) be the category
of $\tilde{G}_F$-equivariant coherent sheaves on $\mathbb{C}^n_*$. Assume that $\pi(\tilde{G}_F)$ contains an endomorphism with all eigenvalues $< 1$. Then all objects of $\text{Coh}_{\tilde{G}_F}(\mathbb{C}^n_*)$ are filtrable by $\tilde{G}_F$-equivariant coherent sheaves of rank at most 1.

We prove Theorem 6.5 in Subsection 6.2.

6.2 $(\mathbb{C}^*)^l$-equivariant coherent sheaves on $\mathbb{C}^n\setminus 0$

We work in assumptions of Theorem 6.5.

Lemma 6.6: Let $R \in \text{Coh}_{\tilde{G}_F}(\mathbb{C}^n_*)$ be a $\tilde{G}_F$-equivariant coherent sheaf over $\mathbb{C}^n_* := \mathbb{C}^n\setminus 0$. Then $R$ is generated over $\mathcal{O}_{\mathbb{C}^n_*}$ by a finite-dimensional $\tilde{G}_F$-invariant space $V \subset \Gamma(R, \mathbb{C}^n_*)$.

Proof: The images of $\mathbb{C}^*$ are dense in $\tilde{G}_F \simeq (\mathbb{C}^*)^l$. Therefore, there exists an embedding $\mathbb{C}^* \overset{\mu}{\to} \tilde{G}_F$ acting on $\mathbb{C}^n$ with all eigenvalues different from 1. This action can be written as

$$t \mapsto \begin{bmatrix} t^{k_1} & 0 & \ldots & 0 \\ 0 & t^{k_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t^{k_n} \end{bmatrix}$$

where all $k_i$ are integers different from 0. Clearly, $\mu$ acts on $\mathbb{C}^n_*$ freely in generic point, and the quotient $\mathbb{C}^n_*/\mu(\mathbb{C}^*)$ is well defined. This quotient is known as a weighted projective space, denoted by $\mathbb{C}P^{n-1}(k_1, k_2, \ldots, k_n)$, and it is a projective orbifold. To give a $\mu$-equivariant coherent sheaf on $\mathbb{C}^n_*$ is by definition the same as to give a coherent sheaf on the orbifold $\mathbb{C}P^{n-1}(k_1, k_2, \ldots, k_n)$. Let $R_0$ be the sheaf on $\mathbb{C}P^{n-1}(k_1, k_2, \ldots, k_n)$ corresponding to $R$, considered as a $\mu$-equivariant sheaf on $\mathbb{C}^n_*$. The sections of $R_0 \otimes \mathcal{O}(i)$ correspond to the sections of $R$ on which $\mu(\mathbb{C}^*)$ acts with the weight $i$. We obtain a sequence of finite-dimensional subspaces

$$\Gamma(R_0 \otimes \mathcal{O}(i)) \subset \Gamma(R).$$

Since $\mathcal{O}(1)$ is ample, the sheaf $R_0 \otimes \mathcal{O}(N)$ is globally generated for $N$ sufficiently big (here we use the Kodaira-Nakano theorem for orbifolds, [Ba]). Then $\oplus_{i \leq N} \Gamma(R_0 \otimes \mathcal{O}(i))$ will generate $\Gamma(R)$ over $\mathcal{O}_{\mathbb{C}^n_*}$. Since $\tilde{G}_F$ commutes with $\mu(\mathbb{C}^*)$, the space $\Gamma(R_0 \otimes \mathcal{O}(i)) \subset \Gamma(R)$ is $\tilde{G}_F$-invariant. This proves Lemma 6.6.
Now we can prove the filtrability of arbitrary $R \in \text{Coh}_{\tilde{G}_F}(\mathbb{C}^n)$. By Lemma 6.6, for any $R \in \text{Coh}_{\tilde{G}_F}(\mathbb{C}^n)$, there exists a surjective $\tilde{G}_F$-equivariant map $R_1 \rightarrow R \rightarrow 0$, where $R_1 = \mathcal{O}_{\mathbb{C}^n} \otimes_c W$, and $W$ is a finite-dimensional representation of $\tilde{G}_F$. Since $\tilde{G}_F$ is commutative, $W = \oplus W_i$, where $W_i$ are $\tilde{G}_F$-invariant 1-dimensional subspaces of $W$. This gives an epimorphism

$$\oplus(\mathcal{O}_{\mathbb{C}^n} \otimes W_i) \rightarrow R$$

where all the summands $\mathcal{O}_{\mathbb{C}^n} \otimes W_i$ are $\tilde{G}_F$-equivariant line bundles. Then, $R$ is clearly filtrable within $\text{Coh}_{\tilde{G}_F}(\mathbb{C}^n)$. This proves Theorem 6.5. Theorem 1.2 is also proven.

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References

[Ba] W.L. Baily, *On the imbedding of V-manifolds in projective spaces*, Amer. J. Math. 79 (1957), 403-430.

[BS] Bando, S., Siu, Y.-T, *Stable sheaves and Einstein-Hermitian metrics*, In: Geometry and Analysis on Complex Manifolds, Festschrift for Professor S. Kobayashi’s 60th Birthday, ed. T. Mabuchi, J. Noguchi, T. Ochiai, World Scientific, 1994, pp. 39-50.

[BM1] Vasile Brinzanescu, Ruxandra Moraru, *Stable bundles on non-Kahler elliptic surfaces*, math.AG/0306192, 15 pages.

[BM2] Vasile Brinzanescu, Ruxandra Moraru, *Twisted Fourier-Mukai transforms and bundles on non-Kahler elliptic surfaces*, math.AG/0309031, 13 pages.

[Br] L. Bruasse, *Harder-Narasimhan filtration on non-Kähler manifolds*, Int. Journal of Maths, 12(5):579-594, 2001.

[DO] S. Dragomir, L. Ornea, *Locally conformal Kähler geometry*, Progress in Mathematics, 155. Birkhäuser, Boston, MA, 1998.

[Ga] P. Gauduchon, *La 1-forme de torsion d’une variété hermitienne compacte*, Math. Ann., 267 (1984), 495–518.

[GO] P. Gauduchon and L. Ornea, *Locally conformally Kähler metrics on Hopf surfaces*, Ann. Inst. Fourier 48 (1998), 1107–1127.

[KO] Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformally Kahler manifolds*, math.DG/0105040, 21 pages.

[Ka1] Kato, Ma. *Some remarks on subvarieties of Hopf manifolds*, A Symposium on Complex Manifolds (Kyoto, 1974). Siyrikaisekikenkyūsho Kōkyūroku No. 240 (1975), 64–87.
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[Ka2] Kato, Ma. *On a characterization of submanifolds of Hopf manifolds*, Complex analysis and algebraic geometry, pp. 191–206. Iwanami Shoten, Tokyo, 1977.

[LY] Li, Jun, and Yau, S.-T., *Hermitian Yang-Mills connections on non-Kahler manifolds*, in “Mathematical aspects of string theory” (S.T. Yau ed.), World Scientific Publ., London, 1987, pp. 560-573.

[LT1] Lübke, M., Teleman, A., *The Kobayashi-Hitchin correspondence*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995. x+254 pp.

[LT2] Lübke, M., Teleman, A., *The universal Kobayashi-Hitchin correspondence on Hermitian manifolds*, math.DG/0402341, 90 pages.

[M1] Moraru, R., *Integrable systems associated to a Hopf surface*, Canad. J. Math. **55** (2003), no. 3, 609–635.

[M2] Moraru, R., *Stable bundles on Hopf manifolds*, preprint (2004).

[OSS] Christian Okonek, Michael Schneider, Heinz Spindler, *Vector bundles on complex projective spaces*. Progress in mathematics, vol. 3, Birkhauser, 1980.

[OV1] L. Ornea, M. Verbitsky, *Structure theorem for compact Vaisman manifolds*, math.DG/0305259, Math. Res. Lett. **10** (2003), no. 5-6, 799–805.

[OV2] L. Ornea, M. Verbitsky, *Immersion theorem for Vaisman manifolds*, math.AG/0306077, 28 pages.

[OV3] L. Ornea, M. Verbitsky, *Locally conformal Kähler manifolds with potential*, math.AG/0407231, 11 pages.

[Va1] I. Vaisman, *Generalized Hopf manifolds*, Geom. Dedicata **13** (1982), no. 3, 231–255.

[Va2] I. Vaisman, *A survey of generalized Hopf manifolds*, Rend. Sem. Mat. Torino, Special issue (1984), 205–221.

[Ve1] M. Verbitsky, *Vanishing theorems for locally conformal hyperkähler manifolds*, 2003, math.DG/0302219, 41 pages.

[Ve2] M. Verbitsky, *Stable bundles on positive principal elliptic fibrations*, 17 pages, 2004, math.AG/0403430.

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