Asymptotically FRW black holes

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Special solutions of the LTB family representing collapsing over-dense regions extending asymptotically to an expanding closed, open, or flat FRW model are found. These solutions may be considered as representing dynamical mass condensations leading to black holes immersed in a FRW universe. We study the dynamics of the collapsing region, and its density profile. The question of the strength of the central singularity and its nakedness, as well as the existence of an apparent horizon and an event horizon is dealt with in detail, shedding light to the notion of cosmological black holes. Differences to the Schwarzschild black hole are addressed.

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I. INTRODUCTION

Let us use the term cosmological black hole for any solution of Einstein equations representing a collapsing overdensity region in a cosmological background leading to an infinite density at its center [1]. There have been different attempts to construct solutions of Einstein equations representing such a collapsing central mass. Gluing of a Schwarzschild manifold to an expanding FRW manifold is one of the first attempts to construct such a cosmological black hole, as done first by Einstein and Straus [2]. Depending of the way the model is constructed, one is led to un-physical behavior of the trajectories [3].

Models not based on a cut-and-paste technology is much more interesting giving more information on the behavior of the mass condensation within a FRW universe model. The first attempt is due to McVittie [4] introducing a spacetime metric that represents a point mass embedded in a Friedmann-Robertson-Walker (FRW) universe. There have been many other attempts to construct cosmological black holes such as Nolan interior solution [4], and Sultana-Dyer solution [1], each of them contrasting some of the features one expect from theory or observation.

The interest for cosmological black holes in the past has been mainly from the theoretical side to understand concepts like black hole, singularity, horizon, and thermodynamics of black holes [6]. Indeed, the conventional definition of black holes implies an asymptotically flat space-time and a global definition of the event horizon. In practice, however, the universe is not asymptotically flat. The need for local definition of black holes and their horizons has led to concepts such as Hayward’s trapping horizon [8], Ashtekar’s isolated horizon [9], Ashtekar and Krishnan’s dynamical horizon [10], and Booth and Fairhurst’s slowly evolving horizon [11].

There are cases where both apparent and event horizon maybe possibly defined. For example, for dynamical black holes one may define the event horizon as the very last ray to reach future null infinity or the light ray that divides those observer who cannot escape the future singularity from those that can [26]. Eardley proposed the conjecture that in such cases trapped surfaces can be deformed to get arbitrarily close to the event horizon [12]. Numerical evidence was provided in [13] and later proved analytically for the Vaidya metric [14].

The precision cosmology has opened a new arena for questions like cosmological black holes and their behavior. New observation of our galactic center allow to resolve phenomena near the Schwarzschild horizon of the central black hole [7]. It is therefore desirable to have black hole models embedded in cosmological environment to see if there may be considerable differences to the familiar Schwarzschild black hole. There have been also increasing interest in the gravitational lensing by a cosmological mass condensation such as a cluster of galaxies in a cosmological background. The simplest cases are the Kottler and the Einstein-Straus model [23]. The more complex situation is lensing by a mass condensation within a dynamical background.

Now, a widely used metric to describe the gravitational collapse of a spherically symmetric dust cloud is the so-called Tolman-Bondi-Lemaitre (LTB) metric [17]. These models have been extensively studied for the validity of the cosmic censorship conjecture [18, 19] and [20]. In particular, we know already [21] that, depending upon the initial conditions defined in terms of the initial density and velocity profiles from which the
collapse develops, the central shell-focusing singularity at \( r = 0 \) can be either a black hole or a locally or globally naked singularity. We may note however, that in all these papers a compact LTB region is glued to the Schwarzschild metric or the FRW outer universe \[22\]. Therefore, the results have to be taken cautiously: any principally existent event horizon is cut off by the outer static or homogeneous space-time. The statement may still be correct that in a dynamic spacetime the cosmic censorship hypotheses is valid, as discussed in \[21\]. It is also possible to glue two different LTB metrics to study the structure formation out of an initial mass condensation or the formation of a galaxy with a central black hole \[25\] and \[26\]. Here again the structure of the metric outside the mass condensation is defined by hand to match with a specific galaxy or cluster feature. Faraoni et al have tried to change McVitie metric so that it resemble a collapsing mass condensation. Their hand to match with a specific galaxy or cluster feature. Faraoni et al have tried to change McVitie metric so that it resemble a collapsing mass condensation. Their solution, however, represents a singularity within a horizon embedded in a universe filled with a non-perfect fluid where the change of the mass is not because of the in-falling matter but the heat flow \[22\]. This metric gives us no clue whatsoever about the dynamics of a possible collapsed mass condensation. Harada et al, being interested in the behavior of primordial black holes within cosmological models with a varying gravitational constant, use a LTB solution to study the evolution of a background scalar field when a black hole forms from the collapse of dust in a flat Friedmann universe probing the gravitational memory \[13\].

Our goal is to look for a model of a cosmological black hole, i.e. a mass condensation leading to a singularity within a FRW universe universe. In this paper we propose the models for closed-, open-, and flat FRW universe studying their density profiles, singularities, and horizon behaviors. There are many nontrivial questions to be answered before understanding in detail the differences of these cosmological black holes to the familiar Schwarzschild ones, which are beyond the scope of this paper and are to be dealt with in future publications.

The question of singularities and the definition of a black hole in such a dynamical environment has been subject of different studies in the last 15 years. We review very shortly different definitions of horizons in section II as a reference to the properties of model solutions we propose. Some initial attempts to model black holes within a FRW universe is introduced in section III. Section IV is devoted to the LTB metric as the generic solution representing a spherically symmetric ideal fluid. Section V is devoted to different models of cosmological black holes, their dynamics, density profile, apparent and event horizons, and singularities. The question of strength and the nakedness of singularities are dealt with in section VI. We then conclude in section VII. Throughout the paper we assume \( 8\pi G = c = 1 \).

II. LOCAL DEFINITIONS OF BLACK HOLES

Standard definition of black holes \[6\] needs some global assumptions such as regular predictability and asymptotic flatness. In the cosmological context concepts of asymptotic flatness and regular predictability have no application. This has already been noticed by Demianski and Lasota \[33\] stressing the fact that in the cosmological context the standard global definition of black holes using event horizons may not be used any more. Tipler \[34\] also present a definition of black hole in non asymptotically flat space time, but these definition did not have comprehensive property of black hole such as thermodynamic laws. In the last decade the interest in a local definition of black holes has led to four different concepts based primarily on the concept of the apparent horizon.

Let us start by assuming a spacelike two surface \( S \) with two normal null vectors \( \ell^a \) and \( n^a \) on it. The corresponding expansions are then defined as \( \theta_{(\ell)} \) and \( \theta_{(n)} \).

**Definition 1** \[8\]. A trapping horizon \( H \) is a hypersurface in a 4-dimensional spacetime that is foliated by 2-surfaces such that \( \theta_{(\ell)} |_{H} = 0 \), \( \theta_{(n)} |_{H} \neq 0 \), and \( \mathcal{L}_n \theta_{(\ell)} |_{H} \neq 0 \). A trapping horizon is called outer if \( \mathcal{L}_n \theta_{(\ell)} |_{H} < 0 \), inner if \( \mathcal{L}_n \theta_{(\ell)} |_{H} > 0 \), future if \( \theta_{(n)} |_{H} < 0 \), and past if \( \theta_{(n)} |_{H} > 0 \). The most relevant case in the context of black holes is the future outer trapping horizon (FOTH).

**Definition 2** \[9\]. A weakly isolated horizon is a three-surface \( H \) such that:

1. \( H \) is null;
2. The expansion \( \theta_{(\ell)} |_{H} = 0 \) where \( \ell^a \), being null and normal to the foliations \( S \) of \( H \);
3. \(-T^a_{\ell\ell} \) is future directed and causal;
4. \( \mathcal{L}_a \omega_n = 0 \), where \( \omega_n = -n_b \nabla_a \ell^a \), and the arrow indicates a pull-back to \( H \).

Weakly isolated horizon is a useful term to be used for characterization of black holes not interacting with their surroundings, and corresponds to isolated equilibrium states in thermodynamics. These definition do not apply to cosmological mass condensations because of their dynamical behavior.

**Definition 3** \[10\]. A marginally trapped tube \( T \) (MTT) is a hypersurface in a 4-dimensional spacetime that is foliated by two-surfaces \( S \), called marginally trapped surfaces, such that \( \theta_{(n)} |_{T} < 0 \) and \( \theta_{(\ell)} |_{T} = 0 \). MTTs have no restriction on their signature, which is allowed to vary over the hypersurface. This is a generalization of the familiar concept of the apparent horizon \[10\]. If a MTT is everywhere spacelike it is referred to as a dynamical horizon. If it is everywhere timelike it is called a timelike membrane (TLM). In case it is everywhere null and non-expanding then we have an isolated horizon. The apparent horizons evolving in the our proposed models will not be everywhere spacelike and will have a complex behavior.
Irrespective of different concepts related to the apparent horizon we may still compromise on a definition of event horizon differing principally from the apparent horizon. We follow the definition of [26] as the very last ray to reach future null infinity or the light ray that divides those observer who cannot escape the future singularity from those that can. We will see in the next sections that cosmological black holes may have distinct apparent and even horizons, in contrast to the Schwarzschild black hole.

III. EXISTING METRICS REPRESENTING OVER-DENSITIES WITHIN A COSMOLOGICAL BACKGROUND AND THEIR DEFICIENCIES

A. McVittie’s solutions

In 1933, McVittie [5] found an exact solution of Einstein equations for a perfect fluid mimicking a black hole embedded in a cosmological background. McVittie’s solutions can be written in the form

\[ ds^2 = -\left(1 - \frac{M}{1 + \frac{M}{2N}}\right)^2 dt^2 + e^{\beta(t)}(1 + \frac{M}{2N})^4(dr^2 + \Omega^2), \quad (1) \]

where \( M = me^{\beta(t)/2} \) and \( m \) is a constant. Functions \( h(r) \) and \( N(r) \) depend on a constant \( k \), and are given, respectively, by

\[ h(r) = \begin{cases} \sinh(r) & k = 1 \\ r & k = 0 \\ \sin(r) & k = -1 \end{cases} \]

\[ N(r) = \begin{cases} 2\sinh(r/2) & k = 1 \\ r & k = 0 \\ 2\sin(r/2) & k = -1 \end{cases} \quad (2) \]

This metric represents a point mass embedded into an isotropic universe. It possesses a curvature singularity at proper radius \( R = 2m \), in contrast to the Schwarzschild metric where there is a coordinate singularity. It has been shown that this singularity is space-like and weak [33]. The interpretation of the metric in the region \( R < 2m \) is also not clear [33]. Therefore, the McVittie’s metric is not a suitable solution of Einstein equations to represent the collapse of a spherical mass distribution with overdensity within a cosmological setting.

B. Sultana-Dyer solution

Recently Sultana and Dyer [1] found an exact solution representing a primordial cosmological black hole. It describes an expanding event horizon in the asymptotic background of the Einstein-de Sitter universe. The black hole is primordial in the sense that it forms ab initio with the big bang singularity and therefore does not represent the gravitational collapse of a matter distribution.

This metric is given by

\[ ds^2 = t^4[(1 - \frac{2m}{r})dt^2 - \frac{4m}{r}dt dr - (1 - \frac{2m}{r})dr^2 - r^2d\Omega^2]. \quad (3) \]

Though the metric has the same causal characteristics as the Schwarzschild spacetime, there are significant differences for timelike geodesics. In particular an increase in the perihelion precession and the non-existence of circular timelike orbits should be mentioned. The matter content is described by a non-comoving two-fluid source, one of which is a dust and the other is a null fluid. At late times the dust becomes superluminal near horizon violating the energy condition.

IV. INTRODUCING REALISTIC MODELS OF COSMOLOGICAL MASS CONDENSATION

There maybe different ways of constructing solutions of Einstein equations representing a collapsing mass concentration in a FRW background, as the preceding sections show. We choose the direct way of a cosmological spherical symmetric isotropic solution, and look for an overdensity mass distribution within the model universe undergoing a collapse to see if and how a singularity representing a black hole emerges. To begin with, we choose a so-called flat LTB metric. This is the simplest spherically symmetric solution of Einstein equations representing an inhomogeneous dust distribution [17].

A. LTB metric

The LTB metric may be written in synchronous coordinates as

\[ ds^2 = dt^2 - \frac{R'^2}{1 + f(r)}dr^2 - R(t,r)^2d\Omega^2. \quad (4) \]

It represents a pressure-less perfect fluid satisfying

\[ \rho(r,t) = \frac{2M'(r)}{R'^2R}, \quad \dot{R}^2 = f + \frac{2M}{R}. \quad (5) \]

Here dot and prime denote partial derivatives with respect to the parameters \( t \) and \( r \) respectively. The angular distance \( R \), depending on the value of \( f \), is given by

\[ R = \frac{M}{f}(1 - \cos \eta(t,r)), \]

\[ \eta - \sin \eta = \frac{(-f)^{3/2}}{M}(t - t_n(r)), \quad (6) \]

\[ \dot{R} = (-f)^{1/2} \frac{\sin(\eta)}{1 - \cos \eta}. \quad (7) \]
for $f < 0$, and
\begin{equation}
R = \left(\frac{9}{4}M\right)^{1/3}(t - t_n)^{2/3},
\end{equation}
for $f = 0$, and
\begin{equation}
R = \frac{M}{f}(\cosh \eta(r, t) - 1),
\end{equation}
\begin{equation}
\sinh \eta - \eta = \frac{f^{3/2}}{M}(t - t_n(r)),
\end{equation}
for $f > 0$.

The metric is covariant under the rescaling $r \rightarrow \tilde{r}(r)$. Therefore, one can fix one of the three free parameters of the metric, i.e. $t_n(r)$, $f(r)$, and $M(r)$. The function $M(r)$ corresponds to the Misner-Sharp mass in general relativity, as shown in the general case of spherically symmetric solutions of Einstein equations [16].

There are two generic singularities of this metric: the shell focusing singularity at $R(t, r) = 0$, and the shell crossing one at $R'(t, r) = 0$. To get rid of the complexity of the shell focusing singularity, corresponding to a non-simultaneous big bang singularity, we will assume $t_n(r) = 0$. This will enable us to concentrate on the behavior of the collapse of an overdensity region in an expanding universe without interfering with the complexity of the inherent bang singularity of the metric.

Now, an expanding universe means generally $\dot{R} > 0$. However, in a region around the center it may happen that $\dot{R} < 0$, corresponding to the collapsing region. It is then easy to show that in this collapsing region $\theta_{(i)} \propto (1 - \frac{\sqrt{4 + f}}{\sqrt{1 + f}})$, $\theta_{(n)} \propto (-1 - \frac{\sqrt{4 + f}}{\sqrt{1 + f}}) < 0$. Therefore, $R = 2M$, is obviously a marginally trapped tube, as defined in section 2, representing an apparent horizon according to the familiar definitions [6, 10]. It will turn out that this apparent horizon is not always spacelike and can have a complicated behavior for different $r$, as was first seen in [36].

\section{Behavior of the curvature function $f(r)$}

Now, we are interested in an expanding universe, meaning generally $\dot{R} > 0$. However, in a region around the center we expect to have a late time behavior $\dot{R} < 0$ corresponding to the collapse phase of the overdensity region. From equations (6), (8), and (9) we infer that to have a collapsing region one has to ask for $f(r) < 0$ in that region. In contrast, the universe outside the collapsing region being expanding leave us to choose $f(r) > 0$, $f(r) = 0$, or $f(r) < 0$ depending on the model. We may have an asymptotically flat FRW universe, however, with $f(r) > 0$ or $f(r) < 0$ tending to zero for large $r$.

Now, we have still to make a choice for $f(r)$ at the center $r = 0$. Expecting the mass $M$ to be zero at $r = 0$ to avoid a central singularity, we see from (6) and (9) that $\frac{M^{3/2}}{\dot{R}} |_{r=0} = \text{const}$, or $f(r = 0) = 0$ [30]. This gives us different possibilities of the function $f(r)$ to behave as shown in Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.9\textwidth]{fig1.png}
\caption{Different behaviors of the curvature function $f(r)$.}
\end{figure}

\section{Construction of models}

We have now the necessary prerequisites to construct our models of mass condensation immersed in FRW models leading to singularities and representing cosmological black holes. Our cosmological black hole solutions evolve from mass condensations within closed, open, or flat FRW universes, leading to singularities having different horizons, and providing us examples of collapsed regions behaving differently to known Schwarzschild ones.

\subsection{Example I: $f < 0$: asymptotically closed LTB metric}

As mentioned before, we are free to choose one of the three parameters of the LTB metric. Assuming a negative $f(r)$, we may choose $r$ such that $f(r) = -M(r)/r$ [18]. Now, let us choose the mass function $M$ such that
\begin{equation}
M(r) = 2^a r^2 + r^3 \frac{\alpha + r^3}{1 + r^3},
\end{equation}
where $a$ and $\alpha$ are constants to be defined properly. We then obtain from (6)
\begin{equation}
R = r(1 - \cos \eta(r, t))
\end{equation}
\begin{equation}
\eta - \sin \eta = \sqrt{\frac{7.2 + r^3}{1 + r^3}} 2^{3/2} at.
\end{equation}

We are free to fix $a$ and $\alpha$ such that for the present time, $t_0$, the region around the center of the overdensity, $r = 0$, is collapsing while far from the center the universe
expands. Note that in contrast to the familiar FRW universe, where the scale factor as a function of time, $t$, is an explicit function having a straightforward behavior. In the LTB case, $R(t)$ playing the role of the scale factor is an implicit function of time and comoving coordinate $r$ given by (6). We now fix $a$ and $\alpha$ such that $r = 0$ corresponds to $\eta = \frac{3\pi}{2}$, and $r \gg 1$ corresponds to $\eta = \frac{5\pi}{6}$ (Fig. 2). We then find $a \simeq 0.75 t_0$ with $t_0$ being the present time, and $\alpha \simeq 7$.

Now, the expansion phase of the model is given by $\dot{R}$ (7). We then see from (7) that the region around $r = 0$, corresponding to $\eta \sim \frac{3\pi}{2}$, is always collapsing for any time $t$, while the regions far from the center, $r >> 0$, at the present time, corresponding to $\eta \sim \frac{5\pi}{6}$, are expanding. Note that this bound LTB model, similar to the closed FRW one, has a maximum comoving radius corresponding to $f(r) = -1$.

FIG. 2: Evolution of the Cauchy surfaces.

The density evolution and the causal structure of the model is shown in Fig. 3. We see clearly how the central overdensity region collapses to a singularity at $r = 0$, while the universe is expanding. Note also how the slope of outgoing null geodesics tend to infinity in the vicinity of the singularity, i.e. $R' \to +\infty$ at $R = 0$.

B. Example II: $f < 0$, $\lim_{r \to \infty} f(r) = 0$; asymptotically flat LTB metric

Our favorite choice is a solution representing a collapsing overdensity region at the center and a flat FRW far from the overdensity region. Of course the overdensity region may take part in the expansion of the universe at early times but gradually reversing the expansion and start collapsing. To achieve this, we require $f(r) < 0$ and $f(r) \to 0$ when $r \to \infty$. This choice give us trivially $M(0) = 0$.

Let us now make the ansatz $f(r) = -re^{-r}$ leading to

$$M(r) = \frac{1}{a}r^{3/2}(1 + r^{3/2}),$$

FIG. 3: The case of the asymptotically closed universe: in the central region the density increases with time indefinitely while far from the center the density is decreasing with time. The apparent horizon and the trapped region is shown in the lower diagram.

where $a$ is a constant having the dimension $[a] = [L]^{-2}$. We fix $a$ by $at_0 = 3\pi/2$. Similar to our previous model I, this value of $a$ corresponds to the collapsing mass condensation around $r = 0$ starting in the expanding phase of the bound LTB model.

Equation (6), (8) then leads to

$$R = \frac{\sqrt{r} (1 + r^{3/2})}{ae^{-r}}(1 - \cos \eta(r, t)),$$

$$\eta - \sin(\eta) = \frac{e^{-\frac{3}{2}r}}{(1 + r^{3/2})^2}at.$$  \hspace{1cm} (11)

We have plotted the density evolution and casual structure of this model in Fig. 4.

As a result of $R' \to +\infty$ near the singularity, the slope of the outgoing null geodesics becomes infinite at the central singularity. Again we see clearly how the collapse of the central region and the evolution of the apparent horizon separates the overdense region from the expanding
Let us make the ansatz density less than the critical one.

FRW at large distances from the center, but having a
is positive? We still have a model which tends to a flat
zonz according to the definition of section 2 for any large
central region. We may, however, define an event hori-

closed but asymptotically flat universe. The causal structure
is shown below. Note the behavior of the event horizon for
arbitrary large but finite $t$.

The negativity of the curvature function $f(r)$ means
that, although the universe is asymptotically flat, wait-
ing enough, every slice $r = \text{constant}$ will collapse to the
central region. We may, however, define an event hori-
zon according to the definition of section 2 for any large
but finite time, as shown in Fig[4]

C. Example III: $f > 0$, $f(r) \to 0$ when $r \to \infty$; asymptotically flat LTB metric 2

What would happen if we choose the curvature function $f(r)$ such that it tends to zero for large $r$ while it
is positive? We still have a model which tends to a flat
FRW at large distances from the center, but having a
density less than the critical one.

Let us make the ansatz $f(r) = -r(e^{-r} - \frac{1}{r^{n+c}})$ with $n = 2$
and $c = 20000$, leading to

$$M(r) = \frac{1}{a}r^{3/2}(1 + r^{3/2}),$$

where $a$ is a constant having the dimension $[a] = [L]^{-2}$.
We fix $a$ by requiring $aL_0 = 3\pi/2$. Equation [6], [8] then
leads to

$$R = \frac{\sqrt{7}(1 + r^{3/2})}{a(e^{-r} - \frac{1}{r^{n+c}})(1 - \cos(\eta(t,r))),}$$
$$\eta - \sin\eta = \frac{(e^{-r} - \frac{1}{r^{n+c}})^{1.5}}{(1 + r^{3/2})} \text{at.}$$ (12)

and for $f > 0$ region,

$$R = \frac{\sqrt{7}(1 + r^{3/2})}{a(e^{-r} - \frac{1}{r^{n+c}})(\cosh(\eta(r,t)) - 1)},$$
$$\eta - \sinh\eta = \frac{(e^{-r} - \frac{1}{r^{n+c}})^{1.5}}{(1 + r^{3/2})} \text{at.}$$ (13)

The solution is continuous at $r = 1$, as can be checked by evaluating $\dot{R}$, $R'$, $\dot{R}$, and $\ddot{R}$ at $r = 1$(see the
appendix).

We have plotted the density evolution and casual struc-
ture of this model in Fig[5]

The term $\frac{1}{\sqrt{1 - \phi_4}}$ is responsible for $f(r)$ being positive
and tending to zero for large $r$ given $n \geq 2$ and $c >> 1$.

Let us check if this may cause shell crossing in the region
where $f'(r) < 0$ while $f > 0$. Using [10] we obtain

$$\frac{R'}{R} = \frac{M'}{M}(1 - \phi_4) + \frac{f'}{f} \left(\frac{3}{2}\phi_4 - 1\right),$$ (14)

where $\frac{3}{2} \leq \phi_4 = \frac{\sinh(\eta(t,r)-n)}{(\cosh(\eta(t,r)-1)^{2}} \leq 1$. The condition for
no shell crossing singularity is then $\frac{Mf'}{M} < \frac{1}{\phi_4 - 1}$. For
$\phi_4 \sim 1$, corresponding to $\eta >> 1$ or $t >> 1$ the inequal-
ity breaks down leading to a shell crossing singularity.
The shell crossing, however, can be shifted to arbitrary
large $t$ by choosing $f' << 1$ corresponding to $n >> 1$ and $c >> 1$ [37]. Therefore, for the model we are proposing
the shell crossing will happen out of the range of appli-
cability of it.

As a result of $R' \to +\infty$ near the singularity, the slope of the outgoing null geodesics become infinite at the cen-
tral singularity. Again we see clearly how the collapse
of the central region and the evolution of the apparent
horizon separates the overdense region from the expand-
ing universe. There is an event horizon defined by the
very last ray to reach future null infinity and separates
those observer who can not escape the future singularity
from those that can. A fixed $r = r_0$ value, being the non-
trivial root of $f(r) = 0$, divides the absolute collapsing
region from the absolute expanding region. We may be
living in a region inside the event horizon but outside the
apparent one without noticing it soon!

This solution represents a collapsing mass within an
asymptotically flat FRW universe. The collapsed region
is dynamical in the sense that its mass is not constant.
In fact the rate of change of the Misner-Sharp energy is
given by $\frac{dM(r)}{dt} \big|_{R = \text{const}} = \frac{dM(r)}{dr} \big|_{R = \text{const}} > 0$ because
$\frac{dM(r)}{dr} > 0$, $R'dr + \dot{R}dt = 0$, $R' > 0$, and $\ddot{R} < 0$ for
collapsing region, so $\frac{dR}{dt} \big|_{R = \text{const}} > 0$. Therefore, it is
D. Example IV: \( f > 0 \): asymptotically open FRW metric

Now we look for a solution which goes to an open FRW metric at distances far from the center. At the same time one should take care of the conditions \( M(0) = 0 \) and \( f(0)^{3/2} \neq \infty \). Let us choose

\[
f(r) = -r(1 - r),
\]

and

\[
M(r) = \frac{1}{a} r^{3/2}(1 + r^{3/2}),
\]

where \( a \) is a constant, which may be fixed by assuming \( r = 0 \) at the present time \( t_0 \) corresponding to \( \eta = \frac{3\pi}{2} \). This leads to \( at_0 = 3\pi/2 + 1 \). We then obtain from (9)

\[
R = \sqrt{r(1 + r^{3/2})} \frac{1 - \cos \eta(r,t)}{a(1 - r)},
\]

\[
\eta - \sin(\eta) = \frac{(1 - r)^{3/2}}{(1 + r^{3/2})} at,
\]

for \( r < 1 \), and

\[
R = \sqrt{r^{3/2}} \frac{1 - \cosh \eta(r,t)}{a(r - 1)} (\cosh \eta(r,t) - 1)
\]

\[
sinh \eta - \eta = \frac{(r - 1)^{3/2}}{(1 + r^{3/2})} at,
\]

for \( r > 1 \). The solution is again continuous at \( r = 1 \), as can be checked by evaluating \( \dot{R}, R', \dot{R}', R \), and \( \dot{R} \) at \( r = 1 \) (see the appendix).

The resulting density profile and the causal structure is plotted in Fig. 6. Obviously a singularity at the origin forms gradually while the universe is expanding. The causal structure is also similar to the open but asymptotically flat case.

This solution represents a collapsing mass within an open FRW universe. The collapsed region is again dynamical in the sense that its mass is not constant, and the rate of change of the Misner-Sharp energy is given by the same amount as the previous model. Therefore, concepts of isolated horizon and slowly evolving horizon do not apply to this case.

VI. CHARACTERISTICS OF SINGULARITIES OF PROPOSED MODELS

We have avoided in the models proposed the shell crossing singularities except example III with a late time shell crossing singularity.

The shell focusing singularities, however, are unavoidable and in fact it is what we are looking for to study characteristics of cosmological black holes. An
A. Strength of the shell focusing singularities

Heuristically, a singularity is termed gravitationally strong, or simply strong, if it destroys by crushing or stretching any object which falls into it. The prototype of such a singularity is the Schwarzschild one: a radially infalling object is infinitely stretched in the radial direction and crushed in the tangential directions, with the net result of crushing to zero volume. Otherwise a singularity is termed weak where no object falling into it is destroyed. To check the strength of singularities of our models we use the criteria defined by Clarke [31]. Let \( k^\mu \) be the tangent vector to the ingoing null geodesic, and \( \lambda \) the corresponding affine parameter being zero at the center. \( R_{\mu\nu} \) being the Ricci tensor, the singularity is said to be strong if

\[
\Psi = \lim_{r \to 0} \lambda^2 k^\mu k^\nu R_{\mu\nu} \neq 0. \quad (17)
\]

For a general LTB metric one obtains easily \( k^\mu k^\nu R_{\mu\nu} = 2(k^t)^2 M' R' \). For the three interesting cases of cosmological black holes in flat and open LTB models we have done the calculation along the lines of the [19] using appropriate coordinates near singularity. For our cases (11-13-15) we obtain after some calculation \( \Psi = 0 \) for \( r \to 0 \). Therefore, shell focusing singularities occurring in the center of the models we are proposing are weak. This is in contrast to the Schwarzschild singularity which is a strong one. We leave it to future studies if this weakness is generic of any cosmological black holes.

B. Nakedness of singularities

We know already from Oppenheimer-Snyder collapse of a homogeneous dust distribution how the shells become singular at the same time, and thus none of them crosses. In the case of spherically symmetric inhomogeneous matter configurations, however, the proper time of collapse depends on the comoving radius \( r \). Thus the piling up of neighboring matter shells at finite proper radius can occur, thereby producing two-dimensional caustics where the energy density and some curvature components diverge. These singularities can be locally naked, but they are gravitationally weak [18, 28], i.e. curvature invariants and tidal forces remain finite. It has also been shown that analytic continuations of the metric, in a distributional sense, can always be found in the neighborhood of the singularity [29].

Models proposed in this paper are, however, free from shell crossing singularities. The shell crossing singularity of example III at late times does not influence the following argumentation. Conditions for the absence of shell crossing singularities have been studied in detail in [37]. In our case these conditions are equivalent to \( M'(r) > 0 \) and \( R' > 0 \), which are satisfied by the models discussed above. We may then conclude that
Therefore, the condition for the apparent horizon \( R = 2M \) to be spacelike is, i.e. \(-1 < \frac{\partial R}{\partial t|_{null}} < 1\), leads to the condition \( R' - M' > 0 \), which is not everywhere satisfied in our model. As a result we notice that apparent horizons of the models proposed here are not spacelike everywhere. Such a behavior has already been discussed in [36].

The case of shell focusing singularities is, however, a different one. Irrespective of the behavior of the apparent and event horizons, it is then a relevant question if the shell focusing singularity could be a naked one. We notice that the slope of the outgoing null geodesics at the singularity are greater than the slope of the singularity itself. Therefore, the singularity is spacelike and no timelike or null geodesic can come out of the singularity. We then conclude that the singularities we are facing can not be naked.

VII. DISCUSSION AND CONCLUSIONS

Unlike models discussed so far in the literature, we have constructed models of mass condensation within the FRW universe leading to cosmological black holes without having the usual pathologies we know from other models: the cosmic fluid is dust and ideal producing a singularity at the center in the course of time. The central singularity is spacelike and not naked. In the case of flat or open universe models the singularity is weak and has distinct apparent and event horizons. The apparent horizons are not everywhere spacelike, to be compared with the Schwarzschild one which is null everywhere. This has already been noticed in a general context by [36]. While the apparent horizon is defined by the surfaces \( R = 2M \), similar to the Schwarzschild horizon, the even horizon is further away. Models we have proposed show that one has to expect new effects while considering dynamical cosmic black holes. The simple Schwarzschild static model may not reflect all the phenomena one may expect in observational cosmology, and the black hole thermodynamics. Even the simple concept of mass is not a trivial one in such a dynamical environment. The answer to these questions are beyond the scope of this paper and will be deal with in future publications.

APPENDIX A

The curvature function \( f(r) \) has a zero point where it changes sign for models III and IV, corresponding to two different solutions. Therefore, we have to take care of joining two solutions across the hypersurface defined by \( f(r) = 0 \) to be continuous. This is done by looking at the metric functions and their derivatives to be continuous.

Let us first look at the model IV. There we have to look at the metric function \( R \) and its derivatives, \( R, R', \bar{R}, \bar{R}' \), at the point \( r = 1 \) where \( f \) vanishes. From the following relations derived from the Einstein equations

\[
\ddot{R} = -\frac{M}{R^2},
\]

\[
\ddot{R}' = \frac{M'}{RR} - \frac{MR'}{RR^2} + \frac{f'}{2R},
\]

and

\[
\ddot{R}' = -\frac{M'}{R^2} + \frac{2MR'}{R^3},
\]

we infer that these second derivatives relevant for the Einstein equations to be continuous on the hypersurface \( f(r) = 0 \) are continuous if the \( f, R, R', \bar{R}, M', \) and \( M \) are continuous. Now, because of the continuity of \( f, M', \) and \( M \), we just have to prove the continuity of \( R, \bar{R}, \) and \( R' \).

Let us look first at \( R \) and its derivative \( R' \). In the case of \( r < 1 \) we have

\[
R = \frac{a(r)}{1-r} (1 - \cos \eta),
\]

\[
\eta - \sin \eta = \frac{(1-r)^{1.5}}{b(r)} t,
\]

where \( a(r) = \sqrt{r} + r^2, b(r) = 1 + r^{1.5} \), and \( a(1) = 2, a'(1) = 2.5, b(1) = 2, b'(1) = 1.5 \), and

\[
\bar{R} = \frac{a\sqrt{1-r}}{b} \frac{\sin \eta}{1 - \cos \eta},
\]

\[
\bar{R}' = \frac{a'(1-r) + a'}{(1-r)^{2.5}} (1 - \cos \eta) - \frac{a}{1-r} \frac{\sin \eta }{1 - \cos \eta}^{1.5} \frac{b + b'(1-r)^{1.5}}{b^2} t.
\]

Defining \( 1-r = x \), we have

\[
\eta - \sin \eta = \frac{\eta^3}{6} - O(\eta^5) = \frac{x^3}{2} t.
\]

Therefore, to first order in \( \eta \) we have \( \eta = \sqrt[3]{3t} \sqrt{x} \). Now taking the limit \( x \rightarrow 0^- \) we obtain

\[
\lim_{x \rightarrow 0^-} R(x) = \lim_{x \rightarrow 0^-} \frac{2}{x} (1 - \cos \eta) = \lim_{x \rightarrow 0^-} \left( \frac{\eta^2}{x} - O(\eta^4)/x \right) = (3t)^{2/3}.
\]
which is a well defined quantity.

In the case of \( r > 1 \) we have

\[
R = \frac{a(r)}{r-1} (\cosh \eta - 1),
\]

\[
\sinh \eta - \eta = \frac{(r-1)^{1.5}}{b(r)} t, \tag{A9}
\]

\[
\dot{R} = \frac{a \sqrt{r - 1}}{b \cosh \eta - 1}. \tag{A10}
\]

and

\[
R' = \frac{a'(r-1) - a (\cosh \eta - 1) + \frac{a}{r-1}}{(r-1)^{2}} + \frac{\sinh \eta}{\cosh \eta - 1} \frac{1.5(r-1)^{0.5} b - b'(r-1)^{1.5}}{b^2} t. \tag{A11}
\]

Now, defining \( r - 1 = x \), and noting that

\[
\sinh \eta - \eta = \frac{\eta^3}{6} + O(\eta^5) = \frac{\eta^3}{2} t, \tag{A12}
\]

we obtain to first order of \( \eta \) the relation \( \eta = 3 \sqrt{t} \sqrt{x} \).

Therefore, the continuity of \( R \) across \( r = 1 \) is established. Similar calculation for the first derivatives shows the continuity of \( R' \) and \( \dot{R} \) having well defined values on both sides of the \( r = r_0 \) hypersurface:

\[
R'(1) = 2.5 (3t)^{2/3} - \frac{3}{4 (3t)^{1/3}} t, \tag{A14}
\]

and

\[
\dot{R}(1) = \frac{2}{(3t)^{1/3}}. \tag{A15}
\]

The case of model III is similar except for the hypersurface defined by \( g(r) = \frac{f(r)}{r} = e^{-r} - \frac{1}{r^2 + 2000} = 0 \) with the root of \( e^{-r_0} = \frac{1}{r_0^2 + 2000} = 0 \) being at a point \( r = r_0 \) different from \( r = 1 \). It is easy to see that \( g(r) \) is an analytic function at \( r = r_0 \), and can be approximated by \( g(r) \approx g'(r_0)(r - r_0) + \frac{g''(r_0)}{2}(r - r_0)^2 + \cdots \). Similar calculations verify the continuity of the metric function \( R \) and its relevant derivatives across the hypersurface \( r = r_0 \).

\[\text{[References]}\]

[1] J. Sultana and C.C. Dyer, Gen. Rel. Grav. 37, 1349 (2005).
[2] A. Einstein and E.G. Straus, Rev. Mod. Phys. 17, 120 (1945); 18, 148 (1946).
[3] G.A. Baker Jr., astro-ph/0003152.
[4] B.C. Nolan, J. Math. Phys. 34, 1 (1993).
[5] G.C. McVittie, Mon. Not. R. Astr. Soc. 93, 325 (1933).
[6] McKee, F. W., in Rev. of Astr. Instruments 13, 113 (1962).
[7] Doeleman, S. S. et al. Nature 455, 7880 (2008).
[8] S. A. Hayward, Phys. Rev. D 49, 6467 (1994).
[9] A. Ashtekar, C. Beetle, O. Dreyer, B. Krishnan, J. Lewandowski and J. Wisniewski, Class. Quantum Grav. 25, 12 (2008).
[10] A. Ashtekar and B. Krishnan, Phys. Rev. Lett. 89, 141101 (2002).
[11] Booth I. and Fairhurst S., Phys. Rev. Lett. 92, 011102 (2004).
[12] D. Eardley, Phys. Rev. D 57, 2299 (1998).
[13] Erik Schnetter and Badri Krishnan, Phys. Rev. D 73, 021502, (2006).
[14] Ishai Ben-Dov, Phys. Rev. D 75, 064007, (2007).
[15] Tomohiro Harada, C. Goymer and B.J. Carr, Phys. Rev. D66, 104023 (2002)
[16] Misner C W and Sharp D H 1964 Phys. Rev. 136 B571.
[17] R. C. Tolman, Proc. Natl. Acad. Sci. U.S.A. 20, 410 (1934); H. Bondi, Mon. Not. R. Astron. Soc. 107, 343 (1947); G. Lemaitre, Ann. Soc. Sci. Bruxelles I A53, 51 (1933).
[18] P. S. Joshi and I. H. Dwivedi, Phys. Rev. D 47, 5357 (1993); R. P. A. C. Newman, Class. Quantum Grav. 3, 527 (1986); D. Christodoulou, Commun. Math. Phys. 93, 171 (1984); D. M. Eardly and L. Smarr, Phys. Rev. D 19, 2239 (1979).
[19] P. S. Joshi and T. P. Singh, Phys. Rev. D 51, 6778 (1995).
[20] A. Chamorro, S.S. Deshingkar, I.H. Dwivedi, and P.S. Joshi, Phys. Rev. D 63, 084018 (2001).
[21] I. H. Dwivedi and P. S. Joshi, Class. Quantum Grav. 14, 1223 (1997); S. S. Deshingkar, S. Jhingan, and P. S. Joshi, Gen. Relativ. Gravit. 30, 1477 (1998).
[22] Khakshournia and R. Mansouri, Phys. Rev. D 65, 027302, (2002); V.A. Berezin, V.A. Kuzmin, and I.I. Tkachev, Phys. Rev. D 36, 2919 (1987).
[23] W. Rindler, M. Ishak, Phys. Rev. D 76 043006 (2007).
[24] R. M. Wald, (gr-qc/9710068).
[25] A. Krasinski and C. Hellaby, Phys. Rev. D 65, 023501 (2002); C. Hellaby and A. Krasinski, Phys. Rev. D 73, 023518 (2004).
[26] A. Krasinski and C. Hellaby, Phys. Rev. D 69, 023502 (2004);
[27] V. Faraoni and A. Jacques, Phys Rev D 78, 024008 (2008).
[28] B. C. Nolan, Phys. Rev. D 60, 02014 (1999).
[29] P. Yodzis, H. J. Seifert, and H. M. zum Hagen, Comm. Math. Phys. 34, 135 (1973); A. Papapetrou and Hamoui, Ann. Inst. Henri Poincare VI, 343 (1967).
[30] N. Mustapha and C. Hellaby, Gen. Rel. Grav. 33, 455-77 (2001).
[31] C. J. S. Clarke, Class. Quant. Grav. 10, 1375 (1993); F. J. Tipler, Phys. Lett. A 64A, 8 (1977); C. J. S. Clarke and A. Krolak, J. Geom. Phys. 2, 127 (1985).
[32] W. B. Bonnor, Class. Quant. Grav. 16, 1313 (1999); J. M. M. Senovilla and R. Vera, Phys. Rev. Lett. 78, 2284 (1997); M. Mars, Phys. Rev. D 57, 3389 (1998).
[33] Demianski M. and Lasota J. P. Nature Phys. Sci. 241, 53 (1973).
[34] Frank J. Tipler. Nature 270, 500 (1977).
[35] B. C. Nolan, Class. Quantum Grav. 16, 1227 (1999); B. C. Nolan, Phys. Rev. D 58, 064006 (1998); Class. Quantum Grav. 16, 3183 (1999).
[36] I. Booth, L. Brits, J. A. Gonzalez, and C. Van Den Broeck. Class. Quant. Grav. 23 413-440 (2006).
[37] C. Hellaby and K. Lake, Astrophys. J. 290, 381 (1985); 300, 461(E) (1985).