EQUIVARIANT COHOMOLOGY OF RATIONALLY SMOOTH GROUP EMBEDDINGS

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Abstract. We describe the equivariant cohomology ring of rationally smooth normal projective embeddings of a reductive group. These embeddings are obtained as projectivizations of reductive monoids. Our main result describes their equivariant cohomology in terms of roots, idempotents, and underlying monoid data. Also, we characterize those embeddings whose equivariant cohomology ring is obtained via restriction to the associated toric variety. Such characterization is given in terms of the corresponding cross section lattice.

Introduction and Statement of the Main Results

Let $G$ be a complex connected reductive algebraic group, $B$ a Borel subgroup of $G$, and $T \subset B$ a maximal torus of $G$. Let $W$ denote the Weyl group of $(G, T)$.

An irreducible complex algebraic variety $X$ is called an embedding of $G$, or a group embedding, if $X$ is a normal $G \times G$-variety containing an open orbit isomorphic to $G$ itself, where $G \times G$ acts on $G$ by left and right multiplication.

Let $M$ be a reductive monoid with zero and unit group $G$. Then there exists a central one-parameter subgroup $\epsilon : \mathbb{C}^* \to G$, with image $Z$, such that $\lim_{t \to 0} \epsilon(t) = 0$. Moreover, the quotient space

$$
\mathbb{P}_\epsilon(M) := (M \setminus \{0\})/Z
$$

is a normal projective variety on which $G \times G$ acts via

$$
G \times G \times \mathbb{P}_\epsilon(M) \to \mathbb{P}_\epsilon(M), \ (g, h, [x]) \mapsto [gxh^{-1}].
$$

Hence, $\mathbb{P}_\epsilon(M)$ is a projective embedding of the quotient group $G/Z$. Embeddings of the form $\mathbb{P}_\epsilon(M)$ are called standard group embeddings. These varieties were introduced by Renner in his study of algebraic monoids (R3, R6, R8). It is known that all projective embeddings of a connected reductive group are standard (Theorem 1.17). More generally, any group embedding is locally isomorphic to an open invariant subset of a standard embedding (Theorem 1.19).

Let $X$ be a complex algebraic variety of dimension $n$. We say that $X$ is rationally smooth at $x$, if there exists a neighborhood $U$ of $x$ (in the complex topology) such that, for all $y \in U$, we have

$$
H^m(X, X - \{y\}) = (0) \text{ if } m \neq 2n, \text{ and } H^{2n}(X, X - \{y\}) = \mathbb{Q}.
$$
Such varieties satisfy Poincaré duality with rational coefficients [M]. See [Br4] for an up-to-date discussion of rationally smooth singularities on complex algebraic varieties with torus action.

Using chiefly methods from the theory of algebraic monoids, Renner ([R6], [R8]) investigated those standard embeddings that are rationally smooth. This class is larger than the class of smooth group embeddings.

Goresky, Kottwitz and MacPherson in their seminal paper [GKM], developed a theory, nowadays called GKM theory, that makes it possible to describe the equivariant cohomology of certain $T$-skeletal varieties: projective algebraic varieties upon which an algebraic torus $T$ acts with a finite number of fixed points and weighted invariant curves. Cohomology, in this article, is considered with rational coefficients. Let $X$ be a $T$-skeletal variety and denote by $X_T$ the fixed point set. The main purpose of GKM theory is to identify the image of the functorial map

$$i^*: H^*_T(X) \to H^*_T(X_T),$$

assuming $X$ is equivariantly formal. This condition is equivalent to the vanishing of the odd dimensional Betti numbers of $X$, for $X_T$ is finite (Theorem 2.5). GKM theory asserts that if $X$ is a GKM variety, i.e. $T$-skeletal and equivariantly formal, then the equivariant cohomology ring $H^*_T(X)$ can be identified with certain ring of piecewise polynomial functions $PP_T^*(X)$ (Theorem 2.7). In the case of standard group embeddings, it is possible to determine $PP^*_{T \times T}(\mathbb{P}_\epsilon(M))$ in terms of combinatorial data obtained directly from the underlying two-sided action $G \times G \times \mathbb{P}_\epsilon(M) \to \mathbb{P}_\epsilon(M)$.

Two subclasses of standard embeddings have been extensively studied via GKM theory: projective regular embeddings and simplicial toric varieties ([Br1], [U]). The former are smooth and, due to the Bialynicki-Birula decomposition [BB], do not have cohomology in odd degrees. The latter have quotient singularities, are rationally smooth, and have non-zero Betti numbers only in even degrees, as proved by Danilov [D]. It follows from these observations that both classes are equivariantly formal. Even more so, the GKM data issued from the $T \times T$-fixed points and $T \times T$-invariant curves has been explicitly obtained [Br3]. As a consequence, the structure of their corresponding equivariant cohomologies has been completely determined. See [DP-1], [BDP], [LP], [Br3], [VV] and [U] for up-to-date information on these compactifications.

In this paper we use the methods developed in our previous work [G-2] to describe combinatorially the equivariant cohomology of rationally smooth standard embeddings. There GKM theory was applied to the study of $Q$-filtrable varieties. We briefly recall the results of [G-2] that are relevant to the case at hand. A normal projective $T$-variety $X$ is called $Q$-filtrable if it has a finite number of $T$-fixed points $x_1, \ldots, x_m$ and the cells

$$W_i = \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x = x_i\}$$
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of the associated Bialynicki-Birula decomposition are all rationally smooth. The BB-cells of a \( \mathbb{Q} \)-filtrable variety are called rational cells. Next, let \( X = \mathbb{P}_\epsilon(M) \) be a rationally smooth standard embedding. Then \( T \times T \) acts on \( X \) with a finite number of fixed points. In fact, \( X^{T \times T} \) corresponds to \( \mathcal{R}_1 \), the rank-one elements of the Renner monoid. Let \( m \) be the cardinality of \( \mathcal{R}_1 \). It follows from Theorem 7.3 of [G-2] that \( X \) is \( \mathbb{Q} \)-filtrable. Therefore, it admits a filtration into closed subvarieties \( X_i, i = 0, \ldots, m \), where

\[ \emptyset = X_0 \subset X_1 \subset \ldots \subset X_{m-1} \subset X_m = X, \]

such that each cell \( C_i = X_i \setminus X_{i-1} \) is a rational cell, for \( i = 1, \ldots, m \). Moreover, since these rational cells behave topologically like even-dimensional cells of a CW complex (when working with rational cohomology), the singular cohomology of \( X_i \) vanishes in odd degrees, for every \( i = 1, \ldots, m \). In other words, each \( X_i \) is equivariantly formal. As a consequence, the canonical map

\[ H^*_T \times_T(\mathbb{P}_\epsilon(M)) \longrightarrow H^*_T \times_T(\mathcal{R}_1) = \bigoplus_{i=1}^m H^*_T \times_T \]

is injective. This is our motivation and starting point.

The purpose of this article is two-fold. First, we show that a rationally smooth standard embedding \( X = \mathbb{P}_\epsilon(M) \) is not only a \( \mathbb{Q} \)-filtrable variety, but also a GKM variety. Secondly, we provide a precise combinatorial description of \( H^*_T \times_T(X) \). Our goals are attained by writing down all the associated GKM data explicitly, in terms of roots, idempotents and the Renner monoid. Our findings increase the applicability of GKM theory in the study of singular group embeddings.

This article is organized as follows. In Section 1 we briefly review the theory of reductive monoids. Using previous results of Brion, Renner and Rittatore, we show that any projective \( G \)-embedding is standard (Theorem 1.17). More generally, we show that any \( G \)-embedding is a union of embeddings that are open invariant subsets of standard embeddings (Theorem 1.19). Section 2 introduces GKM theory and recollects some of its vital properties. Sections 3, 4 and 5 contain our main results. Before stating them, let us introduce some of the notation from Section 1. Let \( M \) be a reductive monoid with zero and unit group \( G \). Let \( E(\mathbb{T}) \) be the idempotent set of the associated torus embedding \( \mathbb{T} \), that is, \( E(\mathbb{T}) = \{ e \in \mathbb{T} | e^2 = e \} \). One defines a partial order on \( E(\mathbb{T}) \) by declaring \( f \leq e \) if and only if \( fe = f \). Denote by \( \Lambda \subset E(\mathbb{T}) \), the cross section lattice of \( M \). The Renner monoid \( \mathcal{R} \subset M \) is a finite monoid whose group of units is \( W \) (the Weyl group) and contains \( E(\mathbb{T}) \) as idempotent set. In fact, any \( x \in \mathcal{R} \) can be written as \( x = fu \), where \( f \in E(\mathbb{T}) \) and \( u \in W \). Recall that \( W \) is generated by reflections \( \{ s_\alpha \}_{\alpha \in \Phi} \). Denote by \( \mathcal{R}_k \) the set of elements of rank \( k \) in \( \mathcal{R} \), that is, \( \mathcal{R}_k = \{ x \in \mathcal{R} | \dim Tx = k \} \). Analogously, one has \( \Lambda_k \subset \Lambda \) and \( E_k \subset E(\mathbb{T}) \).
Let \( X = \mathbb{P}_e(M) \) be a standard group embedding. In Section 3, we devote ourselves to computing the finite GKM data coming from the \( T \times T \)-fixed points and \( T \times T \)-invariant curves of \( X \). The computations in this section are independent of whether or not \( X \) is rationally smooth. Our main results here are summarized as follows.

**Theorem 3.1, 3.5** Let \( X = \mathbb{P}_e(M) \) be a standard group embedding. Then its natural \( T \times T \)-action

\[
\mu : T \times T \times \mathbb{P}_e(M) \to \mathbb{P}_e(M), \quad (s, t, [x]) \mapsto [sx^{-1}t]
\]

is \( T \times T \)-skeletal. Indeed, after identifying the elements \( x \) of \( \mathcal{R}_1 \) with their corresponding images \([x]\) in \( X \), the set \( X^{T \times T} \) corresponds to \( \mathcal{R}_1 \). As for the closed \( T \times T \)-curves of \( X \), they fall into three types:

1. \( \{ U_{l(e)}(w), e \in E_1(T), s_{\alpha} \notin C_W(\epsilon) \text{ and } w \in W. \}
2. \( \{ w, U_{l(e)}(e) \in E_1(T), s_{\alpha} \notin C_W(\epsilon) \text{ and } w \in W. \}
3. \( \{ T \times T[e] = [T x[T] = [x T] = [T x], \text{ where } x \in \mathcal{R}_2. \}

In particular, rationally smooth standard embeddings are GKM varieties. The curves of type 1 and 2 lie entirely in closed \( G \times G \)-orbits, whereas the curves of type 3 do not. Curves of type 3 can be further separated into whether or not the corresponding \( T \times T \)-fixed points are in the same closed \( G \times G \)-orbit.

In Section 4, the main part of this paper, we identify explicitly all the characters associated to the GKM curves of Theorem 3.5. From there we proceed to write down the \( T \times T \)-equivariant cohomology of a rationally smooth standard embedding as a complete combinatorial invariant of the underlying monoid. Let \( \Lambda_1 \) be the set of rank-one idempotents of the cross-section lattice \( \Lambda \). Remarkably, each closed \( G \times G \)-orbit of \( X \) can be written uniquely as \( G[e]G \simeq G/P_e \times G/P_e^{-} \), where \( e \in \Lambda_1 \), and \( P_e, P_e^{-} \) are opposite parabolic subgroups (Propositions 1.16, 2.10). Our main result in Section 4 is a generalization of [1373], Theorem 3.1.1. It asserts the following.

**Theorem 4.10.** Let \( X = \mathbb{P}_e(M) \) be a rationally smooth standard group embedding. Then the natural map

\[
H^*_{T \times T}(X) \to H^*_{T \times T}(\bigsqcup_{e \in \Lambda_1} G[e]G) = \bigoplus_{e \in \Lambda_1} H^*_{T \times T}(G[e]G)
\]

is injective. In fact, its image consists of all tuples \((\varphi_e)_{e \in \Lambda_1}\), indexed over \( \Lambda_1 \) and with \( \varphi_e \in H^*_{T \times T}(G[e]G) \), subject to the additional conditions:

1. If \( f \in E_2(T) \) and \( H_f = \{ f, s_{\alpha_f} f \} \), with \( s_{\alpha_f} f = f s_{\alpha_f} \neq f \), then
   \[
   \varphi_e(f_1 u) \equiv \varphi_e(f_2 u) \mod (\alpha_f, \alpha_f \circ \mathrm{int}(u)),
   \]
   for all \( u \in W \). Here, \( f_1 \) and \( f_2 = s_{\alpha_f} \cdot f_1 \cdot s_{\alpha_f} \) are the two idempotents in \( E_1(T) \) below \( f \), the root \( \alpha_f \) corresponds to the reflection \( s_{\alpha_f} \), and \( e_f \in \Lambda_1 \) is the unique element of \( \Lambda_1 \) which is conjugate to \( f_1 \).
(2) If \( f \in E_2(\mathcal{T}) \) and \( H_f = \{ f \} \), then
\[
\varphi_{e_1}(f_1u) \equiv \varphi_{e_2}(f_2u) \mod (\lambda_f, \lambda_f \circ \text{int}(u)),
\]
for all \( u \in W \). Here, \( \lambda_f \) is the character of \( T \) defined in Lemma 4.3, the idempotents \( f_1, f_2 \) are the unique idempotents below \( f \), and \( e_i \in \Lambda_1 \) is conjugate to \( f_i \), for \( i = 1, 2 \).

Let \( X \) be a \( G \)-variety. A broadly known result of Borel asserts that \( H^*_G(X) \) can be read off from \( H^*_T(X) \) by computing invariants:
\[
H^*_G(X) \simeq H^*_T(X)^W.
\]
This observation and Theorem 4.10 yield to the following.

**Corollary 4.11.** Let \( X = \mathbb{P}_e(M) \) be a rationally smooth standard group embedding. Then the ring \( H^*_{G \times G}(X) \) consists of all tuples \( (\Psi_e)_{e \in \Lambda_1} \), where
\[
\Psi_e : W e W \rightarrow (H^*_T \otimes H^*_T)^{C_W(e) \times C_W(e)},
\]
such that
(a) If \( f \in E_2(\mathcal{T}) \) and \( H_f = \{ f, s_{\alpha_f} f \} \), then
\[
\Psi_e(f_1) \equiv \Psi_e(f_2) \mod (\alpha_f, \alpha_f),
\]
where \( e \in \Lambda_1 \) is conjugate to \( f_1, f_2 = s_{\alpha_f} \cdot f_1 \cdot s_{\alpha_f} \), the reflection \( s_{\alpha_f} \in C_W(f) \) is associated with the root \( \alpha_f \), and \( f_i \leq f \).
(b) If \( f \in E_2 \) and \( H_f = \{ f \} \), then
\[
\Psi_e(f_1) \equiv \Psi_{e'}(f_2) \mod (\lambda_f, \lambda_f),
\]
where \( \lambda_f \in \Xi(T) \), and \( f_1, f_2 \leq f \) are conjugate to \( e \) and \( e' \), respectively.

Associated to \( X = \mathbb{P}_e(M) \), there is a standard torus embedding \( \mathcal{Y} \) of \( T/Z \), namely,
\[
\mathcal{Y} = \mathbb{P}_e(\mathcal{T}) = [\mathcal{T} \setminus \{ 0 \}] / \mathbb{C}^*.
\]
By construction, \( \mathcal{Y} \) is a normal projective torus embedding and \( \mathcal{Y} \subseteq X \).

Our second major theorem in Section 4 allows to compare the equivariant cohomologies of \( X \) and its associated torus embedding \( \mathcal{Y} \subseteq X \). The situation for standard embeddings contrasts deeply with the corresponding one for regular embeddings ([Br3], Corollary 3.1.2; [U], Corollary 2.2.3).

**Theorem 4.12.** The inclusion of the associated torus embedding \( \iota : \mathcal{Y} \hookrightarrow X \) induces an injection:
\[
\iota^* : H^*_{G \times G}(X) \rightarrow H^*_{T \times T}(\mathcal{Y})^W \simeq (H^*_T(\mathcal{Y}) \otimes H^*_T)^W,
\]
where the \( W \)-action on \( H^*_T(\mathcal{Y}) \) is induced from the action of \( \text{diag}(W) \) on \( \mathcal{Y} \). Furthermore, \( \iota^* \) is an isomorphism if and only if \( C_W(e) = \{ 1 \} \) for every \( e \in \Lambda_1 \).

It is also possible to characterize, in terms of closed \( G \times G \)-orbits, those embeddings for which the map \( \iota^* \) of Theorem 4.12 is an isomorphism.
Corollary 4.15. Let $X = \mathbb{P}_\epsilon(M)$ be a rationally smooth standard embedding. Let $\mathcal{Y}$ be the associated torus embedding and $\iota : \mathcal{Y} \to X$ the canonical inclusion. Then the following are equivalent:

(a) The induced map $\iota^* : H^*_G \times G(X) \to H^*_T \times T(\mathcal{Y})^W \simeq (H^*_T(\mathcal{Y}) \otimes H^*_T)^W$ is an isomorphism.

(b) $C^*_W(e) = \{1\}$ for every $e \in E_1(T)$.

(c) All closed $G \times G$-orbits in $X$ are isomorphic to $G/B \times G/B^\perp$.

A rationally smooth standard embedding satisfying any of the equivalent conditions of Corollary 4.15 is called a quasi-regular embedding. The choice of terminology comes from the fact that all projective regular embeddings satisfy Corollary 4.15. It is worth noting, however, that our notion of quasi-regular embedding is of a more combinatorial nature and, for instance, does not require any special conditions on the boundary divisors of $X \setminus (G/Z)$.

Hence, we have supplied the theory of embeddings with an interesting class of test spaces. We conclude Section 4 by extending a result of De Concini and Procesi [DP-2, Theorem 2.2], to quasi-regular embeddings.

Corollary 4.17. Let $M$ be a reductive monoid with zero and unit group $G$. Let $K$ be a maximal compact subgroup of $G$ such that $T_K = T \cap K$ is a maximal compact torus. Suppose that the associated standard embedding $X = \mathbb{P}_\epsilon(M)$ is quasi-regular. Then

$$H^*(X) \simeq H^*((K \times K) \times (T_K \times T_K) \mathcal{Y})^W,$$

where $\mathcal{Y} \subset X$ is the associated toric variety.

Finally, in Section 5, we illustrate the theory with a detailed study of simple projective embeddings (Theorem 5.5).

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1. REDUCTIVE MONOIDS AND STANDARD GROUP EMBDIEeMENTS

Throughout this article, we consider linear algebraic group actions on algebraic varieties over the field $\mathbb{C}$ of complex numbers.

1.1. Algebraic Monoids. We state a few crucial results from the theory of algebraic monoids that will be relevant to our study. For a complete treatment of the subject, the reader is cordially invited to consult [R9] and [Pu].

Definition 1.1. A linear algebraic monoid $M$ is an affine, irreducible, algebraic variety together with an associative morphism $\mu : M \times M \to M$ and an identity element $1 \in M$ for $\mu$. A linear algebraic monoid $M$ is called reductive if it is normal, and its unit group is a reductive algebraic group. A reductive monoid is called semisimple if it has a zero element, and its unit group has a one-dimensional center.
Let $M$ be a linear algebraic monoid. Denote by $G$ its unit group and by $T$ a maximal torus of $G$. There is a natural $G \times G$-action on $M$ given by $(g, h) \cdot a = gah^{-1}$. Let $\mathcal{U}(M)$ be the set of orbits $O = GaG$ which contain an idempotent. The set of idempotents in $M$ is typically denoted by $E(M)$.

The next two results can be found in [R9].

**Theorem 1.2.** Let $M$ be a linear algebraic monoid with zero. Then the following conditions are equivalent:

1. $M$ is reductive,
2. $M = GE(M)$,
3. $\mathcal{U}(M)$ is the set of $G \times G$-orbits in $M$.

**Theorem 1.3.** Let $M$ be a reductive monoid with zero. Let $G$ be its group of units. Then the set of $G \times G$-orbits is finite, and every $G \times G$-orbit contains an idempotent.

Throughout this article we concentrate on reductive monoids.

Let $M$ be a reductive monoid with 0. The results of Putcha ([Pu]) and Renner ([R9]) provide a characterization of the Zariski closure of $T$ in $M$, namely,

$$\overline{T} = C_M(T) = \{x \in M \mid xt = tx, \forall t \in T\}. $$

Notice that $\overline{T}$ is a reductive monoid. Furthermore, $\overline{T}$ is an affine toric variety.

The set of $G \times G$-orbits, $\mathcal{U}(M)$, is often called the set of $J$-classes. In fact, $\mathcal{U}(M)$ is a finite poset:

$$MaM \leq MbM \iff GaG \subset GbG.$$ 

One defines a partial order on $E(\overline{T})$, the set of idempotents of $\overline{T}$, by declaring $f \leq e$ if and only if $ef = f = fe$.

The Weyl group $W = N_G(T)/T$ is a finite group generated by reflections $\{s_\alpha\}_{\alpha \in \Phi}$, where $s_\alpha$ corresponds to reflection with respect to the hyperplane defined by $\alpha$. Here $\Phi$ denotes the set of roots of $G$ relative to $T$ ([Bo2]). By definition, $\Phi \subset \Xi(T)$, where $\Xi(T)$ is the character group of $T$. In this context, there are two important results of Putcha ([Pu]) and Renner ([R9]) that we state here.

**Theorem 1.4.** Any idempotent of $M$ is conjugate to one in $\overline{T}$, that is,

$$E(M) = \bigcup_{g \in G} gE(\overline{T})g^{-1}. $$

Additionally, if $e, f \in E(\overline{T})$ are conjugate under $G$, then they are also conjugate under $W$.

**Theorem 1.5.** Let $M$ be a reductive monoid with zero. Suppose $e$ and $f$ are idempotents of $M$. Then $GeG = GfG$ if and only if $e$ and $f$ are conjugate under $G$. 

All the structures just described are strongly intertwined, as the following theorem shows.

**Theorem 1.6.** Let $M$ be a reductive monoid. Then, there are bijections

$$U(M) \leftrightarrow E(M)/G \leftrightarrow E(T)/W$$

given by

$$GeG \leftrightarrow \{geg^{-1} \mid g \in G\} \leftrightarrow \{wew^{-1} \mid w \in W\}$$

for $e \in E(T)$, where $E(M)/G$ denotes the set of $G$-conjugacy classes in $E(M)$ and $E(T)/W$ denotes the set of $W$-conjugacy classes in $E(T)$.

**Proof.** It follows from Theorems 1.3 and 1.4 that any $G \times G$-orbit can be written as $GeG$, for some idempotent $e \in E(T)$. Now the map on the left is both well-defined and bijective in virtue of Theorems 1.4 and 1.5. Finally, the map on the right is a well-defined bijection due to Theorem 1.4. \[\square\]

Fix a Borel subgroup $B$ of $G$. Define $\Lambda$, the cross section lattice of $M$ relative to $T$ and $B$, by the following formula

$$\Lambda := \{e \in E(T) \mid Be = eBe\}.$$

It turns out that $\Lambda$ can be identified with the set of $G \times G$-orbits in $M$. Therefore,

$$M = \bigsqcup_{e \in \Lambda} GeG.$$

On the other hand, because of Theorem 1.6, we can also identify $\Lambda$ with the set of $W$-orbits in $E(T) = \{e \in T \mid e^2 = e\}$.

Let $R = N_G(T) \subset M$. Then, for all $x \in R$, one has $xT = Tx$ and $x = wt$, where $w \in N_G(T)$ and $t \in T$. Concisely, $R = \{x \in M \mid Tx = xT\}$.

The Renner monoid, $\mathcal{R}$, is defined to be $\mathcal{R} := R/T$. It is a finite regular monoid. More concretely, any $x \in \mathcal{R}$ can be written as $x = fu$, where $f \in E(T)$ and $u \in W$. Besides,

$$\mathcal{R} = \bigsqcup_{e \in \Lambda} WeW,$$

where $\Lambda$ is the cross-section lattice. It is known that $WeW$ has a unique minimal element: there exists a unique $\nu \in WeW$ for which $B\nu = \nu B$. See [R9] for the details.

We should also emphasize that the Renner monoid $\mathcal{R}$ corresponds to the set of $B \times B$-orbits in $M$. In fact, there is an analogue of the Bruhat decomposition for reductive monoids:

$$M = \bigsqcup_{r \in \mathcal{R}} BrB.$$

Denote by $\mathcal{R}_k$ the set of elements of rank $k$ in $\mathcal{R}$, that is,

$$\mathcal{R}_k = \{x \in \mathcal{R} \mid \dim Tx = k\}.$$ 

Analogously, one defines $\Lambda_k \subset \Lambda$ and $E_k \subset E(T)$. 

For any given idempotent \( e \in E(M) \), one can define the following opposite parabolic subgroups of \( G \):

\[
P_e = C^r_G(e) = \{ g \in G \mid ge = ege \},
\]

and

\[
P^{-}_e = C^l_G(e) = \{ g \in G \mid eg = ege \},
\]

they are called right and left centralizer of \( e \), respectively. Their intersection,

\[
C_G(e) = \{ g \in G \mid ge = eg \},
\]

is called the centralizer of \( e \) in \( G \). It can be shown \( [Pu] \) that \( C_G(e) \) is a common Levi factor of \( P_e \) and \( P^{-}_e \); so \( C_G(e) \) is a connected reductive subgroup of \( G \).

**Theorem 1.7** \([R9]\). Let \( M \) be a reductive monoid with unit group \( G \) and cross section lattice \( \Lambda \). Let \( e \in \Lambda \).

1. Define \( eMe = \{ x \in M \mid x = exe \} \). Then \( eMe \) is a reductive algebraic monoid with unit group \( H_e := e \cdot C_G(e) \) and unit element \( e \). A cross section lattice of \( eMe \) is

\[
e\Lambda = \{ f \in \Lambda \mid ef = f \}.
\]

2. Define \( Me = \{ x \in G \mid ex = xe = e \}^\circ \). Then \( Me \) is a reductive algebraic monoid with zero \( e \in M \) and unit group \( G_e = \{ x \in G \mid ex = xe = e \}^\circ \). A cross section lattice for \( Me \) is

\[
\Lambda_e = \{ f \in \Lambda \mid fe = e \}.
\]

**Remark 1.8.** Semisimple monoids are classified numerically. Let \( M \) be a semisimple monoid with unit group \( G \). Associated to \( M \) is its polyhedral root system \((\Xi(T), \Phi, C)\). Here \( \Xi(T) \) is the character group of a maximal torus \( T \subset G \), \( \Phi \) is the set of roots, and \( C \) is the collection of all \( \chi \in \Xi(T) \) that extend to \( \overline{\chi} : T^* \to \mathbb{C} \). Renner has shown that the association

\[
M \longleftrightarrow (\Xi(T), \Phi, C)
\]

between semisimple monoids and polyhedral root systems is bijective on isomorphism classes \( [R1] \).

1.2. **Group Embeddings.** Let \( G \) be a connected reductive group. Consider the action of \( G \times G \) on \( G \) given by left and right multiplication:

\[
(a, b) \cdot g = a g b^{-1}.
\]

This action is transitive, and the isotropy group of \( 1 \) is \( \Delta(G) \), the diagonal of \( G \).

**Definition 1.9.** An *embedding* of \( G \) is a normal irreducible variety \( X \) equipped with an action of \( G \times G \) and containing the homogeneous space \( G = (G \times G) / \Delta(G) \) as an open orbit. In other words, \( X \) is a normal \( G \times G \)-variety containing an open orbit isomorphic to \( G \) itself, where \( G \times G \) acts on \( G \) by left and right multiplication. For brevity, we say that \( X \) is a *group embedding*, or a *\( G \)-embedding* if we wish to specify the group \( G \).
An embedding of $G$ is called simple if it contains only one closed $G \times G$-orbit.

Let $M$ be a reductive monoid with unit group $G$. Then $M$ is a simple affine $G$-embedding (e.g. [Ri]). Even more is true, as the following result of Rittatore [Ri] shows. It places algebraic monoids at the core of embedding theory.

**Theorem 1.10.** Reductive monoids are exactly the affine embeddings of reductive groups. The commutative reductive monoids are exactly the affine embeddings of tori.   

Before characterizing the projective embeddings of reductive groups, we state here two fundamental results of Sumihiro [Su]. They will play a crucial role in the sequel.

**Theorem 1.11.** Let $G$ be a connected linear algebraic group. Let $X$ be a quasi-projective normal $G$-variety. Then there is a finite dimensional rational representation $G \rightarrow GL(V)$ and a $G$-equivariant isomorphism of $X$ with a locally closed $G$-stable subvariety of the projective space $\mathbb{P}(V)$.

**Theorem 1.12.** Let $G$ be a connected linear algebraic group (resp. a torus group) and $X$ a normal $G$-variety. Then $X$ has an open covering which consists of $G$-stable quasi-projective (resp. affine) open subsets of $X$.

Therefore, every algebraic action of a connected linear algebraic group on a normal variety is obtained by patching finitely many linear actions on normal quasi-projective varieties.

**Corollary 1.13.** Let $G$ be a connected reductive group. Let $X$ be a simple $G$-embedding. Then $X$ is quasi-projective.

**Proof.** Let $O_x$ be the unique closed $G \times G$-orbit of $X$. By Theorem 1.12, there exists an invariant quasi-projective open neighborhood $U$ of $O_x$. We claim that $U = X$. Otherwise, $X \setminus U$ would be a non-empty, closed, invariant subvariety of $X$ and, as such, it would contain a closed $G \times G$-orbit different from $O_x$. This is impossible, for $O_x$ is the only closed $G \times G$-orbit in $X$.

1.3. **Standard Group Embeddings.**

**Definition 1.14.** Let $M$ be a reductive monoid with unit group $G$ and zero element $0 \in M$. There exists a central one-parameter subgroup $\epsilon : \mathbb{C}^* \rightarrow G$ with image $Z$, that converges to 0 ([Br6], Lemma 1.1.1). Then $\mathbb{C}^*$ acts attractively on $M$ via $\epsilon$, and hence the quotient $\mathbb{P}_\epsilon(M) = [M \setminus \{0\}] / \mathbb{C}^*$ is a normal projective variety. See Section 1.3 of [Br4]. Notice also that $G \times G$ acts on $\mathbb{P}_\epsilon(M)$ via $G \times G \times \mathbb{P}_\epsilon(M) \rightarrow \mathbb{P}_\epsilon(M)$, $(g, h, [x]) \mapsto [gxh^{-1}]$.

Furthermore, $\mathbb{P}_\epsilon(M)$ is a group embedding of the reductive group $G/Z$. In the sequel, $X = \mathbb{P}_\epsilon(M)$ will be called a **Standard Group Embedding**.
Let $B$ be a Borel subgroup of $G$. Recall that $M$ contains a finite number of $G \times G$-orbits and $B \times B$-orbits, indexed by $\Lambda$ and $R$, respectively. It is clear that $X = \mathbb{P}_e(M)$ inherits such property as well. Indeed, the set of $G \times G$-orbits of $X$ is indexed by $\Lambda \setminus \{0\}$. Similarly, the $B \times B$-orbits of $X$ are indexed by $R \setminus \{0\}$. With these identifications, the set of closed $G \times G$-orbits of $X$ corresponds to $\Lambda_1$.

When $M$ is semisimple (in which case $\epsilon$ is essentially unique), we write $\mathbb{P}(M)$ for $\mathbb{P}_e(M)$. Indeed, for such a monoid, $Z \simeq \mathbb{C}^*$ is the connected center of the unit group $G$ of $M$. Thus, a semisimple monoid with unit group $G$ can be thought of as an affine cone over some projective embedding $\mathbb{P}(M)$ of the semisimple group $G_0 = G/Z$.

For an up-to-date description of these and other embeddings, see [AB].

**Example 1.15.** Let $G_0$ be a semisimple algebraic group over the complex numbers and let $\rho : G_0 \to \text{End}(V)$ be a representation of $G_0$. Define $Y_\rho$ to be the Zariski closure of $G = |\rho(G_0)|$ in $\mathbb{P}(\text{End}(V))$, the projective space associated with $\text{End}(V)$. Finally, let $X_\rho$ be the normalization of $Y_\rho$. By definition, $X_\rho$ is an standard group embedding of $G$. Notice that $M_\rho$, the Zariski closure of $\mathbb{C}^*\rho(G_0)$ in $\text{End}(V)$, is a semisimple monoid whose group of units is $\mathbb{C}^*\rho(G_0)$. Embeddings of this kind will be studied in more detail in Section 5.

Next is a structural description of the $G \times G$-orbits in a standard embedding.

**Proposition 1.16.** Let $M$ be a reductive monoid with zero and $G$ be its unit group. Let $\epsilon \neq 0$ be an idempotent of $E(T)$. Consider $\mathbb{P}_e(M)$ as above. Then the $G \times G$ orbit of $[\epsilon]$ in $X$ fits into the fibration sequence

$$H_\epsilon/\mathbb{C}^* \epsilon \longrightarrow G[\epsilon]G \longrightarrow \pi G/P_\epsilon \times G/P_\epsilon^\perp .$$

Here $H_\epsilon := e \cdot C_G(\epsilon)$. In particular, if $\epsilon$ has rank one, then

$$G[\epsilon]G \simeq G/P_\epsilon \times G/P_\epsilon^\perp ,$$

for, in this case, $eMe \simeq \mathbb{C}$, $H_\epsilon \simeq e \times \mathbb{C}^*$ and $P_\epsilon \cdot e = \mathbb{C}^* \cdot e$.

**Proof.** Notice that $\text{Stab}_{G \times G}(\epsilon)$, the $G \times G$-stabilizer of $\epsilon \in M$, is contained in the subgroup $P_\epsilon \times P_\epsilon^\perp$. To see this, let $(g, h) \in \text{Stab}_{G \times G}(\epsilon)$. Then $geh^{-1} = e$, that is $egeh^{-1} = e^2$, but $e$ is an idempotent, so $egeh^{-1} = e$. The latter yields $ege = eh$, and the term on the right hand side equals $ge$, by assumption. We conclude that $ege = ge$. Analogously, $eh = che$.

Since $\text{Stab}_{G \times G}(\epsilon) \subset P_\epsilon \times P_\epsilon^\perp$, the map $\pi$ is the natural map of homogeneous spaces, and therefore it is a fibration with fibre $(P_\epsilon \times P_\epsilon^\perp)/\text{Stab}_{G \times G}(\epsilon)$. But the fibre it is easily seen to be isomorphic to $e \cdot C_G(\epsilon)$, where

$$C_G(\epsilon) = \{g \in G \mid ge = eg\} .$$

After taking the quotient by the $\mathbb{C}^*$-action, we obtain the result.
It is well-known (e.g. see [Br3]) that the closed $G \times G$-orbits of a regular embedding are all of the form $G/B \times G/B$. Proposition 1.16 makes explicit the difference between standard group embeddings and projective regular embeddings.

The following theorem first appeared in [R3] for the case of semisimple varieties, i.e. projective embeddings of semisimple groups. M. Brion pointed out to us that Renner’s results extend to all normal projective embeddings of reductive groups.

**Theorem 1.17.** Let $X$ be a projective embedding of a connected reductive group $G$. Then $X$ is a standard.

**Proof.** (After [R3] and [Ri]) From Theorem 1.11 we know that $X$ admits a $G \times G$-equivariant closed embedding into a projective space $\mathbb{P}^N$ with a linear action. Let $\pi : \mathbb{C}^{N+1} \setminus \{0\} \rightarrow \mathbb{P}^N$ be the canonical ample $\mathbb{C}^*$-bundle and let $\tilde{X} = \pi^{-1}(X) \cup \{0\}$ be the associated cone over $X$. Put $\tilde{G} = G \times \mathbb{C}^*$. We may regard $\tilde{X}$ as a $\tilde{G} \times \tilde{G}$-variety, where $\mathbb{C}^* \times \mathbb{C}^*$ acts via its morphism $(t, s) \mapsto ts^{-1}$ to $\mathbb{C}^*$. It follows that $\tilde{X}$ is an affine embedding of the reductive group $\tilde{G}$. Moreover, $\tilde{X}$ admits the structure of a reductive monoid (Theorem 1.10). Consequently, $X$ is a standard embedding of $G$. □

**Corollary 1.18.** Standard embeddings are exactly the projective embeddings of connected reductive groups.

Let $M$ and $M'$ be two semisimple monoids with unit group $G$. The standard embeddings $\mathbb{P}(M)$ and $\mathbb{P}(M')$ are (equivariantly) isomorphic if and only if their polyhedral root systems are isomorphic [R3].

Next, we show that any embedding of a connected reductive group is locally standard, as pointed out to us by L. Renner.

**Theorem 1.19.** Any (group) embedding is a union of embeddings that are open invariant subsets of standard embeddings.

**Proof.** Let $G$ be a connected reductive group and let $X$ be a $G$-embedding. First, we claim that any open invariant subset of $X$ contains $G$. Indeed, let $U$ be an open invariant subset of $X$. Clearly, $U \cap G \neq \emptyset$, for $X$ is irreducible and $G$ is an open orbit. So let $x_0 \in U \cap G$. Since $U$ is invariant, it contains the orbit of any of its points; in particular, it contains $G$, the orbit of $x_0$.

Secondly, by Theorem 1.12 and the previous claim, $X$ admits an open covering by quasi-projective embeddings of $G$. Let $\{U_i\}$ be this covering. In view of Theorem 1.11, each $U_i$ can be thought of as a locally closed invariant subvariety of some projective space $\mathbb{P}^{N_i}$ with a linear $G \times G$-action. Denote by $X_i$ the Zariski closure of $U_i$ in $\mathbb{P}^{N_i}$. Now let $\rho_i : Y_i \rightarrow X_i$ be the normalization of $X_i$. It follows from Theorem 1.17 and the universal property of normalizations, that $Y_i$ is a standard embedding of $G$ containing $U_i \cong \rho_i^{-1}(U_i)$ as an open invariant set. □
In conclusion, standard group embeddings form a very natural class from the viewpoint of embedding theory.

2. **Equivariant Cohomology and GKM theory**

Cohomology is always considered with rational coefficients.

### 2.1. Equivariant Cohomology

Let $G$ be a connected reductive group and let $X$ be a $G$-variety, that is, a complex algebraic variety with an algebraic action of $G$. Let $G \rightarrow EG \rightarrow BG$ be a universal principal bundle for $G$. The **equivariant cohomology** of $X$ is defined to be

$$H^*_G(X) := H^*(X_G),$$

where $X_G = (X \times EG)/G$ is the total space associated to the fibration

$$X \xrightarrow{p_X} X_G \rightarrow BG.$$

This construction was introduced by Borel [Bo1]. Here, $BG$ is simply connected, the map $p_X$ is induced by the canonical projection $EG \times X \rightarrow EG$, and $G$ acts diagonally on $EG \times X$. Notice that $H^*_G(X)$ is, via $p_X^*$, an algebra over $H^*_G(pt)$. To simplify notation, we sometimes write $H^*_G$ instead of $H^*_G(X)$. See [Hs] for more details on equivariant cohomology.

**Example 2.1.** Let $T = (\mathbb{C}^*)^r$ be an algebraic torus. Then $BT = (\mathbb{C}P^\infty)^r$, and consequently $H^*_T(pt) = H^*(BT) = \mathbb{Q}[x_1, \ldots, x_r]$, where $\text{deg}(x_i) = 2$. A more intrinsic description of $H^*_T(pt)$ is given as follows. Denote by $\Xi(T)$ the character group of $T$. Any $\chi \in \Xi(T)$ defines a one-dimensional complex representation of $T$ with space $\mathbb{C}_\chi$. Here $T$ acts on $\mathbb{C}_\chi$ via $t \cdot z := \chi(t)z$. Consider the associated complex line bundle

$$L(\chi) := (E_T \times_T \mathbb{C}_\chi \rightarrow BT)$$

and its first Chern class $c(\chi) \in H^2(BT)$. Let $S$ be the symmetric algebra over $\mathbb{Q}$ of the group $\Xi(T)$. Then $S$ is a polynomial ring on $r$ generators of degree 1, and the map $\chi \rightarrow c(\chi)$ extends to a ring isomorphism

$$c : S \rightarrow H^*_T(pt)$$

which doubles degrees: the **characteristic homomorphism** ([Br2]).

The following is a result of Borel (see [Bo1] or [Br2], Prop. 1). It shows that equivariant cohomology for a connected reductive group can be described in terms of equivariant cohomology for a maximal torus.

**Theorem 2.2.** Let $G$ be a connected reductive group and let $T \subset G$ be a maximal torus with Weyl group $W$. Let $X$ be a $G$-variety. Then the group $W$ acts on $H^*_T(X)$ and we have an isomorphism

$$H^*_G(X) \cong H^*_T(X)^W.$$

In particular, $H^*_G(pt)$ is isomorphic to $S^W$, where $S$ denotes the symmetric algebra of the character group $\Xi(T)$ (occurring in degree 2), and $S^W$ the ring of $W$-invariants in $S$. 

□
Thus $H^*_G = H^*(BG)$ is zero in odd degrees.

**Lemma 2.3.** Let $G$ be a connected reductive group with maximal torus $T$. There is a graded $W$-submodule $R$ of $H^*_T$, isomorphic to the regular representation of $W$, such that

$$H^*_T \simeq R \otimes (H^*_T)^W$$

as graded $(H^*_T)^W$-modules.

**Proof.** Let $K$ be a maximal compact subgroup of $G$ such that $T_c = T \cap K$ is a maximal compact torus. Given that $G/K$ is contractible, one concludes that $H^*_G$ is isomorphic to $H^*_K$. Likewise, $T/T_c$ is contractible and so $H^*_T \simeq H^*_T$. Let $B$ be a Borel subgroup of $G$ containing $T$. By the Iwasawa decomposition, $G/B \simeq K/T_c$. Moreover, due to the Bruhat decomposition, $K/T_c$ has no cohomology in odd degrees. It follows from the degeneration of the spectral sequence associated to the fibration $K/T_c \hookrightarrow BT_c \to BK$ that

$$H^*(BT_c) \simeq H^*(K/T_c) \otimes \mathbb{Q} H^*(BK),$$

or, what is the same,

$$H^*_T \simeq H^*(K/T_c) \otimes \mathbb{Q} (H^*_T)^W.$$

That is, $H^*_T$ is a free $(H^*_T)^W$-module and $H^*(K/T_c) = H^*_T / (H^*_T)^W$, where $(H^*_T)^W$ denotes the ideal of $H^*_T$ generated by all homogeneous $W$-invariants of positive degree. A well-known result of Leray ([Bo3], Proposition 20.2) now implies that the representation of $W$ in $H^*(K/T_c)$ is isomorphic to the regular representation. Setting $R = H^*(K/T_c)$ concludes the proof. \[\square\]

2.2. **GKM theory.** GKM theory is a relatively recent tool that owes its name to the work of Goresky, Kottwitz and MacPherson [GKM]. This theory encompasses techniques that date back to the early works of Atiyah, Segal, Borel ([Bo1]) and Chang-Skjelbred ([CS]).

**Definition 2.4.** Suppose an algebraic torus $T$ acts on a (possibly singular) space $X$. Let $p_X : X_T \to BT$ be the fibration associated to the Borel construction. We say that $X$ is **equivariantly formal** if the Serre spectral sequence

$$E_2^{p,q} = H^p(BT) \otimes \mathbb{Q} H^q(X) \implies H_2^{p+\cdot}(X)$$

for this fibration degenerates at $E_2$.

Equivalently, $X$ is equivariantly formal if the $H^*_T$-module $H^*_T(X)$ is free and the map $H^*_T(X) / \mathcal{I} H^*_T(X) \to H^*_T(X)$, induced by restriction to the fiber, is an isomorphism. Here $\mathcal{I}$ denotes the ideal of $H^*_T$ generated by the elements of strictly positive degree.

The following theorem characterizes equivariant formality when the fixed point set is finite. For a proof, see [GKM], Theorem 1.6.2, or [Br5], Lemma 1.2.
Theorem 2.5. Let $X$ be a $T$-variety with a finite number of fixed points. Then the following are equivalent:

(a) $X$ is equivariantly formal.
(b) $H^*_T(X, \mathbb{Q})$ is a free $H^*_T(pt)$-module of rank $|X^T|$, the number of fixed points.
(c) The singular cohomology of $X$ vanishes in odd degrees.

If $X^T$ is finite and $X$ has no cohomology in odd degrees (i.e., $X$ is equivariantly formal), then, by the Borel-Atiyah-Segal localization theorem ([Hs], Theorem III.1), one concludes that the map $i^*: H^*_T(X) \rightarrow H^*_T(X^T)$, induced by the canonical inclusion $i: X^T \hookrightarrow X$, is injective. Moreover, since $H^*_T(X^T) = \bigoplus_{x \in X^T} H^*_T$, then $H^*_T(X)$ is a certain subring of polynomial functions. Identifying the image of $i^*$ is one of the achievements of GKM theory.

Definition 2.6. Let $X$ be a projective $T$-variety. Let $\mu: T \times X \rightarrow X$ be the action map. We say that $\mu$ is a $T$-skeletal action if

1. $X^T$ is finite, and
2. The number of one-dimensional orbits of $T$ on $X$ is finite.

In this context, $X$ is called a $T$-skeletal variety. If a $T$-skeletal variety $X$ has no cohomology in odd degrees, then we say that $X$ is a GKM variety.

Let $X$ be a normal projective $T$-skeletal variety. Then $X$ has an equivariant embedding into a projective space with a linear action of $T$ ([Su], Theorem 1). Moreover, it is possible to define a ring $PP^*_T(X)$ of piecewise polynomial functions. Indeed, let $R = \bigoplus_{x \in X^T} R_x$, where $R_x$ is a copy of the polynomial algebra $H^*_T$. We then define $PP^*_T(X)$ as the subalgebra of $R$ defined by

$$PP^*_T(X) = \{(f_1, \ldots, f_n) \in \bigoplus_{x \in X^T} R_x \mid f_i \equiv f_j \mod(\chi_{i,j})\}$$

where $x_i$ and $x_j$ are the two distinct fixed points in the closure of the one-dimensional $T$-orbit $C_{i,j}$, and $\chi_{i,j}$ is the character of $T$ associated with $C_{i,j}$. The character $\chi_{i,j}$ is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite).

Theorem 2.7 ([CS], [GKM]). Let $X$ be a normal projective $T$-skeletal variety. Suppose that $X$ is a GKM variety. Then the restriction mapping $H^*_T(X) \rightarrow H^*_T(X^T) = \bigoplus_{x \in X^T} H^*_T$ is injective, and its image is the subalgebra $PP^*_T(X)$.

Remark 2.8. Let $T$ be a maximal torus of a connected reductive group $G$. Suppose that $X$ is a $G$ space. Then, by Theorem 2.2, the $G$-equivariant cohomology of $X$ is given by the invariants under the Weyl group, namely, $H^*_G(X) \simeq (H^*_T(X))^W$. 

The formula of Theorem 2.7 is compatible with the action of $W$ given that $W$ permutes the $T$-fixed points and the one-dimensional $T$-orbits. So Theorem 2.7 can be used to calculate the $G$-equivariant cohomology of $X$ as well.

Let $G$ be a connected reductive group with maximal torus $T$. Suppose that $X$ is a $G$-variety such that $X^T$ is finite and $X$ has no cohomology in odd degrees. The next result gives some insight on the module structure of $H^*_G(X)$.

**Lemma 2.9.** Let $X$ be a $G$-variety. Suppose that $X$ has no cohomology in odd degrees and that, for the induced $T$-action, $X^T$ is finite. Then $H^*_G(X)$ and $H^*_T(X)$ are free modules over $H^*_G$ and $H^*_T$, respectively, and their ranks satisfy

$$\text{rank}_{H^*_G} H^*_G(X) = \dim_{\mathbb{Q}} H^*(X) = \text{rank}_{H^*_T} H^*_T(X) = |X^T|.$$ 

**Proof.** The hypotheses imply that $X$ is equivariantly formal for the induced $T$-action, hence $H^*_T(X) \cong H^*_T \otimes_{\mathbb{Q}} H^*(X)$ as free $H^*_T$-modules. Similarly, since both $BG$ and $X$ have cohomology concentrated only in even degrees, then the Serre spectral sequence associated to the fibration $p_X : X_G \to BG$ degenerates and gives $H^*_G(X) \cong H^*(X) \otimes_{\mathbb{Q}} H^*_G$. As a consequence,

$$\text{rank}_{H^*_G} H^*_G(X) = \text{rank}_{H^*_T} H^*_T(X) = \dim_{\mathbb{Q}} H^*(X).$$

Finally, by the localization theorem for torus actions ([Hs], Theorem III.1), we conclude that

$$\text{rank}_{H^*_T} H^*_T(X) = |X^T|.$$

□

Examples of GKM varieties include smooth projective $T$-skeletal varieties, flag varieties, Schubert varieties and, more generally, $T$-skeletal $\mathbb{Q}$-filtrable varieties [G2].

Let $M$ be a reductive monoid with zero and let $X = \mathbb{P}_{\varepsilon}(M)$ be the associated standard group embedding. Let $\Lambda$ be the cross section lattice of $M$. Recall that $\Lambda$ corresponds to the partially ordered set of $G \times G$-orbits in $M$. Under this identification, closed $G \times G$-orbits in $\mathbb{P}_{\varepsilon}(M)$ correspond to idempotents $\varepsilon \in \Lambda_1$. As an application of GKM theory, we finish this section by describing $H^*_{T \times T}(G[\varepsilon]G)$, where $G[\varepsilon]G$ is a closed $G \times G$-orbit of $X$.

**Proposition 2.10.** Let $G[\varepsilon]G$ be a closed $G \times G$ orbit in $X$ (in other words, $\varepsilon \in \Lambda_1$). Then $H^*_{T \times T}(G[\varepsilon]G)$ consists of all maps $\varphi : WeW \to H^*_T \otimes H^*_T$ such that

i) $\varphi(ew) \equiv \varphi(s_\alpha ew) \mod (\alpha, 1)$ for $s_\alpha \notin C_W(\varepsilon)$.

ii) $\varphi(we) \equiv \varphi(wes_\alpha) \mod (1, \alpha)$ for $s_\alpha \notin C_W(\varepsilon)$.

**Proof.** It follows from Proposition 1.16 that $G[\varepsilon]G$ is isomorphic to the complete homogeneous space $G/P_\varepsilon \times G/P_\varepsilon^-$ with vanishing odd cohomology. The $T \times T$-fixed points of $G[\varepsilon]G$ are then given by $WeW$. By Lemma 2.11 below,
the $T \times T$-curves of $G[e]G$ are given by $U_\alpha ew$, with $s_\alpha \notin C_W(e)$ and $weU_\alpha$, with $s_\alpha \notin C_W(e)$. Curves of the former type join the fixed points $ew$ and $s_\alpha ew$. As for the latter type, they join $we$ to $wes_\alpha$. Theorem 2.7 now yields the result.

The following is a result of Carrell [C]. For a proof, see [C] or Lemma 2.2 of [CK].

**Lemma 2.11.** Let $x$ be a $T$-fixed point of the homogeneous variety $G/P$, where $P$ is a parabolic subgroup of $G$. Then every closed irreducible $T$-stable curve $C$ passing through $x$ has the form $C = U_\alpha x$ for some $\alpha \in \Phi$. Moreover, $C_T = \{x, s_\alpha x\}$, and each such $C$ is smooth. □

3. **GKM Data of a Standard Group Embedding**

Let $M$ be a reductive monoid with unit group $G$ and zero element $0 \in M$. Let $\epsilon : \mathbb{C}^* \to Z$ be an attractive one-parameter subgroup in the center of $G$ and consider the standard group embedding $X = \mathbb{P}_\epsilon(M)$. The purpose of this section is to write out the GKM data of $X$ (i.e. the $T \times T$-fixed points and $T \times T$-invariant curves) in terms of the standard combinatorial invariants of $M$. In fact, we will show that any standard group embedding contains only a finite number of $T \times T$-fixed points and $T \times T$-invariant curves. This calculation does not depend on any special property of $M$. Thus there is no harm in such a calculation even though it does not always yield a recipe for $H^*_T(\mathbb{P}_\epsilon(M))$. Later on, we specialize it to the case of rationally smooth embeddings.

Our initial task is to identify the following two sets.

1. $\{x \in M \mid \dim TxT = 1\}$.
2. $\{x \in M \mid \dim TTx = 2\}$.

The first class will determine the set $X^{T \times T}$ of $T \times T$-fixed points and the second one will determine the set $\mathcal{C}(X, T \times T)$ of $T \times T$-invariant curves.

3.1. **Fixed Points.** Let $\mathcal{R} = \{x \in M \mid Tx = xT\}/T = N_G(T)/T$ be the Renner monoid and let $\mathcal{R}_1 = \{x \in \mathcal{R} \mid \dim(Tx) = 1\}$ be the set of rank-one elements of $\mathcal{R}$. We will identify $\mathcal{R}_1$ with its image in $\mathbb{P}_\epsilon(M)$ and simply write $\mathcal{R}_1 \subseteq \mathbb{P}_\epsilon(M)$.

**Theorem 3.1.** $\mathcal{R}_1 \subseteq \mathbb{P}_\epsilon(M)$ is the set of fixed points of $T \times T$ acting on $\mathbb{P}_\epsilon(M)$. Hence, there is only a finite number of $T \times T$-fixed points in $\mathbb{P}_\epsilon(M)$.

**Proof.** The set of fixed points of $T \times T$ on $\mathbb{P}_\epsilon(M)$ corresponds to

$\{x \in M \mid \dim(TxT) = 1\}$.

Notice that if $\dim(Tx) = 1$, then $Tx = Zx$. Similarly, if $\dim(xT) = 1$, then $xT = Zx$. These remarks, together with the fact that $Tx \cup xT \subseteq TTx$, yield the equality

$\{x \in M \mid \dim(TxT) = 1\} = \{x \in M \mid Tx = xT \text{ and } \dim(Tx) = 1\},$
where the latter set is precisely $R_1$.

### 3.2. Invariant Curves

**Proposition 3.2.** Let $x \in M$ and assume that $x \neq 0$. Then the following are equivalent.

1. $\dim TxT = 2$.
2. Either $\dim(xT) = 2$ and $Tx \subseteq xT$, $xT = TxT$; or $\dim(TxT) = 2$ and $xT \subseteq Tx$, $Tx = TTx$; or $\dim(TxT) = 2$ and $xT = TTx$.

**Proof.** It is simple to check that 2. implies 1. For the converse, assume that 1. holds. Now $Tx \cup xT \subseteq TTx$. If $\dim(Tx) = \dim(xT) = 1$, then $Tx = Zx = xT$, where $Z \subseteq T$ is the given attractive one-parameter subgroup of the center of $G$. But then $\dim(TxT) = 1$, a contradiction. Hence at least one of $Tx$ or $xT$ is two-dimensional. If $\dim(TxT) = 2$, then $Tx \subseteq TTx$ yet they have the same dimension. Thus $Tx = TTx$. If $\dim(xT) = 2$, then we end up with $xT = TTx$. □

**Corollary 3.3.** Exactly one of the following assertions is true for $x \in M$ such that $\dim(TxT) = 2$.

1. $xT \subset Tx = TTx$ and $\dim(xT) = 1$.
2. $Tx \subset xT = TTx$ and $\dim(Tx) = 1$.
3. $xT = Tx = TTx$.

□

The following is a result of Renner ([R4], Lemma 3.3). We include a proof for the convenience of the reader.

**Lemma 3.4.** Let $M$ be a reductive monoid with zero and unit group $G$. Let $T \subseteq G$ be a maximal torus. Choose a central one-parameter subgroup $\epsilon : \mathbb{C}^* \to G$, with image $Z$, that converges to 0. Then

$$\{x \in M \setminus \{0\} \mid Zx = Tx\} = \bigsqcup_{e \in E_1(T)} eG.$$

Consequently, if $X = \mathbb{P}_e(M) = (M \setminus \{0\})/\mathbb{C}^*$ and $eX = (eM \setminus \{0\})/\mathbb{C}^* \simeq eG/Z$ then

$$X^T = \bigsqcup_{e \in E_1(T)} eX$$

for the action $T \times X \to X$ given by $(t,[x]) \sim [tx]$. Similar results hold for the right action $([x],t) \sim [tx]$ of $T$ on $X$.

**Proof.** We reproduce Renner’s argument ([R4]). Let $x \in M \setminus \{0\}$ be such that $Zx = Tx$. Since $x \neq 0$ by Theorem 3.4 of [R4] there is an $e \in E_1$ such that $ex \neq 0$ (that $M$ is semisimple is not needed here). By the monoid Bruhat decomposition [R2] we can write $x = brb'$ where $b,b' \in B$ and $r \in R$. Then we let $y = xb'^{-1} = br$. Write $r = fw$ where $f \in E(\overline{T})$ and $w \in W$. Then $fy = fbr = fbfr = fcr = fcw$ for some $c \in C_B(f)$. In particular $fy \in fg$. Thus, by Proposition 3.22 of [R9], if $f \notin E_1$ then $\dim(Tfy) > 1$. Thus $Zfy \subset Tfy$. Thus $Zy \subset Ty$ since $\dim(Ty) \geq \dim(Tfy)$. This is
impossible. We conclude that \( f = e \in E_1 \). Thus, if \( t \in T \) and \( tbe = be \), then \( tebe = etbe = ebe \). In particular \( te = e \). But \( \dim \{ t \in T \mid tbe = be \} = \dim \{ t \in T \mid te = e \} = \dim T - 1 \). In particular \( T_e \subseteq \{ t \in T \mid tbe = be \} \), and consequently \( e \in \{ t \in T \mid tbe = be \} \). Thus \( ebe = be \). Therefore \( y \in eM \), and finally \( x = yb' \in eM \).

**Theorem 3.5.** Notation being as above, there are three types of closed irreducible \( T \times T \)-curves in \( X = P_e(M) \).

1. \( U_\alpha e \), \( s_\alpha \notin C_W(e) \) and \( w \in W \) (fixed pointwise by \( T \) on the right).
2. \( weU_\alpha \), \( s_\alpha \notin C_W(e) \) and \( w \in W \) (fixed pointwise by \( T \) on the left).
3. \( T[x] = [x]T \) where \( x \in R_2 = \{ x \in R \mid \dim(Tx) = 2 \} \).

Thus, there is only a finite number of \( T \times T \)-invariant curves in \( X = P_e(M) \).

**Proof.** Keeping the numeration of Corollary 3.3, we know that the \( T \times T \)-curves of \( X = P_e(M) \) fall into three classes. The first two types correspond, as Lemma 3.4 dictates, to curves that are fixed pointwise by \( T \) on either the left or the right. The former collection lies on \( X^T = \bigsqcup_{e \in E_1(T)} eG/Z \).

Moreover, due to the Bruhat decomposition, for each \( e \in E_1(T) \) the following identity holds

\[
eG/Z = G/P_e = \bigsqcup_{r \in eW} [r]B_u,
\]

where \( B_u \) is the unipotent radical of \( B \).

Our task is to find all the \( T \)-curves of \( eG/Z \), where \( e \) varies over all the rank-one idempotents of \( T \). So fix an idempotent \( e \in E_1(T) \). It follows from the results of Carrell (Lemma 2.11), that the \( T \)-curves of \( eG/Z \) are of the form \([r]U_\alpha\), for some root \( \alpha \) such that \( s_\alpha \notin C_W(f) \) and \( f = w^{-1}ew \). Indeed, since \( f \) is a rank-one idempotent, then \( s_\alpha \in C_W(f) \) if and only if \( U_\alpha f = fU_\alpha = \{ f \} \) (\[G-2\], Lemma 5.1). Because there is no essential difference between \( e \) and \( f \), we conclude that a \( T \times T \)-curve, \( TxT \), is fixed pointwise on the left by \( T \) if and only if \( TxT = wU_\alpha \), where \( \alpha \notin C_W(f) \), \( f \in E_1(T) \), and \( w \in W \). A similar argument disposes of the case when a \( T \times T \)-curve is fixed pointwise by \( T \) on the right.

Finally, if \( Tx = xT = TxT \) and \( \dim(Tx) = 2 \), then \( x \in R_2 \). Identifying \( x \in R_2 \) with its image \([x] \) in \( X = P_e(M) \), it is clear that \( T[x]T \) is a \( T \times T \)-curve in \( X \).

Let us state Theorem 3.1 and Theorem 3.5 in a more compact form.

**Theorem 3.6.** Let \( X = P_e(M) \) be a standard group embedding. Then its natural \( T \times T \)-action

\[
\mu : T \times T \times P_e(M) \to P_e(M), \quad (s, t, [x]) \mapsto [sxt^{-1}]
\]

is \( T \times T \)-skeletal.

As mentioned in the Introduction, it follows from Theorem 7.3 of \[G-2\] and Theorem 3.6 that the following holds.
Corollary 3.7. Let $X = \mathbb{P}_\epsilon(M)$ be a rationally smooth standard embedding. Then the action $\mu$ of $T \times T$ on $X$, given by

$$\mu : T \times T \times X \to X, \quad (s, t, [x]) \mapsto [sxt^{-1}], \quad \text{is a GKM-action. That is, } X \text{ is a GKM variety.}$$

The next result is due to Renner. For a proof, see [R6].

Theorem 3.8. Let $X = \mathbb{P}_\epsilon(M)$ be a standard group embedding. Then the following are equivalent.

1. $X = \mathbb{P}_\epsilon(M)$ is rationally smooth.
2. $M \setminus \{0\}$ is rationally smooth.
3. For any minimal, nonzero, idempotent $e$ of $M$, $M_e$ is rationally smooth.
4. For any maximal torus $T$ of $G$, $\overline{T} \setminus \{0\}$ is rationally smooth.

Notice, in particular, that the condition does not depend on $Z$. Theorem 3.8 provides a combinatorial/numerical description of rationally smooth embeddings. See [R6] for more details.

4. GKM Theory of Rationally Smooth Standard Group Embeddings

It has been shown (Corollary 3.7) that the equivariant cohomology of a rationally smooth standard group embedding can be described in terms of GKM-theory. In this section, for each $T \times T$-invariant curve, we obtain the associated GKM-character explicitly. Theorem 4.10 gives the ultimate description of $H^*_{T \times T}(\mathbb{P}_\epsilon(M))$ in terms of certain characters and the Renner monoid, a finite combinatorial invariant associated to the monoid $M$. We also describe the relation between $H^*_{T \times T}(\mathbb{P}_\epsilon(M))$ and $H^*_T(\mathbb{P}_\epsilon(\mathbb{T}))$, the associated torus embedding (Theorem 4.12).

Let $M$ be a reductive monoid with zero and unit group $G$. Let $T$ be a maximal torus and $\epsilon : \mathbb{C}^* \to Z$ be an attractive one-parameter subgroup in the center of $G$. Consider the standard group embedding $X = \mathbb{P}_\epsilon(M)$.

4.1. Classification of GKM-curves. Let $M$ be a reductive monoid with zero and unit group $G$. Let $T$ be a maximal torus and $\epsilon : \mathbb{C}^* \to Z$ be an attractive one-parameter subgroup in the center of $G$. Consider the standard group embedding $X = \mathbb{P}_\epsilon(M)$. Most of the calculations here do not depend on whether $\mathbb{P}_\epsilon(M)$ is rationally smooth.

Recall that the set of $T \times T$-fixed points in $X$ corresponds to

$$\mathcal{R}_1 = \{x \in \mathcal{R} \mid \dim(Tx) = \dim(xT) = 1\}.$$ 

From Theorem 3.5, we also know that there are three types of $T \times T$-curves in $X$:

1. Curves that are fixed pointwise by $T$ on the right: $\overline{U_\alpha w}$, $e \in E_1(\mathbb{T})$, $s_\alpha \notin C_W(e)$, and $w \in W$. 

Notice, in particular, that the condition does not depend on $Z$. Theorem 3.8 provides a combinatorial/numerical description of rationally smooth embeddings. See [R6] for more details.

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1. Curves that are fixed pointwise by $T$ on the right: $\overline{U_\alpha w}$, $e \in E_1(\mathbb{T})$, $s_\alpha \notin C_W(e)$, and $w \in W$. 

Notice, in particular, that the condition does not depend on $Z$. Theorem 3.8 provides a combinatorial/numerical description of rationally smooth embeddings. See [R6] for more details.
Curves that are fixed pointwise by $T$ on the left: $\overline{we\alpha}$, $e \in E_1(T)$, $s_\alpha \notin C_W(e)$, and $w \in W$.

(3) $tx = \overline{xT} = \overline{TxT}$ where $x \in R_2 = \{x \in R \mid \dim(Tx) = 2\}$.

But which pair of fixed points, i.e. elements of $R_1$, is joined by each of these curves? Preserving the given order, we obtain

1. $ew$ and $s_\alpha ew$
2. $we$ and $wes_\alpha$
3. The two elements $r, s \in R_1$ such that $r, s \in \overline{TxT}$.

**Theorem 4.1.** The set of $T \times T$-curves in $X = \mathbb{P}_e(M)$ is identified as follows, by pairs of $T \times T$-fixed points. Here $Ref(W)$ refers to the set of reflections of $W$ and we assume there is an ambient Borel subgroup (to get the ordering on $R$).

1. $\{(x, sx) \mid x \in R_1, s \in Ref(W) \text{ and } x < sx\}$.
2. $\{(x, xs) \mid x \in R_1, s \in Ref(W) \text{ and } x < xs\}.
3. $R_2 \cong \{A \subseteq R_1 \mid |A| = 2 \text{ and } A = \{ex, fx\} \text{ for some } e, f \in E_1(T) \text{ and some } x \in R_2\}$.

**Proof.** First we recall that the Renner monoid $R$ is partially ordered by the relation $x \leq y$ if $BxB \subseteq ByB$. This is a generalization of the Bruhat-Chevalley order from group theory to the case of reductive monoids. See [R9], Definition 8.32. Bearing this in mind, Assertions 1. and 2. follow from the fact that if $x \neq sx$ and $s \in Ref(W)$, then either $x < sx$ or else $sx < x$ (R9, Section 8.6). For 3. we proceed as follows. Recall that any $x \in R_2$ can be written as $x = fu$, where $f \in E_2(T)$ is a rank-two idempotent, and $u \in W$. Since $u$ is invertible, it is enough to prove the statement for $x = f$. Now notice that $(fT \setminus \{0\})/\mathbb{C}^*$ is isomorphic to $\mathbb{CP}^1$ ([Br4], Corollary 1.4.1). Thus there are exactly two fixed points, they correspond to the unique rank-one idempotents $e, e' \in E_1(T)$ such that $ef \neq 0$ and $e'f \neq 0$. \[\square\]

Any $T \times T$-fixed point is contained in a closed $G \times G$-orbit. The curves identified in 1. and 2. of Theorem 4.1 are the ones that are contained in closed $G \times G$-orbits. The curves identified in 3. of Theorem 4.1 are those that are not contained in any closed $G \times G$-orbit. In [Br3] these curves are further separated into whether or not the corresponding fixed points are in the same closed $G \times G$-orbit. This distinction will become relevant in the next section when we identify the character associated with each $T \times T$-curve of type 3.

Notice that the description in 3. above is just a convenient, indirect way of identifying the elements of $R_2$ as pairs of $T \times T$-fixed points. Notice also that, for each $x \in R_2$, there are exactly two elements $e, f \in E(R_1)$ such that $ex \neq 0$ and $fx \neq 0$.

**Example 4.2.** We illustrate Theorem 4.1 with the example $M = M_n(K)$. Let $E_{i,j}$ denote an elementary matrix. We then obtain (with the ordering as in Theorem 4.1)
(1) \( \{(E_{i,j}, E_{i,k}) \mid j \neq k\} \).
(2) \( \{(E_{i,j}, E_{k,j}) \mid i \neq k\} \).
(3) \( \{(E_{i,j}, E_{k,l}) \mid i \neq k \text{ and } j \neq l\} \).

In each case the associated curve is the \( T \times T \)-orbit of the sum of the given pair of elementary matrices. In case 1. the two elementary matrices are in the same row. In case 2. the two elementary matrices are in the same column. Case 3. determines the remaining cases.

4.2. The Associated Characters. We now identify the character \( \theta_x = (\lambda_x, \rho_x) \) of \( T \times T \) associated with the \( T \times T \)-curve \( c = [TxT] \in \mathcal{C}(X,T) \). Recall that this character, unique up to sign, has been described in Definition 2.6.

As discussed previously (Theorems 3.5 and 4.1), there are three different types of \( T \times T \)-curves. In this section we focus mainly on the third type, that is, when \( c = [TxT] \) and \( x \in \mathbb{R}_2 \). The other \( T \times T \)-curves (where either \( Tx = TxT \) or \( xT = TxT \)) will also be discussed, but recall that these are essentially \( T \)-curves on the complete homogeneous space \( G/P_e \), with \( e \in E_1 \) (Lemma 2.11).

So let \( x \in \mathbb{R}_2 \). Since we are working on the monoid level, the initial step in our discussion is to calculate the map

\[ m_x : T \times T \to TxT, \ (s,t) \mapsto sxt. \]

We then compose \( m_x \) with the canonical map \( \pi_x : TxT \to TxT/Z \cong \mathbb{C}^* \) to obtain

\[ \theta_x = \pi_x \circ m_x \]

where \( Z \subseteq G \) is the given central, attractive, 1-parameter subgroup of the unit group \( G \) of \( M \). Notice that \( \theta_x \) depends on the choice of isomorphism \( TxT/Z \cong \mathbb{C}^* \). The other isomorphism \( TxT/Z \cong \mathbb{C}^* \) yields \( \theta_x^{-1} \). In the calculation of \( \theta_x \) it is important to keep track of this ambiguity. It is also useful to consider the map

\[ t_x : T \to TxT, \ t \mapsto tx \]

and the character \( \lambda_x = \pi_x \circ t_x \). Notice that \( TxT = Tx \), so we wish to express \( \theta_x : T \times T \to \mathbb{C}^* \) as a composition

\[ T \times T \to T \times T \to T \to Tx \to \mathbb{C}^* \]

involving the multiplication \( T \times T \to T \), the action of \( W \) on \( T \), and these other quantities: \( t_x, \pi_x, \lambda_x \).

Also we assess the effect of the \( W \times W \)-action

\[ W \times W \times \mathcal{C}(X,T \times T) \to \mathcal{C}(X,T \times T), \ (v,w,c) \mapsto vcw^{-1} \]

on the associated characters. This will effectively reduce the calculation of \( \theta_x \), with \( x \in \mathbb{R}_2 \), to calculating \( \theta_x \) for a set of representatives of the \( W \times W \)-orbits of \( \mathbb{R}_2 \).
4.2.1. Explicit computations. Denote by $\Xi(T)$ the character group of $T$.

Let $x \in R_2$. Then we can write $x = fu = ug$, where $u \in W$ and $f, g \in E_2(\overline{T})$. An elementary calculation yields that

$$m_x : T \times T \to TxT = xT, \ (s,t) \mapsto sxt$$

is given by $m_x(s,t) = st^u x$ where, by definition, $t^u = utu^{-1}$. Recall that $\lambda_x = \pi_x \circ t_x$, where $t_x : T \to Tx, t \mapsto tx$, and $\pi_x : TxT \to TxT/Z \cong K^*$.

**Lemma 4.3.** Write $\theta_x = (\lambda_x, \rho_x) \in \Xi(T \times T) = \Xi(T) \oplus \Xi(T)$. Then

1. $\lambda_x = \lambda_f$.
2. $\rho_x = \rho_g = \lambda_f \circ \text{int}(u)$, where $\text{int}(u)(t) = utu^{-1}$.

**Proof.** Consider $m : T \times T \to Tf, \ (s,t) \mapsto st^u f$. Then $m(s,t) \in Zf$ if and only if $m_x(s,t) \in Zx$. Thus $\ker(\pi_f \circ m) = \ker(\pi_x \circ m_x)$. So $\lambda_x = \lambda_f$ and $\rho_x = \lambda_f \circ \text{int}(u)$. But $m$ is also the product of $(s,1) \mapsto sf$ and $(1,t) \mapsto t^uf$. The first of these is $\lambda_f$ and the second of these is $\lambda_f \circ \text{int}(u)$. But $t^uf \in Zf$ if and only if $tg \in Zg$ since $ugu^{-1} = f$. Thus $\ker(\lambda_f \circ \text{int}(u)) = \ker(\lambda_g)$. We conclude that $\theta_x = (\lambda_x, \rho_x) = (\lambda_f, \lambda_g) = (\lambda_f, \lambda_f \circ \text{int}(u))$.

Notice that we can also write it as $m_x : T \times T \to TTx = xT, \ m_x(s,t) = sxt = xst^u$. The resulting calculation then yields $\theta_x = (\lambda_x, \rho_x) = (\lambda_f, \lambda_g) = (\lambda_g \circ \text{int}(u^{-1}), \lambda_g)$.

Notice that either $\theta_x = (\lambda_x, \lambda_x \circ \text{int}(u))$ or $\theta_x = (\lambda_x^{-1}, \lambda_x^{-1} \circ \text{int}(u))$ depending on the orientation.

**Lemma 4.4.** Let $x \in R_2$, so that $x = fu = ug$ where $u \in W$ and $f, g \in E_2(\overline{T})$, and write $\theta_x = (\lambda_f, \lambda_g)$ with $\lambda_g = \lambda_f \circ \text{int}(u)$ (as in Lemma 4.3).

1. Let $y = xw$, where $w \in W$. Then $\theta_y = (\lambda_f, \lambda_g \circ \text{int}(w)) = (\lambda_x, \rho_x \circ \text{int}(w))$.
2. Let $y = wx$, where $w \in W$. Then $\theta_y = (\lambda_f \circ \text{int}(w^{-1}), \lambda_g) = (\lambda_x \circ \text{int}(w^{-1}), \rho_x)$.

**Proof.** Assume that $y = xw$, and let $h = (uw)^{-1}fuw$. Then $\theta_y = (\lambda_f, \lambda_h)$ where $\lambda_h = \lambda_f \circ \text{int}(uw) = \lambda_f \circ \text{int}(u) \circ \text{int}(w) = \lambda_g \circ \text{int}(w)$.

Assume that $y = wx$, and let $h = wfw^{-1}$. Then $\theta_y = (\lambda_h, \lambda_g)$ where $\lambda_h = \lambda_f \circ \text{int}(w^{-1})$ (since $h = wfw^{-1}$).

Let $x \in R_2$, and write $x = fu$, where $f \in E_2(\overline{T})$ and $u \in W$. The $H$-class of $x$, denoted by $H_x$, is defined to be $H_x := \{sx \mid s \in C_W(f)\}$. Clearly, $H_x = H_f \cdot u$. See [Pui] for more information on $H$-classes. The following is a result of Putcha.

**Lemma 4.5.** Let $x = fu$ be an element of $R$, the Renner monoid of $M$. Denote by $H_x$ its $H$-class. If $x \in R_2$, then either $H_x$ has two elements or $H_x = \{x\}$. In the former case, $H_x = \{x, y\}$, where $y = s_{\alpha_f} x$ and $s_{\alpha_f} \in C_W(f)$ is the reflection for which $s_{\alpha_f} f = f s_{\alpha_f} \neq f$. In the latter case, any element $s \in C_W(f)$ satisfies $sf = fs = f$. 


Proof. It is enough to check the statement for the rank-two idempotents of \( T \) because, for any \( x = f u \in \mathcal{R}_2 \), one has \( H_x = H_f \cdot u \) with \( f \in E_2(T) \) and \( u \in W \).

So let \( f \in E_2(T) \) and suppose that \( H_f \neq \{ f \} \). Then, there should be a \( s \in C_W(f) \) with the property that \( sf = fs \neq f \). We claim that \( s \) is a reflection. Indeed, consider the inner transformation \( \text{int}(s) : fT \to fT, \)

\[
fx \mapsto fsxs^{-1},
\]

and let’s examine the automorphism \( \sigma \) induced by \( \text{int}(s) \) on \( fT - \{ 0 \}/Z \cong \mathbb{C}P^1 \). Recall that there are exactly two rank-one idempotents \( f_1 \) and \( f_2 \) below \( f \). Denote by \( 0 \) and \( \infty \), respectively, their classes in the orbit space \( fT - \{ 0 \}/Z \). Also, since \( f \) is the identity element of the reductive monoid \( fT \), let us denote its class on \( \mathbb{C}P^1 \) by \( 1 \). Because \( (sf_i s^{-1}) \cdot f = sf_i s^{-1} \) for \( i = 1, 2 \), it is clear that \( \sigma \) permutes the points 0 and \( \infty \). So either \( \sigma(0) = 0 \) and \( \sigma(\infty) = \infty \) or else \( \sigma(0) = \infty \) and \( \sigma(\infty) = 0 \). Moreover, \( \sigma(1) = 1 \) in either case, because \( \sigma \) restricts to an algebraic automorphism of \( \mathbb{C}^* \cong Tf/Z = \mathbb{C}P^1 \setminus \{ 0, \infty \} \). Hence, as a Möbius transformation, \( \sigma \) is either \( z \mapsto z \) or \( z \mapsto z^{-1} \). The former is clearly impossible because, by assumption, \( sf = fs \neq f \). Therefore, by looking at the commutative diagram

\[
\begin{align*}
Tf \xrightarrow{\text{int}(s)} & Tf \\
\pi \downarrow & \pi \\
Tf/Z \xrightarrow{z \mapsto z^{-1}} & Tf
\end{align*}
\]

we conclude that \( s \), when restricted to \( Tf \), is a reflection. Finally, given that the natural map \( T \to Tf \) is \( s \)-equivariant, it follows that \( s \) itself is a reflection in \( W \). So \( s = s_{\alpha_f} \), for some root \( \alpha_f \) in \( \Phi \subseteq \Xi(T) \). Here \( \alpha_f \) equals \( \lambda_f \), the character defined at the beginning of this subsection. It is clear that \( s = s_{\alpha_f} \) is uniquely determined by the commutative diagram above. Hence, \( H_f = \{ f, s_{\alpha_f} \cdot f \} \).

\[\square\]

Lemma 4.6. The following are equivalent for \( x \in \mathcal{R}_2 \).

1. The \( H \)-class of \( x \) contains two elements.
2. The two \( T \times T \)-fixed points in \( X = \mathbb{P}_r(M) \), in the closure of \( T x T \), are in the same \( W \times W \)-orbit.

Proof. Let \( x \in \mathcal{R}_2 \) and let \( a, b \in T x T \) be the two \( T \times T \)-fixed “points” in \( T x T \). Assume that \( H_x = \{ x, y \} \). Then, by Lemma 4.5, there exist \( s = s_{\alpha_f} \)

and \( u \) in \( W \), together with \( f \) and \( g \) in \( E_2(T) \), such that \( x = fu = ug \) and \( y = fsu = sug \). In particular, \( sf = fs \neq f \). Notice also that \( y = fut = utg \) where \( t = u^{-1}su \). In any case, the two fixed points \( a, b \in T x T \)

are \( a = f_1 x = f_1 u \) and \( b = f_2 x = f_2 u \) where \( f_1, f_2 \) are the two rank-one idempotents below \( f \). One checks that \( b = sat \) and \( a = sbt \). Indeed,

\[
sat = s f_1 ut = sf_1 uu^{-1} su = sf_1 su = f_2 u = b.
\]

Notice that \( sf_1 s = f_2 \) since \( sf \neq fs \).

Now let \( x = fu \in \mathcal{R}_2 \) and assume that \( f_1 x = f_1 u \) and \( f_2 x = f_2 u \) are in the same \( W \times W \)-orbit. Then \( f_1 \) and \( f_2 \) are in the same \( W \times W \)-orbit. That
is, $f_1$ and $f_2$ are conjugate (Theorem 3.1.8). Furthermore, Corollary 8.9 and Proposition 10.9 of \[Pu\] assert that $f_1$ and $f_2$ are conjugate by an element $s \in C_W(f) = \{v \in W \mid vf = fv\}$. One then checks that $y = sx$ is the other element in the $H$-class of $x$. \hfill\qed

**Lemma 4.7.** Let $x, y \in R_2$ be distinct and assume that $H_x = \{x, y\}$. Write $x = fu$ and $y = fs_\alpha u$, as in Lemma 4.3. Then $\lambda_f \circ \text{int}(s_\alpha) = \lambda_f^{-1}$. Consequently, $\theta_x = (\lambda_x, \rho_x) \implies \theta_y = (\lambda_x, \rho_x^{-1})$.

Furthermore, $\lambda_x = \alpha_f$ and $\rho_x = \alpha_f \circ \text{int}(u)$ are roots of $G$ with respect to $T$.

**Proof.** From Lemma 4.3 we obtain $\lambda_y = \lambda_f \circ \text{int}(u)$, as well as $\lambda_x = \lambda_f \circ \text{int}(s_\alpha u)$. But $\text{int}(s_\alpha u) = \text{int}(s_\alpha) \circ \text{int}(u)$. Thus, either $\lambda_f = \lambda_f \circ \text{int}(s_\alpha)$ or else $\lambda_f^{-1} = \lambda_f \circ \text{int}(s_\alpha)$ since these characters are unoriented. We must rule out the former case. This amounts to looking at the map induced on $fT/Z$ from the restriction $\text{int}(s_\alpha) : fT \to fT$. By Lemma 4.3, $\text{int}(s_\alpha)\{ft\} = \{ft^{-1}\}$, for all $t \in T$. Thus, $\lambda_f^{-1} = \lambda_f \circ \text{int}(s_\alpha)$. Finally, by Lemma 4.5 again, it follows that $\lambda_x = \lambda_f = \alpha_f$ and $\rho_x = \alpha_f \circ \text{int}(u)$ are roots. \hfill\qed

**Example 4.8.** Let $M = M_n(K)$ and let $T$ be the set of invertible, diagonal matrices. One checks that $R_2 = \{E_{i,j} + E_{k,l} \mid i \neq k$ and $j \neq l\}$.

where $E_{i,j}$ denotes the elementary matrix with a one in the $(i,j)$-position and zeros elsewhere. Let $s = (s_1, ..., s_n) \in T$ denote the obvious diagonal matrix. A simple calculation yields that, for $s, t \in T$ and $x = E_{i,j} + E_{k,l}$,

$$\theta_x(s, t) = s_is_k^{-1}tjt_l^{-1}.$$ 

The other element $y \in R_2$, in the $H$-class of $x = E_{i,j} + E_{k,l}$, is $y = E_{k,j} + E_{i,l}$. Thus,

$$\theta_y(s, t) = s_is_k^{-1}tjt_l^{-1}.$$ 

In the terminology of Lemma 4.3, $\theta_x = (\lambda_x, \rho_x)$ where $\lambda_x = \alpha_{i,k}$ and $\rho_x = \alpha_{j,l}$. Similarly, $\lambda_y = \alpha_{i,k}$ and $\rho_y = \alpha_{i,l}$.

We now discuss the remaining cases (where either $Tx = T x T$ or $xT = T x T$). Again our treatment is somewhat terse because the whole issue reduces to the well-documented situation discussed in [C].

**Lemma 4.9.** Let $x = ew \in R_1$ and let $\alpha \in \Phi$ be such that $U_\alpha x \neq \{x\}$. Then, for $s, t \in T$ and $u \in U_\alpha$,

$$suxt^{-1} = su^{-1}z_x(s, t)x$$

where $z_x : T \times T \to Z$. Thus, the character of the action of $T \times T$ on $C(x, \alpha) = \overline{U_\alpha[x]} \subseteq \mathcal{P}_e(M)$ is the root $(\alpha, 1)$.
Proof. Starting from $suxt^{-1}$, one obtains $suxt^{-1} = sus^{-1}sewt^{-1}w^{-1}w$. Since the quantities $(t^{-1})^w := wt^{-1}w^{-1}$ and $e$ commute, then the term on the right hand side of the identity above becomes $sus^{-1}(s(t^{-1})^w)ew$. This latter expression is, quite simply, equal to $sus^{-1}s(t^{-1})^we$. On the other hand, observe that $Te = Ze$, because $e$ is a rank-one idempotent of $T$. In other words, $s(t^{-1})^we = zx(s,t)e$ where $zx(s,t) \in Z$. From this, it follows that

$$suxt^{-1} = sus^{-1}zx(s,t)x = sus^{-1}xzx(s,t).$$

Hence,

$$s(uxZ)t^{-1} = sus^{-1}xZ,$$

and the result follows. \qed

4.3. The main results. Let $\Lambda$ be the cross section lattice of $M$. Recall that $\Lambda$ corresponds to the partially ordered set of $G \times G$-orbits in $M$. Under this identification, closed $G \times G$-orbits in $P_e(M)$ correspond to idempotents $e \in \Lambda_1$.

We now state the first major result of this article. For the analogous result in the case of (smooth) regular compactifications, see Theorem 3.1.1 of [Br3].

**Theorem 4.10.** Let $X = P_e(M)$ be a rationally smooth standard group embedding. Then the natural map

$$H^*_T(X) \longrightarrow H^*_T(\bigcup_{e \in \Lambda_1} G[e]G) = \bigoplus_{e \in \Lambda_1} H^*_T(G[e]G)$$

is injective. In fact, its image consists of all tuples $(\varphi_e)_{e \in \Lambda_1}$, indexed over $\Lambda_1$ and with $\varphi_e \in H^*_T(G[e]G)$, subject to the additional conditions:

(a) If $f \in E_2(T)$ and $H_f = \{f, s_{\alpha_f}f\}$, with $s_{\alpha_f}f = fs_{\alpha_f} \neq f$, then

$$\varphi_{e_1}(f_1u) \equiv \varphi_{e_2}(f_2u) \mod (\alpha_f, \alpha_f \circ \text{int}(u)),$$

for all $u \in W$. Here, $f_1$ and $f_2 = s_{\alpha_f}f_1 \cdot s_{\alpha_f}f$ are the two idempotents in $E_1(T)$ below $f$, the root $\alpha_f$ corresponds to the reflection $s_{\alpha_f}$, and $e_f \in \Lambda_1$ is the unique element of $\Lambda_1$ which is conjugate to $f_1$.

(b) If $f \in E_2(T)$ and $H_f = \{f\}$, then

$$\varphi_{e_1}(f_1u) \equiv \varphi_{e_2}(f_2u) \mod (\lambda_f, \lambda_f \circ \text{int}(u)),$$

for all $u \in W$. Here, $\lambda_f$ is the character of $T$ defined in Lemma 4.3, the idempotents $f_1, f_2$ are the unique idempotents below $f$, and $e_i \in \Lambda_1$ is conjugate to $f_i$, for $i = 1, 2$.

**Proof.** It is known from Corollary 3.7 that $X$ is a GKM variety, that is, the induced map $i^* : H^*_T(X) \rightarrow H^*_T(X^{T \times T})$ is injective. Now notice that
all the $T \times T$-fixed points of $X$ are contained in the (disjoint) union of the closed orbits. So we have a commutative triangle

$$
\begin{array}{ccc}
H^*_T(X) & \xrightarrow{i^*} & H^*_T(X^{T \times T}) \\
\downarrow j^* & & \downarrow k^* \\
\bigoplus_{e \in \Lambda_1} H^*_T(\text{GeG}) & & \\
\end{array}
$$

where all maps are induced by inclusions. The injectivity of $i^*$ yields at once the injectivity of $j^*$.

We can say even more. Since $\text{GeG} \simeq G/P_e \times G/P_e^-$ (Proposition 1.16), we conclude that each closed orbit is equivariantly formal. What is more, $X^{T \times T} = R_1$ is also the fixed point set of $L = \bigsqcup_{e \in \Lambda_1} \text{GeG}$. Thus, $k^*$ is injective. Now notice that $L$ contains all the curves of type 1 and 2 in $X$ (Theorem 4.1). These curves, in addition, describe the equivariant cohomology of $L$ (Proposition 2.10).

To conclude the proof, we just need to show that the curves of type 3 in Theorem 4.1 give assertions (a) and (b). So let $x = fu \in R_2$ be one of these curves. By Lemma 4.5, the $H$-class $H_x$ of $x$ contains either one or two elements.

If $H_x = \{x, s_{\alpha_f}x\}$, then Lemma 4.6 implies that the two fixed points of $[T \times T]$, namely $f_1x$ and $f_2x$, lie in the same closed $G \times G$-orbit. Here recall that $f_1, f_2$ are the two idempotents below $f$. Moreover, $f_2$ is conjugate to $f_1$ via $s_{\alpha_f}$, namely, $f_2 = s_{\alpha_f} \cdot f_1 \cdot s_{\alpha_f}$. We now use Lemma 4.7 to write the associated character $\theta_x$ as

$$
\theta_x = (\alpha_f, \alpha_f \circ \text{int}(u)),
$$

where $\alpha_f$ is the root associated to the reflection $s_{\alpha_f}$. Since $\Lambda_1$ indexes all closed $G \times G$-orbits in $X$, there exists a unique $e_x \in \Lambda_1$ such that $f_1$ and $e_x$ are conjugate. Assertion (a) is now proved.

Finally, if $H_x = \{x\}$, then $f_1$ and $f_2$ are not conjugate (Lemma 4.6). That is, $f_1x$ and $f_2x$ lie in different closed $G \times G$-orbits. Since $x = fu$, Lemma 4.3 finishes the proof.

The previous result provides a complete combinatorial description of the equivariant cohomology of any rationally smooth standard embedding.

As it was pointed out before, Brion (Br3, Theorem 3.1.1) has obtained a result analogous to Theorem 4.10 for regular compactifications of $G$. These compactifications are characterized, among other properties, by the fact that they are smooth varieties and possess a finite number of closed $G \times G$-orbits, all of them isomorphic to $G/B \times G/B$. There are three main differences between the embeddings studied by Brion in Br3 and our standard group embeddings. First, standard group embeddings are, in general, singular. Second, the closed $G \times G$-orbits of a standard group embedding are usually
of the form $G/P_e \times G/P_e^-$, where $P_e$ and $P_e^-$ are opposite parabolic subgroups (Proposition 4.3.1). Such homogeneous spaces are not necessarily isomorphic to $G/B \times G/B$. Finally, any normal projective group embedding of a connected reductive group is standard (Theorem 1.17). That is, standard group embeddings form a very natural class from the viewpoint of embedding theory. This class is larger than the class of regular compactifications. In particular, our Theorem 4.10 implies Theorem 3.1.1 of [Br3] for the case of projective regular embeddings.

These observations should help the reader to not only understand the importance and scope of our main Theorem 4.10 but also put our results in perspective.

It follows from Theorem 2.2 that the $G \times G$-equivariant cohomology of $X$ is obtained by means of the following formula

$$H^*_{G \times G}(X) \cong (H^*_{T \times T}(X))^{W \times W}.$$  

For the case in hand, we can be more precise, as the following result shows.

**Corollary 4.11.** Let $X = \mathbb{P}(M)$ be a rationally smooth standard group embedding. Then the ring $H^*_{G \times G}(X)$ consists of all tuples $(\Psi_e)_{e \in \Lambda_1}$, where

$$\Psi_e : W e W \to (H^*_T \otimes H^*_T)^{C_W(e) \times C_W(e)},$$

such that

(a) If $f \in E_2(T)$ and $H_f = \{f, s_{\alpha_f} f\}$, then

$$\Psi_e(f_1) \equiv \Psi_e(f_2) \mod (\alpha_f, \alpha_f),$$

where $e \in \Lambda_1$ is conjugate to $f_1, f_2 = s_{\alpha_f} \cdot f_1 \cdot s_{\alpha_f}$, the reflection $s_{\alpha_f} \in C_W(f)$ is associated with the root $\alpha_f$, and $f_i \leq f$.

(b) If $f \in E_2$ and $H_f = \{f\}$, then

$$\Psi_e(f_1) \equiv \Psi_e'(f_2) \mod (\lambda_f, \lambda_f),$$

where $\lambda_f \in \Xi(T)$, and $f_1, f_2 \leq f$ are conjugate to $e$ and $e'$, respectively.

**Proof.** Let $e \in \Lambda_1$. The closed orbit $G[e]G$ is isomorphic to $G/P_e \times G/P_e^-$. Moreover, $P_e = C_G(e) \rtimes U_e$, where $C_G(e)$ is the centralizer of $e$ in $G$, and $U(e)$ is the unipotent part of $P_e$. In fact, $U(e) = R_u(P(e))$ and $C_G(e)$ is a closed connected reductive subgroup, called the Levi subgroup of $P(e)$. It follows from the results of Brion ([Br2], p. 25) that

$$H^*(BP_e) \cong H^*(BC_G(e)) \cong H^*(BT)^{C_W(e)}.$$  

Consequently,

$$H^*_{G \times G}(G[e]G) \cong H^*(BP_e) \otimes H^*(BP_e^-) \cong (H^*_T \otimes H^*_T)^{C_W(e) \times C_W(e)}.$$  

Note that \((u, v) \in W \times W\) acts on a tuple \((f_r)\) in \(H^*_T \times T(\mathcal{R}_1) = \bigoplus_{r \in \mathcal{R}_1} H^*_{T \times T}\) via
\[
(u, v) \cdot (f_r) := ((u, v) \cdot f_{urv^{-1}}).
\]

Inasmuch as the restriction of \(\Psi_e\) to \((u, v) \cdot e = uev^{-1}\) equals \((u, v) \cdot \Psi_e(e)\) for all \((u, v) \in W \times W\), relations (a) and (b) of Theorem 4.10 reduce to the proposed descriptions (a) and (b).

Associated to \(X = \mathbb{P}_\epsilon(M)\), there is a standard torus embedding \(\mathcal{Y}\) of \(T/\mathbb{Z}\), namely, \(\mathcal{Y} = \mathbb{P}_\epsilon(T) = [T \setminus \{0\}] / \mathbb{C}^*\).

By construction, \(\mathcal{Y}\) is a normal projective torus embedding and \(\mathcal{Y} \subseteq X\).

Our next theorem allows to compare the equivariant cohomologies of \(X\) and its associated torus embedding \(\mathcal{Y} \subseteq X\). The situation for standard embeddings contrasts deeply with the corresponding one for regular embeddings ([Br3], Corollary 3.1.2; [U], Corollary 2.2.3). It is worth noting that the idea of comparing \(\mathcal{Y}\) and \(X\) goes back to [LP].

**Theorem 4.12.** The inclusion of the associated torus embedding \(\iota : \mathcal{Y} \hookrightarrow X\) induces an injection:
\[
\iota^* : H^*_{G \times G}(X) \longrightarrow H^*_{T \times T}(\mathcal{Y})^W \simeq (H^*_T(\mathcal{Y}) \otimes H^*_T)^W,
\]
where the \(W\)-action on \(H^*_T(\mathcal{Y})\) is induced from the action of \(\operatorname{diag}(W)\) on \(\mathcal{Y}\). Furthermore, \(\iota^*\) is an isomorphism if and only if \(C_W(e) = \{1\}\) for every \(e \in \Lambda_1\).

**Proof.** Since \(X\) is rationally smooth, then \(\mathcal{Y}\) is rationally smooth as well (Theorem 3.8). Therefore, we have the following commutative diagram
\[
\begin{array}{ccc}
H^*_T(\mathcal{Y}) & \longrightarrow & H^*_T(\mathcal{Y}^{T \times T}) \\
i^* \downarrow & & \downarrow i^* \\
H^*_T(X) & \longrightarrow & H^*_T(X^{T \times T})
\end{array}
\]
where the horizontal maps are injective, because both standard group embeddings are equivariantly formal.

On the other hand, recall that \(\Lambda_1\) provides a set of representatives of both the \(W \times W\)-orbits in \(X^{T \times T} = \mathcal{R}_1\) and the \(W\)-orbits in \(\mathcal{Y}^{T \times T} = E_1(T)\). Thus, after taking invariants, we obtain an injection
\[
H^*_T(\mathcal{R}_1)^W = \bigoplus_{e \in \Lambda_1} (H^*_T)^{C_W(e) \times C_W(e)} \hookrightarrow H^*_T(\mathcal{Y}^{E_1(T)})^W = \bigoplus_{e \in \Lambda_1} (H^*_T)^{C_W(e)}.
\]

Placing this information into the commutative diagram above shows that the restriction map
\[
\iota^* : (H^*_T(X))^W \longrightarrow H^*_T(\mathcal{Y})^W
\]
is injective.
Observe that $H^*_{T \times T}(\mathcal{Y})^W \simeq (H^*_T(\mathcal{Y}) \otimes H^*_T)^W$. Truly, we have a split exact sequence

$$1 \longrightarrow \text{diag}(T) \longrightarrow T \times T \xrightarrow{(t_1, t_2) \mapsto t_1 \cdot t_2^{-1}} T \longrightarrow 1,$$

where the splitting is given by $t \mapsto (t, 1)$. It follows that $T \times T$ is canonically isomorphic to $\text{diag}(T) \times (T \times 1)$. Furthermore, by definition, $\text{diag}(T)$ acts trivially on $\mathcal{Y}$. As a consequence, we have a ring isomorphism $H^*_{T \times T}(\mathcal{Y}) \simeq H^*_{\text{diag}(T)} \otimes H^*_T(\mathcal{Y})$. This isomorphism is further $W$-invariant since the $W$-action on the cohomology rings is induced from the action of $\text{diag}(W)$ on $\mathcal{Y}$.

To prove the second part of the Theorem, we adapt to our situation an argument of Littelmann and Procesi ([LP], Theorem 2.3).

Firstly, assuming that $i^*$ is also surjective, we need to show that $C_W(e) = \{1\}$ for all $e \in \Lambda_1$. Since $X$ is equivariantly formal, then $H^*_{G \times G}(X)$ is a free $(H^*_T)^{W \times W}$-module. And $H^*_{T \times T}(\mathcal{Y})$ is a free $H^*_T$-module, for the same reason. By Corollary 2.3 one can choose a graded $W \times W$-submodule $R$ of $H^*_{T \times T}$, isomorphic to the regular representation of $W \times W$, such that

$$H^*_{T \times T} \simeq R \otimes (H^*_T)^{W \times W}$$

as graded $(H^*_T)^{W \times W}$-module. Accordingly, $H^*_{T \times T}(\mathcal{Y})^{W \times W}$ is in a natural way a free $(H^*_T)^{W \times W}$-module.

Notice that the rank of $H^*_{G \times G}(X)$, as a $H^*_G$-module, equals $|\mathcal{R}_1|$, the number of $T \times T$-fixed points (Lemma 2.9). Since, by assumption, $i^*$ is a graded isomorphism of free $(H^*_T)^{W \times W}$-modules, we conclude that the ranks of $H^*_{G \times G}(X)$ and $H^*_{T \times T}(\mathcal{Y})^W$ must be the same. The next step consists in finding out a more intrinsic formula for the rank of the latter module, so as to compare it with $|\mathcal{R}_1|$.

Let $\mathcal{I}$ denote the ideal in $(H^*_T)^{W \times W}$ of elements of strictly positive degree. Recall that we can find a graded $W$-stable submodule $U$ of $H^*_{T \times T}(\mathcal{Y})$ such that the morphism

$$U \otimes H^*_{T \times T} \longrightarrow H^*_{T \times T}(\mathcal{Y})$$

is a $W$-equivariant isomorphism of graded $H^*_T$-modules. Because $\mathcal{Y}$ is equivariantly formal, we can actually set $U$ to be $H^*(\mathcal{Y})$ (Lemma 2.9). The dimension of $U$ is the Euler characteristic of $\mathcal{Y}$, and hence equal to $|E_1|$, the number of $T \times T$-fixed points in $\mathcal{Y}$. So

$$H^*_{T \times T}(\mathcal{Y})^W / \mathcal{I} H^*_{T \times T}(\mathcal{Y})^W$$

is isomorphic to $(U \otimes R)^W$ as $W$-representation. Since $R$ decomposes into the direct sum of $|W|$-copies of the regular representation of $W$, then Lemma 4.13 below shows that $\dim (U \otimes R)^W = |E_1||W|$. Consequently,

$$\dim H^*_{T \times T}(\mathcal{Y})^W / \mathcal{I} H^*_{T \times T}(\mathcal{Y})^W = |E_1||W|,$$
which, by the graded Nakayama Lemma, also coincides with the rank of $H^*_{T \times T}(\mathcal{Y})^W$ as a free $(H^*_{T \times T})^W$-module.

In summary, the surjectivity of $i^*$ implies that $|R_1| = |E_1||W|$. Now Lemma 4.14 below finally yields $C_W(e) = \{1\}$ for all $e \in \Lambda_1$.

For the converse, suppose that $C_W(e) = \{1\}$ for all $e \in \Lambda_1$. We need to show that $i^*$ is surjective. To achieve our goal, we modify slightly an argument of [LP], Section 4.1, and Brion [Br3], Corollary 3.1.2. Define the variety

$$\mathcal{N} = \bigcup_{w \in W} w\mathcal{Y}.$$  

We claim that this union is, in fact, a disjoint union. Indeed, observe that $\mathcal{N}$ contains all the $T \times T$-fixed points of $X$. That is, $\mathcal{N}$ has $|R_1|$ fixed points. On the other hand, each $w\mathcal{Y}$ has $|E_1|$ fixed points (for its corresponding $T$-action). Now, if it were the case that there is a pair of distinct subvarieties $w\mathcal{Y}$ and $w'\mathcal{Y}$ with non-empty intersection, then this intersection should also contain $T \times T$-fixed points. But then a simple counting argument would yield $|R_1| < |E_1||W|$. This is impossible, by our assumptions and Lemma 4.14. Hence,

$$\mathcal{N} = \bigcup_{w \in W} w\mathcal{Y}.$$  

Clearly, $\mathcal{N}$ is rationally smooth and equivariantly formal (because each $w\mathcal{Y}$ is so, for $w \in W$). Moreover, since $\mathcal{N}$ contains all the $T \times T$-fixed points of $X$, then the restriction map

$$H^*_{T \times T}(X) \rightarrow H^*_{T \times T}(\mathcal{N})$$

is injective.

It follows from Theorem 4.11 that all the $T \times T$-curves of $X$ are contained either in closed $G \times G$-orbits (curves of type 1. and 2.) or in $\mathcal{N}$ (curves of type 3.).

As a consequence, Theorem 2.7 can also be applied to $\mathcal{N}$. After taking $W \times W$-invariants (compare Corollary 4.11), we see that the restriction to $\mathcal{N}$ induces an isomorphism

$$H^*_{T \times T}(X)^W \simeq H^*_{T \times T}(\mathcal{N})^W \simeq \left( \bigoplus_{w \in W} H^*_{T \times T}(\mathcal{Y}) \right)^W \simeq H^*_{T \times T}(\mathcal{Y})^W.$$  

The proof is now complete. $\square$

**Lemma 4.13 ([LP]).** If $N$ is a finite group, and $U$ and $V$ are two finite dimensional representations of $N$ such that $V$ is the sum of copies of the regular representation of $N$, then

$$\dim (V \otimes U)^N = \frac{\dim V \cdot \dim U}{|N|}.$$  

$\square$
Lemma 4.14. Let $\mathcal{R}_1$ be the set of rank one elements of the Renner monoid $\mathcal{R}$. Then $|\mathcal{R}_1| = |E_1| \cdot |W|$ if and only if $C_W(e) = 1$ for every $e \in \Lambda_1$.

Proof. We know, by Theorem 1.6, that $\Lambda_1$ can be identified with a set of representatives of the $W \times W$-orbits in $\mathcal{R}_1$. Likewise, $\Lambda_1$ also corresponds to a set of representatives of the $W$-orbits in $E_1$. Let $k$ be the cardinality of $\Lambda_1$ and let $e_1, \ldots, e_k$ be a complete list of the elements of $\Lambda_1$. Since we are dealing with elements of rank one, it is easy to see that $We_iW \simeq (W/C_W(e_i)) \times (W/C_W(e_i))$, for all $i = 1, \ldots, k$. Thus

$$|\mathcal{R}_1| = \sum_i |We_iW| = \sum_i |W/C_W(e_i)|^2.$$

On the other hand, the orbit $We_i \subset E_1$ satisfies $We_i \simeq W/C_W(e_i)$. This implies the following formula

$$|E_1| = \sum_i |We_i| = \sum_i |W/C_W(e_i)|.$$

Now recall that $\mathcal{R}_1 = E_1W = WE_1$. In other words, $|\mathcal{R}_1| \leq |E_1||W|$ and so

$$\sum_i |W/C_W(e_i)|^2 \leq \sum_i |W/C_W(e_i)||W|.$$

Therefore, $|\mathcal{R}_1| = |E_1||W|$ if and only if

$$\sum_i (|W/C_W(e_i)||W| - |W/C_W(e_i)|^2) = 0.$$

Notice that the latter condition is equivalent to having $|W/C_W(e_i)| = |W|$ for every $i$, because $|W| - |W/C_W(e_i)| \geq 0$. It is now clear that $|\mathcal{R}_1| = |E_1||W|$ if and only if $|C_W(e_i)| = 1$ for all $i = 1, \ldots, k$. \hfill $\Box$

Corollary 4.15. Let $X = \mathbb{P}_e(M)$ be a rationally smooth standard embedding. Let $\mathcal{Y}$ be the associated torus embedding and $\iota : \mathcal{Y} \rightarrow X$ the canonical inclusion. Then the following are equivalent:

(a) The induced map $i^* : H^*_{G \times G}(X) \rightarrow H^*_T(\mathcal{Y})^W \simeq (H^*_T(\mathcal{Y}) \otimes H^*_T)^W$ is an isomorphism.

(b) $C_W(e) = \{1\}$ for every $e \in E_1(\mathcal{T})$.

(c) All closed $G \times G$-orbits in $X$ are isomorphic to $G/B \times G/B$.

Proof. The equivalence between statements (a) and (b) follows at once from Theorem 4.12 since $\Lambda_1$ is the set of representatives of the $W$-orbits in $E_1(\mathcal{T})$. For the equivalence between (b) and (c) remember that every closed $G \times G$-orbit in $X$ is of the form $G/P_e \times G/P_e^{-}$, for $e \in \Lambda_1$. Also, recall that $C_G(e)$, the common Levi subgroup of $P_e$ and $P_e^{-}$, has Weyl group equal to $C_W(e)$. Then $C_W(e) = \{1\}$, for all $e \in \Lambda_1$, if and only if $P_e = B$ and $P_e^{-} = B^{-}$ for all $e \in \Lambda_1$. \hfill $\Box$

Definition 4.16. Let $X = \mathbb{P}_e(M)$ be a rationally smooth group embedding. We say that $X$ is quasi-regular if $X$ satisfies any of the equivalent conditions of Corollary 4.15.
Our choice of terminology comes from the fact that all projective regular embeddings satisfy Corollary 4.15. It is worth noting, however, that our notion of quasi-regular embedding is of a more combinatorial nature and, for instance, does not require any special conditions on the boundary divisors of $X \setminus (G/Z)$. Hence, we have supplied the theory of embeddings with an interesting class of test spaces. In particular, the results of [LP], [U] and [Br3] can be extended to quasi-regular embeddings.

We conclude this section describing the non-equivariant cohomology ring of quasi-regular embeddings. This result is known for regular embeddings, due to the work of De Concini-Procesi [DP-2] (using non-equivariant cohomology), and Littelmann-Procesi [LP] (using equivariant cohomology).

**Corollary 4.17.** Let $M$ be a reductive monoid with zero and unit group $G$. Let $K$ be a maximal compact subgroup of $G$ such that $T_K = T \cap K$ is a maximal compact torus. Suppose that the associated standard embedding $X = \mathbb{P}_e(M)$ is quasi-regular. Then

$$H^*(X) \simeq H^*((K \times K) \times_{(T_K \times T_K)} \mathcal{Y})^W,$$

where $\mathcal{Y} \subset X$ is the associated toric variety.

**Proof.** As $G/K$ is contractible, the functors $H^*_{G \times G}(-)$ and $H^*_{K \times K}(-)$ agree on $G \times G$-spaces. Similar remarks apply to $H^*_{T \times T}(-)$ and $H^*_{T_K \times T_K}(-)$, for $T/T_K$ is also contractible.

By hypothesis, $X$ is quasi-regular, so it follows from Theorem 4.12 that

$$H^*(X) = H^*_{T_K \times T_K}((\mathcal{Y})^W) / \mathcal{I} H^*_{T_K \times T_K}((\mathcal{Y})^W),$$

where $\mathcal{I}$ is the ideal of $(H^*_{T_K \times T_K})^{W \times W}$ generated by the elements of strictly positive degree. As pointed out in [LP], Remark 2.3, the induction formula ([Q], p. 552) implies that

$$H^*_{T_K \times T_K}((\mathcal{Y}) = H^*_{K \times K}((K \times K) \times_{T_K \times T_K} \mathcal{Y}),$$

and the latter is isomorphic to

$$H^*_{K \times K} \otimes_{\mathbb{Q}} H^*((K \times K) \times_{T_K \times T_K} \mathcal{Y}),$$

because $(K \times K) \times_{T_K \times T_K} \mathcal{Y}$ has no cohomology in odd degrees. We conclude that $H^*(X) \simeq H^*((K \times K) \times_{T_K \times T_K} \mathcal{Y})^W$. \qed

5. Simple Embeddings

Let $H$ be a connected reductive group. An embedding of $H$ is called simple if it contains only one closed $H \times H$-orbit.

Let $M$ be a reductive monoid with zero and unit group $G$. Let $B$ be a Borel subgroup of $G$ and $T \subset B$ be a maximal torus of $G$. Denote by $E(T)$ the idempotent set of $T$, the Zariski closure of $T$ in $M$. Let

$$\Lambda = \{ e \in E(T) \mid Be = eBe \}$$
be the cross section lattice of $M$ relative to $T$ and $B$. Recall that $\Lambda$ corresponds to the poset of $G \times G$-orbits in $M$. Also, write $\Lambda_k$ for the collection $\{ e \in \Lambda \mid \dim (Te) = k \}$.

**Definition 5.1.** A reductive monoid $M$ with 0 is called $\mathfrak{J}$-irreducible if $M \setminus \{0\}$ has exactly one minimal $G \times G$-orbit, equivalently, if $|\Lambda_1| = 1$ or if all minimal non-zero idempotents are conjugate.

Let $M$ be a $\mathfrak{J}$-irreducible monoid. It follows from Definition 5.1 that there is a unique, minimal, non-zero idempotent $e_1 \in E(T)$ such that $\Lambda_1 = \{ e_1 \}$, and $fe_1 = e_1$ for all $f \in \Lambda \setminus \{0\}$. Moreover, by Theorem 1.6, the Weyl group $W$ acts transitively on $E_1(T)$, the set of rank-one idempotents of $T$, that is, $E_1(T) \simeq W/C_W(e_1)$.

Any $\mathfrak{J}$-irreducible monoid is also semisimple. See [PR], or Section 7.3 of [R9] for a systematic discussion of this important class of reductive monoids, and for a proof of the following theorem.

**Theorem 5.2.** Let $M$ be a reductive monoid. The following are equivalent.

1. $M$ is $\mathfrak{J}$-irreducible.
2. There is an irreducible rational representation $\rho : M \to \text{End}(V)$ which is finite as a morphism of algebraic varieties.

Let $M$ be a $\mathfrak{J}$-irreducible monoid with $\Lambda_1 = \{ e_1 \}$, as above. We say that $M$ is $\mathfrak{J}$-irreducible of type $J$ if, for this idempotent $e_1$,

$$J = \{ s \in S \mid se_1 = e_1s \},$$

where $S$ is the set of simple involutions of $W$ relative to $T$ and $B$. Notice that $C_W(e_1) = W_J$, the subgroup of $W$ generated by $J$. The set $J$ can be determined in terms of any irreducible representation satisfying condition 2 of Theorem 5.2.

We now relate the study of simple projective embeddings to the theory of $\mathfrak{J}$-irreducible monoids.

**Theorem 5.3.** Standard embeddings obtained from $\mathfrak{J}$-irreducible monoids are exactly the simple projective embeddings of connected reductive groups.

**Proof.** Let $M$ be a $\mathfrak{J}$-irreducible monoid with unit group $G$. Because $M$ is semisimple, the center of $G$ is one-dimensional. So let $Z \simeq \mathbb{C}^*$ be the connected component of the center of $G$. Now consider $\mathbb{P}(M) = [M \setminus \{0\}]/Z$, the associated standard embedding of $G/Z$. We claim that $X$ is simple. Indeed, there is a one-to-one correspondence between closed $G/Z \times G/Z$-orbits in $X$ and minimal $G \times G$-orbits in $M \setminus \{0\}$. Since, by assumption, $M$ is $\mathfrak{J}$-irreducible, we conclude that $X$ has exactly one closed $G/Z \times G/Z$-orbit.

Conversely, let $X$ be a simple projective embedding of a connected reductive group $G'$. Then $X = \mathbb{P}(M)$, where $M$ is a reductive monoid with unit group $G = G' \times \mathbb{C}^*$ (Theorem 1.7). Because $X$ contains only one closed $G' \times G'$-orbit, $M \setminus \{0\}$ has exactly one minimal $G \times G$-orbit, i.e. $M$ is $\mathfrak{J}$-irreducible. □
Let $X = \mathbb{P}(M)$ be a simple projective embedding, where $M$ is a $\mathfrak{g}$-irreducible monoid with $\Lambda_1 = \{e_1\}$ and $J = \{s \in S \mid se_1 = e_1s\}$. In this context,

$$P_{e_1} = C_G^r(e_1) = \bigcup_{w \in W_J} BwB = P_J,$$

where $W_J = C_W(e_1)$, and $P_J \subset G$ is the standard parabolic subgroup associated to $J$. Hence, by Theorem 4.10 the unique closed orbit of $X$ is $G[e_1]G \simeq G/P_J \times G/P_J^-.$ Note that $X$ is quasi-regular only when $J = \emptyset$ (Corollary 4.15).

**Definition 5.4.** Let $X$ be a simple projective embedding. We say that $X$ is **simple of type** $J$ if $X = \mathbb{P}(M)$, where $M$ is a $\mathfrak{g}$-irreducible monoid of type $J$.

The type of a simple embedding is independent of its presentation as a standard embedding. Indeed, let $M$ and $M'$ be two $\mathfrak{g}$-irreducible monoids, with unit group $G$, and of type $J$ and $J'$, respectively. Then $\mathbb{P}(M)$ and $\mathbb{P}(M')$ are (equivariantly) isomorphic if and only if their closed orbits $G/P_J \times G/P_J^-$ and $G/P_J' \times G/P_J'^-$ are isomorphic ([R3], Remark 4.1). The latter holds if and only if the standard parabolic subgroups $P_J$ and $P_J'$ are conjugate, i.e., if and only if $J = J'$. Rationally smooth simple embeddings are classified, according to their type, in [R3]. The reader will find there a complete list of all the subsets $J \subset S$ which yield rationally smooth embeddings.

Let $X = \mathbb{P}(M)$ be a simple embedding of type $J$. Given that $X$ has only one closed orbit, we can associate to any $g \in E_2(T)$ a unique reflection $s_{\alpha_g}$ such that $s_{\alpha_g}g = gs_{\alpha_g} \neq g$ (Lemma 4.5). Put

$$L^J = \{g \in E_2(T) \mid ge_1 = e_1\}.$$

**Theorem 5.5.** Notation being as above, let $X = \mathbb{P}(M)$ be a simple embedding of type $J$. Suppose that $X$ is rationally smooth. Then the natural morphism

$$H^*_T \times T(X) \rightarrow H^*_T \times T(G/P_J \times G/P_J^-)$$

is injective, and its image consists of all maps $\varphi \in H^*_T \times T(G/P_J \times G/P_J^-)$, subject to the condition: for every $g \in L^J$, $u \in W$, and $v \in W$, the following holds

$$\varphi(u e_1 u^{-1} v) \equiv \varphi(u \alpha_g e_1 \alpha_g u^{-1} v) \mod (\alpha_g \circ \text{int}(u^{-1}), \alpha_g \circ \text{int}(u^{-1}) \circ \text{int}(v)),$$

where $\alpha_g$ is the root associated to the reflection $s_{\alpha_g}$.

**Proof.** The first assertion is a direct consequence of Theorem 4.10. Besides, there are no curves of type 3, for curves of that type join necessarily fixed points in different closed $G \times G$-orbits. Consequently, we just need to focus on translating Theorem 4.10 (a) into our situation. Let $f \in E_2(T)$. Then there are exactly two rank-one idempotents $f_1, f_2$, such that $f_1f = f_1$, $ff_1 = f_2$ and $f_2 = s_\alpha f_1 s_\alpha$, where $s_\alpha f = s_\alpha f \neq f$. On the other hand, because
\[ \Lambda_1 = \{e_1\}, \text{ then } f_1 = u e_1 u^{-1}, \text{ for some } u \in W. \] The latter implies that
\[ g = u^{-1} f u \text{ is an idempotent of } T \text{ such that } g e_1 = e_1, \text{ that is, } g \in L^J. \] In short, any \( f \in E_2(T) \), such that \( f e = e \) for some \( e \in E_1(T) \), is conjugate to an element of \( L^J \). This observation and Theorem 4.10 (a) yield the result. □

**Corollary 5.6.** Let \( X = \mathbb{P}(M) \) be a rationally smooth simple embedding of type \( J \). Let \( e_1 \) be the unique rank-one idempotent for which \( \Lambda_1 = \{e_1\} \). Then the ring \( H^*_T \times T(X) \) consists of all tuples \( \Psi \), where
\[
\varphi : W e_1 W \rightarrow (H^*_T \times T)^{W_J \times W_J},
\]
such that
\[
\varphi(e_1) \equiv \varphi(\alpha g e_1 \alpha g) \mod (\alpha g, \alpha g),
\]
for every \( g \in L^J \).

**Proof.** Simply translate Corollary 4.11 into this situation, making use of Theorem 5.5. □

5.1. **Examples.**

5.1.1. The wonderful compactification. Let \( G' \) be a connected semisimple group of adjoint type. Let \( X \) be the wonderful compactification of \( G' \) ([DP-1]). In this case, \( X = \mathbb{P}(M) \), where \( M \) is a \( J \)-irreducible monoid of type \( J = \emptyset \). Let \( G = G' \times C^* \) be the unit group of \( M \) and let \( T \) be a maximal torus of \( G \). Let \( \Lambda_1 = \{e\} \). In this case, our Theorem 5.5 yields a different proof of the results of [Br3] and [U].

**Corollary 5.7.** Let \( X = \mathbb{P}(M) \) be the wonderful compactification of a semisimple group \( G' \) of adjoint type. Then \( H^*_T \times T(X) \) consists of all maps \( \varphi \in H^*_T \times T(G/B \times G/B) \) such that
\[
\varphi(u e u^{-1} v) \equiv \varphi(\alpha g e \alpha g u^{-1} v) \mod (\alpha \circ \text{int}(u^{-1}), \alpha \circ \text{int}(u^{-1}) \circ \text{int}(v)),
\]
for every root \( \alpha \in S \) and \( (u, v) \in W \times W \).

**Proof.** For the wonderful compactification, we have \( GeG \simeq G/B \times G/B \). In addition, since \( J = \emptyset \), then \( L^J \simeq S \) (Proposition 2.16 of [R5]). These observations, and Theorem 5.5, finally imply the result. □

5.1.2. A familiar object: \( \mathbb{P}^{(n+1)^2-1}(C) \). Let \( M = M_{n+1} \) and \( G = GL_{n+1} \). In this case,
\[ W = \langle s_1, \ldots, s_n \rangle, \]
the Weyl group of type \( A_n \). It is known that \( W \) is isomorphic to \( S_{n+1} \), the permutation group of the set \( \{1, 2, \ldots, n + 1\} \), and so \( s_i \) is the simple permutation \( i \leftrightarrow i + 1 \). Recall that \( s_i \) corresponds to the elementary permutation matrix \( E_{i,i+1} \): the identity matrix with the \( i \)-th and \( (i + 1) \)-th rows interchanged.
Clearly, $X = \mathbb{P}(M) = \mathbb{P}^{(n+1)^2-1}$. One also checks that $\Lambda_1 = \{e\}$, where $e = (a_{ij})$, with $a_{11} = 1$ and $a_{ij} = 0$ for any $(i, j) \neq (1,1)$. It follows that $X$ is a simple embedding of type $J = \{s_2, \ldots, s_n\} \subseteq S = \{s_1, \ldots, s_n\}$, and the unique closed orbit in $X$ is $G[e]G = \mathbb{P}^n \times \mathbb{P}^n$. It is worth noting that $X$ is not quasi-regular, even though it is smooth.

For each $i \in \{1, 2, \ldots, n\}$, denote by $g_i$ the $0-1$ diagonal matrix with exactly two non-zero rows: the first row and the $(i+1)$-th row. Let $L^J = \{g \in E_2(T) \mid ge = e\}$. One verifies that $L^J = \{g_1, g_2, \ldots, g_n\}$. Finally, the root associated to $g_i$ is $\alpha_i(t) = t_1 t_{i+1}^{-1}$, where $t = \text{diag}(t_1, \ldots, t_{n+1})$ is an element of $T$.

**Corollary 5.8.** $H^*_{T \times T}(\mathbb{P}^{(n+1)^2-1})$ injects into $H^*_{T \times T}(\mathbb{P}^n \times \mathbb{P}^n)$ and it consists of all maps $\varphi \in H^*_{T \times T}(\mathbb{P}^n \times \mathbb{P}^n)$ subject to the condition that, for every $g_i \in L^J$, $1 \leq i \leq n$, and $(u, v) \in S_n \times S_n$, the following holds:

$$
\varphi(ueu^{-1}v) \equiv \varphi(us_{\alpha_i}es_{\alpha_i}u^{-1}v) \text{ mod } (\alpha_i \circ \text{int}(u^{-1}), \alpha_i \circ \text{int}(u^{-1}) \circ \text{int}(v)),
$$

where $s_{\alpha_i}$ is the reflection associated to the root $\alpha_i = t_1 t_{i+1}^{-1}$. \qed

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