ON THE EXTENDABILITY OF FREE MULTIARRANGEMENTS

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Abstract. A free multiarrangement of rank \( k \) is defined to be extendable if it is obtained from a simple rank \((k+1)\) free arrangement by the natural restriction to a hyperplane (in the sense of Ziegler). Not all free multiarrangements are extendable. We will discuss extendability of free multiarrangements for a special class. We also give two applications. The first is to produce totally non-free arrangements. The second is to give interpolating free arrangements between extended Shi and Catalan arrangements.

1. Introduction

Let \( V = \mathbb{C}^\ell \) be a complex vector space with coordinate \((x_1, \cdots, x_\ell)\), \( \mathcal{A} = \{H_1, \ldots, H_n\} \) be a central arrangement of hyperplanes. Let us denote by \( S = \mathbb{C}[x_1, \ldots, x_\ell] \) the polynomial ring and fix \( \alpha_i \in V^* \) a defining equation of \( H_i \), i.e., \( H_i = \alpha_i^{-1}(0) \). A multiarrangement is a pair \((\mathcal{A}, m)\) of an arrangement \( \mathcal{A} \) with a multiplicity \( m: \mathcal{A} \to \mathbb{Z}_{\geq 0} \), and there exists a basis \( \delta_1, \ldots, \delta_\ell \in D(\mathcal{A}, m) \) such that \( \deg \delta_i = e_i \). Note that the degree \( \deg \delta \) of a derivation \( \delta \) is the polynomial degree, that is defined by \( \deg(\delta f) = \deg \delta + \deg f - 1 \) for a homogeneous polynomial \( f \). An arrangement \( \mathcal{A} \) is said to be free if \((\mathcal{A}, 1)\) is free. Here we recall that the freeness is closed under localization. More precisely, let \( X \subset V \) be a subset and define \( \mathcal{A}_X = \{H \in \mathcal{A} | H \supset X\} \). Then the freeness of \((\mathcal{A}, m)\) implies that of \((\mathcal{A}_X, m|_{\mathcal{A}_X})\).

A multiarrangement naturally appears as a restriction of a simple arrangement \([22]\). Let \( \mathcal{A} \) be an arrangement. The arrangement \( \mathcal{A} \) determines the restricted arrangement \( \mathcal{A}^H = \{H \cap H' | H' \in \mathcal{A}, H' \neq H\} \) on \( H \in \mathcal{A} \). The restricted arrangement \( \mathcal{A}^H \) possesses a natural multiplicity \( m^H: \mathcal{A}^H \to \mathbb{Z} \)

\[
X \mapsto \sharp \{H' \in \mathcal{A} | X = H \cap H'\}.
\]

The freeness of \( \mathcal{A} \) and \((\mathcal{A}^H, m^H)\) are connected by the following theorem due to Ziegler.

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Theorem 1.2. [22] If \( A \) is free with exponents \((1, e_2, \ldots, e_\ell)\), then the restriction \((A^H, m^H)\) is free with exponents \((e_2, \ldots, e_\ell)\).

Recently freeness of multiarrangements are extensively studied [3, 4, 12, 15, 16, 17]. The motivation to this article is to ask whether if a free multiarrangement is obtained as a restriction of a free simple arrangement. Theorem 1.2 leads us to introduce the following notion, which seems to give an important class of free multiarrangements.

Definition 1.3. Let \((A, m)\) be a free multiarrangement in \( K^\ell \). We say \((A, m)\) is extendable if it can be obtained as a restriction of a free simple arrangement in \( K^{\ell+1} \).

Example 1.4. (Non-extendable free multiarrangement) Consider a multiarrangement in \( \mathbb{R}^2 \)

\[
Q(A, m) = x^3y^3(x-y)^1(x-\alpha y)^1(x-\beta y)^1,
\]

with \( \alpha, \beta \neq 0, \pm 1 \) and assume \( \alpha \) and \( \beta \) are algebraically independent over \( \mathbb{Q} \). (Indeed \( \alpha \beta \neq 1 \) is enough.) If the slopes \( \alpha \) and \( \beta \) are generic, then \((A, m)\) is free with exponents \((4,5)\) [17]. So the product of exponents is always \( \leq 20 \). We can prove that it is not extendable. It can be proved as follows (details are left to the reader).

The deconving \( \overline{A} \) (10) with respect to the hyperplane at infinity of an extension of \((A, m)\) is an affine line arrangement \( \mathbb{R}^2 \) having the following defining equations:

\[
\begin{align*}
x &= a_1, a_2, a_3, \\
y &= b_1, b_2, b_3, \\
x - y &= c, \\
x - \alpha y &= d, \\
x - \beta y &= e,
\end{align*}
\]

where \( a_i, b_i, c, d, e \in \mathbb{R} \). The characteristic polynomial \( \chi(\overline{A}, t) \) is of the form \( \chi(\overline{A}, t) = t^2 - 9t + p, \) and we can prove that \( p > 20 \). Thus \( \chi(\overline{A}, t) \) is not factored. It follows from Terao’s factorization theorem (14) that any extension of \( A \) is not free.

Thus a free multiarrangement \((A, m)\) is not necessarily extendable in general. In the next section, we focus on some special kind of multiarrangements.

2. Extendability of locally \( A_2 \) arrangements

Definition 2.1. An arrangement \( A = \{H_1, \ldots, H_n\} \) is said to be locally \( A_2 \) if \(|A_X| \leq 3\) is satisfied for any codimension two intersection \( X \). A system of defining equations \( \{\alpha_1, \ldots, \alpha_n\} \) of a locally \( A_2 \) arrangement \( A \) is called a positive system if the following condition is satisfied: Suppose \( X \) is a codimension two intersection with \(|A_X| = 3\). Setting \( A_X = \{H_i, H_j, H_k\} \). Then one of \( \alpha_i = \alpha_j + \alpha_k, \alpha_j = \alpha_i + \alpha_k \) or \( \alpha_k = \alpha_i + \alpha_j \) holds.

Example 2.2. The following are examples of locally \( A_2 \) arrangements with positive systems.

1. Generic in codimension three. (Equivalently, \(|A_X| = 2\) for any codimension two intersection \( X \).) In this case any system of defining equations is a positive system.
(2) Coxeter arrangement of type ADE. In this case, a positive root system is corresponding to a positive system of defining equations.

(3) Subarrangements or direct products of locally $A_2$ arrangements with positive systems possess the same property. Especially, this class is closed under localization.

(4) Shi arrangement of type $A_2$ $Q = xyz(x + y)(x - z)(y - z)(x + y - z)$.

Remark 2.3. Note that a locally $A_2$ arrangement does not necessarily have a positive system (e. g., $Q = xyz(x + y)(x - z)(y - z)(x + y - 2z)$).

We will discuss the extendability for multiarrangements of this class. More precisely, we consider the following concrete extension $E(A, m)$ for given locally $A_2$ arrangement $A$ with a positive system $(\alpha_H)_H$. Let $(x_1, \ldots, x_\ell, z) \in \mathbb{C}^\ell \times \mathbb{C}$ be a coordinate system of $V \times \mathbb{C}$ and define

$$E(A, m) = \{ z = 0 \} \cup \left\{ \alpha_H = k z \mid k \in \mathbb{Z}, \frac{m(H) - 1}{2} \leq k \leq \frac{m(H)}{2} \right\}.$$ 

Then, if we denote $H_0 = \{ z = 0 \}$, it is obvious that $(E(A, m)^{H_0}, m^{H_0}) = (A, m)$.

Let us define the deconing of $E(A, m)$ as follows:

$$dE(A, m) = \left\{ \alpha_H = k \mid k \in \mathbb{Z}, \frac{m(H) - 1}{2} \leq k \leq \frac{m(H)}{2} \right\}.$$ 

Note that $dE(A, m)$ is an affine arrangement in $V$.

Remark 2.4. The above definition is motivated by that of the extended Catalan and Shi arrangements [8]. Indeed, let $A$ be a Coxeter arrangement of type ADE. Choose the positive root system as the positive system as above. For a given positive integer $k \in \mathbb{Z}_{>0}$, consider constant multiplicities $m = 2k$ and $m = 2k + 1$.

Then $E(A, 2k + 1)$ is so called the extended Catalan arrangement and $E(A, 2k)$ is called the extended Shi arrangement, which are known to be free [20].

Theorem 2.5. Let $A$ be a locally $A_2$ arrangement with a positive system in $V = \mathbb{C}^\ell$. We fix a positive system $\Phi^+ = \{ \alpha_H \mid H \in A \} \subset V^*$ of defining equations. Let $m : A \rightarrow \mathbb{Z}_{\geq 0}$ be a multiplicity. We assume the following condition:

(*) Let $A_X = \{ H_i, H_j, H_k \}$ be a codimension two localization with $\alpha_i = \alpha_j + \alpha_k$.

If $m(H_i)$ is odd, then at least one of $m(H_j)$ or $m(H_k)$ is odd.

Then $(A, m)$ is free, if and only if it is extendable. Indeed, $E(A, m)$ is a free arrangement.

We will give the proof in the next section. Here we notice an immediate corollary.

Corollary 2.6. Let $A$ be a locally $A_2$ arrangement with a positive system. Suppose that the multiplicity $m$ satisfies either $m(H)$ is odd $\forall H \in A$ or $m(H)$ is even $\forall H \in A$. If the multiarrangement $(A, m)$ is free, then it is extendable.

Remark 2.7. The condition (*) in Theorem 2.5 is related to the following phenomenon. Consider a multiarrangement $x^2y^2(x + y)^1$. Then (deconing of) our extension $dE(x^2y^2(x + y)^1)$ is defined by

$$x(x - 1)y(y - 1)(x + y),$$ 

which is not free. However another extension

$$x(x - 1)y(y - 1)(x + y - 1)$$ 

is free.
is free. This shows that even \( E(A, m) \) is not free, \((A, m)\) might have another free extension. The author does not know whether if the extendability can be proved without assuming condition (*). See for a little more complicated example.

**Example 2.8.** Let us consider a multiarrangement \( x^4y^4z^4(x+y)^5(y+z)^5(x+y+z)^4 \). It is known to be free with exponents \((8, 9, 9)\) (see [5, 18] or Proposition 5.2 below). The extension \( E(x^4y^4z^4(x+y)^5(y+z)^5(x+y+z)^4) \) is defined by

\[
\begin{align*}
x, y, z & = kw (k = -1, 0, 1, 2) \\
x + y, y + z & = kw (k = -2, -1, 0, 1, 2) \\
x + y + z & = kw (k = -1, 0, 1, 2) \\
w & = 0,
\end{align*}
\]

which is not free (look at the localization at \( x = y = w = 0 \) and use Lemma 3.1 (3-ii)). However another extension

\[
\begin{align*}
x, y, z & = kw (k = -1, 0, 1, 2) \\
x + y, y + z & = kw (k = -1, 0, 1, 2, 3) \\
x + y + z & = kw (k = 0, 1, 2, 3) \\
w & = 0,
\end{align*}
\]

is free.

We can check the following for \( \ell = 3 \).

**Question 2.9.** Suppose \( A \) is of type \( A_\ell \) and \((A, m)\) is free. Then is \((A, m)\) always extendable?

### 3. Proof

Proof of Theorem 2.5 is done by the induction on the rank \( \ell \). If \( \ell = 2 \), then \( A \) is either \( |A| = 2 \) or type \( A_2 \). Suppose \( |A| = 2 \). Then \( E(A, m) \) is obviously free. Suppose \((A, m)\) is defined by \( x^a y^b(x+y)^c \). The next lemma is elementary.

**Lemma 3.1.** \( \) Assume \( a < b \). Set \( k = a + b + c \) and \( \mathcal{E} = E(x^a y^b(x+y)^c) \).

1. If \( c < b - a + 1 \), then \( \chi(\mathcal{E}, t) = (t-1)(t-b)(t-a-c) \).
2. If \( c \geq a + b + 1 \), then \( \chi(\mathcal{E}, t) = (t-1)(t-a-c) \).
3. If \( b - a \leq c - 1 < a + b \),
   - \( (a, b, c) \neq (even, even, odd) \), then \( \chi(\mathcal{E}, t) = (t-1)(t-|k/2|)(t-|k/2|) \).
   - \( (a, b, c) = (even, even, odd) \), then \( \chi(\mathcal{E}, t) = (t-1)\left((t - \frac{k}{2})^2 + \frac{3}{4}\right) \).

The next result is due to Wakamiko.

**Proposition 3.2.** Let \((A, m) = x^a y^b(x+y)^c\). Assume \( a \leq b \) and set \( k = a + b + c \) as above. Since it is rank two, \((A, m)\) is always free. The exponents are given as follows:

1. If \( c < b - a + 1 \), then \( \exp(\mathcal{A}, m) = (b, a + c) \).
2. If \( c \geq a + b + 1 \), then \( \exp(\mathcal{A}, m) = (c, a + b) \).
3. If \( b - a \leq c - 1 < a + b \), then \( \exp(\mathcal{A}, m) = ([k/2], [k/2]) \).

In [21], a characterization of freeness for rank three arrangements is given. It can be stated as follows.

**Proposition 3.3.** For \( \ell = 2 \), \( E(A, m) \) is free with exponents \((1, d_1, d_2)\) if and only if
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- $\chi(E(\mathcal{A}, m), t) = (t - 1)(t - d_1)(t - d_2)$ and
- $\exp(\mathcal{A}, m) = (d_1, d_2)$.

Combining these results, we can prove Theorem 3.4 for $\ell = 2$. (Note that the condition (*) in the theorem is corresponding to that Lemma 3.1 (3) (ii) does not occur.)

We now consider the case $\ell \geq 3$. Let us first recall the following result.

**Proposition 3.4.** [20] $E(\mathcal{A}, m)$ is free with exponents $(1, e_1, \ldots, e_\ell)$ if and only if $(\mathcal{A}, m)$ is free with exponents $(e_1, \ldots, e_\ell)$ and $E(\mathcal{A}, m)_X$ is free for any positive dimensional intersection $X \subset V$.

It is easily seen that $E(\mathcal{A}, m)_X = E(\mathcal{A}_X, m|_{\mathcal{A}_X})$. Since the localization $(\mathcal{A}_X, m|_{\mathcal{A}_X})$ of a free multiarrangement $(\mathcal{A}, m)$ is free with rank at most $\ell - 1$, it follows from the inductive hypothesis that $E(\mathcal{A}, m)_X$ is free. Hence Proposition 3.4 shows that $E(\mathcal{A}, m)$ is free.

\[\square\]

4. **Totally non-free arrangements**

In a recent paper [4] Abe, Terao and Wakefield observed several phenomena concerning multiplicities and freeness of a multiarrangement $(\mathcal{A}, m)$. In particular they prove that generic four planes $(\mathcal{A}, m)$ defined by $x_1^{m_1}x_2^{m_2}x_3^{m_3}(x_1 + x_2 + x_3)^{m_4}$ will never free for any positive multiplicity $m : \mathcal{A} \to \mathbb{Z}_{>0}$. Such an arrangement $\mathcal{A}$ is called totally non-free. As an application of extendability techniques, we give a straightforward proof of totally non-freeness for generic arrangements.

**Proposition 4.1.** Suppose $\ell = \dim V \geq 3$ and $\mathcal{A}$ is a generic arrangement with $|\mathcal{A}| > \ell$. Let $m : \mathcal{A} \to \mathbb{Z}_{>0}$. Then $(\mathcal{A}, m)$ is not free.

**Proof.** Fix a defining equations $\alpha_H$ for each $H$. As is already noticed, it is a positive system. Since $(\mathcal{A}, m)$ is locally Boolean, $E(\mathcal{A}, m)_X = E(\mathcal{A}_X, m|_{\mathcal{A}_X})$ is free for any nonzero subspace $X \subset V \times \{0\}$. Hence if $(\mathcal{A}, m)$ is free, then Proposition 3.4 shows that $E(\mathcal{A}, m)$ is also free. However, let us consider the restriction to the subspace $X = \{0\} \times \mathbb{C} \subset V \times \mathbb{C}$. Then the localization $E(\mathcal{A}, m)_X$ is isomorphic to $\mathcal{A}$ which is not free. This is a contradiction. \[\square\]

5. **Free interpolations between extended Shi and Catalan arrangements**

Let $\mathcal{A}$ be a crystallographic Coxeter arrangement with a fixed positive system $\Phi^+$ of roots. As is already mentioned, $E(\mathcal{A}, 2k + 1)$ and $E(\mathcal{A}, 2k)$ are free for any $k \in \mathbb{Z}_{>0}$. Obviously these two families of arrangements are related to each other as

$$\cdots \subset E(\mathcal{A}, 2k - 1) \subset E(\mathcal{A}, 2k) \subset E(\mathcal{A}, 2k + 1) \subset \cdots .$$

In [19], it is observed that there exist many free arrangements $\mathcal{B}$ such that $E(\mathcal{A}, 2k) \subset \mathcal{B} \subset E(\mathcal{A}, 2k + 1)$. The purpose of this section is to give a complete description of free arrangements interpolating these families for type ADE.

Let $m : \mathcal{A} \to \{0, 1\}$ be a $\{0, 1\}$-valued multiplicity. Any interpolating arrangement can be described as $E(\mathcal{A}, 2k \pm m)$ for some $m$. We will describe free interpolations by using $\{0, 1\}$-valued multiplicity $m$. Our main result in this section is the following.
Theorem 5.1. Let $\mathcal{A}$ be an irreducible Coxeter arrangement of type ADE with the Coxeter number $h$. Fix $\Phi^+$ a positive root system. Let $k$ be a positive integer. Then the following conditions are equivalent.

1. $m : \mathcal{A} \to \{0, 1\}$ satisfies the following condition.
   
   1-i) $m^{-1}(1) \subset \mathcal{A}$ is a free subarrangement with exponents $(e_1, \ldots, e_\ell)$.
   
   1-ii) if $\alpha_1 = \alpha_2 + 3\alpha_3 (\alpha_i \in \Phi^+)$ and $m(H_1) = 1$, then at least $m(H_2) = 1$ or $m(H_3) = 1$.

2. $E(\mathcal{A}, 2k + m)$ is free with exponents $(1, kh + e_1, \ldots, kh + e_\ell)$.

3. $E(\mathcal{A}, 2k - m)$ is free with exponents $(1, kh - e_1, \ldots, kh - e_\ell)$.

Before going proof of Theorem 5.1 let us recall a result from [5].

Proposition 5.2. [5, Corollary 12] Let $\mathcal{A}$ be the Coxeter arrangement with the Coxeter number $h$, and $m : \mathcal{A} \to \{0, 1\}$ be a multiplicity. Let $k \in \mathbb{Z}_{>0}$. Then the following conditions are equivalent.

- $(\mathcal{A}, m)$ is free with exponents $(e_1, \ldots, e_\ell)$.
- $(\mathcal{A}, 2k + m)$ is free with exponents $(kh + e_1, \ldots, kh + e_\ell)$.
- $(\mathcal{A}, 2k - m)$ is free with exponents $(kh - e_1, \ldots, kh - e_\ell)$.

First we prove (1) $\Rightarrow$ (2). Suppose $m$ satisfies (1-ii). Then the multiplicity $2k + m$ satisfies the condition (*) in Theorem 2.5. Thus the extension $E(\mathcal{A}, 2k + m)$ is free if and only if the multiarrangement $(\mathcal{A}, 2k + m)$ is free. But this is done by the assumption (1-i) and Proposition 5.2.

The implication (1) $\Rightarrow$ (3) is similar.

Finally let us prove (2) $\Rightarrow$ (1). Suppose $E(\mathcal{A}, 2k + m)$ is free. Then by restricting to $H_0$, we have (by Theorem 1.2), the multiarrangement $(\mathcal{A}, 2k + m)$ is free. Again from Proposition 5.2 we have $(\mathcal{A}, m)$ is free, in other words, $m^{-1}(1) \subset \mathcal{A}$ is a free subarrangement. Thus we have (1-i). To prove (1-ii), suppose that there exists $H_1$ such that $\alpha_1 = \alpha_2 + \alpha_3$ and $m(H_1) = 1, m(H_2) = m(H_3) = 0$. Then set $X := H_1 \cap H_2 \cap H_3$, which is a codimension two subspace. From Lemma 5.1(3-ii), the localization $E(\mathcal{A}, 2k + m)_X$ is not free. It is a contradiction. Thus (1-ii) is satisfied.

Using Terao’s factorization theorem, we obtain the following corollary.

Corollary 5.3. Let $\mathcal{A}$ be a Coxeter arrangement with the Coxeter number $h$ and $m : \mathcal{A} \to \{0, 1\}$ be a multiplicity satisfying the condition (1) of Theorem 5.1. Then

$$\chi(\mathcal{A}, 2k \pm m) = \prod_{i=1}^{\ell} (t - kh \mp e_i).$$

The above formula implies

$$\chi(\mathcal{A}, 2k - m) = (-1)^{\ell} \chi(\mathcal{A}, 2k + m, 2kh - t).$$

We should note that the formula [5.1] is very similar to the “functional equation” discovered by Postnikov and Stanley [11]. It might be worth asking whether the formula [5.1] holds for any crystallographic Coxeter arrangement $\mathcal{A}$ and any multiplicity $m : \mathcal{A} \to \{0, 1\}$.

Remark 5.4. Recently Abe, Nuida and Numata obtained more general results for type $A_\ell$ arrangements [2, 9]. Their results suggest that [5.1] holds even for wider class of multiplicities, namely, $m : \mathcal{A} \to \{-1, 0, 1\}$. 

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