On the Fourier Transform of Bessel Functions over Complex Numbers
- II: the General Case

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Abstract. In this paper, we prove an exponential integral formula for the Fourier transform of Bessel functions over complex numbers, along with a radial exponential integral formula. The former will enable us to develop the complex spectral theory of the relative trace formula for the Shimura-Waldspurger correspondence and extend the Waldspurger formula from totally real fields to arbitrary number fields.

1. Introduction

1.1. Representation Theoretic Motivations. It is known by the work of Baruch and Mao (BM1, BM2) that the exponential integral formulae due to Weber and Hardy on the Fourier transform of classical Bessel functions over real numbers realize the Shimura-Waldspurger correspondence between representations of $\operatorname{PGL}_2(\mathbb{R})$ and genuine representations of $\widetilde{\operatorname{SL}}_2(\mathbb{R})$ and constitute the real component of the Waldspurger formula for automorphic forms of $\operatorname{PGL}_2$ and $\widetilde{\operatorname{SL}}_2$ over $\mathbb{Q}$ or a totally real field. For instance, the formula of Weber is as follows

$$\int_0^\infty \frac{1}{\sqrt{x}} J_{\nu} \left(4\pi \sqrt{x}\right) \left(\pm \pi y\right) e^{\pm \left(\frac{1}{2y} - \frac{1}{8} \nu - \frac{1}{8}\right)} J_{\frac{1}{2} \nu} \left(\frac{\pi}{y}\right),$$

for $y > 0$, where $e(x) = \exp(2\pi ix)$ and $J_{\nu}(x)$ is the Bessel function of the first kind of order $\nu$. This formula is valid when $\Re \nu > -1$. Taking $\nu = 2k - 1$ in (1.1), with $k$ a positive integer, the Bessel function of order $2k - 1$, respectively $k - \frac{1}{2}$, is attached to a discrete series representation of $\operatorname{PGL}_2(\mathbb{R})$, respectively $\widetilde{\operatorname{SL}}_2(\mathbb{R})$. Thus, in this case, (1.1) should be interpreted as the local ingredient at the real place of the correspondence due to Shimura, Shintani and Waldspurger between cusp forms of weight $2k$ and cusp forms of weight $k + \frac{1}{2}$.

The purpose of this paper is to prove the complex analogue of Weber and Hardy’s formulae for Bessel functions over complex numbers. As applications of this paper in the future, one may develop the complex spectral theory of the relative trace formula for the

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Shimura-Waldspurger correspondence as the real theory in [BM1], and furthermore extend
the Waldspurger formula from totally real fields as in [BM2] to arbitrary number fields.

1.2. Statement of Results. We now introduce the definition of Bessel functions over
complex numbers (see [Qi2] §15.3, [BM4] (6.21), (7.21)). Let \( \mu \) be a complex number
and \( m \) be an integer. We define

\[
J_{\mu, m}(z) = J_{-2\mu - \frac{1}{2}m}(z) J_{-2\mu + \frac{1}{2}m}(z).
\]

The function \( J_{\mu, m}(z) \) is well defined in the sense that the expression on the right of (1.2) is
independent on the choice of the argument of \( z \) modulo \( 2\pi \). Next, we define

\[
J_{\mu, m}(z) = \begin{cases}
\frac{2\pi^2}{\sin(2\pi \mu)} (J_{\mu, m}(4\pi \sqrt{z}) + J_{4\mu, -m}(4\pi \sqrt{z})) & \text{if } m \text{ is even}, \\
\frac{2\pi^2}{\cos(2\pi \mu)} (J_{\mu, m}(4\pi \sqrt{z}) - J_{4\mu, -m}(4\pi \sqrt{z})) & \text{if } m \text{ is odd},
\end{cases}
\]

where \( \sqrt{z} \) is the principal branch of the square root, and it is understood that in the non-
generic case when \( 4\mu \in \mathbb{Z} + m \) the right hand side should be replaced by its limit. We stress
that \( J_{\mu, m}(z) \) is well defined only when \( m \) is even; nevertheless \( J_{\mu, m}(z^2) \) is always a well
defined function on the complex plane. Moreover, we note that \( J_{-\mu, -m}(z) = J_{\mu, m}(z) \), so
we may assume with no loss of generality that \( m \) is nonnegative.

**Remark 1.1.** According to [Qi2] §17, 18], on choosing the Weyl element \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \),
when \( m \) is even, respectively odd, \( |z| J_{\mu, m}(z) \), respectively \( \sqrt{z} J_{\mu, m}(z) \), is the Bessel func-
tion associated with the principal series representation \( \pi_{\mu, m} \) of \( SL_2(\mathbb{C}) \) (not necessarily
unitary) induced from the character \( \chi_{\mu, m} \left( \begin{pmatrix} a & \cdot \\ \cdot & a^{-1} \end{pmatrix} \right) = |a|^{2\mu}(a/|a|)^m \). In the even case,
the principal series is indeed a representation of \( PGL_2(\mathbb{C}) = PSL_2(\mathbb{C}) \).

For the kernel formula that defines Bessel functions for \( SL_2(\mathbb{C}) \) in representation the-
ory, its proof and applications, see [BM3, Mot, BBA, Qi2, Qi3].

Our main theorem is as follows.

**Theorem 1.2.** Suppose that \( |\text{Re} \mu| < \frac{1}{2} \) and \( m \) is even. We have the identity

\[
(1.4) \int_0^{2\pi} \int_0^{\infty} J_{\mu, m}(xe^{i\phi}) e(-2xy \cos(\phi + \theta))dx d\phi = \frac{1}{4y} e\left( \frac{\cos \theta}{y} \right) J_{\frac{1}{2\mu} - \frac{1}{2}, m}(\frac{1}{16y^2 e^{2\theta}}),
\]

for \( y \in (0, \infty) \) and \( \theta \in [0, 2\pi] \).

**Remark 1.3.** The identity (1.4) reflects the Shimura-Waldspurger correspondence be-
tween the principal series \( \pi_{\mu, m} \) of \( PGL_2(\mathbb{C}) \) and the principal series \( \pi_{\frac{1}{2\mu} - \frac{1}{2}, m} \) of \( SL_2(\mathbb{C}) \). It
should be noted that, unlike \( SL_2(\mathbb{R}) \), there is no nontrivial double cover of \( SL_2(\mathbb{C}) \) and
there do not exist discrete series for \( SL_2(\mathbb{C}) \).

In a previous paper [Qi1], using two formulae for classical Bessel functions, the author
has proved (1.4) in the spherical case when \( m = 0 \). For the nonspherical case, it seems
however that a straightforward proof as in [QH] is almost impossible. In this paper, our proof of (1.4) is in an indirect manner and splits into two steps.

In the first step, we shall prove a radial exponential integral formula (see (1.5) in Theorem 1.4 below), which is considered weaker than the formula (1.4), on the integral of the Bessel function $J_{\mu}(xe^{i\phi})$ against the radial exponential function $\exp(-2\pi cx)$, instead of the Fourier kernel $e(-2xy\cos(\phi + \theta))$. Interestingly, it turns out that the bulk of its proof is combinatorial.

In the second step, we shall prove Theorem 1.2 exploiting a soft method that combines asymptotic analysis of oscillatory integrals and a uniqueness result for ordinary differential equations. The weak exponential integral formula (1.5) is used to determine the constant term in the asymptotic, whereas the method of stationary phase for double integrals is applied for the oscillatory term.

**Theorem 1.4.** Suppose that $|\text{Re} \mu| < \frac{1}{2}$ and $m$ is even. We have

\[
\int_0^{2\pi} \int_0^{\infty} J_{\mu,m}(xe^{i\phi}) \exp(-2\pi cx) dx d\phi = \frac{4\pi^m}{c} K_{2\mu} \left( \frac{4\pi}{c} \right) I_{2m} \left( \frac{4\pi}{c} \right),
\]

for $|\arg c| < \frac{1}{2}\pi$, where $I_\nu(z)$ and $K_\nu(z)$ are the two kinds of modified Bessel functions of order $\nu$.

Although it is not visible in the statement, there is a remarkable distinction between the spherical and nonspherical cases in the proof of Theorem 1.4. It comes from Kummer’s confluent hypergeometric function $M \left( \frac{1}{2}m + 1; m + 1; z \right)$ arising in the proof (see §2.2 and §4.1), and makes the proof of the nonspherical case considerably harder. As alluded to above, when $m = 2k \geq 2$, the identity (1.5) in Theorem 1.4 may be reduced to a complicated combinatorial recurrence identity as follows,

\[
\sum_{n=0}^{k} (-)^n C_{2k}^n \sum_{r=0}^{[n/2]} (-)^r (2/a)^{n-2r} \left( C_{n-r}^r + C_{n-r-1}^{r-1} \right) C_{n-r-1}^{r-1} (I_\nu(aw) v^k (1-v)^{1-k}) = 0,
\]

where $C_n^m$ denotes the binomial coefficient. Moreover, we remark that, when searching for a straightforward proof of Theorem 1.2 such a distinction persists and makes our attempts rather hopeless.

Finally, we would also like to interpret Theorem 1.2 in the theory of distributions. Let $\mathcal{S}(\mathbb{C})$ denote the space of Schwartz functions on $\mathbb{C}$, that is, smooth functions on $\mathbb{C}$ that rapidly decay at infinity along with all of their derivatives. If rapid decay also occurs at zero, then we say the functions are Schwartz functions on $\mathbb{C} \setminus \{0\}$, and the space of such functions is denoted by $\mathcal{S}(\mathbb{C} \setminus \{0\})$.

The Fourier transform $\hat{f}$ of a Schwartz function $f \in \mathcal{S}(\mathbb{C})$ is defined by

\[
\hat{f}(u) = \int_{\mathbb{C}} f(z) e(-\text{Tr}(uz)) idz \wedge d\overline{z}.
\]

with $\text{Tr}(z) = z + \overline{z}$. We have $\hat{\hat{f}}(z) = f(-z)$. 
Corollary 1.5. Let $\mu$ be a complex number and $m$ be an even integer. We have

\[ (1.6) \int_{C \setminus \{0\}} J_{\mu,m}(z) \, \hat{f}(z) \frac{idz \wedge d\overline{z}}{|z|^2} = \frac{1}{2} \int_{C \setminus \{0\}} e \left( \frac{1}{2u^2} \right) J_{\mu,\frac{1}{2}m + \frac{1}{2}} \left( \frac{1}{16u^2} \right) f(u) \frac{idu \wedge d\overline{u}}{|u|^2}, \]

if $\hat{f} \in \mathcal{H}(\mathbb{C})$, under the assumption $|\text{Re}\, \mu| < \frac{1}{2}$. Furthermore, (1.6) remains valid for all values of $\mu$ if one assumes $f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$.

2. Preliminaries

2.1. Classical Bessel Functions.

2.1.1. Basic Properties of $J_{\nu}(z)$, $H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$. Let $\nu$ be a complex number.

Let $J_{\nu}(z)$, $H_{\nu}^{(1,2)}(z)$ denote the Bessel function of the first kind and the Hankel functions of order $\nu$. They all satisfy the Bessel equation

\[ (2.1) \quad z^2 \frac{d^2 w}{dz^2}(z) + z \frac{dw}{dz}(z) + (z^2 - \nu^2) w(z) = 0. \]

$J_{\nu}(z)$ is defined by the series (see [Wat] 3.1 (8))

\[ (2.2) \quad J_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{\nu + 2n}}{n! \Gamma(\nu + n + 1)}. \]

When $\nu$ is not a negative integer, we have the bound (see for instance [Wat] 3.13 (1) or 3.31 (1, 2))

\[ (2.3) \quad |J_{\nu}(z)| \ll |z|^\nu, \quad |z| \leq 1. \]

We have the following connection formulae (see [Wat], 3.61 (1, 2))

\[ (2.4) \quad J_{\nu}(z) = \frac{H_{\nu}^{(1)}(z) + H_{\nu}^{(2)}(z)}{2}, \quad J_{-\nu}(z) = \frac{e^{-i\pi}H_{\nu}^{(1)}(z) + e^{-\pi i}H_{\nu}^{(2)}(z)}{2}. \]

We have the following asymptotics of $H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$ at infinity (see [Wat] 7.2 (1, 2)).

\[ (2.5) \quad H_{\nu}^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z^2 - \frac{1}{2}\pi - \frac{1}{4})} \left(1 + \frac{1 - 4z^2}{8iz} + O\left(\frac{1}{|z|^2}\right)\right), \]

\[ (2.6) \quad H_{\nu}^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z^2 - \frac{1}{2}\pi + \frac{1}{4})} \left(1 - \frac{1 - 4z^2}{8iz} + O\left(\frac{1}{|z|^2}\right)\right), \]

of which (2.5) is valid when $z$ is such that $-\pi + \delta \leq \arg z \leq 2\pi - \delta$, and (2.6) when $-2\pi + \delta \leq \arg z \leq \pi - \delta$, $\delta$ being any positive acute angle. Consequently,

\[ (2.7) \quad J_{\nu}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos \left(z - \frac{1}{2}\pi - \frac{1}{4}\right) + O\left(|z|^{-\frac{1}{2}}\right), \]

for $|\arg z| \leq \pi - \delta$.

According to [Wat] 3.63, $H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$ form a fundamental system of solutions of Bessel’s equation.

2.1.2. Basic Properties of $I_{\nu}(z)$ and $K_{\nu}(z)$. Let $I_{\nu}(z)$ and $K_{\nu}(z)$ denote the modified Bessel function of the first and second kind of order $\nu$, which are defined by [Wat] 3.7 (2,
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6).

\[
I_v(z) = e^{-\frac{1}{2}\pi i}J_v(e^{\frac{1}{2}\pi i}z), \quad K_v(z) = \frac{1}{2\pi i} \frac{I_{-v}(z) - I_v(z)}{\sin(\pi v)}.
\]

We have the following asymptotics of \( I_v(z) \) and \( K_v(z) \) at infinity (\([\text{Wat} \ 7.23 (1, 2, 3)]\)),

\[
I_v(z) = \frac{e^{\frac{1}{2}z}}{(2\pi)^{\frac{1}{2}}} \left( 1 + O \left( |z|^{-1} \right) \right),
\]

\[
K_v(z) = \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} e^{-z} \left( 1 + O \left( |z|^{-1} \right) \right),
\]

of which (2.9) is valid when \( z \) is such that \( \arg z \leq \frac{1}{2}\pi - \delta \), and (2.10) when \( |\arg z| \leq \frac{3}{2}\pi - \delta \).

In addition, we have the following recurrence formulae for \( I_v(z) \) and \( K_v(z) \) (see \([\text{Wat} \ 3.71 (1)]\) and \([\text{AS} \ 9.6.29]\)),

\[
z I_v(z) - v I_v(z) = z I_{v-1}(z),
\]

\[
2^v I_v^{(v)}(z) = \sum_{r=0}^{n} C_r^v I_{v+r-2}(z), \quad (-2)^v K_v^{(v)}(z) = \sum_{r=0}^{n} C_r^v K_{v+r-2}(z),
\]

where \( C_r^v \) is the binomial coefficient.

2.1.3. Integral Formulae. First, we shall need the following integral formula (\([\text{EMOT} \ 8.6 (14)]\)), for \( y > 0, |\arg c| < \frac{1}{2}\pi \) and \( \text{Re } \nu > -2,

\[
\int_0^\infty J_\nu(xy)e^{-cx}dx = \frac{\Gamma \left( \frac{1}{2}\nu + 1 \right) y^{\nu}}{2^{\nu+1}\Gamma(\nu+1)\zeta^{\nu+1}} M \left( \frac{1}{2}\nu + 1, \nu + 1; -\frac{y^2}{4c} \right),
\]

where \( M(a; b; z) \) is Kummer’s confluent hypergeometric function (see \([2.2]\)). Second, when \( |\arg z| < \frac{1}{2}\pi \), we have the integral representation of \( K_v(z) \) (see \([\text{Wat} \ 6.22, (5, 7)]\)),

\[
K_v(z) = \frac{1}{2} \int_0^\infty y^{v-1} e^{-\frac{1}{2}z(y+y^{-1})} dy.
\]

Furthermore, when the order \( \nu = n \) is an integer, we have the integral representations of Bessel for \( J_n(z) \) and \( I_n(z) \) as follows (see \([\text{Wat} \ 2.2 (1)]\)),

\[
J_n(z) = J_{-n}(z) = \frac{1}{2\pi i} \int_{C} e^{-i\phi + i\cos \phi} d\phi,
\]

\[
I_n(z) = I_{-n}(z) = \frac{(-1)^n}{2\pi} \int_0^{2\pi} e^{-i\phi - \cos \phi} d\phi.
\]

2.2. Kummer’s Confluent Hypergeometric Functions. When \( b \) is not a nonpositive integer, Kummer’s confluent hypergeometric Function \( M(a; b; z) \) is defined by

\[
M(a; b; z) = {}_{1}F_{1}(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(b + n)n!} z^n.
\]

It is clear that

\[
M(b; b; z) = e^z.
\]
According to [AS, 13.2.1], when Re \( b > \) Re \( a > 0 \), we have
\[
\frac{\Gamma(b - a)\Gamma(a)}{\Gamma(b)} M(a; b; z) = \int_0^1 e^{\pi y} y^{(b - a - 1)} (1 - y)^{b - a - 1} dy.
\]

**2.3. Preliminaries on the Bessel Function \( J_{\mu, m}(z) \).**

2.3.1. Replacing \( dz/dz \) by \( \partial/\partial z \), we denote by \( \nabla_v \) the differential operator that occurs in (2.1), namely,
\[
\nabla_v = z^2 \frac{\partial}{\partial z} + z \frac{\partial}{\partial z} + z^2 + v^2.
\]
Its conjugation will be denoted by \( \overline{\nabla_v} \).
\[
\overline{\nabla_v} = z^2 \frac{\partial}{\partial z} + \frac{\partial}{\partial z} + z^2 - v^2.
\]
From the definition of \( J_{\mu, m}(z) \) as in [1.2, 1.3], we infer that
\[
\nabla_{2\mu + \frac{1}{2}m} (J_{\mu, m}(z^2/16\pi^2)) = 0, \quad \overline{\nabla_{2\mu - \frac{1}{2}m}} (J_{\mu, m}(z^2/16\pi^2)) = 0.
\]
It follows from (2.23) that if \( \mu \) is generic, that is \( 4\mu \notin 2\mathbb{Z} + m \), then
\[
|J_{\mu, m}(z)| \ll |e^{-2\mu z}| + |z|^{2}\mu|, \quad |z| \leq 1.
\]
In view of the connection formulae in (2.4), we have another expression of \( J_{\mu, m}(z) \) in terms of Hankel functions,
\[
J_{\mu, m}(z) = \pi i \left( e^{2\pi i \mu} H^{(1)}_{\mu, m}(4\pi \sqrt{z}) + (-1)^{m+1} e^{-2\pi i \mu} H^{(2)}_{\mu, m}(4\pi \sqrt{z}) \right),
\]
with the definition
\[
H^{(1, 2)}_{\mu, m}(z) = H^{(1, 2)}_{2\mu + \frac{1}{2}m}(z) H^{(1, 2)}_{2\mu - \frac{1}{2}m}(\overline{z}).
\]
It follows from (2.5) that \( J_{\mu, m}(z) \) admits the following asymptotic at infinity,
\[
J_{\mu, m}(z) = \sum_{\pm} \frac{(-1)^m}{2 \sqrt{|z|}} \left( 2 \left( \sqrt{z} + \sqrt{|z|} \right) \left( 1 \pm \frac{1 - 4 (\mu + \frac{1}{2}m)^2}{8i \sqrt{z}} \pm \frac{1 - 4 (\mu - \frac{1}{2}m)^2}{8i \sqrt{|z|}} \right) + O \left( |z|^{-\frac{3}{2}} \right) \right).
\]

**Lemma 2.1.** Let \( f(z) \) be a solution of the following two differential equations,
\[
\nabla_{2\mu + \frac{1}{2}m} w = 0, \quad \overline{\nabla_{2\mu - \frac{1}{2}m}} w = 0,
\]
with differential operators \( \nabla_{2\mu + \frac{1}{2}m} \) and \( \overline{\nabla_{2\mu - \frac{1}{2}m}} \) defined as in (2.20) and (2.21). Suppose further that \( f(4\pi z) \) admits the same asymptotic of \( J_{\mu, m}(z^2) \), that is,
\[
f(4\pi z) \sim \frac{1}{2|z|} e \left( 2(z + \overline{z}) + \frac{(-1)^m}{2|z|} e \left( -2(z + \overline{z}) \right) \right), \quad |z| \to \infty.
\]
Then \( f(4\pi z) = J_{\mu, m}(z^2) \).

**Proof.** From the theory of differential equations, \( f(z) \) may be uniquely written as a linear combination of \( H^{(k)}_{2\mu + \frac{1}{2}m}(z) \) and \( H^{(l)}_{2\mu - \frac{1}{2}m}(\overline{z}) \), with \( k, l = 1, 2 \), namely,
\[
f(z) = \sum_{k,l=1,2} c_{kl} H^{(k)}_{2\mu + \frac{1}{2}m}(z) H^{(l)}_{2\mu - \frac{1}{2}m}(\overline{z}).
\]
Letting $z = -4\pi ix$, with $x$ positive, we infer from (2.5) and (2.6) that if $c_{12} \neq 0$ then

$$f(-4\pi ix) \sim \frac{c_{12}}{2\pi i mx} \exp(8\pi x), \quad x \to \infty.$$ 

However, the asymptotic of $J_{\mu,m}(-x^2)$ is $(1 + (-1)^m)/2x$, so we must have $c_{12} = 0$ in order for $f(4\pi ix)$ and $J_{\mu,m}(-x^2)$ to have the same asymptotic. Similarly, $c_{21} = 0$. Choosing $z = 2k\pi, (2k - 1)\pi$, with $k$ positive integer, and letting $k \to \infty$, it follows that $c_{11} = \pi^2 e^{2\pi i\mu}$ and $c_{22} = \pi^2 i(-1)^{m+1} e^{-2\pi i\mu}$. Hence, by (2.24), we must have $f(4\pi z) = J_{\mu,m}(z^2)$.

2.3.2. An Integral Representation of $J_{\mu,m}$. In the polar coordinates, we have the following integral representation of $J_{\mu,m}(xe^{i\phi})$ (see [Qi2 Corollary 6.17] and [BM4 Theorem 12.1]),

$$J_{\mu,m}(xe^{i\phi}) = 4\pi i m \int_0^\infty y^{\mu-1} E(ye^{1/4})^{-m} J_m \left(4\pi \sqrt{x} Y(ye^{1/4})\right)dy,$$

with

$$Y(z) = |z + z^{-1}|, \quad E(z) = (z + z^{-1}) / |z + z^{-1}|.$$ 

The integral on the right of (2.27) is absolutely convergent if $|\Re \mu| < \frac{1}{8}$.

2.4. Stationary Phase Integrals. The lemma below is a special case of [Hör Theorem 7.7.5].

**Lemma 2.2.** Let $K \subset \mathbb{C} \setminus \{0\}$ be a compact set, $X$ an open neighbourhood of $K$. In the polar coordinates, if $u(x, \phi) = u(xe^{i\phi}) \in C^2_0(K)$, $f(x, \phi) = f(xe^{i\phi}) \in C^4(X)$ and $f$ is a real valued function on $X$, $f(x_0, \phi_0) = 0$, $f'(x_0, \phi_0) = 0$, $\det f''(x_0, \phi_0) \neq 0$ and $f' \neq 0$ in $K \setminus \{(x_0, \phi_0)\}$, then for $y > 0$

$$\iint_K u(x, \phi) e^{yf(x, \phi)} dx d\phi = \frac{u(x_0, \phi_0) e^{yf(x_0, \phi_0)} + \frac{1}{2}}{y \sqrt{\det f''(x_0, \phi_0)}} + O\left(\frac{1}{y^2}\right).$$

Here the implied constant depends only on $f$, $u$ and $K$.

3. Combinatorial Lemmas and Recurrence Formulae for Classical Bessel Functions

Let $C^n_r$ denote binomial coefficients. By convention, we let $C^n_r = 0$ if either $r < 0$ or $0 \leq n < r$. Throughout this section, unless otherwise specified, we assume that the numbers $k, l, n, r, s...$ are nonnegative integers.

3.1. A Combinatorial Inversion Formula and a Combinatorial Identity. First, we have the following combinatorial inversion formula.

**Lemma 3.1.** For $n \geq 2r$, we define $D^r_n = (-1)^r \left(C^n_{n-r} + C^n_{n-r-1}\right)$. Suppose $\{f_n\}$ and $\{g_n\}$ are two sequences of complex numbers such that

$$f_n = \sum_{r=0}^{[n/2]} C^n_r g_{n-2r},$$

(3.1)
then

\[ g_n = \sum_{r=0}^{[n/2]} D_r^e f_{n-2r}. \]  

Conversely, if \([g_n]\) is constructed from \([f_n]\) by \([3.2]\), then \([3.1]\) holds.

**Proof.** The first statement may be easily proven by induction once the following identity is verified

\[ \sum_{r=0}^{s} \binom{e} r D_{n-2r}^e = \delta_{s,0}, \]

for \(n \geq 2s\), where \(\delta_{s,0}\) is the Kronecker symbol that detects \(s = 0\). The second statement is simply a matter of uniqueness.

We first prove

\[ \sum_{r=0}^{s} (-1)^{s-r} \binom{e} r C_{n-s-r}^r = C_{2s-1}^r. \]  

For this, we consider the identity

\[ (1 - X)^{n-s} \left( 1 + \frac{X}{1 - X} \right)^n = \frac{1}{(1 - X)^r}. \]

The left hand side expands as

\[ \sum_{r=0}^{s} \binom{e} r X^r (1 - X)^{n-s-r} = \sum_{r=0}^{s} \sum_{t,s} (-1)^t \binom{e} t C_{n-s-r}^t X^t, \]

whereas the right hand side expands as

\[ \sum_{r=0}^{s} (-1)^r C_{n-2r}^r X^r = \sum_{r=0}^{s} C_{r+p-1}^r X^r. \]

Then the identity follows immediately from comparing the coefficients of \(X^r\). Similarly, on comparing the coefficients of \(X^{s-1}\) in the identity

\[ (1 - X)^{n-s-1} \left( 1 + \frac{X}{1 - X} \right)^n = \frac{1}{(1 - X)^{s+1}}, \]

we find that

\[ \sum_{r=0}^{s} (-1)^{s-r} \binom{e} r C_{n-s-r-1}^r = -C_{2s-1}^{s-1}. \]  

Summing \([3.4]\) and \([3.5]\) yields \([3.3]\). Q.E.D.

**Lemma 3.2.** For \(0 \leq r \leq n-l\), we define \(B_{i,n}^r = C_{i+r}^r C_{n-l-r}^{n-l} - C_{i+r-1}^r C_{n-l-r-1}^{n-l-1}\). Then

\[ \sum_{r=0}^{s} \binom{e} r B_{i,n-2r}^r = C_{n-i}^l, \]

for \(0 \leq s \leq n-l\).

**Proof.** Consider

\[ P_{l,n}(X,Y) = \frac{(1 - XY)(1 + XY)^n}{(1 - X)(1 - Y)^{l+1}}. \]
First, we expand \( P_{l,n}(X,Y) \) as below,

\[
P_{l,n}(X,Y) = \sum_{r,p,q} \sum_{l} c_{n}^{r} c_{l}^{p} c_{l+q}^{d} (X^{r+p} Y^{r+q} - X^{r+p+1} Y^{r+q+1}).
\]

Hence the left hand side of the identity is exactly the coefficient of \( X^{n-l} Y^{n-l} \) in \( P_{l,n}(X,Y) \).

Second, we write \( P_{l,n}(X,Y) \) in another way,

\[
P_{l,n}(X,Y) = \frac{(1 - XY)(1 + XY)^{n-l-1}}{(1 - (X + Y)/(1 + XY))^{l+1}}
= \sum_{l} a_{l} c_{l}^{n} (1 - XY) (1 + XY)^{n-l-1}.
\]

Thus the degree-\((n-l)\) homogeneous part of \( P_{l,n}(X,Y) \) is equal to

\[
C_{n}(X + Y)^{n-l} + \sum_{l=0}^{n-l} \left( c_{n}^{l} - a_{n-l-1} b_{n-l-1} \right) (X + Y)^{l} (X^{l} Y^{l})^{n-l-1}
= C_{n}(X + Y)^{n-l} = \sum_{s=0}^{n-l} C_{n-s}^{l} X^{s} Y^{n-l-s}.
\]

The proof is complete by comparing the coefficients of \( X^{n-l} Y^{n-l} \). Q.E.D.

We note that \( B_{l,n}^{r} = B_{l,n}^{n-l} \). Moreover, if \((l, r) \neq (0, 0), (0, n)\) it would be preferable to write \( B_{l,n}^{r} = C_{l+r}^{n-l} c_{l}^{r} C_{n-r}^{l} - C_{l+r+l}^{n-l} c_{l}^{r} C_{n-r}^{l-1} \).

### 3.2. Recurrence Formulae for Classical Bessel Functions.

**Lemma 3.3.** Let notations be as above. Define \( I_{r,s}(z) = I_{r}(z) \), \( K_{r,s}(z) = K_{r}(z) \), and, for \( n \geq 1 \), \( I_{r,n}(z) = I_{r+n}(z) + I_{r+n-z}(z) \). \( K_{r,n}(z) = K_{r+n}(z) + K_{r+n-z}(z) \). Then

\[
\sum_{r=0}^{[n/2]} D_{n}^{r} \cdot 2^{r} I_{r}^{(n-2r)}(z) = I_{r,n}(z), \quad \sum_{r=0}^{[n/2]} D_{n}^{r} \cdot (-2)^{r} K_{r}^{(n-2r)}(z) = K_{r,n}(z).
\]

**Proof.** Note that we may reformulate (2.12) as

\[
2^{l} I_{l}^{(r)}(z) = \sum_{r=0}^{[n/2]} c_{n}^{r} I_{r,n-2r}(z), \quad (-2)^{l} K_{l}^{(r)}(z) = \sum_{r=0}^{[n/2]} c_{n}^{r} K_{r,n-2r}(z).
\]

This lemma is therefore a direct consequence of Lemma 3.1 Q.E.D.

With the help of Lemma 3.2, we generalize the first formula in Lemma 3.3 as follows.

**Lemma 3.4.** Let notations be as above. Suppose that \( l \leq n \). We have

\[
\sum_{r=0}^{[(n-l)/2]} D_{n}^{r} \cdot 2^{n-2r} c_{n-2r}^{l} I_{r}^{(n-l-2r)}(z) = 2^{l} \sum_{r=0}^{n-l} B_{l,n}^{r} I_{r-n+l+2r}(z).
\]

**Proof.** In the notations of Lemma 3.1 we let

\[
f_{l,n} = \begin{cases} 2^{l} c_{n}^{l} I_{l}^{(n-l)}(z), & \text{if } n \geq l, \\ 0, & \text{if } n < l, \end{cases}
\]

with the help of Lemma 3.2.
and we only need to prove

\[
g_{l,n} = \begin{cases} 2^l \sum_{s=0}^{[(n-l)/2]} B_{l,n}^s I_{v,n-l-2s}(z), & \text{if } n \geq l, \\ 0, & \text{if } n < l. \end{cases}
\]

By Lemma 3.2,

\[
\sum_{r=0}^{[(n-l)/2]} C_n^r \cdot 2^l \sum_{s=0}^{[(n-l-2r)/2]} B_{l,n-2r}^s I_{v,n-l-2r-2s}(z)
\]

\[
= 2^l \sum_{s=0}^{[(n-l)/2]} I_{v,n-l-2s}(z) \sum_{r=0}^{[(n-l)/2]} C_n^r B_{l,n-2r}^s
\]

\[
= 2^n C_n^l f^{(n-l)}(z).
\]

This verifies the nontrivial case of the identity (3.1) in Lemma 3.1 for \( f_{l,n} \) and \( g_{l,n} \). Therefore, our proof is done with an application of Lemma 3.1. Q.E.D.

Similar recurrence formula holds for \( K_v(z) \), or other types of Bessel functions, but only the formula for \( I_v(z) \) will be needed in the sequel.

### 3.3. Additional Combinatorial Identities

In the following, we collect some combinatorial results that will be used in the proof of Theorem 1.4 or rather Proposition 4.1.

**Lemma 3.5.** Let \( B_{l,n}^r \) be defined as in Lemma 3.2. For \( 0 \leq r \leq k-l \), we define \( A_{k,l}^r = C_{l+r-1}^{l-r} C_{2k-l-1}^{k-r} + C_{l+r}^{l-r} C_{2k-l-1}^{k-r} \). Then

\[
\sum_{n=l+r}^{k} (-)^n C_{2k}^{k-n} B_{l,n}^r = (-)^{l+r} A_{k,l}^r.
\]

**Proof.** In view of the definition of \( B_{l,n}^r \), it suffices to prove the identity

\[
\sum_{n=l+r}^{k} (-)^n C_{2k}^{k-n} C_{n-r}^{n-r} = (-)^{l+r} C_{2k-l-1}^{k-l-r}.
\]

This follows easily from examining the coefficients of \( X^{k-l-r} \) in the identity

\[
\frac{(1 + X) - X^{2k}}{(1 + X)^{k+r}} \cdot \frac{1}{1 + X} = \frac{1}{(1 + X)^{k+r}}.
\]

Q.E.D.

By our conventions on binomial coefficients, \( A_{k,l}^r \) is well defined for all values of \( k, l \) and \( r \), but it vanishes except for \( 0 \leq r \leq k-l \). Moreover, when \( (l, r) \neq (0, 0) \), we shall always write \( A_{k,l}^r = C_{l+r-1}^{l-r} C_{2k-l-1}^{k-r} + C_{l+r}^{l-r} C_{2k-l-1}^{k-r} \).

**Lemma 3.6.** Suppose that \( k \geq 1 \).
(1). We have
\[ A_{k+1,0}^0 = 2A_{k,0}^0 + A_{k,0}^1, \quad A_{k+1,0}^1 = 2A_{k,0}^0 + 2A_{k,0}^1 + A_{k,0}^2, \]
\[ A_{k+1,0}^r = A_{k,0}^{r-1} + 2A_{k,0}^r + A_{k,0}^{r+1}, \quad r \geq 2, \]
(3.6)

(2). We have
\[ A_{k+1,1}^0 = (k+1) (2A_{k,0}^0 - A_{k,0}^1), \quad A_{k+1,1}^r = (k+1) \left( A_{k,0}^r - A_{k,0}^{r+1} \right), \quad r \geq 1. \]
\[ A_{k+1,1}^0 - (2A_{k,0}^0 - A_{k,0}^1 + A_{k,1}^0) = 0, \]
\[ A_{k+1,1}^r - \left( A_{k,0}^r - A_{k,0}^{r+1} + A_{k,1,1}^r \right) = 0, \quad r \geq 1. \]
(3.8)

(3). For \( l \geq 2 \) and \( k - l + 1 \geq r \geq -1 \), we have
\[ (k-1)(l-1)A_{k+1,l}^0 - (k-1)(k+1)(k-l+2)A_{k+1,l-2}^r = -(k+1)(2k-l)(2k-l+1)A_{k-1,l-2}^{r+1}, \]
\[ lA_{k+1,l}^r - (k-l+2) \left( A_{k,l-2}^r + A_{k,l-1}^r - A_{k,l-1}^{r+1} \right) = -(2k-l)A_{k-1,l-2}^{r+1}, \]
\[ A_{k+1,l}^r - \left( A_{k,l-1}^r - A_{k,l-1}^{r+1} + A_{k,l}^r - 2A_{k,l}^r + A_{k,l+1}^r \right) = A_{k-1,l-2}^{r+1}. \]
(3.9)

(3.10)

(3.11)

Proof.
(1). Note that
\[ A_{k,0}^0 = C_{2k}^0 = \frac{1}{2} C_{2k}, \quad A_{k,0}^r = C_{2k}^{k-r} + C_{2k-1}^{k-r-1} = C_{2k}^{k-r}, \quad k, r \geq 1. \]

By Pascal’s rule,
\[ C_{2k+2}^{k-r+1} = C_{2k}^{k-r+1} + 2C_{2k}^{k-r} + C_{2k}^{k-r+1}, \]
which implies (3.6).

(2). We have
\[ A_{k,1}^r = r C_{2k-2}^{k-r-2} + (r+1) C_{2k-2}^{k-r-1}. \]

The simple identity below yields (3.7).
\[ r C_{2k}^{k-r-1} + (r+1) C_{2k}^{k-r} = (k+1) \left( C_{2k}^{k-r} - C_{2k}^{k-r-1} \right). \]

Again, (3.8) follows from Pascal’s rule.

(3). Let us assume \( r \geq 0 \). The degenerated case \( r = -1 \) is much simpler.

The identity (3.9) is equivalent to the vanishing of
\[ (k-1)(r+1)(l+r-1) (r(k-l-r+1) + (l+r)(k+r+1)) \]
\[ - (k-1)(k+1)(k-l+2) \left( (r+1)(k-l-r+1) + (l+r-1)(k+r+1) \right) \]
\[ + (k+1)(k+r+1)(k-l-r+1) \left( (r+1)(k-l-r) + (l+r-1)(k+r) \right), \]
which may be checked directly. As for (3.10), we verify by Pascal’s rule that
\[ A_{k,l-2}^{r+1} + A_{k,l-1}^{r+1} - A_{k,l-1}^r = C_{2k}^{r-1} C_{2k}^{k-r-1} - C_{2k-2}^{r-1} C_{2k-2}^{k-r-1}, \]
(3.12)

(3.13)
Hence, we are left to show (3.14)

Finally, we verify (3.11). Similar to (3.12) and (3.13), we have

Therefore, we are left to show

It is reduced to the following identity, which may be verified directly,

Finally, we verify (3.11). Similar to (3.12) and (3.13), we have

4. Proof of Theorem 1.4

For brevity, we put \( m = 2k \). Without loss of generality, we assume that \( k \geq 0 \), \( |\text{Re}\mu| < \frac{1}{8} \) and \( \mu \neq 0 \). The admissible range of \( \mu \) for (1.5) in Theorem 1.4 may be extended to \( |\text{Re}\mu| < \frac{1}{8} \) by the principal of analytic continuation.

4.1. First Reductions. We insert the integral representation of \( J_{\mu, 2k}(xe^{i\phi}) \) in (2.27) and change the order of integration, then the integral on the left of (1.5) turns into

\[
4\pi(-1)^{\frac{\mu}{2}} \int_{0}^{2\pi} \int_{0}^{\infty} y^{\mu-1} E(ye^{i\phi})^{-2k} \int_{0}^{\infty} J_{2k} \left( 4\pi \sqrt{X} \left(ye^{i\phi}\right) \right) \exp(-2\pi cx) dx dy d\phi.
\]
Note that the triple integral is absolutely convergent as $|\Re \mu| < \frac{1}{2}$. We evaluate the inner integral using the formula (2.13), then the triple integral is equal to
\[
\frac{2(-2\pi)^2}{\Gamma(2k+1)} \int_{0}^{2\pi} \int_{0}^{\infty} y^{2k-1} \left( y e^{-\frac{i}{2} \phi} + y^{-1} e^{\frac{i}{2} \phi} \right)^{2k} M \left( k + 1; 2k + 1; -\frac{2\pi (y^2 + y^{-2} + 2 \cos \phi)}{c} \right) dy d\phi.
\]

When $k = 0$, Weber’s confluent hypergeometric function reduces to the exponential function as in (2.18), so the double integral splits into a product of two integrals and they may be evaluated by (2.14) and (2.16) respectively. Consequently, we obtain
\[
\text{(4.1)} \quad \frac{4\pi}{c} K_{2\mu} \left( \frac{4\pi}{c} \right) I_{0} \left( \frac{4\pi}{c} \right).
\]
then follows the formula (1.5) in Theorem 1.4 in the case $k = 0$.

When $k \geq 1$, we apply the integral representation of Weber’s confluent hypergeometric function in (2.19), expand $\left( y e^{-\frac{i}{2} \phi} + y^{-1} e^{\frac{i}{2} \phi} \right)^{2k}$, change the order of integration, and again evaluate the integrals over $y$ and $\phi$ by (2.14) and (2.16) respectively. It follows that the integral above turns into
\[
\frac{4\pi(-2\pi)^k}{(k-1)!c^{k+1}} \sum_{n=-k}^{k} (-)^n C_{2k}^{k+n} \int_{0}^{1} K_{2\mu+n} \left( \frac{4\pi}{c} \right) I_{n} \left( \frac{4\pi}{c} \right) v^k (1-v)^{k-1} dv.
\]
By Lemma 3.3, this is further equal to
\[
\frac{4\pi(-2\pi)^k}{(k-1)!c^{k+1}} \sum_{n=0}^{k} C_{2k}^{n} \sum_{r=0}^{\lfloor n/2 \rfloor} 2^n 2^{-2r} D_{n} \int_{0}^{1} K_{2\mu+n-2r} \left( \frac{4\pi}{c} \right) I_{n} \left( \frac{4\pi}{c} \right) v^k (1-v)^{k-1} dv.
\]
We now perform integration by parts. Since $|\Re \mu| < \frac{1}{2}$ and $\mu \neq 0$, in view of the series expansions of $K_{2\mu}$ and $I_{n}$ at zero (see (2.2) and (2.8)), all the boundary terms at $v = 0$ vanish. Moreover, we observe that $k - 1$ many differentiations are required to remove the zero of $(1-v)^{k-1}$ at $v = 1$, so all the boundary terms at $v = 1$ are zero except for one, which is,
\[
\text{(4.2)} \quad \frac{4\pi(-1)^k}{c} K_{2\mu} \left( \frac{4\pi}{c} \right) I_{k} \left( \frac{4\pi}{c} \right).
\]
On the other hand, the resulting integral after integration by parts is
\[
\text{(4.3)} \quad \frac{4\pi(-2\pi)^k}{(k-1)!c^{k+1}} \int_{0}^{1} K_{2\mu} \left( \frac{4\pi}{c} \right) S_{k} \left( v, \frac{4\pi}{c} \right) dv,
\]
with
\[
\text{(4.4)} \quad S_{k}(v, a) = \sum_{n=0}^{k} (-)^n C_{2k}^{n} \sum_{r=0}^{\lfloor n/2 \rfloor} (2/a)^{n-2r} D_{r} \left( I_{n}(av)P_{k}(v) \right),
\]
and $P_{k}(v) = v^k (1-v)^{k-1}$. Therefore, the formula (1.5) in Theorem 1.4 follows immediately from the vanishing of $S_{k}(v, a)$. When $k = 1$, for instance, we see from (2.11) that
\[
S_{1}(v, a) = 2I_{0}(av) - (2/a) (I_{1}'(av)v + I_{1}(av)) = 0.
\]
In general, $S_{k}(v, a) \equiv 0$ may be readily proven by the following recursive identity.
PROPOSITION 4.1. Let \( P_k(v) = v^k(1-v)^{k-1} \), \( Q(v) = v(1-v) \) and \( R(v) = 1 - 3v \). Let \( S_k(v, a) \) be defined as in 4.4. We have
\[
a^2 S_{k+1}(v, a) = -4Q(v)\left(c^2_v S_k(v, a) - a^2 S_k(v, a)\right) + R(v)\partial_v S_k(v, a) + (k^2 - 1)S_k(v, a) - (k-1)k S_{k+1}(v, a).
\]

Before starting the proof, let us prove one more lemma on the derivatives of \( P_k(v) = v^k(1-v)^{k-1} \) that will play a very crucial role afterwards. It shows how their derivatives for two consecutive \( k \) are related simply by \( Q(v) = v(1-v) \) and \( R(v) = 1 - 3v \).

**LEMMA 4.2.** Let \( l \geq 0 \).

1. We have
\[
(k - l + 2)P_{k+1}^{(l)} = (k + 2)P_k^{(l)} Q + lP_k^{(l-1)} R + (k-1)(l-1)P_k^{(l-2)}.
\]

It is understood that, when \( l = 0, 1 \), the terms containing \( P_k^{(-1)} \), \( P_k^{(-2)} \) or \( P_{k-1}^{(-1)} \) do not occur in the identities due to the appearance of the factors \( l, l - 1 \).

2. We have
\[
(k - 1)(k - l)P_{k-1}^{(l)} = (k - 2)P_k^{(l+2)} Q + (2k - l - 2)P_k^{(l+1)} R + (k+1)(2k - l - 2)(2k - l - 1)P_k^{(l)}.
\]

**Proof:** First, we have
\[
P_k'(v) = ((k + 1) - (2k + 1)v)P_k(v),
\]
and therefore
\[
P_k^{(l+1)}(v) = ((k + 1) - (2k + 1)v)P_k^{(l)}(v) - (2k + 1)lP_k^{(l-1)}(v).
\]

We prove (4.5) inductively. When \( l = 0, 4.5 \) is simply \( P_{k+1}(v) = P_k(v)Q(v) \). We now assume that the identity (4.5) is valid for \( l \). From differentiating (4.5) and then subtracting (4.7), we obtain the identity (4.5) for \( l + 1 \).

Second, we have
\[
(k - 1)kP_{k-1}(v) = ((k + 1) - (2k + 1)v)P_k'(v) + (2k + 1)^2P_k(v),
\]
and therefore
\[
(k - 1)kP_k^{(l+1)}(v) = ((k + 1) - (2k + 1)v)P_k^{(l+2)}(v) - (2k + 1)(2k - l - 2)P_k^{(l+1)}(v).
\]

With (4.8), we may prove (4.6) in the same fashion as (4.5).

**Q.E.D.**

4.2. **Proof of Proposition 4.1.** In the rest of this section, we shall exploit the combinatorial results established in §3 especially Lemma 3.4, 3.5 and 3.6 along with Lemma 4.2 to prove Proposition 4.1.

First, applying Lemma 3.4 and Lemma 3.5, we simplify \( S_k(v, a) \) as follows,
\[
S_k(v, a) = \sum_{n=0}^{k} \binom{n}{2k} \sum_{r=0}^{\lfloor n/2 \rfloor} 2^{n-2r} D_n^{[r]} \sum_{l=0}^{n-2r} C_{n-2r}^{[l]} \left(1/a\right) t_n^{(l-2r)}(av)P_k^{(l)}(v)
\]
\[
\sum_{l=0}^{k} (-2/a)^l P_k^{(l)}(v) \sum_{n=l}^{k} (-)^n C_{2k}^{k-n} \sum_{r=0}^{[(n-l)/2]} 2^{n-2r} c_r c_{n-2r} f_n^{(n-l-2r)}(av)
\]
\[
= \sum_{l=0}^{k} (2/a)^l P_k^{(l)}(v) \sum_{n=l}^{k} (-)^n C_{2k}^{k-n} \sum_{r=0}^{n-l} B_{r,n}^{k-n} I_{l+2r}(av)
\]
\[
= \sum_{l=0}^{k} (2/a)^l P_k^{(l)}(v) \sum_{n=l}^{k-l} (-)^n C_{2k}^{k-n} B_{n}^{n-k}
\]
\[
= \sum_{l=0}^{k-l} (-2/a)^l P_k^{(l)}(v) \sum_{r=0}^{l} (-) A_{k,l}^{r} I_{l+2r}(av).
\]

Accordingly, we define

\[(4.9) \quad S_{k,l}(z) = \sum_{r=0}^{k-l} (-)^r A_{k,l}^{r} I_{l+2r}(z),\]

so that

\[(4.10) \quad S_k(v,a) = \sum_{l=0}^{k} (-2/a)^l P_k^{(l)}(v) S_{k,l}(av).\]

In view of our conventions on \(A_{k,l}^{r}\), we shall put \(S_{k,l} = 0\) when either \(k < l\) or \(l < 0\). By

\[(4.11) \quad 2S_{k,l-1}^{r} = \sum_{r=-1}^{k-l+1} (-)^r \left( A_{k,l-1}^{r} - A_{k,l-1}^{r+1} \right) I_{l+2r}(z),\]

\[(4.12) \quad 4S_{k,l}^{r} = \sum_{r=-1}^{k-l+1} (-)^r \left( A_{k,l}^{r-1} - 2A_{k,l}^{r} + A_{k,l}^{r+1} \right) I_{l+2r}(z).\]

With the above preparations, we are now ready to finish the proof. Consider

\[S_{k+1} + Q \left( (2/a)^2 \bar{\gamma}^2 S_k - 4S_k \right) + R(2/a)^2 \bar{\gamma} S_k + (k^2 - 1)(2/a)^2 S_k.\]

Using (4.10), straightforward calculations show that it may be partitioned into the sum of

\[(4.13) \quad (-2/a)^{k+2} \left( P_k^{(k+2)} Q + P_k^{(k+1)} R + (k^2 - 1)P_k^{(l)} \right) S_{k,k},\]

\[(4.14) \quad P_{k+1}S_{k+1,0} + 4P_k Q \left( S_{k,0}'' - S_{k,0} \right),\]

\[(4.15) \quad (-2/a) \left( P_{k+1}S_{k,1,0} - 4P_k Q \left( S_{k,0}' - S_{k,1}' + S_{k,1} \right) - 2P_k R S_{k,0}' \right),\]

\[(4.16) \quad \sum_{l=0}^{k+1} (-2/a)^l P_{k+1}^{(l)} S_{k+1,l} + P_k^{(l)} Q \left( S_{k,l-2} - 4S_{k,l-2} + 4S_{k,l-1}'' - 4S_{k,l}'' \right) + P_k^{(l-1)} R \left( S_{k,l-2} - 2S_{k,l-1}' \right) + (k^2 - 1)P_k^{(l-2)} S_{k,l-2}.''

In the above expressions, for succinctness, we have suppressed the arguments \(v, a\) and \(av\) from \(S_k(v,a), ..., P_k^{(l)}(v), ..., Q(v), R(v)\) and \(S_{k,l}(av), ...,\). First, choosing \(l = k + 2\) in (4.5) in Lemma 4.2, it is clear that the sum in (4.13) vanishes. Second, computing with (4.9) (4.11) (4.12), Lemma 4.2 (1), with \(l = 0\) and 1, Lemma 3.6 (1) and (2) imply that both (4.14) and (4.15) yield zero contribution. Note that, for \(S_{k,0}, S_{k,1}'\) and \(S_{k,1}''\) that occur in (4.14)
and \((4.13)\), one needs to combine the terms with \(r = -1\) and \(r = 1\) in \((4.11)\) or \((4.12)\) by
\[I_{-1}(z) = I_1(z)\] and \(I_{-2}(z) = I_2(z)\). Finally, from Lemma \((4.2)(1)\), Lemma \((3.6)(3)\) and at last Lemma \((4.2)(2)\), we infer that \((4.16)\) is equal to
\[-(k - 1)k \sum_{k = 1}^{k+1} (-2/a)P_{k-1}(x) \sum_{r = 0}^{k+1} (-)^r A_{k-1,r}I_{k+2r} = (k - 1)(2/a)^2 S_{k+1}.
\]
This completes the proof of Proposition \((4.1)\) and hence Theorem \((1.4)\).

5. Proof of Theorem \((1.2)\)

Assume for simplicity that \(\mu \neq 0\). Let \(\rho = |\Re \mu| < \frac{1}{4}\). We denote
\(G_{\mu, m}(ye^{i\theta}) = \int_0^{2\pi} \int_0^\infty J_{\mu, m}(xe^{i\theta}) e\left(-\frac{2x \cos(\phi - \theta)}{y}\right) dx d\phi,\)
\((5.1)\)
\(F_{\mu, m}(ye^{i\theta}) = 2 ye^{i\theta} e(-\cos \theta) G_{\mu, m}(ye^{i\theta}).\)
\((5.2)\)

5.1. Asymptotic of \(G_{\mu, m}(ye^{i\theta})\). Our first task is to prove the following asymptotic of \(G_{\mu, m}(ye^{i\theta})\),
\(G_{\mu, m}(ye^{i\theta}) \sim \int_0^{2\pi} \int_0^\infty J_{\mu, m}(xe^{i\theta}) e\left(-\frac{2x \cos(\phi - \theta)}{y}\right) dx d\phi,\)
\((5.3)\)
which yields
\(F_{\mu, m}(ye^{i\theta}) \sim 2 ye^{i\theta} e(-\cos \theta) G_{\mu, m}(ye^{i\theta}).\)
\((5.4)\)

In the sequel, we shall fix a constant \(0 < \delta < \frac{1}{4}\) and let \(y\) be sufficiently large. All the implied constants in our computations will only depend on \(\delta, \rho\) and \(m\).

We split \(G_{\mu, m}(ye^{i\theta})\) as the sum
\(G_{\mu, m}(ye^{i\theta}) = C_{\mu, m}(ye^{i\theta}) + D_{\mu, m}(ye^{i\theta}) + E_{\mu, m}(ye^{i\theta})\)
\(= 2 \int_0^{2\pi} \int_0^\infty u(x) J_{\mu, m}(xe^{i\theta}) e\left(-\frac{2x \cos(\phi - \theta)}{y}\right) dx d\phi + 2 \int_0^{2\pi} \int_0^\infty w(x) \sqrt{2y} J_{\mu, m}(xe^{i\theta}) e\left(-\frac{2x \cos(\phi - \theta)}{y}\right) dx d\phi + 2 \int_0^{2\pi} \int_0^\infty v(x) \sqrt{2y} J_{\mu, m}(xe^{i\theta}) e\left(-\frac{2x \cos(\phi - \theta)}{y}\right) dx d\phi,\)
where \(u(y^2 x^2) + v(x) + w(x) \equiv 1\) is a partition of unity on \((0, \infty)\) such that \(u(x), v(x)\) and \(w(x)\) are smooth functions supported on \([0, 4y^2\]), \([y^2 - 1, 2]\) and \([2, 3]\) respectively, and that \(x^r u(x), x^r v(x), x^r w(x)\) are bounded for \(r = 0, 1, 2\).

5.1.1. Asymptotic of \(C_{\mu, m}(ye^{i\theta})\). When \(x \leq 4y^2\), we write \(e(-x \cos(\phi - \theta)/y) = 1 + O(x/y)\). In view of \((5.23)\) and \((5.26)\), the contribution of the error term to \(C_{\mu, m}(ye^{i\theta})\) is bounded by
\[\frac{1}{y} \int_0^{2\pi} \int_0^{x} x^{-2\sigma+1} dx d\phi + \frac{1}{y} \int_0^{2\pi} \int_0^{x} \sqrt{2y} dx d\phi = O(y^{3\delta-1}).\]
Therefore,
\[ C_{\mu,m} (ye^{\theta}) = 2 \int_{0}^{2\pi} \int_{0}^{\frac{4y2\pi}{y}} u(x) J_{\mu,m} (xe^{i\theta}) \, dx \, d\phi + O(y^{3\delta-1}). \]

On the other hand, choosing \( c = 1/y \) in Theorem 1.4 from the asymptotics of modified Bessel functions in (2.9) and (2.10) we infer that
\[ 2 \int_{0}^{2\pi} \int_{0}^{y} J_{\mu,m} (xe^{i\theta}) \exp \left( -\frac{2\pi x}{y} \right) \, dx \, d\phi = (-1)^{2m} + O \left( \frac{1}{y} \right). \]

With slight abuse of notation, we denote the double integral on the left by \( C_{\mu,m}(y) \), and split \( C_{\mu,m}(y) = A_{\mu,m}(y) + B_{\mu,m}(y) \) according to the partition of unity \( u(x) + (1 - u(x)) \equiv 1 \).

On writing \( \exp (-2\pi x/y) = 1 + O(x/y) \) for \( x \leq 4y2\delta \), similar as above, we find that
\[ A_{\mu,m}(y) = 2 \int_{0}^{2\pi} \int_{0}^{y} u(x) J_{\mu,m} (xe^{i\theta}) \, dx \, d\phi + O(y^{3\delta-1}). \]

As for \( B_{\mu,m}(y) \), we insert the asymptotic of \( J_{\mu,m} (xe^{i\theta}) \) as in (2.26), with an error term contribution \( O(y^{-\delta}) \). For the pair of leading terms, we need to consider
\[ B(y) = \int_{y^{2\delta}}^{\infty} (1 - u(x)) \exp(-2\pi x/y) \frac{1}{\sqrt{x}} \int_{0}^{2\pi} e \left( 4 \sqrt{x} \cos \left( \frac{\pi}{2} \phi \right) \right) \, d\phi \, dx. \]

For the two pairs of lower order terms, the treatments will be similar. In view of (2.15) and (2.7), we see that \( B(y) \) is equal to
\[ 4\pi \int_{y^{2\delta}}^{\infty} (1 - u(x)) \exp(-2\pi x/y) \frac{1}{\sqrt{x}} J_{0}(8\pi \sqrt{x}) \, dx \]
\[ = 2 \int_{y^{2\delta}}^{\infty} (1 - u(x)) x^{-\frac{1}{2}} \exp(-2\pi x/y) \cos \left( 8\pi \sqrt{x} - \frac{\pi}{2} \right) \, dx + O(y^{-\frac{3}{2}\delta}). \]

Applying integration by parts to the oscillatory integral, we get
\[ B(y) = \frac{1}{8\pi y} \int_{y^{2\delta}}^{\infty} (1 - u(x)) \left( x^{-\frac{1}{2}} y + 8\pi x^{-\frac{1}{2}} \right) \exp(-2\pi x/y) \sin \left( 8\pi \sqrt{x} - \frac{\pi}{2} \right) \, dx \]
\[ + \frac{1}{2\pi} \int_{0}^{y^{2\delta}} u'(x)x^{-\frac{1}{2}} \exp(-2\pi x/y) \sin \left( 8\pi \sqrt{x} - \frac{\pi}{2} \right) \, dx + O(y^{-\frac{3}{2}\delta}) \]
\[ = O(y^{-\frac{3}{2}\delta} + y^{-3\delta-1}). \]

Combining the foregoing results, we obtain
\[ (5.5) \quad C_{\mu,m} (ye^{\theta}) = (-1)^{2m} + O(y^{3\delta-1} + y^{-\frac{3}{2}\delta}). \]

5.1.2. Asymptotic of \( D_{\mu,m} (ye^{\theta}) \). Insert the asymptotic of \( J_{\mu,m} (xe^{i\theta}) \) in (2.26) into the integral that defines \( D_{\mu,m} (ye^{\theta}) \). The contribution from the error term is \( O(1/y) \). For the three pairs of main terms, we change the variable of integration from \( xe^{i\theta} \) to \( y^2 \chi^2 e^{2i\theta} \).

Then we obtain an oscillatory integral from the pair of leading terms as below, along with two similar integrals from two pairs of lower order terms,
\[ D(y, \theta) = 4y \int_{0}^{2\pi} \int_{\frac{1}{2}}^{3} w(x) e \left( y f(x, \phi; \theta) \right) \, dx \, d\phi, \]
with the phase function $f(x, \phi, \theta)$ defined by
\[ f(x, \phi, \theta) = 4x \cos \phi - 2x^2 \cos(2\phi - \theta). \]
We have
\[ f'(x, \phi; \theta) = (4 \cos \phi - x \cos(2\phi - \theta)), -4x (\sin \phi - x \sin(2\phi - \theta)) \].
Hence there is a unique stationary point $(x_0, \phi_0) = (1, \theta)$ on the domain of integration.
Moreover, we have $f(x_0, \phi_0; \theta) = 2 \cos \theta, \det f''(x_0, \phi_0; \theta) = -16$ and $w(x_0) = 1$. Applying Lemma 2.2 we obtain
\[ D(y, \theta) = e(2y \cos \phi) + O(1/y) \].
Similarly, we find that two other integrals of lower order are both $O(1/y)$. Therefore,
\[ (5.6) \quad D_{\mu, m}(ye^{it}) = e(2y \cos \phi) + O(1/y) \].
5.1.3. Bound for $E_{\mu, m}(ye^{it})$. Similar as in 5.1.2, from the asymptotic of $J_{\mu, m}(xe^{it})$, the error term contributes $O(y^{-\delta})$ and we need to consider the following oscillatory integral,
\[ E(y, \theta) = 4y \int_0^{2\pi} \int_{-1}^{1} v(x)e(yf(x, \phi; \theta)) \, dx \, d\phi, \]
and two other similar integrals. As the stationary point $(x_0, \phi_0) = (1, \theta)$ does not lie in the support of $v(x)$, according to the method of stationary phase, roughly speaking, one should get a saving of $y$ for each partial integration. Our goal is to show that $E(y, \theta) = o(1)$ after twice of partial integrations. For this, we define
\[ g(x, \phi; \theta) = x^{-1} (\partial_\phi f(x, \phi; \theta))^2 + x^{-3} (\partial_\phi f(x, \phi; \theta))^2 = 16 (x + x^{-1} - 2 \cos(\phi - \theta)). \]
An important observation is that, on the support of $v(x)$, we have
\[ g(x, \phi; \theta) \geq 4 \max \{1/x, 1\} (\geq 8), \quad x \in [y^{\delta-1}, y^{\delta}] \cup [2, \infty). \]
Besides, the following simple upper bounds will also be useful
\[ (1/x) \partial_\phi^2 f(x, \phi; \theta), \partial_\phi \partial_x f(x, \phi; \theta) \ll \max \{1, x\}, \quad \partial_\phi^2 f(x, \phi; \theta) \ll 1, \quad \partial_\phi^3 f(x, \phi; \theta) = 0, \]
\[ \partial_\phi \partial_x g(x, \phi; \theta) \ll 1, \partial_\phi g(x, \phi; \theta) \ll \max \{1/x^2, 1\}, \partial_\phi^2 g(x, \phi; \theta) \ll 1/x^3, \partial_\phi \partial_x g(x, \phi; \theta) = 0. \]
We now apply the elaborated partial integration of Hörmander (see the proof of Theorem 7.7.1). Writing
\[ E(y, \theta) = 2 \pi l \int_0^{2\pi} \int_{-1}^{1} v(x) \left( \frac{\partial_x f(x, \phi; \theta)}{x} \partial_x(e(yf(x, \phi; \theta))) \right) \, dx \, d\phi, \]
and integrating by parts, then $E(y, \theta)$ turns into
\[ - \frac{2}{\pi l} \int_0^{2\pi} \int_{-1}^{1} \left( \frac{\partial}{\partial x} \left( \frac{v(x)}{x^2} \right) \partial_x \left( \frac{v(x)}{x^2} \partial_x \left( \frac{\partial_x f(x, \phi; \theta)}{x g(x, \phi; \theta)} \right) \right) e(yf(x, \phi; \theta)) \right) \, dx \, d\phi. \]
It is routine to calculate the derivatives in the integrand by the product rule for differentiations and estimate each resulting integral by the bounds for $g(x, \phi; \theta)$ and $f(x, \phi; \theta)$ as
above. It is trivially bounded by log y. Indeed, the absolute value of each integrand is bounded by either \(1/x\) or 1 for \(x \in \left[\frac{1}{y^2-1}, \frac{1}{2}\right]\) and by either \(1/x^2\) or \(1/x^3\) for \(x \in [2, \infty)\). In particular, all the integrals are absolutely convergent. Applying partial integration again yields an additional saving of \(y\). Therefore

\[
E(y, \theta) = O(1/y).
\]

Moreover, one application of Hörmander's elaborated partial integration on the two integrals of lower order is sufficient to yield the bound \(1/y\). Hence

\[
(5.7) \quad E_{\mu,m}(ye^{i\theta}) = O\left(y^{-\delta}\right).
\]

Remark 5.1. It should be noted that one application of the elaborated partial integration of Hörmander already addresses the convergence issue of the double integral in (5.1) or (1.4). Furthermore, as will be seen in Lemma 6.1 the same arguments here would yield the compact convergence of the integral with respect to \(ye^{i\theta}\).

5.1.4. Conclusion. Combining (5.5, 5.6, 5.7), the proof of (5.3) is now complete.

5.2. Differential Equations for \(F_{\mu,m}\). We are now going to verify

\[
(5.8) \quad \nabla_{\mu + \frac{1}{4}m}(F_{\mu,m}(u/\pi)) = 0, \quad \nabla_{\mu - \frac{1}{4}m}(F_{\mu,m}(u/\pi)) = 0.
\]

By symmetry, we only need to verify the former, which may be explicitly written as

\[
(5.9) \quad u^2 \frac{\partial^2 F_{\mu,m}(u)}{\partial u^2} + u \frac{\partial F_{\mu,m}(u)}{\partial u} + \left(\pi^2 u^2 - \left(\mu + \frac{1}{4}m\right)^2\right) F_{\mu,m}(u) = 0.
\]

For \(s, r = 0, 1, 2\), with \(s + r = 0, 1, 2\), we introduce

\[
F_{s,r,\mu,m}(u) = \frac{2}{\sqrt{\pi u}} e^{-\frac{u + \frac{\pi}{2}}{2}} \int_{C \setminus \{0\}} z^{s+r-i\frac{\pi}{4}} (\partial/\partial z)^r J_{\mu,m}(z) e^{-\frac{z}{2}} \frac{\partial z}{u} d\zeta \wedge d\zeta.
\]

Here \(F_{s,r,\mu,m}\) are regarded as distributions on \(\mathbb{C} \setminus \{0\}\) and the computations below should be interpreted in the theory of distributions. Note that \(F_{\mu,m} = F_{0,0,\mu,m}\). For brevity, we put \(F_{s,r} = F_{s,r,\mu,m}\) and \(F_s = F_{s,0}\).

First, for \(s = 0, 1\), we have

\[
\frac{\partial F_s}{\partial u} = -\left(\pi i + \frac{1}{2u}\right) F_s + \frac{2\pi i}{u^2} F_{s+1}
\]

and

\[
\frac{\partial^2 F_0}{\partial u^2} = \frac{1}{2u^2} F_0 - \left(\pi i + \frac{1}{2u}\right) \frac{\partial F_0}{\partial u} - \frac{4\pi i}{u^3} F_1 + \frac{2\pi i}{u^2} \frac{\partial F_1}{\partial u}
\]

\[
= \frac{1}{2u^2} F_0 - \left(\pi i + \frac{1}{2u}\right) \left(-\left(\pi i + \frac{1}{2u}\right) F_0 + \frac{2\pi i}{u^2} F_1\right)
\]

\[
- \frac{4\pi i}{u^3} F_1 + \frac{2\pi i}{u^2} \left(-\left(\pi i + \frac{1}{2u}\right) F_1 + \frac{2\pi i}{u^2} F_2\right)
\]

\[
= \left(-\pi^2 + \frac{\pi i}{u} + \frac{3}{4u^2}\right) F_0 + \left(\frac{4\pi^2}{u^2} - \frac{6\pi i}{u^3}\right) F_1 - \frac{4\pi^2}{u^3} F_2.
\]
Finally, combining these, we have
\[
\frac{4\pi i}{u} \int_{\mathbb{C} \setminus \{0\}} z^{s+r+\frac{i}{2}} \frac{1}{z} (\partial / \partial z)^{r+1} J_{\mu, m}(z) e\left(-\left(\frac{z}{u} + \frac{\bar{z}}{u}\right)\right) \, idz \wedge d\bar{z}
\]
\[
= (2s + 2r + 1) \int_{\mathbb{C} \setminus \{0\}} z^{s+r+\frac{i}{2}} \frac{1}{z} (\partial / \partial z)^{r+1} J_{\mu, m}(z) e\left(-\left(\frac{z}{u} + \frac{\bar{z}}{u}\right)\right) \, idz \wedge d\bar{z}
\]
\[
+ 2 \cdot 2 \int_{\mathbb{C} \setminus \{0\}} z^{s+r+\frac{i}{2}} \frac{1}{z} (\partial / \partial z)^{r+1} J_{\mu, m}(z) e\left(-\left(\frac{z}{u} + \frac{\bar{z}}{u}\right)\right) \, idz \wedge d\bar{z}.
\]
Hence,
\[
\frac{4\pi i}{u} F_{s+1, r} = (2s + 2r + 1) F_{s, r} + 2 F_{s, r+1}
\]
and
\[
\frac{-16\pi^2}{u^2} F_2 = \frac{12\pi i}{u} F_1 + \frac{8\pi i}{u} F_{1, 1} = 3 F_0 + 12 F_{0, 1} + 4 F_{0, 2}.
\]
Third, since \(\nabla_{2\mu+\frac{1}{4} m} (J_{\mu, m}(z^2/16\pi^2)) = 0\) (see (2.22)), we have
\[
4 F_{0, 2} + 4 F_{0, 1} + 16\pi^2 F_{1, 0} - \left(2\mu + \frac{1}{2} m\right)^2 F_{0, 0} = 0.
\]
Finally, combining these, we have
\[
u^2 \frac{\partial^2 F_0}{\partial v^2} + u^2 \frac{\partial F_0}{\partial u} + \left(\pi^2 u^2 - \left(\mu + \frac{1}{4} m\right)^2\right) F_0
\]
\[
= \left(-\pi^2 u^2 + \pi u + \frac{3}{4}\right) F_0 + \left(4\pi^2 - \frac{6\pi i}{u}\right) F_1 - \frac{4\pi^2}{u^2} F_2
\]
\[
- \left(\pi u + \frac{1}{2}\right) F_0 + \frac{2\pi i}{u} F_1 + \left(\pi^2 u^2 - \left(\mu + \frac{1}{4} m\right)^2\right) F_0
\]
\[
= \left(\frac{1}{4} - \left(\mu + \frac{1}{4} m\right)^2\right) F_0 + \left(4\pi^2 - \frac{4\pi i}{u}\right) F_1 - \frac{4\pi^2}{u^2} F_2
\]
\[
= \left(\frac{1}{4} - \left(\mu + \frac{1}{4} m\right)^2\right) F_0 + 4\pi^2 F_1 - F_0 - 2 F_{0, 1} + \frac{3}{4} F_0 + 3 F_{0, 1} + F_{0, 2}
\]
\[
= - \left(\mu + \frac{1}{4} m\right)^2 F_0 + 4\pi^2 F_1 + F_{0, 1} + F_{0, 2}
\]
\[
= 0,
\]
which proves (5.9).

### 5.3. Conclusion

Combining (5.4) and (5.8), Lemma 2.3 implies that
\[
(5.10) \quad F_{\mu, m}(4u) = J_{\frac{2\mu}{u}, \frac{1}{4} m}(u^2).
\]
In view of the definition of \(F_{\mu, m}\) given by (5.1) (5.2), this is equivalent to the identity (1.4) in Theorem 1.2.
6. Proof of Corollary [1.5]

In this last section, we shall outline a proof of Corollary [1.5]. For this, we need the compact convergence of the double integral in (1.4) with respect to $ye^{i\theta}$.

**Lemma 6.1.** Suppose that $|\text{Re} \mu| < \frac{1}{2}$ and $m$ is even. The integral

$$
\int_0^{2\pi} \int_0^\infty J_{\mu,m}(xe^{i\phi}) e(-2xy \cos(\phi + \theta)) dx d\phi
$$

is compactly convergent with respect to $ye^{i\theta}$.

**Sketch of Proof.** Fix $Y > 1$. We need to verify that the integral (6.1) converges uniformly on the annulus $\{ye^{i\theta} : y \in [1/Y, Y]\}$. In order to use the arguments in §5.1, let us change $y$ to $1/y$ and $\theta$ to $-\theta$.

First, we make the change of variables from $xe^{i\phi}$ to $y^2 x^2 e^{2i\phi}$. As $y \in [1/Y, Y]$, this does not affect the uniformity of convergence. Second, we introduce a smooth partition of unity $(1 - v(x)) + v(x) \equiv 1$ on $(0, \infty) = (0, 3) \cup (2, \infty)$ and split the integral accordingly into two parts. In view of the asymptotic for $J_{\mu,m}(z)$ near zero and the condition $|\text{Re} \mu| < \frac{1}{2}$, the first integral is obviously uniformly convergent for $y \in [1/Y, Y]$. As for the second integral, we proceed literally in the same way as in §5.1.3 and the uniform convergence becomes clear after one application of Hörmander’s partial integration. Note that the only change is the lower limit of the $x$-integral from $y^6$ to $2$ and that the stationary point $(x_0, \phi_0) = (1, \theta)$ does not lie in the support of the $v(x)$ here.

Q.E.D.

We now return to Corollary [1.5]. Some remarks on the convergence of (1.6) for $f \in \mathcal{S}(\mathbb{C})$ are in order. In view of the asymptotics of $J_{\mu,m}(z)$ at zero and infinity, the right hand side of (1.6) absolutely converges for all $\mu$ and $m$, but the absolute convergence of left hand side only holds for $|\text{Re} \mu| < \frac{1}{2}$.

To prove (1.6) for $f \in \mathcal{S}(\mathbb{C})$ in Corollary [1.5], it suffices to verify the identity,

$$
\int_0^{2\pi} \int_0^\infty \int_0^{2\pi} \int_0^\infty J_{\mu,m}(xe^{i\phi}) e(-2xy \cos(\phi + \theta)) f(ye^{i\theta}) ydyd\theta dxd\phi
$$

$$
= \frac{1}{4} \int_0^{2\pi} \int_0^\infty e \left( \frac{\cos \theta}{y} \right) J_{\frac{3}{2} \mu, \frac{3}{2} m} \left( \frac{1}{16y^2 e^{2i\theta}} \right) f(ye^{i\theta}) dy d\theta.
$$

Note that (6.2) is a direct consequence of the identity (1.4) in Theorem [1.2] if one were able to change the order of integrations. However, the decay of $J_{\mu,m}(z)$ at infinity is too slow to guarantee absolute convergence and the change of integration order. In order to get absolute convergence, our idea is to produce some decaying factor by partial integrations.

Since the idea is straightforward, we shall only give a sketch of the proof as below and leave the details to the readers. Starting from the integral on the left hand side of (6.2), we write the inner integral in the Cartesian coordinates and apply the combination of partial integrations that have the effect of dividing $4\pi^2 x^2 + 1$. To be precise, letting $z = xe^{i\phi}$ and $u = ye^{i\theta}$, define the differential operator $D = -(\partial/\partial u)(\partial/\partial \overline{u}) + 1$ so that $D(e(-\text{Tr}(zu))) = (4\pi^2 z^2 + 1) e(-\text{Tr}(zu))$, then, by partial integrations, the left side of
(6.2) is equal to
\[
\int_0^{2\pi} \int_0^\infty \frac{J_{\mu,m}(xe^{i\phi})}{4\pi^2x^2+1} \left( \frac{1}{2} \int_{C\setminus\{0\}} e(-\text{Tr}(xe^{i\theta}u)x) \frac{Df(u)idu}{d\theta} \right) d\phi dx.
\]
After this, we change the order of integrations, which is legitimate as \(J_{\mu,m}(xe^{i\theta})/(4\pi^2x^2+1)\) is absolutely integrable. Then, we apply the partial integrations reverse to those that we performed at the beginning and rewrite the outer integral in the polar coordinates. In this way, we retrieve the expression on the left hand side of (6.2) but with changed integration order. Here, to prove that differentiations under the integral sign are permissible, we need the compact convergence of the double integral in (1.4) or (6.1) with respect to \(u = ye^{i\theta}\), but this has been verified in Lemma [6.1]. Finally, integrating out the resulting inner integral using (1.4) in Theorem [1.2] we arrive at the integral on the right hand side of (6.2).

When \(f \in \mathcal{S}(\C \setminus \{0\})\), the integral on the left hand side of (1.6) becomes absolutely convergent for all \(\mu\), then follows the second assertion in Corollary [1.5].

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