HEINZ INEQUALITY FOR THE UNIT BALL

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ABSTRACT. In this note we establish the Heinz inequality for the boundary of the unit ball by providing a sharp constant $C_n$ in the inequality: $|\partial_r u(r\eta)|_{r=1} \geq C_n, \|\eta\| = 1$, for every harmonic mapping of the unit ball onto itself satisfying the condition $u(0) = 0$.

1. INTRODUCTION

E. Heinz in his classical paper [3], obtained the following result: If $u$ is a harmonic diffeomorphism of the unit disk $U$ onto itself satisfying the condition $u(0) = 0$, then

$$|u_x(z)|^2 + |u_y(z)|^2 \geq \frac{2}{\pi^2}, \ z \in U.$$ 

The proof uses the following representation of harmonic mappings in the unit disk

(1.1) $u(z) = f(z) + \overline{g(z)},$

where $f$ and $g$ are holomorphic functions with $|g'(z)| < |f'(z)|$. It uses the maximum principle for holomorphic functions and the following sharp inequality

(1.2) $\liminf_{r \to 1^-} \left| \frac{\partial u(re^{it})}{\partial r} \right| \geq \frac{2}{\pi}$

proved by using the Schwarz lemma for harmonic functions. The aim of this paper is to generalize inequality (1.2) for several dimensional case.

If $u$ is a harmonic mapping of the unit ball onto itself, then we do not have any representation of $u$ as in (1.1).

It is well known that a harmonic function (and a mapping) $u \in L^\infty(B^n)$, where $B^n$ is the unit ball with the boundary $S = S^{n-1}$, has the following integral representation

(1.3) $u(x) = \mathcal{P}[f](x) = \int_{S^{n-1}} P(x, \zeta) f(\zeta) d\sigma(\zeta),$

where

$$P(x, \zeta) = \frac{1 - \|x\|^2}{\|x - \zeta\|^n}, \ z \in S^{n-1}$$

is Poisson kernel and $\sigma$ is the unique normalized rotation invariant Borel measure on $S^{n-1}$ and $\| \cdot \|$ is the Euclidean norm.
We have the following Schwarz lemma for harmonic mappings (see e.g. [1]). If \( u \) is a harmonic mapping of the unit ball into itself such that \( u(0) = 0 \) then \( \|u(x)\| \leq U(rN) \), where \( U \) is a harmonic function of the unit ball into \([-1, 1]\) defined by

\[
U(x) = \mathcal{P}[\chi_{S^+} - \chi_{S^-}](x),
\]

where \( \chi \) is the indicator function and \( S^+ = \{x \in S : x_n \geq 0\} \), \( S^- = \{x \in S : x_n \leq 0\} \). Note that, the standard harmonic Schwarz lemma is formulated for real functions only, but we can reduce the previous statement to the standard one by taking \( v(x) = \langle u(x), \eta \rangle \), for some \( \|\eta\| = 1 \), where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product. For Schwarz lemma for the derivatives of harmonic mappings on the plane and space we refer to the papers [6, 5].

By using Hopf theorem it can be proved ([4]) that if \( u \) is a harmonic mapping of the unit ball onto itself such that \( u(0) = 0 \) then

\[
\liminf_{r \to 1} \left\| \frac{\partial u}{\partial r}(r\zeta) \right\| \geq C_n,
\]

where \( C_n \) is a certain positive constant. Our goal is to find sharp constant \( C_n \).

2. Preliminaries and main results

In order to formulate and to prove our results recall the basic definition of hypergeometric functions. For two positive integers \( p \) and \( q \) and vectors \( a = (a_1, \ldots, a_p) \) and \( b = (b_1, \ldots, b_q) \) we set

\[
_{p}F_{q}[a; b, x] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k \cdot k!} x^k,
\]

where \((y)_k := \frac{\Gamma(y+k)}{\Gamma(y)} = y(y+1)\ldots(y+k-1)\) is the Pochhammer symbol. The hypergeometric series converges at least for \(|x| < 1\). For basic properties and formulas concerning trigonometric series we refer to the book [2]. The most important step in the proof of our main results i.e. of Theorem 2.2 and Theorem 2.3 below, is the following lemma

**Lemma 2.1.** The function \( V(r) = \frac{\partial U(rN)}{\partial r} \), \( 0 \leq r \leq 1 \) is decreasing on the interval \([0, 1]\) and we have

\[
V(r) \geq V(1) = C_n := \frac{n! \left(1 + n - (n - 2)_{2F1}[\frac{1}{2}, 1, \frac{3+n}{2}, -1]\right)}{2^{3n/2} \Gamma \left[\frac{3+n}{2}\right] \Gamma \left[\frac{3+n}{2}\right]}.\]

**Proof.** By using spherical coordinates \( \eta = (\eta_1, \ldots, \eta_n) \) such that \( \eta_n = \cos \theta \), where \( \theta \) is the angle between the vector \( x \) and \( x_n \) axis, we obtain from (1.4)

\[
U(rN) = \frac{\Gamma \left[\frac{n}{2}\right]}{\sqrt{\pi} \Gamma \left[\frac{n-1}{2}\right]} \int_0^\pi \frac{(1 - r^2) \sin^{n-2} \theta}{(1 + r^2 - 2r \cos \theta)^{n/2}} (\chi_{S^+}(x) - \chi_{S^-}(x)) d\theta.
\]
and so
\[
U(rN) = \frac{\Gamma \left[ \frac{n}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{n-1}{2} \right]} \int_0^{\pi/2} \left( \frac{(1-r^2) \sin^{n-2} \theta}{(1+r^2-2r \cos \theta)^{n/2}} - \frac{(1-r^2) \sin^{n-2} \theta}{(1+r^2+2r \cos \theta)^{n/2}} \right) d\theta.
\]
Let \( P = 2r/(1+r^2) \). Then
\[
\left( \frac{(1-r^2) \sin^{n-2} \theta}{(1+r^2-2r \cos \theta)^{n/2}} - \frac{(1-r^2) \sin^{n-2} \theta}{(1+r^2+2r \cos \theta)^{n/2}} \right) = \frac{(1-r^2)}{(1+r^2)^{n/2}} \sum_{k=0}^{\infty} \binom{-n/2}{k} ((-1)^k-1) \cos^k \theta \sin^{n-2} \theta P^k.
\]
Since
\[
\int_0^{\pi/2} \cos^k \theta \sin^{n-2} \theta d\theta = \frac{\Gamma \left[ \frac{k+n}{2} \right] \Gamma \left[ \frac{1}{2} (-1+n) \right]}{2 \sqrt{\pi} \Gamma \left[ \frac{k+n}{2} \right]},
\]
we obtain
\[
U(rN) = \frac{\Gamma \left[ \frac{n}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{n-1}{2} \right]} \frac{(1-r^2)}{(1+r^2)^{n/2}} \sum_{k=0}^{\infty} \frac{\Gamma \left[ \frac{k+n}{2} \right] \Gamma \left[ \frac{n-1}{2} \right]}{2 \Gamma \left[ \frac{k+n}{2} \right]} \binom{-n/2}{k} ((-1)^k-1) P^k.
\]
Hence
\[
U(rN) = r (1-r^2) \left( 1 + r^2 \right)^{-1} \frac{4 \Gamma \left[ \frac{1+n}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{1+n}{2} \right]} G(r),
\]
where
\[
G(r) = \text{F}_2 \left[ 1, \frac{2+n}{4}, \frac{3+n}{4}, \frac{1+n}{2} ; \frac{4r^2}{1+r^2} \right].
\]
By [2] Eq. 3.1.8 for \( a = \frac{n}{2} \), \( b = \frac{1}{2} (-1+n) \), \( c = \frac{1}{2} \), we have that
\[
G(r) = \frac{(1-r^2)^{1+n/2}}{1-r^2} \text{F}_3 \left[ \left\{ \frac{n}{2}, \frac{1}{2} (-1+n), \frac{1}{2} ; \frac{1+n}{4} \right\}, \left\{ \frac{n}{2}, \frac{3}{2}, \frac{1+n}{2} \right\}, -r^2 \right].
\]
So
\[
U(rN) = r \frac{4 \Gamma \left[ \frac{1+n}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{1+n}{2} \right]} \text{F}_3 \left[ \left\{ \frac{n}{2}, \frac{1}{2} (-1+n), \frac{1}{2} ; 1 + \frac{n}{4} \right\}, \left\{ \frac{n}{2}, \frac{3}{2}, \frac{1+n}{2} \right\}, -r^2 \right],
\]
which can be written as
\[
U(rN) = \frac{2 \Gamma \left[ \frac{1+n}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{1+n}{2} \right]} r^{2k+1} \sum_{k=1}^{\infty} \frac{2(-1)^k (4k+n) \Gamma \left[ k + \frac{n}{2} \right]}{(1+2k)(-1+2k+n) \sqrt{\pi} \Gamma \left[ 1 + k \right] \Gamma \left[ \frac{1}{2} (n-1) \right]} r^{2k+1}.
\]
Thus
\[
\frac{\partial U(rN)}{\partial r} = \frac{2 \Gamma \left[ \frac{1+n}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{1+n}{2} \right]} + \sum_{k=1}^{\infty} \frac{2(-1)^k (4k+n) \Gamma \left[ k + \frac{n}{2} \right]}{(-1+2k+n) \sqrt{\pi} \Gamma \left[ 1 + k \right] \Gamma \left[ \frac{1}{2} (n-1) \right]} r^{2k}.
\]
Since
\[
\frac{2(-1)^k(4k + n)\Gamma \left[ k + \frac{n}{2} \right]}{(-1 + 2k + n)\sqrt{\pi}\Gamma[1 + k]\Gamma \left[ \frac{1}{2}(n - 1) \right]}
= \frac{(-1)^k2^n\Gamma \left[ 1 + \frac{n}{2} \right] \Gamma \left[ k + \frac{n}{2} \right]}{\pi k!\Gamma[n]}
= \frac{2(-1)^k(-2 + n)\Gamma \left[ k + \frac{n}{2} \right]}{(-1 + 2k + n)\sqrt{\pi}\Gamma[k]\Gamma \left[ \frac{1+n}{2} \right]}
\]
we obtain that
\[
\frac{\partial U(rN)}{\partial r} = \Gamma \left[ 1 + \frac{n}{2} \right] \left( (1 + r^2)^{-n/2}(1 + n) - (n - 2)r^2 \ _2F_1 \left[ \frac{1+n}{2}, \frac{2+n}{2}, \frac{3+n}{2}, -r^2 \right] \right)
\]
which in view of the Kummer quadratic transformation, can be written in the form
\[
\frac{\partial U(rN)}{\partial r} = \Gamma \left[ 1 + \frac{n}{2} \right] \left( 1 + r^2 \right)^{-n/2} \left( 1 + n - (n - 2)r^2 \ _2F_1 \left[ \frac{1}{2}, 1, \frac{3+n}{2}, -r^2 \right] \right)
\]
The function
\[
y_2F_1[1/2, 1, (3 + n)/2, -y]
\]
increases in \( y \). Namely its derivative is
\[
_2F_1[1/2, 2, (3 + n)/2, -y] = \sum_{m=0}^{\infty} (-1)^m a(m) y^m
\]
\[
= \sum_{m=0}^{\infty} (-1)^m (1 + m)\Gamma \left[ \frac{1}{2} + m \right] \Gamma \left[ \frac{3+n}{2} \right] y^m.
\]
Then \( a(m) > 0 \) and
\[
\frac{a(m)}{a(m + 1)} = \frac{(1 + m)(3 + 2m + n)}{(2 + m)(1 + 2m)} > 1
\]
because \( 1 + n + mn > 0 \), and so
\[
_2F_1[1/2, 2, (3 + n)/2, -y] \geq \sum_{m=0}^{\infty} (a(2m) - a(2m + 1))y^{2m} > 0.
\]
The conclusion is that \( \frac{\partial U(rN)}{\partial r} \) is decreasing. In particular
\[
\frac{\partial U(rN)}{\partial r} \geq \frac{\partial U(rN)}{\partial r} \big|_{r=1}.
\]
For \( r = 1 \) we have
\[
\frac{\partial U(rN)}{\partial r} = C_n = \frac{n!(1 + n - (n - 2) \ _2F_1 \left[ \frac{1}{2}, 1, \frac{3+n}{2}, -1 \right])}{2^{3n/2}\Gamma \left[ \frac{1+n}{2} \right] \Gamma \left[ \frac{3+n}{2} \right]}.
\]
Theorem 2.2. If $u$ is a harmonic mapping of the unit ball into itself such that $u(0) = 0$, then for $x \in B$ the following sharp inequality

$$\frac{1 - \|u(x)\|}{1 - \|x\|} \geq C_n$$

holds.

Proof. First we have that $|u(x)| \leq U(rN)$ and so

$$\frac{1 - \|u(x)\|}{1 - \|x\|} \geq \frac{1 - |U(rN)|}{1 - \|x\|}.$$

Further there is $\rho \in (r, 1)$ such that

$$\frac{1 - U(rN)}{1 - \|x\|} = \frac{\partial U(\rho N)}{\partial r},$$

which in view of Lemma 2.1 is $> C_n$. The proof is completed. \hfill \Box

Theorem 2.3. If $u$ is a harmonic mapping of the unit ball onto itself such that $u(0) = 0$, then the following sharp inequality

$$\liminf_{r \to 1} \left| \frac{\partial u}{\partial n}(r\zeta) \right| \geq C_n, \quad \|\zeta\| = 1$$

holds.

Proof. For every $0 < r < 1$, there is a $\rho \in (\|x\|, 1)$ such that

$$\frac{1 - \|u(x)\|}{1 - r} = \frac{\partial \|u(r\zeta)\|}{\partial r} \bigg|_{r=\rho}.$$

On the other hand

$$\left| \frac{\partial u(\zeta)}{\partial r} \right| \geq \frac{\partial \|u(\zeta)\|}{\partial r}.$$

Letting $\|x\| = r \to 1$, in view of Theorem 2.2 and (2.2), we obtain that

$$\liminf_{r \to 1} \left| \frac{\partial u}{\partial n}(r\zeta) \right| \geq C_n.$$

To show that the inequality (2.1) is sharp, let

$$h_m(x) = \begin{cases} 
1 - x/m, & \text{if } x \in (1/m, 1]; \\
(m - 1)x, & \text{if } -1/m \leq x \leq 1/m; \\
-1 - x/m, & \text{if } x \in [-1, -1/m),
\end{cases}$$

and define

$$f_m(x_1, \ldots, x_{n-1}, x_n) = \sqrt{1 - \frac{h_m(x_n)^2}{1 - x_n^2}}(x_1, \ldots, x_{n-1}, 0) + (0, \ldots, 0, h_m(x_n)).$$

Then $f_m$ is a homeomorphism of the unit sphere onto itself, such that

$$\lim_{m \to \infty} f_m(x) = (0, \ldots, 0, \chi_{S^+}(x) - \chi_{S^-}(x)).$$
Further \( u_m(x) = \mathcal{P}[f_m](x) \) is a harmonic mapping of the unit ball onto itself such that \( \lim_{\|x\| \to 1} \|u_m(x)\| = 1 \). Moreover \( u_m(0) = 0 \) and \( \lim_{m \to \infty} u_m(x) = (0, \ldots, 0, U(x)) \). This implies the fact that the constant \( C_n \) is sharp. \( \square \)

Remark 2.4. The following table shows first few constants \( C_n \) and related functions

| \( n \) | \( u(rN) \) | \( \partial_r u(rN) \) | \( C_n \) |
|-------|------------|----------------|--------|
| 2     | \( \frac{4 \arctan(r)}{\pi} \) | \( \frac{4r}{1+r^2} \) | \( \frac{\pi}{2} \) |
| 3     | \( \frac{-1+r^2+\sqrt{4+4r^4}}{r\sqrt{1+r^2}} \) | \( \frac{1-\sqrt{4+4r^4}}{r(1+r^2)^{3/2}} \) | \( \sqrt{2} - 1 \) |
| 4     | \( \frac{2r(-1+r^2)+2(1+r^2)^2 \arctan r}{\pi r^2(1+r^2)} \) | \( \frac{4(r+3r^3-(1+r^2)^2 \arctan r)}{\pi r^3(1+r^2)^2} \) | \( \frac{4-\pi}{\pi} \) |

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