Cycles in Oriented 3-graphs

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Abstract

An oriented 3-graph consists of a family of triples (3-sets), each of which is given one of its two possible cyclic orientations. A cycle in an oriented 3-graph is a positive sum of some of the triples that gives weight zero to each 2-set.

Our aim in this paper is to consider the following question: how large can the girth of an oriented 3-graph (on \( n \) vertices) be? We show that there exist oriented 3-graphs whose shortest cycle has length \( \frac{n^2}{2}(1 + o(1)) \): this is asymptotically best possible. We also show that there exist 3-tournaments whose shortest cycle has length \( \frac{n^2}{3}(1 + o(1)) \), in complete contrast to the case of 2-tournaments.

1 Introduction

An oriented 3-graph on \( n \) vertices consists of a family of triples (3-sets), each of which is given one of its two possible cyclic orientations. A 3-tournament is a complete oriented 3-graph (all triples from the ground set are oriented).

Linial and Morgenstern [4] introduced a notion of ‘cycle’ in an oriented 3-graph. Roughly speaking (we will give a precise definition at the start of Section 2), a cycle in an oriented 3-graph is a positive sum of some of the triples that gives weight zero to each 2-set. Linial and Morgenstern were interested in acyclic 3-tournaments (i.e. 3-tournaments not containing a cycle). They also considered cycles in higher order tournaments. (See also [3] for other results on 3-tournaments and higher order tournaments.)

Our aim in this paper is to consider the following natural question: if an oriented 3-graph (or 3-tournament) on \( n \) vertices contains a cycle, how short a cycle must it contain? The analogous

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question in oriented graphs on \( n \) vertices is trivial: the shortest cycle can have length \( n \). In the case of a tournament (2-tournament), it is straightforward to see that if a tournament has a (directed) cycle, it must contain a directed triangle. See Moon \[5\] for background and many results on tournaments.

To be little more precise, an oriented 3-graph can be denoted by \( G = (V, \mathcal{E}) \), where \( V \) is the vertex set (unless otherwise stated, \( V = [n] = \{1, 2, \ldots, n\} \) with the natural ordering) and \( \mathcal{E} \) is the set of oriented triples. Given a triple \( F = \{a, b, c\} \), it can be oriented (in an oriented 3-graph) either as

\[
\begin{align*}
\text{or}
\end{align*}
\]

We write \( \overrightarrow{abc} \) (\( = \overrightarrow{bca} \) or \( \overrightarrow{cab} \)) for the former and \( \overrightarrow{acb} \) (\( = \overrightarrow{bac} \) or \( \overrightarrow{cba} \)) for the latter. Each oriented triple induces an orientation on each of its 2-sets. Namely, if \( F \) has the former orientation, we have the induced edges \( \overrightarrow{ab}, \overrightarrow{bc}, \overrightarrow{ca} \); and if \( F \) has the latter orientation, we have the induced edges \( \overrightarrow{ac}, \overrightarrow{cb}, \overrightarrow{ba} \). A cycle is a weighted sum (with positive weights) of triples that gives each directed edge a total weight of zero.

For an oriented 3-graph \( G \) that contains a cycle, we are interested in the shortest cycle in \( G \), one with the smallest length. In particular, we want to know how large the girth (the length of the shortest cycle) of \( G \) can be.

The plan of the paper is as follows. We start by considering cycles in 3-tournaments. It is easy to see that if a 3-tournament contains a cycle, then its shortest cycle has length at most \((n - 1)/2 + 1\). We do not know if the upper bound is best possible (or even asymptotically best possible), but we present a construction giving a lower bound of about \( 2^{n/2} \). This is in complete contrast to the case of 2-tournaments, where of course if there is a cycle, there is a cycle of length 3. Our construction is based on some embeddings of complete graphs into surfaces of high genus. This is the content of Section 2.

In Section 3 we turn our attention to general oriented 3-graphs. Here the same upper bound of \((n - 1)/2 + 1\) applies. We show that there exists an oriented 3-graph on \( n \) vertices whose shortest cycle has length \((n/2)(1 + o(1))\), which is asymptotically best possible.

For an oriented 3-graph \( G \), we will usually write \( V(G) \) for its vertex set and \( \mathcal{E}(G) \) for its oriented triples. A triple \( \{a, b, c\} \) in an oriented 3-graph is always oriented, and when its orientation is not important to us we will sometimes refer to it as \( abc \).

Finally, we remark that one could view this notion of cycle as a ‘homological’ version. Of course, as we have 3-sets but no 4-sets there is no notion of ‘boundary’, so that there is no notion of equivalence of cycles. Our paper does not use any homological notions (but for an introduction to homology, see e.g. Armstrong \[1\]).
2 Cycles in 3-tournaments

In this section, we consider $f(n)$, the length of the shortest cycle in a 3-tournament, maximised over all 3-tournaments on $n$ vertices that contain a cycle.

It is often helpful to view cycles in matrix terms. Following [4], the incidence matrix of an oriented 3-graph $G$ is an $\binom{n}{2} \times |E|$ matrix $A$ whose rows and columns correspond to all 2-sets $[n]^{(2)} = \{i < j : i, j \in [n]\}$ and all oriented triples of $E$ respectively. For $E = \{i < j\}$, the $(E, F)$ entry of $A$ is

\[
(A)_{E,F} = \begin{cases} 
1 & \text{if } E \subset F \text{ and } F \text{ induces } \overrightarrow{ij} \text{ on } E, \\
-1 & \text{if } E \subset F \text{ and } F \text{ induces } \overrightarrow{ji} \text{ on } E, \\
0 & \text{otherwise}.
\end{cases}
\]

So each column of $A$ has exactly three non-zero entries.

Given an oriented 3-graph $G = (V,E)$ with its incidence matrix $A$, a non-empty subset $C$ of $E$ is called a cycle if there exists positive real number $\alpha_F$ for every $F \in C$ such that

\[
\sum_{F \in C} \alpha_F x_F = 0,
\]

where $x_F$ is the column vector of $A$ that corresponds to the oriented triple $F$. The length of the cycle $C$ is the number of elements in $C$. For example, the 3-tournament $([4], \{1\overrightarrow{23}, 1\overrightarrow{42}, 1\overrightarrow{34}, 2\overrightarrow{43}\})$ is itself a cycle of length four. (Note that this is called a directed 4-set in [3].)

We first present an easy upper bound on $f(n)$ using standard results from linear algebra. Recall that Carathéodory’s theorem says that if a point $x \in \mathbb{R}^d$ lies in the convex hull of a set of points $P$, there is a subset $P'$ of $P$ consisting of at most $d + 1$ points such that $x$ lies in the convex hull of $P'$.

For a 3-tournament on $n$ vertices $T$, the column vectors of its incidence matrix span a subspace of $\mathbb{R}^{\binom{n}{2}}$, and we denote this subspace by $S_T$.

**Lemma 2.1.** Let $T$ be a 3-tournament that contains a cycle. Suppose that $S_T$ has rank $d$. Then the shortest cycle in $T$ has length at most $d + 1$.

**Proof.** $S_T$ is isomorphic to $\mathbb{R}^d$. A cycle in $T$ corresponds to a set of points (column vectors) $P$, such that its convex hull contains the origin. So by Carathéodory’s theorem, there is a subset $P'$ of $P$ consisting of at most $d + 1$ points such that the origin lies in the convex hull of $P'$, which in turn corresponds to a cycle in $T$ whose length is $|P'| \leq d + 1$.

Together with the fact that $S_T$ has rank at most $\binom{n}{2}$ for $T$ a 3-tournament on $n$ vertices, we can deduce that $f(n) \leq \binom{n}{2} + 1$ from the above lemma. The following easy result gives a better bound for the rank of $S_T$ and hence a better upper bound of $f(n)$.

**Lemma 2.2.** Let $T$ be a 3-tournament on $n$ vertices. Then $S_T$ has rank at most $\binom{n}{2} - n + 1$. 

\[ \]
Proof. Let $A$ be the incidence matrix of $T$. We show that there are $n - 1$ linearly independent vectors of $\mathbb{R}^{(n-1)}$ such that each one of them is orthogonal to every column vector of $A$. And the conclusion of the lemma follows easily from the rank-nullity theorem.

For $1 \leq i \leq n - 1$, let $x_i$ be vectors of length $(n-2)$ indexed by $[n]^{(2)}$ with the following $jk$-th entries ($j < k$).

$$
(x_i)_{jk} = \begin{cases} 
1 & \text{if } i = j < k, \\
-1 & \text{if } j < k = i, \\
0 & \text{otherwise.} 
\end{cases}
$$

It is easy to see that $x_1, x_2, \ldots, x_{n-1}$ are linearly independent as $x_i$ is the only vector with non-zero $in$-th entry among them.

Given a column vector $v$ of $A$, it corresponds to the orientation of a 3-set in $T$, say the 3-set $\{a < b < c\}$. The only nonzero entries of $v$ are $v_{ab}, v_{bc},$ and $v_{ac}$. Either $v_{ab} = 1, v_{bc} = 1, v_{ac} = -1$ or $v_{ab} = -1, v_{bc} = -1, v_{ac} = 1$. In both cases, it is straightforward to check that $v$ is orthogonal to $x_i$ for every $i$. This completes the proof of the lemma.

Combining Lemma 2.1 and Lemma 2.2 we have the following upper bound of $f(n)$.

**Corollary 2.3.** The shortest cycle in a 3-tournament on $n$ vertices that contains a cycle has length at most $(\begin{array}{c}n\end{array}) - n + 2$. That is, $f(n) \leq \left(\begin{array}{c}n\end{array}\right) - n + 2 = (\begin{array}{c}n-1\end{array}) + 1$.

We remark that the bound in Lemma 2.2 is asymptotically best possible. (See the remark at the end of Section 3.)

We now turn our attention to the lower bound of $f(n)$. We will give a construction of a 3-tournament on $n$ vertices whose shortest cycle has length exactly $\frac{1}{3}n(n-1)$, for infinitely many $n$. Our proof is based on some embeddings of the complete graphs in high genus surfaces.

We will also make use of the following lemma by Linial and Morgenstern [4], which is particularly helpful in our construction. For the sake of completeness, we will include the proof here.

**Lemma 2.4** ([4]). Let $C$ be an oriented 3-graph with the following properties.

(i) The only cycle in $C$ consists of all of its triples.

(ii) No additional cycle can be created by addition of any single oriented 3-set.

Then we can orient the remaining 3-sets (namely, the 3-sets from $V(C)^{(3)} \setminus E(C)$) to obtain a tournament $T$ such that $C$ remains as the only cycle in $T$.

**Proof.** We will show that such $T$ can be constructed by orienting the remaining 3-sets one by one. Let $F'$ be a 3-set which was not oriented yet. Suppose that both orientations of $F'$ give rise to new cycles. That is, $\sum \alpha_F x_F + x_{F'} = 0$ and $\sum \alpha'_F x_F - x_{F'} = 0$, where $x_{F'}$ corresponds to $F'$ oriented one of the two ways. Then $\sum (\alpha_F + \alpha'_F) x_F = 0$ is another cycle, which does not involve $F'$. Hence this must be the only cycle $C$, implying the new cycles created use only the 3-sets from $C$ and $F'$, contradicting the properties of $C$ in the lemma.
We are now ready to construct 3-tournaments whose shortest cycle has length $\frac{2}{3}(n)$.

**Theorem 2.5.** Let $n \geq 4$ and $n \equiv 0, 3, 4 \text{ or } 7 \pmod{12}$. Then there is a 3-tournament on $n$ vertices whose shortest cycle has length $\frac{1}{3}n(n-1)$.

**Proof.** It is well known that a complete graph $K_n$ can be embedded in an orientable surface of sufficiently large genus (see, for example, [6]). In the cases where $n \equiv 0, 3, 4 \text{ or } 7 \pmod{12}$, the genus may be chosen such that the embeddings are triangulations. Given any such triangulation, we can induce an oriented 3-graph $C$, which is a cycle of length $\frac{1}{3}n(n-1)$, by orienting every face (a 3-set) on the surface in the same orientation: all oriented clockwise or all oriented anticlockwise, viewing from outside the surface.

We first claim that $C$ does not contain a cycle of a shorter length. Suppose $C' \subset C$ is a cycle. Pick any vertex $v$ in $C'$ and name the remaining vertices $v_1, v_2, \ldots, v_{n-1}$ such that the oriented 3-sets containing $v$ in $C$ are $vv_1v_{i+1}, i \in [n-1]$. (The subscripts are taken mod $n-1$.) By the definition of $C$, it is not too hard to see that if any of these 3-sets is in $C'$, all of them must be in $C'$. Indeed, $vv_{i-1}v_i$ and $vv_{i+1}v_i$ are the only two oriented 3-sets in $C$ containing the the 2-set $vv_i$. This implies that $C'$ contains all of the $n$ vertices. Repeating the above arguments with $v$ replaced by each of $v_i$ shows that all 3-sets of $C$ are in $C'$, proving the claim.

Next, we claim that $C$ has the property that no additional cycle can be created by the addition of any single oriented 3-set. Suppose $C' \cup \rho$ is a cycle, where $C' \subset C$ and $\rho$ is an oriented 3-set not in $C$. That is, there exist positive coefficients $\alpha_F$ such that $(\sum_{F \in C'} \alpha_F x_F) + x_\rho = 0$. As $n \geq 4$, there is a vertex in $C'$ but not in $\rho$. Pick any such vertex $v$, every 3-set containing $v$ must also be in the new cycle (with the same coefficient). Since $\rho \notin C$, all 3-sets of $C$ are in $C'$ and $\alpha_F$ is constant. This immediately implies that $C' \cup \rho = C \cup \rho$ is not a cycle.

Now $C$ satisfies the properties in Lemma 2.4 and there is a tournament such that $C$ remains as the only cycle (hence the shortest cycle). This completes the proof of the theorem. $\square$

Combining Corollary 2.3 and Theorem 2.5, we have $\frac{1}{3}n(n-1) \leq f(n) \leq \binom{n-1}{2} + 1$ for $n \equiv 0, 3, 4 \text{ or } 7 \pmod{12}$. Observe that each 2-set is contained in exactly two 3-sets (each giving a different orientation to the 2-set) in the shortest cycle of the tournament in Theorem 2.5, and for the exact value $f(n)$ to be closer to the upper bound, most 2-sets would have to be in three 3-sets of a shortest cycle, which we believe is unlikely. In fact, we believe that our construction is best possible.

**Conjecture 2.6.** For $n \geq 4$ and $n \equiv 0, 3, 4 \text{ or } 7 \pmod{12}$, we have $f(n) = \frac{1}{3}n(n-1)$.

For the case when $n \not\equiv 0, 3, 4 \text{ or } 7 \pmod{12}$, consider the cycle $C$ (an oriented 3-graph spanning $m$ vertices) induced by the triangulation of $K_m$, as before, where $m$ is the largest integer smaller than $n$ such that $m \equiv 0, 3, 4 \text{ or } 7 \pmod{12}$. Then by almost identical arguments in the proof of Theorem 2.5, there is a tournament on $n$ vertices such that $C$ is the shortest cycle. This, together with Corollary 2.3 and Theorem 2.5, we can bound $f(n)$ for all $n \geq 4$.

**Corollary 2.7.** For $n \geq 4$, we have $\frac{1}{3}m(m-1) \leq f(n) \leq \binom{n-1}{2} + 1$, where $m$ is the largest integer smaller or equal to $n$ such that $m \equiv 0, 3, 4 \text{ or } 7 \pmod{12}$.
Again, we believe that a construction attaining the upper bound is unlikely and that our construction is asymptotically best possible.

**Conjecture 2.8.** \( f(n) = \left( \frac{1}{3} + o(1) \right) n^2 \) for all \( n \geq 4 \).

## 3  Cycles in oriented 3-graphs

Suppose \( G \) is an oriented 3-graph on \( n \) vertices and \( G \) contains a cycle. Using the exact same arguments as those used to derive Corollary 2.3, we know that the shortest cycle in \( G \) has length at most \( \binom{n-1}{2} + 1 \).

Our main aim in this section is to show that there exists an oriented 3-graph \( G \) on \( n \) vertices such that the only cycle in \( G \) consists of all the triples of \( G \), and has length \( \left( \frac{2}{3} \right)(1 - o(1)) \), attaining the upper bound asymptotically. Our construction could be viewed as an attempt to “add as many projective planes as possible to a small starting configuration”.

By a *single cycle*, we shall mean an oriented 3-graph \( G \) where the only cycle in \( G \) consists of all its triples. (Note that it follows from this that all cycles in \( G \) are the same, up to multiplication by positive reals.) The basic idea is to start from a base single cycle \( P \), then delete a triple from it, and attach \( P \) to another single cycle in which a triple is removed. This will result another single cycle, which we can again attach \( P \) onto.

To give a better insight to how our final construction is obtained, we will first start with a simple and symmetric base single cycle, the projective plane. (We assume no knowledge of the projective plane; however, for background on surfaces and the projective plane, see e.g. Armstrong [1] or Hatcher [2].) This will end up giving a single cycle \( G \) of length \( \left( \frac{2}{3} - o(1) \right)(n^2) \). Then we will see how to modify this construction to give girth of \( (1 - o(1))\left( \frac{n}{2} \right) \).

For convenience, we will refer a set of triples \( F \) as a *star system* if there is a vertex \( a \) such that \( F \cap F' = \{a\} \) for every pair of distinct \( F, F' \) in \( F \).

### 3.1 Attaching the projective plane

Consider the standard 6-point triangulation of the projective plane in Fig. 1, where each of the 10 faces (triples) is oriented clockwise. Adding the triple \( \{x, y, z\} \) with anticlockwise orientation \( \overrightarrow{xzy} \) results in a single cycle (the triple \( xyz \) has coefficient 2, while the other triples each has coefficient 1). Now delete the triple \( \overrightarrow{acb} \) and we will denote this oriented 3-graph on six vertices by \( P \) for the rest of the paper. That is, \( V(P) = \{x, y, z, a, b, c\} \) and \( E(P) = \{\overrightarrow{xya}, \overrightarrow{azy}, \overrightarrow{acz}, \overrightarrow{cxy}, \overrightarrow{cyb}, \overrightarrow{byz}, \overrightarrow{bzx}, \overrightarrow{bxa}, \overrightarrow{xzy}\} \).
Given an oriented 3-graph $G$ and an oriented triple $F = \overrightarrow{ijk}$ in $G$, we can form a new oriented 3-graph by attaching $P$ on $G$ via $F$. More precisely, we say $G'$ is a $(P, F)$-attachment on $G$ (or simply $P$-attachment on $G$ when $F$ is understood) where $G'$ has vertex set $V(G') = V(G) \cup V(P)$ with $i, j, k$ identified with $a, b, c$ respectively, and set of triples $E(G') = E(G) \cup E(P) \setminus \overrightarrow{ijk}$. So if $G$ is an oriented 3-graph on $k$ vertices and has $l$ triples, then the $P$-attachment on $G$ is an oriented 3-graph on $k + 3$ vertices and has $l + 9$ triples. It is straightforward to check that if $G$ is a single cycle, then $G'$ is also a single cycle. (Or see the proof of Lemma 3.1.)

We can also attach $P$ on an oriented 3-graph via a set of oriented triples, namely, a star system, as follows. Let $G$ be an oriented 3-graph on $k$ vertices and $\mathcal{F} = \{F_i = \overrightarrow{ab_i c_i} : i = 1, 2, \ldots, d\}$ be a star system in $G$. We will attach $P$ on $G$ via $F_i$ one by one. Set $G_1$ to be the $(P, F_1)$-attachment on $G$, where $G_1$ has vertex set $V(G) \cup \{x_1 = x, y_1 = y, z_1 = z\}$. Now, suppose $G_{i-1}$ is constructed, let $G_i$ be the modified $(P, F_i)$-attachment of $G_{i-1}$: identify the three new vertices of $(P, F_i)$-attachment of $G_{i-1}$ with $\{x, y, z\}$ and delete any repeated triples. That is, if $x_i, y_i, z_i$ are the three new vertices of $(P, F_i)$-attachment of $G_{i-1}$, we identify $x_i$ with $x$, $y_i$ with $y$ and $z_i$ with $z$. (Note that at each stage, we always attach $P$ with the preserved orientations, for example, the triples $\{x_i, y_i, z_i\}$ has orientation $\overrightarrow{x_i y_i z_i}$. So $G_i$ is well defined for all $i$.) And finally, we say $G' = G_d$ is the $(P, \mathcal{F})$-attachment on $G$ (or simply $P$-attachment on $G$ when $\mathcal{F}$ is understood) on $k + 3$ vertices.

In other words, for $i \geq 2$, $G_i$ is obtained from $G_{i-1}$ by deleting the triple $F_i = \overrightarrow{ab_i c_i}$ and adding the set of triples $\{\overrightarrow{az_i c_i}, \overrightarrow{c_i z_i x_i}, \overrightarrow{c_i x_i y_i}, \overrightarrow{b_i c_i y_i}, \overrightarrow{b_i y_i z_i}, \overrightarrow{b_i z_i x_i}, \overrightarrow{a b_i x_i}\}$.

It is not too hard to see that if $G$ is a cycle, then the $P$-attachment on $G$ is also a cycle. In fact, if $G$ is a single cycle with a star system $\mathcal{F}$ of $l$ triples, the $(P, \mathcal{F})$-attachment on $G$ is also a single cycle.
Lemma 3.1. Let $G$ be a single cycle (an oriented 3-graph) on $k$ vertices of length $l$. Suppose $\mathcal{F} = \{F_i = ab_ic_i : i = 1, 2, \ldots, d, b_i \neq c_j \text{ for all } i, j\}$ is a star system in $G$. Then the $(P, \mathcal{F})$-attachment on $G, G'$, is a single cycle on $k + 3$ vertices of length $l + 6d + 3$. Furthermore, $G'$ contains a star system of size $d + 1$.

Proof. Let $G, \mathcal{F}$ and $G'$ be as in the lemma. We first show that $\mathcal{E}(G')$ is a cycle. Let $A$ be the incidence matrix of $G$. Since $G$ is a single cycle, there exists positive $\alpha_F$ for every triple $F$ in $G$ such that

$$\sum_{F \in \mathcal{E}(G)} \alpha_F x_F = 0,$$

where $x_F$ is the column vector of $A$ that corresponds to the oriented triple $F$.

Now let $\mathcal{P}_i = \{\overrightarrow{ax}, \overrightarrow{xy}, \overrightarrow{ybx}, \overrightarrow{bcy}, \overrightarrow{cyx}, \overrightarrow{ayz}, \overrightarrow{azy}, \overrightarrow{zyb} \}$, $\mathcal{X} = \{\overrightarrow{xy}, \overrightarrow{ayz} \}$ and $R = \overrightarrow{xbz}$. With a slight abuse of notation, we now refer $x_F$ to be the column vector of the incidence matrix of $G'$ that corresponds to the triple $F$ in $G'$. It is then straightforward to see that

$$\left( \sum_{F \in \mathcal{E}(G) \setminus \mathcal{F}} \alpha_F x_F \right) + \sum_{i=1}^{d} \left( \alpha_{F_i} \sum_{F \in \mathcal{P}_i} x_F \right) + \left( \sum_{i=1}^{d} \alpha_{F_i} \right) \left( 2x_R + \sum_{F \in \mathcal{X}} x_F \right) = 0.$$

By construction, the $P$-attachment $G'$ is a cycle on $k+3$ vertices and has $l + 6d + 3$ triples. Indeed, $G_1$, the $(P, T_1)$-attachment on $G$, has $l + 9$ triples, and $G_i$, the modified $(P, F_i)$-attachment on $G_{i-1}$, has six additional triples. Also, note that the set of triples $\mathcal{F}' = \{\overrightarrow{yb_i} : i = 1, 2, \ldots, d\} \cup \{\overrightarrow{yx} \}$ in $G'$ is a star system of size $d + 1$.

So we only need to show that $G'$ is a single cycle, that is, the only cycle in $G'$ consists of all the triples in $G'$. We will do this by showing that $G_i$ is a single cycle for each $i \in \{1, 2, \ldots, d\}$. Let $G_0 = G$ and we note that for $i \geq 1$, $G_i$ is obtained from $G_{i-1}$ by deleting the triple $F_i = ab_ic_i$ and adding the set of triples $\mathcal{S}$. Here, $\mathcal{S} = \{\overrightarrow{ax}, \overrightarrow{xy}, \overrightarrow{ybx}, \overrightarrow{bcy}, \overrightarrow{cyx}, \overrightarrow{ayz}, \overrightarrow{azy}, \overrightarrow{zyb} \}$ for $i = 1$ and $\mathcal{S} = \{\overrightarrow{ax}, \overrightarrow{xy}, \overrightarrow{ybx}, \overrightarrow{bcy}, \overrightarrow{cyx}, \overrightarrow{ayz}, \overrightarrow{azy}, \overrightarrow{zyb} \}$ for $i \geq 2$.

Now, for $i \geq 1$, suppose that $G_{i-1}$ is a single cycle and $\mathcal{C}$ is a cycle in $G_i$. If $\mathcal{C} \cap \mathcal{S} = \emptyset$, then $\mathcal{C} \subset \mathcal{E}(G_{i-1})$ is a cycle of length strictly shorter than $|\mathcal{E}(G_{i-1})|$ as $F_i \notin \mathcal{C}$, contradicting $G_{i-1}$ is a single cycle. So we may assume $\mathcal{C}$ contains at least one triple in $\mathcal{S}$, and this will imply that $\mathcal{C} \supset \mathcal{S}$. This is because the only two triples that contain the 2-set $b_ix$ (also the two sets $b_iz, b_iz, b_iz, b_iz, b_iz, b_iz, b_iz$) are both in $\mathcal{S}$, inducing opposite directions of $b_ix$. And if $b_ix$ is contained in a triple in $\mathcal{C}$, both these triples must be in $\mathcal{C}$. Using similar arguments (by considering the 2-sets $ax, ay$ and $az$), we can further claim that the triples $\overrightarrow{xya}, \overrightarrow{ayz}$ and $\overrightarrow{zyb}$ are also in $\mathcal{C}$.

We can then write $\mathcal{C} = \mathcal{C}' \cup \mathcal{S}$, where $\mathcal{C}' \subset \mathcal{E}(G_{i-1})$ and $\mathcal{C}' \supset \{\overrightarrow{xya}, \overrightarrow{ayz}, \overrightarrow{zyb}\}$. It is then straightforward to check that $\mathcal{C}' \cup \{F_i\}$ is a cycle in $G_i$, and hence a cycle in $G_{i-1}$. Since $G_{i-1}$ is a single cycle, necessarily $\mathcal{C}' \cup \{F_i\} = \mathcal{E}(G_{i-1})$, implying $\mathcal{C} = \mathcal{E}(G_i)$, completing the proof of the lemma. 

\hfill \Box
By repeatedly applying Lemma 3.1 to a single cycle, we can construct a single cycle with increasing length.

**Corollary 3.2.** There exists an oriented 3-graph on \( n \) vertices whose shortest cycle has length \( \frac{2}{3} \binom{n}{2} (1 + o(1)) \).

**Proof.** Let \( G_0 \) be a single cycle on 4 vertices with \( E(G_0) = \{123, 142, 134, 243\} \) and let \( F_0 = \{123\} \). For \( i \geq 1 \), suppose \( G_{i-1} \) is a single cycle containing a star system \( F_{i-1} \). Then by Lemma 3.1 there exists a single cycle \( G_i \) containing a star system \( F_i \) of size \( |F_{i-1}| + 1 \).

By construction, \( G_i \) has \( 4 + 3i \) vertices, \( |F_i| = i + 1 \), and \( G_i \) is a single cycle of length

\[
|E(G_i)| = |E(G_{i-1})| + 6|F_{i-1}| + 3 \\
= (|E(G_{i-2})| + 6|F_{i-2}| + 3) + 6|F_{i-1}| + 3 \\
\vdots \\
= |E(G_0)| + 6(|F_0| + |F_1| + \ldots + |F_{i-1}|) + 3i \\
= 4 + 6(1 + 2 + \ldots + i) + 3i \\
= 3i^2 + 6i + 4.
\]

Now, letting \( G = G_k \), we see that \( G \) is a single cycle on \( n = 4 + 3k \) vertices of length

\[
\frac{n^2 - 2n + 4}{3} = \frac{2}{3} \binom{n}{2} (1 - o(1)).
\]

\[ \square \]

### 3.2 Attaching a modified projective plane

In order to improve the lower bound in Corollary 3.2, we can try to use a better base single cycle in the inductive construction. Very strangely, it turns out that this will lead to a much improved construction.

By an increment of 3 vertices, Lemma 3.1 produces a larger single cycle, as well as a star system with one extra triple. It would be better if we could have a base single cycle where the new single cycle produced has less than 6 extra vertices and the star system is enlarged by two extra triples.
Consider the modified triangulation of the projective plane in Fig. 2, where each of the 14 faces (triples) are oriented clockwise. Adding the triple \( \{x, y, z\} \) with anticlockwise orientation \( \overrightarrow{xzy} \) gives a cycle (in fact a single cycle, where the triple \( xyz \) has coefficient 2, while the other triples each has coefficient 1). Now delete the triple \( \overrightarrow{acb} \) and identify \( t_1 \) and \( t_2 \) (denote by \( t_1 = t_2 = t \)) to obtain the oriented 3-graph \( S \). It is straightforward to check that \( S \) is a single cycle with a triple removed.

So \( S \) is an oriented 3-graph on 7 vertices and has 14 triples, where \( V(S) = \{x, y, z, t, a, b, c\} \) and \( E(S) = \{\overrightarrow{xyt}, \overrightarrow{aty}, \overrightarrow{ayz}, \overrightarrow{azt}, \overrightarrow{atc}, \overrightarrow{ctz}, \overrightarrow{czx}, \overrightarrow{cxy}, \overrightarrow{cyb}, \overrightarrow{byz}, \overrightarrow{bzx}, \overrightarrow{bxt}, \overrightarrow{bta}, \overrightarrow{xzy}\} \).

For an oriented 3-graph \( G \) that contains a triple \( F = \overrightarrow{ijk} \), we can define the \( (S, F) \)-attachment on \( G \) as in the previous subsection. That is, we say \( G' \) is the \( (S, F) \)-attachment on \( G \) (or simply \( S \)-attachment on \( G \) when \( F \) is understood) where \( G' \) has vertex set \( V(G') = V(G) \cup V(S) \) with \( i, j, k \) identified with \( a, b, c \) respectively, and set of triples \( E(G') = E(G) \cup E(S) \setminus \{\overrightarrow{ijk}\} \). So if \( G \) is an oriented 3-graph on \( k \) vertices and has \( l \) triples, then the \( S \)-attachment on \( G \) is an oriented 3-graph on \( k + 4 \) vertices and has \( l + 13 \) triples.

Similarly, we define the \( (S, \mathcal{F}) \)-attachment on \( G \) for an oriented 3-graph \( G \) on \( k \) vertices, where \( \mathcal{F} = \{F_i = ab_i c_i : i = 1, 2, \ldots, d\} \) is a star system in \( G \), as follows. Let \( G_1 \) be the \( (S, F_1) \)-attachment on \( G \). And for \( i \geq 2 \), \( G_i \) is obtained from \( G_{i-1} \) by deleting the triple \( F_i = ab_i c_i \) and adding the triples \( \{\overrightarrow{atc_i}, \overrightarrow{c_i tz}, \overrightarrow{c_i x y}, \overrightarrow{c_i y b_i}, \overrightarrow{b_i y z}, \overrightarrow{b_i z x}, \overrightarrow{t b_i x}, \overrightarrow{b_i t a}\} \). Then \( G_d \) is the \( (S, \mathcal{F}) \)-attachment on \( G \).

Given a single cycle with a set of triples with a certain property - a star system - we can attach \( S \) inductively in such a way that each \( S \)-attachment produces a larger single cycle that contains a larger star system. The following lemma, which is very similar to Lemma 3.1, is the key method in our inductive construction. The proof is similar to that of Lemma 3.1 (with extra details) and so is omitted.

![Fig.2 The modified triangulation of the real projective plane](image-url)
Lemma 3.3. Let \( G \) be a single cycle (an oriented 3-graph) on \( k \) vertices of length \( l \). Suppose \( \mathcal{F} = \{ F_i = \overrightarrow{ab_i c_i} : i = 1, 2, \ldots, d, b_i \neq c_j \text{ for all } i, j \} \) is a star system in \( G \). Then the \((S, \mathcal{F})\)-attachment on \( G \), \( G' \), is a single cycle on \( k + 4 \) vertices of length \( l + 8d + 5 \). Furthermore, \( G' \) contains a star system of size \( d + 2 \).

By repeatedly applying Lemma 3.3, we now obtain a single cycle of length \( \binom{n}{2}(1 + o(1)) \). By our earlier remarks (Corollary 2.3), this is asymptotically best possible.

Corollary 3.4. There exists an oriented 3-graph on \( n \) vertices whose shortest cycle has length \( \binom{n}{2}(1 + o(1)) \).

Proof. Let \( G_0 \) be the single cycle on 7 vertices obtained from adding the triple \( \overrightarrow{acb} \) to \( S \). It has 15 triples and a ‘good’ set of triples \( \mathcal{F}_0 = \{ \overrightarrow{ybc}, \overrightarrow{yat}, \overrightarrow{yxz} \} \). For \( i \geq 1 \), suppose \( G_{i-1} \) is a single cycle containing a set of triples \( \mathcal{F}_{i-1} \) satisfying the property in Lemma 3.3. Then there exists a single cycle \( G_i \) containing a set of triples \( \mathcal{F}_i \), again satisfying the property in Lemma 3.3.

By construction, \( G_i \) has \( 7 + 4i \) vertices, \( |\mathcal{F}_i| = 2i + 3 \), and \( G_i \) is a single cycle of length

\[
|\mathcal{E}(G_i)| = |\mathcal{E}(G_{i-1})| + 8|\mathcal{F}_{i-1}| + 5
= (|\mathcal{E}(G_{i-2})| + 8|\mathcal{F}_{i-2}| + 5) + 8|\mathcal{F}_{i-1}| + 5
= \vdots
= |\mathcal{E}(G_0)| + 8(|\mathcal{F}_0| + |\mathcal{F}_1| + \ldots + |\mathcal{F}_{i-1}|) + 5i
= 15 + 8(3 + \ldots + (2i + 1)) + 5i
= 8i^2 + 21i + 15.
\]

Now, letting \( G = G_k \), we see that \( G \) is a single cycle on \( n = 7 + 4k \) vertices of length

\[
\frac{2n^2 - 7n + 11}{4} = \binom{n}{2}(1 - o(1)).
\]

We remark that the inductive construction above is very far from being an optimal single cycle in a 3-tournament. Indeed, for any single cycle \( G \) and any triple \( F \) in \( G \), the \((S, F)\)-attachment on \( G \) has the property that any orientation of \( F \) will give a shorter cycle. Note also that the above construction also shows that the rank of the vector space spanned by its incidence matrix has rank at least \( \binom{n}{2}(1 + o(1)) \), implying that the bound in Lemma 2.2 is asymptotically best possible.
4 Concluding remarks

In this paper we have addressed of the girth of 3-tournaments and oriented 3-graphs. Linial and Morgenstern [4] also considered cycles in higher order tournaments: $d$-tournaments for general $d$. (See [4] for relevant definitions.) It would be interesting to know how girth behaves there. There is again a linear algebra bound of $(n^2 + 1)$: how close is this to being attained?

Finally, although these questions arose naturally in the context of oriented 3-graphs and $d$-graphs, it would be interesting to know what happens in the undirected case.

References

[1] M. A. Armstrong, *Basic Topology*, Springer (1983).
[2] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2001).
[3] I. Leader, T.S. Tan, *Directed simplices in higher order tournaments*, Mathematika 56 (2010), 173-181.
[4] N. Linial, A. Morgenstern, *On high-dimensional acyclic tournaments*, Discrete Comput. Geom. 50 (2013), 1085-1100.
[5] J. W. Moon, *Topics on tournaments*, Holt, Rinehart and Winston (New York, 1968).
[6] G. Ringel, *Map Color Theorem*, Springer (New York, 1974).