ON B. MOSSÉ’S UNILATERAL RECOGNIZABILITY THEOREM

SHIGEKI AKIYAMA, BO TAN, AND HISATOSHI YUASA

Abstract. We complete statement and proof for B. Mossé’s unilateral recognizability theorem. We also provide an algorithm for deciding the unilateral non-recognizability of a given primitive substitution.

1. Introduction

Let $A$ be a finite alphabet consisting of at least two letters. Let $A^+$ denote the set of nonempty words over the alphabet $A$. Every map $\sigma$ from the alphabet $A$ to $A^+$ is called a substitution on the alphabet $A$. The substitution $\sigma$ is said to be primitive if there exists $k \in \mathbb{N} = \{1, 2, \ldots \}$ such that for any pair $(a, b) \in A \times A$, the letter $a$ occurs in the word $\sigma^k(b)$. Throughout the present paper, a given substitution is assumed to be primitive. Suppose that the substitution $\sigma$ has a fixed point $u = u_0u_1u_2 \ldots$ in $A^+\mathbb{Z}$, where $\mathbb{Z}^+ = \{0\} \cup \mathbb{N}$. If the fixed point $u$ is aperiodic under the left shift $T$ on $A^\mathbb{Z}$, i.e. $T^iu = u$ for all $i \in \mathbb{N}$, then the substitution $\sigma$ is said to be aperiodic. There is an algorithm [5, 14] which can check whether a given substitution is aperiodic. We always assume that the substitution $\sigma$ is aperiodic.

For every $p \in \mathbb{N}$, set $E_p = \{0\} \cup \{ |\sigma^p(u_{[i,n])})| \mid n \in \mathbb{N} \}$, where $|w|$ is the length of a word $w$. The elements of $E_p$ are called natural $p$-cutting points; see also [11, §3], [2, §3.4] and [4, §7.2.1]. It is clear that $E_q \subsetneq E_p$ whenever $q > p$. The substitution $\sigma$ is said to be unilaterally recognizable [8, p. 530] if there exists $L \in \mathbb{N}$ such that if $u_{[i,i+L]} = u_{[j,j+L]}$ and $i \in E_1$ then $j \in E_1$. This definition does not depend on the choice of the fixed point $u$ of the substitution $\sigma$. Also, the substitution $\sigma$ is said to be bilaterally recognizable [11, Définition 1.2] if there exists $L \in \mathbb{N}$ such that if $u_{[i-L,i+L]} = u_{[j-L,j+L]}$ and $i \in E_1$ then $j \in E_1$.

The unilateral recognizability is an important notion from viewpoints of subshifts arising from substitutions. If the substitution $\sigma$ is unilaterally recognizable, then a unilateral subshift $X_\sigma$ arising from the substitution $\sigma$ has a Kakutani-Rohlin partition [7] built on a clopen subset $\sigma(X_\sigma)$ of $X_\sigma$. Proposition VI. 6 of [15] states that given a point $x = x_0x_1x_2 \ldots \in X_\sigma$, the first return time of the point $\sigma(x)$ to the clopen subset $\sigma(X_\sigma)$ equals $|\sigma(x_0)|$. This leads to a fact that the first return map on $\sigma(X_\sigma)$ is a topological factor of $X_\sigma$, which shows a self-similarity of $X_\sigma$ if the substitution $\sigma$ is injective on the alphabet $A$; see [15, Corollary VI. 8]. It is also a significant consequence of the unilateral recognizability that $\sigma(X_\sigma)$ is open; see [15, Proposition VI. 3] and [8, Lemme 2]. The unilateral recognizability is a premise of the celebrated theorem of [8], which characterizes eigenvalues and eigenfunctions of the subshift $X_\sigma$.

Date: January 12, 2018.

2000 Mathematics Subject Classification. Primary 68R15; Secondary 37B10.

Key words and phrases. primitive substitution, recognizability, algorithm.
B. Mossé gave [11, Théorème 3.1] to characterize the unilateral non-recognizability. However, if we dare say, it is incomplete and should be formulated as follows.

**Theorem 1.1.** The following are equivalent:

1. the substitution $\sigma$ is not unilaterally recognizable;
2. for each $L \in \mathbb{N}$, there exist $i, j \in \mathbb{Z}_+$ such that
   - $\sigma(u_j)$ is a strict suffix of $\sigma(u_i)$;
   - $\sigma(u_{i+k}) = \sigma(u_{j+k})$ for each integer $k$ with $1 \leq k \leq L$.

Recall that the substitution $\sigma$ is assumed to be aperiodic. The word $B$ appearing in the statement of [11, Théorème 3.1] corresponds to a factor of $\sigma(u_iu_{i+1}\ldots u_{i+L})$.

The other goal is to present an algorithm which determines whether or not a given aperiodic, primitive substitution is unilaterally recognizable. The algorithm is described in terms of the existence of a cycle in a directed, finite graph whose vertex set consists of a pair of those words of constant length which occur in the sequence $u$. In view of Theorem 1.1, it may be of interest to find a computable constant $M$, such that the existence of a word $v = v_1v_2\ldots v_{M+1}$ of length $M + 1$ satisfying that

- $\sigma(v_1)$ is not a strict suffix of $\sigma(w_1)$, or
- $\sigma(v_k) \neq \sigma(w_k)$ for some integer $k$ with $2 \leq k \leq M + 1$

is equivalent to the unilateral recognizability of the substitution $\sigma$, which would give an easier algorithm.

2. $K$-power free sequences

We shall make terminology excepting that done in the preceding section. The empty word is denoted by $\Lambda$. Set $A^* = A^+ \cup \{ \Lambda \}$. We say that a word $w \in A^*$ occurs in a word $v \in A^*$ if there exist $p, s \in A^*$ such that $v = pw_s$. We then write $w \prec v$. More specifically, $w$ is said to occurs at the position $|p|+1$ in $v$. The position is called an occurrence of $w$ in $v$. Let $|v|_w$ denote the number of occurrences of $w$ in $v$. The words $p$ and $s$ are called a prefix and suffix of $v$, respectively. We then write $p \prec_s v$ and $s \prec_s v$, respectively. If $|p| < |v|$ (resp. $|s| < |v|$), then $p$ (resp. $s$) is called a strict prefix (resp. suffix) of $v$, and then we write $p \prec_sp$ (resp. $s \prec_sp$).

We can also define $e \prec_s f$ for $e, f \in A^{2+}$ in such a way that $e = f_{[n, +\infty)}$ for some
A nonnegative square matrix $M$ is said to be primitive if there exists $k \in \mathbb{N}$ for which $M^k$ is positive. The incidence matrix $M_\sigma$ of the substitution $\sigma$ is defined to be an $A \times A$ matrix whose $(a, b)$-entry equals $|\sigma(a)|_b$. The matrix $M_\sigma$ is primitive and has a positive, right eigenvector $\beta = (\beta_a)_{a \in A}$ corresponding to Perron eigenvalue $\lambda$ of $M_\sigma$, i.e. the absolute value of any other eigenvalue is less than $\lambda$. See for example [10] Sections 4.2-4.5. Since for all $a \in A$,

$$\sum_{b \in A} (M_\sigma^n)_{a,b} \beta_b = \lambda^n \beta_a,$$

it follows that for all $a \in A$ and $n \in \mathbb{N}$,

$$\frac{\min_{b \in A} \beta_b}{\max_{b \in A} \beta_b} \cdot \lambda^n \leq \sum_{b \in A} (M_\sigma^n)_{a,b} \leq \frac{\max_{b \in A} \beta_b}{\min_{b \in A} \beta_b} \cdot \lambda^n.$$

Put

$$C = \left[ \frac{\max_{b \in A} \beta_b}{\min_{b \in A} \beta_b} \right].$$

It follows that for all $a \in A$ and $n \in \mathbb{N}$,

$$(2.1) \quad C^{-1} \lambda^n \leq |\sigma^n(a)| \leq C \lambda^n.$$

Given a sequence $v \in A^{\geq}$, set

$$L(v)^+ = \{ v_{[i,j]} := v_i v_{i+1} \ldots v_j \mid i, j \in \mathbb{Z}_+, i \leq j \};$$

$$L(v) = L(v)^+ \cup \{ \Lambda \};$$

$$L_k(v) = \{ w \in L(v) \mid |w| = k \}.$$

We say that a word $w \in A^*$ occurs at a position $i \in \mathbb{Z}_+$ in $v$ if $v_{[i, i+|w|)} = w$. The integer $i$ is called an occurrence of the word $w$. The fixed point $u$ of the substitution $\sigma$ is uniformly recurrent, i.e. given a word $w \in L(u)$, there exists $g \in \mathbb{N}$ so that any interval of length $g$, which is a subset of $\mathbb{Z}_+$, includes an occurrence of the word $w$. For, the primitivity of the substitution $\sigma$ implies the existence of $n \in \mathbb{N}$ such that $w < \sigma^n(a)$ for all $a \in A$. Then, the length $g$ can be chosen to be $2 \max_{a \in A} |\sigma^n(a)|$, because

$$u = \sigma^n(u) = \sigma^n(u_0) \sigma^n(u_1) \ldots$$

We shall refer to the length $g$ as a gap of occurrences of the word $w$ in the sequence $u$. Let $g$ be the maximal value of gaps of occurrences of words belonging to $L_2(u)$.

**Lemma 2.1.** The value $g$ is computable.

**Proof.** Consider an auxiliary substitution $\sigma_2 : L_2(u) \rightarrow L_2(u)^+$ [15] pp. 95-96, where $L_2(u)$ is regarded as a finite alphabet. The substitution $\sigma_2$ is defined by for $w \in L_2(u)$,

$$\sigma_2(w) = \sigma(w)[1, 2] \sigma(w)[2, 3] \ldots \sigma(w)[|\sigma(w_1)|, |\sigma(w_1)| + 1].$$

The substitution $\sigma_2$ is primitive [15] Lemma V.12]. Observe that $\#L_2(u) \leq (\#A)^2$ and $|\sigma_2(w)| = |\sigma(w_1)|$ for all $w \in L_2(u)$. Set

$$u^{(2)} = (u_0 u_1)(u_1 u_2)(u_2 u_3) \ldots.$$
which is a fixed point of $\sigma_2$ in $L_2(u)^{\mathbb{Z}^+}$ [15, Lemma V.11]. Observe that given $w \in L_2(u)$ and $i \in \mathbb{Z}_+$, $i$ is an occurrence of $w$ in $u$ if and only if $u_i^{(2)} = w$. In view of [18, Theorem 2.9], the least $n \in \mathbb{N}$ for which every entry of $(M_{\sigma_2})^n$ is positive has a upper bound $(\#L_2(u))^2 - 2 \cdot \#L_2(u) + 2 \leq (\#A)^4 - 2(\#A)^2 + 2$. Put

$$n_0 = (\#A)^4 - 2(\#A)^2 + 2.$$ 

As done in the proof of [20, Lemma 5.1 (iii)], define a $L_2(u) \times A$-matrix $N$ by letting $N_{w,a} = (M_{\sigma})_{w_1,a}$ for all $(w,a) \in L_2(u) \times A$. Let $v$ be a positive eigenvector of $M_{\sigma}$ corresponding to $\lambda$, as above. Since $M_{\sigma}N = NM_{\sigma}$, whose $(w,a)$-entry is $|\sigma^2(w_1)|_a$ for all $(w,a) \in L_2(u) \times A$, a positive vector $Nv$ is an eigenvector of $M_{\sigma}$ corresponding to its eigenvalue $\lambda$. Consequently, the number $\lambda$ is a dominant eigenvalue of a primitive matrix $M_{\sigma_2}$; see for example [10, p. 108 and Theorem 4.5.11]. We then obtain that for all $w \in L_2(u)$ and $n \in \mathbb{N}$,

$$\left(C \max_{a \in A} |\sigma(a)|\right)^{-1} \lambda^n \leq |\sigma_2^n(w)| \leq C \max_{a \in A} |\sigma(a)|\lambda^n.$$ 

It follows finally that the value $g$ has a upper bound $2C \max_{a \in A} |\sigma(a)|\lambda^{n_0}$. □

A word $v \in A^+$ is said to be primitive [11, Définition 2.2] if it holds that

$$v = w^n, w \in A^+, n \in \mathbb{N} \Rightarrow w = v.$$ 

We say that a sequence $v = v_0v_1v_2\ldots \in A^{\mathbb{Z}^+}$ is ultimately periodic if there exist $n \in \mathbb{Z}_+$ and word $w \in A^+$ for which

$$v = v_{[0,n]}www\ldots$$

If an ultimately periodic sequence $v$ is uniformly recurrent, then $v$ is periodic, i.e. $v$ is written as an infinite repetition of a single word. Recall that the substitution $\sigma$ is assumed to be aperiodic.

**Lemma 2.2** ([11, Lemme 2.5]). There does not exist $N, p \in \mathbb{N}$ and primitive word $v \in A^+$ for which

1. $\sigma^p(w) \prec v^N$ for any $w \in L_2(u)$;
2. $2|v| \leq \min_{a \in A} |\sigma^p(a)|$.

**Proof.** For the sake of completeness, we give a proof. Of course, the idea is due to B. Mossé [11]. Assume that there exists such a triple $N, p$ and $v$. Since $\sigma^p(u_i) \prec v^N$, there exist words $\alpha_i \prec_{sp} v, \beta_i \prec_{sp} v$ and $n_i \in \mathbb{Z}_+$ for which $\sigma^p(u_i) = \alpha_i v^{n_i} \beta_i$. If $n_i = 0$, then $|\sigma^p(u_i)| < 2|v|$, which contradicts the hypothesis. Since

$$\sigma^p(u_i u_{i+1}) = \alpha_i v^{n_i} \beta_i \alpha_{i+1} v^{n_{i+1}} \beta_{i+1} \prec v^N,$$

we see that $v \beta_i \alpha_{i+1} v < v^N$. This implies that $\beta_i \alpha_{i+1}$ is a power of $v$; see [11, Propriété 2.3]. Hence, $u = \sigma^p(u) = \alpha_0 vv v\ldots$. This is a contradiction as $\sigma$ is aperiodic. □

**Lemma 2.3** ([11, Théorème 2.4]). If $n \in \mathbb{N}$ and $w^n \in L(u)^+$, then $n < 2\lambda (g+1) C^2$.

**Proof.** For the sake of completeness, we present a proof. Again, the idea is due to B. Mossé [11]. Suppose that $w$ is a primitive word and $w^n \in L(u)$ for some $n \in \mathbb{N}$. There exists $p \in \mathbb{N}$ for which

$$\frac{1}{2} \min_{a \in A} |\sigma^p(a)| \leq |w| < \frac{1}{2} \min_{a \in A} |\sigma^p(a)|.$$
Recall that \( v < u_{i,i+g} \) for all words \( v \in \mathcal{L}_2(u) \) and \( i \in \mathbb{Z}_+ \). Since \( 2|w| < \min_{a \in A} |\sigma^p(a)| \) and \( \sigma \) is aperiodic, it follows from Lemma 2.2 that
\[
n|w| = |w^n| < (g + 1) \max_{a \in A} |\sigma^p(a)|.
\]
Hence,
\[
n < \frac{1}{2} \frac{(g + 1) \max_{a \in A} |\sigma^p(a)|}{\min_{a \in A} |\sigma^{p-1}(a)|} \leq 2(g + 1)C^2 \lambda.
\]
The last inequality follows from (2.1).

**Lemma 2.4.** \( \#\mathcal{L}_n(u) \leq \lambda C^2(\#A)^2 n \) for every \( n \in \mathbb{N} \).

**Proof.** This proof follows that of [15, Proposition V.19]. Fix \( n \in \mathbb{N} \). Find \( p \in \mathbb{N} \) so that \( \min_{a \in A} |\sigma^{p-1}(a)| \leq n \leq \min_{a \in A} |\sigma^p(a)| \). Then,
\[
\#\mathcal{L}_n(u) \leq (\#A)^2 \min_{a \in A} |\sigma^p(a)| \leq (\#A)^2 \frac{\min_{a \in A} |\sigma^p(a)|}{\min_{a \in A} |\sigma^{p-1}(a)|} n \leq \lambda C^2(\#A)^2 n.
\]
This completes the proof.

See also [3,13] and [2, Theorem 24]. Set
\[
K = \left\lceil \lambda C^2 \max \left\{ 2(g + 1), (\#A)^2 \right\} \right\rceil.
\]
Then, the fixed point \( u \) of the substitution \( \sigma \) is \( K \)-power free, in other words, it holds that if \( v^N \in \mathcal{L}(u) \) and \( N \geq K \) then \( v = \Lambda \). Hence, the constant \( K \) is inevitably greater than or equal to two. In general, if a uniformly recurrent sequence \( v \in A^{\mathbb{Z}_+} \) is aperiodic, then given a word \( w \in \mathcal{L}(v)^+ \) there exists a constant \( L \in \mathbb{N} \) for which \( w^L \notin \mathcal{L}(v) \). However, the constant \( L \) may depend on the choice of the word \( w \).

3. B. MOSSÉ’S CHARACTERIZATION OF THE UNILATERAL NON-RECOGNIZABILITY

**Definition 3.1.**

(1) A finite sequence:
\[
\{ \alpha, \sigma^p(u_\ell), \sigma^p(u_\ell+1), \ldots, \sigma^p(u_\ell+k-1), \beta \}
\]
of words over the alphabet \( A \) is called a natural \( p \)-cutting of \( u_{i,i+\ell} \) if
- \( \alpha \prec_s \sigma^p(u_\ell-1) \);
- \( \beta \prec_p \sigma^p(u_{\ell+k}) \);
- \( u_{i,i+\ell} = \alpha \sigma^p(u_\ell) \sigma^p(u_{\ell+1}) \ldots \sigma^p(u_{\ell+k-1}) \beta \), where \( i \);
- \( i + |\alpha| = |\sigma^p(u_{i,i+\ell})| \).

(2) If a word \( w \) occurs at positions \( i \) and \( j \) in the sequence \( u \), then the word \( w \) is said to have the same natural \( p \)-cutting at the positions \( i \) and \( j \) if
\[
(E_p \cap [i, i+|w|]) + (j - i) = E_p \cap [j, j + |w|],
\]
where \( E + i = \{ e + i | e \in E \} \) if \( E \) is a finite subset of \( \mathbb{Z}_+ \) and \( i \in \mathbb{Z}_+ \).

Compare these definitions with the original ones in [11, § 3]. We do not exclude the possibility that
\[
\alpha = \sigma^p(u_{\ell-1}), \alpha = \Lambda, \beta = \sigma^p(u_{\ell+k}) \text{ or } \beta = \Lambda.
\]
Since we always require \( k \geq 1 \) in Definition 3.1, not every \( u_{i,i+\ell} \) has a natural \( p \)-cutting. It is not necessary that a natural \( p \)-cutting is uniquely determined for given \( i \) and \( \ell \) in Definition 3.1.
Proof of Theorem 1.1 Put

\[ k = C^4(K + 1) + 2C^2 + 1. \]

To show the implication \((1) \Rightarrow (2)\), assume that the substitution \(\sigma\) is not unilaterally recognizable.

**Step 1.** It follows from Lemma 3.2 below that for each \(p \in \mathbb{N}\), there exist integers \(i_p \in E_1, j_p \notin E_1, i_p', j_p' \geq 0, h_p, \ell_p \geq 1\) and words \(\alpha_p, \gamma_p' \in A^*, \gamma_p \in A^+\) such that

- \(u_{[i_p, i_p + \ell_p]} = u_{[j_p, j_p + \ell_p]}\);
- \(u_{[i_p, i_p + \ell_p]}\) has a natural \(p\)-cutting:
  \[
  \{ \alpha_p, \sigma^p(u_{i_p'}), \sigma^p(u_{i_p'+1}), \ldots, \sigma^p(u_{i_p'+k-1}) \};
  \]
- \(u_{[j_p, j_p + \ell_p]}\) has a natural \(p\)-cutting:
  \[
  \{ \gamma_p, \sigma^p(u_{j_p'}), \sigma^p(u_{j_p'+1}), \ldots, \sigma^p(u_{j_p'+h_p-1}), \gamma_p' \};
  \]

Set

\[
\begin{align*}
  m_p & = \min \left\{ m \in \mathbb{N} \mid \alpha_p \gamma_p < \gamma_p' \sigma^p(u_{[j_p' - m, j_p']}) \right\}.
\end{align*}
\]

Since

\[
(m_p - 1)I_p \leq |\sigma^p(u_{[j_p' - m_p + 1, j_p']})| < |\alpha_p \gamma_p| \leq 2S_p,
\]

we obtain that for all \(p \in \mathbb{N}\),

\[
m_p < 2C^2 + 1.
\]

Since

\[
h_pI_p \leq |\gamma_p \sigma^p(u_{[j_p' + h_p]})| = |\sigma^p(u_{[j_p' + h_p]})| \leq kS_p
\]

and

\[
(h_p + 2)S_p \geq |\gamma_p' \sigma^p(u_{[j_p' + h_p]})| = |\sigma^p(u_{[j_p' + h_p]})| \geq kI_p,
\]

we obtain that for all \(p \in \mathbb{N}\),

\[
kC^{-2} - 2 \leq h_p \leq kC^2.
\]

It follows that a set:

\[
\left\{ (m_p, h_p, u_{[i_p' - 1, i_p' + k]}, u_{[j_p' - m_p, j_p' + h_p]}) \mid p \in \mathbb{N} \right\}
\]

has a finite cardinality. Hence, the pigeonhole principle implies that for some infinite set \(I \subset \mathbb{N}\), a set:

\[
\left\{ (m_p, h_p, u_{[i_p' - 1, i_p' + k]}, u_{[j_p' - m_p, j_p' + h_p]}) \mid p \in I \right\}
\]

is a singleton. It allows us to put \(m = m_p\) and \(h = h_p\) for any \(p \in I\).

**Step 2.** Let \(p, q \in I\) with \(p < q\) be arbitrary. We have two natural \(q\)-cuttings:

\[
\begin{align*}
  \{ \gamma_q, \sigma^q(u_{j_q'}), \sigma^q(u_{j_q'+1}), \ldots, \sigma^q(u_{j_q'+h-1}), \gamma_q' \}
\end{align*}
\]

of a word occurring at the position \(j_q + |\alpha_q|\) and

\[
\begin{align*}
  \{ \sigma^{q-p}(\gamma_p), \sigma^q(u_{j_q'}), \sigma^q(u_{j_q'+1}), \ldots, \sigma^q(u_{j_q'+h-1}), \sigma^{q-p}(\gamma_p') \}
\end{align*}
\]

of a word occurring at the position \(j_q + |\alpha_q|\) and \(|\gamma_p| \neq |\sigma^{q-p}(\gamma_p)|\). It would be worthwhile observing that \(\sigma^{q-p}(\gamma_p) \prec_s \sigma^q(u_{j_q'-1})\) and \(\sigma^{q-p}(\gamma_p') \preceq_p \sigma^q(u_{j_q'+h_q})\). Assume that the natural \(q\)-cuttings (3.2) and (3.3) are different. Then, one of the inequalities \(|\gamma_q| \neq |\sigma^{q-p}(\gamma_p)|\) and \(|\gamma_q'| \neq |\sigma^{q-p}(\gamma_p')|\) follows.
Consider the case $|\gamma_q| > |\sigma_q - p(\gamma_p)|$. Since
\[
\gamma_q \sigma^q(u_{[i_q', i_q'^+1]} + h) \gamma_q' = \sigma^q(u_{[i_q', i_q'^+1]}) = \sigma^{q-p}(\sigma^p(u_{[i_q', i_q'^+1]})) = \sigma^{q-p}(\gamma_p) \sigma^q(u_{[i_q', i_q'^+1]}) \sigma^{q-p}(\gamma_p') = \sigma^{q-p}(\gamma_p) \sigma^q(u_{[i_q', i_q'^+1]}) \sigma^{q-p}(\gamma_p'),
\]
a power $v^N$ of a nonempty word $v \prec_{ss} \gamma_q$ occurs in $\sigma^q(u_{[i_q', i_q'^+1]})$ as a prefix. By using the fact that $v \prec_{ss} \sigma^q(u_{[i_q', i_q'^+1]})$, we can see that
\[
(3.4) \quad \max \left\{ N \in \mathbb{N} \left| \ v^N \prec_p \sigma^q(u_{[i_q', i_q'^+1]}) \right. \right\} \geq \frac{hI_q}{S_q} - 1 \geq (kC^{-2} - 2)C^{-2} - 1 = K + C^{-4} > K,
\]
where the equality follows from (3.1). This contradicts the $(K + 1)$-power freeness of the sequence $u$, i.e. Lemma 2.3. The same contradiction emerging in the other cases, we conclude that for any $p, q \in I$ with $p < q$,
\[
\gamma_q = \sigma^{q-p}(\gamma_p).
\]

**Step 3.** Choose integers $p < q$ in $I$ so that
\[
|\sigma^{q-p-1}(\gamma_p)| \geq L.
\]
Observe how $u_{[i_q'+1, i_q'^+1]}$ goes to $\sigma^q(u_{[i_q'+1, i_q'^+1]} + h)$; see Figure 1. Since $\gamma_p \prec_p \sigma^p(u_{[i_q', i_q'^+1]})$, $\gamma_q \prec_p \sigma^q(u_{[i_q', i_q'^+1]})$ and $\sigma^{q-p}(\gamma_p) = \gamma_q$, we can see that
\[
(3.5) \quad \left\{ \sigma(u_{p''}), \sigma(u_{p'' + 1}), \ldots, \sigma(u_{p'' + |\sigma^{q-p-1}(\gamma_p)|}) \right\},
\]
where $p'' = |\sigma^{q-1}(u_{[i_q', i_q']}|$. Remark that
\[
(3.6) \quad u_{[i_q', p'' + |\sigma^{q-p-1}(\gamma_p)|]} = \sigma^{q-p-1}(\gamma_p).
\]

Then, observe how $u_{[i_j'+m, i_j'^+1]}$ goes to $\sigma^q(u_{[i_j'+m, i_j'^+1]} + h)$; see Figure 2. Recalling that the natural $q$-cuttings (3.2) and (3.3) are the same, we can see that $u_{[i_j'' + |\alpha_q|, i_j'' + |\alpha_q\gamma_q|]} = \gamma_q$ has a natural 1-cutting:
\[
(3.7) \quad \left\{ \sigma(u_{j''}), \sigma(u_{j'' + 1}), \ldots, \sigma(u_{j'' + |\sigma^{q-p-1}(\gamma_p)|}) \right\},
\]
where $j'' = |\sigma^{q-1}(u_{[i_q'], |\gamma_p|})|$. Remark that
\[
(3.8) \quad u_{[i_j', j'' + |\sigma^{q-p-1}(\gamma_p)|]} = \sigma^{q-p-1}(\gamma_p).
\]
We are finally in a situation that
- $\alpha_q \gamma_q$ occurs at the positions $i_q \in E_1$ and $j_q \notin E_1$ in $u$;
- $\gamma_q$ has the same natural 1-cutting at the positions $i_q + |\alpha_q|$ and $j_q + |\alpha_q|$; recall (3.5) and (3.7);
- all of the positions $i_q + |\alpha_q|, i_q + |\alpha_q\gamma_q|, j_q + |\alpha_q|$, and $j_q + |\alpha_q\gamma_q|$ are natural 1-cutting points;
- the same natural 1-cutting of $\gamma_q$ consists of at least $L$ words.
Figure 1.

\[ u[i'_{q-1},i'_{q+k-1}] \]

\[ \sigma^q(u[i'_{q-1},i'_{q}+k-1]) \]

\[ \sigma^q(u[i'_{q},i'_{q}+k-1]) \]

\[ u[i'_{q},i'_{q} + k-1] \]

\[ \gamma_p \]

\[ u[i'_{q},i'_{q} + k-1] \]

\[ u[i'_{q},i'_{q} + k-1] \]

\[ \gamma_p \]

\[ i_q \]

\[ i_q + |\alpha_q| \]

\[ i_q + |\alpha_q\gamma_q| \]

\[ \alpha_q \]

\[ \gamma_q \]

\[ \gamma_q \]

\[ \gamma_q \]

\[ j_q \]

\[ j_q + |\alpha_q| \]

\[ j_q + |\alpha_q\gamma_q| \]

\[ \alpha_q \]

\[ \gamma_q \]

\[ \gamma_q \]

\[ \gamma_q \]

\[ \gamma_q \]

Figure 2.
Actually, the second and third conditions are implied by a stronger statement that the same 1-cutting of \( \gamma_q \) at the positions \( i_q + |\alpha_q| \) and \( j_q + |\alpha_q| \) comes from the same word \( (3.6) \) and \( (3.8) \), in other words, the word \( \gamma_q \) has the same ancestor word \( (3.6) \) and \( (3.8) \) at the positions. We will again encounter this kind of fact in the proof of “local unique composition property” (Lemma 4.2).

We reach the desired positions \( i, j \in \mathbb{Z}_+ \) by executing the following procedure in this order:

(P. 1) Set \( \ell = i_q + |\alpha_q| \) and \( m = j_q + |\alpha_q| \).

(P. 2) Let \( \ell' < \ell \) and \( m' < m \) be natural 1-cutting points which are nearest to \( \ell \) and \( m \) respectively.

(P. 3) If \( \ell - \ell' = m - m' \), then set \( \ell = \ell' \) and \( m = m' \) and go back to (P. 2).

(P. 4) In this step, we have that \( \ell - \ell' \neq m - m' \). The desired positions \( i \) and \( j \) are determined by the facts that

(a) \( \ell - \ell' < m - m' \Rightarrow |\sigma(u_{[i_q,j_q]})| = \ell' \) and \( |\sigma(u_{[i_q,j_q]})| = m' \);

(b) \( m - m' < \ell - \ell' \Rightarrow |\sigma(u_{[i_q,j_q]})| = \ell' \) and \( |\sigma(u_{[i_q,j_q]})| = m' \).

The loop between (P. 2) and (P. 3) continues up to \( [\alpha_q|\gamma_q|/I_1] \) times. \( \square \)

**Lemma 3.2.** Let \( k \geq 3C^2 \) be an integer. If the substitution \( \sigma \) is not recognizable, then for each \( p \in \mathbb{N} \) there exist integers \( i_p, j_p \notin E_1, i'_p, j'_p \geq 0, k_p, \ell_p \geq 1 \) and words \( \alpha_p, \gamma'_p \in A^*, \gamma_p \in A^+ \) such that

- \( u_{[i_p,j_p] + \ell_p} = u_{[j_p,j_p+\ell_p]} \);
- \( u_{[i_p,j_p] + \ell_p} \) has a natural \( p \)-cutting:
  \[ \{ \alpha_p, \sigma^p(u_{i_p}), \sigma^p(u_{i_p}+1), \ldots, \sigma^p(u_{i_p+k-1}) \} \];
- \( u_{[j_p,j_p] + \ell_p} \) has a natural \( p \)-cutting:
  \[ \{ \gamma_p, \sigma^p(u_{j_p}), \sigma^p(u_{j_p}+1), \ldots, \sigma^p(u_{j_p+h_p-1}), \gamma'_p \} \].

**Proof.** Fix an integer \( m_p \) with

\[ m_p > (k + 2)S_p. \]

Since \( \sigma \) is not recognizable, there exist integers \( i_p \in E_1 \) and \( j_p \notin E_1 \) such that

\[ u_{[i_p,j_p] + m_p} = u_{[j_p,j_p + m_p]}. \]

The choice of \( m_p \) guarantees that \( u_{[i_p,j_p] + m_p} \) has a natural \( p \)-cutting, say

\[ \{ \alpha_p, \sigma^p(u_{i_p}), \sigma^p(u_{i_p}+1), \ldots, \sigma^p(u_{i_p+k-1}), \beta_p \}. \]

Since

\[ k_p \geq \frac{m_p}{S_p} - 2 > k, \]

we can see that \( u_{[i_p,j_p] + \ell_p} \) has a natural \( p \)-cutting:

\[ \{ \alpha_p, \sigma^p(u_{i_p}), \sigma^p(u_{i_p}+1), \ldots, \sigma^p(u_{i_p+k-1}) \}, \]

where \( \ell_p = |\alpha_p| \sigma^p(u_{[i_p,j_p]+k}) \). Since

\[ \ell_p - |\alpha_p| \geq kI_p \geq kC^{-1}\lambda \geq kC^{-2}S_p \geq 3S_p, \]

\( u_{[j_p,j_p] + \ell_p} \) has a natural \( p \)-cutting:

\[ \{ \gamma_p, \sigma^p(u_{j_p}), \sigma^p(u_{j_p}+1), \ldots, \sigma^p(u_{j_p+h_p-1}), \gamma'_p \} \]

with \( \gamma_p \neq \Lambda \). \( \square \)
Following Property (6) in [19], we make a definition:

**Definition 4.1.** We say that the substitution $\sigma$ has the local unique composition property under an index $L \in \mathbb{N}$ if

- the substitution $\sigma$ is bilaterally recognizable under the index $L$;
- if $u_{i-L,i+L} = u_{j-L,j+L}$, $|\sigma(u_{0,m})| \leq i < |\sigma(u_{0,m+1})|$ and $|\sigma(u_{0,n})| \leq j < |\sigma(u_{0,(n+1)})|$ then $u_m = u_n$.

See also [12, Théorème 2] and [2, Theorem 11]. None has given a computable value of an index under which the local unique composition property holds, which is now done by the following lemma. The lemma excluding the computability is due to [11, Théorème 3.1 bis.], though the theorem do not mention any decidability of the index $L$.

**Lemma 4.2.** The aperiodic, primitive substitution $\sigma$ has the local unique composition property under an index $L_0 = \{C^4(K+1) + 2C^2 + 1\}S_{p_0}$, where

\[ p_0 = K^2 \left\{ C^4(K+1) + 2C^2 + 1 \right\} \times \sum_{n=C^2(K+1)+2}^{(K+1)C^4+2C^2+2} n + 1. \]

**Proof.** Put $k = C^4(K+1) + 2C^2 + 1$. Assume that $u_{i-L_0,i+L_0} = u_{j-L_0,j+L_0}$. The integer $L_0$ is so large that we can choose $\{m_p, n_p \in \mathbb{N} \mid 1 \leq p \leq p_0\}$ so that

- $u_{i-m_p,i+n_p} = u_{j-m_p,j+n_p}$;
- $u_{i-m_p,i+n_p}$ has a natural $p$-cutting:

\[ \{ \sigma^p(u_{ip}), \sigma^p(u_{ip+1}), \ldots, \sigma^p(u_{ip+k-1}) \} . \]

Let

\[ \{ \gamma_p, \sigma^p(u_{jp}), \sigma^p(u_{jp+1}), \ldots, \sigma^p(u_{jp+h_p-1}), \gamma'_p \} \]

be a natural $p$-cutting of $u_{j-m_p,j+n_p}$. Since for every integer $p$ with $1 \leq p \leq p_0$, we have

\[ C^2(K+1) + C^{-2} = kC^{-2} - 2 \leq h_p \leq kC^2 \]

in view of an equality:

\[ |\sigma^p(u_{jp+h_p})| = |\gamma_p \sigma^p(u_{jp+h_p})\gamma'_p| ; \]

it follows from Lemma 2.4 that the cardinality of a set:

\[ \{ (u_{jp+h_p}) \mid 1 \leq p \leq p_0 \} \]

is at most $p_0 - 1$. The pigeonhole principle implies that for some integers $p$ and $q$ with $1 \leq p < q \leq p_0$,

\[ u_{[p,i_p+p+k]} = u_{[q,i_q+k]} \quad \text{and} \quad u_{[p-1,i_p+h_p]} = u_{[q-1,i_q+h_q]} . \]

Hence, $h_p = h_q$. In view of an equality:

\[ \gamma_q \sigma^q(u_{jq+q+h_q}) \gamma_q = \sigma^{q-p}(\gamma_p) \sigma^q(u_{jq+q+h_q}) \sigma^{q-p}(\gamma'_p) ; \]

and the $(K+1)$-power freeness of the sequence $u$; recall (3.4), we obtain that $\gamma_q = \sigma^{q-p}(\gamma_p)$ and $\gamma'_q = \sigma^{q-p}(\gamma'_p)$. Taking account into the positions of the natural $q$-cutting of $u_{[j-m_q,j+n_q]}$, we see that $u_{[j-m_q,j+n_q]}$ and $u_{[j-m_q,j+n_q]}$ have the same natural $(q-p)$-cutting, which is yielded by application of $\sigma^{q-p}$ to identical words:

\[ \sigma^p(u_{[q,i_q+k]}) = \gamma_p \sigma^p(u_{[jq,i_q+h_q]}) \gamma'_p . \]
so that $u_{i-m_q,i+n_q}$ and $u_{j-m_q,j+n_q}$ have the same natural 1-cutting.

To verify [12, Théorème 2], B. Mossé discusses such a constant $p \in \mathbb{N}$ that if $\sigma^{p-1}(a) \neq \sigma(b)^p$ and $a, b \in A$ then $\sigma^k(a) \neq \sigma^k(b)$ for all $k \in \mathbb{Z}_+$. The constant $p$ is formally obtained by setting

$$p = \begin{cases} \max_{(a,b) \in B} \min \{ k \in \mathbb{N} \mid \sigma^k(a) = \sigma^k(b) \} + 1 & \text{if } B \neq \emptyset; \\ 1 & \text{otherwise,} \end{cases}$$

where

$$B = \{ (a, b) \in A \times A \mid a \neq b, \sigma^k(a) = \sigma^k(b) \text{ for some } k \in \mathbb{N} \}.$$  

See also the proof of [9, Theorem 4.36]. As an application of Lemmas 2.4 and 4.2 we can see that

**Proposition 4.3.** The constant $p$ is computable.

**Proof.** Put

$$(4.2) \quad N = K \left( \lfloor L_0 \lambda^{-1}(C - C^{-1}) \rfloor + 1 \right) + \lfloor CL_0 \lambda^{-1} \rfloor + 1.$$  

Choose an integer $k_0$ with $k_0 > \log_\lambda(CN)$. Then, for all letters $a \in A$,

$$|\sigma^{k_0}(a)| \geq C^{-1}k_0 > N.$$  

Following [17], given letter $a \in A$ and integer $k$ with $k \geq k_0$, let $\text{Suf}_N(\sigma^k(a))$ (resp. $\text{Pref}_N(\sigma^k(a))$) denote a suffix (resp. prefix) of $\sigma^k(a)$ whose length is $N$. Fix distinct letters $a_1, a_2 \in A$. Lemma 2.4 together with the pigeonhole principle allows us to find those integers $i_m < j_m$ and $k_m < \ell_m$ which belong to a closed interval $[k_0, k_0 + \lambda C^2(\#A)^2N]$ for which and for all $m = 1, 2$,

- $\text{Pref}_N(\sigma^m(a_m)) = \text{Pref}_N(\sigma^m(a_m));$
- $\# \{ \text{Pref}_N(\sigma^k(a_m)) \mid k_0 \leq k \leq j_m \} = j_m - k_0;$
- $\text{Suf}_N(\sigma^k(a_m)) = \text{Suf}_N(\sigma^k(a_m));$
- $\# \{ \text{Suf}_N(\sigma^k(a_m)) \mid k_0 \leq k \leq \ell_m \} = \ell_m - k_0.$

For $m = 1, 2$, regard

$$\mathcal{P}_m = \{ \text{Pref}_N(\sigma^k(a_m)) \mid k \geq i_m \} \quad \text{and} \quad \mathcal{S}_m = \{ \text{Suf}_N(\sigma^k(a_m)) \mid k \geq k_m \},$$

as sequences of words, which have periods $j_m - i_m$ and $\ell_m - k_m$, respectively. Observe that $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ unless $\mathcal{P}_1 = \mathcal{P}_2$. This fact is also valid for $\mathcal{S}_m$.

If $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ or $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, then $\sigma^k(a) \neq \sigma^k(b)$ for all $k \in \mathbb{N}$. Then, set $p_{a_1,a_2} = 1$. If $\mathcal{P}_1 = \mathcal{P}_2$ and $\mathcal{S}_1 = \mathcal{S}_2$, then set

$$p_{a_1,a_2} = \max \{ i_1, i_2, k_1, k_2, \log_\lambda(CN + C^2L_0 \lambda^{-1}) \} + 1.$$  

Let $p$ denote $p_{a_1,a_2}$ for the simplicity of notation. Let us verify that if $\sigma^{p-1}(a_1) \neq \sigma^{p-1}(a_2)$ then $\sigma(a_1)^k \neq \sigma(a_2)^k$ for all $k \in \mathbb{Z}_+$. To this end, it is enough for us to consider only the case where $\mathcal{P}_1 = \mathcal{P}_2$ and $\mathcal{S}_1 = \mathcal{S}_2$. Let us see that if $\sigma^p(a_1) = \sigma^p(a_2)$ then $\sigma^{p-1}(a_1) = \sigma^{p-1}(a_2)$. For each $m = 1, 2$, let $q_m < r_m$ be unique integers satisfying that

$$|\sigma(\sigma^{p-1}(a_m)[1,q_m])| \leq L_0;$$
$$|\sigma(\sigma^{p-1}(a_m)[1,q_m])| > L_0;$$
$$|\sigma(\sigma^{p-1}(a_m)[1,r_m])| < |\sigma^p(a_m)| - L_0;$$
$$|\sigma(\sigma^{p-1}(a_m)[1,r_m])| \geq |\sigma^p(a_m)| - L_0.$$
These inequalities together with (2.4) imply that for all $m = 1, 2$,

\begin{align}
(4.3) & \quad C^{-1}L_0\lambda^{-1} < q_m \leq CL_0\lambda^{-1}; \\
(4.4) & \quad C^{-1}L_0\lambda^{-1} < |\sigma^{p-1}(a_m)| - r_m + 1 \leq CL_0\lambda^{-1} + 1
\end{align}

and hence

\begin{align*}
|q_1 - q_2| & < L_0\lambda^{-1}(C - C^{-1}); \\
||\sigma^{p-1}(a_1)| - r_1 + 1 - (||\sigma^{p-1}(a_2)| - r_2 + 1)| & < L_0\lambda^{-1}(C - C^{-1}) + 1; \\
|\sigma^{p-1}(a_m)| - q_m + 1 & \geq N + 1; \\
r_m & \geq N.
\end{align*}

Lemma 4.2 allows us to know that

\begin{align}
(4.5) & \quad \sigma^{p-1}(a_{[q_1,r_1]}) = \sigma^{p-1}(a_{[q_2,r_2]}).
\end{align}

Since $\text{pref}_N(\sigma^p(a_1)) = \text{pref}_N(\sigma^p(a_2))$ and $\text{suf}_N(\sigma^p(a_1)) = \text{suf}_N(\sigma^p(a_2))$, it follows from the choice of $p$ that

\begin{align}
(4.6) & \quad \text{pref}_N(\sigma^{p-1}(a_1)) = \text{pref}_N(\sigma^{p-1}(a_2)); \\
(4.7) & \quad \text{suf}_N(\sigma^{p-1}(a_1)) = \text{suf}_N(\sigma^{p-1}(a_2)).
\end{align}

If $q_1 \neq q_2$, then the $K$-power of a word length $|q_1 - q_2|$ must occur at $\min \{q_1, q_2\}$ in $\sigma^{p-1}(a_m)$ if $q_m$ attains the minimum value, which is impossible in virtue of Lemma 2.3. Hence, we obtain that

\begin{align}
(4.8) & \quad q_1 = q_2,
\end{align}

which is less than $N$ in view of (4.2) and (4.3). Similarly, we also obtain that

\begin{align}
(4.9) & \quad |\sigma^{p-1}(a_{[r_1, |\sigma^{p-1}(a)|]})| = |\sigma^{p-1}(a_1)| - r_1 + 1 \\
& = |\sigma^{p-1}(a_2)| - r_2 + 1 = |\sigma^{p-1}(a_{[r_2, |\sigma^{p-1}(a)|]})|,
\end{align}

which is less than $N$ in virtue of (4.2) and (4.4). Now, putting together (4.5)-(4.9), we see that the words $\sigma^{p-1}(a_1)$ and $\sigma^{p-1}(a_2)$ must coincide with each other. \hfill $\Box$

In order to see that a constant $L_1$ appearing in Lemma 4.4 is computable, we will need some facts about Birkhoff contraction coefficients for allowable nonnegative square matrices. We consult [18, Chapter 3] and [6, Subsections 2.1.1 and 2.2.1] for them. Let us consider a projective metric $d$ which is defined by for positive, row vectors $x, y \in \mathbb{R}^A$, i.e. all entries are positive,

\begin{align*}
d(x, y) = \ln \frac{\max_{a \in A} \frac{x_a}{y_a}}{\min_{b \in A} \frac{x_b}{y_b}}.
\end{align*}

Suppose that a primitive matrix $M = (m_{a,b})_{a,b \in A}$ is allowable, i.e. every row and column has a positive entry. Birkhoff contraction coefficient $\tau_B(M)$ is defined by

$$
\tau_B(M) = \sup \left\{ \frac{d(xM, yM)}{d(x, y)} \bigg| x, y \in \mathbb{R}^A \text{ are positive and linearly independent.} \right\}.
$$

There are known properties [3, Lemma 2.1 and Theorem 2.8] that $0 \leq \tau_B(M) \leq 1$ and $\tau_B(MN) \leq \tau_B(M)\tau_B(N)$ if $N$ is another allowable, nonnegative $A \times A$ matrix.
The coefficient $\tau_B(M)$ is computable:

$$
\tau_B(M) = \frac{1 - \sqrt{\phi(M)}}{1 + \sqrt{\phi(M)}}
$$

(4.10)

where

$$
\phi(M) = \min \left\{ \frac{m_{i,j}m_{k,l}}{m_{i,l}m_{k,j}}, \frac{m_{i,l}m_{k,j}}{m_{i,j}m_{k,l}} \right\} \left( \begin{array}{cc} m_{i,j} & m_{i,l} \\ m_{k,j} & m_{k,l} \end{array} \right) \text{ is a submatrix of } M. \right\}
$$

if $M$ is positive, and otherwise, $\phi(M) = 0$. Put

$$
n_0 = (\#A)^2 - 2(\#A) + 2.
$$

(4.11)

Since it follows from [18, Theorem 2.9] that $M_\sigma^n$ is positive for all integers $n$ with $n \geq n_0$, we have that $\phi(M_\sigma^n) > 0$ for all such integers $n$. It follows from (4.10) that for all such integers $n$,

$$
\tau_B(M_\sigma^n) < 1.
$$

(4.12)

As in Section 2, let $\alpha$ be a positive, left eigenvector of $M_\sigma$ corresponding to its Perron eigenvalue $\lambda$. Given a word $w \in A^*$, set

$$
\text{vec}(w) = (|w|_b)_{b \in A},
$$

which is viewed as a row vector in $\mathbb{R}^A$. Define a row vector $e_a$ in $\mathbb{R}^A$ by for each letter $b \in A$,

$$
(e_a)_b = \begin{cases} 1 & \text{if } a = b; \\ 0 & \text{otherwise}. \end{cases}
$$

It is clear that $\text{vec}(\sigma^n(a)) = e_aM_\sigma^n$. It follows from [18, Theorem 2.9] and [6, Theorem 2.3 and Corollary 2.2] that for all letter $a \in A$ and integer $n$ with $n \geq 2n_0$,

$$
\left\| \frac{\text{vec}(\sigma^n(a))}{\|\sigma^n(a)\|_1} - \frac{\alpha}{\|\alpha\|_1} \right\|_1 \leq \exp \left( d(e_aM_\sigma^n, \alpha) \right) - 1
\leq \exp \left( \tau_B(M_\sigma^{n-n_0}) d(e_aM_\sigma^{n_0}, \alpha) \right) - 1
\leq \exp \left( \tau_B(M_\sigma^{n_0}) \left[ \frac{n-n_0}{n_0} \right] \max_{a \in A} d(e_aM_\sigma^{n_0}, \alpha) \right) - 1
\leq \exp \left( \max_{a \in A} d(e_aM_\sigma^{n_0}, \alpha) \right)^{\tau_B(M_\sigma^{n_0})[\frac{n}{n_0}-1]} - 1,
$$

(4.13)

which monotonically decreases to zero in virtue of (4.12) as $n$ increases. As a trivial consequence, we obtain that

$$
\max_{a,b,c \in A} \left| \frac{\sigma^n(a)c}{\|\sigma^n(a)\|_c} - \frac{\sigma^n(b)c}{\|\sigma^n(b)\|_c} \right| \leq 2 \exp \left( \max_{a \in A} d(e_aM_\sigma^{n_0}, \alpha) \right)^{\tau_B(M_\sigma^{n_0})[\frac{n}{n_0}-1]} - 2.
$$

(4.14)

**Lemma 4.4.** For every real number $\rho$ with $1 < \rho < \lambda$, there exists a computable number $L_1 \in \mathbb{N}$ so that for all integers $L \geq L_1$ and $i \geq 0$, we have an inequality:

$$
|\sigma(u_{[i,i+L]})| \geq \rho |u_{[i,i+L]}| = \rho L.
$$

In particular, it holds that for all $i \in \mathbb{Z}_+$,

$$
|\sigma(u_{[i,i+L_1]})| \geq L_1 + 1.
$$
Proof. Let $\epsilon$ denote a number satisfying that $\rho = (1 - \epsilon)\lambda$. This forces that $0 < \epsilon < 1$. Let $n_0$ be as in (4.11). Choose an integer $n$ with $n \geq n_0$ so large that

$$2\exp \left( \max_{a \in A} d(\epsilon_a M_{\sigma}^{n_0}, \alpha) \right)^{\gamma_n(M_{\sigma}^{n_0}) [\frac{1}{n_0} - 1]} - 2 < \frac{\epsilon \lambda}{4(\#A)\|M_{\sigma}\|_1},$$

where $\|M_{\sigma}\|_1$ is a norm of $M_{\sigma}$ defined by

$$\|M_{\sigma}\|_1 = \max \{ \|x M_{\sigma}\|_1 \mid x \in \mathbb{R}^A, \|x\|_1 = 1 \} = \max_{k \in A} \sum (M_{\sigma})_{a,b}.$$

Consequently, the right hand side of (4.13) is also less than (4.15). If the integer $L$ is not less than $2S_n$, then we obtain a natural $n$-cutting:

$$\{ v, \sigma^n(u_{j+1}), \sigma^n(u_{j+2}), \ldots, \sigma^n(u_{j+k}), w \}$$

of $u_{[i,i+L]}$ so that $v \prec_{as} \sigma^n(u_j)$ and $w \prec_{ap} \sigma^n(u_{j+k+1})$. Put

$$\delta_k = 1 - \frac{1}{2C^2k^{-1} + 1},$$

which monotonically decreases as $k$ increases. Since $k \geq S_n^{-1}L - 2$, we have that

$$\delta_k \leq \frac{2C^2}{S_n^{-1}L + 2(C^2 - 1)}.$$

Choose $L_1 \in \mathbb{N}$ so that for all integers $L$ with $L \geq L_1$,

$$\max \{ \delta_k, 2S_nL^{-1} \} < \frac{\epsilon \lambda}{4(\#A)\|M_{\sigma}\|_1}.$$

Suppose that the integer $L$ is not less than $L_1$. Let $i \in \mathbb{Z}_+$ and $c \in A$ be arbitrary. Since

$$\frac{|u_{[i,i+L]}|_c}{L} = \frac{\sum_{j=1}^{j+k+1} |\sigma^n(u_{\ell})|_c}{L} \cdot \frac{\sum_{j=1}^{j+k} |\sigma^n(u_{\ell})|}{L} + \frac{|v|_c + |w|_c}{L},$$

we obtain that

$$\min_{a \in A} \frac{|\sigma^n(a)|_c}{|\sigma^n(a)|} \cdot \frac{\sum_{j=1}^{j+k} |\sigma^n(u_{\ell})|}{L} \leq \frac{|u_{[i,i+L]}|_c}{L} \leq \max_{a \in A} \frac{|\sigma^n(a)|_c}{|\sigma^n(a)|} \cdot \frac{\sum_{j=1}^{j+k} |\sigma^n(u_{\ell})|}{L} + 2S_nL^{-1}.$$

However,

$$1 - \delta_k = \frac{1}{2C^2k^{-1} + 1} \leq \frac{\sum_{j=1}^{j+k} |\sigma^n(u_{\ell})|}{L} \leq 1.$$

Let $b \in A$ be arbitrary. Putting (4.18) and (4.19) together, we obtain that

$$\frac{1 - \delta_k}{\min_{a \in A} \frac{|\sigma^n(a)|_c}{|\sigma^n(a)|} - \frac{|\sigma^n(b)|_c}{|\sigma^n(b)|}} \leq \frac{|u_{[i,i+L]}|_c}{L} - \frac{|\sigma^n(b)|_c}{L} \leq \max_{a \in A} \frac{|\sigma^n(a)|_c}{|\sigma^n(a)|} - \frac{|\sigma^n(b)|_c}{|\sigma^n(b)|} + 2S_nL^{-1},$$

and hence,

$$\frac{|u_{[i,i+L]}|_c}{L} - \frac{|\sigma^n(b)|_c}{|\sigma^n(b)|} \leq \max_{a \in A} \frac{|\sigma^n(a)|_c}{|\sigma^n(a)|} - \frac{|\sigma^n(b)|_c}{|\sigma^n(b)|} + \max \{ \delta_k, 2S_nL^{-1} \}.$$
Using (4.14), (4.15) and (4.17), we hence obtain that
\[
\|\vec{u}_{i,i+L}\|_{L} - \|\vec{\sigma}(b)\|_{L} \leq \sum_{a \in A} \max_{v \in A} \left| \frac{\sigma^v(a)}{\sigma^v(b)} \right| + (\#A) \max \{ \delta_k, 2S_nL^{-1} \}
\]
\[
\leq 2\|M_\sigma\|_1
\]
We finally obtain that
\[
\|\vec{u}_{i,i+L}\|_{L} - \alpha \|\alpha\|_1 \leq \|\vec{u}_{i,i+L}\|_{L} - \|\vec{\sigma}(b)\|_{L} \leq \|\vec{\sigma}(b)\|_{L} - \alpha \|\alpha\|_1 \leq \frac{\epsilon\lambda}{\|M_\sigma\|_1}
\]
and hence,
\[
|\sigma(u_{i,i+L})| = L \|\vec{u}_{i,i+L}\|_{L} M_\sigma \geq L \|\alpha\|_1 M_\sigma - \left( \frac{\vec{u}_{i,i+L}}{\|\alpha\|_1} \right) M_\sigma \geq L(1 - \epsilon)\lambda = \rho L.
\]
It is now clear that $L_1$ is computable, because in virtue of (4.16) and (4.17) it is sufficient to choose $L_1$ so that
\[
\max \left\{ 2C^2 \frac{2\|\vec{u}_{i,i+L}\|_{L}}{S_n^{-1}L_1^2 + 2(C^2 - 1)}, 2S_nL_1^{-1} \right\} < 4\frac{\epsilon\lambda}{\|M_\sigma\|_1}.
\]
This completes the proof. □

Fix $1 < \rho < \lambda$ and $L_1 \in \mathbb{N}$ as in Lemma 4.4. Fix an integer $N$ greater than
\[
\max \left\{ S_1(L_1 + 1) - 1, \frac{1 + L_0(I_1^{-1} + 1)}{\rho - 1} \right\}.
\]
Set
\[
V^* = \{ (xac, ybc) \in \mathcal{L}_{N+L_1+1}(u) \times \mathcal{L}_{N+L_1+1}(u) \mid a, b \in A \ (a \neq b), \ c \in \mathcal{L}_{L_1}(u) \},
\]
which is nonempty, because the sequence $u$ over the finite alphabet $A$ is assumed to be aperiodic; see [1, Theorem 2.11] and [15, Proposition V.18]. Define an equivalence relation $\sim$ on $V^*$ so that $(v, w) \sim (v', w')$ if and only if $(v, w) = (v', w')$ or $(v, w) \sim (w', v')$. Set $V = V^*/\sim$. Let $[v, w]$ denote the equivalence class of a given element $(v, w)$ of $V^*$. Consider a directed, finite graph $G$ with vertex set $V$ and edge set $E$. The edge set $E$ is defined by declaring that there exists an edge from a vertex $v$ to a vertex $v'$ if and only if there exist word $w \in A^+$, representatives $(s, t)$ and $(s', t')$ of the equivalence classes $v$ and $v'$, respectively, such that $s'w \preceq_\sigma \sigma(s)$ and $t'w \preceq_\sigma \sigma(t)$.

**Remark 4.5.** The number of edges leaving a given vertex is at most one.

**Definition 4.6.** We say that a vertex of the directed, finite graph $G$ *generates a gap of natural 1-cutting points* if there exist letters $\alpha, \beta$, words $\gamma, \delta$ of the same length and representative $(v, w)$ of the vertex so that
- $\sigma(\alpha) \preceq_\sigma \sigma(\beta)$;
- $\sigma(\gamma_i) = \sigma(\delta_i)$ for every integer $i$ with $1 \leq i \leq |\gamma|$;
In other words, the unilateral subshift sequence \( u \) is generated by a substitution \( \sigma \). We define a unilateral subshift: \( X_\sigma = \{ x = (x_i)_{i \in \mathbb{Z}_+} \mid x_{[i,j]} \in \mathcal{L}(u) \text{ for all } i, j \in \mathbb{Z}_+ \} \).

In other words, the unilateral subshift \( X_\sigma \) is generated by the language of the sequence \( u \).

**Theorem 4.7.** The following are equivalent:

1. the substitution \( \sigma \) is not unilaterally recognizable;
2. the directed, finite graph \( G \) has a cycle including a vertex generating a gap of natural 1-cutting points.

**Proof.** (2) \( \Rightarrow \) (1): Assume that the directed, finite graph \( G \) includes a cycle:

\[
\{ [x_ia_i c_i, y_i b_i c_i] \in V \mid 0 \leq i \leq \ell \}
\]

of length \( \ell \) so that

- the vertex \( [x_0 a_0 c_0, y_0 b_0 c_0] \) generates a gap of natural 1-cutting points;
- \( x_0 a_0 c_i = x_0 a_0 c_0 \) and \( y_0 b_i c_0 = y_0 b_0 c_0 \);
- for every integer \( i \) with \( 0 \leq i < \ell \), there exists \( w_{i+1} \in A^+ \) satisfying that

\[
x_{i+1} a_{i+1} c_{i+1} w_{i+1} \prec_s \sigma(x_i a_i c_i) \text{ and } y_{i+1} b_{i+1} c_{i+1} w_{i+1} \prec_s \sigma(y_i b_i c_i).
\]

For every integer \( i \) with \( i > \ell \), put \( w_i = w_{i \mod \ell + 1} \). It is straightforward to see that for every \( k \in \mathbb{N} \),

\[
x_0 a_0 c_0 w_k \sigma(w_{k-1}) \sigma^2(w_{k-2}) \ldots \sigma^{k-2}(w_2) \sigma^{k-1}(w_1) \prec_s \sigma^{k\ell}(x_0 a_0 c_0);
\]

\[
y_0 b_0 c_0 w_k \sigma(w_{k-1}) \sigma^2(w_{k-2}) \ldots \sigma^{k-2}(w_2) \sigma^{k-1}(w_1) \prec_s \sigma^{k\ell}(y_0 b_0 c_0).
\]

Hence, Condition (2) in Theorem 1.1 is satisfied.

(1) \( \Rightarrow \) (2): Assume that \( \sigma \) is not unilaterally recognizable. We shall see that the directed, finite graph \( G \) has a vertex which generates a gap of natural 1-cutting points. Using the pigeonhole principle together with Theorem 1.1 and the uniform recurrence of the sequence \( u \), we can find \( x, y \in \mathcal{L}_{N}(u) \), \( a, b \in A \) and \( z, w \in X_\sigma \) so that

- \( xaz, ybw \in X_\sigma \);
- \( \sigma(a) \prec_s \sigma(b) \);
- \( \sigma(z_i) = \sigma(w_i) \) for all \( i \in \mathbb{Z}_+ \).

Lemma 4.2 allows us to find \( \ell \in \mathbb{Z}_+ \) such that

- \( z_{\ell-1} \neq w_{\ell-1} \);
- \( z_{[\ell, +\infty)} = w_{[\ell, +\infty)} \);
- \( |\sigma(z_{[0,\ell)})| \leq L_0 \);
- \( |\sigma(w_{[0,\ell)})| \leq L_0 \),

where we use a convention that \( z_{-1} = a \) and \( w_{-1} = b \).

By the pigeonhole principle again, using the hypothesis that the sequence \( u \) is assumed to be aperiodic, we can find \( e, f \in X_\sigma \) for which

\[
(4.20) \quad xaz \prec_s \sigma(e), \quad xaz \not\prec_s \sigma(e_{[1, +\infty)}), \quad ybw \prec_s \sigma(f) \text{ and } ybw \not\prec_s \sigma(f_{[1, +\infty)}).
\]

In view of Lemma 4.2 again, there exist \( m, n \in \mathbb{N} \) such that

- \( \alpha := e_{m-1} \neq f_{n-1} = \beta \);
- \( \zeta := e_{[m, +\infty)} = f_{[n, +\infty)} \).
Recall that $z_{t-1} \neq w_{t-1}$. There exists a prefix $s \in A^*$ of $z_{[t, +\infty)} = w_{[t, +\infty)}$ such that
\begin{equation}
|s| \leq L_0, \ xaz_{[0, t)}s \prec_s \sigma(e_{[0, m)}) \text{ and } ybw_{[0, t)}s \prec_s \sigma(f_{[0, n]}).
\end{equation}
Since
\[ mS_1 \geq |\sigma(e_{[0, m)})| \geq |xa| = N + 1 \geq S_1(L_1 + 1), \]
we obtain that $m - 1 \geq L_1$. This together with Lemma 4.4 implies that
\[ \rho(m - 1) \leq |\sigma(e_{[1, m)})| < |xaz_{[0, t)}s| \leq N + 1 + L_0(I_1^{-1} + 1) < \rho N, \]
where the second inequality follows from the second property of (4.20), so that $m - 1 < N$. Similarly, we obtain that $n - 1 < N$. These facts allow us to find words $\chi, \tau \in \mathcal{L}_N(u)$ so that
\begin{itemize}
  \item $e_{[0, m-1]} \prec_s \chi$;
  \item $\chi\alpha\zeta \in X_\sigma$;
  \item $f_{[0, n-1]} \prec_s \tau$;
  \item $\tau\beta\zeta \in X_\sigma$.
\end{itemize}
Consequently, we obtain that
\begin{itemize}
  \item $\sigma(e) \prec_s \sigma(\chi)\sigma(e_{[m-1, +\infty)}) = \sigma(\chi\alpha\zeta)$;
  \item $\sigma(f) \prec_s \sigma(\tau)\sigma(f_{[n-1, +\infty)}) = \sigma(\tau\beta\zeta)$.
\end{itemize}
This together with (4.20) shows that
\begin{itemize}
  \item $xaz \prec_s \sigma(\chi\alpha\zeta)$;
  \item $ybw \prec_s \sigma(\tau\beta\zeta)$.
\end{itemize}
Since in virtue of (4.21) we know that $xaz_{[0, t)}s \prec_s \sigma(\chi\alpha)$, letting $\gamma = \zeta_{[0, L_1)}$, we obtain that $xaz_{[0, t)}s\sigma(\gamma) \prec_s \sigma(\chi\alpha\gamma)$, and also that $ybw_{[0, t)}s\sigma(\gamma) \prec_s \sigma(\tau\beta\gamma)$. We have obtained a vertex $[\chi\alpha\gamma, \tau\beta\gamma] \in V$ which generates a gap of natural 1-cutting points.

Now, apply the procedure to $(\chi\alpha\zeta, \tau\beta\zeta)$, which has been applied to $(xaz, ybw)$ for obtaining $(\chi\alpha\zeta, \tau\beta\zeta)$. It yields another vertex where an edge leaves for $(\chi\alpha\gamma, \tau\beta\gamma)$. Applying the procedure inductively yields an infinite path in the directed, finite graph, which results in a cycle in virtue of Remark 1.5.

**Acknowledgments.** The first revision was undertaken during the third author’s visit in 2014 at Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées, CNRS-UMR 6140, Université de Picardie Jules Verne. He is grateful for hospitality and kind support extended to him by the institution. The second revision was done in 2015 after his visit at Institute of Mathematics, University of Tsukuba. This revision was done in 2017 after his visit at Centre International de Rencontres Mathématiques. He is also grateful for their hospitality and discussion. This work was partially supported by JSPS KAKENHI Grant Number 17K05159.

**References**

[1] Ethan M. Coven and G. A. Hedlund, *Sequences with Minimal Block Growth*, Theory Comput. Syst. 7 (1973), 138-153.

[2] F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory Dynam. Systems 19 (1999), no. 4, 953–993. MR 1709427

[3] A. Ehrenfeucht and K. P. Lee and G. Rozenberg, *Subword complexities of various classes of deterministic developmental languages without interactions*, Theoret. Comput. Sci. 1 (1975), no. 1, 59-75.

[4] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002, Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel. MR 1970385
[5] Tero Harju and Matti Linna, *On the periodicity of morphisms on free monoids*, Theor. Inform. Appl. **20** (1986), no. 1, 47-54.

[6] Darald J. Hartfiel, *Nonhomogeneous Matrix Products*, World Scientific, 2002.

[7] Richard H. Herman, Ian F. Putnam and Christian F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. **3** (1992), 827-864.

[8] B. Host, *Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable*, Ergodic Theory Dynam. Systems **6** (1986), no. 4, 529-540. MR 873430

[9] Petr, Kůrka, *Topological and symbolic dynamics*, Cours Spécialisés [Specialized Courses], vol. 11, Société Mathématique de France, Paris, 2003. MR 2041676

[10] Douglas Lind and Brian Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995. MR 1369092

[11] Brigitte Mossé, *Puissances de mots et reconnaissabilité des points fixes d’une substitution*, Theoret. Comput. Sci. **99** (1992), no. 2, 327–334. MR 1168468

[12] Brigitte Mossé, *Reconnaissabilité des substitutions et complexité des suites automatiques*, Bull. Soc. Math. France **124** (1996), no. 2, 329–346. MR 1414542

[13] Jean-Jacques Pansiot, *Complexité des facteurs des mots infinis engendré par morphismes itérés*, Automata, languages and programming (Antwerp, 1984), Lecture Notes in Comput. Sci., vol. 172, Springer, Berlin, 1984, pp. 380-389.

[14] Jean-Jacques Pansiot, *Decidability of periodicity for infinite words*, Theor. Inform. Appl. **20** (1986), no. 1, 43-46.

[15] Martine Queffélec, *Substitution dynamical systems—spectral analysis*, Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 1987. MR 924156

[16] Martine Queffélec, *Substitution dynamical systems—spectral analysis*, second ed., Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 2010. MR 2590264

[17] Grzegorz Rozenberg, *DOL Sequences*, Discrete Math. **7** (1974), no. 3-4, 323-347.

[18] E. Seneta, *Non-negative Matrices and Markov Chains*, Springer Science & Business Media, 2006.

[19] B. Solomyak, *Nonperiodicity implies unique composition for self-similar translationally finite tilings*, Discrete Comput. Geom. **20** (1998), no. 2, 265-279.

[20] Hisatoshi Yuasa, *Invariant measures for the subshifts arising from non-primitive substitutions*, J. Anal. Math., **102** (2007), 143–180. MR 2346556

Institute of Mathematics, University of Tsukuba, Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8571 JAPAN.

E-mail address: akiyama@math.tsukuba.ac.jp

School of Mathematics and Statistics, Huazhong University of Science & Technology, Wuhan 430074, P.R.CHINA.

E-mail address: tanbo@hust.edu.cn

Division of Science, Mathematics and Information, Osaka Kyoiku University, 4-698-1 Asahigaoka, Kashiwara, Osaka 582-8582, JAPAN.

E-mail address: hyuasa@cc.osaka-kyoiku.ac.jp