Longitudinal Network Models and Permutation-Uniform Markov Chains*

William K. Schwartz  
Sonja Petrović  
Hemanshu Kaul

Secretariat Economists LLC  
Illinois Institute of Technology  
Illinois Institute of Technology

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Abstract

Consider longitudinal networks whose edges turn on and off according to a discrete-time Markov chain with exponential-family transition probabilities. We characterize when their joint distributions are also exponential families with the same parameter, improving data reduction. Further we show that the permutation-uniform subclass of these chains permit interpretation as an independent, identically distributed sequence on the same state space. We then apply these ideas to temporal exponential random graph models, for which permutation uniformity is well suited, and discuss mean-parameter convergence, dyadic independence, and exchangeability. Our framework facilitates our introducing a new network model; simplifies analysis of some network and autoregressive models from the literature, including by permitting closed-form expressions for maximum likelihood estimates for some models; and facilitates applying standard tools to longitudinal-network Markov chains from either asymptotics or single-observation exponential random graph models.

Keywords: conditional exponential families; compression; data reduction; dyadic independence; ergms; exponential families; longitudinal networks; Markov chains; permutation uniformity; temporal exponential random graph models.

1 Introduction

Over the last half century, the statistical modeling of networks has ramified. One branch of studies has included exponential random graph models (ergms), which have proved successful for modeling single observations of random networks. Their probabilities are exponential families, parameterized functions of certain observables, called sufficient statistics, such as the number of edges, the degree sequence, or the counts of specified subgraphs. Another branch has modeled longitudinal or dynamic networks, in which edges blossom and die but nodes stay the same,1 as discrete-time Markov chains2 where each row of the matrix of transition probabilities has an exponential-family

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1See, e.g., Holland and Leinhardt 1977; Frank 1991; Snijders 2005.

2First discussed in Katz and Proctor 1959.
representation with the same parameter, as in Robins and Pattison (2001). Hanneke, Fu, and Xing (2010) explored this family of models, calling them temporal ERGMs (TERGMs), and focused on the dyadic independence case.

The present study asks, When are these types of discrete-time Markov chains of networks exponential families with the same parameter as their transition matrices? Our motivation is to re-cast TERGMs as ERGMs since the latter are generative statistical models, allowing scientists to estimate parameters that weight the importance of different sufficient statistics in explaining why a network has a certain topology. Moreover, subsection 1.2 discusses the extended scope for data compression that arises when a TERGM’s joint distribution has an exponential family representation. Finally, statisticians have developed sophisticated techniques for estimating these parameters and testing goodness of fit for ERGMs.  

The answer is, perhaps surprisingly, not “always”. Our theorem 1.2 on page 5 implies that TERGMs have an exponential family joint distribution with the same parameter as their transition matrices if and only if the normalizing term in their transition matrices is the same across all rows. A sufficient condition for this tractable class of TERGMs arises in the case that the Markov chain is also permutation-uniform (p-uniform), meaning that every row of the transition matrix is a permutation of every other row.

The key insight is that when a Markov chain is p-uniform, composing those permutations with the Markov chain itself produces an independent and identically distributed (iid) sequence on the same state space. Readers familiar with Diaconis and Freedman (1999) may recognize p-uniform Markov chains as being induced by sets of permutations on the state space itself. When coupled with existing exponential family models of networks, the iid sequence’s finite-sample joint distribution maintains the exponential family representation with a low-dimensional sufficient statistic interpretable in terms of the underlying Markov chain. This is similar to the autoregressive model of networks in Hoff (2015); see examples 2.3 and 2.4. Moreover, the iid sequence can be viewed as a single observation of a multigraph drawn from an ERGM whose parameter estimates and goodness-of-fit tests apply to the original Markov chain.

In section 2, the main theorem, theorem 2.2, establishes the key identification of p-uniform Markov chains with an iid sequence on the same state space. This identification plays nicely with exponential family representations of the transition matrix and the interpretability of its sufficient statistic. In this way, we can translate certain Markov chains to iid sequences, perform statistical analysis on the latter, and draw conclusions about the former. Applications of p-uniform Markov chains include autoregressive processes on discrete state spaces and several of the examples in subsection 3.3. Shalizi and Rinaldo (2013, § 6) called for ways of analyzing independent random variables in place of dependent random variables in Markov chains of networks. The novelty of the techniques in this paper are disposing of the temporal dependence in a Markov chain, and maintaining interpretability of parameters and sufficient statistics while doing so. For certain Markov chains (see subsection 3.3), dispensing with temporal dependence allows us to compute maximum-likelihood estimators (MLEs) from closed-form expressions where previously the literature contemplated only Markov-chain Monte Carlo (MCMC) or Newton’s method algorithms.

In section 3, we apply the theories of exponential families of transition matrices and of p-uniform Markov chains to network models in the context of the statistical independence of the random connections in the networks. The main result in theorems 3.8 and 3.9 is that we may replace $t$ observations of certain p-uniform Markov chains of graphs with a single observation of

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3See, e.g., Goldenberg et al. 2010; Kolaczyk 2017; Kolaczyk and Csárdi 2020, pp. 88–97.
a corresponding multigraph. We introduce *exponential random t-multigraph models (t-ERGMs)* for this purpose. We expect much of existing ERGM theory to apply without significant modification to t-ERGMs: Fienberg and Rinaldo (2012) thoroughly described parameter estimation, and Gross, Karwa, and Petrović (2022) surveyed the literature for goodness-of-fit testing. Subsections 3.3 and 3.4 illustrate our results with existing Markov-chain network models as well as a new model. The former subsection examines models from Hanneke, Fu, and Xing (2010): their density and stability models are p-uniform, but their reciprocity and transitivity models are not. The latter subsection offers a novel network model whose parameters, one per actor in the network, may be interpreted to measure loyalty in terms of the tendency of each actor to change ties from one time to the next.

Finally, section 4 concludes with a summary of how all our results fit together to constitute an analytical framework for certain kinds of ERGMs.

After this introduction and some notation in subsection 1.1, subsection 1.2 identifies which Markov chains have a joint distribution with a low-dimensional sufficient statistic. This is particularly important when the state space is as large as the set of all \( O(2^n) \) networks on \( n \) vertices. The answer lies in exponential families: theorem 1.2 is a Darmois-Koopman-Pitman-type theorem for discrete-time Markov chains on discrete state spaces.

### 1.1 Notation and Terminology

First let us dispense with some bookkeeping notation. Denote the set \( \{0, 1, 2, \ldots \} \) of natural numbers by \( \mathbb{N} \) and its positive subset by \( \mathbb{N}_{>0} \). \( 1 \) is the vector of all ones of whatever dimension is appropriate. If \( n \in \mathbb{N} \), then \( [n] := \{1, \ldots, n\} \).

Now we turn to statistics terminology. Let \( \mathcal{S} \) be a discrete (i.e., at most countable) state space. Where needed, we endow discrete spaces with the discrete (power set) \( \sigma \)-algebra, allowing us to brush aside abstruse considerations of measurability. Let \( X := \{X_t\}_{t \in \mathbb{N}} \) be an \( \mathcal{S} \)-valued stochastic process, and let \( \Theta \subseteq \mathbb{R}^d \), for some \( d \geq 1 \), be a non-empty parameter space. We assume the existence of a set of probability measures \( \mathbb{P}_\theta \), for each \( \theta \in \Theta \), under each of which \( X \) is a time-homogeneous Markov chain with transition matrix \( P_\theta \in \mathbb{R}^{d \times d} \). Denote \( \mathcal{P} := \{P_\theta\}_{\theta \in \Theta} \). When we say Markov chain, we always assume temporal homogeneity. We typically write an entry of \( P_\theta \) as \( P_\theta(a, b) \) for \( a, b \in \mathcal{S} \).

For all \( \theta \in \Theta \), \( t \in \mathbb{N}_{>0} \), and \( x_0 \in \mathcal{S} \), define \( \mathbb{P}_\theta^t(\cdot \mid x_0) \) to be the *restriction* of \( \mathbb{P}_\theta(\cdot \mid x_0 = x_0) \) to the \( \sigma \)-algebra generated by \( X_0, \ldots, X_t \). Further, define \( L_{\theta, x_0}^t \) to be the probability mass function (pmf) of the law of \( X_1, \ldots, X_t \) under \( \mathbb{P}_\theta^t(\cdot \mid X_0 = x_0) \).

A parameterized set \( \mathcal{M} := \{\mu_\theta\}_{\theta \in \Theta} \) of pmfs on \( \mathcal{S} \) is an *exponential family* if there exist \( \ell \in \mathbb{N}_{>0} \) and functions \( \zeta : \Theta \to \mathbb{R} \), \( \kappa : \mathcal{S} \to [0, \infty) \), \( \eta : \Theta \to \mathbb{R}^\ell \), and \( \tau : \mathcal{S} \to \mathbb{R}^\ell \) such that, for all parameters \( \theta \in \Theta \), the probability mass of any state \( a \in \mathcal{S} \) under \( \mu_\theta \) is

\[
\mu_\theta(a) = \kappa(a) \exp(\eta(\theta) \cdot \tau(a) - \zeta(\theta)).
\]

Equation (1) is an *exponential family representation* of \( \mathcal{M} \), \( \zeta \) the *log-partition function* and thus \( e^{-\zeta} \) the *partition function*, \( \kappa \) the *carrier measure*, \( \eta \) the *parameter function*, and \( \tau \) a sufficient statistic for \( \theta \).

\( \mathcal{M} \)'s *natural parameter space* is \( \{\gamma \in \mathbb{R}^\ell \mid \sum_{b \in \mathcal{S}} \kappa(b) e^{\gamma \cdot \tau(b)} < \infty\} \). We always stipulate that \( \eta(\Theta) \) lies in the natural parameter space;\(^4\) hence \( |\zeta(\theta)| < \infty \) for all \( \theta \in \Theta \).

\(^4\)Wainwright and Jordan 2008, Eq. 3.6.
\(^5\)Casella and Berger 2002, p. 114.
The parameterized set \( \mathcal{P} = \{P_\theta\}_{\theta \in \Theta} \) of transition matrices is a **conditional exponential family** (CEF) if each row \( \{P_\theta(a, \cdot)\}_{\theta \in \Theta} \), where \( a \in \mathcal{S} \), has an exponential family representation with the same parameter function \( \eta: \Theta \to \mathbb{R}^\ell \). Specifically, \( \mathcal{P} \) is a CEF if there exist functions \( \zeta: \mathcal{S} \times \Theta \to \mathbb{R} \), \( \kappa: \mathcal{S} \times \Theta \to [0, \infty) \), \( \tau: \mathcal{S} \times \mathcal{S} \to \mathbb{R}^\ell \) such that

\[
P_\theta(a, b) = \kappa(a, b) \exp(\eta(\theta) \cdot \tau(a, b) - \zeta(a, \theta))
\]

for all \( a, b \in \mathcal{S} \) and all \( \theta \in \Theta \). We call equation (2) a **CEF representation** of \( \mathcal{P} \). Temporal ERGMs are Markov chains on the set of a networks with a given number of nodes whose transition probabilities have a CEF representation.

\( \mathcal{P} \)'s **natural parameter space** is \( \{\gamma \in \mathbb{R}^\ell \mid \sum_{b \in \mathcal{S}} \kappa(a, b) \exp(\gamma \cdot \tau(a, b)) < \infty \text{ for all } a \in \mathcal{S}\} \). We always stipulate that \( \eta(\Theta) \) lies in the natural parameter space;\(^7\) hence \( |\zeta(a, \theta)| < \infty \) for all \( \theta \in \Theta \) and all \( a \in \mathcal{S} \).

The CEF is a **conditionally additive exponential family** (CAEF) if \( \zeta(a, \theta) = \psi(a)\phi(\theta) \) for some functions \( \phi: \Theta \to \mathbb{R} \) and \( \psi: \mathcal{S} \to \mathbb{R} \).\(^8\) Finally, the CAEF is a **Markovian exponential family** (MEF) if \( \psi(a) \) is a non-zero constant for all \( a \in \mathcal{S} \), i.e., \( \zeta(a, \theta) = \zeta(b, \theta) \Rightarrow \zeta(\theta) \) so that,\(^9\) for all \( a, b \in \mathcal{S} \) and all \( \theta \in \Theta \),

\[
P_\theta(a, b) = \kappa(a, b) \exp(\eta(\theta) \cdot \tau(a, b) - \zeta(\theta))
\]

We call equation (3) an **MEF representation** of \( \mathcal{P} \). Equations (2) and (3) defining CEFs and MEFs differ only in that the log-partition function for the CEF in equation (2) depends on which row of the transition matrix it is in, whereas the log-partition function for the MEF in equation (3) is the same in every row of the transition matrix.

**Example 1.1** (Gani 1955, Example iv.3-3, pp. 358–359). The following is an example of an MEF with a scalar parameter. Set \( \mathcal{S} := \{1, 2, 3\} \), \( \Theta := (0, \infty) \), \( \eta(\theta) := \log(\theta) \),

\[
\tau := \begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 3 \end{bmatrix}, \quad \kappa := \begin{bmatrix} 2 & 1 & 1 \\ \frac{2}{3} & \frac{3}{2} & 3 \\ \frac{1}{3} & 1 & \frac{1}{4} \end{bmatrix}, \quad \text{so} \quad P_\theta = \frac{1}{3\theta + 3^{3}} \begin{bmatrix} 2\theta & \theta & \theta^3 \\ \frac{2}{3}\theta^3 & \frac{3}{2}\theta^3 & 3\theta \\ \frac{1}{3}\theta & \theta^3 & \frac{1}{4}\theta^3 \end{bmatrix} \]  \( \Box \)

Heyde and Feigin (1975) first introduced CAEFs, which Feigin (1981) generalized to CEFs. Hudson (1982) coined the name for MEFs. None were aware of the analysis in Gani (1955) decades earlier of what turned out to be MEFs.

### 1.2 Markov Chains with Low-Dimension Sufficient Statistics

**Darmois-Koopman-Pitman (DKP)** theorems assert the uniqueness of exponential families as the only parameterized families of probability distributions whose sufficient statistics do not grow in dimension with the size of the sample (under some conditions). Continuous state spaces have been the usual focus of textbook renditions, e.g., Lehmann and Casella (1998, Thm. 6.18, p. 40). Discrete analogues in the research literature have included Gani (1955, § iv), Andersen (1970), Denny (1972), and Diaconis and Freedman (1984). The former gave three such results for scalar parameters, the Markov-chain version of which we discuss below because it motivates this subsection’s main result, theorem 1.2.

\(^6\)Feigin 1981, Def. A.

\(^7\)Ibid., p. 598.

\(^8\)Ibid., Def. B.

\(^9\)Hudson 1982, Eq. 2.1.
That theorem says roughly that a Markov chain’s parameterized family of finite-sample joint pmfs have an exponential family representation if and only if the corresponding set of transition matrices have an mef representation. As a consequence, \( \mathcal{P} \) is a cef that gives rise to an exponential family of joint pmfs if and only if \( \mathcal{P} \) is an mef. In that case, the joint distribution and transition matrix share the same parameter and the sufficient statistics \( \tau(X_i, X_{i+1}) \) for the transitions sum to form the sufficient statistic \( \sum_{i=0}^{t-1} \tau(X_i, X_{i+1}) \) for the joint distribution. The dimension of both statistics is \( t \), which could be much smaller than the dimension of a sufficient statistic for the joint distribution’s parameter in the general case.

In the general case—even when \( \mathcal{P} \) is not a cef—Markov chains’ joint distributions form exponential families, but with a high-dimensional sufficient statistic. A statistic of \( (X_1, \ldots, X_t) \) that is sufficient for the entries of \( P_\theta \) is the transition-count matrix \( N_t \in \mathbb{N}^{\delta \times \delta} \) whose \( a, b \) entry,

\[
N_t(a, b) := \sum_{i=0}^{t-1} 1(X_i = a) 1(X_{i+1} = b),
\]

where the random variable \( 1(X_i = a) \) is one when \( X_i = a \) and zero otherwise, counts the number of times \( X \) transitioned from state \( a \) to state \( b \) by time \( t \).\(^{10}\) The \( a \)th entry of \( N_t \) gives the number of times that \( X \) has visited state \( a \) during times \( 0, \ldots, t-1 \), inclusive. Even if \( \delta \) is finite, the number of degrees of freedom of the minimal representation of \( N_t \) is \( |\delta|^2 - |\delta| \).\(^{11}\) That is \( O(2^{n^2}) \) if \( \delta \) is the set of graphs on the vertex set \([n]\). \( N_t \) is sufficient for the entries of \( P_\theta \) because the joint pmf of \( (X_1, \ldots, X_t) \) is

\[
L^t_{\theta, X_t}(X_1, \ldots, X_t) = \exp\left( \sum_{a, b \in \delta} N_t(a, b) \log P_\theta(a, b) \right)
\]

(as long as we take 0 log 0 := 0).\(^{12}\)

Discrete-dkp-type results offer a way to fend off \( N_t \)'s dimensional onslaught. In other words, mefs afford greater scope for data compression than cefs. Among the earliest such results in the literature were Gani (1955, § iv.3) and Gani (1956). In the former, Gani assumed that \( \delta \) is finite, \( \Theta \) is a set of scalars in \( \mathbb{R} \), and that every transition matrix \( P_\theta \in \mathcal{P} \) has a stationary distribution\(^{13}\) and is differentiable with respect to \( \theta \).\(^{14}\) He concluded (in slightly different language) that \( X \)'s finite-sample joint pmf admits a one-dimensional statistic sufficient for \( \theta \) if and only if \( \mathcal{P} \) is an mef.

Theorem 1.2 drops Gani’s ergodicity, scalar parameter, and other conditions in the sense that it says that mefs and only mefs make any (discrete space and time homogeneous) Markov chain’s finite-sample joint pmfs an exponential family. Hence the theorem gives a sufficient condition—but no necessary conditions—for the chain’s joint pmf to admit an \( \ell \)-dimensional statistic sufficient for \( \theta \).

The theorem, whose measure theoretic proof we omit, is a consequence of Küchler and Sørensen (1997, Cor. 6.3.4). The analogy between our notation and theirs\(^{15}\) is easiest to see by reading our \( P_\theta(\cdot \mid X_0 = x_0) \) as their probability measure \( Q_{\theta, x_0} \), for which \( Q_{\theta, x_0}(X_0 = x_0) = 1 \) and \( P_\theta(x_0, x_1) = Q_{\theta, x_0}(X_1 = x_1) \).

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\(^{10}\)Cf. Gani 1955, Eqs. 8–9; Stefanov 1995, Eq. 7.

\(^{11}\)Stefanov 1991, § 2.

\(^{12}\)Küchler and Sørensen 1997, Eq. 3.2.5.

\(^{13}\)Schwartz 2021, fn. 9 on p. 15.

\(^{14}\)Though differentiability is inessential—see ibid., fn. 8 on p. 14.

\(^{15}\)Küchler and Sørensen 1997, pp. 19, 65–68.
Theorem 1.2. \( \mathcal{P} \) is the MEF from equation (3) with \( \kappa(a, b) = P_{\theta_0}(a, b) \) for some \( \theta_0 \in \Theta \) and all \( a, b \in \mathcal{S} \) if and only if \( \kappa(a, b) = L_{\theta_0, a}^1(b) \) for some \( \theta \in \Theta \) and all \( a, b \in \mathcal{S} \), and \( \{L_{\theta, x_0}^1\}_{\theta \in \Theta} \) is the exponential family on \( \mathcal{S}^t \) such that

\[
L_{\theta, x_0}^1(x_1, \ldots, x_t) = \exp \left( \eta(\theta) \cdot \sum_{i=0}^{t-1} \tau(x_i, x_{i+1}) - t \zeta(\theta) \right) \prod_{i=0}^{t-1} \kappa(x_i, x_{i+1})
\]

for all \( x_0, x_1, \ldots, x_t \in \mathcal{S}, t \in \mathbb{N}_{>0}, \) and \( \theta \in \Theta \).

While theorem 1.2 gives a general way to recognize when families of transition matrices are MEFs, the next proposition, due to Gani, helps us in subsection 3.3 to recognize when (scalar parameterized) CEFs are not MEFs. It says the rows of a scalar sufficient statistic in an MEF all contain the same set of numbers. For example 1.1 on page 4, that set is \( \{1, 3\} \). We include a sketch of Gani’s proof because the main idea is simple and instructive.\(^{16} \) The proposition foreshadows section 2 in which we study a stronger condition that requires every row of a transition matrix to be a permutation of every other row.

Proposition 1.3 (Gani 1955, pp. 357–358). For the MEF in equation (3), suppose \( \ell = 1 \) (i.e., \( \eta(\theta) \) is a scalar), \( \eta(\Theta) \) contains a number other than zero, and \( a, c \in \mathcal{S} \). Then \( \{\tau(a, b) \mid b \in \mathcal{S}\} = \{\tau(c, b) \mid b \in \mathcal{S}\} \).

Proof sketch. Since \( \sum_{b \in \mathcal{S}} P_{\theta}(a, b) = 1 \), we have, for all \( \theta \in \Theta \),

\[
\exp(\zeta(\theta)) = \sum_{b \in \mathcal{S}} \kappa(a, b) \exp(\eta(\theta) \tau(a, b)) = \sum_{b \in \mathcal{S}} \kappa(c, b) \exp(\eta(\theta) \tau(c, b)).
\]

Exponential functions \( x \mapsto e^{rx} \) are linearly independent for different scalars \( r \),\(^{17} \) so we can set some of the coefficients and exponents equal when \( \eta(\theta) \neq 0 \).

We conclude this subsection with the following statistical theorem for MEFs. It disposes of temporal dependence when we just need the mean parameter \( \mathbb{E}[\tau(X_t, X_{t+1})] \), which is crucial for MLE.\(^{18} \)

Theorem 1.4. Let \( \mathcal{P} \) be the MEF from equation (3). Let \( \theta \in \Theta \) such that \( \eta(\theta) \) is in the interior of the natural parameter space and \( \eta \) is differentiable at \( \theta \) with Jacobian matrix \( J_\theta \). Then, for any \( t \in \mathbb{N} \),

\[
\mathbb{E}_\theta[\tau(X_t, X_{t+1})] = J_\theta^{-1} \nabla \zeta(\theta).
\]

A counterpart to this result is theorem 2.5 on page 12.

Proof. By Feigin (1981, Thm. 1(i)), if \( \eta \) is the identity, and \( X \)’s transition matrix is more generally the CEF given in equation (2), then \( \mathbb{E}_\theta[\tau(X_t, X_{t+1}) \mid X_0, \ldots, X_t] = \nabla \zeta(X_t, \theta) \) for any \( t \in \mathbb{N} \). If \( \eta \) is not the identity function we apply the chain rule, replacing \( \nabla \zeta(X_t, \theta) \) with \( J_\theta^{-1} \nabla \zeta(X_t, \theta) \), as in Lehmann and Casella (1998, Problem 5.6(b), p. 66). However, because \( X \)’s transition matrix is the MEF equation (3), \( \zeta \) is constant with respect to \( X_t \), so we must replace \( J_\theta^{-1} \nabla \zeta(X_t, \theta) \) with \( J_\theta^{-1} \nabla \zeta(\theta) \). Taking expectations on both sides we get, for any \( t \in \mathbb{N} \),

\[
J_\theta^{-1} \nabla \zeta(\theta) = \mathbb{E}_\theta \left( \mathbb{E}_\theta[\tau(X_t, X_{t+1}) \mid X_0, \ldots, X_t] \right) = \mathbb{E}_\theta[\tau(X_t, X_{t+1})]. \tag*{\( \square \)}
\]

\(^{16}\)Similar uses of the idea appeared in Bhat and Gani 1960, p. 454; and in Bhat 1988, Example 2.

\(^{17}\)Axler 2015.

\(^{18}\)Fienberg and Rinaldo 2012; Lehmann and Casella 1998, Thm. 6.2, pp. 125–126.
2 Permutation-Uniform Markov Chains

In this section, we show that if every row of a transition matrix is a permutation of the other rows, then we may identify the corresponding Markov chain with an iid sequence on the same state space. This identification in theorem 2.2, the main theorem of the section, preserves the exponential family representation of the transition matrix and the interpretability of the sufficient statistic when it is transformed into the joint distribution for the iid sequence. Autoregressive processes on discrete state spaces and several of the examples in subsection 3.3 provide applications for this technique.

The main concept in this section is permutation uniformity, the property that “every row is a permutation of every other row.” For example, \([1/θ \ 1-θ]\) and \([1 \ 0]\) are permutation uniform but \([1/2 \ 1/2]\) is not. We make this rough definition precise as follows. Let \(\mathcal{S}\) be any set. For each \(a \in \mathcal{S}\), define the \(a\)th row of a function or matrix \(f : \mathcal{S} \times \mathcal{S} \to \mathcal{S}\) to be the function or row vector \(b \mapsto f(a, b)\). We write permutations on \(\mathcal{S}\) (i.e., bijections \(\mathcal{S} \to \mathcal{S}\)) juxtaposed with other permutations on \(\mathcal{S}\) to denote composition and juxtaposed with elements of \(\mathcal{S}\) to denote application.

**Definition 2.1.** Let \(\pi := \{π_α\}_{α ∈ \mathcal{S}}\) be a set of \(|\mathcal{S}|\) permutations on \(\mathcal{S}\) out of the \(|\mathcal{S}|!\) possible. We say that \(f\) is permutation uniform, or \(p\)-uniform, under \(\pi\) if \(f(a, π_α^{-1}c) = f(b, π_β^{-1}c)\) for all \(a, b, c ∈ \mathcal{S}\). We drop the “under \(\pi\)” part to assert the existence of some permutations under which \(f\) is \(p\)-uniform.

A Markov chain is a permutation-uniform Markov chain, or \(p\)-uniform chain, if its transition matrix is \(p\)-uniform. We derived our definition from Rosenblatt (1959, § 3); a reconciliation of that one with Yano and Yasutomi (2011, Def. 1.4) appears in appendix A.

While our focus is on the statistics of \(p\)-uniform chains, previous investigations of them have largely occurred in the language of dynamic systems. One way this bears out is that some authors have used different terminology for \(p\)-uniform chains, such as \(p\)-uniform stochastic graphs in Rubshtein (2004). Another way is that results on \(p\)-uniformity have focused on Markov chains indexed by \(\mathbb{Z}\) rather than \(\mathbb{N}\), meaning that the chains have no initial states.

The goal of several of these studies—and of the next subsection—has been to connect a \(p\)-uniform chain \(X\) with some iid sequence \(Z\). Rosenblatt (1959, Lem. 3) showed on a countable state space that when \(X\)’s transition matrix \(P\) has distinct entries in a row, \(X_t\) and \(X_{t+1}\) almost surely uniquely determine the iid \(Z_t\)s, each of which are independent of \(X_{t-1}\), \(X_{t-2}\), . . . . Rosenblatt (1960) constructed \(Z_t\) iid uniformly on [0, 1] such that \(X_t\) is a function of all the preceding \(Z_t\)s. Hanson (1963) extended this result to continuous state spaces. The proof of Rosenblatt (1959, Lem. 3) spells out how to construct the \(Z_t\)s, but the distinctness constraint on the entries of \(P\) limits the applicability of the lemma. Rosenblatt (1960) and Hanson (1963) relied on the Markov chain’s having no beginning to prove the existence of \(Z\) without constructing it (e.g., the former used the Borel-Cantelli lemma). In all three models, the \(Z_t\)s take values in a different space from the \(X_t\)s, or at least are not surjective onto the state space of the \(X_t\)s.

2.1 Independence

In this subsection, we derive iid \(Z_t\)s uniquely determined by a \(p\)-uniform Markov chain \(X\) with transition matrix \(P\). In contrast to prior studies of \(p\)-uniform chains, the entries of a row of \(P\) need not be distinct, the \(Z_t\)s take on exactly the same values as the \(X_t\)s, and \(X\) does not go infinitely into the past, but rather starts at time zero with \(X_0, X_1, \ldots\).

The \(Z_t\)s’ distribution can stand in for the \(X_t\)s’, thus making statistical inference on the time-dependent \(X_t\)s easier by using the independent \(Z_t\)s. This makes the Markov chain \(X\) rather like a
random walk. We explore this connection more after the main theorem.

X’s being p-uniform imposes enough structure for us to observe the desired iid sequence Z from X, which we mean in the following sense. We say that the \( \mathcal{S} \)-valued sequence of random variables \( Z_1, \ldots, Z_t, t \in \mathbb{N}_{>0}, \) has a joint distribution similar under \( \mathbb{P} \) to that of the \( \mathcal{S} \)-valued sequence of random variables \( X_0, X_1, \ldots, X_t \)'s for some probability measure \( \mathbb{P} \) if, for all \( x_0, x_1, \ldots, x_t \in \mathcal{S} \), there exist \( z_1, \ldots, z_t \in \mathcal{S} \) such that

\[
\mathbb{P}(X_t = x_t, \ldots, X_1 = x_1 \mid X_0 = x_0) = \mathbb{P}(Z_t = z_t, \ldots, Z_1 = z_1).
\]

For \( \{Z_t\}_{t=1}^{\infty} \) to be iid with a common law (whose pmf under \( \mathbb{P} \) is) \( \mu \) means that \( \mathbb{P}(Z_t = z) = \mu(z) \) for all \( t \in \mathbb{N}_{>0} \) and all \( z \in \mathcal{S} \).

Recall from definition 2.1 that \( \pi = \{\pi_a\}_{a \in \mathcal{S}} \) is a set of permutations on \( \mathcal{S} \).

**Theorem 2.2.** X is a Markov chain on \( \mathcal{S} \) under probability measure \( \mathbb{P} \) with transition matrix \( P \) p-uniform under \( \pi \) if and only if there exist a pmf \( \mu \) on \( \mathcal{S} \), a sequence of \( \mathcal{S} \)-valued random variables \( Z := \{Z_t\}_{t=1}^{\infty} \) and an \( \mathcal{S} \)-valued random variable \( X_0 \) such that

\( \mathcal{S} \)

(a) \( Z \) is iid with common law \( \mu; \)

(b) \( X_{t+1} = \pi_{X_t}^{-1} Z_{t+1} \mathbb{P} \)-almost surely for all \( t \in \mathbb{N} \);

(c) \( P(a, b) = \mu(\pi_a b) \) for all \( a, b \in \mathcal{S} \); and

(d) \( X_0, Z_1, \ldots, Z_t \) are mutually independent for all \( t \in \mathbb{N}_{>0} \).

When such a \( Z \) exists, \( X \) and \( Z \) have similar joint distributions. Further, for all \( t \in \mathbb{N} \), \( Z_{t+1} \) and any random vector of the form \( (X_{i_1}, \ldots, X_{i_n}, X_t) \) for \( i_1, \ldots, i_n \in \{0, 1, \ldots, t-1\} \) and \( n \in \{0, 1, \ldots, t-1\} \) are pairwise independent.

If we are given a p-uniform Markov chain \( X \) with transition matrix \( P \), lemma A.1 already tells us what \( \mu \) has to be: any permutation of any row of \( P \) that is convenient because we can choose any permutations \( \pi \) that make \( P(a, b) = \mu(\pi_a b) \) true. In this sense, once we’re given \( P \), we have no choice over \( \mu \); we can only choose the permutations \( \pi \). The choice of permutations \( \pi \) determines the sequence \( Z \) almost surely from \( X \). Applying theorem 2.2 thus comes down to choosing \( \pi \)s. Conversely if we are given \( Z \) with the common law \( \mu \), different choices of \( \pi \)s give rise to different Markov chains \( X \). Example 3.10 on page 19 illustrates this phenomenon.

Perhaps surprisingly, theorem 2.2 does not care about \( X_0 \)'s distribution. However, \( X_0 \)'s only role in the theorem is to condition other distributions.

**Proof of theorem 2.2. Forward Implication.** Suppose \( X \) is a Markov chain on \( \mathcal{S} \) under probability measure \( \mathbb{P} \) with transition matrix \( P \) p-uniform under \( \pi \).

Lemma A.1 provides for the existence of a pmf \( \mu \) on \( \mathcal{S} \) such that \( P(a, b) = \mu(\pi_a b) \), condition (c). \( X_0 \) is an \( \mathcal{S} \)-valued random variable because \( X \) is a Markov chain.

We can define the \( \mathcal{S} \)-valued random variables \( Z = \{Z_t\}_{t \in \mathbb{N}_{>0}} \) by

\[ Z_{t+1} := \pi_{X_t} X_{t+1} \quad \text{for all } t \in \mathbb{N}. \]

This equality holds everywhere in the sample space and \( \pi_a \) is invertible for all \( a \in \mathcal{S} \), so \( X_{t+1} = \pi_{X_t}^{-1} Z_{t+1} \) holds \( \mathbb{P} \)-almost surely for all \( t \in \mathbb{N} \), establishing condition (b). In the remainder of this direction of the proof, we will make use of the equivalence between \( Z_t = z \) and \( X_t = \pi_{X_{t-1}}^{-1} z \) when \( t \in \mathbb{N}_{>0} \).
Now we must show that $Z$ is IID with common law $\mu$, which is condition (a). The work to prove it will also prove condition (d). To do so, we fix an arbitrary sequence $\{z_t\}_{t \in \mathbb{N}_0} \subseteq \mathcal{S}$ and time $T \in \mathbb{N}_0$, and we show that

$$\mathbb{P}(Z_1 = z_1, \ldots, Z_T = z_T) = \mu(z_1) \cdots \mu(z_T). \quad (4)$$

For each $t \in [T]$, $Z_t = z_t$ determines a transition. Stringing together all $T$ transitions determines the state of the Markov chain through time $T$ if we know the starting position $X_0$. This suggests starting the proof with the left-hand-side expression in equation (4) and using the law of total probability:

$$\mathbb{P}(Z_1 = z_1, \ldots, Z_T = z_T) = \sum_{a \in \mathcal{S}} \mathbb{P}(Z_1 = z_1, \ldots, Z_T = z_T \mid X_0 = a) \mathbb{P}(X_0 = a). \quad (5)$$

We can formalize what we meant by “stringing together” the transitions as follows. For each $t \in \mathbb{N}$, let $x_t : \mathcal{S} \rightarrow \mathcal{S}$ be defined recursively by

$$x_t(a) := \begin{cases} a & t = 0 \\ \pi_{x_{t-1}(a)}^{-1} z_t & t > 0. \end{cases}$$

For a given $a \in \mathcal{S}$, $\{x_t(a)\}_{t \in \mathbb{N}}$ is a deterministic sequence of elements of $\mathcal{S}$. Let $a \in \mathcal{S}$ such that $\mathbb{P}(X_0 = a) > 0$.

As we now prove by induction, the event $A$ in which $Z_1 = z_1, \ldots, Z_T = z_T$, and $X_0 = a$ is almost surely the same event as the event $B$ in which $X_0 = x_0(a), \ldots, X_T = x_T(a)$. Let $t \in [T]$. First suppose we are in event $A$, so $Z_t = z_t$ and $X_0 = a = x_0(a)$. Under the induction hypothesis that $X_{t-1} = x_{t-1}(a)$, we have $X_t = \pi_{x_{t-1}(a)}^{-1} Z_t = x_t(a)$. Thus we are in event $B$ as well. Second suppose we are in event $B$, so $X_0 = x_0(a) = a$, $X_{t-1} = x_{t-1}(a)$, and $X_t = x_t(a)$. Then $Z_t = \pi_{x_{t-1}(a)}X_t = \pi_{x_{t-1}(a)}x_t(a) = \pi_{x_{t-1}(a)}x_{t-1}(a)z_t = z_t$. Thus we are in event $A$ as well. Consequently, we may expand the left term in the summand of equation (5) using these equivalent events for $a \in \mathcal{S}$:

$$\mathbb{P}(Z_1 = z_1, \ldots, Z_T = z_T \mid X_0 = a) = \mathbb{P}(X_1 = x_1(a), \ldots, X_T = x_T(a) \mid X_0 = a). \quad (6)$$

$X$ is a Markov chain so we can expand the right side of equation (6) as

$$\mathbb{P}(X_1 = x_1(a), \ldots, X_T = x_T(a) \mid X_0 = a) = \prod_{t=1}^{T} \mathbb{P}(x_{t-1}(a), x_t(a)). \quad (7)$$

Applying condition (c), we can then write for any $t \in [T]$

$$\mathbb{P}(x_{t-1}(a), x_t(a)) = \mu(\pi_{x_{t-1}(a)}x_t(a)) = \mu(\pi_{x_{t-1}(a)}\pi_{x_{t-1}(a)}^{-1}z_t) = \mu(z_t). \quad (8)$$

Putting together equations (6) to (8), we get

$$\mathbb{P}(Z_1 = z_1, \ldots, Z_T = z_T \mid X_0 = a) = \prod_{t=1}^{T} \mu(z_t). \quad (9)$$

This is almost what we need. Combining equations (5) and (9) yields our goal from equation (4):

$$\mathbb{P}(Z_1 = z_1, \ldots, Z_T = z_T) = \sum_{a \in \mathcal{S}} \mathbb{P}(Z_1 = z_1, \ldots, Z_T = z_T \mid X_0 = a) \mathbb{P}(X_0 = a)$$

$$= \sum_{a \in \mathcal{S}} \left[ \prod_{t=1}^{T} \mu(z_t) \right] \mathbb{P}(X_0 = a) = \prod_{t=1}^{T} \mu(z_t) \sum_{a \in \mathcal{S}} \mathbb{P}(X_0 = a) = \prod_{t=1}^{T} \mu(z_t).$$
This establishes condition (a). Between equations (4) and (9), we see that $X_0, Z_1, \ldots, Z_T$ are independent, establishing condition (d).

**Backward Implication.** We must show that $X = \{X_t\}_{t \in \mathbb{N}}$, defined in condition (b), is a Markov chain with a transition matrix $P$, defined in condition (c), and that $P$ is $p$-uniform under $\pi$. To do so, we fix an arbitrary time $t \in \mathbb{N}$ and vector $x = (x_0, \ldots, x_{t+1}) \in \delta^{t+2}$ such that $P(X_t = x_t) > 0$, and we show that

\[
P(X_{t+1} = x_{t+1} \mid X_0 = x_0, \ldots, X_t = x_t) = \mu(\pi_x, x_{t+1}),
\]

(10)

\[
P(X_{t+1} = x_{t+1} \mid X_t = x_t) = \mu(\pi_x, x_{t+1}).
\]

(11)

Proving equations (10) and (11) together will show that $X$ has the Markov property, and proving equation (11) will show that $X$ is $p$-uniform by lemma A.1 and equation (26). Since the time $t$ is arbitrary, proving equation (11) will also establish that $X$ is homogeneous, so that $P(a,b) = \mu(\pi_a,b)$ is the transition matrix for $X$. By lemma A.1, this will establish that $P$ is $p$-uniform under $\pi$.

As we now prove by induction, the event $C$ in which $X_0 = x_0, \ldots, X_T = x_T$ (on which the left side of equation (10) is conditioned) is, for all $t \in \mathbb{N}$, almost surely (a.s.) the same event as event $D$ in which $X_0 = x_0$ and $Z_1 = \pi_{x_0} x_1, \ldots, Z_T = \pi_{x_T} x_T$. We prove this for the case when $T > 0$. Let $s \in [T]$. First suppose we are in event $C$, so $X_{s-1} = x_{s-1}, X_s = x_s$, and $X_0 = x_0$. We have $x_s = x_s = \pi_{x_{s-1}}^{-1} Z_s = \pi_{x_{s-1}}^{-1} Z_s$ a.s., so $Z_s = \pi_{x_{s-1}} x_s$ a.s. Thus we are in event $D$ as well. Second suppose we are in event $D$, so $Z_s = \pi_{x_{s-1}} x_s, Z_1 = \pi_{x_0} x_1$, and $X_0 = x_0$. Under the induction hypothesis that $X_{s-1} = x_{s-1}$ a.s., we have $x_s = \pi_{x_{s-1}}^{-1} Z_s = \pi_{x_{s-1}}^{-1} \pi_{x_{s-1}} x_s = x_s$ a.s. Thus we are in event $C$ as well.

We prove equation (10) as follows. The first equality uses $C = D$ a.s. for $T = t$, and the second uses $Z$’s pmf $\mu$ from condition (a) and the independence of $X_0, Z_1, \ldots, Z_{t+1}$ from condition (d).

\[
P(X_{t+1} = x_{t+1} \mid X_0 = x_0, X_1 = x_1, \ldots, X_t = x_t)
= P(Z_{t+1} = \pi_{x_t} x_{t+1} \mid X_0 = x_0, Z_1 = \pi_{x_0} x_1, \ldots, Z_t = \pi_{x_{t-1}} x_t) = \mu(\pi_x, x_{t+1}).
\]

We prove equation (11) as follows. The first and third equalities below apply $C = D$ a.s. for $T = t - 1$, and the second uses conditions (a) and (d).

\[
P(X_{t+1} = x_{t+1}, X_t = x_t \mid X_0 = x_0, X_1 = x_1, \ldots, X_{t-1} = x_{t-1})
= P(Z_{t+1} = \pi_{x_t} x_{t+1}, Z_t = \pi_{x_{t-1}} x_t \mid X_0 = x_0, Z_1 = \pi_{x_0} x_1, \ldots, Z_{t-1} = \pi_{x_{t-2}} x_{t-1})
= \mu(\pi_{x_t} x_{t+1}) P(Z_t = \pi_{x_{t-1}} x_t \mid X_0 = x_0, Z_1 = \pi_{x_0} x_1, \ldots, Z_{t-1} = \pi_{x_{t-2}} x_{t-1})
= \mu(\pi_{x_t} x_{t+1}) P(X_t = x_t \mid X_0 = x_0, X_1 = x_1, \ldots, X_{t-1} = x_{t-1}).
\]

Apply the law of total probability with fixed $x_t$ and $x_{t+1}$ and sum over $x = (x_0, \ldots, x_{t-1})$:

\[
P(X_{t+1} = x_{t+1}, X_t = x_t)
= \sum_{x \in \delta^{t}} P(X_{t+1} = x_{t+1}, X_t = x_t \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}) P(X_0 = x_0, \ldots, X_{t-1} = x_{t-1})
= \mu(\pi_{x_t} x_{t+1}) \sum_{x \in \delta^{t}} P(X_t = x_t \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}) P(X_0 = x_0, \ldots, X_{t-1} = x_{t-1})
= \mu(\pi_{x_t} x_{t+1}) P(X_t = x_t).
\]

Dividing both sides by $P(X_t = x_t)$ yields equation (11), as desired.
**Finishing the Proof.** The fact that \( C = D \) a.s. proves \( X \) and \( Z \) have similar distributions. Finally, to prove that \( Z_{t+1} \) and \( (X_{i_1}, \ldots, X_{i_n}, X_t) \) are independent, let \( z \in \mathcal{S} \) and observe that

\[
\mathbb{P}(Z_{t+1} = z \mid X_t = x_t, X_{i_u} = x_{i_u}, \ldots, X_{i_1} = x_{i_1}) = \mathbb{P}(X_{t+1} = \pi_1^{-1}z \mid X_t = x_t) = \mu(\pi_1, \pi_1^{-1}z) = \mathbb{P}(Z_{t+1} = z). \quad \square
\]

Examples 2.3 and 2.4 below translate into theorem 2.2’s notation some random-walk-like models from the Markov-chain literature that turn out to be p-uniform.

**Example 2.3** (Modular Autoregressive Model (Diaconis and Freedman 1999, Example 6.2, p. 66)). Diaconis and Freedman proposed a model of Markov chains induced by iterating random functions. Details of how theorem 2.2 turns out to be a sub-model of their model are in appendix B, but the following example works equally well in the language of either model.

Let \( \mathcal{S} = \mathbb{Z}/n\mathbb{Z} \) be the set of \( n \) integers modulo \( n \). Fix some initial state \( x_0 \in \mathcal{S} \) and set \( X_0 := x_0 \). Then define \( X_{t+1} := X_t + Z_{t+1} \mod n \), where the \( Z_i \)s are uniform, iid random variables taking values zero or one each with probability a half: \( \mu(0) = \mu(1) = 1/2 \). Then \( Z_{t+1} = X_{t+1} - X_t \mod n \), and, since modular subtraction is bijective, \( X \) is a p-uniform chain with \( \pi_{ij} = j - i \mod n \) for each \( i, j \in \mathcal{S} \). The \( i, j \) entry of the transition matrix is defined by \( \mu(\pi_{ij}) = \mu(j - i \mod n) \), which is a half if \( i - j \mod n \) is either zero or one, and is zero otherwise. \( X \) is irreducible and aperiodic, and it converges to its uniform stationary distribution at an exponential rate. \quad \square

**Example 2.4** (Vector Autoregressive Model (Hoff 2015, p. 1171)). Hoff introduced a multilinear tensor autoregressive model for network data similar to VAR models.\(^{19}\) Let \( X_t \) be the vectorization of a network’s weighted adjacency matrix at time \( t \). The bilinear version of Hoff’s model is

\[
X_t = \theta X_{t-1} + Z_t, \quad \mathbb{E}(Z_t) = 0, \quad \mathbb{E}(Z_t Z_j^\top) = \begin{cases} \Sigma & i = j \\ 0 & i \neq j \end{cases}
\]

where \( \theta \) and \( \Sigma \) are matrices of parameters to be estimated. Hoff used variations of this model on a time series of verbal and material diplomatic actions among 25 countries between 2004 and 2014. Geographically nearby countries’ actions were the best predictors of each country’s actions, with the exceptions of the United Kingdom and Australia and of the United States and certain other countries. The analysis also found that the relations between two countries depended on other countries’ relations.

With a couple simple restrictions, this model becomes a p-uniform Markov chain. First, we must restrict all values to rationals so that the state space for the \( X_t \)s is countable. Second, we must assume that the \( Z_t \)s are iid, which is compatible with the assumptions that \( \mathbb{E}(Z_t) = 0 \) and \( \mathbb{E}(Z_t Z_j^\top) = 0 \) if \( i \neq j \). Then the set \( \pi \) of permutations under which the \( X_t \)s are p-uniform are those for which \( \pi_{X_{t-1}} X_t = X_t - \theta X_{t-1} = Z_t \). \quad \square

### 2.2 Convergence

A statistical consequence of a Markov chain’s p-uniformity is that we can apply iid convergence theorems to the Markov chain. In the case of p-uniform Markov chains that are MEFs, theorem 2.5 strengthens results in the literature for CEFs where the function \( \tau \) in the theorem is a sufficient statistic. Stefanov (1995, Prop. 1.1) proved convergence in probability of the time-average of an MEF’s

\[^{19}\]Cf. Wooldridge 2012, p. 657.
Theorem 2.5. Suppose $X$ is a Markov chain on $\mathcal{S}$ under $\mathbb{P}$ with transition matrix $P$ $p$-uniform under $\pi$. Let $\tau: \mathcal{S} \times \mathcal{S} \to \mathbb{R}^l$ be $p$-uniform under $\pi$ such that $\sum_{b \in \mathcal{S}} \tau(a, b)P(a, b)$ converges absolutely for all $a \in \mathcal{S}$. Then, for any $a \in \mathcal{S}$ and any $s \in \mathbb{N}$,

$$\frac{1}{t} \sum_{i=0}^{t-1} \tau(X_i, X_{i+1}) \to \mathbb{E}_\pi[\tau(X_s, X_{s+1})] = \sum_{b \in \mathcal{S}} \tau(a, b)P(a, b) \quad \text{as } t \to \infty,$$

where convergence is both almost sure and in $L^1$ under $\mathbb{P}$. In particular, the limit does not depend on $s$, the expectation does not depend on the initial distribution of $X_0$, and the equation does not depend on $a$.

Proof. By theorem 2.2, $Z_i := \pi X_i, X_i, i \in \mathbb{N}_{>0}$, is a sequence of IID, $\mathcal{S}$-valued random variables with some common law $\mu$. Since $\tau$ is $p$-uniform under $\pi$, lemma A.1 allows us to define $m: \mathcal{S} \to \mathbb{R}^l$ by

$$m(b) := \tau(a, \pi_a^{-1}b)$$

for all $a, b \in \mathcal{S}$.

We claim that $m(Z_1)$ has finite expectation. Since $\sum_{b \in \mathcal{S}} \tau(a, b)P(a, b)$ converges absolutely for all $a \in \mathcal{S}$, the Riemann rearrangement theorem says that every rearrangement converges absolutely to the same value.\(^{20}\) For each $a \in \mathcal{S}$, $\pi_a^{-1}$ is bijective, so

$$\sum_{b \in \mathcal{S}} \tau(a, b)P(a, b) = \sum_{b \in \mathcal{S}} \tau(a, \pi_a^{-1}b)P(a, \pi_a^{-1}b) \quad (12)$$

converges absolutely. Further, theorem 2.2 says that $\mu$ satisfies $P(a, b) = \mu(\pi_a b)$ for all $b \in \mathcal{S}$, so

$$\sum_{b \in \mathcal{S}} \tau(a, \pi_a^{-1}b)P(a, \pi_a^{-1}b) = \sum_{b \in \mathcal{S}} m(b)\mu(b) = \mathbb{E}_\mu[m(Z_1)]. \quad (13)$$

Combining equations (12) and (13) yields that

$$\sum_{b \in \mathcal{S}} \tau(a, b)P(a, b) = \mathbb{E}_\mu[m(Z_1)] \quad (14)$$

converges absolutely. Thus $m(Z_1)$ has finite expectation.

This allows us to apply Kolmogorov’s strong law of large numbers:\(^{21}\)

$$\lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} m(Z_{i+1}) = \mathbb{E}_\mu[m(Z_1)] \quad (15)$$

holds both $\mu$-almost surely and in $L^1$. Since the $Z_i$s have the law $\mu$ under $\mathbb{P}$, equation (15) holds $\mathbb{P}$-almost surely. For any $t \in \mathbb{N}_{>0}$, we have

$$\frac{1}{t} \sum_{i=0}^{t-1} m(Z_{i+1}) = \frac{1}{t} \sum_{i=0}^{t-1} \tau(X_i, \pi_{X_i}^{-1}Z_{i+1}) = \frac{1}{t} \sum_{i=0}^{t-1} \tau(X_i, X_{i+1}). \quad (16)$$

\(^{20}\)Rudin 1976, Thm. 3.55.

\(^{21}\)Jacod and Protter 2004, Thm. 20.2.
For any \( s \in \mathbb{N} \), the definition of expectation and \( X \)'s Markov property yield
\[
\mathbb{E}_\mathcal{P}[\tau(X_s, X_{s+1})] = \sum_{a \in \mathcal{S}} \sum_{b \in \mathcal{S}} \tau(a, b) \mathbb{P}(X_s = a) P(a, b) = \sum_{a \in \mathcal{S}} \mathbb{P}(X_s = a) \sum_{b \in \mathcal{S}} \tau(a, b) P(a, b),
\]
and, by equation (14),
\[
= \sum_{a \in \mathcal{S}} \mathbb{P}(X_s = a) \mathbb{E}_\mu[m(Z_1)]
= \mathbb{E}_\mu[m(Z_1)].
\] (17)

Combining equations (14) to (17) yields the result.

\[\square\]

### 2.3 Symmetry

Sometimes we can deduce that a stationary distribution of a Markov chain \( p \)-uniform under \( \pi \) is the uniform distribution just from the permutations \( \pi \). We say that \( \pi \) is symmetric if \( \pi_a b = \pi_b a \) for all \( a, b \in \mathcal{S} \). Lemma 2.6 carries the symmetry of \( \pi \) through to the Markov chain’s transition matrix. On a finite state space, a symmetric, stochastic matrix is doubly stochastic and thus has the uniform distribution as a stationary distribution; see Marshall, Olkin, and Arnold (2011).

**Lemma 2.6.** Let \( P \) be an \( \mathcal{S} \times \mathcal{S} \) matrix \( p \)-uniform under \( \pi \), and let \( \mu \) be a vector such that \( P(a, b) = \mu(\pi_a b) \).

If \( \pi \) is symmetric, then \( P \) is symmetric. Conversely, if \( P \) is symmetric and the entries of \( \mu \) are all distinct, then \( \pi \) is symmetric.

A common set operator, which we need for subsection 3.3, exemplifies symmetry.

**Example 2.7.** All the elements in exactly one of the sets \( A \) or \( B \) constitute their symmetric difference \( A \triangle B \). If the state space \( \mathcal{S} \) is a field of sets (closed under intersections and unions), then we can define permutations \( \pi = \{\pi_A\}_{A \in \mathcal{S}} \) on \( \mathcal{S} \) by \( \pi_A B := A \triangle B \). \( \triangle \)'s commutativity makes \( \pi \) typical of symmetric sets of permutations in that \( \pi_\emptyset \) is the unique identity permutation: If \( \pi_A \) were also the identity, then \( \pi_A A = A = \pi_\emptyset A \), and, by symmetry, \( \pi_A A = \pi_A \emptyset \), so, by invertibility, \( A = \emptyset \). Lemma 2.6 implies that any Markov chain \( p \)-uniform under the symmetric difference operator has the uniform distribution as a stationary distribution.

\[\square\]

### 2.4 Permutation Uniformity and cefs

In this subsection, we show that \( p \)-uniformity preserves mef structure; we have already discussed convergence of mean parameters of \( p \)-uniform mefs in and around theorem 2.5.

Feigin (1981) presented a theorem about cefs in a spirit similar to our present investigation of cefs’ relationship with iid sequences. It supposed \( X \) is real valued and \( \mathcal{P} \) is a cef as in equation (2) whose natural parameter space is open. Feigin (*ibid.*, Thm. 3) derived an additive process (partial sums of iid real-valued random variables) with the same law as a certain function of \( X \) when \( \tau(a, \cdot) \) is invertible in the second slot. On a finite state space and with \( \kappa(a, \cdot) \) constant, invertibility of \( \tau(a, \cdot) \) in the second slot is equivalent to the transition matrix’s having distinct numbers in every entry of a row. This is similar to the distinctness requirement of Rosenblatt (1959, Lem. 3). Feigin (1981, Thm. 4) derived strong consistency of mle and a central limit theorem for the Fisher information of a cef under the invertibility assumption.
We say that we give the sufficient condition, the proof of which is in appendix D.1. Theorem 2.11. Proposition 2.8 says that, if \( \mathcal{P} \) is the cef given in equation (2), then \( \mathcal{M} \) is an exponential family, which in turn implies that \( \mathcal{P} \) is an MEF.

Next, observation 2.10 and theorem 2.11 give necessary and sufficient conditions for an MEF to be p-uniform in terms of the p-uniformity of the carrier measure \( \kappa \) and sufficient statistic \( \tau \). First we give the sufficient condition, the proof of which is in appendix D.1.

Observation 2.10. If \( \mathcal{P} \) is the cef given in equation (2) and \( \tau \) and \( \kappa \) are p-uniform both under \( \pi \), then \( \mathcal{P} \) is p-uniform under \( \pi \).

The exact converse of the above observation is not quite true. If \( \kappa \) is zero in a couple positions, then the corresponding values of \( \tau \) are unconstrained. We also need \( \Theta \) to be big enough to determine which hyperplanes the different values of \( \tau \) lie in. We manage the latter concern in theorem 2.11 by assuming that \( \eta \) has affinely independent entries, meaning that \( \delta \cdot \eta(\theta) = h \) for all \( \theta \in \Theta \) implies \( \delta = 0 \) and \( h = 0 \).

Theorem 2.11. Suppose that \( \mathcal{P} \) is the MEF from equation (3) and is p-uniform under \( \pi \). Then \( \kappa \) is p-uniform under \( \pi \). Further, if \( \kappa \) is never zero and \( \eta \) has affinely independent entries, then \( \tau \) is p-uniform under \( \pi \).

Proof. Fix \( a, b, c \in \mathcal{S} \). If \( \kappa(a, \pi_a^{-1}b) = 0 \), then 0 = \( \theta \) \( a, \pi_a^{-1}b = \theta \) \( c, \pi_c^{-1}b \) for all \( \theta \in \Theta \). Since exp > 0 on \( \mathbb{R} \), we must have \( \kappa(c, \pi_c^{-1}b) = 0 \) as well.

By the same token, suppose \( \kappa(a, \pi_a^{-1}b) > 0 \), and notice that \( \kappa(c, \pi_c^{-1}b) > 0 \) as well. Then rearranging the equation \( \theta(a, \pi_a^{-1}b) = \theta(c, \pi_c^{-1}b) \) yields

\[
\eta(\theta) \cdot [\tau(c, \pi_c^{-1}b) - \tau(a, \pi_a^{-1}b)] = \log \frac{\kappa(a, \pi_a^{-1}b)}{\kappa(c, \pi_c^{-1}b)} \quad \text{for all } \theta \in \Theta.
\]

Since the right side is constant with respect to \( \theta \), affine independence of \( \eta \)'s entries implies that \( \tau(c, \pi_c^{-1}b) - \tau(a, \pi_a^{-1}b) = 0 \) and \( \log(\kappa(a, \pi_a^{-1}b)/\kappa(c, \pi_c^{-1}b)) = 0 \).

A variety of hypotheses familiar in exponential family theory imply that \( \eta \) has affinely independent entries, justifying the application of theorem 2.11. These hypotheses assume alternately that (a) equation (3) is a minimal representation of the \( a \) th row of \( P \in \mathcal{P} \), meaning that \( \tau(a, \cdot) \) and \( \eta \) both have affinely independent entries;\(^{23}\) (b) \( \eta(\Theta) \) contains an open, \( \ell \)-dimensional set;\(^{24}\) or (c)

\(^{22}\)Küchler and Sørensen 1997, p. 38; Wainwright and Jordan 2008, p. 40.

\(^{23}\)Barndorff-Nielsen 1978, Cor. 8.1, p. 113; Küchler and Sørensen 1997, p. 38; Wainwright and Jordan 2008, p. 40.

\(^{24}\)Casella and Berger 2002, Thms. 5.2.11 or 6.2.25, pp. 217, 288.
$\eta(\Theta)$ contains $\ell + 1$ affinely independent vectors.\textsuperscript{25} Item (b) implies item (c), which in turn implies that $\eta$ has affinely independent entries; the straightforward proof is in appendix D.2.

3 Markov Chains of Graphs

In this section, we apply the theories of CEFs and of p-uniform Markov chains to some of the network models that Hanneke, Fu, and Xing (2010) proposed. We find that some of them are p-uniform and thus CEFs. For two of the models, we can avoid MCMC or Newton’s method for MLE, which is what Hanneke et al. used; instead we give a closed form for the MLE. We also explore the relationships among p-uniformity, CEFs, and statistical independence of the random edges in the graphs.

The main result in theorems 3.8 and 3.9 is that we may replace $t$ observations of a p-uniform Markov chain of graphs with a single observation of a corresponding multigraph. For this purpose, we narrow the scope of the exponential random multigraph model (ERMG) of Shafie (2015) by introducing exponential random $t$-multigraph models, which differ from ERMMs by capping edge multiplicities at $t$.

We consider the state space $\mathcal{S}$ to be the set $\mathcal{G}_{n,t}$ of loopless multigraphs, notation for which we build up as follows. We fix a number $n \geq 2$ of vertices or nodes $[n] := \{1, \ldots, n\}$. Each potential edge comes from the set $\mathcal{D}_n := \binom{[n]}{2}$ of dyads whose elements $\{i, j\} \subseteq \{n\}, i \neq j$, we may write as $ij$ or $ji$ when the meaning is clear. That we do not allow edges from a node to itself makes the multigraphs in $\mathcal{G}_{n,t}$ loopless. Let $N := |\mathcal{D}_n| = \binom{n}{2}$. We fix a maximum multiplicity $t \in \mathbb{N}_{>0}$. A multigraph $g \in \mathcal{G}_{n,t}$ associates the vertex set $[n]$ with the multiplicity $g(f) \in \{0, 1, \ldots, t\}$ of copies of each dyad $f \in \mathcal{D}_n$. When $g(f) > 0$, we say that $f$ is an edge of $g$.

We identify a multigraph $g$ with its edge-multiplicity vector $g \in \{0, 1, \ldots, t\}^{\mathcal{D}_n} \equiv \mathcal{G}_{n,t}$. This is the vectorization of the adjacency matrix of $g$. The edge set $E(g)$ is $\{f \in \mathcal{D}_n \mid g(f) > 0\}$. The complement of $g$ is $\overline{g} := t1 - g$.

An exponential random graph model (ERGM) is an exponential family defined on simple graphs, $\mathcal{G}_{n,1}$. We introduce the analogous family of models for multigraphs.

**Definition 3.1.** We call an exponential family defined on $\mathcal{G}_{n,t}$ an exponential random $t$-multigraph model ($t$-ERMGM).

The choice of which ERGM to use in practice depends on identifying the sufficient statistics appropriate for specific data. Those statistics may incorporate node covariates, leading to models that Fienberg, Meyer, and Wasserman (1985) introduced and whose MLE Yan et al. (2018) investigated. Fienberg et al. contrasted microanalytic studies, such as Snijders (2001), employing node covariates with macroanalytic studies solely of network topology—our focus in the sequel. Many popular macroanalytic models, which Goldenberg et al. (2010) surveyed, rely on statistics built on subgraph counts. The simplest choice of subgraph is the single edge.

**Example 3.2** (Erdős-Rényi Graph Model). This ERGM arises by choosing edges of random graph $G$ independently each with probability $p$. The probability of $G = g$ is $p^{\left|E(g)\right|}(1 - p)^{N - \left|E(g)\right|} = \exp\left[\log(p/(1 - p))\left|E(g)\right| + N \log(1 - p)\right]$. The parameter function is $\eta(p) = \log \frac{p}{1 - p}$, sufficient statistic is $\left|E(g)\right|$, and the log-partition function is $-N \log(1 - p)$.\textsuperscript{26}  

\textsuperscript{25}Lehmann and Casella 1998, Cor. 6.16, p. 39.
\textsuperscript{26}Chatterjee and Diaconis 2013, § 2.2.
The literature on network statistics has studied a variety of other subgraph counts from a variety of viewpoints. Chatterjee and Diaconis (2013) treated subgraph counts in a general context. Specific choices of subgraphs were made in the studies in Bannister, Devanny, and Eppstein (2014) of counts of triangles, in Park and Newman (2004) of counts of two-stars, or in Snijders et al. (2006) of a complicated combination of degree counts, triangles, and two-stars. Holland and Leinhardt (1981), Chatterjee, Diaconis, and Sly (2011), and Rinaldo, Petrović, and Fienberg (2013, § 2) used degree sequences—equivalent to counting labeled $k$-stars. The latter’s generalized $\beta$ model is equivalent to the degree-sequence model for multigraphs of Frank and Shafie (2018).

**Example 3.3 (\( \beta \) Model).** This is the ERGM whose sufficient statistic is the degree sequence $\beta(G)$. The probability of $G = g$ is $\exp(\beta(g) \cdot \log \theta - \zeta(\theta)) = e^{-\zeta(\theta)} \prod_{uv \in E(g)} \theta_v \theta_v$. The parameter $\theta_v$ represents the attractiveness of vertex $v$. The log-partition function is $\zeta(\theta) = \prod_{i=1}^n \prod_{v=1}^{d_i-1} (\theta_v + 1)$. Chatterjee, Diaconis, and Sly (2011, Thm. 1.5) gave an algorithm for the MLE of $\theta$.

Subsection 3.4 introduces a dynamic relative of this $\beta$ model.

### 3.1 Finite Exchangeability

Two multigraphs $b, c \in \mathcal{G}_{n,t}$ are isomorphic, written $b \sim c$, if there exists a bijection $\phi$ on $[n]$ such that $b(\phi(i)\phi(j)) = c(\phi(i)\phi(j))$ for all $i, j \in [n]$. Isomorphism is an equivalence relation. If $\mu$ is any pmf (or any other function) on $\mathcal{G}_{n,t}$, then $\mu$ is finitely exchangeable if

$$b \sim c \implies \mu(b) = \mu(c).$$

Lauritzen, Rinaldo, and Sadeghi (2018) and Lauritzen, Rinaldo, and Sadeghi (2019) defined and analyzed finitely exchangeable pmfs on $\mathcal{G}_{n,1}$. The former showed that the set of all such pmfs on $\mathcal{G}_{n,1}$ form an exponential family whose sufficient statistic counts subgraphs of the random network by isomorphism class. The latter related the finitely exchangeable distributions of random networks to the marginal distributions of their subgraphs, and gave a de Finetti-like theorem for representing those distributions.

We can use proposition 3.4 to extend finite exchangeability from (or to) a transition matrix $P$ on $\mathcal{G}_{n,t} \times \mathcal{G}_{n,t}$ $p$-uniform under $\pi$ to (or from) a pmf $\mu$ on $\mathcal{G}_{n,t}$ such that $P(a, b) = \mu(\pi_a b)$. To do so requires of $\pi$ that $\sim$ be invariant under $\pi$ in the sense that

$$b \sim c \implies \pi_a b \sim \pi_a c \text{ for all } a \in \mathcal{G}_{n,t}. \tag{19}$$

Several simple properties of invariance under $\pi$ for any equivalence relation (not just isomorphism) are straightforward to show on any state space (not just $\mathcal{G}_{n,t}$). Equation (18)’s converse and

$$b \sim c \implies P(a, b) = P(a, c) \text{ for all } a \in \mathcal{G}_{n,t} \tag{20}$$

together imply $\sim$ is invariant under $\pi$; and equation (20)’s converse and equation (18) together imply $\sim$ is invariant under $\pi^{-1} := \{ \pi^{-1}_a \}_{a \in \mathcal{G}_{n,t}}$. (‘$\sim$’s invariance under $\pi^{-1}$ is just the converse of equation (19), and vice versa.) However, since $\mathcal{G}_{n,1}$ is finite (or, even if it weren’t but if $\pi$ were closed under inversion), ‘$\sim$’s invariance under $\pi$ is equivalent to ‘$\sim$’s invariance under $\pi^{-1}$.

---

27Petrović 2017.

28Chatterjee, Diaconis, and Sly 2011, § 1.2.
Proposition 3.4. On $\mathcal{G}_{n,t}$, suppose we have a transition matrix $P$, permutations $\pi$, and a pmf $\mu$ such that $P(a, b) = \mu(\pi a b)$ for all $a, b \in \mathcal{G}_{n,t}$ and such that $\sim$ is invariant under $\pi$. Then $\mu$ is finitely exchangeable if and only if every row of $P$ is (i.e., equation (20) holds), and every row of $P$ is finitely exchangeable if and only if some row of $P$ is.

Proof sketch. Equation (20) and $\sim$’s invariance under $\pi^{-1}$ together imply equation (18); equation (18) and $\sim$’s invariance under $\pi$ together imply equation (20). If there is an $a \in \mathcal{G}_{n,t}$ for which $b \sim c \implies P(a, b) = P(a, c)$, then $\sim$’s invariance under both $\pi$ and $\pi^{-1}$ implies equation (20) holds. □

One permutation set $\pi$ preserving isomorphism classes is the multigraph-complement operation $b \mapsto \overline{b}$. That is, $\sim$ is invariant under $\pi$ when $\pi a b = \overline{b}$ for all $a, b \in \mathcal{G}_{n,t}$. Finitely exchangeable distributions include any in exponential families for which the carrier measure and the sufficient statistic are finitely exchangeable. Edge counts (example 3.2) are constant on isomorphism classes. Degree sequence (example 3.3) is not constant on isomorphism classes, but degree distribution, a contingency table of degree counts, the degree sequence after sorting, or any other of unlabeled subgraphs are.

3.2 Dyadic Independence

Suppose $G$ is a $\mathcal{G}_{n,t}$-valued random variable. If $\{G(f) \mid f \in \mathcal{D}_n\}$ are mutually independent, then $G$ (or equivalently, its distribution) is dyadically independent. 30 Imposing dyadic independence on a model can be appropriate in “settings where the drivers of link formation are predominately bilateral in nature, as may be true in some types of friendship and trade networks as well as in models of (some types of) conflict between nation-states.”. 31

The remainder of this subsection describes conditions under which $G$, the $\mathcal{G}_{n,t}$-valued random variable, is dyadically independent and how dyadic independence interacts with p-uniformity. Sub-subsection 3.2.1 extends some definitions and basic facts about dyadic independence from graphs to multigraphs. Sub-subsection 3.2.2’s main result, theorem 3.9, allows us quickly to convert $t + 1$ samples from a $p$-uniform MEF of simple graphs into a single observation of a $t$-ERMGM.

3.2.1 Exponential Random Multigraph Models

Lemma 3.6 characterizes a $t$-ERMGM’s dyadic independence in terms of its sufficient statistic and carrier measure. We say that a function $\tau: \mathcal{G}_{n,t} \to \mathbb{R}^\ell$ is dyadditive, or factors over edges, 32 if there are functions $\tau_f: \{0, 1, \ldots, t\} \to \mathbb{R}^\ell$ for each dyad $f \in \mathcal{D}_n$ such that

$$\tau(g) = \sum_{f \in \mathcal{D}_n} \tau_f(g(f)).$$ (21)

When $t = 1$ (the ERGM case), $\tau$ is dyadditive if and only if there is a real, $\ell \times D_n$ matrix $Q$ such that $Qg = \tau(g) - \tau(0)$; then the $f$th column of $Q$ is $\tau_f(1) - \tau_f(0)$. When the carrier measure is also constant, it is already known that an ERGM is dyadically independent if and only if its sufficient statistic is dyadditive. 33
Example 3.5. Let $g \in \mathcal{G}_{n,1}$. A sufficient statistic in the Erdős-Rényi graph model is the number $|E(g)|$ of edges. The number of edges is dyadditive because $|E(g)| = 1 \cdot g$.

The $\beta$ model’s sufficient statistic, the degree sequence $\beta(g)$, is dyadditive. To see this, let $K \in \{0, 1\}^{n \times \mathcal{D}_n}$ be the complete graph’s incidence matrix, whose columns are the indicator $n$-vectors of the two-element sets constituting $\mathcal{D}_n$. Then $\beta(g) = K g$.

Statistics that rely on cliques larger than single edges are not dyadditive.

When an ergm’s carrier measure is not constant, lemma 3.6 still specifies some cases where dyadditivity of the sufficient statistic implies dyadic independence. To describe those cases, we say that a function $\kappa : \mathcal{G}_{n,t} \to \mathbb{R}$ is **dyadically multiplicative** if there are functions $\kappa_f : \{0, \ldots, t\} \to \mathbb{R}$ for each dyad $f \in \mathcal{D}_n$ such that

$$
\kappa(g) = \prod_{f \in \mathcal{D}_n} \kappa_f(g(f)).
$$

(22)

Dyadically multiplicative carrier measures in ergms include the common cases in which the carrier measure is constant.

Lemma 3.6. $G$’s pmf is equation (1) such that $\tau$ is dyadditive as in equation (21) and $\kappa$ is dyadically multiplicative as in equation (22) if and only if $G$ is dyadically independent with the pmf for multiplicity $G(f)$ being

$$
\mu_f^\tau(m) = \frac{\kappa_f(m) \exp(\eta(\theta) \cdot \tau_f(m))}{\sum_r \kappa_f(r) \exp(\eta(\theta) \cdot \tau_f(r))},
$$

(23)

$m \in \{0, \ldots, t\}, \ f \in \mathcal{D}_n$.

The proof of the lemma, whose details we relegate to appendix D.3, uses the law of total probability, appendix D’s lemma D.1, and some combinatorics.

Calculating the log-partition function, and thus pmf, of an ergm is computationally intractable for large $n$: it is $\#P$-hard and inapproximable in polynomial time. However, for the special case of dyadically independent ergms, computing the log-partition function requires a number of multiplications merely quadratic in $n$. The following corollary of lemma 3.6, which follows directly from expanding equation (33), generalizes this existing result to $t$-ermgms.

Observation 3.7. Under the conditions of lemma 3.6, the partition function is

$$
e^{\zeta(\theta)} = \prod_{u=1}^n \prod_{v=1}^{u-1} \sum_{r=0}^t \kappa_{uv}^{\tau}(r) \exp(\eta(\theta) \cdot \tau_{uv}(r)).$$

3.2.2 Independent Sequences and Multigraphs

Because of theorem 2.2, we may be able to derive an iid sequence of simple graphs from a Markov chain of simple graphs. In this sub-subsection, we consider how to turn a sequence of $t$ iid draws from an ergm into a single draw from a $t$-ermgm. Define the **multigraph union** of $z_1, \ldots, z_t \in \mathcal{G}_{n,s}$ to be their vector sum $z_1 + \cdots + z_t \in \mathcal{G}_{n,1}$. Let $Z = (Z_1, \ldots, Z_t)$ be an iid sequence of dyadically independent, $\mathcal{G}_{n,1}$-valued random variables with multigraph union $W$. Fix $z = (z_1, \ldots, z_t) \in \mathcal{G}_{n,1}$ with multigraph union $w$.

The following theorem says roughly that the order of the appearance of edges in $Z$ does not matter to $W$. The proof is in appendix D.4.

[^34]: Bannister, Devanny, and Eppstein 2014.
[^35]: Frank and Strauss 1986, Example 1.
Theorem 3.8 (Dyadically Independent Multigraphs). With the notation above, $W$ is dyadically independent and

$$\mathbb{P}(W = w) = \mathbb{P}(Z = z) \prod_{f \in D} \left( t^{f(w)} \right).$$

(24)

We close this subsection by showing that taking multigraph unions preserves exponential family structure. The proof of the result is a straightforward combination of Casella and Berger (2002, Thm. 5.2.11) with equations (1) and (24), using the facts that $\tau$ is dyadditive and that $\kappa$ is dyadically multiplicative. Notice that $Z$ does not appear in equation (25).

Theorem 3.9. Suppose the common pmf of the components of $Z$ is $\mu_{\theta}$ from equation (1) on page 3 such that $\tau$ is dyadditive as in equation (21) and $\kappa$ is dyadically multiplicative as in equation (22). Further, assume that $\eta(\Theta)$ contains an open, $\ell$-dimensional set. Then the pmf of $W$ also has an exponential family representation with the same parameter and parameter function, and with the sufficient statistic and carrier measure respectively equal to

$$\sum_{f \in D} \tau_f(1)W(f) - \tau_f(0)[t - W(f)] \quad \text{and} \quad \prod_{f \in D} \left( t^{W(f)} \right)^{\kappa_f(1)^W(f)\kappa_f(0)^{1-W(f)}}.$$  

(25)

3.3 Examples from the Literature

In this subsection, we run through some of the TERGM sufficient statistics that Hanneke, Fu, and Xing (2010) proposed. Each is scalar, so $\ell = d = 1$ (Hanneke et al. coalesced them into a vector, but we examine each in isolation for simplicity). Throughout, $\mathcal{S} = \mathcal{G}_{n,1}$, and $\mathcal{G} = \{G_i\}_{i \in \mathbb{N}}$ is a $\mathcal{G}_{n,1}$-valued stochastic process that is a Markov chain under the cef $\mathcal{P}$ of transition matrices $P_{\theta}$ from equation (2) on page 4 but with $\kappa \equiv 1$. Let $t \in \mathbb{N}_{>0}$. Notationally, $a$, $b$, and $c$ are generic graphs.

Scaling a sufficient statistic $\tau$ by a constant $c$, often $c = n$ or $c = \frac{1}{n-1}$ so that $\tau$ lies in $[0, n]$, occasionally improves interpretability while being transparent to the probability model because $((\frac{1}{n-1}) \cdot (c\tau)) = \gamma \cdot \tau$.

Example 3.10 (Density and Stability (Hanneke, Fu, and Xing 2010, § 2.1)). If $\tau(a, b) = \frac{1}{n-1} \sum_{ij} b(ij)$, then $\tau$ is called density; if $\tau(a, b) = \frac{1}{n-1} \sum_{ij} b(ij) a(ij) + [1 - b(ij)] [1 - a(ij)]$ then $\tau$ is called stability. Density measures the tendency of an edge to exist in the future irrespective of the present. Stability measures the tendency of an edge to continue existing or not in the future if it is doing so in the present.

With $p$-uniformity as our hint, we notice that we can rewrite the statistics as the number of edges in bijective functions of the present and future networks. $\tau$ is density if $\tau(a, b) = |E(b)|$ or stability if $\tau(a, b) = \frac{1}{n-1} |E(a \Delta b)|$. Either way, $\tau$ is $p$-uniform under $\pi$: In the density case, $\pi_a$ is the identity permutation for all $a$; in the stability case, $\pi_a b = \overline{a \Delta b}$. $\mathcal{P}$ is $p$-uniform by observation 2.10, and an mef by corollary 2.9. By theorem 2.2, in the density case, $Z_i := G_i$ is an iid sequence of $\mathcal{G}_{n,1}$-valued random variables; in the stability case, $Z_i := \overline{G_{i-1} \Delta G_i}$ is.

In both cases, proposition 2.8 says that the common family $\{\mu_{\theta}^{(p)}\}_{p \in (0,1)}$ of pmfs of the $Z_i$s is an exponential family with the dyadditive sufficient statistic $\psi(b) := \psi(a, \pi_a b) = \frac{1}{n-1} |E(b)|$. Hence the $Z_i$s are an iid sequence of Erdős-Rényi graphs (example 3.2) with canonical parameter $p$ and parameter function $\eta(p) = (n - 1) \log \frac{p}{1-p}$. The $Z_i$s are dyadically independent by lemma 3.6.
Theorem 2.5 says that, for any \( j \in \mathbb{N} \), the mean parameter equals

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t} \tau(G_i, G_{i+1}) = \mathbb{E}[\tau(G_j, G_{j+1})] = \sum_{b \in \mathcal{G}_{n,1}} \tau(0, b)P_p(0, b) = p \frac{N}{n-1}.
\]

While Hanneke, Fu, and Xing (2010) relied on MCMC MLE of parameters in the general case and Newton’s method in the dyadditive case, in this particular case we compute the MLE \( \hat{p} \) with a closed-form expression (see also theorem 1.4):

\[
\hat{p} = \frac{n-1}{tN} \sum_{i=0}^{t} \tau(G_i, G_{i+1}).
\]

Applying theorem 3.9, let \( \mathbf{W} := \sum_{i=1}^{t} \mathbf{Z}_i \), which is a single, dyadically independent \( t \)-ERMGM random variable with sufficient statistic \( \frac{1}{n-1} \mathbf{W} \). Then the joint distribution of \( G \) is proportional, per theorem 3.8, to the distribution of \( \mathbf{W} \), in which every edge is added to the multigraph by flipping a \( p \)-weighted coin for each dyad \( t \) times. Put differently, for each dyad \( f \), \( \mathbf{W}(f) \) is an independent, binomially distributed random variable with parameters \( t \) and \( p \).

The main difference between density and stability is that, when \( \tau \) is stability, \( P_p \) is symmetric by lemma 2.6 because symmetric differences commute (see example 2.7). The unique stationary distribution is uniform because \( \kappa \equiv 1 \) implies that all entries of \( P_p \) are positive. Linear algebraically, we can say more. Since \( a \Delta \bar{a} \) is the complete graph with \( N \) edges for any \( a \), the diagonal elements of \( P_p \) are \( P_p(a, a) = p^N \), and the trace of \( P_p \) is \( 2^N p^N \). Every other off-diagonal entry of \( P_p \) has a \( (1-p) \) factor, so \( P_p \to I \) as \( p \to 1^- \) and \( P_p \to \frac{1}{2^{N-1}}(11^T - I) \) as \( p \to 0^+ \). At \( p = \frac{1}{2} \), every row of \( P_p = \frac{1}{2^N} 11^T \) is the uniform distribution.

If \( \tau \) is either of the statistics in the next two examples, then \( \mathcal{P} \) is neither \( p \)-uniform nor an MEF. When \( \mathcal{P} \) is not an MEF, theorem 1.2 says that \( G \)'s finite-sample joint distribution cannot have an exponential-family representation whose parameter is \( \theta \). We are not guaranteed that \( \sum_{i=0}^{t-1} \tau(G_i, G_{i+1}) \) is sufficient for \( \theta \). Even though stability and reciprocity (below) have similar looking formulæ, the latter prevents analyzing \( G \) as an exponential family in the same terms, \( \theta \) and \( \tau \), as its transition matrix was defined. We can interpret a density or stability TERGM either as an iid sequence of Erdős-Rényi graphs or as a single \( t \)-ERMGM, but such a simple interpretation of the statistics below is unavailable.

Example 3.11 (Reciprocity (Hanneke, Fu, and Xing 2010, § 2.1)). Our focus has been on undirected graphs, but we briefly mention a statistic whose interpretation comes from its application to directed graphs, which we represent by “vectors” of adjacency matrices in \( \{0, 1\}^{n \times n} \). Interpreting \( 0/0 \) as zero, \( \tau(a, b) := n[\sum_{i,j=1}^{n} a(i, j)][^{-1} \sum_{i,j=1}^{n} b(j, i)a(i, j) \text{ is called reciprocity.}}

To see that \( \tau \) is not \( p \)-uniform, suppose that \( \eta(\theta) \) contains a number other than zero and that \( n \geq 3 \). Let \( e_1, \ldots, e_n \) be the standard basis of the set of adjacency matrices; these are the graphs containing exactly one edge. Then \( \tau(e_1, b) \in \{0, n\} \) whereas \( \tau(e_1 \cup e_2, e_3) = n/2 \). In fact, by violating the conclusion of proposition 1.3, this also shows that \( \mathcal{P} \) is not an MEF.

Example 3.12 (Transitivity (Hanneke, Fu, and Xing 2010, § 2.1)). Interpreting \( 0/0 \) as zero and taking sums over triples of vertices \( i < j < k \), \( \tau(a, b) := n[\sum_{i,j,k=1} a(ij) a(jk)][^{-1} \sum_{i,j,k=1} b(ik) a(ij) a(jk) \text{ is called transitivity.}}

Transitivity measures the tendency of edge \( ik \) to come into existence in the future
We had hoped that some straightforward calculations of the joint distribution of a finite sample. We needed to develop goodness-of-fit tests for the null model being an exponential family; for a stochastic process, what matters is the degree sequence of the current and next graph. Using the same notation as the previous subsection, replace \( d = \ell : n; \Theta : R^n_{>0}, \) the positive orthant; \( \eta : \log; \) and \( \tau(a, b) := \beta(\pi a b), \) where \( \beta \) is the degree sequence. With these choices, \( \mathcal{P} \) is an MEF p-uniform under any permutations \( \pi \) on \( G_n,1. \) We call the model the \( \beta \) TERGM under \( \pi. \)

For example, we may take use the symmetric difference operator as the permutations:

\[
P_b(a, b) = \exp (\beta(a \Delta b) \cdot \log \theta - \zeta(\theta)) = \frac{\prod_{u,v \in E(a \Delta b)} \theta_u \theta_v}{\prod_{u=1}^n \prod_{v=1}^{u-1} (\theta_u \theta_v + 1)}.
\]

\( Z_i := G_{i-1} \Delta G_i \) is an IID sequence of \( \beta \) ERGMs as in example 3.3. For a given vertex \( v \in [n], \) as \( \theta_v \) increases, so does the probability edges lying on it appear in one or the other, but not both, of the current and next iterations of \( G. \) So an edge lying on \( v \) is in \( G_i \) and it appears in \( G_i \Delta G_{i+1}, \) then that dyad will not appear in \( G_{i+1}. \) Likewise, if a dyad lying on \( v \) is not in \( G_i \) and it appears in \( G_i \Delta G_{i+1}, \) then that edge will appear in \( G_{i+1}. \) Thus \( G \) changes the most in the neighborhoods of vertices \( v \) with high values of \( \theta_v. \) We might say that \( v \) is loyal if it has a low value of \( \theta_v \) (near zero) and is disloyal if it has a high value of \( \theta_v. \)

### 3.4 A Model of Loyalty

Motivated by the theoretical developments of the previous sections, we introduce a new TERGM.

**Example 3.13 (\( \beta \) TERGM).** By analogy with example 3.3 on page 16, we may define an MEF using the degree sequences of functions of the current and next graph. Using the same notation as the previous subsection, replace \( d = \ell : n; \Theta : R^n_{>0}, \) the positive orthant; \( \eta : \log; \) and \( \tau(a, b) := \beta(\pi a b), \) where \( \beta \) is the degree sequence. With these choices, \( \mathcal{P} \) is an MEF p-uniform under any permutations \( \pi \) on \( G_n,1. \) We call the model the \( \beta \) TERGM under \( \pi. \)

For example, we may take use the symmetric difference operator as the permutations:

\[
P_b(a, b) = \exp (\beta(a \Delta b) \cdot \log \theta - \zeta(\theta)) = \frac{\prod_{u,v \in E(a \Delta b)} \theta_u \theta_v}{\prod_{u=1}^n \prod_{v=1}^{u-1} (\theta_u \theta_v + 1)}.
\]

\( Z_i := G_{i-1} \Delta G_i \) is an IID sequence of \( \beta \) ERGMs as in example 3.3. For a given vertex \( v \in [n], \) as \( \theta_v \) increases, so does the probability edges lying on it appear in one or the other, but not both, of the current and next iterations of \( G. \) So an edge lying on \( v \) is in \( G_i \) and it appears in \( G_i \Delta G_{i+1}, \) then that dyad will not appear in \( G_{i+1}. \) Likewise, if a dyad lying on \( v \) is not in \( G_i \) and it appears in \( G_i \Delta G_{i+1}, \) then that edge will appear in \( G_{i+1}. \) Thus \( G \) changes the most in the neighborhoods of vertices \( v \) with high values of \( \theta_v. \) We might say that \( v \) is loyal if it has a low value of \( \theta_v \) (near zero) and is disloyal if it has a high value of \( \theta_v. \)

### 4 Concluding Remarks

Our original goal was to develop goodness-of-fit tests for the TERGM examples in Hanneke, Fu, and Xing (2010) using algebraic statistical techniques from ERGM analysis. These techniques presuppose that the null model is an exponential family; for a stochastic process, what matters is the joint distribution of a finite sample. We needed TERGMs to have exponential-family representations with the same parameter as their transition probabilities because the transitions’ parameters are of ultimate interest. However, Hanneke, Fu, and Xing (ibid.) defined TERGMs as what we have been calling CEFs (though not in this language). For concreteness, consider a TERGM \( X = \{X_t\}_{t \in \mathbb{N}}, \) which is a Markov chain on \( G_n,1 \) whose transition probabilities have the CEF representation in equation (2). We had hoped that some straightforward calculations of the joint PMF using the Markov property would at least show that the statistic \( \sum_{i=0}^{l-1} \tau(X_i, X_{i+1}) \) is sufficient for the transitions’ parameter \( \theta. \)

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The result of these calculations, which Schwartz (2021, Thm. 2.3.13, p. 33) presented, was that the normalizing term of \( \mathbb{P}_\theta(X_1 = x_1, \ldots, X_t = x_t) \) depends on the observed data \( x_1, \ldots, x_t \). This does not happen in exponential-family representations. While not obvious from the formulae that Hanneke, Fu, and Xing (2010) used to define their statistics \( \tau \), this dependence of the normalizing term on the observed data turned out to be because the log-partition function \( \zeta \) in equation (2) depends on the previous state of the Markov chain. As theorem 1.2 turns out to say, to clear the dependence while still testing hypotheses on \( \theta \) requires that a tergm’s transition probabilities have not just a cef but an mef representation.

The challenge remained of identifying when a given tergm (or any cef) is also an mef. Permutation uniformity offers a nice solution because all p-uniform cefs are mefs (corollary 2.9), p-uniformity is discoverable directly from the sufficient statistics around which models are built (observation 2.10), and, most of all, the iid sequences they provide (theorem 2.2) are easy to interpret in the same terms as the original model (proposition 2.8). In the case of a dyadically independent tergm, it is particularly appealing to view a temporally dependent sequence of networks as a single observation of a multigraph (theorems 3.8 and 3.9). In this form, we expect algebraic statistical techniques will apply easily to develop goodness-of-fit tests for dyadically independent, p-uniform tergms and, perhaps with some work, to other tergms that are mefs.

Now that our main results are in hand, one can string them together to obtain a technique for analyzing tergms, namely a general recipe for dispensing with the joint likelihood’s denominator’s dependence on the observed data so that one may rewrite the likelihood in exponential-family form. Specifically, theorem 1.2 says that the joint distribution of an mef has an exponential-family representation with the same parameter as the transition probabilities in equation (3). This permits the usual exponential-family computation of the mle for the joint likelihood’s parameter, which is the same as the transitions’ parameter. Theorem 2.2 says that we can rewrite a Markov chain as a temporally independent sequence (irrespective of dyadic independence), permitting an intuitive interpretation of the mle. The theorem associates a p-uniform tergm with an iid sequence of ergms. We can take the multigraph union of \( t \) samples from that sequence. This multigraph union is then a \( t \)-termgm with the same parameter as the original tergm.\(^{37}\) However, if the distribution of the iid ergms is dyadically independent, then so is the \( t \)-ermgm by theorems 3.8 and 3.9.

Looking back at the literature, Hanneke, Fu, and Xing (ibid.) proposed four statistics for transitions in the Markov chain, and we address each of them in subsection 3.3: Density is such that \( \tau \) already does not depend on the Markov chain’s present state (\( x_i \) in the notation above). Stability does, but, because it is p-uniform, theorem 1.2 permits us to rewrite the joint likelihood of a stability-based tergm in exponential-family form. Reciprocity and transitivity also do depend on \( x_i \), but our results prove that one cannot rewrite the tergms based on them in exponential family form with the same parameter as the transitions. The new model in subsection 3.4 is p-uniform, thus theorem 2.2 applies, providing a more complex Markov-chain-transition sufficient statistic that leads to an exponential-family multigraph model with interpretable parameters.

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\(^{37}\)Lehmann and Casella 1998, p. 26.
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Correspondence
William K. Schwartz <wkschwartz@gmail.com>
Secretariat Economists
2121 K Street NW, Suite 1100
Washington, DC 20037
Appendices

A  Two Equivalent Definitions of Permutation Uniformity

Rosenblatt (1959, § 3) first named and studied p-uniform chains, calling them simply uniform chains. Yano and Yasutomi (2011, Def. 1.4) added the $p$- in their definition of p-uniform chains, which they defined as Markov chains whose transition matrices $f$ satisfy equation (26). The following lemma asserts the equivalence of Yano and Yasutomi’s definition and ours (derived from Rosenblatt’s).

Lemma A.1. $f$ is p-uniform under $\pi$ if and only if there exists a function $g : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$f(a, b) = g(\pi_a b) \quad \text{for all } a, b \in \mathcal{A}. \quad (26)$$

In this case, $g$ is unique up to permutation. If $f$ is a p-uniform stochastic matrix on $\mathcal{A}$, then $g$ is a stochastic vector, i.e., a pmf.

Proof. Equation (26) follows directly from the definition of permutation uniformity.

Suppose $g, h : \mathcal{A} \rightarrow \mathcal{A}$ and $\{\pi_a\}_{a \in \mathcal{A}}$ and $\{s_a\}_{a \in \mathcal{A}}$ are sets of permutations such that $f(a, b) = g(\pi_a b) = h(s_a b)$ for all $a, b$. Then for any $b \in \mathcal{A}$, $g(b) = g(\pi_a b) = f(a, \pi^{-1}_a b) = h(s_a \pi^{-1}_a b)$. That is, $g$ is the composition of $h$ and the permutation $s_a \pi^{-1}_a$.

The last part of the proof follows from the definition of a stochastic matrix. \qed

B  Markov Chains Induced by Random Functions and Theorem 2.2

This appendix translates theorem 2.2’s language for p-uniform chains into Diaconis and Freedman (1999)’s iterated random functions model that example 2.3 alluded to.

Let $\Omega$ be an arbitrary set. Fix some family $F = \{f_{\omega}\}_{\omega \in \Omega}$ of functions from $\mathcal{A}$ to itself indexed by $\Omega$. Let $\mu$ be a pmf defined on $\Omega$. Suppose $X$ is a Markov chain under some probability measure $\mathbb{P}$. $X$ is induced by $F$ and $\mu$ starting at $x_0 \in \mathcal{A}$ if $X_0 = x_0$ and, for $t \in \mathbb{N}$, $X_{t+1} = f_{W_{t+1}}(X_t)$, where $W = \{W_t\}_{t \in \mathbb{N}}$ is an iid sequence of $\Omega$-valued random variables with common law $\mu$ under $\mathbb{P}$. When $\Omega$ is discrete, we have that

$$\mathbb{P}(X_{t+1} = b \mid X_t = a) = \mathbb{P}(f_{W_{t+1}}(a) = b) = \sum_{\omega \in \Omega \atop f_{\omega}(a) = b} \mu(\omega).$$

Adopting the notation of theorem 2.2, let $X$ be a Markov chain p-uniform under $\pi$ with an iid sequence $Z$ with common law $\mu$. Setting $\Omega = \mathcal{A}$ and $f_{\omega}(b) = \pi^{-1}_b a$, we have that $X$ is induced by $F$ and $\mu$:

$$X_{t+1} = \pi^{-1}_{X_t} Z_{t+1} = f_{Z_{t+1}}(X_t).$$

In this case, $\{\omega \in \Omega \mid f_{\omega}(a) = b\} = \{\pi_a b\}$, so that $\mathbb{P}(X_{t+1} = b \mid X_t = a) = \mu(\pi_a b)$ as desired under p-uniformity.

Permutation uniformity forces us to restrict the functions $f_{\omega}$. Since the $\pi_a$s are bijective, $a \mapsto f_{\pi_a}$ is injective in the set of functions $\mathcal{A} \rightarrow \mathcal{A}$, and $a \mapsto f_{\pi_a}(b)$ is bijective in $\mathcal{A}$. The latter assertion follows straight from the definition of $f_{\pi_a}$. To see the former, note that

$$f_a = f_b \iff f_a(c) = f_b(c) \forall c \iff \pi^{-1}_c a = \pi^{-1}_c b \forall c \iff a = b.$$
Other works that consider iterating random maps to form dependent sequences are Wu and Mielniczuk (2010, § 4) and Yano and Yasutomi (2011).

Example 2.3 (continued). Recall from the first part of the example on page 11 that $X_{t+1} = X_t + Z_{t+1}$ (mod $n$), where the $Z_i$s are uniform, i.i.d random variables taking values zero or one, so that $X$ is a $p$-uniform chain under $\pi j = j - i$ (mod $n$) for each $i, j \in \mathcal{S} = \mathbb{Z}/n\mathbb{Z}$.

In terms of iterating random functions, $f_j(i) = \pi_j^{-1} j = j + i$ (mod $n$), and we can write $X_{t+1} = f_{Z_{t+1}}(X_t)$. Notice that we apply only either $f_0(i) = i$ or $f_1(i) = i + 1$ (mod $n$). $f_0$ and $f_1$ are Lipschitz continuous with Lipschitz constant one under the metric $\rho$ on $\mathcal{S}$ defined by $\rho(i, j) = \min\{j - i$ (mod $n), i - j$ (mod $n)\}$, the shortest modular addition distance from $i$ to $j$ in either direction.

\section{C Dyadic Independence and Permutation Uniformity}

Suppose $\tau: \mathcal{G}_{n,t} \times \mathcal{G}_{n,t} \to \mathbb{R}^\ell$ is $p$-uniform under $\pi$, so that, by lemma A.1, there is a function $\nu: \mathcal{G}_{n,t} \to \mathbb{R}^\ell$ such that $\tau(a, b) = \nu(\pi_a b)$ for all $a, b \in \mathcal{G}_{n,t}$. We say that this two-argument function $\tau$ is dyadadditive if $b \mapsto \tau(a, b)$ is dyadadditive for all $a \in \mathcal{G}_{n,t}$. Can we extend dyadadditivity from $\tau$ to $\nu$ or from $\nu$ to $\tau$? Generally no.

Example C.1. For the case of simple graphs ($\ell = 1$) and scalar sufficient statistics ($\ell = 1$), we show that just because $\tau$ is dyadadditive, it is not necessary that $\nu$ is. Suppose $n = 3$ and $\tau(a, b) = |E(b)|$, the number $1 \cdot b$ of edges of $b$, which is dyadadditive (cf. example 3.5). Set $\nu$ and $\pi$ so that

$$\nu(b) := \begin{cases} |E(b)| & \text{if } b \in \{0, 1\} \\ |E(\bar{b})| & \text{else,} \end{cases} \quad \pi_a b := \begin{cases} b & \text{if } b \in \{0, 1\} \\ \bar{b} & \text{else,} \end{cases}$$

for all $a, b \in \mathcal{G}_{3,1}$. Hence $\tau(a, b) = \nu(\pi_a b)$.

Suppose by way of contradiction that $\nu$ is dyadadditive. Per the comment after equation (21), there are real numbers $q_1, q_2$, and $q_3$ such that $\nu(b) = \sum_{f \in E(b)} q_f$. When $b$ has one edge, $\nu(b) = 2$, so $q_1 = q_2 = q_3 = 2$. But when $b$ is the complete graph, we have $3 = \nu(b) = q_1 + q_2 + q_3 = 2 + 2 + 2$, a contradiction. \hfill \square

There is one case where we can guarantee that dyadadditivity passes from $\tau$ to $\nu$: when one of the permutations is the identity permutation, such as in example 2.7.

Proposition C.2. If $\tau$ is dyadadditive and, for some $x \in \mathcal{G}_{n,t}$, $\pi_x$ is the identity permutation, then $\nu$ is dyadadditive.

Proof. For some $\tau$ and all $b$, $\nu(b) = \tau(x, \pi_x^{-1} b) = \tau(x, b) = \sum_{f \in \mathcal{D}_n} \tau_f(x, b(f))$. \hfill \square

\section{D Proofs}

This appendix contains proofs of some of the technical statements in the paper. The proofs of lemma 3.6 and theorem 3.8 rely on the following lemma to facilitate applying the distributive law. Induction on $|\mathcal{T}|$ provides a short proof whose tedious index chasing we spare the reader.
Lemma D.1. Let $\mathcal{T}$ be a non-empty, finite set; \{$B_i \}_{i \in \mathcal{T}}$ be a family of finite sets; $B := \bigcup_{i \in \mathcal{T}} B_i$; $f_i : B_i \to \mathbb{R}$ for each $i \in \mathcal{T}$; and $\mathcal{F}_\mathcal{T} = \{ y \in B^T \mid y_i \in B_i \text{ for all } i \in \mathcal{T} \}$. Then

$$\prod_{i \in \mathcal{T}} \sum_{b \in B_i} f_i(b) = \sum_{y \in \mathcal{F}_\mathcal{T}} \prod_{i \in \mathcal{T}} f_i(y_i).$$

D.1 Proof of Observation 2.10

Proof. For all $a, b, c \in \mathcal{S}$ and all $\theta \in \Theta$,

$$P_\theta(a, \pi_a^{-1}b) = \kappa(a, \pi_a^{-1}b) \exp \left( \eta(\theta) \cdot \tau(a, \pi_a^{-1}b) - \zeta(a, \theta) \right)$$

(27)

$$= \kappa(c, \pi_c^{-1}b) \exp \left( \eta(\theta) \cdot \tau(c, \pi_c^{-1}b) - \zeta(c, \theta) \right).$$

(28)

We set 1 equal to the sum of equation (27) over all $b \in \mathcal{S}$ and rearrange to obtain

$$\exp(\zeta(a, \theta)) = \sum_{b \in \mathcal{S}} \kappa(a, \pi_a^{-1}b) \exp \left( \eta(\theta) \cdot \tau(a, \pi_a^{-1}b) \right).$$

(29)

Doing the same thing to equation (28) and replacing $a$ with $c$ in equation (29) reveals that $\zeta(a, \theta) = \zeta(c, \theta)$. Thus equation (28) equals

$$\kappa(c, \pi_c^{-1}b) \exp \left( \eta(\theta) \cdot \tau(c, \pi_c^{-1}b) - \zeta(c, \theta) \right) = P_\theta(c, \pi_c^{-1}b).$$

D.2 Exponential-Family Assumptions

Here we prove the claim from page 14 after the proof of theorem 2.11.

To prove the first implication, for $i \in [\ell]$, let $e_i$ be the $i$th standard basis vector in $\mathbb{R}^\ell$. The open set contains an open ball centered at some vector $c$ with some radius $r$. The list of $\ell + 1$ vectors $c, e_1 + c, \ldots, e_\ell + c$ is affinely independent.

For the second implication, consider any $\ell + 1$ affinely independent vectors in $\eta(\Theta)$, say, $\eta(\theta_0), \ldots, \eta(\theta_\ell)$. Then $\eta(\theta_1) - \eta(\theta_0), \ldots, \eta(\theta_\ell) - \eta(\theta_0)$ are linearly independent. Form an $\ell \times \ell$ matrix $A$ with these vectors as columns. Suppose $\delta \in \mathbb{R}^\ell$ and $h \in \mathbb{R}$ are such that $\delta \cdot \eta(\theta) = h$ for all $\theta \in \Theta$. Then $\delta \cdot (\eta(\theta_1) - \eta(\theta_0)) = 0$ for all $i \in [\ell]$, and hence $A^T \delta = 0$. By the linear independence of $A$’s columns, $\delta = 0$, and thus $h = 0$. Hence $\eta$ has affinely independent entries.

D.3 Proof of Lemma 3.6

Proof. That $G$ is a $\mathcal{G}_{n,1}$-valued random variable if and only if $G(f)$ is a $\{0, \ldots, t\}$-valued random variable follows directly from the definition of $\mathcal{G}_{n,1}$. For some $\theta \in \Theta$, define $\mu_\theta$ as in equation (1) and $\mu_\theta^f$ as in equation (23). The backward implication will follow if we can show that, for all $g \in \mathcal{G}_{n,1}$,

$$\prod_{f \in \mathcal{D}_n} \mu_\theta^f(g(f)) = \mu_\theta(g),$$

where equations (21) and (22) define $\tau$ and $\kappa$ in terms of $\{\tau_f\}_{f \in \mathcal{D}_n}$ and $\{\kappa_f\}_{f \in \mathcal{D}_n}$, respectively. The forward implication will follow if we can show that, for all $f \in \mathcal{D}_n$,

$$\sum_{g \in \delta \mid g(f) = m} \mu_\theta(g) = \mu_\theta^f(m),$$

30
where equations (21) and (22) define \( \{ \tau_f \}_{f \in \mathcal{D}_n} \) and \( \{ \kappa_f \}_{f \in \mathcal{D}_n} \) in terms of \( \tau \) and \( \kappa \), respectively. Then the dyadic independence of \( G \) will follow from the equality established when proving the backward implication.

Both steps require the following fact. Setting the sum of equation (1) over \( \delta \) to one yields

$$\zeta(\theta) = \log \sum_{x \in \delta} \kappa(x) \exp(\eta(\theta) \cdot \tau(x)).$$

(30)

**Forward Implication.** Fix an arbitrary graph \( G = g \). For tidiness we write \( \delta = \mathcal{G}_{n,t} \). The probability that \( G(f) = m \in \{0, \ldots, t\} \), is

$$\sum_{g \in \delta \atop g(f) = m} \mu(g) = \frac{\sum_{g \in \delta \atop g(f) = m} \kappa(g) \exp(\eta(\theta) \cdot \tau(g))}{\sum_{r=0}^{t} \sum_{x \in \delta \atop x(f) = r} \kappa(x) \exp(\eta(\theta) \cdot \tau(x))},$$

where we have used equation (30). Using equations (21) and (22), define

$$u(g) := \prod_{h \in \mathcal{D}_n \atop h \neq f} \kappa_h(g(h)) \exp(\eta(\theta) \cdot \tau_h(g(h)).$$

\( \tau \) is dyadic and \( \kappa \) is dyadically multiplicative, so

$$\kappa(g) \exp(\eta(\theta) \cdot \tau(g)) = \kappa_f(g(f)) \exp(\eta(\theta) \cdot \tau_f(g(f))) u(g).$$

The expression for \( u(g) \) does not involve \( g(f) \), so \( u(g) = u(x) \) regardless of whether \( g, x \in \delta \) have edge \( f \) the same number of times. Consequently,

$$\sum_{g \in \delta \atop g(f) = m} u(g) = \sum_{x \in \delta \atop x(f) = r} u(x)$$

(31)

for each \( r \in \{0, \ldots, t\} \). Factoring out this sum gives

$$\frac{\sum_{g \in \delta \atop g(f) = m} \kappa(g) \exp(\eta(\theta) \cdot \tau(g))}{\sum_{r=0}^{t} \sum_{x \in \delta \atop x(f) = r} \kappa(x) \exp(\eta(\theta) \cdot \tau(x))} = \frac{\kappa_f(m) \exp(\eta(\theta) \cdot \tau_f(m)) \sum_{g \in \delta \atop g(f) = m} u(g)}{\sum_{r=0}^{t} \kappa_f(r) \exp(\eta(\theta) \cdot \tau_f(r)) \sum_{x \in \delta \atop x(f) = r} u(x)},$$

which, after canceling out equation (31) in the numerator and denominator, equals equation (23).

**Backward Implication.** First off, for any \( x \in \mathcal{G}_{n,t} \), we have

$$\prod_{f \in \mathcal{D}_n} \kappa_f(x(f)) \exp(\eta(\theta) \cdot \tau_f(x(f))) = \exp\left(\eta(\theta) \cdot \sum_{f \in \mathcal{D}_n} \tau_f(x(f))\right) \prod_{f \in \mathcal{D}_n} \kappa_f(x(f))$$

$$= \kappa(x) \exp(\eta(\theta) \cdot \tau(x)),$$

(32)

where we have defined \( \tau \) and \( \kappa \) via equations (21) and (22).

Fix an arbitrary graph \( g \in \mathcal{G}_{n,t} \). Then, using equation (32), the (joint) probability that \( G = g \) is

$$\prod_{h \in \mathcal{D}_n} \mu_{\theta}^h(g(h)) = \frac{\kappa(g) \exp(\eta(\theta) \cdot \tau(g))}{\prod_{h \in \mathcal{D}_n} \sum_{r=0}^{t} \kappa_h(r) \exp(\eta(\theta) \cdot \tau_h(r))}.$$

31
This matches equation (1) if we can show that the denominator equals \( e^{\zeta(\theta)} \). To that end, we exchange \( \prod_{f \in \mathcal{D}_n} \) and \( \sum_{i=0}^{t} \) using lemma D.1. In the language of lemma, set \( \mathcal{T} := \mathcal{D}_n \), and \( B_h := \{0, \ldots, t\} \) and \( f_h(r) := \exp(\eta(\theta) \cdot \tau_h(r)) \) for each \( r \in B_h \) and each \( h \in \mathcal{T} \). Then \( F_T = \mathcal{G}_{n,t} \), and

\[
\prod_{h \in \mathcal{D}_n} \sum_{r=0}^{t} \kappa_h(r) \exp(\eta(\theta) \cdot \tau_h(r)) = \sum_{x \in \mathcal{G}_{n,t}} \prod_{h \in \mathcal{D}_n} \kappa_h(x(h)) \exp(\eta(\theta) \cdot \tau_h(x(h)))
\]

(33)

\[
= \sum_{x \in \mathcal{G}_{n,t}} \kappa(x) \exp(\eta(\theta) \cdot \tau(x)) = \exp \zeta(\theta),
\]

where the second equality follows from equation (32) and the last from equation (30). □

D.4 Proof of Theorem 3.8

Proof. First we show dyadic independence. By the law of total probability,

\[
\mathbb{P}(W = w) = \sum_{x \in \mathcal{G}_{n,t}} \mathbb{P}(Z_1 = x_1, \ldots, Z_t = x_t) = \sum_{x \in \mathcal{G}_{n,t}} \prod_{i=1}^{t} \mathbb{P}(Z_i(f) = x_i(f))
\]

since \( Z \) is iid and dyadically independent.

To apply lemma D.1 on page 30 to swap the sum and product above, notice that \( \mathcal{T} := \mathcal{D}_n \) is a finite set. For each \( f \in \mathcal{D}_n \), take \( B_f \) to be the set of indicator vectors for the time periods \( \leq t \) at which \( f \) could enter the multigraph union: \( B_f := \{b \in \{0,1\}^t \mid \sum_{i=1}^{t} b_i = w(f)\} \). Further, take \( h_f : B_f \to \mathbb{R} \) such that \( h_f(b) = \prod_{i=1}^{t} \mathbb{P}(Z_i(f) = b_i) \). In the notation of lemma D.1, this makes \( \mathcal{F}_{\mathcal{D}_n} = \{x \in \mathcal{G}_{n,t} \mid \sum_{i=1}^{t} x_i = w\} \), so we may interchange the sum and the product as follows.

\[
\mathbb{P}(W = w) = \prod_{f \in \mathcal{D}_n} \sum_{b \in \{0,1\}^t} \prod_{i=1}^{t} \mathbb{P}(Z_i(f) = b_i)
\]

(34)

\[
= \prod_{f \in \mathcal{D}_n} \sum_{b \in \{0,1\}^t} \mathbb{P}(Z_1(f) = b_1, \ldots, Z_t(f) = b_t)
\]

(\( Z \) is iid)

\[
= \prod_{f \in \mathcal{D}_n} \mathbb{P} \left( \sum_{i=1}^{t} Z_i(f) = w(f) \right) = \prod_{f \in \mathcal{D}_n} \mathbb{P}(W(f) = w(f)).
\]

(Law of total prob.)

Therefore the multiplicities of each dyad in \( W \) are independent of each other.

To prove equation (24), let \( \mu_f := \mathbb{P}(Z_i(f) = 1) \) for each \( f \in \mathcal{D}_n \). Since \( Z \) is iid and dyadically independent,

\[
\mathbb{P}(Z = z) = \prod_{f \in \mathcal{D}_n} \prod_{i=1}^{t} \mathbb{P}(Z_i(f) = z_i(f)) = \prod_{f \in \mathcal{D}_n} \mu_f^{z(f)}(1 - \mu_f)^{t-z(f)}.
\]
Likewise, from equation (34) and the combinatorial definition of the binomial coefficient, we have

\[
\mathbb{P}(W = w) = \prod_{f \in \mathcal{D}_n} \sum_{b \in \{0,1\}^t} \prod_{i=1}^t \mathbb{P}(Z_i(f) = b_i) = \prod_{f \in \mathcal{D}_n} \sum_{b \in \{0,1\}^t} \mu_f^{w(f)}(1 - \mu_f)^{t - w(f)}
\]

\[
= \prod_{f \in \mathcal{D}_n} \left( \prod_{i=1}^t \mu_f^{w(f)}(1 - \mu_f)^{t - w(f)} \right) = \mathbb{P}(Z = z) \prod_{f \in \mathcal{D}_n} \left( \prod_{i=1}^t \mu_f^{w(f)}(1 - \mu_f)^{t - w(f)} \right).
\]