Black holes have no short hair

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Abstract

We show that in all theories in which black hole hair has been discovered, the region with non-trivial structure of the non-linear matter fields must extend beyond 3/2 the horizon radius, independently of all other parameters present in the theory. We argue that this is a universal lower bound that applies in every theory where hair is present. This no short hair conjecture is then put forward as a more modest alternative to the original no hair conjecture, the validity of which now seems doubtful.

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The canonical nature of stationary black holes, that is, the fact that they are completely specified by the conserved charges that can be measured at asymptotic infinity, (like mass, angular momentum and electric charge), was proven explicitly in Einstein vacuum theory, Einstein–Maxwell theory \([1]\), and also for various types of theories involving scalar and vector fields \([2]\). The belief that this canonical nature had a universal validity, came to be known as the no hair conjecture (NHC) \([3]\). This belief was based on the rigorous results mentioned above, and on the physical argument that suggested that all matter fields present in a black hole space time would eventually be either radiated to infinity, or “sucked” into the black hole, except when those fields were associated with conserved charges defined at asymptotic infinity.

The discovery of the “color black holes” (i.e. static black hole solutions in Einstein-Yang–Mills (EYM) \([4]\) theory, that require for their complete specification, not only the value of the mass, but also an additional integer that is not, however, associated with any conserved charge), came as a surprise, and it certainly forced us to reassess the status of the NHC.

The fact that these solutions were afterwards proven to be unstable, seemed to allow a resurrection of a more restrictive form of the NHC, which was, then, supposed to apply only to stable black holes. The validity of this new version of the NHC has become highly doubtful, to say the least, since the discovery of static black hole solutions that are linear-perturbation-stable in theories like Einstein-Skyrme (ES) \([5]\), and apparently also Einstein–non Abelian–Procca (ENAP) \([6]\), as discussed in \([7]\). Thus, it seems clear now that there is very little hope for the validity of such a form of the NHC \([8]\).

The lack of validity of the NHC, naturally gives rise to the question: What happened to the physical arguments put forward to support it? It seems clear now that the non-linear character of the matter content of the examples discussed plays an essential role: The interaction between the part of the field that would be radiated away and that which would be sucked in is responsible for the failure of the argument and, thus, for the existence of black hole hair.
On the other hand, this suggests that the non-linear behavior of the matter fields must be present both, in a region very close to the horizon (a region from which presumably the fields would tend to be sucked in) and in a region relatively distant from the horizon (a region from which presumably the fields would tend to be radiated away), with the self interaction being responsible for binding together the fields in these two regions.

We find convenient to introduce the term “Hairosphere” to refer to the loosely defined region where the non-linear behavior of the fields is present, in contrast to the asymptotic region where the behavior of the fields is dominated by linear terms in their respective equations of motion. A slightly more explicit characterization of this region will be given below, after the proof of our main result.

The purpose of this letter is to show a result that gives support to the heuristic argument mentioned above, by showing the existence of a lower bound for the size of the “Hairosphere”. We do this by proving a theorem that applies to all theories in which black hole hair has been found, and which states that this lower bound is parameter and theory independent, and has the universal value given by $3/2$ the horizon radius. Furthermore, the result also applies to black holes in theories whose matter content is any combination of the matter fields corresponding to those theories. Also, it is important to note that the existence of this lower bound is particularly interesting in the cases where the fields in the theory are massive (as in Einstein–Yang–Mills–Higgs (EYMH), ES, ENAP and Einstein–Yang–Mills–Dilaton (EYMD) with an additional potential term $\mathcal{L}$), as one could have naively expected that, by adjusting the mass parameters of the theory, one would obtain a black hole where the fields are substantially different from zero, only within a neighborhood of the horizon that could be made as small as one desires. Our result shows explicitly that this is not what happens.

The first clues about the existence of this bound were obtained in [10], where a procedure was presented, that allowed the construction of a Liapunov function in Einstein-Higgs (EH) theory, giving a proof of a no-hair-theorem. When applied to EYM theory, the same procedure yields instead a lower bound for the region of non-linear behavior of the YM field. The present analysis is motivated by the fact that the same procedure seems to be
applicable to all theories known to exhibit hair, and also by a new formulation [11] of the above mentioned no hair theorem in EH theory, that was suggestive of a way to treat all interesting cases in a unified fashion.

At this point, it seems convenient to define precisely what we call “hair”: We will say that, in a given theory, there is black hole hair when the space time metric and the configuration of the other fields of a stationary black hole solution are not completely specified by the conserved charges defined at asymptotic infinity. Thus, in the language of Ref. [12], we do not consider secondary hair.

We will focus on asymptotically flat static spherically symmetric black hole space–times and write the line element as

$$ds^2 = -e^{-2\delta} \mu \, dt^2 + \mu^{-1} \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \quad (1)$$

where $\delta$ and $\mu = 1 - 2m(r)/r$ are functions of $r$ only, and we assume that there is a regular event horizon at $r_H$, so $m(r_H) = r_H/2$, and $\delta(r_H)$ is finite. Asymptotic flatness requires, in particular, that $\mu \to 1$ and $\delta \to 0$, at infinity.

The matter fields will also respect the symmetries of the space-time, as is the case, in particular, for the specific ansatz employed in each of the cases where black hole hair has been discovered.

The Einstein’s equations, $G_{\mu \nu} = 8\pi T_{\mu \nu}$, together with the equations of motion for the field under consideration, form a dependent set, as they are related by the conservation equation $T^\mu_{\nu;\mu} = 0$. We will use only two of the

Einstein’s equations and this last equation.

The conservation equation has only one non–trivial component:

$$T^\mu_{\nu;\mu} = 0. \quad (2)$$

Einstein’s equations give

$$\mu' = 8\pi r \, T^t_t + \frac{1 - \mu}{r}, \quad (3)$$

$$\delta' = 4\pi r \, \frac{\mu}{\mu} \left( T^t_t - T^r_r \right), \quad (4)$$
where prime stands for differentiation with respect to $r$.

Using Einstein’s equations (3, 4) in (2) it is straightforward to obtain

$$e^\delta (e^{-\delta} r^4 T^r_r)' = \frac{r^3}{2 \mu} \left[ (3\mu - 1) (T^r_r - T^t_t) + 2\mu T \right],$$

(5)

where $T$ stands for the trace of the stress energy tensor.

We will be making the assumption that the matter fields satisfy the weak energy condition (WEC), that in our context means that the energy density, $\rho \equiv -T^t_t$, is positive semi-definite and that it bounds the pressures, in particular, $|T^r_r| \leq -T^t_t$.

Now we are in a position to obtain the following:

**Theorem:** Let equation (1) represent the line element of an asymptotically flat static spherically symmetric black hole space time, satisfying Einstein’s equations, with matter fields satisfying the WEC and such that the trace of the energy momentum tensor is non-positive, and such that the energy density $\rho$ goes to zero faster than $r^{-4}$, then the function $E = e^{-\delta} r^4 T^r_r$ is negative semi-definite at the horizon and is decreasing between $r_H$ and $r_0$ where $r_0 > \frac{3}{2} r_H$ and from some $r > r_0$, the function $E$ begins to increase towards its asymptotic value, namely 0.

**Proof:** First we note that the proper radial distance is given by $dx = \mu^{-\frac{1}{2}} dr$, so equation (3) can be written as

$$\frac{d}{dx}(e^{-\delta} r^4 T^r_r) = \frac{e^{-\delta} r^3}{2} \left[ \mu^\frac{1}{2} (3 (T^r_r - T^t_t) + 2 T) - \mu^{-\frac{1}{2}} (T^r_r - T^t_t) \right].$$

(6)

Since the components $T^r_r, T^t_t, T^\theta_\theta$ must be regular at the horizon, (i. e. the scalar $T^\mu_\nu T^{\mu\nu} = (T^r_r)^2 + (T^t_t)^2 + 2 (T^\theta_\theta)^2$ is regular at the horizon) and, since $x$ is a good coordinate at the horizon, the left hand side of equation (3) must be finite in the limit $r \to r_H$, and since $\mu(r_H) = 0$, we find:

$$T^r_r(r_H) = T^t_t(r_H) = -\rho(r_H) \leq 0,$$

(7)

so $E(r_H) \leq 0$. Next, inspecting equation (3), we note that the right hand side is negative definite, unless $(3\mu - 1) > 0$. This follows from the WEC, which requires $(T^r_r - T^t_t) > 0$, which in turn implies that $\rho(r_H)$ is negative.
and the assumption that \( T < 0 \). Thus \( E \) is a decreasing function at least up to the point
where \( 3 \mu - 1 \) becomes positive, and this occurs at \( r_1 = 3m(r_1) \), therefore \( r_0 > r_1 \). Since
\( m(r) \) is an increasing function, (as follows from the WEC and the fact that eq. (3) can be
written as \( m' = 4\pi r^2 \rho > 0 \)) we then have:

\[
    r_0 > 3m(r_1) > 3m(r_H) = \frac{3}{2}r_H. \tag{8}
\]

Q. E. D.

We see that, under the conditions of the theorem, the asymptotic behavior of the fields
can not start before \( r \) is sufficiently large since this behavior is characterized by the fact
that \( T_{rr} \) approaches zero, at least as \( r^{-4} \). And, in particular, in the asymptotic regime \( E \) is
not simultaneously negative and decreasing.

The physical significance of the behavior of the function \( E \) can be seen from the following
facts:

\( \delta \) From the inequality \( E\left(\frac{3}{2}r_H\right) < E(r_H) \leq 0 \), and from the negativity of the radial pressure,
\( T_{rr} \), it follows that

\[
    |T_{rr}\left(\frac{3}{2}r_H\right)| > \left(\frac{2}{3}\right)^4 |T_{rr}(r_H)| e^{(\delta(\frac{3}{2}r_H)-\delta(r_H))}. \tag{9}
\]

\( \delta \) From the identification \( T_{rr}(r_H) = T_{tt}(r_H) \) in Eq. (7), and using the formula [13]:

\[
    e^{-\delta(r_H)} = \frac{2r_H \kappa}{1 + 8\pi r_H^2 T_{tt}(r_H)}, \tag{10}
\]

where \( \kappa \) is the surface gravity of the black hole, we obtain:

\[
    |T_{rr}\left(\frac{3}{2}r_H\right)| > \left(\frac{2}{3}\right)^4 \frac{2r_H \kappa}{1 + 8\pi r_H^2 T_{tt}(r_H)} |T_{tt}(r_H)| e^{\delta(\frac{3}{2}r_H)}. \tag{11}
\]

Next, note that from Eq. (4) and the weak energy condition, it follows that \( \delta(r) > 0 \) (because
\( \delta(\infty) = 0 \)). Then, we conclude that:

\[
    |T_{rr}\left(\frac{3}{2}r_H\right)| > \left(\frac{2}{3}\right)^4 \frac{2r_H \kappa}{1 + 8\pi r_H^2 T_{tt}(r_H)} |T_{tt}(r_H)|. \tag{12}
\]

We would like to stress here that the bound (12) on the value of the radial pressure at
\( r = \frac{3}{2}r_H \) is expressed completely in terms of physical quantities evaluated at \( r_H \). Thus, we
interpret the theorem as stating that, under the conditions of its hypothesis, the matter fields start their asymptotic behavior at some $r > \frac{3}{2}r_H$, and thus that the “Hairosphere” must extend beyond this point.

It is however worth pointing out that, although Eq. (12) has a clear physical interpretation, it is probably not the best bound that can be put on the value of $T^r_r(\frac{3}{2}r_H)$, since there is no general lower bound on the surface gravity. Nevertheless, it seems clear that if the matter fields have affected the space-time structure in such a way as to dramatically alter the value of $\kappa$ (i. e. a Schwarzschild black hole has a value of $\kappa = \frac{1}{2r_H}$), then the hair can not be regarded as “tiny”. In fact, a much better bound on $T^r_r(\frac{3}{2}r_H)$ can be obtained by considering instead Eq. (9): Integrating Eq. (4) we find

$$\delta(\frac{3}{2}r_H) - \delta(r_H) = 4\pi \int_{r_H}^{\frac{3}{2}r_H} r \frac{(T^t_t - T^r_r)}{\mu} dr$$

one can then substitute the particular form of $T^t_t - T^r_r$ corresponding to the specific theory one is considering. The corresponding expressions are listed in table 1 for all theories known to exhibit hair, and since in all cases $T^t_t - T^r_r$ is proportional to $\mu$, we find that Eq. (13) leads to an explicit bound on $\delta(r_H) - \delta(\frac{3}{2}r_H)$ in terms of the maximal value of the fields in the interval $[r_H, \frac{3}{2}r_H]$. This in turn, when substitute $d$ in Eq. (13), leads to an explicit bound on $T^r_r(\frac{3}{2}r_H)$ in terms of those fields.

Another, perhaps more dramatic example of the physical significance of the result, is provided by the fact that, as a particular case [14], our theorem rules out the possibility of a realistic static shell (with finite thickness), made out of matter satisfying the WEC and the $T \leq 0$ condition, laying completely in the interval $[r_H, r_0]$, i. e. it can not be completely contained within the “Hairosphere”.

Next, we examine the relevance of the theorem. First we note that, in all theories where hair has been found (EYM, ES, EYMH, ENAP, and Einstein–Yang–Mills–Dilaton with or without and additional potential term [9]), the conditions of the theorem hold as shown in table 1, so our results apply to all these theories. Furthermore, the additivity of the
energy momentum tensor ensures that in a theory involving a collection of any number of these fields (with the same type of ansatz), the condition $T \leq 0$ will continue to hold. The requirement that $r^4 T_{rr}$ goes to zero at infinity, does not follow from asymptotic flatness, as can be seen from the fact that it is violated for example by the Reissner-Nordström solution in Einstein–Maxwell theory, but this is not a case where hair is present as there is an additional conserved charge that is needed to complete the specification of the solution. In fact, the charges defined at asymptotic infinity are associated with the $r^{-2}$ behavior of the fields and in general, the energy momentum tensor is at least quadratic in these fields. Thus, this last requirement seems to be the natural way to impose the condition that there are no extra charges associated with the fields.

It appears then, that a suitable definition of the “Hairosphere” could be to take it as the complement of the region where $r^4 T_{rr}$ monotonically approaches zero.

This set of results which have been obtained under the assumption of spherical symmetry, in part because all the cases in which hair has been discovered also involved this simplifying assumption, should, we believe, generalize to the stationary black hole cases where we expect that the ”Hairosphere” should also be characterized by the length $r_{Hair} = \frac{3}{2} \sqrt{\frac{A}{4\pi}}$ where $A$ is the horizon area.

In view of the evidence shown here, and based on the physical arguments described at the beginning of this letter, that suggest the existence of such a universal lower bound, we are lead to conjecture that for all stationary black holes in theories in which the matter content satisfies the WEC as well as the $T \leq 0$ condition, the “Hairosphere”, if it exists, must extend beyond the above mentioned distance. In short: *If a black hole has hair, then it can not be shorter than $3/2$ the horizon radius.*

It is interesting to note that if we take the natural expectation that in theories with massive fields, stationary configurations will correspond to fields that decrease rapidly within a Compton length of the horizon, i. e. that the ”Hairosphere” lies within $r_{hair} < \frac{1}{\text{mass}}$, and combine it with the result presented above, we obtain an upper bound for the size of hairy black holes, namely
\begin{equation}
\frac{r_H}{3} < \frac{r_{\text{hair}}}{3} < \frac{1}{3} \frac{\text{mass}}{\text{mass}}.
\end{equation}

Thus, big black holes will have no hair. Actually, the numerical investigations \cite{7} of all massive theories known to present hair show evidence for such an upper bound on the size of hairy black holes. This gives support to the view that the present \textit{no short hair} results can be taken as an alternative to the no hair conjecture.

The validity of this conjecture as well as the general rigorous definition of the "Hairosphere" should be matters of further research.

One of us, DS, wants to acknowledge helpful discussions with R. M. Wald.
| Theory         | $T$                                                                 | $T_l - T_r$                  | $T_l(r \to \infty) \sim$ |
|---------------|----------------------------------------------------------------------|----------------------------|------------------------------|
| Skyrme        | $-\frac{1}{16\pi}(f^2 \mu F^2 + \sin^2 F/e^2 r^4)$                  | $-\frac{\mu}{16\pi}(f^2 + 2\sin^2 F/e^2 r^2)F^2$ | $\frac{1}{r^6}$                     |
| YM            | 0                                                                    | $-\frac{\mu w^2}{2\pi f^2 r^2}$             | $\frac{1}{r^6}$                     |
| YMD + V       | $-\frac{1}{4\pi}(\mu \phi'^2 + 4V)$                                 | $-\frac{\mu}{4\pi}(\phi'^2 + 2\frac{w'^2}{r^2} e^2 \gamma \phi)$ | $\frac{1}{r^6}$ or $\frac{1}{r^{10}}$ |
| YMH           | $-\frac{1}{4\pi}[\mu \phi'^2 + 4V + \frac{f^2 \phi^2}{2r^2} (1 + w)^2]$ | $-\frac{\mu}{4\pi}(\phi'^2 + 2\frac{w'^2}{r^2})$ | $\frac{e^{-2m_r}}{r^2}$             |
| NAProca       | $-\frac{m^2}{8\pi r^2} (1 + w)^2$                                   | $-\frac{\mu w'^2}{2\pi f^2 r^2}$             | $\frac{e^{-2m_r}}{r^2}$             |

**Table 1.** The values of the trace, $T$, the difference $T_l - T_r$, and the asymptotic behavior of $T_l$, for the different theories known to have hair. The respective Lagrangian and ansatz are: For Skyrme $\mathcal{L}_S = \sqrt{-g} \left(-\frac{f^2}{4} \text{tr}(\nabla_{\mu}U\nabla^\mu U^{-1}) + \frac{1}{32\pi^2} \text{tr}[(\nabla_{\mu}U) U^{-1}, (\nabla_{\nu}U) U^{-1}]^2\right)$, where $\nabla_{\mu}$ is the covariant derivative, $U$ is the $SU(2)$ chiral field, and $f, e$ are the coupling constants. For $U$ we use the hedgehog ansatz $U(r) = e^{i(\sigma \cdot \mathbf{r}) F(r)}$ where $\sigma$ are the Pauli matrices and $\mathbf{r}$ is a unit radial vector. For Yang–Mills $\mathcal{L}_{YM} = -\frac{\sqrt{-g}}{16\pi f^2} F^{\mu \nu \alpha} F_{\mu \nu \alpha}$, where $f$ is the gauge coupling constant, we use the static spherically symmetric ansatz for the potential $A = \sigma_a A_{\mu}^a dx^\mu = \sigma_1 w d\theta + (\sigma_3 \cot \theta + \sigma_2 w) \sin \theta d\phi$, where $w$ is a function of $r$ only. For Yang-Mills–Dilaton with potential $\mathcal{L}_{YMD+V} = \mathcal{L}_{YM} e^{2\gamma \phi} + \frac{\sqrt{-g}}{4\pi} \left(\frac{1}{2} \nabla_{\mu} \phi \nabla^\mu \phi - V(\phi)\right)$, where $\gamma$ is
the dimensionless dilatonic coupling constant, the ansatz for the gauge field configuration is the same as that given in YM case, and $\phi = \phi(r)$. For Yang–Mills–Higgs $\mathcal{L}_{YMH} = \mathcal{L}_{YM} - \frac{g}{4\pi} \left( (D_\mu \Phi)^\dagger (D^\mu \Phi) + V(\Phi) \right)$, where $D_\mu$ is the usual gauge–covariant derivative, $\Phi$ is a complex doublet Higgs field; the ansatz for the Yang–Mills field is the same as before, and for the Higgs field we have $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \phi(r) \end{pmatrix}$, as usual $m = f < \Phi >$. Finally, for Proca $\mathcal{L}_{NAP} = \mathcal{L}_{YM} - \frac{g}{32\pi} m^2 A_\mu^a A_\mu^a$, where $m$ is the mass parameter and the ansatz is defined as in the YM case.
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