Characterization and quantification of symmetric Gaussian state entanglement through a local classicality criterion

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A necessary and sufficient condition for characterization and quantification of entanglement of any bipartite Gaussian state belonging to a special symmetry class is given in terms of classicality measures of one-party states. For Gaussian states whose local covariance matrices have equal determinants it is shown that separability of a two-party state and classicality of one party state are completely equivalent to each other under a nonlocal operation, allowing entanglement features to be understood in terms of any available classicality measure.

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I. INTRODUCTION

Reliable implementations of quantum communication protocols are constrained by the actual ability to manipulate quantum systems to encode both qubits (superpositions of mutually orthogonal states) and quantum channels (non-locally multiparty entangled states) of information. While pure qubits can be prepared under certain conditions on local operations, “good” (pure) quantum channels are hardly achieved due to quantum noise. Efforts have been devoted to both classify and to quantify entanglement in multipartite quantum systems [1], and to quantify the limiting bounds for efficiency of many communication protocols against mixing [2]. While entanglement for qubits systems is nowadays quite well understood, the existent quantifications of entanglement can hardly be calculated, even numerically, for continuous variable states (see [3] and references therein). It is of central importance thus to develop new scenarios of reasoning on entanglement in terms of accessible measures. Recent efforts in that direction include the Horodecki and Ekert [4] direct detection scheme for bipartite qubit states; the Kim et al. [5] proposal in terms of linear operations and homodyning detection for Gaussian continuous variable quantum states; and the P. Marian et al. [6] proposal in terms of Bures [7] distance measures of bipartite Gaussian states.

In this paper we outline an alternative approach to understand and quantify entanglement for Gaussian bipartite input quantum states in terms of one-party output states nonclassicality aspects. By input and output we refer to any bosonic system before and after, respectively, a bilinear Bogoliubov (beam-splitter similar) operation. This approach extends the applicability of a recent theorem providing that if the input state is classical, the output state of a beam splitter must be a separable state [8]. Remarkably, the inverse problem sets a necessary and sufficient condition for entanglement quantification of a special symmetry class of Gaussian states if no other local operations are made on the output ports. Since under the specified operation Gaussian measures for the bipartite input state are transformed into output one-party Gaussian measures, the degree of entanglement of the input can be quantified in terms of any available nonclassicality measure of the Gaussian type for one (or both) of the output ports. We address to the identification of the negativity of the output Glauber P-function [9] for classicality check, and exemplify the above relation by quantifying the entanglement of a two-mode thermal squeezed states in terms of the output one-party nonclassicality measure - the Bures distance from a one-mode squeezed state [6,10].

We begin by reviewing in Sec. II some properties of bipartite Gaussian states, including the necessary and sufficient conditions for the state to be separable. In Sec. III we analyze the classicality of an arbitrary bipartite Gaussian state under a nonlocal bilinear Bogoliubov operation. In Sec. IV we present the way to quantify entanglement in terms of the transformed output one-port classicality measure. In Sec. V we exemplify our discussion by quantifying the amount of entanglement of a two-mode thermal squeezed states by means of the output Bures distance from an one-mode squeezed state. Finally in Sec. VI a conclusion encloses the paper.

II. BIPARTITE GAUSSIAN STATES

A bipartite quantum state ρ is Gaussian if its symmetric characteristic function is given by $C(\eta) \equiv Tr[D(\eta)\rho] = e^{-\frac{1}{2}D(\eta)^{\dagger}D(\eta)}$, where $D(\eta) = e^{-\frac{1}{2}\eta^{\dagger}Ev}$ is a displacement operator in the parameter four-vector $\eta$-space: $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$, $v^\dagger = (a_1^+, a_1, a_2^+, a_2)$, and

$$E = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(1)
where \(a_1(a_1^\dagger)\) and \(a_2(a_2^\dagger)\) are annihilation (creation) operators for party 1 and 2, respectively. \(V\) is a \(4 \times 4\) covariance matrix with elements \(V_{ij} = (-1)^{i+j} \langle \{v_i, v_j^\dagger\} \rangle / 2\), which can be decomposed in four block \(2 \times 2\) matrices,

\[
V = \begin{pmatrix} V_1 & C \\ C^\dagger & V_2 \end{pmatrix} = \begin{pmatrix} n_1 & m_1 & m_s & m_c \\ m_1^* & n_1 & m_c^* & m_s^* \\ m_s & m_c & n_2 & m_2 \\ m_c^* & m_s^* & m_2^* & n_2 \end{pmatrix},
\]

where \(V_1\) and \(V_2\) are Hermitian matrices containing only local elements while \(C\) is the correlation between the two parties. Positivity and separability for bipartite Gaussian quantum states have been largely investigated [9,11,12]. Besides the requirement of positivity

\[
V + \frac{1}{2} E \succeq 0,
\]

which is nothing but the fundamental uncertainty principle, a necessary and sufficient condition for separability of Gaussian states is that they must also follow [12]

\[
\tilde{V} + \frac{1}{2} E \succeq 0,
\]

under a partial phase space mirror reflection \(\tilde{V} = TVT^\dagger\), where \(T = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}\), and \(X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

Through Schur block decomposition [12,13] the physical positivity criterion (3) applies only if

\[
V_1 + \frac{1}{2} Z \succeq 0, \quad \left( V_2 + \frac{1}{2} Z \right) - C^\dagger \left( V_1 + \frac{1}{2} Z \right)^{-1} C \succeq 0,
\]

which explicitly reads

\[
n_1 \geq \sqrt{|m_1|^2 + \frac{1}{4}},
\]

\[
n_2 \geq \frac{s}{d} + \sqrt{\frac{1}{4} \left[ \frac{|m_c|^2 - |m_s|^2}{d} - 1 \right]^2 + |m_2 - e|^2},
\]

respectively, with \(s = n_1 (|m_c|^2 + |m_s|^2) - m_c m_s m_1 + m_c^* m_s^* m_1, c = 2 n_1 m_s m_c - m_c^* m_s^* m_1, \) and \(d = n_1^2 - \frac{1}{4} - |m_1|^2\). Similarly the separability condition (4) writes explicitly into (7) and

\[
n_2 \geq \frac{s}{d} + \sqrt{\frac{1}{4} \left[ \frac{|m_c|^2 - |m_s|^2}{d} + 1 \right]^2 + |m_2 - e|^2}.
\]

### III. Classicality of the Output State

The above general criterion over the two party state observables can be translated into directly measurable quantities of one party under nonlocal operations. Let a Gaussian entangled state \(\rho_{in}\) be transformed under a nonlocal bilinear Bogolubov operation \(B\):

\[
\rho_{out} = B \rho_{in} B^\dagger, \quad B v B^\dagger = M v,
\]

\[
M = \begin{pmatrix} R & S \\ -S^* & R^* \end{pmatrix}
\]

\[
R = \cos \theta \begin{pmatrix} e^{i \phi_0} & 0 \\ 0 & e^{-i \phi_0} \end{pmatrix}, \quad S = \sin \theta \begin{pmatrix} 0 & e^{i \phi_1} \\ e^{-i \phi_1} & 0 \end{pmatrix}
\]

where \(\rho_{out}\) is the density operator for the joint output state. The output symmetric characteristic function is given by

\[
C_{out}(\eta) = Tr[\{D(\eta) \rho_{out}\}] = Tr[B^\dagger D(\eta) B \rho_{in}].
\]

Now with the help of Eqs.(11-13), \(B^\dagger D(\eta) B = e^{-\eta} \rho_{out} M \rho_{out}^{-1} \rho_{out} \equiv D(\zeta)\), with \(\zeta = M \eta\), since \(\rho_{out}\) is the density operator for the joint output state. The output symmetric characteristic function is given by

\[
C_{out}(\eta) = C_{in}(\zeta) = e^{-\frac{i}{2} \zeta^\dagger V \zeta} = e^{-\frac{i}{2} \eta^\dagger V' \eta},
\]

where \(V' = M^{-1} VM\), and analogously to (2), \(V'\) can be block decomposed with

\[
V_1' = R^* V_1 R + S V_2 S^* - SC^* R - R^* C S^*, \quad V_2' = S^* V_1 S + R V_2 R^* + RC^* S + S^* C^* R,
\]

allowing the following important results.

The output state is separable, such that \(C_{out}(\eta) = C_{out}(\eta_1) C_{out}(\eta_2)\), if and only if \(C' = 0\), which explicitly means

\[
sin 2 \theta \left( m_2 e^{i (\phi_0 + \phi_1)} - m_1 e^{-i (\phi_0 + \phi_1)} \right) - 2 \cos 2 \theta m_c = 0, \quad \sin 2 \theta e^{-i (\phi_0 - \phi_1)} (n_1 - n_2) + \cos 2 \theta (m_s e^{-2i \phi_0} + m_s^* e^{2i \phi_1}) + (m_s e^{-2i \phi_0} - m_s^* e^{2i \phi_1}) = 0.
\]

From now on we fix \(\theta = \pi/4\), setting the equivalence of the transformation \(M\) to that of an ideal 50:50 beamsplitter. Thus conditions (19) and (20) reduce to

\[
m_2 e^{i (\phi_0 - \phi_1)} - m_1 e^{-i (\phi_0 - \phi_1)} = 0, \quad e^{-i (\phi_0 - \phi_1)} (n_1 - n_2) + (m_s e^{-2i \phi_0} - m_s^* e^{2i \phi_1}) = 0.
\]

Although restrictive those conditions can be naturally reached for special classes of Gaussian states as we remark below.

**Remark 1:** Important examples of bipartite Gaussian states existent in Nature have a symmetric special form \(n_1 = n_2\), and \(m_1 = m_2 = m_s = 0\) [11,12] that satisfy conditions (21) and (22) automatically. Such is the case for the two-mode thermal squeezed state (TMTSS)
generated in a nonlinear crystal with internal noise. There \( n_1 = n_2 = n = gh_1 \), \( m_1 = m_2 = m = 0 \), and \( m_c = m = gh_2 \), with \( g = h_1 h_2 / (h_1^2 - h_2^2) \) and \( h_1 = [e^{-p_1} + d(2n + 1) ((1 - e^{-p_1})/p_1)] \), where \( p_1 = d + 2r \), and \( p_2 = d - 2r \). \( d = \gamma \) is a diffusion parameter with the relaxation constant \( \gamma \), and \( r = \kappa t \) is the squeezing parameter. \( \bar{n} \) is the mean number of thermal photons introduced by the quantum noise. Latter on we consider this specific situation.

Remark 2: Under a local \( Sp(2, R) \otimes Sp(2, R) \) operation (see \([11,12]\)) an arbitrary covariance matrix \( V \) can be transformed to the special form of Remark 1, without affecting the entanglement if, previously to the local operation, \( V \) is as such that the condition \( n_1^2 - |m_1|^2 = n_2^2 - |m_2|^2 \) applies. Those are Gaussian states whose local covariance matrices \( V_1 \) and \( V_2 \) have the same determinant, i.e., \( \det V_1 = \det V_2 \). We shall call this subset of Gaussian states the set of Gaussian states with Symmetric Local Determinants (SSLD). Thus \((21)\) and \((22)\) restrict the generality of our results to the class of Gaussian states belonging to the SSLD, satisfying \( n_1^2 - |m_1|^2 = n_2^2 - |m_2|^2 \). Naturally this last condition is less restrictive than the one required in \((22)\) and offers a wider range of states that can be covered than those of the special symmetric form of Remark 1. Hereafter we consider only this class of Gaussian states.

Under that condition, both the output state positivity and separability criteria on \( V' \) give equivalently \( V'_1 + \frac{1}{2} Z \geq 0 \) and \( V'_2 + \frac{1}{2} Z \geq 0 \), or explicitly

\[
n'_1 \geq \sqrt{|m'_1|^2 + \frac{1}{4}}, \quad n'_2 \geq \sqrt{|m'_2|^2 + \frac{1}{4}},
\]

in terms of the primed coefficients of the output variance matrix, related to the input coefficients by Eqs. \((16)-(18)\). By writing the inequalities \((23)\) explicitly in terms of the input coefficients we obtain the physical positivity condition \((8)\) and \((9)\) for the input bipartite state.

We are now in position to infer about a quantitative measure of the nature of nonseparable states through the properties of one of the \( M \)-transformed one-party output state. Observe that to any Gaussian nonclassical output state, satisfying \( \sqrt{|m'_i|^2 + \frac{1}{4}} \leq n'_i < |m'_i| + \frac{1}{2}, \ i = 1 \) or \( 2 \), there is a one-to-one corresponding input bipartite mixed entangled Gaussian state belonging to the SSLD. An immediate consequence is that any available quantitative Gaussian measure (a measure that preserves the Gaussian structure of the state) of nonclassicality of one of the one-party output states can be regarded as a quantitative Gaussian measure for the degree of entanglement of the bipartite input state. The veracity of this statement rests on the following proposition:

**Proposition 1:** Let \( \rho \) be a bipartite input Gaussian state of the SSLD and \( \rho'_1 \otimes \rho'_2 \) a \( M \)-transformed separable output state. In terms of Gaussian measures, the less classical are the \( M \)-transformed output states, \( \rho'_1 \) or \( \rho'_2 \), the more entangled is \( \rho \).

To prove proposition 1, an important relation between bipartite and one-party state Gaussian measures will be derived.

**Lemma 1:** Let \( A = Tr[f(A, \rho)] \) be the effect of the measure \( f(A, \cdot) \) over the Gaussian state \( \rho \), being \( f(A, \cdot) \) an analytic function of a trace class Gaussian operation \( A \) and \( \rho \). Any nonlocal Gaussian measure \( f(A, \cdot) \) of a bipartite Gaussian state \( \rho_{in} \) can be converted into local Gaussian measures \( A = Tr[f(A_{out}^{(1)}, \rho_{out}^{(1)})], Tr[f(A_{out}^{(2)}, \rho_{out}^{(2)})] \) of the two one-party output states by nonlocal rotations \( M \).

**Lemma 2:** Any local Gaussian measure of a bipartite separable Gaussian state: \( Tr_1[f(A_{out}^{(1)}, \cdot)] \), or \( Tr_2[f(A_{out}^{(2)}, \cdot)] \), can be converted into a nonlocal Gaussian measure of a nonseparable bipartite Gaussian state under a reduction and an appropriate nonlocal rotation \( M^{-1} \).

**Proof of Lemma 1:** Since unitary transformations do not alter the effect content it is immediate that \( A = Tr[f(BAB' B_{out}' B') \] where \( B \) is the Gaussian characteristic function associated to \( A \) [15] such that \( M \) acts on \( \Gamma \) and \( V \) in similar fashion (Eqs. \((16)-(18)\)). Thus \( A = Tr[f(A_{out}, \rho_{out})], \) where \( A_{out} \) is the output Gaussian operation with covariance matrix \( \Gamma' = M^{-1} \Gamma M \). With the conditions \((21)\) and \((22)\) to bring \( V' \) and \( \Gamma' \) to the (separable) block-diagonal form satisfied, it is immediate that

\[
A = Tr_1[f(A_{out}^{(1)}, \rho_{out}^{(1)})] Tr_2[f(A_{out}^{(2)}, \rho_{out}^{(2)})].
\]
Proof of Lemma 2: The explicit Gaussian forms of $f(A_j^{(i)})$ and $A_j^{(i)}$, itself, are only given when the output measurement is specified, but Lemma 1 must hold even for the situation where the measurement in either of the output ports is operationally equivalent to a reduction: $Tr_2[f(A_2^{(i)}; \rho_{out}^{(i)})] = 1$ or $Tr_1[f(A_1^{(i)}; \rho_{out}^{(i)})] = 1$, exclusively, such that

$$A = Tr_i[f(A_{out}^{(i)}; \rho_{out}^{(i)})], \ (i = 1, \ or \ 2),$$

which by definition is equal to $Tr[f(A, \rho_{in})]$. The input effect can be computed through the corresponding characteristic function of $f(A_j^{(i)}, \rho)$, which by hypothesis is Gaussian, and the proof is completed in similar fashion to the proof of Lemma 1. ■

Proof of Proposition 1: From Lemma 1 and 2, it is immediate that an entanglement quantification via a one-party nonclassicality measure is possible, whenever both the output and input measures are Gaussian and correspond, respectively, to nonclassicality and entanglement quantifications. An important measure falling inside this description is the Uhlmann Fidelity [7] between any two Gaussian states $\rho$ and $\sigma$: $F(\rho, \sigma) = \{Tr[(\sqrt{\rho} \sqrt{\sigma})^{1/2}]^2\}$ which is central for the calculation of the Bures distance [7]: $d_B(\rho, \sigma) = (2 - 2F(\rho, \sigma))$. The Bures distance was recently identified as a quantification for nonclassicality [10] and entanglement [1,6], for one and two parties states, respectively, when $d_B(\rho, \sigma)$ is minimized over the possible referential $\sigma$ belonging to the Gaussian subset of states. In fact from and Lemma 1 it is immediate that $F(\rho_{in}, \sigma_{in}) = F(\rho_{out}, \sigma_{out})F(\rho_{out}, \sigma_{out})$ and the input-output Bures metrics are related as

$$d_B(\rho_{in}, \sigma_{in}) = \frac{1}{2}d_B(\rho_{out}, \sigma_{out})d_B(\rho_{out}, \sigma_{out}). \quad (26)$$

Inversely, if the measurement is made on the port $i$ only, meaning that $\sigma_{out} = \rho_{out}^{(j)}$, $j = i - (1)^{i}$, $(i = 1, \ 2)$, then from Lemma 2, $F(\rho_{out}^{(i)}, \sigma_{out}^{(i)}) = F(\rho_{in}, \sigma_{in})$, and

$$d_B(\rho_{out}^{(i)}, \sigma_{out}^{(i)}) = d_B(\rho_{in}, \sigma_{in}). \quad (27)$$

Here the corresponding $\sigma_{in}$ must necessarily be $B_{i}^{(i)}\sigma_{out}^{(i)}B_{i}$. Both situations reflect the direct connection between two measures of classicality (distance from a reference classical state [6]) and entanglement (distance from a reference separable state [10]), and thus the observance of Proposition 1. ■

V. EXAMPLE

Next we illustrate the above discussion for the TMTSS. To simplify our calculations, instead of using the Bures distance directly, we introduce a simple pictorial entanglement measure $E(\rho, \sigma) = 1 - d_B(\rho, \sigma)/d_B(\rho_{sep}, \sigma)$ (see [6]), where $\sigma$ is a maximally entangled (pure) Gaussian state represented by the equality in (23). $\rho_{sep}$ is the separable density operator, introduced to give $E(\rho, \sigma) = 0$ at the separability boundary, and obtained simply by tracing out one of the parties in $\sigma$. Notice that a $\sigma$ pure does not allows that $E(\rho, \sigma)$ be an entanglement monotone (see [1,6]), once it can always be increased by appropriate local operations on $\rho$. But this choice suffices for our illustrative purpose, simplifying the calculations, since the Uhlmann fidelity is simply given by $F(\rho, \sigma) = Tr(\rho\sigma)$. As such for one party systems the nonclassicality measure is related to the distance from a pure nonclassical one-mode squeezed state, $\sigma_{out}^{(i)} = |\psi_i\rangle\langle\psi_i|$:

$$|\psi_i\rangle = e^{-r(a^{(i)}_1 - a^{(i)}_2)}|0\rangle = \sqrt{1 - \lambda^2} \sum_{k=0}^{\infty} \lambda^{|k|}, \quad (28)$$

while for bipartite systems the entanglement measure is related to the distance from a pure two-mode squeezed state, $\sigma = |\Psi\rangle\langle\Psi|$:

$$|\Psi\rangle = e^{-r(a^{(i)}_1 - ab)}|0, 0\rangle = \sqrt{1 - \lambda^2} \sum_{k=0}^{\infty} \lambda^{|k|, k}, \quad (29)$$

both in absence of noise and with $\lambda = \tanh r$, $n = \cosh 2r/2$, $m = -\sinh 2r/2$, and $r$ the squeezing parameter. Those states maximize our pictorial measures (minimize the Bures distance) for $\lambda \to \infty$ corresponding to irreducible maximal nonclassical and entangled states, respectively.

A nonseparable TMTSS generated in a nonlinear crystal [14] is bounded by $m + \frac{1}{2} > n > \sqrt{m^2 + \frac{1}{4}}$. Now if the two parametric-down-converted beams are mixed at an ideal (lossless) 50 : 50 beam-splitter $(\theta = \pi/4)$, the output port nonclassicality condition of either beams reads $n' < m' + \frac{1}{2}$, which converted into the input parameters corresponds to the input state nonseparability bound. The output $i$-port Uhlmann fidelity is given by

$$F(\rho_{out}^{(i)}, \sigma_{out}^{(i)}) = \left[\frac{1}{r_i^2 - m'_i^2 + n'_i \cosh 2r + m'_i \sinh 2r + \frac{1}{4}}\right]^{-1/2}, \quad (30)$$

which is precisely the Fidelity between the TMTSS and the pure two mode squeezed state (without primes). Both the separability boundary and $E(\rho, \sigma)$ of the two-parametric-down-converted fields are simply given by output local classicality measures based on the Bures distance, which can be achieved by homodyning one of the output ports to a pure squeezed state of reference, as fully described in [5]. In Fig. 1 we plot $E(\rho, \sigma)$ in the $(n, m)$ parameter space for $r = 1$, together with the separability limiting bound (dashed line), as given by the corresponding nonclassicality measure. Notice that although pictorial and simplified the introduced measure is also able to quantify the amount of entanglement in pure states in relation to the maximally entangled pure state, obtained for $r \to \infty$. 4
VI. CONCLUSION

We have given a simple direct relation between entanglement measures for the special class of Gaussian input quantum states belonging to the SSLD in terms of the output states nonclassicality. Under nonlocal rotations both the separability boundary and quantification of the presented amount of entanglement are equivalent to one-party classicality (P-representability) boundary and to a nonclassicality quantification, respectively. It would be certainly interesting to analyze the amount of information about entanglement that can be obtained by appropriate inverse transformation of any other available one-party classicality criteria (such as the hierarchical measure of classicality [16]). Our preliminary results show that the so established inequalities are weaker than the above separability ones, in the same sense that any other classicality criterion is weaker than the P-representability one. This point is left for further investigation.

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Fig. 1. Degree of entanglement $\mathcal{E}(\rho, \sigma)$ for $(r = 1)$-TMTSS.