ABSTRACT. We construct a model structure on the category of cubical sets with connections whose cofibrations are the monomorphisms and whose fibrant objects are defined by the right lifting property with respect to inner open boxes, the cubical analogue of inner horns. We show that this model structure is Quillen equivalent to the Joyal model structure on simplicial sets via the triangulation functor. As an application, we show that cubical quasicategories admit an elegant and canonical notion of a mapping space between two objects.

INTRODUCTION

The category $sSet$ of simplicial sets carries two canonical model structures: the Kan-Quillen model structure [Qui67], presenting the homotopy theory of $\infty$-groupoids, and the Joyal model structure [Joy09], presenting the homotopy theory of $(\infty, 1)$-categories. Both of these model structures have monomorphisms as their cofibrations and their fibrant objects are defined by a more or less restrictive lifting condition, depending on whether or not the 1-simplices of a fibrant object are supposed to be invertible.

The category $cSet$ of cubical sets is also known to carry a model structure, called the Grothendieck model structure, constructed by Cisinski [Cis06, Cis14], presenting the theory of $\infty$-groupoids. This model structure is completely analogous to the Kan-Quillen model structure, but with open boxes replacing horns in the definition of fibrant objects. The goal of the present work is provide a cubical analogue of the Joyal model structure, thus filling the bottom right corner in the table:

| category \ theory | $\infty$-groupoids | $(\infty, 1)$-categories |
|-------------------|---------------------|-------------------------|
| $sSet$            | [Qui67]             | [Joy09]                 |
| $cSet$            | [Cis14]             | present work            |

Our main theorem (cf. Theorem 4.1 and Proposition 4.15 and Theorem 5.1) states

**Theorem.** The category $cSet$ of cubical sets carries a model structure in which:

- the cofibrations are the monomorphisms;
- the fibrant objects are defined by having fillers for all inner open boxes.

Moreover, this model structure is Quillen equivalent to the Joyal model structure on the category $sSet$ of simplicial sets via the triangulation functor $T: cSet \to sSet$. 
A few comments are in order.

First, there are many different notions of a cubical set, depending on the choice of maps in the indexing category $\Box$, called the box category. Here, we are working with the cubical sets with connections (specifically the max-connection), as studied in [Cis14, Mal09, KLV19]. The category of combinatorial cubes with connections is both an EZ-Reedy category and a strict test category, which makes it convenient to work with. Either more restrictive or more lenient choices of maps in the box category (such as the ones studied previously by Cisinski/Jardine [Cis06, Jar06] or in the recent work of Coquand and his group [CCHM18] on cubical type theory) result in a loss of some of these convenient properties.

It is also exactly the category of cubical sets with connections that was recently shown [KLV19] to admit a co-reflective embedding of the category of simplicial sets via the straightening-over-the-point functor $Q : s\text{Set} \to c\text{Set}$, an instance of a more general construction of straightening, studied in [KV18]. And despite its perhaps less clear definition, $Q$ ends up being much easier to work with than the triangulation functor. Indeed, in order to show that $T$ is a Quillen equivalence, we first prove it about $Q$ and establish that the derived functors of $T$ and $Q$ are each other’s inverses.

Lastly, the concept of an inner open box appearing in the statement of our main theorem is the cubical analogue of the notion of an inner horn in simplicial sets. Its definition is somewhat subtle, which is the reason behind our taking a slight detour in the construction of the model structure on cubical sets. At this point however, we shall simply note this subtlety here and give a precise definition in Section 4.

In order to establish a model structure on $c\text{Set}$, we consider first a model structure on marked cubical sets. A marked cubical set is a cubical set with a distinguished subset of edges (to be thought of as “equivalences”), containing all degenerate ones. We then use the minimal marking functor, taking a cubical set to a marked cubical set in which the marked edges are precisely the degeneracies, to left-induce a model structure on cubical sets.

In order to establish that the triangulation functor is a Quillen equivalence between our model structure on cubical sets and the Joyal model structure on simplicial sets, we introduce a cubical theory of cones, which generalizes the straightening-over-the-point construction. Our cubical cones serve as a convenient way of relating simplicial and cubical shapes, and we believe that these tools will find applications beyond present work.

This paper is organized as follows. In Section 1, we collect the necessary results on model categories, cubical sets, and marked cubical sets. Trying to keep the exposition as self-contained as possible, we included statements of frequently used results and those that may be harder to find in the existing literature.

In Section 2 we construct the model structure on the category of marked cubical sets, using Jeff Smith’s theorem. Then, in Section 3 we show that it is right-induced by a model structure on the category of structurally marked cubical sets constructed using the Cisinski theory.

In Section 4 we use the minimal marking functor to construct the desired model structure on the category of cubical sets. We then analyze the resulting classes of maps, characterizing weak equivalences and fibrations between fibrant objects, and construct the mapping space between two 0-cubes in a fibrant object.
Finally, in Section 5 we develop the theory of cones and use it to show that our model structure is Quillen equivalent with the Joyal model structure. This last argument is fairly combinatorial and includes a number of routine computations involving cubical identities. For the clarity of exposition, most of these computations are therefore relegated to appendix A to be verified only by the most masochistic of the readers.

Introduction

1. Cubical sets and marked cubical sets
2. Model structure on marked cubical sets
3. Model structure on structurally marked cubical sets
4. Joyal model structure on cubical sets
5. Comparison with the Joyal model structure

Appendix A. Verification of identities on \( \theta \)

References

1. Cubical sets and marked cubical sets

1.1. Model categories. Here we will review various general results from the theory of model categories which we will use throughout subsequent sections. We begin with a result which allows us to construct model structures having specified classes of cofibrations and weak equivalences.

**Theorem 1.1** (Jeff Smith’s Theorem, [Bar10, Prop. 2.2]). Let \( C \) be a locally presentable category. Let \( W \) be a class of morphisms forming an accessibly embedded, accessible subcategory of \( C \rightarrow \), and \( I \) a set of morphisms in \( C \). Suppose that the following conditions are satisfied.

- \( W \) satisfies the two-out-of-three axiom.
- \( W \) contains all maps having the right lifting property with respect to the maps in \( I \).
- The intersection of \( W \) with the saturation of \( I \) is closed under pushouts and transfinite composition.

Then \( C \) admits a cofibrantly generated model structure with weak equivalences \( W \) and generating cofibrations \( I \).

Next we review some of the machinery of Cisinski theory [Cis06], which allows for the easy construction of model structures on presheaf categories having monomorphisms as cofibrations and weak equivalences defined in terms of homotopy with respect to a cylinder functor.

**Definition 1.2.** Let \( C \) be a small category. A **cylinder functor** on \( C \) consists of an endofunctor \( I \) on the presheaf category \( \text{Set}^{C^{\text{op}}} \), together with natural transformations \( \partial^0, \partial^1 : \text{id} \rightarrow I, \sigma : I \rightarrow \text{id} \), such that:

- \( \partial^0 \) and \( \partial^1 \) are sections of \( \sigma \);
- For all \( X : C^{\text{op}} \rightarrow \text{Set} \), the map \( (\partial^0_X, \partial^1_X) : X \sqcup X \rightarrow IX \) is a monomorphism;
- \( I \) preserves small colimits and monomorphisms;
• For all monomorphisms \( j: X \to Y \) in \( \text{Set}^{C^{\text{op}}} \) and all \( \varepsilon \in \{0, 1\} \), the following square is a pullback:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \varphi^\varepsilon & & \downarrow \varphi^\varepsilon \\
IX & \xrightarrow{Ij} & IY
\end{array}
\]

In what follows, let \( C \) be a small category equipped with a cylinder functor \( I: \text{Set}^{C^{\text{op}}} \to \text{Set}^{C^{\text{op}}} \).

**Definition 1.3.** Let \( f, g: X \to Y \) be maps of presheaves on \( C \). An elementary homotopy from \( f \) to \( g \) is a map \( H: IX \to Y \) such that \( H\varphi^0 = f, H\varphi^1 = g \). A homotopy is a zig-zag of elementary homotopies. The set \([X, Y]\) is the set of maps from \( X \) to \( Y \) modulo the relation of homotopy.

It is easy to see that pre- and post-composition by a fixed map preserve the relation of homotopy; thus a map \( X \to Y \) induces maps \([Z, X] \to [Z, Y]\) and \([Y, Z] \to [X, Z]\) for any \( Z \).

**Definition 1.4.** A cellular model for \( \text{Set}^{C^{\text{op}}} \) is a set \( M \) of monomorphisms in \( \text{Set}^{C^{\text{op}}} \) whose saturation is precisely the class of monomorphisms of \( \text{Set}^{C^{\text{op}}} \).

Let \( M \) be a cellular model for \( \text{Set}^{C^{\text{op}}} \), and \( S \) a set of monomorphisms in \( \text{Set}^{C^{\text{op}}} \). The set of morphisms \( \Lambda(S) \) is defined by the following inductive construction. For a monomorphism \( X \to Y \) in \( \text{Set}^{C^{\text{op}}} \) and \( \varepsilon \in \{0, 1\} \), let \( IX \cup_\varepsilon Y \) and \( IX \cup (Y \sqcup Y) \) be defined by the following pushout squares:

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi^\varepsilon} & Y \\
\downarrow \varphi^\varepsilon & & \downarrow \varphi^\varepsilon \\
IX & \xrightarrow{\varphi^\varepsilon} & IX \cup_\varepsilon Y \\
\downarrow \varphi^\varepsilon & & \downarrow \varphi^\varepsilon \\
IX & \xrightarrow{\varphi^\varepsilon} & IX \cup (Y \sqcup Y)
\end{array}
\]

We now define a set of monomorphisms \( \Lambda(S) \) by an inductive construction. We begin by setting:

\[
\Lambda^0(S) = S \cup \{ IX \cup_\varepsilon Y \to IY | X \to Y \in M, \varepsilon \in \{0, 1\} \}
\]

Now, given \( \Lambda^n(S) \), we define:

\[
\Lambda^{n+1}(S) = \{ IX \cup (Y \sqcup Y) \to IY | X \to Y \in \Lambda^n(S) \}
\]

Finally, we let \( \Lambda(S) = \bigcup_{n \geq 0} \Lambda^n(S) \). We now define several distinguished classes of maps and objects in \( \text{Set}^{C^{\text{op}}} \).

• A cofibration is a monomorphism; a trivial fibration is a map having the right lifting property with respect to the cofibrations.
• An anodyne map is a map in the saturation of $\Lambda(S)$; a naive fibration is a map having the right lifting property with respect to the anodyne maps.

• A fibrant object is a presheaf $X$ such that the map from $X$ to the terminal presheaf is a naive fibration.

• A weak equivalence is a map $X \to Y$ such that the induced map $[Y,Z] \to [X,Z]$ is a bijection for any fibrant $Z$.

• A trivial cofibration is a map which is both a cofibration and a weak equivalence; a fibration is a map having the right lifting property with respect to the trivial cofibrations.

**Theorem 1.5.** The classes above define a cofibrantly generated model structure on $\text{Set}^{C^{\text{op}}}$, in which a map between fibrant objects is a fibration if and only if it is a naive fibration.

**Proof.** The existence of the model structure is [Cis06, Thm. 1.3.22]; the characterization of fibrant objects is [Cis06, Thm. 1.3.36].

**Corollary 1.6.** The homotopy category of $\text{Set}^{C^{\text{op}}}$ with a model structure of Theorem 1.5 can be described as follows:

• its objects are the fibrant presheaves;

• the maps from $X$ to $Y$ are given by $[X,Y]$. □

**Example 1.7.** Let $J$ denote the simplicial set depicted below:

```
1 ----> 0
|     | 1 ----> 0
|      v
1 ----> 0
```

Taking the product with $J$ defines a cylinder functor on $s\text{Set}$, with the natural transformations $\partial^0, \partial^1$ given by taking the product with the endpoint inclusions $\{0\} \hookrightarrow J, \{1\} \hookrightarrow J$. Applying Theorem 1.5 with this cylinder functor, the cellular model $M = \{\partial\Delta^n \to \Delta^n | n \geq 0\}$, and $S = \{\Lambda_i^m | n \geq 2, 1 < i < n\}$ (the set of inner horn inclusions), we obtain the Joyal model structure on $s\text{Set}$, characterized as follows:

• Cofibrations are monomorphisms;

• Fibrant objects are quasicategories, simplicial sets having fillers for all inner horns;

• Fibrations between fibrant objects are characterized by the right lifting property with respect to the inner horn inclusions and the endpoint inclusions $\{\varepsilon\} \hookrightarrow J, \varepsilon \in \{0,1\}$;

• Weak equivalences are maps $X \to Y$ inducing bijections $[Y,Z] \to [X,Z]$ for all quasicategories $Z$.

For more on the Joyal model structure, see [Joy09]; for the details of its construction as a Cisinski model structure, see [Cis19, Sec. 3.3].

Next we review a theorem which allows us to induce one model structure from another using an adjunction between their respective categories.
**Definition 1.8.** Let $F : C \leftrightarrows D : U$ be an adjunction between model categories. The model structure on $C$ is **left induced** by $F$ if $F$ preserves and reflects cofibrations and weak equivalences. Likewise, the model structure on $D$ is **right induced** by $U$ if $U$ preserves and reflects weak equivalences and fibrations.

**Remark 1.9.** Note that for a given adjunction $C \leftrightarrows D$ and a given model structure on $D$, the left-induced model structure is unique, if one exists, since the definition determines the cofibrations and weak equivalences of $C$. Likewise, for a given model structure on $C$, the right-induced model structure is unique, if one exists.

**Theorem 1.10** ([HKRS17, Thm. 2.2.1]). Let $F : C \leftrightarrows D : U$ be an adjunction between locally presentable categories such that $D$ carries a cofibrantly generated model structure with all objects cofibrant. If, for every object $X \in C$, the co-diagonal map admits a factorization $X \sqcup X \overset{F \times F}{\longrightarrow} I X \overset{p}{\longrightarrow} X$, such that $F \times X$ is a cofibration and $F p X$ is a weak equivalence, then $C$ admits a model structure left-induced by $F$ from that of $D$. □

Finally, we review some results which allow us to easily recognize Quillen adjunctions and Quillen equivalences.

**Proposition 1.11** ([JT07, Prop. 7.15]). Let $F : C \leftrightarrows D : U$ be an adjunction between model categories. If $F$ preserves cofibrations and $U$ preserves fibrations between fibrant objects, then the adjunction is Quillen. □

This statement has an immediate corollary, which we will apply in practice:

**Corollary 1.12.** Let $F : C \rightarrow D$ be a left adjoint between model categories and suppose that fibrations between fibrant objects in $C$ are characterized by right lifting against a class $S$. If $F$ preserves cofibrations and sends $S$ to trivial cofibrations, then $F$ is a left Quillen functor. □

**Proposition 1.13** ([Hov99, Cor. 1.3.16]). Let $F : C \leftrightarrows D : U$ be a Quillen adjunction between model categories. Then the following are equivalent.

(i) $F \dashv U$ is a Quillen equivalence.

(ii) $F$ reflects weak equivalences between cofibrant objects and, for every fibrant $Y$, the derived counit $F \tilde{U}Y \rightarrow Y$ is a weak equivalence.

(iii) $U$ reflects weak equivalences between fibrant objects and, for every cofibrant $X$, the derived unit $X \rightarrow U(FX)'$ is a weak equivalence.

Again, in practice we will often apply the following corollary:

**Corollary 1.14.** Let $F : C \leftrightarrows D : U$ be a Quillen adjunction between model categories.

(i) If $U$ preserves and reflects weak equivalences, then the adjunction is a Quillen equivalence if and only if, for all cofibrant $X \in C$, the unit $X \rightarrow UF X$ is a weak equivalence.

(ii) If $F$ preserves and reflects weak equivalences, then the adjunction is a Quillen equivalence if and only if, for all fibrant $Y \in D$, the counit $FUY \rightarrow Y$ is a weak equivalence. □
1.2. The box category and cubical sets. We begin by defining the box category \( \square \). The objects of \( \square \) are posets of the form \([1]^n\) and the maps are generated (inside the category of posets) under composition by the following three special classes:

- **faces** \( \partial_{i,\varepsilon}^n : [1]^{n-1} \to [1]^n \) for \( i = 1, \ldots, n \) and \( \varepsilon = 0, 1 \) given by:
  \[
  \partial_{i,\varepsilon}^n(x_1, x_2, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1});
  \]

- **degeneracies** \( \sigma_i^n : [1]^n \to [1]^{n-1} \) for \( i = 1, 2, \ldots, n \) given by:
  \[
  \sigma_i^n(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n);
  \]

- **connections** \( \gamma_i^n : [1]^n \to [1]^{n-1} \) for \( i = 1, 2, \ldots, n-1 \) given by:
  \[
  \gamma_i^n(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \max\{x_i, x_{i+1}\}, x_{i+2}, \ldots, x_n).
  \]

These maps obey the following co-cubical identities:

\[
\begin{align*}
\partial_{j,\varepsilon} \partial_{i,\varepsilon'} &= \partial_{i+1,\varepsilon'} \partial_{j,\varepsilon} \quad \text{for} \quad j \leq i; \\
\sigma_i \sigma_j &= \sigma_j \sigma_{i+1} \quad \text{for} \quad j \leq i; \\
\sigma_j \partial_{i,\varepsilon} &= \begin{cases} 
\partial_{i-1,\varepsilon} \sigma_j & \text{for} \quad j < i; \\
\text{id} & \text{for} \quad j = i; \\
\partial_{i,\varepsilon} \sigma_{j-1} & \text{for} \quad j > i; 
\end{cases} \\
\gamma_j \gamma_{i+1} &= \gamma_{i} \gamma_j 
\end{align*}
\]

\[
\begin{align*}
\gamma_j \partial_{i,\varepsilon} &= \begin{cases} 
\partial_{i-1,\varepsilon} \gamma_j & \text{for} \quad j < i - 1; \\
\text{id} & \text{for} \quad j = i - 1, \ varepsilon = 0; \\
\partial_{i,\varepsilon} \gamma_j & \text{for} \quad j = i - 1, \ varepsilon = 1; \\
\partial_{i,\varepsilon} \gamma_{j-1} & \text{for} \quad j > i; 
\end{cases} \\
\gamma_j \gamma_i &= \begin{cases} 
\gamma_{i-1} \sigma_j & \text{for} \quad j < i; \\
\sigma_i & \text{for} \quad j = i; \\
\gamma_i \sigma_{j+1} & \text{for} \quad j > i. 
\end{cases}
\end{align*}
\]

**Theorem 1.15** ([GM03 Thm. 5.1]). Every map in the category \( \square \) can be factored uniquely as a composite

\[
(\partial_{k_1,\varepsilon_1} \cdots \partial_{k_t,\varepsilon_t})(\gamma_{j_1} \cdots \gamma_{j_s})(\sigma_{i_1} \cdots \sigma_{i_u}),
\]

where \( i_1 > \ldots > i_r \geq 1, 1 \leq j_1 < \ldots < j_s, \) and \( k_1 > \ldots > k_t \geq 1. \)

**Corollary 1.16.** \( \square \) admits the structure of a Reedy category, in which:

- \( \text{deg}([1]^n) = n; \)
- \( \square_+ \) is generated under composition by the face maps;
- \( \square_- \) is generated under composition by the degeneracy and connection maps.

The category of cubical sets, i.e., contravariant functors \( \square^{op} \to \text{Set} \) will be denoted by \( \text{cSet} \). We will write \( \square^n \) for the representable cubical set, represented by \([1]^n\). We adopt the convention of writing the action of cubical operators on the right. For instance, the \((1,0)\)-face of an \( n \)-cube \( x : \square^n \to X \) will be denoted \( x \partial_{1,0} \).

We write \( \partial \square^n \to \square^n \) for the maximal proper subobject of \( \square^n \), i.e., the union of all of its faces. We will refer to these as the \( n \)-box and the boundary of the \( n \)-box, respectively. The subobject of \( \square^n \) given by the union of all faces except \( \partial_{i,\varepsilon} \) will be denoted \( \cap_{i,\varepsilon}^n \) and referred to as an \((i,\varepsilon)\) open box.

**Definition 1.17.** The **critical edge** of \( \square^n \) with respect to a face \( \partial_{i,\varepsilon} \) is the unique edge of \( \square^n \) which is adjacent to \( \partial_{i,\varepsilon} \) and which, together with \( \partial_{i,\varepsilon} \), contains both of the vertices \((0,\ldots,0)\) and \((1,\ldots,1)\).
More explicitly, the critical edge with respect to $\partial_i^c$ corresponds to the map $f: [1] \to [1]^n$ given by $f_i = \text{id}_{[1]}$, $f_j = \text{const}_{1-\varepsilon}$ for $j \neq i$.

The assignment $([1]^m, [1]^n) \mapsto [1]^{m+n}$ defines a functor $\square \times \square \to \square$. Postcomposing it with the Yoneda embedding and left Kan extending, we obtain the geometric product functor

$$
\begin{array}{c}
\square \times \square \\
\downarrow \\
cSet \times cSet
\end{array} 
\xrightarrow{\otimes} 
\begin{array}{c}
cSet
\end{array}
$$

The standard formula for left Kan extensions gives us the following formula for the geometric product:

$$X \otimes Y = \colim_{x: \square^m \to X, y: \square^n \to Y} \square^{m+n}$$

Note that the geometric product of cubical sets does not coincide with the cartesian product. However, the geometric product implements the correct homotopy type, and is better behaved than the cartesian product – for instance, for $m, n \geq 0$ we have $\square^m \otimes \square^n = \square^{m+n}$. Furthermore, the geometric product is taken to the cartesian product by the geometric realization functor to spaces.

**Proposition 1.18.** The geometric product $\otimes$ defines a monoidal structure on the category of cubical sets, with the unit given by $\square^0$.

This monoidal structure is however not symmetric. Indeed, the existence of a symmetry natural transformation would in particular imply that there is a non-identity bijection $[1]^2 \to [1]^2$ in $\square$.

In particular, for any $X, Y \in \text{cSet}$, the unique maps from $X$ and $Y$ to $\square^0$ induce maps $\pi_X: X \otimes Y \to X, \pi_Y: X \otimes Y \to Y$.

Given a cubical set $A$, we form two non-isomorphic functors $\text{cSet} \to \text{cSet}$: the left tensor $- \otimes A$ and the right tensor $A \otimes -$. As they are both co-continuous, they admit right adjoints and we write $\text{hom}_L(A, -)$ for the right adjoint of the left tensor $- \otimes A$ and $\text{hom}_R(A, -)$ for the right adjoint of the right tensor $A \otimes -$. Thus the monoidal structure on $\text{cSet}$ given by the geometric product is closed, but non-symmetric.

The standard construction of an arbitrary small colimit as a coequalizer of coproducts gives us the following lemma about colimits in presheaf categories.

**Lemma 1.19.** Let $C$ be a category and $D$ a small diagram in $\text{Set}^{\text{C}^{\text{op}}}$. Then any map $C(-, c) \to \text{colim} \, D$ factors through some map in the colimit cone. \qed

This lemma allows us to describe the geometric product of cubical sets explicitly.

**Proposition 1.20.** For $X, Y \in \text{cSet}$, the geometric product $X \otimes Y$ admits the following description.

- For $k \geq 0$, the $k$-cubes of $X \otimes Y$ consist of all pairs $(x: \square^m \to X, y: \square^n \to Y)$ such that $m + n = k$, subject to the identification $(x \sigma_{m+1}, y) = (x, y \sigma_1)$. 


Proof. We begin by noting that for every pair \((x, y)\) such an \(m\)-cube \([x, y]\) the faces, degeneracies, and connections of the \((m + n)\)-cube \((x, y)\) are computed as follows:

\[
- (x, y)\partial_i = \begin{cases} (x\partial_i, y) & 1 \leq i \leq m \\ (x, y\partial_{i-m}) & m + 1 \leq i \leq m + n \end{cases}
\]

\[
- (x, y)\sigma_i = \begin{cases} (x\sigma_i, y) & 1 \leq i \leq n_1 + 1 \\ (x, y\sigma_{i-m}) & m + 1 \leq i \leq m + n + 1 \end{cases}
\]

\[
- (x, y)\gamma_i = \begin{cases} (x\gamma_i, y) & 1 \leq i \leq m \\ (x, y\gamma_{i-m}) & m + 1 \leq i \leq m + n \end{cases}
\]

Likewise, for \(m + 1 \leq i \leq m + n\) we have \(\partial_{i,m}^n = \partial_m^m \otimes \partial_m^n\), implying \((x, y)\partial_i = (x, y\partial_i)\).

Similar proofs hold for degeneracies and connections. In particular, this implies that for any \((x, y)\) we have \((x\sigma_{m+1}, y) = (x, y\sigma_1)\), as both are equal to \((x, y)\sigma_{m+1}\).

To see that all cubes in \(X \otimes Y\) are of this form, note that by Lemma 19 every cube of \(X \otimes Y\) is equal \((x, y)\psi\) for some such pair \((x, y)\) and some map \(\psi\) in \(X\). We have shown that the set of cubes arising from pairs is closed under faces, degeneracies and connections; since these classes generate all maps in \(X\), this proves our claim.

Finally, we must show that the cubes of \(X \otimes Y\) are not subject to any additional identifications, beyond the identification \((x\sigma_{m+1}, y) = (x, y\sigma_1)\) mentioned above. In other words, we must show that for each \(k \geq 0\), \((X \otimes Y)_k\) is the quotient of the set \(\{(x: \square^m \to X, y: \square^n \to Y) | m + n = k\}\) under the smallest equivalence relation \(\sim\) such that \((x'\sigma_{m+1}, y') \sim (x', y'\sigma_1)\) for all \(x': \square^m \to X, y': \square^n \to Y\) such that \(m' + n' = k - 1\).

To that end, let \(x: \square^m \to X, y: \square^n \to Y, x': \square^m' \to X, y': \square^n' \to Y\), such that \(m + n = m' + n'\) and \((x, y) = (x', y')\) in \((X \otimes Y)\). We compute the image of this cube under the map \(\pi_X: X \otimes Y \to X\).

\[
\pi_X(x, y) = \pi_X(x', y')
\]

\[
\therefore x\sigma_{m+1}\sigma_{m+2}...\sigma_{m+n} = x'\sigma_{m'+1}...\sigma_{m+n}
\]
If $m$ or $m'$ is equal to 0, we interpret the corresponding string of degeneracies to be empty. We can apply face maps to both sides of this equation to reduce the left-hand side to $x$. If $m = m'$ then this gives the equation $x = x'$, and a similar calculation shows $y = y'$. Otherwise, we have $x = x' \sigma_{m' + 1} \ldots \sigma_m$. In this case, a similar calculation shows $y' = y \sigma_1 \ldots \sigma_1$, where $\sigma_1$ is applied $m - m'$ times on the right-hand side of the equation. From this we can see that $(x, y) \sim (x', y')$. Thus we see that quotienting the set of pairs $(x, y)$ of appropriate dimensions by $\sim$ does indeed suffice to obtain $(X \otimes Y)_k$. □

**Corollary 1.21.** For cubical sets $X$ and $Y$, we have $(X \otimes Y)_1 \cong (X \times Y_0) \cup (X_0 \times Y_0) (X_0 \times Y_1)$. □

The following lemma, which can be verified by simple computation, allows us to express boundary inclusions and open box inclusions as pushout products with respect to this monoidal structure.

**Lemma 1.22.**

(i) For $m, n \geq 0$, we have

$$(\partial \Box^m \to \Box^m) \otimes (\partial \Box^n \to \Box^n) = (\partial \Box^{m+n} \to \Box^{m+n}).$$

(ii) For $1 \leq i \leq m$ and $\varepsilon \in \{0, 1\}$, the open-box inclusion $\cap_{i, \varepsilon} \hookrightarrow \Box^n$ is the pushout product

$$(\partial \Box^{i-1} \to \Box^{i-1}) \otimes (\{1 - \varepsilon\} \hookrightarrow \Box^1) \otimes (\partial \Box^{m-i} \to \Box^{m-i}).$$

The restriction of the nerve functor defines a functor $\Box \to sSet$; taking the left Kan extension of this functor along the Yoneda embedding, we obtain the triangulation functor $T : cSet \to sSet$.

![Diagram](attachment:image.png)

The triangulation functor has a right adjoint $U : sSet \to cSet$ given by $(UX)_n = sSet(\Delta^1^n, X)$. Intuitively, we think of triangulation as creating a simplicial set $TX$ from a cubical set $X$ by subdividing the cubes of $X$ into simplices.

We now record two basic facts about triangulation. In the given references, these results are proven using a different definition of the category $\Box$, lacking connection maps, but the proofs apply equally well to the cubical sets under consideration here.

**Proposition 1.23 ([Cis06, Ex. 8.4.24]).** The triangulation functor sends geometric products to cartesian products; that is, for cubical sets $X$ and $Y$, there is a natural isomorphism $T(X \otimes Y) \cong TX \times TY$. □

**Corollary 1.24.** Triangulation preserves pushout products; that is, for maps $f, g$ in $cSet$ there is a natural isomorphism $T(f \otimes g) \cong Tf \otimes Ty$. Proof. Immediate by Proposition 1.23 and the fact that $T$ preserves colimits as a left adjoint. □

**Proposition 1.25 ([Cis06, Lem. 8.4.29]).** The triangulation functor preserves monomorphisms. □
1.3. Homotopy theory of cubical sets.

Lemma 1.26. The boundary inclusions \( \partial \square^n \to \square^n \) form a cellular model for \( cSet \).

Proof. This follows from Corollary 1.16.

Definition 1.27. A map of cubical sets is a Kan fibration if it has the right lifting property with respect to all open box fillings. A cubical set \( X \) is a cubical Kan complex if the map \( X \to \square^0 \) is a Kan fibration.

The functor \( \square^1 \otimes - : cSet \to cSet \), together with the natural transformations \( \partial_1 \square^0 \otimes -, \partial_1 \square^1 \otimes - : id \to \square^1 \otimes - \), and \( \pi : \square^1 \otimes - \to id \), defines a cylinder functor on \( cSet \) in the sense of Definition 1.2. Thus, for any \( X,Y \in cSet \) we have a set \([X,Y]\) of homotopy classes of maps from \( X \) to \( Y \) defined by this cylinder functor.

Theorem 1.28 (Cisinski). The category \( cSet \) carries a cofibrantly generated model structure, referred to as the Grothendieck model structure, in which

- cofibrations are the monomorphisms;
- weak equivalences are maps \( X \to Y \) inducing bijections \([Y,Z] \to [X,Z]\) for all cubical Kan complexes \( Z \);
- fibrations are the Kan fibrations.

Proof. The existence of the model structure and characterization of the cofibrations, weak equivalences, and fibrant objects follows from applying Theorem 1.5 with the cylinder functor \( I \), cellular model \( M = \{ \partial \square^n \to \square^n | n \geq 0 \} \), and \( S = \emptyset \). The characterization of the fibrations is given in [Cis14, Thm. 1.7].

The canonical inclusion \( \square \to \Cat \) induces the adjoint pair \( \tau_1 : cSet \rightleftarrows \Cat : N\square \) via hom-out and the left Kan extension. In particular, \( N\square(C)_n = \Cat([1]^n, C) \). The functor \( \tau_1 \) takes a cubical set \( X \) to its fundamental category, which is obtained as the quotient of the free category on the graph \( X_1 \cong X_0 \) modulo the relations: \( \sigma_1 x = \text{id}_x \) and \( gf = qp \) for every 2-cube.

1.4. Marked cubical sets. To define marked cubical sets, we need to introduce a new category \( \square_\sharp \), a slight enlargement of \( \square \). The category \( \square_\sharp \) consists of objects of the form \([1]^n\) for \( n = 0, 1, \ldots \) and an object \([1]_e\). The maps of \( \square_\sharp \) are generated by the usual generating maps of \( \square \) along with \( \varphi : [1] \to [1]_e \) and \( \zeta : [1]_e \to [1]^0 \) subject to an additional identity \( \zeta \varphi = \sigma_1^1 \).

Proposition 1.29. The category \( \square_\sharp \) is a Reedy category with the Reedy structure defined as follows:

- \( \deg([1]^0) = 0, \deg([1]) = 1, \deg([1]_e) = 2 \), and \( \deg([1]^n) = n + 1 \) for \( n \geq 2 \);
- \( (\square_\sharp)_+ \) is generated by face maps and \( \varphi \) under composition;
- \( (\square_\sharp)_- \) is generated by degeneracy maps, connections, and \( \zeta \) under composition. 

□
A structurally marked cubical set is a contravariant functor \( X : \square^{op} \to \text{Set} \) and a morphism of structurally marked cubical sets is a natural transformation of such functors. We will write \( \text{cSet}'' \) for the category of structurally marked cubical sets. When working with the category of structurally marked cubical sets, we will write \( X_n \) for the value of \( X \) at \([1]^n\) and \( X_e \) for the value of \( X \) at \([1]_e\).

Structurally marked cubical sets should be thought of as cubical sets with (possibly multiple) labels on their edges such that for each vertex \( x \), the degenerate edge \( x\sigma_1 \) has, in particular, a distinguished label \( x\zeta \).

A marked cubical set is a structurally marked cubical set for which the map \( X_e \to X_1 \) is a monomorphism. We write \( \text{cSet}' \) for the category of marked cubical sets. Alternatively, we may view a marked cubical set as a pair \( (X, W_X) \) consisting of a cubical set \( X \) together with a subset \( W_X \subseteq X_1 \) of edges that includes all degenerate edges and a morphism of marked cubical sets is a map of cubical sets that preserves marked edges.

The functor taking a (structurally) marked cubical set to its underlying cubical set admits both a left and a right adjoint, given by the minimal and maximal marking respectively. The minimal marking on a cubical set \( X \), denoted \( X^{\flat} \), marks exactly the degenerate edges, whereas the maximal marking, denoted \( X^{\sharp} \), marks all edges of \( X \). If considered as structurally marked cubical sets, the marked edges of \( X^{\flat} \) and \( X^{\sharp} \) are marked exactly once. Altogether we obtain the following adjunctions:

\[
\text{cSet}' \xrightarrow{(-)^\flat} \text{cSet} \xleftarrow{(-)^\sharp} \text{cSet}''
\]

The notation \( \text{cSet}'(\cdot) \) above indicates that the same constructions can be applied to both marked and structurally marked cubical sets. In the context of (structurally) marked cubical sets, we regard a cubical set with its minimal marking by default, writing \( X \) for \( X^{\flat} \).

There is moreover an inclusion \( \text{cSet}'' \to \text{cSet}'' \). This inclusion admits a left adjoint taking \( X \in \text{cSet}'' \) to \( \text{Im}X \) given by \( (\text{Im}X)_n = X_n \) and \( (\text{Im}X)_e = \varphi^*(X_e) \), i.e., the image of \( X_e \) under \( \varphi^* = X(\varphi) \). The inclusion is easily seen to not have a right adjoint, since it fails to preserve the pushout of \( \square^1 \to (\square^1)^{\sharp} \) against itself.

Altogether we obtain the following diagram:

\[
(\star)
\]

A geometric product entirely analogous to that of Subsection 1.2 exists for structurally marked cubical sets. We extend \( \square \times \square \to \text{cSet} \) to \( \square^2 \times \square^2 \to \text{cSet}'' \) by taking \([1]^e \otimes [n]\) to have \( \square^{n+1} \) as the underlying cubical set with edges of the form \((0, x_2, \ldots, x_{n+1}) < (1, x_2, \ldots, x_{n+1}) \) uniquely marked. Similarly, let \([n] \otimes [1]_e \) have \( \square^{n+1} \) as its underlying cubical set, and marked edges those of
the form \((x_1, \ldots, x_n, 0) < (x_1, \ldots, x_n, 1)\). Finally, let \([1]_e \otimes [1]_e := (\square^2)^2\). The left Kan extension yields \(\otimes: \text{cSet}'' \times \text{cSet}'' \to \text{cSet}''\).

This geometric product admits a concrete description analogous to that of Proposition 1.20.

**Proposition 1.30.** For \(X, Y \in \text{cSet}''\), the geometric product \(X \otimes Y\) admits the following description.

- The underlying cubical set of \(X \otimes Y\) is the geometric product of the underlying cubical sets of \(X\) and \(Y\).
- \((X \otimes Y)_e\) is the set of all pairs of the form \((x, y)\)\((\square^1)^2 \to X, y: \square^0 \to Y)\) or \((x: \square^0 \to X, \overline{y}: (\square^1)^2 \to Y)\), subject to the identification \((x, y)\)\((\square^1)^2 \to X, y: \square^0 \to Y)\) for \(x: \square^0 \to X, y: \square^0 \to Y\).
- Structure maps not arising from those of the underlying cubical set are computed as follows:
  - \((x, y)\)\((x, y)\)\(\diamond(X, y)\)\(\varphi = (x, y)\)\(\varphi\).

**Proof.** To compute the underlying cubical set of \(X \otimes Y\), we analyze maps \(\square^k \to X \otimes Y\) exactly as in the proof of Proposition 1.20.

Now we consider maps \((\square^1)^2 \to X \otimes Y\). First observe that for every pair of maps \(\overline{x}: (\square^1)^2 \to X, y: (\square^1)^2 \to Y\), we have a map \(x, y: (\square^1)^2 \otimes (\square^1)^2 \to X \otimes Y\) in the colimit cone, and the same holds for \(x: (\square^1)^2 \to X, \overline{y}: (\square^1)^2 \to Y\). Once again, the stated computations of structure maps follow from the naturality of the colimit cone.

Now we will show that every map \(p: (\square^1)^2 \to X \otimes Y\) has the form described above. By Lemma 1.19 for every such map we have a commuting diagram

\[
\begin{array}{ccc}
\square^1 & \xrightarrow{\psi} & \square^m_n(\square^1) \times \square^m_n(\square^1) \\
\downarrow{p} & & \downarrow{} \\
X \otimes Y & & \\
\end{array}
\]

where the map \(\square^m_n(\square^1) \times \square^m_n(\square^1) \to X \otimes Y\) is part of the colimit cone.

First note that if \(\psi\) factors through \(\zeta\), then \(p = (x, y)\)\(\zeta\) for some \(x: \square^0 \to X, y: \square^0 \to Y\). This takes care of the case \(\square^m_n(\square^1) = \square^m_n, \square^m_n(\square^1) = \square^m\), since any map from \((\square^1)^2\) into these objects factors through \(\zeta\).

Now assume \(\psi\) does not factor through \(\zeta\), implying that at least one of \(\square^m_n(\square^1), \square^m(\square^1)\) is \((\square^1)^2\); then \(\square^m_n(\square^1) \times \square^m_n(\square^1)\) is either \(\square^m(n(\square^1)^2), (\square^1)^2 \otimes \square^m(\square^1)\), or \((\square^2)^2\). Since every map \((\square^1)^2 \to (\square^2)^2\) factors through either \(\square^1(\square^1)^2\) or \((\square^1)^2 \otimes \square^1\), we need only consider the first two cases. If \(\square^m_n(\square^1) \times \square^m_n(\square^1) = \square^m \otimes (\square^1)^2\), then \(\psi\) picks out the unique marking on an edge of the form \((x_1, \ldots, x_m, 0) < (x_1, \ldots, x_m, 1)\). In other words, \(\psi\) factors through the map \((x_1, \ldots, x_m) \otimes (\square^1)^2 : \square^0 \otimes (\square^1)^2 \to \square^m(\square^1)^2\). Thus we have reduced the problem to the case \(\square^m_n(\square^1) = \square^m(n(\square^1)^2), (\square^1)^2 = (\square^1)^2\), but since the only endomorphism of \((\square^1)^2\) which does not factor through \(\zeta\) is the identity, this implies that \(p = (x, y)\) for some \(x: \square^0 \to X, y: (\square^1)^2 \to Y\). In the case \(\square^m_n(\square^1) = (\square^1)^2, \square^m_n(\square^1) = \square^m\), a similar analysis shows \(p = (\overline{x}, y)\) for some \(\overline{x}: (\square^1)^2 \to X, y: \square^0 \to Y\).
To show that the elements of \((X \otimes Y)_e\) are subject to no further identifications, consider two pairs \((x, y), (x', y')\) which are identified in \((X \otimes Y)_e\). Considering the image of the cube corresponding to these pairs under the projections \(\pi_X, \pi_Y\), we see that \(x = x', y = y'\). A similar proof holds for identified pairs of the form \((x, y), (x, y')\). Finally, if \((x, y) = (x, y)\), then applying the projections shows \(x = x_\zeta, y = y_\zeta\). □

Corollary 1.31. For \(X, Y \in \text{cSet}'\), \((X \otimes Y)_e \cong (X_e \times Y_0) \cup_{(X_0 \times Y_0)} (X_0 \times Y_e)\). □

What Proposition 1.30 shows is that for \(x: \Box^1 \to X, y: \Box^0 \to Y\), the set of markings on the edge \((x, y)\) in \(X \otimes Y\) is simply the set of markings on \(x\) in \(X\), that the analogous result holds for \(x: \Box^0 \to X, y: \Box^1 \to Y\), and that for a pair of vertices \(x, y\) the distinguished marking \((x, y)_\zeta\) is identified with both \((x_\zeta, y)\) and \((x, y_\zeta)\).

Remark 1.32. This monoidal structure restricts to a monoidal product on the category \(\text{cSet}'\) of marked cubical sets.

Corollary 1.33. All functors in the diagram (*') are monoidal. □

Finally, as in the case of cubical sets, given a marked cubical set \(A\), we form two non-isomorphic functors \(\text{cSet}'(\cdot) \to \text{cSet}'(\cdot)\): the left tensor \(- \otimes A\) and the right tensor \(A \otimes -\). As they are both co-continuous, they admit right adjoints and we write \(\text{hom}_L(A, -)\) for the right adjoint of the left tensor \(- \otimes A\) and \(\text{hom}_R(A, -)\) for the right adjoint of the right tensor \(A \otimes -\).

2. Model structure on marked cubical sets

The goal of this section is to construct a combinatorial model category structure on the category \(\text{cSet}'\) of marked cubical sets. One would like to that by applying the Cisinski theory, as described in Subsection 1.1, but unfortunately \(\text{cSet}'\) is not a presheaf category. Although there exists a generalization of Cisinski theory to a non-topos case (due to Olschok [Ols09]), we choose to construct the model structure directly, using Jeff Smith’s Theorem 1.1 to obtain a better understanding of it as a result. It is also worth pointing out that our language (e.g., cellular model, cylinder functor) follows the conventions of Cisinski to make the analogy with the Cisinski machinery clear.

2.1. Classes of maps. To begin, we lay out the definitions of the classes of maps that will comprise the model structure.

The cofibrations are the monomorphisms. The trivial fibrations are the maps with the right lifting property with respect to the cofibrations.

Using Lemma 1.26 one obtains:

Lemma 2.1. The cofibrations are the saturation of the set consisting of the boundary inclusions \(\partial \Box^n \to \Box^n\) for \(n \geq 0\) and the inclusion \(\Box^1 \to (\Box^1)^\#\). □

By Lemma 2.1 we have a cofibrantly generated weak factorization system (cofibrations, trivial fibrations).

Definition 2.2. We introduce three classes of maps in \(\text{cSet}'\).
(i) Let the special open box inclusions $i^n_{i,\varepsilon}$ be the marked cubical set maps whose underlying cubical set maps are the open box inclusions $\sqcap^n_{i,\varepsilon} \to \square^n$, with the critical edge marked in each (except for the domain of $i^1_{1,\varepsilon}$, i.e. $\square^0$, in which the critical edge is not present).

(ii) Let $K$ be the cubical set depicted as:

```
• — — —
• — — —
```

Let $K'$ be the marked simplicial set that has the middle edge in the above marked. Define the saturation map to be the inclusion $K \subseteq K'$.

(iii) For each of the four faces of the square, let the 3-out-of-4 map associated to that face be the inclusion of $\square^2$ with all but that face marked into $(\square^2)^\sharp$.

The anodyne maps are defined as the saturation of the set of maps consisting of the special open box inclusions, the saturation map, and the 3-out-of-4 maps. The naive fibrations are those maps that have the right lifting property against anodyne maps. Call an object $X$ of $\text{cSet}'$ a marked cubical quasicategory if the map $X \to \square^0$ is a naive fibration.

Remark 2.3. Viewing marked cubical quasicategories as $(\infty, 1)$-categories, the marked edges represent equivalences. The generating anodyne maps have the following $(\infty, 1)$-categorical meanings.

- The $n$-dimensional special open box fillings for $n \geq 2$ correspond to composition of maps and homotopies, analogous to filling inner and special horns in quasicategories. They also ensure that every morphism presented by a marked edge has a left and right inverse, i.e., is an equivalences.

- The 1-dimensional special open box fillings, $i^1_{1,\varepsilon} : \square^0 \to (\square^1)^\sharp$, are the inclusions of endpoints into the marked interval; thus marked edges may be lifted along naive fibrations, analogous to the lifting of isomorphisms along isofibrations in 1-category theory.

- The saturation map ensures that equivalences, having both left and right inverses, are marked.

- The 3-out-of-4 maps represent the principle that if three maps in a commuting square are equivalences, then so is the fourth. They encode a condition analogous to the two-out-of-three property.

Remark 2.4. For $n \geq 2$, the representable marked cubical set $\square^n$ is not a marked cubical quasicategory, as it lacks fillers for certain special open boxes. This stands in constrast to the case of simplicial sets, in which the representables $\Delta^n$ are quasicategories.

Lemma 2.5. Let $X$ be a marked cubical quasicategory, and $x : \square^1 \to X$ an edge of $X$. Then $x$ is marked if and only if it factors through the inclusion of the middle edge $\square^1 \to K$.

Proof. The inclusions $K \to K'$ and $(\square^1)^\sharp \to K'$ are both anodyne (the latter as a composite of special open box fillings). The stated result thus follows from the fact that $X \to \square^0$ has the right lifting property with respect to both of these maps. \qed
Lemma 2.6. For a marked cubical set $X$ to be a marked cubical quasicategory, it suffices for the map $X \to \Box^0$ to have the right lifting property with respect to special open box fillings and the saturation map.

Proof. Assume that $X$ has the right lifting property with respect to special open box inclusions and the saturation map. The proof of Lemma 2.5 only requires lifting with respect to these maps, so the marked edges of $X$ are precisely those which factor through $K$.

To show that $X \to \Box^0$ lifts against the 3-out-of-4 maps, we must show that, if three sides of a 2-cube in $X$ are marked, then so is the fourth. Using the fact that the three marked sides factor through $K$, we can show that the fourth does as well by a simple exercise in filling three-dimensional special open boxes. Hence the fourth edge is also marked. □

Remark 2.7. In view of Lemma 2.6, whether omitting the 3-out-of-4 maps as generators would change the class of anodyne maps. To see that it is, observe that, using the small object argument, we can factor any three-out-of-four map as a composite of a map in the saturation of the special open box fillings and two-out-of-six map, followed by a map having the right lifting property with respect to these maps. Examining the details of this construction, we can see that the second of these maps will not have the right lifting property with respect to the 3-out-of-4 maps. Thus the 3-out-of-4 maps are not in the saturation of the other two classes of generating anoydes.

One may further note that, without the 3-out-of-4 maps as generators, anodyne maps would not be closed under pushout product with cofibrations. This makes them crucial for our development.

Definition 2.8. Given a map $f: X \to Y$ of marked cubical sets, a naive fibrant replacement of $f$ consists of a diagram as depicted below, with $X$ and $Y$ marked cubical quasicategories, $\iota_X$ and $\iota_Y$ anodyne, and $f$ a naive fibration.

We have a cofibrantly generated weak factorization system (anodyne maps, naive fibrations). This induces a functorial factorization of any map $X \to Y$ as

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_f} & Mf \\
\downarrow f & & \downarrow Qf \\
Y & \xrightarrow{\eta_Y} & Q(\eta_Y f) \\
\end{array}
\]

where $Q$ is an endofunctor on $(\text{cSet}^I)^\to$ sending objects to naive fibrations and $\eta: \text{Id} \to Q$ is pointwise anodyne. Where $f$ is the unique map $X \to \Box^0$, we write $\eta_X$ for $\eta_f$. Given $f: X \to Y$, we can use this factorization to obtain a canonical naive fibrant replacement of $f$:

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_{QYf}} & Y \\
\downarrow \eta_{QYf} & & \downarrow \eta_Y \\
X & \xrightarrow{Q(\eta_Y f)} & Q(\eta_Y f) \\
\end{array}
\]
We declare \( f \) to be a weak equivalence if \( Q(\eta_Y, f) \) is a trivial fibration. A trivial cofibration is a map that is a cofibration and weak equivalence, and a fibration is a map that has the right lifting property against trivial cofibrations.

We now want to show that if \( Y \) is a marked cubical quasicategory, so is \( \hom_R(X, Y) \). The following lemma on pushout-products helps with the proof of this fact.

**Lemma 2.9.** The pushout product of two cofibrations is a cofibration. Furthermore, the pushout product of an anodyne map and a cofibration is anodyne.

**Proof.** Since \( \otimes \) preserves colimits in each variable and anodynes are stable under pushouts and transfinite compositions, we can use induction on skeleta to show that if \( S \to T \) is one of the generating cofibrations (resp. anodynes), then \( (S \to T) \otimes (\partial \square^n \to \square^n) \) and \( (S \to T) \otimes (\square^1 \to (\square^1)^2) \) are cofibrations (resp. anodyne). This will show that if \( i \) and \( j \) are cofibrations, and \( i \) is anodyne, then \( i \otimes j \) is anodyne; the proof for the case where \( j \) is anodyne is entirely analogous.

Several cases can be taken care of by the following fact: If \( f: A \to B \) is an inclusion which is a surjection on vertices and \( p: X \to Y \) is an isomorphism of underlying cubical sets, then \( f \otimes p \) is an isomorphism. This follows because the pushout-product is an isomorphism of underlying cubical sets, and so we need only consider what edges are marked. But the marked edges of \( (B \otimes Y)e = (B_\varepsilon \times Y_\varepsilon) \cup_{B_\varepsilon \times Y_\varepsilon} (B_\varepsilon \times Y_\varepsilon) \), and since each map is a bijection on vertices, all of these edges appear in \( (B \otimes X) \cup_{A \otimes X} (A \otimes Y) \).

This claim, along with the fact that taking the pushout-product with \( \emptyset \to \square^0 \) is the identity, handles all but the following pushout products:

- \( (\partial \square^n \to \square^n) \otimes (\partial \square^n \to \square^n) \): this is the map \( \partial \square^{n+n} \to \square^{n+n} \). This completes the proof of the first statement, concerning the pushout product of two cofibrations; the remaining cases complete the second statement, concerning the pushout product of a cofibration and an anodyne map.

- \( m^{m}_{i,\varepsilon} \otimes (\partial \square^n \to \square^n) \): the underlying cubical set map is the open box inclusion \( m^{m+n}_{i,\varepsilon} \to \square^{m+n} \), with edges in the codomain being marked if and only if they are present and marked in the domain. The critical edge is marked, so this is anodyne as a pushout of a special open box filling.

- \( i^1_{i,\varepsilon} \otimes (\partial \square^1 \to (\square^1)^2) \): this is the 3-out-of-4 map associated to the face \((1, 1 - \varepsilon)\).

\[ \square \]

**Corollary 2.10.** If \( f: A \to B \) is a cofibration and \( g: X \to Y \) is a naive fibration, then the pullback exponential \( f \triangleright g \): \( \hom(A, Y) \to \hom(A, X) \times_{\hom(A, Y)} \hom(B, Y) \) (where \( \hom \) may designate either \( \hom_L \) or \( \hom_R \)) is a naive fibration. Furthermore, if \( f \) is anodyne or \( g \) is a trivial fibration, then \( f \triangleright g \) is a trivial fibration.

In particular, if \( Y \) is a marked cubical quasicategory, then for any \( X \), \( \hom(X, Y) \) is a marked cubical quasicategory.

**Proof.** Let \( i: C \to D \) be anodyne; we wish to show that \( f \triangleright g \) has the right lifting property with respect to \( i \). By a standard duality, it suffices to show that \( g \) has the right lifting property with respect to \( i \otimes f \). This map is anodyne by Lemma 2.9 so the first statement holds.
For the second statement, we can apply the same result with $i$ an arbitrary cofibration. Then $g$ has the right lifting property with respect to $i \otimes f$, either because $f$, and hence also $i \otimes f$, are anodyne, or because $i \otimes f$ is a cofibration and $g$ is a trivial fibration.

The third statement follows from the first by the fact that $\hom(X, Y) \to \square^0$ is the pullback exponential of the cofibration $\emptyset \to X$ with the naive fibration $Y \to \square^0$.

2.2. Homotopies. Next we define the closely-related concepts of connected components in a marked cubical set, and homotopies of maps between cubical sets.

**Definition 2.11.** For a marked cubical set $X$, let $\sim_0$ denote the relation on $X_0$, the set of vertices of $X$, given by $x \sim_0 y$ if there is a marked edge from $x$ to $y$ in $X$. Let $\sim$ denote the smallest equivalence relation on $X_0$ containing $\sim_0$.

**Remark 2.12.** For $x, y \in X_0$, one can easily see that $x \sim y$ if and only if $x$ and $y$ are connected by a zigzag of marked edges.

**Definition 2.13.** For a marked cubical set $X$, the set of connected components $\pi_0(X)$ is $X_0/\sim$.

We may observe that the construction of $\pi_0(X)$ is functorial, since maps of marked cubical sets preserve marked edges, and hence preserve the equivalence relation $\sim$.

**Definition 2.14.** An elementary left homotopy $h: f \sim g$ between maps $f, g: A \to B$ is a map $h: (\square^1)^A \otimes A \to B$ such that $h|\{0\} \otimes A = f$ and $h|\{1\} \otimes A = g$. Note that the elementary left homotopy $h$ corresponds to an edge $(\square^1)^A \to \hom_L(A, B)$ between the vertices corresponding to $f$ and $g$. A left homotopy between $f$ and $g$ is a zig-zag of elementary left homotopies.

A left homotopy from $f$ to $g$ corresponds to a zig-zag of marked edges in $\hom_L(A, B)$ and so maps from $A$ to $B$ are left homotopic exactly if $\pi_0(f) = \pi_0(g)$, where the set of connected components is taken in $\hom_L(A, B)$. We write $[A, B]$ for the set of left homotopy classes of maps $A \to B$.

These induce notions of elementary left homotopy equivalence and left homotopy equivalence. Each of these notions has a “right” variant using $A \otimes (\square^1)^A$ and $\hom_R(A, B)$. Unless the potential for confusion arises or a statement depends on the choice, we will drop the use of “left” and “right”.

**Lemma 2.15.** In a marked cubical quasicategory $X$, the relations $\sim_0$ and $\sim$ conicide.

**Proof.** Using 2-dimensional open box fillers with certain edges degenerate, and the 3-out-of-4 property, we can reduce any zigzag of marked edges connecting $x$ and $y$ in $X$ to a single marked edge from $x$ to $y$. □

By adjointness, we obtain the following corollary.

**Corollary 2.16.** If $f, g: A \to B$ are homotopic and $B$ a marked cubical quasicategory, then $f$ and $g$ are elementarily homotopic. Hence, between marked cubical quasicategories homotopy equivalences coincide with elementary homotopy equivalences.

**Proof.** By Corollary 2.10, $\hom(A, B)$ is a marked cubical quasicategory, and so $\sim_0$ is an equivalence relation on $\hom(A, B)_0$ by Lemma 2.15. Translating what this means for homotopies gives the result. □
Lemma 2.17. If \( f, g: X \to Y \) are left homotopic, then for any \( Z \), the induced maps \( \hom_L(Y, Z) \to \hom_L(X, Z) \) are right homotopic.

Proof. We consider the case of elementary homotopies; the general result follows from this. An elementary left homotopy \( f \sim g \) is given by a map \( H: (\Box^1)^2 \otimes X \to Y \). Pre-composition with \( H \) induces a map \( \hom_L(Y, Z) \to \hom_L((\Box^1)^2 \otimes X, Z) \). Under the adjunction defining \( \hom_L \), this corresponds to a map \( \hom_L(Y, Z) \otimes (\Box^1)^2 \otimes X \to Z \), which in turn corresponds to a map \( \hom_L(Y, Z) \otimes (\Box^1)^2 \to \hom_L(X, Z) \). It is easy to see that this map defines an elementary right homotopy between the pre-composition maps induced by \( f \) and \( g \). □

2.3. Category theory in a marked cubical quasicategory. Let \( X \) be a marked cubical quasicategory and \( x, y \in X_0 \). We will write \( X_1(x, y) \) for the subset of \( X_1 \) consisting of 1-cubes \( f \) with \( f\partial_{1,0} = x \) and \( f\partial_{1,1} = y \). Define an equivalence relation relation \( \sim_X \) on the set \( X_1(x, y) \) of edges from \( x \) to \( y \) as follows: \( f \sim_X g \) if and only if there is a 2-cube in \( X \) of the form

\[
\begin{array}{ccc}
  x & f & y \\
  \downarrow & & \downarrow \\
  x & g & y \\
\end{array}
\]

It is straightforward to verify that this is indeed an equivalence relation: reflexivity follows from degeneracies, whereas symmetry and transitivity are given by filling 3-dimensional open boxes.

We now define three increasingly strong refinements of the concept of a homotopy equivalence.

Definition 2.18. Let \( f: X \to Y \) be a map in \( \mathbf{cSet} \). Then:

- \( f \) is a semi-adjoint equivalence if there exist \( g: Y \to X \) and homotopies \( H: gf \sim \text{id}_X \), \( K: fg \sim \text{id}_Y \) such that \( fH \sim Kf \) as edges of \( \hom(X, Y) \);
- \( f \) is a strong homotopy equivalence if there exist \( g, H, K \) as above with \( fH = Kf \);
- a map \( g: Y \to X \) is a strong deformation section of \( f \) if \( fg = \text{id}_Y \) and there exists a homotopy \( H: gf \sim \text{id}_X \) such that \( fH = \text{id}_f \).

Our next goal will be two show the following:

Lemma 2.19. Let \( f: X \to Y \) be a map of marked cubical quasicategories. The following are equivalent:

(i) \( f \) is a homotopy equivalence;
(ii) \( f \) is a semi-adjoint equivalence.

Furthermore, if \( f \) is a naive fibration, then these are equivalent to:

(iii) \( f \) is a strong homotopy equivalence.

We will prove this by means of a 2-categorical argument.

We now define the homotopy category \( \text{Ho}X \) of a marked cubical quasicategory \( X \) as follows:

- the objects of \( \text{Ho}X \) are the 0-cubes of \( X \);
- the morphisms from $x$ to $y$ in $\text{Ho}X$ are the equivalence classes of edges $X_1(x,y)/\sim_X$;
- the identity map on $x \in X_0$ is given by $x\sigma_1$;
- the composition of $f : x \rightarrow y$ and $g : y \rightarrow z$ is given by filling the open box

Using standard arguments about open box fillings, one verifies the following lemma.

**Lemma 2.20.** The above data define a category.

**Lemma 2.21.** Let $X$ be a marked cubical quasicategory. If there is a 2-cube of the form

then $gf = qp$ in $\text{Ho}X$.

**Proof.** The desired homotopy appears as the top face of the following 3-cube:

The remaining faces of the cube form a special open box in $X$, with critical edge $w\sigma_1$; thus we can obtain the full cube, and in particular the top face, by filling this open box. The result then follows by the fact that composition in $\text{Ho}X$ is well-defined.

**Lemma 2.22.** For a marked cubical quasicategory $X$, the categories $\text{Ho}X$ and $\tau_1X$ are equivalent.

**Proof.** There is a natural inclusion of $\text{Ho}X \rightarrow \tau_1X$, which is the identity on objects and takes a 1-cube $f$ to a string of length 1 consisting of $f$. This is clearly faithful and essentially surjective. To see that it is full, we simply fill in 2-dimensional open boxes with one degenerate edge to reduce a sequence of arbitrary length to a sequence of length 1.
The assignment \( X \mapsto \text{Ho} X \) extends in a straightforward manner to a functor taking a marked cubical quasicategory to its homotopy category. Postcomposing this functor with \( \text{core}: \text{Cat} \to \text{Gpd} \), we obtain a groupoid \( \text{Ho}^2X \).

**Lemma 2.23.** The groupoid \( \text{Ho}^2X \) can be constructed directly as follows:

- Objects are 0-cubes of \( X \);
- Morphisms from \( x \) to \( y \) are equivalence classes of marked edges from \( x \) to \( y \);
- Composition and identities are defined as in \( \text{Ho} X \).

**Proof.** Let \( X \) be a marked cubical quasicategory. It is easy to see that an edge \( f: \Box^1 \to X \) is invertible in \( \text{Ho} X \) if and only if it factors through the map \( \Box^1 \to K \) which picks out the middle edge. Since the inclusions \( (\Box^1)^2 \to K' \) and \( K \to K' \) are anodyne, this holds if and only if \( f \) is marked. \( \square \)

**Definition 2.24.** Define a strict 2-category \( \text{Ho}^2\text{cSet}' \) whose objects are the marked cubical quasi-categories and whose mapping category from \( X \) to \( Y \) is

\[
\text{Ho}^2\text{cSet}'(X,Y) := \text{Ho} \text{hom}_L(X,Y).
\]

This means the 1-morphisms are the usual 1-morphisms \( X \to Y \), and the 2-morphisms are maps \( X \otimes \Box^1 \to Y \), modulo an equivalence relation. Denote the (vertical) composition in \( \text{Ho} \text{hom}_L(X,Y) \) with \( \circ \). The (horizontal) composition

\[
\text{Ho} \text{hom}_L(Y,Z) \times \text{Ho} \text{hom}_L(X,Y) \to \text{Ho} \text{hom}_L(X,Z)
\]

(which will be written by concatenation) is defined on objects by the usual composition. If \( H: Y \otimes \Box^1 \to Z \) and \( K: X \otimes \Box^1 \to Y \) are morphisms \( K: g \to g' \) and \( H: f \to f' \), respectively, define the morphism \( KH: gf \to g'f' \) by choosing a fill for the open box of \( \text{Ho} \text{hom}_L(X,Z) \) depicted by

\[
\begin{array}{c}
gf \quad Kf \quad g'f' \\
gf \\ gH \quad g'H
\end{array}
\]

where the top edge is induced by the composite \( X \otimes \Box^1 \to Y \otimes \Box^1 \to Z \) and the right edge by \( X \otimes \Box^1 \to Y \to Z \). The fact that the \( \text{hom}_L(X,Y) \) are marked cubical quasicategories ensures this defines a well-defined, associative, unital, and functorial operation. For functoriality, note that the morphism \( X \otimes \Box^1 \otimes \Box^1 \xrightarrow{H \otimes \Box^1} Y \otimes \Box^1 \xrightarrow{K} Z \) yields a 2-cube \( \Box^2 \to \text{hom}_L(X,Z) \) which can be depicted as

\[
\begin{array}{c}
gf \quad Kf \quad g'f' \\
gH \quad g'H
\end{array}
\]

and so by Lemma 2.21, we have \((g'H) \circ (Kf) = (Kf') \circ (gH)\), which implies the interchange law.
Definition 2.25. Let $\text{Ho}^\#_{2}\text{cSet}'$ denote the maximal $(2,1)$-category contained in $\text{Ho}_{2}\text{cSet}'$, i.e. the 2-category whose objects are marked cubical sets, with $\text{Ho}^\#_{2}\text{cSet}'(X,Y) = \text{Ho}^\#\text{hom}_{\text{c}}(X,Y)$, and the 2-categorical operations induced by those of $\text{Ho}_{2}\text{cSet}'$.

The $\text{Ho}^\#$ construction, together with the following general results about $(2,1)$-categories, give us the desired result about compatibility of homotopies.

Lemma 2.26 (Undergraduate Lemma). Let $X$ be an object in a $(2,1)$-category $C$, and let $H : p \sim \text{id}_X$ be a morphism in $C(X,X)$. Then $pH = Hp$.

Proof. By the interchange law,

$$H \circ (pH) = (H \text{id}_X) \circ (pH) = (\text{id}_X H) \circ (Hp) = H \circ (Hp).$$

Since $C(X,X)$ is a groupoid, we can cancel $H$. $\Box$

Lemma 2.27 (Graduate Lemma). Let $X, Y$ be objects in a $(2,1)$-category $C$, $f : X \leftrightarrow Y : g$ two morphisms between them, and $H : gf \rightarrow \text{id}_X$ and $K : fg \rightarrow \text{id}_Y$ two 2-cells. Then there is a 2-cell $K' : fg \rightarrow \text{id}_Y$ for which $K'f = fH$.

Proof. Define $K' := K \circ (fHg) \circ (Kgf)^{-1}$. Now, we compute:

$$K'f = Kf \circ (fHgf) \circ (Kgf)^{-1}$$
$$= Kf \circ (fgfH) \circ (Kgf)^{-1} \quad \text{(by 2.26)}$$
$$= fH \quad \text{(by naturality/interchange)}$$

$\Box$

Proof of Lemma 2.19. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are clear. The implication (i) $\Rightarrow$ (ii) follows from applying Lemma 2.27 to the $(2,1)$-category $\text{Ho}^\#_{2}\text{cSet}'$.

Now let $f$ be a naive fibration and a semi-adjoint equivalence. By Corollary 2.10, the map $\text{hom}(X,X) \rightarrow \text{hom}(X,Y)$ is a naive fibration. A simple exercise in 2-dimensional special open box filling, using this fact and the definition of a semi-adjoint equivalence, shows that there exists a homotopy $H' : gf \sim \text{id}_X$ such that $fH' = Kf$. $\Box$

2.4. Fibration category of marked cubical quasicategories.

Lemma 2.28. Every anodyne map between marked cubical quasicategories is a homotopy equivalence.

Proof. Now let $f : X \rightarrow Y$ be anodyne, with $X$ and $Y$ marked cubical quasicategories. We can obtain a retraction $r : Y \rightarrow X$ as a lift in the following diagram:

$$\begin{array}{ccc}
X & \rightarrow & X \\
\downarrow & & \downarrow \\
Y & \rightarrow & \Box^0
\end{array}$$

We can then obtain a left homotopy $fr \sim \text{id}_Y$ as a lift in the following diagram:
The lift exists since the left-hand map is anodyne by Lemma \ref{lem:anodyne}. An analogous proof shows that $f$ is a right homotopy equivalence. \hfill $\blacksquare$

**Lemma 2.29.** Let $f: X \to Y$ be a naive fibration. The following are equivalent:

(i) $f$ is a trivial fibration;

(ii) $f$ has a strong deformation section;

(iii) $f$ is a strong homotopy equivalence.

**Proof.** If $f: X \to Y$ is a trivial fibration, then we can obtain a section $g: Y \to X$ as a lift of the following diagram:

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & X \\
\downarrow & & \downarrow f \\
Y & \longleftarrow & Y
\end{array}
\]

We can then obtain a left homotopy $H: gf \sim \text{id}_X$ satisfying $fH = \text{id}_f$ as a lift in the following diagram:

\[
\begin{array}{cc}
X \sqcup X & \xrightarrow{[s, \text{id}_X]} X \\
\downarrow & \downarrow f \\
(\square^1)^\natural \otimes X & \xrightarrow{f \pi_X} Y
\end{array}
\]

This shows $(i) \Rightarrow (ii)$ and the implication $(ii) \Rightarrow (iii)$ is trivial. To show that $(iii) \Rightarrow (i)$, we first show that $(iii)$ implies the following condition:

(iii)' the canonical map $\iota_{1,0} \circ f \to f$ in $(\text{cSet'})^\to$ admits a section.

To see $(iii) \Rightarrow (iii)'$, suppose $f$ is a strong homotopy equivalence with homotopy inverse $g: Y \to X$ and homotopies $H: gf \sim \text{id}_X, K: fg \sim \text{id}_Y$ satisfying $fH = Kf$. Then we have the following commuting diagram in $\text{cSet'}$:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{hom}(\square^1)^\natural, X} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{\circ \circ \text{id}_X} & X \times_Y \text{hom}(\square^1)^\natural, Y \\
\end{array}
\]

The top-left map is the adjunct of $H$, while the bottom-left map is induced by $g$ and the adjunct of $K$; the right-hand square is as in the statement of condition $(iii)'$. It is easy to see that the composite square is simply the identity square on $f$. 

Finally, note that \( \iota_{f,1} \circ f \) is a trivial fibration by Corollary 2.10. Therefore, if the square given in the statement of condition \((iii)'\) has a section, then \( f \) is a trivial fibration as a retract of a trivial fibration. Thus \((iii)' \Rightarrow (i)\). □

**Corollary 2.30.** A map \( f : X \to Y \) between marked cubical quasicategories is a trivial fibration exactly if it is a homotopy equivalence and a naive fibration.

*Proof.* This follows from Lemmas 2.19 and 2.29 together with the fact that every trivial fibration is a naive fibration since all anodyne maps are cofibrations. □

**Proposition 2.31.** The category of marked cubical quasicategories forms a fibration category, with naive fibrations as the fibrations and homotopy equivalences as the weak equivalences.

*Proof.* The class of homotopy equivalences is closed under 2-out-of-3. Corollary 2.30 shows that the maps between marked cubical quasicategories which are naive fibrations and homotopy equivalences are exactly the trivial fibrations; both fibrations and trivial fibrations are defined via a right lifting property, and hence they are stable under pullback. By Lemma 2.28, each anodyne map between marked cubical quasicategories is a homotopy equivalence, and so the (anodyne, naive fibration)-factorization gives the factorization axiom. □

**Lemma 2.32.** Let \( f : X \to Y \) be a map between marked cubical quasicategories. Then the following conditions are equivalent:

(i) \( f \) is a weak equivalence;

(ii) \( f \) is a left homotopy equivalence;

(iii) \( f \) is a right homotopy equivalence.

*Proof.* Consider the canonical naive fibrant replacement of \( f \) used in the definition of the weak equivalences:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{\iota_X} & & \downarrow_{\iota_Y} \\
X & \xrightarrow{\overline{f}} & Y
\end{array}
\]

(here \( \iota_Y = \eta_Y, \overline{f} = Q(\eta_Y f), \iota_X = \eta_{\eta_Y f} \)).

By Lemma 2.28, \( \iota_X \) and \( \iota_Y \) are left homotopy equivalences. It is easy to show that left homotopy equivalences satisfy the two-out-of-three property, so \( f \) is a left homotopy equivalence if and only if \( \overline{f} \) is one. By Corollary 2.30, \( \overline{f} \) is a left homotopy equivalence if and only if it is a trivial fibration, i.e. if and only if \( f \) is a weak equivalence. So \((i) \iff (ii)\); an analogous argument shows \((i) \iff (iii)\). □

2.5. Cofibration category of marked cubical sets. Our next result shows that the definition of the weak equivalences is not sensitive to the choice of naive fibrant replacement.

**Lemma 2.33.** Let \( f : X \to Y \) be a map of marked cubical sets. The following are equivalent:

(i) \( f \) is a weak equivalence.

*Proof.* Consider the canonical naive fibrant replacement of \( f \) used in the definition of the weak equivalences:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{\iota_X} & & \downarrow_{\iota_Y} \\
X & \xrightarrow{\overline{f}} & Y
\end{array}
\]

(here \( \iota_Y = \eta_Y, \overline{f} = Q(\eta_Y f), \iota_X = \eta_{\eta_Y f} \)).

By Lemma 2.28, \( \iota_X \) and \( \iota_Y \) are left homotopy equivalences. It is easy to show that left homotopy equivalences satisfy the two-out-of-three property, so \( f \) is a left homotopy equivalence if and only if \( \overline{f} \) is one. By Corollary 2.30, \( \overline{f} \) is a left homotopy equivalence if and only if it is a trivial fibration, i.e. if and only if \( f \) is a weak equivalence. So \((i) \iff (ii)\); an analogous argument shows \((i) \iff (iii)\). □
(ii) there exists a naive fibrant replacement of $f$ by a trivial fibration;

(iii) any naive fibrant replacement of $f$ is a trivial fibration.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are immediate from the definition of the weak equivalences. To prove (ii) $\Rightarrow$ (iii), consider a map $f: X \to Y$ having a naive fibrant replacement by a trivial fibration $\overline{f}: \overline{X} \to \overline{Y}$, and an arbitrary naive fibrant replacement $\overline{f}': \overline{X}' \to \overline{Y}'$ of $f$. As depicted below, let $\overline{f}'': \overline{X}'' \to \overline{Y}''$ be a naive fibrant replacement of the induced map between the pushouts $\overline{X} \cup_X \overline{X}' \to \overline{Y} \cup_Y \overline{Y}'$.

The maps $\overline{X} \to \overline{X}'$, $\overline{Y} \to \overline{Y}'$, $\overline{X}' \to \overline{X}''$, $\overline{Y}' \to \overline{Y}''$ are anodyne, as anodyne maps are closed under pushout and composition. Furthermore, $\overline{f}$ is a trivial fibration by assumption. Thus all of these maps are homotopy equivalences by Lemma 2.28 and Corollary 2.30. So we can apply the two-out-of-three property to see that $\overline{f}''$ is a homotopy equivalence; applying it again, we see that $\overline{f}'$ is a homotopy equivalence. Thus $\overline{f}'$ is a trivial fibration by Corollary 2.30. Since $\overline{f}$ was arbitrary, we have shown that $f$ satisfies condition (iii). $\Box$

Corollary 2.34. Every anodyne map is a weak equivalence.

Proof. Let $f: X \to Y$ be anodyne. The following diagram gives a naive fibrant replacement of $f$:

Since $\text{id}_Y$ is a trivial fibration, $f$ is a weak equivalence by Lemma 2.33. $\Box$

Lemma 2.35. The following are equivalent for a marked cubical map $A \to B$:

(i) $A \to B$ is a weak equivalence;

(ii) for any marked cubical quasicategory $X$, the induced map $\text{hom}(B, X) \to \text{hom}(A, X)$ is a homotopy equivalence;
(iii) for any marked cubical quasicategory $X$, the induced map $\pi_0(\text{hom}(B, X)) \to \pi_0(\text{hom}(A, X))$ is a bijection.

Proof. First, suppose that $A \to B$ is a weak equivalence. Thus, there is a square

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
\overline{A} & \to & \overline{B}
\end{array}
\]

with $A \to \overline{A}$ and $B \to \overline{B}$ anodyne, and $\overline{A} \to \overline{B}$ a trivial fibration. By Corollary 2.30, $\overline{A} \to \overline{B}$ is a left homotopy equivalence.

Applying $\text{hom}_L(\cdot, X)$ to the diagram above, we obtain a diagram in which all objects are marked cubical quasicategories by Corollary 2.10:

\[
\begin{array}{ccc}
\text{hom}_L(A, X) & \hookrightarrow & \text{hom}_L(B, X) \\
\uparrow & & \uparrow \\
\text{hom}_L(\overline{A}, X) & \hookrightarrow & \text{hom}_L(\overline{B}, X)
\end{array}
\]

The vertical maps are trivial fibrations by Corollary 2.10, hence homotopy equivalences by Corollary 2.30. By Lemma 2.17, the bottom horizontal map is a right homotopy equivalence, since $\overline{A} \to \overline{B}$ is a left homotopy equivalence. Hence so is the upper horizontal map by 2-out-of-3. Thus we have proven $(i) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (iii)$ is clear, so it remains to show $(iii) \Rightarrow (i)$. For that, we first observe that it suffices to consider $A$ and $B$ marked cubical quasicategories. To see this, consider the canonical naive fibrant replacement $\overline{f}: \overline{A} \to \overline{B}$ of a map $f: A \to B$. By definition, $f$ is a weak equivalence if and only if $\overline{f}$ is a trivial fibration; by Corollary 2.30 and Lemma 2.32, this holds if and only if $\overline{f}$ is a weak equivalence. Furthermore, the anodyne maps $\iota_X, \iota_Y$ are weak equivalences by Corollary 2.34, and therefore satisfy condition $(iii)$; hence $f$ satisfies condition $(iii)$ if and only if $\overline{f}$ does, by the 2-out-of-3 property for bijections.

Hence we can assume $A$ and $B$ are marked cubical quasicategories. Now take $X := A$ and set $g := (\pi_0 f^*)^{-1}[\text{id}_A]$. The verification that a representative of the class $g \in \pi_0 \text{hom}_L(B, A)$ defines a homotopy inverse of $f$ is straightforward; thus $f$ is a weak equivalence by Lemma 2.32. □

Corollary 2.36. The weak equivalences satisfy the 2-out-of-6 property (and hence the 2-out-of-3 property).

Proof. This is immediate from condition $(iii)$ of Lemma 2.35. □

Corollary 2.37. The endpoint inclusions $\square^0 \to K$ are trivial cofibrations.

Proof. The maps in question are clearly cofibrations. To see that they are weak equivalences, consider the following commuting diagram:
The left, right, and bottom maps are anodyne, hence weak equivalences by Corollary 2.34. Thus the top map is a weak equivalence by Corollary 2.36.

**Lemma 2.38.** Trivial fibrations are weak equivalences.

**Proof.** If \( A \to B \) is a trivial fibration, then it is a homotopy equivalence by Corollary 2.30. Hence \( \text{hom}(B, X) \to \text{hom}(A, X) \) is a homotopy equivalence for all marked cubical quasicategories \( X \) by Lemma 2.17 and hence \( A \to B \) a weak equivalence by Lemma 2.35.

**Proposition 2.39.** The category of marked cubical sets forms a cofibration category with the above classes of weak equivalences and cofibrations.

**Proof.** The class of weak equivalences is closed under 2-out-of-3 by Corollary 2.36. The category clearly has an initial object and pushouts. Cofibrations are the left class in a weak factorization system, hence stable under pushout. Using the characterization of weak equivalences given by item (ii) of Lemma 2.35, stability of cofibrations that are weak equivalences under pushout reduces to stability of trivial fibrations under pullback. By Lemma 2.38, trivial fibrations are weak equivalences, so the (cofibration, trivial fibration)-factorization gives the factorization axiom.

### 2.6. Model structure for marked cubical quasicategories.

**Definition 2.40.** A marked cubical set is finite (resp. countable) if it has only finitely (resp. countably) many non-degenerate cubes. The cardinality of a finite marked cubical set is its total number of non-degenerate cubes, in all dimensions.

**Lemma 2.41.** The trivial fibrations form an \( \omega_1 \)-accessible, \( \omega_1 \)-accessibly embedded subcategory of \((\text{cSet}')^\to\).

**Proof.** It suffices to show two things: that filtered colimits (and hence in particular \( \omega_1 \)-filtered colimits) in \( \text{cSet}' \) preserve trivial fibrations, and that any trivial fibration can be expressed as an \( \omega_1 \)-filtered colimit in \( \text{cSet}' \) of trivial fibrations between countable marked cubical sets. The first statement follows from the fact that the domains and codomains of the generating cofibrations are finite.

For the second statement, consider a trivial fibration \( f: X \to Y \). Let \( P \) denote the poset of countable subcomplexes of \( X \); note that we consider edges of subcomplexes of \( X \) to be marked if and only if they are marked in \( X \). This category is \( \omega_1 \)-filtered since any countable union of countable subcomplexes is countable.

Let \( i \) denote the inclusion \( P \hookrightarrow \text{cSet}' \); the colimit of this diagram is \( X \). The images under \( f \) of the countable subcomplexes of \( X \), with the natural inclusions, also define a diagram \( fi: P \to \text{cSet}' \). One can easily show that trivial fibrations are surjective on underlying cubical sets; thus every cube of \( Y \) appears in \( fS \) for some countable subcomplex \( S \subseteq X \). So \( fi \) is a filtered diagram of subcomplexes of \( Y \), in which the maps are inclusions and each cube of \( Y \) is contained in some object in the image of \( fi \).
of the diagram, with every marked edge of $Y$ being marked in some subcomplex in the diagram. From this, one can show that the colimit of $f_i$ is $Y$. The map $f$ induces a natural transformation from $i$ to $f_i$, whose induced map on the colimits is $f$ itself.

However, it may not be the case that for every component of this natural transformation is a trivial fibration. Thus we will replace $i$ by a different diagram, still having colimit $X$, with a natural transformation to $f_i$ which does satisfy this property. For each countable subcomplex $S \subseteq X$, we will define a new countable subcomplex $S' \subseteq X$, such that $fS = fS$, $f_{S'} : S' \to fS$ is a trivial fibration, and for $S' \subseteq S$, we have $S' \subseteq S$.

We first define $S$ for finite $S$, proceeding by induction on cardinality. For $S = \emptyset$, we can simply set $S = \emptyset$. Now assume that we have defined $S$ for $|S| \leq m$, and consider a subcomplex $S$ of cardinality $m + 1$. We will inductively define a family of subcomplexes $S_i$ for $i \geq 0$, each countable and satisfying $fS_i = fS$. Begin by setting $S_0 = S \cup \bigcup_{S' \subseteq S} S'$. Then $S_0$ is countable, $fS_0 = fS$, and for $S' \subseteq S$ we have $S' \subseteq S_0$.

Now assume that we have defined $S_i$ for some $i \geq 0$, and let $D$ be the set of all diagrams $D$ of the form:

$$
\begin{array}{ccc}
\partial \square^n & \partial x & S_i \\
\square^n & y_D & fS
\end{array}
$$

Because $S_i$ and $fS$ are countable, while $\partial \square^n$ and $\square^n$ are finite for any given $n$, there are countably many such diagrams. Because $f$ is a trivial fibration, for each such diagram we may choose a filler in $X$, i.e. an $n$-cube $x_D : \square^n \to X$ whose boundary is $\partial x_D$, such that $f x_D = y_D$. Let $S_i^{i+1} = S_i \cup \bigcup_{D \in D} \{x_D\}$. Then $S_i^{i+1}$ is still countable, since we have added at most countably many cubes to $S$, and its image under $f$ is still $fS$, since each $x_D$ was chosen to map to a specific $y_D \in fS$.

Now let $S = \bigcup_{i \geq 0} S_i$. This is countable, its image is $fS$, and for any $S' \subseteq S$ we have $S' \subseteq S$. Now consider a diagram:

$$
\begin{array}{ccc}
\partial \square^n & \partial x & S \\
\square^n & y & fS
\end{array}
$$

Because $\square^n$ is finite, the image of $\partial x$ is contained in some finite subcomplex of $S$, hence in some $S_i$, so it has a filler in $S_i^{i+1}$ which maps to $y$. Furthermore, $f_{S_i}$ has the right lifting property with respect to the map $\square^1 \to (\square^1)^t$, i.e. an edge $x : \square^1 \to S$ is marked if and only if $x$ is marked, since this is true of edges in $X$. Thus $f_{S_i} : S_i \to fS$ is a trivial fibration.

For a countably infinite $S \subseteq X$ we let $S = \bigcup S_i$, where the union is taken over all finite subcomplexes $S' \subseteq S$. Then $f_{S_i}$ is the filtered colimit of the trivial fibrations $f_{S_i}$, hence it is a trivial fibration.
The subcomplexes $S$ with the natural inclusions define a diagram $\tilde{i}: P \to \text{cSet}'$, and $f$ induces a natural trivial fibration $\tilde{i} \Rightarrow fi$. Observe that $\tilde{i}$ is a filtered diagram of subcomplexes of $X$, in which the maps are inclusions and edges in the objects are marked if and only if they are marked in $X$; furthermore, every cube of $X$ is contained in some finite subcomplex $S$, and hence in $\overline{S}$. From this we can deduce that the colimit of $\tilde{i}$ is $X$, by the same argument we used to show that the colimit of $fi$ is $Y$. The induced map between colimits is $f$; thus we have expressed $f$ as an $\omega_1$-filtered colimit of trivial fibrations between countable marked cubical sets.

**Lemma 2.42.** The weak equivalences form an $\omega_1$-accessible, $\omega_1$-accessibly embedded subcategory of $(\text{cSet}')^\to$.

**Proof.** The (anodyne, naive fibration) factorization gives us a naive fibrant replacement functor $F: (\text{cSet}')^\to \to (\text{cSet}')^\to$. By [Joy09, Prop. D.2.10], this functor is $\omega_1$-accessible, since the domains and codomains of the generating anodyne maps are all countable. By definition, the category of weak equivalences we is given by the following pullback in $\text{Cat}$:

$$
\begin{array}{ccc}
\text{we} & \to & (\text{cSet}')^\to \\
\downarrow & & \downarrow F \\
\text{tfib} & \to & (\text{cSet}')^\to
\end{array}
$$

By Lemma 2.41, tfib is an $\omega_1$-accessible category, and its embedding into $(\text{cSet}')^\to$ is an $\omega_1$-accessible functor. By [MPS9] Thm. 5.1.6, the category of $\omega_1$-accessible categories and $\omega_1$-accessible functors has finite limits, and these are computed in $\text{Cat}$. Thus we is $\omega_1$-accessible, and its embedding into $(\text{cSet}')^\to$ is an $\omega_1$-accessible functor. □

**Theorem 2.43** (Analogue of model structure on marked simplicial sets). The above classes of weak equivalences, cofibrations, and fibrations define a model structure on $\text{cSet}'$. 

**Proof.** We verify the assumptions of Theorem 1.1.

The category of marked cubical sets is locally finitely presentable. Weak equivalences are an $\omega_1$-accessibly embedded, $\omega_1$-accessible subcategory of $(\text{cSet}')^\to$ by Lemma 2.42. Cofibrations have a small set of generators by Lemma 2.1.

Weak equivalences are closed under 2-out-of-3 and weak equivalences that are cofibrations are closed under pushout by Proposition 2.39. Weak equivalences are closed under transfinite composition by Lemma 2.42, implying that the same holds for trivial cofibrations. Every map lifting against cofibrations is a weak equivalence by Lemma 2.38. □

We refer to the model structure constructed above as the *cubical marked model structure*. We will now analyze this model structure, beginning with a strengthening of Lemma 2.38 and Corollary 2.10.

**Lemma 2.44.** If $X \to Y$ is a weak equivalence, then so is $A \otimes X \to A \otimes Y$ for any $A \in \text{cSet}'$.

**Proof.** By the adjunction $A \otimes - \dashv \text{hom}_R(A, -)$, for $Z \in \text{cSet}'$ we have a natural isomorphism $\text{hom}_R(A \otimes X, Z) \cong \text{hom}_R(X, \text{hom}_R(A, Z))$. Let $Z$ be a marked cubical quasicategory; then we have a commuting diagram
By Corollary 2.10, $\text{hom}_R(A, Z)$ is a marked cubical quasicategory, so the bottom map is a homotopy equivalence by Lemma 2.35. Hence the top map is a homotopy equivalence; thus we see that $A \otimes Y \to A \otimes Y$ is a weak equivalence by Lemma 2.35. □

Lemma 2.45. The pushout product of a cofibration and a weak equivalence is a weak equivalence.

Proof. Let $i: A \to B$ be a cofibration and $f: X \to Y$ a weak equivalence; we will show that $i \otimes f$ is a weak equivalence (the case of $f \otimes i$ is similar). Consider the diagram which defines $i \otimes f$:

The maps $A \otimes X \to A \otimes Y$ and $B \otimes X \to B \otimes Y$ are weak equivalences by Lemma 2.44. The map $A \otimes X \to B \otimes X$ is a cofibration by Lemma 2.9. The model structure is left proper, since all objects are cofibrant; thus the map from $B \otimes X$ into the pushout is a weak equivalence. Hence $i \otimes f$ is a weak equivalence by 2-out-of-3. □

Corollary 2.46. Let $i: A \to B, j: A' \to B'$ be cofibrations. If either $i$ or $j$ is trivial, then so is the pushout product $i \otimes j$.

Proof. This is immediate from Lemmas 2.40 and 2.45. □

Corollary 2.47. If $i$ is a cofibration and $f$ is a fibration, then the pullback exponential $i \triangleright f$ is a fibration, which is trivial if $i$ or $f$ is trivial. □

Corollary 2.48. The category $\text{cSet}'$, equipped with the cubical marked model structure and the geometric product, is a monoidal model category. □

Next we will characterize the fibrant objects, and fibrations between fibrant objects, of this model structure.

Proposition 2.49. A map between marked cubical quasicategories is a fibration if and only if it is a naive fibration. In particular, the fibrant objects of the cubical marked model structure are precisely the marked cubical quasicategories.

Proof. It is clear that every fibration is a naive fibration. Now let $f: X \to Y$ be a naive fibration between marked cubical quasicategories, and $i: A \to B$ a trivial cofibration. We wish to show that $f$ has the right lifting property with respect to $i$; for this it suffices to show that $i \triangleright f$ has the right
lifting property with respect to the map \( \emptyset \to \Box^0 \). For this, in turn, it suffices to show that \( i \triangleright f \) is a trivial fibration.

First, note that \( i \triangleright f \) is a naive fibration between marked cubical quasicategories by Corollary 2.10. Therefore, by Corollary 2.30 it is a trivial fibration if and only if it is a homotopy equivalence. Now consider the diagram which defines \( i \triangleright f \):

\[
\begin{array}{ccc}
\text{hom}(B, X) & \xrightarrow{i \triangleright f} & \text{hom}(A, X) \\
\downarrow & & \downarrow \\
\text{hom}(A, X) \times \text{hom}(A, Y) & \xrightarrow{\text{hom}(B, Y)} & \text{hom}(A, Y)
\end{array}
\]

The maps \( \text{hom}(B, X) \to \text{hom}(A, X) \) and \( \text{hom}(B, Y) \to \text{hom}(A, Y) \) are trivial fibrations by Corollary 2.47, the map from the pullback to \( \text{hom}(A, X) \) is a trivial fibration as a pullback of a trivial fibration. Thus \( i \triangleright f \) is a weak equivalence by 2-out-of-3, hence a homotopy equivalence by Lemma 2.32.

3. Model structure on structurally marked cubical sets

The model structure on marked cubical sets described in the previous section resembles the Cisinski model structure on a presheaf category. In this section, we show that the category \( \text{cSet}'' \) of structurally marked cubical sets (see Subsection 1.4) admits a Cisinski model structure which right induces the model structure on marked cubical sets from the previous section via the embedding \( \text{cSet}' \hookrightarrow \text{cSet}'' \), and that the two are Quillen equivalent.

Since \( \text{cSet}'' \) is a presheaf category, we may apply Theorem 1.5 in order to construct a model structure on this category. To do that, we first find a cellular model for \( \text{cSet}'' \), i.e., a generating set of monomorphisms, using the Reedy category structure of \( \Box_\sharp \), established in Proposition 1.29.

**Lemma 3.1.** The monomorphisms of \( \text{cSet}'' \) are the saturation of the set consisting of the boundary inclusions \( \partial \Box^n \hookrightarrow \Box^n \) and the inclusion \( \Box^1 \hookrightarrow (\Box^1)^\sharp \).

The functor \( (\Box^1)^\sharp \otimes - : \text{cSet}'' \to \text{cSet}' \), together with the natural transformations \( \partial_{1,0}^1 \otimes -, \partial_{1,1}^1 \otimes - : \text{id} \to (\Box^1)^\sharp \otimes - \), and \( \pi : (\Box^1)^\sharp \otimes - \to \text{id} \), defines a cylinder functor on \( \text{cSet}'' \) in the sense of Definition 1.2.

Thus we have a notion of homotopy defined in terms of this cylinder functor: an elementary homotopy \( f \sim g : X \to Y \) is a map \( H : (\Box^1)^\sharp \otimes X \to Y \) with \( H|_{\{0\}\otimes X} = f, H|_{\{1\}\otimes X} = g \), and a homotopy is a zigzag of elementary homotopies. In keeping with the notation of Subsection 1.4 we will write \([X, Y]\) for the set of homotopy classes of maps from \( X \) to \( Y \).

**Lemma 3.2.**

(i) The cylinder functors in \( \text{cSet}' \) and \( \text{cSet}'' \) agree, i.e., the latter is the image of the former under the embedding \( \text{cSet}' \to \text{cSet}'' \).

(ii) The model structure on \( \text{cSet}' \) and \( \text{cSet}'' \) are Quillen equivalent.
(ii) For marked cubical sets $X$ and $Y$, the embedding $\text{cSet}' \to \text{cSet}''$ induces a bijection
$$[X,Y]_{\text{cSet}'} \to [X,Y]_{\text{cSet}''},$$
where the subscript indicates which category the homotopy classes are taken in.

Proof. Both of these statements follow easily from the fact that the embedding $\text{cSet}' \to \text{cSet}''$ is monoidal, established in Corollary 1.33. □

Let $S$ be the set of maps in $\text{cSet}''$ consisting of the following maps:
- the special open box inclusions,
- the saturation map, and
- the 3-out-of-4 maps.

Definition 3.3. A map of structurally marked cubical sets is anodyne if it is in the saturation of $S$.

Note that the anodyne generators in $\text{cSet}''$ are precisely those of $\text{cSet}'$, embedded via $\text{cSet}' \to \text{cSet}''$.

Remark 3.4. It might seem natural to include the map $(\square^1)^2 \to (\square^1)^{2\sharp}$, the inclusion of the marked interval into the interval with two distinct markings, in $S$, so that adding a marking to an already-marked edge of a structurally marked cubical set would not change its homotopy type. In fact, however, this map is already anodyne, as it is a pushout of a 3-out-of-4 map.

The following lemma shows that this definition of anodyne maps is consistent with that of Subsection 1.1.

Lemma 3.5. The set $\Lambda(S)$ is contained in the saturation of $S$.

Proof. The proof of Lemma 2.9 applies equally well in this context, showing that a pushout product of a monomorphism with a map in the saturation of $S$ is again in the saturation of $S$. This implies that $\Lambda^n(S)$ is contained in the saturation of $S$; applying the same lemma inductively, we see that each set $\Lambda^n(S)$ is contained in the saturation of $S$. □

Theorem 3.6. The category $\text{cSet}''$ of structurally marked cubical sets carries a cofibrantly generated model structure in which:
- the cofibrations are the monomorphisms;
- the fibrant objects, and fibrations between fibrant objects, are defined by the right lifting property with respect to the set of generating anodyne maps $S$;
- the weak equivalences are maps $X \to Y$ inducing bijections $[Y,Z] \to [X,Z]$ for all fibrant objects $Z$.

Proof. The existence of the model structure follows from Theorem 1.5. Lemma 3.5 shows that the set of generating anodyne maps is exactly $S$. □

The remainder of this section will be devoted to analyzing this model structure and its relationship with the model structure on marked cubical sets of Theorem 2.43. More precisely, we will prove:
Theorem 3.7.

(i) The adjunction $cSet'' \rightleftarrows \text{cSet}'$ is a Quillen equivalence.

(ii) The cubical marked model structure is right induced from the model structure of Theorem 3.6 by the embedding $cSet' \rightarrow cSet''$.

Before proving this theorem, we establish a number of intermediate results.

Proposition 3.8. The adjunction $cSet'' \rightleftarrows \text{cSet}'$ is a Quillen adjunction between the model structure of Theorem 3.6 to the cubical marked model structure.

Proof. By Corollary 1.12, it suffices to show that $\text{Im}$ preserves monomorphisms and takes generating anodynes to anodynes. Both of these statements are immediate. □

Lemma 3.9. A map of structurally marked cubical sets $f : X \rightarrow Y$ is a trivial fibration if and only if the underlying map of cubical sets is a trivial fibration in model structure of Theorem 1.28 (i.e., has the right lifting property with respect to monomorphisms) and, for each edge $x$ of $X$, the map from the set of markings of $x$ to that of $fx$ is surjective.

Proof. By Lemma 3.1, $f$ is a trivial fibration if and only if it has the right lifting property with respect to all boundary inclusions and the inclusion of the interval into the marked interval. Having the right lifting property with respect to all boundary inclusions is equivalent to being a trivial fibration on underlying cubical sets; having the right lifting property with respect to the inclusion of the interval into the marked interval is equivalent to each map of marking sets being surjective. □

Corollary 3.10. For all structurally marked cubical sets $X$, the adjunction unit $X \rightarrow \text{Im}X$ is a trivial fibration. □

Lemma 3.11. The functor $\text{Im} : cSet'' \rightarrow \text{cSet}'$ preserves fibrations and trivial fibrations.

Proof. We will show that if $p : X \rightarrow Y$ is a fibration between structurally marked cubical sets, then $\text{Im}p : \text{Im}X \rightarrow \text{Im}Y$ is also a fibration. Given a trivial cofibration of marked cubical sets $i : A \rightarrow B$ with maps $\alpha : A \rightarrow \text{Im}X$ and $\beta : B \rightarrow \text{Im}Y$ making the square commute, apply Corollary 3.10 to $\emptyset \rightarrow A \rightarrow \text{Im}X$ to get $\alpha' : A \rightarrow X$ with $\alpha = u_X \alpha'$ and then again to the square

\[
\begin{array}{ccc}
A & \rightarrow^p & Y \\
| & \downarrow & \\
B & \rightarrow & \text{Im}Y
\end{array}
\]

to get $\beta' : B \rightarrow Y$ that fits into a square

\[
\begin{array}{ccc}
A & \rightarrow^\alpha & X \\
| & \downarrow & \\
B & \rightarrow & \text{Im}X
\end{array}
\]

to get $\beta' : B \rightarrow Y$ that fits into a square

\[
\begin{array}{ccc}
A & \rightarrow^{\alpha'} & X \\
| & \downarrow & \\
B & \rightarrow & Y
\end{array}
\]

whose lift $L : B \rightarrow X$ yields $u_X L : B \rightarrow \text{Im}X$ which satisfies the equations.
(Imp)\( u_X L = u_Y p L = u_Y \beta' = \beta \) and \( u_x Li = u_x \alpha' = \alpha \)
and so provides the lift. Thus Imp is a fibration.

The proof for trivial fibrations is analogous.

Lemma 3.12. Let \( X \) be a structurally marked cubical set. Then \( X \) is fibrant if and only if \( \text{Im} X \) is fibrant (in the model structure of Theorem 3.6).

Proof. If \( \text{Im} X \) is fibrant, then \( X \) is fibrant by Corollary 3.10. Conversely, if \( X \) is fibrant, then \( \text{Im} X \) is fibrant in \( \text{cSet}' \) by Lemma 3.11 hence also in \( \text{cSet}'' \) by Proposition 3.8.

Proof of Theorem 3.7. First, let us show that the right derived functor of the embedding \( \text{cSet}' \to \text{cSet}'' \) is an equivalence. Since \( [X,Y]_{\text{cSet}'} \to [X,Y]_{\text{cSet}''} \) is bijective by Lemma 3.2, it is full and faithful. For essential surjectivity, by Corollary 1.6 given fibrant \( X \in \text{cSet}'' \), we need fibrant \( Y \in \text{cSet}' \) weakly equivalent to \( X \) in \( \text{cSet}'' \). This is given by Lemmas 3.11 and 3.12.

Now, let us show that cubical marked model structure is right induced. Since \( \text{Im} \) is a left Quillen equivalence and all objects are cofibrant, it preserves and reflects weak equivalences by Proposition 1.13, hence so does the embedding. Since \( \text{Im} \) preserves fibrations, the embedding reflects them.

Preservation of fibrations is part of Proposition 3.8.

4. Joyal model structure on cubical sets

Recall the adjunction \( \text{cSet} \leftrightarrow \text{cSet}' \) of Subsection 1.4, in which the left adjoint is the minimal marking functor and the right adjoint is the forgetful functor. In this section we will use this adjunction to induce a model structure on \( \text{cSet} \) from the model structure on \( \text{cSet}' \) of Theorem 2.43.

Theorem 4.1 (Analogue of Joyal model structure). The category \( \text{cSet} \) of cubical sets carries a model structure in which:

- the cofibrations are the monomorphisms,
- the weak equivalences are created by the minimal marking functor,
- the fibrations are right orthogonal to trivial cofibrations.

Proof. Apply Theorem 1.10 to the adjunction \( \text{cSet} \leftrightarrow \text{cSet}' \) and the cubical marked model structure, with the factorization \( X \sqcup X \to K \otimes X \to X \). That the minimal marking functor sends the first of these maps to a cofibration is clear; that it sends the second to a weak equivalence follows from Corollaries 2.37 and 2.46.

We refer to the model structure constructed above as the cubical Joyal model structure.

Proposition 4.2. The adjunction \( \text{cSet} \leftrightarrow \text{cSet}' \) is a Quillen equivalence.

Proof. The minimal marking functor preserves and reflects weak equivalences by definition, thus we may apply Corollary 1.14 and item (ii). Let \( X \) be a marked cubical quasicategory; abusing notation slightly, let \( X^\circ \) denote the minimal marking of the underlying cubical set of \( X \). We must show that the inclusion \( X^\circ \to X \) is a weak equivalence.
The marked edges of $X^\flat$ are precisely the degenerate edges; by Lemma 2.3, the marked edges of $X$ are precisely those edges $\Box^1 \to X$ which factor through $K$. Thus $X^\flat \to X$ is a (possibly transfinite) composite of pushouts of saturation maps, hence a trivial cofibration.

We define some terminology which will be used in the analysis of this model structure.

- For $n \geq 2$, $1 \leq i \leq \varepsilon \in \{0, 1\}$, the $(i, \varepsilon)$ inner open box, denoted $\flat^n_{i,\varepsilon}$, is the quotient of an open box with the critical edge quotiented to a point. The $(i, \varepsilon)$ inner cube, denoted $\hat{\Box}^n_{i,\varepsilon}$, is defined similarly.
- An inner fibration is a map having the right lifting property with respect to the inner open box inclusions.
- An isofibration is a map having the right lifting property with respect to the endpoint inclusions $\Box^0 \to K$.
- A cubical quasicategory is a cubical set $X$ such that the map $X \to \Box^0$ is an inner fibration.
- For $n \geq 2$, $1 \leq i \leq n, \varepsilon \in \{0, 1\}$, a special open box in a cubical set $X$ is a map $\flat^n_{i,\varepsilon} \to X$ which sends the critical edge to an equivalence.

The concept of homotopy developed in Section 2 adapts naturally to this setting, using equivalences in place of marked edges.

**Definition 4.3.** For a cubical set $X$, let $\sim_0$ denote the relation on $X_0$, the set of vertices of $X$, given by $x \sim_0 y$ if there is an equivalence from $x$ to $y$ in $X$. Let $\sim$ denote the smallest equivalence relation on $X_0$ containing $\sim_0$.

**Remark 4.4.** For $x, y \in X_0$, one can easily see that $x \sim y$ if and only if $x$ and $y$ are connected by a zig-zag of equivalences.

**Definition 4.5.** For a cubical set $X$, the set of connected components $\pi_0(X)$ is $X_0/\sim$.

**Definition 4.6.** An elementary left homotopy $h: f \sim g$ between maps $f, g: A \to B$ is a map $h: K \otimes A \to B$ such that $h|_{\{0\} \otimes A} = f$ and $h|_{\{1\} \otimes A} = g$. Note that the elementary left homotopy $h$ corresponds to an edge $K \to \text{hom}_L(A, B)$ between the vertices corresponding to $f$ and $g$. A left homotopy between $f$ and $g$ is a zig-zag of elementary left homotopies.

A left homotopy from $f$ to $g$ corresponds to a zig-zag of equivalences in $\text{hom}_L(A, B)$ and so maps from $A$ to $B$ are left homotopic exactly if $\pi_0(f) = \pi_0(g)$, where the set of connected components is taken in $\text{hom}_L(A, B)$.

These induce notions of elementary left homotopy equivalence and left homotopy equivalence. Each of these notions has a "right" variant using $A \otimes K$ and $\text{hom}_R(A, B)$. As in Section 2, unless the potential for confusion arises or a statement depends on the choice, we will drop the use of "left" and "right".

**Definition 4.7.** Let $X$ be a cubical set. The natural marking on $X$ is a marked cubical set $X^\natural$ whose underlying cubical set is $X$, with edges marked if and only if they are equivalences.
It is easy to see that this defines a functor \((-)^\natural\) : \(\mathbf{cSet} \to \mathbf{cSet}'\), as maps of cubical sets preserve equivalences.

Many results about the cubical Joyal model structure follow easily from the corresponding results about the cubical marked model structure.

**Lemma 4.8.** If \(i, j\) are cofibrations in \(\mathbf{cSet}\), then the pushout product \(i \hat\otimes j\) is a cofibration. Moreover, if either \(i\) or \(j\) is trivial then so is \(i \hat\otimes j\).

**Proof.** This is immediate from Corollaries 1.33 and 2.46 and Lemma 2.9. □

**Corollary 4.9.** Let \(i, f\) be maps in \(\mathbf{cSet}\). If \(i\) is a cofibration and \(f\) is a fibration, then the pullback exponential \(i \triangleright f\) is a fibration. □

**Corollary 4.10.** The category \(\mathbf{cSet}\), equipped with the cubical Joyal model structure and the geometric product, is a monoidal model category. □

Next we will characterize the fibrant objects, and fibrations between fibrant objects, in the cubical Joyal model structure.

**Lemma 4.11.** The inner open box inclusions \(\mathcal{O}_{i,\varepsilon}^n \to \mathcal{O}_{i,\varepsilon}\), and the endpoint inclusions \(\square^0 \to K\), are trivial cofibrations.

**Proof.** The minimal marking of an inner open box inclusion is a pushout of a special open box inclusion in \(\mathbf{cSet}'\). The minimal marking of \(\square^0 \to K\) is a trivial cofibration by Corollary 2.37. □

**Lemma 4.12.** Cubical quasicategories have fillers for special open boxes.

**Proof.** We only consider positive filling problems; the negative case is dual. We argue by induction on the dimension of the filling problem. For a special open box of dimension 2, it is a simple exercise to explicitly construct a filler by extending the given open box to an inner open box of dimension 3.

Now let \(X\) be a cubical quasicategory, and suppose that \(X\) has fillers for all special open boxes of dimension less than \(n\). Consider a filling problem in \(X\) of dimension \(n\):

\[
(\partial \square^a \times \square^1 \times \square^b) \cup (\square^a \times \{0\} \times \square^b) \cup (\square^a \times \square^1 \times \partial \square^b) \xrightarrow{\square^a \times \square^1 \times \square^b} X
\]

We regard the codomain of the left map as a negative face of a larger cube via the map

\[
\square^a \times \square^1 \times \{0\} \times \square^b \rightarrow \square^a \times \square^1 \times \{0\} \times \square^b
\]
and the domain as the corresponding subobject. The original filling problem then becomes a filling problem in $X$ of the form

$$(\partial \square^n \otimes \square^1 \otimes \{0\} \otimes \square^b)$$

$$(\square^n \otimes \{0\} \otimes \{0\} \otimes \square^b)$$

$$(\square^n \otimes \square^1 \otimes \{0\} \otimes \partial \square^b)$$

$$\rightarrow \square^n \otimes \square^1 \otimes \{0\} \otimes \square^b$$

where the critical edge is

$$(0^n 000^b \rightarrow 0^n 100^b).$$

We will solve this problem by extending the given partial data to the whole of

$$\square^n \otimes \square^1 \otimes \square^1 \otimes \square^b.$$ 

For $n \geq 0$, let $\Gamma^n \subseteq \square^n$ denote the union of the positive faces. We use degeneracies in the new direction to fill

$$(\Gamma^n \otimes \square^1 \otimes \{0\} \otimes \square^b)$$

$$(\Gamma^n \otimes \square^1 \otimes \square^1 \otimes \square^b)$$

$$(\square^n \otimes \{0\} \otimes \{0\} \otimes \square^b) \rightarrow (\square^n \otimes \{0\} \otimes \square^1 \otimes \square^b)$$

$$(\square^n \otimes \square^1 \otimes \{0\} \otimes \Gamma^b) \quad (\square^n \otimes \square^1 \otimes \square^1 \otimes \Gamma^b).$$

Since the critical edge is an equivalence, we can fill the square

$$0^n 000^b \longrightarrow 0^n 010^b$$

$$\downarrow \quad \downarrow$$

$$0^n 100^b \longrightarrow 0^n 110^b$$

where the dotted edge is again an equivalence.

In the following, we will indicate the filling direction of (generalized) open boxes by underlining the appropriate factor in the pushout monoidal product. What this means is that we can factor the given generalized open box inclusion as a series of open box fillings in different dimensions, each of which fills in the specified direction. We now fill the generalized open box

$$(\{0^n\} \otimes (\{0\} \rightarrow \square^1) \otimes (\{0\} \rightarrow \square^1) \otimes (\{0\} \rightarrow \square^b)$$

if $a, b \geq 1$. Here, the critical edges are of the form $uv0w \rightarrow uv1w$ where $u, v, w$ are certain vertices of $\square^n, \square^1, \square^b$, respectively. All of these edges are degenerate except for the bottom edge in (4.1), which is an equivalence. Moreover, this edge only appears as a critical edge in filling problems of lower dimension. So we may indeed fill this generalized open box using fibrancy of $X$ and the induction hypothesis. Dually, we fill the generalized open box

$$((0^n) \rightarrow \square^n) \otimes (\{0\} \rightarrow \square^1) \otimes (\{0\} \rightarrow \square^1) \otimes \{0^b\}$$

if $a, b \geq 1$.

We now fill the generalized open box

$$((0^n) \cup \Gamma^n \rightarrow \partial \square^n) \otimes (\{0\} \rightarrow \square^1) \otimes (\{0\} \rightarrow \square^1) \otimes (\partial \square^b \rightarrow \square^b)$$
if \( a \geq 1 \). Again, the critical edges are of the form as above and we may argue as before. Dually, we fill the generalized open box

\[
(\partial \Box^a \to \Box^a) \otimes (\{0\} \to \Box^1) \otimes (\{0\} \to \Box^1) \otimes (\{0^b\} \cup \Gamma^b \to \partial \Box^b)
\]

if \( b \geq 1 \).

At this stage, we have defined the cube on

\[
(\partial \Box^a \otimes \Box^1 \otimes \Box^1 \otimes \Box^b) \\
\cup (\Box^a \otimes \{0\} \otimes \Box^1 \otimes \Box^b) \\
\cup (\Box^a \otimes \Box^1 \otimes \Box^1 \otimes \partial \Box^b) .
\]

We now fill the open box

\[
(\partial \Box^a \to \Box^a) \otimes (\{0\} \to \Box^1) \otimes (\emptyset \to \{1\}) \otimes (\partial \Box^b \to \Box^b) ,
\]

noting that the critical edge \( 0^a000^b \to 0^a100^b \) is degenerate. We then fill the open box

\[
(\partial \Box^a \to \Box^a) \otimes (\emptyset \to \{1\}) \otimes (\{0\} \to \Box^1) \otimes (\partial \Box^b \to \Box^b) ,
\]

noting that the critical edge \( 0^a000^b \to 0^a010^b \) is degenerate. We finally fill the open box

\[
(\partial \Box^a \to \Box^a) \otimes (\partial \Box^1 \to \Box^1) \otimes (\{0\} \to \Box^1) \otimes (\partial \Box^b \to \Box^b) ,
\]

noting that the critical edge \( 0^a000^b \to 0^a010^b \) is degenerate. This defines the entire cube. \( \square \)

**Lemma 4.13.** Inner fibrations between cubical quasicategories lift against special open box inclusions.

**Proof.** Again we only consider positive filling problems; the negative case is dual. Again we argue by induction on the dimension of the filling problem, with the case for dimension 2 being a simple exercise in filling three-dimensional open boxes, entirely analogous to the base case of the previous proof. Consider a lifting problem

\[
(\partial \Box^a \otimes \Box^1 \otimes \Box^1 \otimes \Box^b) \cup (\Box^a \otimes \{0\} \otimes \Box^1 \otimes \Box^b) \cup (\Box^a \otimes \Box^1 \otimes \partial \Box^b) \to X
\]

\[
\Box^a \otimes \Box^1 \otimes \Box^b \to Y
\]

where the right map is an inner fibration between cubical quasicategories. As before, we regard the codomain of the left map as a negative face of a larger cube via the map

\[
\Box^a \otimes \Box^1 \otimes \Box^b \to \Box^a \otimes \Box^1 \otimes \{0\} \otimes \Box^b
\]

and the domain as the corresponding subobject \( H \). The critical edge is once again \( 0^a000^b \to 0^a100^b \). Let \( H' \) be the union of \( H \) with the subobjects

\[
(\Gamma^a \otimes \Box^1 \otimes \Box^1 \otimes \Box^b) \\
\cup (\Box^a \otimes \{0\} \otimes \Box^1 \otimes \Box^b) \\
\cup (\Box^a \otimes \Box^1 \otimes \Box^1 \otimes \Gamma^b)
\]

and \( H'' \) be the union of \( H' \) with the square

\[
\{0^a\} \otimes \Box^1 \otimes \Box^1 \otimes \{0^b\}.
\]
We use degeneracies in the new direction to extend the map to $X$ from $H$ to $H'$:

\[
\begin{array}{c}
H \rightarrow X,
\end{array}
\]

Since the critical edge is special in $X$, we extend the map to $X$ from $H'$ to $H''$ by filling the square

\[
\begin{array}{c}
0^a000^b \rightarrow 0^a010^b \\
\downarrow \downarrow \\
0^a100^b \rightarrow 0^a110^b
\end{array}
\]

where the dotted edge is again special in $X$. Note that the map $X \rightarrow Y$ preserves special edges.

We construct the dotted arrow in the diagram

\[
\begin{array}{c}
H \rightarrow H'' \rightarrow X \\
\downarrow \downarrow \\
□^a \otimes □^1 \otimes \{0\} \otimes □^b \rightarrow □^a \otimes □^1 \otimes □^1 \otimes □^b \\
\downarrow \\
□^a \otimes □^1 \otimes □^1 \otimes □^b \\
\rightarrow Y
\end{array}
\]

by solving a filling problem

\[
\begin{array}{c}
\langle □^a \otimes □^1 \otimes \{0\} \otimes □^b \rangle \cup H'' \rightarrow Y \\
\downarrow \\
\langle □^a \otimes □^1 \otimes □^1 \otimes □^b \rangle
\end{array}
\]

as follows: the left map factors as a finite composite of open box inclusions of the form

\[
(\partial □^a' \rightarrow □^a') \otimes (\{0\} \rightarrow □^1) \otimes (\{0\} \rightarrow □^1) \otimes (\partial □^b' \rightarrow □^b')
\]

where $□^a'$ and $□^b'$ are faces of $□^a$ and $□^b$, respectively. All critical edges are of the form $uv0w \rightarrow uv1w$ where $u, v, w$ are certain points of $□^a, □^1, □^b$, respectively. All of these edges are degenerate in $Y$ except for the bottom edge in (4.1), which is special. We can thus fill these open boxes using fibrancy of $Y$ and Lemma 4.12.

It remains to construct a lift

\[
\begin{array}{c}
H'' \rightarrow X \\
\downarrow \\
□^a \otimes □^1 \otimes □^1 \otimes □^b \\
\rightarrow Y
\end{array}
\]

which is done exactly as in the proof of Lemma 4.12 using that $X \rightarrow Y$ is a fibration.

\[\square\]

**Lemma 4.14.** If $X$ is a cubical quasicategory, then $X^\natural$ is a marked cubical quasicategory.

**Proof.** Given a cubical quasicategory $X$, we have fillers for special open boxes in $X$ by Lemma 4.12. This implies that $X^\natural$ has fillers for special open boxes in the sense of Definition 2.2. Furthermore, the definition of the natural marking implies that $X^\natural$ has the right lifting property with respect to
the saturation map for any cubical set $X$. By Lemma 2.6 this suffices to show that $X^\natural$ is a marked cubical quasicategory.

**Proposition 4.15.** The fibrant objects of the the cubical Joyal model structure are given by cubical quasicategories. The fibrations between fibrant objects are characterized by lifting against inner open box inclusions and endpoint inclusions $\Box^n \hookrightarrow K$.

**Proof.** By Lemma 4.11 every fibrant object is a cubical quasicategory and every fibration is an inner isofibration.

If $X$ is a cubical quasicategory, then $X^\natural$ is a marked cubical quasicategory by Lemma 4.14. The forgetful functor $\text{cSet}' \to \text{cSet}$ preserves fibrant objects as a right Quillen adjoint, and the underlying cubical set of $X^\natural$ is $X$, thus $X$ is fibrant.

The case of fibrations between fibrant objects proceeds in an analogous way. Let $f : X \to Y$ be an inner isofibration between cubical quasicategories; we will show that $f^\natural$ is a fibration in $\text{cSet}'$. Lifting against one-dimensional special open box inclusions follows from the isofibration property; lifting against higher-dimensional special open box inclusions follows from Lemma 4.13. Since $X^\natural$ is a marked cubical quasicategory, $f^\natural$ has the right lifting property with respect to the saturation and 3-out-of-4 maps by Lemma 2.0 and the fact that these maps are epimorphisms in $\text{cSet}'$. Since $X^\natural$ and $Y^\natural$ are marked cubical quasicategories, this implies that $f^\natural$ is a fibration by Proposition 2.49.

**Corollary 4.16.** Let $f : X \to Y$ be a map between cubical quasicategories. Then $f$ is a weak equivalence if and only if it is a homotopy equivalence.

**Corollary 4.17.** Let $X,Y \in \text{cSet}$, with $Y$ a cubical quasicategory. Then $\text{hom}(X,Y)$ is a cubical quasicategory.

**Proof.** This follows from Corollary 4.9 and Proposition 4.15.

Our next goal will be to characterize the weak equivalences in the cubical Joyal model structure in a manner similar to Lemma 2.35.

**Lemma 4.18.** For $X \in \text{cSet}$, we have a natural isomorphism $\pi_0 X \cong \pi_0 X^\natural$.

**Proof.** It is clear that $X$ and $X^\natural$ have the same set of vertices. To see that the equivalence relations defining $\pi_0 X$ and $\pi_0 X^\natural$ coincide, observe that a pair of vertices are connected by a zigzag of marked edges in $X^\natural$ if and only if they are connected by a zigzag of equivalences in $X$.

**Lemma 4.19.** Let $X,Y \in \text{cSet}$, and let $Y'$ be a marked cubical set whose underlying cubical set is $Y$. The underlying cubical set of $\text{hom}(X^\natural, Y')$ is $\text{hom}(X,Y)$.

**Proof.** The $n$-cubes in the underlying cubical set of $\text{hom}(X^\natural, Y')$ are maps $X^\natural \otimes \Box^n \cong (X \otimes \Box^n)^\natural \to Y'$ (the isomorphism follows from Corollary 1.33). Under the adjunction $\text{cSet} \rightleftarrows \text{cSet}'$, these correspond to maps $X \otimes \Box^n \to Y$.

**Proposition 4.20.** The following are equivalent for a cubical map $A \to B$:

(i) $A \to B$ is a weak equivalence;
(ii) for any cubical quasicategory $X$, the induced map $\hom(B, X) \to \hom(A, X)$ is a homotopy equivalence;

(iii) for any cubical quasicategory $X$, the induced map $\pi_0(\hom(B, X)) \to \pi_0(\hom(A, X))$ is a bijection.

Proof. To see that (i) $\Rightarrow$ (ii), let $A \to B$ be a weak equivalence in $\cSet$, and $X$ a marked cubical quasicategory. Then $X^2$ is a marked cubical quasicategory by Lemma 4.14, so $\hom(B^p, X^2) \to \hom(A^p, X^2)$ is a homotopy equivalence by Lemma 2.35. The underlying cubical set functor preserves weak equivalences between fibrant objects by Ken Brown’s lemma, so $\hom(B, X) \to \hom(A, X)$ is a weak equivalence by Lemmas 4.16 and 4.17. Hence it is a homotopy equivalence by Corollaries 4.16 and 4.17.

The implication (ii) $\Rightarrow$ (iii) is clear, so now we consider (iii) $\Rightarrow$ (i). For this, let $X$ be the underlying cubical set of a marked cubical quasicategory $X'$, and note that by Lemma 4.18 and Lemma 4.19, we have the following commuting diagram in $\Set$:

$$
\begin{array}{ccc}
\pi_0(\hom(B, X)) & \to & \pi_0(\hom(A, X)) \\
\downarrow \cong & & \downarrow \cong \\
\pi_0(\hom(B^p, X')) & \to & \pi_0(\hom(A^p, X'))
\end{array}
$$

Since the underlying cubical set functor preserves fibrant objects, $X$ is a cubical quasicategory. So if (iii) holds then the top map is an isomorphism, hence so is the bottom map. Thus $A^p \to B^p$ is a weak equivalence in $\cSet'$ by Lemma 2.35 meaning that $A \to B$ is a weak equivalence in $\cSet$. □

We now state two straightforward properties of the cubical Joyal model structure.

**Proposition 4.21.**

(i) The Grothendieck model structure on $\cSet$ of Theorem 1.28 is a localization of the cubical Joyal model structure.

(ii) The adjunction $\tau_1 : \cSet \rightleftarrows \Cat : N\square$ is a Quillen adjunction between the canonical model structure on $\Cat$ and the cubical Joyal model structure. □

One of the advantages of working with cubical quasicategories as opposed to their simplicial analogues is a clean definition of a mapping space between two objects in a cubical quasicategory.

**Definition 4.22.** Let $x_0$ and $x_1$ be 0-cubes in a cubical quasicategory $X$. The mapping space from $x_0$ to $x_1$ is the cubical set $\Map_X(x_0, x_1)$ given by

$$
\Map_X(x_0, x_1)_n = \left\{ \Box^{n+1} \xrightarrow{s} X \mid s\partial_{n+1, \epsilon} = x_\epsilon \right\},
$$

with cubical operations given by those of $X$.

There is a clear geometric intuition behind this definition, as the example below shows.

**Example 4.23.** Given a cubical quasicategory $X$ and 0-cubes $x_0, x_1 : \Box^0 \to X$, we have that:

- a 0-cube in $\Map_X(x_0, x_1)$ is a 1-cube from $x_0$ to $x_1$ in $X$;
• a 1-cube in $\text{Map}_X(x_0, x_1)$ is a 2-cube in $X$ of the form

\[
\begin{array}{c}
x_0 \rightarrow x_1 \\
\downarrow \\
x_0 \rightarrow x_1
\end{array}
\]

**Proposition 4.24.** Given a cubical quasicategory $X$ and $0$-cubes $x_0, x_1: \square^{n+1} \rightarrow X$, the mapping space $\text{Map}_X(x_0, x_1)$ is a cubical Kan complex.

**Proof.** By definition of $\text{Map}_X(x_0, x_1)$, a filler for $\cap_{i, ε} \rightarrow \text{Map}_X(x_0, x_1)$ amounts to a filler for $\cap_{i, ε} \rightarrow X$ where the critical edge is contained in the face fully degenerate on $x_1 - ε$. □

We conclude this section with a proof of the following result, relating the cubical Joyal model structure to the Joyal model structure on simplicial sets via the triangulation functor of Subsection 1.2.

**Proposition 4.25.** The adjunction $T : c\text{Set} \rightleftarrows \text{sSet} : U$ is a Quillen adjunction between the cubical Joyal model structure and the Joyal model structure on $\text{sSet}$.

**Lemma 4.26.** $T$ sends the endpoint inclusions $\square^0 \rightarrow K$ to trivial cofibrations in the Joyal model structure.

**Proof.** It is easy to see that $TK$ is the simplicial set depicted below:

![Diagram](image)

Let $Z$ denote the simplicial set defined by the following pushout:

\[
\begin{array}{c}
\Lambda^2_1 \rightarrow \Delta^0 \\
\downarrow \\
\Delta^2 \rightarrow Z
\end{array}
\]

The map $\Delta^0 \rightarrow Z$ is a trivial cofibration, as a pushout of an inner horn inclusion; thus $Z$ is contractible. We have a pair of cofibrations $Z \hookrightarrow TK$, picking out the bottom-left and top-right simplices in the illustration above; the induced map $Z \sqcup Z \rightarrow TK$ is a cofibration since these two simplices have no faces in common. We obtain $J$ as a quotient of $TK$ by contracting each of these two simplices to a point; in other words, we have the following pushout diagram:

\[
\begin{array}{c}
Z \sqcup Z \rightarrow TK \\
\downarrow \\
\Delta^0 \sqcup \Delta^0 \rightarrow J
\end{array}
\]

The left map is a weak equivalence, since coproducts preserve weak equivalences in the Joyal model structure. Thus $TK \rightarrow J$ is a weak equivalence, as a pushout of a weak equivalence along a cofibration. The composite of $\Delta^0 \rightarrow TK$ with this quotient map is an endpoint inclusion $\Delta^0 \rightarrow J$, hence a weak equivalence; thus $\Delta^0 \rightarrow TK$ is a weak equivalence by 2-out-of-3. □
Lemma 4.27. $U$ preserves fibrations between fibrant objects.

Proof. Let $X \to Y$ be a fibration between quasicategories. To show that $UX \to UY$ is a fibration, it suffices, by Proposition 4.15, to show that it has the right lifting property with respect to endpoint inclusions into $K$ and inner open box fillings. The former property follows from Lemma 4.26. For the latter, consider a diagram of the form:

$$
\begin{array}{ccc}
T \cap_{i,\varepsilon}^n & \to & X \\
\downarrow & & \downarrow \\
T \cap_{i,\varepsilon}^n & \to & Y
\end{array}
$$

To obtain a lift in such a diagram, it suffices to obtain a lift in a diagram

$$
\begin{array}{ccc}
T \cap_{i,\varepsilon}^n & \to & X \\
\downarrow & & \downarrow \\
(\Delta^1)^n & \to & Y
\end{array}
$$

where the image of the critical edge in $X$ is degenerate.

By Lemma 1.22, Corollary 1.24, and the symmetry of the cartesian product in sSet, the triangulation of an open box inclusion is the pushout product $(T\partial\square^m \hookrightarrow (\Delta^1)^m) \times(\{\varepsilon\} \hookrightarrow \Delta^1)$. Since $T\partial\square^m \hookrightarrow (\Delta^1)^m$ is a monomorphism of simplicial sets, it can be written as a composite of boundary fillings; since pushout products commute with composition, we can thus rewrite $T\cap_{i,\varepsilon}^m \hookrightarrow (\Delta^1)^m$ as a composite of pushouts of maps of the form $(\partial \Delta^n \to \Delta^n) \times(\{\varepsilon\} \hookrightarrow \Delta^1)$. Thus, to obtain a lift in the diagram above, it suffices to find a lift in each of the induced diagrams

$$
\begin{array}{ccc}
(\Delta^n \times \{\varepsilon\}) \cup (\partial \Delta^n \times \Delta^1) & \to & X \\
\downarrow & & \downarrow \\
\Delta^n \times \Delta^1 & \to & Y
\end{array}
$$

The left-hand map in the diagram above can be explicitly written as a composite of horn fillings. Each of these horn-fillings will be inner except for the last one to be filled, but the critical edge of this horn will be mapped to the critical edge of $\cap_{i,\varepsilon}^n$ by the relevant inclusion $\Delta^n \times \Delta^1 \to (\Delta^1)^n$. Thus its image in $X$ is degenerate, so the horn is special. Hence a lift exists for each of these horn fillings. □

Proof of Proposition 4.25. This follows from Corollary 1.12, together with Proposition 1.25 and Lemma 4.27. □

Corollary 4.28. The triangulation functor preserves weak equivalences.

Proof. Since all cubical sets are cofibrant, this is immediate from Proposition 4.25 and Ken Brown’s lemma. □

5. Comparison with the Joyal model structure

In this section we prove our main theorem:
Theorem 5.1. The adjunction $T : \text{cSet} \rightleftarrows \text{sSet} : U$ is a Quillen equivalence between the cubical Joyal model structure on $\text{cSet}$ and the Joyal model structure on $\text{sSet}$.

Throughout this section, $\text{sSet}$ and $\text{cSet}$ will be equipped with the Joyal and cubical Joyal model structures, respectively, unless otherwise noted.

Due to the difficulty of working directly with the triangulation functor, we first establish a second Quillen adjunction $Q : \text{sSet} \rightleftarrows \text{cSet} : \tilde{f}$; this adjunction was previously studied in [KLW19], but here we will construct it using a general theory of cones in cubical sets. Using this theory of cones, we will prove that $Q \dashv \tilde{f}$ is a Quillen equivalence, and that the left derived functor of $Q$ is an inverse to that of $T$.

5.1. Cones in cubical sets. Before we can define the adjunction $Q \dashv \tilde{f}$, we must first introduce the concept of a cone in a cubical set. We will also prove various lemmas about such cones, which will be of use later on in showing that $Q \dashv \tilde{f}$ is a Quillen equivalence.

Definition 5.2. For $m, n \geq 0$, the standard $(m, n)$-cone $C^{m, n}$ is a cubical set given by the following inductive construction. For a given $m$, let $C^{m, 0} = \square^m$. Then for each $n \geq 1$, $C^{m, n}$ is the pushout of the inclusion $\partial_1 \times C^{m, n-1} : C^{m, n-1} \cong \square^0 \times C^{m, n-1} \hookrightarrow \square^1 \times C^{m, n-1}$ along the unique map $C^{m, n-1} \to \square^0$.

$\square^0 \times C^{m, n-1} \quad \rightarrow \quad \square^1 \times C^{m, n-1}$

Definition 5.3. For $m, n \geq 0$, an $(m, n)$-cone in a cubical set $X$ is a map $C^{m, n} \to X$.

Observe that each cone $C^{m, n} \to X$ corresponds to a unique $(m + n)$-cube of $X$ by pre-composition with the quotient map $\square^{m+n} \to \square^{m+n}$. Thus we will also use the term “$(m, n)$-cone” to refer to a map $\square^{m+n} \to X$ which factors through this quotient map. In particular, when we refer to the $(i, \varepsilon)$ face of a cone $x$, this means the $(i, \varepsilon)$ face of the corresponding cube: $\square^{m+n-1} \xrightarrow{\partial_i \varepsilon} \square^{m+n} \to C^{m, n} \xrightarrow{x} X$.

For $m, n, k \geq 0$, recall that $\square^{m+n}$ is the set of maps $[1]^k \to [1]^{m+n}$ in the box category $\square$; thus we may write such a $k$-cube $f$ as $(f_1, ..., f_{m+n})$ where each $f_i$ is a map $[1]^k \to [1]$. This allows us to describe $C^{m, n}$ explicitly as a quotient of $\square^{m+n}$.

Lemma 5.4. For all $m, n \geq 0$, $C^{m, n}$ is the quotient of $\square^{m+n}$ obtained by identifying two $k$-cubes $f, g$ if there exists $j$ with $1 \leq j \leq n$ such that $f_i = g_i$ for $i \leq j$ and $f_j = g_j = \text{const}_1$ (the constant map $[1]^k \to [1]$ with value 1).

Proof. We fix $m$ and proceed by induction on $n$. For the base case $n = 0$, there cannot exist any $j$ satisfying the given criteria, thus no identifications are to be made; and indeed we have $C^{m, 0} = \square^m$ by definition.

Now suppose that the given description holds for $C^{m, n}$, and let $q$ denote the quotient map $\square^{m+n} \to C^{m, n}$. Then because the functor $\square^l \times -$ preserves colimits, $\square^l \times C^{m, n}$ is a quotient of $\square^{l+m+n}$ with quotient map $\square^l \times q$. From this description we see that $\square^l \times C^{m, n}$ is obtained from $\square^{l+m+n}$ by identifying two $k$-cubes $f, g$ whenever $f_1 = g_1$ and the cubes $(f_2, ..., f_{n+1})$ and $(g_2, ..., g_{n+1})$ are
identified in $C^{m,n}$. In other words, we obtain $\square^1 \otimes C^{m,n}$ from $\square^{1+m+n}$ by identifying $f$ and $g$ if there exists $j$ with $2 \leq j \leq n + 1$ such that $f_i = g_i$ for all $i \leq j$ and $f_j = g_j = \text{const}_1$. Taking the pushout of the inclusion $\partial_{i,1} \otimes C^{m,n} : C^{m,n} \to \square^1 \otimes C^{m,n}$ along the unique map $C^{m,n} \to \square^0$, we then see that $C^{m,n+1}$ is the quotient of $\square^1 \otimes C^{m,n}$ obtained by identifying cubes $f, g$ whenever $f_1 = g_1 = \text{const}_1$. Thus the description holds for $C^{m,n+1}$.

**Corollary 5.5.** For all $n \geq 1$, $C^{0,n} \cong C^{1,n-1}$.

Using the characterization of cones given above, we can show that any face of a given cone is a cone of a specified degree.

**Lemma 5.6.** For $i \leq n$, the image of the composite map $\square^{m+n-1} \xrightarrow{\partial_{\iota,a}} \square^{m+n} \to C^{m,n}$ is isomorphic to $C^{m,n-1}$. For $i \geq n + 1$, $\varepsilon \in \{0,1\}$, the image of the composite map $\square^{m+n-1} \xrightarrow{\partial_{\iota,\varepsilon}} \square^{m+n} \to C^{m,n}$ is isomorphic to $C^{m-1,n}$.

**Proof.** First consider the composite map $\square^{m+n-1} \xrightarrow{\partial_{\iota,a}} \square^{m+n} \to C^{m,n}$. Let $f = (f_1, ..., f_{m+n-1})$ denote a $k$-cube of $\square^{m+n-1}$, as in the proof of Lemma 5.4. We denote the image of this cube under $\partial_{\iota,0}$ by $f' = (f'_1, ..., f'_{m+n-1})$, where $f'_j = f_j$ for $j < i$, $f'_i = \text{const}_0$, and $f'_j = f_{j-1}$ for $j > i$. By Lemma 5.4, given two $k$-cubes $f$ and $g$ in $\square^{m+n-1}$, their images under $\partial_{\iota,0}$ will be identified in the quotient $C^{m,n}$ if and only if there exists $j \leq n$ such that $f'_j = g'_j$ for $l \leq j$ and $f'_j = g'_j = \text{const}_1$. In other words, if there exists $j \leq n$ such that $f_l = g_l$ for $l \leq j$ and $f_j = g_j = \text{const}_1$. The desired isomorphism thus follows from Lemma 5.4.

The analysis of $\partial_{\iota,\varepsilon}$ where $i \geq n + 1$, $\varepsilon \in \{0,1\}$ is similar, except that in that case we have $f'_i = f_j$ for all $j \leq i$. Thus we conclude that the images of $f$ and $g$ in the quotient $C^{m,n}$ are equal if and only if there exists $j \leq n$ such that $f_l = g_l$ for $l \leq j$ and $f_j = g_j = \text{const}_1$.

**Lemma 5.7.** Let $x$ be an $(m,n)$-cone in a cubical set $X$. Then:

- If $n \geq 1$, then for $i \leq n$, $x\partial_{i,0}$ is an $(m, n-1)$-cone;
- If $m \geq 1$, then for $i \geq n + 1$, $x\partial_{i,0}$ is an $(m - 1, n)$-cone;
- If $m \geq 1$, then for all $i$, $x\partial_{i,1}$ is an $(m - 1, n-1)$-cone.

**Proof.** First consider $n \geq 1$, $i \leq n$. By Lemma 5.6, the image of $\square^{m+n+1}$ under the composite map above will be isomorphic to $C^{m,n-1}$; thus the composite map factors through $C^{m,n-1}$, giving a commuting diagram as shown below:

$$
\begin{array}{cc}
\square^{m+n+1} & \xrightarrow{\partial_{i,0}} & \square^{m+n} \\
C^{m,n-1} & \xrightarrow{\partial_{i,1}} & C^{m,n} \\
\end{array}
$$

Now, for an $(m+n)$-cube $x \in X_{m+n}$ to be an $(m,n)$-cone means precisely that the corresponding map $x : \square^{m+n} \to X$ factors through $C^{m,n}$. So the face $x\partial_{i,0}$ can be written as $\square^{m+n+1} \xrightarrow{\partial_{i,0}} \square^{m+n+1} \to C^{m,n} \xrightarrow{x} X$; by the diagram above we can rewrite this as $\square^{m+n-1} \to C^{m,n-1} \to C^{m,n} \xrightarrow{x} X$. So $x\partial_{i,0}$ factors through $C^{m,n-1}$, meaning that it is an $(m,n-1)$-cone.
A similar argument shows that for \(m \geq 1, i \geq n+1\), the composite map \(\Box^{m+n-1} \xrightarrow{\partial_i} \Box^{m+n} \to C^{m,n}\) will factor through \(C^{m-1,n}\), implying that \(x\partial_i\) is an \((m-1,n)\)-cone for any \((m,n)\)-cone \(x\).

Finally, let \(m \geq 1, i \leq n\) and consider the composite \(\Box^{m+n-1} \xrightarrow{\partial_i} \Box^{m+n} \to C^{m,n}\). As above, we let \(f\) denote an arbitrary \(k\)-cube of \(\Box^{m+n-1}\) and let \(f'\) denote its image under \(\partial_i\); then once again we have \(f'_j = f_j\) for \(j \leq i-1\), but now \(f'_j = \text{const}_1\). So let \(f\) and \(g\) be two \(k\)-cubes of \(\Box^{m+n-1}\), and suppose that there exists \(j \leq n\) such that \(f_l = g_l\) for \(l \leq j\) and \(f_j = g_j = \text{const}_1\). Then there exists \(j' \leq n\) such that \(f'_l = g'_l\) for \(l \leq j'\) and \(f'_j = g'_j = \text{const}_1\): if \(j < i\) then \(j' = j\), while if \(j \geq i\) then \(j' = i\). So \(f'\) and \(g'\) are identified in \(C^{m,n}\). Thus the composite map factors through \(C^{m-1,n}\), so for any \((m,n)\)-cone \(x\), \(x\partial_i\) is an \((m-1,n)\)-cone. \(\square\)

**Remark 5.8.** In contrast to Lemma 5.6, for \(i \leq n\) the image of \(\Box^{m+n-1} \xrightarrow{\partial_i} \Box^{m+n} \to C^{m,n}\) is not isomorphic to \(C^{m-1,n}\). For instance, when \(i = 1\) this image is isomorphic to \(\Box^0\).

In some cases it will be more convenient to characterize cones in a cubical set by a set of conditions on their faces. By a direct analysis of the cubes of \(C^{m,n}\), or by an inductive argument similar to that used in the proof of Lemma 5.3 we have the following characterization of \((m,n)\)-cones in \(X\).

**Lemma 5.9.** For \(m, n\) with \(n \geq 1\), and \(X \in cSet\), a cube \(x: \Box^{m+n} \to X\) is an \((m,n)\)-cone if and only if for all \(i\) such that \(1 \leq i \leq n\) we have \(x\partial_{i,1} = x\partial_{m+n,0}\partial_{m+n-1,0}\ldots\partial_{i+1,0}\partial_{i,1}\sigma_{i+1}\ldots\sigma_{m+n-2}\sigma_{m+n-1}\). (In the case \(m = 0, i = n\) we interpret this statement as the tautology \(x\partial_{n,1} = x\partial_{n,1}\).) \(\square\)

**Corollary 5.10.** If \(x: \Box^{m+n} \to X\) is an \((m,n)\)-cone, then \(x\) is also an \((m+k,n-k)\)-cone for all \(k \leq n\). \(\square\)

This characterization allows us to prove some technical lemmas concerning faces and degeneracies of cones.

**Lemma 5.11.** Let \(X\) be a cubical set, and let \(x: C^{m,n} \to X\) be an \((m,n)\)-cone in \(X\). Then:

(i) for \(i \geq n+1\), \(x\sigma_i\) is an \((m+1,n)\)-cone;

(ii) if \(n \geq 1\) then for \(i \leq n\), \(x\gamma_i\) is an \((m,n+1)\)-cone;

(iii) for \(i \geq n+1\), \(x\gamma_i\) is an \((m+1,n)\)-cone;

**Proof.** We will prove item (i); the remaining proofs are similar.

We will show that \(x\gamma_i\) satisfies the conditions of Lemma 5.9 for \((m+1,n)\). For \(j \leq n\), we have \(j < i\). Using this fact and the cubical identities, we can compute:

\[
(x\sigma_i)\partial_{j,1} = x\partial_{j,1}\sigma_{i-1} = x\partial_{m+n,0}\ldots\partial_{j+1,0}\partial_{j,1}\sigma_{i-1} = x\partial_{m+n,0}\ldots\partial_{j+1,0}\partial_{j,1}\sigma_j\ldots\sigma_{m+n} = x\sigma_i\partial_{j,0}\partial_{m+n,0}\ldots\partial_{j+1,0}\partial_{j,1}\sigma_j\ldots\sigma_{m+n} = (x\sigma_i)\partial_{m+n+1,0}\ldots\partial_{j+1,0}\partial_{j,1}\sigma_j\ldots\sigma_{m+n}
\]
Thus $x\sigma_i$ is an $(m+1,n)$-cone.

The proof of (2) requires separate computations for the cases $1 \leq j < i$, $j = i$, $j = i+1$, and $i+1 < j \leq n+1$, while the proof of (3) is essentially identical to the above.

Lemma 5.12. For $m \geq 1, n \geq 0$, let $x$ be an $(m+n-1)$-cube in a cubical set $X$. If $x\gamma_n$ is an $(m,n)$-cone, then it is also an $(m-1,n+1)$-cone.

Proof. By Lemma 5.7 $x\gamma_n \partial_{n+1,0} = x$ is an $(m-1,n)$-cone. Therefore, $x\gamma_n$ is an $(m-1,n+1)$-cone by Lemma 5.11.

We will also have use for the following result, which shows that the standard cones contain many inner open boxes.

Lemma 5.13. For $n \geq 1, 2 \leq i \leq m+n$, the quotient map $\Box^{m+n} \to C^{m,n}$ sends the critical edge with respect to the face $\partial_{i,0}$ to a degenerate edge.

Proof. The critical edge in question corresponds to the function $f: [1] \to [1]^{m+n}$ with $f_i = \text{id}_{[1]}$, $f_j = \text{const}_1$ for $j \neq i$. In particular, $f_1 = \text{const}_1$, so $f$ is equivalent, under the equivalence relation of Lemma 5.3 to the map $[1] \to [1]^{m+n}$ which is constant at $(1,...,1)$.

Theorem 5.15 gives us the following:

Proposition 5.14. Given a cubical set $X$, for any cube $x: \Box^n \to X$ there exist unique (possibly empty) sequences $a_1 < ... < a_p, b_1 < ... < b_q$ and a unique non-degenerate cube $y: \Box^{n-p-q} \to X$ such that $x = y\gamma_{b_1}...\gamma_{b_q}\sigma_{a_1}...\sigma_{a_p}$.

Definition 5.15. For $x: \Box^n \to X$, the expression given by Proposition 5.14 is the standard form of $x$.

For brevity, we will often say that the standard form of a cube $x$ is $zf$, or "ends with $f$", where $f$ is some map in $\Box$; this is understood to mean that $f$ is the rightmost map in the standard form of $x$. For instance, if the standard form of $x$ is $zf$, then $z = y\gamma_{b_1}...\gamma_{b_q}\sigma_{a_1}...\sigma_{a_{p-1}}$ in the notation of Proposition 5.14.

We now prove a lemma regarding the standard forms of cones.

Lemma 5.16. Let $m \geq 1$, and let $x: C^{m,n} \to X$ be a degenerate $(m,n)$-cone whose standard form is $y\gamma_{b_1}...\gamma_{b_q}\sigma_{a_1}...\sigma_{a_p}$, where the string $\sigma_{a_1}...\sigma_{a_p}$ is non-empty. Then $a_p \geq n+1$.

Proof. For $n = 0$ this is trivial, so assume $n \geq 1$. Towards a contradiction, suppose that $a_p \leq n$, and let $z = y\gamma_{b_1}...\gamma_{b_q}\sigma_{a_1}...\sigma_{a_{p-1}}$, so that $z\sigma_{a_p} = x$. Taking the $(a_p, 1)$ faces of both sides of this equation, and applying Lemma 5.9 we see that:

\[ z = x\partial_{m+n,0}...\partial_{a_p,0}\partial_{a_p,1}\sigma_{a_p}...\sigma_{m+n-1} \]
\[ \therefore z\sigma_{a_p} = x\partial_{m+n,0}...\partial_{a_p,0}\partial_{a_p,1}\sigma_{a_p}...\sigma_{m+n-1}\sigma_{a_p} \]
\[ \therefore x = x\partial_{m+n,0}...\partial_{a_p,0}\partial_{a_p,1}\sigma_{a_p}...\sigma_{m+n} \]
In the last step, we have repeatedly used the co-cubical identity $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ for $i \leq j$ to rearrange the string $\sigma_{a_p} \cdots \sigma_{m+n-1} \sigma_{a_p}$ into one whose indices are in strictly increasing order. (We can do this because, by our assumption on $m, m+n-1 \geq n \geq i_1$.) Now let $y' \sigma_{a'} \cdots \sigma_{a'}' \gamma_{b'} \cdots \gamma_{b'}'$ be the standard form of $x \partial_{m+n,0} \cdots \partial_{a_p+1,0} \partial_{a_p,1}$; then we have:

$$x = y' \sigma_{a'} \cdots \sigma_{a'}' \gamma_{b'} \cdots \gamma_{b'}' \sigma_{a_p} \cdots \sigma_{m+n}$$

We can apply further co-cubical identities to re-order the maps on the right-hand side of this equation, obtaining a standard form for $x$ in which the rightmost degeneracy map has index greater than or equal to $m + n$. But as the standard form of $x$ is unique, this contradicts our assumption that $a_p \leq n$. □

We will also require some lemmas concerning subcomplexes of $C^{m,n}$ consisting of specified faces.

**Definition 5.17.** For $m, n \geq 0, k \leq n$, $B^{m,n,k}$ is the subcomplex of $C^{m,n}$ consisting of the images of the faces $\partial_{1,0}$ through $\partial_{k,0}$, as well as all faces $\partial_{i,1}$, under the quotient map $\Box^{m+n} \to C^{m,n}$.

In order to characterize maps out of $B^{m,n,k}$, we will need to prove a couple of lemmas concerning the faces of $C^{m,n}$.

**Lemma 5.18.** For $m, n \geq 0, 1 \leq i_1 < i_2 \leq m+n, \varepsilon_1, \varepsilon_2 \in \{0,1\}$, where $i_1 \geq n+1$ if $\varepsilon_1 = 1$, the intersection of the images of the faces $\partial_{i_1,\varepsilon_1}$ and $\partial_{i_2,\varepsilon_2}$ of $\Box^{m+n}$ under the quotient map $\Box^{m+n} \to C^{m,n}$ is exactly the image of the face $\partial_{i_2,\varepsilon_2} \partial_{i_1,\varepsilon_1} = \partial_{i_1,\varepsilon_1} \partial_{i_2-1,\varepsilon_2}$.

**Proof.** That the intersection of the images of $\partial_{i_1,\varepsilon_1}$ and $\partial_{i_2,\varepsilon_2}$ contains the image of $\partial_{i_2,\varepsilon_2} \partial_{i_1,\varepsilon_1}$ is clear, as this face is the intersection of $\partial_{i_1,\varepsilon_1}$ and $\partial_{i_2,\varepsilon_2}$ in $\Box^{m+n}$. Now we will verify the opposite containment, using description of $C^{m,n}$ from Lemma [5.3]

To this end, consider a map $f : [1]^k \to [1]^{m+n}$ such that the equivalence class $[f] \in C^{m,n}_k$ is contained in the images of faces $(i_1, \varepsilon_1)$ and $(i_2, \varepsilon_2)$. We will construct $f' : [1]^k \to [1]^{m+n}$ such that $f \sim f'$ and $f'$ is contained in the intersection of faces $(i_1, \varepsilon_1)$ and $(i_2, \varepsilon_2)$, thereby showing that $[f] = [f']$ is contained in the image of this intersection under the quotient map.

Since $f$ is in the image of face $(i_1, \varepsilon_1)$, $f \sim g$ for some $g : [1]^k \to [1]^{m+n}$ such that $g_{i_1} = \text{const}_{\varepsilon_1}$. Therefore, at least one of the following holds:

(i) $f_{i_1} = \text{const}_{\varepsilon_1}$;

(ii) $f_j = g_j = \text{const}_1$ for some $j \leq \min(i_1 - 1, n)$.

If (ii) holds, then $f$ is equivalent to any $f'$ such that $f'_l = f_l$ for $l \leq j$; in particular, we can choose such an $f'$ satisfying $f'_{i_1} = \text{const}_{\varepsilon_1}, f'_{i_2} = \text{const}_{\varepsilon_2}$.

Now suppose that (i) holds, but (ii) does not. Then because $f$ is in the image of face $(i_2, \varepsilon_2)$, $f \sim h$ for some $h : [1]^k \to [1]^{m+n}$ such that $h_{i_2} = \text{const}_{\varepsilon_2}$. Therefore, at least one of the following holds:

(i) $f_{i_2} = \text{const}_{\varepsilon_2}$;

(ii) $f_j = h_j = \text{const}_1$ for some $i_1 + 1 \leq j \leq \min(i_2 - 1, n)$. 
In case (i), we have \( f_{i_1} = \text{const}_{\varepsilon_1}, f_{i_2} = \text{const}_{\varepsilon_2} \), so we can simply choose \( f' = f \). In case (ii), \( f \) is equivalent to any \( f' \) such that \( f'_l = f_l \) for \( l \leq j \) (which implies \( f'_1 = \text{const}_{\varepsilon_1} \)); in particular, we can choose such an \( f' \) satisfying \( f'_{i_1} = \text{const}_{\varepsilon_2} \).

\[ \Box \]

**Lemma 5.19.** For \( i \leq n \), the image of the face \( \partial_i,1 \) under the quotient map \( \Box^{m+n} \to C^{m,n} \) is contained in the image of \( \partial_{m+n,1} \).

**Proof.** Let \( f : [1]^k \to [1]^{m+n} \) be a \( k \)-cube of \( \Box^{m+n} \) which factors through \( \partial_i,1 \). Then \( f_i = \text{const}_1 \). Thus \( f \) is equivalent to any \( f' : [1]^k \to [1]^{m+n} \) such that \( f'_j = f_j \) for all \( j \leq i \); in particular, we may choose such an \( f' \) with \( f'_{m+n} = \text{const}_1 \). So \( f' \) factors through \( \partial_{m+n,1} \); thus \([f] = [f']\) is contained in the image of \( \partial_{m+n,1} \) under the quotient map.

**Lemma 5.20.** For a cubical set \( X \), a map \( x : B^{m,n} \to X \) is determined by a set of \((m,n-1)\)-cones \( x_{i_0},\varepsilon : C^{m,n-1} \to X \) for \( 1 \leq i \leq n \) and a set of \((m-1,n)\)-cones \( x_{i,1} \) for \( n+1 \leq i \leq m+n \) such that for all \( i_1 < i_2, \varepsilon_1, \varepsilon_2 \in \{0,1\} \), \( x_{i_2,\varepsilon_2} \partial_{i_1,\varepsilon_1} \) is \( x_{i_1,\varepsilon_1} \partial_{i_2,\varepsilon_2} \), with \( x_{i,\varepsilon} \) being the image of \( \partial_{i,\varepsilon} \) under \( x \).

**Proof.** To define a map \( x : B^{m,n,k} \to X \), it suffices to assign the values of \( x \) on the faces \([\partial_{i,\varepsilon}]\) of \( C^{m,n} \) for which \( i \leq k \) or \( \varepsilon = 1 \), provided that these choices are consistent on the intersections of faces. By Lemma 5.19 it suffices to consider only those faces for which \( i \leq k, \varepsilon = 0 \) or \( i \geq n+1, \varepsilon = 1 \). These faces are isomorphic to \( C^{m,n-1} \) or \( C^{m-1,n} \), respectively, by Lemma 5.6. By Lemma 5.18 to show that these choices are consistent on the intersections of faces, it suffices to show that they satisfy the co-cubical identity for composites of face maps.

**Proposition 5.21.** For all \( m, n \geq 1, n \leq k \leq m+n-1 \), the inclusion \( B^{m,n,k} \hookrightarrow C^{m,n} \) is a trivial cofibration.

**Proof.** We proceed by induction on \( m \). In the base case \( m = 1 \), the only relevant value of \( k \) is \( k = n \). The only face of \( C^{1,n} \) which is missing from \( B^{1,n} \) is \([\partial_{n+1,0}]\), so the inclusion \( B^{1,n} \hookrightarrow C^{1,n} \) is an \((n+1,0)\)-open box filling. By Lemma 5.18 the critical edge for this open box filling is degenerate, hence an equivalence in \( B^{1,n,n} \), so the inclusion is a trivial cofibration.

Now let \( m \geq 2 \), and suppose the statement holds for \( m-1 \). For \( n \leq k \leq m+n-2 \), consider the intersection of the \((k+1,0)\) face of \( C^{m,n} \), \([\partial_{k,0}]\), with the subcomplex \( B^{m,n,k} \). By Lemma 5.18 and Lemma 5.19 this intersection consists of faces \((1,0)\) through \((k,0)\) and \((1,1)\) through \((m+n-1,1)\) of \([\partial_{k+1,0}]\). By Lemma 5.6 it is thus isomorphic to \( B^{m-1,n,k} \).

Thus we can express \( B^{m,n,k+1} \) as the following pushout:

\[
\begin{array}{ccc}
B^{m,n,k} & \xrightarrow{f} & B^{m,n,k} \\
\downarrow & & \downarrow \\
C^{m,n,k} & \xrightarrow{g} & B^{m,n,k+1}
\end{array}
\]

By the induction hypothesis, \( B^{m,n,k} \hookrightarrow C^{m-1,n} \) is a trivial cofibration, since \( n \leq k \leq m+n-2 \). Thus \( B^{m,n,k} \hookrightarrow B^{m,n,k+1} \) is a trivial cofibration, as a pushout of a trivial cofibration. From this we can see that for any \( n \leq k \leq m+n-2 \), the composite inclusion \( B^{m,n,k} \hookrightarrow B^{m,n,k+1} \hookrightarrow \ldots \hookrightarrow B^{m,n,m+n-1} \) is a trivial cofibration.
Thus it suffices to prove that \( B^{m,n,m+n-1} \hookrightarrow C^{m,n} \) is a trivial cofibration. Here, as in the base case, the subcomplex \( B^{m,n,m+n-1} \) is only missing the face \([\partial_{m+n,0}]\), so the inclusion is an \((m+n,0)\)-open box filling. The critical edge of this open box is degenerate by Lemma 5.13, so the inclusion is indeed a trivial cofibration.

Thus we see that the inclusion \( B^{m,n,k} \hookrightarrow C^{m,n} \) is a trivial cofibration for any \( m,n,k \) satisfying the constraints given in the statement. □

5.2. \( Q \dashv \int \). For \( n \geq 0 \), let \( Q^n = C^{0,n} \). These objects were previously studied in [KLW19], in which they were described as quotients of \( \Box^n \) under a certain equivalence relation; this relation is precisely that of Lemma 5.4 in the case \( m = 0 \). We begin by recalling some of the theory developed in that paper.

**Proposition 5.22** ([KLW19, Prop. 2.3]). The assignment \([n] \mapsto Q^n\) extends to a cosimplicial object \( Q^\bullet : \Delta \rightarrow cSet \), in which each simplicial structure map \( Q^m \rightarrow Q^n \) is given by a map \( \Box^m \rightarrow \Box^n \) which descends to a map between the quotients. The correspondence is as follows:

| a map \( Q^{n-1} \rightarrow Q^n \) is induced by a map \( \Box^{n-1} \rightarrow \Box^n \) | \( 0^{th} \) face | 1\(^{st}\) face | 2\(^{nd}\) face | \( \cdots \) | \( j^{th} \) face | \( \cdots \) | \( n^{th} \) face |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 0\(^{th}\) deg. | 1\(^{st}\) deg. | 2\(^{nd}\) deg. | \( \cdots \) | \( j^{th} \) deg. | \( \cdots \) | \( (n-1)^{st} \) deg. |
| \( \sigma_n \) | \( \gamma_{n-1} \) | \( \gamma_{n-2} \) | \( \cdots \) | \( \gamma_{n-j} \) | \( \cdots \) | \( \gamma_1 \) |

Taking the left Kan extension of this cosimplicial object along the Yoneda embedding, we obtain a functor \( Q : sSet \rightarrow cSet \).

This functor has a right adjoint \( \int : cSet \rightarrow sSet \), given by \((\int X)_n = cSet(Q^n,X)\).

**Lemma 5.23** ([KLW19, Lem. 4.2]). For any \( X \in cSet \), the counit \( Q \int X \rightarrow X \) is a monomorphism. □

This lemma shows that for any cubical set \( X \), \( Q \int X \) is a subcomplex of \( X \). Specifically, it is the subcomplex whose non-degenerate \( n \)-cubes, for each \( n \), are those which factor through \( Q^n \) — in other words, they are the non-degenerate \((0,n)\)-cones in \( X \).

**Remark 5.24.** Viewing \( sSet \) as the slice category \( sSet \downarrow \Delta^0 \) and \( cSet \) as the functor category \( cSet^{[0]} \), the adjunction \( Q \dashv \int \) coincides with the cubical straightening-unstraightening adjunction developed in [KV18].

Our next goal is to show the following:

**Proposition 5.25.** The adjunction \( Q \dashv \int \) is Quillen.
To prove this, we will show that this adjunction satisfies the hypotheses of Corollary 1.12. We begin with a simple lemma relating the interval objects in the two model structures of interest.

**Lemma 5.26.** $QJ \cong K$. □

**Lemma 5.27** ([KLW19 Lem. 4.5]). $Q$ preserves monomorphisms. □

**Lemma 5.28.** The image under $Q$ of an inner horn inclusion $\Lambda^n_i \subseteq \Delta^n$ is a trivial cofibration.

**Proof.** Because $Q$ preserves colimits, $Q\Lambda^n_i$ is the subcomplex of $Q^n$ consisting of the images of the maps $Q\partial_j : Q^{n-1} \to Q^n$ for which $j \neq i$. By Proposition 5.22 we can see that this subcomplex is the image of $\Gamma^n_{n-i+1,0}$ under the quotient map $\square^n \to Q^n$. We thus have the following commuting square:

\[
\begin{array}{ccc}
\Gamma^n_{n-i+1,0} & \longrightarrow & Q\Lambda^n_i \\
\downarrow & & \downarrow \\
\square^n & \longrightarrow & Q^n
\end{array}
\]

It is easy to see that this square is a pushout. Furthermore, the critical edge of the open box $\Gamma^n_{n-i+1,0} \to Q\Lambda^n_i$ is degenerate by Lemma 5.13. Thus $Q\Lambda^n_i \hookrightarrow Q^n$ is a trivial cofibration, as an inner open box filling. □

**Proof of Proposition 5.25.** By Lemma 5.27, $Q$ preserves cofibrations. By Lemma 5.28, the image under $Q$ of an inner-horn inclusion is a trivial cofibration. The image under $Q$ of an endpoint inclusion $\Delta^0 \to J$ is an endpoint inclusion $\square^0 \to K$, hence a trivial cofibration by Lemma 4.11. Thus the adjunction is Quillen by Corollary 1.12. □

**Corollary 5.29.** $Q$ preserves weak equivalences.

**Proof.** Since all simplicial sets are cofibrant in the Quillen model structure, this follows from Proposition 5.25 and Ken Brown’s lemma. □

Next we will concern ourselves with the relationship between $Q$ and the triangulation functor. Our goal will be to prove the following:

**Proposition 5.30.** $Q$ reflects weak equivalences.

To do this, we will develop a natural weak equivalence $TQ \Rightarrow \text{id}_{\text{Set}}$.

**Definition 5.31.** For $n \geq 0$, let $a = (a_1, \ldots, a_n)$, where $a_i \in \{0, 1\}$, be an object of $[1]^n$. Then $F(a) = 0$ if $a_i = 0$ for all $i$; otherwise, $F(a) = n - i + 1$, where $i$ is minimal such that $a_i = 1$.

**Proposition 5.32.** For all $n$, $F$ defines a poset map $[1]^n \to [n]$.

**Proof.** Let $a \leq b$ be objects of $[1]^n$. If $a_i = 0$ for all $i$, then $F(a) = 0$. Otherwise, let $i$ be minimal such that $a_i = 1$, and let $j$ be the minimal such value for $b$. Then $b_i = 1$ as well, so $j \leq i$. In either case, we see that $F(a) \leq F(b)$. □

**Proposition 5.33.** For all $n$, $F$ induces a map of simplicial sets $(\Delta^1)^n \to TQ^n$. 
Proof. First, observe that by applying the nerve functor \( N : \text{Cat} \to \text{sSet} \), we get an induced map \( NF : (\Delta^1)^n \to \Delta^n \).

The simplicial set \( TQ^n \) is a quotient of \( T \square^n = (\Delta^1)^n \). Specifically, since \( N \) is fully faithful, we may regard \( n \)-simplices \( \Delta^n \to (\Delta^1)^n \) as poset maps \([n] \to [1]^n\). Then by an argument analogous to the proof of Lemma 5.3 using the fact that \( T \) preserves colimits and sends geometric products to cartesian products, \( TQ^n \) is obtained by identifying two such maps \( f, g \) if there exists \( i \) such that \( f_j = g_j \) for \( j \leq i \) and \( f_i = g_i = \text{const}_{\text{const}} \). \( NF \) then acts on such maps by post-composition with \( F \).

Since \( F \) depends only on the position of the first \( 1 \) in an object of \([1]^n\), it is clear that maps which are identified in \( TQ^n \) agree after post-composition with \( F \). Thus \( NF \) factors through the quotient \( TQ^n \).

Let \( \bar{F} : TQ^n \to \Delta^n \) denote the map constructed above. Then we can show:

**Lemma 5.3.** The maps \( \bar{F} : TQ^n \to \Delta^n \) form a natural transformation of co-simplicial objects in \( \text{sSet} \). That is, for any map \( \phi : [m] \to [n] \) in \( \Delta \), we have a commuting diagram:

\[
\begin{array}{ccc}
TQ^m & \xrightarrow{TQ\phi} & TQ^n \\
\downarrow{\bar{F}} & & \downarrow{\bar{F}} \\
\Delta^m & \xrightarrow{\phi} & \Delta^n
\end{array}
\]

**Proof.** It suffices to show that this holds for the generating morphisms of \( \Delta \), namely the co-face and co-degeneracy maps. For each such map \( \phi : [m] \to [n] \) we have a corresponding map \( \phi' : [1]^m \to [1]^n \) in \( \square \), as described in Proposition 5.22:

- For \( \partial_0 : [n - 1] \to [n], \partial_0' = \partial_{n,1} \);
- For \( i \geq 1, \partial_i : [n - 1] \to [n], \partial_i' = \partial_{n-i+1,0} \);
- For \( \sigma_0 : [n] \to [n - 1], \sigma_0' = \sigma_n \);
- For \( \sigma_i : [n] \to [n - 1], \sigma_i' = \gamma_{n-i} \).

For every such \( \phi \) we have a commuting diagram in \( \text{cSet} \), where the vertical maps \( \square^m \to Q^m \) are the quotient maps:

\[
\begin{array}{ccc}
\square^m & \xrightarrow{\phi'} & \square^n \\
\downarrow{Q^m} & & \downarrow{Q^n} \\
Q^m & \xrightarrow{Q\phi} & Q^n
\end{array}
\]

Furthermore, by direct computation we have commuting diagrams in \( \text{Cat} \):
\( \vdash \)

\[ \begin{array}{c}
\phi \downarrow \downarrow \phi' \\
\downarrow \downarrow TQ \\
\Delta_m \phi \downarrow \downarrow \Delta^n
\end{array} \]

Now consider the following diagram in \( \text{sSet} \):

\[ \begin{array}{c}
\Delta^1_m \xrightarrow{TQ} \Delta^1_n \\
\downarrow \downarrow TQ \phi \\
\Delta^m \phi \downarrow \downarrow \Delta^n
\end{array} \]

The top square commutes, as it is obtained by applying \( T \) to diagram (1); the outer rectangle also commutes, as it is obtained by applying \( N \) to diagram (2). We wish to show that the bottom square commutes, i.e. that \( \phi \circ \bar{F} = \bar{F} \circ TQ \phi \); since the quotient map \( (\Delta^1)^m \to TQ^m \) is an epimorphism, we can show the desired equality by pre-composing with this map and performing a simple diagram chase.

**Corollary 5.35.** \( \bar{F} \) extends to a natural transformation \( \bar{F}: TQ \Rightarrow \text{id}_{\text{sSet}} \).

**Proof.** Let \( X \) be an arbitrary simplicial set. Recall that \( X = \text{colim} \Delta^n \); since \( T \) and \( Q \) both preserve colimits, we have \( TQX = \text{colim} TQ^n \). Thus, by Lemma \( \text{[5.34]} \) we obtain an induced map on the colimits \( \bar{F}: TQX \to X \), natural in \( X \).

**Proposition 5.36.** For every simplicial set \( X \), the map \( \bar{F}: TQX \to X \) is a weak equivalence.

**Proof.** We begin by proving the statement for the case where \( X \) is \( m \)-skeletal for some \( m \geq 0 \), proceeding by induction on \( m \). For \( m = 0, m = 1 \), the map in question is an isomorphism.

Now let \( m \geq 2 \), and suppose that the statement holds for any \((m-1)\)-skeletal \( X \). Then in particular, it holds for any horn \( \Lambda^m_i \). For any \( 0 < i < n \), consider the following commuting diagram:

\[ \begin{array}{c}
TQ \Lambda^m_i \xrightarrow{\sim} TQ^n \\
\downarrow \downarrow \downarrow \cup \cup \\
\Lambda^m_i \xrightarrow{\sim} \Delta^m
\end{array} \]

The left-hand map is a weak equivalence by the induction hypothesis; the bottom map is a trivial cofibration as an inner horn inclusion; and the top map is a trivial cofibration by Proposition \( \text{[5.25]} \) and Proposition \( \text{[4.25]} \). Thus \( \bar{F}: TQ^n \to \Delta^m \) is a weak equivalence by the two-out-of-three property. Extending this result to an arbitrary \( m \)-skeletal simplicial set \( X \) is a straightforward application of the gluing lemma, using the fact that both \( T \) and \( Q \) preserve colimits.
Now let $X$ be an arbitrary simplicial set; then $\tilde{F}$ is a weak equivalence on the $n$-skeleton of $X$ for each dimension $n$. Thus $\tilde{F} : TQX \to X$ is a weak equivalence, using the fact that sequential colimits of cofibrations preserve weak equivalences.

**Proof of Proposition 5.30.** Let $f : X \to Y$ be a map of simplicial sets, such that $Qf$ is a weak equivalence. We have a commuting diagram:

$$
\begin{array}{ccc}
TQX & \xrightarrow{TQf} & TQY \\
\downarrow\tilde{F} & \quad & \quad \downarrow\tilde{F} \\
X & \xrightarrow{f} & Y
\end{array}
$$

The top horizontal map is a weak equivalence by Proposition 1.25 as are the vertical maps by Proposition 5.30. Thus $f$ is a weak equivalence by the 2-out-of-3 property.

5.3. The counit of $Q \dashv f$. We have shown that the adjunction $Q \dashv f$ satisfies the hypotheses of Corollary 1.14 item (3). To show that it is a Quillen equivalence, therefore, we must prove the following:

**Theorem 5.37.** For any cubical quasicategory $X$, the counit $\varepsilon : QfX \to X$ is a trivial cofibration.

Throughout this section, fix a cubical quasicategory $X$; we will build $X$ from $QfX$ via successive inner open-box fillings, thereby showing that the inclusion of $QfX$ into $X$ is a trivial cofibration. We will do this via a three-stage induction: $X$ will be constructed as the sequential colimit of subcomplexes $X^m, m \geq 0$, each of which will be constructed as the sequential colimit of a sequence of subcomplexes $X^{m,n}, n \geq -1$, each of which will be constructed by a kind of induction on skeleta.

We begin by establishing the induction hypothesis which the subcomplexes $X^m$ must satisfy:

**Definition 5.38.** A subcomplex $X^m \subseteq X$ satisfies the induction hypothesis on base dimension for $m$ if:

1. For $0 \leq m' \leq m$ and $n \geq 0$, all $(m', n)$-cones of $X$ are contained in $X^m$;
2. For $i \geq 1$, every non-degenerate cube $\square^{m+i} \to X^m$ is an $(m, i)$-cone;
3. For every $(m', n)$-cone of $X$ with $m' \leq m, n \geq 0$, $X^m$ contains an $(m', n+1)$-cone $\theta^{m', n}(x)$, satisfying the following identities:
   a. For $i \leq n$, $\theta^{m', n}(x)\partial_{i,0} = \theta^{m', n-1}(x\partial_{i,0})$;
   b. $\theta^{m', n}(x)\partial_{n+1,0} = x$;
   c. For $i \geq n + 2$, $\theta^{m', n}(x)\partial_{i,1} = \theta^{m', n-1}(x\partial_{i-1,1})$;
   d. If $x\sigma_i$ is an $(m', n)$-cone for $i \geq n+1$, then $\theta^{m', n}(x\sigma_i) = \theta^{m', n-1}(x)\sigma_{i+1}$;
   e. If $x\gamma_i$ is an $(m', n)$-cone for $i \leq n-1$, then $\theta^{m', n}(x\gamma_i) = \theta^{m', n-1}(x)\gamma_{i+1}$;
   f. If $x\gamma_i$ is an $(m', n)$-cone for $i \geq n+1$, then $\theta^{m', n}(x\gamma_i) = \theta^{m', n-1}(x)\gamma_{i+1}$;
   g. $\theta^{m', n}(\theta^{m', n-1}(x)) = \theta^{m', n-1}(x)\gamma_n$;
Now we must define the functions by Corollary 5.5. Thus \( X \) satisfies Definition 5.38, item [1] follows by Corollary 5.10.

\[
\text{Proof. The non-degenerate } n\text{-cubes of } Q \int X \text{ are the cubes of } X \text{ which factor through } Q^n, \text{ i.e. the } \nondeg{0,n} \text{ of } X, \text{ and for } n \geq 1 \text{ these are also its non-degenerate } (1,n-1)\text{-cubes by Corollary 5.5. Thus } X^1 \text{ satisfies Definition 5.38, item [1].}
\]

Now we must define the functions \( \theta^{0,n} \) and \( \theta^{1,n} \) for all \( n \) and show that they satisfy the identities of item [3]. For a \( (0,n) \)-cone \( x \in X_n \), we set \( \theta^{0,n}(x) = x\sigma_{n+1} \); this is a \( (1,n+1) \)-cone by Lemma 5.11. The hypotheses of item [c], item [d] and item [f] are vacuous here, as there are no cubical structure maps satisfying the given constraints on their indices; item [h] similarly does not apply in this case.

The remaining identities follow easily from the cubical identities:

- For item [a] let \( i \leq n \). Then \( \theta^{0,n}(x)\partial_i,0 = x\partial_{i,0}\sigma_n = \theta^{0,n-1}(x)\sigma_n \).
- For item [b] we have \( \theta^{0,n}(x)\partial_{n+1,0} = x\gamma_{n+1}\partial_{n+1,0} = x \).
- For item [c] let \( 1 \leq i \leq n-1 \). Then \( \theta^{0,n}(x\gamma_i) = x\gamma_i\sigma_{n+1} = x\sigma_{n}\gamma_i = \theta^{0,n-1}(x)\gamma_i \).
- For item [g] we have \( \theta^{0,n+1}(\theta^{0,n}(x)) = x\sigma_{n+1}\sigma_{n+2} = x\sigma_{n+1}\gamma_{n+1} = \theta^{0,n}(x)\gamma_{n+1} \).

Next we define \( \theta^{1,n} \). Because every \( (1,n) \)-cone is a \( (0,n+1) \)-cone, we must have \( \theta^{1,n}(x) = x\gamma_{n+1} \) in order to satisfy item [h]. This is indeed a \( (1,n+1) \)-cone by Lemma 5.11. The hypothesis of item [f] is still vacuous in this case, as there are no connection maps \( \gamma_i : [1]^n \to [1]^{n-1} \) with \( i \geq n + 1 \). Once again, we can verify the remaining identities of item [3] using the cubical identities:

- For item [a] let \( i \leq n \). Then \( \theta^{1,n}(x)\partial_i,0 = x\gamma_{n+1}\partial_{i,0} = x\partial_{i,0}\gamma_n = \theta^{1,n-1}(x)\partial_i,0 \).
- For item [b] we have \( \theta^{1,n}(x)\partial_{n+1,0} = x\gamma_{n+1}\partial_{n+1,0} = x \).
- For item [c] we need only consider the case \( m' = 1, i = n + 2 \). For this case we have \( \theta^{1,n}(x)\partial_{n+2,1} = x\gamma_{n+1}\partial_{n+2,1} = x\partial_{n+1,1}\sigma_{n+1} = \theta^{0,n}(x)\partial_{n+1,1} \).
- For item [d] the only relevant degeneracy is \( \sigma_{n+1} \), and we have \( \theta^{1,n}(x\sigma_{n+1}) = x\sigma_{n+1}\gamma_{n+1} = x\sigma_{n+1}\sigma_{n+2} = \theta^{0,n}(x)\sigma_{n+2} \).
- For item [e] let \( 1 \leq i \leq n-1 \). Then \( \theta^{1,n}(x\gamma_i) = x\gamma_i\gamma_{n+1} = x\gamma_n\gamma_i = \theta^{1,n-1}(x)\gamma_i \).
- For item [g] we have \( \theta^{1,n+1}(\theta^{1,n}(x)) = x\gamma_{n+1}\gamma_{n+2} = x\gamma_{n+1}\gamma_{n+1} + \theta^{1,n}(x)\gamma_{n+1} \).

In view of Proposition 5.39 let \( X^1 = Q \int X \). Now fix \( m \geq 2 \), and assume that we have defined \( X^{m-1} \) satisfying the induction hypothesis on base dimension for some \( m \geq 2 \). We will define \( X^m \) by a further induction; we now establish our induction hypothesis and base case for this inductive construction.

Definition 5.40. For \( n \geq -1 \), a subcomplex \( X^{m,n} \subseteq X \) satisfies the induction hypothesis on degree if:
Given \((m', n')\) where either \(0 \leq m' < m\) or \(m' = m\) and \(0 \leq n' \leq n\), all \((m', n')\)-cones of \(X\) are contained in \(X^{m,n}\);

(2) For \(i \geq 1\), every non-degenerate cube \(\Box^{m+i} \to X^{m,n}\) is an \((m, i)\)-cone;

(3) For every \((m', n')\)-cone in \(X\) with either \(0 \leq m' < m\) or \(m' = m\) and \(0 \leq n' \leq n\), \(X^{m,n}\) contains an \((m', n'+1)\)-cone \(\theta^{m', n'}(x)\) satisfying the identities of Definition 5.38 item \(3\);

(4) Any \((m, n')\)-cone in \(X^{m,n}\) with \(n' > n\) is either degenerate, an \((m-1, n'+1)\)-cone, or \(\theta^{m,n'-1}(x)\) for some \((m, n'-1)\)-cone \(x\);

(5) If \(n \geq 0\) then there is a trivial cofibration \(X^{m,n-1} \hookrightarrow X^{m,n}\).

Proposition 5.41. \(X^{m-1}\) satisfies the induction hypothesis on degree for \(n = -1\).

Proof. This follows immediately from the induction hypothesis on base dimension. \(\square\)

In view of Proposition 5.41 let \(X^{m,n} = X^{m-1}\). Now fix \(n \geq 0\) and assume that we have defined \(X^{m,n-1}\) satisfying the induction hypothesis on degree for \(n - 1\).

Lemma 5.42. \(X^{m,n-1}\) contains all degenerate \((m, n)\)-cones of \(X\).

Proof. If \(y = x\sigma_i\) or \(y = x\gamma_i\) is an \((m, n)\)-cone, then \(x = y\partial_{i,0}\) is an \((m, n-1)\)-cone by Lemma 5.13 and Corollary 5.14. Thus \(x\) is in \(X^{m,n-1}\), and therefore so is \(y\).

Before we can define \(X^{m,n}\), we must prove a lemma involving cones in \(X^{m,n-1}\) of the form \(\theta^{m', 0}(x)\).

Lemma 5.43. Let \(m' \geq 0\), \(x : \Box^{m'} \to X^{m,n-1}\), and \(1 \leq i \leq m+1\). The image under \(\theta^{m', 0}(x) : \Box^{m'+1} \to X^{m,n-1}\) of the critical edge with respect to the face \(\partial_{i,0}\) is degenerate.

Proof. For \(i \geq 2\), this follows from Lemma 5.13 since \(\theta^{m', 0}(x)\) is an \((m', 1)\)-cone. For \(i = 1\), we proceed by induction on \(m'\). For \(m' = 0\), we have \(\theta^{0,0}(x) = x\sigma_1\); so \(\theta^{0,0}(x)\) is a degeneracy of a vertex, thus its unique edge is degenerate.

Now let \(m' \geq 1\), and suppose that the statement holds for \(m'-1\). The edge in question may be written as \(\theta^{m', 0}(x)\partial_{m'+1,1}\partial_{m'+2,2}\). By Definition 5.38 item \(6\), this is equal to \(\theta^{m'-1,0}(x\partial_{m'+1,1})\partial_{m'+1,2,1}\), which is degenerate by the induction hypothesis. \(\square\)

Our next step will be to define \(X^{m,n}\) by a further (transfinite) induction. To that end, let \(T\) be the set of all \((m, n)\)-cones in \(X\) which are not contained in \(X^{m,n-1}\).

Lemma 5.44. The set \(T\) consists of those non-degenerate \((m, n)\)-cones which are not \((m-1, n+1)\)-cones and are not equal to \(\theta^{m,n-1}(x)\) for any \((m, n-1)\)-cone \(x\).

Proof. By Lemma 5.42 every cone in \(T\) is non-degenerate, while all \((m-1, n+1)\)-cones of \(x\) and all cones of the form \(\theta^{m,n-1}(x)\) are contained in \(X^{m,n-1}\) by item \(1\) and item \(3\) of Definition 5.38. That these are the only \((m, n)\)-cones of \(X\) contained in \(X^{m,n-1}\) follows from Definition 5.38 item \(4\). \(\square\)
We now impose an arbitrary well-ordering on \( T \), indexing its elements as \( x_t \) for \( t < \kappa \), for a suitable ordinal \( \kappa \). Similarly to a typical proof by induction on skeleta, we will build \( X^{m,n} \) from \( X^{m,n-1} \) by a series of open-box fillings.

**Proposition 5.45.** For each ordinal number \( t \leq \kappa \), there exists a subcomplex \( X^{m,n,t} \) whose cubes are exactly those of \( X^{m,n-1} \), plus all cubes \( x_{t'} \) for \( t' < t \) and an \( (m,n+1) \)-cone \( \theta^{m,n}(x_{t'}) \) satisfying the identities of item (a) through item (c) of Definition 5.38 for every such \( x_{t'} \). Furthermore, for each \( t < \kappa \), the inclusion \( X^{m,n,t} \to X^{m,n,t+1} \) is a trivial cofibration.

**Proof.** We begin by setting \( X^{m,n,0} = X^{m,n-1} \). Now suppose that we have constructed \( X^{m,n,t} \), and consider the \((m,n)\)-cone \( x_t \). For each \( i \leq n \), the face \( x_t \partial_{i,0} \) is an \((m,n-1)\)-cone by Lemma 5.7; thus \( X^{m,n-1} \) contains an \((m,n)\)-cone \( \theta^{m,n-1}(x_t \partial_{i,0}) \). Similarly, for each \( i \geq n+2 \), the face \( x_t \partial_{i-1,1} \) is an \((m-1,n)\)-cone, and so \( X^{m,n-1} \) contains an \((m-1,n+1)\)-cone \( \theta^{m-1,n}(x_t \partial_{i-1,1}) \), and these cones satisfy the identities of Definition 5.38 item (a) through item (c). Using Lemma 5.20 we will define a map \( y \colon B^{m,n+1,n+1} \to X^{m,n,t} \) with \( y_{i,0} = \theta^{m,n-1}(x_t \partial_{i,0}) \) for \( 1 \leq i \leq n \), \( y_{n+1,0} = x_t \), and \( y_{i,1} = \theta^{m-1,n}(x_t \partial_{i-1,1}) \) for \( i \geq n + 2 \).

To show that we can define such a map, we must verify that our choices of \( y_{i,\varepsilon} \) satisfy the cubical identity for composing face maps.

For \( i_1 < i_2 \leq n \), \( \varepsilon_1 = \varepsilon_2 = 0 \), we have:

\[
y_{i_2,0} \partial_{i_1,0} = \theta^{m,n-1}(d_{i_2,0} x_t) \partial_{i_1,0} = \theta^{m,n-2}(x_t \partial_{j,0} \partial_{i,0}) = \theta^{m,n-2}(x_t \partial_{j,0} \partial_{j-1,0}) = \theta^{m,n-1}(x_t \partial_{j,0}) \partial_{j-1,0} = y_{i,0} \partial_{j-1,0}
\]

For \( i_1 < i_2 = n + 1 \), we have:

\[
y_{n+1,0} \partial_{i_1,0} = x_t \partial_{i_1,0} = \theta^{m,n-1}(x_t \partial_{i_1,0}) \partial_{n,0} = y_{i_1,0} \partial_{n,0}
\]

For \( n + 1 = i_1 < i_2 \) we have:

\[
y_{i_2,1} \partial_{n+1,0} = \theta^{m-1,n}(x_t \partial_{i_2-1,1}) \partial_{n+1,0} = x_t \partial_{i_2-1,1} = y_{n+1,0} \partial_{i_2-1,1}
\]
Finally, for $n + 2 \geq i_1 < i_2$, we have:

$$y_{i_2,1} \partial_{i_2,1} = \theta^{m-1,n}(x_t \partial_{i_2-1,1}) \partial_{i_2,1}$$
$$= \theta^{m-2,n}(x_t \partial_{i_2-1,1} \partial_{i_2-1,1})$$
$$= \theta^{m-2,n}(x_t \partial_{i_1-1,1} \partial_{i_2-1,1})$$
$$= \theta^{m-1,n}(x_t \partial_{i_1-1,1}) \partial_{i_2-1,1}$$
$$= y_{i_1,1} \partial_{i_2-1,1}$$

Thus the $(n + 1)$-tuple $y$ does indeed define a map $B^{m,n+1,n+1} \rightarrow X$. Now consider the following commuting diagram:

$$
\begin{array}{ccc}
B^{m,n+1,n+1} & \overset{y}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
C^{m,n+1} & \overset{\theta}{\longrightarrow} & \boxtimes
\end{array}
$$

The left-hand map is a trivial cofibration by Proposition 5.24, while the right-hand map is a fibration by assumption. Thus there exists a lift of this diagram, i.e. an $(m, n + 1)$-cone $\theta^{m,n}(x_t) : C^{m,n+1} \rightarrow X$ such that for $i \leq n, \theta^{m,n}(x_t) \partial_{i,0} = \theta^{m,n-1}(x_t \partial_{i,0}), \theta^{m,n}(x_t) \partial_{n+1,0} = x_t$, and for $i \geq n + 2, \theta^{m,n}(x_t) \partial_{i,1} = \theta^{m-1,n}(x_t \partial_{i-1,1})$. So each face $\theta^{m,n}(x_t) \partial_{i,0}$ is in $X^{m,n-1} \subseteq X^{m,n,t}$ by assumption. Furthermore, all other faces of $\theta^{m,n}(x_t)$ are $(n - 1, n)$-cones by Lemma 5.7 hence they are in $X^{m-1,n} \subseteq X^{m,n,t}$ by the induction hypothesis on base dimension. Thus the restriction of the cube $\theta^{m,n}(x_t) : \boxtimes^{m,n+1} \rightarrow X$ to the open box $\boxtimes^{m,n+1}$ defines a map $\boxtimes^{m,n+1} \rightarrow X^{m,n,t}$. Furthermore, the critical edge of this open box is degenerate – for $n = 0$ this follows from Lemma 5.13 while for $n \geq 1$ it follows from Lemma 5.14.

So we may define $X^{m,n,t+1}$ by the following pushout diagram:

$$
\begin{array}{ccc}
\boxtimes^{m,n+1} & \overset{\theta^{m,n}(x_t)}{\longrightarrow} & X^{m,n,t} \\
\downarrow & & \downarrow \\
\boxtimes^{m,n+1} & \overset{\theta^{m,n}(x_t)}{\longrightarrow} & X^{m,n,t+1}
\end{array}
$$

Then $X^{m,n,t+1}$ contains $X^{m,n,t}$, plus $x_t$ and $\theta^{m,n}(x_t)$, and no other non-degenerate cubes. Furthermore, the inclusion $X^{m,n,t} \hookrightarrow X^{m,n,t+1}$ is a trivial cofibration, as a pushout of an inner open box inclusion.

For a limit ordinal $t$, we define $X^{m,n,t}$ to be the sequential colimit of the inclusions $X^{m,n,0} \hookrightarrow ... \hookrightarrow X^{m,n,t'} \hookrightarrow ...$ for $t' < t$. □

We define $X^{m,n}$ to be the cubical set $X^{m,n,\omega}$ constructed in Proposition 5.45. Our next task is to show that $X^{m,n}$ satisfies the induction hypothesis on degree. Verifying condition item (3) of Definition 5.40 will take the most work, so we begin with the other conditions.

**Proposition 5.46.** $X^{m,n}$ satisfies item (1), item (2), item (4) and item (5) of Definition 5.40.
Proof. For item (1), we first observe that any \((m', n')\)-cone with \(m' < m\) or \(m' = m, n' < n\) is contained in \(X^{m, n+1} \subseteq X^{m, n}\). Furthermore, each \((m, n)\)-cone of \(X\) which is not in \(X^{m, n+1}\) is equal to \(x_t\) for some \(t < \kappa\), and is thus contained in \(X^{m, n+1} \subseteq X^{m, n}\). The condition of item (2) follows for the cubes of \(X^{m, n-1}\) by the induction hypothesis on degree, while the only new non-degenerate cubes added in the construction of \(X^{m, n}\) are the \((m, n)\)-cones \(x_t\) and the \((m, n+1)\)-cones \(\theta^{m, n+1}(x_t)\). For item (3) we can again apply the induction hypothesis for the cubes of \(X^{m, n-1}\), and observe that the only non-degenerate \((m, n')\)-cones with \(n' > n\) which we have added in \(X^{m, n}\) are those of the form \(\theta^{m, n}(x_t)\). Finally, for item (5) the inclusion \(X^{m, n-1} \hookrightarrow X^{m, n}\) is a trivial cofibration since it is the sequential colimit of the trivial cofibrations \(X^{m, n, t} \hookrightarrow X^{m, n, t+1}\). □

Now we consider item (3). By the induction hypothesis on \(n\), for every \((m, n')\)-cone \(x\) in \(X^{m, n}\) with \(m' < m\) or \(m' = m, n' < n\) there is an \((m', n' + 1)\)-cone \(\theta^{m', n+1}(x)\) in \(X^{m, n+1}\) satisfying the necessary identities; thus we only need to define \(\theta^{m, n}\) and show that it satisfies these identities as well.

Definition 5.47. Let \(x\) be an \((m, n)\)-cone of \(X\). Then \(\theta^{m, n}(x)\): \(\square^{m+n+1} \rightarrow X^{m, n}\) is defined as follows:

1. If the standard form of \(x\) is \(z\sigma_{a_p}\) for some \(a_p \geq n + 1\), then \(\theta^{m, n}(x) = \theta^{m, n-1}(z)\sigma_{a_p+1}\);
2. If the standard form of \(x\) is \(z\gamma_{b_q}\) for some \(b_q \leq n - 1\), then \(\theta^{m, n}(x) = \theta^{m, n-1}(z)\gamma_{b_q}\);
3. If the standard form of \(x\) is \(z\gamma_{b_q}\) for some \(b_q \geq n + 1\), then \(\theta^{m, n}(x) = \theta^{m-1, n}(z)\gamma_{b_q+1}\);
4. If \(x\) is an \((m - 1, n + 1)\)-cone not covered under any of cases (1) through (3), then \(\theta^{m, n}(x) = x\gamma_{n+1}\);
5. If \(x = \theta^{m, n-1}(x')\) for some \(x'\): \(C^{m, n-1} \rightarrow X\) and \(x\) is not covered under any of cases (1) through (4) then \(\theta^{m, n}(x) = x\gamma_n\);
6. If \(x \in T\), then \(\theta^{m, n}(x)\) is as constructed in Proposition 5.45.

Proposition 5.48. Definition 5.47 defines a function \(\theta^{m, n}: \text{cSet}(C^{m, n}, X) \rightarrow \text{cSet}(C^{m, n+1}, X)\).

Proof. There are two things we need to show: first, that each of the constructions of Definition 5.47 produces an \((m, n + 1)\)-cone; second, that at least one of cases (1) through (6) applies to every \((m, n)\)-cone of \(X\).

That the construction of case (6) produces an \((m, n)\)-cone follows from Proposition 5.45; the other cases follow from Lemma 5.11 and Corollary 5.10. To see that every \((m, n)\)-cone of \(X\) falls under one of cases (1) through (6), we first consider degenerate cones. Those whose standard forms end with any map other than \(\gamma_n\) fall under one of cases (1) through (3) (for those whose standard forms end with degeneracy maps, this follows from Lemma 5.11). Those whose standard forms end with \(\gamma_n\) fall under case (4) by Lemma 5.12. Every non-degenerate cone falls under one of cases (4), (5) or (6) by Lemma 5.14. □

The proof that \(\theta^{m, n}\) satisfies the identities of Definition 5.38, item (3) involves many elaborate case analyses; for brevity, these calculations have been relegated to appendix A.

Corollary 5.49. \(X^{m, n}\) satisfies the induction hypothesis on degree for \(n\).
Proof. All criteria of Definition 5.40 are proven in Proposition 5.46, except for item (3), which is proven in Proposition A.1 through Proposition A.5. □

Now, given \( X^{m,n} \) satisfying the induction hypothesis for all \( n \geq -1 \), we let \( X^m \) be the colimit of the sequence of inclusions:

\[
X^{m-1} = X^{m,-1} \hookrightarrow X^{m,0} \hookrightarrow \ldots \hookrightarrow X^{m,n} \hookrightarrow \ldots
\]

Proposition 5.50. \( X^m \) satisfies the induction hypothesis on base dimension.

Proof. For item (1) of Definition 5.38, we may first note that all \((m',n)\)-cones for \( m' < m \) are contained in \( X^{m-1} \subseteq X^m \) by the induction hypothesis on \( X^{m-1} \). Furthermore, if \( x \) is an \((m,n)\)-cone of \( X \) for some \( n \geq 0 \), then \( x \) is contained in \( X^{m,n} \subseteq X^m \). Thus \( X^m \) contains all \((m',n)\)-cones of \( X \) for \( m' \leq m \) and \( n \geq 0 \). Since every cube of \( X^m \) is contained in some \( X^{m,n} \), item (2) and item (3) follow immediately from the corresponding conditions in the induction hypothesis on degree. Finally, by the induction hypothesis on degree, each map \( X^{m,n-1} \rightarrow X^{m,n} \) for \( n \geq 0 \) is a trivial cofibration, hence the sequential colimit \( X^{m-1} \rightarrow X^m \) is a trivial cofibration as well. Thus \( X^m \) satisfies item (4). □

So for every \( m \geq 1 \) we can construct a subcomplex \( X^m \subseteq X \) satisfying the induction hypothesis. By considering the union of all these subcomplexes, we can prove Theorem 5.37.

Proof of Theorem 5.37. Consider the sequence of inclusions

\[
Q \int X = X^1 \hookrightarrow X^2 \hookrightarrow \ldots \hookrightarrow X^m \hookrightarrow \ldots
\]

The colimit of this diagram is the union of all the subcomplexes \( X^m \). But since every cube \( \square^m \rightarrow X \) is contained in \( X^m \) (as an \((m,0)\)-cone), this colimit is \( X \) itself. Because each map in the diagram is a trivial cofibration, the colimit map \( Q \int X \rightarrow X \) is a trivial cofibration as well. □

Theorem 5.51. The adjunction \( Q : \text{sSet} \rightleftarrows \text{cSet} : \int \) is a Quillen equivalence.

Proof. The adjunction is Quillen by Proposition 5.25. \( Q \) preserves and reflects the weak equivalences of the Quillen model structure on \( \text{sSet} \) by Corollary 5.29 and Proposition 5.30. Thus \( Q \dashv \int \) satisfies the hypotheses of Corollary 1.14 item (1) and we can apply Theorem 5.37 to conclude that it is a Quillen equivalence. □

Proof of Theorem 5.1. First note that, because all objects in both \( \text{cSet} \) and \( \text{sSet} \) are cofibrant, the left derived functor \( L(TQ) \) is the composite of the left derived functors \( LT \) and \( LQ \), while the left derived functor of the identity is the identity; this can easily be seen from [Hov99, Def. 1.36]. By Corollary 5.36, we have a natural weak equivalence \( TQ \Rightarrow \text{id}_{\text{sSet}} \). In the homotopy category \( \text{HosSet} \), this natural weak equivalence becomes a natural isomorphism \( LT \circ LQ \cong \text{id}_{\text{HosSet}} \). By Theorem 5.51, \( LQ \) is an equivalence of categories, thus \( LT \) is an equivalence of categories as well. The adjunction \( T \dashv U \) is Quillen by Proposition 1.26 so this implies it is a Quillen equivalence. □
The proofs in this section can easily be adapted to show that $Q \dashv \int$ is a Quillen equivalence between the standard model structures for $\infty$-groupoids on $sSet$ and $cSet$. (This result was essentially stated as [KLW19, Prop. 5.3], but the proof supplied there only shows that $Q$ and $\int$ form a Quillen adjunction.)

**Theorem 5.52.** The adjunction $Q : sSet \rightleftarrows cSet : \int$ is a Quillen equivalence between the Quillen model structure on $sSet$ and the Grothendieck model structure on $cSet$.

**Proof.** Proposition 4.25 and Proposition 5.25 both have natural analogues, showing that $T \dashv U$ and $Q \dashv \int$ are Quillen adjunctions between these model structures (implying in particular that $Q$ preserves weak equivalences). Since every weak equivalence in the Joyal model structure is also a weak equivalence in the Quillen model structure, $F$ is a natural weak equivalence in the Quillen model structure as well. Thus the proof of Proposition 5.30 adapts to show that $Q$ reflects the weak equivalences of the Quillen model structure. Corollary 1.14 item (ii) and Theorem 5.37 then imply the analogue of Theorem 5.51, since every cubical Kan complex is a cubical quasicategory and every weak equivalence in the cubical Joyal model structure is a weak equivalence in the Grothendieck model structure.

The proof of Theorem 5.1 can then be adapted to obtain a new proof that $T \dashv U$ is a Quillen equivalence between the Grothendieck and Quillen model structures, as was previously shown in [Cis06, Prop. 8.4.30].

**Appendix A. Verification of identities on $\theta$**

Here we prove that the construction $\theta^{m,n}$ of Definition 5.47 satisfies all of the necessary identities. We begin with the identities involving faces.

**Proposition A.1.** $\theta^{m,n}$ satisfies the identities of Definition 5.38 item (a) and item (b), that is, for $i \leq n$, $\theta^{m,n}(x) \partial_{i,0} = \theta^{m,n-1}(x \partial_{i,0})$, while $\theta^{m,n}(x) \partial_{n+1,0} = x$.

**Proof.** We will prove this via a case analysis, based on the six cases of Definition 5.47. First, let $x = z \sigma_{a_p}$ in standard form, for $a_p \geq n + 1$. By the induction hypotheses, for $m' < m$ or $m' = m, n' < n$, $\theta^{m',n'}$ satisfies all the identities of Definition 5.38 item (3) (in future computations we will often use this assumption without comment). So for $i \leq n$ we have:

$$\theta^{m,n}(x) \partial_{i,0} = \theta^{m,n-1}(z) \sigma_{a_p+1} \partial_{i,0}$$

$$= \theta^{m-1,n-1}(z) \partial_{i,0} \sigma_{a_p}$$

$$= \theta^{m-1,n-1}(z) \partial_{i,0} \sigma_{a_p}$$

$$= \theta^{m,n-1}(z \partial_{i,0} \sigma_{a_p-1})$$

$$= \theta^{m,n-1}(z \sigma_{a_p} \partial_{i,0})$$

$$= \theta^{m,n-1}(x \partial_{i,0})$$

And for $i = n + 1$ we have:
\[ \theta^{m,n}(x) \partial_{n+1,0} = \theta^{m-1,n}(z) \sigma_{a_p+1} \partial_{n+1,0} \\
= \theta^{m-1,n}(z) \partial_{n+1,0} \sigma_{a_p} \\
= z \sigma_{a_p} \\
= x \]

Now suppose that the standard form of \( x \) is \( z \gamma_{b_q} \), where \( b_q \leq n - 1 \). Note that we must have \( b_q \geq 1 \), so this case can only occur when \( n \geq 2 \). Now for \( i \leq b_q \) we have:

\[ \theta^{m,n}(x) \partial_{i,0} = \theta^{m,n-1}(z) \gamma_{b_q} \partial_{i,0} \\
= \theta^{m,n-1}(z) \partial_{i,0} \gamma_{b_q-1} \\
= \theta^{m,n-2}(z \partial_{i,0}) \gamma_{b_q-1} \\
= \theta^{m,n-1}(z \partial_{i,0} \gamma_{b_q-1}) \\
= \theta^{m,n-1}(z \gamma_{b_q} \partial_{i,0}) \\
= \theta^{m,n-1}(x \partial_{i,0}) \]

For \( i = b_q \) or \( i = b_q + 1 \) we have:

\[ \theta^{m,n}(x) \partial_{i,0} = \theta^{m,n-1}(z) \gamma_{b_q} \partial_{i,0} \\
= \theta^{m,n-1}(z) \\
= \theta^{m,n-1}(z \gamma_{b_q} \partial_{i,0}) \\
= \theta^{m,n-1}(x \partial_{i,0}) \]

For \( b_q + 2 \leq i \leq n \) we have:

\[ \theta^{m,n}(x) \partial_{i,0} = \theta^{m,n-1}(z) \gamma_{b_q} \partial_{i,0} \\
= \theta^{m,n-1}(z) \partial_{i-1,0} \gamma_{b_q} \\
= \theta^{m,n-2}(z \partial_{i-1,0}) \gamma_{b_q} \\
= \theta^{m,n-1}(z \partial_{i-1,0} \gamma_{b_q}) \\
= \theta^{m,n-1}(z \gamma_{b_q} \partial_{i,0}) \\
= \theta^{m,n-1}(x \partial_{i,0}) \]

And for \( i = n + 1 \) we have:
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\[
\theta^{m,n}(x)\partial_{n+1,0} = \theta^{m,n-1}(z)\gamma_{b_q}\partial_{n+1,0} \\
= \theta^{m,n-1}(z)\partial_{n,0}\gamma_{b_q} \\
= z\gamma_{b_q} \\
= x
\]

Next we consider the case where the standard form of $x$ is $z\gamma_{b_q}, b_q \geq n+1$. Then for $i \leq n$ we have:

\[
\theta^{m,n}(x)\partial_{i,0} = \theta^{m-1,n}(z)\gamma_{b_{q+1}}\partial_{i,0} \\
= \theta^{m-1,n}(z)\partial_{i,0}\gamma_{b_q} \\
= \theta^{m-1,n-1}(z\partial_{i,0})\gamma_{b_q} \\
= \theta^{m,n-1}(z\gamma_{b_q}\partial_{i,0}) \\
= \theta^{m,n-1}(x\partial_{i,0})
\]

And for $i = n+1$ we have:

\[
\theta^{m,n}(x)\partial_{n+1,0} = \theta^{m-1,n}(z)\gamma_{b_{q+1}}\partial_{n+1,0} \\
= \theta^{m-1,n}(z)\partial_{n,0}\gamma_{b_q} \\
= z\gamma_{b_q} \\
= x
\]

Next, we consider case (4) of Definition 5.47: let $x$ be an $(m-1,n+1)$-cone not falling under any of cases (1)-(3). By Lemma 5.7, every face $x\partial_{i,0}$ for $i \leq n$ is an $(m-1,n)$-cone, and therefore $\theta^{m,n-1}(x\partial_{i,0}) = x\partial_{i,0}\gamma_n$ by the induction hypothesis. Now, for $i \leq n$, we can compute:

\[
\theta^{m,n}(x)\partial_{i,0} = x\gamma_{n+1}\partial_{i,0} \\
= x\partial_{i,0}\gamma_n \\
= \theta^{m,n-1}(x\partial_{i,0})
\]

And $\theta^{m,n}(x)\partial_{n+1,0} = x\gamma_{n+1}\partial_{n+1,0} = x$.

Next, we consider case (5): consider an $(m,n)$-cone $\theta^{m,n-1}(x')$ not falling under any of cases (1) through (4). Then for $i \leq n - 1$ we have:
\[ \theta^{m,n}(\theta^{m,n-1}(x'))\partial_{i,0} = \theta^{m,n-1}(x')\gamma_n \partial_{i,0} \]
\[ = \theta^{m,n-1}(x')\partial_{i,0} \gamma_{n-1} \]
\[ = \theta^{m,n-2}(x'\partial_{i,0})\gamma_{n-1} \]
\[ = \theta^{m,n-1}(\theta^{m,n-2}(x'\partial_{i,0})) \]
\[ = \theta^{m,n-1}(\theta^{m,n-1}(x')\partial_{i,0}) \]

For \( i = n \) we have:

\[ \theta^{m,n}(\theta^{m,n-1}(x))\partial_{n,0} = \theta^{m,n-1}(x')\gamma_n \partial_{n,0} \]
\[ = \theta^{m,n-1}(x') \]
\[ = \theta^{m,n-1}(\theta^{m,n-1}(x')\partial_{n,0}) \]

And for \( i = n + 1 \) we have \( \theta^{m,n}(\theta^{m,n-1}(x'))\partial_{n+1,0} = \theta^{m,n-1}(x')\gamma_{n+1,0} = \theta^{m,n-1}(x') \).

Finally, we consider case (6). Let \( x \in T \); then the identities hold by Proposition 5.45. □

**Proposition A.2.** \( \theta^{m,n} \) satisfies the identity of Definition 5.38, item (f) that is, for all \( x: C^{m,n} \to X^{m,n}, i \geq n + 2 \), we have \( \theta^{m,n}(x)\partial_{i,1} = \theta^{m,n-1}(x\partial_{i-1,1}) \).

**Proof.** Throughout the proof, we fix \( i \geq n + 2 \). First we consider case (1) of Definition 5.47. Suppose that the standard form of \( x = z\sigma^a_p \), for some \( a_p \geq n + 1 \). Here we must consider various cases based on a comparison of \( i \) with \( a_p \). First suppose that \( i \leq a_p \); note that this implies \( a_p \geq n + 2 \). Then we have:

\[ \theta^{m,n}(x)\partial_{i,1} = \theta^{m,n-1}(z)\sigma^a_{p+1}\partial_{i,1} \]
\[ = \theta^{m,n-1}(z)\partial_{i,1}\sigma^a_p \]
\[ = \theta^{m,n-2}(z\partial_{i-1,1})\sigma^a_p \]
\[ = \theta^{m,n-1}(z\partial_{i-1,1}\sigma^a_{p-1}) \]
\[ = \theta^{m,n-1}(z\sigma^a_p\partial_{i-1,1}) \]
\[ = \theta^{m,n-1}(x\partial_{i-1,1}) \]

To obtain the fourth equality, we have used the identity of item (d) and the fact that \( a_p - 1 \geq n + 1 \).

Next suppose that \( i = a_p + 1 \); then we have:
Finally, suppose $i \geq a_p + 2$; note that this implies $a_p \geq n + 3$. Then we have:

$$\theta^{m,n}(x)\partial_{a_p+1,1} = \theta^{m,n-1}(z)\sigma_{a_p+1} \partial_{a_p+1,1}$$
$$= \theta^{m,n-1}(z)$$
$$= \theta^{m,n-1}(z\sigma_{a_p}\partial_{a_p,1})$$
$$= \theta^{m,n-1}(x\partial_{a_p,1})$$

Next we consider case (2): suppose that $x = z\gamma b_q$ in standard form, where $b_q \leq n - 1$. Then $i \geq b_q + 3$, and we have:

$$\theta^{m,n}(x)\partial_{i,1} = \theta^{m,n-1}(z)\sigma_{a_p+1} \partial_{i,1}$$
$$= \theta^{m,n-1}(z)\partial_{i-1,1}\sigma_{a_p+1}$$
$$= \theta^{m-2,n}(z\partial_{i-2,1})\sigma_{a_p+1}$$
$$= \theta^{m-1,n}(z\partial_{i-2,1}\sigma_{a_p})$$
$$= \theta^{m-1,n}(z\sigma_{a_p}\partial_{i-1,1})$$
$$= \theta^{m-1,n}(x\partial_{i-1,1})$$

Next we consider case (2): suppose that $x = \gamma b_q z$ in standard form, where $b_q \geq n + 1$. Once again, we must perform a case analysis based on a comparison of $i$ with $b_q$. First suppose that $i \leq b_q$, implying $b_q \geq n + 2$. Then we can compute:
Next we consider case (4): let $x\partial_{i,1}$. Furthermore, note that by Lemma 5.9, $x\partial_{b_q+1}$ is not covered under any of cases (1) through (3). Then $x\partial_{b_q+1}$ is an $(m-2, n+1)$-cone by Lemma 5.7, so $\theta^{m-1,n}(x\partial_{b_q+1}) = x\partial_{b_q+1}\gamma_{b_q+1}$ by the identity of item (c) for $\theta^{m-1,n}$. Furthermore, note that by Lemma 5.9, $x\partial_{m+n+1} = x\partial_{m+n+1}\ldots\partial_{n+1,1}\sigma_{n+1}\ldots\sigma_{m+n}$. Using the co-cubical identities, we can rewrite this as $x\partial_{m+n+1}\ldots\partial_{n+1,1}\sigma_{n+1}\ldots\sigma_{n+1}$. Then for $i = n+2$, we can compute:

$$\theta^{m,n}(x)\partial_{i,1} = \theta^{m-1,n}(z)\gamma_{b_q+1}\partial_{i,1}$$

Next suppose that $i = b_q + 1$ or $b_q + 2$; then we have:

$$\theta^{m,n}(x)\partial_{i,1} = \theta^{m-1,n}(z)\gamma_{b_q+1}\partial_{i,1}$$

To obtain the third equality, we used the identity of item (c) for $\theta^{m-1,n}$ and the assumption that $b_q \geq n + 1$. Finally, suppose $i \geq b_q + 3$, implying $i \geq n + 4$. Then we have:

$$\theta^{m,n}(x)\partial_{i,1} = \theta^{m-1,n}(z)\gamma_{b_q+1}\partial_{i,1}$$

Next we consider case (4): let $x$ be an $(m - 1, n + 1)$-cone not covered under any of cases (1) through (3). Then $x\partial_{i-1,1}$ is an $(m - 2, n + 1)$-cone by Lemma 5.7, so $\theta^{m-1,n}(x\partial_{i-1,1}) = x\partial_{i-1,1}\gamma_{n+1}$ by the identity of item (h) for $\theta^{m-1,n}$. Furthermore, note that by Lemma 5.9, $x\partial_{n+1,1} = x\partial_{m+n+1}\ldots\partial_{n+1,1}\sigma_{n+1}\ldots\sigma_{m+n}$. Using the co-cubical identities, we can rewrite this as $x\partial_{m+n+1}\ldots\partial_{n+1,1}\sigma_{n+1}\ldots\sigma_{n+1}$. Then for $i = n+2$, we can compute:
\[ \theta^{m,n}(x) \partial_{n+2,1} = x\gamma_{n+1} \partial_{n+2,1} \]
\[ = x\partial_{n+1,1} \sigma_{n+1} \]
\[ = x\partial_{m+n+1} ... \partial_{n+1,1} \sigma_{n+1} ... \sigma_{n+1} \gamma_{n+1} \]
\[ = x\partial_{n+1,1} \gamma_{n+1} \]

While for \( i \geq n + 3 \), we have:
\[
\theta^{m,n}(x) \partial_{i,1} = x\gamma_{n+1} \partial_{i,1} \\
= x\partial_{i-1,1} \gamma_{n+1} \\
= \theta^{m,n-1}(x\partial_{i-1,1})
\]

Next we consider case (5). Let \( x' : C^{m,n-1} \to X^{m,n} \), and consider \( \theta^{m,n}(\theta^{m,n-1}(x')) \). Then we can compute:
\[
\theta^{m,n}(\theta^{m,n-1}(x')) \partial_{i,1} = \theta^{m,n-1}(x') \gamma_{n} \partial_{i,1} \\
= \theta^{m,n-1}(x') \partial_{i-1,1} \gamma_{n} \\
= \theta^{m,n-1}(x' \partial_{i-1,2,1}) \gamma_{n} \\
= \theta^{m-1,n}(\theta^{m-1,n-1}(x' \partial_{i-1,2,1})) \\
= \theta^{m-1,n}(\theta^{m,n-1}(x') \partial_{i-1,1})
\]

Finally, in case (6), the identity holds by Proposition 5.45.

Next we consider the identities involving degeneracies and connections.

**Proposition A.3.** \( \theta^{m,n} \) satisfies the identities of Definition 5.38, item \([d]\), item \([e]\), and item \([f]\).

That is:
- If \( x\sigma_i \) is an \((m,n)\)-cone for \( i \geq n + 1 \), then \( \theta^{m,n}(x\sigma_i) = \theta^{m-1,n}(x)\sigma_{i+1} \);
- If \( x\gamma_i \) is an \((m,n)\)-cone for \( i \leq n - 1 \), then \( \theta^{m,n}(x\gamma_i) = \theta^{m,n-1}(x)\gamma_i \);
- If \( x\gamma_i \) is an \((m,n)\)-cone for \( i \geq n + 1 \), then \( \theta^{m,n}(x\gamma_i) = \theta^{m-1,n}(x)\gamma_{i+1} \).

**Proof.** For each identity, we will perform a case analysis based on the standard form of \( x \). For item \([d]\) consider an \((m,n)\)-cube \( x\sigma_i \), where \( i \geq n + 1 \) and the standard form of \( x \) is \( y_{b_i} ... \sigma_{a_1} ... \sigma_{a_p} \).
If the string of degeneracy maps in the standard form of \( x \) is empty, or \( a_p < i \), then the standard
form of $x\sigma_i$ ends with $\sigma_i$, so $\theta^{m,n}(x\sigma_i) = \theta^{m-1,n}(x)\sigma_{i+1}$ by definition. So suppose that $a_p \geq i$. Then:

$$\theta^{m,n}(x\sigma_i) = \theta^{m,n}(y\gamma_{b_1} \cdots \gamma_{b_q} \sigma_{a_1} \cdots \sigma_{a_p} \sigma_i)$$

$$= \theta^{m,n}(y\gamma_{b_1} \cdots \gamma_{b_q} \sigma_{a_1} \cdots \sigma_{a_p-1} \sigma_i \sigma_{a_p+1})$$

By assumption, all the indices $a_1, \ldots, a_{p-1}$, are less than $a_p$. Rearranging the expression on the right-hand side of the equation into standard form using the co-cubical identities will not increase any of these indices by more than 1, so the rightmost map in the standard form of $x\sigma_i$, i.e. the degeneracy map with the highest index, is $\sigma_{a_p+1}$. Therefore, we can compute:

$$\theta^{m,n}(y\gamma_{b_1} \cdots \gamma_{b_q} \sigma_{a_1} \cdots \sigma_{a_p-1} \sigma_i \sigma_{a_p+1}) = \theta^{m-1,n}(y\gamma_{b_1} \cdots \gamma_{b_q} \sigma_{a_1} \cdots \sigma_{a_p-1} \sigma_i) \sigma_{a_p+2}$$

$$= \theta^{m-2,n}(y\gamma_{b_1} \cdots \gamma_{b_q} \sigma_{a_1} \cdots \sigma_{a_p-1}) \sigma_i+1 \sigma_{a_p+2}$$

$$= \theta^{m-2,n}(y\gamma_{b_1} \cdots \gamma_{b_q} \sigma_{a_1} \cdots \sigma_{a_p-1}) \sigma_{a_p+1} \sigma_{i+1}$$

$$= \theta^{m-1,n}(y\gamma_{b_1} \cdots \gamma_{b_q} \sigma_{a_1} \cdots \sigma_{a_p-1} \sigma_{a_p}) \sigma_{i+1}$$

$$= \theta^{m-1,n}(x) \sigma_{i+1}$$

So $\theta^{m,n}$ satisfies the identity of item [d].

Next we will verify the identity of item [f]. Consider an $(m, n)$-cube $x\gamma_i$, where $i \geq n+1$ and the standard form of $x$ is as above. If this standard form contains no degeneracy maps, and $b_q < i$ or $x$ is non-degenerate, then the standard form of $x\gamma_i$ ends with $\gamma_i$, so the identity holds by definition. The remaining possibilities for the standard form of $x$ can be divided into various cases. First, suppose that the string of degeneracy maps in the standard form of $x$ is non-empty, i.e. $x = z\sigma_{a_p}$ in standard form. By Lemma 5.7 $x = x\gamma_i \partial_i 0$ is an $(m-1, n)$-cone, so $a_p \geq n+1$ by Lemma 5.16

Now we must break this into further cases based on a comparison between $i$ and $a_p$. If $i < a_p$ then, using the co-cubical identities, item [d] for $\theta^{m,n}$, and item [f] for $\theta^{m-1,n}$, we can compute:

$$\theta^{m,n}(x\gamma_i) = \theta^{m,n}(z\sigma_{a_p} \gamma_i)$$

$$= \theta^{m,n}(z\sigma_{a_p} \gamma_i)$$

$$= \theta^{m,n}(z\gamma_i \sigma_{a_p+1})$$

$$= \theta^{m-1,n}(z\gamma_i) \sigma_{a_p+2}$$

$$= \theta^{m-2,n}(z) \gamma_{i+1} \sigma_{a_p+2}$$

$$= \theta^{m-2,n}(z) \sigma_{a_p+1} \gamma_{i+1}$$

$$= \theta^{m-1,n}(z\sigma_{a_p}) \gamma_{i+1}$$

$$= \theta^{m-1,n}(x) \gamma_{i+1}$$
Next we consider the case \( i = a_p \):

\[
\theta^{m,n}(x\gamma_{a_p}) = \theta^{m,n}(z\sigma_{a_p}\gamma_{a_p}) \\
= \theta^{m,n}(z\sigma_{a_p}\sigma_{a_p+1}) \\
= \theta^{m-1,n}(z\sigma_{a_p}\sigma_{a_p+2}) \\
= \theta^{m-2,n}(z)\sigma_{a_p+1}\sigma_{a_p+2} \\
= \theta^{m-2,n}(z)\sigma_{a_p+1}\gamma_{a_p+1} \\
= \theta^{m-1,n}(z\sigma_{a_p})\gamma_{a_p+1} \\
= \theta^{m-1,n}(z)\gamma_{a_p+1}
\]

Now we consider the case \( i > a_p \). Note that this implies \( i \geq n+2 \), so \( i - 1 \geq n+1 \). Thus we can compute:

\[
\theta^{m,n}(x\gamma_{i}) = \theta^{m,n}(z\sigma_{a_p}\gamma_{i}) \\
= \theta^{m,n}(z\gamma_{i-1}\sigma_{a_p}) \\
= \theta^{m-1,n}(z\gamma_{i-1}\sigma_{a_p+1}) \\
= \theta^{m-2,n}(z)\gamma_{i}\sigma_{a_p+1} \\
= \theta^{m-2,n}(z)\sigma_{a_p+1}\gamma_{i+1} \\
= \theta^{m-1,n}(z\sigma_{a_p})\gamma_{i+1} \\
= \theta^{m-1,n}(z)\gamma_{i+1}
\]

Next we will verify item [f] in the case where the standard form of \( x \) contains no degeneracy maps, and \( i \leq b_q \). In this case we can compute:

\[
\theta^{m,n}(x\gamma_{i}) = \theta^{m,n}(y\gamma_{b_1}...\gamma_{b_q}\gamma_{i}) \\
= \theta^{m,n}(y\gamma_{b_1}...\gamma_{b_q-1}\gamma_{i}\gamma_{b_q+1})
\]

Similarly to what we saw when verifying item [d] the indices \( b_1, ..., b_{q-1} \) are all strictly less than \( b_q \). So after we have rearranged the expression on the right-hand side of this equation into standard form by repeatedly applying the identity \( \gamma_k\gamma_j = \gamma_j\gamma_{k+1} \) for \( j \leq k \), the leftmost map in the expression will still be \( \gamma_{b_q+1} \). Thus we can apply the definition of \( \theta^{m,n} \) to compute:
Thus $\theta^{m,n}$ satisfies item [(f)].

Finally we will verify item [(e)]. Consider an $(m,n)$-cube $x\gamma_i$, where $i \leq n - 1$ and the standard form of $x$ is as above. Once again, we must consider several possible cases based on the standard form of $x$. As with item [(f)], if the standard form of $x$ contains no degeneracy maps, and $b_q < i$ or $x$ is non-degenerate, then $\gamma_i$ is the rightmost map in the standard form of $x\gamma_i$, and the identity holds by definition. Once again, the remaining cases will require computation.

As above, we begin with the case where the string of degeneracy maps in the standard form of $x$ is non-empty. By Lemma 5.7, $x = x\gamma_i \partial_{i,0}$ is an $(m,n-1)$-cone, so $a_p \geq n$ by Lemma 5.10. Then, using the co-cubical identities, item [(d)] for $\theta^{m,n}$, and item [(e)] for $\theta^{m-1,n}$, we can compute:

\[
\theta^{m,n}(x\gamma_i) = \theta^{m,n}(z\sigma_a \gamma_i) = \theta^{m,n}(z) = \theta^{m-1,n}(z) = \theta^{m,n-1}(z) = \theta^{m,n-1}(x) \gamma_i
\]

Next we consider the cases in which the standard form of $x$ contains no degeneracy maps; first, suppose $b_q \geq n$. Then, using the cubical identities, item [(f)] for $\theta^{m,n}$, and item [(e)] for $\theta^{m-1,n}$, we can compute:
\[ \theta^{m,n}(x_{\gamma_i}) = \theta^{m,n}(z_{\gamma_{b_q}} \gamma_i) 
= \theta^{m,n}(z_{\gamma_i} \gamma_{b_q+1}) 
= \theta^{m-1,n}(z_{\gamma_i} \gamma_{b_q+2}) 
= \theta^{m-1,n-1}(z \gamma_i \gamma_{b_q+2}) 
= \theta^{m-1,n-1}(z \gamma_{b_q+1} \gamma_i) 
= \theta^{m,n-1}(\gamma_{b_q} z \gamma_i) 
= \theta^{m,n-1}(x \gamma_i) \]

Next we consider the case \( b_q = n - 1 \). Here we can compute:

\[ x_{\gamma_i} = y_{\gamma b_1 \cdots b_{q-1} \gamma_{n-1} \gamma_i} 
= y_{\gamma b_1 \cdots b_{q-1} \gamma_i \gamma_n} \]

As we have seen in previous cases, all of the coefficients \( b_1, \ldots, b_{q-1} \) are strictly less than \( n - 1 \). So after rearranging this expression into standard form, the rightmost map will still be \( \gamma_n \). Thus \( x_{\gamma_i} \) belongs to case (4), so:

\[ \theta^{m,n}(x_{\gamma_i}) = x_{\gamma_i} \gamma_{n+1} 
= x_{\gamma_n} \gamma_i \]

By Lemma 5.7, \( x = x_{\gamma_i} \partial_i \) is an \((m, n - 1)\)-cone, so the fact that \( b_q = n - 1 \) implies that \( x \) also belongs to case (4). Thus \( x_{\gamma_n} = \theta^{m,n-1}(x) \), so item (e) is satisfied in this case.

Finally, we consider the case \( i \leq b_q \leq n - 2 \). Here we can compute:

\[ \theta^{m,n}(x_{\gamma_i}) = \theta^{m,n}(y_{\gamma b_1 \cdots b_{q-1} \gamma_i}) 
= \theta^{m,n}(y_{\gamma b_1 \cdots b_{q-1} \gamma_{b_q+1}}) \]

As we have done in previous computations, we may observe that all the indices \( b_1, \ldots, b_{q-1} \) are strictly less than \( b_q \), so once the expression on the right-hand side of the equation has been rearranged into standard form, its rightmost map will still be \( \gamma_{b_{q+1}} \). By assumption, \( b_q + 1 \leq n - 1 \), so using the co-cubical identities, the definition of \( \theta^{m,n} \), and item (e) for \( \theta^{m,n-1} \), we can compute:
\[ \theta^{m,n}(y\gamma b_1 \cdots \gamma b_{q-1} \gamma_i \gamma b_q + 1) = \theta^{m,n-1}(y\gamma b_1 \cdots \gamma b_{q-1} \gamma_i \gamma b_q + 1) \]
\[ = \theta^{m,n-2}(y\gamma b_1 \cdots \gamma b_{q-1}) \gamma_i \gamma b_q + 1 \]
\[ = \theta^{m,n-2}(y\gamma b_1 \cdots \gamma b_{q-1}) \gamma b_q \gamma_i \]
\[ = \theta^{m,n-1}(y\gamma b_1 \cdots \gamma b_q) \gamma_i \]
\[ = \theta^{m,n-1}(x) \gamma_i \]

Thus \( \theta^{m,n} \) satisfies item (e). \( \Box \)

**Proposition A.4.** If \( n \geq 1 \) then \( \theta^{m,n} \) satisfies the identity of Definition 5.38, item (g). That is, for any \( x: C^{m,n-1} \to X \), \( \theta^{m,n}(\theta^{m,n-1}(x)) = \theta^{m,n-1}(x) \gamma_n \).

**Proof.** We proceed by a case analysis on \( x \), based on the cases of Definition 5.47. In our computations, we will freely use the identities for \( \theta^{m,n} \) which we have already proven. First suppose that \( x = z \sigma_{a_p} \) in standard form, for some \( a_p \geq n \). Then we can compute:

\[ \theta^{m,n}(\theta^{m,n-1}(x)) = \theta^{m,n}(\theta^{m,n-1}(z \sigma_{a_p})) \]
\[ = \theta^{m,n}(\theta^{m,n-1,n-1}(z)) \sigma_{a_p+1} \]
\[ = \theta^{m,n-1,n-1}(z) \sigma_{a_p+2} \]
\[ = \theta^{m,n-1,n-1}(z) \gamma_n \sigma_{a_p+2} \]
\[ = \theta^{m,n-1,n-1}(z) \gamma_n \gamma_{a_p} \gamma_{b_q} \gamma_n \]
\[ = \theta^{m,n-1}(z \sigma_{a_p}) \gamma_{b_q} \gamma_n \]
\[ = \theta^{m,n-1}(x) \gamma_{b_q} \gamma_n \]

Next let the standard form of \( x \) be \( z \gamma b_q \) where \( b_q \leq n - 2 \). Then we can compute:

\[ \theta^{m,n}(\theta^{m,n-1}(x)) = \theta^{m,n}(\theta^{m,n-1}(z \gamma b_q)) \]
\[ = \theta^{m,n}(\theta^{m,n-2}(z) \gamma b_q) \]
\[ = \theta^{m,n-1}(\theta^{m,n-2}(z)) \gamma b_q \gamma_n \]
\[ = \theta^{m,n-2}(z) \gamma_{n-1} \gamma b_q \gamma_n \]
\[ = \theta^{m,n-2}(z) \gamma b_q \gamma_n \gamma_n \]
\[ = \theta^{m,n-1}(z \gamma b_q) \gamma_n \gamma_n \]
\[ = \theta^{m,n-1}(x) \gamma_n \gamma_n \]

Now let the standard form of \( x \) be \( z \gamma b_q \) where \( b_q \geq n \). Then we can compute:
Next, we consider case (4): suppose that $x$ is an $(m-1,n)$-cone not falling under any of cases (1) through (3). Then $\theta^{m-1,n}(x) = x\gamma_n$. The assumption that $x$ does not belong to any of cases (1) through (3) implies that either it is non-degenerate, or its standard form ends with $\gamma_{n-1}$. Either way, the standard form of $x\gamma_n$ ends with $\gamma_n$, so it falls under case (4) by Lemma 5.12. Thus we can compute:

$$\theta^{m,n}(\theta^{m,n-1}(x)) = \theta^{m,n}(\theta^{m,n-1}(z\gamma_{b_q}))$$
$$= \theta^{m,n}(\theta^{m-1,n-1}(z)\gamma_{b_{q+1}})$$
$$= \theta^{m-1,n}(\theta^{m-1,n-1}(z))\gamma_{b_{q+2}}$$
$$= \theta^{m-1,n-1}(z)\gamma_{n+1}$$
$$= \theta^{m,n-1}(z\gamma_{b_q})\gamma_n$$
$$= \theta^{m,n-1}(x)\gamma_n$$

Finally, case (5) consists of all cubes of the form $\theta^{m,n-1}(x)$ not falling under any of the previous cases, and in this case identity (f) holds by definition.

**Proposition A.5.** $\theta^{m,n}$ satisfies the identity of Definition 5.38 item (h). That is, if $x$ is an $(m-1,n+1)$-cone, then $\theta^{m,n}(x) = x\gamma_{n+1}$.

**Proof.** As in previous proofs, we proceed via case analysis on $x$, based on the cases of Definition 5.47. First suppose that $x$ is an $(m-1,n+1)$-cone whose standard form is $z\sigma_{a_p}$. By Lemma 5.16 $a_p \geq n+2$. Therefore, by Lemma 5.47, $x\partial_{a_p} = z$ is an $(m-2,n+1)$-cone, so $\theta^{m-1,n}(z) = z\gamma_{n+1}$ by item (h) for $\theta^{m-1,n}$. Thus we can compute:

$$\theta^{m,n}(x) = \theta^{m-1,n}(z)\sigma_{a_p+1}$$
$$= z\gamma_{n+1}\sigma_{a_p+1}$$
$$= z\sigma_{a_p}\gamma_{n+1}$$
$$= x\gamma_{n+1}$$
Now let $x$ be an $(m - 1, n + 1)$-cone whose standard form is $z\gamma_{b_q}, b_q \leq n - 1$. Then by Lemma 5.7, $x\partial b_q = z$ is an $(m - 1, n)$-cone. So by item $(h)$ for $\theta^{m,n-1}$, we have $\theta^{m,n-1}(z) = z\gamma_n$. Thus we can compute:

$$\theta^{m,n}(x) = \theta^{m,n-1}(z)\gamma_{b_q}$$

$$= z\gamma_n\gamma_{b_q}$$

$$= z\gamma_{b_q}\gamma_{n+1}$$

$$= x\gamma_{n+1}$$

Next let $x$ be an $(m - 1, n + 1)$-cone whose standard form is $z\gamma_{b_q}$, where $b_q \geq n + 1$. Then by Lemma 5.7, $x\partial b_q+1 = z$ is an $(m - 2, n + 1)$-cone, so $\theta^{m-1,n}(z) = z\gamma_{n+1}$ by item $(h)$. Thus we can compute:

$$\theta^{m,n}(x) = \theta^{m-1,n}(z)\gamma_{b_q+1}$$

$$= z\gamma_{n+1}\gamma_{b_q+1}$$

$$= z\gamma_{b_q}\gamma_{n+1}$$

$$= x\gamma_{n+1}$$

Finally, case (4) consists of all $(m - 1, n + 1)$-cones not falling under any of the previous cases, and in this case the identity of item $(h)$ holds by definition. □

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