Construction of the Hodge-Neumann heat kernel, local Bernstein estimates, and Onsager’s conjecture in fluid dynamics

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Abstract

Most recently, in [Huy19], we introduced the theory of heatable currents and proved Onsager’s conjecture on Riemannian manifolds with boundary, where the weak solution has $B^{1/3}_{3,1}$ spatial regularity. In this sequel, by applying techniques from geometric microlocal analysis to construct the Hodge-Neumann heat kernel, we obtain off-diagonal decay and local Bernstein estimates, and then use them to extend the result to the Besov space $\hat{B}^{1/3}_{3,V}$, which generalizes both the space $\hat{B}^{1/3}_{3,c(N)}$ from [IO14] and the space $\hat{B}^{1/3}_{3,\text{VMO}}$ from [Bar+19b; NNT20] — the best known function space where Onsager’s conjecture holds on flat backgrounds.

Acknowledgments

The author is grateful to Terence Tao, Pierre Albin, Daniel Grieser, András Vasy and Rafe Mazzeo for valuable discussions during the preparation of this work.

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1 Introduction

Recall the incompressible Euler equation in fluid dynamics:

\[
\begin{align*}
\frac{\partial}{\partial t} V + \text{div} (V \otimes V) &= -\text{grad} p \quad \text{in } M \\
\text{div} V &= 0 \quad \text{in } M \\
\langle V, \nu \rangle &= 0 \quad \text{on } \partial M
\end{align*}
\]

(1)

where \((M, g)\) is an oriented, compact smooth Riemannian manifold with boundary, dimension \(\geq 2\)
\(\nu\) is the outwards unit normal vector field on \(\partial M\)
\(I \subset \mathbb{R}\) is an open interval, \(V : I \to X_M\), \(p : I \times M \to \mathbb{R}\).

We observe that the Neumann condition \(\langle V, \nu \rangle = 0\) means \(V \in X_N\), where \(X_N\) is the set of vector fields which are tangent to the boundary.

The last two conditions can also be rewritten as \(V = P V\), where \(P\) is the Leray projection operator.

Roughly speaking, Onsager’s conjecture says that the energy \(\|V(t, \cdot)\|_{L^2(M)}\) is a.e. constant in time when \(V\) is a weak solution whose regularity is at least \(\frac{1}{3}\). Making that statement precise is part of the challenge.

Since then more attention has been directed towards the case with boundary on flat backgrounds [BT18; DN18; Bar+19a; BTW19; NN19; Bar+19b]. The case of manifolds without boundary was first handled via a heat-flow approach in [IO14]. This inspired the consideration of manifolds with boundary in [Huy19], with the weak solution lying in \(L^3_t B^\frac{3}{2}_{3,1}\), the largest space in which the trace theorem applies. However, the best results on flat backgrounds hold in the slightly bigger space \(L^3_t B^\frac{3}{2}_{3,1} VMO\), so this sequel aims to make that improvement.

In essence, the absolute Neumann heat flow, created via functional analysis, is a replacement for the usual convolution on flat spaces, with special properties like commutativity with divergence. However, obtaining a pointwise profile of heat kernels for differential forms (let alone their derivatives) is a difficult problem, so it was hard to reconcile the heat-flow approach with local-type convolution arguments on flat backgrounds. Even the definition of \(B^\frac{3}{2}_{3,1} VMO\) itself is local, and it was not immediately obvious that the heat-flow approach could handle such function spaces.

The solution to this is a manual construction of the Hodge-Neumann heat kernel (Section B), using techniques from microlocal analysis and index theory (in particular, Richard Melrose’s calculus on manifolds with corners [Mel18; Mel92]). The theory mimics the development of pseudodifferential operators, in creating a filtered algebra that quantifies how “nonsingular” an operator is as we approach the edges. In particular, much like the pseudolocality of \(\Psi\)DOs, the construction yields a precise description near the diagonal, as well as rapid decay away from the diagonal. This enables the use of the heat flow as local convolution, and we obtain local Bernstein estimates which allow us to handle VMO-type function spaces.

The addition of local Besov-type estimates also marks another stage of development for the theory of intrinsic harmonic analysis for differential forms (including scalar functions and vector fields) on compact Riemannian manifolds with boundary, originally set forth in the prequel with Hodge theory as the foundation. In particular, we have extended the notion of tempered distributions, and the methods of Littlewood-Paley frequency decomposition (e.g. Bernstein-type estimates), which have proved useful on flat backgrounds for problems in fluid dynamics and dispersive PDEs (cf. [Tao09; Tao13; Tao06; Lem02]), to manifolds with boundary. More history and references can be found in [Huy19].
1.1 Main result

To state the main result, we need some terminology.

The standard Besov spaces $B^s_{p,q}$ and the absolute Neumann heat semigroup $e^{s\Delta_N}$ were discussed in [Huy19].

For $r > 0$, we define $M_{r^+} := \{x \in M : \text{dist}(x, \partial M) > r\}$. Let $\hat{M}$ denote the interior of $M$. For $p \in (1, \infty)$, we say $X \in \widehat{B}^{1/p}_{p,V}\mathcal{X}(M)$ if $X \in L^p\mathcal{X}(M)$ and $\forall r > 0$:

$$
\left(\frac{1}{\sqrt{s}}\right)^\frac{1}{p} \left\| \frac{1}{s} X - e^{s\Delta_N} X \right\|_{L^p(M_{r^+})} \xrightarrow{s \to 0} 0
$$

Or equivalently (by Corollary 30), $(\sqrt{s})^{1 - \frac{1}{p}} \left\| e^{s\Delta_N} X \right\|_{W^{1,p}(M_{r^+})} \xrightarrow{s \to 0} 0$

Similarly, for $p \in (1, \infty)$, we say $\mathcal{X} \in L^p_{t} \widehat{B}^{1/p}_{p,V}\mathcal{X}(M)$ if $\mathcal{X} \in L^p_{t}L^p\mathcal{X}(M)$ and $\forall r > 0$:

$$
\left(\frac{1}{\sqrt{s}}\right)^\frac{1}{p} \left\| \mathcal{X} - e^{s\Delta_N} \mathcal{X} \right\|_{L^p_{t}L^p(M_{r^+})} \xrightarrow{s \to 0} 0
$$

As shown in Lemma 18, $\widehat{B}^{1/3}_{3,V}$ contains the space $\widehat{B}^{1/3}_{3,c(N)}$ from [IO14] (with equality when $\partial M = \emptyset$). While on flat backgrounds, by Theorem 38, $\widehat{B}^{1/3}_{3,V}$ coincides with $\widehat{B}^{1/3}_{3,VMO}$ from [Bar+19b; NNT20; Wie20].

Let $\mathfrak{X}_{00}$ be the space of smooth vector fields compactly supported in the interior of $M$. We say $(V, p)$ is a weak solution to the Euler equation when

- $V \in L^3(I, P L^3\mathfrak{X})$, $p \in L^2_{t}(I, H^{-\beta}(M))$ for any $\beta \in \mathbb{N}_0$
- $\forall \mathcal{X} \in C^\infty_c(I, \mathfrak{X}_{00}) : \int_{I \times M} (V, \partial_t \mathcal{X}) + \langle V \otimes V, \nabla \mathcal{X} \rangle + p \text{ div } \mathcal{X} = 0$.

The last condition means $\partial_t V + \text{ div } (V \otimes V) + \text{ grad } p = 0$ as spacetime distributions.

Remark 1 (Local elliptic regularity). As $V \in L^3_t L^3\mathfrak{X}$, we have $\Delta p = -\text{ div } \left( \text{ div } (V \otimes V) \right)$ in $L^2_t H^{-2,\frac{3}{2}}(M)$. By embedding, there is $\beta \in \mathbb{N}_1$ such that $p \in L^2_{t} \left( I, H^{-\beta,\frac{3}{2}}(M) \right)$. Let $K \subset W \subset M$ where $K$ and $W$ are precompact open sets. Then by interior elliptic regularity (see [Tay11a, Subsection 5.11, Theorem 11.1] and [Tay11b, Subsection 13.6]), we have for a.e. $t \in I$:

$$
\left\| p(t) \right\|_{L^2_{t}(K)} \lesssim_{K,W} \left\| \Delta p(t) \right\|_{H^{-2,\frac{3}{2}}(W)} + \left\| p(t) \right\|_{H^{-\beta,\frac{3}{2}}(W)}
$$

Then we can conclude $p \in L^2_{t} \widehat{B}^{1/2}(K)$, for any $K \subset \hat{M}$ precompact.

As can be seen in [NN19; Bar+19b; NNT20], the correct replacement for the trace theorem is the following “strip decay” hypothesis near the boundary:

$$
\left\| \left( \frac{|V|^2}{2} + p \right) \langle V, \bar{\nu} \rangle \right\|_{L^1_{t}L^1(M_{[r/2, r]}, \text{ avg})} \xrightarrow{r \to 0} 0
$$

where

$$
\bar{\nu} \text{ is the extension of } \nu \text{ near the boundary.}
$$

$M_{[r/2, r]} = \{x \in M : \text{dist}(x, \partial M) \in [r/2, r]\}$.

avg means the measure is normalized to become a probability measure.

Theorem 2. Let $M$ be as in (1). Then $\|V(t, \cdot)\|_{L^2(M)}$ is a.e. constant in time if $(V, p)$ is a weak solution with $V \in L^3_t P L^3\mathfrak{X} \cap L^3_t \widehat{B}^{1}_{3,V}\mathcal{X}$ and (2) being true.
1.2 Outline of the paper

In Section 3, we summarize the key tools from [Huy19], discuss some connections between the heat flow and Besov spaces, and then prove Onsager’s conjecture. However, at certain points we will need some local-type estimates involving the heat flow, which are themselves derived from the construction of the heat kernel. To avoid interrupting the flow of the paper, the local estimates are proved in Section A, while the construction of the kernel, arguably the most technical step of the paper, can be found in Section B.

2 Common notation

Some common notation we use:

- $A \lesssim_{x, \sim y} B$ means $A \leq C B$ where $C > 0$ depends on $x$ and not $y$. Similarly, $A \sim_{x, \sim y} B$ means $A \lesssim_{x, \sim y} B$ and $B \lesssim_{x, \sim y} A$. When the dependencies are obvious by context, we do not need to make them explicit.

- $\mathbb{N}_0, \mathbb{N}_1$: the set of natural numbers, starting with 0 and 1 respectively.

- DCT: dominated convergence theorem, FTC: fundamental theorem of calculus, WLOG: without loss of generality.

- TVS: topological vector space. For TVS $X, Y \leq X$ means $Y$ is a subspace of $X$.

- $L(X, Y)$: the space of continuous linear maps from TVS $X$ to $Y$. Also $L(X) = L(X, X)$.

- $C^0(S \to Y)$: the space of bounded, continuous functions from metric space $S$ to normed vector space $Y$. Not to be confused with $C^0 \text{loc}(S \to Y)$, which is the space of locally bounded, continuous functions.

- $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$ and $\|x\|_{D(A)}^* = \|Ax\|_X$ where $A$ is an unbounded operator on (real/complex) Banach space $X$ and $x \in D(A)$. Note that $\|\cdot\|_{D(A)}^*$ is not always a norm. We also define $D(A^\infty) = \bigcap_{k \in \mathbb{N}_1} D(A^k)$.

- $B(x, r) = B_r(x)$: the open ball of radius $r$ centered at $x$ in a metric space.

3 Onsager’s conjecture

3.1 Summary of preliminaries

We will quickly summarize the key tools that we need for the proof (see [Huy19, Subsection 3.1] for the precise locations where they are proved).

Definition 3. For the rest of the paper, unless otherwise stated, let $M$ be a compact, smooth, Riemannian $n$-dimensional manifold, with no or smooth boundary. We also let $I \subset \mathbb{R}$ be an open time interval. We write $M_{<r} = \{x \in M : \text{dist}(x, \partial M) < r\}$ for $r > 0$ small. Similarly define $M_{\geq r}, M_{<r}, M_{[r_1, r_2]}$ etc. Let $\bar{M}$ denote the interior of $M$.

By the musical isomorphism, we can consider $\mathfrak{X}M$ (the space of smooth vector fields) mostly the same as $\Omega^1(M)$ (the space of smooth 1-forms), mutatis mutandis. We note that $\mathfrak{X}M, \mathfrak{X}(\partial M)$ and $\mathfrak{X}M|_{\partial M}$ are different. Unless otherwise stated, let the implicit domain be $M$, so $\mathfrak{X}$ stands for $\mathfrak{X}M$, and similarly $\Omega^k$ for $\Omega^k M$. For $X \in \mathfrak{X}$, we write $X^\flat$ as its dual 1-form.

Let $\mathfrak{X}_{00}(M)$ denote the set of smooth vector fields of compact support in $\bar{M}$. We define $\Omega^k_{00}(M)$ similarly (smooth differential forms with compact support in $\bar{M}$).
Let $\nu$ denote the outwards unit normal vector field on $\partial M$. $\nu$ can be extended via geodesics to a smooth vector field $\tilde{\nu}$ which is of unit length near the boundary (and cut off at some point away from the boundary).

For $X \in \mathfrak{X}M$, define $nX = \langle X, \nu \rangle \nu \in \mathfrak{X}M|_{\partial M}$ (the normal part) and $tX = X|_{\partial M} - nX$ (the tangential part). We note that $tX$ and $nX$ only depend on $X|_{\partial M}$, so $t$ and $n$ can be defined on $\mathfrak{X}M|_{\partial M}$, and $t (\mathfrak{X}M|_{\partial M}) \hookrightarrow \mathfrak{X}(\partial M)$.

For $\omega \in \Omega^k(M)$, define $t\omega$ and $n\omega$ by

$$t\omega(X_1, ..., X_k) := \omega(tX_1, ..., tX_k) \quad \forall X_j \in \mathfrak{X}M, j = 1, ..., k$$

and $n\omega = \omega|_{\partial M} - t\omega$. Note that $(nX)^b = nX^b \forall X \in \mathfrak{X}$.

Let $\nabla$ denote the Levi-Civita connection, $d$ the exterior derivative, $\delta$ the codifferential, and $\Delta = -(d\delta + \delta d)$ the Hodge-Laplacian, which is defined on vector fields by the musical isomorphism.

Familiar scalar function spaces such as $L^p, W^{m,p}$ (Lebesgue-Sobolev spaces), $B^s_{p,q}$ (Besov spaces), $C^{0,\alpha}$ (Hölder spaces) can be defined on $M$ by partitions of unity and given a unique topology. Similarly, we define such function spaces for tensor fields and differential forms on $M$ by partitions of unity and local coordinates. For instance, we can define $L^2\mathfrak{X}$ or $B^{3}_{3,1}\mathfrak{X}$.

**Fact 4.** $\forall \alpha \in (\frac{1}{3}, 1), \forall p \in (1, \infty) : W^{1,p}\mathfrak{X} \hookrightarrow B^{1}_{p,1}\mathfrak{X} \hookrightarrow \mathfrak{X}$ and $C^{0,\alpha}\mathfrak{X} = B^{0,\alpha,\infty}\mathfrak{X} \hookrightarrow B^{0}_{3,\infty}\mathfrak{X} \hookrightarrow B^{\frac{1}{4}}_{3,3}\mathfrak{X} \hookrightarrow B^{1}_{3,\infty}\mathfrak{X}$

**Definition 5.** We write $\langle \cdot, \cdot \rangle$ to denote the Riemannian fiber metric for tensor fields on $M$. We also define the dot product

$$\langle \langle \sigma, \theta \rangle \rangle = \int_M \langle \sigma, \theta \rangle \text{vol}$$

where $\sigma$ and $\theta$ are tensor fields of the same type, while vol is the Riemannian volume form. When there is no possible confusion, we will omit writing vol.

Define $\Omega(M) = \bigodot_{k=0}^{\infty} \Omega^k(M)$ as the graded algebra of differential forms where multiplication is the wedge product. We then naturally define $W^{m,p}\Omega(M) = \bigoplus_{k=0}^{\infty} W^{m,p}\Omega^k(M)$, and similarly for $B^{s}_{p,q}, F^{s}_{p,q}$ spaces. Spaces like $\Omega^{N}_{N}(M)$, $\Omega^{00}_{00}(M)$ are also defined by direct sums.

We define $\mathfrak{X}_N = \{ X \in \mathfrak{X} : nX = 0 \}$ (Neumann condition). In order to define the Neumann condition for less regular vector fields, we use the trace theorem. We can similarly define $\Omega^{k}_{N}$.

**Fact 6** (Trace theorem). Let $p \in [1, \infty)$. Then

- $B^{\frac{1}{2}}_{p,1}(M) \hookrightarrow L^p(\partial M)$ and $B^{\frac{1}{2}}_{p,1}\mathfrak{X}M \hookrightarrow L^p\mathfrak{X}M|_{\partial M}$ are continuous surjections.

- $\forall m \in \mathbb{N}_1 : B^{m,\frac{1}{2}}_{p,1}\mathfrak{X}M \hookrightarrow B^{m,\frac{1}{2}}_{p,1}\mathfrak{X}M|_{\partial M} \hookrightarrow W^{m,p}\mathfrak{X}M|_{\partial M}$ is continuous.

**Definition 7.** We define $\mathcal{P}$ as the Leray projection, which projects $\mathfrak{X}$ onto $\ker \left( \text{div}|_{\mathfrak{X}_N} \right)$. Note that the Neumann condition is enforced by $\mathcal{P}$.

**Fact 8.** $\forall m \in \mathbb{N}_0, \forall p \in (1, \infty)$, $\mathcal{P}$ is continuous on $W^{m,p}\mathfrak{X}$ and $\mathcal{P}(W^{m,p}\mathfrak{X}) = W^{m,p}\text{-cl}(\mathcal{P}\mathfrak{X})$ (closure in the $W^{m,p}$-topology).

We collect some results regarding our heat flow in one place:

**Fact 9** (Absolute Neumann heat flow). There exists a semigroup of operators $(S(t))_{t \geq 0}$ acting on $\cup_{p \in (1, \infty)} L^p\mathfrak{X}$ such that

1. $S(t_1)S(t_2) = S(t_1 + t_2) \forall t_1, t_2 \geq 0$ and $S(0) = 1$. 


2. \( \forall p \in (1, \infty) , \forall X \in L^p \mathcal{X} : \)
   
   \( (a) \) \( S(t)X \in \mathcal{X}_N \) and \( \partial_t (S(t)X) = \Delta S(t)X \ \forall t > 0. \)
   
   \( (b) \) \( S(t)X \xrightarrow{C^m} S(t_0)X \ \forall t_0 > 0. \)
   
   \( (c) \) \( \|S(t)X\|_{W^{m,p}} \lesssim_{m,p} \left( \frac{1}{t} \right)^{\frac{m}{p}} \|X\|_{L^p} \ \forall m \in \mathbb{N}_0, \forall t \in (0, 1). \)
   
   \( (d) \) \( S(t)X \xrightarrow{L^p} X. \)

3. \( \forall p \in (1, \infty) , \forall X \in W^{1,p} \mathcal{X}_N : \)
   
   \( (a) \) \( \|S(t)X\|_{W^{m+1,p}} \lesssim_{m,p} \left( \frac{1}{t} \right)^{\frac{m}{p}} \|X\|_{W^{1,p}} \ \forall m \in \mathbb{N}_0, \forall t \in (0, 1). \)
   
   \( (b) \) \( S(t)X \xrightarrow{W^{1,p}} X. \)

4. \( S(t)P = PS(t) \) on \( W^{m,p} \mathcal{X} \ \forall m \in \mathbb{N}_0, \forall p \in (1, \infty) , \forall t \geq 0. \)

5. \( \langle \langle S(t)X, Y \rangle \rangle = \langle \langle X, S(t)Y \rangle \rangle \ \forall t \geq 0, \forall p \in (1, \infty) , \forall X \in L^p \mathcal{X}, \forall Y \in L^p \mathcal{X}. \)

These estimates precisely fit the analogy \( e^{i\Delta} \approx P \overline{\mathbf{1}} \) where \( P \) is the Littlewood-Paley projection. Analogous results hold for scalar functions and differential forms.

We observe some basic identities from differential geometry:

- Using Penrose abstract index notation, for any smooth tensors \( T^{a_1...a_k} \), we define \( (\nabla T)^{ia_1...a_k} = \nabla_i T^{a_1...a_k} \) and \( \text{div} T = \nabla^i T_{ia_2...a_k} \).

- For all smooth tensors \( T^{a_1...a_k} \) and \( Q^{a_1...a_{k+1}} \):
  
  \[ \int_M \nabla_i (T^{a_1...a_k} Q^{ia_1...a_k}) = \int_M \nabla_i T^{a_1...a_k} Q^{ia_1...a_k} + \int_M T^{a_1...a_k} \nabla_i Q^{ia_1...a_k} = \int_{\partial M} \nu_i T^{a_1...a_k} Q^{ia_1...a_k} \]

- \( (\nabla_a \nabla_b - \nabla_b \nabla_a) T^{ij} = -R_{ab}T^{ij} + R_{ab}^{ij} T^{kl} + R_{ab}^{ik} T^{jl} + R_{ab}^{il} T^{jk} \) for any tensor \( T^{ij} \), where \( R \) is the Riemann curvature tensor. Similar identities hold for other types of tensors.

When we do not care about the exact indices and how they contract, we can just write the schematic identity \( (\nabla_a \nabla_b - \nabla_b \nabla_a) T^{ij} = R \ast T \). As \( R \) is bounded on compact \( M \), interchanging derivatives is a zeroth-order operation on \( M \). In particular, we have the Weitzenbock formula:

\[ \Delta X = \nabla_i \nabla^i X + R \ast X \ \forall X \in \mathcal{X} M \]  \hspace{1cm} (3)

There is an elementary lemma which is useful for convergence (the proof is straightforward and omitted):

**Lemma 10** (Dense convergence). Let \( X, Y \) be (real/complex) Banach spaces and \( X_0 \leq X \) be norm-dense. Let \( (T_j)_{j \in \mathbb{N}} \) be bounded in \( \mathcal{L}(X, Y) \) and \( T \in \mathcal{L}(X, Y) \).

- If \( T_j x_0 \to T x_0 \ \forall x_0 \in X_0 \) then \( T_j x \to T x \ \forall x \in X \).

As the heat flow does not preserve compact supports in \( \tilde{M} \), it is not defined on distributions. This inspires the formulation of heatable currents.

**Definition 11** (Heatable currents). Define:

- \( \mathcal{Q}_{\tilde{M}} = \mathcal{Q}_{\tilde{M}0} = \text{colim} \{ (\mathcal{K}, C^\infty \text{ topo}) : K \subset \tilde{M} \text{ compact} \} \) as the space of test \( k \)-forms with Schwartz’s topology\(^1\) (colimit in the category of locally convex TVS).

\(^1\)Confusingly enough, “Schwartz’s topology” refers to the topology on the space of distributions, not the topology for Schwartz functions.
• $\mathcal{D}^k = (\mathcal{D}^k)^*$ as the space of $k$-currents (or distributional $k$-forms), equipped with the weak* topology.

• $\mathcal{D}_N \Omega = \{ \omega \in \Omega^k : n\Delta^n \omega = 0, nd^n \omega = 0 \ \forall m \in \mathbb{N}_0 \}$ as the space of heated $k$-forms with the Frechet $C^\infty$ topology and $\mathcal{D}'_N \Omega = (\mathcal{D}_N \Omega)^*$ as the space of heatable $k$-currents (or heatable distributional $k$-forms) with the weak* topology.

In particular, $\mathcal{D}_N \mathcal{X}$ is defined from $\mathcal{D}_N \Omega^1$ by the musical isomorphism, and it is invariant under our heat flow (much like how the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is invariant under the Littlewood-Paley projection). By that analogy, heatable currents are tempered distributions on manifolds, and we can write

$$\langle \langle (S(t) \Lambda, X) \rangle \rangle = \langle \langle \Lambda, S(t) X \rangle \rangle \ \forall \Lambda \in \mathcal{D}'_N \mathcal{X}, \forall X \in \mathcal{D}_N \mathcal{X}, \forall t \geq 0$$

where the dot product $\langle \langle \cdot, \cdot \rangle \rangle$ is simply abuse of notation.

**Fact 12. Some basic properties of $\mathcal{D}_N \Omega (M)$ and $\mathcal{D}'_N \Omega (M)$:**

• $(\langle \langle \Delta X, Y \rangle \rangle) = (\langle \langle X, \Delta Y \rangle \rangle) \ \forall X, Y \in \mathcal{D}_N \mathcal{X}$.

• $\mathcal{D} \Omega \hookrightarrow \mathcal{D}_N \Omega$ and $L^p \Omega \hookrightarrow \mathcal{D}_N \Omega \ \forall p \in (1, \infty)$.

• $S(t) \Lambda \in \mathcal{D}_N \Omega \ \forall t > 0, \forall \Lambda \in \mathcal{D}'_N \Omega$. (a heatable current becomes heated once the heat flow is applied)

• $W^{1,p} - \text{cl}(\mathcal{D}_N \Omega) = W^{1,p} \Omega_N$ and $B_{p,1}^{\frac{1}{p}} - \text{cl}(\mathcal{D}_N \Omega) = B_{p,1}^{\frac{1}{p}} \Omega_N \ \forall p \in (1, \infty)$

• $\forall X \in \mathcal{D}_N \Omega : S(t)X \overset{\mathcal{C}^\infty}{\rightarrow} X$ and $\partial_t (S(t)X) = \Delta S(t)X = S(t)\Delta X \ \forall t \geq 0$.

• $\forall t \in (0, 1), \forall m, m' \in \mathbb{N}_0, \forall p \in (1, \infty), \forall X \in \mathcal{D}_N \Omega :

1. $\|S(t)X\|_{W^{m+m',p}} \lesssim \left( \frac{1}{t} \right)^{\frac{m'}{p}} \|X\|_{W^{m,p}}$

2. $\|S(t)X\|_{W^{m,p}} \lesssim \left( \frac{1}{t} \right) \left( \frac{1}{2} \right)^{m-1} \|X\|_{B_{p,1}^{\frac{1}{p}}} \ \text{when} \ m \geq 1$

3. $\|S(t)X\|_{B_{p,1}^{\frac{1}{p}}} \lesssim \left( \frac{1}{t} \right)^{\frac{1}{2} + \frac{m}{p}} \|X\|_{W^{m,p}}$

### 3.2 Heating the nonlinear term

Recall integration by parts:

$$\langle \langle \text{div} (Y \otimes Z) , X \rangle \rangle = - \langle \langle Y \otimes Z, \nabla X \rangle \rangle + \int_{\partial M} \langle \nu, Y \rangle \langle Z, X \rangle \ \forall X, Y, Z \in \mathcal{X}(M)$$

Let $U, V \in B_{3,1}^4 \mathcal{X}$. Then $U \otimes V \in L^1 \mathcal{X}$ and $\text{div} (U \otimes V)$ is defined as a distribution. So we will define the heatable 1-current $\langle \langle \text{div} (U \otimes V) \rangle \rangle^t$ by

$$\langle \langle \text{div} (U \otimes V) , X \rangle \rangle := - \langle \langle U \otimes V, \nabla X \rangle \rangle + \int_{\partial M} \langle \nu, U \rangle \langle V, X \rangle \ \forall X \in \mathcal{D}_N \mathcal{X} \ \text{(X is heated)}$$

It is continuous on $\mathcal{D}_N \mathcal{X}$ since $|\langle \langle \text{div} (U \otimes V) , X \rangle \rangle| \lesssim \|U\|_{B_{3,1}^{\frac{1}{2}}} \|V\|_{B_{3,1}^{\frac{1}{2}}} \|X\|_{B_{3,1}^{\frac{1}{2}}} + \|U\|_{L^3} \|V\|_{L^3} \\|\nabla X\|_{L^3}$. By the same formula and reasoning, we see that $\langle \langle \text{div} (U \otimes V) \rangle \rangle^t$ is not just heatable, but also a continuous linear functional on $\mathcal{X}(M), C^\infty$ topo.
On the other hand, we can get away with less regularity by assuming \( U \in \mathbb{P}L^2\mathcal{X} \) and \( V \in L^2\mathcal{X} \). Then \((\text{div}(U \otimes V))'\) is heatable as we simply need to define

\[
\langle\langle \text{div}(U \otimes V), X \rangle\rangle = -\langle\langle U \otimes V, \nabla X \rangle\rangle \quad \forall X \in \mathcal{X}
\]

(4)

### 3.3 Besov spaces

For the rest of the proof, we will write \( e^{t\Delta} \) for the absolute Neumann heat flow, as we will not need another heat flow. For \( \varepsilon > 0 \) and vector field \( X \), we will write \( X^\varepsilon \) for \( e^{\varepsilon\Delta}X \).

Now we define a crude version of the Littlewood-Paley projections: \( P_{\leq t} = e^{\frac{t}{2}\Delta} \) for \( t > 0 \) and \( P_N = P_{\leq N} - P_{\leq \frac{N}{2}} \) for \( N > 1, N \in 2\mathbb{Z} \).

The definition of \( P_{\leq t} \) gives a quick Bernstein estimate:

**Theorem 13.** For \( N \geq 1 \) and \( X \in \mathcal{D}_N^N \Omega \), \( \|P_N X\|_p \lesssim \frac{1}{N^2} \left\|P_{\leq \sqrt{2N}} X\right\|_{W^{2,p}} \lesssim \frac{1}{N} \left\|P_{\leq 2N} X\right\|_{W^{1,p}} \)

**Proof.** Recall that \( e^{t\Delta}X \in \mathcal{D}_N^N \Omega \forall \varepsilon > 0 \). Then observe that

\[
P_N X = \left( \exp \left( \frac{\Delta}{2N^2} \right) - \exp \left( \frac{7\Delta}{2N^2} \right) \right) \exp \left( \frac{\Delta}{2N^2} \right) X = \int \frac{\Delta}{2N^2} \Delta e^{t\Delta} \exp \left( \frac{\Delta}{2N^2} \right) X dt
\]

and \( P_{\leq \sqrt{2N}} = P_{\leq 2N} P_{\leq 2N} \).

**Definition 14.** For \( \alpha \in (0,1) \), \( p \in (1, \infty) \), \( q \in [1, \infty] \), we define the Besov heat space \( \hat{B}^\alpha_{p,q} \Omega \) as the space of heatable \( k \)-currents \( X \) where the norm

\[
\|X\|_{\hat{B}^\alpha_{p,q}} = \|X\|_{L^p} + \left\| s^{\frac{\alpha}{2}(1-\alpha)} \|e^{\varepsilon\Delta}X\|_{W^{1,p}} \right\|_{L^q>(0,1)}
\begin{align}
&\sim \|X\|_{L^p} + \|N^{\alpha-1} \|P_{\leq N} X\|_{W^{1,p}} \|l^p_{(N \in 2\mathbb{Z}, N > 1)}
\end{align}

is finite.

Recall the theory of real interpolation. The following fact justifies the name “Besov” in Besov heat space:

**Theorem 15.** \([L^p\Omega, W^{1,p}\Omega]_{\theta,q} = \hat{B}^\alpha_{p,q} \Omega \) for \( q \in [1, \infty] \), \( p \in (1, \infty) \), \( \theta \in (0,1) \).

**Proof.** By definition, \( \hat{B}^\alpha_{p,q} \Omega \rightarrow \mathcal{L}^p\Omega \). We first show \( \hat{B}^\alpha_{p,q} \Omega \rightarrow [L^p\Omega, W^{1,p}\Omega]_{\theta,q} \).

Assume \( \|X\|_{\hat{B}^\alpha_{p,q}} \leq 1 \). Then we decompose \( X \equiv P_{\leq 1} X + \sum_{N \geq 1, N \in 2\mathbb{Z}} P_N X \). Set \( X_0 = P_{\leq 1} X \) and \( X_k = P_{2^{-k}X} \forall k \in \mathbb{Z}, k \leq -1 \). Then by the J-method, and the fact that \( X = \sum_{k \leq 0} X_k \), we have

\[
\|X\|_{[L^p\Omega, W^{1,p}\Omega]_{\theta,q}} \lesssim \left\| 2^{-k\theta} \|X_k\|_{L^p} + 2^{k(1-\theta)} \|X_k\|_{W^{1,p}} \right\|_{l^p_{(k \leq 0)}}
\begin{align}
&\lesssim \|X\|_{L^p} + \left( \frac{1}{2} \right)^{m\theta} \|P_{2^m X}\|_{L^p} + \left( \frac{1}{2} \right)^{m(1-\theta)} \|P_{2^m X}\|_{W^{1,p}}
&\lesssim \|X\|_{L^p} + \left( \frac{1}{2} \right)^{m(1-\theta)} \left\|P_{\leq 2^{m+1} X}\right\|_{W^{1,p}} \leq 1
\end{align}
\]


Now we will show $[L^p \Omega^k, W^{1,p} \Omega^k_N]_{\theta,q} \hookrightarrow \overline{B_{p,q}^\theta \Omega^k}$. Assume $\|Y\|_{[L^p \Omega^k, W^{1,p} \Omega^k_N]_{\theta,q}} \leq 1$, then $\|Y\|_{L^p} \lesssim 1$.

We will use the K-method: for any $N \geq 1$, $Y_0 \in L^p \Omega^k$, $Y_1 \in W^{1,p} \Omega^k_N$ such that $Y = Y_0 + Y_1$, we have

$$\|P_{\leq N} Y\|_{W^{1,p}} \leq \|P_{\leq N} Y_0\|_{W^{1,p}} + \|P_{\leq N} Y_1\|_{W^{1,p}} \lesssim N \|Y_0\|_{L^p} + \|Y_1\|_{W^{1,p}}$$

Note that this is why we need $W^{1,p} \Omega^k_N$ instead of $W^{1,p} \Omega^k$. Then

$$N^{\theta - 1} \|P_{\leq N} Y\|_{W^{1,p}} \lesssim \inf_{Y_0 + Y_1 = Y} N^{\theta} \|Y_0\|_{L^p} + N^{\theta - 1} \|Y_1\|_{W^{1,p}} = N^{\theta} K \left( \frac{1}{N}, Y \right)$$

so

$$\|N^{\theta - 1} P_{\leq N} Y\|_{W^{1,p}} \|_{\mathcal{M}} \lesssim N^{-\theta} K(N,Y) \|_{\mathcal{M}} \leq 1$$

Remark 16. We recover the standard Besov space when the manifold is boundaryless, effectively generalizing the proof in [IO14, Appendix B]. More importantly, in the case with boundary, we have

$$\mathbb{P} \mathcal{B}_{3,1/3}^{1/3} \Omega^k = [\mathbb{P} L^3 \Omega^k, \mathbb{P} W^{1,3} \Omega^k]_{1/3, q} = [\mathbb{P} L^3 \Omega^k, \mathbb{P} W^{1,3} \Omega^k]_{1/3, q} = \mathbb{P} \mathcal{B}_{3,1/3}^{1/3} \Omega^k$$

for $q \in [1, \infty]$. The fact that we need to apply the Leray projection is an important technicality.

Definition 17. For $p \in (1, \infty)$, we say $X \in \mathcal{B}_{p,V} \mathcal{X}(M)$ if $X \in L^p \mathcal{X}(M)$ and $\forall r > 0$:

$$N^{\frac{1}{p} - 1} \|P_{\leq N} X\|_{W^{1,p}(M_{\leq r})} \xrightarrow{N \to \infty} 0$$

Similarly, we say $\mathcal{X} \in L^p_p \mathcal{B}_{p,V} \mathcal{X}(M)$ if $\mathcal{X} \in L^p \mathcal{X}(M)$ and $\forall r > 0$:

$$N^{\frac{1}{p} - 1} \|P_{\leq N} \mathcal{X}\|_{L^p \mathcal{X}(M_{\leq r})} \xrightarrow{N \to \infty} 0$$

Remark. The vanishing property in (6) becomes important for the commutator estimate in Onsager’s conjecture at the critical regularity $\frac{1}{3}$, while higher regularity has enough room for vanishing in norm (which is better).

It is shown in Corollary 30 that (5) is equivalent to

$$N^{\frac{1}{p}} \|P_{\geq N} X\|_{L^p(M_{\leq r})} \xrightarrow{N \to \infty} 0 \forall r > 0$$

We briefly note that when $\partial M = \emptyset$, it is customary to set dist $(x, \partial M) = \infty$, $M_{\leq r} = M = \bar{M}$, $M_{> r} = \emptyset$, and $\mathcal{D}_N \mathcal{X} M = \mathcal{D}_M \mathcal{X} = \mathcal{X} M$.

Recall the space $\mathcal{B}_{3,1/3}^{1/3} \mathcal{X} \mathcal{X} = \overline{\mathcal{B}_{3,1/3}^{1/3} \mathcal{X}}$ from [IO14].

Lemma 18. $\mathcal{B}_{3,1/3}^{1/3} \mathcal{X} \hookrightarrow \mathcal{B}_{3,1/3}^{1/3} \mathcal{X}$. When $\partial M = \emptyset$, $\mathcal{B}_{3,1/3}^{1/3} \mathcal{X} = \mathcal{B}_{3,1/3}^{1/3} \mathcal{X}$.

Proof. Observe that $\mathcal{D}_N \mathcal{X} \hookrightarrow \mathcal{B}_{3,1/3}^{1/3} \mathcal{X}$. For any $r > 0$, $N \geq 1$ and $X \in \mathcal{B}_{3,1/3}^{1/3}$,

$$N^{-2/3} \|P_{\leq N} X\|_{W^{1,3}(M_{\leq r})} \leq N^{-2/3} \|P_{\leq N} X\|_{W^{1,3}(M)} \lesssim \|X\|_{\mathcal{B}_{3,1/3}^{1/3}(M)}$$

Because $\left\{ f \in l^\infty(\mathbb{N}) : f(k) \xrightarrow{k \to \infty} 0 \right\}$ is closed in $l^\infty(\mathbb{N})$, we conclude $\mathcal{B}_{3,1/3}^{1/3} \mathcal{X} \hookrightarrow \mathcal{B}_{3,1/3}^{1/3} \mathcal{X}$.
On the other hand, when $\partial M = \emptyset$, observe that $M_{>r} = M$. Let $X \in \tilde{B}_{3, V}^{1/3} \mathcal{X}$. We aim to show $P_{<K} X \xrightarrow{K \to \infty} X$. For any $N, K \in 2^N_0$:

$$N^{-2/3} \| P_{\leq N} X \|_{W^{1,3}(M)} \lesssim_{K \to \infty} N^{-2/3} \| P_{\leq N} (1 - P_{\leq K}) X \|_{W^{1,3}(M)} \xrightarrow{K \to \infty} 0$$

Let $N_0 \in 2^N_1$. Then observe that

$$\limsup_{K \to \infty} \left\| N^{-2/3} \| P_{\leq N} (1 - P_{\leq K}) X \|_{W^{1,3}(M)} \right\|_{N \geq 1} \leq \limsup_{K \to \infty} \left( \frac{N^{-2/3} \| P_{\leq N} (1 - P_{\leq K}) X \|_{W^{1,3}(M)}}{\| P_{\leq N} X \|_{W^{1,3}(M)}} \right)_{N \in 2^N_0, N < N_0}$$

$$+ \left\| N^{-2/3} \| P_{\leq N} X \|_{W^{1,3}(M)} \right\|_{N \in 2^N_0, N \geq N_0}$$

As $N_0$ is arbitrary, let $N_0 \to \infty$ and we are done. \hfill \Box

Remark 19. On the other hand, Theorem 38 shows that, on flat backgrounds, $\tilde{B}_{3, V}^{1/3}$ coincides with the VMO-type Besov space $B_{3, V}^{1/3, \text{VMO}}$ from [Bar+19b; NNT20].

We will also need to borrow a result from Section A, which allows us to employ cutoffs.

Fact 20 (Pointwise multiplier). If $f \in \mathcal{D}(M)$ and $\mathcal{X} \in L^3_1(B_{3, V}^{1/3}, \mathcal{X})$, then $f \mathcal{X} \in L^3_1 B_{3, V}^{1/3, \mathcal{X}}$.

### 3.4 Proof of Onsager’s conjecture

**Definition 21.** We define the cutoffs

$$\psi_r(x) = \Psi_r(\text{dist}(x, \partial M))$$

where $r > 0$ small, $\Psi_r \in C^\infty((0, \infty), [0, \infty))$ such that $1_{[0, \frac{1}{2} r]} \geq \Psi_r \geq 1_{[0, \frac{1}{2}]}$ and $\| \Psi'_r \|_\infty \lesssim \frac{1}{r}$. Then there is $f_r$ smooth such that $\nabla \psi_r(x) = f_r(x) \vec{v}(x)$ with $|f_r(x)| \lesssim \frac{1}{r}$ and $\text{supp} f_r \subset M_{[\frac{1}{2} r, \frac{3}{2} r]}$.

Let $\chi_r = 1 - \psi_r$. Then $\nabla \chi_r = -f_r \vec{v}$. As usual, there is a **commutator estimate** which we will now assume (leaving the proof to later):

$$\int_I \eta \left\langle \langle \text{div} (U \otimes \chi_r \mathcal{U})^{2\xi} , (\chi_r \mathcal{U})^{2\xi} \rangle \right\rangle - \int_I \eta \left\langle \langle \text{div} (\mathcal{U}^{2\xi} \otimes (\chi_r \mathcal{U})^{2\xi}) , (\chi_r \mathcal{U})^{2\xi} \rangle \right\rangle$$

$$= \int_I \eta \left\langle \langle \text{div} (U \otimes \chi_r \mathcal{U})^{3\xi} , (\chi_r \mathcal{U})^{3\xi} \rangle \right\rangle - \int_I \eta \left\langle \langle \text{div} (\mathcal{U}^{3\xi} \otimes (\chi_r \mathcal{U})^{3\xi}) , (\chi_r \mathcal{U})^{3\xi} \rangle \right\rangle \xrightarrow{\epsilon \downarrow 0} 0$$

for fixed $r > 0$, $U \in L^3_1(B_{3, V}^{1/3}, \mathcal{X} \cap L^3_1 \mathbb{P} L^3 \mathcal{X}, \eta \in C^\infty_c(I)$.

**Remark.** For any $U$ in $\mathbb{P} L^2 \mathcal{X}$ and $V \in L^2 \mathcal{X}$, $\text{div} (U \otimes V)^\xi$ is a heatable 1-current (see Subsection 3.2). In particular, for $\varepsilon > 0$, $\text{div} (U \otimes V)^\xi$ is smooth and

$$\left\langle \langle \text{div} (U \otimes V)^\xi , Y \right\rangle \right\rangle = -\left\langle \langle U \otimes V , \nabla (Y^\varepsilon) \right\rangle \right\rangle \forall Y \in \mathcal{X}$$

Consequently, (8) is well-defined.

Notation: we write $\text{div} (U \otimes V)^\xi$ for $(\text{div} (U \otimes V))^\xi$ and $\nabla \mathcal{U}^\xi$ for $\nabla (\mathcal{U}^\xi)$ (recall that the heat flow does not work on tensors $\mathcal{U} \otimes V$ and $\nabla \mathcal{U}$).
Theorem 22 (Onsager’s conjecture). Let $M$ be a compact, oriented Riemannian manifold with no or smooth boundary. Let $(V,p)$ be a weak solution and $V \in L^1_t \dot{B}^{\frac{2}{3}_x}_\infty \mathbb{X} \cap L^3_t \mathbb{X}^3$.

Assume (8) is true. Also assume strip decay:

$$\left\| \left( \frac{\|V\|^2}{2} + p \right) \langle \dot{V}, \dot{V} \rangle \right\|_{L^1_t L^1(M^{\tau\rightarrow 0_{\mathrm{avg}}})} \rightarrow 0$$

Then we can show

$$\int_I \eta'(t) \langle \langle V(t), V(t) \rangle \rangle \, dt = 0 \quad \forall \eta \in C_c^\infty(I)$$

Consequently, $\langle \langle V(t), V(t) \rangle \rangle$ is constant for a.e. $t \in I$.

Proof. Let $\Phi \in C_c^\infty(\mathbb{R})$ and $\Phi_r \xrightarrow{r \downarrow 0} \delta_0$ be a radially symmetric mollifier. Write $V^\tau$ for $e^{\tau\Delta}V$ (spatial mollification) and $V_r$ for $\Phi_r * V$ (temporal mollification). First, we use the cutoff $\chi_r$ and mollify in time and space

$$\frac{1}{2} \int_I \eta' \langle \langle V, V \rangle \rangle \xrightarrow{\mathrm{DCT}} \lim_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_I \eta' \langle \langle (\chi_r V)^\tau, (\chi_r V)^\tau \rangle \rangle$$

Then for $\varepsilon, \tau$ small, we want to get rid of the time derivative:

$$\frac{1}{2} \int_I \eta' \langle \langle (\chi_r V)^\tau, (\chi_r V)^\tau \rangle \rangle = - \int_I \eta \langle \langle \partial_t (\chi_r V)^\tau, (\chi_r V)^\tau \rangle \rangle = - \int_I \langle \langle \partial_t (\eta (\chi_r V)^\tau), (\chi_r V)^\tau \rangle \rangle + \int_I \eta' \langle \langle (\chi_r V)^\tau, (\chi_r V)^\tau \rangle \rangle$$

We now use the definition of weak solution (WS), and exploit the commutativity between spatial and temporal operators. For the sake of exposition, we will freely cancel the error terms that go to zero upon taking the limits. At the end of the proof, we will show why they can be cancelled.

$$\frac{1}{2} \int_I \eta' \langle \langle (\chi_r V)^\tau, (\chi_r V)^\tau \rangle \rangle = \int_I \langle \langle (\chi_r V)^\tau, (\chi_r V)^\tau \rangle \rangle = \int_I \langle \langle \partial_t (\eta (\chi_r V)^\tau), (\chi_r V)^\tau \rangle \rangle$$

$$\xrightarrow{\text{WS}} - \int_I \langle \langle \nabla \left[ \left( \eta (\chi_r V)^\tau \right)_\tau \right], V \rangle \rangle - \langle \langle \nabla \left[ \left( \eta (\chi_r V)^\tau \right)_\tau \right], \nabla V \rangle \rangle - \langle \langle \nabla \left[ \left( \eta (\chi_r V)^\tau \right)_\tau \right] \nabla V, V \rangle \rangle$$

$$= - \int_I \langle \langle \nabla \left[ \left( \eta (\chi_r V)^\tau \right)_\tau \right] \nabla (\chi_r V), \nabla V \rangle \rangle - \langle \langle \nabla \left[ \left( \eta (\chi_r V)^\tau \right)_\tau \right] \nabla (\chi_r V), V \rangle \rangle$$

As there is no longer a time derivative on $V$, we will get rid of $\tau$ by letting $\tau \downarrow 0$ (fine as $V$ is $L^3$ in time). Also recall Equation (9):
that point fixed. When $t = 0$, we define $N^\prime$ to be the commutator estimate that presents the main difficulty. We proceed similarly as in \cite{IO14}.

Recall from \cite{Huy19} that $\delta_c = \delta |_{\Omega_N}$ and $\delta_c^{\mathcal{D}_N}$ is the extension of $\delta_c$ to heatable currents, defined by $\delta_c^{\mathcal{D}_N} \Lambda (\phi) = \Lambda (d\phi) \ 0 \ \forall \phi \in \mathcal{D}_N \Omega$

Then the fact that $\mathcal{P} \nu^b = \nu^b$ is equivalent to $\delta_c^{\mathcal{D}_N} \nu^b = 0$. This implies:

$$-\text{div} \left( (\chi_r \nu)^{2e} \right) = \delta_c \left( (\chi_r \nu)^{2e} \right) = \left( \delta_c^{\mathcal{D}_N} (\chi_r \nu^b) \right)^{2e} = \left( -\nabla \chi_r \cdot \nu + \chi_r \delta_c^{\mathcal{D}_N} \nu^b \right)^{2e} = (f_r \nu^b)^{2e}$$

With that simplification, and the lack of any time derivatives, \eqref{10} becomes

$$\lim_{r \downarrow 0} \int_{\mathcal{I}} \eta \left( \left\langle (\nabla \cdot \nu)^{2e} \otimes f_r \nu^b, \nu \otimes \nu \right\rangle \right) + \eta \left( \left\langle (\chi_r \nu)^{2e} \cdot f_r \nu^b, p \right\rangle \right) + \eta \left( \left\langle (f_r \nu^b)^{2e} \cdot \chi_r p \right\rangle \right)$$

$$= \lim_{r \downarrow 0} \int_{\mathcal{I}} \eta \left( \left\langle \nu \cdot \nu, \chi_r f_r \nu^b \cdot \nu \right\rangle \right) + 2\eta \left( \left\langle \nu \cdot \nu, f_r \nu^b \cdot \nu \right\rangle \right) = \lim_{r \downarrow 0} \int_{\mathcal{I}} 2\eta \left( \left\langle \frac{|\nu|^2}{2} + p, \chi_r f_r \nu \cdot \nu \right\rangle \right)$$

$$= \lim_{r \downarrow 0} O \left( \int_{\mathcal{I}} |\nu| \int_{M_r \cap \mathcal{I}} \left| \frac{|\nu|^2}{2} + p \right| \right)^{1/2} = 0$$

(remark 23. The proof did not much use the Besov regularity of $\nu$, which is mainly used for the commutator estimate. It is the commutator estimate that presents the main difficulty. We proceed similarly as in \cite{IO14}.

Note that from this point on $r > 0$ is fixed.

Let $\mathcal{U} \in L^2 I \tilde{B}^1_{3,1} \mathcal{X} \cap L^2 I \mathcal{P} L^3 \mathcal{X}$ and $\chi_r$ be as before.

By setting $\mathcal{U}(t)$ to 0 for $t$ in a null set, WLOG we assume $\mathcal{U}(t) \in \mathbb{P}^3 \mathcal{X} \cap \tilde{B}^{1/3}_{1,1} \mathcal{X}$ $\forall t \in I$.

Define the commutator

$$W(t, s) = \text{div} \left( \mathcal{U}(t) \otimes \chi_r \mathcal{U}(t) \right)^{3s} - \text{div} \left( \mathcal{U}^{2s} \otimes (\chi_r \mathcal{U}(t))^{2s} \right)^s$$

When $t$ and $s$ are implicitly understood, we will not write them. As $\text{div} \left( \mathcal{U}(t) \otimes \mathcal{U}(t) \right)^{3s}$ solves $(\partial_s - 3\Delta) \mathcal{X} = 0$, we define $\mathcal{N} = (\partial_s - 3\Delta) W$. Then $W$ and $\mathcal{N}$ obey the Duhamel formula.

\textbf{24 (Duhamel).} For fixed $t_0 \in I$ and $s > 0$:

$$\int_t^s \mathcal{N}(t_0, \sigma)^{3(s-\sigma)} d\sigma \overset{r \downarrow 0}{\rightarrow} W(t_0, s) \text{ in } \mathcal{D}_N^\prime \mathcal{X}.$$
Proof. Let $\varepsilon > 0$. By the smoothing effect of $e^{s\Delta}$, $W(t_0, \cdot)$ and $\mathcal{N}(t_0, \cdot)$ are in $C_{\text{loc}}^1((0,1], \mathcal{D}_N \mathcal{X})$. As $(e^{s\Delta})_{s \geq 0}$ is a $C_0$ semigroup on $(H^m-\text{cl}(\mathcal{D}_N \mathcal{X}), \|\cdot\|_{H^m})$ whenever $m \in \mathbb{N}_0$, and a semigroup basically corresponds to an ODE (cf. [Tay11a, Appendix A, Proposition 9.10 & 9.11]), from $\partial_s W = 3\Delta W + \mathcal{N}$ for $s \geq \varepsilon$ we get the Duhamel formula

$$\forall s > \varepsilon : W(t_0, s) = W(t_0, \varepsilon)^{3(s-\varepsilon)} + \int_{\varepsilon}^{s} \mathcal{N}(t_0, \sigma)^{3(s-\sigma)} \, d\sigma$$

So we only need to show $W(t_0, \varepsilon)^{3(s-\varepsilon)} \overset{\mathcal{G}_x}{\to} 0$. Let $X \in \mathcal{D}_N \mathcal{X}$.

$$\left\langle X, W(t_0, \varepsilon)^{3(s-\varepsilon)} \right\rangle = \left\langle X^{3(s-\varepsilon)}, \text{div} (U(t_0) \otimes \chi_r U(t_0))^{3\varepsilon} - \text{div} (U(t_0)^{2\varepsilon} \otimes (\chi_r U(t_0))^{2\varepsilon}) \right\rangle$$

$$= - \left\langle \nabla (X^{3\varepsilon}), U(t_0) \otimes \chi_r U(t_0) \right\rangle + \left\langle \nabla (X^{3s-2\varepsilon}), U(t_0)^{2\varepsilon} \otimes (\chi_r U(t_0))^{2\varepsilon} \right\rangle \overset{\varepsilon \to 0}{\to} 0.$$

From now on, we write $\int_{0+}^{\varepsilon}$ for $\lim_{\varepsilon \to 0} \int_{0+}^{\varepsilon}$. Then

$$\int_{I} dt \eta(t) \left\langle \left\langle W(t, s), U(t)^s \right\rangle \right\rangle = \int_{I} dt \eta(t) \int_{0+}^{\varepsilon} d\sigma \left\langle \left\langle N(t, \sigma)^{3(s-\sigma)}, U(t)^s \right\rangle \right\rangle$$

Definition 25. Define the $k$-jet fiber norm $|X|_{J^k} = \left( \sum_{j=0}^{k} |\nabla^{(j)} X|^2 \right)^{\frac{1}{2}} \forall X \in \mathcal{X}$.

Let $K(\sigma, x, y)$ be the kernel of the heat flow at time $\sigma > 0$. Then by Section B, we obtain off-diagonal decay for all derivatives:

Fact 26 (Off-diagonal decay). For any multi-index $\gamma$ and $x \neq y$, $D_{\sigma,x,y}^\gamma K(\sigma, x, y) = O(\sigma^\infty)$ as $\sigma \downarrow 0$, locally uniform in \{x $\neq$ y\}.

For convenience, we will write $Y = \chi_r U$. Then for $r > 0, \sigma \in (0,1)$ and $x \in M_{<r/4}$:

$$|Y^{\sigma}(t, x)|_{J^2} \lesssim_{M, r} O_r(\sigma^\infty) \|U(t)\|_{L^3(M_{>r/2})}$$

which implies $\|Y^{\sigma}(t)\|_{W^{2,3}(M_{<r/4})} + \|Y^{\sigma}(t)\|_{W^{2,3}M_{<r/4}} \lesssim_{M, r} O_r(\sigma^\infty) \|U(t)\|_{L^3(M_{>r/2})}$.

We now handle the most important error estimates that will appear in our analysis.

Lemma 27 (2 error estimates). For fixed $r > 0$ small, we have

$$\lim_{s \to 0} \int_I |\eta| \int_{0+}^{\varepsilon} d\sigma \int_{M} |U^{2\sigma}|_{J^1} \left| Y^{2\sigma} \right|_{J^1} \left| Y^{4s-2\sigma} \right|_{J^1} = 0$$

and

$$\lim_{s \to 0} \int_I |\eta| \int_{0+}^{\varepsilon} d\sigma \int_{\partial M} |U^{2\sigma}|_{J^1} \left| Y^{2\sigma} \right|_{J^1} \left| Y^{4s-2\sigma} \right|_{J^2} = 0.$$
Proof. We split (13) into 2 regions: $M_{<r/4}$ and $M_{\geq r/4}$. Observe that

$$\int_I |\eta| \int_{0+}^s d\sigma \int_{M_{<r/4}} |U^{2\sigma}|_{J_1} |Y^{2\sigma}|_{J_1} |Y^{4s-2\sigma}|_{J_1}$$

$$\lesssim \int_I |\eta| \int_{0+}^s d\sigma \|U^{2\sigma}\|_{W^{1,3}(M_{<r/4})} \|Y^{2\sigma}\|_{W^{1,3}(M_{<r/4})} \|Y^{4s-2\sigma}\|_{W^{1,3}(M_{<r/4})}$$

$$\lesssim O_r (s^\infty) \int_I dt |\eta(t)| \|\mathcal{U}(t)\|_{L^2(M)}^3 \int_{0+}^s d\sigma \left( \frac{1}{\sigma} \right) \frac{1}{2} \to 0.$$ 

Define $B(t, s) = s^\frac{3}{2} \| \mathcal{U}(t)^s \|_{W^{1,3}(M_{\geq r/4})}$ and $C(t, s) = s^\frac{3}{2} \| \mathcal{Y}(t)^s \|_{W^{1,3}(M_{\geq r/4})}$. By Fact 20, $Y \in L^\infty_{t,s} B^{1/3}_{3, \infty} \mathcal{X}.$ Therefore, $\|B(t, s)\|_{L^2_t}$ and $\|C(t, s)\|_{L^3_t}$ are continuous in $s$ and converge to 0 as $s \to 0$ by (6).

Observe that

$$\int_I |\eta| \int_{0+}^s d\sigma \int_{M_{\geq r/4}} |U^{2\sigma}|_{J_1} |Y^{2\sigma}|_{J_1} |Y^{4s-2\sigma}|_{J_1}$$

$$\lesssim \int_I |\eta| \int_{0+}^s d\sigma \|U^{2\sigma}\|_{W^{1,3}(M_{\geq r/4})} \|Y^{2\sigma}\|_{W^{1,3}(M_{\geq r/4})} \|Y^{4s-2\sigma}\|_{W^{1,3}(M_{\geq r/4})}$$

$$= \int dt |\eta(t)| \int_{0+}^s d\sigma \left( \frac{1}{\sigma} \right) \frac{3}{2} \left( \frac{1}{2s - \sigma} \right) \frac{1}{2} B(t, 2\sigma) C(t, 2\sigma) C(t, 4s - 2\sigma)$$

$$\lesssim \int_{0+}^s d\sigma \left( \frac{1}{\sigma} \right) \frac{3}{2} \left( \frac{1}{2s - \sigma} \right) \frac{1}{2} \|B(t, 2\sigma)\|_{L^1_t} \|C(t, 2\sigma)\|_{L^2_t} \|C(t, 4s - 2\sigma)\|_{L^3_t}$$

$$\xrightarrow{s \to 0} 0.$$ 

So (13) is proven. For (14), observe that

$$\int_I |\eta| \int_{0+}^s d\sigma \int_{\partial M} |U^{2\sigma}|_{J_1} |Y^{2\sigma}|_{J_1} |Y^{4s-2\sigma}|_{J_2}$$

$$\lesssim O_r (s^\infty) \int_I dt |\eta(t)| \|\mathcal{U}(t)\|_{L^2(M)}^2 \int_{0+}^s d\sigma \|\mathcal{U}(t)^{2\sigma}\|_{W^{1,3}_{t,x,M}}$$

$$\lesssim O_r (s^\infty) \int_I dt |\eta(t)| \|\mathcal{U}(t)\|_{L^2(M)}^2 \int_{0+}^s d\sigma \|\mathcal{U}(t)^{2\sigma}\|_{M^{1/3}_{t,x,M}}$$

$$\lesssim O_r (s^\infty) \int_I dt |\eta(t)| \|\mathcal{U}(t)\|_{L^2(M)}^2 \int_{0+}^s d\sigma \left( \frac{1}{\sigma} \right) \frac{2}{3} \to 0.$$ 

where we used (12) to pass to the second line, and the trace theorem to pass to the third line. \qed

Note that

$$\mathcal{N}(t, \sigma) = (\partial_\sigma - 3\Delta) \left( - \text{div} \left( \mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma} \right) \right) = -2 \text{div} \left( \Delta \mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma} \right) - 2 \text{div} \left( \mathcal{U}^{2\sigma} \otimes \Delta \mathcal{Y}^{2\sigma} \right)$$

$$+ 2\Delta \text{div} \left( \mathcal{U}^{2\sigma} \otimes \mathcal{Y}^{2\sigma} \right).$$
Now, we finally show
\[ \int_I \eta \langle \mathcal{W}(s), Y^s \rangle = \int_I \frac{dt}{t} \eta \langle \mathcal{W}(t, s), Y^s(t) \rangle \xrightarrow{s \downarrow 0} 0 \]

**Proof of the commutator estimate.** First we integrate by parts into three components:

\[
\begin{align*}
\int_I \eta \langle \mathcal{W}(s), Y^s \rangle &= \int_I \frac{dt}{t} \eta \int_{0+}^s d\sigma \left\langle \left\langle \mathcal{N}(t, \sigma) \mathcal{Y}^{3(s-\sigma)} \mathcal{Y}(t)^s \right\rangle \rightangle \\
&= \int_I \frac{dt}{t} \eta \int_{0+}^s d\sigma \left\langle \mathcal{N}(t, \sigma), \mathcal{Y}(t)^{4s-3\sigma} \right\rangle \\
&= 2 \int_I \eta \int_{0+}^s d\sigma \langle \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma}, \nabla (\mathcal{Y}^{4s-2\sigma}) \rangle + 2 \int_I \frac{dt}{t} \eta \int_{0+}^s d\sigma \langle \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma}, \nabla (\mathcal{Y}^{4s-2\sigma}) \rangle \\
&\quad - 2 \int_I \eta \int_{0+}^s d\sigma \langle \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma}, \nabla (\mathcal{Y}^{4s-2\sigma}) \rangle
\end{align*}
\]

Note that for the third component, we used Fact 12 to move the Laplacian.

We now use the Penrose abstract index notation to estimate the three components. To clean up the notation, we only focus on the integral on \( M \), with the other integrals \( 2 \int_I \eta \int_{0+}^s d\sigma (\cdot) \) in variables \( t \) and \( \sigma \) implicitly understood. We also use schematic identities for linear combinations of similar-looking tensor terms where we do not care how the indices contract (recall Equation (3)).

By Lemma 27, it is easy to check that all the terms with \( R \) or \( \nu \) will be negligible (going to 0 in the limit), and interchanging derivatives will be a negligible action. We write \( \approx \) to throw the negligible error terms away.

First component:

\[
\int_M \left\langle \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma}, \nabla (\mathcal{Y}^{4s-2\sigma}) \right\rangle = \int_M R \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma} \mathcal{Y}^{4s-2\sigma} + \int_M \nabla i \mathcal{U}^{2\sigma} (\mathcal{Y}^{2\sigma})^i \nabla j (\mathcal{Y}^{4s-2\sigma})
\]

\[
\approx \int_M \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma} \mathcal{Y}^{4s-2\sigma} - \sum_{\alpha} \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma} \nabla (\mathcal{Y}^{4s-2\sigma})
\]

Second component:

\[
\int_M \left\langle \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma}, \nabla (\mathcal{Y}^{4s-2\sigma}) \right\rangle = \int_M \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma} \mathcal{Y}^{4s-2\sigma} + \int_M (\mathcal{U}^{2\sigma})^i \nabla i (\mathcal{Y}^{2\sigma})^i \nabla j (\mathcal{Y}^{4s-2\sigma})
\]

\[
\approx \int_M \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma} \mathcal{Y}^{4s-2\sigma} - \sum_{\alpha} \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma} \nabla (\mathcal{Y}^{4s-2\sigma})
\]

For the third component, we use the identity \( \nabla (R U) = \nabla R U + R \nabla U \) to compute:

\[
\int_M \left\langle \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma}, \nabla (\mathcal{Y}^{4s-2\sigma}) \right\rangle = -\int_M \mathcal{U}^{2\sigma} \mathcal{Y}^{2\sigma} \nabla (R \mathcal{Y}^{4s-2\sigma}) - \int_M (\mathcal{U}^{2\sigma})^i \nabla i (\mathcal{Y}^{2\sigma})^i \nabla j (\mathcal{Y}^{4s-2\sigma})
\]
Corollary 29. Let $A$ be as in Equation (1). Throughout this section, we write $e^{t\Delta}$ for the absolute Neumann heat flow, as we will not need another heat flow.

Assume the absolute Neumann heat kernel is already constructed, with off-diagonal decay (Fact 26).

As before, define $P_{\leq N} = e^{t\Delta}$ for $N > 0$ and $P_N = P_{\leq N} - P_{\leq \frac{N}{2}}$ for $N > 1, N \in 2\mathbb{Z}$.

Let $\chi_r = 1 - \psi_r$ (see Equation (7)).

Then we have the localized Bernstein estimates:

**Theorem 28.** For any $r > 0$; $m_1, m_2 \in \mathbb{N}_0$; $p \in (1, \infty)$; $N \geq 1$ and $X \in W^{m_1+p, p}(M)$:

$$
||P_{\leq N}X||_{W^{m_1+m_2, p}(M)} \lesssim_{r, m_1, m_2, p} N^{m_2} ||X||_{W^{m_1, p}(M)} + O_r \left( \frac{1}{N^{1/r}} \right) ||X||_{L^p(M)}
$$

**Proof.** Observe that $1 - \chi_{2r} = \psi_{2r} = \psi_{2r} \psi_{4r}$. Then:

$$
||P_{\leq N}X||_{W^{m_1+m_2, p}(M)} \lesssim_{r, m_1, m_2, p} \chi_{2r}X||_{W^{m_1+m_2, p}(M)} + ||P_{\leq N}\left( \psi_{2r} \psi_{4r}X \right)||_{W^{m_1+m_2, p}(M)}
$$

$$
\lesssim_{r, m_1, m_2} N^{m_2} \chi_{2r}X||_{W^{m_1, p}(M)} + O_r \left( \frac{1}{N^{1/r}} \right) ||\psi_{4r}X||_{L^p(M)}
$$

$$
\lesssim_r N^{m_2} ||X||_{W^{m_1, p}(M)} + O_r \left( \frac{1}{N^{1/r}} \right) ||X||_{L^p(M)}
$$

where we have used the standard Bernstein estimate (Theorem 13) and the off-diagonal decay of the heat kernel to pass from the first line to the second line ($\text{supp} \psi_{2r} \subseteq M_{\leq \frac{N}{4r}}$ which does not intersect $M_{\geq 2r}$). \qed

**Corollary 29.** For any $r, C_1, C_2 > 0$; $N \geq 1$; $p \in (1, \infty)$ and $X \in \mathcal{D}'\cap \Omega(M)$:

$$
||\left( P_{\leq C_1 N} - P_{\leq C_2 N} \right)X||_{L^p(M)} \lesssim_{C_1, C_2, r, p} \frac{1}{N^{1/r}} \left( P_{\leq 2 \max(C_1,C_2) N}X \right)||_{W^{2, p}(M)} + O_{C_1, C_2, r} \left( \frac{1}{N^{1/r}} \right) ||X||_{L^p(M)}
$$

$$
\lesssim_{C_1, C_2, r, p} \frac{1}{N} \left( P_{\leq 3 \max(C_1,C_2) N}X \right)||_{W^{1, p}(M_{r/2} \setminus M_{\geq 2r})} + O_{C_1, C_2, r} \left( \frac{1}{N^{1/r}} \right) ||X||_{L^p(M)}
$$

**Proof.** WLOG $C_1 > C_2 > 0$. Let $C = 2 \max(C_1, C_2)$. Then by FTC:

$$
\left( P_{\leq C_1 N} - P_{\leq C_2 N} \right)X ||_{L^p(M)} \leq \int_{\frac{1}{C_1 N^2} \leq \frac{1}{C_2 N^2}} dt \left( e^{t + \frac{t}{C_1 N^2}} \right) ||X||_{W^{2, p}(M_{\geq 2r})}
$$
Corollary 30. Let $p \in (1, \infty)$ and $X \in L^p \Omega (M)$. Then the following conditions are equivalent:

1. \( N^{\frac{1}{p}} \|P_{\leq N} X\|_{W_{1,p}(M_{>r})} \xrightarrow{N \to \infty} 0 \) \( \forall r > 0 \)

2. \( N^{\frac{1}{p}} \|(P_{\leq C_1 N} - P_{\leq C_2 N}) X\|_{L^p(M_{>r})} \xrightarrow{N \to \infty} 0 \) \( \forall r, C_1, C_2 > 0 \)

3. \( N^{\frac{1}{p}} \|P_{> N} X\|_{L^p(M_{>r})} \xrightarrow{N \to \infty} 0 \) \( \forall r > 0 \)

Proof. It is trivial to show (3) \( \implies \) (2) as \( P_{\leq C_1 N} - P_{\leq C_2 N} = P_{> C_2 N} - P_{> C_1 N} \).

Next, we show (2) \( \implies \) (3). Let

\[ w(N) = N^{\frac{1}{p}} \|P_N X\|_{L^p(M_{>r})} = N^{\frac{1}{p}} \|(P_{\leq N} - P_{\leq N/2}) X\|_{L^p(M_{>r})} \xrightarrow{N \to \infty} 0 \]

Then:

\[ N^{\frac{1}{p}} \|P_{> N} X\|_{L^p(M_{>r})} \leq N^{\frac{1}{p}} \sum_{K \in \mathbb{Z}^2, K > N} \|P_K X\|_{L^p(M_{>r})} = N^{\frac{1}{p}} \sum_{K \in \mathbb{Z}^2, K > N} K^{-\frac{1}{p}} w(K) \]

\[ \lesssim \|w(K)\|_{L^\infty(K > N, K \in \mathbb{Z}^2)} \xrightarrow{N \to \infty} 0 \]

We proceed to show (1) \( \implies \) (2). By Corollary 29:

\[ N^{\frac{1}{p}} \|(P_{\leq C_1 N} - P_{\leq C_2 N}) X\|_{L^p(M_{>r})} \lesssim_{C_1, C_2} N^{\frac{1}{p} - 1} \|P_{\leq 3 \max(C_1, C_2) N} X\|_{W_{1,p}(M_{>r/4})} + O_{C_1, C_2, r} \left( \frac{1}{N^{\infty}} \right) \|X\|_{L^p(M)} \]

\[ \xrightarrow{N \to \infty} 0 \]

Finally, we show (2) \( \implies \) (1). Let \( N_0 \geq 1 \) and \( N_0 \in 2\mathbb{Z} \). There are constants \( C_1, C_2 > 0 \) such that \( P_N = P_{\leq 2N} (P_{\leq C_1 N} - P_{\leq C_2 N}) \).

\[ \limsup_{N \to \infty} N^{\frac{1}{p} - 1} \|P_N X\|_{W_{1,p}(M_{>r})} = \limsup_{N \to \infty} N^{\frac{1}{p} - 1} \|(P_{\leq N} - P_{\leq N_0}) X\|_{W_{1,p}(M_{>r})} \]

\[ \lesssim \limsup_{N \to \infty} N^{\frac{1}{p} - 1} \sum_{K \in \mathbb{Z}^2, N_0 < K \leq N} \|P_K X\|_{W_{1,p}(M_{>r})} \]

\[ \lesssim \limsup_{N \to \infty} N^{\frac{1}{p} - 1} \sum_{K \in \mathbb{Z}^2, N_0 < K \leq N} \left( K \|(P_{\leq C_1 K} - P_{\leq C_2 K}) X\|_{L^p(M_{>r/2})} + O_r \left( \frac{1}{K^{\infty}} \right) \|X\|_{L^p(M)} \right) \]

\[ \lesssim \limsup_{N \to \infty} N^{\frac{1}{p} - 1} \sum_{K \in \mathbb{Z}^2, N_0 < K \leq N} K^{1 - 1/p} w(K) + \limsup_{N \to \infty} N^{\frac{1}{p} - 1} O_r \left( \frac{1}{N_0^{\infty}} \right) \|X\|_{L^p(M)} \]  \( (15) \)
Lemma 32 (Loss of norm). Let
\[ w(K) := K^{1/p} \| (P_{\leq C_1 K} - P_{\leq C_2 K}) X \|_{L_p(M_{p,2})} \xrightarrow{K \to \infty} 0. \]
Then we can bound (15) by
\[
\limsup_{N \to \infty} \sum_{K \in 2^\mathbb{Z}, N_0 < K \leq N} (\frac{K}{N})^{1-1/p} \| w(k) \|_{L^\infty(\kappa \geq N_0, \kappa \in 2^\mathbb{Z})} \lesssim \| w(k) \|_{L^\infty(\kappa \geq N_0, \kappa \in 2^\mathbb{Z})}
\]
But \( N_0 \) is arbitrary. Let \( N_0 \to \infty \) and we are done. \( \square \)

Remark 31. By repeating the proof, for \( \mathcal{X} \in L_p^p L_p^p \Omega (M) \):
\[
\forall r > 0 : N^\frac{1}{p} \| P_{\leq N} \mathcal{X} \|_{L_p^p W^{1,p}(M_{p,r})} \xrightarrow{N \to \infty} 0
\]
\[
\iff \forall r > 0 : N^\frac{1}{p} \| P_{> N} \mathcal{X} \|_{L_p^p L_p^p(\mathcal{M}_{p,r})} \xrightarrow{N \to \infty} 0
\]

We now prove a simple lemma from functional analysis.

Lemma 32 (Loss of norm). Let \( X,Y \) be Banach spaces and \( T : X \to Y \) is continuous injection. Let \( (f_j)_{j \in \mathbb{N}_1} \) be a sequence in \( X \) and \( f \in X \). If \( T f_j \to T f \) then
\[
\| f \|_X \leq \liminf_{j \to \infty} \| f_j \|_X
\]

Proof. Note that \( T^* : Y^* \to X^* \) has dense image. Then
\[
\| f \|_X = \sup_{x^* \in X^*} \| (f, x^*) \| = \sup_{y^* \in Y^*} \| (f, T^* y^*) \| = \sup_{y^* \in Y^*} \lim_{j \to \infty} \| (f_j, y^*) \|
\]
\[
= \sup_{y^* \in Y^*} \lim_{j \to \infty} \| (f, T^* y^*) \| \leq \sup_{y^* \in Y^*} \liminf_{j \to \infty} \| f_j \|_X = \liminf_{j \to \infty} \| f_j \|_X
\]
\( \square \)

Theorem 33. Let \( p \in (1, \infty) \), \( f \in D_M \), and \( X \in \hat{B}_{p,V}^{1/p} \mathcal{X} (M) \) (as in Definition 17). Then \( f X \in \hat{B}_{p,V}^{1/p} \mathcal{X} \).

Proof. To show \( f X \in \hat{B}_{p,V}^{1/p} \mathcal{X} \), we just need to show a commutator estimate (much like in the proof of Onsager’s conjecture):
\[
\mathbb{W}(s) := f^s X^s - (f X)^s
\]
\[
(\sqrt{s})^{1-\frac{1}{p}} \mathbb{W}(s) \xrightarrow{s \downarrow 0} 0
\]
where \( X^s \) is short for \( e^{s \Delta} X \). Indeed, assuming this commutator estimate holds true, \( \forall r > 0 \):
\[
\limsup_{t \downarrow 0} \left( \sqrt{t} \right)^{1-\frac{1}{p}} \| e^{t \Delta} (f X) \|_{W^{1,p}(M_{p,r})}
\]
\[
\leq \limsup_{t \downarrow 0} \left( \sqrt{t} \right)^{1-\frac{1}{p}} \| f^t X^t \|_{W^{1,p}(M_{p,r})} + \limsup_{t \downarrow 0} \left( \sqrt{t} \right)^{1-\frac{1}{p}} \| \mathbb{W}(t) \|_{W^{1,p}(M_{p,r})}
\]
\[
\lesssim \limsup_{t \downarrow 0} \left( \sqrt{t} \right)^{1-\frac{1}{p}} \| f^t \|_{C^1(M)} \| X^t \|_{W^{1,p}(M_{p,r})} = 0
\]

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where we have used the fact that $e^{\tau \Delta} f \xrightarrow{t \to 0} f$, as $f \in \mathcal{D}_N(M)$.

Now we prove the commutator estimate. Define $\mathcal{N}(s) = (\partial_s - \Delta) W(s) = (\Delta f^s) X^s + f^s (\Delta X^s) - \Delta (f^s X^s)$. By the Weitzenbock formula, we get

$$\mathcal{N}(s) = (D^1 f^s) * (D^1 X^s)$$

where $D^1$ is schematic for some differential operator of order at most 1, with smooth coefficients (independent of $s$), and $(D^1 f^s) * (D^1 X^s)$ is schematic for a linear combination of similar-looking tensor terms.

On the other hand, by the Duhamel formula for semigroups (cf. [Tay11a, Appendix A, Proposition 9.10 & 9.11]), for any $s > \varepsilon > 0$ we get

$$W(s) = W(\varepsilon)^{s-\varepsilon} + \int_{\varepsilon}^{s} \mathcal{N}(\sigma)^{s-\sigma} d\sigma$$

It is trivial to show that $W(\varepsilon)^{s-\varepsilon} \xrightarrow{\varepsilon \downarrow 0} 0$. Indeed, let $Y \in L^p\mathbb{X}(M)$. Then

$$\left\langle \left\langle W(\varepsilon)^{s-\varepsilon}, Y \right\rangle \right\rangle = \left\langle \left\langle f^s X^\varepsilon - (f X)^\varepsilon, Y^{s-\varepsilon} \right\rangle \right\rangle \xrightarrow{\varepsilon \downarrow 0} \left\langle \left\langle f X - f X, Y \right\rangle \right\rangle = 0$$

Then $\int_{\varepsilon}^{s} \mathcal{N}(\sigma)^{s-\sigma} d\sigma \xrightarrow{\varepsilon \downarrow 0} W(s)$, and by Lemma 32, we conclude

$$\|W(s)\|_{W^{1,p}(M)} \leq \liminf_{\varepsilon \downarrow 0} \int_{\varepsilon}^{s} \mathcal{N}(\sigma)^{s-\sigma} d\sigma \leq \int_{0}^{s} \|e^{(s-\sigma)\Delta} (D^1 f^s * D^1 X^\sigma)\|_{W^{1,p}(M)} d\sigma$$

$$\lesssim \int_{0}^{s} \left( \frac{1}{s-\sigma} \right)^{\frac{1}{2}} \|D^1 f^s * D^1 X^\sigma\|_{L^p(M)} d\sigma \lesssim_{f} \int_{0}^{s} \left( \frac{1}{s-\sigma} \right)^{\frac{1}{2}} \|X^\sigma\|_{W^{1,p}(M)} d\sigma$$

$$\lesssim \|X\|_{L^p(M)} \int_{0}^{1} \left( \frac{1}{s-\sigma} \right)^{\frac{1}{2}} \left( \frac{1}{\tau} \right)^{\frac{1}{2}} d\tau \lesssim_p \|X\|_{L^p(M)}$$

This obviously implies $(\sqrt{s})^{1-\frac{1}{p}} W(s) \xrightarrow{s \downarrow 0} 0$. $\square$

**Remark.** By repeating the proof, with necessary modifications, for any $f \in \mathcal{D}_N(M)$, and $\mathcal{X} \in L^p_\sharp \mathcal{B}^{1/p}_{\beta'} \mathbb{X}(M)$ (as in Definition 17), we have:

$$f \mathcal{X} \in L^p_\sharp \mathcal{B}^{1/p}_{\beta'} \mathbb{X}$$

**A.1 On flat backgrounds**

**Remark 34.** When $M$ is a bounded domain in $\mathbb{R}^n$, the third condition in Corollary 30 takes on a more familiar form. Indeed, let $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\int \phi = 1$ and $\phi_\varepsilon = \frac{1}{\varepsilon^n} \phi(\varepsilon \cdot)$. Then we have the analogy

$$P_{\phi_\varepsilon} f = e^{\varepsilon \Delta} f \approx \phi_\varepsilon \ast f$$

This means

$$N^{\frac{1}{p}} \|P_{\phi N} X\|_{L^p(M_{\phi N})} \xrightarrow{N \to \infty} 0$$

(16)
is analogous to
\[ \frac{1}{\varepsilon^{1/p}} \| X - \phi_x \|_{L^p(M,\gamma)} \xrightarrow{\varepsilon \to 0} 0 \] (17)

**Definition 35.** As in [Bar+19b; NNT20], for \( p \in (1, \infty) \), we say \( X \in B_{p,VMO}^{1/p} \mathcal{X}(M) \) if \( X \in L^p \mathcal{X}(M) \) and \( \forall r > 0 : \)
\[ A_r(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \| X(x - \varepsilon h) - X(x) \|_{L^p_{|h| \leq 1}} \right\|_{L^p_{M,\gamma}(M)} \xrightarrow{\varepsilon \to 0} 0 \] (18)

Similarly, we say \( X \in L^p B_{p,VMO}^{1/p} \mathcal{X}(M) \) if \( X \in L^p_t L^p \mathcal{X}(M) \) and \( \forall r > 0 : \)
\[ A_r(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \| X(t,x - \varepsilon h) - X(t,x) \|_{L^p_{|h| \leq 1}} \right\|_{L^p_{t,L^p(M,M)}(M)} \xrightarrow{\varepsilon \to 0} 0 \] (19)

**Remark 36.** In (18), note that \( A_r(\varepsilon) \) is continuous in \( \varepsilon = (0, r) \). Define
\[ \widetilde{A}_r(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \| X - \varepsilon h \|_{L^p(M)} \right\|_{L^p_{(0,1)}} \leq \varepsilon^{r,p} \left\| X \|_{L^p(M)} + \left\| A_r(\varepsilon) \right\|_{L^p_{(0,1)}} \right. \] for \( \varepsilon \in (0, 1] \) (well-defined). Then \( \widetilde{A}_r(\varepsilon) \) is also continuous in \( \varepsilon \), with \( \widetilde{A}_r(\varepsilon) \leq A_r(\varepsilon) \) \( \forall \varepsilon \in (0, r) \) and \( \widetilde{A}_r(\varepsilon) \leq_r p \| X \|_{L^p(M)} \) \( \forall \varepsilon \in [\frac{r}{2}, 1] \). By [Huy19, Subsection 5.2], we conclude
\[ \| X \|_{B_{p,\infty}^{1/p}(M,\gamma)} \sim \| X \|_{L^p(M,\gamma)} + \left\| A_r(\varepsilon) \right\|_{L^p_{(0,1)}} \leq \varepsilon^{r,p} \left\| X \|_{L^p(M)} + \left\| A_r(\varepsilon) \right\|_{L^p_{(0,1)}} \right. \]

From this we conclude \( B_{p,\infty,loc}^{1/p} \leftrightarrow B_{p,\infty,loc}^{1/p} \) and \( L^p_t B_{p,VMO}^{1/p} \leftrightarrow L^p_t B_{p,\infty}^{1/p} \) where \( B_{p,\infty,loc}^{1/p}(M) := L^p(M) \cap \left( \bigcap_{r > 0} B_{p,\infty}^{1/p}(M,\gamma) \right) \) and \( L^p_t B_{p,\infty,loc}^{1/p}(M) := L^p_t L^p(M) \cap \left( \bigcap_{r > 0} L^p_t B_{p,\infty}^{1/p}(M,\gamma) \right) \).

We observe that (18) trivially implies (17). To relate (18) to (16), we now borrow some results from the construction of the heat kernel (to be proven in Section B).

**Fact 37.** Fix \( r > 0 \). Let \( K(t,x,y) \) be the Hodge-Neumann heat kernel as constructed in Section B.

For \( r' > 0 \), let \( E_{r'} = \{(x,y) \in M \times M : d(x,y) \geq r'\} \). Then \( E_{r'} \) is compact, and by the locally uniform \( \frac{\partial}{\partial t} \)-dilation decay of the heat kernel, we conclude
\[ \forall x,y \in E_{r'}, \forall t \leq 1 : |K(t,x,y)| = O_{r,-\infty}(t^{\infty}) \] (20)

Now let \( F_{r,r'} = \{(x,y) \in M \times M : d(x,y) \leq r'\} \). Then \( F_{r,r'} \) is compact. By interior blow-up, there is \( r' = r'(r) \in (0, t) \) such that
\[ \forall x,y \in F_{r,r'}, \forall t \leq 1 : |K(t,x,y)| = O_{r,-\infty}(t^{\infty}) \] (21)

**Theorem 38.** Let \( M \) be a bounded \( C^\infty \)-domain in \( \mathbb{R}^n \), \( p \in (1, \infty) \) and \( X \in L^p \mathcal{X}(M) \). Then
\[ \forall r > 0 : A_r(\varepsilon) := \frac{1}{\varepsilon^{1/p}} \left\| \| X(x - \varepsilon h) - X(x) \|_{L^p_{|h| \leq 1}} \right\|_{L^p_{M,\gamma}(M)} \xrightarrow{\varepsilon \to 0} 0 \] (22)
Remark. The proof actually shows for \( N \geq 1 \):

\[
N^{\frac{1}{p}} \| P_{> N} X \|_{L^p(M_{> r})} = O_r \left( \| A_r \|_{L^\infty([0, \frac{r}{2}])} + \frac{\| X \|_{L^p(M)}}{N^{\infty}} \right)
\]

Proof. We first show (22) implies (23). Fix \( r > 0 \). Let \( r' = r' (r) \in (0, \frac{r}{2}) \) as in (21). By (20), we can disregard the region \( \{d(x, y) > r'\} \), and just need to show

\[
\left( \frac{1}{\sqrt{t}} \right)^{\frac{1}{p}} \left\| \int_{d(y, x) \leq r'} dy \, K(t, x, y) \, (X(y) - X(x)) \right\|_{L^p_{\tau} (M_{> r})} \to 0 \quad \text{as} \quad t \to 0
\]

But by (21), the left-hand side is bounded by

\[
\left( \frac{1}{\sqrt{t}} \right)^{\frac{1}{p}} \left\| O_r \left( \frac{1}{t^{n/2}} \right)^{\frac{n-x}{2}} \right\|_{L^p_{\tau} (B_{r'}(x))} \left\| X(y) - X(x) \right\|_{L^p_{\tau} (B_{r'}(x))}^{\frac{1}{p}} \quad \text{by (22) and Holder’s inequality, we can disregard the region \( \{ |\zeta| \leq 1 \} \). Then we split} \quad 1 < |\zeta| \frac{\tau_0}{\tau_p} \quad \text{into dyadic rings:}
\]

\[
(24) \lesssim \left( \frac{1}{\sqrt{t}} \right)^{\frac{1}{p}} \sum_{N \in 2^{0 \cup N} \leq \frac{\tau_0}{\tau_p}} \frac{1}{N^{\infty}} \left\| X(x - \sqrt{\tau} \zeta) - X(x) \right\|_{L^p_{|\zeta| \leq \tau_0}}^{\frac{1}{p}} \left\| X(x - \tau) - X(x) \right\|_{L^p_{|\tau| \leq \tau_0}}^{\frac{1}{p}}
\]

where we made the change of variable \( \tau = \sqrt{\tau} \zeta \). Now observe that (22) implies that for \( \varepsilon \leq r/2 \):

\[
\left\| X(x - \tau) - X(x) \right\|_{L^p_{|\tau| \leq \varepsilon}}^{\frac{1}{p}} = \varepsilon^{\frac{1}{p}} A_r (\varepsilon)
\]

where \( 0 \leq A_r (\varepsilon) \leq \| A_r \|_{L^\infty([0, r/2])} \) and \( A_r (\varepsilon) \to 0 \). Then

\[
(24) \lesssim \left( \frac{1}{\sqrt{t}} \right)^{\frac{1}{p}} \sum_{N \in 2^{0 \cup N} \leq \frac{\tau_0}{\tau_p}} \frac{1}{N^{\infty}} \left( \sqrt{\tau} N \right)^{\frac{1}{p}} A_r \left( \sqrt{\tau} N \right)
\]
where the last condition means \( \forall \) such that \( (0, \infty) \) that \( x \), be the standard local coordinates for \([0 \times \mathbb{M}, \pi] \) be a compact Riemannian \( n \)-manifold with boundary. A differential \( k \)-form is a member of \( C^\infty (\mathbb{M}; \Lambda^k \mathbb{M}) \).

In this section, unless otherwise noted, we write \( \Delta \) for the Hodge Laplacian on forms. We also let \((t, x, y)\) be the standard local coordinates for \([0, \infty) \times \mathbb{M} \times \mathbb{M} \). When \( x \) or \( y \) is near the boundary, we can stipulate that \( x_n \) and \( y_n \) stand for the Riemannian distance to the boundary (geodesic normal coordinates).

We aim to construct a unique Hodge-Neumann heat kernel with the absolute Neumann boundary condition. In particular, define \( \text{END} (\Lambda^k \mathbb{M}) = \text{Hom} (\pi^{-}_2 \Lambda^k \mathbb{M}, \pi^{-}_1 \Lambda^k \mathbb{M}) \), where \( \pi_i \) is the projection from \((0, \infty) \times \mathbb{M} \times \mathbb{M} \) onto the \( i \)-th \( \mathbb{M} \). We want

\[
K \in C^\infty_{\text{loc}}((0, \infty) \times \mathbb{M} \times \mathbb{M}; \text{END} (\Lambda^k \mathbb{M}))
\]

such that

\[
(\partial_t - \Delta_x) K (t, x, y) = 0
\]

\[
n_x K (t, x, y) = 0 \quad \text{for} \ x \in \partial \mathbb{M}
\]

\[
n_x d_x K (t, x, y) = 0 \quad \text{for} \ x \in \partial \mathbb{M}
\]

\[
\lim_{t \downarrow 0} K (t, x, y) = \delta_y (x)
\]

where the last condition means \( \forall u \in \mathcal{D} (\mathbb{M}; \Lambda^k \mathbb{M}) \), \( \int K (t, x, y) u(y) \, dy \xrightarrow{t \downarrow 0} u(x) \).
During the construction, we will be able to prove certain properties of the kernel, such as off-diagonal decay for all derivatives.

The construction of the heat kernel comes from [MV13], and we simply discuss the modifications required for our case, to handle the Hodge-Neumann Laplacian on a smooth manifold with smooth boundary.\(^2\)

**Kernel in Einstein sum notation**

Let \( A \in C^\infty_{\text{loc}} ((0, \infty) \times M^2; \text{END}(\Lambda^k M)) \). Let \( U \subset M \) be a coordinate patch. Then, by using Einstein notation, locally for \( x, y \in U \) we have:

\[
A(t, x, y) = A_{I,J}(t, x, y) \, dx^I \otimes \partial_{y^J},
\]

where \( I, J \in \mathcal{I}_k = \{ (i_1, \ldots, i_k) : i_1 < i_2 < \cdots < i_k \} \) and \( \partial_{y^J} \) is dual to the form \( dy^J \). (also in Einstein notation, we write \( x^n \) instead of \( x_n \))

- Note that we are abusing notation, as \( dx^I \) here is a local section of \( \pi_*^* \Lambda^k M \to (0, \infty) \times M^2 \), defined by pulling back the actual form \( dx^I \) on \( M \). We can explicitly write \( A_{I,J}(t, x, y) \, dx^I \big|_x \otimes \partial_{y^J} \big|_y \) to emphasize the pullback.
- Observe that \( d_x A(t, x, y) = d_x (A_{I,J}(t, x, y) \, dx^I) \otimes \partial_{y^J} = \partial_x A_{I,J}(t, x, y) \big( dx^I \wedge dx^I \big) \otimes \partial_{y^J} \).
- If \( u(y) = u_J(y)dy^J \) is a differential form on \( M \), we write \( A(t, x, y) u(y) = A_{I,J}(t, x, y) u_J(y)dx^I \), which is a section of \( \pi_*^* \Lambda^k M \).

As agreed above, when \( U \) touches the boundary, \( \partial_x^n \) is the inwards normal direction, so for \( x \in \partial M \):

\[
\n_x A(t, x, y) = 1_{n \not \in I} A_{I,J}(t, x, y) \, dx^I \otimes \partial_{y^J}.
\]

- If \( \n_x A = 0 \) for all \( x \in \partial M \), then

\[
\n_x d_x A = 1_{n \not \in I} \partial_x^n A_{I,J}(t, x, y) \big( dx^I \wedge dx^I \big) \otimes \partial_{y^J}.
\]

So \( \n_x d_x A = 0 \iff \partial_x^n A_{I,J}(t, x, y) = 0 \) whenever \( x \in \partial M, n \not \in I \). In other words, \( \n_x A = 0 \) and \( \n_x d_x A = 0 \) mean the normal part obeys the Dirichlet boundary condition, while the tangential part obeys the Neumann boundary condition. This will inspire the choice of leading terms later on.

### B.1 Heat calculus

Let \( x = (x', x_n) \) and \( y = (y', y_n) \) be points in \( \mathbb{R}^n \). Recall:

1. The scalar heat kernel on \( \mathbb{R}^n \): \( K(t, x, y) = \left( \frac{1}{4\pi t} \right)^{n/2} \tau^{-n} e^{-\frac{|\zeta|^2}{4\tau}} \) where \( \tau = \sqrt{t}, \zeta = \frac{x-y}{\tau} \).
2. The Dirichlet scalar heat kernel on \( \mathbb{R}^{n-1} \times [0, \infty) \): \( K(t, x, y) = \left( \frac{1}{4\pi t} \right)^{n/2} \tau^{-n} e^{-\frac{|\xi|^2}{4\tau}} \left( e^{-\frac{1}{4}|\xi_n-y_n|^2} - e^{-\frac{1}{4}|\xi_n+y_n|^2} \right) \)

   where \( \xi_n = \frac{x_n-y_n}{\tau}, \xi' = \frac{x'-y'}{\tau} \).
3. The Neumann scalar heat kernel on \( \mathbb{R}^{n-1} \times [0, \infty) \): \( K(t, x, y) = \left( \frac{1}{4\pi t} \right)^{n/2} \tau^{-n} e^{-\frac{|\xi|^2}{4\tau}} \left( e^{-\frac{1}{4}|\xi_n-y_n|^2} + e^{-\frac{1}{4}|\xi_n+y_n|^2} \right) \).

They will inspire the formulation of our boundary heat calculus, which describes heat-type kernels on manifolds.

We assume the reader is familiar with the spaces of conormal and polyhomogeneous distributions on a manifold with corners [Gri01; Mel18].

---

\(^2\)The author thanks Daniel Grieser, András Vasy and Rafe Mazzeo for discussing these ideas.

The original plan was to follow the note [Gri04] which is simpler and does not rely on Melrose’s calculus, but we have decided to clean up the note, modify some steps and publish it at a later date.
B.1.1 Blown-up heat space

We first construct the blown-up heat space $M^2_h$, with the faces lf, ff, td, tf as defined in [MV13] (though our case is simpler).

We start with $[0, \infty) \times M \times M$, with faces tf (temporal face), rf (right face), lf (left face) being defined as $\{0\} \times M \times M$, $[0, \infty) \times \partial M \times M$, $[0, \infty) \times M \times \partial M$ respectively. Then we perform a parabolic blow-up [Mel18, Section 7.4] on the submanifold $\{0\} \times \partial M \times \partial M$ in the time direction $dt$, to create the face ff (front face). This creates an intermediate manifold that we will call $M_1$.

After that, we perform another parabolic blow-up on the lift of the submanifold $\{0\} \times \Delta(M)$ to $M_1$ (to be more precisely defined in (29)), which creates another face td (time diagonal). This is the space $M^2_h$ we need.

![Figure 1: The blown-up heat space $M^2_h$](image)

B.1.2 Local coordinates

By letting $\tau = \sqrt{t}$, we call $(\tau, x, y)$ the ts-coordinate system (time-rescaled) for $[0, \infty) \times M \times M$.

- On $M_1$, near rf and away from lf (i.e. $y_n > 0$), we use the rf-coordinate system

  $T = \frac{t}{y_n^2}, \theta' = \frac{x' - y'}{y_n}, \theta_n = \frac{x_n}{y_n}, y', y_n$ (25)

  where $\theta_n, y_n, T$ are respectively the boundary defining functions for rf, ff, tf. For blow-ups, it is also useful to define the (time-rescaled) tsrf-coordinate system

  $\varsigma = \sqrt{T}, \theta', \theta_n, y', y_n$ (26)

  We observe that as $(\varsigma, \theta', \theta_n, y', y_n) \to (\varsigma, \theta', \theta_n, y', 0)$ in the tsrf-coordinate, in the ts-coordinate we have

  $(0, (y', 0), (y', 0)) + y_n (\varsigma, (\theta', \theta_n), (0, 1)) \to (0, (y', 0), (y', 0))$

  The (time-rescaled) tangent vector $(\varsigma, (\theta', \theta_n), (0, 1))^4$ at $(0, (y', 0), (y', 0))$ (modulo vectors tangent to $\{0\} \times \partial M \times \partial M$, and modulo positive scalar multiplication) corresponds to a point on ff, which

---

3 We are following [MV13] by letting rf be defined by $x_n = 0$. Other authors might prefer $y_n = 0$.

4 Explicitly, the tangent vector is $\varsigma \partial_x + (\theta', \theta_n) \cdot \partial_x + (0, 1) \cdot \partial_y$. 

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is $[(\varsigma, (\vartheta', \theta_n), (0, 1))] = [(\varsigma, (0, \theta_n), (-\vartheta', 1))]$. This is what allows us to extend the (ts)rf-coordinate systems from $[0, \infty) \times M \times M$ to $M_1$, with $\{y_n = 0\}$ being the face ff.

- On $M_1$, near ff and away from tf, we use the ff-coordinate system

$$\tau = \sqrt{t}, x', \xi_n = \frac{x_n}{\sqrt{t}}, \zeta' = \frac{x' - y'}{\sqrt{t}}, \eta_n = \frac{y_n}{\sqrt{t}} \quad (27)$$

where $\tau, \xi_n, \eta_n$ are respectively the boundary defining functions for ff, rf, lf. As $(\tau, x', \xi_n, \zeta', \eta_n) \rightarrow (0, x', \xi_n, \zeta', \eta_n)$ in the ff-coordinate, in the ts-coordinate we have

$$(0, (x', 0), (x', 0)) + \tau (1, (0, \xi_n), (-\zeta', \eta_n)) \rightarrow (0, (x', 0), (x', 0))$$

The (time-rescaled) tangent vector $(1, (0, \xi_n), (-\zeta', \eta_n))$ at $(0, (x', 0), (x', 0))$ corresponds to a point on ff, which is $[(1, (0, \xi_n), (-\zeta', \eta_n))]$.

- On $M^2_h$, near td, near ff, away from tf, we can use the rf-coordinate system from (25) to define the fftd-coordinate system

$$\vartheta = \sqrt{T}, \sigma' = \frac{\theta'}{\sqrt{T}}, \sigma_n = \frac{\theta_n - 1}{\sqrt{T}}, y', y_n \quad (28)$$

where $\vartheta$ is the defining function for td. Note that as $(\vartheta, \sigma', \sigma_n, y', y_n) \rightarrow (0, \sigma', \sigma_n, y', y_n)$ in the fftd-coordinate, in the tsrf-coordinate we have

$$(0, 0, 1, y', y_n) + \tau (1, \sigma', \sigma_n, 0, 0) \rightarrow (0, 0, 1, y', y_n) \quad (29)$$

We observe that the points $(0, 0, 1, y', y_n)$ in the tsrf-coordinate, are precisely the lift of the submanifold $D_0 := \{0\} \times \Delta(M)$ to $M_1$, which we will write as $D_1$. By blowing up $D_1$, we create the face td and $M^2_h$. Note that $\theta_n = 1 > 0$, so td does not intersect rf (or ff). Also, the (time-rescaled) tangent vector $(1, \sigma', \sigma_n, 0, 0)$ at $(0, 0, 1, y', y_n)$ corresponds to a point on the face td.

On the other hand, the point $(\vartheta, \sigma', \sigma_n, y', 0)$ in the fftd-coordinate on $M^2_h$ maps down to the point $(\vartheta, \vartheta \sigma', \vartheta \sigma_n + 1, y', 0)$ in the tsrf-coordinate on $M_1$ (the map being injective on $\{\vartheta > 0\}$), which in turn corresponds to the point $[(\vartheta, 0, \vartheta \sigma_n + 1, (-\vartheta \sigma', 1))]$ on ff.

- The points $(0, 0, 1, y', 0)$ in the (ts)rf-coordinate are precisely the intersection ff $\cap D_1$ in $M_1$.

The points $(0, \sigma', \sigma_n, y', 0)$ in the fftd-coordinate are precisely the intersection ff $\cap$ td in $M^2_h$.

- On $M^2_h$, near td, away from ff and away from tf, we use the td-coordinate system

$$\tau = \sqrt{t}, x, \zeta = \frac{x - y}{\sqrt{t}} \quad (30)$$

where $\tau$ is the defining function for td. As $(\tau, x, \zeta) \rightarrow (0, x, \zeta)$ in td-coordinate, in ts-coordinate we have

$$(0, x, x) + \tau (1, 0, -\zeta) \rightarrow (0, x, x)$$

So we identify the point $(0, x, \zeta)$ in td-coordinate with the (time-rescaled) tangent vector $(1, 0, -\zeta)$ at $(0, x, x) \in D_0$, which gives a point of td (or to be precise, away from the edges, $D_1$ and $D_0$ are locally diffeomorphic, and td being defined as a bundle over $D_1$ is also locally defined over $D_0$).
• Wherever we have both the td-coordinate system and the fftd-coordinate system, the point \((\tau, x, \zeta) = (\tau, (x', x_n), (\zeta', \zeta_n))\) in the td-coordinate (with \(x_n > 0, x_n - \tau \zeta_n > 0\)) corresponds to the point 
\[
\left(\frac{x_n - \tau \zeta_n}{x_n}, \zeta_n, x' - \tau \zeta_n, x_n - \tau \zeta_n\right)
\]
in the fftd-coordinate. Conversely, \((\vartheta, \sigma', \sigma_n, y', y_n)\) in the fftd-coordinate corresponds to \((\vartheta y_n, (y' + \vartheta y_n \sigma', y_n + \vartheta y_n \sigma_n), (\sigma', \sigma_n))\) in the td-coordinate. Consequently,
\[
(0, (x', x_n), (\zeta', \zeta_n)) \quad \text{in td-coordinate corresponds to} \quad (0, \zeta', \zeta_n, x', x_n) \quad \text{in fftd-coordinate}
\]
and we identify the tangent vector \((1, 0, -\zeta)\) at \((0, x, x) \in D_0\) (in the ts-coordinate) with the tangent vector \((1, \zeta', \zeta_n, 0, 0)\) at \((0, 0, 1, x', x_n) \in D_1\) (in the tsrf-coordinate), as the same point in td.

**Remark 39** (Compatibility condition at \(\text{ff} \cap \text{td}\)). For any smooth functions \(u\) on \(\text{ff}\) and \(v\) on \(\text{td}\), the following are equivalent:

1. In the fftd-coordinate:
\[
u(\vartheta, \sigma', \sigma_n, y', 0) \xrightarrow{\vartheta \to 0} v(0, \sigma', \sigma_n, y', 0)\]

2. There is a smooth function \(f\) on \(M^2_k\) such that \(N^0_{\text{ff}}(f) = u, \quad N^0_{\text{td}}(f) = v\).

**B.1.3 Edge calculus**

**Definition 40.** For \(\alpha, \alpha' \in -\mathbb{N}_0\), we define \(\Psi_{e-h}^{\alpha, \alpha', E_{\text{ff}}, E_{\text{td}}} (M; \Lambda^k M)\)\(^5\) as the space of Schwartz kernels \(K\) that are pushforwards of polyhomogeneous kernels \(\tilde{K}\) on \(M^2_k\) (though we will abuse notation and also write \(K\) for \(\tilde{K}\)) such that:

- the index sets at lf and rf are \(E_{\text{lf}} = (E^t_{\text{lf}}, E^n_{\text{lf}})\) and \(E_{\text{rf}} = (E^t_{\text{rf}}, E^n_{\text{rf}})\). Here \(E^t_{\text{lf}}, E^t_{\text{rf}}\) describe the local coefficients of \(t, \tilde{K}\) (the tangent component), while \(E^n_{\text{lf}}, E^n_{\text{rf}}\) describe the local coefficients of \(n, K\).
- the index set at \(\text{ff}\) is \(\{(j - (n + 2 + \alpha), 0) : j \in \mathbb{N}_0\}\) (expansion in \(\tau\) from (27))
- the index set at \(\text{td}\) is \(\{(j - (n + 2 + \alpha'), 0) : j \in \mathbb{N}_0\}\) (expansion in \(\tau\) from (30)). By convention, it is \(\emptyset\) when \(\alpha' = -\infty\).
- the index set at \(\text{tf}\) is \(\emptyset\) (off-diagonal decay).

**Theorem 41.** *The absolute Neumann heat kernel* \(H\) *lies in* \(\Psi_{e-h}^{-2, -2, E_{\text{ff}}, E_{\text{td}}} (M; \Lambda^k M)\) *where*

- \(E^t_{\text{lf}}, E^n_{\text{lf}} = \mathbb{N}_0 \times \{0\}\) *(smoothness at* \(\text{lf})^6*
- \(E^t_{\text{rf}} \subset \mathbb{N}_1 \times \{0\}, \quad E^n_{\text{rf}} \subset (\mathbb{N}_0 \setminus \{1\}) \times \{0\}\) *(absolute Neumann boundary condition)*

We can also write \(\Psi_{e-h}^{-2, -2, N_0, E_{\text{ff}}}\) to describe smoothness at \(\text{lf}\).

**B.2 Proof of Theorem 41**

We proceed exactly as in [MV13, Section 3.2].

For any \(A \in \Psi_{e-h}^{\alpha, \alpha', E_{\text{ff}}, E_{\text{td}}} (M; \Lambda^k M)\), we can expand w.r.t. \(\text{ff}\) (with coordinates as in (27))

\[
A = A^\text{ff}_{-n-2-\alpha}(x, \xi_n, \zeta', \eta_n) \tau^{-n-2-\alpha} + A^\text{ff}_{-n-2-\alpha+1}(x, \xi_n, \zeta', \eta_n) \tau^{-n-2-\alpha+1} + \ldots
\]

\(^5\)To translate to the definition of \(\Psi_{e-h}^{l-p, E_{\text{ff}}, E_{\text{td}}}\) from [MV13, Section 3.2], we can use the formulas \(\alpha = -l, \alpha' = -p-2, n = m, \alpha - 1 = b\).

\(^6\)In fact, due to symmetry, we must have \(E^t_{\text{lf}} = E_{\text{rf}}\), but for this paper we will not need this fact.
We write $N_{\text{ff}}^{-n-2-\alpha} (A)$ for the leading coefficient $A_{-n-2-\alpha}$. We can expand similarly w.r.t. td and define $N_{\text{td}}^{-n-2-\alpha'} (A)$.

Then we note that $t (\partial_t - \Delta_x)$ is a $b$-operator which could be restricted to ff and td. In particular,

$$
\begin{align*}
N_{\text{ff}}^{-n-2-\alpha} (t (\partial_t - \Delta_x) A) &= N_{\text{ff}}^{-n-2-\alpha} (t (\partial_t - \Delta_x)) N_{\text{ff}}^{-n-2-\alpha} (A) \\
N_{\text{td}}^{-n-2-\alpha'} (t (\partial_t - \Delta_x) A) &= N_{\text{td}}^{-n-2-\alpha'} (t (\partial_t - \Delta_x)) N_{\text{td}}^{-n-2-\alpha'} (A)
\end{align*}
$$

where, in the td-coordinate system from (30) and the ff-coordinate system from (27):

$$
\begin{align*}
N_{\text{td}}^{-n-2-\alpha} (t (\partial_t - \Delta_x)) &= -\Delta_x (x) - \frac{1}{2} \zeta \cdot \partial_x - \frac{n+2+\alpha}{2} \\
N_{\text{ff}}^{-n-2-\alpha} (t (\partial_t - \Delta_x)) &= -\Delta (x', 0) - \frac{1}{2} \zeta (x', \eta_n) \cdot \partial_{(x', \xi_n, \eta_n)} - \frac{n+2+\alpha'}{2}
\end{align*}
$$

(33)

Here we have written $\zeta \cdot \partial_x = \sum_i \zeta_i \partial_{x_i}$ and $\Delta_x (x) = \sum_{i,j} g^{ij} (x) \partial_i \partial_j$.

Then we have $t (\partial_t - \Delta_x) \Psi_{\text{e-h}}^{n, \alpha', E_{\text{ff}}, E_{\text{td}}} \subseteq \Psi_{\text{e-h}}^{n, \alpha', N_0, N_0}$.

From this point on, we fix $E_{\text{ff}}, E_{\text{td}}$ to be as in Theorem 41.

**Claim 42.** There is an element $H^{(1)} \in \Psi_{\text{e-h}}^{-3,-\infty, N_0, N_0}$ such that

$$
\begin{align*}
\{ \begin{array}{l}
P^{(1)} := t (\partial_t - \Delta_x) H^{(1)} \in \Psi_{\text{e-h}}^{-3,-\infty, N_0, N_0} \\
\lim_{t \to 0} H^{(1)} (t, x, y) = \delta_y (x)
\end{array} \}
\end{align*}
$$

**Proof.** To prove this claim, we construct $A \in \Psi_{\text{e-h}}^{-2, -E_{\text{ff}}, E_{\text{td}}}$ such that

$$
N_{\text{td}}^{-n} (A) (x, \zeta) = \left( \frac{1}{4 \pi} \right)^{n/2} e^{-\frac{|\zeta|^2}{4} (x)} \text{Id} = \left( \frac{1}{4 \pi} \right)^{n/2} e^{-\frac{|\zeta|^2}{4} (x)} dx |x \otimes \partial_y |x
$$

$$
N_{\text{ff}}^{-n} (A) (x', \xi_n, \zeta', \eta_n) = \left( \frac{1}{4 \pi} \right)^{n/2} e^{-\frac{|\zeta'|^2}{4} (x', 0)} \left( e^{-\frac{1}{2} |\xi_n - \eta_n|^2} (t + n) + e^{-\frac{1}{2} |\xi_n + \eta_n|^2} (t - n) \right)
$$

$$
= \frac{1}{n!} \left( \frac{1}{4 \pi} \right)^{n/2} e^{-\frac{|\zeta'|^2}{4} (x', 0)} \left( e^{-\frac{1}{2} |\xi_n - \eta_n|^2} e^{-\frac{1}{2} |\xi_n + \eta_n|^2} \right) dx |(x', 0) \otimes \partial_y |(x', 0)
$$

$$
+ \frac{1}{n!} \left( \frac{1}{4 \pi} \right)^{n/2} e^{-\frac{|\zeta'|^2}{4} (x', 0)} \left( e^{-\frac{1}{2} |\xi_n - \eta_n|^2} e^{-\frac{1}{2} |\xi_n + \eta_n|^2} \right) dx |(x', 0) \otimes \partial_y |(x', 0)
$$

This choice satisfies the compatibility condition from (32) (with $N_{\text{td}}^{-n} (A) = N_{\text{td}}^{-n} (t \frac{A}{2})$ and $N_{\text{ff}}^{-n} (A) = N_{\text{ff}}^{-n} (t \frac{A}{2})$), since

$$
e^{-\frac{1}{2} |\sigma'|^2 (\nu', 0)} \left( e^{-\frac{1}{2} |\sigma_n|^2} (t + n) + e^{-\frac{1}{2} |\sigma_n + \nu'|^2} (t - n) \right) \quad \xrightarrow{\nu \to 0} \quad e^{-\frac{1}{2} |\sigma'|^2 (\nu', 0)} \left( e^{-\frac{1}{2} |\sigma_n|^2} \right) = e^{-\frac{1}{2} |(\nu', \sigma_n)|^2 (\nu', 0)}
$$

We note that $A$ is smooth on $(0, \infty) \times M \times M$, and we can make $A$ have the same index set for $r$ as $N_{\text{ff}}^{-n} (A)$, therefore satisfying the absolute Neumann condition. Of-diagonal decay is also explicit from these formulas (when $x \neq y$ stay fixed and $t \to 0$, we have $\zeta = \frac{x-y}{\sqrt{t}} \to \infty$).

By direct calculations, we see that $N_{\text{ff}}^{-n} (t (\partial_t - \Delta_x) A) = 0$ and $N_{\text{td}}^{-n} (t (\partial_t - \Delta_x) A) = 0$. Therefore $t (\partial_t - \Delta_x) A \in \Psi_{\text{e-h}}^{-3,-\infty, N_0, N_0}$. We then observe two facts:

- In the expansion of $A$ at td, $A^j_{\text{td}}$ for $j > -n$ can be freely changed.
• For any smooth \( f(x, \zeta) \) that is Schwartz in \( \zeta \) (rapidly decaying) and \( j \geq 1 \), there is a unique \( F(x, \zeta) \) rapidly decaying in \( \zeta \) such that

\[
N_{td}^{-n+j} (t (\partial_t - \Delta_x)) F(x, \zeta) = f(x, \zeta)
\]

In particular, by using the Fourier transform \( \zeta \mapsto z \) (with the convention \( \hat{F}(z) = \int_{\mathbb{R}^n} F(\zeta) e^{-i2\pi \zeta \cdot z} \, d\zeta \)):

\[
\hat{F}(x, z) = \int_0^1 ds \cdot 2^{s-1} \hat{f}(x, sz) e^{-4\pi^2 s (1-s^2) z^2(x)}
\]

See also [Alb17, Section 6.2] for an explanation of this. It boils down to the fact that \( \Delta_\zeta \) is smoothing (elliptic) for \( \zeta \).

Therefore it is possible to change \( (A_{td}^j)_{j>0} \) to make \( t (\partial_t - \Delta_x) \) vanish to infinite order at \( td \). It boils down to solving

\[
N_{td}^j (t (\partial_t - \Delta_x)) A_{td}^j = B_j, \quad j > -n
\]

where \( B_j \) is an inhomogeneous term depending on \( A_{td}^0, \ldots, A_{td}^{j-1} \). Changing \( (A_{td}^j)_{j>0} \) will not affect the index set of \( A \) at \( rf \), since \( td \) does not intersect \( rf \) and \( lf \), by the above reasoning with (28). \( A \) is smooth at \( rf \) and \( lf \), and we therefore obtain \( t (\partial_t - \Delta_x) A \in \Psi^{-3,-\infty,N_0,N_0}_e \).

We finally note that \( \lim_{t \downarrow 0} A(t, x, y) = \delta_y(x) \) due to (34), which is the “universal” formula for the expansion of heat kernels in the interior of manifolds. The claim is then proven. We refer to [MV13, Proposition 3.2] for more details. \( \square \)

So we have solved away the leading coefficient of \( t (\partial_t - \Delta_x) \) \( H \) at \( ff \), as well as all the coefficients at \( td \). Next, we solve away all the coefficients at \( rf \).

**Claim 43.** There is an element \( H^{(2)} \in \Psi^{-2,-2,E_{il},E_{ij}}_e (M; \Lambda^k M) \) such that

\[
\begin{cases}
P^{(2)} := t (\partial_t - \Delta_x) H^{(2)} \in \Psi^{-3,-\infty,N_0,0}_e \\
\lim_{t \downarrow 0} H^{(2)} (t, x, y) = \delta_y(x)
\end{cases}
\]

**Proof.** Let \( r(x) \) be a boundary-defining function for \( rf \) such that \( r(x) = \text{dist}(x, \partial M) = x_n \) near \( rf \). We observe that \( r^2 (\partial_t - \Delta_x) \) is a \( b \)-operator which can be restricted to \( rf \) (defined by \( \theta_n = 0 \) in the \( rf \)-coordinate system from (25)). On the other hand, in the \( ff \)-coordinate system from (27), \( r = \tau \xi_n \), so \( r^2 (\partial_t - \Delta_x) \) is also a \( b \)-operator w.r.t. \( ff \).

We observe that \( (\partial_t - \Delta_x) H^{(1)} \in \Psi^{-1,-\infty,N_0,0}_e \) and we want to obtain \( (\partial_t - \Delta_x) H^{(2)} \in \Psi^{-1,-\infty,N_0,0}_e \).

Therefore it is enough to find \( J \in \Psi^{-3,3,E_{il},E_{ij}}_e (M; \Lambda^k M) \) such that \( r^2 (\partial_t - \Delta_x) (H^{(1)} - J) \) vanishes to infinite order at \( rf \).

Let \( B = r^2 (\partial_t - \Delta_x) H^{(1)} \in \Psi^{-3,-\infty,N_0,N_0+2}_e \). We note that \( B_0^t = B_1^t = 0 \), so it is fine to set \( J_0^t = J_1^t = 0 \).

Recall that \( \Delta_x = \sum_{ij} g^{ij} (x) \partial_x^i \partial_x^j + \sum_i b_i \partial_x^i + c \) where \( b_i, c \) are smooth. Then by translating \( r^2 (\partial_t - \Delta_x) \) into \( rf \)-coordinates, we have to solve the formal expansion at \( rf \):

\[
\theta_n^2 \left( \partial_t - \sum_{i,j \neq n} g^{ij} \partial_t \partial_{\theta_j} - \sum_{i \neq n} y_n b_i \partial_{\theta_i} - \partial_{\theta_n}^2 - y_n b_n \partial_{\theta_n} - c \right) \left( \sum_{j \geq 2} J_j^t \theta_n^j \right) = \sum_{j \geq 2} B_j^t \theta_n^j \tag{35}
\]

Note that near \( rf \), because we have chosen the geodesic normal coordinates, \( g^{in} = \delta_i^n \) for any
For the last step, we consider the formal Volterra series:

\[ N_{i\ell}^j \left( r^2 (\partial_t - \Delta_x) \right) J_{i\ell}^j(T, \theta', y', y_n) = C_j(T, \theta', y', y_n), \quad j \geq 2 \]

where

- \( N_{i\ell}^j \left( r^2 (\partial_t - \Delta_x) \right) = - j \, (j - 1), \quad j \geq 1. \)
- \( C_j \) is an inhomogeneous term depending on \( B_{i\ell}^j \) and \( J_{i\ell}^{j-1} \). In particular, \( C_2 = B_2^j \).

Solving this is trivial (and in fact the solutions are unique), since for \( j \geq 2 \), \( N_{i\ell}^j \left( r^2 (\partial_t - \Delta_x) \right) \) is a nonzero constant. We note that \( \left( J_{i\ell}^j \right)_{j \geq 2} \) inherits many properties from \( \left( B_{i\ell}^j \right)_{j \geq 2} \) by induction:

- In the \( \text{rf} \)-coordinate system, \( B_{i\ell}^j \) is defined from \( \frac{1}{j!} \partial^j_{y_n} \bigg|_{y_n=0} B \) (abuse of notation). But \( y_n \) is the defining function for \( \text{ff} \), so the index set of \( B_{i\ell}^j \) at \( \text{ff} \) is the same as that of \( B \), and therefore this is also true for \( J_{i\ell}^j \). This extends to \( J_{i\ell}^j \forall j \geq 2 \), because we can explicitly derive \( C_j \) from (35), and see that the powers of \( y_n \) never get lowered (no \( \partial \)s or \( 1/y_n \)).
- The index sets of \( B \) at \( \text{td} \) and \( \text{ff} \) are empty (i.e. \( B = O(T^\infty) \) as \( T \to 0 \)), which implies \( J_{i\ell}^j = O(T^\infty) \).

Note that we also have to solve for \( J_{i\ell}^j \) where \( y \) is away from the boundary (which means there is no \( \text{rf} \)-coordinate system). In that case, we use the \( \text{ts} \)-coordinate system and solve the formal expansion at \( \text{rf} \). This proceeds in the same fashion (but it is even simpler, since we are far away from \( \text{ff} \)).

Consequently, constructing \( J \) from \( \left( J_{i\ell}^j \right)_{j \geq 0} \) will give us \( J \in \Psi_{e-h-}^{-3,-\infty,0,0,2} (M; \Lambda^k M) \) such that \( B - r^2 (\partial_t - \Delta_x) J \) vanishes to infinite order at \( \text{rf} \).

With the index set at \( \text{rf} \) being \( N_0 + 2 \), \( J \) satisfies the absolute Neumann boundary condition. Also, because the index sets of \( J \) at \( \text{ff} \) and \( \text{td} \) are higher than those of \( H^{(1)} \), we conclude

\[ \begin{cases} N_{i\ell}^{-n} \left( H^{(1)} - J \right) = N_{i\ell}^{-n} \left( H^{(1)} \right) \\ N_{td}^{-n} \left( H^{(1)} - J \right) = N_{td}^{-n} \left( H^{(1)} \right) \end{cases} \]

By setting \( H^{(2)} = H^{(1)} - J \), the claim is proven.

For the last step, we consider the formal Volterra series:

\[ H = H^{(2)} + H^{(2)} * R^{(2)} + H^{(2)} * R^{(2)} * R^{(2)} + \ldots. \]

where \( R^{(2)} := - (\partial_t - \Delta_x) H^{(2)} \in \Psi^{-1,-\infty,N_0,0}_{e-h} \), and the composition \( A * B \) is defined by

\[ A * B(t, x, y) = \int_0^t ds \int_M dvol_g(z) \, A(t-s, x, z) B(s, z, y) \]

By [MV13, Theorem 5.3], if \( Q_{i\ell} + Q_{i\ell}' > -1; \, \alpha, \gamma, \beta \in -N_1 \), we have the formula

\[ \Psi_{e-h}^{\alpha, \gamma, Q_{i\ell}, Q_{i\ell}'} \subset \Psi_{e-h}^{\alpha + \beta, -\infty, P_{i\ell}, P_{i\ell}'} \]

where \( P_{i\ell} = Q_{i\ell}' \cup (Q_{i\ell} - \beta); \, P_{i\ell} = Q_{i\ell}' \cup (Q_{i\ell}' - \alpha) \). This means that for \( N \in N_1 \):

\[ H^{(2)} * \left( R^{(2)} \right)^N \in \Psi_{e-h}^{-2-N, -\infty, E_{i\ell}, N, E_{i\ell}} \]
where $E_{H,N}$ is defined inductively by $E_{H,1} = \mathbb{N}_0 \cup \{0\}$ and $E_{H,N+1} = \mathbb{N}_0 \cup (E_{H,N} + 1)$ for $N \geq 1$.

Letting $N_j = \{x \in \mathbb{N} : x \geq j\}$ and $\mathcal{N} = \bigcup_{j \in \mathbb{N}_0} N_j$, we conclude that

$$\forall N : E_{H,N} \subset \mathcal{N} = \{(x,y) \in \mathbb{N}_0^2 : y \leq x\}$$

which is a well-defined index set.

A common property of Volterra series is that they converge. We can observe this from the fact that $\forall m \in \mathbb{N}_2$, $L^*(t,x,y)$ is equal to

$$\int_{M^{m-1}} \text{dvol}_g^{m-1} (z_1, \ldots, z_{m-1}) \int_{\Delta_{m-1}^{(t)}} d(s_1, \ldots, s_{m-1}) L(t-s_1-\ldots-s_{m-1},x,z_{m-1}) \ldots L(s_1, y)$$

where $\Delta_{m-1}^{(t)}$ is the simplex defined by $\{0 \leq s_1 \leq s_1 + s_2 \leq \ldots \leq s_1 + \ldots + s_{m-1} \leq t\}$. As the volume of $\Delta_{m-1}^{(t)}$ is $\frac{(m-1)!}{(m-1)!}$, the factorial factor $\frac{(m-1)!}{(m-1)!}$, ultimately forces strong convergence as $m \to \infty$. See [BGV04, Section 2.4], [MV13, Section 3.2], and [Mel18] for more details and estimates.

Consequently, we obtain $H \in \Psi_{e^{-2-2N, \mathcal{N}}, E_{H}}$. Because of the identity $(\partial_t - \Delta_x) \left( H^{(2)} \ast (R^{(2)})^{*N} \right) = (R^{(2)})^{*N} - (R^{(2)})^{*(N+1)}$, we conclude

$$(\partial_t - \Delta_x) H = 0$$

Let us check that $H$ is the true Hodge-Neumann heat kernel.

- The index set $E_{lf}$ satisfies the Neumann boundary condition.
- For any $u \in L^2 (M; \Lambda^k M)$:

$$H(t)u(x) := \int_{M} H(t,x,y) u(y) \text{dvol}_g y \in C^\infty ((0, \infty), \Omega^k_{\text{hom},N})$$

and satisfies $(\partial_t - \Delta_x) (H(t)u(x)) = 0$ on $\{t > 0\}$. In particular, $H(t) \in \text{End} (L^2)$ for all $t > 0$ and

$$\partial_t \left( \|H(t)u\|_{L^2}^2 \right) \leq 0$$  \hspace{1cm} (36)

because the Neumann Laplacian $\Delta_N$ is self-adjoint and dissipative.

- We have $N^{-1}_{\text{nd}} (H) = N^{-1}_{\text{nd}} (H^{(1)})$, therefore $\lim_{t \to 0} H(t,x,y) = \delta_y (x)$. For any $u \in \Omega^k_{\text{hom}} (M) : H(t)u \overset{L^2}{\underset{t \to 0}{\to}} u$, which, along with (36), implies $\|H(t)u\|_{L^2} \leq \|u\|_{L^2}$. By density, we conclude the same for $u \in L^2 \left( M; \Lambda^k M \right)$. Recall that $e^{t\Delta_N}$ is the heat semigroup defined by functional analysis. For any $u \in L^2 \left( M; \Lambda^k M \right)$, $U(t) := H(t)u - e^{t\Delta_N}u$ is a $C^0_t L^2_x$ solution of

$$\begin{cases} 
(\partial_t - \Delta_x) U(t,x) = 0 & \forall t > 0 \\
U(t) \overset{L^2}{\underset{t \to 0}{\to}} 0
\end{cases}$$

By an energy argument just like (36), we must have $U(t) = 0$ for all $t$. Then, $H(t) = e^{t\Delta_N}$.

So $H$ is the true heat kernel, which must be smooth on $(0, \infty) \times M \times M$ by standard parabolic theory. Another way to see this is that the heat kernel must be symmetric, therefore smoothness in $x$ implies smoothness in $y$. Either way, because we have smoothness, there are no log terms on $lf$, and we conclude $H \in \Psi_{e^{-2-2N, \mathcal{N}}, E_{H}}$. 

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B.3 Relevant properties

We extract some key properties from Theorem 41 that we need for this paper, and write them in a language more familiar with analysts.

1. (off-diagonal decay) For any multi-index $\gamma$ and $x \neq y$,

$$D^\gamma_{t,x,y}H(t,x,y) = O(t^\infty)$$ (37)

as $t \downarrow 0$, locally uniform in $(x,y) \notin \Delta(M)$.

2. (interior blow-up) For $x \in \text{int}(M)$, locally in projective coordinates $(\tau,x,\zeta) = \left(\sqrt{t}, x, \frac{x-y}{\sqrt{t}}\right)$, with $\tilde{H}$ being the pullback of $t^{\frac{n}{2}}H$ in these coordinates, we have

(a) $\tilde{H}$ smooth in $\tau,x,\zeta$, up to $\{\tau = 0\}$.

(b) (rapid decay) For any multi-index $\gamma$ and bounded $\tau$:

$$D^\gamma_{\tau,x,\zeta} \tilde{H}(\tau,x,\zeta) = O\left(\langle \zeta \rangle^{-\infty}\right)$$ (38)

Remark 44. Both (37) and (38) come from the empty index set at $t_f$. We also refer to [Kot16, Section 2.3.3] for an explanation of (38).

There are more specific properties from Theorem 41, which we do not currently need.

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