SDES WITH CRITICAL TIME DEPENDENT DRIFTS: STRONG SOLUTIONS

MICHAEL RÖCKNER AND GUOHUAN ZHAO

ABSTRACT. This paper is a continuation of [RZ20]. Based on a compactness criterion for random fields in Wiener-Sobolev spaces, in this paper, we prove the unique strong solvability of time-inhomogeneous stochastic differential equations with drift coefficients in critical Lebesgue spaces, which gives an affirmative answer to a longstanding open problem. As an application, we also prove a regularity criterion for solutions of a stochastic system proposed by Constantin and Iyer (Comm. Pure. Appl. Math. 61(3): 330–345, 2008), which is closely related to the Navier-Stokes equations.

Keywords: Ladyzhenskaya-Prodi-Serrin condition, Malliavin calculus, Wiener chaos decomposition, Kolmogorov equations, Navier-Stokes equations

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1. INTRODUCTION

Let \( W_t \) be a standard \( d \)-dimensional Brownian motion on a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbf{P})\) and let \( b \) be a vector field on \( \mathbb{R}^d \) satisfying the following critical Ladyzhenskaya-Prodi-Serrin (LPS) condition:

\[
b \in \mathbb{L}^{p_1}_{q_1}(T) := L^{q_1}([0,T]; L^{p_1}(\mathbb{R}^d)) \quad \text{with} \quad p_1, q_1 \in [2, \infty] \quad \text{and} \quad \frac{d}{p_1} + \frac{2}{q_1} = 1. \tag{1.1}
\]

Our primary goal is to solve the following longstanding open problem: does the stochastic differential equation (SDE)

\[
X^t_s = x + \int_s^t b(r, X^t_r) \, dr + W_t - W_s, \quad 0 \leq s \leq t \leq T, \quad x \in \mathbb{R}^d
\]

have a unique strong solution under condition (1.1)?

1.1. Main result. Our main result, which gives an affirmative answer to the above open problem, reads as follows. (The notation \( \mathbb{L}^p_q(S,T) \) appearing below is defined in (1.13))

**Theorem 1.1.** Let \( d \geq 3 \). Assume \( b \) satisfies one of following two conditions

(a) \( b \in C([0,T]; L^d) \);

(b) \( b \in \mathbb{L}^{p_1}_{q_1}(T) \) with \( p_1, q_1 \in (2, \infty) \) and \( \frac{d}{p_1} + \frac{2}{q_1} = 1 \).
Then \((1.2)\) admits a unique strong solution such that the following estimate is valid:

\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \int_0^T f(t, X_{s,t}^x) \, dt \right) \leq C \|f\|_{L_p^q(T)},
\]

(1.3)

where \(p, q \in (1, \infty)\) with \(\frac{d}{p} + \frac{2}{q} < 2\) and \(C\) is a constant independent with \(f\). Moreover, the random field \(\{X_{s,t}^x\}_{x \in \mathbb{R}^d, 0 \leq s < t \leq T}\) forms a weakly differentiable stochastic flow and it satisfies the following

1. For any \(p \in (\frac{d}{q-1}, d)\) in case (a) and \(p \in (\frac{p_1}{p_1-1}, p_1)\) in case (b),

\[
\sup_{0 \leq s \leq t \leq T} \int_{\mathbb{R}^d} (\mathbb{E}|\nabla X_{s,t}^x - 1|^{r})^p \, dx < \infty, \quad \text{for any } r \in [2, \infty).
\]

(1.4)

2. For all \(r \in (d, \infty)\), \(\beta \in (0, \frac{1}{2})\), \(R > 0\), \(x_i \in B_R\), \(0 \leq s_i \leq t_i \leq T\), \(i = 1, 2\)

\[
\mathbb{E} \left| X_{s_1,t_1}^{x_1} - X_{s_2,t_2}^{x_2} \right|^{r} \leq C \left( |x_1 - x_2|^{r-d} + |s_1 - s_2|^{\beta(r-d)} + |t_1 - t_2|^{\beta r} \right).
\]

(1.5)

**Remark 1.2.** Except for the case that \(\|b\|_{L_p^q(T)}\) is sufficiently small, our main approach of this paper does not work for the full endpoint case \(p_1 = d\) and \(q_1 = \infty\). However, in the later case the weak well-posedness was proved by Röckner-Zhao in [RZ20], provided that the divergence of \(b\) satisfies an integrability condition. We conjecture that the strong well-posedness of \((1.2)\) holds when \(b \in \mathbb{L}^d_{p_1}(T)\) and \(\text{div} \, b = 0\).

### 1.2. Motivation and Previous results.

The existence of stochastic flows associated with SDEs with singular drifts and their regularity properties have various applications. For instance, in [FGP10], using the stochastic characteristics corresponding to \((1.2)\), Flandoli-Gubinelli-Priola studied the existence and uniqueness for the stochastic transport equation in an \(L^\gamma\)-setting, provided that the drift \(b\) is \(\alpha\)-Hölder continuous uniformly in \(t\) and the divergence of \(b\) satisfies some integrability condition. Later, stochastic continuity equations were also considered in [NO15] when \(b\) is divergence free and it satisfies the supercritical LPS condition

\[
b \in \mathbb{L}^{p_1}_{q_1}(T) \text{ with } p_1, q_1 \in (2, \infty) \text{ and } \frac{d}{p_1} + \frac{2}{q_1} < 1.
\]

(1.6)

The same SPDEs were also investigated by Fedrizzi-Flandoli in [FF13], Mohammed-Nilssen-Proske in [MNP15] and Beck-Flandoli-Gubinelli-Maurelli in [BFGM19] under different settings (see also the reference therein).

Our work is also motivated by the deep connection between singular SDEs and Navier-Stokes equations. The velocity field \(u\) of an incompressible fluid not subject to an external force in \(\mathbb{R}^d\) satisfies the Navier-Stokes equation

\[
\partial_t u - \frac{1}{2} \Delta u + (\nabla u)u + \nabla P = 0 \quad \text{in } [0, T] \times \mathbb{R}^d,
\]

(1.7a)

\[
\text{div} \, u = 0,
\]

(1.7b)

\[
u(0) = \varphi.
\]

(1.7c)

The mathematical studies of Navier-Stokes equations have a long history. In [Ler34], Leray considered \((1.7a)-(1.7c)\) for the initial data \(\varphi \in L^2\). He proved that there exists a global in time Leray-Hopf weak solution \(u \in \mathbb{L}^\infty_{2}(T)\) with \(\nabla u \in \mathbb{L}^2_{2}(T)\). However, to date, the problem of smoothness
of Leray-Hopf weak solutions for the 3D Navier-Stokes equations remains open. Studies by Prodi [Pro59], Serrin [Ser62] and Ladyzhenskaya [Lad67] found that the interior smoothness of Leray-Hopf weak solutions to (1.7a)-(1.7b) is guaranteed, provided that $u \in L^{p_1,q_1}_1(T)$, for $p_1 \in (d,\infty)$ and $d/p_1 + 2/q_1 \leq 1$ (see also [FJR72] and [Gig86]). These conditional regularity results and their generalizations have culminated with the work of Escauriaza-Seregin-Šverák [ESŠ03] for $d = 3$ and then Dong-Du [DD09] for $d \geq 3$. On the other hand, in the corresponding Lagrangian description, a fluid particle motion is described by the SDE

$$dX^x_t = u(t,X^x_t) dt + dW_t, \quad x \in \mathbb{R}^d.$$  

When $u$ is smooth, Constantin-Iyer [CI08] presented an elegant stochastic representation for the solutions to the Navier-Stokes equation, namely

$$u(t,x) = \mathbf{P} \mathbb{E}\left[\nabla^\top (X^x_t)^{-1} \varphi \left((X^x_t)^{-1}\right)\right],$$

where $\mathbf{P}$ is the Leray projection and $(X^x_t)^{-1}$ is the inverse stochastic flow of (1.8a). Conversely, if $u$ is smooth and $(u,X)$ solves the stochastic system (1.8a)-(1.8b), then $u$ also solves (1.7a)-(1.7c). From then on, some researchers started to study (1.7a)-(1.7c) via investigating the corresponding stochastic Lagrangian paths, see [Rez14], [Rez16], [Zha10] and [Zha16], etc. Since the problems of the regularity of solutions to the 3D Navier-Stokes equations are very challenging, two natural questions arise: (1) If the drift term is irregular, when does (1.8a) (or (1.2)) admit a weakly differentiable stochastic flow so that the right hand side of (1.8b) can be defined? (2) Can one also obtain some conditional regularity results for the stochastic system (1.8a)-(1.8b)?

Our Theorem 1.1 shows that (1.8a) has a weakly differentiable stochastic flow if the drifts satisfy the critical LPS conditions. For the second question above, to simplify our presentation, as in [Zha10], in this paper, we study the backward stochastic system

$$\begin{align*}
X^x_{t,s} &= x + \int_s^t u(r,X^x_{r,s}) dr + (W_s - W_t), \quad -T \leq t \leq s \leq 0 \\
u(t,x) &= \mathbf{P} \mathbb{E}\left[\nabla^\top X^x_{t,0} \varphi \left(X^x_{t,0}\right)\right], \quad -T \leq t \leq 0
\end{align*}$$  

(1.9)
corresponding to the backward Navier-Stokes equation

$$\partial_t u + \frac{1}{2} \Delta u + (\nabla u)u + \nabla P = 0, \quad \text{div} u = 0, \quad u(0) = \varphi$$

(1.10)

instead of the forward one (1.8a)-(1.8b). With the help of our new estimate (1.4), we give a regularity criterion for solutions to (1.9) in Theorem 6.1 below, which can be regarded as an analogue of Serrin’s regularity criterion for solutions to the 3D Navier-Stokes equations.

We close this subsection by mentioning some previous work about strong solutions to non degenerate SDEs with singular drifts. The study of strong well-posedness of non degenerate Itô equations with bounded drift coefficients dates back to [Zvo74] and [Ver80]. In [KR05], Krylov-Röckner obtained the existence and uniqueness of strong solutions to (1.2), when $b$ satisfies the subcritical LPS condition. After that a number of papers were devoted to generalize the strong well-posedness result,
as well as the following gradient estimate for $X$:

$$
sup_{x \in \mathbb{R}^d} E \sup_{t \in [0,T]} |\nabla X_{s,t}^x|' < \infty, \quad \forall r \geq 1. \tag{1.11}
$$

The reader is referred to [FF11], [LT17], [Rez14], [XXZZ20], [Zha05], [Zha11], [Zha16] and the reference therein for more details. We also point out that when $u$ is a Leray-Hopf solution to the Navier-Stokes equation, in [Zh19], the second named author also showed that (1.8a) also admits a unique almost everywhere stochastic flow, but the weak differentiability of the flow remained open.

To the best of our knowledge, the strong solvability under the critical condition (1.1) was first touched by Beck-Flandoli-Gubinelli-Maurelli in [BFGM19], where they proved the pathwise uniqueness to SDE (1.2) in a certain class if the initial datum has a diffuse law. Recently, if $b$ belongs to the Orlicz-critical space $L^{d_1}([0,T];L^{p_1}) \subset L^{d_1}_{-q_1}(T)$ for some $p_1,q_1 \in (2,\infty)$ with $d/p_1 + 2/q_1 = 1$, by Zvonkin’s transformation (cf. [Zvo74]), Nam [Nam20] showed the existence and uniqueness of strong solutions for SDE (1.2). The key step in using Zvonkin type of change of variables is to construct a homeomorphism by solving the Kolmogorov equation. If, however, $b$ only satisfies the critical LPS condition (1.1), this strategy seems impossible to implement. Very recently, Krylov [Kry20c] proved the strong well-posedness of (1.2) for the case that $b(t,x) = b(x) \in L^d(\mathbb{R}^d)$ with $d \geq 3$, which is a significant progress on this topic. His approach is based on his earlier work with Veretennikov [VK76] about the Wiener chaos expansion for strong solutions of (1.2), and also some new estimates obtained in [Kry20a] and [Kry20b]. It may be also possible to follow the same procedure as in [Kry20c] to study the time-inhomogeneous case, but one encounters a lot of difficulties due to the fact that there is no good PDE theory for equations with such kind of first order terms so far. In this paper, we use a very different approach from that in [Kry20c], which will be explained briefly in the next subsection.

1.3. Approach and Structure. The approach in this article is probabilistic, employing ideas from the Malliavin calculus coupled with some estimates for parabolic equations. In [RZ20], we obtain weak well-posedness of (1.2) under a slightly more general condition. So, to get the strong well-posedness, one only needs to show the strong existence due to a fundamental result of Cherny [Che02]. Our approach for proving strong existence is quite straightforward. Let $\{b_k\}$ be a smooth approximating sequence of the drift $b$ in $L^{d_1}_{q_1}(T)$ and $X_{s,t}^x(k)$ be the unique strong solution to (1.2) with $b$ replaced by $b_k$. The main effort of the present work is to show that $X_{s,t}^x(k)$ converges to a random field $X_{s,t}^x$, which is a strong solution to equation (1.2). A key ingredient for the convergence of $X_{s,t}^x(k)$ is the fact that for each $s,t \in [0,T]$ and $R > 0$, the sequence $\{X_{s,t}^x(k)\}$ is compact in $L^2(B_R \times \Omega)$. The proof for this assertion is based on a compactness criterion for $L^2$ random fields in Wiener spaces (see Lemma 3.1 below or [BS04]) and the following crucial estimate: for any $\alpha_i \in \{1,2,\cdots,d\}(i \in \mathbb{N}_+)$, $n \in \mathbb{N}_+$ and some $p > 1$,

$$
E \left\| \frac{d}{dt} \prod_{i=1}^n \partial_{\alpha_i} f_i(t_i,X_{s,t}^x(k)) \right\|_{L^p_t} \leq C_{n+1} \prod_{i=1}^n \|f_i\|_{L^p_{q_1}(s,t)}, \tag{1.12}
$$

where $C$ does not depend on $k$ and $\alpha_i$ (see Lemma 4.2 below for the precise statement).
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In fact, the framework mentioned above for proving the strong existence of SDEs with bounded drifts has already been used in [MBP10], [MPMBN+13] and [MNP15]. However, the main techniques in our paper are essentially different in compare with the previous literature. For example, in [MNP15], using Girsanov’s transformation, the main ingredient for the proof of the strong existence result and the gradient estimate for \( X \) was reduced to the following estimate:

\[
\left\| E \int_{0 \leq t_1 \leq \cdots \leq t_n \leq T} \prod_{i=1}^{n} \partial_{t_i} f_i(t_i, x + W_t) \, dt_1 \cdots dt_n \right\|_{L_T^\infty} \leq C^n t^\kappa(n!)^{-\frac{1}{q}} \prod_{i=1}^{n} \|f_i\|_{\infty}.
\]

Such a bound was first obtained by Davie in [Dav07] (cf. [Dav07, Proposition 2.2] and [Sha16, Proposition 2.1]) by proving a bound for certain block integrals. Later, Rezakhanlou [Rez14] also showed that

\[
\left\| E \int_{0 \leq t_1 \leq \cdots \leq t_n \leq T} \prod_{i=1}^{n} \partial_{t_i} f_i(t_i, x + W_t) \, dt_1 \cdots dt_n \right\|_{L_T^\infty} \leq C^n t^\kappa(n!)^{-\frac{1}{q}} \prod_{i=1}^{n} \|f_i\|_{L_T^q(t)},
\]

provided that \( \kappa := 1 - \frac{d}{p} - \frac{2}{q} > 0 \). However, when \( \kappa = 0 \), one can not expect to have bounds that are uniform in \( x \), and the approach used in [MNP15] and [Rez14] seems very hard, if not impossible to deal with the critical case (\( \kappa = 0 \)). To overcome these essential difficulties, due to the fact that we are in the critical case, in this paper, we reduce the desired bounded \( (1.12) \) to a uniform in time \( L^p \)-bound on the solution to a certain parabolic equation with critical drift and a distributional valued inhomogeneous term (see the discussion before Lemma 4.2 below). To obtain such uniform bound, we investigate the PDE mentioned above in Sobolev spaces with mixed norms (see Theorem 2.3 and 2.6) with the aid of some parabolic versions of Sobolev and Morrey inequalities in mixed norm spaces, which are proved by using Sobolevskii Mixed Derivative Theorem.

The rest of this paper is organized as following: In the rest of this section, we list some notations that will be used in this paper frequently. In Section 2, we study Kolmogorov equations with inhomogeneous terms in Sobolev spaces of negative order. In Section 3, give a compactness criterion for \( L^2 \) random fields in Wiener spaces. In Section 4, we derive some crucial uniform estimates for the solutions to certain approximating SDEs. The proof of the main result is presented in Section 5. In Section 6, we apply our main result to prove a regularity criterion for solutions of a stochastic system, which is closely related to the Navier-Stokes equations.

1.4. Notations. We close this section by mentioning some notational conventions used throughout this paper:

- \( \mathbb{N} := \{0, 1, 2, \cdots \} \), \( \mathbb{N}_+ := \{1, 2, \cdots \} \).
- The transpose of a matrix \( A \) is denoted by \( A^\top \).
- For a differentiable map \( X : \mathbb{R}^d \ni x \mapsto (X^1(x), \cdots, X^d(x))^\top \in \mathbb{R}^d \), the matrix \( \nabla X(x) \) is defined by

\[
\nabla X(x) = \\
\begin{pmatrix}
\partial_1 X^1(x) & \partial_2 X^1(x) & \cdots & \partial_d X^1(x) \\
\partial_1 X^2(x) & \partial_2 X^2(x) & \cdots & \partial_d X^2(x) \\
\vdots & \vdots & \ddots & \vdots \\
\partial_1 X^{d_1}(x) & \partial_2 X^{d_1}(x) & \cdots & \partial_d X^{d_1}(x)
\end{pmatrix}.
\]
Given $S, T \in [-\infty, \infty]$, set
\[
\Delta_n(S, T) := \{(t_1, \cdots, t_n) \in \mathbb{R}^n : S \leq t_1 \leq \cdots \leq t_n \leq T\}, \quad \Delta_n(T) := \Delta_n(0, T).
\]
Assume that for each $i \in \{1, 2\}$, $(X_i, \Sigma_i, \mu_i)$ is a measure space. Suppose that $f : X_1 \times X_2 \rightarrow \mathbb{R}$, define
\[
\|f\|_{L_p^1(\mu_1)L_p^2(\mu_2)} := \left[ \int_{X_1} \left( \int_{X_2} |f(x_1, x_2)|^{p_2} \mu_2(dx_2) \right)^{1/p_2} \mu_1(dx_1) \right]^{1/p_1}.
\]
For each $p, q \in [1, \infty]$, the space $L^q([S, T]; L^p(\mathbb{R}^d))$ is denoted by $L^p_q(S, T)$. For any $p, q \in (1, \infty), s \in \mathbb{R}$, define $\mathbb{H}^{s,p}_q(S, T) = L^q([S, T]; H^s(\mathbb{R}^d))$, where $H^{s,p} = (1 - \Delta)^{-s/2}L^p$ is the Bessel potential space.
Throughout this paper, we fix a cutoff function
\[
\chi \in C_c^\infty(\mathbb{R}^d; [0, 1]) \text{ with } \chi_{B_1} = 1 \text{ and } \chi_{B_2} = 0.
\]
For $r > 0$ and $x \in \mathbb{R}^d$, let $\chi^r(x) := \chi\left(\frac{x - z}{r}\right)$. For any $p, q \in [1, \infty]$, define
\[
\widetilde{L}^p := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^d) : \|f\|_{L^p}^p < \infty \right\}
\]
and
\[
\widetilde{L}_q^p(S, T) := \left\{ f \in L^q([S, T]; L^p_{\text{loc}}(\mathbb{R}^d)) : \|f\|_{L^q_{\text{loc}}(S, T)}^q \leq \sup_{z \in \mathbb{R}^d} \|f\chi^r\|_{L^q_{\text{loc}}(S, T)} < \infty \right\}.
\]
The localized Bessel potential space is defined as follows:
\[
\mathbb{H}^{s,p}_q(S, T) := \left\{ f \in L^q([S, T]; H^s_{\text{loc}}(\mathbb{R}^d)) : \|f\|_{\mathbb{H}^{s,p}_q(S, T)} := \sup_{z \in \mathbb{R}^d} \|f\chi^r\|_{\mathbb{H}^{s,p}_q(S, T)} < \infty \right\}.
\]
For simplicity, we set
\[
\mathbb{L}^p_q(T) := \mathbb{L}_q^p(0, T), \quad \mathbb{H}^{s,p}_q(T) := \mathbb{H}^{s,p}_q(0, T), \quad \mathbb{H}^{s,p}_q := \mathbb{L}^q(\mathbb{R}; H^s)\quad \text{and}
\]
\[
\mathbb{H}^{s,p}_q(T) = \mathbb{H}^{s,p}_q(0, T), \quad \mathbb{H}^{s,p}_q := \mathbb{H}^{s,p}_q(0, T).
\]
2. SOME AUXILIARY ANALYTIC RESULTS

In this section, we study the Kolmogorov equations with inhomogeneous terms in Sobolev spaces of negative order. These analytic results, which are of their own interest, will play a crucial role in proofs for the main results.

The following conclusions are variants of Theorem 1.1 and 1.2 in [Kry01].

**Lemma 2.1.** Let $p, q \in (1, \infty)$ and $\alpha \in \mathbb{R}$.

1. Assume $\lambda > 0$, $\mu \geq 0$. For each $u \in L^q(\mathbb{R}; H^\alpha + 2,p) \cap H^{1,q}(\mathbb{R}; H^\alpha)$,
\[
\|\partial_t u\|_{\mathbb{H}^{\alpha,p}_q} + \lambda \|\nabla^2 u\|_{\mathbb{H}^{\alpha,p}_q} + \mu \|u\|_{\mathbb{H}^{\alpha,p}_q} \leq C \|\Delta \lambda - \lambda \Delta + \mu u\|_{\mathbb{H}^{\alpha,p}_q},
\]
where $C$ only depends on $d, p, q$. 

(2) Assume that \( f \in H_q^{\alpha,p}(T) \), then the following heat equation admits a unique solution in \( H_q^{\alpha+2,p}(T) \):

\[
\partial_t u - \frac{1}{2} \Delta u = f \text{ in } (0,T) \times \mathbb{R}^d, \quad u(0) = 0.
\]

Moreover,

\[
\| \partial_t u \|_{H_q^{\alpha,p}(T)} + \| u \|_{H_q^{\alpha+2,p}(T)} \leq C_1 \| f \|_{H_q^{\alpha,p}(T)},
\]

where \( C_1 \) only depends on \( d, p, q, T \).

Consider the following Kolmogorov equation associated with \( (1.2) \):

\[
\partial_t u = \frac{1}{2} \Delta u + b \cdot \nabla u + f, \quad u(0) = 0.
\]

Throughout this paper, we fix a smooth function \( \rho \in C_c^\infty(\mathbb{R}^d) \) satisfying \( \rho \geq 0 \) and \( \int \rho = 1 \), and set \( \rho_m(\cdot) := m\rho(m \cdot) \).

2.1. Case (a): \( b \in C([0,T];L^d(\mathbb{R}^d)) \).

For any \( f \in L_d^q(T) \), define

\[
K_f(m) := \sup_{t \in [0,T]} \| f(t) - f(t) \ast_x \rho_m \|_{L^d}.
\]

**Proposition 2.2.** Suppose that \( f \in C([0,T];L^d(\mathbb{R}^d)) \), then \( K_f(m) \to 0 \) as \( m \to \infty \).

**Proof.** Since the map \( f : [0,T] \to L^d \) is uniformly continuous, for each \( \varepsilon > 0 \) there is a constant \( \delta > 0 \) such that

\[
\sup_{t_1,t_2 \in [0,T]; |t_1 - t_2| \leq \delta} \| f(t_1) - f(t_2) \|_{L^d} < \varepsilon / 2.
\]

Assume that \( k = \{0, 1, 2, \cdots, [T/\delta]\} \) and \( t \in [k \delta, (k + 1) \delta \wedge T] \), then

\[
\limsup_{m \to \infty} \| f(t) - f_m(t) \|_{L^d} \leq \| f(t) - f(k \delta) \|_{L^d} + \limsup_{m \to \infty} \| f(k \delta) - f_m(k \delta) \|_{L^d} + \limsup_{m \to \infty} \| f(k \delta) - f(t) \ast_x \rho_m \|_{L^d}
\]

\[
\leq 2 \| f(t) - f(k \delta) \|_{L^d} \leq 2 \sup_{t_1,t_2 \in [0,T]; |t_1 - t_2| \leq \delta} \| f(t_1) - f(t_2) \|_{L^d} < \varepsilon.
\]

Thus, \( \lim_{m \to \infty} K_f(m) = 0 \). \( \square \)

The following theorem will plays a crucial role in the proof of the first case of our main result.

**Theorem 2.3.** Let \( d \geq 3 \), \( \alpha \in \{0,-1\} \) and \( \{a(m)\}_{m \in \mathbb{N}_+} \) be a sequence converging to zero. Assume \( b \in L_d^q(T) \) and \( K_b(m) \leq a(m) \). Suppose that \( p \in (1,d) \) and \( q \in (1,\infty) \) if \( \alpha = 0 \), or \( p \in (d/(d-1),d) \) and \( q \in (1,\infty) \) if \( \alpha = -1 \). Then for any \( f \in H_q^{\alpha,p}(T) \), equation \( (2.3) \) admits a unique solution in \( H_q^{\alpha+2,p}(T) \). Moreover,

\[
\sup_{t \in [0,T]} t^{-1} \| u \|_{H_q^{\alpha,p}(T)} + \| \partial_t u \|_{H_q^{\alpha,p}(T)} + \| u \|_{H_q^{\alpha+2,p}(T)} \leq C_2 \| f \|_{H_q^{\alpha,p}(T)},
\]

where \( C_2 \) only depends on \( d, p, q, T, \| b \|_{L_d^q(T)}, \{a(m)\} \), and is increasing in \( T \).
Proof. Below we only give the proof for the case that \( \alpha = -1 \) (the case \( \alpha = 0 \) is simpler). To prove the desired result, it suffices to show (2.5) assuming that the solution already exists, since the method of continuity is applicable.

Let \( b_m = b * \rho_m \) and \( \tilde{b}_m = b - b_m \). By Sobolev embedding and Hölder’s inequality, we have

\[
\| \tilde{b}_m \cdot \nabla u \|_{\mathbb{H}^{-1,p}(t)} \leq C_3 \| \tilde{b}_m \cdot \nabla u \|_{L^1(T)} \leq C_3 \| \tilde{b}_m \|_{L^d(T)} \| \nabla u \|_{L^q(T)}
\]

\[
\leq C_3 a(m) \| u \|_{\mathbb{H}^{1,p}(t)},
\]

where \( t \in [0, T] \) and \( C_3 \) only depends on \( d, p \).

Similarly,

\[
\| b_m \cdot \nabla u \|_{\mathbb{H}^{-1,p}(t)} \leq \| \text{div}(b_m u) \|_{\mathbb{H}^{-1,p}(t)} + \| \text{div} b_m \cdot u \|_{L^q(T)} \leq C \| b_m \|_{L^\infty(0,T)C^\gamma_p(T)} \| u \|_{L^q(T)}
\]

\[
\leq C \left( \| \nabla \rho_m \|_{d/d-1} + \| \nabla \rho_m \|_{d/d-1} \right) \| u \|_{L^q(T)} 
\]

\[
\leq C m^2 \| u \|_{L^q(T)},
\]

where \( t \in [0, T] \) and \( C \) only depends on \( d, p, \| b \|_{L^\infty(T)}, \{ a(m) \} \). Thanks to Lemma 2.1, for each \( t \in [0, T] \),

\[
\| \partial_t u \|_{\mathbb{H}^{-1,p}(t)} + \| u \|_{L^q(T)} \leq C_1 \left( \| b \cdot \nabla u \|_{\mathbb{H}^{-1,p}(t)} + \| f \|_{\mathbb{H}^{-1,p}(t)} \right)
\]

\[
\leq C_1 \left( C_3 a(m) \| u \|_{\mathbb{H}^{1,p}(t)} + C m^2 \| u \|_{L^q(T)} + \| f \|_{\mathbb{H}^{-1,p}(t)} \right).
\]

Letting \( m \) be large enough such that \( C_1 C_3 a(m) \leq 1/2 \) and using interpolation, we obtain

\[
I(t) := \| \partial_t u \|_{\mathbb{H}^{-1,p}(t)}^q + \| u \|_{L^q(T)}^q \leq C \left( \| u \|_{\mathbb{H}^{1,p}(t)}^q + \| f \|_{\mathbb{H}^{1,p}(t)}^q \right),
\]

where \( t \in [0, T] \) and \( C \) only depends on \( d, p, q, T, \| b \|_{L^\infty(T)}, \{ a(m) \} \). One the other hand, for any \( t \in (0, T) \), using Hölder’s inequality, we have

\[
\| u \|_{\mathbb{H}^{1,p}(t)}^q = \int_0^t \| u(\tau, \cdot) \|_{L^q(T)}^q \, d\tau = \int_0^t \int_0^\tau \int_0^\tau \| \partial_t u(\sigma, \cdot) \|_{L^q(T)}^q \, d\sigma \, d\tau \leq \int_0^q T^{q-1} \int_0^t I(\tau) \, d\tau \leq C \int_0^t I(\tau) \, d\tau
\]

and together with (2.8), we obtain

\[
I(t) \leq C \| f \|_{\mathbb{H}^{1,p}(T)}^q + C \int_0^q I(\tau) \, d\tau
\]

Gronwall’s inequality yields,

\[
\| \partial_t u \|_{\mathbb{H}^{1,p}(T)} + \| u \|_{L^q(T)} \leq C \| f \|_{\mathbb{H}^{1,p}(T)}.
\]

Noting that (2.9) also implies

\[
\sup_{t \in (0, T]} t^{-1} \| u \|_{\mathbb{H}^{1,p}(t)} \leq C(q) \| \partial_t u \|_{\mathbb{H}^{1,p}(t)}, \quad \forall t \in (0, T],
\]

where \( C(q) \) only depends on \( q \) and \( C \) only depends on \( d, p, q \).
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2.1. \( \sup_{t \in (0,T]} t^{-1} \| u \|_{\mathbb{H}^{-1,p}(t)} + \| \partial_t u \|_{\mathbb{H}^{-1,p}(T)} + \| u \|_{\mathbb{H}^{1,p}(T)} \leq C_2 \| f \|_{\mathbb{H}^{-1,p}(T)} \),

where \( C_2 \) only depends on \( d, p, q, T, \| b \|_{1,\mathcal{L}(T)}, \{ \alpha(m) \} \). So, we complete our proof. \( \square \)

2.2. Case (b): \( b \in L^p_{q_1}(T) \) with \( p_1, q_1 \in (2, \infty) \) and \( d/p_1 + 2/q_1 = 1 \).

In this case, to obtain a result similar to Theorem 2.3, we need to prove some parabolic Morrey and Sobolev inequalities. This can be achieved by using the Mixed Derivative Theorem, which goes back to the work of Sobolevskii (cf. [Sob77]).

Let \( X \) be a Banach space and let \( A : D(A) \to X \) be a closed, densely defined linear operator with dense range. Then \( A \) is called sectorial, if

\[
(0, \infty) \subseteq \rho(-A) \quad \text{and} \quad \| \lambda (\lambda + A)^{-1} \|_{X \to X} \leq C, \quad \lambda > 0,
\]

where \( \rho(-A) \) is the resolvent set of \( -A \). Set

\[
\Sigma_\phi := \{ z \in \mathbb{C} \setminus \{0\} : \arg z < \phi \}.
\]

We recall that

\[
\phi_A := \inf \left\{ \phi \in [0, \pi) : \Sigma_{\pi - \phi} \subseteq \rho(-A), \sup_{z \in \Sigma_{\pi - \phi}} \| z(z + A)^{-1} \|_{X \to X} < \infty \right\}
\]

is the spectral angle of \( A \). For each \( \theta \in (0, 1) \), define

\[
A^\theta x := \frac{\sin \theta \pi}{\pi} \int_0^\infty \lambda^{\theta - 1}(\lambda + A)^{-1}Ax \, d\lambda, \quad x \in D(A)
\]

and

\[
A^{-\theta} x := \frac{\sin \theta \pi}{\pi} \int_0^\infty \lambda^{-\theta}(\lambda + A)^{-1}x \, d\lambda, \quad x \in X.
\]

We need the following Sobolevskii Mixed Derivative Theorem (cf. [Sob77]).

**Lemma 2.4** (Mixed Derivative Theorem). Let \( A \) and \( B \) be two sectorial operators in a Banach space \( X \) with spectral angles \( \phi_A \) and \( \phi_B \), which are commutative and satisfy the parabolicity condition \( \phi_A + \phi_B < \pi \). Then the coercivity estimate

\[
\| Ax \|_X + \lambda \| Bx \|_X \leq M \| Ax + \lambda Bx \|_X, \quad \forall x \in D(A) \cap D(B), \lambda > 0
\]

implies that

\[
\left\| A^{(1-\theta)} B^{\theta} x \right\|_X \leq C \| Ax + Bx \|_X, \quad \forall x \in D(A) \cap D(B), \theta \in [0, 1],
\]

The following parabolic Sobolev and Morrey inequalities will be used frequently in this work.

**Lemma 2.5.** Let \( p, q \in (1, \infty), r \in (p, \infty), s \in (q, \infty) \) and \( \alpha \in \mathbb{R} \). Assume \( \partial_t u \in \mathbb{H}^{\alpha,p}_{q}(T), u \in \mathbb{H}^{\alpha+2,p}_{q}(T) \) and \( u(0) = 0 \).
(1) If \(1 < d/p + 2/q = d/r + 2/s + 1\), then
\[
\|u\|_{\mathcal{E}^{1,r}_q(T)} \leq C_4 \left( \|\partial_t u\|_{\mathcal{E}^{a,p}_q(T)} + \|u\|_{\mathcal{E}^{a+2,p}_q(T)} \right),
\]
where \(C_4\) only depends on \(d, p, q, r, s\).

(2) If \(2 < d/p + 2/q = d/r + 2/s + 2\), then
\[
\|u\|_{\mathcal{E}^{a,r}_q(T)} \leq C_5 \left( \|\partial_t u\|_{\mathcal{E}^{a,p}_q(T)} + \|u\|_{\mathcal{E}^{a+2,p}_q(T)} \right),
\]
where \(C_5\) only depends on \(d, p, q, r, s\).

(3) If \(0 \leq \theta < 1 - 1/q\), for any \(t_1, t_2 \in [0, T]\),
\[
\|u(t_1) - u(t_2)\|_{\mathcal{E}^{\theta+2,p}_q} \leq C_6 |t_1 - t_2|^{-1/q - \theta} \left( \|\partial_t u\|_{\mathcal{E}^{a,p}_q(T)} + \|u\|_{\mathcal{E}^{a+2,p}_q(T)} \right),
\]
where \(C_6\) only depends on \(d, p, q, \theta\).

**Proof.** By considering \((1 - \Delta)^{\alpha/2} u\) instead of \(u\), we see that without loss of generality we may assume \(\alpha = 0\).

Let \(X = L^q(\mathbb{R}; L^p(\mathbb{R}^d))\), \(A = 1 + \partial_t\) and \(B = 1 - \Delta\) in Lemma 2.4. It is well-known that
\[
\phi_A = \frac{\pi}{2} \text{ and } \phi_B = 0.
\]
Due to (2.1), for all \(\lambda > 0\) we have
\[
\|Au\|_X + \lambda \|Bu\|_X = \|u + \partial_t u\|_{L^q} + \lambda \|u - \Delta u\|_{L^q} \\
\leq C \left( \|\partial_t u\|_{L^q} + \lambda \|\nabla^2 u\|_{L^q} + (1 + \lambda) \|u\|_{L^q} \right) \\
\leq C \left( \|u + \partial_t u\| + \lambda (u - \Delta u) \right) \|_{L^q} = C \|Au + \lambda Bu\|_X,
\]
where \(C\) only depends on \(d, p, q\). Thanks to Lemma 2.4, we obtain
\[
\|A^{1-\theta} B^\theta u\|_{L^q} \leq C \|\partial_t u - \Delta u + 2u\|_{L^q} \leq C \left( \|\partial_t u\|_{L^q} + \|u\|_{L^q} \right),
\]
for all \(u \in H^{1,q}(\mathbb{R}; L^p(\mathbb{R}^d)) \cap L^q(\mathbb{R}; H^{2,p}(\mathbb{R}^d))\). For any \(\alpha \in (0, 1), q \in (1, \infty)\) and \(f \in L^q(\mathbb{R})\), we have
\[
\mathcal{F} \left( (1 - \partial_t^2)^{\alpha/2} (1 + \partial_t)^{-\alpha} f \right) = \frac{(1 + 4\pi^2 |\xi|^2)^{\alpha/2}}{(1 + i2\pi \xi)^{\alpha}} \mathcal{F}(f)(\xi) =: m(\xi) \mathcal{F}(f)(\xi),
\]
where \(\mathcal{F}(f)(\xi) := \int_{\mathbb{R}} e^{i2\pi \xi \xi} f(x) dx\) is the Fourier transformation of \(f\). Since \(|\xi|^k m^{(k)}(\xi) \leq C_k < \infty\), by Mikhlin’s multiplier theorem, the operator \((1 - \partial_t^2)^{\alpha/2} (1 + \partial_t)^{-\alpha}\) is bounded on \(L^q(\mathbb{R})\). Therefore,
\[
\|u\|_{H^{1-q,q}(\mathbb{R}; H^{2,p}(\mathbb{R}^d))} = \|(1 - \partial_t^2)^{1-q} (1 - \Delta)^q u\|_{L^q} \leq C \|A^{1-\theta} B^\theta u\|_{L^q} \leq C \left( \|\partial_t u\|_{L^q} + \|u\|_{L^q} \right), \\
\forall \theta \in [0, 1].
\]
If \( u \in H^2_q(T) \), \( \partial_t u \in L^q(T) \) and \( u(0,x) = 0 \), we extend \( u \) by

\[
\bar{u}(t,x) := \begin{cases} 
  u(t,x) & \text{if } t \in [0,T] \\
  -3u(2T-t,x) + 4u \left( \frac{3T}{2} - \frac{t}{2}, x \right) & \text{if } t \in [T, 2T] \\
  4u \left( \frac{3T}{2} - \frac{t}{2}, x \right) & \text{if } t \in [2T, 3T] \\
  0 & \text{otherwise.}
\end{cases}
\]

By the definition of \( \bar{u} \), one sees that

\[
\| \partial_t \bar{u} \|_{L^q} + \| \bar{u} \|_{H^2_q(T)} \leq C \left( \| \partial_t u \|_{L^q(T)} + \| u \|_{H^2_q(T)} \right). \quad (2.16)
\]

Letting \( \theta = \frac{1}{2} + \frac{d}{2p} - \frac{d}{2r} = 1 + \frac{1}{2} - \frac{1}{q} \in \left[ \frac{1}{2}, 1 \right] \), the Sobolev inequality and the above estimates imply

\[
\| u \|_{H^s(T)} \leq \| \bar{u} \|_{H^s(T)} \leq \| \bar{u} \|_{H^{1-th, s}(\mathbb{R}; L^2_q(T))} \leq C \left( \| \partial_t u \|_{L^q(T)} + \| u \|_{H^2_q(T)} \right). \quad (2.15)
\]

So, we complete our proof for (2.11), (2.12) can be proved similarly.

For (2.13), if \( \theta < 1 - 1/q \), by Morrey’s inequality, we have

\[
\sup_{t_1, t_2 \in [0,T]} \frac{\| u(t_1) - u(t_2) \|_{H^{1-th, r}(\mathbb{R}; H^2_q(T))}}{|t_1 - t_2|^{1-1/q-\theta}} \leq C \| \bar{u} \|_{H^{1-th, s}(\mathbb{R}; H^2_q(T))} \leq C_6 \left( \| \partial_t u \|_{L^q(T)} + \| u \|_{H^2_q(T)} \right).
\]

So, we complete our proof.

For any \( f \in L^1_q(T) \), set

\[
K'_f(m) := \| f - f1_{\{|f| \leq m\}} \|_{L^1_q(T)} \quad (2.17)
\]

and

\[
\omega_f(\delta) := \sup_{0 \leq S \leq T - \delta} \| f \|_{L^1_q(S, S+\delta)}.
\]

Obviously, \( K'_f(m) \to 0 \), as \( m \to \infty \) and \( \omega_f(\delta) \to 0 \), as \( \delta \to 0 \).

Next we give an analogue of Theorem 2.3, which is crucial in the proof of the second case of Theorem 1.1.

**Theorem 2.6.** Let \( d \geq 3 \), \( p_1, q_1 \in (2, \infty) \) with \( d/p_1 + 2/q_1 = 1 \), and \( \{a(m)\}_{m \in \mathbb{N}_+} \) be a sequence converging to zero. Assume that \( b \in H^2_q(T) \) and \( K'_f(m) \leq a(m) \),

1. if \( p \in (1, p_1) \) and \( q \in (1, q_1) \), then for any \( f \in L^q(T) \), equation (2.3) admits a unique solution \( u \) in \( H^2_q(T) \) and

\[
\| \partial_t u \|_{L^q(T)} + \| u \|_{H^2_q(T)} \leq C \| f \|_{L^q(T)}, \quad (2.19)
\]

where \( C \) only depends on \( d, p_1, q_1, p, q, T, \{a(m)\} \) and is increasing in \( T \).
To prove the desired result, we only need to prove (2.19), (2.20) and (2.21) assuming that the solution already exists, since the method of continuity is applicable.

(1) Let \( m \) be chosen so that \( \frac{p}{p+1} > 1, q = \frac{p}{q+1} > 1 \), and \( C \) only depends on \( d, p_1, q_1, q, p, q, T, \{a(m)\} \) and is increasing in \( T \).

**Proof.** To prove the desired result, we only need to prove (2.19), (2.20) and (2.21) assuming that the solution already exists, since the method of continuity is applicable.

(1) Let \( m := b1_{\{|b| \leq m\}} \cdot \) Rewrite (2.3) as
\[
\partial_t u - \frac{1}{2} \Delta u = f + b_m \cdot \nabla u + (b - b_m) \cdot \nabla u.
\]

Thanks to Lemma 2.1, for any \( t \in [0, T] \) we have
\[
\left\| \partial_t u \right\|_{L^p_t(\Omega)} + \left\| u \right\|_{L^\infty_t(\Omega)} \leq C_1 \left( \left\| f \right\|_{L^p_t(\Omega)} + m \left\| \nabla u \right\|_{L^q_t(\Omega)} + \left\| (b - b_m) \cdot \nabla u \right\|_{L^q_t(\Omega)} \right),
\]
where \( C_1 = C_1(d, p, q, T) \). Letting \( 1/r = 1/p - 1/p_1 \) and \( 1/s = 1/q - 1/q_1 \), by (2.11) we have
\[
\left\| (b - b_m) \cdot \nabla u \right\|_{L^q_t(\Omega)} \leq C_2 a(m) \left( \left\| \partial_t u \right\|_{L^q_t(\Omega)} + \left\| u \right\|_{L^\infty_t(\Omega)} \right).
\]

We choose \( m \) sufficiently large so that \( C_1 C_2 a(m) \leq 1/2 \). Thus,
\[
I(t) := \left\| \partial_t u \right\|_{L^q_t(\Omega)} + \left\| u \right\|_{L^\infty_t(\Omega)} \leq C \left( \left\| f \right\|_{L^p_t(\Omega)} + N \left\| \nabla u \right\|_{L^q_t(\Omega)} \right).
\]

Noting that
\[
\left\| u \right\|_{L^q_t(\Omega)} \leq \int_0^t \left\| u(\tau, \cdot) \right\|_{L^q} d\tau = \int_0^t \left\| \int_0^\tau \partial_t u(\sigma, \cdot) d\sigma \right\|_{L^q} d\tau \leq \int_0^t \tau^{q-1} \left\| \partial_t u \right\|_{L^q(\tau)} d\tau \leq C(T, q) \int_0^t I(\tau) d\tau,
\]
and using an interpolation inequality, we obtain
\[
\left\| \nabla u \right\|_{L^q_t(\Omega)} \leq \delta \left\| \nabla^2 u \right\|_{L^q_t(\Omega)} + C_\delta \left\| u \right\|_{L^q_t(\Omega)} \leq \delta I(t) + C_\delta \int_0^t I(\tau) d\tau, \quad (\forall \e > 0).
\]

Combing (2.22) and (2.24), we get
\[
I(t) \leq C_7 \delta m^d I(t) + C \left\| f \right\|_{L^q_t(\Omega)} + C_\delta m^d \int_0^t I(\tau) d\tau.
\]
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Letting \( \delta = \delta(m) \) be small enough so that \( C_7 \delta^m \leq 1/2 \), we obtain that for all \( t \in [0, T] \),

\[
I(t) \leq C \|f\|_{L_q^q(T)}^q + C \int_0^t I(\tau) \, d\tau.
\]

Gronwall’s inequality yields

\[
\|\partial_t u\|_{L_q^q(T)} + \|u\|_{H_{\alpha}^{1,p}(T)} \leq C T^{1/q}(T) \leq C \|f\|_{L_q^q(T)}.
\]  

(2.25)

(2.26)

(2.27)

Remark 2.7. Let the assumptions in Theorem 2.3 or Theorem 2.6 hold. Suppose that \( f \in \mathbb{H}^\alpha_{-1,p}(T) \) with \( \alpha \in \{0, -1\} \). Then all the conclusions therein still hold if \( \mathbb{H}_{-\infty} \) and \( \mathbb{L}_{-\infty} \) are replaced by \( \mathbb{H}_{-\infty}^\alpha \) and \( \mathbb{L}_{-\infty}^\alpha \), respectively (cf. [XXZZ20] or [RZ20]).

3. COMPACTNESS CRITERION FOR \( L^2 \) RANDOM FIELDS

In this section, we give a relative compactness criterion for the random fields on the Wiener-Sobolev space, which is essentially a consequence of [BS04, Theorem 1].

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Assume \( \{W_t\}_{t \in [0, T]} \) is a \( d \)-dimensional Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\) and \( \mathcal{F} = \sigma \{W_t : t \in [0, T]\} \). \( T = [0, T] \times \{1, 2, \ldots, d\} \), \( \mu \) is the product of the Lebesgue measure on \([0, T]\) times the uniform measure on \( \{1, 2, \ldots, d\} \). \( H := L^2(T; \mu) \) and the scalar product is

\[
\langle f, g \rangle := \sum_{i=1}^d \int_0^T f((t, i)) g((t, i)) \, dt.
\]
Let $I_m$ denote the multiple stochastic integral
\[ I_m(f_m) = m! \sum_{k_1, \ldots, k_m=1}^d \int_{0<t_1<\cdots<t_m<T} f_m((t_1, k_1), \ldots, (t_m, k_m)) \, dW_{t_1} \cdots dW_{t_m} \tag{3.1} \]
of $L^2(T^m)$ (the collection of all symmetric elements in $L^2(T^m)$). Let $\mathcal{H}_m$ denote the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables $\{H_n(I_1(h)) : h \in H = L^2(T, \mu)\}$, where $H_n$ is the $n$-th Hermite polynomial. The multiple integral $I_m$ is a map from $L^2(T^m)$ onto the Wiener chaos $\mathcal{H}_m$ and any $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ can be expanded into a series of multiple stochastic integrals: $F = \sum_{m=0}^\infty I_m(f_m)$, where $I_0(F) := \mathbb{E}F$. Let $\mathcal{S}_p$ denote the class of smooth random variables $F = f(W(h_1), \ldots, W(h_m))$ and $f \in C_0^\infty(\mathbb{R}^d)$. The Malliavin derivative of a smooth random variable $F$ is the stochastic process $t \mapsto D_tF$ defined by
\[ D_tF := \sum_{i=1}^m \partial_i f(W(h_1), \ldots, W(h_m)) h_i(t) \]
Let $\mathbb{D}^{1,2}$ be the closure of $\mathcal{S}_p$ with respect to the norm
\[ \|F\|_{\mathbb{D}^{1,2}}^2 := \mathbb{E}F^2 + \mathbb{E} \int_0^T |D_tF|^2 \, dt. \]
Assume now $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with smooth boundary and $F_n$ is a sequence of random fields in $L^2(\mathcal{O} \times \Omega)$. The following result is a variant of a compactness criteria for sequences in $L^2(\mathcal{O} \times \Omega)$ due to Bally and Saussereau [BS04].

**Lemma 3.1.** Assume $K > 0$ and that the sequence $\{F_n\}_{n \in \mathbb{N}} \subseteq L^2(\mathcal{O} \times \Omega)$ satisfies the following three conditions, for all $n \in \mathbb{N}$:
\[ \mathbb{E}\|F_n\|_{H^1_0(\mathcal{O})}^2 \leq K, \tag{A1} \]
\[ \mathbb{E} \int_\mathcal{O} \int_0^T |D_x F_n(x)|^2 \, ds \, dx \leq K, \tag{A2} \]
\[ \mathbb{E} \int_\mathcal{O} \int_0^T \int_0^T \frac{|D_x F_n(x) - D_x F_n(x')|^2}{|s-s'|^{1+2\beta}} \, ds' \, dx \leq K, \text{ for some } \beta > 0, \tag{A3} \]
then $\{F_n\}_{n \in \mathbb{N}}$ is relatively compact in $L^2(\mathcal{O} \times \Omega)$.

**Proof.** Since $\mathcal{O}$ is a bounded smooth domain, there exists $\{e_k\}_{k \in \mathbb{N}_+} \subseteq H^1_0(\mathcal{O}) \cap C^\infty(\mathcal{O})$ and a sequence $\{\lambda_k\}_{k \in \mathbb{N}_+}$ of positive real numbers with $\lambda_k \uparrow \infty (k \uparrow \infty)$, such that $\Delta e_k = -\lambda_k e_k$ and $\{e_k\}_{k \in \mathbb{N}_+}$ forms an orthonormal base of $L^2(\mathcal{O})$ (cf. [Eva10]). Moreover, $e_k/\sqrt{\lambda_k}$ forms a basis of $H^1_0(\mathcal{O})$ with norm $\|f\|_{H^1_0(\mathcal{O})} := \left( \int_\mathcal{O} |\nabla f|^2 \right)^{1/2}$. Set $\langle f, g \rangle := \int_\mathcal{O} fg$, then $F_n = \sum_{k=1}^\infty \langle F_n, e_k \rangle e_k$. Integration by parts
and \((A_1)\) yield,
\[
\left\| \sum_{k=K}^{\infty} \langle F_n, e_k \rangle e_k \right\|_{L^2(\Omega \times \Omega)}^2 = E \sum_{k=K}^{\infty} \langle F_n, e_k \rangle^2 = E \sum_{k=K}^{\infty} \lambda_k^{-1} \langle F_n, \Delta e_k \rangle^2
\]
\[
= E \sum_{k=K}^{\infty} \lambda_k^{-1} \langle \nabla F_n, \nabla e_k / \sqrt{\lambda_k} \rangle^2 \leq \lambda_K^{-1} E \sum_{k=K}^{\infty} \langle \nabla F_n, \nabla e_k / \sqrt{\lambda_k} \rangle^2
\]
\[
\leq \lambda_K^{-1} \left\| \nabla F_n \right\|_{L^2(\Omega \times \Omega)}^2 \leq C \lambda_K^{-1} \downarrow 0 \quad (K \uparrow \infty).
\]

Therefore, the relative compactness of the sequence \(\{F_n\}_{n \in \mathbb{N}}\) in \(L^2(\Omega \times \Omega)\) reduces to the relative compactness of the sequence \(\{\langle F_n, e_k \rangle\}_{n \in \mathbb{N}}\) in \(L^2(\Omega)\) for each \(k \in \mathbb{N}_+\). By \((A_1)\), we have
\[
E \langle F_n, e_k \rangle^2 \leq E \|F_n\|^2_{L^2(\Omega)} \leq K. \tag{3.2}
\]

\((A_2)\) and \((A_3)\) yield for all \(n \in \mathbb{N}\)
\[
E \left| D_s \langle F_n, e_k \rangle \right|^2 ds = E \int_0^T \left| D_s \int_0^T F_n(x) e_k(x) dx \right|^2 ds
\]
\[
\leq E \int_0^T \int_0^T |D_s F_n(x) - D_s' F_n(x)|^2 ds ds' \leq K. \tag{3.3}
\]

and
\[
E \int_0^T \int_0^T D_s \langle F_n, e_k \rangle - D_s' \langle F_n, e_k \rangle \left| \frac{\partial_s x}{|s-s'|^{1+2\beta}} \right|^2 ds ds' \leq K. \tag{3.4}
\]

By (3.2)-(3.4), Theorem 1 and Lemma 1 of [DPMN92] (with \(\alpha \in (0, \beta \wedge \frac{1}{2})\) and \(C = A_{\alpha}^{-1}\) therein), one sees that \(\{\langle F_n, e_k \rangle\}_{n \in \mathbb{N}}\) is compact in \(L^2(\Omega)\) for each \(k \in \mathbb{N}_+\). So, we complete our proof. \(\square\)

4. Estimates for the case of regular coefficients

Throughout this section, we assume \(b \in L^\infty([0, T]; C^2_b)\) and the unique strong solution to SDE (1.2) with \(s = 0\) is denoted by \(X^s_t\). Recall that \(K_f(m), K'_f(m)\) and \(\omega_f(m)\) are defined in (2.4), (2.17) and (2.18), respectively. The main purpose of this section is to prove

**Proposition 4.1.** Let \(d \geq 3, \{a(m)\}_{m \in \mathbb{N}_+}\) be a sequence converging to zero and \(\ell(\delta)\) be a monotonically increasing function on \((0, T)\) with \(\lim_{\delta \rightarrow 0} \ell(\delta) = 0\).

(a) Assume that \(b \in L^\infty([0, T]; C^2_b)\) and \(K_0(m) \leq a(m)\). Then for any \(r \geq 2, p \in \left(\frac{d}{d-1}, d\right)\) and \(\gamma \in (0, 1/2),\)
\[
\|\nabla X^s_t - 1\|_{L^p_t L^q_x} \leq C \gamma/2r, \quad \text{for all } 0 \leq t \leq T, \tag{4.1}
\]
\[
\|D_s X^s_t - 1\|_{L^p_t L^q_x} \leq C(t-s)^{\gamma/2r}, \quad \text{for a.e. } s \in [0, T] \text{ with } 0 \leq s \leq t \leq T \tag{4.2}
\]
and
\[
\|D_s X^s_t - D_{s'} X^s_t\|_{L^p_t L^q_x} \leq C|s-s'|^{\gamma/4r}, \quad \text{for a.e. } s, s' \in [0, T] \text{ with } 0 \leq s, s' \leq t \leq T, \tag{4.3}
\]
where \(C\) only depends on \(d, T, r, p, \gamma, \|b\|_{L^p_t}\) and \(\{a(m)\}\).
(b) Assume that $b \in L^q_0(T) \cap L^\infty([0,T];C^2_b)$ with $p_1, q_1 \in (2, \infty)$ and $d/p_1 + 2/q_1 = 1$, $K'_b(m) \leq a(m)$ and $\omega_b(\delta) \leq \ell(\delta)$. Then for any $r \geq 2$, a.e. $s, s' \in [0,T]$ with $0 \leq s, s' \leq t \leq T$, $p \in \left(\frac{p_1}{p_1-1}, p_1\right)$ and $\gamma \in (0, \frac{1}{2} - \frac{1}{q_1})$, the estimates (4.1)-(4.3) still hold, and the constant $C$ only depends on $d, p_1, q_1, T, r, p, \gamma, \{a(m)\}$ and $\ell(\delta)$.

The proof of Proposition 4.1 relies on the following lemma, which contains two key estimates of this paper.

**Lemma 4.2.** Let $d \geq 3, 0 \leq S_0 \leq S_1 \leq T$ and $\{a(m)\}_{m \in \mathbb{N}}$ be a sequence converging to zero.

(a) Suppose $b \in L^d_\infty(T)$ and $K_b(m) \leq a(m)$. Assume that $f_i \in L^\infty([0,T];C^2_b) \ (i \in \mathbb{N}_+)$

$$\sup_{i \in \mathbb{N}_+} \|f_i\|_{L^d(T)} \leq N \text{ and } \sup_{r \in [0,T]} K_{f_i}(m) \leq d_m.$$  

Then for any $p \in \left(\frac{d-n}{d-1}, d\right)$, $\gamma \in (0, \frac{1}{2})$, $\{\alpha_i\}_{i=1}^\infty \subseteq \{1, 2, \cdots d\}$ and all $n \in \mathbb{N}$

$$\left\| \mathbb{E} \int \cdots \int_{\Delta_n(S_0,S_1)} \prod_{i=1}^{n} \partial_{\alpha_i} f_i(t_i, X_{t_i}^s) \, dt_1 \cdots dt_n \right\|_{L^p_T} \leq C^n \left( m^2 N \sqrt{S_1 - S_0} + d_m \right)^{n-1} N(S_1 - S_0)^{\gamma},$$  

where $C$ only depends on $d, p, \gamma, T, \|b\|_{L^d(T)}$ and $\{a(m)\}$.

(b) Suppose $b \in L^q_{\infty}(T)$ with $p_1, q_1 \in (2, \infty)$ and $d/p_1 + 2/q_1 = 1$, and $K'_b(m) \leq a(m)$. Assume that $f_i \in L^p_\infty(T) \cap L^\infty([0,T];C^2_b) \ (i \in \mathbb{N}_+)$. Then for any $p \in \left(\frac{p_1}{p_1-1}, p_1\right)$, $\gamma \in (0, \frac{1}{2} - \frac{1}{q_1})$, $\{\alpha_i\}_{i=1}^\infty \subseteq \{1, 2, \cdots, d\}$ and all $n \in \mathbb{N}$

$$\left\| \mathbb{E} \int \cdots \int_{\Delta_n(S_0,S_1)} \prod_{i=1}^{n} \partial_{\alpha_i} f_i(t_i, X_{t_i}^s) \, dt_1 \cdots dt_n \right\|_{L^p_T} \leq C^{n+1} \prod_{i=1}^{n} \|f_i\|_{L^p_{q_1}(S_0,S_1)} (S_1 - S_0)^{\gamma},$$  

where $C$ only depends on $d, p_1, q_1, p, \gamma, T$ and $\{a(m)\}$.

**Proof.** For fixed $n \in \mathbb{N}_+$ and $\{\alpha_i\}_{i=1}^n \subseteq \{1, 2, \cdots, d\}$, we set $u_{n+1} = 1$ and for any $k \in \{1, 2, \cdots, n\}$, let $g_k := (\partial_{\alpha_i} f_k) u_{k+1}$ and $u_k \in \cap_{p,q \in (1, \infty)} H^2_{q_p} (S_1)$ be the unique function solving equation

$$\partial_t u_k + \frac{1}{2} \Delta u_k + b \cdot \nabla u_k + g_k = 0 \text{ in } (S_0, S_1) \times \mathbb{R}^d, \quad u_k(S_1) = 0 \quad (4.6)$$

(cf. [XXZZ20]). Then the generalized Itô formula yields

$$-u_k(t, X_t^s) = - \int_t^{S_1} g_k(s, X_s^s) \, ds + \int_t^{S_1} \nabla u_k(s, X_s^s) \, dW_s, \quad \forall t \in [0, S_1]$$

which implies

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{S_1} g_k(s, X_s^s) \, ds = u_k(t, X_t^s). \quad (4.7)$$
Recalling that and Case (a):

Therefore, for any \( U \) be the solution to the following PDE:

\[
\partial_t U = \frac{1}{2} \Delta U + B \cdot \nabla U + G \text{ in } (0, S_1) \times \mathbb{R}^d, \quad U(0) = 0,
\]

where

\[
B(t, x) = b(S_1 - t, x) \mathbf{1}_{[0, S_1 - S_0]}(t) + b(t + S_0 - S_1, x) \mathbf{1}_{[S_1 - S_0, S_1]}(t)
\]

and

\[
G(t, x) = g_1(S_1 - t, x) \mathbf{1}_{[0, S_1 - S_0]}(t).
\]

We note that \( u_1(S_1 - t) = U(t) \) for all \( t \in [0, S_1 - S_0] \) and that \( V(t) := U(t + (S_1 - S_0)) \) satisfies

\[
\partial_t V = \frac{1}{2} \Delta V + b \cdot \nabla V \text{ in } (0, S_0) \times \mathbb{R}^d, \quad V(0, x) = U(S_1 - S_0, x) = u_1(S_0, x).
\]

Therefore, for any \( p \in [1, \infty) \),

\[
\left( \int_{\mathbb{R}^d} \left| \mathbb{E} \int_{\Delta_{n-1}(S_0, S_1)} \prod_{i=1}^{n-1} \partial_{\alpha_i} f_i(t_i, X_{t_i}^x) \, dt_1 \cdots dt_{n-1} \right|^p \, dx \right)^{\frac{1}{p}} \overset{(4.8)}{=} \int_{\mathbb{R}^d} \left| \mathbb{E} u_1(S_0, X_{S_0}^x) \right|^p \, dx \overset{(4.8)}{=} \int_{\mathbb{R}^d} |V(S_0, x)|^p \, dx = \|U(S_1)\|^p.
\]

**Case (a):** \( b \in C([0, T]; \mathbb{R}^d) \). Set

\[
f_{k,m}(t) := f_k(t) \ast \rho_m, \quad f_{k,m} := f_k - f_{k,m}.
\]

Let

\[
p \in (d/(d - 1), d) \quad \text{and} \quad q = \gamma^{-1} \in (2, \infty).
\]

By the definitions of \( g_k \) and \( u_{k+1} \),

\[
\|g_k\|_{L_{q^{-1}}^q(S_0, S_1)} = \|\partial_{\alpha} f_{k+1}\|_{L_{q^{-1}}^q(S_0, S_1)} \leq \|\partial_{\alpha} f_{k,m}\|_{L_{q^{-1}}^q(S_0, S_1)} + \|\partial_{\alpha} \tilde{f}_{k,m}\|_{L_{q^{-1}}^q(S_0, S_1)} =: I_1 + I_2.
\]

Recalling that \( u_{k+1} \) solves (4.6) with \( k \) replaced by \( k + 1 \), using (2.5), we get

\[
(S_1 - S_0)^{-1} \|u_{k+1}\|_{L_{q^{-1}}^q(S_0, S_1)} + \|u_{k+1}\|_{L_{q^{-1}}^q(S_0, S_1)} \leq C \|g_{k+1}\|_{L_{q^{-1}}^q(S_0, S_1)}.
\]
An interpolation inequality yields
\[
\|u_{k+1}\|_{E^q_\gamma(S_0,S_1)} \leq C\|u_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)}^{1/2} \|u_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)}^{1/2}
\]
\[
\leq C\sqrt{S_1 - S_0}\|g_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)}.
\]
Thus, like the proof for (2.7), we have
\[
I_1 \leq C\|\partial_{a_x} f_{k,m} u_{k+1}\|_{E^q_\gamma(S_0,S_1)}
\leq C\|f_{k,m}\|_{L^\infty([0,T],C^q)} \|u_{k+1}\|_{L^p(S_0,S_1)}
\leq Cm^2 N\sqrt{S_1 - S_0}\|g_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)}
\]
One the other hand, recalling that \(p \in (d/(d-1),d)\), by the Sobolev embedding,
\[
I_2 \leq \|\partial_{a_x} (\tilde{f}_{k,m} u_{k+1})\|_{H^{1,q}_\gamma(S_0,S_1)} + \|\tilde{f}_{k,m} \partial_{a_x} u_{k+1}\|_{L^q(S_0,S_1)}
\leq C\|f_{k,m}\|_{\mathcal{L}_q(S_0,S_1)} \|u_{k+1}\|_{H^{1,q}_\gamma(T)} + C\|\tilde{f}_{k,m}\|_{L^q(S_0,S_1)} \|u_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)}
\leq Cn\|g_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)}
\]
Combing (4.12)-(4.14), we get
\[
\|g_k\|_{H^{1,q}_\gamma(S_0,S_1)} \leq C\left(m^2 N\sqrt{S_1 - S_0} + d_m\right) \|g_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)},
\]
where \(C\) only depends on \(d, p, \gamma, T, \|b\|_{\mathcal{L}_q(T)}\) and \(\{a(m)\}\). Recalling that \(G\) is defined in (4.10), by the above estimate we obtain
\[
\|G\|_{H^{1,q}_\gamma(S_1)} \leq \|G\|_{H^{1,q}_\gamma(S_0,S_1)}
\leq Cn\left(m^2 N\sqrt{S_1 - S_0} + d_m\right)^{-1} \|g_{k+1}\|_{H^{1,q}_\gamma(S_0,S_1)}
\leq Cn\left(m^2 N\sqrt{S_1 - S_0} + d_m\right)^{-1} N(S_1 - S_0)^\gamma.
\]
Thus,
\[
\|E \int \cdots \int_{\Delta_m(S_0,S_1)} \prod_{i=1}^n \partial_{a_x} f_i(t_i, X_i) \, dt_1 \cdots dt_n\|_{L^p_\gamma}
\leq C\|U(S_1)\|_p \leq C\left(\|\partial_{t_i} U\|_{H^{1,q}_\gamma(S_1)} + \|U\|_{H^{1,q}_\gamma(S_1)}\right)
\]
(taking \(\alpha = -1, \theta = 1/2\), and noticing \(q > 2\))
\[
\leq C\|G\|_{H^{1,q}_\gamma(S_1)} \leq Cn\left(m^2 N\sqrt{S_1 - S_0} + d_m\right)^{-1} N(S_1 - S_0)^\gamma,
\]
where \(C\) only depends on \(d, p, \gamma, \|b\|_{\mathcal{L}_q(T)}\) and \(\{a(m)\}\). So, we complete the proof for (4.4).

Case (b): \(b \in \mathcal{L}_{q_1}(T)\). Set
\[
p \in \left(\frac{p}{p_1}, p_1\right), \quad q = \frac{q_1}{1 + \gamma q_1} \in (2, q_1), \quad p' = \frac{p_1 p}{p_1 + p}, \quad q' = \frac{q_1 q}{q_1 + q}.
\]
Noting that \( p_1 \in (d, \infty), q_1 \in (2, \infty) \) and \( \gamma \in (0, 1/2 - 1/q_1) \), one sees that \( p \in (1, p_1), q \in (2, q_1) \) and \( p' \in (1, p), q' \in (1, q) \).

We claim that for each \( k \in \{n, n - 1, \cdots, 1\} \), we have \( g_k = g_k' + g_k'' \) and

\[
\|g_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} + \|g_k''\|_{L_{q'}^{p'}(S_0, S_1)} \leq C^{n-k+1} \prod_{l=k}^{n} \|f_l\|_{L_{q_l}^{p_l}(S_0, S_1)} (S_1 - S_0)^\gamma, \tag{4.16}
\]

where \( C \) does not depend on \( n \). By the definition of \( g_n \), we have

\[
\|g_n\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} \leq C \|f_n\|_{L_{q_1}^{p_1}(S_0, S_1)} (S_1 - S_0)^\gamma.
\]

Assume \( g_k = g_k' + g_k'' \) and that (4.16) holds for some \( k \in \{n, \cdots, 2\} \). Then \( u_k \) can be decomposed as \( u_k = u_k' + u_k'' \), where \( u_k' \) and \( u_k'' \) solve (4.6) with \( g_k \) replaced by \( g_k' \) and \( g_k'' \), respectively. By Theorem 2.6, one sees that \( u_k \) can be further decomposed as

\[
u_k' \), \( w_k' \) and \( u_k'' \) satisfy

\[
\|\partial_t v_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} + \|v_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} \leq C \|g_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)}, \tag{4.17}
\]

\[
\|\partial_s w_k'\|_{L_{q'}^{p'}(S_0, S_1)} + \|w_k'\|_{L_{q'}^{p'}(S_0, S_1)} \leq C \|g_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} \tag{4.18}
\]

and

\[
\|\partial_t u_k''\|_{L_{q'}^{p'}(S_0, S_1)} + \|u_k''\|_{L_{q'}^{p'}(S_0, S_1)} \leq C \|g_k''\|_{L_{q'}^{p'}(S_0, S_1)}. \tag{4.19}
\]

Let \( r = \frac{p_1 p}{p_1 - p} \) and \( s = \frac{1}{r} = \frac{q_1 q}{q_1 - q} \). Recalling that \( 2 < \frac{d}{p} + \frac{2}{q} = 1 + \frac{d}{p} + \frac{2}{q} = 2 + \frac{d}{r} + \frac{2}{s} \), due to (2.11), (2.12) and (4.17)-(4.19), we have

\[
\begin{aligned}
\|u_k\|_{L_r^r(S_0, S_1)} &\leq \|v_k'\|_{L_r^r(S_0, S_1)} + \|w_k'\|_{L_r^r(S_0, S_1)} + \|u_k''\|_{L_r^r(S_0, S_1)} \\
&\leq C \left( \|\partial_t v_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} + \|v_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} \right) + C \left( \|\partial_t w_k'\|_{L_{q'}^{p'}(S_0, S_1)} + \|w_k'\|_{L_{q'}^{p'}(S_0, S_1)} \right) \\
&\quad + C \left( \|\partial_t u_k''\|_{L_{q'}^{p'}(S_0, S_1)} + \|u_k''\|_{L_{q'}^{p'}(S_0, S_1)} \right) \\
&\leq C \left( \|g_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} + \|g_k''\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} \right) .
\end{aligned} \tag{4.20}
\]

and

\[
\|u_k\|_{\mathbb{E}_r^{1,p}(S_0, S_1)} \leq C \left( \|g_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} + \|g_k''\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} \right) . \tag{4.21}
\]

Set \( g_{k-1}' = \partial_{\alpha_{k-1}}(f_{k-1} u_k) \) and \( g_{k-1}'' = -f_{k-1} (\partial_{\alpha_{k-1}} u_k) \). By Hölder’s inequality, (4.20) and (4.21), we get

\[
\begin{aligned}
\|g_{k-1}'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} + \|g_{k-1}''\|_{L_{q'}^{p'}(S_0, S_1)} \\
\leq \|f_{k-1}\|_{L_{q_1}^{p_1}(S_0, S_1)} \left( \|u_k\|_{L_{r_1}^r(S_0, S_1)} + \|\nabla u_k\|_{L_{r_1}^r(S_0, S_1)} \right) \\
\leq C \|f_{k-1}\|_{L_{q_1}^{p_1}(S_0, S_1)} \left( \|g_k'\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} + \|g_k''\|_{\mathbb{E}_q^{1,p}(S_0, S_1)} \right) .
\end{aligned} \tag{4.20),(4.21}
\]
Similarly, taking

\[ q = 1, \alpha = 0, \theta = \frac{d}{2p_1} \]

in inequality (2.13), and noting that \( 1 - \frac{d}{q} = 1 - \frac{1}{q} = 1 - \frac{d}{2p_1} = \theta \) and \( \frac{1}{p} = \frac{1}{p} - \frac{2d}{p} \), we get

\[
\|W\|_{L^q(S_t)} + \|U\|_{L^q(S_t)} \leq C \left( \|W\|_{L^{q,p}(S_t)} + \|U\|_{L^{q,p}(S_t)} \right).
\]

(4.25) Combining (4.23)-(4.25) with (4.11), we obtain

\[
\left\| E \int_{\Delta_n(S_t)} \prod_{i=1}^n \partial_{\alpha_i} f_i(t_i, X_{t_i}^i) \, dt_1 \cdots dt_n \right\|_{L^p}
\]
Regarding the above equation as a linear random ODE for \( \nabla \) and it is given by

\[ \nabla X^x_t = I + \int_0^t \nabla b(s, X^x_s) \nabla X^x_s \, ds. \]

So, we complete our proof.

Note that \( b \in L^\infty([0, T]; C^2_b) \), the solution to the SDE (1.2) is differentiable with respect to \( x \), and \( \nabla X^x_t \) satisfies

\[ \nabla X^x_t = I + \sum_{n=1}^\infty \int \cdots \int_{\Delta_n(t)} \nabla b(t_n, X^x_{t_n}) \cdots \nabla b(t_1, X^x_{t_1}) \, dt_1 \cdots dt_n, \quad (4.26) \]

provided that this series is convergent (cf. [MNP15]). Moreover, for any \( 0 \leq t_0 \leq t \leq T \),

\[ \nabla X^x_t = \nabla X^x_{t_0} + \sum_{n=1}^\infty \int \cdots \int_{\Delta_n(t_0,t)} \nabla b(t_n, X^x_{t_n}) \cdots \nabla b(t_1, X^x_{t_1}) \, \nabla X^x_{t_0} \, dt_1 \cdots dt_n. \quad (4.27) \]

On the other hand, the Malliavin derivative \( D_s X_t \) is the solution of the linear stochastic equation

\[ D_s X^x_t = I + \int_s^t \nabla b(r, X^x_r) \, D_s X^x_r \, dr, \]

for a.e. \( s \in [0, T] \) with \( s \leq t \), and \( D_s X_t = 0 \) for a.e. \( s \in [0, T] \) with \( s > t \). Thus, one sees that

\[ D_s X^x_t = I + \sum_{n=1}^\infty \int \cdots \int_{\Delta_n(s,t)} \nabla b(t_n, X^x_{t_n}) \cdots \nabla b(t_1, X^x_{t_1}) \, dt_1 \cdots dt_n, \quad (4.28) \]

for a.e. \( s \in [0, T] \) with \( s \leq t \), and

\[ D_s X^x_t - D_t X^x_t = \int_s^t \nabla b(r, X^x_r) \, D_s X^x_r \, dr - \int_s^t \nabla b(r, X^x_r) \, D_t X^x_r \, dr \\
= \int_s^t \nabla b(r, X^x_r) \, D_s X^x_r \, dr + \int_s^t \nabla b(r, X^x_r) \, (D_s X^x_t - D_t X^x_t) \, dr \\
= D_s X^x_t - 1 + \int_s^{t'} \nabla b(r, X^x_r) \, (D_s X^x_t - D_t X^x_t) \, dr \]

for a.e. \( s, s' \in [0, T] \) with \( s < s' \leq t \). Iterating, we get

\[ D_s X^x_t - D_t X^x_t = \left( 1 + \sum_{n=1}^\infty \int \cdots \int_{\Delta_n(s',t)} \nabla b(t_n, X^x_{t_n}) \cdots \nabla b(t_1, X^x_{t_1}) \, dt_1 \cdots dt_n \right) \cdot (D_s X^x_t - 1) \]

(4.29)

for a.e. \( s, s' \in [0, T] \) with \( s < s' \leq t \).

We are now in a position to prove our Proposition 4.1.
Proof of Proposition 4.1. Case (a). We only need to prove the case where \( r \) is a positive even integer. For any \( n \in \mathbb{N}_+ \) and \( 0 \leq S_0 \leq S_1 \leq T \), it is not hard to see that

\[
\left( \int \cdots \int_{\Delta_n(S_0,S_1)} \partial_{\alpha_{n-1}} b^1(t_n,X_{tn}^x) \cdot \partial_{\alpha_{n-2}} b^{\alpha_{n-1}}(t_{n-1},X_{tn-1}^x) \cdots \partial_j b^{\alpha_1}(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n \right)^r
\]
can be written as a sum of at most \( r^n \) terms of the form

\[
\int \cdots \int_{\Delta_n(S_0,S_1)} \partial_{\alpha_{n-1}} b^1(t_1,X_{t_1}^x) \cdot \partial_{\alpha_{n-2}} b^{\alpha_{n-1}}(t_{n-1},X_{tn-1}^x) \cdots \partial_j b^{\alpha_1}(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n.
\]

Fix \( p \in (d/(d-1), d), \gamma \in (0, 1/2) \). By the above discussion and (4.4), we have

\[
\left\| \int \cdots \int_{\Delta_n(S_0,S_1)} \nabla b(t_n,X_{tn}^x) \cdots \nabla b(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n \right\|_{\mathbb{L}^p_{t_{n}} L^\infty_x} \leq C \sum_{i,j=1}^d \sum_{\alpha_{n-1}=1}^d \left\| \int \cdots \int_{\Delta_n(S_0,S_1)} \partial_{\alpha_{n-1}} b^1(t_n,X_{tn}^x) \cdot \partial_{\alpha_{n-2}} b^{\alpha_{n-1}}(t_{n-1},X_{tn-1}^x) \cdots \partial_j b^{\alpha_1}(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n \right\|_{\mathbb{L}^p_{t_{n}} L^\infty_x}.
\]

\[
\leq C \sum_{i,j=1}^d \sum_{\alpha_{n-1}=1}^d \left[ \int_{\mathbb{R}^d} \left( \sum_{\beta,\beta'} \mathbb{E} \int \cdots \int_{\Delta_n(S_0,S_1)} \partial_{\beta_1} b^{\beta_1}(t_1,X_{t_1}^x) \cdot \partial_{\beta_2} b^{\beta_2}(t_2,X_{t_2}^x) \cdots \partial_j b^{\alpha_1}(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n \right)^p \, dx \right]^{1/p}.
\]

\[
\leq C \sum_{i,j=1}^d \sum_{\alpha_{n-1}=1}^d \left[ \sum_{\beta,\beta'} \| \mathbb{E} \int \cdots \int_{\Delta_n(S_0,S_1)} \partial_{\beta_1} b^{\beta_1}(t_1,X_{t_1}^x) \cdot \partial_{\beta_2} b^{\beta_2}(t_2,X_{t_2}^x) \cdots \partial_j b^{\alpha_1}(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n \|_{\mathbb{L}^r_x} \right]^{1/r}.
\]

(4.4)

\[
\leq (rC_8)^n \left( m^2 \|b\|_{\mathbb{L}^\infty_x(T)} \sqrt{T_1 - S_0} + a(m) \right)^{n-1/r} \|b\|_{\mathbb{L}^1_x(T)}^{1/r} (S_1 - S_0)^{\gamma/r}.
\]

Here we also used the fact that the sum \( \sum_{\beta,\beta'} \) contains at most \( r^n \) terms. The constant \( C_8 > 1 \) only depends on \( d, T, \|b\|_{\mathbb{L}^\infty_x(T)}, \{a(m)\}, r, p \) and \( \gamma \). Letting \( m \) be large enough such that \( C_8 ra(m) \leq 1/4 \) and then choosing \( T_r > 0 \) such that \( C_8 kmr^2 \|b\|_{\mathbb{L}^1_x(T)} \sqrt{T_r} = 1/4 \), we have for any \( 0 \leq S_1 - S_0 \leq T_r \),

\[
\sum_{n=1}^\infty \left\| \int \cdots \int_{\Delta_n(S_0,S_1)} \nabla b(t_n,X_{tn}^x) \cdots \nabla b(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n \right\|_{\mathbb{L}^p_{t_{n}} L^\infty_x} \leq \sum_{n=1}^\infty (C_4r)^n \left( m^2 \|b\|_{\mathbb{L}^\infty_x(T)} \sqrt{T} + a(m) \right)^{n-1/r} \|b\|_{\mathbb{L}^1_x(T)}^{1/r} (S_1 - S_0)^{\gamma/r} \leq 2(rC_8)^1/r \|b\|_{\mathbb{L}^1_x(T)}^{1/r} (S_1 - S_0)^{\gamma/r}.
\]

(4.30)

Thus, by (4.26) for each \( r \in [0, T] \),

\[
\|\nabla X_t^x - I\|_{\mathbb{L}^p_{t_{n}} L^\infty_x} \leq C \sum_{n=1}^\infty \left\| \int \cdots \int_{\Delta_n(t)} \nabla b(t_n,X_{tn}^x) \cdots \nabla b(t_1,X_{t_1}^x) \, dt_1 \cdots dt_n \right\|_{\mathbb{L}^p_{t_{n}} L^\infty_x} \leq C t^{\gamma/r}.
\]
For any $t \in [T_2, 2T_2 \wedge T]$, by (4.27), Hölder’s inequality, (4.30) and (4.31) we get
\[
\|\nabla X^s - 1\|_{L_*^p L_*^q} \\
\leq \|\nabla X^s - 1\|_{L_*^p L_*^q} + \sum_{n=1}^{\infty} \left\| \int \cdots \int_{\Delta_n(T_{n-1})} \nabla b(t_n, X^s) \cdots \nabla b(t_1, X^s) \ dt_1 \cdots dt_n \right\|_{L_*^p L_*^q} \\
+ \sum_{n=1}^{\infty} \left\| \int \cdots \int_{\Delta_n(T_{n-1})} \nabla b(t_n, X^s) \cdots \nabla b(t_1, X^s) \ dt_1 \cdots dt_n \right\|_{L_*^p L_*^q} \|\nabla X^s - 1\|_{L_*^p L_*^q} \\
\leq C T^{\gamma/2}.
\]
Iterating, we see that
\[
\|\nabla X^s - 1\|_{L_*^p L_*^q} \leq C T^{\gamma/2}, \quad \forall t \in [0, T].
\]
(4.32)
Using (4.28) one sees that (4.2) can be proved in the same way as (4.1).

For (4.3). Assume $0 \leq s < s' \leq t \leq T$. Combing (4.29) and (4.2), we obtain
\[
\|D_s X^s - D_s X^{s'}\|_{L_*^p L_*^q} \\
\leq \|D_s X^s - 1\|_{L_*^p L_*^q} + \|D_s X^{s'} - 1\|_{L_*^p L_*^q} \|D_s X^s - 1\|_{L_*^p L_*^q} \\
\leq C (s' - s)^{\gamma/4},
\]
for a.e. $s, s' \in [0, T]$ with $0 \leq s < s' \leq t \leq T$. So, we complete our proof for the first case.

Case (b). Let $\gamma \in (0, \frac{1}{r} - \frac{1}{q_1})$. By (4.5) and the argument in the previous case, one can see that for each positive even integer $r$,
\[
\left\| \int \cdots \int_{\Delta_n(S_0, S_1)} \nabla b(t_n, X^s) \cdots \nabla b(t_1, X^s) \ dt_1 \cdots dt_n \right\|_{L_*^p L_*^q} \leq (r C_9)^{n+1/r} \|b\|_{L_{q_1}^p(S_0, S_1)} (S_1 - S_0)^{\gamma/2},
\]
where $C_9 > 1$ only depends on $d, p_1, q_1, T, a(m), r, p$ and $\gamma$. Since $b \in L_{q_1}^p(T)$, for each even integer $r$ there is a positive constant $T_r > 0$ depending on $r, C_9$ and $\ell(\delta)$ such that for any $S_0, S_1 \in [0, T]$ with $0 \leq S_1 - S_0 \leq T_r$
\[
\|b\|_{L_{q_1}^p(S_0, S_1)} \leq (2 r C_9)^{-1}.
\]
Thus,
\[
\left\| \int \cdots \int_{\Delta_n(S_0, S_1)} \nabla b(t_n, X^s) \cdots \nabla b(t_1, X^s) \ dt_1 \cdots dt_n \right\|_{L_*^p L_*^q} \leq (r C_9)^{1/r} 2^{-n} (S_1 - S_0)^{\gamma/2}.
\]
Our desired estimates then can be obtained by the above estimate and the same argument as in the previous case.

5. PROOF OF THE MAIN RESULT

The following lemma is a consequence of Theorem 1.1 in [RZ20].

Lemma 5.1. Let $d \geq 3$. Assume that $b \in C([0, T]; L^d)$ or $b \in L_{q_1}^p(T)$ with $p_1, q_1 \in (2, \infty)$ and $d/p_1 + 2/q_1 = 1$. Then there is a unique weak solution to (1.2) such that for any $p, q \in (1, \infty)$ with $d/p + 2/q < 2$, the Krylov type estimate (1.3) is valid.
Now we are in the position to prove our main result.

**Proof of Theorem 1.1. Case (a):** $b \in C([0, T]; L^d)$.

Recalling that $\rho \in C^\infty(\mathbb{R}^d)$ satisfying $\rho \geq 0$ and $\int \rho = 1$, and let $b_k = b \ast \rho_k$. Since $b \in C([0, T]; L^d)$, by Proposition 2.2 we have

\[ \|b_k - b_k \ast \rho_m\|_{L^q(T)} = \|(b - b \ast \rho_m) \ast \rho_k\|_{L^q(T)} \leq \|b - b \ast \rho_m\|_{L^q(T)} =: a(m) \to 0 \quad (m \to \infty). \]  

(5.1)

It is well-known that for each $k$ there is a unique continuous random field $X(k) : \Delta_2(T) \times \mathbb{R} \to \mathbb{R}^d$ such that

\[ X^k_s(t) = x + \int_s^t b_k(r, X^k_s(r)) \, dr + W_t - W_s, \quad \text{for all} \quad 0 \leq s \leq t \leq T, \, x \in \mathbb{R}^d. \]  

(5.2)

Given $\beta \in (0, 1/2)$, let

\[ p \in (1, d), \, q \in (1, \infty) \quad \text{satisfying} \quad \frac{d}{p} + \frac{2}{q} \in (1, 2 - 2\beta). \]

By estimate (5.1) and Remark 2.7, for any $s \leq t_1 \leq t_2 \leq T$ and $f \in \tilde{L}^q(T)$, there is a unique function $u_k$ in $\tilde{H}^2(T)$ solving

\[ \partial_t u_k + \frac{1}{2} \Delta u_k + b \cdot \nabla u_k + f = 0 \quad \text{in} \quad (s, t_2) \times \mathbb{R}^d, \quad u_k(t_2) = 0 \]

and a constant $C$, which does not depends on $k$, such that

\[ \|\partial_t u_k\|_{\tilde{L}^q(t_1, t_2)} + \|u_k\|_{\tilde{H}^2(t_1, t_2)} \leq C \|f\|_{\tilde{L}^q(t_1, t_2)}. \]  

(5.3)

By the generalized Itô formula (cf. [RZ20]),

\[ -u_k(t_1, X^k_{s,t_1}(k)) = -\int_{t_1}^{t_2} f(t, X^k_{s,t_1}(k)) \, dt + \int_{t_1}^{t_2} \nabla u_k(t, X^k_{s,t_1}(k)) \cdot dW_t. \]

Taking $\alpha = 0$ and $\theta = 1 - \frac{1}{q} - \beta > \frac{d}{2p}$ in (2.13), using Morrey’s inequality and (5.3) we get

\[ \mathbb{E} \left( \int_{t_1}^{t_2} f(t, X^k_{s,t_1}(k)) \, dt \right) = \mathbb{E} \left( u_k(t_1, X^k_{s,t_1}(k)) \right), \]

\[ \leq \|u_k(t_1)\|_\infty \leq C \sup_{z \in \mathbb{R}^d} \|(u_k X^k_{s,t_1})_z(t_1)\|_{H^{2\theta,p}}, \]  

(2.13)

\[ \leq C |t_2 - t_1|^\beta \left( \|\partial_t u_k\|_{\tilde{L}^q(t_1, t_2)} + \|u_k\|_{\tilde{H}^2(t_1, t_2)} \right) \]

(5.3)

\[ \leq C |t_2 - t_1|^\beta \|f\|_{\tilde{L}^q(t_1, t_2)}, \]  

(5.4)

where $C$ only depends on $d, p, q, \beta, T, \|\tilde{b}\|_{L^q(T)}$ and $\{a(m)\}$. Once with (5.4) in hand, it is standard to show that

\[ \mathbb{E} \left( \int_{t_1}^{t_2} f(t, X^k_{s,t_1}(k)) \, dt \right) \leq C_r |t_2 - t_1|^\beta \|f\|_{\tilde{L}^q(t_1, t_2)}, \quad \forall \, f \in \tilde{L}^q(T), \, r > 0, \]  

(5.5)
Therefore, by noting that $p < d, q < \infty$ and $\beta < 1/2$, we get

$$\mathbb{E} \left| X_{s,t_i}^x(k) - X_{s,t_j}^x(k) \right|^r \leq C \mathbb{E} \left( \int_{t_1}^{t_2} |b_k(\tau, X_{\tau}^x)\,d\tau \right)^r + C \mathbb{E} |W_{t_2} - W_{t_1}|^r$$

$$\leq C|t_2 - t_1|^\beta r \left( 1 + \|b\|^r_{L^q(L^d)} \right), \quad \forall r > 0. \tag{5.6}$$

Consequently,

$$\sup_{0 \leq s \leq t \leq T} \mathbb{E} \left| X_{s,t}^x(k) \right|^r \leq C, \quad \forall r > 0, \tag{5.7}$$

which together with (4.1) and Morrey’s inequality implies that for each $r > d$,

$$\sup_{z \in \mathbb{R}^d} \mathbb{E} \left| X_{s,t}^x(k) \right|^r \leq C \sup_{z \in \mathbb{R}^d} \mathbb{E} \left| X_{s,t}^x(k) \right|^r_{H^r(B_1(z))} \leq C. \tag{5.8}$$

Thus, for any $0 \leq s \leq t \leq T, x, y \in \mathbb{R}^d$ and $r > d$,

$$\mathbb{E} \left| X_{s,t}^x(k) - X_{s,t}^y(k) \right|^r \leq C|x - y|^{r - d}, \tag{5.9}$$

where $C$ only depends on $d, p, q, r, \|b\|_{L^q(L^d)}$ and $\{a(m)\}$.

Assume $0 \leq s_1 \leq s_2 \leq t$. By the Markov property and the independence of $X_{s_1,s_2}^x(k)$ and $X_{s_2,t}^y(k)$, for each $r > d$ we obtain

$$\mathbb{E} \left| X_{s_1,s_2}^x(k) - X_{s_2,t}^x(k) \right|^r \leq C \mathbb{E} \left| \int_{s_1}^{s_2} b_k(s, X_{s_1,s_2}^x(k)) \,ds \right|^r + C \mathbb{E} \left| \int_{s_2}^{t} b_k(s, X_{s_2,t}^x(k)) \,ds \right|^r$$

$$\leq C|s_1 - s_2|^\beta r + C \mathbb{E} \left| \int_{s_2}^{t} b_k(s, X_{s_2,t}^x(k)) \,ds \right|^r$$

$$\leq C|s_1 - s_2|^\beta r + C \mathbb{E} \left[ X_{s_2,t}^x(k) - X_{s_2,t}^x(k) \right]^r$$

$$= C|s_1 - s_2|^\beta r + C \mathbb{E} \left| X_{s_2,t}^y(k) - X_{s_2,t}^x(k) \right|^r_{y = X_{s_1,s_2}^x(k)} \tag{5.10}$$

Combing (5.6), (5.8) and (5.9), we obtain that for all $(s, t_i) \in \Delta_2(T), i = 1, 2$,

$$\mathbb{E} \left| X_{s_1,t_i}^x(k) - X_{s_2,t_i}^x(k) \right|^r \leq C \left( |t_1 - t_2|^\beta r + |x - y|^{r - d} + |s_1 - s_2|^\beta(r - d) \right), \tag{5.10}$$

where $r > d$ and $C$ only depends on $d, r, T$ and $b$. On the other hand, noting that

$$\sup_k \|b_k\|_{L^q(L^d)} \leq \|b\|_{L^q(L^d)}, \quad \|b_k - b_k \ast \varphi_m\|_{L^q(L^d)} \leq a(m) \to 0 \quad (m \to \infty),$$

by Lemma 3.1 and Proposition 4.1 one can see that for any fixed $(s, t) \in \Delta_2(T)$ and $R > 0$,

$$\left\{ B_R \times \Omega \ni (x, \omega) \mapsto X_{s,t}^x(k)(\omega) \in \mathbb{R}^d \right\}_{k \in \mathbb{N}^+}.$$
is relatively compact in $L^2(B_T \times \Omega)$. The standard diagonal argument yields that there is a subsequence (still denoted by $X_{s,t}^x(k)$) and a countable dense subset $\mathcal{D}$ of $\mathbb{R}^d$ such that

$$X_{s,t}^x(k) \xrightarrow{L^2(\Omega)} X_{s,t}^x, \forall (s,t) \in \mathbb{Q}^2 \cap \Delta_2(T) \text{ and } x \in \mathcal{D}.$$ 

By (5.7), we also have

$$X_{s,t}^x(k) \xrightarrow{k \to \infty} X_{s,t}^x, \forall r \geq 1, \forall (s,t) \in \mathbb{Q}^2 \cap \Delta_2(T) \text{ and } x \in \mathcal{D}.$$ 

Fatou’s lemma and (5.10) yield that for all $(s_i,t_i) \in \mathbb{Q}^2 \cap \Delta_2(T), i = 1, 2, \text{ and } x \in \mathcal{D},$

$$E \left| X_{s_1,t_1}^{x_1} - X_{s_2,t_2}^{x_2} \right|^r \leq C \left( |x_1 - x_2|^r + |s_1 - s_2|^\beta(r-\delta) + |t_1 - t_2|^\beta r \right), \quad \forall r > d. \tag{5.11}$$

Therefore, $X_{s,t}^x$ can be extended to a continuous random field on $\Delta_2(T) \times \mathbb{R}^d$ satisfying (5.11) due to the Kolmogorov-Chentsov theorem, and up to a subsequence (still denoted by $X_{s,t}^x(k)$),

$$X_{s,t}^x(k, \omega) \xrightarrow{k \to \infty} X_{s,t}^x(\omega), \tag{5.12}$$

for all $(s,t) \in \mathbb{Q}^2 \cap \Delta_2(T), x \in \mathcal{D}$ and $\mathbb{P}$-a.s. $\omega \in \Omega$. Then, since by (5.10) for $\mathbb{P}$-a.s. $\omega \in \Omega$, $X_{s,t}^x(k, \omega), k \in \mathbb{N}_+$, are equicontinuous as functions of $(s,t,x)$, (5.12) holds for all $(s,t) \in \Delta_2(T), x \in \mathbb{R}^d$ and $\omega \in \Omega_0 \subset \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$. Taking limits on both sides of (5.5), we get

$$E \left| \int_{t_1}^{t_2} f(t,X_{s,t}^x) dt \right|^r \leq C|t_2 - t_1|^\beta r \|f\|_{L^r(T)}^r. \tag{5.13}$$

Thus, for each $x \in \mathbb{R}^d$ and $K \in \mathbb{N}_+$,

$$E \left| \int_s^t b(\tau,X_{s,\tau}^x) d\tau - \int_s^t b_k(\tau,X_{s,\tau}^x(k)) d\tau \right| \leq E \int_s^t |b - b_K| (\tau,X_{s,\tau}^x) d\tau + E \int_s^t |b_K - b_k| (\tau,X_{s,\tau}^x(k)) d\tau$$

$$+ E \int_s^t b_K(\tau,X_{s,\tau}^x) d\tau - \int_s^t b_k(\tau,X_{s,\tau}^x(k)) d\tau.$$ 

By our assumption on $b$, it holds that $b - b_k \to 0$ in $L_a^d(T)$. So, the first and second terms on the right hand side of the above inequality converge to 0 as $k$ goes to infinity, due to the fact that $X$ and $X(k)$ satisfy the Krylov type estimates (5.5) and (5.13). On the other hand, by (5.12) and Lebesgue dominated convergence theorem, the third term on the right side of the above inequality also converges to 0 as $k$ goes to infinity. So,

$$E \left| \int_s^t b(\tau,X_{s,\tau}^x) d\tau - \int_s^t b_k(\tau,X_{s,\tau}^x(k)) d\tau \right| \to 0,$$

which together with (5.12) implies

$$X_{s,t}^x - x - \int_s^t b(\tau,X_{s,\tau}^x) d\tau = \lim_{k \to \infty} \left( X_{s,t}^x(k) - x - \int_s^t b(\tau,X_{s,\tau}^x(k)) d\tau \right) = W_t,$$

where $W_t$ is a standard Brownian motion.
i.e. the limit point \( X^+_s \) is a strong solution to (1.2). Hence, we obtain the strong existence of solutions to (1.2). Moreover, by the proof of Theorem 1.1 in [RZ20], we can also see that \( X^+_s \) also satisfies (1.3) for any \( p', q' \in (1, \infty) \) satisfying \( d/p' + 2/q' < 2 \).

Following [Che02], we next show that the limit point of \( X^+_s(k) \) is the unique, and is also the unique strong solution to (1.2) satisfying (1.3). Without loss of generality we may assume \( s = 0 \). Suppose \( X \) is a limit point of \( X^+_s(k) \), which is a strong solution of (1.2) with \( s = 0 \) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\). Then there exists a measurable map \( T : C([0,T]; \mathbb{R}^d) \to C([0,T]; \mathbb{R}^d) \) such that \( X(\omega) = T(W(\omega)) \) for \( \mathbb{P} \)-a.s. \( \omega \). Let \( \{Q_\omega\}_{\omega \in \Omega} \) be the regular conditional expectation of \( X \) with respect to \( \mathcal{F}^W := \sigma\{W_t : t \in [0,T]\} \). Then \( Q_\omega = \delta_{T(W(\omega))} \) for \( \mathbb{P} \)-a.s. \( \omega \). Now, let \( Y \) be another strong solution to (1.2) with \( s = 0 \) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\) satisfying (1.3). Thanks to Lemma 5.1, we have \( \text{law}(X) = \text{law}(Y) \), together with the fact that

\[
W_t = X_t - x - \int_0^t b(r, X_r)dr = Y_t - x - \int_0^t b(r, Y_r)dr,
\]

we obtain \( \text{law}(X, W) = \text{law}(Y, W) \). This implies \( Q_\omega \), the regular conditional expectation of \( Y \) with respect to \( \mathcal{F}^W_T \), equals to \( Q_\omega \) for \( \mathbb{P} \)-a.s. \( \omega \), i.e. \( Q_\omega = \delta_{T(W(\omega))} \). Thus, \( Y(\omega) = T(W(\omega)) = X(\omega) \) for \( \mathbb{P} \)-a.s. \( \omega \).

**Case (b):** \( b \in \mathbb{L}^{p_1}_{q_1}(T) \). Given \( \beta \in (0, 1/2) \). In this case, we take \( p \in (1, p_1) \) and \( q \in (1, q_1) \) such that \( d/p + 2/q < 1/2 - 2\beta \). Define the maximal function of \( b(t, \cdot) \):

\[
M_b(t, x) := \sup_{r > 0} \int_{B_r(x)} |b(t, y)| dy
\]

Define \( b_k = (b_1)_{|b| \leq k} *_x \rho_k \in L^\infty([0,T]; C^2_b) \). Noting that \( b_k \leq |b| *_x \rho_k \leq CM_b \) (cf. [DZ01, Corollary 2.8]), we have \( K^t_{b_k}(m) \leq K^t_{\overline{CM}_b}(m) \). By the basic fact that

\[
\|M_b\|_{\mathbb{L}^{p_1}_{q_1}(T)} \asymp \|b\|_{\mathbb{L}^{p_1}_{q_1}(T)} < \infty,
\]

(cf. [DZ01, Theorem 2.5]), we obtain

\[
\sup_k K^t_{b_k}(m) \leq K^t_{\overline{CM}_b}(m) = a(m) \to 0, \text{ as } m \to \infty.
\]

Also we have

\[
\sup_k \omega_{b_k}(\delta) \leq \omega_b(\delta) = \ell(\delta) \to 0, \text{ as } \delta \to 0.
\]

Then our desired results in the second case can be obtained by the same procedure as for the previous case. \( \square \)

6. **APPLICATION**

In [Zha10], Zhang studied the backward Navier-Stokes equation (1.10) through considering the stochastic system (1.9). As in [Cl08], it was also shown in [Zha10] that the existence of smooth solutions for (1.10) and (1.9) are equivalent (see Theorem 2.3 therein). Therefore, it is quite interesting to find a regularity criterion for solutions of (1.9). Below we give one such conditional regularity result, which is similar to the Serrin criterion for the Navier-Stokes equations.
Theorem 6.1. Let $d \geq 3$, $T > 0$, $p_1, q_1 \in (2, \infty)$, $q > d$ and $k, l \in \mathbb{N}$. Assume $u \in C([-T, 0]; L^d)$ or $u \in L^d_{q_1}(-T, 0)$ with $d/p_1 + 2/q_1 \leq 1$ and $\varphi \in H^{k,q}$. Suppose that $(u, X)$ is a solution to the stochastic system (1.9), then $u \in H^k_{q_1}(-T, 0)$ and for any $l \leq k/2$, $\partial^l_t u \in L^d_{q_1}(-T, 0)$. Consequently, if $\varphi \in C^\infty_c(\mathbb{R}^d)$, then $u \in C^\infty([-T, 0] \times \mathbb{R}^d)$ and it satisfies (1.10).

Proof. Step 1. Assume that $\varphi \in L^d$, $u \in C([-T, 0]; L^d)$, or $u \in L^d_{q_1}(-T, 0)$ with $p_1, q_1 \in (2, \infty)$ and $d/p_1 + 2/q_1 \leq 1$. We claim that

$$\sup_{t \in [-T, 0]} \|u(t)\|_q < \infty. \tag{6.1}$$

Below we only give the proof of (6.1) for the case where $u \in L^d_{q_1}(-T, 0)$ with $p_1, q_1 \in (2, \infty)$ and $d/p_1 + 2/q_1 = 1$, since the other cases are simpler.

Let $a \in (d, q)$ and

$$v(t, x) := \mathbb{E} \left[ (\nabla^T X^x_{t,0} - 1) \varphi \left( X^x_{t,0} \right) \right].$$

Noting that $\frac{1}{q} - \frac{1}{q} \in (0, \frac{1}{a})$, one can always choose $r \geq 2$ and $p \in (\frac{p_1}{q}, p_1)$ such that $\frac{1}{pr} = \frac{1}{a} - \frac{1}{q}$. Thus, for each $t \in [-T, 0]$

$$\|v(t)\|_a \leq \left\| \nabla^T X^x_{t,0} - 1 \right\|_{L^p_{\omega}} \|\varphi(X^x_{t,0})\|_{L^q_{\omega}} \leq C \left\| \nabla X^x_{t,0} - 1 \right\|_{L^p_{\omega} L^q_{\omega}} \|\varphi(X^x_{t,0})\|_{L^q_{\omega} L^q_{\omega}'} \leq C \|\varphi\|_q. \tag{6.2}$$

Here $r' = r/(r - 1)$, and we use the fact that

$$\|\mathbb{E} f(X_{t,0})\|_q \leq \|f\|_q, \quad \forall q \in [1, \infty], \tag{6.3}$$

due to the fact that $u$ is divergence free (cf. [ZZ21, Lemma 3.2]). Recall that $P$ is the Leray projection

$$(PF)_j = F_j - \nabla(\Delta)^{-1} \text{div} F = F_j - \sum_{j=1}^d R_j R_j F_j,$$

where $R_j$ is the Riesz transformation. The $L^q$ boundedness of $R_j$ implies that $P$ is a bounded map on $L^q(\mathbb{R}^d \times \mathbb{R}^d)$ with $q \in (1, \infty)$. By (6.3), we have

$$\|u\|_{L^q_{(-T,0)}} = \sup_{x \in \mathbb{R}^d} \|u X^x_{t,0}\|_{L^q_{(-T,0)}} \leq C \left( \sup_{t \in [-T, 0]} \|P v(t)\|_a + \sup_{t \in [-T, 0]} \|P \mathbb{E} \varphi(X^x_{t,0})\|_q \right) \leq C \sup_{t \in [-T, 0]} \|v(t)\|_a + C \|\varphi\|_q < \infty. \tag{6.4}$$

Noting that $a > d$, combining the above estimate and Theorem 1.1 of [XXZZ20] we get

$$\sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{t \in [-T, 0]} \|\nabla X^x_{t,s}\|_r' < \infty, \quad \forall r \geq 1. \tag{6.4}$$
Therefore, for each \( t \in [-T,0] \),
\[
\|u(t)\|_{q} \leq \left\| \nabla^{T} X_{t,0} \right\|_{L_{w}^{q}} \left\| \phi(X_{t,0}) \right\|_{L_{w}^{q}} \leq C \left\| \nabla^{2} X_{t,0} \right\|_{L_{w}^{q}} \left\| \phi(X_{t,0}) \right\|_{L_{w}^{q}} \leq C \left\| \phi \right\|_{q} < \infty.
\]  
(6.5)

So, we complete our proof for (6.1).

**Step 2.** Now assume that \( \phi \in H^{1,q} \). By [Zha16, Lemma 7.2],
\[
\partial_{t} u(t) = \partial_{t} \text{PE} \left[ \nabla^{T} X_{t,0} \phi(X_{t,0}) \right] = \text{PE} \left[ \nabla^{T} X_{t,0} \nabla \phi(X_{t,0}) - \nabla \phi(X_{t,0}) \right] \partial X_{t,0}.
\]  
(6.6)

Using Hölder’s inequality and (6.4), we get
\[
\left\| \partial_{t} u(t) \right\|_{q} \leq C \left\| \nabla^{T} X_{t,0} \right\|_{L_{w}^{q}} \left\| \nabla \phi(X_{t,0}) \right\|_{L_{w}^{q}} \leq C \left\| \nabla X_{t,0} \right\|_{L_{w}^{q}} \left\| \nabla \phi \right\|_{q} < \infty.
\]
Hence,
\[
\left\| u \right\|_{H_{w}^{1,q}(-T,0)} \leq C \left\| \phi \right\|_{H^{1,q}} < \infty.
\]  
(6.7)

**Step 3.** Assume that \( \phi \in H^{2,q} \). Following [XXZZ20], below we use a Zvonkin type change of variables to convert the first equation in (1.9) to a new SDE. Let \( t \in [-T,0] \) and \( \lambda \geq 0 \) and \( \alpha \in (1,\infty) \) such that \( d/p + 2/\alpha < 1 \). Since \( u \in H_{w}^{1,q}(-T,0) \), there is a unique function \( U \) in \( H_{w}^{\alpha,q}(-T,0) \) satisfying
\[
\partial_{t} U + \left( \frac{\Delta}{2} - \lambda \right) U + u \cdot \nabla U + u = 0 \quad \text{in} \quad (-T,0) \times \mathbb{R}^{d}, \quad U(0) = 0.
\]

Moreover,
\[
\lambda \left\| U \right\|_{H_{w}^{\alpha,q}(-T,0)} + \left\| \partial_{t} U \right\|_{H_{w}^{\alpha,q}(-T,0)} + \left\| \nabla^{2} U \right\|_{H_{w}^{\alpha,q}(-T,0)} \leq C \left\| U \right\|_{H_{w}^{\alpha,q}(-T,0)} < \infty
\]  
(6.8)

(cf. [XXZZ20]). Since \( d/p + 2/\alpha < 1 \), using (6.8), (2.13) and an interpolation inequality one can choose \( \lambda \) large enough so that
\[
\sum_{k=0}^{2} \left\| \nabla^{k} U \right\|_{\infty} \leq 1/2.
\]  
(6.9)

Define
\[
\Phi(s,x) := x + U(s,x).
\]

By (6.9), \( \Phi(s,\cdot) \) is a \( C^{2} \)-diffeomorphism and
\[
\left\| \nabla \Phi \right\|_{\infty}, \left\| \nabla^{2} \Phi \right\|_{\infty}, \left\| \nabla \Phi^{-1} \right\|_{\infty}, \left\| \nabla^{2} \Phi^{-1} \right\|_{\infty} \leq C.
\]  
(6.10)

Set
\[
Y_{t,s}^{x,k} := \Phi^{k}(s, X_{t,s}^{\Phi^{-1}(x,y)}), \quad \sigma_{k}^{y}(s,y) = \partial_{k} \Phi^{k}(s, \Phi^{-1}(s,y)), \quad b^{k}(s,y) = \lambda U^{k}(s, \Phi^{-1}(s,y)).
\]

Then,
\[
Y_{t,s}^{y} = y + \int_{t}^{\tau} b^{y}(\tau, Y_{t,\tau}^{y}) d\tau + \int_{t}^{\tau} \sigma^{y}(\tau, Y_{t,\tau}^{y}) d\tilde{W}_{\tau},
\]
where $\tilde{W}_t := W_t - W_i$ is a standard Brownian motion on $[t, 0]$. By (6.8)-(6.10), (2.13) and the definitions of $\sigma$ and $b$, one sees that

$$\sigma(s,y) - I = \nabla U(s, \Phi^{-1}(s,y)) \in \mathbb{H}^2(\mathbb{R}^d) \cap C([0, T]; C_b^2(\mathbb{R}^d))$$  \hfill (6.11)

and

$$b \in C([0, T]; C_b^2(\mathbb{R}^d)).$$  \hfill (6.12)

By the proof for [XXZZ20, Theorem 1.1], $\partial Y_{t,\tau}^x$ satisfies

$$\partial Y_{t,\tau}^x = e_i + \int_t^s \nabla b(\tau, Y_{t,\tau}^x) \partial Y_{t,\tau}^x d\tau + \int_t^s \partial_t \sigma_b(\tau, Y_{t,\tau}^x) \partial Y_{t,\tau}^x d\tilde{W}_\tau^x.$$  \hfill (6.13)

and

$$\sup_{x \in \mathbb{R}^d} \sup_{t \in [0, T]} |\nabla Y_{t,\tau}^x|^r < \infty, \quad \forall r \geq 1.$$  \hfill (6.14)

For any $h \in \mathbb{R}^d$, define

$$\delta_h^x := \sup_{\tau \in [t, s]} |\partial Y_{t,\tau}^{x+h} - \partial Y_{t,\tau}^x|.$$  \hfill (6.15)

For all $m \in \mathbb{N}_+$, equation (6.13) and the Burkholder-Davis-Gundy inequality yield

$$E(\delta_h^x)^{2m} \leq E\left(\int_t^s |\nabla b(\tau, Y_{t,\tau}^{x+h}) - \nabla b(\tau, Y_{t,\tau}^x)| \partial Y_{t,\tau}^{x+h} d\tau\right)^{2m} + E\left(\int_t^s |\nabla b(\tau, Y_{t,\tau}^x)| \delta_h^x d\tau\right)^{2m} \hfill (6.16)$$

$$+ E\left(\int_t^s \left|\nabla \sigma_b(\tau, Y_{t,\tau}^{x+h}) - \nabla \sigma_b(\tau, Y_{t,\tau}^x)\right|^2 |\partial Y_{t,\tau}^{x+h}| d\tau\right)^m$$

$$+ E\left(\int_t^s |\nabla \sigma_b(\tau, Y_{t,\tau}^x)|^2 (\delta_h^x)^2 d\tau\right)^m = \sum_{i=1}^4 I_i.$$

Using (6.12) and (6.14) one sees that

$$I_1 \leq E\left(\int_t^s |h| \left|\nabla Y_{t,\tau}^{x+h}\right| d\theta |\nabla Y_{t,\tau}^{x+h}| d\tau\right)^{2m} \leq |h|^{2m} \hfill (6.17)$$

and

$$I_2 \leq \int_t^s E(\delta_h^x)^{2m} d\tau.$$  \hfill (6.18)

Similarly, by (6.11) we get

$$I_4 \leq \int_t^s E(\delta_h^x)^{2m} d\tau.$$  \hfill (6.19)

For $I_3$, let

$$Mf(x) := \sup_{r > 0} \int_{B_r(x)} |f(y)| dy$$

be the maximal function of $f$. By the fact that

$$|f(x) - f(y)| \leq C[M|\nabla f|(x) + M|\nabla f|(y)] |x - y|$$
Gronwall’s inequality yields (cf. [CDL08]), we deduce
\[
I_3 \lesssim \mathbb{E}\left\{ \int_0^s \left[ (\mathcal{M}|\nabla^2 \sigma|(\tau,Y_{i,\tau}^{y+h}))^2 + (\mathcal{M}|\nabla^2 \sigma|(\tau,Y_{i,\tau}^{y}))^2 \right] \right. \\
\left. \left( |h| \int_0^1 |\nabla Y_{i,\tau}^{y+\theta h}|d\theta \right)^2 |\partial_i Y_{i,\tau}^{y+h}|^2 d\tau \right\}^m \\
\lesssim |h|^{2m} \mathbb{E}\left\{ \sup_{\tau \in [t,s]} |\nabla Y_{i,\tau}^{y+h}|^{2m} \int_0^1 \sup_{\tau \in [t,s]} |\nabla Y_{i,\tau}^{y+\theta h}|^{2m} d\theta \right\} \\
\left( \int_0^s \left[ (\mathcal{M}|\nabla^2 \sigma|(\tau,Y_{i,\tau}^{y+h}))^2 + (\mathcal{M}|\nabla^2 \sigma|(\tau,Y_{i,\tau}^{y}))^2 \right] d\tau \right)^m \\
\lesssim |h|^{2m} \left[ \mathbb{E} \left( \int_0^s \left[ (\mathcal{M}|\nabla^2 \sigma|(\tau,Y_{i,\tau}^{y+h}))^2 + (\mathcal{M}|\nabla^2 \sigma|(\tau,Y_{i,\tau}^{y}))^2 \right] d\tau \right) \right]^{3m/2} \lesssim |h|^{2m}.
\]

Here we used the facts that $\mathcal{M}|\nabla^2 \sigma| \in L_{q/2}(T)$ with $d/q + 2/a < 1$ and
\[
\sup_{\tau \in [t,s]} \mathbb{E} \int_0^s f(\tau,Y_{i,\tau}^{y}) d\tau \leq C \|f\|_{L_{q/2}(T)}.
\]

(c.f. [XZZZ20]). Thus, combining (6.15)-(6.19), we obtain
\[
\mathbb{E}(\delta^h)^{2m} \lesssim |h|^{2m} + \int_t^s \mathbb{E}(\delta^h)^{2m} d\tau.
\]

Gronwall’s inequality yields
\[
\mathbb{E} \sup_{\tau \in [t,s]} |\partial_i Y_{i,\tau}^{y+h} - \partial_i Y_{i,\tau}^{y}|^{2m} \lesssim |h|^{2m}
\]

Follow the argument of the proof for [XXZZZ20, Theorem 1.1], one sees that
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{s \in [t,0]} |\nabla^2 y_{i,s}^r|^2 < \infty, \quad \forall r \geq 1.
\]

and
\[
\partial_i y_{i,s}^r = \int_t^s \partial_{tt} b(\tau,Y_{i,\tau}^{y}) \partial_i Y_{i,\tau}^{y} \partial_{t} Y_{i,\tau}^{y} d\tau + \int_t^s \partial_t b(\tau,Y_{i,\tau}^{y}) \partial_{i} Y_{i,\tau}^{y} d\tau \\
+ \int_t^s \partial_{tt} \sigma(\tau,Y_{i,\tau}^{y}) \partial_i Y_{i,\tau}^{y} \partial_{t} Y_{i,\tau}^{y} d\tau + \int_t^s \partial_t \sigma(\tau,Y_{i,\tau}^{y}) \partial_{i} Y_{i,\tau}^{y} d\tau
\]

Recalling that $\Phi^{-1} \in C([-T,0];C^2_b(\mathbb{R}^d))$, we see that
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{s \in [t,0]} |\nabla^2 X_{i,s}^r|^2 < \infty, \quad \forall r \geq 1.
\]

By (6.6), we have
\[
\partial_i u(\tau) = \partial_i \mathbb{E} \left[ \nabla^\top X_i^r \phi(X_i^r) \right] = \mathbb{E} \partial_i \left[ \nabla^\top X_i^r [\nabla \phi(X_i^r) - \nabla \phi(X_i^r)] \partial_i X_i^r \right].
\]
Using (6.20) and following the same procedure as in the proof for (6.7), one can verify that
\[ \|u\|_{H^{2,q}_\pm(-T,0)} \leq C\|\varphi\|_{H^{2,q}} < \infty. \]
Repeating the above process higher derivatives can be estimated similarly step by step.

**Step 4.** Assume \( \varphi \in H^{2,q} \) and set
\[ w(t) := E\left[ \nabla^T X^s_{t,0} \varphi(X^s_{t,0}) \right]. \]
By step 3 we can see that
\[ \sup_{x \in \mathbb{R}^d} \sup_{t \leq s \leq 0} |\nabla^2 X^s_{t,0}| < \infty \quad \text{and} \quad w \in H^{2,q}_\pm(-T,0). \]
Following the proof for [Zha10, Theorem 2.1], we see that \( w \) satisfies
\[ \partial_t w = -\frac{A}{T}w - (\nabla w)u - (\nabla^T u)w, \quad w(0) = \varphi. \quad (6.21) \]
Thus, \( \partial_t w \in L^q_\pm(-T,0) \), which also implies \( \partial_t u = \partial_t Pw = P\partial_t w \in L^q_\pm(-T,0) \) due to the \( L^q \) boundedness of \( P \).

If \( \varphi \in H^{4,q} \), following the above discussion we see that \( u, w \in H^{4,q}_\pm(-T,0) \), which implies that the right side of (6.21) is in \( H^{2,q}_\pm(-T,0) \). Hence, \( \partial_t w \in H^{2,q}_\pm(-T,0) \) and \( \partial_t u \in H^{2,q}_\pm(-T,0) \). This means that \( \partial_t [\frac{A}{T}w + (\nabla w)u + (\nabla^T u)w] \in L^q_\pm(-T,0) \), i.e. \( \partial_t w \in L^q_\pm(-T,0) \). Repeating the same process one sees that \( \partial_t^{2k} w \in L^q_\pm(-T,0) \), provided that \( \varphi \in H^{2k,q} \). So, we complete our proof.

\[ \Box \]

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**References**

[BFGM19] Lisa Beck, Franco Flandoli, Massimiliano Gubinelli, and Mario Maurelli. Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. *Electronic Journal of Probability*, 24:1–72, 2019.

[BS04] Vlad Bally and Bruno Saussereau. A relative compactness criterion in Wiener–Sobolev spaces and application to semi-linear stochastic PDEs. *Journal of Functional Analysis*, 210(2):465–515, 2004.

[CDL08] Gianluca Crippa and Camillo De Lellis. Estimates and regularity results for the DiPerna-Lions flow. *Journal für die reine und angewandte Mathematik*, 616:15–46, 2008.

[Che02] Aleksander Semenovich Cherny. On the uniqueness in law and the pathwise uniqueness for stochastic differential equations. *Theory of Probability & Its Applications*, 46(3):406–419, 2002.

[CI08] Peter Constantin and Gautham Iyer. A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 61(3):330–345, 2008.

[Dav07] Alexander M Davie. Uniqueness of solutions of stochastic differential equations. *International Mathematics Research Notices*, 2007, 2007.

[DD09] Hongjie Dong and Dapeng Du. The Navier-Stokes equations in the critical Lebesgue space. *Communications in Mathematical Physics*, 292(3):811–827, 2009.
SDEs WITH CRITICAL DRIFTS

[DPMN92] Giuseppe Da Prato, Paul Malliavin, and David Nualart. Compact families of Wiener functionals. *Comptes rendus de l’Académie des sciences. Série I, Mathématique*, 315(12):1287–1291, 1992.

[DZ01] Javier Duandikoetxea and Javier Duandikoetxea Zuazo. *Fourier analysis*, volume 29. American Mathematical Soc., 2001.

[ESŠ03] Luis Escauriaza, Gregory Seregin, and Vladimir Šverák. Backward uniqueness for parabolic equations. *Archive for Rational Mechanics and Analysis*, 169(2):147–157, 2003.

[Eva10] Lawrence C Evans. *Partial Differential Equations*. The American Mathematical Society, 2010.

[FF11] Ennio Fedrizzi and Franco Flandoli. Pathwise uniqueness and continuous dependence for SDEs with non-regular drift. *Stochastics: An International Journal of Probability and Stochastic Processes*, 83(03):241–257, 2011.

[FF13] Ennio Fedrizzi and Franco Flandoli. Noise prevents singularities in linear transport equations. *Journal of Functional Analysis*, 264(6):1329–1354, 2013.

[FGP10] Franco Flandoli, Massimiliano Gubinelli, and Enrico Priola. Well-posedness of the transport equation by stochastic perturbation. *Inventiones mathematicae*, 180(1):1–53, 2010.

[FJR72] Eugene Barry Fabes, B Frank Jones, and Nestor M Riviere. The initial value problem for the Navier-Stokes equations with data in $L^p$. *Archive for Rational Mechanics and Analysis*, 45(3):193–248, 1972.

[Gig86] Yoshikazu Giga. Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system. *Journal of Differential Equations*, 62(2):186–212, 1986.

[KR05] Nicolai V Krylov and Michael Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probability Theory and Related Fields*, 131(2):154–196, 2005.

[Kry01] Nicolai V Krylov. The heat equation in $L_q((0,T),L_p)$-spaces with weights. *SIAM Journal on Mathematical Analysis*, 32(5):1117–1141, 2001.

[Kry20a] Nicolai V Krylov. On stochastic equations with drift in $L_q$. *arXiv preprint arXiv:2001.04008*, 2020.

[Kry20b] Nicolai V Krylov. On stochastic Itô processes with drift in $L_q$. *arXiv preprint arXiv:2001.03660*, 2020.

[Kry20c] Nicolai V Krylov. On strong solutions of Itô’s equations with $A \in W^1_d$ and $b \in L_d$. *arXiv preprint arXiv:2007.06040v1*, 2020.

[Lad67] Olga Aleksandrovna Ladyzhenskaya. On the uniqueness and on the smoothness of weak solutions of the Navier-Stokes equations. *Zapiski Nauchnykh Seminarov POMI*, 5:169–185, 1967.

[Ler34] Jean Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Mathematica*, 63:193–248, 1934.

[LT17] Haesung Lee and Gerald Trutnau. Existence, uniqueness and ergodic properties for time-homogeneous Itô-SDEs with locally integrable drifts and Sobolev diffusion coefficients. *arXiv*, pages arXiv–1708.01152, 2017.

[MBP10] Thilo Meyer-Brandis and Frank Proske. Construction of strong solutions of SDE’s via Malliavin calculus. *Journal of Functional Analysis*, 258(11):3922–3953, 2010.

[MNP15] Salah-Eldin A Mohammed, Torstein K Nilssen, and Frank N Proske. Sobolev differentiable stochastic flows for SDEs with singular coefficients: Applications to the transport equation. *The Annals of Probability*, 43(3):1535–1576, 2015.

[MPMBN+13] Olivier Menoukeu-Pamen, Thilo Meyer-Brandis, Torstein Nilssen, Frank Proske, and Tusheng Zhang. A variational approach to the construction and Malliavin differentiability of strong solutions of SDE’s. *Mathematische Annalen*, 357(2):761–799, 2013.

[Nam20] Kyeongsik Nam. Stochastic differential equations with critical drifts. *Stochastic Processes and their Applications*, 130(9):5366–5393, 2020.

[NO15] Wladimir Neves and Christian Olivera. Well-posedness for stochastic continuity equations with Ladyzhenskaya–Prodi–Serrin condition. *Nonlinear Differential Equations and Applications NoDEA*, 22(5):1247–1258, 2015.

[Pro59] Giovanni Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Annali di Matematica Pura ed Applicata*, 48(1):173–182, 1959.
[Rez14] Fraydoun Rezakhanlou. Regular flows for diffusions with rough drifts. *arXiv preprint arXiv:1405.5856*, 2014.

[Rez16] Fraydoun Rezakhanlou. Stochastically symplectic maps and their applications to the Navier-Stokes equation. *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 33(1):1–22, 2016.

[RZ20] Michael Röckner and Guohuan Zhao. SDEs with critical time dependent drifts: weak solutions. *arXiv preprint arXiv:2012.04161*, 2020.

[Ser62] James Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Archive for Rational Mechanics and Analysis*, 9:187–195, 1962.

[Sha16] AV Shaposhnikov. Some remarks on Davie’s uniqueness theorem. *Proceedings of the Edinburgh Mathematical Society*, 59(4):1019–1035, 2016.

[Sob77] Pavel Evseyevich Sobolevskii. Fractional powers of coercive-positive sums of operators. *Siberian Mathematical Journal*, 18(3):454–469, 1977.

[Ver80] Alexander Yu’evich Veretennikov. On strong solutions and explicit formulas for solutions of stochastic integral equations. *Matematicheski Sbornik*, 153(3):434–452, 1980.

[VK76] A Ju Veretennikov and Nicolai V Krylov. On explicit formulas for solutions of stochastic equations. *Mathematics of the USSR-Sbornik*, 29(2):239–256, 1976.

[XXZZ20] Pengcheng Xia, Longjie Xie, Xicheng Zhang, and Guohuan Zhao. $L^q(L^p)$-theory of stochastic differential equations. *Stochastic Processes and their Applications*, 130(8):5188–5211, 2020.

[Zha05] Xicheng Zhang. Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients. *Stochastic Processes and their Applications*, 115(11):1805–1818, 2005.

[Zha10] Xicheng Zhang. A stochastic representation for backward incompressible Navier-Stokes equations. *Probability Theory and Related Fields*, 148(1-2):305–332, 2010.

[Zha11] Xicheng Zhang. Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electronic Journal of Probability*, 16:1096–1116, 2011.

[Zha16] Xicheng Zhang. Stochastic differential equations with Sobolev diffusion and singular drift and applications. *The Annals of Applied Probability*, 26(5):2697–2732, 2016.

[Zh19] Guohuan Zhao. Stochastic Lagrangian flows for SDEs with rough coefficients. *arXiv preprint arXiv:1911.05562*, 2019.

[Zvo74] Alexander K Zvonkin. A transformation of the phase space of a diffusion process that removes the drift. *Mathematics of the USSR-Sbornik*, 22(1):129, 1974.

[ZZ18] Xicheng Zhang and Guohuan Zhao. Singular Brownian diffusion processes. *Communications in Mathematics and Statistics*, 6(4):533–581, 2018.

[ZZ21] Xicheng Zhang and Guohuan Zhao. Stochastic lagrangian path for Leray’s solutions of 3d navier–stokes equations. *Communications in Mathematical Physics*, 381(2):491–525, 2021.

MICHAEL RÖCKNER: DEPARTMENT OF MATHEMATICS, BIELEFELD UNIVERSITY, GERMANY, AND ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES (CAS), BEIJING, 100190, P.R.CHINA, EMAIL: roeckner@math.uni-bielefeld.de

GUOHUAN ZHAO: DEPARTMENT OF MATHEMATICS, BIELEFELD UNIVERSITY, GERMANY. EMAIL: zhaoguohuan@gmail.com