The Bennequin Number, Kauffman Polynomial, and Ruling Invariants of A Legendrian Link: the Fuchs Conjecture and Beyond

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Abstract

We show that the ungraded ruling invariants of a legendrian link can be realized as certain coefficients of the Kauffman polynomial which are non-vanishing if and only if the upper bound for the Bennequin number given by the Kauffman polynomial is sharp. This resolves positively a conjecture of Fuchs. Using similar methods a result involving the upper bound given by the HOMFLY polynomial and 2-graded rulings is proved.

1 Introduction

Historically, the first examples of invariants distinguishing between Legendrian links of the same topological type were given by the Bennequin number, \( \beta(K) \), and the rotation number \( r(K) \). The Bennequin number can be negative of arbitrary large magnitude within a topological link class, but perhaps the first substantial result of the theory was Bennequin’s upper bound,

\[
\beta(K) + |r(K)| \leq 2g(K) - 1
\]

where \( g(K) \) is the genus of \( K \) (see [B]).

This result instigated much future research which I divide roughly into 2 directions.
1. Find new upper bounds for $\beta(K)$ in terms of topological link invariants, and clarify the relationship between different bounds for $\beta(K)$.

2. Determine the maximal value for $\beta(K)$ within a given knot type, or more generally characterize classes of knots for which a given bound is sharp.

Many results exist in the direction of 1., see \cite{Ng1} for a list of currently known bounds, most of which are known to be independent of one another. In the direction of 2., at present, the maximal value of $\beta(K)$ within a given topological class has been tabulated for knots of 9 crossings and less. Several partial results also exist clarifying when specific bounds for $\beta(K)$ are sharp (see \cite{EF} citeNg1 \cite{Ng2} \cite{T}).

The main results of this paper give simple necessary and sufficient conditions for the upper bounds on $\beta(K)$ given by Kauffman polynomial, $F_K(z,a)$,

$$\beta(K) < -\deg_a F_K$$

and by the HOMFLY polynomial $P_K(z,a)$,

$$\beta(K) + |r(K)| < -\deg_a P_K(z,a)$$

to be sharp (as estimates for $\beta(K)$).

In the case of the Kauffman polynomial bound, the equivalent condition was conjectured by Fuchs in \cite{F}, and is precisely the existence of an (ungraded) ruling of a front diagram for $K$. Interestingly enough, this condition is itself known to be equivalent to the existence of an augmentation on the Legendrian contact DGA \cite{Ch}, \cite{El1}, \cite{S} as defined by Chekanov and Eliashberg \cite{Ch}, \cite{El2}. The number of ($p$-graded) rulings with the $\#\{\text{switches}\} - \#\{\text{left cusps}\} = n$ fixed (see Section 2) has been considered as a combinatorial invariant \cite{ChP}, with the sequence over all $n$ referred to as the complete ruling invariant in \cite{NgS}. In the case of $p$-graded rulings these invariants can distinguish knots with identical classical invariants. On the contrary, our result shows that in the ungraded case the complete ruling invariant is given by certain coefficients of $F_K$ depending only on $\beta(K)$, hence depends only on topological knot type and classical invariants (as conjectured in \cite{NgS}). For knots, the 2-graded ruling invariant is realized in coefficients of the HOMFLY polynomial.

Together these results clarify the relationship between these two upper bounds in an interesting way. It is known \cite{Per} that no inequality exists in general between $\deg_a P_K$ and $\deg_a F_K$. However, in the case when the HOMFLY estimate is sharp, the Kauffman estimate must be sharp as well.
Finally, together with a proposition of Ng [Ng1], our result implies that the Kauffman bound is sharp for alternating links. As Ng notes, this positively answers a question of Ferrand [Fer], who asked whether for alternating links the estimate coming from the Kauffman polynomial should be better than that given by the HOMFLY polynomial.

1.1 Acknowledgements

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2 Preliminaries and Definitions

2.1 Bennequin numbers and Kauffman polynomials

We study Legendrian knots and links in $\mathbb{R}^3$ with respect to the standard contact structure given by the kernel of the 1-form $ydx - dz$. We adopt a diagrammatic perspective where knots are presented by their projection into the $xz$–plane, hence forth referred to as the front diagram or front. A front diagram is a smooth map of (a disjoint union of several copies of) $S^1$ into the plane with no vertical tangents and no singularities save cusps and non-tangential double points. Two Legendrian links are Legendrian isotopic if there fronts can be transformed into one another through a sequence of the following three Legendrian Reidemeister moves (see Figure 1) and planar isotopies through front diagrams.

Given a front diagram of a link $K$ by smoothing cusps and placing the strand with lesser slope on top at crossings we arrive at a diagram of an ordinary topological knot class. We denote this diagram as $\text{Top}(K)$. All topological invariants of $\text{Top}(K)$ will form Legendrian invariants of $K$, since Legendrian equivalence is strictly stronger.

Given an oriented front $K$ let $c(K)$, $cr(K)$ and $w(K)$ be the number of left cusps of $K$, the number of crossings of $K$, and the writhe of $\text{Top}(K)$.
The following is one of many estimates showing that $\beta(K)$ is bounded above within a topological link type.
Lemma 2.1 ([FT], [Fer]) \(\beta(K) < -\deg_a F_K\) where \(F\) denotes the two variable Kauffman Polynomial.

To define the Kauffman Polynomial (Dubrovnik version) \([K]\) first an auxiliary polynomial, \(D_K(z,a)\), is defined up to regular isotopy. \(D_K\) is characterized by skein relations

\[(i)\quad D \begin{array}{c} \circ \\ \circ \end{array} - D \begin{array}{c} \circ \\ \circ \end{array} = z(D \begin{array}{c} \circ \\ \circ \end{array} - D \begin{array}{c} \circ \\ \circ \end{array}) .\]

\[(ii)\quad D \begin{array}{c} 0 \\ 0 \end{array} = aD;\quad D \begin{array}{c} 0 \\ 0 \end{array} = a^{-1}D;\quad D(\emptyset) = 1.\]

and invariance under type II and type III Reidermeister moves. The Kauffman polynomial \(F_K\) is then defined as \(F_K := a^{-w(K)}D_K\). \(D_K\) depends on the diagram representing \(K\), so if we are given a front \(K\) we let \(D_K := D_{Top(K)}\).

Lemma 2.1 has a simpler equivalent statement in terms of the \(D\) polynomial.

Lemma 2.2 For any front \(K\), \(c(K) - 1 \geq \deg_a D_K\) with equality if and only if \(\beta(K) = -\deg_a F_K - 1\)

Proof.

\[w(K) - c(K) = \beta(K) \leq -\deg_a F_K - 1 = w(K) - \deg_a D_K - 1\]
\[\Leftrightarrow c(K) - 1 \geq \deg_a F_K .\]

Note that Lemma 2.2 implies that the sharpness of Lemma 2.1 for a given Legendrian link class is independent of orientation.

2.2 Rulings and \(R_K\)

The following notion of a ruling was introduced independently (but for entirely different purposes) by Fuchs [F] and Chekanov and Pushkar [ChP]. A similar notion (but without the normality condition (iii)) was considered as early as in 1987 by Eliashberg [EP].

By planar isotopy we may assume that all singularities of a front \(K\) have different \(x\)-coordinates, and we will do so henceforth. Given a subset
\( \rho = \{ \lambda_1, \ldots, \lambda_M \} \) of the set of crossings of \( K \), with the \( x \)-coordinate of \( \lambda_i \) denoted \( x_i \) so that \( x_i < x_{i+1} \), let \( S_\rho(K) \) denote the front obtained from \( K \) by resolving all crossings in \( \rho \) to parallel horizontal lines (see Figure 3).

The set \( \rho \) is called a ruling if

(i) every component \( T_j \) of \( S_\rho(K) \) consists of two horizontal strands having one left cusp and no self crossings. The upper is denoted \( U_j \), and the lower \( L_j \).

(ii) for each \( i \), the strands of \( S_\rho(K) \) meeting where \( \lambda_i \) was in \( K \) belong to different components. Call the upper of these strands \( P_i \) and the lower \( Q_i \).

(iii) one of the following normality conditions holds for each \( i \):

For some \( j_1, j_2 \),

(a) \( P_i = L_{j_1} \) and \( Q_i = U_{j_2} \)

(b) \( P_i = U_{j_1} \) and \( Q_i = U_{j_2} \), with the \( z \)-coordinate of \( L_{j_1} \) less than the \( z \)-coordinate of \( L_{j_2} \) at \( x = x_i \)

(c) \( P_i = L_{j_1} \) and \( Q_i = L_{j_2} \), with the \( z \)-coordinate of \( U_{j_1} \) less than the \( z \)-coordinate of \( U_{j_2} \) at \( x = x_i \). (See Figure 4.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{normality_condition}
\caption{Normality condition}
\end{figure}

For example, the ruling shown in Figure 3 meets the normality condition, but it will fail to do so, if we shift \( b \) to the next crossing at the left.
The elements of a ruling $\rho$ are called *switches* and we denote the number of switches in $\rho$ as $s(\rho) := \#\rho$.

For a front $K$, let $\Gamma(K)$ be the set of rulings of $K$.

For integer $n$, let $f_n := \#\{\rho \in \Gamma(K) | s(\rho) = c(K) + 1 = n\}$. To simplify future notation set

\[ j(\rho) := s(\rho) - c(K) + 1 \]

The sequence \( \{f_n\} \) is a Legendrian Isotopy invariant \cite{ChP}, (this can easily be seen by constructing bijections between rulings under Legendrian Reidemeister moves) and we condense it into the *Ruling Polynomial*,

\[ R_K := \sum_{n \in \mathbb{Z}} f_n z^n = \sum_{\rho \in \Gamma(K)} z^{j(\rho)} \]  

(2)

### 2.3 Definition of the polynomial $B$.

Let $D_K(z, a) = \sum_{n \in \mathbb{Z}} C_n(z) a^n$. Define

\[ B_K(z) = C_{c(K)-1}(z) = \text{coefficient of } a^{-1} \text{ in } a^{\beta(K)} F_K. \]

$B_K$ is a Legendrian isotopy invariant and, by Lemma 2, is non-zero iff the Kauffman estimate for the Bennequin number is sharp.

### 3 Main result

**Theorem 3.1** For any Legendrian link $K$, $R_K = B_K$.

**Lemma 3.1** $R_K$ and $B_K$ both satisfy the following skein relations:

(i) \[ R \begin{array}{c} \nearrow \end{array} - R \begin{array}{c} \nearrow \end{array} = z(R \begin{array}{c} \nearrow \end{array} - R \begin{array}{c} \nearrow \end{array}). \]

(ii) \[ R \begin{array}{c} \nearrow \end{array} = R \begin{array}{c} \nearrow \end{array} = 0; \quad R(\begin{array}{c} \nearrow \end{array}) = 1. \]

(iii) \[ R_{K_1 \cup K_2} = z^{-1}R_{K_1} R_{K_2} \]

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**Proof for** $R_K$. Let $L_1, L_2, L_3, L_4$ denote the link diagrams appearing from left to right in relation (i). Divide the rulings $\Gamma(L_1)$ into two subsets, those where the visible crossing is switched, denoted $A(L_1)$, and those where it is not $B(L_1)$. Do the same for $\Gamma(L_2)$. There are obvious bijections $B(L_1) \leftrightarrow B(L_2), A(L_1) \leftrightarrow \Gamma(L_3),$ and $A(L_2) \leftrightarrow \Gamma(L_4)$. The first preserves the number of switches and the second decrease the number by one. Hence from (2),

$$R_{L_1} - R_{L_2} = \sum_{\rho \in \Gamma(L_1)} z^{j(\rho)} - \sum_{\rho \in \Gamma(L_2)} z^{j(\rho)} = \sum_{\rho \in A(L_1)} z^{j(\rho)} - \sum_{\rho \in A(L_2)} z^{j(\rho)} = \sum_{\rho \in \Gamma(L_3)} z^{j(\rho)+1} - \sum_{\rho \in \Gamma(L_4)} z^{j(\rho)+1} = z(R_{L_3} - R_{L_4}).$$

The relation (ii) is obvious, and the relation (iii) follows from a bijection $\theta : \Gamma(K_1) \times \Gamma(K_2) \mapsto \Gamma(K_1 \sqcup K_2)$ such that $j(\theta(\rho_1, \rho_2)) = j(\rho_1) + j(\rho_2) - 1$.

**Proof for** $B_K$. (i) follows since the corresponding pieces of topological diagrams are precisely those in the skein relation for $D_K$ and all diagrams have the same number of cusps.

(ii) follows since we have observed that $B_K$ vanishes when $\beta(K)$ is not maximal.

(iii) follows from the formula $D_{K_1 \sqcup K_2} = \frac{a - a^{-1}}{z} D_{K_1} D_{K_2}$ [K].

To prove that the relations in Lemma 3.1 uniquely characterize a Legendrian Isotopy invariant, we realize any Legendrian link as the product of certain planar tangles.

![Diagram of elementary tangles](image)

**Figure 5:** Elementary tangles

Let $\sigma^N_m$ be the $m$th generator of the $N$ stranded braid group with no over or understrand specified at the crossing, $1 \leq m \leq N - 1$. Let $i^{N,N+2}_m$ be $N$ horizontal lines with a new left cusp appearing between the $m - 1$th and $m$th strand (increasing the number of strands by 2 in the process). Possible values for $m$ are $1, \ldots, N, N + 1$. $r^{N+2,N}_m$ is its mirror about a vertical line (see Figure
Compositions are defined only when the number of strands agrees, and we will hence omit the upper indices from our notation when it will (hopefully) not cause confusion. Certainly not every well defined product represents a Legendrian link but after a planar isotopy every Legendrian link can be represented by such a product. There are several relations corresponding to Legendrian Reidemeister moves and planar isotopy:

Planar isotopy:

\[
\sigma_m \sigma_m = \sigma_m \sigma_m\text{,} \quad \text{if } |m_1 - m_2| \geq 2;
\]

\[
l_{N,N+m} \sigma_m = l_{N,N+m}\text{,} \quad \text{if } m_1 > m_2 + 1;
\]

\[
l_{N,N+m} \sigma_m = l_{N,N+m}\text{,} \quad \text{if } m_2 > m_1 + 1;
\]

\[
l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_1 > m_2 + 1;
\]

\[
l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_2 > m_1 + 1;
\]

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l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_1 > m_2 + 1;
\]

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l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_2 > m_1 + 1;
\]

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l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_1 > m_2 + 1;
\]

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l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_2 > m_1 + 1;
\]

\[
l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_1 > m_2 + 1;
\]

\[
l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_2 > m_1 + 1;
\]

\[
l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_1 > m_2 + 1;
\]

\[
l_{N,N+m} \tau_{m_1} l_{N,N+m} = l_{N,N+m}\tau_{m_1}, \quad \text{if } m_2 > m_1 + 1;
\]

Type 1:

\[
l_m \sigma_{m-1} r_m = id = l_m \sigma_{m+1} r_m
\]

Type 2:

\[
l_{m-1} \sigma_m \sigma_{m-1} = l_m = l_m \sigma_m \sigma_{m+1}
\]

Type 3:

\[
\sigma_{m+1} \sigma_m \sigma_{m+1} = \sigma_m \sigma_{m+1} \sigma_m
\]

The skein relations can be realized as

(i) \[ R_{\ldots \sigma_{m+1} \sigma_m \ldots} - R_{\ldots \sigma_{m+1} \sigma_m \ldots} = z(R_{\ldots \sigma_{m+1} \sigma_m \ldots} - R_{\ldots \sigma_{m+1} \sigma_m \ldots}) \]

(ii) \[ R_{\ldots \tau_{m+1} \tau_m \ldots} = R_{\ldots \tau_{m+1} \tau_m \ldots} = 0, R_{\ldots \sigma_{m+1} \sigma_m \ldots} = R_{\ldots \sigma_{m+1} \sigma_m \ldots} = R_{\ldots \sigma_{m+1} \sigma_m \ldots} = R_{\ldots \sigma_{m+1} \sigma_m \ldots} = 1 \]

(iii) \[ R_{K_1 \cup K_2} = z^{-1} R_{K_1} R_{K_2} \]

The second entry of (ii) is implied by the first, but we include it for convenience.

**Lemma 3.2** By repeated evaluation of the skein relation a formula for \( R_K \) can be found in terms of the \( R \) polynomials of Legendrian links with less crossings and links whose values are specified by (ii).
Proof. Note that since the fronts on the RHS of (i) have less crossings than the fronts on the LHS, if the theorem holds for one of the fronts on the LHS it must hold for the other. As a consequence, given a link described as a word, \( W \), in the above planar tangles, it is enough to show the following statement.

(A) By substituting into the above relations corresponding to Legendrian isotopy and interchanging \( l_{m+1}\sigma_m \) and \( l_m\sigma_{m+1} \), \( W \) may be reduced to a word with less crossings or to a word whose \( R \)-polynomial is known by (ii).

We refer to the interchange of \( l_{m+1}\sigma_m \) and \( l_m\sigma_{m+1} \) as a skein move.

Statement (A) is proved by nested inductions, the outer being on \( L := \) the number of left cusps of \( W \). The base case is handled entirely by (ii). Now, assuming the statement for \( L - 1 \) we prove it for \( L \) by the following induction.

By looking at the left cusp located farthest to the right in \( W \) we can write \( W = Xl_{m}^{N - 2}NY \) where \( Y \) is a word in the \( \sigma_i \) and \( r_i \). Our inner induction is on \( M := N + cr(Y) \). The base case \((M = 2)\) is handled by (iii) and the outer inductive hypothesis (for then we have a disjoint copy of the Legendrian unknot at the end of \( W \)).

For general \( M \), given a word of the form

\[
Xl_{m}(\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y; N_1, N_2 \geq 0
\]

we give a procedure depending on the first letter of \( Y \) to either reduce one of \( N \) or \( cr((\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y) \) or increase one of \( N_1 \) and \( N_2 \). Since \( N_1 \) and \( N_2 \) can only be increased a finite number of times this will complete the inductive step.

Note that \((\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})\) commutes with \((\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})\) by planar isotopy.

Case 1: \( Y = \sigma_iY' \)

SubCase 1: \( i < m - N_1 - 1 \) or \( i > m + N_2 + 1 \).

By planar isotopy \( \sigma_i \) commutes with

\[
l_{m}(\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})
\]

so can be absorbed into \( X \) decreasing \( cr(Y) \).

SubCase 2: \( i = m - N_1 - 1 \) or \( i = m + N_2 + 1 \).

Add \( \sigma_i \) at the end of the appropriate parenthesis expression increasing \( N_1 \) or \( N_2 \).

SubCase 3: \( i = m - N_1 \) or \( i = m + N_2 \) but \( i \neq m \).
We deal with the first possibility since the second is similar. Note that after a skein move,

\[ l_m(\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2}) \]

becomes

\[ l_{m-1}(\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_m\sigma_{m+1}\ldots\sigma_{m+N_2}). \]

Applying the skein move \( N_1 \) times we arrive at

\[
Xl_{m-N_1}(\sigma_{m-N_1+1}\ldots\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y \\
= Xl_{m-N_1}(\sigma_{m-N_1+1}\ldots\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})\sigma_{m-N_1}Y' \\
= Xl_{m-N_1}(\sigma_{m-N_1+1}\sigma_{m-N_1+2}\ldots\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y'
\]

for which a type II Legendrian Reidemeister move removes two crossings.

SubCase 4: \( m - N_1 > i > m \) or \( m < i < m + N_2 \)

Again we consider just the first possibility,

\[
Xl_{m}(\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y \\
= Xl_{m}(\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})\sigma_i Y' \\
= Xl_{m}(\sigma_{m-1}\ldots\sigma_i\sigma_{i-1}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y' \\
= Xl_{m}(\sigma_{m-1}\ldots\sigma_i\sigma_{i-1}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y' \\
= X\sigma_{i-1}l_{m}(\sigma_{m-1}\ldots\sigma_i\sigma_{i-1}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y'
\]

decreasing \( cr(Y) \) by 1. The 3rd equality is a type 3 Legendrian Reidemeister move. The rest are planar isotopy.

SubCase 5: \( i = m \)

If exactly one of \( N_1 \) or \( N_2 \) is 0 a type 2 Reidemeister can be applied. If they are both 0 the value of the polynomial is 0 by (ii). If neither are zero we have

\[
Xl_{m}(\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})Y \\
= Xl_{m}(\sigma_{m-1}\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+N_2})\sigma_m Y' \\
= Xl_{m}\sigma_{m-1}\sigma_{m+1}\sigma_m(\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+2}\sigma_{m+2}\ldots\sigma_{m+N_2})Y' \\
= Xl_{m-1}\sigma_m\sigma_{m+1}\sigma_m(\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+2}\sigma_{m+2}\ldots\sigma_{m+N_2})Y' \\
= Xl_{m-1}\sigma_{m+1}\sigma_m\sigma_{m+1}(\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+2}\sigma_{m+2}\ldots\sigma_{m+N_2})Y' \\
= X\sigma_{m+1}l_{m-1}\sigma_m\sigma_{m+1}(\sigma_{m-2}\ldots\sigma_{m-N_1})(\sigma_{m+2}\sigma_{m+2}\ldots\sigma_{m+N_2})Y'
\]

decreasing \( c(Y) \) by 1. The equalities are assumption, planar isotopy, skein move (i), type 3 Legendrian Reidemeister move, and planar isotopy respectively.
Case 2. \(Y = r_iY'\)

Again by vertical symmetry of all relations involved we assume without loss of generality that \(i \leq m\).

SubCase 1: \(i < m - N_1 - 1\)

By planar isotopy \(r_i\) commutes with

\[l_m(\sigma_{m-1}\sigma_{m-2}\cdots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\cdots\sigma_{m+N_2})\]

so can be absorbed into \(X\) decreasing \(N\).

SubCase 2: \(i = m - N_1 - 1\)

Applying the skein move \(N_1\) times we arrive at

\[Xl_{m-N_1}r_{m-N_1-1}\]

which is has a zig-zag.

SubCase 3: \(i = m - N_1\)

The inclusion of the factor \(\sigma_{m-N_1}r_{m-N_1}\) shows that the polynomial is 0 by (ii).

SubCase 4: \(m - N_1 > i > m\)

A type 2 Legendrian Reidemeister move may be applied after a planar isotopy.

\[Xl_m(\sigma_{m-1}\sigma_{m-2}\cdots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\cdots\sigma_{m+N_2})r_iY'
= Xl_m(\sigma_{m-1}\cdots\sigma_i\sigma_{i-1}r_i\cdots\sigma_{m-N_1})(\sigma_{m+1-2}\sigma_{m+2-2}\cdots\sigma_{m+N_2-2})Y'.\]

The presence of \(\sigma_i\sigma_{i-1}r_i\) allows two crossings to be removed with a type 2 move.

SubCase 5: \(i = m\)

Again there are 4 cases. \(N_1 = N_2 = 0\) implies we have a disjoint unknot in the middle of the link diagram so we apply (iii) and the outer inductive hypothesis. If exactly 1 is zero we can apply a type 1 Reidemeister move to remove a crossing. In the final case we have

\[Xl_m(\sigma_{m-1}\sigma_{m-2}\cdots\sigma_{m-N_1})(\sigma_{m+1}\sigma_{m+2}\cdots\sigma_{m+N_2})r_mY'
= Xl_m\sigma_{m-1}\sigma_{m+1}r_m(\sigma_{m-2}\cdots\sigma_{m-N_1})(\sigma_{m+2-2}\sigma_{m+2-2}\cdots\sigma_{m+N_2-2})Y'
= Xl_{m-1}\sigma_m\sigma_{m+1}r_m(\sigma_{m-2}\cdots\sigma_{m-N_1})(\sigma_{m+2-2}\sigma_{m+2-2}\cdots\sigma_{m+N_2-2})Y'.\]

where we used planar isotopy and the skein move to arrange the sequence \(\sigma_m\sigma_{m+1}r_m\), allowing a type 2 move.

This concludes the proof.
Corollary 3.1 The skein relations in Lemma 3.2 uniquely characterize a Legendrian Isotopy invariant.

Theorem 3.1 now follows from Lemma 3.2 and Corollary 3.1

4 Oriented Rulings and the HOMFLY Polynomial

4.1 HOMFLY polynomial

With an appropriate strengthening of the notion of ruling a very similar situation holds with regard to the HOMFLY estimate for \( \beta(K) \). First we recall a construction of the HOMFLY polynomial \([13]\). An auxiliary polynomial \( H(z,a) \) is calculated from oriented diagrams according to the skein relations

\[
(i) \quad H \xrightarrow{=} - H \xrightarrow{=} zH \\
(ii) \quad H \xrightarrow{=} aH \xrightarrow{=} H \xrightarrow{=} a^{-1}H \xrightarrow{=} H(\emptyset) = 1.
\]

and invariance under regular isotopy.

The HOMFLY polynomial \( P_K \) is then defined by the normalization \( P_K = a^{-\deg a} P_K \).

Lemma 4.1 ([FT]) \( \beta(K) < -\deg_a P_K \)

We define an invariant of oriented Legendrian links by \( Q_K(z) = C_{c(k)}(-1)(z) \), where \( H_K(z,a) = \sum_{n \in \mathbb{Z}} C_n(z)a^n \). As in Section 2., the non-vanishing of \( \sum_{n \in \mathbb{Z}} C_n(z)a^n \) will be equivalent to the sharpness of Lemma 4.1.

4.2 Oriented Ruling polynomial

We call a ruling of a front \( K \) oriented if all switches are positive crossings (in the sense of writhe). That is, at switches arrows should point in the same
horizontal direction. Let $\Omega(K)$ denote the set of oriented rulings of a front $K$. Define the oriented ruling polynomial of $K$,

$$OR_K = \sum_{\rho \in \Omega(K)} z^{j(\rho)}$$

$OR_K$ can be seen to be a Legendrian isotopy invariant by constructing bijections between oriented rulings under Legendrian Reidemeister moves.

**Remark.** For knots this follows from known results since for knots a ruling is oriented if and only if it is 2-graded.

**Theorem 4.1** $OR_K = Q_K$.

Proof is similar to that of Theorem 1. Both polynomials are easily seen to satisfy the Legendrian skein relations

(i) $Q \begin{array}{c} \rightarrow \\ \leftarrow \end{array} - Q \begin{array}{c} \rightarrow \\ \leftarrow \end{array} = zQ \begin{array}{c} \rightarrow \\ \leftarrow \end{array}$

(ii) $Q \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - Q \begin{array}{c} \rightarrow \\ \leftarrow \end{array} = zQ \begin{array}{c} \rightarrow \\ \rightarrow \end{array}$

(iii) $Q \begin{array}{c} \rightarrow \\ \rightarrow \end{array} - Q \begin{array}{c} \rightarrow \\ \rightarrow \end{array} = zQ \begin{array}{c} \rightarrow \\ \rightarrow \end{array}$

and the same proof shows that these relations uniquely characterize a Legendrian isotopy invariant.
5 Corollaries

Corollary 5.1 If a Legendrian link $K$ admits a ruling then it maximizes $\beta(K)$ within its topological isotopy class.

Corollary 5.2 The sharpness of the estimate for $\beta(K)$ given by the HOMFLY polynomial implies the sharpness of the estimate given by the Kauffman polynomial.

This is somewhat surprising since it is known that no inequality between the two estimates exists in general [Fer]. As a result we can strengthen the estimate given by $P_K$ for some links

Corollary 5.3 If $\deg_a P_K > \deg_a F_K$ then

$$\beta(K) < -\deg_a P_K - 1$$

Corollary 5.4 For an alternating link $L$, $\deg_a P_L \leq \deg_a F_K$.

Proof. In [Ng1], Proposition 11 shows that any alternating link has a Legendrian representative admitting a ruling. Hence, the estimate coming from the Kauffman polynomial is sharp.

This result is conjectured by Ferrand [Fer] using slightly different language ([Fer] uses slightly different conventions for the link polynomials in question as well as different conventions for front diagrams).

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