ON THE MOD $p$ LANNES-ZARATI HOMOMORPHISM

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Abstract. The mod 2 Lannes-Zarati homomorphism was constructed in [21], which is considered as a graded associated version of the mod 2 Hurewicz map in the $E_2$-term of Adams spectral sequence. The map is studied by many authors such as Lannes-Zarati [21], Hưng [14], [15], [16], Hưng et. al. [18], Chơn-Triết [6]. In this paper, we construct an analogue $\varphi_s$ for $p$ odd, and we also investigate the behavior of this map for $s \leq 3$.

1. Introduction and statement of results

For any pointed space $X$, let $QX = \lim_{\longrightarrow} \Omega^n\Sigma^n X$ be the infinite loop space of $X$. An element $\xi \in H_*(QX) = H_*(QX; F_p)$ is called a spherical class if there exists an element $\eta \in \pi_*(QX) = \pi^S_*(X)$ such that $h_*(\eta) = \xi$, where $h_* : \pi_*(QX) \longrightarrow H_*(QX)$ is the Hurewicz map. For $p = 2$, work of Curtis [10] shows that the Hopf invariant one elements and the Kervaire invariant one elements in $\pi^S_*(Q_0S^0)$ (if they exist) are those whose images are nontrivial in $H_*(Q_0S^0)$ under the mod 2 Hurewicz map $h_* : \pi_*(Q_0S^0) \longrightarrow H_*(Q_0S^0)$; and he conjectured that there are only spherical classes in $H_*(Q_0S^0)$ those are detected by the Hopf invariant one elements and the Kervaire invariant one elements, where $Q_0S^0$ is the component of $QS^0$ containing the basepoint. Later, Wellington [30] generalized the Curtis’ result for $p > 2$ and he was led to an analogue conjecture. These conjectures are called the classical conjecture on spherical classes.

An algebraic approach to attack the conjecture is to study the graded associated of the mod $p$ Hurewicz map $h_* : \pi_*(Q_0S^0) \longrightarrow H_*(Q_0S^0)$ in $E_2$-term of the Adams spectral sequence.

For $p = 2$, this homomorphism was constructed by Lannes and Zarati [21], the so-called Lannes-Zarati homomorphism. In more detail, for each $s \geq 1$, there is a homomorphism of Singer’s type

$$\varphi_s : \text{Ext}^s_A(F_2, F_2) \longrightarrow \text{Ann}(D[s]#),$$

where $D[s]$ is the (graded) dual of the Dickson algebra $D[s]$, $D$ is the Singer’s functor (see [21]); and we denote $\text{Ann}(M)$ the subspace of $M$ consisting of all elements annihilated by all positive elements in the Steenrod algebra $A$. The behavior of $\varphi_s$ is actually studied by Lannes-Zarati [21] (for $s \leq 2$), Hưng [14] (for $s = 3$), Hưng [16] (for $s = 4$), Hưng-Quỳnh-Tuân [18] (for $s = 5$) and Chơn-Triết [6] (for $s = 6$ and stem $\leq 114$).

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For $p > 2$, from results of Zarati [31], the $s$th left derived functor $\mathcal{D}_s(\Sigma^{1-s}F_p)$ of the destabilization functor is isomorphic to $\Sigma^sR_s(F_p) \cong \Sigma^s[\mathcal{B}[s]]$, where $\mathcal{B}[s]$ is the image of the restriction from cohomology of the symmetric group $\Sigma_p$ to the cohomology of the elementary $p$-group of rank $s$, $E_s$ [25]. Therefore, there exists an analogue homomorphism, which is also called Lannes-Zarati homomorphism,

$$\varphi_s : \text{Ext}^s_{A}(F_p, F_p) \rightarrow \text{Ann}(\mathcal{B}[s])_t.$$ 

Using Goodwillie towers, Kuhn also pointed out the existence of $\varphi_s$ as a graded associated version of the Hurewicz map in the $E_2$-term of the Adams spectral sequence [20]. In addition, the method of Kuhn can be apply to other generalized cohomology theories.

In the paper, we are interested in the study of the mod $p$ Lannes-Zarati homomorphism for $p$ odd. We show that, up to a sign, the canonical inclusion $B[s] \hookrightarrow \Gamma^+_s$ is the chain-level representation of the dual of $\varphi_s$.

$$\varphi^\#_s : F_p \otimes_A \mathcal{B}[s] \rightarrow \text{Tor}^4_{A}(F_p, F_p),$$

where $\Gamma^+_s = \oplus_{s \geq 0} \Gamma^+_s$ is the Singer-Hung-Sum chain complex [19]. In more detail, we obtain the following theorem, which is the first main result of the paper.

**Theorem 4.6** The inclusion map $\tilde{\varphi}^\#_s : \mathcal{B}[s] \rightarrow \Gamma^+_s$ given by

$$\gamma \mapsto (-1)^{\frac{(s-1)(s+1)}{2} + (s+1) \deg(\gamma)} Q^I$$

is the chain-level representation of the dual of the Lannes-Zarati homomorphism $\varphi^\#_s$.

The theorem is an extended of Theorem 3.9 in [15] for $p$ odd.

Let $\Lambda^{opp}$ be the opposite algebra of the Lambda algebra defined by Bousfield et al. [3], and let $R$ be the Dyer-Lashof algebra, which is the algebra of homology operations acting on the homology of infinite loop spaces. The algebra $R$ is also isomorphic to a quotient of $\Lambda^{opp}$ [19]. It is well-known that $\mathcal{B}[s] \cong R_s$ and $(\Gamma^+_s)^\# \cong \Lambda^{opp}$ [19], where $R_s$ is the subspace of $R$ spanned by all monomials of length $s$. Therefore, in the dual, up to a sign, the canonical projection $\Lambda^{opp} \rightarrow R_s$ is the chain-level representation of the Lannes-Zarati homomorphism $\varphi_s$, which is given by the following corollary.

**Corollary 4.7** The projection $\tilde{\varphi}_s : \Lambda^{opp} \rightarrow R_s$ given by

$$\tilde{\varphi}_s(\lambda_I) = (-1)^{\frac{(s-1)(s+1)}{2} + (s+1) \deg(\lambda_I)} Q^I$$

is the chain-level representation of the Lannes-Zarati homomorphism $\varphi_s$.

From Liulevicius [22], [23] and May [24], there exists the the power operation $P^0$ acting on the cohomology of the Steenrod algebra $\text{Ext}^s_{A}(F_p, F_p)$ whose chain-level representation in the cobar complex is induced from the Frobenius map. Since its representation in $\Lambda^{opp}$ induces naturally an operation in $R$ (seeLemma 5.1), and the latter is compatible with the $A$-action on $R$ (see Lemma 5.2), then there exists an power operation acting on $\text{Ann}(R_s)$, which is also denoted by $P^0$. Furthermore, these power operations commute with each other through the Lannes-Zarati homomorphism $\varphi_s$ (see Proposition 5.3). Using these results to study the behavior of $\varphi_s$, we obtain the following results.
Theorem 6.1. The first Lannes-Zarati homomorphism
\[ \varphi_1 : \text{Ext}^{1,1+t}(F_p, F_p) \longrightarrow \text{Ann}(B[1]^\#)_t \]
is isomorphic.

This result is an analogue of the case \( p = 2 \) [21]. The behavior of \( \varphi_2 \) is given by the theorem.

Theorem 6.2. The second Lannes-Zarati homomorphism
\[ \varphi_2 : \text{Ext}^{2,2+t}(F_p, F_p) \longrightarrow \text{Ann}(B[2]^\#)_t \]
is vanishing for \( t \neq 0 \) and \( t \neq 2(p-1)p^{i+1} - 2, i \geq 0 \).

From the result of Wellington [30], \( \text{Ann}(R_2) \) is nontrivial at stem \( t = 0, t = 2(p-1)p^{i+1} - 2 \) and \( t = 2(p-1)p^{i+1} - 2 \). Therefore, \( \varphi_2 \) is not an epimorphism.

The behavior of \( \varphi_3 \) is given by the following theorem, which is the final result of this work.

Theorem 6.3. The third Lannes-Zarati homomorphism
\[ \varphi_3 : \text{Ext}^{3,3+t}(F_p, F_p) \longrightarrow \text{Ann}(B[3]^\#)_t \]
is vanishing for all \( t > 0 \).

From the above results, we observe that the map \( \varphi_s \), for \( s \leq 3 \), is only nontrivial in positive stem corresponding with the Hopf invariant one and Kervaire invariant one elements (if they exist). Based on these results together with the classical conjecture on spherical classes and Hưng’s conjecture [14, Conjecture 1.2], we are led to a conjecture, which is considered as a graded associated version of the classical one on the spherical classes in \( E_2 \)-term of the Adams spectral sequence, as follows.

Conjecture 1.1. The homomorphism \( \varphi_s \) vanishes in any positive stem \( t \) for \( s \geq 3 \).

Of course, the classical conjecture on spherical class is not a consequence of Conjecture 1.1. But if Conjecture 1.1 were false on a permanent cycle in \( \text{Ext}^{s,s+t}(F_p, F_p) \), then the classical conjecture on spherical classes could be also false.

The Singer transfer was introduced by Singer [29] (for \( p = 2 \) and Crossley [9] (for \( p > 2 \), which is given by, for \( s \geq 1 \),
\[ Tr_s : \text{Ann}(H_*BE_s)_{GL_s} \longrightarrow \text{Ext}^{s,s+t}(F_p, F_p), \]
where \( GL_s \) is the general linear group. The results of Singer [29], Boardman [2], Hà [12], Nam [27], Chơn-Hà [4, 5] (for \( p = 2 \) and Crossley [9] (for \( p \) odd) showed that the image of Singer transfer is a big enough and worthwhile to pursue subgroup of the Ext group. It is well-known that the Ext group is too mysterious to understand, although it is intensively studied. In order to avoid the shortage of our knowledge of the Ext group, we want to restrict \( \varphi_s \) on the image of the Singer transfer. Then we have a weak version of Conjecture 1.1.

Conjecture 1.2. The composition
\[ j_s := \varphi_s \circ Tr_s : \text{Ann}(H_*BE_s)_{GL_s} \longrightarrow \text{Ann}(B[s]^\#)_t \]
is vanishing in any positive degree \( t \) for \( s \geq 3 \).

From Theorem 4.6, Theorem A.2 and Proposition A.12, it is clear that, up to a sign, the canonical inclusion \( B[s] \rightarrow H^*BE_s \) is the chain-level representation of the dual of \( j_s \). Thus, Conjecture 1.2 is equivalent to the following conjecture.
Conjecture 1.3. For $s \geq 3$, $\mathcal{B}[s] \subset \bar{A}H^*BE_s$, where $\bar{A}$ is the augmentation ideal of $A$.

For $p = 2$, Conjecture 1.2 and Conjecture 1.3 appeared in [14, Conjecture 1.3 and Conjecture 1.5], which were showed by Hưng-Nam in [17].

The paper is organized as follows. Section 2 is a preliminary on the Singer-Hưng-Sum chain complex, the Lambda algebra and the Dyer-Lashof algebra. In Section 3 and 4, we construct the mod $p$ Lannes-Zarati homomorphism and its chain-level representation. Section 5 is devoted to develop the power operations. The behavior of the Lannes-Zarati homomorphism is investigated in Section 6. The chain-level representation of the Singer transfer in Singer-Hưng-Sum chain complex is established in Appendix section.

2. Preliminaries

In this section, we recall some preliminaries about the Singer-Hưng-Sum chain complex, the Lambda algebra as well as the Dyer-Lashof algebra (see [19], [3], [28] and [8] for more detail).

2.1. The Singer-Hưng-Sum chain complex. Let $E_s$ be the $s$-dimensional $\mathbb{F}_p$-vector space, where $p$ is an odd prime number. It is well-known that the mod $p$ cohomology of the classifying space $BE_s$ is given by

$$P_s := H^*BE_s = E(x_1, \cdots, x_s) \otimes \mathbb{F}_p[y_1, \cdots, y_s],$$

where $(x_1, \cdots, x_s)$ is the basis of $H^1BE_s = \text{Hom}(E_s, \mathbb{F}_p)$, and $y_i = \beta(x_i)$ for $1 \leq i \leq s$ with $\beta$ the Bockstein homomorphism.

Let $GL_s$ denote the general linear group $GL_s = GL(E_s)$. The group $GL_s$ acts on $E_s$ and then on $H^*BE_s$ according to the following standard action

$$(a_{ij}) y_s = \sum_i a_{is} y_i, \quad (a_{ij}) x_s = \sum_i a_{is} x_i, \quad (a_{ij}) \in GL_s.$$

The algebra of all invariants of $H^*BE_s$ under the actions of $GL_s$ is computed by Dickson [11] and Mùi [25]. We briefly summarize their results. For any $n$-tuple of non-negative integers $(r_1, \cdots, r_s)$, put $[r_1, \cdots, r_s] := \det(y_i^{p^j})$, and define

$$L_{s,i} := [0, \cdots, i, \cdots, s]; \quad L_s := L_{s,s}; \quad q_{s,i} := L_{s,i}/L_s,$$

for any $1 \leq i \leq s$.

In particular, $q_{s,s} = 1$ and by convention, set $q_{s,i} = 0$ for $i < 0$. Degree of $q_{s,i}$ is $2(p^s - p^i)$. Define

$$V_s := V_s(y_1, \cdots, y_s) := \prod_{\lambda_j \in \mathbb{F}_p} (\lambda_1 y_1 + \cdots + \lambda_{s-1} y_{s-1} + y_s).$$

Another way to define $V_s$ is that $V_s = L_s/L_{s-1}$. Then $q_{s,i}$ can be inductively expressed by the formula

$$q_{s,i} = q_{s-1,i-1}^p + q_{s-1,i} V_{s-1}^{p-1}.$$
For non-negative integers $k, r_{k+1}, \ldots, r_s$, set

$$[k; r_{k+1}, \ldots, r_s] := \frac{1}{k!} \begin{vmatrix} x_1 & \cdots & x_s \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_s \\ y^r_{1s} & \cdots & y^r_{s,s} \\ \vdots & \ddots & \vdots \\ y^r_{1s} & \cdots & y^r_{s,s} \end{vmatrix}.$$  

For $0 \leq i_1 < \cdots < i_k \leq s - 1$, we define

$$M_{s;i_1,\ldots,i_k} := [k; 0, \ldots, \hat{i}_1, \ldots, \hat{i}_k, \ldots, s - 1],$$

$$R_{s;i_1,\ldots,i_k} := M_{s;i_1,\ldots,i_k} L_s^{p-2}.$$  

The degree of $M_{s;i_1,\ldots,i_k}$ is $k + 2((p^s - 1) - (p^{s-1} + \cdots + p^{i_k}))$ and then the degree of $R_{s;i_1,\ldots,i_k}$ is $k + 2(p^s - 1) - 2(p^{s-1} + \cdots + p^{i_k})$.  

The subspace of all invariants of $H^*BE_s$ under the action of $GL_s$ is given by the following theorem.

**Theorem 2.1** (Dickson [11], Mùi [25]).  
(1) The subspace of all invariants under the action of $GL_s$ of $F_p[x_1, \ldots, x_s]$ is given by

$$D[s] := F_p[x_1, \ldots, x_s]^{GL_s} = F_p[q_1, \ldots, q_{s-1}].$$  
(2) As a $D[s]$-module, $(H^*BE_s)^{GL_s}$ is free and has a basis consisting of 1 and all elements of $\{R_{s;i_1,\ldots,i_k} : 1 \leq k \leq s, 0 \leq i_1 < \cdots < i_k \leq s - 1\}$. In other words,

$$(H^*BE_s)^{GL_s} = D[s] \oplus \bigoplus_{k=1}^{s} \bigoplus_{0 \leq i_1 < \cdots < i_k \leq s-1} R_{s;i_1,\ldots,i_k} D[s].$$  
(3) The algebraic relations are given by

$$R_{s;i}^2 = 0,$$

$$R_{s;i_1} \cdots R_{s;i_k} = (-1)^{k(k-1)/2} R_{s;i_1,\ldots,i_k} q_{s,0}^{k-1}$$

for $0 \leq i_1 < \cdots < i_k < s$.

Let $\mathcal{B}[s]$ be the subalgebra of $(H^*BE_s)^{GL_s}$ generated by

(1) $q_{s,i}$ for $0 \leq i \leq s - 1$,

(2) $R_{s;i}$ for $0 \leq i \leq s - 1$,

(3) $R_{s;i,j}$ for $0 \leq i < j \leq s - 1$.

Mùi [25] show that the algebra $\mathcal{B}[s]$ is the image of the restriction from the cohomology of the symmetric group $\Sigma_p$ to the cohomology of the elementary abelian $p$-group of rank $s$, $E_s$.

Let $\Phi_s := H^*BE_s[L_s^{-1}]$ be the localization of $H^*BE_s$ obtained by inverting $L_s$. It should be noted that $L_s$ is the product of all non-zero linear forms of $y_1, \ldots, y_s$. So inverting $L_s$ is equivalent to inverting all these forms. The action of $GL_s$ on $H^*BE_s$ extends an action of it on $\Phi_s$. Set

$$\Delta_s := \Phi_s^{T_s}, \quad \Gamma_s := \Phi_s^{GL_s},$$

where $T_s$ is the subgroup of $GL_s$ consisting of all upper triangle matrices with 1’s on the main diagonal.
Put \( u_i := M_{t_{ii}-1}/L_{t_{ii}-1} \) and \( v_i := V_i/g_{t_{ii}-1}. \) Then, from [19], we have
\[
\Delta_s = E(u_1, \ldots, u_s) \otimes \mathbb{F}_p[v_1^{i_1}, \ldots, v_s^{i_s}]
\]
\[
\Gamma_s = E(R_{s,0}, \ldots, R_{s,s-1}) \otimes \mathbb{F}_p[q_1^{i_1}, \ldots, q_{s,s-1}].
\]

Let \( \Delta_+^s \) be the subspace of \( \Delta_s \) spanned by all monomials of the form
\[
v_1^{i_1} v_1^{(p-1)i_1-i_1} \cdots v_s^{i_s} v_s^{(p-1)i_s-i_s}, s \in \{0, 1\}, 1 \leq s, j_1 \geq 1 \text{, and let } \Gamma_+^s := \Gamma_s \cap \Delta_+^s.
\]

From [19], \( \Gamma^+ = \oplus_{s \geq 0} \Gamma_+^s \) is a graded differential \( A \)-algebra with the differential induced by
\[
\partial(u_1^{i_1} v_1^{i_1} \cdots u_s^{i_s} v_s^{i_s}) = \begin{cases} (-1)^{i_1 + \cdots + i_s-1} u_1^{i_1} v_1^{i_1} \cdots u_s^{i_s-1} v_s^{i_s-1}, & \epsilon_s = -i_s = 1; \\ 0, & \text{otherwise,} \end{cases}
\]
where \( \Gamma_0^+ = \mathbb{F}_p. \)

For any \( A \)-module \( M \), we define the stable total power \( St_s(x_1, y_1, \ldots, x_s y_s; m) \), for \( m \in M \), as follows
\[
St_s(x_1, y_1, \ldots, x_s y_s; m) := \sum_{\epsilon=0,1, i_j \geq 0} (-1)^{\epsilon_1 + i_1 + \cdots + \epsilon_s + i_s} u_1^{i_1} v_1^{i_1} \cdots v_s^{i_s-1} v_s^{i_s-1} \eta_1 \cdots v_s^{i_s-1} \eta_m(m).
\]

For convenience, we put \( St_s(m) := St_s(x_1, y_1, \ldots, x_s y_s; m) \). And let \( St_s(M) = \{ St_s(m) : m \in M \} \).

Then \( \Gamma^+ M := \oplus_{s \geq 0} (\Gamma^+ M)_s \), where \((\Gamma^+ M)_0 = M \) and \((\Gamma^+ M)_s = \Gamma_+^s St_s(M) \), is a differential module with its differential given by, for \( \gamma = \sum_{\epsilon, \ell} \gamma_\epsilon, \ell u_1^{i_1} v_1^{i_1} \cdots v_s^{i_s-1} v_s^{i_s-1} \eta_1 \cdots v_s^{i_s-1} \eta_m \in \Gamma_+^s \) and \( m \in M \), where \( \gamma_\epsilon, \ell \in \Gamma_+^s \),
\[
\partial(\gamma St_s(m)) = (-1)^{\deg \gamma + 1} \sum_{\epsilon, \ell} (-1)^{\epsilon_1 + \cdots + \epsilon_s} \gamma_\epsilon, \ell St_{s-1}(\beta^{1-\epsilon} \beta^{\ell} m).
\]

In [19], Hưng and Sum showed that \( H_s(\Gamma^+ M) \cong \text{Tor}^A_1(\mathbb{F}_p, M) \) for any \( A \)-module \( M \). Therefore, \( \Gamma^+ M \) is a suitable complex to compute \( \text{Tor}^A_1(\mathbb{F}_p, \mathbb{F}_p) \).

2.2. The Lambda algebra and the Dyer-Lashof algebra. In [23], Bousfield et. al. defined the Lambda algebra \( \Lambda \), that is a differential algebra for computing the cohomology of the Steenrod algebra. In [28], Priddy showed that the opposite of the Lambda algebra \( \Lambda^{opp} \) is isomorphic to the co-Koszul complex of the Steenrod algebra.

Recall that \( \Lambda^{opp} \) is a graded differential algebra generated by \( \lambda_{i-1} \) of degree \( 2i(p-1) - 1 \) and \( \mu_{i-1} \) of degree \( 2i(p-1) \) subject to the adem relations
\[
\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1} + m \lambda_{j-1} + m = 0,
\]
\[
\sum_{i+j=n} \binom{i+j}{i} (\lambda_{i-1} + m \mu_{j-1} + m - \mu_{i-1} + m \lambda_{j-1} + m) = 0,
\]
\[
\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1} + m \mu_{j-1} + m = 0.
\]
\[
\sum_{i+j=n} \binom{i+j}{i} \mu_{i+pm} \mu_{i-1+m} = 0,
\]
for all \(m \geq 0\) and \(n \geq 0\).

And the differential is given by
\[
d(\lambda_{n-1}) = \sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1} \lambda_{j-1},
\]
\[
d(\mu_{n-1}) = \sum_{i+j=n} \binom{i+j}{i} (\lambda_{i-1} \mu_{j-1} - \mu_{i-1} \lambda_{j-1}),
\]
\[
d(\sigma \tau) = (-1)^{\deg \sigma} \sigma d(\tau) + d(\sigma) \tau.
\]

Let \(\Lambda^{opp}\) be the subspace of \(\Lambda^{opp}\) spanned by all monomials of length \(s\). By the adem relations, \(\Lambda^{opp}\) has an additive basis consisting of all admissible monomials (which are monomials of the form \(\lambda_I = \lambda_{i_1}^{\epsilon_1} \cdots \lambda_{i_s}^{\epsilon_s} \in \Lambda^{opp}_s\) satisfying \(p k - \epsilon_k \geq i_{k-1}\) for \(2 \leq k \leq s\), where \(\lambda_{i_k} = 1\) if \(\epsilon = 1\) and \(\mu_{i_k} = 1\) if \(\epsilon = 0\). Let \((\Lambda^{opp}_s)^\#\) be the dual of \(\Lambda^{opp}_s\) and let \((\lambda_{i_1}^{\epsilon_1} \cdots \lambda_{i_s}^{\epsilon_s})^\#\) be the dual basis of the admissible basis.

By the same method of Hưng-Sum [19], it is easy to show that the map \(\kappa_s : \Gamma^+_s \longrightarrow (\Lambda^{opp}_s)^\#\) given by
\[
\kappa_s(u_1^{\epsilon_1} u_2^{\epsilon_2} \cdots u_s^{\epsilon_s} (p-1)^{i_1} \cdots (p-1)^{i_s} \epsilon_s) = (-1)^{i_1+\cdots+i_s} (\lambda_{i_1}^{\epsilon_1} \cdots \lambda_{i_s}^{\epsilon_s})^\#
\]
is an isomorphism of differential modules over \(A\).

An important quotient algebra of \(\Lambda^{opp}\) is the Dyer-Lashof algebra \(R\), which is also well-known as the algebra of homology operations acting on the homology of infinite loop spaces.

For any admissible monomial \(\lambda_I = \lambda_{i_1}^{\epsilon_1} \cdots \lambda_{i_s}^{\epsilon_s} \in \Lambda^{opp}_s\), we define the excess of \(\lambda_I\) or of \(I\) to be
\[
e(\lambda_I) = e(I) = 2i_1 - \epsilon_1 - \sum_{k=2}^s (2(p-1)i_k + \sum_{k=2}^s \epsilon_k).
\]

Then, the Dyer-Lashof algebra is the quotient of the algebra \(\Lambda^{opp}\) over the ideal generated by all monomials of negative excess [10, 30].

Let \(\beta^* Q^I\) be the image of \(\lambda_{i-1}\) under the canonical projection. A monomial \(Q^I = \beta^{e_1} Q^{i_1} \cdots \beta^{e_s} Q^{i_s}\) is called admissible if \(\lambda_I\) is admissible. Then \(R\) has an addivtive basis consisting of all admissible monomials of nonnegative excess.

Let \(R_s\) be the subspace of \(R\) spanned by all monomials of length \(s\), then \(R_s\) is isomorphic to \(\mathcal{B}[s]^\#\) as \(A\)-coalgebras, where the \(A\)-action on \(R\) is given by the Nishida’s relation (see May [8]).

From the above result, we observe that the restriction of \(\kappa_s\) on \(\mathcal{B}[s]\) is isomorphism between \(\mathcal{B}[s]\) and \(R_s^\#\).

3. The Lannes-Zarati homomorphism

We want to sketch the work of Zarati [31] in this section. And we end this section by the building the mod \(p\) Lannes-Zarati homomorphism.

Let \(\mathcal{M}\) be the category of left \(A\)-modules. A module \(M \in \mathcal{M}\) is called unstable if \(\beta^* P^I x = 0\) for \(\epsilon + 2t > \deg(x)\) and for all \(x \in M\). Let \(\mathcal{U}\) be the full subcategory of \(\mathcal{M}\) consisting of all unstable modules.
The destabilization functor $\mathcal{D} : \mathcal{M} \to \mathcal{U}$ is defined by, for $M \in \mathcal{M}$,
$$
\mathcal{D}(M) = M/EM,
$$
where $EM = \text{Span}_\mathbb{F}_p \{ \beta^i \mathcal{P}^i x : \epsilon + 2i > \deg(x), x \in M \}$, which is a sub-$A$-module of $M$ because of the Adem relations. The functor $\mathcal{D}$ is right exact and admits left derived functors $\mathcal{D}_s, s \geq 0$. Then
$$
\mathcal{D}_s(M) = H_s(\mathcal{D}(F(M))),
$$
for $F(M)$ the free resolution (or projective resolution) of $M$.

Define $\alpha_1(M) : \mathcal{D}_r(\Sigma^{-1}M) \longrightarrow \mathcal{D}_{r-1}(P_1 \otimes M)$ to be the connecting homomorphism of the functor $\mathcal{D}(-)$ associated to the short exact sequence
$$
0 \to P_1 \otimes M \to \hat{P} \otimes M \to \Sigma^{-1}M \to 0,
$$
where $\hat{P}$ is the $A$-module extended of $P_1$ by formally adding a generator $x_1^{-1}u_1$ of degree $-1$. The action of $A$ on $\hat{P}$ is given by setting $\mathcal{P}^n(x_1^{-1}u_1) = (-1)^{n}x_1^{n(p-1)-1}u_1$ and $\beta(x_1^{-1}u_1) = 1$, while the summand $P_1$ has its usual $A$-action. Put
$$
\alpha_s(M) = \alpha_1(P_{s-1} \otimes M) \circ \cdots \circ \alpha_1(\Sigma^{-(s-1)}M),
$$
then $\alpha_s(M) : \mathcal{D}_r(\Sigma^{-s}M) \longrightarrow \mathcal{D}_{r-s}(P_s \otimes M)$.

When $r = s$, we obtain $\alpha_s(M) : \mathcal{D}_s(\Sigma^{-s}M) \longrightarrow \mathcal{D}_0(P_s \otimes M)$.

**Theorem 3.1** ([31 Théorème 2.5]). For any $M \in \mathcal{U}$, the homomorphism $\alpha_s(\Sigma M) : \mathcal{D}_s(\Sigma^{1-s} M) \longrightarrow \Sigma \mathcal{D}_s M$ is an isomorphism of unstable $A$-modules, where $\mathcal{D}_s(-)$ is the Singer functor.

When $M = \mathbb{F}_p$, Hải [13] showed that $\mathcal{D}_s(\mathbb{F}_p) \cong \mathcal{B}[s]$. Therefore, we have the following corollary.

**Corollary 3.2.** For $s \geq 0$, $\alpha_s := \alpha_s(\Sigma \mathbb{F}_p) : \mathcal{D}_s(\Sigma^{1-s} \mathbb{F}_p) \cong \Sigma \mathcal{B}[s]$.

Because of the definition of the functor $\mathcal{D}$, the projection $M \to \mathbb{F}_p \otimes_A M$ factors through $\mathcal{D}_s M$. Then it induces a commutative diagram
$$
\cdots \longrightarrow \mathcal{D}(\mathbb{F}_s M) \longrightarrow \mathcal{D}(\mathbb{F}_{s-1} M) \longrightarrow \cdots
\downarrow i_s \quad \downarrow i_{s-1}
\cdots \longrightarrow \mathbb{F}_p \otimes_A \mathbb{F}_s M \longrightarrow \mathbb{F}_p \otimes_A \mathbb{F}_{s-1} M \longrightarrow \cdots.
$$
Here horizontal arrows are induced by the differential of $FM$, and $i_s$ is given by
$$
i_s([z]) = [1 \otimes_A z].$$
Taking the homology, we get
$$i_s : \mathcal{D}_s(M) \longrightarrow \text{Tor}^A_1(\mathbb{F}_p, M).$$

Since, for $z \in F_s M$ and $a > 0$, $i_s(Sq^a[z]) = i_s([Sq^a z]) = [1 \otimes_A Sq^a z] = [0] \in \mathbb{F}_2 \otimes_A F_s M$, the induced map of $i_s$ in homology factors through $\mathbb{F}_p \otimes_A H_s(\mathcal{D}(F_s M))$.

Therefore, we have following commutative diagram
$$
\begin{array}{ccc}
\mathcal{D}_s(M) & \longrightarrow & \text{Tor}^A_1(\mathbb{F}_p, M) \\
\downarrow i_s & & \downarrow i_s \\
\mathbb{F}_p \otimes_A \mathcal{D}_s(M) & \longrightarrow & \mathbb{F}_p \otimes_A H_s(\mathcal{D}(F_s M))
\end{array}
$$
When $M = \Sigma^{1-s}F_2$, we obtain
\[ \bar{i}_s : F_p \otimes_A D_s(\Sigma^{1-s}F_p) \to \text{Tor}_s^A(F_p, \Sigma^{1-s}F_p). \]
For each $s \geq 1$, we define
\[ \varphi^s := \Sigma^{-1}i_s(1 \otimes_A \alpha_s^{-1}) \Sigma : F_p \otimes_A D[s] \to \text{Tor}_s^A(F_p, \Sigma^{1-s}F_p). \]
In the dual, we have the Lannes-Zarati homomorphism for $p$ odd
\[ \varphi : \text{Ext}^{s+t}_A(F_p, F_p) \to \text{Ann}(D[s]^\#). \]
In [20], Kuhn showed that the map $\varphi_s$, for $s \geq 1$, is the graded associated version of the mod $p$ Hurewicz map $h_\star : \pi_\star(Q_0S^0) \to H_\star Q_0S^0$ in the $E_2$-term of the Adams spectral sequence.

4. The chain-level representation of $\varphi_s$

In this section, we construct the chain-level representation of $\varphi_s$ in the Singer-Hung-Sum chain complex as well as the chain-level representation of $\varphi_s$ in the opposite algebra of the Lambda algebra.

For $M \in \mathcal{M}$, recall that $B_s(M) := \oplus_{s \geq 0} B_s(M)$ is the usual bar resolution of $M$ with
\[ B_s(M) = A \otimes \underbrace{A \otimes \cdots \otimes A}_{s \text{ times}} \otimes M, \]
where $\bar{A}$ is the augmentation ideal of $A$, which is the ideal of $A$ generated by all positive degree elements in $A$.

The element $a_0 \otimes a_1 \otimes \cdots \otimes a_s \otimes m \in B_s(M)$ has homological degree $s$ and internal degree $t = \sum_i \text{deg}(a_i) + \text{deg}(m)$. The total degree is $s + t$, i.e.
\[ \text{deg}(a_0 \otimes a_1 \otimes \cdots \otimes a_s \otimes m) = s + \sum_i \text{deg}(a_i) + \text{deg}(m). \]

The $A$-action on $B_s(M)$ is given by
\[ a(a_0 \otimes a_1 \otimes \cdots \otimes a_s \otimes m) = aa_0 \otimes a_1 \otimes \cdots \otimes a_s \otimes m, \]
and the differential of $B_s(M)$ is given by
\[ \partial(a_0 \otimes a_1 \otimes \cdots \otimes a_s \otimes m) = \sum_{i=0}^{s-1} (-1)^{e_i} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes m \]
\[ - (-1)^{e_{s-1}} a_0 \otimes a_1 \otimes \cdots \otimes a_s m, \]
where $e_i = \text{deg}(a_0 \otimes \cdots \otimes a_i)$.

Since $B_s(M)$ is the free resolution of $M$, by definition one has
\[ \text{Tor}_s^A(N, M) = H_\star(N \otimes_A B_s(M)). \]
As $D[s] \subset \mathcal{G}_s$, for $\gamma \in D[s]$, $\gamma$ has an unique expansion
\[ \gamma = \sum_{l=(\epsilon_1, \epsilon_1, \ldots, \epsilon_s, \epsilon_s) \in \mathcal{I}} u_1^{\epsilon_1} v_1^{(p-1)\epsilon_1 - \epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)\epsilon_s - \epsilon_s}. \]
Put
\[ \tilde{\gamma} := \sum_{l \in \mathcal{I}} (-1)^{\epsilon(I)} \beta^{1-\epsilon_I} P^{\epsilon_1} \otimes \cdots \otimes \beta^{1-\epsilon_s} P^{\epsilon_s} \otimes \Sigma^{1-s} 1 \in B_{s-1}(\Sigma^{1-s}F_p), \]
where $\epsilon(I) = s + \epsilon_1 + \cdots + \epsilon_s + i_1 + \cdots + i_s$. 
Therefore, it is sufficient to show that
\[ \gamma \] given by
\[ \pi \]
It is easy to see that \( \gamma \) gets that the exponents of \( v_i \)'s in the expansion of \( \gamma \) are nonnegative. Therefore \( \tilde{\gamma} \in B_{s-1}(\Sigma^{1-s}F_p) \).

By the action of \( A \),
\[ \tilde{\gamma} = \sum_{l \in I} (-1)^{\ell(l)} \beta_1^{1-\epsilon_l} \cP^\ell_1 (1 \otimes \beta_1^{1-\epsilon_2} \cP^{i_2} \otimes \cdots \otimes \beta_1^{1-\epsilon_s} \cP^{i_s} \otimes \Sigma^{1-s} 1). \]

Therefore, it is sufficient to show that
\[ 2i_1 + (1 - \epsilon_1) > \sum_{k=2}^s (2i_k(p - 1) + (1 - \epsilon_k)) + 1 - s, \]
it is equivalent to
\[ 2i_1 - \epsilon_1 > \sum_{k=1}^s 2i_k(p - 1) - \sum_{k=1}^s \epsilon_k. \]

Also from the proofs of Lemma A.9 Lemma A.10 and Proposition A.12, we observe that \( q_k \) and \( R_{s, i, j} \) can be written in the sum of \( u_{1, q_1}^{(p-1)i_1-\epsilon_1} \cdots v_{s, q_s}^{(p-1)i_s-\epsilon_s} \) where \( (\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \) satisfies (4.1), therefore so is \( \gamma \).

**Lemma 4.2.** The element \( \gamma \) is a cycle in \( EB_{s-1}(\Sigma^{1-s}F_p) \).

**Proof.** Let
\[ \Omega: \Delta_s^+ \longrightarrow A \]
\[ u_1^{(p-1)i_1-\epsilon_1} u_2^{(p-1)i_2-\epsilon_2} \cdots \longrightarrow (-1)^{i_1+i_2+\cdots+i_s+1} \beta_1^{1-\epsilon_1} \cP^{i_1} \beta_1^{1-\epsilon_2} \cP^{i_2}. \]

From the result of Ciampell and Lomonaco [7], one gets \( \Gamma_2 \subset \text{Ker}\Omega \).

Consider the diagonal map \( \psi: \Delta_s^+ \longrightarrow \Delta_{q-1}^+ \otimes \Delta_2^+ \otimes \Delta_{s-q-1}^+ \) defined by
\[ \psi(u_k^{(p-1)i_k-\epsilon_k}) = \begin{cases} u_k^{(p-1)i_k-\epsilon_k} \otimes 1 & k < q, \\ 1 \otimes u_k^{(p-1)i_k-\epsilon_k} \otimes 1 & i \leq k < i + 1, \\ 1 \otimes 1 & k > i + 1. \end{cases} \]

From results of Hùng and Sum [19 Corollary 3.4], \( \psi(\Gamma_s) \subset \Gamma_{i-1} \otimes \Gamma_2 \otimes \Gamma_{s-i-1} \).

Define the homomorphism
\[ \pi_{s, q}: \Delta_s^+ \longrightarrow A^{\otimes (s-1)} = A \otimes \cdots \otimes A, \]
given by
\[ \pi_{s, q}(u_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{(p-1)i_s-\epsilon_s} v_{q+1}^{(p-1)i_{q+1}-\epsilon_{q+1}} \cdots v_s^{(p-1)i_s-\epsilon_s}) \]
\[ = (-1)^{i_1+\cdots+i_q+i_{q+1}+\cdots+i_s} \beta_1^{1-\epsilon_1} \cP^{i_1} \otimes \cdots \otimes \beta_1^{1-\epsilon_q} \cP^{i_q} \beta_1^{1-\epsilon_{q+1}} \cP^{i_{q+1}} \otimes \cdots \otimes \beta_1^{1-\epsilon_s} \cP^{i_s}. \]

It is easy to see that \( \pi_{2, 1} = \Omega \). Moreover, if we define \( \omega_1, \omega_s: \Delta_s^+ \longrightarrow A^{\otimes s} \) given by
\[ \omega_1(u_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{(p-1)i_s-\epsilon_s}) = (-1)^{i_1+\cdots+i_s+1} \beta_1^{1-\epsilon_1} \cP^{i_1} \otimes \cdots \otimes \beta_1^{1-\epsilon_s} \cP^{i_s}, \]
\[ \omega_s(u_1^{(p-1)i_1-\epsilon_1} \cdots u_s^{(p-1)i_s-\epsilon_s}) = (-1)^{i_1+\cdots+i_s+1} \beta_1^{1-\epsilon_1} \cP^{i_1} \otimes \cdots \otimes \beta_1^{1-\epsilon_s} \cP^{i_s}, \]

Thus, \( \omega_1 \) and \( \omega_s \) commute with \( \pi_{s, q} \).
then $\pi_{s,q} = (\omega_{q-1}^* \otimes \pi_{2,1} \otimes \omega_{s-q-1}) \psi$. Since $\pi_{2,1}(\Gamma_2) = 0$, then $\pi_{k,q}(\Gamma_k) = 0$.

By the definition of the differential in the bar resolution, one gets
\[
\partial(\tilde{\gamma}) = (-1)^{\text{deg } \tilde{\gamma} + s} \sum_{q=1}^{s-1} (\pi_{s,q} \otimes id_{\Sigma^1 \otimes \mathbb{F}_p})(\gamma \otimes \Sigma^1 \otimes 1).
\]

Since $\gamma \in \mathcal{D}[s] \subset \Gamma_s$, then $\pi_{s,q}(\gamma) = 0$. Therefore, $\partial(\tilde{\gamma}) = 0$. \hfill \Box

For any $A$-module $M$, from the definition of the functor $\mathcal{D}$, one gets the short exact sequence of chain complexes
\[
0 \longrightarrow EB_s(M) \longrightarrow B_s(M) \longrightarrow \mathcal{D}(B_s(M)) \longrightarrow 0.
\]

Because $B_s(M)$ is acyclic, for $s \geq 1$, the connecting homomorphism
\[
\partial_s : H_s(\mathcal{D}(B_s(M))) \xrightarrow{\cong} H_{s-1}(EB_s(M)) \tag{4.2}
\]
is isomorphic.

Letting $M = \Sigma^{1-s}\mathbb{F}_p$, one gets
\[
\partial_s : \mathcal{D}(\Sigma^{1-s}\mathbb{F}_p) \xrightarrow{\cong} H_{s-1}(EB_{s-1}(\Sigma^{1-s}\mathbb{F}_p)).
\]

**Lemma 4.3.** For $\gamma \in \mathcal{D}[s], \quad \partial_s[1 \otimes \tilde{\gamma}] = [\tilde{\gamma}].$

**Proof.** Suppose that $\gamma = \sum_{L \in \mathcal{L}} u^{\epsilon_1}_L v^{(p-1)i_1-\epsilon_1}_1 \cdots u^{\epsilon_s}_L v^{(p-1)i_s-\epsilon_s}_s \in \mathcal{D}[s]$. Then $[1 \otimes \gamma] \in \mathcal{D}(B_s(\Sigma^{1-s}\mathbb{F}_p))$. Since $\tilde{\gamma}$ is a cycle in $EB_{s-1}(\Sigma^{1-s}\mathbb{F}_p)$, then $[1 \otimes \tilde{\gamma}]$ is a cycle in $\mathcal{D}(B_s(\Sigma^{1-s}\mathbb{F}_p))$. It can be pulled back by the element $1 \otimes \tilde{\gamma} \in B_s(\Sigma^{1-s}\mathbb{F}_p).

In $B_s(\Sigma^{1-s}\mathbb{F}_p)$, we have
\[
\partial(1 \otimes \tilde{\gamma}) = 1 \otimes \partial(\tilde{\gamma}) + \sum_{L \in \mathcal{L}} (-1)^{\|L\|} \beta^{1-\epsilon_1} P^{i_1} \otimes \cdots \otimes \beta^{1-\epsilon_s} P^{i_s} \otimes \Sigma^1 \otimes 1
\]
\[
= \sum_{L \in \mathcal{L}} (-1)^{\|L\|} \beta^{1-\epsilon_1} P^{i_1} \otimes \cdots \otimes \beta^{1-\epsilon_s} P^{i_s} \otimes \Sigma^1 \otimes 1.
\]

Thus, the proof is complete. \hfill \Box

From the short exact sequence
\[
0 \rightarrow \Sigma^{2-s}P_1 \rightarrow \Sigma^{2-s} \tilde{P} \rightarrow \Sigma^{1-s}\mathbb{F}_p \rightarrow 0,
\]
we have the short exact sequence of chain complexes
\[
0 \longrightarrow B_s(\Sigma^{2-s}P_1) \longrightarrow B_s(\Sigma^{2-s} \tilde{P}) \longrightarrow B_s(\Sigma^{1-s}\mathbb{F}_p) \longrightarrow 0.
\]

It induces a short exact sequence (even though $E(-)$ is not exact)
\[
0 \longrightarrow EB_s(\Sigma^{2-s}P_1) \longrightarrow EB_s(\Sigma^{2-s} \tilde{P}) \longrightarrow EB_s(\Sigma^{1-s}\mathbb{F}_p) \longrightarrow 0.
\]

Taking homology, we have the connecting homomorphism
\[
\delta(\Sigma^{2-s}\mathbb{F}_p) : H_{s-1}(EB_s(\Sigma^{-1}\mathbb{F}_p)) \longrightarrow H_{s-2}(EB_s(\Sigma^{2-s}P_1)).
\]

By (4.2), one gets
\[
\alpha_s = \delta(\Sigma\mathbb{F}_p \otimes P_{s-1}) \circ \cdots \circ \delta(\Sigma^{2-s}\mathbb{F}_p) \circ \partial_s
\]

Put $\delta_{s-1} := \delta(\Sigma\mathbb{F}_p \otimes P_{s-1}) \circ \cdots \circ \delta(\Sigma^{2-s}\mathbb{F}_p)$. Then we have the lemma.
Lemma 4.4. For any $\gamma \in \mathcal{B}[s]$,

$$\delta_{s-1}([\gamma]) = (-1)^s \frac{s(s-1)+(s+1)\deg \gamma}{2} [\Sigma \gamma].$$

Proof. Assume that

$$\gamma = \sum_{l \in I} u_{1l}^1 u_{1}^{(p-1)i_1-i_1} \cdots u_{s}^{(p-1)i_s-i_s} \in \mathcal{B}[s].$$

By Lemma 4.2,

$$\tilde{\gamma} = \sum_{l \in I} (-1)^{\epsilon(l)} \beta^{1-\epsilon_1} \mathcal{P}^{i_1} \otimes \cdots \otimes \beta^{1-\epsilon_s} \mathcal{P}^{i_s} \otimes \Sigma^{1-s} \mathcal{B}$$

is a cycle in $EB_{s-1}(\Sigma^{1-s} \mathbb{F}_p)$. It can be pulled back by

$$\mathcal{L} = \sum_{l \in I} (-1)^{\epsilon(l)+\epsilon(l)} \beta^{1-\epsilon_1} \mathcal{P}^{i_1} \otimes \cdots \otimes \beta^{1-\epsilon_s} \mathcal{P}^{i_s} \otimes \Sigma^{2-s} \beta^{1-\epsilon_s} \mathcal{P}^{i_s} (x y_{1}^{-1}),$$

where $\eta(I) = s + \epsilon_1 + \cdots + \epsilon_{s-1} + s \epsilon_s$.

Therefore, $\delta(\Sigma^{2-s} \mathbb{F}_p) ([\tilde{\gamma}])$ is equal to

$$\left[ \sum_{l \in I} (-1)^{\eta(I)+\epsilon(l)} \beta^{1-\epsilon_1} \mathcal{P}^{i_1} \otimes \cdots \otimes \beta^{1-\epsilon_s} \mathcal{P}^{i_s} \otimes \Sigma^{2-s} \beta^{1-\epsilon_s} \mathcal{P}^{i_s} (x y_{1}^{-1}) \right].$$

Repeating this process, finally we have

$$\delta_{s-1}([\gamma]) = \left[ \sum_{l \in I} (-1)^{f_l} \beta^{1-\epsilon_1} \mathcal{P}^{i_1} (x_1 y_1^{-1} \beta^{1-\epsilon_s} \mathcal{P}^{i_s} (x_2 y_2^{-1} \beta^{1-\epsilon_s} \mathcal{P}^{i_s} x y_{1}^{-1})) \right],$$

where $f_l = \epsilon(I) + (1 + \cdots + s) + s(\epsilon_1 + \cdots + \epsilon_s)$.

The lemma follows from Corollary A.13. \hfill \Box

Combining Lemma 4.3 and Lemma 4.4, we have the following corollary.

Corollary 4.5. The map $\mathcal{B}[s] \to \mathbb{B}(\Sigma^{1-s} \mathbb{F}_p)$ given by

$$\gamma \mapsto (-1)^{\frac{s(s-1)+(s+1)\deg \gamma}{2}} [1 \otimes \tilde{\gamma}]$$

is a chain-level representation of the homomorphism

$$(1 \otimes_A \alpha_s)^{-1} \Sigma: \mathbb{F}_p \otimes_A \mathcal{B}[s] \to \mathbb{F}_p \otimes_A \mathcal{B}(\Sigma^{1-s} \mathbb{F}_p).$$

The chain-level representation of the dual of $\varphi_s$ is given by the following theorem.

Theorem 4.6. The inclusion map $\varphi_s^+: \mathcal{B}[s] \to \Gamma^+_s$ given by

$$\gamma \mapsto (-1)^{\frac{s(s-1)+(s+1)\deg \gamma}{2}}$$

is the chain-level representation of the dual of the Lannes-Zarari homomorphism $\varphi_s^\#$. 

Proof. In [28], Priddy showed that the opposite of the lambda algebra \( \Lambda^{\text{opp}} \) is isomorphic to the co-Koszul complex of \( A \), which is the quotient cocomplex of the usual cobar resolution \( C^*(\mathbb{F}_p) := \text{Hom}_A(B_*(\mathbb{F}_p), \mathbb{F}_p) \). The canonical quotient map \( \iota_* : C^*(\mathbb{F}_p) \to \Lambda^{\text{opp}} \) sends \( \tau_0^s \xi_1^1 \otimes \cdots \otimes \tau_0^s \xi_1^j = (-1)^{e_1 + \cdots + e_s} \lambda_{j-1}^1 \cdots \lambda_{j-1}^s \).

In Section 1, we showed that the chain complex \( \Gamma^+ \) is isomorphic to \( (\Lambda^{\text{opp}})^\# \), the dual of \( \Lambda^{\text{opp}} \), via the isomorphism given by
\[
\kappa_s(u_1^{e_1}v_1^{(p-1)j_1-\epsilon_1} \cdots u_s^{e_s}v_s^{(p-1)j_s-\epsilon_s}) = (-1)^{i_1 + \cdots + i_s}(\lambda_{j_1-1}^1 \cdots \lambda_{j_s-1}^s)^e.
\]

Thus, there exists an inclusion \( \nu_* : (\Gamma^+ \Sigma^1 \mathbb{F}_p)_s \to B_s(\Sigma^{1-s} \mathbb{F}_p) \), that sends
\[
\nu_s(u_1^{e_1}v_1^{(p-1)j_1-\epsilon_1} \cdots u_s^{e_s}v_s^{(p-1)j_s-\epsilon_s}) = (-1)^e I \otimes \beta^{1-\epsilon_1} \mathcal{P}^{i_1} \otimes \cdots \otimes \beta^{1-\epsilon_s} \mathcal{P}^{i_s},
\]
where \( e(I) = s + \epsilon_1 + \cdots + \epsilon_s + i_1 + \cdots + i_s \).

This fact together with Lemma 4.3 and Lemma 4.4 we have the assertion of the theorem. \( \square \)

Since \( \Gamma^+_s \cong (\Lambda^{\text{opp}})^\# \) and \( \mathcal{B}[s] \cong R^s_\# \) via \( \kappa_s \), we have the following corollary.

**Corollary 4.7.** The projection \( \hat{\varphi}_s : \Lambda^{\text{opp}} \to R_s \) given by
\[
\hat{\varphi}_s(\lambda_i) = (-1)^{\frac{(e_i-1)}{2} + (s+1) \deg(\lambda_i)} Q^I
\]
is the chain-level representation of the Lannes-Zarati homomorphism \( \varphi_s \).

## 5. The Power Operations

This section is devoted to develop the power operations, these are useful tools in studying the behavior of the Lannes-Zarati homomorphism in the next section.

From Liulevicius [22, 23] and May [24], there exists the power operation \( \mathcal{P}^0 \) : \( \text{Ext}_A^{s,s+1}(\mathbb{F}_p, \mathbb{F}_p) \to \text{Ext}_A^{s+1, p(s+1)}(\mathbb{F}_p, \mathbb{F}_p) \). Its chain-level representation in the cobar complex is given by
\[
\theta_1 \otimes \cdots \otimes \theta_s \mapsto \theta_1^p \otimes \cdots \otimes \theta_s^p,
\]
where \( \theta_i \in A^\# \), the dual of the Steenrod algebra \( A \).

By the projection \( \iota_* : C^*(\mathbb{F}_p) \to \Lambda^{\text{opp}} \), the power operation has a chain-level representation in \( \Lambda^{\text{opp}} \) given by
\[
\tilde{\mathcal{P}}^0(\lambda_{i_1}^1 \cdots \lambda_{i_s}^s) = \begin{cases} 
\lambda_{pi_1}^1 \cdots \lambda_{pi_s}^s, & \epsilon_1 = \cdots = \epsilon_s = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

**Lemma 5.1.** The operation \( \tilde{\mathcal{P}}^0 \) induces an operation \( \theta \) on the Dyer-Lashof algebra \( R \) given by
\[
\theta(\beta^{e_1} Q^I \cdots \beta^{e_s} Q^{p^s}) = \begin{cases} 
\beta^{e_1} Q^{pi_1} \cdots \beta^{e_s} Q^{p^s}, & \epsilon_1 = \cdots = \epsilon_s = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** It is sufficient to show that if \( \lambda_{i_1} \cdots \lambda_{i_s} \) has negative excess then so does \( \lambda_{pi_1} \cdots \lambda_{pi_s} \) for \( s \geq 2 \).
By inspection, one gets
\[
e(\lambda_{p_{i_1}} \cdots \lambda_{p_{i_s}}) = 2p_{i_1} - \sum_{k=2}^{s} 2p(p-1)i_k + (s-2)
= p e(\lambda_{i_1} \cdots \lambda_{i_s}) - (p-1)(s-2).
\]
Therefore, if \(e(\lambda_{i_1} \cdots \lambda_{i_s}) < 0\) then \(e(\lambda_{p_{i_1}} \cdots \lambda_{p_{i_s}}) < 0\). \(\square\)

**Lemma 5.2.** The operation \(\theta\) commutes with the action of \(A\). In particular,
\[
\theta((\beta^{x_1} Q^{i_1} \cdots \beta^{x_s} Q^{i_s}) \mathcal{P}^k) = (\theta(\beta^{x_1} Q^{i_1} \cdots \beta^{x_s} Q^{i_s})) \mathcal{P}^k.
\] (5.1)

**Proof.** It is sufficient to show the lemma in the case \(\epsilon_1 = \cdots = \epsilon_s = 1\).
We will prove the assertion by induction on \(s\).
For \(s = 1\), it is easy to see that
\[
\theta((\beta Q^i) \mathcal{P}^k) = \theta((-1)^k \binom{p-1}{k}(i-k)\beta Q^{i-k})
= (-1)^k \binom{p-1}{k}(i-k)\beta Q^{p_i-p_k},
\]
and \((\theta(\beta Q^i)) \mathcal{P}^k = \beta Q^{p_i} \mathcal{P}^k = (-1)^k \binom{p-1}{pk}(p-1)(i-k)\beta Q^{p_i-p_k} \). \(\Box\)

Since \((-1)^k \binom{p-1}{pk}(i-k) \equiv (-1)^k \binom{p-1}{k} \mod p\), we have the assertion.
For \(s > 1\), by the inductive hypothesis,
\[
\theta((\beta^{x_1} Q^{i_1} \cdots \beta^{x_s} Q^{i_s}) \mathcal{P}^k)
= \theta \left( \sum_t (-1)^{k+t} \binom{p-1}{k+t}(i_1-k-1)\beta Q^{i_1-k+t}(\beta^{x_2} Q^{i_2} \cdots \beta^{x_s} Q^{i_s}) \mathcal{P}^t \right)
+ \theta \left( \sum_t (-1)^{k+t} \binom{p-1}{k+t}(i_1-k-1)\beta Q^{i_1-k+t}(\beta^{x_2} Q^{i_2} \cdots \beta^{x_s} Q^{i_s}) \mathcal{P}^t \right)
= \sum_t (-1)^{k+t} \binom{p-1}{k+t}(i_1-k-1)\beta Q^{p_i(i_1-k+t)}(\beta^{p_{i_2}} Q^{i_2} \cdots \beta^{p_{i_s}}) \mathcal{P}^t.
\]
On the other hand,
\[
(\theta(\beta^{x_1} Q^{i_1} \cdots \beta^{x_s} Q^{i_s})) \mathcal{P}^k = (\beta Q^{p_{i_1}} \cdots \beta Q^{p_{i_s}}) \mathcal{P}^k
= \sum_j (-1)^{p_{j}+k} \binom{p-1}{p_{j}-k-1}(p_{i_1}-p_{j}) \beta Q^{p_{i_1}-p_{j}+k+j}(\beta^{p_{i_2}} Q^{i_2} \cdots \beta^{p_{i_s}}) \mathcal{P}^j
+ \sum_j (-1)^{p_{j}+k} \binom{p-1}{p_{j}-k-1}(p_{i_1}-p_{j}) \beta Q^{p_{i_1}-p_{j}+k+j}(\beta^{p_{i_2}} Q^{i_2} \cdots \beta^{p_{i_s}}) \mathcal{P}^j
= \sum_j (-1)^{k+j} \binom{p-1}{k-j} \beta Q^{p_{i_1}-p_{j}+k+j}(\beta^{p_{i_2}} Q^{i_2} \cdots \beta^{p_{i_s}}) \mathcal{P}^j.
\]
If \( j \) is not divisible by \( p \) then \( (p - 1)(p^i_2 - j) - 1 \equiv j - 1 \mod p \), while \( j - p\ell \equiv j \mod p \). Therefore,

\[
(\beta Q^{p^i_2} \ldots \beta Q^{p^i}) \mathcal{P}^j = \sum_j (-1)^{j+\ell} \left( \frac{(p - 1)(p^i_2 - \ell) - 1}{j - p\ell} \right) (\beta Q^{p^i_3} \ldots \beta Q^{p^i}) \mathcal{P}^\ell
+ \sum_j (-1)^{j+\ell} \left( \frac{(p - 1)(p^i_2 - j) - 1}{j - p\ell - 1} \right) (\beta Q^{p^i_3} \ldots \beta Q^{p^i}) \mathcal{P}^j
= \sum_j (-1)^{j+\ell} \left( \frac{(p - 1)(p^i_2 - \ell) - 1}{j - p\ell} \right) (\beta Q^{p^i_3} \ldots \beta Q^{p^i}) \mathcal{P}^\ell = 0.
\]

Thus,

\[
(\theta(\beta Q^{i_1} \ldots \beta Q^{i_s})) \mathcal{P}^k = \sum_j (-1)^{k+\ell} \left( \frac{(p - 1)(i_1 - k) - 1}{k - pt} \right) (\beta Q^{p(i_1 - k + \ell)} \ldots \beta Q^{p^i}) \mathcal{P}^t.
\]

The lemma is proved. \( \square \)

By Lemma 5.2 the operation \( \theta \) induces an power operation on \( \text{Ann}(R) \), which is also denoted by \( \mathcal{P}^0 \).

**Proposition 5.3.** The power operations \( \mathcal{P}^0 \)'s commute with each other through the Lannes-Zarati homomorphism. In other words, the following diagram is commutative

\[
\begin{array}{ccc}
\text{Ext}_A^{s,s+t}(F_p,F_p) & \xrightarrow{\mathcal{P}^0} & \text{Ext}_A^{s,p(s+t)}(F_p,F_p) \\
\varphi_s & & \varphi_s \\
\text{Ann}(R_s)_{t} & \xrightarrow{\mathcal{P}^0} & \text{Ann}(R_s)_{p(s+t) - s}.
\end{array}
\]

**Proof.** It is immediate from Corollary 4.7. \( \square \)

## 6. Behavior of the Lannes-Zarati Homomorphism

In this section, we use the chain-level representation map of the \( \varphi_s \) constructed in the previous section to investigate its behavior.

### 6.1. The first Lannes-Zarati homomorphism.

**Theorem 6.1.** The first Lannes-Zarati homomorphism

\[
\varphi_1 : \text{Ext}_A^{1,1+t}(F_p,F_p) \longrightarrow \text{Ann}(R[1]^\#)_t
\]

is isomorphic.

**Proof.** As we well-known, \( \text{Ext}_A^{1,1+t}(F_p,F_p) \) spanned by \( \alpha_0 \) of stem 0 and \( h_i \) of stem \( 2i(p - 1) - 1 \). These element are represented in \( \Gamma_1^+ \) respectively be \( v_0^0 \) and \( u_1 v_1^{(p-1)i-1} \) for \( i > 0 \).

On the other hand, \( F_p \otimes R[1] \) is spanned by 1 and \( x_1 y_1^{(p-1)i-1} \) for \( i > 0 \).

Applying Theorem 4.6 one gets

\[
\varphi_1^#([v_0^0]) = [v_0^0]; \quad \varphi_1^#([x_1 y_1^{(p-1)i-1}]) = [u_1 v_1^{(p-1)i-1}].
\]

This fact follows the theorem. \( \square \)
6.2. The second Lannes-Zarati homomorphism.

Theorem 6.2. The second Lannes-Zarati homomorphism

\[ \varphi_2 : \text{Ext}_A^{2,2+t}(F_p, F_p) \longrightarrow \text{Ann}(F[2]^*)_t \]

is vanishing for \( t \neq 0 \) and \( t \neq 2(p-1)p^{i+1} - 2, i \geq 0 \).

Proof. From the results of Liulevicius [23] (see also Aikawa [1]), \( \text{Ext}_A^{2,2+t}(F_p, F_p) \) spanned by the elements

1. \( h_i h_j = [\lambda_{p^{i-1}} \lambda_{p^{j-1}}] \in \text{Ext}_A^{2,2(p-1)(p^{i-1} + p'j)}(F_p, F_p), 0 \leq i < j + 1; \)
2. \( \alpha_0 h_i = [\mu_{-1} \lambda_{p^{i-1}}] \in \text{Ext}_A^{2,2(p-1)p^{i+1}}(F_p, F_p), i \geq 1; \)
3. \( \alpha_0^2 = [\mu_0^2] \in \text{Ext}_A^{2,2}(F_p, F_p); \)
4. \( h_{i;2,1} = (P^0)^i[\lambda_{2p-1} \lambda_0] \in \text{Ext}_A^{2,2(p-1)(2p^{i+1} + p')} (F_p, F_p), i \geq 0; \)
5. \( h_{i;1,2} = (P^0)^i[\lambda_{p-1} \lambda_j] \in \text{Ext}_A^{2,2(p-1)(p^{i+1} + 2p^j)}(F_p, F_p), i \geq 0; \)
6. \( \rho = [\lambda_1 \mu_{-1}] \in \text{Ext}_A^{2,4(p-1)+1}(F_p, F_p); \)
7. \( \delta_i = (P^0)^i \left[\sum_{j=1}^{p-1} (\frac{1}{j})^{i+j} \lambda_{p-j-1} \lambda_{j-1}\right] \in \text{Ext}_A^{2,2(p-1)p^{i+1}}(F_2, F_2), i \geq 0. \)

Here we denote \((P^0)^i = \underbrace{P^0 \cdots P^0}_{i \text{ times}}.\)

It is clear that monomials \( \lambda_{p^{i-1}} \lambda_{p^{j-1}} \) (\( i < j + 1 \)), \( \mu_{-1} \lambda_{p^{i-1}} \), \( \lambda_{p-1} \lambda_1 \) are of negative excess, therefore their images under \( \varphi_2 \) are trivial in \( R_2 \). It implies under \( \varphi_2 \) the images of \( h_i h_j, \alpha_0 h_i, \) and \( h_{0;1,2} \) are trivial. By Proposition 5.3, \( \varphi_2(h_{i;1,2}) = (P^0)^i \varphi_2(h_{0;1,2}) = 0. \)

It is easy to see that \( \varphi_2(\alpha_0^2) = -Q^0 Q^0 \neq 0 \in R_2. \)

By inspection,

\[ \varphi_2(\lambda_{2p-1} \lambda_0) = -\beta Q^{2p} \beta Q^1. \]

Applying adem relation, one gets

\[ \beta Q^{2p} \beta Q^1 = -\sum_{j} (-1)^{2p+j} \left(\frac{p-1(j-1)}{p-1}\right) \beta Q^{p+1-j} \beta Q^j. \]

Since \( pj > 2p + 1 \), then \( e(\beta Q^{p+1-j} \beta Q^j) = 2(2p+1-j) - 2(p-1)j = 2(p+1-pj) < 0. \)

Therefore, \( \beta Q^{2p} \beta Q^1 = 0 \), it implies that \( \varphi_2(h_{0;2,1}) \) and then \( \varphi_2(h_{i;2,1}) = 0. \)

Similarly, \( \varphi_2(\lambda_1 \mu_{-1}) = -Q^2 Q^0. \) Applying adem relation, we obtain \( \beta Q^2 Q^0 = 0 \) and therefore \( \varphi_2(\rho) = 0. \)

Finally, it is easy to verify that

\[ \varphi_2 \left( \sum_{j=1}^{p-1} (-1)^{j+1} \lambda_{p-j-1} \lambda_{j-1}\right) = -\beta Q^{p-1} \beta Q^1 \neq 0 \in R_2. \]

Therefore \( \varphi(\lambda_0) = \beta Q^{p-1} \beta Q^1. \) By Proposition 5.3, one gets

\[ \varphi_2(\lambda_0) = (P^0)^i(\beta Q^{p-1} \beta Q^1) = -\beta Q^{p-1} \beta Q^i \neq 0 \in R_2. \]

The proof is complete. \( \square \)

Remark 6.3. From the result of Wellington [30, Theorem 11.11], \( \text{Ann}(R_2) \) is spanned by \( Q^0 Q^0, \beta Q^{p-1} \beta Q^i, i \geq 0, \) and \( Q^{p-1} Q^s, s = p^i + \cdots + 1, i > 0. \)

Therefore, \( \varphi_2 \) is not an epimorphism.
6.3. The third Lannes-Zarati homomorphism.

**Theorem 6.4.** The third Lannes-Zarati homomorphism

\[ \varphi_3 : \text{Ext}^{3,3+t}_A(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \text{Ann}(\mathcal{A}[3])_t \]

is vanishing for all \( t > 0 \).

**Proof.** By the results of Liulevicius [23] and Aikawa [11], \( \text{Ext}^{3,3+t}_A(\mathbb{F}_p, \mathbb{F}_p) \) is spanned by following elements (for convenience we will write \( \text{Ext}^{s,s+t}_A \) for \( \text{Ext}^{3,3+t}_A(\mathbb{F}_p, \mathbb{F}_p) \))

\[ (1) \ h_{i,j,k} \in \left( \lambda_{p-1} \lambda_{p-1} \lambda_{p-1} \right) \in \text{Ext}^{3,2(p-1)(p^2+p^t+1)}, 0 \leq i < j < k + 2; \]

\[ (2) \ \alpha_0 h_{i,j} = [\mu_1 \lambda_{p-1} \lambda_{p-1}] \in \text{Ext}^{3,2(p-1)(p^3+p^t+1)}, 0 \leq i < j + 1; \]

\[ (3) \ \alpha_0^2 h_i = [\mu_2 \lambda_{p-1}] \in \text{Ext}^{3,2(p-1)p^t+2}, i \leq 0; \]

\[ (4) \ \alpha_0^3 = [\mu_3], \text{Ext}^{3,3}; \]

\[ (5) \ \tilde{\lambda}_i h_j = [L_i \lambda_{p-1}] \in \text{Ext}^{3,2(p-1)(p^{t+1}+p^t)}, i, j \geq 0, j \neq i + 2; \]

\[ (6) \ \alpha_0 \tilde{\lambda} = [\mu_1 L_i], i \geq 0; \]

\[ (7) \ h_{i+1,2}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(p^{t+1}+2p^t+p^t)}, i, j \geq 0, j \neq i + 2, i \leq 1; \]

\[ (8) \ h_{i+1,2}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(p^{t+1}+2p^t+1)}, i \geq 1; \]

\[ (9) \ h_{i+2,1}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{2(p-1)(2p^t+1+p^t+p^t)}, i, j \geq 0, j \neq i + 2, i \leq 1; \]

\[ (10) \ h_{i+2,1}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{2(p-1)(2p^t+1+p^t)}, i \geq 1; \]

\[ (11) \ \alpha_0 \mu_0 = [\lambda_{1, \mu^1} \lambda_{1, \mu^1}] \in \text{Ext}^{3,4(p-1)+2}; \]

\[ (12) \ h_{i+2,1}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p \neq 3, i \geq 0; \]

\[ (13) \ h_{i+2,1}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p \neq 3, i \geq 0; \]

\[ (14) \ h_{i+2,1}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p = 3, i \geq 0; \]

\[ (15) \ h_{i+2,1}^2 h_j = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p = 3, i \geq 0; \]

\[ (16) \ h_{i+3,1} = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p \neq 3, i \geq 0; \]

\[ (17) \ h_{i+3,1} = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p \neq 3, i \geq 0; \]

\[ (18) \ h_{i+2,1} = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p \neq 3, i \geq 0; \]

\[ (19) \ h_{i+2,1} = [\lambda_{p^2+1} \lambda_{p^2-1} \lambda_{p+1-1}] \in \text{Ext}^{3,2(p-1)(3p^{t+1}+2p^t+1+p^t)}, p \neq 3, i \geq 0; \]

\[ (20) \ g_3 = [\lambda_{2, \mu^2+1}] \in \text{Ext}^{3,2(p-1)(p+3)+1}, p \neq 3; \]

\[ (21) \ g_3 = [\lambda_{2, \mu^2+1}] \in \text{Ext}^{3,2(p-1)(p+3)+1}, p \neq 3; \]

\[ (22) \ f_i = (\mathcal{P}^{\pi})_{i-1} [M_1] \in \text{Ext}^{3,2(p-1)(p^{t+1}+2p^t)}, i \geq 1; \]

\[ (23) \ g_i = (\mathcal{P}^{\pi})_{i-1} [N_1] \in \text{Ext}^{3,2(p-1)(2p^t+p^t)}, i \geq 1; \]

where

\[ L_i = (\mathcal{P}^{\pi})_{i-1} \left( \sum_{j=1}^{p-1} \frac{(-1)^{j+1}}{j} \lambda_{(p-j)-1} \lambda_{j-1} \right), i \geq 0; \]

\[ M_1 = \sum_{j=1}^{p-1} \frac{(-1)^{j+1}}{j} \left( \lambda_{j-1} \lambda_{(p-j)} \lambda_{2p-1} - 2 \lambda_{p-1} \lambda_{j-1} \lambda_{2p-j-1} - 2 \lambda_{p-1} \lambda_{p-j} \lambda_{p-j-1} \right); \]
By inspection, we see that $h_i h_j h_k$ ($i < j + 1 < k + 2$), $\alpha_0 h_i h_j$ ($i < j + 1$), $\alpha_0^2 h_i$, $\alpha_0^2 h_i$, $h_{i:1,2} h_{j}$, $h_{i:1,3,1}$, $h_{i:1,3,1}$, $h_{i:1,2,3}$, and $f_i$ are represented by cycles of negative excess. Therefore, their images under $\varphi_3$ are trivial.

It is easy to check that $\varphi_3(\alpha_0^3) = -Q^0 Q^0 Q^0 \neq 0 \in R_3$.

Applying Corollary 4.4, one gets that $\varphi_3(h_i h_j) = -\beta Q^{p-1} Q^p Q^{p'}$. Applying adem relation, we obtain that $\beta Q^{p-1} Q^p = 0$, therefore $\varphi_3(h_i h_j) = 0$.

It is clear that $\varphi_3(h_{i:2,1} h_j) = -\beta Q^{2p+1} Q^{p} Q^{p'}$. Applying adem relation, we obtain that $\beta Q^{2p+1} Q^{p'} = 0$, it implies that $\varphi_3(h_{i:2,1} h_j) = 0$. Similarly, we have $\varphi_3(h_{i:2,1} a_0) = 0$.

By the same argument, we obtain

- $\varphi_3(\rho a_0) = -\beta Q^2 Q^0 Q^0 = 0$;
- $\varphi_3(h_{0:3,2,1}) = -\beta Q^{3p} Q^p Q^1 = 0$;
- $\varphi_3(h_{i:3,2,1}) = -\beta Q^{3p} Q^p Q^2 = 0$;
- $\varphi_3(h_{0:2,2,1}) = -\beta Q^{2p} Q^p Q^1 = 0$;
- $\varphi_3(h_{i:2,2,1}) = -\beta Q^{2p} Q^p Q^2 = 0$;
- $\varphi_3(g_5) = -\beta Q^1 Q^0 Q^0 = 0$;
- $\varphi_3(\tilde{g}_5) = -\beta Q^0 Q^0 Q^0 = 0$.

Finally, by inspection, we have

$$\varphi_3(g_1) = - \sum_{j=1}^{p-1} \frac{(-1)^j}{j} \beta Q^{2p} Q^p Q^{p-j}.$$  

It is clear that $\beta Q^j Q^{p-j} = 0$ if $j < p - 1$. But applying adem relation, we have

$$\beta Q^{2p} Q^{p-1} Q^1 = 0.$$

Combining with Proposition 5.3 we have the assertion of the theorem. ∎

Appendix A. The Singer transfer

The purpose of this section is to establish the chain-level representation of the dual of the mod $p$ Singer transfer in the Singer-Hưng-Sum chain complex. We end this section by the computation of the image of $T[s] \subset \Gamma^*_s$ through the Singer transfer, the result is used in Section 3.

Let $e_1(M) : \text{Tor}_s^A(\mathbb{F}_p, \Sigma^{-1} M) \longrightarrow \text{Tor}^A_{s-1}(\mathbb{F}_p, P_1 \otimes M)$ be the Singer’s element, which is the connecting homomorphism associated with the short exact sequence

$$0 \longrightarrow P_1 \otimes M \longrightarrow \hat{P} \otimes M \longrightarrow \Sigma^{-1} M \longrightarrow 0.$$

Put $e_s(M) := e_1(P_{s-1} \otimes M) \circ \cdots \circ e_1(\Sigma^{-(s-1)} M)$, then

$$e_s(M) : \text{Tor}_s^A(\mathbb{F}_p, \Sigma^{-s} M) \longrightarrow \text{Tor}^A_{s-1}(\mathbb{F}_p, P_s \otimes M).$$

When $M = \mathbb{F}_p$ and $r = s$, we have the dual of the mod $p$ Singer transfer

$$Tr^A_s := e_s(\mathbb{F}_p) \Sigma^{-s} : \text{Tor}^A_s(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \mathbb{F}_p \otimes_A P_s.$$
\textbf{Definition A.1.} The homomorphism \( T_s : \Delta^+_s \longrightarrow E(x_1, \ldots, x_s) \otimes \mathbb{F}_p[y_1^{\pm 1}, \ldots, y_s^{\pm 1}] \) is defined by
\[
T_s(u_1^{\varepsilon_1}v_1^{(p-1)i_1-\varepsilon_1} \cdots u_s^{\varepsilon_s}v_s^{(p-1)i_s-\varepsilon_s}) = (-1)^{f_s} \beta^{1-\varepsilon_1} \mathcal{P}^{i_1}(x_1y_1^{-1}\beta^{1-\varepsilon_1} \mathcal{P}^{i_2}(x_2y_2^{-1} \cdots \beta^{1-\varepsilon_s} \mathcal{P}^{i_s}(x_sy_s^{-1}))),
\]
where \( f_s = (1 + \cdots + s) + s(\varepsilon_1 + \cdots + \varepsilon_s) + i_1 + \cdots + i_s \), and \( i_1, \ldots, i_s \) are arbitrary integers. Here we mean \( \mathcal{P}^j = 0 \) for \( i < 0 \).

\textbf{Theorem A.2.} The restriction of \( T_s \) on \( \Gamma^{+}_s \), \( T_s|_{\Gamma^{+}_s} \) is the chain-level representation of the dual of the mod \( p \) Singer transfer \( T^*_s \).

\textit{Proof.} By the definition, \( e_1(\Sigma^{1-s}\mathbb{F}_p) : \text{Tor}^s_1(\mathbb{F}_p, \Sigma^{-s}\mathbb{F}_p) \longrightarrow \text{Tor}^s_1(\mathbb{F}_p, \Sigma^{1-s}P_1) \) is the connecting homomorphism of the exact sequence of chain complexes
\[
0 \longrightarrow \Gamma^{+}\Sigma^{1-s}P_1 \longrightarrow \Gamma^{+}\Sigma^{1-s}P \longrightarrow \Gamma^{+}\Sigma^{-s}\mathbb{F}_p \longrightarrow 0.
\]
For a cycle \( X = u_1^{\varepsilon_1}v_1^{(p-1)i_1-\varepsilon_1} \cdots u_s^{\varepsilon_s}v_s^{(p-1)i_s-\varepsilon_s} \in (\Gamma^{+}\Sigma^{-s}\mathbb{F}_p)_s \), it can be pulled back to the element \( X' = u_1^{\varepsilon_1}v_1^{(p-1)i_1-\varepsilon_1} \cdots u_s^{\varepsilon_s}v_s^{(p-1)i_s-\varepsilon_s}St_s(\Sigma^{1-s}x_sy_s^{-1}) \in (\Gamma^{+}\Sigma^{-s}P)_s \). Since \( X \) is the cycle in \( (\Gamma^{+}\Sigma^{-s}\mathbb{F}_p) \), in \( \Gamma^{+}\Sigma^{1-s}P \), one gets that \( \partial(X') \) is equal to
\[
(-1)^{k_s+i} u_1^{\varepsilon_1}v_1^{(p-1)i_1-\varepsilon_1} \cdots u_s^{\varepsilon_s}v_s^{(p-1)i_s-\varepsilon_s}St_s(\Sigma^{1-s}\beta^{1-\varepsilon_s} \mathcal{P}^{i_s}(x_sy_s^{-1})),
\]
where \( k_s = s + \varepsilon_1 + \cdots + \varepsilon_{s-1} + s\varepsilon_s \). Therefore, \( e_1(\Sigma^{1-s}\mathbb{F}_p)([X]) \) is equal to
\[
[(-1)^{k_s+i+1} u_1^{\varepsilon_1}v_1^{(p-1)i_1-\varepsilon_1} \cdots u_{s-1}^{\varepsilon_{s-1}}v_{s-1}^{(p-1)i_{s-1}-\varepsilon_{s-1}}St_{s-1}(\Sigma^{1-s}\beta^{1-\varepsilon_s} \mathcal{P}^{i_s}(x_{s-1}y_{s-1}^{-1}))).
\]
Repeating this process, we have the assertion. \( \square \)

For \( M \) is unstable \( A \)-module and \( m \in M^q \), we define
\[
d^* P(x, y; m) := \mu(q) \sum_{\varepsilon = 0, 1, 0 \leq s \leq 2q} (-1)^{\varepsilon+i} \left( \frac{x}{y} \right)^{\varepsilon \frac{(q-2)(q-1)}{2}} \otimes \beta^s \mathcal{P}^{i}(m),
\]
where \( \mu(q) = (h!)^{q}(1)^{q(q-1)/2}, h = (p-1)/2 \).

From Mühl’s [26] and Hung-Sum [19], we have

\textbf{Lemma A.3.} For \( m, n \in H^*BE_1 = \mathbb{F}_p[y] \otimes E(x) \)
(1) \( d^* P(x_1, y_1; V_l-1, y_2, \cdots, y_l) = V_l(y_1, \cdots, y_l) \);
(2) \( d^* P(x_1, y_1; M_{i-1}L^{l-1}_i) = (-h!)M_{i+1}L^{l-1}_i \);
(3) \( d^* P(x, y; mn) = (-1)^{h \deg m \deg n} d^* P(x, y; m)d^* P(x, y; n) \).

\textbf{Lemma A.4.} For \( m, n \in H^*BE_1 = \mathbb{F}_p[y] \otimes E(x) \),
(1) \( St_s(mn) = St_s(m) \cdot St_s(n) \);
(2) \( St_s(x) = (-1)^{s}u_{s+1} \);
(3) \( St_s(y) = (-1)^{s}v_{s+1} \).

\textbf{Corollary A.5.} (1) \( d^* P(x, y, V^{p-1}_i) = \beta P^p(p-1)(xy^{-1} \otimes V^{p-1}_i) \);
(2) \( d^* P(x, y; V^{p-1}_i) = V^{p-1}_i \);
(3) \( y^{\frac{p-1}{2}}d^* P(x, y; R_{i+1}) = (-h!R_{i+1}) \);
(4) \( y^{p-1}d^* P(x, y; q_i, 0) = q_{i+1} \);
(5) \( St_1(u_i) = -u_{i+1} \);
(6) \( St_1(v_i) = -v_{i+1} \).
Lemma A.6. Let \( M \) be an \( A \)-algebra, \( X, Y \in M \). For \( 2a \geq \deg X \) and \( 2b \geq \deg Y \), then

\[
\beta \mathcal{P}^{a+b}(xy^{-1} \otimes XY) = \beta \mathcal{P}^{a}(xy^{-1} \otimes X)\beta \mathcal{P}^{b}(xy^{-1} \otimes Y).
\]

**Proof.** Using Cartan formula, we can verify that

\[
\beta \mathcal{P}^{a+b}(xy^{-1} \otimes XY) = \sum_{\ell, \epsilon} (-1)^{a+b-\ell+\epsilon} x^\epsilon y^{(p-1)(a+b-\ell)-\epsilon} \otimes \beta \mathcal{P}^\ell(XY)
\]

\[
= \sum_{\ell, \epsilon} \sum_{i+j = \ell, \epsilon_1 + \epsilon_2 = \epsilon} (-1)^{a+b-\ell+\epsilon} (-1)^{\epsilon_2} \deg X x^\epsilon y^{(p-1)(a+b-\ell)-\epsilon}
\]

\[
\otimes \beta^{\epsilon_1} P^i(X) \beta^{\epsilon_2} P^j(Y)
\]

\[
= \left( \sum_{i, \epsilon_1} (-1)^{a-i+\epsilon_1} x^\epsilon y^{(p-1)(a-i)-\epsilon_1} \otimes \beta^{\epsilon_1} P^i(X) \right)
\]

\[
\times \left( \sum_{i, \epsilon_1} (-1)^{b-j+\epsilon_2} x^\epsilon y^{(p-1)(b-j)-\epsilon_2} \otimes \beta^{\epsilon_2} P^j(Y) \right)
\]

\[
= \beta \mathcal{P}^{a}(xy^{-1} \otimes X)\beta \mathcal{P}^{b}(xy^{-1} \otimes Y).
\]

The proof is complete. \( \Box \)

Lemma A.7. Let \( M \) be an \( A \)-algebra, \( X, Y \in M \). For \( 2a \geq \deg X \) and \( 2b \geq \deg Y \), then

\[
\mathcal{P}^{a+b}(xy^{-1} \otimes XY) = \mathcal{P}^{a}(xy^{-1} \otimes X)\beta \mathcal{P}^{b}(xy^{-1} \otimes Y).
\]

**Proof.** Using Cartan formula, we can verify that

\[
\mathcal{P}^{a+b}(xy^{-1} \otimes XY) = \left( \sum_{i=0}^{a} (-1)^{a-i} xy^{(p-1)(a-i)-1} \otimes P^i(X) \right)
\]

\[
\times \left( \sum_{j=0}^{b} (-1)^{b-j} y^{(p-1)(b-j)} \otimes P^j(Y) \right)
\]

\[
= \mathcal{P}^{a}(xy^{-1} \otimes X)\beta \mathcal{P}^{b}(xy^{-1} \otimes Y).
\]

The proof is complete. \( \Box \)

Put \( \mathcal{T}_s := (-1)^{1+\cdots+s} \mathcal{T}_s \). Then we have the following result.

Lemma A.8. For elements satisfying

\[
v^I = u_{1}^{\epsilon_1} v_1^{(p-1)j_1-\epsilon_1} \cdots u_{s}^{\epsilon_s} v_s^{(p-1)j_s-\epsilon_s}
\]

and \( v^J = u_{1}^{\sigma_1} v_1^{(p-1)j_1-\sigma_1} \cdots u_{s}^{\sigma_s} v_s^{(p-1)j_s-\sigma_s} \) in \( \Gamma^+ \), one gets

\[
\mathcal{T}_s(v^I \cdot v^J) = \mathcal{T}_s(v^I) \cdot \mathcal{T}_s(v^J).
\]
Proof. We only need to prove for $s = 2$. The case $s > 2$ is proved similarly.

\[ T'_s(v^+ \cdot v^-) = T'_s(u^r_1 v^1_{i_1} \sigma v^e_i v^r_{i_2} \sigma v^e_i v^r_{i_3} \sigma v^e_i v^r_{i_4}) \]

\[ = (-1)^{2(e_1 + e_2 + 2e_2) + i_1 + i_2 + j_1 + j_2} \]

\[ \times \beta^{1-(e_1 + e_2)} p^{i_1 + j_1} (x_1 y_1^{1-1} \otimes \beta^{1-e_2} p^{i_2} (x_2 y_2^{1-1})) \]

\[ = (-1)^{2(e_1 + e_2 + 2e_2) + i_1 + i_2 + j_1 + j_2} \]

\[ \times \beta^{1-(e_1 + e_2)} p^{i_1 + j_1} (x_1 y_1^{1-1} \otimes (\beta^{1-e_2} p^{i_2} (x_2 y_2^{1-1}))(\beta^{1-e_2} p^{i_2} (x_2 y_2^{1-1}))). \]

Since $v^+$ and $v^-$ satisfy condition \ref{eq:A.1}, then applying Lemma \ref{lem:A.6} or Lemma \ref{lem:A.7} one gets

\[ T'_s(v^+ \cdot v^-) = (-1)^{2(e_1 + e_2 + 2e_2) + i_1 + i_2 + j_1 + j_2} \]

\[ \times [\beta^{1-e_2} p^{i_1} (x_1 y_1^{1-1} \otimes (\beta^{1-e_2} p^{i_2} (x_2 y_2^{1-1})))] \]

\[ \times \beta^{1-e_2} p^{i_1} (x_1 y_1^{1-1} \otimes (\beta^{1-e_2} p^{i_2} (x_2 y_2^{1-1}))) = T'_s(v^+) \cdot T'_s(v^-). \]

The lemma is proved. \hfill \box

Lemma A.9. For $1 \leq i \leq s$, $T'_s(V^{(p-1)}_i) = V^{(p-1)}_i$.

Proof. By inspection, we have

\[ V^{(p-1)}_i = v^{i-2}_1 v^{i-3}_1 v^{i-4}_1 \cdots v^{i-p+2}_1 v^{i-p+1}_1. \]

Using \ref{eq:A.1}, one gets

\[ T'_s(V^{(p-1)}_i) = T'_s(v^{i-2}_1 v^{i-3}_1 v^{i-4}_1 \cdots v^{i-p+2}_1 v^{i-p+1}_1) \]

\[ = (-1)^{p-2} v^{i-2}_1 v^{i-3}_1 v^{i-4}_1 \cdots v^{i-p+2}_1 v^{i-p+1}_1 \]

\[ = d^* P(x_1, y_1, \cdots, d^* P(x_{i-1}, y_{i-1}, V^{(p-1)}_i)) = V^{(p-1)}_i. \]

The proof is complete. \hfill \box

Lemma A.10. For $1 \leq i \leq s$, then

\[ T'_s(R_{i;i-1}) = (-1)^s R_{i;i-1}. \]

Proof. By inspection, we have

\[ R_{i;i-1} = v^{i-2}_1 v^{i-3}_1 v^{i-4}_1 \cdots v^{i-p+2}_1 v^{i-p+1}_1. \]

Therefore,

\[ T'_s(R_{i;i-1}) = (-1)^{s+1} v^{i-2}_1 v^{i-3}_1 v^{i-4}_1 \cdots v^{i-p+2}_1 v^{i-p+1}_1 \]

\[ \times \beta^{p-2} v^{1-2}_1 v^{1-3}_1 v^{1-4}_1 \cdots v^{1-p+2}_1 v^{1-p+1}_1 \]

\[ = (-1)^{p-2} v^{i-2}_1 v^{i-3}_1 v^{i-4}_1 \cdots v^{i-p+2}_1 v^{i-p+1}_1 \]

\[ = d^* P(x_1, y_1, \cdots, d^* P(x_{i-1}, y_{i-1}, V^{(p-1)}_i)) = V^{(p-1)}_i. \]

First, we claim that $\beta^{p-2} v^{i-2}_1 v^{i-3}_1 v^{i-4}_1 \cdots v^{i-p+2}_1 v^{i-p+1}_1$. Indeed, by Corollary \ref{cor:A.5} it is easy to see that

\[ \beta^{p-2} v^{1-2}_1 v^{1-3}_1 v^{1-4}_1 \cdots v^{1-p+2}_1 v^{1-p+1}_1 \]

\[ = \frac{1}{\mu(2p^a + 2p^a - 1)} y^{p-1} d^* P(x, y; R_{a+1;a}) \]

\[ = \frac{1}{\mu(2p^a - 2p^a - 1)} (-h) R_{a+2;a+1}. \]
By Wilson’s theorem and the Fermat’s little theorem, one gets
\[-h! \equiv -\left(\frac{p-1}{2}\right)^2 \equiv (-1)^{\frac{p+1}{2}} \equiv 1 \mod p.\]

Since \(x_i y_i^{(p-1)-1} = R_{1,0}\), applying the above claim for \(a\) from 0 to \(i-2\), we have the assertion. \(\square\)

**Corollary A.11.** For \(0 \leq k < i \leq s\),
\[T_s'(R_{i,k}) = (-1)^s R_{i,k}.\]

**Proof.** Using Lemma A.9-A.10 together with the formula
\[R_{i;s} = R_{i-1;s} V_i^{p-1} + q_{i-1,s} R_{i;i-1},\]
we have the assertion. \(\square\)

**Proposition A.12.** Let \(\gamma \in \mathcal{B}[s]\). Then \(T_s(q) = (-1)^{\frac{\deg \gamma + 1}{2} + s} \deg \gamma\).

**Proof.** From Lemma A.8-A.10 it is sufficient to prove \(T_s'(R_{s;i,j}) = R_{s;i,j}\) for \(0 \leq i < j \leq s - 1\).
Since
\[-R_{s;i,j} = R_{s;i} R_{s;j} q_{s-1}^{-1} = R_{s-1;i} R_{s-1;j} q_{s-1,0}^{-1} V_s^{p-1} + (R_{s-1;i} q_{s-1,j} + R_{s-1;j} q_{s-1,i}) R_{s,s-1} q_{s-1,0}^{-1},\]
it is sufficient to show the assertion for \(R_{k;i} R_{k;k-1} q_{k,0}^{-1}\) and \(R_{k-1;i} R_{k;k-1} q_{k-1,0}^{-1}\).

The first case, we have
\[R_{k;i} R_{k;k-1} q_{k,0}^{-1} = q_{k-1,0}^{-2} R_{k-1;i} u_k v_k^{p-2}.\]

By Lemma A.8 and Corollary A.11 one gets
\[T_s'(R_{k;i} R_{k;k-1} q_{k,0}^{-1}) = (-1)^s R_{k-1;i} T_s'(q_{k-1,0}^{-2} u_k v_k^{p-2}).\]

By inspection, we obtain
\[q_{k-1,0}^{-2} u_k v_k^{p-2} = v_k^{p-2} (p-2) \cdots (p-1) u_k v_k^{p-2}.\]
Therefore,
\[T_s'(q_{k-1,0}^{-2} u_k v_k^{p-2}) = \left(-1^{s+k-p-2} (p-2) \cdots (p-2) + 1\right) \times \beta P^{p-2} (x_1 y_1^{-1} \cdots \beta P^{p-2} (x_{k-1} y_{k-1}^{-1} \beta P^{1} (x_k y_k^{-1})))\]
\[= (-1)^{s+k-p-2} (p-2) \cdots (p-2) \times \beta P^{p-2} (x_1 y_1^{-1} \cdots \beta P^{p-2} (x_{k-1} y_{k-1}^{-1} x_k y_k^{p-1}))\]
\[= (-1)^{s+k-p-2} (p-2) \cdots (p-2) \times \beta P^{p-2} (x_{k-1} y_{k-1}^{-1} x_k y_k^{p-1})).\]

It is easy to see that
\[\beta P^{p-2} (x_{k-1} y_{k-1}^{-1} x_k y_k^{p-1}) = (-1)^{(p-1)(p-2)} S_1 (x_k y_k^{p-2}) = (-1)^{(p-1)(p-2)} u_2 v_2^{p-2}.\]
By the same method, one gets
\[ \beta^{p(p^2-1)^2} \sum_{\ell \geq 1} \varepsilon_2^{\ell-1} u_2v_2^{p-2} = (\sum_{\ell \geq 1} \varepsilon_2^{\ell-1} u_2v_2^{p-2}) = (\sum_{\ell \geq 1} \varepsilon_2^{\ell-1} u_2v_2^{p-2}) = (\sum_{\ell \geq 1} \varepsilon_2^{\ell-1} u_2v_2^{p-2}). \]

By induction, we have
\[ T'_s(q_{k-1}^{-1}u_kv_k^{p-2}) = (-1)^s q_{k-1}^{-1}u_kv_k^{p-2}. \]

Thus, \[ T'_s(R_{k;1}R_{k;1-k}^{-1}q_{k,1}^{-1}) = R_{k;1}R_{k;1-k}^{-1}q_{k,1}^{-1}. \]

The final case, we have
\[ R_{k-1;1}R_{k;1-k}^{-1}q_{k,1}^{-1} = R_{k;1-k}^{-1}q_{k,1}^{-1}u_kv_k^{p-2}. \]

By the same argument, we have the assertion. \( \square \)

From this proposition, we have the following corollary.

**Corollary A.13.** For any \( \gamma = \sum_{\ell \geq 1} \varepsilon_2^{\ell-1} u_2v_2^{p-2} \),\( \sum_{\ell \geq 1} \varepsilon_2^{\ell-1} u_2v_2^{p-2} \), then
\[ \sum_{\ell \geq 1} (-1)^{i_1+\cdots+i_\ell} \beta^{1-\ell} \beta^1 u_2v_2^{p-2} = \sum_{\ell \geq 1} (-1)^{i_1+\cdots+i_\ell} \beta^{1-\ell} \beta^1 u_2v_2^{p-2} = \gamma. \]

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