On the N-integrality of instanton numbers

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Abstract

We prove the results announced in [KSV] modulo one general fact on Voevodsky motives that does not exist in the published literature. Namely, we assume that the functor of motivic vanishing cycles commutes with the Hodge and l-adic realizations.

Introduction

This paper is a result of a joint work with Maxim Kontsevich and Albert Schwarz. However, they decided not to sign it in the capacity of authors.

Let $\pi: X \to C$ be a family of Calabi-Yau n-folds over a smooth curve and let $a \in \overline{C} - C$ be a maximal degeneracy point of $\pi$. We assume that the pair $(\pi: X \to C, a)$ is defined over $\mathbb{Z}$. It is predicted that the power series expansion for the canonical coordinate on $C$ at the point $a$ has integral coefficients ([M]). This is a higher-dimensional generalization of the classical fact that the Fourier coefficients of the j-invariant are integers. This conjecture was checked in a number of cases in [LY]. Another related conjecture says the instanton numbers $n_d$, defined from the Picard-Fuchs equation, are integers (see loc. cit.). We do not know how to prove these conjectures. However, in the present paper we indicate a proof of a weaker statement, namely, that these numbers belong to the subring $\mathbb{Z}[N^{-1}] \subset \mathbb{Q}$, where $N$ is an explicitly defined integer.\footnote{i.e. $n_d = \frac{m}{N^k}$, where $m$ and $k$ are integers.}

There are two main ingredients in our proof. The first one is the Frobenius action on the p-adic de Rham cohomology. It easy to see, that under our assumptions, both the coefficients of power series expansion for the canonical coordinates and the instanton numbers are rational. Thus, to prove the integrality statement, it will suffice to show that for almost every prime $p$ they are p-adic integers. To do this we look at the relative de Rham cohomology of our family over p-adic numbers. Then the Frobenius symmetry or, more

\footnote{For example, for the quintic family [COGP], we can take $N = 2 \times 3 \times 5$.}
precisely, the existence of the Fontaine-Laffaille structure on the cohomology bundle imply certain strong integrality properties of the parallel sections (i.e. solutions to the Picard-Fuchs equation).  

The second ingredient is the motivic vanishing cycles. Assume, for the purpose of Introduction, that $\dim H^n(X/C) = n + 1$. Then the limit Hodge structure of the variation $H^n(X/C)$ is a mixed Hodge-Tate structure. Consider the corresponding period matrix $(a_{ij})$. This is a matrix with highly transcendental complex coefficients. On the other hand, we consider the limit Fontaine-Laffaille module and look at the corresponding Frobenius action $(b_{ij})$. This is a matrix with $p$-adic coefficients. To complete the proof of the integrality statement we need to establish a certain relation between the superdiagonal entries of the two matrices. Namely, we have to show, that

$$a_{m,m+1} = (2\pi i)^{m-1}\log c$$

$$b_{m,m+1} = \pm p^{m-1}\log c^{1-p}$$

for some rational number $c$.  

Standard conjectures on motives imply the existence of a mixed Artin-Tate motive $T$ over $\mathbb{Q}$ whose Hodge and $p$-adic realizations are the limit Hodge and Fontaine-Laffaille structure correspondingly. This yields a certain explicit relation between all the coefficients of matrices $(a_{ij})$ and $(b_{ij})$ and, in particular, formula (1). Unfortunately, the motivic conjectures needed to justify this argument are very far from being proved. However, we construct in Section 4 a 1-motive $M_{t,n}(X\mathbb{Q})$ that should be thought of as the maximal 1-motive quotient of $T^*$ and then use it to prove (1).

Still, at one point we have to rely on a general fact that has not yet appeared in the published literature. Namely, we have to assume the compatibility of Ayoub’s motivic vanishing cycles functor with the Hodge and $l$-adic realization functors. Although not published the required compatibility is known to experts [A2], [BOV] and hopefully this piece of a general theory will be written in a matter of time.

The paper is organized as follows. Section 1 contains statements of the results. In Section 2 we recall some well known facts on vector bundles with logarithmic connection, variations of Hodge structure, and give an interpretation of the canonical coordinate as an extension class of certain variations.

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3The idea to use the Frobenius action is due, in a slightly different setting, to Jan Stienstra [Sti].

4Logarithmic functions in these formulas have different meanings: in the first formula $\log$ is the the usual complex-valued function while in the second formula $\log$ takes $p$-adic values.
of Hodge structure. In Section 3 we define, using p-adic Hodge Theory, a p-adic analog of the canonical coordinate and Yukawa coupling (for 1-parameter families of Calabi-Yau varieties over $\mathbb{Z}_p$) and prove, using Dwork’s Lemma, the (p-adic) integrality statements for these objects. Finally, in the last section (the most technical one) we show for families over $\mathbb{Z}$ that the two constructions (complex and p-adic) give the same functions.\footnote{This amounts to proving relation (1).} To do this we give a third geometric definition (i.e. which makes sense over any ground field) of the canonical coordinate. The construction is based on the notion of motivic nearby cycles (due to Ayoub [A1]) and uses the language of Voevodsky motives.

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1 Terminology and statements of the results

1.1. Definition of the canonical coordinate. The following construction is due to Morrison [M]. Let $\pi : X \to C$ be a family of Calabi-Yau varieties of dimension $n$ over a smooth curve over C. This means that locally on C the relative canonical bundle $\Omega^n_{X/C}$ is trivial. Assume that C is embedded into a larger smooth curve $\overline{C} \supset C$ and $a \in \overline{C} - C$ is a boundary point. The point $a$ is called a maximal degeneracy point if the monodromy operator $M : H_n(X_a', \mathbb{Q}) \to H_n(X_a', \mathbb{Q})$, corresponding to a small loop around $a$ is unipotent and $(M - Id)^n \neq 0$.

**Remark.** It is known that for any smooth proper family $\pi : X \to C$ with a unipotent monodromy, $(M - Id)^{n+1} = 0$.

From now on we will assume that $a$ is a maximal degeneracy point. Set $N_B = \log M$. The following remarkable result was derived by Morrison from
the very basic properties of the limit Hodge structure.

**Lemma 1** ([M], Lemma 1.) $\dim \mathbb{Q} \text{Im} N_B^n = 1$ and $\dim \mathbb{Q} \text{Im} N_B^{n-1} = 2$.

Denote by $\mathcal{T}_Z$ the local system over a punctured neighborhood $D^*$ of $a$, whose fiber over a point $a' \in C$ is $\text{Im}(H_n(X_{a'}, \mathbb{Z}) \to H_n(X_{a'}, \mathbb{Q}))$. Let $\delta_1, \delta_2$ be a basis for $\text{Im} N_B^{n-1} \cap (\mathcal{T}_Z)_{a'}$ such that $\delta_1$ generates $\text{Im} N_B^n \cap (\mathcal{T}_Z)_{a'}$ and such that $N_B(\delta_2)$ is a positive multiple of $\delta_1$. We may view $\delta_1$ as a section of $T_Z$ over $D^*$ and $\delta_2$ as a section of the quotient $T_Z/\mathbb{Z}\delta_1$. Choose a non-vanishing section $\omega$ of $\pi_\ast \Omega^n_{X/C}$ over $D^*$. We then see that

$$q = \exp(2\pi i \int_{\delta_2} \omega / \int_{\delta_1} \omega)$$  \hspace{1cm} (2)

is a well defined function on a punctured neighborhood $D^*$ and that $q$ does not depend on the choice of $\delta_1$ and $\omega$ we made. Moreover, the function $q$ extends to $a$ and $\text{ord}_a q = k$, where $k$ is defined from the equation $N_B(\delta_2) = k\delta_1$. We shall say the Betti monodromy of the family $X \to C$ is small if $k = 1$. In this case, the function $q$ is called the canonical (local) coordinate on $C$.

**1.2. Yukawa function.** Denote by

$$\mathcal{H} = (\mathcal{H}_Z = \text{Im}(R^n \pi_* \mathbb{Z} \to R^n \pi_* \mathbb{Q}), \mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^0 = \mathcal{H}_Z \otimes \mathcal{O}_{D^*})$$

the variation of Hodge structure associated to $\pi : X \to C$. The Kodaira-Spencer operator

$$\nabla : \mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{F}^{p-1}/\mathcal{F}^p \otimes \Omega^1_{D^*}$$

extends to a homomorphism of graded algebras

$$S^* T_{D^*} \to \text{End}_{\mathcal{O}_{D^*}}(\bigoplus_p \mathcal{F}^p/\mathcal{F}^{p+1}).$$

Specializing, we get a morphism

$$\kappa : S^* T_{D^*} \to \text{Hom}_{\mathcal{O}_{D^*}}(\mathcal{F}^n, \mathcal{F}^0/\mathcal{F}^1) \simeq (\mathcal{F}^n \otimes \mathcal{F}^n)^*.$$  \hspace{1cm} (3)

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6The proof is reproduced in 2.2.

7 The nontrivial part is to show that $\int_{\delta_1} \omega$ does not vanish on a sufficiently small $D^*$ and that that $q$ has regular singularity at $a$. This is another corollary of the existence of the limit Hodge structure. See 2.2.
The line bundle $F^n \otimes F^n$ is naturally trivialized over $D^*$. To see this, denote by $\omega \in F^n$ the differential form such that
\[
\int_{\delta_1} \omega = (2\pi i)^n.
\]
The section $\omega \otimes \omega \in F^n \otimes F^n$ defines the desired trivialization.

Define the Yukawa function on $D^*$ to be
\[
Y = \kappa((q \frac{d}{dq})^n) \cdot (\omega \otimes \omega).
\]

One can check that $Y$ extends to $D$.

1.3. Statement of main results. Let $S = \text{spec} \mathbb{Z}[N^{-1}]$ be an open subscheme of $\text{spec} \mathbb{Z}$, $\overline{C}_S$ a smooth curve over $S$, and let $a : S \hookrightarrow \overline{C}_S$ be a section. Denote by $t$ a local coordinate on a open neighborhood of $a$ such that $t(a) = 0$. Let $\pi : X_S \rightarrow C_S = \overline{C}_S - a$ be a smooth proper family of Calabi-Yau schemes. We will make the following assumptions:

i) $a_C$ is the maximal degeneracy point of the complex family $X_C \rightarrow C_C$

ii) $\pi : X_S \rightarrow C_S$ extends to a semi-stable morphism $\overline{\pi} : \overline{X}_S \rightarrow \overline{C}_S$

iii) All primes $p \leq \text{dim} X_C$ are invertible on $\mathbb{Z}[N^{-1}]$

iv) The Betti monodromy of the family $X_C \rightarrow C_C$ is small (see 1.1).

\[\text{Theorem 2} \quad \text{Assume that } q'(0) \text{ is a rational number. Then}
\]
\[q(t) \in (\mathbb{Z}[N^{-1}]/(t))^*.
\]

\[\text{Remark: We shall see in Section 4.5 that, for any family } X_\mathbb{Q} \rightarrow C_\mathbb{Q} \text{ over } \mathbb{Q} \text{ with a maximal degeneracy point at } a \in \overline{C}_\mathbb{Q}(\mathbb{Q}), q'(t)^r \in \mathbb{Q}/((t)), \text{for some integer } r.
\]

For the next result we assume that $\text{dim} X_C = 4$ (i.e. $X_C \rightarrow C_C$ is a family of threefolds) and \[\text{for the next result we assume that } \text{dim} X_C = 4 \text{ (i.e. } X_C \rightarrow C_C \text{ is a family of threefolds) and}
\]
\[rk \overline{H}^3_{\overline{DR}}(X_C/C_C) = 4. \quad (4)
\]

8The cycle $\delta_1$ is defined up to sign. But the trivialization of $F^n \otimes F^n$ is independent of this choice.

9This is a corollary of a result of Schmid, which says that the Hodge filtration extends to Deligne’s canonical extension of the underlying vector bundle. (See also 2.1).

10I.e. locally for the etale topology $\overline{\pi} : \overline{X}_S \rightarrow \overline{C}_S$ is isomorphic to $\text{spec} \mathbb{Z}[t, x_1, \cdots x_r]/(x_1 \cdots x_r - t) \rightarrow \text{spec} \mathbb{Z}[t]$, where $r \leq n$.

11Observe that dimension of the space of first order deformations of a Calabi-Yau n-fold $Y$ is equal to $\dim H^1(Y, T_Y) = \dim H^{n+1}(Y, \Omega^n)$. Thus the condition (4) implies that $\pi : X_C \rightarrow C_C$ induces a dominant map from $C_C$ to an irreducible component of the moduli space of Calabi-Yau threefolds. The case of a higher dimensional component in the moduli space of Calabi-Yau threefolds will be considered elsewhere (also see [KSV], Section 3).
We also assume that \( q'(0) \) is a rational number.

**Theorem 3** One has

\[
Y(q) = n_0 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1 - q^d},
\]

(5)

where \( n_i \in \mathbb{Z}[N^{-1}] \).\(^{12}\)

## 2 Hodge Theory

### 2.1 Logarithmic De Rham cohomology

We will need the following construction from [Ste]. Let \( \pi : \overline{X}_S \rightarrow C_S \) be a semi-stable morphism.

\[
Y_S = \overline{X}_S \times_{C_S} S \hookrightarrow \overline{X}_S \hookleftarrow X_S
\]

We then consider the relative logarithmic De Rham complex \( (\Omega^*_{\overline{X}_S/C_S}(\log Y_S), d) \) on \( \overline{X}_S \) defined as follows. Let \( \Omega^*_{\overline{X}_S}(\log Y_S) \) be the sheaf of differential forms with logarithmic singularities along \( Y_S \), and let \( \Omega^*_{\overline{X}_S/C_S}(\log Y_S) \) be the quotient of the sheaf of algebras \( \Omega^*_{\overline{X}_S}(\log Y_S) \) by the ideal generated by \( \pi^* \eta, \eta \in \Omega^1_{\overline{C}_S}(\log S) \). One immediately sees that \( \Omega^*_{\overline{X}_S/C_S}(\log Y_S) \) is a locally free sheaf of \( \mathcal{O}_{\overline{X}_S} \) modules and that the exterior differential \( d : \Omega^i_{\overline{X}_S}(\log Y_S) \rightarrow \Omega^{i+1}_{\overline{X}_S}(\log Y_S) \) descends to \( \Omega^*_{\overline{X}_S/C_S}(\log Y_S) \). We then define the logarithmic De Rham cohomology by

\[
H^i_{log}(\overline{X}_S/C_S) = R^i \pi_* (\Omega^*_{\overline{X}_S/C_S}(\log Y_S), d).
\]

\( H^i_{log}(\overline{X}_S/C_S) \) is a coherent sheaf on \( C_S \) equipped with a logarithmic connection:

\[
\nabla : H^i_{log}(\overline{X}_S/C_S) \rightarrow H^i_{log}(\overline{X}_S/C_S) \otimes \Omega^1_{\overline{C}_S}(\log S).
\]

Assume that all primes \( p \leq n = \text{dim}_{C_S} X_S \) are invertible in \( \mathbb{Z}[N^{-1}] \). Then

i) the coherent sheaf \( H^i_{log}(\overline{X}_S/C_S) \) is locally isomorphic to a direct sum of the sheaves \( \mathcal{O}_{\overline{C}_S}, \mathcal{O}_{\overline{C}_S}/p^e \). In particular, the quotient of \( H^i_{log}(\overline{X}_S/C_S) \)

\(^{12}\)One readily sees that any power series \( Y(q) \in \mathbb{C}[[q]] \) can be written in the form (5), with \( n_d \in \mathbb{C} \). Thus the content of the theorem is the integrality property of the numbers. These are the instanton numbers the title of our paper refers to.
modulo torsion is locally free.

ii) the Hodge spectral sequence (i.e. the spectral sequence associated to the stupid filtration on $(\Omega^\bullet_{X_S/C_S}(log Y_S), d)$) degenerates in the first term. Moreover the induced filtration $\mathcal{F} \subset H^1_{log}(X_S/C_S)$ splits (in the category of $O_{C_S}$-modules) locally on $C_S$. In particular, $R^0\pi_*(\Omega^i_{X_S/C_S}(log Y_S))$ is a locally free $O_{C_S}$-module.

iii) the residue of the connection $N_{DR} = \text{Res} \nabla : a^*H^1_{log}(X_S/C_S) \to a^*H^1_{log}(X_S/C_S)$ is nilpotent. In particular, $H^1_{log}(X_Q/C_Q)$ is the Deligne canonical extension of the vector bundle $H^1_{DR}(X_Q/C_Q)$ equipped with the Gauss-Manin connection.

iv) there is a canonical pairing $\langle \cdot, \cdot \rangle_{DR} : H^i_{log}(X_S/C_S) \otimes H^{2n-i}_{log}(X_S/C_S) \to H^{2n}_{log}(X_S/C_S) \simeq O_{C_S}$ (6)

The induced pairing on the quotient of $H^\bullet_{log}(X_S/C_S)$ modulo torsion is perfect.

Over $\mathbb{C}$ these facts are proven in [Ste]; the integral version is contained in [Fa] (Theorems 2.1 and 6.2).

The De Rham isomorphism

$$H^1_{DR}(X_C/C_C) \simeq R^i\pi^*_{an}\mathbb{Z} \otimes O_C$$

and a choice of a local coordinate $t$ on $\mathbb{C}$ yield an integral structure (of a topological nature) on the vector space $a^*C H^1_{log}(X_C/C_C)$:

$$a^*C H^1_{log}(X_C/C_C) \simeq \Psi^{an,un}(R^i\pi^*_{an}\mathbb{Z}) \otimes \mathbb{Z}$$ (7)

To see this, let $Log^\infty = O_C \cdot [log t]$ be the universal unipotent local system on $\mathbb{C}^*$ and let $Log^{-\infty} = O_C \cdot [log t]$ be the Deligne extension of $Log^\infty$ to $\mathbb{C}$. Let us view $Log^\infty$ as a subsheaf of the direct image $(exp)_*O_C$ of the structure sheaf on the universal cover $exp : \mathbb{C} \to \mathbb{C}^*$. Define a $\mathbb{Z}$-lattice $Log^\infty \subset Log^\infty$ to be $(exp)_*\mathbb{Z} \cap Log^\infty$. We have then

$$Log^\infty \otimes O_C \simeq Log^\infty.$$ 

Let $a_C \in D \subset \overline{C}_C(\mathbb{C})$ be a disk such that the map $t : D \to \mathbb{C}$ defined by the coordinate is an embedding. Given a $\mathbb{Z}$-local system $\mathcal{H}_C$ over $D^*$ we define the space of unipotent vanishing cycles by

$$\Psi^{an,un}_t(\mathcal{H}_C) : = H^0(D^*, \mathcal{H}_C \otimes (t|_{D^*})^*Log^\infty)$$ (13)

This definition is borrowed from [B].
Assume that \( \mathcal{H}_Z \) is unipotent and denote by \( \overline{\mathcal{H}} \) the Deligne extension of \( \mathcal{H} = \mathcal{H}_Z \otimes \mathcal{O}_{D^*} \) to \( D \). We shall define a canonical isomorphism:

\[
a_C^* \overline{\mathcal{H}} \simeq \Psi^{an,un}_t(\mathcal{H}_Z) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{8}
\]

This will induce \( (7) \). To construct \( (8) \) observe that for any vector bundle \( E \) over \( D \) with a logarithmic nilpotent connection (i.e., a logarithmic connection such that \( N_{DR} \) is nilpotent) we have

\[
(E|_{D^*})^\nabla \overset{\sim}{\longrightarrow} E^\nabla \overset{\sim}{\longrightarrow} (a_C^*E)^{N_{DR}=0}. \tag{9}
\]

We apply \( (9) \) to the ind-object \( \overline{\mathcal{H}} \otimes t^*(\text{Log}^\infty) \) and use a canonical isomorphism

\[
(a_C^* \overline{\mathcal{H}} \otimes a_C^*t^*(\text{Log}^\infty))^{N_{DR}=0} \simeq a_C^* \overline{\mathcal{H}},
\]

which takes an element \( v \otimes \log^k t \in (a_C^* \overline{\mathcal{H}} \otimes a_C^*t^*(\text{Log}^\infty))^{N_{DR}=0} \) to \( v \) if \( k = 0 \) and to 0 otherwise.

Denote by

\[
N_B : \Psi^{an,un}_t(R^i \pi_C^{an} \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \Psi^{an,un}_t(R^i \pi_{C*}^{an} \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}
\]

the logarithm of the monodromy operator and by

\[
< \cdot, \cdot >_B : \Psi^{an,un}_t(R^i \pi_C^{an} \mathbb{Z}) \otimes \Psi^{an,un}_t(R^{2n-i} \pi_{C*}^{an} \mathbb{Z}) \to \mathbb{Z}
\]

the pairing induced by the Poincare duality. We then have

\[
N_B = -2\pi i N_{DR}, \quad < \cdot, \cdot >_B = (2\pi i)^n < \cdot, \cdot >_{DR}.
\]

2.2. A variation of mixed Hodge structure. Let \( \pi_C : X_C \to C_C \) be a smooth family of Calabi-Yau schemes with a maximal degeneracy point at \( a_C \in C_C \) and let \( \overline{\pi}_C : X_C \to \overline{C}_C \) be a semi-stable morphism which extends \( \pi_C \). Set

\[
\overline{\mathcal{H}} = H^0_{\text{log}}(X_C/\overline{C}_C), \quad \mathcal{H}_Z = \text{Im}(R^n \pi_{C*} \mathbb{Z} \to R^n \pi_{C*} \mathbb{Q}).
\]

Denote by \( W_\Psi^{an,un}_t(\mathcal{H}_Z) \subset \Psi^{an,un}_t(\mathcal{H}_Z) \) the monodromy filtration. \(^{14}\) We then consider the limit Hodge structure

\[
\Psi^{Hodge,un}_t(\mathcal{H}) = (W_\Psi^{an,un}_t(\mathcal{H}_Z) \subset \Psi^{an,un}_t(\mathcal{H}_Z), \quad a_C^* \mathcal{F} \subset a_C^* \overline{\mathcal{H}}).
\]

\(^{14}\)By definition, this is a unique filtration such that the quotients \( \Psi^{an,un}_t(\mathcal{H}_Z)/W_\Psi^{an,un}_t(\mathcal{H}_Z) \) are torsion free, \( N_B(W_\Psi^{an,un}_t(\mathcal{H}_Z)) \subset W_{i-2} \Psi^{an,un}_t(\mathcal{H}_Z) \otimes \mathbb{Q} \) and \( N_B : \text{Gr}^{W_\Psi^{an,un}_t(\mathcal{H}_Z)} \otimes \mathbb{Q} \simeq \text{Gr}^{W_\Psi^{an,un}_t(\mathcal{H}_Z)} \otimes \mathbb{Q} \).
Since $\mathcal{F}^{n+1} = 0$, we have

$$W_{-1}\Psi_t^{an,un}(\mathcal{H}_Z) = 0, \ W_{2n}\Psi_t^{an,un}(\mathcal{H}_Z) = \Psi_t^{an,un}(\mathcal{H}_Z)$$

$$\text{Im } N_B^n = W_0\Psi_t^{an,un}(\mathcal{H}_Z) \otimes \mathbb{Q}.$$ 

Furthermore, since $rk\mathcal{F}^n = rk\mathcal{F}^n/\mathcal{F}^1 = 1$ and $Im\ N_B^n \neq 0$ we must have

$$rk W_0\Psi_t^{an,un}(\mathcal{H}_Z) = rk W_1\Psi_t^{an,un}(\mathcal{H}_Z) = 1.$$ 

It follows that the map

$$N_B^{n-1} : \Psi_t^{an,un}(\mathcal{H}_Z) \to Im\ N_B^{n-1}/Im\ N_B^n$$

factors through the quotient $\Psi_t^{an,un}(\mathcal{H}_Z)/W_{2n-1}\Psi_t^{an,un}(\mathcal{H}_Z)$ of rank 1. In particular, $\dim\ Im\ N_B^{n-1} = 2$.

Note that for any monodromy invariant lattice $P_Z \subset \Psi_t^{an,un}(\mathcal{H}_Z)$ there is a unique local system $P_Z \subset \mathcal{H}_Z$ over a punctured disk $D^* \subset C_{\mathbb{C}}$ such that $\Psi_t^{an,un}(P_Z) = P_Z$. We apply this remark to $L_Z = Im\ N_B^{n-1}\cap\Psi_t^{an,un}(\mathcal{H}_Z)$ and to $W_0\Psi_t^{an,un}(\mathcal{H}_Z)$. Call the corresponding local systems by $L_Z$ and $\mathcal{H}$. We claim that

$$\mathcal{L}^{Hodge} = (\mathcal{W}L_Z, \mathcal{F}L),$$

where $\mathcal{W}_1L_Z = 0, \mathcal{W}_1L_Z = \mathcal{W}_2L_Z = \mathcal{W}_0\mathcal{H}_Z$, $\mathcal{W}_2L_Z = L_Z$ and $\mathcal{F}^2L = 0, \mathcal{F}^1L = \mathcal{F}^1 \cap L, \mathcal{F}^0L = L = L_Z \otimes \mathcal{O}_{D^*}$, is an admissible variation of mixed Hodge structure over a sufficiently small punctured disk $D^*$. Indeed, over a small disk we have $\mathcal{W}_0\mathcal{H} \oplus \mathcal{F}^1 = \mathcal{H}$. The claim follows.

Thus we get a class

$$[\mathcal{L}^{Hodge}] \in Ext^1_{VMHS}(\mathcal{L}^{Hodge}/\mathcal{W}_0\mathcal{L}^{Hodge}, \mathcal{W}_0\mathcal{L}^{Hodge})$$

$$\simeq Ext^1_{VMHS}(\mathbb{Z}(-1), \mathbb{Z}(0)) \otimes Hom_{\mathbb{Z}}(L_Z/W_0\Psi_t^{an,un}(\mathcal{H}_Z), W_0\Psi_t^{an,un}(\mathcal{H}_Z)).$$

Define

$$q_{\mathcal{C}} \in Ext^1_{VMHS}(\mathbb{Z}(-1), \mathbb{Z}(0)) \otimes \mathbb{Q}$$

(10)

to be the composition of $[\mathcal{L}^{Hodge}]$ with

$$N_B^{-1} : W_0\Psi_t^{an,un}(\mathcal{H}_Z) \otimes \mathbb{Q} \to L_Z/W_0\Psi_t^{an,un}(\mathcal{H}_Z) \otimes \mathbb{Q}.$$ 

If the monodromy is small i.e. that the map $N_B : L_Z/W_0\Psi_t^{an,un}(\mathcal{H}_Z) \to W_0\Psi_t^{an,un}(\mathcal{H}_Z)$ is an isomorphism, the class $q_{\mathcal{C}}$ lifts canonically to

$$\tilde{q}_{\mathcal{C}} \in Ext^1_{VMHS}(\mathbb{Z}(-1), \mathbb{Z}(0)).$$

\textsuperscript{15}Recall that a variation of mixed Hodge structure $(\mathcal{W}L_Z \subset L_Z, \mathcal{F}L \subset L)$ over $D^*$ is called admissible if the Hodge filtration $\mathcal{F}$ extends to the Deligne extension $\tilde{L}$ of $L$. 

9
Recall that the group $\text{Ext}_{V \text{MHS}}^{1}(\mathbb{Z}(-1), \mathbb{Z}(0))$ of admissible extensions is canonically identified with the group of invertible functions on $D^*$ with a regular singularity at the origin. The following lemma immediately follows from the construction.

**Lemma 4** The class $\bar{q}_C$ is equal to the canonical coordinate $q$.

We shall compute the logarithmic derivative

$$q_C \in (\mathbb{C}((t)))^* \otimes \mathbb{Q} \xrightarrow{d \log} \mathbb{C}[[t]] \frac{dt}{t}.$$  

Let $e^0$ be a nonzero parallel section of $\mathcal{W}_0\mathcal{L}$. Then there exists a unique section $e^1$ of $\mathcal{F}^1\mathcal{L}$ such that the projection of $e^1$ to $\mathcal{L}/\mathcal{W}_0\mathcal{L}$ is parallel and

$$-\frac{1}{2\pi i} N_B(e^1(a_C)) = N_{DR}(e^1(a_C)) = e^0(a_C).$$

We have then

$$\nabla e^1 = e^0 \otimes d \log q_C.$$  

Assume that we are in the situation of 1.3. It follows then that

$$\log q_C \in (1 + t\mathbb{Q}[[t]]) \frac{dt}{t}.$$  

Indeed, we can normalize $e^0$ such that $e^0(a_Q) \in a_Q^* H_{\log}^n(X_Q/\overline{C}_Q)$. Then $e^0, e^1 \in H_{\log}^n(X_Q/\overline{C}_Q)$, and we are done by (11).

### 3 p-adic Hodge Theory

**3.1. Fontaine-Laffaille modules.** Fontaine-Laffaille modules over a scheme is a p-adic analog of variations of Hodge structure. Below we recall this notion in the special case of torsion free modules over a punctured disk. This is sufficient for our applications. \(^{16}\) The general definition can be found in [Fa].

Let $(D, a)$ be a formal disk over $\mathbb{Z}_p$ with a point $a : \text{spec } \mathbb{Z}_p \hookrightarrow D$. We view $(D, a)$ as a logarithmic scheme (see [Il]). A logarithmic morphism $G : (D, a) \to (D', a')$ is morphism such that the scheme theoretical preimage of the section $a'$ is supported on $a$ i.e. $G^*(t) = t^m f(t')$, where $t$ and $t'$ are coordinates on $D$ and $D'$ respectively, such that $t(a) = t'(a') = 0$, and $f$ is

\(^{16}\)Except for the last section where we need the category of all Fontaine-Laffaille modules over a point.
an invertible function on $D'$. Denote by $\Omega^1(\log)$ the space of 1-forms on $D$ with logarithmic singularities at $a$.

Let $\mathcal{E}$ be a vector bundle over $D$ with a logarithmic connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(\log)$. For any logarithmic morphism $G : (D', a') \to (D, a)$, $G^*\mathcal{E}$ is endowed with the induced logarithmic connection. Moreover, if two logarithmic morphisms $G$ and $G'$ are equal modulo $p$ we have a canonical parallel isomorphism

$$\theta : G^*\mathcal{E} \simeq G'^*\mathcal{E}. \quad (12)$$

In coordinates $\theta : \mathcal{E} \otimes_G \mathbb{Z}_p[[t']] \to \mathcal{E} \otimes_G' \mathbb{Z}_p[[t']]$ is given by Taylor's formula

$$\theta(e \otimes 1) = \sum_{i=0}^{\infty} (\nabla_\delta)^i e \otimes \frac{(\log(G^*(t)/G'^*(t)))^i}{i!},$$

where $\delta = t d/dt$ is the vector field. One readily checks that $\frac{(\log(G^*(t)/G'^*(t)))^i}{i!} \in \mathbb{Z}_p[[t]]$ and that the series converges.

Let $\tilde{F} : (D, a) \to (D, a)$ be a logarithmic lifting of the Frobenius morphism (i.e. $\tilde{F}^*(t) = t^p(1 + ph(t))$, where $h(t) \in \mathbb{Z}_p[[t]]$). A (torsion free) Fontaine-Laffaille module over the logarithmic disk $(D, a)$ amounts to the following data:

i) a vector bundle $\mathcal{E}$ over $D$ with a filtration by sub-bundles

$$0 = \mathcal{F}^{p-1} \subset \mathcal{F}^{p-2} \subset \cdots \subset \mathcal{F}^0 = \mathcal{E},$$

ii) a logarithmic connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(\log)$ satisfying the Griffiths transversality condition: $\nabla(\mathcal{F}^i) \subset \mathcal{F}^{i-1} \otimes \Omega^1(\log)$,

iii) a parallel morphism (“Frobenius”)

$$\phi : \tilde{F}^*\mathcal{E} \to \mathcal{E}$$

with the following properties

$$\phi(\tilde{F}^*\mathcal{F}^i) \subset p^i\mathcal{E}$$

and

$$\sum_i p^{-i} \phi(\tilde{F}^*\mathcal{F}^i) = \mathcal{E}.$$ 

**Remark.** The definition we gave above depends on the choice of a lifting $\tilde{F}$. Still, the categories corresponding to different liftings are canonically equivalent. To see this let $\tilde{F}'$ be another logarithmic lifting. By (12) there is a canonical parallel isomorphism

$$\theta : \tilde{F}^*\mathcal{E} \simeq \tilde{F}'^*\mathcal{E}. \quad (12)$$
The functor, that provides the equivalence, takes \((E, F_\uparrow, \nabla)\) to the same objects and sends \(\phi\) to \(\phi \theta^{-1}\). The Griffiths transversality condition implies that
\[
\theta^{-1} \hat{F}^* F_k \subset \sum_{i \geq 0} \frac{p^i}{i!} \hat{F}^* F^{k-i} \subset \hat{F}^* E.
\]
Thus, thanks to the assumption on the range of the Hodge filtration (\(F^{p-1} = 0\) and \(F^0 = E\)), \((E, F_\uparrow, \nabla, \phi \theta^{-1})\) satisfies the requirements in iii). Denote the category of Fontaine-Laffaille modules over \((D, a)\) by \(\text{MF}_{[0, p-2]}(D^*)\). A similar construction is used to define the pullback functor
\[
G^* : \text{MF}_{[0, p-2]}(D^*) \rightarrow \text{MF}_{[0, p-2]}(D'^*),
\]
for a logarithmic morphism \(G : (D', a') \rightarrow (D, a)\).

### 3.2. Dwork’s Lemma.

Denote by \(\mathbb{Z}_p(\cdot, k)\), \((k \geq 0)\), the constant variation: \(E = \mathbb{O}, F_k = \mathbb{E}, F^{k+1} = 0, \phi = p^k \text{Id}\). Let \(\mathbb{O}(D^*)\) be the space of functions on the punctured disk (i.e. \(\mathbb{O}(D^*) = \mathbb{Z}_p((t))\)).

**Lemma 5** The group \(\text{Ext}^1_{\text{MF}_{[0, p-2]}(D^*)}(\mathbb{Z}_p(-1), \mathbb{Z}_p(0))\) is canonically isomorphic (i.e. the isomorphism does not depend on the lifting \(\hat{F}\)) to \(p\)-adic completion of the group \(\hat{\mathbb{O}}^*(D^*)\):
\[
\hat{\mathbb{O}}^*(D^*) := \lim_{\leftarrow} \mathbb{O}^*(D^*)/((\mathbb{O}^*(D^*))^{p^i}) \sim \text{Ext}^1(\mathbb{Z}_p(-1), \mathbb{Z}_p(0)).
\]

**Proof:** Let \(t\) be a coordinate on \(D\), and let \(\hat{F} : D \rightarrow D\) send \(t\) to \(tp^i\). Consider an extension \((\mathbb{E}, \mathbb{F}, \phi)\):
\[
0 \rightarrow \mathbb{Z}_p(0) \rightarrow \mathbb{E} \rightarrow \mathbb{Z}_p(-1) \rightarrow 0
\]
Note that \(\mathbb{F}^0 = \mathbb{E}, \mathbb{F}^2 = 0\), and that last map in the exact sequence defines an isomorphism \(\mathbb{F} \simeq \mathbb{O}_D\). Let \(e_1 \in \mathbb{F}^1\) be the preimage of 1 under the above isomorphism, and let \(e_0 \in \mathbb{E}\) be the image of 1 under the first map in the exact sequence. Then \(e_0, e_1\) form a basis for \(\mathbb{E}\). We have:
\[
\nabla e_0 = 0, \nabla e_1 = e_0 \otimes \omega
\]
\[
\phi(\hat{F}^*(e_0)) = e_0, \phi(\hat{F}^*(e_1)) = pe_1 + phe_0,
\]
for some \(\omega \in \Omega^1(\log)\) and \(h \in \mathbb{O}(D)\). Since \(\phi\) is parallel, \(\phi \nabla = \nabla \phi\). This amounts to the following equation
\[
1/p \hat{F}^* \omega - \omega = dh.
\]
Thus the set of extensions is in a bijection with the set of pairs \((\omega, h)\) satisfying the above equation. One can easily see that the above bijection is compatible with the group structure \(^{17}\).

Define a homomorphism

\[
\mathcal{O}^*(D^*) \to Ext^1_{MF_{[0,p-2]}(D^*)}(\mathbb{Z}_p(-1), \mathbb{Z}_p(0)).
\] (15)

sending an invertible function \(q\) to the pair \((d\log q, \frac{1}{p}\log \tilde{F}^* q)\). One readily sees that (15) extends to the \(p\)-adic completion of \(\mathcal{O}^*(D^*)\). This is the map in (14). The injectivity of (14) is clear from the definition and the surjectivity is the content of the Dwork’s lemma \(^{18}\). Let us check that (14) is independent of the choice of the coordinate. Indeed, let \(t'\) be another coordinate and let \(\tilde{F}'\) be the corresponding lifting of the Frobenius. The isomorphism (12):

\[
\theta : \tilde{F}^*E \simeq \tilde{F}'^*E
\]

takes \(\tilde{F}^*(e_0)\) to \(\tilde{F}'^*(e_0)\) and \(\tilde{F}^*(e_1)\) to

\[
\sum_{i=0}^{\infty} \frac{(\log(\tilde{F}^*(t)/\tilde{F}'^*(t)))^i}{i!} \tilde{F}'^*((\nabla_\delta)^i e_1) = \tilde{F}'^* e_1 + \log \tilde{F}'^*(q) e_0.
\] \(^{15}\)

The claim follows.

3.3. Limit Fontaine-Laffaille module. Let \(t\) be a coordinate, \(\tilde{F}\) the corresponding lifting of the Frobenius, and let \((E, F^*, \nabla, \phi) \in MF_{[0,p-2]}(D^*)\) be a Fontaine-Laffaille module. We define

\[
\Psi^{FL}_t((E, F^*, \nabla, \phi)) = (E = a^* E, F^* = a^* F^*, \phi_a).
\]

\(\Psi^{FL}_t((E, F^*, \nabla, \phi))\) is a Fontaine-Laffaille module over the point. The residue of \(\nabla\) is a morphism of Fontaine-Laffaille modules:

\[
N_{DR} : Res \nabla : \Psi^{FL}_t((E, F^*, \nabla, \phi)) \to \Psi^{FL}_t((E, F^*, \nabla, \phi))(-1).
\]

In particular,

\[
N_{DR} \phi_a = p \phi_a N_{DR}.
\] (16)

\(^{17}\) The group structure on the set of pairs \((\omega, h)\) is defined by the formula \((\omega, h) + (\omega', h') = (\omega + \omega', h + h')\).

\(^{18}\) The Dwork’s lemma is the following statement:

Let \(\omega \in \Omega^1_{log} = \mathbb{Z}_p[[t]]/\frac{dt}{t}\) with \(Res_0 \omega \in \mathbb{Z}\). The following two conditions are equivalent:

i) \(1/p \tilde{F}^* \omega - \omega = dh\), for some \(h \in \mathbb{Z}_p[[t]]\)

ii) \(\omega = d \log q\), for some \(q \in \mathcal{O}^*(D^*)\).

\(^{19}\) The last equality follows from the multiplicative version of Taylor’s formula

\[
f(e^b a) = (exp(b \delta)(f))(a) = f(a) + \delta f(a)b + \frac{\delta^2 f(\delta)(a)b^2}{2} + \cdots.
\]
Remark. The functor $\Psi_{t}^{FL}$ depends on the choice of a coordinate $t$. If $t' = bt + \cdots$, $b \in \mathbb{Z}_{p}^{*}$ is another coordinate we have

$$\Psi_{t'}^{FL}(\mathcal{E}, \mathcal{F}, \nabla, \phi \theta^{-1}) \simeq (E, F', \phi_{a} \exp(N_{DR} \log b^{p-1})).$$

In particular, $\Psi_{t}^{FL}$ does not get changed if we replace $t$ by $t'$ with the same derivative.

3.4. $p$-adic canonical coordinate. Let $\pi : \mathcal{X} \to D$ be a proper semi-stable morphism. For any $k < p - 1$, Faltings constructed in [Fa] a Fontaine-Laffaille structure on the logarithmic De Rham cohomology $H_{\log}^{k}(\mathcal{X}/D)$. In the rest of this section we assume that $n := \dim_{\mathbb{D}} X < p - 1$, and let $\mathcal{E} := H_{\log}^{n}(\mathcal{X}/D)/p - \text{torsion} \in MF_{[0, p-2]}(D^*)$. The cup product $H_{\log}^{n}(\mathcal{X}/D) \otimes H_{\log}^{n}(\mathcal{X}/D) \to H_{\log}^{2n}(\mathcal{X}/D) \simeq \mathbb{Z}_{p}(-n)$ induces a perfect paring

$$<\cdot, \cdot>_{DR} : \mathcal{E} \otimes \mathcal{E} \to \mathbb{Z}_{p}(-n). \quad (17)$$

In particular,

$$<\phi(\tilde{F}^{*}v), \phi(\tilde{F}^{*}u)>_{DR} = p^{n} <v, u>_{DR}. \quad (18)$$

Assume that $X$ is a Calabi-Yau scheme over $D^*$ and that $a$ is the maximal degeneracy point. That means, by definition, that $\dim_{\mathbb{D}} E = 1$ and the operator $N_{DR}^{n} : E \to E$ is not equal to 0. Assume, in addition, that $\pi : \mathcal{X} \to D$ extends to a semi-stable scheme over a curve. We have then

$$rk \text{Im} N_{DR}^{n} = 1, \ rk \text{Im} N_{DR}^{n-1} = 2. \quad (19)$$

This follows from Lemma 1 and "the Lefschetz principle".

Lemma 6 The Frobenius operator $\phi_{a}$ restricted to $\text{Im} N_{DR}^{n}$ is equal to $\pm Id$.

Proof: The lemma follows immediately from (16) and (18).

The above lemma implies the existence of a parallel section of $\mathcal{E}$. Namely we have the following result.

Lemma 7 Let $\mathcal{E}$ be a vector bundle over $D$ with a logarithmic connection and $\phi : \tilde{F}^{*} \mathcal{E} \to \mathcal{E}$ be a parallel morphism. For any element $w \in E$ such that $\phi_{a}(w) = \pm w$ there exists a unique parallel section $s$ of $\mathcal{E}$ with $s(a) = w$. The section $s$ satisfies the property $\phi(\tilde{F}^{*}s) = \pm s$. 

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Proof: The uniqueness part is clear. To prove the existence we start with any section $s'$ of $\mathcal{E}$ with $s'(a) = w$ and consider the sections $s'_k = (\phi \tilde{F}^*)^{2k}(s')$. It is easy to see that $\nabla s'_k \in p^{2k} E \otimes \Omega^1(\log a)$ and that $s'_k(a) = w$. This implies that the limit

$$s = \lim s'_k$$

exists and satisfies all the required properties.

The nilpotent operator $N_{DR} : E \to E$ gives rise to a canonical filtration $W_0 = W_1 \subset W_2 \subset \cdots \subset W_{2n} = E$ by Fontaine-Laffaille submodules. It is a unique filtration with torsion free quotients $W_{i+1}/W_i$ such that $W_i \otimes \mathbb{Q}_p$ is the monodromy filtration on $E \otimes \mathbb{Q}_p$. The Frobenius $\phi$ preserves the filtration $W_i$ and $N_{DR}(W_i) \subset W_{i-2}$. Let $L^{FL} := \text{Im} N_{DR}^{n-1} \otimes \mathbb{Q}_p \cap E$. This is a Fontaine-Laffaille submodule of $E$. It follows from (6) that the eigenvalue of $\phi$ on $W_0$ (resp. $L^{FL}/W_0$) is equal to $\pm 1$ (resp. $\pm p$). Lemma (7) implies that the inclusion $W_0 \hookrightarrow E$ extends uniquely to a parallel morphism $\Psi_0 := W_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_D \hookrightarrow \mathcal{E}$. Note that the projection $\Psi_0 \twoheadrightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{L}^{FL}$ is an isomorphism. Thus the Frobenius $\phi : \tilde{F}^*(\mathcal{E}/\Psi_0) \to \mathcal{E}/\Psi_0$ is divisible by $p$. Applying (7) again to $\mathcal{E}$ we conclude that the inclusion $L^{FL}/W_0 \hookrightarrow E/W_0$ extends uniquely to a parallel morphism $\mathcal{L}^{FL} = L^{FL}/W_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_D \hookrightarrow \mathcal{E}/\Psi_0$. Finally, let $\mathcal{L}^{FL} \subset \mathcal{E}$ be the preimage of $L^{FL}/W_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_D$ in $\mathcal{E}$. By construction, $\mathcal{L}^{FL}$ is a unique Fontaine-Laffaille submodule of $\mathcal{E}$ with $\Psi^1_0(\mathcal{L}^{FL}) = L^{FL}$. Thus we get a canonical class

$$[\mathcal{L}^{FL}] \in Ext^1_{MF[0,p-2]}(\mathcal{E}^*) (\mathcal{L}^{FL}/\Psi_0, \Psi_0) \simeq$$

$$Ext^1_{MF[0,p-2]}(\mathcal{E}^*) (\mathcal{L}^{FL}/\Psi_0, \Psi_0) \otimes \text{Hom}_{\mathbb{Z}_p}(L^{FL}/W_0, \Psi_0).$$

Composing this with $N_{DR}^{-1} \in \text{Hom}(\Psi_0 \otimes \mathbb{Q}_p, L^{FL}/W_0 \otimes \mathbb{Q}_p)$ we get the "$p$-adic canonical coordinate":

$$q_{\mathbb{Z}_p} \in Ext^1_{MF[0,p-2]}(\mathcal{E}^*) (\mathcal{L}^{FL}/\Psi_0, \Psi_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \hat{\text{O}}^*(D^*) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$\hspace{1cm} (20)

Observe that the order

$$\text{ord} : \hat{\text{O}}^*(D^*) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Q}_p$$

of $q_{\mathbb{Z}_p}$ is equal to 1. In particular, $q_{\mathbb{Z}_p} \in \text{O}^*(D) \otimes \mathbb{Z} \mathbb{Q}_p$.

Let $e^0$ be a nonzero parallel section of $\Psi_0 \otimes \mathbb{Q}_p$ and let $e^1$ be a section of $(\mathcal{F}^1 \cap \mathcal{L}^{FL}) \otimes \mathbb{Q}_p$ whose projection to $(\mathcal{L}^{FL}/\Psi_0) \otimes \mathbb{Q}_p$ is parallel and such that $N_{DR}(e^1(a)) = e^0(a)$. We then have

$$\nabla e^1 = e^0 \otimes d \log q_{\mathbb{Z}_p}.$$\hspace{1cm} (21)
We shall the p-adic monodromy is small, if the operator $N_{DR} : L^F/\mathcal{W}_0 \to \mathcal{W}_0$ is an isomorphism. If this is the case, one has

$$q_{\mathbb{Z}_p} \in O^*\langle D^\ast \rangle/\mu_{p-1} \subset \hat{O}^*\langle D^\ast \rangle \otimes \mathbb{Q}_p.$$ (22)

3.5. p-adic Yukawa map. In this subsection we assume that the p-adic monodromy is small. Denote by $q \in O_D$ the p-adic canonical coordinate (defined up to a (p-1)th root of unity). Let

$$S^nT_{D,\log} \to \text{Hom}_{O_D}(\mathcal{F}^n, \mathcal{F}^0/\mathcal{F}^1) \simeq (\mathcal{F}^n \otimes \mathcal{F}^m)^*$$ (23)

be the Kodaira-Spenser morphism. Here $T_{D,\log}$ denotes the sheaf dual to $\Omega^1(\log)$ i.e. the sheaf of vector fields on $D$ vanishing at $a$. Choose a generator $e^0$ of $\mathcal{W}_0^n$ and let $e_0 \in \mathcal{F}^m$ be a section with $(e^0, e_0) = 1$. Applying (23) to $(q^d \overline{dq})^\otimes n$ and pairing the result with $e_0 \otimes e_0$ we obtain the p-adic Yukawa function $Y \in O_D$. Observe that $Y(q)$ is well defined up to multiplication by a constant in $\mathbb{Z}_p^\ast$.

**Proposition 8** Assume that $n = 3$ and that $rk E = 4$. Then

$$Y(q) = n_0 + \sum_{d=1}^{\infty} n_d d^3 q^d / 1 - q^d,$$

where $n_d \in \mathbb{Z}_p$.

**Proof:** We shall use the following elementary result:

**Lemma 9** ([KSV]. Lemma 2.) Assume that a formal power series $Y(q) \in \mathbb{Z}_p[[q]]$ is written in the form

$$Y(q) = \sum_{d=1}^{\infty} n_d d^3 q^d / 1 - q^d.$$

Then $n_d \in \mathbb{Z}_p$ if and only if $Y(q) - Y(q^p) = \delta^3(\psi(q))$, for some $\psi(q) \in \mathbb{Z}_p[[q]]$. Here $\delta = q^d dq$.

**Lemma 10** The monodromy filtration $\mathcal{W}_0 = W_1 \subset W_2 = W_3 \subset W_4 = W_5 \subset W_6 = E$, extends to a filtration $\mathcal{W}_i \subset \mathcal{E}$ by Fontaine-Laffaille submodules such that either $W_{2i}/W_{2i-2} \simeq \mathbb{Z}_p(-i)$, for all $0 \leq i \leq 3$, or $W_{2i}/W_{2i-2} \simeq \epsilon \mathbb{Z}_p(-i)$. Here $\epsilon \mathbb{Z}_p(-i)$ denotes the constant Fontaine-Laffaille module with $\mathcal{F}^i = O_D$, $\mathcal{F}^{i+1} = 0$, $\phi = -p^i Id$.

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\(\text{20}\) The pairing $\mathcal{W}_0 \otimes \mathcal{F}^i \to O_D$ is perfect. For the projection $\mathcal{W}_0 \hookrightarrow \mathcal{E} \to \mathcal{F}/\mathcal{F}^1$ is an isomorphism.
Proof: Our assumptions imply that $rk \ W_{2i}/W_{2i-2} = 1$, for $0 \leq i \leq 3$. Thus, by Lemma 6 and 10, the operator $\phi$ acts on $W_{2i}/W_{2i-2}$ as $\pm pId$.

We prove by induction on $i$ that $W_{2i} \subset E$ extends to a subbundle $W_{2i}$ of $\bar{E}$ preserved by the connection and that $\bar{E}^{i+1} \oplus W_{2i} = \bar{E}$. Indeed, for $i = -1$, there is nothing to prove. Assume that we know the result for $i = k$. Then $\bar{E}/W_{2k} = (\bar{E}^{k+1} + W_{2k})/W_{2k} \supset \cdots \supset (\bar{E}^i + W_{2k})/W_{2k} = (\bar{E}^i)/W_{2k}$ is a Fontaine-Laffaille module. Applying Lemma 7 to $\bar{E}/W_{2k} \oplus Z_p(k + 1)$ we see that $W_{2k+2}/W_{2k} \subset E/W_{2k}$ extends to a subbundle of $W_{2k+2}/W_{2k} \subset \bar{E}/W_{2k}$. It remains to show that $W_{2k+2}/W_{2k} \oplus (\bar{E}^{k+2}/W_{2k}) = (\bar{E}/W_{2k})$ is true over the closed point of $D$. Indeed, the operator $p^{-k-1} \phi$ induces an action on $(E/W_{2k}) \otimes F_p$ which is 0 on $((F^{k+2} + W_{2k})/W_{2k}) \otimes F_p$ and invertible on $(W_{2k+2} + W_{2k}) \otimes F_p$. The claim follows.

For the rest of the proof we assume that $\phi$ acts on $W_0$ as $+Id$. The other alternative is considered in a similar way.

By the definition of the canonical coordinate $q$ we can find sections $e^0 \in W_0$, $e^1 \in \bar{E}^1 \cap W_2$ such that

$$\nabla_\delta e^0 = 0, \ \phi e^0 = e^0, \ \nabla_\delta e^1 = e^0, \ \phi e^1 = pe^1,$$

and such that $e^0, e^1$ generate $W_2$. Next, it follows from Lemma 10 that there exist unique $e_1 \in \bar{E}^1 \cap W_4, e_0 \in \bar{E}^1$ such that

$$(e^0, e_0) = 1, \ (e^1, e_1) = -1$$

Observe that $e^i, e_i$ generate $\bar{E}$. Thanks to the self-duality condition we have

$$\nabla_\delta e_1 = Y(q)e^1, \ \nabla_\delta e_0 = e_1$$

$$\phi e_1 = p^2(e_1 + m_{23}(q)e^1 + m_{13}(q)e^0), \ \phi e_0 = p^3(e_0 - m_{13}(q)e^1 + m_{14}(q)e^0),$$

where $Y(q)$ is the Yukawa function. Finally, the relation $\nabla_\delta \phi = p^2 \phi \nabla_\delta$ amounts to

$$Y(q) - Y(q^p) = \delta(m_{23}), \ m_{23} = -\delta(m_{13}), \ \delta(m_{14}) = 2m_{13}.$$ 

Thus

$$Y(q^p) - Y(q) = \frac{1}{2} \delta^3 m_{14},$$

and we are done by Lemma 9.
4 Comparison

4.1. Plan of the proofs of Theorems 2 and 3.

Let \( \pi : \overline{X}_S \rightarrow \overline{C}_S \) be a semi-stable morphism satisfying the conditions i) - iii) from Section 1.3. Denote by \( q_C \in (\mathbb{C}(t))^* \otimes \mathbb{Q} \) the complex canonical coordinate (10) and by \( q_{Z_p} \in (\mathbb{Z}_p((t)))^* \otimes \mathbb{Q} \) the p-adic one (20).

Proposition 11 a) For every prime \( p \) such that \( (p,N) = 1 \), we have

\[ q_C = q_{Z_p} \in (\mathbb{Q}((t)))^* \otimes \mathbb{Q}. \]

b) Assume that the Betti monodromy of the family \( \overline{X}_S \rightarrow \overline{C}_S \) is small (see 1.1). Then, for every prime \( p \) with \( (p,N) = 1 \), the p-adic monodromy is also small (see 3.4). c) Let \( \omega \) be a nonvanishing section of the line bundle \( \mathcal{F}^n = \pi_* \Omega^n_{\overline{X}_S/\overline{C}_S}(\log Y_S) \) over an open neighborhood of the subscheme \( a : S \hookrightarrow \overline{C}_S \). Then

\[ \frac{1}{(2\pi i)^n} \int_{\delta_1} \omega^2 \in (\mathbb{Z}[N^{-1}][[t]])^*. \]

In the remaining part of this section we complete the proofs of Theorems 2 and 3 assuming Proposition 11. A proof of the proposition (which is the hardest technical part of the argument) is given in Sections 4.2-4.5.

Proof of Theorem 2. Since \( q'(0) \in \mathbb{Q}^* \) and \( d\log q(t) \in \mathbb{Q}[[t]] \frac{dt}{t} \) the coefficients of \( q(t) \) are rational numbers. On the other hand, parts a) and b) of Proposition 11 together with formula (22) show that, for every prime \( p \) such that \( (p,N) = 1 \),

\[ q(t) \in (\mathbb{Z}_p((t)))^* \cap (\mathbb{Q}((t)))^* \subset (\mathbb{Q}_p((t)))^*. \]

This completes the proof.

Proof of Theorem 3. Let \( \omega \times \omega \) be a local section of \( \pi_* \Omega^n_{\overline{X}_S/\overline{C}_S}(\log Y_S) \otimes \pi_* \Omega^n_{\overline{X}_C/\overline{C}_C}(\log Y_C) \) defined by the equation

\[ \frac{1}{(2\pi i)^n} \int_{\delta_1} \omega = 1. \]

Part c) of Proposition 11 shows that \( \omega \times \omega \) yields a nonvanishing section of \( \pi_* \Omega^n_{\overline{X}_S/\overline{C}_S}(\log Y_S) \otimes \pi_* \Omega^n_{\overline{X}_S/\overline{C}_S}(\log Y_S) \) over the formal neighborhood \( D_S \).

This together with Theorem 2 imply that the coefficients of the Yukawa function \( Y(q) \) are rational numbers and so are the instanton numbers \( n_d \).
It also follows that $Y(q)$ coincides (up to a constant factor in $\mathbb{Z}_p^*$) with the p-adic Yukawa function from 3.5. Thus by Proposition 3 the numbers $n_d$ are p-adic integers. This completes the proof.

4.2. Recollections on p-adic Comparison Theorem. Recall from [FL] that there is an exact tensor fully faithful functor

$$U : MF_{[0,p-2]} \to Rep(\Gamma)$$

from the category $MF_{[0,p-2]}$ of Fontaine-Laffaille modules over $\text{spec} \mathbb{Z}_p$ to the category $Rep(\Gamma)$ of finitely generated $\mathbb{Z}_p$-modules equipped with an action of the Galois group $\Gamma = \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$. We will use the following properties of $U$:

1) $U$ takes a finite (as a plain abelian group) Fontaine-Laffaille module to a $\Gamma$-module of the same finite order.

2) $U(\mathbb{Z}_p(i)) = \mathbb{Z}_p(i)$ and the induced morphism

$$\lim_{\leftarrow} \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^{p^i} \cong \text{Ext}^1_{MF_{[0,p-2]}}(\mathbb{Z}_p(-1), \mathbb{Z}_p(0)) \xrightarrow{U} \text{Ext}^1_{Rep(\Gamma)}(\mathbb{Z}_p(-1), \mathbb{Z}_p(0))$$

$$\cong \lim_{\leftarrow} \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^{p^i}_K \text{ummer}$$

is identity.

3) Let $\overline{\mathbb{X}}_{\mathbb{Z}_p} \to \overline{\mathbb{C}}_{\mathbb{Z}_p}$ be a proper semi-stable (relative to $\mathbb{Z}_p$) scheme. Assume that $\dim \overline{\mathbb{X}}_{\mathbb{Z}_p} \leq p - 2$. Then there is a canonical isomorphism:

$$U(\Psi_{FL}^t(H^k_{\log}(\overline{\mathbb{X}}_{\mathbb{Z}_p}/\overline{\mathbb{C}}_{\mathbb{Z}_p}))) \cong \Psi_{et}^t(R^k\pi_{Q_p}^*\mathbb{Z}_p)$$

(24)

Here $\pi_{Q_p}$ denotes the projection $X_{Q_p} \to C_{Q_p}$ and $\Psi_{et}^t : Sh^et(C_{Q_p}) \to Sh^et(a_{Q_p}) = Rep(\Gamma)$ is the etale vanishing cycles functor. Moreover, we have the following commutative diagram

$$U(\Psi_{FL}^t(H^k_{\log}(\overline{\mathbb{X}}_{\mathbb{Z}_p}/\overline{\mathbb{C}}_{\mathbb{Z}_p}))) \cong \Psi_{et}^t(R^k\pi_{Q_p}^*\mathbb{Z}_p)$$

$$\xrightarrow{\text{NDR}}$$

$$U(\Psi_{FL}^t(H^k_{\log}(\overline{\mathbb{X}}_{\mathbb{Z}_p}/\overline{\mathbb{C}}_{\mathbb{Z}_p})))_{a} \otimes \mathbb{Z}_p(-1) \cong \Psi_{et}^t(R^k\pi_{Q_p}^*\mathbb{Z}_p) \otimes \mathbb{Z}_p(-1)$$

This follows from the main Comparison Theorem in [Fa].

4.3. 1-motives, the motivic Albanese functor $L\text{Alb}$. The main references here are [D3] and [BK]. Let $k$ be a field of characteristic 0. Fix an algebraic closure $\overline{k} \supset k$. A 1-motive over $k$ is a triple

$$M = (\Lambda, G, \Lambda \xrightarrow{u} G(\overline{k}))$$
where $\Lambda$ is a free abelian group of finite rank equipped with an action of the Galois group $\text{Gal}(\overline{k}/k)$ that factors through a finite quotient, $G$ is an semi-abelian variety over $k$ i.e. an extension

$$0 \to T \to G \to A \to 0$$

(25)
of an abelian variety by a torus, and $u$ is a homomorphism of the Galois modules. We shall denote by $\mathcal{M}_1(k)$ the additive category of 1-motives.

Every 1-motive is equipped with a canonical (weight) filtration:

$$W^{-2}M = (0, T) \subset W^{-1}M = (0, G) \subset W^0M = M.$$

Thus $W^0M/W^{-1}M = (\Lambda, 0)$ and $W^{-1}M/W^{-2}M = (0, A)$. The category $\mathcal{M}_1(k; \mathbb{Q}) := \mathcal{M}_1(k) \otimes \mathbb{Q}$ is abelian ([BK], Proposition 1.1.5) and any morphism in $\mathcal{M}_1(k; \mathbb{Q})$ is strictly compatible with the weight filtration.

We also set

$$\text{T}_{\text{et}}^{\mathbb{Q}_p}(M) = \text{Hom}_{\mathbb{Q}_p}(\text{T}_{\text{et}}^{\mathbb{Z}_p}(M), \mathbb{Q}_p) \in \text{Rep}_{\mathbb{Q}_p}(\text{Gal}(\overline{k}/k)).$$

If $k = \mathbb{C}$ the category $\mathcal{M}_1(\mathbb{C})$ is equivalent to the category of torsion free polarizable mixed Hodge structures of type $\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$ ([D3], 10.1.3):

$$T^{\text{Hodge}} : \mathcal{M}_1(\mathbb{C}) \xrightarrow{\sim} \text{MHS}_1$$

(27)

For $k \subset \mathbb{C}$, $M \in \mathcal{M}_1(k)$, $T^{\text{Hodge}}(M \times_k \text{spec} \mathbb{C}) = (W. \subset V_\mathbb{Z}, F. \subset V_\mathbb{C})$ there is a functorial isomorphism of $\mathbb{Z}_p$-modules

$$V_\mathbb{Z} \otimes \mathbb{Z}_p \simeq T^{\text{et}}_{\mathbb{Z}_p}(M).$$

(28)

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21 The Galois module $\Lambda$ can be viewed as a discrete group scheme over $\text{spec} k$. Giving a homomorphism $\Lambda \longrightarrow G(\overline{k})$ of Galois modules is equivalent to giving a morphism $\Lambda \longrightarrow G$ of the étale sheaves represented by $\Lambda$ and $G$. This remark provides a construction of the category $\mathcal{M}_1(k)$ that is independent of the choice of an algebraic closure $\overline{k}$ ([BK]).
Abusing notation, we shall also denote by $T^{Hodge}$ the equivalence

$$\mathcal{M}_1(C; \mathbb{Q}) \xrightarrow{\sim} MHS^\mathbb{Q}_1 = MHS_1 \otimes \mathbb{Q}$$

induced by (27) and the corresponding equivalence of the derived categories

$$D^b(\mathcal{M}_1(C; \mathbb{Q})) \xrightarrow{\sim} D^b(MHS^\mathbb{Q}_1).$$

Let

$$D^b(\mathcal{M}_1(k; \mathbb{Q})) \hookrightarrow DM_{gm}(k; \mathbb{Q})$$

be embedding of the bounded derived category of 1-motives into the triangulated category of Voevodsky motives ([O]). By [BK] $S$ has a left adjoint functor:

$$LAlb : DM_{gm}(k; \mathbb{Q}) \rightarrow D^b(\mathcal{M}_1(k; \mathbb{Q})).$$

Denote by $MHS^\mathbb{Q}$ the category of mixed polarizable Hodge structures over $\mathbb{Q}$ and by $MHS^\mathbb{Q}_{eff}$ the full subcategory of $MHS^\mathbb{Q}$, whose objects are mixed Hodge structures $(W \subset V_\mathbb{Q}, F \subset V_\mathbb{C})$ with $F^1 = 0$. It is proven in [Vol] that embedding of the derived categories

$$S : D^b(MHS^\mathbb{Q}_1) \rightarrow D^b(MHS^\mathbb{Q}_{eff})$$

admits a $t$-exact left adjoint functor

$$LAlb : D^b(MHS^\mathbb{Q}_{eff}) \rightarrow D^b(MHS^\mathbb{Q}_1)$$

and that

$$T^{Hodge} \circ LAlb \simeq LAlb \circ R^{Hodge} : DM_{gm}(C; \mathbb{Q}) \rightarrow D^b(MHS^\mathbb{Q}_1).$$

Here

$$R^{Hodge} : DM_{gm}(C; \mathbb{Q}) \rightarrow D^b(MHS^\mathbb{Q}_{eff})$$

is the homological Hodge realization functor (i.e. $R^{Hodge}(M) = R_{Hodge}(M)^*$, where $R_{Hodge}$ is Huber’s cohomological realization ([Hu1], [Hu2]).)

4.4. Motivic vanishing cycles. Let $X_k \xrightarrow{\pi} C_k$ be a smooth proper scheme over a punctured curve $C_k \hookrightarrow \overline{C}_k \xrightarrow{a} spec \ k$ over a field $k \subset \mathbb{C}$. Fix a local coordinate $t$ at $a$ and an integer $m \geq 0$. Denote by

$$H^m(X_C/C_\mathbb{C}) = (R^m \pi_* \mathbb{Q}, F^m \subset H^m_{DR}(X_C/C_\mathbb{C})).$$

We say that a triangulated functor $T : D(A) \rightarrow D(B)$ is $t$-exact if $T(A)$ belongs to the essential image of $B$ in $D(B)$. 21
the variation of Hodge structure associated to the family $X_\mathbb{C} \xrightarrow{\pi_\mathbb{C}} \mathbb{C}$ and by $H_m(X_\mathbb{C}/\mathbb{C})$ the dual variation. Let $\Psi_t^{Hodge,un}(H_m(X_\mathbb{C}/\mathbb{C}))$ be the unipotent limiting mixed Hodge structure. The Hodge structure $\overline{LAlb}(\Psi_t^{Hodge,un}(H_m(X_\mathbb{C}/\mathbb{C})))$

can be viewed as a 1-motive over $\mathbb{C}$. In this subsection we explain how this 1-motive canonically descends to a 1-motive $M_{t,m}(X_k) = (\Lambda, G, \Lambda \xrightarrow{\eta} G(\overline{k})) \in \mathcal{M}_1(k; \mathbb{Q})$

over $k$. In addition, $M_{t,m}(X_k)$ comes equipped with a “monodromy” homomorphism $N: \Lambda \otimes \mathbb{Q} \to T_s \otimes \mathbb{Q}$ of $Gal(\overline{k}/k)$-modules. Here $T_s$ denotes the group of cocharacters of the torus $T$: $T_s = \text{Hom}(\mathbb{G}_m, T)(\overline{k})$. Equivalently, $N$ can be viewed as a morphism of 1-motives:

$$N : (W_0 M_{t,m}(X_k)/W_{-1} M_{t,m}(X_k))(1) \to W_{-2} M_{t,m}(X_k).$$

The main properties of $M_{t,m}(X_k)$ are the following.

1) There is a natural isomorphism:

$$\overline{LAlb}(\Psi_t^{Hodge,un}(H_m(X_\mathbb{C}/\mathbb{C}))) \simeq T^{Hodge}(M_{t,m}(X_k))$$

compatible with the monodromy action.

2) There is a natural morphism $Gal(\overline{k}/k)$-modules

$$\alpha : T^{et,un}_q(M_{t,m}(X_k)) \to \Psi_t^{et,un}(R^m \pi^*_s \mathbb{Q}_p)$$

where

$$\Psi_t^{et,un} : Sh_{et}(C_k) \to Sh_{et}(\text{spec } k) \to Rep_{\mathbb{Q}_p}(Gal(\overline{k}/k))$$

denotes the functor of unipotent vanishing cycles (see [B]). The morphism $\alpha$ commutes with the monodromy action.

3) If $k' \supset k$ is any field extension, there is a natural isomorphism

$$M_{t,m}(X_k \times_k \text{spec } k') \simeq M_{t,m}(X_k) \times_k \text{spec } k'$$

compatible in the obvious way with (30).

4) Assume that $k \subset \mathbb{C}$. Set

$$T^{Hodge}(M_{t,m}(X_k)) = (W. \subset V_{\mathbb{Q}}, F. \subset V_{\mathbb{C}}).$$

Abusing notation, we denote by $T^{Hodge}$ the composition of functors $\mathcal{M}_1(k; \mathbb{Q}) \to \mathcal{M}_1(\mathbb{C}; \mathbb{Q}) \xrightarrow{T^{Hodge}} MHS^1$. 

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The following diagram is commutative.

\[
\begin{array}{ccc}
V^* \otimes Q_p & \xrightarrow{\Psi_t^{an,un}} & \Psi_t^{et,un} (R^m_p \pi_* Q) \otimes Q_p \\
\downarrow & & \downarrow \\
T^*_{et} (M_{t,m}(X_k)) & \xrightarrow{\alpha} & \Psi_t^{et,un} (R^m_{et} p) \\
\end{array}
\]

The above properties of \(M_{t,m}(X_k)\) are sufficient for our applications. We shall indicate a conceptual construction of \(M_{t,m}(X_k)\) based on the theory of Voevodsky’s motives. Unfortunately, the construction relies on the following general fact that is not explained in the published literature.

Let

\[\Psi_{t}^{mot,un} : DM_{gm}(\eta; \mathbb{Q}) \to DM_{gm}(k; \mathbb{Q})\]

the functor of (unipotent) motivic vanishing cycles from the triangulated category of motives over the generic point \(\eta \in \mathbb{C}_k\) to the category of motives over \(a\), and let

\[N : \Psi_{t}^{mot,un} (1) \to \Psi_{t}^{mot,un}\]

be the monodromy operator (see [A1]). The fact, we will need, is that the formation \((\Psi_{t}^{mot,un}, N)\) commutes with the etale and Hodge realizations [Hu1], [Hu2]:

\[
\begin{array}{ccc}
DM_{gm}(\eta; \mathbb{Q}) & \xrightarrow{\Psi_{t}^{mot,un}} & DM_{gm}(k; \mathbb{Q}) \\
\downarrow & & \downarrow \\
D^b(Rep_Q_p(Gal(\overline{k(\eta)}/k))) & \xrightarrow{\Psi_{t}^{et,un}} & D^b(Rep_Q_p(Gal(\overline{k}/k))) \\
\downarrow & & \downarrow \\
DM_{gm}(\eta; \mathbb{Q}) & \xrightarrow{\Psi_{t}^{et,un}} & DM_{gm}(\mathbb{C}; \mathbb{Q}) \\
\downarrow R^{Hodge} & & \downarrow R^{Hodge} \\
D^b(VMHSS_{eff}^Q(\eta)) & \xrightarrow{\Psi_{t}^{Hodge,un}} & D^b(MHSS_{eff}^Q). \\
\end{array}
\]

Assuming this fact we construct \(M_{t,m}(X_k)\) as follows. It is proven in [BK] the fully faithful functor \(D^b(M_1(k; \mathbb{Q})) \to DM_{gm}^{eff}(k; \mathbb{Q})\) has a left adjoint:

\[L_{Alb} : DM_{gm}^{eff}(k; \mathbb{Q}) \to D^b(M_1(k; \mathbb{Q})).\]

Set

\[M_{t,m}(X_k) := H_m(L_{Alb} \Psi_{t}^{mot,un}(Q_{tr}[X_\eta])).\]
Let us just explain that $M_{t,m}(X_k)$ has the key property 1. Indeed, by Theorem 2 from [Vol] the Albanese functor commutes with the Hodge realization. Thus, we have

$$T^{\text{Hod}}(M_{t,m}(X_k)) \simeq H_m(L\text{Alb}_t^H \Psi^{\text{mot,un}}(\mathbb{Q}_\text{tr}[X_k]))$$

$$\simeq L\text{Alb}_t^H \Psi^{\text{Hod,un}} H_m(X_k/C).$$

**Example.** Here we explain an elementary construction of the motive $M_{t,1}(X_k)$ for a family $\pi : X_k \to C_k$ of curves with a semi-stable reduction. Choose a semi-stable model $\pi : \bar{X}_k \to \bar{C}_k$, such that all the irreducible components $Y_{k,\gamma}$ of the special fiber $Y_k := \bar{X}_k \times_{\bar{C}_k} \bar{C}$ are smooth. Let $\Gamma$ be the free abelian group whose generators $[\gamma]$ correspond to irreducible components of $Y_k$. For each singular point $y_\mu$ of $Y_k$ we denote by $R_\mu$ the subgroup of $\wedge^2 \Gamma$ generated by $[\gamma_1] \wedge [\gamma_2]$, where $Y_{k,\gamma_1}$ and $Y_{k,\gamma_2}$ are the two components meeting at $y_\mu$ (i.e. $R_\mu$ is isomorphic to $\mathbb{Z}$ but the isomorphism depends on the ordering of the components meeting at $y_\mu$). Define a homomorphism

$$u : R_\mu \to \text{Pic}(Y_k)$$

as follows. Consider the invertible sheaf $\mathcal{O}(y_{\mu,\gamma_1} - y_{\mu,\gamma_2})$ on the normalization $\tilde{Y}_k \to Y_k$, where $y_{\mu,\gamma_1}$, $y_{\mu,\gamma_2}$ are the preimages of $y_\mu$ in $\tilde{Y}_k$. We claim that $\mathcal{O}(y_{\mu,\gamma_1} - y_{\mu,\gamma_2})$ canonically descends to a line bundle $u([\gamma_1] \wedge [\gamma_2])$ over $Y_k$; the descend data are trivial outside of points $y_{\mu,\gamma_1}$, $y_{\mu,\gamma_2}$, and the identification

$$\mathcal{O}(y_{\mu,\gamma_1} - y_{\mu,\gamma_2})_{y_{\mu,\gamma_1}} \simeq \mathcal{O}(y_{\mu,\gamma_1} - y_{\mu,\gamma_2})_{y_{\mu,\gamma_2}}$$

is given by a canonical isomorphism

$$T_{Y_{\bar{k},\gamma_1}} \otimes T_{Y_{\bar{k},\gamma_2}} \simeq T_{\text{Pic}(Y_k)^{\mu}} \simeq \bar{k}^{25}.$$
Thus, it suffices to prove that

\[ q_C(0) = q_{\mathbb{Z}_p}(0) \in Q^* \otimes \mathbb{Q}. \]

Consider the 1-motive \( M_{t,n}(X_{\mathbb{Q}}) = (\Lambda, G, \Lambda \rightarrow G_1) \). By (29) we have

\[ L^{Hodg} \subset T^{*Hodg}(M_{t,n}(X_{\mathbb{Q}})) =: (W \subset V_{\mathbb{Q}}, F^* \subset V_{\mathbb{C}}) \subset \Psi_t^{Hodg}(H^n(X_{\mathbb{C}}/\mathbb{C})). \]

Hence, \( W_0 L^{Hodg} \otimes \mathbb{Q} \simeq \Lambda^* \otimes \mathbb{Q} \) and \( W_{-1} M_{t,n}(X_{\mathbb{Q}}) = W_{-2} M_{t,n}(X_{\mathbb{Q}}) \). We claim that the image of the embedding \( W_2 L^{Hodg} / W_0 L^{Hodg} \otimes \mathbb{Q} \hookrightarrow (T \otimes \mathbb{Q})^* \) is \( Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \)-invariant. Indeed, this is clear from the commutative diagram

\[
\begin{array}{ccc}
L^{Hodg} \otimes \mathbb{Q}_p & \rightarrow & V_{\mathbb{Q}} \otimes \mathbb{Q}_p \\
\downarrow \simeq & & \downarrow \simeq \\
L^{\text{et}} \otimes \mathbb{Z}_p \mathbb{Q}_p := Im N^{n-1} & \rightarrow & T^{*\text{et}}(M_{t,n}(X_{\mathbb{Q}}))
\end{array}
\]

since all the arrows in the bottom row are morphisms of \( Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \)-modules.

It follows that there exists a unique quotient \( L^{\text{mot}} \in M_1(\mathbb{Q}; \mathbb{Q}) \), \( \gamma : M_{t,n}(X_{\mathbb{Q}}) \rightarrow L^{\text{mot}} \) which fits into the following diagram

\[
\begin{array}{ccc}
T^{*Hodg}(L^{\text{mot}}) & \rightarrow & T^{*Hodg}(M_{t,n}(X_{\mathbb{Q}})) \\
\downarrow \simeq & & \downarrow \text{id} \\
L^{Hodg} & \rightarrow & T^{*Hodg}(M_{t,n}(X_{\mathbb{Q}}))
\end{array}
\]

Observe that the operator \( N \) descends to \( L^{\text{mot}} \) and

\[ N : (W_0 L^{\text{mot}} / W_{-2} L^{\text{mot}}) \otimes \mathbb{Q}(1) \simeq W_{-2} L^{\text{mot}} \otimes \mathbb{Q}. \]  

Finally, we have from (30) a canonical isomorphism \( T^{*\text{et}}(L^{\text{mot}}) \simeq L^{\text{et}} \otimes \mathbb{Z}_p \mathbb{Q}_p \) of \( Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \)-modules.

Let \( [L^{\text{mot}}, N^{-1}] \in Ext^1_{M_1(\mathbb{Q}; \mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \) be the class of the extension

\[ 0 \rightarrow W_{-2} L^{\text{mot}} \rightarrow L^{\text{mot}} \rightarrow W_0 L^{\text{mot}} / W_{-2} L^{\text{mot}} \rightarrow 0 \]  

composed with \( N^{-1} \) from (33) and let \( \kappa \) be the corresponding (by (26)) element in \( Q^* \otimes \mathbb{Q} \). The functor \( T^{*Hodg} \) takes this extension to the class \( [L^{Hodg}, N_B^{-1}] \in Ext^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(1)) \simeq \mathbb{C}^* \otimes \mathbb{Q} \). The latter class is equal to \( q_C(0) \). It follows that \( q_C(0) = \kappa^{-1} \).

If we pull back the extension (34) on spec \( \mathbb{Q}_p \) and then apply the etale realization functor \( T^{*\text{et}}_{\mathbb{Q}_p} : M_1(\mathbb{Q}_p; \mathbb{Q}) \rightarrow \text{Rep}_{\mathbb{Q}_p}(\Gamma) \) we get the extension \( L^{\text{et}} \otimes \)

\[ \text{Here } T^{*Hodg}(M_{t,n}(X_{\mathbb{Q}})) \text{ denotes the Hodge structure dual to } T^{Hodg}(M_{t,n}(X_{\mathbb{Q}})). \]

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\( \mathbb{Q}_p, N_{et}^{-1} \) equivalent (by 4.2, 3)) to the one obtained from \([LFL \otimes \mathbb{Q}_p, N_{DR}^{-1}]\) by applying the Fontaine-Laffaille functor \( U \). Hence \( d_{Zp}(0) = \kappa^{-1} \), and we are done.

**Remark:** The above argument shows that for any family \( X_\mathbb{Q} \to C_\mathbb{Q} \) over \( \mathbb{Q} \) with a maximal degeneracy point at \( a \in C_\mathbb{Q}(\mathbb{Q}) \)

\[ q_c \in (\mathbb{Q}((t)))^* \otimes \mathbb{Q}. \]

b) Assume that the Betti monodromy is small i.e.

\[ N_B : W_2L_Z^{Hodge}/W_0L_Z^{Hodge} \simeq W_0L_Z^{Hodge}(-1) \] (35)

We have to show that, for any prime \( p \) in \( S \),

\[ N_{DR} : W_2L^{FL}/W_0L^{FL} \to W_0L^{FL}(-1) \] (36)

is an isomorphism as well. Indeed, by (by 4.2, 3)) the functor \( U \) takes the morphism (36) to

\[ N_B \otimes Id : (W_2L_Z^{Hodge}/W_0L_Z^{Hodge}) \otimes \mathbb{Z}_p \simeq W_0L_Z^{Hodge} \otimes \mathbb{Z}_p(-1). \]

The claim follows.

c) Let \( E \) be the quotient of \( H^\ast_{log}(X_S/S) \) modulo torsion, and let \( \mathcal{F}^\ast \subset \mathcal{E} \), \( \mathcal{W}_0 \subset \mathcal{E} \) be the Hodge and monodromy filtrations (3.3). As we explained in loc. cit. the Poincare duality identifies the line bundle \( \mathcal{F}^\ast \) with the dual to \( \mathcal{W}_0 \). It is also shown there that \( \mathcal{W}_0 \) is generated by a parallel section \( e^0 \in \mathcal{W}_0^\ast \). It suffices to prove the claim for a single nonvanishing section \( \omega \in \mathcal{F}^\ast \). Let us choose \( \omega \) such that \( (e^0, \omega) = 0 \). Then the integral

\[ \frac{1}{(2\pi i)^n} \int_{\delta_1} \omega \]

is a constant function on \( D_C \). We have to show that the square of this constant is in \( \mathbb{Z}[N^{-1}]^* \). The following lemma does the job.

**Lemma 12** Let \( E \) (resp. \( H \)) be the torsion free part of \( a^\ast(H^n_{log}(X_S/S)) \) (resp. \( \Psi^\ast_t(R^n\pi^an_\mathbb{Z}[N^{-1}]) \)), and let

\[ E \leftrightarrow E \otimes \mathbb{C} \simeq H \otimes \mathbb{C} \leftrightarrow H \]

be the isomorphism from (7). Then the two \( \mathbb{Z}[N^{-1}] \)-lattices

\[ (W_0E)^\otimes \leftrightarrow (W_0E)^\otimes \otimes \mathbb{C} \simeq (W_0H)^\otimes \otimes \mathbb{C} \leftrightarrow (W_0H)^\otimes \]

coincide.
Proof: Indeed, consider the monodromy paring

$$\Xi : \langle \cdot, \cdot \rangle_{\text{mon}} : (W_0E)^{\otimes 2} \otimes \mathbb{C} \to \mathbb{C},$$

$$\langle x, y \rangle_{\text{mon}} = \langle x, N_{DR}^{-n} y \rangle_{DR} = \pm \langle x, N_B^{-n} y \rangle_B,$$

where $$N_B^{-n} = (-2\pi i)^{-n} N_{DR}^{-n} : W_0E \otimes \mathbb{C} \to W_{2n}E \otimes \mathbb{C}.$$ The monodromy paring takes $$(W_0E)^{\otimes 2} \otimes \mathbb{Q}$$ and $$(W_0H)^{\otimes 2} \otimes \mathbb{Q}$$ into $$\mathbb{Q} \subset \mathbb{C}.$$ Therefore, since $$\text{rk} W_0E = 1,$$ $$(W_0E)^{\otimes 2} \otimes \mathbb{Q} = (W_0H)^{\otimes 2} \otimes \mathbb{Q}.$$ Moreover, to prove that $$(W_0E)^{\otimes 2} = (W_0H)^{\otimes 2},$$ it is enough to show that $$\Xi((W_0E)^{\otimes 2}) = \Xi((W_0H)^{\otimes 2}).$$ Since the pairings

$$\langle \cdot, \cdot \rangle_{DR} : W_0E \otimes W_{2n}E \to \mathbb{Z}[N^{-1}], \langle \cdot, \cdot \rangle_B : W_0H \otimes W_{2n}H \to \mathbb{Z}[N^{-1}]$$

are perfect, the claim would follow if we prove that, for any prime $$p$$ in $$S,$$ the cokernels of the maps $$N_{DR}^n : W_0E \otimes \mathbb{Z}_p \to W_{2n}E \otimes \mathbb{Z}_p,$$ $$N_{et}^n : W_0H \otimes \mathbb{Z}_p \to W_{2n}H \otimes \mathbb{Z}_p$$ have the same order. This follows from parts 1) and 3) in 4.2.

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