GENERALIZED LINEAR MODELS FOR POPULATION DYNAMICS IN TWO JUXTAPOSED HABITATS

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(Communicated by Irena Lasiecka)

ABSTRACT. In this work we introduce a generalized linear model regulating the spread of population displayed in a $d$-dimensional spatial region $\Omega$ of $\mathbb{R}^d$ constituted by two juxtaposed habitats having a common interface $\Gamma$. This model is described by an operator $L$ of fourth order combining the Laplace and Biharmonic operators under some natural boundary and transmission conditions. We then invert explicitly this operator in $L^p$-spaces using the $H^\infty$-calculus and the Dore-Venni sums theory. This main result will lead us in a later work to study the nature of the semigroup generated by $L$ which is important for the study of the complete nonlinear generalized diffusion equation associated to it.

1. Introduction. The partial differential equations play a natural role in population dynamics, in particular in the reaction-diffusion models which are derived from the well known Fick’s law.

An important problem in population ecology is the effect of environmental changes on the growth and diffusion of the species in areas made up of various habitats. In this situation and in order to understand how populations interact between the habitats, it is necessary to have spatially explicit models incorporating individual behaviour at different boundaries and interfaces of the habitats.

If $u(t,.)$ denotes the population density, the classical Fickian equations in each habitat for these models are typically of the form

$$\frac{\partial u}{\partial t} = l\Delta u + F(u),$$

where $F$ is the nonlinear growth interaction and $l$ is the positive coefficient diffusion (which can be variable).

2010 Mathematics Subject Classification. 35B65, 35C15, 35J40, 35R20, 47A60, 47D06, 92D25.

Key words and phrases. Population dynamics, diffusion equation, semigroups, Landau-Ginzburg free energy functional.

The last author is supported by CIFRE contract 2014/1307 with Qualiom Eco company.

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Now, define the following homogeneous dispersal linear operator equation for growth and dispersal in a population

\[ \frac{\partial u}{\partial t} = -k \Delta^2 u + l \Delta u + F(u), \]

where \( k \) is generally positive and \( l \) is a number which can be negative, see [4], p. 238. In this paper, we only consider the case when \( k \) and \( l \) are positive, but our techniques can be extended to \( k, l \in \mathbb{R} \setminus \{0\} \), satisfying some conditions, this will be done in a forthcoming paper.

Now consider the \( d \)-area \( \Omega = \Omega_- \cup \Omega_+ \) constituted by the two juxtaposed habitats

\[
\begin{aligned}
\Omega_- &:= (a, \gamma) \times \omega \\
\Omega_+ &:= (\gamma, b) \times \omega,
\end{aligned}
\]

with their interface

\[ \Gamma = \{ \gamma \} \times \omega, \]

where \( a, \gamma, b \in \mathbb{R} \) with \( a < \gamma < b \) and \( \omega \) being an open bounded regular set of \( \mathbb{R}^{d-1} \).

Consider the following linear stationary dispersal equations

\[
\begin{align*}
(\text{EQ}_{pde}) &\left\{ \begin{array}{ll}
-k_- \Delta^2 u_- + l_- \Delta u_- = f_- & \text{in } \Omega_- \\
-k_+ \Delta^2 u_+ + l_+ \Delta u_+ = f_+ & \text{in } \Omega_+, \\
\end{array} \right.
\end{align*}
\]

where

\[ u = \begin{cases} 
  u_- & \text{in } \Omega_- \\
  u_+ & \text{in } \Omega_+ 
\end{cases} \quad \text{and} \quad f = \begin{cases} 
  f_- & \text{in } \Omega_- \\
  f_+ & \text{in } \Omega_+, 
\end{cases} \]

with \( f \) given in \( L^p(a, b; L^p(\omega)) = L^p(\Omega) \), and \( k_\pm, l_\pm \) are positive numbers. The spatial variables will be denoted by \((x, y), x \in (a, b) \) and \( y \in \omega \). The above equations will be considered under the following boundary and transmission conditions

\[
\begin{aligned}
(\text{BC}_{pde}) &\left\{ \begin{array}{ll}
u_- (x, \zeta) = 0, & x \in (a, \gamma), \ \zeta \in \partial \omega \\
u_+ (x, \zeta) = 0, & x \in (\gamma, b), \ \zeta \in \partial \omega \\
\Delta u_- (x, \zeta) = 0, & x \in (a, \gamma), \ \zeta \in \partial \omega \\
\Delta u_+ (x, \zeta) = 0, & x \in (\gamma, b), \ \zeta \in \partial \omega \\
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
(\text{TC}_{pde}) &\left\{ \begin{array}{ll}
u_- (a, y) = \varphi_1^-(y), & u_+ (b, y) = \varphi_1^+(y), \ y \in \omega \\
\frac{\partial u_-}{\partial x} (a, y) = \varphi_2^-(y), & \frac{\partial u_+}{\partial x} (b, y) = \varphi_2^+(y), \ y \in \omega,
\end{array} \right.
\end{aligned}
\]

(\( \varphi_1^\pm \) and \( \varphi_2^\pm \) will be given in appropriated spaces) and

\[
\begin{aligned}
\left\{ \begin{array}{ll}
u_- = u_+ & \text{on } \Gamma \\
\frac{\partial u_-}{\partial x} = \frac{\partial u_+}{\partial x} & \text{on } \Gamma \\
k_- \Delta u_- = k_+ \Delta u_+ & \text{on } \Gamma \\
\frac{\partial}{\partial x} (k_- \Delta u_- - l_- u_-) = \frac{\partial}{\partial x} (k_+ \Delta u_+ - l_+ u_+) & \text{on } \Gamma.
\end{array} \right.
\end{aligned}
\]

Now, define the following homogeneous dispersal linear operator

\[
\begin{align*}
D(\mathcal{L}) &= \{ u \in L^p(\Omega) : \Delta u_\pm, \Delta^2 u_\pm \in L^p(\Omega_\pm) \text{ and } u_\pm \text{ satisfies } (\text{BC}_0) \text{ and } (\text{TC}_{pde}) \} \\
\mathcal{L}u &= \begin{cases} 
-k_- \Delta^2 u_- + l_- \Delta u_- & \text{in } \Omega_- \\
k_+ \Delta^2 u_+ + l_+ \Delta u_+ & \text{in } \Omega_+,
\end{cases}
\end{align*}
\]

The variety and the complexity of the habitats and the individuals are not well modeled by spatial effects to be simply Fickian diffusion (as, for example, models of cell motion). An approach based on a the Landau-Ginzburg free energy functional and on the variational derivative consider the more generalized following diffusion equation for growth and dispersal in a population
where \((BC_0)\) corresponds to \((BC_{pde})\) with \(\varphi_1^+ = \varphi_1^- = \varphi_2^+ = \varphi_2^- = 0\).

Therefore, in this work, we will focus ourselves on proving essentially the invertibility of \(L\); this study will be very useful to analyze the following spectral equation
\[
Lu - \lambda u = f, \quad \lambda \in \mathbb{C},
\]
in order to characterize the nature of the semigroup generated by \(L\). On the other hand the same techniques used here will apply for this analysis. We know the importance of this property in the study of the generalized diffusion complete equation quoted above.

Let us comment on the boundary and transmission conditions.

The two first boundary conditions of (1) in \((BC_{pde})\) simply mean that the individuals die when they reach on the other parts of the boundaries \((a, b) \times \partial \omega\) (which means that we have an inhospitable border); the next two others of (1) mean that there is no dispersal in the normal direction. We deduce that the dispersal vanishes on \((a, b) \times \partial \omega\), that is \(\Delta u = 0\) on \((a, \gamma) \times \partial \omega\) and \(\Delta u = 0\) on \((\gamma, b) \times \partial \omega\).

In (2) of \((BC_{pde})\), the population density and the flux are given, for instance on \(\{a\} \times \omega\) and on \(\{b\} \times \omega\). This signifies that the habitats are not segregated.

In \((TC_{pde})\), the two first transmission conditions mean the continuity of the density and its flux at the interface, while the two second express the continuity of the dispersal and its flux (in some sense) at \(\Gamma\).

We can consider more realistic transmission conditions with the noncontinuity of the density and the flux but including the continuity of the generalized dispersal:
\[
-k_- \Delta^2 u_- + l_- \Delta u_- = -k_+ \Delta^2 u_+ + l_+ \Delta u_+ \quad \text{on } \Gamma.
\]
This situation requires to work in spaces built on the continuous functions. We will consider this case in a future work. Note that, when we consider different types of habitats, the response of individuals at the interface is important for the overall movement behaviour.

In many works, a generalized diffusion model is considered. Let us quote a number of them.

In [4] and in [18], the authors have presented in one dimensional case a nonlinear model with spatial structure characterized by a fourth order operator in only one habitat. They used essentially a Landau-Ginzburg free energy functional.

We were essentially inspired by these works to deduce a linear \(d\)-dimensional model set in two bounded juxtaposed cylindrical habitats which requires necessarily boundary and transmission conditions. We will then base ourselves on similar techniques to those used in the works of [7] and [14].

The paper is organized as follows. First, in section 2, we present the PDE transmission problem \((P_{pde})\) and with the help of operator \(A_0\) defined below, we give its operational writing. We will then study problem \((P)\) with a general operator \(A\) instead of \(A_0\).

Then, in section 3, we recall what is a BIP operator, we precise our notations about interpolation spaces, we set our hypotheses and their consequences. We explain how to solve our problem \((P)\) by introducing two auxiliary problems \((P_-)\) and \((P_+)\). We then present our main result in Theorem 3.3. As a consequence of this theorem, we obtain the Corollary 1 which states existence and uniqueness of the solution of problem \((EQ_{pde}) - (BC_{pde}) - (TC_{pde})\) quoted above.

In section 4, we give technical results which help us to prove our main result. In Proposition 1 and in Proposition 2 we solve problems \((P_-)\) and \((P_+)\) provided that the data are in some real interpolation spaces. We establish (see Theorem 4.2) a
useful technical result which allows us to prove Theorem 3.3. Then, we show some technical lemmas which lead us to apply functional calculus.

Section 5 is devoted to the proof of Theorem 3.3. This section is composed of three parts: in the first part, we use Theorem 4.2 to explicit the determinant of the transmission system. In the second part, we inverse the determinant of the transmission system using functional calculus. Finally, in the last part, we show that the general transmission problem has a unique classical solution by establishing the regularity of this solution.

2. Operational formulation. Consider now the problem

\[
\begin{align*}
(P_{pde}) \quad \left\{ \begin{array}{ll}
    k_- \Delta^2 u_+ - l_- \Delta u_- = f_- & \text{in } \Omega_- \\
    k_+ \Delta^2 u_+ - l_+ \Delta u_+ = f_+ & \text{in } \Omega_+ \\
    u_-(x, \xi) = \Delta u_-(x, \xi) = 0, & x \in (a, \gamma), \ \xi \in \partial \omega, \\
    u_+(x, \xi) = \Delta u_+(x, \xi) = 0, & x \in (\gamma, b), \ \xi \in \partial \omega \\
    u_-(a, y) = \varphi_1^-(y), & y \in \omega, \ u_+(b, y) = \varphi_1^+(y), \ y \in \omega \\
    \frac{\partial u_+}{\partial x}(a, y) = \varphi_2^+(y), & y \in \omega, \ \frac{\partial u_+}{\partial x}(b, y) = \varphi_2^-(y), \ y \in \omega \\
    u_- = u_+ & \text{on } \Gamma \\
    \frac{\partial u_-}{\partial x} = \frac{\partial u_+}{\partial x} & \text{on } \Gamma \\
    k_- \Delta u_- = k_+ \Delta u_+ & \text{on } \Gamma \\
    \frac{\partial}{\partial x} (k_- \Delta u_- - l_- u_-) = \frac{\partial}{\partial x} (k_+ \Delta u_+ - l_+ u_+) & \text{on } \Gamma.
    \end{array} \right.
\end{align*}
\]

Let us define $A_0$, the Laplace operator in $\mathbb{R}^{d-1}$, $d \in \mathbb{N} \setminus \{0, 1\}$, as follows

\[
\begin{align*}
D(A_0) := \{ \psi \in W^{2,p}(\omega) : \psi = 0 \text{ on } \partial \omega \} \\
\forall \psi \in D(A_0), \ A_0 \psi = \Delta_y \psi.
\end{align*}
\]

Thus, using operator $A_0$, problem $(P_{pde})$ becomes

\[
\begin{align*}
    u_-^{(4)}(x) + (2 A_0 - \frac{l_-}{k_-} I) u_+''(x) + (A_0^2 - \frac{l_-}{k_-} A_0) u_-(x) &= f_-(x), & x \in (a, \gamma) \\
    u_+^{(4)}(x) + (2 A_0 - \frac{l_+}{k_+} I) u_-''(x) + (A_0^2 - \frac{l_+}{k_+} A_0) u_+(x) &= f_+(x), & x \in (\gamma, b) \\
    u_-(a) &= \varphi_1^-, \quad u_+(b) = \varphi_1^+ \\
    u_-'(a) &= \varphi_2^-, \quad u_+'(b) = \varphi_2^+ \\
    u_-''(\gamma) &= u_+''(\gamma) \\
    u_-'''(\gamma) &= u_+'''(\gamma) \\
    k_+ u_-'''(\gamma) + k_+ A_0 u_+'(\gamma) &= k_- u_+'''(\gamma) + k_- A_0 u_-'(\gamma) \\
    k_+ u_+'''(\gamma) + k_+ A_0 u_-'(\gamma) - l_+ u_+'(\gamma) &= k_- u_-'''(\gamma) + k_- A_0 u_-''(\gamma) - l_- u_-''(\gamma),
\end{align*}
\]

where $f_- \in L^p(a, \gamma; L^p(\omega))$, $f_+ \in L^p(\gamma, b; L^p(\omega))$, $p \in (1, +\infty)$, with $u(x) := u(x, \cdot)$ and $f(x) := f(x, \cdot)$.

Then, we will consider a generalization of this problem with $(-A, D(-A))$, instead of $(-A_0, D(-A_0))$, a BIP operator of angle $\theta \in (0, \pi)$ on a UMD space $X$, see section 3 below for the definitions of BIP operator and UMD spaces, and $f \in L^p(a, b; X)$.
More precisely, we study the following transmission problem (P):

\[
\begin{cases}
\begin{align*}
(EQ) & \quad u^{(4)}_-(x) + (2A - \frac{l}{k_-} I)u''_-(x) + (A^2 - \frac{l}{k_-} A)u_-(x) = f_-(x), \quad x \in (a, \gamma) \\
& \quad u^{(4)}_+(x) + (2A - \frac{l}{k_+} I)u''_+(x) + (A^2 - \frac{l}{k_+} A)u_+(x) = f_+(x), \quad x \in (\gamma, b)
\end{align*}
\end{cases}
\]

\[
\begin{cases}
\begin{align*}
(BC) & \quad u_-(a) = \varphi_-, \quad u_+(b) = \varphi_+ \\
& \quad u'_-(a) = \varphi'_-, \quad u'_+(b) = \varphi'_+
\end{align*}
\end{cases}
\]

\[
\begin{cases}
\begin{align*}
(TC) & \quad u_-(\gamma) = u_+(\gamma) \\
& \quad u'_-(\gamma) = u'_+(\gamma) \\
& \quad k_+u^{(3)}_+(\gamma) + k_+Au'_+(\gamma) - l_+u'_+(\gamma) = k_-u^{(3)}_-(\gamma) + k_-Au'_-(\gamma) - l_-u'_-(\gamma) \\
& \quad k_+u''_+(\gamma) + k_+Au_+(\gamma) = k_-u''_-(\gamma) + k_-Au_-(\gamma).
\end{align*}
\end{cases}
\]

The transmission conditions (TC) will be divided into

\[
\begin{cases}
\begin{align*}
(TC1) & \quad u_-(\gamma) = u_+(\gamma) \\
& \quad u'_-(\gamma) = u'_+(\gamma),
\end{align*}
\end{cases}
\]

and

\[
\begin{cases}
\begin{align*}
(TC2) & \quad k_+u^{(3)}_+(\gamma) + k_+Au'_+(\gamma) - l_+u'_+(\gamma) = k_-u^{(3)}_-(\gamma) + k_-Au'_-(\gamma) - l_-u'_-(\gamma) \\
& \quad k_+u''_+(\gamma) + k_+Au_+(\gamma) = k_-u''_-(\gamma) + k_-Au_-(\gamma).
\end{align*}
\end{cases}
\]

Note that (TC2) is well defined in virtue of Lemma 3.2, see Section 3.2 below.

We will search a classical solution of problem (P), that is a solution \( u \) such that

\[
\begin{cases}
\begin{align*}
& u_- := u_{(\alpha, \gamma)} \in W^{4,p}(a, \gamma; X) \cap L^p(a, \gamma; D(A^2)), \quad u'_- \in L^p(a, \gamma; D(A)), \\
& u_+ := u_{(\gamma, b)} \in W^{4,p}(\gamma, b; X) \cap L^p(\gamma, b; D(A^2)), \quad u'_+ \in L^p(\gamma, b; D(A)),
\end{align*}
\end{cases}
\]

and which satisfies (EQ) - (BC) - (TC).

3. Assumptions, consequences and statement of results.

3.1. The class \( BIP(X, \theta) \). In all the paper, \( (X, \| \cdot \|) \) is a complex Banach space. Recall, see [11], p.19, that a closed linear operator \( T_1 \) is called sectorial of angle \( \alpha \in (0, \pi) \) if

\[
\sigma(T_1) \subset \overline{S_\alpha},
\]

\[
i \forall \alpha' \in (\alpha, \pi), \quad \sup \{ \| \lambda (\lambda I - T_1)^{-1} \|_{\mathcal{L}(X)} : \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}} \} < \infty,
\]

where

\[ S_\alpha := \{ z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \alpha \}. \]

It is known that any injective sectorial operator \( T_1 \) admits imaginary powers \( T_1^{is} \), \( s \in \mathbb{R} \), but, in general, \( T_1^{is} \) is not bounded, see [12], p. 342. Let \( \theta \in [0, \pi) \). We denote by \( BIP(X, \theta) \), see [19], p. 430, the class of sectorial injective operators \( T_1 \) such that

\[
\begin{align*}
& i) \quad D(T_1) = \overline{R(T_1)} = X, \\
& ii) \quad \forall \ s \in \mathbb{R}, \quad T_1^{is} \in \mathcal{L}(X), \\
& iii) \quad \exists \ C \geq 1, \forall \ s \in \mathbb{R}, \quad \| T_1^{is} \|_{\mathcal{L}(X)} \leq Ce^{s\theta}.
\end{align*}
\]

3.2. Generalized linear models for population dynamics.
In this case, \( D(T_1) \cap R(T_1) = X \), see [11], proof of Proposition 3.2.1, c), p. 71.

We will use the well-known Dore-Venni theorem, see [6] and its generalization in [19], which needs to consider a UMD space \( X \). Recall that a Banach space \( X \) is a UMD space if and only if for some \( p \in (1, +\infty) \) and thus for all \( \rho \), the Hilbert transform is bounded from \( L^p(\mathbb{R}, X) \) into itself.

### 3.2. Interpolation spaces

Here we recall some properties of real interpolation spaces in particular cases.

Let \( T_2 : D(T_2) \subset X \to X \) be a linear operator such that

\[
(0, +\infty) \subset \rho(T_2) \quad \text{and} \quad \exists C > 0 : \forall t > 0, \quad \|t(T_2 - tI)^{-1}\|_{L_1(X)} \leq C.
\]  

(4)

Then, for \( \theta \in (0, 1) \) and \( q \in [1, +\infty] \), we can define the real interpolation space

\[
(D(T_2), X)_{\theta,q} := \{ \psi \in X : t \mapsto t^{1-\theta}\|T_2(T_2 - tI)^{-1}\|_{L_1(X)} \in L^q(0, +\infty) \},
\]

see [10], p. 665, Teorema 3.

In [22], p. 78, this space is denoted by \( (X, D(T_2))_{1-\theta,q} \). We set, for any \( k \in \mathbb{N} \setminus \{0\} \)

\[
(D(T_2), X)_{k+\theta,q} := \{ \psi \in D(T_k^2) : T_{k\theta}^2 \psi \in (D(T_2), X)_{\theta,q} \},
\]

\[ (X, D(T_2))_{k+\theta,q} := \{ \psi \in D(T_k^2) : T_{k\theta}^2 \psi \in (X, D(T_2))_{\theta,q} \}. \]

The general situation of the real interpolation space \( (X_0, X_1)_{\theta,q} \) with \( X_0, X_1 \) two Banach spaces such that \( X_0 \hookrightarrow X_1 \), is described in [15].

Note that for an operator \( T_2 \) satisfying (4), \( T_k^2 \) is closed for any \( k \in \mathbb{N} \setminus \{0\} \) since \( \rho(T_2) \neq \emptyset \); consequently we can consider \( (D(T_k^2), X)_{\theta,q} = (X, D(T_k^2))_{1-\theta,q} \).

We have the following lemmas.

**Lemma 3.1.** Let \( \theta, \theta' \in (0, 1) \), \( k, n, m \in \mathbb{N} \setminus \{0\} \), \( p \in [1, +\infty] \) and \( T_2 \) be a linear operator satisfying (4).

i) If \( k\theta \notin \mathbb{N} \), then \( (X, D(T_k^2))_{\theta,p} = (X, D(T_2^m))_{\theta,p} \).

ii) If \( n \leq k \leq m \), then

\[
(D(T_k^2), D(T_2^m))_{\theta,p} = (D(T_k^m), X)_{\theta,p},
\]

where \( \tau \) satisfies \( k(1-\theta) + n\theta = m(1-\tau) \).

iii) If \( k\theta < 1 \), then \( (D(T_k^2), X)_{\theta,p} \subset D(T_2^{k-1}) \).

For statement i), see [16], (2.1.13), p. 43. For ii) see [10], p. 676, Teorema 6. For iii) we apply ii) with \( n = k - 1 \), \( m = k \) and \( \theta = k\theta' \in (0, 1) \), then

\[
(D(T_k^2), D(T_2^{k-1}))_{k\theta',p} = (D(T_k^2), X)_{\theta',p},
\]

which gives \( (D(T_k^2), X)_{\theta',p} \subset D(T_2^{k-1}) \). This inclusion can also be found by writing

\[
(D(T_k^2), X)_{\theta',p} = (X, D(T_k^2))_{1-\theta',p} = (X, D(T_2))_{k-1-\theta',p} = (D(T_2), X)_{(k-1)+k\theta',p} \subset D(T_2^{k-1}).
\]

**Lemma 3.2.** Let \( T_2 \) be a linear operator satisfying (4). Let \( u \) such that

\[
u \in W^{n,p}(a_1, b_1 ; X) \cap L^p(a_1, b_1 ; (D(T_2^k))),
\]

where \( a_1, b_1 \in \mathbb{R} \) with \( a_1 < b_1 \), \( n, k \in \mathbb{N} \setminus \{0\} \) and \( p \in (1, +\infty) \). Then for any \( j \in \mathbb{N} \) satisfying the Poulsen condition \( 0 < \frac{1}{p} + j < n \) and \( s \in \{a_1, b_1\} \), we have

\[
\nu^{(j)}(s) \in (D(T_2^k), X)_{\frac{1}{p} + j, p'}.
\]

This result is proved in [10], p. 678, Teorema 2'.

3.3. **Hypotheses.** In all the sequel, \( k_+, k_-, l_+, l_- \in \mathbb{R}_+ \setminus \{0\} \), \( A \) denotes a closed linear operator in \( X \) and we set
\[
    r_+ = \frac{l_+}{k_+} \quad \text{and} \quad r_- = \frac{l_-}{k_-}.
\]
We assume the following hypotheses:

\( (H_1) \quad X \) is a UMD space,

\( (H_2) \quad 0 \in \rho(A), \)

\( (H_3) \quad -A \in \text{BIP}(X, \theta_A) \) for some \( \theta_A \in (0, \pi), \)

\( (H_4) \quad \sigma(A) \subset (-\infty, 0) \quad \text{and} \quad \forall \theta \in (0, \pi), \quad \sup_{\lambda \in \mathbb{S}_0} \|\lambda(\lambda I - A)^{-1}\|_{L(X)} < +\infty. \)

Note that \( (H_4) \) means that \(-A\) is a sectorial operator of any angle \( \theta \in (0, \pi) \). Let us give some consequences of our assumptions.

3.4. **Consequences.**

1. Note that \( A_0 \) satisfies all the previous hypotheses with \( X = L^q(\omega) \) and \( q \in (1, +\infty) \).
2. To solve each equation of \((EQ)\) in the scalar case (with \(-A > 0\)), it is necessary to introduce the roots \( \pm \sqrt{-A + r_\pm}, \pm \sqrt{-A} \) of the characteristic equations
\[
    x^4 + (2A - r_\pm)x^2 + (A^2 - r_\pm A) = 0,
\]
this is why, in our operational case, we consider the operators
\[
    L_- := -\sqrt{-A + r_- I}, \quad L_+ := -\sqrt{-A + r_+ I} \quad \text{and} \quad M := -\sqrt{-A}. \quad (5)
\]
Due to \( (H_3) \), \(-A - A + r_- I\) and \(-A + r_+ I\) are sectorial operators, so the existence of \( L_- \), \( L_+ \) and \( M \) is ensured, see for instance [11], e), p. 25.
3. Applying Proposition 3.1.9, p. 65, in [11], we have \( D(L_-) = D(L_+) = D(M) \). Thus, for \( n, m \in \mathbb{N} \) and \( m \leq n \)
\[
    D(L_+^m) = D(M^n) = D(L_+^m M^{n-m}) = D(M^m L_-^{n-m}).
\]
4. Due to \( (H_3) \), \(-A + r_- I \in \text{BIP}(X, \theta_A) \) and \(-A + r_+ I \in \text{BIP}(X, \theta_A) \), see [19], Theorem 3, p. 437, from which we deduce that
\[
    -L_-, -L_+, -M \in \text{BIP}(X, \theta_A/2),
\]
see [11], Proposition 3.2.1, e), p. 71. Since \( 0 < \theta_A/2 < \pi/2 \), \( L_-, L_+ \) and \( M \) generate bounded analytic semigroups \((e^{xL_-})_{x \geq 0}, (e^{xL_+})_{x \geq 0} \) and \((e^{xM})_{x \geq 0} \), see [19], Theorem 2, p. 437. Moreover, from [19], Theorem 4, p. 441, we get
\[
    -(L_- + M), -(L_+ + M) \in \text{BIP}(X, \theta_A/2 + \varepsilon),
\]
for any \( \varepsilon \in (0, \pi/2 - \theta_A/2) \). So from [19], Theorem 2, p. 437, \( L_- + M \) and \( L_+ + M \) generate bounded analytic semigroups
\[
    (e^{x(L_- + M)})_{x \geq 0} \quad \text{and} \quad (e^{x(L_+ + M)})_{x \geq 0}.
\]
5. From \( (H_2) \) and \( (H_3) \), we deduce that \( 0 \in \rho(M) \cap \rho(L_-) \cap \rho(L_+) \). Thus, assumptions \( (H_1) \), \( (H_2) \) and \( (H_3) \) lead us to apply the Dore-Venni theorem, see [6], to obtain \( 0 \in \rho(L_- + M) \) and \( 0 \in \rho(L_+ + M) \).
6. It follows from (5) that
\[
    \forall \psi \in D(M^2), \quad \begin{cases}
    (L_+^2 - M^2)\psi = r_+ \psi \\
    (L_-^2 - M^2)\psi = r_- \psi,
\end{cases} \quad (6)
\]
and also
\[ \forall \psi \in D(M), \quad \begin{cases} (L_+ - M)\psi = r_+(L_+ + M)^{-1}\psi \\ (L_- - M)\psi = r_-(L_- + M)^{-1}\psi. \end{cases} \tag{7} \]

3.5. The main results. To solve problem (P), we introduce two problems:
\[ (P_-) \begin{cases} u_-^{(4)}(x) + (2A - r_- I)u_-'(x) + (A^2 - r_- A)u_-(x) = f_-(x), & x \in (a, \gamma) \\ u_-(a) = \varphi_1^-, & u_-(\gamma) = \psi_1 \\ u_-'(a) = \varphi_2^-, & u_-'(\gamma) = \psi_2, \end{cases} \]
and
\[ (P_+) \begin{cases} u_+^{(4)}(x) + (2A - r_+ I)u_+'(x) + (A^2 - r_+ A)u_+(x) = f_+(x), & x \in (\gamma, b) \\ u_+(\gamma) = \psi_1, & u_+(b) = \varphi_1^+ \\ u_+'(\gamma) = \psi_2, & u_+'(b) = \varphi_2^+. \end{cases} \]

Remark 1. \( u \) is a classical solution of (P) if and only if there exist \( \psi_1, \psi_2 \in X \) such that
\begin{enumerate}
  \item \( u_- \) is a classical solution of (P_-),
  \item \( u_+ \) is a classical solution of (P_+),
  \item \( u_- \) and \( u_+ \) satisfy (TC2).
\end{enumerate}

So our goal is to prove that there exists a unique couple \( (\psi_1, \psi_2) \) which satisfies (i), (ii) and (iii). This will lead us to obtain our main result.

Theorem 3.3. Let \( f_- \in L^p(a, \gamma; X) \) and \( f_+ \in L^p(\gamma, b; X) \). Assume that \( (H_1), (H_2), (H_3), (H_4) \) hold. Then, there exists a unique classical solution \( u \), see definition (2), of the transmission problem (P) if and only if
\[ \varphi_1^+, \varphi_1^- \in (D(A), X)_{1+\frac{1}{p}+\frac{1}{p'}, p} \quad \text{and} \quad \varphi_2^+, \varphi_2^- \in (D(A), X)_{1+\frac{1}{p}+\frac{1}{p'}, p}. \tag{8} \]

Remark 2. 1. The proof of Theorem 3.3 uses operators \( L_-, L_+ \), \( M \) and also interpolation spaces \( (D(M), X)_{3-j+\frac{1}{p}+\frac{1}{p'}, p}, j = 0, 1, 2, 3 \). But from Lemma 3.1, we get
\[ \begin{cases} (D(M), X)_{3+\frac{1}{p}+\frac{1}{p'}, p} = (D(A), X)_{1+\frac{1}{p}+\frac{1}{p'}, p} \\ (D(M), X)_{2+\frac{1}{p}+\frac{1}{p'}, p} = (D(A), X)_{1+\frac{1}{p}+\frac{1}{p'}, p} \\ (D(M), X)_{1+\frac{1}{p}+\frac{1}{p'}, p} = (D(A), X)_{\frac{1}{p}+\frac{1}{p'}, p} \\ (D(M), X)_{\frac{1}{p}+\frac{1}{p'}, p} = (D(A), X)_{\frac{1}{p}+\frac{1}{p'}, p}. \end{cases} \tag{9} \]
2. We can generalize this Theorem by considering a transmission problem between \( n \) habitats, with \( n \in \mathbb{N} \setminus \{0\} \). It suffices to use Theorem 3.3 on the two first habitats and then apply it on the transmission problem between the second and the third habitat to solve the problem with \( n = 3 \). By recurrence, we obtain the result.

As consequence of Theorem 3.3, we deduce some results for problem \( (P_{pde}) \) under some necessary boundary conditions. Let us consider the case \( A = A_0 \) (other cases can be treated).
Corollary 1. Assume that \( \omega \) is a bounded open set of \( \mathbb{R}^{d-1} \) where \( d \geq 2 \) with \( C^2 \)-boundary. Let \( f_+ \in L^p(\Omega_+) \) and \( f_- \in L^p(\Omega_-) \) with \( p \in (1, +\infty) \) and \( p > d \); let \( k_+, k_-, l_+, l_- \in \mathbb{R}^+ \setminus \{0\} \). Then, there exists a unique solution \( u \) of \((P_{\text{pde}})\), such that
\[
 u_- \in W^{4,p}(\Omega_-), \quad u_+ \in W^{4,p}(\Omega_+),
\]
if and only if
\[
\begin{align*}
\varphi_{1+}^\pm, \varphi_{2+}^\pm & \in W^{2,p}(\omega) \cap W^{1,p}_0(\omega), \\
\Delta \varphi_{1+}^\pm & \in W^{2-\frac{1}{p},p}(\omega) \cap W^{1,p}_0(\omega), \\
\Delta \varphi_{2+}^\pm & \in W^{1-\frac{1}{p},p}(\omega) \cap W^{1,p}_0(\omega).
\end{align*}
\]

Proof. Let \((x,y) \in (a,b) \times \omega\). Set \( X := L^p(\omega) \). Using \( A_0 \) the linear operator defined by \((1)\), we obtain that problem \((P_{\text{pde}})\) becomes problem \((P)\). From [21], Proposition 3, p. 207, \( X \) satisfies \((H_1)\) and from [8], Theorem 9.15 and Lemma 9.17, p. 241-242, \( A_0 \) satisfies \((H_2)\). Moreover, \((H_3)\) is satisfied from [20], Theorem C, p. 166-167. Moreover, since \( A_0 \) is the Laplace operator, from [11], Chapter 8, section 3, p. 232, \((H_4)\) is satisfied. Finally, all the assumptions of Theorem 3.3 are satisfied. It follows that, there exists a unique classical solution of problem \((P)\) if and only if \((8)\) holds.

Now, it remains to show that if \( \varphi_1^1, \varphi_2, \varphi_3, \varphi_4 \) satisfy \((8)\), then the classical solution \( u_\pm \) satisfies \( u_\pm \in W^{4,p}(\Omega_\pm) \). To this end, we will make explicit the interpolation spaces that appear in \((8)\). We have
\[
(D(A_0), X)_{\frac{1}{p}, p} = \left( W^{2,p}(\omega) \cap W^{1,p}_0(\omega), L^p(\omega) \right)_{\frac{1}{p}, p},
\]
and from [10], p. 683, proposizione 3 and p. 681, 1.10, and [22], p. 317, Theorem 1, since \( 2 - \frac{1}{p} > 1 \) is never integer, we have
\[
\left( W^{2,p}(\omega), L^p(\omega) \right)_{\frac{1}{p}, p} = W^{2-\frac{1}{p},p}(\omega).
\]

Set \( \nu_1 := 2 - \frac{1}{p} - \frac{d-1}{p} = 2 - \frac{d}{p} \). Since \( p > \frac{d}{2} \), we have \( \nu_1 > 0 \). From the Sobolev embedding theorem, see [22], section 4.6.1, p. 327-328, we have:
\[
W^{2-\frac{1}{p},p}(\omega) \hookrightarrow C(\overline{\omega}).
\]
Thus, the traces of the elements of the space described in \((10)\) are well defined. From [9], Proposition 5.9, p. 334, and [22], section 4.3.3, Theorem, p. 321, we deduce that
\[
(D(A_0), X)_{\frac{1}{p}, p} = \left\{ \psi \in W^{2-\frac{1}{p},p}(\omega) : \psi = 0 \text{ on } \partial \omega \right\},
\]
and
\[
(D(A_0), X)_{1+\frac{1}{p}, p} = \left\{ \psi \in D(A_0) : A_0 \psi \in W^{2-\frac{1}{p},p}(\omega) \text{ and } \Delta \psi = 0 \text{ on } \partial \omega \right\}
= \left\{ \psi \in W^{2,p}(\omega) : \Delta \psi \in W^{2-\frac{1}{p},p}(\omega) \text{ and } \psi = \Delta \psi = 0 \text{ on } \partial \omega \right\}.
\]
In the same way, we obtain
\[
(D(A_0), X)_{\frac{1}{2}+\frac{1}{p}, p} = \left( W^{2,p}(\omega) \cap W^{1,p}_0(\omega), L^p(\omega) \right)_{\frac{1}{2}+\frac{1}{p}, p}.
\]

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Lemma 4.1. The operators \( U_+, U_-, V_+, V_- \) in \( \mathcal{L}(X) \) defined by
\[
\begin{align*}
U_+ & := I - e^{d(L_+ + M)} - r_+(L_+ + M)^2 (e^{dM} - e^{dL_+}) \\
U_- & := I - e^{c(L_- + M)} - r_-(L_- + M)^2 (e^{cM} - e^{cL_-}) \\
V_+ & := I - e^{d(L_+ + M)} + r_+(L_+ + M)^2 (e^{dM} - e^{dL_+}) \\
V_- & := I - e^{c(L_- + M)} + r_-(L_- + M)^2 (e^{cM} - e^{cL_-})
\end{align*}
\]
are invertible with bounded inverse.

All these exponentials are well defined, see statement 4 of section 3.4. For a detailed proof, see [13], Proposition 5.4 with \( k = r_- \) or \( k = r_+ \).
4.1. Problem \((P_-)\).

**Proposition 1.** Let \(f_- \in L^p(a, \gamma; X)\). Assume that \((H_1), (H_2), (H_3), (H_4)\) hold. There exists a unique classical solution \(u_-\) of problem \((P_-)\) if and only if

\[
\varphi_1, \psi_1 \in (D(A), X)_{1+\frac{1}{p}, p} \quad \text{and} \quad \varphi_2, \psi_2 \in (D(A), X)_{1+\frac{1}{p}, p}.
\]

Moreover

\[
u_-(x) = \left( e^{(x-a)M} - e^{(\gamma-x)M} \right) \alpha_1^- + \left( e^{(x-a)L_-} - e^{(\gamma-x)L_-} \right) \alpha_2^-
\]

\[
+ \left( e^{(x-a)M} + e^{(\gamma-x)M} \right) \alpha_3^- + \left( e^{(x-a)L_-} + e^{(\gamma-x)L_-} \right) \alpha_4^-
\]

\[
\quad + F_-(x),
\]

where

\[
\begin{align*}
\alpha_1^- & := -\frac{1}{2r_-} (L_- + M) U_-^{-1} \left[ L_- (I + e^{cL_-}) \psi_1 + (I - e^{cL_-}) \psi_2 + \tilde{\varphi}_1^- \right] \\
\alpha_2^- & := \frac{1}{2r_-} (L_- + M) U_-^{-1} \left[ M (I + e^{cM}) \psi_1 + (I - e^{cM}) \psi_2 + \tilde{\varphi}_2^- \right] \\
\alpha_3^- & := -\frac{1}{2r_-} (L_- + M) V_-^{-1} \left[ L_- (I - e^{cL_-}) \psi_1 + (I + e^{cL_-}) \psi_2 + \tilde{\varphi}_3^- \right] \\
\alpha_4^- & := -\frac{1}{2r_-} (L_- + M) V_-^{-1} \left[ M (I - e^{cM}) \psi_1 + (I + e^{cM}) \psi_2 + \tilde{\varphi}_4^- \right],
\end{align*}
\]

and \(F_-\) is the unique classical solution of problem

\[
\begin{align*}
u_-(x) + (2A - r_- I) u''_-(x) + (A^2 - r_- A) u_-(x) &= f_-(x), \quad x \in (a, \gamma) \\
u_-(a) &= u_-(\gamma) = u''_-(a) = u''_-(\gamma) = 0.
\end{align*}
\]

**Proof.** From [13], Theorem 2.5, statement 2, there exists a unique classical solution \(u_-\) of \((P_-)\) if and only if \((13)\) holds. To obtain the representation formula \((14)-(15)-(16)\) of \(u_-\), we have adapted the representation formula \((5.3)-(5.19)-(5.20)\) given in [13], where \(u, L, f, b, F_{0,f}, k, \varphi_1, \varphi_2, \varphi_3, \varphi_4\) are respectively replaced by \(u_-, L_-, f_-, \gamma, F_-, r_-, \varphi_1^-, \psi_1, \varphi_2^-, \psi_2\).

**Remark 3.** In the previous proposition, due to \((13), (15), (16)\), we have

\[
\alpha_i^- \in D(M), \quad \text{for} \quad i = 1, 2, 3, 4.
\]

Moreover, since \(F_-\) is a classical solution of \((17)\), by Lemma 3.2, we deduce that, for \(j = 0, 1, 2, 3\) and \(s = a\) or \(\gamma\)

\[
F_-^{(3)}(s) \in (D(M), X)_{3-j+\frac{1}{p}, p}.
\]

4.2. Problem \((P_+)\).

**Proposition 2.** Let \(f_+ \in L^p(\gamma, b; X)\). Assume that \((H_1), (H_2), (H_3), (H_4)\) hold. There exists a unique classical solution \(u_+\) of \((P_+)\) if and only if

\[
\varphi_1^+, \psi_1 \in (D(A), X)_{1+\frac{1}{p}, p} \quad \text{and} \quad \varphi_2^+, \psi_2 \in (D(A), X)_{1+\frac{1}{p}, p}.
\]

\[
\begin{align*}
u_+(x) + (2A - r_+ I) u''_+(x) + (A^2 - r_+ A) u_+(x) &= f_+(x), \quad x \in (\gamma, b) \\
u_+(b) &= u_+(\gamma) = u''_+(b) = u''_+(\gamma) = 0
\end{align*}
\]
Moreover, for \( u_+ \) we have
\[
\begin{align*}
u_+(x) &= (e^{(x-\gamma)M} - e^{(b-x)M}) \alpha_1^+ + (e^{(x-\gamma)L_+} - e^{(b-x)L_+}) \alpha_2^+ \\
&\quad + (e^{(x-\gamma)M} + e^{(b-x)M}) \alpha_3^+ + (e^{(x-\gamma)L_+} + e^{(b-x)L_+}) \alpha_4^+ + F_+(x),
\end{align*}
\] (19)
where
\[
\begin{align*}
\alpha_1^+ &= \frac{1}{2r_+} (L_+ + M) U_+^{-1} [L_+ (I + e^{dL_+}) \psi_1 - (I - e^{dL_+}) \psi_2 + \tilde{\varphi}_1^+] \\
\alpha_2^+ &= -\frac{1}{2r_+} (L_+ + M) U_+^{-1} [M (I + e^{dB}) \psi_1 - (I - e^{dB}) \psi_2 + \tilde{\varphi}_2^+] \\
\alpha_3^+ &= \frac{1}{2r_+} (L_+ + M) V_+^{-1} [L_+ (I - e^{dL_+}) \psi_1 - (I + e^{dL_+}) \psi_2 + \tilde{\varphi}_3^+] \\
\alpha_4^+ &= -\frac{1}{2r_+} (L_+ + M) V_+^{-1} [M (I - e^{dB}) \psi_1 - (I + e^{dB}) \psi_2 + \tilde{\varphi}_4^+],
\end{align*}
\] (20)
for \( \tilde{\varphi}_1^+, \tilde{\varphi}_2^+, \tilde{\varphi}_3^+, \tilde{\varphi}_4^+ \) defined as
\[
\begin{align*}
\tilde{\varphi}_1^+ &= -L_+ (I + e^{dB}) \varphi_1^+ + (I - e^{dB}) (F_+^r (b) + F_+^r (\gamma) - \varphi_2^+) \\
\tilde{\varphi}_2^+ &= -M (I + e^{dB}) \varphi_1^+ + (I - e^{dB}) (F_+^r (b) + F_+^r (\gamma) - \varphi_2^+) \\
\tilde{\varphi}_3^+ &= -L_+ (I - e^{dB}) \varphi_1^+ - (I + e^{dB}) (F_+^r (b) - F_+^r (\gamma) - \varphi_2^+) \\
\tilde{\varphi}_4^+ &= -M (I - e^{dB}) \varphi_1^+ - (I + e^{dB}) (F_+^r (b) - F_+^r (\gamma) - \varphi_2^+),
\end{align*}
\] (21)
and \( F_+ \) is the unique classical solution of problem
\[
\begin{align*}
u_+(x) + (2A - r_+) I) u_+''(x) + (A^2 - r_+ A) u_+(x) = f_+(x), \quad x \in (\gamma, b) \\
u_+(\gamma) = u_+(b) = u_+''(\gamma) = u_+''(b) = 0.
\end{align*}
\] (22)
Proof. From [13], Theorem 2.5, statement 2, there exists a unique classical solution \( u_+ \) of \((P_+)\) if and only if (18) holds. To obtain the representation formula (19)-(20)-(21) of \( u_+ \), we have adapted the representation formula (5.3)-(5.19)-(5.20) given in [13], where \( u, L, f, \alpha, F_0, j, k, \varphi_1, \varphi_2, \varphi_3, \varphi_4 \) are respectively replaced by \( u_+, L_+, f_+, \gamma, F_+, r_+, \psi_1, \psi_2, \psi_3, \psi_4 \).

\[ \Box \]

Remark 4. In the previous proposition, due to (18), (20) and (21), we have
\[ \alpha_i^+ \in D(M), \quad \text{for } i = 1, 2, 3, 4. \]
Moreover, since \( F_+ \) is a classical solution of (22), by Lemma 3.2, we deduce that, for \( j = 0, 1, 2, 3 \) and \( s = \gamma \) or \( b \)
\[ F_+^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{2},p}. \]

4.3. The transmission system. This section is devoted to the proof of Theorem 4.2 stated below, which gives the link between problem (P) and the following system
\[
\begin{align*}
(P_1^+)^{-1} = k_+ (L_+ + M) L_+ (U_+^{-1} (I + e^{dB}) (I + e^{dB}) + V_+^{-1} (I - e^{dB}) (I - e^{dB})) \\
(P_2^+)^{-1} = k_+ (L_+ + M) (U_+^{-1} (I + e^{dB}) (I - e^{dB}) + V_+^{-1} (I - e^{dB}) (I + e^{dB})) \\
(P_3^+)^{-1} = k_+ (L_+ + M) (U_+^{-1} (I - e^{dB}) (I - e^{dB}) + V_+^{-1} (I + e^{dB}) (I + e^{dB})),
\end{align*}
\] (24)
and similarly
\[
\begin{align*}
P_1^- &= k_-(L_+ + M) (U^-_1(I + e^{-M})(I + e^{L_+}) + V^-_1(I - e^{-M})(I - e^{L_+})) \\
P_2^- &= k_-(L_+ + M) (U^-_1(I + e^{-M})(I + e^{L_+}) + V^-_1(I - e^{-M})(I + e^{L_+})) \\
P_3^- &= k_-(L_+ + M) (U^-_1(I - e^{-M})(I - e^{L_+}) + V^-_1(I + e^{-M})(I + e^{L_+}))
\end{align*}
\]
\tag{25}

The second members are
\[
S_1 = -k_+(L_+ + M) (U^+_1(I + e^{M})\hat{\varphi}_1^+ + V^+_1(I - e^{M})\hat{\varphi}_3^+) \\
- k_-(L_+ + M) (U^-_1(I + e^{M})\hat{\varphi}_1^- + V^-_1(I - e^{M})\hat{\varphi}_3^-) \\
- 2M^{-1}R_1,
\]
with \( R_1 \) given by
\[
R_1 = -k_+ F'''_\gamma(\gamma) + k_+ M^2 F'_\gamma(\gamma) + l_+ F'_\gamma(\gamma) \\
+ k_+ F'''_\gamma(\gamma) - k_- M^2 F'_\gamma(\gamma) - l_- F'_\gamma(\gamma),
\]
and
\[
S_2 = k_+(L_+ + M) (U^+_1(I - e^{L_+})\hat{\varphi}_2^+ + V^+_1(I + e^{L_+})\hat{\varphi}_1^+) \\
- k_-(L_+ + M) (U^-_1(I - e^{L_+})\hat{\varphi}_2^- + V^-_1(I + e^{L_+})\hat{\varphi}_1^-).
\]
\tag{28}

**Theorem 4.2.** Let \( f_+ \in L^p(a, \gamma; X) \) and \( f_+ \in L^p(\gamma, b; X) \). Assume that \( (H_1), (H_2), (H_3), (H_4) \) hold. Then, the transmission problem \((P)\) has a unique classical solution if and only if the data \( \varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^- \) satisfy \((8)\) and system \((23)\) has a unique solution \((\psi_1, \psi_2)\) such that
\[
(\psi_1, \psi_2) \in (D(A), X)_{1 \frac{1}{2}, p} \times (D(A), X)_{1 \frac{1}{2}, p}.
\]
\tag{29}

**Proof.** First, we assume that \((P)\) admits a unique classical solution \( u \). Setting
\[
\psi_1 = u_-(\gamma) = u_+(\gamma) \quad \text{and} \quad \psi_2 = u'_-(\gamma) = u'_+(\gamma),
\]
we get that \( u_- \) (respectively \( u_+ \)) is the classical solution of \((P_-)\) (respectively \((P_+)\)). So, applying Proposition 1 (respectively Proposition 2), we obtain \((8)\) and also \((29)\). It remains to prove that \((\psi_1, \psi_2)\) satisfies \((23)\). To this end we use \((TC2)\) satisfied by \( u \), that is
\[
\begin{align*}
\left\{ \\
k_+ \left( u_+^{(3)}(\gamma) - M^2 u_+'(\gamma) \right) - l_+ u_+'(\gamma) - k_- \left( u_-^{(3)}(\gamma) - M^2 u'_-(\gamma) \right) + l_- u'_-(\gamma) = 0 \\
k_+ \left( u_+''(\gamma) - M^2 u_+'(\gamma) \right) - k_- \left( u_-''(\gamma) - M^2 u'_-(\gamma) \right) = 0
\end{align*}
\]
To make explicit this system, we use the expression of \( u_+ \) given in \((19)\). It follows, for \( x \in (\gamma, b) \)
\[
\begin{align*}
u_+(x) &= (e^{(x-\gamma)M} - e^{(b-x)M}) \alpha_1^+ + (e^{(x-\gamma)L_+} - e^{(b-x)L_+}) \alpha_2^+ \\
&+ (e^{(x-\gamma)M} + e^{(b-x)M}) \alpha_3^+ + (e^{(x-\gamma)L_+} + e^{(b-x)L_+}) \alpha_4^+ \\
&+ F_+(x),
\end{align*}
\]

\[
\begin{align*}
u_+'(x) &= M (e^{(x-\gamma)M} + e^{(b-x)M}) \alpha_1^+ + L_+ (e^{(x-\gamma)L_+} + e^{(b-x)L_+}) \alpha_2^+ \\
&+ M (e^{(x-\gamma)M} - e^{(b-x)M}) \alpha_3^+ + L_+ (e^{(x-\gamma)L_+} - e^{(b-x)L_+}) \alpha_4^+ \\
&+ F_+'(x),
\end{align*}
\]
Then, in virtue of Lemma 3.2, we have
\[ u''_{+}(x) = M^2 \left( e^{(x-\gamma)x} - e^{(b-x)x} \right) \alpha_1^+ + L_+^2 \left( e^{(x-\gamma)x} - e^{(b-x)x} \right) \alpha_2^+ + L_+^3 \left( e^{(x-\gamma)x} + e^{(b-x)x} \right) \alpha_3^+ + L_+^4 \left( e^{(x-\gamma)x} - e^{(b-x)x} \right) \alpha_4^+ + F''_+(x), \]

Furthermore, from (6), we obtain
\[
M^{-2} \left( u^{(3)}_{+}(\gamma) - M^2 u'_{+}(\gamma) \right) = L_+ \left( L_+^2 - M^2 \right) M^{-2} \left( I + e^{dL_+} \right) \alpha_2^+ + L_+ \left( L_+^2 - M^2 \right) M^{-2} \left( I - e^{dL_+} \right) \alpha_4^+ + M^{-2} F''_+(\gamma) - F'_+(\gamma).
\]

Then, we obtain
\[
M^{-2} \left( u^{(3)}_{+}(\gamma) - M^2 u'_{+}(\gamma) \right) = \frac{l_+}{k_+} L_+ M^{-2} \left( I + e^{dL_+} \right) \alpha_2^+ + \frac{l_+}{k_+} L_+ M^{-2} \left( I - e^{dL_+} \right) \alpha_4^+ + M^{-2} F''_+(\gamma) - F'_+(\gamma),
\]

hence
\[
u^{(3)}_{+}(\gamma) - M^2 u'_{+}(\gamma) = \frac{l_+}{k_+} L_+ \left( I + e^{dL_+} \right) \alpha_2^+ + \frac{l_+}{k_+} L_+ \left( I - e^{dL_+} \right) \alpha_4^+ + F''_+(\gamma) - M^2 F'_+(\gamma),
\]

it follows that
\[
k_+ \left( u^{(3)}_{+}(\gamma) - M^2 u'_{+}(\gamma) \right) - l_+ u'_{+}(\gamma) = -l_+ M \left( I + e^{dM} \right) \alpha_1^+ - l_+ M \left( I - e^{dM} \right) \alpha_3^+ + k_+ F''_+(\gamma) - k_+ M^2 F'_+(\gamma) - l_+ F'_+(\gamma).
\]

Note that, from Remark 4 and Lemma 3.2, all the terms in the previous equalities are justified.

By the same arguments and using again (6) and also (22), we have
\[
M^{-1} \left( u''_{+}(\gamma) - M^2 u_{+}(\gamma) \right) = \left( L_+^2 - M^2 \right) M^{-1} \left( I - e^{dL_+} \right) \alpha_2^+ + \left( L_+^2 - M^2 \right) M^{-1} \left( I + e^{dL_+} \right) \alpha_4^+ = \frac{l_+}{k_+} M^{-1} \left( I - e^{dL_+} \right) \alpha_2^+ + \frac{l_+}{k_+} M^{-1} \left( I + e^{dL_+} \right) \alpha_4^+.
\]

Then, we obtain
\[
k_+ \left( u''_{+}(\gamma) - M^2 u_{+}(\gamma) \right) = l_+ \left( I - e^{dL_+} \right) \alpha_2^+ + l_+ \left( I + e^{dL_+} \right) \alpha_4^+.
\]

As previously, for \( u_{-} \), we use (14) and get, for \( x \in (a, \gamma) \)
\[
u(x) = (e^{(x-a)x} - e^{(\gamma-x)x}M) \alpha_1^- + (e^{(x-a)x}M - e^{(\gamma-x)x}L_2) \alpha_2^- + (e^{(x-a)x}M + e^{(\gamma-x)x}M) \alpha_3^- + (e^{(x-a)x}M + e^{(\gamma-x)x}L_2) \alpha_4^- + F_{-}(x),
\]
Then, in virtue of Lemma 3.2, we have

\[ u_\gamma''(x) = M \left( e^{(x-a)M} + e^{(y-x)M} \right) \alpha_1^- + L_2 \left( e^{(x-a)M} - e^{(y-x)M} \right) \alpha_2^- + F''(x), \]

\[ u'_\gamma(x) = M^2 \left( e^{(x-a)M} - e^{(y-x)M} \right) \alpha_3^- + L_2 \left( e^{(x-a)M} - e^{(y-x)M} \right) \alpha_4^- + F''(x), \]

\[ u_\gamma^{(3)}(x) = M^3 \left( e^{(x-a)M} + e^{(y-x)M} \right) \alpha_1^- + L_3 \left( e^{(x-a)M} + e^{(y-x)M} \right) \alpha_2^- + L_3 \left( e^{(x-a)M} - e^{(y-x)M} \right) \alpha_4^- + F'''(x). \]

Then, in virtue of Lemma 3.2, we have

\[ u_\gamma^{(3)}(\gamma) - M^2 u_\gamma' \gamma) = L_2 (L_2^2 - M^2) \left( e^{cL_2} + I \right) \alpha_2^- + L_2 (L_2^2 - M^2) \left( e^{cL_2} - I \right) \alpha_4^- + F'''(\gamma) - M^2 F''(\gamma), \]

hence, due to (6), we have

\[ M^{-2} \left( u_\gamma^{(3)}(\gamma) - M^2 u_\gamma'(\gamma) \right) = L_2 (L_2^2 - M^2) M^{-2} \left( e^{cL_2} + I \right) \alpha_2^- + M^{-2} F'''(\gamma) + L_2 (L_2^2 - M^2) M^{-2} \left( e^{cL_2} - I \right) \alpha_4^- - F''(\gamma) = L_2 \frac{L_2^2 - M^2}{k_-} \left( e^{cL_2} + I \right) \alpha_2^- + M^{-2} F'''(\gamma) + L_2 \frac{L_2^2 - M^2}{k_-} \left( e^{cL_2} - I \right) \alpha_4^- - F''(\gamma). \]

Then, we obtain

\[ k_2 \left( u_\gamma^{(3)}(\gamma) - M^2 u_\gamma'(\gamma) \right) - l_- u_\gamma'(\gamma) = \left( L_2 (L_2^2 - M^2) M^{-2} (e^{cL_2} + I) \alpha_2^- + L_2 (L_2^2 - M^2) M^{-2} (e^{cL_2} - I) \alpha_4^- - l_- M \left( e^{cM} + I \right) \alpha_1^- - l_- \left( e^{cL_2} + I \right) \alpha_2^- - l_- M \left( e^{cM} - I \right) \alpha_3^- - l_- \left( e^{cL_2} - I \right) \alpha_4^- + k_2 F'''(\gamma) - k_2 M^2 F'' \gamma \right) - l_- F''(\gamma) = \left( L_2 (L_2^2 - M^2) M^{-2} (e^{cL_2} + I) \alpha_2^- + L_2 (L_2^2 - M^2) M^{-2} (e^{cL_2} - I) \alpha_4^- - l_- M \left( e^{cM} + I \right) \alpha_1^- - l_- \left( e^{cL_2} + I \right) \alpha_2^- - l_- M \left( e^{cM} - I \right) \alpha_3^- - l_- \left( e^{cL_2} - I \right) \alpha_4^- + k_2 F'''(\gamma) - k_2 M^2 F'' \gamma \right) - l_- F''(\gamma). \]

Furthermore, from (6), we have

\[ M^{-1} \left( u_\gamma''(\gamma) - M^2 u_\gamma(\gamma) \right) = (L_2^2 - M^2) M^{-1} \left( e^{cL_2} - I \right) \alpha_2^- + (L_2^2 - M^2) M^{-1} \left( I + e^{cL_2} \right) \alpha_4^- = \frac{L_2}{k_-} M^{-1} \left( e^{cL_2} - I \right) \alpha_2^- + \frac{L_2}{k_-} M^{-1} \left( I + e^{cL_2} \right) \alpha_4^- \]

Then, we deduce the following equality:

\[ k_2 \left( u_\gamma''(\gamma) - M^2 u_\gamma(\gamma) \right) = \left( e^{cL_2} - I \right) \alpha_2^- + \left( I + e^{cL_2} \right) \alpha_4^- \]

(31)
Note that, from Remark 3 and Lemma 3.2, all the terms in the previous equalities are justified. It follows, from (30) and (31), that system (TC2) becomes

\[
\begin{aligned}
- l_+ M (I + e^{dM}) \alpha_1^+ - l_+ M (I - e^{dM}) \alpha_3^+ &= - l_- M (I + e^{cM}) \alpha_1^- \\
&+ l_- M (I - e^{cM}) \alpha_3^- + R_1 \\
l_+ (I - e^{dL+}) \alpha_2^+ + l_+ (I + e^{dL+}) \alpha_4^+ &= - l_- (I - e^{cL-}) \alpha_2^- \\
&+ l_- (I + e^{cL-}) \alpha_4^- ,
\end{aligned}
\]

where, \( R_1 \) is given by (27). Thus

\[
\begin{aligned}
- l_+ ((\alpha_1^+ + \alpha_3^+) - e^{dM} (\alpha_3^+ - \alpha_1^+)) &= l_- ((\alpha_3^- - \alpha_1^-) - e^{cM} (\alpha_1^- + \alpha_3^-)) \\
&+ M^{-1} R_1 \\
l_+ ((\alpha_2^+ + \alpha_4^+) + e^{dL+} (\alpha_4^+ - \alpha_2^+)) &= l_- ((\alpha_4^- - \alpha_2^-) + e^{cL-} (\alpha_2^- + \alpha_4^-)).
\end{aligned}
\]

But, from (20), (21), (15) and (16), we have

\[
\begin{aligned}
\alpha_1^+ + \alpha_3^+ &= \frac{1}{2r_+} (L_+ + M) U_+^{-1} (L_+ (I + e^{dL+}) \psi_1 - (I - e^{dL+}) \psi_2 + \varphi_1^+) \\
&+ \frac{1}{2r_+} (L_+ + M) V_+^{-1} (L_+ (I - e^{dL+}) \psi_1 - (I + e^{dL+}) \psi_2 + \varphi_3^+) ,
\end{aligned}
\]

\[
\begin{aligned}
\alpha_3^+ - \alpha_1^+ &= - \frac{1}{2r_+} (L_+ + M) U_+^{-1} (L_+ (I + e^{dL+}) \psi_1 - (I - e^{dL+}) \psi_2 + \varphi_1^+) \\
&+ \frac{1}{2r_+} (L_+ + M) V_+^{-1} (L_+ (I - e^{dL+}) \psi_1 - (I + e^{dL+}) \psi_2 + \varphi_3^+) ,
\end{aligned}
\]

and

\[
\begin{aligned}
\alpha_1^- + \alpha_3^- &= - \frac{1}{2r_-} (L_- + M) U_-^{-1} (L_- (I + e^{cL-}) \psi_1 + (I - e^{cL-}) \psi_2 + \varphi_1^-) \\
&+ \frac{1}{2r_-} (L_- + M) V_-^{-1} (L_- (I - e^{cL-}) \psi_1 + (I + e^{cL-}) \psi_2 + \varphi_3^-) ,
\end{aligned}
\]

\[
\begin{aligned}
\alpha_3^- - \alpha_1^- &= \frac{1}{2r_-} (L_- + M) U_-^{-1} (L_- (I + e^{cL-}) \psi_1 + (I - e^{cL-}) \psi_2 + \varphi_1^-) \\
&+ \frac{1}{2r_-} (L_- + M) V_-^{-1} (L_- (I - e^{cL-}) \psi_1 + (I + e^{cL-}) \psi_2 + \varphi_3^-) .
\end{aligned}
\]

So, using (24) and (25), the first line of system (32) writes

\[
(P_1^+ + P_1^-) \psi_1 + (P_2^- - P_2^+) \psi_2 = S_1 ,
\]

where \( S_1 \) is given by (26). By the same way, we have

\[
\begin{aligned}
\alpha_2^+ + \alpha_4^+ &= - \frac{1}{2r_+} (L_+ + M) U_+^{-1} (M (I + e^{dM}) \psi_1 - (I - e^{dM}) \psi_2 + \varphi_2^+) \\
&- \frac{1}{2r_+} (L_+ + M) V_+^{-1} (M (I - e^{dM}) \psi_1 - (I + e^{dM}) \psi_2 + \varphi_4^+) ,
\end{aligned}
\]
\[ \alpha_4^+ - \alpha_2^+ = \frac{1}{2r_+} (L_+ + M) U_+^{-1} (M(I + e^{dM}) \psi_1 - (I + e^{dM}) \psi_2 + \varphi_2^+) \]

\[ - \frac{1}{2r_+} (L_+ + M) V_+^{-1} (M(I - e^{dM}) \psi_1 - (I + e^{dM}) \psi_2 + \varphi_2^+) , \]

\[ \alpha_2^- + \alpha_4^- = \frac{1}{2r_-} (L_- + M) U_-^{-1} (M(I + e^{cM}) \psi_1 + (I - e^{cM}) \psi_2 + \varphi_2^-) \]

\[ - \frac{1}{2r_-} (L_- + M) V_-^{-1} (M(I - e^{cM}) \psi_1 + (I + e^{cM}) \psi_2 + \varphi_2^-) , \]

\[ \alpha_4^- - \alpha_2^- = - \frac{1}{2r_-} (L_- + M) U_-^{-1} (M(I + e^{cM}) \psi_1 + (I - e^{cM}) \psi_2 + \varphi_2^-) \]

\[ - \frac{1}{2r_-} (L_- + M) V_-^{-1} (M(I - e^{cM}) \psi_1 + (I + e^{cM}) \psi_2 + \varphi_2^-) . \]

So, using (24) and (25), the second line of system (32) writes

\[ M \left( P_2^- - P_2^+ \right) \psi_1 + \left( P_3^+ + P_3^- \right) \psi_2 = S_2, \tag{34} \]

where \( S_2 \) is given by (28). Then, due to (33) and (34), \((\psi_1, \psi_2)\) is the expected solution of (23).

Conversely, if (8) holds and system (23) has a unique solution \((\psi_1, \psi_2)\) which satisfies (29), then considering \(u_- \) (respectively \(u_+ \)) the unique classical solution of \((P_-)\) (respectively \((P_+)\)), we get that \(u\) is the unique classical solution of \((P)\). \(\square\)

4.4. Functional calculus. Due to Theorem 4.2, to prove Theorem 3.3, it remains to solve system (23). This will be done by using functional calculus to rewrite the operators defined in (12), (24) and (25) and to inverse the determinant operator of system (23).

To this end, we first recall some classical notations. For \(\theta \in (0, \pi)\), we denote by \(H(S_\theta)\) the space of holomorphic functions on \(S_\theta\) (defined by (3)) with values in \(\mathbb{C}\). Moreover, we consider the following subspace of \(H(S_\theta)\):

\[ \mathcal{E}_\infty(S_\theta) := \{ f \in H(S_\theta) : f = O(|z|^{-s}) \ (|z| \to +\infty) \text{ for some } s > 0 \} . \]

In other words, \(\mathcal{E}_\infty(S_\theta)\) is the space of polynomial decreasing holomorphic functions at \(+\infty\). Let \(T\) be an invertible sectorial operator of angle \(\theta_T \in (0, \pi)\). If \(f \in \mathcal{E}_\infty(S_\theta)\), with \(\theta \in (\theta_T, \pi)\), then we can define, by functional calculus, \(f(T) \in \mathcal{L}(X)\), see [11], p. 45.

Then, we recall a result from [13], section 5.2, Lemma 5.3.

**Lemma 4.3.** Let \(P\) be an invertible sectorial operator in \(X\) with angle \(\theta\), for all \(\theta \in (0, \pi)\). Let \(G \in H(S_\theta)\), for some \(\theta \in (0, \pi)\), such that

1. \(1 - G \in \mathcal{E}_\infty(S_\theta)\),
2. \(G(x) \neq 0\) for any \(x \in \mathbb{R}_+ \setminus \{0\}\).

Then, \(G(P) \in \mathcal{L}(X)\), is invertible with bounded inverse.

Now, in order to apply the previous lemma to inverse the determinant of system (23), we introduce some holomorphic functions and study them on the positive real axis.
Let $r, \delta > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}_-$. We set

\[
\begin{align*}
\begin{cases}
u_{\delta, r}(z) &= 1 - e^{-\delta(\sqrt{z+r} + \sqrt{z})} - \frac{1}{r}(\sqrt{z+r} + \sqrt{z})^2 \left( e^{-\delta \sqrt{z}} - e^{-\delta \sqrt{z+r}} \right) \\
u_{\delta, r}(z) &= 1 - e^{-\delta(\sqrt{z+r} + \sqrt{z})} + \frac{1}{r}(\sqrt{z+r} + \sqrt{z})^2 \left( e^{-\delta \sqrt{z}} - e^{-\delta \sqrt{z+r}} \right),
\end{cases}
\end{align*}
\]

and when $u_{\delta, r}(z) \neq 0$, $v_{\delta, r}(z) \neq 0$

\[
\begin{align*}
\begin{cases}
\begin{align*}
f_{\delta, r, 1}(z) &= (\sqrt{z+r} + \sqrt{z}) \sqrt{z+r} u_{\delta, r}^{-1}(z) \left( 1 + e^{-\delta \sqrt{z}} \right) \left( 1 - e^{-\delta \sqrt{z+r}} \right) \\
&\quad + (\sqrt{z+r} + \sqrt{z}) \sqrt{z+r} v_{\delta, r}^{-1}(z) \left( 1 - e^{-\delta \sqrt{z}} \right) \left( 1 - e^{-\delta \sqrt{z+r}} \right) \\
= - (\sqrt{z+r} + \sqrt{z}) u_{\delta, r}^{-1}(z) \left( 1 - e^{-\delta \sqrt{z}} \right) \left( 1 - e^{-\delta \sqrt{z+r}} \right) \\
&\quad - (\sqrt{z+r} + \sqrt{z}) v_{\delta, r}^{-1}(z) \left( 1 - e^{-\delta \sqrt{z}} \right) \left( 1 - e^{-\delta \sqrt{z+r}} \right)
\end{align*}
\end{cases}
\end{align*}
\]

(35)

\[
\begin{align*}
f_{\delta, r, 2}(z) &= - (\sqrt{z+r} + \sqrt{z}) u_{\delta, r}^{-1}(z) \left( 1 + e^{-\delta \sqrt{z}} \right) \left( 1 - e^{-\delta \sqrt{z+r}} \right) \\
&\quad - (\sqrt{z+r} + \sqrt{z}) v_{\delta, r}^{-1}(z) \left( 1 - e^{-\delta \sqrt{z}} \right) \left( 1 - e^{-\delta \sqrt{z+r}} \right) \\
f_{\delta, r, 3}(z) &= - (\sqrt{z+r} + \sqrt{z}) u_{\delta, r}^{-1}(z) \left( 1 - e^{-\delta \sqrt{z}} \right) \left( 1 - e^{-\delta \sqrt{z+r}} \right) \\
&\quad - (\sqrt{z+r} + \sqrt{z}) v_{\delta, r}^{-1}(z) \left( 1 + e^{-\delta \sqrt{z}} \right) \left( 1 + e^{-\delta \sqrt{z+r}} \right)
\end{align*}
\]

Remark 5. Let $r, \delta, x > 0$. From [13], Lemma 5.2, section 5.2, p. 369, we have $u_{\delta, r}(x) > 0$ and $v_{\delta, r}(x) > 0$. Then, we obtain

\[
\begin{align*}
f_{\delta, r, 1}(x) > 0 \quad \text{and} \quad f_{\delta, r, 2}(x), f_{\delta, r, 3}(x) < 0.
\end{align*}
\]

Moreover, for $z \in \mathbb{C} \setminus \mathbb{R}_-$, we define

\[
g_{\delta, r}(z) = - \sqrt{z+r} \left( 1 - e^{-2\delta(\sqrt{z+r} + \sqrt{z})} \right)^2 - \frac{(\sqrt{z+r} + \sqrt{z})^4}{r^2} \left( e^{-2\delta \sqrt{z}} - e^{-2\delta \sqrt{z+r}} \right)^2
\]

\[
+ \sqrt{z} \left( 1 - e^{-\delta(\sqrt{z+r} + \sqrt{z})} \right)^2 + \frac{1}{r}(\sqrt{z+r} + \sqrt{z})^2 \left( e^{-\delta \sqrt{z}} - e^{-\delta \sqrt{z+r}} \right)^2,
\]

and also, when $u_{\delta, r}(z) \neq 0$, $v_{\delta, r}(z) \neq 0$

\[
h_{\delta, r}(z) = 4(\sqrt{z+r} + \sqrt{z})^2 u_{\delta, r}^{-2}(z) v_{\delta, r}^{-2}(z) g_{\delta, r}(z).
\]

Lemma 4.4. Let $r, \delta, x > 0$. We have

\[
g_{\delta, r}(x) < 0 \quad \text{and} \quad h_{\delta, r}(x) < 0.
\]

Proof. Set $s = \sqrt{x}$ and $t = \sqrt{x+r}$. Then, we have

\[
\begin{align*}
g_{\delta, r}(x) &= - t \left( 1 - e^{-2\delta(s+t)} \right)^2 - \frac{1}{r^2} (s+t)^4 \left( e^{-2\delta s} - e^{-2\delta t} \right)^2 \\
&\quad + s \left( 1 - e^{-\delta(s+t)} \right)^2 + \frac{1}{r} (s+t)^2 \left( e^{-\delta s} - e^{-\delta t} \right)^2 \\
&= - t \left( 1 - e^{-\delta(s+t)} \right)^2 (1 + e^{-\delta(s+t)} )^2 \\
&\quad + \frac{t}{r^2} (s+t)^4 \left( e^{-\delta s} - e^{-\delta t} \right)^2 \left( e^{-\delta s} + e^{-\delta t} \right)^2.
\end{align*}
\]
Furthermore, since $s = \sqrt{x}$ and $t = \sqrt{x + r}$, we have $r = t^2 - s^2 = (t + s)(t - s)$. It follows

$$\frac{(s + t)^2}{r} = \frac{(s + t)}{(t - s)} \quad \text{and} \quad \frac{(s + t)^4}{r^2} = \frac{(s + t)^2}{(t - s)^2}. \quad (36)$$

Then, we have

$$g_{s,r}(x) = \frac{(s + t)^2}{(t - s)^2} \left( e^{-\delta s} - e^{-\delta t} \right)^2 \left( (s + t) \left( e^{-2\delta s} + e^{-2\delta t} \right) + 2(t - s)e^{-\delta(s+t)} \right)$$

$$- \left( 1 - e^{-\delta(s+t)} \right)^2 \left( (t - s)(1 + 2e^{-\delta(s+t)}) + 2(s + t)e^{-\delta(s+t)} \right)$$

$$+ 2 \frac{s(s + t)^2}{(t - s)} \left( 1 - e^{-\delta(s+t)} \right)^2 \left( e^{-\delta s} - e^{-\delta t} \right)^2.$$

Moreover, from [13], Lemma 5.2, we have

$$1 - e^{-\delta(s+t)} - \frac{1}{r} (s + t)^2 \left( e^{-\delta s} - e^{-\delta t} \right) > 0,$$

hence

$$\left( 1 - e^{-\delta(s+t)} \right)^2 > \frac{(s + t)^2}{(t - s)^2} \left( e^{-\delta s} - e^{-\delta t} \right)^2. \quad (36)$$
Then, from (36), we obtain
\[
g_{\delta,r}(x) < \frac{(s + t)^2}{(t - s)^2} \left( e^{-\delta s} - e^{-\delta t} \right)^2 \left( (s + t) \left( e^{-2\delta s} + e^{-2\delta t} \right) + 2(t - s)e^{-\delta(s+t)} \right) \\
- \frac{(s + t)^2}{(t - s)^2} \left( e^{-\delta s} - e^{-\delta t} \right)^2 \left( (t - s)(1 + e^{-2\delta(s+t)}) + 2(s + t)e^{-\delta(s+t)} \right) \\
+ 2 \frac{s(s + t)}{(t - s)} \left( 1 - e^{-\delta(s+t)} \right)^2 \left( e^{-\delta s} - e^{-\delta t} \right)^2
\]
where
\[
g_{\delta,r}(x) = \frac{(s + t)^2}{(t - s)^2} \left( (s + t) \left( e^{-2\delta s} + e^{-2\delta t} \right) + 2(t - s)e^{-\delta(s+t)} \right) \\
- \frac{(s + t)^2}{(t - s)^2} \left( (t - s)(1 + e^{-2\delta(s+t)}) + 2(s + t)e^{-\delta(s+t)} \right) \\
+ 2 \frac{s(s + t)}{(t - s)} \left( 1 - e^{-\delta(s+t)} \right)^2 \left( e^{-\delta s} - e^{-\delta t} \right)^2
\]
Then, from (36), we obtain that \( g_{\delta,r}(x) < 0 \). Finally, we have
\[
g_{\delta,r}(x) < (s + t) \left( e^{-\delta s} - e^{-\delta t} \right)^2 \tilde{g}_{\delta,r}(x) < 0,
\]
from which we deduce that \( h_{\delta,r}(x) < 0 \).

5. Proof of the main result. If (P) has a unique classical solution, from Theorem 4.2, (8) is satisfied.
Conversely, if (8) holds, due to Theorem 4.2, it suffices to prove that system (23) has a unique solution such that (29) holds. The proof is divided in three parts. First, we will make explicit the determinant of system (23). Then, we will show the uniqueness of the solution, to this end, we will inverse the determinant with the help of functional calculus. Finally, we will prove that \( \psi_1 \) and \( \psi_2 \) have the expected regularity.

5.1. **Calculus of the determinant.** Now we have to make explicit the determinant. Recall system (23)

\[
\begin{align*}
(P_1^+ + P_1^-) \psi_1 &+ (P_2^- - P_2^+) \psi_2 = S_1 \\
M(P_2^- - P_2^+) \psi_1 &+ (P_3^+ + P_3^-) \psi_2 = S_2.
\end{align*}
\]

We write the previous system as a matrix equation \( \Lambda \Psi = S \), where

\[
\Lambda = \begin{pmatrix}
P_1^+ + P_1^- & P_2^- - P_2^+ \\
M(P_2^- - P_2^+) & P_3^+ + P_3^-
\end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.
\]

To solve system (23), we will study the determinant

\[
\det(\Lambda) := (P_1^+ + P_1^-)(P_3^+ + P_3^-) - M(P_2^- - P_2^+)^2,
\]

of the matrix \( \Lambda \). We develop it to obtain

\[
\det(\Lambda) = D_1^+ + D_1^- + D_2,
\]

where

\[
D_1^+ = P_1^+P_3^+ - M(P_2^+)^2, \quad D_1^- = P_1^-P_3^- - M(P_2^-)^2,
\]

and

\[
D_2 = P_1^+P_3^- + P_1^-P_3^+ + 2MP_2^+P_2^-.
\]

Using section 3.4, we obtain that

\[
D(\det(\Lambda)) = D(D_1^+ + D_1^- + D_2) = D(M^3),
\]

which justify the equality in (37). In the sequel, we precise the terms \( D_1^+ \) and \( D_1^- \).

**Lemma 5.1.** We have

1. \( D_1^+ = 4k_3^2(L_+ + M)^2U_-^2V_+^{-2}D^+ \), with

\[
D^+ = L_+ \left( I - e^{2d(L_+ + M)} \right)^2 - \frac{1}{r_+^2} (L_+ + M)^2 \left( e^{2dM} - e^{2dL_+} \right)^2.
\]

2. \( D_1^- = 4k_3^2(L_- + M)^2U_+^2V_-^{-2}D^- \), with

\[
D^- = L_- \left( I - e^{2c(L_- + M)} \right)^2 - \frac{1}{r_-^2} (L_- + M)^2 \left( e^{2cM} - e^{2cL_-} \right)^2.
\]

**Proof.** 1. We have

\[
P_1^+P_3^+ = 4k_3^2(L_+ + M)^2L_+U_-^2V_+^{-2}D'_+,
\]

and
where

\[ 4D'_+ = (U^2_+ + V^2_+) (I - e^{2dM}) (I - e^{2dL_+}) + 2U_+V_+ \left[ (I - e^{d(L_+ + M)})^2 + (e^{dM} - e^{dL_+})^2 \right] \]

\[ = (U^2_+ + V^2_+) \left[ (I + e^{d(L_+ + M)})^2 - (e^{dM} + e^{dL_+})^2 \right] + 2U_+V_+ \left[ (I + e^{d(L_+ + M)})^2 + (e^{dM} + e^{dL_+})^2 \right] \]

\[ = (U_+ + V_+)^2 \left[ (I + e^{d(L_+ + M)})^2 - (U_+ - V_+) (e^{dM} + e^{dL_+})^2 \right]. \]

Moreover

\[
\begin{align*}
U_+ + V_+ &= 2 \left( I - e^{d(L_+ + M)} \right) \\
U_+ - V_+ &= -\frac{2}{r_+} (L_+ + M)^2 (e^{dM} - e^{dL_+}) \quad \text{(38)}
\end{align*}
\]

Then

\[ D'_+ = \left( I - e^{d(L_+ + M)} \right)^2 \left( I + e^{d(L_+ + M)} \right)^2 - \frac{1}{r_+^2} (L_+ + M)^4 (e^{dM} - e^{dL_+})^2 (e^{dM} + e^{dL_+})^2. \]

Furthermore, we have

\[ (P^+_2)^2 = 4k_+^2 (L_+ + M)^2 MU_+^{-2} V_+^{-2} D''_+, \]

where

\[ 4D''_+ = \left[ V_+ \left( I + e^{dM} \right) (I - e^{dL_+}) + U_+ \left( I - e^{dM} \right) (I + e^{dL_+}) \right]^2 \]

\[ = V^2_+ \left( I + e^{dM} \right)^2 \left( I - e^{dL_+} \right)^2 + U^2_+ \left( I - e^{dM} \right)^2 \left( I + e^{dL_+} \right)^2 + 2U_+V_+ \left( I - e^{2dM} \right) (I - e^{2dL_+}) \]

\[ = V^2_+ \left[ \left( I - e^{d(L_+ + M)} \right) + (e^{dM} - e^{dL_+}) \right]^2 + U^2_+ \left[ \left( I - e^{d(L_+ + M)} \right) - (e^{dM} - e^{dL_+}) \right]^2 + 2U_+V_+ \left[ (I + e^{d(L_+ + M)})^2 - (e^{dM} + e^{dL_+})^2 \right]. \]

hence

\[ 4D''_+ = V^2_+ \left[ \left( I - e^{d(L_+ + M)} \right)^2 + (e^{dM} - e^{dL_+})^2 \right] + 2V^2_+ \left( I - e^{d(L_+ + M)} \right) (e^{dM} - e^{dL_+}) \]

\[ + U^2_+ \left[ \left( I - e^{d(L_+ + M)} \right)^2 + (e^{dM} - e^{dL_+})^2 \right] - 2U^2_+ \left( I - e^{d(L_+ + M)} \right) (e^{dM} - e^{dL_+}) \]
The determinant of system (23) is invertible with bounded inverse by using functional calculus.

5.2. Inversion of the determinant. In this section, we prove that the determinant of system (23) is invertible with bounded inverse by using functional calculus. From the writing of \( D_1^+ \), \( D_1^- \), given in Lemma 5.1 and the definition of \( D_2 \), we obtain:

\[
D_1^+ = g_1^+ (-A), \quad D_1^- = g_1^- (-A) \quad \text{and} \quad D_2 = g_2(-A),
\]

where we have set, for \( z \in \mathbb{C} \setminus \mathbb{R}_- \):

\[
\begin{align*}
g_1^+(z) &= 4k_2^2 \left( \sqrt{z + r_+ + \sqrt{z}} \right)^2 u_{d,r_+}(-z)v_{d,r_+}(-z)g_{d,r_+}(z) \\
g_1^-(z) &= 4k_2^2 \left( \sqrt{z + r_- + \sqrt{z}} \right)^2 u_{e,r_-}(-z)v_{e,r_-}(-z)g_{e,r_-}(z) \\
g_2(z) &= k_+ f_1^+(z)k_- f_3^-(z) + k_- f_1^-(z)k_+ f_3^+(z) - 2\sqrt{z} k_+ f_2^+(z)k_- f_2^-(z),
\end{align*}
\]

Using again (38), we obtain

\[
D''_+ = \left( I - e^{d(L_+ + M)} \right)^4 + \frac{1}{r_+} (L_+ + M)^4 \left( e^{dM} - e^{dL_+} \right)^4 \\
+ \frac{2}{r_+} (L_+ + M)^2 \left( I - e^{d(L_+ + M)} \right)^2 \left( e^{dM} - e^{dL_+} \right)^2 \\
= \left( I - e^{d(L_+ + M)} \right)^2 + \frac{1}{r_+} (L_+ + M)^2 \left( e^{dM} - e^{dL_+} \right)^2.
\]

Finally, since we have

\[
D_1^+ = P_1^+ P_3^- - M \left( P_2^+ \right)^2,
\]

we obtain

\[
D_1^+ = 4k_2^2 (L_+ + M)^2 U_+^{-2} V_+^{-2} D_+^2,
\]

where \( D_+ = L_+ D_+^2 - M D'_+ \).

2. The result is similarly obtained by replacing respectively \( d \), \( k_+ \), and \( r_+ \) by \( c \), \( k_- \), and \( r_- \) in the proof above.

\( \square \)
(u_{\delta,r}, v_{\delta,r}, g_{\delta,r}, f_i^+ and f_i^- have been defined in section 4.4). So
\[ \det(A) = D_1^+ + D_2^- = f(-A), \]
with \( f = g_1^+ + g_1^- + g_2. \) Note that \( f \in H(S_0) \), for some \( \theta \in (0, \pi) \), moreover from Remark 5 and Lemma 4.4, for \( x > 0 \), we have
\[ f(x) = g_1^+(x) + g_1^-(x) + g_2(x) < 0. \]

Let \( C_1, C_2 \) be linear operators in \( X \). We will write \( C_1 \sim C_2 \) to mean that \( C_1 = C_2 + \Sigma \), where \( \Sigma \) is a finite sum of terms of type \( kL_+^lM_+^m e^{\alpha L_+} e^{\beta L_-} e^{\delta M} \), where \( k \in \mathbb{R}; l, m \in \mathbb{N}; \alpha, \beta, \delta \in \mathbb{R}^+ \) with \( \alpha + \beta + \delta \neq 0 \). Note that \( \Sigma \) is a regular term in the sense:
\[ \Sigma \in \mathcal{L}(X) \quad \text{with} \quad \Sigma(X) \subset (M^\infty) := \bigcap_{k \geq 0} D(M^k). \]

Since \( U_\pm \sim I, V_\pm \sim I \), then setting \( W = U_-U_+V_-V_+ \sim I \), we get that
\[
\begin{align*}
WP_1^+ & \sim 2k_+(L_+ + M)L_+, \quad WP_1^- \sim 2k_-(L_- + M)L_- \\
WP_2^+ & \sim 2k_+(L_+ + M), \quad WP_2^- \sim 2k_-(L_- + M) \\
WP_3^+ & \sim 2k_+(L_+ + M), \quad WP_3^- \sim 2k_-(L_- + M).
\end{align*}
\]
Then
\[
W^2 \det(A) = \left( WP_1^+ WP_3^+ - M(WP_2^+)^2 \right) + \left( WP_1^- WP_3^- - M(WP_2^-)^2 \right) + \left( WP_1^+ WP_3^- + WP_1^- WP_3^+ + 2MW_2^- WP_2^+ \right)
\sim 4k_+^2(L_+ + M)^2(L_+ - M) + 4k_-^2(L_- + M)^2(L_- - M) + 4k_+ k_-(L_+ + M)(L_- + M)(L_+ + L_- + 2M).
\]
Now, due to (7), we have
\[
W^2 \det(A) \sim 4k_+^2 r_+(L_+ + M) + 4k_-^2 r_-(L_- + M) + 4k_+ k_-(L_+ + M)(L_- + M)(L_+ + L_- + 2M).
\]

We set
\[
B = 4(L_+ + M) \left( k_+^2 r_+ + k_-^2 r_-(L_- + M)(L_+ + M)^{1} \right) + 4k_+ k_-(L_+ + M)(L_- + M)(L_+ + L_- + 2M).
\]
Then, we obtain
\[
\det(A) = W^{-2} \left( B + \sum_{j \in J} k_j L_+^{l_j} L_-^{m_j} M^{n_j} e^{\alpha_j L_+} e^{\beta_j L_-} e^{\delta_j M} \right),
\]
where \( J \) is a finite set and for any \( j \in J: \)
\[
k_j \in \mathbb{R}; l_j, m_j, n_j \in \mathbb{N}; \alpha_j, \beta_j, \delta_j \in \mathbb{R}^+ \quad \text{with} \quad \alpha_j + \beta_j + \delta_j \neq 0.
\]

**Lemma 5.2.** Operator \( B \) which is defined above is invertible with bounded inverse.

**Proof.** From \((H_3)\) we have \(-L_+ - L_- - M \in \text{BIP}(X, \theta/2)\) then, from [19], Theorem 5, p. 443, there exists \( \theta' \in (\theta/2, \pi/2) \), such that
\[
-(L_- + M), -(L_+ + M), -(L_+ + M + L_- + M) \in \text{BIP}(X, \theta').
\]
It follows from [19], property (2.7), p. 433, that \(- (L_+ + M)^{-1} \in \text{BIP} (X, \theta')\) and since \(0 \notin \rho(L_- + M)\), from [19], Corollary 3, p. 444, we deduce
\[
\frac{k^2 r_-}{k^2 r_+} (L_- + M)(L_+ + M)^{-1} \in \text{BIP} (X, 2\theta') .
\]
Moreover, from [19], Theorem 3, p. 437, we have
\[
B_1 := k^2 r_+ + k^2 r_-(L_- + M)(L_+ + M)^{-1} \in \text{BIP} (X, 2\theta'),
\]
and \(0 \notin \rho(B_1)\). Finally, from [19], Corollary 3, p. 444, we obtain
\[
B_2 := k_+ k_-(L_- + M)(L_+ + L_- + 2M) \in \text{BIP} (X, 2\theta').
\]
Then, \(B_1 + B_2 \in \text{BIP} (X, \theta'')\), for some \(\theta'' \in [2\theta', \pi)\). Moreover, since \(0 \notin \rho(B_1)\), we deduce from [19], remark at the end of p. 445, that \(0 \notin \rho(B_1 + B_2)\). Since \(4(L_+ + M)\) is invertible, we deduce that \(B = 4(L_+ + M)(B_1 + B_2)\) is invertible with bounded inverse.

From (41) and Lemma 5.2, we deduce that
\[
det(\Lambda) = W^{-2}BF,
\]
with
\[
F = I + \sum_{j \in J} k_j B^{-1} L_j^1 L_j^0 M^1 \alpha_1 L_+ e^\beta_1 L_\rho_\gamma L_\delta M .
\]
For \(z \in \mathbb{C} \setminus \mathbb{R}_-\), we set
\[
\hat{b}(z) = k^2 r_+ + k^2 r_- \frac{\sqrt{z + r_+ + \sqrt{z}}} {\sqrt{z + r_+ + \sqrt{z}}}
+ k_+ k_-(\sqrt{z + r_+ + \sqrt{z}})(\sqrt{z + r_+ + \sqrt{z} + r_- + 2\sqrt{z}}),
\]
(note that, for \(x > 0\), \(\hat{b}(x) > 0\)) and
\[
\hat{f}(z) = 1 + \sum_{j \in J} k_j \hat{b}(z)^{-1} (-\sqrt{z + r_+} L_j^1) (-\sqrt{z + r_-} L_j^0) (-\sqrt{z}) M_j \alpha_j L_+ e^\beta_j L_\rho_\gamma L_\delta M \sqrt{z}.
\]
Then, \(B = 4(L_+ + M)\hat{b}(-A)\) and \(F = \hat{f}(-A)\) and from (39) and (42), we have
\[
f(-A) = det(\Lambda) = W^{-2}B\hat{f}(-A).
\]
By construction, the link between \(f\) and \(\hat{f}\) is
\[
f(z) = -4u_{a,r_+}^2(z)u_{r_+}^2(z)u_{c,r_-}^2(z)u_{c,r_-}^2(z)(\sqrt{z + r_+} + \sqrt{z})\hat{b}(z)\hat{f}(z).
\]

**Proposition 3.** Operator \(F \in \mathcal{L}(X)\) defined above is invertible with bounded inverse.

**Proof.** Note that \(f, \hat{f} \in H(S_0)\), for a given \(\theta \in (0, \pi)\). Moreover, since for \(z \in \mathbb{C} \setminus \mathbb{R}_-\), we have
\[
b_j(z) = k_j \hat{b}^{-1}(z) (-\sqrt{z + r_+} L_j^1) (-\sqrt{z + r_-} L_j^0) (-\sqrt{z}) M_j \alpha_j L_+ e^\beta_j L_\rho_\gamma L_\delta M ,
\]
are polynomial functions, we obtain \(1 - f \in \mathcal{E}_\infty (S_0)\).

From (40), we know that \(f\) do not vanish on \(\mathbb{R}_+ \setminus \{0\}\) and since we have \(u_{d,r_+}, u_{c,r_-}, v_{d,r_+}, v_{c,r_-} > 0\) on \(\mathbb{R}_+ \setminus \{0\}\), we deduce, from (44), that \(f\) do not vanish on \(\mathbb{R}_+ \setminus \{0\}\). Then, applying Lemma 4.3 with \(G = \hat{f}\) and \(P = -A\), we deduce that \(F = f(-A)\) is invertible with bounded inverse. \(\square\)
Finally, we obtain the following result:

**Proposition 4.** Operator \( \det(\Lambda) \) is invertible with bounded inverse.

**Proof.** From (42), Lemma 5.2 and Proposition 3, we have \( \det(\Lambda) = W^{-2}BF \), which is invertible with bounded inverse. \( \square \)

5.3. **Regularity.** From Theorem 4.2, it remains to show that system (23) has a unique solution \((\psi_1, \psi_2)\) satisfying (29). The uniqueness of the solution \((\psi_1, \psi_2)\) is furnished by Proposition 4 and we get

\[
\begin{align*}
\psi_1 &= (P^+_3 + P^-_3) [\det(\Lambda)]^{-1} S_1 - (P^-_2 - P^+_2) [\det(\Lambda)]^{-1} S_2 \\
\psi_2 &= -M (P^-_2 - P^+_2) [\det(\Lambda)]^{-1} S_1 + (P^+_1 + P^-_1) [\det(\Lambda)]^{-1} S_2.
\end{align*}
\]

(45)

To obtain (29), we have first to study \([\det(\Lambda)]^{-1}\).

**Lemma 5.3.** There exists \( R \in \mathcal{L}(X) \) such that

\[
R(X) \subset D(M), \quad [\det(\Lambda)]^{-1} = N^{-1} + N^{-1}R,
\]

where \( N = 4k_+k_- (L_- + M)(L_+ + M)(L_+ + L_- + 2M) \).

**Proof.** From (42), we have

\[
\begin{align*}
\det(\Lambda) &= W^{-2}BF \\
&= NU^{-2}U_+^{-2}V^{-2}V_+^{-2}BN^{-1}F.
\end{align*}
\]

Using (12), (43) and Lemma 5.1 in [13], we get that

\[
F \sim I, \quad U_+^{-1} \sim I, \quad U_-^{-1} \sim I, \quad V_+^{-1} \sim I \quad \text{and} \quad V_-^{-1} \sim I.
\]

So, we have

\[
\det(\Lambda) = NBN^{-1} (I + \Sigma_1),
\]

where \( \Sigma_1 \in \mathcal{L}(X) \) and \( \Sigma_1(X) \subset D(M^\infty) \). But

\[
BN^{-1} = \left( 4k_+^2 r_+ \left( I + \frac{k_+ r_-}{k_+} (L_- + M)(L_+ + M) M^{-3} \right) N^{-1} + 4k_+k_- (L_- + M)(L_+ + M)(L_+ + L_- + 2M) N^{-1} \\
= I + \Sigma_2,
\]

with \( \Sigma_2 \in \mathcal{L}(X) \) and \( \Sigma_2(X) \subset D(M) \). Finally

\[
\det(\Lambda) = N (I + \Sigma_3),
\]

where \( \Sigma_3 \in \mathcal{L}(X) \) and \( \Sigma_3(X) \subset D(M) \). Thus, from Lemma 5.1 in [13], we have

\[
[\det(\Lambda)]^{-1} = N^{-1} (I + \Sigma_3)^{-1}
= N^{-1} (I + R),
\]

with \( R \in \mathcal{L}(X) \) and \( R(X) \subset \Sigma_3(X) \subset D(M) \). \( \square \)

From (26) and (28), we deduce that

\[
\begin{align*}
S_1 &= -k_+ (L_+ + M) (\bar{\varphi}_1^+ + \bar{\varphi}_4^+) - k_- (L_- + M) (\bar{\varphi}_1^- + \bar{\varphi}_3^-) + \tilde{R}_1 \\
S_2 &= k_+ (L_+ + M) (\bar{\varphi}_2^+ + \bar{\varphi}_4^+) - k_- (L_- + M) (\bar{\varphi}_2^- + \bar{\varphi}_3^-) + \tilde{R}_2,
\end{align*}
\]

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where $\tilde{R}_1 \in D(M)$ and $\tilde{R}_2 \in D(M^\infty)$. Using (21), (16), (8), (9), Remark 4 and Remark 3, we obtain that
\[
\tilde{\varphi}_1^+, \tilde{\varphi}_2^-, \tilde{\varphi}_2^+, \tilde{\varphi}_3^+, \tilde{\varphi}_4^+, \tilde{\varphi}_4^- \in (D(M), X)_{2+\frac{1}{p}, p}.
\]
(46)

It follows that $S_1, S_2 \in (D(M), X)_{1+\frac{1}{p}, p}$ and thus
\[
\begin{cases}
[\det(\Lambda)]^{-1} S_1 = N^{-1} (I + R) S_1 \in (D(M), X)_{4+\frac{1}{p}, p} \\
[\det(\Lambda)]^{-1} S_2 = N^{-1} (I + R) S_2 \in (D(M), X)_{4+\frac{1}{p}, p}.
\end{cases}
\]
(47)

Moreover, from (45), we have
\[
\begin{align*}
\psi_1 &= 2 \left( k_+(L_+ + M) + k_- (L_- + M) \right) [\det(\Lambda)]^{-1} S_1 \\
&+ 2 \left( k_+ (L_+ + M) - k_- (L_- + M) \right) [\det(\Lambda)]^{-1} S_2 + \tilde{S}_1 \\
\psi_2 &= 2 \left( k_+(L_+ + M) - k_- (L_- + M) \right) [\det(\Lambda)]^{-1} S_1 \\
&+ 2 \left( k_+ (L_+ + M) L_+ + k_- (L_- + M) L_- \right) [\det(\Lambda)]^{-1} S_2 + \tilde{S}_2,
\end{align*}
\]
(48)

where $\tilde{S}_1, \tilde{S}_2 \in D(M^\infty)$. Finally, (47), (48) and (9) gives
\[
\begin{align*}
\psi_1 &\in (D(M), X)_{3+\frac{1}{p}, p} = (D(A), X)_{1+\frac{1}{p}, p}, \\
\psi_2 &\in (D(M), X)_{2+\frac{1}{p}, p} = (D(A), X)_{1+\frac{1}{p}, p}.
\end{align*}
\]

**Acknowledgments.** We would like to thank the referee for his useful remarks.

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Received September 2018; revised October 2018.

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