Social Influence as a Voting System: a Complexity Analysis of Parameters and Properties

Xavier Molinero¹, Fabián Riquelme², and Maria Serna²

¹ Dept. of Applied Mathematics III. UPC, Manresa, Spain.
² Dept. de Llenguatges i Sistemes Informàtics. UPC, Barcelona, Spain.
xavier.molinero@upc.edu, {farisori,mjserna}@lsi.upc.edu

Abstract. In this paper we propose a new way to analyze influence in networks. There is a common perception that influence is relevant to determine the global behavior of the social network and thus it can be used to determine some normative behavioral rules. From this point of view we propose to analyze influence in networks as a voting systems. For doing so we introduce a new family of simple games: the influence games whose definition is based on the linear threshold model for influence propagation. We show first that any simple game can be described by an influence game and concentrate in the cases in which such construction can be done in polynomial time. We study computational problems related to parameters and properties of simple games. We characterize the computational complexity of various problems for general influence graphs. Finally, we analyze those problems for some particular extremal cases, with respect to the propagation of influence, showing tighter complexity characterizations.

Keywords: Spread of influence, Simple games, Influence games, Computational complexity

1 Introduction

The ways in which people influence each other through their interactions in a social network has received a lot of attention in the last decade. From the point of view of influence propagation a social network can be represented by a graph where each node is an individual, agent or player, and each edge represents the degree of influence of one individual over another individual. Several “motivations” (ideas, trends, fashions, ambitions, rules, etc.) can be initiated by one or more players and eventually be adopted by the system. Social networks have become a huge interdisciplinary research area with important links to sociology, economics, epidemiology, computer science, and mathematics [28,12] (players face the choice of adopting a specific product or not; users choose among competing programs from providers of mobile telephones, having the option to adopt more than one product at an extra cost; etc.). The problem of analyzing how these motivations are propagated within the network, from the influence of only some nodes initially motivated, is called the influence spread problem. A node is active or inactive, depending on whether it has “motivated” or not.

Motivated by viral marketing and other applications the problem that has been usually studied is the influence maximization problem initially introduced by Domingos and Richardson [11,14] and further developed in [30,13]. The problem addresses the question of finding a set of players with a given size that is more influential in the social network, in the sense that after their activation, they could activate as many players as possible. Two general models for spread of influence were defined in [30]. The first one is the linear threshold model, based in the first ideas of [23,43], and the second one is the independent cascade model, created in the context of marketing by [20,21]. The influence maximization problem is NP-hard [11], unless additional restrictions are considered, in which case some generality of the problem is lost [11].

Models of influence spreading in the presence of multiple competing products has also been proposed and analyzed [5,7]. In such a setting there is also work done towards analyzing the problem from the
point of view of non-cooperative game theory. Non-cooperative influence games were defined in 2011 by Irfan and Ortiz [25]. Those games analyze the strategic aspects of two firms competing on the social network and differ from our proposal.

Our objective is to analyze influence spreading from the point of view of decision or voting systems that are mathematically modeled as simple games. Simple games were firstly introduced in 1944 by von Neumann and Morgenstern [38] as a fundamental model for social choice. There are some useful parameters and properties that play a relevant role in the study of simple games. Among those we consider two parameters length and width that are indicators of efficiency for making a decision [38]. The properties defining proper, strong and decisive games have been considered in the context of simple game theory from its origins [47]. Besides those properties we also consider equivalence and isomorphism. Together with properties of the games there are several properties associated to players that are of interest. Among others we consider the critical players which were used at least since 1965 by Banzhaf [4], to describe his popular power index that measures the voting power of players in a simple game. We refer the reader to [47] for a more complete motivation in a viewpoint of simple games and to [2,8] for computational aspects of simple games and in general of cooperative game theory.

To define an influence game in the context of simple games we follow the usual definition of threshold games associated to a set measure [3] (see also [2]). For doing so we take the spread of influence as the value that measures the voting power of a coalition. An influence game is described by an influence graph, modeling a social network, and a quota, indicating the required size of the population to establish a social rule. In an influence game a coalition will be winning if it can influence at least as many individuals as the quota establishes. Such approach reveals the importance of the influence between some players over others in order to form winning coalitions. In this first analysis, we draw upon the deterministic version of the linear threshold model, in which node threshold are fixed, as our model for influence spreading following [9,1]. It will be of interest to analyze influence games under other spreading models in particular in the linear threshold model with random thresholds.

Our results can be summarized as follows. We first show that unweighted influence games capture the complete family of simple games. Although the construction can be computed in polynomial time when the simple game is given in extensive winning or minimal winning form, the number of participants is, in general, exponential in the number of players. Interestingly enough the formalization as weighted influence games allows to implement in polynomial time the operations of intersection and union of weighted simple games thus showing that in several cases simple games that do not admit a succinct representation as weighted games can be represented succinctly as influence games because its codimension is small. As a consequence of the construction and several known results we can set the computational complexity of the above mentioned properties for parameters for influence games in which either the number of participants or the weights are exponential with respect to the number of players. Interestingly enough, in our characterization we make use of a parameter, the minimum size $k$ for which all coalitions with $k$ members are winning, that turns out to be useful to show that the width of a simple game given in extensive winning or minimal winning form can be computed in polynomial time. This settles an open problem from [2].

Our second set of results settles the complexity of the problems related to parameters and properties. Hardness results are obtained for unweighted influence games in which the size of the network is polynomial in the number of players while polynomial algorithms are devised for general influence games. We show that, the problems IsDummy, AreSymmetric, IsProper, IsStrong, IsDecisive and AreEquiv are coNP-complete while AreIso is coNP-hard. Also computing the width, the Banzhaf value and the Shapley-Shubik value are NP-hard. Finally we show that the problems, IsPasser, IsVetoer, IsDictator, IsCritical, IsBlocking and IsSwing belong to P.

Finally we consider two extreme cases of influence spreading in social networks for undirected and unweighted influence graphs. In a maximum influence requirement individuals adopt a behavior only when all its peers have already adopted it. This is opposed to a minimum influence requirement in which an individual get convinced when one of its peers do. In both cases the problems IsProper, IsStrong and IsDecisive as well as computing the Width have polynomial time algorithms. Computing the Length is NP-hard for maximum influence an polynomial for minimum influence.

There are many open lines for future research. On the general topic of influence spreading it will be of interest to analyze the properties of the influence games defined through other influence spreading mechanisms. For the influence games introduced in this paper we have analyzed some extreme situations with respect to the spread of influence following this direction there are many other natural rules,
for example the majority rule when individuals are convinced when a majority of their neighbors are. Also, analyzing the properties of particular types of graphs arising in social networks or other type of organizations in which influence spreading can take part in the decision process.

As we have shown influence games allow capture the family of simple games. As we have shown both, games with low dimension or codimension can be represented by influence games with polynomial size. It remains open to show whether this is true for all the simple games, however the usual construction showing the existence of a simple games with high dimension [17] can be seen to have a short description as influence game.

From the point of view of complexity we have left two main open question: determine the computational complexity of computing the Width of an influence game and the exact complexity of the AREISO problem, although we have shown that the problem is coNP-hard the problems belongs to $\Sigma_2^P$ and could be $\Sigma_2^P$-complete.

2 Definitions and Preliminaries

Simple games were firstly introduced in 1944 by von Neumann and Morgenstern [18], but using a definition that corresponds to the nowadays called strong games, which will be define later. The first definition of actual simple games was given in 1953 by Gillies [19].

From now on, we essentially follow definitions and notations from [47]. As usual, given a finite set $N$, $\mathcal{P}(N)$ denotes its power set, and $n$ its cardinality, i.e., $n := |N|$. A family of subsets $\mathcal{W} \subseteq \mathcal{P}(N)$ is monotonic when $\forall X \in \mathcal{W}$, if $X \subseteq Z$, then $Z \in \mathcal{W}$.

**Definition 1.** A simple game is a tuple $\Gamma := (N, \mathcal{W})$ where $N$ is a finite set of players and $\mathcal{W}$ is a monotonic family of subsets of $N$ formed by the winning coalitions.

In the context of simple games, the subsets of $N$ are called coalitions, $N$ is the grand coalition and $X \in \mathcal{W}$ is a winning coalition. The complement of $\mathcal{W}$, $\mathcal{L} := \neg \mathcal{W} = \{X \subseteq N ; X \notin \mathcal{W}\}$, is the set of losing coalitions.

The games $(N, \emptyset)$ and $(N, \mathcal{P}(N))$ are the trivial simple games. While some authors define the set of simple games excluding the trivial ones, in this paper $\mathcal{W}$ can be empty $\emptyset$ or the power set $\mathcal{P}(N)$, unless otherwise stated.

Other relevant sets associated to a simple game are the family of minimal winning coalitions: $\mathcal{W}^m := \{X \in \mathcal{W} ; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$ which is the set of those winning coalitions such that removing any of its players, the coalition is no longer winning, and the family of maximal losing coalitions: $\mathcal{L}^M := \{Y \in \mathcal{L} ; \forall Z \in \mathcal{L}, Y \not\subseteq Z\}$ which is the subset of losing coalitions for which adding a player, the coalition is no longer losing, i.e. results a winning coalition.

Any of the sets $\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$ or $\mathcal{L}^M$ determine uniquely the game $\gamma$ and constitute the usual forms of representation for simple games (see [47]) although the size of the representation is not, in general, polynomial in the number of players.

There are some parameters and properties that play a relevant role in the study of simple games.

**Definition 2.** Let $\Gamma := (N, \mathcal{W})$ be a simple game.

- The length of $\Gamma$ is $\min\{|S| ; S \in \mathcal{W}\}$.
- The width of $\Gamma$ is $\min\{|S| ; N \setminus S \in \mathcal{L}\}$.
- $\Gamma$ is proper, if $\forall S \subseteq N$, $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$; or improper, otherwise.
- $\Gamma$ is strong, if $\forall S \subseteq N$, $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$; or weak, otherwise.
- $\Gamma$ is decisive, if it is both strong and proper, i.e., $S \in \mathcal{W}$ iff $N \setminus S \notin \mathcal{W}$.

The parameters length and width were introduced in 1990 by Ramamurthy [38], as indicators of efficiency for making a decision. Proper, strong and decisive games have been considered in the context of simple game theory from its origins [17]. For instance, non-proper simple games verify that disjoint winning coalitions can allow contradictory decisions to be made by a voting system [14]. As well as non-strong simple games verify that disjoint losing coalitions can allow issues unresolved [38]. Furthermore, decisive games has application in several areas, such as interactive decision making, distributed computing, logic and linear programming, category theory, social science, hypergraph theory and reliability theory [34].

Together with properties of the games there are several properties associated to players that are of interest.
Definition 3. Let $\Gamma := (N, W)$ be a simple game and $i \in N$ a player.

- $i$ is a dummy if $i \in S \in W$ implies $S \setminus \{i\} \in W$.
- $i$ is a passer if $i \in S$ implies $S \in W$.
- $i$ is a vetoer if $i \notin S$ implies $S \notin W$.
- $i$ is a dictator if $i \in S \iff S \in W$, i.e., $i$ is passer and vetoer.
- Given a coalition $S \subseteq N$, $i$ is critical in $S$ if $S \in W$ and $S \setminus \{i\} \notin W$.
- Given a player $j \in N$, $i$ and $j$ are symmetric if $S \cup \{i\} \in W \iff S \cup \{j\} \in W$.

Dummies were defined in 1944 by von Neumann and Morgenstern [38] and the symmetry of players in 1966 by Maschler and Peleg [32]. Passers, vetoers and dictators are usually used to decide some problems related with solution concepts. The critical property is used at least since 1965 by Banzhaf [4], to describe his popular power index, used to measure the voting power of players in a simple game.

Definition 4. Let $\Gamma := (N, W)$ be a simple game and $S \subseteq N$ be a coalition.

- $S$ is a blocking coalition if $N \setminus S$ is losing.
- $S$ is a swing if $\exists i \in S$ such that $i$ is critical.

The blocking property was firstly defined in 1956 by Richardson [40], as a way to simplify the notation of simple games given in 1953 by Gillies [19], which includes weak games, unlike the original definition appears in 1944 [38]. The swing is a property which emerged from the definition of critical [4].

Definition 5. Let $\Gamma := (N, W)$ be a simple game, and $C_i$ be the set of coalitions where $i$ is critical, i.e.,

$$C_i := \{S \in W ; S \setminus \{i\} \notin W\}.$$

- The Banzhaf value of $i$ on $\Gamma$ is $\eta_i(\Gamma) := |C_i|$.
- The Shapley-Shubik value of $i$ on $\Gamma$ is $\kappa_i(\Gamma) := \sum_{S \in C_i} (|S|! (n - |S| - 1)!)$.

Below we introduce the relevant subfamily of simple games: the weighted games defined in 1944 by von Neumann and Morgenstern [38]. But a similar concept was used one year before by McCulloch and Pitts [34] to define the Threshold Logic Unit (TLU), the first artificial neuron. Some years later, they were deeply studied in 1956 by Isbell [26] in the context of simple game theory, and since then weighted games have been studied in many different contexts under different names, like linearly separated truth function [35] —to contact and to rectifier nets—, linearly separable switching function or threshold Boolean functions [24] —to separate circuits in switching circuit theory and analyze the threshold synthesis problem—, trade robustness [45] —for voting theory and trade exchanges— or threshold hypergraphs [24, 46, 49] —to synchronizing parallel processes—, changing sometimes the name of weight function by threshold criteria.

Definition 6. A simple game $\Gamma := (N, W)$ is a weighted game if there exists a weight function $w : N \to \mathbb{R}^+$ and a real quota $q \in \mathbb{R}$ such that $\forall X \subseteq N, X \in W \iff w(X) \geq q; \ where \ w(X) := \sum_{i \in X} w(i)$.

Alternatively, a weighted game can be represented by a weighted representation, a vector $[q; w_1, \ldots, w_n]$ where $w_i := w(i)$, for any $i \in N$, and $q$ defines a quota, defining a simple game in which $S \in W \iff w(S) \geq q$. According to Hu [24] —see also [15] —the representation can be restricted to integer non-negative weights, i.e., $0 \leq q \leq w(N)$.

The intersection of two simple games is the simple game where a coalition wins if and only if it wins in both games. In a similar way, the union of two simple games is the simple game where a coalition wins if and only if it wins in at least one of the two games. Despite of the fact that weighted games are a strict subclass of simple games, it is known that every simple game can be expressed as an intersection or an union of a finite number of weighted games. The result for intersection (dimension concept) was firstly shown in [29] for hypergraphs, and then expressed for simple games in [40]. The result for union (codimension concept) was introduced for simple games in [27].

Definition 7. A simple game is said to be of dimension (codimension) $k$ if and only if it can be represented as the intersection (union) of exactly $k$-weighted games, but not as the intersection (union) of $(k - 1)$-weighted games.
In this context, it is known that given \( k \) weighted games, decide whether the dimension of their intersection exactly equals \( k \) is \( \mathbb{NP} \)-hard [10].

In this paper, we are interested in analyzing the computation complexity to determine properties and to compute parameters for the family of influence games. We use the following notation for the problems to be considered in what follows:

\[ \text{X:} \quad \text{Given a simple game } \Gamma. \text{ Compute } X(\Gamma), \text{i.e., the parameter } X \text{ for } \Gamma. \]

\[ \text{IsY:} \quad \text{Given a simple game } \Gamma. \text{ Does } \Gamma \text{ satisfy property } Y? \]

In general we extend the notation \( \text{IsY} \) for the problem of deciding a property \( Y \) for games, players or coalitions, considering an input formed by a simple game and players and/or coalitions. We summarize some known computational complexity results (notation of [13]) over parameters and properties in Theorems 1 and 2 [14,33,10,37,2,42,16].

**Theorem 1.** For a simple game \( \Gamma \):

- Given by \((N, W)\) or \((N, W^m)\): \( \text{Lenght, IsDummy, IsPasser, IsVetoer, IsDictator, IsCritical, IsSymmetric, IsBlocking and IsSwing} \) belong to \( \mathbb{P} \). Furthermore, \( \text{Dimension} \) is \( \mathbb{NP} \)-hard.
- Given by \((N, W)\): \( \text{IsProper, IsStrong and IsDecisive} \) belong to \( \mathbb{P} \).
- Given by \((N, W^m)\): \( \text{IsProper} \) belongs to \( \mathbb{P} \), \( \text{IsStrong} \) belongs to \( \text{coNP} \)-complete and \( \text{IsDecisive} \) can be solved in \( \text{quasi-polynomial} \) time.

**Theorem 2.** For a weighted game \( \Gamma = [q; w_1, \ldots, w_n] \):

- \( \text{Lenght, Width, IsPasser, IsVetoer, IsDictator, IsCritical, IsBlocking and IsSwing} \) belong to \( \mathbb{P} \).
- \( \text{IsDummy, IsSymmetric, IsProper, IsStrong and IsDecisive} \) belong to \( \text{coNP} \)-complete.

**Observation 1** Note that for the time being as our knowledge, \( \text{Width} \) remains still open for simple games, even though the simple game is given by \((N, W)\) or \((N, W^m)\).

### 3 Influence Games

The following definitions are based on the linear threshold model [23,13] for spread of influence. We use standard graph notation following [6].

**Definition 8.** An influence graph is a tuple \((G, w, f)\), where \( G := (V, E) \) is a labeled and directed graph (without loops) _—_with \( V \) its set of vertices and \( E \) its set of edges— \( w : E \to \mathbb{N} \) is a weight function and \( f : V \to \mathbb{N} \) is a labeling function that quantify how influenceable each player is. A player \( i \in V \) has influence over another \( j \in V \) iff \((i, j) \in E \). A function \( X : V \to \{0, 1\} \) represents the initial activation of the graph, where \( X(i) = 0 \) (resp. \( X(i) = 1 \)) means that the player \( i \) is active (resp. inactive). Moreover, the size of an influence graph is its number of edges.

Given an influence graph \((G, w, f)\) and an initial activation set \( X \subseteq V \), the spread of influence is denoted by \( F(X) \subseteq V \). \( F(X) \) is obtained by an iterative process in which initially the vertices in \( X \) are activated. At each step those vertices whose influences (sum of weights of nodes connected with them) exceed their label functions (powers of conviction) set activated. The process stops when no additional activation occurs.

We also consider the family of _unweighted influence graphs_ \((G, f)\) in which the weighted function is \( w : E \to \{1\} \).

**Example 1.** Figure 1 shows the spread of influence \( F(X) \) in an unweighted influence graph for the initial activation \( X = \{a\} \). In the first step we obtain \( F(X) = \{a, c\} \), and in the second step (the last one) we obtain \( F(X) = \{a, c, d\} \).

**Observation 2** All results for directed graphs can be applied to undirected graphs.
Next, we introduce the definition of influence game associating a simple game (social choice) to an influence graph (networks).

**Definition 9.** An influence game is a tuple $\Gamma = (G, w, f, q, N)$ where $(G, w, f)$ is an influence graph, $q$ is a quota $0 \leq q \leq |V|$ and $N \subseteq V$ is the set of players. $X \subseteq N$ is a winning coalition if $|F(X)| \geq q$, otherwise $X$ is a loosing coalition.

As it was done for influence graphs, we also consider the family of unweighted influence games in which each edge has weight 1. In such a case we use the notation $\Gamma = (G, f, q, N)$. Note that if $q = 0$, as $F(\emptyset) = \emptyset$, then it is not necessary to activate any player in order to fulfill the condition, i.e., every coalition wins ($\forall X \subseteq N$, then $|F(X)| \geq 0$). Therefore, every influence game $(G, f, 0, N)$ is equivalent to the simple game $(N, \mathcal{P}(N))$. Similarly, every influence game $(G, f, q, N)$ where $q > |V|$ is equivalent to the simple game $(N, \emptyset)$.

Influence games are monotonic because of for any $X \subseteq N$, if $|F(X)| \geq q$ then $|F(X \cup \{i\})| \geq q$, and if $|F(X)| < q$ then $|F(X \setminus \{i\})| < q$. Thus, it is clear that every influence game is a simple game. Moreover, we show that the opposite is also true.

**Theorem 3.** Every simple game can be represented as an unweighted influence game. The construction can be done in polynomial time if the game is given as $(N, W)$ or $(N, W^m)$.

**Proof.** Given a simple game by $(N, W)$ or $(N, W^m)$, first compute $q_{\text{min}} := \min\{k \in \mathbb{N} : \forall S \subseteq N \text{ such that } |S| = k, S \in W\}$ in polynomial time. For the construction of unweighted influence graph, $\forall X \in W^m$, add $q_{\text{min}} - |X|$ nodes with label $|X|$ (i.e., for each new node $j$, then to assign $f(j) := |X|$) and connecting every $i \in X$ to these new nodes. Then $(G, f, q_{\text{min}}, N)$ is the unweighted influence graph of the given simple game.

The following example provides an illustration of the construction.

**Example 2.** Let $\Gamma := (N, W^m)$ be a simple game with $N = \{1, 2, 3, 4\}$ and $W^m = \{\{1, 2, 4\}, \{2, 3\}, \{3, 4\}\}$. Then $q_{\text{min}} = 3$ because of all subsets of $N$ with cardinality 3 are winning, i.e., $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \in W$. For coalition $\{1, 2, 4\}$ we do not need to add nodes, and for both $\{2, 3\}$ and $\{3, 4\}$ we need to add one node with label $3 - 2 = 1$. The resulting unweighted influence graph appears in Figure 2.

![Fig. 1. Spread of influence (colored nodes) for the initial activation $X = \{a\}$.](image1.png)

![Fig. 2. Unweighted influence game $(G, f, q_{\text{min}}, N)$ of the simple game $\Gamma := (N, W)$ with $N = \{1, 2, 3, 4\}$ and $W^m = \{\{1, 2, 4\}, \{2, 3\}, \{3, 4\}\}$](image2.png)
The proof of Theorem 3 shows the completeness of the family of influence games with respect to the class of simple games. However, it cannot be implemented in polynomial time when the simple game is given in a more succinct way as a weighted representation or by means of a monotonic boolean function. Furthermore, the size of the corresponding influence game is in general exponential in the number of players.

For the particular case of weighted games we have an additional result.

**Theorem 4.** Every weighted game can be represented as an influence game. Furthermore, the construction can be done in polynomial time.

**Proof.** Let \([q; w_1, \ldots, w_n]\) be a weighted game, consider the influence game \((G, w, f, n, N)\) of Figure 3.

![Figure 3](image1)

Fig. 3. Influence game \((G, w, f, n, N)\) of a given weighted game \([q; w_1, \ldots, w_n]\).

The \(n\) nodes on the first level correspond to the set \(N\), each of them with weight equal to 1. Each of these nodes \(i\) is connected to another node with weight \(q\), assuming that the corresponding vertex has labeling \(w_i\). Thus, \(X \subseteq N\) is a winning coalition iff \(\sum_{i \in X} w_i \geq q\) iff \(|F(X)| \geq n + 1\).

In this last construction the size of the influence graph is polynomial in the number of players, however the weights can be large. In fact, it is known that \(\max_{i \in N} \{w_i\}\) is at most \((n+1)^{(n+1)/2}/2\). Thus, nevertheless every weighted game can also be represented as an unweighted influence game, the representation can be computed in pseudo-polynomial time.

**Theorem 5.** Every weighted game can be represented as an unweighted influence game. Moreover, such construction is done in pseudo-polynomial time.

**Proof.** Let \([q; w_1, \ldots, w_n]\) be a weighted game. The corresponding unweighted influence graph \((G, f, n + \sum_{i=1}^{n} w_i, N)\) is sketched in Figure 4.

![Figure 4](image2)

Fig. 4. Unweighted influence game \((G, f, n + \sum_{i=1}^{n} w_i, N)\) of a given weighted game \([q; w_1, \ldots, w_n]\).
The $n$ nodes on the first level correspond to the set $N$. Each of these nodes $i$ are connected to $w_i$ nodes on the second level, which represent their respective weights; so, $X \subseteq N$ is a winning coalition iff $\sum_{i \in X} w_i \geq q$ iff $|F(X)| \geq n + \sum_{i=1}^{n} w_i$. Therefore, the influence game $(G, f, n + \sum_{i=1}^{n} w_i, N)$ corresponds to the given weighted game.

It is possible to devise a construction representing intersection or union of weighted games as influence games (see Figures 5 and 6).

Fig. 5. Influence game $(G, w, f, n + 2, N)$ of intersection of two weighted games $[q^{(1)}; w_1^{(1)}, \ldots, w_n^{(1)}] \cap [q^{(2)}; w_1^{(2)}, \ldots, w_n^{(2)}]$.

Fig. 6. Influence game $(G, w, f, n+1, N)$ of union of two weighted games $[q^{(1)}; w_1^{(1)}, \ldots, w_n^{(1)}] \cup [q^{(2)}; w_1^{(2)}, \ldots, w_n^{(2)}]$.

Thus, as any simple game can be represented as the intersection or union (remind the dimension and codimension concepts) of a finite number of weighted games, we have an alternative way to show the completeness of the family of influence games with respect to the class of simple games (Theorem 3). However, as the dimension and the codimension of a simple game might be exponential in the number of players (but bounded by the number of maximal losing and minimal winning coalitions, respectively [17,27]), we are not always getting a representation in which the number of vertices is polynomial in the number of players.

From Theorem 3 and 4 we know that all computational problems related to properties and parameters that are computationally hard for simple games in winning or minimal winning form and for weighted games are also computationally hard for influence games (see Theorems 1 and 2). Nevertheless, the hardness results do not apply to unweighted influence games with polynomial number of vertices.

4 The Complexity of Parameters and Properties

In this section we address the computational complexity of the parameters and properties introduced in Section 2 for influence games. All the hardness proofs are given for the simpler subclass formed by unweighted influence games on undirected influence graphs, which is a subset of all other classes. The
polynomial time algorithms are devised for the biggest class of general influence games, i.e. weighted influence games on directed graphs, that includes all others.

For the hardness result we provide polynomial time reduction from problems based in variations of the vertex cover problem which is known to be NP-hard [13]. The vertex cover problem is stated as:

**Vertex Cover:** Given an undirected graph \( G \) and an integer \( k \).

Is there a vertex subset \( S \) with at \( k \) or less vertices such that each edge in \( G \) has at least one end point in \( S \)?

In our reductions we use basic constructions associating an unweighted influence game to an undirected graph \( G = (V, E) \) and an integer \( k \).

The game \( \Gamma(G) \) is the unweighted influence game \( (G, f, |V|, V) \) where, for any \( v \in V \), we have that \( f(v) \) is the degree of \( v \) in \( G \), i.e., \( f(v) := d_G(v) \).

The game \( \Delta(G, k) \) is the unweighted influence game \( (G', f, q, N) \) defined as follows (Figure 7 shows the corresponding influence graph). Assume that \( V = \{v_1, \ldots, v_n\} \) and \( E = \{e_1, \ldots, e_m\} \). The graph \( G' \)

![Influence graph \( \Delta(G, k) \).](image)

has as vertex set \( V' \) the vertices in \( V \), the edges in \( E \), three new vertices \( x, y, z \) and \( \alpha := m + n + 4 \) additional new vertices, that is

\[
V' = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\}.
\]

The edge set \( E' \) is constructed as follows:

- For any \( e = (v_i, v_j) \in E \) we add the edges \( (e, v_i) \), \( (e, v_j) \) and \( (e, y) \).
- For any \( i, 1 \leq i \leq n \), we add the edge \( (v_i, x) \).
- For any \( j, 1 \leq j \leq m \) we add the edges \( (x, s_i) \) and \( (y, s_i) \).
- Finally, we add the edge \( (z, y) \).

The labeling function \( f \) is the following:

- \( f(v_i) = m + 2, 1 \leq i \leq n \).
- \( f(e_j) = 1, 1 \leq j \leq m \).
- \( f(s_\ell) = 1, 1 \leq \ell \leq \alpha \).
- \( f(z) = 2, f(x) = k + 1, f(y) = m + 1 \).

The quota is defined as \( q := \alpha \) and the set of players is \( N := \{v_1, \ldots, v_n, z\} \).

Observe that by construction a description of the games \( \Gamma(G) \) or \( \Delta(G, k) \) can be obtained in polynomial time. The following result is an immediate consequence of the definitions.

**Lemma 1.** Let \( G = (V, E) \) be an undirected graph. Then we have \( X \) is a winning coalition in \( \Gamma(G) \) iff \( X \) is a vertex in \( G \).
As a consequence of the previous lemma we have that the length of \( I'(G) \) coincides with the size of the minimum vertex cover of \( G \). Moreover, as the minimum vertex cover problem is NP-hard \cite{18} and \( I'(G) \) can be computed in polynomial time we have:

**Theorem 6.** Computing the length of an unweighted influence game is \( \text{NP} \)-hard.

Even though Observation \cite{1} as the width is equal to \( n - (q_{\text{min}} - 1) \) we have:

**Theorem 7.** Given a simple game by \( (N, W) \) or \( (N, W^m) \), width belongs to \( \text{P} \).

The complexity of computing \( q_{\text{min}} \) or width for influence games remains open. Below we present the result for a dummy player.

**Theorem 8.** The \( \text{IsDummy} \) problem for unweighted influence games with polynomial number of vertices belongs to \( \text{coNP} \)-complete.

**Proof.** Membership in coNP follows from the definitions. We provide a reduction form the complement of vertex cover. Let \((G, k)\) be a vertex cover instance. Consider the pair \((\Delta(G), z)\) which is an instance of the IsDummy problem computable in polynomial time.

If \( G \) has a vertex cover \( X \) with size \( k \) or less, by construction we have that \( X \cup \{z\} \) is a winning coalition of \( \Delta(G) \). Furthermore if \( X \) is a vertex cover of minimum size, we have that \( X \cup \{z\} \) is a minimal winning coalition. Therefore, \( z \) is not a dummy player in \( G \).

If \( G \) does not have a vertex cover with size \( k \) or less and \( X \) is a winning coalition containing \( z \), it must hold that \( |X \setminus \{z\}| > k \), therefore \( X \setminus \{z\} \) is a winning coalition. In consequence \( z \) is a dummy player in \( \Delta(G) \).

The remaining results can be summarized in the following theorem.

**Theorem 9.** The problems \( \text{AreSymmetric}, \text{IsProper}, \text{IsStrong} \) and \( \text{IsDecisive} \) belong to \( \text{coNP} \)-complete for unweighted influence games with polynomial number of vertices. The problems \( \text{IsPasser}, \text{IsVetoer}, \text{IsDictator}, \text{IsCritical}, \text{IsBlocking} \) and \( \text{IsSwing} \) for influence games belong to \( \text{P} \).

We split the proof of Theorem 8 in a series of four lemmas.

**Lemma 2.** The \( \text{AreSymmetric} \) problem for unweighted influence games with polynomial number of vertices belongs to \( \text{coNP} \)-complete.

**Proof.** As before the problem belongs trivially top \( \text{coNP} \). To get the hardness we consider a small variation of the previous construction. Given a graph \( G = (V, E) \) with \( n \) vertices and \( m \) edges and an integer \( k \), we define the unweighted influence game \( \Delta(G) = (G'', f, q, N) \). \( G'' \) is obtained from the graph \( G' \) appearing in the construction of \( \Delta(G) \), recall that

\[
V(G') = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_n\},
\]

adding two new vertices \( t \) and \( s \) and the edges \((x, s), (y, s)\) and \((t, s)\). The thresholds for all vertices in \( V(G') \) are the same as in \( \Delta(G) \) and we set \( f(s) = 4 \) and \( f(t) = 2 \). Set \( q := \alpha + 1 = n + m + 5 \) and, with abuse of notation, \( N := \{v_1, \ldots, v_n, z, t\} \).

When \( G \) has a vertex cover \( X \) of size \( k \) or less, by construction the coalition \( X \cup \{z\} \) is winning in \( \Delta'(G) \) while the coalition \( X \cup \{t\} \) is a loosing coalition. Therefore \( z \) and \( t \) are not symmetric.

When \( G \) does not have a vertex cover \( X \) of size \( k \) or less, by construction any winning coalition \( Y \) must contain a subset with at least \( k + 1 \) vertices from \( \{v_1, \ldots, v_n\} \). Therefore both \( Y \cup \{z\} \) and \( Y \cup \{t\} \) are winning coalitions in \( \Delta'(G) \), i.e., vertices \( z \) and \( t \) are not symmetric.

Given \((G, k)\) we can obtain \((\Delta'(G), z, t)\) in polynomial time and the theorem follows.

For the following results we need to prove first that a variant of the vertex cover (called half vertex cover) problem is \( \text{NP} \)-complete. Consider the following problem

**Half-Vertex Cover:** Given an undirected graph \( G \) with an odd number of vertices.

Does \( G \) have a vertex cover of \( G \) with at \( \lceil |V| - 1 \rceil / 2 \) or less vertices?
Lemma 3. The problem \textsc{Half-VertexCover} is \text{NP}-complete.

Proof. By definition the problem belongs to \text{NP}. For proving hardness we show a reduction from vertex cover. Given a graph $G$ with $n$ vertices and an integer $k$, where $0 \leq k \leq n$, we construct a graph $G'$ as follows. $G'$ has vertex set

$$V' = V(G) \cup X \cup Y \cup \{y\},$$

where $X$ has $n - k - 1$ vertices and $Y$ has $k + 1$ vertices, and edge set

$$E' = E \cup \{(x, x') \mid x \neq x' \land x, x' \in X\} \cup \{(x, y) \mid x \in X, y \in Y\} \cup \{(y, z) \mid z \in V \cup X \cup Y\}.$$

By construction $G'$ has $2n + 1$ vertices and can be constructed in polynomial time. Observe that any vertex cover of $G'$ with minimum size has to contain $y$ and all the vertices in $X$ and none of the vertices in $Y$. The rest of the cover is a min vertex cover of $G$. Therefore we have that $G$ has a vertex cover of size $k$ or less iff $G'$ has a vertex cover of size $n$ or less.

Lemma 4. The \textsc{IsProper}, \textsc{IsStrong} and \textsc{IsDecisive} problems for unweighted influence games with polynomial number of vertices belong to \text{coNP}-complete.

Proof. Trivially the problems belong to \text{coNP}. We provide a reduction from the complement of \textsc{Half-VertexCover} which works for the three problems. Let $G$ be an instance of \textsc{Half-VertexCover} with $2k + 1$ vertices, for some value $k$. Consider the unweighted influence game $\Delta(G) = (G', f, n + m + 5, N)$ as defined in the proof of Lemma 2. Recall that

$$V(G') = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\}$$

and $\alpha := n + m + 4$.

If $G$ has a vertex cover $X$ with $|X| \leq k$, the coalition $X \cup \{z\}$ is winning and, as $n + 1 - |X \cup \{z\}| > k$, we have that $N \setminus (X \cup \{z\})$ is also winning. Therefore $\Delta(G)$ is not proper.

When all the vertex covers of $G$ have more than $k$ vertices, any winning coalition $Y$ of $\Delta(G)$ verifies $|Y \cup \{v_1, \ldots, v_n\}| > k$, i.e., $|Y \cup \{v_1, \ldots, v_n\}| \geq k + 1$. For a winning coalition $Y$ we have to consider two cases.

When $z \in Y$: As $z \in Y$, we have $N \setminus Y \subseteq \{v_1, \ldots, v_n\}$ and $|N \setminus Y| < n - k - 1 = k$. In consequence $N \setminus Y$ is a loosing coalition. When $z \notin Y$: As $|N \setminus Y \setminus \{z\}| \leq k + 1$ and $N \setminus Y$ is again a loosing coalition. So we conclude that $\Delta(G)$ is proper.

As $\Delta(G)$ can be obtained in polynomial time the \textsc{IsProper} is \text{coNP}-hard.

When $G$ has a vertex cover $X$ with $|X| \leq k$, $\{v_1, \ldots, v_n\} \setminus X$ is winning and its complement $X \cup \{z\}$ is also winning, therefore $\Delta(G)$ is not strong.

When all the vertex covers of $G$ have more than $k$ vertices, for a loosing coalition $Y$, we need to show that $N \setminus Y$ is winning. When $z \in Y$ we have that $|Y \setminus \{z\}| \leq k$, therefore $|N \setminus Y| \geq k + 1$ and $N \setminus Y$ is winning. When $z \notin Y$ we have $|Y| \leq k$ therefore $|\{v_1, \ldots, v_n\} \setminus Y| \geq k + 1$. Therefore $N \setminus Y$ is winning. In consequence $\Delta(G)$ is strong.

Putting all together we get the claimed results.

The following lemma concludes the proof of Theorem 9.

Lemma 5. The problems, \textsc{IsPasser}, \textsc{IsVetoer}, \textsc{IsDictator} and \textsc{IsCritical} are polynomial time solvable for influence games.

Proof. The result follows from the definitions. Observe that, given an influence game $\Gamma = (G, w, f, q, N)$, a player $i \in N$ and a coalition $S \subseteq N$, we have:

- $i$ is a passer iff $|F((i))| \geq q$.
- $i$ is vetoer iff $|F(N \setminus \{i\})| < q$.
- $S$ is critical for player $i$ iff $|F(S)| \geq q$ and $|F(S \setminus \{i\})| < q$.
- $S$ is blocking iff $|F(N \setminus S)| \leq q$.
- $S$ is swing iff $\exists i \in S$ such that $i$ is critical.

The last set of conditions can be checked in polynomial time.
In addition to the problems in the previous Theorem \([9]\) we also consider the problems AREISO and AREEQUIV corresponding to testing whether two influence games are isomorphic or equivalent according to the following definitions.

**Definition 10.** Let \( \Gamma \) and \( \Gamma' \) be two influence games with the same number of players, \( \Gamma \) and \( \Gamma' \) are isomorphic iff there exists a bijective function \( \varphi : N \rightarrow N' \) such that \( X \in W(\Gamma) \iff \varphi(X) \in W(\Gamma') \). Moreover, if \( N = N' \) and \( \varphi \) is the identity function, then we say that the two influence games are equivalent.

**Theorem 10.** The problems EQUIV and ISO belong to \( \text{coNP} \)-hard for unweighted influence games with polynomial number of vertices.

**Proof.** Consider on one side the game \( \Delta(G, k) \), there \( N = \{v_1, \ldots, v_n, z\} \) and the weighted game \( \Gamma(G) \) defined on the set of players \( N \) with quota \( q = k + 1 \) and, in which the weights are set as follows: \( w(v_i) = 1 \), for \( 1 \leq i \leq n \), and \( w(z) = 0 \). \( \Gamma(G) \) can be given as an unweighted influence game using the construction of Theorem\([8]\). Observe that \( \Delta(G, k) \) is equivalent (isomorphic) to \( \Gamma(G) \) iff \( G \) does not have a vertex cover of size \( k \) or less.

Our last general result settles the complexity of the computation of the Banzhaf and the Shapley-Shubik value.

**Theorem 11.** Computing the Banzhaf value and the Shapley-Shubik value of an influence game with polynomial number of vertices is \( \#P \)-complete.

**Proof.** Let \( G \) be a graph and consider the graph \( G' \) in which we add a new vertex \( x \) connected to all the vertices in \( G \) and the influence game \( \Gamma(G') \) as usual.

Let \( X \) be a winning coalition such that \( x \in X \). When \( X \neq V(G') \) we know that \( X \setminus \{x\} \) must be a vertex cover of \( G \) and furthermore \( x \) is critical as \( X \setminus \{x\} \) is not winning. For the case when \( X = V(G') \), \( X \setminus \{x\} \) is winning and thus \( x \) is not critical for \( X \).

As a consequence of this result we have that \( \eta_b(\Gamma) \) coincides with the number of vertex covers of \( G \) minus one. As computing the number of vertex covers of a graph is \( \#P \)-complete \([13]\), we have that computing the Banzhaf value of an influence game is \( \#P \)-complete.

According to Theorem 3.29 of page 50, \([2]\), to prove that the Shapley-Shubik value is \( \#P \)-complete, it is enough to show that computing the Banzhaf value is \( \#P \)-complete and that influence games verify the property of being a reasonable representation. This last condition is stated as follows: for a simple game \((N, v)\), the new game \((N \cup \{x\}, v')\), where \( v(S) = 1 \) iff \( v'(S \cup \{x\}) = 1 \), is a representation of the new game can be computed with only polynomial blow-up. It remains to show that influence games are a reasonable representation.

For influence games we consider the following construction, let \( \Gamma = (G, w, f, q, N) \). Assume the \( G = (V, E) \) has \( n \) vertices and \( m \) arcs.

Consider the graph \( G' = (V', E') \) and the game \( \Gamma' = (G', w', f', q', N') \) where

- \( V' = V \cup \{x, y\} \cup \{a_1, \ldots, a_{2n}\} \).
- \( E' = E \cup \{(x, y)\} \cup \{(v, y) \mid v \in V\} \cup \{(y, a_i) \mid 1 \leq i \leq 2n\} \).
- \( w'(e) = w(e) \) for \( e \in E \) and \( w'(e) = 1 \) for \( e \in E' \setminus E \).
- \( f'(v) = f(v) \) for \( v \in V, f'(x) = 1, f'(y) = q + 1 \) and \( f'(a_i) = 1 \) for any \( 1 \leq i \leq 2n \).
- \( q' = 2n \) and \( N' = N \cup \{x\} \).

From the construction it follows trivially that \( X \) is a winning coalition in \( \Gamma \) iff \( X \cup \{x\} \) is a winning coalition in \( \Gamma' \). Furthermore, \( \Gamma' \) has polynomial size with respect to the size of \( \Gamma \), therefore influence games are a reasonable representation according to \([2]\), and the result follows.

**5 Unweighted Influence Games on Undirected Graphs**

In this section we analyze the complexity of the proposed problems on some particular subfamilies of unweighted influence games defined on undirected graphs. Note that every undirected graph can be represented by a directed graph because each undirected edge \( e := \{i, j\} \) can be expressed as two directed edges \( e_1 := (i, j) \) and \( e_2 := (j, i) \), such that \( w(e) = w(e_1) = w(e_2) \).
5.1 Maximum Influence

Here we analyze first the case with maximum influence and maximum spread, that is games \( \Gamma = (G,f,|V|,V) \) where \( f(v) := d_G(v) \). Observe that in this case when \( G \) is disconnected the game can be analyzed as the union of the games corresponding to the different components.

**Theorem 12.** In an unweighted influence game with maximum influence and maximum spread on a connected graph \( G \) the following hold:

1. \( X \subseteq V \) is a winning coalition iff \( X \) is a vertex cover of \( G \).
2. \( \Gamma \) is proper iff \( G \) is not bipartite.
3. \( \Gamma \) is weak iff \( G \) has at least two non-incident edges.
4. \( \Gamma \) is decisive iff \( G \) is a triangle.

**Proof.** The first result follows from Lemma 1.

If \( G \) is bipartite, let \( G_1 \cup G_2 = G \) with \( G_1 \cap G_2 = \emptyset \). Then \( G_1 \) is winning in \( G \) and \( G_2 = N \setminus G_1 \) is also winning in \( G \), i.e., \( G \) is not proper. For the opposite: Let be \( G \) not bipartite, for any winning coalition \( X \subseteq V \) then \( N \setminus X \) is loosing, i.e., \( \Gamma \) is proper.

If \( G \) has at least two incident edges \( e_1 \) and \( e_2 \), then \( \{e_1,e_2\} \) wins and \( N \setminus \{e_1,e_2\} \) also wins, i.e., \( G \) is not weak. The opposite has a similar reasoning.

Finally, the non-bipartite graph with two incident edges (proper and strong) is a triangle (decisive).

In regard to the complexity of the two parameters we have the following:

**Theorem 13.** In an unweighted influence game \( \Gamma \) with maximum influence and maximum spread on a connected graph \( G \) the following hold:

1. \( \text{Length} \) is \( \text{NP} \)-hard.
2. \( \text{Width} \) of \( \Gamma \) is 2 if the graph has more than one vertex; and 1, otherwise.

**Proof.** Observe that in the reduction of Theorem 6 the influence game \( \Gamma(G) \) has maximum influence and maximum spread. Thus, \( \text{Length} \) is \( \text{NP} \)-hard.

If \( G \) has more than one vertex, let \( e_1 \) and \( e_2 \) be the edges whose join one vertex, then \( N \setminus \{e_1,e_2\} \) is losing, i.e., the width is 2. It is clear that, if \( G \) has one vertex, the width is 1.

For the case of maximum influence but not maximum spread, that is \( (G,f,q,V) \) where \( f(v) = d_G(v) \) and \( q < n \), the game cannot be directly analyzed from the games on the connected components, as the total quota can be spread in different ways along the components. Nevertheless the winning coalitions can be characterized as follows.

**Lemma 6.** In an unweighted influence game with maximum influence \( \Gamma = (G,d_G,q,V) \) but with non isolated vertices, \( X \subseteq V \) is a winning coalition iff removing \( X \) from \( G \) leaves at least \( q - |X| \) isolated vertices.

This characterization gives rise to the following problem:

**AREISOLATED:** Given a graph \( G := (V,E) \) and \( q,k \in \mathbb{N} \).

Is there \( S \subseteq V \) such that \( |S| \leq k \) and removing \( S \) from \( G \) there are at least \( q - k \) isolated vertices.

Observe that for \( q = n \) we have that \( S \) must be a vertex cover, and thus the problem is \( \text{NP} \)-hard. Thus, computing the minimum size of such a set is equivalent to compute the length of the game. Moreover, for the maximum spread case the width of the game can be computed in polynomial time.

**Theorem 14.** Computing the length and the width of an influence game with maximum influence is \( \text{NP} \)-hard and polynomial time solvable, respectively.
Proof. Taking into a count that the reduction of Vertex Cover described in Theorem 6 has maximum influence and maximum spread, LENGTH is $\mathsf{NP}$-hard.

To compute the width of $\Gamma$ we can restrict to analyze the complements of loosing coalitions $X$ for which $F(X) = X$. For this case we have that $X = V \setminus Y$ is a loosing coalition with $F(X) = X$ iff $|Y| \geq n - q$ and every vertex in $G[Y]$ (induced subgraph of the graph $G$ formed by all vertices adjacent to $Y$) has at least one neighbor.

So, we have to solve the problem of whether it is possible to discard $\alpha$ nodes form $G$ without leaving isolated vertices. For doing so we sort the sizes of the connected components with at least two vertices in increasing order, let $w_1, \ldots, w_k$ be this sorted sequence.

Assume that $w_k > 2$. We compute the first $j$ such that $\sum_{i=1}^{j} w_i = \beta \leq \alpha$ but $\sum_{i=1}^{j} w_i > \alpha$. If $\beta = \alpha$ or $w_{j+1} > \alpha - \beta + 1$ we are done, remove all vertices from components 1 to $j$ and $\alpha - \beta$ form the $j + 1$-th component, and what remains verify the properties. Otherwise, we know that $w_{j+1} = \alpha - \beta + 1$. If $j + 1 < k$ we can remove $\alpha - \beta - 1$ vertices form the $j + 1$-th component and one additional vertex from the $k$-th component leaving two connected vertices in component $j + 1$ to $k$, and we have constructed the desired set. If $j + 1 = k$, the answer will be no. In the case that $w_k = 2$, if $\alpha$ is even and smaller than $n$ the set exists otherwise the set does not exists.

By performing the above test for $\alpha = n - q, n - q + 1, \ldots, n$ we can compute $\alpha_m$, the minimum $\alpha$ for which nodes can be discarded verifying the property. The width of the game is just $n - \alpha_m$, as the complete computation can be performed in polynomial time we get the desired result.

5.2 Minimum Influence
Let be $\Gamma = (G, 1_V, q, N)$ where $1_V(v) := 1$ for any $v \in V$. Observe that if $G$ is connected the game has a trivial structure as any non-empty vertex subset of $N$ is a winning coalition. For the disconnected case we can analyze the game considering an instance of the knapsack problem. Assume that $G$ has $k$ connected components, $C_1, \ldots, C_k$ with non-empty intersection with $N$. Let $w_i := |V(C_i)|$, $1 \leq i \leq k$, be the number of vertices in $C_i$.

Lemma 7. A winning coalition $X$ is minimal iff no more than two vertices in $X$ belong to the same component. Minimal winning coalitions are in correspondence with the minimal winning coalitions of the weighted game $[q; w_1, \ldots, w_k]$.

Moreover, we have the following result.

Theorem 15. In an unweighted influence game with minimum influence $\Gamma = (G, 1_V, q, N)$ the problems LENGTH, WIDTH, ISProPER, ISStrong and ISDecision are polynomial time solvable.

Proof. First we proof that LENGTH and WIDTH belong to $\mathsf{P}$.

Assume that $G$ has $k$ connected components $C_1, \ldots, C_k$ such that each of them contains a vertex in $N$.

To compute the length, sort the connected components of $G$ with non-empty intersection with $N$ in decreasing order of size. Assume that the sizes are $w_1 \geq \ldots \geq w_k$ is the sorted sequence. Then, the length of $\Gamma$ is the minimum $j$ for which $\sum_{i=1}^{j} w_i \geq q$ but $\sum_{i=1}^{j-1} w_i < q$. Of course this value can be computed in polynomial time.

A loosing coalition of maximum size can be obtained by sorting the connected components of $G$ with non-empty intersection with $N$ in increasing order, now the sizes are $w_1 \leq \ldots \leq w_k$, and computing the value $j$ for which $\sum_{i=1}^{j} w_i \geq q$ but $\sum_{i=1}^{j-1} w_i < q$. The width of $\Gamma$ is $n - \sum_{i=1}^{k} |V(C_i) \cap N|$. This last number can be computed in polynomial time.

Below we proof that ISProPER, ISStrong and ISDecision belong to $\mathsf{P}$.

Let $w = \sum_{i=1}^{k} w_i$. Let $\alpha_{\text{max}}$ be the maximum $\alpha \in [q - 1]$ for which there is a set $S \subseteq [k]$ with $\sum_{i \in S} w_i = \alpha$. Let $\alpha_{\text{min}}$ be the minimum $\alpha \in \{q, \ldots, w\}$ for which there is a set $S \subseteq [k]$ with $\sum_{i \in S} w_i = \alpha$. Observe that $\Gamma$ is proper iff $w - \alpha_{\text{min}} < q$ and that $\Gamma$ is strong iff $w - \alpha_{\text{max}} \geq q$.

The values $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$ can be obtained by solving several instances of the knapsack problem, as the weights of the objects are polynomial in $n$, the complete computation can be performed in polynomial time.
References

1. K.R. Apt and E. Markakis. Diffusion in social networks with competing products. CoRR, abs/1105.2434, 2011.
2. H. Aziz. Algorithmic and complexity aspects of simple coalitional games. PhD thesis, Department of Computer Science, University of Warwick, 2009.
3. H. Aziz, F. Brandt, and P. Harrenstein. Monotone cooperative games and their threshold versions. In Proceedings of the 9th Int. Conf. on Autonomous Agent and Multiagent Systems (AAMAS 2010), 2012.
4. J.F. Banzhaf. Weighted voting doesn’t work. Rutgers Law Review, 19:317–343, 1965.
5. S. Bharath, D. Kempe, and M. Salek. Competitive influence maximization in social networks. In WINE 2007, number 4858 in LNCS, pages 306–311, 2007.
6. B. Bollob. Modern graph theory. Springer, 1998.
7. A. Borodin, Y. Flnus, and J. Oren. Threshold models for competitive influence in social networks. In WINE 2010, number 6484 in LNCS, pages 539–550, 2010.
8. G. Chalkiadakis, E. Elkind, and M. Wooldridge. Computational Aspects of Cooperative Game Theory. Morgan and Claypool Publishers, 2012.
9. N. Chen. On the approximability of influence in social networks. SIAM Journal on Discrete Mathematics, 23(3):1400–1415, 2009.
10. V.G. De˘ıneko and G.J. Woeginger. On the dimension of simple monotonic games. European Journal of Operational Research, 170:315–318, 2004.
11. P. Domingos and M. Richardson. Mining the network value of customers. In ACM Press, editor, Proceedings of the Seventh International Conference on Knowledge Discovery and Data Mining, pages 57–66, San Francisco, CA, 2001.
12. D. Easley and J. Kleinberg. Networks Crowds and Markets. Cambridge University Press, 2010.
13. E. Even-Dar and A. Shapira. A note on maximizing the spread of influence in social networks. In Proceedings of the 3rd international conference on Internet and network economics, pages 281–286, 2007.
14. M. Friedman and L. Khachiyan. On the complexity of dualization of monotone disjunctive normal forms. Journal of Algorithms, 21(3):618–628, 1996.
15. J. Freixas and X. Molinero. On the existence of a minimum integer representation for weighted voting systems. Annals of Operation Research, 166(1):243–260, 2009.
16. J. Freixas, X. Molinero, M. Olsen, and M. J. Serna. On the complexity of problems on simple games. RAIRO-Operations Research, 45(4):295–314, 2011.
17. J. Freixas and M.A. Puente. A note about games-composition dimension. Discrete Applied Mathematics, 113:265–273, 2001.
18. M. R. Garey and D. S. Johnson. Computers and intractability, a guide to the theory of NP-Completness. W.H. Freeman and Company, New York, USA, 1999. Twenty-first printing (First printing, 1979).
19. D. Gillies. Some theorems on n-person games. PhD thesis, Department of Mathematics, Princeton University, 1953.
20. J. Goldenberg, B. Libai, and E. Muller. Talk of the network: A complex systems look at the underlying process of word-of-mouth. Marketing Letters, 12(3):211–223, 2001.
21. J. Goldenberg, B. Libai, and E. Muller. Using Complex Systems Analysis to Advance Marketing Theory Development. Academy of Marketing Science Review, 2001.
22. M. Golumbic. Algorithmic graph theory and perfect graphs, volume 57. Annals of Discrete Mathematics, 1980.
23. M. Granovetter. Threshold models of collective behavior. American Journal of Sociology, 83(6):1420–1443, 1978.
24. S. Hu. Threshold logic. University of California Press, Berkeley and Los Angeles, 1965.
25. M.T. Irfan and L.E. Ortiz. A game-theoretic approach to influence in networks. Association for the Advance- ment of Artificial Intelligence, 2011.
26. J. Isbell. A class of majority games. Quarterly Journal of Mathematics. Oxford Scr., 7(1):183–187, 1956.
27. D. Marciniai J. Freixas. On the notion of dimension and codimension of simple games. In Russia Graduate School of Management of Saint Petersburg, editor, Contributions to Game Theory and Management, volume 3, pages 67–81, 2010.
28. M. Jackson. Social and Economic Networks. Princeton University Press, 2008.
29. R.G. Jereslow. On defining sets of vertices of the hypercube by linear inequalities. Discrete Mathematics, 11(2):119–124, 1975.
30. D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In ACM Press, editor, Proceedings of the Ninth International Conference on Knowledge Discovery and Data Mining, pages 137–146, 2003.
31. D.E. Loeb and A.R. Conway. Voting fairly: transitive maximal intersecting family of sets. Journal of Combinatorial Theory, Series A, 91:386–410, 2000.
32. M. Maschler and B. Peleg. A characterization, existence proof and dimension bounds for the kernel of a game. *Pacific J. Math.*, 18(2):289–328, 1966.
33. T. Matsui and Y. Matsui. A survey of algorithms for calculating power indices of weighted majority games. *Journal of the Operations Research Society of Japan*, 43:71–86, 2000.
34. W. McCulloch and W. Pitts. A logical calculus of the ideas immanent in nervous activity. *Bulletin of Mathematical Biophysics*, 5:115–133, 1943.
35. R. McNaughton. Unate truth functions. *IRE Transactions on Electronic Computers*, EC-10(1):1–6, 1961.
36. I. Parberry. *Circuit Complexity and Neural Networks*. The M.I.T. Press, Cambridge, Massachusetts, London, England, 1994.
37. A. Polyméris. Stability of two player game structures. *Discrete applied mathematics*, 156:2636–2646, 2008.
38. K.G. Ramamurthy. *Coherent structures and simple games*. Kluwer, Dordrecht, Netherlands, 1990.
39. J. Reiterman, V. Rodl, E. Sinajova, and M. Tuma. Threshold hypergraphs. *Discrete Mathematics*, 54(2):193–200, 1985.
40. M. Richardson. On finite projective games. In *Proceedings of the American Mathematical Society*, volume 7, pages 458–465, 1956.
41. M. Richardson and P. Domingos. Mining knowledge-sharing sites for viral marketing. In *Proceedings of the Eight International Conference on Knowledge Discovery and Data Mining*, pages 61–70. ACM Press, 2002.
42. F. Riquelme and A. Polyméris. On the complexity of the decisive problem in simple and weighted games. *Electronic Notes in Discrete Mathematics*, 37:21–26, 2011.
43. T. Schelling. *Micromotives and Macrobehavior*. Norton, 1978.
44. L.S. Shapley. Simple games: An outline of the descriptive theory. *Behavioral Science*, 7:59–66, 1962.
45. A. Taylor and W. Zwicker. A characterization of weighted voting. In *Proceedings of the American Mathematical Society*, volume 115, pages 1089–1094, 1992.
46. A. Taylor and W. Zwicker. Weighted voting, multicameral representation, and power. *Games and Economic Behavior*, 5:170–181, 1993.
47. A. Taylor and W. Zwicker. *Simple games: Desirability relations, trading, pseudoweightings*. Princeton University Press, New Jersey, 1st edition, 1999.
48. J. von Neumann and O. Morgenstern. *Theory of games and economic behavior*. Princeton University Press, New Jersey, 1944.