VECTORSI FIELDS ON ORIENTABLE 7-MANIFOLDS

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ABSTRACT. In this paper we prove that for 7-manifolds, stably parallelizable implies parallelizable yet not all orientable manifolds of dimension 7 are parallelizable and that this is unique to dimension 7. Therefore asking whether an orientable 7-manifold is parallelizable is non-trivial and one only needs to show that it is stably parallelizable. As an example, we also present the only orientable 7-dimensional Dold manifold that is not parallelizable. This finding is a result of studying and completing a proof, involving knot theory, of Stiefel’s theorem that all orientable 3-manifolds are parallelizable.

1. Introduction

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1. Introduction

This paper was conceived and motivated by the connections between knot theory and the parallelizability of 3-manifolds discussed in A. T. Fomenko and S. V. Matveev’s proof of Stiefel’s theorem [FM] that all orientable 3-manifolds are parallelizable. A. T. Fomenko and S. V. Matveev proved that all 3-manifolds are stably parallelizable, then R. Benedetti and P. Lisca in [BL] completed the proof by using quasi-framings. In this paper we will provide a different completion of the proof by using the span of a manifold. We then prove that manifolds of dimension \( n = 7 \) have the property that stably parallelizable implies parallelizable and it is the only dimension with this property where there exists non-parallelizable orientable manifolds. Throughout this paper all manifolds are assumed to be connected manifolds.

In Section 2 we will give a brief description of Dehn surgery and integral surgery then present major results from W. B. R. Lickorish [Lic], A. H. Wallace [Wal], R. Thom [Thom], R. Kirby [Kir], and J. Milnor [Mil1] that connect integral surgery to 3-manifolds and parallelizable 4-manifolds. We end the section by discussing how A. T. Fomenko and S. V. Matveev in [FM] use these results in their proof of Stiefel’s theorem. In Section 3 we turn our attention to tangent bundles and the span of a manifold then present two results, one from D. Husemoller in [Hus] and the second from G. E. Bredon and A. Kosinski in [BK], we then use the results to complete A. T. Fomenko
and S. V. Matveev’s proof. In the last portion of the section we ask two questions that arise from
the completion of the proof. We ask for what dimensions does stably parallelizable imply para-
lelizable and if the orientable manifolds of the dimension with this property are all parallelizable?
We then answer the first question with if and only if \( n = 1, 3, 7 \) by using two famous theorems;
one by R. Bott and J. Milnor in [BM] and separately by M. A. Kervaire in [Ker] and the second by
E. Thomas in [Th1] and separately by G. E. Bredon and A. Kosinski in [BK]. We answer the sec-
ond question in the negative in Section 4 by presenting the only orientable 7-dimensional Dold
manifold that is not parallelizable.

1.1. Acknowledgements. This work was supported by the Australian Research Council grant
DP210103136.

2. Integral surgery

Dehn surgery is a well-studied technique that uses knot theory to understand 3-manifolds. A
special case of this technique was introduced in 1910 by M. Dehn [Deh] it was then generalized
in the work of R. H. Bing in the late 1950’s [Bin1, Bin2]. In this section we will give an account
of the various important theorems that provide a clear picture of the connections between knot
theory, 3-manifolds, and parallelizable 4-manifolds.

Definition 2.1. Dehn surgery along a knot \( K \) in an orientable 3-manifold, \( M \), is the process of
taking the tubular neighborhood of the underlying unframed knot \( U(K) \), which is homeomorphic
to \( D^2 \times S^1 \), and removing it from \( M \) so that we obtain two manifolds: \( X_K = M \setminus \text{int}(U(K)) \) and
\( V = U(K) \cong D^2 \times S^1 \). We use an orientation reversing homeomorphism \( \varphi : \partial V \to \partial X_K \)
determined by a curve, \( c \), in \( \partial V \) to add \( V \) back into \( X_K \), that is, adding a 2-handle along \( c \) and then cap it off
with a 3-ball, see Figure 1. The result \( X_K \cup \varphi V \) is a closed orientable manifold, we say this manifold
is obtained by Dehn surgery along the knot from \( M \). We may define Dehn surgery along a link by
applying the process to each component where for each component there is an accompanying curve
on the tubular neighborhood of the component.

\[ \text{Figure 1. An illustration of attaching a 2-handle from } V = (2 - \text{handle}) \cup D^3 \]
along \( c = \varphi(\mu) \) before capping it off with \( D^3 \).

Definition 2.2. For \( M = S^3 \), Integral surgery along a framed knot \( K \) is Dehn surgery along \( K \)
given by a homeomorphism determined by the framing of \( K \). In this case we will denoted by \( M_K \)
the manifold obtained by integral surgery on the framed knot \( K \) from \( S^3 \). We may define integral
surgery along a framed link by applying the process to each component. The manifold obtained by
integral surgery along a framed link from \( S^3 \) is denoted by \( M_L \).

A relationship between all closed orientable 3-manifolds and framed links in \( S^3 \) was first proved
by A. H. Wallace in [Wal], then W. B. R. Lickorish in [Lic] gave a separate elementary and geo-
metric proof that uses Dehn twist homeomorphisms.
Theorem 2.3 (Lickorish-Wallace Theorem). [Lic, Wal] Every closed orientable 3-manifold can be obtained from $S^3$ by performing integral surgery on a framed link.

By using R. Thom’s result in [Thom] we observe Dehn surgery’s strong connections to 4-manifolds.

Theorem 2.4. [Thom] Every closed, connected, 3-manifold is the boundary of some oriented connected 4-manifold.

We turn our attention to even surgery which is integral surgery in $S^3$ along a framed link whose components all have an even framing number. In R. Kirby’s paper on Kirby calculus [Kir], Kirby found a relationship between even surgery and parallelizable 4-manifolds. Recall that a 4-manifold is parallelizable if its tangent bundle is trivial.

Theorem 2.5. [Kir] Let $L$ be an even link, then $W_L$ is parallelizable, where $W_L$ is a 4-manifold determined by a framed link $L$, by adding 2-handles to a 4-ball along $L$.

This relationship becomes more powerful after applying J. Milnor’s work in [Mil1]; in particular the following theorem. (A simplified proof of Theorem 2.6 than that of [Mil1] can be found in [Kap], [Sav], and [FM]).

Theorem 2.6. [Mil1] Every closed orientable 3-manifold can be obtained from $S^3$ by performing integral surgery along an even link.

By combining Theorem 2.6 and Theorem 2.5 we have that every closed orientable 3-manifold is the boundary of a parallelizable 4-manifold. This result is the foundation of Fomenko and Matveev’s proof of Stiefel’s theorem. However, Fomenko and Matveev refer to Husemoller’s book [Hus] for the completion of the proof. This leads us to the next section about tangent bundles, stably parallelizable, and parallelizable manifolds for the completion of Fomenko and Matveev’s proof.

3. Stably parallelizable and parallelizable manifolds

Stably parallelizable manifolds, also known as $\pi$-manifolds, were extensively studied in the 1960’s and a summary of the findings were presented by E. Thomas in [Th2]. In this section we will use a few key theorems on stably parallelizable manifolds in order to complete Fomenko and Matveev’s proof. We finish the section by noting an interesting and unique property of manifolds in dimension 7: while there exists orientable 7-manifolds that are not parallelizable, manifolds of this dimension have the property that stably parallelizable is equivalent to parallelizable.

Definition 3.1. A section of the tangent bundle $TM \xrightarrow{\pi} M$ is called a vector field on $M$. More precisely, a vector field is a smooth function $v : M \rightarrow TM$ which assigns to each point $p \in M$ a vector $v(p)$ tangent to $M$ at $p$ such that $\pi \circ v = id_M$.

In 1966, E. Thomas coined the term “span” of a manifold in [Th1] while working on stably equivalent vector bundles.

Definition 3.2. The vector fields $v_1, \cdots, v_k$ on an $n$-manifold $M$ are called linearly independent if for every $p \in M$ the vectors $\{v_1(p), v_2(p), \cdots, v_k(p)\}$ are linearly independent. The maximal number of linearly independent vector fields on $M$ is called the span of $M$ and is denoted by $\text{span}(M)$. Clearly, $\text{span}(M) \leq n$. 


Definition 3.3. \( M \) is parallelizable if the tangent bundle of \( M \) is trivial. Equivalently, an \( n \)-manifold \( M \) is parallelizable if and only if \( \text{span}(M) = n \).

Definition 3.4. Let \( E_1 \xrightarrow{\pi_1} M \) and \( E_2 \xrightarrow{\pi_2} M \) be two vector bundles. We define the Whitney sum bundle (direct sum bundle) to be the vector bundle \( E_1 \oplus E_2 \xrightarrow{\pi} M \) where \( E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2; \pi_1(v_1) = \pi_2(v_2)\} \) and \( \pi(v_1, v_2) = \pi_1(v_1) = \pi_2(v_2) \). For an arbitrary \( p \in M \), the fiber at \( p \) is \( \pi^{-1}(\{p\}) = \pi_1^{-1}(\{p\}) \oplus \pi_2^{-1}(\{p\}) \).

Definition 3.5. Let \( M \) be an \( n \)-manifold, \( TM \xrightarrow{\pi_1} M \) the tangent bundle over \( M \), and \( M \times \mathbb{R} \xrightarrow{\pi_2} M \) be a trivial line bundle over \( M \), then \( M \) is called stably parallelizable if the Whitney sum bundle \( TM \oplus (M \times \mathbb{R}) \xrightarrow{\pi} M \) is a trivial bundle.

Consider the \( n \)-sphere smoothly embedded into \( \mathbb{R}^{n+1} \) and defined as \( S^n := \{x \in \mathbb{R}^{n+1}; \langle x, x \rangle = 1\} \). Then the normal bundle of \( S^n \) into \( \mathbb{R}^{n+1} \), denoted by \( NS^n \xrightarrow{\pi} S^n \), is a vector bundle where the total space is defined by \( NS^n = \{(x, t) \in S^n \times \mathbb{R}^{n+1}; t \in \mathbb{R}\} \) with projection map defined by \( \pi((x, t)) = x \). The fiber of this bundle is \( \mathbb{R} \) since for each point \( \tilde{p} \in S^n \), we have \( \pi^{-1}(\{\tilde{p}\}) = \{t \in \mathbb{R} \} \). It can be shown that this bundle is bundle isomorphic to the trivial line bundle \( S^n \times \mathbb{R} \). Furthermore, the proof that all \( n \)-spheres are stably parallelizable is usually shown by taking the Whitney sum of the tangent bundle \( TS^n \) with the normal bundle \( NS^n \) and proving that the resulting bundle is trivial. A similar approach (using the normal bundle of an embedded manifold) can be used to prove the next proposition on ambient tangent bundles, however D. Husemoller took a direct approach by constructing a bundle isomorphism.

Definition 3.6. Let \( W \) be an \((n+1)\)-manifold and \( M \) an \( n \)-manifold embedded into \( W \). Consider the tangent bundle of \( W \), \( TW \xrightarrow{\pi} W \), then the ambient tangent bundle over \( M \), denoted by \( TW|_M \), is a vector bundle with total space \( TW|_M = \bigsqcup_{p \in M} T_p W \), the projection map obtained by restricting \( \pi \) to \( TW|_M \), and base space \( M \).

Proposition 3.7. [Hus] Let \( W \) be an \((n+1)\)-dimensional manifold with boundary \( \partial W \neq \emptyset \), and let \( M \) be an \( n \)-manifold such that \( M = \partial W \), then there exists an isomorphism of vector bundles between the ambient tangent bundle over \( M \) and the Whitney sum bundle of the tangent bundle of \( M \) and a trivial line bundle. That is,

\[
TW|_M \cong TM \oplus (M \times \mathbb{R}).
\]

A nice direct corollary to Proposition 3.7 arises when the tangent bundle of a manifold with boundary is trivial. In fact, it is this corollary that Fomenko and Matveev use to claim that every closed orientable 3-manifold is stably parallelizable.

Corollary 3.8. Let \( W \) be an \( n \)-dimensional manifold with boundary \( \partial W \neq \emptyset \). If the tangent bundle of \( W \) is trivial then its boundary is stably parallelizable.

In order to finish the proof, we must prove that for dimension 3, stably parallelizable implies parallelizable. We look to G. E. Bredon and A. Kosinski’s work in [BK] for the desired result.

Theorem 3.9. [BK] Let \( M \) be an \( n \)-dimensional oriented closed stably parallelizable manifold, then \( \text{span}(M) \geq \text{span}(S^n) \).

We now present the completion of Fomenko and Matveev’s proof.

Theorem 3.10. [Sti] Every orientable 3-manifold is parallelizable.
Proof. The following is a completion of the proof found in [FM]. By Theorem 2.6, every closed orientable 3-manifold can be obtained from $S^3$ by integral surgery along an even link. Furthermore, by Theorem 2.5 a closed orientable 3-manifold $M$ obtained from $S^3$ by integral surgery along an even link is the boundary of a parallelizable 4-manifold. Both theorems combined implies that every closed orientable 3-manifold is the boundary of a parallelizable 4-manifold. By Corollary 3.8, every closed orientable 3-manifold is stably parallelizable.

Therefore we only need to show that an arbitrary closed oriented stably parallelizable 3-manifold is parallelizable. We know that this is not generally true for arbitrary dimensions. Suppose $M$ is a closed oriented 3-manifold. Since $S^3$ is parallelizable, then $\text{span}(S^3) = 3$. By Theorem 3.9, $\text{span}(M) \geq 3$, but the span of an $n$-dimensional manifold cannot be greater than $n$, therefore we must have $\text{span}(M) = 3$. Therefore, $M$ is parallelizable.

Now suppose $M$ is an oriented 3-manifold with boundary, then by taking its double we obtain a closed oriented manifold and we can apply the same argument as above.

The finishing touches of the previous proof raise the following questions: in which dimensions does stably parallelizable imply parallelizable? Is this only true if all orientable manifolds of the specified dimension are already parallelizable? In order to answer this we look to two famous theorems. The first is on the parallelizability of $n$-spheres by R. Bott and J. Milnor in [BM] and separately by M. A. Kervaire in [Ker].

**Theorem 3.11.** [BM, Ker] The sphere $S^n$ is parallelizable if and only if $n = 1, 3, \text{ or } 7$.

The second theorem is on the classification of stably parallelizable manifolds by E. Thomas in [Th1] and separately by G. E. Bredon and A. Kosinski in [BK].

**Theorem 3.12.** [Th1, BK] Let $M$ be a stably parallelizable $n$-manifold, then either $M$ is parallelizable or $\text{span}(M) = \text{span}(S^n)$.

As a result of Theorems 3.11 and 3.12, we can classify when stably parallelizable implies parallelizable for all orientable manifolds of a specific dimension.

**Lemma 3.13.** All stably parallelizable $n$-manifolds are parallelizable if and only if $n = 1, 3, 7$.

Proof: Suppose $n = 1, 3, 7$ and let $M$ be a stably parallelizable $n$-manifold. By Theorem 3.12, $\text{span}(M) = n$ or $\text{span}(M) = \text{span}(S^n)$. By Theorem 3.11, $\text{span}(S^n) = n$ which implies that $M$ is parallelizable.

Next, suppose $n \neq 1, 3, 7$. Since $S^n$ is stably parallelizable for all $n$, then by Theorem 3.11, $S^n$ for $n \neq 1, 3, 7$ are examples of stably parallelizable manifolds that are not parallelizable.

This lemma is implied by omission in a summary on stably parallelizable manifolds by E. Thomas in [Th2]. Due to this omission an interesting observation about 7-manifolds was missed. That is, since all orientable 1- and 3-manifolds are parallelizable, then 7-manifolds are the only manifolds with the property that stably parallelizable implies parallelizable yet not all orientable 7-manifolds are parallelizable.

4. Dold manifolds

A. Dold in [Dol] introduced a family of manifolds, now known as the Dold manifolds, in his work on odd-dimensional generators for the unoriented cobordism ring. In this section, we focus on Dold manifolds to present an example of an orientable 7-manifold that is not parallelizable.
**Definition 4.1.** Let \( r \geq 0, s \geq 0, \text{ and } r + s > 0, \) then the Dold manifold, denoted by \( P(r, s) \), is an \( r + 2s \)-dimensional smooth connected manifold. It is constructed from \( S^r \times \mathbb{C}P^s \) by identifying \((x, y) \in S^r \times \mathbb{C}P^s \) with \((-x, \overline{y})\).

The family of Dold manifolds consist of orientable and non-orientable manifolds. This can be shown by letting \( s = 0 \). Since \( P(r, 0) \cong \mathbb{R}P^r \), then \( P(r, 0) \) for even \( r \) are examples of non-orientable Dold manifolds. H. K. Mukerjee, in his work on classifying the homotopy of Dold manifolds [Muk], gave a simple condition on orientable Dold manifolds. By theorem 4.3, two of the four are orientable; three of which are parallelizable.

**Theorem 4.2.** [Muk] The Dold manifold \( P(r, s) \) is orientable if \( r + s + 1 \) is even and is unorientable if \( r + s + 1 \) is odd.

J. Korbaš, in his work on the parallelizability and span of the Dold manifolds [Kor], proved that only six Dold manifolds are stably parallelizable; three of which are parallelizable.

**Theorem 4.3.** [Kor] The Dold manifold \( P(r, s) \) is stably parallelizable if and only if \( (r, s) \in \{(1, 0), (3, 0), (7, 0), (0, 1), (2, 1), (6, 1)\} \).

Furthermore, only \( P(1, 0), P(3, 0), \text{ and } P(7, 0) \) are parallelizable.

There are four 7-dimensional Dold manifolds, \( P(1, 3), P(3, 2), P(5, 1), \text{ and } P(7, 0) \). By Theorem 4.2, two of the four are orientable; \( P(3, 2) \) and \( P(7, 0) \). Furthermore, by applying Theorem 4.3, \( P(3, 2) \) is the only orientable 7-dimensional Dold manifold that is not parallelizable.

5. Main Result

In this section we summarize the findings of this paper in the following theorem.

**Theorem 5.1.** Dimension 7 is the only dimension where there exists non-parallelizable orientable manifolds and stably parallelizable is equivalent to parallelizable.

**Proof:** By Lemma 3.13 stably parallelizable is equivalent to parallelizable for only dimensions \( 1, 3, 7 \). By Milnor’s classification of 1-manifolds [Mil2], all orientable 1-manifolds are parallelizable. Furthermore, by Theorem 3.10, all orientable 3-manifolds are parallelizable. However, by theorems 4.2 and 4.3, the Dold manifold \( P(3, 2) \) is an example of an orientable 7-dimensional manifold that is not parallelizable. Therefore, not all orientable 7-dimensional manifolds are parallelizable.

### References

[BL] R. Benedetti, P. Lisca, Framing 3-manifolds with bare hands, *Enseign. Math.* 64 (2018), no. 3-4, 395-413.

[Bin1] R. H. Bing, Necessary and sufficient conditions that a 3-manifold be \( S^3 \), *Annals of Mathematics* 68 (1958), 17-37.

[Bin2] R. H. Bing, Correction to “Necessary and sufficient conditions that a 3-manifold be \( S^3 \)”, *Annals of Mathematics* 77 (1963), 210.

[BM] R. Bott, J. Milnor, On the parallelizability of the spheres, *Bull. Amer. Math. Soc.* 64 (1958), 87-89.

[BK] G. E. Bredon, A. Kosinski, Vector Fields on \( \pi \)-Manifolds, *Annals of Mathematics*, vol. 84, no. 1, 1966, pp. 85–90.

[Deh] M. Dehn, Über die Topologie des dreidimensionalen Raumes, (German) *Math. Ann.* 69 (1910), no. 1, 137–168.

[Dol] A. Dold, Erzeugende der Thomenschen Algebra \( \mathfrak{F} \), (German) *Math. Z.* 65 (1956), 25–35.

[FM] A. T. Fomenko, S. V. Matveev, Algorithmic and computer methods for three-manifolds. Translated from the 1991 Russian original by M. Tsaplina and Michiel Hazewinkel and revised by the authors. With a preface by Hazewinkel, *Mathematics and its Applications*, 425. Kluwer Academic Publishers, Dordrecht, 1997. xii+334 pp.
[Hus] D. Husemoller, Fibre bundles, Third edition, Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994. xx+353 pp.

[Kap] S. J. Kaplan, Constructing framed 4-manifolds with given almost framed boundaries. Trans. Amer. Math. Soc. 254 (1979), 237–263.

[Ker] M. A. Kervaire, Non-parallelizability of the n-sphere for $n > 7$, Proc. Nat. Acad. Sci. USA 44 (1958), 280–283.

[Kir] R. Kirby, A calculus for framed links in $S^3$, Invent. Math. 45 (1978), no. 1, 35-56.

[Kor] J. Korbasa, On parallelizability and span of the Dold manifolds, (English summary) Proc. Amer. Math. Soc. 141 (2013), no. 8, 2933–2939.

[Lic] W. B. R. Lickorish, A representation of orientable combinatorial 3-manifolds, Ann. of Math., 76 (1962), 531-538.

[Mil1] J. Milnor, Differentiable manifolds which are homotopy spheres, Princeton, 1959.

[Mil2] J. Milnor, Topology from the differentiable viewpoint. Based on notes by David W. Weaver. Revised reprint of the 1965 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. xii+64 pp.

[Muk] H. K. Mukerjee, Classification of homotopy Dold manifolds, (English summary) New York J. Math. 9 (2003), 271-293.

[Sav] N. Saveliev, Lectures on the topology of 3-manifolds. An introduction to the Casson invariant. Second revised edition. De Gruyter Textbook. Walter de Gruyter & Co., Berlin, 2012.

[Sti] E. Stiefel, Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten, (German) Comment. Math. Helv. 8 (1935), no. 1, 305–353.

[Thom] R. Thom, Quelques propriétés globales des variétés différentiables, Comment. Math. Helv. 28, 17-86 (1954).

[Th1] E. Thomas, Cross-sections of stably equivalent vector bundles, Quart. J. Math. Oxford Ser. (2) 17 (1966), 53–57.

[Th2] E. Thomas, Vector fields on manifolds, Bull. Amer. Math. Soc. 75 (1969), 643–683.

[Wal] A. H. Wallace, Modifications and cobounding manifolds, Can. J. Math. 12, 1960, 503-528.

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