Relation between firing statistics of spiking neuron with instantaneous feedback and without feedback

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Abstract

We consider a wide class of spiking neuron models, defined by rather general set of conditions typical for basic models like leaky integrate and fire, or binding neuron model. A neuron is fed with a point renewal process. A relation between the three probability density functions (pdf): (i) pdf of input interspike intervals, (ii) pdf of output interspike intervals of a neuron with instantaneous feedback and (iii) pdf for that same neuron without feedback is derived. This allows to calculate any of the three pdfs provided the another two are given. Similar relation between corresponding means and variances is derived. The relations are checked exactly for the binding neuron model.

Keywords. spiking neuron; renewal stochastic process; probability density function; instantaneous feedback; interspike interval statistics; variance

MSC 60K40; 60K20; 92B20

1 Introduction

Spiking statistics of various neuronal models under a random stimulation is for a long time considered in the framework of diffusion approximation, see Sacerdote and Giraudo (2013) and references therein. In the diffusion approximation, it is difficult to follow the fate of individual impulses, which is required for studying a neural network, or the simplest case of it — a neuron with a feedback. Recently, a progress has been made in description of neuronal firing statistics without using diffusion approximation, see

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e.g. (Arunachalam et al., 2013; Vidybida, 2007, 2008). In the cited papers, output statistics is calculated for binding neuron (Vidybida, 2007) and binding neuron with instantaneous feedback (Arunachalam et al., 2013; Vidybida, 2008) stimulated with Poisson point process and some other processes (Arunachalam et al., 2013). In this letter we offer a simple relation between output statistics of a neuron with instantaneous feedback and that same neuron without feedback, which is valid for any renewal input point process. Also, we do not specify a concrete neuronal model, only formulate a set of conditions the model must satisfy. Most basic neuronal models do satisfy the formulated conditions.

2 Assumptions and definitions

The main function of a neuron is to transform its stream of input impulses (the stimulus) into its stream of output impulses. An output impulse is usually called “spike”. When a neuron emits an output impulse, it is usually said that neuron is triggered and fires a spike. As regards neuronal functioning we assume that the following conditions are satisfied:

COND1 Neuron is stimulated with excitatory input impulses which form a renewal point process. The process is described by means of a probability density function (pdf) of input interspike intervals (ISI), \( p^\text{in}(t) \), where \( t \) denotes an ISI duration.

COND2 Neuron has a deterministic behavior: the same stimulus, which is a sequence of input impulses, gives the same result (the neuron either fires, or does not fire).

COND3 After firing, neuron appears in its resting state, which does not evolve in time until a fresh input impulse comes.

COND4 Neuron may fire only at a moment when an input impulse comes.

COND5 If neuron starts from its resting state, then more than one input impulse is required in order to trigger it.

COND6 Output stream of impulses can be characterized with pdf of the output ISIs, \( p^\text{out}(t) \).

The conditions COND1-COND6, above, are satisfied for basic neuronal models, such as perfect integrate-and-fire model, see (Abbott, 1999), leaky integrate-and-fire model (Stein, 1967), or binding neuron model (Vidybida, ...
Condition COND5 means that the neuron makes some processing of sets of input impulses, instead of channelizing every input impulse into its output stream.

From COND1-COND4 it follows that the output stream of ISIs will be as well a renewal stochastic process.

The above construction can be extended by adding an instantaneous feedback line, see (Vidybida, 2008). The line sends any output impulse to the neuronal input without delay. This impulse is identical to any other input impulse. In this case we have a neuron with instantaneous feedback (IF).

It is worth noticing that immediately after firing the neuron with IF as well appears in a standard initial state. This standard initial state is realized if a neuron being in the resting state gets one input impulse. This state can evolve in time\(^1\). Nevertheless, it is clear that the output stream of ISIs of a neuron with IF as well will be a renewal process. As regards this process, we expect the following:

COND7 The output stream of ISIs of neuron with IF can be described by means of a pdf \(p_{\text{if}}(t)\), where “if” stands for the “instantaneous feedback”.

## 3 Relation between pdfs

Our purpose is to establish a relation between \(p_{\text{in}}(t)\), \(p_{\text{o}}(t)\) and \(p_{\text{o-if}}(t)\). Any of the three processes is a renewal one. If so, it is enough to analyze what happens between two consecutive firings of neuron without feedback.

Expect that neuron without feedback fires at moment 0. In order to fire next time at moment from \([t; t + dt[\), the neuron must obtain an input impulse at this same moment (COND4), and this impulse is not the first one received after the moment 0 (COND5). The first one must be obtained earlier, at some moment \(t' \in ]0; t[\). The probability to receive this impulse in the interval \([t'; t' + dt'][\) is \(p_{\text{in}}(t')dt'\). After receiving this impulse, the neuron appears in the standard initial state of neuron with instantaneous feedback. Now, firing next time at \(t\) means that a neuron with IF fires first time at \(t\) if starts at \(t'\). This event does not depend on the event of receiving first input impulse and has probability \(p_{\text{o-if}}(t - t')dt\). Thus, the compound event of receiving the first impulse at time \(t' < t\) and firing firstly at time \(t\) has the following probability \(p_{\text{in}}(t')dt'p_{\text{o-if}}(t - t')dt\). This expression gives probability of a single alternative, where the whole set of alternatives is parameterized

\(^1\)This state evolves with time for any neuronal model (received with impulse excitation decays in time), except of the perfect integrator.
with \(t'\). To get the \(p^o(t)dt\) one has to add the probabilities of all alternatives:

\[
p^o(t)dt = \int_0^t p^{in}(t')p^{o\alpha}(t-t')dt'dt.
\]

The required relation follows from (1):

\[
p^o(t) = \int_0^t p^{in}(t')p^{o\alpha}(t-t')dt'.
\]

3.1 Inverting Eq. (2)

In general case, the Eq. (2) can be inverted by means of Laplace transform:

\[
\mathcal{L}(p^o)(s) = \mathcal{L}(p^{in})(s)\mathcal{L}(p^{o\alpha})(s),
\]

\[
\mathcal{L}(p^{o\alpha})(s) = \mathcal{L}(p^o)(s)/\mathcal{L}(p^{in})(s).
\]

In order to find exact expression for \(p^{o\alpha}(t)\) it is necessary to apply the inverse Laplace transform to the right hand side of (3). This operation can be accomplished depending on the explicit expressions for the \(p^{in}(t), p^o(t)\).

3.2 Poissonian input case

In this case \(p^{in}(t) = \lambda e^{-\lambda t}\) and its Laplace transform is \(\mathcal{L}(p^{in})(s) = \frac{\lambda}{s+\lambda}\), which gives after using in (3)

\[
\mathcal{L}(p^{o\alpha})(s) = \mathcal{L}(p^o)(s) + s\mathcal{L}(p^o)(s)/\lambda.
\]

Notice that from COND1, COND5 it follows that \(p^o(0) = 0\). If so, then (4) gives

\[
p^{o\alpha}(t) = p^o(t) + \frac{1}{\lambda} \frac{d}{dt}p^o(t).
\]

4 Example - binding neuron with threshold two

The binding neuron (BN) model is characterized with a time interval \(\tau > 0\) during which an input impulse is stored in the neuron. The BN with threshold 2 fires a spike at the moment of receiving an input impulse, provided
at that moment the previous input impulse is still stored in the neuron. Just after firing, BN is free of stored impulses. If BN is stimulated with the Poisson stream, then COND1-COND7 are satisfied.

Exact expression for the $p^o(t)$ can be found in (Vidybida, 2007, Eq. (3)): for $m = 0, 1, 2, \ldots$, if $m\tau < t \leq (m + 1)\tau$ then

$$p^o(t) = e^{-\lambda t} \frac{\lambda^{m+2}}{(m+1)!} (t - m\tau)^{m+1}$$

$$+ e^{-\lambda t} \sum_{2 \leq k \leq m+1} \frac{\lambda^k}{(k-1)!} ((t - (k-2)\tau)^{k-1} - (t - (k-1)\tau)^{k-1}). \quad (6)$$

Exact expression for the $p^{\omega f}(t)$ can be found in (Vidybida, 2008, Eqs. (4),(7)), or in (Arunachalam et al., 2013, Eq. (10)): for $m = 0, 1, 2, \ldots$, if $m\tau < t \leq (m + 1)\tau$ then

$$p^{\omega f}(t) = e^{-\lambda t} \frac{\lambda^{m+1}}{m!} (t - m\tau)^m$$

$$+ e^{-\lambda t} \sum_{2 \leq k \leq m} \frac{\lambda^k}{(k-1)!} ((t - (k-1)\tau)^{k-1} - (t - k\tau)^{k-1}). \quad (7)$$

Now, the validity of (5) for BN can be directly checked by substituting (6), (7) into it.

5 Moments of distribution

Denote $W_n^{(in,o,o,\omega f)}$ the $n$-th moment of the corresponding distribution. With using (22) one has

$$W_n^o = \int_0^\infty dt t^n p^o(t) = \int_0^\infty dt t^n \int_0^\infty dt' p^{in}(t') p^{\omega f}(t - t')$$

$$= \int_0^\infty dt' p^{in}(t') \int_{t'}^\infty dt t^n p^{\omega f}(t - t')$$

$$= \int_0^\infty dt' p^{in}(t') \int_0^{t+t'} dt (t + t')^n p^{\omega f}(t) = \sum_{k=0}^n \binom{n}{k} W_k^{in} W_{n-k}^{\omega f}. \quad (8)$$

In particular, for $n = 1, 2$ one has

$$W_1^o = W_1^{\omega f} + W_1^{in}, \quad (8)$$
\[ W_2^o = W_2^{o,jf} + W_2^{in} + 2W_1^{in}W_1^{o,jf}. \] (9)

For binding neuron with threshold 2 fed with Poisson stream, (8) can be checked explicitly. It is clear that here \( W_1^{in} = \frac{1}{\lambda} \). The \( W_1^{o} \) is found in (Vidybida, 2007, Sec. 2):

\[ W_1^{o} = \frac{1}{\lambda} \left( 2 + \frac{1}{e^{\lambda \tau} - 1} \right). \]

The \( W_1^{o,jf} \) is found in (Vidybida, 2008, Eq. (9)) or (Arunachalam et al., 2013, Eq. (11)):

\[ W_1^{o,jf} = \frac{1}{\lambda(1 - e^{-\lambda \tau})}. \]

The validity of (8) can now be checked by substituting these expressions into it. Similarly, (Vidybida, 2008, Eqs. (12), (13)) give:

\[ W_2^{o} = \frac{2}{\lambda^2} \frac{3e^{2\lambda \tau} + (\lambda \tau - 3)e^{\lambda \tau} + 1}{(e^{\lambda \tau} - 1)^2}, \quad W_2^{o,jf} = \frac{2e^{\lambda \tau}}{\lambda^2} \frac{e^{\lambda \tau} + \lambda \tau}{(e^{\lambda \tau} - 1)^2}. \]

The validity of (9) for binding neuron with threshold 2 fed with Poisson stream can be checked by substituting these expressions into it.

Finally, denote \( \sigma_{in,o,o}^2 \) the variance of corresponding distribution. Then from (8), (9) the following relation can be derived:

\[ \sigma_{o,jf}^2 = \sigma_o^2 - \sigma_{in}^2. \] (10)

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