Forced Vibration in Cutting Process considering the Nonlinear Curvature and Inertia of a Rotating Composite Cutter Bar

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Forced vibration of the cutting system with a three-dimensional composite cutter bar is investigated. The composite cutter bar is simplified as a rotating cantilever shaft which is subjected to a cutting force including regenerative delay effects and harmonic exciting items. The nonlinear curvature and inertia of the cutter bar are taken into account based on inextensible assumption. The effects of the moment of inertia, gyroscopic effect, and internal and external damping are also considered, but shear deformation is neglected. Equation of motion is derived based on Hamilton’s extended principle and discretized by the Galerkin method. The analytical solutions of the steady-state response of the cutting system are constructed by using the method of multiple scales. The response of the cutting system is studied for primary and superharmonic resonances. The results show that nonlinear curvature and inertia imposed a significant effect on the dynamic behavior of the cutting process. The equivalent nonlinearity of the cutting system shows hard spring characteristics. Multiple solutions and jumping phenomenon of typical Duffing system are found in forced response curves.

1. Introduction

In boring or milling machining applications, the chatter problem during the cutting process is increasingly more significant because, for these operations, a flexible cutter bar is used. Chatter is detrimental for surface finish and dimensional accuracy.

The existing modeling and analysis of cutting chatter were established within the framework of linear theory. Although the chatter phenomenon was found by Taylor as early as in 1907 [1] and identified as the limit of productivity, the theoretical explanations for chatter generation were proposed after five to six decades. Regenerative chatter occurs as a result of dynamic interaction between previous and current cuts and remains the most comprehensive explanation for chatter. Several efforts on modeling chatter under linear analysis have been performed, particularly for the purpose of prediction of chatter occurrence and the conditions under which such instability occurs in the milling process [2–8].

Since linear theory cannot predict some important dynamic behaviors in cutting process, nonlinear modeling of the cutting system has received increasing attention. Hanna and Tobias first proposed a time-delay nonlinear model including quadratic and cubic structural stiffness and cutting force [9]; this study triggered a strong interest from many researchers to analyze the global dynamics of the problem [10–18].

Basically, two different approaches for modeling slender cutter bar can be found in the literature [9–18]. One is purely analytical; that is, it is based on physical and dynamic equations governing the dynamic behavior of cutter bar [17, 18]. The other is purely empirical: the simple one-dof or two-dof mass-spring-damper model is used; modal parameters of the cutter bar are measured by performing experimental tests, without explicit connection to the underlying dynamic equations [9–16].

In order to model the dynamic behaviors of a cutter bar, it is necessary to comprehensively examine the influence of various factors, such as cutter bar geometry and material, on
the modal parameters. For this purpose, a great number of repetitive tests have to be performed if the empirical method is used. It will be quite time-consuming and low-efficiency. In view of this, as a more reliable and practical modeling method, dynamic modeling of the continuous system of cutter bar based on the analytical method is needed.

In recent years, fiber-reinforced composite materials have received great interest in the structural design of cutter bars [19–24] because of a much higher specific stiffness and higher damping than conventional metal materials. The superior performance of the composite is beneficial for enhancing the stability against chatter.

However, most of the works emphasized the design of composite boring bars [20–22] or analytical prediction of chatter stability in the cutting process [23–25]; until now, no work seems to have been reported towards the study for the nonlinear dynamics of the cutting system with the cutter bar made up of composite materials.

The aim of this work is to present a dynamic model of the cutting process by considering the geometrical nonlinearity of a rotating composite cutter bar. The composite cutter bar is modeled as an inextensible continuous cantilever rotating shaft with nonlinear curvature and nonlinear inertia. The dynamic model includes the moment of inertia, gyroscopic effect, and external damping. It also is assumed that the cutter bar is subjected to a cutting force including linear regenerative effect, and external damping. It also is assumed that the cutter bar is fixed in one end while another end is free. The rotating composite cutter bar is denoted as an inertia coordinate system, while \((x, y, z)\) denotes the cross-section of the cutter bar. The coordinate origin is located at \(x\) on the centerline of the deformed cutter bar. In addition, \(u(x, t), v(x, t),\) and \(w(x, t)\) denote displacements of point \(x\) on the deformed cutter bar along \(X, Y,\) and \(Z\) directions, respectively, and \(\phi(x, t)\) denotes the torsional deformation of cross section around \(x\)-axis.

The kinetic energy of a composite cutter bar can be written as [26]

\[
T = \frac{1}{2} \left( \int \left[ m \left( \dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) + 2\rho I \dot{\omega}^2 + \rho I \left( \dot{\omega}_x^2 + \dot{\omega}_y^2 \right) \right] dx, \tag{1}
\]

2. Mathematical Model and Solution

2.1. Kinetic Energy and Potential Energy. Figure 1 displays a rotating composite cutter bar with a length of \(L\). The cutter bar is fixed in one end while another end is free. The rectangular coordinate system \((X, Y, Z)\) denotes an inertia coordinate system, while \((x, y, z)\) denotes a local coordinate system. The inertial coordinate axis is consistent with the principal axis of the cross section of the cutter bar. The coordinate origin is located at \(x\) on the centerline of the deformed cutter bar. In addition, \(u(x, t), v(x, t),\) and \(w(x, t)\) denote displacements of point \(x\) on the deformed cutter bar along \(X, Y,\) and \(Z\) directions, respectively, and \(\phi(x, t)\) denotes the torsional deformation of cross section around \(x\)-axis.

The kinetic energy of a composite cutter bar can be written as [26]

\[
m = \pi \sum_{k=1}^{N} \rho^{(k)} \left( r_{k+1}^2 - r_k^2 \right),
\]

\[
\rho I = \frac{\pi}{4} \sum_{k=1}^{N} \rho^{(k)} \left( r_{k+1}^4 - r_k^4 \right),
\]

where \(N\) denotes ply numbers, \(\rho^{(k)}\) denotes the density of the \(k\)-th layer, and \(r_k\) and \(r_{k+1}\) denote both inner and external diameters of the \(k\)-layer.

The rotating angular velocities of the coordinate system \((x, y, z)\) with respect to \((X, Y, Z)\), \(\omega_1, \omega_2,\) and \(\omega_3\) can be written as

\[
\begin{align*}
\omega_1 &= \psi \cdot \sin \psi_z, \\
\omega_2 &= \psi_y \sin \varphi \cos \psi_z + \psi_z \cos \varphi, \\
\omega_3 &= \psi_y \cos \varphi \cos \psi_z - \psi_z \sin \varphi,
\end{align*}
\]

where \(\psi = \psi + \Omega t, \psi_y\) and \(\psi_z\) denote the rotation angles of cross section around \(z\)-axis and \(y\)-axis, respectively, and \(\psi_z\) and \(\psi_y\) can be expressed as [26]

\[
\begin{align*}
\psi_z &= \sin^{-1} \left( \frac{v'}{\sqrt{(1 + u')^2 + v'^2}} \right), \\
\psi_y &= -\sin^{-1} \left( \frac{w'}{\sqrt{(1 + u')^2 + v'^2 + w'^2}} \right),
\end{align*}
\]

where \("'\) represents partial derivative with respect to time \(t\).

The elastic potential energy of a composite Euler-Bernoulli cutter bar can be written as

\[
U = \frac{1}{2} \int_V \left( \sigma_n e_x + \tau_{nx} y_n x_n \right) dV,
\]

where \(dV = r dr d\theta dx\) denotes a differential volume element in the cylindrical coordinate system and \(\alpha\) and \(r\) are polar angle and polar diameter, respectively.
By neglecting shear deformation, the stress-strain relations of the \( k \)-th layer expressed in the cylindrical coordinate system can be expressed as [27]

\[ \sigma_x = Q_{11} \varepsilon_x + Q_{16} \varepsilon_y, \quad \tau_{xa} = Q_{16} \varepsilon_x + Q_{66} \varepsilon_y, \]

(6)

(7)

where \( \sigma_x \) and \( \tau_{xa} \) denote normal stress and shear stress of the point in the cylindrical coordinate \((x, r, \alpha)\) and \( Q_{ij} \) \((i, j = 1, 6)\) denotes the off-axis stiffness coefficient of a single layer of the composite.

The strain-displacement equation can be written as

\[ \varepsilon_x = \varepsilon + \rho_2 r \sin \alpha - \rho_3 r \cos \alpha, \]

(8)

\[ \gamma_{xa} = r \rho_1, \]

(9)

where \( \rho_i \) \((i = 1, 2, 3)\) denotes the curvature of the cutter bar. By virtue of Kirchhoff’s kinetic analog [28], we can obtain the cutter bar’s curvatures \( \rho_1, \rho_2, \) and \( \rho_3 \) by replacing the time derivatives \( \partial \delta / \partial t \) with the spatial derivatives \( \partial \delta / \partial x \) in the angular velocity expressions. Therefore, the curvatures are given by [29]

\[ \rho_1 = \varphi' - \psi_y' \sin \psi_z, \]

(10)

\[ \rho_2 = \psi_y' \sin \varphi \cos \psi_z + \psi_z' \cos \varphi, \]

\[ \rho_3 = \psi_y' \cos \varphi \cos \psi_z - \psi_z' \sin \varphi. \]

The variation of the elastic potential energy can be written as

\[ U = \frac{1}{2} \int_V \left[ (\sigma_x \varepsilon_x + \tau_{xa} \gamma_{xa}) \right] dV. \]

(11)

By substituting equations (6)–(9) into equation (11), the following expression can be derived:

\[ U = \frac{1}{2} \int_0^L \left[ A_{11} \varepsilon^2 + 2B_{16} \rho_1 \varepsilon + D_{66} \rho_1^2 + D_{60} (\rho_2^2 + \rho_3^2) \right] dx, \]

(12)

where

\[ D_{11} = \frac{\pi}{4} \sum_{k=1}^N Q_{11} (r_{kk+1}^4 - r_k^4), \]

\[ D_{66} = \frac{\pi}{2} \sum_{k=1}^N Q_{66} (r_{kk+1}^4 - r_k^4), \]

\[ B_{16} = \frac{2\pi}{3} \sum_{k=1}^N Q_{16} (r_{kk+1}^3 - r_k^3), \]

\[ A_{11} = \pi \sum_{k=1}^N Q_{11} (r_{kk+1}^2 - r_k^2). \]

The strain along the elastic axis of a differential element \( dx \) is defined by

\[ \varepsilon = \sqrt{(1 + u')^2 + v'^2 + w'^2} - 1. \]

(14)

Assuming that the composite cutter bar is inextensible in axial direction, that is, the strain \( \varepsilon \) equals 0, the following expression can be derived:

\[ u' = \sqrt{1 - v'^2 - w'^2} - 1 \equiv -\frac{1}{2} (v'^2 + w'^2) + \cdots. \]

(15)

Assuming that lateral displacements \( v \) and \( w \) are of order \( O(e), \varepsilon \ll 1 \), then \( u = O(e^2) \). Substituting equation (4) into equations (3) and (10), expanding the outcomes in Taylor series, and retaining terms up to \( O(e^3) \), one can obtain the angular velocities and curvatures up to \( O(e^3) \). Substituting the resulted curvatures and angular velocities into equations (1) and (12) and using equation (15), the final expressions for kinetic and strain energy are obtained.

2.2 Virtual Work of the Cutting Force. The virtual work of the cutting force can be written as

\[ \delta W = \int_0^L \left( L_v \delta v + L_w \delta w \right) dx, \]

(16)

where \( L_v = F_y v_D(x - L) \), \( L_w = F_z v_D(x - L) \), and \( v_D \) denotes Delta function.

The regenerative cutting force, \( F_y \) and \( F_z \), can be expressed as [15]

\[
F_y = \frac{N \left[ a_0 (v - v_r) + \beta_0 (w - w_r) + \gamma_0 \right]}{2\pi} + c_f \left[ \frac{c_1 \cos(2\Omega t - (\pi/2)) - \eta_1 \cos 2\Omega t + \eta_1}{2} \right] \\
+ \left[ \eta_2 \cos \left( \Omega t - \frac{\pi}{2} \right) \right] + c_2 \cos \Omega t, \\
F_z = \frac{N \left[ a'_0 (v - v_r) + \beta'_0 (w - w_r) + \gamma'_0 \right]}{2\pi} + c_f \left[ \frac{\varepsilon_1 \cos(2\Omega t - (\pi/2)) + \varepsilon_1 \cos 2\Omega t - \varepsilon_1}{2} \right] \\
+ \left[ \varepsilon_2 \cos \left( \Omega t - \frac{\pi}{2} \right) \right] - \varepsilon_2 \cos \Omega t, 
\]

where

\[
\begin{align*}
\alpha_0 &= 0.5c_1 + 0.25\pi\eta_1, \\
\alpha'_0 &= -0.5\eta_1 + 0.25\pi c_1, \\
\beta_0 &= 0.5\eta_1 + 0.25\pi c_1, \\
\beta'_0 &= 0.5c_1 - 0.25\pi\eta_1, \\
\gamma_0 &= \eta_2 + c_2, \\
\gamma'_0 &= \varepsilon_2 - \eta_2, \\
\varepsilon_1 &= K_v a, \\
\varepsilon_2 &= K_v a, \\
\eta_1 &= K_w a, \\
\eta_2 &= K_w a, \\
\v_r &= v(x, t), \\
v_r' &= v(x, t - \tau), \\
w &= w(x, t), \\
w_r &= w(x, t - \tau).
\end{align*}
\]
Here, \( t \) denotes delay time \( (t = 2\pi/N\Omega) \), \( N \) denotes the number of cutting teeth, \((K_{tc}, K_{ro})\) and \((K_{te}, K_{re})\) are the cutting force coefficients along tangential and radial directions, respectively, \( a \) denotes the axial cutting depth, \( c_f \) denotes the feed per tooth per revolution and corresponds to the static part of the chip thickness \((c_f \sin \phi)\), and \( v_j(t) \) denotes the dynamic chip thickness produced due to vibrations of the cutter bar at the present \((v_j(t))\) and previous \((v_j(t - \tau))\) tooth periods (see Figure 2).

2.3. Equation of Motion in Milling Process. The Lagrangian is defined as \( L = T - U \). Using equations (1) and (12) and introducing the Lagrange multiplier \( \lambda \) to enforce the inextensionality conditions [30], one has

\[
L = \frac{1}{2} \int_0^L \left[ m (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + 2\pi \rho_1 \dot{\omega}_1^2 + \rho I (\dot{\omega}_2^2 + \dot{\omega}_3^2) - \left[ A_{11} \dot{\epsilon}^2 + 2B_{16} \dot{\epsilon} + D_{66} \dot{\rho}_1^2 + D_{66} (\dot{\rho}_2^2 + \dot{\rho}_3^2) \right] + \lambda \left[ 1 - (1 + \dot{u})^2 - \dot{v}^2 - \dot{w}^2 \right] \right] \, dx.
\]

(19)

To derive the governing equations of motion, we use Hamilton’s extended principle as

\[
\int_{t_1}^{t_2} (\delta L + \delta W) \, dt = 0. \tag{20}
\]

The bending-bending coupled nonlinear motion equation of a rotating composite cutter bar can be obtained as

\[
g_1 = 4v'' v' v''' + w^{(4)} v' v'' + w'' v''' + 3w'' v' w''' + w''^2 + \dot{v}^2 + v' \dot{v} + w'' \dot{w} + w''' \dot{w},
\]

(21)

\[
g_2 = 4v'' w' w''' + w^{(4)} v' w'' + v' w''' + 3v'' w' v''' + v'' w'' + w''^2 + v'' \dot{w} + w''' \dot{w},
\]

(23)

\[
h = \int_0^L \left[ \dot{v}^2 + \dot{v}'^2 + \dot{w}^2 + \dot{w}'^2 \right] \, dx.
\]

It should be noted that the torsional fundamental frequency of the cutting bar with circular cross section is much greater than the bending frequency; thus, the torsional inertia item can be neglected [29].

Using the above assumption, the following expression can be obtained:

\[
\phi = -\int_0^L v'' w' \, dx + \ldots. \tag{24}
\]

Since the cutting bar is slender with small rotation inertia, the nonlinear item multiplied with the rotational inertia can also be neglected [26]. Accordingly, bending-bending-torsion coupled nonlinear motion equation can be appropriately simplified into equations (21) and (22).

In equation (21), the terms \( g_1 \) and \( g_2 \) show the effect of the geometric nonlinearities, and the term \( h \) shows the effect of the inertia nonlinearities. In fact, \( h \) is given by \( h = -\lambda = (m/2) \int_0^L \left[ (\partial^2 \phi/\partial t^2) \int_0^L \left( \dot{v}^2 + \dot{w}^2 \right) \, dx \right] \, dx \). One can interpret \( \lambda \) as an axial force acting at the cutter bar’s tip to prevent it from stretching.
The nonlinear equation of motion in dimensionless form can be written as

\[
\ddot{v} + (c_e + c_i)\dot{v} - 2\Omega \rho I \dddot{w} - \rho I \dot{v}' + (v')' + g_1 + c_i \Omega w = F_y \delta(x - 1),
\]

(26)

\[
\ddot{w} + (c_e + c_i)\dot{w} + 2\Omega \rho I \dddot{v} - \rho I \dot{w}' + (w')' + g_2 - c_i \Omega v = F_z \delta(x - 1),
\]

(27)

where

\[
F_y = -\frac{N}{2}\left[\alpha_0(v - v_s) + \beta_0(w - w_s) + \gamma_0\right]
+ \left[c_{f\xi} \cos(2\Omega t - \pi/2) - c_{f\eta} \cos 2\Omega t + c_{f\zeta}\right]
+ \left[\eta_2 \cos \left(\Omega t - \frac{\pi}{2}\right) + \zeta_2 \cos \Omega t\right],
\]

(28)

\[
F_z = -\frac{N}{2}\left[\alpha_0'(v - v_s) + \beta_0'(w - w_s) + \gamma_0'\right]
+ \left[c_{f\xi} \cos(2\Omega t - \pi/2) + c_{f\eta} \cos 2\Omega t - c_{f\zeta}\right]
+ \left[\eta_2 \cos \left(\Omega t - \frac{\pi}{2}\right) - \eta_2 \cos \Omega t\right].
\]

The coefficients in equation (26) can be defined as

\[
\tilde{\alpha}_0 = \frac{L^3 \alpha_0}{D_{11}},
\]

\[
\tilde{\beta}_0 = \frac{L^3 \beta_0}{D_{11}},
\]

\[
\tilde{\gamma}_0 = \frac{L^3 \gamma_0}{D_{11}},
\]

\[
\tilde{c}_{f\xi} = \frac{L^3 c_{f\xi}}{D_{11}},
\]

\[
\tilde{c}_{f\eta} = \frac{L^3 c_{f\eta}}{D_{11}},
\]

\[
\tilde{\alpha}_0' = \frac{L^3 \alpha_0'}{D_{11}},
\]

\[
\tilde{\beta}_0' = \frac{L^3 \beta_0'}{D_{11}},
\]

\[
\tilde{\gamma}_0' = \frac{L^3 \gamma_0'}{D_{11}},
\]

\[
\tilde{\zeta}_2 = \frac{L^3 \zeta_2}{D_{11}},
\]

\[
\tilde{\eta}_2 = \frac{L^3 \eta_2}{D_{11}}.
\]

(29)
For simplicity, the symbol “∼” above all dimensionless variables in equations (26) and (27) has been removed.

Before applying the multiple scales method, the partial differential equations of motion are discretized by using the Galerkin method.

Let
\[ \nu(x, t) = \phi_1(x)V(t), \]
\[ \omega(x, t) = \phi_2(x)W(t). \]

For cantilever beam, the mode shape \( \phi_1(x) \) can be written as
\[ \phi_1(x) = \cos \beta_1 L x - \cosh \beta_1 L x \]
\[ - \frac{\cos \beta_1 L + \cosh \beta_1 L}{\sin \beta_1 L + \sinh \beta_1 L} (\sin \beta_1 L x - \sinh \beta_1 L x), \]
\[ x \in (0, 1), \]
(31)

where \( \beta_1 L = 1.875 \).

By substituting equation (31) into equations (26) and (27), the following ordinary differential equations can be obtained:
\[ A_1 \ddot{V} + A_1 (c_x + c_i)V - A_2 p I (2\Omega W + \ddot{V}) + A_4 V \]
\[ + A_1 c_i \Omega W + A_4 V (V^2 + \dot{V}^2 + W^2 + \dot{W}^2) \]
\[ + 2 A_3 (V^2 + W^2) = \int_0^1 F_1 \delta(x - 1)dx, \]
\[ A_1 \ddot{W} + A_1 (c_x + c_i)\dot{W} + A_2 p I (2\Omega \dot{V} - \dot{W}) + A_4 \dot{W} \]
\[ - A_1 c_i \Omega V + A_4 (\dot{V}^2 + \dot{V} \dot{W} + \dot{W}^2 + \dot{W} \dot{W}) + A_3 \dot{W} (V^2 + W^2) \]
\[ = \int_0^1 F_2 \delta(x - 1)dx, \]
(32)
in which
\[ A_1 = \int_0^1 \phi_1^2 dx, A_2 = \int_0^1 \phi_1 \phi_1'' dx, A_3 = \int_0^1 \phi_1 \phi_1^{(4)} dx, A_4 \]
\[ = \int_0^1 \phi_1 \phi_1' F_1' dx, A_5 = \int_0^1 \phi_1 F_2 dx, \]
\[ F_1 = \left( \int_0^1 \int_0^x (\phi_1')^2 dxdx \right)', F_2 = 4 \phi_1' \phi_1'' + (\phi_1')^3 \]
\[ + \phi_1^{(4)} (\phi_1'). \]
(33)

To simplify the delayed expressions in the cutting force (28), the first-order Pade approximation as \( e^{-\tau t} \equiv 1 - \tau t \) is used for delayed terms (i.e., a truncated Taylor series expansion for \( e^{-\tau t} \) with one delayed term is used) [15].

Substituting the cutting forces in equation (28) into equations (26) and (27), the following expression can be derived:
\[ (A_1 - A_2 p I) \ddot{V} + \mu_V \ddot{V} - 2\Omega p I A_1 \dot{W} + A_3 V + A_1 c_i \Omega W + A_4 V \]
\[ \cdot (V^2 + V \dot{V} + W^2 + W \dot{W}) + A_3 V (V^2 + W^2) = -\lambda_V + F_{z2} \phi_1^{(1)}, \]
\[ (A_1 - A_2 p I) \ddot{W} + \mu_W \ddot{W} + 2\Omega p I A_1 \dot{V} + A_3 W - A_1 c_i \Omega V \]
\[ + A_3 W (V^2 + V \dot{V} + W^2 + W \dot{W}) + A_4 W (V^2 + W^2) = \lambda_W + F_{z2} \phi_1^{(1)}, \]
(34)

where \( F_{z2} \) and \( F_{z2} \) denote the sum of the second and third items in the expressions of cutting forces (28) and the parameters in equation (34) are defined as
\[ \mu_V = A_1 (c_x + c_i) + \frac{N_{\phi_1}}{2\pi}, \]
\[ \mu_W = A_1 (c_x + c_i) - \frac{N_{\phi_1}}{2\pi}, \]
\[ \lambda_V = \frac{N_{\phi_1}}{2\pi}, \]
\[ \lambda_W = \frac{N_{\phi_1}}{2\pi}. \]
(35)

According to equation (35), the terms arisen from regenerative chatter mechanism \( N_{\phi_1} \phi_1^{(1)} / 2\pi \) and \( N_{\phi_1} \phi_1^{(2)} / 2\pi \) are proportional to delay time \( \tau \), and consequently inversely proportional to the spindle speed \( \Omega \). In this paper, a truncated Taylor series expansion for \( e^{-\tau t} \) is used to simplify the regenerative chatter mechanism. Therefore, it must be noticed that the results of this research are valid for small-time delays which occur essentially at high-speed machining. Here, for the considered case study, it can be shown that using the first-order truncated Taylor series for \( e^{-\tau t} \) and modifying only the damping coefficients are sufficient (as done in the works of Moradi et al. [15]).

2.4. Method of Multiple Scales. The multiple scales method is used to find an approximate solution for the proposed nonlinear dynamics of the milling process. In order to apply multiple scale method, \( V \) and \( W \) can be written in the expansion form as
\[ V(t) = e^{\epsilon V_1} (T_0, T_2) + e^{\epsilon V_2} (T_0, T_2) + \cdots, \]
\[ W(t) = e^{\epsilon W_1} (T_0, T_2) + e^{\epsilon W_2} (T_0, T_2) + \cdots, \]
(36)
where \( \epsilon \) is a small dimensionless parameter. \( T_0 = t \) and \( T_2 = e^{\epsilon t} \) are fast and slow time scales, respectively. The damping and exciting force terms should be scaled, so that their effects are balanced with nonlinearities. Thus, \( \mu_V, \mu_W, \lambda_V, \lambda_W, T_{z2}, \) and \( F_{z2} \) are replaced with \( e^{\epsilon \mu_V}, e^{\epsilon \mu_W}, e^{\epsilon \lambda_V}, e^{\epsilon \lambda_W}, e^{\epsilon F_{z2}}, \) and \( e^{\epsilon F_{z2}}. \)

Time derivatives become expansions in terms of the partial derivatives with respect to \( T_0 \) and \( T_2 \) by using the following chain rule:
\[
\frac{d}{dt} = \frac{\partial}{\partial T_1} + \varepsilon_1 \frac{\partial}{\partial T_2} + \cdots; \\
\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_1^2} + 2\varepsilon_2 \frac{\partial}{\partial T_2} + \cdots. 
\]

(37)

By substituting equations (36) and (37) into equation (34) and equating the coefficients of \( \varepsilon \) and \( \varepsilon^2 \), one has

\[O(\varepsilon) \]

\[
(A_1 - A_2 \rho I) \frac{\partial^3 V_3}{\partial T_0^3} - 2A_2 \rho \Omega \frac{\partial W_1}{\partial T_0} + A_3 V_3 = -\mu V_1 \frac{\partial V_1}{\partial T_0} - A_1 V_1 \Omega W_1 - 2(A_1 - A_2 \rho I) \frac{\partial^2 V_1}{\partial T_1^2} + 2A_2 \rho \Omega \frac{\partial W_1}{\partial T_2} 
\]

(38)

\[
- A_4 V_1 \left[ \left( \frac{\partial V_1}{\partial T_0} \right)^2 + V_1 \frac{\partial^2 V_1}{\partial T_0^2} + \left( \frac{\partial W_1}{\partial T_0} \right)^2 + W_1 \frac{\partial^2 W_1}{\partial T_0^2} \right] - A_5 V_1 \left( V_1^2 + W_1^2 \right) - \lambda_V + \Omega_{y2} \varphi_1 (1), 
\]

(39)

\[
(A_1 - A_2 \rho I) \frac{\partial^3 W_3}{\partial T_0^3} + 2A_2 \rho \Omega \frac{\partial V_1}{\partial T_0} + A_3 W_3 = -\mu W_1 \frac{\partial W_1}{\partial T_0} + A_1 V_1 \omega_1 V_1 - 2(A_1 - A_2 \rho I) \frac{\partial^2 W_1}{\partial T_1^2} - 2A_2 \rho \Omega \frac{\partial V_1}{\partial T_2} 
\]

\[
- A_4 W_1 \left[ \left( \frac{\partial V_1}{\partial T_0} \right)^2 + V_1 \frac{\partial^2 V_1}{\partial T_0^2} + \left( \frac{\partial W_1}{\partial T_0} \right)^2 + W_1 \frac{\partial^2 W_1}{\partial T_0^2} \right] - A_5 W_1 \left( V_1^2 + W_1^2 \right) + \lambda_W + \Omega_{z2} \varphi_1 (1). 
\]

The solution to equation (38) can be written as

\[
V_1(T_0, T_2) = \Phi_1(T_2) e^{j\omega_1 T_0} + \Phi_2(T_2) e^{j\omega_2 T_0} + \Phi_{y1}(T_2) e^{-j\omega_1 T_0} + \Phi_{y2}(T_2) e^{-j\omega_2 T_0}, 
\]

(40)

\[
W_1(T_0, T_2) = -iF_1(T_2) e^{j\omega_1 T_0} + iF_2(T_2) e^{j\omega_2 T_0} + iF_{y1}(T_2) e^{-j\omega_1 T_0} - iF_{y2}(T_2) e^{-j\omega_2 T_0}, 
\]

where \( i = \sqrt{-1} \) denotes the imaginary unit; \( \Phi_1(T_2) \) and \( \Phi_2(T_2) \) are the complex-valued functions to be determined, respectively; \( \Phi_{y1}(T_2) \) and \( \Phi_{y2}(T_2) \) are complex conjugates; \( \omega_1 \) and \( \omega_2 \) denote forward and backward linear natural frequencies, respectively, and can be written as

\[\omega_1 = \frac{\sqrt{\left( A_2 \rho \Omega \right)^2 + \left( A_1 - A_2 \rho I \right) A_3}}{(A_1 - A_2 \rho I)}, \]

\[\omega_2 = \frac{-\sqrt{\left( A_2 \rho \Omega \right)^2 + \left( A_1 - A_2 \rho I \right) A_3}}{(A_1 - A_2 \rho I)}. \]

(41)

By substituting (40) into equation (39), one has

\[
(A_1 - A_2 \rho I) \frac{\partial^3 V_3}{\partial T_0^3} - 2A_2 \rho \Omega \frac{\partial W_1}{\partial T_0} + A_3 V_3 = \Omega_{z3} e^{j\omega_1 T_0} + \Omega_{z4} e^{-j\omega_1 T_0} - \lambda_V + \Omega_{y2} \varphi_1 (1) + \text{cc}, 
\]

(42)

where \( \text{cc} \) denotes the complex conjugate.

The coefficients of the harmonic items on the left end of equation (42) have the following form:

\[
\mathcal{P}_3 = \frac{\Gamma F_1 F_2}{2} - 4A_2 \Omega F_1^2 + \frac{iA_2 \Omega}{2} + i\mu V_1 \omega_1 F_1 + iA_1 \omega_2 F_1, 
\]

\[
\mathcal{Q}_3 = \frac{\Gamma F_1 F_2}{2} - 4A_2 \Omega F_2^2 + \frac{iA_2 \Omega}{2} - i\mu V_1 \omega_2 F_2 - iA_1 \omega_2 F_2, 
\]

\[
\mathcal{P}_4 = \frac{iF_1 F_2}{2} + iA_2 \Omega F_1^2 + \frac{A_1 \Omega}{2} - \mu \omega_1 F_1 + A_1 \omega_2 F_1, 
\]

\[
\mathcal{Q}_4 = \frac{iF_1 F_2}{2} - iA_2 \Omega F_2^2 - \frac{A_1 \Omega}{2} + \mu \omega_2 F_2 + A_1 \omega_2 F_2, 
\]

(43)
where
\[
\begin{align*}
\Gamma &= A_4(\omega_1^2 + \omega_2^2)^2 - 16A_5, \\
\Lambda_1 &= 4\left[-(A_1 - A_2\rho I)\omega_1 + A_2\rho I\Omega\right], \\
\Lambda_2 &= -4\left[(A_1 - A_2\rho I)\omega_2 + A_2\rho I\Omega\right].
\end{align*}
\] (44)

In this paper, both types of primary and superharmonic resonances in the cutting process, namely, (I) \( \Omega = \omega_1 \) and (II) \( 2\Omega = \omega_1 \), are investigated.

(I) Primary resonance condition \( (\Omega = \omega_1) \): For this case, the expression \( \Omega = \omega_1 + \epsilon^2\sigma \) is used, where \( \sigma \) is a detuning parameter for controlling the nearness of \( \Omega \) to \( \omega_1 \).

By substituting \( \Omega = \omega_1 + \epsilon^2\sigma \) into equation (42), the following equations can be derived:
\[
\begin{align*}
(A_1 - A_2\rho I) \frac{\partial^2 V_3}{\partial T_0^2} - 2A_2\rho I\Omega \frac{\partial W_3}{\partial T_0} + A_3V_3 &= \bar{\bar{P}}_3e^{i\omega_1T_0} + \bar{\bar{Q}}_3e^{i\omega_2T_0}, \\
(A_1 - A_2\rho I) \frac{\partial^2 W_3}{\partial T_0^2} + 2A_2\rho I\Omega \frac{\partial V_3}{\partial T_0} + A_3W_3 &= \bar{\bar{P}}_4e^{i\omega_1T_0} + \bar{\bar{Q}}_4e^{i\omega_2T_0},
\end{align*}
\] (45)

where
\[
\begin{align*}
\bar{\bar{P}}_3 &= \bar{P}_3 + \bar{\gamma}_1e^{\epsilon\sigma T_0}, \\
\bar{\bar{Q}}_3 &= \bar{Q}_3, \\
\bar{\bar{P}}_4 &= \bar{P}_4 - i\bar{\gamma}_1e^{\epsilon\sigma T_0}, \\
\bar{\bar{Q}}_4 &= \bar{Q}_4.
\end{align*}
\] (46)

Here, \( \bar{\gamma}_1 = \varphi_1(1)(-\bar{\gamma}_2 + \bar{\gamma}_4)/2 \).

The particular solutions of equation (45) are
\[
\begin{align*}
V_3(T_0, T_2) &= F_{11}(T_2)e^{i\omega_1T_0} + F_{12}(T_2)e^{i\omega_2T_0}, \\
W_3(T_0, T_2) &= F_{21}(T_2)e^{i\omega_1T_0} + F_{22}(T_2)e^{i\omega_2T_0}.
\end{align*}
\] (47)

Substituting equation (47) into (45) and equating the coefficient of \( e^{i\omega_1T_0} \) and \( e^{i\omega_2T_0} \) in both sides of equation (45), one has
\[
\begin{align*}
\left( A_3 - (A_1 - A_2I)\omega_1^2 \right) F_{11} - i2A_2\rho I\Omega\omega_1F_{21} &= \bar{\bar{P}}_3, \\
i2A_2\rho I\Omega\omega_1F_{11} + \left( A_3 - (A_1 - A_2I)\omega_1^2 \right) F_{21} &= \bar{\bar{P}}_4,
\end{align*}
\] (48)

\[
\begin{align*}
\left( A_3 - (A_1 - A_2I)\omega_2^2 \right) F_{12} - i2A_2\rho I\Omega\omega_2F_{22} &= \bar{\bar{Q}}_3, \\
i2A_2\rho I\Omega\omega_2F_{12} + \left( A_3 - (A_1 - A_2I)\omega_2^2 \right) F_{22} &= \bar{\bar{Q}}_4.
\end{align*}
\] (49)

Equations (48) and (49) constitute systems of two inhomogeneous algebraic equations for \( F_{11}, F_{21} \), and \( F_{12}, F_{22} \), respectively. Their homogeneous parts have a nontrivial solution. Then their solvability conditions can be written as [33]
\[
\begin{align*}
\bar{\bar{P}}_3 &= -i2A_2\rho I\Omega\omega_1 F_{11}, \\
\bar{\bar{P}}_4 &= A_3 - (A_1 - A_2I)\omega_1^2 F_{21} = 0, \\
\bar{\bar{Q}}_3 &= -i2A_2\rho I\Omega\omega_2 F_{12} = 0, \\
\bar{\bar{Q}}_4 &= A_3 - (A_1 - A_2I)\omega_2^2 F_{22} = 0.
\end{align*}
\] (50)

After simplification, the above four solvability conditions are reduced into two independent equations governing \( F_1 \) and \( F_2 \) in the following form:
\[
\begin{align*}
iA_1\frac{\partial F_1}{\partial T_2} + i2A_1C_1\Omega F_1 - i(\mu_\nu + \mu_\omega)\omega_1 F_1 + \Gamma F_1 F_2 F_2 & = 0, \\
-8A_2 F_1^2 F_2 + 2\eta_1e^{\epsilon\sigma T_0} & = 0,
\end{align*}
\] (51)

\[
\begin{align*}
iA_2\frac{\partial F_2}{\partial T_2} - i2A_1C_2\Omega F_2 - i(\mu_\nu + \mu_\omega)\omega_1 F_2 + \Gamma F_1 F_2 F_1 & = 0, \\
-8A_3 F_2^2 F_1 & = 0.
\end{align*}
\] (52)

Express \( F_1 \) and \( F_2 \) in the polar form; that is, \( F_j = a_j(T_2)e^{i\theta_j(T_2)}/2 \), where \( a_j(T_2) \) and \( \theta_j(T_2) \) \( (j = 1, 2) \) are amplitudes and phase angles of the response, respectively. Substituting \( F_1 \) and \( F_2 \) into equations (51) and (52) and separating the real and imaginary parts, the following expression can be obtained:
\[
\begin{align*}
\frac{\Lambda_1\psi_1a_1}{2} - \frac{\sigma a_1}{8} + \frac{\Gamma a_1^2}{8} - A_3a_1^3 + \varphi_1(1) \\
&\cdot (\bar{\gamma}_2 \sin \psi_1 + \bar{\gamma}_4 \sin \psi_1) = 0, \\
\frac{\Lambda_1a_1}{2} - \frac{(\mu_\nu + \mu_\omega)\omega_1 a_1}{2} + A_1C_1a_1 + \varphi_1(1) \\
&\cdot (\bar{\gamma}_2 \sin \psi_1 - \bar{\gamma}_4 \cos \psi_1) = 0,
\end{align*}
\] (53)

\[
\begin{align*}
\frac{\Lambda_2a_2}{2} - \frac{(\mu_\nu + \mu_\omega)\omega_2 a_2}{2} - A_1C_2a_2 = 0, \\
\frac{\Lambda_2a_2}{2} - \frac{(\mu_\nu + \mu_\omega)\omega_2 a_2}{4} - 2A_3a_2^3 = 0,
\end{align*}
\] (54)

where \( \sigma T_2 - \theta_j = \psi_j \) \( (j = 1, 2) \).

To determine the steady-state forced response, the time derivatives in equation (53) are equated to zero. It can be immediately concluded that only solution for \( a_2(T_2) \) is zero. This shows that, under the primary resonance \( \Omega = \omega_1 \), only the first mode, that is, the forward mode, can be excited, while the second mode, that is, the backward mode, does not participate in the primary resonance and remains stationary.

Substituting \( a_2 = 0 \) into equation (51) and solving \( \sigma \),
\[ \sigma = \frac{2A_c a^2}{A_1} \pm 2 \frac{\varphi_1^2(1)(\psi_1^2 + \eta_2^2)/a_1^2 - c_d^2}{A_1}, \] (54)

where

\[ c_d = A_1 c_i \omega_1 + A_1 c_i (\omega_1 - \Omega) + \frac{\varphi_1^2(1)N\left(\bar{\alpha}_d - \bar{\alpha}_0\right)\omega_1 \tau}{4\pi}. \] (55)

According to equation (55), the total damping is arisen by the external damping \( c_e \), the internal damping \( c_i \), and the induced terms due to the regenerative chatter mechanism.

Damping coefficients \( c_i \) and \( c_e \) are determined by using concepts from a single-degree-of-freedom vibration system; namely,

\[ c_e + c_i = 2c\omega_1\bigg|_{\Omega=0} \frac{(A_1 - A_2 \phi_1)}{A_1}, \] (56)

where \( c \) is the damping ratio. Given a small value of \( c_i, c_e, c_i \) can be obtained from equation (56), and the reasonableness value of \( c_e, c_i \) can be checked by inspecting the decay rate in a numerically obtained solution to an initial value problem.

To study the stability of the steady-state response, consider the following equation:

\[ \frac{\Lambda_1 \psi_1 a_1}{2} - \frac{\sigma a_1}{2} - A_2 a_1^3 + \varphi_1(1)(\psi_2 \cos \psi_1 + \eta_2 \sin \psi_1) = 0, \]
\[ \frac{\Lambda_1 a_1^2}{2} - \left(\mu_1 + \mu_2\right) a_1 + A_1 c_i \Omega a_1 + \varphi_1(1) \]
\[ \cdot (\psi_2 \sin \psi_1 - \eta_2 \cos \psi_1) = 0. \] (57)

The nature of the eigenvalues is investigated by linearization of equation (55) around \((\omega_0, \psi_0)\) as

\[ \begin{pmatrix} a_1' \\ \psi_1' \end{pmatrix} = \frac{2}{\Lambda_1} \begin{pmatrix} c_d & \left(\frac{\sigma}{2} + 3A_2 a_0^2\right) a_0 \\ \frac{\sigma}{2a_0} + 3A_2 a_0 & c_d \end{pmatrix} \begin{pmatrix} a_1 \\ \psi_1 \end{pmatrix}. \] (58)

The stability of the steady-state response depends on the eigenvalue of the coefficient matrix (Jacobian matrix) on the right-hand side of equation (58):

\[ \lambda_{1,2} = c_d \pm \sqrt{\left(\frac{\sigma}{2} + 3A_2 a_0^2\right)\left(\frac{\sigma}{2} + 3A_2 a_0^2\right)} \]. (59)

To have a stable steady-state response, the real part of eigenvalues must be negative. On the other hand, the stable steady-state response will be unstable when the following condition is met:

\[ c_d^2 + \left(\frac{\sigma}{2} + 3A_2 a_0^2\right)^2 < 0. \] (60)

(II) Superharmonic resonance condition \((2\Omega \approx \omega_1)\): For this case, the formulation of steady-state response is the same as case I, but in equation (54), \( \psi_2 \) and \( \eta_2 \) are replaced with \( c_{f1}/2 \) and \( c_{f2}/2 \), respectively.

3. Numerical Results and Discussion

In this study, the composite material of carbon fiber/epoxy resin is selected as the material of the cutter bar. The mechanical properties of the material are shown in Table 1. Coefficients of cutting forces for simulation are given in Table 2. The cutter bar has a hollow structure, the outer diameter of the cross section is \( D = 8 \) mm, the inner diameter is \( D = 40 \) mm, the thickness of the section is \( h = 4 \) mm, and length \( L \) is determined by the given ratio of length to diameter. The composite cutter bar has 16 layers with identical thickness, and the stack sequence is \([\pm \theta]_8\).

Figure 3 shows the natural frequency versus rotating speed when the gyroscopic effect of the cutter bar is considered \((\text{equation } (41))\). In vibration of gyroscopic systems, there are two natural frequencies associated with forward and backward whirling motions. In forward natural frequency, the natural frequency is measured when the rotating cutter bar whirls in direction of the rotation. But, in backward natural frequency, the natural frequency is measured when the cutter bar whirls in the opposite direction of the rotation. The forward natural frequency \((\text{black solid line})\) increases with the increase of the rotational speed, while the backward natural frequency \((\text{blue dashed line})\) decreases with the increase of the rotational speed.

3.1. Primary Resonance \((\Omega \approx \omega_1)\). Figure 4 shows the effect of ratio of length to diameter on the frequency response of the vibration at the cutter tip. The increase of ratio of length to diameter results in less amplitude vibration. This is mathematically because, according to equation (29) and consequently equation (28), by increasing ratio of length to diameter, total damping increases due to the damping term arisen from the regenerative chatter mechanism. Figure 5 shows the effect of the damping ratio on the frequency response of the vibration at the cutter tip. As it is shown and physically expected, when the damping ratio increases, vibration amplitudes of the cutter tip decrease. In order to study the effect of cutting force coefficients, we let \( c_i = K_{c1} \psi_1 \) and \( \eta_2 = K_{c2} \eta_1 \) \((i = 1, 2)\), with nominal parameters \( \psi_1 \) and \( \eta_1 \) given in Table 2. Figure 6 shows the effect of the cutting force coefficient on the frequency response of the vibration at the cutter tip. As the cutting force coefficient increases, the vibration amplitudes of the cutter tip increase. Figure 7 shows the effect of ply angle on the frequency response of the vibration at the cutter tip. The vibration amplitudes of the cutter tip increase as the ply angle increase. This is physically expected. As listed in Table 1, the longitudinal elastic modulus along fiber \( E_{11} \) is significantly greater than the lateral elastic modulus \( E_{22} \). Therefore, at a larger ply angle, the flexural rigidity of the composite cutter bar decreases, thereby causing a larger vibration response at the cutter tip. Figure 8 shows the effect of the values of \( c_i/c_i + c_j \) on the frequency response of the vibration at the cutter tip. The cases of \( \beta = 0, 0.5, \) and 1 refer no internal
As it is shown, vibration amplitudes of the cutter tip are proportional to the internal damping and inversely proportional to the external damping.

The effect of rotating speed on the frequency response of the vibration at the cutter tip is shown in Figure 9. As it is observed, decreasing rotating speed values leads to a decrease in vibration amplitudes. This is physically expected, because total damping increases with the decrease of rotating speed.

**Table 1:** Mechanical properties of carbon fiber/epoxy composite [34].

| Property | Value |
|----------|-------|
| $\rho$ (kg/m$^3$) | 1672 |
| $E_{11}$ (GPa) | 25.8 |
| $E_{22}$ (GPa) | 8.7 |
| $G_{12}$ (GPa) | 3.5 |
| $G_{33}$ (GPa) | 3.5 |
| $\nu_{12}$ | 0.34 |

**Table 2:** Coefficients of cutting forces [15].

| $C_f$ (mm/rev-tooth) | $\xi_1$ (N/mm) | $\eta_1$ (N) | $\xi_2$ (N/mm) | $\eta_2$ (N) | $a_c$ (mm) | $N$ |
|----------------------|----------------|--------------|----------------|--------------|------------|-----|
| 0.2                  | 620            | 43           | 208            | 52           | 3          | 4   |

**Figure 3:** Natural frequency versus rotational speed ($L/d = 10; \theta = 0^\circ$).

**Figure 4:** The effect of ratios of length to diameter on the frequency response of the cutter tip vibration for the primary resonance ($\varsigma = 0.01$, $K_f = 1$, and $\theta = 0^\circ$).

**Figure 5:** The effect of damping ratio on the frequency response of the cutter tip vibration for the primary resonance ($L/d = 10$, $K_f = 1$, and $\theta = 0^\circ$).

**Figure 6:** The effect of cutting force coefficient on the frequency response of the cutter tip vibration for the primary resonance ($L/d = 10$, $\varsigma = 0.01$, and $\theta = 0^\circ$).
speed due to the damping effect from the regenerative chatter mechanism.

It is clear from Figures 4–9 that the curves bend to the right side because of the existence of nonlinear factors. Therefore, the effective nonlinearity for the cutting system is of hardening type. For a specified value of $\sigma$, there are three values for amplitude. Consequently, the phenomena of jumping and bifurcation occur in the system which means that the cutting system works in an unstable cutting condition.

Figures 10–15 show the amplitude versus damping ratio with different ratios of length to diameter, detuning parameter values, cutting force coefficients, ply angles, the values of $\beta$, and rotating speeds, respectively. From these figures, it can be seen that, for some values of $L/d$, $\sigma$, $K_f$, $\theta$, $\beta$, or $\Omega$, there are multivalued curves. For example, as shown in Figure 10, when $L/d = 10$ and $\varsigma < 0.018$, the system has two stable and one unstable branches, but for $L/d = 10$ and $\varsigma > 0.018$, there exists only one stable branch. For large values of $\varsigma$ and arbitrary values of $L/d$, curves are always single-valued.

Figures 16–21 show the amplitude versus ply angle with different ratios of length to diameter, detuning parameter values, cutting force coefficients, damping ratios, the values of $\beta$, and rotating speeds, respectively. Similar to the results in Figures 10–15, for some values of $L/d$, $\sigma$, $K_f$, $\varsigma$, $\beta$, or $\Omega$, multivalued curves appeared.

Figures 22–27 show the amplitude versus cutting force coefficient with different ratios of length to diameter, detuning parameter values, ply angle, damping ratios, the values of $\beta$, and rotating speeds, respectively. Again, for some values of $L/d$, $\sigma$, $\theta$, $\varsigma$, $\beta$, or $\Omega$, multivalued curves can be seen.

3.2. Superharmonic Resonance ($2\bar{\Omega} \approx \omega_1$). For the superharmonic resonance case, similar to case I, both $a_1-\sigma$ curves and the amplitude versus different parameter curves show jump phenomenon and consequently multiple solutions at a given rotating speed. The effective nonlinearity of the hardening type can also be observed. The stability of the fixed points can also be determined by investigating the eigenvalues of the Jacobian matrix of the system. Moreover, comparing primary resonance and superharmonic resonance cases indicates that under superharmonic resonance condition vibration amplitudes are generally lower than those under primary resonance. Accordingly, the
calculated results in case II are not shown here for the sake of space.

To verify the perturbation results, a numerical simulation has been utilized in Figures 28 and 29. The results achieved from multiple scales method show a good agreement with those of numerical simulation. In Figures 28 and 29, it is seen that if $A_5$ (the cubic nonlinearity) is removed from equation (54), the nonlinear solution reduces the (single-valued) linear solution. In addition, an interesting result is observed that vibration amplitudes of the linear system are less than those of nonlinear ones. This is because internal damping provides extra negative damping to the

---

**Figure 10:** Amplitude versus damping ratio with different ratios of length to diameter ($\sigma = 0.25, K_f = 1, \text{and } \theta = 00$).

**Figure 11:** Amplitude versus damping ratio with different detuning parameter values ($L/d = 10, K_f = 1, \text{and } \theta = 00$).
Figure 12: Amplitude versus damping ratio with different cutting force coefficients \((L/d = 10, \sigma = 0.25, \text{ and } \theta = 0^\circ)\).

Figure 13: Amplitude versus damping ratio with different ply angles \((L/d = 10, K_f = 1, \text{ and } \sigma = 0.25)\).

Figure 14: Amplitude versus damping ratio with different values of \(\beta\) \((L/d = 10, K_f = 1, \sigma = 0.25, \text{ and } \theta = 0^\circ)\).

Figure 15: Amplitude versus damping ratio with different rotating speeds \((L/d = 10, K_f = 1, \sigma = 0.25, \text{ and } \theta = 0^\circ)\).
Figure 16: Amplitude versus ply angle with different ratios of length to diameter ($\zeta = 0.01$, $K_f = 1$, and $\sigma = 0.25$).

Figure 17: Amplitude versus ply angle with different detuning parameter values ($L/d = 10$, $K_f = 1$, and $\zeta = 0.01$).

Figure 18: Amplitude versus ply angle with different cutting force coefficients ($L/d = 10$, $\sigma = 0.25$, and $\zeta = 0.01$).

Figure 19: Amplitude versus ply angle with different damping ratios ($L/d = 10$, $\sigma = 0.25$, and $K_f = 1$).
Figure 20: Amplitude versus ply angle with different values of $\beta$ ($L/d = 10$, $\sigma = 0.25$, $K_f = 1$, and $\varsigma = 0.01$).

Figure 21: Amplitude versus ply angle with different rotating speeds ($L/d = 10$, $\sigma = 0.25$, $K_f = 1$, and $\varsigma = 0.01$).

Figure 22: Amplitude versus cutting force coefficient with different ratios of length to diameter ($\theta = 00$, $\varsigma = 0.01$, and $\sigma = -0.25$).

Figure 23: Amplitude versus cutting force coefficient with different detuning parameter values ($\theta = 00$, $\varsigma = 0.01$, and $L/d = 10$).
Figure 24: Amplitude versus cutting force coefficient with different ply angles ($\sigma = 0.25$, $\zeta = 0.01$, and $L/d = 10$).

Figure 25: Amplitude versus cutting force coefficient with different damping ratios ($L/d = 10$, $\theta = 00$, $\sigma = 0.25$, and $K_f = 1$).

Figure 26: Amplitude versus cutting force coefficient with different values of $\beta$ ($L/d = 10$, $\theta = 00$, $\sigma = 0.25$, and $\zeta = 0.01$).

Figure 27: Amplitude versus cutting force coefficient with different rotating speeds ($L/d = 10$, $\theta = 00$, $\sigma = 0.25$, and $\zeta = 0.01$).
Figure 28: Comparison of results between the method of multiple scales and the numerical simulation (case I with internal damping).

Figure 29: Comparison of results between the method of multiple scales and the numerical simulation (case II with internal damping).

Figure 30: Comparison of results between the method of multiple scales and the numerical simulation (case I without internal damping).

Figure 31: Comparison of results between the method of multiple scales and the numerical simulation (case II without internal damping).
peak amplitudes of the nonlinear system, while internal damping has no effect on the peak amplitudes of the linear system when \( \sigma = 0 \). If internal damping is absent, vibration amplitudes of the linear system are larger than those of nonlinear ones, as can be seen in Figures 30 and 31.

4. Conclusions

This study presents the dynamic model of the cutting process considering the nonlinearity of a rotating composite cutter bar. Based on the inextensible assumption, the composite cutter bar was simplified as Euler–Bernoulli cantilever beam with nonlinear curvature and inertia. The cutter tip is subjected to regenerative cutting force. The cutting force includes harmonic exciting items. Meanwhile, the effect of internal and external damping is also considered. Nonlinear ordinary differential motion equation of the cutting system is derived by combining Hamilton’s extended principle and the Galerkin method. The analytical expressions of the nonlinear forced vibration response of the cutting system are derived by means of the method of multiple scales. Furthermore, this study demonstrates that, in primary resonance and superharmonic resonance of the cutting process with nonlinear cutter bar, only forward modes of the cutter bar are excited, the response amplitude of the backward mode equals 0, and the amplitudes of the forced vibration responses along \( y \)-axis and \( z \)-axis directions only depended on \( \alpha_i \). Due to the effect of nonlinearity, the resonance curves show hard spring vibration behavior of Duffing type oscillator; meanwhile, the jumping phenomenon and multiple solutions exist in the response curves. From the perspective of engineering application, cutting conditions and consequently initial conditions can be adjusted to achieve the stable branch with fewer vibration amplitudes. It is also found that the forced vibration resonance amplitude increases with the increase of cutting force, ply angle, internal damping, or rotating speed of composite cutter bar but decreases with the increase of damping ratio or ratio of length to diameter. External damping can reduce or limit the vibration amplitudes. The regenerative chatter mechanism (inversely proportional to rotating speed) and internal damping provide the total damping of the cutting system with positive and negative damping, respectively.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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