Operator Dynamics in Brownian Quantum Circuit

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We explore the operator dynamics in a random $N$-spin model with pairwise interactions (Brownian quantum circuit). We introduce the height $h$ of an operator to characterize its spatial extent, and derive the master equation of the height probability distribution. The study of an initial simple operator with $h = 1$ (minimal nonzero height) shows that the mean height, which is proportional to the squared commutator, has an initial exponential growth. It then slows down around the scrambling time $\sim \log N$ and finally saturates to a steady state in a manner similar to the logistic function. The deviation to the logistic function is due to the large fluctuations (order $N$) in the intermediate time. Moreover, we find that the exponential growth rate (quantum Lyapunov exponent) is smaller for initial operator with $\langle h \rangle \gg 1$. Based on this observation, we propose that the chaos bound at finite temperature can be produced by an initial operator whose height distribution is biased towards higher operators. We numerically test the power law initial height distribution $1/h^\alpha$ in a Brownian circuit with number of spin $N = 10000$ and show that the Lyapunov exponent is linearly constrained by $\alpha$ before reaching the infinite temperature value.

I. INTRODUCTION

Quantum many-body chaos is an interesting phenomenon that has drawn a lot of attention from various fields, including quantum gravity\cite{1}, quantum information\cite{2} and condensed matter physics\cite{3}. The dynamics of chaos can be diagnosed by the squared commutator (out-of-time-order correlator)\cite{4}.

$$
C(t) = -\langle [V(t), W]^2 \rangle_{\beta}
$$

where $V(t) = e^{iHt}V(0)e^{-iHt}$. In some chaotic many-body systems, $C(t)$ can increase exponentially in time, \textit{i.e.}, $C(t) \sim e^{\lambda t}$, where the positive number $\lambda$ is the quantum analogy of Lyapunov exponent\cite{1, 5–9}. These systems are sometimes referred to as “fast scramblers”\cite{10}.

Recently, the Lyapunov exponent has been extensively studied in various fast scramblers, among which a large class are analyzed in the language of field theory in the large $N$ limit\cite{7, 9, 11, 12}. In this paper, we give an alternative way to understand the Lyapunov exponent by directly inspecting the evolution of the Heisenberg operator $V(t)$. In particular, we will focus on the operator dynamics in the Brownian quantum circuit with 2-local interactions (all-to-all interaction)\cite{13}. This model is analytically tractable and has been used to estimate scrambling time and a variety of other time scales\cite{13, 14}.

Our idea is motivated by the recent study of the random unitary circuit with local interaction, in which the operator spreading is found to obey an emergent hydrodynamics in the “operator Hilbert space”\cite{15, 16}. In this model, the typical length of the operator grows linearly in time with the front spreading out diffusively. This physical picture has been successfully applied to the spin-1/2 chain with local interactions\cite{15–17} and is further generalized to spin chain in the presence of conserved quantity\cite{18, 19} or with non-local interactions\cite{20}.

We write down the master equation of the operator height distribution (defined in Sec. II) in a Brownian quantum circuit model. We show that the squared commutator is proportional to the mean height and hence the growth of height reflects the operator spreading. Our solution of the height distribution covers the full range of time which allows us to separate the dynamics of a simple operator into three stages: (1) the initial exponential growth of the mean height during scrambling time of order $\log N$, which has also been observed in other fast scramblers in the large $N$ limit, (2) the slow down of the operator growth caused by finite $N$ and large height fluctuation in the intermediate time and (3) the exponential decay to the equilibrium state with the decay rate the same as the initial Lyapunov exponent.

We further consider an initial condition not localized around $h = 1$ (minimal nonzero height) and demonstrate that the operator scrambling is suppressed with the early time exponential growth rate getting smaller. This is relevant to the generic chaos bound at finite temperature $\lambda \leq \frac{2\pi}{\beta}$ proposed by Ref. 8. We make a connection with the thermal operator $e^{-\beta H/2}V(0)e^{-\beta H/2}$ and argue that this gives a physical interpretation of quantum chaos bound in terms of the operator scrambling. We also perform numerical simulation with non-local power law distribution as initial conditions and discuss the connection with the crossover behaviors from high temperature to low temperature.

The rest of the paper is organized as follows. In Sec. III, we define the operator dynamics and introduce the concept of the height distribution. In Sec. III, we present the master equation of the height distribution in a Brownian quantum circuit and discuss properties of the general solution. In Sec. IV, we discuss the re-
sults of height distribution evolved from a simple operator and power law initial conditions. We conclude in Sec. V. The appendices contain calculation details of the squared commutator and the master equation.

II. OPERATOR DYNAMICS

A quantum wavefunction can be expanded into a linear combination of bases in the Hilbert space

$$\psi(t) = \sum_i c_i \psi_i,$$  \hspace{1cm} (2)

where the coefficient squared $|c_i|^2$ is interpreted as the probability of collapsing to the basis state $\psi_i$ in a measurement. Likewise, quantum operators also live in an operator Hilbert space. It is equipped with the Hilbert-Schmidt inner product

$$\langle O_1, O_2 \rangle = \frac{\text{tr}(O_1^\dagger O_2)}{\text{tr}(\mathbb{I})}$$  \hspace{1cm} (3)

and therefore can have an orthonormal operator basis $B_j$, such that any operator can be expanded as

$$O(t) = \sum_j \alpha_j(t) B_j.$$  \hspace{1cm} (4)

We interpret $|\alpha_j(t)|^2$ as the probability of the basis $B_j$. Their evolution determines the operator dynamics.

We introduce the concept of height for the purpose of studying the spatial extent of the operators. We take a particular set of basis consisting only of tensor products of single site operators (generalized Pauli matrices) $\sigma^\mu = X^\mu_1 Z^\mu_2$ where $\mu_1, \mu_2 \in \mathbb{Z}_q$ and $\mu \in \mathbb{Z}_2$. $X$ and $Z$ are $q \times q$ matrices satisfying $ZX = e^{2\pi i / q} ZX$ and $Z^q = X^q = 1$. The height of each basis is the number of non-identity single site operators. Since $|\alpha_j(t)|$ is the weight of basis $B_j$, the height of a generic operator is the weighted sum

$$f(h, t) = \sum_{\text{height}(B_j) = h} |\alpha_j(t)|^2.$$  \hspace{1cm} (5)

We define a $N + 1$ component unit normalized vector $f$, whose $k$th component $f_k$ is $f(h = k, t)$. In the continuum limit $f(h, t)$ is the height probability density.

The squared commutator of two initially simple operators $O(0)$ and $V$ is defined as

$$C(t) = -\frac{1}{N} \text{tr}([V(t), W]^2).$$  \hspace{1cm} (6)

In the Brownian quantum circuit model, we take $W = \sigma^\mu$ at site 0, and the initial operator $V(0)$ to be evenly distributed into each height space (e.g. taking $V(0) = \sum_{\alpha_i=1}^{q^2-1} \sum_{i=1}^N \sigma_i^\alpha$ for initial distribution $f_k(t = 0) = \delta_{k1}$).

This simplification is reasonable because chaotic evolution will quickly mix the basis if it is not equally distributed in the first place. With this definition, we can show that (App. A)

$$C(t) = \frac{\langle h(t) \rangle}{N},$$  \hspace{1cm} (7)

namely the squared commutator is proportional to the mean height. Therefore the operator scrambling is encoded in the height distribution.

III. OPERATOR DYNAMICS OF BROWNIAN QUANTUM CIRCUIT

A. Brownian Quantum Circuit

Brownian motion is a continuous random walk. In a time interval $\Delta t$, the displacement of the walker is a Gaussian random variable with variance proportional to $\Delta t$. A Brownian quantum circuit can generate continuous random unitary evolution[13]. The evolution operator $U(t)$ performs a random walk on the manifold of the unitary group. The direction of the displacement is specified by the Hamiltonian.

In this paper, we study the Brownian quantum circuit in Ref. 13 that contains only (up to) two-body interactions. More concretely, in a time interval $\Delta t$, the evolution is governed by the Hamiltonian

$$H_s = J \sum_{i<j} \sum_{\mu_1, \mu_2=0}^{q^2-1} \sigma^\mu_i \otimes \sigma^\mu_j \Delta B_{i,j, \mu_1, \mu_2}^s$$  \hspace{1cm} (8)

where $\sigma^\mu_i$ are the generalized Pauli matrices on site $i$ and $\sigma^0_i = \mathbb{I}$. The strength $B_{i,j, \mu_1, \mu_2}^s$ of the spin-spin interactions over sites $i$ and $j$ is a Gaussian random variable with variance proportional to $\Delta t$. The continuum limit of the evolution

$$e^{-iH_s \Delta t} e^{-iH_{s-1} \Delta t} \ldots$$  \hspace{1cm} (9)

defines the Brownian quantum circuit. The “infinitesimal Hamiltonian" $dG(t)$ of an infinitesimal evolution

$$U(t + dt) = \exp(-idG(t)) U(t)$$  \hspace{1cm} (10)

is a standard Brownian motion in the space of two-body interactions

$$dG(t) = J \sum_{i<j} \sum_{\mu_1, \mu_2=0}^{q^2-1} \sigma^\mu_i \otimes \sigma^\mu_j \, dB_{i,j, \mu_1, \mu_2}^s(t)$$  \hspace{1cm} (11)

We will compute all quantities averaged over these Brownian motions.

In the following, we use the normalization (different from Ref. 13)

$$J = \sqrt{\frac{2}{q^4 N}}$$  \hspace{1cm} (12)

where $N$ is the total number of sites.
B. Master Equation of the Height Distribution

The master equation of the height distribution \( f_k(t) \) is

\[
\frac{df(t)}{dt} = A_f f(t),
\]

where \( A_f \) is a tri-diagonal matrix with entries

\[
(A_f)_{k,k} = \frac{4}{n} \left( -k + \frac{1}{q^2} (n - 2k + 1) \right)
\]
\[
(A_f)_{k-1,k} = \frac{4}{n} \frac{k(k-1)}{q^2}
\]
\[
(A_f)_{k+1,k} = \frac{4}{n} k(n-k) \left( 1 - \frac{1}{q^2} \right).
\]

The equation is derived by adopting and developing the Itô calculus techniques from Ref. 13, see details in App. B. It has recently been derived as a special case of the Brownian cluster model in Ref. 21 by a different approach.

The evolution matrix \( A_f \) has zero column sum and preserves the total probability. Its surprisingly simple tri-diagonal structure is a consequence of purely two-body interactions in the Hamiltonian: the interaction terms can only change the height by one in an infinitesimal step. Thus the evolution of the height is local, despite the highly non-local quantum evolution. The height growth rate, as we will see in Sec. IV A, is much faster than the biased diffusion of the 1d chain with local interactions\[15, 16, 21\]. To decrease the height by one, the interaction terms should commute with the operator in the Heisenberg evolution. The chance is \( \frac{1}{q^2} \) which is consistent with the \( \frac{1}{q^2} \) term in the upper diagonal element of \( A_f \). This analysis is essentially the same as the case of a random unitary gate\[15, 16\].

The height probability has two linearly independent stationary solutions. The first one is solely the identity operator

\[
f_0 = 1 \quad f_k = 0 \quad k \geq 1
\]

as the identity is invariant under unitary evolution. In fact the probability of identity and non-identity operators are separately conserved. The non-identity sector containing those operators of non-zero height will be driven to

\[
f_0 = 0 \quad f_k = \binom{n}{k} \frac{(q^2 - 1)^k}{q^{2n} - 1} \quad k \geq 1
\]

This solution can be interpreted as the ratio of the height \( k \) operators with respect to the all. Therefore, the ultimate fate of the non-identity sector is an equal weight superposition of all non-identity operators, i.e., a maximally random operator. The average height saturates to

\[
h_{\text{sat}} = \langle h \rangle = \frac{n(q^2 - 1)q^{2n-2}}{q^{2n} - 1} \approx n \left( 1 - \frac{1}{q^2} \right).
\]

C. General Solutions of the Master Equation

We solve the master equation analytically for any initial conditions. To express the result, we define

\[
\lambda_q = 4 \left( 1 - \frac{1}{q^2} \right)
\]

where \( \lambda_q \) will be shown to be the Lyapunov exponent, see Sec. IV A.

At early time, or large \( N \) limit at fixed time, the master equation simplifies to

\[
\frac{df_k(t)}{dt} = -\lambda_q kf_k + \lambda_q (k-1) f_{k-1}.
\]

In the Sachdev-Ye-Kitaev (SYK) model\[5, 7\], a similar equation for the same quantity was proposed (called \( P(s_k,t) \))\[22\], which was (partly) motivated from a classical bacteria infection problem. Here we write the solution in terms of its generating function

\[
\sum_{k=0}^{\infty} f_k(t) z^k = \left( \frac{ze^{-\lambda_q t}}{1 - z(1 - e^{-\lambda_q t})} \right)^{k'} f_{k'}(t = 0). \tag{20}
\]

All the moments can thus be computed, see Sec. IV A. However, this only works for early time with distributions localized at small height, i.e., \( \langle h \rangle \ll N \).

To understand the physics beyond early time, we take the continuum limit of the master equation and get

\[
\partial_t f(h,t) = -\frac{4}{N} \partial_h [h(h_{\text{sat}} - h)f] \tag{21}
\]

The flux of the probability density vanishes at \( h = 0 \) and \( h = h_{\text{sat}} \), making the total probability conserved\[1\].

Multiplying the equation by \( h \) and integrate, we see that the mean height obeys

\[
\partial_t \langle h \rangle = \frac{4}{N} \langle h_{\text{sat}} - \langle h \rangle \rangle \langle h \rangle - \frac{4}{N} \langle (h^2) - \langle h \rangle^2 \rangle. \tag{22}
\]

The way of writing this equation suggests that if the fluctuation of the height is much smaller than \( N \), the mean height evolves according to the logistic equation. The solution can be written as

\[
\langle h(t) \rangle = y(\langle h(0) \rangle, t) \tag{23}
\]

where \( y(h, t) \) is the logistic function

\[
y(h, t) = \frac{h_{\text{sat}} h e^{\lambda_q t}}{h_{\text{sat}} + h (e^{\lambda_q t} - 1)}. \tag{24}
\]

1 Strictly speaking, this discrete equation allows to generate operators higher than \( h_{\text{sat}} \). However, those exceptions are exponentially small (even for small \( N \)) as can be seen from the steady state distribution.
This was derived in a heuristic way in Ref. 23, under the assumption of vanishing fluctuation and essentially $q = \infty$ limit (the operator will never be shortened). In brief, a two body interaction term that increases the height of a basis by one must have one spin inside the basis and one spin outside. Hence the rate must be proportional to $(N - \langle h \rangle)/\langle h \rangle$. However, the height fluctuation can not be neglected and it reduces the growth rate at intermediate time, see Sec. IV A and Fig. 1d.

The general solution can be obtained by the method of characteristics. It is conveniently expressed as

$$f(h, t) = f(y^{-1}(h, t = 0), t = 0) \frac{dy^{-1}(h, t)}{dh}$$  \hspace{1cm} (25)

whose moments are

$$\langle h^l(t) \rangle = \int_0^N f(h, t) h^l dh = \int_0^N f(h, 0) [y(h, t)]^l dh.$$  \hspace{1cm} (26)

IV. SCRAMBLING AND DYNAMICS OF THE HEIGHT DISTRIBUTION

A. Evolving from a Simple Operator

In this section, we study the height distribution evolved from a simple operator, which is localized at one site. It can be modeled by a delta distribution for the discrete master equation or an exponentially localized site. It can be modeled by a delta distribution for the evolved from a simple operator, which is localized at one
time and then slows down on the way to the steady state. As the evolution goes on, the mean height increases to an appreciable fraction of the maximal height in a very short period of time and then slows down on the way to the steady state distribution.

We obtain an approximate solution\(^2\) by Eq. (26)

$$\langle h(t) \rangle = h_{\text{sat}} (1 - se^{\lambda_q t})$$  \hspace{1cm} (27)

where $Ei[s]$ is the standard exponential integral function and $s$ is a parameter

$$s = h_{\text{sat}} \exp(-\lambda_q t)$$  \hspace{1cm} (28)

that separates the evolution into three different time regimes.

Early time: $s \sim h_{\text{sat}}$. This is the regime where the average height has not felt the existence of the saturation height (or $N$). By using the discrete solution in Eq. (20), the initial condition $f_k(t = 0) = \delta_{k1}$ gives

$$f_k(t) = e^{-\lambda_q t}[1 - e^{-\lambda_q t}(k-1)]$$  \hspace{1cm} (29)

\(^2\) Relative difference $< 0.1\%$ compared to the numerical data of $N = 10^4$

FIG. 1: The height distribution and mean height evolved from simple operator localized at $h = 1$. (a) Comparison of the mean height with logistic function and analytic solution in Eq. (27). The large fluctuation in the intermediate time makes the dynamics slower than the logistic function. (b) Early time: the profile looks like a collapsing sand pile. It is exponentially decreasing in space and the exponent is exponentially decreasing in time. Inset: numerically measured exponent is linearly proportional to $t$ on the semi-log scale with the slope $\lambda_2 = -2.96 \cong -3$ the same as $-\lambda_q$ at $q = 2$. (c) Late time: the height distribution is exponentially decaying to the steady state of a random operator. Inset: The numerical verification of Eq. (32). (d) Intermediate time: the height distribution has appreciable value over almost the whole range of height. The height fluctuation is of order $N$.

The distribution profile looks like a collapsing sandpile whose surface (the probability) is exponentially decreasing in space. As time goes on, the surface rapidly becomes flatter as the exponent log$(1 - e^{-\lambda_q t}) \approx e^{-\lambda_q t}$ also exponentially decays with time. These have been numerically confirmed in Fig. 1a.

The average height is growing exponentially with the same exponent

$$\langle h(t) \rangle = \partial_s \sum_{k=0}^\infty s^k f_k(t) \bigg|_{s=1} = e^{\lambda_q t} \langle h(t = 0) \rangle$$  \hspace{1cm} (30)

regardless of the initial condition (as long as $\langle h \rangle/N \sim 0$). Hence the squared commutator

$$C(t) = \frac{e^{\lambda_q t}}{N} \langle h(t = 0) \rangle$$  \hspace{1cm} (31)

has exactly the same behaviors of many chaotic large-$N$ models [1, 5–9]. This is an alternative way to show the scrambling time $\sim \log N$ and justifies to call $\lambda_q$ the Lyapunov exponent. The scaling can also be obtained by the large $s$ expansion of Eq. (27), which gives the same $\langle h \rangle \sim h_{\text{sat}} 1^\frac{1}{s} = e^{\lambda_q t}$.
Late time: $s \ll 1$. This is the regime when the distribution is close to the steady state defined by a random operator. Fig. 1b shows this approaching process. The small $s$ expansion of Eq. (27) gives

$$
\langle h(t) \rangle \approx h_{\text{sat}}(1 + s \ln s + \gamma s).
$$

It will saturate to $h_{\text{sat}}$ exponentially with the decay rate the same as the initial growth. Notice that there is an extra linear $t$ correction term in front of $e^{-\lambda_q t}$ and is verified numerically in the inset of Fig. 1b.

Intermediate time: $s \sim O(1)$. The is the regime between the early time and late time. $f(h)$ is a broad distribution with a large fluctuation of order $O(N)$ (see Fig. 1c). Because of this, the operator growth is slower than the logistic function (Fig. 1d).

B. Power Law Initial Distribution and Thermal Operator

It would be interesting to understand the bound of the Lyapunov exponent at finite temperature in this context, i.e., the operator dynamics of $e^{-\beta H/2}V(t)e^{-\beta H/2}$. Strictly speaking, there is no way to define finite temperature in a Brownian quantum circuit since the energy is not conserved. Here we interpret the finite temperature effect as a change of the initial height condition from the previously discussed simple operators.

As shown in Sec. IV A, the exponential growth rate in early time is always $\lambda_q$ (see Eq. (30)) when $\langle h \rangle \ll N$. The growth rate starts to become smaller when the ratio $\langle h \rangle / N$ is finite, suggesting a suppression of operator spreading in the Hilbert space. For example, when the initial distribution is $e^{-(h-h_0)}\theta(h-h_0)$ where $h_0 \sim a h_{\text{sat}}$ with $0 < a < 1$, we have

$$
\langle h(t) \rangle = h_{\text{sat}}[1 + s e^{s+h_0}Ei(-s-h_0)].
$$

The small $t$ behavior (large $s$ expansion) is

$$
h(t) \approx h_{\text{sat}} \frac{h_{\text{sat}}}{s+h_0} \approx \frac{1}{\exp(-\lambda_q t) + a} \leq \exp(\lambda_q(1-a)t) \quad (34)
$$

The exponent is therefore bounded by $\lambda_q(1-a)$. We numerically tested the initial condition $f_k = \delta_{k0}$ in Eq. (13) and present the result in Fig. 2a. We find that the exponential growth rate is smaller and saturates the upper bound in Eq. (34) around $t = 0$.

This reveals a possible interpretation for the chaos bound at finite temperature – the thermal operator is no longer localized at small height any more and should have more contribution from large height. At low temperature $\beta \gg 1$, if the Lyapunov exponent scales as $1/\beta$, according to the bound in Eq. (34), we propose that the thermal operator has a large weight around the height $\sim h_{\text{sat}}(1 - 1/\beta)$.

We further investigate the evolution from an initial power-law decreasing distribution $f(h, t = 0) \sim \frac{1}{h}$, which favors longer operators than the localized distributions above. Its relation to the finite temperature physics is not justified, but this can always be viewed as a case study of possible slower operator growth. We test the range of $\alpha \in [1.5, 2.5]$ and fit the early time growth with

$$
\langle h(t) \rangle \approx \exp(\lambda(\alpha)t)).
$$

The inset of Fig. 2b shows that between $\alpha \in [1.5, 2]$, $\lambda$ is linearly proportional to $\alpha$ and approaches $\lambda_q$ when $\alpha > 2$.

V. CONCLUSION AND DISCUSSION

In this work, we study the operator dynamics in a $2$-local Brownian quantum circuit. We first introduce the height probability distribution of an operator and relate its mean value to the squared commutator. Then we derive a continuum master equation and obtain the solutions for the full range of time and numbers of spins $N$.

We find that an initially simple operator (with $h = 1$) will have an exponential growth with Lyapunov exponent $\lambda = 4(1 - \frac{1}{q^2})$, where $q$ is the Hilbert dimension of the spin. This growth will slow down around the scrambling time when the operator has a broad distribution over a large fraction in the operator space. It finally approaches the saturation value in a manner similar to the logistic function. The logistic type growth is exact if fluctuation is negligible compared to $N$. This holds true in the early and late time, partially validates the heuristic arguments discussed in Ref. 23. While in the intermediate stage, the large fluctuation impedes the growth so that the dynamics is slower than the logistic function.

We further give an interpretation of the chaos bound at finite temperature in the language of operator scrambling. We find that higher operators generically have smaller growth rate than shorter operators. Hence the
The extension to the thermal squared commutator is
\[
C(\beta, t) = -\frac{1}{N} \text{tr}([e^{-\frac{\beta H}{2}} O(t)e^{-\frac{\beta H}{2}}, V]^2)
\]
\[
= -\frac{1}{N} \text{tr}([e^{iHt}e^{-\frac{\beta H}{2}} O(0)e^{-\frac{\beta H}{2}}e^{-iHt}, V]^2)
\]
(A4)

The only change is the replacement of the initial operator $O(0)$ to its thermal version $e^{-\frac{\beta H}{2}} O(0)e^{-\frac{\beta H}{2}}$.

Appendix B: Derivation of the Master Equation

In this appendix, we develop techniques in Ref. 13 to derive the discrete master equation of the height distribution. The key observation is that purity equation in Ref. 13 applies not just to density matrix $\rho$ but to any operators.

We take the evolved operator $O(t)$ and compute its partial trace and average “purity”
\[
\phi_A = \frac{1}{q^t} \text{tr}^2_A(O(t))
\]
(B1)

As we assumed in the text, the initial operator is equally distributed on the basis of each height, such that the “purity” $\phi_A$ will only depend on the number of sites in region $A$. All the Itô calculus computation in Ref. 13 follows and we get
\[
\frac{d\phi_k}{dt} = \frac{k(n-k)}{n} \left\{ \frac{4}{q} \phi_{k-1} - 4 \left( 1 + \frac{1}{q^2} \right) \phi_k + \frac{4}{q} \phi_{k+1} \right\}
\]
(B2)

where $\phi_k$ is the “purity” for arbitrary $k$ sites.

We then introduce an intermediate variable: the cut averaged purity
\[
\Phi_k(t) = q^{n-k} \sum_{|A|=k} \phi_A(t) = q^{-(n-k)} \left( \frac{n}{k} \right) \phi_k(t)
\]
(B3)

where the summation is over all the region containing $k$ sites. Through elementary counting, the height distribution is related to $\Phi$ as
\[
\Phi = \text{PL}_{\tau} f
\]
(B4)

where $(\text{PL}_{\tau})_{ij} = \binom{N-j}{N-i}$ is the element (zero based index) of the rotated Pascal’s lower triangle matrix. The master equation in matrix form is
\[
\frac{df}{dt} = A_f f
\]
(B5)

where $A_f = \text{PL}_{\tau}^{-1} A_{\phi} \text{PL}_{\tau}$, the inverse matrix $\text{PL}_{\tau}^{-1}$ has element $(-1)^{i+j} \binom{N-j}{N-i}$ and $A_{\phi}$ is tri-diagonal

\[
(A_{\phi})_{k,k} = -\frac{4}{n} \left[ 1 + \frac{1}{q^2} \right] k(n-k) \quad \text{diagonal}
\]
\[
(A_{\phi})_{k,k-1} = \frac{4}{n} (n-k)(n-k+1) \quad \text{lower diagonal}
\]
\[
(A_{\phi})_{k,k+1} = 4 \frac{1}{n q^2} k(k+1) \quad \text{upper diagonal}
\]
(B6)
To derive the exact matrix element of $A_f$, we notice that the element of $A_f$ can be obtained by taking derivatives, e.g.,

$$k(n-k) = \frac{\partial_x \partial_y x^k y^{n-k}}{x=y=1} \quad (B7)$$

We construct vector $A_{YX} = ZPL^{-1}_r YXPL_r$, where $X = [x^0, x^1, \ldots, x^n]$, $Y = [y^n, y^{n-1}, \ldots, y^0]^\top$, $Z = [z^0, z^1, \ldots, z^n]$. By using the combinatorial property of the Pascal matrix, we have

$$(A_{YX})_k = \sum_{i=1}^{n} (1-z)^{n-i} z^i x^{n-i} (1+x)^{n-k} x_k. \quad (B8)$$

Selecting out terms with the corresponding power (e.g. diagonal element corresponds to terms with total power of $x$ and $y$ to be $n$) and using the derivative trick, we can solve the matrix $A_f$. We find that $A_f$ is a tri-diagonal matrix with elements specified in Eq. (14).

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