A spectrum associated with Minkowski diagonal continued fraction

by Alena Aleksenko

Let $\alpha$ be real irrational number. The function $\mu_\alpha(t)$ is defined as follows. The Legendre theorem states that if

$$\left| \alpha - \frac{A}{Q} \right| < \frac{1}{2Q^2}, \quad (A, Q) = 1$$

then the fraction $\frac{A}{Q}$ is a convergent fraction for the continued fraction expansion of $\alpha$. The converse statement is not true. It may happen that $\frac{A}{Q}$ is a convergent to $\alpha$ but (1) is not valid. One should consider the sequence of the denominators of the convergents to $\alpha$ for which (1) is true. Let this sequence be

$$Q_0 < Q_1 < \cdots < Q_n < Q_{n+1} < \cdots .$$

Then for $\alpha \not\in \mathbb{Q}$ the function $\mu_\alpha(t)$ is defined by

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot ||Q_n\alpha|| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot ||Q_{n+1}\alpha||, \quad Q_n \leq t \leq Q_{n+1} .$$

From the other hand, for every $\nu$ one of the consecutive convergent fractions $\frac{p_\nu}{q_\nu} \to \frac{p_{\nu+1}}{q_{\nu+1}}$ to $\alpha$ satisfies (1). So either

$$(Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1})$$

for some $\nu$, or

$$(Q_n, Q_{n+1}) = (q_{\nu-1}, q_\nu)$$

for some $\nu$.

Actually the function $\mu_\alpha(t)$ was considered by Minkowski [4]. There exists an alternative geometric definition of $\mu_\alpha(t)$. Some related facts were discussed in [3, 6].

The quantity

$$m(\alpha) = \limsup_{t \to +\infty} t \cdot \mu_\alpha(t) .$$

was considered in [6]. An explicit formula for the value of $m(\alpha)$ in terms of continued fraction expansion for $\alpha$ was proved in [6]. It is as follows. Put

$$m_n(\alpha) = \begin{cases} G(\alpha_\nu^*, \alpha_{\nu+2}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1}) \text{ with some } \nu, \\ F(\alpha_{\nu+1}^*, \alpha_{\nu+2}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1}) \text{ with some } \nu, \end{cases}$$

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where
\[ G(x, y) = \frac{x + y + 1}{4}, \quad F(x, y) = \frac{(1 - xy)^2}{4(1 + xy)(1 - x)(1 - y)} \]
and \( \alpha_\nu, \alpha^*_\nu \) come from continued fraction expansion to
\[ \alpha = [a_0; a_1, a_2, \ldots, a_t, \ldots] \]
in such a way:
\[ \alpha_\nu = [a_\nu; a_{\nu+1}, \ldots], \quad \alpha^*_\nu = [0; a_\nu, a_{\nu-1}, \ldots, a_1]. \]

Then
\[ m(\alpha) = \limsup_{n \to +\infty} m_n(\alpha), \]
The spectrum
\[ \mathbb{M} = \{ m \in \mathbb{R} : \exists \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ such that } m = m(\alpha) \}, \]
was studied in [6]. It was proven there that \( \mathbb{M} \subset [\frac{1}{4}, \frac{1}{2}] \) and that \( \frac{1}{4}, \frac{1}{2} \in \mathbb{M} \). However no further structure of the spectrum \( \mathbb{M} \) is known.

In this paper we consider the spectrum
\[ \mathbb{I} = \{ m \in \mathbb{R} : \exists \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ such that } i(\alpha) = m \}, \]
where
\[ i(\alpha) = \liminf_{n \to \infty} m_n(\alpha) \]
(However, compared to \( m(\alpha) \), this quantity has no clear Diophantine sense).

It is clear that
\[ \min \mathbb{I} = \frac{1}{4}, \quad \max \mathbb{I} = \frac{1}{2}. \]

**Theorem.** There exists positive \( \omega_0 \) such that
\[ \left[ \frac{1}{4}, \omega_0 \right] \subset \mathbb{I}. \]

The proof is based on M. Hall’s ideas (see [2]). It uses technique from [5].

**Remark.** An explicit formula for \( \omega_0 \) may be obtained from the proof below. It is interesting to get optimal estimates for the value of \( \omega_0 \).

We need some well known results.

Recall the definition of a \( \tau \)-set \( \mathcal{F} \subset \mathbb{R} \). The set \( \mathcal{F} \) must be of the form
\[ \mathcal{F} = S \setminus \left( \bigcup_{\nu+1}^{\infty} \Delta_\nu \right), \]
where \( S \subset \mathbb{R} \) is a segment, and \( \Delta_\nu \subset S, \nu = 1, 2, 3, \ldots \) is an ordered sequence of disjoint intervals. Moreover for every \( t \) if
\[ S \setminus \left( \bigcup_{\nu+1}^{t-1} \Delta_\nu \right) = \bigcup_{j=1}^{r} \mathcal{M}_j \]
is a union of segments $\mathcal{M}_j$ and $\Delta_t \subset \mathcal{M}_j$ then

$$\mathcal{M}_j = \mathcal{N}^1 \sqcup \Delta_t \sqcup \mathcal{N}^2,$$

and

$$\min(|\mathcal{N}^1|, |\mathcal{N}^2|) \geq \tau |\Delta_t|.$$

Consider the set

$$F_5 = \{ \alpha = [0; b_1, b_2, b_3, \ldots] : b_\nu \leq 5 \ \forall \ \nu \}$$

consisting of all irrational real numbers from the unit interval $(0, 1)$ with partial quotients bounded by $5$. One can easily see that

$$\begin{cases}
A = \min F_5 = [0; 5, 1] = \frac{\sqrt{45} - 5}{10} = 0.1708^+,
B = \max F_5 = [0; 1, 5] = \frac{\sqrt{45} - 5}{2} = 0.85410^+.
\end{cases} \tag{3}$$

Put

$$S_5 = [A, B] \subset [0, 1].$$

The following lemma comes from the results of the papers [1] or [7].

**Lemma 1.** The set $F_5$ is a $\tau$-set with $\tau = \tau_5 = 1.788^+$. Let $H(x, y) : S \times S \to \mathbb{R}$ be a function in two variables of the class $G \in C^1(S \times S)$. Consider the set

$$J = \{ z \in \mathbb{R} : \exists x, y \in S \ z = H(x, y) \}.$$

By continuity argument $J$ is a segment.

**Lemma 2.** Suppose that the derivatives $\partial H/\partial x, \partial H/\partial y$ do not take zero values on the box $S \times S$. Suppose that $F$ is $\tau$-set and $S = [\min F, \max F]$. Suppose that

$$\tau \geq \max_{x, y \in S} \max \left( \left| \frac{\partial H/\partial x}{\partial H/\partial y} \right|, \left| \frac{\partial H/\partial y}{\partial H/\partial x} \right| \right). \tag{4}$$

Then

$$\{ z : \exists x, y \in F \ such \ that \ z = H(x, y) \} = J.$$

Lemma 2 is a straightforward generalization of a result from [5]. We do not give its proof here as the proof follows the argument from [5] word-by-word.

Now we are able to conclude the proof of Theorem.

We consider pairs of integers $(R_1, R_2)$ of the form

$$(R_1, R_2) = (R, R) \ or \ (R, R + 1) \tag{5}$$

with $R \geq 6$. Consider a function

$$H_{R_1, R_2}(x, y) = F \left( \frac{1}{R_1 + x}, \frac{1}{R_2 + y} \right).$$

For $R_1, R_2$ under consideration the function $H_{R_1, R_2}(x, y)$ decreases both in $x$ and in $y$. For $0 < x, y < 1$ put

$$\varphi(x, y) = (1 - 3x + 3xy - x^2y)(1 - x).$$
For any \( y \in (0, 1) \) the function \( \varphi(x, y) \) decreases in \( x \). For any \( x \in (0, 1) \) the function \( \varphi(x, y) \) increases in \( y \). Now
\[
\frac{\partial F}{\partial y} = \varphi(x, y),
\]
and
\[
\frac{\partial H_{R_1, R_2}}{\partial y} = \varphi \left( \frac{1}{R_1 + x}, \frac{1}{R_2 + y} \right) \left( \frac{R_1 + x}{R_2 + y} \right)^2.
\]
Easy calculation shows that for \( R_1, R_2 \geq 6 \) one has
\[
\max_{x, y \in S_5} \left( \frac{\partial H_{R_1, R_2}}{\partial x}, \frac{\partial H_{R_1, R_2}}{\partial y} \right) = \varphi \left( \frac{1}{R_1 + B}, \frac{1}{R_2 + A} \right) \left( \frac{R_1 + B}{R_2 + A} \right)^2 \leq \varphi \left( \frac{1}{R_1 + B}, \frac{1}{R_2 + A} \right) \left( \frac{6 + B}{6 + A} \right)^2 = 1.363 < \tau_5.
\]
Here \( A \) and \( B \) are defined in (3) and in the last inequalities we use the bounds \( 6 \leq R_1 \leq R_2 \) which follows from (5).

We see that for any \( R_1, R_2 \) under consideration and for \( \tau_5 \)-set \( F_5 \) the condition (4) is satisfied. We apply Lemma 2 to see that the image of the set \( F_5 \times F_5 \) under the mapping \( H_{R_1, R_2}(x, y) \) is just the segment
\[
J_{R_1, R_2} = [H_{R_1, R_2}(B, B), H_{R_1, R_2}(A, A)].
\]
But
\[
H_{R,R}(B, B) < H_{R,R+1}(A, A)
\]
and
\[
H_{R,R+1}(B, B) < H_{R+1,R+1}(A, A).
\]
That is why if we put
\[
\omega_0 = H_{R_0,R_0}(A, A).
\]
with \( R_0 \geq 6 \) we get
\[
\bigcup_{R \geq R_0} J_{R,R} \cup \bigcup_{R \geq R_0} J_{R,R+1} = \left( 1/4, \omega_0 \right).
\]
Take \( m \in (0, \omega_0] \). Then there exists \( R_1, R_2 \) such that
\[
m \in J_{R_1, R_2}
\]
and there exist
\[
\beta = [0, b_1, b_2, \ldots, b_\nu, \ldots], \quad \gamma = [0, c_1, c_2, \ldots, c_\nu, \ldots], \quad \beta, \gamma \in F_5,
\]
such that
\[
F \left( \frac{1}{R_1 + \alpha}, \frac{1}{R_2 + \beta} \right) = m.
\]
Now we take
\[ \alpha = [0; a_1, R_1, R_2, b_1, a_2, a_1, R_1, R_2, b_1, b_2, \ldots, a_\nu, a_{\nu-1}, \ldots, a_2, a_1, R_1, R_2, b_1, b_2, \ldots, b_{\nu-1}, b_\nu, \ldots]. \]

Standard argument shows that for \( n_\nu \) defined from
\[ \frac{p_{n_\nu}}{q_{n_\nu}} = [0; a_1, R_1, R_2, b_1, a_2, a_1, R_1, R_2, b_1, b_2, \ldots, a_\nu, a_{\nu-1}, \ldots, a_2, a_1, R_1] \]
one has
\[ \lim_{\nu \to +\infty} F(\alpha_{n_\nu}^*, \alpha_{n_\nu+1}^{-1}) = m. \]

At the same time for \( F(\cdot, \cdot) \) and \( G(\cdot, \cdot) \) we have
\[ \inf_{n \not\in \mathbb{Z}^+} F(\alpha_n^*, \alpha_{n+1}) > \omega_0 \]
and
\[ \inf_{n \not\in \mathbb{Z}^+} G(\alpha_n^*, \alpha_{n+2}^{-1}) > \omega_0, \]
for large \( R_0 \). So \( \mathfrak{i}(\alpha) = m \) and everything is proved. \( \Box \)

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