Abstract

We present the q-deformed multivariate hypergeometric functions related to Schur polynomials as tau-functions of the KP and of the two-dimensional Toda lattice hierarchies. The variables of the hypergeometric functions are the higher times of those hierarchies. The discrete Toda lattice variable shifts parameters of hypergeometric functions. The role of additional symmetries in generating hypergeometric tau-functions is explained.

1 Introduction

Hypergeometric functions play an important role both in physics and in mathematics. Many special functions and polynomials (such as q-Askey-Wilson polynomials, q-Jacobi polynomials, q-Racah polynomials, q-Hahn polynomials, expressions for Clebsch-Gordan coefficients) are just certain hypergeometric functions evaluated in special values of parameters. In physics hypergeometric functions and their q-deformed counterparts sometimes play the role of wave functions and correlation functions for quantum integrable systems. In the present paper we shall construct hypergeometric functions as tau-functions of the Kadomtsev-Petviashvili (KP) hierarchy of equations. It is interesting that the KP equation, which originally serves in plasma physics, now plays a very important role in physics (for modern applications see review) and in mathematics. This peculiarity of Kadomtsev-Petviashvili equation appeared in the paper [3], where L-A pair of the KP equation was presented, and mainly in the paper of V.E.Zakharov and A.B.Shabat in 1974 where this equation was integrated by the dressing method. Actually
it was the paper \[2\] where so-called hierarchy of higher KP equations appeared. Another very important equation is the two-dimensional Toda lattice integrated first in \[8\]. In the present paper we use these equations to construct hypergeometric functions which depend on many variables, these variables are KP and Toda lattice higher times. Here we shall use the general approach to integrable hierarchies of Kyoto school \[3\]. Especially a set of papers about Toda lattice \[4, 7, 12, 18, 23\] is important for us.

We briefly outline the connection of what we do with Zakharov-Shabat dressing method \[1\] and with the nonlocal \(\partial\) problem \[4\]; we mention the related system of orthogonal polynomials.

The lack of space does not allow us to develop these topics.

We devote this paper to Vladimir Evgen’evich Zakharov on his 60 birthday.

\[2\] Notations

There are several well-known different multivariate generalizations of hypergeometric series of one variable \[9, 10\]. Let \(|q| < 1\) and let \(x_{(N)} = (x_1, \ldots, x_N)\) be indeterminates. The multiple basic hypergeometric series \[9, 10\] is

\[
p_\Phi_s \left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_s} \mid q, x_{(N)} \right) = \sum_{l(n) \leq N} \frac{(q^n; q)_n \cdots (q^{a_p}; q)_n}{(q^{b_1}; q)_n \cdots (q^{b_s}; q)_n H_n(q)} s_n(x_{(N)}) .
\]  

(1)

The sum is over all different partitions \(n = (n_1, n_2, \ldots, n_r)\), where \(n_1 \geq n_2 \geq \cdots \geq n_r, r \leq |n|, |n| = n_1 + \cdots + n_r\), and whose length \(l(n) = r \leq N\). Schur polynomial \(s_n(x_{(N)})\), with \(N \geq l(n)\), is a symmetric function of variables \(x_{(N)}\) and defined as follows \[12\]:

\[
s_n(x_{(N)}) = \frac{a_{n+\delta}}{a_\delta}, \quad a_n = \det(x_i^{n_j})_{1 \leq i, j \leq N}, \quad \delta = (N - 1, N - 2, \ldots, 1, 0).
\]  

(2)

Schur function \(s_n(x_{(N)}) = 0\) for \(N < l(n)\). In the theory of the KP hierarchy \[3\] it is convenient to define Schur functions in terms of KP higher times \(t = (t_1, t_2, \ldots)\) as

\[
s_n(t) = \det(p_{n_i - i + j}(t))_{1 \leq i, j \leq r}, \quad \sum_{m=0}^{+\infty} p_m(t)z^m = \exp(\sum_{i=1}^{+\infty} t_i z^i) = e^{\xi(t, z)}.
\]  

(3)

The functions \(s_n(t)\) are related to \(s_n(x_{(N)})\) via the Miwa’s change of variables:

\[
t_m = \sum_{i=1}^{N} \frac{x_i^m}{m}.
\]  

(4)

Each coefficient \((q^a; q)_n\) in \(\Phi\) is expressed in terms of the so-called q-deformed Pochhammer symbols \((q^a; q)_n\):

\[
(q^a; q)_n = (q^a; q)_{n_1}(q^a; q)_{n_2} \cdots (q^a; q)_{n_r}
\]  

(5)

\[
(q^a; q)_{n_i} = (1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+n_i-1}), \quad (q^a; q)_0 = 1
\]  

(6)

q-deformed 'hook polynomials' \(H_n(q)\) are

\[
H_n(q) = \prod_{(i,j) \in n} \left( 1 - q^{h_{ij}} \right), \quad h_{ij} = (n_i + n'_j - i - j + 1),
\]  

(7)

where \(n' = (n'_1 + n'_2 + \cdots + n'_{r'})\) is the conjugated partition \[12\] and \(q^{n(n)} = q^{\sum_{i=1}^{r}(i-1)n_i}\).
The formula
\[ p\Phi_s \left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_s} \bigg| q, x(N), y(N) \right) = \sum_{n=0}^{\infty} \frac{(q^{a_1}; q)_n \cdots (q^{a_p}; q)_n q^n (q^{b_1}; q)_n \cdots (q^{b_s}; q)_n}{(q^{p_1}; q)_n \cdots (q^{p_m}; q)_n H_n(q)} \frac{s_n(x(N)) s_n(y(N))}{n!} \] (8)
defines the multiple basic hypergeometric function of two sets of variables \[\text{[11]}\].

Another generalization of hypergeometric series is so-called hypergeometric function of matrix argument \(X\) with indices \(a\) and \(b\) \[\text{[10]}\] :
\[ pF_s \left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_s} \bigg| X \right) = \sum_n \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_s)_n} Z_n(X) |n|! . \] (9)

Here \(X\) is \(N \times N\) matrix, and \(Z_n(X)\) is zonal spherical polynomial for the symmetric space \(GL(N, C)/U(N)\), see \[\text{[10]}, \text{[11]}\]. Let us note that in the limit \(q \to 1\) series \[\text{[11]}\] coincides with \[\text{[10]}\], see \[\text{[11]}\].

Now let us review some facts from the KP theory \[\text{[5]}\]. We have fermionic fields \(\psi(z) = \sum_{k \in Z} \psi_k z^k\) and \(\psi^*(z) = \sum_{k \in Z} \psi_k^* z^{-k-1} dz\), where fermionic operators satisfy the canonical anticommutation relations:
\[ [\psi_m, \psi_n]_+ = [\psi^*_m, \psi^*_n]_+ = 0; \quad [\psi_m, \psi^*_n]_+ = \delta_{mn}. \] (10)

Let us introduce left and right vacuums by the properties:
\[ \psi_m |0\rangle = 0 \quad (m < 0), \quad \psi^*_m |0\rangle = 0 \quad (m \geq 0), \] \[ \langle 0| \psi_m = 0 \quad (m \geq 0), \quad \langle 0| \psi^*_m = 0 \quad (m < 0). \] (11)

Throughout the text the subscript \(*\) does not denote the complex conjugation. The vacuum expectation value is defined by relations:
\[ \langle 0| 1 \rangle = 1, \quad \langle 0| \psi_m \psi^*_m |0\rangle = 1 \quad m < 0, \quad \langle 0| \psi^*_m \psi_m |0\rangle = 1 \quad m \geq 0, \] \[ \langle 0| \psi_m \psi_n |0\rangle = \langle 0| \psi^*_m \psi^*_n |0\rangle = 0, \quad \langle 0| \psi_m \psi^*_n |0\rangle = 0 \quad m \neq n. \] (13)

Let us notice that if a function \(h\) has no poles and zeroes for integer value of argument, then relations \[\text{[10]}\)-\[\text{[14]}\] are invariant under the following transformation
\[ \psi_n \mapsto \frac{1}{h(n)} \psi_n, \quad \psi^*_n \mapsto h(n) \psi^*_n. \] (15)

Let us denote \(\hat{gl}(\infty) = Lin \{1, : \psi_i \psi^*_j : |i, j \in Z\}\), with usual normal ordering \(\psi_i \psi^*_j := \psi_i \psi^*_j - \langle 0| \psi_i \psi^*_j |0\rangle\). We define the operator \(g\) which is an element of the group \(\hat{GL}(\infty)\) corresponding to the infinite dimensional Lie algebra \(\hat{gl}(\infty)\). The tau-function of the KP equation and the tau-function of the two-dimensional Toda lattice (TL) sometimes are defined as
\[ \tau_{KP}(M, t) = \langle M | e^{H(t)} g | M \rangle, \quad M \in Z, \] \[ \tau_{TL}(M, t, t^*) = \langle M | e^{H(t)} g e^{H(t^*)} | M \rangle, \quad M \in Z, \] (16) (17)
where \( t = (t_1, t_2, \ldots) \) and \( t^* = (t^*_1, t^*_2, \ldots) \) are called higher Toda lattice times \([3,\, 17]\) (the first set \( t \) is in the same time the set of higher KP times). \( H(t) \) and \( H^*(t^*) \) belong to the following \( \hat{gl}(\infty) \) Cartan subalgebras:

\[
H(t) = \sum_{n=1}^{+\infty} t_n H_n, \quad H^*(t^*) = \sum_{n=1}^{+\infty} t^*_n H_{-n}, \quad H_n = \frac{1}{2\pi i} \oint : z^n \psi(z) \psi^*(z) :.
\]

According to \([3]\) the integer \( M \) in \((17)\) plays the role of discrete Toda lattice variable and defines the following charged vacuums

\[
\langle M \rangle = \langle 0 | \psi^*_M, \quad |M \rangle = \psi_M |0\rangle,
\]

\[
\psi_M = \psi^*_{M-1} \cdots \psi^*_1 \psi_0 \quad M > 0, \quad \psi_M = \psi^*_{M-1} \cdots \psi^*_2 \psi^*_1 \quad M < 0,
\]

\[
\psi^*_M = \psi^*_0 \psi^*_1 \cdots \psi^*_{M-1} \quad M > 0, \quad \psi^*_M = \psi^*_1 \psi^*_2 \cdots \psi^*_{M} \quad M < 0.
\]

**Lemma 1**\([3]\) For \(-j_1 < \cdots < -j_r < 0 \leq i_s < \cdots < i_1, \ s - r \geq 0\) the following formula is valid:

\[
\langle s - r | e^{H(t)} \psi^*_{-j_1} \cdots \psi^*_{-j_r} \psi_{i_s} \cdots \psi_{i_1} |0\rangle = (-1)^{j_1 + \cdots + j_r + (r-s)(r-s+1)/2} s_n(t),
\]

where the partition \( n = (n_1, \ldots, n_{s-r}, n_{s-r+1}, \ldots, n_{s-r+j_1}) \) is defined by the following pair of partitions:

\[
(n_1, \ldots, n_{s-r}) = (i_1 - (s - r) + 1, i_2 - (s - r) + 2, \ldots, i_{s-r}),
\]

\[
(n_{s-r+1}, \ldots, n_{s-r+j_1}) = (i_{s-r+1}, \ldots, i_{s_j} - 1, \ldots, j_r - 1).
\]

Here \( (\ldots | \ldots) \) is another notation for a partition \([12]\).

So-called vertex operators \( V(z) \) and \( V^*(z) \) are defined by:

\[
V(z) = z^M e^{\xi(t, z)} e^{-\xi(\partial z^{-1})}, \quad V^*(z) = z^{-M} e^{-\xi(t, z)} e^{\xi(\partial z^{-1})},
\]

where \( \partial = \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \ldots \right) \) and the function \( \xi(t, z) \) is the same as in \((3)\). The Baker-Akhiezer function and the conjugated one are

\[
w(M, t; z) = \frac{V(z) \tau}{\tau} \quad w^*(M, t; z) = \frac{V^*(z) \tau}{\tau}.
\]

The function \( u = 2 \partial^2_t \log \tau \) solves the celebrated KP equation

\[
u_{t_3} = \frac{1}{4} u_{t_1 t_1 t_1} + \frac{3}{2} u u_{t_1} + \frac{3}{4} \int_{t_1}^{t_1} u_{t_2 t_2} dt_1.
\]

It is well-known that the Schur functions \( s_n(t) \) are tau-functions for some rational solutions of the KP hierarchy. It is also known that not any linear combination of Schur functions turns out to be a KP tau-function. In order to find these combinations one should solve bilinear difference equation (see \([3]\)), a version of discrete Hirota equation. Below we shall present KP tau-functions which are infinite hypergeometric series of Schur polynomials \((11), (12)\). We shall use the fermionic representations of tau-functions \([3]\).


3 Hypergeometric tau-functions

3.1 Additional symmetries and tau-function

Let \( r \) be a function of one variable. We shall assume that \( r(n) \) is finite for \( n \in \mathbb{Z} \). Let \( D = \frac{d}{dz} \) acts on the space of functions \( \{ z^n; n \in \mathbb{Z} \} \). Then \( r(D)z^n = r(n)z^n \). All functions of operator \( D \) which we consider below are given via their eigenvalues on this basis.

Let us consider an abelian subalgebra in \( \hat{gl}(\infty) \) formed by the following set of fermionic operators

\[
A_k = \frac{1}{2\pi i} \oint \psi^*(z) \left( \frac{1}{z} r(D) \right)^k \psi(z), \quad k = 1, 2, \ldots,
\]

where the operator \( r(D) \) acts on all functions of \( z \) from the right hand side. For the collection of independent variables \( \beta = (\beta_1, \beta_2, \ldots) \) we denote

\[
A(\beta) = \sum_{n=1}^{\infty} \beta_n A_n.
\]

We set \( r_0(M) = 1 \). Using the notation from (21) we have

**Lemma 2** The following formula holds

\[
\langle 0 | \psi_{i_1}^* \cdots \psi_{i_s}^* \psi_{-j_s} \cdots \psi_{-j_1} e^{-A(\beta)} | 0 \rangle = (-1)^{j_1 + \cdots + j_s} r_n(0) s_n(\beta).
\]

Let us consider the following tau-function of the KP hierarchy

\[
\tau_r(M, t, \beta) := \langle M | e^{H(t)} e^{-A(\beta)} | M \rangle.
\]

**Proposition 1** We have the following expansion:

\[
\tau_r(M, t, \beta) = \sum_{n=0}^{+\infty} \sum_{|n| = n} r_n(M) s_n(t) s_n(\beta).
\]

We shall not consider the problem of convergence of this series. The variables \( M, t \) play the role of KP higher times, \( \beta \) is a collection of group times for a commuting subalgebra of additional symmetries of KP (see \([13, 14, 15]\) and Remark 7 in \([13]\)). From different point of view (32) is a tau-function of two-dimensional Toda lattice \([17]\) with two sets of continuous variables \( t, \beta \) and one discrete variable \( M \). Formula (32) is symmetric with respect to \( t \leftrightarrow \beta \). This 'duality' supplies us with the string equation \([23]\) which characterizes a tau-function of hypergeometric type (see below). In \([18]\) the similar expansions to (32) were considered, without specifying the coefficients and in a different context.

Let us introduce

\[
\tilde{A}_k = -\frac{1}{2\pi i} \oint \psi^*(z) (\tilde{r}(D)z)^k \psi(z), \quad (k = 1, 2, \ldots), \quad \tilde{A}(\beta) = \sum_{n=1}^{\infty} \beta_n A_n.
\]

Then we have the following generalization of Proposition 1:

**Proposition 2**

\[
\langle M | e^{\tilde{A}(\beta)} e^{-A(\beta)} | M \rangle = \sum_n (\tilde{r}r)_n(M) s_n(\beta) s_n(\beta).
\]
For an interpretation of this expansion see Remark 2 below. Let us mark that the function $r(M)$ is connected with the Sato $b$-function \cite{11}.

Let us consider the following difference equations on functions $\tilde{h}(D), h(D)$:

$$\tilde{h}(D)\tilde{r}(D)\tilde{h}^{-1}(D-1) = 1, \quad h^{-1}(D-1)r(D)h(D) = 1.$$ \hfill (35)

The similar equation appeared in the paper of Graev (see formula (6) in \cite{19}) and was used for generating of different hypergeometric series. As in \cite{19}, in terms of the operator $r(D)$, it is possible to construct the differential equations (and q-difference equations) for hypergeometric functions. We shall present some examples below in (50) and (54).

Our fermionic representation is equivalent to that used in Toda lattice theory \cite{17} if $r(n) \neq 0, n \in \mathbb{Z}$. We define a Hamiltonian $H_0(h) \in \mathfrak{gl}(\infty)$ and a set of $C_n, n \in \mathbb{Z}$:

$$H_0(h) := \frac{1}{2\pi i} \oint : \psi^{*}(z) \log(h(D))\psi(z) :$$ \hfill (36)

$$C_n = \frac{1}{h(n-1)} \cdots \frac{1}{h(1)} \frac{1}{h(0)} \frac{1}{h(n-1)} \cdots \frac{1}{h(1)} \frac{1}{h(0)}, \quad n > 0,$$ \hfill (37)

$$C_n = h(n) \cdots h(-2)h(-1)\tilde{h}(n) \cdots \tilde{h}(-2)\tilde{h}(-1), \quad n < 0.$$ \hfill (38)

**Proposition 3** If function $r$ has no zeroes at integer values of argument then

$$\tau(n, \beta, \bar{\beta}) := \langle n|e^{\tilde{A}(\beta)} e^{-A(\bar{\beta})}|n\rangle = \langle n|e^{H(\bar{\beta})} g e^{-H^{*}(\beta)}|n\rangle C_n^{-1},$$ \hfill (39)

$$g = e^{H_0(\tilde{h}h)} = e^{H_0(h)+H_0(h)}, \quad C_n = \langle n|g|n\rangle.$$ \hfill (40)

For $\tilde{r} = 1$ we can put $\beta = t$. Then the following equations hold

$$\partial_{t_{1}} \partial_{t_{2}} \phi_{n} = r(n)e^{\phi_{n-1,\bar{\beta}n} - r(n+1)e^{\phi_{n,\bar{\beta}n+1}, \quad e^{-\phi_{n}} = \frac{\tau(n+1, t, \beta)}{\tau(n, t, \beta)}},$$ \hfill (41)

$$\tau(n)\partial_{t_{1}} \tau(n) = \partial_{t_{1}} \tau(n)\partial_{t_{1}} \tau(n) = r(n)\tau(n-1)\tau(n+1).$$ \hfill (42)

If the function $r$ has no integer zeroes, then after the change $\phi_{n} = -\phi_{n} - \log h(n)$ we obtain Toda lattice equation in standard form \cite{17}:

$$\partial_{t_{1}} \partial_{t_{1}} \varphi_{n} = e^{\varphi_{n+1,\bar{\beta}n} - e^{\varphi_{n,\bar{\beta}n}} - e^{\varphi_{n,\bar{\beta}n}} - e^{\varphi_{n,\bar{\beta}n}}}.$$ \hfill (43)

The main point of the paper is based on the observation that if $r(D)$ is a rational function of $D$ then $\tau_{r}$ is a hypergeometric series. If $r(D)$ is a rational function of $q^{D}$ we obtain q-deformed hypergeometric series. We shall see both cases in the following examples. (In a separate paper the case of rational expressions of elliptic theta-functions will be considered).

### 3.2 Examples of the tau-functions

Now let us consider various $r(D)$.

**Example 1** Let $r(M) = M$ and $\beta = (\beta_{1}, 0, 0, ...)$. For $M = \pm 1$ we get

$$\tau(1, t, \beta_{1}) = e^{\xi(t, \beta_{1})}, \quad \tau(-1, t, \beta_{1}) = e^{-\xi(t, \beta_{1})}.$$ \hfill (44)

Thus $\beta_{1}$ plays the role of a spectral parameter for the vacuum Baker-Akhiezer function. This is in accordance to the meaning of $\beta_{1}$ as a group time for the Galilean transformation \cite{3}.

**Example 2** Let all parameters $b_{k}$ be nonintegers.

$$r_{s}(D) = \frac{(D + a_{1})(D + a_{2}) \cdots (D + a_{p})}{(D + b_{1})(D + b_{2}) \cdots (D + b_{s})}. \hfill (45)$$
If all $a_k$ are also nonintegers the relevant $h(D)$ is:

$$p h_s(D) = \frac{\Gamma(D + b_1 + 1)\Gamma(D + b_2 + 1) \cdots \Gamma(D + b_s + 1)}{\Gamma(D + a_1 + 1)\Gamma(D + a_2 + 1) \cdots \Gamma(D + a_p + 1)}. \quad (46)$$

For the correlator (42) we have:

$$p\tau_s(M, t, \beta) = \sum_{n=0}^{\infty} \sum_{|n|=n} s_n(t) s_n(\beta) \frac{(a_1 + M)_n \cdots (a_p + M)_n}{(b_1 + M)_n \cdots (b_s + M)_n} H_n \quad (47)$$

One can get the hypergeometric function related to Schur functions [11] by putting $\beta_1 = 1$ and $\beta_i = 0$ for $i = (2, 3, \ldots)$ in (17):

$$p F_s \left( \begin{array}{c} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{array} \right)_{t_1, t_2, \ldots} = \sum_{n=0}^{\infty} \sum_{|n|=n} \frac{(a_1 + M)_n \cdots (a_p + M)_n s_n(t)}{(b_1 + M)_n \cdots (b_s + M)_n} H_n \quad (48)$$

with hook polynomial $H_n = s_n(\beta) = \prod_{(i,j) \in n} h_{ij}, h_{ij} = (n_i + n_j' - i - j + 1)$ [12].

We obtain the ordinary hypergeometric function of one variable of type

$$p^{-1} F_s(\pm t_1 \beta_1) = \tau(\pm 1, t, \beta), \quad (49)$$

if we take $a_1 = 0, \ t = (t_1, 0, 0, \ldots), \ \beta = (\beta_1, 0, 0, \ldots)$ [13].

Now we consider the following change of variables $t_m = \sum_{i=1}^{N} \frac{x_i^m}{m}$. In this case the formula (18) turns out to be the hypergeometric function of matrix argument, see [3]. Taking $N = 1$ we obtain the ordinary hypergeometric function of one variable, $x_1$. Formula (67) will explain the connection between this hypergeometric function and the function (19). The ordinary hypergeometric function satisfies well-known hypergeometric equation

$$(\partial_{x_1} - p r_s(D_1)) p F_s(x_1) = 0, \ \ D_1 := x_1 \partial_{x_1}. \quad (50)$$

Example 3 The $q$-generalization of the Example 2:

$$p r^{(q)}_s(D) = \prod_{i=1}^{p} \frac{(1 - q^{a_i + D})}{(1 - q^{b_i + D})}. \quad (51)$$

For the variables $\beta_i = \frac{1}{q(i-1)} \ (i = 1, 2, \ldots)$ and $t_m = \sum_{j=1}^{N} \frac{x_j^m}{m}$ we obtain the formula (11) (see [12])

$$p r^{(q)}_s \left( \begin{array}{c} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{array} \right)_{q, x^{(N)}} = \sum_{n=0}^{\infty} \sum_{|n|=n} \frac{(q^{a_1 + M}; q)_n \cdots (q^{a_p + M}; q)_n q^p(n)}{(q^{b_1 + M}; q)_n \cdots (q^{b_s + M}; q)_n H_n(q)} s_n(x^{(N)}). \quad (52)$$

When $N = 1$ we have the ordinary $q$-deformed hypergeometric function:

$$p \Phi_s \left( \begin{array}{c} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{array} \right)_{q, x_1} = \sum_{n=0}^{\infty} \frac{(q^{a_1 + M}; q)_n \cdots (q^{a_p + M}; q)_n x_1^n}{(q^{b_1 + M}; q)_n \cdots (q^{b_s + M}; q)_n (q; q)_n} \quad (53)$$
which satisfies the following q-difference equation

\[
\left(\frac{1}{x_1} (1 - q^{D_1}) - y r_s^{(q)}(D_1)\right) y \Phi_s(x_1) = 0, \quad D_1 := x_1 \partial_{x_1}.
\]

(54)

There are various applications for series (53), for instance see \[28\], \[29\] and \[30\]. Bosonic representation of (53) was found in \[27\]. Let us note that operator \(q^D\) which acts on fermions \(\psi(z)\) was used in \[31\] in different context.

**Example 4 q-Askey-Wilson polynomials**

Let operator \(4r_3^{(q)}(D)\) be

\[
4r_3^{(q)}(D) = \frac{(1 - q^{-n+D})(1 - abcdq^n+1-D)(1 - ae^{i\eta}q^D)(1 - ae^{-i\eta}q^D)}{(1 - abq^D)(1 - acq^D)(1 - adq^D)}.
\]

(55)

By choosing \(\beta_i = \frac{1}{i(1-q^i)}\) and \(t = (q, q^2, q^3, \ldots)\) we get

\[
4r_3^{(q)}(M, t, \beta) = 4r_3 \left( q^{-n}, q^{M+n-1}abcd, qa, q^{M+1}e^{i\eta}, q^{M+1}e^{-i\eta} \mid q, q \right) = \sum_{m=0}^{+\infty} \frac{(q^{M-n}; q)_m(q^{M+n-1}abcd; q)_m(aq^{M+1}e^{i\eta}; q)_m(aq^{M+1}e^{-i\eta}; q)_m q^m}{(abq^M; q)_m(acq^M; q)_m(adq^M; q)_m(q; q)_m}.
\]

(56)

For \(M = 0\) we obtain q-Askey-Wilson polynomials:

\[
p_n(\cos \eta; a, b, c, d|q) = aq^{-n}(a b|q)_n(a c|q)_n(a d|q) n! r_3^{(q)}(t, \beta, 0).
\]

(57)

**Example 5 Clebsch-Gordan coefficients** \(C_q(l, j)\) see \[10\].

Let us take

\[
3r_2^{(q)}(D) = \frac{(1 - q^{j-l_1+D})(1 - q^{j-l_1+D})(1 - q^{-l_1+m+D})}{(1 - q^{j-l_1+D})(1 - q^{j-l_1+D})(1 - q^{-l_1+m+D})}.
\]

(58)

For the variables \(t = (q, q^2, q^3, \ldots)\) and \(\beta_i = \frac{1}{i(1-q^i)}\)

\[
3r_2^{(q)}(0, t, \beta) = 3\Phi_2 \left( \begin{array}{c} j - l_1, l_1 + j + 1, -l + m \\ l_2 - l + j + 1, -l - l_2 + j \end{array} \mid q, q \right). \]

(59)

Thus we have the following fermionic representation: \(C_q(l, j) = \)

\[
\frac{(-1)^{l_1-j}q^B \Delta(l)[l + l_2 - j]![l_1][l_1][2l + 1]^{1/2}}{[l_1 - l_2 + l]![l + l_2 - l_1]![l_2 - l + j]![l_1 - j]![l_2 + k]![l - m]!^3 r_2^{(q)}(0, t, \beta)}.
\]

(60)

\[
[a] := q^{(1-a)/2} \frac{1 - q^a}{1 - q}, \quad [n] := [1][2] \cdots [n], \quad m = j + k,
\]

\[
\Delta(l) \equiv \Delta(l_1, l_2, l) := \left( \frac{[l_1 + l_2 - l][l_1 - l_2 + l][l - l_1 + l_2]}{[l_1 + l_2 + l + 1]} \right)^{1/2},
\]

\[
[l, j] = [l_1 + j][l_1 - j][l_2 + k][l_2 - k][l + m][l - m]!,
\]

\[
B = \frac{1}{4} (l_2(l_2 + 1) - l_1(l_1 + 1) - l(l + 1) + 2j(m + 1)).
\]
3.3 Different representations

Let us rewrite hypergeometric series in different way representing all Pochhammer coefficients \((q^n; q)_n\) and \((a)_n\) through Schur functions. This gives us the opportunity to interchange the role of Pochhammer coefficients and Schur functions in \([8],[18]\), and to present different fermionic representations of the hypergeometric functions. We have the following relations (see \([12]\)):

\[
\prod_{(i,j) \in \mathbb{N}} (1 - q^{n+j-i}) = \frac{s_n(t(a, q))}{s_n(t(\infty, q))}, \quad \prod_{(i,j) \in \mathbb{N}} (a + j - i) = \frac{s_n(t(a))}{s_n(t(\infty))},
\]

where parameters \(t_m(a, q)\) and \(t_m(a)\) are chosen via generalized Miwa transform \([20]\) with multiplicity \(a\)

\[
t_m(a, q) = \frac{1 - (q^a)^m}{m(1 - q^m)}, \quad t_m(a) = \frac{a}{m}, \quad m = 1, 2, \ldots,
\]

\[
s_n(t(\infty, q)) = \lim_{a \to +\infty} s_n(t(a, q)) = \frac{q^{n(n)}}{H_n(q)},
\]

\[
s_n(t(\infty)) = \lim_{a \to +\infty} s_n \left( \frac{t_1(a)}{a}, \frac{t_2(a)}{a^2}, \ldots \right) = \lim_{a \to +\infty} \frac{1}{a^{|n|}} s_n(t(a)) = \frac{1}{H_n}. \tag{64}
\]

Now we rewrite the series \((52)\) and \((48)\) only in terms of Schur functions:

\[
p_\Phi \left( \begin{array}{c} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \\ \end{array} \right| q, x_{(N)} \right) = \tau_r(M, t(\infty), q),
\]

\[
t_m(a, q) = \frac{1 - (q^a)^m}{m(1 - q^m)}, \quad t_m(a) = \frac{a}{m}, \quad m = 1, 2, \ldots,
\]

\[
s_n(t(\infty, q)) = \lim_{a \to +\infty} s_n(t(a, q)) = \frac{q^{n(n)}}{H_n(q)},
\]

\[
s_n(t(\infty)) = \lim_{a \to +\infty} s_n \left( \frac{t_1(a)}{a}, \frac{t_2(a)}{a^2}, \ldots \right) = \lim_{a \to +\infty} \frac{1}{a^{|n|}} s_n(t(a)) = \frac{1}{H_n}. \tag{64}
\]

We obtain different fermionic representations of hypergeometric functions \((63), (53)\) and they are parametrized by a complex number \(b\):

**Proposition 4.** For \(b \in C\) and for \(r = pr_s\) (see \([13]\)) we have

\[
\tau_r(M, t(\infty), t) = \tau_{r_b}(M, t(b + M), t), \quad r_b = \frac{r}{b + D}. \tag{67}
\]

For \(r = pr_s^{(q)}\) (see \([51]\)) we have

\[
\tau_r(M, t(\infty), t) = \tau_{r_b}(M, t(b + M, q), t), \quad r_b = \frac{r}{1 - q^{b+D}}. \tag{68}
\]

**Remark 1.** There are two ways to restrict the sum \([32]\) to the sum over partitions of length \(l(n) \leq N\). First, if we use Miwa’s change \([4]\), then \(s_n(x_{(N)}) = 0\), for \(n\) with length \(l(n) > N\). The second way is to restrict the Pochhammer coefficients: if we put \(a_i = N\) for one \(i\) from \([41]\) equal to \(N\), then the coefficient \([5]\) vanishes for \(l(n) > N\). Since we expressed Pochhammer coefficients in terms of Schur functions in \([41]\) both ways have the same explanation. Indeed

\[
t_m(N, q) = \frac{1}{m} \frac{1 - (q^N)^m}{1 - q^m} = \frac{1}{m} (1 + (q)^m + (q^2)^m + \ldots + (q^{N-1})^m). \tag{69}
\]
Therefore we obtain for Miwa’s change: \(x_1 = 1, x_2 = q, \ldots, x_N = q^{N-1}\) and
\[
s_n(t(N,q)) = s_n(1, q, \ldots, q^{N-1}) = 0, \quad l(n) > N. \quad (70)
\]
The same we have for the sum over partitions \(n\) such that \(l(n') < K\). Again the first way has to be realized through the following Miwa’s change of variables:
\[
t_m = -\sum_{i=1}^{K} \frac{x_n}{m}, \quad s_n(t) = s_{n'}(x(K)). \quad (71)
\]
The second way is to make one of the parameters, for example \(a_j\) from (51) equal to \((-K)\). In this case
\[
s_n(t(-K,q)) = s_{n'} \left(\frac{1}{q}, \frac{1}{q^2}, \ldots, \frac{1}{q^K}\right) = 0, \quad l(n') > K. \quad (72)
\]

### 3.4 Zakharov-Shabat dressing and string equations

Now we shall get hypergeometric functions from another point of view - the Zakharov-Shabat factorization problem. Let us introduce infinite matrices to describe KP and Toda lattice flows and symmetries \([17]\). We denote the Zakharov-Shabat dressing matrices by \(K\) and \(\tilde{K}\). \(K\) is a lower triangular matrix with unit main diagonal, and \(\tilde{K}\) is an upper triangular matrix. They solve Gauss factorization problem for infinite matrices:
\[
\tilde{K} = KU(M, t, \beta), \quad U(M, t, \beta) = U^+(t)U^-(M, \beta). \quad (73)
\]
Here \(U^\pm\) belong to the different abelian multiparametric subgroups in \(GL(\infty)\) with the infinite sets of group times \(t\) and \(\beta\),
\[
U^+(t) = \exp \left(\xi(t, \Lambda)\right), \quad U^-(M, \beta) = \exp \left(\xi(\beta, \Lambda^{-1}r(\Delta + M))\right), \quad (74)
\]
where the matrices \((\Lambda)_{jk} = \delta_{j,k-1}, (\Delta)_{jk} = j\delta_{j,k}\), for \(\xi\) see \([3]\). The function \(r\) is the same as in \([17]\). Then following Zakharov-Shabat arguments we find that the variables \(-\log(\tilde{K}_{ii}) = \phi_{i+M}\) solve Toda lattice equation in the form \([11]\). At the same time (73) describes a set of KP equations \([17]\) parametrized by integer \(M\).

The tau-function can be obtained as follows. By taking the projection \([17]\) \(U \mapsto U_\pm\) for nonpositive values of matrix indices we obtain a determinant representation of the tau-function \([32]\):
\[
\tau_r(M, t, \beta) = \frac{\det U_-(M, t, \beta)}{\det \left(U^+_-(t)\right) \det \left(U^-_-(M, \beta)\right)} = \det U_-(M, t, \beta), \quad (75)
\]
since both determinants in the denominator are equal to one. Formula (75) is also a Segal-Wilson formula for \(GL(\infty)\) 2-cocycle \([32]\). Choosing the function \(r\) as in Section 3.2 we obtain hypergeometric functions listed in the Introduction.

**Remark 2.** Therefore the hypergeometric functions which were considered above have the meaning of \(GL(\infty)\) two-cocycle on the two multiparametrical group elements \(U^+(t)\) and \(U^-(M, \beta)\). Both elements \(U^+(t)\) and \(U^-(M, \beta)\) can be considered as elements of group of pseudodifferential operators on the circle. The corresponding Lie algebras consist of the multiplication operators \(\{z^n; n \in \mathbb{N}_0\}\) and of the pseudodifferential operators \(\{\left(\frac{1}{z}r(z\frac{d}{dz} + M)\right)^n; n \in \mathbb{N}_0\}\). Two sets of group times \(t\) and \(\beta\) play the role of indeterminates of the hypergeometric functions \([17]\). Formulas \([32]\) and \([34]\) mean the expansion of \(GL(\infty)\) group 2-cocycle in terms of corresponding Lie algebra 2-cocycle
\[
\omega(z, \frac{1}{z}r(D + M)) = r(M), \quad \omega(\tilde{r}(D + M)z, \frac{1}{z}r(D + M)) = \tilde{r}(M)r(M). \quad (76)
\]
For the KP-2 equation we have nonlocal Riemann problem, see [4] Additional symmetries \( z \rightarrow \infty \)

\[ \beta \]

where for function (32). Let us consider an example. We introduce symmetries \( \partial \) as a generating function for the Zakharov-Shabat zero curvature equations for the additional \( \tau \) with the similar constraints on the kernel \( \text{T} \), then the function \( R \) with the Toda lattice times \( \beta \) Let us impose the condition that the group times \( \beta \) and higher flows are given by the following Lax equations

\[ \partial_{t_n} \mathcal{L} = [\mathcal{L}, (r(\overline{\mathcal{L}})L^{-1})^n], \]

\[ \partial_{t_n} \mathcal{L} = [\mathcal{L}, (\overline{\mathcal{L}})^n], \]

Let us impose the condition that the group times \( \beta_m \) of additional symmetries can be identified with the Toda lattice times \( t_m \). Then one can obtain a set of string equations [21, 23], which characterizes the TL hypergeometric solutions:

\[ h(\overline{\mathcal{M}} - 1)L^m = L^m h(\overline{\mathcal{M}} - 1), \quad m = 2, 3, \ldots, \]

where functions \( h \) and \( r \) are connected by (35). The simplest string equation is \( L = \tilde{L}_r(\overline{\mathcal{M}}) \).

When \( r(\mathcal{M}) = M + a \) the string equation describes \( \mathcal{L} = 1 \) string [22].

At last let us mention the \( \partial \) problem [4] for the KP-1 equation

\[ \frac{\partial w(M, t, t^*, z, \bar{z})}{\partial \bar{z}} = w(M, t, t^*, \bar{z}, z)T(z, \bar{z}), \]

where for \( z \rightarrow \infty \) and for \( z \rightarrow 0 \) the asymptotics of Baker function \( w \) are

\[ w = z^M e^{\xi(t, z)} 1 + O(z^{-1}), \quad w = z^{-M} e^{-\xi(t^*, z^{-1})} e^{\varphi_M(t, t^*)} (1 + O(z)) \]

with some function \( \varphi_M \). Instead of writing down the explicit form of the function \( T \), which gives the hypergeometric solution of the KP-1 equation, we only give a set of constraints on \( T \) (which are equivalent to the conditions mentioned above: \( t^* \) may be identified with times of additional symmetries \( \beta \) [13]):

\[ \left( \frac{1}{z} r(D) \right)^m T(z, \bar{z}) = \left( \frac{1}{z} r(-D) \right)^m T(z, \bar{z}), \quad m = 1, 2, \ldots. \]

For the KP-2 equation we have nonlocal Riemann problem, see [4]

\[ w_+(z) - w_-(z) = \int_R w_-(z') R(z', z)dz' \]

with the similar constraints on the kernel \( R \):

\[ \left( \frac{1}{z} r(z \partial_z) \right)^m R(z', z) = \left( \frac{1}{z'} r(-z' \partial_{z'}) \right)^m R(z', z), \quad m = 1, 2, \ldots. \]

The statement is the following. If a hypergeometrical solution belongs to the class described by (31) (or by (33)), then the function \( T \) (respectively \( R \)) solves equation (82) (respectively (84)).

We present the infinitesimal version of the Zakharov-Shabat dressing [11]

\[ \frac{1}{2\pi i} \oint \left( \left( \frac{1}{z} - \frac{1}{z} r(D) \right)^{-1} w(z) \right) \partial_z^{-1} w^*(z)dz = \sum_{n=1}^{\infty} z^{s-n-1} A_n \]

as a generating function for the Zakharov-Shabat zero curvature equations for the additional symmetries \( \left[ \partial_{t_n} - B_n, \partial_{\beta_m} - A_m \right] = 0 \), compare with [13, 14, 13].

From soliton theory and bosonization formula one can obtain various relations for the tau-function [22]. Let us consider an example. We introduce

\[ \Omega_r(\beta) := -\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \oint V^*(z + \epsilon) \sum_{m=1}^{\infty} \beta_m \left( \frac{1}{z} r(z, D) \right)^m V(z)dz, \]

where \( V(z) \) and \( V^*(z) \) are defined by (24). Now we have the following shift argument formula

\[ e^{\Omega_r(\gamma)} \tau_r(M, t, \beta) = \tau_r(M, t, \beta + \gamma). \]
3.5 Orthogonal polynomials

It is known that the hypergeometric functions (4) appear in the group representation theory and are connected with the so-called matrix integrals (10). On the other hand the set of examples reveals a connection between the matrix integrals and the soliton theory. To establish this connection it is useful to consider the related systems of the orthogonal polynomials. Let us briefly describe how to write down these polynomials. Let $M_3$ be the largest integer zero of the function $r$. Then the function

$$f(z^*) = \sum_{n=0}^{\infty} (z^*)^{n+M_3} h(n+M_3)$$

is the eigenfunction of the operator $\frac{1}{r} r(D)$ with the eigenvalue $z^*$. We use this function as weight function for a system of orthogonal polynomials $p_n^\pm, n = 0, 1, 2, \ldots$, related to the hypergeometric solution of KP:

$$\int_p \gamma_n(z, \mathbf{t}, \beta) e^{i(z, \mathbf{t}, \beta)} f(z^*) p_m^+(z^*, \mathbf{t}, \beta) dz = e^{-\phi_M} \delta_{n,m}.$$  

3.6 Further generalization

Formula (39) is related to 'Gauss decomposition' of operators inside vacuums $\langle 0 | \ldots | 0 \rangle$ into diagonal operator $e^{H_0^{(h)}}$, upper triangular operator $e^{H^{(t)}}$ and lower triangular operator $e^{-H^{(t^*)}}$ (the last two have the Toeplitz form). Now let us consider more general two-dimensional Toda lattice tau-function

$$\tau = \langle M | e^{H^{(t)}} | M \rangle, \quad g = e^{\tilde{A}_0} e^{-A_1(\gamma_1)} \cdots e^{-A_k(\gamma_k)}.$$  

where operators $A_k(\gamma_k) = \sum_{j=1}^{+\infty} \gamma_{ij} A_{ij}$ and $\tilde{A}_i(\tilde{\gamma}_i) = \sum_{j=1}^{+\infty} \tilde{\gamma}_{ij} \tilde{A}_{ij}$ are defined like $A(\beta)$ of (27) and $\tilde{A}(\tilde{\beta})$ of (33) respectively and correspond to operators $r^i(D)$ and $\tilde{r}^i(D)$. Collections of variables $\tilde{\gamma} = \{\tilde{\gamma}_{ij}\}, \gamma = \{\gamma_{ij}\}$ play the role of coordinates for some wide enough class of Clifford group elements $g$. Let us calculate this tau-function. We introduce a set consisting of $m+1$ partitions:

$$(n_1, \ldots, n_m, n_{m+1} = n), \quad 0 \leq n_1 \leq n_2 \leq \cdots \leq n_m \leq n_{m+1} = n.$$  

Also we define a set $\Theta_n^m = (n_1, n_2, n_3, \ldots, n_m)$. We take

$$s_{\Theta_n^m}(\mu) = s_{n_1}(\mu_1) s_{n_2}(\mu_2) \cdots s_{n_m}(\mu_{m+1}), \quad \mu_i = \{\mu_{ij}\}.$$  

Here $s_{n_i+1/n_i}(\mu_{i+1})$ is a skew Schur function [12]:

$$s_{n_i+1/n_i}(\mu_{i+1}) = \det(p_{n_j+1-n_i, \rho, \mu_{i+1}}), \quad n_{i+1} = (n_1^{(i+1)}, \ldots, n_r^{(i+1)}).$$

A skew analogy of (29) is

$$r_{\Theta_n^m}(M) = r_{n_1}(M) r_{n_2/n_1}(M) \cdots r_{n_m/n_m}(M),$$

$$r_{n_i+1/n_i}(M) = \prod_{j=1}^{r} r(n^{(i)}_j - j + 1 + M) \cdots r(n^{(i+1)}_j - j + M).$$

If the function $r^i(m)$ has no poles and zeroes at integer points, then the relation $r^i_{\theta_i}(M) = \frac{r_{n_i+1/n_i}(M)}{r_{n_i}(M)} (i = 1, \ldots, m)$ is correct. To calculate the tau function we need the following Lemma

Lemma 3 Let partitions

$$n = (i_1, \ldots, i_s | j_1 - 1, \ldots, j_s - 1), \quad \tilde{n} = (\tilde{i}_1, \ldots, \tilde{i}_r | \tilde{j}_1 - 1, \ldots, \tilde{j}_r - 1).$$

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satisfy the relation \( n \geq \tilde{n} \). Then the next formula is valid:

\[
\langle 0 | \psi^*_{i_1} \cdots \psi^*_{i_r} \psi_{-j_1} \cdots \psi_{-j_r} e^{A_i(\gamma_i)} \psi_{-j_1} \cdots \psi_{-j_r} \psi_{i_1} | 0 \rangle = (-1)^{\tilde{j}_1 + \cdots + \tilde{j}_r + j_1 + \cdots + j_r} s_\theta(\alpha_i) r_\theta(0), \quad \theta = n/\tilde{n}.
\] (97)

The proof is achieved by direct calculation (see [12] for help).

Now we obtain the following generalization of Proposition 1:

**Proposition 5**

\[
\tau(M, t, \beta; \gamma, \tilde{\gamma}) = \sum_{n} \sum_{\Theta_{\tilde{n}}^t} \sum_{\Theta_n} \tilde{r}_{\Theta_{\tilde{n}}^t}(M) r_{\Theta_n}^t(M) s_{\Theta_{\tilde{n}}^t}(t, \tilde{\gamma}) s_{\Theta_n}(\beta, \gamma).
\] (98)

With the help of this series one can obtain different hypergeometric series.

**Example 6.** Let us consider the tau function given by the correlator

\[
\tau(M, \tilde{\beta}, \beta, \gamma) = \langle M | e^{\tilde{A}(\tilde{\beta})} e^{-A_1(\gamma)} e^{-A(\beta)} | M \rangle,
\] (99)

\[
\tilde{r}^1(D) = \frac{\tilde{a}_1 + D}{b_1 + D}, \quad r(D) = \frac{a_1 + D}{b_1 + D}, \quad r^1(D) = 1.
\] (100)

We put \( \tilde{\beta} = (x, \frac{x^2}{2}, \frac{x^3}{3}, \ldots) \), \( \beta = (y_1, 0, 0, \ldots) \), \( \gamma = (y_2, 0, \ldots) \). Thus we have

\[
\tau(M, x, y_1, y_2) = \sum_{n_1=0}^{+\infty} \sum_{n_2=0}^{+\infty} \frac{(\tilde{a}_1 + M)^n_1 + n_2 (a_1 + M)^n_1 + n_2}{(b_1 + M)^n_1 + n_2 (b_1 + M)^n_1 + n_2} e^{n_1 + n_2}.
\] (101)

**Conclusion**

We get multivariate hypergeometric functions as certain tau-functions of the KP hierarchy and also as the ratios of Toda lattice tau-functions considered in [18], [23] evaluated at certain values of higher Toda lattice times. It means that multivariate hypergeometric functions solve a set of continuous and discrete bilinear Hirota equations [4]. We shall write down these equations explicitly in the different paper. Hypergeometric solution of the KP equation is of continuous and discrete bilinear Hirota equations [5]. We shall write down these equations of higher Toda lattice times. It means that multivariate hypergeometric functions solve a set of continuous and discrete bilinear Hirota equations [5]. We can present links between these results, group-theoretic approach to the q-special functions [10, 11] and matrix integrals. We expect to work out connections with matrix models of Kontsevich type [24] and two-matrix models related to 2D Toda lattice [25]. We can present links between our construction in the present paper and so-called generalized Miwa’s change of variables [21] in the three dimensional three-wave systems [1], and with the multicomponent KP hierarchy [5], it will be published separately. We hope to generalize our results to the KP hierarchies of \( B_\infty, C_\infty \) and \( D_\infty \) types [3] to get different hypergeometric series.

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