Algebraic blinding and cryptographic trilinear maps

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Abstract. It has been shown recently that cryptographic trilinear maps are sufficient for achieving indistinguishability obfuscation. In this paper we develop algebraic blinding techniques for constructing such maps. An earlier approach involving Weil restriction can be regarded as a special case of blinding in our framework. However, the techniques developed in this paper are more general, more robust, and easier to analyze. We demonstrate this approach in the construction of trilinear maps based on elliptic curves.

1 Introduction

In this paper we develop algebraic blinding techniques for the construction of cryptographically interesting trilinear maps. Cryptographic applications of \( n \)-multilinear maps for \( n > 2 \) were first proposed in the work of Boneh and Silverberg [3]. However the existence of such maps remains an open problem [3,11]. The problem has attracted much attention recently as multilinear maps and their variants prove to be useful for indistinguishability obfuscation. More recently Lin and Tessaro [12] showed that trilinear maps are sufficient for the purpose of achieving indistinguishability obfuscation (see [12] for references to related works along several lines of investigation). This striking result brought the following question into the spotlight: can a cryptographically interesting trilinear map be constructed?

The results of this paper follow a line of investigation initiated by an observation of Chinburg (at the AIM workshop on cryptographic multilinear maps (2017)) that the following map from \( \acute{e} \text{tale cohomology} \) may serve as the basis of constructing a cryptographically interesting trilinear map:

\[
H^1(A, \mu_{\ell}) \times H^1(A, \mu_{\ell}) \times H^2(A, \mu_{\ell}) \to H^4(A, \mu_{\ell} \otimes 3) \cong \mu_3
\]

where \( A \) is an abelian surface over a finite field \( \mathbb{F} \) and the prime \( \ell \neq \text{char}(\mathbb{F}) \). Following up on Chinburg’s idea, a method for constructing trilinear maps was proposed in [8,9]. It was based on the following map that can be derived from the cohomological map just mentioned: \( (\alpha, \beta, \mathcal{L}) \to \epsilon_{\ell}(\alpha, \varphi_{\mathcal{L}}(\beta)) \), where \( \alpha, \beta \in A[\ell] \), \( \mathcal{L} \) is an invertible sheaf, and \( \varphi_{\mathcal{L}} \) is the map \( A \to A^* = \text{Pic}^0(A) \) so that

\[
\varphi_{\mathcal{L}}(a) = \ell^*_a \mathcal{L} \otimes \mathcal{L}^{-1} \in \text{Pic}^0(A)
\]
for \( a \in A(\bar{\mathbb{F}}) \) where \( t_a \) is the translation map defined by \( a \) (\cite{15} § 1 and § 6). In the map just described one no longer needs to assume that \( A \) is of dimension 2, and the third participant \( L \) in the trilinear map can be identified with an endomorphism of \( A \). With this approach the third group in the pairing is to be constructed from endomorphisms of \( A \), and the challenge is to encode the endomorphisms involved in such a way that the resulting group has hard discrete logarithm problem. The method proposed in \cite{10} tackles this issue by using Weil descent (or Weil restriction) \cite{5,6,7,18}. The trilinear map in \cite{10} is derived from a blinded version of the following trilinear map:

\[
A[\mathcal{L}]^d \times A[\mathcal{L}]^d \times \text{Mat}_d(\mathbb{F}_\ell) \rightarrow \mu_{\ell} \\
(\alpha, \beta, M) \rightarrow e(\alpha, M(\beta))
\]

where \( \alpha, \beta \in A[\mathcal{L}]^d \), \( M \in \text{Mat}_d(\mathbb{F}_\ell) \subseteq \text{End}(A[\mathcal{L}]^d) \), and \( e \) is a non-degenerate bilinear pairing on \( A[\mathcal{L}]^d \) (determined by a non-degenerate bilinear pairing on \( A[\mathcal{L}] \)). The blinding of the map just described involves Weil descent.

In this paper we develop algebraic blinding techniques for constructing trilinear maps. The blinding in \cite{10} involving Weil descent can be regarded as a special case in our framework. In retrospect, blinding using Weil descent is more restrictive, and the analysis is more complicated. However the construction is more efficient in terms of number of secret bits required. The blinding techniques developed in this paper are more robust, more general, and easier to analyze. Under our algebraic blinding system, the relationship between the published entities and the hidden entities is captured by algebraic conditions which in turn determine algebraic sets. The blinding is inherently ambiguous, as reflected in the algebraic sets being triply confusing (see Theorems 7, 10, 11 and the remarks right below Theorem 3). In particular these algebraic sets are of dimension \( \Omega(n) \) in \( n^{O(1)} \) variables, where \( n \) is the security parameter, moreover an affine space of dimension \( \Omega(n) \) can be embedded around every point, and they admit twisted actions by a subgroup of dimension \( \Omega(n) \) of a general linear group. These properties together make it conjecturally infeasible to solve for points in such algebraic sets in order to un-blind hidden objects of interest.

In addition to algebraic blinding, the security of the trilinear maps constructed in this paper also depends on the computational complexity of a trapdoor discrete logarithm problem presented in \S 2.1. The problem involves an associative non-commutative polynomial algebra acting on some torsion points of a blinded product of elliptic curves. The kernel ideal of such action is hidden due to blinding, except polynomially many elements in the kernel are made public. The problem is of independent interest apart from its application to trilinear map construction.

In our construction the blinding parameters are secretly chosen and the elements of the third group in the pairing require private encoding. It remains an interesting open problem whether a trilinear map without private encoding, perhaps along the line of Chinburg’s idea or the approach in \cite{8,9}, can be constructed.
1.1 Algebraic blinding systems

Our blinding maps are composed of the following simple maps: general linear maps, quadratic isomorphisms of affine spaces of small dimension, and Frobenius twists. Let $K$ be a finite extension over some $k = \mathbb{F}_q$.

**General linear maps** Let $\text{GL}_m(K)$ be the set of $m$-dimensional general linear maps defined over $K$. Each $A \in \text{GL}_m(K)$ can be identified with an $m$ by $m$ invertible matrices $(a_{ij})$ with $a_{ij} \in K$ for $1 \leq i, j \leq m$, so that for $x = (x_1, \ldots, x_m) \in k^m$, $A(x) = (\sum_{j=1}^m a_{ij}x_j)_{i=1}^m$.

Let $q$ be a quadratic polynomial in $K[x, y]$ where $\deg p(x) = \deg q(x, y) = 2$. Then $(x, y, z) \rightarrow (x, y + p(x), z + q(x, y))$ defines an isomorphism $k^3 \rightarrow k^3$, denoted as $\lambda_{p, q}$.

Let $A, B \in \text{GL}_3(K)$ and let $p(x) \in K[x, y]$ where $\deg p(x) = \deg q(x, y) = 2$. Then $\lambda = B \circ \lambda_{p, q} \circ A$ defines an isomorphism $k^3 \rightarrow k^3$. For $x \in k^3$, $\lambda(x) = (f_i(x))_{i=1}^3$ where $f_i$ is a quadratic polynomial in $K[x, y, z]$ for $i = 1, 2, 3$. For random choices of $p, q, A, B$, the $f_i$’s are most likely dense.

**Frobenius twists** Suppose $[K : k] = d$. For simplicity we assume $d = O(n)$. Let $\tau$ denote the Frobenius map $x \rightarrow x^q$ for $x \in k$. Let $\tau_\alpha$ also denote $\tau^a$ for $a \in \mathbb{Z}$. For $0 \leq a, b \leq d - 1$, let $\tau_{a, b}$ denote the map $\tau_{a, b} : k^2 \rightarrow k^2$ such that $\tau_{a, b}(x, y) = (\tau_\alpha(x), \tau_\alpha(y)) = (x^{q^a}, y^{q^b})$ for $x, y \in k$.

**Weil descent as a special case** Suppose $V \subset \bar{k}^2$ is an elliptic curve defined over $K$. Suppose $[K : k] = d$ as above. Then a Weil descent of $V$ from $K$ to $k$ can be identified with $\hat{E} = \delta^{-1} \prod_{i=0}^{d-1} V_i$ where $V_i = V^{\tau_i}$ for $i = 0, \ldots, d - 1$, where $\delta \in \text{GL}_d(K)$ is determined by a basis $\theta$ of $K$ over $k$ as follows. Organize the coordinates of $\bar{k}^{2d}$ in two vectors $\hat{x} = x_0, \ldots, x_{d-1}$ and $\hat{y} = y_0, \ldots, y_{d-1}$. Then $\delta = (\delta_i)_{i=0}^{d-1}$ where $\delta_i(\hat{x}, \hat{y}) = (\langle \hat{x}, \theta^{\alpha_i} \rangle, \langle \hat{y}, \theta^{\beta_i} \rangle) \in V_i$ for $i = 0, \ldots, d - 1$, where for $\alpha = (\alpha_i)_{i=0}^{d-1}$ and $\beta = (\beta_i)_{i=0}^{d-1}$ in $\bar{k}^d$, $\langle \alpha, \beta \rangle = \sum_{i=0}^{d-1} \alpha_i \beta_i$. Note that $\delta$ is determined by the matrix in $\text{GL}_d(K)$ with $\theta^n$ as the
§1.6. Let $H$ be the ideal of ambivalence because for polynomials $H$ considered the ideal of ambivalence. The ideal of ambivalence $I$ is isomorphic to $k^2$ such that $\mu(x, y) = (x, y, y)$, and let $W_I$ be the isomorphic image of $k^2$ under $\mu$. Then $W_I$ is isomorphic to $k^2$.

Blinding space and blinding maps Let $\lambda = B \circ \lambda_{p,q} \circ A$ be as discussed above with $A, B \in GL_3(K)$ and $p(x) \in K[x], q(x, y) \in K[x, y]$ where $\deg p(x) = \deg q(x, y) = 2$. Let $\mu = \lambda^{-1}$. Let $\tilde{\mu}$ be the map $k^3 \to k^3$ such that $\tilde{\mu}(x, y) = \mu(x, y, y)$, and let $W_\lambda$ be the isomorphic image of $k^2$ under $\tilde{\mu}$. Then $W_\lambda$ is isomorphic to $k^2$.

We form a blinding space $W \subset k^{3n}$, where $W$ is isomorphic to $k^{2n}$, as follows. For $i = 1, \ldots, n$, choose a random $\lambda_i = B_i \circ \lambda_{p_i,q_i} \circ A_i$ in the manner as discussed above with $A_i, B_i \in GL_3(K)$ and $p_i(x) \in K[x], q_i(x, y) \in K[x, y]$ where $\deg p_i(x) = \deg q_i(x, y) = 2$. Let $W_i = W_{\lambda_i}, W \tilde{\mu}$ is the zero set of $f_{i2} - f_{i3}$ where $\lambda_i(x) = (f_{i2}(x))^3_{j=1}$ where $x = (x, y, z)$. Choose a random $\delta \in GL_{3n}(K)$.

Let $W = \delta^{-1} \prod_{i=1}^n W_i$.

Write $\delta = (\delta_i)_{i=1}^n : k^{3n} \to \prod_{i=1}^n \bar{k}^3$ with $\delta_i : k^{3n} \to \bar{k}^3$ given by linear forms $L_{ij}, j = 1, 2, 3$, in $3n$ variables. For $i = 1, \ldots, n$, $\lambda_i \circ \delta_i = (F_{ij})_{j=1}^3$ where $F_{ij} = f_{ij} \circ (L_{i1}, L_{i2}, L_{i3})$ for $j = 1, 2, 3$. Let $\rho_i = pr \circ \lambda_i \circ \delta_i = (F_{i1}, F_{i2})$ where $pr$ denotes the projection $k^3 \to k^2 : (x, y, z) \to (x, y)$. Then $W_i = \delta_i(W)$ and $W$ is the zero set of $\{F_{i2} - F_{i3} : i = 1, \ldots, n\}$.

The basic blinding map associated with the blinding space $W$ is $\rho : k^{3n} \to \prod_{i=1}^n \bar{k}^2$ where $\rho = (\rho_i)_{i=1}^n$. We see that $\rho$ maps $W$ isomorphically to $k^{2n}$.

A blinding map on $W$ is the basic blinding map twisted by Frobenius locally, that is, $(\tau_{a_i,b_i} \circ \rho_i)_{i=1}^n$ with $0 \leq a_i, b_i \leq d - 1$ for $i = 1, \ldots, n$.

In this paper we will focus on basic blinding maps for the most part since they are sufficient for our purposes. We remark that adding Frobenius twists to basic blinding maps provides an additional layer of protection and allows us to pay attention to the fact that the functions and maps of interest are applied to $K$-rational points. We will briefly discuss blinding maps with Frobenius twists in §1.6.

Ideal of ambivalence The ideal $I$ generated by $F_{i2} - F_{i3}, 1 \leq i \leq n$, is considered the ideal of ambivalence because for polynomials $H, H'$ such that $H - H' \in I$, $H$ and $H'$ define the same map $W \to k$.

Let $I_i$ denote the submodule of $I$ consisting of polynomials of degree bounded by $i$.  

\[ \text{i-th row for } i = 0, \ldots, d - 1. \]
Theorem 1. Let \( \rho = (\rho_i)_{i=1}^n : W \to \bar{k}^{2n} \) be a basic blinding map with \( \rho_i = (F_{i1}, F_{i2}) : W \to \bar{k}^2 \) where \( F_{ij} \) are quadratic polynomials in \( 3n \) variables for \( i = 1, \ldots, n \) and \( j = 1, 2 \). Suppose \( H_{ij} \) are polynomials such that \( H_{ij} - F_{ij} \in I_2 \) for \( i = 1, \ldots, n \) and \( j = 1, 2 \). Then \( \rho_i = (H_{i1}, H_{i2}) : W \to \bar{k}^2 \) for \( i = 1, \ldots, n \). The set of \( (H_{ij})_{1 \leq i \leq n; j=1,2} \) that determine the same map as \( \rho_i : W \to \bar{k}^2 \) is isomorphic to \( \bar{k}^{2n} \). The set of \( (H_{ij})_{1 \leq i \leq n; j=1,2} \) that determine the same basic blinding map as \( \rho \) is isomorphic to \( \bar{k}^{2n^2} \).

Proof The first assertion follows directly from the definition of \( I \). For the second assertion, since the zero set of \( I \), which is \( W \), has dimension \( 2n \), \( F_{i2} = F_{i3} \), \( i = 1, \ldots, n \), are algebraically independent, hence linearly independent as well. Hence there is a linear isomorphism between \( I_2 \) and \( \bar{k}^n \). From this the second and the third assertions follow. \( \Box \)

Lemma 1. For \( \alpha \in \bar{k} \), let \( D_{\alpha} = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \). Consider a local quadratic isomorphism \( \lambda = B \circ \lambda_{p,q} \circ A \) as described above. Let \( \lambda_\alpha = D_\alpha \circ \lambda = B_\alpha \circ \lambda_{p,q} \circ A \) where \( B_\alpha = D_\alpha B \). Then \( W_\lambda = W_{\lambda_\alpha} \) and \( \lambda = \lambda_\alpha : W_\lambda \to \bar{k}^3 \).

Proof Suppose \( \lambda(x) = (f_1(x), f_2(x), f_3(x)) \) for \( x \in \bar{k}^3 \) where \( f_1, f_2, f_3 \) are quadratic polynomials. Then \( \lambda_\alpha(x) = (f_1'(x), f_2'(x), f_3'(x)) \) for \( x \in \bar{k}^3 \) where \( f_1' = f_1 + \alpha(f_2 - f_3) \). Therefore \( W_\lambda = W_{\lambda_\alpha} \). \( \Box \)

Suppose we allow the blinding parameters to take values in \( \bar{k} \), hence \( \delta \in GL_{3n}(\bar{k}) \), \( A_1, B_i \in GL_3(\bar{k}) \), \( p_i \) and \( q_i \) a quadratic polynomials with coefficients from \( \bar{k} \) for \( i = 1, \ldots, n \). Let \( \langle \rho \rangle \) denote the set of parameters \( (\delta, A_i, B_i, p_i, q_i : i = 1, \ldots, n) \) that define the blinding map \( \rho : W \to \bar{k}^{2n} \).

Theorem 2. Let \( \rho : W \to \bar{k}^{2n} \) be a basic blinding map determined by parameters: \( \delta \in GL_{3n}(K) \), \( \lambda_i = B_i \circ \lambda_{p_i,q_i} \circ A_i \), with \( A_i, B_i \in GL_3(K) \), \( p_i \) a quadratic polynomial in one variable, \( q_i \) a quadratic polynomial in two variables, for \( i = 1, \ldots, n \). Let \( \lambda'_i = D_{\alpha_i} \circ \lambda_i \) with \( \alpha_i \in \bar{k} \) for \( i = 1, \ldots, n \). Let \( \rho' = \langle \rho' \rangle_{i=1}^n \) where \( \rho'_i = pr \circ \lambda'_i \circ \delta_i \) for \( i = 1, \ldots, n \). Then \( \rho' = \rho : W \to \bar{k}^{2n} \), hence there is an injective map \( \bar{k}^n \to \bar{k}^{2n} \).

Proof The theorem follows immediately from Lemma 1. \( \Box \)

1.2 Semi-local functions

We say that a rational function \( f : \prod_{i=1}^n V_i \to \bar{k} \) defined over \( K \) is \( c \)-local if there are \( 1 \leq i_1, \ldots, i_c \leq n \) such that for \( x = (x_i) \) with \( x_i \in V_i \), \( f(x) \) depends only on \( x_{i_1}, \ldots, x_{i_c} \). In this paper we only consider \( c \)-local functions of bounded but positive degree. By abuse of notation we also write \( f(x) = f(x_{i_1}, \ldots, x_{i_c}) \). We
consider the function \( g = f \circ \rho \) semi-local, noting that \( g(x) = f(\rho_1(x), \ldots, \rho_n(x)) \).

Suppose the degree of \( f \) is bounded by \( d_j \) at \( x_i \) for \( j = 1, \ldots, c \). Denote by \([f]\) the set of \((h_1, h_2)\) where \( h_1 \) and \( h_2 \) are \( 2c \)-variate polynomials such that the degree of \( h_1 \) and \( h_2 \) is bounded by \( d_j \) at \( x_i \) for \( j = 1, \ldots, c \) and \( f \) as a rational function on in \( 2c \) variables can be defined by \( h_1/h_2 \). Denote by \([\rho]\) the set of \((H_{ij})_{i=1, \ldots, n; j=1,2}\) such that the basic blinding map \( \rho \) can be defined by quadratic polynomials \( H_{ij}(x) \), \( i = 1, \ldots, n \) and \( j = 1, 2 \), so that \( \rho = (\rho_1)_{n}^{n} \) with \( \rho_i = (H_{11}, H_{12}) \). Then for \( x \in W \), \( g(x) = h_{ij}(H_{i1}, H_{i2}, \ldots, H_{in}, H_{jn}, H_{in}, H_{jn}) \).

We say that \( A \) \& \( g \) be the block-diagonal matrix with \( A_1, \ldots, A_n \) as the diagonal blocks. Let \( A \) be a semi-local function as above. If \( g = f \circ \rho \) then \( g = (f \circ A^{-1}) \circ (A \circ \rho) \). Note that \( f \circ A^{-1} : \prod_{i=1}^{n} V'_{i} \to \bar{k} \) where \( V'_i = A_i V_i \), and \( f \circ A^{-1} \) has the same locality \( c \) as \( f \) at \( x_1, \ldots, x_c \). Let \( A_w \) be the block-diagonal matrix with \( A_{i_1}, \ldots, A_{i_c} \) as the diagonal blocks where \( w = \{i_1, \ldots, i_c\} \). If \([f] \circ [\rho]\) is a semi-local decomposition of \( g \) with \([f] = h_1, h_2 \) and \([\rho] = H_{11}, H_{12}, \ldots, H_{in}, H_{jn} \). Then we have semi-local decomposition \([f \circ A^{-1}] \circ [A \circ \rho] \) where \([f \circ A^{-1}] = h'_1, h'_2 \) with \( h'_j = h_j \circ A^{-1} \) for \( j = 1, 2 \), and \([A \circ \rho] = H'_{11}, H'_{12}, \ldots, H'_{in}, H'_{jn} \) with \( (H'_{11}, H'_{12}) = A_j(H_{j1}, H_{j2}) \) for \( j = 1, \ldots, n \). We say that \([f \circ A^{-1}] \circ [A \circ \rho] \) is obtained from \([f] \circ [\rho] \) by the action of the matrix \( A \).

We also consider rational functions \( f : \prod_{i=1}^{n} V_i \times \prod_{i=1}^{n} V_i \to \bar{k} \) defined over \( K \) that is local in the sense that for \( x = (x_i)_{i=1}^{n} \) with \( x_i \in V_i \) and \( y = (y_i)_{i=1}^{n} \), with \( y_j \in V_j \), \( f(x, y) \) depends on \( (x_i, y_i) \) for some \( i \). By abuse of notation we write \( f(x, y) = f(x_i, y_i) \). The function \( g = f \circ (\rho, \rho) : W \times W \to \bar{k} \) is semi-local in the sense that for \( x, y \in W \), \( g(x, y) = f(\rho_i(x), \rho_i(y)) \). Similar to the discussion before, a semi-local decomposition of \( g \) is denoted \([f] \circ [\rho]\) where the local part \([f]\) consists of 4-variate polynomials \( h_1 \) and \( h_2 \), and the blinding part \([\rho]\) consists of quadratic polynomials \( H_{ij} \) in 3n variables, \( i = 1, \ldots, n \), \( j = 1, 2 \), such that for \( x, y \in W \), \( g(x, y) = h_{ij}(H_{i1}(x), H_{i2}(y), H_{i1}(y), H_{i2}(y)) \). Let \( A_j \in GL(\bar{k}) \) for \( j = 1, \ldots, n \). Let \( A \) be the block-diagonal matrix with \( A_1, \ldots, A_n \) as the diagonal blocks. Then we similarly obtain semi-local decomposition \([f \circ A^{-1}] \circ [A \circ \rho] \) from \([f] \circ [\rho]\) by the action of \( A \). In this case \([f \circ A^{-1}] = h'_1, h'_2 \) where \( h'_1 = h_1 \circ (A^{-1}, A^{-1}) \), \( h'_2 = h_2 \circ (A^{-1}, A^{-1}) \), and \([A \circ \rho] = H'_{11}, H'_{12}, \ldots, H'_{in}, H'_{jn} \) with \( (H'_{11}, H'_{12}) = A_j(H_{j1}, H_{j2}) \) for \( j = 1, \ldots, n \).

Similar consideration can be made if \( f \) maps \( V_i \times V_i \times V_i \to \bar{k} \), or more generally if \( f : V_i^c \to \bar{k} \) where \( c \) is a constant.

**Theorem 3.** Suppose we have a set of semi-local functions \( g_i, i=1, \ldots, m \), such that \( g_i \) has semi-local decomposition \([f_i] \circ [\rho] \) for all \( i \), where \( f_i \) is a local function and \( \rho \) is a basic blinding map. Then the following hold.
1. There is an injective map $k^{2n^2} \to \langle \rho \rangle$. More explicitly if $(F_{ij})_{i=1,\ldots,n,j=1,2}$ define $\rho$, then so does $(F'_{ij})_{i=1,\ldots,n,j=1,2}$ if $F'_{ij} - F_{ij} \in I_2$.
2. There is an injective map $k^n \to \langle \rho \rangle$.
3. There is an injective map $\bar{k} \to [f_i]$ if $f_i$ is c-local depending on $V_{i_1} \times \ldots V_{i_t}$ and for some $j$ the degree of $f_i$ at $x_{i_j}$ is greater to equal to the minimum degree of polynomials in the ideal defining $V_{i_j}$.
4. Let $A_j \in \text{Gl}_2(k)$ for $j = 1,\ldots,n$. Let $A$ be the block-diagonal matrix with $A_1, \ldots, A_n$ as the diagonal blocks. Then $g_i$ has semi-local decomposition $[f_i \circ A^{-1}] \circ [A \circ \rho]$ for $i = 1,\ldots,m$.

**Proof** The first assertion follows from Theorem 1. The second assertion follows from Theorem 2. For the third assertion observe that if $(h_1, h_2) \in [f_i]$ then $(h'_1, h'_2) \in [f_i]$ if $h'_1 - h_1$ and $h'_2 - h_2$ are in the ideal defining $V_{i_j}$. The last assertion follows the discussion above. $\square$

**Remarks** Theorem 4 states fundamentally how the algebraic blinding system is designed to be inherently ambiguous. In our setting for constructing trilinear maps using elliptic curves, it will be clear that the condition in the third assertion is satisfied.

1. Let $S = \{g_i : i = 1,\ldots,m\}$ be as in the theorem. Let $V_S$ be the union of $[f_1] \times \ldots [f_m] \times \langle \rho \rangle$, where the union is over all $f_1,\ldots,f_m,\rho$ such that $g_i$ has semi-local decomposition $[f_i] \circ \langle \rho \rangle, i = 1,\ldots,m$. Similarly let $V_{S,\rho}$ be the union of $[f_1] \times \ldots [f_m] \times \langle \rho \rangle$, where the union is over all $f_1,\ldots,f_m,\rho$ such that $g_i$ has semi-local decomposition $[f_i] \circ \langle \rho \rangle, i = 1,\ldots,m$. Then $V_S$ and $V_{S,\rho}$ admit local embedding of affine space of dimension $\Omega(n^2)$ (respectively $\Omega(n)$) around every point by the first two assertions of Theorem 4 and both are acted on by $(\text{Gl}_2(k))^n$.

2. Due to the assertions in the theorem we say that $V_S$ and $V_{S,\rho}$ are triply confusing. Indeed the first two assertions of the theorem states that $V_S$ (resp. $V_{S,\rho}$) is confusing in the binding part $\langle \rho \rangle$ (resp. $\langle \rho \rangle$). The third assertion states that $V_S$ (resp. $V_{S,\rho}$) is confusing in the local part. The fourth assertion states that the GL-action on the decomposition $[f] \circ \langle \rho \rangle$ makes $V_S$ (resp. $V_{S,\rho}$) even more confusing.

3. The fact that $V_S$ and $V_{S,\rho}$ are non-linear of large dimension (respectively $\Omega(n^4)$ and $\Omega(n)$) in $n^{O(1)}$ variables present a challenge to attack. Best known methods for solving a polynomial system of degree $d$ in $n$ variables take at least $d^{O(n^2)}$ when the system has positive dimension, and $d^{O(n)}$ in dimension 0 (see [1] for a comprehensive survey). The three properties that make $V_S$ and $V_{S,\rho}$ triply confusing, seems to make it very difficult to reduce the algebraic sets to dimension zero, say by judicious choice of intersecting hyperplanes.

4. Under our blinding scheme we do not explicitly specify semi-local functions, even though they are critically involved in specifying functions and maps of interest. As we shall see in the following subsections, suppose a set $S$ of
semi-local functions are involved in the specification, then $V_g$ (resp. $V_{g,\rho}$) is a hidden algebraic set that can be locally embedded around every point of the algebraic sets arising from the specification. Therefore these algebraic sets are also triply confusing, satisfying the three properties in Theorem $\ref{thm:property}$.

1.3 Specifying a semi-local sum

For $i > 0$, let $I_i$ be the submodule of $I$ consisting of elements of degree at most $i$. If a map $W \to k$ can be defined a polynomial $h$ of degree $d$, then it is also defined by any polynomial in $h + I_d$. For a random choice of basic blinding map $\rho$, the associated $F_{ij}$ are dense quadratic polynomials in $x$, so are $F_{i2} - F_{i3}$.

Therefore a random element of $h + I_d$ is likely a dense polynomial of degree $d$ in $x$. We write $f \in R h + I_d$ to denote a uniform random selection from $h + I_d$.

To specify a function $f$ that is the sum of hidden semi-local functions, say $f = \sum_{i=1}^m \varphi_i$, where $\varphi_i$ is a hidden semi-local function, we take the following steps to specify $f$ as a sum of $m$ random-looking functions that are not semi-local.

Again we focus on the case $\varphi_i = f_i \circ \rho$ where $f_i$ is $c$-local in that it depends on $V_{i1} \times \ldots \times V_{i_e}$ for some $i_1, \ldots, i_e$. The case that $\varphi_i = f_i \circ (\rho, \rho)$ or $\varphi_i = f_i \circ (\rho, \rho, \rho)$ where $f_i$ is local depending on $V_i \times V_i$ or $V_i \times V_i \times V_i$ is similar.

1. Construct $2m$ random linear forms $\ell_{i,j}(x)$, $i = 1, \ldots, m$, $j = 1, 2$. Put $\ell_{m+1,j} = \ell_{1,j}$ for $j = 1, 2$.

2. Suppose $\varphi_i(x) = \frac{\phi_i(x)}{h_i(x)}$ on $W$ with $g_i, h_i \in K[x]$. Then $\frac{\phi_i}{h_i} + \frac{\ell_{i,1}}{h_i} - \frac{\ell_{i+1,1}}{h_i} = \frac{\delta_i}{h_i}$ where $h_i' = h_i \ell_{i,2} \ell_{i+1,2}$ and $g_i' = g_i \ell_{i,2} + \ell_{i,1} h_i' - \ell_{i+1,1} h_i' \ell_{i,2}$.

3. Choose random $g_i'' \in R h_i' + I_{d',1+2}$ where $d_{i,1} = \deg g_i'$ and $h_i'' \in R h_i' + I_{d',2+2}$ where $d_{i,2} = \deg h_i''$ for $i = 1, \ldots, m$.

4. Publish $\{g_i'', h_i'': i = 1, \ldots, m\}$, and specify $\varphi$ as $\sum_{i=1}^m \frac{g_i''}{h_i''}$ on $W$. (Note that $\sum_{i=1}^m \frac{\ell_{i,1}}{h_i} - \frac{\ell_{i+1,1}}{h_i} = 0$.)

For simplicity assume $f_i$ has locality 2 and depends on $V_{i1} \times V_{i2}$. Let $F_i = (F_{i1,1}, F_{i1,2}, F_{i2,1}, F_{i2,2})$. Suppose $f_i = h_i \ell_{i,1} \ell_{i,2}$ where $h_{i1}$ and $h_{i2}$ are polynomials in 4 variables.

Write $f =_{W} g$ if for rational functions $f$ and $g$ on $k^3$ and $f(x) = g(x)$ for all $x \in W$. Then for $i = 1, \ldots, m$,

\[
\frac{h_{i1} \circ F_i}{h_{i2} \circ F_i} + \frac{\ell_{i,1}}{\ell_{i,2}} - \frac{\ell_{i+1,1}}{\ell_{i+1,2}} =_{W} \frac{g_i''}{h_i''}.
\]

The equation \ref{eq:equation} characterizes the algebraic condition determined by $g_i''$ and $h_i''$ in relation to the unknown $h_{i1}, h_{i2}, F_i, \ell_{i,j}$ and $\ell_{i+1,j}$.
The information contained in the specification is described by \( m \) algebraic conditions of the form \( \{ \} \) giving relations of the specifying polynomials \( g_i', h_i'' \), \( i = 1, \ldots, m \), to the hidden polynomials including \( h_{i1}, h_{i2}, F_{ij}, \) and \( \ell_{ij} \).

Let \( V_f \) be the algebraic set determined by these \( m \) conditions. Let \( V_{f,\rho} \) be the algebraic set determined also by these conditions, however with \( F_{ij} \) expressed in terms of the blinding parameters.

For all local functions \( f_i, i = 1, \ldots, m, \) and basic blinding maps \( \rho \), such that \( \varphi_i \) has semi-local decomposition \( [f_i] \circ [\rho] \) for \( i = 1, \ldots, m \). Every \( (h_{i1}, h_{i2}) \in [f_i] \), \( i = 1, \ldots, m, \) and \( (F_{ij}) \in [\rho] \) satisfies the equations with \( \ell_{ij} \) and \( \ell_{ij+1} \) chosen in the procedure. Let \( S = \{ \varphi_i : i = 1, \ldots, m \} \). Then \( V_S \) (resp. \( V_{S,\rho} \)) is locally embed around every point of \( V_f \) (resp. \( V_{f,\rho} \)), since we have injective maps

\[
\begin{align*}
V_S &= \cup [f_1] \times \cdots \times [f_m] \times [\rho] \rightarrow \cup [f_1] \times \cdots \times [f_m] \times [\rho] \times \{ (\ell_{ij}) \} \subset V_f \\
V_{S,\rho} &= \cup [f_1] \times \cdots \times [f_m] \times (\rho) \rightarrow \cup [f_1] \times \cdots \times [f_m] \times (\rho) \times \{ (\ell_{ij}) \} \subset V_{f,\rho}.
\end{align*}
\]

We have proven the following:

**Theorem 4.** Let \( f : W \rightarrow \bar{k} \) be a function which is the sum of \( m \) semi-local functions of bounded degree. Suppose \( f \) is specified by a set of \( 2m \) polynomials \( g_i', h_i'' \), \( i = 1, \ldots, m \) by following the procedure described above. The information contained in the specification can be described by a set of \( m \) algebraic conditions of the form \( \{ \} \) giving relations of the specifying polynomials \( g_i', h_i'' \), \( i = 1, \ldots, m \), to the hidden polynomials including \( h_{i1}, h_{i2}, F_{ij}, \) and \( \ell_{ij} \). Let \( V_f \) be the algebraic set determined by these \( m \) conditions. Let \( V_{f,\rho} \) be the algebraic set determined also by these conditions, however with \( F_{ij} \) expressed in terms of the blinding parameters. Let \( S = \{ \varphi_i : i = 1, \ldots, m \} \). Then \( V_S \) (resp. \( V_{S,\rho} \)) is locally embed around every point of \( V_f \) (resp. \( V_f,\rho \)).

By Theorem 4, \( V_S \) (resp. \( V_{S,\rho} \)) is locally embedded in \( V_f \) (resp. \( V_{f,\rho} \)). The properties stated in Theorem 3 of \( V_S \) and \( V_{S,\rho} \), and the remarks following Theorem 3 is the basis for the heuristic assumption that when a function which is semi-local sum is specified as above, the specification serves as a black box function.

### 1.4 Specifying a semi-local product

Similarly, to specify a function \( f \) that is the product of semi-local functions, say \( f = \prod_{i=1}^m \varphi_i \) where \( \varphi_i \) is semi-local, we take the following steps to specify \( f \) as a product of \( m \) random-looking functions that are not semi-local. Again we focus on the case \( \varphi_i = f_i \circ \rho \) where \( f_i \) is \( c \)-local in that it non-trivially depends on \( V_i \times \cdots \times V_i \) for some \( i_1, \ldots, i_c \). The case that \( \varphi_i = f_i \circ (\rho, \rho, \rho) \) or \( \varphi_i = f_i \circ (\rho, \rho) \) where \( f_i \) is local depending on \( V_i' \times V_i \) or \( V_i' \times V_i' \times V_i \) is similar.
1. Construct $m$ random linear forms $\ell_i(x)$, $i = 1, \ldots, m$, $j = 1, 2$. Put $\ell_{m+1} = \ell_1$.

2. Suppose $\varphi_i(x) = \frac{g_i(x)}{h_i(x)}$ on $W$ with $g_i, h_i \in K[x]$. Then $\frac{g_i}{h_i} \cdot \ell_i = \frac{g_i'}{h_i'}$ where $h_i' = h_i \ell_{i+1}$ and $g_i' = g_i \ell_i$.

3. Choose random $g_i'' \in R g_i' + I_{d_i,1+1}$ where $d_i,1 = \deg g_i'$ and $h_i'' \in R h_i' + I_{d_i,2+1}$ where $d_i,2 = \deg h_i'$ for $i = 1, \ldots, m$.

4. Publish $\{g_i'', h_i'': i = 1, \ldots, m\}$, and specify $\varphi$ as $\prod_{i=1}^m \frac{g_i''}{h_i''}$ on $W$. (Note that $\prod_{i=1}^m \frac{\ell_i}{\ell_{i+1}} = 1$.)

Suppose $f_i = \frac{h_i}{h_2}$ where $h_{i1}$ and $h_{i2}$ are polynomials in 4 variables. Similar to the case in §1.3, the information contained in the specification of $f$ is captured in $m$ algebraic conditions, $i = 1, \ldots, m$:

$$\frac{h_{i1} \circ F_i}{h_{i2} \circ F_i} \ell_i \ell_{i+1} = w \frac{g_i''}{h_i''}.$$  \hspace{1cm} (2)

The following theorem can be proven in a way that is similar to the proof for Theorem 4.

**Theorem 5.** Let $f : W \to \bar{k}$ be a function which is the product of $m$ semi-local functions of bounded degree. Suppose $f$ is specified by a set of $2m$ polynomials $g''_i, h''_i, i = 1, \ldots, m$ by following the procedure described above. The information contained in the specification can be described by a set of $m$ algebraic conditions of the form (2) giving relations of the specifying polynomials $g''_i, h''_i, i = 1, \ldots, m$, to the hidden polynomials including $h_{i1}, h_{i2}, F_{i1}, F_{i2}$, and $\ell_i$. Let $V_f$ be the algebraic set determined by these $m$ conditions. Let $V_{f,\text{set}}$ be the algebraic set determined also by these conditions, however with $F_{i1}$ expressed in terms of the blinding parameters. Let $S = \{\varphi_i : i = 1, \ldots, m\}$. Then $V_S$ (resp. $V_{S,\rho}$) is locally embedded around every point of $V_f$ (resp. $V_{f,\rho}$).

Theorem 5 and Theorem 6 provide the basis for the heuristic assumption that when a function which is semi-local product is specified as above, the specification serves as a black box function.

### 1.5 Specifying maps of bounded locality

We consider two kinds of rational maps of bounded locality that will be involved in the trilinear map construction.

First suppose $\varphi = (\varphi_i)_{i=1}^n : \prod_{i=1}^n V_i \to \prod_{i=1}^n V_i$ is a rational map of locality bounded by $c$ in the sense that for all $i$, $\varphi_i : \prod_{j=1}^n V_j \to V_i \subset k^2$ consists of two $c_i$-local functions with $c_i \leq c$. For simplicity of discussion we assume $\varphi$ has locality 2. Applying the basic blinding map $\rho$ to $\varphi$ we get $\hat{\varphi} : W \to W$ such that $\rho \hat{\varphi} = \varphi \circ \rho$. 

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Second suppose $\varphi = (\varphi_i)_{i=1}^{3n} : \prod_{i=1}^{n} V_i \times \prod_{i=1}^{n} V_i \to \prod_{i=1}^{n} V_i$ is a rational map such that for all $i$, $\varphi_i : \prod_{j=1}^{n} V_j \times \prod_{j=1}^{n} V_j \to V_i$ is local in that it depends only on $V_i \times V_i$. Applying the basic blinding map $\rho$ to $\varphi$ we get $\hat{\varphi} : W \times W \to W$ such that $\rho \hat{\varphi} = \varphi \circ (\rho, \rho)$.

Let $\hat{\varphi} = (\hat{\varphi}_i)_{i=1}^{3n}$. In the proofs of the results in this section we will focus on the first case where $\varphi = (\varphi_i)_{i=1}^{n} : \prod_{i=1}^{n} V_i \to \prod_{i=1}^{n} V_i$. The argument for the second case is very similar and is omitted.

**Lemma 2.** For $i = 1, \ldots, 3n$, $\hat{\varphi}_i$ is the sum of $3n$ semi-local functions.

**Proof** Let $\rho_i = pr \circ \lambda_i \circ \delta_i$ and let $\mu_i = \lambda_i^{-1}$. Then $\hat{\varphi}_i = \delta^{-1}(\hat{\psi}_i)^{3n}_n$ where $\psi_i = \hat{\mu}_i \circ \varphi_i \circ \rho$ is a map $W \to k^3$ consisting of three semi-local functions. More explicitly if $\varphi_i$ depends on $V_i \times V_2$, then there are polynomials $g_{ij}, g_{ij}'$ of degree $O(1)$ in two variables such that $\psi_i(x) = (g_{ij}(F_{ij}(x)), F_{ij}(x))^3_{i=1}$. \(\square\)

We adopt the following notation. Suppose $f_i$, $i = 1, 2, m$, are polynomials in 3 variables, and $g_i$, $i = 1, 2, 3$, are polynomials. Let $f = (f_1, \ldots, f_m)$ and $g = (g_1, g_2, g_3)$. Then

$$f \circ g := (f_1, \ldots, f_m) \circ (g_1, g_2, g_3) := (f_1(g_1, g_2, g_3), \ldots, f_m(g_1, g_2, g_3)).$$

As before we have

$$\delta_i = (L_{i1}, L_{i2}, L_{i3}) \quad \lambda_i = (f_{i1}, f_{i2}, f_{i3}) \quad F_{ij} = f_{ij} \circ \delta_i \quad \rho_i = pr \circ \lambda_i \circ \delta_i = (F_{i1}, F_{i2})$$

Suppose $\varphi_i$ determined by $V_i \times V_2$. Let $g_{ij}, g_{ij}'$ be polynomials in 4 variables such that $\varphi_i(x) = \frac{g_{ij}(x_1, x_2)}{g_{ij}'(x_1, x_2)}$ where $x = (x_j)_{j=1}^{n} \in \prod_{j=1}^{n} V_j$ with $x_j \in V_j \subset k^2$.

Let $F_i = (F_{i1,1}, F_{i1,2}, F_{i2,1}, F_{i2,2})$. Then

$$\varphi_i \circ \rho = \left(\frac{g_{i1}}{g_{i1}'}, \frac{g_{i2}}{g_{i2}'}\right) \circ F_i.$$

Let $\mu_i = \lambda_i^{-1} = (h_{i1}, h_{i2}, h_{i3})$. Let $\bar{\mu}_i(x, y) = \mu_i(x, y, y)$. Let

$$u_i = (u_{i1}, u_{i2}, u_{i3}) = \bar{\mu}_i \circ \varphi_i = (h_{i1}, h_{i2}, h_{i3}) \circ \left(\frac{g_{i1}}{g_{i1}'}, \frac{g_{i2}}{g_{i2}'}, \frac{g_{i2}}{g_{i2}'}\right).$$

Then $\bar{\mu}_i \circ \varphi_i \circ \rho = u_i \circ F_i$ is semi-local. Let $(v_i)_{i=1}^{3n}$ be such that $u_i \circ F_i = v_{3(i-1)+j}$. Then $\hat{\varphi}_i = \delta^{-1}(v_i)_{i=1}^{3n}$. We have proved the following
Lemma 3. For \( i = 1, \ldots, 3n \), \( \hat{\varphi}_i \) is the sum of \( 3n \) semi-local functions.

We apply the procedure described in § 4.3 to specify each \( \hat{\varphi}_i \) as the sum of \( 3n \) random looking functions from \( W \) to \( k \). Suppose \( \hat{\varphi}_i \) is specified as \( \sum_{j=1}^{3n} \frac{G_{ij}}{G_{ij}'} \) for some polynomials \( G_{ij} \) and \( G_{ij}' \). Therefore \( \hat{\varphi} \) is specified by these \( O(n^2) \) polynomials \( G_{ij} \) and \( G_{ij}' \).

Again, write \( f = W \cdot g \) if for rational functions \( f \) and \( g \) on \( k^{3n} \), \( f(x) = g(x) \) for all \( x \in W \). Now consider information revealed by the specification of \( \hat{\varphi} \) about the hidden map \( \varphi \) and the blinding parameters \( \delta, \lambda_i, i = 1, \ldots, n \).

Suppose \( \delta^{-1} = (w_{ij})_{1 \leq i, j \leq 3n} \). We have \( \hat{\varphi}_i = \sum_{j=1}^{3n} w_{ij} v_j \) where \( v_j = u_{rs} \circ F_r \) with \( 1 \leq r \leq n \) and \( 1 \leq s \leq 3 \) such that \( j = 3(r-1) + s \). Put \( r = r_j \) and \( s = s_j \).

We have \( w_{ij} v_j + \ell_{ij} = \frac{\ell_{ij+1}}{G_{ij}'} = W \frac{G_{ij}}{G_{ij}'} \), and \( \ell_{ij}, \ell_{ij}' \) are linear forms for \( i, j = 1, \ldots, 3n \) with \( \ell_{i,3n+1} = \ell_{i,1} \) and \( \ell_{i,3n+1}' = \ell_{i,1}' \).

Put \( w_{ij} u_{r_j,s_j} = \frac{g}{g'} \) where \( g, g' \) are polynomials in 4 variables. Then as Theorem 4 is applied in this situation we get

\[
\frac{g \circ F_{ij}}{g' \circ F_{ij}} + \frac{\ell_{ij}}{\ell_{ij}'} = W \frac{G_{ij}}{G_{ij}'}
\]

Equation (3) characterizes the condition in specifying \( G_{ij} \) and \( G_{ij}' \), where \( \ell_{ij} \) are unknown linear forms in 3 variables, \( F_{ij} \) are unknown quadratic polynomials in 3 variables, \( f \) and \( g' \) are unknown polynomials in 4 variables of degree 2 deg \( \varphi_i = O(1) \), where \( w_{ij} u_{r_j,s_j} = \frac{g}{g'} \).

The other kind of information provided once \( \hat{\varphi} \) is specified is the information that may be obtained by evaluation of \( \bar{\varphi} \) on \( W \). Suppose \( \alpha \in W \) and \( \bar{\varphi}(\alpha) = \beta \). Then for \( i = 1, \ldots, n \), we have \( \varphi_i \circ \rho(\alpha) = \rho(\beta) \), hence we get the condition

\[
\frac{g_{11}(F_i(\alpha))}{g'_{11}(F_i(\alpha))} = F_{i1}(\beta) \quad (4)
\]

\[
\frac{g_{21}(F_i(\alpha))}{g'_{21}(F_i(\alpha))} = F_{i2}(\beta) \quad (5)
\]

The following two theorems can be proved in a way that is similar to the proof of Theorem 4

Theorem 6. Suppose \( \hat{\varphi} \) is specified using the procedure described in § 4.3 with \( O(n^2) \) polynomials of bounded degree in \( O(n) \) variables. The information contained in the specification can be described by \( n^{O(1)} \) conditions, in the forms of Eqns (3) and (4,5), in \( n^{O(1)} \) unknowns representing the coefficients of the hidden polynomials, including \( F_{ij} \), which determine a basic blinding map, and \( g_{ij}, g'_{ij} \).
which determine the local functions, with \( i = 1, \ldots, n, \ j = 1, 2 \). Let \( V_\varphi \) be the algebraic set determined by these conditions. Let \( V_{\varphi, \rho} \) be the algebraic set determined by these conditions, however with \( F_{ij} \) expressed in terms of the blinding parameters. Let \( S \) be the set of \( 9n^2 \) semi-local functions involved in specifying \( \varphi \). Then \( V_S \) (resp. \( V_{S, \rho} \)) is locally embedded around every point of \( V_\varphi \) (resp. \( V_{\varphi, \rho} \)).

**Theorem 8.** Suppose a random basic blinding map \( \rho \) is chosen and a set of functions each of which is either the sum or product of \( O(n) \) semi-local functions of bounded degree is specified. Then the information contained in the specification can be described by \( n^{O(1)} \) conditions, in the forms of Eqns (1,2), (3) and (4,5), in \( n^{O(1)} \) unknowns representing the coefficients of the hidden polynomials, including \( F_{ij} \), with \( i = 1, \ldots, n \) and \( j = 1, 2 \), which determine a basic blinding map, and polynomials which determine the hidden local functions. Let \( V \) be the algebraic set determined by these conditions. Let \( V_\rho \) be the algebraic set determined by these conditions, however with \( F_{ij} \) expressed in terms of the blinding parameters. Let \( S \) be the set of semi-local functions that are involved. Then \( V_S \) (resp. \( V_{S, \rho} \)) is locally embedded around every point of \( V \) (resp. \( V_\rho \)).

Theorem 7 together with Theorem 3 underscores the difficulty of solving for a closed point of \( V \) or \( V_\rho \) for the purpose of un-blinding.

### 1.6 Blinding maps with Frobenius twists

Fix and publish a randomly chosen basis of \( K/k, \theta = \theta_1, \ldots, \theta_d \). As before, let \( \tau \) denote the Frobenius map \( x \to x^q \) for \( x \in k \) (where \( k = \mathbb{F}_q \)), let \( \tau_a = \tau^a \), and \( \tau_{a,b} \) denote the Frobenious twist \( \bar{k}^2 \to k^2 : (x, y) \to (\tau_a(x), \tau_b(y)) = (x^q, y^q) \). When the context is clear we also denote \( \tau_{a,0} \) as \( \tau_a \).

We consider a blinding map \( \rho' = (\rho'_i)_{i=1}^n \) of the form \( \rho'_i = (\tau_{a_i} \circ \rho_i)_{i=1}^n \) where \( \rho = (\rho_i)_{i=1}^n \) is a basic blinding map. Suppose \( \rho_i = (F_{i1}, F_{i2}) \) where \( F_{ij} \) are quadratic polynomials in \( 3n \) variables.

Let \( S = \{ g_1, \ldots, g_n \} \) where \( g_i \) has semi-local decomposition \( [f_i] \circ [\rho'] \) where \( f_i \) is a local function. In this case we may write the decomposition as \( [f_i] \circ [\tau_a] \circ [\rho] \), where \( \tau_a = (\tau_{a_1}, \ldots, \tau_{a_n}) \). Let \( A \) be a block diagonal matrix with \( A_1, \ldots, A_n \) as the diagonal blocks where \( A_i \in GL_2(k) \). It is easy to check that \( A_i \circ \tau_a = \tau_{a_i} \circ A_i^{-1} \). Using this we verify that \( g_i \) has semi-local decomposition \( [f_i \circ A'] \circ [\tau_a] \circ [\tau^{-1} \circ \rho] \) where \( A' \) is block diagonal matrix consisting of diagonal blocks \( A_1^{-1}, \ldots, A_n^{-1} \). We may write \( A' \) as \( A'^{-1} \). From this it is not hard to see the that we have the following generalization of Theorem 5.

**Theorem 8.** Suppose we have a set of semi-local functions \( g_i, i = 1, \ldots, m \), such that \( g_i \) has semi-local decomposition \( [f_i] \circ [\rho'] \) for all \( i \), where \( f_i \) is a local function and \( \rho' = \tau_a \circ \rho \) where \( \rho \) is a basic blinding map and \( \tau_a = (\tau_{a_1}, \ldots, \tau_{a_n}) \). Then the following hold.
1. There is an injective map $\bar{k}^{2n^2} \to [\rho]$, and an injective map $\bar{k}^n \to (\rho)$.
2. There is an injective map $\bar{k} \to [f]$ if $f_i$ is $c$-local depending on $V_i \times \ldots V_n$, and for some $j$ the degree of $f_j$ at $x_{ij}$ is greater to equal to the minimum degree of polynomials in the ideal defining $V_i$.
3. Let $A_j \in \text{GL}_2(\bar{k})$ for $j = 1, \ldots n$. Let $A$ be the block-diagonal matrix with $A_1, \ldots, A_n$ as the diagonal blocks. Then $g_i$ has semi-local decomposition $[f_i \circ A^{-x}] \circ [\tau_i] \circ [A^{-1} \circ \rho]$ for $i = 1, \ldots, m$.

Let $S = \{ g_i : i = 1, \ldots, m \}$ be as in the theorem. Let $V_S$ be the union of $[f_1] \times \ldots \times [f_m] \times [\rho]$, where the union is over all $f_1, \ldots, f_m, \rho$ such that $g_i$ has semi-local decomposition $[f_i] \circ [\tau_i] \circ [\rho], i = 1, \ldots, m$. Similarly let $V_{S,\rho}$ be the union of $[f_1] \times \ldots \times [f_m] \times [\rho]$, where the union is over all $f_1, \ldots, f_m, \rho$ such that $g_i$ has semi-local decomposition $[f_i] \circ [\tau_i] \circ [\rho]$, $i = 1, \ldots, m$. Then $V_S$ and $V_{S,\rho}$ admit local embedding of affine space of dimension $\Omega(n^2)$ (respectively $\Omega(n)$) around every point by the first two assertions of Theorem [S], and both are acted on by $(\text{GL}_2(\bar{k}))^n$ in a twisted fashion. The two properties combined, and the fact that their dimensions are huge (respectively $\Omega(n^2)$ and $\Omega(n)$), seem to make it difficult to solve for a closed point even if polynomial systems describing $V_S$ and $V_{S,\rho}$ are known.

Suppose $f$ is a polynomial of degree $d$ then $\tau_a \circ f(x) = (f(x))^\rho$ has degree $d\rho^n$. Therefore polynomials of degree exponential in $q$ result as we apply Frobenius twists to blind a semi-local function. This makes it more complicated to describe $V_S$ and $V_{S,\rho}$.

In order to specify blinded maps using low degree polynomials we consider the descent trick which is involved in Weil restriction (descent).

Suppose $f$ is a polynomial with coefficients in $\bar{k}$, let $f^{\tau_a}$ denote the polynomial obtained by applying $\tau_a$ to all coefficients of $f$.

Suppose $f \in \bar{k}[x_1, \ldots, x_{3n}]$. Let $\bar{x}_i = \sum_{j=1}^d x_{ij} \theta_j$, for $i = 1, \ldots, 3n$, where $x_{ij}$ are variables. Let $x = x_1, \ldots, x_{3n}$ and $X = x_{11}, \ldots, x_{3n,d}$. Let $\bar{f}(X) = f(\bar{x}_1, \ldots, \bar{x}_{3n})$.

Let $J$ be the ideal generated by $x_{ij}^q - x_{ij}$ for all $i, j$. Let $f(x) \mod J$ denote the polynomials $g(x)$ with degree less than $q$ in all $x_{ij}^q$ such that $f \equiv g \mod J$. Let $\bar{f}^{(\alpha)}(X) = (\bar{f}(X))^{\rho^n} \mod J$. Note that $\bar{f}^{(\alpha)} = \tau_a \circ f \mod J$.

We call $\bar{f}$ the descent of $f$ with respect to $\theta$, or simply the descent of $f$ when $\theta$ is fixed. We call $\bar{f}^{(\alpha)}$ a twisted $a$-descent of $f$. Note that $\deg \bar{f}^{(\alpha)} \leq \deg f$.

We have an injective homomorphism $\bar{k}[x] \to \bar{k}[^{\bar{x}}] : f \to \bar{f}$ where $\bar{x}$ denotes the sequence $x_{ij}, i = 1, \ldots, 3n, j = 1, \ldots, d$. Let $V_S$ (resp. $V_{S,\rho}$) denote the image
of $V_S$ (resp. $V_{S,\rho}$) under the map naturally induced by the injection $[\rho] \to [\tilde{\rho}]$. From this observation and Theorem 5 we have the following theorem.

**Theorem 9.** Suppose we have a set $S$ of semi-local functions $g_i$, $i = 1, \ldots, n$, such that $g_i$ has semi-local decomposition $[f_i] \circ [\rho']$ for all $i$, where $f_i$ is a local function and $\rho' = \tau_a \circ \rho$ where $\rho$ is a basic blinding map and $\tau_a = (\tau_{a_1}, \ldots, \tau_{a_n})$.

Then the following hold.

1. Around every point of $\tilde{V}_S$ there is an embedding of $\tilde{k}^{2n^2}$ relative to the blinding part, and an embedding of $\tilde{k}$ relative to the local part if there is some $f_i$, $c$-local depending on $V_i \times \ldots V_i$, where for some $j$ the degree of $f_i$ at $x_i$ is greater to equal to the minimum degree of polynomials in the ideal defining $V_i$.

2. Around every point of $\tilde{V}_{S,\rho}$ there is an embedding of $\tilde{k}^n$ relative to the blinding part, and an embedding of $\tilde{k}$ relative to the local part if there is some $f_i$, $c$-local depending on $V_i \times \ldots V_i$, where for some $j$ the degree of $f_i$ at $x_i$ is greater to equal to the minimum degree of polynomials in the ideal defining $V_i$.

3. There is a twisted action of $(GL_2(k))^n$ on $\tilde{V}_S$ and $\tilde{V}_{S,\rho}$.

**Specifying blinded functions and maps**

We have an isomorphism $\lambda : k^d \to K : (a_i)_{i=1}^d \to \sum_{i=1}^d a_i \theta_i$. If $v_i \in k^d$ then

$$f(v_1, \ldots, v_{3n}) = f(\lambda(v_1), \ldots, \lambda(v_{3n}))$$

$$f^{(a)}(v_1, \ldots, v_{3n}) = (f(\lambda(v_1), \ldots, \lambda(v_{3n})))^{\tau_a}$$

Therefore though $\tau_a \circ f(x) = (f(x))^q^a$ has high degree, when the polynomial is applied to $x = \lambda \hat{x}$ where $\hat{x} = (x_{ij})$ with $x_{ij} \in k$, we can instead apply the low-degree $\tilde{f}^{(a)}$ to $\hat{x}$.

Let $W = \rho^{-1}k^{2n}$, the blinding space. Let $\tilde{W}$ consists of $v = (v_i)_{i=1}^{3n}$ with $v_i \in k^d$ for $i = 1, \ldots, 3n$, such that $(\lambda(v_i))_{i=1}^{3n} \in W$. Let $I$ be the ideal generated by $f$ for $f \in I$.

**Lemma 4.** If $f =_{W} 0$ then $\tilde{f}^{(a)} =_{\tilde{W}} 0$ for all $a$. In particular the dimension of $I_2$ is at least the dimension of $I_2$, which is $\Omega(n)$.

The lemma follows since $\tilde{f}^{(a)}(v_1, \ldots, v_{3n}) = (f(\lambda(v_1), \ldots, \lambda(v_{3n})))^{\tau_a}$. \\]

For specifying a function that is the sum of semi-local functions, the same procedure in §13 applies. Then the specifying polynomials $g_i''$ and $h_i''$ are now to be expressed in descent form by applying the substitution $x_i = \sum_{j=1}^d x_{ij} \theta_j$ for $i = 1, \ldots, 3n$. The conditions (1) become in this setting the following:

$$\frac{h_{i1} \circ (\tau_{a_{11}}, \tau_{a_{12}}) \circ \tilde{F}_i \mod J}{h_{i2} \circ (\tau_{a_{11}}, \tau_{a_{12}}) \circ \tilde{F}_i \mod J} + \frac{\tilde{t}_{i,1}}{\tilde{t}_{i,2}} - \frac{\tilde{t}_{i+1,1}}{\tilde{t}_{i+1,2}} = \tilde{W} \frac{g_i''}{h_i''}, \quad (6)$$
Let $\tilde{V}_f$ be the algebraic set determined by these conditions. Then consider as before the same set of $m$ conditions, however with the coefficients of $\tilde{F}_{ij}$ expressed in terms of the blinding parameters. Let $\tilde{V}_{\rho,f}$ be the algebraic set determined by these conditions that involve the blinding parameters. Then $\tilde{V}_{\rho,f}$ is nonlinear of dimension $\Omega(n)$.

A similar analysis can be carried out for products of semi-local functions, maps of bounded locality, and more generally the case where a set of such functions and maps are specified with algebraic blinding with Frobenius twists. In this way analogous theorems to Theorem 5, Theorem 6 and Theorem 7 can be proved. We state below the analogous theorem to Theorem 7.

**Theorem 10.** Suppose a random blinding map $\rho$ is chosen and a set of functions each of which is either the sum or product of $O(n)$ semi-local functions of bounded degree is specified as discussed in this section. Then we have the following.

1. The information contained in the specification can be described by $n^{O(1)}$ conditions, in $n^{O(1)}$ unknowns representing the hidden polynomials in the descent form, including $\tilde{F}_{ij}$, with $i = 1, \ldots, n$ and $j = 1, 2$, where $F_{ij}$'s determine a basic blinding map, and the polynomials which determine the hidden local functions.

2. Let $\tilde{V}$ be the algebraic set determined by these conditions. Let $\tilde{V}_{\rho}$ be the algebraic set determined by these conditions, however with $\tilde{F}_{ij}$ expressed in terms of the blinding parameters. Let $S$ be the set of semi-local functions that are involved. Then $\tilde{V}_S$ (resp. $\tilde{V}_{\rho,S}$) is locally embedded around every point of $\tilde{V}$ (resp. $\tilde{V}_\rho$).

3. Around every point of $\tilde{V}$ there is an embedding of $\bar{k}^{2n^2}$, and around every point of $\tilde{V}_{\rho}$ an embedding of $k^n$. Moreover there is a twisted action of $(G_{l2}(\bar{k}))^n$ on $\tilde{V}$ and $\tilde{V}_\rho$.

## 2 Trilinear map construction

To construct a trilinear map we start by choosing a pairing friendly elliptic curve $E/K$ defined by $y^2 = x^3 + ax + b$ with $a, b \in K$, such that $E[\ell] \subset E(K)$, where $\log \ell$ and $\log |K|$ are linear in the security parameter. The curve $E$ is considered secret.

Our trilinear map is derived from a blinded version of the following map:

$$E[\ell]^n \times E[\ell]^n \times \text{Mat}_n(\mathbb{F}_\ell) \rightarrow \mu_\ell$$

$$(\alpha, \beta, M) \rightarrow e(\alpha, M(\beta))$$

where $\alpha, \beta \in E[\ell]^d$, $M \in \text{Mat}_n(\mathbb{F}_\ell)$ is identified with an element of $\text{End}(E[\ell]^n)$, and $e$ is a non-degenerate bilinear pairing on $E[\ell]^n$ naturally induced by Weil pairing on $E[\ell]$. 

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Consider a family of elliptic curves defined over $K$ which are isomorphic to $E$, which includes for every $A \in GL_2(K)$, the curve that is isomorphic to $E$ under $A$. Choose randomly from this family $E_i$, $i = 1, \ldots, n$.

As in §1.6 suppose $K$ is a finite extension over $k = F_q$. Fix and publish a randomly chosen basis of $K/k$, and let $\tau$ denote the Frobenius map $x \mapsto x^q$ for $x \in k$. Choose a random blinding map $\rho = (\rho_i)_{i=1}^n$ of the form $\rho_i = (\tau a_i \circ \rho_i)'_{n=1}$ where $\rho' = (\rho_i)'_{n=1}$ is a basic blinding map. Let $\hat{E} = \rho^{-1} \prod_{i=1}^n E_i$. Choose $\alpha, \beta \in E[\ell]^n$ such that $e(\alpha, \beta) \neq 1$. Let $\hat{\alpha}, \hat{\beta} \in \hat{E}[\ell]$ such that $\hat{\alpha}$ corresponds to $\alpha$ and $\hat{\beta}$ corresponds to $\beta$ under $\hat{E} \xrightarrow{\beta} \prod_{i=1}^n E_i \simeq E^n$. Let $G_1$ and $G_2$ be the groups generated by $\hat{\alpha}$ and $\hat{\beta}$ respectively. They are the first two groups in the trilinear map. The points $\hat{\alpha}$ and $\hat{\beta}$ are made public, while $\alpha$ and $\beta$ are secret.

The addition map on $\hat{E}[\ell]$, $\hat{\mu}$, which can be securely specified by generalization of Theorem 6 (as discussed in §1.6) see also Theorem 10, serves as the group law in both $G_1$ and $G_2$.

Choose a set of $N = O(n^2)$ matrices $M_i \in GL_n(\mathbb{F}_\ell)$ such that each row of $M_i$ has two non-zero entries, which contain 1, and that the matrices $M_i$ together with the identity matrix $M_0$ generate $Mat_n(\mathbb{F}_\ell)$ as a vector space over $\mathbb{F}_\ell$. Associate $M_i$ with the endomorphism $\varphi_i \in \text{End} \prod_{i=1}^n E_i \simeq \text{End} E^n$ such that for $\alpha \in \prod_{i=1}^n E_i[\ell]$ that under $\prod_{i=1}^n E_i[\ell] \simeq E^n[\ell]$ is identified with $(\alpha_i)_{i=1}^n$ with $\alpha_i \in E[i][\ell]$, $\varphi_i(\alpha)$ is identified with $(\beta_j)_{j=1}^n$ where $\beta_j = m(\alpha_{j1}, \alpha_{j2})$ if $(j, j_1)$ and $(j, j_2)$ are the two non-zero entries of the $j$-th row of $M_j$. Let $\hat{\varphi}_i = \rho^{-1} \circ \varphi_i \circ \rho$ for $i = 1, \ldots, N$. Theorem 6 can be applied to specify $\hat{\varphi}_i$.

Write $R = \mathbb{F}_\ell[z_1, \ldots, z_N]$. Define an action of $R$ on $\hat{E}[\ell]$ so that $z_i$ acts by $\hat{\varphi}_i$ for $i = 1, \ldots, N$. This is compatible with the action of $R$ on $E^n[\ell]$ where $z_i$ acts by $M_i$. Let $\Lambda$ be the kernel of the $\mathbb{F}_\ell$-algebra morphism $\lambda: R \rightarrow Mat_n(\mathbb{F}_\ell)$ determined by $\lambda(z_i) = M_i$, $i = 1, \ldots, N$.

We are ready to describe the trilinear map: $G_1 \times G_2 \times G_3 \rightarrow \mu_\ell \subset K$. Let $K = \mathbb{F}_q$, $E, \ell, \rho, \hat{E} = \rho^{-1} \prod_{i=1}^n E_i$ be as described above where $\log q$, $\log \ell$, $n$ are linear in the security parameter.

$G_1$ and $G_2$ are generated respectively by some $\hat{\alpha}, \hat{\beta} \in \hat{E}[\ell]$ with $\hat{e}(\hat{\alpha}, \hat{\beta}) \neq 1$.

The third group $G_3 = \mathbb{F}_\ell$ in the trilinear map is identified with $(\mathbb{F}_\ell + \Lambda)/\Lambda$. Theoretically $a \in \mathbb{F}_\ell$ is represented by polynomials in $a + \Lambda$. However for efficiency purpose we will only choose polynomials in $a + \Lambda$ of degree $n^{O(1)}$ with number of terms with nonzero coefficients bounded in $n^{O(1)}$. For simplicity let $[a]$ denote the subset of $f \in a + \Lambda$ such that $f$ is a linear polynomial plus a term of degree $n$. We can allow more general $f \in a + \Lambda$ to be included in $[a]$ as long as the
support of \( f \) is polynomially bounded in \( n \). The choice just made is simple but sufficient for our purposes.

**Private encoding** To encode \( a \in \mathbb{F}_\ell \), choose random \( i_1, \ldots, i_n \in \{1, \ldots, N\} \), then find \( c \) and \( b_0, \ldots, b_N \in \mathbb{F}_\ell \) such that \( cM_{i_1} \ldots M_{i_n} + \sum_{i=0}^N b_i z_i = a \). Then set \( f = cz_{i_1} \ldots z_{i_n} + \sum_{i=0}^N b_i z_i \). We have \( f \in [a] \). Note that \( c \) and \( b_i \) can be found by simple linear algebra once \( M = M_{i_1} \ldots M_{i_n} \) is computed.

For \( a, b, c \in \mathbb{F}_\ell \) and \( f \in [c] \), the trilinear map sends \( (a\hat{\alpha}, b\hat{\beta}, f) \) to \( \hat{e}(a\hat{\alpha}, f(b\hat{\beta})) = \zeta^{abc} \), where \( \zeta = \hat{e}(\hat{\alpha}, \hat{\beta}) \).

In addition to \( \hat{\alpha}, \hat{\beta} \in \hat{E}[\ell] \subset \mathbb{K}^{3n} \), a set of maps of bounded locality is specified: the addition morphism \( \hat{\mu} \), and \( \hat{\varphi}_i, i = 1, \ldots, N \) where \( N = O(n^2) \). For the computation of \( \hat{e} \) two functions \( \hat{g} \) and \( \hat{h} \) are specified, both are products of semi-local functions of bounded degree, as will be discussed in §3.

**Theorem 11.** The information contained in the specification of the trilinear map described in this section can be described by \( n^{O(1)} \) algebraic conditions in \( n^{O(1)} \) unknowns. Let \( \hat{V} \) be the algebraic set determined by these conditions. Let \( \hat{V}_\rho \) be the algebraic set determined by these conditions, however with the quadratic polynomials describing the basic blinding map that is involved expressed in terms of the blinding parameters. Let \( S \) be the set of semi-local functions that are involved in specifying the trilinear map. Then the hidden \( \hat{V}_S \) (resp. \( \hat{V}_{S,\rho} \)) is triply confusing and can be locally embedded around every point of \( \hat{V} \) (resp. \( \hat{V}_\rho \)). In particular around every point of \( \hat{V} \) there is an embedding of \( \bar{k}^{2n^2} \), and around every point of \( \hat{V}_\rho \) an embedding of \( \bar{k}^n \), moreover there is a twisted action of \( (Gl_2(\bar{k}))^n \) on \( \hat{V} \) and \( \hat{V}_\rho \).

**Proof** Let \( S \) be the set of semi-local functions involved in \( \hat{\mu}, \hat{\varphi}_i, \hat{g} \) and \( \hat{h} \). The local functions involved in these semi-local functions are all related to the addition law on \( E \) and are of degree 3 in at least one variable (see §3). This implies by Theorem 11, the algebraic set \( V_S \) and \( V_{S,\rho} \) are triply confusing, so are \( \hat{V}_S \) and \( \hat{V}_{S,\rho} \). Theorem 11 follows as Theorem 7 and Theorem 10 are applied to the current context.

### 2.1 Trapdoor discrete logarithm

The security strength of the third group \( G_3 \) is captured in a trapdoor discrete logarithm problem described below. Let \( E \) an elliptic curve defined over \( K = \mathbb{F}_q \) (not necessarily pairing friendly for the problem defined here). Let \( \log q, \log \ell \) and \( n \) are linear in the security parameter The trapdoor secret consists of the following, described earlier in this section.

1. \( \rho \) a randomly chosen blinding map,
2. \( E = \rho^{-1} \prod_{i=1}^n E_i \),

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3. \( M_1, \ldots, M_N \in \text{GL}_n(\mathbb{F}_t) \) with \( N = O(n^2) \), such that each row of \( M_i \) has two non-zero entries, which contain 1, and that the matrices \( M_i \) together with the identity matrix \( M_0 \) generate \( \text{Mat}_n(\mathbb{F}_t) \) as a vector space over \( \mathbb{F}_t \).
4. \( R = \mathbb{F}_t[z_1, \ldots, z_N] \) a non-commutative associative algebra over free variables \( z_1, \ldots, z_N \).

The following are publicly specified:
1. \( \hat{\beta} \in \hat{E}[t] \),
2. \( \hat{\phi}_1, \ldots, \hat{\phi}_N \in \text{End}(\hat{E}[t]), (\rho_i \circ \hat{\phi}_i = M(\rho_i, \rho_i)) \) where \( M(i, i_1) = M(i, i_2) = 1 \) for \( i = 1, \ldots, n \),
3. \( \hat{m} \) where \( m : E \times E \to E \) is the addition morphism.

The discrete logarithm problem is: Given \( f \in R \) supported at \( 1, z_1, \ldots, z_N \) and a monomial of degree \( n \), to determine \( a \in \mathbb{F}_t \) such that \( f(\hat{\phi}_1, \ldots, \hat{\phi}_N)(\hat{\beta}) = a\hat{\beta} \).
In other word, given \( f \in [a] \) with \( a \) unknown, the problem is to find \( a \).

We assume that \( n^{O(1)} \) many random samples of \([0]\) are publicly available. The set \([0]\) contains at least \( N^n \) linearly independent polynomials since there are \( N^n \) (non-commutative) monomials of degree \( n \). Suppose \( f \in [a] \). The probability that \( f - a \) and a random sample of \( s \) elements of \([0]\) are linearly dependent is negligibly small unless \( s \geq N^n \), since these elements most likely involve distinct monomials of degree \( n \). Therefore it seems very unlikely to mount an efficient linear algebra attack, unless the trapdoor secret map \( \lambda : R \to \text{Mat}_n(\mathbb{F}_t) \) is revealed.

Apart from its application to trilinear map construction, the trapdoor discrete logarithm problem formulated above is of independent interest.

2.2 The Decision-Diffie-Hellman (DDH) assumption

It is easy to modify the trilinear map construction so that the Decision-Diffie-Hellman (DDH) assumption is conjecturally satisfied on the pairing groups. We use a random blinding \( \rho \) to construct \( \hat{E} \), and form \( \hat{\beta} \) and \( G_2 \) and \( \hat{\phi}_i \)'s as above. Then we use a different random blinding \( \rho' \) to construct \( \hat{E}' = \rho'^{-1} \prod_{i=1}^{n} E_i \), and form \( \hat{\alpha} \in \hat{E}'[t] \) such that \( \hat{\alpha} \) corresponds to \( \alpha \) under \( \hat{E}' \to \prod_{i=1}^{n} E_i \simeq E^n \). The pairing \( \hat{e} \) is now between \( \hat{E}'[t] \) and \( \hat{E}[t] \).

We note that if the same blinding \( \rho \) is used to form \( G_1 \) as before and \( \hat{e} \) is a pairing between \( \hat{E}[t] \) and \( \hat{E}[t] \), the group \( G_1 \) may not satisfy the DDH assumption. The reason is that we may heuristically assume \( \hat{e}(\hat{\alpha}, \hat{\phi}_i(\hat{\alpha})) \neq 1 \) for some \( i \), so we can use \( \hat{\phi}_i \) to induce a non-degenerate self-pairing on \( G_1 \). We can verify \( ab\hat{a} \) from \( a\hat{a} \) and \( b\hat{a} \) using the following equality:
\[
\hat{e}(a\hat{a}, \hat{\phi}_i(b\hat{a})) = \hat{e}(\hat{\alpha}, \hat{\phi}_i(ab\hat{a})).
\]
When a different blinding \(\rho'\) is used to form \(\hat{\ell}'\), no map from \(\hat{\ell}'[\ell]\) to \(\hat{\ell}['\ell]\) is available to induce a self-pairing on \(G_1\). Similar remarks apply to \(G_2\).

3 Pairing computation

To complete the description of the trilinear map we describe how \(\hat{\ell}\) is defined and specified. To simply notation and make the presentation easier we identify \(E_i\) and \(E\) through isomorphism and denote for example the addition morphism of \(E_i \simeq E\) also as \(m\). We also assume the same blinding map \(\rho\) is used to form the pairing groups, so that \(\hat{\ell}\) is a pairing on \(\hat{\ell}[\ell]\).

Suppose the characteristic of \(K\) is not 2 or 3, and \(E\) is given \(y^2 = x^3 + ax + b\) with \(a, b \in K\). The addition map of \(E\) can be described as follows (see \([12]\)). Let \(P_1 = (x_1, y_1), P_2 = (x_2, y_2)\) be two points on \(E\). If \(x_1 = x_2\) and \(y_1 = -y_2\), then \(P_1 + P_2 = 0\). Otherwise, we can find \(P_3 = (x_3, -y_3)\) such that \(P_1, P_2\) and \(P_3 = (x_3, y_3)\) lie on a line \(y = \lambda x + \nu\), and we have \(P_1 + P_2 = P_3\).

1. If \(x_1 \neq x_2\), then \(\lambda = \frac{y_2 - y_1}{x_2 - x_1}\) and \(\nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}\).

2. If \(x_1 = x_2\) and \(y_1 \neq 0\), then \(\lambda = \frac{3x_1^2 + a}{2y_1}\) and \(\nu = \frac{-x_1^3 + ax_1 + 2b}{2y_1}\).

In both cases \(x_3 = \lambda^2 - x_1 - x_2, y_3 = -\lambda x_3 - \nu\).

Suppose \(P_i = (x_i, y_i)\) with \(P_i \in E\) for \(i = 1, 2, 3, x_1 \neq x_2\) and \(P_1 + P_2 = P_3\). Then \(g_{P_1, P_2} := \frac{y_1 y_2 - y_3 y_1}{x_2 - x_1} + P_3 - O = P_1 + P_2 - 2O\), where \(\lambda = \frac{y_2 - y_1}{x_2 - x_1}\) and \(\nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}\).

Let \(g : E \times E \times E \to \hat{k}\) such that for \(P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E\), and \(Q = (x, y) \in E\), \(g(P_1, P_2, Q) = \frac{y_1 y_2 y - y_3 y_1}{x_2 - x_1} = g_{P_1, P_2}(Q)\).

Let \(\hat{g} : \hat{E} \times \hat{E} \times \hat{E} \to \hat{k}\) so that for \(\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \hat{E}\), \(\hat{g}(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \prod_{i=1}^n g(\alpha_i(\hat{\alpha}), \beta_i(\hat{\beta}), \gamma_i(\hat{\gamma}))\).

For \(D = P_1 - O\) where \(P_1 = (x_1, y_1)\) is not 2-torsion, we have \(2D = (h_D) + D'\) where \(D' = P_3 - O\) with \(2P_1 = P_3 = (x_3, y_3)\) given by the formula above, and \(h_D(x, y) = \frac{L}{x - x_1}\) where \(L = y + \lambda x - \nu, \lambda = \frac{3x_1^2 + a}{2y_1}\) and \(\nu = \frac{-x_1^3 + ax_1 + 2b}{2y_1}\).

So let \(h : E \times E \to \hat{k}\) so that for \(P_1 = (x, y) \in E\) and \(P_2 = (x_2, y_2) \in E\), \(h(P_1, P_2) = h_D(P_2) = h_D(x_2, y_2)\) as above where \(D = P_1 - O\).

Let \(\hat{h} : \hat{E} \times \hat{E} \to \hat{k}\) so that for \(\hat{\alpha}, \hat{\beta} \in \hat{E}\), \(\hat{h}(\hat{\alpha}, \hat{\beta}) = \prod_{i=1}^n h(\alpha_i(\hat{\alpha}), \beta_i(\hat{\beta}))\).

Suppose \(P \in E[\ell]\). Then \(D = P - O\) is an \(\ell\)-torsion divisor. We recall how to efficiently construct \(h\) such that \(\ell D = (h)\) through the squaring trick \([13][14]\).

Let \(D_i = P_i - O\) where \(P_i = 2^i P\) for all \(i\). Apply addition to double \(D\), and get

\[
2D = (h_D) + D_1.
\]
Inductively, we have $H_i$ such that

$$2^i D = (H_i) + D_i.$$

Apply addition to double $D_i$ and get

$$2D_i = (h_{D_i}) + D_{i+1}.$$ We have

$$2^{i+1} D = (H_{D_{i+1}}) + D_{i+1}$$

where $H_{D_{i+1}} = H_{D_i}^2 h_{D_i}$. Write $\ell = \sum a_i 2^i$ with $a_i \in \{0, 1\}$. Let $H_D = \prod_i H_{D_i}^{a_i}$. Then $\ell D = (H_D) + \sum_i a_i D_{i+1}$.

Write $\sum a_i D_{i+1} = D_{i_1} + \ldots + D_{i_m}$ with $i_1 > \ldots > i_m$. Then $P_{i_1} + P_{i_2} = Q_2$, $Q_2 + P_{i_3} = Q_3$, ..., $Q_{i_m-1} + P_{i_m} = O$ with $Q_j \in E$ for $j = 1, \ldots, m-1$. We have

$$\sum a_i D_{i+1} = (G_D) \quad \text{where} \quad G_D = g_{P_1, i_1} g_{Q_2, i_2} g_{Q_3, i_3} \ldots g_{Q_{i_m-1}, i_m}.$$

We have $\ell D = (H_D G_D)$. Let $f_P = H_D G_D$. Then for $P, Q \in E[\ell]$, $e_P(Q) = \frac{f_P(Q)}{f_Q(P)}$.

Suppose $\hat{\alpha} \in \hat{E}[\ell]$. Let $\hat{\alpha}_i = 2^i \hat{\alpha}$. Then for $j = 1, \ldots, n$, $2 \rho_j \hat{\alpha}_i = \rho_j \hat{\alpha}_{i+1}$.

Let $\hat{D} = \hat{\alpha} - O$. Let $h$ be as defined before where $h(P_1, P_2) = h_{P_1 - O}(P_2)$ for $P_1, P_2 \in E$. Inductively define $\hat{H}_{i+1} = \hat{H}_D^2 h$. We can verify inductively

$$\hat{H}_j(\hat{\alpha}, \hat{\beta}) = \prod_{i=1}^n H_{\rho_j(\hat{\alpha})_{i,j}}(\rho_j(\hat{\alpha}), \rho_j(\hat{\beta})).$$

Let $\hat{H} = \prod_i \hat{H}_i^{a_i}$. Then $\hat{H}(\hat{\alpha}, \hat{\beta}) = \prod_{i=1}^n H_{\rho_j(\hat{\alpha}) - O(\hat{\beta})}$, and can be efficiently computed once $h$ is specified.

Write $\sum a_i \hat{\alpha}_{i+1} = \hat{\alpha}_1 + \ldots + \hat{\alpha}_m = O$ with $i_1 > \ldots > i_m$. Let $\hat{\beta}_i$ be such that $\hat{\alpha}_{i_1} + \hat{\alpha}_{i_2} = \hat{\beta}_2$, $\hat{\beta}_2 + \hat{\alpha}_{i_3} = \hat{\beta}_3$, ..., $\hat{\beta}_{i_m-1} + \hat{\alpha}_{i_m} = O$. We have

$$\hat{g}(\hat{\alpha}_{i_1}, \hat{\alpha}_{i_2}, \hat{\beta}_1) = \prod_{i=1}^n g(\rho_j(\hat{\alpha}_{i_1}), \rho_j(\hat{\alpha}_{i_2}), \rho_j(\hat{\beta}_1))$$

$$\hat{g}(\hat{\beta}_2, \hat{\alpha}_{i_3}, \hat{\beta}_2) = \prod_{i=1}^n g(\rho_j(\hat{\beta}_2), \rho_j(\hat{\alpha}_{i_3}), \rho_j(\hat{\beta}_2))$$

$$\ldots$$

$$\hat{g}(\hat{\beta}_{i_m-1}, \hat{\alpha}_{i_m}, \hat{\beta}) = \prod_{i=1}^n g(\rho_j(\hat{\beta}_{i_m-1}), \rho_j(\hat{\alpha}_{i_m}), \rho_j(\hat{\beta}))$$

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\[
\hat{g}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}) \hat{g}(\hat{\beta}_2, \hat{\alpha}_3, \hat{\beta}) \ldots \hat{g}(\hat{\beta}_{m-1}, \hat{\alpha}_m, \hat{\beta}) = \prod_{i=1}^{n} G_{\hat{ρ}_i}(\hat{\alpha} - \hat{O})(\hat{ρ}_i(\hat{β}))
\]

Therefore \( \prod_{i=1}^{n} f_{\hat{ρ}_i}(\hat{α})(\hat{ρ}_i(\hat{β})) \) can be computed efficiently using \( \hat{g} \) and \( \hat{h} \).

Similarly \( \prod_{i=1}^{n} f_{\hat{ρ}_i}(\hat{β})(\hat{ρ}_i(\hat{α})) \) can be computed efficiently using \( \hat{g} \) and \( \hat{h} \). So \( e(\hat{α}, \hat{β}) = \prod_{i=1}^{n} f_{\hat{ρ}_i}(\hat{α})(\hat{ρ}_i(\hat{β})) \prod_{i=1}^{n} f_{\hat{ρ}_i}(\hat{β})(\hat{ρ}_i(\hat{α})) \) can be computed efficiently using \( \hat{g} \) and \( \hat{h} \).

Finally we note that both \( \hat{g} \) and \( \hat{h} \) are products of semi-local functions. They can be specified securely using the procedure described in §1.4.

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