Hypersurfaces of space forms carrying a totally geodesic foliation $^\ast^\dagger$

Marcos Dajczer and Ruy Tojeiro$^\ddagger$

Abstract

In this paper we give a complete local parametric classification of the hypersurfaces with dimension at least three of a space form that carry a totally geodesic foliation of codimension one. A classification under the assumption that the leaves of the foliation are complete was already given in [6] for Euclidean hypersurfaces. We prove that there exists exactly one further class of local examples in Euclidean space, all of which have rank two. We also extend the classification under the global assumption of completeness of the leaves for hypersurfaces of the sphere and show that there exist plenty of examples in hyperbolic space.

In [6] the following basic problem was addressed: Which are the Euclidean hypersurfaces of dimension at least three that carry a totally geodesic foliation of codimension one? Recall that a smooth foliation $\mathcal{F}$ on a Riemannian manifold $M^n$ is totally geodesic if all leaves of $\mathcal{F}$ are totally geodesic submanifolds of $M^n$, that is, if any geodesic of $M^n$ that is tangent to $\mathcal{F}$ at some point is (locally) contained in the leaf of $\mathcal{F}$ through that point.

The main result in [6] states that if $f: M^n \to \mathbb{R}^{n+1}, n \geq 3$, is an isometric immersion of a nowhere flat connected Riemannian manifold that carries a totally geodesic foliation of codimension one with complete leaves, then $f$ is either ruled or a partial tube over a curve, provided that $f(M)$ does not contain any surfacelike strip.

A hypersurface $f: M^n \to \mathbb{R}^{n+1}, n \geq 2$, is ruled if it carries a smooth foliation of codimension one by (open subsets of) affine subspaces of $\mathbb{R}^{n+1}$, whereas it is a partial tube over a smooth regular curve $\gamma: I \subset \mathbb{R} \to \mathbb{R}^{n+1}$ if it is generated by parallel transporting along $\gamma$, with respect to the normal connection, a hypersurface $N^{n-1}$ of the (affine) normal space $N_{\gamma}(t_0)$ to $\gamma$ at some point $t_0 \in I$ (see Section 1.1 for further details). If $n \geq 3$, by a surfacelike strip we mean an open subset $U \subset M^n$ isometric to a product

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$L^2 \times \mathbb{R}^{n-2}$ along which $f$ splits as $f = f_1 \times id$, with $f_1 : L^2 \to \mathbb{R}^3$ an isometric immersion and $id : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ the identity map.

The goal of this paper is to give a full classification of the hypersurfaces that carry a totally geodesic foliation of codimension one without requiring the leaves to be complete. A rather weak necessary condition was provided by Theorem 5 in [6]. Hence, until now the local classification problem remained wide open.

We prove that, apart from ruled hypersurfaces, surfacelike hypersurfaces and partial tubes over curves, all other examples without flat points have rank two and can be completely described in terms of the Gauss parametrization (see Section 1.2) by means of surfaces $g : L^2 \to S^n$ in the unit sphere that we call of type $D$. These are the surfaces that carry a nowhere orthogonal real conjugate system of coordinates $(u, v)$ for which one of the Christoffel symbols vanishes everywhere. More precisely, all of the coordinate functions $g^1, \ldots, g^{n+1}$ of $g$ in $\mathbb{R}^{n+1}$ are solutions of the differential equation of second order
\[ g^{ij}_{uv} + ag^j_{uv} + bg^j = 0 \]
for some $a, b \in C^\infty(L)$, satisfying the constraint $\sum_{j=1}^{n+1} (g^j)^2 = 1$.

Surfaces of type $D$ are closely related to the surfaces of type $C$ that were used in [2] to give a parametrization of all local isometric immersions of $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$. Inspired by a comment in [3] about surfaces of type $C$, we show that any surface of type $D$ in $S^n$ can be generated by a unit principal direction of a surface with flat normal bundle in $\mathbb{R}^{n+1}$ that satisfies some generic regularity conditions. In particular, this also shows that surfaces of type $D$ come in pairs of dual surfaces.

We recall that all Euclidean surfaces with flat normal bundle can be given explicitly in terms of a set of solutions of a completely integrable first order linear system of PDEs associated to the vectorial Ribaucour transformation as shown in [7] and [5]. In particular, since these do not satisfy any constraint, there exists an abundance of surfaces of type $D$.

The other goal of this paper is to study the problem for hypersurfaces of a nonflat space form. In this case, we are also able to give a complete solution of the local version of the problem, as well as of the global version in the spherical case. As for the global version in hyperbolic space, there exist plenty of examples with complete leaves, which include the isometric immersions $f : \mathbb{H}^n \to \mathbb{H}^{n+1}$, the ruled hypersurfaces and those called generalized cones (see Section 4). To characterize for which of the remaining examples all leaves of the foliation are complete is an open problem.

1 Preliminaries

In this section we recall several results on submanifold theory in space forms that will be used in the sequel.
1.1 Partial tubes

We first recall the precise definition of partial tubes over curves, and then state a result from [9] which implies that these are precisely the solutions to our problem for which the totally geodesic distribution is invariant by the shape operator of the hypersurface.

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^{n+1}$ be a smooth unit speed curve defined on an interval and let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal set of parallel normal vector fields along $\gamma$. Consider the parallel vector bundle isometry $\phi : I \times \mathbb{R}^n \to N_\gamma I$ onto the normal bundle of $\gamma$ given by

$$\phi_s(y) = \phi(s, y) = \sum_{i=1}^{n} y_i \xi_i(s)$$

for all $s \in I$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let $f_0 : M_0^{n-1} \to \mathbb{R}^n$ be a hypersurface, denote $M^n = M_0^{n-1} \times I$ and define a map $f : M^n \to \mathbb{R}^{n+1}$ by

$$f(x, s) = \gamma(s) + \phi_s(f_0(x)),$$

One can check that $f$ is an immersion whenever $f_0(M_0) \subset \Omega(\gamma; \phi)$, where

$$\Omega(\gamma; \phi) = \{Y \in \mathbb{R}^n : \langle \gamma''(s), \phi_s(Y) \rangle \neq 1 \text{ for any } s \in I\}.$$

In this case, the hypersurface $f$ is called the partial tube over $\gamma$ with fiber $f_0$. Geometrically, it is generated by parallel transporting $f_0(M_0)$ with respect to the normal connection of $\gamma$.

This construction can be extended as follows to the case in which the ambient space is either the sphere or the hyperbolic space.

Let $\mathbb{R}_{\mu}^{n+2}$ denote either Euclidean space $\mathbb{R}^{n+2}$ or Lorentzian space $\mathbb{I}^{n+2}$, depending on whether $\mu = 0$ or 1, respectively. Denote by $\mathbb{Q}_\epsilon^{n+1} \subset \mathbb{R}_{\mu}^{n+2}$, $\epsilon = 1 - 2\mu$, either the sphere $\mathbb{S}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$. Let $\gamma : I \to \mathbb{R}_{\mu}^{n+2}$ be a smooth unit speed curve such that $\gamma(I)$ is contained in $\mathbb{Q}_\epsilon^{n+1} \subset \mathbb{R}_{\mu}^{n+2}$. Set $\gamma = i \circ \tilde{\gamma}$ where $\tilde{\gamma} : I \to \mathbb{Q}_\epsilon^{n+1}$ and $i : \mathbb{Q}_\epsilon^{n+1} \to \mathbb{R}_{\mu}^{n+2}$ denotes the inclusion. Let $\{\xi_1, \ldots, \xi_n\}$ be a parallel orthonormal frame of the normal bundle $N_\gamma I$ of $\tilde{\gamma}$. Then $\{\xi_1, \ldots, \xi_n, \xi_{n+1} = \epsilon\}$ is a parallel orthonormal frame of the normal bundle $N_\gamma I$ of $\gamma$ and we may define a parallel vector bundle isometry $\phi : I \times \mathbb{R}_{\mu}^{n+1} \to N_\gamma I$ by

$$\phi_s(y) = \phi(s, y) = \sum_{i=1}^{n+1} y_i \xi_i(s).$$

Let $e \in \mathbb{R}_{\mu}^{n+1}$ be such that $\gamma(s) = \phi_s(e)$ for all $s \in I$, and let $f_0 : M_0^{n-1} \to \mathbb{R}_{\mu}^{n+2}$ be an isometric immersion such that

$$f_0(M_0) \subset \mathbb{Q}_\epsilon^{n+1} \cap (e + \Omega(\gamma, \phi)) \subset \mathbb{R}_{\mu}^{n+2}.$$
Then the map \( f: M^n = M_0^{n-1} \times I \to \mathbb{Q}_\epsilon^{n+1} \subset \mathbb{R}^{n+2} \) defined by

\[
f(x, s) = \phi_s(f_0(x))
\]
is called the partial tube over \( \gamma \) with fiber \( f_0 \).

For a partial tube as above, and endowing \( M^n = M_0^{n-1} \times I \) with the induced metric, the distribution on \( M^n \) given by the tangent spaces to the first factor is totally geodesic. Equivalently, the induced metric on \( M^n \) is a twisted product metric

\[
ds^2 = ds_0^2 + \rho^2 dt^2
\]
for some \( \rho \in C^\infty(M) \), where \( ds_0^2 \) and \( dt^2 \) are the metrics on \( M_0^{n-1} \) and \( I \), respectively. Moreover, the tangent vector \( \partial/\partial t \) to the second factor is a principal direction of \( f \) at any point of \( M^n \) (see [9] for details). It follows from more general results given in [9] (see Theorem 16 and Corollary 18) that this property characterizes partial tubes over curves among hypersurfaces of \( \mathbb{Q}_\epsilon^{n+1} \) that carry a smooth totally geodesic distribution of codimension one.

**Proposition 1.** Let \( f: M^n \to \mathbb{Q}_\epsilon^{n+1} \) be an isometric immersion of a twisted product \( M^n = M_0^{n-1} \times_{\rho} I \), where \( I \subset \mathbb{R} \) is an open interval and \( \rho \in C^\infty(M) \). If the tangent vector \( \partial/\partial t \) to the second factor is a principal direction of \( f \) at any point of \( M^n \), then \( f \) is a partial tube over a curve.

### 1.2 The Gauss parametrization

In this section, we recall from [3] or [4] the Gauss parametrization of an oriented hypersurface with constant index of relative nullity in a space form.

Given an oriented hypersurface \( f: M^n \to \mathbb{Q}_\epsilon^{n+1} \) in a simply connected complete space form of sectional curvature \( \epsilon = 0, 1 \) or \( -1 \), we denote by \( \Delta(x) \) the kernel of its shape operator \( A \) with respect to a unit normal vector field \( N \) at any \( x \in M^n \), called the relative nullity subspace. The index of relative nullity at \( x \in M^n \) is \( \nu(x) = \dim \Delta(x) \).

We start with the case of Euclidean hypersurfaces.

**Proposition 2.** Let \( g: L^{n-k} \to S^n \) be an isometric immersion of a Riemannian manifold and let \( \gamma \in C^\infty(L) \). Set \( h = i \circ g \) where \( i: S^n \to \mathbb{R}^{n+1} \) is the inclusion map, and consider the map \( \psi: \Lambda \to \mathbb{R}^{n+1} \) defined on the normal bundle \( \Lambda = N_yL \) of \( g \) by

\[
\psi(y, w) = \gamma(y)h(y) + h_* \text{grad} \gamma(y) + i_*w. \tag{1}
\]

Then, on the open subset of regular points, \( \psi \) is an immersed hypersurface with constant index of relative nullity \( \nu = k \).

Conversely, any hypersurface of \( \mathbb{R}^{n+1} \) with constant index of relative nullity \( \nu = k \) can be parametrized in this way, at least locally. The parametrization is global if the leaves of the relative nullity distribution are complete.
According to the orthogonal decomposition $TM = \Delta \oplus \Delta^\perp$ we decompose $X \in TM$ as $X = X^v + X^h$. Recall that the splitting tensor $C: \Gamma(\Delta) \times \Gamma(\Delta^\perp) \to \Gamma(\Delta^\perp)$ is defined as

$$C(T, X) = - (\nabla_X T)^h$$

for all $T \in \Gamma(\Delta)$ and $X \in \Gamma(\Delta^\perp)$. If $x \in \mathbb{M}^n$ and $T \in \Delta(x)$, then the tensor gives rise to a map $C_T: \Delta^\perp(x) \to \Delta^\perp(x)$. Then we regard $C$ as a map $C: \Gamma(\Delta) \to \Gamma(\text{End}(\Delta^\perp))$.

We now collect several properties of the Gauss parametrization needed in the sequel.

**Proposition 3.** The following assertions hold:

(i) The map $\psi$ is regular at $(y, w) \in \Lambda$ if and only if the selfadjoint operator

$$P_w(y) = \gamma(y)I + \text{Hess} \gamma(y) - A_w$$

on $T_y \mathbb{L}$ is nonsingular, where $A_w$ is the shape operator of $g$ with respect to $w$.

(ii) On the open subset $V$ of regular points, $\psi$ is an immersed hypersurface having the map $G: \Lambda \to \mathbb{S}^n$, given by

$$G(y, w) = g(y),$$

as a Gauss map of rank $n - k$.

(iii) For any $(y, w) \in V$ there exists a map $j = j(y, w): T_y \mathbb{L} \to T_{(y, w)} \Lambda$, which is an isometry onto the orthogonal complement $\Delta^\perp(y, w)$ of the relative nullity subspace $\Delta(y, w) = N_g \mathbb{L}(y)$ of $\psi$ at $(y, w)$, such that

$$\nabla^V_\xi j X = 0$$

for all $\xi \in \Delta(y, w)$ and $X \in \mathfrak{X}(\mathbb{L})$, where $\nabla^V$ denotes the Levi-Civita connection of the metric on $V$ induced by $\psi$, and such that the shape operator $A$ of $\psi$ at $(y, w)$ with respect to $G$ satisfies

$$A_j = -jP_w^{-1}.$$  

(iv) For any $(y, w) \in V$ the splitting tensor $C_\xi: \Delta^\perp(y, w) \to \Delta^\perp(y, w)$ of $\Delta$ with respect to $\xi \in \Delta(y, w) = N_g \mathbb{L}(y)$ is related to the shape operator $A_\xi$ of $g$ at $y$ by

$$C_\xi j = jA_\xi P_w^{-1}. $$

(v) The Levi-Civita connections $\nabla^L$ and $\nabla^V$ of the metrics $\langle \cdot, \cdot \rangle_L$ and $\langle \cdot, \cdot \rangle_V$ on $L^{n-k}$ and $V$ induced by $g$ and $\psi$, respectively, are related by

$$\langle \nabla^L_{P_w^{-1}X}Y, Z \rangle_L = \langle \nabla^V_{jX}jY, jZ \rangle_V$$

for all $X, Y, Z \in \mathfrak{X}(\mathbb{L})$.  

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The Gauss parametrization of oriented hypersurfaces \( f: M^n \to \mathbb{Q}^{n+1}_\epsilon \) with \( \nu = k \) reads as follows for \( \epsilon \in \{-1, 1\} \).

**Proposition 4.** Let \( g: L^{n-k} \to S^{n+1}_\mu \) be an isometric immersion of a Riemannian manifold, where

\[
S^n_\mu = \{ x \in \mathbb{R}^{n+1}_\mu : \langle x, x \rangle = 1 \}.
\]

Denote

\[
\Lambda_\epsilon = \{(y, w) \in \Lambda = N_g L : \langle w, w \rangle = \epsilon \}, \quad \epsilon = 1 - 2\mu,
\]

and define \( \psi: \Lambda_\epsilon \to \mathbb{Q}^{n+1}_\epsilon \) by

\[
\psi(y, w) = w.
\]

Then the following assertions hold:

(i) On the open subset \( V \) of regular points, \( \psi \) is an immersed hypersurface with constant index of relative nullity \( \nu = k \), having the map \( G: \Lambda_\epsilon \to S^{n+1}_\mu \), given by

\[
G(y, w) = g(y),
\]

as a Gauss map of rank \( n - k \).

(ii) Conversely, any hypersurface of \( \mathbb{Q}^{n+1}_\epsilon \) with constant index of relative nullity \( \nu = k \) can be parametrized in this way, at least locally. The parametrization is global if the leaves of the relative nullity distribution are complete.

(iii) The map \( \psi \) is regular at \( (y, w) \in \Lambda_\epsilon \) if and only if the shape operator \( A_w \) of \( g \) at \( y \in L^{n-k} \) is nonsingular.

(iv) At any \( (y, w) \in V \), there exists an isometry \( j = j(y, w): T_y L \to \Delta^\perp(y, w) \) onto the orthogonal complement of the relative nullity subspace

\[
\Delta(y, w) = \{ w' \in N_g L(y) : \langle w', w \rangle = 0 \}
\]

of \( \psi \) at \( (y, w) \), such that (3) holds for all \( \xi \in \Delta(y, w) \) and such that the shape operator \( A \) of \( \psi \) at \( (y, w) \) with respect to \( G \) satisfies

\[
Aj = jA_w^{-1}.
\]

(v) For any \( (y, w) \in V \) the splitting tensor \( C_\xi: \Delta^\perp(y, w) \to \Delta^\perp(y, w) \) of \( \Delta \) with respect to \( \xi \in \Delta(y, w) \) is related to the shape operator \( A_\xi \) of \( g \) at \( y \) by

\[
C_\xi j = jA_\xi A_w^{-1}.
\]

(vi) The Levi-Civita connections \( \nabla^L \) and \( \nabla^V \) of the metrics \( \langle \cdot, \cdot \rangle_L \) and \( \langle \cdot, \cdot \rangle_V \) on \( L^{n-k} \) and \( V \) induced by \( g \) and \( \psi \), respectively, are related by

\[
\langle \nabla^L_{A_{w}^{-1}} X, Y, Z \rangle_L = \langle \nabla^V_{jX} jY, jZ \rangle_V
\]

for all \( X, Y, Z \in \mathfrak{X}(L) \).
2 Surfaces of type $D$

This section is devoted to introduce a class of surfaces in the unit sphere $S^n$ that we call surfaces of type $D$. We will see in the next section that these surfaces are precisely the Gauss images of Euclidean hypersurfaces of rank two that carry a totally geodesic foliation with codimension one.

We say that a surface $g: L^2 \rightarrow S^n$, $n \geq 3$, is a surface of type $D$ if there exist linearly independent vector fields $X, Y \in \mathfrak{X}(L)$ with $\|Y\| = 1$ such that

\[
\begin{align*}
(i) & \quad \langle X, Y \rangle \neq 0 \\
(ii) & \quad \nabla^g_X Y = 0 \\
(iii) & \quad \alpha_g(X, Y) = 0
\end{align*}
\]

where $\nabla^g$ denotes the Levi-Civita connection of the metric on $L^2$ induced by $g$. Notice that conditions $(ii)$ and $(iii)$ together are equivalent to $\nabla_X g_* Y = 0$, where $\nabla$ is the connection of $S^n$, that is, they mean that $Y$ is parallel along $X$ in $S^n$.

If $(u, v)$ are coordinates on $L^2$ such that $X$ and $Y$ are collinear with $\partial u$ and $\partial v$, respectively, then the preceding conditions can be written as

\[
\begin{align*}
(i') & \quad \langle \partial u, \partial v \rangle \neq 0 \\
(ii') & \quad \nabla^g_{\partial u} \partial v \in \text{span}\{\partial v\} \\
(iii') & \quad \alpha_g(\partial u, \partial v) = 0.
\end{align*}
\]

The last two conditions are equivalent to the position vector of $g$ in $\mathbb{R}^{n+1}$ satisfying the differential equation

\[g_{uv} + ag_v + bg = 0\]  

for smooth functions $a = a(u, v)$ and $b = b(u, v) = \langle g_u, g_v \rangle$, where $g_u$ and $g_v$ denote the partial derivatives of $g$.

Surfaces of type $D$ have in common conditions $(ii)$ and $(iii)$ with the surfaces of type $C$ defined in [2] (see also [1]), for which, instead of condition $(i)$, one requires that the vector field $Y$ be nowhere an asymptotic direction for $g$. Surfaces of type $C$ were used in [2] to give a parametrization of all local isometric immersions of $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$.

The following result shows that the existence of linearly independent vector fields $X, Y \in \mathfrak{X}(L)$ with $\|Y\| = 1$ satisfying conditions $(i)$ to $(iii)$ imposes no restrictions on a surface $g: L^2 \rightarrow S^3$ with Gauss curvature $K \neq 1$ everywhere.

**Proposition 5.** Any surface $g: L^2 \rightarrow S^3$ with Gauss curvature $K \neq 1$ everywhere is a surface of type $D$. 


Proof: Since $K \neq 1$ everywhere, a unit normal vector field to $g$ defines an immersion $\hat{g}: L^2 \to S^3$, called the polar surface of $g$. The metrics $\langle , \rangle_g$ and $\langle , \rangle_{\hat{g}}$ on $L^2$ induced by $g$ and $\hat{g}$, respectively, are related by
\[
\langle X, Y \rangle_{\hat{g}} = \langle AX, AY \rangle_g
\]
for all $X, Y \in \mathfrak{X}(L)$, where $A$ is the shape operator of $g$ with respect to $\hat{g}$, and the corresponding Levi-Civita connections $\nabla^g$ and $\nabla^{\hat{g}}$ are related by
\[
A \nabla^g_X Y = \nabla^{\hat{g}}_X AY
\]
for all $X, Y \in \mathfrak{X}(L)$.

Now let $X, \hat{Y}$ be an orthonormal frame of $L^2$ with respect to the metric induced by $\hat{g}$ such that the integral curves of $X$ are geodesics with respect to that metric and such that $X$ is nowhere a principal direction of $\hat{g}$. Thus
\[
\nabla^g_X \hat{Y} = 0 \quad \text{and} \quad \langle \hat{A} X, \hat{Y} \rangle_{\hat{g}} \neq 0
\]
everywhere, where $\hat{A} = A^{-1}$ is the shape operator of $\hat{g}$ with respect to $g$. If $Y = A \hat{Y}$, then
\[
\langle Y, Y \rangle_g = \langle A^{-1} Y, A^{-1} Y \rangle_{\hat{g}} = \langle \hat{Y}, \hat{Y} \rangle_{\hat{g}} = 1,
\]
\[
\nabla^g_X Y = A \nabla^{\hat{g}}_X A^{-1} Y = A \nabla^{\hat{g}}_X \hat{Y} = 0,
\]
\[
\langle AX, Y \rangle_g = \langle X, A^{-1} Y \rangle_{\hat{g}} = \langle X, \hat{Y} \rangle_{\hat{g}} = 0
\]
and
\[
\langle X, Y \rangle_g = \langle A^{-1} X, A^{-1} Y \rangle_{\hat{g}} = \langle A^{-1} X, \hat{Y} \rangle_{\hat{g}} = \langle \hat{A} X, \hat{Y} \rangle_{\hat{g}} \neq 0
\]
everywhere. Thus $X, Y$ satisfy conditions (i) to (iii) for $g$. \hfill \blacksquare

The next result shows that, generically, surfaces of type $D$ have a dual surface of the same type.

**Proposition 6.** Let $g: L^2 \to S^n$ be a surface of type $D$ with Gauss curvature $K \neq 1$ everywhere, and let $X_g, Y_g$ be vector fields satisfying conditions (11) for $g$. Then the map $k: L^2 \to S^n$ given by $k = g_*Y_g$ also defines a surface of type $D$.

**Proof:** We have
\[
k_*X_g = -\langle X_g, Y_g \rangle_g g
\]
and
\[
k_*Y_g = g_* \nabla^{\hat{g}}_X Y_g + \alpha_g(Y_g, Y_g) - g
\]
by the Gauss formula of $g$. Here $\langle , \rangle_g$ stands for the metric induced by $g$ and $\nabla^{\hat{g}}$ by its induced connection. Since $K \neq 1$ everywhere and $\alpha_g(X_g, Y_g) = 0$, then $\alpha_g(Y_g, Y_g) \neq 0$ everywhere by the Gauss equation of $g$. Thus $k$ is an immersion.
Denote $X_k = Y_g$ and $Y_k = X_g/\langle X_g, Y_g \rangle$. Then \((13)\) can be written as

$$k_s Y_k = -g.$$  \hspace{1cm} (14)

In particular, $Y_k$ has unit length with respect to the metric $\langle \cdot, \cdot \rangle_k$ induced by $k$, and

$$\langle X_k, Y_k \rangle_k = \langle k_s Y_g, -g \rangle = 1.$$

Moreover,

$$\tilde{\nabla}_{X_k} k_s Y_k = -\tilde{\nabla}_{X_k} g = -g_s Y_g = -k$$

where $\tilde{\nabla}$ stands for the derivative in $\mathbb{R}^{n+1}$. Thus $\nabla_{X_k} k_s Y_k = 0$, and hence $X_k, Y_k$ satisfy conditions \((11)\) for $k$. \hspace{1cm} \blacksquare

Notice that, by \((14)\), the dual surface of $k$ is $-g$. Notice also that the position vector fields in Euclidean space of the dual surfaces are orthogonal.

The next result, inspired by a comment in \cite{8} about surfaces of type $C$, shows that any surface of type $D$ is spanned by a unit principal direction of a surface with flat normal bundle in $\mathbb{R}^{n+1}$ that satisfies some generic regularity conditions.

**Theorem 7.** Let $h: L^2 \to \mathbb{R}^{n+1}$, $n \geq 3$, be a surface with flat normal bundle and let $X, Y \in \mathfrak{X}(L)$ be an orthonormal frame that diagonalizes the second fundamental form of $h$. Assume that the integral curves of $X$ and $Y$ are nowhere geodesics and that the integral curves of $X$ are nowhere asymptotic. Then the map $g: L^2 \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ given by

$$g = h_s X$$

defines a surface of type $D$. Conversely, any surface of type $D$ can be locally constructed in this way.

**Proof:** Taking derivatives in $\mathbb{R}^{n+1}$ yields

$$g_s X = \tilde{\nabla}_Y h_s X = -\langle \nabla^h Y, X \rangle_h h_s Y,$$

since $\alpha_h(X, Y) = 0$ by assumption. Here $\langle \cdot, \cdot \rangle_h$ denotes the metric induced by $h$ and $\nabla^h$ stands for its Levi-Civita connection. Moreover, since $\nabla^h Y \neq 0$ at any point of $L^2$ by assumption, then also $g_s Y \neq 0$ at any point of $L^2$. We have

$$g_s X = \tilde{\nabla}_X h_s X = \langle \nabla^h X, Y \rangle_h h_s Y + \alpha_h(X, X).$$

Thus the map $g$ is an immersion since, by assumption, $\alpha_h(X, X) \neq 0$ at any point of $L^2$. Moreover, the vector field

$$\dot{Y} = \frac{1}{\langle \nabla^h Y, X \rangle_h} Y$$
has unit length with respect to the metric induced by \( g \), and

\[
\tilde{\nabla}_X g_* \hat{Y} = -\tilde{\nabla}_X h_* Y = \langle \nabla^h X, Y \rangle_h h_* X = \langle \nabla^h X, Y \rangle_h g.
\]

Thus conditions \((ii)\) and \((iii)\) are satisfied for the vector fields \( \hat{X} = X \) and \( \hat{Y} \). Moreover, since also \( \nabla^h_X X \neq 0 \) at any point of \( L^2 \), then

\[
\langle g_* \hat{X}, g_* \hat{Y} \rangle \neq 0,
\]

and hence \( g \) is of type \( D \).

Conversely, let \( g: L^2 \to S^n \subset \mathbb{R}^{n+1} \), \( n \geq 3 \), be a surface of type \( D \) and let \((u,v)\) be coordinates in a neighborhood \( U \) of a point of \( L^2 \) such that conditions \((i')\) to \((iii')\) are satisfied. Thus, the position vector \( g \) satisfies \((12)\) on \( U \). Shrinking \( U \), if necessary, one can find, using for instance the Riemann method, a nowhere vanishing solution \( \varphi \in C^\infty(U) \) of the hyperbolic PDE

\[
\varphi_{uv} - (a + b_u/b) \varphi_v + b \varphi = 0
\]

such that \( \varphi_v \) is also nowhere vanishing on \( U \). By assumption, \( b = \langle g_u, g_v \rangle \neq 0 \). Observe that \((15)\) is an equation of the same type as \((12)\). Now consider the system of PDEs

\[
\begin{cases}
    h_u = \varphi g \\
    h_v = -b^{-1} \varphi_v g_v
\end{cases}
\]

for \( h: U \to \mathbb{R}^{n+1} \). It is easily checked that the integrability conditions of \((16)\) are satisfied by virtue of \((12)\) and \((15)\). Since \( \langle h_u, h_v \rangle = 0 \) and

\[
h_{uv} = \frac{\varphi_v}{\varphi} h_u - \frac{b \varphi_v}{\varphi} h_u \in h_* TU = \text{span}\{g, g_v\},
\]

then \( h \) is a surface with flat normal bundle and \( g = h_* X \), where \( X \) is the unit vector field in the direction of \( \partial u \) with respect to the metric induced by \( h \).

It remains to check that the coordinate curves are nowhere geodesics with respect to the metric on \( U \) induced by \( h \), and that the \( u \)-coordinate curves are nowhere asymptotic for \( h \). In other words, we need to verify that at any point of \( U \) the following conditions hold.

\[
\begin{cases}
    (i'') \quad \nabla^h_{\partial u} \partial u \notin \text{span}\{\partial u\} \\
    (ii'') \quad \nabla^h_{\partial v} \partial v \notin \text{span}\{\partial v\} \\
    (iii'') \quad \alpha_h(\partial u, \partial u) \neq 0.
\end{cases}
\]

For \((i'')\): we have \( \nabla^h_{\partial u} \partial u \in \text{span}\{\partial u\} \) if and only if the component of

\[
h_{uu} = \varphi_u g + \varphi g_u
\]
For (ii): if $\nabla^h_{\partial u} \partial v \in \text{span}\{\partial v\}$, then the $h^* TU$ component of

$$h_{vv} = -(b^{-1} \varphi_v) g_v - b^{-1} \varphi_v g_{vv}$$

is collinear with $h_v = -b^{-1} \varphi_v g_v$. Hence, since $\varphi_v \neq 0$ then $0 = \langle g_{vv}, g \rangle = -\langle g_v, g_v \rangle$, and this is a contradiction.

For (iii): if $\alpha_h(\partial u, \partial u) = 0$ then $h_{uu} = \varphi_u g + \varphi g_u \in \text{span}\{g, g_v\}$, and this is again a contradiction.

Remarks 8. (1) The definition of surfaces of type $D$ can be extended for surfaces in the Lorentzian sphere $S^{n+2}_1$ of constant sectional curvature 1 in the Lorentz space $\mathbb{L}^{n+3}$. We will see in the last section that these are the Gauss images of hypersurfaces with rank two of $\mathbb{H}^{n+1}$ that carry a totally geodesic foliation of codimension one.

(2) In view of Proposition 5, it is natural to exclude from the definition of surfaces $f: L^2 \to S^n$ of type $D$ those with Gauss curvature $K \neq 1$ everywhere that are contained in an umbilical sphere $S^3(c) \subset S^n$ of constant curvature $c \geq 1$. Since it is easily checked that Proposition 5 still holds for surfaces in the Lorentzian sphere $S_1^3$ with Gauss curvature $K \neq 1$ everywhere, we also exclude from the definition of surfaces $f: L^2 \to S_1^{n+2}$ of type $D$ those with Gauss curvature $K \neq 1$ that are contained in an umbilical Lorentzian sphere $S_1^3(c) \subset S_1^{n+2}$ of constant curvature $c \geq 1$.

(3) In the notations of Theorem 7 we have

$$\nabla_Y g_* \tilde{Y} = \nabla_Y h_* Y = \langle \nabla^h_Y Y, X \rangle_h g + \alpha_h(Y, Y).$$

Thus $\tilde{Y}$ is an asymptotic direction of $g$ if and only the vectors $\alpha_h(Y, Y)$ and $\alpha_h(X, X)$ are linearly dependent. Therefore $g$ is also a surface of type $C$ if the first normal spaces of $h$, that is, the subspaces spanned by its second fundamental form, have dimension two at any point.

(4) Notice that the surface given by Proposition 6 is just the other unit principal direction of $h$ in Theorem 7.

3 The case of Euclidean hypersurfaces

In this section we give the local classification of hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ that carry a totally geodesic distribution of codimension one.

First we observe that, besides the ruled hypersurfaces referred to in the introduction and the partial tubes described in Section 1.1, there exist three trivial families of examples.
(1) The first family is that of isometric immersions \( f: U \to \mathbb{R}^{n+1} \) of open subsets \( U \subset \mathbb{R}^n \), since one may consider any foliation of \( U \) by (open subsets of) affine hyperplanes.

(2) The second family consists of cylindrical surfacelike hypersurfaces. For a surface \( g: L^2 \to \mathbb{R}^3 \), let \( D_0 \) be the one-dimensional distribution on \( L^2 \) spanned by the tangent directions to a foliation of \( L^2 \) by geodesics. Set \( M^n = L^2 \times \mathbb{R}^{n-2} \) and define an isometric immersion \( f: M^n \to \mathbb{R}^{n+1} \) by \( f = g \times id \), where \( id: \mathbb{R}^{n-2} \to \mathbb{R}^{n-2} \) is the identity map. Then \( D = D_0 \oplus \mathbb{R}^{n-2} \) is clearly a totally geodesic distribution on \( M^n \) of codimension one. We call \( f \) a \textit{cylindrical surfacelike} hypersurface.

(3) Similarly, for a surface \( g: L^2 \to S^3 \subset \mathbb{R}^4 \) in the sphere, let \( D_0 \) be as before. Define \( M^n = L^2 \times \mathbb{R}_+ \times \mathbb{R}^{n-3} \) and \( f = C(g) \times id \), where \( C(g): L^2 \times \mathbb{R}_+ \to \mathbb{R}^4 \), given by \( C(g)(x,t) = tg(x) \), is the cone over \( g \) in \( \mathbb{R}^4 \). Then the distribution \( D = D_0 \oplus \mathbb{R} \oplus \mathbb{R}^{n-3} \) on \( M^n \) is again totally geodesic of codimension one. We call \( f \) a \textit{conical surfacelike} hypersurface.

Given a surface \( g: L^2 \to S^n \), \( n \geq 3 \), and \( \gamma \in C^\infty(L) \), we say that \((g, \gamma)\) is a \textit{pair of type} \( D \) if \( g \) is a surface of type \( D \) and \( \gamma \) satisfies the same differential equation (12) as the position vector of \( g \). It follows from Proposition 5 that if \( g: L^2 \to S^n \) is a surface with Gauss curvature \( K \neq 1 \) everywhere contained in an umbilical \( S^n(c) \subset S^n \), \( c \geq 1 \), and \( \gamma = 0 \), then \((g, \gamma)\) is always a pair of type \( D \). We show next that this is also the case if \( g: L^2 \to S^n \) is totally geodesic and \( \gamma \in C^\infty(L) \) is any smooth function such that the endomorphism \( P = P_\gamma = \gamma I + \text{Hess} \gamma \) is invertible everywhere.

\textbf{Proposition 9.} If \( g: L^2 \to S^n \) is totally geodesic and \( \gamma \in C^\infty(L) \) is such that \( P = P_\gamma = \gamma I + \text{Hess} \gamma \) is invertible everywhere, then \((g, \gamma)\) is a pair of type \( D \).

\textbf{Proof:} Regard \( g \) as a map \( g: L^2 \to S^2 \) and define \( f: L^2 \to \mathbb{R}^3 \) by

\[ f(y) = \gamma(y)h(y) + h_\ast \text{grad} \gamma. \]

Then \( f_\ast = h_\ast P \), the metrics \( \langle , \rangle_h \) and \( \langle , \rangle_f \) on \( L^2 \) induced by \( h \) and \( f \), respectively, are related by

\[ \langle X, Y \rangle_f = \langle PX, PY \rangle_h \]

for all \( X, Y \in \mathfrak{X}(L) \), the corresponding Levi-Civita connections \( \nabla^h \) and \( \nabla^f \) are related by

\[ P\nabla^f_X Y = \nabla^h_X PY \]

for all \( X, Y \in \mathfrak{X}(L) \), and the shape operator of \( f \) with respect to \( h \) is

\[ A = P^{-1}. \]

Let \( X, \dot{Y} \) be an orthonormal frame of \( L^2 \) with respect to the metric induced by \( f \) such that the integral curves of \( X \) are geodesics with respect to that metric and such that \( X \) is nowhere a principal direction of \( f \). Thus

\[ \nabla_X^f \dot{Y} = 0 \quad \text{and} \quad \langle AX, \dot{Y} \rangle_f \neq 0 \]

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everywhere. Define \( Y = P\tilde{Y} \). Then
\[
\langle Y, Y \rangle_h = \langle P^{-1}Y, P^{-1}Y \rangle_f = \langle \tilde{Y}, \tilde{Y} \rangle_f = 1,
\]
\[
\nabla^h Y = P \nabla_X^f P^{-1}Y = P \nabla_X^f \tilde{Y} = 0,
\]
\[
\langle PX, Y \rangle_h = \langle X, P^{-1}Y \rangle_f = \langle X, \tilde{Y} \rangle_f = 0
\]
and
\[
\langle X, Y \rangle_h = \langle P^{-1}X, P^{-1}Y \rangle_f = \langle P^{-1}X, \tilde{Y} \rangle_f = \langle AX, \tilde{Y} \rangle_f \neq 0
\]
everywhere. Thus \( X, Y \) satisfy conditions (i) to (iii) for \( g \), and (17) implies that \( \gamma \) satisfies the same differential equation (12) as the position vector of \( g \).  

It is therefore natural to exclude from the definition of pairs \((g, \gamma)\) of type \( D \) the cases in which either \( g: L^2 \to S^n \) is a surface with Gauss curvature \( K \neq 1 \) everywhere contained in an umbilical \( S^3(c) \subset S^n \), \( c \geq 1 \), and \( \gamma = 0 \), or \( g: L^2 \to S^n \) is totally geodesic and \( \gamma \in C^\infty(L) \) is any smooth function such that the endomorphism \( P = P_\gamma = \gamma I + \text{Hess} \gamma \) is invertible everywhere. Notice that these are precisely the pairs that give rise, by means of the Gauss parametrization, to conical and cylindrical surfacelike hypersurfaces, respectively.

We are now in a position to state the local classification of Euclidean hypersurfaces of dimension \( n \geq 3 \) without flat points that carry a totally geodesic distribution of codimension one.

**Theorem 10.** Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3 \), be an isometric immersion of a Riemannian manifold without flat points that carries a totally geodesic foliation of codimension one. Then there exists an open and dense subset of \( M^n \) where \( f \) is locally either (cylindrical or conical) surfacelike, ruled, a partial tube over a curve, or given by the Gauss parametrization by a pair \((g, \gamma)\) of type \( D \). Conversely, any of these hypersurfaces carries a totally geodesic foliation of codimension one.

The proof of Theorem 10 relies on the following lemma.

**Lemma 11.** Let \( g: L^2 \to S^n, n \geq 3 \), be a surface, let \( \gamma \in C^\infty(L) \) and let \( \psi: \Lambda \to \mathbb{R}^{n+1} \) be the map defined on the normal bundle \( \Lambda = N_gL \) of \( g \) by
\[
\psi(y, w) = \gamma(y)h(y) + h_* \text{grad} \gamma(y) + i_* w,
\]
where \( h = i \circ g \) is the composition of \( g \) with the inclusion \( i: S^n \to \mathbb{R}^{n+1} \). Assume that there exist \( Y, Z \in \mathfrak{X}(L) \) with \( \|Y\| = 1 \) satisfying the following conditions:
\[
(i) \ \nabla^L_2 Y = 0,
\]
\[
(ii) \ \alpha_g(Y, Z) = 0,
\]

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(iii) \( \langle (\gamma I + \text{Hess} \, \gamma) Y, Z \rangle_L = 0 \).

Then the orthogonal complement of \( \text{span}\{jY\} \) is a totally geodesic distribution on the open subset \( V \) of regular points of \( \psi \) endowed with the induced metric, where \( j = j(y,w) : T_yL \to \Delta^\perp(y,w) \) is the isometry given by part (iii) of Proposition 3 between \( T_yL \) and the orthogonal complement of \( \Delta(y,w) = N_gL(y) \) in \( T_{(y,w)}\Lambda \). Moreover, \( \psi \) is ruled if \( Y \) and \( Z \) are linearly dependent and a partial tube over a curve with fiber a flat hypersurface if \( Y \) and \( Z \) are orthogonal.

Conversely, if \( V \) carries a totally geodesic distribution of codimension one then there exist \( Y, Z \in \mathfrak{X}(L) \) with \( \|Y\| = 1 \) satisfying conditions (i) to (iii).

Proof: Given \((y,w) \in V\), let \( P_w \) be the invertible endomorphism of \( T_yL \) given by (2).

Since conditions (ii) and (iii) are satisfied, then
\[
\langle P_wZ, Y \rangle_L = 0 \tag{18}
\]
for all \( w \in \Gamma(N_gL) \). Thus \( P_wZ \) is collinear with \( X \) for all \( w \in \Gamma(N_gL) \), where \( X \in \mathfrak{X}(L) \) is a unit vector field orthogonal to \( Y \). Hence, for any \((y,w) \in V\) the vector \( P_w^{-1}X(y) \) is collinear with \( Z(y) \). By (ii) we have

\[
\langle A_wZ, Y \rangle_L = \langle \alpha_g(Y, Z), w' \rangle = 0
\]
for all \( w' \in \Gamma(N_gL) \). Using (5) and the fact that \( j \) is an isometry we obtain
\[
\langle \nabla^V_{jX}jY, w' \rangle_V = \langle C_{w'}jX, jY \rangle_V = \langle A_{w'}P_w^{-1}X, Y \rangle_L = 0 \tag{19}
\]
at any \((y,w) \in V\) and for all \( w' \in N_gL(y) \). On the other hand, using (4) we obtain
\[
\langle \nabla^V_{jX}jY, jX \rangle_V = \langle \nabla^L_{P_w^{-1}Y}X, X \rangle_L = 0 \tag{20}
\]
where in the last equality we have used condition (i) and the fact that \( P_w^{-1}X(y) \) is collinear with \( Z(y) \). We conclude from (3), (19) and (20) that the distribution orthogonal to \( jY \) is totally geodesic.

We prove next the last assertion of the direct statement. Suppose first that \( Y \) and \( Z \) are linearly dependent. Without loss of generality, we may assume that \( Z = Y \). As before, let \( X \in \mathfrak{X}(L) \) be a unit vector field orthogonal to \( Y \). By (4), at any \((y,w) \in V\) we have
\[
\langle AjX, jX \rangle_V = -\langle jP_w^{-1}X, jX \rangle_V = -\langle P_w^{-1}X, X \rangle_L.
\]
On the other hand, since (18) holds, then
\[
\langle P_wY, Y \rangle_L = 0,
\]
and hence
\[
P_wY = \langle P_wY, X \rangle_L X.
\]
Therefore

\[ Y = \langle P_w Y, X \rangle_L P_w^{-1} X, \]

which implies using (4) that

\[ 0 = \langle P_w^{-1} X, X \rangle_L = -\langle A j X, j X \rangle_V, \]

and we conclude that \( \psi \) is ruled.

Assume now that \( Z \) and \( Y \) are everywhere orthogonal. Arguing in a similar way, we see that \( A j Z \) is collinear with \( j Z \), which implies from Proposition [1] that \( \psi \) is a partial tube over a curve \( \gamma \), with fiber a hypersurface \( N^{n-1} \) of the affine normal space to \( \gamma \) at some point. Since \( \psi \) has rank two, \( N^{n-1} \) must be flat.

Next we prove the converse. If \( \tilde{Y} \in \Gamma(\Delta^+) \) is a unit vector field such that the distribution \( \{\tilde{Y}\}^\perp \) is totally geodesic, then

\[ \nabla_X^L \tilde{Y} = 0 \]

for any \( \tilde{X} \in \mathfrak{X}(V) \) such that \( \langle \tilde{X}, \tilde{Y} \rangle_V = 0 \). In particular,

\[ \nabla_T^L \tilde{Y} = 0 \]

for all \( T \in \Gamma(\Delta) \). Hence there exists \( Y \in \mathfrak{X}(L) \) such that \( \tilde{Y} = j Y \). From

\[ \nabla_{jX}^V j Y = 0 \]

for \( X \in \mathfrak{X}(L) \) such that \( \langle X, Y \rangle_L = 0 \), using (5) and (6) we obtain, respectively,

\[ 0 = \langle \nabla_{jX}^V j Y, w' \rangle_V = \langle C_w j X, j Y \rangle_V = \langle A_w P_w^{-1} X, Y \rangle_L \tag{21} \]

and

\[ \nabla_{P_w^{-1} X}^L Y = 0 \tag{22} \]

for any \( (y, w) \in V \) and for all \( w' \in N_g L(y) \).

Suppose that at \( y \in L^2 \) we have

\[ T_y L = \text{span}\{P_w^{-1} X(y) : w \in N_g L(y) \text{ with } (y, w) \in V\}. \]

Then this also holds in a neighborhood \( W \) of \( y \), and then (22) implies that \( Y \) is parallel on \( W \). We conclude that \( W \) is flat. On the other hand, by (21) we have

\[ \langle P_w^{-1} X(y), A_w Y \rangle_L = 0 \]

for any \( w \in N_g L(y) \) with \( (y, w) \in V \), and for all \( w' \in N_g L(y) \). It follows that \( A_w Y = 0 \) for all \( w' \in \Gamma(N_g W) \). But then \( Y \) belongs to the relative nullity space of \( g \) at any point of \( W \), and hence the Gauss curvature of \( L^2 \) is equal to one at any point of \( W \), a contradiction.
It follows that there exists $Z \in \mathfrak{X}(L)$ such that, for all $y \in L^2$ and for all $w \in N_g L(y)$ such that $P_w$ is invertible, the vector $P_w^{-1} X(y)$ is collinear with $Z(y)$. This implies that

$$\langle P_w Z, Y \rangle_L = 0.$$  

On the other hand, equation (21) gives

$$\langle A_w Z, Y \rangle_L = 0$$  

for all $y \in L^2$ and $w \in N_g L(y)$, hence

$$\langle (\gamma I + \text{Hess } \gamma) Y, Z \rangle_L = 0.$$  

Finally, from (22) it follows that $\nabla^L_Z Y = 0$.  

Recall that a smooth distribution $D$ on a Riemannian manifold $M^n$ is said to be curvature invariant if

$$R(X,Y)Z \in D$$  

for all $X,Y,Z \in D$ where $R$ denotes the curvature tensor of $M^n$. The next result was obtained in [6].

**Proposition 12.** Let $f: M^n \to \mathbb{Q}^{n+1}_e$ be an oriented hypersurface carrying a curvature invariant distribution $D$ of rank $k$. Then one of the following possibilities holds pointwise:

(i) $A(D) \subset D^\perp$,

(ii) $A(D) \subset D$,

(iii) $\text{rank } D \cap \Delta = k - 1$.

**Proof of Theorem 10:** By Proposition 12, the shape operator $A$ of $f$ satisfies pointwise either of conditions (i) to (iii) in that result. In any oriented open subset where $A$ satisfies condition (ii), $f$ is locally a partial tube over a curve by Proposition 1. On the other hand, $f$ is ruled on an oriented open subset where condition (i) is satisfied. Finally, on an oriented open subset $U$ where $A$ satisfies condition (iii) and neither of conditions (i) or (ii), $f$ has rank two since $M^n$ is nowhere flat. By Lemma 11, $f$ can be parametrized in terms of the Gauss parametrization by a pair $(g, \gamma)$, with $g: L^2 \to S^n$, $n \geq 3$, and $\gamma \in C^\infty(L)$, for which there exist $Y,Z \in \mathfrak{X}(L)$ with $\|Y\| = 1$ satisfying conditions (i) to (iii) in that lemma.

Since $f$ is nowhere ruled or a partial tube over a curve on $U$, the vector fields $Y,Z$ are linearly independent and nowhere orthogonal on $U$. Therefore one of the following possibilities holds locally on $U$: $g$ is totally geodesic and $\gamma \in C^\infty(L)$ is any smooth function such that $P = \gamma I + \text{Hess } \gamma$ is invertible, and hence $f$ is cylindrical surfacelike,
or \(g(L^2)\) is contained in an umbilical \(S^3 \subset S^n\) and \(\gamma = 0\), in which case \(f\) is conical surfacelike, or the pair \(\{Y, Z\}\) satisfies conditions (11) for \(g\) and \(\gamma\) satisfies the same differential equation (12) as the position vector of \(g\), by virtue of condition (iii), and hence \((g, \gamma)\) is a pair of type \(D\). The converse statement follows from the direct statement of Lemma 11.

**Remark 13.** Theorem 10 can be used to derive the result in [6] referred to in the introduction for foliations with complete leaves, once we first prove, as in [6], that if \(U \subset M^n\) is an oriented open subset at each point of which neither of conditions \((i)\) or \((ii)\) in Proposition 12 is satisfied, then any unit speed geodesic starting at a point of \(U\) and tangent to \(\Delta\) at that point remains in \(U\) for any value of the parameter.

In fact, after proving this, since \(f\) has rank two on \(U\) and the leaves of \(\Delta\) are complete therein, one can globally parametrize \(f\) on \(U\) by means of the Gauss parametrization \(\psi\) given by (1) in terms of a pair \((g, \gamma)\) of type \(D\), and then we may identify \(M^n\) with the subset \(V\) of regular points of \(\psi\), endowed with the induced metric, and \(f\) with \(\psi|_V\). Then, given \((y, w) \in U \subset V\), any geodesic \(\gamma\) starting at \((y, w)\) and tangent to \(\Delta(y, w) = N_gL(y)\) at \((y, w)\) is given by \(t \mapsto (y, tw')\) for some \(w' \in N_gL(y)\).

Let \(Y, Z \in \mathfrak{X}(L)\) satisfy conditions \((i)\) to \((iii)\) in the definition of surfaces of type \(D\). Since \((g, \gamma)\) is a pair of type \(D\), the matrix of the endomorphism \(P_{tw'} = \gamma(y)I + \text{Hess} \gamma(y) - tA_{w'}\) of \(T_yL\) with respect to the basis \(\{Y(y), Z(y)\}\) is a diagonal matrix whose diagonal elements are linear in \(t\), and hence have zeros unless the shape operator \(A_{w'}\) is identically zero.

In summary, there always exist points \((y, tw')\) along \(\gamma\) at which \(\psi\) fails to be regular, unless \(A_{w'}\) is identically zero. Thus the latter possibility must hold for all \((y, w) \in U\) and for all \(w' \in N_gL(y)\). But this implies that \(\psi(U)\) is a surfacelike strip, which is ruled out by assumption. The proof then proceeds as in [6] by showing that this implies \(\psi\) to be either ruled or a partial tube over a curve.

### 4 The case of nonflat space forms

As in the Euclidean case, among the hypersurfaces \(f: M^n \to \mathbb{Q}^{n+1}_\epsilon\) in nonflat ambient space forms, that is, \(\epsilon \in \{1, -1\}\), that carry totally geodesic distributions of codimension one, one has the ruled hypersurfaces, that is, the hypersurfaces carrying a smooth foliation by (open subsets of) \((n - 1)\)-dimensional totally geodesic submanifolds of \(\mathbb{Q}^{n+1}_\epsilon\), and the partial tubes over curves described in Section 1.1. There are also two other families of trivial examples.

1. The first family is that of isometric immersions \(f: U \to \mathbb{Q}^{n+1}_\epsilon\) of open subsets \(U \subset \mathbb{Q}^n_\epsilon\), for one may consider any foliation of \(U\) by (open subsets of) totally geodesic
hypersurfaces of constant sectional curvature $\epsilon$. Notice that these examples may have complete leaves if $\epsilon = -1$.

(2) The second family of examples consists of generalized cones. Let $g: L^2 \to Q^3_\epsilon, c \geq \epsilon$, be an isometric immersion, and let $i: Q^3_\epsilon \to Q^{n+1}_\epsilon$ be an umbilical inclusion. Thus, the normal bundle of $\tilde{g} = i \circ g$ splits as

$$N_{\tilde{g}}L = i_* N_g L \oplus N_i Q^3_\epsilon.$$ 

We regard $\Lambda = N_i Q^3_\epsilon$ as a subbundle of $N_{\tilde{g}}L$ and define $f: \Lambda \to Q^{n+1}_\epsilon$ by

$$f(x, v) = \exp_{g(x)} v,$$

where $\exp$ is the exponential map of $Q^{n+1}_\epsilon$. Then $f$ is called the generalized cone over $g$.

Here again, if $\epsilon = -1$ and $c \leq 0$, then generalized cones yield examples of hypersurfaces carrying totally geodesic foliations of codimension one with complete leaves.

The local classification of hypersurfaces of nonflat space forms that carry a totally geodesic foliation of codimension one is as follows.

**Theorem 14.** Let $f: M^n \to Q^{n+1}_\epsilon, n \geq 3$ and $\epsilon \in \{1, -1\}$, be an isometric immersion of a Riemannian manifold without points where all sectional curvatures are equal to $\epsilon$. If $M^n$ carries a totally geodesic foliation of codimension one, then there exists an open dense subset of $M^n$ where $f$ is locally either ruled, a partial tube over a curve, a generalized cone or given in terms of the Gauss parametrization by a surface $g: L^2 \to S^{n+1}_\mu$ of type $D$, where $\mu = (1 - \epsilon)/2$.

Similarly to the case of Euclidean hypersurfaces, the proof relies on the following version of Lemma 11.

**Lemma 15.** Let $g: L^2 \to S^{n+1}_\mu$ be a surface and let $\Lambda_\epsilon$ and $\psi: \Lambda_\epsilon \to Q^{n+1}_\epsilon$ be given by (7) and (8), respectively. Assume that there exist $Y, Z \in \mathfrak{X}(L)$ with $\|Y\| = 1$ satisfying conditions (i) and (ii) in Lemma 11. Then the orthogonal complement of $\text{span}\{jY\}$ is a totally geodesic distribution on the open subset $V$ of regular points of $\psi$, endowed with the induced metric. Here $j = j(y, w): T_y L \to \Delta^1_{y,w}$ is the isometry given by part (iv) of Proposition 4 between $T_y L$ and the orthogonal complement of

$$\Delta(y, w) = \{w' \in N_g L(y) : \langle w', w \rangle = 0\} \subset T_{(y, w)} \Lambda_\epsilon.$$ 

Conversely, if $V$ carries a totally geodesic distribution of codimension one then there exist $Y, Z \in \mathfrak{X}(L)$ with $\|Y\| = 1$ satisfying conditions (i) and (ii) in Lemma 11.

**Proof:** It follows from (ii) that

$$\langle Aw Z, Y \rangle_L = 0$$

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for all \( w \in \Gamma(N_gL) \). Therefore, if \( X \in \mathfrak{X}(L) \) is a unit vector field orthogonal to \( Y \), then \( A_w^1X \) is collinear with \( X \) for all \( w \in \Gamma(N_gL) \). Hence, whenever \( A_w \) is invertible, we see that \( A_w^{-1}X \) is collinear with \( Z \). It follows from (9) that

\[
\langle \nabla^V_{jX}Y, w' \rangle_V = \langle C_w jX, jY \rangle_V = \langle A_w A_w^{-1}X, Y \rangle_L = 0
\]

for any \((y, w) \in V\) and for all \( w' \in \Delta(y, w) \). On the other hand, by (10) we have

\[
\langle \nabla^V_{jX}Y, jX \rangle_V = \langle \nabla^L_{A_w^{-1}X}Y, X \rangle_L = 0
\]

by (i) and the fact that \( A_w^{-1}X \) is collinear with \( Z \). We conclude from (9), (23) and (24) that the distribution orthogonal to \( jY \) is totally geodesic.

We now prove the converse. If \( \tilde{Y} \in \Gamma(\Delta^+) \) is a unit vector field such that the distribution \( \{\tilde{Y}\}^+ \) is totally geodesic, then

\[
\nabla^V_X \tilde{Y} = 0
\]

for any \( \tilde{X} \in \mathfrak{X}(V) \) such that \( \langle \tilde{X}, \tilde{Y} \rangle_V = 0 \). In particular,

\[
\nabla^V_T \tilde{Y} = 0
\]

for all \( T \in \Gamma(\Delta) \), hence there exists \( Y \in \mathfrak{X}(L) \) such that \( \tilde{Y} = jY \). From

\[
\nabla^V_{jX}Y = 0
\]

for \( X \in \mathfrak{X}(L) \) such that \( \langle X, Y \rangle_L = 0 \), using (9) and (10) we obtain, respectively,

\[
0 = \langle \nabla^V_{jX}Y, w' \rangle_V = \langle C_w jX, jY \rangle_V = \langle A_w A_w^{-1}X, Y \rangle_L
\]

and

\[
\nabla^L_{A_w^{-1}X}Y = 0
\]

for any \((y, w) \in V\) and for all \( w' \in \Delta(y, w) \).

Suppose that at \( y \in L^2 \) we have

\[
T_yL = \text{span}\{A_w^{-1}X(y) : w \in N_gL(y) \text{ with } (y, w) \in V\}.
\]

Then this also holds in a neighborhood \( W \) of \( y \), and then \( Y \) is parallel on \( W \) by (26), which implies that \( V \) is flat. On the other hand, by (25) we have

\[
\langle A_w^{-1}X, A_wY \rangle_L = 0
\]

for all \( w, w' \in N_gL(y) \) such that \( A_w \) is invertible and \( \langle w', w \rangle = 0 \). It follows that \( A_wY = 0 \) for all \( w' \in \Gamma(N_gW) \). But then \( Y \) belongs to the relative nullity space of \( g \).
at any point of $W$, and hence the Gauss curvature of $L^2$ is equal to one at any point of $W$, a contradiction.

It follows that there exists $Z \in \mathfrak{X}(L)$ such that, for all $y \in L^2$ and for all $w \in N_y L(y)$ such that $A_w$ is invertible, the vector $A_w^{-1}X(y)$ is collinear with $Z(y)$. Eq. (23) and the fact that $X$ and $Y$ are orthogonal then imply that

$$\langle A_w Z, Y \rangle_L = 0$$

for all $w \in N_y L(y)$. Finally, from (26) we obtain $\nabla^L_2 Y = 0$. ■

Our last result shows that the only hypersurfaces of $S^{n+1}$, $n \geq 3$, that may carry a totally geodesic distribution of codimension one with complete leaves are partial tubes over curves.

**Theorem 16.** Let $f: M^n \to S^{n+1}$, $n \geq 3$, be an isometric immersion of a Riemannian manifold without points where all sectional curvatures are equal to 1. If $M^n$ carries a totally geodesic foliation of codimension one with complete leaves, then $\tilde{f} = f \circ \pi$ is a partial tube over a curve, where $\pi: \tilde{M} \to M^n$ is the universal cover space of $M^n$.

**Proof:** First we prove that there cannot exist any open subset $U \subset M^n$ where neither of conditions (i) or (ii) in Proposition 12 hold. Suppose otherwise and let $U \subset M^n$ be the maximal open subset where neither of conditions (i) or (ii). Then condition (iii) in Proposition 12 holds and $\Delta \subset D$ at any point of $U$. Choose unit vector fields $Y \in \Gamma(D^\perp)$ and $X \in \Gamma(D \cap \Delta^\perp)$ along $U$. Then there exist $\beta, \mu, \rho \in C^\infty(U)$, with $\rho \mu \neq 0$ at any point of $U$, such that

$$AY = \beta Y + \mu X \quad \text{and} \quad AX = \mu Y + \rho X.$$  

Since $D$ and $\Delta$ are both totally geodesic and $\Delta \subset D$, we have

$$\nabla_X Y = \nabla_T Y = \nabla_T X = 0$$

for all $T \in \Gamma(\Delta)$. The Codazzi equation for $(X, T)$ yields

$$T(\rho) = \rho \langle \nabla_X X, T \rangle \quad \text{and} \quad T(\mu) = \mu \langle \nabla_X X, T \rangle$$  

(27)

for all $T \in \Gamma(\Delta)$, whereas the Gauss equation for $(X, T, S, X)$ gives

$$T(\langle \nabla_X X, S \rangle = \langle T, S \rangle + \langle \nabla_X X, \nabla_T S \rangle + \langle \nabla_X X, T \rangle \langle \nabla_X X, S \rangle$$  

(28)

for all $T, S \in \Gamma(\Delta)$.

From (27) we see that $\mu = \varphi \rho$ with $T(\varphi) = 0$ for all $T \in \Gamma(\Delta)$. It follows that the leaves of $\Delta$ through points of $U$ are complete and that condition (iii) remains valid along them. Otherwise, there would exist a geodesic tangent to some $T \in \Gamma(\Delta)$ of unit
length that would reach a point of the boundary of $U$. Since such a point is the limit of a sequence of points where either of conditions (i) or (ii) holds, either $\mu$ or $\rho$ vanishes at that point, and hence both $\mu$ and $\rho$ by the above. But this possibility is ruled out by the assumption that there are no points in $M^n$ where the sectional curvatures are all equal to 1.

Now, for any complete geodesic with unit tangent vector $T$ in a leaf of $\Delta$, by (28) the function $\lambda = \langle \nabla_X X, T \rangle$ satisfies the differential equation

$$T(\lambda) = 1 + \lambda^2,$$

and thus $\arctan(\lambda(t)) = t$, a contradiction with the fact that $\lambda$ is defined for any value of the parameter.

It follows that $U$ is empty, and hence either of conditions (i) or (ii) must hold at any point of $M^n$. Let $S_1$ (respectively, $S_2$) be the subset of $M^n$ where condition (i) (respectively, condition (ii)) is satisfied. Since both $S_1$ and $S_2$ are closed and $M^n = S_1 \cup S_2$, any point $x \in \partial S_1$ belongs to $S_1 \cap S_2$, hence all sectional curvatures are equal to $\epsilon$ at $x$. It follows from our assumption that either $M^n = S_1$ or $M^n = S_2$.

In the first case, $f$ is a ruled hypersurface. But then $f$ has still rank two, $\Delta \subset D$ and (28) still holds, and hence also (29) along each complete geodesic with unit tangent vector $T$ in a leaf of $\Delta$, which leads to a contradiction as before.

We conclude that $M^n = S_2$. Since the leaves of $D$ are complete, then the universal covering space of $M^n$ is isometric to a twisted product $\tilde{M}^n = \tilde{M}_0^{n-1} \times_\rho \mathbb{R}$, where $\rho \in C^\infty(\tilde{M})$ (see Theorem 2.7 of [9]), with the leaves of the lifting $\tilde{D}$ of $D$ corresponding to the slices $\tilde{M}_0^{n-1} \times \{t\}$, $t \in \mathbb{R}$. Moreover, the fact that $M^n = S_2$ means that the vector field $\partial/\partial t$ tangent to the second factor is a principal direction of $\tilde{f}$ at any point. It follows from Proposition 1 that $\tilde{f}$ is a partial tube over a curve. \hfill \blacksquare

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IMPA
Estrada Dona Castorina, 110
22460-320 — Rio de Janeiro
Brazil
marcos@impa.br

Universidade de São Paulo
Av. Trabalhador São-Carlense 400
13560-970 — São Carlos
Brazil
tojeiro@icmc.usp.br