Chaotic behaviour of state variable filters with saturation-type integrators

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The possibility to see route-to-chaos scenarios in common analogue filtering structures known as state variable filters is described. It is shown that strange attractors characterised by a fractal geometrical dimension can be observed in these naturally linear network structures if integrators are supposed to be exclusive nonlinear two-ports.

Introduction: It is well known that second-order driven deterministic dynamical systems with some intrinsic nonlinearity can exhibit a solution which is both extremely sensitive to changes of initial conditions and bounded in finite state space volume. If description of the mathematical model is known, the design procedure towards analogue chaotic oscillators [1, 2] is straightforward and very easy to follow. However, in a reversed approach, analysis of the lumped circuits from the viewpoint of the existence of such unpredictable motion, can explain the so far confusing behaviour of many real electronic systems and is of much important.

The idea behind this Letter is to numerically prove the existence of the strange attractors in KHN filter topologies. Such attractors are dense in state space but, unlikely noise, has a non-integer geometrical dimension. KHN filters are usually composed of a cascade connection of two loss-less integrators with full feedback summed-up at the input port or distributed at the output. Since fundamental structure of KHN filter is supposed it covers an entire class of voltage- current- as well as mixed-mode realisations. Thus, the existence of chaos can be forecasted to general class of filtering two-ports commonly used in everyday signal processing applications.

Linear filtering: The voltage-mode KHN filter belongs to group of filters which can be described in steady-state by a transfer function between the outputs of integrators, the output of a summing inverting amplifier and the input

\[ K_{LP}(s) = \frac{V_{y}}{V_{in}} = \frac{K_{0}a_{0}}{D(s)} \]

\[ K_{HP}(s) = \frac{V_{y}}{V_{in}} = \frac{K_{0}a_{0}/Q_{W}}{D(s)} \]

where \( a_{0} = (t_{1}t_{2})^{-1/2} \) is the pole centre frequency in radians per second, \( Q \) is the pole quality factor, and \( t_{1} \) and \( t_{2} \) are integration constants of first and second integrators, respectively.

Non-ideal integrators: In practice, the two-port active integrator is subject to two major non-ideal effects: saturation of the output variable of the active element and loss integration since the pole of the transfer function is negative rather than located in the origin of the complex plane. Signal limitation or compression caused by the first property can be approximated by a scalar three-segment piecewise-linear (PWL) function with odd-symmetry

\[ h_{(x)} = a_{out}x + \frac{(a_{out} - a_{in})}{2} (|x + \beta| - |x - \beta|) \]

where \( a_{in} \) and \( a_{out} \) are positive slopes of the \( r \)th PWL function in the inner and outer segments and \( \beta \) denotes the breakpoint. True sharp saturation means that \( a_{out} \) approaches positive infinity. For small-signal operation where the amplitude is below the threshold, the filter behaves like a common linear two-port with a limited cycle as the only possible solution.

By considering both integrators being nonlinear, such a situation can be described by a couple of the first-order differential equations

\[ \frac{dx}{dt} = h_{1}(x) \quad \frac{dy}{dt} = h_{2} \left[ -\frac{a_{0}}{Q} y - a_{0}x + u(t) \right] \]

where \( u(t) = A \sin(\omega t) \) is the input harmonic waveform. The right-hand side of the equation represents input inverting voltage summation.

Numerical simulations: The fourth-order Runge-Kutta integration method has been utilised as the tool for trajectory visualisation. Three-dimensional perspective views of the chaos evolution process are given in Fig. 1 where the final time is set to 2000 s, the time step set to 0.01 s, the significant frequency of the KHN filter \( a_{in} = 1.4, a_{out} = 1 \) and \( a_{out} = 90 \) being slopes of the PWL function, and the amplitude and frequency of the input signal is 2 and 1/(2\( \pi \)), respectively.

Standard dynamical motion quantification is based on the calculation of the spectrum of the Lyapunov exponents (LEs) [3]. Positive value of the largest LE indicates dynamical behaviour which is long-term unpredictable, chaotic. Assuming the input waveform and PWL integrators mentioned above, the grey-scale contour-surface plot of the normalised largest LE \( \in (0, 1) \) is provided in Fig. 2. Black regions denote limit cycles, weak chaos is marked by dark grey and strong chaos by light colour. Obviously there is an optimal value of significant filter frequency for the evolution of the chaotic attractor. For LE, calculation time is transformed into third state variable turning analysed system into autonomous. The time interval \( t_{range} \in (100, 1000) \) was chosen in order to obtain an attracting limit set and omit transient solution.

Experimental observations: To verify the theoretical assumptions and the existence of strange attractors in the state variable filter, it was designed as shown in Fig. 3. The PWL nature of the integrators is realised artificially by anti-parallel connections of diodes BAT42. The desired breakpoint voltage can be achieved using several diodes connected in series

\[ |V_{i+}| < 2 \cdot V_{th} \Rightarrow \tau = R_{c}C \]

\[ |V_{i+}| > 2 \cdot V_{th} \Rightarrow \tau = \frac{R_{c}(R_{c} + 2 \cdot R_{d})}{R_{c} + R_{c} + 2 \cdot R_{d}C} \]

where \( V_{i} \) is the voltage at the input terminal of the integrator, \( V_{in} \) is the drop voltage in the forward-biased diode and \( R_{d} \simeq 101 \) is the guessed average value of differential diode resistance. The basic filter parameters can be changed accordingly to formulas

\[ a_{0} = \frac{R_{c}}{R_{c} + R_{d}} \cdot \frac{1}{\pi} \]

\[ \omega_{th} = \frac{R_{c}}{R_{c} + R_{d}} \cdot \frac{T_{1}}{\pi} \]

Fig. 1 Chaotic attractors generated by system with different natural dissipation (see text for details)

\[ a \quad Q = 40 \]

\[ b \quad Q = 70 \]

\[ c \quad Q = 90 \]

\[ d \quad Q = 200 \]

Fig. 2 Largest LE against KHN filter parameters, high-resolution plot having \( \Delta Q \times \Delta a = 201 \times 21 = 4221 \) values

Fig. 3 Largest LE against KHN filter parameters, high-resolution plot having \( \Delta Q \times \Delta a = 201 \times 21 = 4221 \) values

[Image 310x597 to 416x664]

[Image 311x681 to 415x746]
The regular time constant was down-scaled to \( \tau_1 = \tau_2 = R_C \times C = 10^4 \times 10^{-8} = 100 \mu s \) such that the frequency responses of the voltage-feedback TL084 and the current-feedback AD844 operational amplifiers can be considered ideal with negligible parasitic properties. During experiments contactless board resistors \( R_C, R_b, R_f \) were fixed at the value \( 10k \Omega \), \( R_y \) was set to 100 \( \Omega \) and the circuit was fed by a uniform supply voltage of \( \pm 15 \) V.

![Circuit implementation for experimental test of KHN filter structure with saturation-type voltage-mode integrators](image)

**Fig. 3** Circuit implementation for experimental test of KHN filter structure with saturation-type voltage-mode integrators

Experimental validation of the chaotic nature of waveforms generated by the KHN filter is provided in Fig. 4. The frequency of the input harmonic signal is 1.62 kHz and the amplitude is 2.06 V which fully corresponds with the theory.

![Chaotic attractor measured by digital oscilloscope HP54603B](image)

**Fig. 4** Chaotic attractor measured by digital oscilloscope HP54603B

| a | x–y-plane projection |
| b | u–v-plane projection |
| c | w–y-plane projection |
| d | x and y state variables in time domain |

**Conclusion:** It is demonstrated that the presence of two-port integrators with a saturation-type PWL transfer function in KHN circuits can cause noise-like global behaviour known as chaos. This is especially possible in the case of an input signal with large amplitude and frequencies in radians per second close to the geometrical centre of the filter poles and if a filter with a high-quality factor is used. Note that such a circuit configuration is not far away from the common filtering regime. Thus it is a quite plausible situation that if irregular motion is observed in the KHN structure it has more in common with periodically nudged nonlinear dynamics rather than stochastic processes caused by noise or similar uncertainties.

The analysis task presented here can be also reversed into synthesis because it seems that KHN filters, after making few modifications, can be used for modelling vector fields which are typical for driven chaos evolution. It will be of particular interest if there is the possibility to adjust \( \omega_0 \) or \( Q \) by using an external dc quantity (voltage or current control).

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