The 290 fixed-point sublattices of the Leech lattice

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Abstract

We determine the orbits of fixed-point sublattices of the Leech lattice with respect to the action of the Conway group $\text{Co}_0$. There are 290 such orbits. Detailed information about these lattices, the corresponding coinvariant lattices, and the stabilizing subgroups, is tabulated in several tables.

1 Introduction

The *Leech lattice* $\Lambda$ is the unique positive-definite, even, unimodular lattice of rank 24 without roots [La] [Co2]. It may also be characterized as the most densely packed lattice in dimension 24 [CK]. The group of isometries of $\Lambda$ is the *Conway group* $\text{Co}_0$ [Co1]. For a subgroup $H \subseteq \text{Co}_0$ we set

$$\Lambda^H = \{ v \in \Lambda \mid hv = v \text{ for all } h \in H \}.$$  

We call such a sublattice of $\Lambda$ a *fixed-point sublattice*. Let $\mathcal{F}$ be the set of all fixed-point sublattices of $\Lambda$. The Conway group acts by translation on $\mathcal{F}$, because if $g \in \text{Co}_0$, then $g\Lambda^G = \Lambda^{gHg^{-1}}$. In this note, we classify the $\text{Co}_0$-orbits of fixed-point sublattices. We will prove:

**Theorem 1.1.** Under the action of $\text{Co}_0$, there are exactly 290 orbits on the set of fixed-point sublattices of $\Lambda$.

The purpose of the present note is not merely to enumerate the orbits of fixed-point sublattices, but to provide in addition a detailed analysis of their properties. In particular, this includes the *stabilizers* $G$, which are the (largest) subgroups of $\text{Co}_0$.

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that stabilize a given fixed-point sublattice pointwise. Information about the orbits of fixed-point lattices and their fixing groups is given in Table 1 in Section 4. Based on the theory that we present in Sections 2 and 3, this information was obtained by relying on extensive computer calculations using the computer algebra system MAGMA \[Mag\]. We shall say more about this in due course.

There are a number of reasons that make the classification of fixed-point lattices desirable. The group quotient \( \text{Co}_1 = \text{Co}_0/\{\pm 1\} \) is one of the 26 sporadic simple groups. It contains 11 additional sporadic groups, 9 of which can be described in terms of lattice stabilizers. Although these particular realizations have been known for a long time, the complete picture that we provide is new.

The Leech lattice is also the starting point of the construction of interesting vertex operator algebras \[Bo2, FLM\] and generalized Kac-Moody Lie algebras. Such Kac-Moody Lie algebras have root lattices that can often be described in terms of fixed-point lattices inside \( \Lambda \) \[Sch\], and the associated denominator identities provide Moonshine for the corresponding subgroups \[Bo3\].

The geometry of \( K_3 \) surfaces and certain hyperkähler manifolds \( X \), over both the field of complex numbers and in finite characteristic, is controlled (using Torelli-type theorems) by lattices related to \( \Lambda \). In this way, symmetry groups of \( X \) can be mapped into \( \text{Co}_0 \), and properties of the fixed-point lattices control which groups may appear. See \[Nik, Min, Ko, DK\] for \( K_3 \) surfaces and \[Mo1, Hu, HM\] for other hyperkähler manifolds.

Much of the impetus for studying the finite symmetry groups of such manifolds, and recent developments in the related area of Mathieu Moonshine \[EOT\], came from the well-known theorem of Mukai \[Mu\]. This states that a finite group \( G \) of symplectic automorphisms of a K3 surface is isomorphic to a subgroup of the Mathieu group \( M_{23} \) with at least five orbits in its natural permutation representation on 24 letters; furthermore, there are just 11 subgroups (up to isomorphism) which are maximal subject to these conditions. A typical application of our results leads to a simplified approach to this theorem. Indeed, lattice-theoretic arguments \[Ko, Co3\] show that \( G \) can be embedded into \( \text{Co}_0 \) in such a way that \( \text{rk} \Lambda_G \geq 5 \) and \( \alpha(\Lambda_G) := \text{rk} \Lambda_G - \text{rk} A_{\Lambda_G} \geq 2 \) (see Section 2 for notation). The containment \( G \subseteq M_{23} \) follows immediately from Table 1; moreover the 11 maximal such groups are those \( G \) in Table 1 with \( \text{rk} \Lambda_G = 5 \) and \( \alpha(\Lambda_G) \geq 2 \). The advantage of this approach compared to that of Kondo (loc. cit), who initiated the lattice-theoretic approach, is that a case-by-case analysis of the 23 Niemeier lattices with roots is reduced to an analysis of the Leech lattice alone.

The extension of Mukai’s theorem to more general contexts is currently an active research area, and it is widely expected that knowledge of the stabilizers \( G \) with \( \Lambda_G \geq 4 \) will eventually lead to the classification of all finite symplectic automorphism groups of hyperkähler manifolds of type \( K3^{[n]} \) (cf. \[Mo2, HM\]) and symplectic autoequivalences of derived categories of sheaves on K3 surfaces \[AM, GHV, Hu\].

Another application of our tables is to the study of symmetries of the extremal vertex operator superalgebra \( V_{D_{12}^+} \) of central charge 12 \[Ho\]. This SVOA
has a unique \(N=1\) super conformal structure stabilized by the Conway group \([\text{DM}]\). By considering subgroups \(G \subseteq \text{Co}_0\) fixing subspaces of \(\Lambda \otimes \mathbb{R}\) of various dimensions \([\text{CDDHKW}]\)\([\text{CHKW}]\)\([\text{DM}]\) one can define Jacobi forms of weight 0 (and some index), leading to various moonshine phenomena for these \(G\). The groups \(G\) which can occur can directly be read off from Table 1. Only a few examples of such groups have been known and studied before. The resulting moonshine functions are also closely related to the equivariant elliptic genera of hyperkähler manifolds of type \(K3^{[n]}\) (cf. \([\text{CH}]\)\([\text{HM}]\)) and to to Mathieu Moonshine and its generalization called Umbral Moonshine \([\text{CDH}]\); see \([\text{DM}]\).

Notable past work on sublattices of \(\Lambda\) includes that of Curtis \([\text{Cu}]\), who introduced the very useful class of \(S\)-lattices and classified their orbits. We make use of these ideas in the present paper (cf. Section 3). Harada and Lang \([\text{HL}]\) considered the orbits of fixed-point lattices for cyclic subgroups of \(\text{Co}_0\). Fixed-point lattices related to \(K3\) surfaces were classified by Hashimoto \([\text{HM}]\), and the present authors handled those for hyperkähler manifolds of type \(K3^{[2]}\) \([\text{HM}]\). Additional information can also be found in the Atlas of finite groups \([\text{CCNPW}]\).

The present work is based on lattice-theoretic arguments and group-theoretical computations in \(\text{Co}_0\).

The Conway group \(\text{Co}_0\) is presently too large to permit computation of its complete subgroup lattice. (Such a calculation would allow us to list all orbits of fixed-point lattices directly.) Even for the monomial subgroup \(2^{12}:M_{24} \subseteq \text{Co}_0\), the number of conjugacy classes is huge (of order \(10^7\)–\(10^8\)) and we have thus far been unable to determine them all. For the purposes of the present work, however, it is enough to know the conjugacy classes of non-2-groups inside \(2^{12}:M_{24}\) and these were already computed in \([\text{HM}]\). (There are 279,343 of them.)

The paper is organized as follows. Section 2 summarizes some general properties of group actions on lattices. In Section 3 we describe our method to determine the 290 orbits, while Section 4 contains detailed information about the 290 fixed-point lattices. We also discuss several interesting properties of some of the resulting lattices.

Finally, we mention that the corresponding problem of classification of fixed-point lattices and stabilizer subgroups in the case of the \(E_8\)-root lattice and its attendant Weyl group is also of interest. For the convenience of the interested reader, we have stated the main results below as Theorem 3.6. The result is probably well-known to experts.

**Supplemental material**

Supplemental data associated with this article can be found, in the electronic version, at http://dx.doi.org/LOCATION TO BE INSERTED BY PUBLISHER.
2 Integral lattices and their automorphism groups

We introduce some notation related to integral lattices and their automorphism groups and record some results that we will need.

A lattice $L$ is a finitely generated free $\mathbb{Z}$-module together with a rational-valued symmetric bilinear form $(.,.)$. All lattices in this note are assumed to be positive-definite. We let $O(L) := \text{Aut}(L)$ be the group of automorphisms (or isometries) of $L$ considered as lattice, i.e., the set of automorphisms of the group $L$ that preserve the bilinear form. It is finite because of the assumed positive-definiteness of the bilinear form. The lattice $L$ is integral if the bilinear form takes values in $\mathbb{Z}$, and even if the norm $(x,x)$ belongs to $2\mathbb{Z}$ for all $x \in L$. An even lattice is necessarily integral.

A finite quadratic space $A = (A,q)$ is a finite abelian group $A$ equipped with a quadratic form $q : A \rightarrow \mathbb{Q}/2\mathbb{Z}$. We denote the corresponding orthogonal group by $O(A)$. This is the subgroup of $\text{Aut}(A)$ that leaves $q$ invariant.

The dual lattice of an integral lattice $L$ is

$$L^* := \{x \in L \otimes \mathbb{Q} \mid (x,y) \in \mathbb{Z} \text{ for all } y \in L\}.$$ 

The discriminant group $L^*/L$ of an even lattice $L$ is equipped with the discriminant form $q_L : L^*/L \rightarrow \mathbb{Q}/2\mathbb{Z}$, $x + L \mapsto (x,x) \pmod{2\mathbb{Z}}$. This turns $L^*/L$ into a finite quadratic space, called the discriminant space of $L$ and denoted $A_L := (L^*/L, q_L)$.

There is a natural induced action of $O(L)$ on $A_L$, leading to a short exact sequence

$$1 \rightarrow O_0(L) \rightarrow O(L) \rightarrow \overline{O}(L) \rightarrow 1,$$

where $\overline{O}(L)$ is the subgroup of $O(A_L)$ induced by $O(L)$ and $O_0(L)$ consists of the automorphisms of $L$ which act trivially on $A_L$.

A sublattice $K \subseteq L$ is called primitive (in $L$) if $L/K$ is a free abelian group. We set

$$K^\perp := \{x \in L \mid (x,y) = 0 \text{ for all } y \in K\}.$$ 

Assume now that $L$ is even and unimodular, i.e., $L^* = L$. If $K$ is primitive then there is an isomorphism of groups $i : A_K \cong A_K^\perp$ such that $q_{K^\perp}(i(a)) = -q_K(a)$ for $a \in A_K$. We can recover $L$ from $K \oplus K^\perp$ by adjoining the cosets

$$C := \{(a,i(a)) \mid a \in A_K\} \subseteq A_K \oplus A_K^\perp.$$ 

See [Nik] for further details. The following is a special case of another result (Propositions 1.4.1 and 1.6.1, loc. cit).

**Proposition 2.1.** The equivalence classes of extensions of $K \oplus K^\perp$ to an even unimodular lattice $N$ with $K$ primitively embedded into $N$ are in bijective correspondence with double cosets $\overline{O}(K) \backslash O(A_K)/i^* \overline{O}(K^\perp)$, where $i^* : \overline{O}(K^\perp) \rightarrow O(A_K)$ is defined by $g \mapsto i^{-1} \circ g \circ i$. 

\[\square\]
Suppose that \( G \subseteq O(L) \) is a group of automorphisms of a lattice \( L \). The invariant and coinvariant lattices for \( G \) are
\[
L^G = \{ x \in L \mid gx = x \text{ for all } g \in G \},
\]
\[
L_G = (L^G)^\perp = \{ x \in L \mid (x, y) = 0 \text{ for all } y \in L^G \}
\]
respectively. They are both primitive sublattices of \( L \). The restriction of the \( G \)-action to \( L_G \) induces an embedding \( G \subseteq O(L_G) \).

If \( G \subseteq O(L) \), we denote by \( \widetilde{G} \) the pointwise stabilizer of \( L_G \) in \( O(L) \). We always have \( G \subseteq \widetilde{G} \) and \( L^G = L^{\tilde{G}} \). Moreover, \( N_{O(L)}(\widetilde{G}) \) is the setwise stabilizer of \( L^G \), and \( N_{O(L)}(\widetilde{G})/\widetilde{G} \) is a (faithful) group of isometries of \( L^G \).

**Lemma 2.2.** Suppose that \( L \) is even and unimodular. Then \( \widetilde{G} \cong O_0(L_G) \).

**Proof:** As explained above, \( L \) is obtained from \( L^G \oplus L_G \) by adjoining cosets \( C := \{(a, i(a)) \mid a \in A_{L_G} \} \subseteq A_{L_G} \oplus A_{L_G} \). Furthermore, in this case \( \tilde{G} \) necessarily acts trivially on \( A_{L_G} \), so that \( \tilde{G} \subseteq O_0(L_G) \).

On the other hand, we can extend the \( O_0(L_G) \)-action on \( L_G \) to a trivial action on \( L^G \). Since \( O_0(L_G) \) acts trivially on \( A_{L_G} \oplus A_{L_G} \), the action on \( L_G \oplus L^G \) extends to an action on \( L \). The Lemma follows.

A root of \( L \) is a primitive vector in \( v \in L \) such that reflection in \((Zv)^\perp\) is an isometry of \( L \). The root sublattice of \( L \) is the sublattice spanned by all roots.

We also note that the genus of a positive-definite even lattice \( L \) is determined by the quadratic space \( A_L \) together with the rank of \( L \) \[Nik\].

We recall the following fact:

**Lemma 2.3.** A finite group \( G \) has a unique minimal normal subgroup \( N \) such that \( G/N \) is a 2-group. It is the subgroup generated by all elements of odd order.

We follow usual practice and set \( N = O^2(G) \).

**Lemma 2.4.** Let \( L \) be a lattice and assume that \( G \subseteq O(L) \) satisfies \( G = \tilde{G} \). Then
\[
O^2(G) \leq \tilde{O}^2(G) \leq G.
\]

**Proof:** Since \( L^G \subseteq L^{O^2(G)} \) then \( \tilde{O}^2(G) \subseteq \tilde{G} = G \). Moreover, since \( O^2(G) \leq G \) then \( G \) acts on \( L^{O^2(G)} \), and hence normalizes the pointwise stabilizer \( \tilde{O}^2(G) \) of this lattice.

3 Construction of the fixed-point lattices

Recall \[Co2\] that the \( 2^{24} \) cosets comprising \( \Lambda/2\Lambda \) have representatives \( v \) which may be chosen to be short vectors, i.e., \( (v, v) \leq 8 \). More precisely, if \( (v, v) \leq 6 \) then
\{v, -v\} are the only short representatives of \(v + 2\Lambda\); if \((v, v) = 8\) then the short vectors in \(v + 2\Lambda\) comprise a coordinate frame \(\{\pm w_1, \ldots, \pm w_{24}\}\), where the \(w_j\) are pairwise orthogonal vectors of norm 8. In particular, if \(u \in \Lambda\) then \(u = v + 2w\) for some \(v, w \in \Lambda\) and \(v\) a short vector, and if \((v, v) \leq 6\) then \(v\) is unique up to sign.

It is well-known \(\text{[Co2]}\) that \(\text{Co}_0\) acts transitively on coordinate frames, the (set-wise) stabilizer of one such being the monomial group \(2^{12}: M_{24}\).

A sublattice \(S \subseteq \Lambda\) is an \(S\)-lattice if, for every \(u \in S\), the corresponding short vector \(v\) satisfies \((v, v) \leq 6\) and furthermore \(w \in S\). This concept was introduced by Curtis \(\text{[Cu]}\) who showed that there are exactly twelve isometry classes of \(S\)-lattices. The next result is a useful variant of a construction given in the Atlas \(\text{[CCNPW]}\).

**Proposition 3.1.** If \(G = O^2(G) \subseteq \text{Co}_0\), then one of the following holds:

(a) \(\tilde{G} \subseteq 2^{12}: M_{24}\)

(b) \(\Lambda^G\) is an \(S\)-lattice.

**Proof:** Let \(u \in \Lambda^G\) with \(u = v + 2w\), where \(v, w \in \Lambda\), and \((v, v) \leq 8\). Then \(v + 2\Lambda = u + 2\Lambda\) is \(G\)-invariant.

First suppose that for every choice of \(u\), we have \((v, v) \leq 6\). Then \(\{\pm v\}\) are the only short vectors in \(u + 2\Lambda\), so this set is invariant under the action of \(G\). Then every odd order element in \(G\) fixes \(v\), and since \(G = O^2(G)\) then \(v \in \Lambda^G\). Then also \(2w = u - v \in \Lambda^G\), and because \(\Lambda^G\) is primitive then \(w \in \Lambda^G\). So (b) holds in this case.

Otherwise, for some \(u \in \Lambda^G\) we have \(u = v + 2w\) and \((v, v) = 8\). Then because \(\tilde{G}\) fixes \(u\), it acts on \(u + 2\Lambda\) and therefore stabilizes the unique coordinate frame contained in this coset. So in this case (a) holds.

**Remark 3.2.** For the stabilizer \(G = \tilde{G}\) of an \(S\)-lattice one has always \(G = O^2(G)\) and \(G \not\subseteq 2^{12}: M_{24}\).

**Proof:** This can easily be seen directly from the classification of \(S\)-lattices and their stabilizers \(\text{[Cu]}\) (cf. Table I). Note that \(|G|\) does not divide \(2^{12}: M_{24}\) so that part (a) of Proposition 3.1 fails by Lagrange’s Theorem.

Now assume that \(G = \tilde{G} \subseteq \text{Co}_0\) with \(L := \Lambda^G \subseteq M := \Lambda \tilde{O}^2(G)\). By Lemma 2.4 we have \(O^2(G) \subseteq \tilde{O}^2(G) \subseteq G\), and \(\tilde{O}^2(G)\) is the pointwise stabilizer of \(M\). Thus \(L = M^G = M^G/\tilde{O}^2(G)\) is the fixed-point sublattice of \(G/\tilde{O}^2(G)\), which is a faithful \(2\)-group of isometries of \(M\). Furthermore, by Proposition 3.1 and Remark 3.2, either \(\tilde{O}^2(G) \subseteq 2^{12}: M_{24}\) or \(M\) is an \(S\)-lattice.

This leads to the following general approach for finding all fixed-point lattices \(L\):

(a) Find all subgroups \(H = O^2(H) \subseteq 2^{12}: M_{24}\) and all pointwise stabilizers \(H\) of \(S\)-lattices (cf. Remark 3.2).
(b) For each such \(H\), calculate \(L = \Lambda^G\) where \(H \leq G\) and \(G/H\) is a 2-group.

We say that two pairs of lattices \((L_1, L_2)\) and \((L_1', L_2')\) are isometric if there are isometries \(L_i \rightarrow L_i'\) \((i = 1, 2)\).

In order to make the enumeration of the fixed-point lattices outlined above effective, we iteratively compile a list of triples \((G, \Lambda^G, \Lambda_G)\) using the following procedure.

**Step 1:** Select a representative \(G\) from each conjugacy class of subgroups of \(2^{12}:M_{24}\) satisfying \(G = O^2(G)\). Construct \((G, \Lambda^G, \Lambda_G)\). Select one triple for each isometry class of pairs \((\Lambda^G, \Lambda_G)\) of lattices, resulting in a list of such triples.

**Step 2:** For each triple \((G, \Lambda^G, \Lambda_G)\), construct the pointwise-stabilizer \(\tilde{G} = O_0(\Lambda_G)\) in \(Co_0\) and replace \(G\) by \(\tilde{G}\).

**Step 3:** For each triple \((G, \Lambda^G, \Lambda_G)\), compute the normalizer \(N\) of \(G\) in \(Co_0\). For each conjugacy class \([g]\) in \(N/G\), construct the group \(H = \langle G, g \rangle\) and add the triple \((H, \Lambda^H, \Lambda_H)\) to the list if \((\Lambda^H, \Lambda_H)\) is not isometric to a pair of lattices already present.

**Step 4:** Repeat Steps 2 and 3 until the list is saturated.

This results in the list of 290 triples which, along with accompanying data, are described in Table 1.

We explain now why the triples resulting from Steps 1–4 produce the desired list of orbits of fixed-point lattices, thereby proving Theorem 1.1.

First, notice that if \((G, \Lambda^G, \Lambda_G)\) and \((H, \Lambda^H, \Lambda_H)\) are distinct triples on the final list, then \((\Lambda^G, \Lambda_G)\) and \((\Lambda^H, \Lambda_H)\) are not isometric. Therefore, \(\Lambda^G\) and \(\Lambda^H\) certainly lie in distinct \(Co_0\)-orbits, since an element of \(Co_0\) mapping \(\Lambda^G\) onto \(\Lambda^H\) is an isometry that also induces an isometry of \(\Lambda_G\) onto \(\Lambda_H\).

Next we show that every \(Co_0\)-orbit of fixed-point lattices has a representative that occurs in a triple on the final list. First we verify that the isometry classes \((\Lambda^G, \Lambda_G)\) already determine the orbits of fixed-point lattices.

**Proposition 3.3.** For each entry in Table 1, the isometry class of the pair \((\Lambda^G, \Lambda_G)\) uniquely determines the \(Co_0\)-orbit of \(\Lambda^G\).

**Proof:** For each pair \((\Lambda^G, \Lambda_G)\), we determine all isomorphism classes of extensions of \(\Lambda^G \oplus \Lambda_G\) to an even unimodular lattice \(N\) (i.e. the even unimodular overlattices of \(\Lambda^G \oplus \Lambda_G\)) by computing the double cosets for \(O(\Lambda^G) \times i^*(\overline{O}(\Lambda^G))\) in \(O(\Lambda_G)\) (cf. Proposition 2.1). Among the resulting lattices \(N\), it turns out there is always exactly one equivalence class with minimal norm 4, so that it must be isometric to \(\Lambda\).

It follows that \((\Lambda^G, \Lambda_G)\) uniquely determines the \(Co_0\)-orbit of \(\Lambda^G\) since two extensions \(L\) and \(L'\) of \(\Lambda^G \oplus \Lambda_G\) are by definition equivalent if there is an isometry
between $L$ and $L'$ which stabilizes $\Lambda^G \oplus \Lambda_G$ setwise, i.e. after identifications of $L$ and $L'$ with $\Lambda$, the corresponding sublattices $\Lambda^G \oplus \Lambda_G$ of $\Lambda$ can be mapped to each other by an element of $\text{Co}_0$.

Next, all $S$-lattices and their stabilizers appear in Table II. Indeed, the twelve lattices $\Lambda^G$ numbered 35, 101, 122, 163, 167, 222, 223, 225, 230, 273, 274 and 290 have the two properties: $G = O^2(G)$ and $|G|$ does not divide $|2^{12}:M_{24}|$. By Proposition 3.1 each $\Lambda^G$ is an $S$-lattice. According to Curtis [Cu] there are exactly twelve $\text{Co}_0$-orbits of $S$-lattices, so indeed they all appear in Table II.

Along with the $S$-lattices, Step 3 ensures that with a fixed-point lattice $\Lambda^H$, all fixed-point lattices $\Lambda^G$ also occur in a triple whenever $H \unlhd G$ and $G/H$ is a 2-group as the following proposition shows.

**Proposition 3.4.** Assume that for $G \subseteq \text{Co}_0$ the triple $(\tilde{O}^2(G), \Lambda_{\tilde{O}^2(G)}, \Lambda_{\tilde{O}^2(G)})$ is contained on the list in Table II. Then $(\tilde{G}, \Lambda_{\tilde{G}}, \Lambda_{\tilde{G}})$ is also contained in the list.

**Proof:** Because it is a 2-group, $G/O^2(G)$ has a central series

$$O^2(G) = H_0 \unlhd H_1 \unlhd \cdots \unlhd H_n = G$$

with each $H_i \unlhd G$ and $|H_{i+1}/H_i| = 2$, and

$$\Lambda^{O^2(G)} = \Lambda^{H_0} \supseteq \Lambda^{H_1} \supseteq \cdots \supseteq \Lambda^{H_n} = \Lambda^G.$$

$G$ acts on each $\Lambda^{H_i}$, and hence normalizes $\tilde{H}_i$. Using $H_{i+1}/(H_{i+1} \cap \tilde{H}_i) \cong H_{i+1}/\tilde{H}_i$ and $H_i \unlhd H_{i+1} \cap \tilde{H}_i$ we conclude that $[H_{i+1}\tilde{H}_i : \tilde{H}_i] \leq 2$. Thus Steps 2 and 3 guarantee that $(\tilde{H}_{i+1}, \Lambda_{\tilde{H}_{i+1}}, \Lambda_{\tilde{H}_i})$ is on the list whenever $(\tilde{H}_i, \Lambda_{\tilde{H}_i}, \Lambda_{\tilde{H}_i})$ is.

Since, by assumption, the triple $(\tilde{H}_0, \Lambda_{\tilde{H}_0}, \Lambda_{\tilde{H}_0})$ is contained in the list, it follows inductively that $(\tilde{G}, \Lambda_{\tilde{G}}, \Lambda_{\tilde{G}})$ is too.

Together with the results of the computation, we have established Theorem III.

We describe now some more details for the implementation of Steps 1 to 3 with the computer algebra system MAGMA.

We realized the Conway group $\text{Co}_0$ as a matrix group of integral $24 \times 24$-matrices and as a permutation group on the 196,560 vectors of norm 4. We also determined an explicit isomorphism which allows us to evaluate a computation in the most appropriate realization.

For Step 1, we started with the list of conjugacy classes of non-2-groups inside $2^{12}:M_{24}$. In [HM] we had already shown:

**Theorem 3.5.** With respect to conjugation in $2^{12}:M_{24}$, there are 279,343 conjugacy classes of subgroups of $2^{12}:M_{24}$ which are not 2-groups.

From these classes we selected those groups $G$ which satisfy $G = O^2(G)$. This was done by computing $O^2(G)$ as the normal subgroup of $G$ generated by $p$-Sylow
subgroups for all \( p \neq 2 \). This resulted in a list of 3755 groups. For these groups we computed the pairs \((\Lambda^G, \Lambda_G)\) of sublattices inside \( \Lambda \) and checked for isometry by the implemented lattice functions in MAGMA.

For Step 2, we can compute \( \tilde{G} \) abstractly as the group \( O_0(\Lambda_G) \). However, to realize \( \tilde{G} \) as a subgroup of \( C_0 \) we realized in addition \( C_0 \) as a matrix group over the finite field \( \mathbb{F}_2 \) acting on \( \Lambda/2\Lambda \). This allowed us to compute that stabilizer of \( \Lambda^G/2\Lambda \) in \( C_0 \).

Step 3 can easily be done by the implemented group theory functions in MAGMA.

Remarks on the \( E_8 \)-root lattice. The \( E_8 \)-root lattice is the unique even, unimodular, positive-definite lattice of rank 8 and its automorphism group is the corresponding Weyl group. The problem of determining the orbits of fixed-point sublattices and stabilizer groups for this lattice and its automorphism group also has some interest attached to it.

It follows from a Theorem of Steinberg ([St], Thm. 15) that the stabilizer of a sublattice of a root lattice inside the corresponding Weyl group is a reflection group. The conjugacy classes of reflections subgroups for \( W(E_8) \) are known, cf. [DPR], Table 5.

We desist from further discussion, contenting ourselves with the statement of the result, which must be well-known to experts.

**Theorem 3.6.** In its action on the \( E_8 \)-root lattice, the Weyl group of type \( E_8 \) has 41 orbits of fixed-point sublattices. These are in bijective correspondence with the isomorphism types of full subgraphs of the Coxeter graph for \( E_8 \), the lattice-stabilizers being the Coxeter groups determined by these subgraphs. The coinvariant lattices are the corresponding root lattices.

4 The 290 fixed-point lattices

This section describes the fixed-point lattices and discusses several observations regarding the resulting tables. Table provides information about the 290 orbits of fixed-point lattices \( L = \Lambda^G \) inside \( \Lambda \). For a given \( L \), the group \( G \) listed is the full pointwise stabilizer \( C_0 \), i.e., \( G = \tilde{G} \), or \( O_0(\Lambda_G) \).

In addition, we provide the following electronic supplementary material. Supplementary Table 2 consists of the Gram matrices of each \( \Lambda^G \). Supplementary Table 3 gives partial information about the lattice structure of the 290 orbits. We have also added a text file in Magma format which contains a coinvariant lattice from each orbit, the corresponding fixed-point lattice and generator matrices for its stabilizers.
Table 1: Orbits of fixed-point lattices. The columns provide the following information: number of $\Lambda^G$ (no.); rank of $\Lambda^G$ (rk); order of $G$ (order). Information about the group structure of $G$ ($G$). Here, $[n]$ denotes an unspecified group of order $n$ and $p^n$ an elementary abelian group of the same order. Sometimes we list the standard name for the group or the number of $G$ in the database of small groups. The genus symbol for $\Lambda^G$ without the signature information (genus); rank of $\Lambda^G$ minus the rank of $A_{1G}$ ($\alpha$); index of $\overline{O}(\Lambda_G)$ in $O(A_{1G})$ ($\overline{r}_G$); index of $N_{Co_0}(G)/O(\Lambda^G)$ (ind); number of lattices in the genus of $\Lambda^G$ ($h^G$); number of Niemeier lattices with roots into which $\Lambda_G$ embeds ($N$); case type ([M23]; $G \subseteq M_{23}$, [M24]; $G \subseteq M_{24}$ and not [M23]; [Mon2]; $G \subseteq 2^{12}:M_{24}$ but not [M23], [M24] and $G = T:H$ with $H \subseteq M_{24}$ and $T = G \cap 2^{12}$; [Mon3]; $G \subseteq 2^{12}:M_{24}$ but not [M23], [M24], [Mon2]; [-]: $G \subseteq 2^{12}:M_{24}$ but not [S], [S]; $|G| \nmid |2^{12}:M_{24}|$; [*]: $\overline{O}^2(G) = G$ (type).

| no. | rk | order | $G$ | genus | $\alpha$ | $\overline{r}_G$ | $\overline{r}_G$ | ind | $h^G$ | $N$ | G23 |
|-----|----|-------|-----|-------|--------|-------------|-------------|-----|-------|-----|------|
| 1   | 24 | 1     | 1   | 1     | 24     | 1           | 1           | 1   | 24    | 23  | M23* |
| 2   | 16 | 2     | 2   | 2     | $2^{+8}_U$ | 8           | 1           | 2       | 1     | 24  | 17   |
| 3   | 12 | 4     | $2^2$ | $2^{+4}_U$ | 4     | 40         | 40         | 12     | 7     | 7   | M23  |
| 4   | 12 | 3     | 3   | $3^+6$ | 6     | 1          | 1          | 1      | 10    | 8   | M23* |
| 5   | 12 | 2     | 2   | $2^{+12}_U$ | 0     | 104448     | 104448     | 5040   | 3     | 11   | M24  |
| 6   | 10 | 8     | $2^3$ | $2^{+6}_U$ | 2     | 135        | 36         | 30     | 4     | 2   | M23  |
| 7   | 10 | 6     | $S_3$ | $2^{+2}_U$ | 5     | 1          | 2          | 1      | 13    | 7   | M23  |
| 8   | 10 | 4     | $2^2$ | $2^{+8}_U$ | 0     | 45696      | 26112      | 2520   | 2     | 6   | M24  |
| 9   | 10 | 4     | 4   | $2^{+2}_U$ | 4     | 1          | 2          | 1      | 8     | 7   | M23  |
| 10  | 9  | 16    | $2^4$ | $2^{+8}_U$ | 2     | 2          | 1          | 2      | 3     | 1   | M23  |
| 11  | 9  | 16    | $2^4$ | $2^{+8}_U$ | 0     | 2295       | 136        | 270    | 1     | 1   | M24  |
| 12  | 9  | 8     | $2^3$ | $2^{+8}_U$ | 0     | 11200      | 960        | 840    | 2     | 4   | M24  |
| 13  | 9  | 8     | $[2^3]$ (#3) | $4^+5$ | 4     | 1          | 4          | 1      | 8     | 5   | M23  |
| 14  | 8  | 512   | $[2^9]$ | $2^{+8}_U$ | 0     | 2          | 1          | 2      | 1     | -   | Mon2 |
| 15  | 8  | 18    | $[2.3^2]$ (#4) | $3^{+4}_U$ | 3     | 3          | 3          | 2      | 3     | 3   | M23* |
| 16  | 8  | 16    | $2^4$ | $2^{+4}_U$ | 0     | 840        | 64         | 105    | 1     | 2   | M24  |
| 17  | 8  | 16    | $[2^4]$ (#11) | $2^{+4}_U$ | 2     | 6          | 16         | 3      | 3     | 2   | M23  |
| 18  | 8  | 12    | $[2^3]$ (#4) | $4^{+4}_U$ | 4     | 1          | 12         | 1      | 8     | 6   | M23  |
| 19  | 8  | 12    | $2^4$ | $2^{+4}_U$ | 4     | 1          | 1          | 1      | 9     | 4   | M23* |
| 20  | 8  | 10    | $D_{10}$ | $5^+4$ | 4     | 1          | 1          | 1      | 5     | 4   | M23* |
| 21  | 8  | 8     | $[2^4]$ (#3) | $2^{+4}_U$ | 0     | 512        | 8192       | 36     | 1     | 6   | M24  |
| 22  | 8  | 6     | $S_3$ | $3^{+8}$ | 0     | 56862      | 56862      | 1920   | 1     | 7   | M24* |
| 23  | 8  | 4     | $2^4$ | $2^{+4}_U$ | 0     | 16         | 1024       | 6      | 3     | 5   | M24  |
| 24  | 7  | 48    | $[2^4]$ (#50) | $2^{+4}_U$ | 2     | 1          | 1          | 1      | 4     | 1   | M23* |
| 25  | 7  | 32    | $2^5$ | $2^{+8}_U$ | 0     | 28         | 8          | 7      | 1     | 1   | M24  |
| 26  | 7  | 32    | $[2^5]$ (#27) | $2^{+4}_U$ | 2     | 1          | 2          | 1      | 4     | 1   | M23  |
| 27  | 7  | 32    | $2^5$ (#46) | $2^{+4}_U$ | 0     | 60         | 64         | 15     | 1     | 1   | M24  |
| 28  | 7  | 32    | $[2^5]$ (#49) | $4^{+5}$ | 2     | 16         | 2          | 1      | 11    | 4   | M23  |
| 29  | 7  | 24    | $S_4$ | $4^{+3}_U$ | 4     | 1          | 2          | 1      | 11    | 4   | M23  |
| 30  | 7  | 8     | $[2^3]$ (#3) | $2^{+4}_U$ | 0     | 16         | 128        | 6      | 2     | 3   | M24  |
| no. | rk | order | $G$ | genus | $\alpha$ | $\tau_G$ | $\tau^G$ | ind | $h^G$ | $N$ | type |
|-----|----|-------|-----|-------|---------|---------|---------|-----|-------|-----|------|
| 31  | 7  | 8     | $[2^3]$ (#4) | $2_5^3 \cdot 3^2$ | 2 | 1 | 1 | 1 | 4 | 3 | $M_{23}$ |
| 32  | 7  | 8     | $2^3$ | $2_5^2 \cdot 4^2 \cdot 5^2$ | 0 | 32 | 4096 | 6 | 1 | 3 | $M_{24}$ |
| 33  | 6  | 1536  | $[2^3]3$ | $2_6^2 \cdot 3$ | 0 | 1 | 1 | 1 | 1 | 1 | - $Mon_a$ |
| 34  | 6  | 1024  | $[2^{10}]$ | $2_8^2 \cdot 4^2$ | 0 | 1 | 2 | 1 | 1 | - $Mon_a$ |
| 35  | 6  | 486   | $[2^3]3$ (#249) | $3^{5+}$ | 1 | 1 | 1 | 1 | 1 | 1 | - $S^*$ |
| 36  | 6  | 192   | $[2^3]3$ (#1541) | $2_8^2 \cdot 4^2$ | 0 | 2 | 2 | 1 | 1 | 1 | $M_{24}$ |
| 37  | 6  | 192   | $[2^3]3$ (#1023) | $2_8^2 \cdot 8_6^2$ | 2 | 1 | 1 | 1 | 1 | 1 | $M_{23}$ |
| 38  | 6  | 96    | $[2^3] \cdot S_3$ (#227) | $2_5^2 \cdot 2_2^2 \cdot 3^1 \cdot 5^2$ | 2 | 1 | 2 | 1 | 5 | 1 | $M_{23}$ |
| 39  | 6  | 72    | $[2^3]3^2$ (#43) | $4_2^2 \cdot 5^3$ | 3 | 1 | 2 | 1 | 3 | 2 | $M_{23}$ |
| 40  | 6  | 64    | $[2^6]$ (#264) | $2_6^2 \cdot 4^3$ | 0 | 6 | 64 | 3 | 1 | 1 | $M_{24}$ |
| 41  | 6  | 64    | $[2^6]$ (#266) | $2_6^2 \cdot 4^3$ | 0 | 1 | 32 | 1 | 1 | - $Mon_b$ |
| 42  | 6  | 64    | $[2^6]$ (#202) | $2_6^2 \cdot 4^3 \cdot 5^2$ | 0 | 15 | 12 | 5 | 1 | 1 | $M_{24}$ |
| 43  | 6  | 64    | $[2^6]$ (#138) | $4_6^2 \cdot 3^3$ | 2 | 1 | 4 | 1 | 3 | 1 | $M_{23}$ |
| 44  | 6  | 60    | $A_5$ | $2_5^2 \cdot 3^4 \cdot 5^2$ | 4 | 1 | 1 | 1 | 6 | 3 | $M_{23}$ |
| 45  | 6  | 48    | $[2^3]3$ (#48) | $2_5^2 \cdot 4^2 \cdot 3^2$ | 2 | 3 | 6 | 5 | 2 | 2 | $M_{23}$ |
| 46  | 6  | 36    | $[2^3]3^2$ (#9) | $2_6^2 \cdot 2^2 \cdot 3^2$ | 3 | 1 | 1 | 1 | 4 | 2 | $M_{23}$ |
| 47  | 6  | 36    | $[2^3]3^2$ (#10) | $2_6^2 \cdot 3^2 \cdot 3^2$ | 2 | 3 | 18 | 2 | 3 | 3 | $M_{23}$ |
| 48  | 6  | 32    | $[2^3]3$ (#34) | $2_6^2 \cdot 4^2$ | 0 | 30 | 32 | 5 | 1 | 1 | $M_{24}$ |
| 49  | 6  | 32    | $[2^3]3$ (#27) | $2_6^2 \cdot 4^2$ | 0 | 36 | 64 | 9 | 1 | 2 | $M_{24}$ |
| 50  | 6  | 24    | $[2^3]3$ (#14) | $2_6^2 \cdot 3^2$ | 0 | 40 | 240 | 12 | 1 | 2 | $M_{24}$ |
| 51  | 6  | 24    | $S_4$ | $2_6^2 \cdot 2^2 \cdot 4^2$ | 0 | 32 | 32 | 6 | 1 | 3 | $M_{24}$ |
| 52  | 6  | 21    | $F_{21}$ | $7^5$ | 3 | 1 | 1 | 1 | 3 | 2 | $M_{23}$ |
| 53  | 6  | 20    | $[2^2]5$ (#3) | $2_5^2 \cdot 4^3$ | 3 | 1 | 2 | 1 | 6 | 4 | $M_{23}$ |
| 54  | 6  | 16    | $[2^2]3$ (#11) | $2_5^2 \cdot 4^3 \cdot 3^2$ | 0 | 16 | 256 | 6 | 2 | 3 | $M_{24}$ |
| 55  | 6  | 16    | $[2^2]3$ (#8) | $2_5^2 \cdot 4^3 \cdot 3^2$ | 2 | 1 | 2 | 1 | 4 | 3 | $M_{23}$ |
| 56  | 6  | 12    | $A_4$ | $2_6^2 \cdot 4^2$ | 0 | 32 | 32 | 6 | 1 | 3 | $M_{24}$ |
| 57  | 6  | 8     | $2^3$ | $4_6^2$ | 0 | 60 | 2048 | 15 | 1 | 3 | $M_{24}$ |
| 58  | 6  | 8     | $2^3$ | $2_6^2 \cdot 4^3 \cdot 3^2$ | 0 | 1 | 64 | 1 | 2 | 1 | $Mon_a$ |
| 59  | 6  | 8     | $2^3 \times 4$ | $2_6^2 \cdot 4^3$ | 0 | 6 | 16 | 3 | 2 | 3 | $M_{24}$ |
| 60  | 6  | 6     | $S_3$ | $2^6 \cdot 3^2$ | 0 | 1 | 2 | 1 | 2 | 1 | $Mon_b$ |
| 61  | 6  | 6     | $S_3$ | $2^6 \cdot 3^2$ | 0 | 1 | 720 | 1 | 2 | 1 | $Mon_b$ |
| 62  | 6  | 6     | $2^6 \cdot 3^2$ | $2_6^2 \cdot 4^3$ | 0 | 1 | 51840 | 1 | 1 | - $Mon_a$ |
| 63  | 6  | 6     | $2^6 \cdot 3^2$ | $2_6^2 \cdot 4^3$ | 0 | 1 | 48 | 1 | 3 | 3 | $Mon_a$ |
| 64  | 6  | 4     | $4^6$ | $2^6$ | 0 | 64 | 2048 | 6 | 1 | 7 | $M_{24}$ |

Table 1: Orbits of fixed-point lattices
Table 1: Orbits of fixed-point lattices

| no. | rk | order | $G$ | genus | $\alpha$ | $r_G$ | $r^G$ | ind | $h^G$ | $N$ | type |
|-----|----|-------|-----|-------|---------|-------|-------|-----|-------|-----|------|
| 72  | 5  | $A_6$ | $4_5^{1+3+2+5+1}$ | 3 | 1 | 1 | 1 | 4 | 2 | $M_{23}^*$ |
| 73  | 5  | $[5^32^2]$ (#1026) | $2u^2+4+3+2+1$ | 2 | 1 | 2 | 1 | 3 | 1 | $M_{23}^*$ |
| 74  | 5  | $[2^63^2]$ (#955) | $4_2^2+5+3+1$-1 | 2 | 1 | 4 | 1 | 4 | 1 | $M_{23}$ |
| 75  | 5  | $[2^63^2]$ (#1538) | $2u^2+4+3+1$-1 | 0 | 10 | 10 | 4 | 1 | 1 | $M_{24}$ |
| 76  | 5  | $[2^63^2]$ (#1493) | $4_2^3+3+1$-1 | 2 | 1 | 1 | 1 | 2 | 2 | $M_{23}^*$ |
| 77  | 5  | $L_2(7)$ | $4_1^1+1+2$ | 3 | 1 | 1 | 1 | 4 | 2 | $M_{23}^*$ |
| 78  | 5  | $[2^7]$ (#2326) | $4_5^{5}$ | 0 | 1 | 64 | 1 | 1 | - | $Mon_a$ |
| 79  | 5  | $[2^7]$ (#1758) | $2u^2+4+3+1$-1 | 0 | 6 | 16 | 3 | 1 | 1 | $M_{24}$ |
| 80  | 5  | $[2^7]$ (#1759) | $2u^2+4+3+1$-1 | 0 | 1 | 8 | 1 | 1 | - | $Mon_b$ |
| 81  | 5  | $[2^7]$ (#1755) | $2u^2+4+3+1$-1 | 0 | 6 | 16 | 3 | 1 | 1 | $M_{24}$ |
| 82  | 5  | $S_4$ | $4_2^{1+2+5+2}$ | 3 | 1 | 2 | 1 | 3 | 2 | $M_{23}$ |
| 83  | 5  | $[2^7 3^2]$ (#226) | $2u^2+4+1+3+2$-1 | 0 | 15 | 20 | 6 | 1 | 1 | $M_{24}$ |
| 84  | 5  | $[2^7 3^2]$ (#41) | $2u^2+4+3+1$-1 | 2 | 1 | 1 | 1 | 4 | 2 | $M_{23}^*$ |
| 85  | 5  | $[2^7 3^2]$ (#40) | $2u^2+4+3+1$-1 | 2 | 1 | 2 | 1 | 3 | 2 | $M_{23}$ |
| 86  | 5  | $[2^7 3^2]$ (#226) | $4_5^{5}$ | 0 | 30 | 64 | 15 | 1 | 1 | $M_{24}$ |
| 87  | 5  | $[2^7 3^2]$ (#29) | $2u^2+4+3+1$-1 | 2 | 1 | 2 | 1 | 4 | 3 | $M_{24}^*$ |
| 88  | 5  | $[2^7 3^2]$ (#48) | $2u^2+4+3+1$-1 | 0 | 16 | 32 | 6 | 2 | 3 | $M_{24}$ |
| 89  | 5  | $[2^7 3^2]$ (#43) | $2u^2+4+3+1$-1 | 0 | 6 | 8 | 3 | 1 | 1 | $M_{24}$ |
| 90  | 5  | $S_4$ | $2u^2+4+3+1$-1 | 0 | 2 | 48 | 1 | 2 | 2 | $Mon_a$ |
| 91  | 5  | $S_4$ | $2u^2+4+3+1$-1 | 0 | 32 | 64 | 6 | 1 | 3 | $M_{24}$ |
| 92  | 5  | $[2^7 3^2]$ (#226) | $2u^2+4+3+1$-1 | 0 | 1 | 2 | 1 | 2 | 1 | $Mon_b$ |
| 93  | 5  | $[2^7 3^2]$ (#40) | $2u^2+4+3+1$-1 | 0 | 2 | 256 | 1 | 1 | 1 | $M_{24}$ |
| 94  | 5  | $[2^7 3^2]$ (#12) | $2u^2+4+3+1$-1 | 0 | 1 | 1 | 1 | 2 | 1 | $Mon_a$ |
| 95  | 5  | $[2^7 3^2]$ (#11) | $2u^2+4+3+1$-1 | 0 | 1 | 1 | 1 | 2 | 1 | $Mon_a$ |
| 96  | 5  | $[2^7 3^2]$ (#14) | $2u^2+4+3+1$-1 | 0 | 1 | 1 | 1 | 2 | 1 | $Mon_a$ |
| 97  | 5  | $[2^7 3^2]$ (#14) | $2u^2+4+3+1$-1 | 0 | 1 | 4 | 1 | 3 | 1 | $Mon_a$ |
| 98  | 5  | $[2^7 3^2]$ (#14) | $2u^2+4+3+1$-1 | 0 | 1 | 4 | 1 | 3 | 1 | $Mon_a$ |
| 99  | 4  | $[2^7 2^2]$. $A_5$ | $2u^2+4+3+1$-1 | 0 | 1 | 1 | 1 | 1 | - | $Mon_a$ |
| 100 | 4  | $[2^7 2^2]$. $A_5$ | $2u^2+4+3+1$-1 | 0 | 1 | 1 | 1 | 1 | - | $Mon_a$ |
| 101 | 4  | $[2^7 2^2]$. $A_5$ | $2u^2+4+3+1$-1 | 1 | 1 | 1 | 1 | 1 | - | $S^*$ |
| 102 | 4  | $L_3(4)$ | $2u^2+4+3+1$-1 | 0 | 1 | 1 | 1 | 2 | 1 | $M_{23}^*$ |
| 103 | 4  | $L_3(4)$ | $2u^2+4+3+1$-1 | 0 | 1 | 2 | 1 | 1 | - | $Mon_a$ |
| 104 | 4  | $29160$ | $2u^2+4+3+1$-1 | 0 | 1 | 2 | 1 | 1 | - | $Mon_a$ |
| 105 | 4  | $6144$ | $2u^2+4+3+1$-1 | 0 | 1 | 4 | 1 | 1 | - | $Mon_a$ |
| 106 | 4  | $5760$ | $2u^2+4+3+1$-1 | 0 | 1 | 1 | 1 | 1 | - | $Mon_a$ |
| 107 | 4  | $4096$ | $2u^2+4+3+1$-1 | 0 | 1 | 8 | 1 | 1 | - | $Mon_a$ |
| 108 | 4  | $2520$ | $2u^2+4+3+1$-1 | 3 | 1 | 1 | 1 | 2 | 1 | $M_{23}^*$ |
| 109 | 4  | $1944$ | $2u^2+4+3+1$-1 | 1 | 1 | 1 | 1 | 1 | - | $S$ |
| 110 | 4  | $1920$ | $2u^2+4+3+1$-1 | 2 | 1 | 2 | 1 | 3 | 1 | $M_{23}$ |
| 111 | 4  | $1344$ | $2u^2+4+3+1$-1 | 2 | 1 | 1 | 1 | 3 | 1 | $M_{23}^*$ |
| 112 | 4  | $1152$ | $2u^2+4+3+1$-1 | 2 | 1 | 1 | 1 | 2 | 1 | $M_{23}^*$ |
| no. | rk | order | $G$ | genus | $\alpha$ | $r_G$ | $t^G$ | ind | $h^G$ | $N$ | type |
|-----|----|-------|-----|-------|--------|-------|-------|-----|-------|-----|------|
| 113 | 4  | 1152  | $[2^6 3^2]$ | $2\mu_{16}^2 3^{-3}$ | 0 | 2 | 4 | 1 | 1 | 1 | $M_{24}^*$ |
| 114 | 4  | 972   | $[2^3 3^5]$ (9812) | $2\mu_{24}^2 3^{-1}$ | 0 | 1 | 6 | 1 | 1 | - | S |
| 115 | 4  | 768   | $[2^8 3]$ | $2\mu_{12}^2 3^{-2}$ | 0 | 6 | 8 | 3 | 1 | 1 | $M_{24}$ |
| 116 | 4  | 768   | $[2^8 3]$ | $2\mu_{12}^2 3^{-2}$ | 0 | 1 | 4 | 1 | 1 | - | Mon$_b$ |
| 117 | 4  | 768   | $[2^8 3]$ | $4\mu_{4}^2$ | 0 | 12 | 8 | 6 | 1 | 1 | $M_{24}$ |
| 118 | 4  | 720   | $A_6 2$ | $2\mu_{12}^2 3^{-2}$ | 2 | 3 | 3 | 2 | 2 | 1 | $M_{23}$ |
| 119 | 4  | 720   | $A_6 2$ | $2\mu_{12}^2 3^{-2}$ | 2 | 1 | 1 | 1 | 3 | 2 | $M_{23}$ |
| 120 | 4  | 660   | $L_2(11)$ | $11^2$ | 2 | 1 | 1 | 1 | 3 | 2 | $M_{23}$ |
| 121 | 4  | 576   | $[2^6 3^2]$ (98654) | $4\mu_{12}^2 3^{-2}$ | 2 | 1 | 4 | 1 | 3 | 1 | $M_{23}$ |
| 122 | 4  | 500   | $5^{1+2} 2^1$ (#23) | $5^{-3}$ | 1 | 1 | 1 | 1 | 1 | - | S |
| 123 | 4  | 384   | $[2^3 3^6]$ (#90097) | $2\mu_{24}^2 3^{-1}$ | 0 | 1 | 2 | 1 | 1 | - | Mon$_b$ |
| 124 | 4  | 384   | $[2^3 3^6]$ (#18134) | $2\mu_{12}^2 3^{-1}$ | 0 | 1 | 1 | 1 | 1 | - | Mon$_b$ |
| 125 | 4  | 384   | $[2^3 3^6]$ (#17948) | $2\mu_{12}^2 3^{-1}$ | 0 | 3 | 12 | 2 | 1 | 1 | $M_{24}$ |
| 126 | 4  | 384   | $[2^3 3^6]$ (#90098) | $2\mu_{12}^2 3^{-1}$ | 0 | 6 | 4 | 3 | 1 | 1 | $M_{24}$ |
| 127 | 4  | 360   | $A_6 2$ | $2\mu_{12}^2 3^{-2}$ | 0 | 2 | 2 | 1 | 1 | 1 | Mon$_b$ |
| 128 | 4  | 360   | $3 5$ | $3^{-2} 5^{-2}$ | 2 | 1 | 2 | 1 | 3 | 2 | $M_{23}$ |
| 129 | 4  | 336   | $L_2(7)$ | $2\mu_{12}^2 3^{-2}$ | 2 | 1 | 2 | 1 | 3 | 2 | $M_{23}$ |
| 130 | 4  | 256   | $[2^8]$ (95387) | $4\mu_{12}^2 3^{-1}$ | 0 | 1 | 16 | 1 | 1 | - | Mon$_a$ |
| 131 | 4  | 192   | $[2^3 3^6]$ (#1494) | $2\mu_{12}^2 3^{-1}$ | 0 | 2 | 4 | 1 | 1 | 1 | $M_{24}$ |
| 132 | 4  | 192   | $[2^3 3^6]$ (#1493) | $4\mu_{4}^2$ | 0 | 12 | 8 | 3 | 1 | 2 | $M_{23}$ |
| 133 | 4  | 144   | $[2^3 3^2]$ (#187) | $4\mu_{4}^2 3^{-1}$ | 0 | 1 | 1 | 1 | 2 | 1 | Mon$_a$ |
| 134 | 4  | 144   | $[2^3 3^2]$ (#182) | $2\mu_{12}^2 3^{-1}$ | 0 | 16 | 32 | 6 | 1 | 2 | $M_{24}$ |
| 135 | 4  | 144   | $[2^3 3^2]$ (#183) | $2\mu_{12}^2 3^{-1}$ | 0 | 2 | 12 | 1 | 1 | 1 | Mon$_b$ |
| 136 | 4  | 120   | $S_5$ | $2\mu_{12}^2 5^{-2}$ | 0 | 2 | 12 | 1 | 1 | 1 | Mon$_b$ |
| 137 | 4  | 120   | $S_5$ | $2\mu_{12}^2 5^{-2}$ | 0 | 1 | 2 | 1 | 2 | 1 | Mon$_b$ |
| 138 | 4  | 108   | $[2^2 3^3]$ (#17) | $2^2 3^2 5^{-1}$ | 0 | 18 | 54 | 4 | 1 | 3 | $M_{24}$ |
| 139 | 4  | 96    | $[2^2 3^3]$ (#195) | $2\mu_{12}^2 3^{-2}$ | 0 | 2 | 16 | 1 | 1 | 1 | Mon$_a$ |
| 140 | 4  | 72    | $[2^3 3^2]$ (#46) | $2\mu_{12}^2 3^{-1}$ | 0 | 1 | 72 | 1 | 1 | - | Mon$_a$ |
| 141 | 4  | 72    | $[2^3 3^2]$ (#40) | $2\mu_{12}^2 3^{-1}$ | 0 | 1 | 6 | 1 | 2 | 1 | Mon$_b$ |
| 142 | 4  | 72    | $[2^3 3^2]$ (#40) | $2\mu_{12}^2 3^{-1}$ | 0 | 2 | 72 | 1 | 1 | 1 | Mon$_b$ |
| 143 | 4  | 64    | $[2^6]$ (#257) | $2\mu_{12}^2 5^{-1}$ | 0 | 1 | 8 | 1 | 1 | - | Mon$_b$ |
| 144 | 4  | 60    | $A_5$ | $2\mu_{12}^2 3^{-1}$ | 0 | 6 | 6 | 2 | 1 | 2 | $M_{24}$ |
| 145 | 4  | 48    | $[2^4 3]$ (#48) | $2\mu_{12}^2 3^{-1}$ | 0 | 1 | 48 | 1 | 2 | 1 | Mon$_a$ |
| 146 | 4  | 48    | $[2^4 3]$ (#48) | $2\mu_{12}^2 3^{-1}$ | 0 | 1 | 4 | 1 | 2 | 1 | Mon$_a$ |
| 147 | 4  | 48    | $[2^4 3]$ (#29) | $2\mu_{12}^2 3^{-1}$ | 0 | 32 | 64 | 6 | 1 | 3 | Mon$_b$ |
| 148 | 4  | 48    | $[2^4 3]$ (#48) | $2\mu_{12}^2 3^{-1}$ | 0 | 1 | 4 | 1 | 2 | 1 | Mon$_a$ |
| 149 | 4  | 40    | $[2^4 3]$ (#12) | $2\mu_{12}^2 3^{-1}$ | 0 | 3 | 6 | 2 | 2 | 3 | $M_{24}$ |
| 150 | 4  | 36    | $[2^5 3^2]$ (#13) | $2\mu_{12}^2 3^{-1}$ | 0 | 3 | 72 | 2 | 1 | 1 | Mon$_a$ |
| 151 | 4  | 36    | $[2^5 3^2]$ (#10) | $2\mu_{12}^2 3^{-1}$ | 0 | 3 | 6 | 2 | 1 | 1 | Mon$_a$ |
| 152 | 4  | 32    | $[2^6]$ (#40) | $2\mu_{12}^2 3^{-1}$ | 0 | 1 | 2 | 1 | 2 | 1 | Mon$_a$ |
| 153 | 4  | 32    | $[2^6]$ (#46) | $2\mu_{12}^2 3^{-1}$ | 0 | 2 | 64 | 1 | 1 | 1 | Mon$_a$ |
Table 1: Orbits of fixed-point lattices

| no. | rk | order | $G$          | genus | $\alpha$ | $\bar{r}_G$ | $\bar{r}^G$ | ind | $k^G$ | $N$ | type   |
|-----|----|-------|-------------|-------|----------|-------------|-------------|-----|-------|-----|--------|
| 154 | 4  | 24    | $[2^3.3]$  | 0     | 1        | 8           | 1           | 2   | 1     | -   | Mon$_a$ |
| 155 | 4  | 24    | $[2^3.3]$  | 0     | 1        | 1152        | 1           | 1   | 2     | -   | Mon$_a$ |
| 156 | 4  | 24    | $2^2.S_3$  | 0     | 6        | 1152        | 2           | 1   | 2     | -   | $M_{24}^*$ |
| 157 | 4  | 24    | $[2^3.3]$  | 0     | 1        | 48          | 1           | 1   | -     | -   | Mon$_a$ |
| 158 | 4  | 24    | $[2^3.3]$  | 0     | 1        | 96          | 1           | 1   | -     | -   | Mon$_a$ |
| 159 | 4  | 20    | $[2^2.5]$  | 0     | 1        | 120         | 1           | 1   | -     | -   | Mon$_a$ |
| 160 | 4  | 16    | $[2^4]$    | 0     | 4        | 64          | 1           | 1   | 3     | -   | $M_{24}$ |
| 161 | 4  | 12    | $[2^3.3]$  | 0     | 18       | 72          | 4           | 1   | 5     | -   | $M_{24}$ |
| 162 | 3  | 10321920 | $2^9.L_3(4)$ | 0     | 1        | 1           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 163 | 3  | 3265920 | $U_4(3)$   | 0     | 1        | 1           | 1           | 1   | 1     | -   | S$^*$ |
| 164 | 3  | 491520 | $[2^1.2].S_5$ | 0     | 2        | 1           | 1           | 1   | 2     | 1   | Mon$_a^*$ |
| 165 | 3  | 443520 | $M_{22}$   | 0     | 2        | 1           | 1           | 1   | 2     | 1   | $M_{23}^*$ |
| 166 | 3  | 184320 | $[2^3].A_6$ | 0     | 2        | 1           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 167 | 3  | 126000 | $U_3(5)$   | 0     | 2        | 1           | 1           | 1   | 1     | -   | S$^*$ |
| 168 | 3  | 61440  | $[2^3].S_5$ | 0     | 1        | 2           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 169 | 3  | 58320  | $3^4.A_6.2$ | 0     | 1        | 1           | 1           | 1   | 1     | -   | S |
| 170 | 3  | 40320  | $L_3(4).2$ | 0     | 1        | 2           | 1           | 2   | 1     | -   | $M_{23}$ |
| 171 | 3  | 40320  | $L_3(4).2$ | 0     | 1        | 1           | 1           | 1   | 1     | -   | - |
| 172 | 3  | 40320  | $2^4.A_7$  | 0     | 0        | 1           | 1           | 0   | 0     | -   | - |
| 173 | 3  | 36864  | $[2^{11}.2^2]$ | 0     | 2        | 1           | 1           | 1   | 2     | 1   | $M_{23}^*$ |
| 174 | 3  | 24576  | $[2^{11}.3^2]$ | 0     | 1        | 2           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 175 | 3  | 20160  | $A_8$      | 0     | 4        | 4           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 176 | 3  | 18432 | $[2^{11}.3^2]$ | 0     | 1        | 4           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 177 | 3  | 12288  | $[2^{11}.2^2]$ | 0     | 0        | 1           | 1           | 0   | 0     | -   | - |
| 178 | 3  | 11520  | $2^4.S_6$  | 0     | 0        | 1           | 2           | 1   | 1     | -   | Mon$_a^*$ |
| 179 | 3  | 11520  | $2^4.S_6$  | 0     | 0        | 1           | 2           | 1   | 1     | -   | Mon$_a^*$ |
| 180 | 3  | 10752  | $[3^6].L_2(7)$ | 0     | 1        | 1           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 181 | 3  | 10752  | $[3^6].L_2(7)$ | 0     | 1        | 2           | 1           | 1   | 1     | -   | Mon$_a^*$ |
| 182 | 3  | 7920   | $M_{11}$   | 2     | 2        | 1           | 1           | 3   | 2     | -   | Mon$_a^*$ |
| 183 | 3  | 5760   | $[2^{11}.3]$ | 2     | 1        | 1           | 1           | 3   | 2     | -   | Mon$_a^*$ |
| 184 | 3  | 4608   | $[2^9.3]$  | 2     | 0        | 1           | 1           | 2   | 1     | -   | Mon$_a^*$ |
| 185 | 3  | 4608   | $[2^9.3]$  | 0     | 2        | 1           | 1           | 1   | 2     | 1   | Mon$_a^*$ |
| 186 | 3  | 3888   | $[2^{10}.3^2]$ | 0     | 0        | 1           | 2           | 1   | 1     | 1   | S |
| 187 | 3  | 3888   | $[2^{10}.3^2]$ | 0     | 0        | 1           | 2           | 1   | 1     | 1   | S |
| 188 | 3  | 3840   | $[2^9].S_5$ | 0     | 3        | 2           | 1           | 1   | 1     | 1   | Mon$_a^*$ |
| 189 | 3  | 2688   | $[2^9].L_2(7)$ | 0     | 3        | 2           | 1           | 1   | 1     | 1   | Mon$_a^*$ |
| 190 | 3  | 2304   | $[2^7.2]$  | 0     | 6        | 8           | 3           | 1   | 1     | 1   | Mon$_a^*$ |
| 191 | 3  | 1944   | $[2^{11}.5]$ (#3536) | 0     | 1        | 6           | 1           | 1   | 1     | -   | S |
| 192 | 3  | 1536   | $[2^8.3]$  | 0     | 0        | 1           | 8           | 1   | 1     | 1   | Mon$_a^*$ |
| 193 | 3  | 1440   | $2.A_6.2$  | 0     | 0        | 1           | 1           | 1   | 2     | 1   | Mon$_a^*$ |
| 194 | 3  | 1440   | $A_6.2^2$  | 0     | 3        | 3           | 2           | 1   | 1     | 1   | Mon$_a^*$ |
Table 1: Orbits of fixed-point lattices

| no. | rk | order | $G$ | genus | $\alpha$ | $r_G^+$ | $r_G^-$ | ind | $h^G$ | $N$ | type |
|-----|----|-------|-----|-------|--------|--------|--------|-----|--------|-----|------|
| 195 | 3  | 1152  | $[2^3,2^3]$ | $2\cdot 3^3$ | 1       | 1      | 1      | 1   | -      | $\text{Mon}_a$ |
| 196 | 3  | 768   | $[2^3,3]$   | $4\cdot 3^3$ | 1       | 1      | 1      | 1   | -      | $\text{Mon}_a$ |
| 197 | 3  | 768   | $[2^3,3]$   | $2^3\cdot 1^3 + 1^6 + 1_7^3$ | 0 | 1 | 2 | 1 | - | $\text{Mon}_a$ |
| 198 | 3  | 720   | $A_6,2$   | $2^2\cdot 4^2 + 3 + 2$ | 0 | 2 | 4 | 1 | 1 | $\text{Mon}_a$ |
| 199 | 3  | 720   | $A_6,2$   | $2\cdot 2^2 + 1^3 + 3 + 1_5^3$ | 0 | 1 | 6 | 1 | 2 | $\text{Mon}_a$ |
| 200 | 3  | 720   | $A_6,2$   | $2^3\cdot 4^1 + 1^3 + 2$ | 0 | 1 | 2 | 1 | 1 | - | $\text{Mon}_a$ |
| 201 | 3  | 432   | $[2^3,3]$ (#734) | $2^2\cdot 1 + 3 + 2$ | 0 | 6 | 6 | 2 | 1 | 2 | $M_{24}^*$ |
| 202 | 3  | 384   | $[2^3,3]$ (#5602) | $4^1 + 8^2$ | 0 | 2 | 8 | 1 | 1 | 1 | $M_{24}$ |
| 203 | 3  | 288   | $[2^3,3]^2$ (#1027) | $2^2\cdot 4^1 + 1^9 - 1$ | 0 | 1 | 2 | 1 | 2 | 1 | $\text{Mon}_a$ |
| 204 | 3  | 240   | $2.S_5$   | $2^2\cdot 4^1 + 1^5 - 2$ | 0 | 1 | 12 | 1 | 1 | - | $\text{Mon}_a$ |
| 205 | 3  | 240   | $2.S_5$   | $2^2\cdot 4^1 + 3 + 1^5 - 1$ | 0 | 1 | 4 | 1 | 2 | 1 | $\text{Mon}_a$ |
| 206 | 3  | 192   | $[2^3,3]$ (#1472) | $4^1 + 3^2 - 2$ | 0 | 1 | 16 | 1 | 1 | - | $\text{Mon}_a$ |
| 207 | 3  | 168   | $[2^3,3]$ (#43) | $2^1 - 3^2 + 1^2$ | 0 | 2 | 2 | 1 | 1 | 1 | $\text{Mon}_a$ |
| 208 | 3  | 144   | $[2^3,3]^2$ (#183) | $2^2\cdot 4^1 + 3 - 2$ | 0 | 1 | 4 | 1 | 1 | - | $\text{Mon}_a$ |
| 209 | 3  | 144   | $[2^3,3]^2$ (#186) | $2^1\cdot 4^1 + 3 + 1 - 1$ | 0 | 1 | 12 | 1 | 1 | - | $\text{Mon}_a$ |
| 210 | 3  | 144   | $[2^3,3]^2$ (#189) | $2^2\cdot 4^1 + 1^3 - 3$ | 0 | 1 | 48 | 1 | 1 | - | $\text{Mon}_a$ |
| 211 | 3  | 120   | $S_5^2$   | $2^2 - 3^3 - 3$ | 0 | 6 | 6 | 2 | 1 | 2 | $M_{24}$ |
| 212 | 3  | 96    | $[2^3,3]$ (#189) | $2^1\cdot 16 + 1^3 - 1$ | 0 | 1 | 2 | 1 | 2 | 1 | $\text{Mon}_a$ |
| 213 | 3  | 96    | $[2^3,3]$ (#226) | $4^1 + 2^3 + 1^3 - 1$ | 0 | 2 | 16 | 1 | 1 | 1 | $\text{Mon}_a$ |
| 214 | 3  | 72    | $[2^3,3]^2$ (#46) | $2^1\cdot 4^1 + 3 + 1^3 - 3$ | 0 | 3 | 12 | 2 | 1 | 1 | $\text{Mon}_a$ |
| 215 | 3  | 64    | $[2^3,3]$ (#131) | $4^2 + 16^1 + 1$ | 0 | 2 | 8 | 1 | 1 | 1 | $\text{Mon}_a$ |
| 216 | 3  | 64    | $[2^3,3]$ (#73) | $8^2 - 3$ | 0 | 2 | 16 | 1 | 1 | 1 | $\text{Mon}_a$ |
| 217 | 3  | 48    | $[2^3,3]$ (#48) | $2^2\cdot 4^1 + 3^3 - 3$ | 0 | 6 | 48 | 2 | 1 | 2 | $M_{24}$ |
| 218 | 3  | 48    | $[2^3,3]$ (#51) | $4^3 - 3 - 2$ | 0 | 1 | 16 | 1 | 1 | - | $\text{Mon}_a$ |
| 219 | 3  | 48    | $[2^3,3]$ (#38) | $4^1 + 3^3 - 3$ | 0 | 1 | 96 | 1 | 1 | - | $\text{Mon}_a$ |
| 220 | 3  | 42    | $[2,3,7]$ (#1) | $2^1 + 3 - 2$ | 0 | 1 | 6 | 1 | 1 | - | - |
| 221 | 3  | 24    | $[2^3,3]$ (#3) | $8^3 - 3$ | 0 | 4 | 16 | 1 | 1 | 3 | $\text{Mon}_a$ |

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| no. | rk | order | $G$          | genus | $\alpha$ | $\tau_G$ | $\tilde{\tau}_G$ | ind | $h^G$ | $N$ | type      |
|-----|----|-------|-------------|--------|----------|----------|-----------------|-----|--------|-----|-----------|
| 236 | 2  | 322560 | $2^4.A_8$   | $4^1\ 81^1$ | 0        | 2        | 2               | 1   | 1    | 1   | $M_{24}^*$ |
| 237 | 2  | 184320 | $[2^6]3$.S$_5$ | $2^2-23^2-15^{-1}$ | 0 | 1 | 2 | 1 | 1 | - | $Mon_a^*$ |
| 238 | 2  | 147456 | $[2^3]2^3$  | $8^1_2$ | 0        | 1        | 2               | 1   | 1    | 1   | $Mon_a^*$ |
| 239 | 2  | 122880 | $[2^4]A_5. S_5$ | $4_2^2-25^{-1}$ | 0 | 1 | 4 | 1 | 1 | - | $Mon_a^*$ |
| 240 | 2  | 120960 | $L_3(4).[2,3]$ | $3^1\ 2-7^1$ | 0 | 2 | 4 | 1 | 1 | 1 | $M_{24}^*$ |
| 241 | 2  | 116640 | $3^24.2.A_6.2$ | $2^2-20^1$ | 0 | 1 | 1 | 1 | 1 | - | $S$ |
| 242 | 2  | 95040  | $M_{12}$    | $2^2-23^2$ | 0 | 2 | 2 | 1 | 1 | 1 | $M_{24}^*$ |
| 243 | 2  | 80640  | $L_3(4).2^2$ | $2^2_3-14^1-17^{-1}$ | 0 | 1 | 2 | 1 | 1 | - | - |
| 244 | 2  | 73728  | $[2^1]3^3$ | $4^1_2-18^1-13^{-1}$ | 0 | 1 | 4 | 1 | 1 | - | $Mon_a^*$ |
| 245 | 2  | 58320  | $3^4.A_6.2$ | $2^2_2-23^1-9^1$ | 0 | 1 | 6 | 1 | 1 | - | $S$ |
| 246 | 2  | 40320  | $2.A_8$     | $2^2_2^2-23^1-9^1$ | 0 | 1 | 2 | 1 | 1 | - | $Mon_a^*$ |
| 247 | 2  | 36864  | $[2^2]3^2$ | $4^1_2^2-3^2$ | 0 | 1 | 8 | 1 | 1 | - | $Mon_a^*$ |
| 248 | 2  | 23040  | $[2^5].A_6.2$ | $4^1_2-18^1-9^1$ | 0 | 1 | 4 | 1 | 1 | - | $Mon_a^*$ |
| 249 | 2  | 23040  | $[2^6]3$.S$_5$ | $4^1_2^2-25^{-1}$ | 0 | 2 | 4 | 1 | 1 | 1 | $M_{24}^*$ |
| 250 | 2  | 21504  | $[2^7].L_3(7)$ | $4^1_2^1-16^1$ | 0 | 1 | 2 | 1 | 1 | - | $Mon_b^*$ |
| 251 | 2  | 15840  | $2\times M_{11}$ | $2^2_2-21^1-1^1$ | 0 | 1 | 1 | 1 | 2 | 1 | $Mon_a^*$ |
| 252 | 2  | 7920   | $M_{11}$    | $4^1_2^2-3^{-1}$ | 0 | 2 | 2 | 1 | 1 | 1 | $Mon_b^*$ |
| 253 | 2  | 7776   | $[2^2]3^5$ | $2^1_2^2-14^1-13^{-2}$ | 0 | 1 | 2 | 1 | 1 | - | $S$ |
| 254 | 2  | 6912   | $[2^3]3^3$ | $4^1_2^2-23^{-2}$ | 0 | 2 | 8 | 1 | 1 | 1 | $M_{24}^*$ |
| 255 | 2  | 5040   | $S_7$       | $2^2_2^2-25^1+17^{-1}$ | 0 | 1 | 6 | 1 | 1 | - | - |
| 256 | 2  | 5040   | $S_7$       | $2^2_2^2-3^1+17^{-1}$ | 0 | 1 | 2 | 1 | 1 | - | - |
| 257 | 2  | 2880   | $2.A_6.2^2$ | $4^1_2^2-25^1$ | 0 | 2 | 4 | 1 | 1 | 1 | $Mon_b^*$ |
| 258 | 2  | 2688   | $[2^4].L_2(7)$ | $8^2_2$ | 0 | 2 | 2 | 1 | 1 | 1 | $Mon_b^*$ |
| 259 | 2  | 2304   | $[2^6]3^2$ | $2^2_2^2-16^1-13^{-1}$ | 0 | 1 | 2 | 1 | 1 | - | $Mon_b^*$ |
| 260 | 2  | 2160   | $3.A_6.2$ | $2^2_2^2-23^1+9^{-1}$ | 0 | 6 | 6 | 1 | 2 | 1 | $M_{24}^*$ |
| 261 | 2  | 1440   | $2.A_6.2$ | $4^1_2^2-23^2$ | 0 | 2 | 8 | 1 | 1 | 1 | $Mon_b^*$ |
| 262 | 2  | 1000   | $5^1+2^2. [2^3]$ | $5^1_2^2$ | 0 | 1 | 2 | 1 | 1 | - | $S$ |
| 263 | 2  | 864    | $[2^4]3^3$ | $[4661]$ | $2^2_2^2-3^{-1}$ | 0 | 3 | 3 | 2 | 1 | 1 | $Mon_a^*$ |
| 264 | 2  | 720    | $[2^3].S_5$ | $[767]$ | $2^2_2^2-3^{-1}$ | 0 | 1 | 1 | 2 | 1 | 1 | - | - |
| 265 | 2  | 720    | $[2^3].S_5$ | $[767]$ | $2^2_2^2-3^{-1}$ | 0 | 1 | 4 | 1 | 1 | - | - |
| 266 | 2  | 576    | $[2^5]3^2$ | $[8299]$ | $4^1_2^2-20^1$ | 0 | 1 | 4 | 1 | 1 | 1 | $Mon_a^*$ |
| 267 | 2  | 432    | $[2^4]3^3$ | $[523]$ | $4^1_2^2-3^{-1}$ | 0 | 1 | 12 | 1 | 1 | 1 | $Mon_a^*$ |
| 268 | 2  | 432    | $[2^4]3^3$ | $[734]$ | $4^1_2^2-3^{-1}$ | 0 | 2 | 12 | 1 | 1 | 1 | $Mon_b^*$ |
| 269 | 2  | 384    | $[2^7]3$ | $[18127]$ | $8^1_2^1-16^1$ | 0 | 2 | 4 | 1 | 1 | 1 | $Mon_a^*$ |
| 270 | 2  | 288    | $[2^5]3^2$ | $[1028]$ | $4^1_2^2-1^1+3^{-2}$ | 0 | 1 | 8 | 1 | 1 | - | $Mon_a^*$ |
| 271 | 2  | 240    | $2.S_5$    | $4^1_2^2-2^1$ | 0 | 4 | 8 | 1 | 1 | 3 | $M_{24}^*$ |
| 272 | 2  | 80     | $[2^4]5^2$ | $[34]$ | $4^1_2^2+2^1$ | 0 | 1 | 8 | 1 | 1 | - | $Mon_a^*$ |
| 273 | 1  | $[Co_2]$ | $Co_2$ | $4^1_2^1$ | 0 | 1 | 1 | 1 | 1 | - | $S^*$ |
| 274 | 1  | $[Co_3]$ | $Co_3$ | $4^1_2^1$ | 0 | 1 | 1 | 1 | 1 | - | $S^*$ |
| 275 | 1  | 20891566080 | $2^{11}.M_{23}$ | $8^1_2$ | 0 | 1 | 1 | 1 | 1 | - | $Mon_a^*$ |
| 276 | 1  | 18393661440 | $U_6(2).2$ | $4^1_2^1-3^{-1}$ | 0 | 1 | 2 | 1 | 1 | - | $S$ |

Table 1: Orbits of fixed-point lattices
The isometry type of the lattices $\Lambda^G$ and $\Lambda_G$. The isometry class of the coinvariant lattice $\Lambda_G$ determines uniquely the orbit of $\Lambda^G$. However, isometric $\Lambda^G$ may belong to different orbits. In the following table we itemize the isometric orbits (i.e., orbits of isometric fixed-point lattices) which contain more than one orbit of fixed-point lattices.

| Rank | Sets of isometric lattices $\Lambda^G$ |
|------|--------------------------------------|
| 6    | \{34, 36\}, \{40, 49\}, \{41, 48, 56\}, \{57, 64\} |
| 5    | \{67, 71\}, \{78, 86, 91\} |
| 4    | \{104, 127\}, \{105, 113\}, \{107, 117, 132\}, \{114, 144\}, \{116, 131\}, \{140, 142\}, \{150, 161\}, \{153, 160\} |
| 3    | \{155, 156\}, \{164, 181\}, \{174, 185\}, \{176, 198\}, \{177, 190\}, \{184, 207\}, \{191, 211\}, \{192, 202\}, \{210, 217\}, \{216, 221\} |
| 2    | \{226, 236\}, \{228, 242\}, \{231, 252\}, \{238, 258\}, \{239, 249\}, \{245, 260\}, \{247, 254, 261, 271\}, \{267, 268\} |
| 1    | \{276, 278\}, \{280, 288\} |

The lattices $\Lambda^G$ and $\Lambda_G$ are isometric to each other in all three rank 12 cases.
The genus of $\Lambda^G$ and $\Lambda_G$. The genera of $\Lambda_G$ and $\Lambda^G$ determine each other. Two orbits of fixed-point lattices $\Lambda^G$ define the same genus if, and only if, they are isometric.

The isometry classes of lattices in the genus of $\Lambda^G$ have the following property: if the class belongs to a fixed-point lattice then the minimal norm is at least 4; for all other classes, the root sublattice has maximal rank. The root lattice of $\Lambda^G$ itself is zero exactly for orbits no. 1, 2, 4, 7, 18, 20, 39, 52, 53, 82, 108, 120, 128, 129, 227, 243, 251. These lattices were investigated (without explicit classification) in [Bo4]. Most of them are fixed-point lattices of conjugacy classes in $M_{23}$.

As for the isometry classes of lattices in the genus of $\Lambda_G$, if the class belongs to $\Lambda_G$ then the minimal norm is 4. For all other classes it seems that the minimal norm is 2 although the root lattice does not always has maximal rank. However, we checked this only in a small number of cases.

The entry $\alpha$. For an even lattice $L$ we define $\alpha(L) = \text{rk} L - \text{rk} A_L$. Clearly $\alpha(L) \geq 0$.

1. We have $\alpha(\Lambda^G) \geq 2$ if, and only if, $G \subsetneq M_{23}$, i.e., $G = \tilde{G}$ is a proper subgroup of the stabilizer of lattice no. 227.

2. We have $\alpha(\Lambda^G) \geq 1$ if, and only if, $G \subseteq \text{McL}$ or $G \subseteq M_{23}$, i.e., $G$ is contained in the stabilizer of either lattice no. 223 or lattice no. 227.

Niemeier lattices. Let $N$ be a Niemeier lattice in the sense that it is one of the 24 lattices in the genus of $\Lambda$. Its isometry group is a split extension $O(N) = W(N):G$, where $W(N)$ is generated by reflections in hyperplanes orthogonal to the roots of $N$. The coinvariant lattice $N_G$, which is always a lattice without norm 2 vectors, can be embedded into $\Lambda$ in such a way that $G \cong O_0(N_G) \cong O_0(\Lambda_G)$ (cf. [NiK], Remark 1.14.7, Prop. 1.14.8 and [Co3]). The following table lists the no. of the corresponding entry of $\Lambda_G$ in Table I.

| Lattice | $D_{24}$ | $E^4_8$ | $D_{18}E_8$ | $A_{24}$ | $D_{12}^2$ | $D_{10}E_7^2$ | $A_{17}E_7$ | $A_{15}D_9$ | $D_8^3$ | $A_7^3$ | $A_{11}D_7E_6$ | $E_6^4$ |
|---------|---------|---------|-------------|---------|----------|------------|------------|----------|--------|--------|-------------|--------|
| No.     | 1       | 22      | 1           | 5       | 5        | 2          | 2          | 2        | 22     | 64     | 2           | 147    |

| Lattice | $D_6^4$ | $A_5^2D_6$ | $A_5^3$ | $A_5^2D_5^2$ | $A_5^6$ | $D_5^6$ | $A_5^6D_4$ | $A_4^6$ | $A_3^8$ | $A_2^{12}$ | $A_1^{22}$ | $\Lambda$ |
|---------|---------|-----------|---------|-------------|---------|--------|-----------|--------|--------|----------|-----------|---------|
| No.     | 91      | 9         | 161     | 21          | 221     | 260    | 87        | 271    | 258    | 288      | 278       | 290     |

Conversely, to obtain all embeddings of a given $\Lambda_G$ from Table I into Niemeier lattices with roots, we determined all isometry classes of lattices $K$ in the genus of $\Lambda^G$ and all equivalence classes of extensions $K \oplus \Lambda_G$ to an unimodular lattice $N$. There is always a unique lattice $K$ providing a unique extension of $K \oplus \Lambda_G$ to the Leech lattice $\Lambda$. Column $N$ of Table I lists the number of isomorphism classes of Niemeier lattices $N$ with roots obtained in this way. If this number is positive, $G$ embeds into the group $O(N)/W(N)$ of the corresponding Niemeier lattices $N$.
Conjugacy classes of $\text{Co}_0$. There are 72 conjugacy classes $[g]$ in $\text{Co}_0$ such that $\Lambda^g \neq 0$, giving rise to 58 fixed-point lattices $\Lambda^{(g)}$ considered in [HL]. Below we list these lattices, their rank, and the index of the image of $N_{\text{Co}_0}([g])$ in $O(\Lambda^{(g)})$.

| order | 1 2 2 2 3 3 3 4 4 4 4 4 4 4 5 5 6 6 6 6 |
|-------|------------------------------------------|
| rank  | 24 8 16 12 12 6 8 8 8 6 10 4 8 6 8 4 6 6 6 6 |
| no.   | 1 14 2 5 4 35 22 14 41 9 99 21 64 20 122 35 62 33 |
| index | 1 2 1 5040 1 1 1920 240 1 1 2 36 6 1 1 1 1 1 |

| order | 6 6 6 6 6 6 6 7 8 8 8 8 8 8 8 9 9 10 10 10 |
|-------|------------------------------------------|
| rank  | 4 8 4 2 6 4 6 4 4 2 6 4 4 2 4 4 4 4 4 |
| no.   | 104 18 114 222 63 161 52 99 107 224 55 143 147 230 101 100 122 159 |
| index | 2 1 2 1 4 1 6 2 1 1 1 6 1 2 1 1 1 1 |

| order | 10 11 12 12 12 12 12 12 12 12 12 14 14 15 15 16 16 |
|-------|------------------------------------------|
| rank  | 4 4 2 4 2 2 2 4 4 4 4 2 4 2 4 2 2 2 2 |
| no.   | 149 120 222 104 228 222 231 109 123 157 135 271 129 232 128 223 224 226 |
| index | 2 1 2 1 1 1 1 1 6 1 1 1 1 1 1 1 1 1 1 |

| order | 18 18 18 20 20 20 21 22 22 23 23 24 24 24 28 30 30 30 |
|-------|------------------------------------------|
| rank  | 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 |
| no.   | 230 222 245 262 257 225 240 251 251 227 227 229 234 253 232 237 223 246 |
| index | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |

$S$-Lattices. Each of the twelve $S$-lattices $[S]$ arises as a fixed-point lattice in $\Lambda$. The type of $S$, denoted by $2^a3^b$, records the numbers $a$, $b$ of pairs of short representatives $\pm v$ for $S/2S$ of norm 4, 6 respectively. For an $S$-lattice we always have $1 + a + b = 2 \text{rk}(S)$ and $S$ is characterized up to isometry by its type. The $S$-lattices are identified in the following table.

| $S$-Lattice | $2^3 3^0$ $2^1 3^0$ $2^2 3^1$ $2^2 3^2$ $2^2 3^3$ $2^2 3^4$ $2^2 3^5$ $2^2 3^6$ $2^3 3^2$ $2^3 3^3$ $2^3 3^4$ $2^3 3^5$ $2^3 3^6$ $2^3 3^7$ $2^3 3^8$ |
|-------------|------------------------------------------|
| rank        | 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |
| no.         | 290 273 274 222 223 225 230 163 167 101 122 35 |

The stabilizer $H$ of some $S$-lattices can be extended to a stabilizer $G$ with a lower dimensional non-trivial fixed-point lattice $S'$ such that $H = O^2(G)$. The following 18 orbits arise:

| $S$-Lattice | $2^2 3^0$ $2^4 3^1$ $2^3 3^2$ $2^3 3^3$ $2^3 3^4$ $2^3 3^5$ $2^3 3^6$ $2^3 3^7$ $2^3 3^8$ |
|-------------|------------------------------------------|
| $[G/H]$     | 2 2 2 2 2 2 2 2 2 |
| rank $S'$   | 1 1 1 1 1 1 1 1 1 |
| no.         | 276 277 279 282 228 229 169 245 241 287 262 |

| $S$-Lattice | $2^2 3^2$ $2^2 3^1$ $2^3 3^0$ $2^3 3^4$ $2^3 3^5$ |
|-------------|------------------------------------------|
| $[G/H]$     | 2 2 2 2 2 2 |
| rank $S'$   | 5 4 4 3 3 3 |
| no.         | 68 109 114 186 187 191 253 |
Groups related to $2^{12}:M_{24}$. Let $G$ be the full stabilizer of a lattice such that $\Lambda^{G_2}(G)$ is not an $S$-lattice. Using inclusions $M_{23} \subseteq M_{24} \subseteq 2^{12}:M_{24} \subseteq \text{Co}_0$, we define the following five types of $G$:

- $M_{23}$: $G$ is contained in $M_{23}$ (61 cases);
- $M_{24}$: $G$ is contained in $M_{24}$ but not in $M_{23}$ (128 − 61 = 67 cases);
- $\text{Mon}_a$: $G$ is contained in $2^{12}:M_{24}$ but not in $M_{24}$ and $G = T:H$ with $H \subseteq M_{24}$ and $T = G \cap 2^{12}$ (212 − 128 = 84 cases);
- $\text{Mon}_b$: $G$ is contained in $2^{12}:M_{24}$ but not of type $\text{Mon}_a$ (250 − 212 = 38 cases);
- $-$: $G$ is not contained in $2^{12}:M_{24}$ (10 cases).

The type of each $G$ is listed in the last column of Table 1.

If $H \subseteq M_{23}$ then $\tilde{H} \subseteq M_{23}$. For $H$ is contained in $2^{11}.M_{23}$ and $M_{24}$, which are both stabilizers of rank 1 lattices, whence (with an obvious notation) $\tilde{H} \subseteq 2^{11}.M_{23} \cap M_{24} = M_{23}$. Similarly, $H \subseteq M_{24}$ implies $\tilde{H} \subseteq M_{24}$. If $H \subseteq 2^{12}:M_{24}$ but is contained in neither $2^{11}.M_{23}$ nor $M_{24}$, then $\tilde{H}$ is generally not contained in $2^{12}:M_{24}$.

Spherical Designs. The even integral lattices of minimal norm 4 for which the minimal vectors form spherical 6-designs have been classified by Martinet [Mar]. All of them can be obtained from $\Lambda$. In the nomenclature of Table 1 they are as follows: $2\mathbb{Z}$ ($\Lambda^G$ no. 273), $E_8(2)$ ($\Lambda^G$ no. 14 or $\Lambda_G$ no. 2), the Barnes-Wall lattice of rank 16 ($\Lambda^G$ no. 2 or $\Lambda_G$ no. 14), $\Lambda_{23}$ ($\Lambda_G$ no. 273), and $\Lambda$ itself.

A lattice whose minimal vectors and those of its dual form spherical 4-designs is called dual strongly perfect. Using the Molien series of their full automorphism groups, the following additional lattices can be shown to be dual strongly perfect, cf. [Ve]: $A_2$ ($\Lambda^G$ for no. 222), $D_4$ ($\Lambda^G$ for no. 99), $E_6$ ($\Lambda^G$ for no. 33, 35), one lattice of rank 10 ($\Lambda^G$ for no. 7), Coxeter-Todd lattice $K_{12}$ ($\Lambda^G \cong \Lambda_G$ for no. 4), one lattice of rank 18 ($\Lambda_G$ for no. 35), two lattices of rank 22 ($\Lambda_G$ for no. 222 and no. 223), one lattice of rank 23 ($\Lambda_G$ for no. 274).

In addition, further lattices $\Lambda^G$ and $\Lambda_G$ are rescaled versions of the above listed lattices.

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