Buoyancy-induced Lagrangian chaos: the differentially-heated cavity revisited

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Abstract. Natural convection plays a key role in fluid dynamics owing to its ubiquitous presence in nature and industry. Buoyancy-driven flows are prototypical systems in the study of thermal instabilities and pattern formation. The differentially-heated cavity problem has been widely studied for the investigation of buoyancy-induced oscillatory flow. However, far less attention has been devoted to the three-dimensional Lagrangian transport properties in such flows. This study seeks to address this by investigating Lagrangian transport in the steady flow inside differentially-heated cavities. The theoretical and numerical analysis expands on previously reported similarities between the current flow and lid-driven flows. First results reveal that the convective terms in the momentum and energy balances cause non-trivial (and potentially chaotic) Lagrangian transport.

1. Introduction

Natural convection is a canonical subject of investigation in nonlinear dynamics and the various types of buoyancy mechanisms (stemming from different heating and boundary conditions) are archetypal systems in the study of complex phenomena [1]. In particular, the differentially-heated cavity (i.e., no-slip walls with adiabatic horizontal boundaries and isothermal vertical sides) has been important in the study of the onset of unsteadiness and turbulence. This configuration is a prototypical benchmark for numerical simulations and is highly related to applications such as crystal growth, solar collectors, ventilation and cooling [2, 3]. A considerable amount of literature has been published on numerical and experimental heat transfer properties in steady and oscillatory states of such flows in two-dimensional (2D) cavities. However, there is still but a few studies for the case of three-dimensional (3D) domains. Pioneering work on the 3D effects for different box sizes and governing parameters (Prandtl (Pr) and Rayleigh (Ra) numbers) is found in the numerical investigation by Mallinson and de Vahl Davis [4].

Relating dynamical systems theory to fluid dynamics has provided a solid framework for theoretical, computational and experimental studies on Lagrangian transport phenomena over the past few decades. This has revealed important differences between 2D and 3D features and many challenges remain in the latter case, see [5–8] and references therein. Recently, the impact of weak perturbations from the non-inertial limit in the 3D lid-driven cylinder flow has been considered [5, 6, 9, 10], and new types of Lagrangian coherent structures, that have implications for transport phenomena, have been shown in simulations and experiments. Similar properties...
have been found in other systems [11], suggesting that the lid-driven cylinder flow captures the generic features of 3D advection and, thus, is representative of Lagrangian transport in a wide range of realistic 3D flows.

In a recent study, Contreras et al. [12] described the topology of Lagrangian fluid trajectories in a 3D time-periodic flow composed of reorientations of the flow produced by differentially heating (or cooling) opposite vertical walls of a cubic cavity (while the others are thermally insulated). The authors considered the linear problem (i.e., the governing equations without the convective terms) and noted important similarities with other systems studied in the field of fluid mixing, granular flows and volume-preserving systems and, specifically with the previously mentioned lid-driven cylinder flow studied in [6, 8–10, 13].

This work expands on the topological analysis of the natural convective system in square and cubical differentially-heated enclosures. The study aims at deepening the understanding of the impact of nonlinear (convective) terms on 3D Lagrangian transport in natural convective systems for steady flow at moderate values of Ra (i.e., in the laminar regime). The discussion is organized as follows. Section 2 introduces the model of the differentially-heated cavity flow. The numerical analysis is elaborated in Section 3 for the linear and nonlinear problems making emphasis on the streamlines of the flow. Conclusions are in Section 4.

2. Flow model

We consider the flow inside 2D and 3D differentially-heated cavities (i.e., square and cubic, respectively). The working fluid is Newtonian and the system is subject to gravity. The governing equations for the steady problem in non-dimensional form in the Boussinesq approximation are [3, 14–16]:

\[
\frac{\partial u_j}{\partial x_j} = 0, \\
\quad \quad (1)
\]

\[
u \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + Pr \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i, \quad (\vec{b} = (0, 0, PrRaT)) \\
\quad \quad (2)
\]

\[
u \frac{\partial T}{\partial x_j} = \frac{\partial^2 T}{\partial x_j \partial x_j}. \\
\quad \quad (3)
\]

In these expressions \(x_i, u_i \) and \( b_i \) are the \(i\)-components of the position, velocity and the body force, respectively, \(p\) is the pressure and \(T\) the temperature. The non-dimensional parameters are the Prandtl and Rayleigh numbers defined by

\[
Pr = \frac{\nu}{\alpha} \quad \text{and} \quad Ra = \frac{g \beta \Delta T H^3}{\alpha \nu}, \\
\quad \quad (4)
\]

where \(g\) is the acceleration of gravity, \(\Delta T\) is the temperature difference, \(H\) is the vertical characteristic length of the container, \(\beta\) is the coefficient of volume expansion, \(\alpha\) is the thermal diffusivity and \(\nu\) is the kinematic viscosity. The scales for velocity and pressure in (2)-(3) are \(\alpha/H\) and \(\rho \alpha^2/\rho H^2\), respectively. Gravity acts in the negative z-axis and the reference frame is located on the lower left corner (Fig. 1). The fluid motion is generated by the differential heating of two opposite vertical walls of the cavity: the temperature of the right wall (located at \(x = 1\)) is set at a temperature, \(T = 1\), and the opposite wall (at \(x = 0\)) is set at \(T = 0\). All other walls are thermally insulated.

The flow and temperature fields are simulated by the commercial CFD package COMSOL Multiphysics using standard meshing, numerical models and solver settings. Regarding this work, an important feature of COMSOL is the possibility of turning off the inertial terms in the momentum conservation equations (creeping flows) and/or the convective terms in the energy equation. This enables detailed analysis of the departure of the flow from the linear case due to nonlinearities induced by inertia and/or convection.
3. Numerical analysis

3.1. Linear problem

Our starting point is the linear problem, that is, the conservation equations without the inertial and convective terms. In this approximation the temperature field, unaltered by the fluid motion, drives the velocity field. This approximation has been considered in the slow motion of viscous fluids in rectangular and cylindrical enclosures [12, 17, 18]. The equations of motion (1)-(3) become

$$\frac{\partial u_j}{\partial x_j} = 0,$$

(5)

$$\frac{\partial p}{\partial x_i} = Pr \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i, \quad (b_i = (0, 0, Pr Ra T))$$

(6)

and

$$\frac{\partial^2 T}{\partial x_j \partial x_j} = 0.$$  

(7)

With the given boundary conditions, the temperature field is a linear function of $x$, see Fig. 2(a). In this case, the velocity field is determined by the difference between the body force term and the pressure gradient

$$\nabla^2 \vec{u} = \frac{1}{Pr} (\nabla p - \vec{b}).$$

(8)

Fig. 2(b) shows the typical closed streamlines of the linear problem. A symmetric velocity field is found in this case, see [12] and Fig. 4(a).

3.2. Response to nonlinear perturbations

There are two natural extensions to introduce nonlinear phenomena into the linear problem: (i) the inclusion of the inertial effects in the momentum conservation equations; (ii) the convective term in the energy conservation equation. We start by discussing the fully nonlinear problem (i.e., convective terms in both energy and momentum equations (2)-(3)), and subsequently the role of each of the terms separately. Typical examples are shown to illustrate the different behaviors.

Figs. 2(c)-(d) show the temperature field and typical streamlines in the fully nonlinear problem for the 3D cavity. The streamlines now describe toroidal structures and the 2D closed streamlines of the linear case (Fig. 2(b)) are no longer present. Similar toroidal structures in the 3D differentially-heated cavity are found in [4, 19] for different values of the parameters. The toroidal structure described by the streamlines becomes apparent in the cross-sectional view of their intersections with the plane $x = 0.5$ shown in Fig. 3 for three different initial positions in the half of the cavity $y > 0.5$. Important to note is that perturbation of the Stokes limit in
Figure 2. Pr = 5. Temperature distribution (grey scale) (a), (c) and typical streamlines (b), (d). The initial tracer positions are (0.2, 0.6, 0.2) and (0.2, 0.4, 0.2). (a)-(b) Linear problem, Ra = 1 × 10^5. (c)-(d) Nonlinear problem, Ra = 3 × 10^3.

the lid-driven cylinder flow by fluid inertia has the same effect on the streamlines [8, 13]. This reveals the essential similarity between that flow and our natural convective flow.

Figure 3. Ra = 7 × 10^2 and Pr = 5. Cross-sectional view in the plane x = 0.5 for three different initial tracer positions (0.2, 0.53, 0.2), (0.2, 0.6, 0.2), (0.2, 0.7, 0.2).

In order to have a first idea of the role of the nonlinear terms in the governing conservation equations, Fig. 4 shows the velocity field in the 2D cavity for the linear problem and nonlinear cases for Ra = 7 × 10^3 and Pr = 5. Fig. 4(a) and Fig. 4(b) show the linear and nonlinear cases,
respectively. The changes introduced by the convective terms in the momentum equation and energy equation are shown in Fig. 4(c) and Fig. 4(d), respectively. Note that the convective term in the energy equation (Fig. 4(d)) produces a similar qualitative behavior of the velocity field as in the fully nonlinear problem (Fig. 4(b)). While the nonlinear terms in the momentum equation have a much smaller effect in the central region of the cavity (Fig. 4(c)). This can be understood from the different role of the two terms in the equations: the convective term in the energy equation induces nonlinearity in the temperature field and then, via buoyancy, automatically also in the velocity field. Hence, the convective term in the energy equation results in overall nonlinearity. However, the convective term in the momentum equation first of all induces nonlinearity only in the velocity field. This gives nonlinearity in the temperature field only if the convective term in the energy equation is also non-zero (two-way coupling). Thus a nonlinear temperature field always gives a nonlinear velocity field. However, the converse is not true, i.e. a nonlinear velocity field can coexist with a linear temperature field in case convective heat transfer is zero (one-way coupling only via buoyancy).

![Figure 4](image.png)

**Figure 4.** $Ra = 7 \times 10^4$ and $Pr = 5$. Velocity field in the 2D cavity (arrow length represents the velocity magnitude, different proportionality factors were used in each case seeking the clarity of the figures). (a) Linear problem; (b) nonlinear problem; (c) convective terms in the momentum equation; (d) convective term in the energy equation.

Fig. 5 shows the behavior of the 3D streamlines when the different nonlinearities are present, that is, the inertial terms in the momentum equations (Fig. 5(a)), and the convective term in
Figure 5. $Ra = 3 \times 10^3$. Toroidal trajectories resulting from the presence of the different nonlinear terms. The initial tracer position is $(0.2, 0.6, 0.2)$. (a) Convective terms in the momentum equation. (b) Convective term in the energy equation.

The energy equation (Figs. 5(b)). Note that both nonlinearities cause the transition of closed streamlines to toroidal structures as in the fully nonlinear problem.

We consider the behavior of the streamlines for two Prandtl numbers to consider the cases $Pr > 1$ and $Pr \approx 1$. Fig. 6 and Fig. 7 show the cross-sectional views in the plane $x = 0.5$ for three initial conditions and $Pr = 5$ (water) and $Pr = 0.71$ (air), respectively, in the fully nonlinear problem and $Ra = 3 \times 10^3$. In essence the two cases display a similar behavior, though the inner torus in the case of air is slightly bigger and the outer trajectory shows a stronger departure from the well-defined circular-type patterns (Fig. 3). Fig. 8 and Fig. 9 show the cross-sectional views for one tracer for $Ra = 1 \times 10^4$. Tori exhibit a more complex structure with increasing values of $Ra$. Essentially similar behavior has been found in the lid-driven cylinder flow, tori are found and the evolution in the cross-sectional views follow a Poincaré-Birkhoff mechanism for
increasing Reynolds number \[8, 13\]. This further underscores the similarity between both flows. A deeper analysis needs to be done to conclusively determine if chaotic behavior is also present in the natural convective system and this is the subject of ongoing research.

![Graph](image1)

**Figure 8.** \(Ra = 1 \times 10^4, Pr = 5\). Cross-sectional view in the plane \(x = 0.5\), initial tracer position is \((0.2, 0.53, 0.4)\).

**Figure 9.** \(Ra = 1 \times 10^4, Pr = 0.71\). Cross-sectional view in the plane \(x = 0.5\), initial tracer position is \((0.2, 0.53, 0.4)\).

In Fig. 10 we show a side view of the toroidal streamlines induced by the inertial and convective terms for a fixed \(Ra\) and the two considered \(Pr\).

![Graph](image2)

**Figure 10.** Toroidal streamlines stemming from the different convective terms. (a)-(c) \(Pr = 5\); (d)-(f) \(Pr = 0.71\). \(Ra = 6 \times 10^3\), the initial tracer position is \((0.2, 0.53, 0.2)\) in all cases. (a), (d) convective terms in the momentum and energy equations; (b), (e) convective term in the energy equation; (c), (f) convective terms in the momentum equation.
4. Conclusions

Toroidal structures and spiraling streamlines in natural convective flows have been reported in the past for different cavity aspect ratios, and different ranges of Pr and Ra [4, 19]. In this work we show that the different nonlinear terms of the governing equations produce toroidal trajectories and they behave as those found in other systems. Similarities between the nonlinear Lagrangian phenomena arising in this thermally-driven flow and the lid-driven cylinder flow [8–10, 13] have been observed. The main difference between these flows is that in the thermally-driven system there are two natural mechanisms that may induce nonlinear phenomena: (i) inertial effects in the momentum conservation equations; (ii) convective term in the energy conservation equation. In the lid-driven flow only the former is possible. However, despite different mechanism, the Lagrangian transport properties are essentially the same. The study of perturbations of the time-periodic Stokes flows in the lid-driven cylinder system has revealed important mechanisms of Lagrangian transport phenomena. There are indications that the response to nonlinear weak perturbations is generic in the sense that the formation of intricate coherent structures is independent of the nature of the perturbation [10]. An important open question in this context is the universality of the response to perturbations, whether similar behavior can be induced by thermal effects, etc. In this work we give first evidence that indeed the response of the steady flow to nonlinear perturbations is similar.

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