Quantum seaweed algebras and quantization of affine Cremmer–Gervais $r$-matrices

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Abstract

We propose a method of quantization of certain Lie bialgebra structures on the polynomial Lie algebras related to quasi-trigonometric solutions of the classical Yang–Baxter equation. The method is based on an affine realization of certain seaweed algebras and their quantum analogues. We also propose a method of $\omega$-affinization, which enables us to quantize rational $r$-matrices of $\mathfrak{sl}(3)$.

1 Introduction

The aim of this paper is to propose a method of quantizing certain Lie bialgebra structures on polynomial Lie algebras. In the beginning of 90-th Drinfeld in [6] posed the following problem: can any Lie bialgebra be quantized? The problem was solved by Etingof and Kazhdan and the answer was positive. However, another problem of finding explicit quantization formulas remains open.

First results in this direction were obtained in [15], [7] and [9], where the authors quantized the so-called Belavin–Drinfeld list, the list of all quasi-triangular Lie bialgebra structures on finite dimensional simple Lie algebras. It should be noticed that first infinite-dimensional cases were considered in [11]. However, a real break-through in the infinite dimensional case came in [12] and [17], where deformed versions of Yangians $Y(\mathfrak{sl}_N)$ and quantum affine algebras $U_q(\mathfrak{sl}_N)$ were constructed for $N = 2, 3$. In the present paper we solve the problem in the case $U_q(\mathfrak{sl}_N)$. Our solution of the problem is based on a $q$-version of the so-called seaweed algebras. A seaweed subalgebra of $\mathfrak{sl}_N$ is an intersection of two parabolic subalgebras one of which containing the Borel subalgebra $B^+$ and another $B^-$. The case when both parabolic subalgebras are maximal was studied in [19] in connection with the study of rational solutions of the classical Yang-Baxter equation. In particular, a complete answer to the question when such an algebra is Frobenius was obtained there. Later in [4] it was found out when an arbitrary seaweed algebra is Frobenius.
In the present paper we quantize a Lie bialgebra, which as a Lie algebra is \( \mathfrak{gl}_N[u] \), \( (\mathfrak{sl}_N[u]) \). Its coalgebra structure is defined by a quasi-trigonometric solution of the classical Yang-Baxter equation. Quasi-trigonometric solutions were introduced in \cite{10}. We remind briefly some results obtained there.

For convenience we consider the case \( \mathfrak{g} = \mathfrak{gl}_N \) although the results are also valid for an arbitrary simple complex Lie algebra \( \mathfrak{g} \). Let \( e_{ij}, i, j = 1, \ldots, N \), be the standard Cartan-Weyl basis of \( \mathfrak{gl}_N \): \( [e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj} \). The element \( C_2 := \sum_{i,j=1}^{N} e_{ij}e_{ji} \in U(\mathfrak{gl}_N) \) is a \( \mathfrak{gl}_N \)-scalar, i.e. \( [C_2, x] = 0 \) for any \( x \in \mathfrak{gl}_N \), and it is called the second order Casimir element. The element \( \Omega := \frac{1}{2} \left( \Delta(C_2) - C_2 \otimes 1 - 1 \otimes C_2 \right) = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \subset U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_N) \), where \( \Delta \) is a trivial co-product \( \Delta(x) = x \otimes 1 + 1 \otimes x \) \( \forall x \in \mathfrak{gl}_N \), is called the Casimir two-tensor. The two-tensor can be represented in the form \( \Omega = \Omega_+ + \Omega_- \), where \( \Omega_+ = \frac{1}{2} \sum_{1 \leq i \leq N} e_{ii} \otimes e_{ii} + \sum_{1 \leq i < j \leq N} e_{ij} \otimes e_{ji} \) and \( \Omega_- = \frac{1}{2} \sum_{1 \leq i \leq N} e_{ii} \otimes e_{ii} + \sum_{1 \leq i < j \leq N} e_{ji} \otimes e_{ij} \). Note that \( (\omega \otimes \omega)(\Omega_+) = \Omega_-, \) where \( \omega \) is the Cartan automorphism: \( \omega(e_{ij}) = -e_{ji} \).

We say that a solution \( X(u, v) \) of the classical Yang–Baxter equation is quasi-trigonometric if it is of the form:

\[
X(u, v) = \frac{u\Omega_+ + v\Omega_-}{u - v} + p(u, v),
\]

where \( p(u, v) \) is a non-zero polynomial with coefficients in \( \mathfrak{gl}_N \otimes \mathfrak{gl}_N \). If \( p(u, v) = 0 \) then \( X(u, v) \) is the simplest (standard) trigonometric \( r \)-matrix. Any quasi-trigonometric solution of the classical Yang-Baxter equation defines a Lie bialgebra structure on \( \mathfrak{gl}_N[u] \) and the corresponding Lie cobracket on \( \mathfrak{gl}_N[u] \) is given by the formula

\[
\{ A(u) \in \mathfrak{gl}_N[u] \} \rightarrow \{ [X(u, v), A(u) \otimes 1 + 1 \otimes A(v)] \in \mathfrak{gl}_N[u] \otimes \mathfrak{gl}_N[v] \}.
\]

It was proved in \cite{10} that for \( \mathfrak{g} = \mathfrak{sl}_N(\mathfrak{gl}_N) \) there is a one-to-one correspondence between quasi-trigonometric \( r \)-matrices and Lagrangian subalgebras of \( \mathfrak{g} \oplus \mathfrak{g} \) transversal to a certain Lagrangian subalgebra of \( \mathfrak{g} \oplus \mathfrak{g} \) defined by a maximal parabolic subalgebra of \( \mathfrak{g} \). Here we mean the Lagrangian space with respect to the following symmetric non-degenerate invariant bilinear form on \( \mathfrak{g} \oplus \mathfrak{g} \):

\[
Q((a, b), (c, d)) = K(a, c) - K(b, d),
\]

where \( K \) is the Killing form and \( a, b, c, d \in \mathfrak{g} \).

In their famous paper on the classical Yang–Baxter equation, Belavin and Drinfeld listed all the trigonometric \( r \)-matrices. In case \( \mathfrak{sl}_3 \) there exist 4 trigonometric \( r \)-matrices. Two of them relate to the quasi-triangular constant \( r \)-matrices and can be quantized using methods from \cite{7}, \cite{9} and \cite{15}. In our paper we explain how to to quantize one of the two remaining trigonometric \( r \)-matrices found by Belavin and Drinfeld. We also notice that in cases \( \mathfrak{sl}_2 \) and \( \mathfrak{sl}_3 \) our methods lead to quantization of some rational \( r \)-matrices found in \cite{12}. As it was explained in \cite{13}, quantization of the rational \( r \)-matrix for \( \mathfrak{sl}_2 \) has close relations with the Rankin–Cohen brackets for modular forms (see \cite{11}).

Our method is based on finding quantum twists, which are various solutions of the so-called cocycle equation for a number of Hopf algebras:

\[
F^{12}(\Delta \otimes \text{id})(F) = F^{23}(\text{id} \otimes \Delta)(F).
\]

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2 A quantum seaweed algebra and its affine realization

It turns out that it is more convenient to use instead of the simple Lie algebra $\mathfrak{sl}_N$ its central extension $\hat{\mathfrak{sl}}_N$. The polynomial affine Lie algebra $\hat{\mathfrak{sl}}_N[u]$ is generated by Cartan–Weyl basis $e_{ij}^{(n)} := e_{ij}u^n$ ($i, j = 1, 2, \ldots, N$, $n = 0, 1, 2, \ldots$) with the defining relations

$$[e_{ij}^{(n)}, e_{kl}^{(m)}] = \delta_{jk} e_{il}^{(n+m)} - \delta_{il} e_{kj}^{(n+m)}.$$  

The total root system $\Sigma$ of the Lie algebra $\hat{\mathfrak{sl}}_N[u]$ with respect to an extended Cartan subalgebra generated by the Cartan elements $e_{ii}$ ($i = 1, 2, \ldots, N$) and $d = u(\partial/\partial u)$ is given by

$$\Sigma(\hat{\mathfrak{sl}}_N[u]) = \{\epsilon_i - \epsilon_j, n\delta + \epsilon_i - \epsilon_j, n\delta \mid i \neq j; i, j = 1, 2, \ldots, N; n = 1, 2, \ldots\},$$

where $(\epsilon_i, (\epsilon_j) = \delta_{ij}$, and we have following correspondence between root vectors and generators of $\hat{\mathfrak{sl}}_N[u]: e_{ij}^{(n)} = e_{n\delta+i-j}$ for $i \neq j, n = 0, 1, 2, \ldots$. It should be noted that the multiplicity of the roots $n\delta$ is equal to $N$, $\text{Mult}(n\delta) = N$, and the root vectors $e_{ii}^{(n)} (i = 1, 2, \ldots, N)$ "split" this multiplicity. We choose the following system of positive simple roots:

$$\Pi(\hat{\mathfrak{sl}}_N[u]) = \{\alpha_i := \epsilon_i - \epsilon_{i+1}, \alpha_0 := \delta + \epsilon_N - \epsilon_1 \mid i = 1, 2, \ldots, N - 1\}.$$  

Let $\hat{\mathfrak{sw}}_{N+1}$ be a subalgebra of $\hat{\mathfrak{sl}}_{N+1}$ generated by the root vectors: $e_{21}, e_{i,i+1}, e_{i+1,i}$ for $i = 2, 3, \ldots, N$ and $e_{N,N+1}$, and also by the Cartan elements: $e_{11} + e_{22}, e_{ii}$ for $i = 2, 3, \ldots, N$. It is easy to check that $\hat{\mathfrak{sw}}_{N+1}$ has structure of a seaweed Lie algebra (see [4]).

Let $\hat{\mathfrak{sw}}_N$ be a subalgebra of $\hat{\mathfrak{sl}}_N[u]$ generated by the root vectors: $e_{21}^{(0)}, e_{i,i+1}^{(0)}, e_{i+1,i}^{(0)}$ for $i = 2, 3, \ldots, N$ and $e_{N,1}^{(1)}$, and also by the Cartan elements: $e_{ii}^{(0)}$ for $i = 1, 2, 3, \ldots, N$. It is easy to check that the Lie algebras $\hat{\mathfrak{sw}}_N$ and $\hat{\mathfrak{sw}}_{N+1}$ are isomorphic. This isomorphism is described by the following correspondence: $e_{i+1,i}^{(0)} \leftrightarrow e_{i+1,i}^{(0)}$ for $i = 1, 2, \ldots, N - 1$, $e_{i,i+1}^{(0)} \leftrightarrow e_{i+1,i}^{(0)}$ for $i = 2, 3, \ldots, N - 1$, $e_{N,N+1}^{(1)} \leftrightarrow e_{N,1}^{(1)}$, $e_{11}^{(1)}$ for $i = 2, 3, \ldots, N$. We will call $\hat{\mathfrak{sw}}_N$ an affine realization of $\hat{\mathfrak{sw}}_{N+1}$.

Now let us consider $q$-analogs of the previous Lie algebras. The quantum algebra $U_q(\mathfrak{sl}_N)$ is generated by the Chevalley elements\footnote{We denote the generators in the classical and quantum cases by the same letter "$e". It should not cause any misunderstanding.} $e_{i,i+1}, e_{i+1,i}$ ($i = 1, 2, \ldots, N - 1$), $q^{\pm \epsilon_{ii}}$.
According to this ordering we set
\[ q^{e_{ii}} q^{-e_{ii}} = q^{-e_{ii}} q^{e_{ii}} = 1 \, , \]
\[ q^{e_{ii}} q^{e_{jj}} = q^{e_{jj}} q^{e_{ii}} \, , \]
\[ q^{e_{ii}} e_{jk} q^{-e_{ii}} = q^{\delta_{j-k} e_{jk}} \quad (|j - k| = 1) \, , \]
\[ [e_{i,i+1}, e_{j+1,j}] = \delta_{ij} \frac{q^{e_{ii}} - q_{i+1,i+1} - q_{i+1,i+1} - e_{ii}}{q - q^{-1}} \, , \] (8)
\[ [e_{i,i+1}, e_{j,j+1}] = 0 \quad \text{for} \quad |i - j| \geq 2 \, , \]
\[ [e_{i+1,i}, e_{j+1,j}] = 0 \quad \text{for} \quad |i - j| \geq 2 \, , \]
\[ [[e_{i,i+1}, e_{j,j+1}]q, e_{j,j+1}] = 0 \quad \text{for} \quad |i - j| = 1 \, , \]
\[ [[e_{i+1,i}, e_{j+1,j}]q, e_{j+1,j}] = 0 \quad \text{for} \quad |i - j| = 1 \, . \]

where \([e_\beta, e_\gamma]_q\) denotes the \(q\)-commutator:
\[ [e_\beta, e_\gamma]_q := e_\beta e_\gamma - q^{(\beta,\gamma)} e_\gamma e_\beta \, . \] (9)

The Hopf structure on \(U_q(\mathfrak{gl}_N)\) is given by the following formulas for \(q\)-commutation \(\Delta_q\), antipode \(S_q\), and \(q\)-co-unit \(\epsilon_q\):
\[ \Delta_q(q^{\pm e_{ii}}) = q^{\pm e_{ii}} \otimes q^{\pm e_{ii}} \, , \]
\[ \Delta_q(e_{i,i+1}) = e_{i,i+1} \otimes 1 + q^{e_{i+1,i+1} - e_{ii}} \otimes e_{i,i+1} \, , \] (10)
\[ \Delta_q(e_{i+1,i}) = e_{i+1,i} \otimes q^{e_{ii} - e_{i+1,i+1} + 1} \otimes e_{i,i+1} \, ; \]
\[ S_q(q^{\pm e_{ii}}) = q^{\mp e_{ii}} \, , \]
\[ S_q(e_{i,i+1}) = -q^{e_{ii} - e_{i+1,i+1}} e_{i,i+1} \, , \] (11)
\[ S_q(e_{i+1,i}) = -e_{i+1,i} q^{e_{i+1,i+1} - e_{ii}} \, ; \]
\[ \epsilon_q(q^{\pm e_{ii}}) = 1 \, , \quad \epsilon_q(e_{ij}) = 0 \quad \text{for} \quad |i - j| = 1 \, . \] (12)

For construction of the composite root vectors \(e_{ij}\) for \(|i - j| \geq 2\) we fix the following normal ordering of the positive root system \(\Delta_+\) (see 21 [13] [14])
\[ \epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_2 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4 < \ldots < \]
\[ \epsilon_1 - \epsilon_k < \epsilon_2 - \epsilon_k < \ldots < \epsilon_{k-1} - \epsilon_k < \ldots < \epsilon_1 - \epsilon_N < \epsilon_2 - \epsilon_N < \ldots < \epsilon_{N-1} - \epsilon_N \, . \] (13)

According to this ordering we set
\[ e_{ij} := [e_{ik}, e_{kj}]_{q^{-1}} \, , \quad e_{ji} := [e_{jk}, e_{ki}]_q \, , \] (14)
where \(1 \leq i < k < j \leq N\). It should be stressed that the structure of the composite root vectors is not independent on a choice of the index \(k\) in the r.h.s. of the definition (14).

In particular, we have
\[ e_{ij} := [e_{i,i+1}, e_{i+1,j}]_{q^{-1}} = [e_{i,i+1}, e_{i+1,j}]_{q^{-1}} \, , \]
\[ e_{ji} := [e_{j,i+1}, e_{i+1,i}]_q = [e_{j,i+1}, e_{i+1,i}]_q \, , \] (15)
where \(2 \leq i + 1 < j \leq N\).
Using these explicit constructions and the defining relations \([5]\) for the Chevalley basis it is not hard to calculate the following relations between the Cartan–Weyl generators \(e_{ij}\) \((i, j = 1, 2, \ldots, N)\):

\[
q^{e_{kk}}e_{ij}q^{-e_{kk}} = q^{\delta_{ki} - \delta_{kj}}e_{ij} \quad (1 \leq i, j, k \leq N) ,
\]

\[
[e_{ij}, e_{ji}] = q^{e_{ii} - e_{jj}} - q^{e_{jj} - e_{ii}} \quad (1 \leq i < j \leq N) ,
\]

\[
[e_{kl}, e_{ij}] = \delta_{jk}e_{il} \quad (1 \leq i < j < k \leq l \leq N) ,
\]

\[
[e_{ik}, e_{jl}] = (q - q^{-1})e_{jk}e_{il} \quad (1 \leq i < j < k < l \leq N) ,
\]

\[
[e_{kl}, e_{ij}] = 0 \quad (1 \leq i \leq j < k \leq l \leq N) ,
\]

\[
[e_{it}, e_{kj}] = 0 \quad (1 \leq i \leq j < k \leq l \leq N) ,
\]

\[
[e_{ji}, e_{it}] = e_{jt}q^{e_{ji} - e_{jj}} \quad (1 \leq i < j < l \leq N) ,
\]

\[
[e_{kt}, e_{it}] = e_{ki}q^{e_{kk} - e_{it}} \quad (1 \leq i < k < l \leq N) ,
\]

\[
[e_{jt}, e_{ki}] = (q^{-1} - q)e_{kt}e_{ji}q^{e_{jt} - e_{kk}} \quad (1 \leq i < j < k < l \leq N) .
\]

These formulas can be obtained from the relations between elements of the Cartan–Weyl algebra \(U\). The Hopf structure of \(U\) is inherited from the quantum algebras \(U_q(\mathfrak{gl}(N))\) \((N \geq 3)\) is generated (as an unital associative algebra over \(\mathbb{C}[[\log q]]\)) by the algebra \(U_q(\mathfrak{gl}(N))\) and the additional element \(e_{N1}^{(1)}\) with the relations:

\[
q^z e_{N1}^{(0)} e_{N1}^{(1)} = q^{z(\delta_{11} - \delta_{NN})} e_{N1}^{(1)} q^z e_{N1}^{(0)} ,
\]

\[
[e_{i,i+1}^{(0)}, e_{N1}^{(1)}] = 0 \quad \text{for } i = 2, 3, \ldots, N - 2 ,
\]

\[
[e_{1+i,i}^{(0)}, e_{N1}^{(1)}] = 0 \quad \text{for } i = 1, 2, \ldots, N - 1 ,
\]

\[
[e_{12}^{(0)}, e_{12}^{(0)}, e_{N1}^{(1)}]_{q} = 0 ,
\]

\[
[e_{N-1,N}^{(0)}, e_{N-1,N}^{(0)}, e_{N1}^{(1)}]_{q} = 0 ,
\]

\[
[[e_{12}^{(0)}, e_{N1}^{(1)}], e_{N1}^{(1)}]_{q} = 0 ,
\]

\[
[[e_{N-1,N}^{(0)}, e_{N1}^{(1)}], e_{N1}^{(1)}]_{q} = 0 .
\]

The Hopf structure of \(U_q(\mathfrak{gl}(N[u]))\) is defined by the formulas \([10],[12]\) for \(U_q(\mathfrak{gl}(N))\) and the following additional formulas for the comultiplication and the antipode:

\[
\Delta_q(e_{N1}^{(1)}) = e_{N1}^{(1)} \otimes 1 + q^{e_{11}^{(0)} - e_{NN}^{(0)}} \otimes e_{N1}^{(1)} ,
\]

\[
S_q(e_{N1}^{(1)}) = -q^{e_{11}^{(0)} - e_{NN}^{(0)}} e_{N1}^{(1)} .
\]

Quantum analogs of the seaweed algebra \(\mathfrak{sw}_{N+1}\) and its affine realization \(\hat{\mathfrak{sw}}_N\) are inherited from the quantum algebras \(U_q(\mathfrak{gl}(N+1))\) and \(U_q(\mathfrak{gl}(N[u]))\). Namely, the quantum algebra \(U_q(\mathfrak{sw}_{N+1})\) is generated by the root vectors: \(e_{21}, e_{i,i+1}, e_{i+1,i}\) for \(i = 2, 3, \ldots, N\)
and $e_{N,N+1}$, and also by the $q$-Cartan elements: $q^{e_{i+1}+e_{N+1,i+1}}$, $q^{e_{i,i+1}}$ for $i = 2, 3, \ldots, N$ with the relations satisfying (33). Similarly, the quantum algebra $U_q(\mathfrak{sw}_N)$ is generated by the root vectors: $e^{(0)}_{21}$, $e^{(0)}_{i,i+1}$, $e^{(0)}_{i+1,i}$ for $i = 2, 3, \ldots, N$ and $e^{(1)}_{N,1}$, and also by the $q$-Cartan elements: $q^{e^{(0)}_{ii}}$ for $i = 1, 2, 3, \ldots, N$ with the relations satisfying (33) and (34). It is clear that the algebras $U_q(\mathfrak{gl}_{N+1})$ and $U_q(\mathfrak{sw}_N)$ are isomorphic as associative algebras but they are not isomorphic as Hopf algebras. However if we introduced a new coproduct in the Hopf algebra $U_q(\mathfrak{gl}_{N+1})$

$$\Delta_{\tilde{\mathfrak{g}},N+1}(x) = \tilde{\mathfrak{g}}_{1,N+1}\Delta_q(x)\tilde{\mathfrak{g}}^{-1}_{1,N+1} \quad (\forall x \in U_q(\mathfrak{gl}_{N+1})),$$

(29)

where

$$\tilde{\mathfrak{g}}_{1,N+1} := q^{-e_{11}\otimes e_{N+1,N+1}},$$

(30)

we obtain an isomorphism of Hopf algebras

$$U_q(\tilde{\mathfrak{g}},N+1)(\mathfrak{sw}_{N+1}) \simeq U_q(\mathfrak{sw}_N).$$

(31)

Here the symbol $U_q(\tilde{\mathfrak{g}},N+1)(\mathfrak{sw}_{N+1})$ denotes the quantum seaweed algebra $U_q(\mathfrak{sw}_{N+1})$ with the twisted coproduct (20).

### 3 Cartan part of Cremmer-Gervais $r$-matrix

First of all we recall classification of quasi-triangular $r$-matrices for a simple Lie algebra $\mathfrak{g}$. The quasi-triangular $r$-matrices are solutions of the system

$$r_{12}^2 + r_{21}^2 = \Omega,$$  

$$[r_{12}^1, r_{13}^1] + [r_{12}^1, r_{23}^1] + [r_{13}^1, r_{23}^1] = 0,$$

(32)

where $\Omega$ is the quadratic the Casimir two-tensor in $\mathfrak{g} \otimes \mathfrak{g}$. Belavin and Drinfeld proved that any solution of this system is defined by a triple $(\Gamma_1, \Gamma_2, \tau)$, where $\Gamma_1, \Gamma_2$ are subdiagrams of the Dynkin diagram of $\mathfrak{g}$ and $\tau$ is an isometry between these two subdiagrams. Further, each $\Gamma_i$ defines a reductive subalgebra of $\mathfrak{g}$, and $\tau$ is extended to an isometry (with respect to the corresponding restrictions of the Killing form) between the corresponding reductive subalgebras of $\mathfrak{g}$. The following property of $\tau$ should be satisfied: $\tau^k(\alpha) \notin \Gamma_1$ for any $\alpha \in \Gamma_1$ and some $k$. Let $\Omega_0$ be the Cartan part of $\Omega$. Then one can construct a quasi-triangular $r$-matrix according to the following

**Theorem 1 (Belavin–Drinfeld [2]).** Let $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfies the systems

$$r_{01}^2 + r_{02}^2 = \Omega_0,$$

(33)

$$(\alpha \otimes 1 + 1 \otimes \alpha)(r_0) = h_\alpha$$

(34)

$$(\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0$$

(35)

for any $\alpha \in \Gamma_1$. Then the tensor

$$r = r_0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha > 0, k \geq 1} e_{-\alpha} \wedge e_{\tau^k(\alpha)}$$

(36)

satisfies (32). Moreover, any solution of the system (32) is of the above form, for a suitable triangular decomposition of $\mathfrak{g}$ and suitable choice of a basis $\{e_\alpha\}$. 


In what follows, for aim of quantization of algebra structures on the polynomial Lie algebra $gl_N[u]$ we will use the twisted two-tensor $q^{r_0(N)}$ where $r_0(N)$ is the Cartan part of the Cremmer–Gervais $r$-matrix for the Lie algebra $gl_N$ when $\Gamma_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{N-2}\}$, $\Gamma_2 = \{\alpha_2, \alpha_3, \ldots, \alpha_{N-1}\}$ and $\tau(\alpha_i) = \alpha_{i+1}$. An explicit form of $r_0(N)$ is defined by the following proposition (see [32]).

**Proposition 1.** The Cartan part of the Cremmer–Gervais $r$-matrix for $gl_N$ is given by the following expression

$$r_0(gl_N) = \frac{1}{2} \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{1 \leq i < j \leq N} \frac{N + 2(i - j)}{2N} e_{ii} \wedge e_{jj}. \quad (37)$$

It is easy to check that the Cartan part $r_0(N)$, satisfies the conditions

$$(e_k \otimes \text{id} + \text{id} \otimes e_k)(r_0(N)) = \epsilon_{kk} \quad \text{for} \quad k = 1, 2, \ldots, N, \quad (38)$$

$$(e_k \otimes \text{id} + \text{id} \otimes e_k')(r_0(N)) = (k - k')C_1(N) - \sum_{i=k'+1}^{k-1} e_{ii} \quad \text{for} \quad 1 \leq k' < k \leq N, \quad (39)$$

where $C_1(N)$ is the normalized central element:

$$C_1(N) := \frac{1}{N} \sum_{i=1}^{N} e_{ii}. \quad (40)$$

In particular (38) and (39) imply the Belavin–Drinfeld conditions (35) and (35), i.e.

$$(\alpha_k \otimes \text{id} + \text{id} \otimes \alpha_k)(r_0(N)) = h_{\alpha_k} := e_{kk} - e_{k+1,k+1}, \quad (41)$$

$$(\tau(\alpha_k') \otimes \text{id} + \text{id} \otimes \alpha_k')(r_0(N)) = (\alpha_{k'+1} \otimes \text{id} + \text{id} \otimes \alpha_k')(r_0(N)) = 0 \quad (42)$$

for $k = 1, 2, \ldots, N - 1$, $k' = 1, 2, \ldots, N - 2$, where $\alpha_k = \epsilon_k - \epsilon_{k+1}$ and $\alpha_{k'} = \epsilon_{k'} - \epsilon_{k'+1}$ are the simple roots of system $\Pi(gl_N)$ (see [7]).

Now we consider some properties of the two-tensor $q^{r_0(N)}$. First of all it is evident that this two-tensor satisfies cocycle equation (41). Further, for construction of a twisting two-Weyl basis elements $e'_{ij}$ ($i \neq j$) for the quantum algebra $U_q(gl_N)$ as follows

$$e'_{ij} = e_{ij}q^{((\epsilon_i - \epsilon_j) \otimes \text{id})(r_0(N))} = e_{ij}q \sum_{k=1}^{i-1} e_{kk} - (j-i)C_1(N), \quad (43)$$

$$e'_{ji} = q^{(\text{id} \otimes (\epsilon_j - \epsilon_i))(r_0(N))} e_{ji} = q \sum_{k=i+1}^{N} e_{kk} - (j-i)C_1(N) e_{ji}, \quad (44)$$

for $1 \leq i < j \leq N$. Permutation relations for these elements can be obtained from the relations (16)–(25). For example, we have

$$[e'_{ij}, e'_{ji}] = [e_{ij}, e_{ji}]q^{((\epsilon_i - \epsilon_j) \otimes \text{id} + \text{id} \otimes (\epsilon_j - \epsilon_i))(r_0(N))}$$

$$= \frac{2 \sum_{k=i}^{i-1} e_{kk} - 2(j-i)C_1(N)}{q} - \frac{2 \sum_{k=i+1}^{j} e_{kk} - 2(j-i)C_1(N)}{q - q^{-1}}. \quad (45)$$
It is not hard to check that the Chevalley elements $e_{i,i+1}^\prime$ and $e_{i+1,i}^\prime$ have the following coproducts after twisting by the two-tensor $q^{r_0(N)}$:

\[
q^{r_0(N)} q_{i,i+1}^\prime q = e_{i,i+1}^\prime \otimes q^{2(\epsilon_i - \epsilon_{i+1}) \otimes \text{id}}(r_0(N)) + 1 \otimes e_{i,i+1}^\prime
\]

(46)

\[
q^{r_0(N)} q_{i+1,i}^\prime q = e_{i+1,i}^\prime \otimes 1 + q^{-2(\text{id} \otimes (\epsilon_{i+1} - \epsilon_i))}(r_0(N)) \otimes e_{i+1,i}^\prime
\]

(47)

for $1 \leq i < N$. Since the quantum algebra $U_q(\mathfrak{gl}_N)$ is a subalgebra of the quantum affine algebra $U_q(\mathfrak{gl}_N[\hat{u}])$ let us introduce the new affine root vector $e_{N1}^\prime(1)$ in accordance with (43):

\[
e_{N1}^\prime(1) = e_{N1} q^{(\epsilon_N - \epsilon_1) \otimes \text{id}}(r_0(N)) = e_{N1} q^{e_{NN} - C_1(N)}
\]

(48)

The coproduct of this element after twisting by the two-tensor $q^{r_0(N)}$ has the form

\[
q^{r_0(N)} q_{N1}^\prime q = e_{N1}^\prime \otimes q^{2(\epsilon_1 - \epsilon_N) \otimes \text{id}}(r_0(N)) + 1 \otimes e_{N1}^\prime
\]

(49)

Consider the quantum seaweed algebra $U_q(\mathfrak{sw}_{N+1})$ after twisting by the two-tensor $q^{r_0(N+1)}$. Its new Cartan–Weyl basis and the coproduct for the Chevalley generators are given by formulas (43), (44), (45), and (46), where $N$ should be replaced by $N + 1$ and where $i \neq 1$ in (43) and (45), and $j \neq N$ in (43), and $i \neq N$ in (46). In particular, for the element $e_{N,N+1}^\prime$ we have

\[
e_{N,N+1}^\prime = e_{N,N+1} q^{e_{NN} - C_1(N+1)}
\]

(50)

\[
q^{r_0(N+1)} q_{N+1}^\prime q = e_{N,N+1}^\prime \otimes q^{2e_{NN} - 2C_1(N+1)} + 1 \otimes e_{N,N+1}^\prime
\]

(51)

Comparing the Hopf structure of the quantum seaweed algebra $U_q(\mathfrak{sw}_{N+1})$ after twisting by the two-tensor $q^{r_0(N+1)}$, and its affine realization $U_q(\mathfrak{sw}_N)$ after twisting by the two-tensor $q^{r_0(N)}$ we see that these algebras are isomorphic as Hopf algebras:

\[
q^{r_0(N+1)} \Delta_q(U_q(\mathfrak{sw}_{N+1}))q^{-r_0(N+1)} \simeq q^{r_0(N)} \Delta_q(U_q(\mathfrak{sw}_N))q^{-r_0(N)}
\]

(52)

In terms of new Cartan–Weyl bases this isomorphism, “$\iota$”, is arranged as follows

\[
\iota(e_{ij}) = e_{ij}^{(0)} \quad \text{for} \quad 2 \leq i < j \leq N
\]

(53)

\[
\iota(e_{ji}) = e_{ji}^{(0)} \quad \text{for} \quad 1 \leq i < j \leq N - 1
\]

(54)

\[
\iota(e_{ii} - C_1(N + 1)) = e_{ii}^{(0)} - C_1(N) \quad \text{for} \quad 2 \leq i \leq N
\]

(55)

\[
\iota(e_{i,N+1}^\prime) = e_{i1}^{(1)} q^{(\epsilon_i - \epsilon_1 \otimes \text{id})(r_0(N))} = e_{i1}^{(1)} q \sum_{k=1}^{N} e_{kk} - (N+1-i)C_1(N)
\]

(56)

for $2 \leq i \leq N$. Where the affine root vectors $e_{i1}^{(1)} (2 \leq i < N)$ are defined by the formula (cf. 14):

\[
e_{i1}^{(1)} = [e_{iN}^{(0)}, e_{N1}^{(1)}]_{q^{-1}}
\]

(57)
4 Affine realization of a Cremmer-Gervais twist

For construction of a twisting two-tensor corresponding to the Cremmer-Gervais $r$-matrix \((37)\) we will follow to the papers \([7, 9]\).

Let $R$ be a universal $R$-matrix of the quantum algebra $U_q(gl_{N+1})$. According to \([13]\) it has the following form

$$R = R \cdot K \quad \text{(58)}$$

where the factor $K$ is a $q$-power of Cartan elements (see \([13]\)) and we do not need its explicit form. The factor $R$ depends on the root vectors and it is given by the following formula

$$R = R_{12}(R_{13}R_{23})(R_{14}R_{24}R_{34}) \cdots (R_{1,N+1}R_{2,N+1} \cdots R_{N,N+1}) \quad \text{(59)}$$

where

$$R_{ij} = \exp_{q^{-2}}((q - q^{-1})e_{ij} \otimes e_{ji}) \quad \text{(60)}$$

It should be noted that the product of factors $R_{ij}$ in \((59)\) corresponds to the normal ordering \([13]\) where $N$ is replaced by $N + 1$.

Let $R' := q^r_0(N+1)R q^{-r_0(N+1)}$. It is evident that

$$R' = \prod_{j=2}^{N+1} \left( \prod_{i=1}^{j-1} R'_{ij} \right) \quad \text{(62)}$$

where

$$R'_{ij} = \exp_{q^{-2}}((q - q^{-1})e'_{ij} \otimes e'_{ji}) \quad \text{(63)}$$

Here $e'_{ij}$ and $e'_{ji}$ are the root vectors \([43, 44]\) where $N$ should be replace by $N + 1$.

Let $T$ be a homomorphism operator which acts on the elements $e'_{ij}$ ($1 \leq i < j \leq N$) by formulas $T(e'_{ij}) = e'_{\tau(ij)} = e'_{i+1,j+1}$ for $1 \leq i < j \leq N$, and $T(e'_{i,N+1}) = 0$ for all $i = 1, 2, \ldots, N$. We set

$$R'^{(k)} := (T^k \otimes \text{id})(R') = \prod_{j=2}^{N+1-k} \left( \prod_{i=1}^{j-1} R'^{(k)}_{ij} \right) \quad \text{(64)}$$

where

$$R'^{(k)}_{ij} = \exp_{q^{-2}}((q - q^{-1})T^k(e'_{ij}) \otimes e'_{ji}) = \exp_{q^{-2}}((q - q^{-1})e'_{i+k,j+k} \otimes e'_{ji}) \quad \text{(65)}$$

for $k \leq N - j$.

According to \([7, 9]\), the Cremmer-Gervais twist $F_{CG}$ in $U_q(gl_{N+1})$ is given as follows

$$F_{CG} = F \cdot q^r_0(N+1) \quad \text{(66)}$$

where

$$F = R'^{(N-1)}R'^{(N-2)} \cdots R'^{(1)} \quad \text{(67)}$$

It is easy to see that the support of the twisting two-tensor \((67)\) is the quantum seaweed algebra $U_q(sw_{N+1})$ with the coproducts \([46, 47]\) where $N$ should be replace by
Theorem 2. Let $\hat{F}'_{CG}$ be the twist (67) reduced to $U_q(\widehat{\mathfrak{sl}}_N)$, and let $\hat{R}$ be the universal $R$-matrix for $U_q(\widehat{\mathfrak{sl}}_N)$. Then the $R$-matrix $\hat{F}'_{CG} \hat{R} \hat{F}'_{CG}^{-1}$ quantizes the quasi-trigonometric $r$-matrix obtained in Proposition 2.

5 From constant to affine twists

5.1 Affine $r$-matrices of Cremmer-Gervais type and their quantization

In this subsection we are going to describe two classes of the quasi-trigonometric $r$-matrices of Cremmer-Gervais type, which we call affine $r$-matrices of the Cremmer-Gervais type.

We need some results proved in [10] for the case $\mathfrak{sl}_N$. Let us consider the Lie algebra $\mathfrak{sl}_N \otimes \mathfrak{sl}_N$ with the form $Q((a, b), (c, d)) = K(a, c) - K(b, d)$, where $K$ is the Killing form on $\mathfrak{sl}_N$. Let $P_i$ (resp. $P_{i}^{-}$) be the maximal parabolic subalgebra containing all positive (resp. negative) roots and not containing all negative roots which have the simple root $\alpha_i$ in their decomposition. Assume we have two parabolic (or maybe $\mathfrak{sl}_N$) subalgebras $S_1$, $S_2$ of $\mathfrak{sl}_N$ such that their reductive parts $R_1$, $R_2$ are isomorphic and let $T : R_1 \rightarrow R_2$ be an isomorphism, which is an isometry with respect to the reductions of the Killing form $K$ onto $R_1$ and $R_2$. Then the triple $(S_1, S_2, T)$ defines a Lagrangian subalgebra of $\mathfrak{sl}_N \otimes \mathfrak{sl}_N$ (with respect to the form $Q$). It was proved in [10] that any quasi-trigonometric solution is defined by a Lagrangian subalgebra $W \subset \mathfrak{sl}_N \otimes \mathfrak{sl}_N$ such that $W$ is transversal to $(P_i, P_i^{-}, \text{id})$. Now we would like to present two types of $W$, which define quasi-trigonometric $r$-matrices for $\mathfrak{sl}_N$ with $i = N - 1$.

Type 1, which we call affine Cremmer–Gervais I classical $r$-matrix: In this case $S_1 = S_2 = \mathfrak{sl}_N$, $T = \text{Ad}(\sigma_N)$, where $\sigma_N$ is the permutation matrix, which represents the cycle $\{1, 2, \ldots, N\} \rightarrow \{2, 3, \ldots, N, 1\}$.

Proposition 2. The triple $(S_1, S_2, T)$ above defines a quasi-trigonometric $r$-matrix.

Type 2, which we call affine Cremmer–Gervais II classical $r$-matrix: In this case $S_3 = P_i^{-}$, $S_4 = P_{i-1}^{-}$, $T'(e_{ij}) = e_{i-1,j-1}$, where $e_{ij}$ are the matrix units.

Proposition 3. The triple $(S_3, S_4, T')$ above defines a quasi-trigonometric $r$-matrix.

Both propositions can be proved directly.
Now we turn to quantization of the quasi-trigonometric $r$-matrix from Proposition 3. In order to do this we propose another method for construction of affine twists from constant ones. Let $\{\alpha_0, \alpha_1, \ldots, \alpha_{N-1}\}$ be the vertices of the Dynkin diagram of $\mathfrak{sl}_N$. Then the map $\tau$: $\{\alpha_1, \ldots, \alpha_{N-1}\}$ to $\{\alpha_2, \ldots, \alpha_{N-1}, \alpha_0\}$ defines an embedding of Hopf algebras $U_q(\mathfrak{sl}_N) \hookrightarrow U_q(\hat{\mathfrak{sl}}_N)$. Abusing notations we denote this embedding by $\tau$. The map $\tau$ sends each twist $F$ of $U_q(\mathfrak{sl}_N)$ to an affine twist $(\tau \otimes \tau)(F)$. Let us denote $(\tau \otimes \tau)(F_{CG})$ by $\hat{F}_{CG}$.

**Theorem 3.** Let $\hat{R}'$ be the universal $R$-matrix for $U_q(\hat{\mathfrak{sl}}_N)$. Then the $R$-matrix $\hat{F}_{CG}^{(\tau)}(\hat{F}_{CG}^{(\tau)})^{-1}$ quantizes the quasi-trigonometric $r$-matrix obtained in Proposition 3.

Both theorems can be proved straightforward.

### 5.2 $\omega$-affinization and quantization of rational $r$-matrices

Aim of this section is to quantize certain rational $r$-matrices. Of course, we would like to use rational degeneration of the affine twists constructed above. Unfortunately, we cannot do this directly because in both cases $\lim_{q \to 1} F = 1 \otimes 1$. Therefore, propose a new method which we call $\omega$-affinization. We begin with the following result.

**Theorem 4.** Let $\pi : U_q(\mathfrak{sl}_N[u]) \to U_q(\hat{\mathfrak{sl}}_N)$ be the canonical projection sending all the affine generators to zero. Let $\Psi \in U_q(\mathfrak{sl}_N[u]) \otimes U_q(\hat{\mathfrak{sl}}_N)$ be invertible and such that

$$\Psi = \Psi_1 \Psi_2, \quad (73)$$

where $\Psi_2 = (\pi \otimes \text{id})(\Psi)$. Further, let $\Psi_2$ be a twist and let the following two relations hold:

$$\Psi_2^{12} \Psi_2^{12} = \Psi_1^{12} \Psi_2^{23}, \quad (\pi \otimes \text{id})(\Delta \otimes \text{id})(\Psi_1) = \Psi_1^{23} \quad (74)$$

Finally, let there exist $\omega \in U_q(\mathfrak{sl}_N[u])[\zeta]$ such that

$$\Psi_\omega := (\omega \otimes \omega)(\Psi_\Delta(\omega^{-1}) \in U_q(\mathfrak{sl}_N) \otimes U_q(\hat{\mathfrak{sl}}_N). \quad (75)$$

Then $\Psi$ is a twist.

**Proof.** Let us consider the Drinfeld associator

$$\text{Assoc}(\Psi) = \Psi^{12} (\Delta \otimes \text{id})(\Psi)(\text{id} \otimes \Delta)(\Psi^{-1})(\Psi^{-1})^{23} \quad (76)$$

and the following one equivalent to it

$$\text{Assoc}(\Psi_\omega) = (\omega \otimes \omega \otimes \omega)(\text{Assoc}(\Psi))(\omega^{-1} \otimes \omega^{-1} \otimes \omega^{-1}) \quad (77)$$

By (76) we have

$$\text{Assoc}(\Psi_\omega) = (\pi \otimes \text{id} \otimes \text{id})(\text{Assoc}(\Psi_\omega)) \quad (78)$$

Further, we take into account the following considerations:

$$(\pi \otimes \text{id})(\Psi_2) = (\pi \otimes \text{id})(\pi \otimes \text{id})(\Psi_1 \Psi_2) = (\pi \otimes \text{id})(\Psi_1)(\pi \otimes \text{id})(\Psi_2), \quad (79)$$

what implies that $(\pi \otimes \text{id})(\Psi_1) = 1 \otimes 1$. Moreover, the latter also implies that

$$(\pi \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta)(\Psi_1^{-1}) = (\text{id} \otimes \Delta)(\pi \otimes \text{id})(\Psi_1^{-1}) = 1 \otimes 1 \otimes 1. \quad (80)$$
Thus,
\[
\text{Assoc}(\Psi_\omega) = \text{Ad}(\pi(\omega) \otimes \omega)(\Psi_2) \circ (\Psi_2^{-1}) \circ (\Psi_1^{-1}) \circ (\Psi_1) \circ (\Psi_2^{-1}) \circ (\Psi_2) \circ (\Psi_2^{-1}) \circ (\Psi_1^{-1}) \circ (\Psi_1) \circ (\Psi_2)
\]
Thus, we have
\[
\text{Assoc}(\Psi_\omega) = \text{Ad}(\pi(\omega) \otimes \omega)(\Psi_2) \circ (\Psi_2^{-1}) \circ (\Psi_1^{-1}) \circ (\Psi_1) \circ (\Psi_2^{-1}) \circ (\Psi_2) \circ (\Psi_2^{-1}) \circ (\Psi_1^{-1}) \circ (\Psi_1) \circ (\Psi_2)
\]
\[
\text{Assoc}(\Psi_\omega) = \text{Ad}(\pi(\omega) \otimes \omega)(\Psi_2) \circ (\Psi_2^{-1}) \circ (\Psi_1^{-1}) \circ (\Psi_1) \circ (\Psi_2^{-1}) \circ (\Psi_2) \circ (\Psi_2^{-1}) \circ (\Psi_1^{-1}) \circ (\Psi_1) \circ (\Psi_2)
\]

Since \( \Psi_2 \) is a twist we deduce that \( \text{Assoc}(\Psi_\omega) = 1 \otimes 1 \otimes 1 \) and therefore \( \text{Assoc}(\Psi) = 1 \otimes 1 \otimes 1 \) what proves the theorem.

An element \( \omega \) satisfying conditions of Theorem 4 we will call affinizator.

**Corollary 1.** Let \( \Psi_1, \Psi_2, \omega \) satisfy the conditions of Theorem 4. Then
\[
\Psi_1 = (\omega^{-1} \pi(\omega) \otimes \id) \Psi_2 \{ (\pi \otimes \id) \Delta(\omega^{-1}) \} \Delta(\omega) \Psi_2^{-1}.
\]

**Proof.** We have
\[
\Psi_\omega = (\pi \otimes \id)(\Psi_\omega) = (\pi(\omega) \otimes \omega)\Psi_2(\pi \otimes \id)\Delta(\omega^{-1})
\]
because \( \Psi_2 = (\pi \otimes \id)(\Psi) \). Now we see that
\[
(\omega \otimes \omega)(\Psi_1 \Psi_2)\Delta(\omega^{-1}) = (\pi(\omega) \otimes \omega)\Psi_2(\pi \otimes \id)\Delta(\omega^{-1})
\]
what yields the required expression for \( \Psi_1 \).

Now we would like to explain how \( \omega \)-affinization can be used to find a Yangian degeneration of the affine Cremmer–Gervais twists. Let us consider the case \( \mathfrak{s} l_3 \). We set
\[
\Psi_2 := \mathcal{F}_{CG} := \exp q^2(-q - q^{-1}) \zeta \varepsilon_{12} \otimes \varepsilon_{32} \cdot \mathcal{K}_{\mathfrak{s} l_3}
\]
where
\[
\mathcal{K}_{\mathfrak{s} l_3} = q^{\frac{1}{3}h_{12} \otimes h_{12} + \frac{2}{3}h_{12} \otimes h_{23} + \frac{1}{2}h_{23} \otimes h_{12} + \frac{1}{2}h_{23} \otimes h_{23}}
\]
with \( h_{ij} := e_{ii} - e_{jj} \). The twisting two-tensor \( \mathcal{K}_{\mathfrak{s} l_3} \) belongs to \( U_q(\mathfrak{s} l_3) \otimes U_q(\mathfrak{s} l_3)[[\zeta]] \).

The following affinizator \( \omega_3^{\text{long}} \) was constructed in [17]. It is given by the following formula
\[
\omega_3^{\text{long}} = \exp q^2 \left( \frac{\zeta}{1 - q^2} e_{12}(0) \right) \exp q^2 \left( \frac{-q^2 \z^2}{1 - q^2} e_{21}(1) \right) \exp q^2 \left( \frac{q^2 \z^2}{1 - q^2} e_{31}(1) \right) \exp q^2 \left( \frac{-q^2 \z^2}{1 - q^2} e_{32}(0) \right) \exp q^2 \left( \frac{q^2 \z^2}{1 - q^2} e_{32}(0) \right)
\]
where \( h^\perp_\alpha = \frac{1}{3}(e_{11} + e_{22}) - \frac{2}{3}e_{33} \) and \( h^\perp_\alpha = \frac{2}{3}e_{11} - \frac{1}{3}(e_{22} + e_{33}) \).

For convenience sake we remind the reader that
\[
\begin{align*}
\varepsilon_{12} &= q^{-h^\perp_\alpha} e_{12}, \\
\varepsilon_{21} &= q^{h^\perp_\alpha} e_{21}, \\
\varepsilon_{32} &= q^{-h^\perp_\alpha} e_{32}, \\
\varepsilon_{31} &= q^{h^\perp_\alpha} e_{31}, \\
\varepsilon_{12} &= q^{\frac{1}{3}h_{12} \otimes h_{12} + \frac{2}{3}h_{12} \otimes h_{23} + \frac{1}{2}h_{23} \otimes h_{12} + \frac{1}{2}h_{23} \otimes h_{23}} \mathcal{K}_{\mathfrak{s} l_3}
\end{align*}
\]
Theorem 5. The elements \( \omega^\text{long}_3 \), \( \Psi_2 = F_C^{(r)} \), and \( \Psi_1 \) constructed according to Corollary 1 satisfy the conditions of Theorem 4 and consequently \( \Psi_\omega = (\pi \otimes \text{id})(\omega \otimes \omega)\Psi_2 \Delta(\omega^{-1}) \) is a twist.

Proof. Straightforward.

It turns out that \( \Psi_\omega \) has a rational degeneration. To define this rational degeneration we have to introduce the so-called \( f \)-generators:

\[
\begin{aligned}
    f_0 &= (q - q^{-1})e_{31}^{(0)}, \\
    f_1 &= q^{2h_3^\perp}e_{31}^{(1)} + q^{-1}\zeta e_{31}^{(0)}, \\
    f_2 &= (1 - q^{-2})e_{32}^{(0)}, \\
    f_3 &= q^{h_2^\perp}e_{32}^{(1)} - \zeta e_{32}^{(0)}.
\end{aligned}
\]

Let us consider the Hopf subalgebra of \( U_q^{\mathfrak{e}_+}(\mathfrak{sl}_3) \) generated by

\[ \{h_{12}, h_{23}, f_0, f_1, f_2, f_3, e_{12}, e_{21} \}. \]

When \( q \rightarrow 1 \) we obtain the following Yangian twist (see [17]):

\[
\begin{aligned}
    \mathcal{F}_1 &= (1 \otimes 1 - \zeta 1 \otimes \mathcal{F}_3 - \zeta^2 h_3^\perp \otimes \mathcal{F}_2)^{(-h_3^\perp \otimes 1)}(1 \otimes 1 + \zeta 1 \otimes e_{21})^{(-h_3^\perp \otimes 1)} \\
    &\quad \times \exp(\zeta^2 e_{12}^{(0)} h_{13} \otimes \mathcal{F}_0) \exp(-\zeta e_{12}^{(0)} \otimes \mathcal{F}_1) \cdot \exp(-\zeta e_{12}^{(0)} \otimes \mathcal{F}_2) \\
    &\quad \times (1 \otimes 1 - \zeta 1 \otimes \mathcal{F}_3 - \zeta^2 h_2^\perp \otimes \mathcal{F}_2)^{(h_2^\perp - h_3^\perp) \otimes 1},
\end{aligned}
\]

where the overlined generators are the generators of \( Y(\mathfrak{sl}_3) \). In the evaluation representation we have:

\[
\begin{aligned}
    \mathcal{F}_0 &\mapsto e_{31}, \quad \mathcal{F}_1 \mapsto u e_{31} \\
    \mathcal{F}_2 &\mapsto e_{32}, \quad \mathcal{F}_3 \mapsto u e_{32} \\
    -e_{21} &\mapsto e_{21}, \quad -e_{12} \mapsto e_{12}.
\end{aligned}
\]

Therefore we have obtained the following result:

Theorem 6. The Yangian twist \( \mathcal{F}_1 \) quantizes the following classical rational \( r \)-matrix

\[
r(u, v) = \frac{\Omega}{u - v} + h_\alpha^\perp \otimes ve_{32} - ve_{32} \otimes h_\alpha^\perp + h_\beta^\perp \wedge e_{21} + e_{21} \otimes ve_{31} - ve_{31} \otimes e_{21} + e_{12} \wedge e_{32}.
\]

To obtain a quantization of the second non-trivial rational \( r \)-matrix for \( \mathfrak{sl}_3 \) we take the following affinizator \( \omega^\text{short}_3 \) and apply it to \( \Psi_2 = q^\omega^{(3)} \), where the Cartan part of the Cremmer-Gervais constant \( r \)-matrix for \( \mathfrak{sl}_3 \) has the form:

\[
r_0(3) = \frac{2}{3}(h_{\alpha_1} \otimes h_{\alpha_1} + h_{\alpha_2} \otimes h_{\alpha_2}) + \frac{1}{3}(h_{\alpha_1} \otimes h_{\alpha_2} + h_{\alpha_2} \otimes h_{\alpha_1}) + \frac{1}{6}h_{\alpha_1} \wedge h_{\alpha_2}.
\]

We have

\[
\omega^\text{short}_3 = \exp q^{-2}(\zeta e_{21}^{(0)}) \exp q^2 \left( -\frac{\zeta}{1 - q^2} q^{2h_3^\perp} e_{31}^{(1)} \right) \exp q^{-2} \left( \frac{\zeta}{1 - q^2} e_{32}^{(0)} \right),
\]

where

\[
\begin{aligned}
    e_{21}^{(0)} &= q^{-\frac{1}{2}(h_{12} - h_{23})} e_{21}^{(0)}, \\
    e_{32}^{(0)} &= q^{-h_2^\perp} e_{32}^{(0)}, \\
    e_{31}^{(1)} &= q^{-h_3^\perp} e_{31}^{(1)}.
\end{aligned}
\]

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We have to calculate
\[
\text{Aff}_{\omega_{\text{short}}}^{}(q^{r_0(3)}) := (\pi \otimes \text{id}) \circ \left( (\omega_{\text{short}}^{} \otimes \omega_{\text{short}}^{}) q^{r_0(3)} \Delta(\omega_{\text{short}}^{})^{-1} \right).
\]  
(95)

Using standard commutation relations between \(q\)-exponents, the formula (76) can be brought to the following form:
\[
\left( 1 \otimes 1 + \zeta \ 1 \otimes q^{2h_{\alpha}^+} \hat{e}_{31}^{(1)} + \zeta q^{-2h_{\alpha}^+} \otimes (\text{Ad exp}_{q^2}(\hat{e}_{21}^{(0)})/(\hat{e}_{32}^{(0)}))^{(-h_{\alpha}^+ \otimes 1)} \right) q^2
\times \left( 1 \otimes 1 + \zeta (1 - q^2) 1 \otimes \hat{e}_{21}^{(0)} \right)^{(-\frac{1}{3}(h_{12} - h_{23}) \otimes 1)} q^{r_0(3)}.
\]  
(96)
The \(q\)-Hadamard formula allows us to calculate the \(\text{Ad}\)-term explicitly:
\[
(\text{Ad exp}_{q^{-2}}(\zeta \hat{e}_{21}^{(0)}))/(\hat{e}_{21}^{(0)}) = \hat{e}_{21}^{(0)} + \zeta q^{-h_{\alpha}^+} \hat{e}_{31}^{(0)},
\]  
(97)
where \(\hat{e}_{31}^{(0)} := e_{31}^{(0)} e_{32} - q e_{32} e_{21}\). To define a rational degeneration we introduce \(g\)-generators, which satisfy the Yangian relations as \(q \to 1\):
\[
g_0 = (q - q^{-1}) q^{-h_{\alpha}^+} e_{31}^{(0)}, \quad g_1 = q^{2h_{\alpha}^+} \hat{e}_{31}^{(1)} + \zeta q^{-h_{\alpha}^+} \hat{e}_{31}^{(0)}, \quad g_2 = (q^2 - 1) \hat{e}_{21}^{(0)}.
\]
Using \(g\)-generators we can calculate the rational degeneration of the twist \(\text{Aff}_{\omega_{\text{short}}}^{}(q^{r_0(3)})\):
\[
\mathcal{F}_2 = \left( 1 \otimes 1 + \zeta \ 1 \otimes (g_1 + \hat{e}_{32}^{(0)}) - \zeta^2 h_{\alpha}^+ \otimes g_0 \right)^{(-h_{\alpha}^+ \otimes 1)}
\times \left( 1 \otimes 1 - \zeta \ 1 \otimes g_2 \right)^{(-\frac{1}{3}(h_{12} - h_{23}) \otimes 1)}.
\]  
(98)

**Theorem 7.** This Yangian twist \(\mathcal{F}_2\) quantizes the following rational \(r\)-matrix:
\[
r(u, v) = \frac{\Omega}{u - v} - u e_{31} \otimes h_{\alpha}^+ + v h_{\alpha}^+ \otimes e_{31} + h_{\alpha}^+ \wedge e_{32} - \frac{1}{3}(h_{12} - h_{23}) \wedge e_{21}.
\]  
(99)

Therefore we have quantized all non-trivial rational \(r\)-matrices for \(\mathfrak{sl}_3\) classified in [19].

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