Lacunary Generating Functions for Laguerre Polynomials

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Abstract

Symbolic methods of umbral nature are playing an important and increasing role in the theory of special functions and in related fields like combinatorics. We discuss an application of these methods to the theory of lacunary generating functions of Laguerre polynomials for which we give a number of new close form expressions. We present furthermore the different possibilities offered by the method we have developed, with particular emphasis on their link with new family of special functions and with previous formulations, associated with the theory of quasi monomials.
I. INTRODUCTION

It has been shown that symbolic methods of umbral nature provide powerful tools to deal with the properties of special polynomials and functions [1, 2]. These techniques greatly simplify the problems underlying such studies and allow to reduce the derivation of the relevant properties to straightforward algebraic manipulations.

The key element of the game is the introduction of a symbolic operator used as an ordinary algebraic quantity. Within such a framework the ordinary Laguerre polynomials [3, 4]

\[ L_n(x, y) = n! \sum_{r=0}^{n} \frac{(-x)^r y^{n-r}}{(r!)^2(n-r)!} \]

are expressed as

\[ L_n(x, y) = (y - \hat{c}x)^n \varphi_0, \quad (1.1) \]

where \( \hat{c} \) is an operator satisfying the property \( \hat{c}^{\alpha} \hat{c}^{\beta} = \hat{c}^{\alpha+\beta} \), and acting on the "vacuum" \( \varphi_0 \) in such a way that

\[ \hat{c}^{\alpha} \varphi_0 = \frac{1}{\Gamma(1+\alpha)}. \]

The properties of Laguerre polynomials could have accordingly been reduced to those of a Newton binomial, containing an operator treated, in all the manipulations, as an ordinary algebraic quantity.

We also introduce the family of associated Laguerre polynomials

\[ L_n^{(\alpha)}(x, y) = \frac{\Gamma(1+\alpha+n)}{n!} \Lambda_n^{(\alpha)}(x, y) \quad (1.2) \]

with

\[ \Lambda_n^{(\alpha)}(x, y) = \hat{c}^{\alpha} (y - \hat{c}x)^n \varphi_0 = n! \sum_{r=0}^{n} \frac{(-x)^r y^{n-r}}{r!(n-r)\Gamma(1+\alpha+r)}. \quad (1.3) \]

Using the umbral method, the double- and triple-lacunary Laguerre polynomials can be expressed via \( \Lambda_n^{(\alpha)}(x, y) \) as follows

\[ L_{2n}(x, y) = (y - \hat{c}x)^{2n} \varphi_0 = (y - \hat{c}x)^n \left[ (y - \hat{c}x)^n \varphi_0 \right] = \sum_{r=0}^{n} \binom{n}{r} (-x)^r y^{n-r} \Lambda_n^{(r)}(x, y) \quad (1.4) \]

1 Strictly speaking Laguerre polynomials are one-variable polynomials and are defined as \( L_n(x) = L_n(x, 1) \). It is also easily checked that \( L_n(x, y) = y^n L_n \left( \frac{x}{y} \right) \). We will use two-variable forms for future convenience. In final results we shall sometime use again the one-variable form. This comment applies to two-variable Hermite polynomials, see Eq. (2.4), too.
and

\[ L_{3n}(x, y) = (y - \hat{c}x)^{3n} \varphi_0 = \sum_{k,r=0}^{n} \binom{n}{k}(x)^{r+k}y^{2n-r-k}L_n^{(r+k)}(x, y). \] (1.5)

A further family of polynomials, introduced by the same means as before, is provided by what we will call, for reasons which will be clear in the following, Laguerre-Wright polynomials, namely

\[ \Lambda_n^{(\alpha, \beta)}(x, y) = \hat{c}^\alpha(y - \hat{c}^\beta x)^n \varphi_0 = n! \sum_{r=0}^{n} \frac{(-x)^ry^{n-r}}{r!(n-r)! \Gamma(\beta r + 1 + \alpha)}. \] (1.6)

From Eq. (1.3) \( \Lambda_n^{(\alpha, 1)}(x, y) = \Lambda_n^{(\alpha)}(x, y) \) follows.

An interesting example for illustrating the flexibility and the usefulness of the symbolic methods is derivation of the generating functions of special polynomials and of special functions as well. We will consider in the following two types of generating functions, namely the exponential and the ordinary ones. In the first case, i.e. the exponential generating function of polynomials in Eq. (1.6) leads to the following form

\[ E_{n}^{(\alpha, \beta)}(x, y|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda_n^{(\alpha, \beta)}(x, y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{c}^\alpha(y - \hat{c}^\beta x)^n = e^{yt} \hat{c}^\alpha e^{\hat{c}^\beta xt} \varphi_0 = e^{yt} W^{(\beta, \alpha + 1)}(-tx), \] (1.7)

where

\[ W^{(\beta, \alpha)}(x) = \hat{c}^{\alpha - 1} \exp(\hat{c}^\beta x) \varphi_0. \] (1.8)

In the case of ordinary generating function, Eq. (1.6) gives

\[ O_{n}^{(\alpha, \beta)}(x, y|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda_n^{(\alpha, \beta)}(x, y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{c}^\alpha(y - \hat{c}^\beta x)^n = \frac{1}{1 - ty + \frac{tx}{1 - ty}} \hat{c}^{\beta x} \varphi_0 = \frac{1}{1 - ty} E_{\beta, \alpha + 1} \left( \frac{-tx}{1 - ty} \right), \] (1.9)

where

\[ E_{\beta, \alpha}(x) = \frac{\hat{c}^{\alpha - 1}}{1 - \hat{c}^\beta x} \varphi_0. \] (1.10)

In Eqs. (1.8) and (1.10) we have formally defined the Bessel-Wright and Mittag-Leffler
functions, respectively \( \hat{c} \), which in explicit form read

\[
W^{(\beta,\alpha)}(x) = \hat{c}^{\alpha-1} e^{\hat{c}x} \varphi_0 = \hat{c}^{\alpha-1} \sum_{r=0}^{\infty} \frac{\hat{c}^r}{r!} x^r \varphi_0 = \sum_{r=0}^{\infty} \frac{x^r}{r!} \Gamma(\beta r + \alpha),
\]

\[
E_{\beta,\alpha}(x) = \frac{\hat{c}^{\alpha-1}}{1 - \hat{c}^x} \varphi_0 = \hat{c}^{\alpha-1} \sum_{r=0}^{\infty} x^r \hat{c}^r \varphi_0 = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\beta r + \alpha)}.
\]

The generating function of the associated Laguerre polynomials are accordingly derived as the special case of previous demonstrations:

\[
oG_1^{(\alpha,1)}(x, y|t) = \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x, y) \quad \text{and} \quad E_{\beta,\alpha}^{(\alpha,1)}(x, y|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(\alpha)}(x, y),
\]

\[
oG_1^{(\alpha,1)}(x, y|t) = \frac{\Gamma(1 + \alpha)}{(1 - ty)^{1+\alpha}} \hat{c}^{\alpha} \left(1 + \frac{tx\hat{c}}{1 - ty}\right)^{-1-\alpha} \varphi_0
\]

\[
= \frac{1}{(1 - ty)^{1+\alpha}} \left[ \sum_{n=0}^{\infty} \left(1 - ty\right)^n \frac{\Gamma(n + \alpha + 1)}{n!} \hat{c}^{n+\alpha} \right] \varphi_0
\]

\[
= \frac{1}{(1 - ty)^{1+\alpha}} \sum_{n=0}^{\infty} \frac{(-tx)^n}{n!} \exp\left(\frac{-tx}{1 - ty}\right), \quad (1.11)
\]

which for \( L_n^{(\alpha)}(x) = L_n^{(\alpha)}(x, 1) \) gives the well-known formula (8.975.1) on p. 1002 of [8]. Furthermore, repeating similar calculation for exponential generating function for integer \( \alpha = m, m \in \mathbb{N} \), we have

\[
E_{\beta,\alpha}^{(m,1)}(x, y|t) = \hat{c}^m \left[ \sum_{n=0}^{\infty} \frac{(n + m)!}{(n!)^2} [t(y - \hat{c}x)]^n \right] \varphi_0
\]

\[
= \hat{c}^m e^{t(y - \hat{c}x)} \left[ \sum_{r=0}^{m} \binom{m}{r} \frac{m!}{r!} [t(y - \hat{c}x)]^r \right] \varphi_0
\]

\[
= e^{ty} \sum_{r=0}^{m} \binom{m}{r} \frac{m!}{r!} \left[ \sum_{s=0}^{r} \binom{r}{s} (ty)^{r-s} (-x)^s \hat{c}^s \exp(-tx) \right] \varphi_0
\]

\[
= e^{ty} \sum_{r=0}^{m} \binom{m}{r} \frac{m!}{r!} \left[ \sum_{s=0}^{r} \binom{r}{s} (ty)^{r-s} (-xt)^s W^{(1,m+s+1)}(-tx) \right], \quad (1.12)
\]

where we used the binomial formula and the well-known relation

\[
(1 - z)^{-a} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(n + a)}{\Gamma(a)}, \quad |z| < 1.
\]
II. LACUNARY GENERATING FUNCTIONS OF LAGUERRE POLYNOMIALS

The double-lacunary exponential generating function of ordinary Laguerre polynomials

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}(x, y) = E_G^{(1,1)}(x, y|t) \]  

(2.1)

is apparently not known. A related expansion given in literature does not appear to be correct, see Eq. (5.11.2.10) p. 704 of [9]. In the following we obtain the explicit form of Eq. (2.1) in terms of known functions.

According to our procedure, we rewrite Eq. (2.1) using Eq. (1.4) as

\[ E_G^{(1,1)}(x, y|t) = \left[ \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( y - \hat{c} x \right)^{2n} \right] \varphi_0 = e^{y^2 t} e^{2 y^2 t - 2 \hat{c} x t} \varphi_0. \]  

(2.2)

We can provide a definite meaning for the previous expression in terms of known special functions by recalling the following expansion

\[ e^{b z^2 + a z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n^{(2)}(a, b), \]  

(2.3)

with

\[ H_n^{(2)}(a, b) = n! \sum_{r=0}^{[n/2]} \frac{a^{n-2r} b^r}{(n-2r)! r!}, \]

where \([n]\) the floor function. The polynomials \(H_n^{(2)}(a, b)\) are two-variable Hermite polynomials [10], also defined through the operational rule [11]

\[ H_n^{(2)}(a, b) = e^{b z^2} e^{a z}, \]

(2.4)

They reduce to the ordinary Hermite polynomials \(H_n(z)\) through the relation

\[ H_n^{(2)}(x, y) = (-i \sqrt{y})^n H_n \left( \frac{i x}{2 \sqrt{y}} \right). \]

The above family of polynomials provides a basis for the definition of the so-called \(H\)-based functions [1]; for example the \(H\)-based cylindrical Bessel functions are defined as

\[ H^J_n(x, y) = \sum_{r=0}^{\infty} (-1)^r \frac{H_n^{(2)}(x, y)}{2^{n+2r} r! (n + r)!}. \]

They have been obtained by replacing \(x^{n+2r}\) in the relevant series expansion by \(H_n^{(2)}(x, y)\).
According to Eq. (2.3), we obtain
\[
e^{e^{2x^2t-2y\sqrt{t}}} \varphi_0 = \left[ \sum_{r=0}^{\infty} \frac{\hat{c}_r}{r!} H_r^{(2)}(-2yxt, x^2t) \right] \varphi_0
\]
\[
= \sum_{r=0}^{\infty} \frac{1}{(r!)^2} H_r^{(2)}(-2yxt, x^2t)
\]
\[
= \sum_{r=0}^{\infty} \frac{(ix\sqrt{t})^r}{(r!)^2} H_r(iyx\sqrt{t})
\]
\[
= HC_0(-2yxt, x^2t),
\]
where
\[
HC_\alpha(x, y) = \sum_{r=0}^{\infty} \frac{H_r^{(2)}(-x, y)}{r! \Gamma(1 + \alpha + r)}
\]
is the $H$-based version of the Bessel-Tricomi function
\[
C_\alpha(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r! \Gamma(1 + \alpha + r)}.
\]
Thus we get in conclusion
\[
EG^{(1,1)}_2(x, y|t) = e^{y^2t} HC_0(-2yxt, x^2t).
\]

In terms of standard Laguerre polynomials $L_n(x) = L_n(x, 1)$ Eq. (2.1) can thus be rewritten as
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}(x) = e^t \sum_{r=0}^{\infty} \frac{(ix\sqrt{t})^r}{(r!)^2} H_r(i\sqrt{t}),
\]
which constitutes one of key results of the present investigation. The validity of identity (2.7) as well as of further results of this type for other generating functions (see Eqs. (1.12), and (2.8), (2.14), (3.1), (3.3) below) can be independently proven by the substitution $t \rightarrow -t^2$, followed by coefficient extraction and the use of Eqs. (1.11) and (2.3), see [12].

Using the above technique, see Eqs. (2.1)-(2.5), we derive the exponential double-lacunary generating function for associated Laguerre polynomials $L_n^{(\alpha)}(x)$, for $\alpha = m = 1, 2, \ldots$ and establish that
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}^{(m)}(x, y) = e^{ty} \sum_{r=0}^{\infty} \frac{p_{2m}(r; x, y, t)}{r!(r + 3m)!} H_r^{(2)}(-2yxt, tx^2),
\]
which for standard associated Laguerre polynomials \( L_n^{(m)}(x) = L_n^{(m)}(x, 1) \) and using Eq. (2.4) has the form

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(m)}(x) = e^t \sum_{r=0}^{\infty} \frac{p_{2m}(r; x, 1, t)}{r!(r + 3m)!} (ix\sqrt{t})^r H_r(i\sqrt{t}).
\] (2.8)

The polynomials \( p_{2m}(r; x, y, t) \) are of degree \( 2m \) in the variable \( r \), with the coefficients depending on \( x, y \) and \( t \). For \( m = 1, 2 \), \( p_{2m}(r; x, y, t) \) have the explicit form

\[
p_2(r; x, y, t) = (1 + 2ty^2)r^2 + (5 - 4xyt + 10y^2)r + (6 + 12ty^2 - 12xyt + 2tx^2),
\]

\[
p_4(r; x, y, t) = (2 + 10ty^2 + 4t^2y^4)r^4 + [36 + (180y^2 - 20yx)t + (72y^4 - 16y^2)x^2]r^3
\]

\[
+ [238 + (10x^2 + 1190y^2 - 300yx)t + (-240y^3x + 24y^2x^2 + 476y^4)t^2]r^2
\]

\[
+ [684 + (110x^2 - 1480yx + 3420y^2)t + (-1184y^3x + 264y^2x^2 - 16yx^3 + 1368y^4)t^2]r
\]

\[
+ [720 + (-2400yx + 3600y^2 + 300x^2)t
\]

\[
+ (-1920y^3x - 96yx^3 + 720y^2x^2 + 1440y^4 + 4x^4)t^2].
\]

With a moderate effort the polynomials \( p_{2m}(r; x, y, t) \) for \( m > 2 \) can be also obtained.

The method can be extended to the more general cases like

\[
\mathcal{E}G_2^{(\alpha, \beta)}(x, y|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda_{2n}^{(\alpha, \beta)}(x, y),
\]

which, according to our procedure, can be written as

\[
\mathcal{E}G_2^{(\alpha, \beta)}(x, y|t) = \hat{c}^\alpha \sum_{r=0}^{\infty} \frac{t^n}{n!} (y - \hat{c}^\beta x)^{2n} \varphi_0 = e^{y^2t} \hat{c}^\alpha e^{-2\hat{c}^\beta xyt + \hat{c}^2b x^2t} \varphi_0
\]

\[
= \left[ e^{y^2t} \hat{c}^\alpha \sum_{r=0}^{\infty} \frac{\hat{c}^{2r}b^r}{r!} H_r^{(2)}(-2xyt, x^2t) \right] \varphi_0
\]

\[
= e^{y^2t} \mathcal{H}W^{(\beta, \alpha+1)}(-2xyt, x^2t)
\] (2.9)

with

\[
\mathcal{H}W^{(\alpha, \beta)}(x, y) = \sum_{r=0}^{\infty} \frac{H_r^{(2)}(x, y)}{r!\Gamma(\alpha r + \beta)}
\]

being the \( H \)-based version of the Bessel-Wright function.
We give here without demonstration the following two formulas of similar type
\[ \sum_{n=0}^{\infty} t^n L_{2n}(x) = \frac{1}{1-t} \sum_{r=0}^{\infty} \frac{L_r(x/2)}{(1/2)_r} \left[ -\frac{tx}{2(1-t)} \right]^r \]  
(2.10)
and
\[ \sum_{n=0}^{\infty} t^n L_{3n}(x) = \frac{1}{1-t} \sum_{r=0}^{\infty} \left( -\frac{3tx}{1-t} \right)^r \left[ \sum_{s=0}^{r} \frac{r! (-x)^s}{(r-s)! (r+2s)!} L_{s+r} \left( \frac{x}{3} \right) \right]. \]  
(2.11)

In this section we have shown that the tools of employing the symbolic method to derive lacunary generating functions for Laguerre polynomials can easily be applied with a minimum of computation effort.

We now employ the symbolic method to derive old and new generating function for the associated Laguerre polynomials. We start with simple observation, namely
\[ \hat{c}^\alpha e^{bc^{-1}} \varphi_0 = \sum_{n=0}^{\infty} \frac{b^n}{n!} \hat{c}^{\alpha-n} \varphi_0 = \frac{(1+b)^\alpha}{\Gamma(1+\alpha)}. \]  
(2.12)
Thereafter, we calculate \( G_1^{(\alpha)}(x, y|t) \) which after applying Eqs. (1.2), (1.3) and (2.12) has the following form
\[ G_1^{(\alpha)}(x, y|t) = \sum_{n=0}^{\infty} t^n L_n^{(\alpha-n)}(x, y) = \hat{c}^\alpha \Gamma(1+\alpha) \left[ \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \frac{y-\hat{c}x}{\hat{c}} \right)^n \right] \varphi_0 \]
\[ = = e^{-tx} \Gamma(1+\alpha) \hat{c}^\alpha e^{yt \hat{c}^{-1}} \varphi_0 = (1+yt)^\alpha e^{-tx}, \]  
(2.13)

Formula (2.13) for the ordinary associated Laguerre polynomials, i.e. for \( y = 1 \), is equal to (5.11.4.8) on p. 706 of [9]. In the case of generating function
\[ G_2^{(\alpha)}(x, y|t) = \sum_{n=0}^{\infty} t^n L_{2n}^{(\alpha-2n)}(x, y) \]
Eqs. (1.2) and (1.3) yield
\[ G_2^{(\alpha)}(x, y|t) = \hat{c}^\alpha \Gamma(1+\alpha) \left[ \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} \left( \frac{y-\hat{c}x}{\hat{c}} \right)^{2n} \right] \varphi_0 \]
\[ = \hat{c}^\alpha \Gamma(1+\alpha) \cosh \left( \frac{y\sqrt{t}}{\hat{c}} - x\sqrt{t} \right) \varphi_0 \]
\[ = \Gamma(1+\alpha) \left[ \cosh(x\sqrt{t}) \cosh \left( \frac{\sqrt{t}y}{\hat{c}} \right) - \sinh(x\sqrt{t}) \sinh \left( \frac{\sqrt{t}y}{\hat{c}} \right) \right] \varphi_0 \]
\[ = (1-ty^2)\frac{1}{2} \left\{ \cosh(\sqrt{t}x) \cos(T) - i \sinh(\sqrt{t}x) \sin(T) \right\} \]
\[ = (1-ty^2)\frac{1}{2} \cosh(\sqrt{t}x - iT), \]  
(2.14)
where

\[ T(t, y) = \alpha \arcsin \left( \frac{\sqrt{ty}}{\sqrt{ty^2 - 1}} \right). \]

### III. FURTHER DEVELOPMENTS

The method we have developed so far can be extended to obtain slightly more complicated expressions like

\[ E_G^{(1, 1)}(x, y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n+1}(x, y) \quad \text{and} \quad E_G^{(1, 1)}(x, y|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_{3n+1}(x, y), \]

for \( l = 1, 2, \ldots \). Note that corresponding formulas for the Hermite polynomials were obtained in [13]. Applying Eqs. (1.1), (2.2), and (2.3) for the generating function of double-lacunary Laguerre polynomials, we have

\[ E_G^{(1, 1)}(x, y|t) = e^{y t} (y - \hat{c} x)^l \left[ \sum_{r=0}^{\infty} \frac{\hat{c}^r}{r!} H_r^{(2)}(-2yxt, x^2t) \right] \varphi_0 = e^{y t} \sum_{s=0}^{l} \binom{l}{s} y^{l-s} (-x)^s H_s(2xty, x^2t), \]

which, when written in terms of standard associated Laguerre and Hermite polynomials, is equal to

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n+1}(x) = e^t \sum_{r=0}^{\infty} \frac{(ix \sqrt{t})^r}{r!} H_r(i \sqrt{t}) \left[ \sum_{s=0}^{l} \binom{l}{s} \binom{(l-s)}{(r+s)} \right] \]

\[ = e^t \sum_{r=0}^{\infty} \frac{(ix \sqrt{t})^r}{r!(l+r)!} L^{(r)}_l(x) H_r(i \sqrt{t}). \quad (3.1) \]

Using the analogous procedure we obtain also \( E_G^{(1, 1)}(x, y|t) \) in the form

\[ E_G^{(1, 1)}(x, y|t) = e^{y t} (y - \hat{c} x)^l \left[ \sum_{r=0}^{\infty} \frac{\hat{c}^r}{r!} H_r^{(3)}(-3xy^2t, 3x^2yt, -x^3t) \right] \varphi_0 \]

\[ = e^{y t} \sum_{s=0}^{l} \binom{l}{s} y^{l-s} (-x)^s H_C^{(3)}(3x^2yt, x^3t), \quad (3.2) \]

where

\[ H_C^{(3)}(x, y, z) = \sum_{r=0}^{\infty} \frac{H_r^{(3)}(-x, y, -z)}{r!(r+s)!} \]

is a third order \( H \)-based Tricomi function, with

\[ H_r^{(3)}(x, y, z) = n! \sum_{r=0}^{[n/3]} \frac{z^{r} H_{n-3r}^{(2)}(x, y)}{(n - 3r)!r!} \]
being a third order three variable Hermite polynomial, with generating function

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(3)}(x, y, z) = e^{xt+yt^2+zt^3}. \]

Eq. (3.2) expressed via standard Laguerre and Hermite polynomials is given as

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{l}(x) = e^{t!} \sum_{r=0}^{\infty} \frac{q_3(r; x, t)}{r!(r+4)!} H_r(-3tx, 3tx^2, -tx^3), \]

\[ = e^{t!} \sum_{r=0}^{\infty} \frac{q_3(r; x, t)}{r!(r+4)!} \left[ \sum_{r=0}^{n/3} \frac{(-tx^3)^r(i\sqrt{3}t)^{n-3r}}{r!(n-3r)!} H_{n-3r} \left( i\sqrt{3}t \right) \right]. \quad (3.3) \]

In the case of the associated Laguerre polynomials for \( l = 0 \), \( L_{3n}^{(1)}(x, 1) = L_{3n}^{(1)}(x) \), the generating function is given as

\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} L_{3n}^{(1)}(x) = e^{t!} \sum_{r=0}^{\infty} \frac{q_3(r; x, t)}{r!(r+4)!} H_r(-3tx, 3tx^2, -tx^3), \]

\[ = e^{t!} \sum_{r=0}^{\infty} \frac{q_3(r; x, t)}{r!(r+4)!} \left[ \sum_{r=0}^{n/3} \frac{(-tx^3)^r(i\sqrt{3}t)^{n-3r}}{r!(n-3r)!} H_{n-3r} \left( i\sqrt{3}t \right) \right]. \quad (3.4) \]

with

\[ q_3(r; x, t) = (1 + 3t)r^3 + (9 + 27t - 9tx)r^2 + (26 + 78t - 63tx + 9tx^2)t \]

\[ + (24 + 72t - 108tx + 36tx^2 - 3tx^3), \]

which is a third order polynomial in \( r \), with the coefficients depending on \( x \) and \( t \). The extension to the case

\[ E_G^{(1,1)}_{m,l}(x, y|t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_{mn+l}(x, y) \]

can be straightforwardly accomplished within the present framework and writes

\[ E_G^{(1,1)}_{m,l}(x, y|t) = e^{ty^n}(y - \hat{c}x)^l \left[ \sum_{r=0}^{\infty} \frac{\tilde{c}^r}{r!} H_{r}^{(m)}(-\alpha_1, \alpha_2, \ldots, (-1)^m \alpha_m) \right] \varphi_0 \]

\[ = e^{ty^n} \sum_{s=0}^{l} \binom{l}{s} y^{l-s}(-x)^s H_{c}^{(m)}(\alpha_1, \alpha_2, \ldots, \alpha_m), \]

\[ \alpha_p = \binom{m}{p} x^p y^{m-p} t, \quad p = 1, 2, \ldots, m, \quad (3.5) \]

where \( H_{c}^{(m)}(\alpha_1, \alpha_2, \ldots, \alpha_m) \) is an \( m \)-th order based Tricomi function defining as basis the \( m \)-th variable Hermite polynomials \( H_{n}^{(m)}(-\alpha_1, \alpha_2, \ldots, (-1)^m \alpha_m) \) specified by the generating
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(x_1, x_2, \ldots, x_m) = \exp \left( \sum_{s=1}^{m} x_s t^s \right). \tag{3.6}
\]

Let us finally consider the bilateral generating function
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x, y) L_n(z, u) = \left[ \sum_{n=0}^{\infty} \frac{t^n}{n!} [(y - \hat{c}_1 x)(u - \hat{c}_2 z)]^n \right] \varphi_{1,0} \varphi_{2,0} \tag{3.7}
\]
\[c_q^{\alpha} \varphi_{q,0} = \frac{1}{\Gamma(1 + \alpha)}, \quad q = 1, 2.
\]

Accordingly we obtain
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x, y) L_n(z, u) = e^{tyz} \exp \left( -\hat{c}_1 x u t - \hat{c}_2 y z t + \hat{c}_1 \hat{c}_2 x z t \right) \varphi_{1,0} \varphi_{2,0}
\]
\[= e^{tyz} H_{C_{0,0}}(xu t, -y z t | x z t), \tag{3.8}
\]
where
\[H_{C_{0,0}}(x, y | \tau) = \sum_{r,s,k=0}^{\infty} \frac{x^r y^s \tau^k}{r! s! k! (r + k)! (s + k)!}.
\]

Closely related considerations devoted to combinatorics of Laguerre polynomials are developed in \[15\].

Using the operational method we can also obtain the other interesting identities of the generating function of the lacunary associated Laguerre polynomials. These formula are listed below:
\[
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{3} \right)_n \frac{t^n}{(1 + \frac{2}{3})_n}}{n!} L_n^{(\alpha)}(x) = (1 - t)^{-1 + \alpha} \sum_{r=0}^{\infty} \frac{L_n^{(r+\alpha)} \left( \frac{2}{3} \right)}{(1 + \frac{2}{3})_r} \left[ - \frac{tx}{2(1 - t)} \right]^r; \tag{3.9}
\]
\[
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{3} \right)_n \frac{t^n}{(1 + \frac{2}{3})_n} L_n^{(2m)}(x)}{n!} = \frac{1}{\sqrt{1 - t}} \left( \frac{x \sqrt{t}}{2} \right)^{-m} \exp \left( - \frac{tx}{1 - t} \right) I_m \left( \frac{x \sqrt{t}}{1 - t} \right), \tag{3.10}
\]
where \(m = 0, 1, 2, \ldots\) and \(I_m(z)\) is the modified Bessel function;
\[
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{3} \right)_n \frac{t^n}{(1 + \frac{2}{3})_n} L_n^{(3n)}(x)}{n!} = (1 - t)^{-1 + \alpha} \sum_{r=0}^{\infty} \frac{\Gamma(3r + \alpha + 1)}{(1 + \frac{4}{3})_r (\frac{2}{3} + \frac{\alpha}{3})_r} \left[ - \frac{tx}{9(1 - t)} \right]^r
\]
\[\times \left[ \sum_{s=0}^{r} \frac{(-x)^s L_s^{(s+\alpha+r)} \left( \frac{2}{3} \right)}{(r - s)! \Gamma(2s + \alpha + r + 1)} \right]. \tag{3.11}
\]

Eq. (3.10) corrects the formula (5.10.2.10), p. 704 of \[11\].
Before concluding this paper it is worth presenting some further comments to reconcile the present results with previous approaches based on the monomiality principle \[16\]–\[18\]. Within that context the so called Laguerre derivative \(L\hat{D}_x\) has been introduced, so that

\[
L\hat{D}_x L_n(x, y) = n L_{n-1}(x, y). \tag{3.12}
\]

The Laguerre derivative in differential terms has been defined as \(L\hat{D}_x = -\partial_x e^x\partial_x\). It is evident that, according to the umbral technique proposed in this paper, the Laguerre derivative can alternatively be introduced as

\[
L\hat{D}_x = -\hat{c}^{-1} \partial_x. \tag{3.13}
\]

On the basis of such a definition we find

\[
e^{yL\hat{D}_x} e^{-x} = e^{-y\hat{c}^{-1} \partial_x} \frac{1}{1 + \hat{c}x} \varphi_0 = \frac{1}{1 + \hat{c}(x - y\hat{c}^{-1})} \varphi_0 = \frac{1}{1 - y} \frac{1}{1 + \frac{y}{1-y}} \varphi_0,
\]

thus recovering the well-known result \[18\]

\[
e^{yL\hat{D}_x} e^{-x} = \frac{1}{1 - y} e^{-\frac{x}{1-y}}. \tag{3.14}
\]

As a further example of application we consider the action of the exponential containing the Laguerre derivative acting on the Tricomi function, namely

\[
e^{yL\hat{D}_x} C_0(x) = e^{-y\hat{c}^{-1} \partial_x} e^{-\hat{c}x} \varphi_0 = e^y C_0(x), \tag{3.15}
\]

in agreement with the fact that \(C_0(x)\) is an eigenfunction of the Laguerre derivative \[16\].

The formal procedure we have envisaged can even be pushed further.

The use of the following identity

\[
K^{x\partial_x} f(x) = f(Kx). \tag{3.16}
\]

and the definition of the 0-th order cylindrical Bessel function as a pseudo-Gaussian \[1\], namely

\[
J_0(x) = e^{-\hat{c}\left(\frac{x}{2}\right)^2} \varphi_0 \tag{3.17}
\]

allow the derivation of the following identity

\[
\hat{c}^{-\frac{1}{2}x\partial_x} J_0(x) = e^{-\left(\frac{x}{2}\right)^2}, \tag{3.18}
\]

which, in terms of integral transforms, can be interpreted as a kind of Borel transform \[17\]

\[
\hat{c}^{-\frac{1}{2}x\partial_x} J_0(x) = \int_0^{\infty} e^{-s} J_0\left(\sqrt{s}x\right) ds \tag{3.19}
\]
Finally let us note that having expressed the cylindrical Bessel functions in terms of Gaussian, it is also possible to "reduce" a Gaussian to a Lorentzian, according to the identity
\[
e^{-x^2} = \frac{1}{1 + \hat{c}x^2}\varphi_0.
\] (3.20)

According to this last identity we can write the relevant integral as
\[
\int_0^x e^{-\xi^2} d\xi = \hat{c}^{-\frac{1}{2}} \arctan \left( \sqrt{\hat{c}}x \right) \varphi_0.
\] (3.21)

Here we give for completeness the list of new close form expressions obtained in the present investigation. These are equations: (1.7), (1.9), (1.12), (2.7), (2.8), (2.9), (2.10), (2.11), (2.14), (3.1), (3.3), (3.4), (3.5), (3.9), (3.10), and (3.11). The methods we have illustrated in this paper appear fairly flexible and amenable for further implementations, as will be shown in a future investigation.

We thank A. Bostan for important discussions. This work has been supported by Agence Nationale de la Recherche (Paris, France) under Program PHYSCOMB No. ANR-08-BLAN-0243-2.

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