ON LOCALLY SOLVABLE SUBGROUPS IN DIVISION RINGS

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Abstract. Let $D$ be a division ring with center $F$, and $G$ a subnormal subgroup of $D^*$. We show that if $G$ is a locally solvable group such that a derived subgroup $G^{(i)}$ is algebraic over $F$, then $G$ must be central. Also, if $M$ is a non-abelian locally solvable maximal subgroup of $G$ with $M^{(1)}$ algebraic over $F$, then $D$ is a cyclic algebra of prime degree over $F$.

1. Introduction

A well-known result of L. K. Hua says that if the multiplicative subgroup $D^*$ of a division ring $D$ is solvable, then $D$ is a field. The subnormal subgroups of $D^*$ have been studied for a long time by many authors. Subnormal subgroups of some special types were considered, such as nilpotent, solvable, and locally nilpotent subgroups.

In this direction, Stuth [10] asserted that every solvable subnormal subgroup of $D^*$ is central, i.e., it is contained in the center $F$ of $D$. In [7], Huzurbazar showed that this result remains true if the word “solvable” is replaced by “locally nilpotent”. It is more difficult to handle the case “locally solvable”, and it is unknown whether or not every locally solvable subnormal subgroup in a division ring is central.

Relating to this problem, the authors in [4] showed that the question has the positive answer in the case when $D$ is algebraic over $F$. To the best of our knowledge there has been no better results until now. In this note, we show that every locally solvable subnormal subgroup whose $i$-th derived subgroup (for some $i \geq 1$) is algebraic over $F$ is central, which is a slight generalization of those results.

In another direction, maximal subgroups of a subnormal subgroup in division rings were also considered (see e.g. [4, 5]). A remarkable result in [2] asserted that if $M$ is a non-abelian locally solvable maximal subgroup of $D^*$ such that $M'$ is algebraic over $F$, then $[D : F] < \infty$. This result was generalized by the authors in [3], where it is shown that $D$ is even a cyclic algebra of prime degree over $F$ (Theorem 4.2). We show that the result is also true if $D^*$ is replaced by an arbitrary subnormal subgroups and we only need the condition that $M^{(1)}$ is algebraic over $F$ instead of $M'$.

Throughout this note, for a ring $R$ with the identity $1 \neq 0$, the symbol $R^*$ stands for the group of units of $R$. If $D$ is a division ring with the center $F$ and $S \subseteq D$, then $F[S]$ and $F(S)$ denotes respectively the subring and the division subring of $D$ generated by $F \cup S$. For a group $G$ and a positive integer $i$, the symbol $G^{(i)}$ is the $i$-th derived subgroup of $G$. If $H$ and $K$ are two subgroups in a group $G$, then $N_K(H)$ denotes the set of all elements $k \in K$ such that $k^{-1}Hk \leq H$, i.e., $N_K(H) = K \cap N_G(H)$. If $A$ is a ring or a group, then $Z(A)$ denotes the center of $A$.

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2. Results

Lemma 2.1 ([13 3.2]). Let $R$ be a ring, $J$ a subring of $R$, and $H \leq K$ subgroups of the group of units of $R$ normalizing $J$ such that $R$ is the ring of right quotients of $J/H \leq R$ and $J[K]$ is a crossed product of $J[B]$ by $K/B$ for some normal subgroup $B$ of $K$. Then $K = HB$.

Lemma 2.2 ([12 Corollary 24]). Let $A$ be a one-sided Artinian ring. Suppose that $S$ is a right Goldie subring of $A$ and $G$ a locally solvable subgroup of the group of units of $A$ normalizing $S$. Set $R = S[G] \leq A$ and assume that $R$ is prime. Then $R$ is right Goldie.

Lemma 2.3. Let $D$ be a division ring with center $F$, and $G$ a subnormal subgroup of $D^*$. If $G$ is abelian-by-locally finite, then $G \subseteq F$.

Proof. Let $A$ be an abelian normal subgroup of $G$ such that $G/A$ locally finite. Then $A$ is an abelian subnormal subgroup of $D^*$, from which it follows by [10 Theorem 2] that $A \subseteq F$. Therefore $A \subseteq G \cap F^* \subseteq Z(G)$, from which it follows that $G/Z(G)$ is locally finite. Consequently, we conclude that $G'$ is locally finite. Now, $G'$ is a torsion subnormal subgroup of $D^*$, and thus $G' \subseteq F$ by [10 Theorem 8]. This implies that $G$ is solvable, and hence it is contained in $F$ by [10 Theorem 2]. □

Lemma 2.4. Let $D$ be a division ring with center $F$, and $G$ a subgroup of $D^*$. If $G/G \cap F^*$ is a locally finite group, then $F(G)$ is locally finite dimensional (as a vector space) over $F$.

Proof. For any finite subset $\{x_1, x_2, \ldots, x_k\} \subseteq F[G]$, we may write

$$x_i = f_{i1} g_{i1} + f_{i2} g_{i2} + \cdots + f_{it} g_{it},$$

where $f_{ij} \in F$ and $g_{ij} \in G$. Let $A = \langle g_{ij} \rangle$ the subgroup of $G$ generated by all $g_{ij}$. By hypothesis, the group $AF^*/F^*$ is finite. Let $\{y_1, y_2, \ldots, y_n\}$ be a transversal of $F^*$ in $AF^*$ and let

$$R = F y_1 + F y_2 + \cdots + F y_n.$$ 

It is clear that $R$ is a division ring containing the set $\{x_1, x_2, \ldots, x_k\}$ and is finite dimensional over $F$. □

Theorem 2.5. Let $D$ be a division ring with center $F$, and $G$ a locally solvable subnormal subgroup of $D^*$. If $G^{(i)}$ is algebraic over $F$ for some $i$, then $G \subseteq F$.

Proof. First, we prove that $F[G^{(i)}]$ is a division ring. Since $F[G^{(i)}]$ is normalized by $G$, it follows by Stuth’s Theorem ([10 Theorem 1]) that either $F[G^{(i)}] \subseteq F$ or $F[G^{(i)}] = D$. If the first case occurs, then $F[G^{(i)}] = F$; we are done. Assume that $F[G^{(i)}] = D$. Let $T = \tau(G^{(i)})$ be the unique maximal periodic normal subgroup of $G^{(i)}$. By [8 Theorem 8], we conclude that $T \subseteq F$. It follows by [13 Theorem 1.1(c)] that $F[G^{(i)}]$ is a crossed product over an abelian characteristic subgroup $A$ of $G^{(i)}$. Let $B$ be a maximal abelian normal subgroup of $G^{(i)}$ containing $A$. We shall show that $G^{(i)}/B$ is a simple group. For, let $C$ be a normal subgroup of $G^{(i)}$ properly containing $B$. It is clearly that $C$ is a non-abelian subnormal subgroup of $D^*$, hence $F(C) = D$ by Stuth’s Theorem. Moreover, Lemma 2.2 says that $F[C]$ is an Ore domain whose skew field of fractions coincided with $F(C) = D$. Thus, we may apply Lemma 2.4 to conclude that $G^{(i)} = CA = C$; recall that $A \subseteq C$. It follows that $G^{(i)}/B$ is simple, as claimed. Since $G^{(i)}/B$ is locally solvable and simple
Next, we claim that \( G^{(i)} \subseteq F \). Indeed, by what we have proved, it follows that \( F[G^{(i)}] \) is a division ring, which is clearly normalized by \( G \). By Stuth’s Theorem ([11, Theorem 1]), either \( F[G^{(i)}] \subseteq F \) or \( F[G^{(i)}] = D \). If the first case occurs, then we are done. Now suppose that \( F[G^{(i)}] = D \). It follows from [11, Theorem 1.1] that \( G^{(i)} \) is abelian-by-locally finite. Since \( G \) is a subnormal subgroup of \( D^* \), so is \( G^{(i)} \). By Lemma 2.5, we have \( G^{(i)} \subseteq F \). Therefore, in any case \( G^{(i)} \subseteq F \), as claimed. In other words, we have \( G \) is solvable, and the result follows from [10, Theorem 2]. □

**Lemma 2.6.** Let \( D \) be a division ring with center \( F \), and \( G \) a subnormal subgroup of \( D^* \). Assume that \( M \) is a non-abelian locally solvable maximal subgroup of \( G \) such that \( M^{(i)} \) is algebraic over \( F \) for some \( i \). Then \( F[M^{(i)}] \) is a division ring.

**Proof.** Since \( F(M^{(i)}) \) is normalized by \( M \), it follows that \( M \subseteq N_G(F(M^{(i)})) \subseteq G \). Since \( M \) is maximal in \( G \), either \( M = N_G(F(M^{(i)})) \) or \( N_G(F(M^{(i)})) = G \). If the first case occurs, then \( M^{(i)} \subseteq N_G(F(M^{(i)})) \cap G \) is subnormal in \( F(M^{(i)})) \subseteq M \). By Theorem 2.4, we conclude that \( M^{(i)} \) is abelian. The algebraicity of \( M^{(i)} \) implies that \( F[M^{(i)}] = F(M^{(i)}) \) is a field; we are done. If the second case occurs, then \( F(M^{(i)})) = D \) by Stuth’s Theorem. Let \( T = \tau(M^{(i)}) \) be the unique maximal periodic normal subgroup of \( M^{(i)} \). The local solvability of \( T \) implies that it is actually a locally finite group ([8, Lemma 2.2]). Since \( T \) is a characteristic subgroup of \( M^{(i)} \), it is normal in \( M \). It follows that \( M \subseteq N_G(F(T)) \subseteq G \), which yields either \( G = N_G(F(T)) \) or \( M = N_G(F(T)) \). The former case implies \( F(T) = D \), hence \( D \) is a locally finite division ring by Lemma 2.4. Since \( M \) is locally solvable, it contains no non-cyclic free subgroups, and thus \( [D : F] < \infty \) by [3, Theorem 3.1]. If the latter case occurs, then \( T \subseteq F(T) \cap G \) is subnormal in \( F(T) \). In view of [6, Theorem 8], we conclude that \( T \) is contained in the center of \( F(T) \), which means \( T \) is abelian. There are two possible cases.

Case 1. \( T \nsubseteq F \).

Take \( x \in T \setminus F \). Since \( x \) is algebraic over \( F \), the elements of the set \( x^M = \{ m^{-1}xm | m \in M \} \subseteq F(T) \) have the same minimal polynomial over \( F \); recall that \( F(T) \) is a field. This implies that \( |x^M| < \infty \), which says that \( |M : C_M(x)| < \infty \). If we set \( H = \mathrm{Core}_M(C_M(x)) \), then \( H \) is a normal subgroup of finite index in \( M \). The normality of \( H \) in \( M \) implies that \( M \subseteq N_G(F(H)) \subseteq G \). Therefore, we have either \( N_G(F(H)) = G \) or \( N_G(F(H)) = M \). The first case implies that \( F(H) = D \), which means \( x \in F \), a contradiction. We may therefore assume that \( N_G(F(H)) = M \), from which it follows that \( H \subseteq G \cap F(H) \) is a subnormal subgroup of \( F(H) \) contained in \( M \). By Theorem 2.5, we have \( H \) is abelian. If we set \( K = F(H) \), then the finiteness of \( M/H \) implies that \( D = F(M) \) is finite dimensional over \( K \). This fact yields \( [D : F] < \infty \), hence \( F[M^{(i)}] = F(M^{(i)}) \) is a division ring.

Case 2. \( T \subseteq F \).
It follows by [13, Theorem 1.1(c)] that $F[M^{(i)}]$ is a crossed product over an abelian characteristic subgroup $A$ of $M^{(i)}$. Let $B$ be a maximal abelian normal subgroup of $M^{(i)}$ containing $A$. We shall show that $M^{(i)}/B$ is a simple group. For, let $C$ be a normal subgroup of $M^{(i)}$ properly containing $B$. It is clearly that $C$ is non-abelian. Since $C$ is normal in $M$, we have $M \subseteq N_G(F(C)^*) \subseteq G$, hence either $M = N_G(F(C)^*)$ or $G = N_G(F(C)^*)$. The former case implies that $C$ is abelian, a contradiction. Thus we have $G = N_G(F(C)^*)$, from which it follows that $F(C) = D$. Moreover, Lemma 2.2 says that $F[C]$ is an Ore domain whose skew field of fractions coincided with $F(C) = D$. Thus, we may apply Lemma 2.4 to conclude that $M^{(i)} = CA = C$; recall that $A \subseteq C$. This fact shows that $M^{(i)}/B$ is simple. Since $M^{(i)}/B$ is locally solvable and simple group, it is finite. Setting $K = F[B]$, then the algebraicity of $B$ implies that $K = F(B)$ is a subfield of $D$ contained in $F[M^{(i)}]$. Because $M^{(i)}/B$ is finite, we conclude that $[F[M^{(i)}] : K]_F < \infty$. Now, $[F[M^{(i)}]]$ is a domain which is finite dimensional over the subfield $K$, hence $F[M^{(i)}]$ is a division ring. \hfill \Box

Recall that for a group $G$, the set of all elements with finite conjugate classes forms a subgroup of $G$. Such subgroup is called the FC-center of $G$.

**Theorem 2.7.** Let $D$ be a division ring with center $F$, and $G$ a normal subgroup of $D^*$. Assume that $M$ is a non-abelian locally solvable maximal subgroup of $G$ such that $M^{(i)}$ is algebraic over $F$ for some $i$. Then, the following hold:

(i) There exists a maximal subfield $K$ of $D$ such that $K/F$ is a finite Galois extension with $\text{Gal}(K/F) \cong M/K^* \cap G \cong \mathbb{Z}_p$ for some prime $p$, and $[D : F] = p^2$.

(ii) The subgroup $K^* \cap G$ is the FC-center. Also, $K^* \cap G$ is the Fitting subgroup of $M$. Furthermore, for any $x \in M \setminus K$, we have $x^p \in F$ and $D = F[M] = \bigoplus_{i=1}^p Kx^i$.

**Proof.** First, we show that $[D : F] < \infty$. If we set $R = F[M^{(i)}]$, then $M \subseteq N_G(R^*) \subseteq G$. By the maximality of $M$ in $G$, it follows that either $N_G(R^*) = M$ or $N_G(R^*) = G$. We need to consider two possible cases:

**Case 1:** $N_G(R^*) = M$.

In this case, we see that $R^* \cap G \subseteq M$, from which we conclude that $M^{(i)} \trianglelefteq R^* \cap G$. This implies that $M^{(i)}$ is a subnormal subgroup of $R^*$. Moreover, by Lemma 2.6 $R$ is a division ring. According Lemma 2.8 we deduce that $M^{(i)}$ is abelian. Hence $M$ is solvable, and we are done by [3, Theorem 3.2] or [1].

**Case 2:** $N_G(R^*) = G$.

Since $R$ is a division ring normalized by $G$, by Stuth’s Theorem, either $F[M^{(i)}] \subseteq F$ or $R = F[M^{(i)}] = F[M] = D$. The first case implies that $M$ is solvable, and hence $[D : F] < \infty$ by [3, Theorem 3.2] or [1]. Now, suppose the second case occurs, from which it follows by [11, Theorem 1.1] that $M$ is abelian-by-locally finite. Let $A$ be an abelian normal subgroup of $M$ such that $M/A$ is locally finite. We have the two following subcases:

**Subcase 2.1:** $M^{(i)} \cap A \subseteq F$. 

We know that $M/(i) / M/(i) \cap A \cong M/(i) A/A \leq M/A$ is a locally finite group. By Lemma 2.4, it follows that $D = F[M/(i)]$ is a locally finite division ring. Since $M$ is locally solvable, it contains no non-cyclic free subgroups. Thus, by [3, Theorem 3.1], we have $[D : F] < \infty$.

Subcase 2.2: $M/(i) \cap A \nsubseteq F$.

In this case, there is an element $x \in (M/(i) \cap A) \setminus F$ which is algebraic over $F$. By the same arguments used in Case 1 of the proof of Lemma 2.6, we have $[D : F] < \infty$.

Setting $n = [D : F]$, we know that $D \otimes_F D^\text{op} \cong M_n(F)$. Thus, by viewing $M$ as a subgroup of $GL_n(F)$, we conclude that $M$ is a solvable group and the results follow from [8, Theorem 3.2] or [1].

□

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