Nilpotent Classical Mechanics

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March 27, 2022

Abstract

The formalism of nilpotent mechanics is introduced in the Lagrangian and Hamiltonian form. Systems are described using nilpotent, commuting coordinates $\eta$. Necessary geometrical notions and elements of generalized differential $\eta$-calculus are introduced. The so called $s-$geometry, in a special case when it is orthogonally related to a traceless symmetric form, shows some resemblances to the symplectic geometry. As an example of an $\eta$-system the nilpotent oscillator is introduced and its supersymmetrization considered. It is shown that the $R$-symmetry known for the Graded Superfield Oscillator (GSO) is present also here for the supersymmetric $\eta$-system. The generalized Poisson bracket for $(\eta, p)$-variables satisfies modified Leibniz rule and has nontrivial Jacobiator.

March 27, 2022

1 Introduction

Supersymmetric mechanics is well established theory with numerous applications and interesting models, not all of them being just a toy models for a supersymmetric field theory, but interesting for themselves [1, 2, 3]. Structurally it is related to the classical limit of theories containing fermions. Two algebraical properties of the $\theta$-variables used in description of the supersymmetric mechanics are essential for that: anticommutativity and resulting from it nilpotency. As it is noted by Freed [4] very important issue in describing fermions classically is nilpotency in the ring of functions of $\theta$-variables. He notes that anticommutativity is less important.

*This work is partially supported by Polish KBN Grant # 1PO3B01828
In the present work we wish to study systems where only nilpotency condition is assumed, we relax the demand to have anticommuting (spinors) in the basic nilpotent theory. The anticommuting variables will only appear after supersymmetrization of such a nilpotent systems as an additional sector of the relevant model. In the nilpotent mechanics the Pauli exclusion principle is present only via nilpotency without other relations to spin and statistic (anticommutativity). However having commuting and nilpotent variables is not harmless. We have to use the modified calculus [5], where the usual Leibniz rule is no valid. This fact enters in various points of the generalization. The nilpotent mechanics in some extend indirectly shows, that for classical description of fermions as nilpotency as anticommutativity are equally important.

In the present work we firstly, in Sec.2 and Sec.3, introduce notion of the commuting nilpotent $\mathcal{N}$-numbers and elements of generalized differential calculus for functions of nilpotent commuting variables. In Sec.4 we describe the $s$-geometry in the $\mathcal{N}$-modules. In a general case $s$-forms are related to the pseudo-Euclidean metrics with arbitrary signature. Here we shall restrict to the even dimensional $s$-forms orthogonally related to the pseudo-Euclidean forms of the zero signature. This is motivated by the present formulation of the nilpotent $\eta$-mechanics. In such a case we describe a symmetry group of the $s$-form denoted here by the $Ap(n)$. This group is conjugated to the $O(n,n)$, however due to the restrictions inherent in the $s$-geometry only specific matrix realizations are possible for the elements of the $Ap(n)$ (we shall use the $Ap(n)$ symbol to differentiate such matrices from the generic $O(n,n)$ elements). Then, in Sec.5, we introduce the notion of $\eta$-nilpotent mechanics in terms of the Lagrangian and then Hamiltonian formalisms, and we show properties of the $\eta$-Poisson bracket. This generalized bracket is linear and antisymmetric, but satisfies modified Leibniz rule and has nontrivial Jacobitator. Finally in Sec.6 we describe the nilpotent oscillator and we give its $N=2$ supersymmetrization. In addition to the supersymmetry invariance there is present interesting $R$-symmetry, already known for the Graded Superfield Oscillator (GSO) [6]. This symmetry intertwines two sectors of the total system which are otherwise separately invariant under $N=2$ SUSY.

2 $\mathcal{N}$-numbers

Our aim is to work with spaces where the coordinates are given by nilpotent commuting $\eta$ ‘numbers’ - an analog of the anticommuting $\theta$ variables used in supersymmetry. Because of the nilpotency of the $\eta$-variables we can expect some similarities to the superanalysis, but this two formalisms are different. The nilpotency is natural for anticommuting (odd) elements but for commuting (even) elements it is an additional restrictive condition. It makes for example, that the usual Leibniz rule is not valid and we have to modify it [5]. We firstly introduce the notion of commuting nilpotent $\mathcal{N}$-numbers then functions of $\mathcal{N}$-variables and discuss, based on them, generalized differential and integral calculus.

Let $\mathcal{N}_n(\mathcal{K})$ be an associative commutative free algebra over $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$
generated by the set $\{\xi^i\}_1^m$ of nilpotent elements, $(\xi^i)^2 = 0$, $i = 1, 2, \ldots, m$. We define the unital algebra $\mathcal{N}_m(\mathcal{K}) = \mathcal{K} \oplus \overline{\mathcal{N}}_m$. The $\mathcal{N}_m$ as a vector space is isomorphic to the Grassmann algebra $\wedge^{\mathcal{N}}_m$ over $m$-dimensional space $\mathcal{V}_m$, however $\mathcal{N}$ is a commutative algebra, but still graded with the dimension equal $2^m$. For the $\mathcal{N}_m$ we have $2^m$ monomials spanning the algebra
\begin{equation}
1, \xi^i_1, \xi^i_1 \xi^i_2, \ldots, \xi^i_1 \xi^i_2 \ldots \xi^i_k, \ldots, \xi^1 \ldots \xi^m \tag{1}
\end{equation}
Products of $\xi^i$ are symmetric and indices, due to nilpotency of $\xi^i$, cannot repeat therefore we will write such product of generators $\xi^i$ using strictly ordered multi-indices, namely
\begin{equation}
\xi^I_k = \xi^i_1 \xi^i_2 \ldots \xi^i_k, \quad \text{and} \quad \xi^I_0 \equiv 1, \quad I_0 = \emptyset, \tag{2}
\end{equation}
where $I_k = (i_1, i_2, \ldots, i_k)$ and $i_1 < i_2 < \ldots < i_k$. Any element $\nu$ of $\mathcal{N}_m(\mathcal{K})$ is of the form
\begin{equation}
\nu = \sum_{k=0}^{\infty} v_{I_k} \xi^I_k, \quad v_{I_k} \in \mathcal{K}, \tag{3}
\end{equation}
where $v_0 \equiv v_0$. Analogously to the Grassmannian case, we will refer the decomposition $\mathcal{N} = \mathcal{K} \oplus \overline{\mathcal{N}}$ to the body and soul of an element $\nu = v_B + v_S$, $v_B \in \mathcal{K}$, $v_S \in \overline{\mathcal{N}}$. In the finite dimensional $\mathcal{N}_m$ any $v_S$ is nilpotent of some order i.e. there exists $n < m$ that $(v_S)^n = 0$. One can also consider the infinite dimensional version of this algebra which will be denoted by $\mathcal{N}$ (analogously to the Banach-Grassmann algebra considered in Ref.\cite{14, 15}). In the following we will consider the infinite number of algebraically independent nilpotent generators in the $\mathcal{N}$. Let $\mathcal{M} \subset \mathcal{N}$, the anihilator $\mathcal{M}^\perp$ of $\mathcal{M}$ is defined as the following set
\begin{equation}
\mathcal{M}^\perp = \{ \nu' \in \mathcal{N} | \forall \nu \in \mathcal{M}, \nu \nu' = 0 \} \tag{4}
\end{equation}
Analogously to the Grassmann algebra the $\mathcal{N}$ algebra we shall call effective when $\mathcal{N}^\perp = \{ 0 \}$. Infinite dimensional $\mathcal{N}$ algebra is effective and in the finite dimensional case anihilator is nontrivial and contains the ‘last’ element $\xi^1 \ldots \xi^m$. We shall use effective $\mathcal{N}$ algebra to avoid degeneracy related to the presence of the nontrivial anihilator. The $\mathcal{N}$ algebra will play the role of “field of numbers”. Elements of $\mathcal{N}$ with nontrivial body are invertible and in such case inverse element is given in the following form
\begin{equation}
\nu^{-1} = v_B^{-1} \sum_{n=0}^{\infty} (-v_B^{-1} v_S)^n \tag{5}
\end{equation}
Let $\eta \in \overline{\mathcal{N}}$ denotes nilpotent element whose square vanishes, $\eta^2 = 0$. We shall call such elements first order nilpotents, and denote them by $\eta$. Let $\mathcal{D} = \{ \eta \in$
\( \mathcal{N} | \eta^2 = 0 \) \( \subset \mathcal{N} \) be a set of such first order nilpotents. Pair of the first order nilpotent elements \( \eta \) and \( \eta' \) we shall call independent nilpotents if \( \eta \eta' \neq 0 \). Let us put, for \( k \in \mathbb{N} \)

\[
\hat{D}_k = \{(\eta_1, \eta_2, \ldots, \eta_k) \in \mathcal{D}^k = \mathcal{D} \times \ldots \times \mathcal{D} \mid \eta_1 \eta_2 \ldots \eta_k \neq 0\}
\]  

(6)

Nilpotent of the first order elements do not need be monomials in \( \xi^i \). For example one can take \( \eta = \nu_1 \xi^1 + \nu_1^3 \xi^1 \xi^3 \), then \( \eta^2 = 0 \). As we describe it below, the \( \mathcal{N}_m \) algebras have matrix realizations, but since we want to use this structure as the basics "field" of numbers, such realizations will not be used in our approach.

**Examples:**

1. The simplest possible \( \mathcal{N} \) algebra has one nilpotent generator. Arbitrary \( v \in \mathcal{N}_1 \) has the form

\[
v = v_0 \cdot 1 + v_1 \xi, \quad v_0, v_1 \in \mathcal{H}
\]  

(7)

Matrix realization is the following

\[
\xi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix} v_0 & v_1 \\ 0 & v_0 \end{pmatrix}.
\]  

(8)

This structure is isomorphic as an algebra to the dual numbers as well as to the Grassmann algebra with one odd generator.

2. Matrix realization from the above example can be generalized to the \( \mathcal{N}_m \) case. The nilpotent generators \( \{\xi^i\}_1^m \) can be taken in the following form

\[
\xi_i = I_2 \otimes \ldots I_2 \otimes \xi \otimes I_2 \ldots \otimes I_2,
\]  

(9)

where \( I_2 \) is two dimensional unit matrix and \( \xi \) is located at the \( i \)-th position of the tensor product. Obviously \( \xi^i \xi^j = \xi^j \xi^i \neq 0 \) for \( i \neq j \) and \( (\xi^i)^2 = 0 \), \( i, j = 1, 2, \ldots, m \).

Let us note that in the commutative algebra sum of nilpotents is again nilpotent of some order, but for noncommutative algebras this is not true. Obviously in the algebra \( \mathcal{N}_m \) for \( m > 2 \) we have nilpotents of order higher then one, on the other hand for our formulation nilpotents of the first order play distinguished role. In the following we will explicitly mention the degree of nilpotency when necessary, to avoid misunderstanding.

### 3 \( \eta \)-calculus

One can introduce the notion of differential and integral for functions of the \( \eta \)-variables, in parallel to these used in superanalysis. However, due to different properties of the multiplication in the algebra \( \mathcal{N} \) they behave differently. As in the approach to the superalgebras and superanalysis considered e.g in [14][15] we shall treat the \( \eta \)-variables, \( \eta^2 = 0 \), as not fixed.
3.1 $\eta$-functions

Let $\eta_i, i = 1, 2, \ldots, n$ be from the set of independent nilpotent first order elements $\eta_i \in \tilde{D}_n$, namely

$$(\eta^i)^2 = 0 \ \forall i, \ \eta^1 \cdot \eta^2 \cdot \ldots \cdot \eta^n \neq 0$$

and $\bar{\eta} = (\eta^1, \eta^2, \ldots, \eta^n)$. Moreover let, as before, the $I_k$ denotes a strictly ordered multi-index. We shall define the function $f(\bar{\eta}) \in F[\bar{\eta}]$ of $n$ $\eta$-variables by the following expansion

$$f(\bar{\eta}) = \sum_{k=0}^{n} f_{I_k} \eta^{I_k}, \quad (11)$$

where $f_{I_k} \in \mathcal{N}$ are constant elements. The expansion (11) gives explicitly the dependence of a function $f$ on the $\eta$-variables. When the function $f$ depends also on the $x \in \mathcal{K}^n$, then $f_{I_k} : \mathcal{K}^n \mapsto \mathcal{N}, f \in F[x, \bar{\eta}]$. In the following we will consider $f(t, \bar{\eta}) = \sum f_{I_k}(t) \eta^{I_k}$, with $t \in \mathbb{R}$. In particular one can consider the set of $\eta$-functions with coefficients in the expansion (11) from the $\mathcal{K}^n$, $f_{I_k} \in \mathcal{K}$. The such of such a functions will be denoted by $\mathcal{F}_0[x, \bar{\eta}]$. We want to stress that $\eta$-variables are treated here analogously to the $\theta$ anticommuting variables used in superdifferential calculus.

3.2 $\eta$-derivative

Let us introduce the $\eta$-derivative analogously as it is done in superanalysis, namely as a contraction defined on the basic variables

$$\partial_i \eta^j = \delta^j_i, \quad \partial_i 1 = 0, \quad (12)$$

where

$$\partial_j = \frac{\partial}{\partial \eta^j} \quad (13)$$

By linearity over $\mathcal{N}$ we extend it to the $\mathcal{F}[\bar{\eta}]$ i.e. $\partial_i (a \eta^k + b \eta^j) = a \partial_i \eta^k + b \partial_i \eta^j$, where $a, b \in \mathcal{N}$. Naturally,

$$\partial_i \partial_j = \partial_j \partial_i \quad (14)$$

Immediate conclusion from this definition is that the conventional Leibniz rule is not valid, instead for $f(\bar{\eta}), g(\bar{\eta}) \in \mathcal{F}[\bar{\eta}]$ we have the following relation

$$\partial_i (f \cdot g) = \partial_i f \cdot g + f \cdot \partial_i g - 2 \eta^i \partial_i f \partial_i g \quad (15)$$

This is an example of generalized Leibniz rule with the so called Leibnizian term considered in Ref. [5]. The following relations are direct consequence of the Eq. (15).
(i) \[ \partial_i(\eta_i f) = f - \eta_i \partial_i f \] 

(ii) \[ [\partial_i, \eta_i] = 1 - 2\eta_i \partial_i, \quad [\partial_i, \eta_i]_+ = 1 \] 

(iii) \[ \nabla_i(f g) = \nabla_i f g + f \nabla_i g, \quad \text{for} \quad \nabla_i = \eta_i \partial_i \quad (\text{no sum}) \] 

Let us observe that one can consider "square root" of \( \eta \)-functions and derivatives within the superalgebra, in the following sense. In the simplest case we can treat \( \eta \) as a composite object \( \eta = \theta_1 \theta_2 \), where \( \theta_i, i = 1, 2 \) are Grassmannian anticommuting variables. Taking the even in \( \theta \) functions of the form \( f(\theta_1, \theta_2) = f_0 + f_{12} \theta_1 \theta_2 \) and second order derivative \( D = \partial^2 / \partial \theta_2 \partial \theta_1 \) we can write

\[ D(f(\theta_1, \theta_2) g(\theta_1, \theta_2)) = Df \cdot g + f \cdot Dg - (-1)^{|f|} \left( \frac{\partial}{\partial \theta_1} f \frac{\partial}{\partial \theta_2} g - \frac{\partial}{\partial \theta_2} f \frac{\partial}{\partial \theta_1} g \right) \] 

where graded Leibniz rule for \( \theta \)-derivatives was used. For even functions, \(|f| = 0\), using expansion of functions \( f, g \) finally we get

\[ D(f \cdot g) = Df \cdot g + f \cdot Dg - 2\theta_1 \theta_2 Df \cdot Dg. \] 

Expressing in above functions product of \( \theta \)'s by \( \eta \) we can also identify \( D \) and \( \partial / \partial \eta \). In this context modified Leibniz rule for \( \eta \) calculus can be interpreted as the effect of the second order superdifferential calculus. However, this observation does not mean that the nilpotent mechanics, we are going to introduce, can be reduced to the pseudo-mechanics or supersymmetric mechanics. Treating the \( \eta \)-variables as a composite - even product of the Grassmannian variables - does not automatically reduce one model to an another.

### 3.3 \( \eta \)-integration

Let us use again the analogy to the superanalysis and define \( \eta \)-integral by the following contractions

\[ \int \eta_i d\eta_j = \delta_{ij}, \quad \int d\eta_i = 0 \] 

and by linearity over \( \mathcal{N} \) extend it to the \( \mathcal{F}[\bar{\eta}] \). Such defined \( \eta \)-integral has the following properties

(i) \[ \int \bar{\eta} d\bar{\eta} = 1, \quad \bar{\eta} = \eta_1 \eta_2 \ldots \eta_n, \quad d\bar{\eta} = d\eta_1 d\eta_2 \ldots d\eta_n \]
or equivalently for strictly ordered multi-indices $I_k$

$$\int \eta^k d\bar{\eta} = \begin{cases} 
0, & k < n \\
1, & k = n
\end{cases} \quad (23)$$

(ii) \hspace{1cm} \int \partial_i f(\bar{\eta}) d\eta_i = 0, \quad \text{and} \quad \int \partial_i f(\bar{\eta}) d\bar{\eta} = 0, \quad (24)

where $f(\bar{\eta}) = f(\eta_1, \eta_2, \ldots, \eta_n)$

(iii) the integration by part formula

$$\left( \int f d\eta_i \right) \left( \int g d\eta_i \right) = \frac{1}{2} \left( \int (\partial_i f) \cdot g d\eta_i + \int f \cdot (\partial_i g) d\eta_i \right) \quad (25)$$

(iv) Let matrix $A$ represents permutation and scaling transformation, $\bar{\eta} = A\eta'$

$$\int f(\bar{\eta}) d\bar{\eta} = (\text{Per} A)^{-1} \int f(A\eta') d\eta' \quad (26)$$

where $\text{Per} A$ is the permanent of the matrix $A$.

(v) $\delta$-function

$$\int f(\bar{\eta}) \delta(\bar{\eta} - \bar{\rho}) d\bar{\eta} = f(\bar{\rho}) \quad (27)$$

has the following resolution

$$\delta(\bar{\eta} - \bar{\rho}) = \prod_{i=1}^{n} (\eta_i + \rho_i) \quad (28)$$

Note the plus sign in the above resolution of the $\delta$ function for $\eta$ variables (what differs it from the $\delta$-function known from superanalysis, for anticommuting $\theta$-variables).

The proofs of above facts are straightforward. Let us prove the integration by part formula. Indeed, using modified Leibniz rule we can write

$$\int \partial_i (f(\bar{\eta}) \cdot g(\bar{\eta})) d\eta_i = \int \partial_i f(\bar{\eta}) \cdot g(\bar{\eta}) d\eta_i + \int f(\bar{\eta}) \cdot \partial_i g(\bar{\eta}) d\eta_i$$

$$-2 \int \eta_i \partial_i f(\bar{\eta}) \partial_i g(\bar{\eta}) d\eta_i \quad (29)$$

$$-2 \int \eta_i \partial_i f(\bar{\eta}) \partial_i g(\bar{\eta}) d\eta_i \quad (30)$$
The left-hand side of the above formula vanishes, due to the (24), and
\[ \int \eta_i \partial_i f(\tilde{\eta}) \partial_i g(\tilde{\eta}) d\eta_i = \partial_i f(\tilde{\eta}) \partial_i g(\tilde{\eta}), \] (31)
moreover \( \int f d\eta_i = \partial_i f \), therefore
\[ \int \partial_i f(\tilde{\eta}) \cdot g(\tilde{\eta}) d\eta_i = 2 \int f(\tilde{\eta}) d\eta_i \int g(\tilde{\eta}) d\eta_i - \int f(\tilde{\eta}) \cdot \partial_i g(\tilde{\eta}) d\eta_i \] (32)
or in in equivalent form (25).

4 The geometry of \( s \)-forms

The notion of a configuration space that we will use in the definition of the nilpotent mechanics has two ingredients: one is the free bimodule over commutative algebra \( \mathcal{N} \), the second is the \( s \)-form with values in \( \mathcal{N} \) defining geometry in such a bimodule (the geometry which is, in some sense, compatible with nilpotency of the coordinates). Firstly let us describe necessary notions from the \( s \)-geometry \[8\], then we will consider \( \mathcal{N} \)-module which allows to use commutative nilpotent \( \eta \) variables. Main idea behind \( s \)-geometry is to have a non-degenerate symmetric form, which cannot be diagonalized, to avoid its triviality after generalization to the coordinates which are nilpotent. Unlike the Grassmannian case, where coordinates anticommute and natural geometry is given by antisymmetric form, here construction is not general-linearly covariant. We have to restrict the set of admissible bases. Obtained description resembles the light cone formalism for pseudo-Euclidean spaces. Because of further applications to the nilpotent mechanics we shall confine ourselves to the case when such an \( s \)-form can be related to the diagonal pseudo-Euclidean one by an orthogonal transformation. What means that we shall restrict discussion to the \( s \)-forms related to pseudo-Euclidean metrics of zero signature. We will discuss the group of symmetries of such \( s \)-form in dimension \( 2n \), which will be denoted \( Ap(n) \). It is conjugated to the \( O(n,n) \) but because of the restrictions on the set of bases we have to use special matrix representation. It can be obtained from the hyperbolic \( c - s \) form for the pseudo-orthogonal matrices or in the special case, from the solution analogous to the one known for the symplectic group \( Sp(n) \).

4.1 \( s \)-geometry

We shall begin with the definition of the \( s \)-form in a vector space. Let \( \mathbb{V} \) be a vector space \( \text{dim}\mathbb{V} = 2n \), over \( \mathbb{R} \). The \( s \)-form is a \( \mathbb{R} \)-bilinear, symmetric mapping
\[ s : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R} \]
which is weakly nondegenerate (i.e. if \( s(v, v') = 0 \) for all \( v' \in \mathbb{V} \), then \( v = 0 \)) and strictly traceless for some set of bases \( \mathcal{B} \), where we say that metric is strictly traceless in a basis \( \{ b_i \}_{i=1}^{2n} \) in \( \mathbb{V} \) if

\[
s(b_i, b_i) = 0, \quad \forall i = 1, 2, \ldots, 2n.
\] (33)

Let us note here that an antisymmetric form is automatically strictly traceless in any basis, for a symmetric form the condition of being strictly traceless in any basis makes it trivial. Therefore we have to restrict the set of bases admitted by \( s \)-form. In the usual linear geometry terms the \( s \)-form would be diagonalized in a suitable basis to the form (within the \( s \)-geometry such transformation is not allowed)

\[
diag(+,+,\ldots,+,,-,-,\ldots,-)
\] (34)

where \( p + q = 2n \) and in general \( p \neq q \). However, if a form \( s \) can be related to the diagonal pseudo-Euclidean by an orthogonal transformation, then \( p = q = n \).

In the rest of this paper we shall confine to this case and related pseudo-Euclidean metrics will be of signature zero.

We will say that the \( s \) is given in the the standard form if

\[
s(v, v') = \sum_{i=1}^{n} v_i v'_{n+i} + v'_i v_{n+i}
\] (35)

The name "standard" is used here in analogous sense as the standard form of symplectic metric, moreover as it is known from the Witt’s theorem it is always possible to put nondegenerate symmetric form in the standard form, in the so called Witt’s or hiperbolic basis [7]. A basis \( \{ e_i \}_{i=1}^{2n} \) is the \( s \)-admissible basis iff \( s(e_i, e_i) = 0, \forall i \). By the \( s \)-space we shall understand the triple: a linear space (module) \( \mathbb{V}_{2n} \) with a \( s \)-form and the set of all \( s \)-admissible bases \( \mathcal{B} \) i.e. \( (\mathbb{V}_{2n}, s, \mathcal{B}) \).

Two \( s \)-spaces \( (\mathbb{V}, s, \mathcal{B}) \) and \( (\mathbb{V}', s', \mathcal{B}') \) are isomorphic if there exists a vector space isomorphism \( \Phi \) such that

\[
s'(\phi(v), \phi(w)) = s(v, w), \quad \text{where} \ v, w \in \mathbb{V}
\] (36)

In particular we have that \( \phi(\mathcal{B}) = \mathcal{B}' \). Let \( Aut_s(\mathbb{V}) \) be the set of automorphisms of \( (\mathbb{V}, s, \mathcal{B}) \)[8].

Such defined \( s \)-form is not positive definite. In the usual linear geometry terms it is related to the traceless hyperbolic geometry in the light-cone formalism. We shall use "relative length" of vectors, in the following sense: \( s(b_i, b_i) \) vanishes but we can normalize non-vanishing product of vectors from a fixed set, e.g. \( s(b_i, b_{i+n}) = 1, i = 1, 2, \ldots, n \).
4.2 The $Ap(n)$ group

As we have already noted, the $s$-geometry is obtained as the restriction of usual hyperbolic geometry in spaces with pseudo-orthogonal metric. There are twofold restrictions here: even dimensionality of space, smaller set of allowed bases and transformations. The group of transformations preserving the $s$-form (orthogonally related to the pseudo-Euclidean metric of zero signature) we shall call the $s$-plectic group and denote it $Ap(n)$. This name is justified by some resemblances to the symplectic group $Sp(n)$ [9, 10]. To describe this symmetry group let us consider the $s$ in the defined by (35) standard form. It is represented by the following matrix

$$s = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}; \quad s^2 = I_{2n}; \quad s^T = s; \quad Tr(s)|_k = 0, k = 1, 2, \ldots, 2n$$ \hspace{1cm} (37)

The $I_n$ denotes $n$-dimensional unit matrix and $Tr(s)|_k$ the trace of a principal $k \times k$ block of the matrix $s$. Now, $D \in Ap(n)$ when

$$D^T s D = s.$$ \hspace{1cm} (38)

Because the $s$ is given by Eq.(37), then writing $D \in Ap(n)$ in a block form

$$D = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$ \hspace{1cm} (39)

with $P, Q, R, S$ being $n \times n$-blocks, we get that the condition (38) enforces the following relations

$$R^T P = -P^T R, \quad R^T Q + P^T S = I_n \hspace{1cm} (40)$$
$$S^T Q = -Q^T S, \quad S^T P + Q^T R = I_n \hspace{1cm} (41)$$

and moreover $(det D)^2 = 1$. Above conditions resemble the analogous ones for the symplectic group, but unlike the symplectic case here the sign of $det D$ depends on dimension of $s$-space).

Let us observe that $Ap(n)$ is isomorphic to the pseudo-orthogonal group $O(n,n)$. If $J$ is the matrix of the pseudo-Euclidean metric in the canonical form

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$ \hspace{1cm} (42)

then $G \in O(n,n)$ if $G^T J G = J$. Because there exists the orthogonal matrix $U$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & -I_n \\ I_n & I_n \end{pmatrix}$$ \hspace{1cm} (43)

then $G \in O(n,n)$ if $G^T J G = J$. Because there exists the orthogonal matrix $U$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & -I_n \\ I_n & I_n \end{pmatrix}$$ \hspace{1cm} (43)
such that $U^{-1}SU = J$ we have isomorphism $\phi_U$ of $O(n, n)$ and $Ap(n)$ groups

$$D = \phi_U(G) = UGU^{-1}$$  \hfill (44)

Let us note that matrix $s \notin O(n, n)$ and $J \notin Ap(n)$. The matrix group $O(n, n)$ is well known and its explicit description can be obtained from the description of the orthogonal group $O(2n)$ by means of the exchange mapping [12]. The exchange mapping is defined generally on block matrices by the following formula

$$\text{exc} \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \left( \begin{array}{cc} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{array} \right),$$  \hfill (45)

where $A_{ij}, i, j = 1, 2$ are blocks of a matrix and $A_{11}$ is assumed to be invertible.

There is the ”hyperbolic” version of the $c - s$ theorem [12] which states that any pseudo-orthogonal $G$ matrix can be written in one of the following forms (we consider here only the case $O(n, n)$ not general $O(p, q)$) showing explicitly four connected components of this group $[13, 12]$

$$G = \pm \left( \begin{array}{cc} \tilde{c} & \pm \tilde{s} \\ \pm \tilde{c} & \tilde{s} \end{array} \right),$$  \hfill (46)

where $\tilde{c} = \text{diag}(\tilde{c}_1, \ldots, \tilde{c}_n)$, $\tilde{s} = \text{diag}(\tilde{s}_1, \ldots, \tilde{s}_n)$ and $\tilde{c}^2 - \tilde{s}^2 = I_n$. Now using the mapping $\phi_U$ we can get general description of matrices from $Ap(n)$

$$\pm \left( \begin{array}{cc} \tilde{c} - \tilde{s} & 0 \\ 0 & \tilde{c} + \tilde{s} \end{array} \right), \quad \pm \left( \begin{array}{cc} 0 & \tilde{c} - \tilde{s} \\ \tilde{c} + \tilde{s} & 0 \end{array} \right)$$  \hfill (47)

$\tilde{c} - \tilde{s} = e^{-\beta}$, $\tilde{c} + \tilde{s} = e^{\beta}$, $\beta \in \mathbb{R}$.

It is interesting to observe that for the $n = 2k$ conditions (40) and (41) can be solved generically in analogous way as it is done in the case of the symplectic group $Sp(n)$ [11]. Namely, let $n = 2k$, then subsets

$$\mathcal{N} = \left\{ \left( \begin{array}{cc} I_n & B_1 \\ 0 & I_n \end{array} \right) : B_1^T = -B_1 \right\}$$  \hfill (48)

$$\tilde{\mathcal{N}} = \left\{ \left( \begin{array}{cc} I_n & 0 \\ B_2 & I_n \end{array} \right) : B_2^T = -B_2 \right\}$$  \hfill (49)

and

$$\mathcal{D} = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & (A^T)^{-1} \end{array} \right) : A \in GL(n) \right\}$$  \hfill (50)

are subgroups of $Ap(n)$ and

$$\tilde{\mathcal{N}}\mathcal{D}\mathcal{N} = \left\{ \left( \begin{array}{cc} P & Q \\ R & S \end{array} \right) \in Ap(n) : \det P \neq 0 \right\}$$  \hfill (51)
It is easy to check that $\mathfrak{N}$, $\bar{\mathfrak{N}}$ and $\mathfrak{D}$ solve the conditions (40) and (41) and form subgroups. Now taking product

\[
\begin{pmatrix}
\mathbb{I}_n & 0 \\
B_2 & \mathbb{I}_n
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & (A^T)^{-1}
\end{pmatrix}
\begin{pmatrix}
\mathbb{I}_n & B_1 \\
0 & \mathbb{I}_n
\end{pmatrix} = 
\begin{pmatrix}
A & AB_1 \\
B_2A & B_2AB_1 + (A^T)^{-1}
\end{pmatrix}
\]  

(52)

We see that for $A, \det A \neq 0$ we get the following solution

\[
P = A, \quad Q = AB_1, \quad R = B_2A, \quad S = B_2AB_1 + (A^T)^{-1}
\]  

(53)

again satisfying the conditions (40) and (41). Therefore the $Ap(n)$ is generated by the sets $\mathfrak{D} \cup \mathfrak{N} \cup \{s\}$ or $\mathfrak{D} \cup \bar{\mathfrak{N}} \cup \{s\}$, where $s$ is as usual the following matrix

\[
s = \begin{pmatrix}
0 & \mathbb{I}_n \\
\mathbb{I}_n & 0
\end{pmatrix}
\]  

(54)

Moreover

\[
D^{-1} = sD^T = \begin{pmatrix}
S^T & Q^T \\
R^T & P^T
\end{pmatrix}
\]  

(55)

Finally, let us note that the solution given by the formula (52) can be written by means of the exchange operator

\[
\text{exc}\left( \begin{pmatrix}
A^{-1} & -B_1 \\
B_2 & (A^T)^{-1}
\end{pmatrix} \right) = 
\begin{pmatrix}
A & AB_1 \\
B_2A & B_2AB_1 + (A^T)^{-1}
\end{pmatrix},
\]  

(56)

where $A$ is invertible and $B_i, i = 1, 2$ are antisymmetric $n \times n$ matrices. Exchange operator maps such class of a matrices into the $Ap(n)$ group.

5 Nilpotent mechanics

In the present section we will discuss basic issues of the formalism of nilpotent mechanics. Lagrangian and Hamiltonian picture is possible. We propose a generalization of the Poisson bracket. Such an $\eta$-Poisson bracket satisfies the modified Leibniz rule and has nontrivial Jacobiator, which is not a subject of the Malcev identity.

5.1 Configuration space description

The nilpotent mechanical system will be defined on the configuration space being the free $\mathcal{N}$-bimodule $\mathbb{V}_{\mathcal{N}}$ with the $\mathcal{N}$-valued $s$-form.

\[
s : \mathbb{V}_{\mathcal{N}} \times \mathbb{V}_{\mathcal{N}} \mapsto \mathcal{N}
\]  

(57)
Since this structure is commutative, the generalization of definitions form the Sec. 4 to the free $\mathcal{N}$-module case, is straightforward.

We can introduce a Lagrangian with terms being analogs of the kinetic and potential energy. This entities are $\mathcal{N}$-valued therefore, no positive definitness can be established. Moreover the $s$-form itself is not positive definite either [8] (even on usual $\mathbb{R}^{2n}$ space). The Lagrangian of an $\eta$-system is defined as a $\mathcal{N}$-valued function on configuration $\mathcal{N}$-bimodule, given by

$$L = \frac{m}{2} s(\dot{\eta}, \eta) - V(\eta)$$  \hspace{1cm} (58)

In particular we shall consider a ”quadratic” potential for the $\eta$-oscillator.

$$V(\eta) = \frac{m\omega}{2} s(\eta, \eta) = \frac{m\omega}{2} s_{ij} \eta^i \eta^j.$$  \hspace{1cm} (59)

As it is known, usual variational principle can be generalized algebraically to the case of supersymmetric systems described by anticommuting variables. It turns out that for the nilpotent mechanics it is possible to reach this goal too, despite the lack of the usual Leibniz rule.

Let us consider the $\mathcal{N}$-valued action of the form

$$I[\eta^i, \dot{\eta}^i; \alpha] = \int_{t_1}^{t_2} L(\eta^i(t, \alpha), \dot{\eta}^i(t, \alpha)) dt, \quad \alpha \in \mathbb{R}$$  \hspace{1cm} (60)

One can consider generalization of the conventional [16][17] variations

$$\eta^i(t, \alpha) = \eta^i(t) + \alpha \zeta^i(t), \quad \zeta^i(t_1) = \zeta^i(t_2) = 0$$  \hspace{1cm} (61)

$$\dot{\eta}^i(t, \alpha) = \dot{\eta}^i(t) + \alpha \dot{\zeta}^i(t)$$  \hspace{1cm} (62)

$$\eta^i(t)^2 = \zeta^i(t)^2 = 0, \quad \dot{\eta}^i(t)^2 = \dot{\zeta}^i(t)^2 = 0$$  \hspace{1cm} (63)

However $\eta^i(t, \alpha)^2 \neq 0$ in general. This property is essentially different from the supersymmetric case where the first order nilpotency is automatical. We shell consider two cases

(i) $\eta^i$ and $\zeta^i$ are algebraically independent i.e. $\eta^i \zeta^i \neq 0$.

Then $\partial^i \zeta = 0$ and $\eta^i(t, \alpha)^2 = 2\alpha \eta^i \zeta^i \neq 0$. Where, as before, $\partial_i = \partial / \partial \eta^i$

(ii) $\eta^i$ and $\zeta^i$ are algebraically dependent i.e. $\eta^i \zeta^i = 0$.

Hence $\partial_i (\eta^i \zeta^i) = \zeta_i - \eta^i \partial_i \zeta_i = 0$. This means that $\zeta_i = \eta^i \partial_i \zeta_i$ and $\zeta_i = \eta^i \eta^i$ for some $\eta^i \neq 0$, $\eta^i \eta^i \neq 0$

The variation of the action is of the following form

$$\delta I = \frac{dI}{d\alpha} \bigg|_{\alpha=0} \alpha = \int_{t_1}^{t_2} \sum_k \left( \zeta^k \frac{\delta}{\delta \eta^k} L + \frac{d}{dt} \left( \zeta^k \frac{\delta L}{\delta \dot{\eta}^k} \right) \right) \alpha dt$$  \hspace{1cm} (64)
but for \( \eta \)-functions we have the following extension of the time ‘derivative’

\[
\frac{d}{dt} f(\vec{\eta}, \vec{\dot{\eta}}, t) = \frac{\partial}{\partial t} f(\vec{\eta}, \vec{\dot{\eta}}, t) + \sum_i \frac{\partial}{\partial \eta_i} f(\vec{\eta}, \vec{\dot{\eta}}, t) \dot{\eta}_i + \sum_i \frac{\partial}{\partial \dot{\eta}_i} f(\vec{\eta}, \vec{\dot{\eta}}, t) \ddot{\eta}_i
\]  

(65)

what gives for the product of functions in Eq. (64) the following relation

\[
\frac{d}{dt} \left( \sum_k \zeta^k \frac{\partial L}{\partial \dot{\eta}^k} \right) = \sum_k \zeta^k \frac{\partial L}{\partial \dot{\eta}^k} + \sum_k \zeta^k \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\eta}^k} \right) - 2 \sum_{ki} \dot{\eta}_i \dot{\eta}_i \frac{\partial \zeta^k}{\partial \eta^i} \frac{\partial}{\partial \eta^i} \left( \frac{\partial L}{\partial \dot{\eta}^k} \right)
\]  

(66)

Now for algebraically independent variations last term in (66) vanishes and for algebraically dependent variations we have

\[
\zeta^k = \eta^k \frac{\partial}{\partial \eta^k} \zeta^k \quad \text{(no sum)}
\]  

(67)

using this result in the variation of action (60) we finally get the analog of the Euler-Lagrange equations of motion for both cases

\[
\text{EL}^{(a)}: \quad \frac{\partial L}{\partial \eta^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\eta}^k} \right) = 0, \quad \text{for} \quad \eta^k(\alpha)^2 \neq 0
\]  

(68)

\[
\text{EL}^{(b)}: \quad \frac{\partial L}{\partial \eta^k} - \left( \frac{d}{dt} - 2 \eta^k \frac{\partial}{\partial \eta^k} \right) \frac{\partial L}{\partial \dot{\eta}^k} = 0, \quad \text{for} \quad \eta^k(\alpha)^2 = 0
\]  

(69)

For Lagrangians quadratic in velocities and coordinates these two types of equations coincide. In the present work we will restrict ourselves to such a case. It is interesting to observe that the function analogous to the energy of the conventional system, despite the modified Leibniz rule of the time derivative, is still "preserved" for both types of the E-L equations of motion for the systems with quadratic Lagrangians. Namely, we introduce the following function \( E \)

\[
E(\eta, \dot{\eta}) = \dot{\eta}^k \frac{\partial L}{\partial \dot{\eta}^k} - L(\eta, \dot{\eta})
\]  

(70)

and by inspection we get that

\[
\frac{d}{dt} E(\vec{\eta}, \vec{\dot{\eta}}) = 0.
\]  

(71)
5.2 Phase space description

To pass to the $\mathcal{N}$-phase space description of the system we introduce the momenta

$$p_k = \frac{\partial L}{\partial \eta^k},$$  \hspace{1cm} (72)

($p_k$ are first order nilpotents for $\eta$-systems) and adapt the idea of Legendre transformation. As in the conventional case we have to assume the regularity of a Lagrangian, in the following sense. Namely, the matrix $W$

$$W_{kl} = \frac{\partial^2 L}{\partial \eta^k \partial \eta^l},$$  \hspace{1cm} (73)

should be invertible for all $(\eta^k, \dot{\eta}^k)$ (for this it is not necessary that kinetic term is positive definite). For the Lagrangians with the kinetic term defined within the $s$-geometry in the $\mathcal{N}$-module we have that $W_{ij} = \frac{m^2}{2} s_{ij}$, and $det(s) \neq 0$. In such a case the EL$^{(a)}$ equations can be solved for accelerations

$$\ddot{\eta}_i = \sum_k W_{ik}^{-1} \left( \frac{\partial L}{\partial \eta^k} - \sum_j \dot{\eta}_j \frac{\partial^2 L}{\partial \eta^j \partial \dot{\eta}^k} \right),$$  \hspace{1cm} (74)

The space $\mathcal{P}_N$ with coordinates $(\eta^i, p_i)$ is $4K$-dimensional, where dimension of configuration space is even $N = 2K$, due to the nonsingular $s$-form. The generalization of the Hamiltonian function to the $\mathcal{P}_N$ case is the following

$$H = \sum_k p_k \dot{\eta}^k - L,$$  \hspace{1cm} (75)

and we get for this formalism the generalized Hamilton’s equations of motion

$$\dot{p}_k = - \frac{\partial H}{\partial \eta^k},$$  \hspace{1cm} (76)

$$\ddot{\eta}^k = \frac{\partial H}{\partial p_k}.$$  \hspace{1cm} (77)

The extension of the time derivative to the phase space $\mathcal{P}_N$ functions $f(\eta, p)$ is given in the following form

$$\frac{d}{dt} = \partial_t + \sum_k \eta^k \partial_k + \sum_k p_k \bar{\partial}^k,$$ \hspace{1cm} where \( \partial_k = \frac{\partial}{\partial \eta^k} \) and \( \bar{\partial}^k = \frac{\partial}{\partial p_k} \) \hspace{1cm} (78)

For further convenience let us introduce the following notations

$$\nabla_i = \eta_i \partial_i, \quad \bar{\nabla}^i = \bar{p}^i \bar{\partial}^i, \quad \text{(no summation!)}$$ \hspace{1cm} (79)
It is obvious that \( \langle \nabla_i, \tilde{\nabla}^j \rangle \) are derivations but \( \langle \tilde{\partial}_i, \tilde{\partial}^j \rangle \) are not derivations and they satisfy the \( \text{para-Leibniz} \) rule. Using above notation this rule can be written for the functions on \( \mathcal{P}, \mathcal{N} \) in the following forms (for \( \eta \) derivatives, \( \tilde{\partial}_i \))

\[
\tilde{\partial}_i (f \cdot g) = \tilde{\partial}_i f \cdot g + f \cdot \tilde{\partial}_i g - 2 \eta_i \tilde{\partial}_i f \partial_i g = \partial_i f \cdot g + f \cdot \partial_i g - \nabla_i f \partial_i g - \partial_i f \nabla_i g
\]

(81)

and analogously for the derivatives with respect to the momenta, \( \tilde{\partial}_i \). The extension of the time derivative to the functions on \( \eta \)-phase space is not usual derivation, but operation which satisfies modified Leibniz rule. We have

\[
\frac{d}{dt}(g(\eta, p) \cdot h(\eta, p)) = \dot{g} \cdot h + g \cdot \dot{h} - 2 \sum_k \eta^k \eta_k \partial_k g \cdot \partial_k h - 2 \sum_k \tilde{p}_k \tilde{p}^k g \cdot \tilde{p}^k h
\]

(83)

Using the equations of motion, the time derivative can be written as

\[
\frac{d}{dt}f(\eta, p) = \sum_k (\tilde{\partial}^k H \cdot \tilde{\partial}_k f(\eta, p) - \partial_k H \cdot \tilde{\partial}^k f(\eta, p))
\]

(82)

and for the product of functions we have

\[
\frac{d}{dt}(g(\eta, p) \cdot h(\eta, p)) = \dot{g} \cdot h + g \cdot \dot{h} - 2 \sum_k (\tilde{\partial}^k H \cdot \tilde{\nabla}_k g \cdot \partial_k h - \partial_k H \cdot \tilde{\nabla}^k g \cdot \tilde{\partial}^k h)
\]

(83)

Defining \( \eta \)-Poisson bracket as

\[
\{f(\eta, p), g(\eta, p)\}_0 = \sum_k (\tilde{\partial}^k f(\eta, p) \cdot \partial_k g(\eta, p) - \partial_k f(\eta, p) \cdot \tilde{\partial}^k g(\eta, p))
\]

(84)

we can finally realize the time derivative in the following form

\[
\frac{d}{dt}f(\eta, p) = \{H, f(\eta, p)\}_0
\]

(85)

This realization is consistent with the modified Leibniz rule for the time ‘derivative’, since using (85) we have

\[
\{H, g \cdot h\}_0 = \{H, g\}_0 \cdot h + g \cdot \{H, h\}_0 - 2 \sum_k (\tilde{\partial}^k H \cdot \tilde{\nabla}_k g \cdot \partial_k h - \partial_k H \cdot \tilde{\nabla}^k g \cdot \tilde{\partial}^k h)
\]

(86)

what agrees with the formula (83) and we can write the Hamilton’s equations of motion using the bracket \( \{\cdot, \cdot\}_0 \), in the form

\[
\dot{p}_k = \{H, p_k\}_0, \quad \dot{\eta}^k = \{H, \eta^k\}_0.
\]

(87)
5.3 \(\eta\)-Poisson bracket

Despite the similar definition \([84]\) the \(\eta\)-Poisson bracket for the commuting nilpotent variables differs from the usual or graded Poisson bracket we shall call it \(\eta\)-Poisson bracket or para-bracket. It is linear, antisymmetric, satisfies modified Leibniz rule. However it does not satisfy the Jacobi identity. We obtain nontrivial Jacobiator \([18]\) and moreover, the Jacobiator itself does not satisfy the Malcev identity.

Let \(f, g, h : \mathcal{P}_N \to \mathcal{N}\) be the \(\mathcal{P}_N\) phase space functions i.e. \(f, g, h \in \mathcal{F}(\mathcal{P}_N)\). The \(\eta\)-Poisson bracket

\[
\{f, g\}_0 \equiv \sum_k (\bar{\partial}^k f \cdot \partial_k g - \partial_k f \cdot \bar{\partial}^k g)
\]

has the following properties

(i) \(\{f, g\}_0 = -\{g, f\}_0\)  
(ii) \(\{f_1 + f_2, g\}_0 = \{f_1, g\}_0 + \{f_2, g\}_0\)  
(iii) \(\{f, g \cdot h\}_0 = \{f, g\}_0 \cdot h + g \cdot \{f, h\}_0 - 2\diamond(f|g,h)\)  
(iv) \(\sum_{cycl}\{\{f, g\}, h\}_0 = J(f, g, h)\)

where the para-Leibniz term is of the form

\[
\diamond(f|g,h) = \sum_k (\bar{\partial}^k f \cdot \nabla_k g \cdot \partial_k h - \partial_k f \cdot \bar{\partial}^k g \cdot \bar{\partial}^k h).
\]

The skew-symmetric operator \(J\) appearing in the Eq. (92) is called Jacobiator cf. \([18]\) and for our \(\eta\)-bracket has explicitly the following form

\[
J(f, g, h) = 2\sum_{cycl} i (\eta^i \{\partial_i f, \partial_i g\} \bar{\partial}^i h - p_i \{\bar{\partial}^i f, \bar{\partial}^i g\} \partial_i h).
\]

To see that the Jacobi identity is violated let us consider the following functions

\(f = \eta^i, g = p_i \eta^k, h = p_k p_i, k \neq i.\)

\[
J(\eta^i, p_i \eta^k, p_k p_i) = 2p_i
\]

The same example shows that the Malcev identity is not satisfied as well. Namely,

\[
\{J(f, g, h), f\} \neq J(f, g, \{f, h\}).
\]
6 Supersymmetric nilpotent oscillator

In the present section we shall consider the simple system of the two dimensional nilpotent \( \eta \)-oscillator (i.e. two is the lowest dimension of the nontrivial \( s \)-space). Then its supersymmetrization within the \( N = 2 \) SUSY multiplet. The nilpotent oscillator has its very close counterpart in the fermionic oscillator which is described using anticommuting variables and antisymmetric form \( \epsilon^{ij} \). Here again the simplest example is two dimensional. Using the \( N = 2 \) SUSY multiplet we supersymmetrize the the fermionic oscillator and then combine both independently supersymmetric systems into one. Such a new total system exhibits the \( R \)-symmetry known from the GSO [6]. We also comment the issue of the ‘parity duality’ between systems.

6.1 Nilpotent harmonic \( \eta \)-oscillator

As the simplest system realizing the nilpotent mechanics in \( \mathcal{N} \)-module \( \mathbb{V}_\mathcal{N} \) with nontrivial \( s \)-geometry, let us consider the \( \eta \)-oscillator in two dimensions. It is defined by the Lagrangian of the form

\[
\mathcal{L} = \frac{m}{2} s_{ij} \dot{\eta}^i \dot{\eta}^j - \frac{m \omega}{2} s_{ij} \eta^i \eta^j, \quad i, j = 1, 2
\]  

(97)

The properties of the \( s \)-form provide that this Lagrangian is nontrivial and gives equations of motion analogous to the conventional case.

\[
\ddot{\eta}_i = -\omega^2 \eta_i
\]  

(98)

The passage to the phase space description yields the \( \eta \)-Hamiltonian containing \( s \)-form on the \( \mathcal{P}_\mathcal{N} \)

\[
H = \frac{1}{2m} (s^{-1})^{ij} p_i p_j + \frac{m \omega^2}{2} s_{ij} x^i x^j
\]  

(99)

and generalized Hamilton’s equations of motion

\[
\begin{cases} 
\dot{x}^i = \frac{1}{m} (s^{-1})^{ij} p_j \\
\dot{p}_i = -m \omega^2 s_{ij} x^j
\end{cases}
\]  

(100)

6.2 Supersymmetric nilpotent oscillator

To supersymmetrize the nilpotent oscillator we introduce analogously to the usual bosonic oscillator case, the multiplet of functions with values in the \( \mathcal{N} \) algebra for commuting components and with values in the odd part of the Banach-Grassmann algebra \( \mathcal{Q}_1 \) [15] for anticommuting components. Therefore we have

\[
\eta_i \mapsto (\eta_i(t), \psi_i^\alpha(t), F_i(t)),
\]  

(101)
where the parity of functions is: \(|\eta_i| = |F_i| = 0, |\psi_i^\alpha| = 1\). Now the supersymmetric Lagrangian has fully analogous form to the usual supersymmetric model, namely

\[
\mathcal{L}_\eta = \frac{1}{2} s^{ij}(\dot{\eta}_i F_j + F_i \dot{\eta}_j) + \frac{1}{2} \delta_{\alpha\beta} s^{ij} \psi_i^\alpha \psi_j^\beta - \frac{\omega}{2} s^{ij}(\eta_i F_j + F_i \eta_j - \epsilon_{\alpha\beta} \psi_i^\alpha \psi_j^\beta)
\]  

(102)

Action defined by this Lagrangian is invariant under the following supersymmetry transformations (obviously we have usual s-geometry \(Ap(2)\) invariance as well)

\[
\delta \epsilon \eta_i = \epsilon \alpha \psi_i^\alpha
\]

(103)

\[
\delta \epsilon \psi_i^\alpha = \epsilon \alpha\beta \epsilon \alpha F_i - \epsilon \alpha \dot{\eta}_i
\]

(104)

\[
\delta \epsilon F_i = \epsilon \alpha\beta \epsilon \alpha \dot{\psi}_i^\beta
\]

(105)

The Lagrangian \(\mathcal{L}_\eta\) transforms by the total time derivative term, like in the usual supersymmetric case. Let us note that \((\eta_i)^2 = 0\) but \((\delta \epsilon \eta_i)^2 \neq 0\). The equations of motion that one obtains for the present system have the form

\[
\omega \eta_i - F_i = 0
\]

(106)

\[
\omega \psi_i^\alpha - \epsilon \alpha\beta \psi_i^\beta = 0
\]

(107)

\[
\omega F_i + \dot{\eta}_i = 0,
\]

(108)

and after eliminating the auxiliary function \(F_i\) we get that \(\dot{\eta}_i = -\omega^2 \eta_i\).

### 6.3 Graded nilpotent oscillator and \(R\)-symmetry

As it is known [6], for the even dimensional configuration space it is possible to form a composite system of supersymmetric bosonic oscillator and supersymmetric fermionic one. This system is separately supersymmetric in both parts, but it exhibits also a nontrivial so called \(R\)-symmetry [6] which intertwines both parts of such a composite Lagrangian. Moreover it relates kinetic terms with potential ones. We will see here that this symmetry survives also in the generalization to the graded nilpotent oscillator.

To fix the notation, let us firstly recall the Lagrangian of the supersymmetric fermionic oscillator [6]. Here we depart from the anticommuting coordinates of the basic configuration space \(\psi_i\) and associate the following multiplet of functions

\[
\psi_i \mapsto (\psi_i(t), \eta_i^\alpha(t), A_i(t)),
\]

(109)

where the parity of functions is: \(|\psi_i| = |A_i| = 1, |\eta_i^\alpha| = 0\). Now the \(N = 2\) SUSY transformations on this multiplet have the following form [6]

\[
\delta \epsilon \psi_i = \epsilon \alpha \eta_i^\alpha
\]

(110)

\[
\delta \epsilon \eta_i^\alpha = \epsilon \alpha\beta \epsilon \alpha A_i - \epsilon \alpha \psi_i
\]

(111)

\[
\delta \epsilon A_i = \epsilon \alpha \epsilon \alpha\beta \eta_i^\beta
\]

(112)
As before we can write the Lagrangian yielding the action invariant under SUSY transformations (110), namely

\[ L_\psi = -\frac{1}{2} \varepsilon^{ij} (\dot{\psi}_i \dot{\psi}_j + A_i A_j) + \frac{1}{2} \delta_{\alpha\beta} \varepsilon^{ij} \dot{\eta}_i^\alpha \dot{\eta}_j^\beta - \frac{\omega}{2} \varepsilon^{ij} (\dot{\psi}_i A_j + A_i \dot{\psi}_j - \varepsilon_{\alpha\beta} \eta_i^\alpha \eta_j^\beta) \]

(113)

The equations of motion for the supersymmetric fermionic oscillator are of the following form

\[ \omega \psi_i + A_i = 0 \quad (114) \]
\[ \omega \eta_i^\alpha + \varepsilon^{\alpha\beta} \dot{\eta}_i^\beta = 0 \quad (115) \]
\[ \omega A_i - \dot{\psi}_i = 0, \quad (116) \]

and after eliminating the auxiliary function \( F_i \) we get that \( \dot{\psi}_i = -\omega^2 \psi_i \).

Let us observe that there is a kind of an informal ‘parity duality’ between the \( N = 2 \) supersymmetric nilpotent oscillator and \( N = 2 \) supersymmetric fermionic oscillator, in the sense that

\[ (\eta_i)^2 = 0, \quad \eta_i \longleftrightarrow \psi_i, \quad (\psi_i)^2 = 0 \quad (117) \]
\[ \psi_i^\alpha \longleftrightarrow \eta_i^\alpha \quad (118) \]
\[ F_i \longleftrightarrow A_i \quad (119) \]
\[ s^{ij} \longleftrightarrow \varepsilon^{ij} \quad (120) \]

But such relation is not working on the level of derivatives with respect to the nilpotent co-ordinates. We cannot directly relate

\[ \frac{\partial}{\partial \psi_i} \]

(121)

which is \( \mathbb{Z}_2 \) graded derivative and

\[ \frac{\partial}{\partial \eta_i} \]

(122)

which is para-derivative, with nontrivial para-Leibniz term.

Now we can combine two independently invariant under \( N = 2 \) SUSY Lagrangians (102) and (113) and form the graded nilpotent oscillator

\[ \mathcal{L} = \mathcal{L}_\eta + \mathcal{L}_\psi \]

(123)

It turns out that such a system is invariant under the generalization of the \( R \)-symmetry introduced in Ref.[6]. This transformation is parametrized by the odd
parameter $\xi$. For the $\eta$-multiplet (101) it takes the form

$$
\delta_\xi \eta_i = \xi s^i_j \frac{1}{\omega} \psi^j
$$

(124)

$$
\delta_\xi \psi_i^\alpha = -\xi s^i_j (\epsilon^{\alpha \beta} \eta^j_\beta - \frac{1}{\omega} \delta^{\alpha \beta} \eta^j_\beta)
$$

(125)

$$
\delta_\xi F_i = \xi s^i_j (\frac{1}{\omega} \dot{A}_j + 2 \psi_j)
$$

(126)

and for the $\psi$-multiplet (109)

$$
\delta_\xi \psi_i = -\xi \epsilon^{ij} \frac{1}{\omega} \dot{\eta}^j
$$

(127)

$$
\delta_\xi \eta_i^\alpha = -\xi \epsilon^{ij} (\epsilon^{\alpha \beta} \psi^j_\beta + \frac{1}{\omega} \delta^{\alpha \beta} \psi^j_\beta)
$$

(128)

$$
\delta_\xi A_i = -\xi \epsilon^{ij} (\frac{1}{\omega} \dot{F}_j - 2 \dot{\psi}_j)
$$

(129)

The full Lagrangian (123) transforms by total time derivative

$$
\mathcal{L} = -2 \omega \frac{d}{dt} (\psi_i \eta^i) + \frac{d}{dt} \left( F_i \psi^i - \eta^i_\alpha A_i - \eta^i \epsilon^{\alpha \beta} \psi^i_\beta \right) + \frac{1}{\omega} \frac{d}{dt} \left( \eta_i \psi^i + F_i A^i + \eta^i_\alpha \psi^{\alpha i} - \eta^i_\alpha \psi^i_\alpha \right)
$$

(130)

It is worth noting, that in each term of above total time derivative, there are contributions or cancellations coming from both parts of the Lagrangian (123). The $R$-symmetry mixes both sectors defined by multiplets $(\eta_i(t), \psi_i^\alpha(t), F_i(t))$ and $(\psi_i(t), \eta^\alpha_i(t), A_i(t))$. Moreover, it also mixes the kinetic and potential terms. The $R$-symmetry fixes the scale between both parts of these otherwise, independent systems.

**Conclusions**

In the present work we have discussed the possibility of constructing formalism based on the nilpotent commuting coordinates. Such generalization is tempting, having in mind the supersymmetric systems, where the classical formalism is based on the superspaces and supergeometry with anticommuting nilpotent variables. There $\mathbb{Z}_2$-graded structures provide the generalized description and allow to study bosons and fermions in unified language. The present approach shows that we can consider nontrivial systems defined by nilpotent commuting coordinates, but their description is less natural then that of...
graded systems given by anticommuting coordinates. In his book [4] Freed considers, in the context of supermanifolds, the idea of "nilpotent cloud" or "nilpotent fuzz" around points stressing nilpotency as its primary property. Here we tried to find some consequences of using solely nilpotency without anticommutativity. Let us observe that in the simplest possible case of the $\mathcal{N}_1$ algebra, from the point of view of the set of functions $f: \mathcal{N}_1 \mapsto \mathcal{N}_1$ there is no difference between $\mathbb{Z}_2$-graded version and commutative version of the algebra. One can say that in both cases there is no difference between these two kinds of the "nilpotent fuzz". But when introducing the differential calculus in the $\mathbb{Z}_2$-graded case we obtain superderivative anticommuting with supervariables, in the commutative nilpotent case we have differential calculus with the deformed Leibniz rule and both constructions are different.

We have also shown that $\eta$-nilpotent systems can be supersymmetrized and exhibit interesting (super)symmetries. Phase space description of $\eta$-mechanical systems is not ordinary. The analysis given in the present work shows that it is not obvious how to define generalization of the Poisson or even Poisson-Malcev [19] structure.

As we have mentioned in the Sec.3.2 one can consider $\eta$ variables as a composite objects $\eta = \theta_i \theta_j$, with $\theta$'s being Grassmannian variables, but from the point of view of the nilpotent mechanics reduction of the $\eta$-model to the pseudo-mechanical (model with Grassmannian co-ordinates) or supersymmetric one is not automatical, and in general, not useful. For example the $\eta$-Lagrangian written in terms of the $\theta$-variables does not define pseudo-mechanical system of any interest.

\section*{Acknowledgements}

The author wants to thank Andrzej Borowiec for discussions and helpful comments.

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