Orientation and Connectivity Based Criteria for Asymptotic Consensus

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Abstract

In this article, we establish orientation and connectivity based criteria for the agreement algorithm to achieve asymptotic consensus in the context of time-varying topology and communication delays. These criteria unify and extend many earlier convergence results on the agreement algorithm for deterministic and discrete-time multiagent systems.

1 Asymptotic consensus in a multiagent system

Let us consider a set of autonomous agents that interact with each other by exchanging values and perform instantaneous operations on received values. Agents each start with a real value and must reach agreement on a value which is a convex combination of the initial values. The agents are not required to agree exactly as in the decision problem called consensus in fault-tolerance [13], but ought to iteratively compute values that all converge to the same limit.

The motivation for this asymptotic consensus problem comes from a variety of contexts involving distributed systems. For example, sensors (altimeters) on board an aircraft could be attempting to reach agreement about the altitude. Or a collection of clocks that are possibly drifting apart have to maintain clock values that are sufficiently close.

For the multiagent systems described above, the asymptotic consensus problem has been proposed a solution which consists in an iterative linear procedure, classically referred to as the agreement algorithm. It has been introduced by DeGroot [5] for the synchronous and time-invariant case, and then has been extended by Tsitsiklis et al. [18, 19] to the case of asynchronous communications and time-varying environment. A related algorithm has been later proposed by Vicsek et al. [20] as a model of cooperative behavior. The subject has recently attracted considerable interest within the context of flocking and multiagent coordination (see for instance [12, 14, 4, 15] for surveys and references).

In this article, we establish orientation and connectivity based criteria for the agreement algorithm to achieve asymptotic consensus in the context of time-varying topology and communication delays. These criteria unify and extend earlier convergence results, namely the one in [18, 19, 14, 10, 2, 3, 17, 11], and notably concern the coordinated and the decentralized models of multiagent systems that we define by simple orientation and connectivity properties on their communication graphs.

Our proofs of convergence rely on uniform techniques. They share the same core, and only differ in the control of the convergence speed: in coordinated systems, the convergence speed depends quadratically on the number of agents while it is finite but unbounded in decentralized systems.
1.1 The agreement algorithm with time-varying topology and communication delays

We briefly recall the model for the agreement algorithm, and the set of assumptions that are usually made. We consider a set of \( N \) agents denoted \( 1, \ldots, N \). We assume the existence of a discrete global clock and we take the range of the clock’s ticks to be the set \( \mathbb{N} \) of natural numbers. The state of the agent \( i \) is captured by a scalar variable \( x_i \), and the value held by \( i \) at time \( t \) is denoted \( x_i(t) \). Each agent \( i \) starts with an initial value \( x_i(0) \), and the evolution of the local variable \( x_i \) is described by the linear transition function:

\[
x_i(t + 1) = \sum_{j=1}^{N} A_{i,j}(t)x_j(\tau_{ij}(t)) .
\]  

(1)

Equation (1) corresponds to the fact that at each time \( t + 1 \), the agent \( i \) updates \( x_i \) with a weighted average of the values it has received at time \( t \). In the presence of communication delays, the value received by \( i \) from \( j \) at time \( t \) may be an outdated value, i.e., sent by \( j \) at some time \( \tau_{ij}(t) \) with \( 0 \leq \tau_{ij}(t) \leq t \). For each time \( t \), we form the \( N \times N \) matrix \( A(t) \) and the communication graph \( G(t) = ([N], E(t)) \) which is the directed graph with a node for each agent in \( [N] = \{1, \ldots, N\} \) and where there is an edge from \( i \) to \( j \) if and only if \( A_{i,j}(t) > 0 \). In other words, the agent \( i \) is connected to the agent \( j \) in \( G(t) \) if \( i \) hears of \( j \) at time \( t \).

We now formulate a series of assumptions on the matrices \( A(t) \) and the delays \( \tau_{ij}(t) \), which hold naturally in the context of a multiagent system running the agreement algorithm.

A1: Each matrix \( A(t) \) is stochastic.

A2: Each communication graph \( G(t) \) contains all possible self-loops, i.e., \( A_{i,i}(t) > 0 \) for all \( i \in [N] \).

A3: The positive entries of the matrices \( A(t), t \in \mathbb{N} \), are uniformly lower bounded, i.e., there exists \( \alpha \in [0,1] \) such that \( A_{i,j}(t) \in \{0\} \cup [\alpha, 1] \) for all \( i, j \in [N] \) and all \( t \in \mathbb{N} \).

Concerning the delays, we assume

B1: \( \tau_{ij}(t) \leq t \), for all \( i, j \in [N] \) and all \( t \in \mathbb{N} \).

B2: \( \tau_{ii}(t) = t \), for all \( i \in [N] \) and all \( t \in \mathbb{N} \).

B3: There exist some positive integer \( \Delta \) such that \( \tau_{ij}(t) \geq \max(0, t - \Delta + 1) \) for all \( i, j \in [N] \) and all \( t \in \mathbb{N} \).

Assumption A1 corresponds to the updating rules of the \( x_i \)’s in terms of weighted averages discussed above. Assumptions A2 and B2 express the fact that an agent has an immediate access to its own current value. Assumption A3 is obviously fulfilled when the set of matrices \( A(t) \) is finite. Assumption B1 captures the fact communication does not violate causality: a future value of agent \( j \) cannot influence the computation of agent \( i \)’s value. Finally, with assumption B3 we suppose the multiagent system to be partially synchronous, namely communication delays are bounded. However since we do not require the functions \( \tau_{ij} \) to be either non-decreasing, surjective, or injective, communications between agents may be non-FIFO and unreliable (duplication and loss). The case of zero communication delays is captured by assumptions A and B with \( \Delta = 1 \), and equation (1) corresponds in this case to the evolution of the agreement algorithm for a synchronous multiagent system.
1.2 The coordinated and decentralized models

All the above assumptions are classical, contrary to the conditions C and D we introduce now, that lie at the heart of our main convergence theorems.

Let us recall that for a directed graph $G$, its strongly connected components are, in general, strictly included into its connected components defined as the connected components of the undirected version of $G$. Let us also recall that the directed graph $G = (V, E)$ is said to be $j$-oriented, for $j \in V$, if for every node there exists a directed path originating at this node and terminating at $j$ [7]. If $G$ is $j$-oriented for some node $j$, then $G$ is said to be oriented. We now introduce the following condition on sequences of communication graphs.

**C :** At every time $t \in \mathbb{N}$, the communication graph $G(t)$ is oriented.

Intuitively, while the communication graph is $j$-oriented, the agent $j$ gathers the values in its strongly connected component, computes some average value, and attempts to impose this value to the rest of the agents as long as the communication graph remains $j$-oriented. In other words, its particular position in the communication graph makes $j$ to play the role of system coordinator for the agreement algorithm. Accordingly, we define the coordinated model as the model of multiagent systems which, in addition to assumptions A and B, satisfy condition C.

From the above discussion about the role of coordinator, it is easy to grasp why in the particular case of a steady coordinator, all the agents converge to a common value when running the agreement algorithm. Our first theorem shows that asymptotic consensus is actually achieved even when coordinators change over time.

**Theorem 1.** In the coordinated model, the agreement algorithm guarantees asymptotic consensus.

We introduce a second model of multiagent systems, the decentralized model, in which the orientation condition C of the coordinated model is replaced by two connectivity conditions D1 and D2. Before stating them, let us recall that a directed graph is said to be completely reducible if all its connected components are strongly connected.

**D1:** For every time $t \in \mathbb{N}$, the directed graph $(\mathbb{N}, \cup_{s \geq t} E(s))$ is strongly connected.

**D2:** At every time $t \in \mathbb{N}$, the communication graph $G(t)$ is completely reducible.

The second main result of this paper is the following convergence theorem for the agreement algorithm.

**Theorem 2.** In the decentralized model, the agreement algorithm guarantees asymptotic consensus.

The convergence mechanism behind this result can be understood as follows in intuitive terms: there is no source (that is, no node without incoming edge) in a completely reducible directed graph, and thus for the decentralized model, there is no dead end in the information flow of each connected component of the communication graph. The strong connectivity condition D1 guarantees that the values computed in each connected component are then spread out over the whole system. Even if at a given time, the roles performed by agents may be not at all equivalent since the communication graph may be non-symmetric, all the agents eventually play the same role over time and converge to the same value.

Because of the self-loop assumption A2, the decentralized model corresponds to a weak form of ergodicity, namely each matrix $A(t)$ is block ergodic. Similarly, a close inspection of our proof
of Theorem reveals that in the coordinated model, each matrix $A(t)$ is “partially ergodic” in the
sense that there exist some index $j$ in $[N]$ and some positive integer $n$ such that all the entries in
the $j$-th column of the matrix $(A(t))^n$ are positive. Clearly a matrix that is both block ergodic
and partially ergodic is ergodic. Since a strongly connected directed graph is oriented with respect
to each of its nodes, the intersection of the coordinated model and the decentralized model indeed
coincides with the model where communication graphs are all strongly connected.

1.3 Some strengthenings of our convergence theorems

Theorems and admit generalizations — which turn out to be useful in applications — where
conditions C and D are weakened in diverse directions while maintaining convergence.

Firstly, convergence is maintained when condition C (resp. D) holds only eventually: C (resp. D)
may be violated during a finite period, but is supposed to hold from some time onward. Roughly
speaking, that corresponds to a realistic situation where the multiagent system stabilizes after some
transient phase during which the communication graph is arbitrary, of duration unknown to the
agents.

Secondly, we may address the issue of the granularity at which condition C (resp. D) holds:
instead of observing the multiagent system at each time $t \in \mathbb{N}$, we might have access to its state
only at the end of each period of time of a fixed duration $\Phi$. Formally, that corresponds to the
introduction of a new time scale $u \in \mathbb{N}$, and to let $t = \Phi u$. In the synchronous case where we
assume $\tau_{ij}(t) = t$, the evolution equation takes the form

$$x_i(\Phi(u+1)) = \sum_{j=1}^{N} \tilde{A}_{i,j}(u)x_j(\Phi u)$$

where $\tilde{A} = A(\Phi u + \Phi - 1) \cdots A(\Phi u)$. In other words, in the synchronous model, the change
of granularity amounts to grouping the matrices in blocks of length $\Phi$, and in replacing each block
by the product of the matrices in the block. It is remarkable that, in the general non synchronous
case captured by assumption B, where the maximum delay $\Delta$ is any positive integer and where
the system evolution does not follow equation anymore, asymptotic consensus is still guaranteed
under a simple extension of condition C involving matrix products of length $\Phi$.

Actually, these two types of weakening of conditions C and D can be combined in the following
formulations, respectively.

◊C: There exist a time $T_0 \in \mathbb{N}$ and a positive integer $\Phi$ such that at every time $t \geq T_0$, the
communication graph $H(t)$ of the product $A(t + \Phi - 1) \cdots A(t)$ is oriented.

◊D2: There exist a time $T_0 \in \mathbb{N}$ and a positive integer $\Phi$ such that at every time $t \geq T_0$, the
communication graph $H(t)$ of the product $A(t + \Phi - 1) \cdots A(t)$ is completely reducible.

Simple extensions of our proofs will allow us to show the following generalizations of Theorems
and 2.

**Theorem 3.** Under assumptions A and B, the agreement algorithm guarantees asymptotic consen-
sus when condition ◊C holds.

**Theorem 4.** Under assumptions A and B, the agreement algorithm guarantees asymptotic consen-
sus when condition ◊D, the conjunction of D1 and ◊D2, holds.
Beside the above variants of conditions C and D (derived from C and D by a weakening procedure standard in temporal logic), a close inspection of the proof of Theorem 2 leads us to introduce another weakening of condition D. Indeed, our proof shows that the agreement algorithm achieves asymptotic consensus when replacing condition D by the following weaker condition.

\[ D^* : \text{There is some } j \in [N] \text{ such that at every time } t \in \mathbb{N}, \]

1. the directed graph \([N], \cup_{s \geq t} E(s)\) is \(j\)-oriented;

2. the connected component of \(j\) in the communication graph \(G(t)\) is \(j\)-oriented, and every other connected component of \(G(t)\) is strongly connected.

**Theorem 5.** Under assumptions A and B, the agreement algorithm guarantees asymptotic consensus when condition \(D^*\) holds.

Contrary to the coordinators in condition C, the agent \(j\) in condition \(D^*_2\) is fixed in time. Indeed, the combination of conditions C and \(D^*_2\) defines a simple model of a steady coordinator. One point of interest in Theorem 5 is that it demonstrates that asymptotic consensus can be achieved in a hybrid model: agents can be disconnected from the coordinator provided they are still clustered into independent and strongly connected groups.

### 1.4 Related work

Numerous convergence results for the agreement algorithm have been established in the literature. Presumably, the first one, which is a straightforward corollary of the classical Frobenius’ theorem [6], concerns the case of a fixed ergodic matrix in the synchronous setting. Wolfowitz’s theorem [21] extends this result to the sequences of varying matrices taken from a finite set of ergodic matrices such that any finite product of matrices in that set is ergodic. We refer the reader to [16] for historical references and variants of these theorems.

Bertsekas and Tsitsiklis [1] introduced the set of assumptions A and B to relax the finiteness hypothesis on the set of matrices and to handle communication delays. Moreover they defined a condition on the sequence of communication graphs, the condition of \(\Phi\)-bounded intercommunication intervals, where \(\Phi\) denotes a positive integer: if \((i, j)\) is an edge of the communication graph infinitely often, then \((i, j)\) is required to be an edge of the communication graph at least once during each period of duration \(\Phi\). Tsitsiklis [18] proved that under assumptions A, B, D1 and on the condition of bounded intercommunication intervals, the agreement algorithm guarantees asymptotic consensus. It is easy to see that assumptions A2 and D1 combined with the condition of \(\Phi\)-bounded intercommunication intervals imply that from some time onward, any product of \(\Phi N\) consecutive matrices in the sequence \((A(t))_{t \in \mathbb{N}}\) is a positive matrix. Thereby condition \(D\) holds and the convergence result in [18] appears as a special case of our Theorem 4.

Moreau [14] and Hendrickx and Blondel [10] independently proved that in the synchronous case (\(\Delta = 1\)), asymptotic consensus is still guaranteed when replacing the condition of bounded intercommunication intervals by a symmetry condition on the edges of the communication graphs, namely \((i, j) \in E(t)\) iff \((j, i) \in E(t)\) for any \(t \in \mathbb{N}\). The latter condition corresponds to a particular case of the decentralized model, and thus the convergence result in [14, 10], as well as its extension in [2] to the case of bounded communication delays (assumption B), are contained in Theorem 2.

Cao et al. [3] proved that in the case of stochastic matrices with equal positive entries in each row (usually referred to as the equal neighbor model), and under assumptions A and B with
Δ = 1, the agreement algorithm achieves asymptotic consensus when all the communication graphs are oriented. Their convergence result thus coincides with Theorem 1 in the particular case of a synchronous multiagent system and with the equal neighbor model.

After writing the proof of Theorem 1, we became aware of two recent papers both containing Theorem 2 in the synchronous case. In [11], Hendrickx and Tsitsiklis showed that the agreement algorithm achieves asymptotic consensus under assumptions A, B with Δ = 1, D1 and the so-called cut-balance condition. In light of Proposition 27 below, the latter condition turns out to correspond to the decentralized model. Independently, Touri and Nedić [17] established a general convergence result for the infinite product of random stochastic matrices, and to do that, they first proved that this result holds in the deterministic case; see Lemma 5 in [17]. It is easy to see that this lemma actually coincides with our Theorem 2 in the particular case of a synchronous multiagent system. In fact, our technique for proving the lemmas in Sections 3.2 and 3.4 infra specialized to Δ = 1, is similar to the one used for the proof of the deterministic result in [17].

2 A seminorm for multiagent dynamics

In this section, we discuss some auxiliary convergence results that will enter enter in our proofs in Section 3. To illustrate their usefulness for the study of convergence of product of stochastic matrices, we also present a short proof of Wolfowitz’s classical theorem based on them.

2.1 An operator seminorm

For any integer N ≥ 2, we consider the seminorm on R^N defined as the difference between the maximum and minimum entry of vector x

\[ \| x \|_\perp = \max_i (x_i) - \min_i (x_i) . \]

Thus \( \| x \|_\perp \) is null if and only if \( x \in R^1 \), where 1 is the vector whose components are all equal to 1. For any square matrix A with 1 as an eigenvector, the induced matrix seminorm \( \| A \|_\perp \) is

\[ \| A \|_\perp = \sup_{x \neq 1} \frac{\| Ax \|_\perp}{\| x \|_\perp} . \]

One key property of the seminorm \( \| \cdot \|_\perp \) is that it is sub-multiplicative, i.e.,

\[ \| AB \|_\perp \leq \| A \|_\perp \| B \|_\perp . \]

Another one is about the vectors that realize \( \| A \|_\perp \).

Lemma 6. Let A be a square matrix with 1 as an eigenvector, and let \( \{ e_i \mid i \in [N] \} \) denote the standard basis of R^N. There exists a nonempty subset I of [N] such that the vector \( e_I = \sum_{i \in I} e_i \) realizes \( \| A \|_\perp \), i.e.,

\[ \| A \|_\perp = \| A(e_I) \|_\perp . \]

Proof. By linearity, we have

\[ \| A \|_\perp = \sup_{\| x \|_\perp = 1} \| A(x) \|_\perp . \]
Let $x$ be a vector such that $\|x\|_\perp = 1$, and let $\mathbf{F} = x - x_{i_0} \mathbf{1}$ where $x_{i_0} = \min_i(x_i)$. Then $\|\mathbf{F}\|_\perp = 1$, and $\mathbf{F}_{i_0} = \min_i(\mathbf{F}_i) = 0$. Since $A(\mathbf{1}) = \mathbf{1}$,

$$\|A\|_\perp = \sup \left\{ \|Ax\|_\perp \mid x \in \mathbb{R}^N \text{ with } \max_i(x_i) = 1 \text{ and } \min_i(x_i) = 0 \right\}.$$  

Moreover by compactness, there exists a vector $x$ with $\max_i(x_i) = 1$ and $\min_i(x_i) = 0$, and such that $\|A\|_\perp = \|Ax\|_\perp$.

Now suppose that some entries of $x$ are not in $\{0, 1\}$, and let $x_j \in [0, 1[$ be one of them. We consider the two vectors $x^- = x - x_j \cdot e_j$ and $x^+ = x - (1-x_j) \cdot e_j$, and we denote $y = Ax$, $y^- = Ax^-$, and $y^+ = Ax^+$. Then for any index $i$, we have $y^-_i = y_i - x_j A_{ij}$, $y^+_i = y_i + (1-x_j) A_{ij}$, and so $y^-_i \leq y_i \leq y^+_i$. Since $\|x\|_\perp = \|x^+\|_\perp = \|x^-\|_\perp = 1$, $\|y^-\|_\perp$ and $\|y^+\|_\perp$ are both less or equal to $\|A\|_\perp$. Let $i_0$ and $i_1$ be two indices such that $\|y\|_\perp = y_{i_1} - y_{i_0}$. Then

$$y^-_{i_1} - y^-_{i_0} = \|A\|_\perp x_j(A_{i_1,j} - A_{i_0,j}) \text{ and } y^+_{i_1} - y^+_{i_0} = \|A\|_\perp (1-x_j)(A_{i_1,j} - A_{i_0,j}).$$

From $x_j \in [0, 1[$, $y^-_{i_1} - y^-_{i_0} \leq \|A\|_\perp$, and $y^+_{i_1} - y^+_{i_0} \leq \|A\|_\perp$, we derive that $A_{i_1,j} = A_{i_0,j}$, and so $\|y^-\|_\perp = \|y^+\|_\perp = \|A\|_\perp$. One by one, we thus eliminate all the entries of $x$ different from 0 and 1, and obtain a vector of the desired form. \hfill $\square$

Observe that if $e_I = \sum_{i \in I} e_i$ realizes $\|A\|_\perp$, then so does $e_{\overline{I}}$, where $\overline{I}$ denotes the complement of $I$ within $[N]$.

The vector $\mathbf{1}$ is an eigenvector of each stochastic matrix, and we easily check that for any stochastic matrix $A$ and any vector $x \in \mathbb{R}^N$,

$$\|Ax\|_\perp \leq \|x\|_\perp.$$  

It follows that the induced matrix seminorm of a stochastic matrix is less or equal to 1.

Interestingly we can compare $\|A\|_\perp$ with the coefficients of ergodicity of $A$ previously introduced in the literature \cite{9, 21}, namely

$$\delta(A) = \max_j \max_{i_1, i_2} |A_{i_1,j} - A_{i_2,j}|,$$

and

$$\lambda(A) = 1 - \min_{i_1, i_2} \sum_{j=1}^{N} \min(A_{i_1,j}, A_{i_2,j}).$$

**Proposition 7.** Let $A$ be a stochastic matrix. Then

$$\delta(A) \leq \|A\|_\perp \leq \lambda(A).$$

Moreover $\lambda(A) = 1$ if and only if $\|A\|_\perp = 1$.

**Proof.** First we observe that

$$\delta(A) = \max_{j=1, \ldots, N} \|Ae_j\|_\perp,$$

and the inequality $\delta(A) \leq \|A\|_\perp$ immediately follows.
For the second inequality, we consider a realizer of \( \|A\|_\perp \) that we denote \( e_I = \sum_{i \in I} e_i \) (cf. Lemma 6). Let \( f = Ae_I, f_\alpha = \max_i f_i \), and \( f_\beta = \min_i f_i \). Then we have

\[
\|A\|_\perp = \sum_{j \in I} (A_{\alpha j} - A_{\beta j}) .
\]

Since \( A \) is a stochastic matrix, we get

\[
1 - \|A\|_\perp = \sum_{j=1}^{N} A_{\alpha j} - \sum_{j \notin I} (A_{\alpha j} - A_{\beta j}) = \sum_{j \notin I} A_{\alpha j} + \sum_{j \in I} A_{\beta j} .
\]

Hence

\[
1 - \|A\|_\perp \geq \sum_{j=1}^{N} \min(A_{\alpha j}, A_{\beta j}) ,
\]

and the inequality \( \|A\|_\perp \leq \lambda(A) \) immediately follows.

Suppose now that \( \lambda(A) = 1 \), i.e., there exist two indices \( i_1, i_2 \) such that for each \( j \in [N] \), either \( A_{i_1 j} = 0 \) or \( A_{i_2 j} = 0 \). Let \( I = \{ j \in [N] \mid A_{i_1 j} \neq 0 \} \), \( e_I = \sum_{i \in I} e_i \), and \( f = Ae_I \). Since \( A \) is stochastic, its \( i_1 \)-th and \( i_2 \)-th rows each contains a non-null entry, and thus neither \( I \) or its complement is empty. Hence

\[
f_{i_1} = 1 \text{ and } f_{i_2} = 0 ,
\]

which shows \( \|A\|_\perp = 1 \). \( \square \)

We now give a corollary that is useful for convergence proofs.

**Corollary 8.** Let \( A \) be a \( N \times N \) stochastic matrix, \( j \) be any index in \([N]\), and let \( \beta_j \) be the minimum of all the entries in the \( j \)-th column, i.e., \( \beta_j = \min \{ A_{i,j} \mid i \in [N] \} \). Then,

\[
\|A\|_\perp \leq 1 - \sum_{j=1}^{N} \beta_j .
\]

In particular, if all the entries of one column of \( A \) are positive, then \( \|A\|_\perp < 1 \).

**Proof.** Since all \( A \)'s entries are non-negative, we have

\[
\lambda(A) \leq 1 - \sum_{j=1}^{N} \beta_j .
\]

Using Proposition 7, we derive that \( \|A\|_\perp \leq 1 - \beta_j \). In the case \( \beta_j > 0 \) for some index \( j \), we obtain \( \|A\|_\perp < 1 \). \( \square \)

### 2.2 A simple criterion for convergence

Now we give a seminorm based condition on a sequence of stochastic matrices under which their product converges to a rank one stochastic matrix. This criterion lies implicitly behind several convergence proofs (for instance, see [1, Section 7.3] or [2, 17] and also [8] for related results concerning Markov operators on cones).
Proposition 9. For each integer $t \in \mathbb{N}$, let $A(t)$ be a stochastic matrix, and let $P(t) = A(t) \ldots A(0)$. The following conditions are equivalent:

1. The sequence $(P(t))_{t \in \mathbb{N}}$ converges to a matrix of the form $1\pi^T$ where $\pi$ is a probability vector in $\mathbb{R}^N$.

2. The sequence $(\|P(t)\|_{\perp})_{t \in \mathbb{N}}$ converges to 0.

3. For each vector $v \in \mathbb{R}^N$, the sequence $(P(t)v)_{t \in \mathbb{N}}$ converges to some vector in the line $\mathbb{R}1$.

4. For each vector $v \in \mathbb{R}^N$, the sequence $(\|P(t)v\|_{\perp})_{t \in \mathbb{N}}$ converges to 0.

Proof. The implications $(1) \Rightarrow (2)$, $(3) \Rightarrow (4)$, $(1) \Rightarrow (3)$, and $(2) \Rightarrow (4)$ are obvious.

To show that $(4) \Rightarrow (3)$, we consider a sequence of vectors $x(t) = P(t)v$ where $v$ is some vector in $\mathbb{R}^N$. We denote

$$M(t) = \max_i (x_i(t)) \quad \text{and} \quad m(t) = \min_i (x_i(t)).$$

Hence $\|x(t)\|_{\perp} = M(t) - m(t)$, and $(4)$ is equivalent to $\lim_{t \to \infty} M(t) - m(t) = 0$. Since each matrix $A(t)$ is stochastic, the sequences $(M(t))_{t \in \mathbb{N}}$ and $(m(t))_{t \in \mathbb{N}}$ are non-increasing and non-decreasing, respectively. It follows that the latter two sequences as well as all the sequences $(x_i(t))_{t \in \mathbb{N}}$ are convergent to the same limit, which shows $(3)$.

For the implication $(3) \Rightarrow (1)$, suppose that $(3)$ holds. In particular with $v = e_j$, each sequence $(P(t))_{t \in \mathbb{N}}$ converges to some scalar $j$ independent of index $i$. Therefore, the sequence $(P(t))_{t \in \mathbb{N}}$ converges, and $\lim_{t \to +\infty} P(t) = 1\pi^T$. Since each matrix $P(t)$ is stochastic and the set of stochastic matrices is closed, $\pi$ is a probability vector.

As an immediate consequence of Corollary \[8\] the sub-multiplicativity of the seminorm $\|\cdot\|_{\perp}$, and Proposition \[9\], we obtain the well-known result that for any ergodic stochastic matrix $A$, $\lim_{t \to \infty} A^t$ exists and is a rank one stochastic matrix. In fact, we can even derive Wolfowitz’s theorem \[21\] which generalizes the latter result to infinite products of matrices taken from a finite set of ergodic stochastic matrices.

Theorem 10 (Wolfowitz). Let $\mathcal{M}$ be a nonempty finite set of stochastic matrices such that any finite product of matrices in this set is ergodic. For each $t \in \mathbb{N}$, let $A(t)$ be a matrix in $\mathcal{M}$. Then $\lim_{t \to +\infty} A(t) \cdots A(0)$ exists, and the limit is of the form $1\pi^T$ where $\pi$ is a probability vector in $\mathbb{R}^N$.

Proof. We begin by mimicking the first steps of the proof in \[21\] with the seminorm $\|\cdot\|_{\perp}$ to be substituted for the coefficient of ergodicity $\lambda$. Then instead of Theorem 2 of \[9\] used by Wolowitz, we just need the sub-multiplicativity of the seminorm to conclude.

In more detail, we first show the following lemma.

Lemma 11. The seminorm of any product of $N^2 + 1$ matrices in $\mathcal{M}$ is less than 1.

Proof. Let $\sim$ be the equivalence relation defined on the set of square stochastic matrices by $A \sim B$ iff $A$ and $B$ have the same communication graph. We easily check that $\sim$ is preserved by (right) multiplication with stochastic matrices. Moreover, the conditions $\lambda(A) = 1$ and $\lambda(B) = 1$ are clearly equivalent when $A \sim B$. Using Proposition \[7\] twice, we derive that if $A \sim B$ and $\|A\|_{\perp} = 1$, then $\|B\|_{\perp} = 1$. 

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Let $A_0, \ldots, A_{N^2}$ be $N^2 + 1$ matrices in $\mathcal{M}$. Since there are at most $N^2$ equivalence classes under the relation $\sim$, there exist two indices $k, \ell$, $0 \leq k < \ell \leq N^2$, such that

$$A_{N^2} \cdots A_\ell \sim A_{N^2} \cdots A_k.$$ 

Let $A = A_{N^2} \cdots A_\ell$ and $B = A_{\ell - 1} \cdots A_k$; we have $AB \sim A$. It follows that for any positive integer $n$, $AB^n \sim A$. Moreover by assumption on $\mathcal{M}$, the matrix $B$ is ergodic, i.e., $B^{n_0} > 0$ for some positive integer $n_0$.

Now suppose that $\|A\|_\perp = 1$. The above argument shows that $\|AB^{n_0}\|_\perp = 1$ and by the sub-multiplicativity of $\|\cdot\|_\perp$, we get $\|B^{n_0}\|_\perp = 1$, a contradiction with Corollary 8. Therefore, $\|A\|_\perp < 1$. Using the sub-multiplicativity of $\|\cdot\|_\perp$ again, we obtain $\|A_{N^2} \cdots A_0\|_\perp < 1$ as required.

Let $\delta$ denote the supremum of the seminorms of the matrices that are a product of $N^2 + 1$ matrices in $\mathcal{M}$. Since $\mathcal{M}$ is finite, there is a finite number of such matrices and Lemma 11 shows that $\delta < 1$. Using the sub-multiplicativity of the seminorm, the theorem immediately follows.

In the next section, we show how to use the criterion in Proposition 9 to prove convergence theorems where the finiteness assumption of the set $\mathcal{M}$ is weakened by assuming a uniform positive lower bound on positive entries (assumption A3), and where the ergodicity assumption is replaced by conditions that basically guarantee “eventual positivity” of some finite products of matrices in $\mathcal{M}$.

### 3 Convergence with bounded delays and topological changes

This section is devoted to the proof of Theorems 1 and 2. In Section 3.1, we introduce a family of stochastic matrices $(A^\Delta(t))_{t \in \mathbb{N}}$ of size $\Delta N$, which allows us to express our original evolution equation (1) as a “zero delay” evolution equation of the form

$$X(t + 1) = A^\Delta(t)X(t)$$

on suitably defined vectors $X(t)$. In Section 3.2, we investigate the time evolution of the sets of positive entries in each column of the successive products of the matrices $A^\Delta(t)$. The content of this subsection constitutes the core of our proofs of Theorems 1 and 2. Combined with a simple combinatorial argument, it allows us to conclude in Section 3.3 for the case of the coordinated model. Section 3.4 is dedicated to the study of the decentralized model. We start by deriving from condition D1 the eventual positivity of some columns of the successive products of the matrices $A^\Delta(t)$. Then we refine the “stationarity” condition elaborated in Section 3.2 to prove that asymptotic consensus in the decentralized model.

#### 3.1 Reduction to the zero delay case

We mimic the classical reduction of a $\Delta$-th order ordinary differential equation to a system of $\Delta$ ordinary differential equations of first order. For any time $t \geq \Delta - 1$, let $X(t)_{(\delta, i)}$ denote the family in $\mathbb{R}^{[\Delta] \times [N]}$ defined by

$$X(t)_{(\delta, i)} = x_i(t - \delta + 1),$$
where $\delta \in [\Delta]$ and $i \in [N]$. Letting $\delta_{ij}(t) = t - \tau_{ij}(t) + 1 \in \{0, \cdots, \Delta - 1\}$, equation (1) can be rewritten in the following way

$$X(t+1)_{(\delta, i)} = \begin{cases} X(t)_{(\delta-1, i)} & \text{if } \delta \in \{2, \cdots, \Delta\} \\ \sum_{j=1}^{N} A_{i,j}(t) X(t)_{(\delta_{ij}(t), j)} & \text{if } \delta = 1 \end{cases},$$

which is equivalent to

$$X(t+1)_{(\delta, i)} = \sum_{(\delta', j) \in [\Delta] \times [N]} A^\Delta_{(\delta, i), (\delta', j)}(t) X(t)_{(\delta', j)}$$

with

$$A^\Delta_{(\delta, i), (\delta', j)}(t) = \begin{cases} 1 & \text{if } i = j, \delta' = \delta - 1, \text{ and } \delta \in \{2, \cdots, \Delta\} \\ A_{i,j}(t) & \text{if } \delta = 1 \text{ and } \delta' = \delta_{ij}(t) \\ 0 & \text{otherwise.} \end{cases}$$

The key point here is that from time $\Delta - 1$ on, the vector $X$ is updated according to the linear equation with “zero delay”

$$X(t+1) = \mathcal{A}(t) X(t).$$

For ease of notation, in what follows we encode each ordered pair $(\delta, i)$ in $[\Delta] \times [N]$ into the integer $k = \Delta i - \delta + 1$ in $[\Delta N]$. Then the vector $X(t)$ is in $\mathbb{R}^{\Delta N}$ and $A^\Delta(t)$ is a $\Delta N \times \Delta N$ matrix.

The updating rule for $X(t)$ can be rewritten

$$X_k(t+1) = \begin{cases} X_{k+1}(t) & \text{if } k \text{ is not a multiple of } \Delta \\ \sum_{j=1}^{N} A_{i,j}(t) X_{\Delta j - \delta_{ij}(t)+1}(t) & \text{otherwise.} \end{cases}$$

Using the Kronecker delta $\delta_{m,n}$ (not to be confused with the delays $\delta_{ij}(t)$), the above expression of $A^\Delta_{(\delta', j), (\delta, i)}(t)$ implies that

1. $A^\Delta_{m,n}(t) = \delta_{m+1,n}$ if $m$ is not a multiple of $\Delta$;
2. all the entries $A^\Delta_{\Delta i, \Delta (j-1)+1}(t), \ldots, A^\Delta_{\Delta i, \Delta j}(t)$ are null except one which is equal to $A_{i,j}(t)$;
3. $A^\Delta_{\Delta i, \Delta i}(t) = A_{i,i}(t)$.

We observe that each matrix $A^\Delta(t)$ is stochastic (A1), and every positive entry of $A^\Delta(t)$ is at least equal to $\alpha$ (A3). However neither A2 nor D1 holds for the matrices $A^\Delta(t)$, and the above reduction does not allow us to limit ourselves to the zero delay case $\Delta = 1$ while maintaining the basic assumptions A and B.

We now choose some time $t_0 \geq \Delta - 1$ which stays fixed in the rest of the section except in the very last steps of the proofs of Theorems 1 and 2. For any time $t \geq t_0$, we let

$$P(t) = A^\Delta(t) \cdots A^\Delta(t_0).$$

Because of the above mentioned properties of the matrix $A^\Delta(t)$, we have the following recurrence relations for $P(t)$’s entries:

**R1:** $P_{m,\Delta j}(t+1) = P_{m+1,\Delta j}(t)$, for each index $m$ that is not a multiple of $\Delta$;

**R2:** $P_{\Delta i,\Delta j}(t+1) = \sum_{k=1}^{N} A_{i,k}(t+1) P_{m_k,\Delta j}(t)$, where for each index $k$ in the sum, $m_k$ is some integer in $\{\Delta(k-1)+1, \ldots, \Delta k\}$, and where $m_k = \Delta k$ for $k = i$. 


3.2 Positive entries in the $\Delta j$-th column

In this section, we fix some index $j \in [N]$, and study the entries in the $\Delta j$-th column of $P(t)$. We define the two sets

$$S^\Delta_j(t) = \{ m \in [\Delta N] \mid P_{m,\Delta j}(t) > 0 \} \quad \text{and} \quad S_j(t) = \{ i \in [N] \mid P_{\Delta i,\Delta j}(t) > 0 \} \ .$$

**Lemma 12.** $\Delta j \in S^\Delta_j(t_0)$.

**Proof.** By definition, $P(t_0) = A^\Delta(t_0)$. The lemma directly follows from property (3) of the matrix $A^\Delta(t_0)$, and from assumption A2 on the matrix $A(t_0)$.

**Lemma 13.** For all $t \geq t_0$, $S_j(t) \subseteq S_j(t+1)$.

**Proof.** From the recurrence relation R2, we deduce that

$$P_{\Delta i,\Delta j}(t+1) \geq A_{i,i}(t+1) P_{\Delta i,\Delta j}(t) \ ,$$

and the lemma follows from assumption A2 on the matrix $A(t+1)$.

**Lemma 14.** For all $t \geq t_0$, $S^\Delta_j(t) \subseteq S^\Delta_j(t+1)$.

**Proof.** First we give a more precise description of $S^\Delta_j(t_0)$. Since $P(t_0) = A^\Delta(t_0)$, properties (1) and (2) of the matrix $A^\Delta(t_0)$ implies that

$$m \in S^\Delta_j(t_0) \Leftrightarrow \begin{cases} m = \Delta i \quad \text{and} \quad A^\Delta_{i,\Delta j}(t_0) > 0 \\ m = \Delta j - 1 \ . \end{cases}$$

Let $t \geq t_0$, and $m \in S^\Delta_j(t)$. We denote $t = t_0 + \theta$, and $m = \Delta i - \ell$ with $i \in [N]$ and $\ell \in \{0, \cdots, \Delta - 1\}$. We consider two cases:

1. $\theta \geq \ell$.

   By as many applications as possible of the recurrence relation R1, we obtain

   $$P_{m,\Delta j}(t) = P_{\Delta i,\Delta j}(t_0 + \theta - \ell) \quad \text{and} \quad P_{m,\Delta j}(t+1) = P_{\Delta i,\Delta j}(t_0 + 1 + \theta - \ell) \ .$$

   The lemma follows from Lemma 13 in this case.

2. $\theta < \ell$.

   Then $\ell \geq 1$, i.e., $m$ is not a multiple of $\Delta$, and we can apply the recurrence relation R1 to obtain

   $$P_{m,\Delta j}(t) = P_{\Delta i-\ell+\theta,\Delta j}(t_0) \quad \text{and} \quad P_{m,\Delta j}(t+1) = P_{\Delta i-\ell+\theta-1,\Delta j}(t_0) \ .$$

   In this case, $\Delta i - \ell + \theta$ is not a multiple of $\Delta$, and from the above description of $S^\Delta_j(t_0)$ follows that $\Delta i - \ell + \theta = \Delta j - 1$. Hence $P_{m,\Delta j}(t+1) = A_{j,j}(t_0)$. The lemma follows from assumption A2 on the matrix $A(t_0)$ in this case.

By Lemmas 12 and 13 each set $S^\Delta_j(t)$ is non-empty. We define $\pi_j(t)$ as the minimum positive entry in the $\Delta j$-th column of the matrix $P(t)$, or

$$\pi_j(t) = \min\{ P_{m,\Delta j}(t) \mid m \in S^\Delta_j(t) \} \ .$$

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Lemma 15. For all \( t \geq t_0 \), \( \pi_j(t + 1) \geq \alpha \pi_j(t) \).

Proof. Let \( m \in S_j^\Delta (t + 1) \), i.e., \( P_{m,\Delta_j}(t + 1) > 0 \). There are two cases to consider:

1. \( m \) is a multiple of \( \Delta \), i.e., \( m = \Delta i \) for some \( i \in [N] \). Relation R2 implies that

\[
P_{\Delta i,\Delta_j}(t + 1) \geq A_{i,i}(t + 1) P_{\Delta i,\Delta_j}(t).
\]

Since the matrix \( A(t + 1) \) satisfies A2 and A3, we have \( P_{\Delta i,\Delta_j}(t + 1) \geq \alpha \pi_j(t) \).

2. \( m \) is not a multiple of \( \Delta \). By relation R1, we obtain

\[
P_{m,\Delta_j}(t + 1) = P_{m+1,\Delta_j}(t).
\]

Hence \( m + 1 \in S_j^\Delta (t) \), and thus \( P_{m,\Delta_j}(t + 1) \geq \pi_j(t) \).

In both cases we have \( P_{m,\Delta_j}(t + 1) \geq \alpha \pi_j(t) \), as needed. \( \square \)

Lemma 16. Let \( t \geq t_0 \). If \( S_j^\Delta (t) = S_j^\Delta (t + 1) \), then for each index \( i \) in \( S_j(t) \), all the entries \( P_{\Delta(i-1)+1,\Delta_j}(t), \ldots, P_{\Delta i,\Delta_j}(t) \) are positive.

Proof. Let \( i \in S_j(t) \). By decreasing induction on \( m \), \( \Delta (i - 1) \leq m \leq \Delta i \), we prove that

\[
P_{m,\Delta_j}(t) > 0.
\]

1. The basic case \( m = \Delta i \) corresponds to \( i \in S_j(t) \).

2. For the inductive step, let us assume that \( P_{m,\Delta_j}(t) > 0 \) with \( m \in \{ \Delta (i-1)+1, \cdots, \Delta i \} \).

Therefore, \( m - 1 \) is not a multiple of \( \Delta \) and by relation R1, we have

\[
P_{m-1,\Delta_j}(t + 1) = P_{m,\Delta_j}(t).
\]

Hence \( m - 1 \in S_j^\Delta (t + 1) \). Since \( S_j^\Delta (t) = S_j^\Delta (t + 1) \), we obtain \( P_{m,\Delta_j}(t) > 0 \), as required. \( \square \)

Lemma 17. If \( S_j^\Delta (t) = S_j^\Delta (t + 1) \), then \( S_j(t) \) has no incoming edge in the graph \( G(t + 1) \).

Proof. By contradiction, suppose that \( S_j^\Delta (t) = S_j^\Delta (t + 1) \), and that \( S_j(t) \) has an incoming edge in the graph \( G(t + 1) \). Hence there exist \( i \) and \( k \) in \( [N] \) such that \( P_{\Delta i,\Delta_j}(t) = 0 \), \( P_{\Delta k,\Delta_j}(t) > 0 \), and \( A_{i,k}(t + 1) > 0 \). Since \( S_j^\Delta (t) = S_j^\Delta (t + 1) \), we have

\[
P_{\Delta i,\Delta_j}(t + 1) = 0.
\]

Moreover Lemma 16 states that

\[
\forall m \in \{ \Delta (k - 1) + 1, \cdots, \Delta k \} : P_{m,\Delta_j}(t) > 0,
\]

and relation R2 implies that

\[
P_{\Delta i,\Delta_j}(t + 1) \geq A_{i,k}(t + 1) P_{\ell,\Delta_j}(t)
\]

for some \( \ell \in \{ \Delta (k - 1) + 1, \cdots, \Delta k \} \). Hence \( P_{\Delta i,\Delta_j}(t + 1) > 0 \), a contradiction. \( \square \)
3.3 Convergence in the coordinated model

In this section, we consider the coordinated model (assumptions A, B, and C), and prove our first convergence theorem.

We start by specializing Lemma 17 to the case of oriented communication graphs. Let us define

\[ S_{\Delta}(t) = \{(m, j) \in [\Delta N] \times [N] \mid m \in S_{\Delta j}(t)\} . \]

Lemma 18. If \( G(t + 1) \) is \( j \)-oriented, then either \( S_{\Delta}(t) \neq S_{\Delta}(t + 1) \), or \( S_{\Delta j}(t) = [\Delta N] \).

Proof. Suppose that \( S_{\Delta}(t) = S_{\Delta}(t + 1) \). Therefore \( S_{\Delta j}(t) = S_{\Delta j}(t + 1) \). Lemma 17 ensures that \( S_{j}(t) \) has no incoming link in \( G(t + 1) \). Since \( G(t + 1) \) is \( j \)-oriented, \( S_{j}(t) = [N] \). By Lemma 16 we conclude that \( S_{\Delta j}(t) = [\Delta N] \), as required.

The latter lemma allows us to show that there is an agent \( j \) such that \( S_{\Delta j} \) is equal to \( [\Delta N] \) by time \( t_0 + \Delta N^2 - 2N + 1 \).

Proposition 19. In the coordinated model, there exists an agent \( j \) such that

\[ \forall m \in [\Delta N] : P_{m,j}(t_0 + \Delta N^2 - 2N + 1) > 0 . \]

Proof. Let \( t \geq t_0 \). As immediate consequences of Lemma 12 and Lemma 14 respectively, we have

\[ |S_{\Delta}(t_0)| \geq N \]

and for all \( t \geq t_0 \),

\[ S_{\Delta}(t) \subseteq S_{\Delta}(t + 1) . \]

Using Lemmas 18 and 14 we obtain that either \( |S_{\Delta}(t)| \geq N + t - t_0 \) or \( S_{\Delta j}(t) = [\Delta N] \) for some \( j \) in \([N]\). We observe that if the cardinality of \( S_{\Delta}(t) \) is greater than \( \Delta N^2 - N \), then the matrix \( P(t) \) has at least one of its \( \Delta j \)-th columns with positive entries to complete the proof of the lemma.

We are now in position to prove Theorem 1. Let us consider an agent \( j \) such that

\[ S_{\Delta j}(t_0 + \Delta N^2 - 2N + 1) = [\Delta N] \]

the existence of which is ensured by Proposition 19. By Lemma 15 we have

\[ \pi_j(t_0 + \Delta N^2 - 2N + 1) \geq \alpha^{\Delta N^2 - 2N + 1} . \]

By Corollary 8 we derive \( \|P(t_0 + \Delta N^2 - 2N + 1)\| \leq 1 - \alpha^{\Delta N^2 - 2N + 1} \). In other words, we have shown that for any \( t_0 \geq \Delta - 1 \)

\[ \|A_{\Delta}(t_0 + \Delta N^2 - 2N + 1) \cdots A_{\Delta}(t_0)\| \leq 1 - \alpha^{\Delta N^2 - 2N + 1} . \]

Together with the sub-multiplicativity of the seminorm \( \|\cdot\| \), this implies that

\[ \lim_{t \to +\infty} \|A_{\Delta}(t) \cdots A_{\Delta}(0)\| = 0 . \]

Theorem 1 then follows from Proposition 9 and the definition of vector \( X(t) \).
3.4 Convergence in the decentralized model

We now consider the decentralized model (assumptions A, B, and D). Under the sole assumptions A and B, the set $\Delta^\lambda$ may remain small forever: for example, in the case of the sequence of the powers of the unit matrix, $\Delta^\lambda$ is constantly equal to the diagonal in $[N]^2$. Firstly, we show that assumption D1 ensures that $\Delta^\lambda$ is eventually equal to $[\Delta N] \times N$. However, D1 provides no bound on the time required the set $\Delta^\lambda$ to be maximal, and so under assumptions A, B, and D1, no positive lower bound on the positive entries of the matrix $P$ are guaranteed. In the second part of this section, we show that assumption D2 allows us to control how functions $\pi_j$ can decrease in time.

We start by refining the strong connectivity property of the graph $G_0 = ([N], \cup_{t \geq t_0} E(t))$ ensured by D1.

**Lemma 20.** For each $i \in [N]$, there exists a path $i_n, \cdots, i_0$ in the graph $G_0$ starting at $i_n = i$, ending at $i_0 = j$, and such that each edge $(i_k, i_{k-1})$ is in $E(t_k)$ with $t_k \geq t_{k-1} + \Delta$.

**Proof.** We consider the graph $G^\infty = ([N], E^\infty)$ where $E^\infty$ is the set of edges in $G_0$ that occur infinitely often. The lemma immediately follows from D1. \hfill \Box

**Proposition 21.** In the decentralized model, for every $j \in [N]$, there exists some time $\theta_j \geq t_0$ such that for each

$$\forall m \in [\Delta N] : P_{m, \Delta j}(\theta_j) > 0 .$$

**Proof.** At each time $t \in \mathbb{N}$, let us consider the directed graph $G^\Delta(t)$ to be the communication graph of $A^\Delta(t)$, i.e.,

$$G^\Delta(t) = ([\Delta N], E^\Delta(t)) ,$$

where $(m, n) \in E^\Delta(t)$ if and only if $A^\Delta_{m,n}(t) > 0$, and let

$$G^\Delta_0 = ([\Delta N], \cup_{t \geq t_0} E^\Delta(t)) .$$

Properties (1) and (3) of the matrix $A^\Delta(t)$ imply that for each index $i \in [N]$, the sequence of nodes $\Delta(i - 1) + 1, \cdots, \Delta i$ is a path in $G^\Delta(t)$, and that there is a self-loop at node $\Delta i$. Further if $A_{i_1,i_2}(t) > 0$, then property (2) for $A^\Delta(t)$ guarantees that there is an edge in $G^\Delta(t)$ from node $\Delta i_1$ to some node in the path $\Delta(i_2 - 1) + 1, \cdots, \Delta i_2$ (namely, node $\Delta i_2 - 1$ in Figure 1).

Let $m \in [\Delta N]$; using Lemma 20 and the above properties of the graphs $G^\Delta(t)$, we inductively construct a path $m_{\ell}, \cdots, m_0$ in the graph $G^\Delta_0$ starting at $m_{\ell} = m$, ending at $m_0 = \Delta j$, and such
that each edge \((m_k, m_{k-1})\) is in \(E^\Delta(t_k)\) with \(t_k = t_0 + k\). Let us denote \(\theta_j(m) = t_k\); by definition of the matrix \(P(\theta_j(m))\), the existence of this path is equivalent to \(P_{m,\Delta j}(\theta_j(m)) > 0\). Finally, we let

\[
\theta_j = \max_{m \in [\Delta N]} (\theta_j(m))
\]

and we use Lemma 14 to complete the proof.

We now give a condition which ensures that the function \(\pi_j\) does not decrease when the set \(S^\Delta_j\) remains stationary.

**Lemma 22.** If \(S^\Delta_j(t) = S^\Delta_j(t+1)\) and \(S_j(t)\) has no outgoing edge in the graph \(G(t+1)\), then \(\pi_j(t+1) \geq \pi_j(t)\).

**Proof.** Let \(m \in S^\Delta_j(t)\), i.e., \(P_{m,\Delta j}(t) > 0\). We consider two cases:

1. \(m\) is not a multiple of \(\Delta\). By relation R1, we have

\[
P_{m,\Delta j}(t+1) = P_{m+1,\Delta j}(t) .
\]

By Lemma 14, we get \(m \in S^\Delta_j(t+1)\), i.e., \(P_{m,\Delta j}(t+1) > 0\). Therefore, \(m+1 \in S^\Delta_j(t)\) and \(P_{m,\Delta j}(t+1) \geq \pi_j(t)\).

2. \(m\) is a multiple of \(\Delta\), i.e., \(m = \Delta i\) for some index \(i \in [N]\). By relation R2, we have

\[
P_{\Delta i,\Delta j}(t+1) = \sum_{k=1}^{N} A_{i,k}(t+1) P_{n_k,\Delta j}(t) ,
\]

for some \(n_k \in \{\Delta(k-1) + 1, \ldots, \Delta k\}\). It follows that

\[
P_{\Delta i,\Delta j}(t+1) \geq \sum_{k \in S_j(t)} A_{i,k}(t+1) P_{n_k,\Delta j}(t) .
\]

Using Lemma 16 and the definition of \(\pi_j(t)\), we obtain

\[
P_{\Delta i,\Delta j}(t+1) \geq \pi_j(t) \times \sum_{k \in S_j(t)} A_{i,k}(t) .
\]

Since \(i \in S_j(t)\) and \(S_j(t)\) has no outgoing edge in the graph \(G(t+1)\), we have

\[
\sum_{k \in S_j(t)} A_{i,k}(t+1) = \sum_{k=1}^{N} A_{i,k}(t+1) .
\]

The latter sum is equal to 1 as the matrix \(A(t+1)\) is stochastic. Hence \(P_{\Delta i,\Delta j}(t+1) \geq \pi_j(t)\). \(\square\)
We now put it all together to prove Theorem 2. Let
\[ \theta = \max_{j=1,\ldots,N} \theta_j \]
where the \( \theta_j \)'s are defined with regard to Proposition 21. Combining Lemma 12, assumption A3 with Lemmas 15, 17, and 22 we obtain
\[ \forall t \geq \theta : \forall j \in [N] : \pi_j(t) \geq \alpha \Delta N. \]
By Corollary 8 we derive \( \|P(\theta)\|_\perp \leq 1 - N\alpha \Delta N. \) In other words, we have shown that for any \( t_0 \geq \Delta - 1 \), there exists \( \theta \geq t_0 \) such that
\[ \|A^\Delta(\theta) \cdots A^\Delta(t_0)\|_\perp \leq 1 - N\alpha \Delta N. \]
Together with the sub-multiplicativity of the seminorm \( \|\cdot\|_\perp \), this implies that
\[ \lim_{t \to +\infty} \|A^\Delta(t) \cdots A^\Delta(0)\|_\perp = 0. \]
Theorem 1 follows from Proposition 9 and the definition of vector \( X(t) \).

4 Generalizations and remarks

In this section, we present diverse strengthenings of Theorems 1 and 2 which are obtained from direct generalizations of the arguments developed in the proofs of these theorems, or just simply by closely examining the proofs. We conclude by a discussion of examples which demonstrate the role of the various assumptions in Theorems 1–5.

4.1 Eventual condition and coarser granularity

Our proofs of theorems 3 and 4 are similar in the way we generalize the arguments in the proofs of Theorems 1 and 2, respectively. We present only one of them, the proof of Theorem 4.

Suppose that condition ♦D holds for some time \( T_0 \) and some positive integer \( \Phi \). We use the notation introduced in Section 3, Lemmas 12–17, and Lemma 22 for some time parameter \( t_0 \geq \max(T_0, \Delta - 1) \).

Assume that
\[ S_j^\Delta(t) = \cdots = S_j^\Delta(t + \Phi). \]
By definition of the sets \( S_j^\Delta \) and \( S_j \), all the sets \( S_j(t), \cdots, S_j(t + \Phi) \) are then equal to some set of nodes, which we denote by \( S \). Repeated application of Lemma 17 show that \( S \) has no incoming edge in each of the communication graphs \( G(t+1), \cdots, G(t+\Phi) \). Hence \( S \) has no incoming edge in \( H(t+1) \). Condition ♦D then guarantees that \( S \) has no outgoing edge in \( H(t+1) \).

Suppose now that \( S \) has an outgoing edge \((i, k)\) in some communication graph \( G(t + \varphi) \) with \( \varphi \in [\Phi] \). Because of the self-loop assumption A2, we deduce that \((i, k)\) is an outgoing edge of \( S \) in \( H(t+1) \), a contradiction. Therefore, the set of nodes \( S \) has no outgoing edge in each of the communication graphs \( G(t+1), \cdots, G(t+\Phi) \), and Lemma 22 implies that \( \pi(t + \Phi) \geq \cdots \geq \pi(t) \).

Using the same arguments as for Theorem 2, we conclude that there exists \( \theta \geq t_0 \) such that
\[ \|A^\Delta(\theta) \cdots A^\Delta(t_0)\|_\perp \leq 1 - N\alpha \Phi \Delta N. \]
Theorem 4 then follows from sub-multiplicativity of the seminorm \( \|\cdot\|_\perp \), from Proposition 9 and from the definition of \( X(t) \).
4.2 Partial complete reducibility

The proof of Theorem 5 is based on the remark that it suffices that one column of a stochastic matrix $A$ be positive to ensure that $A$ is contracting (with respect to the seminorm $||·||_\perp$); see Corollary 8.

Suppose that condition D* holds for some $j_0 \in [N]$. We use the notation introduced in Section 3, Lemmas 12–17, and Lemma 22 for some time parameter $t_0 \geq \Delta - 1$ and for node $j_0$. A close examination of the proof of Proposition 21 reveals that if the directed graph $([N], \cup_{s \geq t_0} E(s))$ is $j_0$-oriented (first part in condition D*), then the $\Delta j_0$-th column of the matrix $P$ is eventually positive, i.e., there exists some time $\theta_0 \geq t_0$ such that for each

$$\forall m \in [\Delta N] : P_{m, \Delta j_0}(\theta_0) > 0 .$$

Then Lemmas 17 and 22 lead us to relax the complete reducibility property into the following property for a directed graph $G = (V, E)$, and a node $j_0 \in V$.

$P_{j_0}$: There exists no subset of $V$ containing node $j_0$ with an outgoing edge and no incoming edge.

We now study property $P_{j_0}$ and give an equivalent, but more tractable expression of it. For that, we consider the condensation $G^*$ of $G$ defined as the directed acyclic graph obtained by contracting each strongly connected component of $G$ into a single node. We denote by $i^*$ the strongly connected component of some node $i$.

The next lemma, whose proof is obvious, allows us to restrict ourselves to the case of acyclic graphs.

**Lemma 23.** A directed graph $G$ satisfies $P_{j_0}$ if and only if the condensation $G^*$ of $G$ satisfies $P_{j_0^*}$.

In turn, property $P_{j_0}$ on a directed acyclic graph $G$ admits a simple equivalent expression in terms of the connected components of $G$.

**Lemma 24.** If $G$ is a directed acyclic graph, then $G$ satisfies $P_{j_0}$ if and only if (a) node $j_0$ is the one and only sink of its own connected component, and (b) every other connected component of $G$ reduces to a single isolated node.

**Proof.** For any node $i$, let $I$, $I^-$, and $I^+$ denote the connected component of $i$, the set of $i$’s ancestors, and the set of $i$’s descendants (both including $i$), respectively.

First assume that (a) and (b) both hold. Let $S$ be any subset of nodes with $j_0 \in S$, and suppose that $(k_0, k_1)$ is an outgoing edge of $S$. Then, $k_0$ and $k_1$ are in the same connected component. Condition (b) implies that $k_1 \in J_0$. Since $k_1 \notin S$, we have $k_1 \neq j_0$. By condition (a), $k_1$ is not a sink, and let $k_2$ be an outgoing neighbor of $k_1$. If $k_2 \in S$, then $(k_1, k_2)$ is an incoming edge of $S$; otherwise, we repeat the argument with $k_2$ instead of $k_1$. In this way, we construct a sequence of nodes $k_1, k_2, \cdots$ in the complement of $S$. Because $G$ is acyclic, this sequence is finite, i.e., $S$ has an incoming edge.

Conversely, suppose that either (a) or (b) does not hold. We consider the following three cases, and show that for each of them, $G$ does not satisfy $P_{j_0}$.

1. The node $j_0$ is not a sink, i.e., $j_0$ has at least an outgoing neighbor $k$. Then, the set $J_0^-$ has no incoming edge, but an outgoing edge, namely $(j_0, k)$. 

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2. The node \( j_0 \) is a sink and there is another sink, denoted \( i \), in \( J_0 \). Then, \( i \) is not an isolated node and has an incoming neighbor \( k \). The complement of \( \{i\} \) contains \( j_0 \), has no incoming edge, but an outgoing edge, namely \((k, i)\).

3. There exists some edge \((i, k)\) with \( i \) and \( k \) both outside \( J_0 \). Then, the set \( J_0 \cup I^- \) has no incoming edge, but an outgoing edge, namely \((i, k)\).

As observed in [7] (Section 2, page 12), condition (a) in Lemma 24 can be expressed in terms of \( j_0 \)-orientation.

**Lemma 25.** Let \( j_0 \) be any node of a connected and directed acyclic graph \( G \). Then, node \( j_0 \) is the one and only sink of \( G \) if and only if \( G \) is \( j_0 \)-oriented.

We leave the simple proof of Lemma 25 to the reader.

Finally, we compare orientation in a directed graph and in its condensation in the following lemma whose proof is trivial.

**Lemma 26.** The directed graph \( G \) is \( j_0 \)-oriented if and only if the condensation of \( G \) is \( j_0^* \)-oriented.

By combining the above four lemmas, we obtain an equivalent form of \( P_{j_0} \).

**Proposition 27.** Let \( G \) be a directed graph, and let \( j_0 \) be any node of \( G \). The following two properties are equivalent.

1. \( G \) satisfies \( P_{j_0} \).
2. The connected component of \( j_0 \) in \( G \) is \( j_0 \)-oriented, and every other connected component is strongly connected.

The end of the proof of Theorem 5 is similar to the one of Theorem 2 except that the upper bound on the seminorm of matrix \( P(\theta_0) \) is now

\[
\|A^\Delta(\theta_0) \cdots A^\Delta(t_0)\| \leq 1 - \alpha^{\Delta N}.
\]

### 4.3 Examples

We now present two examples that demonstrate the roles of the self-loop assumption A2 and of the conditions C, D, or D* in our convergence theorems. In both we consider the case of a synchronous system with 3 agents; in other words, \( N = 3 \) and \( \Delta = 1 \).

In our first example, the sequence of matrices \((A(t))_{t \in \mathbb{N}}\) is 3-periodic with

\[
A(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

The sequence of matrices corresponds to the following scenario in the synchronous case. Agent 1 communicates only with itself while agent 2 communicates with agent 3 and agent 3 communicates with agent 2, but none of the two agents 2 and 3 takes into account their own values. This first
round is then repeated infinitely often while rotating the communication graph. We easily check that the algorithm actually keeps executing the instruction \((x_1, x_2, x_3) := (x_3, x_2, x_1)\). Therefore, the algorithm does not achieve consensus, and does not even converge. However, condition D and all the assumptions considered so far hold except A2. In fact, we even observe that the self-loops which occur at each node during every period of duration 3 units of time do not help to achieve asymptotic consensus.

We now briefly recall an example given in [2], which shows how crucial is the fact that the agent \(j\) in conditions C or C* does not change over time. Here, \(x(0) = (0, 1, 0)\) and the matrices in the sequence \((A(t))_{t \in \mathbb{N}}\) are taken in the set \(\{A_1, A_2, A_3\}\) where

\[
A_1 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Given an increasing sequence of integers \((t_n)_{n \in \mathbb{N}}\) with \(t_0 = 0\) and to be chosen suitably later, we let \(A(t) = A_k\) when \(t_{3i+k-1} \leq t < t_{3i+k}\) for some non-negative integer \(i\). This means, for instance, that until time \(t_1\), agent 3 communicates with agent 1, and agent 1 forms the average of its own value and the value received from agent 3.

Let \((\epsilon_n)_{n \in \mathbb{N}}\) be a sequence of positive reals such that \(\ell = \sum_{n=1}^{\infty} \epsilon_n < 1/2\). Times \(t_1\) and \(t_2\) are chosen large enough to have \(x_1(t_1) \geq 1 - \epsilon_1\) and \(x_2(t_2) \leq \epsilon_1\). Similarly, \(t_3\) and \(t_4\) are chosen large enough to have \(x_2(t_3) \geq 1 - \epsilon_1 - \epsilon_2\) and \(x_1(t_4) \leq \epsilon_1 + \epsilon_2\), and so on. The resulting vector \(x(t)\) is such that each of its three entries is infinitely often at least equal to \(1 - \ell\) and infinitely often at most equal to \(\ell\). Since \(\ell < 1/2\), this proves that the sequence \((x(t))_{t \in \mathbb{N}}\) is not convergent.

In this example, all the assumptions A, B, and D1 hold. Moreover, the following weakening of D* is satisfied: at every time \(t \in \mathbb{N}\), there is some \(j \in [N]\) such that the connected component of \(j\) in the communication graph \(G(t)\) is \(j\)-oriented, and every other connected component of \(G(t)\) is strongly connected. Indeed, all the communication graphs have two connected components: one component is reduced to a single node with a self-loop, and the other one is oriented with respect to one single node (namely, node \(k\) for the matrix \(A_k\)).

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