Biharmonic Constant $\Pi_1$–Slope Curves according to Type-2 Bishop Frame in Heisenberg Group $\text{Heis}^3$

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Abstract: In this paper, we introduce constant $\Pi_1$–slope curves according to type-2 Bishop frame in the Heisenberg group $\text{Heis}^3$. We characterize the biharmonic constant $\Pi_1$–slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the Heisenberg group $\text{Heis}^3$. Additionally, we illustrate our main theorem.

Key Words: Biharmonic curve, type-2 Bishop frame, Heisenberg group.

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1. Introduction

A smooth map $\phi : N \rightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{J}(\phi)|^2 \, dv_h,$$

where $\mathcal{J}(\phi) := \text{tr} \nabla^\phi d\phi$ is the tension field of $\phi$.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{J}_2(\phi) = 0$. Here the section $\mathcal{J}_2(\phi)$ is defined by

$$\mathcal{J}_2(\phi) = -\Delta_{\phi} \mathcal{J}(\phi) + \text{tr} R(\mathcal{J}(\phi), d\phi) d\phi,$$

and called the bitension field of $\phi$. Non-harmonic biharmonic maps are called proper biharmonic maps, [5,6,7].

This study is organised as follows: Firstly, we introduce constant $\Pi_1$–slope curves according to type-2 Bishop frame in the Heisenberg group $\text{Heis}^3$. We characterize the biharmonic constant $\Pi_1$–slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the Heisenberg group $\text{Heis}^3$. Additionally, we illustrate our main theorem.

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2. The Heisenberg Group Heis\(^3\)

Heisenberg group Heis\(^3\) can be seen as the space \(\mathbb{R}^3\) endowed with the following multiplication:
\[
(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (x + \bar{x}, y + \bar{y}, z + \frac{1}{2}\bar{x}y + \frac{1}{2}x\bar{y}) \tag{2.1}
\]
Heis\(^3\) is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric \(g\) is given by
\[
g = dx^2 + dy^2 + (dz - xdy)^2.
\]

The Lie algebra of Heis\(^3\) has an orthonormal basis
\[
e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \tag{2.2}
\]
for which we have the Lie products
\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0
\]
with
\[
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\]

We obtain
\[
\nabla e_1 e_1 = \nabla e_2 e_2 = \nabla e_3 e_3 = 0,
\]
\[
\nabla e_1 e_2 = -\nabla e_2 e_1 = \frac{1}{2}e_3,
\]
\[
\nabla e_1 e_3 = \nabla e_3 e_1 = \frac{1}{2}e_2,
\]
\[
\nabla e_2 e_3 = \nabla e_3 e_2 = \frac{1}{2}e_1.
\]

3. Biharmonic Constant \(\Pi_1\)–Slope Curves according to New Type-2 Bishop Frame in Heisenberg Group Heis\(^3\)

Assume that \(\{T, N, B\}\) be the Frenet frame field along \(\gamma\). Then, the Frenet frame satisfies the following Frenet–Serret equations:
\[
\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B, \quad \nabla_T B = -\tau N, \tag{3.1}
\]
where \(\kappa\) is the curvature of \(\gamma\) and \(\tau\) its torsion and
\[
g(T, T) = 1, \quad g(N, N) = 1, \quad g(B, B) = 1,
\]
\[
g(T, N) = g(T, B) = g(N, B) = 0.
\]
The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

\[ \nabla_T T = k_1 M_1 + k_2 M_2, \]
\[ \nabla_T M_1 = -k_1 T, \]
\[ \nabla_T M_2 = -k_2 T, \]

where

\[ g(T, T) = 1, \quad g(M_1, M_1) = 1, \quad g(M_2, M_2) = 1, \]
\[ g(T, M_1) = g(T, M_2) = g(M_1, M_2) = 0. \]

Here, we shall call the set \{T, M_1, M_2\} as Bishop trihedra, \(k_1\) and \(k_2\) as Bishop curvatures and \(U(s) = \arctan \frac{k_2}{k_1}\), \(\tau(s) = U'(s)\) and \(\kappa(s) = \sqrt{k_1^2 + k_2^2}\).

Bishop curvatures are defined by

\[ k_1 = \kappa(s) \cos U(s), \]
\[ k_2 = \kappa(s) \sin U(s). \]

**Theorem 3.1.** \(\gamma : I \rightarrow Heis^3\) is a unit speed biharmonic curve with Bishop frame if and only if

\[ k_1^2 + k_2^2 = \text{constant} = C \neq 0, \]
\[ k_1'' - Ck_1 = k_3 \left[ \frac{1}{4} - (M_3^2)^2 \right] - k_2 M_1^3 M_2^3, \]
\[ k_2'' - Ck_2 = k_3 M_1^3 M_2^3 + k_2 \left[ \frac{1}{4} - (M_3^2)^2 \right]. \]

Let \(\gamma\) be a unit speed regular curve in Heis\(^3\) and (3.1) be its Frenet–Serret frame. Let us express a relatively parallel adapted frame:

\[ \nabla_T \Pi_1 = -\epsilon_1 B, \]
\[ \nabla_T \Pi_2 = -\epsilon_2 B, \]
\[ \nabla_T B = \epsilon_1 \Pi_1 + \epsilon_2 \Pi_2, \]

where

\[ g(B, B) = 1, \quad g(\Pi_1, \Pi_1) = 1, \quad g(\Pi_2, \Pi_2) = 1, \]
\[ g(B, \Pi_1) = g(B, \Pi_2) = g(\Pi_1, \Pi_2) = 0. \]

We shall call this frame as Type-2 Bishop Frame. In order to investigate this new frame’s relation with Frenet–Serret frame, first we write

\[ \tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}. \]
The relation matrix between Frenet–Serret and type-2 Bishop frames can be expressed

\[
\begin{align*}
T &= \sin \mathfrak{A}(s) \Pi_1 - \cos \mathfrak{A}(s) \Pi_2, \\
N &= \cos \mathfrak{A}(s) \Pi_1 + \sin \mathfrak{A}(s) \Pi_2, \\
B &= B.
\end{align*}
\]

So by (3.5), we may express

\[
\begin{align*}
\epsilon_1 &= -\tau \cos \mathfrak{A}(s), \\
\epsilon_2 &= -\tau \sin \mathfrak{A}(s).
\end{align*}
\]

By this way, we conclude

\[
\mathfrak{A}(s) = \arctan \frac{\epsilon_2}{\epsilon_1}.
\]

The frame \( \{\Pi_1, \Pi_2, B\} \) is properly oriented, and \( \tau \) and \( \mathfrak{A}(s) = \int_0^s \kappa(s) ds \) are polar coordinates for the curve \( \gamma \). We shall call the set \( \{\Pi_1, \Pi_2, B, \epsilon_1, \epsilon_2\} \) as type-2 Bishop invariants of the curve \( \gamma \), [19].

With respect to the orthonormal basis \( \{e_1, e_2, e_3\} \), we can write

\[
\begin{align*}
\Pi_1 &= \pi_1^1 e_1 + \pi_2^1 e_2 + \pi_3^1 e_3, \\
\Pi_2 &= \pi_1^2 e_1 + \pi_2^2 e_2 + \pi_3^2 e_3, \\
B &= B^1 e_1 + B^2 e_2 + B^3 e_3.
\end{align*}
\]

(3.6)

**Theorem 3.2.** Let \( \gamma : I \rightarrow \text{Heis}^3 \) be a unit speed non-geodesic biharmonic constant \( \Pi_1 \)-slope curves according to type-2 Bishop frame in the \( \text{Heis}^3 \). Then, the parametric equations of \( \gamma \)

\[
\begin{align*}
x(s) &= \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 - \kappa)} - \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 + \kappa)} - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2]}{2 (\mathcal{R}_1 - \kappa)} - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 + \kappa)} + \mathcal{L}_3, \\
y(s) &= \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 - \kappa)} + \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 + \kappa)} + \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2]}{2 (\mathcal{R}_1 - \kappa)} - \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 + \kappa)} + \mathcal{L}_4,
\end{align*}
\]
The vector $\Pi_1$ is a unit vector, we have the following equation

$$\Pi_1 = \sin \mathcal{A} \cos [L_1 s + L_2] e_1 + \sin \mathcal{A} \sin [L_1 s + L_2] e_2 + \cos \mathcal{A} e_3,$$  \hspace{1cm} (3.7)

where $L_1, L_2 \in \mathbb{R}$.

Then by type-2 Bishop formulas (3.4) and (2.1), we have

$$\Pi_2 = \cos \mathcal{A} \cos [L_1 s + L_2] e_1 + \cos \mathcal{A} \sin [L_1 s + L_2] e_2 - \sin \mathcal{A} e_3.$$
Applying above equation and (3.9), we get
\[ B = - \sin [\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_1 + \cos [\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_2. \]

Then, a combination of these equations with the second equation of (5.1) would give us
\[ T = \frac{\sin [\kappa s] \sin \mathfrak{A} \cos [\mathcal{L}_1 s + \mathcal{L}_2] - \cos [\kappa s] \cos \mathfrak{A} \cos [\mathcal{L}_1 s + \mathcal{L}_2]}{2 (\mathcal{L}_1 - \kappa)} \mathbf{e}_1 \\
+ \frac{\sin [\kappa s] \sin \mathfrak{A} \sin [\mathcal{L}_1 s + \mathcal{L}_2] - \cos [\kappa s] \cos \mathfrak{A} \sin [\mathcal{L}_1 s + \mathcal{L}_2]}{2 (\mathcal{L}_1 + \kappa)} \mathbf{e}_2 \\
+ \frac{\sin [\kappa s] \cos \mathfrak{A} - \cos [\kappa s] \sin \mathfrak{A}}{2} \mathbf{e}_3. \quad (3.8) \]

From (2.2) and (3.8), we have
\[ x(s) = \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 - \kappa)} - \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 + \kappa)} + \mathcal{L}_3, \]

where \( \mathcal{L}_3 \) is constant of integration.

Also,
\[ y(s) = \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 - \kappa)} + \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2]}{2 (\mathcal{L}_1 + \kappa)} + \mathcal{L}_4, \]

where \( \mathcal{L}_4 \) is constant of integration.

Again, by combining (2.2) and (3.8) we have
\[ \frac{dz}{ds} = \frac{1}{2 (\mathcal{L}_1 - \kappa)} \sin^2 \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2] \sin [\kappa s] \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
- \frac{1}{2 (\mathcal{L}_1 - \kappa)} \sin \mathfrak{A} \cos \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2] \cos [\kappa s] \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
- \frac{1}{2 (\mathcal{L}_1 + \kappa)} \sin^2 \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2] \sin [\kappa s] \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
+ \frac{1}{2 (\mathcal{L}_1 + \kappa)} \sin \mathfrak{A} \cos \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2] \cos [\kappa s] \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
+ \frac{1}{2 (\mathcal{L}_1 - \kappa)} \cos \mathfrak{A} \sin \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa) s + \mathcal{L}_2] \sin [\kappa s] \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
- \frac{1}{2 (\mathcal{L}_1 + \kappa)} \cos \mathfrak{A} \sin \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2] \sin [\kappa s] \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
+ \frac{1}{2 (\mathcal{L}_1 + \kappa)} \cos^2 \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa) s + \mathcal{L}_2] \cos [\kappa s] \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
+ \mathcal{L}_3 \sin [\kappa s] \sin \mathfrak{A} \sin [\mathcal{L}_1 s + \mathcal{L}_2] - \cos [\kappa s] \cos \mathfrak{A} \sin [\mathcal{L}_1 s + \mathcal{L}_2] \\
+ \sin [\kappa s] \cos \mathfrak{A} - \cos [\kappa s] \sin \mathfrak{A}. \]
Integrating both sides, we have theorem. Thus, the proof of theorem is completed.

\[ \text{Theorem 3.3. Let } \gamma : I \rightarrow \text{Heis}^3 \text{be a unit speed non-geodesic biharmonic constant } \Pi_1 \text{-slope curves according to type-2 Bishop frame in } \text{Heis}^3. \text{ Then, the position vector of } \gamma \text{ is} \]

\[
\gamma(s) = \left[ \frac{\sin \alpha \cos \left[ (\mathcal{L}_1 - \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \alpha \cos \left[ (\mathcal{L}_1 + \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 + \kappa)} \right] + \left[ \frac{\cos \alpha \sin \left[ (\mathcal{L}_1 - \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 - \kappa)} + \frac{\cos \alpha \sin \left[ (\mathcal{L}_1 + \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 + \kappa)} \right] \right] e_1
\]

\[
+ \left[ \frac{\sin \alpha \sin \left[ (\mathcal{L}_1 - \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \alpha \sin \left[ (\mathcal{L}_1 + \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 + \kappa)} \right] + \left[ \frac{\cos \alpha \cos \left[ (\mathcal{L}_1 - \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 - \kappa)} + \frac{\cos \alpha \cos \left[ (\mathcal{L}_1 + \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 + \kappa)} \right] \right] e_2
\]

\[
+ \left[ \frac{\sin \alpha \sin \left[ (\mathcal{L}_1 - \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \alpha \sin \left[ (\mathcal{L}_1 + \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 + \kappa)} \right] + \left[ \frac{\sin \alpha \cos \left[ (\mathcal{L}_1 - \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 - \kappa)} + \frac{\sin \alpha \cos \left[ (\mathcal{L}_1 + \kappa) s + \mathcal{L}_2 \right]}{2(\mathcal{L}_1 + \kappa)} \right] \right] e_3
\]

\[
+ \frac{\sin^2 \alpha}{2(\mathcal{L}_1 - \kappa)} e_4 + \frac{\sin^2 \alpha}{2(\mathcal{L}_1 + \kappa)} e_5
\]
where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are constants of integration.

**Proof:** Substituting (2.2) in Theorem 3.2, we have above equation. This completes the proof. □

We can use Mathematica in Theorem 3.2, yields
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