HECKE ALGEBRAS FOR PROTONORMAL SUBGROUPS

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Abstract. We introduce the term protonormal to refer to a subgroup $H$ of a group $G$ such that for every $x$ in $G$ the subgroups $x^{-1}Hx$ and $H$ commute as sets. If moreover $(G, H)$ is a Hecke pair we show that the Hecke algebra $\mathcal{H}(G, H)$ is generated by the range of a canonical partial representation of $G$ vanishing on $H$. As a consequence we show that there exists a maximum $C^*$-norm on $\mathcal{H}(G, H)$, generalizing previous results by Brenken, Hall, Laca, Larsen, Kaliszewski, Landstad and Quigg. When there exists a normal subgroup $N$ of $G$, containing $H$ as a normal subgroup, we prove a new formula for the product of the generators and give a very clean description of $\mathcal{H}(G, H)$ in terms of generators and relations. We also give a description of $\mathcal{H}(G, H)$ as a crossed product relative to a twisted partial action of the group $G/N$ on the group algebra of $N/H$. Based on our presentation of $\mathcal{H}(G, H)$ in terms of generators and relations we propose a generalized construction for Hecke algebras in case $(G, H)$ does not satisfy the Hecke condition.

1. Introduction.

After the pioneering work of Bost and Connes [BC], several authors started a systematic investigation of $C^*$-algebras obtained as completions of Hecke algebras. It was quickly realized [ALR], [B], [LR1] that the Hecke $C^*$-algebra which plays the central role in [BC] may be successfully described as the crossed product algebra relative to a semigroup of endomorphisms, prompting a large interest in the application of crossed product techniques to study Hecke algebras. See also [LL1], [LL2] and [LF].

The objective of the present paper is to study Hecke algebras from a similar point of view, namely the theory of partial group representations [E3; 6.2] and twisted partial crossed products [E2], [DE2]. See also [DE1], [DEP], [E1], [E4], [EL1], [EL2], and [ELQ].

If $H$ is a subgroup of a group $G$ recall that $(G, H)$ is said to be a Hecke pair if for each $x$ in $G$ the double coset $HxH$ is the disjoint union of finitely many right cosets; the number of right cosets involved usually being denoted in the literature by $R(x)$. Some authors [BC] also express the fact that $(G, H)$ is a Hecke pair by saying that $H$ is an almost-normal subgroup of $G$.

Given a Hecke pair and a field $F$ one defines the Hecke algebra $\mathcal{H}(G, H)$ as being the $F$-algebra formed by all $F$-valued finitely supported functions on the double coset space $H\backslash G/H$, under a certain convolution product.

This algebra is therefore obviously linearly generated by the simplest possible functions $1_{HxH}$ (the characteristic function of the singleton $\{HxH\}$), where $x$ ranges in a family of representatives for the double coset space $H\backslash G/H$. For technical purposes we assume that the characteristic of $F$ is zero and use

$$\sigma_x = \frac{1}{R(x)} 1_{HxH}, \quad \forall x \in G.$$ 

The starting point for our research can be subsumed by the question as to what extent the map

$$x \in G \mapsto \sigma_x \in \mathcal{H}(G, H)$$

is a group representation. The most naive form of this question, namely expecting that $\sigma$ be a genuine group representation, is not too interesting since this holds if and only if $H$ is a normal subgroup of $G$, in which case $\mathcal{H}(G, H)$ trivializes, being just the group algebra of the quotient group.

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This is where the theory of partial group representations comes into play. Recall that a partial representation of a group $G$ in a unital algebra $A$ is a map $u : G \to A$, such that $u(1) = 1$, and the usual group law \(u(xy) = u(x)u(y)\) holds after it is left-multiplied by $u(x^{-1})$ or right-multiplied by $u(y^{-1})$. See (2.1) below for a detailed definition.

It therefore makes sense to ask when is $\sigma$ a partial representation. Unfortunately the answer is again negative for many Hecke pairs, including most examples associated to the modular group $SL_2(\mathbb{Z})$ discussed e.g. in [Kr].

But, on the fortunate side, there are interesting examples for which the answer is affirmative. Among these is the Hecke pair appearing in the already mentioned work by Bost and Connes [BC], as well as some, but not all, Hecke pairs appearing in the papers that came in its wake.

Our first major effort is therefore directed at classifying the Hecke pairs for which $\sigma$ is a partial representation. In pursuit of this goal I have been led to considering a very weak normality property: let us say that a subgroup $H$ of a group $G$ is \textit{protonormal} if for every $x$ in $G$ the conjugate subgroup

\[ H^x = x^{-1}Hx \]

commutes with $H$ in the sense that the products of sets $H^xH$ and $HH^x$ coincide.

There is not much in the literature about this property except for some conditions for subnormality based on it for finite groups; see [W] and the references given there for more details. Also, it seems to me that this condition is related to Drinfeld’s notion of \textit{quantum double} (see [Ka: Chapter IX]) and perhaps it is interesting to explore this relationship further, a task I have not undertaken.

In what I believe is the main contribution of the present work, Theorems (8.1) and (8.2) prove that $\sigma$ is a partial representation if and only if $H$ is protonormal.

It is elementary to check, for instance, that for the Hecke pair in [BC] this condition is fulfilled. That Hecke pair is in fact a “bit more normal than protonormal”. Recall from [W] that the subgroup $H \subseteq G$ is said to be $n$-subnormal if there exists a normal chain

\[ H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_n = G, \]

of length $n$. If $H$ is 2-subnormal in $G$ then for every $x$ in $G$ and every $h$ in $H$ one has that $x^{-1}hx \in H_1$, so that $H^x \subseteq H_1$. Since $H$ is normal in $H_1$ one has that $yH = Hy$, for all $y \in H^x$, and hence $H^xH = HH^x$. In other words, 2-subnormal subgroups are necessarily protonormal.

Given the relevance of 2-subnormal subgroups in this work we shall call these simply \textit{subnormal}.

The first proof I found of the fact that $\sigma$ is a partial representation assumed that $H$ is subnormal, but in trying to prove that subnormality is a necessary condition for $\sigma$ being a partial representation I could only prove that $H$ must be protonormal. So the desire to generalize to protonormal groups came naturally. Having been born in such a roundabout way, I wonder how relevant the notion of protonormal subgroups will ever be. After fiddling a bit with this notion I was able to find a curious example of a Hecke pair $(G, H)$ such that $H$ is protonormal in $G$ but not subnormal. This seems to be based on the exceptional properties of the prime number 2. The reader will find the relevant results in (14.2) and (14.3) below.

Although Brenken does not mention the word “subnormal” in [B], he often works under the assumption that there exists a normal subgroup $N$ of $G$, containing $H$, and contained in the normalizer of $H$. Clearly the existence of such a subgroup $N$ is tantamount to the fact that $H$ is subnormal in $G$. Our results therefore generalize some of the results in [B]. See also [KLO: Theorem 8.5].

When the base field $F$ is equipped with an involution (as defined precisely in the next section) such as the usual involution on the field of complex numbers, Hecke algebras over $F$ can be made into $*$-algebras by considering the involution (as in [BC])

\[ f^\#(x) = \overline{f(x^{-1})}, \quad \forall x \in G, \]

for all finitely supported functions $f$ on $H \backslash G/H$. 

As in the theory of unitary group representations, most partial representations of interest taking values in a *-algebra satisfy the identity
\[ u(x)^* = u(x^{-1}) \]
Since \( \mathcal{H}(G, H) \) is a *-algebra it is natural to ask if this is the case for \( \sigma \). The answer is no but there exists another involution on \( \mathcal{H}(G, H) \) with respect to which \( \sigma \) satisfies the condition above. This involution was already used in [KLQ] and is defined by
\[ f^*(x) = \Delta(t^{-1}) \overline{f(x^{-1})} \]
where \( \Delta(x) = R(x)/R(x^{-1}) \) (recall from [Kr: I.3.6] that \( \Delta \) is a group homomorphism). En passant, the similarity with the formula for the adjoint in the C*-algebra of a locally compact group given in [Pedersen: 7.1], \( \Delta \) representing the modular function there, is nicely explained in [KLQ: Section 4].

If our field allows for taking square roots, or more precisely if there exists a multiplicative map \( \lambda \) from \( G \) to \( F \) such that \( \lambda(x)^2 = \Delta(x) \) for all \( x \) in \( G \) (which is clearly the case if \( F \) is the field of complex numbers) then the two involutions are isomorphic (see (5.6)). Assuming that \( F = \overline{\mathbb{Q}} \) and that \( H \) is protonormal observe that \( \sigma \) being a partial representation gives
\[ \sigma_x \sigma_x^* \sigma_x = \sigma_x \sigma_{x^{-1}} \sigma_x = \sigma_x \sigma_{x^{-1}} = \sigma_x \sigma_1 = \sigma_x, \]
so that any *-representation of \( \mathcal{H}(G, H) \) on a Hilbert space must send the generating elements \( \sigma_x \) to partial isometries, and hence to operators with norm no bigger than 1. Therefore, for every \( a \in \mathcal{H}(G, H) \) the supremum of \( \| \pi(a) \| \), as \( \pi \) range in the collection of all *-representations of \( \mathcal{H}(G, H) \), is a finite real number. This supremum defines a C*-norm on \( \mathcal{H}(G, H) \) which is obviously the maximum among all such. This solves a problem which has been addressed by many authors [B: Proposition 2.8], [H: Corollary 4.6], [LL1: Proposition 1.4], [KLQ: Theorem 8.5].

Our next main effort has got to do with the formula for the product \( \sigma_x \sigma_y \). Since \( \mathcal{H}(G, H) \) is linearly generated by the \( \sigma_x \), its multiplication operation is completely described by the "structure constants" \( \lambda_{x,y}^z \) implicitly defined by
\[ \sigma_x \sigma_y = \sum_{H \in \mathcal{H}(G/H)} \lambda_{x,y}^z \sigma_z. \]
The reader will find formulas for these constants in [Kr: I.4.4] and [KLQ].

Based on the techniques we developed we were able to find a significant simplification for these formulas under the hypothesis that \( H \) is subnormal. In fact, given \( x \) and \( y \) in \( G \) it is easy to show, based on the defining property of Hecke pairs, that \( HxHyH \) is the disjoint union of finitely many double cosets, say
\[ HxHyH = \bigcup_{1 \leq i \leq n} Hz_i H. \]
We prove in Theorem (10.2) that
\[ \sigma_x \sigma_y = \frac{1}{n} \sum_{i=1}^{n} \sigma_{z_i}. \]
Thus, viewing the Hecke algebra as the algebra generated by double cosets, as some authors have it, we see that the product of the double cosets \( HxH \) and \( HyH \) in the Hecke algebra is very closely related to the set theoretic product \( HxH \cdot HyH \) in \( G \); the former is precisely the average of the double cosets contained in the latter. In particular there is no mention to right or left cosets as in most other product formulas.

Based on concrete examples we were able to determine that (†) does not hold in general. It is therefore an interesting question (see (10.3)) to precisely determine for which Hecke pairs does this hold. I would very much like to know, for instance, whether (†) holds for protonormal subgroups, a question I have tried to solve without success.
Back to the subnormal situation a straightforward but interesting aspect about the above product formulas is that they encompass the whole algebraic structure of $\mathcal{H}(G, H)$. Precisely speaking we show in Theorem (10.5) that $\mathcal{H}(G, H)$ is the universal $F$-algebra generated by symbols $\{\sigma_x\}_{x \in G}$ under relations (†). This should be compared to other descriptions of Hecke algebras in terms of generators and relations, e.g.

Motivated by [LR1] we then take up the problem of describing Hecke algebras as crossed products. In order to describe our results in that direction let $(G, H)$ be a Hecke pair and suppose that there exists a subgroup $N$ of $G$ such that $H \subseteq N \subseteq G$. Clearly this implies that $H$ is subnormal in $G$.

One may motivate the desire to describe $\mathcal{H}(G, H)$ as a crossed product as follows: since this algebra arises as an attempt to make sense of the group algebra of the quotient $G/H$ (which is only a group if $H$ is normal in $G$), it should be obtained somehow as a product of $G/N$ by $N/H$.

In [LR1] and [B] it is assumed that $G$ is a semidirect product $N \rtimes K$ for some group $K$ (in which case $K$ is clearly isomorphic to $G/N$) and it is proved, under suitable hypothesis, that $\mathcal{H}(G, H)$ is a crossed product of the group algebra of $H/N$ by a semigroup of endomorphisms somehow based on $K$.

Our description of $\mathcal{H}(G, H)$ as a crossed product is based not on the theory of crossed products by endomorphisms, but on the recent theory of crossed products by partial actions [TPA], [DE2] briefly described in the next section. Precisely because this theory allows for a “twisting cocycle” we do not need to assume a semidirect product structure on $G$. Our main result in that direction, Theorem (11.9), then provides an isomorphism

$$\mathcal{H}(G, H) \simeq F\left(\frac{N}{H}\right) \times \frac{G}{N}$$

where $F\left(\frac{N}{H}\right)$ is the group algebra of the quotient group $N/H$ and the crossed product is with respect to a certain twisted partial action of the quotient group $G/N$ on $F\left(\frac{N}{H}\right)$.

If $G$ does have a semidirect product structure we may get rid of the cocycle, a result we prove in Corollary in (11.10).

We should mention that [LL1] proves a similar result in which $H \subseteq N \subseteq G$, but $H$ is not supposed to be normal in $N$ (the Hecke algebra for the pair $(N, H)$ replaces the group algebra $F\left(\frac{N}{H}\right)$), although it is still assumed that $G$ is a semidirect product. A common generalization therefore seems a worthwhile project.

Another interesting crossed product description for Hecke algebras, based on Green’s twisted crossed products, may be found in [KLQ].

Perhaps an advantage of the partial crossed product description over endomorphism crossed products is that we need not care at all about the existence of certain generating subsemigroups required in [LL1: Theorem 1.9] or [B: Theorem 3.12].

Recall that our description of Hecke algebras in terms of generators and relations in (10.5) refers to the decomposition of $HxHyH$ as a disjoint union of finitely many double cosets. One could then be tempted to do away with the Hecke condition, namely that every double cosets contains finitely many right cosets, and introduce a generalized condition by saying that $(G, H)$ is a pseudo Hecke pair if for every $x$ and $y$ in $G$ one has that $HxHyH$ is made out of finitely many double cosets. Unfortunately though, at least in the case of a subnormal $H \subseteq G$, one may prove with the aid of Propositions (10.1) and (3.2) that every pseudo Hecke pair is a true Hecke pair and vice-versa, so no extension of the usual concept is obtained.

Nevertheless, based on some insight provided by Cuntz-Krieger algebras for infinite matrices [EL1], we risk to introduce a generalized Hecke algebra for a group-subgroup pair $(G, H)$ which does not satisfy the Hecke condition. See Definition (13.1). Not having taken a single step in the description of the beast thus brought into existence, we at least give an example which might be of interest to some.

I would like to express my gratitude to a number of colleagues who, in a way or another, knowingly or not, were instrumental for the completion of this work. Those include, but are not limited to, M. Dokuchaev, D. Evans, and A. Zalesski who, over a short lunch, showed me a smooth path to the basic theory of Hecke algebras.
For the readers’s convenience this work is divided up into the following sections:

1. Introduction.
2. Generalities about partial representations.
3. Generalities about Hecke pairs.
4. The Hecke algebra.
5. $$^\ast$$-algebra structure.
6. Commuting subgroups.
7. Protonormal subgroups.
8. The canonical partial representation.
9. Generalities about subnormal groups.
10. A formula for the product and relations for the Hecke algebra.
11. Hecke algebra as a crossed product.
12. Hecke $$C^\ast$$-algebras.
13. A possible generalization of Hecke algebras.
14. An example.

2. Generalities about partial representations.

Let $$F$$ be a field of characteristic zero\(^1\). We will assume that $$F$$ has a conjugation, that is, an involutive automorphism $$z \in F \mapsto \bar{z} \in F$$, which will be fixed form now on. In the absence of a more interesting conjugation one could take the identity map by default. Clearly when $$F$$ is the field of complex numbers the conjugation of choice should be the standard one.

A map $$\phi : U \to V$$ between $$F$$-vector spaces $$U$$ and $$V$$ will be called conjugate-linear when it is additive and $$\phi(\lambda u) = \bar{\lambda} \phi(u)$$ for all $$\lambda \in F$$ and $$u \in U$$.

A $$^\ast$$-algebra is by definition an algebra $$A$$ over $$F$$ equipped with an involution 

$$a \in A \mapsto a^\ast \in A$$

which is conjugate-linear and such that $$(ab)^\ast = b^\ast a^\ast$$, for all $$a$$ and $$b$$ in $$A$$.

Whenever we speak of the group algebra $$F(G)$$, for a given group $$G$$, we will think of it as a $$^\ast$$-algebra with the the unique involution such that 

$$(\delta_t)^\ast = \delta_{t^{-1}}, \quad \forall t \in G,$$

where $$\delta_t$$ refers to the group element $$t$$ interpreted as an element of $$F(G)$$.

By a sesqui-linear form on an $$F$$-vector space $$V$$ we will mean a function 

$$\phi : V \times V \to F,$$

which is linear in the first variable and conjugate-linear in the second variable. We will say that $$\phi$$ is a hermitian form if $$\phi$$ moreover satisfies 

$$\phi(u, v) = \overline{\phi(v, u)}, \quad \forall u, v \in V.$$

A non-degenerate hermitian form will be one for which 

$$(\forall v \phi(u, v) = 0) \Rightarrow u = 0.$$

We shall now list a few definitions of relevance to the later sections for the convenience of the reader. See the references given for more information.

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\(^1\) One may perhaps generalize our results to other fields by tracking that its characteristic does not divide the order of certain coset spaces to be considered later.
2.1. Definition. [E3, DEP]. A partial representation of a group $G$ in a unital algebra $A$ is, by definition, a map $\sigma : G \to A$ such that

(i) $\sigma_1 = 1$,
(ii) $\sigma_{x^{-1}} \sigma_x \sigma_y = \sigma_{x^{-1}} \sigma_{xy}$,
(iii) $\sigma_x \sigma_y \sigma_{y^{-1}} = \sigma_{xy} \sigma_{y^{-1}}$,

for all $x, y$ in $G$. If moreover $A$ is a *-algebra we will say that $\sigma$ is a *-partial representation if

(iv) $(\sigma_x)^* = \sigma_{x^{-1}}$ for all $x$ in $G$.

Observe that under (2.1.iv) one has that (2.1.ii) and (2.1.iii) become equivalent. Given a partial representation $\sigma$ of $G$ on an algebra $A$ one has the following useful commutation relation

$$\sigma_x e_y = e_{xy} \sigma_x,$$

(2.2)

where $e_y := \sigma_y \sigma_{y^{-1}}$ and $e_{xy}$ is similarly defined (see [E3:2.4] for a proof).

2.3. Definition. [E2,DE2]. A twisted partial action of a group $G$ on an algebra $A$ is a triple

$$\Theta = \{(D_t)_{t \in G}, \{\theta_t\}_{t \in G}, \{w_{r,s}\}_{(r,s) \in G \times G}\},$$

where, for each $t$ in $G$, $D_t$ is a closed two sided ideal in $A$, $\theta_t$ is an isomorphism from $D_{t^{-1}}$ onto $D_t$, and for each $(r,s)$ in $G \times G$, $w_{r,s}$ is an invertible multiplier of $D_r \cap D_s$, satisfying the following postulates, for all $r, s$ and $t$ in $G$:

(i) $D_1 = A$ and $\theta_1$ is the identity automorphism of $A$,
(ii) $\theta_t(D_{r^{-1}} \cap D_s) = D_r \cap D_{rs}$,
(iii) $\theta_t(\sigma_s(a)) = w_{r,s} \theta_s(a) w_{r,s}^{-1}$, \quad $\forall a \in D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$,
(iv) $w_{t,e} = e_{t,e} = 1$,
(v) $\theta_t(aw_{s,t})w_{r,st} = \theta_t(a)w_{r,s}w_{r,s,t}$, \quad $\forall a \in D_{r^{-1}} \cap D_r \cap D_{st}$.

If moreover $A$ is a *-algebra we will say that the above is a *-twisted partial action if for all $t, r, s \in G$

(vi) $(D_t)^* = D_t$
(vii) $\theta_t(a^*) = (\theta_t(a))^*$, for all $a$ in $D_t$,
(viii) $(w_{r,s})^{-1} = (w_{r,s})^*$.

2.4. Definition. [E2,DE2]. Given a twisted partial action, as above, the crossed product algebra, denoted $A \rtimes_\Theta G$, is defined to be the direct sum

$$A \rtimes_\Theta G = \bigoplus_{g \in G} D_g,$$

with multiplication

$$(a_g \delta_g)(a_h \delta_h) = \theta_g(\theta^{-1}_g(a_g))w_{g,h}\delta_h,$$

for all $a_g \in D_g$ and $a_h \in D_h$, where we denote by $a_g \delta_g$ the element $a_g$ viewed in the factor $D_g$ of the above direct sum.

See [E2] and [DE2] for more details, including a proof of associativity of the above algebra under suitable hypotheses.
3. Generalities about Hecke pairs.

Throughout this section $G$ will be a group and $H$ a subgroup. We will denote by $G/H$ (respectively $H\setminus G$) the quotient of $G$ by the equivalence relation according to which $g_1 \sim g_2$ if and only if $g_1^{-1}g_2 \in H$ (respectively $g_1g_2^{-1} \in H$). Thus the equivalence classes relative to $G/H$ are the so called left cosets $gH$, for $g \in G$. Speaking of $H\setminus G$ one similarly has the right cosets $Hg$.

We will also consider the equivalence relation according to which the elements $g_1$ and $g_2$ of $G$ are equivalent when there exist $h,k \in H$ such that $g_1 = hg_2k$. The corresponding double cosets therefore have the form $HgH$, for $g \in G$, and the coset space will be denoted $H\setminus G/H$.

When $H$ is normal in $G$ then all notions coincide but, having developed a bias towards right coset spaces, we will insist in using the notation $H\setminus G$ while most people would prefer to use $G/H$. Moreover, we will adopt the standard fraction notation for right coset spaces, especially in displayed formulas:

### 3.1. Definition
If $A$ is a subgroup of a group $B$ we will let

$$
\frac{B}{A} := A\setminus B.
$$

A subset $S$ of $G$ will be called a family of representatives for a coset space (such as the ones above) if there is exactly one member of $S$ in each equivalence class.

### 3.2. Proposition
Let $H$ be a subgroup of a group $G$. For every $x \in G$ let

$$
H^x = x^{-1}Hx.
$$

Given $x \in G$, let $S$ be a family of representatives for the coset space $(H \cap H^x)\setminus H$. Then

$$
HxH = \bigcup_{h \in S} Hxh,
$$

where the symbol "∪" stands for disjoint union. Conversely, if $S$ is any subset of $H$ such that (3.2.1) holds then it is a family of representatives for $(H \cap H^x)\setminus H$.

**Proof.** The inclusion "⊇" in (3.2.1) is obvious so let’s prove "⊆". Given $y \in HxH$ write $y = k_1xk_2$, with $k_1, k_2 \in H$. By assumption there exists $h \in S$ such that $k_2h^{-1} \in H \cap H^x$, so that $k_2h^{-1} = x^{-1}kx$, for some $k \in H$. Therefore

$$
y = k_1xk_2 = k_1x(x^{-1}kxh) = k_1kxh \in Hxh.
$$

In order to prove disjointness suppose that $Hxh = Hk$, for $h, k$ in $S$. Then $xh = \ell xk$ for some $\ell \in H$ whence

$$
hk^{-1} = x^{-1}\ell x \in H \cap H^x,
$$

which implies that $h = k$. We leave the converse statement for the reader. ⊓⊔

### 3.3. Definition
Let $H$ be a subgroup of a group $G$. We will say that $(G, H)$ is a Hecke pair if for every $x$ in $G$ one (and hence all) of the following equivalent conditions hold:

(i) $HxH$ is a finite union of right cosets,

(ii) $(H \cap H^x)\setminus H$ is finite.

One could as well add two other equivalent conditions to the above, namely that (iii) $HxH$ is a finite union of left cosets, and (iv) $H/(H \cap H^x)$ is finite; but these will not be used here.
3.4. Definition. Let \((G, H)\) be a Hecke pair.
(i) We will denote by \(R : G \rightarrow \mathbb{N}\), the function defined by
\[
R(x) = |(H \cap H^x)\backslash H|, \quad \forall x \in G.
\]
(ii) We will denote by \(\Delta : G \rightarrow \mathbb{Q}\), the function defined by
\[
\Delta(x) = \frac{R(x)}{R(x^{-1})}, \quad \forall x \in G.
\]

By (3.2) we have that \(R(x)\) is also the number of right cosets in \(HxH\). It should also be noticed that \(R(x^{-1})\) is the number of left cosets in \(HxH\). Recall from [Krieg: I.3.6] that \(\Delta\) is a homomorphism into the additive group of rational numbers.

From now on we fix a Hecke pair \((G, H)\).

3.5. Definition. Denote by \(F(H\backslash G)\) any \(F\)-vector space having a basis with as many elements as \(H\backslash G\). Fix such a basis and denote it by
\[
B = \{\delta_u : u \in H\backslash G\}.
\]
For each right coset \(Hg\) we will denote by \(\delta_{Hg}\) the linear functional on \(F(H\backslash G)\) given by
\[
\langle \delta_{Ht}, \delta_{Hg} \rangle = \begin{cases} 1, & \text{if } Ht = Hg, \\ 0, & \text{otherwise,} \end{cases}
\]
for every \(t \in G\), where we denote the duality between \(F(H\backslash G)\) and its dual space by \(\langle \cdot, \cdot \rangle\), as usual.

3.6. Proposition. Given any \(x \in G\) there exists a unique linear operator \(\sigma_x\) on \(F(H\backslash G)\) such that
\[
\sigma_x(\delta_{Ht}) = \frac{1}{|S_x|} \sum_{h \in S_x} \delta_{Hxht}, \quad \forall t \in G, \tag{3.6.1}
\]
where \(S_x\) is any (necessarily finite) family of representatives for \((H \cap H^x)\backslash H\).

Proof. Observing that
\[
HxHt = \bigcup_{h \in S_x} Hxht,
\]
by (3.2.1), we see that the expression given for \(\sigma_x(\delta_{Ht})\) in the statement is just the average of the basis elements corresponding to the right cosets making up \(HxHt\). It is therefore immediate that \(\sigma_x\) is well defined and does not depend on the choice of \(S_x\). \qed

It is clear that \(\sigma_h\) is the identity operator for each \(h\) in \(H\). In fact this is a special case of the following more general fact:

3.7. Proposition. For every \(x, y \in G\) one has that \(\sigma_x = \sigma_y\) if and only if \(HxH = HyH\).

Proof. Suppose that \(HxH = HyH\), so \(y = k_1xk_2\) for some \(k_1, k_2 \in H\). Letting \(S_x\) be a family of representatives for \((H \cap H^y)\backslash H\) observe that
\[
HyH = Hk_1xk_2H = HxH = \bigcup_{h \in S_x} Hxh = \bigcup_{h \in S_x} Hk_1^{-1}yk_2^{-1}h = \bigcup_{h \in S_x} Hyk_2^{-1}h,
\]
so \(k_2^{-1}S_x\) is a family of representatives for \((H \cap H^y)\backslash H\). Therefore for every \(t \in G\),
\[
\sigma_y(\delta_{Ht}) = \frac{1}{|S_x|} \sum_{h \in S_x} \delta_{Hyk_2^{-1}ht} = \frac{1}{|S_x|} \sum_{h \in S_x} \delta_{Hk_1xht} = \frac{1}{|S_x|} \sum_{h \in S_x} \delta_{Hxht} = \sigma_x(\delta_{Ht}).
\]
Conversely suppose that \(\sigma_x = \sigma_y\). Then, since \(\langle \sigma_x(\delta_{H}), \delta_{Hx} \rangle \neq 0\), one necessarily also has \(\langle \sigma_y(\delta_{H}), \delta_{Hx} \rangle \neq 0\), hence there exists some \(k \in S_y\) (a family of representatives for \((H \cap H^y)\backslash H\)), such that \(Hyk = Hx\), so that \(HyH = HxH\). \qed
Let \((G,H)\) be a Hecke pair, fixed throughout this section.

4.1. Definition. The Hecke algebra of the pair \((G,H)\), denoted by \(\mathcal{H}(G,H)\), or simply \(\mathcal{H}\) if the pair \((G,H)\) is understood, is defined to be the sub-algebra of linear operators on \(F(H\setminus G)\) generated by the set \(\{\sigma_x : x \in G\}\).

For every \(g \in G\) denote by \(\rho(g)\) the “right multiplication” operator on \(F(H\setminus G)\) given by

\[
\rho(g)(\delta_{Ht}) = \delta_{Htg}, \quad \forall t \in G.
\]

It is apparent that \(\sigma_x\) commutes with \(\rho(g)\) for every \(x\) and \(g\) in \(G\). It therefore follows that each \(a \in \mathcal{H}\) commutes with every \(\rho(g)\).

4.2. Proposition. Let \(a, b \in \mathcal{H}\). If for some \(t \in G\) one has that \(a(\delta_{Ht}) = b(\delta_{Ht})\), then \(a = b\).

Proof. For every \(s \in G\) one has that

\[
a(\delta_{Hs}) = a(\rho(t^{-1}s)(\delta_{Ht})) = \rho(t^{-1}s)(a(\delta_{Ht})) = a(\rho(t^{-1}s)(\delta_{Ht})) = \rho(t^{-1}s)(b(\delta_{Ht})) = b(\delta_{Hs}),
\]

and hence \(a = b\). \(\square\)

In our next definition we will again make use of the linear functionals \(\delta_{Hg}\) introduced in (3.5).

4.3. Definition. For each \(a \in \mathcal{H}\), let \(f_a\) be the \(F\)-valued function on \(G\) defined by

\[
f_a(t) = \langle a(\delta_H), \delta_{Ht}^\prime \rangle, \quad \forall t \in G,
\]

so that

\[
a(\delta_H) = \sum_{Ht \in H\setminus G} f_a(t)\delta_{Ht}.
\]

By (4.2) we see that \(a\) is completely determined by \(f_a\). Observe also that by its very definition, \(f_a\) is constant on right cosets.

4.4. Proposition. For every \(a\) in \(\mathcal{H}\) one has that \(f_a\) is constant on each double coset. Moreover \(f_a\) is supported in the union of finitely many such double cosets.

Proof. Given \(x \in H\), let \(\rho'(x)\) be the dual operator of \(\rho(x)\). It is immediate to verify that \(\rho'(x)(\delta_{Ht}) = \delta_{Htx^{-1}}^\prime\).

For \(h, k \in H\) we therefore have

\[
f_a(kgh) = \langle a(\delta_H), \delta_{Hkgh}^\prime \rangle = \langle a(\delta_H), \delta_{Hgh}^\prime \rangle = \langle a(\delta_H), \rho'(h^{-1})(\delta_{Hg}^\prime) \rangle = \langle \rho(h^{-1})(a(\delta_H)), \delta_{Hg}^\prime \rangle = \langle a(\delta_{Hh^{-1}}), \delta_{Hg}^\prime \rangle = \langle a(\delta_H), \delta_{Hg}^\prime \rangle = f_a(g).
\]

As for the last part observe that, since \(a(\delta_H)\) is a vector in \(F(H\setminus G)\), it is a finite linear combination of the \(\delta_{Hg}\) and hence \(f_a\) is in fact supported in the union of finitely many right cosets, which must obviously involve an even smaller number of double cosets. \(\square\)

We can make use of \(f_a\) to describe the matrix of each operator \(a \in \mathcal{H}\):

4.5. Proposition. Let \(a \in \mathcal{H}\). Then, for each \(s, t \in G\) one has that

\[
\langle a(\delta_{Hs}), \delta_{Ht} \rangle = f_a(ts^{-1}).
\]
4.6. Proposition. If \( x \in G \) then the function \( f_{\sigma_x} \) coincides with the characteristic function of \( HxH \) divided by \( |(H \cap H^x) \setminus H| \).

**Proof.** Let \( S_x \) be a family of representatives for \( (H \cap H^x) \setminus H \). For every \( t \in G \) we have

\[
\begin{align*}
    f_{\sigma_x}(t) &= \langle \sigma_x(\delta_H), \delta'_H \rangle \\
    &= \left( \frac{1}{|S_x|} \sum_{h \in S_x} \delta_{H \cap H'h}, \delta'_H \right) \\
    &= \frac{1}{|S_x|} \left[ \exists h \in S_x, \ \delta_{H \cap H'h} = \delta_H \right] = \ldots
\end{align*}
\]

where the brackets correspond to the boolean value of the logical statement inside. Still making use of brackets, the above equals

\[
\ldots = \frac{1}{|S_x|} [Ht \subseteq HxH] = \frac{1}{|S_x|} [t \in HxH].
\]

Since \(|S_x|\) coincides with \(|(H \cap H^x) \setminus H|\), the proof is complete.

We therefore have a description of \( H \), at least as far as its linear structure is concerned:

4.7. Proposition. The correspondence \( a \mapsto f_a \) establishes a bijective linear correspondence between \( \mathcal{H} \) and the space of functions on \( G \) which are constant on double cosets and whose support consist of a finite union of such cosets.

**Proof.** By (4.4) we have that \( f_a \) does belong to the indicated set, while (4.2) shows that the correspondence is one-to-one. That our map is surjective follows from (4.6).

As an easy consequence we have:

4.8. Corollary. Let \( S \) be a family of representatives for \( H \setminus G/H \). Then the set \( \{ \sigma_x : x \in S \} \) is a linear basis for the Hecke algebra \( \mathcal{H}(G,H) \).

In order to describe the multiplicative structure of \( \mathcal{H} \) in terms of doubly invariant functions we need the following:

4.9. Proposition. If \( a, b \in \mathcal{H} \) then

\[
    f_{ab}(t) = \sum_{Hs \in H \setminus G} f_a(t^{-1}s)f_b(s), \quad \forall t \in G.
\]

**Proof.** We have

\[
    f_{ab}(t) = \langle ab(\delta_H), \delta'_H \rangle = \left( \sum_{Hs \in H \setminus G} f_b(s)\delta_{Hs} \right) \delta'_H = \sum_{Hs \in H \setminus G} f_a(t^{-1}s)f_b(s),
\]

concluding the proof.

We thus reconcile our point of view with the classical definition of Hecke algebras (see e.g. [Krieg: I.4]):

4.10. Corollary. \( \mathcal{H} \) is isomorphic to the algebra of doubly invariant functions on \( G \) which are supported in the union of finitely many double cosets, equipped with the convolution product defined, for every \( f \) and \( g \) in said algebra, by

\[
    (f * g)(t) = \sum_{Hs \in H \setminus G} f(ts^{-1})g(s), \quad \forall t \in G.
\]
5. \textit{*}-ALGEBRA STRUCTURE.

In this section we will turn the Hecke algebra into a \textit{*}-algebra. The reader might be familiar with the involution given by

\[ f^\#(t) = f(t^{-1}), \]

used by many authors (see e.g. [BC]). However the involution used in [KLQ] is better suited for our purposes, given our emphasis on the operators \( \sigma_x \), as we shall see shortly.

Below we will use the rational homomorphism \( \Delta(x) = R(x)/R(x^{-1}) \) defined in (3.4.ii).

5.2. Definition. We will denote by \( \langle \langle \cdot, \cdot \rangle \rangle \) the unique sesqui-linear form on \( F(H \setminus G) \) such that for every \( t, s \in G \),

\[ \langle \langle \delta_{Ht}, \delta_{Hs} \rangle \rangle = \begin{cases} \Delta(s), & \text{if } Ht = Hs, \\ 0, & \text{if } Ht \neq Hs. \end{cases} \]

Observe that when \( Ht = Hs \), then we have that \( HtH = HsH \) so \( R(t)/R(t^{-1}) = R(s)/R(s^{-1}) \) and we see that our form is hermitian. It is elementary to verify that it is non-degenerate as well.

5.3. Proposition. For every \( x \in G \) one has

\[ \langle \langle \sigma_x(\xi), \eta \rangle \rangle = \langle \langle \xi, \sigma_{x^{-1}}(\eta) \rangle \rangle, \quad \forall \xi, \eta \in F(H \setminus G). \]

Proof. It is obviously enough to consider \( \xi = \delta_{Ht} \), and \( \eta = \delta_{Hs} \), where \( t, s \in G \).

Choose families of representatives \( S_x \) and \( S_{x^{-1}} \) for the coset spaces \( (H \cap H^x) \setminus H \) and \( (H \cap H^{x^{-1}}) \setminus H \), respectively, so that \( |S_x| = R(x) \) and \( |S_{x^{-1}}| = R(x^{-1}) \). We have

\[ \langle \langle \sigma_x(\delta_{Ht}), \delta_{Hs} \rangle \rangle = \frac{1}{R(x)} \sum_{h \in S_x} \langle \langle \delta_{Hxt}, \delta_{Hs} \rangle \rangle = \frac{R(s)}{R(x)R(s^{-1})} \left[ st^{-1} \in HxH \right], \]

where the brackets denote boolean value, as before. On the other hand

\[ \langle \langle \delta_{Ht}, \sigma_{x^{-1}}(\delta_{Hs}) \rangle \rangle = \frac{1}{R(x^{-1})} \sum_{k \in S_{x^{-1}}} \langle \langle \delta_{Ht}, \delta_{Hsx^{-1}k} \rangle \rangle = \frac{R(t)}{R(x^{-1})R(t^{-1})} \left[ ts^{-1} \in Hx^{-1}H \right] = \]

\[ = \frac{R(t)}{R(x^{-1})R(t^{-1})} \left[ st^{-1} \in HxH \right]. \]

In order to complete the proof it is then enough to prove that

\[ \frac{R(s)}{R(x)R(s^{-1})} = \frac{R(t)}{R(x^{-1})R(t^{-1})} \quad (5.3.1) \]

whenever \( st^{-1} \in HxH \).

Write \( st^{-1} = hkk \), with \( h, k \in H \), so that \( x = h^{-1}st^{-1}k^{-1} \) and, given that \( R \) is clearly a doubly invariant function we have that \( R(x) = R(h^{-1}st^{-1}k^{-1}) = R(st^{-1}) \) and hence (5.3.1) boils down to

\[ \frac{R(s)}{R(st^{-1})R(s^{-1})} = \frac{R(t)}{R(ts^{-1})R(t^{-1})} \]

which follows immediately from [Krieg:1.3.6]. \( \square \)
5.4. **Corollary.** For every \( a \) in \( \mathcal{H}(G, H) \) there exists a unique \( a^* \) in \( \mathcal{H}(G, H) \) such that
\[
\langle a(\xi), \eta \rangle = \langle \xi, a^*(\eta) \rangle, \quad \forall \xi, \eta \in F(H \setminus G).
\]
In addition \( \mathcal{H}(G, H) \) becomes a *-algebra under the operation \( a \mapsto a^* \) and, for every \( x \in G \), we have \( \sigma_x^* = \sigma_{x^{-1}} \).

With this we may improve the description of \( \mathcal{H}(G, H) \) in terms of doubly invariant functions given in (4.10):

5.5. **Proposition.** \( \mathcal{H}(G, H) \) is *-isomorphic to the algebra of doubly invariant functions described in (4.10) once the latter is made a *-algebra by the involution given by
\[
f^*(t) = \Delta(t^{-1}) \overline{f(t^{-1})}, \quad \forall t \in G,
\]
for every \( f \) in said function algebra.

**Proof.** Observe that for all \( t \) and \( s \) in \( G \) we have that
\[
\langle \delta_{Ht}, \delta_{Hs} \rangle = \Delta(s) \langle \delta_{Ht}, \delta'_{Hs} \rangle,
\]
where the duality in the right-hand-side is given by (3.5). So obviously
\[
\langle \xi, \delta_{Hs} \rangle = \Delta(s) \langle \xi, \delta'_{Hs} \rangle, \quad \forall \xi \in F(H \setminus G).
\]
Given \( a \) in \( \mathcal{H}(G, H) \) we have by definition (4.3) that
\[
f_{a^*}(t) = \langle a^*(\delta_H), \delta'_{Ht} \rangle = \Delta(t^{-1}) \langle a^*(\delta_H), \delta_{Ht} \rangle = \Delta(t^{-1}) \langle a(\delta_H), \delta_{Ht} \rangle = \Delta(t^{-1}) \langle a(\delta_H), \delta_H \rangle = \Delta(t^{-1}) f_a(t^{-1}) = (f_a)^*(t). \quad \square
\]

Our last result of this section shows that, under certain hypotheses about the field \( F \), the two involutions are essentially the same:

5.6. **Proposition.** Suppose there exists a group homomorphism \( \lambda \) from \( G \) to the multiplicative group of \( F \) such that \( \lambda(x)^2 = \Delta(x) \), for all \( x \) in \( G \). Then the *-algebras \( (\mathcal{H}(G, H), *) \) and \( (\mathcal{H}(G, H), \#) \) are isomorphic.

**Proof.** It is elementary to check that the map
\[
\Lambda : (\mathcal{H}(G, H), *) \to (\mathcal{H}(G, H), \#)
\]
given by
\[
\Lambda(f)|_x = \lambda(x)f(x), \quad \forall x \in G,
\]
for all \( f \) in \( \mathcal{H}(G, H) \), is an isomorphism of *-algebras. \quad \square
6. Commuting subgroups.

We will now develop a few basic facts about commuting subgroups in preparation for our study of protonormal subgroups.

6.1. Definition.
(i) If $A$ and $B$ are subsets of a group $G$ we will denote by $AB$ the set

$AB = \{ab : a \in A, \ b \in B\}$.

(ii) If $A$ and $B$ are subgroups of $G$ we will say that $A$ and $B$ commute if $AB = BA$.

The following lists useful alternative characterizations of the concept above:

6.2. Proposition. Given subgroups $A$ and $B$ of a group $G$ the following are equivalent

(i) $A$ and $B$ commute,
(ii) $BA \subseteq AB$,
(iii) $AB$ is closed under multiplication,
(iv) $AB$ is a subgroup of $G$.

Proof. 
(i)$\Rightarrow$(ii): obvious.
(ii)$\Rightarrow$(iii): we have $ABAB = A(BA)B \subseteq A(AB)B = AABB = AB$.
(iii)$\Rightarrow$(iv): we have $(AB)^{-1} = B^{-1}A^{-1} = BA \subseteq ABAB \subseteq AB$,
so $AB$ is closed under taking inverses and hence is a subgroup.
(iv)$\Rightarrow$(i). We have $BA \subseteq ABAB = AB$.

Taking inverses we get $AB \subseteq BA$, so $AB = BA$. $\Box$

We now list two elementary results for future reference, in which the fraction notation introduced in (3.1) is used.

6.3. Lemma. If the subgroups $A$ and $B$ commute there is a natural bijection

$$\frac{B}{A \cap B} \rightarrow \frac{AB}{A}$$

which sends the right coset $(A \cap B)b$ to the right coset $Ab$, for every $b \in B$.

Proof. Left to the reader. $\Box$

6.4. Lemma. Let $A, B,$ and $C$ be groups with $A \subseteq B \subseteq C$ and let $\{b_i : i \in I\}$ and $\{c_j : j \in J\}$ be families of representatives for the coset spaces $A \backslash B$ and $B \backslash C$, respectively. Then $\{b_ic_j : (i, j) \in I \times J\}$ is a family of representatives for $A \backslash C$. In particular, if $A \backslash C$ is finite, then $A \backslash B$ and $B \backslash C$ are both finite and

$$\left| \frac{C}{A} \right| = \left| \frac{B}{A} \cdot \frac{C}{B} \right|.$$

Proof. Left to the reader. $\Box$

Let us fix, for the time being, a group $G$ and a subgroup $H$ and let $F(H \backslash G)$ be as defined in (3.5).
6.5. Definition. If $S$ is any finite subset of $H \setminus G$ we will denote by $\mu(S)$ the average of the elements of $S$ computed in $F(H \setminus G)$. Precisely speaking,

$$\mu(S) = \frac{1}{|S|} \sum_{s \in S} \delta_s.$$ 

6.6. Definition. Given a subgroup $K$ of $G$ which commutes with $H$ and such that $(H \cap K) \setminus K$ is finite, observe that $H \setminus HK$ is a finite subset of $H \setminus G$ by (6.3). We therefore denote by $q_K$ the element of $F(H \setminus G)$ defined by

$$q_K = \mu(H \setminus HK).$$

If $S \subseteq K$ is a family of representatives for the coset space $(H \cap K) \setminus K$, then by (6.3) we have that the elements of the form $Hk$, with $k \in S$, are precisely all of the (pairwise distinct) elements of $H \setminus HK$ and hence

$$q_K = \frac{1}{|S|} \sum_{k \in S} \delta_{Hk}.$$ (6.6.1)

In addition to $H$ we will now fix a subgroup $K$ of $G$ as above, that is, such that $K$ commutes with $H$ and $(H \cap K) \setminus K$ is finite.

As before let us denote by $\rho$ the right-regular (anti-)representation of $G$ on $F(H \setminus G)$.

6.7. Proposition. For all $g$ in $HK$ one has that

$$\rho_g(q_K) = q_K.$$ 

Proof. If $g \in HK$ then the operator $\rho_g$ clearly leaves $H \setminus HK$ invariant and hence it must consist of a permutation of the elements in the latter set, therefore leaving $q_K$ unchanged. \hfill \Box

If $x, y \in G$ are such that $Hx = Hy$, then $x = hy$ for some $h \in H$ and hence

$$\rho_x(q_K) = \rho_{hy}(q_K) = \rho_y(\rho_h(q_K)) \overset{(6.7)}{=} \rho_y(q_K),$$

so the expression $\rho_x(q_K)$ depends only on the right coset where $x$ lies. This proves the following:

6.8. Proposition. The correspondence

$$x \in G \mapsto \rho_x(q_K) \in F(H \setminus G)$$

drops to the quotient providing a well defined map from $H \setminus G$ to $F(H \setminus G)$ which, when linearized, gives an operator $Q_K$ on $F(H \setminus G)$ satisfying

$$Q_K(\delta_{Hx}) = \rho_x(q_K).$$ (6.8.1)

If $S \subseteq K$ is a family of representatives for the coset space $(H \cap K) \setminus K$ as in (6.6.1), notice that for all $x \in G$,

$$Q_K(\delta_{Hx}) = \rho_x(q_K) = \frac{1}{|S|} \sum_{k \in S} \rho_x(\delta_{Hk}) = \frac{1}{|S|} \sum_{k \in S} \delta_{HKx}.$$ (6.8.2)

This should be compared to the identity $(\sum_{k \in S} \delta_{Hk})\delta_{Hx} = \sum_{k \in S} \delta_{Hkx}$, which would only make sense if $H$ were a normal subgroup of $G$ and we were using of the group-algebra structure of $F(H \setminus G)$.

Denote by $\pi : H \setminus G \to HK \setminus G$, the quotient map and let

$$\tilde{\pi} : F(H \setminus G) \to F(HK \setminus G)$$

be its linearization.
6.9. Proposition. The restriction of \( \tilde{\pi} \) to the range of \( Q_\kappa \) is a linear isomorphism onto \( F(HK\setminus G) \). In addition
\[
\tilde{\pi}(Q_\kappa(\delta_{Hx})) = \delta_{HKx}, \quad \forall x \in G,
\]
so that
\[
\tilde{\pi} \circ Q_\kappa = \tilde{\pi}.
\]

Proof. Denote by \( \hat{\pi} \) the restriction of \( \tilde{\pi} \) to the range of \( Q_\kappa \). Let \( S \) be as in (6.8.2) so that for all \( x \in G \) one has
\[
\hat{\pi}(Q_\kappa(\delta_{Hx})) = \frac{1}{|S|} \sum_{k \in S} \tilde{\pi}(\delta_{Hkx}) = \frac{1}{|S|} \sum_{k \in S} \delta_{HKkx} = \delta_{HKx},
\]
where the last step holds because \( S \subseteq K \). This proves the identity in the statement and also that \( \hat{\pi} \) is surjective. In order to prove injectivity consider the map
\[
\phi : x \in G \mapsto Q_\kappa(\delta_{Hx}) \in F(H\setminus G),
\]
and observe that if \( x, y \in G \) are such that \( x = gy \), with \( g \in HK \), then
\[
\phi(x) = Q_\kappa(\delta_{Hx}) = \rho_x(q_K) = \rho_y(q_K) = \rho_y(\rho_g(q_K)) = Q_\kappa(\delta_{Hy}) = \phi(y).
\]
Therefore \( \phi \) drops to the quotient \( HK\setminus G \) and the corresponding linearization is a map
\[
\tilde{\phi} : F(HK\setminus G) \to F(H\setminus G)
\]
satisfying
\[
\tilde{\phi}(\delta_{HKx}) = Q_\kappa(\delta_{Hx}), \quad \forall x \in G.
\]
Therefore we have for all \( x \in G \), that
\[
\tilde{\phi}\left(\hat{\pi}(Q_\kappa(\delta_{Hx}))\right) = \tilde{\phi}(\delta_{HKx}) = Q_\kappa(\delta_{Hx})
\]
showing that \( \tilde{\phi} \circ \hat{\pi} \) is the identity map on the range of \( Q_\kappa \). Thus \( \hat{\pi} \) is injective. \( \Box \)

In the last result of this section we shall again refer to the quotient map \( \pi : H\setminus G \to HK\setminus G \), as well as to its linearized version \( \tilde{\pi} \).

6.10. Proposition. If \( L \) is yet another subgroup of \( G \) which commutes with both \( H \) and \( K \), and such that \( (H \cap L)\setminus L \) is finite, then
\[
\tilde{\pi}(q_L) = \mu(HK\setminus HKL).
\]
(6.10.1)

In particular \( \tilde{\pi}(q_L) \) only depends on the image of \( H\setminus HL \) under \( \pi \).

Proof. Consider the chain of subgroups
\[
H \cap L \subseteq HK \cap L \subseteq L
\]
and let \( \{b_i : i \in I\} \) and \( \{c_j : j \in J\} \) be families of representatives for the coset spaces \((H \cap L)\setminus (HK \cap L)\) and \((HK \cap L)\setminus L\), respectively. By (6.4) we then have that \( \{b_i c_j : (i, j) \in I \times J\} \) is a family of representatives for \((H \cap L)\setminus L\), which is a finite set by hypothesis hence implying that both \( I \) and \( J \) must be finite sets as well. By (6.6.1) we have that
\[
\tilde{\pi}(q_L) = \tilde{\pi}\left( \frac{1}{|I||J|} \sum_{i \in I} \sum_{j \in J} \delta_{Hb_ic_j} \right) = \frac{1}{|I||J|} \sum_{i \in I} \sum_{j \in J} \delta_{HKb_ic_j} = \frac{1}{|J|} \sum_{j \in J} \delta_{HKc_j} = \mu(HK\setminus HKL),
\]
where the last step follows from the natural equivalence between \((HK\cap L)\setminus L\) and \(HK\setminus HKL\) given by (6.3). \( \Box \)
7. Protonormal subgroups.

7.1. Definition. Let $H$ be a subgroup of a group $G$. We will say that $H$ is a protonormal\(^\text{2}\) subgroup if $H^x$ and $H$ commute for every $x \in G$ (recall from (3.2) that $H^x$ means $x^{-1}Hx$).

Observe that every normal subgroup $H$ is protonormal since $H^x = H$ for all $x \in G$. More generaly, suppose that there exists a subgroup $N$ of $G$ containing $H$ such that $H \triangleleft N \triangleleft G$ (the symbol “$\triangleleft$” standing for “is normal in”), in which case it is sometimes customary to say that $H$ is 2-subnormal, which we shall shorten to subnormal. Then for every $x \in G$ and $h \in H$ we have that $x^{-1}hxH = Hx^{-1}hx$.

It easily follows that $H^x$ and $H$ commute. In other words, every subnormal subgroup is protonormal.

Given $y \in G$ and assuming that $H^yx$ and $H$ commute we conclude, upon applying the inner automorphism, $Ad_{x^{-1}}: g \in G \mapsto x^{-1}gx \in G$, that $H^y$ and $H^x$ also commute. Thus, if $H$ is a protonormal subgroup then all of its conjugates commute among themselves. It is also evident that the subgroups of the form

$$H^{x_1}H^{x_2} \ldots H^{x_n},$$

where $x_1, x_2, \ldots, x_n \in G$, all commute with each other. Since for all $y \in G$ we have that

$$y^{-1}(H^{x_1}H^{x_2} \ldots H^{x_n})y = H^{x_1y}H^{x_2y} \ldots H^{x_ny},$$

we see that $H^{x_1}H^{x_2} \ldots H^{x_n}$ is also protonormal.

When $(G, H)$ is a Hecke pair such that $H$ is protonormal in $G$ we have by definition that $(H \cap H^x) \setminus H$ is finite and therefore so is $H^x \setminus H^xH$ by (6.3). This allows for a slightly different but useful description for the operators $\sigma_x$ of (3.6):

7.2. Proposition. If $T_x$ is a family of representatives for the coset space $H^x \setminus H^xH$, then

$$\sigma_x(\delta_{Ht}) = \frac{1}{|T_x|} \sum_{k \in T_x} \delta_{Hxkt}, \quad \forall t \in G.$$ 

Proof. Let $T_x = \{k_1, \ldots, k_n\}$ and write each $k_i$ as $\ell_i h_i$, with $\ell_i \in H^x$ and $h_i \in H$. It is then easy to prove that $\{h_1, \ldots, h_n\}$ is a family of representatives for $(H \cap H^x) \setminus H$. In addition notice that $x\ell_i x^{-1} \in H$, so that

$$Hxk_i t = Hx\ell_i h_i t = Hx\ell_i x^{-1}xh_i t = Hxh_i t,$$

from where the result follows. \(\square\)

\(^2\) From “dictionary.reference.com”: proto- (pref.) 4. Having the least amount of a specified element or radical.
8. The canonical partial representation.

Throughout this section we will fix a Hecke pair \((G, H)\) such that \(H\) is a protonormal subgroup of \(G\). Our major goal will be to show that \(\sigma\) is a partial group representation.

8.1. Theorem. If \((G, H)\) is a Hecke pair with \(H\) protonormal in \(G\) then the correspondence

\[ x \in G \mapsto \sigma_x \in \mathcal{H}(G, H) \]

is a partial representation.

Proof. Axiom (2.1.i) is obviously verified so we begin by proving that for every \(x\) and \(y\) in \(G\) one has that

\[ \sigma^{-1}_x \sigma_x \sigma_y = \sigma^{-1}_x \sigma_{xy}. \]

By (4.2) it is enough to show that these operators coincide on \(\delta_{H^y}^{-1}\).

For every \(u \in \{x^{-1}, x, y, xy\}\), pick a family of representatives \(S_u\) for the coset space \((H \cap H^u) \setminus H\). We therefore have

\[ \sigma^{-1}_x \sigma_x \sigma_y (\delta_{H^y}^{-1}) = \frac{1}{|S_x|} \sum_{k \in S_x} \sum_{k \in S_x} \sum_{\ell \in S_y} \delta_{H^x}^{-1} \delta_{H^y}^{-1} = \ldots \]

Recalling that \(\rho\) denotes the right regular representation of \(G\) on \(F(\langle H \setminus G\rangle)\) we may write the above as

\[ \ldots = \frac{1}{|S_x|} \sum_{k \in S_x} \sum_{\ell \in S_y} \rho_k \delta_{H^y}^{-1} (\delta_{H^x}^{-1}) = \frac{1}{|S_x|} \sum_{k \in S_y} \sum_{\ell \in S_x} \rho_k \delta_{H^x}^{-1} (\rho_k (\delta_{H^y}^{-1})) = \ldots \]

Given that \(S_x^{-1}\) is a family of representatives for \((H \cap H^{x^{-1}}) \setminus H\), it is evident that \(\{x^{-1} h x : h \in S_x^{-1}\}\) is a family of representatives for \((H \cap H^x) \setminus H^x\), so that the term within the big pair of parenthesis above coincides with \(q_{H^x}\) by (6.6.1). Here we are using the results of section (6) with the role of the groups \(H\) and \(K\) mentioned there played by \(H\) and \(H^x\), respectively. The above then equals

\[ \ldots = \frac{1}{|S_x|} \sum_{k \in S_x} \sum_{\ell \in S_y} \rho_k \delta_{H^y}^{-1} (q_{H^x}) = \frac{1}{|S_x|} \sum_{k \in S_y} \sum_{\ell \in S_x} \rho_k \delta_{H^x}^{-1} (\rho_k (q_{H^y}^{-1})) = \ldots \]

where the last identity again follows from (6.6.1) since \(\{\ell y^{-1} : \ell \in S_y\}\) is a family of representatives for \((H \cap H^{y^{-1}}) \setminus H^{y^{-1}}\).

On the other hand

\[ \sigma^{-1}_x \sigma_{xy} (\delta_{H^y}^{-1}) = \frac{1}{|S_x|} \sum_{k \in S_x} \sum_{m \in S_{xy}} \delta_{H^x}^{-1} \delta_{H^y}^{-1} = \]

\[ = \frac{1}{|S_{xy}|} \sum_{m \in S_{xy}} \rho_{ym}^{-1} \left( \frac{1}{|S_x|} \sum_{h \in S_x} \delta_{H^x}^{-1} \delta_{H^y}^{-1} \right) = \frac{1}{|S_{xy}|} \sum_{m \in S_{xy}} \rho_{ym}^{-1} (q_{H^x}) = \]

\[ = Q_{H^x} \left( \frac{1}{|S_{xy}|} \sum_{m \in S_{xy}} \delta_{H^y}^{-1} \right), \]
Denoting by $q'$ the element of $F(H\backslash G)$ enclosed by the last big pair of parenthesis above our task is therefore reduced to proving the identity
\[ Q_{H^x} (q_{H^{y^{-1}}}) = Q_{H^x} (q'). \]

Employing (6.9) we see that the above identity holds if and only if $\bar{\pi} (Q_{H^x} (q_{H^{y^{-1}}})) = \bar{\pi} (Q_{H^x} (q'))$), which is to say that
\[ \bar{\pi} (q_{H^{y^{-1}}}) = \bar{\pi} (q'), \] (8.1.1)

by the last part of (6.9).

Consider the diagram below in which we use the notation described in (3.1):

\[
\begin{array}{cccc}
H & \xrightarrow{Ad_y} & H^{y^{-1}} & \xrightarrow{(6.3)} & HH^{y^{-1}} & \xrightarrow{} & G \\
H \cap H^y & \downarrow \pi & H^{y^{-1}} \cap H & \downarrow \pi & H & \downarrow \pi \\
\end{array}
\]

where the arrows “→” refer to inclusion and the vertical arrows are quotient mappings.

We now intend to apply (6.10) for the two situations outlined in the rows in our diagram. Precisely, with respect to the top row, the triple $(H, K, L)$ of groups referred to in (6.10) will be taken to be $(H, H^x, H^{y^{-1}})$. Identity (6.10.1) is then translated to
\[ \bar{\pi} (q_{H^{y^{-1}}}) = \mu (HH^x \backslash H H^x H^{y^{-1}}). \]

Speaking of the bottom row, take the triple $(H, K, L)$ of (6.10) to be $(H^x, H, H^{y^{-1}})$. In order to distinguish from the previous application of (6.10), we will use $q^2$ in place of $q$.

Observe that since $S_{xy}$ is a family of representatives for $(H \cap H^{xy}) \backslash H$, we have that $Ad_y (S_{xy})$ is a family of representatives for $(H^x \cap H^{y^{-1}}) \backslash H^{y^{-1}}$. Therefore
\[ q^2_{H^{y^{-1}}} = \frac{1}{|S_{xy}|} \sum_{m \in S_{xy}} \delta_{H^x y m y^{-1}}. \]

Applying (6.10) we therefore deduce that
\[ \bar{\pi}_x (q^2_{H^{y^{-1}}}) = \mu (H H^x \backslash H H^x H^{y^{-1}}). \]

Since the $\mu$'s of our two situations coincide, as they both correspond to averaging within $F(H H^x \backslash G)$, we then conclude that
\[ \bar{\pi} (q_{H^{y^{-1}}}) = \bar{\pi}_x (q^2_{H^{y^{-1}}}). \]

It follows that
\[ \bar{\pi} (q_{H^{y^{-1}}}) = \bar{\pi}_x (q^2_{H^{y^{-1}}}) = \frac{1}{|S_{xy}|} \sum_{m \in S_{xy}} \bar{\pi}_x (\delta_{H^x y m y^{-1}}) = \]
\[ = \frac{1}{|S_{xy}|} \sum_{m \in S_{xy}} \delta_{H H^x y m y^{-1}} = \frac{1}{|S_{xy}|} \sum_{m \in S_{xy}} \bar{\pi} (\delta_{H y m y^{-1}}) = \bar{\pi} (q'), \]
proving (8.1.1) and hence showing that \( \sigma \) satisfies (2.1.ii). With respect to (2.1.iii) observe that for all \( \xi, \eta \in F(\mathcal{H} \setminus G) \) one has

\[
\langle \langle \sigma x \sigma y \sigma y^{-1} (\xi), (\eta) \rangle \rangle \overset{(5.3)}{=} \langle \langle \xi, \sigma y \sigma y^{-1} \sigma x^{-1} (\eta) \rangle \rangle \overset{(5.3)}{=} \langle \langle \sigma y \sigma y^{-1} (\xi), (\eta) \rangle \rangle.
\]

Given that \( \langle \langle \cdot, \cdot \rangle \rangle \) is nondegenerated we conclude that

\[
\sigma x \sigma y \sigma y^{-1} = \sigma y \sigma y^{-1} x \sigma x^{-1}.
\]

We would now like to show that it is necessary to assume that \( H \) is protonormal in \( G \) in order to conclude that \( \sigma \) is a partial representation.

**8.2. Theorem.** Let \( (G, H) \) be a Hecke pair such that

\[
\sigma x \sigma x^{-1} \sigma x = \sigma x, \quad \forall x \in G,
\]

(which obviously holds in case \( \sigma \) is a partial representation). Then \( H \) is protonormal in \( G \).

**Proof.** For every \( u \in \{ x^{-1}, x \} \), pick a family of representatives \( S_u \) for the coset space \((H \cap H^u) \setminus H\). Then

\[
\sigma x \sigma x^{-1} \sigma x (\delta_H) = \frac{1}{|S_x|} \sum_{h \in S_x} \sum_{k \in S_{x^{-1}}} \sum_{t \in S_x} \delta_{Hxhx^{-1}kxt}.
\]

It is easy to see that every right coset contained in \( HxHx^{-1}HxH \) occurs with a nonzero coefficient in the sum above\(^3\). It must therefore occur as well in the sum describing \( \sigma x (\delta_H) \), namely

\[
\sigma x (\delta_H) = \frac{1}{|S_x|} \sum_{h \in S_x} \delta_{Hxh}.
\]

It follows that \( HxHx^{-1}HxH \subseteq HxH \). Multiplying this on the left by \( x^{-1} \) gives

\[
x^{-1}HxHx^{-1}HxH \subseteq x^{-1}HxH,
\]

or equivalently that \( H^x H H^x H \subseteq H^x H \). Using (6.2.iii) it follows that \( H^x \) and \( H \) commute, so \( H \) is protonormal in \( G \), as desired. \( \square \)

Observe that the kernel of \( \sigma \), namely

\[
\text{Ker}(\sigma) = \{ x \in G : \sigma x = 1 \}
\]

is precisely \( H \). This shows that, while the kernel of a partial representation is always a subgroup, it needs not be normal. This motivates the general question as to which subgroups of a group \( G \) coincide with the kernel of a partial representation. The answer is very simple, all subgroups do. Given any subgroup \( H \subseteq G \) consider the map \( u : G \to F \) given by

\[
u(x) = \begin{cases} 1, & \text{if } x \in H, \\ 0, & \text{otherwise}. \end{cases}
\]

It is easy to see that \( u \) is a partial representation and clearly \( \text{Ker}(u) = H \).

\(^3\) Observe that we are using, in a non-trivial manner, that the characteristic of \( F \) is zero.
9. Generalities about subnormal groups.

Some of our results can only be proved for subgroups which are a bit more normal than protonormal. We shall briefly describe this class in what follows referring the reader to [W] for more information.

9.1. Definition. Let $H$ be a subgroup of a group $G$. We will say that $H$ is subnormal in $G$ if for every $x \in G$ and $h, k \in H$ one has that

$$xhx^{-1}kxh^{-1}x^{-1} \in H.$$  

Writing the above as $(xhx^{-1})k(xhx^{-1})^{-1}$, this says that $H$ is closed under conjugation by elements $g$ in $G$ of the form $g = xhx^{-1}$ (which itself is the conjugation of the element $h \in H$ by the arbitrary element $x \in G$).

9.2. Proposition. If $H$ is a subgroup of a group $G$ then the following are equivalent:

(i) $H$ is subnormal in $G$.

(ii) For every $x \in G$ and $h \in H$ one has that $Hxhx^{-1} = xhx^{-1}H$.

(iii) $H$ is normal in the intersection of all normal subgroups of $G$ containing $H$.

(iv) There exists a subgroup $N$ of $G$ such that $H \trianglelefteq N \trianglelefteq G$.

Proof. Observe that any normal subgroup of $G$ containing $H$ must contain the set

$$Y = \{xhx^{-1} : x \in G, h \in H\},$$

and hence also the subgroup $N$ generated by $Y$.

Since $Y$ is obviously invariant under conjugation by elements of $G$, one sees that the same applies to $N$, that is, $N \trianglelefteq G$. This said it becomes clear that $N$ is the intersection of all normal subgroups of $G$ containing $H$ mentioned in (ii).

Assuming (i) notice that $yHy^{-1} = H$, for every $y$ in $Y$. Therefore the same holds for every $y$ in $N$. So $H \trianglelefteq N$. This proves that (i) $\Rightarrow$ (iii).

It is obvious that (iii) $\Rightarrow$ (iv). In order to show that (iv) $\Rightarrow$ (i) let $N$ be as in (iv) and let $x \in G$ and $h, k \in H$. Observe that the element $n = xhx^{-1}$ satisfies

$$n = xhx^{-1} \in xHx^{-1} \subseteq xNx^{-1} = N,$$

so that

$$(xhx^{-1})k(xhx^{-1})^{-1} = nkn^{-1} \in nHn^{-1} \subseteq H,$$

because $H \trianglelefteq N$.

We leave the elementary implication (i) $\iff$ (ii) for the reader.  

Recall from the introduction that $H \subseteq G$ is said to be $n$-subnormal if there exists a normal chain

$$H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_n = G,$$

of length $n$. Thus, our concept of subnormality is equivalent to 2-subnormality. Also observe that every subnormal subgroup is protonormal.

9.3. Proposition. If $H$ is subnormal in $G$ let $N$ be a subgroup of $G$ such that $H \trianglelefteq N \trianglelefteq G$. Then for every $x \in G$ one has that $H^x \trianglelefteq N$. In particular

(i) $H \cap H^x \trianglelefteq H$, and

(ii) $H \trianglelefteq HH^x$.

Proof. $H^x$ is contained in $N$ because

$$H^x = x^{-1}Hx \subseteq x^{-1}Nx = N.$$

Moreover $H^x$ is the image of $H$ under the (not necessarily internal) automorphism $Ad_{x^{-1}}$ of $N$, and hence $H^x$ is normal in $N$. (i) and (ii) are elementary consequences of the first part.
10. A Formula for the Product and Relations for the Hecke Algebra.

From now on we assume that \((G, H)\) is a Hecke pair such that \(H\) is subnormal in \(G\). One of our main goals is to obtain what we believe are the cleanest formulas ever for the product \(\sigma_x \sigma_y\) of generators of \(\mathcal{H}(G, H)\). See also [Kr: I.4.4] and [KLQ].

We begin by studying certain aspects of double cosets in a little more detail. As usual let us view double cosets as the orbits of the action \(\beta\) of \(H \times H\) on \(G\) given by

\[
\beta_{(h, k)}(x) = hxk^{-1}, \quad \forall (h, k) \in H \times H, \quad \forall x \in G.
\]

Given double cosets \(HxH\) and \(HyH\), observe that their product \(HxHyH\) is invariant under \(\beta\) and hence may be written as the disjoint union of orbits of the form \(HxhyH\), for certain elements \(h\) in \(H\). Observe moreover that, for \(h, k \in H\), one has that

\[
HxkyH = HxhyH \quad \text{(†)}
\]

if and only if

\[
xky \in HxhyH \iff k \in x^{-1}HxhyHy^{-1} = H^xH^y^{-1}.
\]

Since \(H\) is subnormal in \(G\) we have that \(H^xh = hH^x\) and hence \((†)\) holds if and only if \(h\) and \(k\) define the same coset modulo \(H \cap H^xH^y^{-1}\). We therefore have:

10.1. Proposition. Suppose that \(H\) is a subnormal subgroup of a group \(G\). Given \(x, y \in G\), let \(S_{x,y}\) be a family of representatives for \((H \cap H^xH^y^{-1})\setminus H\). Then

\[
HxHyH = \bigcup_{h \in S_{x,y}} HxhyH.
\]

(10.1.1)

Conversely, if \(S_{x,y}\) is any subset of \(H\) such that (10.1.1) holds then it is a family of representatives for \((H \cap H^xH^y^{-1})\setminus H\).

Proof. If \(h, k \in H\) notice that

\[
HxhyH = HxkyH \iff xhy \in HxkyH \iff h \in x^{-1}HxkyHy^{-1} = H^xkH^y^{-1}.
\]

Under the hypothesis that \(H\) is subnormal we have that \(H^xk = kH^x\) so the above holds if and only if \(hk^{-1} \in H \cap H^xH^y^{-1}\). \(\square\)

We are now ready to prove an important result, namely that the product of two elements \(\sigma_x\) and \(\sigma_y\), corresponding to the double cosets \(HxH\) and \(HyH\), is the average of the \(\sigma_z\) for the double cosets \(HzH\) which make up \(HxHyH\).

10.2. Theorem. Let \((G, H)\) be a Hecke pair such that \(H\) is subnormal in \(G\). Given \(x, y \in G\), let \(S_{x,y}\) be any subset of \(H\) such that \(HxHyH = \bigcup_{h \in S_{x,y}} HxhyH\). Then

\[
\sigma_x \sigma_y = \frac{1}{|S_{x,y}|} \sum_{h \in S_{x,y}} \sigma_{xhy}.
\]
Proof. Consider the following diagram in which all horizontal maps are defined to be the inclusion of the group appearing in the corresponding numerator, moded out by the corresponding denominators:

\[
\begin{array}{cccccc}
0 & \to & H \cap H^x & \to & H & \to & H \cap H^y \\
& & \uparrow{Ad_{y^{-1}}} & & & \\
0 & \to & H^y \cap H^z & \to & H & \to & H^y \cap H^z \\
& & & & & & \\
0 & \to & H^y \cap H^z H^y & \to & H^y & \to & H^y \cap H^z H^y \\
\end{array}
\]

It is elementary to check that all rows are exact and all vertical maps are isomorphisms.

We will refer to these groups by the cardinal points so that for instance \(H \cap H^x\) will be called the northwest group.

Recalling that \(S_{x,y}\) is a family of representatives for the northeast group by (10.1), let \(A\) be a family of representatives for the northwest group, so that the set

\[S_x := S_{x,y}A = \{ba : b \in S_{x,y}, a \in A\}\]

is a family of representatives for the north group. It is also worth noticing that for distinct pairs \((a_1, b_1)\) and \((a_2, b_2)\) in \(A \times S_{x,y}\) one has that \(b_1a_1 \neq b_2a_2\), so that \(|S_x| = |A||S_{x,y}|\).

Similarly let \(C\) and \(D\) be families of representatives for the southwest and southeast groups, respectively.

Notice that the equivalence class of an element in \(D\) is unaltered upon multiplication by an element from \(H^x H^y\) so we may suppose that

\[D \subseteq H.\]

(10.2.1)

It follows that \(CD \subseteq HH^y\) and it is then clear that \(T_y := CD\) is a family of representatives for the south group. As above \(|T_y| = |C||D|\).

Also, observe that the equivalence class of an element in \(C\) is unaffected under multiplication by an element from \(H^y\) and hence we may assume that

\[C \subseteq H^z.\]

(10.2.2)

Using (7.2) for the description of \(\sigma_y\) we have

\[
\sigma_x \sigma_y (\delta_H) = \frac{1}{|S_x||T_y|} \sum_{b \in S_x} \sum_{k \in T_y} \delta_{Hx^by^k} = \frac{1}{|A||S_{x,y}||C||D|} \sum_{b \in S_{x,y}} \sum_{a \in A} \sum_{c \in C} \sum_{d \in D} \delta_{Hxbaycd}.
\]

For each \(b, a, c,\) and \(d\) as above notice that

\[xbaycd = (xbay)c(xbay)^{-1}xbayd = c'xbayd,
\]

where

\[c' = (xbay)c(xbay)^{-1} \in xbay H^x H^y a^{-1} b^{-1} x^{-1} = H.\]
Therefore $H_{xbayd} = H_{xbyd}$, so that

$$
\sigma_x \sigma_y (\delta_H) = \frac{1}{|A||S_{x,y}| |D|} \sum_{b \in S_{x,y}} \sum_{a \in A} \sum_{d \in D} \delta_{H_{xbayd}}.
$$

Next write

$$
H_{xbayd} = H_{xbyy^{-1}ayd} = H_{xbya'd},
$$
where $a' = y^{-1}ay$. Denoting by $A' = y^{-1}A_y$, we see that $A'$ is a family of representatives for the west group and thus, by (10.2.1), we have that

$$
T_{xy} := A'D
$$
is a subset of $HH^{xy}$ as well as a family of representatives for the center group. So

$$
\sigma_x \sigma_y (\delta_H) = \frac{1}{|A'||S_{x,y}| |D|} \sum_{b \in S_{x,y}} \sum_{a' \in A'} \sum_{d \in D} \delta_{H_{xbya'd}} = \frac{1}{|S_{x,y}| |T_{xy}|} \sum_{b \in S_{x,y}} \sum_{\ell \in T_{xy}} \delta_{H_{xby\ell}} = \frac{1}{|S_{x,y}|} \sum_{b \in S_{x,y}} \sigma_{xby}(\delta_H).
$$

Based on examples we have been able to determine that the above product formulas do not hold for general Hecke pairs. We therefore leave open the following:

10.3. Question. For which Hecke pairs do the product formulas of (10.2) hold? Do they hold when $H$ is protonormal?

Back to the subnormal realm we obtain the following universal property of Hecke algebras:

10.4. Theorem. Suppose that $(G, H)$ is a Hecke pair such that $H$ is subnormal in $G$ and let $\tau$ be any map from $G$ into a unital $F$-algebra $B$ such that $\tau_1 = 1$, and for every $x, y \in G$ and for every finite set $S_{x,y} \subseteq G$ such that

$$
HxHyH = \bigcup_{h \in S_{x,y}} HxHyH,
$$
one has that

$$
\tau_x \tau_y = \frac{1}{|S_{x,y}|} \sum_{h \in S_{x,y}} \tau_{xhy}.
$$

(10.4.1)

Then:

(i) There exists a unique unital homomorphism $\phi : \mathcal{H}(G, H) \to B$ such that $\phi(\sigma_x) = \tau_x$, for all $x \in G$.

(ii) If moreover $B$ is a *-algebra and $\tau_{x^{-1}} = \tau_x^*$, for all $x \in G$, then $\phi$ is a *-homomorphism.

Proof. Given $x$ in $G$ and $h$ in $H$ we have that

$$
HxHhH = HxH = Hxh^{-1}hH = Hx1hH,
$$
which means that the singletons $\{h^{-1}\}$ and $\{1\}$ are acceptable choices for $S_{x,h}$. Therefore we have by (10.4.1) that

$$
\tau_x \tau_h = \tau_{xh^{-1}h} = \tau_{x1h},
$$
which implies that $\tau_x = \tau_{xh}$. Beginning with $HhHxH = HxH$ one may similarly conclude that $\tau_x = \tau_{hx}$. It therefore follows that $\tau$ is a doubly invariant function on $G$. Employing (4.8) we therefore see that there exists a unique linear map $\phi : \mathcal{H}(G, H) \to B$ such that $\phi(\sigma_x) = \tau_x$, for all $x \in G$. 
So we need only prove that \( \phi \) is a homomorphism in order to establish (i). In order to do this it is obviously enough to prove that \( \phi(\sigma_x \sigma_y) = \phi(\sigma_x) \phi(\sigma_y) \), for all \( x \) and \( y \) in \( G \). Given \( S_{x,y} \) as in the statement we have
\[
\phi(\sigma_x \sigma_y) = \frac{1}{|S_{x,y}|} \sum_{h \in S_{x,y}} \phi(\tau_{xhy}) = \frac{1}{|S_{x,y}|} \sum_{h \in S_{x,y}} \tau_{xhy} = \tau_{x \tau_y} = \phi(\sigma_x) \phi(\sigma_y),
\]
proving our claim that \( \phi \) is a homomorphism.

If \( B \) is a *-algebra and \( a \in \mathcal{H}(G,H) \) is the finite sum \( a = \sum_{x \in G} \lambda_x \sigma_x \), then
\[
\phi(a^*) = \phi \left( \sum_{x \in G} \lambda_x^* \sigma_x^* \right) = \phi \left( \sum_{x \in G} \lambda_x^* \tau_{x^{-1}} \right) = \sum_{x \in G} \lambda_x^* \tau_{x^{-1}} = \sum_{x \in G} \lambda_x \tau_{x^{-1}} = \phi(a)^*.
\]

Putting together (10.2) and (10.4) we arrive at the following presentation of the Hecke algebra.

**10.5. Theorem.** Let \((G,H)\) be a Hecke pair such that \( H \) is subnormal in \( G \). Then the Hecke algebra \( \mathcal{H}(G,H) \) admits the following presentation in the category of unital \( F \)-algebras:

(a) **GENERATORS:** any set indexed by \( G \), say \( \{ \tau_x : x \in G \} \),

(b) **RELATIONS:**
   (i) \( \tau_1 = 1 \),
   (ii) \( \tau_x \tau_y = \frac{1}{|S_{x,y}|} \sum_{h \in S_{x,y}} \tau_{xhy} \), whenever \( S_{x,y} \) is a subset of \( H \) such that \( HxHyH = \bigcup_{h \in S_{x,y}} HxHyH \).

If we add
   (iii) \( \tau_{x^{-1}} = \tau_x^* \), for every \( x \) in \( G \),
we arrive at a presentation of \( \mathcal{H}(G,H) \) in the category of unital *-algebras over \( F \).

Let us now study some simple properties shared by maps \( \tau \) satisfying the above relations:

**10.6. Proposition.** Let \((G,H)\) be a Hecke pair with \( H \) subnormal in \( G \) and let \( B \) be a unital \( F \)-algebra. Given any map \( \tau : G \rightarrow B \) satisfying (10.5.b.i-ii) we have

(i) \( \tau \) is a partial representation,
(ii) if \( xH \subseteq Hx \) then \( \tau_x \tau_{x^{-1}} = 1 \), and \( \tau_x \tau_y = \tau_{xy} \), for all \( y \) in \( G \),
(iii) if \( Hx \subseteq xH \), then \( \tau_{-x} \tau_x = 1 \), and \( \tau_{-y} \tau_x = \tau_{yx} \), for all \( y \) in \( G \),
(iv) if \( x \) lies in the normalizer of \( H \) then \( \tau_x \) is invertible, \( (\tau_x)^{-1} = \tau_{x^{-1}} \), \( \tau_x \tau_y = \tau_{xy} \), and \( \tau_y \tau_x = \tau_{yx} \), for all \( y \) in \( G \),
(v) \( \tau_x = \tau_{hxk} \), for all \( h \) and \( k \) in \( H \).

**Proof.** In order to prove (i) let \( \phi : \mathcal{H}(G,H) \rightarrow B \) be the homomorphism given by (10.4.i). Then for every \( x \) and \( y \) in \( G \) we have
\[
\tau_{x^{-1}} \tau_x \tau_y = \phi(\sigma_{x^{-1}}) \phi(\sigma_x) \phi(\sigma_{y}) = \phi(\sigma_{x^{-1}} \sigma_x \sigma_{y}) = \phi(\sigma_{x^{-1}} \sigma_{xy}) = \tau_{x^{-1}} \tau_{xy},
\]
while a similar argument proves that \( \tau_{x} \tau_{y} \tau_{y^{-1}} = \tau_{xy} \tau_{y^{-1}} \).

Supposing that \( xH \subseteq Hx \), we have that \( HxHx^{-1}H = H = H1H \), so we may take \( S_{x,x^{-1}} = \{ 1 \} \) in (10.5.b.ii) to conclude that \( \tau_x \tau_{x^{-1}} = \tau_1 = 1 \). Moreover
\[
\tau_x \tau_y = \tau_x \tau_{x^{-1}} \tau_x \tau_y = \tau_x \tau_{x^{-1}} \tau_{xy} = \tau_{xy}.
\]
Clearly (iii) follows from (ii) by taking inverses, while (iv) follows from (ii) and (iii). As for (v) first notice that taking \( x \in H \) and \( y = 1 \) in (10.5.b.ii) we have that \( HxHyH = H = Hx^{-1}yH \), so, taking \( S_{x,y} = \{ x^{-1} \} \) we get
\[
\tau_x = \tau_x \tau_1 = \tau_{xx^{-1}} = 1,
\]
proving that \( \tau \) is constantly equal to 1 on \( H \). Since \( H \) obviously normalizes itself we have that (v) follows from (iv). \( \square \)
Throughout this section we fix a Hecke pair \((G, H)\) and a subgroup \(N\) of \(G\) such that \(H \subseteq N \subseteq G\), in which case \(H\) is necessarily subnormal in \(G\). Our goal will be to show that there exists a twisted partial action of \(N \setminus G\) on the group algebra \(F(\text{H} \setminus \text{N})\) such that the corresponding crossed-product is isomorphic to \(\text{H}(G, H)\).

By (10.6.iv) we have that the restriction of \(\sigma\) to \(N\) is a global (as opposed to partial) representation of \(N\) on \(\text{H}(G, H)\). Since \(\sigma\) vanishes on \(H\) we in fact get a group representation of \(H \setminus N\) on \(\text{H}(G, H)\) which maps each right (= left) coset \(Hn\) in \(H \setminus N\) to \(\sigma_n\).

11.1. Proposition. The homomorphism \(\iota : F(\text{H} \setminus \text{N}) \to \text{H}(G, H)\) obtained by linearizing the above representation of \(H \setminus N\) is injective.

Proof. Given \(n \in N\) it is evident that \(\{1\}\) is a family of representatives for \((H \cap H^n) \setminus H\), so \(\sigma_n(\delta_H) = \delta_{Hn}\). We thus see that for a general element

\[ a = \sum_{i=1}^{n} \lambda_i \delta_{Hn_i} \in F(\text{H} \setminus \text{N}) \]

(by abuse of language we denote by \(\delta_{Hn}\) the canonic basis elements of \(F(\text{H} \setminus \text{N})\) as well) one has that

\[ \iota(a) = \sum_{i=1}^{n} \lambda_i (\delta_{Hn_i}) = \sum_{i=1}^{n} \lambda_i \sigma_{n_i}, \]

whence

\[ \iota(a)|_{\delta_H} = \sum_{i=1}^{n} \lambda_i \sigma_{n_i}(\delta_H) = \sum_{i=1}^{n} \lambda_i \delta_{Hn_i}, \]

from which the statement follows.

Using \(\iota\) we will identify, from now on, \(F(\text{H} \setminus \text{N})\) with a sub-algebra of \(\text{H}(G, H)\), namely the linear span of the set \(\{\sigma_n : n \in N\}\).

11.2. Lemma. Let \(x, y \in G\) be such that \(xy \in N\). Then \(\sigma_x \sigma_y \in F(\text{H} \setminus \text{N})\).

Proof. By (10.2) it is enough to show that \(xhy \in N\), for every \(h \in H\). Let \(n = xy\), so that \(y = x^{-1}n\). Thus, given \(h \in H\), we have

\[ xhy = xhx^{-1}n \in xHx^{-1}n \subseteq xNx^{-1}n = Nn = N. \]

11.3. Lemma. For every \(x \in G\) one has that \(e_x := \sigma_x \sigma_{x^{-1}}\) is a central idempotent in \(F(\text{H} \setminus \text{N})\). Moreover if \(Nx = Ny\) then \(e_x = e_y\).

Proof. From (8.1) it follows that \(e_x\) is an idempotent and from (11.2), that \(e_x \in F(\text{H} \setminus \text{N})\).

Let \(S_x\) be a family of representatives for \((H \cap H^x) \setminus H\). Plugging \(y = x^{-1}\) in (10.1) we have that \(HxH^{-1} = \bigcup_{h \in S_x} Hxhx^{-1}H\) so, by (10.2),

\[ e_x = \sigma_x \sigma_{x^{-1}} = \frac{1}{|S_x|} \sum_{h \in S_x} \sigma_xhx^{-1}. \]

In order to prove that \(e_x\) is central it is enough to show that \(e_x\) commutes with \(\sigma_n\), for all \(n \in N\). For this observe that \(m := x^{-1}nx \in N\), so \(Ad_m\) is an inner automorphism of \(N\), which therefore leaves invariant the

\[ 4\text{ We seem to be irremediably biased towards right coset spaces so we will keep using the notation for right cosets even when they coincide with left cosets.} \]
normal subgroups $H$ and $H^x$. We conclude that $Ad_m(S_x)$ is another family of representatives for $(H \cap H^x) \setminus H$
so we can alternatively compute $e_x$ as
\[
    e_x = \sigma_x \sigma_{x^{-1}} = \frac{1}{|S_x|} \sum_{h \in S_x} \sigma_{xhx^{-1}x^{-1}} = \frac{1}{|S_x|} \sum_{h \in S_x} \sigma_{n\sigma_{hx^{-1}x^{-1}} = \sigma_n e_x \sigma_{x^{-1}} = \sigma_n e_x (\sigma_n)^{-1},}
\]
proving that $e_x$ commutes with $\sigma_n$. If $Nx = Ny$ we may write $y = nx$, with $n \in N$, so
\[
    e_y = e_n x = \sigma_{n\sigma_{x^{-1}n^{-1}}} = \sigma_n \sigma_x \sigma_{x^{-1}n^{-1}} = \sigma_n e_x \sigma_{n^{-1}} = e_x. \quad \square
\]

11.4. Definition. For each $x \in G$ we will let:
(i) $D^x$ be the ideal of $F(H \setminus N)$ generated by $e_x$, that is $D^x = e_x F(H \setminus N)$,
(ii) $\psi_x$ be the linear operator on $H(G, H)$ given by
\[
    \psi_x : a \in H(G, H) \mapsto \sigma_x a \sigma_{x^{-1}} \in H(G, H).
\]
By the last part of (11.3) it is clear that $D^x$ only depends on the class of $x$ in $N \setminus G$. If $t \in N \setminus G$ we will
therefore denote by
\[
    D_t := D^x, \tag{11.4.1}
\]
where $x$ is any element of $G$ such that $Nx = t$, so $D_t$ is independent of the choice of $x$.

11.5. Proposition. For every $x \in G$ one has that $\psi_x(F(H \setminus N)) = D^x$. Moreover the restriction of $\psi_x$ to $D^{x^{-1}}$ is an isomorphism onto $D^x$.

Proof. In order to verify that $\psi_x(F(H \setminus N)) \subseteq D^x$ it is enough to show that $a := \sigma_x \sigma_{n\sigma_{x^{-1}}} \in D^x$, for all $n \in N$. Notice that
\[
    a = \sigma_x \sigma_{n\sigma_{x^{-1}}} \overset{(10.6.4v)}{=} \sigma_{x\sigma_n \sigma_{x^{-1}}} \overset{(11.2)}{=} F(H \setminus N).
\]
Since $a = e_x a$, by (8.1), we conclude that $a \in D^x$.

Observe that for $a \in D^{x^{-1}}$ we have
\[
    \psi_{x^{-1}}(\psi_x(a)) = \sigma_{x^{-1}} \sigma_x a \sigma_{x^{-1}} \sigma_x = e_{x^{-1}} a e_{x^{-1}} = a
\]
from which it follows that $\psi_x$ is a bijection from $D^{x^{-1}}$ to $D^x$. From this we also obtain that $\psi_x(F(H \setminus N)) = D^x$. Finally, in order to show that the restriction of $\psi_x$ to $D^{x^{-1}}$ is multiplicative, let $a, b \in D^{x^{-1}}$. Then
\[
    \psi_x(ab) = \sigma_x a \sigma_{x^{-1}} = \sigma_x a \sigma_{x^{-1}} b \sigma_{x^{-1}} = \sigma_x a \sigma_{x^{-1}} b \sigma_{x^{-1}} = \psi_x(a) \psi_x(b). \quad \square
\]
Fix, once and for all, a section $\xi$ for the quotient map $\pi : G \to N \setminus G$, that is, $\xi$ is a map (not necessarily a homomorphism) from $N \setminus G$ to $G$ such that $\pi \circ \xi$ is the identity map on $N \setminus G$. For the special case of the coset $N1$ we will force the choice
\[
    \xi(N1) = 1.
\]
Given $r, s \in N \setminus G$, observe that
\[
    \pi(\xi(r)\xi(s)\xi(rs)^{-1}) = rs(rs)^{-1} = 1,
\]
so the element $\xi(r)\xi(s)\xi(rs)^{-1}$ lies in $N$. 
11.6. Definition. For every \( r \) and \( s \) in \( N\setminus G \) we let
\[
wr,s = \sigma_{\xi(r)}(r)s(\xi(rs))^{-1}.
\]

Clearly \( wr,s \) is an invertible element in \( F(H\setminus N) \) by (10.6.iv).

11.7. Lemma. Given \( r \) and \( s \) in \( N\setminus G \), let \( x = \xi(r) \), \( y = \xi(s) \), and \( z = \xi(rs) \). Then
\[
(i) \ \sigma_x\sigma_y\sigma_y^{-1} = wr,s\sigma_z\sigma_y^{-1},
(ii) \ e_{y^{-1}}\sigma_{y^{-1}}\sigma_{x^{-1}} = e_{y^{-1}}\sigma_{y^{-1}}(wr,s)^{-1}.
\]
Proof. Letting \( n = \xi(r)\xi(s)\xi(rs)^{-1} = xzy^{-1} \) we have that \( xzy = nz \) and \( wr,s = \sigma_n \). So
\[
\sigma_x\sigma_y\sigma_y^{-1} = \sigma_x\sigma_y\sigma_y^{-1}\sigma_y = \sigma_{xy}\sigma_{y^{-1}}\sigma_y = \sigma_{ny}\sigma_{y^{-1}} \quad (10.6.iv) \Rightarrow \sigma_n\sigma_z\sigma_y^{-1} = wr,s\sigma_z\sigma_y^{-1}.
\]

As for (ii) we have
\[
e_{y^{-1}}\sigma_{y^{-1}}\sigma_{x^{-1}} = e_{y^{-1}}\sigma_{y^{-1}}\sigma_{y^{-1}}\sigma_{x^{-1}} = e_{y^{-1}}\sigma_{y^{-1}}\sigma_{y^{-1}}(10.6.iv) \Rightarrow e_{y^{-1}}\sigma_{z^{-1}}(\sigma_n)^{-1} = e_{y^{-1}}\sigma_{z^{-1}}(wr,s)^{-1}. \quad \Box
\]

11.8. Theorem. For each \( t \in N\setminus G \), let \( D_t \) be as in (11.4.1) and let \( \theta_t \) be the isomorphism from \( D_{t^{-1}} \) to \( D_t \) given by restricting \( \psi(t) \) to \( D_{t^{-1}} \) as in (11.5). Then the triple
\[
\{(D_t)_{t \in N\setminus G}, \ \{\theta_t\}_{t \in N\setminus G}, \ \{wr,s\}_{t,s \in N\setminus G}\}
\]
is a twisted partial action of \( N\setminus G \) on \( F(H\setminus N) \).
Proof. During the course of this prove we will let \( A := F(H\setminus N) \).

Since \( \sigma_1 = 1 \), it is evident that \( D_1 = A \) and \( \theta_1 \) is the identity map on \( A \). In order to verify \((2.3.ii)\) let \( r, s \in N\setminus G \), and put \( x = \xi(r) \) and \( y = \xi(s) \). So
\[
\theta_r(D_{r^{-1}} \cap D_s) = \sigma_x(e_{x^{-1}}A \cap e_yA)\sigma_{x^{-1}} = \sigma_x(e_{x^{-1}}e_yA)\sigma_{x^{-1}} = \sigma_x\sigma_{x^{-1}}\sigma_x\sigma_y\sigma_{y^{-1}}A\sigma_{x^{-1}} = \sigma_{xy}\sigma_{y^{-1}}A\sigma_{x^{-1}} = \sigma_{xy}\sigma_{y^{-1}}\sigma_{x^{-1}}\sigma_x\sigma_{y^{-1}}A\sigma_{x^{-1}} = \sigma_{xy}\sigma_{y^{-1}}\sigma_{x^{-1}}\sigma_x\sigma_{y^{-1}}A\sigma_{x^{-1}} = e_{xy}\sigma_xA\sigma_{x^{-1}} \quad (11.5) \Rightarrow e_{xy}D^y = D^{xy} \cap D^x = Drs \cap Dr.
\]

As for \((2.3.iii)\) let \( x = \xi(r) \) and \( y = \xi(s) \) as above and put \( z = \xi(rs) \). Take \( a \in D_{s^{-1}} \cap D_{s^{-1}}r^{-1} \), which we may clearly suppose has the form
\[
a = e_{y^{-1}}e_{(xy)}\sigma_{n^{-1}}.
\]
where \( n \in N \). Then
\[
\theta_r(\theta_s(a)) = \sigma_x\sigma_y(a) = \sigma_{xy}\sigma_{y^{-1}}\sigma_{x^{-1}} = \sigma_{x\sigma_y}(y^{-1}e_{(xy)}\sigma_{n}^{-1} \quad (11.7.i) \Rightarrow \sigma_{wr,s}(e_{y^{-1}}e_{(xy)}\sigma_{n}^{-1} \quad (11.7.ii) \Rightarrow \sigma_{wr,s}(e_{xy}^{-1}\sigma_{z^{-1}}(w_{r,s})^{-1} = \sigma_{w_{r,s}}(a) \quad (w_{r,s})^{-1}.
\]
The forced choice of $\xi(N1) = 1$ clearly gives (2.3.iv) so it remains to check (2.3.v). So let $r, s, t \in N \setminus G$ and $a \in D_{r-1} \cap D_s \cap D_{st}$. Put $x = \xi(r)$, $y = \xi(s)$, $z = \xi(t)$, $\alpha = \xi(rs)$, $\beta = \xi(st)$, and $\gamma = \xi(rst)$. We then have

$$a = e_{x^{-1}}a = e_ya = e_\beta a,$$

while

$$w_{r,s} = \sigma_{xya^{-1}}, \quad w_{s,t} = \sigma_{y\beta^{-1}}, \quad w_{r,st} = \sigma_{x\beta^{-1}}, \quad w_{y,ts} = \sigma_{\alpha \beta^{-1}}.$$

Therefore we have

$$\theta_r(aw_{s,t})w_{r,ts} = \sigma_xa\sigma_{y\beta^{-1}}\sigma_{x^{-1}}\sigma_{\alpha \beta^{-1}}(\xi) = \sigma_xa\sigma_{(y\beta^{-1})x^{-1}(\alpha \beta^{-1})} =$$

$$= \sigma_xa\sigma_{\alpha \beta^{-1}} = \sigma_xa\sigma_{(xya^{-1})(\alpha \beta^{-1})}(\xi) = \sigma_xa\sigma_{xya^{-1}}\sigma_{\alpha \beta^{-1}} = \theta_r(a)w_{r,s}w_{r,ts}.$$

Observe that the passages marked “(!)” are justified by (10.6.iv) and the fact that the elements $yz\beta^{-1}$, $x\beta^{-1}$, $xya^{-1}$, and $\alpha \beta^{-1}$ lie in $N$.

11.9. Theorem. The crossed product

$$F(H\setminus N) \rtimes N \setminus G$$

relative to the above twisted partial action is isomorphic to the Hecke algebra $\mathcal{H}(G, H)$.

Proof. Let

$$\Phi : F(H\setminus N) \rtimes N \setminus G \to \mathcal{H}(G, H)$$

be the unique linear map such that

$$\Phi(\alpha \delta_t) = a\sigma_{\xi(t)}, \quad \forall t \in N \setminus G, \quad \forall a \in D_t.$$ 

In order to show that $\Phi$ is multiplicative let $r, s \in N \setminus G$ and take $a \in D_r$ and $b \in D_s$. Putting $x = \xi(r)$ and $y = \xi(s)$ we have

$$\Phi(\alpha \delta_r)\Phi(\beta \delta_s) = a\sigma_xb\sigma_y = e_xa\sigma_xb\sigma_y = \sigma_x\sigma_a^{-1}a\sigma_xb\sigma_y = \sigma_x\theta_r^{-1}(a)b\sigma_y =$$

$$= \sigma_x\theta_r^{-1}(a)be_{x^{-1}}\sigma_y = \sigma_x\theta_r^{-1}(a)b\sigma_{x^{-1}}\sigma_x\sigma_y = \theta_r(\theta_r^{-1}(a)b)\sigma_x\sigma_y =$$

$$= \theta_r(\theta_r^{-1}(a)b)\sigma_x\sigma_ype_{y^{-1}} = \ldots$$

Putting $z = \xi(rs)$ and applying (11.7.i) we find that the above equals

$$\ldots = \theta_r(\theta_r^{-1}(a)b)w_{r,s}\sigma_z e_{y^{-1}} = \theta_r(\theta_r^{-1}(a)b)w_{r,se_{y^{-1}}\sigma_z} = \ldots$$

Notice that $\pi(zy^{-1}) = \pi(z)\pi(y)^{-1} = (rs)s^{-1} = r = \pi(x)$ which implies that $N zy^{-1} = N x$. Hence $e_{zy^{-1}} = e_x$ by (11.3) and the above equals

$$\ldots = \theta_r(\theta_r^{-1}(a)b)w_{r,s}e_x\sigma_z = \theta_r(\theta_r^{-1}(a)b)e_xw_{r,s}\sigma_z = \theta_r(\theta_r^{-1}(a)b)w_{r,s}\sigma_z.$$

On the other hand, since $(a \delta_r)(b \delta_s) = \theta_r(\theta_r^{-1}(a)b)w_{r,s} \delta_{rs}$, we have that

$$\Phi((a \delta_r)(b \delta_s)) = \theta_r(\theta_r^{-1}(a)b)w_{r,s} \delta_{rs},$$

proving that $\Phi$ is a homomorphism. In order to prove that $\Phi$ is bijective we will now provide an inverse for it based on the universal property (10.4) of the Hecke algebra.

Consider the map

$$\tau : G \to F(H\setminus N) \rtimes N \setminus G$$
given by
\[ \tau(x) = e_x \sigma_x \xi(x) \delta_{\tau(x)}. \]
In order to simplify the above expression we will often write it as
\[ \tau(x) = e_x \sigma_x \delta_r, \]
where \( r = \pi(x), \tilde{x} = \xi(r), \) and \( n = x \tilde{x}^{-1}. \) Observe that \( n \) is necessarily in \( N. \)

CLAIM: Given \( x \) and \( y \) in \( G \) let
\[ r = \pi(x), \quad \tilde{x} = \xi(r), \quad n = x \tilde{x}^{-1}, \]
\[ s = \pi(y), \quad \tilde{y} = \xi(s), \quad m = y \tilde{y}^{-1}, \quad \tilde{z} = \xi(rs). \]

Then
\[ \tau_x \tau_y = \sigma_x \sigma_y \sigma_{\tilde{z}}^{-1} \delta_{rs}. \]

In fact we have
\begin{align*}
\tau_x \tau_y & = (e_x \sigma_x \delta_r) \big( e_y \sigma_m \delta_s \big) = \theta_r \big( \theta_s^{-1} (e_x \sigma_n \delta_r \big) w_{rs} \delta_{rs} = \\
& = e_x \sigma_{\tilde{z}} \big( \sigma_{\tilde{z}}^{-1} (e_x \sigma_n \sigma_{\tilde{z}} e_y \sigma_m \sigma_{\tilde{z}}^{-1} w_{rs} \delta_{rs} = e_x e_x \sigma_{\tilde{z}} e_y \sigma_m \sigma_{\tilde{z}}^{-1} \sigma_{\tilde{z}}^{-1} \delta_{rs} = \\
& = e_x \sigma_n \sigma_{\tilde{z}} e_y \sigma_m \sigma_{\tilde{z}}^{-1} \delta_{rs} = e_x \sigma_n \sigma_{\tilde{z}} e_y \sigma_{\tilde{z}}^{-1} \delta_{rs} = e_x \sigma_y \sigma_{\tilde{z}}^{-1} \delta_{rs} = e_x \sigma_y \sigma_{\tilde{z}}^{-1} \delta_{rs} = e_x \sigma_y \sigma_{\tilde{z}}^{-1} \delta_{rs},
\end{align*}

proving our claim. Next let us show that \( \tau \) satisfies (10.4.1). For this let \( S_{x,y} \) be a family of representatives for \( (H \cap H^x H^y) \setminus H. \) Using our claim and (10.2) we conclude that
\[ \tau_x \tau_y = \frac{1}{|S_{x,y}|} \sum_{h \in S_{x,y}} \sigma_{xh} y \sigma_{\tilde{z}}^{-1} \delta_{rs}. \]

On the other hand, in order to compute the right-hand-side of (10.4.1), namely the sum
\[ \frac{1}{|S_{x,y}|} \sum_{h \in S_{x,y}} \tau_{xh}, \]
we observe that \( \pi(xh) = \pi(x) \pi(y) = rs, \) so that \( \xi(\pi(xh)) = \xi(rs) = \tilde{z}. \) This implies that
\[ \tau_{xh} = e_{xh} \sigma_{xh} \sigma_{\tilde{z}}^{-1} \delta_{rs} = \sigma_{xh} \sigma_{\tilde{z}}^{-1} \delta_{rs} = \sigma_{xh} \sigma_{\tilde{z}}^{-1} \delta_{rs}. \]

This shows that (10.4.1) holds and hence by the universal property of \( \mathcal{H}(G, H) \) we conclude that there exists a homomorphism
\[ \Psi : \mathcal{H}(G, H) \to F \left( H \setminus N \right) \times N \setminus G \]
such that \( \Psi(\sigma_x) = \tau_x, \) for all \( x \) in \( G. \) We claim that \( \Psi \) is the inverse of \( \Phi. \) In fact, using (11.9.1), we have
\[ \Phi(\Psi(\sigma_x)) = \Phi(\tau_x) = \Phi(e_x \sigma_x \sigma_x \xi(r)) = e_x \sigma_x \sigma_x \xi(r) = e_x \sigma_x \sigma_x \tilde{z} = e_x \sigma_n \tilde{z} = e_x \sigma_n \tilde{z} = e_x \sigma_x = \sigma_x. \]

This shows that \( \Phi \circ \Psi \) is the identity on \( \mathcal{H}(G, H). \) To show that \( \Psi \circ \Phi \) is also the identity on \( F \left( H \setminus N \right) \times N \setminus G \) it is clearly enough to check that \( \Psi(\Phi(\sigma_x)) = \sigma_x, \) for every \( a \) in \( F \left( H \setminus N \right) \times N \setminus G \) of the form \( a = e_x \sigma_p \sigma_r, \) where \( p \) is in \( N, \) and we are again using (11.9.1). We have
\[ \Phi(e_x \sigma_p \sigma_r) = e_x \sigma_p \sigma_x \tilde{z} = e_x \sigma_p \sigma_x \tilde{z} = e_x \sigma_p \sigma_x \tilde{z} =
As discussed in the introduction, many authors have considered the problem of describing Hecke algebras as crossed products by semigroups, assuming that $G$ has a semi-direct product structure. It is easy to see that $G$ can be written as a semi-direct product $G = N \rtimes K$, where $K$ is another group (necessarily $K = N \setminus G$), if and only if there exists a section $\xi : N \setminus G \to G$ for the quotient map which is a group homomorphism. In this case notice that the cocycle $w$ defined in (11.6) becomes trivial. We therefore have:

11.10. Corollary. Let $G = N \rtimes K$ be a semidirect product of groups and let $H$ be a normal subgroup of $N$ such that $(G, H)$ is a Hecke pair\(^5\). Then there is an (untwisted) partial action of $K$ on $F(H \setminus N)$ such that

$$\mathcal{H}(G, H) \simeq F(H \setminus N) \rtimes K.$$

12. Hecke C*-algebras.

In this section we take $F$ to be the field of complex numbers and consider the existence of a maximum C*-norm on $\mathcal{H}(G, H)$. See the introduction for references to similar results in the literature. The completion of $\mathcal{H}(G, H)$ relative to this norm, when it exists, is sometimes called the Hecke C*-algebra of the pair $(G, H)$ and its *-representation theory is equivalent to the *-representation theory of $\mathcal{H}(G, H)$. Observe that by (5.6) it does not matter which involution we take on $\mathcal{H}(G, H)$.

12.1. Proposition. Let $F = \mathbb{C}$ and let $(G, H)$ be a Hecke pair with $H$ protonormal in $G$. Then there exists a maximum C*-norm on $\mathcal{H}(G, H)$.

Proof. Given $a \in \mathcal{H}(G, H)$ let $\|a\|$ be defined as the supremum of $\|\pi(a)\|$, where $\pi$ ranges in the set of all *-representations of $\mathcal{H}(G, H)$. To see that $\|a\|$ is finite write $a$ as a finite sum $a = \sum_{x \in G} a_x \sigma_x$. Observe that if $\pi$ is any *-representation of $\mathcal{H}(G, H)$ then, given that

$$\sigma_x \sigma_x^* \sigma_x = \sigma_x \sigma_x^{-1} \sigma_x = \sigma_x \sigma_x^{-1} = \sigma_x \sigma_1 = \sigma_x,$$

we see that $\pi(\sigma_x)$ is a partial isometry and hence $\|\pi(\sigma_x)\| \leq 1$. It follows that

$$\|\pi(a)\| \leq \sum_{x \in G} |a_x| \|\pi(\sigma_x)\| \leq \sum_{x \in G} |a_x|.$$

This proves that $\|a\| \leq \sum_{x \in G} |a_x|$ and hence $\|a\|$ is finite as claimed. It is now easy to see that $\|\cdot\|$ defines a C*-norm which dominates all others. \hfill \Box

The completion of $\mathcal{H}(G, H)$ relative to this norm is a C*-algebra sometimes denoted by $C^*_u(G, H)$ and called the full Hecke C*-algebra of the pair $(G, H)$. It is elementary to see that the *-representation theory of this algebra coincides with that of $\mathcal{H}(G, H)$.

On the other hand, as some authors have already done, one could consider the reduced Hecke C*-algebra $C^*_r(G, H)$, namely the completion of $\mathcal{H}(G, H)$, normed as operators on the inner-product space defined in (5.2). The question as to whether $C^*_u(G, H)$ coincides with $C^*_r(G, H)$ is then at least as rich as the corresponding question for group C*-algebras.

\(^5\) See [LL1: Proposition 1.7] for sufficient conditions for $(G, H)$ to be a Hecke pair.
13. A possible generalization of Hecke algebras.

In this short section we wish to propose a generalization for the definition of Hecke algebras for a group-subgroup pair \((G,H)\) which is not a Hecke pair, namely, such that not all double cosets are finite union of right cosets.

Initially observe that the relations (10.5.b) make sense as long as every “triple coset” \(HxHyH\) is a finite union of double cosets. One could then be tempted to say that the pair \((G,H)\) is a pseudo Hecke pair if for every \(x\) and \(y\) in \(G\) this finiteness condition holds.

However observe that at least in the case of a subnormal \(H \subseteq G\), we have by (10.1) that \(HxHyH\) is a finite union of double cosets if and only if \((H \cap H^xH^y^{-1})\backslash H\) is finite. If this is so for every \(x\) and \(y\) then, plugging \(y = x^{-1}\) we conclude that \((H \cap H^x)\backslash H\) is finite and hence \(HxH\) is a finite union of right cosets by (3.2). In other words every pseudo Hecke pair is a true Hecke pair.

However there is a lesson to be learned from \([EL1]\) which could perhaps yield a true generalization. That lesson is that, when a collection of relations involves summations, some of which refuse to converge, it is sensible to simply ignore the divergent ones. A well known instance of this phenomenon takes place when one considers Cuntz algebras. The relation “\(\sum_{i=1} S_i S_i^* = 1\)” in the usual presentation of \(O_n\) is simply ignored in the definition of \(O_\infty\).

One could then risk the following:

13.1. Definition. Let \(H\) be a subnormal subgroup of a group \(G\). The generalized Hecke algebra \(\widetilde{H}(G,H)\) is the universal \(F\)-algebra generated by a collection of elements \(\{\sigma_x : x \in G\}\) subject to the relations declaring that \(\sigma\) is a partial representation in addition to the following: whenever \(HxHyH\) happens to be a finite union of double cosets (and only in this case) we require that (10.5.b.ii) holds as well.

While we have nothing of interest to say at the moment about the algebra so defined, it is not hard to give an example of a group-subgroup pair \((G,H)\) which is not a Hecke pair although there are many pairs of elements \(x\) and \(y\) for which \(HxHyH\) is a finite union of double cosets. Consider for example

\[
G = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & \mathbb{R}_+ \end{pmatrix} = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \in GL_2(\mathbb{R}) : a,b \in \mathbb{R}, a > 0 \right\},
\]

with \(H = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}\). If \(x = \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \in G\), it is easy to see that \(H^x = \begin{pmatrix} 1 & a\mathbb{Z} \\ 0 & 1 \end{pmatrix}\) hence, if \(y = \begin{pmatrix} 1 & d \\ 0 & c \end{pmatrix}\) we have

\[
H^x H^y^{-1} = \begin{pmatrix} 1 & a\mathbb{Z} + d^{-1}\mathbb{Z} \\ 0 & 1 \end{pmatrix}.
\]

Quite often one would have that \(H \cap H^x H^y^{-1} = \{0\}\) in which case there are infinitely many double cosets in \(HxHyH\). However if the rational vector space generated by \(a\) and \(d^{-1}\) contains a nonzero rational number then there will be an integral solution \((n,m,p)\) to the equation

\[
an + d^{-1}m = p,
\]

with nonzero \(p\), in which case \(\begin{pmatrix} 1 & p\mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq H \cap H^x H^y^{-1}\) so that \(HxHyH\) will contain no more than \(p\) double cosets and then relation (10.5.b.ii) would apply.

In Definition (13.1) we have restricted ourselves to the situation in which \(H\) is subnormal in \(G\) so that, when \((G,H)\) is a Hecke pair, one recovers the usual Hecke algebra \(H(G,H)\) by (10.5). However there does not seem to be any immediate technical difficulty in adopting Definition (13.1) for a general (non-subnormal) group-subgroup pair \((G,H)\) although this would most definitely depart from the usual theory of Hecke algebras.
14. An example.

In this section we shall give an example of a Hecke pair \((G,H)\), such that \(H\) is protonormal in \(G\) but not subnormal.

Let \(\mathcal{P} \subseteq \mathbb{N}\) be a set of prime numbers and let \(\mathbb{A}_\mathcal{P}\) be the subset of all rational numbers \(n/m\), with \(n, m \in \mathbb{N}, m \neq 0\), such that no prime in \(\mathcal{P}\) divides \(m\). It is clear that \(\mathbb{A}_\mathcal{P}\) is a subring of \(\mathbb{Q}\). We will denote by \(\mathbb{A}_\mathcal{P}^*\) the set of invertible elements in \(\mathbb{A}_\mathcal{P}\), so that a rational number \(\xi\) lies in \(\mathbb{A}_\mathcal{P}^*\) if and only if \(\xi = n/m\) and no prime in \(\mathcal{P}\) divide either \(n\) or \(m\).

Denote by \(G\) the group
\[
G = \begin{pmatrix} 1 & \mathbb{Q}^* \\ 0 & \mathbb{Q}^* \end{pmatrix},
\]
meaning the set of all matrices \(\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \in GL_2(\mathbb{Q})\), such that \(a \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}\), and \(b \in \mathbb{Q}\), and let \(H_\mathcal{P}\) be the subgroup
\[
H_\mathcal{P} = \begin{pmatrix} 1 & \mathbb{A}_\mathcal{P} \\ 0 & \mathbb{A}_\mathcal{P}^* \end{pmatrix}.
\]

14.1. Proposition. For any set \(\mathcal{P}\) of primes one has that \((G,H_\mathcal{P})\) is a Hecke pair.

Proof. Let \(x = \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \in G\). We shall prove that \((H_\mathcal{P} \cap H_\mathcal{P}^{-1})\setminus H_\mathcal{P}\) is finite. As a first step let us try to identify certain elements in \(H_\mathcal{P} \cap H_\mathcal{P}^{-1}\). Given \(h = \begin{pmatrix} 1 & \eta \\ 0 & \xi \end{pmatrix} \in H_\mathcal{P}\), notice that \(h \in H_\mathcal{P}^{-1}\), if and only if \(x^{-1}hx \in H_\mathcal{P}\). We have
\[
x^{-1}hx = \begin{pmatrix} 1 & -ba^{-1} \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & \eta a + (1 - \xi)b \\ 0 & \xi \end{pmatrix}.
\]
Therefore \(x^{-1}hx \in H_\mathcal{P}\) if and only if
\[
\eta a + (1 - \xi)b \in \mathbb{A}_\mathcal{P}.
\]

Since neither \(a\) or \(b\) have been assumed to lie in \(\mathbb{A}_\mathcal{P}\) their denominator may contain factors in \(\mathcal{P}\). Factoring these out we may write
\[
a = \frac{a_0}{p}, \quad \text{and} \quad b = \frac{b_0}{q},
\]
where \(a_0, b_0 \in \mathbb{A}_\mathcal{P}\), and \(p\) and \(q\) are products of primes in \(\mathcal{P}\). Writing \(a = qa_0/pq\) and \(b = pb_0/pq\), we may assume without loss of generality that \(p = q\).

Let \(\mathbb{Z}_q\) denote the ring \(\mathbb{Z}/q\mathbb{Z}\). Given \(\zeta \in \mathbb{A}_\mathcal{P}\) write it in reduced form \(\zeta = n/m\), so that no prime in \(\mathcal{P}\) divides \(m\) and hence \(\gcd(m,q) = 1\) (greatest common divisor). Therefore \(m\) is invertible modulo \(q\) and hence it makes sense to set
\[
\phi(\zeta) = nm^{-1}(\text{mod } q).
\]
This therefore gives a well defined map \(\phi : \mathbb{A}_\mathcal{P} \to \mathbb{Z}_q\), which can be easily proven to be a homomorphism of rings. Let \(G_q\) be the subgroup of \(GL_2(\mathbb{Z}_q)\) defined by
\[
G_q = \begin{pmatrix} 1 & \mathbb{Z}_q \\ 0 & \mathbb{Z}_q \end{pmatrix},
\]
and set
\[
\overline{\phi} : \begin{pmatrix} 1 & \eta \\ 0 & \xi \end{pmatrix} \in H_\mathcal{P} \mapsto \begin{pmatrix} 1 & \phi(\eta) \\ 0 & \phi(\xi) \end{pmatrix} \in G_q.
\]
Since \(G_q\) is a finite group we have that \(\ker(\overline{\phi})\) is a normal subgroup of \(H_\mathcal{P}\) of finite index.
Recall that a while ago we concluded that the element $h = \begin{pmatrix} 1 & \eta \\ 0 & \xi \end{pmatrix}$ (introduced near the beginning of this proof) lies in $\mathcal{H}_P \cap H^*_P$ if and only if (14.1.2) holds. We claim that this is the case for all elements $h \in \text{Ker}(\tilde{\phi})$. In fact, if $h \in \text{Ker}(\tilde{\phi})$, we have that $\phi(\eta) = 0$, and $\phi(\xi) = 1$. Therefore there are $\eta_0, \xi_0 \in \mathcal{A}_P$, such that $\eta = q\eta_0$, and $\xi = 1 + q\xi_0$. Plugging this in (14.1.2) we conclude that

$$\eta a + (1 - \xi)b = q\eta_0a - q\xi_0b = q\eta_0\frac{a_0}{q} - q\xi_0\frac{b_0}{q} = \eta_0a_0 - \xi_0b_0 \in \mathcal{A}_P.$$ 

This proves that $\text{Ker}(\tilde{\phi}) \subseteq \mathcal{H}_P \cap H^*_P$, and hence the index of the latter group in $\mathcal{H}_P$ is finite. $\square$

Observe that if $\mathcal{P}$ is the empty set then $\mathcal{A}_P = \mathbb{Q}$, and hence $H_P = G$. In all other cases we have:

**Proposition.** If $\mathcal{P}$ is a nonempty set of primes then $H_P$ is not subnormal in $G$.

**Proof.** We will show that there exists $h, k \in \mathcal{H}_P$, and $x \in G$ such that

$$(x^{-1}hx)^{-1}k(x^{-1}hx) \notin \mathcal{H}_P,$$

thus violating (9.1). Let $a \in \mathbb{Q}$ and put

$$x = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Then

$$x^{-1}hx = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$ 

So

$$(x^{-1}hx)^{-1}k(x^{-1}hx) = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & -1 \end{pmatrix}.$$ 

Choosing $a = 1/2p$, where $p$ is any prime in $\mathcal{P}$, we conclude that this is not in $\mathcal{H}_P$. $\square$

Among these Hecke pairs we can identify at least one for which $H_P$ is protonormal.

**Theorem.** If $\mathcal{P} = \{2\}$, that is, $\mathcal{P}$ consists of the single prime 2, then $H_P$ is protonormal in $G$.

**Proof.** Given $x = \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}$ in $G$ we need to prove that $H^*_P$ commutes with $H_P$. For this let $h = \begin{pmatrix} 1 & \eta \\ 0 & \xi \end{pmatrix}$ and $k = \begin{pmatrix} 1 & \nu \\ 0 & \mu \end{pmatrix}$ be in $H_P$ and notice that by (14.1.1) we have that

$$x^{-1}hxk = \begin{pmatrix} 1 & \eta a + (1 - \xi)b \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 1 & \nu + \eta a \mu + (1 - \xi)b \mu \\ 0 & \xi \mu \end{pmatrix}.$$ 

We want to write this as $k'x^{-1}h'x$, where $h' = \begin{pmatrix} 1 & \eta' \\ 0 & \xi' \end{pmatrix}$ and $k' = \begin{pmatrix} 1 & \nu' \\ 0 & \mu' \end{pmatrix}$ are in $H_P$. We have

$$k'x^{-1}h'x = \begin{pmatrix} 1 & \nu' \\ 0 & \mu' \end{pmatrix} \begin{pmatrix} 1 & \eta' a + (1 - \xi')b \\ 0 & \xi' \end{pmatrix} = \begin{pmatrix} 1 & \eta' a + (1 - \xi')b + \nu' \xi' \\ 0 & \mu' \xi' \end{pmatrix}.$$ 

Thus, given $\eta, \nu \in \mathcal{A}_P$, and $\xi, \mu \in \mathcal{A}_P^*$, we need to find $\eta', \nu' \in \mathcal{A}_P$ and $\xi', \mu' \in \mathcal{A}_P^*$ such that

$$\begin{cases} \nu + \eta a \mu + (1 - \xi)b \mu = \eta' a + (1 - \xi')b + \nu' \xi' \\ \xi \mu = \mu' \xi' \end{cases}$$

(*)
CLAIM: Setting ξ′ = 1 + (ξ − 1)μ, we have that ξ′ ∈ A_p^+.
In fact, let ξ = x/y and μ = z/w, where x, y, z and w are odd integers. Then

\[ ξ′ = 1 + (ξ − 1)μ = 1 + \left( \frac{x}{y} − 1 \right) \frac{z}{w} = 1 + \left( \frac{x − y}{y} \right) \frac{z}{w} = \frac{yw + (x − y)z}{yw}. \]

Notice that yw is odd and (x − y)z is even so yw + (x − y)z is odd, hence proving the claim.

In order to solve (⋆) it is then enough to set

\[
\begin{aligned}
\xi' &= 1 + (\xi - 1)\mu, \\
\mu' &= \xi\mu\xi'^{-1}, \\
\eta' &= \eta\mu, \\
\nu' &= \nu\xi'^{-1}.
\end{aligned}
\]

□

REFERENCES

[ALR] J. Arledge, M. Laca, and I. Raeburn, “Semigroup crossed products and Hecke algebras arising from number fields”, *Doc. Math.*, 2 (1997), 115–138 (electronic).

[BC] J.-B. Bost and A. Connes, “Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory”, *Selecta Math. (N.S.)*, 1 (1995), 411–457.

[B] B. Brenken, “Hecke algebras and semigroup crossed product C*-algebras”, *Pacific J. Math.*, 187 (1999), 241–262.

[DE1] M. Dokuchaev and R. Exel, “Associativity of crossed products by partial actions, enveloping actions and partial representations”, *Trans. Amer. Math. Soc.*, to appear, [arXiv:math.RA/0212056].

[DE2] M. Dokuchaev and R. Exel, “Crossed products by twisted partial actions”, preprint.

[DEP] M. Dokuchaev, R. Exel, and R. Piccione, “Partial representations and partial group algebras”, *J. Algebra*, 226 (2000), 505–532, [arXiv:math.GR/9903129], MR 2001m:16034.

[E1] R. Exel, “Circle actions on C*-algebras, partial automorphisms and a generalized Pimsner-Voiculescu exact sequence”, *J. Funct. Analysis*, 122 (1994), 361–401, [arXiv:funct-an/9211001], MR 95g:46122.

[E2] R. Exel, “Twisted partial actions, a classification of regular C*-algebras bundles”, *Proc. London Math. Soc.*, 74 (1997), 417–443, [arXiv:funct-an/9405001], MR 98d:46075.

[E3] R. Exel, “Partial actions of groups and actions of inverse semigroups”, *Proc. Amer. Math. Soc.*, 126 (1998), 3481–3494, [arXiv:funct-an/9511003], MR 99b:46102.

[E4] R. Exel, “Partial representations and amenable Fell bundles over free groups”, *Pacific J. Math.*, 192 (2000), 39–63, [arXiv:funct-an/9706001], MR 2001c:46090.

[EL1] R. Exel and M. Laca, “Cuntz-Krieger algebras for infinite matrices”, *J. reine angew. Math.*, 512 (1999), 119–172, [arXiv:funct-an/9712008], MR 2000e:46064.

[EL2] R. Exel and M. Laca, “Partial dynamical systems and the KMS condition”, *Commun. Math. Phys.*, 232 (2003), 223–277, [arXiv:math.OA/0006169].

[ELQ] R. Exel, M. Laca, and J. Quigg, “Partial dynamical systems and C*-algebras generated by partial isometries”, *J. Operator Theory*, 47 (2002), 169–186, [arXiv:funct-an/9712007].

[GW] H. Glockner and G. A. Willis, “Topologization of Hecke pairs and Hecke C*-algebras”, Proceedings of the 16th Summer Conference on General Topology and its Applications (New York), *Topology Proc.*, 26 (2001/02), 565–591.

[H] R. W. Hall, “Hecke C*-algebras”, Ph.D. thesis, Pennsylvania State University, 1999.

[KLQ] S. Kaliszewski, M. B. Landstad, and J. Quigg, “C*-algebras, Schlichting completions, and Morita-Rieffel equivalence”, preprint, [arXiv:math.OA/0311222].

[Ka] C. Kassel, “Quantum Groups”, Springer Verlag, 1995.

[Kr] A. Krieg, “Hecke algebras”, *Mem. Amer. Math. Soc.*, 87 (1990), no. 435, x+158 pp.

[L] M. Laca, “Semigroups of *-endomorphisms, Dirichlet series, and phase transitions”, *J. Funct. Anal.*, 152 (1998), 330–378.

[LF] M. Laca and M. van Frankenhuijsen, “Phase transitions on Hecke C*-algebras and class-field theory over Q”, preprint, [arXiv:math.OA/0410302].

[LL1] M. Laca and N. S. Larsen, “Hecke algebras of semidirect products”, *Proc. Amer. Math. Soc.*, 131 (2003), 2189–2199 (electronic), [arXiv:math.OA/0106261].
[LL2] M. Laca and N. S. Larsen, “Errata to: "Hecke algebras of semidirect products"”, Proc. Amer. Math. Soc., 131 (2003), 1255–1256 (electronic).

[LR1] M. Laca and I. Raeburn, “A semigroup crossed product arising in number theory”, J. London Math. Soc., 59 (1999), 330–344.

[LR2] M. Laca and I. Raeburn, “The ideal structure of the Hecke C*-algebra of Bost and Connes”, Math. Ann., 318 (2000), 433–451.

[W] J. J. Ward, “A survey of subnormal subgroups”, Irish Math. Soc. Bull., (1990), 38–50.