Analytical solution of $k$th price auction

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Abstract

We provide an exact analytical solution of the Nash equilibrium for the $k$th price auction by using inverse of distribution functions. As applications, we identify the unique symmetric equilibrium where the valuations have polynomial distribution, fat tail distribution and exponential distributions.

key words: Vickrey auctions, $k$th price auctions, game theory

1 Introduction

In a $k$th price auction with $k$ or more bidders, the highest bidder wins the bid and pays the $k$th highest bid as price. The $k$th price auction has been studied by many researchers in recent years. Readers can refer to [2, 3, 4, 10] for related literature. In particular, Roger B. Myerson [6] established the Revenue Equivalence Theorem (known as RET theorem) in 1981, which characterized the equilibrium strategy. Later in 1998, Monderer and Tenenholtz [5] proved the uniqueness of the equilibrium strategies in $k$th price auctions for $k = 3$. Under some regularity assumptions, they also provided sufficient conditions for the existence of the equilibrium. In 2000, Wolfsteller [11] solved the equilibrium $k$th price auctions for a uniform distribution. Recently in 2018, Nawar and Sen [12] proved that the bid function of $k$th price auction can be represented as a finite series involving Catalan number. With their representation of the bid function and properties of Catalan number, they provided a closed form of the unique symmetric, increasing equilibrium of a $k$th price auction for a second degree polynomial distribution.

In this paper, using a method involving inverse of distribution functions, we provide a new representation of the equilibrium bid function of $k$th price auction with this representation. For applications, we extend Nawar and Sen’s results and provide a closed form solution of a $k$th price auction for a class of polynomial distribution and other classical distributions.

After recalling the framework of the problem in a first part, we demonstrate our main result in a second part. Then, in a third part, we compare our result with those of Nawar and Sen. Finally in a fourth part, we provide closed form solutions of a $k$th price auction for polynomial distributions, fat-tail distributions and exponential distributions.

2 Notations and assumptions

In this section we present our assumptions and remind us of the result on the uniqueness of the equilibrium strategies provided by Monderer and Tenenholtz. The $k$th price auction problem can be formulated as follows: consider a $k$th price auction with $n$ bidders, where the highest bidder wins, and pays only the $k$-th highest bid. Let’s assume that $k \geq 2$ and $n \leq k$. We make the following assumptions [8]:

1. The valuations $X_i, i = 1, \cdots, n$ of the bidders are independent and identically distributed with distribution function $F$. 

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2. The distribution function $F$ is with values in $I$ where $I = [0, 1]$ or $I = \mathbb{R}^+$. We also assume that:

(A) $F$ is $k - 2$ times continuously differentiable and the density function $f := F'$ satisfying $\forall x \in I, f(x) > 0$.

We remark that under the assumption of the density function $f$, the quantile function $Q := F^{-1}$ exists and is well-defined on $(0, 1)$. Remember that for an analysis of the 3-price auction in the literature, the existence and the continuity of the density function are often assumed. It is thus natural to assume (A) holds for the case of general k-price auctions. Now we denote $\beta_i$ the strategy profile of the bidder $X_i$ which determines its bid for any value. A strategy profile for $n$ bidders is given by $(\beta_1, \ldots, \beta_n)$. We point out that a strategy profile is symmetric if $\beta_i$ are all equal to a common strategy $\beta$. We also point out that a symmetric strategy is increasing if $\beta$ is an increasing function. The equilibrium of a $k$th price auction with a symmetric strategy profile is characterized by the following theorem:

**Theorem 2.1.** [Monderer and Tennenholtz] Let $\beta : [0, 1] \mapsto \mathbb{R}^+$. A symmetric strategy profile with common strategy $\beta$ is an equilibrium of the $k$th price auction if and only if the following two conditions hold.

(E1) $\beta$ is an increasing function.

(E2) For all $x \in [0, 1]$:

$$\int_{0}^{x} (x - \beta(y)) F(y)^{n-k}(F(x) - F(y))^{k-3} f(y) dy = 0. \quad (1)$$

Moreover (1) has at most one solution and for such a solution $\beta$ is differentiable for all $x \in \text{Supp}(F)$.

According to Theorem 2.1, for a given $F$, if we can compute $\beta$ and show that $\beta$ is differentiable and increasing, then $\beta$ is the unique equilibrium bid function. If the equilibrium bid function $\beta$ exists, $\beta(X)$ is the random variable representing the equilibrium strategy of bidders with valuation $X$. Furthermore, if $\beta$ is strictly increasing, together with differentiability of $\beta$ and the assumption that $F$ has a density, we can deduce that $\beta(X)$ also has a continuous density function. We denote $\hat{F}$ the distribution function of $\beta(X)$ ($\hat{F}(\beta(x)) = F(x)$) and $Q := \hat{F}^{-1}$ the quantile function.

## 3 Analysis of equilibrium

Here we present our main result. We give a closed form solution of equation (1) for $k \geq 3$. With this solution, we are able to find a closed form expression of the bid function for some non-linear distributions. The key idea is, instead of working directly with the distribution function $F$ as in the literature, we use the quantile function $Q$, which can largely simplify the calculus and have a better insight.

**Theorem 3.1.** Assume that $\beta : [0, 1] \mapsto \mathbb{R}^+$ is an increasing function and $\beta(X)$ has an increasing distribution function with a continuous density. Then (1) has a unique solution given by

$$\beta(x) = \frac{\gamma^{k-2}(\hat{x})}{(k-2)! (n-k)^{-1} F(x)^{n-k}}, \quad (2)$$

where $\hat{x} = F(x)$ and $\gamma(\hat{x}) := Q(\hat{x}) \hat{x}^{n-2}$.

**Proof.** Denote

$$s(x) := x \int_{0}^{x} [F(x) - F(y)]^{k-3} F(y)^{n-k} f(y) dy, \quad (3)$$

$$w(x) := \int_{0}^{x} \beta(y) [F(x) - F(y)]^{k-3} F(y)^{n-k} f(y) dy. \quad (4)$$

Note that (1) can be written as $s(x) = w(x)$.

Since $F'(y) = f(y)$, making the transformation $z = F(y)/F(x)$ in (3) we have

$$s(x) = x F(x)^{n-2} \int_{0}^{1} (1 - z)^{k-3} z^{n-k} dz = x F(x)^{n-2} B(n - k + 1, k - 2), \quad (5)$$

where $B(c, d)$ is the beta function with $B(c, d) = (c-1)!(d-1)!/(c+d-1)!$ for positive integers $c, d$. It follows that

$$s(x) = \frac{x F(x)^{n-2}}{(k-2)(n-2)}, \quad (6)$$

...
Make the transformation $z = F(y)$ in (1). Recall $\hat{F}$ is the distribution function of $\beta(X_i)$. As $\beta$ is increasing, we have $F(y) = \hat{F}(\beta(y))$, so that

$$\beta(y) = \hat{F}^{-1}(F(y)) = \hat{Q}(F(y)) = \hat{Q}(z).$$

Thus,

$$w(x) = \int_0^{F(x)} \hat{Q}(z)(F(x) - z)^{k-3}z^{n-k} dz. \tag{7}$$

As $\hat{x} = F(x)$, we have $x = Q(\hat{x})$. Denote

$$S(\hat{x}) := s(Q(\hat{x})) \text{ and } W(\hat{x}) := w(Q(\hat{x})).$$

Also recall that $\gamma(\hat{x}) := Q(\hat{x})\hat{x}^{n-2}$. Then from (8):

$$S(\hat{x}) = s(Q(\hat{x})) = s(x) = \frac{\gamma(\hat{x})}{(k-2)(n-2)}. \tag{8}$$

By (7):

$$W(\hat{x}) := w(Q(\hat{x})) = w(x) = \int_0^{x} \hat{Q}(z)(\hat{x} - z)^{k-3}z^{n-k} dz. \tag{9}$$

According to (1), $S(\hat{x}) = W(\hat{x})$. Noticing that assumption (A) ensures that $S(\hat{x})$ is $k - 2$ times continuously differentiable and denoting by superscript $(t)$ the $t^{th}$ order derivative with respect to $\hat{x}$, we have:

$$S^{(k-2)}(\hat{x}) = W^{(k-2)}(\hat{x}). \tag{10}$$

According to (1), $S^{(k-2)}(\hat{x})$ is known by (10). To find $W^{(k-2)}(\hat{x})$, apply Lemma 7.1 of the Appendix. Taking $m = n - k$ in Lemma 7.1, we have $W(\hat{x}) = S_{k-3}(\hat{x})$. By (2) it follows that

$$W^{(k-2)}(\hat{x}) = (k - 3)!\hat{Q}(\hat{x})\hat{x}^{n-k}. \tag{11}$$

Hence, (11) becomes:

$$\frac{\gamma(\hat{x})}{(k-2)(n-2)} = (k - 3)!\hat{Q}(\hat{x})\hat{x}^{n-k}. \tag{11}$$

As $\hat{F}(\beta(x)) = F(x) = \hat{x}$, we have $\hat{Q}(\hat{x}) = \hat{F}^{-1}(\hat{x}) = \beta(x)$. Using this, (2) follows from (11).

**Remark 3.1.** Asymptotically our equation (2) gives:

$$\beta(x) = \sum_{i=0}^{k-2} \binom{k-2}{i} (\hat{x}^{n-2-i})Q(\hat{x})(k-2-i).$$

by developing the two terms for $i = k - 2$ and $i = k - 3$ we obtain:

$$\beta(x) = x + \frac{k - 2}{n - k + 1} \frac{F(x)}{f(x)} + O(\frac{1}{n^2}).$$

This result coincides with the result of Wofstellel (11) in $O(\frac{1}{n})$. Moreover it agrees with the expression in proposition 3 of Gadi Fibich and Arieh Gavious’s work in (11).

### 4 Comparison with Nawar and Sen’s result

Applying the revenue equivalence principle for expected payment of a bidder in a $k$th price auction with $n$ bidders, Nawar and Sen (2018) [12] have obtained the following expression of $\beta(x)$:

$$\beta(x) = \frac{\psi_{k-1}(x)}{(k-2)!\binom{n-2}{k-2}F(x)^{n-k}}, \tag{12}$$

where
\[
\psi_0(x) = \int_0^x yF(y)^{n-2}f(y)dy \quad \text{and} \quad \psi_{t+1}(x) = \frac{\psi_t'(x)}{f(x)} \quad \text{for} \quad t = 0, 1, \ldots
\] (13)

By making the transformation \( z = F(y) \) then \( y = F^{-1}(z) = Q(z) \) we have \( \psi_0(x) = \int_0^x Q(z)z^{n-2}dz. \) Thus

\[
\frac{d\psi_0(x)}{dx} = Q(\hat{x})\hat{x}^{n-2} = \gamma(\hat{x}).
\]

Also note that as \( \hat{x} = F(x) \), we have \( \frac{d\hat{x}}{dx} = F'(x) = f(x). \) By (13), we have

\[
\psi_1(x) = \frac{d\psi_0(x)}{dx} = \frac{1}{f(x)} \frac{d\psi_0(x)}{d\hat{x}} \frac{d\hat{x}}{dx} = \gamma(\hat{x}).
\]

Applying the iterative definition of (13) again, we have:

\[
\psi_2(x) = \frac{d\psi_1(x)}{dx} = \frac{1}{f(x)} \frac{d\psi_1(x)}{d\hat{x}} \frac{d\hat{x}}{dx} = \frac{d\gamma(\hat{x})}{d\hat{x}} = \gamma^{(1)}(\hat{x}).
\]

By induction, if \( \psi_t(x) = \gamma^{(t-1)}(\hat{x}) \), then

\[
\psi_{t+1}(x) = \frac{d\psi_t(x)}{dx} = \frac{1}{f(x)} \frac{d\psi_t(x)}{d\hat{x}} \frac{d\hat{x}}{dx} = \frac{d\gamma^{(t)}(\hat{x})}{d\hat{x}}.
\]

This shows that \( \psi_t(x) = \gamma^{(t-1)}(\hat{x}) \) for \( t = 1, 2, \ldots \) So \( \psi_{k-1}(x) = \gamma^{(k-2)}(\hat{x}) \) and the expression (12) coincides with our expression (2). Therefore, our expression (2) is an equivalence representation of Nawar and Sens result. Instead of expanding \( \psi \) with series about Catalan’s number, we compute it with the quantile function. From this expression (2), it is easy to establish the equilibrium for some non-linear distributions, and we will detail them in the next section.

5 Examples of Equilibrium for some distributions

5.1 Equilibrium for non linear distribution

In this section we study the equilibrium bid function for some non-linear distributions. We first provide a sufficient condition for existence and uniqueness of the equilibrium bid function as a corollary of theorem 5.1. We will then provide close form solution of the equilibrium bid function for polynomial distribution, exponential distribution, a class of fat tail distribution.

According to theorem 2.1, for some distribution \( F \), if one can show that \( \beta \) found by (2) is an strictly increasing function, it will follow that the symmetric strategy profile with common strategy \( \beta \) is an equilibrium of the \( k \)th price auction.

**Corollary 5.1.** Consider a \( k \)th price auction where each \( X_i \) is i.i.d. on the interval \([0, 1]\), with distribution function quantile function \( Q \). Assume that it holds

\[
\forall i \in [0, k-1], \forall \tilde{x} \in [0, 1], Q^{(i)}(\tilde{x}) > 0.
\] (14)

Then the existence and uniqueness of equilibrium bid function is given by (2).

**Proof.** By using the Leibniz rule for derivation,

\[
\gamma^{(k-1)}(\tilde{x}) = \sum_{i=0}^{k-1} \binom{k-2}{i} (\tilde{x}^{n-2})(k-1-i)Q^{(i)}(\tilde{x}).
\]

Therefore, by equation (14), for all \( \hat{x} > 0, \gamma^{(k-1)}(\hat{x}) > 0 \) and we deduce that \( \gamma^{(k-2)}(\hat{x}) \) is strictly increasing. Together with the fact that the distribution \( F \) is increasing, we deduce that \( \beta \) found in equation (2) is strictly increasing. The conclusion follows.
For some distributions, the condition \( Q^{(i)}(\tilde{x}) > 0 \) for all \( \tilde{x} \in [0,1] \) is easy to check but for some it is not. However, for many non-linear distributions, the quantile function \( Q \) is analytic on \([0,1]\). According to the latter Corollary, we provide another sufficient condition as follows, the condition \((P^+)\), which is easy to check.

**Condition** \((P^+)\): The function \( Q \) is analytic on \([0,1]\) with positive Taylor coefficients in the representation of power series. More precisely, there exist positive \( \alpha_i \), such that for \( x \in [0,1] \),

\[
Q(\tilde{x}) = \sum_{i=0}^{\infty} \alpha_i \tilde{x}^i.
\]

The equality is defined in the sense of power series, reader can refer to any power series literature for a complete justification of technical convergence details. From condition \((P^+)\), deriving the equation successively, it is easy to see that equation \((15)\) holds. According to corollary 5.1, \( \beta \) is an equilibrium bid function. To check the condition \((P^+)\), we only need to calculate the Taylor expansion of the quantile function \( Q \) on 0, then check the sign of the Taylor coefficient. Here are some examples of applications of corollary 5.1.

**Example 5.1** (Exponential distribution.). Let \( F(x) := 1 - e^{-\lambda x} \) for \( \lambda > 0 \) and \( x \in \mathbb{R}^+ \). The equilibrium bid function is given by \([2]\). In fact \( Q(x) := \frac{\lambda}{\lambda+1}\ln(1-x) \). Moreover \( Q(\tilde{x})(x) = \frac{1}{\lambda+1}\tilde{x}^\beta \), which is positive on \([0,1]\).

Applying \([2]\) with simplifications, the equilibrium \( \beta(x) \) has the expression:

\[
\beta(x) = \frac{1}{(k-2)!}\left(\frac{1}{n-2}\right) \left[ \frac{-1}{\lambda} \ln(1-\tilde{x}) \right] (n-2)! \left( \frac{1}{(1-\tilde{x}^\alpha)} \sum_{i=1}^{k-2} \frac{(i-1)!}{(n-\tilde{x}^\alpha)} \left( \frac{1-\tilde{x}^\alpha}{i} \right) \right),
\]

with \( \tilde{x} = F(x) \).

**Example 5.2** (Fat tail distribution.). Let \( F(x) := 1 - \frac{1}{x^c} \) for some \( c > 0 \) and \( x \in \mathbb{R}^+ \). The equilibrium bid function is given by \([2]\). In fact \( Q(x) := 1 + \ln(1-x) \). Moreover \( Q(\tilde{x}) \) is analytic on \([0,1]\) with positive Taylor coefficients proving the condition \((14)\).

Applying \([2]\) with simplifications, equilibrium \( \beta(x) \) has the expression:

\[
\beta(x) = \frac{1}{(k-2)!}\left(\frac{1}{n-2}\right) \left[ c(c+1)...(c+i-1)\tilde{x}^i \right] \left( \frac{1}{(n-k+i)!} \left( \frac{1-\tilde{x}^\alpha}{i} \right) \right),
\]

where \( \tilde{x} = F(x) \).

**Theorem 5.2.** Consider a \( k \)-th price auction where each \( X_i \) is iid on the interval \([0,1]\), with distribution function \( F(x) = x^\alpha \) where \( \alpha > 0 \). Then there is a unique symmetric equilibrium. The equilibrium common strategy \( \beta : [0,1] \rightarrow \mathbb{R}^+ \) is

\[
\beta(x) = \frac{\Gamma(n-k+1)\Gamma(n-1+1/\alpha)}{\Gamma(n-1)\Gamma(n-k+1+1/\alpha)} x,
\]

where \( \Gamma \) is the Gamma function. In particular, if \( \alpha = \frac{1}{m} \) where \( m \) is a positive integer,

\[
\beta(x) = \frac{(n-2+m)...(n-k+m+1)}{(n-2)...(n-k+1)} x.
\]

**Proof.** To prove this, we find \( \beta \) using theorem 3.1 and show that it is a strictly increasing function of \( x \). As \( \tilde{x} = F(x) = x^\alpha \), it follows \( Q(\tilde{x}) = F^{-1}(\tilde{x}) = \tilde{x}^{1/\alpha} \). Then

\[
\gamma(\tilde{x}) = Q(\tilde{x})\tilde{x}^{n-2} = \tilde{x}^{n-2+1/\alpha}.
\]

Therefore,

\[
\gamma(\tilde{x})^{(k-2)} = (n-2+1/\alpha)(n-3+1/\alpha)...(n-k+1/\alpha)\tilde{x}^{n-k+1/\alpha} = (n-2+1/\alpha)(n-3+1/\alpha)...(n-k+1/\alpha)F(x)^{n-k}x.
\]

Together with \([2]\), \((15)\) follows.
5.2 Counter-example

We saw in the previous part that for a wide range of distributions the \[ 2 \] gives the solution of the equilibrium bid function. Then a natural question is: are there any counter-examples? We will build one in this part.

Let’s take \( a \in (0, 1) \) and define

\[
F(x) := -\frac{(1-a) + \sqrt{(1-a)^2 + 4ax}}{2a} \quad \text{for} \ x \in [0, 1].
\]

Thus \( F \) satisfies \( F'(x) > 0 \) for all \( x \in [0, 1] \) and \( Q(x) := ax^2 + (1-a)x. \)

Then \( \frac{(x^n Q(x))^{(k-2)} }{x^n} \) simplifies well and takes the form:

\[
\frac{(x^n Q(x))^{(k-2)} }{x^n} = a \frac{n!}{(n-k+2)!} \hat{x}^{2} + (1-a) \frac{(n-1)!}{(n-k+1)!} \hat{x}.
\]

By taking the derivation of the last expression and by evaluating in \( \hat{x} = 1 \) we find:

\[
\left( \frac{(x^n Q(x))^{(k-2)} }{x^n} \right)'(1) = \frac{(n-1)!}{(n-k+2)!} (2an + (1-a)(n-k+2)).
\]

When \( a \) approaches \( 1^- \) the last expression is equal to:

\[
\lim_{a \to 1^-} \frac{(n-1)!}{(n-k+2)!} (2an + (1-a)(n-k+2)) = \frac{(n-1)!}{(n-k+2)!} 2(2-k),
\]

which is strictly negative for \( k > 2 \).

Finally there exist an infinite number of \( a \) close enough to \( 1 \) such that the expression \[ 2 \] is decreasing on a neighborhood of \( 1 \). This provides a counter example.

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7 Appendix

Lemma 7.1. Consider a real valued bounded function \( \hat{Q} : \mathbb{R} \to [0, 1] \). For positive integer \( r \) and positive real number \( m \), let \( A_r(u, z) := \hat{Q}(z)(u-z)^r z^m \) and \( S_r(u) := \int_0^u A_r(u, z)dz \). Then the \((r+1)\)th derivative of \( S_r(u) \) is

\[
S_r^{r+1}(u) = r!\hat{Q}(u)u^m.
\]

Proof. By the Leibniz rule of differentiating an integral, if \( S(u) := \int_{t_0(u)}^{t_1(u)} A(u, z)dz \), under assumption of integrability of \( \partial_u A(u, z) \), it holds:

\[
S'(u) = [A(u, l_1(u))l'_1(u) - A(u, l_0(u))l'_0(u)] + \int_{t_0(u)}^{t_1(u)} \partial_u A(u, z)dz.
\]

It is easy to check the integrability of \( \partial_u \hat{Q}(z)(u-z)^r z^m \), thus taking \( t_0(u) = 0, t_1(u) = u \) in the previous equation:

\[
S'_r(u) = A_r(u, u) + \int_0^u \partial_u A_r(u, z)dz.
\]

For \( r = 0 \), \( A_0(u, u) = \hat{Q}(u)u^m \) and \( \partial_u A_r(u, z) = 0 \). For \( r \geq 1 \), \( A_r(u, u) = 0 \) and \( \partial_u A_r(u, z) = \hat{Q}(z)r(u-z)^{r-1}z^m = rA_{r-1}(u, z) \). Therefore \[ 17 \] implies

\[
S'_r(u) = rS_{r-1}u \quad \text{for} \ r \geq 1 \ \text{and} \ S'_0(u) = \hat{Q}(u)u^m.
\]

Thus for \( r' \leq r \):

\[
S_{r'}^{(r')}(u) = r(r-1)...(r-r'+1)S_{r-r'}u,
\]

so \( S_r^{(r)} = r!S_0(u) \) and therefore \( S_r^{(r+1)} = r!S'_0(u) = r!\hat{Q}(u)u^m. \)
Lemma 7.2. For all \( p, m \in \mathbb{N} \), it holds: For \( n > k \geq 2 \), it holds:

\[(i) \quad \sum_{p=0}^{k-2} \frac{(-1)^{k-2-p}}{n-1-p} \binom{k-2}{p} = \frac{1}{(n-1)\binom{n-2}{k-2}} \]

and

\[(ii) \quad \sum_{p=0}^{k-2} \frac{(-1)^{k-2-p}}{n-1-p} \binom{k-2}{p} \frac{1}{p} = \frac{1}{(n-2)\binom{n-2}{k-2}}. \]

Proof. (i) Consider

\[r(u) = \frac{k-2}{\sum_{p=0}^{k-2} (-u)^{n-2-p} \binom{k-2}{p}}. \tag{18}\]

Note that

\[r(u) = (-1)^{n-k}u^{n-k} \sum_{p=0}^{k-2} (-u)^{k-2-p} \binom{k-2}{p} = (-1)^{n-k}u^{n-k}(1-u)^{k-2}. \tag{19}\]

From (18):

\[
\int_{0}^{1} r(u) du = \sum_{p=0}^{k-2} (-1)^{n-2-p} \binom{k-2}{p} \int_{0}^{1} u^{n-2-p} du \\
= (-1)^{n-k} \sum_{p=0}^{k-2} (-1)^{k-2-p} \binom{k-2}{p} \frac{1}{n-1-p}. \tag{20}\]

Now let From (19):

\[
\int_{0}^{1} r(u) du = (-1)^{n-k} \int_{0}^{1} u^{n-k} \frac{(1-u)^{k-2}}{du} \\
= (-1)^{n-k} B(n-k+1, k-1) = \frac{1}{(n-1)\binom{n-2}{k-2}}. \tag{21}\]

Then (i) follows by (20) and (21). \( \square \)

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