Gravitomagnetic Field and Time-Dependent Spin-Rotation Coupling

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The Kerr metric of spherically symmetric gravitational field is analyzed through the coordinate transformation from the rotating frame to fixing frame, and consequently that the inertial force field (with the exception of the centrifugal force field) in the rotating system is one part of its gravitomagnetic field is verified. We investigate the spin-rotation coupling and, by making use of Lewis-Riesenfeld invariant theory, we obtain exact solutions of the Schrödinger equation of a spinning particle in a time-dependent rotating reference frame. A potential application of these exact solutions to the investigation of Earth’s rotating frequency fluctuation by means of neutron-gravity interferometry experiment is briefly discussed in the present paper.

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I. INTRODUCTION

One can easily verify that the field equation of general relativity in low-motion weak-field approximation is somewhat analogous to Maxwell’s equation of electromagnetic field. It is the most outstanding point that the former field (gravitational field) also possesses both the gravitoelectric potential written as \( g_{00} - \frac{1}{2} g_{ij} \) and the gravitomagnetic potentials as \( \vec{A} = (g_{01}, g_{02}, g_{03}) \), and the corresponding gravitomagnetic field strength is of the form \( \vec{B} = -\frac{1}{2} \nabla \times \vec{A} \).

A particle with intrinsic spin possesses a gravitomagnetic moment of such magnitude that it equals the spin of this particle. The interaction between the gravitomagnetic moment and the gravitomagnetic field is thus also called the spin-gravity coupling \([1,2]\), of which the Hamiltonian is given by

\[
H = \frac{1}{2} \vec{B} \cdot \vec{S}.
\]  

(1.1)

It is shown in what follows that the strongest gravitomagnetic field that we can find on the Earth arises from the Earth’s rotation, that is, the Earth’s rotation gives rise to an inertial gravitomagnetic force field observed in the rotating frame. Since the Earth is a noninertial reference frame due to its rotation, a spinning particle is coupled to a more strong gravitomagnetic field (i.e., Earth’s rotation frequency), which represents the coupling of spin-noninertial frame in addition to the interaction expressed by Eq. (1.1). It is apparently seen that the interaction of angular momentum of a particle with noninertial frame is related to the Coriolis force \([3]\). These two gravitomagnetic fields (see in Sec.2 for detailed differences between them) have different origins and properties: the gravitomagnetic field caused by mass current, expressed by \( \vec{B} = -\frac{1}{2} \nabla \times \vec{A} \), is similar to the magnetic field produced by electric current, and its strength is dependent on the Newtonian gravitational constant \( G \), while the gravitomagnetic field associated with the Coriolis force depends on the choice of the coordinates and in consequence its strength is independent of the Newtonian gravitational constant. That is, in accordance with Newton’s law the coordinate transformation from the rotating frame to the fixing frame results in this inertial force observed by the observer fixed in the rotating reference frame. Apparently, due to the smallness of \( G \), the coupling of the latter gravitomagnetic field with intrinsic spin is \( 10^{20} \) times stronger than that of Eq. (1.1) \([4]\).

In the present paper, we further investigate the interaction between this inertial force field and the intrinsic spin of a particle. According to the equivalence principle, the nature of the inertial force is gravitational force, and consequently both expressions of these two gravitomagnetic forces (namely, the gravitomagnetic Lorentz force and Coriolis force) can be derived from the equation of gravitational field. This work is given in what follows and we thus obtain the Hamiltonian of the spin-rotation coupling. It is known that Mashhoon’s approach to deriving the intrinsic spin-rotation coupling is suggested by analyzing the Doppler’s effect of wavelength in the rotating frame with a respect to the fixing frame \([2,4]\). In this paper, however, the transformation of the gravitomagnetic potentials is studied through the coordinate transformation, and as a result, the Hamiltonian of the coupling of the intrinsic spin of a particle with the rotating frequency of a rotating reference frame is then obtained.
The reason why the coupling of spin (or gravitomagnetic moment) with noninertial frame is of great importance lies in that, with the development of laser technology and their applications to the gravitational interferometry experiment [5–7], it becomes possible for us to investigate quantum mechanics in weak-gravity field. The utilization of these relativistic quantum gravitational effects enables physicists to test the fundamental principles of general relativity in microscopic areas. Although the equivalence principle still holds in the relativistic quantum gravitational effect [4], there are some physically interesting phenomena such as the violation of the principle of free falling body for the spinning particle [4,8] moving in, for instance, the Kerr spacetime.

Since the analogy can be drawn between gravity and electrodynamics [3]. The Kerr metric of the exterior gravitational field of the rotating spherically symmetric body is of the form

\[
ds^2 = (1 - \frac{2GMr}{c^2(r^2 + a^2 \cos^2 \theta)})c^2 dt^2 - \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 - \frac{2GMr}{c^2}} dr^2 - (r^2 + a^2 \cos^2 \theta) d\theta^2 - \sin^2 \theta(\frac{2a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{GMr}{c^2}) + r^2 + a^2 d\varphi^2 + \frac{2a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{GMr}{c} dt d\varphi ,
\]

where \( r, \theta, \varphi \) are the displacements of spherical coordinate, \( a \) is so defined that \( ac \) is the angular momentum of unit mass of the gravitational body, and \( M \) denotes the mass of this gravitational body. Since the space-time coordinate of Kerr metric (2.1) is in the fixing reference frame, we can transform it into that in the rotating reference frame. Because of the smallness of the Earth’s rotating velocity, one can apply the following Galileo transformation to the coordinates of a particle moving radially in the rotating frame

\[
\begin{align*}
\dot{r}' + v \dot{t}' &= dr, \quad \dot{\theta}' = d\theta, \quad \dot{\varphi}' = d\varphi + \omega dt, \quad \dot{t}' = dt
\end{align*}
\]

with \( v \) being the radial velocity of the particle relative to the rotating reference frame, \((r', \theta', \varphi', t')\) and \((r, \theta, \varphi, t)\) the space-time coordinates of the rotating frame and fixing frame, respectively. \( \omega \) denotes the rotating frequency of the rotating frame with respect to the fixing reference frame. For the simplicity of calculation, the radial velocity \( v \) is taken to be much less than \( \omega r \), then substitution of Eq. (2.2) into Eq. (2.1) yields

\[
ds^2 = [1 - \frac{2GMr}{c^2(r^2 + a^2 \cos^2 \theta)} - \frac{(r^2 + a^2 \cos^2 \theta) v^2}{r^2 + a^2 - \frac{2GMr}{c^2}} - \sin^2 \theta(r^2 + a^2 \frac{2a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{GMr}{c^2})^2]
\]

II. GRAVITOMAGNETIC FIELD AND SPIN-ROTATION COUPLING

The Kerr metric (2.1) is in the fixing reference frame, we can transform it into that in the rotating reference frame. Because of the smallness of the Earth’s rotating velocity, one can apply the following Galileo transformation to the coordinates of a particle moving radially in the rotating frame

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\]
with $\omega^2 \sin^2 \theta$ in $g_{\mu\nu}$ results in the inertial centrifugal force written as $\vec{F} = m\vec{\omega} \times (\vec{\omega} \times \vec{r})$. Ignoring the terms associated with $\frac{a^2}{r^2} \ll 1$ in $g_{\mu\nu}'$, one can obtain

$$g_{\mu\nu}' \, d\varphi' \, dt' = \left(\frac{2aGMr \sin^2 \theta}{c^2} + 2\omega r^2 \sin^2 \theta\right) dt' \, d\varphi'$$

Thus the gravitomagnetic potentials can be written as

$$A_\varphi = \frac{2aGM \sin \theta}{c^2 r^2} + 2\omega r \sin \theta, \quad A_r = -2v, \quad A_\theta = 0.$$  \hfill (2.5)

It follows that the first term $\frac{2aGM \sin \theta}{c^2 r^2}$ of $A_\varphi$ is exactly analogous to the magnetic potential $\frac{\mu_0 \sigma_\varphi}{4\pi r^2} \sin \theta$ of the rotating charged spherical shell in the electrodynamics. Then we can calculate the exterior gravitomagnetic strength of the charged spherical shell in the electrodynamics. Then we can calculate the exterior gravitomagnetic strength of the rotating gravitational body, and the result is $\vec{B}_\varphi = \frac{2aGMr \sin^2 \theta}{c^2} \vec{e}_\varphi - \frac{2\omega aGMr \cos \theta}{c^2} \vec{e}_r \vec{e}_\vartheta$ \hfill [5].

In accordance with the equation of geodesic line of a particle in the post-Newtonian approximation, the gravitomagnetic strength can be defined by $-\frac{1}{2} \nabla \times \vec{A}$ with $\vec{A} = (g_{01}, g_{02}, g_{03})$ as assumed above. Set $\beta_\varphi = 2\omega r \sin \theta, \beta_r = -2v, \beta_\theta = 0$, then the gravitomagnetic strength that arises from the choice of the reference frames is given as follows:

$$-\frac{1}{2} \nabla \times \vec{\beta} = -2\omega \cos \theta e_r + 2\omega \sin \theta e_\theta$$

(2.6)

with $e_r$, $e_\theta$ being the unit vector. It follows from Eq.(2.6) that this gravitomagnetic strength is related to the rotation of noninertial frame and independent of the Newtonian gravitational constant $G$. From the point of view of Newtonian mechanics, it is the inertial force field in essence rather than the field that is produced by mass current. Since we have assumed that the velocity of a particle is parallel to $e_r$, i.e., $\vec{v} = ve_r$, the gravitational Lorentz force acting on the particle in the gravitomagnetic field is thus given by

$$\vec{F} = m\vec{v} \times (-\frac{1}{2} \nabla \times \vec{\beta}) = 2v \omega \sin \theta e_\varphi = 2m\omega \sin \theta \vec{\omega},$$

(2.7)

We conclude from Eq. (2.7) that the gravitational Lorentz force in rotating reference frame is the familiar Coriolis force and the rotating frequency $\vec{\omega}$ can be regarded as the gravitomagnetic field strength.

In the following we will derive the Hamiltonian of spin-rotation coupling by investigate the Dirac equation with spin connection

$$[\gamma^\mu (\partial_\mu - \frac{i}{4} \sigma^{\lambda\tau} \omega_{\lambda\tau\mu}) - mc] \psi = 0$$

(2.8)

with $\sigma^{\lambda\tau} = \frac{i}{2} (\gamma^\lambda \gamma^\tau - \gamma^\tau \gamma^\lambda)$. In the rotating frame, we have the following form of the line element of spacetime

$$ds^2 = (1 - \frac{\omega^2}{c^2} \vec{\omega} \cdot \vec{x}) c^2 dt^2 - d\vec{x} \cdot d\vec{x} - 2(\vec{\omega} \times \vec{x}) \cdot d\vec{x} \, dt$$

(2.9)

by ignoring the gravitational effect associated with the gravitational constant $G$ and utilizing the weak-field low-motion approximation. Then further calculation yields the following connections [18]
\[ \omega_{\lambda \tau 0} = -\epsilon_{\lambda \tau \eta} \frac{\omega_\eta}{c}, \quad \omega_{0 \tau 0} = -\omega_{\tau 0 0} = 0, \]
\[ \omega_{\lambda \tau \mu} = 0 (\mu = 1, 2, 3) \] (2.10)
with \( \epsilon_{\lambda \tau \eta} \) being three-dimensional Levi-Civita tensor. By making use of Eq. (2.8), Eq. (2.9) and Eq. (2.10), one can arrive at the following Dirac equation
\[ i \frac{\partial}{\partial t} \psi = H \psi \] (2.11)
with
\[ H = \beta mc^2 + c\vec{a} \cdot \vec{p} + \vec{\omega} \cdot \vec{L} + \vec{\omega} \cdot \vec{S}. \] (2.12)

We thus obtain the Hamiltonian of spin-rotation coupling
\[ H_{s-r} = \vec{\omega} \cdot \vec{S} \] (2.13)
which is consistent with Mashhoon’s result [2].

III. EXACT SOLUTIONS OF TIME-DEPENDENT SPIN-ROTATION COUPLING

The variation of the Earth’s rotating frequency may be caused by the motion of interior matter, tidal force, and the motion of atmosphere as well. Once we have information concerning the Earth’s rotating frequency, it is possible to investigate the motion of matter on the Earth. For the sake of detecting the fluctuation of the Earth’s time-dependent rotation conveniently, we suggest a potential approach to measuring the geometric phase factor arising from the interaction of neutron spin with the Earth’s rotation by using the neutron interferometry experiment. First we should exactly solve the time-dependent Schrödinger equation of a spinning particle in the rotating system.

The Schrödinger equation which governs the interaction of neutron spin with Earth’s rotation is
\[ i \frac{\partial}{\partial t} |\Psi(t)\rangle_s = H_{s-r}(t) |\Psi(t)\rangle_s. \] (3.1)
Set \( \vec{\omega}(t) = \omega_0(t) [\sin \theta(t) \cos \varphi(t), \sin \theta(t) \sin \varphi(t), \cos \theta(t)] \), and \( \sigma_{\pm} = \sigma_1 \pm i \sigma_2 \) with \( \sigma_1, \sigma_2 \) being Pauli matrices, then the expression (2.13) for \( H_{s-r}(t) \) can be rewritten as
\[ H_{s-r}(t) = \omega_0(t) \left\{ \frac{1}{4} \sin \theta(t) \exp[-i \varphi(t)] \sigma_+ + \frac{1}{4} \sin \theta(t) \exp[i \varphi(t)] \sigma_- + \frac{1}{2} \cos \theta(t) \sigma_3 \right\}. \] (3.2)

In accordance with the invariant theory, an invariant which satisfies the following invariant equation [14]
\[ \frac{\partial I(t)}{\partial t} + \frac{i}{4} [I(t), H_{s-r}(t)] = 0 \] (3.3)
should be constructed often in terms of the generators of Hamiltonian (3.2). Then it follows from Eq. (3.3) that the invariant may be written in terms of Pauli matrices as follows
\[ I(t) = \frac{1}{4} \sin \lambda(t) \exp[-i \gamma(t)] \sigma_+ + \frac{1}{4} \sin \lambda(t) \exp[i \gamma(t)] \sigma_- + \frac{1}{2} \cos \lambda(t) \sigma_3, \] (3.4)
where the time-dependent parameters \( \lambda(t) \) and \( \gamma(t) \) satisfy the following two auxiliary equations
\[ \dot{\lambda}(t) = \omega_0(t) \sin \theta \sin(\varphi - \gamma), \quad \dot{\gamma}(t) = \omega_0(t) \cos \theta - \sin \theta \cot \lambda \cos(\varphi - \gamma) \] (3.5)
with dot denoting the time derivative. It is readily verified by using Eq. (3.5) that the invariant \( I(t) \) has time-independent eigenvalue \( \sigma = \pm \frac{1}{2} \) and its eigenvalue equation is
\[ I(t) |\sigma, t\rangle = \sigma |\sigma, t\rangle. \] (3.6)
According to the Lewis-Riesenfeld invariant theory, the particular solution \(|\sigma,t\rangle_s\) of Eq. (3.1) is different from the eigenfunction \(|\sigma,t\rangle\) of the invariant \(I(t)\) only by a phase factor \(\exp[i\phi_\sigma(t)]\). Then the general solution of the Schrödinger equation (3.1) can be written as
\[
|\Psi(t)\rangle_s = \sum_\sigma C_\sigma \exp[i\phi_\sigma(t)] |\sigma,t\rangle, \tag{3.7}
\]
where
\[
\phi_\sigma(t) = \int_0^t \langle \sigma,t' | i \frac{\partial}{\partial t'} - H_{s-r}(t') | \sigma,t' \rangle \, dt', \\
C_\sigma = \langle \sigma,t = 0 | \Psi(0) \rangle_s. \tag{3.8}
\]

In order to obtain the analytic solution of the time-dependent Schrödinger equation (3.1), we introduce an invariant-related unitary transformation operator \(V(t)\)
\[
V(t) = \exp[i\frac{\beta(t)}{2} \sigma_+ - i\frac{\beta^*(t)}{2} \sigma_-], \tag{3.9}
\]
where the time-dependent parameter
\[
\beta(t) = -\frac{\lambda(t)}{2} \exp[-i\gamma(t)], \quad \beta^*(t) = -\frac{\lambda(t)}{2} \exp[i\gamma(t)]. \tag{3.10}
\]

\(V(t)\) can be easily shown to transform the time-dependent invariant \(I(t)\) to \(I_V(t)\) which is time-independent:
\[
I_V \equiv V^\dagger(t)I(t)V(t) = \frac{1}{2} \sigma_3. \tag{3.11}
\]
The eigenstate of the \(I_V = \frac{1}{2}\sigma_3\) corresponding to the eigenvalue \(\sigma\) is denoted by \(|\sigma\rangle\) which is of the form
\[
|\sigma\rangle = V^\dagger(t)|\sigma,t\rangle. \tag{3.12}
\]
By making use of \(V(t)\) in expression (3.9) and the Baker-Campbell-Hausdorff formula [19], one can obtain \(H_V(t)\) from \(H_{s-r}(t)\) [15]
\[
H_V(t) = V^\dagger(t)H_{s-r}(t)V(t) - V^\dagger(t)i\frac{\partial V(t)}{\partial t}
= \frac{1}{2} \{[\cos \lambda \cos \theta + \sin \lambda \sin \theta \cos(\gamma - \varphi)] + \dot{\gamma}(1 - \cos \lambda)\}\sigma_3.
\]
From Eqs.(3.5), it is shown that
\[
\cos \lambda \cos \theta + \sin \lambda \sin \theta \cos(\gamma - \varphi) = 0, \tag{3.13}
\]
thus, the expression (3.9) can be rewritten as
\[
H_V(t) = \frac{1}{2} \dot{\gamma}(t)[1 - \cos \lambda(t)]\sigma_3. \tag{3.14}
\]
Based on (3.8) and (3.14), the geometric phase of the neutron whose eigenvalue of spin is \(\sigma\) can be expressed by
\[
\phi_\sigma(t) = -\frac{1}{2} \{\int_0^t \dot{\gamma}(t')[1 - \cos \lambda(t')] \, dt'\} \langle \sigma | \sigma_3 | \sigma \rangle. \tag{3.15}
\]
Since we know the eigenvalues and eigenstates of \(I_V(t) = \frac{1}{2}\sigma_3\), with the help of (3.7), (3.8), and (3.15), it is easy to get the general solution of the time-dependent Schrödinger equation which governs the neutron spin-rotation coupling is given
\[
|\Psi(t)\rangle_s = \sum_\sigma C_\sigma \exp[i\phi_\sigma(t)]V(t)|\sigma\rangle \tag{3.16}
\]
with the coefficients \(C_\sigma = \langle \sigma,t = 0 | \Psi(0) \rangle_s\).

It follows from the expression (3.13) that the dynamical phase of solutions of Eq. (3.1) vanishes, and the geometric phase is expressed by Eq. (3.15). Since geometric phase appears only in systems whose Hamiltonian is time-dependent or possessing some evolution parameters, this enables us to obtain the information concerning the variation of the Earth’s rotation by measuring the geometric phase difference of spin polarized vertically down and up in the neutron-gravity interferometry experiment.
IV. CONCLUDING REMARKS

This paper obtains the expression for the Hamiltonian of spin-rotation coupling by coordinate transformation of Kerr metric from the fixing reference frame to the rotating reference frame. By making use of the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation, we obtain exact solutions of the time-dependent Schrödinger equation governing the interaction of neutron spin with Earth's rotation. We propose a potential method to investigate the time-varying rotating frequency of the Earth by measuring the phase difference between geometric phases of neutron spin down and up. In view of the above discussions, the invariant-related unitary transformation formulation is a useful tool for treating the geometric phase factor and the time-dependent Schrödinger equation. This formulation replaces the eigenstates of the time-dependent invariants with those of the time-independent invariants through the unitary transformation. Additionally, it should be pointed out that the time-dependent Schrödinger equation is often seen in the literature, whereas the exact solutions of time-dependent Klein-Gordon equation is paid less attention to. Work in this direction is under consideration and will be published elsewhere.

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