AN INTRODUCTION TO PULL-BACKS OF BUNDLES AND HOMOTOPY INVARIANCE

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Abstract. We define the pull-back of a smooth principal fibre bundle, and show that it has a natural principal fibre bundle structure. Next, we analyse the relationship between pull-backs by homotopy equivalent maps. The main result of this article is to show that for a principal fibre bundle over a paracompact manifold, there is a principal fibre bundle isomorphism between pull-backs obtained from homotopic maps. This enables simple proofs of several results on the structure of principal fibre bundles. No new results are obtained—this is simply an accessible presentation of an important idea.

1. The pull-back

In this section we give a brief definition of the pull-back of a principal fibre bundle [1, 2]. For brevity many details are left to the references [3]. Suppose $\xi = G \rightarrow P \xrightarrow{\pi} N$ is a principal fibre bundle with structure group $G$ over the base space $N$. If $f : M \rightarrow N$, we define the pull-back of $\xi$ by $f$ as

$$E = \{(x, p) \in M \times P \mid f(x) = \pi(p)\},$$

$$\pi_E : E \rightarrow M \quad \text{by} \quad \pi_E(x, p) = x,$$

$$f^* \xi = G \rightarrow E \xrightarrow{\pi_E} M.$$  

The right action of $G$ on $E$ is given simply by $(x, p)g = (x, pg)$. The fibre bundle structure is also simple to describe. If $\varphi : \pi^{-1}(U) \rightarrow U \times G$ is a local trivialisation of $\xi$ over $U \subset N$, then let $V = f^{-1}(U) \subset M$, and define $\psi : \pi_E^{-1}(V) \rightarrow V \times G$ by $\psi(x, p) = (x, \varphi_2(p))$, where $\varphi_2$ denotes the projection of $\varphi$ onto the $G$ factor. The condition $f(x) = \pi(p)$ in the definition of the pull-back ensures that this prescription is valid.

2. Homotopies

Let $M$ and $N$ be topological spaces, and $f, g, h : M \rightarrow N$ be continuous maps. We say $f$ and $g$ are homotopic if there exists a map $H : I \times M \rightarrow N$ so $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in M$. The map $H$ is called a homotopy from $f$ to $g$. Then homotopy equivalence is in fact an equivalence relation. Reflexivity is simple, using the homotopy $H_f$ defined by $H_f(t, x) = f(x)$. If $H$ is a homotopy from
f to g, define $H^{-1}$ by $H^{-1}(t, x) = H(1 - t, x)$. This appears unusual notation, in that at this point $H^{-1}$ is not an ‘inverse’ in any obvious sense. This will become clear later. Then $H^{-1}$ is a homotopy from $g$ to $f$, and so homotopy equivalence is reflexive. Transitivity is seen by defining the composition of homotopies. If $H$ is a homotopy from $f$ to $g$, and $K$ is a homotopy from $g$ to $h$, then we can define a homotopy from $f$ to $h$, denoted $K \circ H$, defined by

$$(K \circ H)(t, x) = \begin{cases} H(2t, x) & \text{if } t \in [0, \frac{1}{2}] \\ K(2t - 1, x) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$ 

This is clearly continuous, and thus establishes the homotopy equivalence of $f$ and $h$.

We would now like to specialise to the situation of smooth manifolds and smooth maps between them. We define homotopy equivalence the same way, requiring that the homotopy is also a smooth map. Unfortunately the formula given above for composition of homotopies fails, because there is no guarantee that this map will be smooth. This is because the derivatives may not match up when $t = \frac{1}{2}$. For our purposes, it is possible to avoid this problem by considering a restricted class of homotopies. We define a steady smooth homotopy from $f$ to $g$ to be a map $H : I \times M \to N$ so that, for some $\varepsilon > 0$, $H(t, x) = f(x)$ for all $x \in M$ and $t \in [0, \varepsilon]$, and $H(t, x) = g(x)$ for all $x \in M$ and $t \in [1 - \varepsilon, 1]$. It is easy to see that steady smooth homotopies can be composed to form a steady smooth homotopy. Further, if there is a smooth homotopy from $f$ to $g$, then there is a steady smooth homotopy, by the following prescription. Firstly define $\varphi_0 : I \to I$ by

$$\varphi_0(s) = \begin{cases} \exp \left( \frac{-1}{(s - \frac{1}{3})(s - \frac{2}{3})} \right) & \text{if } s \in \left( \frac{1}{3}, \frac{2}{3} \right) \\ 0 & \text{otherwise} \end{cases},$$

and $\varphi : I \to I$ by $\varphi(t) = \int_0^t \varphi_0(s) ds$. $\varphi_0$ is smooth, and so $\varphi$ is smooth also. Then given a smooth homotopy $H$ from $f$ to $g$, define the steady smooth homotopy $H'$ by $H'(t, x) = H(\varphi(t), x)$. From this result, it is seen that we lose nothing by passing to the steady smooth homotopies, and so homotopy equivalence is in fact an equivalence relation, regardless of whether we use smooth homotopies or steady smooth homotopies.

3. Connections and paracompactness

A paracompact manifold is a manifold with a countable basis for its topology, or, equivalently, a manifold on which there exists a partition of unity subordinate to any open covering. Using the existence of partitions of unity, we can define a connection form on any principal fibre bundle defined over the paracompact manifold. Further, from this connection, we obtain a rule for equivariant path lifting. That is,
given a path in the base space \( \alpha : I \to M \), and a point \( p \in \pi^{-1}(\alpha(0)) \), we can form a lifted path, \( \tilde{\alpha}_p : I \to P \), so \( \pi \circ \tilde{\alpha}_p = \alpha \), and \( \tilde{\alpha}_p(0) = p \). The equivariance of the path lifting is expressed by \( \tilde{\alpha}_p(t)g = \tilde{\alpha}_p(t)g \) for all \( g \in G \). We can use this path lifting to parallel transport a point of the total space. Furthermore, the parallel transport is independent of the parametrisation of the path, as the path lifting is defined as the integral curve of the lifting of the tangent to the path in the base space to the total space using the connection. That is, if \( \alpha \) is a path in \( M \) and \( \beta \) is a reparametrisation of \( \alpha \) such that \( \alpha(0) = \beta(0) \) and \( \alpha(t_0) = \beta(t_1) \), then \( \tilde{\alpha}_p(t_0) = \tilde{\beta}_p(t_1) \). The main result about parallel transport which we will rely on is that when two paths can be composed to give a smooth path, the parallel transports compose, in the sense that \( (\tilde{\alpha} \circ \beta)_p = \tilde{\alpha}_{\tilde{\beta}_p(1)} \circ \tilde{\beta}_p \). This fact is easily obtained from the definition of the path lifting as an integral curve.

4. Bundle morphisms induced by homotopies

4.1. A categorical approach. In this section we prove the main result of this article, that if \( M \) and \( N \) are smooth manifolds, \( \xi \) is a principal fibre bundle over \( N \), and \( f \) and \( g \) are homotopic maps from \( M \) to \( N \), then there is a principal fibre bundle isomorphism between the pull-back bundles, \( f^*\xi \) and \( g^*\xi \). We’ll use several notions from category theory to organise the proof. We fix smooth manifolds \( M \) and \( N \), and a principal fibre bundle \( \xi \) over \( N \) with structure group \( G \). Firstly we consider a category \( H \) whose objects are smooth maps from \( M \) to \( N \), and whose morphisms are steady smooth homotopies, with composition as defined above. We consider a second category \( B \), whose objects are principal fibre bundles over \( M \) with structure group \( G \), and whose morphisms are principal fibre bundle morphisms. Notice that there is a natural map from the objects of \( H \), to the objects of \( B \), given by taking the pull-back of \( \xi \) by the smooth map in question. This motivates our goal of constructing a functor from \( H \) to \( B \), that is, constructing a principal fibre bundle morphism for each homotopy. This construction will be lead primarily by the requirement that we obtain a functor. This motivates the use of connections and path liftings, because of the functorial character of the path lifting with respect to path composition described above. The existence of functor then establishes the final result, as follows. Since a functor takes isomorphisms to isomorphisms, and each homotopy has an inverse (its ‘time reversal’), the principal fibre bundle morphisms obtained are also invertible, and so isomorphisms. Notice that because of the arbitrary choice of the connection, this does not establish a canonical isomorphism between the pull-backs.

4.2. The construction. Let \( E \) be the total space of \( f^*\xi \), and \( F \) be the total space of \( g^*\xi \). From the homotopy \( H \) of \( f \) to \( g \), we want to
construct a map \( \lambda_H : E \to F \). Fix \( e = (x, p) \in E \). Define \( \alpha : I \to N \) by \( \alpha(t) = H(t, x) \). Now according to the definition of the pull-back bundle, \( \pi(p) = \alpha(0) = f(x) \). Thus, using the connection, we can parallel transport \( p \) to \( \tilde{\alpha}_p(1) \), and \( \pi(\tilde{\alpha}_p(1)) = \alpha(1) = g(x) \). Therefore define \( \lambda_H(p) = (x, \tilde{\alpha}_p(1)) \). \( \lambda_H \) as defined is smooth, because the solutions of differential equations, and hence parallel transport, depend smoothly on initial conditions. Further, \( \lambda_H \) clearly respects the fibration of \( E \) and \( F \), and the group action, because \( \lambda_H(pg) = (x, \tilde{\alpha}_p(1)) = (x, \tilde{\alpha}_p(1)g) = \lambda_H(p)g \). Thus \( \lambda_H \) is in fact a principal fibre bundle morphism.

4.3. **Functoriality.** The remaining step is to ensure that the association we have given \( f \leadsto f^*\xi, H \leadsto \lambda_H \) is in fact functorial, in the sense that if \( H \) is a homotopy from \( f \) to \( g \), and \( K \) is a homotopy from \( g \) to \( h \), so \( K \circ H \) is a homotopy from \( f \) to \( h \), then \( \lambda_{K \circ H} = \lambda_K \circ \lambda_H \). To check this, we fix \( e = (x, p) \in E \), and construct \( \alpha^H, \alpha^K \) and \( \alpha^{K \circ H} \) as

\[
\begin{align*}
\alpha^H(t) &= H(t, x) \\
\alpha^K(t) &= K(t, x) \\
\alpha^{K \circ H}(t) &= (K \circ H)(t, x) = \begin{cases} H(2t, x) & \text{if } t \in [0, \frac{1}{2}] \\ K(2t - 1, x) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}
\end{align*}
\]

Thus \( \alpha^{K \circ H} = \alpha^K \circ \alpha^H \), in the usual sense of composition of paths. Then \( \lambda_H(x, p) = (x, \tilde{\alpha}_p^H(1)) \), and

\[
\lambda_K(\lambda_H(x, p)) = \lambda_K(x, \tilde{\alpha}_p^H(1)) = (x, \tilde{\alpha}_p^K(1)) = \lambda_K(\lambda_H(x, p)).
\]

This completes the construction, and established the result. The above result also holds for general fibre bundles.

4.4. **An application.** Our result enables a simple proof of the theorem that principal fibre bundles over contractible base spaces are trivial. Thus, let \( \xi \) be a principal fibre bundle over \( M \), a contractible smooth manifold. Define \( f, g : M \to M \) by \( f(m) = m, g(m) = m_0 \), for some fixed \( m_0 \in M \). Since \( M \) is contractible, \( f \) and \( g \) are homotopic, so the pull-back bundles \( f^*\xi \) and \( g^*\xi \) are isomorphic. However \( f^*\xi \) is simply \( \xi \), and it is not hard to see that \( g^*\xi \) is trivial. This type of theorem can be used to classify bundles. A simple extension of the theorem shows that if the base space factors as \( M = A \times B \), and \( B \) is contractible, then for every point \( m \in M \), then is an open set \( U \subset A \) so \( m \in U \times B \), and \( \xi \) is trivial over \( U \times B \). The following is a deep theorem based on these ideas, which is described in [4]. We denote by \( \mathcal{B}_G(N) \) is set of isomorphism classes of principal \( G \) bundles over the
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base space $N$, and by $[P,Q]$ the set of homotopy classes of maps from $P$ to $Q$.

Theorem 4.1. If $G$ is a group, then there exists a universal bundle $UG = G \hookrightarrow PG \twoheadrightarrow MG$ such that for any smooth manifold $M$ the mapping from $[N, MG]$ to $BG(N)$ described above is one-to-one and onto. That is,

1. If $f, g : N \to MG$, then $f^*(UG)$ and $g^*(UG)$ are isomorphic if and only if $f$ and $g$ are homotopic maps.
2. If $\xi$ is any principal fibre bundle over $N$ with structure group $G$ then there is some map $f : N \to MG$ so $\xi$ is isomorphic to $f^*UG$.

Different universal bundles with the group $G$ necessarily have homotopy equivalent base spaces.\footnote{That is, they have base spaces which are isomorphic in the category whose objects are smooth manifolds, and whose morphisms are homotopy classes of smooth maps.}

This theorem indicates that the problem of classifying all $G$ bundles over $N$ is identical that of finding all homotopy classes of maps from $N$ to $BG$, or, $[N, MG] \cong BG(N)$. Particular examples include the classification result $B_{U(1)}(N) \cong [N, CP^\infty] \cong H^2(N, \mathbb{Z})$, that the $U(1)$ bundles over a manifold are in one-to-one correspondence with the second cohomology (with integer coefficients) classes of $N$. The theory of characteristic classes is also related to this result.

References

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