Ensemble computation approach to the Hough transform

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February 19, 2018

Abstract

It is demonstrated that the classical Hough transform with shift-elevation parametrization of digital straight lines has additive complexity of at most $O(n^3/\log n)$ on an $n \times n$ image. The proof is constructive and uses ensemble computation approach to build summation circuits. The proposed method has similarities with the fast Hough transform (FHT) and may be considered a form of the “divide and conquer” technique. It is based on the fact that lines with close slopes can be decomposed into common components, allowing generalization for other pattern families. When applied to FHT patterns, the algorithm yields exactly the $\Theta(n^2 \log n)$ FHT asymptotics which might suggest that the actual classical Hough transform circuits could smaller size than $\Theta(n^3/\log n)$.

Keywords: Hough transform (HT), fast Hough transform (FHT), additive complexity, ensemble computation, partition tree, summation circuit, digital straight line.

1 Introduction

The Hough transform is a well-known procedure in the area of image processing. It is one of discrete analogues of the integral Radon transform and is widely used for solving numerous tasks, the obvious one being line detection. A good (albeit incomplete) review is given in [1], it shows that a variety of techniques may be meant under this term. Here we concentrate on its simplest and most straightforward form which we shall call the classical Hough transform (in [1] it would be named the “standard Hough transform” (SHT) for the case of straight lines). Supposing that an $n \times n$ image is a numerical function $f = f_{x,y}$ on $\mathbb{Z}^2$ with bounded support $\{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-1\}$, the classical Hough transform $HT(f)$ maps quantizations $L$ of continuous lines to sums

$$HT(f): L \mapsto \sum_{(x,y) \in L} f_{x,y}. \quad (1)$$

Patterns $L$, also called digital straight lines (DSL) [2], are taken from some specific family $\mathcal{L}(n)$, the choice of which defines the particular classical Hough transform type. The total number of all possible DSLs is $\Theta(n^4)$ [3] but in practice $|\mathcal{L}(n)| = \Theta(n^2)$ is sufficient providing dense enough covering of continuous lines. Since the case of mostly vertical lines is symmetrical to the case of mostly horizontal lines, w.l.o.g. one can assume that line slopes belong to the interval $[0,1]$ (the $[-1,0]$ case is also symmetrical) and consider only the lines which are “mostly horizontal inclined
to the right” (i.e. all lines are split into four symmetrical “quadrants”). In this case one of possible \( \mathcal{L}(n) \) choices would be to take all lines of form

\[
L_s^e: \quad y = \left(\frac{e}{n-1}\right)x + s, \quad e = 0, 1, ..., n - 1, \quad s \in \mathbb{Z},
\]

i.e. all lines which pass through pairs of integer points \((0, s)\) and \((n - 1, s + e)\), lying on continuations of the left and right image borders. For otherwise line \((2)\) does not intersect with the image, intercept (or shift) \(s\) can be assumed to be in \((-n, n) \cap \mathbb{Z}\). \(e\) is called elevation. This elevation-intercept parametrization produces \( \Theta(n^2) \) digital straight lines. In fact such splitting of lines into two families is one way of overcoming the issue of slope unboundedness in \((2)\) when the \(x\) axis intersection angle approaches \(\frac{\pi}{2}\) \([1, \text{section 2.1}]\). Another possible approach is to use normal line parametrization \([4]\).

Computing the Hough transform \((1)\) for all patterns \((2)\) would require \( \Theta(n^3) \) additions when a straightforward independent summation along all lines is used. For performance-heavy applications this complexity might be a critical limitation, raising a natural question of reducing the number of binary operations by a careful choice of the summation order. In general form this task is known as the ensemble computation problem \([5]\), it is more familiar with a different formulation in the boolean circuits theory \([6]\). The ensemble computation problem is NP-complete \([5]\) making it extremely hard to devise an optimal algorithm.

One possible workaround is to use a specific approximation to the Hough transform by replacing digital straight lines \((2)\) with a different ensemble of patterns which would allow recursive computation (see fig. 5). The algorithm constructed in this manner is known (ambiguously) as the fast Hough transform (FHT) and was reinvented several times \([7, 8, 9, 10, 11]\). Being extremely convenient for computation, it requires only \( \Theta(n^2 \log n) \) summations, so certain lack of attention to this method is surprising (for example, published in 2015 survey \([1]\) does not mention it).

Using a specific result about boolean linear operators complexity \([6, \text{theorem 3.12}]\) it has recently been proved \([12]\) that neither the classical \((2)\) nor the fast Hough transform can be computed in less than \( \Theta(n^2 \log n) \) additions, but non-trivial \((o(n^3))\) upper bounds were unknown for the classical transform. In this paper we make an improvement in this direction by suggesting a method for building a computational circuit computing the Hough transform in \( O(\frac{n^3}{\log n}) \) additions. The key idea of the complexity estimation is to combine lines with consequent slopes (elevations) from their common subpatterns, then repeat this for lines with slopes differing by 2 and so forth finally arriving to the ultimate single elevation. It was inspired by the FHT algorithm but is more sophisticated. The method constructs a tree consisting of image partitions with each parent node being a common refinement of its children. An interesting fact is that when applied to the FHT patterns, this tree produces exactly the \( \Theta(n^2 \log n) \) circuit with optimal size, suggesting that the complexity of the classical Hough transform computational circuits produces by the proposed algorithm might be smaller than the proven upper bound.

As in the case of the FHT, we consider “cyclic” lines from a single quadrant (mostly horizontal inclined to the right). Their “wrapping over image border” property is convenient because it provides a fixed line length and guarantees that every pattern is a function graph defined on the whole domain \(\{0, 1, ..., n - 1\}\). The general case is of course reduced to this one, see the discussion section.

The rest of the paper is organized as follows. Section 2 introduces basic notations and reproduces several useful common facts, section 3 establishes the ensembles framework (not connected with images) and provides the method for constructing computational circuits, section 4 introduces an important concept of span partitions and investigates its properties, section 5 formally defines the
Hough ensemble and proves the main complexity estimation theorem, while section 6 analyzes the obtained results and suggests a few directions for further research.

2 Notations and useful common facts

We use the following notations for operating with numbers and sets. For $t \in \mathbb{R}$ symbols $\lfloor t \rfloor$ and $\lceil t \rceil$ denote the usual floor and ceiling operations, $\lfloor t \rfloor \overset{\text{def}}{=} \lfloor t + \frac{1}{2} \rfloor$ is rounding to the nearest integer. For $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $n \in \mathbb{Z}$ we denote by $\mod_p(n)$ or $\mod_p n$ the remainder of dividing $n$ by $p$, satisfying condition $0 \leq \mod_p(n) < p$. Symbols $\subset$ and $\supset$ mean proper set inclusion, while $\subseteq$ and $\supseteq$ allow set equality. $A \cup B$ means disjoint union, i.e. $C = A \cup B$ if $C = A \cup B$ and $A \cap B = \emptyset$. Set cardinality is denoted as $|A|$ and $2^A$ is the set of all $A$ subsets.

Function $f : U \rightarrow V$ is an injection if $f(x_1) = f(x_2)$ yields $x_1 = x_2$. Injections have the following important easily verifiable property:

Proposition 2.1. Injective function $f : U \rightarrow V$ preserves set structure on $U$, i.e. for $R, S \subseteq U$ the following statements are true:

1. $f(R) \cap f(S) = f(R \cap S)$, where $\square \in \{\cap, \cup, \setminus\}$.
2. $f(R) \cap f(S) \iff R \cap S$, where $\circ \in \{=, \subset, \supset, \subseteq, \supseteq\}$.
3. $|f(R)| = |R|$.

Any function $f : U \rightarrow V$ is naturally extended to a function $f_1 : 2^U \rightarrow 2^V$ by rules $f_1(P) \overset{\text{def}}{=} \{f(x) \mid x \in P\} \subseteq 2^V$ for $\emptyset \neq P \subseteq U$ and $f_1(\emptyset) \overset{\text{def}}{=} \emptyset$. It is usually clear which set the argument belongs to, so by standard practice we use the same symbol $f$ for both cases. This extension can further be performed for $2^{2^U}$ and so forth, if $f : U \rightarrow V$ is injective then all such extensions are also injective.

We actively use the concept of a function graph, so recall the necessary terms. Any function $f : U \rightarrow V$ induces injective embedding $\hat{f} : U \rightarrow U \times V$ by the rule

$$\hat{f}(x) \overset{\text{def}}{=} (x, f(x)).$$

A graph of function $f$ on a subset $A \subseteq U$ is the set $G = \hat{f}(A) \subseteq A \times V$. Projection $\pi : U \times V \rightarrow U$ is defined as $\pi((u, v)) \overset{\text{def}}{=} u$. Restriction $\hat{f}_0 \overset{\text{def}}{=} \hat{f}|_A : A \rightarrow G$ is a bijection with $\hat{f}_0^{-1} = \pi_0 \overset{\text{def}}{=} \pi|_G$. Where it does not lead to confusion, symbols $\hat{f}$ and $\pi$ are used in place of $\hat{f}_0$ and $\pi_0$. The following property is obvious yet so useful that we formulate it separately:

Proposition 2.2. For two functions $f, g : U \rightarrow V$ and sets $A, B \subseteq U$ let $G = \hat{f}(A) \cap \hat{g}(B)$. Then the following statements are true:

1. $f|_{\pi(G)} = g|_{\pi(G)}$.
2. $G = \hat{f}(\pi(G)) = \hat{g}(\pi(G))$.
3. $\pi(G) = A \cap B \cap X^{f \cdot g}$, where $X^{f \cdot g} = \pi(\hat{f}(U) \cap \hat{g}(U)) = \{x \in U \mid f(x) = g(x)\}$.
3 Ensemble computation

3.1 Ensembles, partitions and combinations

Consider some finite set \( U \) which we shall call domain, its subsets will be called patterns. An ensemble on \( U \) is a non-empty collection \( A \subseteq 2^U \setminus \{\emptyset\} \) of non-empty patterns. Pattern \( C \) is composed (combined) of patterns \( A \) and \( B \) when \( C = A \sqcup B \). Support \( \text{supp} A \overset{\text{def}}{=} \bigcup_{A \in A} A \). Ensemble \( A \) is a \( U \)-partition, domain partition or simply partition, if \( U = \bigsqcup_{A \in A} A \). Partitions have the following obvious property:

**Proposition 3.1.** If \( A \) is a partition and \( B = \bigcup_{A \in \sigma} A \) then \( B = \bigsqcup_{A \in \sigma} A \) and this presentation is unique.

Partition \( A \) is finer than partition \( B \) (and \( B \) is coarser than \( A \)) if any pattern \( A \in A \) is contained in some pattern \( B \in B \) (such \( B \) is unique because \( B \) patterns do not intersect). We also say that \( A \) is a refinement of \( B \) and denote this partial order relation between partitions as \( A \preceq B \) or \( A \to B \) (the latter variant for diagrams). This notation tacitly implies that both ensembles \( A \) and \( B \) are partitions of the appropriate domain (which in this section is \( U \)). Of course, \( \text{supp} A = U \) for any partition \( A \) and \( A \preceq B \) yields \( |A| \geq |B| \). There is one finest partition

\[ U = U^* \overset{\text{def}}{=} \{\{u\} \mid u \in U\}, \]

i.e. \( U \preceq A \) for any partition \( A \), so always

\[ |A| \leq |U| = |U|. \]

Partition refinement is a fragmentation and vice versa:

**Proposition 3.2.** Let \( A \) be a partition. Then ensemble \( B \) is a partition and \( A \preceq B \) iff for any pattern \( B \in B \)

\[ B = \bigsqcup_{A \in \alpha(B)} A, \]

where \( \alpha(B) \overset{\text{def}}{=} \{A \in A \mid A \subseteq B\} \subseteq A \) and

\[ A = \bigsqcup_{B \in B} \alpha(B). \]

**Proof.** Suppose \( B \) is a partition and \( A \preceq B \). Then for any \( A \in A \) there is a unique \( B_A \in B \) such that \( A \subseteq B_A \) and for all other \( B \in B \) \( A \cap B = \emptyset \), so \( \alpha(B_1) \cap \alpha(B_2) = \emptyset \) for \( B_1 \neq B_2 \). Any \( B \in B \) is decomposed as \( B = B \cap U = B \cap \bigsqcup_{A \in A} A = \bigcup_{A \in \alpha(B)} (B \cap A) = \bigsqcup_{A \in \alpha(B)} A \) and since any \( A \in A \) is contained in some \( B \), \( A = \bigcup_{B \in B} \alpha(B) = \bigcup_{B \in B} \alpha(B) \). The converse implication proof is even easier. \( \square \)

Partition \( C \) is a common refinement of partitions \( A \) and \( B \) if \( C \preceq A \) and \( C \preceq B \). Of course, \( U \) is always a common refinement for any two partitions but there is always a unique coarsest common refinement

\[ A \lor B \overset{\text{def}}{=} \{A \cap B \mid A \in A, B \in B\} \setminus \{\emptyset\} \preceq A, B. \]
The $\lor$ operation is obviously associative and commutative and $A \preceq B$ yields $A \lor B = A$. Directly from the definition,

$$|A \lor B| \leq |A| \cdot |B|. \quad (6)$$

We say that ensemble $A$ combines ensemble $B$ ($A, B$ need not necessarily be partitions) if any pattern $B \in B$ can be combined (possibly in multiple ways) using some patterns from $A$: $B = \bigcup_{A \in \sigma(B)} A, \sigma(B) \subseteq A$. We denote this relation between ensembles as $A \triangleright B$. Obviously, $U \triangleright A$ for any ensemble $A$. The combination relation is reflexive and transitive: $A \triangleright A$ and from $A \triangleright B, B \triangleright C$ (for short) follows $A \triangleright C$. However, it is not antisymmetric and thus is not a partial order: $A = \{\{u\}, \{v\}, \{u, v\}\}, B = \{\{u\}, \{v\}\}$ is an example of two ensembles which combine each other yet differ.

If $A \triangleright B$ then for every pattern $B \in B$ we define its combination weight $\omega_A(B)$ with respect to $A$ as the minimal number of binary $\lor$ operations needed to construct $B$ from $A$ patterns:

$$\omega_A(B) \overset{\text{def}}{=} \min_{\sigma \subseteq A, \bigcup_{A \in \sigma} A = B} |\sigma| - 1. \quad (7)$$

Consider a binary combination tree of pattern $B \in B$ with its $n = \omega_A(B) + 1$ leaves being patterns from one of $\sigma_0 \subseteq A$ minimizing the expression above and all levels full except maybe the last one. The depth of such tree is obviously $\lceil \log_2 n \rceil$. If all its non-leaf nodes are associated with the binary $\lor$ operation, such tree represents a possible way of building $B$ using binary disjoint unions and has the minimal depth among all binary trees combining $B$. This justifies defining pattern combination depth as

$$d_A(B) \overset{\text{def}}{=} \lceil \log_2 (\omega_A(B) + 1) \rceil. \quad (8)$$

Combination weight $\omega_A(B)$ and depth $d_A(B)$ of ensemble $B$ with respect to $A$ are

$$\omega_A(B) = |A \triangleright B| \overset{\text{def}}{=} \sum_{B \in B} \omega_A(B),$$

$$d_A(B) \overset{\text{def}}{=} \max_{B \in B} d_A(B).$$

$\omega_A(B)$ is the minimal number of binary $\lor$ operations needed to assemble all $B$ patterns from ensemble $A$ patterns directly, i.e. without composing and reusing intermediate patterns. From combination relation and weight and depth definitions directly follow:

**Proposition 3.3.** Any ensemble $A$ combines itself with $|A \triangleright A| = d_A(A) = 0$.

**Proposition 3.4.** If $A \triangleright B$ then

1. $|A \triangleright B| \leq |B| \cdot (|A| - 1)$.

2. For any $B \in B$ its combination weight $\omega_A(B) \leq \min(|A| - 1, |A \triangleright B|)$.

3. $d_A(B) \leq \lceil \log_2 \min(|A|, |A \triangleright B| + 1) \rceil$.

**Proposition 3.5.** If $A \triangleright B, C \triangleright D$ and $P = A \cup C, Q = B \cup D$ then $P \triangleright Q$ and

1. $|P \triangleright Q| \leq |A \triangleright B| + |C \triangleright D|$.

2. $d_P(Q) \leq \max(d_A(B), d_C(D))$. 


If (a) \( \text{supp } A \cap \text{supp } C = \emptyset \) or (b) \( A = C \) and \( B \cap D = \emptyset \), then these inequalities transform to equalities.

**Corollary 3.6.** If \( A_i, B_i \) are ensembles on domains \( U_i \) and each \( A_i \triangleright B_i \) then for ensembles \( A = \bigcup_i A_i, B = \bigcup_i B_i \) on domain \( U = \bigcup_i U_i \):

1. \( A \triangleright B \).
2. \( |A \triangleright B| \leq \sum_i |A_i \triangleright B_i| \).
3. \( d_A(B) \leq \max_i d_{A_i}(B_i) \).

If (a) domains \( U_i \) do not intersect pairwise or (b) all \( A_i = A \) and \( B = \bigsqcup_i B_i \), then the inequalities transform to equalities.

The refinement partial order is stronger than the combination relation:

**Proposition 3.7.** If \( A \preceq B \) then \( A \succ B \) and \( |A \succ B| = |A| - |B| \).

**Proof.** \( A \succ B \) follows from (1). By proposition 3.1 with \( \alpha(B) \) from proposition 3.2, \( \omega_A(B) = |\alpha(B)| - 1 \) for \( B \in B \), so using (3), \( |A \succ B| = \sum_{B \in B} (|\alpha(B)| - 1) = |A| - |B| \).

Combination and refinement relations as well as the corresponding weights and depths are preserved by injections:

**Proposition 3.8.** If \( f: U \to V \) is an injection then

1. \( A \succ B \) yields \( f(A) \succ f(B) \) with \( |f(A) \succ f(B)| = |A \succ B| \) and \( d_{f(A)}(f(B)) = d_A(B) \).
2. If \( A \) is a \( U \)-partition then \( f(A) \) is an \( f(U) \)-partition and \( A \preceq B \) yields \( f(A) \preceq f(B) \).

For both cases decomposition structure is preserved, i.e. for every \( B \in B \)

\[
f(B) = \bigsqcup_{A \in \gamma} f(A) \leftrightarrow B = \bigsqcup_{A \in \gamma} A.
\]

**Proof.** Follows from proposition 2.1.

Obviously, unions of partitions on non-intersecting domains (such unions are of course disjoint) are again partitions and refinement relation is preserved, so from corollary 3.6 follows

**Proposition 3.9.** If \( A_i, B_i \) are \( U_i \)-partitions each \( A_i \preceq B_i \) and domains \( U_i \) do not intersect pairwise then

1. Ensembles \( A = \bigsqcup_i A_i \) and \( B = \bigsqcup_i B_i \) are \( U \)-partitions with \( U \overset{\text{def}}{=} \bigcup_i U_i \).
2. \( A \preceq B \).
3. \( |A \succ B| = \sum_i |A_i \succ B_i| \).
4. \( d_A(B) = \max_i d_{A_i}(B_i) \).
3.2 Ensemble computation complexity

Computation chain of length \( n \) is a sequence \( \mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}_0 \triangleright \mathcal{C}_1 \triangleright \ldots \triangleright \mathcal{C}_{n-1} \triangleright \mathcal{C}_n \). We say that \( \mathcal{C} \) computes \( \mathcal{A}_n \) from \( \mathcal{A}_0 \) and write \( \mathcal{A}_0 \overset{\mathcal{C}}{\triangleright} \mathcal{A}_n \). Computation chain weight and depth are defined as

\[
\omega(\mathcal{C}) = |\mathcal{A}_0 \overset{\mathcal{C}}{\triangleright} \mathcal{A}_n| \stackrel{\text{def}}{=} |\mathcal{A}_0 \triangleright \ldots \triangleright \mathcal{A}_n| = \sum_{i=0}^{n-1} |\mathcal{A}_i \triangleright \mathcal{A}_{i+1}|, \\
d(\mathcal{C}) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} d(\mathcal{A}_i)(\mathcal{A}_{i+1}).
\]

Let us say that \( \mathcal{A} \) computes \( \mathcal{B} \) and write \( \mathcal{A} \overset{\mathcal{C}}{\triangleright} \mathcal{B} \) with some chain \( \mathcal{C} \).

There is a natural interpretation of the computation relation hence its name. Suppose \( U = \{u_1, u_2, \ldots\} \) and associate each \( u_i \in U \) with a variable containing, for example, integer values. Consider the task of computing \( n \) sums

\[
s_j = \sum_{u_i \in A_j} u_i, \quad j = 1, \ldots, n,
\]

where patterns \( A_j \in \mathcal{A}, \ |\mathcal{A} \| = n \). What is the minimal binary addition operations count needed? We may assume that \( \text{supp} \mathcal{A} = U \) (it does not change operations count because all variables from \( U \setminus \text{supp} \mathcal{A} \) would remain unused). Computing \( (10) \) directly takes \( \sum_{j=1}^{n} (|A_j| - 1) = \omega_U(\mathcal{A}) \) operations which corresponds to the trivial chain \( \mathcal{U} \triangleright \mathcal{A} \), here \( \mathcal{U} = U_\ast \) again as in \( (9) \). If, however, we first once calculate certain \( u_i \) combinations (\( \mathcal{B} \)) and then reuse them, we might get a smaller operations count corresponding to chain \( \mathcal{U} \triangleright \mathcal{B} \triangleright \mathcal{A} \). If we also compute and reuse combinations of \( \mathcal{B} \) patterns, we might reduce additions number even further (\( \mathcal{U} \triangleright \mathcal{B} \triangleright \mathcal{C} \triangleright \mathcal{A} \)). The ultimate question is to find the computation chain \( \mathcal{C} = [\mathcal{U} = \mathcal{A}_1 \triangleright \ldots \triangleright \mathcal{A}_m = \mathcal{A}] \) with minimal \( \omega(\mathcal{C}) \), suggesting the following

**Definition 3.10.** Computation complexity of ensemble \( \mathcal{B} \) with respect to ensemble \( \mathcal{A} \) is the number

\[
\mu(\mathcal{A}, \mathcal{B}) = |\mathcal{A} \overset{\mathcal{C}}{\triangleright} \mathcal{B}| \stackrel{\text{def}}{=} \min_{\mathcal{A} \overset{\mathcal{C}}{\triangleright} \mathcal{B}} \omega(\mathcal{C}).
\]

If \( \mathcal{A} \) is omitted, then we presume that \( \mathcal{A} = (\text{supp} \mathcal{B})_\ast \), i.e. the (“internal”) computation complexity of ensemble \( \mathcal{B} \) is

\[
\mu(\mathcal{B}) \stackrel{\text{def}}{=} |(\text{supp} \mathcal{A})_\ast \overset{\mathcal{C}}{\triangleright} \mathcal{A}|.
\]

**Proposition 3.11.** Suppose \( \mathcal{B} \) is a \( U \)-partition and some ensemble \( \mathcal{A} \triangleright \mathcal{B} \). Then there is a partition \( \mathcal{A}_0 \subseteq \mathcal{A} \) such that \( \mathcal{A}_0 \overset{\mathcal{C}}{\triangleright} \mathcal{B} \) and \( |\mathcal{A}_0 \triangleright \mathcal{B}| = |\mathcal{A} \triangleright \mathcal{B}| \), \( d(\mathcal{A}_0)(\mathcal{B}) = d(\mathcal{B}) \).

**Proof.** According to \( (7) \), for every \( B \in \mathcal{B} \) we can choose \( \alpha(B) \subseteq \mathcal{A} \) such that \( \omega(\mathcal{B}) = |\alpha(B)| - 1 \) and \( B = \bigcup_{A \in \alpha(B)} A \). Since \( \mathcal{B} \) is a partition, \( \mathcal{A}_0 = \bigcup_{B \in \mathcal{B}} \alpha(B) \subseteq \mathcal{A} \) is also a partition and satisfies the desired conditions. \( \square \)

**Proposition 3.12.** If partitions \( \mathcal{A} \overset{\mathcal{C}}{\triangleright} \mathcal{B} \), then \( \mu(\mathcal{A}) = |\mathcal{A}| - |\mathcal{B}| \).

**Proof.** Suppose \( \mu(\mathcal{A}) = |\mathcal{A} \overset{\mathcal{C}}{\triangleright} \mathcal{B}| \) for computation chain \( \mathcal{C} = [\mathcal{A} \triangleright \mathcal{A}_1 \triangleright \ldots \triangleright \mathcal{A}_m = \mathcal{B}] \). We will transform \( \mathcal{C} \) to \( \mathcal{C}' \) without increasing the computation weight so that each \( \mathcal{C}' \) element would be a
partition refining the next element. We start from the last pair \( \mathcal{A}_{m-1} \triangleright \mathcal{A}_m \). By proposition 3.11 there is a partition \( \mathcal{A}'_{m-1} \subseteq \mathcal{A}_{m-1} \) such that \( \mathcal{A}'_{m-1} \triangleright \mathcal{A}_m \) and \( |\mathcal{A}'_{m-1} \triangleright \mathcal{A}_m| = |\mathcal{A}_{m-1} \triangleright \mathcal{A}_m| \).

Obviously, still \( \mathcal{A}_{m-2} \triangleright \mathcal{A}'_{m-1} \) with \( |\mathcal{A}_{m-2} \triangleright \mathcal{A}'_{m-1}| \leq |\mathcal{A}_{m-2} \triangleright \mathcal{A}_{m-1}| \), so the computation chain \( \mathcal{C}_1 \) with \( \mathcal{A}_{m-1} \) changed to \( \mathcal{A}'_{m-1} \) is still valid and its weight is not greater than that of \( \mathcal{C} \). Continuing in this manner by replacing \( \mathcal{A}_{n-2} \) and etc., we will get computation chain \( \mathcal{C}' \) consisting of partitions with \( \omega(\mathcal{C}') \leq \mu_A(\mathcal{B}) \). Since \( \mathcal{C} \) has minimal possible weight, \( \omega(\mathcal{C}') = \mu_A(\mathcal{B}) \) The statement then follows from (9) and proposition 3.7.

3.3 The classical ensemble computation problem and circuits complexity

This section briefly establishes a connection between the the concept of computation complexity introduced in definition 3.10, the classical ensemble computation problem and the concept of additive circuit complexity.

The ensemble computation problem is NP-complete and formulated as follows [5]. Given a collection \( \mathcal{A} \) of \( U \) subsets and some number \( c \in \mathbb{N} \), is there a sequence \( z_1 = x_1 \sqcup y_1, z_2 = x_2 \sqcup y_2, ..., z_n = x_n \sqcup y_c \) of \( n \leq c \) disjoint unions, where each \( x_i, y_i \) is either \( \{u\} \) for some \( u \in U \) or \( z_j \) for some \( j < i \), and for any \( A \in \mathcal{A} \) there is some \( z_j = A \)? In other words, one asks, whether an ensemble of \( \mathcal{A} \) can be obtained from \( \mathcal{U} = \{\{u\} \mid u \in U\} \) using not more than \( c \) disjoint union operations.

An equivalent formulation originates from the circuits theory [13]. Suppose \( \mathcal{U} = \{u_i\} \) represents a set of number-valued variables, \( \mathcal{A} = \{A_j\} \subseteq 2^U \) and we need to compute all sums (10) using a circuit, i.e. an acyclic directed graph which has exactly \( |U| \) fanin-0 nodes \( u_i \), exactly \( |\mathcal{A}| \) fanout-0 nodes \( A_j \) and other nodes representing the + operation (performing summation over its input edges and distributing the result over its output edges). What is the minimal size of such circuit? When restricted to using only fanin-0,1,2 nodes and after defining the circuit size to be the number of its fanin-2 nodes, we come to the classical ensemble computation problem.

Remark 3.13. This is one of several possible circuit complexity definitions. One may also count the number of edges and consider unlimited fanin nodes and use different binary operations set [6].

It is obvious, that if the complexity defined in section 3.2 for ensemble \( \mu(\mathcal{A}) = n \), then the classical ensemble computation problem (and, hence, the corresponding minimal circuit complexity) does not exceed \( n \). Indeed, take the computation chain \( \mathcal{C} = [\mathcal{U} = \mathcal{C}_0 \triangleright \mathcal{C}_1 \triangleright ... \triangleright \mathcal{C}_d = \mathcal{A}] \), such that \( \mathcal{U} \triangleright \mathcal{A} \) and \( \omega(\mathcal{C}) = n \). We should simply transform every segment \( \mathcal{C}_k \triangleright \mathcal{C}_{k+1} \) of the computation chain into a set of \( |\mathcal{C}_{k+1}| \) binary combination trees computing patterns from \( \mathcal{C}_{k+1} \) (see the paragraph right above (8)) and then join them for \( k = 1, 2, ..., d \). The resulting circuit will have exactly \( n \) disjoint union nodes by \( \mu(\mathcal{A}) \) definition.

The reverse is also true. Take the minimal circuit computing \( \mathcal{A} \). We can always transform it so that the distance (number of edges in a connecting path) any output node and all its input (fanin-0) nodes is exactly \( d \), where \( d \) is the maximal such distance in the original circuit (its depth), and that the number of fanin-2 nodes does not change. We can then define ensembles \( \mathcal{C}_k \), consisting of nodes which have distance from the input nodes exactly \( k \). Obviously, \( \mathcal{U} = \mathcal{C}_0 \triangleright \mathcal{C}_1 \triangleright ... \triangleright \mathcal{C}_d = \mathcal{A} \) with the weight equal to the number of fanin-2 nodes.

3.4 Partition trees

Proposition 3.14. Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are \( U \)-partitions. Then

\[
\mu(\mathcal{A} \cup \mathcal{B}) \leq \mu(\mathcal{A}) + \mu(\mathcal{B}) - \mu(\mathcal{A} \vee \mathcal{B}) = |\mathcal{U}| + |\mathcal{A} \vee \mathcal{B}| - |\mathcal{A}| - |\mathcal{B}|.
\]
Proof. Consider computation chain $\mathcal{C} = U \triangleright A \lor B \triangleright A \cup B$. By proposition 3.5

\[ |A \lor B \triangleright A| + |A \lor B \triangleright B| = 2 \cdot |A \lor B| - |A| - |B|, \]

so

\[ \omega(\mathcal{C}) = |U \triangleright A \lor B| + |A \lor B \triangleright A \cup B| \leq |U| - |A \lor B| + 2 \cdot |A \lor B| - |A| - |B| = \mu(A) + \mu(B) - \mu(A \lor B), \]

the last equality follows from proposition 3.12.

Proposition 3.14 shows that the cost of simultaneously computing two partitions $A$ and $B$ is at least by $\mu(A \lor B)$ smaller than the weight of the trivial chain $U \triangleright A \cup B$, so the smaller $|A \lor B|$, the better is the two-step chain $U \triangleright A \lor B \triangleright A \cup B$. Given some thought, it seems obvious. Indeed, we can efficiently compute $A$ and $B$ when they have many patterns with big intersections – we first combine the intersections and then assemble the rest.

Suppose now that our computation target is a union of four partitions: $T = A \cup B \cup C \cup D$. In a similar manner we could arrange $U \leadsto T$ computation in a hierarchical order (recall that notation $A \rightarrow B$ is equivalent to $A \preceq B$) in many ways, for example as in fig. 1. This tree corresponds to

\[ \mu(T) \leq \mu(A) + \mu(B) + \mu(C) + \mu(D) - \mu(A \lor B \lor C \lor D) - \mu(A \lor B) - \mu(C \lor D), \]

or, using cardinalities,

\[ \mu(T) \leq |U| + |A \lor B \lor C \lor D| + |A \lor B| + |C \lor D| - |A| - |B| - |C| - |D|. \]

It means that this particular order is good when each pair $A, B$ and $C, D$ has a lot of well intersecting patterns. In a different case another order might be more efficient. Inserting the $A \lor B \lor C \lor D$ node between $U$ and the rest cannot spoil anything: when $A \lor B \lor C \lor D = U$ (the worst case), the inequality transforms to

\[ \mu(T) \leq 2|U| + |A \lor B| + |C \lor D| - |A| - |B| - |C| - |D|, \]

corresponding to a shorter diagram with $A \lor B \lor C \lor D$ replaced by $U$.

These examples can be generalized using the concept of partition trees. Call $\mathcal{L} \overset{\text{def}}{=} \bigcup_i \mathcal{L}_i$ a partition-union ensemble if all $\mathcal{L}_i$ are partitions. A binary partition tree $\mathcal{T}$ is a binary tree rooted at $U$ where each node is a partition which refines its children. $\mathcal{T}$ computes partition-union ensemble $\mathcal{L}$ if its leaves are exactly $\mathcal{L}_i$, in this case we write $U \leadsto \mathcal{L}$. No additional restrictions on tree structure are imposed — some nodes might have one child and leaves may reside at different levels.

Remark 3.15. Any ensemble $A$ can be extended to a partition-union ensemble by adding $|A|$ complement patterns $U \setminus A$, $A \in A$. This number can be reduced if we group non-intersecting patterns of $A$. 

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Figure 2: Partition tree of depth 4 computing five partitions $L_i$. Nodes of one type are numbered right to left because we build such trees starting from leaves.

Let us define weight of an edge $A \to B$ as $|A \downarrow B| = |A| - |B|$ (proposition 3.7) and the tree weight $\omega(T)$ as the sum of all edges weights:

$$\omega(T) \overset{\text{def}}{=} \sum_{A \to B} (|A| - |B|).$$

Suppose cumulative level ensemble $R_k$ is a union of all partition nodes up to level $k$, so for the example in fig. 2, $R_0 = \mathcal{U}$, $R_1 = \mathcal{U} \cup L_4 \cup D_2$, $R_2 = R_1 \cup D_1 \cup L_3$, $R_3 = R_2 \cup S_0 \cup D_0$, $R_4 = R_3 \cup L_0 \cup L_1 \cup L_2$. The computation chain $c(T)$ associated with tree $T$ is

$$c(T) \overset{\text{def}}{=} [U = R_0 \downarrow R_1 \downarrow \ldots \downarrow R_{d-1} \downarrow L],$$

where $d$ is tree depth (number of edges in a longest path). The chain is valid (indeed a computation chain) by corollary 3.6 applied to every segment $R_{k-1} \downarrow R_k$. The last segment $R_{d-1} \downarrow L$ is valid because $R_{d-1} \supset R_d$ and $L \subseteq R_d$, so $R_{d-1} \downarrow L$ and $|R_{d-1} \downarrow L| = |R_{d-1} \downarrow R_d|$. By this same corollary,

$$\omega(c(T)) \leq \omega(T),$$

implying

$$\mu(L) \leq \omega(T). \tag{14}$$

These facts make the term "$T$ computes $L$" more clear.

We define tree computation depth (not to be confused with ordinary depth $d$) as the depth of its associated chain:

$$d(T) \overset{\text{def}}{=} d(c(T)).$$

The tree computation depth may not necessarily be equal to the depth of the longest chain $\mathcal{U} \downarrow P_1 \downarrow \ldots \downarrow P_d$. There may be a depth-1 branch $\mathcal{U} \downarrow P$ which has greater computational depth than any other branch because one of the patterns in $P$ needs very many $\mathcal{U}$ components to be constructed.

**Proposition 3.16.** If a binary tree $T$ nodes are labeled with numbers and every edge $\alpha \to \beta$ is assigned weight $\beta - \alpha$, then the sum of all edge weights

$$\sum_{\alpha \to \beta} (\alpha - \beta) = \rho + \sum_j \delta_j - \sum_i \lambda_i, \tag{15}$$

where $\rho$ is the root label, $\lambda_i$ are leaves labels and $\delta_j$ are labels of nodes with exactly two children.

**Proof.** For convenience we denote nodes and the corresponding labels by the same symbol. Let us use induction over tree depth $d$. Induction base $d = 0$ is obvious, both sides of the equality are zero. Suppose now it is valid for all trees with depth $d$ and consider a tree with depth $d+1$. If its root $\rho$ has only one child $\rho'$, then we apply the induction hypothesis to subtree rooted at $\rho'$. It has the same
set of leaves and two-children nodes, so the new weight is expressed as \((\rho - \rho') + (\rho' + \sum_j \delta_j - \sum_i \lambda_i)\) which transforms to (15) for the new root.

Suppose now that root \(\rho\) has two children, \(\rho'\) and \(\rho''\) and the corresponding subtrees have leaves and two-children nodes \(\lambda'_k, \delta'_l\) and \(\lambda''_m, \delta''_n\) resp. By induction hypothesis the new weight is

\[
(\rho - \rho') + (\rho' + \sum_l \delta'_l - \sum_k \lambda'_k) + (\rho'' - \rho') + (\rho'' + \sum_n \delta''_n - \sum_m \lambda''_m) = \\
\rho + (\rho + \sum_l \delta'_l + \sum_n \delta''_n) - (\sum_k \lambda'_k + \sum_m \lambda''_m).
\]

After noting that \(\{\lambda_i\} = \{\lambda'_k\} \cup \{\lambda''_m\}\) and \(\{\delta_j\} = \{\rho\} \cup \{\delta'_l\} \cup \{\delta''_n\}\) we get (15).

**Proposition 3.17.** For any binary tree the number of leaves is the number of two-children nodes plus one.

**Proof.** Use the same tree depth induction technique as for for (15).

**Corollary 3.18.** If a binary partition tree \(\mathcal{T}\) computes partition-union ensemble \(\mathcal{L} = \bigcup_i \mathcal{L}_i\) and its nodes (partitions) with exactly two children are \(\mathcal{D}_j\), then

\[
\omega(\mathcal{T}) = |\mathcal{U}| + \sum_j |\mathcal{D}_j| - \sum_i |\mathcal{L}_i| = \sum_i \mu(\mathcal{L}_i) - \sum_j \mu(\mathcal{D}_j).
\]

Due to (14), this equality generalizes (11), (12) and (13).

**Proof.** The first equality directly follows from (15). The second one follows from propositions 3.17 and 3.12.

### 3.5 Algorithms for building trees

Corollary 3.4 naturally suggests a greedy approach for building a binary partition tree (and hence the associated computation chain), see algorithm 1 which computes partition-union ensemble \(\mathcal{L} = \bigcup_{i=0}^{E-1} \mathcal{L}_i\) consisting of \(E\) partitions and hopefully has weight smaller than the trivial chain \(\mathcal{U} \vartriangleright \mathcal{L}\) has.

**Data:** partition-union ensemble \(\mathcal{L} = \bigcup \mathcal{L}_i\)

**Result:** partition tree \(\mathcal{T}\) such that \(\mathcal{U} \overset{\mathcal{T}}{\rightarrow} \mathcal{L}\)

\[
Q := \{\mathcal{L}_0, ..., \mathcal{L}_{E-1}\}
\]

**while** \(|Q| > 1\) **do**

- Take distinct \(A, B \in \mathcal{M}\) with minimal \(|A \lor B|\)
- \(C := A \lor B\)
- \(Q := Q \cup \{C\} \setminus \{A, B\}\)
- Create node \(C\) with edges \(C \rightarrow A\) and \(C \rightarrow B\)

**end**

Create root \(\mathcal{U}\) and edge \(\mathcal{U} \rightarrow A_0\) for the single remaining \(A_0 \in Q\)

**Algorithm 1:** Greedy construction of a partition tree computing \(\mathcal{L}\).
One flaw of this algorithm is a lack of tree depth control. Another flaw is its complexity. The most costly procedure here is the computation of $|Q| - 2$ common refinement cardinalities $|C \lor X|$, $X \in Q$ after removing $A, B$ but before adding $C$. The loop is executed $E - 1$ times, so with a straightforward implementation $\Theta(E^2)$ refinements should be computed in total, which might be too much.

Should this be the case or if some a priori information is known about the initial $L_i$ partitions, a fixed order based on this information might be used. A trivial order would be to arrange the $L_i$ partitions sequentially, refine the consequent pairs and repeat this procedure until a single partition is left, then connect it with $U$. Precisely this method works with the Hough patterns because with the natural elevation numbering consequent lines have small $|A \lor B|$ (see (19)). See algorithm 2 and its sample output in fig. 3. The total number of partition refinements here is $E - 1$ which a lot better than $\Theta(E^2)$.

Algorithm 2: Building a depth $[\log_2 E] + 1$ partition tree computing $L$ by a predefined order.

All $L^k_i$ reside on the same level, let us denote their number as $E_k$ and the number of $L^k_i$ having two children (i.e. created inside the inner loop) as $E^*_k \leq E_k$. In the beginning of every iteration of the outer loop $\text{prev.level.size} = E_{k-1}$ and the inner loop makes $E^*_k$ iterations. The distribution of single-child nodes depends on the binary representation of $E$ but it is guaranteed that for any $k$ it can only be the last node $L^k_i$ (i.e. with the biggest possible $i$).

**Proposition 3.19.** $E_k = \lceil \frac{E_k}{2} \rceil$ and $E^*_k = \lfloor \frac{E_k}{2} \rfloor$.

**Proof.** $E_k$ and $E^*_k$ obviously satisfy recurrence relations $E_{k+1} = \lceil \frac{E_k}{2} \rceil$ and $E^*_{k+1} = \lfloor \frac{E_k}{2} \rfloor$.

For any $R \ni t \geq 0$ holds $\lceil \frac{t}{2} \rceil = \lceil \frac{t}{2} \rceil$. Indeed, case $t \in \mathbb{Z}$ is trivial, for other $t$ use representation $t = \lfloor t \rfloor + \{t\}$, $0 < \{t\} < 1$ and consider cases of $\lfloor t \rfloor$ being odd or even. The first formula follows.
from this fact and the recurrence relation by induction. The same proof works for the second one using the equality $\left\lfloor \frac{t}{2} \right\rfloor = \left\lfloor \frac{t^2}{2} \right\rfloor$.

**Corollary 3.20.** The partition tree constructed by algorithm 2 has depth $\lceil \log_2 E \rceil + 1$.

We can finally formulate the important statement for estimating the Hough patterns complexity.

**Lemma 3.21.** Suppose the cardinalities of one level nodes $L^k_i$ of the partition tree $T$ constructed by algorithm 2 for ensemble $L = \bigcup_{i=0}^{E-1} L_i$ are bounded by a sequence $a_k$:

$$\max_i |L^k_i| \leq a_k, \quad k = 0, 1, ..., \lceil \log_2 E \rceil,$$

Then the tree weight

$$\omega(T) \leq |U| + E \sum_{k=1}^{\lceil \log_2 E \rceil} a_k - \sum_{i=0}^{E-1} |L_i|, \quad (16)$$

and the computational depth

$$d(T) \leq \log_2 |U| + \sum_{k=1}^{\lceil \log_2 E \rceil} \log_2 a_k. \quad (17)$$

**Proof.** The first statement follows from corollary 3.18 and proposition 3.19 after noting that $E^*_{k} \leq \frac{|E|}{2^k}$. The second statement follows from proposition 3.4 applied to all edges $L^k_i \rightarrow L^{k-1}_j$ and corollary 3.6 applied sequentially to consecutive unions of single-level partitions. □

## 4 Ensembles and partitions on images

### 4.1 Image, shifts and spans

*Image* of width $w \in \mathbb{N}$ and height $h \in \mathbb{N}$ is a set $I \overset{\text{def}}{=} X \times Y = \{p_{x,y} \overset{\text{def}}{=} (x,y) \mid x \in X, y \in Y\} \subset \mathbb{R}^2$ of $|I| = w \cdot h$ elements $p_{x,y}$ called *pixels*, where $X \overset{\text{def}}{=} \{0, 1, ..., w-1\}$, $Y \overset{\text{def}}{=} \{0, 1, ..., h-1\}$. We consider numbers $w$ and $h$ fixed. Image subsets are also called *patterns*. Here and further *projection* is a function $\pi: I \rightarrow X, \pi(p_{x,y}) \overset{\text{def}}{=} x$. The finest image partition $I$ of course consists of pixel-singletons: $I \overset{\text{def}}{=} \{\{p\} \mid p \in I\}$.  

![Partition tree constructed by algorithm 2 to compute ensemble $L_0 \cup L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$.](image-url)
Additive commutative group \( \mathbb{Z} \) acts on \( \mathbb{I} \) by (vertical) shifts according to the rule \( p_{x,y} + s \overset{\text{def}}{=} p_{x, \text{mod}_h(y+s)} \) for shift \( s \in \mathbb{Z} \). This action also induces action on patterns:

\[
P + s \overset{\text{def}}{=} \{ p + s \mid p \in P \},
\]
supposing \( \emptyset + s = \emptyset \), and functions \( F: X \to \mathbb{I} \):

\[
(F + s)(x) \overset{\text{def}}{=} F(x) + s.
\]

Projection function \( \pi \) is obviously shift-invariant: for any \( P \) and \( s \in \mathbb{Z} \), \( \pi(P + s) = \pi(P) \). For both cases we denote shift orbit as

\[
\langle z \rangle \overset{\text{def}}{=} \{ z + s \mid s \in \mathbb{Z} \} = \{ z + s \mid s \in Y \}.
\]

Shift span of ensemble \( \mathcal{A} \) on \( \mathbb{I} \) is the set of \( \mathcal{A} \) patterns shifted to all possible positions (i.e. the union of pattern orbits):

\[
[\mathcal{A}] \overset{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \langle A \rangle = \{ A + s \mid A \in \mathcal{A}, s \in Y \}.
\]

Obviously, \([\{ A \}] = \langle A \rangle \) for any pattern \( A \).

**Remark 4.1.** As any group action, shifts on \( \mathbb{I} \) and \( 2^\mathbb{I} \) are bijections, hence injections and proposition 2.1 statements also hold for shifts (with \( f(x) = x + s \)).

### 4.2 Function equality sets

**Definition 4.2.** For any two functions \( f, g: X \to Y \) and \( n \in \mathbb{Z} \), \( n \)-equality set \( X_{n}^{f,g} \) is defined as

\[
X_{n}^{f,g} \overset{\text{def}}{=} \{ x \in X \mid \hat{g}(x) = \hat{f}(x) + n \} = \{ x \in X \mid \text{mod}_h(g(x) - f(x) - n) = 0 \}.
\]

\( f,g \)-equality partition is ensemble

\[
\mathcal{X}^{f,g} \overset{\text{def}}{=} \{ X_{n}^{f,g} \mid n \in \sigma(f,g) \}
\]

with the \( f,g \)-equality index set

\[
\sigma(f,g) \overset{\text{def}}{=} \{ n \in Y \mid X_{n}^{f,g} \neq \emptyset \}.
\]

This definition can be visualized as cutting graph \( \hat{g}(X) \) by consequent slices \( \hat{f}(X), \hat{f}(X) + 1, \hat{f}(X) + 2, \ldots \) and taking projections. By projection shift-invariance property, \( X_{n+s,g+s}^{f} = X_{n}^{f,g} \). The \( \mathcal{X}^{f,g} \) definition is consistent by the following

**Proposition 4.3.** \( \mathcal{X}^{f,g} \) is always an \( X \)-partition, i.e.

\[
X = \bigcup_{n \in \sigma(f,g)} X_{n}^{f,g}.
\]

**Proof.** For any \( x \in X \) \( \hat{g}(x) = \hat{f}(x) + n \) with \( n = \text{mod}_h(g(x) - f(x)) \), so by definition \( x \in X_{n}^{f,g} \) implying \( \text{supp} \mathcal{X}^{f,g} = X \). Suppose now \( x \in X_{n_1}^{f,g} \cap X_{n_2}^{f,g} \), \( n_1, n_2 \in Y \). Then by definition, \( \hat{f}(x) + n_1 = \hat{f}(x) + n_2 \) which is possible only when \( n_1 = n_2 \) if both \( n_1, n_2 \in Y \). \( \square \)
Proposition 4.4. Restrictions \( g|_B = (f + n)|_B \) for any \( B \subseteq X^X \).

Proof. Immediately follows from \( X^X \) definition.

Proposition 4.5. For any set \( A \subseteq X \)

\[ g(A) = \bigcup_{n \in \sigma} f(A \cap X^X) + n. \]

Proof. Follows from decomposition \( g(A) = g(A \cap X) = g(A \cap \bigcup_{n \in \sigma} X^X) = \bigcup_{n \in \sigma} g(A \cap X^X) \) by proposition 4.4, \( \sigma = \sigma(f,g) \).

Corollary 4.6. \( \hat{g}(X) = \bigcup_{n \in \sigma(f,g)} (f(X^X) + n) \).

4.3 Partition spans

We will now investigate how span operation interacts with domain \( X \)-partitions via functions \( X \rightarrow Y \). The first obvious property allows to “span” \( \mathbb{I} \)-partitions from \( X \)-partitions using a graph function:

Proposition 4.7. If \( f : X \rightarrow Y \) and \( A \) is an \( X \)-partition, then

1. \( \hat{f}(A) \) is an \( \mathbb{I} \)-partition.
2. \( \pi(\hat{f}(A)) = A \).
3. \( ||\hat{f}(A)|| = h \cdot |A| \).

\( \hat{f}(A) \) is a partition span or a span partition and \( \hat{f} \) is a spanning function.

Proof. By proposition 3.8, slices \( Q_s = \hat{f}(A) + s, s \in Y \) are partitions in \( Q_s = \text{supp} Q_s = f(X) + s \).

Supports \( Q_s \) obviously do not intersect pairwise, so \( \bigcup_{s \in Y} Q_s = \mathbb{I} \) and statement 1 and 3 follow from proposition 3.9. Statement 2 follows from \( \pi(Q_s) = \pi(\hat{f}(A)) = A \), shift-invariance of projection and \( Q_s = \bigcup_{s \in Y} Q_s \).

Proposition 4.8. If \( f : X \rightarrow Y \) and \( X \)-partitions \( A \preceq B \), then \( \hat{f}(A) \preceq \hat{f}(B) \) and

1. Decomposition structure on every slice is the same as in \( A \preceq B \), i.e. \( B = \bigcup_{A \in \alpha} A \) corresponds to \( \hat{f}(B) + s = \bigcup_{A \in \alpha} \hat{f}(A) + s \) for any shift \( s, \alpha = \alpha(B) \subseteq A \) is the unique \( B \) decomposition from proposition 3.2.
2. \( ||\hat{f}(A)|| > ||\hat{f}(B)|| = h \cdot (|A| - |B|) \).
3. \( d_{\hat{f}(A)}(\hat{f}(B)) = d_A(B) \).

Proof. The fact that \( \hat{f}(B) \) is an \( X \)-partition and the computation weight equality follow from propositions 3.7 and 3.4. Suppose \( P \in \hat{f}(B) \), so \( P = \hat{f}(B) + s \) for some \( B \in B \) and shift \( s \). Since \( A \preceq B \), we decompose \( B = \bigcup_{A \in \alpha} A \), so \( P = \hat{f}(\bigcup_{A \in \alpha} A) + s = \bigcup_{A \in \alpha} (\hat{f}(A) + s) \), where each component is an element of \( \hat{f}(A) \). This proves \( \hat{f}(A) \preceq \hat{f}(B) \) and provides decomposition structure on slices.
Proposition 4.9. For any shifts $s_1, s_2$ and sets $A, B \subseteq X$
\[
(f + s_1)(A) \cap (g + s_2)(B) = (f + s_1)(C) = (g + s_2)(C),
\]
where $C = A \cap B \cap X^{f,g}_{s_1-s_2}$.

Proof. It immediately follows from proposition 2.2 after noticing that $X^{f,g}_{s_1-s_2} = \{x \in X \mid (f + s_1)(x) = (g + s_2)(x)\}$. \qed

Corollary 4.10. For any $X$-partitions $A$ and $B$
\[
[f(A)] \lor [g(B)] = [f(C)] = [g(C)],
\]
where $C = \pi([f(A)] \lor [g(B)]) = A \lor B \lor X^{f,g}$.

This corollary says that the common refinement of two span-partitions is also a span-partition with its spanning function being either of the two spanning functions used. It also shows that to construct this common refinement one may perform a one-dimensional $\lor$ procedure with mixing in an additional component, equality set of the span functions. This is much easier than building a refinement directly in $I$.

The next corollary helps visualize it as well as better understand the nature of $X^{f,g}$:

Corollary 4.11. \[
\langle f(X) \rangle \lor \langle g(X) \rangle = [f(X)] = [g(X)]\]
and \[
X^{f,g} = \pi(\langle f(X) \rangle \lor \langle g(X) \rangle).
\]

5 The Hough ensemble

5.1 Definition and basic properties

We now arrived to the primary target of our research – the Hough patterns. Consider the following base functions $f_e : X \to Y$ with the number $e \in E \overset{\text{def}}{=} \{0, 1, 2, \ldots, |E| - 1\}$ called elevation:
\[
f_e(x) \overset{\text{def}}{=} \text{mod}_h \left[ \frac{e}{w-1} x \right]. \tag{18}
\]
The term “elevation” originates from the fact that without the modulo-operation the $f_e$ graphs would be elevated by $e$ pixels at the image border comparing to the origin, i.e. they would pass through points $(0, 0)$ and $(w - 1, e)$.

Definition 5.1. The Hough ensemble or the ensemble of discrete lines is the partition-union ensemble $L \overset{\text{def}}{=} \{\hat{f}_e(X) \mid e \in E\} = \bigcup_{e \in E} \langle f_e(X) \rangle$.

By proposition 4.7 every \[
L_e \overset{\text{def}}{=} \langle f_e(X) \rangle = [\hat{f}_e(X)]
\]
is an $I$-partition of size $h$, consisting of parallel lines with the same elevation. We say that any pattern $\hat{f}_e(X) + s \in L_e$ is a line with elevation $e$ and shift $s$. If $L_e$ do not intersect pairwise, i.e. all line patterns with various elevations are distinct, then \[
|L| = |E| \cdot h.
\]
This is always the case when $|E| \leq h$ since every value $f_e(w - 1)$ would be unique. Anyway, in all cases $|L| \leq |E| \cdot h$.

Our point of interest is to find a non-trivial bound for $\mu(L)$. The trivial bound follows from $I \triangleright L$ by corollary 3.6 and propositions 3.7, 4.7:

$$\mu(L) \leq |E| \cdot h \cdot (w - 1).$$

To get a better result we will use one nice property of the Hough patterns – as elevations difference decreases, lines similarity grows.

**Proposition 5.2.** For any two elevations $e_1, e_2$

$$|X^{e_1, e_2}| \leq |e_1 - e_2 + 1|,$$

here and further we for short use notations

$$X^{e_1, e_2} \overset{\text{def}}{=} X^{f_{e_1}, f_{e_2}}, \quad X^{e_1, e_2}_n \overset{\text{def}}{=} X^{f_{e_1}, f_{e_2}}_n.$$

**Proof.** Assume that $e_2 > e_1$ (always $|X^{f, g}| = |X^{g, f}|$) and denote linear $X \to \mathbb{R}$ functions $g_e(x) = \frac{e}{w-1}x$, so $f_e(x) = \text{mod}_h [g_e(x)]$. One can easily see that for any $x \in X$

$$g_{e_1}(x) \leq g_{e_2}(x) \leq g_{e_1}(x) + e_2 - e_1.$$

Since $t \leq t'$ yields $[t] \leq [t']$ and $[t + s] = [t] + s$ for $t, t' \in \mathbb{R}, s \in \mathbb{Z}$ we get

$$[g_{e_1}(x)] \leq [g_{e_2}(x)] \leq [g_{e_1}(x)] + e_2 - e_1.$$

This implies that for any $x_0 \in X$, $[g_{e_2}(x_0)] = [g_{e_1}(x_0)] + n(x_0)$ and, hence, $f_{e_2}(x_0) = \text{mod}_h (f_{e_1}(x_0) + n(x_0))$ where $n(x_0) \in \{0, 1, ..., e_2 - e_1\}$, which by definition means that $x_0 \in X^{e_1, e_2}_n$. The proposition statement follows from proposition 4.3 and the fact that $n(x_0)$ takes $e_2 - e_1 + 1$ values. \qed

**Corollary 5.3.** For any elevation $e > 0$ the Hough ensemble has

$$|X^{e-1, e}| \leq 2.$$  \hspace{1cm} (19)

This corollary is illustrated in fig. 4: elevation-5 line (dark circles) is contained in two consecutive elevation-4 lines (variously shaded cells), generating the two-element $X$-partition $X^{4, 5} = \{X_0^{4, 5}, X_1^{4, 5}\}$.

![Figure 4: Equality partition for elevations 4 and 5 on an image of width 15.](image)
5.2 Building the partition tree and $X$ domain reduction

By definition, Hough patterns $L$ is a partition-union ensemble consisting of $I$-partitions $L_e$ and we can apply algorithm 2 to build the corresponding partition tree. Let’s assume that its produced nodes are again denoted as $L^k_i$, $k = 0, 1, \ldots, \lceil \log_2 |E| \rceil$, $i = 0, 1, \ldots, E_k = \left\lfloor \frac{|E|}{2^k} \right\rfloor$.

We say that $L^k_i$ covers elevation $e$ if node $L_e = L^0_i$ resides in the subtree rooted at $L^k_i$. Obviously, for a fixed $k$ nodes $L^k_i$ cover consequent intervals each consisting of $2^k$ consequent numbers, except maybe the last $i = E_k - 1$ which may cover fewer elevations (only one in the worst case for $|E| = 2^n + 1$, $n \in \mathbb{N}$). If we denote the covering set of node $L^k_i$ as $C(k, i)$, then

$$C(k, i) = \{i2^k, i2^k + 1, i2^k + 2, \ldots, i2^k + 2^k - 1\} \cap E.$$  

Of course, for $L^k_i$ having two children, $C(k, i) = C(k - 1, 2i) \cup C(k - 1, 2i + 1)$ and if $L^k_i$ has one child, then $C(k, i) = C(k - 1, 2i)$.

By design of the algorithm and the $\lor$ operation associativity and commutativity,

$$L^k_i = \bigvee_{e \in C(k, i)} L_e = \bigvee_{e \in C(k, i)} \langle l^e(X) \rangle = \bigvee_{e \in C(k, i)} \langle l^e(X) \rangle.$$  

(20)

**Proposition 5.4.** All $L^k_i$ are span-partitions with span functions $l^e$ for any $e \in C(k, i)$:

$$L^k_i = \langle l^e(P^k_i) \rangle,$$

where

$$P^k_i \overset{\text{def}}{=} \pi(L^k_i).$$

**Proof.** It follows from (20) by applying corollary 4.10 multiple times. \hfill \Box

**Proposition 5.5.** For any $L^k_i$ having to children (i.e. when $i < E^k_i$) and any $e_1 \in C(k - 1, 2i)$, $e_2 \in C(k - 1, 2i + 1)$ holds

$$P^k_i = \pi(L^k_i) = P^{k-1}_{2i} \lor P^{k-1}_{2i+1} \lor X^{e_1, e_2}.$$  

(21)

If $L^k_i$ had no children then

$$P^k_i = P^{k-1}_{2i}.$$  

(22)

**Proof.** The second part of the statement is obvious, so let’s concentrate on the first one. By two-children assumption, $L^k_i = L^{k-1}_{2i} \lor L^{k-1}_{2i+1} = [l^{e_1}(P^{k-1}_{2i})] \lor [l^{e_2}(P^{k-1}_{2i+1})]$ for any $e_1 \in C(k - 1, 2i)$, $e_2 \in C(k - 1, 2i + 1)$ (proposition 5.4). The first statement then immediately follows from corollary 4.10. \hfill \Box

This proposition means that domain $X$ projection partitions $P^k_i$ can be retrieved independently of $L^k_i$ by assuming $P^0_e = \{X\}$ for all $e \in E$ and using (21) or (22). $P^k_i$ also form a partition tree with the structure identical to the one generated by algorithm 2. The $L^k_i$ partitions can in turn be obtained from $P^k_i$ using the spanning procedure. It effectively reduces the complex task of getting a common refinement of two $L^k_i$-sets to constructing the common refinement of the two corresponding $P^k_i$ sets with one additional component, an equality set, mixed in. The arbitrary choice of $e_1$ and $e_2$ allows us to get a convenient bound on $|P^k_i|$.

**Proposition 5.6.** For all $k$ and $i$

$$|P^k_i| \leq \min(2^{2k-1}, w).$$  

(23)
Proof. By proposition 5.5, either $\mathcal{P}_i^k = \mathcal{P}_i^{k-1} \lor \mathcal{P}_i^{k-1} \lor \mathcal{P}_i^{e_1,e_2}$, with $e_1 \in C(k-1,2i)$, $e_2 \in C(k-1,2i+1)$, or $\mathcal{P}_i^k = \mathcal{P}_i^{k-1}$. For the first case we may always choose $e_1$ and $e_2$ so that $e_2 = e_1 + 1$, explicitly $e_1 = 2i \cdot 2^{k-1} + 2^{k-1} - 1$ is the greatest element of $C(k-1,2i)$ and $e_2 = e_1 + 1$ is the smallest element of $C(k-1,2i+1)$. With such choice, using (6), we have $|\mathcal{P}_i^k| \leq 2(\max_j |\mathcal{P}_j^{k-1}|)^2$, which is of course also true for the second case $\mathcal{P}_i^k = \mathcal{P}_i^{k-1}$. Since $|\mathcal{P}_i^0| = |\{X\}| = 1$ for all $i$ and $\mathcal{P}_i^{e_1,e_2} \leq 2$ by (9), $|\mathcal{P}_i^k|$ is bounded by sequence $C_k$ defined as $c_0 = 1$, $c_k = 2c_{k-1}^2$. One can verify that $c_k = 2^{2k-1}$. Inequality $|\mathcal{P}_i^k| \leq w$ is trivial, as $\mathcal{P}_i^k$ is an $X$-partition.

Using proposition 5.7, we deduce

Corollary 5.7. For all $k$ and $i$

$$|\mathcal{L}_i^k| \leq h \cdot \min(2^{2k-1}, w).$$

5.3 Complexity bound

Lemma 5.8. For any $n \in \mathbb{N}$

$$\sum_{k=1}^{n} 2^{2k-1} < 2^{2n-1} + 2^{2n-1} + 1.$$  \hspace{1cm} (24)

Proof. We prove this by induction. For $n = 1, 2, 3$ the inequality holds. Suppose that $m > 3$ and it is valid for all $n < m$. After denoting the sum on the left side of (24) as $s_n$, the induction hypothesis implies

$$s_m = s_{m-1} + 2^{2m-1} < 2^{2m-1} + 2^{2m-1} + 2^{2m-1}. $$

Since the second term $2^{2m-1} < 2^{2m-1}$ for $m > 3$ we get

$$s_m < 2^{2m-1} + 2^{2m-1} + 2^{2m-1} = 2^{2m-1} + 2^{2m-1}. $$

\[\square\]

At last we can state the main theorem estimating the Hough ensemble complexity.

Theorem 5.9. The complexity of the Hough ensemble $\mathcal{L}$ produced by $E$ base lines on an image with width $w$ and height $h$ is bounded by the following inequality:

$$\mu(\mathcal{L}) \leq \frac{4whE}{\log_2 w + 1} \left(1 + \sqrt{\frac{2}{w}}\right) + h(w - E).$$  \hspace{1cm} (25)

The corresponding computation chain has depth at most $\log_2 w \cdot (\lceil \log_2 E \rceil + 1)$.

Proof. Suppose we have built the partition tree computing $\mathcal{L}$ as described in section 5.2. Because of (14) it is enough to estimate the weight of the constructed tree $\mathcal{T}$. Denote $t = \log_2(\log_2 w + 1)$ and let $k_0 = \lceil t \rceil - 1$, in this case $t - 1 \leq k_0 < t$. Taking into account that for the Hough ensemble $|\mathcal{L}_i| = h$ and $|\mathcal{T}| = w \cdot h$, using corollary 5.7 we can rewrite (16) as

$$\omega(\mathcal{T}) \leq w \cdot h + E \cdot h\left(\sum_{k=1}^{k_0} \frac{2^{2k-1}}{2^k} + \sum_{k=k_0+1}^{\lceil \log_2 E \rceil} \frac{w}{2^k} - E \cdot h.\right) \hspace{1cm} (26)$$
Using (24) the first sum is bounded by

\[
2^{2k_0-k_0-1} + 2^{2k_0-1-k_0+1} < 2^{2t-(t-1)-1} + 2^{2t-1-(t-1)+1} = \frac{2^{2t}}{2t} + \frac{4 \cdot 2^{2t-1}}{2t} = \frac{2w + 4\sqrt{2w}}{\log_2 w + 1}.
\]

The second sum is bounded by

\[
\sum_{k=k_0+1}^{\infty} \frac{w}{2^k} = \frac{w}{2^{k_0}} \leq \frac{w}{2^{t-1}} = \frac{2w}{\log_2 w + 1}.
\]

After substitution of these terms to (26) we obtain (25).

To prove the depth inequality note that by proposition 4.8 depth \(d_{\mathcal{L}_1}(\mathcal{L}_{2i}^{k-1}) = d_{\mathcal{P}_i}(\mathcal{P}_{2i}^{k-1})\) (same for \(\mathcal{L}_{2i+1}^{k-1}\)). The bound then follows from (17) for the \(\mathcal{P}_i^{k}\) partition tree (obviously \(|\mathcal{P}_i^{k}| \leq w\)).

**Corollary 5.10.** On a square \(n \times n\) image \((n > 1)\) the Hough ensemble \(\mathcal{L}\) generated from \(n\) base lines has complexity

\[
\mu(\mathcal{L}) < \frac{4n^3}{\log_2 n} \left(1 - \frac{1}{\log_2 n + 1} + \sqrt{\frac{2}{n}}\right).
\]

(27)

The corresponding computation chain has depth at most \(\log_2 n \cdot (\log_2 n + 1)\).

**Remark 5.11.** Inequalities (25) and (27) hold for any span generated patterns ensemble satisfying (19), because we used only this property to prove the theorem. In fact a weaker condition like \(|X^{e,e+1}| < C\) would also work with the same asymptotics but a different constant.

### 6 Discussion

In the introduction we initially defined the classical Hough transform not for “cyclic” (wrapping over image border) lines (18) but for digital line segments (2). The latter case can always be reduced to the former one by extending the original image vertically (for mostly horizontal lines) and padding it with zero values which changes the computation asymptotics at most by a fixed factor. One can also then prune the redundant (constant zero) input pixels and all the nodes which depend only on them from the generated computation circuit. This justifies the “cyclic” \(\mod_b(\cdot)\) approach used in sections 4 and 5.

The Hough ensemble complexity bound in theorem 5.9 is constructive as its proof uses an explicit binary partition tree and thus the associated computation chain which is straightforwardly transformed into a computation circuit (section 3.3). Of course, in practice one should not try to directly compute \(\mathcal{L}_{2i}^{k-1} \lor \mathcal{L}_{2i+1}^{k-1}\) in algorithm 2 but rather use recurrence relation (21) for the \(X\)-domain.

It is worth noting that applying the proposed algorithm to the fast Hough transform patterns would produce the \(n^2 \log_2 n\) asymptotics – precisely the FHT algorithm complexity! Indeed, consequent FHT patterns (fig. 5) for \(w = h = E = n = 2^m\) overlap either on \([0, \frac{n}{2})\) or on \([\frac{n}{2}, n)\), blocks of patterns with four consequent elevations overlap on segments \([0, \frac{2n}{4}), \left[\frac{2n}{4}, \frac{3n}{4}\right), \left[\frac{3n}{4}, \frac{4n}{4}\right)\) or \([\frac{4n}{4}, n)\) and so on, so \(|\mathcal{P}_i^{k}| = 2^k\), which by (16) in the same manner as (26) gives

\[
\omega(\mathcal{X}) \leq n^2 + n^2 \sum_{k=1}^{m} \frac{2^k}{2^k} - n^2 = n^2 \log_2 n.
\]
Figure 5: Pairs of FHT patterns with consecutive elevations: 0-1, 2-3, 4-5, 6-7 (top to bottom, left to right). Vertical lines help distinguish common components. Image width is 8.

It gives hope that in practice Hough ensemble circuits might yield a much better result than (25). Indeed, the cardinality bound in (23) is very rough, in practice the \( P^k \) sets might grow by far not that fast. Performing the necessary computational experiments as well as assessing the number of operations and memory requirements of algorithm (2) applied to the Hough ensemble is the plan of our next research.

Another area of interest is generalizing or modifying the suggested approach to handle shift-invariant ensembles consisting of patterns which are not function graphs so their pattern orbits do not per se produce image partitioning.

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