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Schwinger Bosons Approaches to Quantum Antiferromagnetism

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1.1 SU(N) Heisenberg Models

The use of large N approximations to treat strongly interacting quantum systems been very extensive in the last decade. The approach originated in elementary particles theory, but has found many applications in condensed matter physics. Initially, the large N expansion was developed for the Kondo and Anderson models of magnetic impurities in metals. Soon thereafter it was extended to the Kondo and Anderson lattice models for mixed valence and heavy fermions phenomena in rare earth compounds [1, 2].

In these notes we shall formulate and apply the large N approach to the quantum Heisenberg model [3–6]. This method provides an additional avenue to the static and dynamical correlations of quantum magnets. The mean field theories derived below can describe both ordered and disordered phases, at zero and at finite temperatures, and they complement the semiclassical approaches.

Generally speaking, the parameter N labels an internal SU(N) symmetry at each lattice site (i.e., the number of “flavors” a Schwinger boson or a constrained fermion can have). In most cases, the large N approximation has been applied to treat spin Hamiltonians, where the symmetry is SU(2), and N is therefore not a truly large parameter. Nevertheless, the 1/N expansion provides an easy method for obtaining simple mean field theories. These have been found to be surprisingly successful as well.

The large N approach handles strong local interactions in terms of constraints. It is not a perturbative expansion in the size of the interactions but rather a saddle point expansion which usually preserves the spin symmetry of the Hamiltonian. The Hamiltonians are written as a sum of terms $O_{ij}O_{ij}^\dagger$, which are biquadratic in the Schwinger boson creation and annihilation opera-
tors, on each bond on the lattice. This sets up a natural mean field decoupling scheme using one complex Hubbard Stratonovich field per bond.

At the mean field level, the constraints are enforced only on average. Their effects are systematically reintroduced by the higher-order corrections in \(1/N\).

It turns out that different large \(N\) generalizations are suitable for different Heisenberg models, depending on the sign of couplings, spin size, and lattice. Below, we describe two large \(N\) generalizations of the Heisenberg antiferromagnet.

### 1.2 Schwinger Representation of SU(\(N\)) Antiferromagnets

The SU(2) algebra is defined by the familiar relations \([S^\alpha, S^\beta] = i\epsilon_{\alpha\beta\gamma}S^\gamma\). The spin operators commute on different sites, and admit a bosonic representation. Since the spectrum of a bosonic oscillator includes an infinite tower of states, a constraint is required in order to limit the local Hilbert space dimension to \(2S + 1\). In the Holstein-Primakoff representation, one utilizes a single boson \(h\), writing \(S^+ = h\sqrt{2S - h^\dagger h}, S^- = (S^+)^\dagger\), and \(S^z = h - S\), together with the non-holonomic constraint \(0 \leq h^\dagger h \leq 2S\). The square roots prove inconvenient, and practically one must expand them as a power series in \(h^\dagger h/2S\). This generates the so-called spin-wave expansion.

Another representation, due to Schwinger, makes use of two bosons, \(a\) and \(b\). We write

\[
S^+ = a^\dagger b \quad S^- = b^\dagger a \quad S^z = \frac{1}{2}(a^\dagger a - b^\dagger b),
\]

along with the holonomic constraint,

\[
a^\dagger a + b^\dagger b = 2S,
\]

where the boson occupation, \(2S\), is an integer which determines the representation of SU(2). This scheme is depicted graphically in fig. 1.1.

There are three significant virtues of the Schwinger representation. The first is that there are no square roots to expand. The second is that the holonomic constraint (1.2) can be elegantly treated using a Lagrange multiplier. The third is that it admits a straightforward and simple generalization to SU(\(N\)). That generalization involves adding additional boson oscillators – \(N\) in all for SU(\(N\)) – which we write as \(b_\mu\) with \(\mu = 1, \ldots, N\). The generators of SU(\(N\)) are then

\[
S_{\mu\nu} = b_\mu^\dagger h_\nu .
\]

These satisfy the SU(\(N\)) commutation relations

\[
[S_{\mu\nu}, S_{\mu'\nu'}] = S_{\mu\nu'} \delta_{\mu'\nu} - S_{\mu'\nu'} \delta_{\mu\nu'} .
\]

The constraint is then
which specifies the representation of SU(N). The corresponding Young tableau is one with $n_b$ boxes in a single row.

### 1.2.1 Bipartite Antiferromagnet

We consider the case of nearest neighbor SU(2) antiferromagnet, with interaction strength $J > 0$, on a bipartite lattice with sublattices $A$ and $B$. A bond $\langle ij \rangle$ is defined such that $i \in A$ and $j \in B$. The antiferromagnetic bond operator is defined as

$$A_{ij} = a_i b_j - b_i a_j.$$  \hfill (1.6)

This is antisymmetric under interchange of the site indices $i$ and $j$, and transforms as a singlet under a global SU(2) rotation.

Consider now a rotation by $\pi$ about the $y$ axis on sublattice $B$ only, which sends

$$a_j \rightarrow -b_j, \quad b_j \rightarrow a_j.$$  \hfill (1.7)

This is a canonical transformation which preserves the constraint (1.5). The antiferromagnetic bond operator takes the form

$$A_{ij} \rightarrow a_i a_j + b_i b_j.$$  \hfill (1.8)
The \( \text{SU}(2) \) Heisenberg model is written in the form
\[
\mathcal{H} = J \sum_{\langle ij \rangle} S_i \cdot S_j
\]
\[
= -\frac{J}{2} \sum_{\langle ij \rangle} \left( A_{ij}^\dagger A_{ij} - 2S^2 \right) .
\]
(1.9)

The extension to \( \text{SU}(N) \) for \( N > 2 \) is straightforward. With \( N \) species of bosons, (1.8) generalizes to
\[
A_{ij} = \sum_{\mu=1}^{N} b_{i\mu} b_{j\mu} .
\]
(1.10)

The nearest-neighbor \( \text{SU}(N) \) antiferromagnetic Heisenberg model is then
\[
\mathcal{H} = -\frac{J}{N} \sum_{\langle ij \rangle} \left( A_{ij}^\dagger A_{ij} - NS^2 \right)
\]
\[
= \frac{J}{N} \sum_{\langle ij \rangle} \left( \sum_{\mu,\nu} S^{\mu\nu}_i \tilde{S}^{\nu\mu}_j - NS^2 \right) ,
\]
(1.11)
where
\[
\tilde{S}^{\mu\nu}_j = -b_{j\nu}^\dagger b_{j\mu}.
\]
(1.12)
are the generators of the conjugate representation on sublattice \( B \). One should note that \( \mathcal{H} \) of (1.11) is not invariant under uniform \( \text{SU}(N) \) transformations \( U \) but only under staggered conjugate rotations \( U \) and \( U^\dagger \) on sublattices \( A \) and \( B \), respectively.

1.2.2 Non-bipartite (Frustrated) Antiferromagnets

For the group \( \text{SU}(2) \), one can always form a singlet from two sites in an identical spin-\( S \) representation. That is, the tensor product of two spin-\( S \) states always contains a singlet:
\[
S \otimes S = 0 \oplus 1 \oplus \cdots \oplus 2S .
\]
(1.13)

For \( \text{SU}(N) \) this is no longer the case. For example, for two \( \text{SU}(3) \) sites in the fundamental representation, one has \( 3 \otimes 3 = \overline{3} \oplus 6 \). One needs three constituents to make an \( \text{SU}(3) \) singlet, as with color singlets in QCD, and \( N \) constituents in the case of \( \text{SU}(N) \). This is why, in the case of the antiferromagnet, one chooses the conjugate representation on the \( B \) sublattice – the product of a representation and its conjugate always contains a singlet.

But what does one do if the lattice is not bipartite? This situation was addressed by Read and Sachdev [7], who extended the Schwinger boson theory
to the group $\text{Sp}(N)$. This amounts to generalizing the link operator $A_{ij}$ in (1.6) to include a flavor index $m$:

$$A_{ij} = \sum_{m=1}^{N} (a_{im} b_{jm} - b_{im} a_{jm})$$

$$\equiv \sum_{\alpha, \alpha' = 1}^{2N} A_{\alpha \alpha'} b_{i \alpha} b_{j \alpha'} .$$

(1.14)

Here, the indices $\alpha$ and $\alpha'$ run from 1 to $2N$. They may be written in composite form as $\alpha \rightarrow (m, \mu)$, where $m$ runs from 1 to $N$ and $\mu$ from 1 to 2 (or $\uparrow$ and $\downarrow$).

In this case, on each site one has $b_{m \uparrow} = a_{m \uparrow}$ and $b_{m \downarrow} = b_{m \downarrow}$. The matrix $\Lambda_{\alpha \alpha'}$ is then defined by $\Lambda_{m \mu, n \nu} = \delta_{mn} \epsilon_{\mu \nu}$, where $\epsilon_{\mu \nu} = i \sigma^y_{\mu \nu}$ is the rank two antisymmetric tensor.

If we make a global transformation on the Schwinger bosons, with $b_{i \alpha} \rightarrow U_{\alpha \alpha'} b_{i \alpha'}$, then we find

$$A_{ij} \rightarrow (U^t \Lambda U)_{\alpha \alpha'} b_{i \alpha} b_{j \alpha'} .$$

(1.15)

Thus, the link operators remain invariant under the class of complex transformations which satisfy $U^t \Lambda U = \Lambda$. This is the definition of the group $\text{Sp}(2N, \mathbb{C})$. If we further demand that $U \in \text{U}(2N)$, which is necessary if the group operations are to commute with the total occupancy constraint, we arrive at the group

$$\text{Sp}(N) = \text{Sp}(2N, \mathbb{C}) \cap \text{U}(2N) .$$

(1.16)

For $N = 1$ one has $\text{Sp}(1) \simeq \text{SU}(2)$. The particular representation is again specified by the local boson occupation, $n_\alpha = \sum_\alpha b^\dagger_{i \alpha} b_{i \alpha}$. The Hamiltonian is

$$\mathcal{H} = -\frac{1}{2N} \sum_{i < j} J_{ij} A^{\dagger}_{ij} A_{ij} .$$

(1.17)

Here, we have allowed for further neighbor couplings, which can be used to introduce frustration in the square lattice antiferromagnet, e.g. the $J_1 - J_2 - J_3$ model [7]. For each distinct coupling $J_{ij}$ (assumed translationally invariant), a new Hubbard-Stratonovich decomposition is required.

One can also retain the definition in (1.10) even for frustrated lattices. In this case, under a global transformation $b_{i \mu} \rightarrow U_{\mu \nu} b_{i \nu}$, the link operator $A_{ij}$ transforms as $A_{ij} \rightarrow (U^t U)_{\mu \nu} b_{i \mu} b_{j \nu}$, and invariance of $A_{ij}$ requires $U^t U = 1$. This symmetry is that of the complex orthogonal group $\text{O}(N, \mathbb{C})$. Once again, we require $U \in \text{SU}(N)$ so that the constraint equation remains invariant. We then arrive at the real orthogonal group $\text{O}(N) = \text{O}(N, \mathbb{C}) \cap \text{SU}(N)$. For $N = 2$, in terms of the original spin operators, we have

$$-\frac{1}{N} A^{\dagger}_{ij} A_{ij} = \frac{1}{N} b^\dagger_{i \mu} b_{i \nu} b^\dagger_{j \mu} b_{j \nu}$$

$$= -\left( S^x_i S^x_j - S^y_i S^y_j + S^z_i S^z_j + S^2 \right) \quad (N = 2) .$$

(1.18)
On a bipartite lattice, one can rotate by $\pi$ about the $y$-axis on the $B$ sublattice, which recovers the isotropic Heisenberg interaction $S_i \cdot S_j$. On non-bipartite lattices, the $N = 2$ case does not correspond to any isotropic $SU(2)$ model, and so one loses contact with the original problem.

### 1.3 Mean Field Hamiltonian

Within a functional integral approach, one introduces a single real field $\lambda_i(\tau)$ on each site to enforce the occupancy constraint, and a complex Hubbard-Stratonovich field $Q_{ij}(\tau)$ on each link to decouple the interaction. At the mean field level it is assumed that these fields are static. This results in the mean field Hamiltonian

$$H_{\text{MF}} = \frac{pN}{J} \sum_{i<j} |Q_{ij}|^2 + \sum_{i<j} (Q_{ij} A^\dagger_{ij} + Q^*_ij A_{ij})$$

$$+ \sum_i \lambda_i (b^\dagger_{i\alpha} b_{i\alpha} - n_b) + (VN)^{-1/2} \sum_{i\alpha} (\phi^*_{i\alpha} b_{i\alpha} + \phi_{i\alpha} b^\dagger_{i\alpha}) ,$$

where $V$, the volume, is the number of Bravais lattice sites, and where $\alpha$ runs from 1 to $N$ for the $SU(N)$ models (for which $p = 1$), and from 1 to $2N$ for the $Sp(N)$ models (for which $p = 2$). The field $\phi_{i\alpha}$, which couples linearly to the Schwinger bosons, is conjugate to the condensate parameter $\langle b_{i\alpha}^\dagger \rangle$, which means

$$\frac{\partial F}{\partial \phi^*_{i\alpha}} = \langle b_{i\alpha} \rangle \sqrt{\frac{V}{N}} .$$

Let us further assume that the mean field solution has the symmetry of the underlying lattice, and that the interactions are only between nearest neighbor sites on a Bravais lattice. Then, after Fourier transforming, we have

$$H_{\text{MF}} = VN \left( \frac{p^2}{2J} |Q|^2 - \frac{n_b}{N} \lambda \right) + \frac{z}{2} \sum_{k,\alpha,\alpha'} \left[ Q \xi_k K_{\alpha\alpha'} b_{k,\alpha}^\dagger b^\dagger_{-k,\alpha'} + Q^* \xi_k^* K_{\alpha\alpha'} b_{k,\alpha} b_{-k,\alpha'} \right]$$

$$+ \lambda \sum_{k,\alpha} b^\dagger_{k,\alpha} b_{k,\alpha} + (VN)^{-1/2} \sum_{k,\alpha} (\phi^*_{k,\alpha} b_{k,\alpha} + \phi_{k,\alpha} b^\dagger_{k,\alpha}) ,$$

where $z$ is the lattice coordination number, and where $K_{\alpha\alpha'} = \delta_{\alpha\alpha'}$ for $SU(N)$ and $K_{\alpha\alpha'} = A_{\alpha\alpha'}$ for $Sp(N)$. We define

$$\xi_k = \frac{2}{z} \sum_{\delta} \eta_{\delta} e^{i k \cdot \delta} ,$$

where the sum is over all distinct nearest neighbor vectors in a unit cell. That is, $-\delta$ is not included in the sum. The quantity $\eta_{\delta} = \pm 1$ is a sign about which we shall have more to say presently. On the square lattice, for example,
\[ \xi_k = \eta_x e^{ik_x} + \eta_y e^{ik_y}. \]

For symmetric \(K_{ab}\), owing to the sum on \(k\), we can replace \(\xi_k\) with its real part, while for antisymmetric \(K_{ab}\) we must replace it with \(i\) times its imaginary part. We therefore define

\[ \gamma_k = \frac{2}{z} \sum_{\delta} \eta_{\delta} \cos(k \cdot \delta) \] if \(K = K^t\)

\[ = \frac{2i}{z} \sum_{\delta} \eta_{\delta} \sin(k \cdot \delta) \] if \(K = -K^t\)

The sign \(\eta_{\delta}\) is irrelevant on bipartite lattices, since it can be set to unity for all \(\delta\) simply by choosing an appropriate center for the Brillouin zone. But on frustrated lattices, the signs matter.

It is now quite simple to integrate out the Schwinger bosons. After we do so, we make a Legendre transformation to replace the field \(\phi_{i\alpha}\) with the order parameter \(\beta_{i\alpha} = \langle b_{i\alpha} \rangle / \sqrt{V N}\), by writing

\[ G = F - \sum_{i\alpha} \left( \phi_{i\alpha} \beta_{i\alpha}^* + \phi_{i\alpha}^* \beta_{i\alpha} \right). \]

The final form of the free energy per site, per flavor, is

\[ g = \frac{G}{VN} = \frac{p^2}{2J} |Q|^2 - (\kappa + \frac{\eta}{2}) \lambda + p \int_{BZ} \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{2} \omega_k + T \ln \left( 1 - e^{-\omega_k/T} \right) \right] + E_{con}, \]

where \(\kappa = n_b/N\), and \(E_{con}\) is the condensation energy,

\[ E_{con} = \lambda \sum_{k,\alpha} |\beta_{k,\alpha}|^2 + \frac{1}{2} z \sum_{k,\alpha,\alpha'} K_{\alpha\alpha'} \left( Q \gamma_k \beta_{k,\alpha}^* \beta_{-k,\alpha'}^* + Q^* \gamma_k^* \beta_{k,\alpha} \beta_{-k,\alpha'} \right). \]

The dispersion is given by

\[ \omega_k = \sqrt{\lambda^2 - |zQ\gamma_k|^2}. \]

The fact that \(g\) is formally of order \(N^0\) (assuming \(\kappa\) is as well) allows one to generate a systematic expansion of the free energy in powers of \(1/N\).

### 1.3.1 Mean Field Equations

The mean field equations are obtained by extremizing the free energy \(G\) with respect to the parameters \(\lambda, Q,\) and \(\beta_{k,\alpha}^*\). Thus,

\[ \kappa + \frac{\eta}{2} = p \int_{BZ} \frac{d^d k}{(2\pi)^d} \frac{\lambda}{\omega_k} \left( n_k(T) + \frac{1}{2} \right) + \sum_{k,\alpha} |\beta_{k,\alpha}|^2 \]

\[ \omega_k = \sqrt{\lambda^2 - |zQ\gamma_k|^2}. \]
\[ \frac{p^z}{|Q|^2} \int_{BZ} d^d k (2\pi)^d \frac{|zQ\gamma_k|^2}{\omega_k} (n_k(T) + \frac{1}{2}) + \lambda \sum_{k,\alpha} |\beta_{k,\alpha}|^2 = 0 = \lambda \beta_{k,\alpha} + zQ\gamma_k \sum_{\alpha'} K_{\alpha\alpha'} \beta_{-k,\alpha'}^* \cdot \] (1.31)

Here, \( n_k(T) = (e^{\omega_k/T} - 1)^{-1} \) is the thermal Bose occupancy function. In deriving the second of the above mean field equations, we have also invoked the third. Assuming that the condensate occurs at a single wavevector \( k \), the last equation requires that \( \omega_k = 0 \) at the ordering wavevector, ensuring gaplessness of the excitation spectrum. When there is no condensate, \( \beta_{k,\alpha} = 0 \) for all \( k \) and \( \alpha \).

It is instructive to compute \( \langle S_{\alpha\alpha'} \rangle = \langle b^\dagger_{R,\alpha} b_{R',\alpha'} - \kappa \delta_{\alpha\alpha'} \rangle \), which serves as the local order parameter. After invoking the mean field equations, one finds

\[ \langle S_{\alpha\alpha'} \rangle = N \sum_{k, k'} e^{i(k' - k) \cdot R} \beta_{k,\alpha}^* \beta_{k',\alpha'} - \sum_{k, \alpha'} |\beta_{k,\alpha'}|^2 \delta_{\alpha\alpha'} . \] (1.33)

Note that the trace of the above expression vanishes on average (i.e. upon summing over \( R \)), and vanishes locally provided that the condensate satisfies the orthogonality condition

\[ \sum_{\alpha} \beta_{k,\alpha}^* \beta_{k',\alpha} = \delta_{kk'} \sum_{\alpha} |\beta_{k,\alpha}|^2 . \] (1.34)

In the case of an SU(\( N \)) antiferromagnet on a (bipartite) hypercubic lattice, the condensate occurs only at the zone center \( k = 0 \) and the zone corner \( k = \pi \). One then has

\[ \langle S_{\alpha\alpha'} \rangle = N (\beta_{0,\alpha}^* \beta_{\pi,\alpha'} + \beta_{\pi,\alpha}^* \beta_{0,\alpha'}) e^{i\pi \cdot R} . \] (1.35)

Thus, Bose condensation of the Schwinger bosons is equivalent to long-ranged magnetic order.

At \( T = 0 \), there is a critical value of \( \kappa \) above which condensation occurs. To find this value, we invoke all three equations, but set the condensate fraction to zero. For the SU(\( N \)) models, the minimum of the dispersion occurs at the zone center, \( k = 0 \). Setting \( \omega_{k=0} = 0 \), we obtain the relation \( \lambda = z|Q| \). The first equation then yields

\[ \kappa_c = \frac{1}{2} \int_{BZ} \frac{d^d k}{(2\pi)^d} \left( 1 - |\gamma_k|^2 \right)^{-1/2} - \frac{1}{2} . \] (1.36)

For \( d = 1 \), there is no solution, and there is never a condensate. For \( d = 2 \), one finds \( \kappa_c = 0.19 \) on the square lattice [3]. Since \( \kappa = S \) for the SU(2) case, this suggests that even the minimal \( S = \frac{1}{2} \) model is Néel ordered on the square lattice, a result which is in agreement with quantum Monte Carlo studies.
Consider next the $\text{Sp}(N)$ model on the triangular lattice. We first must adopt a set of signs $\eta_\delta$. There are three bonds $\delta_{1,2,3}$ per unit cell, along the directions $a_1$, $a_2$, and $a_1 - a_2$, where the primitive direct lattice vectors are $a_1 = a \hat{x}$ and $a_2 = \frac{\sqrt{3}}{2} a \hat{x} + \frac{\sqrt{3}}{2} a \hat{y}$. Lattice symmetry suggests $\eta_1 = +1$, $\eta_2 = -1$, and $\eta_3 = +1$ (as opposed to all $\eta_\delta = 1$), resulting in

$$
\gamma_k = \frac{1}{3} \sin \theta_1 - \frac{1}{3} \sin \theta_2 + \frac{1}{3} \sin(\theta_2 - \theta_1),
$$

(1.37)

where the wavevector is written as

$$
k = \frac{\theta_1}{2\pi} G_1 + \frac{\theta_2}{2\pi} G_2,
$$

(1.38)

with $G_1, G_2$ being the two primitive reciprocal lattice vectors for the triangular lattice. The maximum of $|\gamma_k|^2$, corresponding to the minimum of the dispersion $\omega_k$, occurs when $\gamma_k$ lies at one of the two inequivalent zone corners. In terms of the $\theta_i$, these points lie at $(\theta_1, \theta_2) = (\frac{4\pi}{3}, \frac{2\pi}{3})$, where $\gamma_k = -\sqrt{3}$, and at $(\theta_1, \theta_2) = (\frac{2\pi}{3}, \frac{4\pi}{3})$, where $\gamma_k = \sqrt{3}$. Sachdev [8] has found $\kappa_c = 0.34$ for the triangular structure. As one would guess, frustration increases the value of $\kappa_c$ relative to that on the square lattice. On the Kagomé lattice, which is even more highly frustrated, he finds $\kappa_c = 0.53$.

### 1.4 The Mean Field Antiferromagnetic Ground State

For a finite system (no long range order or Bose condensation) one can explicitly write down the ground state of the $\text{SU}(N)$ Schwinger Boson Mean Field Theory $\Psi^{MF}$. It is simply the vacuum of all the Bogoliubov operators

$$
\beta_{k,\alpha} \Psi^{MF} = 0 \quad \forall \, k, \alpha.
$$

(1.39)

where,

$$
\beta_{k\alpha} = \cosh \theta_k b_{k\alpha} - \sinh \theta_k b_{-k\alpha}^\dagger,
$$

(1.40)

and

$$
\tanh 2\theta_k = -\frac{zQ\gamma_k}{\lambda}.
$$

(1.41)

The ground state wavefunction $\Psi^{MF}$ can be explicitly written in terms of the original Schwinger bosons as

$$
\Psi^{MF} = C \exp \left[ \frac{1}{2} \sum_{ij} u_{ij} \sum_m b_{ia}^{\dagger} b_{ja}^{\dagger} \right] |0\rangle,
$$

$$
u_{ij} = \frac{1}{V} \sum_k e^{jkR_{ij}} \tanh \theta_k.
$$

(1.42)
For \( N = 2 \), using the \textit{unrotated} operators \( a^\dagger \) and \( b^\dagger \), the mean field Schwinger boson ground state \( \Psi^{MF} \) is

\[
\Psi^{MF}_{N=2} = \exp \left[ \sum_{i \in A} \sum_{j \in B} u_{ij} \left( a_i^\dagger b_j^\dagger - b_j^\dagger a_i^\dagger \right) \right] |0\rangle.
\] (1.43)

\( \Psi^{MF} \) contains many configurations with occupations different from \( 2S \) and is therefore not a pure spin state. As shown in [9], under Gutzwiller projection it reduces to a valence bond state. Since \( \tanh \theta_{k+\pi} = -\tanh \theta_k \), where \( \pi = (\pi, \pi, \ldots) \), the bond parameters \( u_{ij} \) only connect sublattice \( A \) to \( B \). Furthermore, one can verify that for the nearest neighbor model above, \( u_{ij} \geq 0 \), and therefore the valence bond states obey Marshall’s sign.

Although \( \Psi^{MF} \) is manifestly rotationally invariant, it may or may not exhibit long-ranged antiferromagnetic (Néel) order. This depends on the long-distance decay of \( u_{ij} \). As was shown [9, 10] the SBMFT ground state for the nearest neighbor model is disordered in one dimension, and can exhibit long-range order in two dimensions for physically relevant values of \( S \).

For further calculations, it is convenient to introduce the parametrizations:

\[
\omega_k \equiv c \sqrt{(2\xi)^{-2} + \frac{z}{2}(1 - \gamma_k^2)}
\]
\[c \equiv \sqrt{2z|Q|}
\]
\[\xi^{-1} \equiv \frac{2}{c} \sqrt{\lambda^2 - (z|Q|)^2}
\]
\[t = \frac{T}{z|Q|}.
\] (1.44)

Here, \( c, \xi, \) and \( t \) describe the spin wave velocity, correlation length, and the dimensionless temperature, respectively. In Fig. 1.2 the dispersion for the one-dimensional antiferromagnet is drawn.

At the zone center and zone corner the mean field dispersion is that of free massive relativistic bosons,

\[
\omega_k \approx c \sqrt{(2\xi)^{-2} + |k - k_j|^2}, \quad k_j = 0, \pi.
\] (1.45)

When the gap (or “mass” \( c/2\xi \)) vanishes, \( \omega_k \) are Goldstone modes which reduce to dispersions of antiferromagnetic spin waves.

### 1.5 Staggered Magnetization in the Layered Antiferromagnet

Consider now a layered antiferromagnet on a cubic lattice where the in-plane nearest neighbor coupling is \( J \) and the interlayer coupling is \( \alpha J \), with \( \alpha \ll 1 \).
We expect long range magnetic order at a finite Néel temperature $T_N$. The order parameter, which is the staggered magnetization $M = \langle (−1)^l e^{i\pi \cdot R} S_{R, l} \rangle$ becomes finite when the in-plane correlation length $\xi$, which diverges exponentially at low $T$, produces an effective coupling between neighboring layers $\alpha \xi (T_N)$ which is of order unity. Here $R$ locates the site within a plane, and $l$ is the layer index. This means in effect that the coarse grained spins start to interact as if in an isotropic three dimensional cubic lattice which orders at $T_N$. The interlayer mean field theory, introduced by Scalapino, Imry and Pincus (SIP) [13] in the 1970’s, can be applied within the SBMFT. Here we follow Keimer et al. [14], and Ofer et al. [15], to compute the temperature dependent staggered magnetization, in the range $T \in [0, T_N]$.

The Hamiltonian is given by

$$\mathcal{H} = \sum_{R,l} \left( S_{R, l} \cdot S_{R+\hat{x}, l} + S_{R, l} \cdot S_{R+\hat{y}, l} + \alpha S_{R, l} \cdot S_{R, l+1} \right)$$

(1.46)

The interplane coupling is decomposed using Hartree-Fock staggered magnetization field:

$$\alpha S_{R, l}^z S_{R, l+1}^z \rightarrow (-1)^l e^{i\pi \cdot R} \left( S_{R, l}^z - S_{R, l+1}^z \right) \frac{h_{R,l}}{\alpha},$$

(1.47)

where it is assumed that $M = M \hat{z}$. Here $h_{R,l}$ is the local Néel field due to any ordering in the neighboring layers. Assuming a uniform solution, $h_{R,l} = h$, self-consistency is achieved when

$$\frac{h}{\alpha} = 2 M(T, h) = \langle a_R^\dagger a_R \rangle - \langle b_R^\dagger b_R \rangle,$$

(1.48)

where $M(T, h)$ is the staggered magnetization response to an ordering staggered field $h$ on a single layer.
Fig. 1.3. Numerical solution, from Ofer et al. [15], of the staggered magnetization \( M(T) \) of the layered antiferromagnet for various values of anisotropy parameter \( \alpha_{\text{eff}} \). \( M_0 = S - 0.19660 \) is the zero temperature staggered magnetization, and \( T_N \) is the Néel temperature.

Extracting \( T_N \) is relatively easy, since as \( T \to T_N \), \( h \to 0 \), and the expressions for \( \langle a_R^\dagger a_R \rangle \) and \( \langle b_R^\dagger b_R \rangle \). In this limit, one finds that the second mean field equation in (1.31) is not affected by the staggered field, which simplifies the calculation. At \( T_N \) one finds

\[
2\alpha \chi_s^2(T_N) = 1. \tag{1.49}
\]

Since we know that \( \chi_s^2 \propto \xi_{2D}(T_N) \), we recover the ordering temperature of the SIP theory. The more precise calculation yields (restoring the Heisenberg exchange energy scale \( J \)),

\[
T_N = \frac{J}{\log \alpha} \left( \frac{2\pi M_0}{\log \left( \frac{1}{8} \pi^2 \log(4\alpha/\pi) M_0^2 \right)} \right) \tag{1.50}
\]

The numerically determined \( M(T) = h(T)/2\alpha \) is shown in Fig 1.3 for various anisotropy parameters.

One can also analyze the layered antiferromagnet using the SBMFT’s native decoupling scheme, without proceeding via the interlayer mean field theory of (1.47). Starting from an anisotropic Heisenberg model with in-plane exchange \( J_\parallel \) and interlayer exchange \( J_\perp \), one assumes a mean field solution where \( Q_{ij} = Q_\parallel \) when \( \langle ij \rangle \) is an in-plane bond, and \( Q_{ij} = Q_\perp \) when \( \langle ij \rangle \) is an out-of-plane bond. The second mean field equation, (1.31), then becomes two
equations. The Néel temperature can be written \( T_N = J_\| f(J_\perp / J_\|) \), where \( f(\alpha) \) is a dimensionless function. To find \( f(\alpha) \), we demand that the spectrum be gapless, but the condensate vanishes. This results in two coupled equations,

\[
\kappa + \frac{1}{2} = \frac{1}{2} \left[ \int_{-1}^{-1} d\gamma_\| \rho_\| (\gamma_\|) \int_{-1}^{-1} d\gamma_\perp \rho_\perp (\gamma_\perp) \frac{1 + \epsilon}{\Omega(\gamma_\|, \gamma_\perp)} \left( n(\gamma_\|, \gamma_\perp) + \frac{1}{2} \right) \right] \quad (1.51)
\]

\[
\frac{J_\perp}{2J_\|} = \frac{1}{2} \left[ \int_{-1}^{-1} d\gamma_\| \rho_\| (\gamma_\|) \int_{-1}^{-1} d\gamma_\perp \rho_\perp (\gamma_\perp) \left( \frac{\gamma_\|^2}{\Omega} \right) (n + \frac{1}{2}) \right], \quad (1.52)
\]

where \( \epsilon = |Q_\perp / 2Q_\| \) and \( \Omega(\gamma_\|, \gamma_\perp) = \sqrt{(1 + \epsilon)^2 - (\gamma_\| + \epsilon \gamma_\perp)^2} \), and where \( n = (e^{\Omega/t_c} - 1)^{-1} \), with \( t_c = T_N/4|Q_\| \). Once the above two equations are solved for \( \epsilon \) and \( t_c \), we determine \( Q_\| \) from

\[
Q_\| = J_\| \left[ \int_{-1}^{-1} d\gamma_\| \rho_\| (\gamma_\|) \int_{-1}^{-1} d\gamma_\perp \rho_\perp (\gamma_\perp) \frac{\gamma_\|^2}{\Omega} \right] \left( n + \frac{1}{2} \right). \quad (1.53)
\]

The functions \( \rho_\| \) and \( \rho_\perp \) are given by

\[
\rho_\| (\gamma_\|) = \frac{2}{\pi^2} K(1 - \gamma_\|^2) \quad , \quad \rho_\perp (\gamma_\perp) = \frac{1}{\pi \sqrt{1 - \gamma_\|^2}}. \quad (1.54)
\]

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