A mathematical model to investigate the transmission of COVID-19 in the Kingdom of Saudi Arabia

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Abstract
Since the first confirmed case of SARS-CoV-2 coronavirus (COVID-19) in the 2nd day of March, Saudi Arabia has not report a quite rapid COVID-19 spread compared to America and many European countries. Possible causes include the spread of asymptomatic cases. To characterize the transmission of COVID-19 in Saudi Arabia, this paper applies a susceptible, exposed, symptomatic, asymptomatic, hospitalized, and recovered dynamical model, along with the official COVID-19 reported data by the Ministry of Health in Saudi Arabia. The basic reproduction number $R_0$ is estimated to range from 2.87 to 4.9.

Keywords: COVID-19 in Saudi Arabia, SEIR model, Simulation.

1 Introduction
As of April 22, 2020, more than 12772 cases and 114 deaths of coronavirus disease 2019 (COVID-19) caused by the SARS-CoV-2 virus had been confirmed in Saudi Arabia. Since the 4th of March [17], control measures have been implemented within Saudi Arabia to try to control the spread of the disease. Isolation of confirmed cases and contact tracing are crucial part of these measures, which are common interventions for controlling infectious disease outbreaks [26–28]. For example, the severe acute respiratory syndrome (SARS) outbreak SARS and Middle East respiratory syndrome (MERS), were controlled through tracing suspected cases and isolating confirmed cases because the majority of transmission occurred concurrent or after symptom onset [27–29].

However, it is unknown if transmission of COVID-19 can occur before symptom onset, which could decrease the effectiveness of isolation and contact tracing [26,27,29].

NOTE: This preprint reports new research that has not been certified by peer review and should not be used to guide clinical practice.
In this paper, the impact of asymptomatic COVID-19 cases on the spread of the disease will be considered using a modified version of the susceptible-exposed-infected-recovered (SEIR) dynamical model, along with the official COVID-19 data reported by the ministry of health in Saudi Arabia. Other main objectives of this paper include: estimating the basic reproduction number ($R_0$) of COVID-19 in Saudi Arabia and how interacting with infected individual (symptomatic and asymptomatic) affect the estimated number, estimating the maximum required number of hospital beds and intensive care units (ICU).

2 Model establishment

The population will be divided into six categories: susceptible ($S$), exposed ($E$), symptomatic ($Y$), asymptomatic ($N$), hospitalized ($H$), and recovered ($R$) individuals (SEYNHR). Individuals moves from the susceptible compartment $S$ to the exposed compartment $E$ after interacting with infected individuals with transmission rates $\beta_1$, $\beta_2$, and $\beta_3$ as shown in Figure 1. COVID-19 is known to have an incubation period, from 2 to 14 days, between exposure and development of symptoms [6, 30]. After this period, exposed individual transits from the compartment $E$ to either compartment $Y$ at a rate $\alpha$, or to compartment $N$ at a rate $\alpha(1-\gamma)$. An individual could move from compartment $N$ to $Y$ at a rate $K$ if they show symptoms. Once an individual becomes infected with the coronavirus that causes COVID-19, that individual develops immunity against the virus with a rate $\Phi_Y$ or the individual will be hospitalized with a rate of $\epsilon$ or die because of the disease with a rate of $\mu_1$. When individual becomes hospitalized, that individual receives treatment and develops immunity against the virus with a rate $r$ or die because of the disease with a rate of $\mu_2$.

As shown in Figure 1, the SEYNHR model has six compartments, and there-
fore a discrete dynamical system consisting six non-linear differential equations will be formed as the following:

\[
\begin{align*}
\frac{dS}{dt} &= A - (\beta_1 Y + \beta_2 N + \beta_3 H)S - \mu S, \\
\frac{dE}{dt} &= (\beta_1 Y + \beta_2 N + \beta_3 H)S - (\alpha + \mu)E, \\
\frac{dY}{dt} &= \alpha(1 - \gamma)E - (\Phi_Y + \epsilon + \mu_1 + \mu)Y + KN, \\
\frac{dN}{dt} &= \alpha\gamma E - (\Phi_N + K + \mu)N, \\
\frac{dH}{dt} &= \epsilon Y - (r + \mu_2 + \mu)H, \\
\frac{dR}{dt} &= \Phi_Y Y + \Phi_N N + rH - \mu R,
\end{align*}
\]

where \(N(t) = S(t) + E(t) + Y(t) + N(t) + H(t) + R(t)\). The next-generation matrix will be used to derive an analytical expression for the basic reproduction number \((R_0)\), for the compartmental model above. Calculating \(R_0\) is a useful metric for assessing the transmission potential of an emerging COVID19 in Saudi Arabia.

3 Basic reproduction number \(R_0\)

An important concept in epidemiology is the basic reproduction number, defined as “the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual” [10]. The next-generation method will be used to calculate \(R_0\) [11]. The system in Equation 1 can be rewritten as follows

\[
\frac{dw}{dt} = \Omega(w) = F(w) - V(w),
\]

where \(F := (F1, F2, F3, F4, F5, F6)^T\) and \(V := (V1, V2, V3, V4, V5, V6)^T\), or more explicitly

\[
\begin{pmatrix}
\dot{E} \\
\dot{Y} \\
\dot{N} \\
\dot{H} \\
\dot{R} \\
\dot{S}
\end{pmatrix}
= \begin{pmatrix}
\beta_1 Y + \beta_2 N + \beta_3 H \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
- \begin{pmatrix}
(\alpha + \mu)E \\
(\gamma + \epsilon + \mu_1 + \mu)Y - \alpha(1 - \gamma) * E - KN \\
(\Phi_N + K + \mu)N - \alpha\gamma E \\
(r + \mu_2 + \mu)H - \epsilon Y \\
(P + \mu)R - \Phi_Y - \Phi_N - rH \\
(\beta_1 Y + \beta_2 N + \beta_3 H)S - A - PR
\end{pmatrix}.
\]
The Jacobian matrices of \( F \) and \( V \) evaluated at the disease-free equilibrium (DFE) of the system in Equation 1, \( M = (\frac{A}{\mu}, 0, 0, 0, 0) \), are given by

\[
F = \left( \frac{F}{x_j} \big|_{M} \right)_{1 \leq i, j \leq 4} = \begin{pmatrix}
0 & \frac{\beta_1 A}{\mu} & \frac{\beta_2 A}{\mu} & \frac{\beta_3 A}{\mu} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

and

\[
V = \left( \frac{V}{x_j} \big|_{M} \right)_{1 \leq i, j \leq 4} = \begin{pmatrix}
-\alpha + \mu & 0 & 0 & 0 \\
-\alpha(1-\gamma) & \Phi_Y + \epsilon + \mu_1 + \mu & -K & 0 \\
-\alpha(1-\gamma) & 0 & \Phi_N + K + \mu & 0 \\
0 & -\epsilon & 0 & r + \mu_2 + \mu
\end{pmatrix}.
\]

Direct calculations show that

\[
V^{-1} = \begin{pmatrix}
\frac{1}{\alpha + \mu} & 0 & 0 & 0 \\
m_1 & m_4 & m_6 & 0 \\
m_2 & 0 & m_7 & 0 \\
m_3 & m_5 & m_8 & \frac{1}{r + \mu_2 + \mu}
\end{pmatrix},
\]

where

\[
\begin{align*}
m_1 &= \frac{\alpha(1-\gamma)K + \alpha(1-\gamma)\mu + \alpha(1-\gamma)\Phi_N + K\alpha}{(\Phi_Y + \epsilon + \mu_1 + \mu)(\Phi_N + K + \mu)(\alpha + \mu)}, \\
m_2 &= \frac{(\alpha + \mu)(\Phi_Y + K + \mu)}{\epsilon(\alpha(1-\gamma)K + \alpha(1-\gamma)\mu + \alpha(1-\gamma)\Phi_N + K\alpha)}, \\
m_3 &= \frac{(\Phi_Y + \epsilon + \mu_1 + \mu)(\Phi_N + K + \mu) + (\epsilon \mu)(\alpha + \mu)(r + \mu_2 + \mu)}{(\Phi_Y + \epsilon + \mu_1 + \mu)(\Phi_N + K + \mu)(\alpha + \mu)(r + \mu_2 + \mu)}, \\
m_4 &= \frac{1}{(\Phi_N + K + \mu)}, \\
m_5 &= \frac{\Phi_Y + \epsilon + \mu_1 + \mu}{(\Phi_Y + \epsilon + \mu_1 + \mu)(\alpha + \mu)(r + \mu_2 + \mu)}, \\
m_6 &= \frac{(\Phi_N + K + \mu)\Phi_Y + \epsilon + \mu_1 + \mu}{k}, \\
m_7 &= \frac{1}{\Phi_N + K + \mu}, \\
m_8 &= \frac{\epsilon K}{(\Phi_Y + \epsilon + \mu_1 + \mu)(\Phi_N + K + \mu)(r + \mu_2 + \mu)}.
\end{align*}
\]

Denoting the 4x4 identity matrix by \( \mathbb{I} \), the characteristic polynomial \( \Gamma(\lambda) \) of the matrix \( FV^{-1} \) is given by

\[
\Gamma(\lambda) = \text{det}(FV^{-1} - \lambda \mathbb{I}),
\]

\[
= -\lambda^3 \left( (\Phi_N + K + \mu)(\beta_1 \mu + (r + \mu_2)\beta_1 + \beta_3 \epsilon)A\alpha(1-\gamma) + (B_1 + B_2)A\alpha\gamma \right) + \lambda^4
\]

\[
= \frac{-\lambda^3 ((\Phi_N + K + \mu)(\beta_1 \mu + (r + \mu_2)\beta_1 + \beta_3 \epsilon)A\alpha(1-\gamma) + (B_1 + B_2)A\alpha\gamma) + \lambda^4}{\mu(\Phi_Y + \epsilon + \mu_1 + \mu)(\Phi_N + K + \mu)(\alpha + \mu)(r + \mu_2 + \mu)}.
\]

where

\[
\begin{align*}
B_1 &= \beta_2 \mu^2 + ((\mu_1 + \mu_2 + \Phi_Y + \epsilon + r)\beta_2 + \beta_1 K)\mu + K(r + \mu_2)\beta_1, \\
B_2 &= (r + \mu_2)(\Phi_Y + \epsilon + \mu_1)\beta_2 + \beta_3 \epsilon K.
\end{align*}
\]
The solutions $\lambda_{1,2,3,4}$ are given by

$$\left\{0,0,0, \frac{(\Phi_N + K + \mu)(\beta_1 \mu + (r + \mu_2)\beta_1 + \beta_3 \epsilon)A\alpha(1 - \gamma) + (B_1 + B_2)A\alpha\gamma}{\mu(\Phi_Y + \epsilon + \mu_1 + \mu)(\Phi_N + K + \mu)(\alpha + \mu)(r + \mu_2 + \mu)}\right\}.$$ 

Therefore, the reproduction number for the $SEYNHR$ model in Equation 1 is given by

$$R_0 = \max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\},$$

$$= \frac{(\Phi_N + K + \mu)(\beta_1 \mu + (r + \mu_2)\beta_1 + \beta_3 \epsilon)A\alpha(1 - \gamma) + (B_1 + B_2)A\alpha\gamma}{\mu(\Phi_Y + \epsilon + \mu_1 + \mu)(\Phi_N + K + \mu)(\alpha + \mu)(r + \mu_2 + \mu)}.$$ 

### 4 Results

At the initial time $t = 0$, if we set the initial population $N(0) = 1$, then $N(0) = S(0) + E(0) + Y(0) + N(0) + H(0) + R(0) = 1$. The transition probabilities between states are all in the range of $[0, 1]$, i.e.,

$$0 \leq A, \beta_1, \beta_2, \beta_3, \mu, \alpha, \gamma, \Phi_Y, \epsilon, \mu_1, K, \Phi_N, r, \mu_2 \leq 1.$$ 

I fitted the model in Equation 1 to the published data from the Ministry of Health in Saudi Arabia (MOH) from the 2nd day of March until the 14th of April to estimate values for the unknown parameters $\beta_1, \beta_2, \beta_3$ and $K$ using MATLAB. The results are shown in the upper panel of Figure 2. Increasing the value of $K$ from 0.009/day requires increasing the value of $\beta_1, \beta_2$ and $\beta_3$ from 0.267, 0.53 and 0.13 (day x individual)$^{-1}$ to 0.5, 1 and 0.26 (day x individual)$^{-1}$, respectively. All other parameters and their descriptions are given in Table 1.

I assumed that the mean asymptomatic infectious period is the same as the mean symptomatic infectious period because there is no estimation available in the literature [9, 29]. Based on those estimated, assumed and measured values, the basic reproduction number $R_0$ is estimated to range from 2.87 to 4.9. The variation of the basic reproduction number $R_0$ for different values of $\beta_1, \beta_2, \Phi_Y, \Phi_N$ and $K$ are shown in the heat maps in Figure 3. The upper heat map of Figure 3 shows that practicing physical distancing could significantly reduce the value of $R_0$ and hence control the spread of the disease.

The center panel of Figure 2 shows that about 18% of the entire Saudi population will be asymptomatic in the last week of May 2020 and about 17% will be exposed in the third week of May. The percentage of the entire population being symptomatic at anytime will not exceed 1%, which is estimated to occur in the third week of May. Moreover, about 60000 hospital beds and 18000 ICU beds are required (30% of the hospitalized cases [6]) immediately after the second week of May. Currently, the Ministry of Health designated 25 hospitals for COVID-19 infected patients with up to 80,000 beds and 8000 intensive care units (ICU) beds [25] and therefore extra 10000 ICU beds could be required.
Table 1: Parameters used in the simulations.

| Parameter | Description                                           | Dimension | Value       | Source |
|-----------|-------------------------------------------------------|-----------|-------------|--------|
| $N$       | Population of Saudi Arabia in 2019.                   | Individual| 34218169    | [12]   |
| $A$       | Birth rate in Saudi Arabia in 2019.                   | Individual/day | 1650     | [12]   |
| $\mu$     | Death rate in Saudi Arabia in 2019.                   | day$^{-1}$ | 3.7x10$^{-5}$ | [12]   |
| $\beta_1$ | Effective contact rate from symptomatic to susceptible.| day$^{-1}$ | [0.267, 0.5] | fitted |
| $\beta_2$ | Effective contact rate from asymptomatic to susceptible.| day$^{-1}$ | [0.53, 1]   | fitted |
| $\beta_3$ | Effective contact rate from hospitalized to susceptible.| day$^{-1}$ | [0.13, 0.26] | fitted |
| $\alpha$  | Mean latent period.                                   | days      | 5.1         | [13]   |
| $\gamma$  | Probability of becoming asymptomatic upon infection.  | n/a       | 0.86        | [14]   |
| $\Phi_Y$  | Mean symptomatic infectious period.                   | days      | 8           | [15]   |
| $\epsilon$ | Rate of symptomatic becomes hospitalized.             | day$^{-1}$ | 0.125       | [17]   |
| $\mu_1$   | Death rate of symptomatic patients.                   | day$^{-1}$ | 0.392       | [16]   |
| $K$       | Rate of asymptomatic becomes symptomatic.            | day$^{-1}$ | [0.009, 0.125] | fitted |
| $\Phi_N$  | Mean asymptomatic infectious period.                  | days      | 8           | assumed|
| $r$       | Rate of recovered hospitalized patients.              | day$^{-1}$ | 0.1         | [6]    |
| $\mu_2$   | Death rate of hospitalized patients.                  | day$^{-1}$ | 0.392       | [17]   |

5 Discussion

The parameter with the highest degree of uncertainty are the effective contact rates from symptomatic to susceptible $\beta_1$, from asymptomatic to susceptible $\beta_2$ and from hospitalized to susceptible $\beta_3$ (expected to be a fraction of $\beta_1$ because of the protective measures in hospitals), as well as the mean infectious periods for symptomatic $\Phi_Y$ and asymptomatic $\Phi_N$ individuals. I estimated a maximum value of $\beta_1$ to be 0.5 which is one half of the value reported by Li et al. [18]. This could be a reasonable estimation as we have not seen similar scenario in Saudi Arabia after 5 weeks since reaching 100 confirmed cases on the 14th of March (week 7 since the first case) as we have seen in many other countries like China, America and different European countries in the same timescale. This could be a result of the precautionary measures taken by the Saudi authorities, including closure of schools and universities that started as early as the 8th of March (six days after the first confirmed COVID-19 case in Saudi Arabia). Based on the above estimation for $R_0$, the center panel of Figure 3 suggests that the infectious period for symptomatic patient could be in the range from 6.6 to 12.5 days (i.e., $\Phi_Y \in [0.08, 0.15]$) and the infectious period for asymptomatic patient could be in the range from 6.6 to 25 days (i.e., $\Phi_N \in [0.04, 0.15]$). The infectious period for symptomatic cases are consistent with what is being observed clinically [30].

In reality, $R_0$ is not a biological constant; it could fluctuate daily depending on environmental and social factors such as percentage of entire susceptible
population wearing suitable medical mask and practicing physical distancing. In the literature, estimates of $R_0$ vary greatly: from 1 to 6 [18–24] up to 26.5 [9]. This variation is because of different assumptions and factors they had considered in the calculations. In general, considering asymptomatic infection sub-population will increase the estimated values of $R_0$.

6 Conclusion

The contribution of undocumented COVID-19 infections (asymptomatic cases) on the transmission of the disease deserves further studies and investigations. This paper shows that asymptomatic cases of COVID-19 will drive the growth of the pandemic in Saudi Arabia. Therefore, more testing is needed to identify COVID-19 patients (symptomatic and asymptomatic) and to contain the spread of the disease.
Figure 2: Numerical results of the SEYNHR model with the parameter listed on Table 1, where $\beta_1 = 0.35$, $\beta_2 = 0.7$, $\beta_3 = 0.18$ and $K = 0.009$. The upper figure shows the estimated number of symptomatic COVID-19 cases, with the published data by the Ministry of Health in Saudi Arabia of confirmed COVID-19 cases (blue circles). The center figure shows the estimated susceptible, exposed, symptomatic, asymptomatic, hospitalized, and recovered sub-populations. The lower figure shows estimations of the hospitalized cases and the required ICU beds (black dashed line). The red line represents the number of hospital beds available, while the red dashed line represents the number of ICU beds available in Saudi Arabia.
Figure 3: Heat maps showing the variation of $R_0$ for different parameter values: the upper heat map shows the variation of $R_0$ for different values for $\beta_1$ and $\beta_2$, in the center the heat map shows the variation of $R_0$ for different values for $\Phi_Y$ and $\Phi_N$ and the lower heat map shows the variation of $R_0$ for different values for $\beta_1$ and $K$. 
References

[1] Bernoulli D. “Essai d’une nouvelle analyse de la mortalit cause par la petite vrole.” Mm. Math Phys Acad Roy Sci Paris.1:1-45 (1766).

[2] Kermack, W. O. and McKendrick, A. G. “A Contribution to the Mathematical Theory of Epidemics.” Proc. Roy. Soc. Lond. A 115, 700-721 (1927).

[3] Bernoulli, D., Blower, S. “An attempt at a new analysis of the mortality caused by smallpox and of the advantages of inoculation to prevent it.” Reviews in Medical Virology, 14, 275-288 (2004).

[4] Jia Wangping, Han Ke, Song Yang, Cao Wenzhe, Wang Shengshu, Yang Shanshan, He Yao. “Extended SIR prediction of the epidemics trend of COVID-19 in Italy and compared with Hunan, China”. https://doi.org/10.1101/2020.03.18.20038570 (2020).

[5] Chinazzi M, Davis JT, Ajelli M, Gioannini C, Litvinova M, Merler S. “The effect of travel restrictions on the spread of the 2019 novel coronavirus (COVID-19) outbreak”. Science (New York, NY) (2020).

[6] World Health Organization. Coronavirus disease (COVID-19) situation reports. https://www.who.int/emergencies/diseases/novel-coronavirus-2019/situation-reports/. Retrieved Mar. 14 (2020).

[7] Hellewell J, Abbott S, Gimma A, Bosse N, Jarvis C., Munday J, Kucharski A, Edmunds W. “Feasibility of controlling COVID-19 outbreaks by isolation of cases and contacts.” Lancet Glob Health; 8: e488-96 (2004).

[8] W. Guan, et al. “New England Journal of Medicine” (2020).

[9] Aguilar A.B, Faust, J.S, Westafer L.M, Gutierrez, J.B. “Investigating the Impact of Asymptomatic Carriers on COVID-19 Transmission” https://doi.org/10.1101/2020.03.18.20037994 (2020).

[10] Driessche P., James Watmough. “Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission”. Physica A 29-48 (2002).

[11] Van den Driessche P., Watmough J., Mathematical biosciences 180, 29 (2002).

[12] General authority for statistics in Saudi Arabia, https://www.stats.gov.sa.

[13] Lauer S. A., et al., Annals of Internal Medicine (2020).

[14] Nishiura H., et al. “The rate of underascertainment of novel coronavirus (2019-ncov) infection: Estimation using japanese passengers data on evacuation flights” (2020).
[15] Zhou F., et al. (2020).

[16] Baud D., et al., The Lancet Infectious Diseases (2020).

[17] Ministry of health in Saudi Arabia, https://www.moh.gov.sa.

[18] Li R. et al. “Substantial undocumented infection facilitates the rapid dissemination of novel coronavirus (SARS-CoV2)”. Science (2020).

[19] Wu P. et al. “Real-time tentative assessment of the epidemiological characteristics of novel coronavirus infections in Wuhan, China, as at 22 January 2020”. Eurosurveillance 25 (2020).

[20] Zhang S. et al. “Prevalence of comorbidities and its effects in coronavirus disease 2019 patients: A systematic review and meta-analysis”. International Journal of Infectious Diseases (2020).

[21] Li R. et al. “Clinical characteristics of 2019 novel coronavirus infection in China”, medRxiv (2020).

[22] Wu JT, Leung K, Leung GM. “Nowcasting and forecasting the potential domestic and international spread of the 2019-nCoV outbreak originating in Wuhan, China: a modelling study”. The Lancet (2020).

[23] Shen M, Peng Z, Xiao Y, Zhang L. “Modelling the epidemic trend of the 2019 novel coronavirus outbreak in China”. bioRxiv doi: https://doi.org/10.1101/2020.01.23.916726 (2020).

[24] Read JM., Bridgen JRE., Cummings DAT., Jewell CP. “Novel coronavirus 2019-nCoV: early estimation of epidemiological parameters and epidemic predictions”. medRxiv. doi: https://doi.org/10.1101/2020.01.23.20018549 (2020).

[25] World Health Organization, Regional Office for the Eastern Mediterranean. Available from: http://www.emro.who.int/media/news/who?saudi-arabia-join-forces-to-fight-covid-19-nationally-regionally-and-globally.html. [Last retrieved on 2020 Mar 31].

[26] Fraser C, Riley S, Anderson RM, Ferguson NM. “Factors that make an infectious disease outbreak controllable”. Proc Natl Acad Sci USA; 101: 6146?51 2004.

[27] Peak CM, Childs LM, Grad YH, Buckee CO. “Comparing nonpharmaceutical interventions for containing emerging epidemics”. Proc Natl Acad Sci USA; 114: 4023?28 (2017).

[28] Kang M, Song T, Zhong H, et al. “Contact tracing for imported case of Middle East respiratory syndrome”. China (2015).

[29] Hellewell J. et al. “Feasibility of controlling COVID-19 outbreaks by isolation of cases and contacts” The lancet (2020).
[30] Q. Li, X. Guan, P. Wu, X. Wang, L. Zhou, Y. Tong, et al. “Early transmission dynamics in Wuhan, China, of novel coronavirus-infected pneumonia”. N Engl J Med (2020).