CONSTRUCTING COHERENT STATES FOR THE RATIONAL EXTENSIONS OF THE HARMONIC OSCILLATOR POTENTIAL

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Abstract. Using the formalism of Maya diagrams and ladder operators, we describe the algebra of annihilating operators for the class of rational extensions of the harmonic oscillators. This allows us to construct the corresponding coherent state in the sense of Barut and Girardello. The resulting time-dependent function is an exact solution of the time-dependent Schrödinger equation and a joint eigenfunction of the algebra of annihilators.

Introduction

Supersymmetric quantum mechanics (SUSYQM) has proven to be a key technique in the construction of exactly solvable potentials and in the understanding of shape-invariance. The supersymmetric partners of the harmonic oscillator Hamiltonian are known as rational extensions because the corresponding potentials consist of the harmonic oscillator potential plus a rational term that vanishes at infinity.

There has been recent interest in rational extensions possessing ladder operators, which may furnish higher-order analogues of annihilation operators introduced in the second quantization of the harmonic oscillator. Ladder operators have applications in the study of superintegrable systems and rational solutions of Painlevé equations; they also provide a clear avenue for generalizing one of the defining properties of the canonical coherent states.

In this work, we present a condition, expressed in terms of the Maya diagram associated with a rational extension, for such a ladder operator to be designated an annihilation operator. We then construct the coherent states of the rational extensions as eigenstates of these annihilation operators.

Note: this is a preliminary version of this article, and will be revised before publication.
Preliminaries

Partitions and Maya diagrams. A partition of a natural number $N$ is a non-increasing integer sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ such that

$$|\lambda| := \sum_{i=1}^{\infty} \lambda_i = N.$$ Implicit in this definition is the assumption that $\lambda_i = 0$ for $i$ sufficiently large. The length $\ell$ of $\lambda$ is the number of non-zero elements of the sequence. It is useful to represent a partition using a Young diagram $Y_\lambda = \{(i,j): 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$, consisting of $\lambda_i$ cells in rows $i = 1, \ldots, \ell$. The hook

$$H_{\lambda}(i,j) = \{(i,k) \in Y_\lambda: j \leq k\} \cup \{(k,j) \in Y_\lambda: i \leq k\}$$

is the set of cells connecting cell $(i,j)$ to the rim of the diagram. The hooklength $hk_{\lambda}(i,j)$ is the cardinality of hook $(i,j) \in Y_\lambda$; the number

$$d_{\lambda} = \frac{N!}{\prod_{(i,j) \in Y_\lambda} hk_{\lambda}(i,j)}$$

counts the number of standard Young tableaux of shape $\lambda$ and corresponds to the dimension of an irreducible representation of the symmetric group $\mathfrak{S}_N$.

Partitions are closely related to a concept called a Maya diagram. We say that a set of integers $M \subset \mathbb{Z}$ is a Maya diagram if

$$K_M^+ = \{m \in M: m \geq 0\}, \quad K_M^- = \{m \in \mathbb{Z} \setminus M: m < 0\}$$

are finite sets. In other words, a Maya diagram is a subset of $\mathbb{Z}$ that contains a finite number of positive integers and excludes a finite number of negative integers. The group $\mathbb{Z}$ acts on $\mathcal{M}$ by translations, since

![Young diagram and corresponding hook-lengths for the partition $(5, 5, 4, 2, 2)$](image)
for \( M \in \mathcal{M} \) and \( n \in \mathbb{Z} \),

\[
M + n = \{ m + n : m \in M \}
\]
is also a Maya diagram. We will refer to the equivalence class \( M/Z \) of a partition modulo translations as an unlabelled Maya diagram. An unlabelled Maya diagram can be represented as a horizontal sequence of filled \( \bullet \) and empty \( \circ \) states beginning with an infinite segment of \( \bullet \) and terminating with an infinite segment of \( \circ \), where \( \bullet \) in position \( m \) is taken to indicate membership \( m \in M \). A choice of origin serves to convert an unlabelled Maya diagram to a subset of \( \mathbb{Z} \).

There is a natural bijection between the set of partitions and the set of unlabelled Maya diagrams. For a given partition \( \lambda \), the set

\[
M^{(\lambda)} = \{ \lambda_i - i : i = 1, 2, \ldots \}
\]
is a Maya diagram.

**Proposition 1.** Every Maya diagram is a translate of a unique \( M^{(\lambda)} \).

**Proof.** Indeed, let \( M \subset \mathbb{Z} \) be a Maya diagram and \( m_1 > m_2 > \cdots \) the strictly decreasing sequential ordering of the elements \( m \in M \). Set

\[
\sigma_M = \#K^+_M - \#K^-_M
\]
where \( \# \) denotes the cardinality of a finite set. By construction,

\[
\sigma_{M+n} = \sigma_M + n, \quad n \in \mathbb{Z}.
\]

Hence, there exists a sufficiently large \( \ell \) such that \( m_i = -i + \sigma_M \) for all \( i > \ell \). Next, let

\[
\lambda_i = m_i + i - \sigma_M, \quad i = 1, 2, \ldots.
\]

It follows that \( \lambda_i = 0 \) for all \( i > \ell \); i.e., \( \lambda \) is a partition. Furthermore, by construction,

\[
M = M^{(\lambda)} + \sigma_M.
\]

\[\square\]

We will refer to the integer \( \sigma_M \) defined in (1) as the index of \( M \), and to the set

\[
K_M = K^+_M \cup K^-_M
\]
as the index set of \( M \).

The hooklength formula (1) can be re-expressed in terms of a Maya diagram as follows. Let \( K^{(\lambda)} \) denote the index set of \( M^{(\lambda)} + \ell \), with
Figure 2. The bent Maya diagram with index set \( K = \{4, 3, 1, -1, -4, -5\} \) is the rim of the Young diagram of the corresponding partition \( \lambda = (5, 5, 4, 2, 2) \).

\[
k_i = \lambda_i - i + \ell \quad (i = 1, 2, \ldots, \ell)
\]

the decreasing enumeration of \( K^{(\lambda)} \). If \( \lambda \) is the partition corresponding to \( M \), then

\[
\prod_{i,j} h_{\lambda}(i, j) = \frac{\prod_i k_i!}{\prod_{i<j}(k_i - k_j)}.
\]

The bijection between unlabelled Maya diagrams and partitions can be visualized by representing a filled state with a unit downward arrow and an empty state with a unit right arrow. As can be seen in Figure 2, the resulting “bent” Maya diagram traces out the boundary of the Young diagram of the corresponding partition \( \lambda \); see [3] for more details.

Let \( \mathcal{M} \) denote the set of all Maya diagrams. The flip \( f_k \) at position \( k \in \mathbb{Z} \) is the involution \( f_k : \mathcal{M} \to \mathcal{M} \) defined by

\[
f_k : M \mapsto \begin{cases} M \cup \{k\}, & k \notin M \\ M \setminus \{k\}, & k \in M. \end{cases}
\]
In the event that \( k \notin M \), the flip \( f_k \) is said to act on \( M \) by a state-deleting transformation \( \circ \to \bullet \), while in the opposite scenario \( (k \in M) \), it is said to act by a state-adding transformation \( \bullet \to \circ \).

Let \( Z \) denote the set of all finite subsets of \( \mathbb{Z} \). For a finite set of integers \( K = \{k_1, \ldots, k_p\} \in Z \) we define the corresponding multi-flip to be the transformation \( f_K : \mathcal{M} \to \mathcal{M} \) defined according to

\[
(9) \quad f_K(M) = (f_{k_1} \circ \cdots \circ f_{k_p})(M).
\]

Observe that multi-flips are also involutions. This means that Maya diagrams together with multi-flips have the natural structure of a complete graph \( (\mathcal{M}, Z) \). The unique edge connecting Maya diagrams \( M_1, M_2 \) is the integer set

\[
(10) \quad K = M_1 \ominus M_2 = M_2 \ominus M_1,
\]

where

\[
M_1 \ominus M_2 := (M_1 \setminus M_2) \cup (M_2 \setminus M_1)
\]

is the symmetric difference operation.

Equivalently, we may identify Eq. (10) as giving the edge between \( M_1 \) and \( M_2 \); i.e., the unique multiflip satisfying \( f_K(M_1) = M_2 \) and \( f_K(M_2) = M_1 \).

Since \( (\mathcal{M}, Z) \) is a complete graph, we can define a bijection \( Z \to \mathcal{M} \) given by \( K \mapsto f_K(Z_{\ominus}) \), where

\[
(11) \quad Z_{\ominus} = \{m \in \mathbb{Z} : m < 0\}
\]

denotes the trivial Maya diagram. Inversely, for \( M \in \mathcal{M} \), the unique \( K \in Z \) such that \( M = f_K(Z_{\ominus}) \) is simply the index set \( K = K_M \) defined in (7).

**Vertex operators and Schur functions.** For \( k \in \mathbb{N}_0 \), define the ordinary Bell polynomials \( B_k(t_1, \ldots, t_k) \in \mathbb{Q}[t_1, \ldots, t_k] \) as the coefficients of the power generating function

\[
(12) \quad \exp \left( \sum_{k=1}^{\infty} t_k z^k \right) = \sum_{k=0}^{\infty} B_k(t_1, \ldots, t_k) z^k,
\]

where \( \mathbf{t} = (t_1, t_2, \ldots) \). The multinomial formula implies that

\[
(13) \quad B_k(t_1, \ldots, t_k) = \sum_{\|\mu\| = k} \frac{t_1^{\mu_1} t_2^{\mu_2} \cdots t_k^{\mu_k}}{\mu_1! \mu_2! \cdots \mu_k!}, \quad \|\mu\| = \mu_1 + 2\mu_2 + \cdots + k\mu_k
\]

\[
= \frac{t_k}{k!} + \frac{t_{k-2} t_2}{(k-2)!} + \cdots + t_{k-1} t_1 + t_k.
\]
For a partition $\lambda$ of $N$, define the Schur function $S^{(\lambda)}(t_1, \ldots, t_N) = \det(B_{m_i+j})_{i,j=1}^\ell$,

\begin{equation}
S^{(\lambda)} = \det(B_{m_i+j})_{i,j=1}^\ell,
\end{equation}

where

\[ m_i = \lambda_i - i, \]

and where $B_k = 0$ when $k < 0$. Moreover, since

\[ \partial_i B_j(t_1, \ldots, t_j) = B_{j-i}(t_1, \ldots, t_{j-i}), \quad j \geq i, \]

we may re-express (14) in terms of a Wronskian determinant,

\begin{equation}
S^{(\lambda)} = \text{Wr}[B_{m_{\ell+1}}, \ldots, B_{m_{\ell+\ell}}],
\end{equation}

where the Wronskian is taken with respect to $t_1$.

Let $X_m = X_m(t, \partial_t), m \in \mathbb{Z}$, be the operators defined by the generating function

\begin{equation}
V(t, \partial_t, z) = \exp \left( \sum_{k=1}^{\infty} t_k z^k \right) \exp \left( \sum_{j=1}^{\infty} -j^{-1} \partial_t z^{-j} \right)
= \sum_{m=0}^{\infty} X_m(t, \partial_t) z^m.
\end{equation}

Expanding the above formulas gives

\begin{align}
X_m &= \sum_{j=0}^{\infty} B_{j+m}(t_1, \ldots, t_k) B_j \left( \partial_{t_1}, \ldots, j^{-1} \partial_{t_j} \right), \quad m \geq 0; \\
X_m &= \sum_{j=0}^{\infty} B_j(t_1, \ldots, t_k) B_{j-m} \left( -\partial_{t_1}, \ldots, -j^{-1} \partial_{t_j} \right), \quad m < 0.
\end{align}

It can be shown that the above operators obey the fundamental relation

\begin{equation}
X_m X_n + X_{n-1} X_{m+1} = 0.
\end{equation}

Despite the fact that the $X_m(t, \partial_t)$ are differential operators involving infinitely many variables, they have a well-defined action on polynomials. In particular, when applied to Schur functions, they function as multi-variable raising operators.

**Proposition 2.** For every partition $\lambda$ of length $\ell$, we have

\begin{equation}
S^{(\lambda)} = X_{\lambda_1} \cdots X_{\lambda_\ell} 1,
\end{equation}

where 1 is the Schur function corresponding to the trivial partition.

The proof of (19)–(20) can be found in [7, Appendix A]. As an immediate corollary we obtain the following.
Proposition 3. Let $\lambda$ be a partition, $M^{(\lambda)} \subset \mathbb{Z}$ the corresponding Maya diagram (3), and

\begin{equation}
J^{(\lambda)} = \mathbb{Z} \setminus M^{(\lambda)}
\end{equation}

the corresponding integer complement. Then, for every $m \in \mathbb{Z}$ we have

\begin{equation}
X_m S^{(\lambda)} = \begin{cases} 
-1 \#\{k \in M^{(\lambda)} : k > m\} S_{m \triangleright \lambda} & \text{if } m \in J^{(\lambda)} \\
0 & \text{if } m \in M^{(\lambda)}.
\end{cases}
\end{equation}

By construction, the action of $V(t, z)$ on a polynomial $P(t) \in \mathbb{C}[t_1, \ldots, t_n]$ can be given as

\begin{equation}
V(t, z)P(t) = \exp \left( \sum_{k=1}^{\infty} t_k z^k \right) P \left( t_1 - z^{-1}, t_2 - \frac{z^{-2}}{2}, \ldots, t_n - \frac{z^{-n}}{n} \right).
\end{equation}

Proposition 3 allows the action of $V(t, z)$ on a Schur polynomial to be conveniently written in terms of the “insertion” procedure:

**Theorem 4.** Let $\lambda$ be a partition. With the above notation, we have

\begin{equation}
V(t, z)S^{(\lambda)}(t) = \sum_{m \in J^{(\lambda)}} (-1)^{\#\{k \in M^{(\lambda)} : k > m\}} S_{m \triangleright \lambda}(t) z^m.
\end{equation}

**Hermite polynomials.** Hermite polynomials $H_n(x)$, $n = 0, 1, \ldots$, are univariate polynomials defined by

\begin{equation}
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, n = 0, 1, 2, \ldots.
\end{equation}

The $H_n(x)$ are known as classical orthogonal polynomials because they satisfy a second-order eigenvalue equation

\begin{equation}
y'' - 2xy' = 2ny, \quad y = H_n(x),
\end{equation}

as well as a 3-term recurrence relation

\begin{equation}
H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),
\end{equation}

which together imply the following orthogonality relation:

\begin{equation}
\int_{\mathbb{R}} H_m(x)H_n(x)e^{-x^2}dx = \sqrt{\pi} 2^n n! \delta_{n,m}.
\end{equation}

The generating function for the Hermite polynomials is

\begin{equation}
e^{xz - \frac{1}{2}z^2} = \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{2^n n!}.
\end{equation}

To prove this statement (29), write

\begin{equation}
e^{xz - \frac{1}{2}z^2} = \sum_{n=0}^{\infty} f_n(x) \frac{z^n}{2^n n!}
\end{equation}
and observe that
\[(\partial_x + 2\partial_z) e^{xz - \frac{1}{4} x^2 - z^2} = (\partial_x + 2\partial_z) e^{-\frac{1}{4} (2x-z)^2} = 0.\]
Hence,
\[\partial_x e^{xz - \frac{1}{4} x^2 - z^2} = -2\partial_z e^{xz - \frac{1}{4} x^2 - z^2} = - \sum_{n=0}^{\infty} f_{n+1}(x) e^{-x^2} \frac{z^n}{2^n n!}.\]
It follows that
\[\frac{d}{dx} \left( f_n(x) e^{-x^2} \right) = -f_{n+1}(x) e^{-x^2},\]
so since \(f_0(x) = 1\), we conclude that \(f_n(x) = H_n(x)\) for all \(n = 0, 1, \ldots\).
Comparison of (29) with (12) shows that the Hermite polynomials are specializations of Bell polynomials:
\[H_n(x) = n! 2^n B_n(x, -\frac{1}{4}, 0, \ldots).\]
Applying (13) then gives the well-known formula
\[H_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n!}{(n-2j)! j!} (2x)^{n-2j}.\]
In the sequel, we will also make use of the conjugate Hermite polynomials. These can be defined using the following equivalent relations:
\[\tilde{H}_n(x) = e^{x^2} d^n \frac{d^n e^{-x^2}}{dx^n} = i^{-n} H_n(ix).\]

The harmonic oscillator. Write \(p = i\partial_x\), so that
\[T(x, \partial_x) = p^2 + x^2 = -\partial_x^2 + x^2\]
is the Hamiltonian of the quantum harmonic oscillator. We say that a function \(\psi(z)\) is quasi-rational if its log-derivative, \(\psi'(z)/\psi(z)\), is a rational function. The quasi-rational eigenfunctions of \(T\) are
\[\psi_n(x) = \begin{cases} e^{-\frac{x^2}{2}} H_n(x), & n \geq 0, \\ e^{\frac{x^2}{2}} \tilde{H}_{-n-1}(x), & n < 0 \end{cases},\]
where \(H_n(x)\) and \(\tilde{H}_n(x)\) are the Hermite and conjugate Hermite polynomials defined in (25) and (30). The eigenfunctions \(\psi_n, \ n \geq 0\), represent the bound states of the harmonic oscillator, while those with \(n < 0\)
do not satisfy the boundary conditions at $\pm \infty$ and instead represent virtual states. The corresponding eigenvalue relation is

\begin{equation}
T \psi_n = (2n + 1) \psi_n, \quad n \in \mathbb{Z}.
\end{equation}

It will be instructive to prove (32) using generating functions. Multiplication of (29) by $e^{-x^2/2}$ yields

\begin{equation}
\Psi_0(x, z) = e^{-\frac{1}{2}(x-z)^2 + \frac{1}{4}z^2},
\end{equation}

which serves as the generating function for the bound states of the harmonic oscillator:

\begin{equation}
\Psi_0(x, z) = \sum_{n=0}^{\infty} \psi_n(x) z^n 2^n n!.
\end{equation}

By a direct calculation, we have

\begin{equation}
T(x, \partial_x) \Psi_0(x, z) = (2z \partial_z + 1) \Psi_0(x, z).
\end{equation}

Applying the above relation to (34) and comparing the coefficients of the resulting power series then returns (32).

The classical ladder operators

\begin{equation}
L_\pm(x, \partial_x) := \partial_x \pm x
\end{equation}

satisfy the intertwining relations

\begin{equation}
TL_- = L_-(T - 2), \quad TL_+ = L_+(T + 2).
\end{equation}

As a consequence, we have the following lowering and raising relations for the bound states:

\begin{equation}
L_- \psi_n = 2n \psi_{n-1}, \quad n = 0, 1, 2, \ldots
\end{equation}

\begin{equation}
L_+ \psi_n = \psi_{n+1}.
\end{equation}

Relations (37) can also be established using generating functions; it suffices to observe that

\begin{equation}
L_-(x, \partial_x) \Psi_0(x, z) = z \Psi_0(x, z) = \sum_{n=0}^{\infty} 2n \psi_{n-1}(x) \frac{z^n}{2^n n!};
\end{equation}

\begin{equation}
L_+(x, \partial_x) \Psi_0(x, z) = 2 \partial_z \Psi_0(x, z) = \sum_{n=0}^{\infty} \psi_{n+1}(x) \frac{z^n}{2^n n!}.
\end{equation}
The canonical coherent state. The canonical coherent state
\[
\Phi_0(x, t; \alpha) := \exp \left(-it + \frac{1}{2}x^2 - \left(x - \frac{1}{2}e^{-2it}\alpha\right)^2\right)
\]
is a time-dependent eigenfunction of the lowering operator
\[
L_- (x, \partial_x) \Phi_0(x, t; \alpha) = \alpha e^{-2it} \Phi_0(x, t; \alpha),
\]
as well as an exact solution to the time-dependent Schrödinger equation:
\[
( -i \partial_t + T(x, \partial_x) ) \Phi_0(x, t; \alpha) = 0.
\]
Observe that
\[
\Phi_0(x, t; \alpha) = e^{-it} \Phi_0(x, \alpha e^{-2it}).
\]
Hence, the canonical coherent state may also be regarded a generating function for the bound states of the harmonic oscillator. Indeed, the change of variables (43) transforms the eigenvalue relation (41) into (38), and the TDSE (42) into relation (35).

Pseudo-Wronskians. Let \( M \in \mathcal{M} \) be a Maya diagram. Also, let \( 0 > s_1 \geq \cdots \geq s_p \) be the elements of \( K^-_M \) and \( 0 \leq t_1 \leq \cdots \leq t_q \) the elements of \( K^+_M \) arranged in the indicated order. Define
\[
H_M = e^{-pa} \text{Wr}[e^{2s_1} \tilde{H}_{s_1-1}, \ldots, e^{2s_p} \tilde{H}_{s_p-1}, H_{t_1}, \ldots, H_{t_q}],
\]
where \( H_n(x) \) and \( \tilde{H}_n(x) \), \( n = 0,1,2,\ldots \), are the classical Hermite and conjugate Hermite polynomials, and where \( \text{Wr} \) denotes the Wronskian determinant of the indicated functions. The polynomial nature of \( H_M \) becomes evident once we represent it as the following pseudo-Wronskian determinant [3]:
\[
H_M = \begin{vmatrix}
\tilde{H}_{s_1} & \tilde{H}_{s_1+1} & \cdots & \tilde{H}_{s_1+p+q-1} \\
\vdots & \ddots & \ddots & \vdots \\
\tilde{H}_{s_p} & \tilde{H}_{s_p+1} & \cdots & \tilde{H}_{s_p+p+q-1} \\
H_{t_1} & D_xH_{t_1} & \cdots & D_x^{p+q-1}H_{t_1} \\
\vdots & \ddots & \ddots & \vdots \\
H_{t_q} & D_xH_{t_q} & \cdots & D_x^{p+q-1}H_{t_q}
\end{vmatrix}.
\]
The proof of (45) can be found in [3]. The same article also showed that the normalized polynomial
\[
\hat{H}_M = \frac{(-1)^{pq}H_M}{\prod_{i<j}(2(s_j - s_i)) \prod_{i<j}(2(t_j - t_i))}
\]
is translation invariant:
\[
\hat{H}_M = \hat{H}_{M+n}, \quad n \in \mathbb{Z}.
\]
It follows that the normalized Hermite pseudo-Wronskian (46) has the following expression in terms of Schur functions:

\[(48) \quad \hat{H}_M(x) = 2^N N! d_\lambda S^{(\lambda)}(x, -\frac{1}{4}, 0, \ldots),\]

where \(N = |\lambda|\) and where \(d_\lambda\) is the combinatorial factor defined in (1).

**Rational Extensions.** In this section, we describe the correspondence between Maya diagrams and rational extensions of the harmonic oscillator.

Let \(M \in \mathcal{M}\) be a Maya diagram. The pseudo-Wronskian defined in (44) can now be expressed simply as

\[(49) \quad H_M(x) = e^{\sigma_M \frac{x^2}{2}} \text{Wr}[\psi_{k_1}(x), \ldots, \psi_{k_p}(x)],\]

where \(\psi_n(x), n \in \mathbb{Z},\) are the quasi-rational eigenfunctions (31), and where \(\sigma_M\) is the index defined in (4). The potential

\[(50) \quad U_M(x) = x^2 - 2 \frac{d^2}{dx^2} \log \text{Wr}[\psi_{k_1}, \ldots, \psi_{k_p}]\]

\[= x^2 + 2 \left( \frac{H'_M}{H_M} \right)^2 - \frac{2H''_M}{H_M} - 2\sigma_M\]

is a rational extension of the harmonic oscillator potential, so called because the terms following the \(x^2\) in (50) are all rational.

The corresponding Hamiltonian operator

\[(51) \quad T_M := -\frac{d^2}{dx^2} + U_M\]

is exactly solvable [2] with eigenfunctions

\[(52) \quad \psi_{M,m} = e^{\epsilon_M(m) \frac{x^2}{2}} \frac{\hat{H}_{pm(M)}}{H_M}, \quad \epsilon_M(m) = \begin{cases} -1 & \text{if } m \notin M \\ +1 & \text{if } m \in M \end{cases}, \quad m \in \mathbb{Z}\]

and eigenvalues

\[(53) \quad T_M \psi_{M,m} = (2m + 1) \psi_{M,m}, \quad m \in \mathbb{Z}.\]

(In (52), \(\hat{H}_M\) is the normalized pseudo-Wronskian defined in (46).) Relation (47) together with Eq. (5) implies that \(T_M\) and the corresponding eigenfunctions are translation covariant:

\[(54) \quad T_{M+n} = T_M + 2n, \quad n \in \mathbb{Z} \quad \psi_{M+n,m+n} = \psi_{M,n}.\]

As regards integrability, it should be noted that by the Krein-Adler theorem [2], \(H_M\) has no real zeros if and only if all finite \(\bullet\) segments
of $M$ have even size. It is precisely for such $M$ that $T_M$ corresponds to a self-adjoint operator and that the eigenfunctions $\psi_{M,m}$ are square-integrable for $m \in \mathbb{Z} \setminus M$. The set
\begin{equation}
I_M := \mathbb{Z} \setminus M = J^{(\lambda)} + \sigma_M
\end{equation}
of empty boxes of $M$ then serves as the index set for the bound states.

The generating function for the bound states of a rational extension can be given as follows:

**Proposition 5.** For a partition $\lambda$, define
\begin{equation}
\Psi^{(\lambda)}(x, z) = \frac{S^{(\lambda)}(x - z^{-1}, -\frac{1}{4} - \frac{1}{2}z^{-2}, -\frac{1}{3}z^{-3}, \ldots)}{S^{(\lambda)}(x, -\frac{1}{4}, 0, \ldots)} \Psi_0(x, z).
\end{equation}

Let $M \in \mathcal{M}$ be a Maya diagram and $\lambda$ the corresponding partition as per Proposition 1. Then
\begin{equation}
\Psi^{(\lambda)}(x, z) = \sum_{m \in I_M} \psi_{M,m}(x) \prod_{i=1}^{\ell} \frac{(m - m_i)}{(m - \sigma_M + \ell)!} \left(\frac{z}{2}\right)^{m - \sigma_M},
\end{equation}
where $m_1 > m_2 > \cdots$ is the decreasing enumeration of $M$.

**Proof.** This follows from (16), (22) and (23). \qed

Observe that if $M = \mathbb{Z}_-$ is the trivial Maya diagram, then (57) reduces to the classical generating function shown in (34).

**Ladder Operators.** In this section, we introduce ladder operators for the rational extensions $T_M$, $M \in \mathcal{M}$, defined above.

Intertwining relations have their origins in supersymmetric quantum mechanics (SUSYQM). For differential operators $A$, $T_1$, $T_2$, we say that $A$ intertwines $T_1$ and $T_2$ if
\begin{equation}
AT_1 = T_2 A.
\end{equation}
We will refer to $A$ as a ladder operator if $T_2 = T_1 + \lambda$ for some constant $\lambda$. As a direct consequence of (58), $A$ maps eigenfunctions of $T_1$ to eigenfunctions of $T_2$, possibly annihilating finitely many eigenfunctions. This means that if $A$ is a ladder operator, then $A$ has a well-defined raising or lowering action on the states of $T_1$.

Within the class of rational extensions, the intertwiners take the form
\begin{equation}
A_{M,K}[y] = \frac{\text{Wr}[\psi_{M,m_1}, \ldots, \psi_{M,m_p}, y]}{\text{Wr}[\psi_{M,m_1}, \ldots, \psi_{M,m_p}]},
\end{equation}
where $M \in \mathcal{M}$ is a Maya diagram, $K \in \mathbb{Z}$ is a finite set of integers, and where the $\psi_{M,m}$ are the quasi-rational eigenfunctions of $T_M$ defined in
By construction, \( A_{M,K} \) is a monic differential operator of order \( p \) that intertwines \( T_M \) and \( T_{f_K(M)} \).

Intertwiners \( A_{M_1,K_1} \) and \( A_{M_2,K_2} \) such that \( M_2 = f_K(M_1) \) can be composed according to
\[
A_{M_2,K_2} \circ A_{M_1,K_1} = A_{M_1,K_1 \ominus K_2} \circ p_{K_1,K_2}(T_M),
\]
where
\[
p_{K_1,K_2}(x) = \prod_{k \in K_1 \cap K_2} (2k + 1 - x).
\]

Since \( T_{M+n} = T_M + 2n \), the above intertwiners are also translation invariant:
\[
A_{M+n,K+n} = A_{M,K}, \quad n \in \mathbb{Z}.
\]

For additional details, see Section 4 of [4].

Now fix an integer \( n \in \mathbb{Z} \), and let
\[
K_{M}^{(n)} := (M + n) \ominus M,
\]
\[
L_{M}^{(n)} := A_{M,K_{M}^{(n)}}.
\]

Theorem 4.1 of [4] implies that, in this particular case, the intertwining relation takes the form
\[
L_{M}^{(n)} T_M = (T_M + 2n)L_{M}^{(n)}.
\]

We will refer to such an \( L_{M}^{(n)} \) as a ladder operator for the rational extension \( T_M \). The action of ladder operators on states is that of a lowering or raising operator according to
\[
L_n[\psi_{M,k}] = C_{M,n,k} \psi_{M,k-n}, \quad k \notin M,
\]
where \( C_{M,n,k} \) is zero if \( \psi_{M,k-n} \) is not a bound state, i.e., if \( k - n \notin M \). Otherwise, \( C_{M,n,k} \) is a rational number whose explicit form is given in [4].

The annihilator algebra. In this section we describe the annihilation operators of a rational extension. The situation is more complicated than in the canonical case, where the annihilation operators are generated by \( L_- = \partial_x + x \). For a non-trivial rational extension, the annihilation operators form a non-trivial ring of commuting operators with a structure determined by the combinatorics of the corresponding Maya diagram, as we will now show.

For \( q \in \mathbb{N} \), we say that a Maya diagram \( M \in \mathcal{M} \) is a \( q \)-core if \( M \subset M + q \). In such a case, the symmetric difference \((M + q) \ominus M\) is nothing but the set difference \((M + q) \setminus M\). It follows that the kernel of the ladder operator \( L_M^{(q)} \) is spanned by \( \psi_{M,m}, \quad m \in (M + q) \setminus M \).
We say that $q \in \mathbb{N}$ is a critical degree of a Maya diagram $M \in \mathcal{M}$ if $M$ is a $q$-core. Observe that if $q$ is a critical degree of $M$, then $q$ is a critical degree of $M + n$ for every $n \in \mathbb{N}$. Thus, the $q$-core property is an attribute of an unlabelled Maya diagram. The set of unlabelled Maya diagrams is naturally bijective to the set of partitions, and so we use $D(\lambda)$, where $\lambda$ is the partition defined in (6), to denote the set of all critical degrees of the unlabelled Maya diagram $M/\mathcal{Z}$. This definition is consistent with the definition of the $q$-core partition used in combinatorics; see [6] for more details.

A $q \in \mathbb{N}$ fails to be in $D(\lambda)$ if and only if there exists an occupied position on a Maya diagram $m \in M$ and an unoccupied position $k \in I_M$ such that $k = m - q$. The smallest empty position on a Maya diagram occurs at position $m_{\ell+1} + 1 = \sigma_M - \ell$, while the largest occupied position occurs at $m_1 = \lambda_1 - 1 + \sigma_M$. It then follows that

$$q_c := m_1 - (\sigma_M - \ell) + 1 = \lambda_1 + \ell$$

is a threshold critical degree, in the sense that $q \in D(\lambda)$ for all $q \geq q_c$ and $q_c - 1 \notin D(\lambda)$. See Figure 3 for an example.

Let $M \in \mathcal{M}$ be a Maya diagram and $\lambda$ the corresponding partition. Set $R(\lambda) = \text{span}\{z^q : q \in D(\lambda)\}$, and observe that if $q_1, q_2 \in D(\lambda)$, then $q_1 + q_2 \in D(\lambda)$ also. It follows that $R(\lambda)$ is closed with respect to multiplication; i.e. $R(\lambda)$ is an operator algebra. Also note that composition of annihilation operators on the left of (70) is equivalent to multiplication of eigenvalues on the right; it follows that $R(\lambda)$ is isomorphic to the ring of annihilation operators associated with the rational extension $T_M$.

Relations (57) and (70) entail the following action of the annihilators on the bound states:

$$L^{(q)}_M(x, \partial_x)\psi_{M,m}(x) = 2^q \gamma^{(q)}_M(m)\psi_{M,m-q}(x),$$

where $m \in I_M$, $q \in D(\lambda)$, and where

$$\gamma^{(q)}_M(x) = \prod_{k \in (M+q) \setminus M} (x - k).$$

Observe that $m - q \notin I_M$ if and only if $m \in M + q$. Hence, $\gamma^{(q)}_M(m) = 0$ when $\psi_{M,m}(x)$ is a well-defined bound state, but $\psi_{M,m-q}(x)$ fails to satisfy the boundary conditions at $\pm \infty$.

A Coherent State for Rational Extensions. We are now ready to present the construction wherein one may generalize the notion of a coherent state to any rational extension $T_M$ of the harmonic oscillator potential. We proceed, as in the canonical case, by constructing the coherent state in terms of the generating function. In [5], it was shown
that, in terms of the generating function $\Psi^{(\lambda)}(x, z)$, the eigenvalue relation (53) is equivalent to
\begin{equation}
T_M(x, \partial_x) \Psi^{(\lambda)}(x, z) = (z \partial_z + 1 + 2 \sigma_M) \Psi^{(\lambda)}(x, z).
\end{equation}
Using the same change of variables as in (43), let us therefore set
\begin{equation}
\Phi^{(\lambda)}(x, z) = e^{- (1 + 2 \sigma_M) t} \Psi^{(\lambda)}(x, e^{-2it} z).
\end{equation}
Then by construction, $\Phi^{(\lambda)}(x, t)$ is an exact solution of the time-dependent Schrödinger equation corresponding to the rational extension $T_M$:
\begin{equation}
i \partial_t \Phi^{(\lambda)}(x, t) = T_M(x, \partial_x) \Phi^{(\lambda)}(x, t).
\end{equation}
By Theorem 6.1 of [5], for every critical degree $q \in D^{(\lambda)}$, we have
\begin{equation}
L_M^{(q)}(x, \partial_x) \Psi^{(\lambda)}(x, z) = z^q \Psi^{(\lambda)}(x, z).
\end{equation}
In other words, the generating function is a joint eigenfunction of the annihilator algebra. From here, it isn’t difficult to modify (70) to obtain an eigenrelation for the coherent state $\Phi^{(\lambda)}(x, z)$ defined in (68).
\begin{equation}
L_M^{(q)}(x, \partial_x) \Phi^{(\lambda)}(x, t; \alpha) = \alpha^q e^{-2iqt} \Phi^{(\lambda)}(x, z).
\end{equation}
Hence, $\Phi^{(\lambda)}(x, t; \alpha)$ is a joint eigenfunction of the annihilator algebra and satisfies the definition of a coherent state in the sense of Barut-Girardello [1].

**Example.** As an example, we construct the coherent state corresponding to the index set $K = \{2, 3\}$. The corresponding Maya diagram, partition, and index are
\begin{align*}
M &= f_K(\mathbb{Z}_-) = \mathbb{Z}_- \cup \{2, 3\}, \quad \lambda = (2, 2), \quad \sigma_M = 2,
\end{align*}
while the corresponding rational extension is
\begin{equation}
T_M(x, \partial_x) = -\partial_x^2 + \left( x^2 + 4 + \frac{32 x^2}{4 x^4 + 3} - \frac{384 x^2}{(4 x^4 + 3)^2} \right).
\end{equation}

Table II shows the numerator polynomials of the first few bound states of $T_M$. These bound states are indexed by
\begin{equation}
I_M = \{0, 1, 4, 5, 6, \ldots \},
\end{equation}
and explicitly, the bound state with eigenvalue $2m + 1$, $m \in I_M$, can be written as
\begin{equation}
\psi_{M,m} = e^{\frac{1}{2} x^2} \frac{H_{M,m}(x)}{4 x^4 + 3}, \quad m \in J^{(\lambda)} + 2,
\end{equation}
Table 1. Bound states of the rational extension corresponding to the index set $K = \{2, 3\}$.

| $m$ | $H_{M,m}(x)$ |
|-----|--------------|
| 0   | $x^2 + \frac{1}{2}$ |
| 1   | $2x^3 + 3x$ |
| 4   | $16x^6 + 24x^4 + 36x^2 - 18$ |
| 5   | $32x^7 + 16x^5 + 40x^3 - 60x$ |
| 6   | $64x^8 - 64x^6 - 240x^2 + 60$ |

where

$$H_{M,m} = \frac{\text{Wr}(2x^2 - 1, 2x^3 - 3, H_m)}{4(m-2)(m-3)(4x^4 + 3)}, \quad m \geq 0, \ m \neq 2, 3.$$ 

In this case, the Schur function is

$$\Psi^{(\lambda)}(t_1, t_2, t_3) = \frac{t_1^4}{12} + t_2^2 - t_1 t_3.$$ 

Using (56), the generating function for the bound states is therefore

$$\Psi^{(\lambda)}(x, z) = \left(1 - \frac{16x^3}{4x^4 + 3}z^{-1} + \frac{12(2x^2 + 1)}{4x^4 + 3}z^{-2}\right)e^{\frac{1}{2}(x-z)^2 + \frac{1}{4}x^2}.$$ 

The set of critical degrees, meanwhile, is $D^{(\lambda)} = \{4, 5, \ldots\}$. Note that there are no critical degrees below the threshold $q_c = 2 + 2 = 4$. Figure 3 illustrates the fact that $q = 4$ is a critical degree and that $q = 3$ fails to be a critical degree since $0 + 3 \in M$ but $0 \notin M$.

The coherent state wave function

$$\Phi^{(\lambda)}(x, t; \alpha) = e^{-5it}\Psi^{(\lambda)}(x, \alpha e^{-2it})$$

is an exact solution of the corresponding time-dependent Schrödinger equation (69). The annihilators

$$L^{(4)}_M = A_{M,\{0,1,6,7\}},$$

$$L^{(5)}_M = A_{M,\{0,1,4,7,8\}},$$

$$L^{(6)}_M = A_{M,\{0,1,4,5,8,9\}},$$

$$L^{(7)}_M = A_{M,\{0,1,4,5,6,9,10\}}$$

are commuting differential operators that generate the annihilator algebra of this rational extension. In each case, direct calculation shows that $\Phi^{(\lambda)}(x, t; \alpha)$ is an eigenfunction of $L^{(q)}_M$ with eigenvalue $\alpha^q e^{-2qit}$. 

Figure 3. Top: The Maya diagram $M$ corresponding to index set $K = \{2, 3\}$. The corresponding partition and index are $\lambda = (2, 2)$ and $\sigma_M = 2$, respectively, while the threshold critical degree is $q_c = 4$. Middle: $M + 3$. Bottom: $M + 4$. Note that 4 is a critical degree since $M \subset M + 4$. However, 3 fails to be a critical degree since $3 \in M$ but $3 \notin M + 3$.

References

[1] A.O. Barut and L. Girardello. New coherent states associated with non-compact groups. *Communications in Mathematical Physics* **21.1** (1971): 41-55.

[2] David Gómez-Ullate, Yves Grandati, and Robert Milson, Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials, *J. Phys. A* **47** (2013), no. 1, 015203.

[3] D. Gómez-Ullate, Y. Grandati, and R. Milson. Durfee rectangles and pseudo-Wronskian equivalences for Hermite polynomials. *Studies in Applied Mathematics* **141** (2018): 596-625.

[4] D. Gómez-Ullate, Y. Grandati, Z. McIntyre and R. Milson, Ladder operators and rational extensions, *Proceedings of QTS 11*, CRM Series on Mathematical Physics, 2020.

[5] A. Kasman and R. Milson, The Adelic Grassmannian and Exceptional Hermite Polynomials, preprint arXiv:2006.10025

[6] Macdonald IG. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.

[7] M. Noumi and Y. Yamada, Symmetries in the fourth Painlevé equation and Okamoto polynomials, *Nagoya Math. J.* **153** (1999), 53-86.

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