TWO-BODY DIRAC EQUATIONS FOR
RELATIVISTIC BOUND STATES OF QUANTUM FIELD THEORY

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Abstract

We review a little-known treatment of the relativistic two-body bound-state problem - that provided by Two-Body Dirac Equations obtained from constraint dynamics. We describe some of its more important results, its relation to older formulations and to quantum field theory. We list a number of features crucial for the success of such a formulation, many of which are missing from other methods; we show how the treatment provided by Two-Body Dirac Equations encompasses each of them.

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I. INTRODUCTION

The correct formulation of relativistic two-body bound state wave equations and their connection to quantum field theory is an old problem going back to papers by Eddington and Gaunt \[1\] in 1928 (bases on conjectures by Heisenberg). But judging from the large variety of approaches \[2\] attempted even in recent years, this problem has no generally agreed-upon solution. Perhaps for this reason, recent field theory books have avoided this topic. For example, Steven Weinberg \[3\] states: “It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in satisfactory shape.”

In \[2\] we have listed many of the attempts at solving this problem made over the past 47 years. At the head of this list, appear the work of Nambu and the better-known work of Bethe and Salpeter, both developed over 20 years after the partly successful, semirelativistic equation of Breit. The Bethe-Salpeter equation is an integral equation in momentum space that is manifestly covariant, obtained directly from relativistic quantum field theory. Over the years, however, many problems have turned up to impede its direct implementation. These are the sources of the numerous attempts to reformulate the two body problem of relativistic quantum field theory. For example as pointed out by Weinberg \[3\] : “The uncrossed ladders can be summed by solving an integral equation known as the Bethe-Salpeter equation, but there is no rationale for selecting out this subset of diagrams unless both particles are non-relativistic, in which case the Bethe-Salpeter equation reduces to the ordinary nonrelativistic Schrödinger equation, plus relativistic corrections associated with the spin-orbit couplings that can be treated as small perturbations.” Furthermore, the Bethe-Salpeter equation in the ladder approximation possesses negative norm or ghost states, due to its treatment of the relative time degree of freedom - spoiling the naive interpretation of it as a quantum wave equation.

Salpeter and many others \[2\] have developed noncovariant instantaneous truncations of this equation. For a variety of reasons most of the attempts we have cited are not appropriate for the treatment of highly relativistic effects like those necessary for the calculation of quark-
antiquark bound states.

Among the authors who have tried to rectify this problem are Professor Robson and his collaborator Dr. Stanley who gave one of the first comprehensive attempts to obtain a potential-model description of the entire meson spectrum, combining exact relativistic kinematics with a non-perturbative treatment of the effects of the important spin-dependent interactions. On the other hand, our work treats much the same spectral phenomenology but remains close to Dirac’s own one body work by starting with a pair of covariant two-body Dirac equations (one for each particle) which forces certain restrictions on the various spin-dependent interactions that can appear.

II. TWO-BODY DIRAC EQUATIONS FROM CONSTRAINT DYNAMICS

The list of references on the relativistic two-body problem includes diverse alternative approaches. On the other hand the one-body Dirac equation [4] has no serious rivals. That equation is a well defined wave equation that can be solved nonperturbatively and serves as an example of a successful bound state equation.

A. The One-Body Dirac Equation

The free Dirac equation

\[(\gamma \cdot p + m)\psi = 0 \tag{2.1}\]

provides a relativistic version of Newton’s first law, with associated relativistic second law appearing as the four-vector substitution

\[p_\mu \rightarrow p_\mu - A_\mu \tag{2.2}\]

for electromagnetic interaction and as the minimal mass substitution

\[m \rightarrow m + S \tag{2.3}\]
for scalar interactions combining to give

$$(\gamma \cdot (p - A) + m + S)\psi = 0.2.4$$  \hspace{1cm} (2.4)$$

As we shall see, the two-body Dirac equations take simple forms that generalize this one to the interacting two-body system.

**B. The Breit and Eddington-Gaunt Equations**

The Breit equation \[5\] and most three dimensional truncations of the Bethe-Salpeter equation are neither well defined wave equations, nor manifestly covariant. Breit proposed his equation

$$E\Psi = \left\{ \alpha \cdot p_1 + \beta_1 m_1 + \hat{\alpha}_2 \cdot \hat{p}_2 + \beta_2 m_2 - \frac{\alpha}{r}[1 - \frac{1}{2}(\hat{\alpha}_1 \cdot \hat{\Omega} + \hat{\alpha}_1 \cdot \hat{r} \cdot \hat{\alpha}_2 \cdot \hat{r})] \right\} \Psi \hspace{1cm} (2.5)$$

in 1929 as a correction to an earlier defective equation which he called the Eddington-Gaunt equation.

The Eddington-Gaunt equation \[1\] has the form

$$E\Psi = \left\{ \alpha \cdot p_1 + \beta_1 m_1 + \hat{\alpha}_2 \cdot \hat{p}_2 + \beta_2 m_2 - \frac{\alpha}{r}(1 - \alpha \cdot \alpha) \right\} \Psi. \hspace{1cm} (2.6)$$

This equation was defective because it failed to include the semirelativistic electrodynamic interaction of Darwin \[3\], together with the Coulomb interaction in the combination

$$- \frac{\alpha}{r}[1 - \frac{1}{2}(\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \hat{r} \vec{v}_2 \cdot \hat{r})]. \hspace{1cm} (2.7)$$

This structure arises from the semirelativistic expansion of the Green function, including retardative terms through order \(1/c^2\).

$$\int \int J_1 G J_2 = \int d\tau_1 \int d\tau_2 \hat{x}_1 \hat{x}_2\delta[(x_1 - x_2)^2]$$

$$\rightarrow - \frac{\alpha}{r}[1 - \frac{1}{2}(\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \hat{r} \vec{v}_2 \cdot \hat{r})]. \hspace{1cm} (2.8)$$
Replacing velocities in the Darwin interaction by $\vec{\alpha}$’s led Breit to terms missing from Eddington’s approach.

In modern language the Eddington-Gaunt interaction is most closely connected to QED in the Feynman gauge while the Breit interaction is most closely connected to QED in the Coulomb gauge. Eddington’s incorrect implementation of the Feynman gauge produces the wrong QED spectrum while Breit’s correct implementation of the Coulomb gauge produces the correct QED spectrum for positronium and muonium.

Since the Breit equation is not covariant and not a well defined wave equation it must be handled by semirelativistic perturbative methods. For example, for QED it is standardly rearranged as the semirelativistic elaboration of the Schrodinger Equation:

$$H\psi = w\psi.$$  \hfill (2.9)

($w$ is the total c.m. energy) in which the Hamiltonian is the first-order perturbative form

$$H = (m_1 + \frac{\vec{p}^2}{2m_1} - \frac{(\vec{p}_1^2)^2}{8m_1^2}) + (m_2 + \frac{\vec{p}^2}{2m_2} - \frac{(\vec{p}_2^2)^2}{8m_2^2}) - \frac{\alpha}{r} +$$

$$-\alpha(-\frac{\vec{p}^2}{m_1m_2} + \frac{1}{2m_2m_2r}\vec{p}\cdot(1-\hat{r}\hat{r})\cdot\vec{p})_{ordered}$$

$$-\frac{1}{2}(\frac{1}{m_1} + \frac{1}{m_2})\delta(\vec{r}) - \frac{1}{4r^3}\cdot\Bigl[\{(\frac{1}{m_1} + \frac{2}{m_1m_2})\vec{\sigma}_1 + (\frac{1}{m_2} + \frac{2}{m_1m_2})\vec{\sigma}_2\}\]$$

$$+ \frac{1}{4m_1m_2}\{(\frac{8\pi}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\delta(\vec{r}) + \frac{\vec{\sigma}_1\cdot\vec{\sigma}_2}{r^3} - \frac{3\vec{\sigma}_1\cdot\vec{r}\vec{\sigma}_2\cdot\vec{r}}{r^5})\}. \hfill (2.10)$$

C. Manifestly Covariant Two-Body Dirac Equations

In the 1970’s, several authors used Dirac’s constraint mechanics to attack the relativistic two-body problem at its classical roots \[7\] successfully evading the so-called no interaction theorem \[8\]. Using this method, the present authors extended the constraint approach to pairs of spin one half particles to obtain two-body quantum bound state equations that
correct the defects in the Breit equation and more importantly in the ladder approximation to the Bethe-Salpeter equation, exorcising quantum ghosts by covariantly controlling the relative time variable. Those equations are the two-body Dirac equations of constraint dynamics. They possess a number of important features some of which are unique and which correct defects in patchwork approaches. They are manifestly covariant while yielding simple three-dimensional Schrödinger-like forms similar to those of their nonrelativistic counterparts. Their spin dependence is not put in by hand, as in patchwork approaches, but is determined naturally by the Dirac-like structure of the equations. These equations have passed numerous tests showing that they reproduce correct QED perturbative results when solved nonperturbatively. They thus qualify as bona fide wave equations. In addition, the Dirac forms of the equations automatically make unnecessary the ad hoc introduction of cut-off parameters. The relativistic potentials appearing in these equations are related directly to the interactions of perturbative quantum field theory or (for QCD) may be introduced semiphenomenologically. In QCD one can regard these manifestly covariant equations as an anticipation of those that may eventually emerge from lattice gauge theory as applied to meson spectroscopy.

As in the ordinary one-body Dirac equation Eq.(2.4), for particles interacting through world vector and scalar interactions the two-body Dirac equations equations take the general minimal coupling form

\[ S_1 \psi \equiv \gamma_{51}(\gamma_1 \cdot (p_1 - \tilde{A}_1) + m_1 + \tilde{S}_1)\psi = 0 \]  

\[ S_2 \psi \equiv \gamma_{52}(\gamma_2 \cdot (p_2 - \tilde{A}_2) + m_2 + \tilde{S}_2)\psi = 0. \]

These equations provide a non-perturbative framework for extrapolating perturbative field theoretic results into the highly relativistic regime of bound light quarks, in a quantum mechanically well defined way. That framework incorporates of three main properties:

I: Exact Lorentz covariance,

II: Minimal interaction structure
III: Compatibility of the two-equations (which leads to a relativistic 3rd law, covariantly restricting the relative momentum and energy while correctly structuring spin-dependent recoil):

\[ [S_1, S_2]\psi = 0. \quad (2.12) \]

The satisfaction of this requirement originates in part from the presence of supersymmetries in the (pseudo-)classical limit of the Two-Body Dirac Equations. The potentials in these equations for four-vector and world scalar interactions are intimately connected to those of Wheeler-Feynman Electrodynamics (and its scalar counterpart) \[^{11,10,12}\] and have been obtained systematically from perturbative quantum field theory \[^{13,14}\].

For vector interactions alone, their momentum and spin dependences take the simple "hyperbolic" forms \[^{15,16}\]

\[
\tilde{A}_1 = [1 - \cosh(\mathcal{G})]p_1 + \sinh(\mathcal{G})p_2 - \frac{i}{2}(\partial e^\mathcal{G} \cdot \gamma_2)\gamma_2
\]

\[
\tilde{A}_2 = [1 - \cosh(\mathcal{G})]p_2 + \sinh(\mathcal{G})p_1 + \frac{i}{2}(\partial e^\mathcal{G} \cdot \gamma_1)\gamma_1
\]

in which

\[
\mathcal{G} = -\frac{1}{2} \ln(1 - 2\mathcal{A}/w) \quad (2.14)
\]

(with \(w\) the total c.m. energy). We originally found the logarithm form in a derivation from the Wheeler-Feynman classical field theory. In fact, in quantum electrodynamics that form turns out to embody an eikonal summation of ladder and cross-ladder diagrams.

In Eqs.(2.14), the invariant \(\mathcal{A}\) is a function of the covariant spacelike particle separation

\[
x^\mu_{\perp} = x^\mu + \hat{P}^\mu(\hat{P} \cdot x)
\]

perpendicular to the total four-momentum, \(P\). (\(\hat{P} \equiv \frac{P}{w}\) is a time-like unit vector.) Its appearance signifies that the dynamics is independent of the relative time in the c.m system.

For lowest order electrodynamics,
\[ A = A(x_\perp) = -\frac{\alpha}{r} \]  

(2.16)

in which

\[ r \equiv \sqrt{x_\perp^2}. \]  

(2.17)

The form of the covariant spin-dependent terms and the fact that \( A \) depends on \( x_\perp \) are consequences of compatibility of the two Dirac equations \( ([S_1, S_2] \psi = 0) \). In quark-model calculations, the invariant \( A \) and its counterpart for the scalar interaction are chosen on semiphenomenological grounds.

These two-body Dirac equations bypass most of the difficulties of the Bethe-Salpeter equation that arise from the presence relative time and energy variables, and yield a three-dimensional but manifestly covariant rearrangement of the Bethe-Salpeter equation. The three dimensional character is partially embodied in the invariant \( r \) which reduces to the interparticle separation only in the c.m. system. The fact that the interaction is instantaneous in the c.m. system is a direct consequence of the compatibility of the two equations. It is not an ad hoc restriction imposed on the equation as is done for various instantaneous approximations of the Bethe-Salpeter equation \[14\].

Just as Dirac arrived at his equation by ”taking the square root of the Klein-Gordon equation” so these equations can be derived by rigorously ”taking the square root” of the corresponding compatible ”two body Klein-Gordon equations” \[10\]

\[ S_1 \equiv \gamma_{51} (\gamma_1 \cdot (p_1 - A_1) + m_1 + \tilde{S}_1) = \sqrt{(p_1 - A_1)^2 + (m_1 + S_1)^2 + ...} \]  

(2.18a)

\[ S_2 \equiv \gamma_{51} (\gamma_1 \cdot (p_1 - A_1) + m_1 + \tilde{S}_1) = \sqrt{(p_1 - A_1)^2 + (m_1 + S_1)^2 + ...}. \]  

(2.18b)

It is in this sense that it is most natural to call Eqs.(2.11) ”Two-Body Dirac equations” .

These equations are not only manifestly covariant but are also quantum-mechanically well-defined. That is, their covariant Schrödinger-like forms for the effective particle of relative motion
\[ (p^2 + \Phi_w(\sigma_1, \sigma_2, A(r), S(r)))\psi = b^2(w)\psi \]  \hspace{1cm} (2.19)

in which
\[ b^2(w) = \frac{1}{4w^2}(w^4 - 2(m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2), \]  \hspace{1cm} (2.20)

can be solved nonperturbatively for both QED and QCD bound state calculations since every term in \( \Phi_w(\sigma_1, \sigma_2, A(r), S(r)) \) is well defined (less singular than \(-1/4r^2\)) [17,14]. Furthermore, recent work has shown that the Schrödinger-like forms can be transformed into equations that, like their nonrelativistic counterparts, involve at most 2 coupled wave functions [18,19] even when non-central tensor forces or spin-difference-orbit interactions are present. Note that in the decoupled form (or any other convenient form) the specific forms of the spin dependent potentials are dictated (through the reduction process) by the interaction structure of the original Dirac equation Eq.2.11 and are not put in by hand.

We have checked the nonperturbative validity of these equations as well defined wave equations by solving them analytically and numerically to obtain the standard fine and hyperfine spectra of QED. For example, we obtained an exact spectral solution for the singlet positronium system \( A = -\alpha/r \)

\[ w = m \sqrt{2 + 2/ \left[ \sqrt{1 + \frac{\alpha^2}{(n + \sqrt{(l + \frac{1}{2})^2 - \alpha^2 - l - \frac{1}{2})^2}} \right] - \alpha^2} \]  \hspace{1cm} (2.21)

of the fully coupled system of 16-component equations [15,20]

\[ S_1\psi = S_2\psi = 0. \]

Such validation we claim ought to be required of all candidate equations for nonperturbative quark model calculations. No others on that list of 60, including the Salpeter reduction (the no-pair Breit equation) and the Blankenbecler-Sugar equation, have yet met this demand. If not required to meet this demand two body formalisms may lead to possibly spurious nonperturbative predictions. For example Spence and Vary [22] have shown that in their treatment, the no-pair Breit equation and the Blankenbecler-Sugar equation predict low
energy electron-positron resonances between 1.4 and 2.2 MeV in Bhabha scattering. Such states have not been observed in low energy electron-positron collisions nor are they predicted by our equations. With no rigorous check on nonperturbative solutions for ordinary QED bound states, how can the QCD results of such treatments be trusted?

Given a static potential model $V = V(r)$ for the quark-antiquark interaction we can incorporate it in a covariant way into our equations by [17,14]

a) replacing nonrelativistic $r$ by $\sqrt{x_1^2}$

b) parcelling out the static potential $V$ into the invariant functions $A(r)$ and $S(r)$. This step remains a partially phenomenological one. However, in our approach once $A$ and $S$ are fixed so are all the accompanying spin dependences. One cannot adjust the various parts independently, as is done in many approaches [21] which just add plausible potential energy terms to two-body relativistic kinetic energy operators.

D. Comparison of Two-Body Dirac with Breit, Eddington-Gaunt and other Approaches

Since the Two-Body Dirac Equations make quantum mechanical sense they can be solved nonperturbatively in QCD bound state calculations, just like the ordinary one-body Dirac equation (to which they reduce in the limit that either particle becomes infinitely massive. The well-defined potential structures function as a natural smoothing mechanism that yields the correct spectrum while avoiding singular effective potentials like delta functions that appear in the Breit reductions and most competing approaches. For example, the Pauli, i.e. Schrödinger-like forms of our equations make quantum mechanical sense in the strong potential, nonperturbative regime where relativistic effects of the wave operator on the wave function are not negligible. This claim is easiest to see by examining the connection [17] of the main spin-spin term in our equation with that of Breit.

$$\left(\text{Two – Body Dirac}\right) - \frac{1}{6} \sigma_1 \cdot \sigma_2 \partial^2 \ln(1 - \frac{2A}{w}) \rightarrow \frac{1}{3} \frac{\sigma_1 \cdot \sigma_2 \partial^2 A}{m_1 + m_2} \text{ Breit} \quad (2.22)$$
For $\mathcal{A}$’s that have singular short range behaviors like $-\alpha/r$ (QED) and $8\pi/27rlnr$ (QCD) the weak potential form which appears in the reduced Breit equation and most patchwork approaches can only be used in a perturbative calculation. The two-body Dirac form on the left can be used nonperturbatively with the logarithm term providing a natural smoothing mechanism avoiding the necessity of introducing singularity softening parameters in phenomenological approaches [17,14].

For QED, the Two-Body Dirac equations work naturally in the covariant Feynman gauge, organizing the diagrammatic summation in a more efficient way than does the BSE. They naturally produce an equation closely connected in form to the defective equation of Eddington but yielding a correct spectrum. They achieve this result by effectively summing an infinite number of ladder and cross ladder diagrams in a kind of eikonal approximation [23]. Can we see this effect directly? Since they form a compatible pair they can be combined in any number of equivalent ways, in particular in a Breit-like form [12,9,16,24]:

$$w\Psi = \{\alpha_1 \cdot \vec{p}_1 + \beta_1 m_1 + \alpha_2 \cdot \vec{p}_2 + \beta_2 m_2 + w(1 - \exp[-G(x_\perp)(1 - \alpha_1 \cdot \alpha_2)])\} \Psi , \hspace{1cm} (2.23)$$

$$G = -\frac{1}{2} ln(1 + \frac{2\alpha}{rw}) = -\frac{\alpha}{wr} + ... \hspace{1cm} (2.24)$$

Comparing this to Eddington and Gaunt’s

$$w\Psi = \{\bar{\alpha}_1 \cdot \bar{\vec{p}}_1 + \beta_1 m_1 + \bar{\alpha}_2 \cdot \bar{\vec{p}}_2 + \beta_2 m_2 - \frac{\alpha}{r}(1 - \bar{\alpha}_1 \cdot \bar{\alpha}_2)\} \Psi , \hspace{1cm} (2.25)$$

we see that by effectively stopping at lowest order, the Eddington-Gaunt equation was doomed to failure.

In detail, to all orders in the potential, our equations produce [16]

$$w(1 - \exp[-G(x_\perp)(1 - \alpha_1 \cdot \alpha_2)]) \hspace{1cm} (2.26)$$

$$= \mathcal{A}(1 - \bar{\alpha}_1 \cdot \bar{\alpha}_2) - \frac{\mathcal{A}^2}{w}(1 - \bar{\sigma}_1 \cdot \bar{\sigma}_2) - \frac{\mathcal{A}^3}{w^2(1 - 2\mathcal{A}/w)}(1 - \gamma_5 \gamma_5 + \bar{\alpha}_1 \cdot \bar{\alpha}_2 - \bar{\sigma}_1 \cdot \bar{\sigma}_2) . \hspace{1cm} (2.27)$$

From the point of view of the Breit form of our equations this means that not only do they contain vector interactions but other covariant interactions as well (e.g. pseudovector, scalar...
and pseudoscalar interactions), which plausibly can be viewed as originating in the sorts of products of vector interactions that occur in multiparticle exchange. Using a transformation due to Schwinger, we have shown elsewhere that the extra terms that remain in the weak potential limit the are canonically equivalent to Breit’s retardative terms [16].

E. Connection to Quantum Field Theory

The invariant forms $A$ and $S$ in the Two-Body Dirac Equations Eq.(2.11) may be systematically obtained from the corresponding quantum field theories [13,14]. The connection is

$$\Phi_w(\sigma_1,\sigma_2, A(r), S(r)) = \pi i \delta(\hat{P} \cdot p) K (1 + \bar{K})^{-1}$$

(2.28)

giving the quasipotential in terms of the Bethe-Salpeter kernel $K$ and its projection

$$\bar{K} = G K$$

(2.29)

in which

$$G \equiv \left( \frac{1}{p_1^2 + m_1^2 - i0} \frac{1}{p_2^2 + m_2^2 - i0} - \pi i \delta(P \cdot p) \frac{w}{p_1^2 - b^2(w) - i0} \right)$$

(2.30)

is the difference between forms of two-body propagators as given by the Bethe-Salpeter equation on the one hand and the constraint equations on the other (with the relativistic third law delta function.) ($p$ is the relative momentum and $P$ is the total momentum.) This connection is a sort of covariant version of the three-dimensional Lippman-Schwinger equation. The difference is that in this equation, derived by Sazdjian [13] as "quantum mechanical transform of the Bethe-Salpeter equation", the potential follows from the irreducible scattering matrix rather than the other way around as done in the nonrelativistic case. The present authors have also derived the effective potentials from classical field theory in the form of Fokker-Tetrode actions through comparison with the classical limit of the constraint equations [25,12].
III. TWO-BODY DIRAC EQUATIONS AND MESON SPECTROSCOPY

We have formulated a constraint version of the naive quark model for mesons by using the static Adler-Piran quark-antiquark potential \[\text{[26]}\], (covariantly reinterpreted) in our Two-Body Dirac Equations. Adler and Piran obtained their static quark potential from an effective non-linear field theory derived from QCD. It has the general form

\[ V_{AP}(r) = \Lambda(U(\Lambda r) + U_0) (= A + S). \] (3.1)

Since their potential is nonrelativistic it cannot distinguish between world scalar and vector potentials, simply representing the effect of their sum in the nonrelativistic limit. It incorporates asymptotic freedom analytically through

\[ \Lambda U(\Lambda r << 1) \sim 1/(r \ln \Lambda r) \] (3.2)

and linear confinement through

\[ \Lambda U(\Lambda r >> 1) \sim \Lambda^2 r. \] (3.3)

At long distances their potential includes not only the linear confinement piece but also subdominant logarithm terms among others

\[ V_{AP}(r) = \Lambda(c_1 x + c_2 \ln(x) + c_3/\sqrt{x} + c_4/x + c_5), \quad x \equiv \Lambda r > 2. \] (3.4)

The \( c_i \)'s are given by the Adler-Piran leading log-log model. The realistic Adler-Piran potential or ones like it such as the Richardson potential \[\text{[27]}\], fail miserably for light mesons when used in the nonrelativistic Schrödinger equation \[\text{[28]}\]. Exact covariance is essential to handle the light mesons if one insists, as we do, on using potentials closely tied to QCD.

We apportion the potential between the relativistic invariants \( S \) and \( A \) that determine the scalar and vector potentials according to the scheme

\[ A = e^{\exp(-\beta r)}[V_{AP} - \frac{c_4}{r}] + \frac{c_4}{r} + \frac{e_1 e_2}{r}, \] (3.5a)

\[ S = V_{AP} + \frac{e_1 e_2}{r} - A. \] (3.5b)
In this way we impose the requirement that at short distance the potential is strictly vector while at long distance the confining portion is scalar with Coulomb vector portions. The relativistic invariance of $S$ and $A$ follows by reinterpreting the variable $r$ as $r \equiv \sqrt{x^2}$.

Our quark model is a naive quark model in that we ignore flavor mixing and the effects on the bound state energies of decays. The results we obtain, using the same potentials for all of the mesons, are spectrally quite accurate, from the heaviest upsilonium states to the lowly pion.

**TABLE I - MESON MASSES FROM COVARIANT CONSTRAINT TWO-BODY DIRAC EQUATIONS**

| NAME | EXP. | THEORY |
|------|------|--------|
| $\Upsilon : b\bar{b} \ 1^3 S_1$ | 9.460( 0.2) | 9.453( 0.6) |
| $\Upsilon : b\bar{b} \ 1^3 P_0$ | 9.860( 1.3) | 9.842( 1.4) |
| $\Upsilon : b\bar{b} \ 1^3 P_1$ | 9.892( 0.7) | 9.889( 0.1) |
| $\Upsilon : b\bar{b} \ 1^3 P_2$ | 9.913( 0.6) | 9.921( 0.5) |
| $\Upsilon : b\bar{b} \ 2^3 S_1$ | 10.023( 0.3) | 10.022( 0.0) |
| $\Upsilon : b\bar{b} \ 2^3 P_0$ | 10.232( 0.6) | 10.227( 0.2) |
| $\Upsilon : b\bar{b} \ 2^3 P_1$ | 10.255( 0.5) | 10.257( 0.0) |
| $\Upsilon : b\bar{b} \ 2^3 P_2$ | 10.269( 0.4) | 10.277( 0.8) |
| $\Upsilon : b\bar{b} \ 3^3 S_1$ | 10.355( 0.5) | 10.359( 0.1) |
| $\Upsilon : b\bar{b} \ 4^3 S_1$ | 10.580( 3.5) | 10.614( 0.9) |
| $\Upsilon : b\bar{b} \ 5^3 S_1$ | 10.865( 8.0) | 10.826( 0.2) |
| $\Upsilon : b\bar{b} \ 6^3 S_1$ | 11.019( 8.0) | 11.013( 0.0) |
| $B : b\pi \ 1^1 S_0$ | 5.279( 1.8) | 5.273( 0.1) |
| $B : b\bar{d} \ 1^1 S_0$ | 5.279( 1.8) | 5.274( 0.1) |
| $B^* : b\pi \ 1^3 S_1$ | 5.325( 1.8) | 5.321( 0.1) |
| $B_s : b\bar{s} \ 1^1 S_0$ | 5.369( 2.0) | 5.368( 0.0) |
| $B_s : b\bar{s} \ 1^3 S_1$ | 5.416( 3.3) | 5.427( 0.1) |
|        |         |         |         |
|--------|---------|---------|---------|
| $\eta_c : cc$ 1$^1S_0$ | 2.980 ( 2.1) | 2.978 ( 0.0) |
| $\psi : cc$ 1$^3S_1$ | 3.097 ( 0.0) | 3.129 ( 12.6) |
| $\chi_0 : cc$ 1$^1P_1$ | 3.526 ( 0.2) | 3.520 ( 0.4) |
| $\chi_0 : cc$ 1$^3P_0$ | 3.415 ( 1.0) | 3.407 ( 0.4) |
| $\chi_1 : cc$ 1$^3P_1$ | 3.510 ( 0.1) | 3.507 ( 0.2) |
| $\chi_2 : cc$ 1$^3P_2$ | 3.556 ( 0.1) | 3.549 ( 0.6) |
| $\eta_c : cc$ 2$^1S_0$ | 3.594 ( 5.0) | 3.610 ( 0.1) |
| $\psi : cc$ 2$^3S_1$ | 3.686 ( 0.1) | 3.688 ( 0.1) |
| $\psi : cc$ 1$^3D_1$ | 3.770 ( 2.5) | 3.808 ( 2.0) |
| $\psi : cc$ 3$^3S_1$ | 4.040 ( 10.0) | 4.081 ( 0.2) |
| $\psi : cc$ 2$^3D_1$ | 4.159 ( 20.0) | 4.157 ( 0.0) |
| $\psi : cc$ 3$^3D_1$ | 4.415 ( 6.0) | 4.454 ( 0.4) |
| $D : cc$ 1$^1S_0$ | 1.865 ( 0.5) | 1.866 ( 0.0) |
| $D : cc$ 1$^3S_0$ | 1.869 ( 0.5) | 1.873 ( 0.1) |
| $D^* : cc$ 1$^3S_1$ | 2.007 ( 0.5) | 2.000 ( 0.4) |
| $D^* : cc$ 1$^3S_1$ | 2.010 ( 0.5) | 2.005 ( 0.3) |
| $D^* : cc$ 1$^3P_1$ | 2.422 ( 1.8) | 2.407 ( 0.6) |
| $D^* : cc$ 1$^3P_1$ | 2.428 ( 1.8) | 2.411 ( 0.5) |
| $D^* : cc$ 1$^3P_2$ | 2.459 ( 2.0) | 2.382 ( 11.3) |
| $D^* : cc$ 1$^3P_2$ | 2.459 ( 4.0) | 2.386 ( 3.5) |
| $D_s : cc$ 1$^1S_0$ | 1.968 ( 0.6) | 1.976 ( 0.5) |
| $D_s^* : cc$ 1$^3S_1$ | 2.112 ( 0.7) | 2.123 ( 0.9) |
| $D_s^* : cc$ 1$^3P_1$ | 2.535 ( 0.3) | 2.511 ( 6.2) |
| $D_s^* : cc$ 1$^3P_2$ | 2.574 ( 1.7) | 2.514 ( 9.6) |
| $K : ss$ 1$^1S_0$ | 0.494 ( 0.0) | 0.492 ( 0.0) |
| $K : ss$ 1$^1S_0$ | 0.498 ( 0.0) | 0.492 ( 0.4) |
| $K^* : ss$ 1$^3S_1$ | 0.892 ( 0.2) | 0.910 ( 0.6) |
| $K^* : ss$ 1$^3S_1$ | 0.896 ( 0.3) | 0.910 ( 0.3) |
| State | Quantum Numbers | Energy Difference | Statistical Error |
|-------|-----------------|------------------|------------------|
| $K_1$ : $s \bar{u} \, 1^3 P_1$ |  | 1.273 (7.0) | 1.408 (3.2) |
| $K_1^*$ : $s \bar{u} \, 1^3 P_0$ |  | 1.429 (4.0) | 1.314 (0.7) |
| $K_1$ : $s \bar{u} \, 1^3 P_1$ |  | 1.402 (7.0) | 1.506 (1.0) |
| $K_2^*$ : $s \bar{u} \, 1^3 P_2$ |  | 1.425 (1.3) | 1.394 (0.5) |
| $K_2^*$ : $s \bar{u} \, 1^3 P_2$ |  | 1.432 (1.3) | 1.394 (0.6) |
| $K^*$ : $s \bar{u} \, 2^1 S_0$ |  | 1.460 (30.0) | 1.591 (0.2) |
| $K^*$ : $s \bar{u} \, 2^3 S_1$ |  | 1.412 (12.0) | 1.800 (6.7) |
| $K_2$ : $s \bar{u} \, 1^1 D_2$ |  | 1.773 (8.0) | 1.877 (0.8) |
| $K^*$ : $s \bar{u} \, 1^3 D_1$ |  | 1.714 (20.0) | 1.985 (1.4) |
| $K_2$ : $s \bar{u} \, 1^3 D_2$ |  | 1.816 (10.0) | 1.945 (1.3) |
| $K_3$ : $s \bar{u} \, 1^3 D_3$ |  | 1.770 (10.0) | 1.768 (0.0) |
| $K^*$ : $s \bar{u} \, 3^1 S_0$ |  | 1.830 (30.0) | 2.183 (1.4) |
| $K_2^*$ : $s \bar{u} \, 2^3 P_2$ |  | 1.975 (22.0) | 2.098 (0.2) |
| $K_4^*$ : $s \bar{u} \, 1^3 F_4$ |  | 2.045 (9.0) | 2.078 (0.1) |
| $K_4$ : $s \bar{u} \, 2^3 D_2$ |  | 2.247 (17.0) | 2.373 (0.5) |
| $K_5^*$ : $s \bar{u} \, 1^3 G_5$ |  | 2.382 (33.0) | 2.344 (0.0) |
| $K_3^*$ : $s \bar{u} \, 2^3 F_3$ |  | 2.324 (24.0) | 2.636 (1.9) |
| $K_4^*$ : $s \bar{u} \, 2^3 F_4$ |  | 2.490 (20.0) | 2.757 (1.6) |
| $\phi$ : $s \bar{u} \, 1^3 S_1$ |  | 1.019 (0.0) | 1.033 (2.2) |
| $f_0$ : $s \bar{u} \, 1^3 P_0$ |  | 1.370 (40.0) | 1.319 (0.0) |
| $f_1$ : $s \bar{u} \, 1^3 P_1$ |  | 1.512 (4.0) | 1.533 (0.3) |
| $f_2$ : $s \bar{u} \, 1^3 P_2$ |  | 1.525 (5.0) | 1.493 (0.3) |
| $\phi$ : $s \bar{u} \, 2^3 S_1$ |  | 1.680 (20.0) | 1.850 (0.8) |
| $\phi$ : $s \bar{u} \, 1^3 D_3$ |  | 1.854 (7.0) | 1.848 (0.0) |
| $f_2$ : $s \bar{u} \, 2^3 P_2$ |  | 2.011 (69.0) | 2.160 (0.1) |
| $f_2$ : $s \bar{u} \, 3^3 P_2$ |  | 2.297 (28.0) | 2.629 (1.6) |
| $\tau$ : $u \bar{d} \, 1^1 S_0$ |  | 0.140 (0.0) | 0.144 (0.2) |
| $\rho$ : $u \bar{d} \, 1^3 S_1$ |  | 0.767 (1.2) | 0.792 (0.1) |
\begin{align*}
b_1 : u\bar{d} & 1^1P_1 & 1.231(10.0) & 1.392(2.1) \\
a_0 : u\bar{d} & 1^3P_0 & 1.450(40.0) & 1.491(0.0) \\
a_1 : u\bar{d} & 1^3P_1 & 1.230(40.0) & 1.568(0.7) \\
a_2 : u\bar{d} & 1^3P_2 & 1.318(0.7) & 1.310(0.0) \\
\pi : u\bar{d} & 2^1S_0 & 1.300(100.0) & 1.536(0.1) \\
\rho : u\bar{d} & 2^3S_1 & 1.465(25.0) & 1.775(1.4) \\
\rho : u\bar{d} & 1^1D_2 & 1.670(20.0) & 1.870(0.9) \\
\rho : u\bar{d} & 1^3D_1 & 1.700(20.0) & 1.986(1.9) \\
\rho_3 : u\bar{d} & 1^3D_3 & 1.691(5.0) & 1.710(0.0) \\
\pi : u\bar{d} & 3^1S_0 & 1.795(10.0) & 2.166(7.9) \\
\rho : u\bar{d} & 3^3S_1 & 2.149(17.0) & 2.333(0.7) \\
\rho_4 : u\bar{d} & 1^3F_4 & 2.037(26.0) & 2.033(0.0) \\
\pi_2 : u\bar{d} & 2^1D_2 & 2.090(29.0) & 2.367(0.5) \\
\rho_3 : u\bar{d} & 2^3D_3 & 2.250(45.0) & 2.305(0.0) \\
\rho_5 : u\bar{d} & 1^3G_5 & 2.330(35.0) & 2.307(0.0) \\
\rho_6 : u\bar{d} & 1^3H_6 & 2.450(130.0) & 2.547(0.0) \\
\chi^2 & & & 101.0
\end{align*}

(The numbers in parentheses represent experimental uncertainties and \(\chi^2\) contributions for each meson.)

The heavy upsilon fits to the ground and excited states are due more to the specific form of the Adler-Piran potential than to the relativistic features of the equations, since the heavy quark motions are nearly nonrelativistic. The spin-orbit splittings arise from our apportionment of the Adler potential into scalar and vector parts. The good fits to the heavy \(B\) mesons are due primarily to the fact that in the infinite quark mass limit our equations reduce to the one body Dirac equation.

For charmonium, the semirelativistic effects of the formalism become important, the higher order relativistic effects becoming more important in the \(D\) and then \(\phi\) mesons. The
only significant weakness in our fits (due to limitations of apportioning our potential only between scalar and vector potentials) manifests itself at the $LS$ multiplet level for the lighter mesons. For those states, partial inversion takes place.

However, the equations work well in the extreme relativistic domain of the very light mesons and their lower excitations - $(K, K^*, \pi, \rho)$ and first radial excitations.

In our equations, the pion is a Goldstone boson in the sense that its mass tends toward zero numerically in the limit in which the quark mass numerically goes toward zero. This may be seen in the accompanying plot. Note that the $\rho$ meson mass approaches a finite value in the chiral limit. This also holds true for the excited pion states.

![Graph showing the masses of pion and rho vs. quark mass](image)

Fig. 1. $\pi$ and $\rho$ masses versus quark mass (in MeV)

Our results have room for improvement in the spin-orbit splittings of the light mesons and other orbital and radial excitations. This is a consequence of our assumption that at long distance the confining interaction is pure scalar.

With just two parametric functions, (amounting to an ad hoc division of the Adler potential into vector and scalar parts), we are able to obtain a fit about as good as that obtained by Godfrey and Isgur [21], who use six parametric functions, basically one for each type of spin dependence. The Godfrey and Isgur results seem more accurate than ours for the spin-orbit splitting of the light mesons, likely due to their use of different parameters for
each spin-dependent part. Note that they must also introduce parameters that smooth the potential, a procedure that is not necessary in the two-body Dirac equations because of the connection of their structure to that of the one-body Dirac equation.

IV. TWO-BODY DIRAC EQUATIONS FOR GENERAL COVARIANT INTERACTIONS

One may generalize the interactions in the Two-Body Dirac Equations to ones other than vector and scalar. How does one determine the forms of these equations for arbitrary interactions? We use this opportunity to discuss the role played by supersymmetry in the origins of the constraint formalism for two-body Dirac equations \[10,11\].

A. The Role of Supersymmetry

We first rewrite one-body Dirac equation in terms of theta matrices defined by

$$\theta^\mu \equiv i\sqrt{\frac{1}{2}}\gamma^\mu, \quad \mu = 0, 1, 2, 3, \quad \theta_5 \equiv i\sqrt{\frac{1}{2}}\gamma_5,$$

(4.0a)

$$[\theta^\mu, \theta^\nu] = -\eta^{\mu\nu}, \quad [\theta_5, \theta^\mu] = 0, \quad [\theta_5, \theta_5] = -1.$$ (4.0b)

In terms of these the free Dirac equation and its Klein-Gordon square become

$$\mathcal{S}\psi \equiv (p \cdot \theta + m\theta_5)\psi = 0$$ (4.1)

in which $\mathcal{S}$ is literally the “operator square root” of Einstein’s mass shell condition:

$$\mathcal{S}^2\psi = -\frac{1}{2}(p^2 + m^2)\psi = 0.$$ (4.2)

The supersymmetry is most easily revealed through examination of ”pseudoclassical” mechanics - the ”correspondence-principle” limit of the Dirac equation. In that limit, these theta matrices become Grassmann variables obeying “pseudoclassical” Berezin-Marinov brackets which upon quantization generate the Dirac algebra \[29\]: \{\theta^\mu, \theta^\nu\} = i\eta^{\mu\nu}, \{\theta_5, \theta^\mu\} = 0, \{\theta_5, \theta_5\} = i.$
In this limit, the Dirac equation becomes a constraint imposed on both bosonic \((p)\) and fermionic \((\theta, \theta_5)\) variables:

\[
S \equiv (p \cdot \theta + m \theta_5) \approx 0 \quad (4.3)
\]

while the mass shell condition comes from the bracket

\[
\mathcal{H} \equiv \frac{1}{i} \{S, S\} = p^2 + m^2 \approx 0. \quad (4.4)
\]

We first examine the supersymmetry of the free Dirac constraint under transformations generated by the self-abelian \(\mathcal{G}\) defined by

\[
\mathcal{G} = p \cdot \theta + \sqrt{-p^2} \theta_5, \quad (4.5a)
\]

\[
\{\mathcal{G}, \mathcal{G}\} = 0. \quad (4.5b)
\]

That is,

\[
\{\mathcal{G}, S\} \approx 0. \quad (4.6)
\]

\(\mathcal{G}\) is a supersymmetry generator in the sense that it mixes bosonic variables like the momentum with the Grassmann or fermionic variables. To maintain this supersymmetry in the presence of interaction, we must introduce interactions that have zero brackets with \(\mathcal{G}\). The position four-vector \(x\) is not supersymmetric and in fact displays pseudoclassical zitterbewegegung. However, the “zitterbewegungless” position variable

\[
\tilde{x}^\mu = x^\mu + \frac{i \theta^\mu \theta_5}{m} \quad (4.7)
\]

is supersymmetric. But, because of the presence of \(m\) in this variable, this object must itself be modified in the presence of scalar interaction \(M = m + S\). We accomplish this by generalizing this position variable to the self-referent form

\[
\tilde{x}^\mu = x^\mu + \frac{i \theta^\mu \theta_5}{M(\tilde{x})}. \quad (4.8)
\]
This is a supersymmetric position variable appropriate for scalar interactions in the sense that

$$\{G, \bar{x}\} \approx 0. \quad (4.9)$$

The supersymmetric constraints (both fermionic and bosonic) then become

$$S = p \cdot \theta + M(\bar{x})\theta_5 \approx 0, \quad \frac{1}{i}\{S, S\} \equiv \mathcal{H} = p^2 + M^2(\bar{x}) \approx 0. \quad (4.10)$$

Since $\theta_5^2 = 0$, the expansion of the self-referent form truncates

$$M(\bar{x}) = M(x) + \frac{i\partial M(x) \cdot \theta_5}{M(x)} \quad (4.11)$$

so that the $S$ constraint assumes the standard form

$$S = p \cdot \theta + M(x)\theta_5 \approx 0 \quad (4.12)$$

while the mass-shell constraint becomes the familiar square

$$\frac{1}{i}\{S, S\} \equiv \mathcal{H} = p^2 + M^2(\bar{x}) = p^2 + M^2(x) + 2i\partial M(x) \cdot \theta_5. \quad (4.13)$$

One arrives back at the Dirac equation and its square for scalar interaction by replacing Grassmann variables with theta matrices and dynamical variables $x$ and $p$ with their operator forms.

$$S\psi = [p \cdot \theta + M(x)\theta_5]\psi = 0 \quad (4.14a)$$

$$\mathcal{H}\psi = [p^2 + M^2(x) + 2i\partial M(x) \cdot \theta_5]\psi = 0. \quad (4.14b)$$

These contain the usual spin-dependent corrections expected for scalar interactions. We have gone through this exercise in order to show that this structure is present in the usual result. The supersymmetry generated by $G$ and realized through the presence of $\bar{x}$ is a natural feature of both the free Dirac equation and its standard form for external scalar interaction.
How does one implement this supersymmetry in the case of two interacting particles? For two pseudoclassical free particles we begin with

\[ S_{i0} = p_i \cdot \theta_i + m_i \theta_{5i} \approx 0, \quad i = 1, 2 \tag{4.15} \]

in which the two sets of \( \theta \)'s are independent Grassmann variables so that the mutual \( S_{i0} \) bracket vanishes strongly:

\[ \{ S_{10}, S_{20} \} = 0. \tag{4.16} \]

In the presence of interaction, we require the preservation of supersymmetry for each spinning particle. For scalar interactions we accomplish this through

\[ m_i \rightarrow M_i (x_1 - x_2) \rightarrow M_i (\tilde{x}_1 - \tilde{x}_2) \equiv \tilde{M}_i, \quad i = 1, 2 \tag{4.17} \]

depending on a supersymmetric variable for each particle position

\[ \tilde{x}_i^\mu = x_i^\mu + \frac{i \theta_\mu \theta_{5i}}{M_i}, \quad i = 1, 2. \tag{4.18} \]

Note that the Grassmann Taylor expansions of the \( \tilde{M}_i \) truncate. Carrying out those expansions, we obtain the following two-body Dirac constraints:

\[ S_1 = p_1 \cdot \theta_1 + \tilde{M}_1 \theta_{51} = p_1 \cdot \theta_1 + M_1 \theta_{51} - \frac{i \partial M_1 \cdot \theta_1 \theta_{52} \theta_{51}}{M_2} \approx 0, \tag{4.19a} \]

\[ S_2 = p_2 \cdot \theta_2 + \tilde{M}_2 \theta_{52} = p_2 \cdot \theta_2 + M_2 \theta_{52} + \frac{i \partial M_2 \cdot \theta_2 \theta_{51} \theta_{52}}{M_1} \approx 0. \tag{4.19b} \]

whose “squares” become

\[ \frac{1}{i} \{ S_i, S_i \} \equiv \mathcal{H}_i = \tilde{p}_i^2 + \tilde{M}_i^2, \quad i = 1, 2. \tag{4.20} \]

One can show that all four \( S \) and \( \mathcal{H} \)'s are mutually compatible provided that two other conditions are met. Straightforward computation leads to

\[ \{ S_1, S_2 \} = -p_1 \frac{\partial M_2}{M_1} \theta_{51} \theta_{52} - p_2 \frac{\partial M_1}{M_2} \theta_{51} \theta_{52}. \tag{4.21} \]
This bracket vanishes strongly provided that

$$\partial(M_1^2 - M_2^2) = 0,$$

(4.22)

(the relativistic "third law" condition), and

$$M_i = M_i(x_\perp)$$

(4.23)

in which

$$x^\mu_\perp = (\eta^{\mu\nu} - \frac{(p_1 + p_2)^{\mu}(p_1 + p_2)^\nu}{(p_1 + p_2)^2})(x_1 - x_2)_\nu$$

(4.24)

which, eliminates covariantly the troublesome "relative time". Now we use these results to rewrite the constraints in a form which we can easily generalize to interactions other than scalar. The third law condition Eq. (4.22) has the solution

$$M_1^2 - M_2^2 = m_1^2 - m_2^2$$

(4.25)

which has the convenient invariant hyperbolic parametrization

$$M_1 = m_1 \cosh L + m_2 \sinh L, \quad M_2 = m_2 \cosh L + m_1 \sinh L,$$

(4.26)

in which

$$L = L(x_\perp).$$

(4.27)

In this case just one invariant function suffices to define the two-body interaction. For such interactions

$$\mathcal{H}_1 - \mathcal{H}_2 = p_1^2 + M_1^2 - p_2^2 - M_2^2 = p_1^2 + m_1^2 - p_2^2 - m_2^2 \approx 0$$

(4.28)

just as in the noninteracting case (and indeed also in the spinless interacting case [30,10]). When the potentials satisfy these conditions, the Dirac-like constraints $S_1$ and $S_2$ are compatible and generate a consistent "pseudoclassical" dynamics. Note that the combination of the constraints corresponding to the difference of the squares eliminates the relative energy in the c.m. rest frame. To see this write the constraints in terms of the total four-momentum
\[ P^{\mu} = p_{1}^{\mu} + p_{2}^{\mu}; \quad -P^{2} \equiv w^2; \quad \hat{P}^{\mu} \equiv P^{\mu}/w, \quad (4.29) \]

and relative four-momentum
\[ p^{\mu} = (\epsilon_{2} p_{2}^{\mu} - \epsilon_{1} p_{1}^{\mu})/w; \quad \epsilon_{1} + \epsilon_{2} = w, \quad \epsilon_{1} - \epsilon_{2} = (m_{1}^{2} - m_{2}^{2})/w. \quad (4.30) \]

(Note that \( p \) is canonically conjugate to \( x_{\perp} \). The \( \epsilon_{i} \)'s are the invariant c.m. energies of each of the (interacting) particles.) Thus the constraint difference
\[ \mathcal{H}_{1} - \mathcal{H}_{2} = -2 \mathcal{P} \cdot p \approx 0 \quad (4.31) \]

places an invariant restriction on the relative energy. The other linearly independent combination
\[ (\epsilon_{2} \mathcal{H}_{1} + \epsilon_{1} \mathcal{H}_{2})/w = p_{\perp}^{2} + \Phi_{w} - b^2(w) \approx 0, \quad (4.32) \]
in which
\[ b^2(w) = (w^4 - 2w^2(m_{1}^2 + m_{2}^2) + (m_{1}^2 - m_{2}^2)^2)/4w^2 \quad (4.33) \]

incorporates exact relativistic two-body kinematics and governs the dynamics through the quasipotential
\[ \Phi_{w} \equiv \tilde{M}_{1}^2 - m_{1}^2 = \tilde{M}_{2}^2 - m_{2}^2. \quad (4.34) \]

Just as in the one-body case, we canonically quantize this system by replacing the Grassmann variables \( \theta_{\mu i}, \bar{\theta}_{5i} \) \( i = 1, 2 \) by two mutually commuting sets of theta matrices, and
\[ \{x^{\mu}, p^{\nu}\} \rightarrow [x^{\mu}, p^{\nu}] = i\eta^{\mu\nu}. \quad (4.35) \]

This leads to strongly compatible \([S_{1}, S_{2}] = 0\) two-body Dirac equations (for scalar interactions) in minimal interaction form
\[ S_{1}\psi = (\theta_{1} \cdot p + \epsilon_{1} \theta_{1} \cdot \hat{P} + M_{1} \theta_{51} - i\partial L \cdot \theta_{3} \theta_{52} \theta_{51})\psi = 0, \quad (4.36a) \]
\[ S_{2}\psi = (-\theta_{2} \cdot p + \epsilon_{2} \theta_{2} \cdot \hat{P} + M_{2} \theta_{52} + i\partial L \cdot \theta_{1} \theta_{52} \theta_{51})\psi = 0, \quad (4.36b) \]
in which

\[ \partial L = \frac{\partial M_1}{M_2} = \frac{\partial M_2}{M_1}, \]  

\[ M_1 = m_1 \, chL + m_2 shL, \quad M_2 = m_2 \, chL + m_1 shL. \]  

We see that in these coupled Dirac equations the remnants of pseudoclassical supersymmetries are the extra spin dependent recoil corrections to the ordinary one-body Dirac equations. Without those terms the two equations would not be compatible. When \( M_i = m_i + S_i \), these extra terms vanish when one of the particles becomes infinitely heavy.

**B. Hyperbolic form of the two-body Dirac equations for General Covariant Interactions**

How do we introduce general interactions? We accomplish this by recasting the minimal interaction forms of the two-body Dirac equations into a more general form, one that generalizes the hyperbolic forms we encountered above. Simple identities such as

\[ ch^2(\Delta) - sh^2(\Delta) = 1 \]  

lead to

\[ S_1 \psi = (ch(\Delta)S_1 + sh(\Delta)S_2)\psi = 0, \]  

\[ S_2 \psi = (ch(\Delta)S_2 + sh(\Delta)S_1)\psi = 0, \]  

in which appear auxiliary constraints defined by

\[ S_1 \psi \equiv (S_{10}ch(\Delta) + S_{20}sh(\Delta))\psi = 0, \]  

\[ S_2 \psi \equiv (S_{20}ch(\Delta) + S_{10}sh(\Delta))\psi = 0, \]  

with
\[ \Delta = -\theta_{51}\theta_{52}L(x_\perp). \] (4.42)

Note that in this form, the interaction enters only through an invariant matrix function \( \Delta \) with all other spin-dependence contained in the kinetic free Dirac operators \( S_{10}, S_{20} \). One can show \[31,9\] that both \( S_i \) and \( S_i \) constraints are compatible for general \( \Delta \):

\[ [S_1, S_2] \psi = 0 \text{ and } [S_1, S_2] \psi = 0 \] (4.43)

provided only that

\[ \Delta = \Delta(x_\perp). \] (4.44)

For the polar interactions we find

\[ \Delta(x_\perp) = -L(x_\perp)\theta_{51}\theta_{52} \text{ scalar} \] (4.45a)

\[ \Delta(x_\perp) = J(x_\perp)\hat{P} \cdot \theta_1\hat{P} \cdot \theta_2 \text{ time like vector} \] (4.45b)

\[ \Delta(x_\perp) = G(x_\perp)\theta_{1\perp} \cdot \theta_{2\perp} \text{ space like vector} \] (4.45c)

\[ \Delta(x_\perp) = F(x_\perp)\theta_{1\perp} \cdot \theta_{2\perp} \theta_{51}\theta_{52}\hat{P} \cdot \theta_1\hat{P} \cdot \theta_2 \text{ tensor (polar)} \] (4.45d)

The constraint equations for QED presented at the beginning of this paper are generated in this form by taking \( L = 0 = F \) and \( G = -J = -\frac{1}{2}ln(1 - 2A/w) \). For the QCD calculations which we performed above \( L \) is non zero and determined \[32\] by the \( S \) of Eq.(3.5b) while the vector interaction is written in terms of \( A \) of Eq.(3.5a) and just as for QED has the Feynman gauge combination \( J = -G \). For the axial counterparts the hyperbolic forms of our constraints are (note the minus sign)

\[ S_1\psi = (ch(\Delta)S_1 - sh(\Delta)S_2)\psi = 0 \] (4.46a)

\[ S_2\psi = (ch(\Delta)S_2 - sh(\Delta)S_1)\psi = 0, \] (4.46b)
in which \( S_1 \) and \( S_2 \) are defined as before while the interactions appear through

\[
\Delta(x_{\perp}) = C(x_{\perp})/2 \text{ pseudoscalar} \tag{4.47a}
\]

\[
\Delta(x_{\perp}) = H(x_{\perp}) \hat{P} \cdot \theta_1 \hat{P} \cdot \theta_2 \theta_{51} \theta_{52} \text{ time like pseudovector} \tag{4.47b}
\]

\[
\Delta(x_{\perp}) = I(x_{\perp}) \theta_{1\perp} \cdot \theta_{2\perp} \theta_{51} \theta_{52} \text{ space like pseudovector} \tag{4.47c}
\]

\[
\Delta(x_{\perp}) = Y(x_{\perp}) \theta_{1\perp} \cdot \theta_{2\perp} \hat{P} \cdot \theta_1 \hat{P} \cdot \theta_2 \text{ tensor (axial).} \tag{4.47d}
\]

Future research will determine the relative importance of the interactions other than scalar and vector in meson spectroscopy.

We conclude by listing important features realized by these two body Dirac equations which are incompletely realized if present at all in other approaches to the relativistic two-body problem. A) manifest covariance, B) simple three-dimensional Schrödinger-like forms similar to their nonrelativistic counterparts, C) spin dependence dictated by the Dirac-like structure of the equations (not put in by hand), D) thoroughly tested in QED: the equations reproduce the correct perturbative results when solved nonperturbatively (they qualify as bona fide wave equations less likely to produce spurious nonperturbative effects), E) the close connection to the one body Dirac equations automatically eliminates the ad hoc introduction of cutoff parameters, G) its potentials are determined through straightforward connection to perturbative quantum field theory or introduced semiphenomenologically.
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