In this work we shall study the late-time behavior of $k$-Essence $f(R)$ gravity without scalar potential, in the presence of matter and radiation perfect fluids. We quantify the late-time study by using the statefinder function $Y_{H}(z) = \frac{\dot{H}}{H} - 1$, which is a function of the redshift and of the Hubble rate. By appropriately rewriting the Friedmann equation in terms of the redshift and of the function $Y_{H}(z)$, we numerically solve it using appropriate initial conditions, and we critically examine the effects of the $k$-Essence higher order kinetic terms. As we demonstrate, the effect of the higher order scalar field kinetic terms on the late-time dynamics is radical, since the dark energy oscillations are absent, and in addition, the cosmological physical quantities are compatible with the latest Planck data and also the model is almost indistinguishable from the $\Lambda$ Cold Dark Matter model. This is in contrast to the standard $f(R)$ gravity case, where the oscillations are present. Furthermore, by choosing a different set of values of two of the free parameters of the model, and specifically the coefficient of the higher order kinetic term and of the exponent of $R^2$ appearing in the $f(R)$ action, we demonstrate that it is possible to obtain $\rho_{DE} < 0$ for redshifts $z \sim 2-3.8$, which compiles phenomenologically with, and seems to explain, the observational data for the same redshifts, and also to obtain a viable cosmological evolution at $z \sim 0$, at least when the dark energy equation of state parameter and the dark energy density parameters are considered.

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I. INTRODUCTION

The last 25 years were crucial for the development of cosmology, since the observations elevated theoretical cosmology to be what we now know as precision cosmology. The observational data coming from the cosmic background radiation (CMB) confirmed a nearly scale invariant power spectrum of primordial curvature perturbations, while the observational data coming from SNIa standard candles indicated a striking late-time phenomenon, the currently accelerating Universe. The late-time era, is usually dubbed dark energy era, due to the fact that the physical process that drives this late-time acceleration era is still unknown. In order to have acceleration in standard Einstein-Hilbert gravity, the equation of state (EoS) parameter of the fluid that drives the acceleration must be $w < -\frac{1}{3}$, and a negative pressure is the main characteristic of the fluid that drives late-time acceleration. The cosmological constant $\Lambda$, is the simplest quantity that may generate the late-time acceleration, and up to date, the so-called $\Lambda$ Cold Dark Matter (ACDM) model is the most successful description of late-time physics, being quite compatible with the CMB data.

Apart from the successes of the ACDM model, there are several questions unanswered, mainly having to do with the dynamical nature of dark energy. In the ACDM model, the EoS parameter of dark energy is constant, however although this is compatible with the observations, it is not certain that the EoS parameter is constant. In fact, it might be evolving from a quintessential value to a phantom value. Apart from the above issue, the $H_0$-tension turns out to be a serious troubling problem that needs to be explained in a theoretical way, and has recently been discussed in the literature. Another quite important issue is the discrepancy between the CMB based value of $\Omega_m h^2 \simeq 0.12 \pm 0.001$ and the one which is evaluated from the Friedmann equation at $z = 2.34$, if one substitutes the observed value $H(z = 2.34) = 222 \pm 7 Km/Mpc/s$, which indicates that $\Omega_m h^2 \simeq 0.132 \pm 0.008$ if dark energy is absent. The description given in Ref. perfectly describes this issue, so we now share the description of Ref. in order to clarify how the findings of indicate a possible tension with the value of $\Omega_m h^2$ obtained from the CMB or the ACDM model. If the general relativistic Friedmann equation is assumed, $H^2(z) = \frac{8\pi G}{3} \rho_{m}(1+z)^3 (\kappa^2 = 8\pi G)$, or equivalently, $h^2(z) = \Omega_m h^2$, then by substituting $H(z = 2.34) = 222 \pm 7 Km/Mpc/s$, then we obtain $h^2(z) \simeq 0.132 \pm 0.008$, which is different from the CMB value $\Omega_m h^2 \simeq 0.12 \pm 0.001$. This is a serious issue, which should be theoretically explained.

It is conceivable that such theoretical issues cannot be harbored by standard Einstein-Hilbert gravity, and require formal extensions that may describe such involved physical behaviors. Modified gravity provides a solid theoretical

$f(R)$ Gravity $k$-Essence Late-time Phenomenology

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framework in the context of which phenomena such as the dark energy era, and also the dark matter issue can be consistently be described accurately, for reviews see [16–23], however with regard to dark matter, the particle dark matter description [24–26] still seems to be supported from observations, like in the bullet cluster. In fact, it is possible to describe in a unified way both the inflationary era and the late-time acceleration eras, using the same theoretical framework. This was firstly demonstrated in the context of \( f(R) \) gravity in Ref. [31], and several other \( f(R) \) gravity unified cosmologies appeared in the literature [33–35]. However, although \( f(R) \) gravity can describe a viable late-time era, compatible with the observational data and the \( \Lambda \)CDM model, there is a feature that haunts the \( f(R) \) gravity description of the dark energy era, namely the dark energy oscillations at larger redshifts [35]. Particularly, it is known that, due to the presence of higher derivatives of the Hubble rate, the \( f(R) \) gravity description of late-time evolution is plagued with dark energy oscillations during the last stages of the matter domination [35]. These oscillations are even more enhanced if statefinder quantities are considered, such as the deceleration parameter and the jerk. In general if quantities that contain higher derivatives of the Hubble rate are considered, the oscillations are more pronounced.

In this work we shall study the late-time behavior of \( k \)-Essence \( f(R) \) gravity models, with the \( k \)-Essence part containing only a canonical kinetic term for the scalar field, and higher order kinetic terms, without the presence of a scalar potential. \( k \)-Essence theories themselves are quite interesting phenomenologically, since firstly these survived after the striking GW170817 event [39] in 2017 (see Ref. [40] for a complete list of the viable modified gravity theories), which indicated that the gravitational wave speed is equal to one in natural units. Apart from this important feature, \( k \)-Essence theories can describe in a viable way both inflation and the late-time era, and for an important stream of papers on this issue see [41–62]. For the purposes of this work, we shall choose an appropriate \( f(R) \) gravity, in the presence of dust and radiation perfect fluids, which is extensively studied in Ref. [32], see also [13], which can describe in a unified way both the inflationary era and the dark energy era, and also can describe an early dark energy era, by adding an appropriate early dark energy term. We shall call the \( f(R) \) gravity model of Ref. [37], power-law corrected \( R^2 \) model, just for the purposes of this paper, in order to discriminate it from other power-law \( f(R) \) gravity models which contain powers of the curvature. As it is shown in Ref. [37], the power-law corrected \( R^2 \) model produces a viable late-time phenomenology, compatible with the Planck 2018 data [63], and mimics to a great extent the \( \Lambda \)CDM model [57]. We shall incorporate to the theory the \( k \)-Essence terms, and by numerically solving the Friedmann equation, we shall explore the effects of the \( k \)-Essence terms on the \( f(R) \) gravity late-time phenomenology. For our study, we shall express all the physical quantities in terms of the statefinder function \( Y_H(z) = \frac{\rho_m}{\rho_{DE}} \), which is a function of the redshift and of the Hubble rate. As we demonstrate, for a specific set of values of the free parameters of the model, the dark energy oscillations at large redshifts of the order \( z \sim 10 \), which are present in the simple \( f(R) \) gravity model, are absent in the case of the \( k \)-Essence \( f(R) \) gravity model, while at the same time the cosmological evolution remains viable and compatible with the Planck 2018 observational data and the \( \Lambda \)CDM model. Moreover, by using another set of values of the free parameters, we show that it is possible to comply with the observations of [14] on the value of the Hubble rate at \( z \sim 2.34 \). Our results indicate that the \( k \)-Essence terms may actually act as a compensating dark energy mechanism of the \( f(R) \) gravity effective fluid, and at the same time a viable evolution at \( z \sim 0 \) is obtained. For our analysis we investigate the behavior of several well-known statefinder functions, and we compare the results with the \( \Lambda \)CDM values and with the observational data.

This paper is organized as follows: In section II we present and discuss the theoretical model of \( k \)-Essence \( f(R) \) gravity. In section III we introduce the function \( Y_H(z) \) and by expressing the physical quantities in terms of \( Y_H(z) \) and the redshift, we rewrite the Friedmann equation in terms of \( Y_H(z) \) and its derivatives. In addition, in section III, we study numerically the late-time behavior of a specific \( k \)-Essence \( f(R) \) gravity model and we compare the results to the power-law \( f(R) \) gravity model and the \( \Lambda \)CDM model. Accordingly, we demonstrate how the \( k \)-Essence \( f(R) \) gravity model can explain the 2014 results on the Hubble rate value for redshifts \( z \sim 2.34 \) without the need for introducing a compensating dark energy term. Finally, the conclusions of our work follow at the end of the paper.

### II. \( f(R) \) GRAVITY \( k \)-ESSENCE FRAMEWORK

The \( k \)-Essence \( f(R) \) gravity theory belongs to the general class of theories of the form \( f(R, X, \phi) \), with \( X = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \). We shall assume that the gravitational action is,

\[
S = \int d^4 x \sqrt{-g} \left( \frac{f(R)}{2\kappa^2} + G(X) + \mathcal{L}_{\text{matter}} \right),
\]

1. We used Mathematica 9®
where \( f(R) \) is an arbitrary function of the Ricci scalar to be specified later on, \( G(X) \) is a function depending solely on the kinetic term \( X = \frac{1}{2} \partial \rho \partial \phi \) and \( \kappa^2 = 8\pi G = \frac{8\pi G}{M_p^2} \), where \( G \) is Newton’s constant and \( M_p \) is the reduced Planck mass. In addition, \( \mathcal{L}_{\text{matter}} \) denotes the Lagrangian of the perfect matter fluids that are present. Moreover, the background geometry will be assumed to be a flat Friedmann-Robertson-Walker (FRW) metric, with line element,

\[
 ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2 ,
\]

where \( a(t) \) is the scale factor. From now on, we assume that the scalar field is homogenous, meaning it is only time dependent. Recalling the definition of the kinetic term \( X \) and the line element, we get,

\[
 X = -\frac{1}{2} \dot{\phi}^2 ,
\]

In order to find the equations of motion, we vary the gravitational action \(^1\) with respect to the metric tensor and to the scalar field, and the gravitational equations of motion are,

\[
 \begin{align*}
 \kappa^2 \rho + \frac{1}{2}(FR - f) + \kappa^2 G_X(X)X - 3H \dot{F} &= 3FH^2 , \\
 \kappa^2 (\rho + P) + \ddot{F} - H \dot{F} + 2H \dot{F} + \kappa^2 G_X(X)X &= 0 , \\
 \frac{1}{a^3} \frac{d}{dt}(a^3 G_X(X)\dot{\phi}) &= 0 ,
\end{align*}
\]

where \( F = \frac{\partial f}{\partial R} \), \( G_X = \frac{\partial G}{\partial X} \), and \( \rho \) stands for the energy density of the matter perfect fluids that are present, and \( P \) is the corresponding pressure. Also

For the purposes of this work we shall assume that both non-relativistic matter (cold dark matter and baryons) and relativistic matter (radiation) are present, so \( \rho \) is equal to,

\[
 \rho = \rho_m^{(0)} \left( \frac{1}{a^3} + \chi \frac{1}{a^3} \right) ,
\]

where \( \chi = \frac{\rho_m^{(0)}}{\rho_m} \). As we already mentioned in the introduction, the gravitational wave speed (speed of tensor metric perturbations) is for the \( f(R, X, \phi) \) theory at hand \( c_T^2 = 1 \), but we need to mention for the sake of completeness that the sound wave speed of the perturbations of the theory is non-trivial,

\[
 c_A^2 = \frac{XG_X + \frac{3 \dot{F}^2}{2F}}{XG_X + 2X^2G_{XX} + \frac{3 \dot{F}^2}{2F}} ,
\]

with \( G_{XX} = \frac{\partial^2 G}{\partial X^2} \). However, this wave speed affects the scalar and tensor perturbations, and will not affect the late-time behavior of the model. Having presented in brief the theoretical framework we shall consider, in the next section we shall express the gravitational equations in terms of suitable statefinder functions and in terms of the redshift and we shall consider the late-time behavior of a specific \( f(R, X, \phi) \) model, with quite interesting late-time phenomenology.

### III. A VIABLE \( f(R) \) GRAVITY \( k \)-ESSENCE MODEL AND COMPARISON WITH STANDARD \( f(R) \) GRAVITY

In order to study the late-time era of the \( k \)-Essence \( f(R) \) gravity model, we shall introduce appropriate functions that will quantify our study accurately. Firstly, we shall use the redshift as a dynamical variable, defined as,

\[
 1 + z = \frac{1}{a} ,
\]

where we took that the present scale factor of the Universe is unity, so present time corresponds to \( z = 0 \). By using,

\[
 \frac{d}{dt} = -H(1 + z) \frac{d}{dz} ,
\]
and $H = H(z)$ we shall express all the quantities in the gravitational equations as functions of the redshift. The derivatives with respect to the cosmic time, correspond to the following derivatives with respect to the redshift,

$$
\begin{align*}
\dot{F} &= -H(1 + z)F_z, \\
\dot{H} &= -H(1 + z)H_z, \\
\dot{\phi} &= -H(1 + z)\phi_z, \\
\end{align*}
\tag{9}
$$

where $F_z = \frac{dF}{dz}$ and $F_{zz} = \frac{d^2F}{dz^2}$. We shall use this notation hereafter, so the subscript to a function will mean the total or partial derivative of the function with respect to the variable appearing to the subscript. The Ricci scalar for a flat FRW spacetime is,

$$
R = 12H^2 + 6\dot{H},
\tag{10}
$$

so this can be expressed as a function of the redshift as follows,

$$
R = 12H^2 - 6HH_z(1 + z).
\tag{11}
$$

Also, the equation of motion of the scalar field is easily obtained,

$$
H\left(\frac{dG_X}{dz}H\phi_z + G_XH_z\phi_z + G_XH\phi_{zz} - \frac{2G_XH\phi_z}{1 + z}\right) = 0.
\tag{12}
$$

In order to better quantify the late-time behavior of the $k$-Essence $f(R)$ gravity model, we shall introduce the following function $Y_H(z)$ \cite{38, 67},

$$
Y_H(z) = \frac{\rho_{DE}}{\rho_m^0},
\tag{13}
$$

with $\rho_m^0$ being the present time energy density of non-relativistic matter. In the above equation, $\rho_{DE}$ is the energy density of the dark energy fluid, which now consists of the $k$-Essence terms and of the $f(R)$ gravity terms. Actually, the field equations can be cast in an Einstein-Hilbert form for a flat FRW metric as follows,

$$
\begin{align*}
3H^2 &= \kappa^2\rho_{tot}, \\
-2\dot{H} &= \kappa^2(\rho_{tot} + P_{tot}),
\end{align*}
\tag{14}
$$

with $\rho_{tot} = \rho_m + \rho_{DE} + \rho_r$ denoting the total energy density of the effective cosmological fluid and correspondingly, $P_{tot} = P_r + P_{DE}$ stands for the total pressure of the cosmological fluid. The effective cosmological fluid in our case receives contributions from the cold dark matter ($\rho_m$) and radiation fluids ($\rho_r$), but also from the combined $k$-Essence and $f(R)$ gravity fluids ($\rho_{DE}$), with the latter fluids being responsible for the late-time evolution. The energy density of dark energy fluid, which has combined contributions from the $k$-Essence and $f(R)$ gravity fluids, is equal to,

$$
\kappa^2\rho_{DE} = \frac{1}{2}(FR - f) + \kappa^2G_X(X)X - 3H\dot{F} - 3H^2(F - 1),
\tag{15}
$$

which can easily be read off from the Friedmann equation \cite{41}. Accordingly, from the Raychaudhuri equation \cite{44}, the pressure of the dark energy fluid is,

$$
\kappa^2P_{DE} = \dot{F} - H\dot{F} + 2\dot{H}(F - 1) + \kappa^2G_X(X)X - \kappa^2\rho_{DE}.
\tag{16}
$$

In the way chosen in the above equations, all the fluids that constitute the total cosmological effective fluid, are perfect fluids, non-interacting, and satisfy the continuity equations,

$$
\begin{align*}
\dot{\rho}_a + 3H(\rho_a + P_a) &= 0, \\
\dot{\rho}_r + 3H(\rho_r + P_r) &= 0, \\
\dot{\rho}_{DE} + 3H(\rho_{DE} + P_{DE}) &= 0.
\end{align*}
\tag{17}
$$

Having the fluid descriptions at hand, the function $Y_H(z)$ defined in Eq. \cite{13} can be written in terms of the Hubble rate by using the Friedman equation,

$$
3H^2 = \kappa^2(\rho(z) + \rho_{DE}),
\tag{18}
$$
and it reads,
\[ \mathcal{Y}_H(z) = \frac{H^2}{m_z^2} - (1 + z)^3 - \chi (1 + z)^4. \] (19)

Obviously, the function \( \mathcal{Y}_H(z) \) is a statefinder function since it depends only on the Hubble rate and the redshift. Also the parameter \( m_z^2 \) appearing in Eq. (19) is \( m_z^2 = \frac{\kappa^2 \rho^{(0)}_m}{3} = H_0 \Omega_m = 1.87101 \times 10^{-67} \text{eV}^2 \), and we used the current observational data coming from Planck 2018 [66] for the definition of the Hubble rate and \( \Omega_m \) (see also later on in this section the discussion on the values of the cosmological parameters). We shall express every quantity entering the Friedmann equation in terms of the function \( \mathcal{Y}_H(z) \), so practically, the Hubble rate is expressed in terms of the function \( \mathcal{Y}_H(z) \), and we have,
\[ H^2 = m_z^2 \left( \mathcal{Y}_H + \frac{\rho}{\rho^{(0)}_m} \right), \] (20)

and accordingly, by differentiating the above with respect to \( z \) we get,
\[ H H_z = \frac{m_z^2}{2} \left( \frac{d \mathcal{Y}_H}{dz} + \frac{\rho_{zz}}{\rho^{(0)}_m} \right), \] (21)

where the subscript “\( z \)” denotes differentiation with respect to \( z \). Assuming that the \( f(R) \) gravity function is written as \( f(R) = R + f_0(R) \) where again, \( f_0(R) \) is an arbitrary function of the Ricci scalar, then \( F = \frac{df}{dR} \) reads,
\[ F = 1 + \frac{df_0}{dR}. \] (22)

Similarly, the derivative of \( F \) with respect to \( z \) is equal to,
\[ F_z = \frac{d}{dz} \left( \frac{df_0}{dR} \right). \] (23)

It is vital to derive an expression for the derivative of the Ricci scalar with respect to redshift \( z \), so we have,
\[ \frac{dR}{dz} = 18 H H_z - 6(1 + z)(H_z^2 + H H_{zz}). \] (24)

In addition, by further differentiating Eq. (21) we get,
\[ H_z^2 + H H_{zz} = \frac{m_z^2}{2} \left( \frac{d^2 \mathcal{Y}_H}{dz^2} + \frac{\rho_{zz}}{\rho^{(0)}_m} \right). \] (25)

Finally, another useful expression is the derivative of the kinetic term \( X \) with respect to the redshift \( z \), which is,
\[ \frac{dX}{dz} = -(1 + z) \phi_z ((1 + z) H H_z \phi_z + H^2 ((1 + z) \phi_{zz} + \phi_z)). \] (26)

Thus the gravitational equations that we will solve numerically have the following form,
\[ \kappa^2 \rho + \frac{1}{2} (FR - f) + \kappa^2 G_X(X) X + 3H^2 \left( (1 + z) F_z - F \right) = 0, \] (27)
\[ H \left( \frac{dG_X}{dz} H \phi_z + G_X H_z \phi_z + G_X H \phi_{zz} - \frac{2G_X H \phi_z}{1 + z} \right) = 0, \]

and the following definitions and expressions shall be used, firstly \( \rho(z) \) as a function of the redshift,
\[ \rho = \rho^{(0)}_m ((1 + z)^3 + \frac{\rho_{zz}^{(0)}}{\rho^{(0)}_m} (1 + z)^4) = \rho^{(0)}_m ((1 + z)^3 + \chi (1 + z)^4), \] (28)

and then, the Hubble rate and its derivatives with respect to the redshift,
\[ H^2 = m_z^2 \left( \mathcal{Y}_H + \frac{\rho}{\rho^{(0)}_m} \right), \] (29)
\[ HH_z = \frac{m_s^2}{2} \left( \frac{dY_H}{dz} + \frac{\rho_z}{\rho_m^{(0)}} \right), \]
\[ H_z^2 + HH_{zz} = \frac{m_s^2}{2} \left( \frac{d^2Y_H}{dz^2} + \frac{\rho_{zz}}{\rho_m^{(0)}} \right). \]

Furthermore, the Ricci scalar, its derivative with respect to the redshift and \( F \) and \( F_z \) are,
\[ R = 12H^2 - 6HH_z(1 + z), \]
\[ \frac{dR}{dz} = 18HH_z - 6(1 + z)(H_z^2 + HH_{zz}), \]
\[ F = 1 + \frac{df_0}{dR}, \]
\[ F_z = \frac{d^2f_0}{dR^2} \frac{dR}{dz}. \]

At this point, we shall specify the \( G(X) \) function appearing in Eq. \( 31 \), so we assume that,
\[ G(X) = \beta(X + \frac{1}{2} f_1 X^m), \]
where \( \beta \) is a dimensionless parameter which will be set equal to \( \beta = -1 \) in order to have a canonical kinetic term. We chose to leave this in general form in \( G(X) \), and not equal to \(-1\), in order to have the phantom scalar case available, but this is not our case though. Also \( f_1 \) has mass dimensions \([m]^{4-m}\). For the model \( 31 \) we have,
\[ G_X(X) = \beta(1 + \frac{m}{2} f_1 X^{m-1}), \]
\[ \frac{dG_X(X)}{dz} = \beta \frac{m(m-1)}{2} f_1 X^{m-2} \frac{dX}{dz}. \]

with the derivative of the kinetic term with respect to the redshift being equal to,
\[ \frac{dX}{dz} = -(1 + z)\phi_z(\Sigma H_z\phi_z + H^2((1 + z)\phi_{zz} + \phi_z)), \]
and the functions \( H \) and \( H_z \) are given in terms of the function \( Y_H(z) \) in Eqs. \( 29 \).

\[ \text{FIG. 1: Plots of the statefinder function } Y_H \text{ (left plot), and of } R/m_s^2 \text{ (right plot) for the } k \text{-Essence } f(R) \text{ gravity (blue curves) and for the power-law corrected } R^2 \text{ model (red curves) as functions of the redshift.} \]

**A. Late-time \( k \)-Essence \( f(R) \) Gravity Dynamics**

At this point, let us specify the \( f(R) \) gravity function in order to quantify the effect of the \( k \)-Essence terms on the late-time dynamics of the \( f(R) \) gravity theory. We shall choose the following \( f(R) \) gravity \( 37 \),
\[ f(R) = R + \frac{1}{M^2} R^2 - \gamma \Lambda \left( \frac{R}{3m_s^2} \right) ^\delta. \]
In the above equation, the parameter $m^2$ was defined below Eq. (10), and also $\delta$ is freely chosen in the interval $0 < \delta < 1$, while $\gamma$ is equal to $\gamma = 2$. The parameter $\delta$ shall be chosen equal to $\delta = 1/100$ for late-time phenomenological reasons. Moreover, the value of the parameter $\Lambda$ will be given later on in this section. Also, the parameter $M$ is chosen for inflationary phenomenological reasons equal to $M = 1.5 \times 10^{-5} \left( \frac{\rho}{c^2} \right)^{-1} M_p$, with $N$ being the $e$-foldings number during the inflationary era. The phenomenology of the model (34) is thoroughly investigated in Ref. [37].

Furthermore, the parameter denoted as $f_1$ appearing in the $G(X)$ function (31) of the $k$-Essence part of the Lagrangian shall be chosen proportional to the parameter $\Lambda$,

$$f_1 \sim \Lambda^{2-2m},$$

where $m$ is the power of the kinetic term in Eq. (31). When necessary, the results of this section shall be compared to the ones corresponding to the $\Lambda$CDM model, so the Hubble rate of the $\Lambda$CDM model is,

$$H_\Lambda(z) = H_0 \sqrt{\Omega_\Lambda + \Omega_m(z+1)^3 + \Omega_r(1+z)^4},$$

which we also used earlier for the definition of the parameter $\Lambda$. Moreover, the value of the parameter $\Lambda$ appearing in Eq. (31) will be taken equal to $\Lambda \simeq 11.895 \times 10^{-67}eV^2$ and in addition, the parameter $m^2$ expressed in eV $m^2 \simeq 1.87101 \times 10^{-67}eV^2$, while $M$ is $M \simeq 3.04375 \times 10^{22}eV$ for $N \sim 60$. Also, the fraction of the present time radiation to dark matter densities is,

$$\frac{\rho_r(0)}{\rho_m(0)} = \chi = 3.1 \times 10^{-4}.$$

Hence, we shall take into account only the CMB based value of the Hubble rate only, disregarding the Cepheid based value. In addition, according to the CMB extracted observational data $\Omega_c h^2$ is,

$$\Omega_c h^2 = 0.12 \pm 0.001,$$

so $H_0 = 67.4 km/sec/Mpc$ or equivalently $H_0 = 1.37187 \times 10^{-33}$eV, therefore $h \simeq 0.67$. Hence, we shall assume that the Hubble rate is 66,

$$H_0 = 67.4 \pm 0.5 \frac{km}{sec \times Mpc},$$

FIG. 2: Plots of $\phi(z)$ (left plot) and of $\phi'(z)$ (right plot) for the $k$-Essence $f(R)$ gravity as functions of the redshift.
with $H_0$ the present value of the Hubble rate, while $\Omega_\Lambda \simeq 0.681369$ and $\Omega_m \sim 0.3153$\[66\]. In addition, while $\Omega_r/\Omega_m \simeq \chi$, and we defined the parameter $\chi$ below Eq. (37).

Let us proceed to the choice of the initial conditions. Our numerical analysis will be focused on the redshift interval $z = [0, 10]$, and in the following the final redshift value will be $z_f = 10$. For the function $Y_H(z)$ the initial conditions are chosen to be\[37, 38\],

\[
Y_H(z = z_f) = \frac{\Lambda}{3m_s^2}(1 + \frac{1 + z_f}{1000}),
\]

\[
\frac{dY_H}{dz} \bigg|_{z=z_f} = \frac{\Lambda}{3m_s^2} \frac{1}{1000},
\]

where $z_f$ is the final redshift $z_f = 10$. In addition, for the scalar field the initial conditions are chosen to be,

\[
\phi(z = z_f) = 10^{-20} M_p, \quad \frac{d\phi}{dz} \bigg|_{z=z_f} = -10^{-20} M_p.
\]

Also, we shall assume that $f_1$ takes the value $f_1 = 3 \times 10^{-40} \Lambda^{2-2m}$, and also we shall take $m = 2$, so we have a quadratic higher order kinetic term in the Lagrangian of the $k$-Essence $f(R)$ gravity and also recall that $\beta = -1$ in order to have a canonical kinetic term for the scalar field. In effect, the quadratic higher order kinetic term appearing in the $k$-Essence $f(R)$ gravity is $\sim -f_1X^2$.

![FIG. 3: Plot of the dark energy EoS parameter $\omega_{DE}(z)$ for the $k$-Essence $f(R)$ gravity (blue curve) and for the power-law corrected $R^2$ model (red curve) as functions of the redshift.](image)

At this point let us present in detail the results of our numerical analysis. We shall focus on the behavior of the most important cosmological quantities, and of the most important statefinder functions that are used in the literature. Let us start with the statefinder function $Y_H(z)$ and the curvature $R$, and in Fig. 1 we present the plots of $Y_H$ (left plot), and of $R/m_s^2$ (right plot) for the $k$-Essence $f(R)$ gravity (blue curves) and for the power-law corrected $R^2$ model (red curves) as functions of the redshift. As it is obvious in both the plots, the dark energy oscillations are completely absent from the $k$-Essence $f(R)$ gravity theory, at least for redshifts up to $z \sim 10$. Also, it might seem that $Y_H$ for the $k$-Essence $f(R)$ gravity theory, is constant, however this is not true, for example at $z = 10$ we have $Y_H(10) = 2.14249$ while at redshift $z = 0$ we have $Y_H(0) = 2.1213$. Also in Fig. 2 we present the plots of $\phi(z)/M_p$ (left plot) and of $\phi'(z)/M_p^2$ (right plot) for the $k$-Essence $f(R)$ gravity model as functions of the redshift. As it can be seen, the values of the scalar field increase as the redshift drops. It is also notable that if we choose an initial condition $\phi'(z) > 0$ at $z = 10$, the scalar field takes negative values, but we did not study more this case. Let us proceed to the behavior and values of some important cosmological quantities, starting with the dark energy EoS parameter $\omega_{DE} = \frac{\rho_{DE}}{\rho_{DE}}$, which in terms of $Y_H$ is given below,

\[
\omega_{DE}(z) = -1 + \frac{1}{3}(z + 1) \frac{1}{Y_H(z)} \frac{dY_H(z)}{dz}.
\]

Notably, the dark energy EoS parameter is also a statefinder quantity since it depends implicitly on $H(z)$ and its higher first order derivatives. The value of the EoS parameter at present time is evaluated to be $\omega_{DE} = -0.999667$ for the $k$-Essence $f(R)$ gravity model, which is compatible with the latest Planck 2018 data\[66\] values $\omega_{DE} = -1.018 \pm 0.031$. 
Furthermore, in Fig. 3 we present the plot of the dark energy EoS parameter \( \omega_{DE}(z) \) for the k-Essence \( f(R) \) gravity (blue curve) and for the power-law corrected \( R^2 \) model (red curve) as functions of the redshift. In the plot we can see clearly that in the power-law corrected \( R^2 \) model case (red curve) the oscillations are strongly pronounced as the redshift increases, and in contrast, in the k-Essence \( f(R) \) gravity case, the oscillations are completely absent. Also it is notable that in the k-Essence \( f(R) \) gravity case, the dark energy EoS parameter is slowly varying, with \( \omega_{DE}(10) = -0.99967 \), while as we mentioned, the value at redshift zero is \( \omega_{DE} = -0.999667 \).

Let us now consider the behavior of several well-known statefinder quantities, and we shall be interested in the deceleration parameter \( q \), the jerk parameter \( j \), the parameter \( Om(z) \) and finally the parameter \( s \), which are given below,

\[
q = -1 - \frac{\dot{H}}{H^2}, \quad j = \frac{\ddot{H}}{H^3} - 3q - 2, \quad s = \frac{j - 1}{3(q - \frac{1}{2})}, \quad Om(z) = \frac{H(z)^2 - 1}{(1 + z)^3 - 1}.
\]

All the statefinder quantities are valuable for the study of the dark energy era, since they depend solely on the Hubble rate and its higher derivatives, hence they depend explicitly on the geometry of spacetime via the Hubble rate. The values of the aforementioned statefinder quantities for the \( \Lambda \)CDM model are presented in Table I where we also present the corresponding values for the k-Essence \( f(R) \) gravity model and for the power-law corrected \( R^2 \) model, for several redshifts. Let us firstly consider the statefinder \( Om(z) \) and in Fig. 4 we present the plots of \( Om(z) \) for the k-Essence \( f(R) \) gravity (blue curve), for the power-law corrected \( R^2 \) model (red curve) and the \( \Lambda \)CDM model (black dashed curve) as functions of the redshift. As it can be seen, for this specific statefinder quantity, there are differences between the three models, and also no oscillations are observed in the \( f(R) \) gravity related models, as expected since \( Om(z) \) depends only on the Hubble rate and not on its derivatives. Accordingly in Fig. 5 we present the plot of the deceleration parameter \( q \) as a function of the redshift, for the k-Essence \( f(R) \) gravity (blue curve), for the power-law corrected \( R^2 \) model (red curve). In this case, the oscillations in the k-Essence \( f(R) \) gravity model are completely eliminated, while these are present for the power-law corrected \( R^2 \) model. It is notable that both models are almost indistinguishable from the \( \Lambda \)CDM model. Finally, in Fig. 6 we present the plots of the statefinder function jerk \( j \) (left plot) and the of \( s \) (right plot) for the k-Essence \( f(R) \) gravity (blue curves), for the power-law corrected \( R^2 \) model (red curves). In this case too, the oscillations are completely absent in the k-Essence \( f(R) \) gravity case. Also, we need to
Note that the jerk for the $k$-Essence $f(R)$ gravity case is almost indistinguishable from the $\Lambda$CDM value, however it is not constant, as is probably inferred from Fig. it is slowly varying though. For example its value at a redshift $z = 10$ is $j(10) = 1.00677$ while at $z = 0$ is $j(0) = 0.99952$, which are both very close to the $\Lambda$CDM value $j = 1$. In Table I we gather the values of several cosmological quantities and statefinders for various redshifts values, for the $k$-Essence $f(R)$ gravity and the power-law corrected $R^2$ models, and we also quote the corresponding $\Lambda$CDM values, the latest Planck constraints or SNe Ia constraints applying for the deceleration parameter. All the models are viable, however the $k$-Essence $f(R)$ gravity model seems to be more close to the $\Lambda$CDM model for most of the quantities considered. In conclusion, our results indicate that the effect of the $k$-Essence terms on the late-time phenomenology

\[ \Omega_{DE}(0) \]
\[ \omega_{DE}(0) \]
\[ h(0) \]
\[ q(0) \]
\[ s(0) \]

\[ a \]

\[ b \]

\[ c \]

\[ d \]

\[ e \]

\[ f \]

\[ g \]

\[ h \]

\[ i \]

\[ j \]

\[ k \]

\[ l \]

\[ m \]

\[ n \]

\[ o \]

\[ p \]

\[ q \]

\[ r \]

\[ s \]

\[ t \]

\[ u \]

\[ v \]

\[ w \]

\[ x \]

\[ y \]

\[ z \]

\[ \phi(z = z_f) = 10^{-2} M_p \]
\[ \left. \frac{d\phi}{dz} \right|_{z=z_f} = -10^{-10} M_p \]

\[ 45 \]
but it notable that a stiff system is obtained if \( \phi'(z) \bigg|_{z=10} \sim -M_p \). Another important issue worthy of mentioning is the combined effect that possibly the \( R^2 \) and the \( k \)-Essence terms have. Particularly, the \( R^2 \) is known to eliminate the dark energy singularities \[68, 69\] and refines in general the behavior of terms that contain higher derivatives of the Hubble rate, so perhaps the combined effect of the \( k \)-Essence terms with the \( R^2 \) term eliminates completely the dark energy oscillations. Let us note that the dark energy singularities are connected with non-linear oscillations of the curvature scalar, during which a finite-time (sudden) singularity occurs in the curvature. For more details on this issue and on the way that the \( R^2 \) term cures the singularities, we refer the reader to \[68, 69\].

Finally, the parameters that strongly affect the late-time phenomenology are \( f_1 \) and \( \delta \), with \( f_1 \) being the coefficient of the higher order kinetic term in the Lagrangian \( X^2 \), and \( \delta \) is the exponent of \( \sim R^\delta \) in the \( f(R) \) gravity of Eq. \[54\]. In fact, for a specific range of values of these two parameters, the late-time phenomenology of the \( k \)-Essence \( f(R) \) gravity dramatically changes, and interesting physical results are obtained. In the next section we study in brief a phenomenologically interesting situation.

Let us note here that we performed all the above calculations by using the Planck 2018 value for the Hubble rate, namely the one appearing in Eq. \[55\], that is \( H_0 = 67.4 \text{ km/sec/Mpc} \) or equivalently \( H_0 = 1.37187 \times 10^{-34} \text{ eV} \), but in principle one could use the value predicted by other sources different from the CMB, like the ones in Refs. \[2, 3\], which predict a tension in the value of \( H_0 \sim 72 \text{ km/sec/Mpc} \). If we use the value \( H_0 \sim 72 \text{ km/sec/Mpc} \), the whole analysis we performed in this section shall change as it is conceivable, and in order for a correct viable late-time phenomenology to be obtained, the values of the free parameters must change. For example, if we use the value \( H_0 \sim 72 \text{ km/sec/Mpc} \) or equivalently \( H_0 = 2.13512 \times 10^{-33} \text{ eV} \), the plots we presented in this section will change. As it can be seen, there are some changes, and this will be more apparent if we evaluate the value of \( \omega(0) \), which by choosing \( H_0 \sim 72 \text{ km/sec/Mpc} \) we get \( \omega_{DE}(0) \sim -0.994612 \), which is slightly different from the value found in Table II for \( H_0 = 67.4 \text{ km/sec/Mpc} \), namely \( \omega_{DE}(0) = -0.995205 \). The slight difference is caused by the change in the parameter \( m_\gamma^2 \), with \( \delta \) in Eq. \[54\], and the parameters \( f_1 \) and \( m \) related to the coupling and the exponent of the higher order kinetic term \( \sim f_1 X^m \) in Eq. \[31\]. The choice of \( m \) we assumed in the text, was \( m = 2 \), and this was for simplicity reasons, because the quadratic higher order term is the simplest case we can have. Now the parameter \( \Lambda \) was chosen \( \Lambda \approx 11.895 \times 10^{-67} \text{ eV}^2 \) for two reasons, firstly in order for it to be of the order of the present day cosmological constant, and secondly in order the fine tuning on this parameter leads to more aesthetically optimal values for the parameters \( \gamma \) and \( \delta \), and more importantly all the choices lead to a viable cosmology. The parameter \( \delta \) must be in the interval \( 0 < \delta < 1 \), so if we choose \( \Lambda \approx 11.895 \times 10^{-67} \text{ eV}^2 \), the values \( \gamma = 2 \) and \( \delta = 1/100 \) result to a phenomenologically viable late-time phenomenology \[57\]. A slight change in \( \Lambda \) might require not so aesthetically appealing values for \( \gamma \) and \( \delta \) in order to achieve a viable late-time phenomenology, for instance \( \gamma = 0.561 \) and \( \delta = 1/103 \) for \( \Lambda \approx 7 \times 10^{-67} \text{ eV}^2 \). Now the
parameter $f_1$ is the only one that requires fine tuning, in order to obtain a viable late-time phenomenology for the $k$-Essence $f(R)$ gravity model, and the choice was $f_1 \sim \Lambda^{2-2m}$ appearing in Eq. (38).

IV. DYNAMICALLY SCREENED DARK ENERGY ERA AT LOW REDSHIFTS

One of the latest observations in the last decade was the measurement of the Hubble rate at low redshift $z \sim 2.34$, with value $H(z = 2.34) = 222 km/Mpc/sec$ [14]. It must be mentioned the result of [11], which indicates that the Hubble rate at higher redshift increases, is also supported by other groups in the literature, see for example [72–75], however, other measurements at the same redshift do not exist to our knowledge, so caution is needed. For the purposes of this article, we shall assume that the measurement of [14] is correct, but in principle, in order to consider this result legitimate, this value has to be confirmed.

There are two ways to interpret the result of [14], if it is assumed to be correct, one to substitute $\rho_{DE} = 0$ in the Friedmann equation (23), which would imply that $\Omega_m h^2 = 0.142$ which is obviously in conflict with the CMB value reported by the Planck data $\Omega_m h^2 = 0.12 \pm 0.001$ [66]. Obviously, this would be a curious result, in the absence of dark energy, so the second way to interpret it would be to assume that dark energy terms are present, and these would result to negative $\rho_{DE}$ at $z \sim 2.34$ [12]. It is not the first time that negative dark energy density appears in the literature, see for example Refs. [72, 77] and references therein. What we would like to briefly demonstrate in this subsection is the possibility to generate a negative $\rho_{DE}$ contribution for redshifts $z \sim 2$, without the need of introducing a compensating dark energy term by hand, as it is done in Ref. [15]. In fact, the $k$-Essence $f(R)$ gravity framework generates such a phenomenological behavior, by simply choosing appropriately the parameters of the model. We shall again consider the $k$-Essence $f(R)$ gravity model of the previous section, with the same conventions for the cosmological parameters, and the same initial conditions, with the difference that we choose $f_1 = 4.01 \times 10^{-47} \Lambda^{2-2m}$ and also the power of the $\sim R^3$ term to be $\delta = 1/15$. Now, a negative $\rho_{DE}$ contribution in the context of our work, would imply negative values of the statefinder function $Y_H(z)$, and this is the aim of this subsection, to demonstrate that this is possible by using the $k$-Essence $f(R)$ gravity theory framework. We numerically solved the Friedmann equation, and the results of the behavior of the statefinder function $Y_H(z)$ as a function of the redshift are presented in the left plot of Fig. 8 while in the right plot we present the behavior of $\phi(z)/M_p$. As it can be seen, at $z = 2-3.8$ the function $Y_H(z)$ develops negative values, and as the redshift decreases, $Y_H(z)$ increases until present time. In Table II we quote the values of the statefinder $Y_H(z)$ for various redshifts for both the compensating $k$-Essence $f(R)$ gravity and for the power-law corrected $R^2$ model of the previous section. As it can be seen, for redshifts $z > 2$ the differences are quite significant. Thus the $k$-Essence $f(R)$ gravity theoretical framework provides a natural compensating dark energy mechanism, which we do not introduce by hand. We shall call it for the purposes of this paper compensating $k$-Essence $f(R)$ gravity. It is worth investigating further the phenomenology of this case, so we evaluated the dark energy EoS parameter $\omega_{DE}(0)$ at present time and also the dark energy density parameter $\Omega_{DE}$, and the results are,

$$\Omega_{DE}(0) = 0.685836, \quad \omega_{DE}(0) = -0.995346,$$

which are both compatible with the latest Planck constraints $\Omega_{DE} = 0.6847 \pm 0.0073$ and $\omega_{DE} = -1.018 \pm 0.031$. In general, by analyzing the statefinder quantities it turns out that the resulting phenomenology is marginally appealing, though peculiar and quite different from the case studied in the previous section. In order to see this more clearly,
we chose to present in Fig. 10 the plots of the deceleration parameter $q$ (left plot) and the $Om(z)$ statefinder (right plot), for the compensating $k$-Essence $f(R)$ gravity model (blue curves) and the $Λ$CDM model (red curves). As it is obvious from the plots, the compensating $k$-Essence $f(R)$ gravity model is quite different from the $Λ$CDM model, and only at very small redshifts near the present-time era there is some overlap between the two models, at least when the deceleration parameter is considered. It is worth quoting here the values of the deceleration parameter and $Om(z)$ statefinder for $z = 0$ for the compensating $k$-Essence $f(R)$ gravity, and these are,

$$q(0) = -0.523917, \quad Om(0.0000000001) = 0.317389,$$

which are quite close to the $Λ$CDM values $q = -0.535$ and $Om(z) = 0.3153$. We refrain from going into further details, since the general picture is obvious, the result is that we obtain a marginally compatible to the $Λ$CDM phenomenology, only at low redshifts, while at larger redshifts, there are differences. However, with the compensating $k$-Essence $f(R)$ gravity model we obtain negative dark energy density at nearly $z \sim 2.34$, which can explain the observational values of the Hubble rate \cite{14} at the same redshift. Our model presented in this section offers a phenomenological description which can provide phenomenologically acceptable values to some of the cosmological quantities of interest at redshift zero (present day), like the $Ω_{DE}(0)$, $ω_{DE}(0)$, the deceleration parameter and the statefinder $Om(z)$, and also can yield negative values for the statefinder $Y_H(z)$ at redshift $z \sim 2.34$. However, the model cannot be considered a fully correct description of the Universe, neither at present time, nor at higher redshifts, since the overall behavior of some observable quantities is not phenomenologically acceptable. For example, in Fig. 8 we plot $ω_{DE}(z)$ for redshifts $z = [0,10]$ for the compensating $k$-Essence $f(R)$ gravity model, and the result is rather unappealing and phenomenologically not acceptable, although the value of $ω_{DE}(z)$ at redshift zero is phenomenologically acceptable. Also the blue curves in Fig. 8 do not provide an optimal fit to the $Λ$CDM model. What we aimed in this section is to demonstrate that negative values of the statefinder $Y_H(z)$ can be obtained by the $k$-Essence $f(R)$ gravity model, by appropriately tuning some parameters of the model, and at the same time obtaining compatibility with the observational data for some of the observable quantities at present time. Our description though cannot be considered as a fully viable description of the Universe, and also the results of Ref. \cite{14} must also be widely accepted in order to further study how our model can be a viable description of the Universe, up to redshifts $z \sim 3$.
FIG. 9: Plot of $\omega_{DE}(z)$ for the compensating $k$-Essence $f(R)$ gravity model for $z = [0, 10]$.

FIG. 10: Plots of the deceleration parameter $q$ (left plot) and the $\Omega_m(z)$ statefinder (right plot), for the compensating $k$-Essence $f(R)$ gravity model (blue curves) and the $\Lambda$CDM model (red curves).

V. CONCLUSIONS

In this paper we studied the effects of $k$-Essence terms in the late-time phenomenology of $f(R)$ gravity in the presence of cold dark matter and radiation perfect fluids. We chose an $f(R)$ gravity model which has quite phenomenologically appealing late-time properties, and we assumed that a canonical scalar field term and a higher order kinetic term are also present in the gravitational Lagrangian. With regard to the higher order kinetic term, we studied the case that this term is a quadratic term of the form $\sim f_1 X^2$. The dimensionful parameter plays an important role on the phenomenology as it turns, and in fact, this term in conjunction with an $R^2$ term appearing in the $f(R)$ function, crucially affect the late-time phenomenology of the $k$-Essence $f(R)$ gravity model. The power-law corrected $R^2$ model apart from the fact that it is similar to the $\Lambda$CDM model, it was plagued with the issue of dark energy oscillations at large redshifts. These dark energy oscillations are more pronounced when physical quantities that contain higher derivatives of the Hubble rate are considered. Our aim in this work was to investigate whether these dark energy oscillations are eliminated from the late-time era in the context of $k$-Essence $f(R)$ gravity, and as it turns, this occurs for a wide range of initial conditions imposed on the scalar field, and for some values of the parameter $f_1$ which is the coefficient of the quadratic kinetic term $X^2$. As we demonstrated, a viable late-time phenomenology is produced by the $k$-Essence $f(R)$ gravity, without the presence of dark energy oscillations, at least up to a redshift $z \sim 10$. We studied several cosmological quantities of cosmological interest, such as the dark energy EoS parameter, the dark energy density parameter, and several statefinder quantities. In all the cases, the $k$-Essence $f(R)$ gravity model was almost indistinguishable from the $\Lambda$CDM model, as for example in the case of the deceleration parameter $q$ and the jerk, and in all cases the dark energy oscillations were absent. We also compared directly the power-law corrected $R^2$ model and the $k$-Essence $f(R)$ gravity models, to see the difference between the two models and the complete absence of dark energy oscillations for the $k$-Essence $f(R)$ gravity model. Apart from this major issue, we also investigated how the $k$-Essence $f(R)$ gravity model could explain the 2014 observational data concerning the Hubble
rate at redshift \( z \sim 2.34 \). In the presence of dark energy, this observation would require a negative energy density for dark energy. As we demonstrated, by using a specific set of values for the parameter \( f_1 \) and the exponent of \( R^2 \), we achieved a negative energy density for redshifts \( z \sim 2 - 3.8 \), and also we demonstrated that the resulting values of the dark energy EoS parameter and of the dark energy density parameter were compatible with the latest Planck data, however the model produced quite different behavior of the statefinder parameters in comparison to the \( \Lambda \text{CDM} \) model. Our findings support the idea that the data, however the model produced quite different behavior of the statefinder parameters in comparison to the \( \Lambda \text{CDM} \) of the dark energy EoS parameter and of the dark energy density parameter were compatible with the latest Planck (S.D.O). Finally, let us note that in our study we did not take into account the presence of a potential for the scalar field, and we did not investigate at all the case that the scalar field is a phantom scalar. We hope to address these issues in a future work.

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