Distributed Convex Optimization in Networks of Agents with Single-integrator Dynamics

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Abstract—In this paper, distributed convex optimization problem over undirected dynamical networks is studied. Here, networked agents with single-integrator dynamics are supposed to rendezvous at a point that is the solution of a global convex optimization problem with some local inequality constraints. To this end, all agents shall cooperate with their neighbors to seek the optimum point of the network’s global objective function. A distributed optimization algorithm based on the interior-point method is proposed, which combines an optimization algorithm with a nonlinear consensus protocol to find the optimum value of the global objective function. We tackle this problem by addressing its subproblems, namely a consensus problem and a convex optimization problem. First, we propose a saturation protocol for the consensus subproblem. Then to solve the distributed optimization part, we implement a centralized control law, which yields the optimum value of the global objective function, in a distributed fashion with the help of a distributed estimator. Convergence analysis for the proposed protocol based on the Lyapunov stability theory for time-varying nonlinear systems is included. A simulation example is given at the end to illustrate the effectiveness of the proposed algorithm.

I. INTRODUCTION

In recent years, developing distributed paradigms for solving optimization problems among interconnected agents has attracted attention of researchers. We briefly review some of the existing works in this area.

The authors of [12] used the dual decomposition scheme, which maintains a small duality gap, to solve optimization problems in a network of dynamical nonlinear agents. The reference [9] exploited a subgradient-based distributed method to find the approximation of an optimal point associated with a collective convex function over a network of interconnected agents. The paper [8] proposed a zero-gradient-sum continuous-time algorithm to drive the states of a weight-balanced directed network to the optimal point of a global objective function along an invariant zero-gradient-sum manifold. The references [7], [15], [16] exploited the interior-point method to solve the convex optimization problems and attain the saddle point of the corresponding Lagrangian function, proper dynamics associated with primal and dual variables were developed in [4] that yield the optimal solution. In the above mentioned works, the complexity associated with the implementation of proposed algorithms increases as the number of agents or that of constraints corresponding with them becomes greater.

In this paper, we consider the constrained distributed optimal problem for single-integrator networks, where each agent has a convex objective function and personalized inequality constraints. To solve the problem, we divide it into a consensus problem and a distributed optimization one. To deal with the former problem, we utilize a continuous consensus protocol based on local information sharing. Then, we exploit a distributed optimization algorithm based on the interior-point method to solve the convex optimization problem. In the proposed algorithm, no Lagrangian variables associated with the consensus constraint, which is needed so that all agents attain the same optimum solution, and the local inequality constraints are required. This reduces the complexity of the proposed solution and its implementation. Moreover, the proposed algorithm can handle the limitations associated with actuator saturation that commonly occurs in practice.

This paper is structured as follows. The next section reviews some background materials required in this paper. The problem formulation and main results are introduced in Section III. Finally, Section IV concludes the paper.

II. PRELIMINARIES AND NOTATIONS

In this section, we briefly review some background materials required in this paper.

A. Graph Theory

In graph theory, \( G = (\mathcal{N}, E, A) \) denotes an undirected network, where \( \mathcal{N} = \{1, \cdots, N\} \) is the set of nodes. An edge between node \( i \) and node \( j \) is denoted by the pair \((i, j) \in E\) that indicates mutual communication between two nodes \( i \) and \( j \). The set \( E \subseteq \mathcal{N} \times \mathcal{N} \) represents the

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set of edges, and $A = \{a_{ij}\}_{N \times N}$ is the adjacency matrix. $A$ is symmetric and $a_{ij} = 1$ when $(i, j) \in E$, and $a_{ij} = 0$ indicates $(i, j) \notin E$. It is assumed that there is no repeated edge and no self-loop, i.e. $a_{ii} = 0$. The set of neighbors of node $i$ is denoted by $N_i = \{j \in V : (i, j) \in E\}$. Assume an arbitrary orientation for each edge in $G$, then $D = [d_{ik}] \in \mathbb{R}^{N \times |E|}$ is the incidence matrix associated with $G$, in which $d_{ik} = -1$ if the edge $(i, j)$ leaves node $i$, $d_{ik} = 1$ if it enters the node, and $d_{ik} = 0$ otherwise. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ associated with the graph $G$ is defined as $l_{ij} = \sum_{k=1}^{N} a_{ij}$ and $l_{jj} = -\sum_{k=1}^{N} a_{ij}$ for $i \neq j$. Note that $L = DD^T$. The Laplacian matrix $L$ is semi-positive definite, and if $1 \in \mathbb{R}^N$ denotes a vector of which elements are all 1, then, $L1 = 0$ and $1^T L = 0$. The Laplacian matrix $L$ has one zero eigenvalue if the graph $G$ is connected. All eigenvalues of $L$ are non-negative. We define consensus error in a network by $\bar{e}_x = \Pi \bar{x}$ where $\Pi = I_N - \frac{1}{N} 1_N 1_N^T$, and $\bar{x}$ denotes the aggregate state of the network as $\bar{x} = [x_1 \ldots x_N]^T$. Note that $1^T \Pi = 0$ and $\Pi 1 = 0$.

B. Stability of Perturbed Systems

Consider the nominal system

$$\dot{x} = f(x, t),$$  

(1)

where $f : D \times [0, \infty) \to \mathbb{R}^N$ is piecewise continuous in $t$ and locally Lipschitz in $x$ on $D \times [0, \infty)$, and $D \subset \mathbb{R}^N$ is a domain that contains the origin $x = 0$. Suppose that the system (1) is perturbed by the term $d(x, t)$, where $d : D \times [0, \infty) \to \mathbb{R}^N$ denotes perturbation and is called unvanished perturbation if $d(0, t) \neq 0$. Then, the perturbed system corresponding to (1) is given by

$$\dot{x} = f(x, t) + d(x, t).$$  

(2)

**Lemma 2.1.**  [6, Theorem 5.1] Let $V : D \times [0, \infty) \to \mathbb{R}^N$ be a continuously differentiable function such that

$$\frac{\partial V(x, t)}{\partial t} + \frac{\partial V}{\partial x} f(x, t) \leq -W_3(x), \quad \forall \|x\| \geq \mu > 0,$$

$\forall t \geq 0$, $\forall x \in D$, where $W_1(x), W_2(x),$ and $W_3(x)$ are continuous positive definite functions on $D$. Take $r > 0$ such that $B_r \subset D$. Suppose that $\mu$ is small enough such that

$$\max_{\|x\| \leq \mu} W_2(x) < \min_{\|x\|=r} W_1(x).$$

Consider $\rho = \max_{\|x\| \leq \rho} W_2(x)$ and take $\rho$ such that $\eta < \rho < \min_{\|x\|=r} W_1(x)$. Then, there exists a finite time $t_1$ (dependent on $x(t_0)$ and $\mu$) such that $\forall x(t_0) \in \{x \in B_r : W_2(x) \leq \rho\}$, the solutions of $\dot{x} = f(x, t)$ satisfy $x(t) \in \{x \in B_r : W_1(x) \leq \rho\}, \forall t \geq t_1$. Moreover, if $D = \mathbb{R}^N$ and $W_1(x)$ is radially unbounded, then this result holds for any initial state and any $\mu$.

C. Notations

Throughout this paper, $\|\cdot\|_1$ and $\|\cdot\|$ denote 1-norm and 2-norm operators, respectively. $\mathbb{R}$ represents the real numbers set and $\mathbb{R}^+$ implies the positive real numbers subset. $\mathbb{R}^N$ includes all vectors with $N$ real elements. The term $\mathbb{R}^{N \times N}$ represents the set of all $N \times N$ matrices with real entries. Furthermore, $[\mathcal{M}_{ij}]_{N \times N}$ represents an $N \times N$ matrix with entries $\mathcal{M}_{ij}$, where the index $i$ stands for the $i$-th row and $j$ refers to $j$-th column.

III. Problem Statement and Main Results

Consider the single-integrator dynamics

$$\dot{x}(t) = u(t),$$  

(3)

where $u \in \mathbb{R}$ and $x \in \mathbb{R}$ denote the state and control input, respectively. Assume an objective function, say $Q(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, that is twice continuously differentiable and strictly convex in $x$.

**Lemma 3.1:** The following control input will make the dynamics (3) converge to the minimizer of the time-varying convex objective function $Q(x, t)$.

$$u(t) = -\left(\frac{\partial^2 Q(x, t)}{\partial x^2}\right)^{-1} \left(\frac{\partial Q(x, t)}{\partial x} + \frac{\partial^2 Q(x, t)}{\partial x \partial t}\right)$$  

(4)

**Proof:** Choose a Lyapunov function as $V(x, t) = \frac{1}{2} \left(\frac{\partial Q(x, t)}{\partial x}\right)^2$ and take its time derivative along the trajectories of the dynamics (3). Then, we have

$$\dot{V}(x, t) = -\left(\frac{\partial Q(x, t)}{\partial x}\right) \left(\frac{\partial^2 Q(x, t)}{\partial x^2}\right) \dot{x} + \frac{\partial^2 Q(x, t)}{\partial x \partial t}.$$  

By substituting $\dot{x}$ in the above relation from (3), the following is obtained

$$\dot{V}(x, t) = -\left(\frac{\partial Q(x, t)}{\partial x}\right)^2 \leq 0$$  

(5)

From the above inequality, it follows that $\frac{\partial Q(x, t)}{\partial x}$ remains bounded in $\mathbb{R}^n \cup \{\infty\}$, i.e. it belongs to $L^\infty$ space. With integrating from both sides of equality (5), in the view of passivity of $V(x, t)$, we have

$$\int_0^R \left(\frac{\partial Q(x, t)}{\partial x}\right)^2 dt = \int_0^R \dot{V}(x, t) dt = -V(x(R), R) + V(x(0), 0) \leq V(x(0), 0).$$  

(6)

So, $\frac{\partial Q(x, t)}{\partial x} \in L^2$. Now, we invoke Barbalat’s Lemma [13] and obtain that $\frac{\partial Q(x, t)}{\partial x}$ asymptotically converges to zero as $t \to \infty$. Thereby, the optimality condition is certified, i.e.

$$\frac{\partial Q(x, t)}{\partial x} = 0.$$  

Now, consider the following autonomous agents under the topology $\mathcal{G}$. Each agent is described by the continuous-time single-integrator dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{N}$$  

(7)
where \( x_i(t) \) and \( u_i(t) \in \mathbb{R} \) represent the position and the control input to agent \( i \), respectively. For the sake of notational brevity, we will use \( x_i \) and \( u_i \) in the rest of this paper. We suppose that agent \( i, \forall i \in \mathcal{N} \), can share its state’s information with agents within its neighborhood set, i.e. \( \mathcal{N}_i \), according to the communication graph \( \mathcal{G} \).

The agents are supposed to rendezvous at a point that shall minimize the aggregate convex function \( \sum_{i=1}^{N} f_i(x_i) \) with regards to individual convex inequalities \( g_i(x_i) \leq 0, i = 1, \ldots, N \). This problem can be described by

\[
\underset{x_i}{\text{min}} \sum_{i=1}^{N} f_i(x_i), \quad i \in \mathcal{N} \tag{8}
\]

subject to \( g_i(x_i) \leq 0, i \in \mathcal{N} \)

in which \( f_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is the local objective function associated with node \( i \) and \( g_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) represents a constraint on the optimal position imposed by \( i \)-th agent. It is supposed that each agent only has the information of its own local objective function and states of those agents within the set of its neighbors.

We express the above explained problem as the following convex optimization problem,

\[
\underset{x_i \in \mathcal{N}}{\text{min}} \sum_{i=1}^{N} f_i(x_i), \quad i \in \mathcal{N} \tag{9}
\]

subject to \( g_i(x_i) \leq 0, i \in \mathcal{N} \)

In the minimization problem \( \text{(9)} \), the consensus constraint, i.e. \( x_i = x_j, i,j = 1, \ldots, N \), is imposed to guarantee that the same decision is made by all agents eventually. In order to find the solution of the problem \( \text{(9)} \), each agent seeks the minimum of its own objective function, \( f_i(x_i) \), fulfilling its associated inequality constraint \( g_i(x_i) \leq 0 \). Furthermore, all agents reach consensus on their final states by exchanging states’ information under the graph \( \mathcal{G} \).

The following assumptions are considered in relation to the optimization problem \( \text{(9)} \).

**Assumption 3.2.** \( \text{a.} \) The objective functions \( f_i, i = 1, \ldots, N \), are strictly convex and twice continuously differentiable on \( \mathbb{R} \). The constraint functions \( g_i, i = 1, \ldots, N \), are convex and twice continuously differentiable on \( \mathbb{R} \).

\( \text{b.} \) The team objective function \( \sum_{i=1}^{N} f_i(x_i) \) is radially unbounded.

**Assumption 3.3.** (Slater’s Condition) There is some \( x^* \in \mathbb{R} \) such that \( g_i(x^*) \leq 0 \).

**Assumption 3.4.** The graph \( \mathcal{G} \) is undirected and has a spanning tree.

Intuitively, the problem \( \text{(9)} \) consists of a constrained convex optimization problem and a consensus problem. The convex constrained optimization problem can be defined as

\[
\underset{x_i \in \mathcal{N}}{\text{min}} \sum_{i=1}^{N} f_i(x_i), \quad i \in \mathcal{N} \tag{10}
\]

subject to \( g_i(x_i) \leq 0, \forall i \in \mathcal{N} \).

The consensus problem is

\[
\lim_{t \to \infty} (x_i - x_j) = 0, \quad i,j = 1, \ldots, N. \tag{11}
\]

Based on interior-point method [1], the convex optimization problem \( \text{(10)} \) can be reformulated as follows,

\[
\min_{x_i \in \mathcal{N}} \sum_{i=1}^{N} f_i(x_i) - \frac{\alpha}{\tau} \ln (-g_i(x_i)) \tag{12}
\]

where \( \tau \in \mathbb{R}^+ \) and \( \alpha > 1 \). The term \(-\ln (-g_i(x_i))\) is referred to as logarithmic barrier function. Note that the domain of the logarithmic barrier is the set of strictly feasible points, i.e. \( x_i \in \{ z \in \mathbb{R} : g_i(z) < 0 \} \). The logarithmic barrier is a convex function; hence, the new optimization problem remains to be convex.

Consider the objective function given in \( \text{(12)} \). As \( x_i \) approaches the line \( g_i(x_i) = 0 \), the logarithmic barrier \(-\ln (-g_i(x_i))\) becomes extremely large. Thus, it keeps the search domain within the strictly feasible set. Note that the initial estimate shall be feasible, i.e. \( g_i(x_i(0)) < 0, i = 1, \ldots, N \).

**Remark 3.5.** Suppose that the solutions to the optimization problem \( \text{(10)} \) and \( \text{(12)} \) are \( x^* \) and \( \bar{x}^* \), respectively. Then, it can be shown that \( f_i(x^*) - f_i(\bar{x}^*) = \frac{\alpha}{2} \) [1], [14]. This suggests a very straightforward method for obtaining the solution to \( \text{(10)} \) with an accuracy of \( \epsilon \) by choosing \( \tau \geq \frac{\alpha}{2} \) and solving \( \text{(12)} \). Consequently, as \( \tau \) increases, the solution to the optimization problem \( \text{(12)} \) becomes closer to the solution of \( \text{(10)} \), i.e. as \( \tau \to \infty \), \( f_i(x^*) - f_i(\bar{x}^*) \to 0 \) is concluded [1, pp. 568-571].

The optimality conditions (so-called centrality conditions) for the convex optimization problem \( \text{(12)} \) are expressed as [1]

\[
\sum_{i=1}^{N} \frac{\partial f_i(\bar{x}^*_i)}{\partial x_i} - \frac{\alpha}{\tau} \frac{\partial g_i(\bar{x}^*_i)}{\partial x_i} = 0, \tag{13}
\]

\[
g_i(\bar{x}^*_i) \leq 0.
\]

We now redefine the problem \( \text{(12)} \) as

\[
\min_{x_i \in \mathcal{N}} \sum_{i=1}^{N} f_i(x_i) - \frac{\alpha}{t+1} \ln (-g_i(x_i)) \tag{14}
\]

that yields the solution of \( \text{(12)} \) asymptotically.

**A. Centralized Algorithm**

In this subsection, we propose a central paradigm to find the solution of the problem \( \text{(14)} \). Later, in the next subsection, we realize this centralized protocol via a distributed algorithm.

We utilize the strategy stated in [4] and propose the following centralized control law to find the optimal solution for the optimization problem \( \text{(14)} \).
that the positions of agents, i.e. $x_i$, practical consensus under the control law (15). | dynamics as (7) is said to reach a \[
\bar{L}(x_i, t) = f_i(x_i) - \frac{\alpha}{t+1} \ln(-g_i(x_i)), \]
and \[
r_i = -\beta_1 \sum_{j \in N_i} \tanh \beta_2 (x_i - x_j),
\]
in which $t$ represents time and $\beta_1, \beta_2 \in \mathbb{R}^+$. 
Note that the control command (15) consists of two parts: the first term is to minimize the local objective function, and the second part is a saturation term associated with the consensus error.

**Definition 3.6:** A network of agents with single-integrator dynamics as (7) is said to reach a practical consensus if \[|x_i(t) - x_j(t)| \leq \delta_0 \quad \forall i, j \in N \]
in an arbitrarily small $\delta_0$.

In the sequel, we will show through the following lemma that the positions of agents, i.e. $x_i$, reach the practical consensus under the control law (15).

**Lemma 3.7:** Consider Assumptions 3.2a and 3.4 If $|\omega_i - \omega_j| < \omega_0$, $i, j = 1, \ldots, N$, where $\omega_i = \left( \sum_{i=1}^{N} \frac{\partial L_i}{\partial x_i} \right)^{-1} \left( \sum_{i=1}^{N} \frac{\partial L_i}{\partial x_i} + \sum_{i=1}^{N} \frac{\partial^2 L_i}{\partial x_i \partial t} \right)$, and $\beta_1 \sqrt{\lambda_2(L)} > \omega_0$, then, there exist $t_0$ and $\delta_0 > 0$ such that the positions of all the agents with dynamics (7) under the control law (15) satisfy practical consensus, i.e. $|x_i(t) - x_j(t)| \leq \delta_0$, $i, j = 1, \ldots, N$, for $t > t_0$.

**Proof:** The aggregate dynamics of the agents in (7) under the control law (15) can be written as
\[
\dot{x} = -\beta_1 D \tanh (\beta_2 D^T \bar{x}) + \Omega,
\]
where $\Omega = [\omega_1 \ldots \omega_N]^T$. Let the network’s consensus error be defined as $\bar{e}_x = \Pi \bar{x}$. Hence,
\[
\dot{\bar{e}}_x = -\beta_1 D \tanh (\beta_2 D^T \bar{e}_x) + \Pi \Omega.
\]
Choose the Lyapunov candidate function
\[
V(\bar{e}_x) = \frac{1}{2} \bar{e}_x^T \bar{e}_x.
\]
By taking time derivative from $V(\bar{e}_x)$ along the trajectories of $\bar{e}_x$, it can be obtained that
\[
\dot{V}(\bar{e}_x) = -\beta_1 \bar{e}_x^T D \tanh (\beta_2 D^T \bar{e}_x) + \bar{e}_x^T \Pi \Omega.
\]
Define $\bar{y} = D^T \bar{e}_x, \bar{y} = [y_1 \ldots y_N]^T$. Then, one can say that $-\bar{y}^T \tanh(\beta_2 \bar{y}) = \sum_i y_i \tanh(\beta_2 y_i)$. From the inequality $-\eta \tanh(\frac{\eta}{2}) + |\eta| < 0.2785\epsilon$ for some $\epsilon, \eta \in \mathbb{R}$ [10], it is straightforward to establish that $-\bar{e}_x^T D \tanh (\beta_2 D^T \bar{e}_x) < -\frac{1}{2} \bar{e}_x^T \bar{e}_x + \frac{\alpha}{t+1} 0.2785$. Thus, the following inequalities hold
\[
\dot{V}(\bar{e}_x) \leq -\beta_1 \frac{\partial L_i}{\partial x_i} + \frac{\lambda_n}{2} \frac{0.2785}{2} \|\bar{e}_x\|_1 + \frac{\lambda_n}{2} \|\bar{e}_x\|_1 + \|\Pi \Omega\| \leq \beta_1 \frac{\partial L_i}{\partial x_i} + \frac{\lambda_n}{2} \frac{0.2785}{2} \|\bar{e}_x\|_1 + \|\Pi \Omega\|
\]
The second inequality arises from the fact that $\|p\| \leq \|p\|_1$ which holds for any $p \in \mathbb{R}^n$. Then, from the assumption $\|\omega_i - \omega_j\| < \omega_0$, $\forall i, j \in N$, one can attain
\[
\dot{V}(\bar{e}_x) \leq -\beta_1 \sqrt{\frac{\lambda_n}{2}} \frac{0.2785}{2} \|\bar{e}_x\|_1 + \|\Pi \Omega\|
\]
According to Courant-Fischer Formula [5], one can observe that $\bar{e}_x^T D \tanh (\beta_2 D^T \bar{e}_x) \geq \lambda_2(L) \|\bar{e}_x\|_1^2$, so,
\[
\dot{V}(\bar{e}_x) \leq -\beta_1 \sqrt{\frac{\lambda_2}{2}} \|\bar{e}_x\|_1^2 + \|\Pi \Omega\| \leq \frac{\lambda_n}{2} \frac{0.2785}{2} \|\bar{e}_x\|_1 \|\Pi \Omega\|
\]
From the statement of Lemma, we have $\beta_1 \sqrt{\lambda_2(L)} > \omega_0$. Furthermore, for $\|\bar{e}_x\| > \frac{\omega_0}{\beta_1 \sqrt{\lambda_2(L)} - \omega_0}$, we obtain $V(\bar{e}_x) \leq 0$. Now, we are ready to invoke Lemma 2.1 that guarantees that by choosing $\beta_2$ large enough, one can make the consensus error $\delta_0$ as small as desired.

**Remark 3.8:** Assumption $|\omega_i - \omega_j| < \omega_0$ in Lemma 3.7 may seem unreasonable since, under some mild conditions, it implies boundedness of agents’ positions, $x_i, i = 1, \ldots, N$. By the following lemma, we will prove that the agents’ positions stay bounded.

**Lemma 3.9:** Consider the dynamics (7) driven by the control command (15). Then, under Assumptions 3.2a and 3.4 the solutions of (7) are globally bounded.

**Proof:** We study boundedness of the solutions of dynamics (7) under the control law (15) via the Lyapunov stability analysis. Let us consider the following quadratic Lyapunov function
\[
W(\bar{x}) = \frac{1}{2} (\bar{x} - \bar{x}^*)^T (\bar{x} - \bar{x}^*),
\]
where $\bar{x}^* \in \mathbb{R}^n$ is the optimum point for the convex function $\sum_{i=1}^{N} L_i(x_i, t)$. Let us take derivative from both sides of (25) along the trajectories (7) under the control law (15) with respect to time. Then, we obtain
\[
\dot{W}(\bar{x}) = (\bar{x} - \bar{x}^*)^T \dot{\bar{x}}
\]
\[
= - \sum_{i=1}^{N} (x_i - x_i^*) \left( \sum_{i=1}^{N} \frac{\partial L_i}{\partial x_i} + \sum_{i=1}^{N} \frac{\partial^2 L_i}{\partial x_i \partial t} \right)
\]
\[
+ \left( \sum_{i=1}^{N} \frac{\partial^2 L_i}{\partial x_i^2} \right)^{-1} \left[ -\beta_1 (\bar{x} - \bar{x}^*)^T D \tanh (\beta_2 D^T \bar{x}) \bar{x} + \sum_{i=1}^{N} \frac{\partial f_i(x_i)}{\partial x_i} \frac{x_i - x_i^*}{(t+1)^2} \sum_{i=1}^{N} \frac{\partial g_i(x_i)}{\partial x_i} \right]
\]
\[
= - \sum_{i=1}^{N} (x_i - x_i^*) \left( \sum_{i=1}^{N} \frac{\partial L_i}{\partial x_i} - \frac{\alpha t}{t+1} \sum_{i=1}^{N} \frac{\partial g_i(x_i)}{\partial x_i} \right)
\]
\[
+ \left( \sum_{i=1}^{N} \frac{\partial^2 L_i}{\partial x_i^2} \right)^{-1} \left[ -\beta_1 (\bar{x} - \bar{x}^*)^T D \tanh (\beta_2 D^T \bar{x}) \bar{x} \right]
\]
(24)
Define $\hat{L}_i(x_i, t) = f_i(x_i) - \frac{\alpha t}{(t+1)} \ln(-g_i(x_i))$. Note that $\hat{L}_i(x_i, t)$ is strictly convex as $\alpha > 1$. Let the minimizer of $\hat{L}_i(x_i, t)$ be $\hat{x}_i^*$. One can observe that the minimizers of $\hat{L}_i(x_i, t)$ and $L_i(x_i, t)$, $i = 1, \ldots, N$, are identical, i.e., $\hat{x}_i^* = x_i^*$. On the other hand, due to convexity of $\hat{L}_i(x_i, t)$ in $x_i$, it holds that $-(x_i - x_i^*) \frac{\partial \hat{L}_i(x_i, t)}{\partial x_i} < \hat{L}_i(x_i, t) - \hat{L}_i(x_i, t)$, $i = 1, \ldots, N$. As the inequality $\hat{L}_i(x_i^*, t) \leq \hat{L}_i(x_i, t)$ holds for any $x_i$, from the definition of convexity, it can be inferred that the first term on the right side of the equality (24) is non-positive. Thus, one obtains

$$
\dot{W}(\bar{x}) \leq -\beta_1 \|D^T \bar{x}\| + \frac{0.2785\beta_1}{\beta_2} + \beta_1 \|D^T \bar{x}\|
$$

The last inequality arises from the inequalities $-\eta \tanh(\frac{\theta}{2}) + [\eta] < 0.2785\epsilon$ [10], with $\epsilon, \eta \in \mathbb{R}$, and $\|\tanh(\epsilon)\| \leq 1$. Furthermore, one can easily find $m \in \mathbb{R}$ such that $\|D^T \bar{x}\| \leq m$. This discussion leads to

$$
\dot{W}(\bar{x}) \leq -\beta_1 \|D^T \bar{x}\| + \frac{0.2785\beta_1}{\beta_2} \beta_1 m
$$

$$
= -\beta_1 \sqrt{\beta_2} D^T \bar{x} + \frac{0.2785\beta_1}{\beta_2} \beta_1 m
$$

$$
\leq -\theta \|\bar{x}\| + \left(\theta - \beta_1 \sqrt{\beta_2} \sqrt{\lambda_2(DD^T)}\right) \|\bar{x}\|
$$

$$
+ \frac{0.2785\beta_1}{\beta_2} \beta_1 m, 0 < \theta < 1
$$

$$
\leq -\theta \|\bar{x}\|, \forall \bar{x} \in B.
$$

where $B = \left\{ \bar{x} \in \mathbb{R}^N | \|\bar{x}\| \geq \frac{0.2785\epsilon}{\theta - \beta_1 \sqrt{\beta_2}} \right\}$. Now, by Lemma 2.1 it will be certified that $\bar{x}$ remains bounded. $\blacksquare$

B. Distributed Algorithm

It is obvious that the control law (15) is not locally implementable since it requires the knowledge of the whole network as aggregate objective function $\sum_{i=1}^N f_i(x)$ as well as all inequality constraints $g_i(x) \leq 0, i = 1, \ldots, N$. Through the following algorithm, we estimate (15) in a distributed manner and adopt it to solve the distributed optimization problem (10).

As it follows, each agent generates an internal dynamics to obtain the estimates of collective objective function’s gradients and other terms which are required for computation of (15) via only local information in a cooperative fashion. Consider the following estimator dynamics,

$$
\dot{\kappa}_i(t) = -c \sum_{j \in N_i} \text{sgn}(\nu_i(t) - \nu_j(t)),
$$

where

$$
\nu_i(t) = \kappa_i(t) + \left[ \frac{\partial L_i(x_i, t)}{\partial x_i} \right] V_i(t),
$$

From (25), one obtains $\sum_{i=1}^N \kappa_i(t) = 0$. Assume that $\kappa_i, i = 1, \ldots, N$, are initialized such that $\sum_{i=1}^N \kappa_i(0) = 0$. Then, $\sum_{i=1}^N \kappa_i(t) = 0$ is concluded for all $t > 0$.

Hence, $\sum_{i=1}^N \nu_i(t) = \sum_{i=1}^N \left[ \frac{\partial L_i(x_i, t)}{\partial x_i} \right] V_i(t)$. It follows from Theorem 1 in [2] that if $c > \sup_t \{\|\nu_i(x_i(t))\|\}, \forall i \in N$, then consensus on $\nu_i, i = 1, \ldots, N$, i.e. $|\nu_i(t) - \nu_j(t)| = 0 \forall i, j \in N$, is achieved over a finite time say $T$. With $\nu_i(t) = \nu_j(t)$, the following holds,

$$
\nu_i(t) = \frac{1}{N} \sum_{i=1}^N \left[ \frac{\nu_{i1} \nu_{i2}}{\nu_{i3}} \right],
$$

where $\nu_{i1} = \frac{\partial L_i(x_i, t)}{\partial x_i}, \nu_{i2} = \frac{\partial^2 L_i(x_i, t)}{\partial x_i \partial t}, \text{ and } \nu_{i3} = \frac{\partial^2 L_i(x_i, t)}{\partial x_i \partial t}$.

Theorem 3.10: Suppose that Assumptions 3.2, 3.3 and 3.4 hold. Moreover, $\sum_{i=1}^N \kappa_i(0) = 0$ and $c > \sup_t \{\|\kappa_i(x_i(t))\|\}, \forall i \in N$. Then, the protocol (28) will drive the agents to the solution of the distributed convex optimization problem (10).

Proof: Let us define the following Lyapunov candidate function $V = \frac{1}{2} \left( \sum_{i=1}^N \nu_{i1}^2 \right)$. After calculating time derivate of $V$, the following holds,

$$
\dot{V} = \left( \sum_{i=1}^N \nu_{i1} \right) \left( \sum_{i=1}^N \nu_{i3} \nu_{i1}^2 + \nu_{i2} \right).
$$

From (28), we have

$$
\dot{V} = -\left( \sum_{i=1}^N \nu_{i1} \right)^2,
$$

in which we used the equalities $\nu_{i3} = \nu_{i3}, \forall i, j \in N$ for $t > T$, and $\sum_{i=1}^N r_i = 0$. Hence, $\dot{V} \leq 0, \forall t > T$. On the other hand, we assert that $x_i, i = 1, \ldots, N$, stay bounded after a finite time as the agents’ dynamics are locally Lipschitz and their inputs are bounded. This means that for $t \leq T$, we have $x_i \in \mathbb{R}, \forall i \in N$. Now, we can do stability analysis from $T$ onwards.

From the above inequality, it follows that $\sum_{i=1}^N \nu_{i1}$ remains bounded in $\mathbb{R}^n \cup \{\infty\}$, i.e. it belongs to $L^\infty$ space. With integrating from both sides of equality (5), in the view of passivity of $V(x, t)$, we have

$$
\int_0^T \sum_{i=1}^N \nu_{i1} dt = -\int_0^T \dot{V} dt
$$

$$
= -V(R) + V(0) \leq 0.
$$

Therefore, $\sum_{i=1}^N \nu_{i1} \in L^2$. By means of Barbalat’s lemma [13], we have $\sum_{i=1}^N \nu_{i1} = 0$ as $t \to \infty$. Thereby, the first optimality condition in (13) is asymptotically satisfied.
We now illustrate that the second optimality condition in (13) also holds. Suppose that \( g_i(x_i(0)) < 0 \) for all \( i \). We do the proof by contradiction to establish that \( g_i(x_i(t)) \) is less than 0 for \( t > 0 \). Assume that we had \( g_i(x_i(t_1)) > 0 \) for some \( i \) and a finite \( t_1 > 0 \). Due to continuity of the function \( g_i(x) \), \( g_i(x_i(t_1)) \) would be zero. This implies that \( \sum_{i=1}^{N} \nu_{i1} \) becomes unbounded at \( t_1 \) that contradicts the fact that \( \sum_{i=1}^{N} \nu_{i1} \in L^\infty \). Hence, the inequality \( g_i(x_i(t)) \) is less than 0 with \( g_i(x_i(t)) \) holds for \( t > 0 \). This ends the proof.

C. Numerical Example

This section presents simulation studies using Matlab/Simulink software for a network of four agents with dynamics according to (3) driven by the proposed distributed algorithm (26). We consider the constrained convex optimization problem (8) with objective functions \( f_1(x) = (x + 2)^2 \), \( f_2(x) = x^2 \), \( f_3(x) = (x - 10)^2 \), and \( f_4(x) = (x - 2)^2 \) and constraints \( g_1(x) = x - 1 \), \( g_2(x) = x - 2 \), and \( g_3(x) = x - 4 \). Note that these functions are all smooth and convex. In our simulation, the information sharing graph \( \mathcal{G} \) is set as: \( 1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \). The initial conditions are set as \( x_i(0) = [0 \ 1 \ 3 \ 0]^T \). The evolution of agents’ trajectories is depicted in the Fig 1. As shown, all agents meet each other and, then, converge towards the optimal point of the collective objective function which is one in this case.

IV. CONCLUSIONS

We investigated the problem of distributed optimization for undirected networks of single-integrator agents. Here, agents shall reach an agreed point that minimizes a collective convex objective function with respect to local inequality constraints. A centralized control law, which yields the optimal solution to this problem, and consists of a saturation consensus part and an optimization part based on the interior-point method was proposed. To illustrate the convergence of the proposed algorithm, we first established that the proposed consensus protocol provides practical consensus, i.e., all agents will have the same decision eventually, perhaps with a small admitted error. We then suggested a distributed estimator as a tool to estimate some terms within the protocol associated with the global knowledge, which is only partially available to agents with the network. It was proved that the presented distributed algorithm converges to the solution of the original constrained convex optimization problem. Finally, to evaluate the performance of our work, a numerical example was presented.

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