Framed M-branes, corners, and topological invariants

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Abstract

We uncover and highlight relations between the M-branes in M-theory and various topological invariants: the Hopf invariant over \( \mathbb{Q}, \mathbb{Z} \) and \( \mathbb{Z}_2 \), the Kervaire invariant, the \( f \)-invariant, and the \( \nu \)-invariant. This requires either a framing or a corner structure. The canonical framing provides a minimum for the classical action and the change of framing encodes the structure of the action and possible anomalies. We characterize the flux quantization condition on the C-field and the topological action of the M5-brane via the Hopf invariant, and the dual of the C-field as (a refinement of) an element of Hopf invariant two. In the signature formulation, the contribution to the M-brane effective action is given by the Maslov index of the corner. The Kervaire invariant implies that the effective action of the M5-brane is quadratic. Our study leads to viewing the self-dual string, which is the boundary of the M2-brane on the M5-brane worldvolume, as a string theory in the sense of cobordism of manifolds with corners. We show that the dynamics of the C-field and its dual are encoded in unified way in the 4-sphere, which suggests the corresponding spectrum as the generalized cohomology theory describing the fields. The effective action of the corner is captured by the \( f \)-invariant, which is an invariant at chromatic level two. Finally, considering M-theory on manifolds with \( G_2 \) holonomy we show that the canonical \( G_2 \) structure minimizes the topological part of the M5-brane action. This is done via the \( \nu \)-invariant and a variant that we introduce related to the one-loop polynomial.

Contents

1 Introduction and statement of results

2 M-branes as framed submanifolds and topological invariants
   2.1 The framing and stable framing on the M2-brane
   2.2 The M5-brane as a framed submanifold
   2.3 The connection to the Hopf invariant at the rational level
   2.4 Hopf invariant 2 over \( \mathbb{Z} \) and refinement via the dual of the C-field
   2.5 M-branes via cohomotopy

3 M-branes as corners and topological invariants
   3.1 M-branes and corners
   3.2 The \( \mathbb{Z}_2 \) Hopf invariant
   3.3 The M5-brane and the Kervaire invariant
   3.4 M-branes and the Maslov index
   3.5 The M-branes and the \( f \)-invariant
   3.6 M5-brane, \( G_2 \) holonomy and the \( \nu \)-invariant

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1 Introduction and statement of results

The branes in M-theory, i.e. the M-branes, possess rich geometric and topological structures. Examples of such structures have been uncovered recently [51] [54] [55] [56]. In this paper we propose and highlight explicit relations to various topological invariants, including: the mod 2 and the integral Hopf invariants, the Kervaire invariant, the $f$-invariant [37], and the (newly constructed) $\nu$-invariant [15]. A relation to the Hopf invariant at the rational level was also proposed in [33], and the connection of the M5-brane to the Kervaire invariant is implicit in the work of Hopkins-Singer [30], which is a precursor to the solution of the Kervaire invariant problem [28]; a description of the effective action of the M5-brane is given in [30] (along the lines of [76]). The $f$-invariant appears in [60] in the description of anomalies in heterotic string theory. Combining that with the approach of [53] [57] leads naturally to our discussion on the relation of the M-branes to the $f$-invariant. This generalizes previous connections to the Adam’s $e$-invariant and the Atiyah-Patodi-Singer $\eta$-invariant [52] [51]. Recently, a description of M-branes via higher geometry and higher Chern-Simons theory was given in [20] [21] [22] [24].

Making the connections to the above topological invariants requires certain topological structures to be admitted by the M-branes. For example, the connection to the Kervaire invariant requires the M-brane to have a framing, that is, a trivialization of the tangent bundle (see [51]). Closely related is the notion of String structure and its variations, such as Atiyah’s 2-framing, captured essentially by the C-field $C_3$ and its field strength $F_4$ [52] [51]. On the other hand, the connection to the $f$-invariant requires a corner structure on the M-branes. For the M5-brane the corner will be the six-dimensional worldvolume itself, while for the M2-brane the corner will be either the three-dimensional worldvolume or the boundary given by the self-dual string. Furthermore, requiring a $G_2$ holonomy structure on the 7-dimensional extension of the M5-brane worldvolume leads to the direct connection to the $\nu$-invariant. We will illustrate these points in detail. The discussion will naturally lead along the way to a twisted notion of $p_1$-structure (see Def. 3).

Once the worldvolumes are endowed with geometric and topological structures, it is natural to consider the space of such structures and to investigate whether the physical entities depend on the specific structure chosen. It then makes sense to ask whether there is a distinguished such structure in that set. For instance, the space of Spin structures and the space of framings are both affine spaces, and hence have no natural base points. However, one can distinguish specific such structures which minimize some corresponding invariant; these are the canonical structures. An example of the minimization of the volume of the M2-brane, captured by the String structure, is described in [51]. Analogously, we consider a distinguished $\nu$-invariant, which characterizes a minimum for (a properly interpreted) topological part of the action of the M5-brane. The M2-brane will be described either in three dimensions or by moving up/down by one dimension. The M5-brane will be described either in six dimensions or by moving up by one or two dimensions. Therefore, the relevant dimensions are 2, 3, and 4 for the M2-brane and 6, 7, and 8 for the M5-brane.

The main content of this paper can be summarized in the following two theorems, on the relation of the M2-brane and the M5-brane, respectively, to various topological invariants.

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2Some of the statements might be known to the experts but we have not seen them in print, and we believe that, at any rate, it is useful to have such explicit statements.
Theorem 1  For the M2-brane in M-theory with oriented Riemannian worldvolume $M^3$ we have:

(i) For $M^3$ as framed manifold, the canonical framing provides a minimum for the classical action.

(ii) Under the change of framing of the bounding 4-manifold $W^4$, the flux quantization condition of the C-field takes the form $2[F_4] + \lambda = 2a$ for $\phi \mapsto \phi + \sigma$ and $[F_4] + \frac{1}{2}\lambda = a$ for $\phi \mapsto \phi + \rho$, where $\sigma$ and $\rho$ are the generators of $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$.

(iii) The M2-brane partition function is anomaly-free if the framing of the worldvolume is even.

(iv) The combination $[F_4] + \frac{1}{2}\lambda$ is an integral class if the Hopf invariant of the extended worldvolume of the M2-brane vanishes.

(v) The dynamics of the C-field and its dual are encoded in the rational homotopy of the 4-sphere.

(vi) In the formulation of the action of the M2-brane using the signature, the corner correction from the self-dual string is given by the Maslov index $\mu$ as $\exp(2\pi i \frac{1}{8} \mu)$.

(vii) The self-dual string as the boundary of the M2-brane on the M5-brane worldvolume defines a string theory in the sense of cobordism of manifolds with corners.

Theorem 2  For the M5-brane in M-theory with oriented Riemannian worldvolume $M^6$ we have:

(i) The topological action for a framed M5-brane is given by the change of framing formula. The resulting action depends on the choice of framing.

(ii) The degree eight cohomology class, given by the lift of the electric current of the C-field and describing the dynamics on the M5-brane, is given by an element of Hopf invariant two.

(iii) The topological action of the M5-brane in eight dimensions, both at the rational and integral levels, is given by (a differential refinement of) the Hopf invariant.

(iv) For a Spin worldvolume $M^6$ and a normal bundle with a Membrane structure, the vanishing of the Kervaire invariant implies that the effective action of the M5-brane is quadratic.

(v) The contribution to the effective action due to the corner is given by the $f$-invariant.

(vi) In the signature formulation of the action of the M5-brane in eight dimensions, the corner correction from the six-dimensional worldvolume $M^6$ is given by the Maslov index $\mu$ as $\exp(2\pi i \frac{1}{8} \mu)$.

(vii) The canonical $G_2$ structure minimizes the topological part of the M5-brane action, via the $\nu$-invariant.

The field strength $F_4$ and its dual satisfy quantization conditions characterized by the Pontrjagin classes of the M-theory target space $Y^{11}$ [75] [18] [17]. The author has previously asked the question of whether these two fields can be viewed as components of one field strength which satisfies a general quantization condition, the components of which yields the expected conditions. It was also proposed that the total field strength might live in a generalized cohomology theory (see [51]). In this direction, building on Theorem 1(v), we have

Proposal/conjecture. The dynamics of the form fields in M-theory is described via the spectrum $S^4$ corresponding to the 4-sphere $S^4$.

A discussion on justification of this statement, as well as Theorem 1(v) is given in Section 2.5. The paper is divided into two main sections: Sec. 2 on invariants associated with framings and Sec. 3 on those associated with corners. The M2-brane with its various notations of framings are discussed in Sec. 2.1 where also the statements of Theorem 1(i–iii) are addressed. The M5-brane as a framed submanifold is described in Sec. 2.2 where Theorem 2(i) is explained. Theorem 1(iv) and Theorem 2(ii–iii) are described in Sec. 2.3 Sec. 2.4; and Sec. 3.2. The general discussion on relevance of corners is taken in Sec. 3.1. The statements about the signature, i.e. Theorem
1(vi-vii) and Theorem 2(vi) are elaborated in Sec. 3.4. The remaining statements in Theorem 2, i.e. parts (iv) (v), and (vii) are explained in Sec. 3.3, Sec. 3.5 and Sec. 3.6, respectively. In the latter we also define a variant of the $\nu$-invariant arising from the one-loop polynomial.

2 M-branes as framed submanifolds and topological invariants

We start by outlining the various structures which an M-brane can possess. We will consider framings for both the M2-brane and the M5-brane. Furthermore, the M2-brane will allow for more than one type/notion of framing. Using \cite{38}, we consider the set of framings $\phi$, stable framings $\varphi$, and 2-framings $2\varphi$, which are compatible with a Spin structure $s$ on the worldvolume.

1. Framing: A framing $\phi$ of the tangent bundle of the M2-brane worldvolume $M^3$, as an oriented bundle, is a homotopy class of sections of the associated frame bundle with structure group $SO(3)$. Considering $M^3$ to be oriented or Spin, such a framing always exists. Denote the set of framings by $F$, and by $F_s$ the set of framings which are compatible with a Spin structure $s$ on $M^3$. The latter is an affine space with translation group $\pi_3(SO(3)) = Z$ and is (non-canonically) isomorphic to $H^1(M^3;\mathbb{Z}_2) \oplus Z$.

2. Stable framing: The topological study of the M2-brane is often facilitated by extending to four dimensions. One can then consider structures on this coboundary, whose tangent bundle is $\varepsilon^1 \oplus TM^3$, where $\varepsilon^1$ is an oriented line bundle. A framing of this Whitney sum is a stable framing $\varphi$ of the worldvolume $M^3$. Denote the set of stable framings by $\Phi$ and the ones compatible with a Spin structure $s$ by $\Phi_s$. Similarly, $\Phi_s$ is an affine space with translation group $\pi_3(SO(4)) = Z \oplus Z$ and is (non-canonically) isomorphic to $H^1(M^3;\mathbb{Z}) \oplus Z \oplus Z$. This is analogous to the statement that a Spin$^c$ structure on a manifold $X$ is equivalent to a Spin structure on $\varepsilon \oplus TX$ (see \cite{59}). Note that each framing $\phi$ of the M2-brane worldvolume $M^3$ can be identified with the stable framing $\varphi = \phi \oplus \varphi_1$, where $\varphi_1$ is a framing of $\varepsilon^1$.

3. 2-framing: A 2-framing $2\varphi$ of the worldvolume $M^3$ is a homotopy class of trivializations of the twice the tangent bundle $2TM^3 = TM^3 \oplus TM^3$. This corresponds to the inclusion of structure groups $SO(3) \times SO(3) \hookrightarrow SO(6)$. The set of 2-framings $2\Phi$ is an affine space with translation group $\pi_3(SO(6)) \cong Z$. This is related to String structures \cite{13} and applied to describe the C-field and the M2-brane in \cite{51} \cite{52}. The main connection is via the quantization condition of the C-field \cite{75}, which implies that $C_3$ is essentially (but not literally) the difference of two Chern-Simons forms; see \cite{51}.

We have seen that a framing is a trivialization of (some variant of) the tangent bundle of the worldvolume. This is the tangential framing. There is another trivialization that is also important to the M-branes, namely the normal framing.

The M-branes as framed submanifolds. A normally framed $m$-submanifold of an $n$-dimensional manifold $N$ is a submanifold $M$ with a given framing $f : N_\nu M N \cong M \times \mathbb{R}^{n-m}$ of the normal bundle $N_\nu M$. Let us consider the situation in M-theory on $Y^{11}$. For a normally framed M2-brane

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\footnote{This is strictly speaking a trivialization of a class in real $K$-theory $KO$. However, by what seems like an accepted abuse, we will take this to be a trivialization of a particular bundle when no confusion arises.}
with worldvolume $M^3$, we have $f_3 : \mathcal{N}_{M^3} Y^{11} \cong M^3 \times \mathbb{R}^8$. The main examples of framed three-dimensional worldvolumes are 3-tori, 3-spheres and their quotients, such as lens spaces. Similarly, for a normally framed M5-brane with worldvolume $M^6$, we have $f_6 : \mathcal{N}_{M^6} Y^{11} \cong M^6 \times \mathbb{R}^5$. Main examples of such M5-brane worldvolumes are the 6-sphere, the products $G_1 \times G_2$ of any of the Lie groups $SO(3)$ and $Sp(1) = SU(2) = Spin(3)$ and their quotients by finite groups (see [50]).

The M-branes when $Y^{11}$ is a $\pi$-manifold. We have previously discussed geometric and topological consequences of having the target spacetime $Y^{11}$ to be a framed manifold or a $\pi$-manifold [52] [60]. We now extend an aspect of the description in the presence of M-branes. Consider $Y^{11}$ as a $\pi$-manifold and take $M^m$ to be the worldvolume of the M2-brane ($m = 3$) or of the M5-brane ($m = 6$). Since $TM^m \oplus \mathcal{N} M^m = T M Y^{11}$ then $TM^m \oplus \mathcal{N} M^m \oplus \varepsilon^k = T Y^{11} \oplus \varepsilon = \varepsilon^{11+k}$ so that the normal bundle $\mathcal{N} M^m$ is stably trivial if and only if the worldvolume $M^m$ is a $\pi$-manifold. Next we consider the disk bundle, needed for the description of tubular neighborhood for the M5-brane (see [17] [53]). Let $X$ be the total space of the disk bundle over the worldvolume $M^m$, which we take to be a $\pi$-manifold, associated to a vector bundle $\eta$. We identify $M^m$ with the zero section of $X$. Then $TX$ is stably trivial if and only if $TX|_M$ is stably trivial. Since $TX|_M = TM^m \oplus \mathcal{N} M^m = TM^m \oplus \eta$, and since $TM^m$ is stably trivial, then $X$ is parallelizable if and only if $\eta$ is stably trivial.

2.1 The framing and stable framing on the M2-brane

We consider the M2-brane worldvolume $M^3$ with a Spin structure. $M^3$ as a framed manifold is discussed extensively in [51].

Chern-Simons theory and $p_1$-structure. A closely related structure to both a 2-framing and a String structure is the notion of $p_1$-structure [6], or rigging [60], arising in the context of topological field theory, especially Chern-Simons theory. Analyzing the worldvolume anomalies of the M2-brane leads to the quantization condition on the C-field [75], given by $[F_4] - \frac{1}{2} \lambda = a$, where $F_4$ is the field strength – curvature, in the description as a 2-gerbe/3-circle bundle [20] [21] of $C_3$, $\lambda = \frac{1}{2} p_1$ is the first Spin characteristic class of the Spin bundle, and $a$ is the degree four characteristic class of an $E_8$ bundle. Therefore, a main aspect of the C-field is captured by Chern-Simons theory. The path integral in this theory depends on the orientation as well as on the $p_1$-structure. Such a structure arises since the invariants considered turn out to have a framing anomaly [74], i.e. the invariants themselves depend on the $p_1$-structure. This has an interpretation in terms of String structures on the M-branes [51].

Let $X = BO(p_1)$ be the homotopy fiber of the map $p_1 : BO \to K(\mathbb{Z}, 4)$ corresponding to the first Pontrjagin class of the universal stable bundle $\gamma$ over the classifying space $BO$. Let $\gamma_X$ be the pullback of $\gamma$ over $X$. A $p_1$-structure on the worldvolume $M^3$ is a fiber map from the stable tangent bundle $TM^3$ of $M^3$ to $\gamma_X$. That is, there is the following lifting diagram

\[
\begin{array}{ccc}
X = BO(p_1) & \xrightarrow{p_1} & K(\mathbb{Z}, 4) \\
M^3 & \downarrow & \\
& BO & \\
& \xrightarrow{p_1} & 
\end{array}
\]

The Spin/String version of this construction is explained in our context in [51]. The study of the M2-brane anomalies require the extension to four dimensions. The above discussion shows that
The group \( \mathbb{Z}_{24} \) from eleven dimensions. Consider the M2-brane worldvolume \( M^3 \) embedded in the ambient 11-dimensional spacetime via \( i : M^3 \hookrightarrow Y^{11} \). The simplest situation is a product \( Y^{11} \cong M^3 \times \mathbb{R}^8 \). Write \( M^3_+ \) for the disjoint union of \( M^3 \) with a disjoint base-point. Then there is a canonical homeomorphism \( T(M^3 \times \mathbb{R}^8) \cong \Sigma^8(M^3_+) \) between the Thom space of the trivial 8-dimensional vector bundle and the 8-fold suspension of \( M^3_+ : (S^8 \times M^3_+) / (S^8 \vee M^3_+) = S^8 \vee M^3_+ \) is given with a choice of trivialization of the normal bundle \( N(M^3, i) \). Now suppose that \( M^3 \) is given with a choice of trivialization of the normal bundle \( N(M^3, i) \). This choice is a choice of homoeomorphism \( T(N(M^3, i)) \cong \Sigma^8M^3_+ \), which is a framing of \((M^3, i)\). Now identify the 11-dimensional sphere \( S^{11} \) with the one-point compactification \( \mathbb{R}^{11} \cup \{ \infty \} \). The Pontrjagin-Thom construction is the map \( S^{11} \to T(N(M^3, i)) \) given by collapsing the complement of the interior of the unit disk bundle \( \mathbb{D}(N(M^3, i)) \) to the point corresponding to \( S(N(M^3, i)) \) and by mapping each point of \( \mathbb{D}(N(M^3, i)) \) to itself. Identify the 8-dimensional sphere with the 8-fold suspension \( \Sigma^8S^0 \) of the zero-dimensional sphere (i.e. two points, one of which is the base-point). The map which collapses \( M^3 \) to the non-basepoint yields a base-point preserving map \( \Sigma^8(M^3_+) \to S^8 \) whose homotopy class defines an element of \( \pi_{11}(S^8) \cong \mathbb{Z}_{24} \).

Extending the C-field framing to the bounding 4-manifold. The extension of the C-field to the bounding 4-manifold \( W^4 \) in the Spin case is independent of the choice of this 4-manifold [70, 75]. A similar observation holds in the case of String structures [51], where the relation to gerbes is highlighted. We now provide a description in terms of (variants of) framings. Since \( H^4(W^4) = 0 \), the Pontrjagin class \( p_1(E) \) for all real vector bundles over \( W^4 \) vanishes. Therefore, we should instead consider relative characteristic classes, as done in [51]. The relative Pontrjagin number is defined as the pairing of the relative Pontrjagin class with the fundamental class \( p_1(W^4, \varphi) = p_1(TW^4, \varphi)[W^4, M^3] \) associated to any given tangential framing \( \varphi \) over \( M^3 \). Thus \( p_1(W^4, \varphi) \) is an integer invariant which measures the obstruction to extending \( \varphi \) to a framing of the tangent bundle of \( W^4 \). The Hirzebruch defect of a framing \( \varphi \) of the closed oriented 3-manifold \( M^3 \) is

\[
h(\varphi) = p_1(W^4, \varphi) - 3\text{sign}(W^4) .
\] (2.2)

It follows from the Novikov additivity of the signature and the signature formula for closed manifolds that \( h(\varphi) \) is independent of the choice of \( W^4 \). This is a multiple of the Atiyah-Patodi-Singer eta-invariant of the (odd) signature operator is given by \( \eta(M^3) := \int_{W^4} L - \text{sign}(W^4) \), where the metric on some collar neighborhood of \( \partial W^4 \) is isometric to the product metric on \( M^3 \times [0, \epsilon) \), and \( L \) is the Hirzebruch \( L \)-class as a degree four class \( \frac{1}{2\pi} p_1(W^4) \) on \( W^4 \).
The relation between the eta-invariant $\eta(M^3)$ and the Chern-Simons invariant $CS(M^3)$ for a compact 3-manifold $M^3$ is $3\eta(M^3) \equiv 2CS(M^3) + \tau$ (mod $\mathbb{Z}$), where $\tau$ is the number of 2-primary summands of the first homology group $H_1(M^3;\mathbb{Z})$. Thus the eta-invariant completely determines the Chern-Simons invariant once the homology of the worldvolume $M^3$ is known. When $\tau = 0$ the Hirzebruch defect $h(\varphi)$ is given by a multiple of the Chern-Simons invariant

$$2CS(M^3) \equiv h(\varphi) \mod \mathbb{Z}.$$  

Part of the M2-brane partition function is captured by the holonomy of the C-field, which is essentially given by Chern-Simons theory. The holonomy is given by (see [17])

$$\chi(C_3) = \exp\left[2\pi i \left(\frac{1}{2}CS(A) + c\right)\right],$$

where $CS(A)$ is the Chern-Simons invariant of a connection $A$ of an $E_8$ bundle, $CS(g)$ is the ‘gravitational’ Chern-Simons invariant corresponding to a metric $g$ on $M^3$ (or to the corresponding Levi-Civita connection on $M^3$), and $c$ is a constant background 3-form.

**The M2-brane effective action and canonical tangential framings.** We will see that there is a framing of the M2-brane worldvolume $M^3$ which is, in a sense, preferred. In [51] [52] the canonical String structure of [47] [13] was highlighted as the one preferred by the M2-brane. Here we elaborate, highlighting some dynamical aspects. A Spin structure $s$ over $M^3$ can be considered as a framing of $TM^3$ over the 2-skeleton of $M^3$, and consider the set $\mathcal{F}_s$ of framings of $M^3$ which are compatible with $s$. From obstruction theory, the difference between two such framings is specified by an element of the cohomology group $H^3(M^3;\pi_3(SO(3))) = \mathbb{Z}$. A framing $\varphi$ of the worldvolume $M^3$ is canonical for the Spin structure $s$ if it is compatible with $s$, and $|h(\varphi)| \leq |h(\varphi')|$ for all other framings $\varphi'$ which are compatible with the Spin structure $s$. This means that $\varphi$ is a minimum for the invariant $|h|$ on $\mathcal{F}_s$ [38]. The discussion in [51] [52] shows that this invariant is the effective action of the M2-brane. Therefore, we have a minimization of this effective action when we choose the canonical framing. This is a classical statement. A corresponding quantum (or at least semi-classical) statement might involve anomalies; this is what we consider next, but using stable framings instead. Dependence on Spin structure is considered in [59] while dependence of the M2-brane on String structures is discussed in [51]. We now consider dependence on framing.

**A possible M2-brane anomaly via canonical stable framings.** We will discuss this from the point of view of both the C-field $C_3$ and its field strength $F_4$. The first involves dealing with secondary classes such as the eta-invariant and the Chern-Simons invariant, while the second involves the primary classes, mainly the first Pontrjagin class. First, concentrating on the latter, we consider the effect of change of ‘usual’ framings and show how that is anomaly-free. Now we extend to the bounding 4-manifold and ask whether there is a dependence on framing or whether there is a corresponding potential anomaly. For two framings $\varphi_1$ and $\varphi_2$ of $M^3$, the Pontrjagin class has the property $p_1(W^4,\varphi_1 \oplus \varphi_2) = p_1(W^4,\varphi_1) + p_1(W^4,\varphi_2)$. Correspondingly, for any two framings $\varphi_1$ and $\varphi_2$ of $M^3$, the Hirzebruch defect has the the additivity property $h(\varphi_1 \oplus \varphi_2) = h(\varphi_1) + h(\varphi_2)$, which induces a similar property for the Chern-Simons invariant. The effect of such a change on the holonomy of the C-field is multiplicative and there is no potential anomaly.

Next we look at stable framings. Let $\phi$ be a stable framing of $M^3 = \partial W^4$. Then, as shown in
the effect of translation by the generators $\rho$ and $\sigma$ of $\pi_3(\text{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$ is given by

$$p_1(W^4, \phi + \rho) = p_1(W^4, \phi) + 4, \quad p_1(W^4, \phi + \sigma) = p_1(W^4, \phi) + 2. \quad (2.5)$$

In the flux quantization condition on the C-field, $[F_4] + \frac{1}{2} \lambda = a \in H^4(Y^{11}; \mathbb{Z})$, the 3-sphere $S^3$ was used as a representative case for the M2-brane worldvolume $M^3$. This is further considered and generalized in $[31]$ from the point of view of framed 3-manifolds. In the general case, one has that the flux quantization holds on $W^4$ by considering the extension of spacetime $Y^{11}$ to bounding $Z^{12}$ in such a way that the extra directions in both cases can be identified. Then the transformations $[23]$ suggest that the change of stable framing by $\rho$ is allowed, while that by $\sigma$ leads to a potential anomaly since $\frac{1}{2} \lambda$ is not defined. However, this is not a drastic anomaly since one can modify what one means by the quantization condition in such situations, i.e. multiply through by 2. Nevertheless, this shows that at least the transformation $\sigma$ requires care. Note that a stable framing $\phi$ on $M^3$ which is compatible with a Spin structure $s$ on $M^3$ extends to a framing of a compact 4-manifold $W^4$ bounding $M^3$, then $p_1(W^4, \phi) = 0$. $[38]$. The consequence of divisibility of $\lambda$ by 2 is highlighted in $[55]$ leading to the notion of a Membrane structure, that is having $w_4 = 0$.

Next we consider one effect of the change of framing. Since the C-field is essentially a Chern-Simons form, we can rely on the effect of change of framing on the Chern-Simons form itself. Under a change of framing $\varphi \mapsto \varphi + s$, the gravitational Chern-Simons form transforms as $CS(\omega) \mapsto CS(\omega) + 2\pi s$, and the partition function transforms as $Z \mapsto Z \cdot \exp \left(\frac{2\pi i}{\hbar} \int \omega \right) [24] [33].$ This is described using String cobordism in $[51]$. Taking $C_3 = CS(A) - \frac{1}{2} CS(\omega)$, we get $C_3 \mapsto C_3 - \pi s$ and $e^{iC_3} \mapsto e^{iC_3} \cdot e^{-i\pi s}$. This shows that, unless the other factor in the partition function, namely the Pfaffian $\text{Pfaf}(D)$ of the Dirac operator $D$, cancels this anomalous factor, we see that $s$ has to be even.

**Restriction to the heterotic boundary.** We now consider the M2-brane with boundary $\partial M^3$, where this boundary does not lie on the M5-brane in eleven dimensions, but rather restricts to the heterotic string on the boundary of the ambient 11-manifold. Let $Y^{11}$ be a manifold with boundary $X^{10} = \partial Y^{11}$ and let $L$ be a subbundle of $T_X Y^{11}$ spanned by a vector field $v$ pointing inside $Y^{11}$. Then $T_X Y^{11} = TX^{10} \oplus L$. Now we take $M^3 \subset Y^{11}$ to be a neat submanifold, that is, the intersection of the tubular neighborhood of the M2-brane worldvolume $M^3$ with $X^{10}$ is a tubular neighborhood of the string worldsheet $\Sigma = \partial M^3$ in $X^{10}$. Then we can assume that along $Y^{11}$ the vector $v$ points inside $M^3$. Then the tangent bundle to the M2-brane worldvolume splits as $T_{\Sigma} M^3 = T_{\Sigma} \oplus L$, where $\Sigma$ is the heterotic string on the boundary. There is a natural identification $T_{\Sigma} Y^{11} / T_{\Sigma} M^3 = TX^{10} / T_{\Sigma}$ of bundles restricted to the string worldsheet $\Sigma$. That is, the normal bundle of the M2-brane worldvolume $M^3$ restricted to its string boundary worldsheet $\Sigma$ can be identified with the normal bundle of the string worldsheet $\Sigma$ in the heterotic target space $X^{10} = \partial Y^{11}$.

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4 Explicit generators $\rho$ and $\sigma$ for $\pi_3(\text{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$ can be found in $[67]$ (see also $[38]$). View $S^3$ as the unit sphere in the quaternions $\mathbb{H}$, oriented by the ordered basis $1, i, j, k$, and view $\text{SO}(4)$ as the rotation group of $\mathbb{H}$. Then, for $q \in \mathbb{H}$ and $x \in S^3$, the maps $\rho$ and $\sigma : S^3 \to \text{SO}(4)$ defined by $\rho(q)x = qxq^{-1}$ and $\sigma(q)x = qx$ represent generators of $\pi_3(\text{SO}(4))$. By restricting to $1\mathbb{H}\mathbb{H}$ of pure (imaginary) quaternions, $\rho$ also represents a generator of $\pi_3(\text{SO}(3))$, and these two $\rho$’s correspond under the natural map $\pi_3(\text{SO}(3)) \to \pi_3(\text{SO}(4))$ induced by the inclusion $\text{SO}(3) \hookrightarrow \text{SO}(4)$. 

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7
2.2 The M5-brane as a framed submanifold

We now turn to the framing for the M5-brane. The discussion will rely not only on that worldvolume but also on the interplay with the normal bundle in eleven-dimensional spacetime. Other notions of framings has been used before, but they are different from the (standard) one we use here. For example, [9] consider the M5-brane to be framed, whereby the normal bundle splits off a one-dimensional trivial summand so that the structure group reduces from SO(5) to SO(4), as in a case in [76]. We will consider more general and standard decompositions, where the normal bundle completely decomposed into a Whitney sum of trivial line bundles.

The M5-brane need not necessarily be Spin, but can be taken to be oriented. Therefore, the discussion in [54] [55] can be adapted to this case, resulting in twisted versions of $p_1$-structures, rather than twisted versions of String and String$^c$ structures, on the worldvolume. Note that such a situation can be interpreted also in terms of elliptic cohomology [39]. Therefore, instead of conditions of the form $\frac{1}{2}p_1 + \alpha = 0$ for various degree 4 classes $\alpha$, we will have a condition of the form $p_1 + \alpha = 0$. In the spirit of the approach of [72] [64] [51] [55] [56], we will interpret this latter condition as a twisted structure:

**Definition 3** An $\alpha$-twisted $p_1$-structure on a brane $\iota : M \to Y$ with a Riemannian structure classifying map $f : M \to BO$, is a 4-cocycle $\alpha : Y \to K(\mathbb{Z}, 4)$ and a homotopy $\eta$ in the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & BO(n) \\
\downarrow{\iota} & \nearrow{\eta} & \downarrow{p_1} \\
Y & \xrightarrow{\alpha} & K(\mathbb{Z}, 4).
\end{array}
$$

The obstruction is then $p_1(M) + [\alpha] = 0 \in H^4(M; \mathbb{Z})$. As in the twisted String case, the set of such structures will be a torsor for $H^3(M; \mathbb{Z})$.

Framing of the M5-brane as a submanifold. A framing of the M5-brane worldvolume as a submanifold $M^6 \subset Y^{11}$ of dimension 6 is the assignment $\xi$ of 5 linearly independent vectors $(\xi^1(x), \ldots, \xi^5(x))$ in $T_xY^{11}$ that are normal to $M^6$. The pair $(M^6, \xi)$ is then a Pontrjagin framed manifold. A framing may be thought of as an isomorphism between the normal bundle of the embedding and the Whitney sum of five copies of the (unique) oriented line bundle over $M^6$. The importance of this arises from the fact that the spinors on the M5-brane take values in the Spin bundle of the normal bundle. With the latter being trivial, the former is also trivial and hence admits a maximal number of sections; that is, such a configuration is maximally supersymmetric.

Let $i : M^6 \to \mathbb{R}^{6+r}$ be an embedding in Euclidean space for large enough $r$. The normal bundle $N(M^6, i)$ of $i$ is the quotient of the pullback of the tangent bundle of $\mathbb{R}^{6+r}$ by the sub-bundle given by the tangent bundle of $M^6$, $N(M^6, i) = iT\mathbb{R}^{6+r}/TM^6$, so that $N(M^6, i)$ is an $r$-dimensional real vector bundle over $M^6$. If we give $T\mathbb{R}^{6+r} = \mathbb{R}^{6+r} \times \mathbb{R}^{6+r}$ the Riemannian metric obtained from the usual inner product in Euclidean space, $N(M^6, i)$ may be identified with the orthogonal complement of $TM^6$ in $iT\mathbb{R}^{6+r}$. That is, the fiber at $z \in M^6$ may be identified with the subspace of vectors $(z, x) \in \mathbb{R}^{6+r} \times \mathbb{R}^{6+r}$ such that $x$ is orthogonal to $i_z(TM^6)_z$, where $i_z$ is the induced embedding of $TM^6$ into $T\mathbb{R}^{6+r}$. If $M^6$ admits an embedding with trivial normal bundle, we say...
that $M^6$ has a stably trivial normal bundle. Different embeddings lead to different normal bundles, but stably they all coincide with the stable normal bundle which is classified by the Gauss map $N(M^6) \to BO = \lim_k BO(k)$. If $r$ is sufficiently large and $i_1, i_2 : M^6 \to \mathbb{R}^{6+r}$ are two embeddings, then $N(M^6, i_1)$ is trivial (i.e. $N(M^6, i_1) \cong M \times \mathbb{R}^r$) if and only if $N(M^6, i_2)$ is trivial.

In general, $r$ has to be more than twice the dimension of the submanifold, that is $r > 12$ in the case of the fivebrane. However, one can take the topology of $M^6$ to be such that this still works for lower codimension. Therefore, we will assume that we are in such a setting and will concentrate on the case when we have a physical embedding in the 11-dimensional M-theory target space $Y^{11}$. Note that for the case of the M2-brane, this is automatically satisfied because of the already large codimension. Further discussion related to embeddings of the M-branes can be found in [48].

The M5-brane via the Pontrjagin-Thom construction. Similarly to the M2-brane, the M5-brane can be described straightforwardly using the Pontrjagin-Thom construction. There is a canonical homeomorphism $T(M^6 \times \mathbb{R}^5) \cong \Sigma^5(M^6_+)$ between the Thom space of the trivial 5-dimensional vector bundle and the 5-fold suspension of $M^6_+$. The map which collapses $M^6$ to the non-basepoint yields a base-point preserving map $\Sigma^5(M^6_+) \to S^5 = \Sigma^5 S^2$ whose homotopy class defines an element of $\pi_{11}(S^5) \cong \mathbb{Z}_2$. Therefore, instead of $\mathbb{Z}_{24}$ for the M2-brane we have $\mathbb{Z}_2$ for the M5-brane. This is appropriate for considering the relation to the Kervaire invariant rather than to String structures, although there are relations of the M5-brane to the latter manifested in a different way (see [54] [20] [21]). We will now highlight a consequence of that.

The String $E_8$ bundle and the octic invariant. We now consider the effect of having an $E_8$ bundle in six dimensions, as suggested by the presence of the C-field. We take $M^6$ to be a compact framed 6-manifold without a boundary admitting a String structure (see [54] [20] for justification). More precisely, we take a String $E_8$ bundle on $M^6$, that is, a bundle of the 3-connected cover $E_8(3)$ of Lie group $E_8$. Thus there is no transgressed degree three class that would characterize the bundle; instead, there will be a class in dimension 15 corresponding to the generator of $H^{16}(BE_8; \mathbb{Z}) \cong \mathbb{Z}$. This octic invariant has an explicit description (see [14]), to which we now propose a connection. Consider a map $f : M^6 \to BE_8(4)$ to the classifying space $BE_8(4)$ of $E_8(3)$ String bundles on $M^6$. For such a map, the Pontrjagin-Thom construction yields the isomorphism $S^{6+5} \to T(N(M^6, i)) \cong \Sigma^5 M^6_+ \to \Sigma^5 BE_8(4)_+$, whose homotopy class defines an element of $\pi_{11}(\Sigma^5(K(Z, 16))$. Now, to ‘lowest order’, the homotopy type of $BE_8(4)$ can be ‘approximated’ by that of the Eilenberg-MacLane space $K(Z, 16)$. Then, taking $M^6$ to be framed and embedded in $\mathbb{R}^{11}$, we get a generator of $\pi_{11}(\Sigma^5(K(Z, 16))$. This is $\pi_{11}(K(Z, 11)) \cong \mathbb{Z}$ and corresponds to the octic invariant of $E_8$. Therefore, we can detect the generator of $\pi_{15}(E_8)$ or of $H^{15}(E_8; \mathbb{Z})$, albeit rather indirectly and essentially in the topologically trivial sector.

Framed bordism of M-branes and corresponding invariants. Two framed $n$-manifolds $(N_i, f_i)$ are bordant if there is a framed $(n + 1)$-manifold $(B, g)$ with $\partial B = \{0\} \times N_0 \cup \{1\} \times N_1$, $\partial g = f_0 \cup f_1$. The bordism classes $[N, f]$ of framed submanifolds form an abelian group $\Omega_n^B N$. For M2 and M5 branes with trivial normal bundles embedded in spacetime $Y^{11}$, we consider

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One usually has the approximation $E_8 \sim K(Z, 3)$, but then considering 3-connected covers kills topology in dimension 3, and if one considers the next nontrivial class, this would be in dimension 15, though this resulting structure will be topologically trivial.
the M-branes as boundaries or as admitting boundaries. For example, we can use the above bordism picture to describe framed open M2-branes with boundaries on M5-branes [69] [70] or framed M5-branes with boundaries on M9-branes [5]. This is discussed in [51] for the case of the M2-brane. Certain expressions such as the effective action or the partition function sometimes are themselves topological invariants, or depend on topological invariants; for instance, framed cobordism invariants. This implies that theses expressions will have the same form or value for every element (i.e. space/worldvolume) in the cobordism class. Therefore, it is enough for this purpose to evaluate the expression, i.e., the effective action or the partition function, on generators.

For the case of the M2-brane, the framed cobordism group is \( \Omega^6_{fr} \cong \mathbb{Z}_{24} \), generated by \( S^3 = SU(2) \) with the Lie group framing. For the M5-brane the corresponding cobordism group is \( \Omega^6_{fr} \cong \mathbb{Z}_2 \), generated by \( S^3 \times S^3 = SU(2) \times SU(2) \) with the product Lie group framing.

### The topological pairing and the action in the presence of a framing.

It is useful to formulate the M5-brane theory in seven dimensions [20] and consider an appropriate restriction to six dimensions [20] [21] [22] [23]. We now consider the effect of framing on the topological part of the action, making use of results of Lannes [40] [43]. We start with the M5-brane worldvolume \( M^6 = \partial N^7 \) as a framed manifold and study mod 2 torsion cohomology classes of middle degree \( H^3(M^6; \mathbb{Z}_2) \), corresponding to a lift of the class of the worldvolume gerbe. We take the inclusion of the boundary \( i : M^6 \hookrightarrow N^7 \), and discuss to which extent the gerbe on \( M^6 \) arises from the one on the extension \( N^7 \) when studying the effective action. The self-duality of the exact sequence

\[
H^3(N^7; \mathbb{Z}_2) \xrightarrow{i^*} H^3(M^6; \mathbb{Z}_2) \xrightarrow{\delta} H^4(N^7, M^6; \mathbb{Z}_2) \longrightarrow H^4(N^7; \mathbb{Z}_2)
\]

implies that \( \mathcal{I} := \text{Im}(i^* : H^3(N^7; \mathbb{Z}_2) \to H^3(M^6; \mathbb{Z}_2)) \) is a Lagrangian subspace in the inner product space \( E := H^3(M^6; \mathbb{Z}_2) \). That is, \( \mathcal{I} = \mathcal{I}^\perp \) with respect to the symplectic form on \( E \).

An element \( x \in H^3(M^6; \mathbb{Z}_2) \) can be thought of as a homotopy class of maps of the one-point compactification of the worldvolume \( M^6 \to K(\mathbb{Z}_2, 3) \), and so determines an element \( S^6 \to \Sigma^\infty M^6 \to \Sigma^\infty K(\mathbb{Z}_2, 3) \) of the stable homotopy group \( \pi_6(K(\mathbb{Z}_2, 3)) \). This group is of order 2, so that the framing \( f \) determines a map

\[
q_f : H^3(M^6; \mathbb{Z}_2) \to \mathbb{Z}_2,
\]

i.e. \( E \) can be equipped with this quadratic form. If \( N^7 \) admits a framing extending that of \( M^6 \), then the quadratic form is trivial on \( \mathcal{I} \), and so the Witt class of the quadratic form is a framed cobordism invariant. In this case one can characterize the elements \( u \in E \) such that \( q(x) = u \cdot x \) for \( x \in \mathcal{I} \), in terms of the relative Wu class \( v_4(\nu, f) \in H^4(N^7, M^6; \mathbb{Z}_2) \). This class restricts on \( N^7 \) to \( v_4(\nu) \in H^4(N^7; \mathbb{Z}) \), due to dimension. Let \( u \in H^3(M^6; \mathbb{Z}_2) \) be the class such that \( u = v_4(\nu, f) \) \( \in H^4(N^7, M^6; \mathbb{Z}_2) \). This is well-defined modulo \( \mathcal{I} \), and \( q(x) = x \cdot u \) for any \( x \in \mathcal{I} \). Take \( x = i^*y \) for \( y \in H^3(N^7; \mathbb{Z}_2) \). By self-duality of the sequence (2.6), this expression can be rewritten as

\[
q(i^*y) = i^*y \cup u = y \cup \delta u = y \cup v_4(\nu, f).
\]

### Effect of change of framing on the quadratic form.

The intersection pairing determined by the framing leads to a quadratic refinement in the sense of Browder-Brown [12] [11]. Now

\[6\] The arguments we make here can be extended to differential cohomology. While this is interesting, we will not pursue it here as we are interested in topological invariants. See also the end of Section 2.
we consider the effect of change of framing via gauge transformations. That is, we consider two
framings $f_1, f_2$ differing by a gauge transformation $g : M^6 \to O(6)$. We can also consider this
in the stable range, i.e. replace $O(6)$ by $O$. Then one has the change of framing formula \[ q_{f_1}(x) + q_{f_2}(x) = x \cdot g^* z, \]
where $z$ is the degree three class in the diagram

\[
\begin{array}{ccc}
O & \xrightarrow{z} & K(\mathbb{Z}_2, 3) \\
\downarrow & & \downarrow \\
EO & \xrightarrow{BO(v_4)} & K(\mathbb{Z}_2, 3) \\
\downarrow & & \downarrow \\
BO & \xrightarrow{BO} & BO.
\end{array}
\] (2.9)

The gauge group of smooth maps from $M^6$ to $O$ acts transitively on framings (with respect to this
embedding), and the corresponding group of maps, the real K-group $KO^{-1}(M^6) = [M^6, O]$, acts
transitively on the set of stable framings. This is depicted in the following diagram

\[
\begin{array}{ccc}
EO & \xrightarrow{O} & K(\mathbb{Z}_2, 3) \\
\downarrow & & \downarrow \\
M^6 & \xrightarrow{f} & BO \\
\downarrow & & \downarrow \\
BO & \xrightarrow{BO} & K(\mathbb{Z}_2, 4). \quad (2.10)
\end{array}
\]

For $g$ a gauge transformation, Brown’s theorem \[11\] implies the transformation law

\[ q_{gf}(x) = q_f(x) + \langle x \cup g^* v_4, [M^6] \rangle, \] (2.11)

where $v_4 = z$ denotes the image in $H^3(O; \mathbb{Z}_2)$ of the Wu class $v_4$ under the map $\omega : \Sigma O \to BO$, which is adjoint to the equivalence map $O \to \Omega BO$. Therefore, this gives a formula for the action
in six dimensions. Furthermore, the action in six dimensions depends on the choice of framing, since $\frac{6}{2} - 1$ is a power of 2.

\textbf{Example.} Consider the case when the worldvolume is a product $M^6 = M^3 \times M^3$ of two 3-
dimensional framed manifolds. Consider two gauge transformations $f_i : M^3_i \to O$, $i = 1, 2$, such
that the pullback of the generator $z$ in (2.9) is nonzero $f^*_i z = 0$. With $\pi_i : M^3_i \times M^3_i \to M^3_i$, $i = 1, 2$, the projection to the two factors, we consider the composite maps $g_i = f_i \circ \pi_i$. Then we use these
maps to pull back $z$ as $\alpha_i = g^*_i z$, $i = 1, 2$ and also define a third map as the product $g_3 = g_1 g_2$ with
$\alpha_3 = g_3^* z$. With this, $\alpha_3 = \alpha_1 + \alpha_2$, so that $q(\alpha_3) = q(\alpha_1) + q(\alpha_2) + 1$. This relation implies that,
upon using the change of framing formula \[36\], one of the maps $g_i$, $i = 1, 2, 3$ changes the value of
the Kervaire invariant. This demonstrates that the action in this example depends on the choice
of framing.

\subsection{2.3 The connection to the Hopf invariant at the rational level}

We identify the topological part of the action of the M5-brane in seven dimensions with the Hopf
invariant. We start at the rational level, making connection to \[33\], and then at the integral level
in Section 2.4 and also the $\mathbb{Z}_2$ level in Section 3.2. As far as we know, the latter two are new.

\footnote{This criterion works in higher dimensions as well.}
We consider the M5-brane effective action lifted to seven dimensions, alternatively viewed as a reduction of the M-theory action on a 4-manifold, as in [20]. We will concentrate first on the case of the 7-sphere and then on more general 7-manifolds. The relevant term is the topological coupling \( \int C_3 \wedge F_4 \), which can be written as \( \int *J \wedge C_3 \), i.e. as an interaction of the potential \( C_3 \) with the (conserved) topological current \( J_3 = *F_4 \) [71].

For an arbitrary smooth map \( f : S^7 \to S^4 \), we describe the Hopf invariant \( H(f) \in \mathbb{R} \) in our context as follows. This will be a straightforward generalization of the case of the second Hopf fibration presented in [45]. We start by choosing a 4-form \( F_4 \in \Omega^4(S^4) \) on \( S^4 \) normalized as \( \int_{S^4} F_4 = 1 \). This is satisfied, notably, for the Freund-Rubin ansatz [25], where \( F_4 \) is taken to be constantly proportional to the volume form on \( S^4 \). Then, since \( d(f^*F_4) = f^*(dF_4) = 0 \) by the Bianchi identity, we have that \( f^*F_4 \) is a closed form on \( S^7 \). From \( H^4(S^7; \mathbb{R}) = 0 \), by the de Rham theorem there exists a 3-form \( C_3 \in \Omega^3(S^7) \) such that \( f^*F_4 = dC_3 \). Then the Hopf invariant of \( f \) is defined as

\[
H(f) = \int_{S^7} C_3 \wedge dC_3 .
\]

(2.12)

The value of \( H(f) \) is determined independently of the choices of \( F_4 \) and \( C_3 \), and thus depends only on \( f \). In fact, it can be shown that the value of \( H(f) \) depends only on the homotopy class of \( f \), i.e., if two smooth maps \( f_0, f_1 : S^7 \to S^4 \) are homotopic, then \( H(f_0) = H(f_1) \) (see [45] for an illustration for the similar second Hopf fibration). We highlight that the above statement is the gauge invariance of the 7-dimensional Chern-Simons term for \( C_3 \).

If we replace \( S^7 \) with an arbitrary closed compact orientable 7-manifold \( M^7 \), we may still obtain an invariant of \( f : M^7 \to S^4 \) and we still have

\[
H(f) = \int_{M^7} C_3 \wedge f^*\omega_4 = \int_{M^7} C_3 \wedge F_4 = \int_{M^7} C_3 \wedge dC_3 ,
\]

(2.13)

where \( \omega_4 \) is the volume 4-form on \( S^4 \) and \( C_3 \) satisfies \( F_4 = f^*\omega_4 = dC_3 \). Then a straightforward generalization of the discussion for \( S^7 \) gives the following statement: Let \( M^7 \) be a closed Riemannian extended worldvolume and \( \omega_4 \in \Omega^4(S^4) \) be the area form on \( S^4 \). Then \( H(f) \) provides a Riemannian invariant for a map \( f : M^7 \to S^4 \) if the 4-form \( f^*\omega_4 \) is exact. This may be viewed as a (generalized) Freund-Rubin ansatz.

**The Hopf invariant as a linking number.** The Hopf invariant is geometrically given by the linking number of the pre-images \( \ell_1 = f^{-1}(r_1) \) and \( \ell_2 = f^{-1}(r_2) \) of two distinct regular values \( r_1 \) and \( r_2 \) of the map \( f \). Choose open neighborhoods \( W_{\ell_1} \) and \( W_{\ell_2} \) of \( \ell_1 \) and \( \ell_2 \) and choose representatives \( F_4^{\ell_1} \) and \( F_4^{\ell_2} \) of the compact Poincaré duals of \( \ell_1 \) and \( \ell_2 \) in the cohomology groups with compact supports \( H^4_c(W_{\ell_1}) \) and \( H^4_c(W_{\ell_2}) \). Now, since \( H^4_{dR}(S^7) = 0 \), the extensions of \( F_4^{\ell_1} \) and \( C_3^{\ell_2} \) by zero to all of \( S^7 \) are exact, i.e. there are 3-forms \( C_3^{\ell_1} \) and \( C_3^{\ell_2} \) on \( S^7 \) such that \( dC_3^{\ell_1} = F_4^{\ell_1} \) and \( dC_3^{\ell_2} = F_4^{\ell_2} \). The differential form definition of the linking number is

\[
\text{Linking number} = L(\ell_1, \ell_2) = \int_{S^7} C_3^{\ell_1} \wedge F_4^{\ell_2} .
\]

(2.14)

This is well-defined, as is shown e.g. in [10] for the case of \( S^3 \), and the proof for our case of \( S^7 \) is similar and, likewise, known.
We now consider the case when our extended worldvolume is a product of a 3-manifold $M^3$ and a 4-manifold $N^4$, relying on [73] [71]. Consider two continuous maps $f(M^3)$ and $g(N^4)$ from the two smooth, oriented, non-intersecting manifolds $M^3$ and $N^4$ into Euclidean space $\mathbb{R}^8$. Let $S^7$ be the unit 7-sphere centered at the origin of $\mathbb{R}^8$ and $e^*\Omega_7$ be the pullback of the volume form of $S^7$ under the map $e : M^3 \times N^4 \to S^7$, where we associate to each pair of points $(m, n) \in M^3 \times N^4$ the unit vector $e$ in $\mathbb{R}^8$ given by $e(m, n) = (g(n) - f(m))/|g(n) - f(m)|$. The degree of this map is the generalized Gauss linking number of $M^3$ and $N^4$

$$L(f(M^3), g(N^4)) := L(M^3, N^4) = \frac{1}{\Omega_7} \int_{M^3 \times N^4} e^*\Omega_7 ,$$

(2.15)

which is a special case of the Gauss integral. Note that $L(M^m, N^n)$ obeys the graded-commutativity law $L(M^m, N^n) = (-1)^{(m-1)(n-1)}L(N^n, M^m)$, so that it is zero for even-dimensional manifolds $M^m$ and $N^n$, and in our case this is symmetric so that it does not matter in which order we take $M^3$ and $N^4$. In the special case when we take $M^3 = S^3$ and $N^4 = S^4$ we get a linking number on the product $S^3 \times S^4 \subset \mathbb{R}^8$, which can be viewed as a result of ‘untwisting’ the total space of the 7-sphere.

**Differential refinement of the Hopf invariant.** The differential form that appears in the definition of the Hopf invariant has, in the context of M-theory, a refinement to a class in differential cohomology. Then the action, which is of Chern-Simons type, can be written as the refinement of the Hopf invariant

$$\hat{H}(f) = \int_{M^7} \hat{C}_3 \cup \hat{G}_4 ,$$

(2.16)

which is exactly the type of expressions studied in [20] [21] [22]. Therefore, the M5-brane action in seven dimension can be seen as a differential (and even a stacky) refinement of the Hopf invariant.

### 2.4 Hopf invariant 2 over $\mathbb{Z}$ and refinement via the dual of the C-field

We now consider the integral version of the Hopf invariant and relate it to the M-branes as well as to the C-field and its dual. It is also useful to extend to eight dimensions, as in [75] [30]. The Hopf invariant $H(f) \in \mathbb{Z}$ of a map $f : S^7 \to S^8$ is determined by the cup product structure of the mapping cone $X = S^4 \cup_f \mathbb{D}^8$ with

$$a \cup a = H(f)b \in H^8(X; \mathbb{Z}) ,$$

(2.17)

for generators $a \in H^4(X; \mathbb{Z}) = \mathbb{Z}$ and $b \in H^8(X; \mathbb{Z}) = \mathbb{Z}$. Therefore, the Hopf invariant defines a morphism of groups $H : \pi_7(S^4) \to \mathbb{Z}$ given by $f \mapsto H(f)$ . This can be characterized as follows. Since $S^3$ is 2-connected, then the EHP exact sequence of homotopy

$$\cdots \to \pi_6(S^3) \xrightarrow{\Sigma} \pi_7(\Sigma S^3) \xrightarrow{H} \pi_6(S^3 \vee S^3) \xrightarrow{P} \pi_5(S^3) \to \cdots$$

(2.18)

gives that the suspension $\Sigma$ and the Hopf invariant map $H$ fit into the exact sequence

$$\pi_6(S^3) \xrightarrow{\Sigma} \pi_7(S^4) \xrightarrow{H} \mathbb{Z} \xrightarrow{P} \pi_5(S^3) .$$

(2.19)

Note that the standard Hopf map $\eta : S^7 \to S^4$ has Hopf invariant one: $H(\eta) = 1$.

---

*This is in contrast to the $f$-invariant that we consider in Section 3.5.*
Refinement of the Hopf invariant via the dual of the C-field. We consider the dual $C_6$ of the C-field $C_3$ associated to the M5-brane in a way that is ‘dual’ to the way the C-field is associated to the M2-brane. The two fields $C_3$ and $C_6$ are related as follows. The first is a 3-connection for a curvature $F_4$. At the level of differential forms, the Hodge dual $*_{11}F_4$ is a field $F_7$, which admits $C_6$ as a trivialization (or a 6-connection in the higher geometric language of [52] [54]).

The M2-brane has a coupling $\int_{M^2} C_3$, while the M5-brane has a coupling $\int_{M^5} C_6$. The dual field is given in full generality by the right hand side of the equation of motion $d \ast F_4 = \frac{1}{2}F_4 \wedge F_4 + I_8(g)$, where $\ast$ is the Hodge duality operator with respect to the metric $g_Y$ on the eleven-dimensional target space $Y^{11}$, and $I_8(g)$ is polynomial in the Pontrjagin forms of $Y^{11}$ given by $I_8 = \frac{1}{8}(p_2(g) - (\frac{1}{2}p_1(g))^2)$. Lifted to cohomology, the corresponding degree eight class is $[17]$

$$\Theta_Y = \frac{1}{2}a \cup a - \frac{1}{2}a \cup \lambda + 30\hat{A}_8(Y^{11}),$$

(2.20)

where $a$ is the class of an $E_8$ bundle on $Y^{11}$ and $\lambda$ is the first Spinc characteristic class $\frac{1}{2}p_1$, related via the quantization condition $[F_4] + \frac{1}{2}\lambda = a \in H^4(Y^{11};\mathbb{Z})$ [75].

Now we observe that when the Pontrjagin classes vanish, or more precisely when we have String and Fivebrane structures (cf. [63]), then we get Hopf invariant 2

$$a \cup a = 2\Theta_Y.$$  

(2.21)

Next, if we have the vanishing of the $\hat{A}$-genus then we get

$$a \cup a - a \cup a = 2\Theta_Y.$$  

(2.22)

The right hand side is a quadratic refinement of the bilinear form on the cohomology group $H^4$ given by $\lambda$. Therefore, this suggests that we have a quadratic refinement of the Hopf invariant. In the general case, we have the nonlinear term proportional to $\lambda^2$, and so we view (2.20) as both a quadratic refinement and a nonlinear modification.

First example of the dual of the C-field and Hopf invariant 2. Let $f : S^7 \rightarrow S^4$ be any continuous map. Consider the 8-dimensional cell complex $X := \mathbb{D}^8 \cup_f S^4$ obtained by identifying a point on the boundary $x \in \partial \mathbb{D}^8 = S^7$ with $f(x) \in S^4$. Thus $X$ is a CW complex with one cell each in dimensions 0, 4, and 8. Then the cohomology groups of $X$ are $H^i(X;\mathbb{Z}) \cong \mathbb{Z}$ if $i = 0, 4, 8$ and are 0 otherwise. Denote by $\sigma_i$ the generator of $H^i(X;\mathbb{Z})$ determined by the cell (endowed with an orientation) in dimension $i$ for $i = 4, 8$. In terms of generators, the Hopf invariant of $f$ is the integer $H(f)$ such that $\sigma_2^2 = H(f)\sigma_8$. If $H(f) = h \in \mathbb{Z}$ then the cohomology ring of $X$ is isomorphic to $\mathbb{Z}[\sigma_4, \sigma_8]/\langle \sigma_4^2 = h\sigma_8, \sigma_4^3, \sigma_8^2 \rangle$. In particular, if $h = 1$, then $H^*(X;\mathbb{Z}) = \mathbb{Z}[\sigma_4]/\langle \sigma_4^2 \rangle$, and this is the quaternionic projective plane $\mathbb{H}P^2$. In our case, when we have Hopf invariant 2, the mapping cone is not a projective plane. It is obvious that we can allow $h$ to take any value we like; in particular, we can have the value $h = 2$ appropriate for the dual of the C-field.

Second example with Hopf invariant 2. We consider the Hopf fibration and make use of two distinct copies of $S^4$. Take $S^4 \times S^4$ as the cell complex formed by attaching an 8-cell to the wedge of two spheres $S^4 \vee S^4$, using the attaching map $S^7 \rightarrow S^4 \vee S^4$. This can be described using the Whitehead product, by forming the composition of $g$ with the folding map $F : S^4 \vee S^4 \rightarrow S^4$. Starting with two base-point preserving maps $f : S^4 \rightarrow X$ and $g : S^4 \rightarrow X$, let $[f, g] : S^7 \rightarrow X$ be the
composition \( S^7 \to S^4 \vee S^4 \xrightarrow{f \vee g} X \). This gives a well-defined product \( \pi_4(X) \times \pi_4(X) \to \pi_7(X) \), which generalizes the commutator product for the (nonabelian) fundamental group. Then the map \( S^7 \to S^4 \vee S^4 \) is the Whitehead product \([\iota_x, \iota_x]\) of the two inclusions of \( S^4 \) into \( S^4 \vee S^4 \). Now let \( X = e^8 \cup [\iota, \iota] \) be the space obtained from \( S^4 \) by attaching an 8-cell \( e^8 \) via the map representing \([\iota, \iota]\). Let \( u \in H^4(X^8; \mathbb{Z}) \) and \( v \in H^8(X^8; \mathbb{Z}) \) be the cohomology generators in degree four and eight, respectively. Then, if we take \( \iota \) to be the class of the identity map, then \( u^2 = 2v \), so that the Whitehead product \([\iota, \iota]\) has Hopf invariant \( \pm 2 \).

**Third example with Hopf invariant 2.** Let \( U^4_+ \) be the northern hemisphere of \( S^4 \) and let \( g : (U^4_+, S^3) \to (S^4, (1,0,0,0)) \) be any map whose restriction to the open upper-hemisphere \( U^4_+ - S^3 \) is a homeomorphism. Now view \( S^7 \) as the boundary of \( D^4 \times D^4 \). On the other hand, the simplicial boundary of \( D^4 \times D^4 \) is also \( \partial D^4 \times D^4 + D^4 \times \partial D^4 \). Then define the map \( f : \partial D^4 \times D^4 + D^4 \times \partial D^4 \to S^7 \) by \( (y, (x_1, x_2, x_3, x_4)) \in \partial D^4 \times \partial D^4 \mapsto g \left( x_1, x_2, x_3, x_4, \sqrt{1 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} \right) \) on the first factor and \( ((y_1, y_2, y_3, y_4), x) \in D^4 \times \partial D^4 \mapsto g \left( y_1, y_2, y_3, y_4, \sqrt{1 - (y_1^2 + y_2^2 + y_3^2 + y_4^2)} \right) \) on the second factor. Then this map \( f \) has degree 2 and hence Hopf invariant 2.

**Remarks.** The Hopf invariant is \( H(f) = -2 \) if we choose the orientation of \( S^7 \) determined by the generator \(-a\) instead of \( a\). Moreover, starting with a map with Hopf invariant one, we can get a map of Hopf invariant \( k \) by composing with a map on the seven sphere of degree \( k \). Let \( h : S^7 \to S^7 \) be a map of degree \( k \), i.e. \( H^*(a) = ka \), and \( f : S^7 \to S^4 \) a continuous map. Then \( H(fh) = kH(f) \). Taking \( k = 2 \) we get a map with \( H(f) = 2 \). Note that, in contrast, this cannot be generated by composing with a map of degree 2 on the 4-sphere. Indeed, let \( f : S^7 \to S^7 \) be a continuous map, and \( h : S^4 \to S^4 \) a map of degree \( k \), i.e. \( h^*(u) = ku \). Then \( H(hf) = k^2 H(f) \) and, of course, \( k^2 \) cannot be 2.

### 2.5 M-branes via cohomotopy

The topological study of the C-field amounts to an interpretation of a degree three cohomology class together with some extra structure. One could consider this as an \( E_8 \) bundle or as a \( K(\mathbb{Z}, 3) \) bundle. The latter can be viewed as an approximation of the former in the range of dimension of M-theory. We will propose another interpretation, whereby we use the loop space \( \Omega S^4 \) of the 4-sphere. Note that \( BE_8 \simeq K(\mathbb{Z}, 4) \) and that \( \pi_4(K(\mathbb{Z}, 4)) = \pi_4(S^4) \) with obvious injection \( \pi_4(K(\mathbb{Z}, 4)) \to \pi_4(S^4) \). Therefore, we can account for the degree four class corresponding to the class of \( F_4 \) using the 4-sphere as a model. However, we will see that the connection is much more precise. We have a degree four field \( F_4 \) and a degree seven field \( F_7 := *F_4 \), satisfying the Bianchi identity and the equation of motion of the C-field (without correction terms)

\[
dF_4 = 0 \quad \text{and} \quad d*F_4 = \frac{1}{2}F_4 \wedge F_4. \tag{2.23}
\]

**Rational homotopy of \( S^4 \).** Consider the 4-sphere \( S^4 \) with its volume form \( \omega_4 \). The de Rham cohomology \( H^{*}_{\text{DR}}(S^4) \) is not an exterior algebra, and one has to impose the condition \( \omega_4 \wedge \omega_4 = 0 \). Correspondingly, let \( x_4 \) be the generator of de Rham cohomology in degree four. In addition to \( dx_4 = 0 \), this generator has to be such that \( x_4^2 \) is zero in the cohomology of the minimal model. This means that there is a generator \( y_7 \) in degree seven such that a differential \( d \) can be defined with \( x_4^2 = dy_7 \). This then gives that \( x_4 \) and \( y_7 \) are the only generators because of the above imposed
relation $x_2 = 0$ and the automatic relation $y_7 = 0$. Sullivan’s minimal model (see [19]) for $S^4$ is then given by the map $\tau : (\wedge (x_4, y_7), d) \to A(S^4)$, where $dy_7 = x_4^3$ and $\tau(x_4)$ represents the unique cohomology generator of degree eight. The map from the minimal model to the de Rham complex of $S^4$ is given by sending $x_4$ to the volume form $\omega_4$ of $S^4$ and sending $y_7$ to zero. The point here is that these expressions correspond to equations (2.23).

Note that $\tau$ is a quasi-isomorphism of free commutative differential graded algebras so that the rational homotopy $\pi_\ast(S^4) \otimes \mathbb{Q}$ is concentrated in degrees 4 and 7; furthermore, the rank in each of these degrees is one. In fact, one can consider the Whitehead product $\pi_4(S^4) \otimes \pi_4(S^4) \to \pi_7(S^4)$, which we used in the second example in Section 2.4 and which is given by $f, g : S^7 \to S^4$ are the corresponding maps. Since $S^4$ is simply-connected then the vector space $\pi_\ast^\mathbb{Q}(S^4) := \pi_\ast^\mathbb{Q}(S^4)$ turns into a graded Lie algebra with the Whitehead product as the Lie bracket of degree $-1$.

The rational 4-sphere. A simply connected space $X$ is rational if $\pi_\ast(X), \tilde{H}_\ast(X; \mathbb{Z})$, or $\tilde{H}_\ast(\Omega X; \mathbb{Z})$ is a $\mathbb{Q}$-vector space, with the three conditions being equivalent, where $\Omega X$ is the space of based loops on $X$. Then the rational 4-sphere is defined to be (see [27]) $S^4_\mathbb{Q} := \left( \bigvee_{k \geq 1} S^4_k \bigcup \prod_{k \geq 2} D^5_k \right)$, where $D^5_k$ is attached to $S^4_k \vee S^4_{k+1}$ via a representative $S^4 \to S^4_k \vee S^4_{k+1}$ of $\iota_{4,k} - (k+1)\iota_{4,k+1}$, where $\iota_{4,k}$ denotes the homotopy class of the inclusion of $S^4$ as the $k$-th summand $S^4_k$ of $\bigvee_{k \geq 1} S^4_k$. Then the (reduced) homology of this space is $\tilde{H}_\ast(S^4_\mathbb{Q}; \mathbb{Q}) = \mathbb{Q}$ for $k = 4$ and zero otherwise.

Cohomotopy. There is the Borsuk cohomotopy functor $\pi^M = [-, M] : \text{Top} \to \text{Set}^{\text{op}}$ from the category of topological spaces to the (opposite) category of sets, given by $\pi^M(X) = [X, M]$, where $M$ is an arbitrary topological space. This is a homotopy invariant. When $M = S^n$, an $n$-sphere, then this is the usual cohomotopy functor $\pi^n(X) = [X, S^n]$. When $X$ is of dimension less or equal to $2n - 1$ then the set $\pi^n(X)$ has a canonical abelian group structure arising from taking suspension $\Sigma : [X, S^n] = [\Sigma X, S^{n+1}]$ and using suspension coordinates; see [22].

We propose that the C-field and its dual on $Y^{11}$ are captured by the 4-sphere, via homotopy classes of maps $[Y^{11}, S^4]$. This cohomotopy set $\pi^4(Y^{11})$ is in canonical bijection with the set of cobordism classes of codimension-4 framed submanifolds $N^7$ of $Y^{11}$. The set $\pi^4(Y^{11})$ is in general not a group, but we will take $Y^{11}$ to have a topologically nontrivial factor with relatively low dimension. In order to be in the stable range, we will take $Y^{11} = N^7 \times \mathbb{R}^4$, so that the homotopy classes of maps become $[N^7, S^4] = \pi^4(N^7)$. When we take $N^7 = S^7$ then, by duality, we have $\pi^4(S^7) = \pi^{-7}(S^4)$. This reduces to the discussion above on the Hopf invariant. In terms of the sphere spectrum $S$ we have the stable cohomotopy

$$\pi^{-s}(X) = \pi_{\ast}[X, S].$$

Relation to the Pontrjagin-Thom construction. Let $\mathcal{F}_7(Y^{11})$ denote the set of closed 7-dimensional submanifolds of $Y^{11}$ with a framing on their normal bundle, up to normally framed bordism in $Y^{11} \times [0, 1]$. We have the following diagram

$$\begin{array}{ccc}
\pi^4(Y^{11}) & \to & \mathcal{F}_7(Y^{11}) \\
\downarrow h^4 & & \downarrow h_7 \\
H^4(Y^{11}) & \to & H_7(Y^{11})
\end{array}$$
where the map $h^4$ pulls back the cohomology class of $S^4$. The forgetful map $h_7$ uses the normal framing to orient the submanifold $N^7$ and then push forward its homology fundamental class.

**Examples.** We can take $Y^{11}$ to be of the form $\text{AdS}_4 \times S^7$ or $\text{AdS}_7 \times S^4$. If we take the anti de Sitter factors to be Euclideanized, $\text{AdS}^\text{Euc}_7$, then the cohomotopy $[Y^{11}, S^4]$ becomes homotopy equivalent to $[S^7, S^4] = \pi_7(S^4) = \mathbb{Z}$ and $[S^4, S^4] = \pi_4(S^4) = \mathbb{Z}$. One can also allow for more general situations leading to fractions of supersymmetry, namely to allow for more general Einstein manifolds $M^4$, or even orbifolds $M^4/\Gamma$, where $\Gamma$ is a finite subgroup of $\text{SO}(4)$, most notably a cyclic group $\mathbb{Z}_k$ of order $k$. Then the Hopf isomorphism for cohomotopy (dual to the Hurewicz isomorphism for homotopy) gives

$$
\pi^4(Y^{11}) = [\text{AdS}^\text{Euc}_7 \times M^4, S^4] \cong H^4(M^4). \tag{2.26}
$$

For example, with $M^4 = S^4/\mathbb{Z}_2 = \mathbb{R}P^4$, this captures the 2-torsion part of the C-field, as in e.g. [75] [31].

### 3 M-branes as corners and topological invariants

We now consider corner structures on the M2-brane and the M5-brane, as in [57] [53]. This uncovers some structures and connections to invariants, such as the $\mathbb{Z}_2$ Hopf invariant and the $f$-invariant. In addition, we will consider the connection to a geometric/topological invariant, namely the $\nu$-invariant for 7-manifolds with a $G_2$-structure.

#### 3.1 M-branes and corners

We start with the M2-brane and then consider the M5-brane. In each case we characterize the corner structure as it relates to the topological action.

**The M2-brane and corners.** We have considered the 3-dimensional worldvolume $M^3$ as the boundary of a 4-dimensional manifold $W^4$ in order to consider anomalies via the field strength $F_4$, as in [73]. On the other hand, we have also considered boundaries of M2-branes – effectively the self-dual strings– ending on the M5-brane worldvolume. As in the approach of [57] [53] [60], putting the two together leads to viewing the self-dual string as a corner of codimension-2. The structure of the manifolds involved and corresponding topological actions take the form

\[
\begin{align*}
W^4 : & \quad \int_{W^4} F_4, \quad \text{Manifold with corners of codimension-2.} \tag{3.1} \\
M^3 : & \quad \int_{M^3} C_3, \quad \text{Boundary.} \\
\Sigma_2 : & \quad \int_{\Sigma_2} B_2 \quad \text{Corner of codimension-2.}
\end{align*}
\]
The M5-brane and corners. We have discussed in [57] [53] how the M5-brane worldvolume can be viewed as a corner, essentially due to the presence of tubular neighborhood. The main source of spaces with corners of codimension-2 is the product of two manifolds with boundary. We will focus on the representative case of the product of two closed disks. The fivebrane worldvolume, taken as \( S^3 \times S^3 \), can be viewed as a corner in two different ways, schematically

\[
\begin{array}{c}
S^3 \times S^3 \\
\downarrow \partial_1 \quad \downarrow \partial_2 \\
\partial_2 \downarrow \quad \partial_1 \downarrow \\
S^2 \times D^4 \\
\downarrow \quad \downarrow \\
D^4 \times D^4 \\
\end{array}
\]  

The corresponding topological action decomposes accordingly. Starting with the quadratic action in eight dimensions and keeping track of two different components we have a reduction pattern schematically of the form

\[
\begin{array}{c}
C_3 \cup C'_3 \\
\downarrow \partial_1 \quad \downarrow \partial_2 \\
F_4 \cup F'_4 \\
\downarrow \quad \downarrow \\
B_2 \cup F'_4 \quad \text{or} \quad H_3 \cup C'_3 \\
\downarrow \quad \downarrow \\
C_3 \cup C'_3 \\
\end{array}
\]  

where we take \( B_2 = \pi_*(C_3) \) or \( H_3 = \pi_*(F_4) \) obtained by integration over the circle fiber (as in the 11-dimensional case, see [42]). The entries in the middle row correspond to Chern-Simons extensions and are examples of (higher) cup-product Chern-Simons theories [22] [23]. The extension of such theories to the bottom line, corresponding to the corners, is currently being developed.

### 3.2 The \( \mathbb{Z}_2 \) Hopf invariant

In Section 2.3 and Section 2.4 we considered the relation to the rational and integral Hopf invariants, respectively. Here we start with the relation of the mod 2 Hopf invariant to both the M2-brane and the M5-brane.

#### The \( \mathbb{Z}_2 \) Hopf invariant and the M2-brane.

The mod 2 Hopf invariant \( H_2(g) \in \mathbb{Z}_2 \) of a map \( g : S^7 \to S^4 \) defines an isomorphism \( H_2 : \pi_5(S^4) \to \mathbb{Z}_2 \) given by \( g \mapsto H_2(g) \). This is determined by the Steenrod square on the mod 2 cohomology of the mapping cone \( X^8 = S^4 \cup g \mathbb{D}^8 \) with

\[
S_q^4 = H_2(g) : H^4(X^8; \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H^8(X^8; \mathbb{Z}_2) \cong \mathbb{Z}_2 .
\]  

\footnote{At the face of it, it seems like we have a major problem: we need Stokes theorem for manifolds with corners, in the sense of having double boundary operators on the manifolds and double exterior derivatives the corresponding forms, which we do not have. However, we approach the corners from the topology point of view via cobordism, which avoids the need for boundary operator calculus, which may or may not exist. See the end of [22]; this will be expanded in the context of topological field theory in a work in progress [24]. The main point is that higher connections can be themselves be viewed as curvatures.}
In this case, the mod 2 Hopf invariant \( H_2(g) \in \mathbb{Z}_2 \) is the mod 2 reduction of the integral Hopf invariant \( H(g) \in \mathbb{Z} \). We will make use of the nice description for this invariant given in [43].

Consider a vector bundle \( E \to M \) associated to the \( O(n) \)-universal bundle \( EO(n) \to BO(n) \) over the classifying space of the orthogonal group \( O(n) \). The corresponding Thom spectrum is

\[
MO(n) := (BO^E)_n = T(\mathbb{R}^n \oplus E) \cong S^n \wedge T(E) \cong \Sigma^n T(E) .
\] (3.5)

The inclusions \( O(n) \hookrightarrow O(n+1) \) induce a directed system of such spectra, and the Thom spectrum \( MO \) is the direct limit \( MO := \lim_\to MO(n) \). An element of the homotopy group \( \pi_n(MO) \) is naturally identified with the set of cobordism classes of closed \( n \)-manifolds. Now consider the quotient \( \overline{MO} = MO/S \) by the sphere spectrum \( S \), which is the suspension spectrum of a point and which detects whether or not the manifold is framed. Of interest to us is an element of the fourth homotopy group of this space \( \pi_4(\overline{MO}) \), which represents a class of triples \((W^4, M^3, f)\) in which \( W^4 \) is a 4-manifold with boundary \( M^3 = \partial W^4 \), and \( f \) is a trivialization of the normal bundle \( \nu_M \). As before, \( M^3 \) represents the worldvolume of the M2-brane, \( W^4 \) is its bounding 4-manifold, and the normal bundle \( \mathcal{N}_M \) to \( W^4 \) can be identified with the normal bundle in the embedding in 11-dimensional spacetime \( Y^{11} \), as here we are already in the stable range.

Such an \((O, f, r)\)-manifold represents zero if it is a boundary, i.e. if it embeds in a 5-dimensional manifold with corner \((N^5, W^4, W^4, M^3, f)\). This means that \( N^5 \) is a 5-manifold whose boundary is given by \( W^4 \cup_{M^3} W^4 \), with \( W^4 \) and \( W^4 \) are manifolds with boundary the 3-manifold \( M^3 \), \( \partial W^4 = M^3 = \partial W^4 \), and \( f' \) is a trivialization of the normal bundle of \( W^4 \) which restricts to the given trivialization of the normal bundle of \( M^3 \). The map \( \pi_4(\overline{MO}) \to \pi_4(S) \) sends \((W^4, M^3, f)\) to its boundary \((M^3, f)\), with \( f = f'|_{M^3} \).

![Diagram](image)

\( W^4 \longrightarrow \cdots \longrightarrow EO \longrightarrow O \)

\( M^3 \)

\( \overline{MO} \)

We now consider classes in \( \overline{H}^*(BO; \mathbb{Z}_2) \) corresponding to the spectrum \( \overline{MO} \). Let us denote by \( \mathcal{N} = \mathcal{N}_N \) the normal bundle of \( W^4 \). The trivialization \( f \) of \( \mathcal{N}|_{M^3} \) provides a factorization of \( W^4 \to BO \) through \( W^4/M^3 \). Hence, as explained more generally in [43], any \( c \in \overline{T}^k(BO; \mathbb{Z}_2) \) gives rise to a relative class \( c(\mathcal{N}, t) \in H^k(W^4, M^3; \mathbb{Z}_2) \). In particular, we consider the relative Stiefel-Whitney class \( w_4(\mathcal{N}, f) \in H^4(W^4, M^3; \mathbb{Z}_2) \). The \( \mathbb{Z}_2 \) Hopf invariant is captured by the Hurewicz map on \( \pi_4(\overline{MO}) \) [58]. In our case,

\[
\text{Hopf}(M^3, f) = \langle w_4(\mathcal{N}_W, f), [W^4, M^3] \rangle .
\] (3.7)

Note that the Stiefel-Whitney class \( w_4 \) is related to the first Spin characteristic class \( \lambda = Q_1 = \frac{1}{2}p_1 \) via mod 2 reduction; this is used extensively in [51], [54], [55], [56]. Thus we have

\[
\text{Hopf}(M^3, f) = 0 \iff w_4(\mathcal{N}_N, f) = 0
\]

\[
\iff \lambda(\mathcal{N}_N, f) \in 2\mathbb{Z}
\]

\[
\iff \frac{1}{2}\lambda(\mathcal{N}_N, f) \in \mathbb{Z} .
\]

We see that this captures the divisibility of \( \lambda \) by 2. Therefore, we arrive at the conclusion that the quantization condition on the C-field, \([F_4] + \frac{1}{2}\lambda \in \mathbb{Z}\), holds if the Hopf invariant vanishes, \(\text{Hopf}(M^3, f) = 0\).
The $\mathbb{Z}_2$ Hopf invariant and the M5-brane. We now consider the situation for the M5-brane; the argument here will essentially be a ‘shift up of degree four’ of the argument presented for the M2-brane above. That is, we will deal with manifolds of dimensions 6, 7, and 8, replacing those of dimension 2, 3, and 4, respectively. However, unlike the case of the M2-brane which could be taken as the boundary, the M5-brane will in fact be the corner, i.e. the bottom manifold in the above hierarchy of three manifolds of consecutive dimensions. We consider the M5-brane worldvolume as the boundary of a seven-manifold $N^7$, which in turn is the boundary of an eight-manifold $W^8$, making the latter a manifold with corners of codimension-2. The Hopf invariant in this case is

$$\text{Hopf}(N^7, f) = \langle w_8(N, t), [W^8, N^7] \rangle. \quad (3.8)$$

On the other hand, the one-loop polynomial that appears in the degree eight cohomology class $\Theta$ can be written in terms of the second Spin characteristic class $Q_2$ as \[I_8 = \frac{1}{24} Q_2, \quad (3.9)\]

where we have $w_8 = Q_2 \mod 2$. This means that if $I_8$ is integral, which is the case for Spin ten-manifolds with $w_4 = 0$ (see [75]), then certainly $Q_2$ is divisible by 2, and hence $w_8 = 0$. This then gives zero Hopf invariant. In the general case, the requirement is that the second Spin characteristic class $Q_2$ is divisible by 2. This can be viewed as an analog of the requirement of the first Spin characteristic class $Q_1 = \frac{1}{2} p_1$ to be divisible by 2, in order for the flux quantization condition on the C-field to hold precisely.

Note that if $N_N$ admits a Fivebrane structure, in the sense of [63] [64], then $I_8$ is zero in cohomology, and we have (2.21), so that the dual of the C-field defines a Hopf invariant 2. As we indicated earlier, in general the dual of the C-field provides a refinement of the Hopf invariant.

3.3 The M5-brane and the Kervaire invariant

In Section 3.1 we motivated the corner structure for the M-branes and described how the corresponding topological action decomposes in various dimensions. Here we continue the discussion by seeking an explicit relation to the Kervaire invariant. To that end, we will consider the 6-dimensional worldvolume for the M5-brane, viewed as a corner/codimension-2 defect, while for the M2-brane the corner will be the 2-dimensional boundary of the corresponding worldvolume.

Consider again the reduced Thom spectrum $\overline{MO} = MO/S$. We are interested in considering classes in the wedge sum $\overline{MO} \vee \overline{MO}$; this is the quotient of the disjoint union of two copies of $\overline{MO}$ by the equivalence obtained by identification of points. In the case of the sphere, this operation gives a bouquet of spheres. \[11\] An element of the homotopy group $\pi_8(\overline{MO} \vee \overline{MO})$ is represented by a $(O, fr)^2$-manifold, i.e. a framed manifold with corners of codimension 2. This consists of the data $(W^8, N_1^7, N_2^7, N_1, N_2, f_1, f_2)$, where $W^8$ is an 8-manifold with boundary $\partial W^8 = N^7 = N_1^7 \cup_{M^6} N_2^7$, with corner $\partial N_1^7 = M^6 = \partial N_2^7$. The normal bundle $N_{W^8}$ comes with a splitting $N_{W^8} = N_1 \oplus N_2$, and here $f_1$ is a trivialization of $N_1|_{N_1^7}$ and $f_2$ is a trivialization of $N_2|_{N_2^7}$. The normal bundle of the corner $M^6$ thus acquires a trivialization $f$ as well. The map $\pi_8(\overline{MO} \vee \overline{MO}) \to \pi_7(\overline{MO})$ carries the above data to $(N_1^7, M^6, f)$.

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10See expression (2.20) and the discussion around it.
11There is an analogous discussion in the case of Eilenberg-MacLane spectrum in [18] (section 3.2), which can be recast in the above language.
The M5-brane and the Kervaire invariant. Let \((W^8, N^7_1, N^7_2, N_1, N_2, f_1, f_2)\) be an 8-dimensional \((O, fr)^2\)-manifold representing the extended M5-brane worldvolume, as in \([57] [53] [60]\). Then, by the Lannes-Miller theorem \([40] [43]\), the Kervaire invariant is given by

\[
Kervaire(M^6, f) = \sum_{i=0}^{3} \langle v_{4-i}(N_1, f_1) \cup v_i(N_2) \cup v_4(N_2, f_2), [W^8, N^7] \rangle \ ,
\]

where \(v_j\) is the \(j\)-th Wu class. We take \(N_1\) and \(N_2\) to be (at least) oriented, i.e. \(v_1(N_1, f_1) = 0 = v_1(N_2, f_2)\), so that there are two contributions \(v_4(N_1, f_1) \cup v_4(N_2, f_2)\) and \(v_4(N_1, f_1) \cup v_2^2(N_2, f_2)\) in the above sum, that is

\[
\langle w_2(N_1, f_1)w_2(N_2)(w_4(N_2, f_2) + w_2^2(N_2, f_2)), [W^8, N^7] \rangle \ .
\]

If we, furthermore, assume the Spin condition \([12]\) then \(v_4 = w_4\) and so

\[
Kervaire(M^6, f) = \langle w_4(N_1, f_1) \cup w_4(N_2, f_2), [W^8, N^7] \rangle \ .
\]

If \(w_4\) of either \(N_1\) or \(N_2\) is zero then this Kervaire invariant vanishes. One implication of this is that the action (see \([76] [30]\))

\[
\int_{W^8} F_4 \cup F_4 + \lambda \cup F_4 \mod 2
\]

becomes only quadratic without a refinement.

The M2-brane with boundary and the Kervaire invariant. Now consider the boundary of the M2-brane as a corners, as in \([57]\). That is, we have a 2-manifold \(X^2\), which is the boundary of the M2-brane worldvolume \(M^3\). This in turn is the boundary of a 4-manifold, in the above sense. The self-dual string \(\Sigma = X^2 \subset M^6\) lies inside the M5-brane worldvolume. Then, assuming the normal bundles to be oriented we have

\[
Kervaire(X^2, f) = \langle w_2(N_1, f_1) \cup w_2(N_2, f_2), [W^4, M^3] \rangle \ .
\]

We see that a Spin structure on either factor, \(N_1\) or \(N_2\), is equivalent to the vanishing of the Kervaire invariant of the self-dual string. Note, furthermore, that in the sigma model description spinors take values in the normal bundle, and hence the above statement is a natural one to have.

3.4 M-branes and the Maslov index

We have seen that the M2-brane action can be described in terms of the signature of a bounding 4-manifold. In the case of the M5-brane we have something similar: the effective action and partition function of the M5-brane can be described via the signature in eight dimensions, as described in \([30]\), which builds on \([76]\). The relevant term in the partition function is \([30] [4]\)

\[
\exp \left[ 2\pi ik_8 \int_{W^8} \bar{\lambda} \cup \lambda - L_8 \right] , \quad \text{(3.15)}
\]

12The distinction between orientation and Spin structure on the M5-brane has an interpretation via elliptic cohomology \([59]\).
where $L_8$ is the degree eight component of the Hirzebruch L-polynomial, and $\lambda$ is a differential integral lift of the degree four Wu class $v_3$. When $W^8$ is a manifold with boundary, then the contribution to the boundary is given by the signature defect, which is essentially (a differential refinement of) the eta-invariant. We study consequences to this formula of having a corner instead. Note that the $f$-invariant is the right answer for the replacement of the eta-invariant for manifolds with corners. However, before explicitly considering the $f$-invariant, we will make connection to the Maslov index. This can be viewed as the signature defect in the presence of corners, which leads to identifying the contribution to the action due to the presence of corners.

The Maslov correction to the M2-brane. Now consider the M2-brane worldvolume $M^3$, viewed as a boundary of a compact 4-manifold $W^4$. In fact, we take three copies $M_i^3$, $i = 0, 1, 2$ of $M^3$ and take $W^4$ to be decomposed along a component $M_0^3$ of $M^3$ into two parts $W_1^4$ and $W_2^4$ such that

$$\partial W_1^4 = (-M_0^3) \cup M_1^3, \quad \partial W_2^4 = (-M_0^3) \cup M_2^3, \quad \partial M_0^3 = \partial M_1^3 = \partial M_2^3 = X^2.$$  

(3.16)

Then the signature of $W^4$ is given in terms of the signatures of $W_1^4$ and $W_2^4$ and a correction term $\mu$ that arises from the way the homology classes of the spaces involved are related. That is,

$$\text{sign}(W^4) = \text{sign}(W_1^4) + \text{sign}(W_2^4) + \mu(K_0, K_1, K_2),$$  

(3.17)

where $K_i = \ker (H_1(X^2; \mathbb{R}) \rightarrow H_1(M_i^3; \mathbb{R}))$ for $i = 0, 1, 2$; so we are considering nontrivial 1-cycles in the self-dual string $X^2$ that become homologically trivial when lifted to $M^3$. The importance of this, from a physical point of view, is that the dynamics of the BPS states in Seiberg-Witten theory for which $X^2$ is the defining curve, are governed by such cycles. The Maslov index $\mu(K_0, K_1, K_2)$ is defined as follows. We will restrict $H_1(X^2)$ to the domain of 1-cycles satisfying $c_0 + c_1 + c_2$, with $c_i \in K_i$, $i = 0, 1, 2$, that is, giving a zero total cycle. This domain is the intersection $K_0 \cap (K_1 + K_2)$. Now we would like to quotient this space by 1-cycles that are made of pairs of 1-cycles that sum to the zero 1-cycle, i.e. by elements of $K_0 \cap K_1$ and of $K_1 \cap K_2$ with $c_0 + c_1 = 0$ and $c_0 + c_2 = 0$, respectively, with obvious symmetries. Then we form the vector space

$$V = \frac{K_0 \cap (K_1 + K_2)}{(K_0 \cap K_1) + (K_0 \cap K_2)}.$$  

(3.18)

with elements denoted by $[c_0]$, representing those 1-cycles that satisfy $c_0 + c_1 + c_2 = 0$. Corresponding to the skew-symmetric bilinear form $\omega$ on the first homology group $H_1(X^2)$ is a symmetric bilinear form on $V$ defined by $\rho([c_0], [c_1]) = \omega(c_0, c_1)$. The Maslov index is then the signature of this form

$$\mu(K_0, K_1, K_2) := \text{sign}(\rho).$$  

(3.19)

This is the contribution to the topological action of the M2-brane arising from the Maslov index.

The Maslov correction to the M5-brane. We now consider the M5-brane in the setting that we have had: as a corner of codimension-2 of an 8-dimensional manifold $W^8$. A similar study from an analytic point of view is taken in [57]. The signature of an oriented 8-manifold with boundary $(W^8, N^7)$ is the signature of the symmetric intersection form on $H^4(W^8, \mathbb{R})$, i.e.

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13Comprehensive literature and references on the Maslov index can be found in [46].
sign($W^8$) = sign($H^4(W^8; \mathbb{R})$) ∈ $\mathbb{Z}$. We now decompose $W^8$ into three pieces $W^8_i$, $i = 1, 2, 3$, such that $W^8 = W^8_1 \cup W^8_2 \cup W^8_3$ is the union of the three codimension-0 manifolds with boundary meeting transversally with $W^8_1 \cap W^8_2 \cap N^7 = W^8_2 \cap W^8_3 \cap N^7 = W^8_3 \cap W^8_1 \cap N^7 = \emptyset$. The corner $M^6 := W^8_1 \cap W^8_2 \cap W^8_3$ is a codimension-2 submanifold and hence is identified as the M5-brane worldvolume.

The bisecting segments correspond to the double intersections of the corresponding $W^8_i$’s. The nonsingular intersection form on $H^3(M^6; \mathbb{R})$ comes equipped with three Lagrangian subspaces

\[
\begin{align*}
L_1 &= \text{Image } (H^3(W^8_2 \cap W^8_3; \mathbb{R}) \to H^3(M^6; \mathbb{R})) , \\
L_2 &= \text{Image } (H^3(W^8_1 \cap W^8_3; \mathbb{R}) \to H^3(M^6; \mathbb{R})) , \\
L_3 &= \text{Image } (H^3(W^8_1 \cap W^8_2; \mathbb{R}) \to H^3(M^6; \mathbb{R})) .
\end{align*}
\]

The signature defect is given by Wall’s non-additivity invariant $\mu(L_1, L_2, L_3) \in \mathbb{Z}$, which coincides with the Maslov index,

\[
\text{sign}(W^8) = \text{sign}(W^8_1) + \text{sign}(W^8_2) + \text{sign}(W^8_3) + \mu(L_1, L_2, L_3) .
\]

The interpretation is that we are looking at those classes of gerbes on the worldvolume which arise from corresponding classes of gerbes in seven dimensions. Following the discussion similar to the case of the M2-brane above, but now for degree three cohomology rather than for degree one homology, we have the following interpretation. We are looking at those gerbes whose H-fields have the symmetry $h_1 + h_2 + h_3 = 0$, $h_i \in L_i$, on the M5-brane worldvolume, modulo those which have the cancellation in pairs property, i.e. $h_1 + h_2 = 0 = h_1 + h_3$. This then gives a vector space as in (3.18), with the $K$’s replaced by the $L$’s and the definition of the Maslov index is similar to (3.19). The contribution to the exponentiated action is then of the form $e^{2\pi i \frac{\mu}{2}}$.

### 3.5 The M-branes and the $f$-invariant

We have indicated that the correct replacement for manifolds with corners of the eta-invariant (or $e$-invariant) for manifolds with boundary is the $f$-invariant of [37]. In the context of M-theory this is realized in [60]. Other variants of this are also used in [52].

We have so far been dealing with signature operators. We would like to consider the $f$-invariant in terms of Dirac operators instead; physically this is justified by having spinors on the worldvolume, and mathematically this is possible because of the direct relation between the indices of the two types of operators. In fact, as explained in [26] and utilized in this context in [57], a Dirac operator can be viewed as a (generalized) signature operator. From another angle and explicitly, for the case
of the M2-brane, we have in degree four the L-genus \( L_4 = \frac{1}{2} p_1 \) and the Dirac genus \( \hat{A}_4 = -\frac{1}{22} p_1 \), which can be directly related as \( L_4 = -8 \hat{A}_4 \). For the M5-brane we have degree eight components \( L_8 = \frac{1}{2 \pi^2} (7p_2 - p_2^2) \) and \( \hat{A}_8 = \frac{1}{2 \cdot 3 \pi^2} (7p_2^2 - 4p_2) \) related as \( \frac{1}{8} (\lambda^2 - L_8) = 28 \hat{A}_8 \). Therefore, expression (3.15) can be written using solely the Dirac index as

\[
\exp \left[ 2\pi ik \cdot 28 \int_{W_8} \hat{A}_8 \right].
\]

Note that an analogous analysis in degree twelve is useful in the case of M-theory [58].

The real and complex \( e \)-invariant. As the expression of the \( f \)-invariant will involve the \( e \)-invariant, we start by describing the latter in our setting, following [13] [7] [8]. Consider the canonical Spin\(^c\) Dirac operator \( D \) on our \((U,fr)\)-manifold \( W^4 \) with Levi-Civita connections \( \nabla^{TW,LC} \) on its tangent bundle \( TW^4 \), and with boundary the M2-brane worldvolume \( M^3 \). Consider also a bundle \( E \) over \( W^4 \) with a connection \( \nabla^E \) and a restriction to \( M^3 \). Then the reduced eta-invariant \( \xi(D_M) := \frac{1}{2} (\eta + \dim \ker D_M) \) for \( D_M = D|_M \) is given by \( \xi(D_M) \equiv \int_W Td(\nabla^{TW,LC}) \mod Z \), where \( Td \) is the Todd genus. The complex \( e \)-invariant is \( e_C(M^3) \equiv \xi(D_M) + \int_W \{ Td(\nabla^E) - Td(\nabla^{TW,LC}) \} \mod Z \). By Stokes’ theorem, this can be reduced to an integral over \( M^3 \). With the identification of the C-field with the Chern-Simons form this becomes \( e_C(M^3) \equiv \xi(D_M) + \int_{M^3} C_3 \mod Z \). On the other hand, the real \( e \)-invariant is \( e_R(M^3) \equiv \frac{1}{2} (\hat{A}(TW^4), [W^4, M^3]) \equiv \frac{1}{2} \int_{M^3} C_3 + \xi(D_M) \mod Z \). The real and complex \( e \)-invariants are related as follows. The Spin cobordism \( MS\text{Spin}_3 = 0 \) and the almost complex cobordism \( MSU_3 = 0 \) are both trivial, and since the first Chern class of an SU-manifold is trivial, the Todd genus coincides with the Dirac genus, so that \( \frac{1}{2} e_R \equiv e_C \mod Z \). For example, for the 3-sphere, \( e_R(S^3) = -\frac{1}{12} \) and \( e_C(S^3) = -\frac{1}{24} \) or \( \frac{11}{24} \).

The M2-brane with a corner and the \( f \)-invariant. We will study a representative example. Consider the open M2-brane worldvolume as the manifold with boundary \( S^1 \times D^2 \). This can be lifted to four dimensions by considering it as the boundary of \( W^4 = D^2 \times D^2 \), the product of two closed 2-disks. On the other hand, one can take the boundary of the open M2-brane leading to a corner \( S^1 \times S^1 \). Now the generator \( \eta \in \pi^3_1 \cong \mathbb{Z}_2 \) can be represented by the circle with its non-bounding framing. Using a Fourier decomposition, the Spin\(^c\) Dirac operator has symmetric spectrum and a single zero mode, so \( e_C(\eta) = \frac{1}{2} \). Then, using [7] [8], the \( f \)-invariant of the corner \( S^1 \times S^1 \) is

\[
f(\eta^2) \equiv \frac{1}{2} \frac{E_1 - 1}{2} = \frac{1}{2} \sum_n \sum_{d|n} \left( \frac{d}{q} \right) q^n = \frac{1}{2} q + \frac{1}{2} q^3 + \mathcal{O}(q^4),
\]

where \( E_1 = 1 + 6 \sum_n \sum_{d|n} \left( \frac{d}{q} \right) q^n \) is the modular form of weight one for the congruence subgroup \( \Gamma_1(3) \). Note that if \( q = 0 \) then \( f = 0 \), so this expression is, in some sense, nonclassical. This is compatible with the fact that the \( f \)-invariant is associated with chromatic level two in homotopy theory. Expression (3.22) is the \( q \)-expansion contribution to the effective action of the M2-brane due to the corner.

The self-dual string as a string theory via the M2-brane. We have seen above that the corner structure is not merely a mathematical tool. We will further describe a setting where it is in fact a very useful part of the structure of the physical theory. One natural question is to what

\[14\text{Note that an explicit proposal for viewing the C-field as an index gerbe is given in [49].}\]
extent the self-dual string forms an actual string theory, in the sense of perturbative expansion and cobordism. The usual way that this string is constructed does not immediately allow for such a description. We propose that the concept of corners naturally gives such a desirable type of string theory. The M2-brane has a boundary $\partial M2 = \Sigma_2$ that lies on the M5-brane. In order for this $\Sigma_2$ to qualify for a string theory in the sense of cobordism, we must have a nontrivial boundary $\partial \Sigma_2 \neq \emptyset$. Therefore, we take the M2-brane itself to be a manifold with corners of codimension-2. This point of view, although straightforward and obvious in light of the above discussion, does in fact solve a problem. This allows for the construction of (nontrivial) elliptic objects needed for the elliptic cohomology description of the M2-branes [51]. We will provide the corresponding details and further consequences elsewhere.

**The M5-brane as a corner and the $f$-invariant.** We now consider the M5-brane and concentrate on the decomposable case, $M^6 = M^3_1 \times M^3_2$, where $M^3_1$ and $M^3_2$ are 3-dimensional framed manifolds. Let $m(M^3)$ be any modular form of weight 2 with respect to the fixed congruence subgroup $\Gamma = \Gamma_1(N)$ with $m(M^3) = m(M^3_1) - e_C(M^3_1) \in \mathbb{Z}[[q]]$. Then the (geometric) $f$-invariant of the product is [8]

$$f(M^3_1 \times M^3_2) = m(M^3_1) + e_C(M^3_1) \equiv e_C(M^3_2) e_C(M^3_1). \tag{3.23}$$

In particular, the geometric $f$-invariant of a product is antisymmetric under exchange of the factors, in contrast to the case of the Hopf invariant. We will concentrate on a representative situation, which is the product of two 3-spheres, $M^6 = S^3 \times S^3$; this is the generator of the framed cobordism group in six dimensions. The study of the elliptic genus of the products of two closed disks $Ell(D^4 \times D^4)$ in M-theory and the relation elliptic cohomology is taken in [52]. Now consider $S^3$ as the sphere bundle $S(L)$ of the Hopf line bundle over $S^2$. The framing of the base and the fiber leads to a framing of the total space. The complex Adams $e$-invariant is $e_C(S(L)) = -\frac{1}{12}$. Since $\text{MSpin}_3 = 0$, the framed 3-manifold $M^3$ is the boundary of a Spin 4-manifold $W^4$. This disk bundle $D(S(L))$ represents the element $\nu$ in the homotopy groups of spheres. Then, using [7] [8], the $f$-invariant of the product is given by

$$f(\nu^2) \equiv \frac{1}{2} \left( \frac{E_2 - 1}{12} \right)^2 = \frac{1}{2} q^2 + 3q^3 + \frac{11}{2} q^4 + O(q^5). \tag{3.24}$$

This essentially corresponds to two copies of the M2-brane with no boundary theory, and can be viewed as an $f$-invariant description of the cup product Chern-Simons theory described in [22] [23]. Note that $f = 0$ when $q = 0$, so that this expression is purely modular or, in some sense, quantum. Expression (3.24) is the contribution to the effective action of the M5-brane due to the corner in the representative case.

**3.6 M5-brane, $G_2$ holonomy and the $\nu$-invariant**

We consider the M5-brane worldvolume $M^6$ as the base of a 7-dimensional cone, which admits a $G_2$-holonomy structure. Topologically, a cone is a product $(M^6 \times I)/(M^6 \times \{0\})$ with a metric making the size of the interval go to zero on one end. That is, we have a cylinder with one end collapsed to a point. We then take this 7-manifold to be the boundary of an 8-dimensional manifold, thus making $M^6$ a codimension-2 corner and at the same time admitting extra geometric structures. A standard example is our favorite, namely $M^6 = S^3 \times S^3$; see e.g. [16] for an application in M-theory on $G_2$-manifolds.
Another setting in which special holonomy arises is when we take the 7-dimensional $G_2$-holonomy manifold to be the total space of a circle bundle with base the M5-brane worldvolume $M^6$, with $SU(3)$-structure. The relation between the 3-form $\omega_3$ defining the $G_2$-structure in seven dimensions and the 2-form in six dimensions is explained generally in [29]. We will identify the C-field with that 3-form, i.e. $C_3 = \omega_3$.

We now make connection to the $\nu$-invariant [15] of [15]. We consider $W^8$ to be an eight-dimensional manifold with Spin(7) holonomy. For any Spin(8) bundle on $W^8$ the Euler class for the positive and negative chirality is given by [24]

$$e_\pm(W^8) = \frac{1}{16}(p_1^2 - 4p_2^2 \pm 8e)$$

from which, using the Spin(7) structure, one has the following relation between degree eight genera $48\hat{A}(W^8) + \chi(W^8) - 3\text{sign}(W^8) = 0$. When there is a boundary $N^7 = \partial W^8$, the Spin(7) structure induces a boundary $G_2$-structure. In this case, one can consider the $\hat{A}$-defect $\nu(\omega_3) := \chi(W^8) - 3\text{sign}(W^8) \mod 48 \in \mathbb{Z}_{48}$, which depends only on the $G_2$-structure on $N^7$. For example, we can take $W^8 = \mathbb{D}^8$ with its flat Spin(7)-structure, having a boundary $S^7$ with a standard $G_2$-structure $\omega_3$. Then the invariant is given by $\nu(\omega_3) \cong \chi(\mathbb{D}^8) - 3\text{sign}(\mathbb{D}^8) \cong 1 \mod 48$. We propose to view this canonical $G_2$-structure, as minimal (nontrivial) among the set of $G_2$ structures; this set is isomorphic to $H^7(N^7; \pi_7(S^7)) \cong \mathbb{Z}$. That is, the canonical $G_2$-structure minimizes the classical topological action for the M5-brane. This is analogous to the canonical (2-)framings, discussed earlier, as well as canonical String structures, discussed in [51] [54].

We now see that the $\nu$-invariant can be deduced from the one-loop polynomial $I_8$, and hence from (part of) the dynamics of the M-branes. Observe that the combination of the first two terms in (3.25) is a multiple of the one-loop polynomial. This is used in [50] to write the one-loop polynomial in terms of the second Spin characteristic class $Q_2$, as indicated in (3.29). This can also be written as a linear combination of the $L$-genus and the $\hat{A}$-genus as $I_8 = -\frac{1}{8}L - 2\hat{A}$. In terms of $I_8 = \int_{W^8} I_8$, expression (3.25) for closed $W^8$ takes the form

$$24I_8 - \chi(W^8) = 0 .$$

This then allows us to have the following definition in the case when $w_4(W^8) = 0$, i.e. when we have a Membrane structure, in the sense of [55].

**Definition 4** The “$I_8$-defect” in the case when $W^8$ has a boundary with a $G_2$-holonomy structure is

$$i := \chi(W^8) \mod 24 \in \mathbb{Z}_{24} .$$

We view this also as an 8-dimensional analog of the canonical String structure in three dimensions. Note that the presence of a 7-dimensional boundary, allows for a Chern-Simons interpretation of the one-loop term [63] [51] leading to a description via higher bundles [20] [21] [22]. We highlight that the $I_8$-defect (3.27) depends both on

1. the $G_2$ structure on the boundary $N^7$, and
2. a choice of one half the first Spin characteristic class $\frac{1}{2}Q_1(N^7)$.

More precisely, this means that we provide an M-brane interpretation of a specific linear combination of the $\nu$-invariant of the $G_2$-structure and the Gauss sum of quadratic refinement associated to

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15 Of course this is not the generator $\nu$ encountered previously in Section 3.5.
a choice of \( \frac{1}{2}Q_1(N^7) \). A somewhat analogous situation arises in the phase of the partition function in [61] (without the \( \nu \)-invariant interpretation).

An explanatory remark is in order. A closed 8-manifold obtained by gluing two 8-manifolds with \( w_4 = 0 \) along the boundary need not necessarily have \( w_4 = 0 \). However, any 7-dimensional Spin manifold will have a Spin coboundary with \( w_4 = 0 \). The analogy in the simpler case of dimension four is that gluing two Spin 4-manifolds along their boundary gives a Spin 4-manifold only if the Spin structure on the boundary is preserved. If \( W_1^8 \) and \( W_2^8 \) are two different Spin coboundaries of \( M^7 \) with \( w_4(W_1^8) = 0 \), then being able to choose \( p_1(W_i^8)/4 \) with equal image in \( H^4(M^7) \) is enough to show that the gluing \( X^8 := W_1^8 \cup_M W_2^8 \) has signature divisible by 8. That this is indeed the case can be shown using Milgram’s theorem and appropriate quadratic refinements.

We have indicated how the Hopf invariant can be differentially refined. It is obvious that the other invariants, including the Kervaire, \( f \)-invariant, and the \( \nu \)-invariant can be refined. The Kervaire invariant involves the differential refinement of the Wu classes, the \( f \)-invariant involves essentially refined Chern-Simons theory, and the \( \nu \)-invariant involves refining the 3-form to a gerbe with connection, again obvious from Chern-Simons theory, along the lines of the above references.

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