The evolution from BCS to BEC superconductivity is an old one but recently it has received considerable attention in connection with high temperature superconductors, where strong deviations from the BCS behavior have been observed experimentally in spectroscopic quantities at low temperatures. Furthermore, the recent discovery of BEC in atomic Fermi systems and the exciting possibility of BEC in atomic systems raises the question of the evolution from BCS to BEC in atomic Fermi systems as well. In this work we address the question of whether the evolution of spectroscopic properties in the s-wave and d-wave channels is analyzed from BCS to BEC in atomic Fermi systems as well. [12]

In this work we address the question of whether the evolution of spectroscopic properties from a BCS to a BEC superconductor is smooth at zero temperature. For this purpose we study the single quasiparticle properties (excitation spectrum, momentum distribution, spectral function, and density of states) as a function of attraction strength or particle density for the s-wave and d-wave cases. In anticipation of the main results, we must say that the evolution of spectroscopic properties in the s-wave case is smooth, while in the d-wave it is not. The main reasoning for this statement is as follows.

Quite generally the evolution from BCS to BEC superconductivity can be characterized by two parameters: the chemical potential \( \mu \) and the Cooper pair size \( \xi_{\text{pair}} \). The BCS limit is characterized by a positive chemical potential \( \mu = \epsilon_F \) and a large size of Cooper pairs \( (\xi_{\text{pair}} \gg k_F^{-1}) \), while the BEC regime is characterized by a large and negative chemical potential \( \mu = -E_b^{(\ell)} \), where \( E_b^{(\ell)} \) is the binding energy of the two-body problem in the \( \ell \)th angular momentum channel, and by a small size of pairs \( (\xi_{\text{pair}} \ll k_F^{-1}) \). Here \( \ell = 0 \) (or s) indicates the s-wave channel, while \( \ell = 2 \) (or d) indicates the d-wave channel. The excitation spectrum at zero temperature has the form \( E_{\ell}(k) = ((\epsilon_k - \mu)^2 + \Delta_{\ell}(k)^2)^{1/2} \), where \( \epsilon_k = k^2/2m \) and \( \Delta_{\ell}(k) = \Delta_{\ell} h_\ell(k) \cos(\ell \phi) \), with \( k = |k| \). In the s-wave case the excitation spectrum \( E_s(k) \) is gapped for all \( k \), and it increases smoothly from the BCS to the BEC limit. As a result the quantities that depend directly on the excitation spectrum \( E_s(k) \) also evolve smoothly. For instance, the quasiparticle density of states \( N_s(\omega) \) at low frequencies is always zero, since there are no available states inside the gap. Thus, contributions from single quasiparticle excitations to thermodynamic quantities are always exponentially small at low temperatures. In the d-wave case the situation is qualitatively different. For \( \mu > 0 \) the superconductor is gapless at the Dirac points \( k = k_\mu = \sqrt{2m\mu} \), \( \phi = \pm \pi/4, \pm 3\pi/4 \), while for \( \mu < 0 \) the superconductor acquires a finite gap. The line \( \mu = 0 \) separates two regimes with qualitatively different behavior. This has important consequences for the momentum distribution, spectral function, and density of states. The quasiparticle density of states \( N_d(\omega) \) changes discontinuously at low frequencies from linear in \( \omega \) for \( \mu > 0 \) (where \( E_d(k) \) is linear in momentum close to the Dirac points), to a constant at \( \mu = 0 \) (where \( E_d(k) \) is quadratic for small momenta), to zero for \( \mu < 0 \) (where \( E_d(k) \approx |\mu| + O(k^2) \) for small \( k \)). Thus, contributions from single quasiparticle excitations to thermodynamic quantities at low temperatures also exhibit singular behavior in the vicinity of \( \mu = 0 \).

In order to analyze how the spectroscopic quantities change from the BCS to BEC limit, we start with the two dimensional Hamiltonian

\[
H = \sum_{k\tau} \epsilon_k \psi_{k\tau}^\dagger \psi_{k\tau} + \sum_{kk'q} V_{kk'} b_{kq}^\dagger b_{k'q}
\]

where \( b_{kq} = \psi_{-k+q/2j}^\dagger \psi_{k+q/2j} \). The interaction potential \( V_{kk'} \) is expanded in its angular momentum components as \( V_{kk'} = \sum_{\ell=-\infty}^{+\infty} V_{kk'}^{(\ell)} \exp(i\ell \phi_{kk'}) \), where \( \phi_{kk'} = \cos(\hat{k} \cdot \hat{k}') \) is the angle between the vectors \( k \) and \( k' \) and \( V_{kk'}^{(\ell)} = 2\pi \int_{-\infty}^{\infty} drr J_{\ell}(kr) J_{\ell}(k'r) V(r) \). The index \( \ell \) labels angular momentum states in two spatial dimensions, with \( \ell = 0, \pm 1, \pm 2, \ldots \), corresponding to s,p,d,..., channels respectively. A possible choice of the real space potential can be \( V(r) = V_1 \Theta(R_1 - r) - V_0 \Theta(r - R_1)\Theta(R_0 - r) \), which is repulsive at short distances \( r < R_1 \), attractive at intermediate distances \( R_1 < r < R_0 \), and vanishes for \( r > R_0 \). Quite generally any short ranged real space potential \( V(r) \), with range \( R_0 \), leads to a \( V_{kk'}^{(\ell)} \) which is separable for small momenta, provided that \( kR_0 \ll 1 \) or \( k'R_0 \ll 1 \). In the simpler limit when both \( kR_0 \ll 1 \) and

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distribution, provided that the system is near the phase transition, and that the wave function is the order parameter. For a given interaction range $\ell$, the single particle excitation energy, and $\Delta_0$ to $\Delta_\ell$ for fixed density $n = 1$, and changing $\lambda_\ell$. Similar plots can also be made for fixed interaction and varying density $n$.

Under these circumstances, quite generally it is not possible to find a separable potential in momentum space $V_{kk'} = -\lambda w^*(k)w(k')$, nevertheless in the spirit of ref. 4, we choose to study a separable potential that contains most of the general features described above. In addition, we consider only single particle excitations, where the $s$-wave and the $d$-wave channels are studied separately. For this purpose, we use the separable potential $V_{kk'} = -\lambda w^*(k)w(k')$. The interaction term $w^*(k)$ can be written as the product of two functions, $w^*(k) = h_\ell(k)g_\ell(k)$, where $h_\ell(k) = (k/k_0)^\ell/[1 + (k/k_0)]^\ell$ and $g_\ell(k) = \cos(\ell\phi)$ is the angular dependence of the interaction. Here $k_0 \sim R_0^{-1}$ and $k_1$ sets the scale at low momenta. At zero temperature, we assume that pairing occurs with the same total momentum $q = 0$ only. This simplifying feature leads to the following saddle point and number equations,

$$\frac{1}{\lambda_\ell} = \sum_k \frac{|w_\ell(k)|^2}{2E_\ell(k)},$$

$$n = 2 \sum_k n_\ell(k),$$

where $n_\ell(k) = [1 - (\epsilon_k - \mu)/E_\ell(k)]/2$ is the momentum distribution, $E_\ell(k) = ((\epsilon_k - \mu)^2 + |\Delta_\ell(k)|^2)^{1/2}$ is the single particle excitation energy, and $\Delta_\ell(k) = \Delta_0 w_\ell(k)$. Is the order parameter. For a given interaction range $R_0 \sim k_0^{-1}$, the transition from the BCS limit (largely overlapping pairs) to the BEC limit of (weakly overlapping pairs) may occur either by changing the attraction strength $\lambda_\ell$ or the density $n$. In either case, this evolution can be safely analyzed with the approximations used here provided that the system is dilute enough, i.e., $n \ll k_0^3$. This means that below a maximum density $n_{\text{max}} \sim k_{F_{\text{max}}}^2$, the interaction range $R_0$ is much smaller than the interparticle spacing $k_{F_{\text{max}}}^{-1}$, or equivalently $k_0/k_{F_{\text{max}}} \gg 1$. Thus we choose to scale all energies with respect to the maximal Fermi energy $\epsilon_{F_{\text{max}}}$, which fixes the maximum density $n = n_{\text{max}} = 2\rho e_{F_{\text{max}}}$, and all momenta with respect to $k_{F_{\text{max}}} = \sqrt{2\rho e_{F_{\text{max}}}}$. The coupling constant is scaled with respect to the two-dimensional density of states $\rho$. From now on we use this scaling. The numerical solutions for $\Delta_0$ and $\mu$, when $k_1 = k_0 = 10$ are shown in Fig. 1 for fixed density $n = 1$, and changing $\lambda_\ell$. Similar plots can also be made for fixed interaction and varying density $n$.

In the BCS limit the amplitude of the order parameter ($\phi = 0$) is given by

$$\Delta_\ell(k_\mu) \sim \exp \left[2(\lambda_0^{-1}(k_\mu) - \lambda_\ell^{-1})/h_\ell^2(k_\mu) \right].$$

With our choice of $h_\ell(k)$, $\lambda_0(k_\mu) \simeq 8 + \mu/2\epsilon_1^2 + O(|\mu/\epsilon_1|^3)$, valid for $\mu/\epsilon_1 \ll 1$, where $\epsilon_1 = k_1^2$. The ratios between $\Delta_\ell(k_\mu)$, and the critical temperature $T_{\text{cs}}$ satisfy the usual relations $\Delta_\ell(k_\mu)/T_{\text{cs}} = 1.76$, and $\Delta_\ell(k_\mu)/T_{\text{cd}} = 2.14$. In Fig. 1 $\Delta_0$ and $\mu$ have a second order discontinuity as a function of $\lambda_\ell$. This discontinuity occurs when $\mu = 0$ in both $\Delta_0$ and $\mu$, for varying interaction $\lambda_\ell$ or varying density $n$. The line $\mu = 0$ for a $d$-wave system is very special as it will be seen in the following discussion of spectroscopic quantities.

The first spectroscopic quantity to be analyzed is the single quasiparticle excitation spectrum $E_\ell(k)$. Let us discuss first the $s$-wave case in the zero range interaction limit $k_0 \to \infty$. For $\mu > 0$ the excitation spectrum has an isotropic gap at $k = k_\mu$, $E_\ell(k_\mu) = |\Delta_\ell(k_\mu)|$. This gap is completely isotropic in the vicinity of $k_\mu$. At the intermediate regime, when $\mu = 0$, the gap takes the value $E_\ell(0) = |\Delta_\ell(0)|$, when the chemical potential $\mu$ becomes
negative towards the BEC limit, the minimum of the energy gap remains at \( k = 0 \), \( E_g(0) = (\mu^2 + |\Delta_\mu(0)|^2)^{1/2} \). When \( k_0 \) is finite, the position of the minimum gap changes, but the excitation spectrum is always gapped.

In the d-wave case the situation is qualitatively different. For \( \mu > 0 \), including the BCS limit, the excitation spectrum is gapless at \( k_\mu \) along the special directions \( \phi = \pm \pi/4, \pm 3\pi/4 \), near which the excitation spectrum disperses linearly with momentum. The energy gap at \( k = k_\mu \) and \( \phi = 0 \), \( E_g(k_\mu) = |\Delta_\mu(k_\mu)| \) is a nonmonotonic function of \( k_\mu \) for fixed density, and thus a nonmonotonic function of \( \lambda_d \). The maximum \( E_g(k_\mu) \) is reached at intermediate values of \( \mu > 0 \). At \( \mu = 0 \), the minimal gap is \( E_g(0) = |\Delta_0(0)| = 0 \), and occurs at the single point \( k = 0 \). In this case the excitation spectrum is \( E_d(k) = (\epsilon_k^2 + |\Delta_d(k)|^2)^{1/2} \), which behaves quadratically for small momenta at any given angle \( \phi \), since \( \Delta_d(k) \sim k^2 \cos(2\phi) \) and \( \epsilon_k = k^2/2m \). The shrinking of the energy gap to zero at \( k = 0 \) is a consequence of the diminishing pairing interaction \( h_d(k_\mu) \) for \( \mu \to 0 \). As soon as \( \mu < 0 \), including the BEC limit, a full gap in the excitation spectrum appears, but the minimal gap remains at \( k = 0 \) with value \( E_g(0) = |\mu| \) since \( \Delta_d(0) = 0 \). Thus, the \( \mu = 0 \) line separates a gapless d-wave superconductor (\( \mu > 0 \)) from a fully gapped d-wave superconductor (\( \mu < 0 \)). Fig. 2 shows the lines where \( \mu = 0 \) on the graph of \( n \) vs. \( \lambda_d \). Notice in Fig. 2 that the low density limit of the s-wave system is always Bose-like, i.e., a two-body bound state appears at arbitrarily small \( \lambda_s \). On the other hand, the d-wave system is qualitatively different: it is BCS-like for \( \lambda_d < \lambda_{sd} \) and Bose-like for \( \lambda_d > \lambda_{sd} \), where the critical coupling \( \lambda_d \) separating the two regimes is finite, i.e., the appearance of a two-body bound state in the d-wave case requires finite \( \lambda_d \).

In the vicinity of \( k_\mu \) the momentum distribution is \( n_d(k_\mu, \delta k) \approx \left[ 1 - k_\mu^2/\kappa \right] \delta k/\Delta_\mu(k_\mu) \). At low \( k \) it behaves as \( n_d(k) \approx [1 + \gamma_p(1 + \alpha k^2/2k_0^2)]/2 \), where \( \gamma_p = \mu/\sqrt{\mu^2 + \Delta_\mu^2} \) and \( \alpha = \Delta^2/(\mu^2 + \Delta_\mu^2) \). When \( \mu = 0 \), the momentum distribution at small momenta is \( n_d(k) \approx (1 - k^2/\Delta_\mu)/(1 + \alpha k^2/2k_0^2) \). For negative \( \mu \), \( n_d(k) = [1 - \gamma_n(1 + \alpha k^2/2k_0^2)]/2 \) for small \( k \), with \( \gamma_n = |\mu|/\sqrt{\mu^2 + \Delta_\mu^2} \). Notice that \( n_d(0) \) is a continuous function of \( \mu \). In fact \( n_d(k) \) is a smooth function of \( \mu \) for all momenta. This is not the case for a d-wave system, which shall be discussed next.

**FIG. 3.** The momentum distribution of quasiparticles for \( \phi = 0 \), \( n = 1 \), \( k_1 = k_0 = 10 \), and several values of \( \mu \) for a d-wave order parameter. The inset shows results for \( \mu \leq 0 \).

The momentum distribution in the d-wave case is anisotropic, having the form \( n_d(k) = [1 - \text{sgn}(k^2 - \mu)] \) along the direction of the nodes \( (\phi = \pm \pi/4, \pm 3\pi/4) \). This behavior already signals discontinuity of \( n_d(k) \) as a function of \( \mu \) at \( k = 0 \), a suspicion further confirmed by analyzing the more interesting direction \( \phi = 0 \) and its equivalents \( \phi = \pm \pi/2, \pi \). Near \( k_\mu \) the momentum distribution is \( n_d(k_\mu, \delta k) \approx (1 - k_\mu^2/\kappa \delta k/\Delta_d(k_\mu))/\Delta_d(k_\mu) \). On the other hand, the momentum distribution behaves as \( n_d(k) \approx 1 - (\Delta_d^2/\mu^2)(k^4/k_\mu^2) \) for small \( k \). When \( \mu = 0 \) the momentum distribution at \( k = 0 \) is \( n_d(0) \approx 1 - \kappa/2 \), where \( \kappa = (1 + \Delta_\mu^2/4k_0^2)^{-1/2} \). When \( \mu < 0 \), then \( n_d(k) \approx (\Delta_\mu^2/\mu^2)(k^4/k_\mu^2) \) for small \( k \). Notice the discontinuity of the momentum distribution at low \( k \), when chemical potential crosses zero. This discontinuity, which is illustrated in Fig. 3, coincides with the collapse of the four Dirac points to a single point at \( k_\mu = 0 \), and with the appearance of a full gap as soon as \( \mu < 0 \).

The qualitative changes in \( E_d(k) \) and \( n_d(k) \), as a function of \( \mu \), affect substantially the quasiparticle density of states \( N_d(\omega) = N_{d+}(\omega) + N_{d-}(\omega) \), where

\[
N_{d+}(\omega) = [2\pi]^{-1} \int d^2 \mathbf{k} \left[ 1 - n_d(\mathbf{k}) \right] \delta(\omega - E_d(\mathbf{k})),
\]

corresponds to adding a quasiparticle, and

**FIG. 2.** The line \( \mu = 0 \) for both s- and d-wave order parameters for \( n = 1 \) and \( k_1 = k_0 = 10 \).
\[ N_d(\omega) = [2\pi]^{-1} \int d^2 k n_d(k) \delta(\omega + E_d(k)), \]

(5)
corresponds to removing a quasiparticle. In the s-wave case \( N_d(\omega) \) is always zero at low frequencies, since the excitation spectrum is gapped for all \( \mu \). On the other hand, \( N_d(\omega) \) changes discontinuously at low frequencies, from linear in \( \omega \) for \( \mu > 0 \), where \( E_d(k) \) is linear in momentum close to the nodes, to a constant at \( \mu = 0 \) (where \( E_d(k) \propto k^2 \) at low k), to zero for \( \mu < 0 \) (where \( E_d(k) \propto |\mu| + O(k^2) \) for small k), as can be seen in Fig. [4].

Lastly, it is important to point out qualitative differences in thermodynamic quantities, e.g. specific heat \( C \) and spin susceptibility \( \chi \), during the evolution from BCS to BEC superconductivity at low temperatures. The contributions from quasiparticles to \( C \) and \( \chi \) are exponentially small at low temperatures in the s-wave case for all \( \mu \), given that the excitation spectrum is always gapped. The situation is qualitatively different in the d-wave case where \( C \propto T^2 \), and \( \chi \propto T \) for \( \mu > 0 \); \( C \propto T \), and \( \chi \propto \text{const.} \) for \( \mu = 0 \); \( C \propto T^{-1} \exp(-|\mu|/T) \), and \( \chi \propto \exp(-|\mu|/T) \) for \( \mu < 0 \).

In summary we studied the low temperature evolution from BCS to BEC superconductivity for varying density and interaction strength in both s-wave and d-wave channels. In the s-wave case the excitation spectrum is always gapped, and the momentum distribution is a continuous function of \( \mu \). However, in the d-wave case the excitation spectrum is gapless for \( \mu > 0 \) and acquires a full gap for \( \mu < 0 \). Furthermore, the momentum distribution is discontinuous at low \( k \), as \( \mu \) crosses zero. As a result, the changes in spectroscopic and thermodynamic properties near \( \mu = 0 \) are dramatic at low temperatures. The line \( \mu = 0 \) in the \( n \) vs. \( \lambda_d \) plane seems to correspond to a quantum critical line.

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