The six-vertex model on random lattices

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In this letter, the 6-vertex model on dynamical random lattices is defined via a matrix model and rewritten (following I. Kostov) as a deformation of the $O(2)$ model. In the large $N$ planar limit, an exact solution is found at criticality. The critical exponents of the model are determined; they vary continuously along the critical line. The vicinity of the latter is explored, which confirms that we have a line of $c = 1$ conformal field theories coupled to gravity.
1. Introduction

Starting from the observation that the combinatorial properties of large $N$ matrix models \cite{1} allow them to reproduce summations over discretized surfaces \cite{2,3} it is possible to study various 2D statistical models on random dynamical lattices, that is consider systems in which both the “spin” degrees of freedom sitting on the lattice and the lattice itself are allowed to fluctuate. Among the models that were solved this way, let us cite the Ising model \cite{4}, the $O(n)$ model \cite{5,6}, the Potts model \cite{7,8,9}. All these models have critical points which correspond to conformal field theories of central charge $c$ less or equal to 1 coupled to gravity, the limiting case $c = 1$ being of particular interest. One well-known statistical model has until now resisted attempts at an exact solution (on random lattices): the 8-vertex model and its critical version, the 6-vertex model. On a flat lattice, the 6-vertex model displays an infra-red behavior which spans a whole semi-infinite line of $c = 1$ theories; a similar behavior is expected when put on random lattices \cite{10}. In fact, a special 2-parameter slice of the 8-vertex model was solved exactly in \cite{11}; the 6-vertex point of the model was shown to indeed exhibit a $c = 1$ behavior.

In this letter, we shall perform a study of a matrix model which describes the 6-vertex model. We shall be concerned with its large $N$ limit, which correspond to selecting the spherical topology for the lattices. We shall solve the model exactly when the renormalized cosmological constant vanishes, that is in the limit where the average size of the graphs goes to infinity, and give explicit expressions for some averages. We shall then show how to explore the vicinity of this critical region and compute the first correction, which yields the string susceptibility.

2. The model and its saddle point equation

The model is defined by the following partition function:

$$Z = \int dX dX^\dagger \exp \left[ N \text{tr} \left( -XX^\dagger + bX^2X^\dagger^2 + \frac{c}{2}(XX^\dagger)^2 \right) \right]$$  \hfill (2.1)

where $X$ is a general $N \times N$ complex matrix. The Feynman rules of the model reproduce the configurations of the \textit{six-vertex model} (figure 1) on a random four-valent lattice.
In the parametrization of e.g. [12], \(c/b = 2 \cos(\lambda \pi/2)\); the parameter \(\lambda\) can, on random surfaces, vary in the range \(0 \leq \lambda < 2\). For \(\lambda = 0\) we recover the usual \(O(2)\) model [3]; for \(\lambda = 1\) we recover the critical \(ABAB\) model [11]. Finally, the singular limit \(\lambda \to 2\) corresponds to the three-colouring problem [13,14]. When \(\lambda\) is fixed, we can still vary \(b\), which plays the role of (bare) cosmological constant. For any \(\lambda\) we expect that there is a value \(b_{\text{crit}}(\lambda)\) for which the average size of the graphs diverges.

In order to solve the model, we use the following trick: we decouple the quartic interaction [13]

\[
Z = \int \text{d}A \text{d}X \text{d}X^\dagger \exp \left[ \text{Ntr} \left( -XX^\dagger - \frac{1}{2} A^2 + \sqrt{b} A (XX^\dagger e^{i\lambda \pi/4} + X^\dagger X e^{-i\lambda \pi/4}) \right) \right] 
\]

(2.2)

where \(A\) is a hermitean matrix. This new model is a deformation of the \(O(2)\) model. Its Feynman rules (figure 2) allow to interpret it as a model of oriented loops in which each left/right turn costs \(\omega^\pm = e^{\pm i\lambda \pi/4}\); this means that each loop, taking into account its two possible orientations, contributes a factor of \(2 \cos(p\lambda \pi/4)\), where \(p\) is the number of right turns minus the number of left turns performed when going around the loop. For a regular infinite lattice, this number is fixed and we recover this way the usual \(O(n)\) model with \(n = 2 \cos(p\lambda \pi/4)\); but this connection between the 6-vertex and the \(O(n)\) models breaks down on a random lattice (because of curvature). It only remains at \(\lambda = 0\), of course, where we recover, as mentioned previously, the \(O(2)\) model.

\[
\sqrt{b} \ e^{-i\lambda \pi/4} \\
\sqrt{b} \ e^{+i\lambda \pi/4}
\]

Fig. 2: Part of a diagram generated by the perturbative expansion of (2.2).
The next step to solve the model is to integrate out $X$ and $X^\dagger$, and shift $A$ by the constant $\gamma = \frac{1}{\sqrt{b(\omega+\omega^{-1})}}$:

$$Z = \int dA \det^{-1}(\omega \otimes A + A \otimes \omega^{-1}) \exp\left(-\frac{N}{2} \text{tr}(A - \gamma)^2\right)$$  \hspace{1cm} (2.3)

In terms of the eigenvalues $a_i$ of $A$:

$$Z = \int \prod_{i=1}^{N} da_i e^{-\frac{N}{2}(a_i - \gamma)^2} \prod_{i \neq j} (a_i - a_j) \prod_{i,j} (\omega a_i + \omega^{-1} a_j)$$  \hspace{1cm} (2.4)

In the large $N$ limit, the $a_i$ form a saddle point distribution characterized by a continuous density $\rho_0(a)da$ which fills an interval $[\alpha, \beta]$ of the real line. It will be convenient to make the following change of variable: $a = \beta e^u$. Up to an overall constant we find:

$$Z = \int \prod_{i=1}^{N} du_i e^{-\frac{N}{2}(\beta e^{u_i} - \gamma)^2} \prod_{i \neq j} e^{u_i - u_j} \prod_{i,j} (\omega e^{u_i} + \omega^{-1} e^{u_j})$$  \hspace{1cm} (2.5)

One notices that the two-body interaction now only depends on the difference $u_i - u_j$. More explicitly, we are now trying to minimize an action of the form

$$S = \int du \rho(u) V(u) + \frac{1}{2} \int \int du \rho(u) dv \rho(v) V_2(u - v)$$  \hspace{1cm} (2.6)

with the density $du \rho(u) = da \rho_0(a)$, a potential $V(u) = \frac{1}{2}(\beta e^u - \gamma)^2$ and an interaction

$$V_2(u) = -2 \log(1 - e^u) + \log(1 + \omega^2 e^u) + \log(1 + \omega^{-2} e^u)$$  \hspace{1cm} (2.7)

The density $\rho(u)$ satisfies $\rho(u) \geq 0$ and $\int du \rho(u) = 1$. It is worth remarking the similarity of this problem with the determination of the ground state in the presence of magnetic field in Bethe Ansatz solvable models. We shall comment on this analogy later.

On the support of $\rho(u)$, the minimization of $S$ leads to the saddle point equation:

$$K \ast \rho(u) = \beta e^u (\beta e^u - \gamma)$$  \hspace{1cm} (2.8)

where $\ast$ means convolution product and $K$ is the derivative of $-V_2$:

$$K(u) = \frac{2}{1 - e^{-u}} - \frac{1}{1 + \omega^2 e^{-u}} - \frac{1}{1 + \omega^{-2} e^{-u}}$$  \hspace{1cm} (2.9)

Principal part at $u = 0$ is implied.

We shall now proceed to solve this equation.
3. Exact results at criticality

The analytic problem (2.8) is well known to be exactly solvable using the Wiener–Hopf technique when the support of \( \rho(u) \) is semi-infinite. In our case this corresponds to \( \alpha = 0 \), which is precisely the critical regime (i.e. when the renormalized cosmological constant vanishes and the area of the lattices becomes large). The support of \( \rho(u) \) is then \([-\infty, 0]\) (figure 3). Note that the situation is then analogous to the Bethe Ansatz study of the ground state in the presence of magnetic field of 2D integrable models with only one chirality, see for example [16].

![Fig. 3: Potential \( V(u) \). The eigenvalues fill the well; criticality is attained when the eigenvalues start “overflowing” at minus infinity.](image)

We introduce the Fourier transform \( \hat{K}(k) \):

\[
\hat{K}(k) = \int_{-\infty}^{+\infty} K(u)e^{iku}du
\]

\[
= 4\pi \frac{\sinh(\frac{1}{2} + \frac{\lambda}{4})\pi k \sinh(\frac{1}{2} - \frac{\lambda}{4})\pi k}{\sinh \pi k} \tag{3.1}
\]

Similarly we have the Fourier transform \( \hat{\rho}(k) \) of \( \rho(u) \), which can be defined alternatively as an average in our model:

\[
\hat{\rho}(k) = \frac{1}{\beta^{ik}} \left\langle \frac{1}{N} \text{tr} A^{ik} \right\rangle \tag{3.2}
\]

It is clear that \( \hat{\rho}(k) \) is an analytic function in the lower half-plane \( \text{Im} k \leq 0 \). We now Fourier transform the equation (2.8):

\[
\hat{K}(k)\hat{\rho}(k) = \hat{f}(k) \tag{3.3}
\]

where \( \hat{f}(k) \) is a function whose inverse Fourier transform \( f(u) \) satisfies \( f(u) = \beta e^u(\beta e^u - \gamma) \) for \( u \leq 0 \).
In order to determine $\hat{\rho}$ and $\hat{f}$, we decompose $\hat{K}(k)$ as $\hat{K}(k) = \hat{K}_-(k) / \hat{K}_+(k)$ where $\hat{K}_-$ (resp. $\hat{K}_+$) is holomorphic in the lower (resp. upper) half-plane. Explicitly, we choose

$$
\hat{K}_-(k) = i(k - i0) \frac{\Gamma(1 + ik)}{\Gamma(1 + iu_+ k) \Gamma(1 + iu_- k)} e^{i\epsilon k} \\
\hat{K}_+(k) = \frac{4}{\pi^2} \frac{1}{1 - \frac{\lambda}{4}^2} \frac{\Gamma(1 -iu_+ k) \Gamma(1 -iu_- k)}{\Gamma(1 - i k)} e^{i\epsilon k}
$$

(3.4)

where $u_\pm = \frac{1}{2} \pm \frac{\lambda}{4}$ and $\epsilon = u_+ \log u_+ + u_- \log u_-$. From (3.3) we infer

$$
\hat{K}_-(k) \hat{\rho}(k) = \hat{K}_+(k) \hat{f}(k)
$$

The left hand side is a function which is holomorphic in the lower half-plane, and therefore the right hand side must also be. It is now an easy exercise to find it starting from $f(u) = \beta (\beta e^u - \gamma)$, $u \leq 0$. The result is:

$$
\hat{K}_-(k) \hat{\rho}(k) = \beta \left( \frac{K_2}{i(k - 2i)} - \frac{K_1}{i(k - i)} \right)
$$

(3.5)

where $K_1$ and $K_2$ are $\hat{K}_+(k)$ evaluated at $k = i$ and $k = 2i$: $K_1 = \frac{1}{4 \pi \cos(\lambda \pi / 4)} e^{-\epsilon}$ and $K_2 = \frac{\lambda}{4 \pi \sin(\lambda \pi / 2)} e^{-2\epsilon}$. Dividing by $\hat{K}_-(k)$ gives $\hat{\rho}(k)$. Note however that one must impose the normalization condition $\hat{\rho}(k = 0) = 1$. This in fact imposes two constraints since generically the function $\hat{\rho}(k)$ given by (3.3) has a pole at $k = 0$. The pole cancellation condition reads

$$
\gamma = \beta \frac{K_2}{2 K_1}
$$

(3.6)

while the normalization condition reads

$$
\hat{\rho}(0) = \frac{1}{4} \beta^2 K_2 = 1
$$

(3.7)

These two conditions determine the critical values of $\beta$ and $\gamma$. In terms of the original coupling constant $b$, we find:

$$
b_{\text{crit}} = \frac{1}{32} \frac{\sin(\lambda \pi / 4)}{\lambda \pi / 4} \frac{1}{\cos^3(\lambda \pi / 4)}
$$

(3.8)

Finally we obtain the expression for $\hat{\rho}(k)$:

$$
\hat{\rho}(k) = \frac{2 \Gamma(1 + iu_+ k) \Gamma(1 + iu_- k)}{\Gamma(3 + i k)} e^{-i\epsilon k}
$$

(3.9)
The usual correlation functions $\langle \frac{1}{N} \text{tr} A^n \rangle$ are simply given, according to (3.2), by $\beta^n \hat{\rho}(k = -in)$. However, to extract the universal information from the correlation functions, it is simpler to consider the singularities of the density.

The density $\rho_0(a)$ has two singularities: one at $a = \alpha = 0$ and one at $a = \beta$, which correspond to $u = -\infty$ and $u = 0$.

The singularity of $\rho_0(a)$ at $a = \beta$ is determined by the behavior of $\hat{\rho}(k)$ as $k \to \infty$. We find that $\hat{\rho}(k) \overset{k \to \infty}{\sim} k^{-3/2}$, or

$$\rho_0(a) \sim (a - \beta)^{1/2}$$

i.e. the usual square root singularity.

On the other hand, the singularity of $\rho_0(a)$ at $a = 0$ can be inferred from the poles of $\hat{\rho}(k)$, which lie at

$$k = \frac{in}{u_{\pm}} \quad n = 1, 2, \ldots$$

(3.11)

For $\lambda > 0$, the pole closest to the real axis is $k = i/u_+$, which corresponds to a behavior $\rho(u) \overset{u \to -\infty}{\sim} e^{u/u_+}$ and therefore $\rho_0(a) \sim a^{\frac{1}{u_+} - 1}$ or more explicitly a leading singularity of the form

$$\rho_0(a) \sim a^{\frac{2 - \lambda}{2 + \lambda}}$$

(3.12)

Therefore we have found one critical exponent of our model, which depends continuously on $\lambda$. The other poles give subleading terms in the expansion around $a = 0$.

In the limiting case $\lambda = 0$, we find a double pole at $k = 2i$, which results in a behavior $\rho(u) \sim u e^{2u}$, or

$$\rho_0(a) \sim a \log a$$

(3.13)

4. Vicinity of the critical line

Outside the critical regime, i.e. when the support $[\alpha, \beta]$ of the density $\rho(u)$ is finite, it is not clear how to solve exactly the equation (2.8). This is completely analogous to the Bethe Ansatz equations for 2D integrable models with 2 chiralities (like massive relativistic models). However, what one can do is an exact expansion as $\alpha \to 0$. Here we shall follow the method of [17]. We shall only compute the first correction to the calculation of the previous section, up to some constants which could be determined by a more careful study.

Let us denote $B = \log(\beta/\alpha)$. We are interested in the limit where $B$ is large (in the Bethe Ansatz language, the limit where the two chiralities are almost decoupled from each
other). Then we know that we can split our equation (2.8) into two equations for the regions \( u \approx -B \) and \( u \approx 0 \). The first equation is after the shift \( u \rightarrow u + B \):

\[
K * \rho(u) = -\beta \gamma e^{-B} e^u \quad \forall u \in [0, B]
\]  

(4.1)

We introduce the function \( g(u) \) with support \([-\infty, 0]\) such that

\[
K * \rho(u) = -\beta \gamma e^{-B} e^u + e^{-B/u+} g(u) \quad \forall u \in [-\infty, B]
\]  

(4.2)

We have included the prefactor \( e^{-B/u+} \) so that \( g(u) \) has a finite limit when \( B \rightarrow \infty \), as a simple calculation shows (for \( \lambda = 0 \) one should replace \( e^{-B/u} + \) with \( Be^{-2B} \)). The second equation is our usual saddle point equation; we can insert in it the first correction:

\[
K * \rho(u) = \beta e^u (\beta e^u - \gamma) + e^{-B/u+} g(u + B) \quad \forall u \in [-\infty, 0]
\]  

(4.3)

so that the equation is valid for all negative \( u \). Therefore this is again a Wiener–Hopf problem. We shall not bother to solve it explicitly; let us simply write down the form of the solution at leading order as \( B \rightarrow \infty \):

\[
\hat{K}_-(k) \hat{\rho}(k) = \beta \left( \frac{\beta K_2}{i(k - 2i)} - \frac{\gamma K_1}{i(k - i)} \right) + e^{-B/u+} e^{-ikB} \hat{h}(k)
\]  

(4.4)

When dividing by \( \hat{K}_-(k) \) we are again faced with the problem of the behavior at \( k = 0 \), which leads to the two conditions:

\[
\gamma = \beta \frac{K_2}{2K_1} + c_1 e^{-B/u+} + \ldots
\]

\[
1 = \frac{1}{4} \beta^2 K_2 (1 + c_2 Be^{-B/u+} + c_3 e^{-B/u+} + \ldots)
\]

(4.5)

where \( c_1, c_2, c_3 \) are constants. The term \( Be^{-B/u+} \) comes from differentiation of \( e^{-ikB} \).

This allows to solve for \( \beta \) and \( \gamma \) as a function of \( B \). The renormalized cosmological constant \( \Delta \) is defined as the variation of \( \gamma \). From (4.5) we infer

\[
\Delta \equiv \gamma - \gamma_{\text{crit}} \sim Be^{-B/u+}
\]  

(4.6)

Next, we consider any correlation function \( \langle \frac{1}{N} \text{tr} A^n \rangle \) (obtained by evaluating \( \hat{\rho}(k) \)). It is clear from Eq. (4.4) that its variation is of the form

\[
\hat{\rho}(k) = c_4 + c_5 Be^{-B/u+} + c_6 e^{-B/u+} + \ldots
\]  

(4.7)

and therefore its singular part is

\[
\hat{\rho}(k)_{\text{sing}} \sim \frac{\Delta}{\log \Delta}
\]  

(4.8)

This last result displays a zero string susceptibility exponent and a logarithmic correction characteristic of a \( c = 1 \) theory [18].

7
5. Summary of results and prospects

We have given in this letter the exact solution of the six vertex model on random planar graphs at zero cosmological constant and described the vicinity of this critical line. Note that all the expressions found agree with the known results in the $\lambda = 0$ and $\lambda = 1$ cases. In particular, for $\lambda = 0$ the critical value $b_{\text{crit}} = \frac{1}{32}$ and the behaviors (3.13) coincide with earlier calculations [5], while for $\lambda = 1$ the critical value $b_{\text{crit}} = \frac{1}{4\pi}$ and some correlation functions such as $\langle \frac{1}{N} \text{tr} XX^\dagger \rangle = \frac{2}{3}(4 - \pi)$ coincide with [11]. The limit $\lambda \to 2$ is singular, as can be seen in (3.8), and therefore we cannot compare directly our results with those of [13,14]; in fact, even though the $\lambda = 2$ probably does have a critical point for a finite value $b_{\text{crit}}$ [14], it is believed to belong to the universality class of pure gravity, which is different from the critical behavior found for $\lambda < 2$ ($c = 1$ theories).

The solution is explicit enough to allow us to give exact expressions for correlation functions at criticality, and the method can be used to generate an exact expansion around the critical region. Two critical exponents have been computed this way: the exponent governing the singularity of the density of eigenvalues, and the string susceptibility exponent; the latter turned out to be zero plus a logarithmic scaling violation, confirming the central charge $c = 1$ of the infra-red CFT.

This raises the hope that it is possible to solve much more general models. For example, one can replace the quadratic potential of $A$ with a more general one. This could be used to simulate e.g. vortices in the 6-vertex model or dilution in the deformed $O(2)$ model. There is in principle no problem to finding the critical properties of these models, and we expect a rich structure of multi-critical points. All this should appear in a future publication, as well as a more detailed description of the off-critical region.

Finally, let us mention that these results should have an interesting application to knot theory and more precisely the counting of alternating links, see [19]. This is currently under study.

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