Ashtekar Variables
in Classical General Relativity

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Introduction

The task I was given for this lecture is to present Ashtekar’s connection variables for classical General Relativity in a pedagogical manner. My intention is to describe a possible route towards a reformulation of General Relativity in terms of Ashtekar’s connection variables. I try to give self-contained and hopefully painless derivations of the crucial steps, without entering too much into those details not directly relevant for this purpose. There exists a comprehensive monograph on this subject (Ashtekar 1991) which contains many applications, as well as a periodically updated bibliography (Brügmann 1993), so that the interested reader should have no difficulties to find his/her way into the subject and the current developments. I make no attempt to give a full account of the current status.

In these lectures I proceed as follows: In the first chapter, I very briefly review some elementary concepts from differential geometry, mainly to fix notation and conventions. Chapter two introduces the variational principle which we use to derive the field-equations of General Relativity. Chapter three considers complex General Relativity and shows how its field equations can be obtained from a variational principle involving only the self-dual part of the connection. In chapter four the (3+1)-decomposition is presented in as much detail as seemed necessary for an audience that does not consist entirely of canonical relativists. It is then applied to complex General Relativity in chapter five, where for the first time Ashtekar’s connection variables are introduced. The Hamiltonian of complex General Relativity is presented in terms of connection variables. In chapter six the constraints that follow from the variational principle are analyzed and their Poisson brackets are presented. In chapter seven we discuss the reality conditions that have to be imposed by hand to select real solutions, and briefly sketch the geometric interpretation of the new variables. In chapter 8 we indicate how the Hamiltonian has to be amended by surface integrals in the case of open initial data hypersurfaces with asymptotically flat data. It ends with a demonstration of the positivity of the mass at spatial infinity for maximal hypersurfaces. Throughout we will make no use of spinors.
Chapter 1. Some Basic Differential Geometry

Let $M$ be a connected orientable and time orientable Lorentz 4-manifold with topology $\Sigma \times \mathbb{R}$, where $\Sigma$ can be any connected orientable 3-manifold. Our signature convention is “mostly plus”, $(-, +, +, +)$. Greek indices refer to coordinate bases, latin indices to frame bases. If they are taken from the beginning of the alphabet, i.e., $(\alpha, \beta, \ldots; a, b, \ldots)$, their range is $\{0, 1, 2, 3\}$, whereas for the middle of the alphabet, i.e., $(\mu, \nu, \ldots; i, j, \ldots)$, their range is only $\{1, 2, 3\}$. Square brackets including a string of $n$ indices denote full antisymmetrization including the factor $1/n!$. In the same way, round brackets denote symmetrization. The Lorentz metric is denoted by $g$ (components $g_{ab}, g_{\alpha\beta}$). $\eta_{ab}$ denotes the matrix $\text{diag}(-1, 1, 1, 1)$. The components of the curvature tensor are written $R^a_{\quad bcd}$, $R_{ab} = R^c_{\quad acb}$ are the components of the Ricci-tensor and $R = g^{ab}R_{ab}$ denotes the Ricci scalar.

The structure group for the real frame bundle on $M$ is $GL(4, \mathbb{R})$, and $SO(1, 3)$ if one restricts to orthonormal frames. We shall adopt this restriction throughout which is equivalent to imposing an orientation and a metric structure. We denote the Lie algebra of $SO(1, 3)$ by $so(1, 3)$. Due to the assumption that $M$ is topologically $\Sigma \times \mathbb{R}$ and orientable, the frame bundle is necessarily trivial so that we can always assume the existence of a globally defined tetrad. This is also true if one considers the complexified tangent bundle, which we shall need later on. We stress that this is particular to four dimensions and will generally not hold in dimensions three or higher than four. Let

\begin{align*}
\{ e_a \} & \quad \text{orthonormal tetrad} \\ \{ e^a \} & \quad \text{orthonormal co-tetrad dual to (1.1)}, \tag{1.1} \\
\text{so that} & \quad g(e_a, e_b) = \eta_{ab} \tag{1.2} \\
\text{and} & \quad e^a(e_b) = \delta^a_b. \tag{1.3}
\end{align*}

The volume form, $\epsilon$, on $M$ induced by $g$ is given by ($\wedge$ denotes the antisymmetric tensor product)

$$
\epsilon = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \frac{1}{4!} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d, \tag{1.5}
$$

where

$$
\epsilon_{abcd} = 4! \delta^0_a \delta^1_b \delta^2_c \delta^3_d. \tag{1.6}
$$

Once we have restricted the structure group to $SO(1, 3)$ we also restrict the connections to be metric-preserving. The connection 1-form can be represented by a globally defined $so(1, 3)$-valued 1-form, $\omega^a_b$, and the curvature by an $so(1, 3)$-valued 2-form $\Omega^a_{\quad b}$. If $\nabla$ denotes the covariant derivative, one has

\begin{align*}
\nabla_{e_a} e_b & = \omega^c_b(e_a) e_c \tag{1.7} \\
\eta_{ac} \omega^c_b =: \omega_{ab} & = -\omega_{ba}, \tag{1.8}
\end{align*}
and
\[
(\nabla_{e_a} \nabla_{e_b} - \nabla_{e_b} \nabla_{e_a} - \nabla_{[e_a,e_b]} ) e_c = \Omega^d_c(e_a,e_b) e_d = R^d_{cab} e_d
\]

\[
\eta_{ac} \Omega^c_b =: \Omega_{ab} = - \Omega_{ba}.
\]  

(1.9)

(1.10)

Throughout we identify the Lie algebra with tangent space tensors \( \sigma^b_a \) such that \( \eta_{ac} \sigma^c_b = \sigma^b_a = - \sigma^a_b \). The torsion is represented by an \( R^4 \)-valued 2-form \( T^a \) which is defined through

\[
\nabla_{e_a} e_b - \nabla_{e_b} e_a - [e_a,e_b] = T^c(e_a,e_b) e_c.
\]

(1.11)

Let \( V \) denote a vector space that carries a representation, \( \rho \), of \( SO(1,3) \), and \( \lambda \) a \( V \)-valued \( n \)-form on \( M \) which under change of tetrads transforms via \( \rho \). Then the exterior covariant derivative, \( D \), on vector-valued forms is defined via

\[
D\lambda := d\lambda + \rho(\omega) \wedge \lambda,
\]

so that

\[
D^2\lambda = \rho(\Omega) \wedge \lambda
\]

(1.12)

(1.13)

Here the symbol \( \rho(\omega) \wedge \lambda \) is to be understood in the following way: The representation \( \rho \) of \( SO(1,3) \) on \( V \) induces a representation of its Lie algebra \( so(1,3) \) on \( V \), which we also denote by \( \rho \). Via this representation \( \omega \) \( (so(1,3)\)-valued) acts on \( \lambda \) \( (V\)-valued), while as forms the exterior product is taken. The exterior covariant derivatives with respect to two different connections \( \omega \) and \( \omega' \) (denoted by \( D^\omega \) and \( D^{\omega'} \)) are related via

\[
D^{\omega'} \lambda = D^\omega \lambda + \rho(\omega' - \omega) \wedge \lambda.
\]

(1.14)

Applying (1.11-13) to the \( R^4 \)-valued 1-form \( \{ e^a \} \) (carrying the defining representation) one obtains the first Cartan structure equation and the first Bianchi identity:

\[
D \omega^a = d \omega^a + \omega^a_b \wedge e^b = T^a, 
\]

\[
D^2 \omega^a = \Omega^a_b \wedge e^b = DT^a.
\]

(1.15)

(1.16)

By direct calculation, using the definition of the curvature, and regarding \( \Omega^a_b \) as an \( so(1,3) \)-valued 2-form, we obtain the second Cartan structure equation and second Bianchi identity:

\[
d \omega^a_b + \omega^a_c \wedge \omega^c_b = d \omega^a_b + \frac{1}{2} [\omega,\omega]^a_b = \Omega^a_b, 
\]

\[
D \Omega^a_b = d \Omega^a_b + [\omega,\Omega]^a_b = 0.
\]

(1.17)

(1.18)

Here \([,] \) denotes the commutator if the product of the form degrees is even, and the anticommutator if it is odd. Note also that it would neither be algebraically correct nor
meaningful to write (1.17) as $D\omega^a_b$ (in the adjoint representation): firstly, this expression would not have the factor 1/2 that appears in (1.17) and, secondly, $\omega^a_b$ does not transform with any linear transformation under change of tetrads, as would be necessary for the covariant derivative to be meaningful. Rather, from (1.7) one easily proves that under a change of frames: $e_a \mapsto e'_a = R^b_a e_b$, with $SO(1,3)$-valued matrix function $R$, one has (in matrix notation):
\[ \omega \mapsto R \omega' = R^{-1} \omega R + R^{-1} dR. \] (1.19)

Finally we mention the Hodge-duality map, *, which provides a linear isomorphism between $n$ forms and $(4-n)$-forms at each point on $M$. It can be defined via
\[ *\left(e^{a_1} \wedge \ldots \wedge e^{a_n}\right) := \frac{1}{(4-n)!} \epsilon^{a_1 \ldots a_n}_{a_{n+1} \ldots a_4} e^{a_{n+1}} \wedge \ldots \wedge e^{a_4}, \] (1.20)
and linear extension, where the indices on $\epsilon$ are raised using $\eta^{ab}$ (the inverse matrix to $\eta_{ab}$, which has the same entries). Since $\epsilon_{a_1 a_2 a_3 a_4}$ is invariant under $SO(1,3)$, the exterior covariant derivative and the *-operation are compatible in the following sense:
\[ D * (e^{a_1} \wedge \ldots \wedge e^{a_n}) = \frac{1}{(4-n)!} \epsilon^{a_1 \ldots a_n}_{a_{n+1} \ldots a_4} D(e^{a_{n+1}} \wedge \ldots \wedge e^{a_4}). \] (1.21)

Applying * twice results in plus or minus the identity. In fact, on any $n$-form $\lambda$ one has
\[ *(*)\lambda = (-1)^n(4-n) \lambda. \] (1.22)

Analogous formulae hold in any spacetime dimensions. Here, the first minus sign on the right hand side of (1.21) is the sign of the determinant of $\eta_{ab}$. In four dimensions and Lorentz signature * squares to $-1$ on two forms, so that eigenforms only exist on the complexified tangent bundle with eigenvalues $+i$ (self-dual 2-forms) and $-i$ (anti-self-dual 2-forms). Given two $n$-forms $\lambda$ and $\sigma$ at some point in $M$:
\[ \lambda = \frac{1}{n!} \lambda_{a_1 \ldots a_n} e^{a_1} \wedge \ldots \wedge e^{a_n}, \] (1.23)
\[ \sigma = \frac{1}{n!} \sigma_{a_1 \ldots a_n} e^{a_1} \wedge \ldots \wedge e^{a_n}, \] (1.24)
we define their inner product, induced by $g$, via
\[ \langle \lambda, \sigma \rangle := \frac{1}{n!} \lambda_{a_1 \ldots a_n} g^{a_1 b_1} \ldots g^{a_n b_n} \sigma_{b_1 \ldots b_n} = \lambda_{a_1 \ldots a_n} \sigma^{a_1 \ldots a_n}, \] (1.25)
and have
\[ \lambda \wedge *\sigma = \langle \lambda, \sigma \rangle \epsilon. \] (1.26)
Chapter 2. The Variational Principle

We first show how the well known Einstein-Hilbert action* is written in terms of the curvature 2-form $\Omega_{ab}$ and the co-tetrads $e^a$.

$$\text{Action} = S := \int \Omega_{ab} \wedge \ast (e^a \wedge e^b)$$

$$= \frac{1}{2} \int R_{abcd} (e^c \wedge e^d) \wedge \ast (e^a \wedge e^b)$$

$$= \frac{1}{2} \int R_{abcd} (e^c \wedge e^d, e^a \wedge e^b) \epsilon$$

$$= \frac{1}{2} \int R_{abcd} (\eta^{ac} \eta^{bd} - \eta^{ad} \eta^{bc}) \epsilon$$

$$= \int R \epsilon = \text{Einstein-Hilbert Action.}$$

We now take the co-tetrads $e^a$ and connection 1-forms $\omega^a_b$ as independent variables with respect to which we vary the action $S$. We stress that since we restricted the connection 1-form to be $so(1,3)$-valued we have put in metricity of the connection by hand. Varying the curvature with respect to the connection 1-form yields:

$$\delta \Omega_{ab} = d(\delta \omega_{ab}) + [\omega, \delta \omega]_{ab} = D(\delta \omega_{ab}),$$

so that the variation of $S$ with respect to $\omega$, denoted by $\delta^\omega$, is given by

$$\delta^\omega S = \frac{1}{2} \int \delta \omega_{ab} \wedge \epsilon_{cd} \, D(e^c \wedge e^d),$$

where we have performed an integration by parts and used the fact that the variations vanish at the boundary of the integration domain. The requirement of stationarity is thus equivalent to:

$$\delta^\omega S = 0 \Leftrightarrow D(e^a \wedge e^b) = 0 = T^a \wedge e^b - e^a \wedge T^b.$$

It is not difficult to show directly from the rightmost expression that this implies vanishing torsion. But, keeping an eye towards later generalizations, we shall proceed differently. We write $\omega = \Gamma + \Lambda$ where $\Gamma$ is the Levi-Civita connection, i.e., the unique metric, torsion

* Throughout this article we write all actions in their simplest form neglecting prefactors. One way to normalize the action within pure gravity is to evaluate the energy of the Schwarzschild solution and require it to be equal to the standard expression. The Einstein-Hilbert action, as written down in (2.1), gives the right energy expression in units where $16\pi G/c^4 = 1$. Note also that in the Hamiltonian formulation the canonical momenta scale with the prefactor.
free connection. If we denote the exterior covariant derivative with respect to \( \Gamma (\omega) \) by \( D^\Gamma (D^\omega) \), we have, using (1.13) and the vanishing torsion of \( \Gamma \),

\[
D^\omega (e^a \wedge e^b) = D^\Gamma (e^a \wedge e^b) + \Lambda^a_c \wedge e^c \wedge e^b + \Lambda^b_c \wedge e^a \wedge e^c
\]

\[
= (\Lambda^a_{[cd]} \delta^b_c + \Lambda^b_{[ce]} \delta^a_c) e^c \wedge e^d \wedge e^e ,
\]

where we used the expansion \( \Lambda^a_b = \Lambda^a_{cb} e^c \). This last expression vanishes precisely if the cyclic sum in \( c - d - e \) of the coefficient does:

\[
\Lambda^a_{[cd]} \delta^b_c + \Lambda^b_{[ce]} \delta^a_c + \Lambda^a_{[de]} \delta^b_c + \Lambda^b_{[ec]} \delta^a_c + \Lambda^a_{[ed]} \delta^b_c + \Lambda^b_{[dc]} \delta^a_c = 0 .
\]

Contraction on \( b \) and \( e \) yields:

\[
\Lambda^a_{[cd]} + \Lambda^b_{[ce]} \delta^a_d - \Lambda^b_{[db]} \delta^a_c = 0 .
\]

Another contraction on \( a \) and \( d \) implies \( \Lambda^a_{[ca]} = 0 \), which, after reinsertion into (2.7), implies \( \Lambda^a_{[cd]} = 0 \), i.e. symmetry of \( \Lambda^a_{bc} \) in the lower index pair. In its purely covariant form: \( \eta_{bd} \Lambda^d_{ac} =: \Lambda_{abc} \), we obtain a tensor which is symmetric in the first and third, and antisymmetric in the second and third slot, where the antisymmetry follows from the metricity of \( \omega \) and \( \Gamma \). But such a tensor is necessarily zero as three consecutive applications of interchangements of the first with third followed by second with third index show. Note that \( \Lambda^a_{[bc]} = 0 \) is just the condition for \( \omega \) and \( \Gamma \) having the same torsion, so that the last arguments may be seen as a uniqueness proof for metric, torsion free connections. As result of the variation \( \delta^\omega \) we thus obtain

\[
\delta^\omega S = 0 \iff \omega = \Gamma = \text{Levi-Civita connection}.
\]

For the Levi-Civita connection \( \Gamma \) we can invert the first Cartan structure equation (1.15) and express it purely in terms of the co-tetrad \( e^a \):

\[
\Gamma^{ab} = - * [(\ast de^a) \wedge e^b - (\ast de^b) \wedge e^a + \frac{1}{2} (e^a \wedge e^b) \wedge (\ast (de^c \wedge e^d) \eta_{cd})] .
\]

Next, we take the variation with respect to the co-tetrad \( e^a \), which we denote by \( \delta^e \). The variation of the integrand yields:

\[
\delta^e (\Omega_{ab} \wedge \ast (e^a \wedge e^b)) = (e^{ab}_{cd} \Omega_{ab} \wedge e^c) \wedge \delta e^d
\]

\[
= \frac{1}{2} (e^{ab}_{cd} R_{abgh} e^g \wedge e^h \wedge e^c) \wedge \delta e^d
\]

\[
= \frac{1}{2} (\epsilon_{abcd} R^{ab}_{gh} e^{ghce} \eta_{ef} \ast e^f) \wedge \delta e^d
\]

\[
= - \frac{1}{2} \left( 3! \delta^g_{[a] \delta^h_{[b]} \delta^e_{[d]} R^{ab}_{gh} \eta_{ef} \ast e^f \right) \wedge \delta e^d
\]

\[
= 2 (R_{fd} - \frac{1}{2} \eta_{fd} R) \ast e^f \wedge \delta e^d ,
\]

(2.10)
so that

$$\delta e S = 0 \iff \frac{\delta S}{\delta e} = 2 \left( R_{ab} - \frac{1}{2} \eta_{ab} R \right) \ast e^a = 0 \quad (2.11)$$

$$\iff G_{ab} = R_{ab} - \frac{1}{2} \eta_{ab} R = 0. \quad (2.12)$$

If we finally insert into this variation formula the result of the first variation, $\omega = \Gamma$, where $\Gamma$ is to be understood as a function of the co-tetrad according to (2.9), then we obtain equations for the co-tetrads which are equivalent to Einstein’s vacuum equations.

**Chapter 3. Complex GR and Self-Dual Variational Principle**

We now extend the differential-geometric framework and consider the complexified tensor bundle

$$T_C = \bigoplus_{r,s} T^r_s(M) \otimes \mathbb{C}, \quad (3.1)$$

over the real manifold $M$. Here $T^r_s(M)$ denotes the (real) tensor bundle of $r$-fold contravariant and $s$-fold covariant tensors over $M$. The structure group is now $SO(1,3)_C \cong SO(4)_C$. The fields $e^a, \omega^a_b, \Omega^a_b, T^a$ and $\eta_{ab}$ extend to sections in the complexified bundle. Everything we have said so far is still valid in the complex case. In particular, the variational principle and the resulting equations of motion transcribe verbatim with all quantities now being complex. The theory so obtained may be called complex General Relativity. Since the complex tensor bundle was obtained by complexification of a real tensor bundle, there is a natural reality structure. In other words, we know how to complex conjugate, namely by complex conjugation of the components with respect to a real basis which is naturally given to us since we work over a real manifold. We can thus speak of real sections in the complex tensor bundle.

On the space of complex 2-forms we can now consider eigenforms to the $\ast$-operator (remember, on 2-forms: $\ast^2 = -\text{identity}$)

$$\ast \lambda = i \lambda \quad \text{self-dual}, \quad (3.2a)$$

$$\ast \lambda = -i \lambda \quad \text{anti-self-dual}. \quad (3.2b)$$

Similarly, in the complexified Lie algebra $so(1,3)_C$, we can define a dualization operation (denoted by $\ast$):

$$\ast \sigma_{ab} := \frac{1}{2} \epsilon_{ab}^{\;cd} \sigma_{cd} \quad (3.3)$$

that also squares to minus the identity and thus decomposes the complexified Lie algebra into self- and anti-self-dual part:

$$so(1,3)_C^{(+)} = \{ \sigma \in so(1,3) / \ast \sigma = i \sigma \} \quad \text{self-dual}, \quad (3.4a)$$

$$so(1,3)_C^{(-)} = \{ \sigma \in so(1,3) / \ast \sigma = -i \sigma \} \quad \text{anti-self-dual}. \quad (3.4b)$$
Using the standard formula for multiplying two $\epsilon$-tensors one easily shows the following relations for commutators:

\[
\star [\sigma, \star \tau] = -[\sigma, \tau] = \star [\sigma, \tau],
\]

(3.5)

hence

\[
\star [\sigma, \tau] = [\star \sigma, \tau] = [\sigma, \star \tau]
\]

(3.6)

and

\[
[\star \sigma, \star \tau] = -[\sigma, \tau].
\]

(3.7)

If we denote the projectors onto the self-dual and anti-self-dual parts of $P$ relations for commutators:

\[
P^{(\pm)} = \frac{1}{2} (1 \mp i \star),
\]

(3.8)

then (3.5-7) imply

\[
P^{(\pm)}[\sigma, \tau] = [P^{(\pm)} \sigma, \tau] = [\sigma, P^{(\pm)} \tau] = [P^{(\pm)} \sigma, P^{(\pm)} \tau].
\]

(3.9)

This tells us that $so(1,3)_{C}^{(\pm)}$ form ideals in $so(1,3)_{C}$ with projection homomorphisms $P^{(\pm)}$. We thus have a decomposition of the Lie algebra into two ideals:

\[
so(1,3)_{C} = so(1,3)_{C}^{(+)} \oplus so(1,3)_{C}^{(-)}.
\]

(3.10)

so(1,3) and $so(1,3)_{C}^{(\pm)}$ are simple Lie algebras whereas $so(1,3)_{C}$ is only semi-simple.

We can now decompose $\omega$ and $\Omega$ according to (3.9):

\[
\omega = (+)\omega + (-)\omega = P^{(+)} \omega + P^{(-)} \omega,
\]

(3.11)

\[
\Omega = (+)\Omega + (-)\Omega = P^{(+)} \Omega + P^{(-)} \Omega = \Omega^{(+)} + \Omega^{(-)}
\]

(3.12)

where $\Omega^{(\pm)\omega}$ is the curvature built from $(\pm)\omega$ according to (1.17). It should be noted that $(\pm)\omega$ are not connections on the original bundle with structure group $SO(1,3)_{C}$ (e.g. a general gauge transformation in $SO(1,3)_{C}$ will not leave it $so(1,3)_{C}^{(\pm)}$-valued), but on associated bundles whose structure groups are given by the subgroups in $SO(1,3)_{C}$ that correspond to the Lie sub-algebras $so(1,3)_{C}^{(\pm)}$. Noting that $\Omega_{ab} \wedge *(e^{a} \wedge e^{b}) = \star \Omega_{ab} \wedge (e^{a} \wedge e^{b})$, we can decompose the action of complex General Relativity:

\[
S_{C} = S_{C}^{(+)} + S_{C}^{(-)} = \int (+)\Omega_{ab} \wedge *(e^{a} \wedge e^{b}) + \int (-)\Omega_{ab} \wedge *(e^{a} \wedge e^{b})
\]

\[
= i \int (+)\Omega_{ab} \wedge (e^{a} \wedge e^{b}) - i \int (-)\Omega_{ab} \wedge (e^{a} \wedge e^{b}).
\]

(3.13)

The variation of the actions $S_{C}$ and $S_{C}^{(\pm)}$ with respect to $\omega$, respectively $(\pm)\omega$, will in each case result in a determination of these connections in terms of the co-tetrad fields. The variation with respect to the co-tetrad fields, evaluated at the particular value of the connections just determined, then result in equations for the co-tetrad fields only. Their solutions will then determine connections $\omega$ (if $S_{C}$ is varied) or $(\pm)\omega$ (if $S_{C}^{(\pm)}$ is varied) in the corresponding bundles. The crucial property of these different variational principles is that they result in the same equations for the co-tetrad. Let us summarize this in the following...
Theorem. The stationary points of $S_C$ and $S^{(+)}_C$ lie over the same co-tetrad fields. Both actions assume the value zero at their stationary points.

Proof. The variation of $S_C$ proceeds identically to the variation of the real action $S$. With $S^{(±)}_C$ we also proceed as in the real case. We shall only deal with $S^{(+)}_C$ since the anti-self-dual case is entirely analogous. First varying $(+)_ω$ we obtain (where we denote the exterior covariant derivative with respect to $(±)_ω$ by $(±)D$)

$$\delta^{(+)}ω S^{(+)}_C = \int^{(+)}D(δ^{(+)}ω_{ab}) \wedge *(e^a \land e^b)$$

$$= i \int δ^{(+)}ω_{ab} \land (+)D(e^a \land e^b)$$

$$= 0 ⇔ (+)D(e^a \land e^b)^{(+)0} = 0$$

where $(e^a \land e^b)^{(+)0}$ denotes the self-dual part of $(e^a \land e^b)$ considered as a Lie algebra valued 2-form. We set $(+)_ω = (+)_Γ + (+)Λ = Γ - (-)_Γ + (+)Λ$ and have, using that $Γ$ has vanishing torsion and that the anti-self-dual $(-)_Γ^{ab}$ commutes with the self-dual $(e^a \land e^b)^{(+)0}$, analogous to formula (2.5),

$$δ^{(+)}ω S^{(+)} = 0 ⇔ (+)Λ^a_c \land (e^c \land e^b)^{(+)0} + (+)Λ^b_c \land (e^a \land e^c)^{(+)0}$$

$$= (+)Λ^a_c \land (e^c \land e^b)^{(+)0} + (+)Λ^b_c \land (e^a \land e^c)$$

$$= (+)Λ^a_c [cd] δ_{e}^b + (+)Λ^b_c [ce] δ_{d}^a \land e^c \land e^d \land e^e,$$

where we used the expansion $(+)_Λ^{ab} = (+)_Λ^{ac}_b e^c$. We can now conclude exactly as in (2.5) and (2.6), with $(+)_Λ$ replacing $Λ$, and obtain the analogous formula to (2.7). This leads to

$$δ^{(+)}ω S^{(+)} = 0 ⇔ (+)_ω = (+)_Γ = \text{self-dual part of the Levi-Civita connection.}$$

As in the real case, $(+)_ω$ is now determined as function of the co-tetrad by taking the self dual part of (2.9).

Next we perform the variation with respect to the co-tetrad. The variational derivative is easily obtained:

$$\frac{δS^{(+)}_C}{δe^b} = 2i^{(+)_Ω_{ab} \land e^a}.$$ (3.17)

Evaluated at $(+)_ω = (+)_Γ$, this is equal to

$$\frac{δS^{(+)}_C}{δe^b}\big|_{(+)_ω = (+)_Γ} = i \left( Ω_{ab} \land e^a - i \frac{1}{2} ε^{ab}_{cd} Ω_{cd} \land e^a \right)$$

$$= \frac{1}{2} ε^{ab}_{cd} Ω_{cd} \land e^a$$

$$= \left( R_{ab} - \frac{1}{2} η_{ab} R \right) * e^a$$

$$= \frac{1}{2} δS_C \big|_{ω = Γ},$$ (3.21)
where from (3.18) to (3.19) we have used the first Bianchi identity (1.16) and the step from (3.19) to (3.20) is entirely analogous to the calculation leading to (2.10). Equation (3.20) then follows from comparison with (2.11), simply transcribed for the complex case. Finally, we observe that at the stationary points the actions $S_{C}^{(\pm)}$ and $S_{C}$ assume the value zero. Note also the factor $\frac{1}{2}$ in (3.21), which is just the $\frac{1}{2}$ from the projection operation (formula (3.8)) onto the self-dual part. It implies that the properly normalized self-dual action should have twice the prefactor of the action $S_{C}$. This concludes our proof of the theorem $\bullet$.

Chapter 4. The 3+1-Decomposition

Let $M \cong \mathbb{R} \times \Sigma$ be foliated by the images of a one-parameter family of embeddings, $\mathcal{E}_{t} : \Sigma \rightarrow M$, and let $\Sigma_{t}$ denote the image of $\Sigma$ under $\mathcal{E}_{t}$ in $M$. We have the following vector field over $\mathcal{E}_{t}$:

$$\frac{\partial}{\partial t} := \frac{d}{dt}\mathcal{E}_{t} \quad p \mapsto \frac{d}{dt}\mathcal{E}_{t}(p) \in T_{\mathcal{E}_{t}(p)}(M),$$

(4.1)

whose spacetime interpretation is to generate the motion of $\Sigma$ through $M$. We decompose it with respect to an orthonormal tetrad $\{e_{\perp}, e_{k}\}$ which is adapted to the foliation. The dual co-tetrad is $\{e_{\perp}^\ast, e_{k}^\ast\}$. Note that the $e_{k}$’s are chosen tangential to $\Sigma$, which does not imply the $e_{k}^\ast$’s to be tangent as well (i.e. a section in the cotangent bundle of $\Sigma$). Rather, the dual basis to $\{e_{k}\}$, intrinsic to $\Sigma$, is given by the projections of the $e_{k}$’s parallel to $\Sigma$ and will instead be called $\{\theta^{k}\}$. We will not use it until the last section. We now have:

$$\frac{\partial}{\partial t} = Ne_{\perp} + N^{k}e_{k},$$

(4.2)

where $N$ and $N^{k}e_{k}$ are the so-called lapse function and shift vector field respectively. Let further $\{x^{\mu}\}$ be a coordinate system on $\Sigma$ so that $\{t, x^{\mu}\}$ is an adapted coordinate system on $M$ and where $\partial/\partial t$ is given by (4.2). We shall from now on write a $\perp$ for the zeroth frame index and $t$ for the zeroth coordinate to distinguish the two and to remind on the special choice made.

The relation between the spatial coordinate and frame bases is given by

$$\frac{\partial}{\partial x^{\mu}} = e_{\mu}^{k}e_{k} , \quad e_{k} = e_{k}^{\mu}\frac{\partial}{\partial x^{\mu}},$$

(4.3)

where $\{e_{k}^{\mu}\}$ and $\{e_{k}^{\mu}\}$ are relative inverse matrices. We have ($N^{\mu}$ denote the components of the shift vector with respect to the coordinate basis)

$$\begin{pmatrix} \partial_{t} \\ \partial_{\mu} \end{pmatrix} = \begin{pmatrix} N & N^{k} \\ 0 & e_{\mu}^{k} \end{pmatrix} \begin{pmatrix} e_{\perp} \\ e_{k} \end{pmatrix}$$

(4.4)

and

$$\begin{pmatrix} e_{\perp} \\ e_{k} \end{pmatrix} = \begin{pmatrix} 1/N & -N^{\mu}/N \\ 0 & e_{k}^{\mu} \end{pmatrix} \begin{pmatrix} \partial_{t} \\ \partial_{\mu} \end{pmatrix}$$

(4.5)

10
and

\[(dt, dx^\mu) = (e^\perp, e^k) \begin{pmatrix} 1/N & -N^\mu/N \\ 0 & e^\mu_k \end{pmatrix} \]

\[(e^\perp, e^k) = (dt, dx^\mu) \begin{pmatrix} N & N^k \\ 0 & e^\mu_k \end{pmatrix}.\]  

The four-dimensional Lorentz metric then decomposes as follows:

\[ds^2 = \eta_{ab} e^a \otimes e^b = g_{\alpha\beta} dx^\alpha \otimes dx^\beta\]

\[= -e^\perp \otimes e^\perp + \sum_{k=1}^3 e^k \otimes e^k\]

\[= -N^2 dt \otimes dt + \sum_{k=1}^3 (N^k dt + e^k_\mu dx^\mu) \otimes (N^k dt + e^k_\nu dx^\nu),\]

where the coordinate components of the spatial part are given by

\[h_{\mu\nu} = \sum_{k=1}^3 e^k_\mu e^k_\nu (\partial_\mu, \partial_\nu) = \sum_{k=1}^3 e^k_\mu e^k_\nu\]  

and inverse \[h^{\mu\nu} = \sum_{k=1}^3 e^{\mu}_k e^{\nu}_k,\]

so that \[e^k_\mu = h^{\mu\nu} \delta_{kl} e^l_\nu \quad \text{and} \quad e^k_\mu = h_{\mu\nu} \delta^{kl} e^l_\nu.\]

In words: \[e^k_\mu\] is obtained from \[e^i_\nu\] by raising the (latin) frame index with the Kronecker delta \[\delta^{kl} (= \delta_{kl})\] and lowering the (greek) coordinate index with \[h_{\mu\nu}.\] Using (4.10-11) and the definition \[N^\mu := h_{\mu\nu} N^\nu,\] we can write the Lorentz metric and its “inverse” in the following form:

\[g_{\alpha\beta} dx^\alpha \otimes dx^\beta = -e^\perp \otimes e^\perp + \sum_{k=1}^3 e^k \otimes e^k\]

\[= -(N^2 - N^\mu N_\mu) dt \otimes dt + N_\mu (dt \otimes dx^\mu + dx^\mu \otimes dt) + h_{\mu\nu} dx^\mu \otimes dx^\nu\]

\[g^{\alpha\beta} \partial_\alpha \otimes \partial_\beta = -e^\perp \otimes e^\perp + \sum_{k=1}^3 e^k \otimes e^k\]

\[= -\frac{1}{N^2} \partial_t \otimes \partial_t + \frac{N^\mu}{N^2} (\partial_t \otimes \partial_\mu + \partial_\mu \otimes \partial_t) + \left(h^{\mu\nu} - \frac{N^\mu N^\nu}{N^2}\right) \partial_\mu \otimes \partial_\nu,\]

or simply:

\[\{g_{\alpha\beta}\} = \begin{pmatrix} -N^2 + N^\mu N_\mu & N_\mu \\ N_\mu & h_{\mu\nu} \end{pmatrix}\]

\[\{g^{\alpha\beta}\} = \begin{pmatrix} -1/N^2 & N^\mu/N^2 \\ N^\mu/N^2 & h^{\mu\nu} - N^\mu N^\nu/N^2 \end{pmatrix}.\]
The square-root of the determinant of \( \{g_{\alpha\beta}\} \), denoted by \( \sqrt{g} \), is most easily obtained from (4.7,9) and the definition of the volume 4-form (1.5) (the square-root of the determinant of \( \{h_{\mu\nu}\} \) is called \( \sqrt{h} \)):

\[
\epsilon = e_1^\perp \wedge e^1 \wedge e^2 \wedge e^3 = N \det\{e^k_\mu\} \ dt \wedge dx^1 \wedge dx^2 \wedge dx^3
\]
\[
= N\sqrt{h} \ dt \wedge dx^1 \wedge dx^2 \wedge dx^3
\]
\[
= \sqrt{g} \ dt \wedge dx^1 \wedge dx^2 \wedge dx^3 , \quad \text{hence} \quad \sqrt{g} = N\sqrt{h}
\]

Note that

\[
\sqrt{h} = \det\{e^k_\mu\} = 1/\det\{e^k_\mu\}.
\]

If we were to follow the standard route and put the action principle (2.1) into Hamiltonian form, we would decompose the co-tetrad variables according to (4.7), i.e., we would use the variables \((e^k_\mu, N^\mu, N)\).

A first step in arriving at an algebraically simpler form of the Hamiltonian action integral is to take the frame rather than co-frame components as variables and redefine these variables by partially densitizing them (i.e. multiplying by powers of \( \sqrt{h} \)). The new variables are:

\[
E^\mu_k := \sqrt{h} e^\mu_k = \frac{1}{\det\{e^k_\mu\}} e^\mu_k
\]
\[
N'^\mu := N^\mu ,
\]
\[
N' := \frac{1}{\sqrt{h}} N = \det\{e^k_\mu\} N.
\]

We have

\[
\{dx^\alpha(e_a)\} = \begin{pmatrix} 1/N & 0 \\ -N^\mu/N & e^\mu_k \end{pmatrix} = \frac{1}{\sqrt{h}} \begin{pmatrix} 1/N' & 0 \\ -N'^\mu/N' & E^\mu_k \end{pmatrix} =: \frac{1}{\sqrt{h}} E^\alpha_a
\]

and

\[
\sqrt{g} = N\sqrt{h} = N'h
\]
\[
h = [\det\{e^k_\mu\}]^{-2} = \det\{E^k_\mu\}.
\]

Now we have all the necessary formulae to write the action

\[
S = \int \Omega_{ab} \wedge *(e^a \wedge e^b)
\]
in 3+1-decomposed form using the densitized variables just introduced. Since from now on
the old variables will never appear, we remove the dashes from these variables. But
remember: from now on $E^\mu_k$ are the components of a triad of density weight-one, $N^\mu$ is
a spatial vector field and $\Omega$ is a spatial scalar of density weight minus one. The action
integrand then decomposes as follows:

$$\Omega_{ab} \wedge \ast (e^a \wedge e^b) = \frac{1}{2} R_{ab\alpha\beta} dx^\alpha \wedge dx^\beta \wedge \ast (e^a \wedge e^b)$$

$$= \frac{1}{2} R_{ab\alpha\beta} \langle dx^\alpha \wedge dx^\beta, e^a \wedge e^b \rangle \epsilon$$

$$= \frac{1}{2} R_{ab\alpha\beta} E^\alpha_c E^\beta_d \langle e^c \wedge e^d, e^a \wedge e^b \rangle \frac{1}{\hbar} \epsilon$$

$$= R^{cd}_{\alpha\beta} E^\alpha_c E^\beta_d N dtd^3 x$$

$$= \left( N R^{kl}_{\mu\nu} E^\mu_k E^\nu_l - 2N^\mu R^{k\perp\mu}_{\mu\nu} E^\nu_k + 2R^{k\perp\mu}_{t\nu} E^\nu_k \right) dtd^3 x , \quad (4.26)$$

where we simply write $dtd^3 x$ for $dt \wedge dx^1 \wedge dx^2 \wedge dx^3$. The first thing to observe is the
purely polynomial dependence on all field variables $N, N^\mu, E^\mu_k$ and $\omega^{ab}_{kl}$. The reason for
this lies in the usage of the densitized variables (4.19-21). In a next step we would like to
calculate the Hamiltonian. For this we have to calculate the momenta by differentiating
with respect to the time derivatives of the field variables. Obviously, $N, N^\mu$ and $E^\mu_k$
appear undifferentiated. As far as the connection variables are concerned, the first and
second terms in (4.26) only contain $\omega^{kl}_{\mu}$ and $\omega^{\perp\mu}_{\nu}$ and spatial derivatives thereof. The
third term, however, contains $\omega^{kl}_{\mu}, \omega^{k\perp\mu}_{\nu}, \omega^{k\perp\mu}_{\nu}$ and spatial derivatives thereof, but also $\omega^{k\perp\mu}_{\nu}$
with its time derivative. No other quantities appear with their time derivatives.

In calculating the Hamiltonian we would then have to go through a constraint analysis
along the standard algorithm as outlined in (Dirac 1967) and spelled out in more
detail in (Hanson, Regge and Teitelboim 1976) or (Henneaux and Teitelboim 1992). We
also recommend (Gotay, Nester, Hinds 1978) and (Marmo, Mukunda and Samuel 1983)
for a more geometric approach. A good summary may be found in the lecture by Andreas
Wipf in this volume. Here we cannot go through the full constraint analysis as it would
present itself when started at this point. We merely want to point out one of its impor-
tant results that may bee seen as being responsible for the strategy of complexification.
More details may be found in (Romano 1993).

In our case, the constraint analysis would start with the primary constraints of vanishing momenta for all field variables except $\omega^{k\perp\mu}_{\nu}$, and then continue with secondary
constraints to ensure preservation of these during Hamiltonian evolution. This system of
constraints then turns out not to be first class. Solving the purely second-class constraints
reintroduces a complicated dependence on the field variables, which essentially brings us
back to the standard ADM treatment of canonical General Relativity. This undesirable
feature can be located as being due to the nondynamical nature of the field components.
\( \omega^\mu \), i.e. their vanishing momenta. The way complex General Relativity circumvents this problem is via the self dual representation (the anti-self-dual representation would just be as good), where

\[
\frac{1}{2} \varepsilon_{lk}^{\perp} l_m (\omega^\mu \omega_{\alpha} \omega^m_{\alpha} = \frac{i}{2} \omega_{\alpha}^\perp \omega^m_{\alpha},
\]

so that the time derivatives of \( \omega_{\mu} \omega_{\mu} \) now appear through those of \( \omega_{\mu} \omega_{\mu} \). To see how much is achieved this way we have to explicitly 3+1-decompose the self dual action of complex General Relativity. This is done in the next section.

Chapter 5. 3+1-Decomposition of Self-Dual Action in Ashtekar’s Variables

From now on we only consider complex General Relativity with self dual action. We omit the \((+)\) on \( \omega \) and simply write \( \omega \) for the self dual connection. Unless stated otherwise, all the connections and curvatures are self-dual from now on. We have

\[
\omega_{\alpha}^\perp = -\frac{i}{2} \varepsilon_{kl}^{\perp} \omega_{\alpha} \omega_{m} \omega_{m} \omega_{m} \omega_{m} \omega_{m},
\]

\[
\omega_{\alpha}^m = -i \varepsilon_{m}^{\perp} \omega_{\alpha} \omega_{p} \omega_{p} \omega_{p} \omega_{p} \omega_{p},
\]

\[
R_{\alpha}^{\perp m} \omega_{\alpha}^m = -\frac{i}{2} \varepsilon_{l}^{\perp} \omega_{l} \omega_{l} \omega_{l} \omega_{l} \omega_{l} \omega_{l},
\]

which we use to convert all \( R_{\alpha}^{\perp m} \omega_{\alpha}^m \) into \( R_{m}^{\alpha} \omega_{m} \omega_{m} \omega_{m} \omega_{m} \omega_{m} \omega_{m} \) and \( \omega_{\alpha}^\perp \omega_{\alpha}^m \) into \( \omega_{\alpha}^m \omega_{\alpha}^m \) in the expression for the 3+1-decomposed self-dual action. Moreover, we write antisymmetric complex self-dual matrices as 3-dimensional complex vectors, i.e., if \( \lambda^{kl} \) is such a matrix, we set:

\[
\lambda_{m} = -\frac{i}{2} \varepsilon_{m}^{\perp} \lambda_{kl} \lambda_{kl},
\]

\[
\lambda^{kl} = -i \varepsilon^{kl} \lambda_{m} \lambda_{m} = -i \varepsilon^{kl} \lambda_{m} \lambda_{m},
\]

and abbreviate the 3-tuple \((\lambda^1, \lambda^2, \lambda^3)\) by \( \vec{\lambda} \). We also use “scalar-” and “vector-product” notations:

\[
\sum_{k=1}^{3} \lambda^{k} \sigma^{k} =: \vec{\lambda} \cdot \vec{\sigma}
\]

\[
\varepsilon_{n}^{k} \lambda^{n} \sigma^{m} := (\vec{\lambda} \times \vec{\sigma})^{k}.
\]

This allows us to use convenient identities like

\[
\vec{\lambda} \cdot (\vec{\sigma} \times \vec{\rho}) = (\vec{\lambda} \times \vec{\sigma}) \cdot \vec{\rho},
\]

\[
\vec{\lambda} \times (\vec{\sigma} \times \vec{\rho}) = \vec{\rho} (\vec{\lambda} \cdot \vec{\sigma}) - \vec{\sigma} (\vec{\lambda} \cdot \vec{\rho}).
\]

Note that due to the factor of \( i \) (which we included for later convenience) in (5.6), the right hand side of the second formula (5.7) carries the opposite sign to the familiar rule
in vector calculus. Using this notation, we have e.g. \( \lambda^k_l = \delta_{ln} \lambda^{kn} \), that is, indices are raised and lowered with the Kronecker delta)

\[
\lambda^k_l \sigma^l = (\vec{\lambda} \times \vec{\sigma})^k \tag{5.8}
\]

\[
[\lambda, \sigma]^{kl} := (\lambda^k_r \sigma^r_l - \sigma^k_r \lambda^r_l) = \lambda^k_l - \lambda^l_k \tag{5.9}
\]

hence

\[-i \frac{1}{2} \epsilon_{klm} [\lambda, \sigma]^{kl} = (\vec{\lambda} \times \vec{\sigma})^m \tag{5.10} \]

Using these formulae we can simply convert the curvature components into a convenient 3-dimensional notation. The connection becomes a 3-vector valued 1-form \( \vec{\omega} \), and by \( \vec{\omega} \times \vec{\omega} \) we shall denote the vector valued 2-form whose components are \( i \epsilon^{k}_{lm} \omega^l \wedge \omega^m \).

\[
R^{\perp k}_{\mu \nu} = (d\vec{\omega} + \vec{\omega} \times \vec{\omega})^k_{\mu \nu} \tag{5.11}
\]

\[
R^{kl}_{\mu \nu} = -i \epsilon^{kl}_{\mu \nu} (d\vec{\omega} + \vec{\omega} \times \vec{\omega})^{k\mu \nu} \tag{5.12}
\]

\[
R^{\perp k}_{t \nu} = \partial_t \omega^k_{\nu} - \partial_\nu \omega^k_t - 2(\vec{\omega} \times \vec{\omega})^k_t \tag{5.13}
\]

The next step is to bring these expressions into a 3-dimensional covariant form, that is, into a form involving only geometric quantities on the 3-dimensional manifold \( \Sigma \). Note, for example, that the expression on the right hand side of (5.11) differs by a relative factor of 2 from what would be the curvature of a 3-dimensional connection \( \vec{\omega} \); it cannot be, since \( \vec{\omega}_{\mu} \) does not even define a 3-dimensional connection. But \( 2\vec{\omega} \) does! (connections form an affine but not a linear space and cannot generally be added and multiplied with numbers). An elementary way to see this is the following (during this argument we reemploy the old notation using \((+)\omega\): the spatial part of the self-dual projection for the curvature 1-form is given by, \((+\omega)_k^{kl} = \frac{1}{2}(\omega_{k}^{kl} - i \epsilon_{klm} \omega_{m}^{\perp n}). \omega_{k}^{kl} \) is easily seen to form the components of a spatial connection and \( \omega_{m}^{\perp n} = K_{m}^{n}, \) form the components of the 3-dimensional tensor of extrinsic curvature (as we shall independently show below in (7.11-12)). Thus their sum (but not half of it) form a 3-dimensional connection. A geometrically more satisfying argument, which we suppress at this point, can be given using general bundle language. It is however important to realize that the spacetime structure group \( SO(1,3)C \) is reduced to the spatial structure group \( SO(3)C \) which is that subgroup that stabilizes the normal \( \vec{e}_\perp \). It is only with respect to this reduction that the components \( \omega_{\mu}^{kl} \) form a 3-connection and \( \omega_{\mu}^{\perp n} \) a tensor.

We thus define

\[
\vec{A}_{\mu} := 2\vec{\omega}_{\mu} \tag{5.14}
\]

\[
\vec{\Lambda} := -2\vec{\omega}_{t} \tag{5.15}
\]
(the minus sign in (5.15) being chosen for later convenience) and have

\[ R^{\perp \mu \nu}_{\kappa \lambda} = \frac{1}{2} F^\kappa_{\mu \nu} \]  
\[ R^{\perp \mu \nu}_{\kappa \lambda} = -i \frac{1}{2} \epsilon^{\kappa \lambda \rho} F_{\rho \mu \nu}, \]  
where \( F^\mu_{\mu \nu} = (d \vec{A} + \frac{1}{2} \vec{A} \times \vec{A})_{\mu \nu} \)  
and \( R^{\perp \mu \nu}_{\kappa \lambda} = \frac{1}{2} (\partial_\kappa \vec{A}^\nu + D_\nu \Lambda^\kappa) \),

where \( D_\nu \vec{A} := \partial_\nu \vec{A} + \vec{A}_\nu \times \vec{A} \).

We finally also abbreviate \((E^\mu_1, E^\mu_2, E^\mu_3) = \vec{E}^\mu\) and regard the action as functional in the variables \(\vec{A}_\mu, \vec{E}^\mu, N\) and \(N^\mu\). These are Ashtekar’s variables, or connection variables, since \(\vec{A}_\mu\) is a connection 1-form for a \(SO(3)_C\)-connection over the 3-dimensional manifold \(\Sigma\). Given the relations above and the decomposition (4.26) of the action (4.25) (now considered for the complex case), we can write the complex action in the following form:

\[ S^C = \int \left\{ (\partial_t \vec{A}_\mu) \cdot \vec{E}^\mu - \left[ \frac{1}{2} N \vec{F}_{\mu \nu} \cdot (\vec{E}^\mu \times \vec{E}^\nu) + N^\mu \vec{F}_{\mu \nu} \cdot \vec{E}^\nu + \vec{\Lambda} \cdot D_\mu \vec{E}^\mu \right] \right\} dt d^3x, \]  
(5.21)

where the last term has already been integrated by parts without keeping the surface term. The action is now seen to be already in Hamiltonian form, i.e., of the form

\[ \int (\dot{q} \dot{p} - H(q, p)) dt, \]  
(5.22)

thus identifying the canonical conjugate pair of variables as

\[ (\vec{A}_\mu, \vec{E}^\mu) \]  
(5.23)

and the Hamiltonian

\[ H = \int_{\Sigma} \left\{ \frac{1}{2} N \vec{F}_{\mu \nu} \cdot (\vec{E}^\mu \times \vec{E}^\nu) + N^\mu \vec{F}_{\mu \nu} \cdot \vec{E}^\nu + \vec{\Lambda} \cdot (D_\nu \vec{E}^\nu) \right\} d^3x. \]  
(5.24)

In making this direct identification of the canonical variables from the Hamiltonian variational principle we have jumped over a large part of constraints analysis. Note that originally both, the connection and the tetrad, were configuration-space variables and now turn out to be canonical conjugates. This would have been seen in the constraint analysis as an outcome of the explicit reduction of certain second-class constraints (e.g., Romano 1993). On the other hand, we know of the equivalence of the original action with the reformulated action (5.21), which can be employed to circumvent a tedious piece of explicit reduction of second-class constraints by starting directly from (5.21). This leaves us with a purely first-class system, as will be shown in the following chapter.
Chapter 6. Constraint Algebra

If we vary the action with respect to the fields $N$, $N^\mu$ and $\vec{\Lambda}$ we obtain equations in the canonical variables without time derivatives, that is, we obtain the constraints:

\begin{align}
H &:= \frac{1}{2} \vec{F}_{\mu\nu} \cdot (\vec{E}^\mu \times \vec{E}^\nu) = 0, \\
H_\mu &:= \vec{F}_{\mu\nu} \cdot \vec{E}^\nu = 0, \\
\vec{H} &:= D_\mu \vec{E}^\mu = 0.
\end{align}

The Hamiltonian is just the sum of smeared constraints with the smearing functions $N$, $N^\mu$ and $\vec{\Lambda}$:

\begin{equation}
\mathcal{H} = \int_\Sigma \left\{ NH + N^\mu H_\mu + \vec{\Lambda} \cdot \vec{H} \right\} d^3x =: \mathcal{S}[N] + \mathcal{D}[N^\mu] + \mathcal{G}[\vec{\Lambda}],
\end{equation}

where the expressions on the right hand side are just abbreviations for the integrals in their appearing order. The constraint equations on phase space are properly interpreted as saying that for specified test-function spaces for $N$, $N^\mu$ and $\vec{\Lambda}$ the expressions $\mathcal{S}(N)$, $\mathcal{D}(N^\mu)$ and $\mathcal{G}(\vec{\Lambda})$ have to vanish for all test fields $N$, $N^\mu$ and $\vec{\Lambda}$. We call them the scalar-, diffeomorphism-, and gauge-constraints respectively. Note that there are infinitely many of each of them.

The actions of the scalar-, gauge-, and diffeomorphism-constraint on the phase space are easily determined. Let $\{,\}$ denote the Poisson bracket with respect to the symplectic structure given to us with the action (5.21) (in the standard way, namely by telling us that $\vec{A}_\mu$ and $\vec{E}^\mu$ are Darboux coordinates on phase space for the symplectic 2-form):

\begin{align}
\{ \vec{A}_\mu, \mathcal{S}[N] \} &= N \vec{E}^\nu \times \vec{F}_{\mu\nu} \tag{6.5} \\
\{ \vec{E}^\mu, \mathcal{S}[N] \} &= D_\nu (N \vec{E}^\nu \times \vec{E}^\mu), \tag{6.6}
\end{align}

and

\begin{align}
\{ \vec{A}_\mu, \mathcal{G}[\vec{\Lambda}] \} &= -D_\mu \vec{\Lambda} \tag{6.7} \\
\{ \vec{E}^\mu, \mathcal{G}[\vec{\Lambda}] \} &= \vec{\Lambda} \times \vec{E}^\mu, \tag{6.8}
\end{align}

and

\begin{align}
\{ \vec{A}_\mu, \mathcal{D}[N^\mu] \} &= L_N \vec{A}_\mu + \{ \vec{A}_\mu, \mathcal{G}[\vec{A}(N)] \} \tag{6.9} \\
\{ \vec{E}^\mu, \mathcal{D}[N^\mu] \} &= L_N \vec{E}^\mu + \{ \vec{E}^\mu, \mathcal{G}[\vec{A}(N)] \}, \tag{6.10}
\end{align}

where $L_N$ denotes the Lie derivative with respect to the vector field $N^\mu \partial_\mu$. Note that Lie derivatives of densities differ from those of the undensitized quantities by divergence terms in $N^\mu$. The Lie derivative of the connection $\vec{A}_\mu$ is defined componentwise by treating the connection as three one-forms. Note that this definition makes only sense with respect to a global trivialization of the underlying bundle (which we do have here), but not generally. We also set $\vec{A}(N) = \vec{A}_\mu N^\mu$.  

17
The first observation we make is, that due to the simple polynomial form of the Hamiltonian we never had to invert the matrix \( \{e_k^\mu\} \) in order to calculate the Hamiltonian flow. This formally gives rise to the possibility of generalized, degenerate solutions, which have no counterpart in the metric formulation.

Formulae (6.7-10) show that \( \mathcal{G} \) generates \( SO(3)_C \) frame rotations and that \( \mathcal{D} \) generates diffeomorphisms. It might appear that \( \mathcal{D} \) also generates frame rotations, if one defines a “pure” diffeomorphism by having only a Lie derivative on the right hand side of (6.9-10). But such a definition does not make unambiguous sense since it is gauge-dependent. In fact, the right hand side of (6.9-10) is gauge independent. Nevertheless, with respect to a chosen trivialization, such a definition of a pure diffeomorphism is possible and might even be convenient for applications. In the present case, this amounts to a redefinition of the diffeomorphism generator:

\[
H_\mu \mapsto H'_\mu := H_\mu - \vec{A}_\mu \cdot \vec{H}, \quad \text{or} \quad \mathcal{D}'[N^\mu] = \mathcal{D}[N^\mu] - \mathcal{G}[\vec{A}(N)],
\]

so that

\[
\{\vec{A}_\mu, \mathcal{D}'[N^\mu]\} = L_N \vec{A}_\mu \quad \text{(6.9')}
\]

\[
\{\vec{E}_\mu, \mathcal{D}'[N^\mu]\} = L_N \vec{E}_\mu \quad \text{(6.10')}
\]

With a little bit of further work one can calculate the Poisson brackets for all constraint functions:

\[
\{\mathcal{G}[\vec{A}], \mathcal{G}[\vec{M}]\} = \mathcal{G}[\vec{A} \times \vec{M}] \quad \text{(6.12)}
\]

\[
\{\mathcal{G}[\vec{A}], \mathcal{D}[N^\mu]\} = 0 \quad \text{(6.13)}
\]

or

\[
\{\mathcal{G}[\vec{A}], \mathcal{D}'[N^\mu]\} = -\mathcal{G}[L_N \vec{A}] \quad \text{(6.13')}
\]

\[
\{\mathcal{G}[\vec{A}], \mathcal{S}(N)\} = 0 \quad \text{(6.14)}
\]

\[
\{\mathcal{D}[N^\mu], \mathcal{D}[M^\mu]\} = \mathcal{D}[L_N M^\mu] - \mathcal{G}[\vec{F}(N, M)] \quad \text{(6.15)}
\]

or

\[
\{\mathcal{D}'[N^\mu], \mathcal{D}'[M^\mu]\} = \mathcal{D}'[L_N M^\mu] \quad \text{(6.15')}
\]

\[
\{\mathcal{D}[N^\mu], \mathcal{S}[M]\} = \mathcal{S}[L_N M] + \mathcal{G}[MN^\mu \vec{F}_{\mu\nu} \times \vec{E}^\nu] \quad \text{(6.16)}
\]

or

\[
\{\mathcal{D}'[N^\mu], \mathcal{S}[M]\} = \mathcal{S}[L_N M] \quad \text{(6.16')}
\]

\[
\{\mathcal{S}[N], \mathcal{S}[M]\} = \mathcal{D}[K^\mu] = \mathcal{D}'[K^\mu] + \mathcal{G}[\vec{A}(K)] \quad \text{(6.17)}
\]

where

\[
K^\mu := \vec{E}^\mu \cdot \vec{E}^\nu (M \partial_\nu N - N \partial_\nu M). \quad \text{(6.18)}
\]

First of all, we observe that the right sides are all proportional to constraint functions with coefficients that generally depend on phase space. Phase space dependencies appear in (6.15)(6.16) and (6.17). In the first two cases these can be removed by using \( \mathcal{D}' \) instead of \( \mathcal{D} \) (equations (6.15') and (6.16')). So the only phase space dependence appears in the Poisson bracket of two scalar constraints in the bilinear combination \( \vec{E}^\mu \cdot \vec{E}^\nu \) on the
right hand side of (6.17). This prevents $\mathcal{S}, \mathcal{D}, \mathcal{G}$ from generating an (infinite dimensional) Lie algebra. For example, we cannot calculate higher Poisson brackets using only the relations (6.12-17), since we would need to know their Poisson-bracket with the term (6.18), which cannot be inferred from the relations (6.12-17). For this we would need to know the explicit functional forms of the generators.

Relation (6.17) also tells us that the scalar constraints do not close onto themselves, as it is the case with the $\mathcal{G}$ and the $\mathcal{D}'$ generators individually. Moreover, the Poisson brackets for $\mathcal{G}$ with any other generator is proportional to $\mathcal{G}$. We say that $\mathcal{G}$ generates an ideal whereas the $\mathcal{D}'$ only generate a subalgebra (note the abuse of language here). For them to form an ideal too we should have the right hand side of (6.16') to be proportional to a diffeomorphism-, not a scalar-constraint. The geometric meaning of $\mathcal{G}$ generating an ideal is the following: the phase space flow generated by $\mathcal{D}$ (or $\mathcal{D}'$) and $\mathcal{S}$ restricted to the constraint hypersurface $\mathcal{G} = 0$ is tangential to this surface and therefore leaves it invariant. This allows us to reduce the constraints in steps, that is, to first go to the $\mathcal{G} = 0$ hypersurface and then reduce the action generated by all other constraints. That such a separate phase space reduction for the gauge generators exists is clear from the existence of a formulation of the theory in terms of gauge invariant variables, like the metric formulation. However, a similar procedure with the diffeomorphism constraint is impossible, since its flow does not leave the $\mathcal{S} = 0$ hypersurface invariant. On the other hand, for example, if one were to determine functions in phase space (or subsets thereof) which are invariant under the action of the constraints, it would be sufficient to look for functions annihilated by a generating set of constraints. But due to the intertwinment of diffeomorphism- with scalar-constraints, generating sets much smaller than the defining set may actually be found. For example, functions on the constraint hypersurface $\mathcal{S}[N] = 0 = \mathcal{D}[N^\mu]$, with zero derivative along the Hamiltonian vector fields of $\mathcal{S}[N]$ for all $N$, have also zero derivative along the Hamiltonian vector fields of $\mathcal{D}[K^\mu]$, due to (6.17-18). But it is easy to see that every vector field can locally be written as sum of vector fields of the form (6.18) so that $\mathcal{S}[N]$-invariant functions are automatically $\mathcal{D}[N^\mu]$-invariant. Similar observations of this type where first made in (Moncrief 1972) and (Moncrief and Teitelboim 1972).
Chapter 7. Reality Conditions and Geometric Interpretation

So far we have reformulated complex General Relativity. But, of course, ultimately we are interested in the ordinary real case. A central question therefore is how to recover the real theory within the formalism developed, that is, how to derive a real 3-metric from the complex equations of motion for $\vec{A}_\mu$ and $\vec{E}_\mu$ with real lapse and shift functions $N$ and $N^\mu$. In the real case it is also easy to give some geometric interpretation to the variables $\vec{A}_\mu$ and $\vec{E}_\mu$. We shall briefly discuss these points in this chapter.

As outlined in chapter 3, we are working on complex tensor bundles with reality structure. Real sections have real component functions with respect to real bases and real bases are provided since we work on a real manifold. Note that we only require the 3-metric and not the (densitized) tetrad to be real. The former is derived from the latter according to:

$$h_{\mu\nu} = \frac{1}{h} \sum_{k=1}^{3} \vec{E}_\mu \cdot \vec{E}_\nu,$$

so that

$$h^{\mu\nu} = \left[ \det \{ \vec{E}_\mu \cdot \vec{E}_\nu \} \right]^{-\frac{1}{2}} \vec{E}_\mu \cdot \vec{E}_\nu. \quad (7.1)$$

Suppose now that $\vec{E}_\mu \cdot \vec{E}_\nu$ is real. The Hamiltonian flow should not disturb this reality, that is

$$\{ \vec{E}_\mu \cdot \vec{E}_\nu, S[N] + D'[N^\mu] + G[\vec{A}] \} = \text{real}, \quad (7.3)$$

for all $\vec{A}$ and all real $N$ and $N^\mu$. The gauge generator Poisson-commutes with $\vec{E}_\mu \cdot \vec{E}_\nu$ (compare (6.8)) and therefore drops out of discussion. The diffeomorphisms generator simply yields (compare (6.10'))

$$\{ \vec{E}_\mu \cdot \vec{E}_\nu, D'[N^\mu] \} = L_N \left( \vec{E}_\mu \cdot \vec{E}_\nu \right) \quad (7.4)$$

which is real for real $N^\mu$ (the components with respect to the real basis $\partial_\mu$). So the diffeomorphism generator as well does not disturb reality. We can thus focus on the scalar constraint. We have

$$\{ \vec{E}_\mu \cdot \vec{E}_\nu, S[N] \} = \left[ \vec{E}_\mu \cdot (\vec{E}_\nu \times \vec{E}_\sigma) + \vec{E}_\nu \cdot (\vec{E}_\mu \times \vec{E}_\sigma) \right] \partial_\sigma N$$

$$+ N \left( D_\sigma \vec{E}_\sigma \right) \cdot (\vec{E}_\mu \times \vec{E}_\nu + \vec{E}_\nu \times \vec{E}_\mu)$$

$$+ N \left[ \vec{E}_\sigma \cdot (\vec{E}_\mu \times D_\sigma \vec{E}_\nu + \vec{E}_\nu \times D_\sigma \vec{E}_\mu) \right]. \quad (7.5)$$

The first two expressions vanish identically, so that the resulting reality condition is:

$$\left( \vec{E}_\sigma \times D_\sigma \vec{E}_\nu(\mu) \right) \cdot \vec{E}_\nu = \text{real}. \quad (7.6)$$

20
One may check that there are no further constraints, i.e., the Poisson bracket of this expression with the Hamiltonian is real due to the reality conditions already posed. So we have the result that the Hamiltonian flow preserves reality provided the initial data are further constrained by the reality of $h_{\mu\nu}$ and (7.6).

Now we turn to the geometric interpretation of the canonical variables. We begin with the momenta $\vec{E}^\mu$. We write (7.1) in the form

$$\det\{h_{\mu\nu}\}h^{\mu\nu} = \vec{E}^\mu \cdot \vec{E}^\nu$$

(7.7)

and consider e.g. the 3-3 component of this equation. The left hand side is just the subdeterminant $h_{11}h_{22} - h_{12}^2$ which for real $h_{\mu\nu}$ measures the area content of the parallelogram in tangent-space spanned by the vectors $\partial_1$ and $\partial_2$. The area content of an embedded 2-surface ( coordinatized by $\sigma$ and $\tau$):

$$(\sigma, \tau) \mapsto x^\mu(\sigma, \tau),$$

(7.8)

is thus given by (no summation convention here):

$$\int \sum_{\lambda, \mu, \nu} \sqrt{\vec{E}^\lambda \cdot \vec{E}^\lambda} \varepsilon_{\lambda \mu \nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \sigma} \ d\tau d\sigma,$$

(7.9)

where the integral is taken over the surface. $\varepsilon_{\lambda \mu \nu}$ denotes the 3-form density on $\Sigma$ of weight $-1$, whose components in all coordinate systems are 0 if two indices coincide and $\pm 1$ if they form an even, respectively odd permutation of 123. In that sense the variables $\vec{E}^\mu$ are geometrically connected to areas rather than length.

Next we consider the variables $\vec{A}_\mu$. For this, we restore our old notation using $\omega$ for the connection 1-form and $^{(+)}\omega$ for the self-dual part, and have

$$A^k_\mu = 2^{(+)}\omega^k_\mu = -i\frac{1}{2} \epsilon^{\perp k}_{\ l}m (\omega^l_\mu - i \epsilon^l m_{\perp p} \omega^p_\mu).$$

(7.10)

If the equations of motion are satisfied we have $\omega = \Gamma$ (the Levi-Civita connection). Let its spacetime covariant derivative be $^{(4)}\nabla$ and $^{(3)}\nabla$ its induced space covariant derivative on $\Sigma$ (compatible with $h_{\mu\nu}$). We recall the definition of extrinsic curvature:

$$^{(4)}\nabla_{e_k} e_l = :^{(3)}\nabla_{e_k} e_l + K_{kl} e_\perp$$

(7.11)

and have $\omega^{\perp k} = K^{k}_\mu = \delta^{kl} e^l_\mu K_{k l}$. 

(7.12)

Inserting this into (7.10) yields

$$A^k_\mu = \Gamma^k_\mu + K^k_\mu,$$

(7.13)

where $\Gamma^k_\mu := -i\frac{1}{2} \epsilon^{\perp k}_{\ l}m \Gamma^m_{\mu \perp}$.

(7.14)
The spatial components, $\Gamma^{ln}_{\mu}$, are of course just the components for the Levi-Civita connection on $\Sigma$ compatible with $h_{\mu\nu}$. Relation (7.13) clearly shows the non-triviality of the transformation to the connection variables: the connection 1-form $\vec{A}$ involves the metric and the extrinsic curvature and therefore both conjugate sets of variables of the metric formulation. We also note that (7.13) gives rise to an alternative way of expressing a reality condition which proves useful in a variety of applications, in particular when compared with the standard metric formulation. The reality of $K^k_{\mu}$ and $\Gamma^{ln}_{\mu}$ (hence $\Gamma^k_{\mu}$ is purely imaginary) implies (a bar denotes complex conjugation)

$$A^k_{\mu} - \bar{A}^k_{\mu} = 2\Gamma^k_{\mu}. \quad (7.15)$$

Finally, we point out some further interesting decomposition of the connection 1-form $\vec{A}$. First of all, using the dual basis $\{\theta^k\}$ of $\{e_k\}$ on $\Sigma$ (as introduced before equation (4.2)), we have $A^k = A^k_{\mu} dx^\mu = A^k_\mu \theta^\mu$. The latin indices are now raised and lowered using $\delta_{kl}$, i.e., $A_{kl} = \delta_{kn} A^n_l$. Correspondingly, (7.13) reads: $A_{kl} = \Gamma_{kl} + K_{kl}$. Due to the symmetry of $K_{kl}$ (which follows from (7.11) and the vanishing torsions of $^{(4)}\nabla$ and $^{(3)}\nabla$) the antisymmetric part of $A_{kl}$ just involves $\Gamma_{kl}$. Defining the quantity $\Gamma_l := \frac{i}{2} \epsilon^{nm} \Gamma_{nm}$, which is real, we have

$$A_{kl} = A_{(kl)} - i \epsilon_{kl} \Gamma_n = \Gamma_{(kl)} + K_{kl} - i \epsilon_{kl} \Gamma_n. \quad (7.16)$$

We further set

$$A := \delta^{kl} A_{kl} = \delta^{kl} \Gamma_{(kl)} + \delta^{kl} K_{kl} =: \Gamma + K, \quad (7.17)$$

where $K$, the trace of the extrinsic curvature, is zero, iff $\Sigma$ is a maximal hypersurface (i.e. a stationary point of the area functional). Using the first Cartan structure equation for the basis $\{\theta^k\}$ and the torsion-free Levi-Civita connection on $\Sigma$, one easily shows the following identities:

$$\Gamma_k \theta^k = \frac{1}{2} \ast \delta_{kl} (\theta^k \wedge \ast d\theta^l), \quad (7.18)$$

$$\Gamma = \frac{1}{2} i \ast \delta_{kl} (\theta^k \wedge d\theta^l), \quad (7.19)$$

where $\ast$ is now the 3-dimensional Hodge-duality map on $\Sigma$ (defined analogously to (1.20)). $\Gamma_k \theta^k$ and $\Gamma$ are a frame dependent 1-form and a frame dependent function respectively (recall the transformation law for connections (1.19)) and have been employed in (Nester 1989) to fix the gauge freedom in the choice of frames by imposing the gauge conditions:

$$d(\Gamma_k \theta^k) = 0, \quad (7.20)$$

$$d\Gamma = 0. \quad (7.21)$$
If we insert the Levi-Civita connection with respect to an arbitrary rotated frame by using formula (1.19) to parameterize it by the rotation matrix $R$, these conditions can be seen to form a system of non-linear elliptic differential equations for $R$ which under certain circumstances fixes it up to a constant rotation. Since we assume $\Sigma$ to be connected, (7.20) is clearly equivalent to $\Gamma = \text{const.}$. If we make the additional assumption that $\Sigma$ has trivial first DeRahm Cohomology (equivalently, the space of harmonic 1-forms is zero dimensional. A sufficient condition for this is that $\Sigma$ has finite fundamental group.), then all closed 1-forms are exact so that $\Gamma_k = e_k(\rho)$ for some globally defined function $\rho$ on $\Sigma$. In this integrated form the gauge conditions have been shown to be equivalent to a massive 2-spinor equation (the mass being given by the value $\Gamma$), which is linear and elliptic (Dimakis and Müller-Hoissen 1989). For asymptotically flat $\Sigma$, the fall-off conditions require $\Gamma = 0$, in which case existence and uniqueness of the massless spinor equation has been demonstrated, up to possible isolated singularities, in (Ashtekar and Horowitz 1983). For a neighborhood of flat 3-space, existence and uniqueness can be shown directly (Nester 1989, also Nester 1992).

Another interesting question that plays a significant rôle in the investigation of the initial data problem is, what spacetimes can be evolved from maximal (i.e. $K = 0$) hypersurfaces $\Sigma$. It has been shown (Bartnik 1984) that all spacetimes sufficiently close (in a sense specified by Bartnik) to Minkowski space allow such maximal slicings. Conversely, it is also known that there are even topological restrictions on $\Sigma$ in order to allow for any maximal initial data (Witt 1987). The manifolds ruled out are those which do not allow for metrics with everywhere positive Ricci scalar (as is directly seen from the Hamiltonian constraint in the standard metric representation). They all have infinite fundamental group.

At the end of the next section we will show how the special gauge conditions (7.20-21) can be used to prove a restricted version of the positive mass theorem which applies to maximal $\Sigma$ (i.e. spacetimes admitting such hypersurfaces).
Chapter 8. Surface Integrals

If the spatial manifold $\Sigma$ is not closed one has to specify fall-off conditions for the field $\vec{A}_\mu$ and $\vec{E}_\mu$. These must be chosen wide enough to accommodate physically interesting solutions, with, say, non-vanishing energy, momentum and angular momentum etc. at spatial infinity (we call these quantities the Poincaré charges). In this case the Hamiltonian must be amended by surface integrals in order to make it a continuously differentiable function on phase space. Note that the point is not that otherwise we would get the wrong equations of motion, but rather that we would not get any equations at all. Also, the variational principle need a careful restatement in order to ensure stationary points in the right class of functions. However, in order to arrive at the right Hamiltonian it is not necessary to restart from a modified variational principle to calculate the modified Hamiltonian by keeping all non-vanishing boundary integrals. A much simpler strategy is to first calculate a formal Hamiltonian “brute force” without keeping track of any surface integrals, and then amend it “by hand” with the right surface terms that make it a continuously differentiable function. For the metric formulation of General Relativity this is explained and carried through in (Regge and Teitelboim 1979) and in more detail in (Beig and Ó Murchadha 1987), which also corrects a mistake in the first reference concerning the generator of boosts. Using the connection variables, an analysis similar to the one by Beig and Ó Murchadha was performed in (Glößner 1992) and also in (Thiemann 1993). An alternative approach not within the canonical framework is presented in (Ashtekar and Romano 1993). Here we cannot give the full details of the general procedure but only want to outline the underlying ideas.

We already have the (formal) Hamiltonian without the surface terms. Let us restrict to $\vec{A} = 0 = N^\mu$ so that the (formal) Hamiltonian equals the scalar constraint:

$$S[N] = \frac{1}{2} \int_\Sigma N F_{\mu\nu} \cdot (\vec{E}_\mu \times \vec{E}_\nu) \, d^3x .$$  \hspace{1cm} (8.1)

This expression is clearly differentiable with respect to $\vec{E}_\mu$ (since it enters undifferentiated), but if we try to differentiate it with respect to $\vec{A}_\mu$ we encounter the surface integral

$$- \int_{S_\infty} N \vec{A}_\mu \cdot (\vec{E}_\mu \times \vec{E}_\nu) \, d\sigma_\nu ,$$ \hspace{1cm} (8.2)

where $S_\infty$ is the 2-sphere at spatial infinity. The integral over it is understood in the usual way: take the integral over 2-spheres in the finite, such that the result depends on the radius, and then take the limit as the radius approaches infinity. Here we assume of course that it be finite. Now, if this surface term is not zero, the functional $S[N]$ is not differentiable with respect to $\vec{A}_\mu$. We restore differentiability by simply subtracting (8.2)
from (8.1) and obtain:

\[ \mathcal{H}_s[N] := \frac{1}{2} \sum \left( N \vec{F}_{\mu\nu} \cdot (\vec{E}^\mu \times \vec{E}^\nu) \right) d^3x + \int_{S_\infty} N \vec{A}_\mu \cdot (\vec{E}^\mu \times \vec{E}^\nu) d\sigma_\nu, \]  

(8.3)

which we call the scalar Hamiltonian. The constraints are still given by (6.1), or, in smeared form, by the requirement that the first term in (8.3) vanishes. On the constraint surface the scalar Hamiltonian is thus non-vanishing and assumes the values of minus the surface integral (8.2). In the non-closed case the Hamiltonian is not the sum of the constraints and does not vanish on the constraint surface. Rather, it defines non-trivial functions which descend to functions on the reduced phase space, i.e., to non-trivial observables. For the case at hand one expects this quantity to be the total energy of the isolated system described by asymptotically flat data, and one may wonder whether the quantity above is actually real. This can be quite easily demonstrated. But to do so, we have to first make slightly more precise what we define as asymptotically flat data. Asymptotically we require \( N(r \to \infty) \to \text{const.} \) (so we leave out boosts from the discussion given here since for them \( N \propto x^\mu \) for large \( r \)).

\[ E_\mu^k = \delta_\mu^k + \frac{\alpha_\mu^k(\theta, \varphi)}{r} + O\left( \frac{1}{r} \right), \]  

(8.4)

\[ A_\mu^k = \frac{\beta_\mu^k(\theta, \varphi)}{r^2} + O\left( \frac{1}{r^2} \right), \]  

(8.5)

where \( \alpha_\mu^k \) and \( \beta_\mu^k \) are functions on the 2-sphere which also obey the conditions to be of even-, respectively odd parity under the antipodal map of the sphere. We refer to (Beig, Ó Murchadha 1987) for a discussion of these conditions in the metric case. \( O\left( \frac{1}{r^n} \right) \) stands for terms with asymptotic radial fall-off faster than \( \propto 1/r^n \). Using formulae (7.13-14) we can write the surface contribution to (8.3) “on shell” (i.e., the equations of motion being satisfied) in the form

\[ \int_{S_\infty} N E_\mu^k E_\nu^l \left( \Gamma^p_{kl} - \iota \varepsilon^{kl}_{\perp p} K^p_{l} \right) d\sigma_\mu. \]  

(8.6)

In a real basis the first term is real and the second purely imaginary. But the second vanishes due to the symmetry of \( K_{kl} \) and the asymptotic conditions (8.4). The first term can be shown to be just one-half times the ADM energy expression (which means that the properly normalized action is twice ours in units where \( 16\pi G/c^4 = 1 \), as was already remarked in the discussion following equation (3.21)).

Let us also have a look at the diffeomorphism constraint. In the same way as above one immediately sees that differentiability with respect to \( \vec{A}_\mu \) is obstructed by the surface integral

\[ -2 \int_{S_\infty} N^{[\mu} \vec{E}_{\nu]} \cdot \vec{A}_\mu d\sigma_\nu, \]  

(8.7)
which we must subtract to obtain the differentiable diffeomorphism-Hamiltonian

\[
\mathcal{H}_d[N^\mu] := \int_\Sigma N^\mu \bar{F}_{\mu\nu} \bar{E}^\nu \, d^3x + 2 \int_{S_\infty} N^{[\mu} \bar{E}^{\nu]} \cdot \bar{A}_\mu \, d\sigma_\nu .
\] (8.8)

Note that due to the parity conditions on \( \beta^k_\mu \) the surface integral is also finite for asymptotic rotations, where \( N^\mu \propto \varepsilon_{\mu\nu\lambda} n^\nu x^\lambda \) with constant \( n^\mu \). “On shell”, (8.8) becomes

\[
\mathcal{H}_d[N^\mu] = 2 \int_{S_\infty} (K^k_\nu - i e^{\frac{1}{\ell m} \Gamma_{\ell m}} \delta^k_\nu K_\lambda^\lambda) N^{[\mu} \bar{E}^{\nu]} \, d\sigma_\mu .
\] (8.9)

Again it can be shown that the imaginary part vanishes. The real part is

\[
- \int_{S_\infty} N^\nu (K^\mu_\nu - \delta^\mu_\nu K^\lambda_\lambda) \, d\sigma_\mu
\] (8.10)

which is again one-half times the ADM-expression for momentum and angular momentum.

Let us finally mention the gauge constraint. Here the only surface contribution comes from the requirement of differentiability with respect to \( \bar{E}^\mu \) and is given by:

\[
\int_{S_\infty} \bar{\Lambda} \cdot \bar{E}^\mu \, d\sigma_\mu .
\] (8.11)

But here one probably does not want to have an extra charge associated with the asymptotic gauge (frame) rotations for which the following fall-off is sufficient (\( n_\mu \) is the normal to the asymptotic 2-sphere)

\[
\Lambda^k \delta^\mu_\nu n_\mu \propto O \left( \frac{1}{r^2} \right)
\] (8.12)

Essentially the same condition follows, for example, from the requirement of invariance of angular momentum under gauge transformations, i.e., frame rotations. Since the Poincaré charges are given by integrals with non manifestly gauge-invariant integrands (due to the explicit appearance of \( \bar{A}_\mu \)) one generally has to explicitly check compatibility of the chosen fall-off conditions with the requirement of gauge invariance.

We remark that it is indeed possible to write down surface integrals that make the Hamiltonian continuously differentiable and where the lapse and shift function spaces are chosen wide enough to accommodate for all asymptotic Poincaré charges. As expected, it turns out that the boost generator with asymptotic behaviour \( N \propto x^\mu \) requires the most subtle discussion to establish finiteness and differentiability. This is as in the metric case – compare (Beig and Ó Murchadha) –, and has in fact a similar solution (Glößner 1992). There is thus an important difference between the closed and open case. In the
former, all of dynamics is entirely generated by constraints, whereas in the latter the constraints generate only asymptotically (at spatial infinity) trivial changes, in contrast to the Hamiltonian, which also generates dynamics on quantities at infinity. No effective changes remain in the closed case after having divided out those generated by the constraints. In contrast, in the asymptotically flat case, there are still residual motions which are generated by the Poincaré group.

We want to end this section by demonstrating positivity of the surface integral in (8.3) for the special cases where $\Sigma$ is maximal (i.e. $K=0$). In doing this we essentially follow (Nester 1994) with some adaptations in notation. The idea is to rewrite the scalar Hamiltonian (8.3) into a form in which positivity properties can be deduced, possibly by suitably choosing the lapse density $N$ in the interior of $\Sigma$. But since on the constraint set the surface integral equals in value the scalar Hamiltonian and is independent of the lapse density in the interior of $\Sigma$, this will also show positivity of the surface integral. We start by simply rewriting the scalar Hamiltonian (8.3) in terms of forms by using the dual basis $\{\theta^k\}$, equation (4.19) and $\sqrt{h} d^3 x = \theta^1 \wedge \theta^2 \wedge \theta^3$:

$$H_s[N] = i \int_{\Sigma} N \sqrt{h} \delta_{kl} F^k \wedge \theta^l - i \int_{S_{\infty}} N \sqrt{h} \delta_{kl} A^k \wedge \theta^l .$$ (8.13)

Note that $N \sqrt{h}$ is a scalar function and equal to the lapse before the redefinition (4.21) was made. Converting the surface integral into a volume integral and expanding the forms in terms of $\theta^k$s yields

$$H_s[N] = \frac{1}{2} \int_{\Sigma} N \sqrt{h} (A^{kl} A_{lk} - A^2) \sqrt{h} d^3 x$$

$$- \int_{\Sigma} N \sqrt{h} (A^{kl} \Gamma_{lk} - A \Gamma) \sqrt{h} d^3 x$$

$$- \int_{\Sigma} \epsilon^{klm} e_k (N \sqrt{h}) i A_{lm} \sqrt{h} d^3 x .$$ (8.14)

using the notation defined at the end of the previous section. We now impose the reality condition (7.15), just transcribed to the components $A_{kl}$ with respect to the real basis $\{\theta^k\}$, and decompose $A_{kl}$ into symmetric and antisymmetric part according to (7.16) and obtain the real expression:

$$H_s[N] = \frac{1}{2} \int_{\Sigma} N \sqrt{h} \left( \bar{A} (kl) A_{(kl)} - \bar{A} A \right) \sqrt{h} d^3 x$$

$$+ \int_{\Sigma} N \sqrt{h} \Gamma_k \Gamma^k \sqrt{h} d^3 x$$

$$- 2 \int_{\Sigma} \epsilon_k (N \sqrt{h}) \Gamma^k \sqrt{h} d^3 x .$$ (8.15)
Finally, we impose the special orthonormal frame gauge (7.20-21) in integrated form: Σ = 0 and Σ_k = 2e_k(ln(λ)), where the way of writing the gradient is just chosen for convenience with an everywhere positive and asymptotically constant function λ. We then arrive at

\[ H_s[N] = \frac{1}{2} \int_{\Sigma} N \sqrt{h} \left( \tilde{A}^{(kl)} A_{(kl)} - K^2 \right) \sqrt{h} d^3x \]

\[ - 4 \int_{\Sigma} \delta^{kl} e_k \left( \frac{N \sqrt{h}}{\lambda} \right) e_l(\lambda) \sqrt{h} d^3x. \]  

(8.16)

The second integral can be made to vanish with the choice \( N \sqrt{h} = \lambda \), in which case the Hamiltonian simply reads

\[ H_s[\lambda] = \frac{1}{2} \int_{\Sigma} \lambda \left( \tilde{A}^{(kl)} A_{(kl)} - K^2 \right) \sqrt{h} d^3x. \]  

(8.17)

For maximal hypersurfaces Σ one has K = 0 and the integrand is non-negative and zero only for \( A_{(kl)} = 0 \), which implies (real and imaginary part vanish separately) \( \Gamma_{(kl)} = 0 \) and \( K_{kl} = 0 \). Geometrically, the latter equation just says that Σ is totally geodesic.
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