Transfer of quantum states using finite resources

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We discuss the problem of transferring a qubit from Alice to Bob using a noisy quantum channel and
only finite resources. As the basic protocol for the transfer we apply quantum teleportation. It turns
out that for a certain quality of the channel direct teleportation combined with qubit purification
is superior to entanglement purification of the channel. If, however, the quality of the channel is
rather low one should simply apply an estimation-preparation scheme.

I. INTRODUCTION

Ideal quantum information protocols \(1\) do usually not consider the resources needed for performing a certain
task. However, in quantum information processing the question of resources automatically comes into play. The
reason for this is that the measurement process is an important part of all considerations and each quantum
measurement changes the state of the system. Hence a quantum measurement sequence is fundamentally differ-
ent from a classical one used in classical computing (e.g. readout of a quantum register versus readout of a clas-
sical register): Collecting information about a quantum system \(2\) usually requires the preparation of an ensemble
of those systems. The accuracy needed for this information typically determines the size of the ensemble, that
is, the resources for this task.

Typical examples for this situation have been discussed recently: Quantum state estimation \(3\) asks for the
optimal exploitation of \(N\) identically prepared quantum systems in order to find their state. This problem is
closely related to optimal quantum cloning \(4,7\) which considers the map of a finite resource of \(N\) qubits onto
\(M\) clones. Furthermore, the question of finite resources and accuracy poses itself when one investigates optimal
frequency standards \(8\) and optimal quantum clocks \(9\).

In the present paper we shall discuss the transfer of quantum states from Alice to Bob under the assump-
tion that they have only finite resources at hand. This problem becomes very important when one considers con-
cepts of distributed quantum computing \(10\) where quantum computations are performed by spatially separated
quantum processors that communicate with each other via quantum channels. The ideal solution to the state-
transfer problem is known: quantum teleportation \(11\). If Alice and Bob share a perfect Bell state, they can apply
this protocol and Bob will perfectly receive Alice’s qubit. As a resource they just need the Bell state which
sets up the quantum channel between the two parties.

This ideal situation changes drastically when the quantum channel becomes noisy, that is, when Alice and Bob
only share a mixed entangled state. Then it turns out that they have different possibilities to transfer a qubit.

In the following we shall investigate three versions. First, they can start from a resource of \(N\) non-maximally en-
tangled states and purify \(\mathbb{I}\) them until they find a “relatively pure” state for teleportation. Second, they
can transfer the qubits using the teleportation protocol with the \(N\) unpurified entangled states. The question
then is whether Bob can extract the original qubit from the \(N\) mixed qubits he gets. For this purpose he can use a qubit purification protocol \(12,13\). Third, Alice and Bob can perform an obvious protocol. Alice simply esti-
mates \(\mathbb{I}\) \(\mathbb{I}\) the state of the qubit from the \(N\) systems she possesses. Then she classically sends this information to
Bob who can use it to prepare a corresponding qubit. We shall compare and contrast these methods by calculating
the average fidelities in each case.

The paper is organized as follows: In Sec. II we present the physical scenario that we want to investigate and in-
roduce the corresponding notation. Sec. III contains the description of the three different methods for the
quantum-state transfer. These methods are then compared with each other in Sec. IV. We conclude with Sec.
V.

II. TRANSFER OF QUANTUM STATES

We shall investigate the following scenario: Imagine that Alice wants to send an unknown quantum state \(|\psi\rangle\),
which will be a qubit state in our case, to Bob. The state can be parameterized in terms of the basis states \(|0\rangle\) and
\(|1\rangle\) as

\[
|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle
\]

with Bloch angles \(\theta \in [0, \pi]\) and \(\phi \in [0, 2\pi]\). At this point the question arises which type of quantum chan-
nel the two parties should use. In the present paper we shall consider quantum teleportation as a possible can-
didate. We assume that Alice and Bob possess all the necessary ingredients for doing quantum state teleporta-
tion \(11\). If these ingredients were perfect we could have an ideal transmission scheme: Alice and Bob set up the
quantum channel using a single copy of a maximally en-
tangled state, i.e., they use one of the four Bell states.
Then Alice sends her copy of \(|\psi\rangle\) to Bob according to the
well-known protocol of quantum state teleportation \(11\).
In the present paper, however, we assume that the channel is noisy and hence the corresponding entangled pair of qubits (ebit) is not given by one of the Bell states but by a density operator. Reasons for this non-ideal entanglement could be decoherence, an imperfect ebit source or imperfect transfer channels from the ebit source to Alice and Bob.

Consequently, the state of Bob’s qubit will differ from $|\psi\rangle$ due to the imperfect ebit and has to be described by a mixed state. However, as Bob wants to use his qubits for further experiments his aim will be to possess at least one qubit in a state as close as possible to the the initial state $|\psi\rangle$. In order to achieve this goal Alice can send not only one copy of $|\psi\rangle$ but has $N$ copies available which she can send to Bob. This, of course, also requires that Alice and Bob can set up the channel with $N$ ebits. Furthermore, Bob is allowed to perform any operations on his qubits. His final state will then be called $\hat{\rho}_B$.

The quality of Bob’s state will be measured in terms of the fidelity $\langle \psi | \hat{\rho}_B | \psi \rangle$ which quantifies the overlap between the initial and the final state. Our question therefore is how we can improve the fidelity using the given finite resources, i.e., using the $N$ ebits for the channel and the $N$ copies of $|\psi\rangle$ which Alice can send.

Before we consider the different strategies with which Bob can improve the state $\hat{\rho}_B$, i.e., improve the fidelity, let us first look at Bob's state resulting from a teleportation with a mixed ebit state. We assume that the ebit setting up the channel is in a Werner state $|\lambda\rangle$,

$$\hat{\rho}_\lambda = \lambda |\phi^+\rangle \langle \phi^+| + \frac{1 - \lambda}{3} (|\phi^+\rangle \langle \phi^-| + |\phi^-\rangle \langle \phi^-| + |\psi^-\rangle \langle \psi^-|)$$

with the parameter $\lambda \in [1/4, 1]$ and the four Bell states $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$. The parameter $\lambda$ defines the quality of the channel. For $\lambda = 1$ we recover the ideal teleportation again whereas for $\lambda = 1/4$ the Werner state is completely mixed and consequently no teleportation is possible. Note that the assumption of a Werner state does not restrict the generality of our results because any ensemble of quantum states can be converted into a Werner state by applying local random rotations onto the two qubits of the ebit.

As a result of the teleportation Bob will get the state

$$\hat{\rho}_B = \frac{1 + 2\lambda}{3} |\psi\rangle \langle \psi| + \frac{2}{3} (1 - \lambda) |\bar{\psi}\rangle \langle \bar{\psi}|,$$

where the state

$$|\bar{\psi}\rangle = \frac{\sin \frac{\theta}{2} |0\rangle - \cos \frac{\theta}{2} e^{i\phi} |1\rangle}$$

is orthogonal to $|\psi\rangle$. That is, we find a classical mixture of $|\psi\rangle$ and $|\bar{\psi}\rangle$ in which the desired $|\psi\rangle$ component prevails if the ebit parameter fulfills the condition

$$\lambda > \lambda_{\text{crit}} = \frac{1}{4}.$$  

From this representation we can easily determine the fidelity

$$F_B(\lambda) = \langle \psi | \hat{\rho}_B(\lambda) | \psi \rangle = \frac{2\lambda + 1}{3}.$$  

of Bob’s output state. This fidelity is of course the fidelity that we get if we run the imperfect teleportation apparatus only once. However, we have $N$ ebits and $N$ qubits at our disposal and thus can perform the teleportation at most $N$ times. Therefore, we can ask the question: What is the best way to use our finite resources to achieve an output state $\hat{\rho}_B$ as close as possible to the initial state $|\psi\rangle$? The goal of the following sections will be to discuss and compare several methods designed for this purpose.

### III. METHODS TO IMPROVE THE TRANSFER FIDELITY

In this section we will discuss three methods with which Alice and Bob can improve the fidelity of the final output state $\hat{\rho}_B$. To some of these methods there exist a number of related versions. We will, however, concentrate on only one specific method in each case.

#### A. Entanglement purification

The first strategy to improve the transport quality of $|\psi\rangle$ is to use one of the existing entanglement purification protocols to get one highly entangled pair of qubits out of the available $N$ mixed ebits. This final pair of qubits can then be used to teleport $|\psi\rangle$ from Alice to Bob. As an example of such a protocol we will use the one proposed by Deutsch et al. [12], which for Werner states yields the same results as the original entanglement purification protocol [13], but is conceptually simpler and can also be applied to ebits in a general Bell-diagonal state. This purification scheme works for all Werner states with $\lambda > 1/2$. In addition, the purification protocol is quite simple with respect to experimental realizability because it only requires a low number of fundamental quantum operations to be performed.

The purification scheme of Deutsch et al. [12] consists of the following steps. First, Alice and Bob both take the qubits of a pair of ebits and perform a unitary operation on each of these qubits separately. Second, they perform a controlled-not operation on their pairs of qubits and make sure that for each of them the control and target qubits belong to the same ebit. Third, Alice and Bob measure the target qubits in the computational basis $|0\rangle$ and $|1\rangle$ and tell each other the results. If the results coincide they keep the corresponding control qubits. The control qubit of Bob and the control qubit of Alice form the purified ebit. Otherwise, if their measurement results are not the same, they have to discard their control.
qubits. This, of course, means that the purification was not successful and hence Bob and Alice lost two ebits.

If the purification steps are successful one finds that for two Werner states, Eq. (2), with parameter \( \lambda \) the resulting density operator of the purified ebit reads

\[
\hat{\rho} = \frac{1}{9p_{pass}} \left( 10\lambda^2 - 2\lambda + 1 \right) \left| \psi^+ \right\rangle \left\langle \psi^+ \right| + \frac{2}{3p_{pass}} (\lambda^2 - 2\lambda) \left| \phi^- \right\rangle \left\langle \phi^- \right| + \frac{2}{9p_{pass}} (1 - \lambda)^2 \left( \left| \psi^+ \right\rangle \left\langle \psi^+ \right| + \left| \psi^- \right\rangle \left\langle \psi^- \right| \right)
\]

with the probability

\[
p_{pass} = \frac{1}{5} (8\lambda^2 - 4\lambda + 5)
\]

to pass the measurement test at the end of the purification scheme. Of course, this is no longer a Werner state but can be transformed into one by local random rotations again [13].

As the aim of our purification scheme is the preparation of one highly entangled pair we propose the following algorithm that is also depicted in Fig. 1. We start with our finite set of \( N \) initial ebits, all in the same Werner state \( \hat{\rho}_c(\lambda_0) \), Eq. (2), with \( \lambda_0 > 1/4 \). We recall that \( \lambda_0 = \lambda_{prit} = 1/4 \) was the critical value which limits the use of teleportation as a transport channel for the state \( |\psi\rangle \). It is, however, obvious that purification of our ebits will not improve the fidelity of Bob’s output state if we start in the range \( 1/4 < \lambda_0 \leq 1/2 \). The reason for this simply is that the purification protocol does not work in this range since \( \hat{\rho}_c(\lambda_0) \), Eq. (2), is then separable. We will, however, include this interval in our calculations in order to see explicitly how the output fidelity of Bob’s state changes with growing \( \lambda_0 \).

Let us now go through the steps of our repeated purification algorithm, see Fig. 1:

If \( i = N \) is even, we directly perform the first purification step using all \( i \) ebits. For odd \( i \) we first store one ebit and then continue with the purification using only \( i - 1 \) ebits. The result of the \( n \)-th purification step performed on \( i \) Werner states with parameter \( \lambda_{n-1} \) will be a number of \( j \in \{0, 1, \ldots, i/2\} \) successfully purified ebits, where each possible \( j \) occurs with the probability

\[
p_{ij}(\lambda_{n-1}) = \binom{i/2}{j} p_{pass}(\lambda_{n-1}) [1 - p_{pass}(\lambda_{n-1})]^{i/2-j}.
\]

(9)

The purified ebits can now be converted into Werner states with the parameter

\[
\lambda_n = \frac{10\lambda_{n-1}^2 - 2\lambda_{n-1} + 1}{8\lambda_{n-1}^2 - 4\lambda_{n-1} + 5}
\]

(10)

after the \( n \)-th purification step. If we still have at least two ebits left, i.e., \( j > 1 \) we repeat our purification using the already purified ebits. For \( j = 1 \) we only have one ebit left and use it to teleport our initial qubit to Bob. The resulting fidelity of Bob’s qubit in state \( \hat{\rho}_B(\lambda_n) \), Eq. (3), will then be \( F_B(\lambda_n) \), Eq. (6), if the purification has been performed \( n \) times. In the case of \( j = 0 \), however, we have to look for previously stored ebits. If we have stored ebits we use the lastly stored ebit for the teleportation [14]. The teleportation fidelity will be \( F_B(\lambda_k) \), if we have stored the last ebit after the \( k \)-th purification step. The worst case occurs when we have no stored ebits at all. In this case we have lost all our ebits and thus cannot use them for teleportation. As Bob has no information about the initial qubit state, he can only achieve a fidelity \( F_B(1/4) = 1/2 \). That is, his information must be described by a completely mixed state.

Hence, if we start with \( N \) ebits defined by \( \lambda_0 \) we can now calculate, using the algorithm above, the average fidelity

\[
F_B^{(1)}(N, \lambda_0) = \left\langle F_B \right\rangle
\]

(11)
of Bob’s output state. Note that the averaging \( \langle \ldots \rangle \) here means to average over all possible paths through the algorithm depending on the probabilities, Eq. (9). The resulting average fidelities \( F_B^{(1)}(N, \lambda_0) \) of our purification scheme are shown in Fig. 2. As one would expect the fidelities always increase with growing \( \lambda_0 \) and approach 1 for \( \lambda_0 \to 1 \).

The dependency of \( F_B^{(1)}(N, \lambda_0) \) on \( N \) is more complicated. As expected the fidelity shows the behaviour \( F_B^{(1)}(N = 1, \lambda_0) > F_B^{(1)}(1 < N \leq 32, \lambda_0) \) in the range \( 1/4 < \lambda_0 < 1/2 \). There the Werner state is separable and the entanglement purification method yields no improvement as argued before. Or, in other words, in this \( \lambda_0 \) range one could simply use one of the unpurified states \( \hat{\rho}_c(\lambda_0) \) and perform the teleportation with it. On the other hand if we start with a larger initial Werner parameter \( \lambda_0 > 1/2 \) we clearly see an increase of \( F_B^{(1)} \) with growing \( N \).

However, there is a clear difference between even and odd \( N \). The fidelities for odd \( N \) are always considerably higher than in the case of adjacent even \( N \). This is a consequence of the fact that we never lose all the ebits for odd \( N \). Thus we see that on average we get a better quality of Bob’s output state if we only use the highest odd number of ebits for the entanglement purification. On the other hand this means that for an even \( N \) one should discard one ebit first and perform the purification with the remaining \( N - 1 \) ebits. For this reason we will always use this modified method for the remaining parts of the paper, i.e., in the case of even \( N \) we will only use \( N - 1 \) ebits for the entanglement-purification method so that the effective average fidelity is \( F_B^{(1)}(N - 1, \lambda_0) \) for even \( N \).
B. Qubit purification

Instead of purifying the \( N \) ebits that are used for the teleportation Alice could simply use all of the \( N \) ebits in state \( \rho_e(\lambda_0) \) to teleport \( N \) states \( |\psi\rangle \) so that Bob would get \( N \) qubits in the state \( \rho_B(\lambda_0) \), Eq. (3). Bob can then apply a qubit purification protocol \([3]\) to his \( N \) states described by the product \( \rho_B^{\otimes N} \equiv \rho_B^{(0)}(\lambda_0) \otimes \cdots \otimes \rho_B^{(0)}(\lambda_0) \). The qubit purification protocol performs a projection of the qubit product state \( \rho_B^{\otimes N} \) which yields an entangled state \( \hat{\rho}_M \) made up of \( M \) qubits and a product state of Bell-\( |\psi^-\rangle \) states. Thus the effective transformation of the qubit purification procedure can be written as

\[
P : \rho_B^{\otimes N} \rightarrow \hat{\rho}_M \otimes (|\psi^-\rangle\langle\psi^-|)^{\otimes (N-M)/2}.
\]

(12)

For even \( N \) values one obtains even values of \( M \in \{0, 2, ..., N\} \), whereas for odd \( N \) one finds odd \( M \in \{1, 3, ..., N\} \). Any value of \( M \) with a probability

\[
p_M(\lambda_0) = d_M \left[ c_0 c_1^{(N-M)/2} c_1^{M+1} - c_0^{M+1} \right] / c_1 - c_0
\]

(13)

where we used the notation \( c_1 \equiv (1 + 2\lambda_0)/3, c_0 \equiv (1 - \lambda_0)/3 \) and the combinatorial prefactor

\[
d_M = \left\{ \begin{array}{ll} \left( \frac{N}{3} \right) - \left( \frac{N-M}{3} \right) & \text{for } M < N \\ 1 & \text{for } M = N \end{array} \right.
\]

(14)

The density operator \( \hat{\rho}_M \) can also be calculated \([4]\) and one finds

\[
\hat{\rho}_M(\lambda_0) = \frac{c_1 - c_0}{c_1^{M+1} - c_0^{M+1}} (M + 1)
\]

\[
\times \int \frac{d\theta' d\phi'}{4\pi} \langle \Psi(\theta', \phi') | \Psi(\theta', \phi') \rangle^{\otimes M}
\]

(15)

where the unnormalized states

\[
|\Psi(\theta', \phi')\rangle = \sqrt{c_1} \cos \frac{\theta'}{2} |\psi\rangle + \sqrt{c_0} \sin \frac{\theta'}{2} e^{i\phi'} |\psi\rangle
\]

(16)

are a superposition of the original qubit state \( |\psi\rangle \), Eq.(1), and the corresponding orthogonal state \( |\bar{\psi}\rangle \), Eq.(4).

After having performed the qubit purification procedure we discard the \( N-M \) qubits in the \( |\psi^-\rangle \) state and just keep the \( M \) qubits in the entangled state \( \hat{\rho}_M \). The final goal of our scheme is to get one output qubit with a maximal fidelity compared to the initial state \( |\psi\rangle \). Thus we have to look at the reduced density operator \( \hat{\rho}_M^{\otimes M}(\lambda_0) \) of \( \hat{\rho}_M(\lambda_0) \) that can be evaluated by tracing over all qubits except of one. The average fidelity of \( \hat{\rho}_M^{\otimes M} \) then reads \([4]\)

\[
f_M(\lambda_0) = \langle \psi | \hat{\rho}_M^{\otimes M}(\lambda_0) |\psi\rangle
\]

\[
= \left\{ \begin{array}{ll} \frac{(M+1)c_1^{M+1} - c_0^{M+1}}{2} & \text{for } M > 0 \\ 0 & \text{for } M = 0 \end{array} \right.
\]

(17)

This fidelity \( f_M \) is larger than the initial fidelity \( f_B(\lambda_0) \) for \( 1/4 < \lambda_0 < 1 \) and \( M > 0 \). This means that we have improved the quality of the output qubit by the qubit purification. In contrast to the entanglement purification scheme the qubit purification leads to an improvement for all parameters \( \lambda_0 > 1/4 \). The average fidelity of our output qubit for this second method is then given by

\[
f_B^{(2)}(N, \lambda_0) = \left\{ \begin{array}{ll} \sum_{i=0}^{N/2} p_{2i} f_{2i} & \text{for } N \text{ even} \\ \sum_{i=0}^{(N-1)/2} p_{2i+1} f_{2i+1} & \text{for } N \text{ odd} \end{array} \right.
\]

(18)

with the probabilities \( p_M = p_M(\lambda_0) \), Eq.(13), and the single qubit fidelities \( f_M = f_M(\lambda_0) \), Eq.(17). The resulting average fidelities are plotted in Fig. 3. In contrast to the entanglement purification scheme the best performance is always obtained by using all available qubits here.

C. State estimation and preparation

Alice and Bob have a third possibility to transfer a qubit state. This possibility consists of two very basic ingredients and avoids any quantum teleportation at all. Alice simply has to perform measurements on her \( N \) quantum systems and to estimate the quantum state from her measurement results. Then she tells Bob the parameters of her estimated quantum state via a classical communication channel. Bob can then prepare the qubits in the desired quantum state on his side. For qubits the state \( |\psi\rangle \), Eq.(1), can be described by the two Bloch-parameters \( (\theta, \phi) \) which Alice has to tell Bob. This straightforward scheme avoids the use of a noisy quantum channel.

However, Alice cannot accurately estimate the quantum state from a finite ensemble of \( N \) qubits. The optimal state estimation limit has been found \([3]\) and yields the optimal estimation fidelity

\[
f_B^{(3)}(N) = \frac{N + 1}{N + 2}
\]

(19)

This is, of course, also the fidelity with which Bob can then prepare the corresponding output state. Note that this fidelity does not depend on \( \lambda_0 \) since no quantum channel is involved in this method. In addition this optimal estimation scheme requires a simultaneous joint measurement to be performed on all \( N \) qubits \([17]\).

IV. COMPARISON OF THE METHODS

For the transfer of a qubit state Alice and Bob will of course try to use the most efficient of the three methods described above. This efficiency will be measured in terms of the average fidelity of Bob’s final output state
with respect to the initial input state $|\psi\rangle$. The fidelity will in general depend on the quality of the quantum channel which will be characterized by the parameter $\lambda_0$ of the ebit state $\hat{\rho}_e$, Eq.(2), and on the number $N$ of available ebits.

Before we start with the more general comparison let us first look at a typical example of the behaviour of the three proposed methods, namely for $N = 9$. The average fidelities of the output state are plotted versus $\lambda_0$ in Fig. 4. As the estimation-preparation method does not depend on the quantum channel its fidelity stays constant over the whole range of $\lambda_0$ values. The qubit-purification scheme increases the average fidelity for any value $1/4 < \lambda_0 < 1$ and offers a better result than the entanglement-purification scheme for $N = 9$ in the whole range. In addition to that we find crossing points of the two purification schemes and the estimation-preparation method and denote them by $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$, respectively. This means that for quantum channels with $\lambda_0 < \lambda_0^{(2)}$ the estimation-preparation method yields better results than the qubit-purification method. Moreover, the qubit-purification method is always superior to the entanglement-purification method in the case of $N = 9$.

This rises the question how the crossing points change with $N$. The answer to this question is shown in Fig. 5. The crossing points $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$ lie at the same position for $N = 1$ and also for $N = 2$ [18]. For larger $N$ we find that $\lambda_0^{(2)}$ is always smaller than $\lambda_0^{(1)}$, again indicating the better results of the qubit-purification scheme. Moreover, the value of $\lambda_0^{(1)}$ increases with growing $N$, whereas $\lambda_0^{(2)}$ asymptotically converges towards the value $5/8$ [15]. Thus we can conclude that Alice and Bob should use the qubit-purification method for their quantum state transfer whenever their quantum channel yields ebits with a Werner-state parameter larger than $\lambda_0^{(2)}$. If they, however, only get ebits with $\lambda_0 < \lambda_0^{(2)}$ then Alice should estimate her quantum state and only send her result to Bob via a classical channel. As the entanglement-purification scheme never works better than the qubit-purification methods, its application should be avoided in any case.

Due to the averaging process that is necessary to calculate $F_B^{(1)}$, Eq.(11), and $F_B^{(2)}$, Eq.(18), it is hard to see this superiority of the qubit-purification scheme from the analytic expressions. However, the higher quality of the presented qubit-purification scheme can be seen qualitatively from basic properties of the purification schemes. The number of possible purification results that correspond to a failure of the purification process is small for the qubit-purification scheme [14]. In this scheme only the outcome $M = 0$ that occurs with probability $p_0$, Eq.(13), leads to a failure of the purification. For the entanglement-purification scheme, on the other hand, several sequences of purification steps result either in a loss of all ebits or no improvement due to the purification. This is even true for our scheme in which we take into account the possibility to store ebits during the purification process. Hence the presented scheme will be superior to standard entanglement-purification schemes which are known to have rather low output yields [13, 20]. It seems to be impossible to construct an efficient entanglement purification algorithm based on finite resources due to the low efficiency of every single purification step. In contrast to this the qubit-purification scheme can lead to much larger improvements in the output fidelities, Eq.(17), because it is only a one-step process.

V. CONCLUSION

In this paper we have studied the problem of how to transfer a qubit state efficiently from a sender to a receiver when only finite resources are available. In this context we investigated three different methods to achieve this transfer. It turned out that one can never achieve better results by an entanglement-purification scheme than by qubit purification. Furthermore, there exists a threshold for the quality of the quantum channel through which the qubits are sent. If the quality of the channel is below this threshold then Alice should estimate her quantum states and tell Bob the results classically so that he can prepare the quantum state himself. If the channel quality is above the threshold then Alice should send her qubits to Bob without purifying the entanglement of the channel. Rather Bob should apply the qubit-purification method to get his final output state.

This state transfer problem can achieve practical importance in the context of distributed quantum computing [11] where quantum states have to be exchanged between spatially separated quantum processors. As there is a cost (computation time and number of resources) associated with each computational step, it is extremely important to use improved protocols for the quantum state transfer in this case.

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[16] We have also considered the possibility to use some of the previously stored ebits and purify further. It turns out, however, that the average fidelity $F_B^{(1)}$ of the output qubit cannot be increased by further purification steps of already stored ebits.

[17] This disadvantage can be overcome by restricting oneself to simple spin measurements in connection with an adaptive algorithm. The adaptive algorithm is used to select the direction in which the spin is measured from measurement to measurement. With such schemes fidelities very close to the optimal one, Eq.(19), can be achieved. For details see Ref. [3].

[18] We recall that due to the chosen strategy (see Sec. III.A) we effectively have the average fidelity $F_B^{(1)}(N = 1, \lambda_0)$ for the case of $N = 2$ initial ebits. Also for the qubit-purification method there is no difference on average between $N = 1$ and $N = 2$ as can be seen from Fig. 3.

[19] Taking the asymptotic expressions [14] for the average fidelity of qubit purification one can easily calculate the asymptotic value of $\lambda_0^{(2)}$. As result we find $\lambda_0^{(2)} \to 5/8 = 0.625$ for $N \to \infty$. 

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FIG. 1. Flow chart describing the repeated entanglement-purification algorithm. The aim of the algorithm is to generate one highly entangled ebit from an initial supply of $N$ ebits. If we get one purified ebit in the last purification step, i.e., $j = 1$ then we use this ebit for the teleportation. If no purified ebit is left over ($j = 0$) we look for previously stored ebits. The ebit that has been stored lastly is then used for the teleportation. Only if no ebit has been stored during the purification procedure we have no ebit available for the teleportation. Thus no qubit state can be transferred to Bob and the fidelity of his output qubit will be 1/2.

FIG. 2. Plot of Bob’s qubit fidelity $F_B^{(1)}$ resulting from an application of the entanglement-purification method. The dependency of $F_B^{(1)}$ on the number $N$ of available ebits and the quantum channel quality, represented by $\lambda_0$, is shown. For all $\lambda_0 \in (0.25, 1)$ the fidelities for odd $N$ are higher than for the adjacent even values. In addition, the fidelities decrease with growing $N$ for $\lambda_0 < 0.5$ and increase with $N$ for $\lambda_0 > 0.5$. As expected the output fidelity always improves if the parameter $\lambda_0$ is increased.
FIG. 3. Output fidelity $F_B^{(2)}$ of Bob’s qubit after applying the qubit-purification scheme. The fidelity is plotted versus the number $N$ of available qubits and the parameter $\lambda_0$. In contrast to the entanglement-purification method, cf. Fig. 2, $F_B^{(2)}$ always increases with growing $N$ and $\lambda_0$ for $N > 2$ and $1/4 < \lambda_0 < 1$. Note the remarkably large increase of the fidelity that already occurs for small $N$.

FIG. 4. Comparison of the three state transfer schemes for $N = 9$. Over the whole range of possible $\lambda_0$ values the fidelities $F_B^{(2)}$ (*) of the qubit-purification method are always bigger than the fidelities $F_B^{(1)}$ (▲) of the entanglement-purification method. As the fidelity $F_B^{(3)}$ (■) of the estimation-preparation method does not depend on the channel quality it is represented by a constant line that crosses the curves for methods based on purification: We denote the crossing point of $F_B^{(1)}$ with $F_B^{(3)}$ by $\lambda_0^{(1)}$ and the one of $F_B^{(2)}$ with $F_B^{(3)}$ by $\lambda_0^{(2)}$. Obviously, for $\lambda_0$ values smaller than the crossing-point value $\lambda_0^{(2)}$ the estimation-preparation method should be used for transferring the state. For larger $\lambda_0$ values the qubit-purification method should be used since it always results in higher fidelities than the entanglement-purification method. We emphasize that this general behavior, shown here for $N = 9$, is generic, see Fig. 5.
FIG. 5. Plot of the crossing point values $\lambda_0^{(i)}$ versus number $N$ of available ebits. The crossing point values $\lambda_0^{(1)}$ (▲) of the entanglement-purification scheme increases monotonically with $N$. For $N > 2$ the values are always bigger than the corresponding values $\lambda_0^{(2)}$ (★) for the qubit-purification scheme. Moreover, $\lambda_0^{(2)}$ stays almost constant for $N > 6$. 