Distributed Mean Estimation with Optimal Error Bounds

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Abstract

Motivated by applications to distributed optimization and machine learning, we consider
the distributed mean estimation problem, in which $n$ nodes are each assigned a multi-
dimensional input vector, and must cooperate to estimate the mean of the input vectors,
while minimizing communication. In this paper, we provide the first tight bounds for this
problem, in terms of the trade-off between the amount of communication between nodes
and the variance of the node estimates relative to the true value of the mean.

1 Introduction

Several problems in distributed machine learning and optimization can be reduced to variants
of the following distributed mean estimation problem. We are given a set $M$ of $n$ machines such
that each machine $v$ receives as input a $d$-dimensional vector $x_v \in \mathbb{R}^d$, and a bound $y$ on how
far any other input vector can be from $x_v$, i.e. $\|x_u - x_v\|_2 < y, \forall u \in M$. Each machine should
return an accurate estimate of the mean of the input vectors $\mu = \frac{1}{n} \sum_{v=1}^{M} x_v$, while minimizing
the total communication. Given its simplicity, variants of this problem have been considered
ever since the seminal work of Tsitsiklis and Luo [20]. Alternatively, this problem can be seen as
a $d$-dimensional instance of approximate consensus [7], where nodes are additionally constrained
to agree on an estimate in the vicinity of the true mean.

This problem has seen renewed attention recently, given the surge of interest in distributed
optimization for machine learning. Notably, distributed mean estimation is a key component
in data-parallel distributed stochastic gradient descent (SGD), the standard way to parallelize
the training of deep neural networks, e.g. [4] [14], where it is used to estimate the average of
gradient updates obtained in parallel at the nodes. A trivial solution to the problem would be
for each node to broadcast its input in an all-to-all fashion: this obtains no error, but has high
communication cost, especially in the context of training large-scale neural networks, where $d$
can be in the order of millions or even billions [11]. Consequently, several references propose
efficient compression schemes for mean estimation, see e.g. [19] [2] [17] [8] for recent work, and [3]
for a general survey. The main idea behind the upper bounds is to quantize every node’s input
coordinate-wise, via a carefully-designed procedure, and only transmit the quantized values,
possibly using some efficient encoding. References then characterize the trade-off between the
number of bits sent per entry, and the added variance due of quantization. Surprisingly, despite
considerable work on this problem, its optimal complexity thresholds are not yet known in
general: previous results either require input restrictions, or are sub-optimal with respect to
the communication-variance trade-off.
Contributions. In this paper, we provide the first tight bounds for this problem without any restrictions on the input. Specifically, we study the classic distributed mean estimation problem, defined above, and a natural probabilistic variant which arises in machine learning applications, which we call variance reduction. In the latter problem, the input vectors are themselves random estimates of a single true vector, and the goal is to output a new estimate with minimal variance.

Our results are as follows: first, for distributed mean estimation, we show that, to achieve a reduction of a factor $q$ in the input variance $y$, it is necessary and sufficient for machines to communicate $\Theta(d \log q)$ bits. Second, for variance reduction, we show tight $\Theta(d \log n)$ bounds on the worst-case amount of communication bits required to achieve optimal $O(n)$-factor variance reduction (and, indeed, any variance reduction at all). Furthermore, to achieve an $q$-factor reduction in variance, machines must communicate $\Omega(d \log q)$ bits in expectation. Our results are based on new techniques, which we detail in Section 4, and are backed by real-world experimental results (Section 8) showing non-trivial practical gains relative to state-of-the-art compression methods.

2 Preliminaries

We now formally define our problems and setting:

Mean estimation The problem of MeanEstimation is as follows: we have a set $M$ of $n$ machines $v$, each of which receives as input a vector $x_v \in \mathbb{R}^d$. We also assume that all machines receive a common value $y$, with the guarantee that for any machines $u,v$, $||x_u - x_v||_2 \leq y$. Our goal is for all machines to output the same value $EST \in \mathbb{R}^d$, which is an unbiased estimator of the mean $\mu = \frac{1}{n} \sum_{v \in M} x_v$, i.e. $E[EST] = \mu$, with variance as low as possible. Notice that any randomness in the output arises only from the algorithm; the input specification is entirely deterministic.

Variance reduction A variant motivated by machine learning applications is VarianceReduction: we now have an unknown true vector $\nabla$. We again have a set $M$ of $n$ machines $v$, each of which receives as input a vector $x_v$; we now assume that $x_v$ is an independent unbiased estimator of $\nabla$ (i.e., $E[x_v] = \nabla$) with variance $E[||x_v - \nabla||^2_2] \leq \sigma^2$. Machines are assumed to have knowledge of $\sigma$. Our goal is for all machines to output the same value $EST \in \mathbb{R}^d$, which is an unbiased estimator of $\nabla$, i.e., $E[EST] = \nabla$, with variance as low as possible. Output randomness now stems from randomness in the inputs as well as (potentially) random choices made by the algorithm used.

VarianceReduction is common in the context of gradient-based optimization of machine learning models, where we assume that each node $v$ processes local samples in order to obtain a stochastic gradient $\tilde{g}_v$, which is an unbiased estimator of the true gradient $\nabla$, with variance bound $\sigma^2$. If we directly averaged the local stochastic gradients $\tilde{g}_v$, we can obtain an unbiased estimator of the true gradient $G$ with variance bound $\sigma^2/n$, which leads to faster convergence. We describe this setting in detail in Section 3.

2.1 Discussion on Problem Definitions

We pause to make a few notes on these definitions:

1. Input Variance. It is common in machine learning applications of VarianceReduction to assume that an estimate of the variance $\sigma^2$ is known [2, 8]. To study both problems in a common framework, we make the analogous assumption about MeanEstimation, and
assume knowledge of the input ‘variance’ \( y^2 \). For consistency of terminology we use the term \textit{input variance} for both (though the input to \textsc{MeanEstimation} is deterministic).

2. \textit{Parameter Regime.} We will concentrate on the case when machines use \( \Omega(d) \) bits of communication; this is because, as we show in Theorems \[\text{10}\] and \[\text{11}\], one cannot obtain an asymptotic improvement in output variance over input variance using \( O(d) \) communication bits per machine.

3. \textit{Relationship Between Problems.} If one allows unrestricted communication between machines, the best solution to both problems is to output the average of the inputs. This is, by definition, an exact solution to \textsc{MeanEstimation} with variance 0, and is also the best possible solution to \textsc{VarianceReduction}, giving an unbiased estimator of \( \nabla \) with variance at most \( \frac{\sigma^2}{n} \). However, doing so would require the exchange of infinite precision real numbers. So, we will instead communicate quantized values of bounded bit-length, which will engender additional variance caused by random choices within the quantization method. The resulting estimates will therefore have variance \( \text{Var}_{\text{quant}} \) for \textsc{MeanEstimation}, and \( \frac{\sigma^2}{n} + \text{Var}_{\text{quant}} \) for \textsc{VarianceReduction}. In the former case, we will show a trade-off between bits of communication and output variance, while in the latter we will minimize the communication required to achieve the asymptotically optimal variance of \( O(\frac{\sigma^2}{n}) \), and show that this is also the asymptotic amount of bits that some machine must receive in order to achieve \textit{any} variance reduction.

The other major difference between the two problems is that in \textsc{MeanEstimation}, distances between inputs are bounded by \( y \) with certainty, whereas in \textsc{VarianceReduction} they are instead bounded by \( O(\sigma) \) only in expectation. This causes extra complications for quantization, and, as we will see, introduces a gap between average and worst-case communication cost.

2.2 \textit{Distributed Model}

In this work we focus primarily on quantization, and aim to provide a widely applicable method. Therefore we avoid relying on the specifics of particular distributed models, and instead assume that the basic communication structures we use (binary trees) can be used as an overlay without significant overhead. This setting is supported by machine learning applications, which have very high input dimension (i.e., \( d \gg n \)), and so the costs of synchronization or construction of an overlay (which do not depend on \( d \)), will be dominated by the communication costs incurred subsequently.

Hence, we will present our algorithms within a basic synchronous fault-free message-passing model, in which machines can send arbitrary messages to any other machine, but they could naturally be extended to asynchronous and shared-memory models of communication.

Our aim will be to minimize the number of bits sent and received by any machine during the course of the algorithm.

2.3 \textit{Notation}

When dealing with vectors in \( \mathbb{R}^d \), we will use names in bold, e.g. \( \mathbf{x}, \mathbf{y} \). We will use the notation \( B_\delta(x) \) to mean the (open) ball of radius \( \delta \), centered at \( x \), i.e., \( \{ y \in \mathbb{R}^d : ||x - y||_2 < \delta \} \). We will often deal with the \textit{volume} of balls, which we will denote \( \text{Vol}(B_\delta) \) (we need not specify the center since it does not affect the volume). When we say an event occurs \textit{with high probability}, we mean that it has probability at least \( 1 - n^{-c} \), for some positive constant \( c \).
2.4 Paper Structure

We cover previous work in Section 3 and discuss our general approach in Section 4. In Section 5, we present our method of quantizing vectors, consisting of an encoding procedure to map vectors to a short bit-string, and a decoding procedure to recover something close to the original vector. In Section 6, we show how to employ this quantization procedure in order to compute an approximation of the mean over a distributed system, providing upper bounds for MeanEstimation and VarianceReduction. In Section 7, we show matching lower bounds for the problems, demonstrating the optimality of our algorithm. We provide experimental results in Section 8.

3 Related Work

The communication costs of various forms of distributed optimization problems have been widely studied since at least the seminal work of Tsitsiklis and Luo [20]. The specific problems of distributed mean estimation and variance reduction, though, have recently risen to particular prominence due to applications to distributed stochastic gradient descent, widely used as an optimization procedure in machine learning. Several recent works have approached the problems from this perspective [22, 18, 21], and provide quantization schemes for estimating a true gradient \( \nabla \) using limited communication. However, these works made assumptions on input structure, and evaluated primarily the practical performance of SGD rather than isolating the variance reduction step. As a result, they do not provide theoretical bounds on the problems.

The work of Alistarh et al. [2] is closer to our definition of VarianceReduction; the major remaining difference is that co-ordinates of the input vectors are assumed to be specified by 32-bit floats, rather than arbitrary real values. Hence, transmitting input vectors exactly already requires only \( O(d) \) bits. Alistarh et al. therefore focus on reducing the constant factor (and thereby improving practical performance for SGD), rather than providing asymptotic results on communication cost. They show that the expected number of bits per machine can be reduced from 32 to expected 2.8, at the expense of obtaining an output variance bound in terms of second moment rather than input variance.

Using similar techniques, Suresh et al. [19] study the MeanEstimation problem as defined in Section 2 (i.e., on real-valued input vectors). They present a series of quantization methods, providing an \( \Theta(n) \) upper bound, and an \( \Omega(1) \) lower bound on the problem which is claimed to be asymptotically tight (though as we will discuss in more detail shortly, this claim is not, in general, true). Finally, very recent work by Gandikota et al. [8] studies VarianceReduction, and uses multi-dimensional quantization techniques. However, their focus is on protocols using \( o(d) \)-bit messages per machine (which therefore cannot reduce input variance), though they do give two quantization methods using \( \Theta(d) \)-bit messages. Of these, one gives an \( O(d) \) bound on output variance, similar to the bound of [19] for MeanEstimation (the other is much less efficient since it is designed to achieve a privacy guarantee). All the above references work under the assumption that the mean to be estimated is centered around zero—an assumption contradicted by our experimental results in Section 3. In the same section, we show that lattice quantization can achieve superior variance-communication trade-off relative to [2, 19]; in particular, the quantization variance can be more than two orders of magnitude lower relative to these schemes, at the same communication cost, even for sub-optimal lattices.
3.1 Variance Versus Second Moment

As mentioned, several previous works have provided quantization schemes for use in similar problems, particularly variants arising in machine learning applications [2, 19, 8]. However, to our knowledge, all such works obtain an output variance bound which is only in terms of the second moment of the input, rather than its variance (recall that by the input variance of a \textsc{MeanEstimation} instance, we mean the variance in the input of a randomly chosen machine, i.e., at most \( y^2 \)). In some cases it is claimed that these bounds are optimal, but such claims should carry a major caveat: they assume that the input second moment is asymptotically equivalent to its variance.

We highlight this crucial point using the bounds of [19] for \textsc{MeanEstimation} as an illustrative example. There, the authors show that using \( O(d) \) expected bits of communication per machine, one can solve \textsc{MeanEstimation} with an output variance of

\[
\Theta \left( \sum_{i<n} \|x_i\|_2^2 \right).
\]

Theorem 5 of [19] provides (as a corollary of [24]) a corresponding lower bound, stating that any algorithm in which machines receive \( O(d) \) bits in expectation must have \( \Omega(1) \) output variance. It is claimed that these bounds are tight, and the algorithm optimal, but this is only true under the (extremely) strong assumption that the \( \|x_i\|_2^2 \) values are \( O(1) \) (recall that the input vectors are high-dimensional, so even the all-ones vector \( 1 \) has \( \|1\|_2^2 = d >> 1 \)). Without this assumption, it is easy to construct input examples for which the gap between upper and lower bounds is arbitrarily bad, simply by increasing the norms of the input vectors \( x_i \).

We contend that the reason for this discrepancy is that output quality is being bounded by the wrong parameter, and that the correct approach is to bound output variance in comparison to input variance \( y^2 \) (or \( \sigma^2 \) for \textsc{VarianceReduction}) rather than \( \|x_i\|_2^2 \). We note that \( y \) is at most twice the maximum input norm, but it can be (and, as we will see, often is) far lower, when the average \( \mu \) (or, in the case of \textsc{VarianceReduction}, the true vector \( \nabla \)) is not the \( 0 \) vector. Indeed, in applications of \textsc{VarianceReduction} to stochastic gradient descent (SGD), the case that \( \nabla \approx 0 \) is precisely when convergence has been reached and the procedure terminated, so the only case of interest is \( \nabla \neq 0 \).

4 Our Approach

In this work, we challenge the assumptions made by previous works, and argue that it is both stronger and more natural to bound output variance in terms of input variance, rather than second moment. To this end, we devise truly optimal quantization schemes for \textsc{MeanEstimation} and \textsc{VarianceReduction}, and prove matching lower bounds for all cases, regardless of input norms. We also provide experimental results to demonstrate that in machine learning applications, it is indeed the case that input variance is much lower than its second moment, and so our algorithms provide greatly reduced output variance.

4.1 Novel Technical Concepts

We here summarize the major conceptual ideas that allow us to achieve these results.

**Covering \( \mathbb{R}^d \) with quantization points** The reason that all prior works obtain output variance bounds in terms of second moment rather than input variance is that they employ sets of quantization points which are centered around the origin \( 0 \). There is, in effect, an implicit assumption that the desired output is close to \( 0 \), and performance suffers when it is not. However, without specific grounds for this assumption, \( 0 \) is essentially an arbitrary vector, and the desired output could just as easily lie anywhere in \( \mathbb{R}^d \). So, we instead cover the entire
space $\mathbb{R}^d$ with quantization points that are, in some sense that we will define, *uniformly spaced.* This is what allows us to give upper bounds on output variance which are independent of the norms of the input vectors.

**Lattice-based quantization** To fill $\mathbb{R}^d$ with uniformly-spaced quantization points, we will employ the concept of lattices, subgroups of $\mathbb{R}^d$ consisting of the integer combinations of a set of basis vectors. It has been known for over a century \[15\] that certain lattices have desirable properties for covering and packing Euclidean space, and lattices have been previously used for some other applications of quantization (see, e.g., \[9\]), though mostly only in low dimension.

By choosing the correct family of lattices for our setting, we will show that any vector in $\mathbb{R}^d$ can be rounded (in a randomized, unbiased fashion) to a nearby lattice point, but also that there are not too many nearby lattice points, so the correct one can be specified using few bits.

**Decoding using own input** The space $\mathbb{R}^d$, of course, has infinite volume, and therefore if we encode points using bounded bit-length, and wish to have any distance guarantee independent of input norm, we must map an infinite subset of points to the same bit-string. So, the question remains of how we decode a bit-string representing an infinite subset of points to the correct point intended by the encoder.

Our solution is to utilize the fact that we have a bound on the distance between any two machines’ inputs ($y$ for \textsc{MeanEstimation}, and $\sigma n$, with high probability, for \textsc{VarianceReduction}). Therefore, if all of the points that map to the same bit-string are sufficiently far apart, a machine can decode the correct one simply by picking the closest to its own input. This is the reason we must assume knowledge of $y$ and $\sigma$, since if our quantization scheme did not depend on these parameters, we could not guarantee that machines could correctly decode messages.

### 4.2 Paper Structure

In Section 5, we present our method of quantizing vectors, consisting of an encoding procedure to map vectors to a short bit-string, and a decoding procedure to recover something close to the original vector. In Section 6, we show how to employ this quantization procedure in order to compute an approximation of the mean over a distributed system, providing upper bounds for \textsc{MeanEstimation} and \textsc{VarianceReduction}. In Section 7, we show matching lower bounds for the problems, demonstrating the optimality of our algorithm. Finally, in Section 8, we give experimental results providing evidence for the efficacy of our approach in machine learning applications.

### 5 Quantization Method

We will define a parameterized quantization procedure $Q_{\epsilon,q} : \mathbb{R}^d \to \{0,1\}^{d \log q}$ which maps an input vector to a subset of $\mathbb{R}^d$ specified with $d \log q$ bits. We will also define a corresponding decoding procedure $R_{\epsilon,q} : \{0,1\}^{d \log q} \times \mathbb{R}^d \to \mathbb{R}^d$, which uses the decoder’s own input vector to recover a vector in $\mathbb{R}^d$ from the quantized value.

Our procedure is based upon filling the space $\mathbb{R}^d$ with quantization points specified by a lattice. It has been long been known (since at least the Minkowski–Hlawka theorem \[15,12\]) that there exist lattices with properties that make them well-suited for use as quantization point sets. We will first describe the specific properties of lattices that we will require, and then refer to a construction of lattices which have these properties.
5.1 Lattices

A lattice $\Lambda$ in $d$ dimensions is an additively-closed subgroup of $\mathbb{R}^d$, defined by a basis $b_1, \ldots, b_d \in \mathbb{R}^d$, and consisting of all integer combinations of the basis vectors.

The cover radius $r_c > 0$ of a lattice is the minimum distance such that all points in $\mathbb{R}^d$ are within distance $r_c$ of a lattice point (i.e., it is the minimum such that balls of radius $r_c$, centered at the lattice points, cover $\mathbb{R}^d$).

The packing radius $r_p > 0$ is the maximum distance such that no points in $\mathbb{R}^d$ are within distance $r_p$ of two distinct lattice points (or equivalently, all lattice points are at least distance $2r_p$ apart, so balls of radius $r_p$, centered at the lattice points, have no intersection). Notice that we must have $r_p \leq r_c$.

We will be using the points in a lattice as our quantization points, and so we prove bounds on the number of such points that fall inside a ball, in terms of cover and packing radius of the lattice:

**Lemma 1.** Let $\Lambda$ be a lattice in $d$ dimensions, for even $d$, with cover radius $r_c$ and packing radius $r_p$. Then, For any $x \in \mathbb{R}^d$, $\delta > 0$, $(\frac{\delta - r_c}{r_p})^d \leq |\Lambda \cap B_\delta(x)| \leq (\frac{\delta + r_p}{r_p})^d$.

**Proof.** We fix a point $x \in \mathbb{R}^d$ and upper-bound the number of points in $\Lambda$ within distance $\delta$ of it: consider the ball $B_{\delta + r_p}(x)$. For any point $y$ within distance $\delta$ of $x$, $B_{r_p}(y) \subseteq B_{\delta + r_p}(x)$. It is also the case that for any $z \neq y \in \Lambda$, $B_{r_p}(z) \cap B_{r_p}(y) = \emptyset$. So,

$$|B_\delta(x) \cap \Lambda| \leq \frac{Vol(B_{\delta + r_p})}{Vol(B_{r_p})}.$$

The formula for the volume of a $d$-dimensional ball of radius $r$ is $\frac{\pi^{d/2}r^d}{2^d}$. The ratio $\frac{Vol(B_{\delta + r_p})}{Vol(B_{r_p})}$ is therefore $(\frac{\pi^{d/2}r}{2^d})^d$. So:

$$|B_\delta(x) \cap \Lambda| \leq \left(\frac{\delta + r_p}{r_p}\right)^d.$$

The lower bound follows similarly: $B_{\delta - r_c}(x) \subseteq \bigcup_{y \in \Lambda \cap B_\delta(x)} B_{r_c}(y)$, so

$$|\Lambda \cap B_\delta(x)| \geq \frac{Vol(B_{\delta - r_c})}{Vol(B_{r_c})} = \left(\frac{\delta - r_c}{r_c}\right)^d.$$

\[\square\]

For our purposes, we will require a lattice for which $r_c = O(r_p)$. Such lattices are known to exist (see, e.g., [3]):

**Theorem 2.** [Theorems 1, 2 of [3]] There is a sequence of distributions $D_d$ of lattices of dimension $d$ such that for any $\phi > 0$,

$$\lim_{d \to \infty} \Pr_{\Lambda \sim D_d}[r_c \leq (2 + \phi)r_p] = 1.$$

We will call a lattice $\Lambda$ an $\epsilon$-lattice if $\epsilon = r_p \leq r_c \leq 2.1\epsilon$. Theorem 2 implies that for sufficiently large $d$, such lattices exist. We note that $r_p, r_c$ scale linearly with the lattice basis vectors, and therefore for any $\epsilon_1, \epsilon_2 > 0$, if we have an $\epsilon_1$-lattice we can obtain an $\epsilon_2$-lattice simply by multiplying all basis vectors by $\frac{\epsilon_2}{\epsilon_1}$.
5.2 Lattice Coloring

Rounding a vector $\mathbf{x} \in \mathbb{R}^d$ to a nearby lattice point is not sufficient for it to be communicated using a bounded number of bits, since there are still an infinite number of lattice points (though we have reduced from the uncountably infinite $\mathbb{R}^d$ to the countably infinite $\Lambda$). So, we must encode an infinite subset of lattice points using the same bit-string, and we want to ensure that the points in this subset are far apart, so that a receiver can identify which point the encoder intended. There is a natural way to do this: we simply take a coordinate-wise (with respect to the lattice basis) operation coordinate-wise to each of the entries $\alpha_i$. We can then encode $c(s)$ using $d\log q$ bits, since there are $q$ possible values for each co-ordinate.

**Lemma 3.** For any $\epsilon$-lattice $\Lambda$, two points $s_1 \neq s_2 \in \Lambda$ such that $c_q(s_1) = c_q(s_2)$ must have $||s_1 - s_2||_2 \geq 2\epsilon q$.

**Proof.** Since $c_q(s_1) = c_q(s_2)$, the vector $\frac{1}{q}(s_1 - s_2)$ must have integer co-ordinates under the canonical basis of $\Lambda$. Therefore it is a point in $\Lambda$. So, $||\frac{1}{q}(s_1 - s_2)|| \geq 2\epsilon$, since otherwise $B(q\frac{1}{q}(s_1 - s_2)) \cap B(0) \neq \emptyset$, which cannot happen since $r_p \geq \epsilon$. Then $||s_1 - s_2||_2 \geq 2q\epsilon$.

5.3 Encoding Procedure

We are now ready to define our quantization method $Q_{\epsilon,q} : \mathbb{R}^d \rightarrow \{0,1\}^b$ as follows: let $\Lambda$ be an $\epsilon$-lattice, and let $\mathbf{x}$ be a fixed vector to quantize. We show that the convex hull of nearby lattice points contains $\mathbf{x}$:

**Lemma 4.** Any $\mathbf{x}$ is within the convex hull of $B_{5\epsilon}(\mathbf{x}) \cap \Lambda$.

**Proof.** We show that for any nonzero vector $y \in \mathbb{R}^d$, there exists $s \in B_{5\epsilon}(\mathbf{x}) \cap \Lambda$ such that $\langle y, s \rangle > \langle y, \mathbf{x} \rangle$. Let $s$ be the closest point in $\Lambda$ to $\mathbf{x} + 2.2\epsilon \frac{y}{||y||_2}$. Since $r_c \leq 2.1\epsilon$, $||s - (\mathbf{x} + 2.2\epsilon \frac{y}{||y||_2})||_2 < 2.1\epsilon$ (and so $s \in B_{5\epsilon}(\mathbf{x})$). So,

$$
\langle y, s \rangle = \langle y, \mathbf{x} + 2.2\epsilon \frac{y}{||y||_2} + s - (\mathbf{x} + 2.2\epsilon \frac{y}{||y||_2}) \rangle = \langle y, \mathbf{x} + \langle y, 2.2\epsilon \frac{y}{||y||_2} \rangle \rangle + \langle y, s - (\mathbf{x} + 2.2\epsilon \frac{y}{||y||_2}) \rangle \\
\geq \langle y, \mathbf{x} + 2.2\epsilon ||y||_2 - ||y||_2 \cdot ||s - (\mathbf{x} + 2.2\epsilon \frac{y}{||y||_2})||_2 \\
> \langle y, \mathbf{x} + 2.2\epsilon ||y||_2 - 2.1\epsilon ||y||_2 \rangle > \langle y, \mathbf{x} \rangle.
$$

Therefore, if $\mathbf{x}$ is in the external half-space of a face of the convex hull, we can take $y$ to be orthogonal to the face and show that a member of $B_{5\epsilon}(\mathbf{x}) \cap \Lambda$ is also in this external half-space, causing a contradiction. So, $\mathbf{x}$ must be within the convex hull of $B_{5\epsilon}(\mathbf{x}) \cap \Lambda$.

We can therefore show that we can probabilistically map $\mathbf{x}$ to these nearby lattice points in such a way that the expectation of the result is $\mathbf{x}$.
Enumerate the points in \( B_{\epsilon}(x) \cap \Lambda \) as \( s_1, \ldots, s_z \). Since \( x \) is within the convex hull of \( B_{\epsilon}(x) \cap \Lambda \), there must be some sequence of non-negative co-efficients \( a_1, \ldots, a_z \) such that
\[
\sum_{i=1}^z a_i s_i = x.
\]
Let \( z \) be a random vector taking value \( s_i \) with probability \( \frac{a_i}{\sum_i a_i} \), for all \( i \).

Then, \( E[z] = x \), and \( ||x - z||_2 < 5\epsilon \) (and so \( z \) is an unbiased estimator of \( x \) with variance most \((5\epsilon)^2 \leq 19\epsilon^2\)).

We then set \( Q_{\epsilon,q}(x) \) to be the color class of \( z \) in the coloring \( c_q \) of \( \Lambda \), which can be specified in \( d \log q \) bits. \( Q_{\epsilon,q}(x) \) now corresponds to a set containing \( z \), such that any two elements of \( Q \) are of distance at least \( 2q\epsilon \) apart.

We summarize the properties of our quantization scheme in the following lemma:

**Lemma 5.** There is a function \( Q_{\epsilon,q} : \mathbb{R}^d \to \{0,1\}^b \) which maps each \( x \in \mathbb{R}^d \) to a subset \( Q_{\epsilon,q}(x) \subset \mathbb{R}^d \), specified with \( b = d \log q \) bits, with the following properties:

- \( \exists z \in Q_{\epsilon,q}(x) \) such that \( z \) is an unbiased estimator of \( x \), with \( ||z - x||_2 < 5\epsilon \).
- \( \forall w \in Q_{\epsilon,q}(x) \setminus \{z\}, ||z - w||_2 \geq 2q\epsilon \).

### 5.4 Decoding Procedure

We must now also define a procedure \( R_{\epsilon,q} : \{0,1\}^{d \log q} \times \mathbb{R}^d \to \mathbb{R}^d \) to decode quantized values, using a machine’s own input \( x_v \) (as the second input to the function). The procedure is simple: we simply take the point in the subset encoded by the received quantized value \( Q_{\epsilon,q}(x) \) which is closest to \( x_v \).

**Lemma 6.** If \( q \geq 9 \) and \( ||x - x_v||_2 < (q - 5)\epsilon \), then the decoding procedure \( R_{\epsilon,q}(Q_{\epsilon,q}(x), x_v) \) returns the vector \( z \).

**Proof.** We upper-bound the distance to \( z \):

\[
||x_v - z||_2 \leq ||x_v - x||_2 + ||x - z||_2 < (q - 5)\epsilon + 5\epsilon = q\epsilon ,
\]

and lower-bound the distance to any other point \( y \in Q_{\epsilon,q}(x) \setminus \{z\} \):

\[
||x_v - y||_2 \geq ||y - z||_2 - ||x_v - z||_2 > 2q\epsilon - q\epsilon = q\epsilon .
\]

Therefore \( z \) is the closest point in \( Q_{\epsilon,q}(x) \) to \( x_v \), and will be returned by the decoding procedure. \( \square \)

### 6 Mean Estimation Algorithm

Now that we have our quantization procedure, it remains to show an algorithm applying it to MEANESTIMATION and VARIANCEREDUCTION.

Our algorithm for MEANESTIMATION (parameterized by \( m \), which dictates the output variance) is given below (Algorithm 1). We will later show how it can also be applied to VARIANCEREDUCTION.

**Algorithm 1 MEANESTIMATION(\( m \))**

1. Sample a set \( T \) of \( \min(m,n) \) machines uniformly at random.
2. Arrange nodes into a complete binary tree, with nodes in \( T \) as leaves.
3. Collect estimates from \( T \) to root of tree, averaging and encoding with \( Q_{\frac{\epsilon}{m^n}, m^3} \) at every step.
4. Compute final average at root, and broadcast to all machines (via a binary tree) encoded with \( Q_{\frac{\epsilon}{m^n}, m^3} \).
5. Decode and output result at all machines.
We now describe in more detail the steps of the algorithm:

**Sampling a set of machines** We begin by sampling a set $T$ of $\min(m, n)$ machines (so if $m \geq n$, $T$ is simply the set $M$ of all machines). Our goal will then be to estimate the average of the inputs of the leaf nodes (machines in $O$). We may assume it is a power of order to convey estimates of the inputs of the leaf nodes (machines in $O$). The communication tree may be taken by any arbitrary machines, so long as all machines take only $O(1)$ roles. We will then send messages up the tree (i.e., from child nodes to their parents), in order to convey estimates of the inputs of the leaf nodes (machines in $T$) to the root, which can then compute the final average.

**Collecting estimates from leaves to root** When a node in the communication tree receives and decodes a vector from both of its children, it takes the average, encodes with $Q_{m^3, m^3}$, and sends the result to its parent. We will denote by $A_v$, the average input of all descendant leaves of a tree node $v$. Our goal is then to show that the average computed by node $v$ is an unbiased estimator of $A_v$ with low variance. Since $A_r = \mu_T$ for the root $r$ of the communication tree, we will then have an unbiased estimator of $\mu_T$ (and therefore $\mu$) as desired.

To bound the estimator error at each step, we will employ an inductive argument. Considering nodes by their depth in the tree (with leaves at depth $0$ and the root $r$ at depth $\log \min(m, n)$), we show the following:

**Lemma 7.** A node $v$ at depth $i$ sends to its parent an unbiased estimator $a_v$ of $A_v$ with $||A_v - a_v||_2 \leq \frac{5i y}{m^2}$, encoded with $Q_{m^3, m^3}$.

**Proof.** By induction. Clearly the claim is true for leaf nodes at $0$, which encode exactly $A_v$ (i.e. their own input). Assuming the claim is true for $i$, we prove for $i + 1$:

Node $v$ receives two values $Q_{m^3, m^3}(a_u)$ and $Q_{m^3, m^3}(a_w)$ from its children $u$ and $w$. By the inductive assumption, $||A_u - a_u||_2 \leq \frac{5y}{m^2}$, and so $||x_v - a_u||_2 \leq ||A_u - a_u||_2 + ||x_v - A_u||_2 \leq \frac{5i y}{m^2} + y < (m^3 - 5) \frac{y}{m^2}$. By Lemma 6, therefore, $v$ correctly decodes the message from $u$, to recover an unbiased estimator $z_u$ of $a_u$ with $||z_u - a_u||_2 < \frac{5y}{m^2}$. The same argument holds for $w$. Node $v$ then takes the average of $z_u$ and $z_w$. Since $||z_u - A_u||_2 \leq ||A_u - a_u||_2 + ||z_u - a_u||_2 < \frac{5i y}{m^2} + \frac{5y}{m^2} = \frac{5(i + 1)y}{m^2}$ (and the same holds for $w$):

$$||A_v - a_v||_2 \leq \frac{||z_u - A_u||_2 + ||z_w - A_w||_2}{2} < \frac{5(i + 1)y}{m^2}.$$ 

This completes the proof by induction.

**Computing the final average at the root** By Lemma 7, the root node $r$, at depth $\log \min(m, n)$, computes an unbiased estimator $a_r$ of $\mu_T$ with $||a_r - \mu_T||_2 \leq \frac{5y \log m}{m^2}$. It then encodes this vector with $Q_{m^3, m^3}$, and broadcasts it to all other machines via an
arbitrary binary tree (all machines performing the role of one node in the tree). Nodes in the tree relay the same message to their children until all nodes have received the message. Then, all machines decode the message and output the resulting vector.

**Decoding and outputting** *EST* All machines have now received an unbiased estimator $a_r$ of $\mu_T$ with $||a_r - \mu_T||_2 \leq \frac{5y\log m}{m^2}$, encoded with $Q_{\frac{y}{m^2},m}$. Any machine $v$ has

$$||a_r - x_v||_2 \leq ||a_r - \mu_T||_2 + ||x_v - \mu_T||_2 \leq \frac{5y\log m}{m^2} + y < (m^3 - 5)\frac{y}{m^2},$$

and so by Lemma 6, $v$ correctly decodes the message to recover an unbiased estimator $z_r$ of $a_r$ with $||z_r - a_r||_2 < \frac{5y}{m^2}$.

All nodes therefore output $z_r$, which is an unbiased estimator of $\mu_T$, with

$$||z_r - \mu_T||_2 \leq ||z_r - a_r||_2 + ||\mu_T - a_r||_2 \leq \frac{5y}{m^2} + \frac{5y\log m}{m^2} = O\left(\frac{y}{m}\right).$$

As noted earlier, an unbiased estimator of $\mu_T$ is also an unbiased estimator of $\mu$ (with $O\left(\frac{y^2}{m}\right)$ additional variance), and $z_r$ is therefore a correct solution to **MeanEstimation** with variance

$$\text{Var}[z_r] \leq ||z_r - \mu_T||_2^2 + O\left(\frac{y^2}{m^2}\right) = O\left(\frac{y^2}{m^2}\right).$$

We can bound the communication cost as follows: all machines have sent and received $O(1)$ vectors encoded with $Q_{\frac{y}{m^2},m}$. Therefore, each machine has used $O(d\log m)$ communication bits in total. Setting $b = \Theta(d\log m)$ then completes the proof of Theorem 8.

**Theorem 8.** For any $B \geq d$, **MeanEstimation** can be performed with each machine using $O(B)$ communication bits in total, and with output variance

$$E \left[ ||\text{EST} - \mu||_2^2 \right] = O\left(y^2 2^{-\frac{2B}{7}}\right).$$

In Section 7 we will prove Theorem 10 which shows that this bound is the optimal trade-off between communication bits and output variance.

### 6.1 Extension to VarianceReduction

We can apply Algorithm 1 also to the problem of **VarianceReduction** as follows: first, we note that the probability that any pair of machines $u,v$ have $P_{\text{Pr}}[||x_u - x_v||_2 \geq n^2\sigma] \leq \frac{1}{n^2}$ by Chebyshev’s inequality, and therefore, by a union bound, with high probability, all pairs of inputs are within distance $n^2\sigma$.

Furthermore, the average $\mu$ of the inputs is an unbiased estimator of $\nabla$ with $O\left(\frac{\sigma^2}{n}\right)$ variance. Therefore, with high probability we can reduce a **VarianceReduction** instance with parameter $\sigma$ to a **MeanEstimation** instance with parameter $\sigma n^2$, at the cost of an additive $O\left(\frac{\sigma^2}{m}\right)$ term in output variance (and this term is necessary, since it is well known that the best possible output is $\mu$).

So, choosing $b = cd\log n$ for sufficiently large constant $c$, Theorem 8 states that we can obtain a solution with output variance $(\sigma n^2)^2 2^{-O\left(\frac{d}{7}\right)} + O\left(\frac{\sigma^2}{m}\right) = O\left(\frac{\sigma^2}{m}\right)$. We thereby obtain Theorem 9.

**Theorem 9.** **VarianceReduction** can be performed, with high probability, with each node using $O(d\log n)$ communication bits in total, and with output variance

$$E \left[ ||\text{EST} - \nabla||_2^2 \right] = O\left(\frac{\sigma^2}{n}\right).$$
Here, the maximum number of bits needed per machine is the same to achieve optimal $O\left(\frac{\sigma^2}{n}\right)$ output variance as it is to achieve any variance reduction at all, so there is no trade-off. By Theorem 11, this bound is optimal in a worst-case sense (i.e., any VARIANCEREDUCTION algorithm achieving any variance reduction must cause at least one node to receive $\Omega(d \log n)$ bits).

7 Lower Bounds

We prove matching lower bounds for MEANESTIMATION and VARIANCEREDUCTION, using an argument based on bounding the volume of space for which a node can output a good estimate, if it receives a limited number of communication bits.

In a well-known approach, we will prove our lower bounds for deterministic algorithms on a distribution of inputs; by Yao’s minimax principle [23], they then immediately apply also to randomized algorithms on a worst-case input.

7.1 Lower Bound for MeanEstimation

The bound we prove is the following:

**Theorem 10.** For any MEANESTIMATION algorithm in which each machine receives at most $B \geq d$ bits in expectation,  
\[ \mathbb{E} \left[ ||EST - \mu||^2 \right] = y^2 2^{-O(y)} . \]

**Proof.** We define an input distribution as follows: choose a vector $\mu$ uniformly at random from $B_{d,y}(0)$. The probability that $||\mu||_2 \leq d^2 y - y$ is:
\[ \frac{Vol(B_{d^2y-y})}{Vol(B_{d^2y})} \geq \left( \frac{d^2 y - y}{d^2} \right)^d = \left( 1 - \frac{1}{d^2} \right)^d \geq \frac{2}{3} - \frac{2}{d} . \]

Here we have used the inequalities $1 - \frac{1}{d^2} \geq 3^{-\frac{1}{d^2}}$ for $d^2 \geq 6$, and $3^{-\frac{1}{d^2}} \geq 1 - \frac{2}{d}$ for all $d > 0$.

Therefore, with probability at least $1 - \frac{1}{d^2}$, $||\mu||_2 \leq d^2 y - y$. We will call this event $A$.

We partition machines into pairs $u,v$. We choose the input $x_u$ independently uniformly at random from $B_{\frac{1}{d^2}y}(\mu)$, and then choose $x_v = 2\mu - x_u$. In this way we guarantee that the average of all inputs is indeed $\mu$, and that $||x_u - x_v||_2 \leq y$ for any $w,v$.

We now fix a machine $v$ to examine. Conditioning on $A$ and the value of $x_v$, $\mu$ is distributed uniformly at random in $B_{\frac{1}{d^2}y}(x_v)$ (we need to condition on $A$ since otherwise $B_{\frac{1}{d^2}y}(x_v)$ could intersect $B_{\frac{1}{d^2}y}(0)$, i.e., there would be some region in which $\mu$ cannot fall).

Machine $v$’s output $EST$ is deterministically dependent on $x_v$ and the bits it receives during the course of the algorithm. We denote by $b_v$ the number of such bits. If $b_v < b$ for some $b$, then $v$ has at most $\sum_{i=0}^{b-1} 2^i = 2^b - 1$ possible outputs. We denote the set of these possible outputs $OUT_v$. The volume of the set of point of $\mathbb{R}^d$ which are within distance $dist$ of an element of $OUT_v$ is at most $|OUT_v| \cdot Vol(B_{dist})$. Therefore,
\[ \Pr \left[ ||EST - \mu||_2 \leq dist \mid A \right] \leq \frac{|OUT_v| \cdot Vol(B_{dist})}{Vol(B_{dist})} \leq |OUT_v| \cdot \left( \frac{2 \cdot dist}{y} \right)^d . \]

We will set $dist = \frac{y^3}{2} 2^{-\frac{d}{2}}$. Then,
\[ \Pr \left[ ||EST - \mu||_2 \leq dist \mid A \right] \leq |OUT_v| \cdot \left( \frac{y^{3-b}}{y} \right)^d \leq |OUT_v| \cdot 3^{-b} . \]

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Finally, we condition on \( b_v < b \) and remove the conditioning on \( A \):

\[
\Pr \left[ \|EST - \mu\|_2 \leq \text{dist} \mid b_v < b \right] \leq \Pr \left[ \|EST - \mu\|_2 \leq \text{dist} \mid A, b_v < b \right] + \Pr \left[ \overline{A} \right]
\]

\[
\leq (2^b - 1) \cdot 3^{-b} + \frac{2}{d} = \left( \frac{2}{3} \right)^b - 3^{-b} + \frac{2}{d} < \frac{1}{2} .
\]

So, \( \mathbb{E} \left[ \|EST - \mu\|_2^2 \mid b_v < b \right] \geq \frac{1}{2} \text{dist}^2 = \Omega(y^2 9^{-\frac{b}{4}}) \).

We now set \( b = 2B \); by Markov’s inequality, \( \Pr \left[ b_v < b \right] \geq \frac{1}{2} \). Then,

\[
\mathbb{E} \left[ \|EST - \mu\|_2^2 \right] \geq \mathbb{E} \left[ \|EST - \mu\|_2^2 \mid b_v < b \right] \mathbb{P} \left[ b_v < b \right] = \Omega(y^2 9^{-\frac{b}{4}}) = y^2 2^{-O(\frac{B}{d})}
\]

\( \square \)

Theorem \( 10 \) shows that the trade-off given by Theorem \( 8 \) is asymptotically tight; of particular note is that using \( O(d) \) bits per machine gives \( \Theta(y^2) \) output variance, so to asymptotically reduce input variance, we need \( \omega(d) \) bits per machine.

### 7.2 Lower Bound for VarianceReduction

We further prove a similar bound (using similar techniques) for VarianceReduction. The main difference is that we use the fact that, in VarianceReduction, machines only have a probabilistic distance bound from each other, to construct a distribution of inputs in which one particular node must receive extra bits of communication.

**Theorem 11.** For any VarianceReduction algorithm in which each machine receives at most \( B \geq d \) bits in expectation,

\[
\mathbb{E} \left[ \|EST - \nabla\|_2^2 \right] = \sigma^2 2^{-O(\frac{B}{d})} .
\]

Furthermore, if the algorithm ensures that all machines receive strictly at most \( B \) bits,

\[
\mathbb{E} \left[ \|EST - \nabla\|_2^2 \right] = n \sigma^2 2^{-O(\frac{B}{d})} .
\]

**Proof.** We will define an input distribution, and begin similarly by choosing vector \( \nabla \) uniformly at random from \( B_{\ell^2} \nabla (0) \). The probability that \( ||\mu||_2 \leq d^2 n \sigma - n \sigma \) is (similarly to the previous proof):

\[
\frac{\text{Vol}(B_{\ell^2} \nabla \sigma)}{\text{Vol}(B_{\ell^2} \sigma)} \geq \left( \frac{d^2 n \sigma}{n \sigma} \right)^d = \left( 1 - \frac{1}{d^2} \right)^d \geq 1 - \frac{2}{d^2} .
\]

Therefore, with probability \( 1 - \frac{2}{d^2} \), \( ||\mu||_2 \leq d^2 n \sigma - n \sigma \). We call this event \( A \).

We now choose the input \( x_v \) for each machine \( v \) as follows: we first choose one machine uniformly at random to designate as \( \xi \). This is the machine that we will show requires extra communication bits. We generate \( x_\xi \) uniformly from \( B_{\ell^2} \nabla (\nabla) \); for all other machines \( v \) we choose \( x_v \) independently uniformly from \( B_{\ell^2} \nabla (\nabla) \). We will denote by \( E_v \) the event that machine \( v \) is designated as \( \xi \). We note that the inputs, are, as required, unbiased estimators of \( \nabla \), and (since each machine has a \( \frac{1}{n} \) chance of being \( \xi \)) they have variance at most \( \frac{1}{n} \left( \left( \frac{\sigma \sqrt{d^2}}{d} \right)^2 + \left( 1 - \frac{1}{n} \right) \cdot \left( \frac{\sigma \sqrt{d^2}}{d} \right)^2 < \sigma^2 \right) \).

We fix \( v \), and first consider the case when \( E_v \) does not occur. Then, conditioning on \( A, \overline{E_v} \), and the value of \( x_\xi, \nabla \) is uniformly distributed in \( B_{\ell^2} (x_\xi) \). As before, if \( v \) receives fewer than \( b \)
bits then it has only $2^b - 1$ possible outputs, and the volume of the space within distance $\text{dist}$ of one of these outputs is at most $(2^b - 1)\text{Vol}(B_{\text{dist}})$. If we set $\text{dist} = \frac{\sigma}{2}3^{-\frac{b}{2}}$ then:

$$\Pr \left[ \|EST - \mu\|_2 \leq \text{dist} \mid E_v, b_v < b \right] \leq \Pr \left[ \|EST - \mu\|_2 \leq \text{dist} \mid E_v, A, b_v < b \right] + \Pr \left[ A \right]$$

$$\leq \frac{(2^b - 1)\text{Vol}(B_{\text{dist}})}{\text{Vol}(B_{\frac{\sigma}{2}})} + \frac{2}{d}$$

$$\leq (2^b - 1) \left( \frac{\sigma 3^{-\frac{b}{2}}}{\frac{\sigma}{2}} \right)^d + \frac{2}{d}$$

$$= \left( \frac{2}{3} \right)^b - 3^{-b} + \frac{2}{d} < \frac{1}{2}.$$  

We can then set $b = 2B$ and obtain

$$\mathbb{E} \left[ \|EST - \nabla\|_2^2 \right] = \Omega(\text{dist}^2) = y^2 2^{-O(B)}.$$

by the same argument as in Theorem $[10]$ proving the first part of Theorem $[11]$.

We now consider what happens if the event $E_v$ occurs. Then, conditioning on $A, E_v,$ and the value of $x_v, \nabla$ is uniformly distributed in $B_{\frac{\sigma}{2}}(x_v)$. As before, if $v$ receives fewer than $b$ bits then it has only $2^b - 1$ possible outputs, and the volume of the space within distance $\text{dist}$ of one of these outputs is at most $(2^b - 1)\text{Vol}(B_{\text{dist}})$. Using $\text{dist} = \frac{\sigma 3^{-\frac{b}{2}}}{\frac{\sigma}{2}}$,

$$\Pr \left[ \|EST - \mu\|_2 \leq \text{dist} \mid E_v, b_v < b \right] \leq \Pr \left[ \|EST - \mu\|_2 \leq \text{dist} \mid E_v, A, b_v < b \right] + \Pr \left[ A \right]$$

$$\leq \frac{(2^b - 1)\text{Vol}(B_{\text{dist}})}{\text{Vol}(B_{\frac{\sigma}{2}})} + \frac{2}{d}$$

$$\leq (2^b - 1) \left( \frac{\sigma 3^{-\frac{b}{2}}}{\frac{\sigma}{2}} \right)^d + \frac{2}{d}$$

$$= \left( \frac{2}{3} \right)^b - 3^{-b} + \frac{2}{d} < \frac{1}{2}.$$  

So, $\mathbb{E} \left[ \|EST - \mu\|_2^2 \mid E_v, b_v < b \right] \geq \frac{1}{2} \text{dist}^2 = \Omega(n\sigma^29^{-\frac{b}{2}})$, and (again using $b = 2B$) we can conclude that:

$$\mathbb{E} \left[ \|EST - \nabla\|_2^2 \mid E_v \right] = \Omega(\text{dist}^2) = n\sigma^22^{-O(B)}.$$

Furthermore, with certainty one of the events $E_v$ occurs. Therefore, any VARIANCEREDUCTION algorithm in which all machines receive at most $B$ bits has $\mathbb{E} \left[ \|EST - \mu\|_2^2 \right] = \Omega(n\sigma^29^{-\frac{B}{2}})$, completing proof of the second part of the theorem.

This result implies that to achieve any variance reduction at all (i.e. for the variance of the output $EST$ to be lower than the input variance $\sigma^2$), at least one machine must receive $\Omega(d \log n)$ bits, and this is also the amount of bits required to achieve the optimal output variance $O(\sigma^2)$. Therefore, if we measure cost by the maximum number of bits communicated by a machine, there is no trade-off between communication cost and output quality; one should always use $\Theta(d \log n)$ bits. If we instead measure the average number of bits communicated per machine, we have the same trade-off as for ESTIMATEMEAN, except that the best output variance possible is $\Theta(\frac{\sigma^2}{n})$, and so using more than $\Theta(d \log n)$ bits in expectation does not provide any further improvement.
8 Experimental Validation

In this section, we validate our results experimentally, in the context of the popular and resource-intensive task of training large-scale neural networks in a distributed environment. We consider the task of training a deep neural network (ResNet20 [10]) for image classification on the CIFAR-10 dataset [13], in a distributed environment. This dataset consists of 60,000 images partitioned into 10 classes, 10,000 of which are kept for model validation, while the ResNet20 model is a convolutional neural network with 20 layers and approximately 270K parameters. Due to the computational cost of training this model/dataset combination, it is common for training to be distributed onto more than one GPU node.

Distributed SGD and Variance Reduction. Mathematically, the function $f$ we wish to minimize in training is the total error of the model $\vec{x}$ on the given dataset $D$. For each sample $s$ in $D$, the classification error is encoded via the loss $\ell(s, \vec{x})$. Training therefore minimizes the function

$$f(\vec{x}) = \frac{1}{m} \sum_{s \in D} \ell(s, \vec{x}),$$

where $m$ is the size of the dataset. In this case, the gradient of the loss at a randomly chosen datapoint, which we denote by $\tilde{g}$, is an unbiased estimator of the true gradient of the loss. This which inspires the following iteration, corresponding to the stochastic gradient descent (SGD) algorithm:

$$\vec{x}_{t+1} = \vec{x}_t - \alpha \tilde{g} (\vec{x}_t), \tag{1}$$

where $\alpha$ is the learning rate parameter.

A standard way of solving this minimization problem in parallel is to process a set of samples (usually called a batch) in parallel, dividing the computation of gradient updates among processors. Assume for simplicity that each processor is allotted one sample, whose corresponding gradient it computes with respect to the current model $\vec{x}_t$. Processors then sum their stochastic gradients, and update their local models by the resulting sum, leading to the following global iteration:

$$\vec{x}_{t+1} = \vec{x}_t - \alpha \sum_{i=1}^{n} \tilde{g}^i (\vec{x}_t), \tag{2}$$

where $\tilde{g}^i$ is the stochastic gradient obtained at the processor $i$ at the given step, and $p$ is the batch size, equal to the number of processors. Since this sum is the same at every processor, this procedure yields the same model at each processor at the end of every parallel iteration. The average $(1/n) \sum_{i=1}^{n} \tilde{g}^i (\vec{x}_t)$ is still a stochastic gradient, but with lower variance than gradients at single processors. This in turn should imply a faster rate of convergence for this batched variant of the algorithm, see e.g. [5]. Since samples are processed in parallel, parallelization should result in a reduction of the wall-clock running time, since the number of samples processed per second is in theory multiplied by $n$.

However, in practice, due to the high dimensionality of the models/gradients, the parallel summation procedure can easily become a system bottleneck. The generic solution to this problem, e.g. [2, 8, 17] has been to quantize the individual components of the gradient vector $\tilde{g}^i$ at every processor $i$.

Q1: Is the Second Moment Bound of Gradients Small? As discussed in Section 3 a common assumption made by this previous work on gradient quantization was that the second moment of the gradients is small, that is, there exists a small constant $M^2$ such that:
∀ ⃗x ∈ R^d, E_{⃗x ∼ D} \left[ \| \tilde{g}(⃗x) \|^2 \right] \leq M^2.

(3)

Recall, that, instead, we argue that a more natural assumption in this context is to assume that the variance of the stochastic gradients is well-bounded. After all, if the true gradient were zero, then the procedure would not actually have to optimize anything—the stochastic gradients amount to noise.

Our first experiment investigates whether the assumption that the true gradient is close to zero is valid. For this, we sample the true gradient during the standard training procedure of the ResNet20 model on CIFAR10 (after each epoch, which is a full training iteration over the entire dataset), and measure its two-norm. Under the above assumptions, this norm value should be close to zero. The result in Figure 1 clearly shows that the norm of the gradient is not close to zero. Moreover, this norm is not even monotonically decreasing during the first part of training.

Our second experiment compares the variance of the gradients against their second moment during the first part of the training process. For this, we sample stochastic gradients at the beginning of each training epoch and estimate their variance and second moment bound (average squared 2-norm). The results from Figure 2 exhibit a clear gap between the variance and the second moment, which confirms our choice of assumptions.

Q2: Does Lattice Quantization Outperform Previous Methods? The second key question we wish to examine is whether our method can provide lower variance for the same number of bits sent, compared against other state-of-the-art distributed mean estimation methods, notably QSGD [2] and Hadamard-based unbiased quantization [19]. For this, we implemented all these schemes in Pytorch [16], a popular machine learning framework. Concretely, we fixed the number of bits and the granularity at which we quantize (i.e., how many sub-components are quantized jointly), and compared the variance of the quantized gradients generated, by examining multiple quantizations of the same real stochastic gradient.

For lattice-based quantization (LQSGD in the results), we employ the natural cubic lattice in order to minimize computational cost. (We note that our results can only improve with a better choice of lattice.) Each node uses its own stochastic gradient in order to decode. Further, in terms of parametrization, we set \( b = 2 \log d \), where \( b \) is the number of bits and \( d \) is the size...
of the quantized vector (the granularity at which we quantize). So, in these experiments, the number of bits used and the granularity of the quantization for LQSGD are linked, although this does not have to be the case in general.

The experiments in Figures 3 and 4 display the variance of the quantized gradients computed using these three schemes, when using 4 bits and 8 bits per entry, respectively. (These bit-widths are common in quantization schemes, due to efficient computational support, e.g., [2].) We note that lattice-based quantization can ensure significantly lower variance relative to these state-of-the-art schemes, even for this simple choice of lattice. In particular, the variance induced by our scheme in the case where 4 bits are used per entry can be two orders of magnitude lower than that induced by any of the other methods. We therefore conclude that our method can ensure significant improvements over previous work in practice.

9 Conclusions

We have argued in this work that for the problems of distributed mean estimation and variance reduction, one should measure the output variance in terms of the input variance, rather than the second moment as used by previous works. Through this change in perspective, we have shown truly optimal algorithms, and matching lower bounds, for both problems, independently of the norms of the input vectors. This improves significantly over previous work (for which there was a large gap between upper and lower bounds) whenever the inputs are not known to be concentrated around the origin, both theoretically and in terms of practical performance.

Regarding potential future directions, Theorem 11 demonstrates that any VARIANCE REDUCTION algorithm achieving optimal $O(\frac{\sigma^2}{n})$ output variance must cause at least one machine to receive $\Omega(d \log n)$ bits, and we have presented an algorithm for which the communication cost for all machines matches this bound. However, there remains open the possibility of an algorithm in which any fixed machine $v$ receives only $O(d)$ bits in expectation, and this would be an exciting avenue of research. Such an algorithm would necessitate an efficient method for machines to estimate the distance between their input and the average input, since it is precisely those nodes whose input is far from the average that must receive more bits.
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