SCATTERING FOR THE NON-RADIAL 3D CUBIC NONLINEAR
SCHRÖDINGER EQUATION

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Abstract. Scattering of radial $H^1$ solutions to the 3D focusing cubic nonlinear Schrödinger equation below a mass-energy threshold $M[u]E[u] < M[Q]E[Q]$ and satisfying an initial mass-gradient bound $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, where $Q$ is the ground state, was established in Holmer-Roudenko [7]. In this note, we extend the result in [7] to non-radial $H^1$ data. For this, we prove a non-radial profile decomposition involving a spatial translation parameter. Then, in the spirit of Kenig-Merle [10], we control via momentum conservation the rate of divergence of the spatial translation parameter and by a convexity argument based on a local virial identity deduce scattering. An application to the defocusing case is also mentioned.

1. Introduction

We consider the Cauchy problem for the cubic focusing nonlinear Schrödinger (NLS) equation on $\mathbb{R}^3$:

\begin{align*}
\frac{i}{\partial_t}u + \Delta u + |u|^2 u &= 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \\
u(x, 0) &= u_0 \in H^1(\mathbb{R}^3).
\end{align*}

It is locally well-posed (e.g., see Cazenave [3]). The equation has 3 conserved quantities; namely, the mass $M[u]$, energy $E[u]$ and momentum $P[u]$:

\begin{align*}
M[u] &= \int |u(x, t)|^2 \, dx = M[u_0], \\
E[u] &= \frac{1}{2} \int |\nabla u(x, t)|^2 \, dx - \frac{1}{4} \int |u(x, t)|^4 \, dx = E[u_0], \\
P[u] &= \text{Im} \int \bar{u}(x, t) \nabla u(x, t) \, dx = P[u_0].
\end{align*}

The scale-invariant Sobolev norm is $\dot{H}^{1/2}$ and the scale-invariant Lebesgue norm is $L^3$. Let $u(x, t) = e^{it}Q(x)$; then $u$ solves (1.1) provided $Q$ solves the nonlinear elliptic equation

\begin{equation}
-Q + \Delta Q + |Q|^2 Q = 0.
\end{equation}

This equation has an infinite number of solutions in $H^1(\mathbb{R}^3)$. The solution of minimal mass, hereafter denoted by $Q(x)$, is positive, radial, exponentially decaying, and is called the ground state. For further properties of $Q$, we refer to Weinstein [14], Holmer-Roudenko [7], Cazenave [3].

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In Holmer-Roudenko [7, Theorem 1.1] (see also Holmer-Roudenko [8]), it was proved that under the condition $M\|u\|E[u] < M\|Q\|E[Q]$, solutions to (1.1)-(1.2) globally exist if $u_0$ satisfies

\begin{equation}
\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2},
\end{equation}

and radial solutions with initial data satisfying (1.4) scatter in $H^1$ in both time directions. This means that there exist $\phi_{\pm} \in H^1$ such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta}\phi_{\pm}\|_{H^1} = 0.$$ 

In this note we extend the scattering result to include non-radial $H^1$ data.

**Theorem 1.1.** Let $u_0 \in H^1$ and let $u$ be the corresponding solution to (1.1) in $H^1$. Suppose

\begin{equation}
M\|u\|E[u] < M\|Q\|E[Q].
\end{equation}

If $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, then $u$ scatters in $H^1$.

The argument of [7] in the radial case followed a strategy introduced by Kenig-Merle [9] for proving global well-posedness and scattering for the focusing energy-critical NLS. The argument begins by contradiction: suppose the threshold for scattering is strictly below that claimed. A profile decomposition lemma based on concentration compactness principles (and analogous to that of Keraani [11]) was invoked to prove the existence of a global but nonscattering solution $u_c$ standing exactly at the threshold between scattering and nonscattering. The profile decomposition lemma is again invoked to prove that the flow of $u_c$ is a precompact subset of $H^1$, which then implies that $u_c$ remains spatially localized uniformly in time. This uniform localization enabled the use of a local virial identity to establish, with the aid of the sharp Gagliardo-Nirenberg inequality, a strictly positive lower bound on the convexity (in time) of the local mass of $u_c$. Mass conservation is then violated at a sufficiently large time.

In this paper, we show that the above program carries over to the non-radial setting with the addition of two key ingredients. First, in §2, we introduce a profile decomposition lemma that applies to non-radial $H^1$ sequences. To compensate for the lack of localization at the origin induced by radiality, a spatial translation sequence is needed. We also here adapt the proof given in [7] of the energy Pythagorean expansion (Lemma 2.3) to apply to non-radial sequences; in [7], an inessential application of the compact embedding $H^1_{rad} \to L^4$ was used at one point. The profile decomposition and concentration compactness techniques are previously used in works of Keraani [11], Gerard [5], see also Bahouri and Gerard [1]-[2], and originate from P.-L. Lions [12]-[13].

The application of the non-radial profile decomposition to time slices of the flow of the critical solution $u_c$ yields the existence of a continuous time translation parameter $x(t)$ such that the translated flow $u_c(\cdot - x(t),t)$ is precompact in $H^1$ (Prop. 3.2). This implies the localization of $u_c(\cdot, t)$ near $x(t)$ (as opposed to the radial case, in which localization is obtained near the origin).

Obtaining suitable control on the behavior of $x(t)$ is the main new step beyond [7]. This is done by following a method introduced by Kenig-Merle [10] (who applied it to the energy-critical nonlinear wave equation). First, we argue that by Galilean
invariance, the solution $u_c$ must have zero momentum (see §4). An appropriate selection of the phase shift is possible in our case since our solution belongs to $L^2$. This zero-momentum solution is then shown in §5 to have a near-conservation of localized center-of-mass, which provides the desired control on the rate of divergence of $x(t)$ (specifically, $x(t)/t \to 0$ as $t \to \infty$).

In §7, we remark on the adaptation of these techniques to the defocusing cubic NLS in 3D.

2. Non-radial profile and energy decompositions

We will make use of the Strichartz norm notation used in [7]. We say that $(q, r)$ is $H^s$ Strichartz admissible (in 3D) if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s.$$ 

Let

$$\|u\|_{S(L^2)} = \sup_{(q, r) \text{ admissible}} \|u\|_{L^q_t L^r_x}.$$ 

Define

$$\|u\|_{S(\dot{H}^{1/2})} = \sup_{(q, r) \text{ admissible}} \|u\|_{L^q_t L^r_x},$$

where $6^-$ is an arbitrarily preselected and fixed number $< 6$; similarly for $4^+$.  

Lemma 2.1 (Profile expansion). Let $\phi_n(x)$ be a uniformly bounded sequence in $H^1$. Then for each $M$ there exists a subsequence of $\phi_n$, also denoted $\phi_n$, and

1. for each $1 \leq j \leq M$, there exists a (fixed in $n$) profile $\psi^j(x)$ in $H^1$,
2. for each $1 \leq j \leq M$, there exists a sequence (in $n$) of time shifts $t^j_n$,
3. for each $1 \leq j \leq M$, there exists a sequence (in $n$) of space shifts $x^j_n$,
4. there exists a sequence (in $n$) of remainders $W^M_n(x)$ in $H^1$,

such that

$$\phi_n(x) = \sum_{j=1}^M e^{-it^j_n} \psi^j(x - x^j_n) + W^M_n(x).$$

The time and space sequences have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq M$, we have

$$\lim_{n \to +\infty} |t^j_n - t^k_n| + |x^j_n - x^k_n| = +\infty.$$  

(2.1)

The remainder sequence has the following asymptotic smallness property$^2$:

$$\lim_{M \to +\infty} \left[ \lim_{n \to +\infty} \|e^{it^j_n} W^M_n \|_{S(\dot{H}^{1/2})} \right] = 0.$$

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$^1$It could not be applied in the Kenig-Merle paper [9] on the energy critical NLS since the argument there takes place in $H^1$.

$^2$We can always pass to a subsequence in $n$ with the property that $\|e^{it^j_n} W^M_n \|_{S(\dot{H}^{1/2})}$ converges. Therefore, we use lim and not lim sup or lim inf. Similar remarks apply for the limits that appear in the Pythagorean expansion.
For fixed $M$ and any $0 \leq s \leq 1$, we have the asymptotic Pythagorean expansion

$$
\|\phi_n\|_{H^s}^2 = \sum_{j=1}^{M} \|\psi_j\|_{H^s}^2 + \|W_n^M\|_{H^s}^2 + o_n(1).
$$

Remark 2.2. If the assumption that $\phi_n$ is uniformly bounded in $H^1$ is weakened to the assumption that $\phi_n$ is uniformly bounded in $H^{1/2}$, then the above profile decomposition remains valid provided a scaling parameter $\lambda$ is also involved, similar to the theorem in [11]. However, it is not needed for the results of this note and for simplicity of exposition the proof is omitted.

Proof. The proof is very close to the one of [7, Lemma 5.2]. We also refer to [11] for a similar result in the energy-critical case.

Step 1. Construction of $\psi_n^1$. Let $A_1 = \limsup_n \|e^{it\Delta} \phi_n\|_{L_t^q L_x^r}$. If $A_1 = 0$, we are done. Indeed, for an arbitrary $H^{1/2}$-admissible couple $(q, r)$ we have

$$
\|e^{it\Delta} \phi_n\|_{L_t^q L_x^r} \leq \|e^{it\Delta} \phi_n\|_{L_t^1 L_x^2}^{\theta} \|e^{it\Delta} \phi_n\|_{L_t^1 L_x^2}^{1-\theta} \quad \text{with} \quad \theta = \frac{4}{q} \in (0, 1).
$$

Noting that $\|e^{it\Delta} \phi_n\|_{L_t^1 L_x^2} \leq C\|\phi_n\|_{H^{1/2}}$, we get that $\limsup_n \|e^{it\Delta} \phi_n\|_{S(H^{1/2})} = 0$, and we can take $\psi^j = 0$ for all $j$.

If $A_1 > 0$, let

$$
c_1 = \limsup_n \|\phi_n\|_{H^1}.
$$

Extracting a subsequence from $\phi_n$, we show that there exist sequences $t_n^1$, $x_n^1$ and a function $\psi^1 \in H^1$ such that

$$
e^{it_n^1 \Delta} \phi_n(\cdot + x_n^1) \rightharpoonup \psi^1 \quad \text{weakly in} \quad H^1,
$$

$$
Kc_1^4 \|\psi^1\|_{H^{1/2}} \geq A_1^5,
$$

where $K > 0$ is a constant independent of all parameters.

Let $r = \frac{16c_1^2}{A_1^2}$ and $\chi_r$ be a radial Schwartz function such that $\hat{\chi}_r(\xi) = 1$ for $\frac{1}{r} \leq |\xi| \leq r$, and supp $\chi_r \subset \left[\frac{1}{2r}, 2r\right]$. By the arguments of [7], there exists sequences $t_n^1$, $x_n^1$ such that

$$
|\chi_r * e^{it_n^1 \Delta} \phi_n(x_n^1)| \geq \frac{A_1^3}{32c_1^2}.
$$

Pass to a subsequence so that $e^{it_n^1 \Delta} \phi_n(\cdot + x_n^1) \rightharpoonup \psi^1$ weakly in $H^1$. In [7] the functions $\phi_n$ are radial, and thus, by the radial Gagliardo-Nirenberg inequality, one can show that $x_n^1$ is bounded in $n$, which is not necessarily the case here. As in [7], the estimate $\|\chi_r\|_{H^{-1/2}} \leq r$ yields, together with Plancherel and Cauchy-Schwarz inequalities, the estimate (2.5).

Next, define $W_n^1(x) = \phi_n(x) - e^{-it_n^1 \Delta} \psi^1(x - x_n^1)$. Since $e^{it_n^1 \Delta} \phi_n(\cdot + x_n^1) \rightharpoonup \psi^1$ in $H^1$, expanding $\|W_n^1\|_{H^s}^2$ as an inner product and using the definition of $W_n^1$, we obtain

$$
\lim_{n \to \infty} \|W_n^1\|_{H^s}^2 = \lim_{n \to \infty} \|e^{it_n^1 \Delta} \phi_n\|_{H^s}^2 - \|\psi^1\|_{H^s}^2, \quad 0 \leq s \leq 1,
$$

which yields (2.3) for $M = 1$. 
Step 2. Construction of $\psi^j$ for $j \geq 2$. We construct the functions $\psi^j$ inductively, applying Step 1 to the sequences (in $n$) $W_n^{j-1}$. Let $M \geq 2$. Assuming that $\psi^j$, $x_n^j$, $t_n^j$ and $W_n^j$ are known for $j \in \{1, \ldots, M-1\}$, we consider

$$A_M = \limsup_n \|W_n^{M-1}\|_{L^8_t L^2_x}.$$  

If $A_M = 0$, we take, as in Step 1, $\psi^j = 0$ for $j \geq M$. Assume $A_M > 0$. Applying Step 1 to the sequence $W_n^{M-1}$, we obtain, extracting if necessary, sequences $x_n^M$, $t_n^M$ and a function $\psi^M \in H^1$ such that

$$(2.6) \quad e^{it_n^M \Delta} W_n^{M-1}(\cdot + x_n^M) \rightharpoondown \psi^M \text{ weakly in } H^1,$$

$$(2.7) \quad Kc_M \|\psi^M\|_{H^{1/2}}^2 \geq A_M^2,$$

where $c_M = \limsup_n \|W_n^{M-1}\|_{H^1}$. We then define $W_n^M(x) = W_n^{M-1}(x) - e^{-it_n^M \Delta} \psi^M(x - x_n^M)$.

We next show (2.1) and (2.3) by induction. Assume that (2.3) holds at rank $M-1$. Expanding

$$\|W_n^M\|_{H^s}^2 = \|e^{it_n^M \Delta} W_n^M(\cdot + x_n^M)\|_{H^s}^2 = \|e^{it_n^M \Delta} W_n^{M-1}(\cdot + x_n^M) - \psi^M\|_{H^s}^2$$

and using the weak convergence (2.6), we obtain directly (2.3) at rank $M$.

Assume that the condition (2.1) holds for $j, k \in \{1, \ldots, M-1\}$. Let $j \in \{1, \ldots, M-1\}$. Then (here, $W_n^0 = \phi_n$),

$$-e^{it_n^M \Delta} W_n^{M-1}(x + x_n^j) + e^{it_n^M \Delta} W_n^{j-1}(x + x_n^j) - \psi^j(x) = \sum_{k=j+1}^{M-1} e^{it_n^{t_k} - t_n^k} \Delta \psi^k(x + x_n^j - x_n^k).$$

By the orthogonality condition (2.1), the right hand side converges to 0 weakly in $H^1$ as $n$ tends to infinity. Furthermore, by the definition of $W_n^j$,

$$e^{it_n^M \Delta} W_n^{j-1}(x + x_n^j) - \psi^j(x) \rightharpoondown_{n \to +\infty} 0 \text{ weakly in } H^1.$$  

Thus, $e^{it_n^M \Delta} W_n^{M-1}(x + x_n^j)$ must go to 0 weakly in $H^1$. From (2.6), we deduce, if $\psi^M \neq 0$

$$\lim_{n \to +\infty} |x_n^j - x_n^M| + |t_n^j - t_n^M| = +\infty,$$

which shows that (2.1) must also holds for $k = M$.

It remains to show (2.2). Note that by (2.3), $c_M \leq c_1$ for all $M$. If for all $M$, $A_M > 0$, we have by (2.3)

$$\sum_{M \geq 1} A_M^{10} \leq K^2 c_1 \sum_{n \geq 1} \|\psi^M\|_{H^{1/2}}^2 \leq K^2 c_1 \limsup_n \|\phi_n\|_{H^{1/2}}^2 < \infty,$$

which shows that $A_M$ tends to 0 as $M$ goes to $\infty$, yielding (2.2) and concluding the proof of Lemma 2.1. □

Lemma 2.3 (Energy Pythagorean expansion). In the situation of Lemma 2.1, we have

$$(2.8) \quad E[\phi_n] = \sum_{j=1}^M E[e^{-it_n^M \Delta} \psi^j] + E[W_n^M] + o_n(1).$$
Proof. According to (2.3), it suffices to establish for all $M \geq 1$,
\begin{equation}
\|\phi_n\|_{L^4}^4 = \sum_{j=1}^{M} \|e^{-it_n^j\Delta}\psi^j\|_{L^4}^4 + \|W_n^{M}\|_{L^4}^4 + o_n(1).
\end{equation}

Step 1. Pythagorean expansion of a sum of orthogonal profiles. We show that if $M \geq 1$ is fixed, orthogonality condition (2.1) implies
\begin{equation}
\left\| \sum_{j=1}^{M} e^{-it_n^j\Delta}\psi^j \right\|_{L^4}^4 = \sum_{j=1}^{M} \|e^{-it_n^j\Delta}\psi^j\|_{L^4}^4 + o_n(1).
\end{equation}
By reindexing, we can arrange so that there is $M_0 \leq M$ such that
- For $1 \leq j \leq M_0$, we have that $t_n^j$ is bounded in $n$.
- For $M_0 + 1 \leq j \leq M$, we have that $|t_n^j| \to \infty$ as $n \to \infty$.
By passing to a subsequence, we may assume that for each $1 \leq j \leq M_0$, $t_n^j$ converges (in $n$), and by adjusting the profiles $\psi^j$ we can take $t_n^j = 0$.
Note that
\begin{equation}
M_0 + 1 \leq k \leq M \implies \lim_{n \to +\infty} \|e^{-it_n^k\Delta}\psi^k\|_{L^4} = 0.
\end{equation}
Indeed, in this case $|t_n^k| \to \infty$ as $n \to \infty$. For a function $\tilde{\psi} \in \dot{H}^{3/4} \cap L^{4/3}$, from Sobolev embedding and the $L^p$ space-time decay estimate of the linear flow, we obtain
$$
\|e^{-it_n^k\Delta}\psi^k\|_{L^4} \leq c \|\psi^k - \tilde{\psi}\|_{\dot{H}^{3/4}} + \frac{c}{|t_n^k|^{3/2}} \|\tilde{\psi}\|_{L^{4/3}}.
$$
By approximating $\psi^k$ by $\tilde{\psi} \in C_{0}^{\infty}$ in $\dot{H}^{3/4}$ and sending $n \to \infty$, we obtain (2.11).
By (2.1), if $1 \leq j < k \leq M_0$, $\lim_n |x_n^j - x_n^k| = +\infty$, and thus, it implies
$$
\left\| \sum_{j=1}^{M_0} \psi^j \right\|_{L^4}^4 = \sum_{j=1}^{M_0} \|\psi^j\|_{L^4}^4 + o_n(1),
$$
which yields, together with (2.11), expansion (2.10).

Step 2. End of the Proof. We first note
\begin{equation}
\lim_{M_1 \to +\infty} \left( \lim_{n \to +\infty} \|W_n^{M_1}\|_{L^4} \right) = 0.
\end{equation}
Indeed,
\begin{align*}
\|W_n^{M_1}\|_{L^4} &\leq \|e^{it\Delta}W_n^{M_1}\|_{L_t^\infty L_x^4} \\
&\leq \|e^{it\Delta}W_n^{M_1}\|_{L_t^{1/2} L_x^2}^{1/2} \|e^{it\Delta}W_n^{M_1}\|_{L_t^{1/2} H_x^{1/2}}^{1/2} \\
&\leq \|e^{it\Delta}W_n^{M_1}\|_{L_t^{1/2} L_x^2} \sup_n \|\phi_n\|_{H^1}^{1/2}.
\end{align*}
By (2.2), we get (2.12).
Let $M \geq 1$ and $\varepsilon > 0$. Note that $\{\phi_n\}_n$ is uniformly bounded in $L^4$, since it is uniformly bounded in $H^1$ by the hypothesis; furthermore, by (2.12) $\{W_n^{M}\}_n$ is also
uniformly bounded in $L^4$. Thus, we can choose $M_1 \geq M$ and $N_1$ such that for $n \geq N_1$, we have
\begin{equation}
(2.13) \quad \left\| \phi_n - W^{M_1} - \phi_n \right\|_{L^4}^4 + \left\| W^{M_1} - W^{M_1}_n \right\|_{L^4}^4 + \left\| W^{M_1}_n \right\|_{L^4}^4 \leq C \left( \sup_n \| \phi_n \|_{L^4}^3 + \sup_n \left\| W^{M_1}_n \right\|_{L^4}^3 \right) \left\| W^{M_1}_n \right\|_{L^4} + \left\| W^{M_1}_n \right\|_{L^4}^4 \leq \varepsilon.
\end{equation}

By (2.10), we get $N_2 \geq N_1$ such that for $n \geq N_2$,
\begin{equation}
(2.14) \quad \left\| \phi_n - W^{M_1}_n \right\|_{L^4}^4 - \sum_{j=1}^{M_1} \left\| e^{-it_n \Delta} \psi_j \right\|_{L^4}^4 \leq \varepsilon.
\end{equation}

Using the definition of $W^{M}_n$, expand $W^{M}_n - W^{M_1}_n$ to obtain
\[ W^{M}_n - W^{M_1}_n = \sum_{j=M+1}^{M_1} e^{-it_n \Delta} \psi_j (-x_j). \]

By (2.10) there exists $N_3 \geq N_2$ such that for $n \geq N_2$,
\begin{equation}
(2.15) \quad \left\| W^{M}_n - W^{M_1}_n \right\|_{L^4}^4 - \sum_{j=M+1}^{M_1} \left\| e^{-it_n \Delta} \psi_j \right\|_{L^4}^4 \leq \varepsilon.
\end{equation}

By (2.13), (2.14) and (2.15), we obtain that for $n \geq N_3$,
\[ \left\| \phi_n \right\|_{L^4}^4 - \sum_{j=1}^{M} \left\| e^{-it_n \Delta} \psi_j \right\|_{L^4}^4 - \left\| W^{M}_n \right\|_{L^4}^4 \leq 4\varepsilon, \]
which concludes the proof of (2.9).

3. Outline of the proof of the main result

Let $u(t)$ be the corresponding $H^1$ solution to (1.1)-(1.2). By Theorem 1.1(1)(a) in [7] the solution is globally well-posed, so our goal is to show that
\begin{equation}
(3.1) \quad \| u \|_{S(H^{1/2})} < \infty.
\end{equation}

This combined with Proposition 2.2 from [7] will give $H^1$ scattering. We will use the strategy of [9]. We shall say that $SC(u_0)$ holds if (3.1) is true for the solution $u(t)$ generated from $u_0$.

By the small data theory there exists $\delta > 0$ such that if $M[u]E[u] < \delta$ and $\| u_0 \|_{L^2} \| \nabla u_0 \|_{L^2} < \| Q \|_{L^2} \| \nabla Q \|_{L^2}$, then (3.1) holds. For each $\delta > 0$ define the set $S_\delta$ to be the collection of all such initial data in $H^1$:
\[ S_\delta = \{ u_0 \in H^1 \mid M[u]E[u] < \delta \quad \text{and} \quad \| u_0 \|_{L^2} \| \nabla u_0 \|_{L^2} < \| Q \|_{L^2} \| \nabla Q \|_{L^2} \}. \]

Next define $(ME)_c = \sup \{ \delta : u_0 \in S_\delta \Rightarrow SC(u_0) \text{ holds} \}$. If $(ME)_c = M[Q]E[Q]$, then we are done, so we assume
\begin{equation}
(3.2) \quad (ME)_c < M[Q]E[Q].
\end{equation}

Then there exists a sequence of solutions $u_n$ to (1.1) with $H^1$ initial data $u_{n,0}$ (rescale all of them to have $\| u_n \|_{L^2} = 1$ for all $n$) such that $\| \nabla u_{n,0} \|_{L^2} < \| Q \|_{L^2} \| \nabla Q \|_{L^2}$ and $E[u_n] \backslash (ME)_c$ as $n \to +\infty$, for which $SC(u_{n,0})$ does not hold for any $n$. 
The next proposition gives the existence of an $H^1$ solution $u_c$ to (1.1) with initial data $u_{c,0}$ such that $\|u_{c,0}\|_{L^2} \|\nabla u_{c,0}\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ and $M[u_c]E[u_c] = (ME)_c$ for which SC($u_{c,0}$) does not hold. This will imply that $K = \{ u_c(-x(t), t) \mid 0 \leq t < +\infty \}$ is precompact in $H^1$ (Proposition 3.2). As a consequence (see Corollary (3.3)) we obtain that for each $\epsilon > 0$, there is an $R > 0$ such that, uniformly in $t$, we have

$$ \int_{|x+x(t)| > R} |\nabla u_c(t, x)|^2 dx \leq \epsilon. $$

This together with the hypothesis of zero momentum (which can always be achieved by Galilean invariance – see §4) provides a control on the growth of $x(t)$ (Lemma 5.1). Finally, the rigidity theorem (Theorem 6.1), which appeals to this control on $x(t)$ and the uniform localization (3.3), will lead to a contradiction that such critical element exists (unless it is identically zero) which will conclude the proof.

**Proposition 3.1** (Existence of a critical solution). Assume (3.2). Then there exists a global ($T^* = +\infty$) solution $u_c$ in $H^1$ with initial data $u_{c,0}$ such that $\|u_{c,0}\|_{L^2} = 1,$

$$ E[u_c] = (ME)_c < M[Q]E[Q], $$

$$ \|\nabla u_c(t)\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2} \quad \text{for all } 0 \leq t < +\infty, $$

and

$$ \|u_c\|_{S(H^1/2)} = +\infty. $$

**Proof.** The proof closely follows the proof of [7, Prop 5.4]. \qed

**Proposition 3.2** (Precompactness of the flow of the critical solution). With $u_c$ as in Proposition 3.1, there exists a continuous path $x(t)$ in $\mathbb{R}^3$ such that

$$ K = \{ u_c(-x(t), t) \mid t \in [0, +\infty) \} \subset H^1 $$

is precompact in $H^1$ (i.e., $K$ is compact).

**Proof.** For convenience, we write $u = u_c$. We argue by contradiction. By the arguments in Appendix A, we can assume that there exists $\eta > 0$ and a sequence $t_n$ such that for all $n \neq n'$,

$$ \inf_{x_0 \in \mathbb{R}^3} \| u(-x_0, t_n) - u(-x_0, t_{n'}) \|_{H^1} \geq \eta. $$

Take $\phi_n = u(t_n)$ in the profile expansion lemma (Lemma 2.1). The remainder of the argument closely follows the proof of [7, Prop 5.5]. \qed

**Corollary 3.3** (Precompactness of the flow implies uniform localization). Let $u$ be a solution to (1.1) such that

$$ K = \{ u(-x(t), t) \mid t \in [0, +\infty) \} $$

is precompact in $H^1$. Then for each $\epsilon > 0$, there exists $R > 0$ so that

$$ \int_{|x+x(t)| > R} |\nabla u(x, t)|^2 + |u(x, t)|^2 + |u(x, t)|^4 \, dx \leq \epsilon, \quad \text{for all } 0 \leq t < +\infty. $$
Proof. If not, then there exists \( \epsilon > 0 \) and a sequence of times \( t_n \) such that
\[
\int_{|x + x(t_n)| > R} |\nabla u(x, t_n)|^2 + |u(x, t_n)|^2 + |u(x, t_n)|^4 \, dx \geq \epsilon,
\]
or, by changing variables,
\[
(3.5) \int_{|x| > R} |\nabla u(x - x(t_n), t_n)|^2 + |u(x - x(t_n), t_n)|^2 + |u(x - x(t_n), t_n)|^4 \, dx \geq \epsilon.
\]
Since \( K \) is precompact, there exists \( \phi \in H^1 \) such that, passing to a subsequence of \( t_n \), we have \( u(\cdot - x(t_n), t_n) \to \phi \) in \( H^1 \). By (3.5)
\[
\forall R > 0, \quad \int_{|x| > R} |\nabla \phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^4 \geq \epsilon,
\]
which is a contradiction with the fact that \( \phi \in H^1 \). \( \square \)

4. Zero momentum of the critical solution

Proposition 4.1. Assume (3.2) and let \( u_c \) be the critical solution constructed in Section 3. Then its conserved momentum \( P[u_c] = \text{Im} \int \bar{u} \nabla u_c \, dx \) is zero.

Proof. Consider for some \( \xi_0 \in \mathbb{R}^3 \) the transformed solution
\[
w_c(x, t) = e^{ix \xi_0 e^{-it} |\xi_0|^2} u_c(x - 2\xi_0 t, t).
\]
We compute
\[
\|\nabla w_c\|_{L^2}^2 = |\xi_0|^2 M[u_c] + 2\xi_0 \cdot P[u_c] + \|\nabla u_c\|_{L^2}^2.
\]
Observe that \( M[w_c] = M[u_c] \) and
\[
E[w_c] = \frac{1}{2} |\xi_0|^2 M[u_c] + \xi_0 \cdot P[u_c] + E[u_c].
\]
To minimize \( E[w_c] \), we take \( \xi_0 = -P[u_c]/M[u_c] \).

Assume \( P[u_c] \neq 0 \). Choose \( \xi_0 = -\frac{P[u_c]}{M[u_c]} \). Then \( P[w_c] = 0 \) and
\[
(4.1) \quad M[w_c] = M[u_c], \quad E[w_c] = E[u_c] - \frac{1}{2} \frac{P[u_c]^2}{M[u_c]}, \quad \|\nabla w_c\|_{L^2}^2 = \|\nabla u_c\|_{L^2}^2 - \frac{P[u_c]^2}{M[u_c]}.
\]
Thus, \( M[w_c]E[w_c] < M[u_c]E[u_c] \), \( \|w_c\|_{L^2} \|\nabla w_c\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2} \). By Proposition 3.1, \( \|u_c\|_{s(\dot{H}^{1/2})} = +\infty \), and hence, \( \|w_c\|_{s(\dot{H}^{1/2})} = +\infty \), which contradicts the definition of \( u_c \). \( \square \)

5. Control of the spatial translation parameter

Observe that
\[
(5.1) \quad \frac{\partial}{\partial t} \int x |u(x, t)|^2 \, dx = 2 \text{Im} \int \bar{u} \nabla u \, dx = 2P[u].
\]
Since \( P[u_c] = 0 \) (see Prop. 4.1), it follows that \( \int x |u_c(x, t)|^2 \, dx = \text{const} \), provided it is finite. We will replace this identity with a version localized to a suitably large radius \( R > 0 \). Provided the localization \( R \) is taken large enough over an interval \( [t_0, t_1] \) to envelope the entire path \( x(t) \) over \( [t_0, t_1] \), we can exploit the localization of \( u_c \) in \( H^1 \) around \( x(t) \) (induced by the precompactness of the translated flow \( u_c(\cdot - x(t), t) \))
and the zero-momentum property to prove that the localized center of mass is nearly conserved. The parameter $x(t)$ is then constrained from diverging too quickly to $+\infty$ by the localization of $u_\epsilon$ in $H^1$ around $x(t)$ and the near conservation of localized center of mass. We refer to [10, Lemma 5.5] for a similar proof in the case of the energy-critical non-radial wave equation.

**Lemma 5.1.** Let $u$ be a solution of (1.1) defined on $[0, +\infty)$ such that $P[u] = 0$ and $K = \{u(\cdot - x(t), t) | t \in [0, \infty)\}$ is precompact in $H^1$, for some continuous function $x(\cdot)$. Then

$$\frac{x(t)}{t} \to 0 \quad \text{as} \quad t \to +\infty.$$  

**Proof.** Assume that (5.2) does not hold. Then there exists a sequence $t_n \to +\infty$ such that $|x(t_n)|/t_n \geq \epsilon_0$ for some $\epsilon_0 > 0$. Without loss of generality we may assume $x(0) = 0$. For $R > 0$, let

$$t_0(R) = \inf \{t \geq 0 : |x(t)| \geq R \},$$

i.e., $t_0(R)$ is the first time when $x(t)$ reaches the boundary of the ball of radius $R$. By continuity of $x(t)$, the value $t_0(R)$ is well-defined. Moreover, the following properties hold: (1) $t_0(R) > 0$; (2) $|x(t)| < R$ for $0 \leq t < t_0(R)$; and (3) $|x(t_0(R))| = R$.

Define $R_n = |x(t_n)|$ and $\hat{t}_n = t_0(R_n)$. Note that $t_n \geq \hat{t}_n$, which combined with $|x(t_n)|/t_n \geq \epsilon_0$ gives $R_n/\hat{t}_n \geq \epsilon_0$. Since $t_n \to +\infty$ and $|x(t_n)|/t_n \geq \epsilon_0$, we have $R_n = |x(t_n)| \to +\infty$. Thus, $\hat{t}_n = t_0(R_n) \to +\infty$. At this point, we can forget about $t_n$; we will work on the time interval $[0, \hat{t}_n]$ and the only data that we will use in the remainder of the proof is:

1. for $0 \leq t < \hat{t}_n$, we have $|x(t)| < R_n$;
2. $|x(\hat{t}_n)| = R_n$;
3. $\frac{R_n}{\hat{t}_n} \geq \epsilon_0$ and $\hat{t}_n \to +\infty$.

By the precompactness of $K$ and Corollary 3.3, it follows that for any $\epsilon > 0$ there exists $R_0(\epsilon) > 0$ such that for any $t \geq 0$,

$$\int_{|x+x(t)| \geq R_0(\epsilon)} (|u|^2 + |\nabla u|^2) \, dx \leq \epsilon.$$  

We will select $\epsilon > 0$ appropriately later.

For $x \in \mathbb{R}$, let $\theta(x) \in C_0^\infty(\mathbb{R})$ be such that $\theta(x) = x$, for $-1 \leq x \leq 1$, $\theta(x) = 0$ for $|x| \geq 2^{1/3}$, $|\theta(x)| \leq |x|$, $\|\theta\|_\infty \leq 4$, and $\|\theta\|_\infty \leq 2$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let $\phi(x) = (\theta(x_1), \theta(x_2), \theta(x_3))$. Then $\phi(x) = x$ for $|x| \leq 1$ and $\|\phi\|_\infty \leq 2$. For $R > 0$, set $\phi_R(x) = R\phi(x/R)$. Let $z_R : \mathbb{R} \to \mathbb{R}^3$ be the truncated center of mass given by

$$z_R(t) = \int \phi_R(x) \, |u(x, t)|^2 \, dx.$$  

Then $z_R'(t) = ([z'_R(t)]_1, [z'_R(t)]_2, [z'_R(t)]_3)$, where

$$[z'_R(t)]_j = 2 \text{Im} \int \theta'(x_j/R) \partial_j u \, dx.$$
Note that \( \theta'(x_j/R) = 1 \) for \(|x_j| \leq 1\). By the zero momentum property,
\[
\text{Im} \int_{|x_j| \leq R} \partial_j u \bar{u} = - \text{Im} \int_{|x_j| > R} \partial_j u \bar{u},
\]
and thus,
\[
[z'_R(t)]_j = -2 \text{Im} \int_{|x_j| \geq R} \partial_j u \bar{u} dx + 2 \text{Im} \int_{|x_j| \geq R} \theta'(x_j/R) \partial_j u \bar{u} dx,
\]
from which we obtain by Cauchy-Schwarz,
\[\tag{5.4} |z'_R(t)| \leq 5 \int_{|x| \geq R} (|\nabla u|^2 + |u|^2).\]

Set \( \tilde{R}_n = R_n + R_0(\epsilon) \). Note that for \( 0 \leq t \leq \tilde{t}_n \) and \(|x| > \tilde{R}_n\), we have \(|x + x(t)| \geq \tilde{R}_n - R_n = R_0(\epsilon)\), and thus, (5.4) and (5.3) give
\[\tag{5.5} |z'_{\tilde{R}_n}(t)| \leq 5 \epsilon.\]

Now we obtain an upper bound for \( z_{\tilde{R}_n}(0) \) and a lower bound for \( z_{\tilde{R}_n}(t) \).
\[
z_{\tilde{R}_n}(0) = \int_{|x| < R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u_0(x)|^2 dx + \int_{|x + x(t)| \geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u_0(x)|^2 dx,
\]
and hence, by (5.3), we have
\[\tag{5.6} |z_{\tilde{R}_n}(0)| \leq R_0(\epsilon)M[u] + 2 \tilde{R}_n \epsilon.\]

For \( 0 \leq t \leq \tilde{t}_n \), we split \( z_{\tilde{R}_n}(t) \) as
\[
z_{\tilde{R}_n}(t) = \int_{|x + x(t)| \geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u(x,t)|^2 dx + \int_{|x + x(t)| \leq R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u(x,t)|^2 dx.
\]

To estimate I, we note that \(|\phi_{\tilde{R}_n}(x)| \leq 2 \tilde{R}_n\) and use (5.3) to obtain \(|I| \leq 2 \tilde{R}_n \epsilon\). For II, we first note that \(|x| \leq |x + x(t)| + |x(t)| \leq R_0(\epsilon) + R_n = \tilde{R}_n\), and thus \( \phi_{\tilde{R}_n}(x) = x \).

We now rewrite II as
\[
\text{II} = \int_{|x + x(t)| \leq R_0(\epsilon)} (x + x(t)) |u(x,t)|^2 dx - x(t) \int_{|x + x(t)| \leq R_0(\epsilon)} |u(x,t)|^2 dx
= \int_{|x + x(t)| \leq R_0(\epsilon)} (x + x(t)) |u(x,t)|^2 dx - x(t) \int_{|x + x(t)| \geq R_0(\epsilon)} |u(x,t)|^2 dx
= \text{IIA} + \text{IIB} + \text{IIC}.
\]

Trivially, \(|\text{IIA}| \leq R_0(\epsilon)M[u]\), and by (5.3), \(|\text{IIB}| \leq |x(t)| \epsilon \leq \tilde{R}_n \epsilon\). Thus,
\[
|z_{\tilde{R}_n}(t)| \geq |\text{IIB|} - |I - |\text{IIA|} - |\text{IIC|}
\geq |x(t)|M[u] - R_0(\epsilon)M[u] - 3 \tilde{R}_n \epsilon.
\]

Taking \( t = \tilde{t}_n \), we get
\[\tag{5.7} |z_{\tilde{R}_n}(\tilde{t}_n)| \geq \tilde{R}_n(M[u] - 3 \epsilon) - R_0(\epsilon)M[u].\]
Combining (5.5), (5.6), and (5.7), we have

\[ 5 \varepsilon \tilde{t}_n \geq \int_0^{\tilde{t}_n} \left| z'_{\tilde{R}_n} (t) \right| dt \geq \left| \int_0^{\tilde{t}_n} z'_{\tilde{R}_n} (t) \right| dt \geq \left| z_{\tilde{R}_n} (\tilde{t}_n) - z_{\tilde{R}_n} (0) \right| \]

\[ \geq \tilde{R}_n (M[u] - 5 \varepsilon) - 2 R_0 (\varepsilon) M[u] . \]

Dividing by \( \tilde{t}_n \) and using that \( \tilde{R}_n \geq R_n \) (assume \( \varepsilon \leq \frac{1}{5} M[u] \)), we obtain

\[ 5 \varepsilon \geq \frac{R_n}{\tilde{t}_n} (M[u] - 5 \varepsilon) - \frac{2 R_0 (\varepsilon) M[u]}{\tilde{t}_n} . \]

Since \( R_n/\tilde{t}_n \geq \varepsilon_0 \), we have

\[ 5 \varepsilon \geq \varepsilon_0 (M[u] - 5 \varepsilon) - \frac{2 R_0 (\varepsilon) M[u]}{\tilde{t}_n} . \]

Take \( \varepsilon = M[u] \varepsilon_0 / 16 \) (assume \( \varepsilon_0 \leq 1 \)), and then send \( n \to +\infty \). Since \( \tilde{t}_n \to +\infty \), we get a contradiction. \( \square \)

6. Rigidity theorem

We now prove the following rigidity, or Liouville-type, theorem.

**Theorem 6.1 (Rigidity).** Suppose \( u_0 \in H^1 \) satisfies \( P[u_0] = 0 \),

\( (6.1) \quad M[u_0] E[u_0] < M[Q] E[Q] \)

and

\( (6.2) \quad \| u_0 \|_{L^2} \| \nabla u_0 \|_{L^2} < \| Q \|_{L^2} \| \nabla Q \|_{L^2} . \)

Let \( u \) be the global \( H^1 \) solution of (1.1) with initial data \( u_0 \) and suppose that

\[ K = \{ u(\cdot - x(t), t) | t \in [0, +\infty) \} \text{ is precompact in } H^1 . \]

Then \( u_0 = 0 \).

Before beginning the proof, we recall in Lemma 6.2 below a few basic facts proved in [7]. These facts are consequences of the Gagliardo-Nirenberg inequality

\[ \| u \|_{L^4}^4 \leq c_{GN} \| u \|_{L^2} \| \nabla u \|_{L^2}^3 \]

with the sharp value of \( c_{GN} \) expressed as

\[ c_{GN} = \frac{4}{3 \| Q \|_{L^2} \| \nabla Q \|_{L^2}^2} . \]

One also uses the relation

\[ M[Q] E[Q] = \frac{1}{6} \| Q \|_{L^2}^2 \| \nabla Q \|_{L^2}^2 , \]

which is a consequence of the Pohozaev identities.

**Lemma 6.2.** If \( M[u] E[u] < M[Q] E[Q] \) and \( \| u_0 \|_{L^2} \| \nabla u_0 \|_{L^2} < \| Q \|_{L^2} \| \nabla Q \|_{L^2} \), then for all \( t \),

\( (6.3) \quad \| u(t) \|_{L^2} \| \nabla u(t) \|_{L^2} \leq \omega \| Q \|_{L^2} \| \nabla Q \|_{L^2} , \)
where \( \omega = \left( \frac{M[u]E[u]}{M[u]E[u]} \right)^{1/2} \). We also have the bound, for all \( t \)
\[ 8\|\nabla u(t)\|^2 - 6\|u(t)\|_4^4 \geq 8(1 - \omega)\|\nabla u(t)\|^2 \geq 16(1 - \omega)E[u]. \]
We remark that under the hypotheses here, \( E[u] > 0 \) unless \( u \equiv 0 \). In fact, one has the bound \( E[u] \geq \frac{1}{8} \|\nabla u_0\|^2 \).

**Proof of Theorem 6.1.** In the proof below, all instances of a constant \( c \) refer to some absolute constant. Let \( \varphi \in C_0^\infty \) be radial with
\[ \varphi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}. \]
For \( R > 0 \), define
\[ z_R(t) = \int R^2\varphi \left( \frac{x}{R} \right) |u(x,t)|^2 \, dx. \]
Then, by direct calculation,
\[ z_R'(t) = 2 \text{Im} \int R\nabla\varphi \left( \frac{x}{R} \right) \cdot \nabla u(t) \, \bar{u}(t) \, dx. \]
By the Hölder inequality,
\[ (6.5) \quad |z_R'(t)| \leq cR \int_{|x| \leq 2R} |\nabla u(t)||u(t)| \, dx \leq cR\|\nabla u(t)\|_2\|u(t)\|_2. \]
Also by direct calculation, we have the local virial identity
\[ z_R''(t) = 4 \sum_{j,k} \int \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \left( \frac{x}{R} \right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} - \frac{1}{R^2} \int (\Delta^2 \varphi) \left( \frac{x}{R} \right) |u|^2 - \int (\Delta \varphi) \left( \frac{x}{R} \right) |u|^4. \]
Since \( \varphi \) is radial, we have
\[ (6.6) \quad z_R''(t) = \left( 8 \int |\nabla u|^2 - 6 \int |u|^4 \right) + A_R(u), \]
where
\[ A_R(u) = 4 \sum_j \int \left( (\Delta^2 x_j \varphi) \left( \frac{x}{R} \right) - 2 \right) |\frac{\partial u}{\partial x_j}|^2 + 4 \sum_{j \neq k R \leq |x| \leq 2R} \int \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \left( \frac{x}{R} \right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} \]
\[ - \frac{1}{R^2} \int (\Delta^2 \varphi) \left( \frac{x}{R} \right) |u|^2 - \int (\Delta \varphi) \left( \frac{x}{R} \right) - 6) |u|^4. \]
From this expression, we obtain the bound
\[ (6.7) \quad |A_R(u(t))| \leq c \int_{|x| \geq R} \left( |\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^4 \right) \, dx. \]

We want to examine \( z_R(t) \), for \( R \) chosen suitably large, over a suitably chosen time interval \([t_0, t_1]\), where \( 1 < t_0 < t_1 < \infty \). By (6.6) and (6.4), we have
\[ (6.8) \quad |z_R''(t)| \geq 16(1 - \omega)E[u] - |A_R(u(t))|. \]
Set \( \epsilon = \frac{1 - \omega}{c} E[u] \) in Corollary 3.3 to obtain \( R_0 \geq 0 \) such that \( \forall t \),
\[ (6.9) \quad \int_{|x + x(t)| \geq R_0} (|\nabla u|^2 + |u|^2 + |u|^4) \leq \frac{(1 - \omega)}{c} E[u]. \]
If we select \( R \geq R_0 + \sup_{t_0 \leq t \leq t_1} |x(t)| \), then (6.8) combined with the bounds (6.7) and (6.9) will imply that, for all \( t_0 \leq t \leq t_1 \),
\[
|z''_R(t)| \geq 8(1 - \omega)E[u].
\]

By Lemma 5.1, there exists \( t_0 \geq 0 \) such that for all \( t \geq t_0 \), we have \( |x(t)| \leq \eta t \), with \( \eta > 0 \) to be selected later. Thus, by taking \( R = R_0 + \eta t_1 \), we obtain that (6.10) holds for all \( t_0 \leq t \leq t_1 \). Integrating (6.10) over \([t_0, t_1] \), we obtain
\[
|z'_R(t_1) - z'_R(t_0)| \geq 8(1 - \omega)E[u](t_1 - t_0).
\]

On the other hand, for all \( t_0 \leq t \leq t_1 \), by (6.5) and (6.3), we have
\[
|z'_R(t)| \leq cR\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \leq cR\|Q\|_{L^2}\|\nabla Q\|_{L^2}
\]
\[
\leq c\|Q\|_{L^2}\|\nabla Q\|_{L^2}(R_0 + \eta t_1).
\]
Combining (6.11) and (6.12), we obtain
\[
8(1 - \omega)E[u](t_1 - t_0) \geq 2c\|Q\|_{L^2}\|\nabla Q\|_{L^2}(R_0 + \eta t_1).
\]
Recall that \( \omega \) and \( R_0 \) are constants depending only upon \((M[u]E[u])/(M[Q]E[Q])\), while \( \eta > 0 \) is yet to be specified and \( t_0 = t_0(\eta) \). Put \( \eta = (1 - \omega)E[u]/(c\|Q\|_{L^2}\|\nabla Q\|_{L^2}) \) and then send \( t_1 \to +\infty \) to obtain a contradiction unless \( E[u] = 0 \) which implies \( u \equiv 0 \).

To complete the proof of Theorem 1.1, we just apply Theorem 6.1 to \( u_c \), constructed in Proposition 3.1, which by Propositions 3.2 and 4.1, meets the hypotheses in Theorem 6.1. Thus \( u_{c,0} = 0 \), which contradicts the fact that \( \|u_c\|_{S(R^{1/2})} = \infty \). We have thus obtained that if \( \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2} \) and \( M[u]E[u] < M[Q]E[Q] \), then SC(\( u_0 \)) holds, i.e. \( \|u\|_{S(R^{1/2})} < \infty \). By Proposition 2.2 [7], \( H^1 \)-scattering holds.

7. Remarks on the defocusing equation

One may use the above arguments to show \( H^1 \)-scattering of solutions of the defocusing equation
\[
\begin{align*}
\dot{i}u(t) + \Delta u - |u|^2 u &= 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \\
u(x, 0) &= u_0 \in H^1(\mathbb{R}^3).
\end{align*}
\]
In this case, scattering is already known, as a consequence of Morawetz [6], or interaction Morawetz [4] inequalities.

We argue by contradiction. If scattering does not hold, there exists a critical solution \( u_c \), which does not scatter, and such that \( M[u_c]E[u_c] \) is minimal for non-scattering solutions of (7.1). As before, one shows that \( P[u_c] = 0 \), and that there exists \( x(t) \) such that the set \( K = \{u_c(t, \cdot - x(t)) \}, t \in \mathbb{R}\) is precompact in \( H^1 \). Note that because of the defocusing sign of the non-linearity, we do not need to assume \( M[u_c]E[u_c] < M[Q]E[Q] \) and \( \|u_c(0)\|_{L^2} \|\nabla u_c(0)\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2} \). The control of the spatial translation \( x(t) \) works as in Section 5, and one concludes as in Section 6, by a localized virial argument, using that in the defocusing case, the second derivative of the localized variance \( z_R(t) \) is
\[
z''_R(t) = \left( 8 \int |\nabla u|^2 + 6 \int |u|^4 \right) + B_R(u(t))
\]
where $B_R$ satisfies the bound
\[
|B_R(u(t))| \leq c \int_{|x| \geq R} \left( |\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^4 \right) dx.
\]

Note that the use of the virial identity is potentially more robust since one might be able to handle variants of the NLS equation (for example with a linear potential) that might be out of reach for Morawetz based proofs.

**Appendix A. A lifting lemma**

In this appendix, we discuss some basic analysis facts needed in the very beginning of the proof of Prop. 3.2.

Let $G \cong \mathbb{R}^3$ act on $H^1$ by translation, i.e., $(x_0 \cdot \phi)(x) = \phi(x - x_0)$. Write $G \backslash H^1$ for the quotient space endowed with the quotient topology. We represent elements of $G \backslash H^1$ (the equivalence classes) by $[\phi]$, and let $\pi : H^1 \to G \backslash H^1$ be the natural projection.

**Lemma A.1.** $G \backslash H^1$ is metrizable with metric
\[
d([\phi], [\psi]) = \inf_{x_0 \in \mathbb{R}^3} \|\phi(\cdot - x_0) - \psi\|_{H^1}.
\]

With respect to this metric, $G \backslash H^1$ is complete. (Caution that $G \backslash H^1$ is not a vector space, however.)

**Proof.** First, we establish that the orbits of $G$ are closed in $H^1$. The orbit of 0 is 0. Suppose $\phi \neq 0$, $\{x_n\} \subset \mathbb{R}^3$ and $\phi(\cdot - x_n)$ converges to $\psi$ in $H^1$. Then we claim that $x_n$ converges. Indeed, if not, then either $x_n$ is unbounded and there is a subsequence $x_n$ such that $|x_n| \to \infty$, or $x_n$ is bounded and there are two subsequences $x_n \to x_0$ and $x_{n'} \to x_0'$. In the first case, we obtain that $\psi = 0$ (by examining, for fixed $R > 0$, the convergence on $B(0, R)$), which implies $\phi = 0$, a contradiction. In the second case, we obtain that $\phi(\cdot - x_0) = \phi(\cdot - x_0')$, only possible if $\phi = 0$, a contradiction.

Next, we verify that $d$ is a metric. Suppose $d([\phi], [\psi]) = 0$. Then $\inf_{x_0 \in \mathbb{R}^3} \|\phi(\cdot - x_0) - \psi\|_{H^1} = 0$, and thus $\psi$ is a point of closure (in $H^1$) of the orbit of $\phi$. But since the orbits are closed, $\psi$ belongs to this orbit, and thus, $[\phi] = [\psi]$. The triangle inequality is a straightforward exercise dealing with infima, and symmetry is obvious.

Suppose $[\phi_n]$ is a Cauchy sequence; to show that it converges, it suffices to show that a subsequence converges. We can pass to a subsequence $[\phi_n]$ so that $d([\phi_n], [\phi_{n+1}]) \leq 2^{-n}$. Take $x_1 = 0$. Construct a sequence $x_n$ inductively as follows: given $x_{n-1}$, select $x_n$ so that $\|\phi_{n-1}(\cdot - x_{n-1}) - \phi_n(\cdot - x_n)\|_{H^1} \leq 2^{-n+1}$. Then $\phi_n(\cdot - x_n)$ is a Cauchy sequence in $H^1$, and hence, converges to some $\phi$. It is then clear that $[\phi_n] \to [\phi]$ in $G \backslash H^1$.

It can be checked that for each $\phi \in H^1$ and $r > 0$, $\pi(B(\phi, r)) = B([\phi], r)$. Therefore, the topology induced by the metric $d$ on $G \backslash H^1$ is the quotient topology. □

The following two lemmas will reduce Prop. 3.2 to proving that the set
\[
\pi\{u(\cdot, t) \mid t \in [0, +\infty)\}
\]
is precompact in $G \backslash H^1$. 

Lemma A.2. Let $K$ be a precompact subset of $G\setminus H^1$. Assume

(A.1) \[ \exists \eta > 0 \text{ such that } \forall \phi \in \pi^{-1}(K), \quad \eta \leq \|\phi\|_{H^1}. \]

Then there exists $\tilde{K}$ precompact in $H^1$ such that $\pi(\tilde{K}) = K$.

Proof. Let $B(0,1)$ be the unit ball in $\mathbb{R}^3$. We first show by contradiction that there exists $\varepsilon > 0$ such that for all $p$ in $K$, there exists $\psi = \psi(p) \in \pi^{-1}(p)$ such that

(A.2) \[ \|\psi(p)\|_{H^1(B(0,1))} \geq \varepsilon. \]

If not, there exists a sequence $\phi_n$ in $\pi^{-1}(K)$ such that

(A.3) \[ \sup_{x_0 \in \mathbb{R}^3} \|\phi_n(\cdot - x_0)\|_{H^1(B(0,1))} \leq \frac{1}{n}. \]

The precompactness of $K$ implies, extracting a subsequence from $\phi_n$ if necessary, that there exists $\phi \in H^1$ such that $\pi(\phi_n) \to p$ in $G\setminus H^1$. In other words, if $\phi$ is fixed in $\pi^{-1}(p)$, \( \inf_{x_0 \in \mathbb{R}^3} \|\phi_n(\cdot - x_0) - \phi\|_{H^1} \) tends to $0$ as $n$ tends to infinity. Thus, one may find a sequence $x_n$ in $\mathbb{R}^3$ such that

(A.4) \[ \|\phi_n(\cdot - x_n) - \phi\|_{H^1} \xrightarrow{n \to +\infty} 0. \]

Now, by (A.3), for all $x_0 \in \mathbb{R}^3$, $\|\phi_n(\cdot - x_0 - x_n)\|_{H^1(B(0,1))} \leq \frac{1}{n}$. Hence, by (A.4), for all $x_0$, $\phi$ vanishes on $B(x_0,1)$. But then $\phi = 0$, which contradicts assumption (A.1), concluding the proof of the existence of $x(\phi)$.

Let $\tilde{K} = \{\psi(p) \mid p \in K\}$, where $\psi(p)$ satisfies (A.2). Of course, $\pi(\tilde{K}) = K$. By the definition of $x(\phi)$,

(A.5) \[ \forall \phi \in \pi^{-1}(K), \quad \|\phi\|_{H^1(B(0,1))} \geq \varepsilon. \]

Let us show that $\tilde{K}$ is precompact. Let $\phi_n$ be a sequence in $\tilde{K}$. Then by the precompactness of $K$, there exists (extracting subsequences) $\phi \in H^1$ and a sequence $x_n$ of $\mathbb{R}^3$, such that

(A.6) \[ \lim_{n \to +\infty} \|\phi_n(\cdot - x_n) - \phi\|_{H^1} = 0. \]

Note that $K$ being precompact, $\phi_n$ is bounded in $H^1$, thus, we may assume (extracting again)

(A.7) \[ \lim_{n \to +\infty} \|\phi_n\|_{H^1} = \ell \in (0, +\infty). \]

Let us show that $x_n$ is bounded. If not, we may assume that $|x_n| \to +\infty$. By (A.5) and (A.7), we have

\[ \limsup_{n \to -\infty} \|\phi_n(\cdot - x_n)\|_{H^1(B(0,|x_n|-1))} \leq \ell - \varepsilon. \]

As $|x_n| \to \infty$, we conclude that $\|\phi\|_{H^1} \leq \ell - \varepsilon$, contradicting (A.7). Therefore, $x_n$ is bounded. Extracting if necessary, we may assume that $x_n$ converges, which shows by (A.6) that $\phi_n$ converges. This concludes the proof of the precompactness of $\tilde{K}$. \( \square \)

Lemma A.3. Let $u$ be a global $H^1$ solution to (1.1). Suppose

\[ \pi(\{ u(\cdot, t) \mid t \in [0, +\infty) \}) \]

is precompact in $G\setminus H^1$. Then there exists $x(t)$, a continuous path in $\mathbb{R}^3$, such that

\[ \{ u(\cdot - x(t), t) \mid t \in [0, +\infty) \} \]
is precompact in $H^1$.

Proof. By taking $K = \pi(\{u(\cdot, t) \mid t \in [0, +\infty)\})$ in Lemma A.2, we obtain a $\tilde{K}$ precompact in $H^1$ such that $\pi(\tilde{K}) = K$. For each $N$, the map $u : [N, N+1] \to H^1$ is uniformly continuous. Thus, for each $N$, there exists $\delta_N > 0$ such that if $t, t' \in [N, N+1]$ and $|t - t'| \leq \delta_N$, then $\|u(t, \cdot) - u(t', \cdot)\|_{H^1} \leq 1/N$. Let $t_n$ be the increasing sequence of times $\to +\infty$ defined to include evenly spaced elements with density $\delta_N$ in $[N, N+1]$ for each $N$. Thus, $t_n$ is an increasing sequence with possibly more elements per unit interval as we move out to $+\infty$.) For each $n$, select $x(t_n) \in \mathbb{R}^3$ such that $u(\cdot - x(t_n), t_n) \in \tilde{K}$. Now define $x(t)$ to be the continuous function that connects $x(t_n)$ to $x(t_{n+1})$ by a straight line in $\mathbb{R}^3$.

We claim that $\{u(\cdot - x(t), t) \mid t \in [0, +\infty)\}$ is precompact in $H^1$. Indeed, let $s_k$ be a sequence in $[0, +\infty)$. Then there exists a subsequence (also labeled $s_k$) such that either $s_k$ converges to some finite $s_0$ or $s_k \to +\infty$. In the first case, $u(\cdot - x(s_k), s_k) \to u(\cdot - x(s_0), s_0)$ by the continuity of $u(t)$ and $x(t)$. In the second case, for each $k$, obtain the unique index $n(k)$ such that $t_{n(k) - 1} \leq s_k < t_{n(k)}$. By the precompactness of $\tilde{K}$, we can pass to a subsequence (in $k$) such that both $u(\cdot - x(t_{n(k)-1}), t_{n(k)-1})$ and $u(\cdot - x(t_{n(k)}), t_{n(k)})$ converge in $H^1$. By the density of the $t_n$ sequence and uniform continuity of $u$, we obtain that $u(\cdot - x(t_{n(k)-1}), t_{n(k)})$ converges and that it suffices to show that $u(\cdot - x(s_k), t_{n(k)})$ has a convergent subsequence. But since both $u(\cdot - x(t_{n(k)-1}), t_{n(k)})$ and $u(\cdot - x(t_{n(k)}), t_{n(k)})$ converge, we have that $x(t_{n(k)-1}) - x(t_{n(k)})$ converges. Recall that $x(s_k)$ lies on the line segment joining $x(t_{n(k)-1})$ and $x(t_{n(k)})$, and thus, $x(s_k) - x(t_{n(k)-1})$ converges (after passing to a subsequence). Hence, $u(\cdot - x(s_k), t_{n(k)})$ converges in $H^1$.

Thus, to prove Prop. 3.2, it suffices to prove that

$$\pi(\{u(\cdot, t) \mid t \in [0, +\infty)\})$$

is precompact in $G \setminus H^1$. Since $G \setminus H^1$ is complete, if we assume that (A.8) is not precompact in $G \setminus H^1$, then there exists a sequence $\{u(t_n)\}$ in $G \setminus H^1$ and $\eta > 0$ such that $d(\{u(t_n)\}, \{u(t_n')\}) \geq \eta$, or equivalently, (3.4) in the proof of Prop 3.2 holds.

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