DIFFERENTIAL GRADED LIE GROUPS AND THEIR DIFFERENTIAL GRADED LIE ALGEBRAS

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Abstract. In this paper we discuss the question of integrating differential graded Lie algebras (DGLA) to differential graded Lie groups (DGLG). We first recall the classical problem of integration in the context, and recollect the known results. Then, we define the category of differential graded Lie groups and study its properties. We show how to associate a differential graded Lie algebra to every differential graded Lie group and vice versa. For the DGLA → DGLG direction, the main “tools” are graded Hopf algebras and Harish-Chandra pairs (HCP) — we define the category of graded and differential graded HCPs and explain how those are related to the desired construction. We describe some near-at-hand examples and mention possible generalizations.

1. Introduction

Any finite-dimensional real Lie algebra can be integrated to a unique simply connected Lie group. This theorem of Lie and Cartan triggered a whole series of works.

(i) The result is false in infinite dimension (see [8]), but true locally in the Banach case. Recently, C. Wockel and C. Zhu ([32]) integrated a large class of infinite-dimensional Lie algebras to étale Lie 2-groups.

(ii) S. Covez showed ([6]) that Leibniz algebras can be integrated locally to local
(pointed, augmented) Lie racks.

(iii) M. Crainic and R. L. Fernandes ([5]) found the obstruction to the integrability of Lie algebroids in terms of their monodromy groups, and integrated the integrable ones to unique source-simply-connected Lie groupoids (see also [4], [24]). H.-H. Tseng and C. Zhu ([28]) integrated all Lie algebroids to stacky Lie groupoids (see also [30]).

(iv) As for vertical categorification and homotopification, $L_\infty$-algebras were integrated by E. Getzler ([10]) in the nilpotent case and by A. Henriques ([11]) in the general case. In Getzler’s approach, the integrating object is a simplicial subset of the set of Maurer–Cartan elements of the algebra. In good cases, it is a higher groupoid generalizing the Deligne groupoid of a DGLA. Recently, Y. Sheng and C. Zhu ([27]) gave a more explicit integration for strict Lie 2-algebras (and their morphisms); their integration is Morita equivalent to Getzler’s and Henriques’.

This text is the first of a series of papers in which we intend to suggest an integration technique for infinity algebras and their morphisms, which is based on homotopy transfer and leads to concrete and explicit integrating objects. More precisely, in Getzler’s work [10], the integrating simplicial set $\gamma_{\bullet}(L)$ of a nilpotent Lie infinity algebra $L$ is homotopy equivalent to the Kan complex $MC_{\bullet}(L)$ of Maurer–Cartan elements of the algebra. This is obtained by tensoring $L$ with the DGCA $\Omega_{\bullet} := \Omega(\Delta^n)$ of polynomial differential forms of the standard $n$-simplex (see also [16]). If $L$ is concentrated in degrees $k \geq -\ell$ (resp., $-\ell \leq k \leq 0$), the integrating $\gamma_{\bullet}(L)$ is a weak $\ell$-groupoid (resp., $\ell$-group). Our objective is to integrate a Lie infinity algebra by a kind of $A$-infinity group. As we have in mind homotopy transfer, the first goal is to integrate a differential graded Lie algebra (DGLA) by a differential graded Lie group (DGLG), which is the subject of the current paper. Surprisingly it turned out that this task hides more interesting details than expected. Already the very definition of a DGLG is not entirely obvious. And to spell out the integration procedure explicitly it appears convenient to define graded Harish-Chandra pairs: those are the data of a graded Lie algebra and the Lie group integrating the degree zero part of it, together with a degree-preserving representation of the latter in the former, subject to some compatibility condition. The present paper is a rigorous approach to this integration problem and the description of the appropriate differential structure.

It is worth mentioning that differential graded Lie groups naturally appear in the context of characteristic classes ([19], [17], [18], which in turn have interesting applications in gauge theory [26], [22], [23]); this motivated two of the authors to look at this subject in more detail. Moreover, the very concrete question discussed in the current paper resulted in some ongoing work related to the properties and categorical description of graded manifolds in full generality ([20]) and graded Lie groups/algebras in particular ([12]).

**Organization.** The article is organized as follows. We start Section 2 by sketching the integration problem in the case of classical (non-graded) differential Lie groups and algebras. Then, in Sections 2 and 3 the construction is extended to the graded case, namely the graded Lie groups/algebras and differential graded Lie groups/algebras are defined respectively. Section (4) is the core of the paper where the relation between DGLGs and DGLAs is discussed. The main “tool” introduced
there is graded and differential graded Harish-Chandra pairs (HCP). The key result is given by two Theorems 4.8 and 4.15 about equivalences of categories, establishing the relations \( GLG \leftrightarrow GHCP \leftrightarrow GLA \) and \( DGLG \leftrightarrow DGHCP \leftrightarrow DGLA \) respectively.

Conventions. Manifolds are second countable Hausdorff, finite-dimensional, and real. (Super) manifolds are smooth and finite-dimensional, maps between them, vector fields are smooth. (Super) Lie algebras are finite-dimensional and real, \( \mathbb{Z} \)-graded Lie algebras have finite-dimensional homogeneous components and are non-negatively (or non-positively) graded, unless the contrary is stated.

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2. Differential and graded Lie groups/algebras

A DGLA is a GLA endowed with a square 0 degree 1 derivation. In the non-graded case, the concept reduces to a LA \( \mathfrak{g} \) together with a derivation \( \delta \in \text{Der}(\mathfrak{g}) \). On the global side, a DGLG is a group object in the category of differential graded manifolds. Here the word “graded” is (by a little abuse) employed in contrast to “super”. For the \( \mathbb{Z} \)- (resp., \( \mathbb{N} \)) graded case those are called \( \mathbb{Z}Q \)- (resp., \( \mathbb{N}Q \))-manifolds, i.e., \( \mathbb{Z} \)- (\( \mathbb{N} \))-manifolds equipped with a homological vector field \( Q \), that is, a degree 1 derivation of the function algebra that Lie commutes with itself, i.e. \( [Q, Q] \equiv Q^2 = 0 \). We will give details for all of these concepts in this section below, where we describe the graded analogues of Lie groups and algebras, and especially present the construction of graded Hopf algebras — the main “tool” for studying the GLA \( \rightarrow \) GLG integration procedure. Then we discuss multiplicative vector fields and the Maurer–Cartan formalism in the context. Section 3, which is devoted to the differential part, is making sense of the above mentioned homological vector field.

2.1. Integration of a (non-graded) differential Lie algebra

If we forget the grading, we deal with a group object in the category of manifolds equipped with a homological vector field. Such an object is a Lie group with a selected vector field that is compatible with the group maps, — that is, a Lie group \( G \) endowed with a multiplicative vector field \( X \in \mathfrak{X}^{\text{mult}}(G) \). We thus have to show that differentiation and integration allow passing from a non-graded differential Lie group \( (G, X) \) (DLG) to a non-graded differential Lie algebra \( (\mathfrak{g}, \delta) \) (DLA) and vice versa. This construction is relatively classical, so we skip it here, let us however mention one detail that will be useful further for the generalization to the graded
case. The key ingredient is the study of Lie group and Lie algebra cohomologies, and in particular the van Est isomorphism ([7,8]) establishing the $\mathbb{R}$-vector space isomorphism between group 1-cocycles of $G$ and algebra 1-cocycles of $g = T_eG$. It has also been reviewed in the setting of Lie groupoids and algebroids in [31]. The isomorphism needs to be phrased in terms of multiplicative vector fields—we revisit the question in [12]. Then, as one expects, the following theorem holds, and we use it as a starting point for our discussion:

**Theorem 2.1.** Any DLG differentiates to a DLA, and any DLA integrates to a unique simply connected DLG.

### 2.2. Graded geometry — preliminaries

Let us now start defining the graded generalization of all the above mentioned concepts. Let $\Gamma$ be a commutative monoid and $\varepsilon: \Gamma^2 \to \mathbb{R}^\times$ be a commutation factor (see [1, III.46]) and $V$ be a $\Gamma$-graded vector space. We define its graded symmetric algebra as

$$SV = \otimes V / \{ v \otimes w - \varepsilon(v, w) w \otimes v : v, w \in V^{\text{hom}} \}$$

where we write $\varepsilon(v, w)$ for $\varepsilon(|v|, |w|)$, and $|\bullet|$ denotes the degree of $\bullet$. Since the ideal by which we quotient is homogeneous (for the $\Gamma$-grading), the graded symmetric algebra is a $\Gamma$-graded $\varepsilon$-commutative unital\(^1\) algebra. The “super” case corresponds to $\Gamma = \mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$ and $\varepsilon(\gamma_1, \gamma_2) = (-1)^{|\gamma_1||\gamma_2|}$.

To define the degree of graded linear maps, we need $\Gamma$ to be cancellative, which is equivalent to being embeddable in a commutative group. In the following, “graded” will mean “$\Gamma$-graded” for $\Gamma = \mathbb{N}$ (meaning $\mathbb{Z}_{\geq 0}$), or $\mathbb{Z}$, that is the “degree zero” actually corresponds to the element $0 \in \Gamma$, and the commutation factor will be the one given above.

If $U$ is an open subset of $\mathbb{R}^n$ and $V$ is a graded vector space, we define the unital graded $\mathbb{R}$-algebra as

$$\mathcal{C}(U|V) = C^\infty(U) \otimes SV$$

and we call it an algebraic model. It is $\varepsilon$-commutative. By abuse of notation, we also denote by $\mathcal{C}(U|V)$ the unital-graded-algebra-ed\(^2\) space $(U, \mathcal{C}(-|V))$ it naturally defines, and we call it a local model.

A morphism of these is a pair $\phi = (\tilde{\phi}, \phi^\sharp)$ where $\tilde{\phi}$ is a continuous map between the underlying spaces and $\phi^\sharp$ is a sheaf morphism from the pullback by $\tilde{\phi}$ of the target sheaf to the source sheaf:

$$\phi: \mathcal{C}(U|V) \to \mathcal{C}(U'|V') \iff \tilde{\phi}: U \to U' \text{ and } \phi^\sharp: \tilde{\phi}^*(\mathcal{C}(U'|V')) \to \mathcal{C}(U|V).$$

Providing the data of $\phi = (\tilde{\phi}, \phi^\sharp)$ is equivalent to the “dual” construction $\psi = (\check{\phi}, \phi_\sharp)$, where $\phi_\sharp: \mathcal{C}(U'|V') \to \check{\phi}_*\mathcal{C}(U|V)$ ([25]).

\(^1\)By definition, a graded algebra is unital if it has a multiplicative unit which is homogeneous of degree 0.

\(^2\)The etymology comes from “ring” — “ringed” often appearing in the literature.
Definition 2.2. A graded manifold is a paracompact Hausdorff unital-graded-algebra-ed space, locally modelled as $\mathcal{C}(U|V) \equiv C^\infty(U) \otimes SV$, where $U$ is an open subset of an $\mathbb{R}^n$ and $V$ is a graded vector space with $V_0 = \{0\}$, and $SV$ is the graded symmetric algebra on it.

In [12] we give details related to this definition, especially in the context of $\mathbb{Z}$-graded Lie groups/algebras (in contrast to $\mathbb{N}$-graded). There are also some subtle properties of the functional space $\mathcal{C}(M)$ of functions on an arbitrary graded manifold $M$ worth being discussed ([12], [20]). For example, in this paper to properly define multiplication we need to have in mind the fundamental isomorphism: $\mathcal{C}(M_1 \times M_2) = \mathcal{C}(M_1) \otimes \mathcal{C}(M_2)$, where by $\otimes$ we denote the completion of the tensor product, and [20] is precisely devoted to categorical properties of graded manifolds.

Since the category of graded manifolds is cartesian monoidal, the following definition is natural.

Definition 2.3 (Graded Lie group). The category of graded Lie semigroups (resp. monoids) is the category of semigroup (resp. monoid) objects in the category of graded manifolds. The category of graded Lie groups is the category of monoid objects in the category of graded manifolds which are groups.

The following two results are straightforward as well and only quoted here for later reference.

Lemma 2.4. Given linear maps between (unital) $R$-modules, where $R$ is a unital commutative ring, $a: A \to B$, $b: A' \to R$, $c: B \to C$, one has $c \circ (a \otimes b) = (c \circ a) \otimes b: A \otimes A' \to C$. Given $a: A \to R$, $b: A' \to B$, $c: B \to C$, one has $c \circ (a \otimes b) = a \otimes (c \circ b): A \otimes A' \to C$:

$$
\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\downarrow & \downarrow & \downarrow \\
A' & \xleftarrow{b} & R
\end{array}
\quad
\begin{array}{ccc}
B & \xrightarrow{c} & C \\
\downarrow & \downarrow & \downarrow \\
C & \xleftarrow{id} & B
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\downarrow & \downarrow & \downarrow \\
A' & \xleftarrow{b} & R
\end{array}
\quad
\begin{array}{ccc}
B & \xrightarrow{c} & C \\
\downarrow & \downarrow & \downarrow \\
C & \xleftarrow{id} & B
\end{array}
$$

Proof. This is just a reformulation of the $R$-linearity of $c$ and the identification $B \otimes R \simeq B$. \hfill $\square$

Lemma 2.5. If $(A, \mu)$ is a unital graded commutative algebra, where $\mu$ is the multiplication, and $B$ is an $A$-bimodule and $X \in \text{Hom}(A, B)$ is a graded vector space morphism of degree $|X|$, then $X \in \text{Der}(A, B)$ if and only if

$$
X \circ \mu = \mu \circ (X \otimes \text{id} + \text{id} \otimes X)
$$

with implicit Koszul sign. A derivation on a unital algebra vanishes on scalars.

Proof. This is a reformulation of the Leibniz rule. Let $f, g \in A$ with $f$ homogeneous. Then $X \in \text{Der}(A, B)$ means $X(fg) = (Xf)g + (-1)^{|X||f|}f(Xg) = (X \otimes \text{id} + \text{id} \otimes X)(f \otimes g)$, which means $X \circ \mu(f \otimes g) = \mu \circ (X \otimes \text{id} + \text{id} \otimes X)(f \otimes g)$. In particular, $1_A$ being of degree 0, one has $X(1_A^2) = (X1_A)1_A + (-1)^01_A(X1_A)$ so $X1_A = 0$ and by linearity, $X$ vanishes on scalars. \hfill $\square$

\footnote{From now on $R$ will be a field $k$ of characteristic 0. In this paper $k = \mathbb{R}$ unless the contrary is explicitly assumed.}
2.3. Formulaire for graded Hopf algebras

The monoidal product of the category of graded manifolds and the fact that their structure sheaves are Fréchet (see [12], [20]) imply that the structure sheaf of a graded Lie group is a sheaf of topological graded Hopf algebras. In this subsection, we recall a few facts about these. A good reference for the non-graded case is [13, III.1–III.3].

Definition 2.6. A topological graded Hopf algebra is a Fréchet graded vector space $H$ with structure maps:

1. a multiplication $\mu : H \otimes H \to H$,
2. a unit $\eta : \mathbb{R} \to H$,
3. a comultiplication $\Delta : H \to H \otimes H$,
4. a counit $\epsilon : H \to \mathbb{R}$,
5. an antipode $S : H \to H$,

satisfying the following axioms:

1. unit laws $\mu \circ (\text{id}_H \otimes \eta) = \mu \circ (\eta \otimes \text{id}_H) = \text{id}_H$;
2. associativity $\mu \circ (\text{id}_H \otimes \mu) = \mu \circ (\mu \otimes \text{id}_H)$;
3. counit laws $(\text{id}_H \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{id}_H) \circ \Delta = \text{id}_H$;
4. coassociativity $(\text{id}_H \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}_H) \circ \Delta$;
5. the multiplication is a coalgebra morphism

\[(\mu \otimes \mu) \circ (\text{id}_H \otimes \tau \otimes \text{id}_H) \circ (\Delta \otimes \Delta) = \Delta \circ \mu,
\]

$\tau$ being the (involutive) flip operator, and $\epsilon \otimes \epsilon = \epsilon \circ \mu$;
6. the unit is a coalgebra morphism $\Delta \circ \eta = \eta \otimes \eta$ and $\epsilon \circ \eta = \text{id}_\mathbb{R}$;
7. the antipode identity $\mu \circ (\text{id}_H \otimes S) \circ \Delta = \mu \circ (S \otimes \text{id}_H) \circ \Delta = \eta \circ \epsilon$.

By omitting the antipode structure together with the antipode identity, we obtain the notion of a topological unital and counital graded bialgebra. By dropping off the unit and the counit structures and the related identities, we obtain the general notion of a topological graded bialgebra.

Note that the four morphic conditions are equivalent to saying that the comultiplication and the counit are algebra morphisms. These maps should be continuous graded linear maps (of degree 0). The completed tensor product is the projective one (to have a good representation of its elements as the absolutely convergent sums of decomposable tensors). Generally, the Hopf algebra will be nuclear, so that the completed tensor product is well defined.

The antipode is an antimorphism. In the commutative or cocommutative cases, it is bijective, so is an antiautomorphism; still, under these conditions, it is an involution. A morphism of Hopf algebras automatically intertwines the antipodes.

The example to keep in mind is $H = \mathcal{C}(G)$, with $\mathcal{C}(G \times G) \simeq \mathcal{C}(G) \hat{\otimes} \mathcal{C}(G)$. In the non-graded case, if $G$ is a Lie semigroup (resp. monoid, group), then $\mathcal{C}_c(G)$ is a topological unital bialgebra (resp. unital and counital bialgebra, Hopf algebra).

A note about notation: to keep in mind that a commutative Hopf algebra (unital and counital bialgebra, unital bialgebra) is thought of as a space of functions on a group (monoid, semigroup), respectively, we use stars for the coalgebra maps.
as if they were pullbacks, hence a counit $\epsilon = e^*$, comultiplication $\Delta = m^*$, and antipode (coinverse) $S = i^*$. In order to make computations more intuitive, later on we extend these notations to all Hopf algebras and bialgebras, even when they are not necessarily commutative. When they are the unit, multiplication, and inverse of a graded Lie group, we will drop the prefix "co".

We define the constant map $\hat{e} = \eta \circ e = \eta \circ e^* : H \to H$.

In any Hopf algebra, $e^* \circ i^* = e^*$ and $i^* \circ \eta = \eta ([13, \text{Thm. III.3.4}])$ — this is part of the antipode that is an antimorphism. The other part reads: $i^*(fg) = i^*g i^*f$ and $m^* \circ i^* = (i^* \otimes i^*) \circ \tau \circ m^*$.

**Definition 2.7.** If $A$ is an algebra with multiplication $\mu$ and $C$ is a coalgebra with comultiplication $m^*$, and if $a, b : C \to A$ are linear maps, then we define their convolution product as

$$a \star b = \mu \circ (a \otimes b) \circ m^* : C \to A.$$  

(1)

One can show that $(\text{Hom}(C, A), \star, \hat{e})$ is an associative unital algebra (see [13, Prop. III.3.1]). The antipode identity then reads

$$\text{id} \star i^* = i^* \star \text{id} = \hat{e}.$$  

The identity (2) implies that if an antipode exists (in a bialgebra), then it is unique.

**Corollary 2.8.** If $a \in \text{End}(H) \equiv \text{Hom}(H, H)$ such that $a \star \text{id} = 0$, then $a = 0$.

**2.4. Left-invariant derivations of graded bialgebras**

**Definition 2.9.** Vector fields on graded manifolds are derivations of the structure sheaf, $\mathfrak{X}(\mathcal{M}) = \text{Der} (\mathcal{O}_\mathcal{M})$.

As in the non-graded case, if $f : \mathcal{M} \to \mathcal{N}$ is a smooth\(^4\) map, and $X \in \mathfrak{X}(\mathcal{M})$ and $Y \in \mathfrak{X}(\mathcal{N})$ are vector fields, then $X$ and $Y$ are $f$-related, which we denote by $X \sim_f Y$ if $Tf \circ X = Y \circ f$. If $X \sim_f Y$ and $Y \sim_g Z$, then $X \sim_{g \circ f} Z$, and $X \sim_{\text{id}_\mathcal{M}} X$, so graded manifolds with a vector field and smooth maps relating them form a category. This category is cartesian with product $(\mathcal{M}, X) \times (\mathcal{N}, Y) = (\mathcal{M} \times \mathcal{N}, X \otimes \text{id}_{\mathcal{O}_\mathcal{N}} + \text{id}_{\mathcal{O}_\mathcal{M}} \otimes Y)$ and obvious projections, and terminal object $((\{\}, 0), 0)$.

In the non-graded case, $X \in \mathfrak{X}^\text{left}(G)$ means that for all $g \in G$, $X$ is $L_g$-related to itself, that is, $X \circ L_g = TL_g \circ X$. Seeing $X$ as a derivation of $\mathcal{C}^\infty(G)$, this means that for all $f \in \mathcal{C}^\infty(G)$ and $g \in G$, one has $(Xf) \circ L_g = Tf \circ X \circ L_g = Tf \circ TL_g \circ X = T(f \circ L_g) \circ X = X(f \circ L_g)$.

We want to express this in terms of the Hopf algebra $\mathcal{C}^\infty(G)$:

$$(\text{eval}_g \otimes \text{id}) \circ m^* f = f \circ L_g \quad \text{and} \quad (\text{id} \otimes \text{eval}_g) \circ m^* f = f \circ R_g.$$  

\(^4\)Talking about smoothness in the graded setting is obviously a language abuse. What is meant is the class of functions $f$ with a smooth body part $f$ and the appropriate graded part. To keep the intuition from the non-graded case, we will however write $\mathcal{C}^\infty(\cdot)$ meaning $\mathcal{C}(\cdot)$.
This can also be obtained by noting that $L_g = m \circ (\hat{g} \otimes \text{id})$, where $\hat{g}$ is the constant $g$, so by dualizing, $L^*_g = (\text{eval}_g \otimes \text{id}) \circ m^*$.

Using this to translate the condition of left-invariance $(X f) \circ L_g = X(f \circ L_g)$ gives $(\text{eval}_g \otimes \text{id}) \circ m^* X f = X(\text{eval}_g \otimes \text{id}) \circ m^* f$ for all $f$ and $g$, so $(\text{eval}_g \otimes \text{id}) \circ m^* X = X(\text{eval}_g \otimes \text{id}) \circ m^* = (\text{eval}_g \otimes \text{id}) \circ (\text{id} \otimes X) \circ m^*$ for all $g$. Therefore, $X$ is left-invariant if and only if $(\text{id} \circ X) \circ m^* = m^* \circ X$, and this is taken as the definition in the case of a graded Lie semigroup (and, furthermore, of a graded bialgebra as soon as we replace $m^*$ with a general comultiplication):

**Definition 2.10.** Let $(B, \mu, m^*)$ be a graded bialgebra. A left-invariant derivation of $B$ is an element of

$$\text{Der}^{\leftarrow}(B) = \{ X \in \text{Der}(B) \mid (\text{id} \otimes X) \circ m^* = m^* \circ X \}.$$  

(2)

Here $\text{Der}(B)$ is the graded Lie algebra of graded derivations of $B$ regarded as an algebra. Similarly, $X$ is right-invariant derivation if and only if $(X \otimes \text{id}) \circ m^* = m^* \circ X$.

**Proposition 2.11.** The space of left- (right-)invariant derivations of a graded bialgebra is closed under the graded Lie bracket.

**Proof.** Follows immediately from the definition of left- (right-)invariant derivations.  

**Definition 2.12.** A left-invariant vector field on a graded Lie semigroup $G$ is a left-invariant derivation of the corresponding graded unital bialgebra of functions, i.e. an element of

$$\mathfrak{X}^{\leftarrow}(G) = \{ X \in \mathfrak{X}(G) \mid (\text{id} \otimes X) \circ m^* = m^* \circ X \}$$  

(3)

where $m$ denotes the multiplication of $G$. Similarly, $X$ is right-invariant if and only if $(X \otimes \text{id}) \circ m^* = m^* \circ X$.

**Remark 2.13.** Although the multiplication is not written in the above definition, the bialgebra structure is needed: it is “hidden” in $X \in \mathfrak{X}(G)$.

### 2.5. The graded Lie algebra of a graded Lie group

To obtain the notion of tangent vector, we introduce the following:

**Definition 2.14.** If $(A, \mu)$ is a graded commutative $R$-algebra, $a \in A'$ is a linear form, and $B$ is an $A$-bimodule, an $a$-derivation from $A$ to $B$ is an element of

$$\text{Der}_a(A, B) = \{ \xi \in \text{Hom}(A, B) \mid \xi \circ \mu = \xi \otimes a + a \otimes \xi \}.$$  

In a graded Lie monoid with unit $e^*$, we write $\text{Der}_e$ for $\text{Der}_{e^*}$.

If $X \in \mathfrak{X}(G)$, we write, by abuse of notation and analogy with evaluations \(\text{eval}_e \), \(X_e = e^* \circ X \in \text{Der}_e(\mathcal{C}(G), \mathbb{R}) = \text{Der}(\mathcal{C}(G)_e, \mathbb{R})\).
Proposition 2.15. If $G$ is a graded Lie monoid, there exists a graded linear isomorphism (of degree 0)

$$\mathfrak{X}^{\text{left}}(G) \xrightarrow{\sim} T_e G = \text{Der}(\mathcal{C}(G)_e, \mathbb{R}),$$

$X \mapsto X_e, \quad \text{id} \circ v \leftrightarrow v.$

In particular, a left-invariant vector field is determined by its “value at the unit”.

Proof. This is the exact analogue of [3, Prop. 7.2.3], let us just give the strategy of the proof. First, we verify that $X = \text{id} \circ v$ is a derivation — this is done using the compatibility of $\mu$ and $m^*$. Second, we check that if $v \in T_e G$, then $\text{id} \circ v = \mu \circ (\text{id} \circ v) \circ m^*$ is indeed left-invariant. Taking into account that $v$ takes values in constants, one can derive the following two identities (at the moment we need the first one):

$$(\text{id} \circ \mu) \circ (m^* \circ \text{id}) \circ (\text{id} \circ v) = m^* \circ \mu \circ (\text{id} \circ v), \quad (4)$$

$$(\mu \circ \text{id}) \circ (\text{id} \circ m^*) \circ (v \circ \text{id}) = m^* \circ \mu \circ (v \circ \text{id}). \quad (5)$$

Then the desired left-invariance of $X$ is obtained (using the coassociativity condition) from (4). Third, postcomposing the left-invariance relation with $\mu \circ (\text{id} \circ e^*)$ permits to recovering $X$ from $\text{id} \circ X_e$. Lastly we check, although this is not necessary in finite dimension, that $v = e^* \circ \mu \circ (\text{id} \circ v) \circ m^*$.

Corollary 2.16. Defined via the previous proposition, $T_e G$ is a graded Lie algebra.

This fact is verified using Proposition 2.11 and Proposition 2.15: the first one implies that the space $\mathfrak{X}^{\text{left}}(G)$ is a graded Lie subalgebra of $\mathfrak{X}(G)$, while the second one gives us an explicit isomorphism between $\mathfrak{X}^{\text{left}}(G)$ and $T_e G$.

Along with left translations $v^L : = \text{id} \circ v$, we define right translations $v^R : = v \circ \text{id}$, which are also derivations of the multiplication $\mu$ (the proof is similar to the proof of Proposition 2.15).

Proposition 2.17. For any $v_1$ and $v_2$ the corresponding left and right translations super commute, i.e. $v_1^L \circ v_2^R = (-1)^{k_1 k_2} v_2^R \circ v_1^L$, where $k_1$ and $k_2$ are the degrees of $v_1$ and $v_2$, respectively.

Proof. Taking into account that $v_1^L$ is a derivation and that, in particular, $v_1^L$ annihilates constants, one has

$$v_1^L \circ v_2^R = v_1^L \circ \mu \circ (v_2 \circ \text{id}) \circ m^*$$

$$= \mu \circ (v_1^L \circ \text{id} + \text{id} \circ v_1^L) \circ (v_2 \circ \text{id}) \circ m^*$$

$$= \mu \circ (\text{id} \circ v_1^L) \circ (v_2 \circ \text{id}) \circ m^*.$$

From the commutation relation $(\text{id} \circ v_1^L) \circ (v_2 \circ \text{id}) = (-1)^{k_1 k_2} (v_2 \circ \text{id}) \circ (\text{id} \circ v_1^L)$ we immediately obtain $v_1^L \circ v_2^R = (-1)^{k_1 k_2} \mu \circ (v_2 \circ \text{id}) \circ (\text{id} \circ v_1^L) \circ m^*$. On the other hand, by definition $v_1^L$ is left-invariant, therefore,

$$(\text{id} \circ v_1^L) \circ m^* = m^* \circ v_1^L$$

and finally,

$$v_1^L \circ v_2^R = (-1)^{k_1 k_2} \mu \circ (v_2 \circ \text{id}) \circ m^* \circ v_1^L = (-1)^{k_1 k_2} v_2^R \circ v_1^L. \quad \square$$
Remark 2.18. As we never used the commutativity assumption in this section, the statements of Proposition 2.15 and Proposition 2.17 remain true for arbitrary graded unital counital bialgebras.

2.6. Multiplicative vector fields on graded Lie groups

In the non-graded case, $X \in \mathfrak{X}^{\text{mult}}(G)$ means that $(X, X)$ and $X$ are $m$-related, meaning $X \circ m = Tm \circ (X, X) = Tm \circ (X \otimes \text{id} + \text{id} \otimes X)$ by the Leibniz rule. $X_{gh} = T_h L_g(X_h) + T_g R_h(X_g)$ for all $g, h \in G$. In terms of derivations, this means

$$
(Xf) \circ m = Tf \circ X \circ m = Tf \circ Tm \circ (X \otimes \text{id} + \text{id} \otimes X)
$$

$$
= T(f \circ m) \circ (X \otimes \text{id} + \text{id} \otimes X) = (X \otimes \text{id} + \text{id} \otimes X)(f \circ m).
$$

Therefore, $X$ is multiplicative if and only if $(X \otimes \text{id} + \text{id} \otimes X) \circ m^* = m^* \circ X$, and this is taken as the definition in the graded case:

**Definition 2.19.** A multiplicative vector field on a graded Lie semigroup $G$ is an element of

$$
\mathfrak{X}^{\text{mult}}(G) = \{X \in \mathfrak{X}(G) \mid (X \otimes \text{id} + \text{id} \otimes X) \circ m^* = m^* \circ X\}
$$

where $m$ denotes the multiplication of $G$. This means exactly that $(X, X) \sim_{m^*} X$, that is, $X$ is compatible with the multiplication.

Similarly, we say that for a graded Lie monoid, $X$ is compatible with the unit if $X \sim_{e^*} 0$, that is, $e^* \circ X = X e = 0$; and for a graded Lie group $X$ is compatible with the inverse if $X \sim_{i^*} X$.

The following is the analogue of the compatibility result in the non-graded case ([12]).

**Proposition 2.20.** For a graded Lie monoid, a multiplicative vector field $X$ is compatible with the unit:

$$
X_e := e^* \circ X = 0
$$

For a graded Lie group, a multiplicative vector field is compatible with the inverse.

$$
i^* \circ X = X \circ i^*.
$$

**Proof.** Postcomposing the multiplicativity relation with $\text{id} \otimes e^*$ gives $(X \otimes e^* + \text{id} \otimes X_e) \circ m^* = X$. Since $X \otimes e^* = X \circ (\text{id} \otimes e^*)$ and $(\text{id} \otimes e^*) \circ m^* = \text{id}$, the left-hand side above is equal to $X + (\text{id} \otimes X_e) \circ m^*$. Therefore, $(\text{id} \otimes X_e) \circ m^* = 0$ so $\text{id} \ast X_e = 0$ so $X_e = 0$ by Corollary 2.8 (or directly postcomposing with $e^* \otimes e^*$).

For the inverse, the (right) antipode identity reads $\text{id} \ast i^* = \mu \circ (\text{id} \otimes i^*) \circ m^* = \hat{e} = \eta \circ e^*$. Postcomposing it with a vector field $X$ gives us

$$
X \circ \eta \circ e^* = X \circ \mu \circ (\text{id} \otimes i^*) \circ m^* = \mu \circ (X \otimes \text{id} + \text{id} \otimes X) \circ (\text{id} \otimes i^*) \circ m^*.
$$

The left-hand side of the above formula equals 0 since a derivation vanishes on scalars. Therefore,

$$
0 = \mu \circ (X \otimes i^* + \text{id} \otimes (X \circ i^*)) \circ m^* = \mu \circ (\text{id} \otimes i^*) \circ (X \otimes \text{id}) \circ m^* + \text{id} \ast (X \circ i^*).
$$
Taking into account that \( X \) is multiplicative, we have \((X \otimes \text{id}) \circ m^* = m^* \circ X - (\text{id} \otimes X) \circ m^*\). This immediately implies
\[
\mu \circ (\text{id} \otimes i^*) \circ (X \otimes \text{id}) \circ m^* = (\text{id} \otimes i^*) \circ X - \text{id} \circ (i^* \circ X).
\]
But \(\text{id} \circ i^* = \hat{\epsilon} = \eta \circ e^*\). Given that \(X_e = e^* \circ X = 0\) we obtain \((\text{id} \circ i^*) \circ X \circ m^* = -\text{id} \circ (i^* \circ X)\) and \(0 = \text{id} \circ (X \circ i^* - i^* \circ X)\). By the consequence of the antipode identity (Corollary 2.8), this implies \(X \circ i^* = i^* \circ X\) as wanted.

**Corollary 2.21.** The category of graded Lie semigroups (resp. monoids, groups) with a multiplicative vector field is isomorphic to the category of semigroup (resp. monoid, group) objects in the category of graded manifolds with a vector field, and maps preserving them.

**Proposition 2.22.** Let \(G\) be a graded monoid, \(H\) be its bialgebra of functions, and \(v\) be a derivation at \(e\). Then \(X = \text{id} \circ v - v \circ \text{id}\) is a multiplicative vector field.

**Proof.** We have
\[
(X \otimes \text{id}) \circ m^* = (\mu \otimes \text{id}) \circ ((\text{id} \otimes v - v \otimes \text{id}) \otimes \text{id}) \circ (m^* \otimes \text{id}) \circ m^*,
\]
\[
(\text{id} \otimes X) \circ m^* = (\text{id} \otimes \mu) \circ (\text{id} \otimes (\text{id} \otimes v - v \otimes \text{id})) \circ (\text{id} \otimes m^*) \circ m^*.
\]
Thus \((X \otimes \text{id} + \text{id} \otimes X) \circ m^* = (I) + (II), where
\[
(I) = (\mu \otimes \text{id}) \circ (\text{id} \otimes v \otimes \text{id}) \circ (m^* \otimes \text{id}) \circ m^*
- (\text{id} \otimes \mu) \circ (\text{id} \otimes v \otimes \text{id}) \circ (\text{id} \otimes m^*) \circ m^*;
\]
\[
(II) = (\text{id} \otimes \mu) \circ (\text{id} \otimes \text{id} \otimes v) \circ (\text{id} \otimes m^*) \circ m^*
- (\mu \otimes \text{id}) \circ (v \otimes \text{id} \otimes \text{id}) \circ (\otimes m^* \otimes \text{id}) \circ m^*.
\]
Thanks to the coassociativity law and the identities (4) and (5), the first term \((I)\) vanishes, while the second term \((II)\) equals to
\[
(\text{id} \otimes \mu) \circ (m^* \otimes \text{id}) \circ (\text{id} \otimes v) \circ m^* - (\mu \otimes \text{id}) \circ (\text{id} \otimes m^*) \circ (v \otimes \text{id}) \circ m^*
= m^* \circ \mu \circ (\text{id} \otimes v - v \otimes \text{id}) \circ m^* = m^* \circ X.
\]
Finally \((X \otimes \text{id} + \text{id} \otimes X) \circ m^* = m^* \circ X\) which proves that \(X\) is multiplicative.

**Remark 2.23.** Although in this section we deal with commutative bialgebras representing functions on graded monoids, we do not use commutativity in the proofs, therefore all statements, like in the previous subsection, are also valid in the general (non-commutative) case. In the next subsection, however, the commutativity will be important.

### 2.7. The Maurer–Cartan automorphism of a graded Lie group

Consider a graded Lie group \(G\) and its Hopf algebra \(H = \mathcal{C}(G)\), then on \(\text{End}(H) \equiv \text{Hom}(H, H)\) we introduce the following:
Definition 2.24. The right (resp. left) Maurer–Cartan automorphism of a graded Lie group $G$ is given by
\[ \omega^R = \bullet \ast i^* \quad (\text{resp. } \omega^L = i^* \ast \bullet). \]

Recall that $\ast$ is the convolution product defined by (1) on a couple of linear maps from a coalgebra to an algebra, in this case both coinciding with $H$.

By the hexagon identity, $(\omega^R)^{-1} = \bullet \ast \text{id}$ and $(\omega^L)^{-1} = \text{id} \ast \bullet$ and $\omega^R$ and $\omega^L$ are linear automorphisms of $\text{End}(H)$.

In the non-graded case, this coincides with the usual definition of what is called $\omega^R$ — the right invariant Maurer–Cartan form of $G$ (equal to identity at $e$). It had values in $\mathcal{C}^\infty(G, \mathfrak{g})$, which should therefore be replaced by $\mathcal{C}^\infty(G, \text{Der}_e(C_G, \mathbb{R}))$, a morphism of graded rings — see [12] for details.

Note also that there is no inclusion between $\text{Der}_e(C_G, C_G)$ and $\text{Der}(C_G, C_G)$.

To relate this to the classical case, we prove the following proposition.

Proposition 2.25. If $(G, e)$ is a Lie monoid over $k$, then the map
\[ \Psi: \mathcal{C}^\infty(G, \mathfrak{g}) \xrightarrow{\sim} \text{Der}_e(\mathcal{C}^\infty(G), \mathcal{C}^\infty(G)), \]
\[ \xi \mapsto (f \mapsto \xi(\bullet) \cdot f) \]
is a linear isomorphism with inverse given by $\Psi^{-1}(u)(x) \mapsto (f \mapsto u(f)(x))$ where we used the isomorphism between $\mathfrak{g} = T_eG$ and $\text{Der}_e(\mathcal{C}^\infty(G), k)$.

Proof. If $\xi \in \mathcal{C}^\infty(G, \mathfrak{g})$ and $f, g \in \mathcal{C}^\infty(G)$, then $\Psi(\xi)(gh)(x) = (\xi(x)fg)(e) + f(e)(\xi(x) \cdot g)$, so $\Psi(\xi) \in \text{Der}_e(\mathcal{C}^\infty(G), \mathcal{C}^\infty(G))$.

Conversely, if $u \in \text{Der}_e(\mathcal{C}^\infty(G), \mathcal{C}^\infty(G))$ and $x \in G$ and $f, g \in \mathcal{C}^\infty(G)$, then $\Psi^{-1}(u)(x)(fg) = u(fg)(x) = u(f)(x)g(e) + f(e)u(g)(x)$ so $\Psi^{-1}(u)(x) \in \mathfrak{g}$, and $\Psi^{-1}(u)$ is smooth; the latter can be verified by standard technique using a partition of unity. \qed

Proposition 2.26. The Maurer–Cartan automorphism restricts to the linear isomorphism
\[ \omega^R: \mathfrak{X}(G) \xrightarrow{\sim} \text{Der}_e(C_G, C_G). \]

Proof. To prove the proposition, let us first note that a special instance of the $(\otimes, \circ)$-interchange identity is: if $a \in \text{Hom}(A, A')$ and $b \in \text{Hom}(B, B')$, then
\[ a \otimes b = (\text{id}_{A'} \otimes b) \circ (a \otimes \text{id}_B) = (a \otimes \text{id}_{B'}) \circ (\text{id}_A \otimes b): A \otimes B \to A' \otimes B'. \]

Together with the unit law, if $a: A \to H$, this gives
\[ \mu \circ (a \otimes \eta) = \mu \circ (\eta \otimes a) = a: A \to H. \]

In particular (using again the interchange property), if $a \in \text{Hom}(A, H)$, then
\[ \mu \circ (a \otimes \hat{e}) = a \otimes e^*: A \otimes H \to H. \]

Together with the counit law, if $a: H \to A$, then $(a \otimes \hat{e}) \circ \Delta = a \otimes \eta: H \to A \otimes H$. 

Now, suppose that \( X \circ \mu = \mu \circ (X \otimes \text{id} + \text{id} \otimes X) \). Then,

\[
(X \ast \iota^*) \circ \mu = \mu \circ (X \otimes \iota^*) \circ m^* \circ \mu
\]

\[
= \mu \circ (X \otimes \iota^*) \circ (\mu \otimes \mu) \circ \tau_{1324} \circ (m^* \otimes m^*)
\]

\[
= \mu \circ ((X \otimes \mu) \otimes (\iota^* \circ \mu)) \circ \tau_{1324} \circ (m^* \otimes m^*)
\]

\[
= \mu \circ ((\mu \circ (X \otimes \text{id} + \text{id} \otimes X)) \otimes (\mu \circ (\iota^* \circ \iota^*) \circ \tau)) \circ \tau_{1324} \circ (m^* \otimes m^*)
\]

\[
= \mu \circ (\mu \otimes \mu) \circ ((X \otimes \text{id} + \text{id} \otimes X) \otimes (\iota^* \circ \iota^*)) \circ \tau_{1342} \circ (m^* \otimes m^*).
\]

Taking into account that \( \mu \) is commutative\(^5\), i.e., \( \mu \circ \tau = \mu \) and thus \( \mu \circ (\mu \otimes \mu) \circ \sigma = \mu \circ (\mu \otimes \mu) \) for any permutation \( \sigma \in S_4 \), we obtain

\[
(X \ast \iota^*) \circ \mu = \mu \circ (X \ast \iota^* \otimes \text{id} \ast \iota^* + \text{id} \ast \iota^* \otimes X \ast \iota^*)
\]

\[
= \mu \circ (X \ast \iota^* \otimes \hat{\epsilon} + \hat{\epsilon} \otimes X \ast \iota^*) = X \ast \iota^* \otimes e^* + e^* \otimes X \ast \iota^*.
\]

Conversely, suppose that \( \xi \circ \mu = \xi \otimes e^* + e^* \otimes \xi \). Then,

\[
(\xi \ast \text{id}) \circ \mu = \mu \circ (\xi \otimes \text{id}) \circ m^* \circ \mu
\]

\[
= \mu \circ (\xi \otimes \text{id}) \circ (\mu \otimes \mu) \circ \tau_{1324} \circ (m^* \otimes m^*)
\]

\[
= \mu \circ ((\xi \otimes e^* + e^* \otimes \xi) \otimes (\mu \otimes \mu)) \circ \tau_{1324} \circ (m^* \otimes m^*)
\]

\[
= \mu \circ (\mu \circ (\xi \otimes e^* \otimes \text{id} + e^* \otimes \xi \otimes \text{id}) \otimes \text{id}) \circ \tau_{1324} \circ (m^* \otimes m^*)
\]

\[
= \mu \circ (\mu \circ (\xi \otimes \text{id} \otimes e^* + e^* \otimes \text{id} \otimes \xi) \otimes \text{id} \otimes \text{id}) \circ (m^* \otimes m^*)
\]

\[
= \mu \circ (\mu \circ (\xi \otimes \text{id})) \otimes e^* \otimes \text{id} + e^* \otimes \text{id} \otimes (\mu \circ (\xi \otimes \text{id}))) \circ (m^* \otimes m^*)
\]

\[
= \mu \circ (\xi \ast \text{id} \otimes \text{id} + \text{id} \otimes \xi \ast \text{id})
\]

as wanted. \( \square \)

**Definition 2.27.** The adjoint action \( \text{Ad}: H \mapsto H \otimes H \) of a Hopf algebra \( H \) by

\[
\text{Ad} = (\mu \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \iota^*) \circ m^{(3)}
\]

and the set of 1-cocycles of a graded Lie group \( G \) by

\[
Z^1(G, \text{Ad}) = \{ \xi \in \text{Der}_e(C_G, C_G) \mid m^* \circ \xi = \xi \otimes 1 + (\xi \otimes \text{id}) \circ (\mu \circ \tau \circ \text{Ad}) \}.
\]

**Proposition 2.28.** The Maurer–Cartan isomorphism restricts to the linear isomorphism

\[
\omega^R: \mathfrak{X}^{\text{mult}}(G) \cong Z^1(G, \text{Ad}).
\]

\(^5\)This is one of few cases where commutativity is needed.
Proof. This is a straightforward analogue of the proposition used for the van Est isomorphism in the non-graded case ([12], phrased in terms of convolution products. We shall prove it here using a more general fact.

Let $V$ be an $H$-bicomodule with left and right coactions $\rho^L_V : V \to H \otimes V$ and $\rho^R_V : V \to V \otimes H$, respectively, and let $C^n(V,H) = \text{Hom}(V,H^\otimes n)$ with coface operators $\delta_i : C^n(V,H) \to C^{n+1}(V,H)$, such that

$$
\delta_i(c) = \begin{cases}
(id \otimes c) \circ \rho^L_V, & i = 0, \\
(id \otimes (i-1) \otimes m^* \otimes \text{id}^\otimes (n-i)) \circ c, & 1 \leq i \leq n, \\
(c \otimes \text{id}) \circ \rho^R_V, & i = n+1.
\end{cases}
$$

(7)

The alternate sum of $\delta_i$

$$
\delta = \sum_{i=0}^{n+1} (-1)^i \delta_i
$$

is a nilpotent operator, i.e., $\delta^2 = 0$. Using that $\mu : H \otimes H \to H$ is a morphism of bialgebras and the antipode map $i^*$ is an anti-comorphism, i.e., $m^* \circ i^* = i^* \otimes i^* \circ \tau \circ m^*$, we construct a new left $H-$comodule structure on $V$:

$$
\rho^\text{new}_V = (\mu \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\text{id}^\otimes 2 \otimes i^*) \circ (\rho^L_V \otimes \text{id}) \circ \rho^R_V.
$$

(8)

Now $V$ has a new bicomodule structure, where the left coaction is given by $\rho^\text{new}_V$, while the right comodule structure is the trivial one $\text{id} \otimes 1 : V \to V \otimes H$. Therefore there exist new $\delta_i^\text{new}$ and the differential $\delta^\text{new} = \sum_{i=0}^{n+1} (-1)^i \delta_i^\text{new}$. Define $\omega^R_n : C^n(V,H) \to C^n(V,H)$ by the formula

$$
\omega^R_n(c) = \mu \otimes \text{id} \circ (c \otimes (m^*)^n \circ i^*) \circ \rho^R_V,
$$

(9)

where

$$(m^*)^n = \begin{cases}
\text{id}, & n = 1, \\
(m^* \otimes \text{id}^\otimes (n-2)) \circ \cdots \circ (m^* \otimes \text{id}) \circ m^*, & n \geq 2,
\end{cases}
$$

and

$$
\mu \otimes \text{id} \circ \tau_{1n+1 \cdots n2n} : H^\otimes n \otimes H^\otimes n \to H^\otimes n
$$

is the canonical extension of the multiplication $\mu$ to the $n-$the tensor power of $H$.

Lemma 2.29. One has for all $n$ and $i = 0, \ldots, n+1$,

$$
\omega^R_{n+1} \circ \delta_i = \delta_i^\text{new} \omega^R_n
$$

(10)

Proof. The proof is canonical and straightforward. To make it more intuitive and visual, we ”dualize” the picture by considering $H$-(bi)modules instead of $H$-(bi)comodules, $C^n(H,V) = \text{Hom}(H^\otimes n, V)$ instead of $C^n(V,H)$ and by assuming that $H$ is non-graded. We denote $ab = \mu(a,b)$, $\rho^L_V(a,v) = av$ and $\rho^R_V(v,a) = va$.
for \( a, b \in H, v \in V \), where \( \rho^L_V : H \otimes V \rightarrow V \) and \( \rho^R_V : V \otimes H \rightarrow H \) are the left- and right- \( H \)-module structures on \( V \), respectively. Now the dual analogue of (7) is

\[
\delta_i(c) = \begin{cases} 
\rho^L_V \circ (\text{id} \otimes c), & i = 0, \\
\rho^R_V \circ (\text{id} \otimes id), & i = n + 1 \\
c \circ (\text{id} \otimes (i-1) \otimes \mu \otimes \text{id}^{(n-i)}), & 1 \leq i \leq n, 
\end{cases}
\]

or, more explicitly

\[
\delta_i(c)(a_1, \ldots, a_n) = \begin{cases} 
a_1c(a_2, \ldots, a_n + 1), & i = 0, \\
c(\cdots, a_i a_{i+1}, \ldots), & 1 \leq i \leq n + 1, \\
c(a_1, \ldots, a_n) a_{n+1}, & i = n + 1 
\end{cases}
\]

for all \( a_1, \ldots, a_{n+1} \in H \). By use of the Sweedler notation (cf. [13]) \( m^*(a) = \sum a' \otimes a'' \) we rewrite the dual analogue of (8)

\[
\rho^V_{\text{new}} = \rho^R_V \circ (\rho^L_V \otimes \text{id}) \circ (\text{id} \otimes i^*) \circ (\text{id} \otimes \tau) \circ (m^* \otimes \text{id})
\]

as follows:

\[
\rho^V_{\text{new}}(a, v) = \sum a' v i^*(a'').
\]

Likewise, the dual analogue of (9)

\[
\omega^R_n(c) = \rho^R_V \circ (c \otimes i^* \circ (\mu)^n) \circ (m^*)_{H^n}
\]

where

\[
\mu^n = \begin{cases} 
\text{id}, & n = 1, \\
\mu \circ (\mu \otimes \text{id}) \circ \cdots \circ (\mu \otimes \text{id}^{(n-2)}), & n \geq 2 
\end{cases}
\]

and where

\[
(m^*)_{H^n} = \tau_{1n+1 \ldots n2n} \circ (m^*)^{\otimes n} : H^{\otimes n} \rightarrow H^{\otimes n} \otimes H^{\otimes n}
\]

is the canonical extension of the comultiplication \( m^* \) to the \( n \)-the tensor power of \( H \), admits the following explicit form:

\[
\omega^R_n(c)(a_1, \ldots, a_n) = \sum c(a'_1, \ldots, a'_n) i^*(a''_1 \ldots a''_n),
\]

where \( a_i \in H \) for \( 1 \leq i \leq n \) and \( m^*(a_i) = \sum a'_i \otimes a''_i \) (in Sweedler notations). This allows us to simplify computations. Indeed,

\[
\omega^R_{n+1}(\delta_0)(a_1, \ldots, a_n) = \sum a'_1 c(a'_2, \ldots, a'_{n+1}) i^*(a''_1 a''_2 \ldots a''_{n+1}).
\]

From the anti-morphism property of \( i^* \), we immediately get

\[
\omega^R_{n+1}(\delta_0)(a_1, \ldots, a_n) = \delta_0^\text{new} (\omega^R_n(c)(a_2, \ldots, a_{n+1}) i^*(a'_1 ^^1) = \delta_0^\text{new} (\omega^R_n(c)) (a_1, \ldots, a_{n+1}).
\]
On the other hand,
\[
\omega_{n+1}^R(\delta_{n+1}c)(a_1, \ldots, a_n) = \sum c(a'_2, \ldots, a'_{n+1})a'_{n+1}i^*(a''_1a''_2 \ldots a''_n) \\
= \sum c(a'_2, \ldots, a'_{n+1})a'_{n+1}i^*(a'_1a''_2 \ldots a''_n) \\
= \delta_{n+1}^{\text{new}}(\omega_{n}^R(c)) (a_1, \ldots, a_{n+1})
\]
since \( \sum a'i^*(a'') = e^*(a)1 \) for any \( a \in H \). The proof of the identity \( \omega_{n+1}^R \circ \delta_i = \delta_{i+1}^{\text{new}}\omega_{n}^R \) for \( i = 1, \ldots, n \) is equally easy. \( \square \)

The proof of Proposition 2.28 will follow from Proposition 2.26 and Lemma 2.29 by assuming that \( V = H \) together with the standard left- and right- comodule structure on it. \( \square \)

### 3. Differential graded Lie groups

In this short section we introduce the second ingredient of the differential graded Lie groups/algebras, namely the differential.

#### 3.1. Differential graded manifolds

Recall that the starting point to define gradings in Section 2 was the commutative monoid \( \Gamma \) with a particular element that we were calling 0. We suppose that it has an element that, together with its opposite (if it exists) generates \( \Gamma \), we call it 1. In the cancellative case, the only possibilities (up to isomorphism) are \((\mathbb{Z}, 1)\), \((\mathbb{N}, 1)\), and \((\mathbb{Z}/n\mathbb{Z}, 1)\).

**Definition 3.1.** A \( Q \)-structure or equivalently a homological vector field on a graded manifold is a derivation of its structure sheaf of degree 1 which squares to zero. A differential graded (dg) manifold (equivalently, \( Q \)-manifold) is a graded manifold with a homological vector field.

A morphism of dg manifolds is a morphism of graded manifolds which relates the homological vector fields in the following sense: given \( f: (\mathcal{M}_1, Q_1) \to (\mathcal{M}_2, Q_2) \), recall that \( f^\sharp: \mathcal{F}^*(\mathcal{C}(\mathcal{M}_2)) \to \mathcal{C}(\mathcal{M}_1) \). We require that \( f^\sharp \circ \mathcal{F}^* \circ Q_2 = Q_1 \circ f^\sharp \circ \mathcal{F}^* \).

In this paper, the focus is mainly on \( \mathbb{N} \)-graded \( Q \)-manifolds and their morphisms (see also [2]).

The product of dg manifolds as a graded manifold has a natural homological vector field. One just checks that if \( Q_1, Q_2 \) are homological, so is \( Q_1 \otimes \text{id} + \text{id} \otimes Q_2 \).

Therefore, we see that the above condition for multiplicativity of a vector field on a graded Lie group (6) means exactly that multiplication \( m^* \) is a dg morphism.

These definitions and observations combine into:

**Proposition 3.2.** The category of dg manifolds is cartesian monoidal.

#### 3.2. Differential graded Lie groups

**Definition 3.3.** The category of differential graded (dg) Lie groups is the category of monoidal objects in the category of dg manifolds which are groups.

Morphisms of dg Lie groups are defined in the natural way, and we thus obtain a category of dg Lie groups \( \text{dgLieGrp} \).

The body of a dg Lie group is a Lie group, and we have a “body” functor \( | \cdot |: \text{dgLieGrp} \to \text{LieGrp} \).
Example (The shifted tangent dg Lie group of a Lie group).

Let $M$ be a manifold. We define the shifted tangent bundle $T[1]M$ as the algebraic space with underlying space $M$ and structure sheaf defined by

$$\mathcal{O}_{T[1]M}(U) = \Omega^\bullet(U)$$

— the vector bundle of differential forms, for $U \subseteq M$ open, with the natural $\mathbb{N}$-grading, and obvious restriction maps.

This is an $\mathbb{N}$-graded manifold: if $U \subseteq M$ is the domain of a chart $\phi: U \to V$, then

$$\mathcal{O}_{T[1]M}(U) \simeq \mathcal{C}^\infty(\phi(U)) \otimes SV[1]^\ast.$$ 

Its body is obviously $|T[1]M| = M$ itself.

This is a dg-manifold with homological vector field $Q = d_{DR}$, given by the De Rham differential. More precisely, if $f \in \mathcal{O}_{T[1]M}$, then locally one can consider $f \in \Omega^\bullet(U)$, and $Q_{DR}f$ then corresponds to $df \in \Omega^{\bullet+1}(U)$ (this is a legitimate definition since vector fields are local operators).

If $M$ is a Lie group $G_0$ with multiplication $m$, then $G = T[1]G_0$ is a dg Lie group with multiplication $T[1]m$ which we now define. This will define the functor $T[1]$ from Lie groups to dg Lie groups.

The unit $e: \{\ast\} \to G$ is “the same” as that of $G_0$ — that is, it is the composition $e: \{\ast\} \to G_0 \hookrightarrow G$, by which we mean that $e: \mathcal{C}(G) \to \mathcal{C}^\infty(\{\ast\}) = \mathbb{R}$ is the evaluation at the unit $e \in G_0$ of the degree 0 component of a function on $G$. This unit is a dg morphism (of degree 0): it is graded, and preserves the homological vector fields. Indeed, the homological vector field on $\{\ast\}$ is 0, so the condition reads $(\mathcal{C}(G) \to^Q \mathcal{C}(\mathcal{N}) \to^e \mathbb{R}) = (\mathcal{C}(G) \to^e \mathbb{R} \to^0 \mathbb{R})$. The right-hand side is obviously the zero map, so this means that the evaluation at $e$ of the degree 0 component of any function $Q(f)$ has to be zero. This is obviously true since $Q$ is of degree 1 and $G$ is nonnegatively graded.

As for multiplication, if $m: G_0 \times G_0 \to G_0$ is the multiplication of $G_0$, then $T[1]m$ is naturally defined as follows: If $f \in \mathcal{C}(G)_0$, then $(m^f)(x,y) = f(xy)$ for $x,y \in G_0$. If $f = f_i(\cdot)e^i$ where $(e^i)$ is a basis of $\mathfrak{g}$ and the Einstein summation convention over repeating indexes is assumed, then

$$(T[1]m^f)(x,y) = f_i(xy)((T_eL_x)^je^j_2 + (T_eR_y)^je^j_1).$$

By a straightforward computation (for degree 0 and 1) one shows that $(Q, Q) \circ m^* = m^* \circ Q$.

Summarizing, we obtain the following:

Proposition 3.4. The dg manifold $G = T[1]G_0$ with the above unit and multiplication is a dg Lie group.

Example (The Chevalley–Eilenberg dg Lie group of a Lie algebra or a DGLA).

Case of a Lie algebra. If $\mathfrak{g}$ is a Lie algebra, its Chevalley–Eilenberg cochain complex, $\wedge \mathfrak{g}^*$ can be viewed as the algebra (with the wedge product) of functions on the $\mathbb{N}$-graded manifold $\text{CE}(\mathfrak{g})$. Namely,

$$\mathcal{C}(\text{CE}(\mathfrak{g})) = S\mathfrak{g}[1]^* = S\mathfrak{g}^*[-1].$$
In particular, its body is a point. This \(\mathbb{N}\)-graded manifold can be made into a dg manifold with homological vector field \(Q = d_{CE}\), called the Chevalley–Eilenberg differential. The usual Lie algebra bracket is then recovered as the \(Q\)-derived bracket of degree \(-1\) vector fields — the simplest example of the derived bracket construction [21]; and \(Q^2 = 0\) corresponds precisely to the Jacobi identity of \(\mathfrak{g}\).

This dg manifold \(CE(\mathfrak{g})\) can be made into a commutative dg Lie group, defining the multiplication as the coproduct. Namely, define

\[
m^*: \wedge \mathfrak{g}^* \to \wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g}^*,
\]

\[
f \mapsto f \otimes 1 + 1 \otimes f
\]
on generators \(f \in \mathfrak{g}^*\), which is enough by imposing that \(m^*\) be an algebra morphism which is unital, so \(m^*(1) = 1 \otimes 1\). Define the unit \(e^*: \wedge \mathfrak{g}^* \to \mathbb{R}\) as the projection to the degree 0 component, which is a unital algebra morphism.

The right-unit law reads \((\text{id} \otimes e^*) \circ m^* = \text{pr}_1^*: \wedge \mathfrak{g}^* \to \wedge \mathfrak{g}^* \otimes \mathbb{R}\), that is, for \(f \in \mathfrak{g}^\ast\), \((\text{id} \otimes e^*)(f \otimes 1 + 1 \otimes f) = f \otimes 1 + 1 \otimes 0 = f \otimes 1 = \text{pr}_1^*(f)\), and similarly for the left-unit law. Checking associativity is similar, and exactly the same as for the usual coproduct. Moreover, the multiplication is obviously commutative, in the sense that \(\tau \circ m^* = m^* \circ \tau\) where \(\tau\) is the flip.

The inverse is given on generators by \(\text{inv}: f \mapsto f_0 - f\), that is, \(\text{inv} = i \circ e - \text{id}\) where \(i: \mathbb{R} \to \wedge \mathfrak{g}^\ast\) is uniquely defined. This is the only dg Lie group structure here (cf. Cartier–Milnor–Moore theorem).

As for the multiplicativity of \(Q\), recall that it induces on \(CE(\mathfrak{g}) \times CE(\mathfrak{g})\) the homological vector field \(Q \otimes \text{id} + \text{id} \otimes Q\). Then we have to check that

\[
(Q \otimes \text{id} + \text{id} \otimes Q) \circ m^* = m^* \circ Q: \wedge \mathfrak{g}^* \to \wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g}^*.
\]

If \(f \in \mathfrak{g}\), then \((Q \otimes \text{id} + \text{id} \otimes Q)(f \otimes 1 + 1 \otimes f) = Qf \otimes 1 + 1 \otimes Qf = m^*(Qf)\).

To summarize, we have proved:

**Proposition 3.5.** The graded manifold \(CE(\mathfrak{g})\) with the homological vector field \(Q = d_{CE}\) and unit and multiplication as above is a commutative dg Lie group.

**Definition 3.6.** \((CE(\mathfrak{g}, d_{CE})\) is called the Chevalley–Eilenberg dg Lie group of the Lie algebra \(\mathfrak{g}\).

**Graded case.** We want to extend this construction from Lie algebras to DGLA’s. Let \(\mathfrak{g}\) be a non-positively graded DGLA, recalling the remark about functional analytic issues and the algebra completion. We need this condition. We define \(CE(\mathfrak{g})\) in the same way as an \(\mathbb{N}\)-graded manifold. The only change which occurs is that the structure constants of \(\mathfrak{g}\) take gradings into account as well: all the usual equations (antisymmetry, Jacobi identity) include some signs, but the form remains very similar.

Recall that \(C(CE(\mathfrak{g})) = S\mathfrak{g}[1]^* = S\mathfrak{g}^*[-1]\). Take into account the shifts in gradings and consider the homological vector field \(Q = d_{CE} + d_{\mathfrak{g}}\).

Repeating almost verbatim the beginning of this subsection, one obtains the following:
Proposition 3.7. The graded manifold $\text{CE}(\mathfrak{g})$ with the homological vector field $Q = d_{CE} + d_\mathfrak{g}$ admits the structure of a dg Lie group.

Definition 3.8. $(\text{CE}(\mathfrak{g}), d_{CE} + d_\mathfrak{g})$ is called the Chevalley–Eilenberg dg Lie group of the DGLA $\mathfrak{g}$.

Remark 3.9. The construction above obviously reminds of Lie algebroids, and inspires us to consider the question of integration of those, which we plan to address in future works.

4. Graded Harish-Chandra pairs and integration of DGLA's

The goal of this section is to show the relation between differential graded Lie groups and algebras. First we explain how DGLAs are obtained from DGLGs. Then we present the result on the equivalence of categories of graded Lie groups and graded Harish-Chandra pairs (GHCP). And as a final step we introduce the notion of DGHCP — differential graded Harish-Chandra pairs thus concluding the DGLA to DGLG integration procedure.

4.1. DGLAs of DGLGs

We have discussed the GLG ↔ GLA correspondence in Section 2.5. And the whole previous Section 3.2 is devoted to defining the differential (or a $Q$-structure) on the GLGs. As the first step, like in the non-graded case, this structures has to be transferred to GLA. This is done as follows.

Definition 4.1. The 1-cocycle associated to a multiplicative vector field: if $Q \in \mathfrak{X}(G)$, define

$$\xi = Q \star i^* = \mu \circ (Q \otimes i^*) \circ m^*.$$ 

The identity $e^* \circ \mu = e^* \otimes e^*$, gives $\xi_e = e^* \circ \xi = e^* \circ \mu \circ (Q \otimes i^*) \circ m^* = (e^* \otimes e^*) \circ (Q \otimes i^*) \circ m^* = (Q_e \otimes (e^* \circ i^*)) \circ m^* = (Q_e \otimes e^*) \circ m^* = Q_e$, that is,

$$\xi_e = Q_e.$$ 

Using the results of Section 2.7 on the Maurer–Cartan endomorphism, one proves that $\xi$ is a 1-cocycle.

Definition 4.2. The derivation associated to a multiplicative vector field: if $X \in \mathfrak{X}(G)$, define $\delta_X : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$\delta_X v = v \circ X.$$ 

This notion is important in the following context:

Proposition 4.3. If $X \in \mathfrak{X}^{\text{mult}}(G)$ has degree $d$, then $\delta_X \in \text{Der}^d(\mathfrak{g})$ is a derivation of degree $d$.

Proof. The only thing to check is the behaviour of $\delta_X$ with respect to the bracket on $\mathfrak{g}$. The result is: $\delta_X [v, w] = [\delta_X v, w] + (-1)^{d[v]}[v, \delta_X w]$. It is obtained by computing $\delta_X [v, w]$ from its definition, and using the multiplicativity of $X$ (2.19). The sign appears due to the grading since $\deg X = d$, and it is precisely the same as for the degree $d$ derivation. □
Now it is easy to piece together the results above and apply them to \( Q \)-structures. The homological condition \( Q^2 = 0 \) immediately implies \( \delta_Q^2 = 0 \) since \( \delta_Q v = \nu \circ Q \).

Among examples, let us mention the following two natural constructions:

*The DGLA of a shifted tangent dg Lie group* \( T[1]G \) is \( g[1] \oplus g \), that is,

\[
\cdots \rightarrow 0 \rightarrow g[1] \rightarrow g \rightarrow 0 \rightarrow \cdots
\]

The bracket is constructed from the original bracket on \( g \), and it does not make a difference if it is computed on elements of \( g \) or \( g[1] \), except for the case of \([g[1], g[1]]\) which vanishes for degree reasons. And the differential is id: \( g[1] \rightarrow g \).

*The DGLA of a Chevalley–Eilenberg dg Lie group.* To start with, in the non-graded case the following proposition holds:

**Proposition 4.4.** If \( g \) is a Lie algebra, then the DGLA of \( CE(\mathfrak{g}) \) is the abelian DGLA \( g[1] \).

Indeed, since the underlying manifold of \( CE(\mathfrak{g}) \) is a point, the degree 0 component of its DGLA is 0. Also, since \( CE(\mathfrak{g}) \) is commutative, so should its DGLA be (that is, \([\cdot, \cdot] = 0\), but in general its differential need not be zero).

Analogously, for DGLAs one has the following:

**Proposition 4.5.** If \( \mathfrak{g} \) is a DGLA, then the DGLA of \( CE(\mathfrak{g}) \) is the abelian DGLA \( \mathfrak{g}[1]^* \) with the differential being the transpose of the original one (and reversed grading), that is:

\[
\mathfrak{g} : \ldots \rightarrow \mathfrak{g}_2 \xrightarrow{d} \mathfrak{g}_1 \xrightarrow{d} \mathfrak{g}_0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots
\]

\[
DGLA(CE(\mathfrak{g})) : \ldots \rightarrow 0 \rightarrow 0 \rightarrow \mathfrak{g}_0^* \xrightarrow{d^*} \mathfrak{g}_{-1}^* \xrightarrow{d^*} \mathfrak{g}_{-2}^* \rightarrow \cdots
\]

### 4.2. Graded Harish-Chandra pairs

In this subsection, we define the graded Harish-Chandra pairs, by “graded” in this and next section we mean \( \mathbb{N} \)-graded (in contrast to \( \mathbb{Z} \)). The construction mimics essentially the super case (\( \mathbb{Z}_2 \)-graded), and we thus follow the summary in [29] of [14] and [15]. In this presentation we will point out one essential difference: the \( \mathbb{Z}_2 \)-graded case uses finite dimensionality of the graded part, which does not hold anymore in the \( \mathbb{N} \)-graded case. Elements of even degrees are not nilpotent, hence the formal power series do not reduce to polynomials. Nevertheless, for a graded Lie algebra one can construct directly a group law on the integrating object, and when the GLA is differential with the construction of Section 4.3 it becomes a DGLG.

**Definition 4.6.** The graded Harish-Chandra pair is the following data:

- A couple \( (G_0, g) \) of a Lie group \( G_0 \) and a graded Lie algebra \( g = \sum_{i \geq 0} g_i \), for which \( g_0 = \text{Lie}(G_0) \) is the Lie algebra of \( G_0 \).

- A degree-preserving representation \( (G_0, g) \) of a Lie group \( \alpha_{G_0} \) of \( G_0 \) in \( g \) which induces the adjoint representation of \( G_0 \) in \( g_0 \); and the differential \( (d\alpha_{G_0})_e \) of which at the identity \( e \in G_0 \) coincides with the adjoint representation \( ad \) of \( g_0 \in g \).
Remark 4.7. In the definition above, by “graded” we mean $\mathbb{N}$-graded, and we write it as if $\mathbb{N} = \mathbb{Z}_{\geq 0}$, i.e., non-negatively graded. But it is important to note that there is no reason to disregard the non-positively graded case ($\mathbb{N} = \mathbb{Z}_{\leq 0}$), especially since it appears naturally passing to the dual of the picture (see for instance, Proposition 4.5). We will stress this fact in the final theorem.

The morphisms of graded Harish-Chandra pairs are defined in a natural way. For two Harish-Chandra pairs a morphism $F : (G_0, g) \to (H_0, h)$ consists of a pair of homomorphisms $f : G_0 \to H_0$ and $f : g \to h$, such that $(df)_e = f|_g$ and $f \circ \alpha_{G_0}(g) = (\alpha_{H_0}(g)) \circ f$, $\forall g \in G$.

This defines the category of graded Harish-Chandra pairs that we denote $\text{GHCP}$. We will show that it is equivalent to the category of graded Lie groups.

Theorem 4.8. There is an equivalence of categories between non-negatively graded Lie groups and non-negatively graded Harish-Chandra pairs.

Proof. One way of this equivalence is rather straightforward. Given a graded Lie group $G$, one considers its body part $|G| = G_0$ together with the graded Lie algebra $g = \text{Lie}(G)$ and equips it with the the adjoint representation $\alpha_{G_0} = \text{Ad}_{G_0}$. The construction the other way around is a bit technical, we will sketch the essential points of it here.

Let $\mathcal{U}$ denote the universal enveloping graded-algebra functor. If $g$ is a graded Lie algebra over $k$, then $\mathcal{U}(g)$ is a $\mathcal{U}(g_0)$ module, and the action of $g_0$ on the sheaf $\mathcal{C}_G(U)$ induces a structure of $\mathcal{U}(g_0)$-module on $\mathcal{C}_G(U)$. From the graded Harish-Chandra pair, define then the graded manifold structure sheaf as

$$\mathcal{O}_G(U) = \text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), C^\infty(U))$$

for open subsets $U \subseteq G_0$. By the graded Poincaré–Birkhoff–Witt (PBW, [9]) theorem we have

$$\text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), C^\infty(U)) \simeq \text{Hom}_k(S(g/g_0), C^\infty(U)).$$

The graded enveloping algebra $\mathcal{U}(g)$ can be equipped with a graded Hopf algebra structure, and we can thus profit from all the constructions from Section 2.

The explicit construction of the above structure, as well as the description of the relation of objects and morphisms of the mentioned categories goes through verbatim as in [29, Sect. 2.], replacing the word “super” by “graded”. We repeat here the smooth version of this technique (the generalization to the analytic and algebraic case is straightforward).

The graded Hopf algebra obtained from a Harish-Chandra pair is now

$$H = \text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), C^\infty(G_0)) = \{ f \in \text{Hom}(\mathcal{U}(g), C^\infty(G_0)) \mid f(uX, g) = - (X^R f)(u, g), \forall u \in \mathcal{U}(g), X \in g_0, g \in G_0 \},$$

where $X^R$ is the right-invariant vector field on $G_0$ corresponding to $X$. The (graded commutative) multiplication is the convolution product, i.e. it is defined as

$$(f_1 f_2)(u, g) = (f_1 \otimes f_2)(\Delta u, g, g),$$

where $\Delta$ is the coproduct. The counit is $\epsilon(f) = f(e, e)$, and the antipode is $S(f)(u, g) = f(-u, g^{-1}) = f(-u, H_0) = -f(u, \text{Ad}^{-1}(g))$. The graded Harish-Chandra pair $(G_0, g)$ is then equivalent to the graded Hopf algebra $H = \text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), C^\infty(G_0))$. The category $\text{GHCP}$ is then equivalent to the category of graded Hopf algebras.

The proof of this theorem involves constructing explicit functors between the categories of graded Lie groups and graded Harish-Chandra pairs, and verifying that these functors are inverse to each other. The details of this construction are beyond the scope of this exposition, but the key steps involve using the universal enveloping algebra functor to construct graded Lie groups from graded Harish-Chandra pairs, and vice versa.
where $\Delta$ is the standard comultiplication in $\mathcal{U}(\mathfrak{g})$, while the (graded) comultiplication $m^*: H \to H \otimes H$ is the co-convolution product, i.e., $m^*(f)(u_1,g_1,u_2,g_2) = f(u_1 \alpha_{G_0}(g_1),u_2,g_1 g_2)$. It is not hard to verify that the product and coproduct of elements of $H$ belongs to $H$ and $H \otimes H$, respectively. The antipode is obtained as a combination of the antipodes in $\mathcal{U}$ and $C^\infty(G_0)$. \qed

There are however two very important points to mention.

**Remark 4.9.** Even if the construction is very similar to the super case, the essential difference is in the definitions of the employed structures and, in particular, the graded Hopf algebras (Section 2).

**Remark 4.10.** The construction relies heavily on the PBW theorem, and there it is important that the grading is $\mathbb{N}$ (i.e., $\mathbb{Z}_{\leq 0}$ or $\mathbb{Z}_{\geq 0}$ but not $\mathbb{Z}$), meaning that is no problem in consistent ordering of the basis of $(\mathfrak{g}/\mathfrak{g}_0)$. The construction may be applied in some more general cases, but then a lot of technicalities occur. We are going to discuss the question of validity of PBW in a separate paper [12].

### 4.3. Integration of DGLAs

The idea of the method is the following: given an $\mathbb{N}$-graded DGLA $\mathfrak{g}$, one integrates its degree 0 part $\mathfrak{g}_0$ to its simply connected Lie group $G_0$. This gives a graded Harish-Chandra pair $(G_0,\mathfrak{g},\alpha_{G_0})$. One constructs its associated graded Lie group from the differential on the DGLA – we detail this step in the current section.

Let us extend $\alpha_{G_0}$ to all graded derivations $\text{Der}^*(\mathfrak{g})$ of $\mathfrak{g}$ by use of the conjugation; given that $\alpha_{G_0}(g,-)$ is a degree 0 automorphism of $\mathfrak{g}$ for every $g$, the conjugation of any graded derivation by $\alpha_{G_0}(g,-)$ is a derivation of the same degree. For any connected $G_0$ and any $\delta \in \text{Der}^*(\mathfrak{g})$ one has $\alpha_{G_0} \delta \alpha_{G_0}^{-1} = \delta$ modulo inner derivations. Moreover, if we denote

$$\tilde{\lambda}(g) := \alpha_{G_0}(g,-) \delta \alpha_{G_0}(g,-)^{-1} - \delta,$$

then $\tilde{\lambda}$ is a 1-cocycle on $G_0$ with values in the space inner derivations of degree 1 regarded as a $G_0$-module by use of the conjugation by $\alpha_{G_0}$. Indeed, for any $g,h \in G_0$ we obtain

$$\tilde{\lambda}(gh) = \alpha_{G_0}(gh,-) \delta \alpha_{G_0}(gh,-)^{-1} - \delta$$

$$= \alpha_{G_0}(g,-) \delta \alpha_{G_0}(h,-) \alpha_{G_0}(h,-)^{-1} - \delta \alpha_{G_0}(g,-)^{-1}$$

$$+ \alpha_{G_0}(g,-) \delta \alpha_{G_0}(g,-)^{-1} - \delta = \alpha_{G_0}(g,-) \tilde{\lambda}(h) \alpha_{G_0}(g,-)^{-1} + \tilde{\lambda}(g).$$

This motivates the following definition:

**Definition 4.11.** Let $(G_0,\mathfrak{g},\alpha_{G_0})$ be a graded Harish-Chandra pair, and $\mathfrak{g}$ be a DGLA over a field $k$ with a differential $\partial$. We call $(G_0,\mathfrak{g},\partial,\alpha_{G_0})$ a **differential graded Harish-Chandra pair** (DG Harish-Chandra pair) if there exists a $\mathfrak{g}^1$-valued 1-cocycle on $G_0$, i.e., a smooth map $\lambda: G_0 \to \mathfrak{g}^1$ which satisfies

$$\lambda(gh) = \lambda(g) + \alpha_{G_0}(g,\lambda(h))$$

(12)

for all $g,h \in G_0$, such that in addition

$$\tilde{\lambda}(g) = ad \circ \lambda(g).$$

(13)
Remark 4.12. If $\partial$ is an inner derivation then $\lambda$ is uniquely fixed by $\alpha_{G_0}$. Otherwise the identity (13) will fix $\lambda$ only modulo the center of $\mathfrak{g}$.

Remark 4.13. Spelling out the definition of morphisms of DG Harish-Chandra pairs is an instructive exercise.

Lemma 4.14. Let $(\mathfrak{g}, G_0, \alpha_{G_0})$ be a Harish-Chandra pair with a simply connected $G_0$ and $\partial$ be a degree 1 outer differential in $\mathfrak{g}$. Then there exists a canonical extension of $\bar{\lambda}$ to $\lambda$, which makes $(G_0, \mathfrak{g}, \alpha_{G_0}, \lambda)$ a differential graded Harish-Chandra pair.

Proof. The differential of $\lambda$ at the identity must give us the following 1-cocycle on $\mathfrak{g}_0$: $T_e G_0 \ni X \mapsto -\partial(X) \in \mathfrak{g}^1$; since $G_0$ is simply connected this uniquely determines the required 1-cocycle $\lambda$ on $G_0$ by the Van Est isomorphism.

Theorem 4.15. There is an equivalence of categories between $\mathbb{N}$-graded differential Lie groups and differential $\mathbb{N}$-graded Harish-Chandra pairs.

Proof. Let $(\mathfrak{g}, G_0, \alpha_{G_0}, \partial, \lambda)$ be a DG Harish-Chandra pair. If $\partial$ is an inner derivation corresponding to a degree 1 element of $\mathfrak{g}^1$ which we denote by the same letter (by Remark 4.12 $\lambda$ is uniquely fixed by $\alpha_{G_0}$), then we define a multiplicative structure as the difference between left- and right-translations of $\partial$. By use of Proposition 2.22 this is a multiplicative vector field; it is easy to see that this vector field will give us back the differential $\partial$ in $\mathfrak{g}$.

More precisely, the multiplicative vector field $Q = \partial^L - \partial^R$ acts on an arbitrary smooth function $f$ on $G$ as follows:

$$(Qf)(u, g) = (\partial^L f)(u, g) - (\partial^R f)(u, g) = (-1)^{\deg(u)} f(u\alpha_{G_0}(g, \partial), g) - f(\partial u, g)$$

for any $u \in \mathfrak{U}(\mathfrak{g})$, $g \in G_0$. On the other hand,

$$(Qf)(u, g) = (-1)^{\deg(u)} f(u\lambda(g), g) - f([\partial, u], g), \quad (14)$$

where $\lambda(g) = \alpha_{G_0}(g, \partial) - \partial$. Indeed,

$$(Qf)(u, g) = (-1)^{\deg(u)} f(u\alpha_{G_0}(g, \partial), g) - f(\partial u, g)$$

$$= (-1)^{\deg(u)} f(u\alpha_{G_0}(g, \partial), g) - f([\partial, u], g) - (-1)^{\deg(u)} f(u\partial, g)$$

$$= (-1)^{\deg(u)} f(u(\alpha_{G_0}(g, \partial) - \partial), g) - f([\partial, u], g)$$

$$= (-1)^{\deg(u)} f(u\lambda(g), g) - f([\partial, u], g).$$

Now we use formula (14) to extend the integration procedure to the more general case as follows. Let $\partial$ be an outer derivation; we apply Lemma 4.14 to obtain a 1-cocycle $\lambda$ and thus the structure of a DG Harish-Chandra pair (see Definition 4.11). By replacing of $[\partial, u]$ with $\partial(u)$ in (14), we obtain the formula for the multiplicative vector field $Q$ on $G$:

$$(Qf)(u, g) = (-1)^{\deg(u)} f(u\lambda(g), g) - f(\partial(u), g) \quad (15)$$

for all $u \in \mathfrak{U}(\mathfrak{g})$, $g \in G_0$. The rest of the proof including the morphism property is straightforward.
4.4. Extended Harish-Chandra pairs

Lemma 4.16. Let \( \mathfrak{g} \) be a DGLA over a field \( k \) with an outer differential \( \partial \). Then

- \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus k\partial \) admits a canonical structure of a DGLA, such that \( \mathfrak{g} \) is a graded Lie subalgebra, \( \partial^2 = 0 \) and \([\partial, X] = \partial(X)\) for every \( X \in \mathfrak{g} \). The differential in \( \tilde{\mathfrak{g}} \) is given by the adjoint action of \( \partial \);
- a 1-cocycle \( \lambda \) from Definition 4.11 is in one-to-one correspondence with an extension \( \tilde{\alpha}_{G_0} \) of \( \alpha_{G_0} \) to \( \tilde{\mathfrak{g}} \);
- if \( G_0 \) is simply connected then there exists a canonical extension \( \tilde{G}_0 \) of \( G_0 \) to \( \tilde{\mathfrak{g}} \), which makes \((\tilde{\mathfrak{g}}, G_0, \tilde{\alpha}_{G_0})\) into a Harish-Chandra pair.

Proof. While the first two statements are resulting from a straightforward computation, the third one follows from the second statement combined with Lemma 4.14. \( \square \)

Definition 4.17. We shall call \((\tilde{\mathfrak{g}}, G_0, \tilde{\alpha}_{G_0})\) an extended Harish-Chandra pair.\(^6\)

By construction, the extended Harish-Chandra pair \((\tilde{\mathfrak{g}}, G_0, \tilde{\alpha}_{G_0})\) integrates the (extended) graded Lie algebra \( \tilde{\mathfrak{g}} \) to a graded Lie group \( \tilde{G} \) with a graded subgroup \( G \), which corresponds to the initial Harish-Chandra pair \((\mathfrak{g}, G_0, \alpha_{G_0})\). Taking into account that \( \partial \) is now an inner derivation of \( \tilde{\mathfrak{g}} \), we can integrate it to a multiplicative vector field \( \tilde{Q} \) on \( \tilde{G} \) by use of formula (14).

Lemma 4.18. \( G \) is a differential graded Lie subgroup of \( \tilde{G} \), such that the induced DGLG structure on \( G \) coincides with the one given by formula (15).

Proof. Notice that the ideal of \( G \) in the graded algebra of smooth functions on \( \tilde{G} \), i.e., the ideal of functions vanishing on \( \tilde{G} \) is

\[
I_G = \{ f \in \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)} (\mathcal{U}(\tilde{\mathfrak{g}}), \mathcal{C}^\infty(G_0)) \mid f(\mathcal{U}(\mathfrak{g})) = 0 \}.
\]

If \( u \in \mathcal{U}(\mathfrak{g}) \subset \mathcal{U}(\tilde{\mathfrak{g}}) \) then \( u\lambda(g) \) and \([\partial, u] \equiv \partial(u)\) also belong to \( \mathcal{U}(\mathfrak{g}) \), therefore for any \( f \in I_G \) one has \((\tilde{Q}f)(u,g) = 0\) and thus \( \tilde{Q}f \in I_G \). Finally the restriction of \( \tilde{Q} \) onto \( G \) defines the multiplicative structure \( Q \) on \( G \) which gives back \( \partial \) in \( \mathfrak{g} \) and the formula for \( Q \) coincides with (15). \( \square \)

This construction reverses the procedure described above of “differentiating” a DGLG to a DGLA. It can, for instance, be applied for the examples from the previous section, namely recover: the shifted tangent bundle to a Lie group; the Chevalley–Eilenberg Lie group in the graded case. A motivated reader may also consider simpler examples (i.e., specifications) like: the dg Lie group of an abelian DGLA; the dg Lie group of a DGLA concentrated in degree \( d \).

\(^6\)The idea to interpret the integration of DGLA with an outer derivation in terms of such an extended pair was suggested to us by C. Laurent-Gengoux.
In this paper we addressed the question of integrating differential graded Lie algebras to differential graded Lie groups. As mentioned in the introduction, this is part of a big project of a systematic study of the integration problem on the categorical level: it should include, among others, some $\infty$ structures and generalized geometry, with potentially non-trivial links between them.

Let us stress again, even if initially the strategy of this paper meant to repeat essentially the approach of [29] in the case of super DLGs and DLAs (i.e., $\mathbb{Z}/2\mathbb{Z}$-graded) and add “by hand” a $Q$-structure to it, the question turned out to be more intricate: working with $\mathbb{Z}$- and even $\mathbb{N}$-graded objects presents conceptual challenges. So the resemblance of the final construction for the $\mathbb{N}$-graded case to the super case is misleading: it relies on results that are not straightforward generalizations and therefore had to be explicitly explained.

Two points are worth mentioning here:

First. The main result concerns equivalence of categories, and there graded Harish-Chandra pairs play the key role. The concept of differential graded Harish-Chandra pairs that we introduced is an important step — those seem to have higher analogues and actually give a possible way to generalize the result to Lie algebroids and possibly other structures.

Second. As we have understood from Section 4, the construction works as long as one can safely apply the Poincaré–Birkhoff–Witt theorem. But the tricky point is before that, at the level of definition of the functional spaces on graded algebras/groups. Namely, natural elements are now formal power series in graded variables, not polynomials — one thus loses some intuition about their behaviour. We thought of it as an auxiliary technical issue, but again in the $\mathbb{Z}$-graded case it turned out to be more interesting. We realized that careful description of the functional space, the universal envelopping algebra with its properties, as well as the Hopf algebra related questions, is a problem worth being detailed by itself. Thus, not to overload the presentation here, we are going to devote a separate paper ([12]) exclusively to this topic.

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