The shift bound for abelian codes and generalizations of the Donoho-Stark uncertainty principle

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Dedicated to the memory of J.H. van Lint.

Abstract

Let $G$ be a finite abelian group. If $f : G \to \mathbb{C}$ is a nonzero function with Fourier transform $\hat{f}$, the Donoho-Stark uncertainty principle states that $|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G|$. The purpose of this paper is twofold. First, we present the shift bound for abelian codes with a streamlined proof. Second, we use the shifting technique to prove a generalization and a sharpening of the Donoho-Stark uncertainty principle. In particular, the sharpened uncertainty principle states, with notation above, that $|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G| + |\text{supp}(f)| - |H(\text{supp}(f))|$, where $H(\text{supp}(f))$ is the stabilizer of $\text{supp}(f)$ in $G$. 

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I. Introduction

In a breakthrough paper [22], Van Lint and Wilson developed a technique called shifting to obtain lower bounds for the minimum distance of cyclic codes. The best bound obtainable by this technique is called the shift bound (or, sometimes, the Van Lint-Wilson bound), which depends only on the defining zeros of the cyclic code. The shift bound can be difficult to compute, but given the corresponding shifting steps, it is easy to verify correctness of the bound. So it can be used as a proof certificate that the minimum distance of a given code has (at least) a certain value.

The shifting technique for cyclic codes can be easily generalized to abelian codes, which are ideals of a group algebra $\mathbf{F}[G]$, with $G$ being a finite abelian group and $\mathbf{F}$ some finite field. If the field has characteristic 0 or if the characteristic of the field does not divide $|G|$ then a Fourier-type transform can be defined on $\mathbf{F}[G]$, and such a code can then be characterized in terms of the vanishing of certain Fourier transform coefficients for all codewords. The shift bound for abelian codes generalizes the ordinary shift bound and now only depends on the set of coefficients that vanish for all codewords. After Section III which contains some background on characters of abelian groups, we present the shift bound for abelian codes in Section IV with a streamlined proof. In Section IV, we give some examples to illustrate applications of the shift bound. In Section V, we present an alternative, simpler derivation of the shift bound, based on ideas from [28].

The Donoho-Stark uncertainty principle for finite abelian groups states: If $G$ is a finite abelian group, and $f : G \to \mathbb{C}$ is a nonzero complex-valued function with Fourier transform $\hat{f}$, then $|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G|$; equality holds if and only if $f$ is a nonzero multiple of the restriction of a character to a coset of a subgroup of $G$ (see, e.g., [25], [38]).

It seems natural to try to generalize this principle to all fields for which a Fourier-type transform exists. The original proof in [25] as well as the simple proofs in [32] or [41] crucially depend on the existence of an absolute value, and hence do not generalize to finite fields. It is not too difficult to see that the elementary induction proof in [26] does generalize (indeed, note that for cyclic groups, the principle can be seen to follow from the BCH bound); however, the resulting proof is still rather complicated. Since the shift bound for abelian codes provides a lower bound on the weight $w(f) = |\text{supp}(f)|$ of a nonzero function $f : G \to \mathbf{F}$ in terms of the support of its Fourier transform $\hat{f}$, it seems reasonable to investigate whether the Donoho-Stark uncertainty principle can be obtained as a consequence of the shift bound. In Section VII, we show that this is indeed the case. Here, we use the shift bound to derive a generalization of the Donoho-Stark uncertainty principle. We note that a similar approach was used in [28] for a generalization to non-abelian groups.

In Section VII, we use the shifting technique to prove a sharpening of the Donoho-Stark uncertainty principle. Let $G$ be a finite abelian group, $\mathbf{F}$ a field of characteristic 0 or characteristic $p$ with $p \nmid |G|$, and let $f : G \to \mathbf{F}$ be a nonzero function with Fourier transform $\hat{f}$. We obtain a pair of inequalities which are stronger than the Donoho-Stark uncertainty principle: $|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G| + |\text{supp}(\hat{f})| - |H(\text{supp}(f))|$ and $|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G| + |\text{supp}(f)| - |H(\text{supp}(f))|$, where $H(\text{supp}(f))$ is the stabilizer of $\text{supp}(f)$ in $G$, and $H(\text{supp}(f))$ is similarly defined as a subset of $G$.

II. Preliminaries

In this section we summarize some of the theory of $\mathbf{E}$-valued characters and Fourier transforms over finite abelian groups, for general (possibly finite) fields $\mathbf{E}$. Readers who are familiar with Fourier theory might skip this section on first reading. Further background on harmonic analysis and Fourier analysis on groups can be found, e.g., in [18], [24], [35], [42]. For the use of characters and Fourier transforms in relation to coding theory, see, e.g., [4], [6], [15], where most of the results below can be found. Abelian codes were first investigated in [2], [3].

Let $(G, +)$ be a finite abelian group. We write $|G|$ to denote the order of $G$. The exponent $\exp(G)$ of $G$ is defined as the smallest positive integer $N$ for which $Nx = 0$ for all $x \in G$, where 0 denotes the identity element of $(G, +)$.

In the remainder of this paper, $\mathbf{F}$ is a field of characteristic $\text{char}(\mathbf{F}) = 0$ or $\text{char}(\mathbf{F}) = p$ with $p \nmid |G|$, and $\mathbf{E}$ is an extension of $\mathbf{F}$ containing a primitive $N$-th root of unity $\xi$, an element of multiplicative order $N$ in $\mathbf{E}$, where $N = \exp(G)$. Note that our assumption on $\text{char}(\mathbf{F})$ is necessary and sufficient to guarantee that such an
extension exists. A character \( \chi \) of \( G \) is a homomorphism of \((G, +)\) to the cyclic group of order \( N \) generated by \( \xi \), and hence takes its values in the field \( E \). We will refer to such a character as a \( E \)-valued character. These characters form a group \((\hat{G}, +)\) under the operation of pointwise multiplication defined by

\[
(\chi + \phi)(g) = \chi(g)\phi(g)
\]

for \( \chi, \phi \in \hat{G} \) and \( g \in G \).

The abelian group \( G \) is isomorphic to a direct product

\[
G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}
\]

of cyclic groups; note that

\[
N = \exp(G) = \text{lcm}(n_1, \ldots, n_r).
\]

For each \( e = (e_1, \ldots, e_r) \in G \), define the map \( \chi_e : G \to E \) by

\[
\chi_e(x) = \xi^\sum_{i=1}^r \frac{x_i}{e_i} x_i,
\]

for all \( x = (x_1, \ldots, x_r) \in G \). It is easy to see that each \( \chi_e \) is a character, and \( \chi_a + \chi_b = \chi_{a+b} \) for all \( a, b \in G \). Moreover, since \( \xi \) is a primitive \( N \)-th root of unity, all the \( \chi_e \)'s are distinct and, in fact, it is easily shown that each character is of this form. Hence the group \((\hat{G}, +)\) of characters of \( G \) is isomorphic to \((G, +)\). Note that the identity element of \( \hat{G} \) is \( \chi_0 : x \mapsto 1 \) for all \( x \in G \). An important property is that for every \( x \in G \setminus \{0\} \) there exists a character \( \chi \in \hat{G} \) for which \( \chi(x) \neq 1 \).

For \( a \in G \), define \( \Phi_a : \hat{G} \to E \) by letting

\[
\Phi_a(\chi) = \chi(a)
\]

for all \( \chi \in \hat{G} \). Note that \( \Phi_a \) is a character on \( \hat{G} \), that is, an element of \( \hat{\hat{G}} \). In fact, it turns out that \( G \) and \( \hat{\hat{G}} \) are isomorphic (Pontryagin duality), with the map \( a \to \Phi_a \) being an isomorphism.

Given the group \((\hat{G}, +)\) of \( E \)-valued characters and a function \( f : G \to E \), we define the Fourier transform \( \hat{f} \) of \( f \) by

\[
\hat{f}(\chi) = \sum_{x \in G} f(x)\chi(-x),
\]

for all \( \chi \in \hat{G} \). The supports \( \text{supp}(f) \) and \( \text{supp}(\hat{f}) \) of \( f \) and \( \hat{f} \) are defined respectively by

\[
\text{supp}(f) = \{x \in G \mid f(x) \neq 0\}, \quad \text{supp}(\hat{f}) = \{\chi \in \hat{G} \mid \hat{f}(\chi) \neq 0\}.
\]

The group algebra \( F[G] \) consists of all formal sums

\[
f = \sum_{x \in G} f(x)x,
\]

with \( f(x) \in F \). In what follows, we will not distinguish between the element \( f \) in \( F[G] \) written as a vector

\[
f = (f(x_1), \ldots, f(x_n))
\]

in \( F^n \), where \( G = \{x_1, \ldots, x_n\} \), and the function \( f : G \to F \) given by

\[
f : x \mapsto f(x),
\]

for all \( x \in G \). Addition and scalar multiplication in \( F[G] \) are defined by the corresponding vector operations, and multiplication \( * \) in \( F[G] \) is the convolution operation defined by

\[
(f * g)(z) = \sum_{x \in G} f(x)g(z-x)
\]

for all \( z \in G \).
The characters \( \chi \in \hat{G} \), considered as elements in \( E[G] \), constitute a basis of \( E[G] \); the Fourier transform can then be understood in terms of a base change since
\[
f = \sum_{x \in G} f(x)x = |G|^{-1} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi
\]
holds for all \( f \in E[G] \). In fact, the characters constitute an orthogonal basis of eigenfunctions, with
\[
\chi \ast \chi' = \begin{cases} |G|\chi, & \text{if } \chi = \chi'; \\ 0, & \text{otherwise}, \end{cases}
\]
and
\[
f \ast \chi = \hat{f}(\chi) \chi
\]
for all \( \chi, \chi' \in \hat{G} \) and \( f \in E[G] \).

For our investigation of the case of equality in the generalized Donoho-Stark principle in Section III, we need some additional facts. Firstly, the Inverse Fourier Transform for functions \( g : \hat{G} \to E \) defined by
\[
g^*(x) = |G|^{-1} \sum_{\chi \in \hat{G}} g(\chi) \chi(x)
\]
for \( x \in G \) is the inverse of the Fourier transform, that is, \((\hat{f})^* = f \) for all functions \( f : G \to E \).

Next, let \( H \) and \( K \) be subgroups of \( G \) and \( \hat{G} \), respectively. We define \( H^\perp \) and \( K^\perp \) by
\[
H^\perp = \{ \chi \in \hat{G} \mid \chi(y) = 1 \text{ for all } y \in H \}
\]
and
\[
K^\perp = \{ x \in G \mid \eta(x) = 1 \text{ for all } \eta \in K \}.
\]
Then \( H^\perp \) and \( K^\perp \) are subgroups of \( \hat{G} \) and \( G \), respectively, with
\[
|H^\perp| = |G|/|H|, \quad |K^\perp| = |\hat{G}|/|K|.
\]
Moreover, if \( H = K^\perp \), then \( H^\perp = K \). Finally, we have that
\[
\sum_{y \in H} \chi(y) = \begin{cases} |H|, & \text{if } \chi \in H^\perp; \\ 0, & \text{otherwise}, \end{cases}
\]
and
\[
\sum_{\chi \in H^\perp} \chi(y) = \begin{cases} |H^\perp|, & \text{if } y \in H; \\ 0, & \text{otherwise}. \end{cases}
\]

Using the above facts it is not difficult to show the following.

**Theorem 2.1:** Let the function \( f : G \to E \) be such that its Fourier Transform \( \hat{f} \) has support equal to a coset \( \phi + K \) of a subgroup \( H \) of \( G \). Then \(|\text{supp}(f)||\text{supp}(\hat{f})| = |G|\) holds if and only if \( f \) and \( \hat{f} \) are of the form
\[
f = \lambda \phi I_{\phi + H}, \quad \hat{f} = \mu \Phi_{-a} I_{\phi + K},
\]
for some \( a \in G \), with \( \lambda = \phi(-a)f(a) \in E \setminus \{0\} \) and \( \mu = |K|f(a) \); here \( H = K^\perp, \Phi_{-a} : \chi \mapsto \chi(-a) \) is the character in \( \hat{G} \) associated with \(-a\), and \( I_{\phi + H} \) and \( I_{\phi + K} \) denote the indicator functions of \( \phi + H \) and \( \phi + K \), respectively.
Proof: By the inversion formula, we have \( f(x) = |G|^{-1} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(x) \). If \( \hat{f}(\chi) = 0 \) for all \( \chi \notin \phi + \mathcal{K} \), then for any \( x \in G \) and \( h \in H \), we have

\[
\begin{align*}
f(x + h) &= |G|^{-1} \sum_{\chi \in \phi + \mathcal{K}} \hat{f}(\chi) \chi(x + h) \\
&= |G|^{-1} \phi(h) \sum_{\chi \in \phi + \mathcal{K}} \hat{f}(\chi) \chi(x) \\
&= \phi(h) f(x),
\end{align*}
\]

where we used that \( H = \mathcal{K} \) and \( \chi(h) = \phi(h) \) for \( \chi \in \phi + \mathcal{K} \) and \( h \in \mathcal{K} \). Since \( \phi(x) \neq 0 \) for all \( x \in G \), we conclude from the above that the support of \( f \) is a union of cosets of \( H \). Since \( |H| = |\mathcal{K}^+| = |\hat{G}|/|\mathcal{K}| \), we have \( |\text{supp}(f)| \cdot |\text{supp}(\hat{f})| = |G| \) only if the support of \( f \) is \( a + H \), for some \( a \in G \), so that \( f \) is of the form \( f = \lambda \phi I_{a + \mathcal{K}} \) with \( \lambda = \phi(-a) f(a) \). Moreover, given that \( f \) is of this form, using the fact that \( H^+ = \mathcal{K} \), we obtain that \( f = \mu \Phi - a I_{a + \mathcal{K}} \) with \( \mu = |H| f(a) \), as claimed.

\[ \square \]

### III. Shifting for abelian codes

We now discuss a technique called shifting to find lower bounds on the minimum weight of abelian codes. This technique is a straightforward generalization of the shifting technique introduced in [22] to obtain lower bounds on the minimum distance of cyclic codes.

As before, \((G, +)\) is a finite abelian group, \( F \) is a field with \( \text{char}(F) = 0 \) or \( \text{char}(F) = p \) with \( p \nmid |G| \), and \( E \supseteq F \) is a field extension of \( F \) containing a primitive \( N \)-th root of unity, where \( N \) denotes the exponent \( \exp(G) \) of \( G \). Recall that the collection of \( E \)-valued characters on \( G \) forms a group \((G, +)\) isomorphic to \((G, +)\).

Now let \( f : G \to F \) be an \( F \)-valued function with \( f \) as its Fourier transform. We define

\[
Z(f) = \{ \chi \in \hat{G} \mid \hat{f}(\chi) = 0 \}, \quad \mathcal{N}(f) = \hat{G} \setminus Z.
\]

We call \( Z(f) \) and \( \mathcal{N}(f) \) the zeros and nonzeros of \( f \) (in \( \hat{G} \)), respectively. Note that \( \text{supp}(\hat{f}) = \mathcal{N}(f) \).

Let \( Z \subseteq \hat{G} \). The ideal \( C \) in \( F[\hat{G}] \) consisting of all \( f \in F[\hat{G}] \) whose zeros include \( Z \), i.e.,

\[
C = \{ f \in F[\hat{G}] \mid Z(f) \supseteq Z \},
\]

is called the abelian code with \( Z \) as defining zeros. Note that if \( G \) is cyclic of order \( n \), then \( \hat{G} \) essentially is the collection \( E_n \) of \( n \)-th roots of unity in \( E \) and \( C \) is just the cyclic code with defining zeros \( Z \subseteq E_n \).

Any field automorphism \( \sigma : E \to E \) of \( E \) that fixes \( F \) pointwise (that is, \( \sigma \in \text{Aut}(E/F) \)) induces a map on \( \hat{G} \) (which we denote again by \( \sigma \)) defined by \( \sigma : \chi \mapsto \chi^\sigma \), where \( \chi^\sigma(x) = \chi(x)^\sigma \) for \( x \in G \). A subset of \( \hat{G} \) that is closed under all field automorphisms in \( \text{Aut}(E/F) \) will be called \( F \)-closed. Note that the set of zeros \( Z(f) \) of an \( F \)-valued function \( f : G \to F \) is \( F \)-closed. Similarly, if the ideal \( C \) in \( F[\hat{G}] \) has the set \( Z \) as defining zeros, then the collection \( Z' \) of common zeros of elements of \( C \), called the complete set of zeros of \( C \), is just the \( F \)-closure of \( Z \), the smallest \( F \)-closed superset of \( Z \).

Now let \( f \) be a nonzero function in \( F[\hat{G}] \), and let \( Z = Z(f) \). Assume that \( f \) have support \( \text{supp}(f) = S \), where

\[
\text{supp}(f) = \{ x \in G \mid f(x) \neq 0 \}.
\]

Write \( S = \{ x_1, \ldots, x_w \} \), where \( w = w(f) = |\text{supp}(f)| \) is the weight of the vector \( f \). With each \( \chi \in \hat{G} \) we associate a vector \( v(\chi) \) in \( E^w \) defined by

\[
v(\chi) = (\chi(-x_1), \ldots, \chi(-x_w))^T;
\]

also, we define \( \gamma = \gamma(f) \) in \( E^w \) by

\[
\gamma = (f(x_1), \ldots, f(x_w))^T.
\]

As a consequence of these definitions, we have \( \chi \in Z \) if and only if \( \gamma \perp v(\chi) \). Finally, for \( \psi \in \hat{G} \), write \( D(\psi) \) to denote the diagonal matrix

\[
D(\psi) = \text{diag}(\psi(-x_1), \ldots, \psi(-x_w)).
\]
Note that
\[ D(\psi)v(\chi) = v(\psi + \chi), \]
where $\psi + \chi$ is the character in $\hat{G}$ defined by $(\psi + \chi)(x) = \psi(x)\chi(x)$ for all $x \in G$.

We say that the set $A \subseteq \hat{G}$ is independent if the corresponding set of vectors $V(A) = \{v(\chi) \mid \chi \in A\}$ is independent in $E^w$. Our interest in independent subsets of $\hat{G}$ stems from the fact that if $A \subseteq \hat{G}$ is independent, then $w(f) = w(\hat{f}) \geq |A|$. The next lemma, which is the key result for the shift bound, provides a means to construct independent sets in $\hat{G}$.

**Lemma 3.1:**
1) [initialize] $\emptyset$ is independent;
2) [shifting] If $A$ is independent and if $\psi \in \hat{G}$, then $\psi + A = \{\psi + \chi \mid \chi \in A\}$ is independent;
3) [extension] If $A \subseteq Z$ is independent and if $\eta \notin Z$, then $A \cup \{\eta\}$ is independent;
4) [field automorphisms] If $\sigma \in \text{Aut}(E/F)$, then $\sigma(A) = \{\sigma \chi \mid \chi \in A\} \subseteq Z$, and $\sigma(A)$ is independent.

**Proof:**
1. Evident.
2. Since $\psi(x) \neq 0$ for all $x \in G$, the diagonal matrix $D(\psi)$ is nonsingular. Since $D(\psi)V(A) = V(\psi + A)$, the result follows.
3. Since $A \subseteq Z$ and $\eta \notin Z$, we see that the vector $\gamma$ is orthogonal to all vectors in $V(A)$ and $\gamma = \gamma(f)$ is not orthogonal to $v(\eta)$; hence $v(\chi)$ cannot be contained in the linear span of $V(A)$.
4. Evident. \(\square\)

The rules 1–3 in Lemma 3.1 inductively define a family of independent subsets of $\hat{G}$ that only depend on the subset $Z = Z(f)$ of $\hat{G}$. We will call such sets independent with respect to $Z$ or, more briefly, $Z$-independent. Lemma 3.1 has the following immediate consequence.

**Theorem 3.2:** Let $f : G \to F$ be a nonzero function from an abelian group $(G, +)$ to some field $F$ of characteristic zero or of characteristic $p$ relatively prime to $|G|$, $Z = \{\chi \in G \mid f(\chi) = 0\}$ be the set of zeros of $f$, and let $w(f) = |\text{supp}(f)|$. Then
\[ w(f) \geq |A| \]
for every $Z$-independent subset $A$ of $\hat{G}$.

For a subset $Z$ of an abelian group $\hat{G}$, we denote by $\delta(\hat{G}, Z)$ the largest size of a $Z$-independent subset of $\hat{G}$. Then Theorem 3.2 has the following consequence.

**Theorem 3.3 (the shift bound for abelian codes):** Let $(G, +)$ be an abelian group and let $F$ be a field of char$(F) = 0$ or char$(F) = p$ not dividing $|G|$. If $C$ is an abelian code in $F[G]$ with set of defining zeros $Z$, then the minimum weight $d(C)$ of the code $C$ satisfies
\[ d(C) \geq \min \delta(\hat{G}, Z'), \]
where the minimum is over all $F$-closed proper subsets $Z'$ of $\hat{G}$ such that $Z' \supseteq Z$.

**IV. Some Examples of Shifting**

In this section we illustrate the shifting method by discussing a couple of applications. Readers who are mainly interested in the Donoho-Stark uncertainty principle can skip this section.

**Example 4.1:** [The BCH-bound for abelian codes] Let $C \subseteq F[G]$ be an abelian code with defining zeros $Z$ in $\hat{G}$, where $F$ is a field of characteristic $p$ not dividing $|G|$. If there is a character $\chi \in \hat{G}$ and integers $d, a$ such that $Z$ contains all zeros $\chi^i$ for $a \leq i \leq a + d - 2$ and if $D$ is the abelian code with defining zeros
\[ Z \cup \{\chi^i \mid i \geq 0\}, \]
then the minimum weight $d(C)$ of $C$ satisfies
\[ d(C) \geq \min(d, d(D)). \]
To see this, consider a word $c$ in $C$; then either $c$ is contained in $D$ and $w(c) \geq d(D)$, or there is a $e \geq \hat{d}$ such that all $\chi^i$ for $a \leq i \leq a + e - 2$ are zeros of $c$ but $\chi^{a+e-1}$ is a nonzero of $c$. In the latter case, we can use the shifting rules from Lemma 5.1 to construct independent sets as follows:

$$
\emptyset \mapsto \emptyset \cup \{\chi^{a+e-1}\} \mapsto \{\chi^{a+e-2}\} \cup \{\chi^{a+e-1}\} \\
\mapsto \ldots \mapsto \{\chi^a, \ldots, \chi^{a+e-2}\} \cup \{\chi^{a+e-1}\};
$$

hence the set $\{\chi^a, \ldots, \chi^{a+e-1}\}$ is independent, of size $e \geq d$, so that by Theorem 5.2 we have that $w(c) \geq d$.

An application of this bound can be found for example in [10].

**Example 4.2:** Let $F = F_2$, and consider the binary abelian code $C$ over $G = \mathbb{Z}_7 \times \mathbb{Z}_7$ with defining zeros

$$(0, 0), (0, 1), (0, 3), (1, 0), (3, 0), (1, 1), (1, 2), (1, 4), (3, 3), (3, 5), (3, 6)$$

in the dual group $\hat{G} = \mathbb{Z}_7 \times \mathbb{Z}_7$. This code is not equivalent to a cyclic code. Note that if $(x, y)$ is a zero, then $(2x, 2y)$ and $(4x, 4y)$ are also (conjugate) zeros. So the full set of zeros of codewords is

$$Z = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (4, 1), (4, 2), (4, 4), (3, 3), (3, 5), (3, 6), (5, 3), (5, 5), (5, 6), (6, 3), (6, 5), (6, 6)\}.$$

Note that this code is the collection of all polynomials

$$c(x, y) = \sum_{i=0}^{6} \sum_{j=0}^{6} c_{i,j} x^i y^j$$

for which $c(\alpha^i, \alpha^j) = 0$ for all pairs $(i, j)$ in the above list, where $\alpha$ is primitive in $E = F_8$. Possible zeros of codewords are $(1, 3), (1, 6), (1, 5)$, their conjugates, and their symmetric counterparts $(3, 1), (6, 1), (5, 1)$ and their conjugates.

This code has length $n = 49$, dimension $k = 18$, and minimum distance $d = 12$. Note that in this case a BCH bound can be at most seven since each non-identity element of $E^*$ has order seven. To prove that the minimum distance $d$ of the code satisfies $d \geq 12$ with shifting, first we assume that $(1, 3)$, and hence also $(2, 6)$ and $(4, 5)$, are nonzeros of a codeword $c$. Then shift as follows:

$$
\emptyset \mapsto \emptyset \cup \{(1, 3)\} \mapsto \{(0, 2)\} \cup \{(1, 3)\} \mapsto \{(0, 0), (1, 1)\} \cup \{(1, 3)\} \mapsto \\
{(6, 5), (0, 6), (0, 1)} \cup \{(2, 6)\} \mapsto \{(6, 0), (0, 1), (0, 3), (2, 1)\} \cup \{(4, 5)\} \mapsto \\
{(6, 3), (0, 4), (0, 6), (2, 4), (4, 1)} \cup \{(2, 6)\} \mapsto \\
{(4, 4), (5, 5), (5, 0), (0, 5), (2, 2), (0, 0)} \cup \{(2, 6)\} \mapsto \\
{(6, 6), (0, 0), (0, 2), (2, 0), (4, 4), (2, 2), (4, 1)} \cup \{(4, 5)\} \mapsto \\
{(4, 2), (5, 3), (5, 5), (0, 3), (2, 0), (0, 5), (2, 4), (2, 1)} \cup \{(2, 6)\} \mapsto \\
{(2, 4), (3, 5), (3, 0), (5, 5), (0, 2), (5, 0), (0, 6), (0, 3), (0, 1)} \cup \{(1, 3)\} \mapsto \\
{(1, 1), (2, 2), (2, 4), (4, 2), (6, 6), (4, 4), (6, 3), (6, 0), (6, 5), (0, 0)} \cup \{(1, 3)\} \mapsto \\
{(2, 2), (3, 3), (3, 5), (5, 3), (0, 0), (5, 5), (0, 4), (0, 1), (0, 6), (1, 1), (2, 4)} \cup \{(2, 6)\},
$$

thus proving that $w(c) \geq 12$ in this case.

So we may assume that $(1, 3), (2, 6), (4, 5)$ are also zeros of $c$. Now assume that $(1, 5)$, and hence also $(2, 3)$ and $(4, 6)$, are nonzeros of $c$. Then shifting proves that first the sets

$$\emptyset, \{(1, 5)\}, \{(1, 4), (1, 5)\}, \ldots, \{(1, 0), (1, 1), \ldots, (1, 5)\},$$

and then the sets

$$\{(0, 0), (0, 1), \ldots, (0, 5), (1, 5)\}, \ldots, \{(0, 2), (0, 3), \ldots, (0, 6), (0, 0), (1, 0), \ldots, (1, 5)\}$$

are all independent, thus again proving that $w(c) \geq 12$ in this case.
So we may assume that (1, 5), (2, 3) and (4, 6) are also zeros of \( c \). Now assume that (1, 6), and hence also (2, 5) and (4, 3), are nonzeros of \( c \). Then a similar shifting procedure proves first that the sets
\[
\emptyset, \{(1, 6)\}, \{(1, 5), (1, 6)\}, \ldots, \{(1, 0), (1, 1), \ldots, (1, 6)\},
\]
and then the sets
\[
\{(0, 0), (0, 1), \ldots, (0, 6), (1, 6)\}, \ldots, \{(0, 1), (0, 2), \ldots, (0, 6), (0, 0), (1, 0), \ldots, (1, 6)\}
\]
are all independent, thus proving that now \( w(c) \geq 14 \). Now use the fact that the original zero set is symmetric under \( (x, y) \mapsto (y, x) \) as follows: if one of (1, 3) or (3, 1) is a nonzero, then \( w(c) \geq 12 \); otherwise both (1, 3) and (3, 1) (and all their conjugates) are zeros. In that case, if one of (1, 5) or (5, 1) is a nonzero, then again \( w(c) \geq 12 \); otherwise also both (1, 5) and (5, 1) (and all their conjugates) are zeros. Finally, in that case, if one of (1, 6) or (6, 1) is a nonzero, then \( w(c) \geq 16 \); otherwise also both (1, 6) and (6, 1) (and all their conjugates) are zeros. But then all elements of \( \hat{G} \) are zeros, and the codeword is the all-zero word.

This code was investigated in \([6]\), where it was shown that the distance is 12 by other means.

V. An alternative derivation of the shift bound

In this section we relate the shift bound to a method from \([28]\). We begin by recalling some notions from that paper. Let \( G \) be a finite abelian group, \( f : G \to F \) be a nonzero \( F \)-valued function, and let \( S = \text{supp}(f) \) be the support of \( f \). As we did before, we identify \( f \) with the element \( f = \sum_{x \in G} f(x) x \) in the group algebra \( F[G] \). Define a linear map \( T_f : F[G] \to F[G] \) by \( T_f(u) = fu \) for all \( u \in F[G] \). Since \( T_f(\chi) = \hat{f}(\chi) \chi \) for every character \( \chi \in \hat{G} \), we see that the rank of \( T_f \) equals \( |\text{supp}(\hat{f})| \). Now suppose that \( x_1, \ldots, x_t \in G \) have the property that \( x_i + S \not\subseteq (x_1 + S) \cup (x_2 + S) \cup \cdots \cup (x_{t-1} + S) \) for \( i = 2, 3, \ldots, t \), where \( x + S = \{x + s \mid s \in S\} \).

Then from this property of the support \( S \) and the fact that \( T_f(x_i) = \sum_{x \in x_i + S} f(z - x_i)z \), it follows that no linear combinations \( \lambda_1 T_f(x_1) + \cdots + \lambda_t T_f(x_t), \) with \( \lambda_1 \in F \) for all \( i \), can be 0 unless \( \lambda_1 = \cdots = \lambda_t = 0 \); that is, \( T_f(x_1), \ldots, T_f(x_t) \) are independent in \( F[G] \), and hence \( \text{rank}(T_f) = |\text{supp}(\hat{f})| \geq t \). We can formalize the above as follows.

**Definition 5.1:** Let \((G, +)\) be an abelian group, and let \( S \subseteq G \) be a nonempty subset of \( G \). We say that the sequence \( x_1, \ldots, x_t \in G \) has \( S \)-rank \( t \) in \( G \) if \( x_1 + S \not\subseteq (x_1 + S) \cup (x_2 + S) \cup \cdots \cup (x_{t-1} + S) \) for \( i = 2, 3, \ldots, t \). Then the discussion preceding the above definition can be stated as follows.

**Proposition 5.2:** Let \( f : G \to F \) be nonzero and let \( S = \text{supp}(f) \). If there exists a sequence in \( G \) with \( S \)-rank \( t \), then \( |\text{supp}(\hat{f})| \geq t \).

By dualizing, we obtain the following.

**Corollary 5.3:** Let \( G \) be an abelian group, \( F \) be a field of characteristic 0 or with \( \text{char}(F) \) not dividing \( |G| \), \( f : G \to F \) be a nonzero function, and let \( N = \text{supp}(f) \). If there exists a sequence in \( G \) with \( N \)-rank \( t \), then \( |\text{supp}(\hat{f})| \geq t \).

**Proof:** Immediate consequence of the fact that \( \hat{G} = G \) and \( (\hat{f})^* = f \).

We now show that the lower bound on the minimum distance of an abelian code afforded by Corollary 5.3 is equivalent to the shift bound, an observation that seems to be new. We need some preparation.

**Definition 5.4:** Let \((G, +)\) be an abelian group and let \( Z \subseteq G \). We say that the sequence \( \alpha_1, \ldots, \alpha_t \) in \( G \) is \( Z \)-independent in \( G \) if there are \( \psi_1, \psi_2, \ldots, \psi_t \in \hat{G} \) such that \( \psi_1 + \alpha_1 \notin Z \) and for \( 2 \leq i \leq t \), we have
\[
\psi_i + \alpha_i + \alpha_1 \notin Z \text{ and } \psi_i + \alpha_i + \alpha_2, \ldots, \psi_i + \alpha_i + \alpha_{i-1} \in Z.
\]

**Proposition 5.5:** Let \( Z \subseteq G \) be a proper subset of \( G \). A set \( A \subseteq G \) is \( Z \)-independent if and only if, for some ordering \( A = \{\alpha_1, \ldots, \alpha_t\} \) of the elements of \( A \), the sequence \( \alpha_1, \ldots, \alpha_t \) is \( Z \)-independent in \( G \).

**Proof:** Since \( N \neq \emptyset \), the statement holds for \( t = 1 \). Since the only way to enlarge independent sets is through rule 2, we see that a set \( A \) of size \( t \geq 2 \) is \( Z \)-independent if and only if it can be written as \( A = A' \cup \{\alpha_t\} \) with \( A' \) being \( Z \)-independent and with \( \psi_t + A' \subseteq Z \) and \( \psi_t + \alpha_t \in N \) for some \( \psi_t \in \hat{G} \). Now the statement follows by induction on \( t \).

**Proposition 5.6:** Let \( Z \subseteq \hat{G} \), and let \( N = \hat{G} \setminus Z \). The sequence \( \alpha_1, \ldots, \alpha_t \) has \( N \)-rank \( t \) in \( \hat{G} \) if and only if the sequence \( -\alpha_1, \ldots, -\alpha_t \) is \( Z \)-independent.
Proof: Write \( Z = \hat{G} \setminus \mathcal{N} \). By definition, a sequence \( \alpha_1, \ldots, \alpha_t \) has \( \mathcal{N} \)-rank \( t \) in \( \hat{G} \) precisely when \( \alpha_i + \mathcal{N} \leq \mathcal{N} \cup \cdots \cup \mathcal{N} \) for \( i = 2, \ldots, t \), or, equivalently, if there exist \( \psi_2, \ldots, \psi_t \) such that \( \psi_i \in \alpha_i + \mathcal{N} \) and \( \psi_i \in (\alpha_1 + \mathcal{N} \cup \cdots \cup \alpha_{i-1} + \mathcal{N})^c \) for \( i = 2, \ldots, t \). Since \( \mathcal{N} \neq \emptyset \) by assumption, the claim now follows from Proposition 5.5.

As a consequence of Proposition 5.5 and 5.6, we have the following alternative description of the shift bound.

**Theorem 5.7:** Let \( (G, +) \) be abelian and let \( Z \subset \hat{G} \) be a proper subset of \( \hat{G} \). Put \( \mathcal{N} = \hat{G} \setminus Z \). Then \( \delta(\hat{G}, Z) \) is equal to the largest integer \( t \) for which there exists a sequence of \( \mathcal{N} \)-rank \( t \) in \( \hat{G} \).

In view of this theorem, Corollary 5.3 provides an alternative derivation of the shift bound for abelian codes in Theorem 5.3.

VI. A GENERALIZATION OF THE DONOHO-STARK UNCERTAINTY PRINCIPLE

The Donoho-Stark uncertainty principle for finite abelian groups states: If \( f : G \to \mathbb{C} \) is a nonzero complex function on a finite abelian group \( (G, +) \) with \( \hat{f} \) its complex Fourier transform, then

\[
|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G|
\]

equality holds if and only if \( f \) takes the form

\[
f = c\chi I_{a+H}
\]

for some \( c \in \mathbb{C}^* \) and some complex character \( \chi \), where \( I_{a+H} \) denotes the indicator function of some coset \( a + H \) of a subgroup \( H \) of \( G \). This uncertainty principle has an interesting history. The principle, in a more general form for locally compact abelian (LCA) groups, seems to have been discovered first by Matolcsi and Szűcs [23] in 1973; the case of equality was handled by K.T. Smith [38] in 1990. In the case where the group is cyclic an elementary proof was given in 1989 by Donoho and Stark [9], essentially using the BCH bound. They also treat the more general case where \( f \) is highly concentrated on a subset of the group. Similar investigations can already be found in the work of Slepian in [37]. For further work on uncertainty relations, see for example [43], [44], [13].

A still somewhat complicated elementary induction proof for finite abelian groups was given in [26], see also [31], [30]. Their proof uses the Donoho-Stark principle for cyclic groups that is proved in [9], essentially by using the BCH bound. More recently, the principle has been recovered in [22], where a simpler elementary proof was given. Basically the same proof has been given by Tao in [41]. In that paper, an interesting sharpening of this inequality has been obtained: if \( G \) is cyclic of prime order \( p \), and \( f : G \to \mathbb{C} \) is a nonzero function, then

\[
|\text{supp}(f)| + |\text{supp}(\hat{f})| \geq p + 1.
\]

The proof depends on an old result of Chebotarëv (see, e.g. [49]) that is not always valid for finite fields, see [19], [20]. This work has been generalized to all abelian groups by Meshulam in [27]. Tao later observed that the generalization signifies that for an abelian group \( (G, +) \), the points \( (\text{supp}(f), \text{supp}(\hat{f})) \) are in the convex hull of the points \( (|H|, |G/H|) \) for subgroups \( H \) of \( G \), see for example [19], [20].

Some different type of generalizations are discussed in the next section.

Here it is our aim to present several generalizations of the Donoho-Stark uncertainty principle for nonzero functions \( f : G \to \mathbb{F} \), for any field \( \mathbb{F} \) of characteristic zero or characteristic \( p \) not dividing \( |G| \). As an aid, we show that for \( Z \subset \hat{G} \) the shift bound \( \delta(\hat{G}, Z) \) satisfies

\[
\delta(\hat{G}, Z) \geq |\hat{G}|/(|\hat{G}| - |Z|),
\]

and we determine when equality holds. We start with the following.

**Theorem 6.1:** Let \( G \) be a finite abelian group with \( \hat{G} \) its group of E-characters, let \( \mathcal{K} \) be a subgroup of \( \hat{G} \), and let \( Z \) be a proper subset of \( \mathcal{K} \). Write \( \mathcal{N} = \mathcal{K} \setminus Z \). Then the following hold.

(i) There exists a \( \mathcal{Z} \)-independent subset of \( \mathcal{K} \) of size at least \( |\mathcal{K}|/|\mathcal{N}| \), that is, \( \delta(\mathcal{K}, Z) \geq |\mathcal{K}|/(|\mathcal{K}| - |Z|) \).

(ii) If \( |\mathcal{N}| \) divides \( |\mathcal{K}| \) and the maximum size of a \( \mathcal{Z} \)-independent subset of \( \mathcal{K} \) is \( |\mathcal{K}|/|\mathcal{N}| \), then \( \mathcal{N} \) is a coset of a subgroup of \( \mathcal{K} \).
Proof: (i) We claim that if $\mathcal{A} \subseteq \mathcal{K}$ and $|\mathcal{A}| < |\mathcal{K}|/|\mathcal{N}|$, then there exists some $\psi \in \mathcal{K}$ such that $\mathcal{A} - \psi = \{\alpha - \psi \mid \alpha \in \mathcal{A}\} \subseteq \mathcal{Z}$. Indeed, for all $\eta \in \mathcal{N}$ we have $\eta \in \mathcal{A} - \psi$ if and only if $\psi \notin \mathcal{A} - \eta$; hence $\mathcal{A} - \psi \subseteq \mathcal{Z}$ if and only if $\psi \notin \bigcup_{\eta \in \mathcal{N}}(\mathcal{A} - \eta)$. Now if $|\mathcal{A}| < |\mathcal{K}|/|\mathcal{N}|$, then

$$| \bigcup_{\eta \in \mathcal{N}} (\mathcal{A} - \eta) | \leq \sum_{\eta \in \mathcal{N}} |\mathcal{A} - \eta| \leq |\mathcal{N}||\mathcal{A}| < |\mathcal{K}|;$$

hence there exists a $\psi \in \mathcal{K}$ such that $\psi \notin \bigcup_{\eta \in \mathcal{N}}(\mathcal{A} - \eta)$, and the claim follows. Since $\mathcal{N}$ is nonempty by assumption, part (i) of the lemma follows by induction from parts 2 and 3 of Lemma [3.1].

(ii) Now suppose that $|\mathcal{N}|$ divides $|\mathcal{K}|$, that $\mathcal{A} \subseteq \mathcal{K}$ is $\mathcal{Z}$-independent with $|\mathcal{A}| = |\mathcal{K}|/|\mathcal{N}|$, and that no $\mathcal{Z}$-independent set in $\mathcal{K}$ is larger than $|\mathcal{K}|/|\mathcal{N}|$. Since $\mathcal{A}$ is $\mathcal{Z}$-independent, there exists a $\mathcal{Z}$-independent set $B \subseteq \mathcal{Z}$ and some $\eta \in \mathcal{N}$ such that $\mathcal{A} = (B \cup \{\eta\}) - \psi$ for some $\psi \in \mathcal{K}$. Consider the $\mathcal{Z}$-independent sets in $\mathcal{K}$ of the form $A_\eta = B \cup \{\eta\}$ for $\eta \in \mathcal{N}$. Each of these sets has the maximum size $|\mathcal{K}|/|\mathcal{N}|$, so by our assumptions none of these sets can be shifted inside $\mathcal{Z}$. Hence from the analysis in part (i), we see that for each $\eta \in \mathcal{N}$ we have that

$$\mathcal{K} = \bigcup_{\eta \in \mathcal{N}}(B \cup \{\eta\}) - \eta'.$$

Hence if $\mathcal{L} = \mathcal{K} \setminus (\bigcup_{\eta' \in \mathcal{N}}B - \eta')$, then $|\mathcal{L}| = |\mathcal{N}|$ and $\eta - \mathcal{N} = \mathcal{L}$ for all $\eta \in \mathcal{N}$. In particular, $\mathcal{N} - \mathcal{N} = \mathcal{L}$. So we immediately have that $\mathcal{L} = (\eta - \mathcal{N}) - (\eta - \mathcal{N}) = \mathcal{N} - \mathcal{N} = \mathcal{L}$, that is, $\mathcal{L}$ is a subgroup of $\mathcal{K}$, and $\mathcal{N} = \eta - \mathcal{L}$ is a coset of $\mathcal{L}$. \qed

As an immediate consequence we obtain the following generalization of the Donoho-Stark uncertainty principle.

Theorem 6.2 (Generalized Donoho-Stark): Let $(G, +)$ be a finite abelian group and let $F$ be a field of characteristic zero or characteristic $p$ not dividing $|G|$. Let $E$ denote the extension of $F$ containing a primitive $N$-th root of unity, where $N = \exp(G)$, and let $\hat{G}$ denote the group of $E$-valued characters of $G$. Then for any subgroup $\mathcal{K}$ of $\hat{G}$ and for any function $f : G \to F$ with Fourier Transform $\hat{f}$ that is nonzero on $\mathcal{K}$ we have that

$$|\text{supp}(f)||\text{supp}(\hat{f}) \cap \mathcal{K}| \geq |\mathcal{K}|.$$

Equality holds only if $\text{supp}(\hat{f}) \cap \mathcal{K}$ is a coset of a subgroup of $\mathcal{K}$. In particular, if $f : G \to F$ is nonzero, then

$$|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G|,$$

with equality if and only if $f$ is of the form

$$f = \lambda \chi I_{a+H},$$

for some $\lambda \in E \setminus \{0\}$, some $\chi \in \hat{G}$, some $a \in G$, and some subgroup $H$ of $G$; here $I_{a+H}$ denotes the indicator function of the coset $a + H$.

Proof: If $\mathcal{N} = \text{supp}(\hat{f}) \cap \mathcal{K}$ is nonempty and if $\mathcal{Z} = \mathcal{Z}(f) \cap \mathcal{K}$ is the collection of zeros of $f$ on $\mathcal{K}$, then by part (i) of Theorem [6.1] there is a $\mathcal{Z}$-independent set in $\mathcal{K}$ of size at least $|\mathcal{K}|/|\mathcal{N}|$; hence from Theorem [6.2] we conclude that $|\text{supp}(f)| \geq |\mathcal{K}|/|\mathcal{N}|$.

Furthermore, part (ii) of Theorem [6.1] shows that if this bound cannot be improved, then the support $\mathcal{N}$ of $\hat{f}$ on $\mathcal{K}$ is a coset of a subgroup of $\mathcal{K}$. Now the last part of Theorem [6.2] follows by applying Theorem [2.1] with $G = \mathcal{K}$. \qed
VII. A sharpening of the Donoho-Stark uncertainty principle

Our aim in this section is to obtain a sharpening of the Donoho-Stark uncertainty principle by the shifting technique. We need some preparations. Let $G$ be a finite abelian group, and let $S$ be a subset of $G$. We define the stabilizer $H(S)$ of $S$ by $H(S) = \{ g \in G \mid g + S = S \}$. A few simple properties of the stabilizers are given below.

**Lemma 7.1:** With the above notation, we have the following:

(i) $S$ is a union of cosets of $H(S)$. So $|H(S)|$ divides $|S|$, and in particular, $|H(S)| \leq |S|$.
(ii) If $S$ is a coset of a subgroup $K$ of $G$, then $H(S) = K$.
(iii) For $S \subseteq G$, we have $H(S) = H(G \setminus S)$.
(iv) For $S \subseteq G$, we have $H(S) = \cap_{s \in S}(s - S)$.
(v) Let $A$ and $S$ be subsets of $G$ such that $A \cap S = \emptyset$. Then $(A - S) \cap H(S) = \emptyset$.

**Proof:** (i) Write $H$ for $H(S)$. Since $s + H \subseteq S$ for every $s \in S$, the set $S$ is a union of cosets of $H$. So $|H(S)|$ divides $|S|$, and equality holds if and only if $S$ is a coset of $H$.

(ii) Let $S = a + K$ for a subgroup $K$ of $G$. Clearly we have $K \subseteq H(S)$. On the other hand, since $H(S) \subseteq S - S$, we see that $H(S) \subseteq K$. Hence $H(S) = K$.

(iii) Evident.

(iv) Note that $g \in s - S$ for all $s \in S$ precisely when $s - g \in S$ for all $s$, that is, when $S = g - S$, i.e., when $g \in H(S)$.

(v) Let $a - s \in H(S)$ with $a \in A$ and $s \in S$; then $a + s \in H(S)$. \hfill $\square$

We are now ready to prove the following improvement of the Donoho-Stark uncertainty principle.

**Theorem 7.2 (Sharpened Donoho-Stark):** Let $(G, +)$ be a finite abelian group and let $F$ be a field of characteristic zero or characteristic $p$ not dividing $|G|$. Let $E$ denote an extension of $F$ containing a primitive $N$-th root of unity, where $N = \exp(G)$, and let $\hat{G}$ denote the group of $E$-valued characters of $G$. Then for any nonzero function $f : G \rightarrow F$ with Fourier transform $\hat{f}$, we have that

$$|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G| + |\text{supp}(\hat{f})| - |H(\text{supp}(\hat{f}))|;$$

and dually,

$$|\text{supp}(f)||\text{supp}(\hat{f})| \geq |G| + |\text{supp}(f)| - |H(\text{supp}(f))|.$$

**Proof:** Let $\mathcal{Z} \subseteq \hat{G}$ and write $\mathcal{N} = \hat{G} \setminus \mathcal{Z}$. Assume that $\mathcal{N} \neq \emptyset$. Suppose that $A$ is a $\mathcal{Z}$-independent set of maximum size. Then $A$ is of the form $A = B \cup \{ \eta_0 \} - \chi_0$ with $B \subseteq \mathcal{Z}$ being $\mathcal{Z}$-independent and $\eta_0 \in \mathcal{N}$, and by the assumption that $A$ has maximum size, no $\mathcal{Z}$-independent set $B \cup \{ \eta \}$ (with $\eta \in \mathcal{N}$) can be extended any further; that is, for each $\chi \in \hat{G}$ and each $\eta \in \mathcal{N}$, we have $B \cup \{ \eta \} - \chi \not\subseteq \mathcal{Z}$. So for every $\eta \in \mathcal{N}$ and $\chi \in G$, there exists $\eta' \in \mathcal{N}$ such that $\eta' \in B \cup \{ \eta \} - \chi$; that is, $\chi \in B \cup \{ \eta \} - \mathcal{N}$. Hence if $\chi \not\in B - \mathcal{N}$, then $\chi \in \eta - \mathcal{N}$; since this holds for every $\eta \in \mathcal{N}$, we have $\chi \in \cap_{\eta \in \mathcal{N}}(\eta - \mathcal{N}) = H(\mathcal{N})$, where the equality follows from Lemma 7.1 part (iv). Since $B \subseteq \mathcal{Z} = \hat{G} \setminus \mathcal{N}$ by assumption, we have $B \cap \mathcal{N} = \emptyset$; hence according to Lemma 7.1 part (v), we have $(B - \mathcal{N}) \cap H(\mathcal{N}) = \emptyset$, and it now follows that

$$B - \mathcal{N} = \hat{G} \setminus H(\mathcal{N}).$$

Using $|B - \mathcal{N}| \leq |B||\mathcal{N}|$ and $|G| = |\hat{G}|$, we conclude that

$$|G| - |H(\mathcal{N})| = |B - \mathcal{N}| \leq |B||\mathcal{N}|.$$

Now suppose that $f : G \rightarrow F$ is nonzero, and let $\mathcal{N} = \text{supp}(\hat{f})$ and $\mathcal{Z} = \mathcal{Z}(f)$. Suppose that $A$ is a $\mathcal{Z}$-independent set of maximum size. By the shifting bound, we have $|\text{supp}(f)| \geq |A|$. As we have shown above, we can write $A = B \cup \{ \eta_0 \} - \chi_0$ with $B \subseteq \mathcal{Z}$ being $\mathcal{Z}$-independent and $\eta_0 \in \mathcal{N}$, and $B - \mathcal{N} = \hat{G} \setminus H(\mathcal{N})$. It follows that

$$|\text{supp}(f)||\text{supp}(\hat{f})| \geq (|B| + 1)|\mathcal{N}| \geq |G| - |H(\mathcal{N})| + |\mathcal{N}| = |G| - |H(\text{supp}(\hat{f}))| + |\text{supp}(\hat{f})|.$$

Since $(\hat{f})^* = f$ and $|G| = |\hat{G}|$, the second inequality in the theorem follows by dualizing (i.e., replacing $G$ by $\hat{G}$ and interchanging $f$ and $\hat{f}$). \hfill $\square$
Remark 7.3: Note that by Lemma 7.1 part (i), we have $|H(\text{supp}(\hat{f}))| \leq |\text{supp}(\hat{f})|$. It follows that the right hand side of (3) is greater than or equal to $|G|$. So (3) is a sharpening of the Donoho-Stark uncertainty principle. When $N = \text{supp}(\hat{f})$ is a coset of $H(N)$, then $H(N) = N$ and the inequality (3) reduces to the Donoho-Stark inequality. We also remark that the lower bound on $|\text{supp}(f)|$ arising from (3) improves the bound obtained from the Donoho-Stark inequality provided that $|G|$ (mod $|\text{supp}(\hat{f})|$), the least non-negative remainder, is greater than $|H(\text{supp}(\hat{f}))|$. For an example of this situation, see Example 7.4.

Example 7.4: Let $n = 2^d - 1$, $G = \mathbb{Z}_n$, and $G' = \{1, \alpha, \ldots, \alpha^{n-1}\}$, where $\alpha$ is a primitive element of $\mathbb{F}_{2^d}$. Let $\mathcal{Z} = \{\alpha, \alpha^2, \alpha^{2^2}, \ldots, \alpha^{2^{d-1}}\} \subseteq G$ and $N = G' \setminus \mathcal{Z}$. Then $H(N) = H(\mathcal{Z}) = \{1\}$. Let $f : \mathbb{Z}_n \to \mathbb{F}_2$ be a nonzero function with $f(\alpha) = 0$. Note that the condition on $f$ implies that the associated codeword $(f(0), f(1), \ldots, f_{n-1})$ is contained in the $[n = 2^d - 1, k = 2^d - 1 - d, 3]$ binary Hamming code. Now if $f \neq 0$ and $f$ has no additional zeros, then for $d \geq 3$ the Donoho-Stark bound for $f$ gives

$$w(f) \geq \left\lfloor \frac{2^{d-1}}{2^d - 1 - d} \right\rfloor = 1 + \left\lfloor \frac{d}{2^d - 1 - d} \right\rfloor = 2,$$

while our improved bound gives

$$w(f) \geq \left\lfloor \frac{2^{d-1} + 2^{d-1} - 1 - 1}{2^d - 1 - d} \right\rfloor = 2 + \left\lfloor \frac{d - 1}{2^d - 1 - d} \right\rfloor = 3,$$

showing that our new bound can improve the weight estimate afforded by the Donoho-Stark bound.

In the remainder of this section, we investigate the case of equality in (3). Note that if we have equality in the Donoho-Stark inequality, then $\text{supp}(f)$ is a coset of a subgroup $K$ of $G$, hence $|H(\text{supp}(\hat{f}))| = |K| = |\text{supp}(f)|$ and we also have equality in the sharpened versions (3) and (4) of that inequality. We will call this the classical case of equality. Note that when one of $|H(\text{supp}(\hat{f}))| = |\text{supp}(f)|$ or $|H(\text{supp}(\hat{f}))| = |\text{supp}(f)|$ holds, they both hold and then (and only then) we are in the classical case. We next describe a simple non-classical example.

Example 7.5: Let $(G, +)$ be a finite abelian group with identity 0, and let $F$ be a field of characteristic zero or characteristic $p$ not dividing $|G|$. Let $a \in G \setminus \{0\}$, and define $f : G \to F$ by letting $f(0) = 1$, $f(a) = -1$, and $f(g) = 0$ for $g \neq 0, a$. We claim that for this $f$ equality in (3) holds. Obviously, $\text{supp}(f) = \{0, a\}$, for $\chi \in G$, we have that $\hat{f}(\chi) = \sum_{x \in G} f(x)\chi(-x) = 1 - \chi(a)$; hence $\chi \in \text{supp}(f)$ iff $\chi(a) \neq 1$, i.e., iff $\chi \notin \langle a \rangle$; so $\text{supp}(\hat{f}) = G \setminus \langle a \rangle$. Finally, $H(\text{supp}(\hat{f})) = H(G \setminus \langle a \rangle) = H(\langle a \rangle^\perp) = \langle a \rangle^\perp$ by Lemma 7.1.

In the previous example, we have $f(a) = -1$, so we indeed have equality in (3). This example is a classical one if and only if $|G| = 2|\langle a \rangle|$. Since $a \neq 0$, we have $\langle a \rangle^\perp \neq G$; so using (1), we see that this example is a classical one if and only if $|\langle a \rangle| = 2$, that is, if and only if $\langle a \rangle = \{0, a\}$ and $2a = 0$.

Let us now investigate when we also have equality in (4). Using (1), we have that $|\text{supp}(f)||\text{supp}(\hat{f})| = 2(|G| - |G|/|\langle a \rangle|)$, and $|G| + |\text{supp}(f)| - |H(\text{supp}(\hat{f}))| = |G| + 2 - |H(\langle a \rangle)|$. We now distinguish two cases. First, if $H(\{0, a\}) = \{0, a\}$, that is, if $2a = 0$, then $\{0, a\} = \{a\}$ and we always have equality in (4); this is a classical example. Second, if $H(\{0, a\}) = \{0\}$, that is, if $2a \neq 0$, then $|\langle a \rangle| > 2$ and we have equality precisely when $G = \langle a \rangle$ with $|G| = 3$; we can take $G = \mathbb{Z}_3$ and $a = 1$. This is again a non-classical example.

In order to have equality in (4), we need sets $B, N \subseteq G$ with $B \cap N = \emptyset$ satisfying

$$G - H(N) = B - N, \quad |B - N| = |B||N|,$$

where $N$ is a union of cosets of $H(N)$. (Note that here, for convenience, we have dualized.) We ask whether from such sets, we can construct a function $f$ for which $\text{supp}(f) = N$ and $|\text{supp}(f)| = |B| + 1$? Such sets give rise to what is called a near-factorization [40, Section 9.3], [5], [29], [36], [1]. Indeed, put $G_0 = G/H(N)$; since $N$ is a union of cosets of $H(N)$, we have that $N = H(N) - D$ for some set $D$ of size $|N/H(N)|$. Then $G_0 \setminus \{0\} = B_0 + D$ (direct sum), where $B_0 = \{b + H(N) \mid b \in B\}$. It is conjectured that near-factorizations of abelian groups exist only for cyclic groups [40].

Example 7.6: Let $G = \mathbb{Z}_n$ with $n - 1 = uv$. Take $N = \{0, 1, \ldots, u - 1\}$ and $A = \{u, 2u, \ldots, (v - 1)u\}$. Then $A - N = G \setminus \{0\}$, so $(N, -A)$ is a near-factorization of $G$.

For other examples of near-factorizations and further discussions, we refer to the references given above.
While the equality case in the Donoho-Stark uncertainty principle can be characterized completely, it seems not easy to characterize the equality case in these new inequalities. We leave this as a problem for further research.

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