A Principled Approximation for Optimal Control of Semi-Markov Jump Linear Systems using Pseudo-Markovianization

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Abstract

High-performance control of semi-Markov jump linear systems requires an accurate model for the underlying stochastic jump process. In practice, however, a reasonable compromise between the control quality and computational costs should be made and the tractability of the control design problem has to be established. This paper considers the problem of finite-horizon optimal quadratic control of semi-Markov jump linear systems and investigates how the modeling quality of the underlying jump process may affect the control performance. The problem is first examined using the phase-type distribution approach by approximating an arbitrary semi-Markov jump linear system with fully observable jumps by a Markov jump linear system with partially observable jumps. Then, through a process called pseudo-Markovianization, a technique for low-order approximation of the jump process is proposed that models a semi-Markovian process by a Markov-like model with possibly negative transition rates. It is shown that in modeling of the holding-time distributions the probabilistic interpretation of the model does not need to be preserved. The flexibility provided by the technique enables us to obtain a more accurate, yet low-order approximate model for the jump process for control design. Several examples are given throughout the paper to demonstrate the strengths and effectiveness of the technique.

I. INTRODUCTION

In many engineering applications, complex dynamical systems may experience random abrupt variations in their parameters and structure that change the system’s dynamic and operating condition. Examples include power systems with randomly varying loads \cite{1}, systems subject to

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unpredictable variations due to environmental factors, networked control systems with sudden changes due to random variations in the network topology [2], and avionic systems in the presence of electromagnetic disturbances from both natural and man-made sources [3]. See [4, §1.3] and [5, §1.2] and references therein for other applications. Figure 1 shows a controlled dynamical system whose characteristics and operating conditions depend on variables governed by random processes. Although the states of these stochastic processes may be observed using sensing devices, they are unknown beforehand; and hence an appropriate stochastic model is needed for control design.

Fig. 1. Illustration of a controlled dynamical system whose parameters, structure, and operating conditions may abruptly change according to the behavior of some stochastic processes. The control performance can be improved by an appropriate modeling of these random processes using historical data.

Due to the tractability of linear models for control and optimization purposes, systems subject to random changes are often modeled by *stochastic jump linear systems*. This class of stochastic switching systems consist of a finite number of linear models where switching among them is governed by a random process. Each linear model describes the behavior of the dynamical system at a particular mode of operation and is obtained using classical system-identification techniques and by linearization around each operating point. The jump times are random and unknown beforehand, and hence the time spent in each mode before jump to another mode is a random variable. Modeling of the jump process is carried out by fitting a suitable probability model to historical data about sequence of jump times and the waiting times in each mode. Then, a linear deterministic dynamical model for each mode of operation is known and a probability model for the jump process is available. Obviously, a real random process can never be perfectly modeled and there always exist inevitable modeling errors between a theoretical probability model and the actual random process. It is therefore desired to investigate how the modeling error of a
jump process may affect the performance of a control scheme and how to improve the modeling quality and achieve a better performance.

The homogeneous Markov chain due to its mathematical tractability is the most commonly used stochastic model for control design and reliability analysis. However, the memoryless property of homogeneous Markov models forces the holding time in each state to be exponentially distributed with a constant hazard rate, while many features of real systems are not memoryless. Hence, such models may produce unrealistic and too conservative results in many practical applications. Therefore, a more realistic stochastic model is required to incorporate general probability distributions in the model of the jump process. The semi-Markov process is a generalization of the Markov chain where the distribution of the time the process spends in any state before jumping to another state is allowed to be non-exponential (with a non-constant hazard rate). In many applications, the semi-Markov process is a natural stochastic model to describe random processes with a discrete state space. The semi-Markov process is a suitable model to describe the operating characteristics of power plants and to assess the reliability of power systems [6]. For optimization and reliability analysis of wind turbines, the wind speed process is modeled by a semi-Markov process, as it more accurately reproduces the statistical properties of wind speed data compared to a Markov process [7], [8].

The mathematical analysis of controlled dynamical systems consisting of a non-Markovian process is often very difficult. In order to arrive at a tractable method for analysis and design, one approach is to transform the stochastic process into a finite-state homogeneous Markov model by including sufficient supplementary state variables to model some part of the process history [9, §2.3.7]. A commonly-used approach to model non-exponential distributions, yet retain the Markovian structure of the process is approximation by a class of distributions called phase-type distribution (or PH distribution, for short) [9]. The PH distribution is a generalization of the exponential distribution with a partial memoryless property. It is defined in terms of an underlying Markov chain with a finite number of discrete phases (or stages) as the distribution of the time to enter an absorbing state from a set of transient states in a finite-state Markov chain. The PH distributions are dense in the set of all probability distributions on $[0, \infty)$, that is they can be used to approximate any given non-negative distribution, with nonzero density in $(0, \infty)$, to any desired accuracy [10]. Moreover, the matrix representation of PH distributions makes them suitable for theoretical analysis.
The PH distribution approach enables us to include more information about the characteristics of the process in the model, yet it preserves the analytical tractability of the exponential distribution, then one can employ powerful tools and techniques developed for Markovian models to analyze non-Markovian processes. The PH distribution has various applications in reliability and queueing theory. It has been also used in [12], [13] for stochastic stability analysis of a class of jump linear systems with a finite-phase semi-Markov jump process.

The steady-state properties and the finite-horizon control of Markov jump linear systems (MJLSs) have been extensively studied in the literature during the past decades under different assumptions of full-state feedback, output-feedback, completely observable modes, clustered observations, and no observable modes, and several control design issues and challenges have been discussed [5], [14], [15]. The stability and stabilization of semi-Markov jump linear systems (S-MJLSs) have been the subject of much research during the past years. More recently in [16], [17], the problem of stability of a class of S-MJLSs is studied and numerically testable criteria for stability and stabilization are provided. However, finite-horizon optimal control problem for general S-MJLSs has not been adequately addressed in the control literature. In many practical applications, the system is utilized over a bounded period of time and the performance of the system is concerned during its finite utilization period. It is therefore desired to control and analyze the transient behavior of the system and optimize the system’s operation process over a finite horizon. The optimal control problem for stochastic jump linear systems deals with optimization of a performance index which is usually the expected value of an evaluation function, that is the average behavior of the system is to be controlled by applying an appropriate control action to the system.

In this paper, we focus on finite-horizon optimal control of S-MJLSs and investigate how the quality of the jump process model may affect the control performance. We first show that how the PH distribution approach can be employed to approximate an arbitrary S-MJLS with fully observable jumps by a MJLS with partially observable jumps. A design procedure is provided to find the optimal control gains and to evaluate the control cost. Then, a model reduction technique for the jump process is proposed to provide a compromise between computational burden and the control quality.

The rest of the paper is organized as follows. Section II gives some preliminary definitions, notations, and a series of lemmas used throughout the paper. The problem is defined and
formulated in Section III. The Markovianization process using PH distributions is discussed in Section IV and subsequently the optimal control of S-MJLSs and the effects of modeling error in the jump process on the control quality are investigated in Section V. In Section VI, the problem of jump process model reduction is explored and some model reduction techniques are described. To illustrate the ideas presented in the paper, a numerical case study is given in Section VII, and finally concluding remarks are summarized in Section VIII.

II. Preliminaries and Notations

For a continuous random variable $T$, the probability density function (pdf), cumulative distribution function (cdf), and the complementary cumulative distribution function (ccdf) are respectively denoted by $f_T(t)$, $F_T(t)$, and $\bar{F}_T(t) = 1 - F_T(t)$. The hazard rate function of $T$ is defined as $h_T(t) = f_T(t)/\bar{F}_T(t)$. For the exponential distribution, the hazard rate function is constant. Associated with a (semi-) Markov process with discrete state space $V = \{1, 2, \ldots, m\}$, there is a directed graph $G = (V, E)$ having the set of vertices $V$ and the set of edges $E$. There is a directed arc from vertex $i$ to vertex $j$ denoted by $(i, j) \in E$ if and only if direct transition from state $i$ to state $j$ is possible. The in-neighborhood of state $i$ is defined as $N_i^- = \{j \in V | (j, i) \in E, j \neq i\}$ whose elements are called the in-neighbors of state $i$. Similarly, the out-neighborhood of state $i$ is defined as $N_i^+ = \{j \in V | (i, j) \in E, j \neq i\}$ whose elements are called the out-neighbors of state $i$. The probability that a (semi-) Markov process is in state $i$ at time $t$ is denoted by $\mu_i(t)$. The mode indicator of a random process is denoted by $\delta_i(t)$ which is equal to 1 when the process is in state $i$ at time $t$ and is 0 otherwise, then $\mu_i(t) = \mathbb{E}[\delta_i(t)]$, where $\mathbb{E}[\cdot]$ denotes the expectation operator. The transition rate matrix of a continuous-time homogeneous Markov chain is denoted by $\Pi = [\pi_{ij}]$, where $\pi_{ij}$ is the rate at which transitions occur from state $i$ to state $j$ and $\pi_{ii} = -\sum_{i \neq j} \pi_{ij}$. The off-diagonal elements of $\Pi$ are finite non-negative and the sum of all elements in any row of $\Pi$ is zero. A state $i$ with $\pi_{ii} = 0$ is called absorbing. A PH distribution is defined as the distribution of the time to enter an absorbing state from the set of transient states of a time-homogeneous Markov chain [18]. Analogous to state-space representation of LTI systems with strictly proper transfer functions, an $m$-phase PH distribution model can be represented by a triple $(\Pi, \eta, \alpha)$, where $\Pi \in \mathbb{R}^{m \times m}$ is referred to as the sub-generator matrix which is an invertible matrix with non-negative off-diagonal elements, negative elements on the main diagonal, and non-positive row sums, $\eta = -\Pi 1_m \in \mathbb{R}^m$ is called
the exit rate vector (or the closing vector) of the model which is a column vector of outgoing transition rates with non-negative elements, where \( \mathbb{I}_m \) is a column vector with all elements equal to 1. The elements of \( \eta \) and the off-diagonal elements of \( \Pi \) are transition rates. The column vector \( \alpha \in \mathbb{R}^m \) is called the starting vector of the model which is a sub-stochastic vector with non-negative elements such that \( \alpha^\top \mathbb{I}_m \leq 1 \) [11, §1.2].

**Lemma 1:** [18, §5.1] Let \( T \) be a non-negative random variable with an \( m \)-phase PH distribution represented by \((\Pi, \eta, \alpha)\), then

(i) The pdf of \( T \) is given by \( f_T(t) = \alpha^\top \exp(\Pi t) \eta \), \( t \geq 0 \), with the Laplace transform \( \mathcal{L}[f_T(t)] = \alpha^\top (s \mathbb{I}_m - \Pi)^{-1} \eta \), where \( \mathbb{I}_m \) is the \( m \times m \) identity matrix.

(ii) The cdf of \( T \) is given by \( F_T(t) = \mathbb{P}[T \leq t] = \int_0^t f_T(\tau) d\tau = 1 - \alpha^\top \exp(\Pi t) \mathbb{I}_m \), \( t \geq 0 \), where \( \mathbb{I}_m \) is a column vector with all elements equal to 1. Then, the ccdf (or survival function) of \( T \) is \( \bar{F}_T(t) = \mathbb{P}[T > t] = 1 - F_T(t) = \alpha^\top \exp(\Pi t) \mathbb{I}_m \), \( t \geq 0 \).

(iii) The \( n \)-th moment of \( T \) is \( \mathbb{E}[T^n] = \int_0^\infty t^n f_T(t) dt = (-1)^n n! \alpha \Pi^{-n} \mathbb{I}_m \).

An \( m \)-state time-homogeneous continuous-time semi-Markov process \( \{r(t)\} \) is described by three components: (i) an initial probability vector \( \mu(0) \in \mathbb{R}^m \), where \( \mu_i(0) = \mathbb{P}[r(0) = i] = \mathbb{E}[\delta_i(0)] \), (ii) a discrete-time embedded Markov chain with one-step transition probability matrix \( P = [p_{ij}] \in \mathbb{R}^{m \times m} \) (with no self-loop, \( p_{ii} = 0 \)) which determines the mode to which the process will next go after leaving mode \( i \), and (iii) the conditional distribution function \( F_{ij}(t) = \mathbb{P}[T_{ij} \leq t] \), where the random variable \( T_{ij} \) is the time spent in mode \( i \) from the moment the process last entered that mode, given that the next mode to visit is mode \( j \) [19, §9.11]. The random variable \( T_{ij} \) is called a conditional holding time of mode \( i \). It has also been referred to by various other names such as sojourn time or residence time. So, a semi-Markov jump process is completely specified by \((\mu(0), [p_{ij}], [F_{ij}])\). Sample paths of a semi-Markov process are specified as \((r_0, t_0), (r_1, t_1), (r_2, t_2), \ldots \) , where the pair \((r_k, t_k)\) indicates that the process jumps to mode \( r_k \) at time \( t_k \) and remains there over the period \([t_k, t_{k+1})\). Let \( T_i \) denote the time spent in mode \( i \) before making a transition (the successor mode is unknown), then \( T_i = \sum_j p_{ij} T_{ij} \) with the distribution function \( \mathbb{P}[T_i \leq t] = \sum_j p_{ij} F_{ij}(t) \). The random variable \( T_i \) is referred to as the unconditional holding time of mode \( i \). Obviously, if mode \( j \) is the only out-neighbor of mode \( i \), then \( p_{ij} = 1 \) and \( T_i = T_{ij} \). In a semi-Markov process, once the system enters mode \( i \), the process randomly selects the next mode \( j \neq i \) according to a probability transition matrix \( P = [p_{ij}] \). If mode \( j \) is selected, the time spent in mode \( i \) before jumping to mode \( j \) is determined by the
distribution function $F_{ij}(t)$.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be a $p \times q$ matrix. The Kronecker product of $A$ and $B$ is an $mp \times nq$ matrix defined as $A \otimes B = [(a_{ij}B)]$. The following lemma gives some properties of the Kronecker product.

**Lemma 2:** [20] The Kronecker product satisfies the following properties
(i) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{n \times s}$, and $D \in \mathbb{R}^{q \times r}$.
(ii) $\alpha(A \otimes B) = (\alpha A) \otimes B = A \otimes (\alpha B)$, where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, and $\alpha$ is a scalar.
(iii) $I_n \otimes A$ and $B \otimes I_m$ commute, for any $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{n \times n}$.
(iv) $\exp(A \otimes I_s) = \exp(A) \otimes I_s$, for any $A \in \mathbb{R}^{m \times m}$ and any positive integer $s$.

Let $A(t) \in \mathbb{R}^{n \times n}$ be a piecewise continuous, locally bounded function of time $t$. The *evolution matrix* of the system $\dot{X}(t) = A(t)X(t)$ is denoted by $\Phi_A(t, \tau)$, which is also referred to as the *state transition matrix* associated with $A(t)$.

**Lemma 3:** Let $M_1(t)$ and $M_2(t)$ be square matrices, piecewise continuous, locally bounded function of time $t$, with state transition matrices $\Phi_{M_1}(t, \tau)$ and $\Phi_{M_2}(t, \tau)$, respectively.

(i) For a block-diagonal matrix $M(t) = \text{diag}(M_1(t), M_2(t))$, the state transition matrix is given by $\Phi_M(t, \tau) = \text{diag}(\Phi_{M_1}(t, \tau), \Phi_{M_2}(t, \tau))$. In general, if $M(t) = \text{diag}(M_1(t), M_2(t), \ldots, M_n(t))$, then $\Phi_M(t, \tau) = \text{diag}(\Phi_{M_1}(t, \tau), \Phi_{M_2}(t, \tau), \ldots, \Phi_{M_n}(t, \tau))$.

(ii) The state transition matrix of $M(t) = M_1(t) + M_2(t)$ is given by $\Phi_M(t, \tau) = \Phi_{M_1}(t, 0)\Phi_{M_2}(t, 0)$.

**Proof:** The proof is given in the Appendix.

**Lemma 4:** Let $A \in \mathbb{R}^{m \times m}$ be a constant matrix and $B(t) \in \mathbb{R}^{n \times n}$ be a piecewise continuous, locally bounded function of time $t$. Then, the state transition matrix of $M(t) = (A \otimes I_n) + (I_m \otimes B(t))$ is given by $\Phi_M(t, \tau) = \exp(A(t - \tau)) \otimes \Phi_B(t, \tau)$.

**Proof:** The proof is given in the Appendix.

**Lemma 5:** [21] §1.1 Let $M(t) \in \mathbb{R}^{m \times m}$, $N(t) \in \mathbb{R}^{n \times n}$, and $R(t) \in \mathbb{R}^{m \times n}$ be piecewise continuous and locally bounded functions of time $t$. The unique solution of the differential equation $\dot{X}(t) = M(t)X(t) + X(t)N(t) + R(t)$, $X(t_0) = X_0$, is given by $X(t) = \Phi_M(t, t_0)X_0 \Phi_{N^\top}(t, t_0) + \int_{t_0}^t \Phi_M(t, \tau)R(\tau)\Phi_{N^\top}(t, \tau)d\tau$, $\forall t$, where $\Phi_M(t, \tau)$ and $\Phi_{N^\top}(t, \tau)$ are the state transition matrices associated with square matrices $M(t)$ and $N^\top(t)$, respectively.
III. Problem Statement

Consider a semi-Markov jump linear system whose behavior over its utilization period $[0, t_f]$ is described by the following stochastic state-space model

$$\dot{x}(t) = A(r(t), t)x(t) + B(r(t), t)u(t), \quad x(0) = x_0$$

where $t \in [0, t_f]$, the final time $t_f$ is finite, known and fixed, $x(t)$ is a measurable state vector, $\{r(t)\}$ is a continuous-time semi-Markov process with a finite discrete state space $\mathcal{V}_s$ that autonomously determines the system mode at each time $t$, and $u(t)$ is a control input. The state vector $x(t)$ is the continuous component of the system’s state and $r(t)$ is the discrete component of the system’s state. If $r(t) = i \in \mathcal{V}_s$, we write $(A(r(t), t), B(r(t), t)) = (A_i(t), B_i(t))$, where for each $i$, $A_i(t)$ and $B_i(t)$ are known, bounded, continuous and deterministic matrices representing the linearized model of the system at an operating point. The state of the jump process $r(t)$ is assumed to be observable and statistically independent of $x(t)$ which are reasonable assumptions in many applications. For example, load level in power systems, wind speed, and solar irradiance can be measured online using sensing devices and are independent of the state vector of the dynamical system. Also, in modeling of an aircraft dynamics with multiple flight modes, when no information about the aircraft intent is available, the mode transitions are independent of the continuous dynamics [22].

The optimal regulation problem is to find a state-feedback control law of the form

$$u(t) = K(r(t), t)x(t)$$

such that the following cost functional subject to (1) and (2) is minimized.

$$J = \mathbb{E} \left[ \int_0^{t_f} \left( x^\top(t)Q(r(t), t)x(t) + u^\top(t)R(r(t), t)u(t) \right) dt + x^\top(t_f)S(r(t_f))x(t_f) \right],$$

where $Q$ is symmetric positive semi-definite matrix $\forall t$, $R$ is a symmetric positive definite matrix $\forall t$, and $S$ is constant symmetric positive semi-definite matrix that penalizes deviations of the final state from zero. Linear feedback controllers due to their simple structure and low complexity are of practical interest; hence, it is desired to find the best controller of this class in the sense that it optimizes a certain performance index.

The problem of quadratic optimal control Markov jump linear systems, i.e. when the jump process $\{r(t)\}$ is modeled by a Markov chain and $r(t)$ is fully observable, has been solved long
time ago and it has been proved that the optimal control law is in a state-feedback form with a switching feedback gain [23]. Since the Markovian assumption is not realistic in many practical applications, it is desired to deal with the problem with non-Markovian jump process. A ‘suitable model’ for the jump process is therefore needed to deal with analysis and control design problems for a random jump dynamical system. By ‘suitable model’, we mean a probability model that can capture the actual process characteristics and retain the tractability of the control problem. Moreover, the trade-offs between improving the quality of the model and the computational cost associated with control design should be investigated.

**Remark 1:** Since this paper aims to examine the problem of jump process modeling and to explore the effect of modeling uncertainties in the probability model on the control quality, then the problem is studied under simplifying assumptions on the linearized plant dynamics, i.e., the plant full-state \( x(t) \) is assumed to be measurable and there is no process/measurement noise and no unmodeled dynamics. Also, it is assumed that the state vector \( x(t) \) and the control input \( u(t) \) are not constrained by any boundaries. The results of the paper are still applicable if these assumptions are relaxed.

**IV. MARKOVIANIZATION OF S-MJLSs USING PH MODELS**

The PH distribution approach is a technique to exactly or approximately transform a semi-Markov process to a Markov process which can be used to facilitate analysis of S-MJLSs. In order to Markovianize a semi-Markov process, the holding-time distribution of each mode is approximated by a finite-phase PH model. For a mode with multiple out-neighbors, there are multiple conditional holding times with possibly different distributions, then in this case several PH models are to be designed, all corresponding to the same mode. For simplicity of representations, we consider a class of PH distributions called **Coxian distribution** whose sub-generator matrix has an upper bi-diagonal structure, the results, however, are applicable to any PH model with a single starting phase. The Coxian model due to its simple structure and mathematical tractability is often used for analysis and computation. Many PH distributions have an equivalent Coxian representation; for example, any PH distribution with triangular, symmetric, or tri-diagonal sub-generator matrix has an equivalent Coxian representation of the same order [11, §1.4]. Moreover, both PH and Coxian distributions are dense in the class of non-negative distributions, and in general any non-negative distribution can be approximated
arbitrarily closely by a network of exponential stages representable by a Coxian network \([\Pi]\). In a Cox model the initial phase is the first phase, then the starting vector is always of the form 
\(\alpha = [1, 0, \ldots, 0]^\top\). Figure 2 shows the state transition diagram of a third-order Cox model whose state-space representation \((\Pi, \eta, \alpha)\) is given by

\[
\begin{pmatrix}
\pi_{11} & \pi_{12} & 0 \\
0 & \pi_{22} & \pi_{23} \\
0 & 0 & \pi_{33}
\end{pmatrix}, \quad
\begin{pmatrix}
\pi_{10} \\
\pi_{20} \\
\pi_{30}
\end{pmatrix}, \quad
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\]  
(4)

and from the parametric constraints of PH models we have \(\pi_{ii} < 0, \pi_{ij} \geq 0, i \neq j, \pi_{11} = -(\pi_{12} + \pi_{10}), \pi_{22} = -(\pi_{23} + \pi_{20}),\) and \(\pi_{33} = -\pi_{30}\). Then, from Lemma 1, the pdf of the holding time of mode \(a\) is 
\(f_a(t) = \alpha^\top \exp(\Pi t) \eta, \ t \geq 0\). Since the initial phase of the Cox model is concentrated in the first phase, then if \(\mu_a(0)\) is the probability that the system is in mode \(a\) at initial time \(t = 0\), then in the corresponding Cox model, \(\mu_1(0) = \mu_a(0)\) and \(\mu_2(0) = \mu_3(0) = 0\).

Analogous to LTI systems where the transfer function (i.e. the Laplace transform of the impulse response) is unique while the state-space realizations are not, any PH distribution has a unique pdf, but there is not a unique state-space representation \((\Pi, \eta, \alpha)\). Similarly, a PH model is called minimal, if no pdf-equivalent PH model of smaller order exists. For example, it can easily verify that in the Cox model (4) if \(\pi_{10} = \pi_{20} = \pi_{30}\), then the three-phase Cox model is not minimal as it is equivalent to a single-phase Cox model with pdf
\(f_a(t) = \pi_{30} \exp(-\pi_{30} t), \ t \geq 0\).

Figure 3(a) shows a three-mode semi-Markov process with general conditional holding-time distributions \(F_{ab}, F_{bc},\) and \(F_{bd}\). Mode \(b\) of this process has two out-neighbors. If the process enters mode \(b\), it jumps to mode \(c\) after the time determined by the distribution function \(F_{bc}\), and jumps to mode \(d\) after the time determined by the distribution function \(F_{bd}\). A Markovian approximation of the process by Coxian distributions is shown in Figure 3(b), where each holding-time distribution is approximated by a three-phase Cox model. The holding-time distributions \(F_{ab}, F_{bc},\) and \(F_{bd}\) are
respectively approximated by phases 1-3, 4-6, and 7-9. The one-step transition probability matrix of the embedded discrete-time Markov chain of the semi-Markov process shown in Figure 3(a) is

$$ P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_{bc} & p_{bd} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

where

$$ p_{bc} = \mathbb{P}[T_{bd} > T_{bc}] = \int_0^\infty \mathbb{P}[T_{bd} > t]f_{bc}(t)dt = \int_0^\infty (1 - F_{bd}(t))f_{bc}(t)dt $$

and

$$ p_{bd} = 1 - p_{bd}, $$

where $f_{bc}(t)$ is the pdf of the holding time of mode $b$, given that the next mode to visit is mode $c$. Then, if the process with probability $\gamma$ is in mode $b$ at the initial time, i.e., $\mu_b(0) = \gamma$, then the initial probabilities of the two Cox models of mode $b$ in Figure 3(b) are $\mu_4(0) = \gamma p_{bc}$ and $\mu_7(0) = \gamma p_{bd}$. If the exit rate vector of the Cox model of mode $a$ is $\eta_a = [\beta_1, \beta_2, \beta_3]^\top$, then the transition rates of the incoming links to phase 4 are $p_{bc}\eta_a$, and those of phase 7 are $p_{bd}\eta_a$. It should be noted that any PH model with starting vector $\alpha = [1, 0, \ldots, 0]^\top$ can be used in this approximation approach.

When the underlying jump process of a S-MJLS is transformed to a Markov chain by PH approximation, then all phases of any PH model associated with a particular mode of the S-MJLS.
share the same dynamic. For example, in Figure 3(a), if the pair \((A_b(t), B_b(t))\) represents the dynamic of mode \(b\), then in the corresponding MJLS with the Markovian jump process shown in Figure 3(b), all phases 4 to 9 share the same dynamic \((A_b(t), B_b(t))\). It should be noted that the transitions between the internal phases of a PH model cannot be observed or estimated. The sensing devices can only detect transitions between the modes of the semi-Markov process, i.e., only the jumps between modes \(a, b, c, d\) can be observed. Then, a completely mode-observable S-MJLS is approximated by a partially mode-observable MJLS.

The PH distribution approach, however, suffers from a drawback. Although in theory, any non-negative distribution function can be approximated arbitrarily well by a PH distribution, modeling of many distributions by a PH distribution with an acceptable level of accuracy may need a very large number of phases. This can make design and analysis of the S-MJLS computationally infeasible. Then, in practice, this approach may not allow us to fully incorporate actual distribution functions in the stochastic model. This is because for a wide class of distributions, the best PH approximate model of a reasonable size may result in an unacceptably large error, while capturing time-dependent characteristics of the jump process and its transient behavior with high-enough sufficiently may be crucial to achieve a satisfactory control performance. Therefore, it is necessary to find a compromise between modeling accuracy and computational cost. We address this key limitation of the PH distribution approach in Section VI and propose a new technique for low-order approximation of non-exponential holding-time distributions.

V. Optimal Control of S-MJLSs

Consider the problem formulated in Section III and assume that the underlying semi-Markov jump process of the system is approximated with high accuracy by a Markov chain using Cox (or PH) models of sufficiently high order. In this section, we first present a control design procedure based on the necessary optimality condition for the Markovianized version of the process with partially observable transitions, then we investigate how the optimal control gains and the cost functional value are related to the characteristics of the jump process.

The following theorem gives a necessary optimality condition based on the matrix minimum principle \([24]\) that provides a set of potentially optimal gains for the problem of optimal state-feedback control design of the form (2) that minimized the cost functional (3) for a MJLS with partially observable transitions. That is, the state of the underlying Markov jump process is
partitioned into a number of clusters and only transitions between the clusters can be observed and all transitions between the internal states of each cluster are unobservable. Then, associated with each cluster a state-feedback control gain is to be designed such that the cost (3) is minimized. The following theorem from [25] gives an optimality condition for MJLSs with partially observable modes.

**Theorem 1:** [25] Consider a continuous-time MJLS of the form (1) and assume \( \{r(t)\} \) is a continuous-time homogeneous Markov chain with state space \( \mathcal{V} \) and transition rate matrix \( \Pi = [\pi_{ij}] \). The system’s dynamic in mode \( i \in \mathcal{V} \) is represented by the pair \( (A_i(t), B_i(t)) \) and for any \( i, j \in \mathcal{V} \), the transition rate from mode \( i \) to mode \( j \) is denoted by \( \pi_{ij} \). Assume \( \mathcal{V} \) is partitioned into \( N \) disjoint subsets (clusters) \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N \subseteq \mathcal{V} \), where \( \bigcup_{i=1}^{N} \mathcal{C}_i = \mathcal{V} \) and that only transitions between the clusters can be observed. Then, an optimal switching state-feedback control law of the form \( u(t) = K_i(t)x(t) \), where \( K_i(t) = \Gamma_k(t) \), \( \forall i \in C_k \), that minimizes the cost (5) (if it exists) must satisfy the following equations.

\[
\sum_{i \in C_k} \left( R_i(t)\Gamma_k(t) + B_i^\top(t)\Lambda_i(t) \right) X_i(t) = 0,
\]

(5)

for \( k = 1, 2, \ldots, N \) and all \( t \in [0, t_f] \), where

\[
-\dot{\Lambda}_i(t) = \bar{A}_i^\top(t)\Lambda_i(t) + A_i(t)\bar{A}_i(t) + Q_i(t) + K_i^\top(t)R_i(t)K_i(t) + \sum_{j \in \mathcal{V}} \pi_{ij} \Lambda_j(t)
\]

(6)

for all \( i \in \mathcal{V} \), is the co-state equation with terminal condition \( \Lambda_i(t_f) = S_i \), where \( \bar{A}_i(t) = A_i(t) + B_i(t)K_i(t) \) is the closed-loop matrix of mode \( i \), and

\[
\dot{X}_i(t) = \bar{A}_i(t)X_i(t) + X_i(t)\bar{A}_i^\top(t) + \sum_{j \in \mathcal{V}} \pi_{ji} X_j(t)
\]

(7)

for all \( i \in \mathcal{V} \), is the covariance equation with initial condition \( X_i(0) = x_0x_0^\top \mu_i(0) \), where \( \mu_i(0) \) is the probability that the system is in mode \( i \) at \( t = 0 \).

**Proof:** The proof has been presented in [25]. It is also included in the Appendix for the sake of completeness. Similar results to Theorem [1] are obtained in [4] §3.5 for continuous-time MJLSs with fully non-observable modes. Also, a discrete-time version of Theorem [1] for MJLSs with fully non-observable mode as well as partially observable modes is given in [14], [26].

In Theorem [1], the internal states of a cluster \( C_k \) may have different dynamics but they share the same control gain \( \Gamma_k(t) \). The covariance variable associated with each \( i \in \mathcal{V} \) is defined as

\[
X_i(t) = \mathbb{E}[x(t)x^\top(t)\delta_i(t)]
\]

which is used to transform the stochastic optimization problem to an
equivalent deterministic one in an average sense. In the case that every transition in the Markov jump process is observable, the covariance variables will not appear in the controller equation (5). This is because in this case every cluster is a singleton \( C_i = \{i\} \) and (5) must be satisfied for all \( X_i(t) \), then the controller equation reduces to \( K_i(t) = -R_i^{-1}(t)B_i^\top(t)\Lambda_i(t) \). Using the dynamic programming approach, it has been proved [23], [4, §3.2] that \( u(t) = -R_i^{-1}(t)B_i^\top(t)\Lambda_i(t)x(t) \) is indeed the optimal control law over all admissible controls for the all-mode observable case; a result analogous to the classical deterministic LQR problem, but with switching feedback gains and coupling terms in the co-state and covariance equations.

We now return to the original problem and use the above result for optimal control of S-MJLSs with a Markovianized jump process. After Markovianizing of a semi-Markov process using the PH distribution approach, each cluster \( C_k \) in Theorem 1 will correspond to a mode of the S-MJLS and hence all internal states of each cluster share the same dynamic, weighting matrices, and control gain.

**Lemma 6:** Let \( \mathcal{V} \) be the state space of the Markovianized jump process. For a control law of the form \( u(t) = K_i(t)x(t) \), the cost function (3) can be written as

\[
J = \sum_{i \in \mathcal{V}} \left( \int_0^{t_f} \text{tr}[L_i(t)X_i(t)]dt + \text{tr}[S_iX_i(t_f)] \right) = \sum_{i \in \mathcal{V}} \text{tr}[\Lambda_i(0)X_i(0)] = x_0^\top \left( \sum_{i \in \mathcal{V}} \mu_i(0)\Lambda_i(0) \right)x_0,
\]

where \( L_i(t) = Q_i(t) + K_i(t)R_i(t)K_i(t) \) and the co-state variable \( \Lambda_i(t) \) satisfies (6) and the covariance variable \( X_i(t) \) satisfies (7).

**Proof:** The proof follows by using the cyclic permutation invariance property of matrix trace and the linearity property of the expectation operator. For the sake of completeness, the details are included in the Appendix.

**Remark 2:** From Lemma 6, to evaluate the cost for a given set of of control gains \( \{K_i(t), i \in \mathcal{V}, t \in [0, t_f]\} \), we just need to solve the co-state equation (6) which is a linear matrix differential equation and can be solved numerically backward in time. However, to compute the optimal control gains, we have to solve a set of nonlinear coupled matrix differential equations (5)-(7). These equations can be solved using the iterative procedures proposed in the literature for this class of equations [4, §3.6], [21, §6.9], [14].

As mentioned earlier, a PH distribution does not have a unique state-space realization. Then,
in the optimal control problem, when a semi-Markov process is Markovianized using the PH
distribution approach, it is desired to investigate how the properties of the closed-loop system
depend on the structure and parameters of state-space realizations of the holding-times distribution
and whether changing the realizations may alter the behavior of the closed-loop system.

**Definition 1:** In a controlled S-MJLS, any signal or quantity that depends on the holding-time
distribution models of the jump process only through their pdfs and is invariant with respect
to the selection of the state-space realization of distribution models is said to be **pdf-dependent**.

We first show that for a given set of control gains $K_i$’s, the cost value $J$ is pdf-dependent. Since the initial phase of each Cox model is concentrated in the first phase, then from (8) and
(6) it suffices to show that the co-state variable corresponding to the first phase of each Cox
model is pdf-dependent.

**Theorem 2:** Consider an $m$-phase Cox model $(\Pi, \eta, \alpha)$ corresponding to mode $a$ of a S-
MJLS as shown in Figure 4. Assume the $n$-th order dynamic of mode $a$ is represented by the
pair $(A(t), B(t))$ and the state-feedback control gain is $K(t)$. The co-state variable of the first

![Fig. 4. The holding-time distribution of mode $a$ is represented by an $m$-phase Cox model. All internal phases of the Cox model share the same dynamics, the same control gain, and the same weighting matrices in the quadratic cost functional.](image)

phase of the Cox model is given by

$$
\Lambda_1(t) = \Phi(t_f - t)\Phi(A_1(t_f, t)S_1\Phi(A_1(t_f, t))
+ \int_t^{t_f} f_a(\tau - t)\Phi(A_1(\tau, t)\Lambda_k(\tau)\Phi(A_1(\tau, t))d\tau
+ \int_t^{t_f} \bar{F}_a(\tau - t)\Phi(A_1(\tau, t)L_1(\tau)\Phi(A_1(\tau, t))d\tau, (9)
$$

where $f_a(t)$ and $\bar{F}_a(t) = 1 - \int_0^t f_a(\sigma)d\sigma$ are the pdf and the ccdf of the holding time of mode $a$, respectively, $\Phi(A_1(t, \tau)$ is the state transition matrix associated with the closed-loop state matrix
$
\tilde{A}_1(t) = A(t) + B(t)K(t), L_1(t) = Q(t) + \tilde{K}_1(t)R(t)K(t),$
and $\Lambda_k(t)$ is the co-state variable of the external state $k$.

**Proof:** The proof is given in the Appendix.
It should be noted that the external state labeled $k$ is an out-neighbor of mode $a$ whose co-state variable equation may depend on $\Lambda_1(t)$ but not $\Lambda_i(t)$ for $i = 2, \ldots, m$. If state $k$ is the first phase of another Cox model, representing mode $b$, then it satisfies an equation similar to (9) for mode $b$. Therefore, the co-state variable of the initial phase of every Cox model is pdf-dependent, then from (8) it follows that the dependency of the cost value on the holding-time distributions is through their pdfs; hence, replacing each Cox model by any pdf-equivalent model (with different order and parameters) keeps the cost value invariant.

Let us assume mode $a$ is the initial mode of a S-MJLS and the actual pdf of the holding time of this mode is denoted by $f_a(t)$ which can be realized with high accuracy by a sufficiently high-order Cox model. Also, let $\hat{f}_a(t)$ be an estimate of $f_a(t)$ represented by a low-order Cox model. From (8) and (9), the error in the cost due to the error in the pdfs is given by

$$\tilde{J} = J - \tilde{J} = x_0^T \tilde{\Lambda}_1(0)x_0,$$

where $\tilde{\Lambda}_1(0) = \Lambda_1(0) - \hat{\Lambda}_1(0)$. From Theorem 2 it follows that

$$\tilde{\Lambda}_1(t) = - \left( \int_t^{t-t} \varepsilon_a(\sigma)d\sigma \right) \Phi_{\tilde{A}_1}(t_i, t)S_1\Phi_{\tilde{A}_1}(t_i, t) + \int_t^{t-t} \varepsilon_a(\tau - t)\Phi_{\tilde{A}_1}(\tau, t)\Lambda_k(\tau)\Phi_{\tilde{A}_1}(\tau, t)d\tau + \int_t^{t-t} f_a(\tau - t)\Phi_{\tilde{A}_1}(\tau, t)\tilde{\Lambda}_k(\tau)\Phi_{\tilde{A}_1}(\tau, t)d\tau$$

$$- \left( \int_t^{t-t} \varepsilon_a(\sigma)d\sigma \right) \Phi_{\tilde{A}_1}(\tau, t)\Lambda_1(\tau)\Phi_{\tilde{A}_1}(\tau, t)d\tau,$$

where $\varepsilon_a(t) = f_a(t) - \hat{f}_a(t)$ is the pdf error and $\int_0^t \varepsilon_a(\sigma)d\sigma = F_a(t) - \hat{F}_a(t)$ is the cdf error, and $\tilde{\Lambda}_i(t_i) = S_i - S_i = 0$. The matrix $\tilde{\Lambda}_k(t_i) = \Lambda_k(t_i) - \hat{\Lambda}_k(t_i)$ in the third term in the right-hand side of (11) is due to the error is the pdf of the holding time of other modes of the S-MJLS. Then, if the error in the pdf of the holding time of all modes goes to zero, then the error in cost goes to zero. Also, it is clear from (11) that for a given approximate pdf for each holding-time of a S-MJLS, the amount of change in the cost due to the modeling error of holding-time distributions depends on the dynamics of the modes, control gains, and the weighting matrices, i.e., $\Phi_{\tilde{A}_1}(t, \tau)$, $S_i$, and $L_i(t) = Q_i(t) + K_i(t)R_i(t)K_i(t)$. So, in general, if there is a relatively large error in pdfs over some periods of time, the relative change in the cost may or may not be large. Another implication of Theorem 2 is that the use of a distribution model obtained by matching the first few moments may lead to a large error in the cost value. For example, the rate equivalent
(or insensitivity) approach is a simple method of Markovianizing a semi-Markov process in which any holding-time distribution of a semi-Markov process is replaced by an exponential distribution with the same mean [27], [28]. This approach is proposed to study the steady-state behavior of some class of semi-Markov processes; however, the resulting error in the transient behavior can be arbitrarily large. The use of such approaches in optimal control of S-MJLSs may cause large errors in the pdfs and hence a drastic change in the cost value.

**Example 1:** Consider a two-mode S-MJLS with a scalar dynamic. Let the holding time of mode 1 before jumping to mode 2 be denoted by $T_{12}$. Suppose $T_{12}$ has a non-exponential distribution represented by a 3-phase PH model with the following sub-generator matrix

$$
\Pi = \begin{bmatrix}
-10 & 1 & 0 \\
0 & -5 & 1 \\
0 & 0 & -0.01 \\
\end{bmatrix}.
$$

For simplicity we assume that the holding time of mode 2 is exponentially distributed with the rate parameter 1. The dynamic and weighting matrices of modes 1 and 2 are respectively $(A_1, B_1, Q_1, R_1, S_1) = (1, 0.1, 1, 1, 0)$ and $(A_2, B_2, Q_2, R_2, S_2) = (-10, 10, 1, 1, 0)$. The system is initially in mode 1 ($\mu_1(0) = 1$), the initial condition of the system is $x_0 = 1$, and constant control gains $K_1 = -12$ and $K_2 = -6$ are given for mode 1 and 2, respectively. Let $J$ be the cost corresponding to the actual semi-Markov process, where $T_{12}$ has a 3-phase PH distribution with pdf $f_{12}(t)$ and with mean $E[T_{12}] = -\alpha \Pi^{-1} 1$. In order to see the effect of modeling error in the distribution of $T_{12}$ on the cost value, let $\hat{J}$ be the cost value for the case when $f_{12}(t)$ is replaced by an exponential pdf $\hat{f}_{12}(t)$ (1-phase PH distribution) with the same statistical mean $E[T_{12}] = E[\hat{T}_{12}]$, i.e., $\hat{f}_{12}(t) = \lambda \exp(-\lambda t)$ with $\lambda = 1/E[T_{12}]$. For the given control gains and final time $t_f = 30$ sec, the cost values are $J = 23.22$ and $\hat{J} = 167.12$. That is, even though the first moment of the two distributions are exactly the same, the error in the holding-time pdf leads to about 620% relative change in the control cost.

In what follows, we show that the optimal control gains are also pdf-dependent. Before showing this fact, we state and prove a lemma.

**Lemma 7:** Consider any Cox model with $m$ phases labeled $\{1, 2, \ldots, m\}$. Then, $\sum_{i=1}^{m} X_i(t)$ and $\sum_{i=1}^{m} \Lambda_i(t)X_i(t)$ are pdf-dependent matrices, where $X_i(t)$ and $\Lambda_i(t)$ are the covariance and co-state variables of phase $i$, respectively.

**Proof:** The proof is given in the Appendix.
Theorem 3: In a S-MJLS, the optimal control gains are pdf-dependent.

Proof: The proof is given in the Appendix.

Theorem 3 implies that if the holding-time distribution model of any mode is replaced by a different but pdf-equivalent model, the optimal gains remain unchanged. Due to the pdf-dependency of the cost value and the optimal control gains, the use of distribution models obtained by moment-matching techniques that only match the first few moments may adversely affect the control quality, because the error between the pdfs may be large.

Example 2: Consider the system in Example 1. For the actual system with pdf \( f_{12}(t) \), the optimal cost value is \( J^* = 10.25 \) which is obtained by solving (5)-(7). Now, we replace \( f_{12}(t) \) by an exponential pdf \( \hat{f}_{12}(t) \) (of the same mean). This means that the two-mode S-MJLS is approximated by a two-mode MJLS. We consider the approximated model as a nominal model for optimal control design. Let the optimal gains for the approximated model be denote them by \( \hat{K}_{1}^*(t) \) and \( \hat{K}_{2}^*(t) \), \( t \in [0, t_f] \). If we apply the control law \( u(t) = \hat{K}_{1}^*(t)x(t) \) to the actual S-MJLS, the achieved cost is \( \hat{J} = 28.79 \). That is, computing the optimal control gains based on a low-order approximate jump process model leads to about 180% relative increase in the cost value.

It should be noted that the results presented in this section hold true for any PH model with a single starting phase, not only for Coxian distributions, i.e., the sub-generator matrix of each distribution model can be a full matrix (not necessarily a bi-diagonal matrix). In the following section, the modeling and model reduction of the jump process are examined.

VI. JUMP PROCESS MODELING AND MODEL REDUCTION

The optimal control problem involves trade-offs between control effectiveness and control effort. The weighting matrices in the cost functional should be properly selected to achieve a desirable response. In practice, trial-and-error tuning is used to determined the most appropriate weighting matrices [29, §2.1]. For each choice of weighting matrices, the optimal control is computed and applied to the plant model and the response is evaluated. If the behavior of the closed-loop system is not satisfactory, the whole procedure is repeated with different weighting matrices until an acceptable response is achieved. In problem of optimal control of S-MJLSs, corresponding each state of the Markovianized jump process there is a co-state and a covariance equation, then for an \( N \)-state Markov process there are \( 2N \) nonlinear matrix
differential equations to be solved iteratively. Hence, if a high-order PH distribution is needed to model each holding-time distribution of the jump process, the computational burden associated with solving the nonlinear differential equations significantly increases. A reduced-order model is therefore to be developed to make the control design computationally feasible. The reduced-order model for the jump process should approximate the full-order model with sufficiently high accuracy in the sense that the distance between the holding-time pdfs is sufficiently small, but with a smaller state-space dimension. Once the design is done based on the reduced-order model and appropriate weighting matrices are determined, the performance of the resulting controller on the full-order system has to be examined before implementation on the real system.

Modeling of a general distribution by a PH distribution using empirical data is a complex non-linear optimization problem [30]. There has been much research done on developing numerical algorithms to fit a class of PH distributions to empirical data containing a large number of measurements [31]–[33]. The method of maximum likelihood estimation due to its desirable statistical properties has been widely used to estimate parameters of probability distributions, and expectation maximization algorithms have been developed to find the maximum-likelihood estimate of the parameters of a distribution [33]. For instance, the function ‘PHFromTrace’ in MATLAB toolbox Butools [34] performs PH distribution fitting to empirical data using the algorithm proposed in [32].

Analogous to the LTI systems, after fitting a model to empirical data (modeling step), we need to develop a new model by appropriately reducing the order of the full-order model (model-reduction step) to be used for simulation and control design. The model reduction problem for PH distributions is defined as follows: Given an $m$-phase PH model $(\Pi, \eta, \alpha)$ with pdf $f(t)$, find an $\hat{m}$-phase PH model $(\hat{\Pi}, \hat{\eta}, \hat{\alpha})$ with pdf $\hat{f}(t)$, where $\hat{m} < m$, such that the distance between $\hat{f}(t)$ and $f(t)$ is kept sufficiently small or even minimized.

The connection between PH distributions and positive LTI systems has been discussed in [35]. One may use this connection to solve the problem of PH model reduction. A continuous-time LTI system with state-space realization $(A, B, C)$ is said to be positive, if the off-diagonal elements of $A$ and all the elements of $B, C$ are non-negative [36]. In positive systems, for any non-negative initial state vector and non-negative input, the output and the state trajectory are non-negative for all values of time. The following two theorems give characterizations of PH distributions and positive LTI systems.
Theorem 4: A non-negative probability distribution with continuous pdf \( f(t) \) and rational Laplace transform \( \mathcal{L}[f(t)] \), which is not a point mass at zero is a PH distribution if and only if (i) \( f(t) > 0 \) for all \( t > 0 \) (and \( f(0) \geq 0 \)), and (ii) \( \mathcal{L}[f(t)] \) has a unique pole of maximal real part (possibly with multiplicity greater than one).

It should be noted that general PH distributions may have non-dominant complex poles, but Coxian distributions have only real poles.

Theorem 5: An LTI system with impulse response \( h(t) \) has a positive realization if and only if (i) \( h(t) > 0 \) for all \( t > 0 \) (and \( h(0) \geq 0 \)), and (ii) \( \mathcal{L}[h(t)] \) has a unique pole of maximal real part (possibly with multiplicity greater than one).

From Theorem 4, Theorem 5, and Lemma 1 it follows that the pdf and cdf of a PH distribution are respectively equivalent to the impulse response and the step response of a positive LTI system with state-space realization \((\Pi, \eta, \alpha)\). The similarity between PH distributions and positive systems can be exploited to deal with the problem of PH model reduction by using positivity-preserving model reduction techniques. Since we are interested in minimizing the distance between the pdfs, \( \varepsilon(t) = f(t) - \hat{f}(t) \), the \( \ell_2 \)-norm is an appropriate metric for PH model reduction which is defined as \( \|\varepsilon\|_2^2 = \int_0^\infty |\varepsilon(\tau)|^2 d\tau \). From the Parseval’s relation, minimizing \( \|\varepsilon\|_2 \) is equivalent to minimizing the \( \mathcal{H}_2 \)-norm of the error in the frequency domain, i.e.,

\[
\|\varepsilon\|_2^2 = \|\mathcal{E}(s)\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathcal{E}(j\omega)|^2 d\omega,
\]

where \( \mathcal{E}(s) = \mathcal{L}[f(t) - \hat{f}(t)] \). Minimizing the \( \mathcal{H}_2 \)-norm of a transfer function, however, is a non-convex problem and finding a global minimizer is a hard task [38]; moreover, imposing the constraints of the PH models makes the problem more complex. Then, the PH model reduction problem can be formulated as a \( \gamma \)-suboptimal \( \mathcal{H}_2 \) model reduction defined as follows: Consider an \( m \)-phase PH model \((\Pi, \eta, \alpha)\) with pdf \( f(t) \). For a given \( \gamma > 0 \), find (if it exists) an \( \hat{m} \)-phase PH model \((\hat{\Pi}, \hat{\eta}, \hat{\alpha})\) with pdf \( \hat{f}(t) \), where \( \hat{m} < m \), such that \( \|\mathcal{E}(s)\|_2 < \gamma \).

The positivity-preserving \( \gamma \)-suboptimal \( \mathcal{H}_2 \) model reduction problem has been studied in [39] where the problem is formulated as an LMI feasibility problem. In order to find a reduced-order PH model, one may use the proposed algorithm in [39] by imposing the constraints of PH models as a set of LMIs. The MATLAB toolbox YALMIP with solver SeDuMi can be used to solve the LMIs. In general, however, LMI-based algorithms due to their computational complexity, are not applicable to high dimensional systems. It is, therefore, desired to develop more efficient techniques with low computational complexity for distribution model reduction.
As mentioned earlier, the PH distribution is defined in terms of an underlying Markov chain and due to its parametric constrains (i.e., the constraints on the elements of $(\Pi, \eta, \alpha)$ given in Section II), accurate approximation by PH distribution may result in a very high-order model, especially when the density function has abrupt variations or has minima close to zero. Moreover, there are many distributions with rational Laplace transform that are not phase-type. For example, there is no finite-phase PH model that exactly represents distributions with pdfs $f(t) = e^{-t}(t-1)^2$ and $f(t) = 0.4e^{-t}(1 + t + \sin t)$, because the first one violates condition (i) and the second one violates condition (ii) of Theorem 4. Therefore, due to the restrictions of the PH models, to achieve a desired level of modeling accuracy, the use of PH distribution approach may give a very high-order Markovianized jump process.

In Section V it is shown that in optimal control of S-MJLSs, the cost value and optimal control gains are pdf-dependent. These properties are independent of the sign of the elements of the sub-generator matrix and those of the exit rate vector of the PH models. In modeling of holding-time distributions the primary objective is to capture the behavior of pdfs with sufficiently high accuracy and it is not necessary for the distribution model to have any probabilistic interpretation. Relaxing the sign constraints of PH distributions can considerably reduce the order of the models. Indeed, the PH distribution approach is used as a mathematical tool to compute the optimal control gains and to evaluate the control cost, and it is not necessary to force the transition rates of the model to be non-negative.

The matrix-exponential (ME) distributions are generalizations of PH distributions in which the sign constraints on the elements of $(\Pi, \eta, \alpha)$ are removed [11, §1.7], then unlike the PH distribution they do not necessarily have a probabilistic interpretation. The ME distributions can approximate more complicated distributions at a significantly lower order compared to the PH distributions.

Example 3: Consider $f_a(t) = e^{-t}(t-1)^2, t \geq 0$ with $\mathcal{L}[f_a(t)] = (s^2 + 1)/(s + 1)^3$ as the pdf of the holding time of mode $a$ of a S-MJLS. Since $f_a(1) = 0$, this distribution cannot be exactly realized by a finite-order PH model. However, it can be exactly represented by a third-order ME
distribution with a triple \((\Pi_a, \eta_a, \alpha)\) given by

\[
\begin{bmatrix}
-3 & 2 & 0 \\
-\frac{3}{2} & 0 & \frac{3}{2} \\
-\frac{1}{3} & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
\frac{1}{3}
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\] .

(12)

Figure 5 shows the state transition diagram of the model. Since some transition rates are negative, the model has no stochastic interpretation, but it can be used to compute control gains and the cost value in the optimal control problem. Fitting a third-order PH model using function 'PHFromTrace' [34] to a 50000-sample data set obtained by inverse transform sampling gives a density function with about 30\% fit to the actual pdf, while the above third-order ME model gives a 100\% fit. The ‘fit percent’ is defined in terms of the normalized root mean squared error expressed as a percentage, i.e., FitPercent = 100\((1 - \|f_a - \hat{f}_a\|/\|f_a - c\|)\), where \(f_a(t)\) and \(\hat{f}_a(t)\) are time series of the actual pdf and the estimated pdf, respectively, the constant \(c\) is the arithmetic mean of \(f_a\), and \(\|\cdot\|\) indicates the 2-norm of a vector.

A pseudo-Markov chain is a Markov chain with possibly negative transition rates [40]. Then, we call the above process of holding-time distribution modeling pseudo-Markovianization, as a technique for low-order approximation of non-exponential holding-time distributions.

The fitting techniques for ME distributions, however, are very challenging [41]. The main difficulty is to ensure that the resulting ME representation corresponds to a true distribution model with a non-negative density function. The sign constraints of the PH distribution guarantee the non-negativity of the density function and relaxing these constraints makes the fitting problem a hard task. A number of algorithms have been proposed for ME distribution fitting. Moment matching methods are developed in [42], [43], however, they do not necessarily give a valid ME distribution. The function ‘MEFromMoments’ in MATLAB toolbox Butools [34] is based on
the algorithm in [43] which returns an ME distribution of order $n$ from an given set of $2n - 1$ moments, the density function, however, is not guaranteed to be non-negative. A semi-infinite programming approach is proposed in [41], [44] which requires some approximation to ensure that the result is a valid ME distribution.

In the optimal control problem, we are looking for the lowest-order realization $(\hat{\Pi}, \hat{\eta}, \alpha)$ such that $\hat{f}(t) = \alpha^T \exp(\hat{\Pi}t)\hat{\eta}$ closely approximates the pdf of a given distribution (and hence $\hat{F}(t) = 1 - \alpha^T \exp(\hat{\Pi}t) \mathbb{1}$ closely approximates its cdf). For a distribution model with state-space representation $(\hat{\Pi}, \hat{\eta}, \alpha)$ the following assumption are made: (i) $\hat{\Pi}$ is Hurwitz, (ii) $\hat{\eta} = -\hat{\Pi} \mathbb{1}$, (iii) the starting vector is of the form $\alpha = [1, 0, \ldots, 0]^T$, and (iv) $\hat{f}(t) = \alpha^T \exp(\hat{\Pi}t)\hat{\eta} \geq 0, \forall t \geq 0$. As is shown the following lemma, assumptions (i)-(iii) are not restrictive assumptions. The main difficulty is to ensure that condition (iv) holds true.

**Lemma 8:** Any stable LTI system with a strictly proper rational transfer function $H(s)$ of order $m$ with unit DC gain ($H(0) = 1$) can be represented by a triple $(\Pi, \eta, \alpha)$, where $\alpha = [1, 0, \ldots, 0]^T \in \mathbb{R}^m$, $\eta = -\Pi \mathbb{1} \in \mathbb{R}^m$, and $\Pi \in \mathbb{R}^{m \times m}$ is a Hurwitz matrix. The impulse response of the system is expressed as $h(t) = \alpha^T \exp(\Pi t)\eta$ and the step response is given by $s(t) = \int_0^t h(\tau) d\tau = 1 + \alpha^T \exp(\Pi t)(\Pi^{-1}\eta) = 1 - \alpha^T \exp(\Pi t) \mathbb{1}_m, \ t \geq 0$.

**Proof:** The proof is given in the Appendix.

Considering the connection between LTI systems and ME distributions, we can employ the sophisticated tools and techniques developed for LTI systems to deal with the problems of model fitting and model reduction for non-exponential distributions. For example, one may use the available algorithms for transfer function fitting on time-domain input/output data. In the modeling step, the empirical cdf and pdf can be viewed respectively as the step and impulse responses of an LTI system. In the model reduction step, any model reduction technique that preserves stability and DC gain may be employed; or, one may fit a low-order transfer function to a sample time series of the pdf or cdf of a given full-order model. Some modifications, however, may be needed to make the impulse response of the obtained LTI model non-negative.

The function ‘tfest(data, $n_p$, $n_z$)’ in MATLAB System Identification Toolbox estimates a transfer function with $n_p$ poles and $n_z$ zeros for a given input/output time-domain data set. This function utilizes efficient algorithms for initializing the parameters of the transfer function, then the parameters are updated using a nonlinear least-squares search method. This algorithm does not guarantee the non-negativity of the impulse response. Even for a transfer function that
provides a very high fit percent, the impulse response may become slightly negative over some periods of time. However, using classical filter design techniques, some modifications can be applied to the obtained transfer function to make its impulse response non-negative.

Example 4: Consider a Weibull random variable with pdf \( f(t) = 4t^3 \exp(-t^4), \quad t \geq 0 \). Using MATLAB function ‘PHFromTrace’ [34], fitting a six-phase PH distribution to a 50000-sample data set obtained by inverse transform sampling gives a density function with 60% fit to the actual pdf. Now, we fit a sixth-order transfer function to sample time series of \( f(t) \) as the impulse response of the system. The MATLAB function ‘tfest’ gives a sixth-order transfer function whose impulse response has a 99% fit to actual pdf \( f(t) \); however, this impulse response is slightly negative at the beginning and the tail behavior is oscillating around zero. To obtain a sixth-order model that corresponds to a valid ME distribution, we first fit a fifth-order transfer function with two zeros. Since \( f(0) = 0 \), the relative degree of the transfer function is set to 3 to have a smooth initial response. The dominant poles of resulting transfer function are complex with real part \(-1.72\), then to ensure that the oscillations in the tail behavior of the impulse response decay faster than the dominant terms, we add a real negative pole and zero to the transfer function, such that the transfer function has a unique pole of maximal real part. By multiplying the transfer function by \((s + 1.7)/(s + 1.5)\) we obtain the following transfer function. The DC gain is set to one by scaling the transfer function.

\[
H(s) = \frac{24.24s^3 - 182.3s^2 + 1411s + 3045}{s^8 + 12.02s^6 + 117.8s^4 + 642.3s^3 + 2264s^2 + 4326s + 3045}.
\]

The impulse response of \( H(s) \) is non-negative and has a 90% fit to \( f(t) \). Figure 6 shows the actual pdf, the pdf of the 6-phase PH distribution, and the impulse response of \( H(s) \).

Therefore, given a distribution model or empirical data of holding times of a S-MJLS, one may use the efficient algorithms developed for transfer function estimation to find a reduced-order model for holding-time distributions.

VII. NUMERICAL SIMULATION

Power systems have nonlinear dynamics and their operating conditions vary with the load level. A typical control design procedure is to partition the load range into several intervals, each representing a mode of operation, then an LTI approximation to the plant associated with each mode is obtained [45]. For randomly varying loads, a S-MJLS is well suited for describing the behavior of such systems.
Fig. 6. Approximation of the pdf of a Weibull random variable. The solid curve is the true pdf and the two dashed curves are its approximations by fitting a six-phase PH distribution (60% fit) and fitting a sixth-order transfer function (ME distribution) (90% fit).

Let us consider the load process of the ship engine in [46, §8.2] which is modeled by a semi-Markov process. The load range [0, 3500] kW in partitioned into eight intervals [0, 250), [250, 270), [270, 280), [280, 300), [300, 350), [350, 560), [560, 1270), and [1270, 3500] kW each representing an operational mode of the engine. Figure 7 shows the state transition diagram of the load process. The embedded Markov chain of the semi-Markov process is given by

Fig. 7. State transition diagram of the semi-Markov model of a ship engine load process.
which is obtained from empirical data as $p_{ij} = n_{ij}/\sum_j n_{ij}$, where $n_{ij}$ denotes the number of direct jumps from mode $i$ to mode $j$, $i \neq j$. The holding-time of each mode is a Weibull distribution whose parameters are obtained through statistical analysis of the data. For mode $i \in \{1, 2, \ldots, 8\}$, the cdf of the conditional holding-times are $F_{i j}(t) = F_i(t)$, $\forall j \in N_i^+$, where $F_i(t)$ is a Weibull distribution with shape parameter $\kappa_i$ and scale parameter $\lambda_i$ given in Table I.

Using the MATLAB function ‘tfest’ we fit a transfer function of order $m_i$ to the holding-time pdf of each mode $i$ and evaluate the fitting quality. Some modifications are applied as in Example 4 to make the impulse responses non-negative. The normalized root mean squared errors between the actual and identified model for both pdf and cdf are listed in Table I.

**TABLE I**

Fitting a transfer function of order $m_i$ with unit DC gain to Weibull distributions with parameters $\kappa_i$ and $\lambda_i$. The normalized root mean squared error between the actual and identified model is given for both pdf and cdf to assess the fitting quality.

| mode | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| $\kappa_i$ | 2.5 | 1.4 | 1.2 | 1.0 | 1.8 | 2.2 | 2.0 | 1.2 |
| $\lambda_i$ | 11.25 | 8.99 | 12.88 | 11.34 | 20.56 | 9.01 | 8.52 | 15.54 |
| $m_i$ | 4 | 3 | 3 | 1 | 3 | 4 | 4 | 3 |
| pdf Fit% | 98.35 | 96.00 | 98.00 | 100 | 98.79 | 92.45 | 96.49 | 98.05 |
| cdf Fit% | 99.64 | 96.00 | 99.03 | 100 | 99.19 | 95.03 | 96.89 | 99.32 |

Since for each mode the conditional holding times are identically distributed, then we can transform the given eight-state semi-Markov process into a pseudo-Markov chain with 25 states. For simplicity of presentation, we assign a scalar dynamic $(A_i, B_i)$ to each mode of the process
and compute the optimal control gains $K_i^*(t)$, $i \in \{1, 2, \ldots, 8\}$, and the corresponding cost $J(K^*)$. The system parameters are $(A_1, B_1) = (0.5, -10)$, $(A_2, B_2) = (20, 0.2)$, $(A_3, B_3) = (10, 1)$, $(A_4, B_4) = (5, -1)$, $(A_5, B_5) = (8, 2)$, $(A_6, B_6) = (4, 2)$, $(A_7, B_7) = (3, 1)$, $(A_8, B_8) = (5, -2)$, and weighting matrices are $Q_i = 100I$, $R_i = 100$, $S_i = 0$, for $i = 1, 2, \ldots, 8$, and $t_f = 100$. We assume that the system is initially in mode 1 and the initial state is $x_0 = 1$. By solving (5)-(7) for the 25-state pseudo-Markov chain, we obtain $J(K^*) = 11.63$. Now, let's assume that the pdf of the holding-time distributions of the semi-Markov process are unknown and only the mean of each holding time is available. Then, each holding-time is assumed to be exponentially distributed with the same statistical mean as that of the actual Weibull distribution. That is, an eight-state Markov chain is available as an approximate nominal model of the jump process for control design. Let optimal control gains designed based on the nominal jump process be denoted by $\hat{K}_i(t)$. By apply the control law $u(t) = \hat{K}_i(t)x(t)$ to the actual S-MJLS, the cost value is $J(\hat{K}) = 42.79$, i.e., the error in the modeling of the jump process leads to 268% relative change in the cost value. This example demonstrates the importance of accurate modeling of the jump process for control of S-MJLSs.

VIII. Conclusion

This paper studies finite-horizon optimal control of S-MJLSs and investigates modeling of the underlying semi-Markov jump process of the system for control design and analysis. The effect of the quality of the jump process model on the control performance is explored and the trade-offs between control quality and computational cost are examined. It is shown that the dependency of the optimal gains and the control cost to the holding-time distributions are through their pdfs. Therefore, as long as the pdfs remain invariant, any realization of the holding-time distributions yields the same control gains and cost value. To design an optimal linear feedback control law for a S-MJLS, the underlying jump process is transformed into a suitable model that captures the characteristics of holding-time distributions with high enough accuracy. This model does not need to have any probabilistic interpretation, as it just serves as a computational tool to find the optimal gains and to evaluate the control cost. The proposed modeling technique provides flexibility to arrive at low-order, yet accurate models for holding-time distributions. Therefore, the computational cost associated with control design can be significantly reduced without sacrificing the control quality.
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Proof of Lemma 3: The proof follows by using the Peano-Baker series or by showing that $\Phi_M(t, \tau)$ satisfies the equation $\partial \Phi_M(t, \tau)/\partial t = M(t)\Phi_M(t, \tau)$ with $\Phi_M(\tau, \tau) = I$, for any $t, \tau$. In the proof of (ii), the invertibility property of the state transition matrix is to be used, i.e., $\Phi_A^{-1}(t, \tau) = \Phi_A(\tau, t)$, for any $t, \tau$. □

Proof of Lemma 4: From Lemma 3(ii) with $M_1 = A \otimes I_n$ and $M_2(t) = I_m \otimes B(t)$, we have, $\Phi_M(t, \tau) = \Phi_{M_1}(t, 0)\Phi_Z(t, \tau)\Phi_{M_1}(0, \tau)$, where $Z(t) = \Phi_{M_1}(0, t)M_2(t)\Phi_{M_1}(t, 0)$. Since $M_1$ is a constant matrix, then

$$\Phi_{M_1}(t, \tau) = \exp(M_1(t - \tau)) \overset{*}{=} \exp(A(t - \tau) \otimes I_n)$$

$$\overset{**}{=} \exp(A(t - \tau)) \otimes I_n,$$

where the identities ($*$) and ($**$) follow from Lemma 2(ii) and Lemma 2(iv), respectively. Then, we can write

$$Z(t) = \Phi_{M_1}(0, t)M_2(t)\Phi_{M_1}(t, 0)$$

$$= (\exp(-At) \otimes I_n)(I_m \otimes B)(\exp(At) \otimes I_n)$$

$$\overset{*}{=} (\exp(-At) \otimes I_n)(\exp(At) \otimes I_n)(I_m \otimes B)$$

$$\overset{**}{=} (I_m \otimes I_n)(I_m \otimes B) = I_m \otimes B,$$

where the identity ($*$) follows from Lemma 2(iii) and the identity ($**$) follows from Lemma 2(i) and the fact that for square matrices $A_1, A_2$, $\exp(A_1)\exp(A_2) = \exp(A_1 + A_2)$, if and only if $A_1$ and $A_2$ commute. Since $Z(t) = I_m \otimes B = \text{diag}(B, \ldots, B)$ is a block diagonal matrix, then
from Lemma 3(i), \( \Phi_Z(t, \tau) = \text{diag}(\Phi_B(t, \tau), \ldots, \Phi_B(t, \tau)) = I_m \otimes \Phi_B(t, \tau) \). Then,
\[
\Phi_M(t, \tau) = \Phi_{M_1}(t, 0)\Phi_Z(t, \tau)\Phi_{M_1}(0, \tau)
\]
\[
= (\exp(At) \otimes I_n)(I_m \otimes \Phi_B(t, \tau))(\exp(-A\tau) \otimes I_n)
\]
\[
= (\exp(At) \otimes I_n)(\exp(-A\tau) \otimes I_n)(I_m \otimes \Phi_B(t, \tau))
\]
\[
= (\exp(A(t - \tau)) \otimes I_n)(I_m \otimes \Phi_B(t, \tau))
\]
\[
= \exp(A(t - \tau)) \otimes \Phi_B(t, \tau).
\]
This completes the proof. \( \square \)

**Proof of Theorem 1.** Following the matrix minimum principle [24], the Hamiltonian function \( H \) is defined as
\[
H(t) = \sum_{i \in \mathcal{V}} \text{tr}[L_i(t)X_i(t)]dt
\]
\[
+ \text{tr}[(\dot{A}_i(t)X_i(t) + X_i(t)\dot{A}_i(t) + \sum_{j \in \mathcal{V}} \pi_{ji}X_j(t))\Lambda_i(t)]
\]
Then, for optimal gains \( K_i(t) \), the following condition must hold
\[
-\dot{\Lambda}_i(t) = \frac{\partial H(t)}{\partial X_i(t)}, \quad \dot{X}_i(t) = \frac{\partial H(t)}{\partial \Lambda_i(t)}, \quad \frac{\partial H(t)}{\partial K_i(t)} = 0,
\]
for any \( i \in \mathcal{V} \). The first two equations conditions give (6) and (7), and the third equation is because the control gains are assumed to be unconstrained. Then, since \( K_i(t) = \Gamma_k(t), \forall i \in \mathcal{C}_k \), the third condition gives (5). This completes the proof. \( \square \)

**Proof of Lemma 6.** With \( u(t) = K_i(t)x(t) \), the cost function (3) can be written as
\[
J = \mathbb{E}\left[ \int_0^{t_f} x^\top(t)L(r(t), t)x(t)dt + x^\top(t_f)S(r(t_f))x(t_f) \right]
\]
\[
= \mathbb{E}\left[ \int_0^{t_f} \text{tr}[x^\top(t)L(r(t), t)x(t)]dt + \text{tr}[x^\top(t_F)S(r(t_f))x(t_f)] \right]
\]
\[
= \mathbb{E}\left[ \int_0^{t_f} \text{tr}[L(r(t), t)x(t)x^\top(t)]dt + \text{tr}[S(r(t_f))x(t_f)x^\top(t_f)] \right]
\]
\[
= \sum_{i \in \mathcal{V}} \left( \int_0^{t_f} \text{tr}[L_i(t)x_i(t)]dt + \text{tr}[S_iX_i(t_f)] \right),
\]
where the second equality is because the cost functional is scalar and the trace of a scalar is itself, the third equality is obtained from the cyclic permutation invariance property of matrix trace, and the forth equality is due to the linearity of the expectation operator and that \( L(r(t), t)x(t)x^\top(t) = \frac{\partial}{\partial X_i(t)} \)
$\sum_{i \in V} L_i(t)x(t)x^T(t)\delta_i(t)$ and $X_i(t) = \mathbb{E}[x(t)x^T(t)\delta_i(t)]$, where $\delta_i(t) \in \{0, 1\}$ is the mode indicator. Now, by post-multiplying both sides of (6) by $X_i(t)$ and pre-multiplying both sides of (7) by $\Lambda_i(t)$, and then adding them up we obtain

$$L_iX_i = -\frac{d}{dt}(\Lambda_iX_i) + \Lambda_iX_i\bar{A}_i^T - \bar{A}_i \Lambda_iX_i + \sum_{j \in V}(\pi_{ji}\Lambda_iX_j - \pi_{ij}\Lambda_jX_i).$$

Since $\text{tr}[\Lambda_iX_i\bar{A}_i^T] = \text{tr}[\bar{A}_i^T \Lambda_iX_i]$ and $\sum_{i \in V}\sum_{j \in V}(\pi_{ji}\Lambda_iX_j - \pi_{ij}\Lambda_jX_i) = 0$, then

$$\sum_{i \in V} \int_0^{t_i} \text{tr}[L_i(t)X_i(t)]dt = \sum_{i \in V} \left(\text{tr}[\Lambda_i(0)X_i(0)] - \text{tr}[\Lambda_i(t_i)X_i(t_i)]\right),$$

where $\Lambda_i(t_i) = S_i$. Then, the cost can be expressed as in (8). This completes the proof. \hfill $\square$

Proof of Theorem 2 Since all the $m$ phases of the Cox model share the same dynamic $(A_1(t), B_1(t))$, the same control gain $K_1(t)$, and the same weighting matrices $Q_1(t), R_1(t)$, and $S_1$, then from (6) we have

$$\dot{\Lambda}_1 = -\bar{A}_1^T \Lambda_1 - \Lambda_1 \bar{A}_1 - \pi_{11}\Lambda_1 - \pi_{12}\Lambda_2 - \pi_{1(k)}\Lambda_k - L1$$

$$\dot{\Lambda}_2 = -\bar{A}_1^T \Lambda_2 - \Lambda_2 \bar{A}_1 - \pi_{22}\Lambda_2 - \pi_{23}\Lambda_3 - \pi_{2(k)}\Lambda_k - L1$$

$$\vdots$$

$$\dot{\Lambda}_m = -\bar{A}_1^T \Lambda_m - \Lambda_m \bar{A}_1 - \pi_{mm}\Lambda_m - \pi_{mk}\Lambda_k - L1$$

where $\Lambda_i(t_i) = S_i$, for $i = 1, 2, \ldots, m$. The above equation can be written in a matrix form as

$$\begin{bmatrix}
\dot{\Lambda}_1 \\
\dot{\Lambda}_2 \\
\vdots \\
\dot{\Lambda}_m
\end{bmatrix} = -
\begin{bmatrix}
\bar{A}_1^T & & & \\
& \bar{A}_1^T & & \\
& & \ddots & \\
& & & \bar{A}_1^T
\end{bmatrix}
\begin{bmatrix}
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_m
\end{bmatrix} -
\begin{bmatrix}
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_m
\end{bmatrix}
\begin{bmatrix}
\bar{A}_1 \\
\bar{A}_1 \\
\vdots \\
\bar{A}_1
\end{bmatrix} -
\begin{bmatrix}
\pi_{11}I_n & \pi_{12}I_n \\
\pi_{22}I_n & \pi_{23}I_n \\
\vdots & \ddots & \ddots \\
\pi_{(m-1)m}I_n & \pi_{mm}I_n
\end{bmatrix}
\begin{bmatrix}
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_m
\end{bmatrix} -
\begin{bmatrix}
\pi_{1k}I_n \\
\pi_{2k}I_n \\
\vdots \\
\pi_{mk}I_n
\end{bmatrix}
\begin{bmatrix}
L_1 \\
L_1 \\
\vdots \\
L_1
\end{bmatrix}.$$

This equation can be expressed in terms of the Kronecker product as follows: Let $\Lambda_a = [\Lambda_1^T, \Lambda_2^T, \ldots, \Lambda_m^T]^T$, then

$$\dot{\Lambda}_a = - (I_m \otimes \bar{A}_1^T)\Lambda_a - \Lambda_a \bar{A}_1 - (\Pi_a \otimes I_n)\Lambda_a - (\eta_a \otimes I_n)\Lambda_k - (I_m \otimes L_1),$$
with terminal condition $\Lambda_a(t_f) = I_m \otimes S_1$, which can be rearranged to

$$
\dot{A}_a(t) = -\left( (\Pi_a \otimes I_n) + (I_m \otimes \tilde{A}^T_1(t)) \right) \Lambda_a(t) - \Lambda_a(t) \tilde{A}_1(t) \\
- \left( (\eta_a \otimes I_n) A_k(t) + (I_m \otimes L_1(t)) \right), \\
\Lambda_a(t_f) = I_m \otimes S_1.
$$

From (13) and Lemma 5 with $M(t) = -((\Pi_a \otimes I_n) + (I_m \otimes \tilde{A}^T_1(t)))$, $N(t) = -\tilde{A}_1(t)$, and $R(t) = -((\eta_a \otimes I_n) A_k(t) + (I_m \otimes L_1(t)))$, we obtain

$$
\Lambda_a(t) = \Phi_M(t, t_f) (I_m \otimes S_1) \Phi_{N^T}(t, t_t) + \int_{t_t}^{t_f} \Phi_M(t, \tau) R(\tau) \Phi_{N^T}(t, \tau) d\tau, \quad \forall t.
$$

Using the property of the state transition matrix that $\Phi_{-A^T}(t, \tau) = \Phi_{A^T}(t, \tau)$, $\forall t, \tau$ [21, §1.1], we have

$$
\Phi_{N^T}(t, \tau) = \Phi_{-A^T_1}(t, \tau) = \Phi_{\tilde{A}_1}(t, \tau).
$$

From Lemma 4 the state transition matrix of $M(t) = (-\Pi_a) \otimes I_n + I_m \otimes (-\tilde{A}_1^T(t))$ is given by

$$
\Phi_M(t, \tau) = \exp((\Pi_a(t - \tau)) \otimes \Phi_{-A_1^T}(t, \tau) \\
= \exp((\Pi_a(t - \tau)) \otimes \Phi_{\tilde{A}_1}(t, \tau).
$$

The co-state variable associated with the first phase of the Cox model can be expressed as

$$
\Lambda_1(t) = (\alpha^T \otimes I_n) \Lambda_a(t), \quad \text{where } \alpha = [1, 0, \ldots, 0]^T \in \mathbb{R}^m,
$$

then from (14) and (15) we have

$$
\Lambda_1(t) = (\alpha^T \otimes I_n) \Phi_M(t, t_f)(I_m \otimes S_1) \Phi_{\tilde{A}_1}(t_f, t) \\
- \int_{t_t}^{t_f} (\alpha^T \otimes I_n) \Phi_M(t, \tau) R(\tau) \Phi_{\tilde{A}_1}(\tau, t) d\tau.
$$

From (16), we can write

$$
(\alpha^T \otimes I_n) \Phi_M(t, t_f)(I_m \otimes S_1) = (\alpha^T \otimes I_n)(\exp((\Pi_a(t_f - t)) \otimes \Phi_{\tilde{A}_1^T}(t_f, t)))(I_m \otimes S_1) \\
= (\alpha^T \exp((\Pi_a(t_f - t))) I_m) \otimes (\Phi_{\tilde{A}_1^T}(t_f, t)) S_1 \\
= \bar{F}_a(t_f - t) \otimes (\Phi_{\tilde{A}_1^T}(t_f, t)) S_1 \\
= \bar{F}_a(t_f - t) \Phi_{\tilde{A}_1^T}(t_f, t) S_1.
$$
where the identity (*) follows from Lemma 1. Also, we have

\[(\alpha^T \otimes I_n)\Phi_M(t, \tau)R(\tau) = -(\alpha^T \otimes I_n)(\exp(\Pi_a(\tau - t)) \otimes \Phi_{A_1}^T(\tau, t))R(\tau)\]

\[= -\left(\alpha^T \exp(\Pi_a(\tau - t)) \otimes \Phi_{A_1}^T(\tau, t)\right) \left(\eta_a \otimes I_n\right)\Lambda_k(\tau) + (I_m \otimes L_1(\tau))\]

\[= -\left(\alpha^T \exp(\Pi_a(\tau - t))\eta_a \otimes \Phi_{A_1}^T(\tau, t)\right)\Lambda_k(\tau)\]

\[= -\left(\alpha^T \exp(\Pi_a(\tau - t))\eta_a \otimes \Phi_{A_1}^T(\tau, t)\right)\Lambda_k(\tau) - \left(F_a(\tau - t) \otimes (\Phi_{A_1}^T(\tau, t)L_1(\tau))\right)\]

\[= -f_a(\tau - t)\Phi_{A_1}^T(\tau, t)\Lambda_k(\tau) - \bar{F}_a(\tau - t)\Phi_{A_1}^T(\tau, t)L_1(\tau).\]

Therefore, from (17), (18), and (19) we obtain (9). This completes the proof.

**Proof of Lemma 7:** Consider the Cox model described in Theorem 2. Since all phases of the Cox model share the same dynamic, control gain, and weighting matrices, then from (7) we have

\[\dot{X}_i(t) = \bar{A}_1(t)X_i(t) + X_i(t)\bar{A}_1^T(t) + \sum_{j=1}^m \pi_{ji}X_j(t),\]

with initial condition \(X_1(0) = \mu_a(0)x_0x_0^T\) and \(X_i(0) = 0\), for \(i = 2, \ldots, m\), where \(\mu_a(0)\) is the probability that the system is in mode \(a\) at \(t = 0\), this is because in each Cox model the initial phase is concentrated in the first phase. We can write the above system of equations in a matrix form as

\[
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\vdots \\
\dot{X}_m
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_1 \\
\vdots \\
\bar{A}_1
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_m
\end{bmatrix} +
\begin{bmatrix}
\pi_{11}I_n \\
\pi_{12}I_n \\
\vdots \\
\pi_{(m-1)m}I_n
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_m
\end{bmatrix} +
\begin{bmatrix}
I_n \\
0 \\
\vdots \\
0
\end{bmatrix}W,
\]

\(W(t) = (\eta_0^T \otimes I_n)X_0(t)\) and \(X_0(t)\) is the covariance variable of the external state 0 and \(\eta_0\) denotes the exit rate vector from state 0 to mode \(a\). This equation can be expressed in terms of
the Kronecker product as

\[ \dot{X}_a = (I_m \otimes \bar{A}_1)X_a + X_a\bar{A}_1^\top + (\Pi_a^\top \otimes I_n)X_a + (\alpha \otimes I_n)W, \]

with initial condition \( X_a(0) = (\alpha \otimes I_n)X_1(0) \), where \( X_a = [X_1^\top, \ldots, X_m^\top]^\top \). From Lemma 5 with \( M(t) = \Pi_a^\top \otimes I_n + I_m \otimes \bar{A}_1(t) \), \( N(t) = \bar{A}_1(t) \), and \( R(t) = (\alpha \otimes I_n)W(t) \), we obtain

\[ X_a(t) = \Phi_M(t, 0)(\alpha \otimes I_n)X_1(0)\Phi_{\bar{A}_1}^\top(t, 0) + \int_0^t \Phi_M(t, \tau)(\alpha \otimes I_n)W(\tau)\Phi_{\bar{A}_1}^\top(t, \tau)d\tau. \]

From Lemma 4, \( \Phi_M(t, \tau) = \exp((\Pi_a^\top t - \tau)) \otimes \Phi_{\bar{A}_1}(t, \tau) \), then using the properties of the Kronecker product given in Lemma 2 we can write

\[
X_a(t) = (\exp(\Pi_a^\top t))\otimes \Phi_{\bar{A}_1}(t, 0)X_1(0)\Phi_{\bar{A}_1}^\top(t, 0) \\
+ \int_0^t (\exp(\Pi_a^\top (t - \tau)))\otimes \Phi_{\bar{A}_1}(t, \tau)W(\tau)\Phi_{\bar{A}_1}^\top(t, \tau)d\tau.
\]

Then, we have

\[
(1^\top \otimes I_n)X_a(t) = (1^\top\exp(\Pi_a^\top t))\otimes \Phi_{\bar{A}_1}(t, 0)X_1(0)\Phi_{\bar{A}_1}^\top(t, 0) \\
+ \int_0^t (1^\top\exp(\Pi_a^\top (t - \tau)))\otimes \Phi_{\bar{A}_1}(t, \tau)W(\tau)\Phi_{\bar{A}_1}^\top(t, \tau)d\tau
\]

and

\[
(\eta_a^\top \otimes I_n)X_a(t) = (\eta_a^\top\exp(\Pi_a^\top t))\otimes \Phi_{\bar{A}_1}(t, 0)X_1(0)\Phi_{\bar{A}_1}^\top(t, 0) \\
+ \int_0^t (\eta_a^\top\exp(\Pi_a^\top (t - \tau)))\otimes \Phi_{\bar{A}_1}(t, \tau)W(\tau)\Phi_{\bar{A}_1}^\top(t, \tau)d\tau.
\]

Since the pdf and ccdf are scalar functions, then from Lemma 1 \( f_a(t) = \alpha^\top\exp(\Pi_a^\top t)\eta_a = \eta_a^\top\exp(\Pi_a^\top t)\alpha \) and \( \bar{F}_a(t) = \alpha^\top\exp(\Pi_a^\top t)1_m = 1^\top\exp(\Pi_a^\top t)\alpha \). Therefore, from (21) and (22) we obtain

\[
(1^\top \otimes I_n)X_a(t) = \bar{F}_a(t)\Phi_{\bar{A}_1}(t, 0)X_1(0)\Phi_{\bar{A}_1}^\top(t, 0) \\
+ \int_0^t \bar{F}_a(t - \tau)\Phi_{\bar{A}_1}(t, \tau)W(\tau)\Phi_{\bar{A}_1}^\top(t, \tau)d\tau
\]

and

\[
(\eta_a^\top \otimes I_n)X_a(t) = f_a(t)\Phi_{\bar{A}_1}(t, 0)X_1(0)\Phi_{\bar{A}_1}^\top(t, 0) \\
+ \int_0^t f_a(t - \tau)\Phi_{\bar{A}_1}(t, \tau)W(\tau)\Phi_{\bar{A}_1}^\top(t, \tau)d\tau.
\]

As shown in Figure 4, the state labeled 0 is an in-neighbor of mode \( a \) and \( \eta_0 \) is the transition rate from state 0 to state 1. Then from (24). If instead of a single state, a Cox model with realization
From Theorem 2 and equations (23) and (24), \( \Lambda \) completes the proof.

\[
(\Pi_b, \eta_b, \alpha), \text{ representing mode } b, \text{ is in the in-neighborhood of mode } a, \text{ then the matrix } W(t) \text{ in (23) and (24) can be expressed as } W(t) = (\eta_b^\top \otimes I_n)X_b(t), \text{ where } X_b \text{ is the vector of all covariance variables of the phases of the Cox model of mode } b \text{ and } \eta_b \text{ is the corresponding exit rate vector. The pdf-dependency of } W(t) \text{ is clear as it satisfies an equation similar to (24). It is obvious that } (I_n^\top \otimes I_n)X_a(t) = \sum_{i=1}^m X_i(t) \text{ which is pdf-dependent as shown above.}

For the Cox model described in Theorem 2, the co-state and covariance equations are given by

\[
\dot{\Lambda}_i(t) = -\bar{A}_i^\top(t)\Lambda_i(t) - \Lambda_i(t)\bar{A}_i(t) - \sum_{j \in V} \pi_{ij}\Lambda_j(t) - L_1(t), \quad (25a)
\]

\[
\dot{X}_i(t) = \bar{A}_i(t)X_i(t) + X_i(t)\bar{A}_i^\top(t) + \sum_{j \in V} \pi_{ji}X_j(t), \quad (25b)
\]

where \( \bar{A}_i(t) = A_i(t) + B_i(t)K_1(t) \) and \( L_1(t) = Q_i(t) + K_i^\top(t)R_1(t)K_1(t), \) \( \Lambda_i(t) = S_1, \) \( i = 1, \ldots, m, \) and \( X_1(0) = \mu_0(0)x_0x_0^\top \) and \( X_i(0) = 0, \) for \( i = 2, \ldots, m. \) By post-multiplying both sides of (25a) by \( X_i(t) \) and pre-multiplying both sides of (25b) by \( \Lambda_i(t), \) and then adding them up we obtain

\[
\dot{A}_iX_i + \Lambda_i\dot{X}_i = -\bar{A}_i^\top\Lambda_iX_i + \Lambda_iX_i\bar{A}_i^\top + \Lambda_i \left( \sum_{j \in V} \pi_{ji}X_j \right) - \left( \sum_{j \in V} \pi_{ij}\Lambda_j \right) X_i - L_1X_i, \quad (26)
\]

for \( i = 1, 2, \ldots, m. \) Let \( Z = \sum_{i=1}^m \Lambda_iX_i, \) then adding up the equations in (26) we can write

\[
\dot{Z} = -\bar{A}_1^\top Z + Z\bar{A}_1^\top + \sum_{i=1}^m \left[ \Lambda_i \left( \sum_{j \in V} \pi_{ji}X_j \right) - \left( \sum_{j \in V} \pi_{ij}\Lambda_j \right) X_i \right] - L_1 \left( \sum_{i=1}^m X_i \right). \quad (27)
\]

It is easy to verify that the third term in the right-hand side of (27) can be expressed as

\[
\sum_{i=1}^m \left[ \Lambda_i \left( \sum_{j \in V} \pi_{ji}X_j \right) - \left( \sum_{j \in V} \pi_{ij}\Lambda_j \right) X_i \right] = \Lambda_1W - (\eta_a^\top \otimes \Lambda_k)X_a,
\]

where \( W(t) = (\eta_0^\top \otimes I_n)X_0(t). \) Then

\[
\dot{Z}(t) = -\bar{A}_1^\top(t)Z(t) + Z(t)\bar{A}_1^\top(t) + \Lambda_1(t)W(t)
\]

\[
- (\eta_a^\top \otimes \Lambda_k(t))X_a(t) + L_1(t)(I_n^\top \otimes I_n)X_a(t).
\]

From Theorem 2 and equations (23) and (24), \( \Lambda_1(t), \) \( \Lambda_k(t), \) \( (I_n^\top \otimes I_n)X_a(t), \) and \( W(t) = (\eta_0^\top \otimes I_n)X_0(t) \) are pdf-dependent. Also, by pre-multiplying both sides of (20) by \( (\eta_a^\top \otimes \Lambda_k(t)), \) it follows that \( (\eta_a^\top \otimes \Lambda_k(t))X_a(t) \) and hence \( Z(t) = \sum_{i=1}^m \Lambda_i(t)X_i(t) \) are pdf-dependent. This completes the proof. \( \square \)
**Proof of Theorem 3**: The proof directly follows from Lemma 7. Consider the model shown in Figure 4 and let $K_1(t)$ be the optimal control gain of mode $a$. From (5), we have

$$R_1(t)K_1(t)\sum_{i=1}^{m} X_i(t) + B_1^\top \sum_{i=1}^{m} (\Lambda_i(t) X_i(t)) = 0.$$  \hfill (28)

From Lemma 7, the matrices $\sum_{i=1}^{m} X_i$ and $\sum_{i=1}^{m} \Lambda_i X_i$ are pdf-dependent, therefore the optimal control gains are pdf-dependent. This completes the proof. \hfill \Box

**Proof of Lemma 8**: It is well known that any strictly proper rational transfer function

$$H(s) = \frac{b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{s^m + a_1 s^{m-1} + \ldots + a_{m-1} s + a_m}$$

can be represented by a triple $(\Pi_c, \eta_c, \alpha)$ in the following canonical form:

$$\Pi_c = \begin{bmatrix} -a_1 & \vdots & 0 \\ -a_2 & I_{m-1} & \vdots \\ \vdots & \ddots & \vdots \\ -a_m & 0 & \end{bmatrix}, \quad \eta_c = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \alpha = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$ 

If the $H(0) = 1$, i.e., $a_m = b_m \neq 0$, then there always exists a similarity transformation $T$ such that $H(s) = \alpha^\top (sI - \Pi)^{-1} \eta$, where $(\Pi, \eta, \alpha) = (T^{-1} \Pi_c T, T^{-1} \eta_c, T^\top \alpha)$ and $\eta = -\Pi \eta_c$. An example of such a transformation matrix is a unit lower triangular matrix $T = [t_{ij}]$ with $t_{ii} = 1$, $\forall i$, and $t_{i1} = a_{i-1} - b_{i-1} - 1$, for $i = 2, \ldots, m$, and all other elements are zero. The expression for the step response follows from the properties of matrix exponential function, that for any invertible square matrix $\Pi$, we have $\int_0^t \exp(\Pi \tau) d\tau = (\exp(\Pi \tau) - I) \Pi^{-1}$. This completes the proof. \hfill \Box