Global and local aspects of spectral actions

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Abstract

The principal object in noncommutative geometry is the spectral triple consisting of an algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and a Dirac operator $D$. Field theories are incorporated in this approach by the spectral action principle, which sets the field theory action to $\text{Tr} f \left( \frac{D^2}{\Lambda^2} \right)$, where $f$ is a real function such that the trace exists and $\Lambda$ is a cutoff scale. In the low-energy (weak-field) limit, the spectral action reproduces reasonably well the known physics including the standard model. However, not much is known about the spectral action beyond the low-energy approximation. In this paper, after an extensive introduction to spectral triples and spectral actions, we study various expansions of the spectral actions (exemplified by the heat kernel). We derive the convergence criteria. For a commutative spectral triple, we compute the heat kernel on the torus up to the second order in gauge connection and consider limiting cases.

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1. Introduction

Spectral functions of (pseudo)differential operators have been used in mathematical physics for a long time. For example, the heat kernel was introduced in the context of quantum physics by Fock [33] already in 1930s. The zeta regularization, first suggested by Dowker and Critchley [26] and then developed by Hawking [44], is now the most advanced regularization technique for a quantum field theory on curved or otherwise complicated backgrounds. A good overview of the spectral zeta functions can be found in [76].

It is widely known that geometry of a manifold is intimately connected to the spectrum of natural differential operators. A proposal by Connes goes further [21, 42]. In his approach, geometry of $M$ is defined by a differential operator, or, more precisely, by a triple consisting
of an algebra $\mathcal{A}$ that plays the role of an algebra of functions on $M$, of a Hilbert space $\mathcal{H}$ which is an analog of the space of square integrable spinors and of an unbounded operator $D$ which corresponds to a Dirac operator. Since the algebra $\mathcal{A}$ is not required to be commutative, one naturally incorporates noncommutative geometries in this approach.

Moreover, even the field theory actions in noncommutative geometry [9] are defined through a trace of a function of $D$. Noncommutative geometry and noncommutative field theory share a common technique with quantum field theory. This technique is the spectral geometry (see monographs [39, 41, 27, 53, 72]). This unity between different branches of science goes far beyond the pure technical level. In a sense, noncommutative geometry is always quantum.

The algebra $\mathcal{A}$ need not be noncommutative. A lot of efforts was spent on analyzing commutative and almost commutative spectral triples in the context of noncommutative geometry, see [24] for an overview. It has been realized that at scales much smaller than a certain cutoff scale $\Lambda$, the spectral action is represented by an expansion, with all terms being integrals of local expressions essentially coinciding with the heat kernel coefficients. Among those terms, one can find the Einstein–Hilbert action, the Yang–Mills action, the Higgs potential and all other ingredients of the standard model. Even the coefficients come out correctly, so that one can speak now about checking phenomenological consequences of the spectral action principle. We can say therefore that local aspects of (almost) commutative spectral actions are quite well understood. The study of global aspects, including nonlocal terms, rapidly varying fields, etc, has been initiated rather recently. These aspects are the main subject of this paper.

We start below with a long pedagogical introduction to spectral triples (section 2) and spectral actions (section 3). We shall introduce the spectral dimension, the Wodzicki residue, the noncommutative integral and other useful notions. We describe in detail the large-$\Lambda$ expansion of the spectral action, which is essentially a generalization of the heat kernel expansion. For the readers who do not want to go deep into mathematics, we derive again this expansion in the commutative case by rather elementary methods in section 3.4. This type of expansion assumes that the fields are small and slowly varying, i.e. they are weak-field approximations. A few references on physical applications of spectral action are given in section 3.5, while the difference between the action and its asymptotic is recalled is section 3.6. Section 4 is devoted to the convergence of the Dyson–Phillips (also called Duhamel) expansion, i.e. when a perturbation of a generator of the heat kernel gives rise, taking its trace, to an expansion series in terms of the perturbation, see (14).

Various approximations to the spectral actions are based on corresponding expansions of the heat kernel. It is interesting therefore to construct an expansion of the heat kernel in the fields (potential, curvatures, etc) without assuming that they vary slowly. On the plain $\mathbb{R}^d$ this was done long ago [7, 8] and recently translated into expansions for the spectral action [49]. In section 5, we consider this problem on the torus $\mathbb{T}^d$, which is considerably more complicated due to the presence of extra dimensionful parameters—circumferences of the torus. We compute the heat kernel to the second order in the potential and gauge field strength and consider various limits, including large and small proper time, slowly and rapidly varying external fields. We also discuss implications for the spectral action.

2. Notion of spectral triple

The main properties of a compact spin Riemannian manifold $M$ can be recaptured using the following triple: $(\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(M, S), D = D)$. The coordinates $x = (x^1, \ldots, x^d)$ are exchanged with the algebra $C^\infty(M)$, and the Dirac operator $D$ acting on the space $\mathcal{H}$ of square
The idea of noncommutative geometry is to forget about the commutativity of the algebra $C^\infty(M)$ and to impose axioms on a triplet $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an algebra, $\mathcal{H}$ is a separable Hilbert space and $D$ is an (unbounded) self-adjoint operator, to generalize the above one in order to be able to obtain appropriate definitions of important notions: pseudodifferential operators, measure and integration theory, etc.

This scheme is quite minimalist. Clearly, one needs the algebra of functions to be able to talk about a manifold. The Dirac operator defines derivations and thus length scales. The idea of noncommutative geometry is the spectral triple. The operator of the triplet are necessary to define a meaningful geometry and/or physics. As we shall see below, they are also sufficient.

The idea of the construction outlined below is to formulate basic properties of the triple $(\mathcal{A}, \mathcal{H}, D)$ and then use these properties in a more abstract setting when commutativity of $\mathcal{A}$ is not assumed. On this way, one arrives at the following.

**Definition 2.1.** A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the data of an involutive (unital) algebra $\mathcal{A}$ with a faithful representation $\pi$ on a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D$ with compact resolvent such that $[D, \pi(a)]$ is bounded for any $a \in \mathcal{A}$.

This definition may be supplemented by more detailed restrictions on the spectral triple:

for the commutative spin manifold case $C^\infty(M)$ (see precise definition in [42]), one shows that $D$ is a (matrix-valued) differential operator, so compactness of the resolvent ensures that the operator $D$ is no less than a first-order operator. Boundedness of the commutator means that $D$ is at most of the first order. So $D = \mathcal{D} + \rho$ where $\rho$ is a smooth section on $M$ of endomorphisms of the spin bundle and $\mathcal{D}$ appears to be a minimum of some action functional on all operators $\mathcal{D}'$ generating the same metric as $\mathcal{D}$ [42, theorem 11.2].

The triple is even if there is a grading operator $\chi$ such that $\chi = \chi^*$.

The properties of the gamma matrices depend periodically on the dimensionality of space, with the period equal to 8. Nonequivalent dimensions are distinguished by the properties of the resolvent.

The spectral triple is 'real of KO-dimension $d$' if there is an antilinear isometry $J: \mathcal{H} \to \mathcal{H}$ such that $J D = e J D J$, $J^2 = e'$, $J \chi = e'' \chi J$ with a table of signs $\epsilon, \epsilon', \epsilon''$ given in [24, 42] and the following commutation rules: $[\pi(a), \pi(b)] = 0$, $[[D, \pi(a)], \pi(b)] = 0$, $\forall a, b \in \mathcal{A}$, where $\pi(a)^\gamma = J \pi(a^*) J^{-1}$ is a representation of the opposite algebra $\mathcal{A}^\gamma$.

The spectral triple is $d$-summable (or has the metric dimension $d$) if the singular values $\mu_n$ of $D^{-1}$ behave for large $n$ as $\mu_n(D^{-1}) = O(n^{-1/d})$. In the commutative case, this is just the Weyl formula for the distribution of eigenvalues.

The key result is the reconstruction theorem [22, 23], telling us that given a commutative spectral triple satisfying the above axioms and some more requirements, there exists a compact spin manifold $M$ such that $\mathcal{A} \simeq C^\infty(M)$ and $D$ is just a perturbation of the Dirac operator $\mathcal{D}$.

Let us continue with constructing relevant objects of noncommutative geometry through the spectral triple. The operator $D$ tells us what the abstract analog of a first-order
pseudodifferential operator is and generates naturally the set of all related pseudodifferential operators $\Psi(A)$ [24]. For $P \in \Psi(A)$, we define the zeta function associated with $P$ (and $D$) by

$$\zeta^P_D(s) : s \in \mathbb{C} \rightarrow \text{Tr}(P[D]^{-s})$$

(1)

which makes sense since for $\Re(s) > 1$, $P[D]^{-s}$ is trace class.

The dimension spectrum $S_d$ of the triple is the set of poles of $\zeta^P_D(s)$ $\forall P \in \Psi(A)$. It is said to be simple if the poles have order at most 1.

**Proposition 2.2.** Let $S_d(M)$ be the dimension spectrum of a commutative geometry of dimension $d$. Then $Sp(M)$ is simple and $S_d(M) = [d - k \mid k \in \mathbb{N}]$.

Finally, one can define a trace, called a noncommutative integral of $P$, that is given by

$$\int P = \text{Res}_{s=0} \zeta^P_D(s).$$

In (1), we assume $D$ invertible since otherwise one can replace $D$ by the invertible operator $D + \text{Pr}$, with $\text{Pr}$ being the projection on $\text{Ker}D$. As noted by Wodzicki, $\int P$ is equal to $-2$ times the coefficient in log $t$ of the asymptotics of $\text{Tr}(P e^{-tD^2})$ as $t \to 0$. It is remarkable that this coefficient is independent of $D$ and this gives a close relation between the $\zeta$ function and the heat kernel expansion with the Wodzicki residue $\text{WRes}$ [77, 78]. Actually,

$$\text{Tr}(P e^{-tD^2}) \sim \sum_{k=0}^{\infty} a_k t^{(k-\text{ord}(P)-d)/2} + \sum_{k=0}^{\infty} (-a_k^* \log t + b_k) t^k,$$

so $\int P = 2a_0^*$. Since $\int$ and $\text{WRes}$ are traces on $\Psi(C^\infty(M))$, thus by uniqueness $\int P = c \text{WRes} P$. The Wodzicki residue is known in quantum field theory due to its relation to the multiplicative anomaly [28].

Let us proceed with differential forms. The algebra $A$ gives us smooth functions, and thus the 0-forms. The set of 1-forms may be defined as $[21, 42]

$$\Omega^1_D(A) = \text{span}\{adb \mid a, b \in A\}, \quad db = [D, b].$$

In the commutative case, this defines 1-forms contracted with $\gamma$-matrices rather than usual forms. One can add a 1-form $A$ to $D$ to obtain $D_A := D + A$, but when a reality operator $J$ exists, we also want $D_J = eJD_A$, so we choose

$$D_A = D + \tilde{A}, \quad \tilde{A} = A + eJAJ^{-1}, \quad A = A^*.$$

(2)

In the commutative case, $A^0 \simeq JAJ^{-1} \simeq A$ which also gives

$$JAJ^{-1} = -eA^*, \quad \forall A \in \Omega^1_D(A)$$

thus $\tilde{A} = 0$ when $A = A^*$.

This does not mean that $D$ cannot fluctuate in commutative geometries, but that one has to consider only the non-symmetrized fluctuations $D_A$.

3. Spectral action

We would like to obtain a good action for any spectral triple and for this it is useful to look at some examples in physics. In any physical theory based on geometry, the interest of an action functional is, by a minimization process, to exhibit a particular geometry, for instance, trying to distinguish between different metrics. This is the case in general relativity with the Einstein–Hilbert action (with its Riemannian signature).
The Einstein–Hilbert action is
\[ S_{\text{EH}}(g) = -\int_M R_g(x) \, d\text{vol}_g(x), \]
where \( R \) is the scalar curvature (chosen positive for the sphere). This is nothing else in dimension 4 (up to a constant) than \( f \partial^2 \).

But in the search for invariants under diffeomorphisms, there are more quantities than the Einstein–Hilbert action, a trivial example being \( \int_M f(R_g(x)) \, d\text{vol}_g(x) \), and there are others. In this desire to implement gravity in noncommutative geometry, the eigenvalues of the Dirac operator look as natural variables [55]. However, we look for invariants which add up under disjoint unions of different geometries.

### 3.1. Quantum approach and spectral action

In a way, a spectral triple fits quantum field theory since for (Euclidean) fermions and we can compute Feynman graphs with fermionic internal lines. The gauge bosons are only derived objects obtained from internal fluctuations described by a choice of a connection which is associated with a 1-form in \( \Omega^1_D(A) \). Thus, the guiding principle followed by Chamseddine and Connes is to use a theory which is of pure geometric origin with a functional action based on the spectral triple, namely which depends only on the spectrum of \( D \). They proposed the following.

**Definition 3.1.** The spectral action of a spectral triple \((A, \mathcal{H}, D)\) is defined by
\[
S(D, f, \Lambda) := \text{Tr}(f(D^2/\Lambda^2)),
\]
where \( \Lambda \in \mathbb{R}^+ \) plays the role of a cutoff and \( f \) is any positive function (such that \( f(D^2/\Lambda^2) \) is a trace-class operator).

One can also define as in [24] \( S(D, g, \Lambda) = \text{Tr} \left( g(D/\Lambda) \right) \) where \( g \) is positive and even. With this second definition, \( S(D, g, \Lambda) = \text{Tr} \left( f(D^2/\Lambda^2) \right) \) with \( g(x) = f(x^2) \).

As an example for \( f \), one can take the characteristic function of \([0, 1]\]. Then \( \text{Tr}(f(D^2/\Lambda^2)) \) is nothing else but the number of eigenvalues of \( D \) within \([-\Lambda, \Lambda]\).

When this action has an asymptotic series in \( \Lambda \to \infty \), we deal with an effective theory. Naturally, \( D \) has to be replaced by \( D_{\Lambda} \) which is just a decoration. To this bosonic part of the action, one adds a fermionic term \( \frac{1}{2} \langle \bar{\psi}, D\psi \rangle \) for \( \psi \in \mathcal{H} \) to obtain a full action. In the standard model of particle physics, this latter corresponds to the integration of the Lagrangian part for the coupling of gauge bosons and Higgs bosons with fermions.

#### 3.1.1. Yang–Mills action

This action plays an important role in physics so it is natural to consider it in the noncommutative framework. Recall first the classical situation: let \( G \) be a compact Lie group with its Lie algebra \( g \) and let \( A \in \Omega^1(M, g) \) be a connection. If \( F = da + \frac{1}{2}[A, A] \in \Omega^2(M, g) \) is the curvature (or field strength) of \( A \), then the Yang–Mills action is \( S_{\text{YM}}(A) = \int_M \text{tr}(F \wedge \star F) \). In the Abelian case \( G = U(1) \), it is the Maxwell action and its quantum version is the quantum electrodynamics (QED) since the ungauged \( U(1) \) of electric charge conservation can be gauged, and its gauging produces electromagnetism [68]. It is conformally invariant when the dimension of \( M \) is \( d = 4 \).

The Yang–Mills action can be defined in the context of noncommutative geometry for a spectral triple \((A, \mathcal{H}, D)\) of dimension \( d \) [20, 21]. Let \( A \in \Omega^1_D(A) \) and curvature \( \theta = dA + A^2 \). Then it is natural to consider
\[
I(A) = \text{Tr}_{\text{Dix}}(\theta^2|D|^{-d})
\]
since it coincides (up to a constant) with the previous Yang–Mills action in the commutative case. Here $\text{Tr}_{D_{x}}$ is the Dixmier (singular) trace [25]: if $P = \theta^{2} |D|^{-d}$, for the principal symbol, $\text{tr} (\sigma P (x, \xi)) = c \text{tr} (F \wedge \star F) (x)$, and the Dixmier trace is also related to the Wodzicki residue for PDOs on compact manifolds (uniqueness of traces up to a constant). The key observation regarding the Yang–Mills action is that it appears in the $1/\Lambda$ expansion of the spectral action, see section 3.4.

The spectral action is more conceptual than the Yang–Mills action since it gives no fundamental role to the distinction between gravity and matter in the artificial decomposition $D_{A} = D + A$. For instance, for the minimally coupled standard model, the Yang–Mills action for the vector potential is part of the spectral action, as well as the Einstein–Hilbert action for the Riemannian metric [10].

As quoted in [15], the spectral action has conceptual advantages:

- simplicity: when $f$ is a cutoff function, the spectral action is just the counting function;
- positivity: when $f$ is positive (which is the case for a cutoff function), the action $\text{Tr} (f (D^{2}/\Lambda^{2})) \geq 0$ has the correct sign for a Euclidean action: the positivity of the function $f$ will ensure that the actions for gravity, Yang–Mills, Higgs couplings are all positive and the Higgs mass term is negative;
- invariance: the spectral action has a much larger invariance group than the usual diffeomorphism group as for the gravitational action; this is the unitary group of the Hilbert space $\mathcal{H}$.

However, this action is non-local. It only becomes so when it is replaced by the asymptotic expansion.

### 3.2. Asymptotic expansion for $\Lambda \to \infty$

The heat kernel method already used in previous sections will give a control of spectral action $S(D, f, \Lambda)$ when $\Lambda$ goes to infinity.

**Theorem 3.2** ([24]). Let $(A, \mathcal{H}, \mathcal{D})$ be a spectral triple with a simple dimension spectrum $S_{d}$. We assume that

$$\text{Tr} (e^{-t D^{2}}) \simeq \sum_{\alpha \in S_{d}} a_{\alpha} t^{\alpha}.$$

(i) If $a_{\alpha} \neq 0$ with $\alpha < 0$, then the zeta function $\zeta_{\mathcal{D}}$ defined in (1) has a pole at $-2\alpha$ with

$$\text{Res} \ z = -2\alpha.$$

(ii) For $\alpha = 0$, we obtain $\zeta_{\mathcal{D}} (0) = a_{0} - \dim \text{Ker} \mathcal{D}$.

(iii) One has the following asymptotic expansion over the positive part $S_{d}^{+}$ of $S_{d}$:

$$\text{Tr} (f (D/\Lambda)) \sim \sum_{\beta \in S_{d}^{+}} f_{\beta} \Lambda^{\beta} \int |D|^{-\beta} + f (0) \zeta_{\mathcal{D}} (0) + \cdots,$$

where the dependence of the even function $f$ is $f_{\beta} := \int_{0}^{\infty} f(x) x^{\beta - 1} \, dx$ and $\cdots$ involves the full Taylor expansion of $f$ at 0.

Here, one assumes sufficient hypothesis on $f$ like $f$ is a Laplace transform with $|f_{\beta}| < \infty$, etc. It can be useful to make a connection between the spectral action and heat expansion [12, 39, 41].
Corollary 3.3. Assume that the spectral triple \((A, \mathcal{H}, \mathcal{D})\) has dimension \(d\).
If \(\text{Tr}(e^{-tD^2}) \simeq \sum_{k=0}^{d} f_k(D^2) + \cdots\), then
\[
S(\mathcal{D}, f, \Lambda) \simeq \sum_{k=0}^{d} f_k \Lambda^k a_d(D^2) + f(0)a_d(D^2) + \cdots
\]
with \(f_k \equiv \frac{1}{(2\pi)^{d/2}} \int_{0}^{\infty} e^{-s} s^{d/2-1} \text{ds}.\) Moreover,
\[
a_k(D^2) = \frac{1}{2} \Gamma \left( \frac{d-k}{2} \right) \int |D|^{-d+k} \quad \text{for} \quad k = 0, \ldots, d-1,
\]
\[
a_d(D^2) = \text{dim Ker}\mathcal{D} + \zeta_{\mathcal{D}}(0).
\]

Asymptotics \((5)\) uses the value of \(\zeta_{\mathcal{D}}(0)\) in the constant term \(\Lambda^0\), so it is fundamental to look at its variation under a gauge fluctuation \(\mathcal{D} \to \mathcal{D} + A\). For instance, \(\zeta_{\mathcal{D}}(0) = \zeta_D(0) + \sum_{q=1}^{d} \frac{(-1)^q}{q} \int (\Lambda D^{-1})^q\) \([11, 29]\).

3.3. Remark on the use of Laplace transform

In \((6)\), the spectral action asymptotic behavior
\[
S(\mathcal{D}, f, \Lambda) \simeq \sum_{n=0}^{\infty} c_n \Lambda^{-d-n} a_n(D^2)
\]
has been proved for a smooth function \(f\) which is a Laplace transform for an arbitrary spectral triple (with simple dimension spectrum) satisfying \((4)\). However, this hypothesis is too restrictive since it does not cover the heat kernel case where \(f(x) = e^{-x}\).

When the triple is commutative and \(D^2\) is a generalized Laplacian on sections of a vector bundle over a manifold of dimension \(d\), hypothesis \((4)\) can be proved \([39]\) since the spectrum dimension is given by proposition 2.2 and previous asymptotics \([31]\) for \(d = 4\) is (see also the next section)
\[
\text{Tr} \left( f(D^2/\Lambda^2) \right) \simeq \frac{1}{(4\pi)^2} \left[ \left( \text{rk}(E) \right) \int_{0}^{\infty} x f(x) \text{d}x \right] \Lambda^4 + \left( b_2(D^2) \int_{0}^{\infty} x f(x) \text{d}x \right) \Lambda^2
\]
\[
+ \sum_{m=0}^{\infty} \left( (-1)^m f^{(m)}(0) b_{2m+4}(D^2) \right) \Lambda^{-2m} \quad \Lambda \to \infty,
\]
where \((-1)^m b_{2m+4}(D^2) = \frac{4\pi^2}{m!} \mu_m(D^2)\) are suitably normalized, integrated moment terms of the spectral density of \(D^2\).

The main point is that this asymptotics is meaningful in the Cesàro sense (see \([30–32]\) for definition) for \(f \in \mathcal{K}'(\mathbb{R})\), which is the dual of \(\mathcal{K}(\mathbb{R})\). This latter is the space of smooth functions \(\phi\) such that for some \(a \in \mathbb{R}\), \(\phi^{(k)}(x) = O(|x|^{a-k})\) as \(|x| \to \infty\), for each \(k \in \mathbb{N}\). In particular, the Schwartz functions are in \(\mathcal{K}(\mathbb{R})\) (and even dense).

Of course, the counting function is not smooth but is in \(\mathcal{K}'(\mathbb{R})\), so such behavior \((8)\) is wrong beyond the first term, but is correct in the Cesàro sense. Actually there are more derivatives of \(f\) at 0 as explained on examples in \([31, p \, 243]\).

3.4. Commutative case

It is instructive to rederive the asymptotics of the spectral action in a commutative case where one does not need any of the new notions, like the Wodzicki residue or the dimension spectrum. Let us take a compact spin manifold \(M\) of dimension \(d = 4\) without a boundary and
\[ D = D^\text{Dirac} \equiv i\gamma^\mu (\nabla_\mu + A_\mu), \]

being the standard Dirac operator with a spin connection \( \nabla \) and a gauge field \( A \). Let us suppose that \( f \) is a Laplace transform,

\[ f(z) = \int_0^\infty dt\, e^{-zt} \phi(t). \]

We also have an asymptotic expansion

\[ \text{Tr}(e^{-tD^2}) \sim t \sum_{k=0}^{\infty} t^{-2k} a_{2k}(D^2). \]

\( (9) \)

Therefore, assuming commutation of sums, traces and integrals (this point is investigated later), we can write

\[ S(D, f, \Lambda_1) = \int_0^\infty dt\, \text{Tr}(e^{-tD^2}) \phi(t) \approx \int_0^\infty dt\, \sum_{k=0}^{\infty} t^{-2k} a_{2k}(D^2) \phi(t) \]

\[ \approx \sum_{k=1}^{\infty} \Lambda^{2(2-k)} \phi^2_k a_{2k}(D^2), \]

\( (10) \)

where

\[ \phi_{2k} = \int_0^\infty dt\, t^{-2k}\phi(t). \]

\( (11) \)

The heat kernel coefficients for \( D^2 \) may be found in a textbook, see [35, section 4.4]:

\[ a_0(D^2) = \frac{1}{4\pi^2} \int_M d^4x \sqrt{g} \, \text{tr}(1), \]

\[ a_2(D^2) = -\frac{1}{48\pi^2} \int_M d^4x \sqrt{g} \, \text{tr}(R), \]

\[ a_4(D^2) = \frac{1}{24\pi^2} \int_M d^4x \sqrt{g} \, \text{tr}(-F_{\mu\nu}F^{\mu\nu} + (R^2\text{-terms})), \]

where the trace is taken over gauge indices and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). Thus, we obtained an expansion of the spectral action in \( 1/\Lambda^2 \), where we can find many relevant terms. The \( a_0 \) term is the cosmological term, while \( a_2 \) is the Einstein–Hilbert action. The \( a_4 \) term contains the Yang–Mills action and curvature-squared corrections to the gravity action. Note that the Einstein–Hilbert and Yang–Mills actions came out with correct signs (assuming \( \phi_{2k} \) positive).

Each of the heat kernel coefficients \( a_{2k} \) is an integral of a polynomial of the canonical dimension \( 2k \), i.e. it contains a limited number of fields and derivatives. Therefore, we deal with a weak-field expansion of the spectral action.

In [71], it was suggested to choose a function \( f \) such that expansion (8) or (12) contains a finite number of terms, and to use this expansion instead of the full spectral action. Although such a theory may have some good properties, they do not carry over to the full theory [49].

3.5. About the meaning of the spectral action via its asymptotics

We have discussed above the spectral action for compact spectral triples. In a non-compact case, which is more physically motivated frequently, the definition of spectral action has to be
modified \[49\],

\[ S(D, f, \Lambda) := \text{Tr} \left( f(D^2/\Lambda^2) - f(D_0^2/\Lambda^2) \right), \]

(12)

where \(D_0\) is the unperturbed Dirac operator with \(\Lambda = 0\).

Almost commutative geometry, which is a commutative geometry times a finite one (where the finite one is a sum of matrices), has been deeply and intensively investigated for the noncommutative approach to the standard model of particle physics, see [19, 24]. This approach offers a lot of interesting perspectives. For instance, the possibility of computing the Higgs representations and mass (for each noncommutative model) is particularly instructive [9, 16, 14, 56, 50, 51, 60]. The choice of the Dirac operator is dictated by coupling of the fermions in the standard model. It is interesting that the whole bosonic action is then reproduced automatically.

The spectral action has been computed in [47] for the quantum group \(SU_q(2)\) which is not a deformation of \(SU(2)\) of the type considered on the Moyal plane [36–38]. It is quite peculiar since (5) has only a finite number of terms. Also on the Moyal spaces, the heat kernel expansion and expansion (8) of the spectral action were calculated in a number of papers [36–38, 73–75]. The properties of both expansions depend crucially on the number of compact noncommutative dimensions, as discussed in [69].

In the light of the difficulties in dealing with non-compact manifolds, the case of spheres \(S^4\) or \(S^3 \times S^1\) has been investigated in [15, 18] for instance in the case of Robertson–Walker metrics.

An approach to the spectral action based on quantum anomalies has been suggested in [2] and applied to Higgs dilaton interactions in [1].

All the machinery of spectral geometry has recently been applied to cosmology, computing the spectral action in a few cosmological models related to inflation, see [54, 57–59, 61, 66].

Spectral triples associated with manifolds with boundary have been considered in [13, 17, 45, 46, 48]. The main difficulty is precisely to put good boundary conditions to the operator \(D\) to still obtain a self-adjoint operator and then, to define a compatible algebra \(\mathcal{A}\). This is probably a must to obtain a result in a noncommutative Hamiltonian theory in dimension 1 + 3.

The case of manifolds with torsion has also been studied in [43, 64, 65], and even with boundary in [48]. These works show that the Holst action appears in spectral actions and that torsion could be detected in a noncommutative world.

Somewhat similar ideas that the gravity is a low-energy effect of quantized matter field rather than a fundamental force were suggested long ago by Zeldovich [81] and Sakharov [67], see [62] for a review. The spectral action approach extends much further: the spectral action is valid for all energies. This is why one is interested in the effects which are not seen at the asymptotic expansions.

3.6. About convergence and divergence, local and global aspects of the asymptotic expansion

The asymptotic expansion series (6) of the spectral action may or may not converge. It is known that each function \(g(\Lambda^{-1})\) defines at most a unique expansion series when \(\Lambda \to \infty\) but the converse is not true since several functions have the same asymptotic series. We give here examples of convergent and divergent series of this kind.

When \(M\) is the torus \(\mathbb{T}^d\) as in section 5.2 with \(\Lambda = g^{\mu\nu} \partial_\mu \partial_\nu\),

\[ \text{Tr}(e^{\lambda A}) = \frac{(4\pi)^{-d/2} \text{Vol}(T^d)}{p^{d/2}} + O(t^{-d/2} e^{-1/4 t}); \]

thus, the asymptotic series \(\text{Tr}(e^{\lambda A}) \simeq \frac{(4\pi)^{-d/2} \text{Vol}(T^d)}{p^{d/2}}, t \to 0\), has only one term.
In the opposite direction, let now $M$ be the unit 4-sphere $S^4$ and $\mathcal{D}$ be the usual Dirac operator. By proposition 2.2, equation (4) yields (see [18])

$$\text{Tr}(e^{-t\mathcal{D}^2}) = \frac{1}{t^2} \left( \frac{2}{3} + \frac{2}{3} t + \sum_{k=0}^{\infty} a_k t^{k+2} + O(t^{n+3}) \right),$$

with Bernoulli numbers $B_{2k}$. Thus, $t^2 \text{Tr}(e^{-t\mathcal{D}^2}) \approx \frac{2}{3} t + \frac{2}{3} t + \sum_{k=0}^{\infty} a_k t^{k+2}$ when $t \to 0$ and this series is not convergent but only asymptotic: $a_k > \frac{2}{12} \frac{2^k}{k^2} \frac{B_{2k}}{(2k)!} > 0$ and $|B_{2k+4}| = 2 \frac{(2k+4)!}{(2k+4)^2} \xi (2k + 4) \approx 4 \sqrt{\pi} (k + 2) \left( \frac{k + 2}{2\pi} \right)^{3k+4} \to \infty$ when $k \to \infty$.

More generally, in the commutative case considered above and when $\mathcal{D}$ is a differential operator—like a Dirac operator, the coefficients of the asymptotic series of $\text{Tr}(e^{-t\mathcal{D}^2})$ are locally defined by the symbol of $\mathcal{D}^2$ at point $x \in M$ but this is not true in general: in [40], a positive elliptic pseudodifferential operator is given such that non-locally computable coefficients especially appear in (9) when $2k > d$. Nevertheless, all coefficients are local for $2k \leq d$.

Recall that a locally computable quantity is the integral on the manifold of a local frame-independent smooth function of one variable, depending only on a finite number of derivatives of a finite number of terms in the asymptotic expansion of the total symbol of $\mathcal{D}^2$. For instance, some nonlocal information contained in the ultraviolet asymptotics can be recovered if one looks at the (integral) kernel of $e^{-t\mathcal{D}^2}$ in $\mathbb{T}^d$, with $\text{Vol}(\mathbb{T}^d) = 2\pi$, we obtain [34]

$$\text{Tr}(e^{-t\sqrt{-\Delta}}) = \frac{\sinh(t)}{\cosh(t) - 1} = \coth \left( \frac{t}{2} \right) = \frac{2}{t} \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k} = \frac{2}{t} \left[ 1 + \frac{t^2}{12} - \frac{t^4}{720} + O(t^6) \right]$$

and the series converges when $t < 2\pi$, since $\frac{B_{2n}}{(2n)!} = (-1)^{n+1} \frac{(2n)!}{2^{2n}n!}$; thus $\frac{|B_{2k}|}{(2k)!} \approx \frac{2}{(2\pi)^{2k}}$ when $k \to \infty$.

Thus, we have an example where $t \to \infty$ cannot be used with the asymptotic series.

Thus, the spectral action (6) precisely encodes the local and nonlocal behavior which appears or not in its asymptotics for different $f$. The coefficient of the action for the positive part (at least) of the dimension spectrum corresponds to renormalized traces, namely the noncommutative integrals of (7). In conclusion, asymptotic (8) of spectral action may or may not have nonlocal coefficients.

For the flat torus $\mathbb{T}^d$, the difference between $\text{Tr}(e^{i\Delta})$ and its asymptotic series is a term which is related to periodic orbits of the geodesic flow on $\mathbb{T}^d$. Similarly, the counting function $N(\lambda)$ (number of eigenvalues including multiplicities of $\Delta$ less than $\lambda$) obeys Weyl’s law: $N(\lambda) = \frac{(4\pi)^{-d/2} \text{Vol}(\mathbb{T}^d)}{\lambda^{d/2}} + o(\lambda^{d/2})$—see [4] for a good historical review on these fundamental points. The relationship between the asymptotic expansion of the heat kernel and the formal expansion of the spectral measure is clear: the small-$t$ asymptotics of the heat kernel is determined by the large-$\lambda$ asymptotics of the density of eigenvalues (and eigenvectors). However, the latter is defined modulo some average: Cesaro sense as reminded in section 3.3, or Riesz mean of the measure which washes out ultraviolet oscillations, but also gives information on intermediate values of $\lambda$ [34].

In [15, 57], examples are given of spectral actions on (compact) commutative geometries of dimension 4 whose asymptotics have only two terms. In the quantum group $SU_q(2)$, the spectral action (3) itself has only four terms, independently of the choice of function $f$. 
4. Trace-class convergence of Dyson–Phillips series (Duhamel expansion)

We review here a few facts about the Gibbs semigroups, the Duhamel or Dyson–Phillips expansion and related convergence questions. We refer to [52, 6, 80] for more information on this subject.

A $C_0$-semigroup (or strongly continuous semigroup) is a family $(G(t))_{t \in \mathbb{R}_+}$ of bounded operators on a Hilbert space $\mathcal{H}$, such that, $G(0) = \text{Id}$, for any $t, t' \in \mathbb{R}_+$, $G(t+t') = G(t)G(t')$, and $t \mapsto G(t)$ is a continuous function in the strong operator topology sense, or in other words, $(G(t))(\psi)$ is a continuous function of $t$ for any fixed $\psi \in \mathcal{H}$.

Given a $C_0$-semigroup $(G(t))_{t \geq 0}$, the generator of the semigroup is the operator $T$ on $\mathcal{H}$, defined on $\text{Dom} T := \{ \psi \in \mathcal{H} : T\psi := \lim_{h \to 0_+} h^{-1}(G(h)\psi - \psi) \}$ exists. It turns out that $T$ determines the semigroup uniquely and is a closed densely defined operator on $\mathcal{H}$. Moreover, if $T$ is self-adjoint, $G(t) = e^{itT}$, where $e^{itT}$ is defined thanks to the spectral theorem. When $T$ generates $(G(t))_{t \geq 0}$, the map $u : t \mapsto G(t)\psi$ on $\mathbb{R}_+$ for a given $\psi \in \mathcal{H}$ is the unique solution to the following abstract Cauchy problem:

$$ u'(t) = Tu(t), \quad u(0) = \psi. $$

It is therefore not surprising that operator semigroup theory is particularly useful for the description of phenomena associated with linear evolution equations.

A good part of this theory is devoted to the generation problem, which is finding conditions on a closed densely defined operator $T$ so that it is the generator of a semigroup with given desired properties. The general case is solved by the Feller–Miyadera–Phillips theorem: fix $w \in \mathbb{R}$, $M \in \mathbb{R}_+$ and let $T$ be a closed densely defined operator such that for any $\lambda \in \mathbb{C}_{-w} := \{ \mu \in \mathbb{C} : \Re(\mu) > w \}$, the number $\lambda$ is in the resolvent set of $T$ and the following estimate holds for any $n \in \mathbb{N}$:

$$ \|(T - \lambda)^{-n}\| \leq M(\Re(\lambda) - w)^{-n}. $$

Then, $T$ is the generator of a $C_0$-semigroup $(G(t))_{t \geq 0}$ satisfying \( \| G(t) \| \leq M e^{\omega t} \) for all $t \geq 0$. Since any $C_0$-semigroup satisfies an estimate of this type, this result provides a general description of all possible semigroup generators. This theorem is a generalization of the Hille–Yosida theorem, which concerns the generation of contractive semigroups $(\| G(t) \| \leq 1, \forall t \in \mathbb{R}_+)$, and is obtained by considering $w = 0$ and $M = 1$ in the previous formulation.

As an application, one can show that any self-adjoint operator $T$ generates a $C_0$-semigroup if and only if it is bounded above. For example, any self-adjoint bounded perturbation of the Laplacian on $\mathbb{R}^n$, acting on $L^2(\mathbb{R}^n)$, yields a $C_0$-semigroup.

If the operator $H = -T$ is interpreted as an unperturbed Hamiltonian, the operator $H + P$ obtained by a suitable perturbation $P$ can be seen as a Hamiltonian of an interacting system. This fact motivates the perturbation problem of operator semigroup theory: given a generator $T$ of a strongly continuous semigroup, what are the conditions on an operator $B$ so that $T + B$ is the generator of a strongly continuous semigroup?

It turns out that any bounded perturbation $B$ of a generator $T$ of a $C_0$-semigroup $(G_T(t))_{t \geq 0}$ is a generator of a strongly continuous semigroup $(G_{T+B}(t))_{t \geq 0}$. Moreover, the perturbed semigroup $G_{T+B}$ can be obtained as a Dyson–Phillips series (also called Duhamel expansion):

$$ G_{T+B}(t) = \sum_{n=0}^{\infty} G_n(t), \quad t \geq 0, $$

where the sequence of operators $(G_n)$ is inductively defined by $G_0(t) = G_T(t)$ and

$$ G_{n+1}(t) := \int_0^t G_T(t-s)BG_n(s) \, ds $$
in the strong operator sense. The convergence of (13) is here to be understood in the norm topology (uniform convergence), and is obtained by iterative application of the Duhamel formula:

\[ G_{T+B}(t) - G_T(t) = \int_0^t G_{T+B}(t-s)BG_T(s) \, ds, \quad t \geq 0, \]

in the strong operator sense.

Many natural semigroups \((G(t))\) that appear in quantum statistical mechanics or in heat kernel theory are actually families of operators which are not only bounded but also trace-class when the parameter \(t\) is nonzero. This means that for any \(t > 0\), \(\|G_t\|_1 := \sum_{k \in \mathbb{N}} |\langle e_k, G_te_k \rangle|\) is finite and the trace of \(G_t\) exists: \(\text{Tr}(G_t) = \sum_{k \in \mathbb{N}} \langle e_k, G_te_k \rangle\), where \((e_k)\) is any orthonormal basis of \(\mathcal{H}\). A strongly continuous semigroup which has this property is called a Gibbs semigroup [70]. It turns out that the condition of finiteness of the trace-norm \(\|G(t)\|_1\) for each \(t > 0\) automatically implies continuity of the map \(t \rightarrow G(t)\) in the topology of the \(\|\cdot\|_1\)-norm [70, proposition 2].

The natural question related to perturbation theory is now in this setting: Can we extend the Dyson–Phillips expansion formula (13) to the trace of \(G_{T+B}(t)\) and \(G_T(t)\)? This question has been answered positively by Uhlenbrock [70, theorem 3.2]. He proved that if \(G_T\) is a Gibbs semigroup with generator \(T\) and if the perturbation \(B\) is bounded, then the perturbed \(C_T\)-semigroup \(G_{T+B}\) is a Gibbs semigroup, and the Dyson–Phillips series (13) converges in the trace-norm \(\|\cdot\|_1\) sense. This implies in particular that for all \(t > 0\),

\[ \text{Tr}(G_{T+B}(t)) = \sum_{n=0}^{\infty} \int_{R_n^t} \text{Tr} \left( G_T(t-s_{n-1})BG_T(s_{n-1} - s_{n-2}) \cdots G_T(s_1 - s_0)BG_T(s_0) \right) \, ds, \]

(14)

where \(R_n^t := \{ (s_0, \ldots, s_{n-1}) : 0 \leq s_0 \leq \cdots \leq s_{n-1} \leq t \}\).

In some physical situations, the boundedness condition on the perturbation \(B\) is too strong. The question on possible generalization to unbounded perturbation was first investigated in [3], where the trace-norm convergence of the Dyson–Phillips series has been obtained for perturbations in the \(\mathcal{P}_0\)-class, that is, for closed operators \(B\) with domain containing \(\bigcup_t G_T(t)(\mathcal{H})\), and such that \(\int_0^t \|BG_T(t)\| \, dt < \infty\). This class is actually included in the set of operators which are relatively bounded with respect to \(T\) with relative bound equal to 0 [79, theorem 2.2]. Recall that an operator \(B\) is relatively bounded to \(T\) (or \(T\)-bounded) if the domain of \(B\) contains the domain of \(T\) and \(\|B\psi\| \leq a \|\psi\| + b \|T\psi\|\) for all \(\psi\) in the domain of \(T\), for constants \(a, b \geq 0\). The infimum of all possible values of \(b\) in the previous estimate is called the relative bound of \(B\) with respect to \(T\) [52, p 190].

As classical examples of perturbation operators \(B\) with zero relative bound with respect to the Laplace operator \(\Delta\) on \(\mathbb{R}^n\), one can consider the Kato–Rellich class of potential \(V \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)\), where \(p = 2\) if \(n = 3\) and \(p > n/2\) if \(n \geq 4\). Note also that any first-order differential operator is \(\Delta\)-bounded with zero relative bound.

The trace-norm convergence (as well as other analyticity questions) of the Dyson–Phillips expansion in the case of a nonzero relative bound has been investigated by Zagrebnov [79, 80]. In particular, if \(T\) is bounded above with \(p\)-summable resolvent for a finite \(p \geq 1\), then \(T\) is the generator of a Gibbs semigroup, any \(T\)-bounded perturbation \(B\) with relative bound \(b < 1\) yields a Gibbs semigroup \(G_{T+B}\), and the associated Dyson–Phillips expansion (13) holds in the trace-norm sense and thus (14) holds too [79, theorem 4.1].
5. Perturbation theory on $\mathbb{T}^d$

Here we develop a perturbation theory for the heat kernel up to the second order in the potential and in the connection on $\mathbb{T}^d$. On the plane, corresponding results were obtained long ago [7, 8]. Some interesting expansions of the evolution operator (that is, the heat operator at imaginary $t$) can be found in [63].

5.1. Basic example

The purpose of this subsection is to give the simplest example with which one can illustrate the technique and discuss the global properties of the heat kernel.

Let us take a scalar Laplace operator

$$L = -\left(\partial_x^2 + E(x)\right)$$

on the unit circle $\mathbb{T}^1$. $E$ is a smooth periodic function, $E \in C^\infty(\mathbb{T}^1)$. According to section 4, the heat trace can be expanded using the Duhamel expansion (14) since it is a bounded perturbation of minus the Laplacian:

$$K(L, t) = \text{Tr} (\exp(-tL)) = \sum_{k=0}^\infty K_k(t) \quad \text{(15)}$$

giving at the first and second orders in $E$

$$K_1(t) = t \text{Tr} \left( e^{tE} \right),$$

$$K_2(t) = \frac{t^2}{2} \text{Tr} \left( \int_0^1 dq \, e^{(1-\xi)qt^2} E e^{tE} \right).$$

After expanding in the Fourier series, one obtains

$$K_1(t) = t \sum_{q \in \mathbb{Z}} e^{-qt^2} \hat{E}(0) = \frac{t}{2\pi} \sum_{q \in \mathbb{Z}} e^{-qt^2} \int_{\mathbb{T}^1} dx E(x),$$

$$K_2(t) = \frac{t^2}{4\pi} \int_0^1 dq \sum_{p, q \in \mathbb{Z}} \hat{E}(-p) \hat{E}(p) e^{-q^2 + (1-\xi)pq + (1-\xi)p^2 t},$$

where the form-factor $v(p, t)$ is

$$v(p, t) := \frac{t^2}{4\pi} \int_0^1 dq \sum_{q \in \mathbb{Z}} e^{-q^2 + (1-\xi)pq + (1-\xi)p^2 t}. \quad \text{(16)}$$

This is the analog of $w_A(p^2)$, $\Lambda^{-2} = t$, in [49].

The form-factor (16) may be more explicitly evaluated for small or for large $t$.

Let us first consider the case of small $t$. Physically, this means that the regularization parameter $\Lambda = t^{1/2}$ is much larger than the inverse radius of $\mathbb{T}^1$ (which we put equal to 1 in this subsection). We rewrite

$$e^{-q^2 + (1-\xi)pq + (1-\xi)p^2 t} = e^{-(q + (1-\xi)p)^2 + (1-\xi)p^2 t}$$

and use the Poisson summation formula to represent

$$\sum_{q \in \mathbb{Z}} e^{-q^2 + (1-\xi)p^2 t} = \sqrt{\frac{\pi}{t}} \sum_{k \in \mathbb{Z}} e^{-2i(1-\xi)p^2 t} e^{-k^2/2t}.$$

By dropping the terms that are exponentially small for small $t$ (uniformly in $p$), we arrive at

$$v(p, t) \simeq \frac{t^{3/2}}{4\pi^{1/2}} \int_0^1 dq \, e^{-p^2 q (1-\xi)t}, \quad t \to 0,$$  \quad \text{(17)}
which is nothing else than the Barvinsky–Vilkovisky formula [7, 8], that was obtained on the plane, but, as we see now, is valid also on $\mathbb{T}^1$ for small $t$.

Similarly, in the first order of $E$ we have for small $t$

$$K_1(t) \simeq \frac{t^{1/2}}{2\pi^{1/2}} \int dx E(x),$$

which is the only term in the heat kernel asymptotics linear in $E$.

The expansion of (17) for small $p$ reproduces the usual heat kernel expansion, while for large $p$, we obtain an analog of the formula obtained in [49]

$$v(p, t) \simeq \frac{1}{2p^2} \sqrt{\frac{t}{\pi}}, \quad t \to 0, \quad p \to \infty.$$  \hspace{1cm} (18)

The order of limits here is with $t \to 0$ first, and then $p \to \infty$. Below we shall see that the order is not important.

Let us consider the opposite, large-$t$, asymptotic. We have

$$K_1(t) \simeq \frac{t}{2\pi} \int dx E(x)$$

modulo exponentially small terms.

To analyze $K_2$, let us integrate over $\xi$. Suppose $p \neq 0$,

$$\int_0^1 d\xi e^{-i(q^2 + (1-\xi)(2pq+p^2))} = \frac{1}{t(2pq + p^2)} (e^{-i\xi^2} - e^{-i(p+q)^2})$$  \hspace{1cm} (20)

for $q \neq -p/2$. If $q \neq 0$ and $q \neq -p$, the corresponding contributions to the sum over $q$ are exponentially small. If $p$ is even, there is a term with $q = -p/2$ which has to be treated separately. It does not contribute to the large-$t$ asymptotics since the right-hand side of (20) then equals $e^{-i\xi^2/4}$, which is also exponentially small as we assumed $p \neq 0$. By summing up the contributions from $q = 0$ and $q = -p$, one obtains

$$v(p, t) \simeq \frac{t}{2\pi p^2}, \quad t \to \infty, \quad p \neq 0,$$  \hspace{1cm} (21)

up to exponentially small terms. Of course, equation (21) cannot be obtained from (17).

Another interesting asymptotics of $v(p, t)$ is when $p \to \infty$ at a fixed $t$. To obtain this asymptotics, we integrate over $\xi$ in (16) with the help of (20) and drop all terms with $e^{-i\xi^2}$, but keep $e^{-i\xi^2}$ yielding

$$v(p, t) \simeq \frac{t}{2\pi} \sum_{q=-p/2} e^{-iq^2} \frac{e^{-i(p+q)^2}}{(q+p)^2 - q^2}, \quad p \to \infty.$$  \hspace{1cm} (22)

Taking then a $t \to \infty$ asymptotics gives us back (21), as expected. In the limit $t \to 0$, one arrives at (18) where the limits are taken in the reverse order. This was not guaranteed since the terms $e^{-i\xi^2}$ are not small in the $t \to 0$ limit.

5.2. General case

Consider a Laplace-type operator $L$ acting on sections of a vector bundle over $\mathbb{T}^d$. By choosing an appropriate covariant derivative $\nabla_\mu = \partial_\mu + \omega_\mu$ and an endomorphism $E$, one can bring this operator to the form [39]

$$L = -(g^{\mu\nu} \nabla_\mu \nabla_\nu + E).$$  \hspace{1cm} (22)

We suppose that the metric $g^{\mu\nu}$ is constant. The coordinates on $\mathbb{T}^d$ are supposed to be $2\pi$-periodic. Normalization of the Fourier modes is fixed in a metric-independent way,

$$\phi(x) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \hat{\phi}(k) e^{ikx}.$$
We expand the heat kernel as (the notation $\text{tr}$ refers to the trace of operators on the vector bundle)

$$K(L, t) = \text{tr} \left( \sum_{k \in \mathbb{Z}^d} (\hat{E}(k) v_1(k, t) + \hat{E}(-k) \hat{E}(k) v_2(k, t) + \hat{\omega}_\mu(-k) \hat{\omega}_\nu(k) v_{ij}^{\mu \nu}(k, t)) + \cdots \right),$$

(23)

where the dots denote the terms that are of higher than quadratic order in $E$ and $\omega$. We also dropped a non-interesting constant term. The form-factors read

$$v_1(k, t) := \delta_k, 0 (2\pi)^{-d/2} \sum_{q \in \mathbb{Z}^d} e^{-i\xi q^2},$$

$$v_2(k, t) := \frac{t^2}{2(2\pi)^d} \int_0^1 d\xi \sum_{q \in \mathbb{Z}^d} e^{-i(\xi(q+k)^2 + (1-\xi)q^2)},$$

$$v_{ij}^{\mu \nu}(k, t) := \frac{t^2}{2(2\pi)^d} \int_0^1 d\xi \sum_{q \in \mathbb{Z}^d} \left[ (k^\mu k^\nu - 4q^\mu q^\nu) e^{-i(\xi(q+k)^2 + (1-\xi)q^2)} + \frac{2}{t} g^{\mu \nu} e^{-\xi q^2} \right].$$

Here, the vectors with subscripts $q_\mu, k_\mu$ belong to $\mathbb{Z}^d$, while $q^\mu \equiv g^{\mu \nu} q_\nu$, etc.

In the small-$t$ asymptotic,

$$K(L, t) \simeq (4\pi t)^{-d/2} \int_{\mathbb{R}^d} \sqrt{g} \text{d}x \text{tr} \left[ tE + t^2E \frac{1}{2} h(-t\partial^2)E + t^2 \Omega_{\mu \nu} q(-t\partial^2) \Omega_{\mu \nu} \right] + \cdots$$

(24)

with

$$q(z) = -\frac{1}{2} h(z) + \frac{1}{z}, \quad h(z) = \int_0^1 d\alpha e^{-\alpha(1-\alpha)z}. \quad (25)$$

For large $t$,

$$K(L, t) \simeq \frac{t}{(2\pi)^d} \text{tr} \int_{\mathbb{R}^d} \text{d}x \left[ E(x) + E(-\partial^2)^{-1} E + \frac{1}{2} \Omega_{\mu \nu} (-\partial^2)^{-1} \Omega_{\mu \nu} \right] + \cdots. \quad (26)$$

In both (24) and (26), the dots denote higher order terms in $E$ and $\omega$ which were dropped already in (23) and also the terms that are exponentially small in the limits $t \to 0$ and $t \to \infty$ in (24) and (26), respectively. By considering different asymptotic regimes in (23), one arrives at expressions (24) and (26) with $\Omega_{\mu \nu} = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu$. However, one can also use the gauge-covariant expression

$$\Omega_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\mu \omega_\nu - \omega_\nu \omega_\mu,$$

as the difference is in the higher order terms which are neglected anyway.

The following remarks are in order.

(i) Derivation of the formulas above uses the same methods as were employed in section 5.1 though it is considerably more lengthy. Equation (23) follows from the Duhamel expansion. To obtain (24), one uses the Poisson summation formula and then drops exponentially small terms. In the opposite limit, $t \to \infty$, one first integrates over $\xi$ and then neglects exponentially small contributions.

(ii) As discussed in section 4, the Duhamel expansion is convergent in the trace-norm. No further conditions on $E$ or $\omega_\mu$ (supposed to be $C^\infty(\mathbb{T}^d)$) are required by (24) and (26). These formulas are good approximations to the heat kernel if $t^{1/2}$ is much smaller (larger) than the smallest (largest) radius of $\mathbb{T}^d$ defined by the metric $g_{\mu \nu}$. It is interesting to note that (24) contains $\sqrt{g}$, while (26) does not. It is important that the metric is constant. Although the Duhamel expansion is valid for any smooth metric, it is most useful when we have a closed expression for the unperturbed heat kernel $e^{t \delta_{\mu \nu} \delta_\mu \delta_\nu}$. 

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Expression (25) is the same as obtained for the heat kernel on $\mathbb{R}^d$ [7, 8] where it is valid without the assumption that $t$ is small. $\mathbb{R}^d$ is, in a sense, an infinite-radius limit of the torus, so that the small-$r$ approximation is always valid (see (ii) above).

5.3. Consequences for the spectral action

We shall not consider here the spectral actions in general form, but shall restrict ourselves to a particular, though rather typical, choice of $f$. Let us take $f = f_e$ with $f_e(z) = e^{-z}$. Then

$$S(D, f_e, \Lambda) = K(D^2, \Lambda^{-2}),$$

so that most of the results obtained above are valid if one takes $L = D^2A$. This corresponds to setting $\omega_{\mu} = A_{\mu}$, $E = \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}$ in (22). The $\gamma$-matrices are Hermitian $m \times m$ matrices with $m = \lceil d/2 \rceil$ satisfying $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$.

Let us consider the large-momentum limit of the spectral action. After lengthy but elementary algebra, one obtains

$$S(D, f_e, \Lambda) = -\frac{2^m \Lambda^{-2}}{(2\pi)^d} \sum_{p,q \in \mathbb{Z}^d} e^{-p^2/\Lambda^2} \text{tr}(A p^{-6} p_{\rho} \hat{F}_{\rho\mu} (-p) p_{\sigma} \hat{F}_{\sigma\nu} (p) q_{\mu} q_{\nu}),$$

$$+ 2 p^{-6} p_{\rho} \hat{F}^{\rho\nu} (-p) p_{\sigma} \hat{F}_{\mu\nu} (p) q^\rho q^\sigma + O(p^{-6}).$$

We see that the spectral action behaves as $p^{-4}$ at large momenta, as well as on the plane, cf [49], though on the torus, the structure of the action is much more complicated. If we now take the limit $\Lambda \to \infty$, we obtain

$$S(D, f_e, \Lambda) \simeq \sum_{p \in \mathbb{Z}^d} \text{tr}[\hat{F}^{\mu\nu} (-p) w_\Lambda (p^2) F^{\mu\nu} (p)],$$

$$w_\Lambda (p^2) = -\frac{2^{m+1} \sqrt{\sqrt{d}}}{(4\pi)^m} \frac{1}{p^2}, \quad p \to \infty, \Lambda \to \infty.$$
here for a generic Laplace-type operator on $T^d$ up to the second order in $E$ and $\omega$, giving also an expansion for the heat kernel of $D^2$ up to the second order in $A$. Also discussed are various limiting cases, including large-/small-$t$ (small-/large-$\Lambda$ is the spectral action), slowly/rapidly varying $A$.

Another meaningful expansion of the spectral action is the one when the curvatures are assumed to be almost covariantly constant (that is, derivatives of the curvatures are assumed to be small, though the curvatures themselves are of order of unity). Such expansions of the heat kernel were extensively studied by Avramidi [5].

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