Threshold Estimate for Fault Tolerant Quantum Computing

Christof Zalka

e-mail zalka@t6-serv.lanl.gov

March 3, 2008

Abstract

It has been show [2, 3] that once elementary unitary operations ("gates") in a quantum computer can be carried out with more than some threshold accuracy, then it is possible to carry out arbitrary precision operations on suitably encoded "computational" qubits. In this paper I will take a more practical, engineer-like point of view. I will follow P.Shor [1] for fault tolerant error correction (FTEC) and the fault tolerant implementation of elementary operations on states encoded by the 7-bit code. I will use (and try to justify) the most simple and natural error model. Computer simulation of an optimized version of Shors techniques gives an (astonishingly high) threshold of $\epsilon \approx 1/300$ on the tolerable error probability. For comparison I also provide a very rough calculation by hand.

1 Introduction

In the past 2 years ever better results have been achieved concerning what in principle can be done with the help of quantum error correcting codes. The most promising result yet says that once one can carry out elementary unitary operations with a certain precision, then suitable encoding schemes allow to carry out operations on encoded "computational" qubits with an arbitrary precision. From this it follows that one can then carry out computations with any number of steps without errors accumulating too much. Actually previously proposed methods seem not to make quantum computation possible on a computationally interesting scale with realistic assumptions on achievable gate accuracies.

2 On the error model

I would like to sketch a justification for the simple error model I use. The model is as follows: First of all I assume that errors happen with a certain probability
(per gate or per time). I also assume that these probabilities are independent for different qubits, except when an operation acts on several (2 to 3) qubits at once (like an XOR). For 1-bit errors I assume that the three possible errors (bit-flip, phase-flip and combined bit and phase-flip) occur all with probability $\frac{1}{3}$ (depolarizing channel). This is very natural as it is equivalent to the admixture of the unit density matrix to the original pure state. The analogous assumption will be made for several qubits taking part in an operation. Thus the 15 different errors of the two qubits which took part in an XOR are equally probable. The size is chosen such, that when we look only at one of the two qubits, we get the same probabilities as for a 1-qubit error. Actually one sees that the quantity $\frac{1}{3}$ would be more natural than the usual $\epsilon$. For simplicity I also assume that the error probability is the same for all operations and that such errors are the only ones occurring. Thus I e.g. don’t consider decoherence on “resting” qubits.

For decoherence a probabilistic error model immediately seems reasonable, as a mixed state (which decoherence produces) can be viewed as an ensemble of pure states together with their probabilities of occurrence. For this to be possible, the ensemble of states, taken as a basis, must diagonalize the density matrix. For depolarizing channel type of decoherence (which we assume here) this is true for any orthogonal set of states which includes the undisturbed original state.

A second source of errors are gate inaccuracies (unitary errors), thus deviations of the parameters of the unitary operation from the desired value. Let’s imagine a whole ensemble of QCs carrying out the same operation on the same state and assume that the unitary errors in different QCs are independent and have average 0 (=statistical errors). Provided we know nothing about the actual errors in the individual QCs, the resulting ensemble of states can at best be described by a density matrix. For the appropriate natural assumption on the distribution of the unitary errors, this (at least for small errors) leads to the same kind of mixed state as decoherence. The “natural” assumption is obviously that the error is “isotropic”, thus is’s the same for any three (for SU(2)) orthonormal parameters.

Quite another problem are systematic unitary errors (which we neglect here), e.g. when a rotation on a qubit tends to overrotate in all QCs. Such errors could e.g. be diminished by some feedback mechanism which would correct gates after several preliminary test runs.

3 Review of Shors fault tolerant error correction

3.1 The 7-bit code

The codes used by Shor are of the type described by Calderbank and Shor [4] and Steane [5], where a quantum code is constructed from two (classical) binary linear codes $C_1$ and $C_2$ with $C_2^\perp \subset C_1$. For Shors code we have $C_1 = C$ and $C_2 = C^\perp$, 

and also \( \dim(C) - \dim(C^\perp) = 1 \), thus these codes encode 1 qubit. Such a code can be obtained from a self dual \( (C = C^\perp) \) Reed-Muller code by leaving away one of the \( 2^m \) bits. For the threshold the smallest such code that still can correct for 1 error is relevant. This is the 7-bit code given by the following 4 basis elements:

\[
\begin{align*}
1011100 \\
1101010 \\
1110001 \\
0100011
\end{align*}
\]

The dual code \( C^\perp \) is given by the first 3 of these elements, thus \( l = \dim(C) = 4 \), \( \dim(C^\perp) = 3 \), and the length \( n = 7 \). In the so called “s-basis” the quantum codewords are

\[
|s_v\rangle = 2^{-(n-l)/2} \sum_{w \in C^\perp} |w + v\rangle \quad \text{where} \quad v \in C
\]

Actually there are only 2 different codewords, and they will represent an encoded 0 resp. 1:

\[
|0_L\rangle = |s_v\rangle \quad \text{for any} \quad v \in C^\perp
\]

and

\[
|1_L\rangle = |s_v\rangle \quad \text{for any} \quad v \in C/C^\perp \quad \text{(thus} \quad v \in C, \quad \text{but} \quad v \not\in C^\perp).\]

Where \( L \) stands for “logical” which means the same as “computational”. By applying a Hadamard transformation to each of the 7 qubits we go to the “c-basis”. One can show that:

\[
|c_v\rangle = H^7 |s_v\rangle = 2^{-l/2} \sum_{w \in C} (-1)^{v \cdot w} |w\rangle
\]

### 3.2 Syndrom measurement

In classical coding theory the syndrom of a received (and possibly distorted) codeword is obtained by scalar multiplication with 3 basis elements of \( C^\perp \). Clearly when all 3 syndrom bits are 0 then there is no error in the codeword. The other 7 possibilities each correspond to a particular one of the 7 bits having flipped. Thus knowledge of the syndrom allows to restore the original codeword (provided there is at most 1 error). For quantum error correction it is crucial that one can measure the syndrom without measuring the encoded qubit. Encoded qubits \( \alpha |0_L\rangle + \beta |1_L\rangle \) are in either basis linear combinations of basis states \( |v\rangle \) with \( v \in C \). A bit-flip in one qubit will cause the same syndrom in all \( |v\rangle \)’s, thus measuring the syndrom will not collapse the encoded qubit. Phase-flips don’t affect the syndrom and combined bit and phase-flips will look like bit-flips. Now a Hadamard transform on a qubit will change a bit-flip into a phase-flip and vice versa. Thus by measuring the syndrom in both, the \( s \)- and the \( c \)-basis
we get the full “quantum syndrom” which allows us to restore the original state by applying appropriate flips to the state. We see here that the 7-bit code can actually handle more than just 1-bit errors as long as there is only one bit-flip and one phase-flip (combined bit and phase-flips count as both).

To measure the syndrom bit corresponding to some \( v \in \mathcal{C}^\perp \), we could take an auxiliary qubit in the initial state \( |0\rangle \) and then XOR all qubits in the encoded state at positions where \( v \) has a 1 into it. Note that all non-zero elements of \( \mathcal{C}^\perp \) have four 1’s.

### 3.3 Fault tolerant syndrom measurement

Shor proposes to measure syndrom bits in a different way. For every such measurement we need an auxiliary 4-qubit “cat” state

\[
|\text{cat}\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle)
\]

which we Hadamard transform to get

\[
H^4 |\text{cat}\rangle = 2^{-3/2} \sum_{|x| \text{ even}} |x\rangle
\]

thus a linear combination of 4-bit words with an even number of 1’s. Now instead of XORing the 4 qubits of the state into one auxiliary qubit, we XOR each one into one of the 4 auxiliary qubits. Then we observe this state. The parity bit of the observed 4-bit word will then be the syndrom bit. One can wonder whether we collapse the encoded state in an unwanted way by observing 4 qubits instead of just the syndrom bit, but one can show that this is not so.

### 3.4 What is fault tolerance?

If computation with encoded states is to pay off, the error probability (of making a state uncorrectable), should be smaller than the fundamental error rate \( \epsilon \) one would have without encoding. This can be achieved by making sure that an error in error correction affects at most one bit of the state. Then the probability of introducing errors into several qubits is of order \( \epsilon^2 \). Thus by making \( \epsilon \) small enough we can make sure that encoding pays off. A conventional syndrom measurement with just 1 auxiliary qubit isn’t fault tolerant, because, as one can check, the probability of affecting several qubits is of order \( \epsilon \), not \( \epsilon^2 \) as with Shors method.

### 4 Fault tolerant quantum computation (FTQC)

To achieve fault tolerant quantum computation, besides FTEC we need a way to apply computational operations to the encoded states such that again the
error probability is of order $\epsilon^2$. For the 7-bit code several operations can be carried out bitwise, that is, to apply an operation to an encoded qubit we have to apply it individually to all 7 qubits. Pauli matrix operations (Clifford group) and the XOR between 2 computational qubits can be carried out in this way. To be able to carry out any calculation we need an additional operation, e.g. the Toffoli gate. Applying a Toffoli gate to three encoded qubits is not this simple, but it can also be done fault tolerantly (see [1]).

4.1 Interleaving operations and error correction steps

To compute with encoded states, we are going to apply operations fault tolerantly and once in a while we will apply an FTEC step to an encoded qubit. There is an optimal number of operations on should carry out between error correction steps. This can be seen as follows: When we apply a long series of operations without FTEC, the probability of errors in 1 bit will increase approximately linearly. Thus the probability of uncorrectable errors will increase quadratically, so the error probability per operation increases linearly with the number of operations. If we would apply too many FTECs we would introduce unnecessary errors, as there is some probability (of order $\epsilon^2$) that even an undisturbed state will be destroyed. Somewhere in between these extreme cases there will be an optimum. In the following let’s only consider the operations which can be carried out bitwise (“transversally”) on encoded states. One such operation will introduce error probability $\epsilon$ per qubit. It turns out that the optimal number of such operations between FTECs is approximately independent of $\epsilon$ (for small $\epsilon$) and is roughly 15 for my implementation of Shors FTEC. As the fault tolerant implementation of the Toffoli gate corresponds to rather more than this number of bitwise operations, we may want to introduce FTECs in the middle of a Toffoli gate application. We see that the performance of some FTEC scheme can be summarized by giving the amount of destroyed states per 1-bit error probability introduced by operations. For this I get approximately 300 $\epsilon$, thus 1 bitwise operation will cause the destruction of the state with probability $300 \epsilon^2$.

5 My (optimized) implementation of Shors FTEC

Here I describe how I have implemented FTEC in a computer program to get goods performance. First a cat state has to be constructed. This can be done by resetting 4 qubits to $|0\rangle$ and then Hadamard transform the first one to get $1/\sqrt{2} (|0\rangle + |1\rangle) \otimes |000\rangle$. Then I apply an XOR from the 1. to the 2. qubit then from 2. $\rightarrow$ 3. and finally from 3. $\rightarrow$ 4. Now this state would have a probability of order $\epsilon$ to have more than 1 bit-flip, which would be harmful. Thus, as Shor points out, we have to verify this state. I take an additional auxiliary qubit in state $|0\rangle$ and XOR the 1. and the 4. cat qubit into it. If upon observation
this qubit isn’t 0 we have to try again to construct a cat state. Hadamard transforming the 4 qubits in the last step is of course fault tolerant.

5.1 Syndrom measuring strategies

As Shor also points out, it is not enough to measure the syndrom just once to get error probability $O(\epsilon^2)$. On the other hand we should minimize the number of syndrom bit measurements as they threaten to destroy the encoded state. Instead of repeating the whole syndrom measurement (in one basis, say the $s$-basis), I measure the parity bit of the 3 just measured syndrom bits. Thus if $v_1, v_2, v_3$ are the 3 basis elements of $C^\perp$, I next measure the syndrom bit corresponding to $v_4 = v_1 + v_2 + v_3$. If this parity check turns out to be wrong, I again measure the parity bit of the last three measurements, thus

$v_5 = v_2 + v_3 + (v_1 + v_2 + v_3) = v_1$, and so on till the last 4 measurements are consistent. This strategy has the advantage that errors which have been introduced into the encoded state by early syndrom bit measurements, may be detected and corrected. Another improvement can be made when the syndrom indicates an undisturbed state, as will be the case most of the time. In this case I don’t measure the parity bit, as no erroneous error correction attempt threatens to introduce an additional error into the state. Actually we can even go further and abandon the error correction step as soon as the first syndrom bit measurement yields 0. Of course which one we take as the first one should then cyclicly be changed between $v_1, v_2, v_3$ from error correction step to error correction step. In a way this is like carrying out only $1/3$ of an error correction step, but as actually 4 out of 7 errors will show up in a given syndrom bit, it pays off. Between such $1/3$ FTEC steps the optimal number of operations will also roughly be divided by 3, thus we get 5, which I have used in my simulation.

5.2 Reducing the number of Hadamard transforms

Schematically a syndrom bit measurement in the $s$-basis looks as follows:

$$
\begin{align*}
|\psi\rangle & \xrightarrow{H} |\text{cat}\rangle \\
& \xrightarrow{\text{XOR}} M
\end{align*}
$$

where $H$ stands for a Hadamard transformation on each qubit and $M$ means measurement. The XOR applies only to the 4 qubits of the encoded state $|\psi\rangle$ which are at the positions of 1s in $v \in C^\perp$. 

6
Measuring a single syndrome bit in the $c$-basis is more complicated, as we first have to transform the state to the $c$-basis and then back again:

$$|\psi\rangle \xrightarrow{H} \xrightarrow{H} |\mathrm{cat}\rangle \xrightarrow{H} \xrightarrow{\mathrm{M}}$$

But an XOR conjugated with Hadamard transforms is simply an XOR in the opposite direction, thus we get the same result with:

$$|\psi\rangle \xrightarrow{\mathrm{XOR}} |\mathrm{cat}\rangle \xrightarrow{\mathrm{H}} \xrightarrow{\mathrm{M}}$$

6 Iterated encoding (concatenated codes)

If the error per operation is smaller on the encoded level than on the fundamental level, it seems that by iterating this step we can achieve an arbitrary precision. So on the next level we would encode states with the 7-bit code where each qubit would again be encoded with this code (49 qubits per computational qubit). This requires that all operations needed for FTEC can also be carried out on the encoded level (with the reduced error probability). The necessary operations are: Hadamard transformation, XOR, preparation of $|0\rangle$ and observation of a qubit. For my threshold result it will be crucial that the Toffoli gate isn’t needed for FTEC. The Hadamard transform and the XOR can be carried out bitwise (transversally), thus they can be carried out with the new “fundamental” error $\epsilon_1 = 300 \epsilon_0^2$ ($\epsilon_0 = \epsilon$).

6.1 Making $|0_L\rangle$ and observing encoded qubits

For preparing encoded states $|0_L\rangle$ I propose to start with an arbitrary 7-qubit state and then essentially apply FTEC to it till we get $|0_L\rangle$, where the possibility to throw away states that don’t seem to have come out right, is important. To be able to discriminate between $|0_L\rangle$ and $|1_L\rangle$, we need “syndrom” bits other than $v \in C^\perp$, namely we need at least one $v \in C/C^\perp$. We could proceed as
follows: We take an arbitrary 7-qubit state and apply error correction to it. In addition to the usual $2 \times 3 = 6$ syndrom bits we have to measure a “0/1-syndrom bit” telling us whether we have $|0_L\rangle$ or $|1_L\rangle$. If we get $|1_L\rangle$ we can either throw it away or we can change it to $|0_L\rangle$ by negating all 7 qubits. Then we have to verify the thus obtained state, e.g. by measuring all 6+1 syndrom bits. If they are not all 0 we start over. One can see that the error probability of such verified states is actually less than $\epsilon_1$. I have also checked it by computer simulation.

The observation of an encoded qubit should be done in the $s_v$ basis as only there one can tell $|0_L\rangle$ from $|1_L\rangle$ by observing a 7-bit word belonging to the superposition. Because these words are in $C$ we get an error probability $O(\epsilon^2)$ which is actually clearly smaller than $\epsilon_1$.

So the bitwise operations plus preparation of $|0_L\rangle$ and qubit observation can be done with error $\epsilon_l$ (or less) on the $l$th level where $\epsilon_l = 300 \epsilon_{l-1}^2$. This recursion relation leads to a very fast diminution of $\epsilon_l$ with increasing $l$ provided the threshold condition $\epsilon_0 = \epsilon < 300$ is fulfilled.

6.2 The error per Toffoli gate

It turns out that the Toffoli gate doesn’t play a role in the determination of the threshold, but to achieve a certain precision, it may be necessary to increase the number of encoding levels. Shors fault tolerant implementation of the Toffoli gate \[1\] consists of some 20 bitwise operations plus on average about 2 bitwise (transversal) applications of the (next lower level) Toffoli gate. This leads to the following recursion relation for the error $\epsilon_{T,l}$ per Toffoli gate on level $l$:

$$\epsilon_{T,l} = 20 \epsilon_l + 300 \epsilon_{l-1}^2 \cdot 2 \epsilon_{T,l-1}$$

Where I have used the number 300 $\epsilon_{l-1}$ for the performance of FTEC on level $l$. Given the fast convergence of $\epsilon_l$ this recursion relation also leads to an asymptotically fast convergence $\epsilon_{T,l} \to 0$, but $\epsilon_{T,l}$ may first become larger than its starting value $\epsilon_{T,l=0} = \epsilon$. Actually the error per Toffoli gate will become larger than the amount of error after which we would like to make an FTEC step. This will give a larger Toffoli error than predicted by the above recursion relation, but the error will still converge fast to 0, as one can check.

7 Calculations

7.1 Some calculations by hand

Consider the following diagram representing $n$ bitwise operations in a row followed by an error correction step.
The lines represent transitions between the 3 different “error states” in which an encoded state can be. I have marked only the transitions which will play a role in the following calculation. In the “n operations” step of course most (100% to order $\epsilon^0$) states will remain error free. The same is true for the error correction step. I also assume that we have a “full” error correction step, which means that to order $\epsilon^0$ all correctable errors get corrected. In my “optimized” implementation of FTEC where most of the time I measure only one syndrome bit in the $s$-basis and one in the $c$-basis, we would also have to take into account that $p_{11} > 0$, actually $p_{11} = 3/7$ or so. It is also a simplification that I put the correctable errors in one category. Actually one should make a difference between single bit-flip or phase-flip errors and 1 bit-flip plus 1 phase-flip errors, as they behave differently. I hope the simplified picture is enough to give an impression of the full calculation. Another approximation is that I assume $p_{01}$ to be small compared to the amount of correctable errors introduced by the $n$ operations, thus I set $p_{01} = 0$.

The quantity we are looking for is the probability that a state gets an uncorrectable error. As the diagram shows, such errors happen during the operations stage as well as during the error correction step. As the $n$ operations are applied bitwise, the error probabilities for different qubits are independent. This gives $\binom{7}{2}(n\epsilon)^2$ for the probability that two errors occurred in different qubits.

With all this we get for the probability of uncorrectable errors per bitwise operation:

$$\epsilon_1 = \frac{1}{n} \left( \binom{7}{2}(n\epsilon)^2 + 7n\epsilon \cdot p_{12} + p_{02} \right)$$

For fault tolerant error correction we can write in leading order:

$$p_{12} = K_{12} \epsilon \quad p_{02} = K_{02} \epsilon^2$$

where the $K$’s are characteristic numbers for the FTEC scheme. With this we get:

$$\epsilon_1 = \frac{1}{n} \left( 21n^2 + 7n\epsilon \cdot K_{12} + pK_{02} \right) \epsilon^2$$
We now have to find the optimal number \( n \) of bitwise operations per error correction step by minimizing this expression. We get:

\[
  n = \sqrt{K_{02}/21}, \quad \text{so} \quad \epsilon_1 = (2\sqrt{21K_{02}} + 7K_{12})\epsilon^2 = K\epsilon^2
\]

Thus the real interesting number for characterizing the FTEC scheme is \( K \).

### 7.2 Code combinatorics

The hard part of the calculation is to determine the combinatoric constants \( K_{02} \) and \( K_{12} \), which is why eventually I used a computer simulation. Nevertheless I try to show how in principle one could proceed by presenting a very crude calculation. The interested reader is encouraged to fill in the gaps (and to correct the errors).

The quantity \( K_{02} \cdot \epsilon^2 \) is the probability that an initially undisturbed state gets destroyed by an FTEC step, given that 2 errors occurred. Thus \( K_{02} \) is the number of possibilities how two errors in FTEC (both of probability \( \epsilon \)) can destroy a state. Let’s look at \( K_{02} \) and \( K_{12} \) for the case where we only try to correct errors in one basis (thus e.g. only bit flips). Most of the time we will get 0 for the first syndrom bit and stop there. Thus both errors would have to be introduced into the state by the syndrom bit measurement. One can estimate that this doesn’t give a big contribution.

It seems that the main contribution to \( K_{02} \) comes from the case where one error causes the first syndrom bit measurement to yield 1 and at the same time the state to get an error.

To estimate these contributions, we should first know the probabilities for errors in the cat state. Here I take these probabilities from my computer simulation, although with some patience it could be done by hand. A cat state has at most 1 phase-flip (pf). I get the following numbers:

\[
  p_{1pf} \approx 1.6\epsilon, \quad p_{1bf} \approx 0.7\epsilon, \quad p_{1pf+1bf} \approx 0.7\epsilon, \quad p_{2bf} \approx 10\epsilon^2
\]

\( \text{From this we next calculate the probabilities of a wrong syndrom bit measurement and of introducing an error into the state. Note that under the Hadamard transformation the cat phase-flips become bit-flips which give an erroneous syndrom bit. So the syndrom bit error is:} \)

\[
  p_{sb} = p_{1pf} + p_{1pf+1bf} + 4 \cdot 2/3 \approx 5\epsilon
\]

where the 3. term is the XOR error. The probability of introducing an error into the state during a syndrom bit measurement is:

\[
  p_{state} = p_{1bf} + p_{1pf+1bf} + 4 \approx 4.5\epsilon
\]

The first two terms are phase-flips that are transported from \( H^4 \text{cat} \) to state by the XOR. The 3. term is again from the XOR error. The probability that both, the syndrom bit and the state get wrong is:
\[ p_{sb+state} = p_{bf+1bf} + 4 \cdot 2/3 \epsilon \approx 3.4 \epsilon. \]

Say this happened in the first syndrom bit measurement. Then in the following 3 to 4 such measurements either the state gets another error or another syndrom bit is wrong which may lead to an erroneous error correction attempt that makes the state uncorrectable. With \( 3 \frac{1}{2} \) measurements these two probabilities are \( 3 \frac{1}{2} \cdot p_{sb} \) and \( 3 \frac{1}{2} \cdot p_{state} \). Taking only these contributions we get very roughly:

\[ K_{02} \approx 3.4 (3 \frac{1}{2} + 3 \frac{1}{2} \cdot 4.5) \approx 113 \]

For \( K_{12} \) we take the probability that a second error gets introduced into the state during the 4 syndrom bit measurements. But only a fraction of about \( \frac{2}{3} \cdot \frac{4}{7} \) of the correctable errors will show up in the first syndrom bit and will thus lead to more measurements, thus we get:

\[ K_{12} \approx \frac{2}{3} \cdot \frac{4}{7} \cdot 4.5 \approx 7 \]

Now we have estimated these values for 1/6 FTEC and there is really no clean way to obtain from this the corresponding values for a full FTEC step without knowing more constants. But as we want to get some answer, we simply multiply both constants with 6 and calculate \( K \) from this. We get:

\[ K \approx 2 \sqrt{21} \cdot 6 \cdot 113 + 7 \cdot 6 \cdot 7 \approx 240 + 300 \approx 550 \]

which confirms the order of magnitude of the threshold given by the computer simulation.

### 7.3 The computer simulation

As mentioned, in the simulation I use the probabilistic error model with 3 equally possible 1-bit errors. I simulate neither a pure state (with \( 2^2 \) qubits amplitudes) nor the whole density matrix (with even \( 2^2 \cdot 2^2 \) matrix elements) but I simply store for each qubit whether it is undisturbed or which one of the three 1-bit errors it has undergone. This can be done by storing a bit-flip bit and a phase-flip bit for each qubit. For a combined bit and phase-flip both bits will then be 1. Note that by repeating my simulation often enough, I actually get the density matrix (up to some statistical error).

After each operation, including observations and preparations of \( |0 \rangle \), I introduce 1-bit errors with the appropriate probability. When doing a Hadamard transform I interchange the bit-flip and the phase-flip bits. For XORs the bit-flip bit has to be “propagated” forward, thus we have to add the bit-flip bit of the 1. qubit to the bit-flip bit of the target qubit. Phase-flips have to be “propagated” in the opposite direction.
In the simulation I subject an initially undisturbed encoded state alternat-
ingly to 1/3 FTEC steps and to the optimal number (5) of bitwise operations,
thus I introduce with a probability of $5\epsilon$ 1-bit errors into each qubit. I then look
how many operations I can carry out like that till the state has more than 1
bit-flip or 1 phase-flip. Taking the average of $10^4$ to $10^5$ such runs gives a value
for the error per bitwise operation on the encoded level with a small statistical
uncertainty.

In my simulation I also took into account that some complicat-
ed error pat-
terns (e.g. 3 bit-flips) actually are equivalent (give the same state) to simpler,
correctable errors. To do this, I had to consider the so called stabilizer of the
code. That is the multi qubit errors that leave the state unchanged. It turns
out that the stabilizer consists of 4 bit-flips or 4 phase-flips corresponding to a
$v \in \mathbb{C}^\perp$. The implementation of “stabilizer-reduction” of states into my simula-
tion has improved the threshold result from $1/500$ to $1/300$.

I would be glad to provide interested people with more details of my simu-
lation (or also the C-program), so that the result could be checked.

8 Summary

I claim that the threshold is determined only by the operations needed in a fault
tolerant error correction step. For the 7 bit code these are: XOR, Hadamard
transformation, qubit measurement and preparation of $|0\rangle$. As long as the Toffoli
gate can be applied fault tolerantly to encoded states, its error can eventually
be brought down very fast by sufficient concatenation, though it may remain
much larger than e.g. the error of an XOR.

An improved implementation of FTEC with the 7 bit code allows me to
achieve a threshold of $\epsilon \approx 1/300$. The improvements are: I carry out several
operations (effectively 15) between error correction steps. This gives an
improvement by about a factor of 4. Then instead of repeating the whole syndrom
measurement, I just measure the syndrom bit corresponding to the parity bit of
the last 3 syndrom measurements and stop if it comes out right. Also I try to
minimize the threat to undisturbed states by stopping if the first syndrom bit is
0. Like that a fraction of 3/7 of correctable states is going to be overlooked, but
they will be corrected in following FTEC steps where I will measure another
syndrom bit first.

9 Outlook

The present scheme is not thought to be applied directly to a physical quantum
computer as much more efficient methods may take care of the special error
patterns usually occurring. The result $\epsilon = 1/300$ may rather be seen as a
guideline about what kind of accuracy is required.
My scheme can certainly be improved further. The following observation won’t help to improve the threshold, but may allow to reduce the number of encoding levels: If the fundamental error $\epsilon$ is below the threshold such that the error on the 1. encoded level is sufficiently reduced, we may use another code than the 7-bit one for the 2. level, as it may reduce the error faster for small enough errors, even so its threshold is worse. In particular we may want to use a code which allows a simpler implementation of the Toffoli gate (or an equivalent one, see e.g. [3]) and thus a faster reduction of the Toffoli error.

A more general observation is the following: On a lower level of encoding one could use a shorter mere error detection code which would signal to the higher level which encoded qubit is in error. This would also allow a simpler code on the higher level. Eventually of course a compact unified understanding of threshold FTQC may allow more progress.

10 Acknowledgements

I did this work at the Quantum Computation and Quantum Coherence program at the Institut of Theoretical Physics, University of California, Santa Barbara and at LANL in November 96. I would like to thank the swiss national science foundation (Schweizerischer Nationalfonds) for its support.

References

[1] P. Shor
   “Fault-Tolerant Quantum Computation” quant-ph/9605011

[2] E. Knill, R. Laflamme and W. Zurek
   “Accuracy Threshold for Quantum Computation” quant-ph/9610011

[3] D. Aharonov, M. Ben-Or
   “Fault-Tolerant Quantum Computation With Constant Error” quant-ph/9611025

[4] A. Calderbank, P. Shor
   “Good quantum error-correcting codes exist” quant-ph/9512032

[5] A. Steane
   “Multiple particle interference and quantum error correction” quant-ph/9601029