Strings and skyrmions on domain walls

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Abstract

We address a simple model allowing the existence of domain walls with orientational moduli localised on them. Within this model, we discuss an analytic solution and explore it in the context of the previous results. We discuss the existence of one-dimensional domain walls localised on two-dimensional ones, and construct a corresponding effective action. In the low-energy limit, which is $O(3)$ sigma-model, we discuss the existence of skyrmions localized on domain walls, and provide a solution for a skyrmion configuration, based on an analogy with instantons. We perform symmetry analysis of the initial model and low-energy theory on the domain wall world volume.
1 Introduction

In certain field theories existence of non-Abelian moduli, localized on topological defects, is possible; this discovery was first made for Super-Yang-Mills theories with matter for non-Abelian topological strings [1, 2, 3]. Then, inspired by a work [4] on cosmic strings, a similar construction was developed [5, 6] for domain walls. In [6] existence of such a construction was proved by numerical calculations, and its low-energy dynamics - translational moduli of the domain wall and rotational non-Abelian moduli, localized on the wall - was analysed.

Here we expand the previous results on non-Abelian moduli in several directions. First we establish correspondence between recently found analytic solution [7, 8] of the domain wall profile, which supports non-Abelian moduli, and previous results from [6], discussing the possibility to compare numeric and analytic solutions and the energies of lowest modes. Then we explore a simplified case of the same model, where $O(3)$ symmetry of it is replaced by $Z_2$ and there are only two scalar fields, one of which creates a domain wall and another one gets a non-zero expectation value on it only. We find that such a localized field can have a domain wall by itself, which in low-energy limit behaves as a one dimensional string-like object, localized on a two-dimensional surface. We discuss its properties and low-energy dynamics, deriving a corresponding effective action.

Then in a construction which supports non-Abelian moduli, described by an $O(3)$ sigma-model, we discuss the possibility of existence of skyrmions of a type which is usually explored in condensed matter physics, and give a solution for such a skyrmion, based on the analogy with instantons.

Both for the initial model and the model of non-Abelian moduli on the domain wall we perform a symmetry analysis, keeping in mind possible applications of this method to more complex problems. The systems under consideration are very convenient for testing this method, since they allow us to compare its results with some of the known answers, and at the same time allow to consider a well-known set-up from a new viewpoint. Review of the symmetry analysis method is provided.

Paper is organized as follows: section 2 reviews construction of moduli on domain walls, section 3 discusses an analytical solution of such a system, conditions of its existence and its properties, comparing it with the previous results in [6], section 4 describes one-dimensional wall, localized on a two-dimensional one, and in section 5 skyrmion, localized on a domain wall, is
constructed. In sections 2 and 5 we provide a symmetry analysis of the systems under consideration, and Appendix is devoted to the review of the method of symmetry analysis.

2 Moduli on a domain wall

Let us review the construction [5, 6] of moduli on domain wall. First consider a model with a real scalar field, described by the Lagrangian

$$L_0 = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi), \quad V(\varphi) = \lambda(\varphi^2 - v^2)^2. \quad (1)$$

It has a $\mathbb{Z}_2$ symmetry ($\varphi \to -\varphi$) which is spontaneously broken in the vacuum, where $\varphi$ can be equal to $v$ or $-v$. So this model supports domain walls created in the region which connects two vacua. Let us assume that a wall is parallel to the $(x, y)$ plane and $\varphi = \pm v$ for $z = \mp \infty$ (of course, the inverse case is also possible). Then the solution for $\varphi$ can be found analytically

$$\varphi(z) = -v \tanh\left[ \frac{m_\varphi}{2} (z - z_0) \right], \quad (2)$$

where $m_\varphi = \sqrt{8\lambda v^2}$ is the mass of the $\varphi$ field and $z_0$ is the wall center.

Then let us add to the above model a triplet of fields $\chi^i$, $i = 1, 2, 3$, described by the Lagrangian

$$L_\chi = \frac{1}{2} \partial_\mu \chi^i \partial^\mu \chi^i - U(\chi, \varphi),$$

$$U(\varphi, \chi^i) = \gamma \left[ (\varphi^2 - \mu^2)\chi^i \chi^i + \beta(\chi^i \chi^i)^2 \right], \quad v^2 > \mu^2, \quad (3)$$

so the new model has the Lagrangian $\mathcal{L} = L_0 + L_\chi$. Then for a choice of parameters$^1$ for which the vacua are given by $\varphi^2 = v^2, \chi^i = 0$, we still can have a domain wall, connecting these two vacua, and $\chi$, getting a non-zero expectation value only inside of that wall. Since some energy in the wall is taken by the kinetic term of $\chi$ and still the total energy should be smaller than without $\chi$ condensation, kinetic term of $\varphi$ should be smaller and the transition region between vacua (i.e. the wall) should become wider.

$^1$For more details about constraints on the set of parameters and stability of the solution under consideration see [6, 7, 9].
To get a domain wall solution explicitly, we need Euler-Lagrange equations, corresponding to $\mathcal{L}$:

\[
\begin{align*}
\varphi'' &= 4\lambda \varphi (\varphi^2 - v^2) + 2\gamma \chi^2 \varphi, \\
\chi^i'' &= 2\gamma (\varphi^2 - \mu^2) \chi^i + 4\beta \gamma \chi^2 \chi^i.
\end{align*}
\] (4)

The solution which minimizes domain wall energy is degenerate and still has an $O(3)$ symmetry of the Lagrangian (3): in all points of the physical space all orientations in the target space will have the same energy, as far as $\chi^i\chi^i$ is equal to the value, determined by the energy minimization, i.e. the live on sphere $S^2$. So we can find that solution, fixing orientation of $\chi$ in target space everywhere, for example, making only one component $\chi^3$ being non-zero and denoting it just as $\chi$; corresponding system of Euler-Lagrange equations will be just (4) with $\chi^i = \chi$, and we’ll also refer to it as (4). It possesses the first integral

\[
\frac{1}{2} \left( \frac{d\varphi}{dz} \right)^2 + \frac{1}{2} \left( \frac{d\chi}{dz} \right)^2 - U[\varphi(z),\chi(z)] - V[\varphi(z),\chi(z)].
\] (5)

Solution of (4) with domain wall, created by $\varphi$, and $\chi$, localized on it, was obtained in [6]. After energy-minimizing configuration was obtain, we can restore the $O(3)$ symmetry of the $\chi$ field, allowing its position in target space to depend on the space-time coordinates on the wall, while $\chi^i\chi^i$ is still determined by the energy-minimization and so localized on sphere $S^2$ in target space. In this way we get non-Abelian moduli on domain wall. Corresponding effective Lagrangian was also derived in [6].

Both in the problem of this section and in the consideration of an effective theory on domain wall world volume later in this paper we have a relatively simple Lagrangian, which allows us to see immediately the symmetries of the system under consideration and moduli which appear after breaking of some of these case. But in general this may not be the case. Some symmetries may be much harder to observe, and systems which don’t allow a Lagrangian description may be considered as well. For such cases a mathematical apparatus which allows to analyze symmetries of differential equations and integrate them using those symmetries may be a very powerful tool. Here and later in analysis of skyrmions we will perform such an analysis for the purpose of checking the results, exploration of symmetries which may be unnoticed by initial investigation and practicing with a method of symmetry analysis on a simple and well-understood set-up.
For details of the method, we refer the reader to the Appendix below, and to the book by Peter Olver [39]. For heavy-duty calculations that are often needed in the symmetry analysis, we recommend to use the software provided with the book [40].

In the case of the system (4), the symmetry analysis renders only the translational symmetry along the $z$-direction. So there are no other symmetries that we might had missed in our discussion. Quest for generalised symmetries furnishes no new results either.

### 3 Analytic solution

The numerical method, developed in [6], can give a solution for any set of parameters for which such a solution exists. But it is worth looking for an analytic solution of the same problem, since it can give a result, which is easier to analyse qualitatively. The price of such simplification may be some loss of generality, since the existence of an analytic solution may impose an additional constraints to the problem under consideration.

Following [7], let us look to the solution of the system for $\varphi$ field in the same form (2) as it had without additional field:

$$\varphi(z) = -v \tanh(\alpha z),$$

where the coefficient $\alpha$, which determines the new width of the wall, is to be determined later. We also placed the center of the wall at $z_0 = 0$. Then from the first equation in (4) we have:

$$\lambda^2 = 2\gamma v^2 - \alpha^2 \gamma^2 \cosh^2 \alpha z \equiv \frac{A^2}{\cosh^2 \alpha z}.$$

Plugging in this result into the second equation of (4), we get a polynomial function of $(\cosh \alpha z)^{-1}$; equality always holds if all the coefficients are zero, which leads to the following result:

$$\alpha^2 = 2\gamma (v^2 - \mu^2),$$

and

$$\alpha^2 = \gamma (v^2 - 2\beta A^2).$$

So for a field $\varphi$, which creates a domain wall, and a field $\chi$, localized on it, we obtained the analytic solution, given by (6) and (7) respectively; $\alpha$ is
given by (8). Since $\alpha$ and $A$ are functions of the parameters of the model, (9) gives one constraint for a set of parameters, under which such a solution exists.

Note that for $\varphi$ and $\chi$, provided above, the following holds:

$$\frac{\varphi^2}{v^2} + \frac{\chi^2}{A^2} = 1,$$

i.e. the solution corresponds to an ellipse in $(\varphi, \chi)$ plane, where vacua of the model correspond to the ends of the major axis, where $\varphi = \pm v$ and $\chi = 0$.

It is worth noting that the solution (6) for $\varphi$ has the same form as (2), which was obtained without condensing of $\chi$, but now the width $\alpha^{-1}$ of the wall, created by $\varphi$, is determined by parameters $\gamma$ and $\mu$ from $\chi$ part of the Lagrangian. The derivation above is inverse to the one provided in [10], where a similar result is produced, starting from an assumption that the solution in $(\varphi, \chi)$ plane has an elliptic shape.

It would be interesting to compare the numerical procedure, applied in [6], which may help to check the precision of the numerical solution, but this can’t be done for the set of parameters in [6], since they don’t satisfy (9).

The main difficulty in calculating numerically the domain wall configuration is the fact that boundary conditions are given only at infinity, and they correspond to the equilibrium state, i.e. if we start solving the differential equation from it, we’ll always remain at the same state. To omit this difficulty, the following approach was used in [6]: both fields were slightly changed from their equilibrium values, which corresponded to the values near infinity, and then shooting method was used: initial deflection was varied, until it gave the expected results near the other infinity.

This approach required asymptotics near infinity, which were obtained from (4) by leaving only the terms on the first order in $\varphi$ and $\chi$. But for the solution in (6, 7) this approach won’t work, since variation of $\varphi$ from the vacuum $\eta \equiv v - \varphi$ will be of the order of $\chi^2$, which is clear from (10). So the simple asymptotics of [6] don’t work. Of course, it is possible to obtain the asymptotics directly from the analytic solution [7], but it will be to a some extent different calculation procedure. When we have neither analytical solution nor ability to simplify [4], a configuration can be found only in more complicated numerical procedure, involving variation not only fields, but their derivatives too.

Another strategy of solving the system (4) numerically may turn out to be helpful. The form of the analytic solution written down in (6, 7)
suggests such a change of variables that the new variable is defined on a finite interval. Under this kind of transformation, the vacuum values of the fields at infinity become the boundary values at the ends of the said finite interval. Specifically, (4) compels us to try the change of variables

\[ z = \kappa \arctan \xi , \]  

(11)

with \( \kappa \) being a free parameter, and \( \xi \) being a new argument defined on the interval \([-1, 1]\). However, under this substitution, the equations acquire singular points at the ends of the interval, even though the desired solutions stay nonsingular there. Having performed a substitution (11) — or another transformation to a variable belonging to a finite domain — one can choose from an arsenal of methods developed to solve the boundary values problem.

In [7], an important problem of energy levels of a "bare" (with \( \chi = 0 \) everywhere) domain wall was discussed. Bases on a known spectrum of modified Pöschl-Teller potential it provided a much better estimate than [6], where the potential was just approximated by a parabolic well. But in general both results can’t give an exact result for energy levels of a non-"bare" wall, since condensation of \( \chi \) also changes the profile of \( \varphi \).

4 1d wall on a 2d wall

We are still working with the Lagrangian with the fixed orientation of \( \chi \) in the target space:

\[ \mathcal{L}_{z_2} = \frac{1}{2}(\partial \varphi)^2 + \frac{1}{2}(\partial \chi)^2 - \lambda(\varphi^2 - \nu^2)^2 - \gamma \left[(\varphi^2 - \mu^2)\chi^2 + \beta \chi^4\right]. \]  

(12)

Note that \( \chi \) possesses a \( \mathbb{Z}_2 \) symmetry: its Lagrangian and equations of motion are invariant under substitution \( \chi \rightarrow -\chi \). So if we have some configuration of the fields \( \varphi \) and \( \chi \), satisfying the equations of motion (4), configuration with the same \( \varphi \) and \( \chi \), which has the opposite sign everywhere, will also satisfy (4). Of course, this is correct for the construction described above: if we have a domain wall, created by the field \( \varphi \), and a field \( \chi \), localized on it, then the solution for \( \chi \) with opposite sign is possible and will have exactly the same energy. In particular, this is visible from [7], where \( \chi \)

\[ ^2 \text{In principle, one can use any function mapping an infinite interval onto a finite segment, like } z = \kappa \tan \frac{\xi \pi}{2} \text{ or } z = \frac{\xi}{(1 - \xi)}, \]
is defined up to a sign, but also holds in a more general case, when we don’t have an analytic solution. Static configuration on the wall breaks this $\mathbb{Z}_2$ symmetry, picking one of the two configurations with the same energy, and this allows appearance of topological defects.

Let us assume that a $\varphi$ field creates a domain wall in $(x, y)$ plane with the center at $z = 0$. Then assuming that at $y = -\infty$ we have a solution with positive $\chi$ and for $y = +\infty$ with a negative $\chi$, we’ll have a transition region between them - a domain wall for $\chi$, localized on a domain wall of $\varphi$. A similar construction was considered in [11].

It is clear that in the one-dimensional wall not only the profile of $\chi$, but also the profile of $\varphi$ will be different from the one at infinity. Previously we showed that non-zero expectation value of the former makes the wall along $z$ for the latter bigger, but in the transition region of $\chi$ in is close to zero, so here $\varphi$ penetrates deeper and the wall is narrower.

From a bigger scale the wall of $\varphi$ can be considered as a two-dimensional surface and the wall of $\chi$ - as a one-dimensional string, localized on it. Let us look to the low-energy transitional moduli of it. In a low-energy limit the profile of the system can be considered as unchanged, but the position of the string will depend on time and a coordinate $x$ along it. Plugging in this result into the action of $L = L_0 + L_\chi$ gives

$$
(\partial_\mu \chi)(\partial^\mu \chi) = - \left( \frac{\partial \chi}{\partial y} \right)^2 - \left( \frac{\partial \chi}{\partial z} \right)^2 - \left( \frac{\partial \psi}{\partial y} \right)^2 \partial_y \partial^y \partial^y \partial^0 - \left( \frac{\partial \chi}{\partial z} \right)^2 \partial_z \partial^z \partial^z \partial^0
$$

(13)

and analogous expression for $\psi$. Here we left only relevant terms (integral of the potential term will be same since the profile of the domain walls is the same). Omitting the inessential constants, which are obtained after the integration of the first two terms in (13), we get the following string effective action:

$$
\Delta S = \int dt dx \left[ B_y (\partial_y \partial^0 y_0) + B_z (\partial_z \partial^0 z_0) \right],
$$

(14)

where

$$
B_y = - \int dy dz \left[ \left( \frac{\partial \chi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right],
$$

(15)
\[ B_z = -\int dydz \left[ \left( \frac{\partial \chi}{\partial z} \right)^2 + \left( \frac{\partial \varphi}{\partial z} \right)^2 \right]. \]  \tag{16}

The second term in (14) should be taken into account due to the fact noted above: in the one-dimensional wall of \( \chi \) profile of two-dimensional wall of \( \varphi \) is different, so the effective action here is also different from the low-energy domain wall action, derived in [6]. If we interested in an effective action of a string on a wall, only the first term in (14) should be considered.

Let us look to the conditions which ensure the possible existence of the domain line solution described above and its stability. There are two different aspects of the problem. First of all, there should exist the solution which makes condensation of the \( \chi \) field inside of a domain wall energetically advantageous and stable. Such an existence and stability lay constraints on the parameters of the model which were extensively investigated in [6, 7, 9] and are applicable here too. This comes from the fact that the energy-minimizing solution with \( \chi \) condensation really defines only a profile of \( \chi(z)^2 \) but doesn’t affect any structure that \( \chi^2 \) can have: it works for an \( O(3) \) triplet \( \chi^2 = \chi^i \chi^i, i = 1, 2, 3 \) which was described above, for a \( \mathbb{Z}_2 \) symmetric singlet we are interested in in this section, and will work for any other model with a symmetry which would allow \( \chi^2 \) to be fixed.

When the constraints in parametric space are imposed, for the \( \mathbb{Z}_2 \) symmetric model (12) we are considering here the stability of a domain line follows from a topological grounds in the well-known way (see e.g. [12]): we have a topologically non-trivial mapping of an infinity in real space to the vacuum manifold, and it’s impossible so deform it to a topologically trivial mapping without making the energy of the system infinite in process.

The stability of the moduli configuration was proven in [6]. The stability of the domain-line solution within an effective field theory on a domain wall is guaranteed by topological arguments. Now, what if such a domain-line solution, albeit stable within the effective field theory, is unstable within the general theory with the Lagrangian (12) we start with? If a domain-line defect is unstable, it should have an energetically favourable mode of decay. The only possibility for the decay would be the region with \( \chi = 0 \) in the domain-line core spreading infinitely wide and leaving no \( \chi \) condensate on the wall. This, however, contradicts with the fact (proven by [6]), that the solution with non-zero \( \chi \) is energetically favourable. Therefore, the domain line must be stable.
Skyrmions

Now let us restore the $O(3)$ symmetry of the $\chi$ field. Magnitude of the vector $\vec{\chi}$ in the target space in each point of the real space is defined by energy minimization, but it has a freedom of rotations in target space, so we can write $\chi^i = \chi(z) S^i(t, x, y)$. In [6] it was shown that the corresponding two-dimensional effective action on domain wall is

$$S_{eff} = \frac{1}{2} \frac{\lambda^2}{m^2} I_1 \int dt \, dx \, dy \left( \partial_p S^i \partial^p S^i \right), \quad S^i S^i = 1, \quad i = 1, 2, 3, \quad p = 0, 1, 2,$$

(17)

where

$$I_1 = \frac{m^2}{\lambda^2} \int \chi(z)^2 dz.$$  \hspace{1cm} (18)

Topic of skyrmions on a domain wall has attracted a considerable attention [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. It is important both from theoretical viewpoint (see, for example, interesting results in [24] about non-commuting momenta) and in relation to the active experimental research on magnetic skyrmions on thin films and domain walls and their possible practical applications for data storage and in logic gates [25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. The model described in Chapter 2 with the Lagrangian
\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\chi \] actually supports skyrmions on domain walls within a setup very similar to the one, proposed in the pioneering work [35]. Its main advantage is that it describes both the bulk and a domain wall with skyrmions within a unified approach, where the properties of lower-dimensional topological defects follow from the general description of the (3+1)-dimensional physics. The domain wall itself arises as a symmetry-breaking solution ensuing from the translationally invariant Lagrangian. The domain wall solution allows to derive the effective field theory on that wall from the initial model and the skyrmion, in turn, arises as the solution of that effective field theory. This way, we do not need to manually include symmetry-breaking terms like the anisotropy term in [23] or the commonly employed Dzyaloshinskii-Moriya interaction. In the same way as for the domain line in the \( \mathbb{Z}_2 \) model stability of such a skyrmion is of topological nature.

It is convenient to visualize such a defect, imagining a target space position as a 3-dimensional vector, attached to each point of the domain wall plane, whose \( x, y \) and \( z \) components are \( S^1, S^2 \) and \( S^3 \) respectively. This representation becomes exact, when target space position represents a spin vector. Let us introduce spherical coordinates \((\theta, \psi)\) in target space and polar coordinates \((r, \phi)\) on the plane with \( r = 0 \) at the center of the defect. For spin systems investigated configurations include the simplest case, depicted on Figure 1 when vector in target space goes from the north pole in the

Figure 2: Spiral skyrmion: target space vector winds around while going from one pole at the center of the defect to another one at infinity. Image from [26].
Figure 3: (a) - (c) - components of $S$ from \((20)\) as functions of coordinates on the domain wall plane, measured in the units of $|\vec{y}|$, (d) - $S$, visualised as a three-dimensional unit vector on the domain wall plane, whose $x$, $y$ and $z$ components are $S^1$, $S^2$ and $S^3$ respectively.
center of the defect do the south pole at infinity, while its projection to the
domain wall plain alway goes along the radius-vector. In a more complicated
spiral case (Figure 2) projection of a target space vector to the plane winds
around the center while $r$ increases.

Topologically the defects we are looking for are the easiest to understand
through stereographic projections of the the real plane of domain wall to a
sphere, where the center of the defect corresponds to one pole and vacuum
at infinity - to the other one. The topologically non-trivial mappings are
possible between real and target space spheres, which can be classified, using
the winding number

$$N = \frac{1}{8\pi} \int \varepsilon_{ij}(\partial^i \vec{x} \times \partial^j \vec{x})dxdy.$$  \hspace{1cm} (19)

The simplest way to get an exact solution for $S^i$ as functions of space
coordinates for a skyrmion is based on it being topologically identical to the
instanton in 1+1 dimensions, when the latter one is constructed in Euclidean
time. Then for a skyrmion with $N = 1$ which goes from north pole at the
center to the south pole at infinity we can immediately write the answer
[36, 37]:

$$S^1(\vec{x}) = \frac{2(\vec{x} - \vec{x}_0) \cdot \vec{y}}{(\vec{x} - \vec{x}_0)^2 + \vec{y}^2}, \quad S^2(\vec{x}) = \frac{2(\vec{x} - \vec{x}_0) \times \vec{y}}{(\vec{x} - \vec{x}_0)^2 + \vec{y}^2},$$

$$S^3(\vec{x}) = \frac{\vec{y}^2 - (\vec{x} - \vec{x}_0)^2}{(\vec{x} - \vec{x}_0)^2 + \vec{y}^2}, \quad (\vec{a} \times \vec{b} = a_1 b_2 - a_2 b_1).$$  \hspace{1cm} (20)

Here $\vec{x}$ represents coordinates in plane, $\vec{x}_0$ it the skyrmion center and
vector $\vec{y}$ defines orientation and size ($|\vec{y}|$) of the defect. These functions for
$\vec{y} = (1, 0)$ are shown on the Figure 3 both as three different components of
$S$ and as three-dimensional vectors on the domain wall plane. We see that
$S^3$ decreases continuously and the regions with positive and negative $S^1$ and
$S^2$ are symmetric with respect to $y$ and $x$ axes respectively.

The skyrmion (20) has four degrees of freedom: two coordinates of the
center, orientation and size. Considering them as functions of time while
keeping the profile of the system fixed, we can derive an appropriate low-
energy effective action in the same way as it was done above for the one-
dimensional domain wall. Since this was done for a very similar model in
[38], we refer reader to it.
Skyrmion solution again gives us a convenient set-up to apply the method of symmetry analysis. Such a solution was obtained from the system (36, 37):

$$\partial_\mu S^i = -\epsilon^{ijk} S^j \epsilon_{\mu\nu} \partial_\nu S^k$$

(21)

with boundary conditions for which vector in target space goes from one pole at the center of the defect to another pole at infinity.

The system (21) permits for the symmetry group generated by the following operators:

$$X_1 = \partial_x$$
$$X_2 = \partial_y$$
$$X_3 = y \partial_x - x \partial_y$$
$$X_4 = x \partial_x + y \partial_y$$
$$X_5 = S^1 \partial_{S^2} - S^2 \partial_{S^1}$$
$$X_6 = S^2 \partial_{S^3} - S^3 \partial_{S^2}$$
$$X_7 = S^3 \partial_{S^1} - S^1 \partial_{S^3}$$

(22)

The first four are responsible for the translational, boost and scaling invariance in the coordinate space, while the three latter — for the rotational invariance in the target space, see the discussion in Appendix.

6 Conclusions

After reviewing the construction of domain walls with orientational moduli, localized on them, we compared new results on analytic solutions for such a system with previous numeric results. Then in a simplified model with two scalar fields, both possessing a $\mathbb{Z}_2$ symmetry, we shooed a possibility of one-dimensional domain walls, localized on two-dimensional ones, and derived a corresponding effective action. Then for an effective $O(3)$ sigma-model, localized on a domain wall, we considered the possibility of existence of skyrmions and provided a solution in analogy with the known one for instantons. We analysed symmetries of the initial model which supports domain walls and effective low-energy theory of the domain wall world volume.
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Appendix A  Symmetry analysis

We start out with addressing a general approach to eliminating all continuous symmetries of a system of differential equations of the form

\[ X = \xi^j(x^k, u^l) \partial_{x^j} + \eta^s(x^k, u^l) \partial_{u^s} , \]

\[ j, k = 1, 2, \ldots, m \quad , \quad l, s = 1, 2, \ldots, n \ . \quad (23) \]

Here \( x^j \) and \( u^l(x^j) \) stand for independent variables and their functions, respectively.

The form of the symmetry operator (23) indicates that we are seeking so-called point symmetries — those for which the functions \( \xi^j \) and \( \eta^l \) depend only on the coordinates \( x^j \) and functions \( u^l(x^j) \). In principle, nothing should prohibit us from allowing the functions \( \xi^k \) and \( \eta^l \) to depend on the derivatives of functions as well. However, in the current paper we limit ourselves to point symmetries solely.

All the symmetries of a system of differential equations can be found with the aid of a general algorithm described in [39]. Being straightforward but quite lengthy, this algorithms is often implemented by computer packages. These are especially needed when the system is complicated, i.e., contain a large number of variables or/and dependent functions, or contains higher-order derivatives.

Oftentimes, the knowledge of symmetries helps to perform integration of the system of differential equations. This task is more challenging than the search for symmetries, and each specific problem has to be treated uniquely. Still, various methods exist, of which the most universal one relies on the employment of differential invariants of the symmetries, which are the differential manifolds on which the operator (26) acts invariantly.

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3 Such symmetries (referred to as generalised) are not necessarily unphysical. A textbook example of this kind of symmetry having a physical meaning is the conservation of the Runge-Lenz vector. It can be demonstrated that the conservation of this vector stems from a generalised symmetry of the Kepler problem.

4 We recommend using the software that comes with the book [40].
Below we remind the definition of a continuous symmetry for a given manifold, and extrapolate the construction to differential manifolds.

**Appendix A.1  Manifolds, invariants, symmetries**

In an ambient space of coordinates $x^i$, $i = 1, \ldots, m$, we define a manifold $\mathcal{M}$ as a set of all points satisfying a certain system of $s$ equations, with $s < m$:

$$\mathcal{M} : \quad F^\sigma(x^i) = 0 \quad , \quad \sigma = 1, 2, \ldots, s \quad .$$  \hfill (24)

A symmetry of the manifold is its property of staying invariant under the action of a differential operator $X$:

$$XF^\sigma(x^k)\big|_\mathcal{M} = 0 \quad , \quad \sigma = 1, 2, \ldots, N \quad ,$$  \hfill (25)

where

$$X = \xi^j(x^k) \partial_{x^j} \quad , \quad j, k = 1, \ldots, m \quad .$$  \hfill (26)

The subscript $\mathcal{M}$ in the equation (25) serves to emphasise that the expression $XF^\sigma(x^k)$ vanishes only in the points of the manifold.

Let $r$ stand for the rank of Wronskian of the system (24):

$$r = \text{rank} \left( \frac{\partial F^\sigma}{\partial x^k} \right) \quad .$$  \hfill (27)

When $r = s$, the system is said to have a full rank, while the manifold is termed regular. To evaluate (25) in a point of the manifold, we solve (24) for some $r$ coordinates, i.e., express them through the other $m - r$ coordinates. Insertion of the so-obtained $r$ expressions into (25) furnishes $XF^\sigma(x^k)$ as functions of the said $m - r$ coordinates.

As an example, one can consider the manifold $\mathcal{M}_0$ defined by the equation

$$\mathcal{M}_0 : \quad F_0(x, y, z) = e^{-\arctan x/y} - \frac{x^2 + y^2}{z} = 0 \quad .$$  \hfill (28)

It stays invariant under the action of the operator

$$X_0 = y \partial_x - x \partial_y + z \partial_z \quad ,$$  \hfill (29)

since

$$X_0 F_0 = -F_0 \quad .$$  \hfill (30)
The equation
\[ X_0 I(x) = 0 \] (31)
renders all the functionally-independent invariants of the operator \(X_0\):
\[ I_1 = x^2 + y^2, \quad I_2 = \log z - \arctan x/y. \] (32)
Any other invariant of \(X_0\) can be expressed as a function of those. For example, the manifold \(\mathcal{M}_0\) reads as:
\[ \mathcal{M}_0 : e^{I_2} - I_1 = 0. \] (33)

**Appendix A.2 Symmetries of differential manifolds**

Let us now introduce, in addition to the independent variables \(x^j\), functions \(u^l(x^j)\). Their partial derivatives will be denoted as:
\[ p_j^l \equiv \frac{\partial u^l}{\partial x^j}, \quad q_{jk}^l \equiv \frac{\partial u^l}{\partial x^j \partial x^k}, \quad \ldots \] (34)
Then a system of differential equations for the functions \(u^l(x^j)\) and their derivatives can be treated as a differential manifold:
\[ \mathcal{M} : F^\sigma [x^j, u^l, p_j^l, q_{jk}^l, \ldots] = 0, \quad \sigma = 1, 2, \ldots, N. \] (35)

In order to calculate the action of the symmetry operator
\[ X = \xi^j(x^k, u^l) \partial_{x^j} + \eta^s(x^k, u^l) \partial_{u^s}, \] (36)
on the equations of the system (35), we have to define its action on the partial derivatives (34). This requires the construction of the so-called prolonged operator. When looking for the symmetries of the differential equation (35) of the order \(\kappa\), one first has to construct the \(\kappa\)-th prolongation of the symmetry operator. Such prolongations have the form of
\[ \begin{align*}
X_1 &= X + \xi^l_j \partial_{p_j^l}, \\
X_2 &= X_1 + \xi_{jk}^l \partial_{q_{jk}^l}, \\
\cdots
\end{align*} \] (37a, 37b, 37c)
While the functions \( \xi^j \) and \( \eta^s \) in (36) are allowed to depend on \( x^j \) and \( u^l \) only, the functions \( \zeta \) depend on the partial derivatives as well:

\[
\begin{align*}
\zeta^l_j &= \zeta^l_j (\xi^j, \eta^s, p^l_j) , \\
\zeta^l_{jk} &= \zeta^l_{jk} (\xi^j, \eta^s, p^l_j, q^l_{jk}) , \\
\ldots & .
\end{align*}
\]

(38a)

(38b)

(38c)

More importantly, these functions are fully determined by the form of operator \( X \): given the functions \( \xi^j \) and \( \eta^s \), one can find the prolongated operator. For convenience of the further derivations, it will be convenient to define the covariant derivative as

\[
D_j = \partial_{x^j} + p^l_j \partial_{u^l} .
\]

(39)

Then, the first and higher prolongations can be calculated using

\[
\begin{align*}
\zeta^l_j &= D_j \eta^l - p^l_j D_j \xi^s , \\
\zeta^l_{i_1 i_2 \ldots i_k j} &= D_j \zeta^l_{i_1 i_2 \ldots i_k} - p^l_{i_1 i_2 \ldots i_k} D_j \xi^s .
\end{align*}
\]

(40a)

(40b)

The result of action of a prolongated operator on the differential equations of the system will have the form of a polynomial in variables \( p^l_j, q^l_{jk}, \ldots \).

By setting all its coefficients equal to zero, one obtains a system of differential equations for the functions \( \xi^j \) and \( \eta^s \), which are called the determining equations. In most cases, these equations are easy so solve, and the set of their solutions determines the full group of the continuous symmetries of the equation.

Let us briefly summarise the steps of the described procedure.

1. For a given set of independent variables and unknown functions, write down the most general form of the symmetry operator, prolonged up to the highest order of the derivative in the system.

For instance, the for system (21), the variables and functions are \( \{ x^1 \equiv x, x^2 \equiv y \} \) and \( \{ S^1, S^2, S^3 \} \), while the symmetry operator, prolonged once, has the form

\[
X = \underbrace{\xi^1 \partial_x + \xi^2 \partial_y + \eta^1 \partial_{S^1} + \eta^2 \partial_{S^2} + \eta^3 \partial_{S^3}}_{1} \\
+ \zeta^1_{i_1 i_2} \partial_{p^l_{i_1 i_2}} + \zeta^2_{i_1 i_2} \partial_{p^l_{i_1 i_2}} + \zeta^3_{i_1 i_2} \partial_{p^l_{i_1 i_2}} + \zeta^1_{i_1 i_2} \partial_{p^l_{i_1 i_2}} + \zeta^2_{i_1 i_2} \partial_{p^l_{i_1 i_2}}.
\]

(41)
With aid of (40a), the functions $\zeta_j^l$ are found to be:

$$
\zeta_j^l = \eta_j^l + p_j^1 \eta_{S_1}^l + p_j^2 \eta_{S_2}^l + p_j^3 \eta_{S_3}^l
- (\xi_{x_j}^1 + p_j^1 \xi_{S_1}^1 + p_j^2 \xi_{S_2}^1 + p_j^3 \xi_{S_3}^1)p_1^l
- (\xi_{x_j}^2 + p_j^1 \xi_{S_1}^2 + p_j^2 \xi_{S_2}^2 + p_j^3 \xi_{S_3}^2)p_2^l.
$$

(42)

2. Act with the prolongated operator on each equation of the system.

For skyrmions, the system reads as:

$$
p_j^l = -\epsilon_{jst}^l \epsilon_{j}^{sk} S_{k}^l.
$$

(43)

The result of action of the operator (41) on the functions (43) is straightforward to obtain, but is too long to present here.

3. Solve (algebraically) the original system of equations for any $N$ partial derivatives, and substitute these into the expression resulting from the action of the prolongated operator on the equations (35). See also the discussion after the equation (27).

4. Obtain the system of determining equations by setting zero all the coefficients next to monomials in partial derivatives. Be mindful, that in the preceding step we have already restricted ourselves to the manifold.

5. Solve the determining equations. This step is usually very technical but straightforward. The more complicated the system of equations is and the less symmetries it respects, the more determining equations can be obtained, and the simpler they are.

Above we assumed the system of $N$ differential equations to be non-degenerate, which is often not the case. For example, of the six equations in the system (43) not all are independent. One can solve four of them for some partial derivatives, and the other two equations will be satisfied automatically (if one keeps in mind that the functions $S^l$ are

5 By monomials in partial derivatives we mean expressions of the form $(p_j^l)^\alpha (q_{jk}^s)^\beta \ldots$.}

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defined on a unit sphere):

\[
\begin{align*}
    p_1^2 &= ((S^1)^2 - 1) \left( p_1^1 S^1 S^2 + p_2^1 S^3 \right), \\
    p_2^2 &= ((S^1)^2 - 1) \left( p_2^1 S^1 S^2 - p_1^1 S^3 \right), \\
    p_3^1 &= ((S^1)^2 - 1) \left( p_1^1 S^3 - p_2^1 S^2 \right), \\
    p_3^2 &= ((S^1)^2 - 1) \left( p_1^1 S^2 + p_2^1 S^1 S^3 \right).
\end{align*}
\]

(44)

For this reason, one should only analyse four instead of six equations. The equations (44) can also serve for restricting to the manifold in the step 3.

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