Sheaves on $\mathcal{T}$-topologies

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Abstract. The aim of this paper is to give a unifying description of various constructions of sites (subanalytic, semialgebraic, o-minimal) and consider the corresponding theory of sheaves. The method used applies to a more general context and gives new results in semialgebraic and o-minimal sheaf theory.

Introduction.

Sheaf theory in some tame contexts such as semi-algebraic geometry ([10]), subanalytic geometry ([28], [35]) and o-minimal geometry ([19]) has had recently different applications in various fields of mathematics such as model theory [4], [5], [20], analysis [28], [30], [31], [36] and representation theory [1], [2], [37]. Each one of the above theories is very useful for the mentioned applications but has some elements which are missing in the other ones: the aim of this paper is to give a unifying description of all these various constructions (subanalytic, semialgebraic, o-minimal) using a modification of the notion of $\mathcal{T}$-topology introduced by Kashiwara and Schapira in [28].

The idea is the following: on a topological space $X$ one chooses a subfamily $\mathcal{T}$ of open subsets of $X$ satisfying some suitable hypothesis, and for each $U \in \mathcal{T}$ one defines the category of coverings of $U$ as the topological coverings $\{U_i\}_{i \in I} \subset \mathcal{T}$ of $U$ admitting a finite subcover. In this way one defines a site $X_\mathcal{T}$ and studies the category of sheaves on $X_\mathcal{T}$ (called Mod($k_\mathcal{T}$)). This idea was already present in [28]. However in [28], the space $X$ is assumed to be Hausdorff, locally compact and the elements of $\mathcal{T}$ are assumed to have finitely many connected components.

The exigence to treat in a unifying way all the previous constructions, to treat also some non Hausdorff cases (as conic subanalytic sheaves which are related to the extension of the Fourier-Sato transform [36]) and the non-standard setting which appears naturally in the o-minimal context (where the elements of $\mathcal{T}$ are totally disconnected and never locally compact), motivates a modification of the definition of [28]. In particular, in our definition we replace “connectedness” by the notion of $\mathcal{T}$-connectedness (which in the standard o-minimal context is connectedness). Remark that there are many important o-minimal expansions

$$\mathcal{M} = (\mathbb{R},<,0,1,+,\cdot,(f)_{f \in \mathcal{F}})$$

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of the ordered field of real numbers. For example $\mathbb{R}_{an}$, $\mathbb{R}_{exp}$, $\mathbb{R}_{an,exp}$, $\mathbb{R}_{an}^*$, $\mathbb{R}_{an}^{*,exp}$ see resp., [12], [40], [15], [17], [18]. For each such we have $2^\kappa$ many non-isomorphic non-standard o-minimal models for each $\kappa > \text{cardinality of the language}$. There is however a non-standard o-minimal structure

$$\mathcal{M} = \left( \bigcup_{n \in \mathbb{N}} \mathbb{R}((t^{1/n})), <, 0, 1, +, \cdot, (f_p)_{p \in \mathbb{R}[[\zeta_1, \ldots, \zeta_n]]} \right)$$

which does not came from a standard one ([32], [23]).

With this more general notion of $T$-space $X$ we study the category of sheaves on the site $X_T$. The natural functor of sites $\rho : X \to X_T$ induces relations between the categories of sheaves on $X$ and $X_T$, given by the functors $\rho_*$ and $\rho^{-1}$. The functor $\rho_*$ is fully faithful. Moreover when $X$ is locally weakly quasi-compact there is a right adjoint to the functor $\rho^{-1}$, denoted by $\rho_i$. The functor $\rho_i$ is exact, commutes with $\lim$ and $\otimes$ and is fully faithful. We introduce the category of $T$-flabby sheaves (known as sa-flabby in [10] and as quasi-injective in [35]): $F \in \text{Mod}(k_T)$ is $T$-flabby if the restriction $\Gamma(U; F) \to \Gamma(V; F)$ is surjective for each $U, V \in T$ with $U \supseteq V$. We prove that $T$-flabby sheaves are stable under $\lim$ and $\otimes$ and are acyclic with respect to the functor $\Gamma(U; \bullet)$, for $U \in T$. More generally, if one introduces the category $\text{Coh}(T) \subset \text{Mod}(k_X)$ of coherent sheaves (i.e. sheaves admitting a finite resolution consisting of finite sums of $k_U$, $U \in T$), then $T$-flabby sheaves are acyclic with respect to $\text{Hom}_k(\rho_* G, \bullet)$, for $G \in \text{Coh}(T)$. Coherent sheaves also give a description of sheaves on $X_T$: for each $F \in \text{Mod}(k_T)$ there exists a filtrant inductive family $\{F_i\}_{i \in I}$ such that $F \simeq \lim_i F_i$. In fact, we have an equivalence between the categories $\text{Mod}(k_T)$ and $\text{Ind}(\text{Coh}(T))$ the indization of the category $\text{Coh}(T)$.

All of the above results and methods are new in the o-minimal context and most of them are new even in the semialgebraic case as well. On the other hand, we also introduce a method for studying the category $\text{Mod}(k_T)$ of sheaves on $T$-spaces which is the fundamental tool in the semialgebraic and o-minimal case, namely, we prove that as in [19] the category of sheaves on $X_T$ is equivalent to the category of sheaves on a locally quasi-compact space $\tilde{X}_T$, the $T$-spectrum of $X$, which generalizes the notion of o-minimal spectrum as well as the real spectrum of commutative rings from real algebraic geometry. In particular, sheaves on the subanalytic site are sheaves on the $T$-spectrum associated to the family of relatively compact subanalytic subsets. Such a result was not present in [28].

This theory can then be specialized to each of the examples we mentioned above: when $T$ is the category of semialgebraic open subsets of a locally semialgebraic space $X$ we obtain the constructions (and the generalizations) of results of [10], in particular, when $X$ is a Nash manifold, we recover the setting of [37]. When $T$ is the category of relatively compact subanalytic open subsets of a real analytic manifold $X$ we obtain the constructions and results of [28], [35]. Moreover, when $T$ is the category of conic subanalytic open subsets of a real analytic manifold $X$ we obtain a suitable category of conic subanalytic sheaves considered in [36]. Finally, when $T$ is the category of definable open subsets of a locally definable space $X$ we obtain in the definable case the
constructions of [19] and we obtain new results in the o-minimal context generalizing those of the two previous cases.

The paper is organized in the following way. In Section 1 we introduce the locally weakly quasi-compact spaces and study some properties of sheaves on such spaces. The results of this section will be used in two crucial ways on the theory of sheaves on \( T \)-spaces, they are required to show that: (i) when a \( T \)-space \( X \) is locally weakly quasi-compact, then there is a right adjoint \( \rho! \) to the functor \( \rho^{-1} \) induced by the natural functor of sites \( \rho : X \to X_T \); (ii) for a \( T \)-space \( X \), the category of sheaves on \( X_T \) is equivalent to the category of sheaves on a locally quasi-compact space \( \tilde{X}_T \), the \( T \)-spectrum of \( X \).

In Section 2 we introduce the \( T \)-spaces and develop the theory of sheaves on such spaces as already described above.

1. Sheaves on locally weakly quasi-compact spaces.

Let \( X \) be a non necessarily Hausdorff topological space. One denotes by \( \text{Op}(X) \) the category whose objects are the open subsets of \( X \) and the morphisms are the inclusions. In this section we generalize some classical results about sheaves on locally compact spaces. For classical sheaf theory our basic reference is [26]. We refer to [39] for an introduction to sheaves on Grothendieck topologies.

1.1. Locally weakly quasi-compact spaces.

**Definition 1.1.1.** An open subset \( U \) of \( X \) is said to be relatively weakly quasi-compact in \( X \) if, for any covering \( \{ U_i \}_{i \in I} \) of \( X \), there exists \( J \subset I \) finite, such that
\[
U \subset \bigcup_{i \in J} U_i.
\]

We will write for short \( U \subset\subset X \) to say that \( U \) is a relatively weakly quasi-compact open set in \( X \), and we will call \( \text{Op}_c(U) \) the subcategory of \( \text{Op}(U) \) consisting of open sets \( V \subset\subset U \). Note that, given \( V, W \in \text{Op}_c(U) \), then \( V \cup W \in \text{Op}_c(U) \).

**Definition 1.1.2.** A topological space \( X \) is locally weakly quasi-compact if satisfies the following hypothesis for every \( U, V \in \text{Op}(X) \)

LWC1. Every \( x \in U \) has a fundamental neighborhood system \( \{ V_i \} \) with \( V_i \in \text{Op}_c(U) \).
LWC2. For every \( U' \in \text{Op}_c(U) \) and \( V' \in \text{Op}_c(V) \) one has \( U' \cap V' \in \text{Op}_c(U \cap V) \).
LWC3. For every \( U' \in \text{Op}_c(U) \) there exists \( W \in \text{Op}_c(U) \) such that \( U' \subset\subset W \).

Of course an open subset \( U \) of a locally weakly quasi-compact space \( X \) is also a locally weakly quasi-compact space. Let us consider some examples of locally weakly quasi-compact spaces:

**Example 1.1.3.** A locally compact topological space \( X \) is a locally weakly quasi-compact. In this case, for \( U, V \in \text{Op}(X) \) we have \( V \subset\subset U \) if and only if \( V \) is relatively compact subset of \( U \).

**Example 1.1.4.** Let \( X \) be a topological space with a basis of quasi-compact (i.e. each open covering admits a finite subcover) open subsets closed under taking finite intersections. Then \( X \) is locally weakly quasi-compact and, for \( U, V \in \text{Op}(X) \) we have \( V \subset\subset U \) if and only if \( V \) is contained in a quasi-compact subset of \( U \). In this situation
we have the following particular cases:

(i) $X$ is a Noetherian topological space (each open subset of $X$ is quasi-compact). This includes in particular: (a) algebraic varieties over algebraically closed fields; (b) complex varieties (reduced, irreducible complex analytic spaces) with the Zariski topology.

(ii) $X$ is a spectral topological space (in addition: (i) $X$ is quasi-compact; (ii) $T_0$; (iii) every irreducible closed subset is the closure of a unique point). This includes in particular: (a) real algebraic varieties over real closed fields; (b) the o-minimal spectrum of a definable space in some o-minimal structure.

(iii) $X$ is an increasing union of open spectral topological spaces $X_i$’s, i.e. $X = \bigcup_{i \in I} X_i$. This space $X$ has a basis of quasi-compact open subsets closed under taking finite intersections and in addition is: (i) not quasi-compact in general unless $I$ is finite; (ii) $T_0$. This includes in particular: (a) the semialgebraic spectrum of a locally semialgebraic space; (b) more generally, the o-minimal spectrum of a locally definable space in some o-minimal structure.

Example 1.1.5. Let $E$ be a real vector bundle over a locally compact space $Z$, endowed with the natural action $\mu$ of $\mathbb{R}^+$ (the multiplication on the fibers). Let $\hat{E} = E \setminus Z$, and for $U \in \text{Op}(E)$ set $U_Z = U \cap Z$ and $\hat{U} = U \cap \hat{E}$. Let $E_{\mathbb{R}^+}$ denote the space $E$ endowed with the conic topology i.e. open sets of $E_{\mathbb{R}^+}$ are open sets of $E$ which are $\mu$-invariant. With this topology $E_{\mathbb{R}^+}$ is a locally weakly quasi-compact space and, for $U, V \in \text{Op}(E_{\mathbb{R}^+})$ we have $V \subset U$ if and only if $V_Z \subset U_Z$ in $Z$ and $\hat{V} \subset \hat{U}$ in $\hat{E}_{\mathbb{R}^+}$ (the later is $\hat{E}$ with the induced conic topology).

1.2. Sheaves on locally weakly quasi-compact spaces.

Recall that $X$ is a non necessarily Hausdorff topological space.

Definition 1.2.1. Let $U = \{U_i\}_{i \in I}$ and $U' = \{U'_i\}_{i \in I}$ be two families of open subsets of $X$. One says that $U'$ is a refinement of $U$ if for each $U_i \in U$ there is $U'_i \in U'$ with $U'_i \subseteq U_i$.

One denotes by Cov$(U)$ the category whose objects are the coverings of $U \in \text{Op}(X)$ and the morphisms are the refinements, and by Cov$^f(U)$ its full subcategory consisting of finite coverings of $U$.

Given $V \in \text{Op}(U)$ and $S \in \text{Cov}(U)$, one sets $S \cap V = \{U \cap V_{i \in S} \in \text{Cov}(V)$.

Definition 1.2.2. The site $X^f$ on the topological space $X$ is the category $\text{Op}(X)$ endowed with the following topology: $S \subset \text{Op}(U)$ is a covering of $U$ if and only if it has a refinement $S^f \in \text{Cov}(U)$.

Definition 1.2.3. Let $U, V \in \text{Op}(X)$ with $V \subset U$. Given $S = \{U_i\}_{i \in I} \in \text{Cov}(U)$ and $T = \{V_j\}_{j \in J} \in \text{Cov}(V)$, we write $T \subset S$ if $T$ is a refinement of $S \cap V$, and $V_j \subset U_i$ if and only if $V_j \subset U_i$.

Let us recall the definitions of presheaf and sheaf on a site.
Definition 1.2.4. A presheaf of $k$-modules on $X$ is a functor $\text{Op}(X)^{op} \to \text{Mod}(k)$. A morphism of presheaves is a morphism of such functors. One denotes by $\text{Psh}(k_X)$ the category of presheaves of $k$-modules on $X$.

Let $F \in \text{Psh}(k_X)$, and let $S \in \text{Cov}(U)$. One sets

$$F(S) = \ker \left( \prod_{W \in S} F(W) \Rightarrow \prod_{W', W'' \in S} F(W' \cap W'') \right).$$

Definition 1.2.5. A presheaf $F$ is separated (resp. is a sheaf) if for any $U \in \text{Op}(X)$ and for any $S \in \text{Cov}(U)$ the natural morphism $F(U) \to F(S)$ is a monomorphism (resp. an isomorphism). One denotes by $\text{Mod}(k_X)$ the category of sheaves of $k$-modules on $X$.

Let $F \in \text{Psh}(k_X)$, one defines the presheaf $F^+$ by setting

$$F^+(U) = \lim_{S \in \text{Cov}(U)} F(S).$$

One can show that $F^+$ is a separated presheaf and if $F$ is a separated presheaf, then $F^+$ is a sheaf. Let $F \in \text{Psh}(k_X)$, the sheaf $F^{++}$ is called the sheaf associated to the presheaf $F$.

Lemma 1.2.6. For $F \in \text{Psh}(k_X)$, and let $U \in \text{Op}(X)$. If $F$ is a sheaf on $X^f$, then for any $V \in \text{Op}^c(U)$ the morphism

$$F^+(U) \to F^+(V)$$

factors through $F(V)$.

Proof. Let $S \in \text{Cov}(U)$, and set $S \cap V = \{ W \cap V \}_{W \in S}$. Since $V \in \text{Op}^c(U)$, there is a finite refinement $T^f \in \text{Cov}^f(V)$ of $S \cap V$. Then the morphism (1.1) is defined by

$$F^+(U) \simeq \lim_{S \in \text{Cov}(U)} F(S)$$

$$\to \lim_{S \in \text{Cov}(U)} F(S \cap V)$$

$$\to \lim_{T^f \in \text{Cov}^f(V)} F(T^f)$$

$$\to \lim_{T \in \text{Cov}(V)} F(T)$$

$$\simeq F^+(V).$$

The result follows because $F(T^f) \simeq F(V)$. \qed
Corollary 1.2.7. With the hypothesis of Lemma 1.2.6, we consider two coverings $S \in \text{Cov}(U)$ and $T \in \text{Cov}(V)$. If $T \subset S$, then the morphism

$$F^+(S) \to F^+(T)$$

factors through $F(T)$. In particular, if $T$ is finite, then the morphism (1.2) factors through $F(V)$.

From now on we will assume the following hypothesis:

the topological space $X$ is locally weakly quasi-compact. (1.3)

Lemma 1.2.8. Let $U \in \text{Op}(X)$, and consider a subset $V \subset U$. Then for any $S^f \in \text{Cov}^f(U)$ there exists $T^f \in \text{Cov}^f(V)$ with $T^f \subset S^f$.

Proof. Let $S^f = \{U_i\}$. For each $x \in U$ and $U_i \ni x$, consider a $V_{x,i} \in \text{Op}^c(U_i)$ containing $x$. Set $V_x = \bigcap_i V_{x,i}$, the family $\{V_x\}$ forms a covering of $U$. Then there exists a finite subfamily $\{V_j\}$ containing $V$. By construction $V_j \cap V \subset U_i$ whenever $V_j \subset U_i$.

Lemma 1.2.9. Let $F \in \text{Psh}(k_X)$, and let $U \in \text{Op}(X)$. If $F$ is a sheaf on $X^f$, then for any $V \in \text{Op}^c(U)$ the morphism

$$F^{++}(U) \to F^{++}(V)$$

factors through $F(V)$.

Proof. Since $X$ is locally weakly quasi-compact, there exists $W \in \text{Op}^c(U)$ with $V \subset W$. As in Lemma 1.2.6 we obtain a diagram

$$
\begin{array}{ccc}
F^{++}(U) & \to & F^{++}(W) \\
\downarrow & & \downarrow \\
\lim_{S^f \in \text{Cov}^f(W)} F^+(S^f) & \longrightarrow & \lim_{T^f \in \text{Cov}^f(V)} F^+(T^f).
\end{array}
$$

Since $X$ is locally weakly quasi-compact then by Lemma 1.2.8 for any $S^f \in \text{Cov}^f(W)$ there exists $T^f \in \text{Cov}^f(V)$ with $T^f \subset S^f$. By Corollary 1.2.7 the morphism

$$F^+(S^f) \to F^+(T^f)$$

factors through $F(T^f) \simeq F(V)$. Then the morphism

$$\lim_{S^f \in \text{Cov}^f(W)} F^+(S^f) \to \lim_{T^f \in \text{Cov}^f(V)} F^+(T^f)$$
factors through $F(V)$ and the result follows. \hfill \Box

**Corollary 1.2.10.** Let $F \in \text{Psh}(k_X)$. If $F$ is a sheaf on $X^I$, then:

(i) for any $V \in \text{Op}^c(X)$ one has the isomorphism $\lim_{U \supseteq V} F(U) \cong \lim_{U \supseteq V} F^+(U)$.

(ii) for any $U \in \text{Op}(X)$ one has the isomorphism $\lim_{V \subseteq U} F(V) \cong \lim_{V \subseteq U} F^+(V)$.

**Proof.** (i) By Lemma 1.2.9 for each $U \in \text{Op}(X)$ with $U \supseteq V$ we have a commutative diagram

\[
\begin{array}{ccc}
F^+(U) & \rightarrow & F^+(V) \\
\downarrow & & \downarrow \\
F(U) & \rightarrow & F(V)
\end{array}
\]

This implies that the identity morphism of $\lim_{U \supseteq V} F(U)$ factors through $\lim_{U \supseteq V} F^+(U)$. On the other hand this also implies that the identity morphism of $\lim_{U \supseteq V} F^+(U)$ factors through $\lim_{U \supseteq V} F(U)$. Then $\lim_{U \supseteq V} F(U) \cong \lim_{U \supseteq V} F^+(U)$.

The proof of (ii) is similar. \hfill \Box

**Corollary 1.2.11.** Let $X$ be a quasi-compact and locally weakly quasi-compact space, and let $F \in \text{Psh}(k_X)$. If $F$ is a sheaf on $X^I$, then the natural morphism

$$F(X) \rightarrow F^+(X) \quad (1.5)$$

is an isomorphism.

**Proof.** It follows immediately from Corollary 1.2.10 (i) with $V = X$. \hfill \Box

Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\text{Mod}(k_X)$. One sets

\[\text{"} \lim" F_i = \text{inductive limit in the category of presheaves,} \]

\[\lim F_i = \text{inductive limit in the category of sheaves.} \]

Recall that $\lim F_i = (\text{"} \lim" F_i)^{++}$.

**Proposition 1.2.12.** Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\text{Mod}(k_X)$ and let $U \in \text{Op}(X)$. Then for any $V \in \text{Op}^c(U)$ the morphism

$$\Gamma(U; \lim F_i) \rightarrow \Gamma(V; \lim F_i)$$
factors through $\varprojlim \Gamma(V; F_i)$.

**Proof.** By Lemma 1.2.9 it is enough to show that $\varprojlim F_i$ is a sheaf on $X$. Let $U \in \text{Op}(X)$ and $S \in \text{Cov}^I(U)$. Since $\varprojlim$ commutes with finite projective limits we obtain the isomorphism $(\varprojlim F_i)(S) \simeq \varprojlim F_i(S)$. The result follows because $F_i \in \text{Mod}(k_X)$ for each $i \in I$. \hfill $\square$

**Corollary 1.2.13.** Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\text{Mod}(k_X)$.

(i) For any $V \in \text{Op}(X)$ one has the isomorphism $\varinjlim \Gamma(U; F_i) \simeq \varprojlim \Gamma(U; \lim F_i)$.

(ii) For any $U \in \text{Op}(X)$ one has the isomorphism $\varprojlim \lim \Gamma(V; F_i) \simeq \varinjlim \Gamma(V; \lim F_i)$.

**Proof.** It follows from Corollary 1.2.10 with $F = \varprojlim F_i$. \hfill $\square$

**Corollary 1.2.14.** Let $X$ be a quasi-compact and locally weakly quasi-compact space. Then the natural morphism

$$\varprojlim \Gamma(X; F_i) \to \Gamma(X; \varprojlim F_i)$$

is an isomorphism.

**Proof.** It follows from Corollary 1.2.11 with $F = \varprojlim F_i$. \hfill $\square$

**Example 1.2.15.** Let us consider the formula

$$\varprojlim \Gamma(U; F_i) \simeq \varinjlim \Gamma(U; \varprojlim F_i) \quad (1.6)$$

(i) Let $X$ be a Noetherian space and let $V \in \text{Op}(X)$. Then $\Gamma(V; F) \simeq \varinjlim \Gamma(U; F)$, since every open set is quasi-compact and (1.6) becomes $\varprojlim \Gamma(V; F_i) \simeq \Gamma(V; \varprojlim F_i)$.

(ii) Assume that $X$ has a basis of quasi-compact open subsets and let $V \in \text{Op}^c(X)$. Then $V$ is contained in a quasi-compact open subset of $X$ and $\varprojlim \Gamma(U; F) \simeq \varinjlim \Gamma(W; F)$, where $W$ ranges through the family of quasi-compact subsets of $X$.

(iii) Let $X$ be a locally compact space and let $V \in \text{Op}^c(X)$. Then $\Gamma(V; F) \simeq \varinjlim \Gamma(U; F)$, and (1.6) becomes $\varprojlim \Gamma(V; F_i) \simeq \Gamma(V; \varprojlim F_i)$.
(iv) Let $E_{\mathbb{R}^+}$ be a vector bundle endowed with the conic topology, and let $V \in \text{Op}^c(E_{\mathbb{R}^+})$. Then $\lim_{U \supset V} \Gamma(U; F) \simeq \Gamma(K; F)$, where $K$ is the union of the closures of $V_Z$ in $Z$ and $V$ in $E_{\mathbb{R}^+}$, and (1.6) becomes $\lim_{i} \Gamma(K_i; F_i) \simeq \Gamma(K; \lim_{i} F_i)$.

**Lemma 1.2.16.** Let $F \in \text{Psh}(k_X)$. Then we have the isomorphism

$$\lim_{V \subset W} \lim_{V \subset W} F(W) \cong \lim_{V \subset W} F(V).$$

**Proof.** The result follows since for each $V \in \text{Op}^c(X)$ there exists $W \in \text{Op}^c(X)$ such that $V \subset W$ since $X$ is locally weakly compact. Let $U, V \subset X$ such that $U \supset V$. The restriction morphism $F(U) \to F(V)$ factors through $\lim_{W \supset V} F(W)$. Taking the projective limit we obtain the result. \hfill \Box

**Remark 1.2.17.** The notion of locally weakly quasi-compact can be extended to the case of a site, by generalizing the hypothesis LWC1–LWC3. For our purpose we are interested in the topological setting and we refer to [34] for this approach.

### 1.3. $c$-soft sheaves on locally weakly quasi-compact spaces.

Let $X$ be a locally weakly quasi-compact space, and consider the category $\text{Mod}(k_X)$.

**Definition 1.3.1.** We say that a sheaf $F$ on $X$ is $c$-soft if the restriction morphism $\Gamma(W; F) \to \lim_{U \supset V} \Gamma(U; F)$ is surjective for each $V, W \in \text{Op}^c(X)$ with $V \subset W$.

It follows from the definition that injective sheaves and flabby sheaves are $c$-soft. Moreover, it follows from Corollary 1.2.13 that filtrant inductive limits of $c$-soft sheaves are $c$-soft.

**Proposition 1.3.2.** Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\text{Mod}(k_X)$, and assume that $F'$ is $c$-soft. Then the sequence

$$0 \to \lim_{U \supset V} \Gamma(U; F') \to \lim_{U \supset V} \Gamma(U; F) \to \lim_{U \supset V} \Gamma(U; F'') \to 0$$

is exact for any $V \in \text{Op}^c(X)$.

**Proof.** Let $s'' \in \lim_{U \supset V} \Gamma(U; F'')$. Then there exists $U \supset V$ such that $s''$ is represented by $s''_U \in \Gamma(U; F'')$. Let $\{U_i\}_{i \in I} \in \text{Cov}(U)$ such that there exists $s_i \in \Gamma(U_i; F)$ whose image is $s''_U|_{U_i}$ for each $i$. There exists $W \in \text{Op}^c(U)$ with $W \supset V$, a finite covering $\{W_j\}_{j=1}^n$ of $W$ and a map $\varepsilon : J \to I$ of the index sets such that $W_j \subset U_{\varepsilon(j)}$. We may argue by induction on $n$. If $n = 2$, set $U_i = U_{\varepsilon(i)}$, $i = 1, 2$. Then $(s_1 - s_2)|_{U_i \cap U_2}$ belongs to $\Gamma(U_i \cap U_2; F')$, and its restriction defines an element of $\lim_{W \supset W_1 \cap W_2} \Gamma(W'; F')$, hence it extends to $s' \in \Gamma(U; F')$. By replacing $s_1$ with $s_1 - s'$ on $W_1$ we may assume that
$s_1 = s_2$ on $W_1 \cap W_2$. Then there exists $s \in \Gamma(W_1 \cup W_2; F)$ with $s|_{W_i} = s_i$. Thus the induction proceeds. \qed

**Proposition 1.3.3.** Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\text{Mod}(k_X)$, and assume $F', F$ $c$-soft. Then $F''$ is $c$-soft.

**Proof.** Let $V, W \in \text{Op}^c(X)$ with $V \subset \subset W$ and let us consider the diagram below

$$
\begin{array}{ccc}
\Gamma(W; F) & \to & \Gamma(W; F'') \\
\downarrow \alpha & & \downarrow \gamma \\
\lim_{U \supset \supset V} \Gamma(U; F) & \to & \lim_{U \supset \supset V} \Gamma(U; F'').
\end{array}
$$

The morphism $\alpha$ is surjective since $F$ is $c$-soft and $\beta$ is surjective by Proposition 1.3.2. Then $\gamma$ is surjective. \qed

**Proposition 1.3.4.** The family of $c$-soft sheaves is injective respect to the functor $\lim_{U \supset \supset V} \Gamma(U; \bullet)$ for each $V \in \text{Op}^c(X)$.

**Proof.** The family of $c$-soft sheaves contains injective sheaves, hence it is cogenerated. Then the result follows from Propositions 1.3.2 and 1.3.3. \qed

Assume the following hypothesis

**Lemma 1.3.5.** Assume (1.7). Then there exists a covering $\{V_n\}_{n \in \mathbb{N}}$ of $X$ such that $V_n \subset \subset V_{n+1}$ and $V_n \in \text{Op}^c(X)$ for each $n \in \mathbb{N}$. \hfill (1.7)

**Proof.** Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable cover of $X$ with $U_n \in \text{Op}^c(X)$ for each $n \in \mathbb{N}$. Set $V_1 = U_1$. Given $\{V_i\}_{i=1}^n$ with $V_{i+1} \supset \supset V_i$, $i = 1, \ldots, n-1$, let us construct $V_{n+1} \supset \supset V_n$. Consider $x \notin V_n$. Up to take a permutation of $\mathbb{N}$ we may assume $x \in U_{n+1}$. Since $X$ is locally weakly quasi-compact there exists $V_{n+1} \in \text{Op}^c(X)$ such that $V_n \cup U_{n+1} \subset \subset V_{n+1}$. \qed

**Proposition 1.3.6.** Assume (1.7). Then the category of $c$-soft sheaves is injective respect to the functor $\Gamma(X; \bullet)$.

**Proof.** Take an exact sequence $0 \to F' \to F \to F'' \to 0$, and suppose $F'$ $c$-soft. By Lemma 1.3.5 there exists a covering $\{V_n\}_{n \in \mathbb{N}}$ of $X$ such that $V_n \subset \subset V_{n+1}$ (and $V_n \in \text{Op}^c(X)$) for each $n \in \mathbb{N}$. All the sequences

$$
0 \to \lim_{U_n \supset \supset V_n} \Gamma(U_n; F') \to \lim_{U_n \supset \supset V_n} \Gamma(U_n; F) \to \lim_{U_n \supset \supset V_n} \Gamma(U_n; F'') \to 0
$$

are exact.
are exact by Proposition 1.3.2, and the morphism \( \lim_{n} \Gamma(U_{n+1}; F') \to \lim_{n} \Gamma(U_{n}; F') \) is surjective for all \( n \). Then by Proposition 1.12.3 of [26] the sequence

\[
0 \to \lim_{n} \lim_{n+1 \supset V_{n}} \Gamma(U_{n}; F') \to \lim_{n} \lim_{n+1 \supset V_{n}} \Gamma(U_{n}; F) \to \lim_{n} \lim_{n+1 \supset V_{n}} \Gamma(U_{n}; F'') \to 0
\]

is exact. By Lemma 1.2.16 \( \lim_{n} \lim_{n+1 \supset V_{n}} \Gamma(U_{n}; G) \simeq \Gamma(X; G) \) for any \( G \in \text{Mod}(kX) \) and the result follows. □

**Example 1.3.7.** Let us consider some particular cases

(i) When \( X \) is Noetherian c-soft sheaves are flabby sheaves.

(ii) When \( X \) has a basis of quasi-compact open subsets, then \( F \in \text{Mod}(kX) \) is c-soft if the restriction morphism \( \Gamma(U; F) \to \Gamma(V; F) \) is surjective, for any quasi-compact open subsets \( U, V \) of \( X \) with \( U \supseteq V \).

(iii) When \( X \) is a locally compact space countable at infinity, then we recover c-soft sheaves as in chapter II of [26].

(iv) When \( E_{\mathbb{R}+} \) is a vector bundle endowed with the conic topology, then \( F \in \text{Mod}(kE_{\mathbb{R}+}) \) is c-soft if the restriction morphism \( \Gamma(E_{\mathbb{R}+}; F) \to \Gamma(K; F) \) is surjective, where \( K \) is defined as in Example 1.2.15.

2. **Sheaves on \( T \)-spaces.**

In the following we shall assume that \( k \) is a field and \( X \) is a topological space. Below we give the definition of \( T \)-space, adapting the construction of Kashiwara and Schapira [28]. We study the category of sheaves on \( X_{T} \) generalizing results already known in the case of subanalytic sheaves. Then we prove that as in [19] the category of sheaves on \( X_{T} \) is equivalent to the category of sheaves on a locally weakly-compact topological space \( \tilde{X}_{T} \), the \( T \)-spectrum, which generalizes the notion of o-minimal spectrum.

2.1. **\( T \)-sheaves.**

Let \( X \) be a topological space and let us consider a family \( T \) of open subsets of \( X \).

**Definition 2.1.1.** The topological space \( X \) is a \( T \)-space if the family \( T \) satisfies the hypotheses below

\[
\begin{cases}
\text{(i) } T \text{ is a basis for the topology of } X, \text{ and } \emptyset \in T, \\
\text{(ii) } T \text{ is closed under finite unions and intersections,}
\text{(iii) every } U \in T \text{ has finitely many } T\text{-connected components},
\end{cases}
\]

where we define:

- a \( T \)-subset is a finite Boolean combination of elements of \( T \);
- a closed (resp. open) \( T \)-subset is a \( T \)-subset which is closed (resp. open) in \( X \);
- a \( T \)-connected subset is a \( T \)-subset which is not the disjoint union of two proper
$T$-subsets which are closed and open.

**Example 2.1.2.** Let $R = (R, <, 0, 1, +, \cdot)$ be a real closed field. Let $X$ be a locally semialgebraic space ([10], [11]) and consider the subfamily of $\text{Op}(X)$ defined by $T = \{U \in \text{Op}(X) : U$ is semialgebraic}. The family $T$ satisfy (2.1). Note also that the $T$-subsets of $X$ are exactly the semialgebraic subsets of $X$ ([7]).

**Example 2.1.3.** Let $X$ be a real analytic manifold and consider the subfamily of $\text{Op}(X)$ defined by $T = \text{Op}^c(X_{sa}) = \{U \in \text{Op}(X_{sa}) : U$ is subanalytic relatively compact}. The family $T$ satisfies (2.1).

**Example 2.1.4.** Let $X$ be a real analytic manifold endowed with a subanalytic action $\mu$ of $\mathbb{R}^+$. In other words we have a subanalytic map

$$\mu : X \times \mathbb{R}^+ \to X,$$

which satisfies, for each $t_1, t_2 \in \mathbb{R}^+$:

$$\begin{cases}
\mu(x, t_1 t_2) = \mu(\mu(x, t_1), t_2), \\
\mu(x, 1) = x.
\end{cases}$$

Denote by $X_{\mathbb{R}^+}$ the topological space $X$ endowed with the conic topology, i.e. $U \in \text{Op}(X_{\mathbb{R}^+})$ if it is open for the topology of $X$ and invariant by the action of $\mathbb{R}^+$. We will denote by $\text{Op}^c(X_{\mathbb{R}^+})$ the subcategory of $\text{Op}(X_{\mathbb{R}^+})$ consisting of relatively weakly quasi-compact open subsets. Consider the subfamily of $\text{Op}(X_{\mathbb{R}^+})$ defined by $T = \text{Op}^c(X_{sa, \mathbb{R}^+}) = \{U \in \text{Op}^c(X_{\mathbb{R}^+}) : U$ is subanalytic}. The family $T$ satisfies (2.1).

**Example 2.1.5.** Let $\mathcal{M} = (M, <, (c)_{c \in C}, (f)_{f \in F}, (R)_{R \in R})$ be an arbitrary o-minimal structure. Let $X$ be a locally definable space ([3]) and consider the subfamily of $\text{Op}(X)$ defined by $T = \text{Op}(X_{\text{def}}) = \{U \in \text{Op}(X) : U$ is definable}. The family $T$ satisfies (2.1). Note also that the $T$-subsets of $X$ are exactly the definable subsets of $X$ (by the cell decomposition theorem in [13], see [19, Proposition 2.1]).

Let $X$ be a $T$-space. One can endow the category $T$ with a Grothendieck topology, called the $T$-topology, in the following way: a family $\{U_i\}_i$ in $T$ is a covering of $U \in T$ if it admits a finite subcover. We denote by $X_T$ the associated site, write for short $k_T$ instead of $X_T$, and let $\rho : X \to X_T$ be the natural morphism of sites. We have functors

$$\text{Mod}(k_X) \xrightarrow{\rho_*} \text{Mod}(k_T). \quad (2.2)$$

**Proposition 2.1.6.** We have $\rho^{-1} \circ \rho_* \simeq \text{id}$. Equivalently, the functor $\rho_*$ is fully faithful.

**Proof.** Let $V \in \text{Op}(X)$ and let $G \in \text{Mod}(k_T)$. Then $\rho^{-1}G = (\rho^{-}F)^{++}$, where $\rho^{-}G \in \text{Psh}(k_X)$ is defined by
Sheaves on $T$-topologies

\[ \text{Op}(X) \ni V \mapsto \lim_{U \supseteq V, U \in T} G(U). \]

In particular, when $U \in T$, $\rho^{-1}G(U) = G(U)$.

Let $F \in \text{Mod}(k_X)$ and denote by $\iota: \text{Mod}(k_X) \to \text{Psh}(k_X)$ the forgetful functor. The adjunction morphism $\rho^{-1} \circ \rho_* \to \text{id}$ in $\text{Psh}(k_X)$ defines $\rho^{-1} \rho_* F \to \iota F$. This morphism is an isomorphism on $T$, since $\rho^{-1} \rho_* F(U) \simeq \rho_* F(U) \simeq F(U) \simeq \iota F(U)$ when $U \in T$. By (2.1) (i) $T$ forms a basis for the topology of $X$, hence we get an isomorphism

$$\rho^{-1} \rho_* F \simeq (\rho^{-1} \rho_*)^+ \simeq (\iota F)^+ \simeq F$$

and the result follows. □

**Proposition 2.1.7.** Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\text{Mod}(k_T)$ and let $U \in T$. Then

$$\lim_i \Gamma(U; F_i) \simeq \Gamma(U; \lim_i F_i).$$

**Proof.** Denote by $\text{"lim"}_i F_i$ the presheaf $V \mapsto \lim_i \Gamma(V; F_i)$ on $X_T$. Let $U \in T$ and let $S$ be a finite covering of $U$. Since $\lim_i$ commutes with finite projective limits we obtain the isomorphism $(\text{"lim"}_i F_i)(S) \sim \lim_i F_i(S)$ and $F_i(U) \sim F_i(S)$ since $F_i \in \text{Mod}(k_T)$ for each $i$. Moreover the family of finite coverings of $U$ is cofinal in $\text{Cov}(U)$. Hence $\text{"lim"}_i F_i \sim (\text{"lim"}_i F_i)^+$. Applying once again the functor $(\cdot)^+$ we get

$$(\text{"lim"}_i F_i) \sim (\text{"lim"}_i F_i)^+ \sim (\text{"lim"}_i F_i)^{++} \sim \lim_i F_i.$$ 

Hence applying the functor $\Gamma(U; \cdot)$ we obtain the isomorphism $\lim_i \Gamma(U; F_i) \sim \Gamma(U; \lim_i F_i)$ for each $U \in T$. □

**Proposition 2.1.8.** Let $F$ be a presheaf on $X_T$ and assume that

(i) $F(\emptyset) = 0$,
(ii) For any $U, V \in T$ the sequence $0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$ is exact.

Then $F \in \text{Mod}(k_T)$.

**Proof.** Let $U \in T$ and let $\{U_j\}_{j=1}^n$ be a finite covering of $U$. Set for short $U_{ij} = U_i \cap U_j$. We have to show the exactness of the sequence

$$0 \to F(U) \to \bigoplus_{1 \leq k \leq n} F(U_k) \to \bigoplus_{1 \leq i < j \leq n} F(U_{ij}),$$

...
where the second morphism sends \((s_k)_{1 \leq k \leq n}\) to \((t_{ij})_{1 \leq i < j \leq n}\) by \(t_{ij} = s_i|_{U_{ij}} - s_j|_{U_{ij}}\).

We shall argue by induction on \(n\). For \(n = 1\) the result is trivial, and \(n = 2\) is the hypothesis. Suppose that the assertion is true for \(j \leq n - 1\) and set \(U' = \bigcup_{1 \leq k < n} U_k\).

By the induction hypothesis the following commutative diagram is exact

\[
\begin{array}{ccc}
0 & \rightarrow & F(U) \\
\downarrow & & \downarrow \\
0 & \rightarrow & F(U') \oplus F(U_n) \\
\downarrow & & \downarrow \\
\bigoplus_{i<n} F(U_i) \oplus F(U_n) & \rightarrow & \bigoplus_{i<n} F(U_{in}) \\
\downarrow & & \downarrow \\
\bigoplus_{i<j<n} F(U_{ij}) & . &
\end{array}
\]

Then the result follows. \(\square\)

**Example 2.1.9.** Let us see some examples of sites associated to \(\mathcal{T}\)-topologies:

(i) When \(\mathcal{T}\) is the family of Example 2.1.2 we obtain the semi-algebraic site of \([10]\), \([11]\).

(ii) When \(\mathcal{T}\) is the family of Example 2.1.3 we obtain the subanalytic site \(X_{sa}\) of \([28]\), \([35]\).

(iii) When \(\mathcal{T}\) is the family of Example 2.1.4 we obtain the conic subanalytic site of \([36]\).

(iv) When \(\mathcal{T}\) is the family of Example 2.1.5 we obtain the o-minimal site \(X_{def}\). It is the one considered in \([19]\) when \(X\) is a definable space.

**2.2. \(\mathcal{T}\)-coherent sheaves.**

Let us consider the category \(\text{Mod}(k_X)\) of sheaves of \(k_X\)-modules on \(X\), and denote by \(\mathcal{K}\) the subcategory whose objects are the sheaves \(F = \bigoplus_{i \in I} kU_i\), with \(I\) finite and \(U_i \in \mathcal{T}\) for each \(i\). The following definition is extracted from \([28]\).

**Definition 2.2.1.** Let \(\mathcal{T}\) be a subfamily of \(\text{Op}(X)\) satisfying (2.1), and let \(F \in \text{Mod}(k_X)\).

(i) \(F\) is \(\mathcal{T}\)-finite if there exists an epimorphism \(G \rightarrow F\) with \(G \in \mathcal{K}\).

(ii) \(F\) is \(\mathcal{T}\)-pseudo-coherent if for any morphism \(\psi : G \rightarrow F\) with \(G \in \mathcal{K}\), \(\ker \psi\) is \(\mathcal{T}\)-finite.

(iii) \(F\) is \(\mathcal{T}\)-coherent if it is both \(\mathcal{T}\)-finite and \(\mathcal{T}\)-pseudo-coherent.

Remark that (ii) is equivalent to the same condition with “\(G\) is \(\mathcal{T}\)-finite” instead of “\(G \in \mathcal{K}\)”. One denotes by \(\text{Coh}(\mathcal{T})\) the full subcategory of \(\text{Mod}(k_X)\) consisting of \(\mathcal{T}\)-coherent sheaves. It is easy (see \([29\), Exercise 8.23]) to prove that \(\text{Coh}(\mathcal{T})\) is additive and stable by kernels.
Lemma 2.2.2. Let $F, G \in \mathcal{K}$. Then, given $\varphi : F \to G$, we have $\ker \varphi \in \mathcal{K}$.

Proof. We have $F = \bigoplus_{i=1}^l k_{W_i}; \ G = \bigoplus_{j=1}^m k_{W'_j}$. Composing with the projection $p_j$, $j = 1, \ldots, m$ on each factor of $G$, $\ker \varphi$ will be the intersection of the $\ker p_j \circ \varphi$ so that, if each one has the desired form, the same will happen to their intersection. Therefore it is sufficient to assume $m = 1$, let us say, $G = k_{W}$. A morphism $\varphi : F \to G$ is then defined by a sequence $v = (v_1, \ldots, v_l)$, where $v_i$ is the image by $\varphi$ of the section of $k_{W_i}$ defined by 1 on $W_i$, so $v_i = 0$ if $W_i \not\subset W$. More precisely, if $s = (s_1, \ldots, s_l)$ is a germ of $F$ in $y$, we have $\varphi(s_1, \ldots, s_l) = \sum_{i=1}^l v_i s_i$. So, given $s = (s_1, \ldots, s_l) \in \ker \varphi$, if, for a given $i$, we have $v_i s_i \neq 0$, then $s$ defines a germ of $H_i =: \bigoplus_{i \neq i} k_{W_i \cap W_i}$ in $y$.

Accordingly, $\ker \varphi \simeq \bigoplus_{i=1}^l H_i$. \hfill \Box

Therefore, according to the definition of $\text{Coh}(\mathcal{T})$ and to Lemma 2.2.2, any $F \in \text{Coh}(\mathcal{T})$ admits a finite resolution

$$K^\bullet := 0 \to K^1 \to \cdots \to K^n \to F \to 0$$

consisting of objects belonging to $\mathcal{K}$.

Proposition 2.2.3. Let $U \in \mathcal{T}$ and consider the constant sheaf $k_{U_X^T} \in \text{Mod}(k_{\mathcal{T}})$. We have $k_{U_X^T} \simeq \rho_* k_U$.

Proof. Let $F$ be the presheaf on $X_T$ defined by $F(V) = k$ if $V \subset U$, $F(V) = 0$ otherwise. This is a separated presheaf and $k_{U_X^T} \simeq F^{++}$. Moreover there is an injective arrow $F(V) \hookrightarrow \rho_* k_U(V)$ for each $V \in \text{Op}(X_T)$. Hence $F^{++} \hookrightarrow \rho_* k_U$ since the functor $(\cdot)^{++}$ is exact. Let $S \subseteq T$ be the sub-family of $T$-connected elements. Then $S$ forms a basis for the Grothendieck topology of $X_T$. For each $W \in S$ we have $F(W) \simeq \rho_* k_U(W) \simeq k$ if $W \subset U$ and $F(W) = 0$ otherwise. Then $F^{++} \simeq \rho_* k_U$. \hfill \Box

Proposition 2.2.4. The restriction of $\rho_*$ to $\text{Coh}(\mathcal{T})$ is exact.

Proof. Let us consider an epimorphism $G \to F$ in $\text{Coh}(\mathcal{T})$, we have to prove that $\psi : \rho_* G \to \rho_* F$ is an epimorphism. Let $U \in \mathcal{T}$ and let $0 \neq s \in \Gamma(U; \rho_* F) \simeq \text{Hom}_{k_{X}}(k_U, F)$ (by adjunction). Set $G' = G \times_F k_U = \ker(G \oplus k_U \Rightarrow F)$. Then $G' \in \text{Coh}(\mathcal{T})$ and moreover $G' \to k_U$. There exists a finite $\{U_i\}_{i \in I} \subset \mathcal{T}$ of $T$-connected elements such that $\bigoplus_{i \in I} k_{U_i} \to G'$. The composition $k_{U_i} \to G' \to k_U$ is given by the multiplication by $a_i \in k$. Set $I_0 = \{k_{U_i}; a_i \neq 0\}$, we may assume $a_i = 1$. We get a diagram

\[
\begin{array}{ccc}
\bigoplus_{i \in I_0} k_{U_i} & \longrightarrow & G' \longrightarrow G \\
\downarrow & & \downarrow \\
k_{U} & \to & F.
\end{array}
\]

The composition $k_{U} \to G' \to G$ defines $t_i \in \text{Hom}_{k_{X}}(k_{U_i}, G) \simeq \Gamma(U_i; \rho_* G)$. Hence for each $s \in \Gamma(U; \rho_* F)$ there exists a finite covering $\{U_i\}$ of $U$ and $t_i \in \Gamma(U_i; \rho_* G)$ such that
\[ \psi(t_i) = s|_{U_i}. \] This means that \( \psi \) is surjective. \[ \square \]

**Notation 2.2.5.** Since the functor \( \rho_* \) is fully faithfull and exact on \( \text{Coh}(T) \), we will often identify \( \text{Coh}(T) \) with its image in \( \text{Mod}( k_T ) \) and write \( F \) instead of \( \rho_* F \) for \( F \in \text{Coh}(T) \).

**Theorem 2.2.6.** The following hold:

(i) The category \( \text{Coh}(T) \) is stable by finite sums, kernels, cokernels and extensions in \( \text{Mod}( k_T ) \).

(ii) The category \( \text{Coh}(T) \) is stable by \( \bigotimes_{k_T} \) in \( \text{Mod}( k_T ) \).

**Proof.** (i) The result follows from a general result of homological algebra of [27, Appendix A.1]. With the notations of [27] let \( P \) be the set of finite families of elements of \( T \), for \( U = \{ U_i \}_{i \in I} \in P \) set

\[ L(U) = \bigoplus_i k_{U_i}, \]

for \( V = \{ V_j \}_{j \in J} \in P \) set

\[ \text{Hom}_P(U, V) = \text{Hom}_{k_T}(L(U), L(V)) = \bigoplus_i \bigoplus_j \text{Hom}_{k_T}(k_{U_i}, k_{V_j}) \]

and for \( F \in \text{Mod}(k_T) \) set

\[ H(U, F) = \text{Hom}_{k_T}(L(U), F) = \bigoplus_i \text{Hom}_{k_T}(k_{U_i}, F). \]

By Proposition A.1 of [27] in order to prove (i) it is enough to prove the properties (A.1)–(A.4) below:

(A.1) For any \( U = \{ U_i \} \in P \) the functor \( H(U, \bullet) \) is left exact in \( \text{Mod}(k_T) \).

(A.2) For any morphism \( g : V \to W \) in \( P \), there exists a morphism \( f : U \to V \) in \( P \) such that \( U \xrightarrow{f} V \xrightarrow{g} W \) is exact.

(A.3) For any epimorphism \( f : F \to G \) in \( \text{Mod}(k_T) \), \( U \in P \) and \( \psi \in H(U, G) \), there exists \( V \in P \) and an epimorphism \( g \in \text{Hom}_P(V, U) \) and \( \varphi \in H(V, F) \) such that \( \psi \circ g = f \circ \varphi \).

(A.4) For any \( U, V \in P \) and \( \psi \in H(U, L(V)) \) there exists \( W \in P \) and an epimorphism \( f \in \text{Hom}_P(W, U) \) and a morphism \( g \in \text{Hom}_P(W, U) \) such that \( L(g) = \psi \circ f \) in \( \text{Hom}_{k_T}(L(W), L(V)) \).

It is easy to check that the axioms (A.1)–(A.4) are satisfied.

(ii) Let \( F \in \text{Coh}(T) \). Then \( F \) has a resolution

\[ \bigoplus_{j \in J} k_{U_j} \to \bigoplus_{i \in I} k_{U_i} \to F \to 0 \]
with $I$ and $J$ finite. Let $V \in \mathcal{T}$. The sequence

$$\bigoplus_{j \in J} k_{V \cap U_j} \to \bigoplus_{i \in I} k_{V \cap U_i} \to F_V \to 0$$

is exact. Then it follows from (i) that $F_V$ is coherent. Let $G \in \text{Coh}(\mathcal{T})$. The sequence

$$\bigoplus_{j \in J} G_{U_j} \to \bigoplus_{i \in I} G_{U_i} \to G \bigotimes_{k_{\mathcal{T}}} F \to 0$$

is exact. The sheaves $G_{U_i}$ and $G_{U_j}$ are coherent for each $i \in I$ and each $j \in J$. Hence it follows by (i) that $G \bigotimes_{k_{\mathcal{T}}} F$ is coherent as required. □

**Corollary 2.2.7.** The following hold:

(i) The category $\text{Coh}(\mathcal{T})$ is stable by finite sums, kernels, cokernels in $\text{Mod}(k_X)$.

(ii) The category $\text{Coh}(\mathcal{T})$ is stable by $\bigotimes_{k_X}$ in $\text{Mod}(k_X)$.

**Proof.** (i) The stability under finite sums and kernels is easy, see [29, Exercise 8.23]. Let $F, G \in \text{Coh}(\mathcal{T})$ and let $\varphi : F \to G$ be a morphism in $\text{Mod}(k_X)$. Then $\rho_*(\varphi)$ is a morphism in $\text{Mod}(k_{\mathcal{T}})$ and $\text{coker}(\rho_*\varphi) \in \text{Coh}(\mathcal{T})$ by Theorem 2.2.6. We have $\text{coker}(\rho_*\varphi) \simeq \rho_* \text{coker} \varphi$ since $\rho_*$ is exact on $\text{Coh}(\mathcal{T})$ by Proposition 2.2.4. Composing with $\rho^{-1}$ and applying Proposition 2.1.6 we obtain $\text{coker} \varphi \in \text{Coh}(\mathcal{T})$.

(ii) The proof of the stability by $\bigotimes_{k_X}$ is similar to that of Theorem 2.2.6. □

**Theorem 2.2.8.** (i) Let $G \in \text{Coh}(\mathcal{T})$ and let $\{F_i\}$ be a filtrant inductive system in $\text{Mod}(k_{\mathcal{T}})$. Then we have the isomorphism

$$\lim_i \text{Hom}_{k_{\mathcal{T}}}(\rho_* G, F_i) \simeq \text{Hom}_{k_{\mathcal{T}}}(\rho_* G, \lim_i F_i).$$

(ii) Let $F \in \text{Mod}(k_{\mathcal{T}})$. There exists a small filtrant inductive system $\{F_i\}_{i \in I}$ in $\text{Coh}(\mathcal{T})$ such that $F \simeq \lim_i \rho_* F_i$.

**Proof.** (i) There exists an exact sequence $G_1 \to G_0 \to G \to 0$ with $G_1, G_0$ finite direct sums of constant sheaves $k_U$ with $U \in \mathcal{T}$. Since $\rho_*$ is exact on $\text{Coh}(\mathcal{T})$ and commutes with finite sums, by Proposition 2.2.3 we are reduced to prove the isomorphism $\lim_i \Gamma(U; F_i) \simeq \Gamma(U; \lim_i F_i)$. Then the result follows from Proposition 2.1.7.

(ii) Let $F \in \text{Mod}(k_{\mathcal{T}})$, and define $I_0 := \{(U, s) : U \in \mathcal{T}, s \in \Gamma(U; F)\}$

$$G_0 := \bigoplus_{(U, s) \in I_0} \rho_* k_U$$

The morphism $\rho_* k_U \to F$, where the section $1 \in \Gamma(U; k_U)$ is sent to $s \in \Gamma(U; F)$
defines an epimorphism \( \varphi : G_0 \to F \). Replacing \( F \) by \( \ker \varphi \) we construct a sheaf \( G_1 = \bigoplus_{(V,t) \in I_1} \rho_*k_V \) and an epimorphism \( G_1 \to \ker \varphi \). Hence we get an exact sequence \( G_1 \to G_0 \to F \to 0 \). For \( J_0 \subseteq I_0 \) set for short \( G_{J_0} = \bigoplus_{(U,s) \in J_0} \rho_*k_U \) and define similarly \( G_{J_1} \). Set

\[ J = \{ (J_1, J_0); J_k \subseteq I_k, J_k \text{ is finite and } \text{im} \varphi |_{G_{J_1}} \subseteq G_{J_0} \} \]

The category \( J \) is filtrant and \( F \simeq \varprojlim_{(J_1, J_0) \in J} \text{coker}(G_{J_1} \to G_{J_0}) \).

**Corollary 2.2.9.** Let \( G \in \text{Coh}(T) \) and let \( \{ F_i \} \) be a filtrant inductive system in \( \text{Mod}(k_T) \). Then we have an isomorphism

\[
\varprojlim_i \text{Hom}_{k_T}(G, F_i) \simeq \text{Hom}_{k_T}(G, \varprojlim_i F_i).
\]

**Proof.** Let \( U \in T \). We have the chain of isomorphisms

\[
\Gamma(U; \varprojlim_i \text{Hom}_{k_T}(G, F_i)) \simeq \varprojlim_i \Gamma(U; \text{Hom}_{k_T}(G, F_i)) \\
\simeq \varprojlim_i \text{Hom}_{k_T}(G_U, F_i) \\
\simeq \text{Hom}_{k_T}(G_U, \varprojlim_i F_i) \\
\simeq \Gamma(U; \text{Hom}_{k_T}(G, \varprojlim_i F_i)),
\]

where the first and the third isomorphism follow from Theorem 2.2.8 (i). The fact that \( G_U \in \text{Coh}(T) \) follows from Theorem 2.2.6 (ii).

As in [28], we can define the indization of the category \( \text{Coh}(T) \). Recall that the category \( \text{Ind}(\text{Coh}(T)) \), of \( \text{ind-}T \)-coherent sheaves is the category whose objects are filtrant inductive limits of functors

\[
\varprojlim_i \text{Hom}_{\text{Coh}(T)}(\bullet, F_i) \quad (\text{"}\varprojlim_i F_i\text{" for short}),
\]

where \( F_i \in \text{Coh}(T) \), and the morphisms are the natural transformations of such functors. Note that since \( \text{Coh}(T) \) is a small category, \( \text{Ind}(\text{Coh}(T)) \) is equivalent to the category of \( k \)-additive left exact contravariant functors from \( \text{Coh}(T) \) to \( \text{Mod}(k) \). See [29] for a complete exposition on indizations of categories. We can extend the functor \( \rho_* : \text{Coh}(T) \to \text{Mod}(k_T) \) to \( \lambda : \text{Ind}(\text{Coh}(T)) \to \text{Mod}(k_T) \) by setting \( \lambda(\text{"}\varprojlim_i F_i\text{"}) := \varprojlim_i \rho_* F_i \).

**Corollary 2.2.10.** The functor \( \lambda : \text{Ind}(\text{Coh}(T)) \to \text{Mod}(k_T) \) is an equivalence of categories.
Proof. Let $F = \lim_{\rightarrow} F_j, G = \lim_{\rightarrow} G_i \in \text{I}(\text{Coh}(T))$. By Theorem 2.2.8 (i) and the fact that the functor $\rho_*$ is fully faithfull on $\text{Coh}(T)$ we have
\[
\text{Hom}_{kT}(\lambda(F), \lambda(G)) \simeq \text{Hom}_{kT}(\lim_{\rightarrow} \rho_* F_j, \lim_{\rightarrow} \rho_* G_i)
\]
\[
\simeq \lim_{\rightarrow} \lim_{\rightarrow} \text{Hom}_{kT}(\rho_* F_j, \rho_* G_i)
\]
\[
\simeq \lim_{\rightarrow} \lim_{\rightarrow} \text{Hom}_{\text{Coh}(T)}(F_j, G_i)
\]
\[
\simeq \text{Hom}_{\text{Ind}(\text{Coh}(T))}(F, G),
\]
hence $\lambda$ is fully faithful. By Theorem 2.2.8 (ii) for each $F \in \text{Mod}(kT)$ there exists $G = \lim_{\rightarrow} G_i \in \text{Ind}(\text{Coh}(T))$ such that $\lambda(G) = \lim_{\rightarrow} \rho_* F_i \simeq F$, hence $\lambda$ is essentially surjective.

2.3. $T$-flabby sheaves.

Definition 2.3.1. We say that an object $F \in \text{Mod}(kT)$ is $T$-flabby if for each $U, V \in T$ with $V \supseteq U$ the restriction morphism $\Gamma(V; F) \to \Gamma(U; F)$ is surjective.

Remark 2.3.2. Remark that the category $\text{Mod}(kT)$ is a Grothendieck category, hence it has enough injectives. It follows from the definition that injective sheaves are $T$-flabby. This implies that the family of $T$-flabby objects is cogenerating in $\text{Mod}(kT)$.

Example 2.3.3. Let us see some examples of $T$-flabby sheaves:

(i) When $T$ is the family of Example 2.1.2 we obtain the family of $sa$-flabby objects of $[10]$.

(ii) When $T$ is the family of Example 2.1.3 we obtain the family of quasi-injective objects of $[35]$.

Proposition 2.3.4. The following hold:

(i) Let $F_i$ be a filtrant inductive system of $T$-flabby sheaves. Then $\lim_{i} F_i$ is $T$-flabby.

(ii) Products of $T$-flabby objects are $T$-flabby.

Proof. We will only prove (i) since the proof of (ii) is similar since taking products is exact and commutes with taking sections. Let $U \in T$. Then for each $i$ the restriction morphism $\Gamma(V; F_i) \to \Gamma(U; F_i)$ is surjective. Applying the exact $\lim_{i}$ and using Proposition 2.1.7, the morphism
\[
\Gamma(V; \lim_{i} F_i) \simeq \lim_{i} \Gamma(V; F_i) \to \lim_{i} \Gamma(U; F_i) \simeq \Gamma(U; \lim_{i} F_i)
\]
is surjective.
Proposition 2.3.5. The full additive subcategory of \( \text{Mod}(k_T) \) of \( T \)-flabby object is \( \Gamma(U; \bullet) \)-injective for every \( U \in T \), i.e.:

(i) For every \( F \in \text{Mod}(k_T) \) there exists a \( T \)-flabby object \( F' \in \text{Mod}(k_T) \) and an exact sequence \( 0 \to F \to F' \).

(ii) Let \( 0 \to F' \to F \to F'' \to 0 \) be an exact sequence in \( \text{Mod}(k_T) \) and assume that \( F' \) is \( T \)-flabby. Then the sequence

\[
0 \to \Gamma(U; F') \to \Gamma(U; F) \to \Gamma(U; F'') \to 0
\]

is exact.

(iii) Let \( F', F, F'' \in \text{Mod}(k_T) \), and consider the exact sequence

\[
0 \to F' \to F \to F'' \to 0.
\]

Suppose that \( F' \) is \( T \)-flabby. Then \( F \) is \( T \)-flabby if and only if \( F'' \) is \( T \)-flabby.

Proof. (i) It follows from the definition that injective sheaves are \( T \)-flabby. So (i) holds since it is true for injective sheaves. Indeed, as a Grothendieck category, \( \text{Mod}(k_T) \) admits enough injectives.

(ii) Let \( s'' \in \Gamma(U; F'') \), and let \( \{V_i\}_{i=1}^n \in \text{Cov}(U) \) be such that there exists \( s_i \in \Gamma(V_i; F) \) whose image is \( s''|_{V_i} \). For \( n \geq 2 \) on \( V_1 \cap V_2 \) \( s_1 - s_2 \) defines a section of \( \Gamma(V_1 \cap V_2; F') \) which extends to \( s' \in \Gamma(U; F') \) since \( F' \) is \( T \)-flabby. Replace \( s_1 \) with \( s_1 - s' \) (identifying \( s' \) with its image in \( F \)). We may suppose that \( s_1 = s_2 \) on \( V_1 \cap V_2 \). Then there exists \( t \in \Gamma(V_1 \cap V_2, F) \) such that \( t|_{V_i} = s_i, \ i = 1, 2 \). Thus the induction proceeds.

(iii) Let \( U, V \in T \) with \( V \supseteq U \) and let us consider the diagram below

\[
\begin{array}{ccc}
0 & \longrightarrow & \Gamma(V; F') \\
\downarrow{\alpha} & & \downarrow{\beta} \\
0 & \longrightarrow & \Gamma(U; F')
\end{array}
\begin{array}{ccc}
\Gamma(V; F) & \longrightarrow & \Gamma(V; F'') \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\Gamma(U; F) & \longrightarrow & \Gamma(U; F'')
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & 0 \\
\downarrow{\gamma} & & \\
0 & \longrightarrow & 0
\end{array}
\]

where the row are exact by (ii) and the morphism \( \alpha \) is surjective since \( F' \) is \( T \)-flabby. It follows from the five lemma that \( \beta \) is surjective if and only if \( \gamma \) is surjective.

\[\square\]

Theorem 2.3.6. Let \( F \in \text{Mod}(k_T) \). Then the following hold:

(i) \( F \) is \( T \)-flabby if and only if the functor \( \text{Hom}_{k_T}(\bullet, F) \) is exact on \( \text{Coh}(T) \).

(ii) If \( F \) is \( T \)-flabby then the functor \( \text{Hom}_{k_T}(\bullet, F) \) is exact on \( \text{Coh}(T) \).

Proof. (i) is a consequence of a general result of homological algebra (see Theorem 8.7.2 of [29]). For (ii), let \( F \in \text{Mod}(k_T) \) be \( T \)-flabby. There is an isomorphism of functors

\[
\Gamma(U; \text{Hom}_{k_T}(\bullet, F)) \cong \text{Hom}_{k_T}((\bullet)_U, F)
\]

for each \( U \in T \). By Theorem 2.2.6 and (i) the functor \( \text{Hom}_{k_T}((\bullet)_U, F) \) is exact on
Coh(T) and so the functor $\mathcal{H}om_{k_T}(\bullet, F)$ is also exact on Coh(T).

**Theorem 2.3.7.** Let $G \in \text{Coh}(T)$. Then the following hold:

(i) The family of $T$-flabby sheaves is injective with respect to the functor $\text{Hom}_{k_T}(G, \bullet)$.

(ii) The family of $T$-flabby sheaves is injective with respect to the functor $\mathcal{H}om_{k_T}(G, \bullet)$.

**Proof.** (i) Let $G \in \text{Coh}(T)$. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\text{Mod}(k_T)$ and assume that $F'$ is $T$-flabby. We have to show that the sequence

$$0 \to \text{Hom}_{k_T}(G, F') \to \text{Hom}_{k_T}(G, F) \to \text{Hom}_{k_T}(G, F'') \to 0$$

is exact.

There is an epimorphism $\varphi : \bigoplus_{i \in I} kU_i \to G$ where $I$ is finite and $U_i \in T$ for each $i \in I$. The sequence $0 \to \ker \varphi \to \bigoplus_{i \in I} kU_i \to G \to 0$ is exact. We set for short $G_1 = \ker \varphi$ and $G_2 = \bigoplus_{i \in I} kU_i$. We get the following diagram where the first column is exact by Theorem 2.3.6 (i)

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}_{k_T}(G, F') \to \text{Hom}_{k_T}(G, F) \to \text{Hom}_{k_T}(G, F'') \to 0 \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}_{k_T}(G_2, F') \to \text{Hom}_{k_T}(G_2, F) \to \text{Hom}_{k_T}(G_2, F'') \to 0 \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}_{k_T}(G_1, F') \to \text{Hom}_{k_T}(G_1, F) \to \text{Hom}_{k_T}(G_1, F'') \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

The second row is exact by Proposition 2.3.5 (ii), hence the top row is exact by the snake lemma.

(ii) Let $G \in \text{Coh}(T)$. It is enough to check that for each $U \in T$ and each exact sequence $0 \to F' \to F \to F'' \to 0$ with $F'$ $T$-flabby, the sequence

$$0 \to \Gamma(U; \mathcal{H}om_{k_T}(G, F')) \to \Gamma(U; \mathcal{H}om_{k_T}(G, F)) \to \Gamma(U; \mathcal{H}om_{k_T}(G, F'')) \to 0$$

is exact. We have

$$\Gamma(U, \mathcal{H}om_{k_T}(G, \bullet)) \simeq \text{Hom}_{k_T}(G_U, \bullet),$$

and, by (i) and the fact that $G_U \in \text{Coh}(T)$ (Theorem 2.2.6 (ii)), $T$-flabby objects are injective with respect to the functor $\text{Hom}_{k_T}(G_U, \bullet)$ for each $G \in \text{Coh}(T)$, and for each $U \in T$. \hfill $\square$
Proposition 2.3.8. Let $F \in \text{Mod}(k_T)$. Then $F$ is $T$-flabby if and only if $\text{Hom}_{k_T}(G, F)$ is $T$-flabby for each $G \in \text{Coh}(T)$.

Proof. Suppose that $F$ is $T$-flabby, and let $G \in \text{Coh}(T)$. We have

$$\text{Hom}_{k_T}(\bullet, \text{Hom}_{k_T}(G, F)) \cong \text{Hom}_{k_T}(\bullet \bigotimes_{k_T} G, F)$$

and $\text{Hom}_{k_T}(\bullet \bigotimes_{k_T} G, F)$ is exact on $\text{Coh}(T)$ by Theorems 2.2.6 (ii) and 2.3.6 (i).

Suppose that $\text{Hom}_{k_T}(G, F)$ is $T$-flabby for each $G \in \text{Coh}(T)$. Let $U, V \in T$ with $V \supseteq U$. For each $W \in T$ the morphism $\Gamma(V; \Gamma_W F) \rightarrow \Gamma(U; \Gamma_W F)$ is surjective. Hence the morphism

$$\Gamma(V; F) \cong \Gamma(V; \Gamma_V F)$$

$$\rightarrow \Gamma(U; \Gamma_V F)$$

$$\cong \Gamma(U; F)$$

is surjective. □

Let us consider the following subcategory of $\text{Mod}(k_T)$:

$$\mathcal{P}_{X_T} := \{ G \in \text{Mod}(k_T); G \text{ is } \text{Hom}_{k_T}(\bullet, F)\text{-acyclic for each } F \in \mathcal{F}_{X_T} \},$$

where $\mathcal{F}_{X_T}$ is the family of $T$-flabby objects of $\text{Mod}(k_T)$.

This category is generating. In fact if $\{ U_j \}_{j \in J} \in T$, then $\bigoplus_{j \in J} kU_j \in \mathcal{P}_{X_T}$ by Theorem 2.3.7 (and the fact that $\Pi \text{Hom}_{k_T}(\bullet, \bullet) \cong \text{Hom}_{k_T}(\bigoplus \bullet, \bullet)$) and products are exact). Moreover $\mathcal{P}_{X_T}$ is stable by $\bullet \bigotimes_{k_T} K$, where $K \in \text{Coh}(T)$. In fact if $G \in \mathcal{P}_{X_T}$ and $F \in \mathcal{F}_{X_T}$ we have

$$\text{Hom}_{k_T}(G \bigotimes_{k_T} K, F) \cong \text{Hom}_{k_T}(G, \text{Hom}_{k_T}(K, F))$$

and $\text{Hom}_{k_T}(K, F)$ is $T$-flabby by Proposition 2.3.8. In particular, if $G \in \mathcal{P}_{X_T}$ then $G_U \in \mathcal{P}_{X_T}$ for every $U \in \text{Op}(X_T)$.

Theorem 2.3.9. The category $(\mathcal{P}_{X_T}^{\text{op}}, \mathcal{F}_{X_T})$ is injective with respect to the functors $\text{Hom}_{k_T}(\bullet, \bullet)$ and $\text{Hom}_{k_T}(\bullet, \bullet)$.

Proof. (i) Let $G \in \mathcal{P}_{X_T}$ and consider an exact sequence $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ with $F'$ $T$-flabby. We have to prove that the sequence

$0 \rightarrow \text{Hom}_{k_T}(G, F') \rightarrow \text{Hom}_{k_T}(G, F) \rightarrow \text{Hom}_{k_T}(G, F'') \rightarrow 0$
is exact. Since the functor $\text{Hom}_{kT}(G, \bullet)$ is acyclic on $T$-flabby sheaves we obtain the result.

Let $F$ be $T$-flabby, and let $0 \to G' \to G \to G'' \to 0$ be an exact sequence on $\mathcal{P}_{X_T}$. Since the objects of $\mathcal{P}_{X_T}$ are $\text{Hom}_{kT}(\bullet, F)$-acyclic the sequence

$$0 \to \text{Hom}_{kT}(G'', F) \to \text{Hom}_{kT}(G, F) \to \text{Hom}_{kT}(G', F) \to 0$$

is exact.

(ii) Let $G \in \mathcal{P}_{X_T}$, and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence with $F'$ $T$-flabby. We shall show that for each $U \in T$ the sequence

$$0 \to \Gamma(U; \text{Hom}_{kT}(G, F')) \to \Gamma(U; \text{Hom}_{kT}(G, F)) \to \Gamma(U; \text{Hom}_{kT}(G, F'')) \to 0$$

is exact. This is equivalent to show that for each $U \in T$ the sequence

$$0 \to \text{Hom}_{kT}(G'_U, F') \to \text{Hom}_{kT}(G_U, F) \to \text{Hom}_{kT}(G'_U, F'') \to 0$$

is exact. This follows since $G'_U \in \mathcal{P}_{X_T}$ as we saw above. The proof of the exactness in $\mathcal{P}_{X_T}^{op}$ is similar. □

**Proposition 2.3.10.** Let $F \in \text{Mod}(k_T)$. The following assumptions are equivalent

(i) $F$ is $T$-flabby,
(ii) $F$ is $\text{Hom}_{kT}(G, \bullet)$-acyclic for each $G \in \text{Coh}(T)$,
(iii) $R^1 \text{Hom}_{kT}(k_{V \setminus U}, F) = 0$ for each $U, V \in T$.

**Proof.** (i) $\Rightarrow$ (ii) follows from Theorem 2.3.7, (ii) $\Rightarrow$ (iii) setting $G = k_{V \setminus U}$ with $U, V \in T$, (iii) $\Rightarrow$ (i) since if $R^1 \text{Hom}_{kT}(k_{V \setminus U}, F) = 0$ for each $U, V \in T$ with $V \supseteq U$, then the restriction $\Gamma(V; F) \to \Gamma(U; F)$ is surjective. □

Let $X, Y$ be two topological spaces and let $T \subset \text{Op}(X), T' \subset \text{Op}(Y)$ satisfy (2.1). Let $f : X \to Y$ be a continuous map. If $f^{-1}(T') \subset T$ then $f$ defines a morphism of sites $f : X_T \to Y_{T'}$.

**Proposition 2.3.11.** Let $f : X_T \to Y_{T'}$ be a morphism of sites. $T$-flabby sheaves are injective with respect to the functor $f_*$. The functor $f_*$ sends $T$-flabby sheaves to $T'$-flabby sheaves.

**Proof.** Let us consider $V \in T'$. There is an isomorphism of functors $\Gamma(V; f_* \bullet) \simeq \Gamma(f^{-1}(V); \bullet)$. It follows from Proposition 2.3.5 that $T$-flabby are injective with respect to the functor $\Gamma(f^{-1}(V); \bullet)$ for any $V \in T'$.

Let $F$ be $T$-flabby and let $U, V \in T'$ with $V \supset U$. Then the morphism

$$\Gamma(V; f_* F) = \Gamma(f^{-1}(V); F) \to \Gamma(f^{-1}(U); F) = \Gamma(U; f_* F)$$

is surjective. □
2.4. \textbf{T}-sheaves on locally weakly quasi-compact spaces.

Assume that $X$ is a locally weakly quasi-compact space.

**Lemma 2.4.1.** For each $U \in \text{Op}^c(X)$ there exists $V \in T$ such that $U \subseteq \subseteq V \subseteq \subseteq X$.

**Proof.** Since $X$ is locally weakly quasi-compact we may find $W \in \text{Op}^c(X)$ such that $U \subseteq \subseteq W$. By (2.1) (i) we may find a covering $\{W_i\}_{i \in I}$ of $X$ with $W_i \in T$ and $W_i \subseteq \subseteq X$ for each $i \in I$. Then there exists a finite family $\{W_j\}_{\ell = 1}^\ell$ whose union $V = \bigcup_{\ell = 1}^\ell W_j$ contains $W$. Then $V \in T$ and $U \subseteq \subseteq V \subseteq \subseteq X$. \hfill \Box

When $X$ is locally weakly quasi-compact we can construct a left adjoint to the functor $\rho^{-1}$.

**Proposition 2.4.2.** Let $F \in \text{Mod}(k_T)$, and let $U \in \text{Op}(X)$. Then

$$\Gamma(U; \rho^{-1} F) \simeq \lim_{V \subseteq \subseteq U, V \in T} \Gamma(V; F)$$

**Proof.** By Theorem 2.2.8 we may assume $F = \lim_{i} \rho_* F_i$, with $F_i \in \text{Coh}(T)$. Then $\rho^{-1} F \simeq \lim_{i} \rho^{-1} \rho_* F_i \simeq \lim_{i} F_i$. We have the chain of isomorphisms

$$\begin{align*}
\Gamma(U; \rho^{-1} F) &\simeq \lim_{V \subseteq \subseteq U, V \in T} \lim_{W \subseteq \subseteq V} \Gamma(W; \rho^{-1} F) \\
&\simeq \lim_{V \subseteq \subseteq U, V \in T} \lim_{W \subseteq \subseteq V} \Gamma(W; \lim_{i} \rho^{-1} \rho_* F_i) \\
&\simeq \lim_{V \subseteq \subseteq U, V \in T} \Gamma(V; \rho_* F_i) \\
&\simeq \lim_{V \subseteq \subseteq U, V \in T} \Gamma(V; F),
\end{align*}$$

where the first and the fourth isomorphisms follow from Lemma 1.2.16, the third isomorphism is a consequence of Corollary 1.2.13, and the last isomorphism follows from Proposition 2.1.7. \hfill \Box

**Proposition 2.4.3.** The functor $\rho^{-1}$ admits a left adjoint, denoted by $\rho_!$. It satisfies

(i) for $F \in \text{Mod}(k_X)$ and $U \in T$, $\rho_! F$ is the sheaf associated to the presheaf $U \mapsto \lim_{U \subseteq \subseteq V} \Gamma(V; F)$,

(ii) For $U \in \text{Op}(X)$ one has $\rho_! k_U \simeq \lim_{V \subseteq \subseteq U, V \in T} k_V$.

**Proof.** Let $\tilde{F} \in \text{Psh}(k_T)$ be the presheaf $U \mapsto \lim_{U \subseteq \subseteq V} \Gamma(V; F)$, and let $G \in \text{Mod}(k_T)$. We will construct morphisms

$$\text{Hom}_{\text{sh}(k_T)}(\tilde{F}, G) \xrightarrow{\xi} \text{Hom}_{k_X}(F, \rho^{-1} G).$$
To define $\xi$, let $\varphi : \tilde{F} \to G$ and $U \in \text{Op}(X)$. Then the morphism $\xi(\varphi)(U) : F(U) \to \rho^{-1}G(U)$ is defined as follows

$$F(U) \simeq \lim_{V \subseteq U, V \in T} \lim_{V \subseteq W} F(W) \xrightarrow{\varphi} \lim_{V \subseteq U, V \in T} G(V) \simeq \rho^{-1}G(U).$$

On the other hand, let $\psi : F \to \rho^{-1}G$ and $U \in T$. Then the morphism $\vartheta(\psi)(U) : \tilde{F}(U) \to G(U)$ is defined as follows

$$\tilde{F}(U) \simeq \lim_{U \subseteq V \in T} F(V) \xrightarrow{\psi} \lim_{U \subseteq V \in T} \rho^{-1}G(V) \to G(U).$$

By construction one can check that the morphism $\xi$ and $\vartheta$ are inverse to each other. Then (i) follows from the chain of isomorphisms

$$\text{Hom}_{\text{Psh}(kT)}(\tilde{F}, G) \simeq \text{Hom}_{kT}(\tilde{F}^{++}, G) \simeq \text{Hom}_{kT}(\tilde{F}^{++}, G).$$

To show (ii), consider the following sequence of isomorphisms

$$\text{Hom}_{kT}(\rho_!k_U, F) \simeq \text{Hom}_{kX}(k_U, \rho^{-1}F) \simeq \lim_{V \subseteq U, V \in T} \text{Hom}_{kT}(k_V, F) \simeq \text{Hom}_{kT}(\lim_{V \subseteq U, V \in T} k_V, F),$$

where the second isomorphism follows from Proposition 2.4.2.

**Proposition 2.4.4.** The functor $\rho_!$ is exact and commutes with $\lim$ and $\otimes$.

**Proof.** It follows by adjunction that $\rho_!$ is right exact and commutes with $\lim$, so let us show that it is also left exact. With the notations of Proposition 2.4.3, let $\tilde{F} \in \text{Mod}(k_X)$, and let $\tilde{F} \in \text{Psh}(k_T)$ be the presheaf $U \mapsto \lim_{V \subseteq U, V \in T} \Gamma(V; F)$. Then $\rho_!F \simeq \tilde{F}^{++}$, and the functors $F \mapsto \tilde{F}$ and $G \mapsto G^{++}$ are left exact.

Let us show that $\rho_!$ commutes with $\otimes$. Let $F, G \in \text{Mod}(k_X)$, the morphism

$$\lim_{U \subseteq V \in T} F(V) \otimes_k \lim_{U \subseteq V \in T} G(V) \to \lim_{U \subseteq V \in T} \left( F(V) \otimes_k G(V) \right)$$

defines a morphism in $\text{Mod}(k_T)$

$$\rho_!F \otimes_{k_T} \rho_!G \to \rho_! \left( F \otimes_{k_X} G \right)$$

by Proposition 2.4.3 (i). Since $\rho_!$ commutes with $\lim$ we may suppose that $F = k_U$ and

**□**
\(G = k_V\) and the result follows from Proposition 2.4.3 (ii).

**Proposition 2.4.5.** The functor \(\rho_!\) is fully faithful. In particular one has \(\rho^{-1}\circ \rho_! \simeq \text{id}\). Moreover, for \(F \in \text{Mod}(k_X)\) and \(G \in \text{Mod}(k_T)\) one has

\[
\rho^{-1}\text{Hom}_{k_T}(\rho_! F, G) \simeq \text{Hom}_{k_X}(F, \rho^{-1} G).
\]

**Proof.** For \(F, G \in \text{Mod}(k_X)\) by adjunction we have

\[
\text{Hom}_{k_X}(\rho^{-1} \rho_! F, G) \simeq \text{Hom}_{k_X}(F, \rho_! \rho^{-1} G) \simeq \text{Hom}_{k_X}(F, G).
\]

This also implies that \(\rho_!\) is fully faithful, in fact

\[
\text{Hom}_{k_T}(\rho_! F, \rho_! G) \simeq \text{Hom}_{k_X}(F, \rho^{-1} \rho_! G) \simeq \text{Hom}_{k_X}(F, G).
\]

Now let \(K, F \in \text{Mod}(k_X)\) and \(G \in \text{Mod}(k_T)\), we have

\[
\text{Hom}_{k_X}(K, \rho^{-1} \text{Hom}_{k_T}(\rho_! F, G)) \simeq \text{Hom}_{k_T}(\rho \rho_!, \text{Hom}_{k_T}(\rho_! F, G))
\]

\[
\simeq \text{Hom}_{k_T}(\rho_! \rho K \bigotimes_{k_T} \rho_! F, G)
\]

\[
\simeq \text{Hom}_{k_T}(\rho_! K \bigotimes_{k_X} F, G)
\]

\[
\simeq \text{Hom}_{k_X}(K \bigotimes_{k_X} F, \rho^{-1} G)
\]

\[
\simeq \text{Hom}_{k_X}(K, \text{Hom}_{k_X}(F, \rho^{-1} G)).
\]

\[\square\]

Finally let us consider sheaves of rings in \(\text{Mod}(k_T)\). If \(\mathcal{A}\) is a sheaf of rings in \(\text{Mod}(k_X)\), then \(\rho_* \mathcal{A}\) and \(\rho \mathcal{A}\) are sheaves of rings in \(\text{Mod}(k_T)\).

Let \(\mathcal{A}\) be a sheaf of unitary \(k\)-algebras on \(X\), and let \(\mathcal{A} \in \text{Psh}(k_T)\) be the presheaf defined by the correspondence \(T \ni U \mapsto \lim_{U \subseteq C} \Gamma(V; \mathcal{A})\). Let \(F \in \text{Psh}(k_T)\), and assume that, for \(V \subset U\), with \(U, V \in T\), the following diagram is commutative:

\[
\begin{array}{ccc}
\Gamma(U; \mathcal{A}) \bigotimes_k \Gamma(U; F) & \longrightarrow & \Gamma(U; F) \\
\downarrow & & \downarrow \\
\Gamma(V; \mathcal{A}) \bigotimes_k \Gamma(V; F) & \longrightarrow & \Gamma(V; F).
\end{array}
\]

In this case one says that \(F\) is a presheaf of \(\mathcal{A}\)-modules on \(T\).

**Proposition 2.4.6.** Let \(\mathcal{A}\) be a sheaf of \(k\)-algebras on \(X\), and let \(F\) be a presheaf of \(\mathcal{A}\)-modules on \(X_T\). Then \(F^{++} \in \text{Mod}(\rho_! \mathcal{A})\).
PROOF. Let $U \in \mathcal{T}$, and let $r \in \varprojlim_{U \subseteq V} \Gamma(V; \mathcal{A})$. Then $r$ defines a morphism
\[
\lim_{U \subseteq V} \varprojlim_{U \subseteq V} \Gamma(V; \mathcal{A}) \otimes_k \Gamma(W; F) \to \Gamma(W; F)
\]
for each $W \subseteq U$, $W \in \mathcal{T}$, hence an endomorphism of $(F^{++})_{|U_{X,T}} \simeq (F|_{U_{X,T}})^{++}$. This morphism defines a morphism of presheaves $\tilde{A} \to \mathcal{E}nd(F^{++})$ and $\tilde{A}^{++} \simeq \rho_{\ast}A$ by Proposition 2.4.3. Then $F^{++} \in \text{Mod}(\rho_{\ast}A)$.

PROPOSITION 2.4.7. Assume that $X$ is locally weakly quasi-compact. Let $F \in \text{Mod}(k_{\mathcal{T}})$ be $\mathcal{T}$-flabby. Then $\rho^{-1}F$ is $c$-soft.

PROOF. Recall that if $U \in \text{Op}(X)$ then $\Gamma(U; \rho^{-1}F) \simeq \varinjlim_{V \subseteq U} \Gamma(V; F)$, where $V \in \mathcal{T}$. Let $W \in \text{Op}(X)$, $W \subset \subset X$. It follows from Lemma 2.4.1 that every $U' \supset W$, $U' \in \text{Op}(X)$ contains $U \subseteq W$. Hence
\[
\varinjlim_{U'} \Gamma(U'; F) \simeq \varinjlim_{U} \Gamma(U; F),
\]
where $U' \supset W$, $U' \in \text{Op}(X)$ and $U \in \mathcal{T}$ such that $U \supset W$. We have the chain of isomorphisms
\[
\varinjlim_{U} \Gamma(U; \rho^{-1}F) \simeq \varinjlim_{U} \varinjlim_{V \subseteq U} \Gamma(V; F)
\]
\[
\simeq \varinjlim_{U} \Gamma(U; F)
\]
where $U \in \mathcal{T}$, $U \supset W$ and $V \in \mathcal{T}$. The first isomorphism follows from Proposition 2.4.2 and second one follows since for each $U \supset W$, $U \in \mathcal{T}$, there exists $V \in \mathcal{T}$ such that $U \supset V \supset W$.

Let $V, W \in \text{Op}^c(X)$ with $V \subset \subset W$. Since $F$ is $\mathcal{T}$-flabby and filtrant inductive limits are exact, the morphism $\varinjlim_{U} \Gamma(U; \rho^{-1}F) \simeq \varinjlim_{W'} \Gamma(W'; F) \to \varinjlim_{U} \Gamma(U; F) \simeq \Gamma(W'; \rho^{-1}F)$, where $W', U \in \mathcal{T}$, $W' \supset W$, $U \supset V$, is surjective. Hence $\Gamma(W'; \rho^{-1}F) \to \varinjlim_{U} \Gamma(U; \rho^{-1}F)$ is surjective.

2.5. $\mathcal{T}_{\text{loc}}$-sheaves.

Let $X$ be a $\mathcal{T}$-space and let
\[
\mathcal{T}_{\text{loc}} = \{U \in \text{Op}(X) : U \cap W \in \mathcal{T} \text{ for every } W \in \mathcal{T}\}.
\]

Clearly, $\emptyset, X \in \mathcal{T}_{\text{loc}}$, $\mathcal{T} \subseteq \mathcal{T}_{\text{loc}}$ and $\mathcal{T}_{\text{loc}}$ is closed under finite intersections.

DEFINITION 2.5.1. We make the following definitions:

- a subset $S$ of $X$ is a $\mathcal{T}_{\text{loc}}$-subset if and only if $S \cap V$ is a $\mathcal{T}$-subset for every $V \in \mathcal{T}$;
- a closed (resp. open) $\mathcal{T}_{\text{loc}}$-subset is a $\mathcal{T}_{\text{loc}}$-subset which is closed (resp. open) in $X$;
- a $\mathcal{T}_{\text{loc}}$-connected subset is a $\mathcal{T}_{\text{loc}}$-subset which is not the disjoint union of two proper
clopen $T_{loc}$-subsets.

Observe that if $\{S_i\}_i$ is a family of $T_{loc}$-subsets such that $\{i : S_i \cap W \neq \emptyset\}$ is finite for every $W \in T$, then the union and the intersection of the family $\{S_i\}_i$ is a $T_{loc}$-subset. Also the complement of a $T_{loc}$-subset is a $T_{loc}$-subset. Therefore the $T_{loc}$-subsets form a Boolean algebra.

**Example 2.5.2.** Let us see some examples of $T_{loc}$ subsets:

(i) Let $T$ be the family of Example 2.1.2. Then the $T_{loc}$ subsets are the locally semi-algebraic subsets of $X$.

(ii) Let $T$ be the family of Example 2.1.3. Then the $T_{loc}$ subsets are the subanalytic subsets of $X$.

(iii) Let $T$ be the family of Example 2.1.4. Then the $T_{loc}$ subsets are the conic subanalytic subsets of $X$.

(iv) Let $T$ be the family of Example 2.1.5. Then the $T_{loc}$ subsets are the locally definable subsets of $X$.

One can endow $T_{loc}$ with a Grothendieck topology in the following way: a family $\{U_i\}_i$ in $T_{loc}$ is a covering of $U \in T_{loc}$ if for any $V \in T$, there exists a finite subfamily covering $U \cap V$. We denote by $X_{T_{loc}}$ the associated site, write for short $k_{T_{loc}}$ instead of $k_{X_{T_{loc}}}$, and let

$$
\begin{array}{c}
X \\
\downarrow \rho_{T_{loc}} \quad \rho \\
X_{T_{loc}} \quad X_T
\end{array}
$$

be the natural morphisms of sites.

**Remark 2.5.3.** The forgetful functor, induced by the natural morphism of sites $X_{T_{loc}} \to X_T$, gives an equivalence of categories

$$\mathrm{Mod}(k_{T_{loc}}) \sim \mathrm{Mod}(k_T).$$

The quasi-inverse to the forgetful functor sends $F \in \mathrm{Mod}(k_T)$ to $F_{loc} \in \mathrm{Mod}(k_{T_{loc}})$ given by $F_{loc}(U) = \lim_{V \in T} F(U \cap V)$ for every $U \in T_{loc}$.

Therefore, we can and will identify $\mathrm{Mod}(k_{T_{loc}})$ with $\mathrm{Mod}(k_T)$ and apply the previous results for $\mathrm{Mod}(k_T)$ to obtain analogues results for $\mathrm{Mod}(k_{T_{loc}})$.

Recall that $F \in \mathrm{Mod}(k_T)$ is $T$-flabby if the restriction $\Gamma(V; F) \to \Gamma(U; F)$ is surjective for any $U, V \in T$ with $V \supseteq U$. Assume that

$$X_{T_{loc}} \text{ has a countable cover } \{V_n\}_{n \in \mathbb{N}} \text{ with } V_n \in T, \forall n \in \mathbb{N}. \quad (2.4)$$

**Proposition 2.5.4.** Let $F \in \mathrm{Mod}(k_T)$. Then $F$ is $T$-flabby if and only if the
restriction $\Gamma(X; F) \to \Gamma(U; F)$ is surjective for any $U \in T_{loc}$.

**Proof.** Suppose that $F$ is $\mathcal{T}$-flabby. Consider a covering $\{V_n\}_{n \in \mathbb{N}}$ of $X_{T_{loc}}$ satisfying (2.4). Set $U_n = U \cap V_n$ and $S_n = V_n \setminus U_n$. All the sequences

$$0 \to k_{U_n} \to k_{V_n} \to k_{S_n} \to 0$$

are exact. Since $F$ is $\mathcal{T}$-flabby the sequence

$$0 \to \text{Hom}_{\mathcal{T}}(k_{S_n}, F) \to \text{Hom}_{\mathcal{T}}(k_{V_n}, F) \to \text{Hom}_{\mathcal{T}}(k_{U_n}, F) \to 0$$

is exact. Moreover the morphism $\text{Hom}_{\mathcal{T}}(k_{S_{n+1}}, F) \to \text{Hom}_{\mathcal{T}}(k_{S_n}, F)$ is surjective for all $n$ since $S_n = S_{n+1} \cap V_n$ is open in $S_{n+1}$. Then by Proposition 1.12.3 of [26] the sequence

$$0 \to \varprojlim_n \text{Hom}_{\mathcal{T}}(k_{S_n}, F) \to \varprojlim_n \text{Hom}_{\mathcal{T}}(k_{V_n}, F) \to \varprojlim_n \text{Hom}_{\mathcal{T}}(k_{U_n}, F) \to 0$$

is exact. The result follows since $\varprojlim \Gamma(U_n; G) \simeq \Gamma(U; G)$ for any $G \in \text{Mod}(k_T)$ and $U \in T_{loc}$. The converse is obvious. \hfill $\Box$

**Proposition 2.5.5.** The full additive subcategory of $\text{Mod}(k_T)$ of $\mathcal{T}$-flabby object is $\Gamma(U; \bullet)$-injective for every $U \in T_{loc}$.

**Proof.** Take an exact sequence $0 \to F' \to F \to F'' \to 0$, and suppose that $F'$ is $\mathcal{T}$-flabby. Consider a covering $\{V_n\}_{n \in \mathbb{N}}$ of $X_{\mathcal{T}_{loc}}$ satisfying (2.4). Set $U_n = U \cap V_n$. All the sequences

$$0 \to \Gamma(U_n; F') \to \Gamma(U_n; F) \to \Gamma(U_n; F'') \to 0$$

are exact by Proposition 2.3.5, and the morphism $\Gamma(U_{n+1}; F') \to \Gamma(U_n; F')$ is surjective for all $n$. Then by Proposition 1.12.3 of [26] the sequence

$$0 \to \varprojlim_n \Gamma(U_n; F') \to \varprojlim_n \Gamma(U_n; F) \to \varprojlim_n \Gamma(U_n; F'') \to 0$$

is exact. Since $\varprojlim \Gamma(U_n; G) \simeq \Gamma(U; G)$ for any $G \in \text{Mod}(k_T)$ the result follows. \hfill $\Box$

Let $X, Y$ be two topological spaces and let $\mathcal{T} \subset \text{Op}(X)$, $\mathcal{T}' \subset \text{Op}(Y)$ satisfy (2.1). Let $f : X \to Y$ be a continuous map. If $f^{-1}(\mathcal{T}'_{loc}) \subseteq \mathcal{T}_{loc}$ then $f$ defines a morphism of sites $f : X_{\mathcal{T}_{loc}} \to Y_{\mathcal{T}'_{loc}}$.

**Corollary 2.5.6.** Let $f : X_{\mathcal{T}_{loc}} \to Y_{\mathcal{T}'_{loc}}$ be a morphism of sites. $\mathcal{T}$-flabby sheaves are injective with respect to the functor $f_*$. The functor $f_*$ sends $\mathcal{T}$-flabby sheaves to $\mathcal{T}'$-flabby sheaves.
Proof. Let us consider $V \in T'_\text{loc}$. There is an isomorphism of functors $\Gamma(V; f_* \bullet) \cong \Gamma(f^{-1}(V); \bullet)$. It follows from Proposition 2.5.5 that $T$-flabby are injective with respect to the functor $\Gamma(f^{-1}(V); \bullet)$ for any $V \in T'_\text{loc}$.

Let $F$ be $T$-flabby and let $U, V \in T'$ with $V \supset U$. Then the morphism

$$\Gamma(V; f_* F) = \Gamma(f^{-1}(V); F) \to \Gamma(f^{-1}(U); F) = \Gamma(U; f_* F)$$

is surjective by Proposition 2.5.4. □

Remark 2.5.7. An interesting case is when $X$ is a locally weakly quasi-compact space and there exists $S \subseteq \text{Op}(X)$ with $T = \{ U \in S : U \subset X \}$ satisfying (2.1).

Assume that $X$ satisfies (1.7). Then $X$ has a covering $\{ V_n \}_{n \in \mathbb{N}}$ of $X$ such that $V_n \in T$ and $V_n \subset V_{n+1}$ for each $n \in \mathbb{N}$. By Lemma 1.3.5 we may find a covering $\{ U_n \}_{n \in \mathbb{N}}$ of $X$ such that $U_n \in \text{Op}^c(X)$ and $U_n \subset U_{n+1}$ for each $n \in \mathbb{N}$. By Lemma 2.4.1 for each $n \in \mathbb{N}$ there exists $V_n \in T$ such that $U_n \subset V_n \subset U_{n+1}$.

In this situation Proposition 2.5.4 and 2.5.5 are satisfied.

2.6. $T$-spectrum.

Let $X$ be a topological space and let $\mathcal{P}(X)$ be the power set of $X$. Consider a subalgebra $\mathcal{F}$ of the power set Boolean algebra $\langle \mathcal{P}(X), \subseteq \rangle$. Then $\mathcal{F}$ is closed under finite unions, intersections and complements. We refer to [25] for an introduction to this subject.

The Boolean algebra $\mathcal{F}$ has an associated topological space, that we denote by $\tilde{S}(\mathcal{F})$, called its Stone space. The points in $\tilde{S}(\mathcal{F})$ are the ultrafilters $\alpha$ on $\mathcal{F}$. The topology on $\tilde{S}(\mathcal{F})$ is generated by a basis of open and closed sets consisting of all sets of the form

$$\tilde{A} = \{ \alpha \in \tilde{S}(\mathcal{F}) : A \in \alpha \},$$

where $A \in \mathcal{F}$. The space $\tilde{S}(\mathcal{F})$ is a compact totally disconnected Hausdorff space. Moreover, for each $A \in \mathcal{F}$, the subspace $\tilde{A}$ is Hausdorff and compact.

Definition 2.6.1. Let $X$ be a $T$-space and let $\mathcal{F}$ be the Boolean algebra of $T'_\text{loc}$-subsets of $X$ (i.e. Boolean combinations of elements of $T'_\text{loc}$). The topological space $\tilde{X}_T$ is the data of:

- the points of $\tilde{S}(\mathcal{F})$ such that $U \in \alpha$ for some $U \in T$,
- a basis for the topology is given by the family of subsets $\{ \tilde{U} : U \in T \}$.

We call $\tilde{X}_T$ the $T$-spectrum of $X$.

With this topology, for $U \in T$, the set $\tilde{U}$ is quasi-compact in $\tilde{X}_T$ since it is quasi-compact in $\tilde{S}(\mathcal{F})$. Hence $\tilde{X}_T$ is locally weakly quasi-compact with a basis of quasi-compact open subsets given by $\{ \tilde{U} : U \in T \}$. Note that if $X \in T$, then $\tilde{X}_T = \tilde{X}$ which is a spectral topological space.

Remark 2.6.2. We may also define $\tilde{X}_T$ by means of prime filters of elements of $T$. This is because $T$-subsets can be written as finite unions and intersections of $T$-open
and $T$-closed subsets. In this situation an ultrafilter is determined by the prime filter contained in it.

**Proposition 2.6.3.** Let $X$ be a $T$-space. Then there is an equivalence of categories $\text{Mod}(k_T) \simeq \text{Mod}(k_{\tilde{X}_T})$.

**Proof.** Let us consider the functor

$$\zeta^t : T \to \text{Op}(\tilde{X}_T)$$

$$U \mapsto \tilde{U}.$$ 

This defines a morphism of sites $\zeta : \tilde{X}_T \to X_T$. Indeed, if $V \in T$, $S \in \text{Cov}(V)$, then $\tilde{S} = \{ \tilde{V}_i : V_i \in S \} \in \text{Cov}(V)$. Let $F \in \text{Mod}(k_T)$ and consider the presheaf $\zeta^t F \in \text{Psh}(k_{\tilde{X}_T})$ defined by $\zeta^t F(U) = \lim_{\tilde{U} \subseteq \tilde{V}} F(V)$. In particular, if $U = \tilde{V}$, $V \in T$, $\zeta^t F(U) \simeq F(V)$. In this case, by Corollary 1.2.11 we have the isomorphisms

$$\zeta^{-1} F(\tilde{V}) = (\zeta^{-1} F)^{++}(\tilde{V}) \simeq \zeta^{-1} F(\tilde{V}) \simeq F(V).$$

Then for $V \in T$ we have

$$\zeta_* \zeta^{-1} F(V) \simeq \zeta^{-1} F(\tilde{V}) \simeq F(V).$$

This implies $\zeta_* \circ \zeta^{-1} \simeq \text{id}$. On the other hand, given $\alpha \in \tilde{X}_T$ and $G \in \text{Mod}(k_{\tilde{X}_T})$,

$$\langle \zeta^{-1} \zeta_* G, \alpha \rangle \simeq \lim_{\tilde{U} \ni \alpha, U \in T} \zeta^{-1} \zeta_* G(\tilde{U})$$

$$\simeq \lim_{\tilde{U} \ni \alpha, U \in T} \zeta_* G(U)$$

$$\simeq \lim_{\tilde{U} \ni \alpha, U \in T} G(\tilde{U})$$

$$\simeq G_\alpha$$

since $\{ \tilde{U} : U \in T \}$ forms a basis for the topology of $\tilde{X}_T$. This implies $\zeta^{-1} \circ \zeta_* \simeq \text{id}$. □

**Example 2.6.4.** Let us see some examples of $T$-spectra.

(i) When $T$ is the family of Example 2.1.2 the $T$-spectrum $\tilde{X}_T$ of $X$ is the semialgebraic spectrum of $X$ ([10]). When $X$ is semialgebraic, then $\tilde{X}_T = \tilde{X}$, the semialgebraic spectrum of $X$ from [9].

(ii) When $T$ is the family of Example 2.1.3 the $T$-spectrum $\tilde{X}_T$ of $X$ is the subanalytic spectrum of $X$. The equivalence $\text{Mod}(k_{\tilde{X}_{sa}}) \simeq \text{Mod}(k_{X_{sa}})$ was used in [38] to bound the homological dimension of subanalytic sheaves.

(iii) When $T$ is the family of Example 2.1.5 the $T$-spectrum $\tilde{X}_T$ of $X$ is the $\alpha$-minimal
spectrum of $X$. When $X$ is a definable space, then $\tilde{X}_T = \tilde{X}$, the o-minimal spectrum of $X$ from [33], [19].

3. Examples.

In this section we recall our main examples of $T$-sheaves. Good references on o-minimality are, for example, the book [13] by van den Dries and the notes [8] by Coste. For semialgebraic geometry relevant to this paper the reader should consult the work by Delfs [10], Delfs and Knebusch [11] and the book [7] by Bochnak, Coste and Roy. For subanalytic geometry we refer to the work [6] by Bierstone and Milman.

3.1. The semialgebraic site.

Let $R = (R, <, 0, 1, +, \cdot)$ be a real closed field. Let $X$ be a locally semialgebraic space and consider the subfamily of $\text{Op}(X)$ defined by $\mathcal{T} = \{U \in \text{Op}(X) : U$ is semialgebraic$\}$. The family $\mathcal{T}$ satisfies (2.1) and the associated site $X_\mathcal{T}$ is the semialgebraic site on $X$ of [10], [11]. Note also that: (i) the $\mathcal{T}$-subsets of $X$ are exactly the semialgebraic subsets of $X$ ([7]); (ii) $\mathcal{T}_{\text{loc}} = \{U \in \text{Op}(X) : U$ is locally semialgebraic$\}$ and (iii) the $\mathcal{T}_{\text{loc}}$-subsets of $X$ are exactly the locally semialgebraic subsets of $X$ ([11]).

One can show (using triangulation of semialgebraic sets, as in [26]) that the family $\text{Coh}(\mathcal{T})$ corresponds to the family of sheaves which are locally constant on a locally semialgebraic stratification of $X$. For each $F \in \text{Mod}(k_\mathcal{T})$ there exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\text{Coh}(\mathcal{T})$ such that $F \simeq \lim_{i} F_i$.

The subcategory of $T$-flabby sheaves corresponds to the subcategory of $sa$-flabby sheaves of [10] and it is injective with respect to $\Gamma(U; \bullet), U \in \text{Op}(X_\mathcal{T})$ and $\text{Hom}_{k_\mathcal{T}}(G, \bullet)$, $G \in \text{Coh}(\mathcal{T})$. Our results on $T$-flabby sheaves generalize those for $sa$-flabby sheaves from [10].

We call in this case the $T$-spectrum $\tilde{X}_\mathcal{T}$ of $X$ the semialgebraic spectrum of $X$. The points of $\tilde{X}_\mathcal{T}$ are the ultrafilters $\alpha$ of locally semialgebraic subsets of $X$ such that $U \in \alpha$ for some $U \in \text{Op}(X_\mathcal{T})$. This is a locally weakly quasi-compact space with basis of quasi-compact open subsets given by $\{U : U \in \text{Op}(X_\mathcal{T})\}$ and there is an equivalence of categories $\text{Mod}(k_\mathcal{T}) \simeq \text{Mod}(k_{\tilde{X}_\mathcal{T}})$. When $X$ is semialgebraic, then $\tilde{X}_\mathcal{T} = \tilde{X}$, the semialgebraic spectrum of $X$ from [9], and there is an equivalence of categories $\text{Mod}(k_\mathcal{T}) \simeq \text{Mod}(k_{\tilde{X}})$ ([10]).

3.2. The subanalytic site.

Let $X$ be a real analytic manifold and consider the subfamily of $\text{Op}(X)$ defined by $\mathcal{T} = \text{Op}^c(X_{sa}) = \{U \in \text{Op}(X_{sa}) : U$ is subanalytic relatively compact$\}$. The family $\mathcal{T}$ satisfies (2.1) and the associated site $X_\mathcal{T}$ is the subanalytic site $X_{sa}$ of [28], [35]. In this case the $\mathcal{T}_{\text{loc}}$-subsets are the subanalytic subsets of $X$.

The family $\text{Coh}(\mathcal{T})$ corresponds to the family $\text{Mod}_{\mathbb{R}_{\text{c}}}(k_X)$ of $\mathbb{R}$-constructible sheaves with compact support, and for each $F \in \text{Mod}(k_{X_{sa}})$ there exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\text{Mod}_{\mathbb{R}_{\text{c}}}(k_X)$ such that $F \simeq \lim_{i} F_i$.

The subcategory of $T$-flabby sheaves corresponds to quasi-injective sheaves and it is injective with respect to $\Gamma(U; \bullet), U \in \text{Op}(X_{sa})$ and $\text{Hom}_{k_{X_{sa}}}(G, \bullet), G \in \text{Mod}_{\mathbb{R}_{\text{c}}}(k_X)$.
We call in this case the $T$-spectrum $\tilde{X}_T$ of $X$ the subanalytic spectrum of $X$ and denote it by $\tilde{X}_{sa}$. The points of $\tilde{X}_{sa}$ are the ultrafilters of subanalytic subsets of $X$ such that $U \in \alpha$ for some $U \in \text{Op}^c(X_{sa})$. Then there is an equivalence of categories $\text{Mod}(k_{X_{sa}}) \simeq \text{Mod}(k_{\tilde{X}_{sa}})$.

Let $U \in \text{Op}(X_{sa})$ and denote by $U_{X_{sa}}$ the site with the topology induced by $X_{sa}$. This corresponds to the site $X_T$, where $T = \text{Op}^c(X_{sa}) \cap U$. In this situation (2.1) is satisfied.

### 3.3. The conic subanalytic site.

Let $X$ be a real analytic manifold endowed with a subanalytic action $\mu$ of $\mathbb{R}^+$. In other words we have a subanalytic map

$$\mu : X \times \mathbb{R}^+ \to X,$$

which satisfies, for each $t_1, t_2 \in \mathbb{R}^+$:

$$\begin{cases} \mu(x, t_1 t_2) = \mu(\mu(x, t_1), t_2), \\ \mu(x, 1) = x. \end{cases}$$

Denote by $X_{\mathbb{R}^+}$ the topological space $X$ endowed with the conic topology, i.e. $U \in \text{Op}(X_{\mathbb{R}^+})$ if it is open for the topology of $X$ and invariant by the action of $\mathbb{R}^+$. We will denote by $\text{Op}^c(X_{\mathbb{R}^+})$ the subcategory of $\text{Op}(X_{\mathbb{R}^+})$ consisting of relatively weakly quasi-compact open subsets.

Consider the subfamily of $\text{Op}(X_{\mathbb{R}^+})$ defined by $T = \text{Op}^c(X_{sa,\mathbb{R}^+}) = \{U \in \text{Op}^c(X_{\mathbb{R}^+}) : U \text{ is subanalytic}\}$. The family $T$ satisfies (2.1) and the associated site $X_T$ is the conic subanalytic spectrum $X_{sa,\mathbb{R}^+}$. In this case the $T_{loc}$-subsets are the conic subanalytic subsets.

Set $\text{Coh}(X_{sa,\mathbb{R}^+}) = \text{Coh}(T)$. For each $F \in \text{Mod}(k_{X_{sa,\mathbb{R}^+}})$ there exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\text{Coh}(X_{sa,\mathbb{R}^+})$ such that $F \simeq \varprojlim_i F_i$.

The subcategory of $T$-flabby sheaves is injective with respect to $\Gamma(U; \bullet), U \in \text{Op}(X_{sa,\mathbb{R}^+})$ and $\text{Hom}_{k_{X_{sa,\mathbb{R}^+}}}(G, \bullet), G \in \text{Coh}(X_{sa,\mathbb{R}^+})$.

We call in this case the $T$-spectrum $\tilde{X}_T$ of $X$ the conic subanalytic spectrum of $X$ and denote it by $\tilde{X}_{sa,\mathbb{R}^+}$. The points of $\tilde{X}_{sa,\mathbb{R}^+}$ are the ultrafilters $\alpha$ of conic subanalytic subsets of $X$ such that $U \in \alpha$ for some $U \in \text{Op}^c(X_{sa,\mathbb{R}^+})$. Then there is an equivalence of categories $\text{Mod}(k_{X_{sa,\mathbb{R}^+}}) \simeq \text{Mod}(k_{\tilde{X}_{sa,\mathbb{R}^+}})$.

### 3.4. The o-minimal site.

Let $\mathcal{M} = (M, <, (c)_{c \in C}, (f)_{f \in F}, (R)_{R \in \mathcal{R}})$ be an arbitrary o-minimal structure. Let $X$ be a locally definable space and consider the subfamily of $\text{Op}(X)$ defined by $T = \text{Op}(X_{\text{def}}) = \{U \in \text{Op}(X) : U \text{ is definable}\}$. The family $T$ satisfies (2.1) and the associated site $X_T$ is the o-minimal site $X_{\text{def}}$ of [19]. Note also that: (i) the $T$-subsets of $X$ are exactly the definable subsets of $X$ (by the cell decomposition theorem in [13], see [19, Proposition 2.1]); (ii) $T_{loc} = \{U \in \text{Op}(X) : U \text{ is locally definable}\}$ and (iii) the $T_{loc}$-subsets of $X$ are exactly the locally definable subsets of $X$. 
Set $\text{Coh}(X_{\text{def}}) = \text{Coh}(T)$. For each $F \in \text{Mod}(k_{X_{\text{def}}})$ there exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\text{Coh}(X_{\text{def}})$ such that $F \simeq \lim_{\rightarrow} \rho_\ast F_i$.

The subcategory of $T$-flabby sheaves (or definably flabby sheaves) is injective with respect to $\Gamma(U; \bullet)$, $U \in \text{Op}(X_{\text{def}})$ and $\text{Hom}_{\text{Coh}(X_{\text{def}})}(G, \bullet)$, $G \in \text{Coh}(X_{\text{def}})$.

We call in this case the $T$-spectrum $\tilde{X}_T$ of $X$ the definable or o-minimal spectrum of $X$ and denote it by $\tilde{X}_{\text{def}}$. The points of $\tilde{X}_{\text{def}}$ are the ultrafilters $\alpha$ of the Boolean algebra of locally definable subsets of $X$ such that $U \in \alpha$ for some $U \in \text{Op}(X_{\text{def}})$. This is a locally weakly quasi-compact space with basis of quasi-compact open subsets given by $\{\tilde{U} : U \in \text{Op}(X_{\text{def}})\}$ and there is an equivalence of categories $\text{Mod}(k_{X_{\text{def}}}) \simeq \text{Mod}(k_{\tilde{X}_{\text{def}}})$.

Finally observe that since locally semialgebraic spaces are locally definable spaces in a real closed field and real closed fields are o-minimal structures and, relatively compact subanalytic sets are definable sets in the o-minimal expansion of the field of real numbers by restricted globally analytic functions, both the semialgebraic and subanalytic sheaf theory are special cases of the o-minimal sheaf theory.

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