Microscopic Derivation of Ginzburg-Landau Equations for Coexistent States of Superconductivity and Magnetism

Kazuhiro KUBOKI* and Keiji YANO

Department of Physics, Kobe University, Kobe 657-8501

Ginzburg-Landau (GL) equations for the coexistent states of superconductivity and magnetism are derived microscopically from the extended Hubbard model with on-site repulsive and nearest-neighbor attractive interactions. In the derived GL free energy a cubic term that couples the spin-singlet and spin-triplet components of superconducting order parameters (SCOP) with magnetization exists. This term gives rise to a spin-triplet SCOP near the interface between a spin-singlet superconductor and a ferromagnet, consistent with previous theoretical studies based on the Bogoliubov de Gennes method and the quasiclassical Green’s function theory. In coexistent states of singlet superconductivity and antiferromagnetism it leads to the occurrence of \( \pi \)-triplet SCOPs.

KEYWORDS: GL theory, unconventional superconductivity, coexistence, proximity effect

1. Introduction

The coexistence and competition of superconductivity and magnetism have been important issues in various strongly correlated electron systems, \( e.g., \) high-\( T_C \) cuprate superconductors.\(^1\) This is because these two ordered states originate from the same interaction; thus, understanding their relation may give insight into the mechanism of superconductivity.

Heterostructures composed of superconductors and magnetic materials may be useful systems for studying these problems. The properties of the states near an interface strongly depend on the materials used, especially the symmetry of superconducting (SC) states and the underlying electronic states,\(^2-16\) namely, the shape of Fermi surfaces and the type of interactions. The interface states of heterostructures may have quite different characters from those in the bulk. Not only the coexistence of the original order parameters (OPs), but also new ordered states may arise depending on the constituent material. For example, spin-triplet SCOPs can occur near the interface between a spin-singlet superconductor and a ferromagnet. This was theoretically found using the Bogoliubov de Gennes (BdG) method\(^5\) and the

*E-mail address: kuboki@kobe-u.ac.jp
quasiclassical Green’s functions theory.\cite{9,10} Therefore, by combining various types of superconductors with ferromagnets or antiferromagnets, we may know the conditions under which a particular SC state can be realized.

In this paper, we derive GL equations and the GL free energy microscopically from a tight-binding model on a square lattice with on-site repulsive and nearest-neighbor attractive interactions, \textit{i.e.}, the extended Hubbard model. Although this model is a minimal one for treating magnetism and unconventional superconductivity, it exhibits $s$-, $d$-, and chiral $(p_x \pm ip_y)$-wave superconductivity,\cite{17,18} and ferro- and anti-ferromagnetism\cite{19} for different choices of the parameters, especially the electron density (in other words, the shape of the Fermi surface). For this reason this model may be used to examine the material dependence of the interface states of heterostructures composed of superconductors and magnetic materials. The method of deriving GL equations is based on that by Gor’kov\cite{20} with an extension to include magnetic OPs. The resulting GL equations are coupled equations for all types of OPs including magnetization. (The GL equations for superconductors with $s$- and $d$-wave SCOPs have already been obtained from a similar model.\cite{21,22})

Although the GL theory is reliable only qualitatively except near $T_C$, it can give a simple and clear description of the coexistence and competition of multiple OPs. Thus, it is complementary to more sophisticated methods such as the BdG and quasiclassical Green’s function theory.

This paper is organized as follows. In §2, we present the model and treat it by mean-field approximation (MFA). In §3, GL equations and the GL free energy are derived for the coexistent states of superconductivity and ferromagnetism. The case of antiferromagnetism and superconductivity is examined in §4. Section 5 is devoted to summary and discussion.

2. Model and Mean-Field Approximation

We consider the extended Hubbard model on a square lattice, \textit{i.e.}, a tight-binding model that has on-site repulsive and nearest-neighbor attractive interactions. (We use the units \(h = k_B = 1\), and the lattice constant is taken to be unity.) By treating the latter interaction using MFA, $s$-, $d$-, and chiral $(p_x \pm ip_y)$-wave SC states can be realized depending on the electron density. Namely, the symmetry of the SC state may be determined by the shape of the Fermi surface.\cite{17,18} Similarly, the magnetic order, either ferromagnetic and antiferromagnetic (AF), can be obtained by treating the repulsive interaction by MFA.\cite{19}
The Hamiltonian of our model is given by

\[ H = -t \sum_{j \sigma} \sum_{\delta = \pm \delta_j} (c_{j+\delta \sigma}^r c_{j \sigma} e^{i \phi_{j+\delta \sigma}} + h.c.) - \mu_0 \sum_{j \sigma} c_{j \sigma}^r c_{j \sigma} + U \sum_j n_{j \uparrow} n_{j \downarrow} - V \sum_j \sum_{\delta = \pm \delta_j} (n_{j \uparrow} n_{j+\delta \downarrow} + n_{j \downarrow} n_{j+\delta \uparrow}), \]

where \( t, \mu_0, U, \) and \( V \) are the transfer integral, chemical potential, and the on-site repulsive and nearest-neighbor attractive interactions, respectively; \( \sigma = \uparrow, \downarrow \) is the spin index. The magnetic field is taken into account using the Peierls phase \( \phi_{j \ell} = \frac{A}{\phi_0} \int_{j}^{\ell} A \cdot dl \), with \( A \) and \( \phi_0 = \frac{hc}{2e} \) being the vector potential and flux quantum, respectively. We treat this Hamiltonian using the standard procedure of MFA:

\[ \begin{align*}
    n_{j \uparrow} n_{j \downarrow} & \rightarrow \langle n_{j \uparrow} \rangle n_{j \downarrow} + \langle n_{j \downarrow} \rangle n_{j \uparrow} - \langle n_{j \uparrow} \rangle \langle n_{j \downarrow} \rangle, \\
    n_{j \uparrow} n_{\ell \downarrow} & \rightarrow \Delta_{j \ell} c_{j \uparrow}^\dagger c_{\ell \downarrow} + \Delta_{j \ell}^* c_{\ell \downarrow}^\dagger c_{j \uparrow} - |\Delta_{j \ell}|^2.
\end{align*} \]

The SCOPs and magnetization (i.e., the OP for magnetism) are defined as

\[ \Delta_{j \ell} = \langle c_{j \uparrow} c_{\ell \downarrow} \rangle, \quad m_j = \frac{1}{2} (n_{j \uparrow} - n_{j \downarrow}). \]

Then the mean-field Hamiltonian is written as

\[ H_{MF} = -t \sum_{j \sigma} \sum_{\delta = \pm \delta_j} c_{j+\delta \sigma}^r c_{j \sigma} e^{i \phi_{j+\delta \sigma}} + \sum_j [(\mu - \mu_0 - Um_j) c_{j \uparrow}^\dagger c_{j \downarrow} - (\mu + Um_j) c_{j \downarrow}^\dagger c_{j \uparrow}] - V \sum_j \sum_{\delta = \pm \delta_j} [\Delta_{j \downarrow + \delta} c_{j \downarrow + \delta}^\dagger c_{j \uparrow} + h.c.] + E_0, \]

where

\[ E_0 = U \sum_j [m_j^2 - \frac{1}{4} (n_j^{(0)})^2] + V \sum_j \sum_{\delta = \pm \delta_j} |\Delta_{j \downarrow + \delta}|^2, \]

with \( n_j^{(0)} = \langle n_{j \uparrow} + n_{j \downarrow} \rangle \) being the electron density at the site \( j \). Here, \( \mu = \mu_0 - Un_0/2 \) is the renormalized chemical potential with \( n_0 \) being the average electron density of the system.

In order to derive the GL equations for OPs, we introduce the following thermal Green’s functions:

\[ G_{\sigma}(j, \ell, \tau) = -\langle T_\tau c_{j \sigma}(\tau) c_{\ell \sigma}^\dagger \rangle, \quad F_{\sigma \sigma'}^\dagger(j, \ell, \tau) = -\langle T_\tau c_{j \sigma}(\tau) c_{\ell \sigma'}^\dagger \rangle. \]

The equations of motion for \( G_\sigma \) and \( F_{\sigma \sigma'}^\dagger \) (i.e., the Gor’kov equations) are obtained by taking their \( \tau \) derivatives and carrying out Fourier transformation to the Matsubara frequency \( ie_n = (2n+1)i\pi T; \) \( T \) being the temperature). These equations can be transformed to the following
coupled equations for $G_\sigma$ and $F_{\sigma\sigma'}^\dagger$:

\begin{align}
G_\uparrow(j, \ell, i\epsilon_n) &= \tilde{G}_0(j, \ell, i\epsilon_n) + V \sum_{k,\delta} \tilde{G}_0(j, k, i\epsilon_n)\Delta_{k,\delta+k} F_{1\uparrow}^\dagger(k + \delta, \ell, i\epsilon_n) \\
&\quad - U \sum_k \tilde{G}_0(j, k, i\epsilon_n)m_k G_\uparrow(k, \ell, i\epsilon_n), \\
G_\downarrow(j, \ell, i\epsilon_n) &= \tilde{G}_0(j, \ell, i\epsilon_n) - V \sum_{k,\delta} \tilde{G}_0(j, k, i\epsilon_n)\Delta_{k,\delta+k} F_{1\downarrow}^\dagger(k + \delta, \ell, i\epsilon_n) \\
&\quad + U \sum_k \tilde{G}_0(j, k, i\epsilon_n)m_k G_\downarrow(k, \ell, i\epsilon_n), \\
F_{1\uparrow}^\dagger(j, \ell, i\epsilon_n) &= - V \sum_{k,\delta} \tilde{G}_0(k, j, -i\epsilon_n)\Delta_{k,\delta+k}^* G_\uparrow(k + \delta, \ell, i\epsilon_n) \\
&\quad + U \sum_k \tilde{G}_0(k, j, -i\epsilon_n)m_k F_{1\downarrow}^\dagger(k, \ell, i\epsilon_n), \\
F_{1\downarrow}^\dagger(j, \ell, i\epsilon_n) &= V \sum_{k,\delta} \tilde{G}_0(k, j, -i\epsilon_n)\Delta_{k,\delta+k}^* G_\downarrow(k + \delta, \ell, i\epsilon_n) \\
&\quad - U \sum_k \tilde{G}_0(k, j, -i\epsilon_n)m_k F_{1\uparrow}^\dagger(k, \ell, i\epsilon_n),
\end{align}

where the summation on $\delta$ ($k$) is over $\pm \hat{x}$ and $\pm \hat{y}$ (all sites). Here, $\tilde{G}_0(j, \ell, i\omega_n)$ is Green’s function for the system without $\Delta$ and $m$ but with $A$ satisfying

$$(i\epsilon_n + \mu)\tilde{G}_0(j, \ell, i\epsilon_n) + t \sum_\delta \tilde{G}_0(j + \delta, \ell, i\epsilon_n)e^{i\phi_{j+\delta}} = \delta_{j,\ell}.$$ 

$\tilde{G}_0$ is related to Green’s function for the system without $A$, $G_0$, as $\tilde{G}_0(j, \ell, i\epsilon_n) = G_0(j, \ell, i\epsilon_n)e^{i\phi_{j\ell}}$. $G_0(j, \ell, i\epsilon_n)$ is the Fourier transform of $G_0(p, i\epsilon_n) = 1/(i\epsilon_n - \xi_p)$ with $\xi_p = -2t(\cos p_x + \cos p_y) - \mu$.

Spin-singlet and spin-triplet SCOPs on the bond $(j, j+\eta)$ are expressed in terms of Green’s functions $F_{1\uparrow}$ and $F_{1\downarrow}$:

\begin{align}
(A^{(S)}_\eta(j))^* &= \frac{1}{2}(c_{j\uparrow}c_{j+\eta\uparrow} - c_{j\downarrow}c_{j+\eta\downarrow})^* = \frac{1}{2}(\Delta_{j,\eta+\eta} + \Delta_{j,\eta,\eta})^* \\
&= \frac{1}{2}T \sum_{\epsilon_n} [F_{1\uparrow}^\dagger(j + \eta, j, i\epsilon_n) - (F_{1\downarrow}^\dagger(j + \eta, j, i\epsilon_n)), \\
(A^{(T)}_\eta(j))^* &= \frac{1}{2}(c_{j\uparrow}c_{j+\eta\uparrow} + c_{j\downarrow}c_{j+\eta\downarrow})^* = \frac{1}{2}(\Delta_{j,\eta+\eta} - \Delta_{j,\eta,\eta})^* \\
&= -\frac{1}{2}T \sum_{\epsilon_n} [F_{1\downarrow}^\dagger(j + \eta, j, i\epsilon_n) + F_{1\uparrow}^\dagger(j + \eta, j, i\epsilon_n)],
\end{align}

and the magnetization is similarly given using $G_\uparrow$ and $G_\downarrow$ as

\begin{align}
m_j &= \frac{1}{2}c_{j\uparrow}c_{j\downarrow} - \frac{1}{2}T \sum_{\epsilon_n} [G_\uparrow(j, j, i\epsilon_n) - G_\downarrow(j, j, i\epsilon_n)].
\end{align}

We substitute eq. (7) into eqs. (9) and (10) iteratively and keep the terms up to the third order.
in OPs to get the following GL equations:

\[
\begin{align*}
(\Delta^n_\eta(j))^* &= \sum_{k,\delta} L^{(1)}(j, k, \eta, \delta)(\Delta^n_\delta(k))^* + \sum_{k,k',\delta} L^{(2)}(j, k, k', \eta, \delta)(\Delta^n_\delta(k))^* m_{k'} \\
&\quad + \sum_{k,k',k'',\delta,\delta',\delta''} L^{(3)}(j, k, k', k'', \eta, \delta, \delta', \delta'')(\Delta^n_\delta(k))^*(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) \\
&\quad - (\Delta^n_\delta(k))^*(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) - (\Delta^n_\delta(k)^*)(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) \\
&\quad + (\Delta^n_\delta(k)^*)(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) \\
&\quad + \sum_{k,k',k'',\delta} L^{(4)}(j, k, k', k'', \eta, \delta)(\Delta^n_\delta(k))^* m_{k'} m_{k''} ,
\end{align*}
\]

\[
(\Delta^n_\eta(j))^* = - \sum_{k,\delta} L^{(1)}(j, k, \eta, \delta)(\Delta^n_\delta(k))^* - \sum_{k,k',\delta} L^{(2)}(j, k, k', \eta, \delta)(\Delta^n_\delta(k))^* m_{k'} \\
&\quad + \sum_{k,k',k'',\delta,\delta',\delta''} L^{(3)}(j, k, k', k'', \eta, \delta, \delta', \delta'')(\Delta^n_\delta(k))^*(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) \\
&\quad - (\Delta^n_\delta(k))^*(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) - (\Delta^n_\delta(k)^*)(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) \\
&\quad + (\Delta^n_\delta(k)^*)(\Delta^n_\delta(k')^*)(\Delta^n_\delta(k'')^*) \\
&\quad - \sum_{k,k',k'',\delta} L^{(4)}(j, k, k', k'', \eta, \delta)(\Delta^n_\delta(k))^* m_{k'} m_{k''} ,
\]

\[
m_j = \sum_k L^{(5)}(j, k)m_k + \sum_{k,k',\delta,\delta'} L^{(6)}(j, k, k', \delta, \delta')(\Delta^n_\delta(k)(\Delta^n_\delta(k')^* - \Delta^n_\delta(k)(\Delta^n_\delta(k')^*) \\
&\quad + \sum_{k,k',k''} L^{(7)}(j, k, k', k'', \delta, \delta')(\Delta^n_\delta(k)(\Delta^n_\delta(k')^* - \Delta^n_\delta(k)(\Delta^n_\delta(k')^*) \\
&\quad + \sum_{k,k',k'',\delta} L^{(8)}(j, k, k', k'', \delta, \delta')(\Delta^n_\delta(k)(\Delta^n_\delta(k')^* - \Delta^n_\delta(k)(\Delta^n_\delta(k')^*) m_{k'} m_{k''} .
\]

where the functions \(L^{(n)}(n = 1, \ldots, 8)\) are given in Appendix A.

From eqs. (11)-(13), it is seen that the equations for \(\Delta^S, \Delta^T, \) and \(m\) have the second-order terms of the forms \(m\Delta^T, m\Delta^S, \) and \(\Delta^S\Delta^T, \) respectively. This implies that the GL free energy should have the cubic term of the form \(m\Delta^S\Delta^T, \) and it is actually the case as we will see in the following sections. It should be noted that eqs. (11)-(13) are valid even when the OPs have rapid spatial variations, because we have not yet taken a continuum limit. This property is important when we consider the antiferromagnetic case in §4.

3. GL Equations for Coexistent States of Superconductivity and Ferromagnetism

In this section, we consider the coexistent states of superconductivity and ferromagnetism. The GL equation for the SCOP of each symmetry can be obtained by making a linear combi-
necation of eqs. (11) and (12):
\[ \Delta_s(j) = \frac{1}{4} \sum_{\eta = \pm \hat{x}, \pm \hat{y}} \Delta^{(S)}_{\eta}(j), \quad \Delta_d(j) = \frac{1}{4} \sum_{\eta = \pm \hat{x}} \Delta^{(S)}_{\eta}(j) - \sum_{\eta = \pm \hat{y}} \Delta^{(S)}_{\eta}(j), \]
\[ \Delta_{\rho x}(j) = \frac{1}{2} \left[ \Delta^{(T)}_{\gamma}(j) - \Delta^{(T)}_{\gamma}(j) \right], \quad \Delta_{\rho y}(j) = \frac{1}{2} \left[ \Delta^{(T)}_{\gamma}(j) - \Delta^{(T)}_{\gamma}(j) \right]. \]  
(14)

Assuming that the SCOPs and magnetization are slowly varying, we take a continuum limit. The SCOPs in the linear and quadratic terms are expanded in powers of derivatives by denoting \( r_j \rightarrow r, r_k \rightarrow r' \):
\[ \Delta_s(k) \rightarrow \Delta_s(r') \]
\[ \sim \Delta_s(r) + (r' - r)\nabla_{\rho} \Delta_s(r) + \frac{1}{2} (r' - r)_{\mu} (r' - r)_{\nu} \nabla_{\rho} \nabla_{\rho} \Delta_s(r), \]
where the summations over \( \mu \) and \( \nu \) are assumed, and a similar approximation is carried out for \( m \). The Peierls phase coming from \( \tilde{G}_0 \) is also expanded in powers of \( A \). Using the approximation \( \phi_{k,j} \sim -\frac{\tilde{G}_0}{\phi_0} (r' - r) \cdot A(r) \), the derivatives and \( A \) are combined to construct the gauge-invariant gradient acting on \( \Delta \). \( D \equiv \nabla + \frac{2\pi i}{\phi_0} A \), and we keep the terms up to the second order in \( D \). As a typical example for treating the derivative terms, the derivation of the second-order term for \( \Delta_s \) is presented in Appendix B. In the third-order terms, we neglect the derivative terms and the vector potential \( A \) as usual, namely, \( \Delta^{(S)} \), \( \Delta^{(T)} \), and \( m \) with the arguments \( k, k' \), and \( k'' \) are replaced with \( \Delta^{(S)}(r), \Delta^{(T)}(r) \), and \( m(r) \), respectively. Rewriting \( \Delta^{(S)} \) \( (\Delta^{(T)}) \) using \( \Delta_s \) and \( \Delta_d \) \( (\Delta_{\rho x} \) and \( \Delta_{\rho y} \)), we carry out straightforward but lengthy calculations to get the following GL equations for SCOPs and magnetization:
\[ \alpha_s \Delta_s + 2\beta_s |\Delta_s|^2 \Delta_s - K_s(D_s^2 + D_y^2) \Delta_s - K_{sd}(D_s^2 - D_y^2) \Delta_d + K_{sym} \left[ (\nabla_{\rho} m) \Delta_{\rho x} + (\nabla_{\rho} m) \Delta_{\rho y} + 2m(D_s \Delta_{\rho x} + D_y \Delta_{\rho y}) \right] \]
\[ + \gamma_1 |\Delta_s|^2 \Delta_s + 2\gamma_2 |\Delta_s|^2 \Delta_d + \gamma_3 (|\Delta_{\rho x}|^2 + |\Delta_{\rho y}|^2) \Delta_s + 2\gamma_5 (\Delta_{\rho x}^2 + \Delta_{\rho y}^2) \Delta_d^* \]
\[ + \gamma_7 (|\Delta_{\rho x}|^2 - |\Delta_{\rho y}|^2) \Delta_d + \gamma_8 (\Delta_{\rho x}^2 - \Delta_{\rho y}^2) \Delta_d^* + \gamma_m m^2 \Delta_s = 0, \]
\[ \alpha_d \Delta_d + 2\beta_d |\Delta_d|^2 \Delta_d - K_d(D_s^2 + D_y^2) \Delta_d - K_{sd}(D_s^2 - D_y^2) \Delta_s + K_{dym} \left[ (\nabla_{\rho} m) \Delta_{\rho x} - (\nabla_{\rho} m) \Delta_{\rho y} + 2m(D_s \Delta_{\rho x} - D_y \Delta_{\rho y}) \right] \]
\[ + \gamma_1 |\Delta_s|^2 \Delta_d + 2\gamma_2 |\Delta_s|^2 \Delta_d + \gamma_3 (|\Delta_{\rho x}|^2 + |\Delta_{\rho y}|^2) \Delta_d + 2\gamma_5 (\Delta_{\rho x}^2 + \Delta_{\rho y}^2) \Delta_d^* \]
\[ + \gamma_7 (|\Delta_{\rho x}|^2 - |\Delta_{\rho y}|^2) \Delta_s + \gamma_8 (\Delta_{\rho x}^2 - \Delta_{\rho y}^2) \Delta_d^* + \gamma_m m^2 \Delta_d = 0, \]  
(16) 
(17)
\[ 
\alpha_m \Delta_{px(y)} + 2\beta_m |\Delta_{px(y)}|^2 \Delta_{px(y)} - K_{p1} D_{x(y)}^2 \Delta_{px(y)} - K_{p2} D_{x,y}^2 \Delta_{px(y)} \\
- (K_{p3} + K_{p4}) D_x D_y \Delta_{px(y)} - K_{spm} \left( (\nabla_x m) \Delta_x + 2m D_{x(y)} \Delta_x \right) \\
- K_{dpm} \left[ (\nabla_x m) \Delta_d + 2m D_{x(y)} \Delta_d \right] + \gamma_{p1} |\Delta_{px(y)}|^2 \Delta_{px(y)} + 2\gamma_{p2} \Delta_{py(x)}^2 \Delta_{px(y)}^* \\
+ \gamma_3 |\Delta_d|^2 \Delta_{px(y)} + \gamma_4 |\Delta_d|^2 \Delta_{px(y)} + 2\gamma_5 \Delta_d^2 \Delta_{px(y)}^* + 2\gamma_6 \Delta_d^2 \Delta_{px(y)}^* \\
\pm \gamma_7 (\Delta_d \Delta_{px(y)}^* + \text{c.c.}) \Delta_{px(y)} \pm 2\gamma_8 \Delta_d \Delta_{px(y)}^* + \gamma_{mp} m^2 \Delta_{px(y)} = 0, \\
\] 

\[ 
\alpha_m m + 2\beta_m m^3 - K_m (\nabla_x^2 + \nabla_y^2) m \\
+ \frac{1}{2} K_{spm} \left[ \Delta_{x} ((\Delta_x \Delta_{px})^* + (\Delta_y \Delta_{py})^*) - \{ \Delta_{px} D_x \Delta_x + \Delta_{py} D_y \Delta_y \} + \text{c.c.} \right] \\
+ \frac{1}{2} K_{dpm} \left[ \Delta_{d} ((\Delta_x \Delta_{px})^* - (\Delta_y \Delta_{py})^*) - \{ \Delta_{px} D_x \Delta_d - \Delta_{py} D_y \Delta_d \} + \text{c.c.} \right] \\
+ \gamma_{ms} m |\Delta_d|^2 + \gamma_{md} m |\Delta_d|^2 + \gamma_{mp} m (|\Delta_{px}|^2 + |\Delta_{py}|^2) = 0, \\
\] 

where the coefficients appearing in eqs. (16)-(19) are given in Appendix C.

Equations (16)-(19) are the coupled equations that determine the SCOPs and magnetization self-consistently. The most important point is that the second-order terms with a first-order derivative exist in the GL equations. They can induce triplet (singlet) SCOPs in a singlet (triplet) superconductor once the magnetization coexists inhomogeneously. It should also be noted that the coefficients in GL equations are determined microscopically, reflecting the nature of the electronic states of the original model, e.g., the shape of the Fermi surface. This property can be used to study the coexistent states of realistic materials to be considered.

The GL free energy \( F \) up to the fourth order in OPs can be obtained from the above GL equations in such a way that the variations of \( F \) with respect to OPs reproduce eqs. (16)-(19).
The results are written as follows:

\[
F = F_S + F_T + F_{ST} + F_M + F_{SM} + F_{TM} + F_{STM}.
\]

\[
F_S = \int d^2r \left[ \alpha_s |\Delta_s|^2 + \beta_s |\Delta_s|^4 + K_s |\tilde{D}\Delta_s|^2 + \alpha_d |\Delta_d|^2 + \beta_d |\Delta_d|^4 + K_d |\tilde{D}\Delta_d|^2 \\
+ \gamma_1 |\Delta_s|^2 |\Delta_d|^2 + \gamma_2 (\Delta_s^2)^2 + c.c. \\
+ K_{ds} ((D_s\Delta_d)(D_s\Delta_s)^* - (D_d\Delta_d)(D_s\Delta_s)^*) + c.c. \right],
\]

\[
F_T = \int d^2r \left[ \alpha_p (|\Delta_{px}|^2 + |\Delta_{py}|^2) + \beta_p (|\Delta_{px}|^4 + |\Delta_{py}|^4) \\
+ \gamma_p_1 |\Delta_{px}|^2 |\Delta_{py}|^2 + \gamma_p_2 (\Delta_{px}^2)^2 + c.c. \\
+ K_{p1} (|D_s\Delta_{px}|^2 + |D_s\Delta_{py}|^2) + K_{p2} (|D_s\Delta_{px}|^4 + |D_s\Delta_{py}|^4) \\
+ K_{p3} ((D_s\Delta_{px})^* (D_s\Delta_{py}) + c.c.) + K_{p4} ((D_s\Delta_{px})^* (D_s\Delta_{py}) + c.c.) \right],
\]

\[
F_{ST} = \int d^2r \left[ \gamma_3 (|\Delta_{px}|^2 + |\Delta_{py}|^2) |\Delta_s|^2 + \gamma_4 (|\Delta_{px}|^2 + |\Delta_{py}|^2) |\Delta_d|^2 \\
+ \gamma_5 (|\Delta_{px}^2 + \Delta_{py}^2)(\Delta_s^2)^2 + c.c.) + \gamma_6 (|\Delta_{px}^2 + \Delta_{py}^2)(\Delta_s^2)^2 + c.c.) \\
+ \gamma_7 (|\Delta_{px}|^2 - |\Delta_{py}|^2)(\Delta_s^2)^2 + c.c.) + \gamma_8 (|\Delta_{px}|^2 - |\Delta_{py}|^2)(\Delta_s^2)^2 + c.c.) \right],
\]

\[
F_M = \int d^2r \left[ \alpha_m m^2 + \beta_m m^4 + K_m (\nabla m)^2 \right],
\]

\[
F_{SM} = \int d^2r \left[ \gamma_{ms} m^2 |\Delta_s|^2 + \gamma_{md} m^2 |\Delta_d|^2 \right],
\]

\[
F_{TM} = \int d^2r \left[ \gamma_{mp} m^2 (|\Delta_{px}|^2 + |\Delta_{py}|^2) \right],
\]

\[
F_{STM} = \int d^2r \left[ K_{spm} m^2 \left[ \Delta_s ((D_s\Delta_{px})^* + (D_s\Delta_{py})^*) - ((D_s\Delta_s)\Delta_{px}^* + (D_s\Delta_s)\Delta_{py}^*) \right] \\
+ K_{dpm} m^2 \left[ \Delta_d ((D_s\Delta_{px})^* - (D_s\Delta_{py})^*) - ((D_s\Delta_d)\Delta_{px}^* - (D_s\Delta_d)\Delta_{py}^*) \right] + c.c. \right].
\]

Here, \(F_S, F_T, \) and \(F_M\) are the free energy for the singlet and triplet SCOPs and the magnetization, respectively, while \(F_{ST}, F_{SM}, F_{TM},\) and \(F_{STM}\) describe their couplings. Note that \(F\) is invariant under all the symmetry operations of the square lattice.\(^2\) The cubic term \(F_{STM}\) has derivative couplings of singlet and triplet SCOPs with magnetization, so that triplet (singlet) SCOPs would be induced once ferromagnetism coexists with singlet (triplet) superconductivity inhomogeneously, as already noted. (In other words, the triplet (singlet) SCOP would not be induced if the coexistence occurs uniformly.) This gives a simple and clear interpretation for previous theoretical results using the BdG method\(^5\) and quasiclassical Green’s functions,\(^9,10\) in which the occurrence of \(p\)-wave SCOPs near the interface between a singlet superconductor and a ferromagnet was pointed out.

Dahl and Sudbø\(^24\) derived the GL free energy from a model with a spin generalized BCS term and a Heisenberg exchange term, which is different from ours. They found a cubic term in the GL free energy that couples a nonunitary SCOP with magnetization.
4. Case of Superconductivity and Antiferromagnetism

Next, we consider the coexistent states of superconductivity and antiferromagnetism. In the AF state the magnetization \( m_j \) is oscillating, so we expect that the triplet SCOP will be induced even in a uniform AF state once the coexistence occurs. As a slowly varying OP to be considered in the continuum theory, we define the staggered magnetization \( M_j \equiv M_j e^{iQ \cdot r_j} \) with \( Q \equiv (\pi, \pi) \). If we assume that the singlet component of SCOP, \( \Delta^{(S)} \), is also slowly varying, then the triplet component \( \Delta^{(T)} \) should oscillate, as can be seen from eqs. (11)-(13). Therefore, we define the \( \pi \)-triplet SCOP \( \Delta^{(\pi T)}_q(j) \equiv \Delta^{(T)}_q(j) e^{iQ \cdot r_j} \). Rewriting eqs. (11)-(13) in terms of \( M \) and \( \Delta^{(\pi T)} \), we find that all terms in these equations do not have staggered oscillations. Defining the \( p_x \) and \( p_y \) components of the \( \pi \)-triplet SCOP as

\[
\Delta^{(\pi T)}_{px}(j) = \frac{1}{\gamma} [\Delta^{(\pi T)}_x(j) + \Delta^{(\pi T)}_x(-j)], \\
\Delta^{(\pi T)}_{py}(j) = \frac{1}{2} [\Delta^{(\pi T)}_y(j) + \Delta^{(\pi T)}_y(-j)],
\]

we carry out calculations similar to those in the ferromagnetic case to get GL equations and the GL free energy. Here, we present only the resulting expressions for the free energy \( F^{AF} \):

\[
F^{AF} = F_S + F^{AF}_T + F^{AF}_S + F^{AF}_M + F^{AF}_{ST} + F^{AF}_{TM} + F^{AF}_{STM},
\]

\[
F^{AF}_T = \int d^2r \left[ \tilde{\alpha}_p (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) + \tilde{\alpha}_p (\Delta^{(\pi T)}_{px} \Delta^{(\pi T)}_{px})^* + c.c. \right] + \tilde{\beta}_p (|\Delta^{(\pi T)}_{px}|^4 + |\Delta^{(\pi T)}_{py}|^4)
\]

\[
+ \tilde{\gamma}_p (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) (\Delta^{(\pi T)}_{px} \Delta^{(\pi T)}_{px}) + \tilde{\gamma}_p (\Delta^{(\pi T)}_{px} \Delta^{(\pi T)}_{py})^* + c.c. \right] + \tilde{K}_p (|D_x \Delta^{(\pi T)}_{px}|^2 + |D_y \Delta^{(\pi T)}_{py}|^2)
\]

\[
+ \tilde{K}_p (D_x \Delta^{(\pi T)}_{px} (\Delta^{(\pi T)}_{px})^* + (D_y \Delta^{(\pi T)}_{py}) (\Delta^{(\pi T)}_{py})^* + c.c.) \right] ,
\]

\[
F^{AF}_S = \int d^2r \left[ \tilde{\gamma}_s (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) |\Delta_x|^2 + \tilde{\gamma}_s (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) |\Delta_y|^2
\]

\[
+ \tilde{\gamma}_s (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) (\Delta^*_x \Delta_x + c.c.) + \tilde{\gamma}_s (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) (\Delta^*_y \Delta_y + c.c.) \right] + \tilde{\gamma}_s (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2)
\]

\[
+ \tilde{\gamma}_s (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) (\Delta^*_x \Delta_y + c.c.) + \tilde{\gamma}_s (|\Delta^{(\pi T)}_{px}|^2 + |\Delta^{(\pi T)}_{py}|^2) (\Delta^*_y \Delta_x + c.c.) \right] ,
\]

\[
F^{AF}_M = \int d^2r \left[ \tilde{\alpha}_m \Delta^*_x \Delta_x + \tilde{\alpha}_m \Delta^*_y \Delta_y + \tilde{\alpha}_n (\nabla \Delta)^2 \right],
\]

\[
F^{AF}_{ST} = \int d^2r \left[ \tilde{\alpha}_m \Delta_x M_x^2 + \tilde{\alpha}_m \Delta_x M_y^2 + \tilde{\gamma}_m (\nabla M)^2 \right],
\]

\[
F^{AF}_{TM} = \int d^2r \left[ \tilde{\gamma}_m \Delta_x M_x \Delta^*_x + \tilde{\gamma}_m \Delta_y M_y \Delta^*_y \right],
\]

\[
F^{AF}_{STM} = \int d^2r \left[ \tilde{\gamma}_m \Delta_x M_x \Delta^*_x + \tilde{\gamma}_m \Delta_y M_y \Delta^*_y \right] + \tilde{\gamma}_m \Delta_x M_x \Delta^*_x + \tilde{\gamma}_m \Delta_y M_y \Delta^*_y + c.c. \right] ,
\]

(22)
where $F_S$ is the same as that in the ferromagnetic case. The expressions for the coefficients appearing in $F^{AF}_{STM}$ are summarized in Appendix D. The cubic term $F^{AF}_{STM}$ in this case couples $\Delta^{(S)}$, $\Delta^{(\alpha T)}$, and $M$ directly without derivatives. Then the $(p_x + p_y)$-wave [(($p_x - p_y$)-wave] $\pi$-triplet component would be induced when $s$-wave ($d$-wave) superconductivity and antiferromagnetism coexist, even in a uniform case. This is consistent with the results of previous mean-field calculations that predict the occurrence of the $\pi$-triplet component in uniformly coexistent states of $d$-wave superconductivity and antiferromagnetism.\(^{25-29}\) This term also gives a simple explanation for the occurrence of the $\pi$-triplet SCOP near the interface between a singlet superconductor and an antiferromagnet, which was found in numerical calculations based on the BdG method.\(^5\)

5. Summary and Discussion

We have derived GL equations and the GL free energy for the coexistent states of superconductivity and magnetism microscopically from the extended Hubbard model with on-site repulsive and nearest-neighbor attractive interactions. It was found that, in the GL free energy, a cubic term that couples singlet and triplet SCOPs with magnetization exists. Owing to this term, triplet SCOPs would be induced when ferromagnetism coexists with singlet superconductivity inhomogeneously. This gives a simple explanation for previous theoretical studies on bilayer systems composed of a ferromagnet and a singlet superconductor.\(^5, 9, 10\) In the coexistent state of antiferromagnetism and singlet superconductivity, $\pi$-triplet SCOPs would be induced. This occurs not only in inhomogeneous cases but also in spatially uniform states.

The validity of the model employed in this paper is limited because of the absence of the $SU(2)$ symmetry in spin space. For more general and precise argument of the symmetry of the induced OPs, theoretical investigations based on the model that respects this symmetry will be necessary, although the present study may capture some of the important aspects.

In order to study the material dependence of interface states more generally, it is necessary to derive GL equations from other microscopic models. For example, the low-energy electronic states of high-$T_C$ cuprate superconductors are described by the $t - J$ model,\(^30\) and so the interface state of heterostructures made of high-$T_C$ cuprates and magnetic materials may be studied using the GL equations derived from this model.

Numerical study of the GL equations derived from different microscopic models may clarify the material dependence of the interface states of heterostructures composed of various superconductors and magnetic materials. This problem will be examined separately.
Acknowledgment

K. K. thanks H. Yamase for useful discussions.

Appendix A: Functions Appearing in GL Equations

The functions \( L^{(n)} (n = 1, \ldots, 8) \) appearing in eqs. (12)-(14) are defined as follows:

\[
L^{(1)}(j, k, \eta, \delta) = VT \sum_{\epsilon_n} \sum_{k, \delta} \tilde{G}_0(k, j + \eta, -i\epsilon_n) \tilde{G}_0(k + \delta, j, i\epsilon_n),
\]

\[
L^{(2)}(j, k, k', \eta, \delta) = VUT \sum_{\epsilon_n} \left[ \tilde{G}_0(k, j + \eta, -i\epsilon_n) \tilde{G}_0(k + \delta, k', i\epsilon_n) \tilde{G}_0(k', j, i\epsilon_n) - \tilde{G}_0(k', j + \eta, -i\epsilon_n) \tilde{G}_0(k, k', -i\epsilon_n) \tilde{G}_0(k + \delta, j, i\epsilon_n) \right],
\]

\[
L^{(3)}(j, k, k', k'', \eta, \delta, \delta', \delta'') = -V^3T \sum_{\epsilon_n} \left[ \tilde{G}_0(k, j + \eta, -i\epsilon_n) \tilde{G}_0(k + \delta, k', i\epsilon_n) \times \tilde{G}_0(k'', k' + \delta', -i\epsilon_n) \tilde{G}_0(k'' + \delta'', j, i\epsilon_n), \right.
\]

\[
L^{(4)}(j, k, k', k'', \eta, \delta) = VU^2T \sum_{\epsilon_n} \left[ \tilde{G}_0(k, j + \eta, -i\epsilon_n) \tilde{G}_0(k + \delta, k', i\epsilon_n) \times \tilde{G}_0(k', k'', j, i\epsilon_n)
\]

\[
\times \tilde{G}_0(k'', k' + \delta', -i\epsilon_n) \tilde{G}_0(k'' + \delta'', j, i\epsilon_n),
\]

\[
L^{(5)}(j, k) = -UT \sum_{\epsilon_n} \tilde{G}_0(j, k, i\epsilon_n) \tilde{G}_0(k, j, i\epsilon_n),
\]

\[
L^{(6)}(j, k, k', \delta, \delta') = V^2T \sum_{\epsilon_n} \tilde{G}_0(j, k, i\epsilon_n) \tilde{G}_0(k', k + \delta, -i\epsilon_n) \tilde{G}_0(k' + \delta', j, i\epsilon_n),
\]

\[
L^{(7)}(j, k, k', k'') = -U^3T \sum_{\epsilon_n} \tilde{G}_0(j, k, i\epsilon_n) \tilde{G}_0(k', i\epsilon_n) \tilde{G}_0(k'', k + \delta, -i\epsilon_n) \tilde{G}_0(k', j, i\epsilon_n),
\]

\[
L^{(8)}(j, k, k', k'', \delta, \delta') = V^2UT \sum_{\epsilon_n} \left[ \tilde{G}_0(j, k, i\epsilon_n) \tilde{G}_0(k', k + \delta, -i\epsilon_n) \times \tilde{G}_0(k' + \delta', k'', j, i\epsilon_n)
\]

\[
\times \tilde{G}_0(k' + \delta'', k', i\epsilon_n) \tilde{G}_0(k'' + \delta'', j, i\epsilon_n),
\]

\[
+ \tilde{G}_0(j, k'', i\epsilon_n) \tilde{G}_0(k', k, i\epsilon_n) \tilde{G}_0(k', k + \delta, -i\epsilon_n) \tilde{G}_0(k' + \delta', j, i\epsilon_n)
\]

\[
- \tilde{G}_0(j, k, i\epsilon_n) \tilde{G}_0(k'', k + \delta, -i\epsilon_n) \tilde{G}_0(k', -i\epsilon_n) \tilde{G}_0(k' + \delta', j, i\epsilon_n) \right].
\]

Appendix B: Derivation of the Second Order Terms in GL Equations

In this appendix, we show how to calculate the second-order terms in the GL equations for ferromagnetism and superconductivity. Here, the equation for \( \Delta_x \) is treated as an example. (Other OPs can be treated similarly.) The term to be considered is

\[
\frac{1}{4} \sum_{\eta} \sum_{\epsilon_n} \sum_{k, k', \delta} \left( \Delta_\delta^{(3)}(k) \right)^* m_k \left[ \tilde{G}_0(k, j + \eta, -i\epsilon_n) \tilde{G}_0(k + \delta, k', i\epsilon_n) \tilde{G}_0(k', j, i\epsilon_n) \right. \right.
\]

\[
- \tilde{G}_0(k', j + \eta, -i\epsilon_n) \tilde{G}_0(k, k', -i\epsilon_n) \tilde{G}_0(k + \delta, j, i\epsilon_n) \right].
\]
We substitute eq. (15) for $\Delta^{(T)}_{\delta}(k)$ and use a similar approximation to $m_{\kappa'}$, and denote $r_f \rightarrow r$, $r_k \rightarrow r'$ and $r_{\kappa} \rightarrow r''$. The term without derivatives and $A$ is given by

$$
\frac{1}{4} V U m(r) \sum_{p} \frac{1}{N} \sum_{\eta} \sum_{\delta} e^{-i\eta \delta} \sum_{\mu} (\Delta^{(T)}_{\delta}(r))^{*} e^{-i\mu \delta} \left[ G_{0}^{2}(p, i\epsilon_{n}) G_{0}(p, -i\epsilon_{n}) \right]
$$

where $N$ is the total number of lattice sites. This term is seen to vanish by putting $\epsilon_{n} \rightarrow -\epsilon_{n}$ in the second line. Next the terms that are first order in derivatives and $A$ are given as

$$
\frac{1}{4} V U \sum_{p} \sum_{\eta} \sum_{\delta} \int d^{2}r' \int d^{2}r'' \left[ \mathbf{m}(r)(r'-r)_{\mu} [\nabla_{\mu} - \frac{2\pi i}{\phi_{0}} A_{\mu}(r)] \Delta^{(T)}_{\delta}(r) \right]
$$

Next the terms that are first order in derivatives and $A$ are given as

$$
+ \frac{1}{4} VU \sum_{p} \sum_{\mu} \sum_{\eta} \sum_{\delta} \left[ G_{0}(p, -i\epsilon_{n}) G_{0}(p, i\epsilon_{n}) \right] \left[ e^{-i\eta \delta} - e^{i\eta \delta} \right]
$$

We carry out the integrations over $r'$ and $r''$ after performing the partial integration on $p_{\mu}$. Then eq. (B.3) becomes

$$
\frac{1}{4} V U \sum_{p} \sum_{\mu} \sum_{\eta} \sum_{\delta} \left[ \mathbf{m}(r)(r'-r)_{\mu} \right]
$$

Substituting the relations

$$
(r' - r)_{\mu} = e^{i(p_{1} + p_{2}) \cdot r'} \left( -i \frac{\partial}{\partial p_{\mu}} \right) e^{i(p_{1} + p_{2}) \cdot (r' - r)},
$$

we carry out the integrations over $r'$ and $r''$ after performing the partial integration on $p_{\mu}$. Then eq. (B.3) becomes

$$
\frac{1}{4} V U \sum_{p} \sum_{\mu} \sum_{\eta} \sum_{\delta} \left[ \mathbf{m}(r)(r'-r)_{\mu} \right]
$$

With the definitions of $K_{spu}$ and the function $I_{3}(p)$ in Appendix C, the last expression is seen to give the second-order term appearing in eq. (16).

Terms that are second order in derivatives and $A$ can be shown to vanish by carrying out similar calculations.
Appendix C: Coefficients in GL free energy for ferromagnetism and superconductivity

The coefficients appearing in GL equations [eqs. (16)-(19)] and the GL free energy [eq. (20)] are given as follows:

\[
\alpha_{s(d)} = 4V(1 - \frac{V}{N} \sum_p I_1(p) \omega^2_{s(d)}),
\]

\[
\beta_{s(d)} = 8V^4 \frac{1}{N} \sum_p I_2(p) \omega^4_{s(d)},
\]

\[
\gamma_1 = 32V^4 \frac{1}{N} \sum_p I_2(p) \omega^2_s \omega^2_d, \quad \gamma_2 = \frac{1}{4} \gamma_1,
\]

\[
K_{s(d)} = V^2 \frac{1}{N} \sum_p I_2(p) \left( \frac{\partial \xi_p}{\partial p_x} \right)^2 \omega^2_{s(d)},
\]

\[
K_{sd} = 2V^2 \frac{1}{N} \sum_p I_2(p) \left( \frac{\partial \xi_p}{\partial p_x} \right)^2 \omega_s \omega_d,
\]

\[
\alpha_p = 2V \left(1 - \frac{2V}{N} \sum_p I_1(p) \omega^2_s\right),
\]

\[
\beta_p = 8V^4 \frac{1}{N} \sum_p I_2(p) \omega_s,
\]

\[
\gamma_{p1} = 32V^4 \frac{1}{N} \sum_p I_2(p) \omega^2_s \omega^2_s, \quad \gamma_{p2} = \frac{1}{4} \gamma_{p1},
\]

\[
K_{p1(2)} = V^2 \frac{1}{N} \sum_p I_2(p) \left( \frac{\partial \xi_p}{\partial p_x} \right)^2 \omega^2_{s(d)},
\]

\[
K_{p3} + K_{p4} = 4V^2 \frac{1}{N} \sum_p I_2(p) \left( \frac{\partial \xi_p}{\partial p_x} \right) \left( \frac{\partial \xi_p}{\partial p_y} \right) \omega_s \omega_s,
\]

\[
\{\gamma_3, \gamma_4, \gamma_7\} = 32V^4 \frac{1}{N} \sum_p I_2(p) \{ \omega^2_s \omega^2_s, \omega^2_s \omega^2_d, \omega_s \omega_d \omega^2_s \},
\]

\[
\gamma_5 = -\frac{1}{4} \gamma_3, \quad \gamma_6 = -\frac{1}{2} \gamma_4, \quad \gamma_8 = -\frac{1}{2} \gamma_7,
\]

\[
\alpha_m = U \left(1 + \frac{U}{N} \sum_p f'(p)\right),
\]

\[
\beta_m = -\frac{U^4}{12N} \sum_p f'''(p),
\]

\[
K_m = -\frac{U^2}{12N} \sum_p \left( \frac{\partial \xi_p}{\partial p_x} \right)^2 f'''(p),
\]

\[
\{\gamma_{ms}, \gamma_{md}, \gamma_{mp}\} = -4V^2 U^2 \frac{1}{N} \sum_p \{2I_3(p) - I_2(p)\} \{ \omega^2_s, \omega^2_d, \omega^2_s \},
\]

\[
K_{spm} = 8V^2 U \frac{1}{N} \sum_p I_3(p) \frac{\partial \xi_p}{\partial p_x} \omega_s \omega_s,
\]

\[
K_{dpm} = 8V^2 U \frac{1}{N} \sum_p I_3(p) \frac{\partial \xi_p}{\partial p_x} \omega_d \omega_s,
\]

\[
(C1)
\]
where \( \omega_s = \cos p_x + \cos p_y \), \( \omega_d = \cos p_x - \cos p_y \), and \( \omega_{x(y)} = \sin p_{x(y)} \), and the summation on \( p \) is taken over the first Brillouin zone. The functions \( I_1 \), \( I_2 \), and \( I_3 \) are defined as

\[
I_1(p) = T \sum_{\epsilon_n} G_0(p, i\epsilon_n) G_0(p, -i\epsilon_n), \quad I_2(p) = T \sum_{\epsilon_n} G_0^2(p, i\epsilon_n) G_0^2(p, -i\epsilon_n), \quad I_3(p) = T \sum_{\epsilon_n} G_0^3(p, i\epsilon_n) G_0(p, -i\epsilon_n),
\]

with \( f(\xi_p) \) being the Fermi distribution function.
Appendix D: Coefficients in GL Free Energy for Antiferromagnetism and Superconductivity

The expressions of the coefficients in $F^{AF}$ [eq. (22)] are given as follows:

$$\tilde{\alpha}_{p1} = 2V \left(1 - \frac{2V}{N} \sum_p I_4(p) \cos^2 p_x \right),$$

$$\tilde{\alpha}_{p2} = -\frac{4V^2}{N} \sum_p I_4(p) \cos p_x \cos p_y,$$

$$\tilde{\beta}_p = 8V^4 \frac{1}{N} \sum_p I_5(p) \cos^4 p_x,$$

$$\tilde{\gamma}_{p1} = 32V^4 \frac{1}{N} \sum_p I_5(p) \cos^2 p_x \cos^2 p_y, \quad \tilde{\gamma}_{p2} = \frac{1}{4} \tilde{\gamma}_{p1},$$

$$\tilde{\gamma}_{p3} = 16V^4 \frac{1}{N} \sum_p I_5(p) \cos^3 p_x \cos p_y,$$

$$\tilde{K}_{p1(2)} = -V^4 \frac{1}{N} \sum_p I_5(p) \left(\frac{\partial \xi_p}{\partial p_x}\right)^2 \cos^2 p_{x(y)},$$

$$\tilde{K}_{p3} = -2V^2 \frac{1}{N} \sum_p I_5(p) \left(\frac{\partial \xi_p}{\partial p_x}\right)^2 \cos p_x \cos p_y,$$

$$\tilde{\gamma}_{3(4)} = 32V^4 \frac{1}{N} \sum_p I_6(p) \omega_{s(d)}^2 \cos^2 p_x,$$

$$\tilde{\gamma}_{5(6)} = 8V^4 \frac{1}{N} \sum_p I_7(p) \omega_{s(d)}^2 \cos^2 p_x,$$

$$\tilde{\gamma}_{7} = 32V^4 \frac{1}{N} \sum_p I_6(p) \omega_x \omega_d \cos^2 p_x,$$

$$\tilde{\gamma}_{8} = 16V^4 \frac{1}{N} \sum_p I_7(p) \omega_x \omega_d \cos^2 p_x,$$

$$\tilde{\gamma}_{9(10)} = 32V^4 \frac{1}{N} \sum_p I_6(p) \omega_{s(d)} \cos p_x \cos p_y,$$

$$\tilde{\gamma}_{11(12)} = 16V^4 \frac{1}{N} \sum_p I_7(p) \omega_{s(d)} \cos p_x \cos p_y,$$

$$\tilde{\alpha}_m = U \left(1 + \frac{U}{N} \sum_p I_8(p) \right),$$

$$\tilde{\beta}_m = \frac{U^4}{2N} \sum_p I_9(p),$$

$$\tilde{K}_m = \frac{U^2}{N} \sum_p I_9(p) \left(\frac{\partial \xi_p}{\partial p_x}\right)^2,$$

$$\{\tilde{\gamma}_{ms}, \tilde{\gamma}_{md}\} = -4V^2 U^2 \frac{1}{N} \sum_p \left[2I_{10}(p) + I_9(p)\right] \{\omega_x, \omega_d^2\},$$

$$\{\tilde{\gamma}_{mp1}, \tilde{\gamma}_{mp2}\} = -4V^2 U^2 \frac{1}{N} \sum_p \left[2I_{11}(p) + I_7(p)\right] \{\cos^2 p_x, \cos p_x, \cos p_y\},$$

$$\{\tilde{\gamma}_{spm}, \tilde{\gamma}_{dpm}\} = 8V^2 U^2 \frac{1}{N} \sum_p I_{12}(p) \cos p_x \{\omega_x, \omega_d\}.$$

(D-1)
where the functions appearing in the integrands are defined as

\[
I_4(p) = T \sum_{\epsilon_n} G_0(p, -i\epsilon_n)G_0(p + Q, i\epsilon_n),
\]

\[
I_5(p) = T \sum_{\epsilon_n} G_0^2(p, i\epsilon_n)G_0^2(p + Q, -i\epsilon_n),
\]

\[
I_6(p) = T \sum_{\epsilon_n} G_0^2(p, i\epsilon_n)G_0(p, -i\epsilon_n)G_0(p + Q, -i\epsilon_n),
\]

\[
I_7(p) = T \sum_{\epsilon_n} G_0(p, i\epsilon_n)G_0(p, -i\epsilon_n)G_0(p + Q, i\epsilon_n)G_0(p + Q, -i\epsilon_n),
\]

\[
I_8(p) = T \sum_{\epsilon_n} G_0(p, i\epsilon_n)G_0(p + Q, i\epsilon_n),
\]

\[
I_9(p) = T \sum_{\epsilon_n} G_0^2(p, i\epsilon_n)G_0^2(p + Q, i\epsilon_n),
\]

\[
I_{10}(p) = T \sum_{\epsilon_n} G_0^2(p, i\epsilon_n)G_0(p, -i\epsilon_n)G_0(p + Q, i\epsilon_n),
\]

\[
I_{11}(p) = T \sum_{\epsilon_n} G_0^2(p, i\epsilon_n)G_0(p + Q, i\epsilon_n)G_0(p + Q, -i\epsilon_n),
\]

\[
I_{12}(p) = T \sum_{\epsilon_n} G_0(p, i\epsilon_n)G_0(p, -i\epsilon_n)G_0(p + Q, i\epsilon_n).
\]
References

1) Y. Kitaoka, S. Shimizu, H. Mukuda, S. Tabata, P. M. Shirage, and A. Iyo: J. Phys. Chem. Solids 72 (2011) 486.
2) A. I. Buzdin: Rev. Mod. Phys. 77 (2005) 935.
3) F.S. Bergelet, A. F. Volkov, and K. B. Efetov: Rev. Mod. Phys. 77 (2005) 1321.
4) E. A. Demler, G. B. Arnold, and M. R. Beasley: Phys. Rev. B55 (1997) 15174.
5) K. Kuboki: J. Phys. Soc. Jpn. 68 (1999) 3150.
6) S. Kashiwaya, Y. Tanaka, N. Yoshida, and M. R. Beasley: Phys. Rev. B60 (1999) 3572.
7) J.-X. Zhu and C. S. Ting: Phys. Rev. B61 (2000) 1456.
8) K. Halterman and O. T. Valls: Phys. Rev. B65 (2001) 014509.
9) F. S. Bergeret, A. F. Volkov, and K. B. Efetov: Phys. Rev. Lett. 86 (2001) 4096.
10) M. Eschrig, J. Kopu, J. C. Cuevas, and G. Schönh: Phys. Rev. Lett. 90 (2003) 137003.
11) V. Braude and Yu. V. Nazarov: Phys. Rev. Lett. 98 077003 (2007) 077003.
12) K. Kuboki and H. Takahashi: Phys. Rev. B70 214524 (2004) 214524.
13) M. Krawiec, B. L. Györffy, and J. F. Annett: Phys. Rev. B70 (2004) 134519.
14) P. M. R. Brydon, D. Manske, and M. Sigrist: J. Phys. Soc. Jpn. 77 (2008) 103714.
15) P. M. R. Brydon: Phys. Rev. B80 (2009) 224520.
16) M. Cuoco, A. Romano, C. Noce, and P. Gentile: Phys. Rev. B78 (2008) 054503.
17) K. Kuboki: J. Phys. Soc. Jpn. 70 (2001) 2698.
18) R. Micnas, J. Ranninger, and S. Robaszkiewicz: Rev. Mod. Phys. 62 113 (1990) 113.
19) J. E. Hirsch: Phys. Rev. B31 (1985) 4403.
20) L. P. Gor’kov: Sov. Phys. JETP 9 (1959) 1364.
21) Y. Ren, J.-H. Xu, and C. S. Ting: Phys. Rev. Lett. 74 (1995) 3680.
22) D. L. Feder and C. Kallin: Phys. Rev. B55 (1997) 559.
23) For a review on the symmetry arguments in the GL theory, see M. Sigrist and K. Ueda: Rev. Mod. Phys. 63 (1991) 269.
24) E. K. Dahl and A. Sudbø: Phys. Rev. B75 (2007) 144504.
25) G. C. Psaltakis and E. W. Fenton: J. Phys. C16 (1983) 3913.
26) M. Murakami and H. Fukuyama: J. Phys. Soc. Jpn. 67 (1998) 2784.
27) M. Murakami: J. Phys. Soc. Jpn. 69 (2000) 1113.

28) B. Kyun: Phys. Rev. B 62 (2000) 9083.

29) A. Aperis, G. Varelogiannis, P. B. Littlewood, and B. D. Simons: J. Phys.: Condens. Matter 20 (2008) 434235.

30) M. Ogata and H. Fukuyama: Rep. Prog. Phys. 71 (2008) 036501.