Abstract

The notion of a measure on the space of connections modulo gauge transformations that is invariant under diffeomorphisms of the base manifold is important in a variety of contexts in mathematical physics and topology. At the formal level, an example of such a measure is given by the Chern-Simons path integral. Certain measures of this sort also play the role of states in quantum gravity in Ashtekar’s formalism. These measures define link invariants, or more generally multiloop invariants; as noted by Witten, the Chern-Simons path integral gives rise to the Jones polynomial, while in quantum gravity this observation is the basis of the loop representation due to Rovelli and Smolin. Here we review recent work on making these ideas mathematically rigorous, and give a rigorous construction of diffeomorphism-invariant measures on the space of connections modulo gauge transformations generalizing the recent work of Ashtekar and Lewandowski. This construction proceeds by doing lattice gauge theory on graphs analytically embedded in the base manifold.

1 Introduction

In physics, diffeomorphism-invariant “measures” on the space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations play an important role in the quantization of generally covariant gauge theories. However, these “measures” are often purely formal in nature; it is a challenge to find a formulation of them that is both mathematically rigorous and sufficiently flexible. In this paper we begin by reviewing of work by Ashtekar, Isham, Lewandowksi and the author [2, 3, 4] on holonomy C*-algebras as an approach to this problem. Here the heuristic notion of a “measure” on $\mathcal{A}/\mathcal{G}$ is replaced by the concept of a continuous linear functional on a particular algebra of functions on $\mathcal{A}/\mathcal{G}$. Then, generalizing the work of Ashtekar and Lewandowski, we construct diffeomorphism-invariant states on holonomy C*-algebras from certain invariants of “multiloops,” that is, finite collections of loops. This construction makes
it clear that to understand the relation between diffeomorphism-invariant gauge theories and knot theory one should treat the space of all links as a subset of the space of multiloops and attempt to extend link invariants to multiloop invariants. Interestingly, the same idea has recently shown up both in the theory of Vassiliev invariants \[4, 8, 9, 32, 34\] and in work on quantum gravity \[12, 10, 16, 18, 19\].

Two very important diffeomorphism-invariant gauge theories are Chern-Simons theory in 3 dimensions and general relativity in 4 dimensions. In the path-integral approach to Chern-Simons theory, we take a 3-manifold \(M\) as spacetime, consider a principal bundle \(G \to P \to M\), and calculate vacuum expectation values as integrals with respect to the “measure”

\[
e^{iS(A)} \mathcal{D}A,
\]

where \(\mathcal{D}A\) is the purely formal - that is, nonexistent - Lebesgue measure on \(A/G\), and

\[
S(A) = \frac{k}{4\pi} \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)
\]

is the Chern-Simons action, where \(k\) is an integer. The Chern-Simons action is preserved by the action of orientation-preserving diffeomorphisms of \(M\) and, up to integers, by the action of gauge transformations. Thus at a formal level, \(e^{iS(A)} \mathcal{D}A\) is regarded as a measure on \(A/G\) that is invariant under orientation-preserving diffeomorphisms of \(M\). Given a loop \(\gamma: S^1 \to M\), let \(T(\gamma, A)\) denote the trace of the holonomy of \(A\) around \(\gamma\), taking the trace in some finite-dimensional representation of \(G\). This is a gauge-invariant function of \(A\), so it can be regarded as a function \(T(\gamma)\) on \(A/G\). Thus given a collection of loops \(\{\gamma_i\}\) and an orientation-preserving diffeomorphism \(g: M \to M\), we expect that

\[
\int T(\gamma_1) \cdot \ldots \cdot T(\gamma_n) e^{iS(A)} \mathcal{D}A = \int T(g \circ \gamma_1) \cdot \ldots \cdot T(g \circ \gamma_n) e^{iS(A)} \mathcal{D}A
\]

In other words, we expect Chern-Simons theory to give an “ambient isotopy invariant” of multiloops \(\gamma = \{\gamma_i\}\), and in particular of links.

Due to the formal nature of the Chern-Simons “measure” it is not surprising that there are complications. For links, a formal calculation by Witten \[35\] using conformal field theory indicates that the integral must be regularized using a framing of the link. Taking the trace in the fundamental representation of \(G = SU(2)\), the result is an ambient isotopy invariant of framed links known as the Kauffman bracket \[20\], closely related to the Jones polynomial \[23\]. Witten’s result has been checked perturbatively by various authors \[10, 8, 14, 17\]. For other groups \(G\) and other representations, one obtains other link invariants. All these invariants may be constructed rigorously using the corresponding quantum group representations \[24, 33\]. Thus we expect a close relationship between any rigorous formulation of the Chern-Simons path integral and the representation theory of quantum groups.

In the connection representation of quantum gravity in 4 dimensions \[1\], we take a 3-manifold \(M\) as space, rather than spacetime, and we take \(P \to M\) to be the trivial
bundle with $G = SL(2, \mathbb{C})$, or in Euclidean quantum gravity, $SU(2)$. At a formal level, states of quantum gravity are given by “measures” on $\mathcal{A}/\mathcal{G}$ of the form

$$\Psi(A) \mathcal{D}A,$$

that are invariant under the identity component $\text{Diff}_0(M)$ of the diffeomorphism group and also annihilated by operators known as the Hamiltonian constraints. (We ignore, in this brief sketch, the important but subtle issue of the “reality conditions” in this approach to quantum gravity.) Alternatively, following Rovelli and Smolin [30], we can work in the loop representation and think of the state as a function $\hat{\Psi}$ of multiloops in $M$, where $\hat{\Psi}$ is the “loop transform” of $\Psi$:

$$\hat{\Psi}(\gamma) = \int T(\gamma_1) \cdots T(\gamma_n) \Psi(A) \mathcal{D}A.$$ 

Here $\hat{\Psi}$ will automatically be an ambient isotopy invariant of multiloops. A very interesting problem is to describe the Hamiltonian constraints in terms of the loop representation and find all multiloop invariants that are annihilated by these operators [10, 12, 18, 16].

In fact, there is a deep relation between Chern-Simons theory and quantum gravity, noticed by Kodama [24] and subsequently explored in many papers [6, 10, 15, 16, 28, 31]. This is that the Chern-Simons “measure,” in addition to being $\text{Diff}_0(M)$-invariant, is annihilated by the Hamiltonian constraint for quantum gravity with cosmological constant $\Lambda = \frac{24\pi i}{k}$.

This fact finds its deepest explanation so far in terms of the Capovilla-Dell-Jacobson formulation of general relativity, in which the action is closely related to the 2nd Chern class [13, 26].

Unfortunately, much of the aforementioned work, while very interesting, is not quite mathematics yet, because the “measures” in question have not been constructed in any rigorous sense. They are unlikely to be Borel measures on the space $\mathcal{A}/\mathcal{G}$ with any of its standard topologies. In order to address this issue, Ashtekar and Isham [2] introduced a generalization of measures on $\mathcal{A}/\mathcal{G}$, namely, continuous linear functionals on an algebra called the holonomy C*-algebra. This algebra the completion in the $L^\infty$ norm of the algebra generated by the Wilson loops $T(\gamma)$. Before describing this algebra in Section 3, we review some general ideas on functional integration in Section 2. This material is standard but perhaps phrased in a somewhat new manner. In Section 4 we give a characterization of $\text{Diff}_0(M)$-invariant continuous linear functionals on the holonomy C*-algebra in terms of lattice gauge theory on graphs embedded in $M$. Recently, Ashtekar and Lewandowski constructed such a continuous linear functional - in fact, a state - using a technique that depends crucially on working with piecewise analytic loops. In Section 5 we use the results of the previous section to construct many $\text{Diff}_0(M)$-invariant states on the holonomy algebras of
bundles over manifolds $M$ of any dimension. These examples involve an interesting interplay between the singularity theory of analytic curves in $M$ and the harmonic analysis of measures on spaces of the form $G^d$. In Section 6 we sketch an extension of the work in the previous section that is applicable only in the case of 3-dimensional manifolds. This extension, which has not been fully worked out, is very similar to Reshetikhin and Turaev’s construction of graph invariants from quantum group representations. We also briefly discuss the relation between regularization and framing-dependence of link invariants. This section may be regarded as a program for rigorously constructing the Chern-Simons “measure.”

2 Generalized Measures

In order to understand what the diffeomorphism-invariant “measures” on $\mathcal{A}/\mathcal{G}$ in physics might really be, it is useful to take the stance that the use of a measure is to integrate functions. Thus, we downplay the notion of a measure as a set function, and emphasize its role as a linear functional:

$$f \mapsto \int f d\mu.$$ 

For spaces that are not “too big,” there is a well-known correspondence between measures as set functions and measures as linear functionals. This is the Riesz-Markov theorem: if $X$ is a compact Hausdorff space, there is a one-to-one correspondence between measures on $X$ and continuous linear functionals on $C(X)$, the algebra of continuous complex-valued functions on $X$ equipped with the $L^\infty$ norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$ 

Here, and in all that follows, by measure we really mean a finite regular Borel measure. The Riesz-Markov theorem assigns to each measure $\mu$ the functional

$$f \mapsto \int_X f d\mu.$$ 

The deep part of the theorem is that all continuous linear functionals $f: C(X) \to \mathbb{C}$ are of this form.

Note that $C(X)$ is a unital $C^*$-algebra, that is, a Banach space over $\mathbb{C}$ that is an algebra with multiplicative identity equipped with a $*$ operation, in this case pointwise complex conjugation, satisfying

$$(f + g)^* = f^* + g^*$$
$$(\lambda f)^* = \overline{\lambda} f^*$$
$$(fg)^* = g^* f^*$$
$$f^{**} = f$$
\[ \|fg\| \leq \|f\|\|g\| \]
\[ \|f^*f\| = \|f\|^2. \]

The Gelfand-Naimark theorem says that conversely, every commutative unital C*-algebra is isomorphic to \( C(X) \) for some compact Hausdorff space, unique up to homeomorphism. Taken together, the Riesz-Markov and Gelfand-Naimark theorems allow us to treat all of measure theory on compact Hausdorff spaces in terms of C*-algebras and continuous linear functionals. For example, a positive measure on \( X \) corresponds to a positive linear functional on \( C(X) \), that is, a linear map \( f: C(X) \to \mathbb{C} \) such that
\[
\int f \geq 0.
\]
(Such functionals are automatically continuous.) Similarly, a probability measure on \( X \) corresponds to a state on \( C(X) \), that is, a positive linear functional \( f \) with \( f(1) = 1 \).

These results can be generalized to the case of Hausdorff spaces that are only locally compact. However, the infinite-dimensional spaces arising in physics are typically not even locally compact. For example, if we take \( A/G \) to consist of smooth connections modulo smooth gauge transformations, with the \( C^\infty \) topology, it will not be locally compact. The same holds if we work with connections and gauge transformations lying in appropriate Sobolev spaces. It is certainly possible to construct measures on spaces that are not locally compact, Wiener measure being a famous example, but it is sometimes more simple to consider a generalization of the notion of measure.

Let \( X \) be an arbitrary Hausdorff space and let \( C_b(X) \) denote the C*-algebra of all bounded complex continuous functions on \( X \). While a measure would enable us to integrate all functions in \( C_b(X) \), we may well be satisfied with being able to integrate functions in some subalgebra of \( C_b(X) \), which in physics terminology is a class of “distinguished observables.” Assume that \( A_0 \subseteq C_b(X) \) is a *-subalgebra, that is, a subalgebra such that \( f \in A_0 \) implies \( \overline{f} \in A_0 \). Assume also that there is a linear functional \( \int: A_0 \to \mathbb{C} \), and that \( \int \) is bounded, that is, for some \( C > 0 \)
\[ |\int f| \leq C\|f\|_{\infty} \]
for all \( f \in A_0 \). Then we say that \( f \) is a \textit{generalized measure} on \( X \), or that \( (X, A_0, f) \) is a \textit{generalized measure space}.

We can do a large amount of measure theory abstractly in this context. For example, if the generalized measure is positive:
\[ \int \overline{f} f \geq 0, \]
we can define \( L^2(X, f) \) to be the completion of \( A_0 \) in the norm
\[ \|f\|_2 = \left( \int \overline{f} f \right)^{1/2}. \]
Suppose that $G$ is a group acting as homeomorphisms of $X$. Then $G$ acts on $C_b(X)$ by
\[ gf(x) = f(g^{-1}x). \]
Suppose that this action preserves the generalized measure $f$, that is, $f \in A_0$ implies $gf \in A_0$, and $\int f = \int gf$. Then the action of $G$ on $A_0$ extends to a unitary action of $G$ on $L^2(X, f)$.

On the other hand, if we wish, we can translate the theory of generalized measures back into the theory of measures on compact Hausdorff spaces, allowing us to use all the standard tools of measure theory. Suppose $(X, A_0, \int)$ is a generalized measure space. Let $A$ denote the completion of $A_0$ in the $L^\infty$ norm, or equivalently, the closure of $A_0$ in $C_b(X)$. Then by the Gelfand-Naimark theorem there is a compact Hausdorff space $\overline{X}$ such that $A \cong C(\overline{X})$. There is also a continuous map from $X$ to $\overline{X}$ with dense range. If $A_0$ is sufficient to separate points in $X$, that is, if $x \neq y$ implies there exists $f \in A_0$ with $f(x) \neq f(y)$, then the map from $X$ to $\overline{X}$ is one-to-one. Every function $f \in A$ canonically defines a function $\tilde{f}$ on $\overline{X}$. On the other hand, since the functional $\int$ is bounded, it extends uniquely to a continuous linear functional $\tilde{\int}: A \to C(\overline{X})$. Thus by the Riesz-Markov theorem there is a unique measure $\mu$ on $\overline{X}$ such that
\[ \int f = \int_{\overline{X}} \tilde{f} d\mu \]
for all $f \in A$.

In short, generalized measure theory on $X$ can be translated into ordinary measure theory on a compact space $\overline{X}$ containing certain “limits” of points of $X$. This way of thinking also extends to the case when there is a group action present. If $G$ acts as homeomorphisms of $X$ and preserves the generalized measure $f$, there is a unique action of $G$ as homeomorphisms of $\overline{X}$ such that the map from $X$ to $\overline{X}$ is $G$-equivariant, and this action on $\overline{X}$ preserves the measure $\mu$. In the next sections we will apply this philosophy to the case in which $X$ is the space $A/G$ of connections modulo gauge transformations for some bundle over a manifold $M$, and seek generalized measures on $A/G$ that are invariant under the action of Diff$_0(M)$.

### 3 The Analytic Holonomy C*-algebra

Let $G$ be a compact Lie group, and $\rho$ a $k$-dimensional unitary representation of $G$. Let $M$ be a real-analytic $n$-manifold and $P \to M$ a principal $G$-bundle over $M$. Define $\tau: G \to C$ by
\[ \tau(g) = \frac{1}{k} \text{tr}(\rho(g)). \]
Given a smooth connection $A$ on $P$ and a piecewise smooth loop $\gamma: S^1 \to M$, let $T(\gamma, A)$ denote the trace of the holonomy of $A$ around the loop $\gamma$, computed using the trace $\tau$. Let $A$ denote the space of smooth connections on $P$ and $G$ the group of smooth gauge transformations of $P$. The functions $T(\gamma) = T(\gamma, \cdot)$, known as Wilson
loops, are $G$-invariant bounded continuous functions on $A$. Alternatively, they may be thought of as elements of $C_b(A/G)$.

In general, a holonomy algebra is a subalgebra of $C_b(A/G)$ generated in some manner by the functions $T(\gamma)$ for some class of loops $\gamma$. There are a variety of versions of holonomy algebra. The original holonomy algebra due to Ashtekar and Isham [2] was generated by the traces of holonomies around all piecewise smooth loops, and their holonomy $C^*$-algebra was the completion of the holonomy algebra in the $L^\infty$ norm. The topology of a piecewise smooth loop can be extremely complicated, which makes it difficult to construct diffeomorphism-invariant states on this holonomy $C^*$-algebra. Here we will work with the holonomy algebra defined by Ashtekar and Lewandowski [3], which involves only piecewise analytic loops. In Section 6 of this paper we will mention some other sorts of holonomy algebra that involve “regularized” or “smeared” Wilson loops. It appears necessary to consider such holonomy algebras to treat the Chern-Simons path integral as a generalized measure.

Henceforth we will assume $M$ to be a real-analytic manifold. We say that $\gamma: S^1 \to M$ is piecewise analytic if $\gamma$ is continuous and we can write $S^1$ as a finite union of closed intervals $I_i$ such that $\gamma|_{I_i}$ extends to a real-analytic function from an open interval containing $I_i$. From now on we will use the word loop to mean a piecewise analytic loop.

Let $H_0$ denote the algebra of functions on $A$ generated by the functions $T(\gamma)$ for all such loops, and let $H$ denote the closure of $H_0$ in the norm

$$\|f\|_\infty = \sup_{A \in A} |f(A)|.$$ 

Note that the pointwise complex conjugate of the function $T(\gamma)$ is the function $T(\gamma^{-1})$, where $\gamma^{-1}$ is $\gamma$ with its orientation reversed. Thus $H_0$ is closed under complex conjugation, so $H$ is a $C^*$-subalgebra of the $C^*$-algebra $C_b(A/G)$.

The algebra $H$ is called the holonomy $C^*$-algebra of the associated bundle $P \times_\rho C^n$. By the general theory of the previous section, $H$ is isomorphic to $C(\overline{A/G})$ for some compact Hausdorff space $\overline{A/G}$, and there is a continuous map from $A/G$ to $\overline{A/G}$ with dense range. When the the functions in $H_0$ separate the points of $A/G$, the map from $A/G$ to $\overline{A/G}$ is one-to-one. This is true when $\rho$ is the fundamental representation of $SU(n)$, and probably much more generally. In these cases, points of $\overline{A/G}$ may be regarded as gauge equivalence classes of “distributional connections” on $P$. Ashtekar, Isham and Lewandowski have given some explicit examples of such distributional connections, as well as a very clean abstract description of all of them when $\rho$ is the fundamental representation of $SU(2)$ [2, 3].

The group $\text{Diff}(M)$ of real-analytic diffeomorphisms of $M$ acts as homeomorphisms of $A/G$, as $*$-automorphisms of $H$ by

$$gT(\gamma) = T(g \circ \gamma),$$

and dually on the space of continuous linear functionals $H^*$. Let us write $H^*_{\text{inv}}$ for the elements of $H^*$ that are invariant under the action of $\text{Diff}_0(M)$, the identity component of $\text{Diff}(M)$. 

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In fact, elements of $H^*_\text{inv}$ are classified by multiloop invariants, where by a multiloop we mean an $n$-tuple of loops. Let us say that two multiloops $\gamma = \{\gamma_i\}$ and $\eta = \{\eta_i\}$ are ambient isotopic if for some $g \in \text{Diff}_0(M)$ we have $\gamma_i = g \circ \eta_i$.

**Proposition 1.** Suppose that $\mu \in H^*_\text{inv}$. Then there is an ambient isotopy invariant of multiloops $L_\mu$ given by

$$L_\mu(\gamma) = \mu(T(\gamma_1) \cdots T(\gamma_n))$$

for any multiloop $\gamma = \{\gamma_i\}$. Moreover, the map $\mu \mapsto L_\mu$ is one-to-one.

Proof - That $L_\mu$ is a multiloop invariant follows directly from the definitions, and the map $\mu \mapsto L_\mu$ is one-to-one because $H_0$ is dense in $H$. \qed

A fundamental unsolved problem in the theory of holonomy algebras is to determine which multiloop invariants are of the form $L_\mu$ for some $\mu \in H^*_\text{inv}$, for a fixed $G$-bundle $P$ and representation $\rho$. One obvious constraint is that $L_\mu(\gamma) = L_\mu(\eta)$ if $T(\gamma_i) = T(\eta_i)$ for all $i$. This constraint has been studied by Ashtekar and Isham [2]. The fundamental representation of $SU(2)$ is special in this regard, in that $T(\gamma) = T(\gamma^{-1})$ for all loops $\gamma$.

Every multiloop invariant restricts to an ambient isotopy invariant of oriented links in $M$, but this restriction map is not one-to-one. Thus multiloop invariants have more information than link invariants; they depend not only on the topology of the mappings $\gamma_i: S^1 \to M$ but also on the structure of the singularities of these mappings. In what follows we will construct elements of $H^*_\text{inv}$ from data corresponding to the various possible singularities, or “vertex types,” admitted by analytic multiloops. Our method generalizes that of Ashtekar and Lewandowksi, who constructed a state $\mu \in H^*_\text{inv}$ in the case when $\rho$ is the fundamental representation of $G = SU(2)$. To construct examples states in $H^*_\text{inv}$, in the next section we develop a characterization of all such states.

### 4 Reduction to the Finite-Dimensional Case

To tackle the problem of constructing elements of $H^*_\text{inv}$, we can adapt a technique used in functional integration on infinite-dimensional vector spaces. This is an algebraic formulation of the idea of “cylinder measures,” which goes back to Kolmogorov and has subsequently received many formulations [7, 22]. If $V$ is a real vector space, let $A_0$ be the algebra of tame functions on $V$, that is, functions $f: V \to \mathbb{C}$ such

$$f = \tilde{f} \circ j$$

where $j: V \to \mathbb{R}^d$ is a quotient map, that is, an onto linear map, and $\tilde{f} \in C_b(\mathbb{R}^d)$. In other words, the tame functions are the bounded continuous functions depending on
only finitely many coordinates of \( V \). We can construct generalized measures on \( V \) as follows. Suppose that for each quotient map \( j: V \to \mathbb{R}^d \) (\( d \) arbitrary) there is given a measure \( \mu_j \) on \( \mathbb{R}^d \). We may attempt to define \( \int f = \int_{\mathbb{R}^d} Fd\mu_j \)

if \( f = F \circ j \). For \( f \) to be well-defined, the following consistency condition is sufficient: if \( f = F \circ j \) and also \( f = F' \circ j' \) for some other quotient map \( j': V \to \mathbb{R}^{d'} \), we must have

\[
\int_{\mathbb{R}^d} Fd\mu_j = \int_{\mathbb{R}^{d'}} F'd\mu_j'.
\]

If in addition there is a constant \( C > 0 \) such that

\[ \left| \int_{\mathbb{R}^d} Fd\mu_j \right| \leq C \| f \|_{\infty} \]

for all quotient maps \( j \) and all \( F \in C_b(\mathbb{R}^d) \), \( (V, A_0, f) \) is a generalized measure space.

The advantage of this construction is that it reduces integration on \( V \) to a problem only involving finite-dimensional spaces.

Similarly, we will show how to construct elements of \( H^*_{\text{inv}} \) from a family of measures on spaces of the form \( G^d \) that satisfies consistency and uniform boundedness conditions. Conversely, we show that \( \text{all} \) elements of \( H^*_{\text{inv}} \) arise this way. In the next section we construct some examples.

First, we define the notion of an embedded graph in \( M \). We define an *embedded graph* in \( M \) to be a nonempty set \( \phi \subseteq M \) such that there exist finitely many maps \( \phi_i: [0, 1] \to M \) with:

1) \( \phi = \bigcup_i \phi_i[0, 1] \),
2) each \( \phi_i \) is one-to-one,
3) \( \phi_i|_{(0,1)} \) is an embedding,
4) for all \( i \) and \( j \), \( \phi_i[0, 1] \cap \phi_j[0, 1] \subseteq \{ \phi_i(0), \phi_j(0) \} \).

This implies that \( \phi \) has the topology of a finite graph. We call the maps \( \phi_i \) edges. Note that for a given embedded graph \( \phi \), which is just a set, there is not a unique choice of edges \( \phi_i \) satisfying 1) - 4). We call a finite set of maps \( \phi_i: [0, 1] \to M \) an *edge decomposition* of \( \phi \) if 1) - 4) hold.

Given an embedded graph \( \phi \subseteq M \) we define the subalgebra \( A_0(\phi) \) of \( H_0 \) to be the subalgebra generated by the elements \( T(\gamma) \) for all loops \( \gamma: S^1 \to M \) with range lying in \( \phi \). Let \( A(\phi) \subseteq H \) denote the closure of \( A_0(\phi) \) in the \( L^\infty \) norm. Note that \( A_0(\phi) \) is a *-algebra and \( A(\phi) \) is a C*-algebra.

If \( \phi, \psi \) are embedded graphs with \( \phi \subseteq \psi \), then \( A(\phi) \subseteq A(\psi) \). We will use the notation \( \{ f_{\phi} \} \) to denote a family of functionals \( f_{\phi} \in A(\phi)^* \), one for each embedded graph \( \phi \). We say such a family is consistent if whenever \( \phi \subseteq \psi \), then \( f_{\phi} \) is the restriction of \( f_{\psi} \) to \( A(\phi) \). We say the family is uniformly bounded if for some \( C > 0 \), \( \| f_{\phi} \| \leq C \) for all \( \phi \). We have:
Theorem 1. Suppose that $f \in H^*$. Given an embedded graph $\phi$, let $f_\phi$ be the restriction of $f$ to $A(\phi)$. Then $\{f_\phi\}$ is a consistent and uniformly bounded family. Conversely, given a consistent and uniformly bounded family $\{f_\phi\}$ there exists a unique $f \in H^*$ such that for all embedded graphs $\phi$, $f_\phi$ is the restriction of $f$ to $A(\phi)$.

Proof - Suppose that $f \in H^*$ and let $f_\phi$ be the restriction of $f$ to $A(\phi)$. Then the $f_\phi$ are consistent and uniformly bounded by the norm of $f$. Conversely suppose we are given a consistent and uniformly bounded family $\{f_\phi\}$. In order to construct a functional $f \in H^*$ whose restriction to each $A(\phi)$ is $f_\phi$, we use two lemmas whose proofs we omit:

Lemma 1. If $\phi$, $\psi$ are embedded graphs, so is $\phi \cup \psi$.

Lemma 2. If $\gamma : S^1 \to M$ is a loop, the range of $\gamma$ is an embedded graph.

Any element $a \in H_0$ is a finite linear combination of products of elements $T(\gamma_i)$, where $\{\gamma_i\}$ is a finite set of loops. By Lemmas 1 and 2, there is an embedded graph $\phi$ such that $T(\gamma_i) \in A(\phi)$ for all $i$. Define

$$\int f = \int_\phi f.$$  

We need to check that $f$ is well-defined, linear, and extends to a continuous linear functional on $H$. Clearly the extension is unique since $H_0$ is dense in $H$.

For well-definedness, suppose that $a \in A(\phi)$ and $a \in A(\psi)$ as well. By Lemma 1, $\phi \cup \psi$ is an embedded graph with $\phi, \psi \subseteq \phi \cup \psi$. Thus by consistency,

$$\int_\phi f = \int_{\phi \cup \psi} f = \int_\psi f.$$  

For linearity, suppose $f, g \in H_0$. Then there are embedded graphs $\phi, \psi$ such that $f \in A(\phi)$ and $g \in A(\psi)$. Then $f, g, f + g \in A(\phi \cup \psi)$ and

$$\int (f + g) = \int_{\phi \cup \psi} (f + g) = \int_{\phi \cup \psi} f + \int_{\phi \cup \psi} g = \int f + \int g.$$  

Clearly $f(\lambda f) = \lambda f$ for all $\lambda \in C$. Finally, to show that $f$ extends to a continuous linear functional on $H$ it suffices to note that for any $f \in H_0$ we can choose $\phi$ with $f \in A(\phi)$, so

$$|\int f| = |\int_\phi f| \leq C\|f\|.  

\Box$$  

In physics one is especially interested in states on $H$, so it is useful to note that a consistent family of states on the subalgebras $A(\phi)$ determines a unique state on $H$:
Proposition 2. If the family functionals \( \{ f_{\phi} \} \) are consistent and \( f_{\phi} \) is positive for every embedded graph \( \phi \), then the family \( \{ f_{\phi} \} \) is uniformly bounded.

Proof - Note that the unit \( 1 \in H \) is in \( A(\phi) \) for all \( \phi \). Since \( f_{\phi} \) is positive, \( \| f_{\phi} \| = f_{\phi} 1 \). So it suffices to show that for any two embedded graphs \( \phi \) and \( \psi \), \( f_{\phi} 1 = f_{\psi} 1 \). This follows from consistency:

\[
\int_\phi 1 = \int_{\phi \cup \psi} 1 = \int_\psi 1.
\]

\( \Box \)

We now lay the groundwork for constructing elements of \( H_{\text{inv}}^* \). First note that any diffeomorphism \( g: M \to M \) with \( g(\phi) \subset \psi \) for embedded graphs \( \phi, \psi \) induces a *-homomorphism \( g_*: A_0(\phi) \to A_0(\psi) \) such that

\[
g_* T(\gamma) = T(g \circ \gamma)
\]

for all loops \( \gamma: S^1 \to M \) with range in \( \phi \). This homomorphism is norm-preserving, so it is one-to-one, and extends to a one-to-one *-homomorphism from \( A(\phi) \) to \( A(\psi) \), which we also call \( g_* \). By duality we obtain a linear map

\[
g^*: A(\psi)^* \to A(\phi)^*.
\]

Given a family \( \{ f_{\phi} \} \) of functionals on the algebras \( A(\phi) \), we say it is covariant if for any \( g \in \text{Diff}_0(M) \), given embedded graphs \( \phi \) and \( \psi \) with \( g(\phi) \subseteq \psi \), then

\[
g^* \int_\psi = \int_\phi.
\]

Note that a covariant family is automatically consistent.

Theorem 2. Suppose that \( f \in H_{\text{inv}}^* \). Given an embedded graph \( \phi \), let \( f_{\phi} \) be the restriction of \( f \) to \( A(\phi) \). Then \( \{ f_{\phi} \} \) is a covariant and uniformly bounded family. Conversely, given a covariant and uniformly bounded family \( \{ f_{\phi} \} \) there exists a unique \( f \in H_{\text{inv}}^* \) such that for all embedded graphs \( \phi \), \( f_{\phi} \) is the restriction of \( f \) to \( A(\phi) \).

Proof - Using Theorem \( \bullet \) the only substantial point to check is that the element \( f \in H^* \) determined by a covariant and uniformly bounded family is \( \text{Diff}_0(M) \)-invariant. For this it suffices to check that \( f f = f g f \) for all \( f \in H_0 \) and \( g \in \text{Diff}_0(M) \). By Lemmas 1 and 2 we can find an embedded graph \( \phi \) such that \( f \in A(\phi) \). The image \( g\phi \) of \( \phi \) under \( g \) is again an embedded graph and \( g\phi \in A(g\phi) \), so by covariance

\[
\int f = \int_{\phi} f = \int_{g\phi} g f = \int g f.
\]

\( \Box \)
Note that an embedded graph \( \phi \) has the topology of a finite graph with the points \( \phi_i(0), \phi_i(1) \) as vertices and the sets \( \phi_i[0,1] \) as edges. Thus the study of states on \( \mathcal{A}(\phi) \) essentially amounts to doing lattice gauge theory on a graph. (For a review of applications of Wilson loops to lattice gauge theory, see Loll [27].) Let \( \pi_1 = \pi_1(\phi) \) denote the fundamental group of \( \phi \), which we define as the free product of the fundamental groups of the components of \( \phi \) if \( \phi \) is not connected. Then \( \pi \) is a finitely generated free group, and for any edge decomposition of \( \phi \) we may find loops generating \( \pi_1 \) that are products of the edges \( \phi_i \) and their inverses. Since \( \phi \) is a graph, the holonomy of a connection \( A \in \mathcal{A} \) around any loop \( \gamma : S^1 \to \phi \) depends only on its homotopy class. Thus we have a map

\[
p : \mathcal{A} \to \text{Hom}(\pi_1,G),
\]

and as noted by Ashtekar and Lewandowski [3] this map is onto when \( G \) is connected, which we assume henceforth. Note that \( \text{Hom}(\pi_1,G) \) is a manifold diffeomorphic to \( G^d \), where \( d \) is the rank of \( \pi_1 \). Moreover, any element of \( \mathcal{A}(\phi) \), regarded as a function on \( \mathcal{A} \), is of the form \( f \circ p \) for some \( f \in C(\text{Hom}(\pi_1,G)) \). Since \( p \) is onto, \( f \) is unique. Thus we may identify \( \mathcal{A}(\phi) \) with a subalgebra of \( C(\text{Hom}(\pi_1,G)) \). It follows that any measure on \( \text{Hom}(\pi_1,G) \) determines an element \( \int \in \mathcal{A}(\phi)^* \).

In the next section we sketch how to actually construct elements of \( H^{inv}_* \) using Theorem 2. The results from this section that we will need can be expressed as follows. We say that \( \{\mu_\phi\} \) is a covariant family of measures if for each embedded graph \( \phi, \mu_\phi \) is a measure on \( \text{Hom}(\pi_1(\phi),G) \), and for all \( g \in \text{Diff}_0(M) \) with \( g : \phi \to \psi \) we have

\[
g^* \mu_\psi = \mu_\phi,
\]

where \( g^* \) is the map from measures on \( \text{Hom}(\pi_1(\psi),G) \) to measures on \( \text{Hom}(\pi_1(\phi),G) \) induced by \( g \). We have:

**Corollary 1** Suppose \( G \) is connected, and suppose that \( \{\mu_\phi\} \) is a covariant family of probability measures. Then there exists a unique state \( \int \in H^{inv}_* \) such that for all embedded graphs \( \phi \) and all \( f \in \mathcal{A}(\phi) \),

\[
\int f = \int_{\text{Hom}(\pi_1(\phi),G)} f d\mu
\]

where on the right side we identify \( f \) with a function on \( \text{Hom}(\pi_1(\phi),G) \).

Proof - This follows immediately from Proposition [2], Theorem [2], and the remarks above. \( \square \)

It is convenient to think of probability measures on \( \text{Hom}(\pi_1(\phi),G) \) in terms of \( G \)-valued random variables. A probability measure on \( \text{Hom}(\pi_1(\phi),G) \) is the same as an function from elements \( \gamma \in \pi_1(\phi) \) to \( G \)-valued random variables \( g(\gamma) \) such that whenever \( \gamma, \eta \) are homotopy classes of loops in the same component of \( \phi \),

\[
g(\gamma)g(\delta) = g(\gamma\delta).
\]
Since $\pi_1(\phi)$ is a free group with $d$ generators, we may also think of a probability measure on $\text{Hom}(\pi_1(\phi), G)$ as a $d$-tuple of $G$-valued random variables, one for each generator.

5 Constructing Diffeomorphism-invariant States

The space of multiloops is a stratified space. Embeddings, or links, form an open dense subset of this space, while strata of increasingly high codimension correspond to multiloops with ever more complicated self-intersections and other singularities. It seems likely that a deeper understanding of the connection between knot theory and gauge theory will require studying the whole space of multiloops, and involve a blend of singularity theory and group representation theory. There is quite an amount of work that points in this direction, even though the full picture has not yet emerged.

On the one hand, Vassiliev’s study of the space of loops [34] led him to formulate the notion of knot invariants of finite type. Then Bar-Natan, Birman and Lin [8, 9] found that such knot invariants may be constructed from the perturbative Chern-Simons theory, or alternatively via group representations theory. Further work showed that there really is a theory of graph invariants of finite type [32], and that finite type invariants may be related to perturbative quantum gravity [4].

On the other hand, in the loop representation of quantum gravity it appears that multiloop invariants having trivial behaviour on multiloops with singularities form only a small part of the space of states [10, 18]. Thus it is important to devise systematic constructions of multiloop invariants, and in particular, natural ways of extending known link invariants to multiloops with self-intersections. This has been pursued for the Chern-Simons link invariants by Brügmann, Gambini, Pullin, Kauffman, and others [11, 16, 20].

Here we outline a procedure to construct diffeomorphism-invariant generalized measures on $A/G$ - or in other words, elements of $H^*_{\text{inv}}$ - using Corollary 1. First we define an equivalence relation on points in embedded graphs. Given embedded graphs $\phi, \psi$ and points $x \in \phi, y \in \psi$, we say that $(x, \phi)$ and $(y, \psi)$ have the same vertex type if there is an element $g \in \text{Diff}_0(M)$ such that $g(x) = y$ and there are neighborhoods $U \ni x, V \ni y$ such that $g(\phi \cap U) = \psi \cap V$. We call an equivalence class of pairs $(x, \phi)$ a vertex type, and write the vertex type containing $(x, \phi)$ as $[x, \phi]$. Some of the simpler vertex types (for $\dim M \geq 2$) are the half-arc:

```
\[]
```

the arc:
the cusp:

\[ \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{cusp.png}}
\end{array} \]

the corner:

\[ \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{corner.png}}
\end{array} \]

the T:

\[ \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{T.png}}
\end{array} \]

and the transverse double point:

\[ \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{double_point.png}}
\end{array} \]

It is important to note that more complicated vertex types can come in parametrized families. For example, in two dimensions there is a continuous one-parameter family of vertex types that look roughly like:

\[ \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{continuous.png}}
\end{array} \]

Given a edge decomposition $\phi_i$ of an embedded graph $\phi$, we call the points $\phi_i(0), \phi_i(1)$ the vertices. Note that the edges and vertices of $\phi$ depend on the choice of edge decomposition. For example, the following embedded graph $\phi$ has an edge decomposition with 5 edges and 4 vertices:

\[ \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{edge_decomposition.png}}
\end{array} \]
We could, however, insert extra vertices, hence extra edges.

To construct $\text{Diff}_0(M)$-invariant states on the holonomy algebra $H$, it suffices by Corollary 1 to construct a covariant family $\{\mu_\phi\}$, where $\mu_\phi$ is a probability measures on $\text{Hom}(\pi_1(\phi), G)$. Defining $\pi_1(\phi)$ requires a choice of basepoint for each component of $\phi$. We may always choose these basepoints to be vertices; in the example above we have arbitrarily chosen the vertex $\phi_2(1)$ as a basepoint for $\phi$.

We define the *valence* of a vertex type as in graph theory, so that associated to each $n$-valent vertex type $v$ there is a set $E(v)$ of $n$ (equivalence classes of) edges. We will construct $\text{Diff}_0(M)$-invariant states on the holonomy algebra $H$ by a procedure that involves fixing for each vertex type $v$ a probability measure $m_v$ on $G^{E(v)}$, the set of maps from $E(v)$ to the group $G$. We may think of $m_v$ as a collection of $G$-valued random variables, one for each edge in $E(v)$.

Suppose now that $x$ is any vertex of the embedded graph $\phi$ (relative to fixed edge decomposition $\{\phi_i\}$). Let $v$ be the vertex type of $x$, that is, let $v = [x, \phi]$. We say an edge $\phi_i$ is *incident* to $x$ if $x = \phi_i(0)$ or $x = \phi_i(1)$. There is an obvious one-to-one correspondence between the edges incident to $x$ and the set $E(v)$. This allows us to associate to each edge incident to $x$ a $G$-valued random variable, such that the random variables for all the edges incident to $x$ are distributed according to the probability measure $m_v$ on $G^{E(v)}$. We draw these random variables as dots near $x$ on the edges incident to $x$, as follows:

We do this for all the vertices, and require that the random variables associated to different vertices are independent.

For example, the vertices $\phi_2(0)$ and $\phi_2(1)$ above have the vertex type of the crossing, so we attach four $G$-valued random variables to each, with distribution equal to the the probability measure $m_+$ on $G^4$ associated to the crossing. The vertices $\phi_4(0)$ and $\phi_5(1)$ have the vertex type of the half-arc, so we attach one $G$-valued random variable to each, with distribution equal to the probability measure $m_-$ associated to the half-arc. Since the random variables near different vertices are independent, the ten variables $g_j$ have distribution $m_- \otimes m_+ \otimes m_+ \otimes m_-$ on $G^{10}$.

This construction allows us to associate a $G$-valued random variable to each loop in $\phi$ that is a product of the edges $\phi_i$ and their inverses. Going around such a loop
we write down a factor of $g_j$ for each dot we come to while exiting a vertex, and a factor of $g_j^{-1}$ for each dot we come to while entering a vertex. For example, to the loop $\phi_1 \phi_2$ we associate the product $g_7 g_3^{-1} g_4 g_6^{-1}$, while to the loop $\phi_5 \phi_5^{-1}$ we associate the product $g_8 g_{10}^{-1} g_9^{-1} g_8^{-1} = 1$. Note that homotopic loops automatically receive the same random variable. Moreover, this procedure associates to each homotopy class $\gamma \in \pi_1(\phi)$ a random variable $g(\gamma)$ in such a way that $g(\gamma) g(\eta) = g(\gamma \eta)$ whenever the product of homotopy classes $\gamma, \eta$ is defined. Thus, by the remark at the end of the previous section, we have constructed a probability measure $\mu_\phi$ on $\text{Hom}(\pi_1(\phi), G)$.

It remains to state conditions on the measures $m_v$ such that the measure $\mu_\phi$ is independent of a choice of edge decomposition for the embedded graph $\phi$, and such that the family $\{\mu_\phi\}$ is covariant in the sense of Corollary 1. We will need two conditions.

We define an *inclusion* of one vertex type in another, written $i: [x, \phi] \to [y, \psi]$, to be an equivalence class of diffeomorphisms $g \in \text{Diff}_0(M)$ such that $g(\phi) \subseteq \psi$ and $g(x) = y$, where we say two diffeomorphisms are equivalent if they give rise to the same map from $E[x, \phi]$ to $E[y, \psi]$. For example, there are two inclusions of the half-arc in the arc:

![Half-arc in the arc](image)

and

![Half-arc in the cusp](image)

but no inclusions of the half-arc in the cusp. Similarly, there are four inclusions of the arc in the transverse double point, but no inclusions of the arc in the corner. Note that given an inclusion $i: v \to w$, there is a natural inclusion of sets $E(v) \hookrightarrow E(w)$. This in turn gives rise to a natural surjection $G^{E(w)} \to G^{E(v)}$, and we can push forward a measure $\mu_w$ on $G^{E(w)}$ to a measure which we call $i^* \mu_w$ on $G^{E(v)}$. Our first condition is that given any inclusion $i: v \to w$,

$$i^* m_w = m_v.$$  \hspace{1cm} (1)

A given embedded graph typically has many edge decompositions. However, given any two edge decompositions of $\phi$ with vertices $\{x_i\}$, $\{y_i\}$ respectively, there is an edge decomposition with vertices $\{x_i\} \cup \{y_i\}$. In fact, one can go between any two edge decompositions by a series of local moves in which one replaces

![Local move](image)
by

or vice versa. Here the vertex on the left stands for one of any type and the vertex on the right has the type of an arc. Our second condition is therefore as follows. Suppose the random variables \((g, h_1, h_2)\) have distribution \(m_v \otimes m \downarrow\) on \(G^3\), where \(m_v\) is the measure on \(G\) associated to any 1-valent vertex type and \(m \downarrow\) is the measure on \(G^2\) associated to the arc. Let \(p: G \times G^2 \to G\) be given by

\[
p(g, h_1, h_2) = gh_1^{-1}h_2.
\]

Then we require

\[
p_*(m_v \otimes m \downarrow) = m_v
\]

Pictorially, this says that

\[
= 
\]

From conditions (1) and (2), it follows that for any \(n\)-valent vertex type \(v\), labelling the edges of \(v\) arbitrarily with integers \(\{1, \ldots, n\}\), so that \(m_v\) becomes a probability measure on \(G^n\), and defining

\[
p(g_1, \ldots, g_n, h_1, h_2) = (g_1, \ldots, g_{n-1}, g_nh_1^{-1}h_2),
\]

we have

\[
p_*(m_v \otimes m \downarrow) = m_v.
\]

Pictorially, this says that

\[
= 
\]

This guarantees that the measures \(\mu_\phi\) are independent of edge decomposition.

In fact, condition (3) also implies the measures \(\{\mu_\phi\}\) are a covariant family. Suppose \(\phi, \psi\) are embedded graphs and \(g \in \text{Diff}_0(M)\) has \(g(\phi) \subseteq \psi\). Then we can find an edge decomposition of \(\psi\) such that a subset of the edges \(\{\psi_i\}\) give an edge decomposition of \(g\phi\). Then for each vertex \(x\) of \(\phi\), \(g(x)\) is a vertex of \(\psi\), and \(g\) determines an inclusion \(i: [x, \phi] \to [g(x), \psi]\). Using these facts one easily sees from (3) that \(g^*\mu_\phi = \mu_\phi\).

The simplest example of a family of probability measures \(\{m_v\}\) meeting conditions (1) and (2) is that assigning to each vertex type \(v\) the Dirac delta at the identity of
This gives the flat state in $H^*_{inv}$. When the principal bundle $P$ admits a connection $A_0$ such that the holonomy around every loop is trivial, i.e., when $P$ is trivial, the flat state may be identified with Dirac delta measure at $[A_0] \in \mathcal{A}/\mathcal{G}$. Curiously, the flat state is well-defined for any principal bundle $P$, even if $P$ admits no flat connections. The real reason for this is that the restriction of $P$ to any embedded graph $\phi$ is trivial, since we are assuming that $G$ is connected. This permits a remarkable degree of "bundle-independence" for elements of $H^*_{inv}$, a phenomenon noted by Ashtekar and Lewandowski [3] that deserves further study.

A more interesting example of a family $\{m_v\}$ meeting conditions (1) and (2) is that assigning to each vertex type $v$ the normalized Haar measure on the Lie group $G_{E(v)}$. This gives the Ashtekar-Lewandowski state, constructed by these authors in the case where $G = SU(2)$ and $\rho$ is the fundamental representation [3]. It is easy to check that this state is faithful, that is, if $f \in H$ has $f \geq 0$ and $\int_{T} f = 0$, then $f = 0$.

A complete analysis of the solutions of (1) and (2) would require a deep understanding of the interplay of singularity theory for curves and harmonic analysis on $G$ involved in these equations. We do not have such an understanding yet - indeed, one might worry that the solutions we have mentioned are the only ones! To allay such fears, we present a few new solutions, but leave as an open problem a thorough analysis of the conditions (1) and (2).

It is easiest to describe solutions to (1) and (2) in terms of the $n$-tuple of $G$-valued random variables assigned by $m_v$ to each $n$-valent vertex type $v$. Given two edges $e, f \in E(v)$, let us say that $e$ and $f$ form an arc if there is an inclusion $i: \bullet \to v$ of the arc in $v$ such that image of $E(\bullet)$ under the corresponding inclusion $E(\bullet) \hookrightarrow E(v)$ is $\{e, f\}$. For example, for the vertex below, a combination of the cusp and the T, only edges $e_3$ and $e_5$ form an arc.

Now fix a probability measure $m$ on $G$. To $v$ assign $n$ $G$-valued random variables $g_e$, one for each edge $e \in E(v)$, that are all distributed according to the measure $m$, and such that $g_e = g_f$ if the edges $e$ and $f$ form an arc, while $g_e$ and $g_f$ are independent otherwise. Conditions (1) and (2) are easy to check. We thus obtain $\text{Diff}_0(M)$-invariant states on $H$. Note that if

$$\gamma = \circlearrowright, \quad \eta = \triangledown,$$

then $\int T(\gamma) = 1$ but generally $\int T(\eta) \neq 1$ in these states. In the flat state, $\int T(\gamma) = 1$.
for any loop $\gamma$. In the Ashtekar-Lewandowski state for $\rho$ the fundamental representation of $G = SU(2)$, $\int T(\gamma) = \int T(\eta) = 0$.

### 6 Generalizations and Conclusions

When $M$ is 3-dimensional, the construction of states in the previous section may be regarded as the “trivially braided” case of a more general construction. If $M = \mathbb{R}^3$ or $S^3$, for example, we may assign $G$-valued random variables not only to vertices but also to crossings in a planar diagram of the embedded graph:

![Diagram](image)

The conditions for the measure $\mu_\phi$ associated to a given embedded graph $\phi$ to be independent of the choice of diagram will involve the Yang-Baxter equation and relations allowing us to move vertices past crossings, e.g.:

![Diagram](image)

These conditions are, in fact, very similar to the Reidemeister-type moves for rigid-vertex isotopy of graphs in the sense of Kauffman [21], as well as the moves Reshetikhin and Turaev discuss in their work on ribbon graphs [29]. Reshetikhin and Turaev systematically construct invariants of ribbon graphs using braided tensor categories, of which categories of representations of quantum groups are the main example. In particular, the quantum group link invariants are precisely those one would hope to obtain from the Chern-Simons path integral. The quantum group link invariants depend on a parameter $q$, which is related to the integer $k$ appearing in the Chern-Simons action by

$$q = e^{2\pi i/(k + c_2(G)/2)}$$

where $c_2(G)$ is the value of the quadratic Casimir operator for $G$ in the adjoint representation, normalized so that it equals $2n$ for $SU(n)$. Ideally, one would hope to be able to construct the Chern-Simons “measure” as a generalized measure on $A/\mathcal{G}$ using the sort of construction of the previous section, but with a nontrivial braiding. In particular, one would hope that, at least for certain values of the quantization parameter $q$, the universal $R$-matrix for the quantum group associated to a semisimple Lie group $G$ could be expressed as a measure on $G \times G$. This appears to be an open question.
The main difficulty in trying to construct the Chern-Simons “measure” as an element of $H^*_{\text{inv}}$ is that the link invariants associated to the Chern-Simons path integral are framing-dependent, while elements of $H^*_{\text{inv}}$ determine framing-independent link invariants. In certain cases this problem can be sidestepped, since the framing dependence typically enters via an exponential of the writhe, or self-linking number, of the framed link. Namely, if one works with a Lie group of the form $G \times U(1)$, one can arrange that the $U(1)$ factor provides an exponential of the writhe that cancels that coming from $G$ \[8\]. For example, Chern-Simons theory in the fundamental representation of $G = SU(2)$ gives the Kauffman bracket, an invariant of framed unoriented links, but Chern-Simons theory with $SU(2) \times U(1)$ can be used to obtain the Jones polynomial, an framing-independent invariant of oriented links that differs from the Kauffman bracket by a factor of an exponential of the writhe.

For more general Chern-Simons path integrals in which we cannot arrange a cancellation of factors involving the writhe, we will need to replace the holonomy algebra $H$ by one for which diffeomorphism-invariant continuous linear functionals yield invariants of framed links. This is was the motivation for earlier work \[5\] in which we described a modified holonomy algebra generated by regularized Wilson loops, or tubes: functions on $\mathcal{A}/\mathcal{G}$ of the form

$$
\int_{D^{n-1}} T(\gamma^x, A) \omega(x)
$$

where $\omega$ is a smooth $(n-1)$-form compactly supported in the interior of $D^{n-1}$, $\gamma: S^1 \times D^{n-1} \to M$ is a smoothly embedded torus in the smooth manifold $M$, and for each $x \in D^{n-1}$ the loop $\gamma^x$ is given by $\gamma^x(t) = \gamma(t, x)$.

The completion of this algebra in the $L^\infty$ norm is called the tube $C^*$-algebra. There is a linear map from $\text{Diff}_0(M)$-invariant continuous linear functionals on the tube $C^*$-algebra to ambient isotopy invariants of framed links. What is more, this map is one-to-one. Thus the regularization involved in working with the tube algebra has two good effects: it reduces the amount of information contained in a $\text{Diff}_0(M)$-invariant continuous linear functional from a multiloop invariant to a link invariant, and it introduces the possibility of framing-dependence.

Unfortunately, it appears difficult to construct $\text{Diff}_0(M)$-invariant continuous linear functionals on the tube algebra by methods analogous to those of the present paper, essentially because tubes are too “thick” for the method of embedded graphs to apply. A promising compromise currently under investigation is the “strip algebra” based on analytically embedded annuli $\gamma: S^1 \times [0, 1] \to M$.

In addition to these directions for further investigation, it is tempting to try to construct the Chern-Simons measure not just on $S^3$, but on general compact 3-manifolds using the machinery of modular tensor categories \[15, 29\]. Interestingly,
and not at all coincidentally, the category of representations of a quantum group gives rise to a modular tensor category precisely when

\[ q = e^{\pm 2\pi i / (k + c_2(G)/2)} \]

with \( k \) a nonnegative integer. It would also be interesting, and comparatively straightforward, to extend the theory developed in this paper to the case of manifolds with boundary, generalizing from Wilson loops to include also Wilson lines with endpoints at the boundary, in order to make contact with the theory of tangles and braid group representations [4].

To conclude, it should be clear that holonomy algebras offer a promising route to doing diffeomorphism-invariant gauge theory in a rigorous way. There is much to be done to explore the connections between topology, singularity theory, representation theory and category theory that arise in the study of diffeomorphism-invariant states on holonomy C*-algebras.

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