Geometric transitions and integrable systems

D.-E. Diaconescu\textsuperscript{b}, R. Dijkgraaf\textsuperscript{z}, R. Donagi\textsuperscript{†}, C. Hofman\textsuperscript{♯} and T. Pantev\textsuperscript{†}

\textsuperscript{b} New High Energy Theory Center, Rutgers University
126 Frelinghuysen Road, Piscataway, NJ 08854

\textsuperscript{z} Institute for Theoretical Physics and KdV Institute for Mathematics
University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

\textsuperscript{♯} The Weizmann Institute for Science, Department of Particle Physics
Herzl Street 2, 76100 Rehovot, Israel

\textsuperscript{†} Department of Mathematics, University of Pennsylvania
David Rittenhouse Lab., 209 South 33rd Street, Philadelphia, PA 19104-6395

June 2005

Abstract

We consider B-model large N duality for a new class of noncompact Calabi-Yau spaces modeled on the neighborhood of a ruled surface in a Calabi-Yau threefold. The closed string side of the transition is governed at genus zero by an $A_1$ Hitchin integrable system on a genus $g$ Riemann surface $\Sigma$. The open string side is described by a holomorphic Chern-Simons theory which reduces to a generalized matrix model in which the eigenvalues lie on the compact Riemann surface $\Sigma$. We show that the large $N$ planar limit of the generalized matrix model is governed by the same $A_1$ Hitchin system therefore proving genus zero large $N$ duality for this class of transitions.

Contents

1 Introduction 2

2 Review of Dijkgraaf-Vafa transitions 3

3 Dijkgraaf-Vafa limits of compact Calabi-Yau spaces 12
   3.1 Stratification of moduli 12
   3.2 Examples of nested moduli 13
1 Introduction

Large $N$ duality [1] has been at the center of many recent developments in topological string theory. In particular B-model transitions [2, 3] have revealed a fascinating interplay of random matrix models, integrable systems and Calabi-Yau geometry.

In this paper we generalize the results of [2, 3] reviewed in section two to a new class of conifold transitions among noncompact Calabi-Yau threefolds. As explained in section three, the starting point of our construction is a configuration

\[
\begin{align*}
\widetilde{S} & \subset \widetilde{M} \\
\downarrow & \\
S & \subset M \subset L.
\end{align*}
\]

of moduli spaces which generalizes the essential geometric features of the local transitions studied in [2]. Here $L$ is a component of the moduli space of projective or quasi-projective Calabi-Yau threefolds and $M$ is a subspace of $L$ parameterizing Calabi-Yau manifolds with isolated conifold singularities which admit a (quasi-)projective small resolution. The deepest stratum $S$, which is a key element of the construction, parameterizes Calabi-Yau spaces with a genus $g$ curve $\Sigma$ of $A_1$ singularities. The spaces $\widetilde{S}, \widetilde{M}$ are moduli spaces of the resolution of Calabi-Yau spaces in $S$ and respectively $M$. Such geometric structures have been considered before in the physics literature [4, 5, 6] in relation to $N = 2$ gauge theories and open string superpotentials. Here we will show that they play a key role in B-model geometric transitions.

Our main construction is carried out in section four. We consider noncompact Calabi-Yau spaces fibered by affine quadrics over a fixed genus $g$ curve $\Sigma$. A special feature of this
model is that the moduli spaces $\tilde{M}, L$ are isomorphic to the total spaces of vector bundles over $S$.

Large $N$ duality is an equivalence between $B$-type open-closed topological strings on a resolved threefold corresponding to a point in $\tilde{M}$ and closed topological strings on a generic threefold in $L$. In the present paper we establish this result for genus zero topological amplitudes in the geometric framework described above. The proof involves two parts.

The genus zero dynamics for closed $B$-topological strings on Calabi-Yau spaces is usually encoded in the intermediate Jacobian fibration over the moduli space, which supports an integrable system structure [7, 8]. In our case we show that the relevant integrable system for a family of threefolds parameterized by a normal slice to $S$ in $L$ is the $A_1$ Hitchin integrable system. In particular, the normal slice $L_s$ at a point $s \in S$ is isomorphic to the space of quadratic differentials on $\Sigma$, which is the base of the Hitchin systems. This follows from a structure result for the intermediate Jacobians of the Calabi-Yau threefolds in $L$ proved in section five.

The second part of the proof is more physical in nature and involves $B$-topological open string dynamics on a small resolution parameterized by a generic point in $\tilde{M}$. In section six we construct the holomorphic Chern-Simons theory which captures open string target space dynamics using the formalism of D-brane categories. Then we argue that the holomorphic Chern-Simons functional integral reduces to a finite dimensional integral on a real cycle in the product $\text{Sym}_N^N(\Sigma) \times \text{Sym}_N^N(\Sigma)$. This can be regarded as a generalized matrix model in which the eigenvalues are parameterized by a compact Riemann surface. The final result of this section is that the large $N$ planar limit of this generalized matrix model is captured by the same $A_1$ Hitchin system that was found in section five. This concludes the physical proof of genus zero large $N$ duality.

Acknowledgments. We are very grateful to Bogdan Florea and Antonella Grassi for collaboration at an early stage of the project and many useful discussions. We would also like to thank Jacques Distler, Sheldon Katz, Balázs Szendrői and Cumrun Vafa for helpful discussions. D.-E. D. would also like to acknowledge the partial support of the Alfred P. Sloan foundation and the hospitality of KITP Santa Barbara and The Aspen Center for Physics where part of this work was performed. The research of R.D. was supported by a NWO Spinoza grant and the FOM program String Theory and Quantum Gravity. R.D. was partially supported by NSF grant DMS 0104354 and FRG grant 0139799 for “The Geometry of Superstrings”. T.P. was partially supported by NSF grants FRG 0139799 and DMS 0403884. The work of C.M.H. was supported in part by a Marie Curie Fellowship under contract MEIF-CT-2003-500687, the Israel-US Binational Science Foundation, the ISF Centers of Excellence Program and Minerva.

2 Review of Dijkgraaf-Vafa transitions

In this section we will review large $N$ duality for a class of geometric transitions among noncompact Calabi-Yau threefolds first studied in [2]. Adopting the common terminology in the physics literature, for us a geometric transition will be an extremal transition connecting
two different components of a moduli space of Calabi-Yau threefolds through a degeneration. The degenerations usually considered in this context are nodal Calabi-Yau threefolds with isolated ODP singularities. More complicated singularities, such as rational double points, can also appear as junctions of geometric transitions and support very interesting large $N$ physics [9]. We will not look at these more complicated geometries here but they certainly deserve a thorough investigation from the point of view of Dijkgraaf-Vafa quantization.

In the situation considered in [2], we have a moduli space $L$ of noncompact Calabi-Yau hypersurfaces in $X_l \subset \mathbb{C}^4$ defined by equations of the form

\begin{equation}
uv + y^2 - l(x) = 0
\end{equation}

where $l(x)$ is an arbitrary polynomial of degree $2n$. The moduli space $L$ is the complex vector space of dimension $2n + 1$ parameterizing the coefficients of $l(x)$. The degeneration takes place along a subvariety $M \subset L$ characterized by the property that

\begin{equation}
m(x) = (W'_m(x))^2
\end{equation}

for $m \in M$, where $W_m(x)$ is an arbitrary polynomial of degree $n + 1$. Therefore $M$ is a $(n + 1)$-dimensional subvariety in $L$. In the following we will call $W_m(x)$ the classical superpotential for reasons that will shortly become clear. It is easy to check that if the roots of $W'_m(x)$ are distinct, $X_m$ has $n$ isolated ODPs given by solutions of the equations

\begin{align*}
u = v = y = 0, & \quad W'_m(x) = 0
\end{align*}

We will refer to such points in $M$ as generic points. If $W'_m(x)$ has coincident roots, $X_m$ develops more complicated singularities. A special role in the theory will be played by the singular point $s$ of $M$ for which the polynomial $s(x)$ (and hence $W'_m(x)$) is identically zero.

For a generic point $m \in M$ we can easily construct a quasi-projective crepant resolution of $X_m$ by blowing-up $\mathbb{C}^4$ along the subvariety

\begin{align*}
u = 0, & \quad y - W'_m(x) = 0
\end{align*}

This resolution is not unique since we can obtain a different one for example by blowing up $\mathbb{C}^4$ along

\begin{align*}
v = 0, & \quad y + W'_m(x) = 0,
\end{align*}

and we can also consider obvious variations. However all these resolutions are related by flops. Therefore we will have a moduli space $\tilde{M}$ of smooth Calabi-Yau threefolds $\tilde{X}_{\tilde{m}}$ and a finite-to one (in fact in this case two to one) surjective map $\rho : \tilde{M} \to M$ so that $\tilde{X}_{\tilde{m}}$ is a quasi-projective crepant resolution of $X_{\rho(\tilde{m})}$. We will denote by $C_1, \ldots, C_n$ the exceptional curves on $\tilde{X}_{\tilde{m}}$.

Resolving the singular threefolds $X_m$ corresponding to the special points in $M$ where $W'_m(x)$ acquires multiple roots is more involved. Here we will only discuss the extreme case of the threefold $X_s$ corresponding to the singular point $s \in M$. Note that $s$ is a branch
point for the cover $\rho: \tilde{M} \to M$. The inverse image $\rho^{-1}(s)$ consists of a single point $\tilde{s} \in \tilde{M}$. Since $W_s(x) \equiv 0$, the singular threefold $X_s$ is determined by the equation

$$uv + y^2 = 0.$$ 

Therefore $X_s$ is isomorphic to a direct product of the form $\mathbb{C} \times Y$ where $Y$ is the singular quadric surface described by the same equation in $\mathbb{C}^3$ with coordinates $(u, v, y)$. In particular $X_s$ has a line $u = v = y = 0$ of $A_1$ singularities. The resolution $\tilde{X}_s$ is isomorphic to the direct product $\mathbb{C} \times \tilde{Y}$, where $\tilde{Y}$ is the minimal resolution of $Y$; $\tilde{Y}$ is isomorphic to the total space of the line bundle $\mathcal{O}(-2)$ over $\mathbb{P}^1$, and the map $\tilde{Y} \to Y$ is the contraction of the zero section.

Note that a resolution $\tilde{X}_m$ corresponding to a generic point is related to $\tilde{X}_s$ by a complex structure deformation and has the structure of a fibration in quadrics over the complex line. Consider the projection map $\tilde{\pi}: \tilde{X}_m \to \mathbb{C}$ defined in terms of local coordinates by forgetting $(y, u, v)$. Using the equation (1), the fibers of this projection are easily seen to be affine quadrics in $\mathbb{C}^3$. The fibers over points in $\mathbb{C}$ different from the zeroes of $W_m(x)$ are smooth affine quadrics which can be described as smoothings of the $A_1$ singularity. The fiber over a zero of $W_m(x)$ is isomorphic to the resolution $\tilde{Y}$ of the $A_1$ singularity. In particular these fibers contain the exceptional curves $C_1, \ldots, C_n$ which may be identified with the zero section of $\tilde{Y}$.

In the framework of topological B-branes, large $N$ duality relates topological open strings on a resolution $\tilde{X}_m$ to topological closed strings on a smoothing $X_l$. The topological open string B-model is constructed by wrapping $N_i$ topological B-branes on the $i$-th exceptional curve $C_i$, $i = 1, \ldots, n$ in $\tilde{X}_m$. The precise statement for extremal transitions among general projective or quasi-projective Calabi-Yau varieties is not known\(^1\). Here we will explain how large $N$ duality works for the special class of transitions introduced above. Our discussion follows closely [2].

First we should explain what we mean by a topological open B-model defined by wrapping branes on the exceptional curves. From a rigorous mathematical point of view, topological B-branes should be described in terms of derived objects – or, more concretely complexes of vector bundles – on the threefold $\tilde{X}_m$. Although this formalism will be very useful for generalizations, in the present case it suffices to think informally of a D-brane with multiplicity $N_i$ wrapping an exceptional curve $C_i$ as a rank $N_i$ vector bundle $E_i$ on $C_i$. More specifically for the purpose of large $N$ duality we will consider trivial bundles of the form $E_i = C_i \times \mathbb{C}^{N_i}$. In the following we fix a threefold $\tilde{X}_m$ with $m = \rho(\tilde{m})$ generic.

The dynamics of such a brane should be described in terms of a set of off-shell fields which in this case are $C^\infty$ bundle valued differential forms on $C_i$ of the form

$$A^{0,p} \left( \text{End}(E_i) \otimes \Lambda^q N_{C_i/\tilde{X}_m} \right)$$

subject to the constraint $p + q = 1$. In principle, one would like to write down a holomorphic Chern-Simons action functional for such fields using first principles [11] and work out the rules\(^1\).
for quantization. However, this is quite difficult to do in practice because the quantization of holomorphic Chern-Simons theories is typically untractable.

One of the main insights of [2] is that this program can actually be carried out in the present geometric situation as follows. First construct the holomorphic Chern-Simons action for \( B \)-branes on the resolution of the singular threefold \( \tilde{X}_s \). Then construct the holomorphic Chern-Simons action for branes on a generic threefold \( \tilde{X}_m \) by adding a superpotential deformation. The end result is that the dynamics of \( B \)-branes at a generic point \( \tilde{m} \in \tilde{M} \) is captured by a holomorphic matrix model.

The first step is easy. The resolved threefold \( \tilde{X}_s \) is isomorphic to the product \( \mathbb{C} \times \tilde{Y} \), where \( \tilde{Y} \) is isomorphic to the total space of \( \mathcal{O}(-2) \) over \( \mathbb{P}^1 \). Therefore \( \tilde{X}_s \) contains a ruled surface \( S = \mathbb{C} \times \mathbb{P}^1 \) where the rational fibers of the ruling are exceptional curves on \( \tilde{X}_s \) obtained by resolving the line of \( A_1 \) singularities. In particular all fibers of \( S \to \mathbb{P}^1 \) are \((0, -2)\) curves on \( \tilde{X}_s \). We consider a system of \( N \) topological \( B \)-branes wrapping a given fiber \( C \) of the ruling, where

\[
N = \sum_{i=1}^{n} N_i
\]

Informally this means that we pick up a trivial bundle \( E \) of the form \( \mathbb{C} \times \mathbb{C}^N \). We want to write down a holomorphic Chern-Simons action functional on the set of off-shell fields

\[
A^{0,p} \left( \text{End}(E) \otimes \Lambda^q \mathcal{N}_{C/\tilde{X}_s} \right) = A^{0,p} \left( \Lambda^q \mathcal{N}_{C/\tilde{X}_s} \right) \otimes M_N(\mathbb{C})
\]

with \( p + q = 1 \). Since \( \mathcal{N}_{C/\tilde{X}_s} \simeq \mathcal{O}_C \oplus \Omega^1_C \) we are left with three off-shell fields

\[
\phi^{0,1} \in A^{0,1} \otimes M_N(\mathbb{C}), \quad \phi^{0,0} \in A^{0,0} \otimes M_N(\mathbb{C}), \quad \phi^{1,0} \in A^{0,0} \otimes \Omega^1_C \otimes M_N(\mathbb{C}).
\]

In order to write down the holomorphic Chern-Simons action, we have to regard the holomorphic bundle \( E \) as a \( C^\infty \) bundle equipped with a \((0,1)\) connection which in this case can be taken to be the trivial Dolbeault operator \( \overline{\partial} \) on \( C \). The field \( \phi^{0,1} \) represents an off-shell deformation of the background connection which can be eliminated by performing a gauge transformation. Therefore it suffices to write down an action for the remaining fields \( \phi^{0,0}, \phi^{1,0} \). This is simply given by

\[
S_s = \int_C \text{Tr} \left( \phi^{1,0} \wedge \overline{\partial} \phi^{0,0} \right).
\]

This is the holomorphic Chern-Simons action for \( B \)-branes on \( \tilde{X}_s \). The Chern-Simons actions for the branes on an arbitrary threefold \( \tilde{X}_m \) is constructed in [2] by adding a superpotential term to the functional \[(3)\].

From our perspective, this construction can be best summarized as follows. Our final goal is to construct the partition function for \( B \)-branes on \( \tilde{X}_m \), at least as a formal perturbative expansion. The partition function of any gauge theory is obtained by formally integrating over all fields in the action and dividing by the volume of the gauge group. Usually, if the
fluctuations of the theory are described by some complex fields $\psi$, the path integral can be formally written as an integral over the space of fields $\int D\psi D\overline{\phi} e^{-S(\phi, \overline{\phi})}$. However, the holomorphic Chern-Simons action depends only on the holomorphic part of the fields; the antiholomorphic part is absent. Then the quantum theory should be formally defined by integrating the holomorphic measure $D\phi e^{-S(\phi)}$ over a suitable middle dimensional real cycle $\Gamma$ in the space of fields. Therefore the formal expression of the functional integral is

$$Z = \frac{1}{\text{vol}(\mathcal{G})} \int_{\Gamma} D\phi e^{-S(\phi)}.$$  

The cycle $\Gamma$ should be regarded as part of the data specifying the quantum theory. The dependence of the physical quantities of the choice of $\Gamma$ has been thoroughly investigated for holomorphic matrix models in \cite{13}. Since we are interested only in a semiclassical expansion around the critical points, the choice of the contour is irrelevant. We will only assume that one can find such a cycle so that the functional integral is (at least formally) well defined and the usual perturbative techniques are valid.

As always, the functional integral reduces to an integral over (a middle dimensional real cycle in) the moduli space $\mathcal{M}$ of critical points of the action modulo gauge transformations. The holomorphic measure of the moduli space integral should be determined in principle by integrating out the massive modes, provided that the original measure $D\phi$ is at least formally well defined.

For the action written down in equation (3), the critical point equations are

$$\overline{\partial}\phi^{0,0} = 0, \quad \overline{\partial}\phi^{1,0} = 0.$$  

The set of solutions is parameterized by the complex vector space of $N \times N$ complex matrices. As explained above in order to formulate the quantum theory we have to specify a real middle dimensional cycle in the space of solutions $M_N(\mathbb{C})$. A convenient choice is to restrict the functional integral to the subspace of hermitian matrices. Therefore the partition function of $B$-branes on $\tilde{X}_z$ is given by the matrix integral

$$Z_z = \frac{1}{\text{vol}(U(N))} \int d\phi$$  

where $d\phi$ is the linear measure on the space of hermitian $N \times N$ matrices. The coefficient $1/\text{vol}(U(N))$ is due to a residual $U(N)$ gauge symmetry which acts on $\phi$ by conjugation.

Now, recall that our goal is to find an effective description for the dynamics of off-shell fluctuations around a $B$-brane configuration on $\tilde{X}_\vec{m}$ specified by the multiplicities $N_i$, $i = 1, \ldots, n$. According to \cite{2}, the quantum fluctuations about such a background are governed by the perturbative expansion of a deformed matrix integral of the form

$$Z_{\vec{m}} = \frac{1}{\text{vol}(U(N))} \int d\phi e^{-\frac{1}{g_s} W_{\vec{m}}(\phi)}$$

where $d\phi$ is the linear measure on the space of hermitian $N \times N$ matrices. The coefficient $1/\text{vol}(U(N))$ is due to a residual $U(N)$ gauge symmetry which acts on $\phi$ by conjugation.
where $m = \rho(\tilde{m})$, and $W_m(x)$ is the polynomial function introduced in equation (2). Although this deformation has been recently derived by topological open string computation in [14], a more geometric treatment is better suited for our purposes.

The main observation is that on a generic threefold $\tilde{X}_m$ there exists a family of transverse holomorphic deformations of the exceptional curves $C_i$, $i = 1, \ldots, n$ parameterized by the complex line. Transverse holomorphic deformations of a holomorphic cycle are $C^\infty$-deformations of the cycle which depend holomorphically on the deformation parameters. To construct a transverse holomorphic family which includes the $C_i$'s, we will use the special geometry of the fibration $\tilde{X}_m \to \mathbb{C}$. We constructed $\tilde{X}_m$ as a quasi-projective small resolution of the singular threefold $X_m$ defined by $uv + y^2 = (W'_m(x))^2$, given by a generic polynomial $W_m$ of degree $n + 1$. Viewing the polynomial $W'_m(x)$ as a degree $n$ map from $\mathbb{C}$ to $\mathbb{C}$ we can identify $X_m$ with the fiber product

$$
\begin{array}{ccc}
X_m & \longrightarrow & Z \\
\uparrow & & \downarrow \\
\mathbb{C} & \longrightarrow & \mathbb{C}
\end{array}
$$

where $Z$ is the conifold hypersurface $uv + y^2 = w^2$ in $\mathbb{C}^4$ with coordinates $(w, y, u, v)$. Hence we can obtain the quasi-projective small resolution $\tilde{X}_m$ of $X_m$ as the fiber product

$$
\begin{array}{ccc}
\tilde{X}_m & \longrightarrow & \tilde{Z} \\
\uparrow & & \downarrow \\
\mathbb{C} & \longrightarrow & \mathbb{C}
\end{array}
$$

with a resolved conifold. Explicitly $\tilde{Z}$ is the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$ and the map $\tilde{Z} \to \mathbb{C}$ corresponds to the natural epimorphism of vector bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}$. In this picture for $\tilde{X}_m$, the exceptional curves $C_i$, $i = 1, \ldots, n$ are simply the preimages of the zero section $C \subset \tilde{Z}$ of

$$
\tilde{Z} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1.
$$

Now, the fibration $\tilde{Z} \to \mathbb{C}$ is well known [15] to be the complement of the fiber at infinity of the twistor family for the Taub-NUT hyperkähler metric on the surface $Y = \text{tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$. In particular, the family of twistor lines on $\tilde{Z}$ gives a $C^\infty$ trivialization $\tau: \tilde{Y} \times \mathbb{C} \to \tilde{Z}$ of the fibration $\tilde{Z} \to \mathbb{C}$, which is transverse holomorphic by construction. Note that the twistor-line trivialization identifies the surface $\tilde{Y} \times \{0\} \subset \tilde{Y} \times \mathbb{C}$ holomorphically with the zero fiber of $\tilde{Z} \to \mathbb{C}$. and $C \subset \tilde{Y} \cong \tilde{Z}_0$ can be viewed either as the zero section of $\mathcal{O}(-2)$ or as the zero section of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The $\tau$-image of the holomorphic family

$$
\begin{array}{ccc}
C \times \mathbb{C} & \hookrightarrow & \tilde{Y} \times \mathbb{C} \\
\uparrow & & \downarrow \\
\mathbb{C}
\end{array}
$$

8
gives a transverse holomorphic family of two spheres

\[ \tau(C \times \mathbb{C}) \xrightarrow{\sim} \tilde{Z} \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ C \]

which over \( 0 \in \mathbb{C} \) specializes to the holomorphic \((-1, -1)\) curve \( C \subset \tilde{Z} \). Finally, we can pull back this family to the fiber product \( \tilde{X}_m \) to obtain a transverse holomorphic family

\[ \mathcal{C} := \rho_{\tilde{Z}}^{-1}(\tau(C \times \mathbb{C})) \xrightarrow{\sim} \tilde{X}_m \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \mathbb{C} \]

of 2-spheres, which is parameterized by \( \mathbb{C} \) and includes the exceptional curves \( C_i \).

Given such a family, there is a pure geometric construction for a holomorphic function \( \mathcal{W} \) on the parameter space whose critical points coincide with the locations of the holomorphic fibers. Such a function is called a superpotential and is determined by the Abel-Jacobi map associated to the transverse holomorphic family. To construct this function for the present model, pick an arbitrary reference point \( p_0 \) in \( \mathbb{C} \) and for any point \( p \in \mathbb{C} \setminus \{p_0\} \) pick an arbitrary path \( \gamma \) joining \( p_0 \) and \( p \). For each path \( \gamma \) we can construct a canonical three-chain \( \Gamma, i = 1, \ldots, n \) with boundary

\[ \partial \Gamma = \mathcal{C}_p - \mathcal{C}_{p_0} \]

which is swept by the cycles \( \mathcal{C}_q \) in the family with \( q \in \gamma \). Then we have

\[ (7) \quad \mathcal{W} = \int_{\gamma} \Omega_{\tilde{X}_m} \]

where \( \Omega_{\tilde{X}_m} \) is a global holomorphic three-form on \( \tilde{X}_m \). Usually such a function would be multivalued because of the choices involved, but this complication does not appear in the present example because \( H_3(\tilde{X}_m, \mathbb{Z}) = 0 \). Finally, one can check by an explicit computation that \( \mathcal{W} \) agrees with \( W_m \) up to an irrelevant additive constant. This is the geometric interpretation of (6).

Using standard matrix model technology, the matrix integral (8) can be rewritten as an integral over the eigenvalues \( \{\lambda_a\}, a = 1, \ldots, N \) of \( \phi \)

\[ (8) \quad Z_m = \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{a=1}^{N} d\lambda_a \Delta(\lambda_a) e^{-\frac{1}{g_s} \sum_{a=1}^{N} W_m(\lambda_a)} \]

where

\[ \Delta(\lambda_a) = \prod_{a \neq b} (\lambda_a - \lambda_b). \]
The critical points of the classical superpotential are in one to one correspondence with partitions of the form

\[ N = N_1 + N_2 + \cdots + N_n. \]

Each such partition corresponds to a distribution of eigenvalues among the \( n \) zeros of \( W_m \). For the purpose of large \( N \) duality we are interested in the perturbative expansion of the matrix integral around such a critical point in the large \( N \) limit. The duality predicts a precise relation between this expansion and the perturbative expansion of the closed topological string on a smooth threefold \( X_l \).

Here we will only explain how duality works for genus zero amplitudes. The genus zero amplitudes of the topological open string are captured by the large \( N \) planar limit of the matrix integral. This means that we take the limits

\[ N \to \infty, \quad g_s \to 0 \quad \text{while keeping } \mu = Ng_s \text{ fixed.} \]

In this limit, the perturbative expansion of the matrix model free energy about a classical vacuum is encoded in a geometric structure associated to the semiclassical equations of motion. The semiclassical equations of motion can be obtained by applying the variational principle to the effective superpotential

\[ W^{scl} = \sum_{a=1}^{N} W_m(\lambda_a) - g_s \log \Delta(\lambda_a). \]

The distribution of eigenvalues is characterized by the resolvent

\[ \omega(x) = \frac{1}{N} \text{Tr} \left( \frac{1}{x - \phi} \right) \]

which is a rational function on the complex plane with poles at the locations of the eigenvalues. In the \( N \to \infty \) limit, the eigenvalues behave as a continuous one dimensional fluid with density \( \rho(\lambda) \) normalized so that

\[ \int_{\mathbb{R}} \rho(\lambda) d\lambda = 1. \]

The large \( N \) limit of the resolvent is

\[ \omega_{\infty}(x) = \int_{\mathbb{R}} \frac{\rho(\lambda)d\lambda}{x - \lambda}. \]

Then one can show that the semiclassical vacua at large \( N \) are in one to one correspondence to solutions to the algebraic equation

\[ \omega_{\infty}^2 - \frac{1}{\mu} W_m'(x) \omega_{\infty} = g(x) \]
where \( g(x) \) is a polynomial of degree smaller or equal to \( n - 1 \). More precisely, for each choice of \( f(x) \), one can find a large \( N \) semiclassical vacuum by solving equation (9). The distribution of eigenvalues in such a semiclassical vacuum is supported on a disjoint union of line segments \( \Gamma_1, \ldots, \Gamma_n \) centered around the roots of \( W_m(x) \). Note that equation (9) determines a hyperelliptic curve, which is a double cover of the complex plane with branch points situated at the endpoints of the segments \( \Gamma_i, i = 1, \ldots, n \). The line segments \( \Gamma_i, i = 1, \ldots, n \) are branch cuts for the double cover.

Next we claim that the genus zero free energy can be expressed in terms of period integrals of the meromorphic one-form \( \eta \) on the hyperelliptic curve. Given the branch cuts \( \Gamma_i, i = 1, \ldots, n \) we can choose \( 2(n-1) \) contours \((A_1, \ldots, A_{n-1}, B_1, \ldots, B_{n-1})\) in the complex plane \( \mathbb{C} \), which give rise to a symplectic basis of cycles on the hyperelliptic curve. The periods of \( \eta \) on the \( A \)-cycles

\[
 s^i = \frac{1}{2\pi i} \int_{A_i} \omega(x) dx
\]
determines the filling fractions associated to the cuts \( \Gamma_1, \ldots, \Gamma_{n-1} \). Note that there are only \( n - 1 \) independent filling fractions since their sum should be 1. According to [2], the integrals of the same differential \( \omega(x) dx \) compute the amount of energy necessary for moving an eigenvalue from the \( i \)-th branch cut \( \Gamma_i, i = 1, \ldots, n-1 \) to \( \Gamma_n \). Therefore, if we denote by \( \mathcal{F}^{op} \) the free energy of the matrix model, we have

\[
 \frac{\partial \mathcal{F}^{op}}{\partial s^i} = \int_{B_i} \omega(x) dx, \quad i = 1, \ldots, n-1.
\]

The main claim of large \( N \) duality is that the large \( N \) limit of the open string theory on \( \widetilde{X}_m \) is equivalent to a closed string theory on a threefold \( X_l \) of the form (1) for some \( l(x) \) of the form \( l(x) = f(x) + (W_m')(x)^2 \). The above geometric interpretation of the matrix model free energy makes this correspondence very transparent. To the spectral curve of the matrix model (9) we can associate a noncompact Calabi-Yau threefold \( X_{m,f} \) determined by the equation

\[
 uv + y^2 - (W_m')(x)^2 = f(x)
\]

where

\[
 y = 2\mu \omega(x) - W_m'(x), \quad f(x) = 4\mu g(x).
\]

This threefold is a fibration in quadrics over the complex plane with coordinate \( x \). The generic fiber is a smooth affine quadric in \( \mathbb{C}^3 \), while the fibers over the roots of the polynomial \( (W_m')(x)^2 + f(x) \) are singular quadrics with isolated \( A_1 \) singularities. As explained above, one can construct a smooth two cycle homeomorphic to a two-sphere on each smooth fiber. This cycle shrinks to a point on the singular fibers. Therefore \( X_{m,f} \) can be viewed as the total space of a family of affine quadrics which degenerate to singular \( A_1 \) quadrics at finitely many points. The smooth cycles in the smooth fibers are vanishing cycles with respect to the degenerations.
Each contour $A_i$ in the complex plane gives rise to closed three-cycle $S_i$ which is swept by the two-cycles in the fibers over the points of $A_i$. We can perform the same construction for the contours $B_i$. In this case the fiber two-cycles shrink at the endpoints of the contour resulting again in a closed three-cycle $T_i$ on $X_{m,f}$. With a suitable normalization of the global holomorphic three-form on $X_{m,f}$, one now gets that

\begin{equation}
  s^i = \int_{S_i} \Omega_{X_{m,f}}, \quad \frac{\partial F^{\text{op}}}{\partial s^i} = \int_{T_i} \Omega_{X_{m,f}}
\end{equation}

for $i = 1, \ldots, n - 1$. However these are precisely the defining relations for the special geometry holomorphic prepotential associated to the family of smooth Calabi-Yau threefolds $L$. Therefore we can conclude that the open string free energy $F^{\text{op}}$ can be interpreted in the large $N$ planar limit as the genus zero closed string free energy for a B-model on the threefold $X_{m,f}$.

# 3 Dijkgraaf-Vafa limits of compact Calabi-Yau spaces

In the previous section we have studied large N duality for a special class of extremal transitions among noncompact Calabi-Yau threefolds. The geometric features of those models enabled us to formulate and give a physical proof of the duality conjecture. In this section we will develop a general geometric framework for geometric transitions by generalizing the essential features of the local models. The first important aspect of Dijkgraaf-Vafa transitions which can be taken as a guiding principle for more general geometric situations is a stratified structure of the moduli space which will be described in detail below.

## 3.1 Stratification of moduli

The general setup for geometric transitions consists of the following data. We take $L$ to be a fixed component of the moduli space of Calabi-Yau threefolds. Let $M$ be the subvariety of $L$ parameterizing the singular threefolds with a fixed number $n$ of isolated ODPs which admit a crepant projective resolution. We will denote by $l$, $m$ the points of $L$ and respectively $M$ and by $X_l$, $X_m$ the corresponding Calabi-Yau threefolds. For a fixed generic point $m \in M$, $X_m$ may have several distinct projective crepant resolutions related by flops. Therefore the moduli space $\tilde{M}$ of the resolution is a finite cover of $M$. We denote by $\rho : \tilde{M} \rightarrow M$ the finite to one map which associates to any smooth crepant resolution the singular threefold obtained by contracting the exceptional locus. We will also denote by $\tilde{m}$ a point in $\tilde{M}$ so that $\rho(\tilde{m}) = m$. Then the extremal transition can be represented by a diagram of the form

\begin{equation}
  \begin{array}{c}
  \tilde{X}_{\tilde{m}} \\
  X_l \sim \sim \sim X_m.
  \end{array}
\end{equation}
where $X_l$ is a smooth Calabi-Yau space corresponding to a point $l \in L$, $X_l \sim \sim X_m$ is a degeneration of $X_l$ to a Calabi-Yau variety $X_m$, $m \in M$ having $n$ ordinary double points, and $\tilde{X}_m \to X_m$, $\tilde{m} \in \tilde{M}$ is a crepant quasi-projective resolution of $X_m$.

In the previous section, the connection between holomorphic Chern-Simons theory and matrix models was based on the existence of a maximally degenerate point $s \in M$ where the Calabi-Yau threefold develops a curve of $A_1$ singularities. In a general situation, we should be looking for a singular subspace $S \subset M$ characterized by the property that the Calabi-Yau threefolds parameterized by points $s \in S$ have curves of singularities. It is not known if such a boundary locus of $L$ exists in general, but from now on we will restrict our considerations to components with this property. We will discuss several examples later in this section.

Note that the threefolds $X_s$, $s \in S$ have a unique projective crepant resolution, as opposed to Calabi-Yau varieties $X_m$ parameterized by generic points $m \in M$. Therefore the moduli space $\tilde{S}$ of the resolution is a subspace of the ramification locus of the map $\rho : \tilde{M} \to M$ isomorphic to $S$. Therefore we obtain a stratified structure of the moduli spaces $L$, $\tilde{M}$ described by the following diagram

\[
\begin{array}{c}
\tilde{S} \subset \tilde{M} \\
\downarrow \downarrow \\
S \subset M \subset L.
\end{array}
\]

From now on we will refer to extremal transitions with this structure as stratified extremal transitions. Let us now discuss some examples.

### 3.2 Examples of nested moduli

Several examples of stratified extremal transitions among compact Calabi-Yau threefolds have been constructed in the literature [4, 5, 6, 16]. In all these examples, the Calabi-Yau threefolds are hypersurfaces or complete intersections in weighted projective spaces. Other examples can be obtained using the Borcea-Voisin construction.

Here we will discuss in detail one of the complete intersection models. We consider the moduli space $L$ of complete intersections in $\mathbb{P}^5$ of the form

\[
Q_2(z_i) = 0, \quad Q_4(z_i) = 0
\]

where $Q_2(z_i), Q_4(z_i)$ are homogeneous polynomials of degree 2, 4 in the projective coordinates $[z_1 : z_2 : \ldots : z_6]$. The stratification we are interested is induced by a stratification of the space of quadric polynomials $Q_2(z_i)$. Let us denote by $Z_2$, $Z_4$ the four dimensional quadric and quartic in $\mathbb{P}^5$ cut by the equations (14). If $Q_2$ is a quadric of rank three, then $Z_2$ will be singular along a linear subspace $\mathbb{P}^2 \subset \mathbb{P}^5$ which intersects a generic $Z_4$ along curve $\Sigma$ of genus $g = 3$. The resulting Calabi-Yau threefold has $A_1$ singularities at the points of $\Sigma$. If $Q_2$ is a quadric of rank four, $Z_2$ will be singular along a line $\mathbb{P}^1 \subset \mathbb{P}^4$, which intersects a generic quartic $Z_4$ in four points. Therefore the complete intersection has four isolated ODPs. If $Q_2$ is a quadric of rank greater or equal to 5, the generic complete intersection is smooth.
Therefore we obtain a stratified moduli space

\[ S \subset M \subset L \]

where \( S, M, L \) have dimensions 83, 86 and respectively 89. The general point in \( S, M, L \) is a complete intersection of a generic \( Z_4 \) and a \( Z_2 \) defined by a quadric \( Q_2 \) of rank 3, 4, and \( \geq 5 \) respectively. For future reference let us write down equations for the generic threefolds in each stratum. Up to automorphisms of \( \mathbb{P}^5 \), we can write the equation of a rank three quadric in the form

\[ z_1^2 - z_2 z_3 = 0. \]

The singular locus is cut by the equations \( z_1 = z_2 = z_3 = 0 \). The quadrics corresponding to points in \( M \) can be written as

\[ z_1^2 - z_2 z_3 + Q_1(z_4, z_5, z_6)^2 = 0 \]

where \( Q_1(z_4, z_5, z_6) \) is a homogeneous polynomial of degree one in \( z_4, z_5, z_6 \). The singular locus is given in this case by \( z_1 = z_2 = z_3 = Q_1(z_4, z_5, z_6) = 0 \). The quadrics corresponding to points in \( L \) can be written as

\[ z_1^2 - z_2 z_3 + Q_2(z_4, z_5, z_6) = 0 \]

where \( Q_2(z_4, z_5, z_6) \) is a homogeneous polynomial of degree two in \( z_4, z_5, z_6 \) which is not necessarily a perfect square.

Given a stratified extremal transition among compact Calabi-Yau threefolds, one can obtain a similar transition among noncompact spaces by linearization. This is an algebraic-geometric process which embodies the notion of local limit of a Calabi-Yau space often used in the physics literature. For a stratified extremal transition as above, we would like to perform linearization of the singular threefolds \( X_s \) along the curves of singularities \( \Sigma \). Taking into account the complete intersection presentation of these models, we can achieve this goal by linearizing the ambient projective space \( \mathbb{P}^5 \) along the subspace \( \mathbb{P}^2 \subset \mathbb{P}^5 \) cut by the equations \( z_1 = z_2 = z_3 = 0 \).

Consider the direct product \( \mathbb{C} \times \mathbb{P}^5 \) regarded as a trivial fibration over the complex line, and perform a blow-up along the subspace \( \{0\} \times \mathbb{P}^2 \). The resulting space has the structure of a fibration over \( \mathbb{C} \) in which the central fiber is reducible and consists of two components. One component \( \overline{P} \) is isomorphic to the total space of the projective bundle

\[(15) \quad \mathbb{P}(\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}) \]

over \( \mathbb{P}^2 \). The second component is isomorphic to the blow-up of \( \mathbb{P}^5 \) along \( \mathbb{P}^2 \). The two components intersect transversely along the section of \( \overline{P} \) over \( \mathbb{P}^2 \) defined by the trivial summand in \( (13) \). The linearization of \( \mathbb{P}^5 \) along \( \mathbb{P}^2 \) is given by the complement \( P \) of this section in \( \overline{P} \).

By construction, \( P \) is isomorphic to the total space of the bundle \( \mathcal{O}(1)^{\oplus 3} \) over \( \mathbb{P}^2 \).

Now let us consider a complete intersection \( X_s \) of the form \( (14) \) with a quadric polynomial of the form \( Q_2(z_i) = z_1^2 - z_2 z_3 \). The curve \( \Sigma \) of \( A_1 \) singularities on \( X_s \) is cut by the equations

\[ z_1 = z_2 = z_3 = 0, \quad Q_4(z_i) = 0. \]
Therefore $\Sigma$ is a quartic in the projective plane $\mathbb{P}^2 \subset \mathbb{P}^5$ defined by

$$P_4(z_4, z_5, z_6) \equiv Q_4(0, 0, 0, z_4, z_5, z_6) = 0.$$ 

The linearization of $X_s$ along $\Sigma$ can be accordingly described as a complete intersection in $P$. Let $x_1, x_2, x_3$ denote the canonical coordinates along the fibers of $P$, that is tautological sections of $O(1)^{\oplus 3}$ pulled back to $P$. Then the linearization of $X_s$ is given by the following equations in $P$

\begin{equation}
(16) \quad x_1^2 - x_2 x_3 = 0, \quad P_4(z_4, z_5, z_6) = 0.
\end{equation}

Therefore we obtain a moduli space $S$ of singular threefolds isomorphic to the space of quartics in $\mathbb{P}^2$. The higher strata $M, L$ can be described as moduli spaces of smoothings of $X_s$ by deforming the quadric equation as explained above.

Note that in this case the linearized threefolds $X_s, s \in S$ are isomorphic to a direct product of the form $\Sigma \times Y$ where $Y$ is the canonical $A_1$ surface singularity. The resolution $\bar{X}_s$ will therefore be isomorphic to a direct product $\Sigma \times \bar{Y}$, just as in the previous section. Moreover, the deformations $X_m, X_l$, as well as the resolution $\bar{X}_m$ have the structure of affine quadric bundles over $\Sigma$. Therefore the local transition obtained by linearization is a direct generalization of the Dijkgraaf-Vafa transitions. This motivates calling such models Dijkgraaf-Vafa limits of compact Calabi-Yau spaces. We will construct below a more general class of extremal transitions among noncompact Calabi-Yau threefolds which exhibit the same properties.

### 4 Linear transitions

The local extremal transition obtained in the previous section by linearization is a special case of a general abstract construction which will be the main focus of this section. These transitions will be called linear transitions because the moduli spaces $\bar{M}$ and $L$ are vector bundles over the bottom stratum $S$ as will become clear below. First we present the geometric construction, and then explain the connection between our noncompact Calabi-Yau threefolds and Hitchin’s integrable system.

Let $\Sigma$ denote a smooth projective curve of genus $g \geq 2$. Take $S$ to be the moduli space of pairs $(\Sigma, V)$, where $V \to \Sigma$ is a semi-stable rank two bundle equipped with a fixed isomorphism $\Lambda^2 V \simeq K_\Sigma$. For each such pair $s = (\Sigma, V)$, we can construct a singular Calabi-Yau threefold $X_s$ as follows. Let $T_s$ be the total space of $V$ and let $\xi_s : T_s \to T_s$ be the holomorphic involution acting by multiplication by $(-1)$ on each fiber. Then we take $X_s$ to be the quotient $T_s/(\xi_s)$, where $(\xi_s)$ denotes the finite group with two elements generated by $\{1, \xi_s\}$. Note that $\xi_s$ fixes the zero section of $V \to \Sigma$, therefore $X_s$ has a curve of $A_1$ singularities isomorphic to $\Sigma$. Blowing up $X_s$ along its singular locus $\Sigma \subset X_s$ yields a canonical quasi-projective crepant resolution $\bar{X}_s \to X_s$ of the singularities of $X_s$. The exceptional locus $S_s$ of the map $\bar{X}_s \to X_s$ is the projectivization of the normal cone of $\Sigma \subset X_s$ and so can be identified with the geometrically ruled surface $\mathbb{P}(V)$. For future
reference, note that \( \tilde{X}_s \) can be naturally identified with the total space of a line bundle over \( S_s \), so that the exceptional divisor \( S_s \subset \tilde{X}_s \) becomes the zero section of this line bundle. Indeed, this follows by noting the blow-up of the vertex of a two dimensional affine quadric is the total space of \( \mathcal{O}_{P^1}(-2) \). Explicitly we get \( \tilde{X}_s \cong \text{Tot}(\mathcal{O}_{S_s}(-2)) \).

In order to describe \( M \) and \( L \) we have first to understand the deformation theory of a singular threefold \( X_s \) as well as the deformation theory of its resolution \( \tilde{X}_s \).

### 4.1 Deformations of singular threefolds

For simplicity we will drop the subscript \( s \) in this subsection, denoting a singular threefold by \( X \). The small resolution \( \tilde{X}_s \) will be denoted by \( \tilde{X} \). The space of infinitesimal deformations of \( X \) is \( \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \). The local to global spectral sequence yields the following exact sequence

\[
0 \to H^1(\text{Ext}^0_X(\Omega^1_X, \mathcal{O}_X)) \to \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(\text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)) \\
\to H^2(\text{Ext}^0(\Omega^1_X, \mathcal{O}_X))
\]

The first term of this sequence \( H^1(\text{Ext}^0_X(\Omega^1_X, \mathcal{O}_X)) \) parameterizes equisingular deformations of \( X \), therefore it is isomorphic to the tangent space \( T_S S \) to the bottom stratum \( S \). Our main interest in this section is in the quotient space

\[
\frac{\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)}{H^1(\text{Ext}^0_X(\Omega^1_X, \mathcal{O}_X))} \cong \ker \left( H^0(\text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)) \to H^2(\text{Ext}^0(\Omega^1_X, \mathcal{O}_X)) \right)
\]

which parameterizes general deformations of \( X_s \) modulo equisingular deformations.

We claim that for the singular threefolds \( X = T/\langle \xi \rangle \) constructed above, this space is isomorphic to the space of holomorphic quadratic differentials on \( \Sigma \)

\[
H^0(\Sigma, K^\otimes 2_{\Sigma}).
\]

In order to prove this claim, it will be useful to find a presentation of \( X \) as a singular hypersurface in the total space of a holomorphic bundle over \( \Sigma \). We will first explain this construction in a simplified situation when \( V \) is complex vector space of dimension two and \( \xi \) is a holomorphic involution acting on \( V \) by multiplication by \( -1 \). In this case \( V/\langle \xi \rangle \) can be realized as a hypersurface in the complex vector space \( W = S^2(V) \). The map \( V/\langle \xi \rangle \to W \) is given by the \( \xi \)-invariant polynomials on \( V \). Explicitly we can describe this map as follows.

The second symmetric tensor power \( S^2(V) \) is a subspace of \( \text{Hom}(V^\vee, V) \). Given a linear map \( \phi : V^\vee \to V \), there is an induced determinant map

\[
det(\phi) := \wedge^2 \phi \Lambda^2(V^\vee) \to \Lambda^2(V).
\]

Let us write an arbitrary element of \( S^2(V) \) in the form

\[
\phi = u e_1 \otimes e_1 + v (e_1 \otimes e_2 + e_2 \otimes e_1) + w e_2 \otimes e_2
\]
where \( \{ e_1, e_2 \} \) is a basis of \( V \). Let \( \{ f^1, f^2 \} \) be the dual basis of \( V^\vee \). Then a straightforward linear algebra computation shows that

\[
\det(\phi)(f^1 \wedge f^2) = (uw - v^2)e_1 \wedge e_2.
\]

Therefore the hypersurface

\[
\det(\phi) = 0
\]

is isomorphic to the canonical \( A_1 \) singularity \( V/(\xi) \).

If \( V \) is a rank two bundle over \( \Sigma \), we can perform this construction fiberwise, obtaining a hypersurface in the total space \( W \) of the second symmetric power \( S^2(V) \). Let us denote by \( \pi_W : W \to \Sigma \) the projection map. Then the above computation shows that there is a tautological section \( \det_W \) of the pull-back bundle

\[
\pi_W^* \text{Hom}_W(\Lambda^2 V^\vee, \Lambda^2 V) = \pi_W^*(\Lambda^2 V)^{\otimes 2} \simeq \pi_W^* K_{\Sigma}^{\otimes 2}
\]

to the total space \( W \). The zero locus of this section is a singular hypersurface \( X \) in \( W \) isomorphic to \( V/(\xi) \). The singular locus of \( X \) is isomorphic to \( \Sigma \).

Using this presentation of \( X \), we can compute the space of normal deformations \( [18] \).

We have an exact sequence

\[
\mathcal{J}_X/\mathcal{J}_X^2 \to \Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{O}_X \to \Omega^1_X \to 0.
\]

Since \( X \) is a Cartier divisor in \( W \), it follows that the first map in this sequence is injective. Therefore we obtain an exact sequence

\[
0 \to \mathcal{O}_W(-X)|_X \to \Omega^1_W|_X \to \Omega^1_X \to 0.
\]

Thus the complex \([\mathcal{O}_W(-X)|_X \to \Omega^1_W|_X]\) is a resolution of the sheaf of Kähler differentials \( \Omega^1_X \) by locally free \( \mathcal{O}_X \)-modules. The Ext sheaves \( \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \) are the sheaf cohomology groups of the dual complex

\[
\Theta_{W|X} \xrightarrow{f} \mathcal{O}_W(X)|_X
\]

obtained from \([\mathcal{O}_W(-X)|_X \to \Omega^1_W|_X]\) by applying the functor \( \text{Hom}_X(\ , \mathcal{O}_X) \). Here \( \Theta_W := \text{Hom}(\Omega^1_X, \mathcal{O}_W) \) denotes the sheaf of holomorphic tangent vectors to \( W \). Computing locally we see that the first local Ext sheaf (= the sheaf cokernel of \( f \)) is given by

\[
\text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \simeq \mathcal{O}_X(X)|_{\Sigma}.
\]

By construction,

\[
\mathcal{O}_X(X)|_{\Sigma} \simeq K_{\Sigma}^{\otimes 2}
\]

therefore we find that

\[
H^0(X, \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)) \simeq H^0(\Sigma, K_{\Sigma}^{\otimes 2})
\]
is the space of holomorphic quadratic differentials on $\Sigma$.

In order to finish the computation, we have to determine the kernel of the map

$$(23) \quad H^0(X, \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)) \to H^2(X, \text{Ext}^0_X(\Omega^1_X, \mathcal{O}_X)).$$

Note that the projection $\pi : X \to \Sigma$ is an affine map, therefore we have

$$H^2(X, \mathcal{F}) = H^2(\Sigma, \pi_* \mathcal{F}) = 0$$

for any coherent sheaf $\mathcal{F}$ on $X$. Therefore the map (23) is trivial, and we obtain the desired result.

### 4.2 Deformations of the resolution

We now turn to the deformation theory of the resolution $\tilde{X}_s$. By analogy with the previous subsection, our aim is to understand all deformations of $\tilde{X}_s$ modulo the deformations of the pair $(\tilde{X}_s, S)$. We will keep using the notation of the previous paragraph omitting the subscript $s$.

The infinitesimal deformations of the pair $(\tilde{X}, S)$ are parameterized by the first hypercohomology group of the two term complex

$$\mathcal{T} : \Theta_{\tilde{X}} \to i_* S_{/\tilde{X}}$$

where the first term is placed in degree zero and $i_S : S \hookrightarrow \tilde{X}$ is the natural inclusion. The hypercohomology spectral sequence reduces to an exact sequence which reads in part

$$\cdots \longrightarrow H^0(\tilde{X}, N_{S/\tilde{X}})$$

$$\mathbb{H}^1(\mathcal{T}) \longrightarrow H^1(\tilde{X}, \Theta_{\tilde{X}}) \longrightarrow H^1(\tilde{X}, N_{S/\tilde{X}})$$

$$\mathbb{H}^2(\mathcal{T}) \longrightarrow H^2(\tilde{X}, \Theta_{\tilde{X}}) \longrightarrow \cdots$$

By construction, the normal bundle $N_{S/\tilde{X}}$ is isomorphic to the canonical bundle $K_S$ of $S$. Since $S = \mathbb{P}(V) = \text{Proj}(S^* V^\vee)$ is a projective bundle over $\Sigma$, we have

$$K_S = q^* K_\Sigma \otimes \omega_{S/\Sigma}$$

$$(24) \quad \simeq q^* (K_\Sigma \otimes \wedge^2 V^\vee) \otimes O_S(-2) = O_S(-2)$$

where $O_S(1)$ denotes the relative hyperplane bundle of $S$ over $\Sigma$, and we have used the relation [17, Chapter III.8, exercise 8.4]

$$\omega_{S/\Sigma} \simeq q^* \Lambda^2 V^\vee \otimes O_S(-2).$$
This shows that

\[ H^0(\tilde{X}, N_{S/\tilde{X}}) \simeq H^0(S, K_S) = 0. \]

In addition, it is not hard to check that the map

\[(25) \quad \mathbb{H}^2(\tilde{X}, T) \to H^2(\tilde{X}, \Theta_{\tilde{X}})\]

is injective. Indeed, by definition \( \Theta_{\tilde{X}} \to i_{S*}N_{S/\tilde{X}} \) is surjective. The kernel sheaf \( \Theta_{X,S} = \ker[\Theta_{\tilde{X}} \to i_{S*}N_{S/\tilde{X}}] \) is locally free and is the sheaf of germs of vector fields on \( \tilde{X} \) that at the points of \( S \) are tangent to \( S \). Write \( a : X \to S \) for the natural projection. As explained at the beginning of section [4], the resolution \( \tilde{X} \) of \( X \) is the total space of the line bundle \( \mathcal{O}_S(-2) \cong K_S \cong N_{S/\tilde{X}} \) on \( S \). In particular, the vertical tangent bundle \( \Theta_{\tilde{X}/S} \) can be identified with the line bundle \( a^*K_S \) and so the tangent sequence for the map \( a : \tilde{X} \to S \) reads:

\[(26) \quad 0 \to a^*K_S \to \Theta_{\tilde{X}} \xrightarrow{da} a^*T_S \to 0.\]

Combining (26) with the normal-to-\( S \) sequence

\[ 0 \to \Theta_{\tilde{X},S} \to \Theta_{\tilde{X}} \to i_{S*}N_{S/\tilde{X}} \to 0 \]

we obtain a commutative diagram with exact rows and columns:

\[(27) \quad \begin{array}{ccc}
0 & \to & a^*K_S(-S) \\
\downarrow & & \downarrow \\
\Theta_{\tilde{X},S} & \xrightarrow{da} & a^*T_S \\
\downarrow & & \downarrow \\
i_{S*}N_{S/\tilde{X}} & \xrightarrow{i_{S*}N_{S/\tilde{X}}} & i_{S*}N_{S/\tilde{X}} \\
0 & & 0
\end{array} \]

Since \( S \subset \tilde{X} = \text{Tot}(K_S) \) is identified with the zero section of \( K_S \), it follows that the tautological section of \( a_*K_S \) vanishes precisely at \( S \), and so \( a^*K_S \cong \mathcal{O}_{\tilde{X}}(S) \). Consequently,
Looking at the long exact sequences in cohomology for the first two rows we get the diagram (27) becomes

\[
\begin{array}{cccccc}
0 & 0 & \Theta_{\tilde{X},S} & a^*T_S & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & a^*K_S & \Theta_{\tilde{X}} & a^*T_S & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
i_S*NS/\tilde{X} & i_S*NS/\tilde{X} & 0 & 0 & \\
\end{array}
\]

Looking at the long exact sequences in cohomology for the first two rows we get

\[
\begin{array}{cccc}
H^1(a^*T_S) & \rightarrow & H^2(O_{\tilde{X}}) & \rightarrow \Theta_{\tilde{X},S} & H^2(a^*T_S) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(a^*T_S) & \rightarrow & H^2(a^*K_S) & \rightarrow \Theta_{\tilde{X}} & H^2(a^*T_S) \\
\end{array}
\]

and so the map \(\mathbb{H}^2(T) = H^2(\Theta_{\tilde{X},S}) \rightarrow H^2(\Theta_{\tilde{X}})\) will be injective if the map \(H^2(O_{\tilde{X}}) \rightarrow H^2(a^*K_S)\) is injective. Since \(a : \tilde{X} \rightarrow S\) is an affine map we have

\[
\begin{align*}
H^2(\tilde{X}, O_{\tilde{X}}) &= H^2(S, a_*O_{\tilde{X}}) = H^2(S, \oplus_{i \geq 0} K_S^{-i}) \\
H^2(\tilde{X}, a^*K_S) &= H^2(S, a_*a^*K_S) = H^2(S, \oplus_{i \geq 1} K_S^{-i}).
\end{align*}
\]

The map \(H^2(O_{\tilde{X}}) \rightarrow H^2(a^*K_S)\) is induced from the obvious inclusion of sheaves \(\oplus_{i \geq 0} K_S^{-i} \rightarrow \oplus_{i \geq 1} K_S^{-i}\) and so we have an exact sequence

\[
H^1(S, \oplus_{i \geq 1} K_S^{-i}) \rightarrow H^1(S, K_S) \rightarrow H^2(O_{\tilde{X}}) \rightarrow H^2(a^*K_S).
\]

Since the map \(H^1(S, \oplus_{i \geq 1} K_S^{-i}) \rightarrow H^1(S, K_S)\) is induced from the projection \(K_S \oplus (\oplus_{i \geq 0} K_S^{-i}) \rightarrow K_S\), it is clearly surjective and hence \(H^2(O_{\tilde{X}}) \rightarrow H^2(a^*K_S)\) must be injective. This implies the injectivity of \(\mathbb{H}^2(T) \rightarrow H^2(\Theta_{\tilde{X}})\) and so we get a short exact sequence of the form

\[
(28) \quad 0 \rightarrow \mathbb{H}^1(T) \rightarrow H^1(\tilde{X}, \Theta_{\tilde{X}}) \rightarrow H^1(\tilde{X}, N_{S/\tilde{X}}) \rightarrow 0.
\]

The first term in (28) parameterizes infinitesimal deformations of the pair \((\tilde{X}, S)\), hence it is isomorphic to the tangent space \(T_s\tilde{S}\) to \(\tilde{S}\) at \(s\). The middle term parameterizes infinitesimal deformations of \(\tilde{X}\) regardless of the behavior of \(S\). Therefore the quotient

\[
(29) \quad \frac{H^1(\tilde{X}, \Theta_{\tilde{X}})}{\mathbb{H}^1(T)} \simeq H^1(\tilde{X}, i_S*NS/\tilde{X})
\]
parameterizes infinitesimal deformations of $\tilde{X}$ not preserving $S$ modulo deformations of the pair $(\tilde{X}, S)$. Using the fact that $K_S = O_S(-2)$ and the Leray spectral sequence, we can easily compute
\begin{equation}
H^1(\tilde{X}, N_{S/\tilde{X}}) \simeq H^1(S, K_S) \simeq H^0(\Sigma, K_\Sigma).
\end{equation}

4.3 Higher strata

Let us now construct the higher strata $\tilde{M}, L$. Note that assuming these spaces exist, the quotient spaces $[13], [29]$ are isomorphic to the fibers of the normal bundle $N_{S/L}$ and respectively $N_{\tilde{S}/\tilde{M}}$ at $s$. In this section we will show that these infinitesimal normal deformations can be integrated to finite linear deformations. More precisely, we will construct families of noncompact Calabi-Yau manifolds parameterized by

$$L = S \times H^0(\Sigma, K_\Sigma^\otimes 2), \quad \tilde{M} = \tilde{S} \times H^0(\Sigma, K_\Sigma)$$

and the quadratic map $\Pi : \tilde{M} \to L$ which form a diagram of the form $[13]$.

We start with the construction of $\tilde{M}$. The main observation here is that the space $H^1(S, K_S) = H^0(\Sigma, K_\Sigma)$ parameterizes deformations of the canonical bundle of $S$ as an affine bundle over $S$. For a fixed $S$ there is a linear family of such deformations which can be constructed synthetically as follows.

An element $\alpha \in H^0(\Sigma, K_\Sigma) = H^1(S, K_S) \simeq \text{Ext}^1(O_S, K_S)$ determines (up to isomorphism) an extension
\begin{equation}
0 \to K_S \to E_\alpha \to O_S \to 0,
\end{equation}
where $E_\alpha$ is a locally free sheaf of rank two on $S$. Let $\tilde{X}_\alpha$ be the total space of the $\mathbb{P}^1$ bundle $\mathbb{P}(E_\alpha)$ over $S$, and let $\tilde{X}_\alpha$ be the complement of the infinity section $H_\alpha := \mathbb{P}(K_S) \subset \mathbb{P}(E_\alpha)$. Then $\tilde{X}_\alpha$ is an affine bundle over $S$, or more formally an $K_S$-torsor over $S$. The threefolds $\tilde{X}_\alpha$ form a linear family

$$\tilde{M} \to \tilde{M}_s, \quad \tilde{M}_s := H^0(\Sigma, K_\Sigma).$$

of noncompact Calabi-Yau manifolds.

For future reference let us summarize some elementary geometric properties of the generic fiber $\tilde{X}_\alpha$. Assume that $\alpha$ has distinct simple zeroes. Let $\pi_\alpha : E_\alpha \to \Sigma$ denote the projection map to $\Sigma$. By construction, the restriction of $E_\alpha$ to a generic fiber $S_p$, $p \notin \text{div}(\alpha)$ is the unique (up to isomorphism) nontrivial extension

$$0 \to O(-2) \to O(-1) \oplus O(-1) \to O \to 0$$

of $O$ by $O(-2)$ over $\mathbb{P}^1$. The restriction of $E_\alpha$ to a special fiber $S_p$ of the ruling with $p \in \text{div}(\alpha)$ is the trivial extension

$$0 \to O(-2) \to O(-2) \oplus O \to O \to 0$$

21
Therefore the generic fibers of $\pi_\alpha : E_\alpha \to \Sigma$ are isomorphic to the total space of the rank two bundle

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1,$$

whereas the special fibers are isomorphic to the total space of the rank two bundle

$$\mathcal{O}(-2) \oplus \mathcal{O} \to \mathbb{P}^1.$$

It follows that the projective bundle $\mathbb{P}(E_\alpha)$ is a projective quadric fibration $\pi_\alpha : \mathbb{P}(E_\alpha) \to \Sigma$ with generic fibers isomorphic to $\mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and special fibers isomorphic to the Hirzebruch surface $\mathbb{F}_2$.

Recall that the noncompact threefold $\widetilde{X}_\alpha$ is the complement in $\overline{X}_\alpha$ of the section at infinity $H_\alpha = \mathbb{P}(K_\Sigma)$. From the point of view of the fibration structure over $\Sigma$, $H_\alpha$ intersects the generic $\mathbb{F}_0$ fiber of $\pi_\alpha : \mathbb{P}(E_\alpha) \to \Sigma$ along a $(1,1)$ curve and the special $\mathbb{F}_2$ fibers along a section of $\mathbb{F}^2 \to \mathbb{P}^1$ of self-intersection $+2$. Therefore the noncompact threefold $\widetilde{X}_\alpha$ contains $2g - 2$ projective rational curves $C_1, \ldots, C_{2g-2}$ which can be identified with the negative sections of the special $\mathbb{F}_2$ fibers. A straightforward local computation confirms that each of these curves is a $(-1,-1)$ curve on $\widetilde{X}_\alpha$. Therefore for generic $\alpha$, $\widetilde{X}_\alpha$ contains exactly $2g - 2$ isolated $(-1,-1)$ curves as expected.

If $\alpha$ is non generic, that is it has multiple zeroes, a similar analysis shows that $\widetilde{X}_\alpha$ contains a projective rational curve for each zero of $\alpha$. However the curve corresponding to a double zero is a rigid $(0,-2)$ curve as opposed to a $(-1,-1)$ curve as in the generic case.

Next we will show that one can contract the exceptional curves constructed above on each $\widetilde{X}_\alpha$ obtaining a singular threefold $X_\alpha$ which depends only on $\alpha^2 \in H^0(\Sigma, K_\Sigma \otimes \mathbb{O}_\Sigma^2)$. We claim that there exists a $\mathbb{P}^2$-bundle $\overline{\pi}_\alpha : \overline{W} \to \Sigma$, so that for any $\alpha \in H^0(\Sigma, K_\Sigma)$ there exists a canonical map $\phi_\alpha : \overline{X}_\alpha \to \overline{W}$ which contracts the exceptional curves. The image $\phi_\alpha(\overline{X}_\alpha)$ is a singular hypersurface $\overline{X}_\alpha^2$ in $\overline{W}$ depending only on $\alpha^2$ sitting in a fixed (independent of $\alpha$) linear system on $\overline{W}$.

Moreover, there is a preferred hyperplane at infinity $h_\infty$ in $\overline{W}$ so that the restriction

$$\phi_\alpha|_{\overline{X}_\alpha} : \overline{X}_\alpha \to X_\alpha^2$$

is a small contraction of $\overline{X}_\alpha$ onto the noncompact nodal Calabi-Yau threefold

$$X_\alpha^2 = \overline{X}_\alpha^2 \setminus (\overline{X}_\alpha^2 \cap h_\infty).$$

To prove this claim, take $\overline{W}$ to be the $\mathbb{P}^3$-bundle $\mathbb{P}(S^2V \oplus \mathcal{O}_\Sigma)$ over $\Sigma$, and take $h_\infty$ to be the hyperplane $\mathbb{P}(S^2V)$. In order to construct the map $\phi_\alpha : \overline{X}_\alpha \to \overline{W}$ it suffices to exhibit a line bundle $\xi_\alpha$ on $\overline{X}_\alpha$ so that

$$\overline{\pi}_\alpha, \xi_\alpha \simeq (S^2V \oplus \mathcal{O}_\Sigma)^\vee = S^2(V^\vee) \oplus \mathcal{O}_\Sigma.$$

Let $\xi_\alpha$ be the relative hyperplane bundle $\xi_\alpha = \mathcal{O}_{\overline{X}_\alpha}(H_\alpha)$ for $r_\alpha : \overline{X}_\alpha \to S$. The restriction of $\xi_\alpha$ to each fiber of $r_\alpha : \overline{X}_\alpha \to S$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$ and $r_\alpha, \xi_\alpha \simeq E_\alpha^\vee$. Moreover, the
restriction of $\xi_\alpha$ to a generic $F_0$ fiber of the quadric fibration $\pi_\alpha : \mathcal{X}_\alpha \to \Sigma$ is isomorphic to $\mathcal{O}_{F_0}(1, 1)$ whereas the restriction to a special $F_2$ fiber is isomorphic to $\mathcal{O}_{F_2}(\Delta_0)$ where $\Delta_0$ denotes the zero section on $F_2$, $\Delta_0^2 = 2$.

Let us compute

$$\pi_\alpha^* \xi_\alpha = q_*(r_\alpha^* \xi_\alpha) \simeq q_* E_\alpha^\vee.$$ 

By construction, $E_\alpha^\vee$ is an extension of the form

$$0 \to \mathcal{O}_S \to E_\alpha^\vee \to K_S^{-1} \to 0$$

on $S$. Taking direct images, we find the following extension on $\Sigma$

$$0 \to \mathcal{O}_\Sigma \to q_! E_\alpha^\vee \to q_! (K_S^{-1}) \to 0$$

where we have used $R^1 q_* \mathcal{O}_S = 0$. Now $K_S \simeq \mathcal{O}_S(-2)$, hence $q_* K_S^{-1} \simeq S^2 V^\vee$. Therefore we have an extension of the form

$$(32) \quad 0 \to \mathcal{O}_\Sigma \to q_! E_\alpha^\vee \to S^2 V^\vee \to 0$$

on $\Sigma$. In order to construct a contraction map $\phi_\alpha$ we have to prove that this extension splits and we also have to choose a splitting.

First note that if $V$ is a stable rank two bundle on $\Sigma$, 

$$\text{Ext}^1(\mathcal{O}_\Sigma, S^2 V) \simeq H^0(\Sigma, S^2 V^\vee \otimes K_\Sigma) = 0$$

since $S^2 V^\vee \otimes K_\Sigma$ is a stable bundle with trivial determinant. Therefore in that case, the extension $32$ splits. If $V$ is semistable, but not stable, the extension group $\text{Ext}^1(\mathcal{O}_\Sigma, S^2 V)$ is not necessarily trivial. However, we claim that the extension (32) is still split for an arbitrary semistable bundle $V$. This claim is equivalent to the statement that the pushforward map

$$q_* : H^1(S, K_S) \to H^1(\Sigma, (q_* K_S^{-1})^\vee)$$

is trivial. Using Serre duality and respectively relative Serre duality for the map $q : S \to \Sigma$, we obtain a dual map

$$q_*^\vee : H^0(\Sigma, q_* (K_S^{-1}/\Sigma)) \to H^1(S, \mathcal{O}_S) \simeq H^1(\Sigma, \mathcal{O}_\Sigma)$$

where $K_{S/\Sigma}$ is the relative dualizing sheaf for $q : S \to \Sigma$. This map is the connecting homomorphism for the short exact sequence of sheaves

$$(33) \quad 0 \to \mathcal{O}_\Sigma \to q_*(q^* V \otimes \mathcal{O}_S(1)) \to q_*(K_{S/\Sigma}^{-1}) \to 0.$$ 

obtained by pushing forward the dual of the relative Euler sequence on $S$. We can easily compute the terms in the above exact sequence obtaining

$$0 \to \mathcal{O}_\Sigma \to V^\vee \otimes V \to \text{End}_0(V) \to 0$$
where \( \text{End}_0(V) \) is the bundle of traceless endomorphisms of \( V \). However this sequence is canonically split, hence the connecting homomorphism vanishes. Therefore we can conclude that the extension \( \text{(32)} \) is split. Choosing a splitting we obtain a (non-canoncial) isomorphism

\[
q_\ast(E^\vee_\alpha) \simeq S^2V \oplus \mathcal{O}_\Sigma
\]

which defines a map \( \phi_\alpha : \overline{X}_\alpha \to \overline{W} \) as claimed above. Since the divisor \( \mathcal{H}_\alpha \) does not intersect the exceptional curves on \( \overline{X}_\alpha \), it follows that these curves are contracted by \( \phi_\alpha \). Therefore \( \phi_\alpha \) maps \( \overline{X}_\alpha \) onto a nodal hypersurface in \( \overline{W} \).

Next, we will show that the image of \( \phi_\alpha \) moves in a fixed linear system on \( \overline{W} \). Let us fix \( \alpha \in H^0(\Sigma, K_\Sigma) \). By construction, the image of \( \phi_\alpha \) must be the zero locus of a section of a line bundle \( \mathcal{L} \) on \( \overline{W} \). Since \( \phi_\alpha : \overline{X}_\alpha \to \overline{W} \) is a small contraction, we can apply the adjunction formula obtaining

\[
K_{\overline{X}_\alpha} = \phi_\ast(K_{\overline{W}} \otimes \mathcal{L}).
\]

A routine computation using the relative Euler sequence yields

\[
K_{\overline{X}_\alpha} \simeq \xi_\alpha^{-2}, \quad K_{\overline{W}} \simeq \mathcal{O}_{\overline{W}}(-4) \otimes \pi_\ast K_\Sigma^{\otimes 2}.
\]

By direct substitution in \( \text{(34)} \), we obtain

\[
\xi_\alpha^{-2} \simeq \xi_\alpha^{-4} \otimes \pi_\ast K_\Sigma^{\otimes 2} \otimes \phi_\ast \mathcal{L}.
\]

Since this equation is valid for any value of \( \alpha \) it follows that

\[
\mathcal{L} \simeq \mathcal{O}_{\overline{W}}(2) \otimes \pi_\ast K_\Sigma^{\otimes 2}
\]

Therefore we can conclude that the hypersurfaces \( \phi_\alpha(\overline{X}_\alpha) \) belong to the linear system \( |2h_\infty + \pi_\ast K_\Sigma^{\otimes 2}| \) for any \( \alpha \).

Let us compute the space of global sections of \( \mathcal{L} \). We have

\[
\pi_\ast \mathcal{O}_{\overline{W}}(2) = S^2(\mathcal{O}_\Sigma \oplus S^2V^\vee) = \mathcal{O}_\Sigma \oplus S^2V^\vee \oplus S^2(S^2V^\vee) \]

\[
= \mathcal{O}_\Sigma \oplus S^2V^\vee \oplus S^2(\Lambda^2V^\vee) \oplus S^4V^\vee
\]

hence, using the isomorphisms \( \Lambda^2V \simeq K_\Sigma \), and \( S^4V^\vee \otimes K_\Sigma^{\otimes 2} \simeq S^4V \otimes K_\Sigma^{\otimes 2} \), we find

\[
H^0(\overline{W}, \mathcal{O}_{\overline{W}}(2) \otimes \pi_\ast K_\Sigma^{\otimes 2}) = H^0(\Sigma, \mathcal{O}_\Sigma) + H^0(\Sigma, K_\Sigma^{\otimes 2}) + H^0(\Sigma, S^2V^\vee) + H^0(\Sigma, S^4V \otimes K_\Sigma^{\otimes 2}).
\]

Let us interpret the terms in the right hand side of equation \( \text{(37)} \). By construction, a non-zero element in \( H^0(\Sigma, \mathcal{O}_\Sigma) \) viewed as a section in \( \mathcal{O}_{\overline{W}}(2) \otimes \pi_\ast K_\Sigma^{\otimes 2} \) will vanish exactly along the image \( \phi_0(\overline{X}) \) of the undeformed threefold \( \overline{X} \) in \( \overline{W} \). This is the projective completion of the singular affine hypersurface \( X \subset W \) constructed in section \( \text{4.1} \) as the zero locus of the
determinant $\det W \in H^0(W, \pi_W^* K_\Sigma^{\otimes 2})$. We will denote by $\underline{\det W}$ the section of $\mathcal{L}$ whose zero locus is $\overline{X}$. The second term parameterizes hypersurface deformations of $\overline{X}$ of the form

$$\det \underline{W} - \beta = 0$$

where $\beta \in H^0(\Sigma, \mathcal{O}_\Sigma(2K))$ is a quadratic differential on $\Sigma$. By construction, the defining equation of the hypersurface $\phi_\alpha(\overline{X}_\alpha)$ is in the affine subspace

$$(38) \quad H^0(\Sigma, \mathcal{O}_\Sigma) \oplus \{-\alpha^2\} \oplus H^0(\Sigma, S^2V).$$

All divisors in this affine subspace are isomorphic since any two divisors are related by a global automorphism of $W$ which is a translation by a section in $H^0(\Sigma, S^2V)$. In fact the affine space (38) also parameterizes the choice of a splitting of the exact sequence (32). Therefore each point $a$ in this affine space represents the image of $\overline{X}_\alpha$ through a map $\phi_{\alpha,a}$ which depends on the choice of the splitting. In particular for some point $a$ we will obtain a hypersurface in $\overline{W}$ with defining equation

$$\det \underline{W} - \alpha^2 = 0$$

We will denote this hypersurface by $\overline{X}_{\alpha^2}$. Finally, the last term in the right hand side of the decomposition (37) corresponds other deformations of $\overline{X}$ which will not be considered in this paper (note that for stable $V$ these extra deformations vanish).

This gives rise to the following picture. For a fixed point $s \in S \simeq \tilde{S}$ we obtain a linear deformation space $L_s = H^0(\Sigma, K_\Sigma^{\otimes 2})$ parameterizing noncompact Calabi-Yau threefolds $X_\beta$ determined by equations of the form

$$(39) \quad \det W - \pi_W^* \beta = 0$$

We also have a linear deformation space $\tilde{M}_s = H^0(\Sigma, K_\Sigma)$ parameterizing the threefolds $\tilde{X}_\alpha$. Moreover we have a quadratic map

$$(40) \quad \Pi_s : \tilde{M}_s \to L, \quad \alpha \to \alpha^2$$

which corresponds to a small contraction of $\tilde{X}_\alpha$. The image of this quadratic map is a singular subvariety in $H^0(\Sigma, K_\Sigma^{\otimes 2})$ isomorphic to $H^0(\Sigma, K_\Sigma)/(\pm 1)$. Note that $M_s, L_s, \tilde{M}_s$ do not depend on the point $s \in S$.

Then we can construct the higher strata $M, L, \tilde{M}$ as direct products

$$(41) \quad M = S \times H^0(\Sigma, K_\Sigma)/(\pm 1)$$

$${\tilde{M}} = \tilde{S} \times H^0(\Sigma, K_\Sigma)$$

$${M} = S \times H^0(\Sigma, K_\Sigma^{\otimes 2})$$. 25
Remark 4.1 The explicit geometric description of the non-compact Calabi-Yau spaces $X_s$, $X_l$ and $\tilde{X}_s$ above can be extended to “linear” Calabi-Yau varieties with an arbitrary $ADE$ singularity along a curve $\Sigma$. More precisely suppose $R \subset SL(2, \mathbb{C})$ is a fixed finite subgroup. We now can look at the moduli space $S$ of pairs $(\Sigma, V)$, where $\Sigma$ is a smooth curve of genus $g \geq 2$ and $V \to \Sigma$ is a rank two holomorphic vector bundle which has canonical determinant and is equipped with a fiberwise $R$-action. Again for each $s = (\Sigma, V)$ we can form the non-compact Calabi-Yau variety $X_s = \text{tot}(V)/R$. The variety $X_s$ has a curve of singularities if type $R$ and canonical minimal crepant resolution $\tilde{X}_s$. We can again look at the moduli spaces $\tilde{M}$ and $L$ of $\tilde{X}_s$ and $X_s$ and try to describe them explicitly. A uniform description of these spaces for all groups $R$ was given in [18]. Again, it turns out that $L$ and $\tilde{M}$ are total spaces of vector bundles on $S$. In [18] Szendrői identifies the fibers $L_s$ and $\tilde{M}_s$ over a point $s = (\Sigma, V)$ with the vector spaces

$$
\tilde{M}_s = H^0(\Sigma, K_\Sigma \otimes t),
L_s = H^0(\Sigma, (K_\Sigma \otimes t)/W),
$$

where $t$ and $W$ denote the Cartan algebra and the Weyl group of the complex $ADE$ group corresponding to $R$ under the McKay correspondence.

Furthermore [18] describes explicitly the universal families of deformations of $\tilde{X}_s$ and $X_s$ over $\tilde{M}_s$ and $L_s$ and shows that $\tilde{M}_s \subset L_s$ is naturally isomorphic to the cone

$$
H^0(\Sigma, (K_\Sigma \otimes t))/W \subset H^0(\Sigma, (K_\Sigma \otimes t)/W).
$$

We will analyze the large $N$ physics of these more general transversal geometries in the forthcoming paper [19] but for now we concentrate on the case $R = \mathbb{Z}/2$.

5 Intermediate Jacobians and Hitchin Pryms

As we saw in the previous section the (normal to $S$) loci $L_s \subset L$ are isomorphic to the base $H^0(\Sigma, K_\Sigma^{\otimes 2})$ of the $A_1$-Hitchin system on $\Sigma$. This raises the question whether there is a more intrinsic geometric connection between our noncompact Calabi-Yau threefolds and Hitchin systems. In this section we will give a positive answer to this question developing an intrinsic geometric relation between Hitchin Pryms and the intermediate Jacobians of the Calabi-Yau threefolds $X_\beta$, $\beta \in H^0(\Sigma, K_\Sigma^{\otimes 2})$. We begin with a brief review of the Hitchin system.

5.1 Hitchin integrable systems and Pryms

For simplicity we will consider here only the $SL(2, \mathbb{C})$ Hitchin system which is relevant for our problem. Recall that an algebraically completely integrable Hamiltonian system (ACIHS) [7,8] is defined by the following data

(i) a nonsingular complex algebraic variety $\mathcal{N}$ equipped with a non-degenerate global holomorphic $(2, 0)$ form $\sigma$
(ii) a projection \( h : \mathcal{N} \to \mathcal{B} \) where \( \mathcal{B} \) is a nonsingular complex algebraic variety so that the fibers \( \mathcal{N}_\beta \) of \( h \) are abelian varieties satisfying

(iii) \( \mathcal{N}_\beta \) is a Lagrangian subvariety of \( \mathcal{N} \) for any point \( \beta \in \mathcal{B} \).

Let us now recall the construction of the Hitchin integrable system \([20]\) following \([7, 8]\). A \( SL(2, \mathbb{C}) \) Higgs bundle \( (E, \phi) \) on \( \Sigma \) consists of a rank two holomorphic bundle on \( \Sigma \) with trivial determinant and a global section \( \phi \in H^0(\Sigma, \text{End}_0(E) \otimes K_\Sigma) \). Here \( \text{End}_0(E) \) denotes the bundle of traceless endomorphisms of \( E \). Such a pair is called stable (semistable) if there are no \( \phi \)-invariant subbundles of \( E \) that violate the usual slope inequality \([20]\). In this case, there exists a quasi-projective moduli variety \( \mathcal{M} := \mathcal{M}_{SL(2, \mathbb{C})} \) of semistable Hitchin pairs of complex dimension \( 6g - 6 \). More generally \([20]\), we can consider \( G \) Higgs bundles for a general complex reductive group \( G \). By definition these are semistable pairs \( (P, \phi) \) with \( P \) a principal \( G \)-bundle on \( \Sigma \) and \( \phi \in H^0(\Sigma, \text{ad}(P) \otimes K_\Sigma) \). Again there is a quasi-projective moduli space of such Higgs bundles and much of the discussion below generalizes to these moduli spaces. For the purposes of this paper we will ignore these more general moduli spaces with one exception, namely the moduli space \( \mathcal{M}_{\mathbb{P}GL(2, \mathbb{C})} \) of topologically trivial \( \mathbb{P}GL(2, \mathbb{C}) \) Higgs bundles on \( \Sigma \). As we will see below, the spaces \( \mathcal{M}_{SL(2, \mathbb{C})} \) and \( \mathcal{M}_{\mathbb{P}GL(2, \mathbb{C})} \) are closely related. They are the only moduli spaces of Hitchin pairs corresponding to a structure group of type \( A_1 \) and will not reappear in the next section as families of intermediate Jacobians for Calabi-Yaus in the moduli space \( L \).

The key element in the construction of the ACIHS is the notion of a spectral cover introduced in \([20]\). The spectral cover \( p_\beta : \tilde{\Sigma} = \tilde{\Sigma}_\beta \to \Sigma \) of a pair \( (E, \phi) \) is a curve in the total space of the cotangent bundle \( T^*\Sigma \) of \( \Sigma \) defined by the eigenvalue equation

\[
\det(y \cdot \text{id} - p^*\phi) = 0.
\]

Here \( p : T^*\Sigma \to \Sigma \) is the natural projection and \( y \in H^0(T^*\Sigma, p^*T^*\Sigma) \) is the tautological section. Since \( \phi \) is a traceless endomorphism of \( E \), this equation can be rewritten as

\[
y^2 - \beta = 0
\]

where \( \beta = \det(\phi) \in H^0(\Sigma, K^\otimes_\Sigma) \) is the determinant of \( \phi \). This shows that the spectral cover \( \tilde{\Sigma} \) is smooth reduced and irreducible if and only if \( \beta \) has distinct simple zeroes. Moreover, \( \tilde{\Sigma} \) is invariant under the holomorphic involution \( \iota : T^*\Sigma \to T^*\Sigma \) which acts by multiplication by \((-1)\) on each fiber.

According to \([20]\), the map

\[
h : \mathcal{M} \to H^0(\Sigma, K^\otimes_\Sigma), \quad (E, \phi) \to \det(\phi)
\]

is proper and surjective. We will denote by \( B = H^0(\Sigma, K^\otimes_\Sigma) \) the space of quadratic differentials on \( \Sigma \) and by \( \mathcal{B} \subset B \) the open subset consisting of quadratic differentials \( \beta \) with simple zeroes. Let \( \mathcal{N} = h^{-1}(\mathcal{B}) \) denote the inverse image of \( \mathcal{B} \) in \( \mathcal{M} \).
In order to determine the fiber of $h$ at a point $\beta \in \mathcal{B}$, note that a pair $(E, \phi) \in \mathcal{N}$ gives rise to a pair $(\widetilde{\Sigma}, L)$ where $L$ is a complex holomorphic line bundle on $\widetilde{\Sigma}$ of degree $2g - 2$. $L$ is defined by the property that the fiber $L_\lambda$ over a point $\lambda \in \widetilde{\Sigma}$ is the eigenspace of $\phi$ corresponding to the eigenvalue $\lambda$. Since $E$ is a $SL(2, \mathbb{C})$ rather than $GL(2, \mathbb{C})$ bundle, the resulting line bundle $L \to \Sigma_\beta$ satisfies the equivariance condition

$$\iota^* L = L^\vee \otimes p^*_0 K_{\Sigma}.$$  

One can show that there is a one-to-one correspondence between pairs $(E, \phi)$ in $\mathcal{N}$ with fixed spectral cover $\widetilde{\Sigma}$ and line bundles $L$ on $\widetilde{\Sigma}$ satisfying condition (45). This correspondence can be extended to singular, reducible or non-reduced spectral covers by allowing $L$ to be a torsion free rank one sheaf on $\widetilde{\Sigma}$.

This shows that the fiber $\mathcal{N}_\beta = h^{-1}(\beta)$ is isomorphic to the subvariety of the Picard variety $\text{Pic}^{2g-2}(\Sigma_\beta)$ of line bundles on $\widetilde{\Sigma}$ of degree $2g - 2$. This subvariety is defined by the equation (45). Equivalently it can be identified with the fiber $Nm^{-1}(\mathbb{K}_{\Sigma})$ of the norm map $Nm : \text{Pic}^{2g-2}(\Sigma) \to \text{Pic}^{2g-2}(\Sigma)$ over the canonical point $\mathbb{K}_{\Sigma} \in \text{Pic}^{2g-2}(\Sigma)$. According to [21], the degree zero Picard $\text{Pic}^0(\widetilde{\Sigma})$ decomposes up to isogeny into a direct product of Abelian varieties

$$\text{Pic}^0(\Sigma) \simeq \text{Pic}^0(\Sigma)^+ \times \text{Pic}^0(\Sigma)^-$$

where $\text{Pic}^0(\Sigma)^+$ is the fixed locus of the inversion involution $L \mapsto L^\vee$ on $\text{Pic}^0(\Sigma)$ and $\text{Pic}^0(\Sigma)^-$ is the anti-invariant part of this involution. The abelian subvariety $\text{Pic}^0(\Sigma_\beta)^-$ is usually denoted by $\text{Prym}(\Sigma/\Sigma)$ and is called the Prym variety of the spectral cover. The natural translation action of $\text{Pic}^0(\Sigma)$ on $\text{Pic}^{2g-2}(\Sigma)$ intertwines the inversion involution with the involution $\iota$ and realizes the Hitchin fiber $\mathcal{N}_\beta \subset \text{Pic}^{2g-2}(\Sigma)$ as a principal homogeneous space over the Prym variety $\text{Prym}(\Sigma/\Sigma) \subset \text{Pic}^0(\Sigma)$. So $\mathcal{N}_\beta$ is (non-canonically) isomorphic to $\text{Prym}(\Sigma/\Sigma)$.

Since we have an isomorphism $\text{Pic}^0(\Sigma) \to J(\Sigma)$ determined by the Abel-Jacobi map, it follows that the Jacobian

$$J(\Sigma) = H^0(\Sigma, \Omega^1_{\Sigma})^\vee / H_1(\Sigma, \mathbb{Z})$$

also decomposes up to isogeny into a direct sum of invariant and anti-invariant parts. The Abel-Jacobi map maps the Prym to the abelian subvariety of $J(\Sigma)^- \subset J(\Sigma)$, given by

$$J(\Sigma)^- = (H^0(\Sigma, \Omega^1_{\Sigma})^-)^\vee / H_1(\Sigma, \mathbb{Z})^-$$

where the superscript "\(^-\)" denotes the anti-invariant part. Note that

$$H^0(\Sigma, K_{\Sigma}) = H^0(\Sigma, p^* K_{\Sigma}^{\otimes 2})
= H^0(\Sigma, K_{\Sigma}^{\otimes 2} \otimes p_* O_{\Sigma})
= H^0(\Sigma, K_{\Sigma}) \oplus H^0(\Sigma, K_{\Sigma}^{\otimes 2}).$$

Under this isomorphism, the space $H^0(\Sigma, \Omega^1_{\Sigma})^- = H^0(\Sigma, K_{\Sigma})^-$ of anti-invariant holomorphic differentials on $\Sigma$ is identified with $H^0(\Sigma, K_{\Sigma}^{\otimes 2})$. Therefore $\dim \mathcal{N}_\beta = \dim \text{Prym}(\Sigma/\Sigma)$ equals the dimension $3g - 3$ of the space of quadratic differentials on $\Sigma$. 

28
So far we have constructed a family $h : \mathcal{N} \to \mathcal{B}$ of Abelian varieties over the open subset $\mathcal{B} \subset B$ so that the dimension of the fibers equals the dimension of the base. To complete the data of an ACIHS we have to construct a holomorphic symplectic form $\sigma$ on $\mathcal{N}$ so that the fibers $\mathcal{N}_\beta$ are Lagrangian cycles with respect to $\sigma$. This can be seen easily by noticing that the cotangent space to the moduli space of rank two stable bundles with trivial determinant at a point $E$ is given by the cohomology group $H^0(\Sigma, \text{End}_0(E) \otimes K_\Sigma)$. In other words the total space of the cotangent bundle to the moduli of stable bundles (of rank two and with trivial determinant) is a Zariski open and dense set in $\mathcal{M}$. Since $\mathcal{N} \subset \mathcal{M}$ is also Zariski open and dense and since cotangent bundles are naturally symplectic, we obtain a holomorphic symplectic form defined on an open dense set in $\mathcal{N}$. Hitchin showed [22] that this form extends to a holomorphic symplectic form on all of $\mathcal{N}$ and argued that all the fibers $\mathcal{N}_\beta = h^{-1}(\beta)$ are necessarily Lagrangian (see also [20, 7]).

**Remark 5.1** The holomorphic symplectic structure $\sigma$ admits also an explicit interpretation in terms of the spectral data $(\widetilde{\Sigma}, L)$. We recall this interpretation since it has a direct physical significance: it is related to the natural special Kähler geometry on the base $\mathcal{B}$ of the ACIHS $h : \mathcal{N} \to \mathcal{B}$. Let us denote by $\mathcal{V}$ the vector bundle over $\mathcal{B}$ whose sections are vertical vector fields on $\mathcal{N}$ which are constant on each torus fiber, i.e. $\mathcal{V} = h^*T_{\mathcal{N}/\mathcal{B}}$. Note that the fiber $\mathcal{V}_\beta$ is isomorphic to the space $(H^0(\widetilde{\Sigma}, \Omega^1_{\Sigma})^\vee)$, which is isomorphic in turn to $B^\vee = H^0(\Sigma, K^\otimes_\Sigma)^\vee$. On the other hand the tangent space $T_\beta \mathcal{B}$ to the base at the point $\beta$ is isomorphic to $B = H^0(\Sigma, K^\otimes_\Sigma)$. Therefore we have the isomorphisms

$$T_{\mathcal{B}}^\vee \simeq \mathcal{V} \simeq B^\vee \otimes \mathcal{O}_\mathcal{B}.$$  

The integrable structure can be characterized by the cubic criterion of Donagi and Markman [7, 8]. Let us choose a marking of the Abelian varieties $\mathcal{N}_\beta$, i.e. a continuously varying symplectic basis of $H_1(\mathcal{N}_\beta, \mathbb{Z})$, $\beta \in \mathcal{B}$. For example we can choose a symplectic basis of anti-invariant cycles in $H_1(\Sigma_\beta, \mathbb{Z})^-$, for $\beta \in \mathcal{B}$.

Then, locally on $\mathcal{B}$ the family $\mathcal{N} \to \mathcal{B}$ determines a period map $\varrho : \mathcal{B} \to \mathbb{H}_{3g-3}$ where $\mathbb{H}_{3g-3}$ denotes the Siegel upper half space

$$\mathbb{H}_{3g-3} = \{(3g - 3) \times (3g - 3) \text{ symmetric complex matrices } Z \text{ with } \text{Im}(Z) > 0\}$$

Note that we can identify $\mathbb{H}_{3g-3}$ with a subspace of $S^2B$ by choosing a basis of holomorphic quadratic differentials on $\Sigma$. Then the following conditions are equivalent [7, Lemma 7.4]

(i) There exists a holomorphic symplectic form $\sigma$ on $\mathcal{N}$ so that the fibers of $h : \mathcal{N} \to \mathcal{B}$ are Lagrangian, and $\sigma$ induces the identity isomorphism

$$\text{Id} \in \text{Hom}(T_{\mathcal{N}/\mathcal{B}}, h^*T_{\mathcal{B}}^\vee) \simeq \text{Hom}(h^*\mathcal{V}, h^*\mathcal{V}).$$
(ii) The period map $\varrho : \mathcal{B} \to S^2V$ can be locally written in $\mathcal{B}$ as the Hessian of a holomorphic function on $\mathcal{B}$ (the holomorphic prepotential.)

(iii) The differential of the period map $d\varrho \in \text{Hom}(T_{\mathcal{B}}, S^2V) \simeq V \otimes S^2V$ is a section of $S^3V$ (the cubic condition).

For the family $\mathcal{N} \to \mathcal{B}$ of $\text{SL}(2, \mathbb{C})$-Hitchin Pryms the cubic $d\varrho$ can be computed explicitly and is given by

$$d\varrho_{\beta} : S^3(T_{\beta} \mathcal{B}) \to \mathbb{C}, \quad \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \to \text{Res}^2 \left( \frac{\gamma_1 \cdot \gamma_2 \cdot \gamma_3}{\beta^2} \right).$$

where $\text{Res}^2 \left( \frac{\gamma_1 \cdot \gamma_2 \cdot \gamma_3}{\beta^2} \right)$ is the quadratic residue of a quadratic differential.

For future reference we recast the Hodge theoretic interpretation of the spectral Prym in terms of data on the base curve $\Sigma$. Recall that $\text{Prym}(\tilde{\Sigma}/\Sigma)$ was naturally identified with the kernel of the norm map $\text{Nm} : \text{Pic}^0(\tilde{\Sigma}) \to \text{Pic}^0(\Sigma)$ between the degree zero Picard varieties of $\tilde{\Sigma}$ and $\Sigma$. Topologically we have identifications $\text{Pic}^0(\tilde{\Sigma}) = H_1(\tilde{\Sigma}, \mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})$ and $\text{Pic}^0(\Sigma) = H_1(\Sigma, \mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})$. In fact these identifications can be thought of as isomorphisms of abelian varieties if we endow the right hand sides with the complex structures coming from the Hodge structures on the first homology of $\tilde{\Sigma}$ and $\Sigma$. This gives a natural topological identification

$$\text{Prym}(\tilde{\Sigma}/\Sigma) = \ker \left[ \text{Pic}^0(\tilde{\Sigma}) \xrightarrow{\text{Nm}} \text{Pic}^0(\Sigma) \right]$$

$$= \ker \left[ H_1(\tilde{\Sigma}, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z}) \right] \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z})$$

$$= H^1(\Sigma, \mathcal{K}) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}),$$

where

$$\mathcal{K} := \ker \left( p_{\beta*} \mathbb{Z}_{\tilde{\Sigma}} \xrightarrow{T_{\beta}} \mathbb{Z}_{\Sigma} \right)$$

is the kernel of the natural trace map. In the last step of this sequence:

$$0 \to H^0(\Sigma, \mathbb{Z}/n) \to H^1(\Sigma, \mathcal{K}) \to H^1(\tilde{\Sigma}, \mathbb{Z}) \to H^1(\Sigma, \mathbb{Z})$$

which implies that $H^1(\Sigma, \mathcal{K})$ agrees with $\ker(H_1(\tilde{\Sigma}, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z}))$ up to torsion, which disappears when tensoring with $\mathbb{R}/\mathbb{Z}$.

Alternatively, the sheaf $\mathcal{K}$ can be described as follows. Write $\mathcal{B} \subset \Sigma$ and $\mathcal{R} \subset \tilde{\Sigma}$ for the branch and ramification divisors of $p_{\beta}$. Let $\Sigma^o := \Sigma - \mathcal{B}$, $\tilde{\Sigma}^o := \Sigma - \mathcal{R}$, and let $p_{\beta}^o := \tilde{\Sigma}^o \to \Sigma^o$ be the restriction of the projection. If we now denote by $j : \Sigma^o \hookrightarrow \Sigma$, then $\mathcal{K}$ can be viewed
as $j_*$ of the local system on $\Sigma^o$ of anti-invariant $\mathbb{Z}$-valued functions on $\tilde{\Sigma}^o$. More invariantly, we have

$$K = (p_\beta^* \Lambda_r)^W = j_*(p_\beta^* \Lambda_r)^W,$$

where $\Lambda_r$ and $W$ denote the root lattice and Weyl group of of $SL(2, \mathbb{C})$ respectively, and we view the covering involution of $\tilde{\Sigma} \to \Sigma$ as the generator of $W$.

The analysis of the moduli space $\mathcal{M}_{PGL(2, \mathbb{C})}$ of topologically trivial $PGL(2, \mathbb{C})$ Higgs bundles is similar. In fact, the moduli space $\mathcal{M}$ determines $\mathcal{M}_{PGL(2, \mathbb{C})}$. To see this we first note that with any Hitchin pair $(E, \phi)$ consisting of a rank two vector bundle $E$ with trivial determinant and a Higgs field $\phi: E \to E \otimes K_\Sigma$ gives rise to a $PGL(2, \mathbb{C})$ Hitchin pair $(P_E, ad_\phi)$, where $P_E$ is the $PGL(2, \mathbb{C})$-bundle associated with the frame bundle of $E$ via the adjoint representation of $SL(2, \mathbb{C})$. It is easy to check that this procedure preserves semistability and so one gets a well defined morphism $ad : \mathcal{M} \to \mathcal{M}_{PGL(2, \mathbb{C})}$. Furthermore, since a principal $PGL(2, \mathbb{C})$ bundle can be viewed as a $\mathbb{P}^2$ bundle, and since on a curve all projective bundles are projectivizations of vector bundles, we see that the morphism $\mathcal{M} \to \mathcal{M}_{PGL(2, \mathbb{C})}$ is surjective. A more careful analysis (see e.g. [23]) shows that $\mathcal{M}_{PGL(2, \mathbb{C})}$ is in fact the quotient of $\mathcal{M}$ by the finite group $H^1(\Sigma, \mathbb{Z}/2) = Pic^0(\Sigma)[2]$ of 2-torsion line bundles on $\Sigma$. Here an element $\xi \in Pic^0(\Sigma)[2]$ acts as $(E, \phi) \mapsto (E \otimes \alpha, \phi \otimes id)$ and so this action preserves the fibers $N_\beta$ of the Hitchin map $h$ and the symplectic form on $\mathcal{M}$. Thus $\mathcal{M}$ and $\mathcal{M}_{PGL(2, \mathbb{C})}$ are ACIHS which are fibered by Lagrangian tori over the same base space $B = H^0(\Sigma, K_\Sigma^{\otimes 2})$ and related by a finite map that respects the symplectic forms and the Lagrangian fibrations:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{ad} & \mathcal{M}_{PGL(2, \mathbb{C})} \\
\downarrow h & & \downarrow h \\
B & & B
\end{array}$$

In particular, the finite map $ad$ gives an explicit identification of the fiber of the $PGL(2, \mathbb{C})$ Hitchin map over $\beta \in B$ with the quotient $N_\beta/ \text{Pic}^0(\Sigma)[2]$. Thus the fibers of the $SL(2, \mathbb{C})$ and $PGL(2, \mathbb{C})$ Hitchin maps over the same point $\beta \in B$ are isogenous abelian varieties.

This concludes our review of the $A_1$-Hitchin system. Next, we will explain the connection between this ACIHS and the family of noncompact threefolds constructed in the previous section.

### 5.2 Intermediate Jacobians and the Calabi-Yau integrable system

Let us start with some general considerations regarding Jacobian fibrations and integrable systems associated to families of Calabi-Yau threefolds [4, 8].

Suppose $\mathcal{B}$ is a component of an enlarged moduli space of smooth projective Calabi-Yau threefolds supporting a universal family $\mathcal{X} \to \mathcal{B}$. Recall that the enlarged moduli space
parameterizes Calabi-Yau threefolds together with a choice of a nontrivial global holomorphic three-form. The family $\mathcal{X} \to \mathcal{B}$ determines a complex torus fibration $\mathcal{J} \to \mathcal{B}$ whose fiber over a point $\beta \in \mathcal{B}$ is the intermediate Jacobian

$$J_3(X_\beta) = F^1H^3(X_\beta, \mathbb{C})^\vee / H_3(X_\beta, \mathbb{Z}).$$

Here

$$F^1H^3(X_\beta) = H^{3,0}(X_\beta, \mathbb{C}) \oplus H^{2,1}(X_\beta)$$

denotes the first step in the Hodge filtration on the third cohomology of $X_\beta$. The intermediate Jacobian has a natural non-degenerate indefinite polarization corresponding to the intersection pairing on $H_3(X_\beta, \mathbb{Z})$. Therefore $J_3(X_\beta)$ are complex analytic tori, but not Abelian varieties. It is also possible to describe the intermediate Jacobian without passing to quotients. Namely we have a natural identification:

$$J_3(X_\beta) = H_3(X_\beta, \mathbb{Z}) \otimes_\mathbb{Z} S^1,$$

where $S^1$ is the circle $S^1 = \mathbb{R}/\mathbb{Z}$. This point of view is particularly good for studying the algebraic properties of $J_3(X_\beta)$ as a torus but the complex structure on $J_3(X_\beta)$ is disguised in this interpretation.

According to [7, 8] the resulting fibration $\mathcal{J} \to \mathcal{B}$ underlies an analytically completely integrable Hamiltonian system. This is very similar to the ACIHS structure encountered in the previous section, except that we have to employ analytic spaces instead of algebraic varieties in the defining properties (i)-(iii) listed on page 26. The existence of an analytic integrable structure follows again from the cubic criterion of [7, 8]. The required cubic form is in this case the normalized Yukawa coupling (see for example [24].)

In the section 4.3 we have constructed a family of noncompact Calabi-Yau threefolds $\mathcal{X} \to \mathcal{L}$ parameterized by $\mathcal{L} = S \times B$ where $S$ is the moduli space of rank two bundles on $\Sigma$ with canonical determinant, and $B = H^0(\Sigma, K_{\Sigma}^\otimes 2)$. The threefold $X_l$, $l = (s, \beta) \in S \times B$ is smooth if and only if $\beta \in B$ has distinct simple zeroes, that is if and only if $\beta \in \mathcal{B}$. Since the dependence on the point $s \in S$ is inessential, throughout this section we will consider the family obtained by restricting $\mathcal{X} \to \mathcal{L}$ to a subspace of the form $\{s\} \times B \subset S \times B$. From now on, until the end of this section, we will drop the point $s$ from the labeling. Moreover, abusing notation we will denote by $\mathcal{X} \to \mathcal{B}$ the restriction of the family $\mathcal{X}$ to the open subset $\mathcal{B} \subset \{s\} \times B$ parameterizing smooth threefolds.

Our main goal is to establish a connection between the intermediate Jacobian fibration of the family $\mathcal{X} \to \mathcal{B}$ and the Hitchin integrable system. In contrast with the case of compact threefolds described above, there is no general result concerning the existence of an integrable structure on the intermediate Jacobian fibration of a family of noncompact Calabi-Yau manifolds. In general, these Jacobians are noncompact and may not even have the same dimension as the base of the fibration.

However in our case the fibers of the family $\mathcal{X} \to \mathcal{B}$ are simple enough so that we can analyze their intermediate Jacobians in detail. Fix a point $\beta \in \mathcal{B}$ and let $X_\beta$ be the corresponding smooth non-compact Calabi-Yau threefold. The intermediate Jacobians of
$X_\beta$ are Hodge theoretic invariants of the complex structure of $X_\beta$. They are generalized tori (= quotients of a vector space by a discrete abelian subgroup) defined in terms of the mixed Hodge structure on the cohomology or the homology of $X_\beta$. More precisely, by the work of Deligne [25] we know that the abelian group $H = H^k(X_\beta, \mathbb{Z})$ (respectively $H = H_k(X_\beta, \mathbb{Z})$) is equipped with a mixed Hodge structure $(W_\bullet, F^\bullet)$, where $W_\bullet$ is an increasing weight filtration on the rational vector space $H_\mathbb{Q} := H \otimes \mathbb{Q}$ and $F^\bullet$ is a decreasing Hodge filtration on the complex vector space $H_{\mathbb{C}} := H \otimes \mathbb{C}$.

The weight and Hodge filtrations should be compatible in the sense that $F^\bullet$ induces a Hodge decomposition of weight $\ell$ on the $\ell$-th graded piece of $\text{gr}_\ell^W = W_\ell/W_{\ell-1}$ of the weight filtration. This means that $\text{gr}_\ell^W \otimes \mathbb{C} = \oplus_{p+q=\ell} H^{p,q}$, where

$$H^{p,q} = [(F^p W_\ell + W_{\ell-1})/W_{\ell-1}] \cap [(\overline{F^p} W_\ell + W_{\ell-1})/W_{\ell-1}].$$

Equivalently the three filtrations $W_\bullet$, $F^\bullet$ and $\overline{F^\bullet}$ should satisfy

$$\text{gr}_\ell^p \text{gr}_\ell^q \text{gr}_\ell^W H = 0, \quad \text{for all } p + q \neq \ell.$$

Given a mixed Hodge structure $(H, W_\bullet, F^\bullet)$ we can consider the smallest interval $[a, b]$ such that $\text{gr}_\ell^W = 0$ for $\ell \notin [a, b]$. The integer $b - a$ is called length of the mixed Hodge structure and $a$ and $b$ are the lowest and highest weight respectively. A mixed Hodge structure of length zero is pure of weight $a = b$.

With every mixed Hodge structure $(H, W_\bullet, F^\bullet)$ one associates a sequence of intermediate Jacobians. If $(H, W_\bullet, F^\bullet)$ is a mixed Hodge structure and $p$ is any integer satisfying

$$p > \frac{1}{2}(\text{highest weight of } (H, W_\bullet, F^\bullet)),$$

then the level $p$ intermediate Jacobian of $(H, W_\bullet, F^\bullet)$ is

$$H_{\mathbb{C}}/(F^p H_{\mathbb{C}} + H).$$

The condition on $p$ here is imposed to ensure that $H_\mathbb{Z}$ projects to a discrete subgroup in $H_{\mathbb{C}}/F^p H$, i.e. that the Jacobian is a generalized torus.

Since $X_\beta$ is non-compact we will have to take extra care in distinguishing the intermediate Jacobians associated with the mixed Hodge structures on $H^3(X_\beta, \mathbb{Z})$ and $H_3(X_\beta, \mathbb{Z})$. We will denote these generalized tori by $J^3(X_\beta)$ and $J_3(X_\beta)$ respectively. Explicitly

\begin{align*}
(51) \quad J^3(X_\beta) &= H^3(X_\beta, \mathbb{C})/(F^2 H^3(X_\beta, \mathbb{C}) + H^3(X_\beta, \mathbb{Z})), \\
(52) \quad J_3(X_\beta) &= H_3(X_\beta, \mathbb{C})/(F^{-1} H_3(X_\beta, \mathbb{C}) + H_3(X_\beta, \mathbb{Z})), \\
(53) \quad &= H^3(X_\beta, \mathbb{C})/(F^2 H^3(X_\beta, \mathbb{C}) + H_3(X_\beta, \mathbb{Z})),
\end{align*}

where in the formula $\text{[53]}$ the inclusion $H_3(X_\beta, \mathbb{Z})/(\text{torsion}) \hookrightarrow H^3(X_\beta, \mathbb{C})$ is given by the intersection pairing map on three dimensional cycles in $X_\beta$. More precisely, by the universal
coefficients theorem we can identify $H^3(X_\beta, \mathbb{Z})/(\text{torsion})$ with the dual lattice $H_3(X_\beta, \mathbb{Z})^\vee := \text{Hom}_\mathbb{Z}(H_3(X_\beta, \mathbb{Z}), \mathbb{Z})$. Combining this identification with the intersection pairing on the third homology of $X_\beta$ we get a well defined map

$$i : \quad H_3(X_\beta, \mathbb{Z}) \to H^3(X_\beta, \mathbb{Z})/(\text{torsion})$$

$$L \quad \to \quad (L, \bullet)$$

which is injective on the free part of $H_3(X_\beta, \mathbb{Z})$. Combining $i$ with the natural inclusion $H^3(X_\beta, \mathbb{Z})/(\text{torsion}) \subset H^3(X_\beta, \mathbb{C})$ we obtain the map appearing in (53). Furthermore since $i$ is injective modulo torsion, it follows that the induced surjective map on intermediate Jacobians

$$(54) \quad J^3(X_\beta) \to J^3(X_\beta)$$

is a finite isogeny of generalized tori. Note that when $X_\beta$ is compact the unimodularity of the Poincare pairing implies that (54) is an isomorphism and so we do not have to worry about the distinction between $J^3(X_\beta)$ and $J^3(X_\beta)$. In fact, we will see below that for our non-compact $X_\beta$, $\beta \in \mathscr{B}$ the mixed Hodge structure on $H^3(X, \mathbb{Z})$ is actually pure and of weight 3. This implies that

$$J^3(X_\beta) = H_3(X_\beta, \mathbb{Z}) \otimes \mathbb{Z} S^1$$

$$J^3(X_\beta) = H^3(X_\beta, \mathbb{Z}) \otimes \mathbb{Z} S^1$$

and so the two Jacobians are compact complex tori. Furthermore the isogeny (54) can be identified explicitly as

$$i \otimes \text{id} : \quad H_3(X_\beta, \mathbb{Z}) \otimes \mathbb{Z} S^1 \to H^3(X_\beta, \mathbb{Z}) \otimes \mathbb{Z} S^1$$

However we will also check that the map $i \otimes \text{id}$ is not an isomorphism and has a finite kernel that can be identified explicitly.

To demonstrate the purity of the Hodge structure on $H^3(X, \mathbb{Z})$ we look at the map $\pi_\beta : X_\beta \to \Sigma$ onto the compact Riemann surface $\Sigma$. As we saw in the previous section, the fibers $X_{\beta,t} := \pi_\beta^{-1}(t)$ are smooth affine quadrics for $t$ not in the divisor of the quadratic differential $\beta \in H^0(\Sigma, K_\Sigma^\otimes 2)$ and are irreducible quadratic cones for those $t$ for which $\beta(t) = 0$. On the other hand, every two dimensional smooth affine quadric $Q \subset \mathbb{C}^3$ is deformation equivalent\(^2\) to the surface $\text{tot}(O_{\mathbb{P}^1}(-2))$. Thus by Ehresmann’s fibration theorem $Q$ is homeomorphic to $\text{tot}(O_{\mathbb{P}^1}(-2))$ which in turn is homotopy equivalent to $\mathbb{P}^1$. Therefore

$$H^0(Q, \mathbb{Z}) = \mathbb{Z} \quad H^2(Q, \mathbb{Z}) = \mathbb{Z}$$

$$H_0(Q, \mathbb{Z}) = \mathbb{Z} \quad H_2(Q, \mathbb{Z}) = \mathbb{Z}$$

\(^2\)The deformation equivalence of $Q$ and $\text{tot}(O_{\mathbb{P}^1}(-2))$ is realized explicitly by the resolved conifold $\tilde{Z}$ discussed in detail in section \(2\).
and the rest of the cohomology and homology of $Q$ vanishes. Also, under the deformation equivalence $Q \sim \text{tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$ the generator $c$ of $H_2(Q, \mathbb{Z})$ can be identified with the zero section of $\mathcal{O}_{\mathbb{P}^1}(-2)$ and so the intersection form on $H_2(Q, \mathbb{Z})$ is given by $c \cdot c = -2$. Thus the second homology $H_2(Q, \mathbb{Z})$ can be intrinsically identified with the root lattice $\Lambda_r$ of $SL(2, \mathbb{C})$. If we now use the universal coefficients theorem to identify $H^2(Q, \mathbb{Z})$ with $H_2(Q, \mathbb{Z})^\vee$ we get a natural map

$$
\begin{array}{c}
H_2(Q, \mathbb{Z}) \\
\alpha \mapsto \alpha \cdot \bullet
\end{array}
$$

given by the intersection pairing on two cycles. Since $c \cdot c = -2$, it follows that the image of $H_2(Q, \mathbb{Z})$ is a subgroup of index two in $H^2(Q, \mathbb{Z})$. In other words we get a natural isomorphism of $H^2(Q, \mathbb{Z})$ with the weight lattice $\Lambda_w$ of $SL(2, \mathbb{C})$. Now fix a base point $t_0 \in \Sigma^o = \Sigma - (\text{divisor of } \beta)$ and identify the fiber $\pi_\beta^{-1}(t_0)$ with $Q$. Choose a collection $\{\gamma_x\}_{x \in \text{div}(\beta)}$ of non-intersecting paths in $\Sigma$ connecting $t_0$ with each zero $x$ of $\beta$. Since $\pi_\beta : X_\beta \to \Sigma$ is a Lefschetz fibration, it follows that the sphere $c \subset \pi_\beta^{-1}(t_0) \cong Q$ vanishes along each $\gamma_x$ and that the local monodromy on $H_2(Q, \mathbb{Z})$ is given by the Picard-Lefschetz transformation $c \to c + (c, c) \cdot c = c - 2c = -c$. Thus the local monodromy action on $H^2(Q, \mathbb{Z}) = \Lambda_w$ is naturally equal to the action of the Weyl group $W$ on the weight lattice of $SL(2, \mathbb{C})$. Furthermore, it is not hard to compute the global monodromy

$$
\text{mon} : \pi_1(\Sigma^o, t_0) \to \{\pm 1\} = \text{Aut}(H^2(\pi_\beta^{-1}(t_0), \mathbb{Z})).
$$

Indeed, as explained at the end of section 13, the threefold $X_\beta$ is given by the equation

$$
det_W - \pi_\beta^* \beta = 0
$$

in the total space $W$ of the rank three vector bundle $S^2V$. In particular, the global monodromy (56) is the same as the monodromy of a double cover of $\Sigma$ which is branched precisely at the zeroes of $\beta$. If we further look at the compactification $\overline{X}_\beta$ we can identify the representation (55) with the monodromy on the two families of rulings in the fibers of $\overline{X}_\beta|_{\Sigma^o} \to \Sigma^o$. However from the equation (56) it is manifest that the covering parameterizing the two families of rulings is given by the equation $y^2 - \beta$ in $T^*\Sigma$, i.e. is the spectral cover $p_\beta : \tilde{\Sigma}_\beta \to \Sigma$.

This description of the local and global monodromies implies that we have a natural identification

$$
R^2\pi_\beta^* \mathbb{Z} = j_*((p_\beta^o \Lambda_w)^W) = (p_\beta^* \Lambda_w)^W,
$$

where $j : \Sigma^o \hookrightarrow \Sigma$ and $p_\beta^o : \tilde{\Sigma}^o \to \Sigma^o$ are the maps described at the end of the previous section.

Finally from the Leray spectral sequence for the map $\pi_\beta : X_\beta \to \Sigma$ we immediately see that $H^3(\Sigma, \mathbb{Z}) = H^1(\Sigma, R^2\pi_\beta^* \mathbb{Z})$ or equivalently

$$
H^3(\Sigma, \mathbb{Z}) = H^1(\Sigma, (p_\beta^* \Lambda_w)^W).
$$

Similarly, we have $R^2\pi_\beta^* \mathbb{C} = j_*((R^2\pi_\beta^o \mathbb{C})$ and $H^3(X_\beta, \mathbb{C}) = H^1(\Sigma, j_*((R^2\pi_\beta^o \mathbb{C}))$. Since the Leray filtration is compatible with mixed Hodge structures and for the affine quadric $Q$
the cohomology $H^2(Q, \mathbb{C})$ is spanned by a single class of type $(1, 1)$, it follows that the Hodge structure on $H^3(X, \mathbb{C})$ is pure of weight three. The last statement follows from the fact that the local system $R^2\pi^*_\beta \mathbb{C}$ is a variation of pure Hodge structures of Tate type and weight two, and from the fact that for every complex local system $\mathcal{U}$ on $\Sigma^o$ we have $H^1(\Sigma, j_* \mathcal{U}) = \text{im} [H^1(\Sigma^o, \mathcal{U}) \to H^1(\Sigma^o, \mathcal{U})]$. In particular the only non-trivial pieces in the Hodge decomposition on $H^3(X, \mathbb{C})$ are of Hodge types $(2, 1)$ and $(1, 2)$. Twisting $H^3(X, \mathbb{C})$ by the Tate Hodge structure of weight $(-2)$ we get a Hodge decomposition on $H^3(X, \mathbb{C})$ which involves only $(1, 0)$ and $(0, 1)$ components. Thus $J^3(X)$ and $J_3(X)$ are both abelian varieties which are dual to each other. Finally, the two intermediate Jacobians come with a canonical isogeny $J_3(X) \to J^3(X)$ which combined with the duality gives natural polarizations on the abelian varieties $J^3(X)$ and $J_3(X)$.

With all of this in place, we are now ready to compare the intermediate Jacobians $J^3(X)$ and $J_3(X)$ with the Hitchin fiber $\mathcal{N}_\beta = \text{Prym}(\bar{\Sigma}_\beta/\Sigma_\beta)$. According to equations (18), (19) and (57) we have

$$\mathcal{N}_\beta = H^1(\Sigma, (p_{\beta^*} \Lambda^r)^W) \otimes_{\mathbb{Z}} S^1,$$

$$J^3(X) = H^1(\Sigma, (p_{\beta^*} \Lambda^w)^W) \otimes_{\mathbb{Z}} S^1,$$

$$J_3(X) = H^1(\Sigma, (p_{\beta^*} \Lambda^w)^W) \otimes_{\mathbb{Z}} S^1,$$

with complex structures all coming from the Hodge decomposition on the space $H^1(\Sigma, (p_{\beta^*} \mathcal{C}_{\Sigma})^W)$, where the $W$ action on the constant sheaf $\mathcal{C}_{\Sigma} \to \bar{\Sigma}$ corresponds to the diagonal action of $W$ on $\bar{\Sigma} \times \mathbb{C}$ which is the covering involution on $\bar{\Sigma}$ and the multiplication by $(-1)$ on $\mathbb{C}$. Therefore the comparison of $J^3(X)$ and $J_3(X)$ with $\mathcal{N}_\beta$ as polarized abelian varieties amounts to a comparison of the torsion free parts of the abelian groups $H^1(\Sigma, (p_{\beta^*} \Lambda^w)^W)$ and $H^1(\Sigma, (p_{\beta^*} \Lambda^w)^W)$.

To make the comparison more explicit we choose a basis in $H^1(\bar{\Sigma}_\beta, \mathbb{Z}) = H_1(\bar{\Sigma}_\beta, \mathbb{Z})$ which is adapted to the double cover $p_{\beta} : \bar{\Sigma}_\beta \to \Sigma$. Up to homotopy we can bring all the branch points of $p_{\beta}$ inside a fixed disk $D \subset \Sigma$. Thinking of $D$ as a genus zero surface with a single boundary component, we can build the cover $\bar{\Sigma}_\beta$ topologically by first taking a double cover of $D$ corresponding to branch cuts between pairs of branch points and then attaching two copies of $\Sigma - D$ to the two boundary circles of this double cover as depicted on Figure 4 below.

Now choose a basis $\{\alpha_i\}_{i=1}^{2g}$ in $H_1(\Sigma - D, \mathbb{Z}) = H_1(\Sigma, \mathbb{Z})$ consisting of loops contained in $\Sigma - D$. The inverse image of an $\alpha_i$ in $\bar{\Sigma}_\beta$ consists of two independent disjoint loops $\alpha_i'$ and $\alpha_i''$. Adding to $\alpha_i'$ and $\alpha_i''$ the extra loops $\gamma_j$ coming from the branch cuts on $D$ we get a basis

$$\{\alpha_i', \alpha_i'', \gamma_j | i = 1, \ldots, 2g, j = 1, \ldots, 4g - 6\}$$

of $H^1(\bar{\Sigma}_\beta, \mathbb{Z})$ for which:

- Under the map $p_{\beta^*} : H^1(\bar{\Sigma}_\beta, \mathbb{Z}) \to H^1(\Sigma, \mathbb{Z})$ we have $p_{\beta}(\alpha_i') = p_{\beta^*}(\alpha_i'') = \alpha_i$, and $p_{\beta^*}(\gamma_j) = 0$. 

36
Figure 1: The $p_\beta$-adapted basis in the homology of $\tilde{\Sigma}_\beta$

- The covering involution $\iota : \tilde{\Sigma}_\beta \rightarrow \tilde{\Sigma}_\beta$ for the cover $p_\beta$ transforms this basis as $\iota(\alpha'_i) = \alpha''_i$, $\iota(\gamma_j) = -\gamma_j$.

- The classes $\alpha'_i$, $\alpha''_i$, and $\gamma_j$ do not intersect with each other.

With the basis \[\text{at our disposal we can now proceed with the comparison of} \ H^1(\Sigma, (p_\beta*\Lambda_r)^W) \text{ and } H^1(\Sigma, (p_\beta*\Lambda_r)^W). \]

Choose a generator of $\Lambda_r$ and identify $\Lambda_r$ with $\mathbb{Z}$ and $\Lambda_r \otimes \mathbb{Q}$ with $\mathbb{Q}$. Using the natural injection $\Lambda_r \rightarrow \Lambda_w$ given by the Cartan form on $\Lambda_r$ we can identify $\Lambda_w$ with the sub-lattice $\frac{1}{2}\mathbb{Z} \subset \mathbb{Q}$. With this identification the generator of the Weyl group $W$ acts as multiplication by $(-1)$. Now by the calculation we did at the end of section 5.1 we get that

$$H^1(\Sigma, (p_\beta*\Lambda_r)^W) = \left( H^1(\tilde{\Sigma}_\beta, \mathbb{Z}) \otimes \Lambda_r \right)^W = H^1(\tilde{\Sigma}_\beta, \mathbb{Z})^- = (\oplus_i \mathbb{Z}(\alpha'_i - \alpha''_i)) \oplus (\oplus_j \gamma_j)$$

$$H^1(\Sigma, (p_\beta*\Lambda_w)^W) = \left( H^1(\tilde{\Sigma}_\beta, \mathbb{Z}) \otimes \Lambda_w \right)^W = \frac{1}{2} H^1(\tilde{\Sigma}_\beta, \mathbb{Z})^- = \frac{1}{2} ((\oplus_i \mathbb{Z}(\alpha'_i - \alpha''_i)) \oplus (\oplus_j \gamma_j)).$$

Here $H^1(\tilde{\Sigma}_\beta, \mathbb{Z})^-$ denotes the anti-invariants of the action of the covering involution $\iota$.

From this identification it follows that the abelian varieties

$$\mathcal{N}_\beta = H^1(\Sigma, (p_\beta*\Lambda_r)^W) \otimes S^1$$
$$\mathcal{J}^3(X) = H^1(\Sigma, (p_\beta*\Lambda_w)^W) \otimes S^1$$

are isomorphic but the natural map between them is given by a composition of an isomorphism and a multiplication-by-two map.
Similarly we get that the dual abelian varieties $N^\vee_\beta$ and $J_3(X_\beta)$ are also isomorphic. Explicitly we compute

$$H^1(\Sigma, (p_{\beta*}\Lambda_t)^W) = \left( H^1(\tilde{\Sigma}_\beta, \mathbb{Z}) \otimes \Lambda_w \right)_W = \left( \bigoplus_i \mathbb{Z} \frac{a_i' - a_i''}{2} \right) \oplus \left( \bigoplus_j \gamma_j \right),$$

$$H^1(\Sigma, (p_{\beta*}\Lambda_w)^W) = \left( H^1(\tilde{\Sigma}_\beta, \mathbb{Z}) \otimes \Lambda_t \right)_W = 2 \left( \bigoplus_i \mathbb{Z} \frac{a_i' - a_i''}{2} \right) \oplus \left( \bigoplus_j \gamma_j \right).$$

Here for a $W$-module $\Gamma$ we write $\Gamma_W$ for the group of $W$-coinvariants in $\Gamma$. That is, $\Gamma_W$ is the quotient of $\Gamma$ by the additive subgroup in $\Gamma$ generated by all elements of the form $\gamma - w \cdot \gamma$, $w \in W$, $\gamma \in \Gamma$. In particular we see that $N^\vee_\beta = J_3(X_\beta)$ is also isomorphic to the quotient of $N_\beta = J^2(X_\beta)$ by the group Pic$^0(\Sigma)[2]$ of 2-torsion points on the Jacobian of $\Sigma$. On the other hand, as we saw in the previous section (see also [23]), the quotient $N_\beta / \text{Pic}^0(\Sigma)[2]$ is the Hitchin fiber for the Hitchin system of the Langlands dual group $\text{PGL}(2, \mathbb{C})$.

To summarize: The family of homology intermediate Jacobians associated with the family of non-compact Calabi-Yau manifolds $\mathcal{X} \to \mathcal{B}$ is an ACIHS isomorphic to the Hitchin system for the group $\text{PGL}(2, \mathbb{C})$. The family of cohomology intermediate Jacobians associated with $\mathcal{X} \to \mathcal{B}$ is an ACIHS isomorphic to the Hitchin system for the group $\text{SL}(2, \mathbb{C})$.

In [19] we will show how to generalize the above statement to an isomorphism of the Calabi-Yau integrable system of the Calabi-Yau varieties described in Remark [14] and the Hitchin integrable system for the corresponding $ADE$ group.

### 6 Large $N$ quantization for linear transitions

In this section we give a physical proof of genus zero large $N$ duality for the linear transitions constructed in section four. The open string side of the duality is constructed by wrapping topological $B$-branes on the exceptional curves in a threefold $\tilde{X}_\alpha$ corresponding to a generic point in $\tilde{M}$. The resulting topological open-closed string theory is expected to be related to closed topological string theory on a smoothing $X_l$, $l \in L$ of the nodal singularities. We will give a physical proof for this equivalence at genus zero by showing that the large $N$ dynamics of topological $B$-branes in the planar limit is governed by the Hitchin integrable system constructed in section five.

Following the strategy of the previous sections, we will fix a point $s \in S$ and work only along the normal to $S$ slice $L_s = H^0(\Sigma, K_{\Sigma}^{\otimes 2})$ of the open stratum $L$ of the moduli space. We are interested in topological open-closed string theory on a resolution $\tilde{X}_\alpha$ of a nodal threefold $X_\alpha$. Assuming $\alpha$ to be generic, let us denote by $C_1, \ldots, C_{2g-2}$ the exceptional curves on $\tilde{X}_\alpha$. We construct an open-closed topological string theory by wrapping $N_i$ $B$-type branes on the curve $C_i$, $i = 1, \ldots, 2g-2$ so that the net D-brane number $\sum_{i=1}^{2g-2} N_i$ is zero. This means that on each curve $C_i$ we may have either branes or anti-branes depending on the sign of
\(N_i\). We will denote by \(N\) the total number of branes in the system, which equals the total number of anti-branes i.e.

\[
N = \sum_{N_i > 0} N_i = -\sum_{N_i < 0} N_i.
\]

For a more precise mathematical definition of the boundary B-model, recall that topological B-branes on a Calabi-Yau space should be thought of as derived objects \([26, 27, 28]\) (see also \([29, 30, 31]\)). The off-shell dynamics of B-branes is captured by a topological string field theory whose action is a graded version of holomorphic Chern-Simons theory on \(\tilde{X}_\alpha\). For a physical proof of large \(N\) duality, we need to solve this theory at least in the large \(N\) planar limit. We will employ a strategy inspired from \([2]\). Since we are working with linearized deformations, we can write the holomorphic Chern-Simons theory on \(\tilde{X}_\alpha\) as a perturbation of holomorphic Chern-Simons on \(\tilde{X}\). The latter can be reformulated in terms of a holomorphic gauge theory on the curve \(\Sigma\) by dimensional reduction. The effect of complex structure deformations of \(\tilde{X}\) can then be taken into account by a perturbation of the gauge theory on \(\Sigma\) which is more tractable than holomorphic Chern-Simons theory on \(\tilde{X}_\alpha\).

There is however an important subtlety in this approach. The holomorphic Chern-Simons theory only captures the open string background in a fixed closed string background. The dynamics of the full-open closed topological string theory should be described in terms of holomorphic Chern-Simons theory coupled to Kodaira-Spencer theory. Although we do not have a rigorous justification, we will assume that the Kodaira-Spencer theory decouples from the holomorphic Chern-Simons theory in the genus zero sector of the theory. Therefore if we are only interested in the large \(N\) planar limit of the theory we can quantize open strings in a fixed closed string background. This assumption will be a posteriori justified by the results modulo a subtle caveat related to the integration measure for the open string theory which will be discussed in section 6.3.

### 6.1 Holomorphic Chern-Simons theory and twisted Higgs complexes

Following the general outline, let us start with holomorphic Chern-Simons theory on \(\tilde{X}\). Recall that \(\tilde{X}\) contains a ruled surface \(S\) obtained by resolving the curve \(\Sigma\) of \(A_1\) singularities of \(X\). We consider a D-brane configuration consisting of \(N\) branes and \(N\) anti-branes wrapping fibers of \(S\). For simplicity let us consider the generic case in which the branes and antibranes wrap distinct fibers of \(S\). More precisely we specify two divisors

\[
D_+ = \sum_{a=1}^{N} p_a, \quad D_- = \sum_{a=1}^{N} q_a
\]

on \(\Sigma\) so that the branes are supported on the fibers \(S_{p_a}\) and the antibranes are supported on the fibers \(S_{q_a}\). Generically we will have \(p_a \neq p_b, q_a \neq q_b\) for any \(a, b = 1 \ldots N, a \neq b\), and \(p_a \neq q_b\) for any \(a, b = 1, \ldots, N\).
The corresponding boundary topological $B$-model is given by the complex $Q = Q^+ \oplus Q^- [-1]$ where

$$Q^+ = \bigoplus_{a=1}^N O_{s_{pa}}, \quad Q^- = \bigoplus_{a=1}^N O_{s_{qa}}.$$  

The boundary chiral ring is isomorphic to the $\text{Ext}$ algebra

$$\bigoplus_{k=0}^3 \text{Ext}^k_{\widetilde{X}}(Q, Q).$$

In order to write down a physical action for off-shell fluctuations around this open string background, it is more convenient to work with a locally free resolution $E$ of $Q$. Since $E$ and $Q$ are quasi-isomorphic complexes, $E$ defines an equivalent boundary $B$-model. Then the space of off-shell open string states is given by

$$A_{\widetilde{X}} = \bigoplus_{k=0}^3 \bigoplus_{m,n} \Omega_{m,n}^k (E_m \otimes E_n).$$

Note that there is a integral ghost number grading on this vector space defined by $k + (n - m)$. The physical states are elements of ghost number $k + (n - m) = 1$. The string field theory action for the physical states is a graded version of holomorphic Chern-Simons theory [31]. This action is not very tractable for concrete practical applications.

However, since $\widetilde{X}$ is isomorphic to the total space of a line bundle over the ruled surface $S$ we can find a better starting point for large $N$ quantization invoking Koszul duality. Very briefly, in this situation Koszul duality establishes an equivalence between coherent sheaves on $\widetilde{X}$ finite over $S$ and Higgs sheaves on $S$. For convenience, recall that a Higgs sheaf on $S$ is a pair $(\mathcal{Q}, \Phi)$ where $\mathcal{Q}$ is a coherent sheaf on $S$ and $\Phi : \mathcal{Q} \to \mathcal{Q} \otimes K_S$ is a morphism from $\mathcal{Q}$ to $\mathcal{Q} \otimes K_S$. In general $\Phi$ should satisfy an integrability condition which is empty in our particular case.

From a physical point of view, a Higgs sheaf on $S$ can be interpreted as a topological $B$-brane wrapping the surface $S \subset \widetilde{X}$ as follows. For simplicity suppose that $\mathcal{Q}$ is locally free and denote by $Q$ the underlying vector bundle. Then the data $(S, Q)$ determines a topological boundary $B$-model with a nilpotent BRST symmetry. Such models have been analyzed in great detail in [32]. Their results will be very useful in the following. In particular, according to [32], the spectrum of $B$-model boundary physical states is realized as the limit of a local to global spectral sequence with second term

$$E_2^{p,q} = H^{0,p}(\text{End}(Q) \otimes \Lambda^q(N_{S/X}))$$

which converges to $\text{Ext}^k_{\widetilde{X}}(Q, Q)$, $k = p + q$. Koszul duality implies that this spectral sequence collapses at the second term, and it has a canonical split filtration so that

$$\text{Ext}^k_{\widetilde{X}}(Q, Q) = \bigoplus_{p+q=k} H^{0,p}(\text{End}(Q) \otimes \Lambda^q(N_{S/X})).$$

Since $N_{S/\widetilde{X}} \cong K_S$, this shows that instead of working with bundles on $\widetilde{X}$, it suffices to consider bundles on the compact surface $S$ as long as we take into account the Higgs field data.
It is also helpful to discuss this data from the point of view of holomorphic Chern-Simons theory. Adopting a differential geometric point of view, we can think of $Q$ as a $C^\infty$ bundle on $S$ equipped with a connection $A$ satisfying the integrability condition $F_A^{0,2} = 0$. Then the covariant Dolbeault operator $\overline{\partial}_A$ determines a holomorphic structure on $Q$. The off-shell open string states of the boundary $\mathbf{B}$-model are elements of the infinite dimensional vector space

\[(59) \quad \mathcal{A}_S = \bigoplus_{p=0}^2 \bigoplus_{q=0}^1 \Omega^0,\overline{\chi}(\text{End}(Q) \otimes \Lambda^q(N_S/\overline{X}))\]

In order to construct a string field action for the off-shell fluctuations around this background we define a DG-algebra structure on $\mathcal{A}$ as follows. We first construct the $\mathbb{Z}$-graded superalgebra

\[(60) \quad \Omega_S = \bigoplus_{p=0}^2 \bigoplus_{q=0}^1 \Omega^{0,p}_S \otimes \Omega^0,0_S \left( \Lambda^q(N_S/\overline{X}) \right)\]

where the two factors are the exterior algebras of the antiholomorphic cotangent bundle of $S$ and respectively the holomorphic normal bundle to $S$ in $\overline{X}$. The grading $\deg : \Omega_S \rightarrow \mathbb{Z}$ is defined by

\[(61) \quad \deg(\omega) = p + q, \quad \text{for } \omega \in \Omega^{0,p}_S(\Lambda^q(N_S/\overline{X})).\]

Since $\Lambda^1(N_S/\overline{X}) \simeq \Omega^2_S$, we have an isomorphism of graded vector spaces

\[(62) \quad \Omega_S \simeq \bigoplus_{p=0}^2 \bigoplus_{q=0}^1 \Omega^{2,p}_S.\]

Using this isomorphism, the superalgebra structure on $\Omega_S$ can be explicitly written in the form

\[(63) \quad \omega \omega' = (-1)^{p'q}\omega \wedge \omega'\]

where $\omega \in \Omega^{2,p}_S$, $\omega' \in \Omega^{2,q'}_S$. Then we give the space $\mathcal{A}_S$ defined in (59) a tensor product superalgebra structure of the form

\[(64) \quad \mathcal{A}_S = \Omega_S \otimes_{\Omega^{0,0}_S} \Omega^{0,0}_S(\text{End}(Q))\]

where the last factor can be regarded as a superalgebra with trivial odd component. The grading (61) extends trivially to $\mathcal{A}_S$.

Next, note that the covariant Dolbeault operator with respect to the background connection $A$ defines a degree one differential operator $\overline{\partial}^{(0)} : \mathcal{A}_S \rightarrow \mathcal{A}_S$ satisfying the Leibnitz rule

$$\overline{\partial}^{(0)}(\omega \omega') = (\overline{\partial}^{(0)} \omega) \omega' + (-1)^{\deg(\omega)} \omega (\overline{\partial}^{(0)} \omega')$$

for any $\omega, \omega' \in \mathcal{A}_S$. Therefore we obtain an associative DG-algebra structure on $\mathcal{A}_S$. 

41
In order to write down a holomorphic Chern-Simons action for off-shell fluctuations we also need a trace \( \text{tr} : \mathcal{A} \rightarrow \mathbb{C} \), which in this case is given by

\[
\text{tr} = \int_S \text{str}
\]

where \( \text{str} : \mathcal{A} \rightarrow \Omega_S \) is the supertrace. The string field action is defined for ghost number one fields \( \phi \in \mathcal{A}_S, \text{deg}(\phi) = 1 \) which parameterize arbitrary deformations \( \overline{\partial}^{(0)} \rightarrow \overline{\partial}^{(0)} + \phi \) of the \( DG \) structure on \( \mathcal{A}_S \). More precisely, \( \phi \) can be written as a sum of homogeneous elements

\[
\phi = \phi^{0,1} + \phi^{2,0},
\]

where \( \phi^{0,1} \in \Omega^{0,1}_S(\text{End}(Q)) \) is an arbitrary deformation of the background Dolbeault operator \( \overline{\partial}^{(0)} \) on \( Q \) and \( \phi^{2,0} \in \Omega^{0,0}_S(\text{End}(Q) \otimes N_{S/\tilde{X}}) \) is a Higgs field on \( S \). These are the expected off-shell \( C_\infty \) deformations of a topological \( B \)-brane supported on \( S \). Applying the general reasoning of [11] to the present case, it follows that the string field action reduces to a holomorphic Chern-Simons action on \( S \) of the form

\[
\mathcal{S}_{CS} = \int_S \text{str} \left( \frac{1}{2} \phi \overline{\partial}^{(0)} \phi + \frac{1}{3} \phi^3 \right).
\]

Substituting equation (65) in this expression we obtain

\[
\mathcal{S}_{CS} = \int_S \text{Tr} \left( \phi^{2,0} \wedge (\overline{\partial}^{(0)} \phi^{0,1} + \phi^{0,1} \wedge \phi^{0,1}) \right)
\]

\[
= \int_S \text{Tr} \left( \phi^{2,0} \wedge F^{0,2} \right)
\]

where \( F^{0,2} \) is the \((0,2)\) component of the curvature of the deformed connection \( A + \phi^{0,1} \).

Note that the action \( \mathcal{S}_{CS} \) is left invariant by gauge transformations of the string field \( \phi \) of the form

\[
\delta \phi = \overline{\partial}^{(0)} \lambda + [\phi, \lambda]
\]

where \( \lambda \in \mathcal{A} \) is an arbitrary ghost number zero field.

The solutions to the equations of motion for \( \phi \) modulo gauge transformations parameterize deformations of the boundary topological \( B \)-model. Applying the variational principle to the action (66) yields the Maurer-Cartan equation

\[
\overline{\partial}^{(0)} \phi + \phi \phi = 0.
\]

In components, we obtain

\[
\overline{\partial} \phi^{2,0} = 0, \quad F^{0,2} = 0
\]

hence the solutions are in one to one correspondence to Higgs bundle structures on a fixed underlying \( C_\infty \) bundle \( Q \). Gauge equivalent solutions correspond to isomorphic Higgs bundles, therefore we can conclude that deformations of the boundary \( B \)-model are in one to one correspondence to isomorphism classes of Higgs bundles.
In our case, we have to extend the above construction to an open string background specified by a complex \( \mathcal{Q} \) of coherent sheaves on \( S \). We first pick a locally free resolution \( E \) of \( \mathcal{Q} \) on \( S \). Note that \( \mathcal{Q} \) and \( E \) are quasi-isomorphic as complexes of coherent sheaves on \( \tilde{X} \), therefore they define equivalent boundary B-models. Then we construct a holomorphic Chern-Simons action on \( S \) for off-shell fluctuations around this open string background following the same steps. The main difference is that we will have to take into account the \( \mathbb{Z} \)-grading of the complex \( E \), as explained in a similar context in [31]. The discussion is fairly general, so in the following we can take \( \mathcal{E} \) to be an arbitrary complex of locally free sheaves.

In differential-geometric language, the open string background is specified by a finite sequence of smooth complex bundles and maps

\[
\cdots \to E_{n-1} \xrightarrow{e_{n,n-1}} E_n \xrightarrow{e_{n+1,n}} E_{n+1} \to \cdots
\]

The bundles are equipped with background connections \( A_n \) subject to the integrability condition \( F_{A_n}^{0,2} = 0 \) for all \( n \in \mathbb{Z} \). Therefore the covariant Dolbeault operators \( \overline{\partial}_n^{(0)} = \overline{\partial}_{A_n} \) define holomorphic structures on the bundles \( E_n \). The maps \( e_{n+1,n}, n \in \mathbb{Z} \) are required to be holomorphic with respect to the resulting complex structures and satisfy the condition

\[
e_{n+1,n} e_{n,n-1} = 0
\]

for all \( n \in \mathbb{Z} \).

In this context, the space of off-shell open string states is given by

\[
\mathcal{A}_S = \bigoplus_{p=0}^2 \bigoplus_{q=0}^1 \bigoplus_{m,n \in \mathbb{Z}} \Omega_S^{0,p}(\text{Hom}(E_m, E_n) \otimes \Lambda^q(N_{S/\tilde{X}}))
\]

The ghost number grading is defined by \( \text{deg} : \mathcal{A}_S \to \mathbb{Z} \),

\[
\text{deg}(\phi) = p + q + (n - m)
\]

for \( \phi \in \Omega_S^{0,p}(\text{Hom}(E_m, E_n) \otimes \Lambda^q(N_{S/\tilde{X}})) \). In order to define the correct superalgebra structure on \( \mathcal{A}_S \), we have to regard the \( \mathbb{Z} \)-graded vector bundle \( \{E_n\} \) as a \( \mathbb{Z} \)-graded supervector bundle \( \{\tilde{E}_n\} \) [31], where

\[
\tilde{E}_n = \begin{cases} (E_n, 0), & \text{for } n \text{ even} \\ (0, E_n), & \text{for } n \text{ odd.} \end{cases}
\]

Then we have a superalgebra \( \Omega_S^{0,0}(\text{End}(\tilde{E})) \) where \( \tilde{E} \) is the supervector bundle \( \tilde{E} = (E^+, E^-) \) where

\[
E^+ = \bigoplus_{n \in \mathbb{Z}} E_{2n}, \quad E^- = \bigoplus_{n \in \mathbb{Z}} E_{2n+1}.
\]

The superalgebra structure on \( \mathcal{A}_S \) is then defined by writing \( \mathcal{A}_S \) as a tensor product of superalgebras

\[
\mathcal{A}_S = \Omega_S \otimes_{\Omega_S^{0,0}} \Omega_S^{0,0}(\text{End}(\tilde{E}))
\]

Note that the Dolbeault operators \( \{\overline{\partial}_n^{(0)}\} \) define a degree one differential operator \( \overline{\partial}^{(0)} : \mathcal{A}_S \to \mathcal{A}_S \) satisfying the Leibnitz rule with respect to superalgebra multiplication. Moreover
the maps $e_{n+1,n} : E_n \to E_{n+1}$ define a degree one element $e \in A_S$. Then we define the BRST operator $D^{(0)} : A_S \to A_S$ in the background specified by the complex (68) to be

$$D^{(0)} = \delta^{(0)} + e.$$ 

Given a field $\phi \in A_S$, we have

$$D^{(0)} \phi = \delta^{(0)} \phi + [e, \phi]$$

where $[,]$ is the supercommutator

$$[\phi, \phi'] = \phi \phi' - (-1)^{\deg(\phi)\deg(\phi')} \phi' \phi.$$

Now we can write down the graded holomorphic Chern-Simons action for ghost number one open string fields $\phi \in A_S$

$$\mathcal{S}_{CS} = \int_S \text{str} \left( \frac{1}{2} \phi D^{(0)} \phi + \frac{1}{3} \phi^3 \right).$$

This action is left invariant by infinitesimal gauge transformations of the form

$$\delta \phi = (D^{(0)} + \phi) \lambda$$

where $\lambda$ is an arbitrary ghost number zero element of $A_S$. Note that the $D = D^{(0)} + \phi$ is an arbitrary off-shell deformation of the BRST operator. The equations of motion derived from the holomorphic Chern-Simons action can be written in compact form

$$D^{(0)} \phi + \phi^2 = 0,$$

which is equivalent with the integrability condition

$$D^2 = 0$$

for the deformed BRST operator. The solutions to these equations modulo gauge transformations parameterize deformations of the open string background specified by the complex (68). In the ungraded case, we identified these deformations with Higgs bundle structures on a fixed $C^\infty$ bundle up to isomorphism. The equations (72) yield a generalization of the Higgs bundle conditions which is better understood in the framework of D-brane categories, which we explain next.

So far we have been studying fluctuations around a fixed open string background. In principle one can consider more general situations in which we have several topological D-branes wrapping the surface $\hat{S}$. In that case the algebraic structure of the resulting open string theory is encoded in a triangulated D-brane category $[26, 27, 28, 29, 30]$. This category is a physical variant of the Bondal-Kapranov construction [33] introduced in the context of cubic string field theory in [29, 30].
In the present context, we start with a DG-category $C$ given as follows. The objects of $C$ are holomorphic vector bundles $E \to S$. The space of morphisms between two objects $E, E'$ is given by the complex

$$\Hom_C(E, E') = \bigoplus_{p=0}^2 \bigoplus_{q=0}^1 \Omega^0_S(p) \otimes \Lambda^q(N_{S/X})$$

where the grading is given by $p + q$ and the differential is given by the covariant Dolbeault operator $\partial(0)$. Here $E, E'$ are regarded again as $C^\infty$ bundles equipped with $(0, 1)$-connections.

Next we construct a new DG-category $\tilde{C}$ by taking the shift completion of $C$. The objects of $\tilde{C}$ are pairs $(E, n)$, where $E$ is an object of $C$ and $n \in \mathbb{Z}$. The space of morphisms between two objects $(E, n), (E', n')$ is the shifted complex

$$\Hom_{\tilde{C}}((E, n), (E', n')) = \Hom_C(E, E')[n - n'].$$

From a physical point of view, the integer $n$ represents the D-brane grading introduced in [26]. For future reference, we should also keep in mind that composition of morphisms in $\tilde{C}$ is given by

$$(gf) \tilde{C} = (-1)^{(k+n-n')(n''-n)}(gf)c,$$

for any $f \in \Hom_C^k((E, n), (E', n'))$ and any $g \in \Hom_C^l((E', n'), (E'', n'')$).

The D-brane category is the triangulated category $\text{Tr}(\tilde{C})$ of twisted complexes over $\tilde{C}$ defined in [33]. Twisted complexes over $\tilde{C}$ are finite collections of objects $\{(E_i, n_i)\}$ and degree one morphisms $\Phi_{ji} \in \Hom_{\tilde{C}}^1((E_i, n_i), (E_j, n_j))$ satisfying the Maurer-Cartan equation

$$\partial(0) \Phi_{ji} + \sum_k \Phi_{jk} \Phi_{ki} = 0.$$

Twisted complexes form a DG-category denoted by $\text{Pre-Tr}(\tilde{C})$ in [33]. The morphisms between two such objects $(E_i, n_i, \Phi_{ji})$ and $(E'_i, n'_i, \Phi'_{ji})$ are DG-complexes

$$\Hom_{\text{Pre-Tr}(\tilde{C})}^k((E_i, n_i, \Phi_{ji}), (E'_i, n'_i, \Phi'_{ji})) = \bigoplus_{i,j} \Hom_{\tilde{C}}^k((E_i, n_i), (E'_j, n'_j)).$$

The action of the differential on morphisms $\eta \in \Hom_{\tilde{C}}^1((E_i, n_i), (E'_j, n'_j))$ is defined by

$$d\eta = \partial(0)\eta + \sum_k \Phi'_{kj}\eta - (-1)^k \eta \Phi_{ik}.$$

Then one can obtain a triangulated category with same objects as $\text{Pre-Tr}(\tilde{C})$ by taking the space of morphisms between two objects to be the degree zero cohomology of the complex $\partial(0)$. For our purposes it is more convenient to work with the $\mathbb{Z}$ graded category obtained from $\text{Pre-Tr}(\tilde{C})$ by taking the space of morphisms between two objects to be the
full cohomology of the complex \( (75), (76) \). We will denote the resulting enriched triangulated category by \( \text{Tr}(\tilde{C}) \).

In order to see the connection between twisted complexes and graded holomorphic Chern-Simons theory, note that the solutions to the equation of motion \( (72) \) are in fact diagonal twisted complexes characterized by \( n_i = i \) for all \( i \).

More precisely, if \( \phi \in \mathcal{A}^1 \) is a solution to \( (72) \), one can easily check that \( \Phi = \phi + e \) satisfies the equation

\[
(77) 
\overline{\partial}^{(0)} \Phi + \Phi^2 = 0. 
\]

By construction the collection of fields

\[
\Phi_{nm} \in \Omega^{2q,p}_S(\text{Hom}(E_m, E_n))
\]

with \( p + q = 1 + m - n \) can be regarded as morphisms

\[
\Phi_{nm} \in \text{Hom}_{\tilde{C}}(E_m, E_m) \simeq \text{Hom}_{\tilde{C}}^{1+m-n}(E_m, E_n).
\]

between the bundles \( E_m, E_n \) in the category \( \tilde{C} \). Moreover, the equation of motion \( (77) \) is identical to the Maurer-Cartan equation \( (74) \). In order to check this equivalence, notice that the sign rule \( (73) \) for composition of morphisms in \( \tilde{C} \) is compatible with multiplication in the superalgebra \( (69) \). Therefore the collection \( \{E_n, \Phi_{nm}\} \) is a twisted complex with \( n_i = i \).

Keeping this correspondence in mind, from now on we will refer to the objects of \( \text{Tr}(\tilde{C}) \) as twisted Higgs complexes.

In order to apply this construction to D-branes wrapping fibers of the ruling \( S \to \Sigma \), recall that we have canonical locally free resolutions of divisors on \( S \)

\[
0 \to \mathcal{O}_S \left( -\sum_{a=1}^N S_{pa} \right) \xrightarrow{e_1} \mathcal{O}_S \to \bigoplus_{a=1}^N \mathcal{O}_{S_{pa}} \to 0
\]

\[
0 \to \mathcal{O}_S \left( -\sum_{a=1}^N S_{qa} \right) \xrightarrow{e_2} \mathcal{O}_S \to \bigoplus_{a=1}^N \mathcal{O}_{S_{qa}} \to 0.
\]

Then we can construct a complex \( \mathcal{E} \) of locally free sheaves which is quasi-isomorphic to \( \mathcal{Q} \) and has the form

\[
(78) 
0 \to \mathcal{O}_S \left( -\sum_{a=1}^N S_{qa} \right) \xrightarrow{(e_1)} \mathcal{O}_S \oplus \mathcal{O}_S \left( -\sum_{a=1}^N S_{pa} \right) \xrightarrow{(0,e_2)} \mathcal{O}_S \to 0.
\]

The graded holomorphic Chern-Simons action we are searching for given by \( (70) \) in which the locally free complex \( \mathcal{E} \) is taken to be \( (78) \).

According to the general approach explained in section 2, we need to understand the moduli space of solutions to the equations of motion of the Chern-Simons action modulo gauge transformations. In this case the problem can be further simplified by taking dimensional reduction of the action along the fibers of the ruling \( q : S \to \Sigma \). More precisely, we
pick up a Kähler metric on $S$ so that the volume of the fibers is very small compared to the volume of the base. Then we can reduce the Chern-Simons action along the fibers obtaining a two-dimensional field theory. Strictly speaking this procedure is employed in the physics literature only when $S$ is a direct product $S = \Sigma \times \mathbb{P}^1$. In fact this restriction is too severe. Dimensional reduction can be applied equally well in all cases when the projective bundle $S = \mathbb{P}(V)$, where $V \to \Sigma$ is a rank 2 holomorphic bundle, having locally constant transition functions. This will be the case if $V \to \Sigma$ is a polystable holomorphic bundle. For simplicity, we will assume in the following that $S$ is a direct product, that is $V$ is a trivial rank 2 bundle. If $V$ is non-trivial, but polystable, the result remains unchanged.

Let $\mathcal{E} = q^* \mathcal{F}$ be the pull-back of a complex of holomorphic vector bundles $\mathcal{F}$

\[ \cdots \to F_{n-1} \xrightarrow{f_{n,n-1}} F_n \xrightarrow{f_{n+1,n}} F_{n+1} \to \cdots \]

As above the bundles $F_n$ are regarded as $C^\infty$ bundles equipped with $(0, 2)$ connections $B_n$ which determine covariant Dolbeault operators $\bar{\partial}_n$. In order to perform the dimensional reduction we write the off-shell field $\phi = \sum \phi_{n,m}^{2q,p} \in \mathcal{A}$ as

\[ \phi_{n,m}^{2q,p} = \psi_{n,m}^{2q,p} \eta_0^0 + \chi_{n,m}^{2q,p-1} \eta_0^1 + \chi_{n,m}^{2q,p-1} \eta_0^0 + \psi_{n,m}^{2q-1,p-1} \eta_1^1 \]

where $\psi_{n,m}^{k,l}, \chi_{n,m}^{k,l} \in \Omega_{\Sigma}^{k,l}(\text{Hom}(F_m, F_n))$, $\eta_0^r \in \Omega_{\mathbb{P}^1}^r$. In the right hand side of (80), all forms should be pulled back to $S$. Next, we take $\eta_0^r \to$ to be solutions to the linearized equations of motion modulo gauge transformations. Since the background complex is pulled back from $\Sigma$, the linearized equations of motion read

\[ \bar{\partial}_n \eta_0^r = 0 \]

for any $r, s = 0, 1$. Therefore the space of solutions to the linearized equations of motion modulo gauge transformations is parameterized by $H^{r,s}(\mathbb{P}^1)$. Let us choose generators $1 \in H^{0,0}(\mathbb{P}^1)$, $[\eta] \in H^{1,1}(\mathbb{P}^1)$ of the nontrivial cohomology groups. Then the dimensional reduction ansatz (80) reduces to

\[ \phi_{n,m}^{2q,p} = \psi_{n,m}^{2q,p} + \eta \psi_{n,m}^{2q-1,p-1} \]

Therefore we obtain the following space of off-shell fields on $\Sigma$

\[ \mathcal{B} = \bigoplus_{q=0}^1 \bigoplus_{p=0}^1 \bigoplus_{m,n \in \mathbb{Z}} \Omega_{\Sigma}^{q,p}(\text{Hom}(F_m, F_n)) \]

where the ghost number grading is given by $\text{deg}(\psi_{n,m}^{q,p}) = 2q + p + (n - m)$. We can give $\mathcal{B}$ a $\mathbb{Z}$-graded superalgebra structure by adopting the construction used below equation (68). Let $\Omega_\Sigma$ be the $\mathbb{Z}$-graded superalgebra obtained by reducing $\Omega_S$ along the fibers of the ruling. We have

\[ \Omega_\Sigma = \bigoplus_{q=0}^1 \bigoplus_{p=0}^1 \Omega_{\Sigma}^{q,p} \]

where the multiplication is defined by

\[ \omega \omega' = (-1)^p q / \omega \land \omega' \].
Let \( \mathcal{F} = (F^+, F^-) \) be the supervector bundle obtained by rolling the \( \mathbb{Z} \)-graded vector space \( \bigoplus_{n \in \mathbb{Z}} F_n \). Then we take

\[
(83) \quad \mathcal{B} = \Omega_\Sigma \otimes \Omega^0_\Sigma \Omega^{0,0}_\Sigma (\text{End}(\mathcal{F})).
\]

The background connections \( B_n \) together with the maps \( f_{n+1, n} : F_n \to F_{n+1} \) define a differential operator \( D^{(0)} = \mathcal{B}^{(0)} + [f, ] : \mathcal{B} \to \mathcal{B} \) satisfying the Leibnitz rule. By dimensional reduction, the Chern-Simons action (70) yields the following action on \( \Sigma \)

\[
(84) \quad \mathcal{S} = \int_\Sigma \text{str} \left( \frac{1}{2} \psi D^{(0)} \psi + \frac{1}{3} \psi^3 \right).
\]

This is again left invariant by infinitesimal gauge transformations of the form

\[ \delta \psi = D\lambda \]

where \( \lambda \) is an arbitrary ghost number zero element and \( D = D^{(0)} + \psi \). The equations of motion of this action are again of the form

\[
(85) \quad D^{(0)} \psi + \psi^2 = 0.
\]

In order to facilitate the construction of the moduli space of solutions to the equations modulo gauge transformations, it is very helpful to rephrase this construction in terms of D-brane categories. One can construct a triangulated category of twisted complexes on \( \Sigma \) by performing dimensional reduction on twisted complexes on \( S \). More precisely, let us start with the DG category \( \mathcal{D} \) of holomorphic vector bundles \( F \to \Sigma \) so that the space of morphisms between two objects \( F, F' \) is given by the complex

\[
(86) \quad \text{Hom}_{\mathcal{D}}(F, F') = \bigoplus_{q=0}^{1} \bigoplus_{p=0}^{1} \Omega^{q,p}(\text{Hom}(F, F')).
\]

The grading is defined by \( p + 2q \) and the differential is given by the covariant Dolbeault operator as in the previous case. The D-brane category in question is the enriched triangulated category \( \text{Tr}(\mathcal{D}) \) of twisted complexes on \( \Sigma \) associated to the shift completion \( \mathcal{D} \). It is a straightforward exercise to check that twisted complexes on \( \Sigma \) can be obtained by dimensional reduction of twisted complexes on \( S \).

This concludes our discussion of holomorphic D-branes on \( \tilde{X} \) from a classical point of view. In order to reach our goal we have to understand the quantum dynamics of holomorphic branes on \( \tilde{X} \), at least in the large \( N \) limit.

### 6.2 Quantization and moduli space

As explained in section (2) the quantization of the holomorphic Chern-Simons theory involves a formal integral of a holomorphic measure on a middle dimensional real cycle in the space of fields. Exploiting the topological symmetry of this theory, this functional integral localizes
to a finite dimensional integral on a middle dimensional cycle in the moduli space \( \mathcal{M} \) of solutions to the classical field equations of motion.

We would like to carry out this construction for the holomorphic Chern-Simons action associated to the complex \( \mathcal{C} \). Since this complex is pulled back from \( \Sigma \), the classical moduli space can be determined using dimensional reduction and truncation to zero modes. Therefore it suffices to consider the cubic action \( \mathcal{S} \) for a complex of the form

\[
0 \to \mathcal{O}(\Sigma)(-D_+) \xrightarrow{(f)} \mathcal{O}(\Sigma) \oplus \mathcal{O}(\Sigma)(-D_-) \xrightarrow{(f_2)} \mathcal{O} \to 0
\]

where \( D_+ = \sum_{a=1}^N p_a, \quad D_- = \sum_{a=1}^N q_a \) are disjoint divisors on \( \Sigma \) and

\[
0 \to \mathcal{O}(\Sigma)(-D_+) \xrightarrow{f_1} \mathcal{O} \to 0 \quad \text{and} \quad 0 \to \mathcal{O}(\Sigma)(-D_-) \xrightarrow{f_2} \mathcal{O} \to 0
\]

are canonical maps. On common grounds the functional integral \( \mathcal{H} \) reduces to an integral over a middle dimensional real cycle on the moduli space \( \mathcal{M} \) of critical points of the action modulo gauge transformations. The holomorphic measure of the moduli space integral should be determined in principle by integrating out the massive modes, provided that the original measure \( D\psi \) is at least formally well defined. A direct approach to this problem is beyond the purpose of the present paper so we will employ a different technique. We first determine the moduli space and then find an expression for the measure using holomorphy and physical constraints. In the process we will discover a new aspect of this problem involving spin structures on \( \Sigma \).

In order to find the moduli space it is convenient to write the action \( \mathcal{S} \) in terms of the field \( \Psi = f + \psi - \psi^{0,1} \) and the Dolbeault operator \( \bar{\partial} = \bar{\partial}^{(0)} + \psi^{0,1} \). Then we have

\[
\mathcal{S}_{CS} = \int_S \text{Tr}\left( \Psi^{1,0}_0 \bar{\partial} \Psi^{0,0}_0 + \Psi^{1,0}_2 \bar{\partial} \Psi^{0,0}_2 + \Psi^{0,1}_2 \bar{\partial} \Psi^{0,0}_0 \right).
\]

The equations of motion \( \mathcal{S}_{CS} \) become

\[
\begin{align*}
\bar{\partial}_0 \Psi^{0,0}_0 &= 0 & \Psi^{0,0}_0 \Psi^{0,0}_0 &= 0 \\
\bar{\partial}_2 \Psi^{0,0}_2 &= 0 & \Psi^{0,0}_1 \Psi^{0,0}_0 &= 0 \\
\bar{\partial}_0 \Psi^{1,0}_0 + \Psi^{0,1}_2 \Psi^{0,0}_2 &= 0 & \Psi^{0,0}_0 \Psi^{1,0}_0 - \Psi^{1,0}_2 \Psi^{0,0}_2 &= 0 \\
\bar{\partial}_2 \Psi^{1,0}_2 + \Psi^{0,0}_1 \Psi^{1,1}_0 &= 0 & \Psi^{0,0}_0 \Psi^{1,0}_1 &= 0 \\
\end{align*}
\]

As explained above, our goal is to determine the moduli space of solutions to the equations of motion modulo gauge transformations. A better formulation of the problem can be achieved in the framework of D-brane categories developed in the previous subsection. We have to find the moduli space of three term twisted complexes of the form \( \mathcal{T} = (F_i, \Psi_{ji}) \), \( i, j = 0, 1, 2 \) where

\[
F_0 = \mathcal{O}(\Sigma)(-D_+), \quad F_1 = \mathcal{O}(\Sigma) \oplus \mathcal{O}(\Sigma)(-D_-), \quad F_2 = \mathcal{O}(\Sigma)
\]
as $C^\infty$-bundles. By convention we fix the degrees of $F_0, F_1, F_2$ to be $-1, 0, 1$. Two such twisted complexes are said to be equivalent if there exist morphisms $\eta \in H^0_{\text{Tr}(\mathcal{D})}(\mathcal{T}, \mathcal{T}')$, $\rho \in H^0_{\text{Tr}(\mathcal{D})}(\mathcal{T}', \mathcal{T})$ such that $\rho \circ \eta = \mathbb{I}_T$ and $\eta \circ \rho = \mathbb{I}_{T'}$.

In order to formulate a well defined moduli problem, we have to translate this data into algebraic-geometric language. We start with the three term twisted complexes $T$. The fields $\Psi_{01}, m = 0, 1, 2$ correspond to integrable complex structures on the bundles $F_m, m = 0, 1, 2$. We will denote by $F_m, m = 0, 1, 2$ the associated locally free sheaves. $\Psi_{01}, m = 0, 1, 2$ determine morphisms $\Psi_{10} : F_0 \to F_1, \Psi_{21} : F_1 \to F_2$ of locally free sheaves which compose to zero. Therefore we obtain a complex of locally free sheaves

$$0 \to F_0 \xrightarrow{\Psi_{10}} F_1 \xrightarrow{\Psi_{21}} F_2 \to 0.$$  

The remaining fields $\Psi_{10, m+1}, m = 0, 1, 2, \Psi_{02, 1}$ can be interpreted as follows. $\Psi_{02, 1}$ determines an extension

$$0 \to F_0 \otimes O(K_S) \to S \to F_2 \to 0.$$  

and $\Psi_{01}, \Psi_{12}$ determine splittings

$$0 \to F_0 \otimes O(K_S) \to S \to F_2 \to 0.$$  

where $\Psi_{12} S, \Psi_{10} S$ denote the pullback and respectively pushforward extensions. For further reference note that the splitting $\Psi_{01}$ induces a map $\Psi_{21} : F_1 \to S$ lifting $\Psi_{21} : F_1 \to F_2$. The splitting $\Psi_{12}$ induces a canonical projection $\Pi_{12} : \Psi_{10} S \to F_1 \otimes O(K_S)$ which induces in turn a map $\tilde{\Psi}_{10} : S \to F_1 \otimes O(K_S)$ extending $\Psi_{10} \otimes \mathbb{I}_{O(K_S)} : F_0 \otimes O(K_S) \to F_1 \otimes O(K_S)$.

In order to interpret the remaining constraints

$$\Psi_{01}^0 \Psi_{10}^0 = 0$$  

$$\Psi_{10}^0 \Psi_{01}^1 - \Psi_{12}^0 \Psi_{21}^0 = 0$$  

$$\Psi_{21}^0 \Psi_{12}^1 = 0$$  

50
Note for example that $\Psi_{10} : \mathcal{F}_0 \to \mathcal{F}_1$ induces a splitting

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{F}_0 \otimes \mathcal{O}(K_\Sigma) & \Psi_{10}^* \Psi_{21}^* S & \to & \mathcal{F}_0 & \to & 0 \\
\Psi_{10} \Psi_{01} & \uparrow & \uparrow & \Psi_{01} & \downarrow & \Psi_{10} & \downarrow & \Psi_{21} S & \to & \mathcal{F}_1 & \to & 0.
\end{array}
\]

The first equation in (94) is equivalent to the condition

\[
(95) \quad \Psi_{10}^* \Psi_{21}^* S = \mathcal{F}_0 \oplus \mathcal{F}_0 \otimes \mathcal{O}(K_\Sigma) \quad \text{and} \quad \Psi_{10}^* \Psi_{01} = (\mathbb{I}_{\mathcal{F}_0}, 0).
\]

Similarly, $\Psi_{21} : \mathcal{F}_1 \to \mathcal{F}_2$ induces a splitting

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{F}_2 \otimes \mathcal{O}(K_\Sigma) & (\Psi_{21} \otimes \mathbb{I}_{\mathcal{O}(K_\Sigma)}) S & \to & \mathcal{F}_2 & \to & 0 \\
\Psi_{10} \Psi_{01} & \uparrow & \uparrow & \Psi_{10} S & \to & \mathcal{F}_2 & \to & 0.
\end{array}
\]

The last equation is equivalent to

\[
(96) \quad (\Psi_{21} \otimes \mathbb{I}_{\mathcal{O}(K_\Sigma)}) S = \mathcal{F}_2 \oplus \mathcal{F}_2 \otimes \mathcal{O}(K_\Sigma) \quad \text{and} \quad (\Psi_{21} \otimes \mathbb{I}_{\mathcal{O}(K_\Sigma)}) \Psi_{12} = (\mathbb{I}_{\mathcal{F}_2}, 0).
\]

Finally, in order to find the algebraic interpretation of the second equation in (94) note that we have two induced splittings

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{F}_1 \otimes \mathcal{O}(K_\Sigma) & (\Psi_{10} \otimes \mathbb{I}_{\mathcal{O}(K_\Sigma)}) S & \to & \mathcal{F}_1 & \to & 0 \\
\Psi_{10} \Psi_{01} & \uparrow & \uparrow & \Psi_{01} & \downarrow & \Psi_{10} S & \to & \mathcal{F}_1 & \to & 0.
\end{array}
\]

Note also that the extensions $(\Psi_{10} \otimes \mathbb{I}_{\mathcal{O}(K_\Sigma)}) S$ and $\Psi_{21}^* S$ are canonically isomorphic. Then the middle equation in (94) is equivalent to the condition that the two splittings agree

\[
(97) \quad (\Psi_{10} \otimes \mathbb{I}_{\mathcal{O}(K_\Sigma)}) S = \Psi_{21}^* S.
\]
To summarize, we have shown that the data given by a three term twisted complex $\mathcal{T} = (\mathcal{F}_i, \Psi_{ji})$, $i, j = 0, 1, 2$ on $\Sigma$ is equivalent to the following algebraic data

i) a three term complex $\mathcal{F}$ of locally free sheaves $\mathcal{F}_{10}$,

ii) an extension of the form $\mathcal{F}_{10}$, and

iii) splittings $\mathcal{F}_{10}$ satisfying conditions $\mathcal{F}_{10}$, $\mathcal{F}_{10}$ and $\mathcal{F}_{10}$.

Next we have to specify equivalence relations between two sets of algebraic data. This will be achieved by finding an algebraic description for the morphism space between three term twisted complexes in $\text{Tr}(\mathcal{D})$. Let $(\mathcal{F}_i, \Psi_{02}, \Psi_{01}, \Psi_{12})$ and $(\mathcal{F}'_i, \Psi'_{02}, \Psi'_{01}, \Psi'_{12})$ $i = 0, 1, 2$ be two sets of algebraic data satisfying conditions $\mathcal{F}_{10}$ above.

Given the data $(\mathcal{F}_i, \Psi_{ji})$ as above, we construct the five term complex $\mathcal{K}$

$\mathcal{F}_0 \xrightarrow{\Psi_{10}} \mathcal{F}_1 \xrightarrow{\Psi_{12}} \mathcal{S} \xrightarrow{\Psi_{10}} \mathcal{F}_1 \otimes \mathcal{O}(K_{12}) \xrightarrow{\Psi_{12} \otimes \mathcal{O}(K_{12})} \mathcal{F}_2 \otimes \mathcal{O}(K_{12})$.

in which the first term has degree $-2$. We also have a similar complex $\mathcal{K}'$ for the data $(\mathcal{F}'_i, \Psi'_{ji})$

$\mathcal{F}'_0 \xrightarrow{\Psi'_{10}} \mathcal{F}'_1 \xrightarrow{\Psi'_{12}} \mathcal{S}' \xrightarrow{\Psi'_{10}} \mathcal{F}'_1 \otimes \mathcal{O}(K_{12}) \xrightarrow{\Psi'_{12} \otimes \mathcal{O}(K_{12})} \mathcal{F}'_2 \otimes \mathcal{O}(K_{12})$.

Note that we have short exact sequences of complexes

$0 \to \mathcal{F} \otimes \mathcal{O}(K_{12})[-2] \to \mathcal{K} \to \mathcal{F} \to 0$

$0 \to \mathcal{F}' \otimes \mathcal{O}(K_{12})[-2] \to \mathcal{K}' \to \mathcal{F}' \to 0$

Applying the functor $\text{Hom}(\mathcal{F}, \mathcal{F}') \otimes \mathcal{O}(K_{12})[-2]$ to the first complex in $\mathcal{F}_{10}$, we obtain the short exact sequence of complexes

$0 \to \text{Hom}(\mathcal{F}, \mathcal{F}') \otimes \mathcal{O}(K_{12})[-2] \to \text{Hom}(\mathcal{K} \otimes \mathcal{O}(-K_{12})[2], \mathcal{F}') \to \text{Hom}(\mathcal{F}, \mathcal{F}') \to 0$.

Applying the functor $\text{Hom}(\mathcal{F}, \mathcal{F})$ to the second complex in $\mathcal{F}_{10}$ we obtain the short exact sequence of complexes

$0 \to \text{Hom}(\mathcal{F}, \mathcal{F}') \otimes \mathcal{O}(K_{12})[-2] \to \text{Hom}(\mathcal{F}, \mathcal{K}') \to \text{Hom}(\mathcal{F}, \mathcal{F}') \to 0$.

Therefore we have produced two extensions of the complex $\text{Hom}(\mathcal{F}, \mathcal{F}')$ by $\text{Hom}(\mathcal{F}, \mathcal{F}') \otimes \mathcal{O}(K_{12})[-2]$. Next we construct a new complex $C(\mathcal{K}, \mathcal{K}')$ by taking the difference of the two extensions. This results in a five term complex whose terms are explicitly computed in appendix A. We claim the $\mathbb{Z}$ graded space of morphisms between the twisted complexes $\mathcal{T}, \mathcal{T}'$ in $\text{Tr}(\mathcal{D})$ is isomorphic to the hypercohomology of $C(\mathcal{K}, \mathcal{K}')$ on $\Sigma$

$H_0^\bullet(\mathcal{T}, \mathcal{T}') \simeq H^\bullet(C(\mathcal{K}, \mathcal{K}'))$.

The proof of this assertion reduces to a rather lengthy homological algebra computation which is performed in appendix A.
Using the algebraic formulation we can set up a well defined moduli problem for a three term complex $\mathcal{F}$ of the form (87) in the category of twisted complexes. In principle, one can construct a moduli stack associated to this moduli problem, but we will perform such a construction here. For a semiclassical analysis it suffices to identify the irreducible component of the moduli space which contains equivalence classes of complexes of the form $\mathcal{F}$ up to birational equivalence. This is an easier task, which can be accomplished as follows.

First note that we can construct a family of equivalence classes of complexes $\mathcal{F}$ in the category of twisted complexes by varying the divisors $D_+,D_-$ on $\Sigma$. This gives rise to a map $\phi : \text{Sym}^N(\Sigma)^2 \to \mathcal{M}$ from $\text{Sym}^N(\Sigma)^2$ to the moduli space sending a point $(D_+,D_-) \in (\Sigma)^2$ to a three term complex of the form (87).

Now let us compute the space of infinitesimal first order deformations $H^1_{\text{Tr}(\mathcal{D})}(\mathcal{F},\mathcal{F})$ of the complex $\mathcal{F}$ in the category of twisted complexes. According to equation (103), we have

$$H^1_{\text{Tr}(\mathcal{D})}(\mathcal{F},\mathcal{F}) \simeq \mathbb{H}^1(C(\mathcal{K},\mathcal{K}'))$$

where $\mathcal{K},\mathcal{K}'$ are the five term complexes defined in (88),(99). Since $\Psi_{02} = 0$ in this case, the extension (10) is canonically split. Then one can check that

$$\mathcal{K} = \mathcal{F} \oplus \mathcal{F} \otimes \mathcal{O}(K_\Sigma)[-2]$$

and

$$C(\mathcal{K},\mathcal{K}) = \text{End}(\mathcal{F}) \oplus \text{End}(\mathcal{F}) \otimes \mathcal{O}(K_\Sigma)[-2].$$

Therefore we have to evaluate the hypercohomology group

$$\mathbb{H}^1(\text{End}(\mathcal{F}) \oplus \text{End}(\mathcal{F}) \otimes \mathcal{O}(K_\Sigma)[-2]) = \text{Hom}_{D^b(\Sigma)}(\mathcal{F},\mathcal{F}[1]) \oplus \text{Hom}_{D^b(\Sigma)}(\mathcal{F},\mathcal{F} \otimes \mathcal{O}(K_\Sigma)[-1]).$$

The complex $\mathcal{F}$ is quasi-isomorphic to its cohomology $\mathcal{O}_{D_+} \oplus \mathcal{O}_{D_-}[-1]$, hence we have

$$\text{Hom}_{D^b(\Sigma)}(\mathcal{F},\mathcal{F}[1]) \oplus \text{Hom}_{D^b(\Sigma)}(\mathcal{F},\mathcal{F} \otimes \mathcal{O}(K_\Sigma)[-1]) =$$

$$= \text{Hom}_{D^b(\Sigma)}(\mathcal{O}_{D_+} \oplus \mathcal{O}_{D_-}[-1],\mathcal{O}_{D_+}[1] \oplus \mathcal{O}_{D_-}) \oplus$$

$$\text{Hom}_{D^b(\Sigma)}(\mathcal{O}_{D_+} \oplus \mathcal{O}_{D_-}[-1],\mathcal{O}_{D_+}(K_\Sigma)[-1] \oplus \mathcal{O}_{D_-}(K_\Sigma)[-2]) =$$

$$\text{Hom}_{D^b(\Sigma)}(\mathcal{O}_{D_+},\mathcal{O}_{D_+}[1]) \oplus \text{Hom}_{D^b(\Sigma)}(\mathcal{O}_{D_-},\mathcal{O}_{D_-}[1]).$$

Using Serre duality, one can show that all other terms in the right hand side of the above equation vanish provided that $D_+,D_-$ have disjoint supports. Therefore, if $\text{Supp}(D_+) \cap \text{Supp}(D_-) = \emptyset$, we find that

$$\mathbb{H}^1(C(\mathcal{K},\mathcal{K})) = \text{Ext}^1(\mathcal{O}_{D_+},\mathcal{O}_{D_+}) \oplus \text{Ext}^1(\mathcal{O}_{D_-},\mathcal{O}_{D_-}),$$

which is the space of infinitesimal deformations of the pair $(D_+,D_-)$ i.e. the tangent space $T_{(D_+,D_-)}\text{Sym}^N(\Sigma)^2$.

Let $\Delta_\pm$ denote the big diagonals in the two factors of the direct product $\text{Sym}^N(\Sigma) \times \text{Sym}^N(\Sigma)$; the points in $\text{Sym}^N(\Sigma) \setminus \Delta_\pm$ parameterize effective divisors $D_\pm$ with distinct simple points. We will also denote by

$$\Delta \subset \text{Sym}^N(\Sigma)^2, \quad \Delta = \{(D_+,D_-) \in \text{Sym}^N(\Sigma)^2|\text{Supp}(D_+) \cap \text{Supp}(D_-) \neq \emptyset\}.$$
the big diagonal of the direct product.

The above considerations show that $\phi$ is an isomorphism between the open subset
\[(\text{Sym}^N(\Sigma) \times \text{Sym}^N(\Sigma)) \setminus (\Delta_+ \times \text{Sym}^N(\Sigma) \cup \text{Sym}^N(\Sigma) \times \Delta_- \cup \Delta)\]
and an open subset of the moduli space $\mathcal{M}$. Note that $\phi$ is also well defined and induces an isomorphism of virtual tangent spaces along the divisor
\[\Delta_+ \times \text{Sym}^N(\Sigma) \cup \text{Sym}^N(\Sigma) \times \Delta_- .\]
However, $\phi$ does not extend to this divisor as in isomorphism of stacks, since the moduli space $\mathcal{M}$ is a higher stack. In principle, it could extend as an isomorphism of stacks in the presence of a suitable stability condition on twisted complexes which would make $\mathcal{M}$ an stack. We will not attempt to formulate such a condition here, but we will assume that with right choice of a stability condition, we can identify an open subset of the D-brane moduli space with the open subset
\[\mathcal{M}_0 = [\Sigma^N/S_N] \times [\Sigma^N/S_N] \setminus \Delta,\]
where $[\Sigma^N/S_N]$ denotes the (stacky) orbifold quotient of the cartesian product $\Sigma^N$ by the action of the symmetric group $S_N$.

### 6.3 The measure

The next step in the quantization process requires a holomorphic measure on the moduli space. In the following we will concentrate on the open subset $\mathcal{M}_0$ of the moduli space constructed in the previous subsection. Note that $\mathcal{M}_0$ has a finite cover of the form
\[\mathcal{M}'_0 = (\Sigma^N \times \Sigma^N) \setminus \Delta\]
where $\Delta$ is defined again as the divisor of $\Sigma^N \times \Sigma^N$ where a point $p_a$ coincides with a point $q_b$ for some $a, b = 1, \ldots, N$. In fact $\mathcal{M}_0$ is a quotient of $\mathcal{M}'_0$ by the obvious action of $S_N \times S_N$ which from a physical point of view should be thought of as a residual discrete gauge action.

Following the common practice in gauge theories, we will write the measure on a finite cover of the moduli space of the form and divide the resulting functional integral by $|S_N \times S_N| = (N!)^2$.

The restriction of the measure to $\mathcal{M}'_0$ should be a holomorphic $2N$ differential form. Since $\mathcal{M}_0$ is isomorphic to the complement of the diagonal $\Delta$ in $\Sigma^{2N}$, any such form can be extended to a meromorphic $2N$ form $\Omega$ on $\Sigma^{2N}$.

In principle one should be able to derive a formula for $\Omega$ starting from the path integral formulation of holomorphic Chern-Simons theory. However, this approach may be quite cumbersome in practice, so it is more economical to find a formula for $\Omega$ based on holomorphy and physical constraints. The main idea is that physical constraints specify the polar structure of $\Omega$ along the divisor
\[(\Delta_+ \times \Sigma^N) \cup (\Sigma^N \times \Delta_-) \cup \Delta.\]
Although these conditions do not determine \( \Omega \) uniquely, with some additional physical insight we can write down the measure uniquely up to a scale.

The physical constraints on \( \Omega \) are imposed by the universal character of the local effective interactions among branes at very short distances. This means that when the branes are very close to each other the dominant interaction terms are identical to their local counterparts discussed in section two. Therefore we should have Coulomb repulsion between two branes or two anti-branes approaching each other and also Coulomb attraction between a brane anti-brane pair \([2, 35]\). This means that \( \Omega \) should have a zero of order two along each divisor in \( \mathcal{M}_0 \) given by \( p_a = p_b \) or \( q_a = q_b \) for any \( a, b = 1, \ldots, N, a \neq b \) and a pole of order two along divisors given by \( p_a = q_b \) for all \( a, b = 1, \ldots, N \). It is straightforward to check that any such differential must have extra zeroes on \( \mathcal{M}_0 \). These zeroes do not have a direct physical interpretation.

Although a natural meromorphic form \( \Omega \) with these properties does not exist, we can construct such a preferred form once we choose a spin structure on \( \Sigma \). More precisely, let us pick a theta-characteristic \( \epsilon \), and take \( \Omega \) to be the square of the free fermion correlator

\[
\langle \psi(p_1) \ldots \psi(p_N) \bar{\psi}(q_1) \ldots \bar{\psi}(q_N) \rangle.
\]

where \( \psi \) is a complex spinor on \( \Sigma \) with respect to the spin structure \( \epsilon \). This is a well defined meromorphic top form on \( \Sigma^N \times \Sigma^N \) exhibiting the physical behavior explained in the last paragraph.

Now, one may legitimately ask why this peculiar construction is the correct moduli space measure for the holomorphic Chern-Simons theory. A short answer to this question is that although the physical constraints do not fix the measure uniquely, they do fix the relevant part of the measure in the large \( N \) planar limit. That is the structure of the semiclassical large \( N \) vacua of the theory is insensitive to changing the measure by adding a holomorphic top form without zeroes or poles along the divisor \((106)\). This will be manifest in the calculations performed in the next section. Therefore this choice of the measure is as good as any other choice exhibiting identical polar structure along the divisor \((106)\).

However this is not a conceptually satisfying answer since the full quantum theory involves much more than the large \( N \) planar limit. To gain a different perspective on this problem, recall that so far we have ignored the coupling between holomorphic Chern-Simons theory and Kodaira-Spencer theory. Indeed, although this coupling does not play an important role in the large \( N \) planar limit, it is certainly expected to play an important role in the full quantum theory. Without getting into too much detail at this point, let us mention that in the present set-up Kodaira-Spencer theory can be described in terms of a chiral boson on the Riemann surface \( \Sigma \) by analogy with the examples considered in \([30, 37]\). It is well known that in order to define the theory of a chiral boson at quantum level, one has to choose a spin structure on \( \Sigma \). Moreover, the branes should be regarded as fermion operators in this theory related to the Kodaira-Spencer field by bosonization \([30, 37]\). This explains the choice of a spin structure. The construction of the measure in terms of fermionic correlators can be justified starting from the identification between the Kodaira-Spencer field and the collective field of the D-branes in the large \( N \) limit \([3, 37]\). We will fully develop these ideas elsewhere.
It is also worth noting that the choice of a spin structure has a natural geometric interpretation in the topological closed string theory discussed in section five. Recall that the genus zero structure of the theory was shown to be encoded in an $A_1$ Hitchin integrable system $\varpi : \mathcal{N} \to \mathcal{B}$. The fiber $\mathcal{N}_\beta$ are torsors over the Prym variety $\text{Prym}(\Sigma_\beta/\Sigma)$. The choice of a spin structure on $\Sigma$ determines a section of the fibration $\varpi : \mathcal{N} \to \mathcal{B}$, hence also an isomorphism between each fiber $\mathcal{N}_\beta$ and the Prym. It would be very interesting to understand the connection between this section and Kodaira-Spencer theory on the open-closed side of the transition.

For computational purposes it is helpful to write the fermionic correlator \[^{\text{(107)}}\] in terms of $\theta$-functions and prime forms. To keep the technical complications to a minimum, we will choose $\epsilon$ so that the corresponding Dirac operator has no zero modes: $h^0(\Sigma, \epsilon) = h^1(\Sigma, \epsilon) = 0$. In particular $\epsilon$ has to be an even spin structure.

Let us briefly recall the construction of the prime form associated to the Riemann surface $\Sigma$. We denote by $\tilde{\Sigma}$ the universal cover of $\Sigma$ and by $\tilde{p}, \tilde{q}, \ldots$ points on the universal cover projecting to $p, q, \ldots$ on $\Sigma$. The prime form \[^{[38, \text{pg 17}]}\] is a $(-\frac{1}{2}, -\frac{1}{2})$ differential on $\tilde{\Sigma} \times \tilde{\Sigma}$ so that $E(\tilde{p}, \tilde{q})$

$$E(\tilde{p}, \tilde{q}) = 0 \iff p = q,$$

and the order of vanishing along $p = q$ is one. One can construct such a form by picking up a nonsingular odd theta characteristic $\delta$ with $h^0(\Sigma, \delta) = 1$. Then we have

\[^{\text{(108)}}\]

$$E(\tilde{p}, \tilde{q}) = \frac{\vartheta(\tilde{p} - \tilde{q} + \delta)}{h(\tilde{p}) h(\tilde{q})}$$

where $\vartheta$ is Riemann’s theta function and $h$ is the unique (up to multiplication by a nonzero constant) section of $\delta$. Note that the function $E$ does not depend on the choice of $\delta$ but it depends on the choice of a homology basis for $\Sigma$ \[^{[38, \text{pg 17}]}\]. Somewhat more invariantly we can view $E$ as a section in a line bundle on the surface $\Sigma \times \Sigma$. Let $\theta \to \text{Pic}^{g-1}(\Sigma)$ be the canonical theta line bundle on the degree $(g - 1)$ Jacobian. Let $AJ_\delta : \Sigma \times \Sigma \to \text{Pic}^{g-1}(\Sigma)$ be the Abel-Jacobi map $AJ_\delta(p, q) = p - q + \delta$. Now the theta function $\vartheta(\tilde{p} - \tilde{q} + \delta)$ descends to a section of the line bundle $AJ_\delta^* \theta$ and so $E$ descends to a section (which will be denoted again by $E$) in the line bundle $(AJ_\delta^* \theta) \otimes p_1^* \delta \otimes p_2^* \delta$. It is easy to check that this section is holomorphic. Furthermore a straightforward application of the see-saw principle shows that $(AJ_\delta^* \theta) \otimes p_1^* \delta \otimes p_2^* \delta = \mathcal{O}_{\Sigma \times \Sigma}(\text{Diag})$ and so up to scale $E$ is the unique holomorphic section in the line bundle $\mathcal{O}_{\Sigma \times \Sigma}(\text{Diag})$ corresponding to the diagonal divisor $\text{Diag} := \{(x, y) \in \Sigma \times \Sigma | x = y\}$. Since the self-intersection $\text{Diag}^2 = 2 - 2g$ of the divisor $\text{Diag}$ is negative, it follows that $H^0(\Sigma \times \Sigma, \mathcal{O}(\text{Diag}))$ is one dimensional and so $E$ necessarily vanishes along the diagonal.

Now consider the following top degree meromorphic differential on $\Sigma^{2N}$

\[^{\text{(109)}}\]

$$\Omega(\tilde{p}_a, \tilde{q}_a) = \frac{\vartheta(\sum_{a=1}^N \tilde{p}_a - \sum_{a=1}^N \tilde{q}_a + \epsilon)^2}{\vartheta(\epsilon)^2} \prod_{a, b=1}^N \frac{E(\tilde{p}_a, \tilde{p}_b) E(\tilde{q}_a, \tilde{q}_b)}{\prod_{a, b=1}^N E(\tilde{p}_a, \tilde{q}_b)^2}.$$  

Using the modular transformation properties of $E(\tilde{p}, \tilde{q})$ and the $\vartheta$-function, one can check that $\Omega$ descends to a differential (denoted by the same letter) on $\Sigma^{2N}$. Again we can interpret
Ω as a section in a line bundle on Σ^{2N}. If we write a_ε : Σ^{2N} → \text{Pic}^{g-1}(Σ) for the Abel-Jacobi map given by a_ε(\{(p_a)_{a=1}^N, \{q_a\}_{a=1}^N\}) := \epsilon + \sum_{a=1}^N (p_a - q_a), then Ω is by definition a meromorphic section in the line bundle a_ε^*θ \otimes O_{Σ^{2N}}(2\Delta_+ + 2\Delta_- - 2\Delta) with a double pole along the divisor Δ. Again a quick computation with the see-saw principle identifies a_ε^*θ \otimes O_{Σ^{2N}}(2\Delta_+ + 2\Delta_- - 2\Delta) with the line bundle K_{Σ^{2N}}. In other words, Ω is a meromorphic 2N form on Σ^{2N} which has a double pole precisely along the “diagonal” divisor Δ ⊂ Σ^N × Σ^N.

According to [40], the formula (109) represents the square of the fermionic correlator (107). In the following we will adopt (109) as the measure for the moduli space integral.

For a complete definition of the quantum theory, we have to specify an integration contour in Σ^{2N}. A priori, there is no canonical choice of contour, and at the moment it is unclear what physical constraints should be imposed on such a contour. In all holomorphic Chern-Simons theories studied so far [2, 12], the moduli space has a natural antiholomorphic involution τ : M → M so that τ^*Ω = Ω. In these case the contour Φ is chosen to be the fixed point set of this involution, which is a special Lagrangian cycle calibrated by the measure Ω. In particular, the restriction of Ω to Φ is a real differential form up to multiplication by a nonzero complex number.

This reality condition is important for a semiclassical analysis for the following simple reason. The semiclassical vacua are determined as the critical points of a complex function on Γ obtained by dividing Ω by some reference classical measure Ω_0, which is real. Generically, a complex valued function on the cycle Φ does not have any critical points. Therefore, if we choose Φ to be an arbitrary cycle, we will not be able to develop a semiclassical expansion of the theory. While this is not a consistency requirement of a nonperturbative quantum field theory, we will adopt the reality condition as a selection criterion for Φ. The issue of the dependence of the action on the choice of a contour has been raised in the physics literature before. In particular a reality condition appears in [2], and the subtleties of extending this condition to the context of holomorphic matrix models are analyzed in detail in [41].

In our case, for a generic Σ, the moduli space does not admit antiholomorphic involutions, therefore there is no canonical choice for Φ. Following the arguments of the previous paragraph, we will choose Φ to be any special Lagrangian cycle with respect to the measure Ω. If Σ admits an antiholomorphic involution τ : Σ → Σ, we can construct such a cycle as Φ = Γ^{2N} where Γ is a component of the fixed point set of τ on Σ. In the absence of a real structure, we will simply assume that we can find a closed contour Γ on Σ so that Γ^{2N} is special Lagrangian with respect to Ω.

6.4 Deformations and classical superpotential

So far we have formulated holomorphic Chern-Simons theory on \tilde{X} in terms of a contour integral of a meromorphic top form on the classical moduli space M_0. This is only a first step in our program since we are interested in B-branes on a deformed threefold \tilde{X}_α.

Recall that the threefold \tilde{X}_α constructed in section four has an affine bundle structure over the ruled surface S. If α is generic, that is

\text{div}(α) = v_1 + \ldots + v_{2g-2}
with \( v_1, \ldots, v_{2g-2} \) distinct points on \( \Sigma \), then only the \( 2g - 2 \) fibers of the ruling \( q : S \to \Sigma \) sitting over the points \( \{ v_i \} \) will lift to holomorphic isolated \((-1,-1)\) curves \( C_1, \ldots, C_{2g-2} \) curves in \( \tilde{X}_\alpha \).

The classical vacua of this theory consist of configurations of \( N^+_k, k = 1, \ldots, r \) branes wrapped on \( r \) curves \( C_{sk}, k = 1, \ldots, r \), and \( N^-_k, k = r + 1, \ldots, 2g-2 \) antibranes wrapped on the remaining curves \( C_{sk}, k = r + 1, \ldots, 2g-2 \) for some \( 1 \leq r \leq 2g-3 \). The multiplicities are subject to the constraint

\[
\sum_{k=1}^{r} N^+_k = \sum_{k=r+1}^{2g-2} N^-_k = N.
\]

In terms of the holomorphic gauge theory on \( \Sigma \), these D-brane configurations correspond to points of the form

\[
D_+ = \sum_{k=0}^{r} N^+_k v_{sk}, \quad D_- = \sum_{k=r+1}^{2g-2} N^-_k v_{sk}
\]

of the undeformed moduli space. Without loss of generality we can set \( s_k = k, k = 1, \ldots, 2g-2 \) in the following.

Following the strategy of Dijkgraaf-Vafa transitions, holomorphic Chern-Simons theory on \( \tilde{X}_\alpha \) can be defined as a superpotential deformation of the \( \alpha = 0 \) theory. The superpotential in question should be a (possibly multivalued) holomorphic function on the finite cover \( \mathcal{M}_0^\prime \) of the moduli space whose critical points are in one-to-one correspondence with classical D-brane configurations.

As explained in section 2, such a function has a natural geometric origin. Indeed, one can construct a transverse holomorphic family \( \mathcal{C} \) of two-cycles on \( \tilde{X}_\alpha \) parameterized by \( \Sigma \) including the holomorphic curves \( C_1, \ldots, C_{2g-2} \) at the points \( v_1, \ldots, v_{2g-2} \). The generic fiber \( C_p \) of this family over a point \( p \in \Sigma \) is a smooth non holomorphic two-cycle on the affine quadric \( (\tilde{X}_\alpha)_p \), similarly to the local situation described in section 2.

This family determines a Donaldson-Thomas superpotential on the moduli space via the Abel-Jacobi map by analogy with the considerations of section 2. The only difference is that in the present case, the Donaldson-Thomas superpotential is multivalued since \( \Sigma \) contains nontrivial homology one-cycles. Therefore in order to obtain a single valued expression we have to work on the universal cover \( \tilde{\Sigma}^{2N} \) of the direct product \( \Sigma^{2N} \). The superpotential is then given by

\[
W_\alpha(\tilde{p}_a, \tilde{q}_a) = \sum_{a=1}^{N} \int_{\tilde{q}_a} \alpha.
\]

It is a simple exercise to check that the critical points of \( W_\alpha \) are classical vacua of the form \( \{ v_i \} \).
Summarizing this discussion, the deformed holomorphic Chern-Simons action will be defined by a measure

\[ \Omega_\alpha(\tilde{p}_a, \tilde{q}_a) = \Omega(\tilde{p}_a, \tilde{q}_a)e^{-\frac{1}{g_s}W_\alpha(\tilde{p}_a, \tilde{q}_a)} \]

on \( \tilde{\Sigma}^{2N} \). To complete the construction, we have to specify a middle dimensional cycle \( \mathcal{G} \) on \( \tilde{\Sigma}^{2N} \). According to the discussion of the previous subsection, \( \mathcal{G} \) should be a special Lagrangian cycle with respect to the deformed measure (113). In the following we will assume that the cycle \( \mathcal{G} \) can be chosen of the form \( \mathcal{G} = \Gamma^{2N} \) where \( \Gamma \) is a closed one-cycle on \( \Sigma \) passing through the zeroes of \( \alpha \). This assumption is not unreasonable. For instance if \( \Sigma \) is a real curve equipped with an antiholomorphic involution \( \tau : \Sigma \rightarrow \Sigma \), and \( \alpha \) satisfies an appropriate reality condition, \( \Gamma \) can be chosen to be a component of the fixed point set of \( \tau \).

### 6.5 Semiclassical vacua at large \( N \) and Hitchin systems

In this subsection we determine the semiclassical vacua of the deformed holomorphic Chern-Simons theory in the large \( N \) limit. The main result is that the semiclassical vacua are in one-to-one correspondence to \( A_1 \) Hitchin spectral covers of \( \Sigma \). This establishes a direct connection with the algebraic integrable systems found in the previous section.

In order to derive the semiclassical equations of motion we will rewrite the measure \( \Omega_\alpha \) in a more explicit form. Let \( U \) denote the complement of the divisor \( h = 0 \) in \( \tilde{\Sigma} \). We define a local coordinate function \( z : U \rightarrow \mathbb{C} \) so that

\[ dz(\tilde{p}) = h(\tilde{p})^2 \]

as in [39, pg. 3.207-3.211]. The restriction of \( \Omega \) to \( U^{2N} \) can then be written as

\[ \Omega|_U = \frac{\vartheta(\sum_{a=1}^{N} \tilde{p}_a - \sum_{a=1}^{N} \tilde{q}_a + \epsilon)^2}{\vartheta(\epsilon)^2} \cdot \frac{\prod_{a,b=1, a \neq b}^{N} \vartheta(\tilde{p}_a - \tilde{p}_b + \delta) \vartheta(\tilde{q}_a - \tilde{q}_b + \delta)}{\prod_{a,b=1}^{N} \vartheta(\tilde{p}_a - \tilde{q}_b + \delta)^2} \prod_{a=1}^{N} dz(\tilde{p}_a) dz(\tilde{q}_a) \]

The deformed measure \( \Omega_\alpha \) gives rise to an effective semiclassical superpotential

\[ \frac{1}{g_s} W_\alpha^{\text{eff}}(\tilde{p}_a, \tilde{q}_a) = \frac{1}{g_s} W_\alpha(\tilde{p}_a, \tilde{q}_a) - \sum_{a,b=1}^{N} (\log \vartheta(\tilde{p}_a - \tilde{p}_b + \delta) + \log \vartheta(\tilde{q}_a - \tilde{q}_b + \delta)) \]

\[ + \sum_{a,b=1}^{N} \log \vartheta(\tilde{p}_a - \tilde{q}_b + \delta) - 2 \log \vartheta \left( \sum_{a=1}^{N} \tilde{p}_a - \sum_{a=1}^{N} \tilde{q}_a + \epsilon \right) \].

Applying the variational principle to \( W_\alpha^{\text{eff}}(\tilde{p}_a, \tilde{q}_a) \) we obtain the following semiclassical equa-
tions of motion

\[-\frac{1}{g_s} \alpha(\tilde{p}_a) + 2 \sum_{\substack{b=1 \\
b \neq a}}^N d_{\tilde{p}_b} \log \vartheta(\tilde{p}_b - \tilde{p}_a + \delta) - 2 \sum_{\substack{b=1 \\
b \neq a}}^N d_{\tilde{p}_b} \log \vartheta(\tilde{q}_a - \tilde{q}_b + \delta) + 2d_{\tilde{p}_a} \log \vartheta \left( \sum_{b=1}^N \tilde{p}_b - \sum_{b=1}^N \tilde{q}_b + \epsilon \right) = 0 \]

(116)

\[-\frac{1}{g_s} \alpha(\tilde{q}_a) + 2 \sum_{\substack{b=1 \\
b \neq a}}^N d_{\tilde{q}_b} \log \vartheta(\tilde{q}_b - \tilde{q}_a + \delta) - 2 \sum_{\substack{b=1 \\
b \neq a}}^N d_{\tilde{q}_b} \log \vartheta(\tilde{p}_a - \tilde{q}_a + \delta) + 2d_{\tilde{q}_a} \log \vartheta \left( \sum_{b=1}^N \tilde{p}_b - \sum_{b=1}^N \tilde{q}_b + \epsilon \right) = 0. \]

In order to study the large $N$ limit of these equations, let us introduce the meromorphic differential

\[\omega(\tilde{p}) = \frac{1}{N} d_{\tilde{p}} \sum_{a=1}^N \log \frac{E(\tilde{p}, \tilde{p}_a)}{E(\tilde{p}, \tilde{q}_a)}. \]

(117)

It is easy to check that $\omega$ is invariant under the action of the fundamental group, therefore it descends to a meromorphic one-form on $\Sigma$. This is an Abelian differential of the third kind with simple poles with residue $\frac{1}{N}$ at $p_a, a = 1, \ldots, N$ and simple poles with residue $-\frac{1}{N}$ at $q_a, a = 1, \ldots, N$. In the present context $\omega$ plays the same role as the resolvent in the large $N$ solution of matrix models. Over $U$ we can write

\[\omega(\tilde{p}) = \frac{1}{N} \sum_{a=1}^N d_{\tilde{p}} \log \frac{\vartheta(\tilde{p} - \tilde{p}_a + \delta)}{\vartheta(\tilde{p} - \tilde{q}_a + \delta)}. \]

(118)

Our goal is to show that in the large $N$ limit the equations of motion (116) give rise to an algebraic equation of the form

\[\omega^2 - \frac{1}{\mu} \alpha \omega = \text{holomorphic deformation} \]

analogous to the loop equation of matrix models. Here $\mu = Nq_s$ is the ’t Hooft coupling constant which is kept finite in the large $N$ limit. Note that the theta function $\vartheta(\tilde{p} - \tilde{p}_a + \delta)$ has a first order zero at $\tilde{p} = \tilde{p}_a$ [39 pg. 311]. Therefore locally we can write

\[\vartheta(\tilde{p} - \tilde{p}_a + \delta) = (z(\tilde{p}) - z(\tilde{p}_a)) \tilde{\vartheta}(\tilde{p}, \tilde{p}_a) \]

(119)

where $\tilde{\vartheta}(\tilde{p}, \tilde{p}_a)$ is a holomorphic function over $U$ non vanishing at $\tilde{p} = \tilde{p}_a, a = 1, \ldots, N$. Then $\omega$ can be further rewritten in the form

\[\omega(\tilde{p}) = \frac{1}{N} \sum_{a=1}^N \left( \frac{dz(\tilde{p})}{z(\tilde{p}) - z(\tilde{p}_a)} - \frac{dz(\tilde{p})}{z(\tilde{p}) - z(\tilde{q}_a)} + d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{p}_a) - d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{q}_a) \right). \]

(120)
The equations of motion (116) yield

\[
- \frac{1}{g_s} \sum_{a=1}^{N} \frac{1}{z(\tilde{p}) - z(\tilde{p}_a)} d\tilde{z}(\tilde{p}_a) + 2 \sum_{a,b=1 \atop b \neq a}^{N} \frac{1}{z(\tilde{p}) - z(\tilde{p}_a)} d\tilde{p}_a \log \vartheta(\tilde{p}_a - \tilde{p}_b + \delta) \\
- 2 \sum_{a,b=1 \atop b \neq a}^{N} \frac{1}{z(\tilde{p}) - z(\tilde{p}_a)} d\tilde{p}_a \log \vartheta(\tilde{p}_a - \tilde{q}_b + \delta) \\
+ 2 \sum_{a=1}^{N} \frac{1}{z(\tilde{p}) - z(\tilde{q}_a)} d\tilde{q}_a \log \vartheta(\tilde{q}_a - \tilde{q}_b + \delta) = 0
\]

Using (120), (121), a somewhat tedious computation yields

\[
\left( \frac{\omega(\tilde{p})}{dz(\tilde{p})} \right)^2 - \frac{1}{N g_s} \frac{\omega(\tilde{p})}{dz(\tilde{p})} \frac{\alpha(\tilde{p})}{dz(\tilde{p})} = \\
- \frac{1}{N^2 g_s} \sum_{a=1}^{N} \left[ \frac{1}{z(\tilde{p}) - z(\tilde{p}_a)} \left( \frac{\alpha(\tilde{p})}{dz(\tilde{p})} - \frac{\alpha(\tilde{p}_a)}{dz(\tilde{p}_a)} \right) - \frac{1}{z(\tilde{p}) - z(\tilde{q}_a)} \left( \frac{\alpha(\tilde{p})}{dz(\tilde{p})} - \frac{\alpha(\tilde{q}_a)}{dz(\tilde{q}_a)} \right) \right] \\
- \frac{1}{N^2 g_s} \sum_{a=1}^{N} \left( \frac{d\tilde{p}_a \log \tilde{\vartheta}(\tilde{p}, \tilde{p}_a) - d\tilde{p} \log \tilde{\vartheta}(\tilde{p}, \tilde{p}_a)}{dz(\tilde{p})} \frac{\alpha(\tilde{p})}{dz(\tilde{p})} \right) \\
+ \frac{1}{N^2} \sum_{a=1}^{N} \left( \frac{d\tilde{p}_a \log \tilde{\vartheta}(\tilde{p}, \tilde{p}_a + \delta)}{dz(\tilde{p})} \right)^2 + \left( \frac{d\tilde{p}_a \log \vartheta(\tilde{p}_a, \tilde{p}_b)}{dz(\tilde{p}_a)} \right)^2 \\
+ \frac{2}{N^2} \sum_{a,b=1 \atop a \neq b}^{N} \frac{1}{z(\tilde{p}) - z(\tilde{p}_a)} \left( \frac{d\tilde{p}_a \log \tilde{\vartheta}(\tilde{p}, \tilde{p}_b)}{dz(\tilde{p}_a)} - \frac{d\tilde{p}_a \log \tilde{\vartheta}(\tilde{p}_a, \tilde{p}_b)}{dz(\tilde{p}_a)} \right) \\
+ \frac{2}{N^2} \sum_{a,b=1 \atop a \neq b}^{N} \frac{1}{z(\tilde{p}) - z(\tilde{q}_a)} \left( \frac{d\tilde{q}_a \log \tilde{\vartheta}(\tilde{q}, \tilde{q}_b)}{dz(\tilde{q}_a)} - \frac{d\tilde{q}_a \log \tilde{\vartheta}(\tilde{q}_a, \tilde{q}_b)}{dz(\tilde{q}_a)} \right)
\]
nonzero, but the poles cancel even at finite $N$.

The form

$$\sum_{a,b=1}^N \frac{1}{z(\tilde{p}) - z(\tilde{p}_a)} \left( \frac{d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{q}_b)}{dz(\tilde{p})} - \frac{d_{\tilde{p}_a} \log \tilde{\vartheta}(\tilde{p}_a, \tilde{q}_b)}{dz(\tilde{q}_a)} \right)$$

$$- \frac{2}{N^2} \sum_{a,b=1}^N \frac{1}{z(\tilde{p}) - z(\tilde{q}_a)} \left( \frac{d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{q}_b)}{dz(\tilde{p})} - \frac{d_{\tilde{q}_a} \log \tilde{\vartheta}(\tilde{q}_a, \tilde{p}_b)}{dz(\tilde{q}_a)} \right)$$

$$+ \frac{1}{N^2} \sum_{a,b=1 \atop a \neq b}^N \left( \frac{d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{p}_a)}{dz(\tilde{p})} \frac{d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{q}_b)}{dz(\tilde{p})} + \frac{d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{q}_a)}{dz(\tilde{p})} \frac{d_{\tilde{p}} \log \tilde{\vartheta}(\tilde{p}, \tilde{q}_b)}{dz(\tilde{p})} \right)$$

Now we take the limit $N \to \infty$, $g_s \to 0$ keeping the 't Hooft coupling $\mu = N g_s$ fixed. The behavior of the terms in the equation (122) is determined by their scaling with $N$. Terms of the form

$$\frac{1}{N} \sum_{a=1}^N \ldots \quad \text{and} \quad \frac{1}{N^2} \sum_{a,b=1}^N \ldots$$

are expected to have finite limit when $N \to \infty$, whereas terms of the form

$$\frac{1}{N^2} \sum_{a=1}^N \ldots$$

tend to zero because they are suppressed by an extra power of $N$. Using these rules, we can check that the right hand side of equation (122) is a holomorphic function on $U$ in the large $N$ limit. The polar part of that expression – that is the terms in the second and last two lines – scales as $1/N$, therefore it vanishes in the large $N$ limit. The remaining terms are nonzero, but the poles cancel even at finite $N$.

The limit of the left hand side of equation (122) is a quadratic expression of the form

$$\omega_\infty^2 - \frac{1}{\mu} \omega_\infty \alpha$$

where $\omega_\infty$ is the large $N$ limit of the meromorphic form $\omega$ which characterizes the distribution of branes on $\Sigma$. We will assume that the limit $\omega_\infty$ exists and it is a well defined mathematical object on $\Sigma$. By construction, the expression (122) can only have poles at the locations of the branes on $\Sigma$. However, we have shown in the previous paragraph that these poles are
absent from the left hand side of equation (122). Therefore we can conclude that the large $N$ limit of equation (122) is of the form

\begin{equation}
\omega_\infty - \frac{1}{\mu} \alpha \omega_\infty = \beta
\end{equation}

where $\beta$ is a global holomorphic quadratic differential on $\Sigma$.

Then $\omega_\infty$ must be a multivalued holomorphic differential on $\Sigma$ with branch points located at the zeroes of $\beta$. Note that by construction $\omega_\infty$ characterizes the distribution of branes on $\Sigma$ in the large $N$ limit. For finite $N$, $\omega$ has poles at the locations of the branes, therefore we would naively expect $\omega_\infty$ to have infinitely many poles on $\Sigma$. The only way such a mathematical object can be well defined is if the collection of poles of $\omega$ becomes a collection of branch cuts $\Gamma_1, \ldots, \Gamma_{2g-2}$ in the large $N$ limit, and $\omega_\infty$ is a multivalued differential. This is consistent with the large $N$ limit (124) of the equation of motion and it also shows that the branch points must be located on the contour $\Gamma$ and the branch cuts must be some line segments contained in $\Gamma$. The filling fraction associated to each branch cut is determined by the period

\begin{equation}
\int_{\gamma_s} \Omega_\infty
\end{equation}

where $\gamma_s$ is a contour on $\Sigma$ surrounding the branch cut $\Gamma_s$, $s = 1, \ldots, 2g-2$.

To summarize: we have found that the large $N$ semiclassical vacuum configurations are in one-to-one correspondence with quadratic holomorphic differentials $\beta$. The distribution of branes in a semiclassical vacuum determined by $\beta$ is encoded in the multivalued differential $\omega_\infty$ which solves (124).

Now the connection with Hitchin spectral covers becomes manifest. Equation (124) is the defining equation of a spectral cover $\tilde{\Sigma}_\beta$ and $\omega_\infty$ is the canonical holomorphic differential on $\tilde{\Sigma}_\beta$. Each contour $\gamma_s$ on $\Sigma$ lifts to an anti-invariant closed one-cycle $\tilde{\gamma}_s$ on $\Sigma_\beta$, and the filling fractions (125) are given by the periods of $\omega_\infty$ on the cycles $\tilde{\gamma}_s$.

Taking into account the relation between Hitchin Pryms and homology intermediate Jacobians proved in section 5.2, the filling fractions (125) can be related to periods of the holomorphic three-form $\Omega_{X_\beta}$ on the threefold $X_\beta$ constructed in section 4.

In conclusion, we have shown that the large $N$ limit of the holomorphic Chern-Simons theory is governed by a Hitchin integrable system, which is in turn isomorphic to the Calabi-Yau integrable system for the universal family of Calabi-Yau threefolds over the of the moduli space $L$. This is a physical proof of large $N$ duality at genus zero.

**Appendix A Morphisms of Twisted Complexes**

In this appendix we prove equation (113) in section 6.2. For convenience, recall that we are given two sets of algebraic data $(\mathcal{F}_i, \Psi_{ji}), (\mathcal{F}'_i, \Psi'_{ji}), i, j = 0, 1, 2$ satisfying conditions
(i)-(iii) below equation (A.131). In particular we have two three term complexes \((F, F')\) and we construct the extensions
\[
\begin{align*}
0 & \rightarrow F \otimes \mathcal{O}(K_S)[-2] \rightarrow \mathcal{K} \rightarrow F \rightarrow 0, \\
0 & \rightarrow F' \otimes \mathcal{O}(K_S)[-2] \rightarrow \mathcal{K}' \rightarrow F' \rightarrow 0
\end{align*}
\]
which yield the short exact sequences of complexes
\[
\begin{align}
0 & \rightarrow Hom(F, F') \otimes \mathcal{O}(K_S)[-2] \rightarrow Hom(\mathcal{K}, \mathcal{O}(-K_S)[2], F') \rightarrow Hom(F, F') \rightarrow 0. \\
0 & \rightarrow Hom(F, F') \otimes \mathcal{O}(K_S)[-2] \rightarrow Hom(F, \mathcal{K}') \rightarrow Hom(F, F') \rightarrow 0.
\end{align}
\]
More explicitly, using the notation of section 5.2, we have
\[
\begin{align}
F_0 \xrightarrow{\Psi_1} F_1 \xrightarrow{\Psi_2} S \xrightarrow{\Psi_3} F_1 \otimes \mathcal{O}(K_S) \xrightarrow{\Psi_4} F_2 \otimes \mathcal{O}(K_S).
\end{align}
\]
where the sheaves \(S, S'\) are given by extensions
\[
\begin{align}
0 & \rightarrow F_0(K_S) \xrightarrow{i} S \xrightarrow{\rho} F_2 \rightarrow 0 \\
0 & \rightarrow F'_0(K_S) \xrightarrow{i'} S' \xrightarrow{\rho'} F'_2 \rightarrow 0.
\end{align}
\]
Our problem is to construct the difference \(C(\mathcal{K}, \mathcal{K'})\) of the two extensions (A.127) and compute its hypercohomology.

Suppose we have two exact sequences of complexes of coherent sheaves on a smooth projective variety
\[
\begin{align}
0 & \rightarrow A \rightarrow C \rightarrow B \rightarrow 0, \\
0 & \rightarrow A \rightarrow C' \rightarrow B \rightarrow 0.
\end{align}
\]
The difference \(C' - C\) can be constructed in two steps. First take the pull-back extension
\[
\begin{align}
0 & \rightarrow A \oplus A \xrightarrow{0} C \oplus C' \xrightarrow{\rho} B \oplus B \rightarrow 0 \\
0 & \rightarrow A \xrightarrow{\iota^-} C \oplus C' \xrightarrow{\rho} B \oplus B \rightarrow 0
\end{align}
\]
where \(\iota^- : A \rightarrow A \oplus A\) is the anti diagonal embedding. Then take a second pullback
\[
\begin{align}
0 & \rightarrow A \xrightarrow{\iota^- C \oplus C'} B \oplus B \rightarrow 0 \\
0 & \rightarrow A \xrightarrow{\iota^+} C' - C \rightarrow B \rightarrow 0
\end{align}
\]
where \(\iota^+ : B \rightarrow B \oplus B\) is the diagonal embedding.

We have to carry out this construction for the extensions (A.127). In order to keep the formulas short, we will use the notation \(H_{nm} \equiv Hom(F_m, F'_n)\). We will first write down explicitly the complexes \(Hom(F, F'), Hom(F, F') \otimes \mathcal{O}(K_S)[-2]\) and then write down the complex \(C(\mathcal{K}, \mathcal{K'})\) as an extension of complexes, obtaining the following diagrams.
(A.133) \[ \text{Hom}(\mathcal{F}, \mathcal{F}'): \]  \[ \text{Hom}(\mathcal{F}, \mathcal{F}') \otimes O(K_{\Sigma})[-2]: \]  \[ H_{02} \]  \[ H_{12} \oplus H_{01} \]  \[ \text{Hom}(\mathcal{F}_2, S') \oplus H_{11} \oplus H_{00} \]  \[ H_{12}(K_{\Sigma}) \oplus \text{Hom}(\mathcal{F}_1, S') \oplus H_{10} \]  \[ H_{22}(K_{\Sigma}) \oplus H_{11}(K_{\Sigma}) \oplus \text{Hom}(\mathcal{F}_0, S') \]  \[ H_{21}(K_{\Sigma}) \oplus H_{10}(K_{\Sigma}) \]  \[ H_{20}(K_{\Sigma}) \]  

(A.134)  \[ \begin{array}{c}
0 \rightarrow C(\mathcal{K}, \mathcal{K}')^{-2} \rightarrow H_{02} \\
\downarrow^{c_{-1,-2}} \\
0 \rightarrow C(\mathcal{K}, \mathcal{K}')^{-1} \rightarrow H_{12} \oplus H_{01} \\
\downarrow^{c_{0,-1}} \\
H_{02}(K_{\Sigma}) \rightarrow C(\mathcal{K}, \mathcal{K}')^{0} \rightarrow H_{22} \oplus H_{11} \oplus H_{00} \\
\downarrow^{c_{1,0}} \\
H_{12}(K_{\Sigma}) \oplus H_{01}(K_{\Sigma}) \rightarrow C(\mathcal{K}, \mathcal{K}')^{1} \rightarrow H_{21} \oplus H_{10} \\
\downarrow^{c_{21}} \\
H_{22}(K_{\Sigma}) \oplus H_{11}(K_{\Sigma}) \oplus H_{00}(K_{\Sigma}) \rightarrow C(\mathcal{K}, \mathcal{K}')^{2} \rightarrow H_{20} \\
\downarrow^{c_{32}} \\
H_{21}(K_{\Sigma}) \oplus H_{10}(K_{\Sigma}) \rightarrow C(\mathcal{K}, \mathcal{K}')^{3} \rightarrow 0 \\
\downarrow^{c_{43}} \\
H_{20}(K_{\Sigma}) \rightarrow C(\mathcal{K}, \mathcal{K}')^{4} \rightarrow 0
\end{array} \]
This determines the terms of degrees $-2, -1, 3, 4$. The remaining terms can be determined following the steps (A.131)-(A.132) described above. In degree $0$, the first step yields

\[ C \oplus C' = \frac{\operatorname{Hom}(F_2, S') \oplus \operatorname{Hom}(S, F'_0)(K_{\Sigma})}{\iota^-(H_{02}(K_{\Sigma}))} \oplus H_{22} \oplus H_{11}^\oplus \oplus H_{00} \]

and the final result is

\[ C(K', K)^0 = \frac{\operatorname{Hom}(F_2, S') \oplus \operatorname{Hom}(S, F'_0)(K_{\Sigma})}{\iota^-(H_{02}(K_{\Sigma}))} \oplus H_{11}. \]

In degree one, we obtain at the first step

\[ C \oplus C' = \frac{H_{12}(K_{\Sigma}) \oplus \operatorname{Hom}(F_1, S') \oplus H_{10} \oplus H_{21} \oplus \operatorname{Hom}(S, F'_1)(K_{\Sigma}) \oplus H_{01}(K_{\Sigma})}{\iota^-(H_{12}(K_{\Sigma}) \oplus H_{01}(K_{\Sigma}))} \]

\[ = \frac{H_{12}(K_{\Sigma}) \oplus \operatorname{Hom}(F_1, S') \oplus \operatorname{Hom}(S, F'_1)(K_{\Sigma}) \oplus H_{01}(K_{\Sigma})}{\iota^-(H_{12}(K_{\Sigma}) \oplus H_{01}(K_{\Sigma}))} \oplus H_{10} \oplus H_{21} \]

\[ \simeq \operatorname{Hom}(F_1, S') \oplus \operatorname{Hom}(S, F'_1)(K_{\Sigma}) \oplus H_{10} \oplus H_{21}. \]

After the second step we obtain

\[ C(K, K')^1 \simeq \operatorname{Hom}(F_1, S') \oplus \operatorname{Hom}(S, F'_1)(K_{\Sigma}). \]

The term of degree two can be similarly determined to be

\[ C(K, K')^2 \simeq \operatorname{Hom}(F_0, S') \times_{H_{20}} \operatorname{Hom}(S, F'_2)(K_{\Sigma}) \oplus H_{11}(K_{\Sigma}). \]

Now let us determine the differentials. The first and the last differentials are standard, hence we will focus on the remaining ones. We have

\[ c_{0,-1} : H_{12} \oplus H_{01} \rightarrow \frac{\operatorname{Hom}(F_2, S') \oplus \operatorname{Hom}(S, F'_0)(K_{\Sigma})}{\iota^-(H_{02}(K_{\Sigma}))} \oplus H_{11} \]

\[ (s_{12}, s_{01}) \rightarrow \left(\tilde{\Psi}_{s_{12}}' s_{12}, + (s_{01} \otimes 1_{K_{\Sigma}}) \tilde{\Psi}_{10}\right), s_{12} \tilde{\Psi}_{21} + \Psi'_{10}s_{01} \]

where we use the notation $[,]$ for equivalence classes in the quotient $(\operatorname{Hom}(F_2, S') \oplus \operatorname{Hom}(S, F'_0)(K_{\Sigma}))/\iota^-(H_{02}(K_{\Sigma}))$. Next,

\[ c_{10} : \frac{\operatorname{Hom}(F_2, S') \oplus \operatorname{Hom}(S, F'_0)(K_{\Sigma})}{\iota^-(H_{02}(K_{\Sigma}))} \oplus H_{11} \rightarrow \operatorname{Hom}(F_1, S') \oplus \operatorname{Hom}(S, F'_1)(K_{\Sigma}) \]

\[ ([u, v], w) \rightarrow \left(-u \tilde{\Psi}_{21} - i'v \tilde{\Psi}_{21} + \tilde{\Psi}_{21}', \tilde{\Psi}_{21}' w, \tilde{\Psi}_{10}' u \rho + (\Psi_{10}' \otimes 1_{K_{\Sigma}}) v - w \tilde{\Psi}_{10}\right) \]

using the notation of equation (A.129). Let us check that the map $c_{10}$ is well defined on equivalence classes $[u, v]$. It suffices to show that $c_{10}([i's, -s\rho], 0) = 0$ for any element
For concreteness, we will write down explicit formulas for the subcomplex \( c \) where

\[
H \in \mathcal{O}_{K \Sigma}
\]

and show that \( H \) is well defined and \( H \) following from diagrams (92), (93). One can similarly check that \( c \) is well defined and \( c \) following from (A.128). Proceeding in a similar manner, we find

\[
c_{10}: \text{Hom}(\mathcal{F}_1, S') \oplus \text{Hom}(\mathcal{S}, \mathcal{F}'_3)(K_\Sigma) \to \text{Hom}(\mathcal{F}_0, S') \times_{H_{20}} \text{Hom}(\mathcal{S}, \mathcal{F}'_3)(K_\Sigma) \oplus H_{11}(K_\Sigma)
\]

\[
(x, y) \to \left( x \Psi_{10} + (\Psi_{21} \otimes I_{K_\Sigma}) y, (\rho' \otimes I_{K_\Sigma}) x \Psi_{10} + (\Psi_{21} \otimes I_{K_\Sigma}) y, \Psi_{10} x + y \Psi_{21} \right)
\]

and

\[
c_{32} : \text{Hom}(\mathcal{F}_0, S') \times_{H_{20}} \text{Hom}(\mathcal{S}, \mathcal{F}'_3)(K_\Sigma) \oplus H_{11}(K_\Sigma) \to H_{21}(K_\Sigma) \oplus H_{10}(K_\Sigma)
\]

\[
(r, t, z) \to \left( t \Psi_{21} - \Psi_{21} z, \Psi_{10} r + 2 \Psi_{10} \right)
\]

It is a straightforward exercise to check that \( c_{21} \) is well defined and \( c_{21} c_{10} = c_{32} c_{21} = 0 \).

In the remaining part of this section, we will write down the hypercohomology double complex of the sheaf complex \( C(\mathcal{K}, \mathcal{K}') \) and prove formula (103). We will regard all locally free sheaves \( \mathcal{F}_i, \mathcal{F}'_i, i = 0, 1, 2 \) as \( C^\infty \) vector bundles \( F_i, F'_i, i = 0, 1, 2 \) equipped with Dolbeault operators and use Dolbeault resolutions. The hypercohomology double complex is if the form

\[
\mathcal{H}^{p, q} = \Omega^{0, p}(C(\mathcal{K}, \mathcal{K}'))^q, \quad D = \overline{\partial} + (-1)^p c
\]

where \( c \) is the differential of \( C(\mathcal{K}, \mathcal{K}') \) and it gives rise to a single complex

\[
\mathcal{H}^n = \bigoplus_{p+q=n} \Omega^{0, p}(C(\mathcal{K}, \mathcal{K}'))^q, \quad D = \overline{\partial} + (-1)^p c.
\]

For concreteness, we will write down explicit formulas for the subcomplex

\[
\mathcal{H}^{-1} \overset{D_{01}}{\longrightarrow} \mathcal{H}^0 \overset{D_{10}}{\longrightarrow} \mathcal{H}^1
\]

and show that \( H^0(\mathcal{H}, D) \) is isomorphic to the space of degree zero morphisms between three term twisted complexes as defined in section 5.1. One can prove the same result for degree one morphisms by a very similar computation.
Let us write down the terms of the complex (A.144). In degree $-1$ we have
\begin{equation}
\Omega^{0,0}(C(\mathcal{K}, \mathcal{K'})^{-1}) \oplus \Omega^{0,1}(C(\mathcal{K}, \mathcal{K'})^{-2}) \simeq \Omega^{0,0}(F_2, F_0') \oplus \Omega^{0,0}(F_1, F_0') \oplus \Omega^{0,1}(F_2, F_0')
\end{equation}
where we have used the shorthand notation $\Omega^{p,q}(\text{Hom}(F_m, F'_n)) \equiv \Omega^{p,q}(F_m, F'_n)$. The Dolbeault operator is the direct sum of the Dolbeault operators for the individual terms in (A.145). In order to write down the degree zero term, note that the extensions (A.129) are split as exact sequences of $C^\infty$ bundles. Therefore we have
\begin{equation}
\Omega^{0,0}(C(\mathcal{K}, \mathcal{K'})^0) \oplus \Omega^{0,1}(C(\mathcal{K}, \mathcal{K'})^{-1}) \simeq \Omega^{0,0}(F_2, F_0'(K_\Sigma)) \oplus \Omega^{0,1}(F_2, F_0'(K_\Sigma)) + \Omega^{0,0}(F_2, F_0') \oplus \Omega^{0,1}(F_2, F_0') \oplus \Omega^{0,0}(F_2, F_0') \oplus \Omega^{0,1}(F_2, F_0') \oplus \Omega^{0,1}(F_1, F_0').
\end{equation}

The Dolbeault operator
\begin{equation}
\bar{\partial} : \Omega^{0,0}(C(\mathcal{K}, \mathcal{K'})^0) \rightarrow \Omega^{0,1}(C(\mathcal{K}, \mathcal{K'})^0)
\end{equation}
is given by
\begin{equation}
\bar{\partial} = \begin{bmatrix}
\lambda_{02}^{1,0} \\
\lambda_{22}^{0,0} \\
\lambda_{00}^{0,0} \\
\lambda_{11}^{0,0}
\end{bmatrix}
= \begin{bmatrix}
\bar{\partial} \lambda_{02}^{1,0} + \psi_{02}^{1,0} \lambda_{22}^{0,0} - \lambda_{00}^{0,0} \psi_{02}^{1,1} \\
\bar{\partial} \lambda_{22}^{0,0} \\
\bar{\partial} \lambda_{00}^{0,0} \\
\bar{\partial} \lambda_{11}^{0,0}
\end{bmatrix}
\end{equation}
where $\lambda_{n,m}^{p,q}$ denotes an arbitrary element of $\Omega_{n,m}^{p,q}$. The degree one term is
\begin{equation}
\Omega^{0,0}(C(\mathcal{K}, \mathcal{K'})^1) \oplus \Omega^{0,1}(C(\mathcal{K}, \mathcal{K'})^0) \simeq \Omega^{0,0}(F_1, F_0')(K_\Sigma) \oplus \Omega^{0,1}(F_2, F_0') \oplus \Omega^{0,1}(F_1, F_0') \oplus \Omega^{0,0}(F_2, F_0')(K_\Sigma) \oplus \Omega^{0,1}(F_2, F_0') \oplus \Omega^{0,1}(F_0, F_0') \oplus \Omega^{0,1}(F_1, F_1')
\end{equation}
Now let us compute the differentials. We have
\begin{equation}
D_{0,-1} : \Omega^{0,0}(C(\mathcal{K}, \mathcal{K'})^{-1}) \oplus \Omega^{0,1}(C(\mathcal{K}, \mathcal{K'})^{-2}) \rightarrow \Omega^{0,0}(C(\mathcal{K}, \mathcal{K'})^0) \oplus \Omega^{0,1}(C(\mathcal{K}, \mathcal{K'})^{-1})
\end{equation}
\begin{equation}
D_{0,-1} = \begin{bmatrix}
\lambda_{00}^{0,0} \\
\lambda_{01}^{0,0} \\
\lambda_{02}^{0,0} \\
\lambda_{01}^{0,1} \\
\lambda_{02}^{0,1} \\
\lambda_{01}^{0,2}
\end{bmatrix}
= \begin{bmatrix}
\lambda_{01}^{0,0} \psi_{12}^{1,0} + \psi_{12}^{1,0} \lambda_{12}^{0,0} \\
\psi_{02}^{0,1} \lambda_{12}^{0,0} \\
\lambda_{00}^{0,0} \psi_{01}^{0,0} \\
\psi_{10}^{0,1} \lambda_{01}^{0,0} + \lambda_{10}^{0,0} \psi_{01}^{0,0} \\
\overline{\partial} \lambda_{12}^{0,0} - \psi_{01}^{0,1} \lambda_{12}^{0,0} \\
\overline{\partial} \lambda_{01}^{0,0} + \lambda_{02}^{0,1} \psi_{21}^{0,0}
\end{bmatrix}
\end{equation}
In order to evaluate
\[ D_{10} : \Omega^{0,0}(C(\mathcal{K}, \mathcal{K}')) + \Omega^{0,1}(C(\mathcal{K}, \mathcal{K})^{-1}) \to \Omega^{0,0}(C(\mathcal{K}, \mathcal{K}')) + \Omega^{0,1}(C(\mathcal{K}, \mathcal{K}')) \]
we need some auxiliary results. The relevant Dolbeault operator has been written down in (A.147). We also have to determine the action of differentials \( c_{0,-1}, c_{10} \) on Dolbeault elements \( \lambda_{n,m}^{p,q} \). Given the isomorphism (A.146), the differential
\[ c_{10} : \Omega^{0,0}(C(\mathcal{K}, \mathcal{K}')) \to \Omega^{0,0}(C(\mathcal{K}, \mathcal{K}')) \]
can be determined by making the substitutions
\[
\begin{aligned}
  u &= \frac{1}{2} \lambda_{02}^{1,0} + \lambda_{22}^{0,0}, \\
v &= \frac{1}{2} \lambda_{02}^{1,0} + \lambda_{00}^{0,0}, \\
w &= \lambda_{11}^{0,0}
\end{aligned}
\]
in (A.141). Using the fact that the extensions (A.129) are split, we have (A.150)
\[
- u \Psi_{21} - i' v \bar{\Psi}_{21} + \bar{\Psi}_{21}' w = - \left( \frac{1}{2} \lambda_{02}^{1,0} + \lambda_{22}^{0,0} \right) \Psi_{21}^{0,0} - \frac{1}{2} \lambda_{02}^{1,0} \Psi_{21}^{0,1} - \lambda_{00}^{0,0} \Psi_{01}^{0,1} + \Psi_{21}^{1,0} \lambda_{11}^{0,0} \\
+ \Psi_{01}^{0,0} \lambda_{11}^{0,0} \\
= - \lambda_{02}^{1,0} \Psi_{21}^{0,0} - \lambda_{00}^{0,0} \Psi_{01}^{0,1} + \Psi_{01}^{1,0} \lambda_{11}^{0,0} - \lambda_{22}^{0,0} \Psi_{21}^{0,1} + \Psi_{21}^{0,0} \lambda_{11}^{0,0}
\]
\[
\bar{\Psi}_{10}' u \rho + (\Psi_{10}' \boxtimes I_{K_{x}}) v - w \bar{\Psi}_{10} = \frac{1}{2} \Psi_{10}' \lambda_{02}^{1,0} + \Psi_{12}^{1,0} \lambda_{22}^{0,0} + \Psi_{10}^{0,0} \left( \frac{1}{2} \lambda_{02}^{1,0} + \lambda_{00}^{0,0} \right) - \lambda_{11}^{0,0} \Psi_{12}^{0,0}
\]
Then we obtain
\[
(A.151) \quad D_{10} = \begin{bmatrix}
\lambda_{02}^{1,0} \\
\lambda_{22}^{0,0} \\
\lambda_{00}^{0,0} \\
\lambda_{11}^{0,1} \\
\lambda_{01}^{0,1} \\
\lambda_{01}^{0,0}
\end{bmatrix}
= \begin{bmatrix}
\Psi_{01}^{1,0} \lambda_{11}^{1,0} - \lambda_{02}^{1,0} \Psi_{21}^{0,1} - \lambda_{00}^{0,0} \Psi_{01}^{0,1} \\
\Psi_{21}^{0,0} \lambda_{11}^{0,0} - \lambda_{22}^{0,0} \Psi_{21}^{0,1} \\
\Psi_{00}^{0,0} \lambda_{00}^{0,0} - \lambda_{11}^{0,0} \Psi_{00}^{0,1} \\
\Psi_{10}^{0,0} \lambda_{00}^{0,0} + \Psi_{12}^{1,0} \lambda_{22}^{0,0} - \lambda_{11}^{0,0} \Psi_{12}^{0,1} \\
\Psi_{10}^{0,0} \lambda_{02}^{1,0} + \Psi_{12}^{1,0} \lambda_{22}^{0,0} - \lambda_{11}^{0,0} \Psi_{12}^{0,1} \\
\Psi_{00}^{0,0} \lambda_{00}^{0,0} + \Psi_{01}^{0,0} \lambda_{01}^{0,0} - \lambda_{01}^{0,1} \Psi_{02}^{1,0} - \lambda_{00}^{0,0} \Psi_{02}^{1,0}
\end{bmatrix}
\]

The hypercohomology group \( \mathcal{H}^0(\mathcal{K}, \mathcal{K}') \) is isomorphic to the quotient \( \text{Ker}(D_{10})/\text{Im}(D_{0,-1}) \). We would like to compare this space with the space of degree zero morphisms \( H^0_{\text{Tr}(\mathcal{D})}(\mathcal{T}, \mathcal{T}') \). The later is the 0-th cohomology of the complex \( H_{\text{Pre-Tr}(\mathcal{D})}(\mathcal{T}, \mathcal{T}') \) defined in section 5.1, equations (75), (76). Given equations (A.145), (A.146) and (A.148), it is straightforward to check that
\[
(A.152) \quad H^l_{\text{Pre-Tr}(\mathcal{D})}(\mathcal{T}, \mathcal{T}') \simeq \mathcal{H}^l
\]
for \( l = -1, 0, 1 \). One can also show that the differentials are identical by specializing formulas (76) to the case at hand taking into account (73). Then, for \( \lambda_{n',n}^{q,p} \in H^l_D((F_n, n-1), (F_{n'}, n'-1)) \), \( n, n' = 0, 1, 2 \), \( p, q = 0, 1 \), \( l = 2q + p + n' - n \), we obtain

\[
(A.153) \quad d\lambda_{n',n}^{q,p} = \delta \lambda_{n,m}^{q,p} + \sum_{m=0}^2 \sum_{r,s=0}^1 \left[ (-1)^{p(n' - m)} \Psi_{n',n}^{s,r} \lambda_{m,n}^{q,p} - (-1)^{(r+1)(n-n') + p} \Psi_{n',n}^{s,r} \lambda_{n,m}^{q,p} \right]
\]

where by convention \( \Psi_{n,m}^{s,r} = 0 \) unless \( 2s + r + m - n = 1 \) and \( \Psi_{n',n}^{s,r} = 0 \) unless \( 2s + r + n - m = 1 \). It is a simple exercise to check that the differentials \( (A.153) \) agree with \( (A.149) \) and \( (A.151) \). This proves formula (103) in degree zero. The proof in degree one is very similar.

References

[1] R. Gopakumar and C. Vafa, On the gauge theory/geometry correspondence, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math. Vol. 23, pp. 45–63, Amer. Math. Soc., Providence, RI, 2001.

[2] R. Dijkgraaf and C. Vafa, Matrix models, topological strings, and supersymmetric gauge theories, Nucl. Phys. B 644 (2002), hep-th/0206255.

[3] R. Dijkgraaf and C. Vafa, On geometry and matrix models, Nucl. Phys. B 644 (2002), hep-th/0207106.

[4] S. Katz, D. Morrison and M.R. Plesser, Enhanced gauge symmetry in type II string theory, Nuclear Phys. B 477 (1996) 105.

[5] S. Kachru et al., Open string instantons and superpotentials, Phys. Rev. D (3) 62 (2000) 026001, 9.

[6] S. Kachru et al., Mirror symmetry for open strings, Phys. Rev. D (3) 62 (2000) 126005, 14.

[7] R. Donagi and E. Markman, Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math. Vol. 1620, pp. 1–119, Springer, Berlin, 1996.

[8] R. Donagi and E. Markman, Cubics, integrable systems, and Calabi-Yau threefolds, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc. Vol. 9, pp. 199–221, Bar-Ilan Univ., 1996.

[9] F. Ferrari, Planar diagrams and Calabi-Yau spaces, Adv. Theor. Math. Phys. 7 (2004) 619, hep-th/0309151.
[10] D. Diaconescu et al., Geometric transitions and mixed Hodge structures, 2005, preprint.

[11] E. Witten, Chern-Simons gauge theory as a string theory, The Floer Memorial Volume, edited by A.W. H. Hofer, C.H. Taubes and E. Zehnder, p. 637, Birkhäuser, 1995, hep-th/9207094.

[12] E. Witten, Perturbative gauge theory as a string theory in twistor space, Comm. Math. Phys. 252 (2004) 189, hep-th/0312171.

[13] C. Lazaroiu, Holomorphic matrix models, JHEP 0305 (2003), hep-th/0303008.

[14] P. Aspinwall and S. Katz, Computation of superpotentials for D-branes, 2004, hep-th/0412209.

[15] N. Hitchin, Twistor construction of Einstein metrics, Global Riemannian geometry (Durham, 1983), Ellis Horwood Ser. Math. Appl., pp. 115–125, Horwood, Chichester, 1984.

[16] B. Szendrői, Calabi-Yau threefolds with a curve of singularities and counterexamples to the Torelli problem, Internat. J. Math. 11 (2000) 449.

[17] R. Hartshorne, Algebraic geometry (Springer-Verlag, New York, 1977), Graduate Texts in Mathematics, No. 52.

[18] B. Szendrői, Artin group actions on derived categories of threefolds, J. Reine Angew. Math. 572 (2004) 139.

[19] D.E. Diaconescu, R. Donagi and . T.Pantev, Dijkgraaf-Vafa transitions of type $ADE$, in preparation.

[20] N. Hitchin, Stable bundles and integrable systems, Duke Math. J. 54 (1987) 91.

[21] R. Donagi, Decomposition of spectral covers, Astérisque (1993) 145, Journées de Géométrie Algébrique d’Orsay (Orsay, 1992).

[22] N.J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) 55 (1987) 59.

[23] T. Hausel and M. Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, Invent. Math. 153 (2003) 197.

[24] D. Morrison, Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians, J. Amer. Math. Soc. 6 (1993) 223.

[25] P. Deligne, Théorie de Hodge II, Publications Mathématiques de l’I.H.E.S. 40 (1972) 5.
[26] M. Douglas, D-branes, categories and N=1 supersymmetry, J. Math. Phys. 42 (2001) 2818, hep-th/0011017.
[27] P. Aspinwall and A. Lawrence, Derived categories and zero-brane stability, JHEP 0108 (2001), hep-th/0104147.
[28] E. Sharpe, D-branes, derived categories, and Grothendieck groups, Nucl.Phys.B 561 (1999), hep-th/9902116.
[29] C. Lazaroiu, Generalized complexes and string field theory, JHEP 0106 (2001).
[30] C. Lazaroiu, Unitarity, D-brane dynamics and D-brane categories, JHEP 0112 (2001), hep-th/0102183.
[31] D.E. Diaconescu, Enhanced D-brane categories from string field theory, JHEP 06 (2001), hep-th/0104200.
[32] S. Katz and E. Sharpe, D-branes, open string vertex operators, and Ext groups, Adv. Theor. Math. Phys. 6 (2002) 979.
[33] A. Bondal and M. Kapranov, Enhanced triangulated categories, Math. USSR. Sbornik 70 (1991) 93.
[34] S. Bradlow, J. Glazebrook and F. Kamber, A new look at the vortex equations and dimensional reduction, Geometry, topology and physics (Campinas, 1996), pp. 83–106, de Gruyter, Berlin, 1997.
[35] R. Dijkgraaf, Geometry and matrix models, Lectures at the KITP miniprogram: Geometry, Topology and Strings, Santa Barbara, 2003, http://online.itp.ucsb.edu/online/mp03/.
[36] M. Aganagic et al., The topological vertex, Commun.Math.Phys. 254 (2005) 425, hep-th/0305132.
[37] M. Aganagic et al., Topological strings and integrable hierarchies, 2003, hep-th/0312085.
[38] J. Fay, Theta functions on Riemann surfaces (Springer-Verlag, Berlin, 1973), Lecture Notes in Mathematics, Vol. 352.
[39] D. Mumford, Tata lectures on theta. II, Progress in Mathematics Vol. 43 (Birkhäuser Boston Inc., Boston, MA, 1984), Jacobian theta functions and differential equations, With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura.
[40] E. Verlinde and H. Verlinde, Chiral bosonization, determinants and the string partition function, Nuclear Phys. B 288 (1987) 357.
[41] A. Bilal and S. Metzger, Special geometry of local Calabi-Yau manifolds and superpotentials from holomorphic matrix models, 2005, preprint, hep-th/0503173.