Fraïssé sequences – a category-theoretic approach to universal homogeneous structures

WIESŁAW KUBIŚ

Instytut Matematyki, Akademia Świętokrzyska,
ul. Świętokrzyska 15, 25-406 Kielce, POLAND
wkubis@pu.kielce.pl

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Abstract

We present a category-theoretic approach to universal homogeneous objects, with applications in the theory of Banach spaces and in set-theoretic topology.

Disclaimer: This is only a draft, full of gaps and inaccuracies, put to the Math arXiv for the sake of reference. More complete versions are coming soon.

Contents

1 Introduction 2

2 Definitions and notation 2
  2.1 Arrows between sequences ............................................. 3

3 Fraïssé sequences 3
  3.1 Basic properties .......................................................... 4
  3.2 The existence ............................................................. 5
  3.3 Cofinality ................................................................. 6
  3.4 The back-and-forth principle .......................................... 8

4 Fraïssé sequences and functors 10

5 Retractive pairs 11
6 Applications
6.1 Universal Banach spaces ................................................................. 17
6.2 Binary trees .............................................................................. 20

1 Introduction

[...]

2 Definitions and notation

Categories will usually be denoted by letters \( \mathcal{K}, \mathcal{L}, \mathcal{M} \), etc. Let \( \mathcal{K} \) be a category. We shall write “\( a \in \mathcal{K} \)” for “\( a \) is an object of \( \mathcal{K} \)”. Given \( a, b \in \mathcal{K} \), we shall denote by \( \mathcal{K}(a, b) \) the set of all \( \mathcal{K} \)-morphisms from \( a \) to \( b \). The composition of two compatible arrows \( f \) and \( g \) will be denoted by \( g \circ f \). A subcategory of \( \mathcal{K} \) is a category \( \mathcal{L} \) such that each object of \( \mathcal{L} \) is an object of \( \mathcal{K} \) and each arrow of \( \mathcal{L} \) is an arrow of \( \mathcal{K} \) (with the same domain and codomain). We write \( \mathcal{L} \subseteq \mathcal{K} \). Recall that a subcategory \( \mathcal{L} \) of \( \mathcal{K} \) is full if \( \mathcal{L}(a, b) = \mathcal{K}(a, b) \) for every objects \( a, b \in \mathcal{L} \). We say that \( \mathcal{L} \) is cofinal in \( \mathcal{K} \) if for every object \( x \in \mathcal{K} \) there exists an object \( y \in \mathcal{L} \) such that \( \mathcal{K}(x, y) \neq \emptyset \). The opposite category to \( \mathcal{K} \) will be denoted by \( \mathcal{K}^{\text{op}} \). That is, the objects of \( \mathcal{K}^{\text{op}} \) are the objects of \( \mathcal{K} \) and arrows are reversed, i.e. \( \mathcal{K}^{\text{op}}(a, b) = \mathcal{K}(b, a) \).

Let \( \mathcal{K} \) be a category. We say that \( \mathcal{K} \) has the amalgamation property if for every \( a, b, c \in \mathcal{K} \) and for every morphisms \( f \in \mathcal{K}(a, b) \), \( g \in \mathcal{K}(a, c) \) there exist \( d \in \mathcal{K} \) and morphisms \( f' \in \mathcal{K}(b, d) \) and \( g' \in \mathcal{K}(c, d) \) such that \( f' \circ f = g' \circ g \). If, additionally, for every arrows \( f'' \), \( g'' \) such that \( f'' \circ f = g'' \circ g \) there exists a unique arrow \( h \) satisfying \( h \circ f' = f'' \) and \( h \circ g' = g'' \) then the pair \( \langle f', g' \rangle \) is a pushout of \( \langle f, g \rangle \). Reversing the arrows, we define the reversed amalgamation and the pullback. We say that \( \mathcal{K} \) has the joint embedding property if for every \( a, b \in \mathcal{K} \) there exists \( g \in \mathcal{K} \) such that both sets \( \mathcal{K}(a, g), \mathcal{K}(b, g) \) are nonempty.

Fix a category \( \mathcal{K} \) and fix an ordinal \( \delta > 0 \). An inductive \( \delta \)-sequence in \( \mathcal{K} \) is formally a covariant functor from \( \delta \) (treated as a set category) into \( \mathcal{K} \). In other words, it could be described as a pair of the form \( \langle \{a_\xi\}_{\xi<\delta}, \{a_\eta^\xi\}_{\xi<\eta<\delta} \rangle \), where \( \delta \) is an ordinal, \( \{a_\xi\}_{\xi<\delta} \subseteq \mathcal{K} \) and \( a_\eta^\xi \in \mathcal{K}(a_\xi, a_\eta) \) are such that \( a_\eta^\xi a_\xi^\eta = a_\eta^\xi \) for every \( \xi < \eta < \varphi < \delta \).

We shall denote such a sequence shortly by \( \vec{a} \). The ordinal \( \delta \) is the length of \( \vec{a} \).

Let \( \kappa \) be an infinite cardinal. A category \( \mathcal{K} \) is \( \kappa \)-continuous if all inductive sequences of length \( \varphi < \kappa \) have colimits in \( \mathcal{K} \). Every category is \( \aleph_0 \)-continuous, since the colimit of a finite sequence is its last object. More generally, we say that a category \( \mathcal{K} \) is relatively \( \kappa \)-continuous in \( \mathcal{L} \), if \( \mathcal{K} \subseteq \mathcal{L} \) and every sequence in \( \mathcal{K} \) of length \( \varphi < \kappa \) has a colimit in \( \mathcal{L} \).

A category \( \mathcal{K} \) is \( \kappa \)-bounded if for every inductive sequence \( \vec{x} \) in \( \mathcal{K} \) of length \( \varphi < \kappa \) there exist \( y \in \mathcal{K} \) and a cocone of arrows \( \{y_\alpha\}_{\alpha<\lambda} \) such that \( y_\alpha: x_\alpha \to y \) and \( y_\beta \circ x_\alpha^\beta = y_\alpha \) for
every $\alpha < \beta < \lambda$. Obviously, every $\kappa$-continuous category is $\kappa$-bounded. We shall write
“$\sigma$-continuous” and “$\sigma$-bounded” for “$\aleph_1$-continuous” and “$\aleph_1$-bounded” respectively.

We shall need the following notion concerning families of arrows. Fix a family of arrows
$F$ in a given category $\mathcal{K}$. We shall write $\text{Dom}(F)$ for the set \{dom($f$): $f \in F$\}. We say
that $F$ is dominating in $\mathcal{K}$ if the family of objects $\text{Dom}(F)$ is cofinal in $F$ and moreover
for every $a \in \text{Dom}(F)$ and for every arrow $f: a \to x$ in $\mathcal{K}$ there exists an arrow $g$ in $\mathcal{K}$
such that $g \circ f \in F$.

2.1 Arrows between sequences

Fix a category $\mathcal{K}$ and denote by $\text{Seq}_{<\kappa}(\mathcal{K})$ the class of all sequences in $\mathcal{K}$
which have length $<\kappa$. We shall write $\text{Seq}_{\leq\kappa}(\mathcal{K})$ instead of $\text{Seq}_{<\kappa+}(\mathcal{K})$ and $\sigma\mathcal{K}$
instead of $\text{Seq}_{<\aleph_0}(\mathcal{K})$.

We would like to turn $\text{Seq}_{<\kappa}(\mathcal{K})$ into a category in such a way that an arrow from a
sequence $\vec{a}$ into a sequence $\vec{b}$ induces an arrow from $\lim \vec{a}$ into $\lim \vec{b}$, whenever $\mathcal{K}$ is
embedded into a category in which sequences $\vec{a}$, $\vec{b}$ have colimits.

Fix two sequences $\vec{a}$ and $\vec{b}$ in a given category $\mathcal{K}$. Let $\lambda = \text{dom}(\vec{a})$, $\rho = \text{dom}(\vec{b})$. A
transformation from $\vec{a}$ to $\vec{b}$ is, by definition, a natural transformation from $\vec{a}$ into $\vec{b} \circ \varphi$,
where $\varphi: \lambda \to \rho$ is an order preserving map (i.e. a covariant functor from $\lambda$ to $\rho$).

In order to define an arrow from $\vec{a}$ to $\vec{b}$ we need to identify some transformations. Fix
two natural transformations $F: \vec{a} \to \vec{b} \circ \varphi$ and $G: \vec{a} \to \vec{b} \circ \psi$. We shall say that $F$ and
$G$ are equivalent if the following conditions hold:

(1) For every $\alpha \leq \beta$ such that $\varphi(\alpha) \leq \psi(\beta)$ we have that $b_{\psi(\beta)}^{\psi(\beta)} \circ F(\alpha) = G(\beta) \circ a_{\alpha}^{\beta}$.

(2) For every $\alpha \leq \beta$ such that $\psi(\alpha) \leq \varphi(\beta)$ we have that $b_{\varphi(\alpha)}^{\varphi(\alpha)} \circ G(\alpha) = F(\beta) \circ a_{\alpha}^{\beta}$.

It is rather clear that this defines an equivalence relation. Every equivalence class of
this relation will be called an arrow (or morphism) from $\vec{a}$ to $\vec{b}$. It is easy to check that
this indeed defines a category structure on all sequences in $\mathcal{K}$. The identity arrow of $\vec{a}$
is the equivalence class of the identity natural transformation $1_{\vec{a}}: \vec{a} \to \vec{a}$.

3 Fraïssé sequences

Below we introduce the key notion of this work.

Let $\mathcal{K}$ be a category and let $\kappa$ be a cardinal. A Fraïssé sequence of length $\kappa$ in $\mathcal{K}$ (briefly:
a $\kappa$-Fraïssé sequence) is an inductive sequence $\vec{u}$ satisfying the following conditions:

(U) For every $x \in \mathcal{K}$ there exists $\xi < \kappa$ such that $\mathcal{K}(x, u_{\xi}) \neq \emptyset$.

(A) For every $\xi < \kappa$ and for every arrow $f \in \mathcal{K}(u_{\xi}, y)$, where $y \in \mathcal{K}$, there exist $\eta \geq \xi$
and $g \in \mathcal{K}(y, u_{\eta})$ such that $u_{\xi}^{\eta} = g \circ f$.  

3
An inductive sequence satisfying (U) will be called $\mathcal{K}$-cofinal. More generally, a collection $U$ of objects of $\mathcal{K}$ is $\mathcal{K}$-cofinal if for every $x \in \mathcal{K}$ there is $u \in U$ such that $\mathcal{K}(x, u) \neq \emptyset$. Condition (A) will be called amalgamation property.

### 3.1 Basic properties

Let $\vec{v}$ be a $\kappa$-sequence in a category $\mathcal{K}$. We say that $\vec{v}$ has the extension property if the following holds:

(E) For every arrows $f: a \to b$, $g: a \to v_\alpha$ in $\mathcal{K}$, where $\alpha < \kappa$, there exist $\beta \geq \alpha$ and an arrow $h: b \to v_\beta$ such that $i_\alpha^\beta \circ g = h \circ f$.

Clearly, this condition implies (A).

**Proposition 3.1.** Let $\vec{u}$ be a $\kappa$-Fraïssé sequence in a category $\mathcal{K}$. Then $\mathcal{K}$ has the joint embedding property. Moreover, the following conditions are equivalent:

(a) $\vec{u}$ has the extension property.

(b) $\mathcal{K}$ has the amalgamation property.

**Proof.** The first statement is trivial. Assume (a) and fix arrows $f: z \to x$, $g: z \to y$. Using (U), find $h: x \to u_\alpha$, $\alpha < \kappa$. Using (E), find $\beta \geq \alpha$ and $k: y \to u_\beta$ such that $k \circ g = u_\alpha^\beta \circ h \circ f$. Thus (b) holds.

Finally, assume (b) and fix arrows $f: a \to b$ and $g: a \to u_\alpha$, $\alpha < \kappa$. Using (b), find arrows $f_1: b \to w$ and $g_1: u_\alpha \to w$ so that $f_1 \circ f = g_1 \circ g$. Using (A) for the sequence $\vec{u}$, we find $\beta \geq \alpha$ and $h: w \to u_\beta$ so that $h \circ g_1 = u_\alpha^\beta$. Thus $(h \circ f_1) \circ f = h \circ g_1 \circ g = u_\alpha^\beta \circ g$, which shows that (a) holds.

**Proposition 3.2.** Assume $\mathcal{K}$ is a category with the joint embedding property. Then every sequence in $\mathcal{K}$ satisfying condition (A) is Fraïssé.

**Proof.** Let $\vec{u}$ be a sequence in $\mathcal{K}$ satisfying (A). Fix $x \in \mathcal{K}$. Using the joint embedding property, there exist $w \in \mathcal{K}$ and arrows $f: u_0 \to w$, $g: x \to w$. Using (A), we find an arrow $h: w \to u_\xi$ such that $h \circ f = u_\xi^\xi$. Thus $\mathcal{K}(x, u_\xi) \neq \emptyset$, which shows (U).

**Proposition 3.3.** Let $\mathcal{K}$ be a category, let $\vec{u}$ be an inductive sequence of length $\kappa$ in $\mathcal{K}$ and let $S \subseteq \kappa$ be unbounded in $\kappa$.

(a) If $\vec{u}$ is a Fraïssé sequence in $\mathcal{K}$ then $\vec{u} \upharpoonright S$ is Fraïssé in $\mathcal{K}$.

(b) If $\mathcal{K}$ has amalgamation and $\vec{u} \upharpoonright S$ is a Fraïssé sequence in $\mathcal{K}$ then so is $\vec{u}$.
Proof. Assume $\vec{u}$ is a Fra"issé sequence. Then $\vec{u} \upharpoonright S$ clearly satisfies (U). In order to check (A), fix $f: u_\xi \to y$ with $\xi \in S$. Then $u_\eta^\alpha = g \circ f$ for some arrow $g$ and for some $\eta \geq \xi$. Since $S$ is unbounded in $\kappa$, there is $\alpha \in S$ such that $\alpha \geq \eta$. Then $u_\alpha^\alpha = u_\eta^\alpha \circ g \circ f$, which shows that $\vec{u} \upharpoonright S$ satisfies (A).

Now assume $\vec{u} \upharpoonright S$ is a Fra"issé sequence. Clearly, $\vec{u}$ satisfies (U). Fix $f: u_\xi \to y$, $\xi < \kappa$. Find $\alpha \in S$ with $\alpha \geq \xi$. Using the amalgamation property of $K$, find $f': u_\alpha \to z$ such that the diagram

\[
\begin{align*}
\xymatrix{ & z \ar[ld]_{u_\alpha} \ar[d]_{u_\eta} \ar[rd]^{f'} & \\
 u_\xi \ar[rr]_{g} & & y }
\end{align*}
\]

commutes for some arrow $g$ in $K$. Now, using (A) for $\vec{u} \upharpoonright S$, we can find $\beta \in S$ such that $\beta \geq \alpha$ and $h \circ f' = u_\beta^\beta$ holds for some $h: z \to u_\beta$. This shows that $\vec{u}$ satisfies (A).

A Fra"issé sequence can possibly have finite length. In that case, by Proposition 3.3(a), there is also a Fra"issé sequence of length one -- it is an object $u$ which is cofinal in $\mathfrak{K}$ and which satisfies the following version of (A): given $f \in \mathfrak{K}(u, x)$, where $x \in \mathfrak{K}$, there exists $g \in \mathfrak{K}(x, u)$ such that $g \circ f = \mathfrak{id}_u$. We shall call $u$ a Fra"issé object in $\mathfrak{K}$. Given a Fra"issé object $u$, the sequence $u \to u \to \ldots$, where each arrow is identity, is a Fra"issé sequence of length $\omega$. Thus, it follows from Theorem 3.9 below that a possible Fra"issé object is unique, up to isomorphism. Below we give a direct proof of this fact.

Proposition 3.4. Assume $u, v$ are Fra"issé objects in a category $\mathfrak{K}$. Then $u \approx v$. If moreover all arrows in $\mathfrak{K}$ are monomorphisms then every arrow $f: u \to x$ is an isomorphism.

Proof. Applying (U) for $v$, we find a morphism $f_0: u \to v$ which, using (A) for $u$, has a left inverse $g_0: v \to u$, i.e. $g_0 \circ f_0 = \mathfrak{id}_u$. Now, using (A) for $v$, we obtain an arrow $f_1: u \to v$ such that $f_1 \circ g_0 = \mathfrak{id}_u$. Observe that

\[
f_1 = f_1 \circ \mathfrak{id}_u = f_1 \circ (g_0 \circ f_0) = (f_1 \circ g_0) \circ f_0 = \mathfrak{id}_u \circ f_0 = f_0.
\]

Hence $f_0 \circ g_0 = \mathfrak{id}_u$, which shows that $f_0$ is an isomorphism.

Finally, let $f: u \to x$ be a morphism in $\mathfrak{K}$. Again by (A), $f$ has a left inverse $g: x \to u$. Assuming $g$ is a monomorphism, we deduce that $f \circ g = \mathfrak{id}_x$, because $g \circ (f \circ g) = g \circ \mathfrak{id}_x$. Thus $f$ is an isomorphism. 

3.2 The existence

We present below a simple yet useful criterion for the existence of a Fra"issé sequence. In case of sequences of length $\leq \aleph_1$, this criterion becomes a characterization.
Theorem 3.5 (Existence). Let $\kappa$ be an infinite regular cardinal and let $\mathcal{K}$ be a $\kappa$-bounded category which has the amalgamation property and the joint embedding property. Assume further that $\mathcal{F} \subseteq \text{Arr}(\mathcal{K})$ is dominating in $\mathcal{K}$ and $|\mathcal{F}| \leq \kappa$. Then there exists a Fraïssé sequence $\vec{u}$ of length $\kappa$ in $\mathcal{K}$.

Proof. Let $\text{Dom}(\mathcal{F}) = \{a_\alpha\}_{\alpha<\kappa}$ and enumerate $\mathcal{F}$ as $\{f_\alpha\}_{\alpha<\kappa}$ so that for each $f \in \mathcal{F}$ the set $\{\alpha : f = f_\alpha\}$ has cardinality $\kappa$. We shall construct inductively the sequence $\vec{u}$, so that the following conditions are satisfied:

(i) $u_\eta^\alpha \circ u_\xi^\eta = u_\xi^\alpha$ for every $\eta < \xi < \alpha$.

(ii) $u_\alpha \in \text{Dom}(\mathcal{F})$ and $\mathcal{K}(a_\alpha, u_\alpha) \neq \emptyset$.

(iii) Given $\xi < \alpha$, if $\text{dom}(f_\alpha) = u_\xi$ then there exists an arrow $h$ in $\mathcal{K}$ such that $h \circ f_\alpha = u_\alpha^\xi$.

We start with $u_0 = a_0$. Assume that $\beta < \kappa$ is such that $u_\xi$ and $u_\eta^\alpha$ have been constructed for all $\xi < \eta < \beta$. Using the fact that $\mathcal{K}$ is $\kappa$-bounded, find $v \in \mathcal{K}$ and $j_\beta : u_\alpha \to v$ such that $j_\xi = j_\eta \circ u_\xi^\eta$ holds for every $\xi < \eta < \beta$. Using the joint embedding property, we may ensure that $\mathcal{K}(a_\beta, v) \neq \emptyset$. Now, if $f_\beta : u_\alpha \to y$ and $\alpha < \beta$ then using amalgamation we may find arrows $h : v \to w$ and $g : y \to w$ so that $g \circ f_\beta = h \circ j_\alpha$ holds. Using (D1), we may further assume that $w \in \text{Dom}(\mathcal{F})$. Finally, set $u_\beta := w$ and $u_\xi^\beta := h \circ j_\xi$ for $\xi < \beta$. It is clear that conditions (i) – (iii) hold.

It follows that the construction can be carried out. It remains to check that $\vec{u} : \kappa \to \mathcal{K}$ is a Fraïssé sequence. Condition (i) says that $\vec{u}$ is indeed an inductive sequence. Conditions (D1) and (ii) imply (U). In order to justify (A), fix $\xi < \kappa$ and $f \in \mathcal{K}(u_\xi, x)$, where $x \in \mathcal{K}$. We need to find $\alpha > \xi$ and an arrow $g$ so that $g \circ f = u_\alpha^\xi$. Since $u_\xi \in \text{Dom}(\mathcal{F})$, using (D2), we can find $g \in \mathcal{F}$ such that $g = k \circ f$ for some arrow $k$. Now find $\alpha > \xi$ such that $f_\alpha = g$. By (iii), $h \circ g = u_\alpha^{\xi+1}$ for some arrow $h$. Hence $(h \circ k) \circ f = u_\alpha^{\alpha+1}$, which completes the proof.

3.3 Cofinality

Below we discuss the crucial property of a Fraïssé sequence: cofinality in the category of sequences.

Theorem 3.6 (Countable Cofinality). Assume $\vec{u}$ is a Fraïssé sequence in a category with amalgamation $\mathcal{K}$. Then for every countable inductive sequence $\vec{x}$ in $\mathcal{K}$ there exists a morphism of sequences $F : \vec{x} \to \vec{u}$.
Proof. We use the extension property (property (E)) of the sequence $\vec{u}$, which is equivalent to the amalgamation property of $\mathfrak{K}$ (Proposition 3.1). Let $\vec{x}$ be an $\omega$-sequence in $\mathfrak{K}$. Using (U), find an arrow $f_0 : x_0 \rightarrow u_{\alpha_0}$. Now assume that arrows $f_0, \ldots, f_{n-1}$ have been defined so that $f_m : x_m \rightarrow u_{\alpha_m}$ and the diagram

$$
\begin{array}{ccc}
x_\ell & \xrightarrow{f_\ell} & u_{\alpha_\ell} \\
\downarrow & & \downarrow \\
x_k & \xrightarrow{f_k} & u_{\alpha_k}
\end{array}
$$

commutes for every $k < \ell < n$ (in particular $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{n-1}$). Using (E), find $\alpha_n \geq \alpha_{n-1}$ and an arrow $f : x_n \rightarrow u_{\alpha_n}$ so that $f \circ x_n = u_{\alpha_{n-1}} \circ f_{n-1}$ and define $f_n := f$. Given $m < n - 1$, by the induction hypothesis, we get

$$f_n \circ x_m = f_n \circ x_n \circ x_{n-1} = u_{\alpha_{n-1}} \circ f_{n-1} \circ x_m = u_{\alpha_{n-1}} \circ u_{\alpha_{n-1}} \circ f_m = u_{\alpha_m} \circ f_m.$$

Finally, setting $F = \{f_n\}_{n \in \omega}$, we obtain the required morphism $F : \vec{x} \rightarrow \vec{u}$. \qed

The above proof can be easily extended to uncountable sequences, assuming continuity:

**Theorem 3.7.** Let $\mathfrak{K}$ be a category with the amalgamation property and let $\vec{u}$ be a Fraïssé sequence of a regular length $\kappa$ in $\mathfrak{K}$. Then for every continuous sequence $\vec{x} \in \text{Seq}_{\leq \kappa}(\mathfrak{K})$ there exists an arrow of sequences $F : \vec{x} \rightarrow \vec{u}$.

**Proof.** We repeat the construction from the proof of Theorem 3.6. In the case of a limit ordinal $\delta$, we let $\alpha_\delta$ to be the supremum of $\{\alpha_\xi : \xi < \delta\}$ and we define $f_\delta$ to be the unique arrow satisfying $f_\delta \circ x_\alpha = f_\alpha$ for every $\alpha < \delta$. This is possible, because $x_\delta$ together with the cocone of arrows $\{x_\xi\}_{\xi < \delta}$ is, by assumption, the colimit of $\vec{x} \upharpoonright \delta$. Thus, the construction from the proof of Theorem 3.6 can be carried out, obtaining the desired arrow $F : \vec{x} \rightarrow \vec{u}$. \qed

We shall see later that an uncountable Fraïssé sequence may not be cofinal for $\omega_1$-sequences. From Theorem 3.6 we immediately get the following characterization of the existence of a Fraïssé sequence of length $\omega_1$.

**Corollary 3.8.** Let $\mathfrak{K}$ be a category with the amalgamation property. There exists a Fraïssé sequence of length $\omega_1$ in $\mathfrak{K}$ if and only if $\mathfrak{K}$ is $\sigma$-bounded and dominated by a family of at most $\aleph_1$ arrows.

**Proof.** The “if” part is a special case of Theorem 3.5. Let $\vec{u}$ be an $\omega_1$-Fraïssé sequence in $\mathfrak{K}$. Then $\mathfrak{K}$ has the joint embedding property and the family $\{u_\beta : \alpha \leq \beta < \omega_1\}$ is dominating in $\mathfrak{K}$. Fix $\vec{x} \in \sigma \mathfrak{K}$. Theorem 3.6 says that there exists an arrow of sequences $f : \vec{x} \rightarrow \vec{u}$, so some $u_\alpha$ provides a bound for $\vec{x}$. Thus, every countable sequence is bounded in $\mathfrak{K}$. \qed
3.4 The back-and-forth principle

Fix a category \( \mathcal{K} \) and let \( \vec{u}, \vec{v} \) be Fraïssé sequences in \( \mathcal{K} \). We shall say that \( \langle \vec{u}, \vec{v} \rangle \) satisfies the back-and-forth principle if for every \( \alpha \) below the length of \( \vec{u} \), for every arrow \( f: u_\alpha \to \vec{v} \) there exists an isomorphism of sequences \( h: \vec{u} \to \vec{v} \) such that \( h \circ i_\alpha = f \), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
\vec{u} & \xrightarrow{h} & \vec{v} \\
\downarrow{i_\alpha} & & \downarrow{f} \\
u_\alpha & & \vec{v}
\end{array}
\]

Since there exists at least one arrow \( f: u_0 \to \vec{v} \), this implies that \( \vec{u} \approx \vec{v} \). It turns out that countable Fraïssé sequences always satisfy the back-and-forth principle. We shall see in Section 6.2 that this not true for sequences of length \( \omega_1 \).

**Theorem 3.9** (Uniqueness). Assume that \( \vec{u}, \vec{v} \) are \( \omega \)-Fraïssé sequences in a given category \( \mathcal{K} \). Assume further that \( k, \ell < \omega \) and \( f: u_k \to v_\ell \) is an arrow in \( \mathcal{K} \). Then there exists an isomorphism \( F: \vec{u} \to \vec{v} \) in \( \sigma \mathcal{K} \) such that the diagram

\[
\begin{array}{ccc}
\vec{u} & \xrightarrow{F} & \vec{v} \\
\downarrow{u_k} & & \downarrow{v_\ell} \\
u_k & & v_\ell \\
\end{array}
\]

commutes. In particular \( \vec{u} \approx \vec{v} \).

Notice that in the above statement we do not assume that the given category has the amalgamation property.

**Proof.** We construct inductively arrows \( f_n: u_{k_n} \to v_{\ell_n} \), \( g_n: v_{\ell_n} \to u_{k_{n+1}} \), where \( k_0 \leq \ell_0 < k_1 \leq \ell_1 < \ldots \) and for each \( n \in \omega \) the diagram

\[
\begin{array}{ccc}
u_{k_n} & \xrightarrow{f_n} & u_{k_{n+1}} \\
\downarrow{f_{n+1}} & & \downarrow{g_n} \\
v_{\ell_n} & & u_{k_{n+1}} \\
\end{array}
\]

commutes.

We start with \( f_0 := f \), \( k_0 := k \), \( \ell_0 := \ell \), possibly replacing \( f \) by some arrow of the form \( j_{k_0}^m \circ f \) to ensure that \( k_0 \leq \ell_0 \). Using property (A) of the sequence \( \vec{u} \), find \( k_1 > k_0 \) and \( g_1: v_0 \to u_{k_1} \) such that \( g_1 \circ f = u_{k_1}^0 \). Assume that \( f_m, g_m \) have already been constructed for \( m \leq n \). Using the amalgamation of \( \vec{v} \), find \( \ell_{n+1} \geq k_{n+1} \) and an arrow \( f_{n+1}: u_{k_{n+1}} \to v_{\ell_{n+1}} \) such that \( f_{n+1} \circ g_n = v_{\ell_{n+1}}^{k_{n+1}} \). Now, using the amalgamation of \( \vec{u} \), we
find \( k_{n+2} > \ell_{n+1} \) and an arrow \( g_{n+1} : u_{\ell_{n+1}} \to u_{k_{n+2}} \) such that \( g_{n+1} \circ f_{n+1} = u_{k_{n+1}}^n \). By the induction hypothesis, \( g_n \circ f_n = u_{k_n}^n \), therefore the above diagram commutes. This finishes the construction.

Finally, set \( F = \{ f_n \}_{n \in \omega} \) and \( G = \{ g_n \}_{n \in \omega} \). Then \( F : \vec{u} \to \vec{v} \), \( G : \vec{v} \to \vec{u} \) are morphisms of sequences and by a simple induction we show that

\[
(*) \quad g_n \circ v_{\ell_n}^m \circ f_m = u_{k_m}^n \quad \text{and} \quad f_n \circ u_{k_m+1}^n \circ g_m = v_{\ell_m}^n
\]

holds for every \( m < n < \omega \). This shows that \( F \circ G = \text{id}_\vec{v} \) and \( G \circ F = \text{id}_\vec{u} \), therefore \( F \) is an isomorphism. The equality \( v_0^\infty \circ f = F \circ u_0^\infty \) means that \( v_0^\infty \circ f = f_n \circ u_0^k_n \) should hold for every \( n \in \omega \). Fix \( n > 0 \). Applying \((*)\) twice (with \( m = 0 \) and \( m = n - 1 \) respectively), we get

\[
f_n \circ u_0^k_n = f_n \circ g_{n-1} \circ v_0^\infty \circ f = v_{\ell_{n-1}}^\infty \circ v_{\ell_{n-1}}^\infty \circ f = v_{\ell_n}^\infty \circ f.
\]

Thus \( v_0^\infty \circ f = F \circ u_0^\infty \).

Finally, notice that, by property (U) of the sequence \( \vec{v} \), for some \( \ell < \omega \) there exists an arrow \( f : u_0 \to v_\ell \), so applying the first part we see that \( \vec{u} \approx \vec{v} \).

In general, there are examples of incomparable Fraïssé sequences of length \( \omega_1 \), so the back-and-forth principle may fail. However, in case of a continuous category the above arguments are easily generalized – this has already been done in [2]. Since our approach differs from that in [2], we shall give a detailed proof.

**Theorem 3.10.** Let \( \kappa > \aleph_0 \) be a regular cardinal and let \( \mathcal{R} \) be a \( \kappa \)-continuous category. Then every two Fraïssé sequences of length \( \kappa \) in \( \mathcal{R} \) satisfy the back-and-forth principle.

**Proof.** [...] 

***

TO DO:

- Straightforward applications: Fraïssé-Jónsson theory, reversed Fraïssé limits, the results of Droste & Göbel.

- Some examples (not too many!).
4 Fraïssé sequences and functors

Assume $\mathcal{K}$ is a category with amalgamation which has a Fraïssé sequence $\vec{u}$ of an uncountable regular length $\kappa$, but the category itself is not $\kappa$-continuous. Then there is no direct way to show that $\vec{u}$ is cofinal in $\text{Seq}_{\leq \kappa}(\mathcal{K})$. The same applies to the back-and-forth principle. In fact, both cofinality and the back-and-forth principle for uncountable sequences sometimes fails. However, in some situations we can “move” our Fraïssé sequence to a different category showing its cofinality in the new category. This makes sense only if after moving the sequence we do not use too much information. In this section we discuss preservation of Fraïssé sequences with respect to functors and we introduce the notion of a Fraïssé sequence over a functor, which is useful in some applications. We explain our motivation below.

***

Let $\mathcal{L}$ be the category of nonempty compact metric lines with increasing quotients (such maps are automatically continuous). Consider two natural subcategories of $\mathcal{L}$. Let $\mathcal{K} \subseteq \mathcal{L}$ have the same and objects as $\mathcal{L}$, while an arrow $f: X \to Y$ belongs to $\mathcal{K}$ if and only if it is right-invertible in the category of compact spaces. In other words, $f: X \to Y$ is an arrow in $\mathcal{K}$ iff $f$ is an increasing quotient and there exists a continuous (necessarily increasing) map $j: Y \to X$ such that $f \circ j = \text{id}_Y$. Finally, let $\mathcal{K}_0$ be the full subcategory of $\mathcal{K}$ whose objects are all 0-dimensional (metric compact) lines. The last category is dominated by a single arrow (see [...] and hence it has a (reversed) Fraïssé sequence $\vec{u}$ of length $\omega_1$. Now observe that $\vec{u}$ is no longer Fraïssé in $\mathcal{K}$, because it fails property (U). Further, $\vec{u}$ has property (U) when considered in $\mathcal{L}$, but it clearly fails (A) in $\mathcal{L}$. On the other hand, $\vec{u}$ satisfies the following variation of (A): given $\xi < \omega_1$ and an arrow $f: u_\xi \to y$ in $\mathcal{K}$, there are $\eta \geq \xi$ and an arrow $g: y \to u_\eta$ in $\mathcal{L}$ so that $u_\xi^g = g \circ f$. Moreover, $\mathcal{K}$ satisfies the following version of amalgamation: given arrows $f: z \to x$, $g: z \to y$ such that $f \in \mathcal{K}$ and $g \in \mathcal{L}$, there are arrows $f' \in \mathcal{L}$ and $g' \in \mathcal{K}$ with $f' \circ f = g' \circ g$. Adding the fact that the Fraïssé sequence $\vec{u}$ can be made continuous in $\mathcal{L}$, it turns out that these properties are sufficient to conclude that $\vec{u}$ is cofinal in $\mathcal{L}$ for all $\omega_1$-sequences from $\mathcal{K}$. Since every isomorphism in $\mathcal{L}$ is also an isomorphism in $\mathcal{K}_0$, we shall further conclude that $\vec{u}$ satisfies the back-and-forth principle in $\mathcal{K}_0$.

We shall come back to this example later.

***

TO DO:

- Preserving Fraïssé sequences.
- Functors with amalgamation.
- Back-and-forth Principle revisited.
5 Retractive pairs

In this section we describe a general construction on a given category, which is suitable for applications to the theory of Valdivia compacta and Banach spaces. This construction had been used by D. Scott [...] for getting certain models of unsigned $\lambda$-calculus.

We fix a category $\mathcal{K}$. Define $\mathcal{K}'$ to be the category whose objects are the objects of $\mathcal{K}$ and a morphism $f: X \to Y$ is a pair $\langle e, r \rangle$ of arrows in $\mathcal{K}$ such that $e: X \to Y$, $r: Y \to X$ and $r \circ e = \text{id}_X$. We set $e(f) := e$ and $r(f) := r$, so $f := \langle e(f), r(f) \rangle$. Given morphisms $f: X \to Y$ and $g: Y \to Z$ in $\mathcal{K}$, we define their composition in the obvious way:

$$g \circ f := \langle e(g), r(g) \rangle \circ \langle e(f), r(f) \rangle := \langle e(g) \circ e(f), r(f) \circ r(g) \rangle.$$

It is clear that this defines an associative operation on compatible arrows. Further, given an object $a \in \mathcal{K}$, pair of the form $(\text{id}_a, \text{id}_a)$ is the identity morphism in $\mathcal{K}'$. Thus, $\mathcal{K}'$ is indeed a category. Note that $f \mapsto e(f)$ defines a covariant functor $e$ from $\mathcal{K}'$ into $\mathcal{K}$ and $f \mapsto r(f)$ defines a contravariant functor $r: \mathcal{K}' \to \mathcal{K}$.

Now let $f: Z \to X$ and $g: Z \to Y$ be arrows in $\mathcal{K}'$. We say that arrows $h: X \to W$, $k: Y \to W$ provide a proper amalgamation of $f, g$ if $h \circ f = k \circ g$ and moreover $e(g) \circ r(f) = r(k) \circ e(h)$, $e(f) \circ r(g) = r(h) \circ e(k)$ hold. Translating it back to the original category $\mathcal{K}$, this means that the following four diagrams commute:

We draw arrows $\longrightarrow$ and $\longleftarrow$ in order to indicate mono- and epimorphisms respectively. We shall say that $\mathcal{K}'$ has proper amalgamations if every pair of arrows in $\mathcal{K}'$ with common domain can be properly amalgamated in $\mathcal{K}'$.

Below is a useful criterion for the existence of proper amalgamations.

Lemma 5.1. Let $\mathcal{K}$ be a category and let $f, g$ be arrows in $\mathcal{K}'$ with the same domain. If $e(f), e(g)$ have a pushout in $\mathcal{K}$ then $f, g$ can be properly amalgamated in $\mathcal{K}'$.

Proof. Let $h: X \to W$ and $k: Y \to W$ form a pushout of $e(f), e(g)$. Consider the following diagram:
The dotted arrows indicate unique morphisms completing appropriate diagrams, i.e. \( j \) is the unique arrow satisfying equations \( j \circ h = e(g) \circ r(f), j \circ k = e(h) \circ r(g) \) and \( \ell \) is the unique arrow satisfying equations \( \ell \circ k = e(f) \circ r(g), \ell \circ h = e(g) \circ r(f) \). Consequently, \( \langle k, j \rangle \) and \( \langle h, \ell \rangle \) are morphisms in \( \downarrow \mathfrak{K} \). Set \( s = r(f) \circ \ell \). Then

\[
(1) \quad s \circ k = r(f) \circ \ell \circ k = r(f) \circ e(f) \circ r(g) = r(g) \quad \text{and} \quad s \circ h = r(f) \circ \ell \circ h = r(f).
\]

Recall that \( r(f) e(f) = \text{id}_Z = r(g) e(g) \). Since \( k, h \) is a pushout of \( e(f), e(g) \), we deduce that \( s \) must be the unique arrow satisfying (1). Now let \( t = r(g) \circ j \). Similar computations show that \( t \circ k = r(g) \) and \( t \circ h = r(f) \), therefore by uniqueness we deduce that \( s = t \) or, in other words, \( r(f) \circ \ell = r(g) \circ j \). This shows that the full diagram is commutative and hence \( \langle k, j \rangle \) and \( \langle h, \ell \rangle \) provide a proper amalgamation of \( f, g \) in the category \( \downarrow \mathfrak{K} \) \( \Box \).

As an example, if \( \mathfrak{K} \) is the category of nonempty sets, then Lemma 5.1 says that category \( \downarrow \mathfrak{K} \) has proper amalgamations. We show below that a typical amalgamation in \( \downarrow \mathfrak{K} \) may not be proper.

**Example 5.2.** Consider the category of nonempty sets \( \mathcal{Set}^+ \). Let \( a, b, c, d \) be pairwise distinct elements and set \( Z = \{ a \}, X = \{ a, b \}, Y = \{ a, c \} \) and \( W = \{ a, b, c \} \). We are going to define arrows \( f: Z \to X, g: Z \to Y, h: X \to W \) and \( k: Y \to W \) in the category \( \downarrow \mathcal{Set}^+ \). Let \( e(f), e(g), e(h) \) and \( e(k) \) be the inclusion maps and let \( r(f) \) and \( r(g) \) be the obvious constant maps. Finally, let \( r(h)(c) = a \) and \( r(k)(b) = c \). This already defines \( r(h) \) and \( r(k) \), since these maps must be identity on the ranges of \( e(h) \) and \( e(k) \) respectively. It is clear that \( h \circ f = k \circ g \), i.e. \( h, k \) amalgamate \( f, g \) in the category \( \downarrow \mathcal{Set}^+ \). On the other hand, \( e(g) \circ r(f)(b) = a \) and \( r(k) \circ e(h)(b) = c \), therefore \( e(g) \circ r(f) \neq r(k) \circ e(h) \). Note that actually \( e(f) \circ r(g) = r(h) \circ e(k) \) holds, although redefining \( r(h)(c) \) to \( b \) we can get \( e(f) \circ r(g) \neq r(h) \circ e(k) \).

Let \( \mathfrak{K} \) be a fixed category. We have already seen that if left-invertible arrows have a pushout in \( \mathfrak{K} \), then \( \mathfrak{K} \) has proper amalgamations. This is still insufficient to get cofinality for uncountable sequences. It turns out that usually \( \downarrow \mathfrak{K} \) is not continuous, however one can consider the following weakening of continuity, which is good enough for applications.

We say that a sequence \( \vec{x} \) in \( \downarrow \mathfrak{K} \) is *semicontinuous* if \( e[\vec{x}] \) is continuous in \( \mathfrak{K} \). The dual notion of semicontinuity with respect to the functor \( r \) can be obtained by considering \( \downarrow \mathfrak{K}^{\text{op}} \) instead of \( \downarrow \mathfrak{K} \).

**Theorem 5.3.** Assume \( \mathfrak{K} \) is a category and \( \vec{u} \) is a semicontinuous Fraïssé sequence in \( \downarrow \mathfrak{K} \) of a regular length \( \kappa \). If \( \downarrow \mathfrak{K} \) has proper amalgamations then for every semicontinuous sequence \( \vec{x} \in \text{Seq}_{\leq \kappa}(\downarrow \mathfrak{K}) \) there exists an arrow of sequences \( \vec{f}: \vec{x} \to \vec{u} \).

**Proof.** We construct a sequence of arrows \( f_\alpha: x_\alpha \to u_{\varphi(\alpha)} \) so that \( \{ \varphi(\alpha) \}_{\alpha < \kappa} \) is strictly increasing and
(i) $\xi < \eta \implies u^\varphi(\eta)_{\varphi(\xi)} \circ f_\xi = f_\eta \circ x^\eta_\xi$,

(ii) $\xi < \eta \implies r \left( u^\varphi(\eta)_{\varphi(\xi)} \right) \circ e(f_\eta) = e(f_\xi) \circ r(x^\eta_\xi)$. 

We start with $f_0: x_0 \to u^\varphi(0)$ obtained from the fact that $\vec{u}$ is Fraïssé. Fix an ordinal $\beta > 0$ and assume $f_\xi$ have been defined for all $\xi < \beta$.

Suppose first that $\beta = \alpha + 1$. Find arrows $g: x_{\alpha+1} \to y$, $h: u^\varphi(\alpha) \to y$ which provide a proper amalgamation of $x^\alpha_{\alpha+1}$ and $f_\alpha$. Using the amalgamation property of $\vec{u}$, find $\varphi(\alpha+1) > \varphi(\alpha)$ and an arrow $k: y \to u^\varphi(\alpha+1)$ such that $x^\varphi(\alpha+1) = k \circ h$. The situation is described in the following diagram:

```
\[ \cdots \u_{\varphi(\alpha)} \u_{\varphi(\alpha)} \u_{\varphi(\alpha+1)} \u_{\varphi(\alpha+1)} \cdots \]
```

\[ f_\alpha \]

\[ k \]

\[ \cdots x_\alpha \]

\[ x^\alpha_{\alpha+1} \]

\[ x_{\alpha+1} \]

\[ \cdots \]

Define $f_{\alpha+1} = k \circ g$. By the induction hypothesis, it suffices to check (i) and (ii) with $\xi = \alpha$ and $\eta = \alpha + 1$. That (i) holds follows from the above commutative diagram. It remains to check that $r \left( u^\varphi(\alpha+1)_{\varphi(\alpha)} \right) \circ e(f_{\alpha+1}) = e(f_\alpha) \circ r(x^\alpha_{\alpha+1})$. We know that

\[
(1) \quad r(h) \circ e(g) = e(f_\alpha) \circ r(x^\alpha_{\alpha+1}),
\]

because $g, h$ properly amalgamate $x^\alpha_{\alpha+1}$ and $f_\alpha$. On the other hand,

\[
(2) \quad r \left( u^\varphi(\alpha+1)_{\varphi(\alpha)} \right) \circ e(k) = r(h) \circ r(k) \circ e(k) = r(h).
\]

Using (1) and (2) we get

\[
r \left( u^\varphi(\alpha+1)_{\varphi(\alpha)} \right) \circ e(f_{\alpha+1}) = r \left( u^\varphi(\alpha+1)_{\varphi(\alpha)} \right) \circ e(k) \circ e(g) = r(h) \circ e(g) = e(f_\alpha) \circ r(x^\alpha_{\alpha+1}).
\]

Suppose now that $\beta$ is a limit ordinal. Define $\varphi(\beta) = \sup_{\xi < \beta} \varphi(\xi)$. Then $\varphi(\beta) < \kappa$ is a limit ordinal, because the sequence $\{ \varphi(\xi) \}_{\xi < \beta}$ was assumed to be strictly increasing. Thus, $u^\varphi(\beta)$ together with the cocone of arrows $\{ \theta^\beta_\xi \}_{\xi < \beta}$ is the colimit of the sequence $\vec{v}: \beta \to \mathfrak{K}$, where $v_\xi = u^\varphi(\xi)$ and $v^\eta_\xi = e \left( x^\varphi(\eta)_{\varphi(\xi)} \right)$. Thus, by (i), there exists a unique arrow $e: x_\beta \to u^\varphi(\beta)$ such that the diagram

\[
\begin{array}{ccc}
\u_{\varphi(\xi)} & \xrightarrow{e(v^\eta_\xi)} & \u_{\varphi(\beta)} \\
\v_{\xi} \downarrow & & \downarrow \v_{\beta} \\
x_\xi & \xrightarrow{e(x^\eta_\xi)} & x_\beta
\end{array}
\]

(3)
commutes in $\mathcal{K}$ for every $\xi < \beta$. Similarly, using (ii) and the fact that $x_\beta$ is the colimit of $e[x \upharpoonright \beta]$ in $\mathcal{K}$, we find a unique arrow $r: u_{\varphi(\beta)} \to x_\beta$ such that the diagram

\[
\begin{array}{ccc}
\alpha \downarrow & u_{\varphi(\xi)} & \rightarrow & u_{\varphi(\beta)} \\
\downarrow r(f_\xi) & \downarrow & \downarrow r \\
\alpha \downarrow & x_\xi & \rightarrow & x_\beta \\
\end{array}
\]

commutes for each $\xi < \beta$. By uniqueness, we get $r \circ e = \mathbb{1}_{x_\beta}$, so setting $f_\beta := \langle e, r \rangle$ we define an arrow in $\mathcal{K}$. Diagram (4) says that (ii) holds with $\eta := \beta$. Fix $\xi < \beta$. Diagram (3) says that $e(f_\beta) \circ e(x_\xi^\beta) = e\left(u_{\varphi(\xi)}^{\varphi(\beta)}\right) \circ e(f_\xi)$ holds. It remains to show that $r(x_\xi^\beta) \circ r(f_\beta) = r(f_\xi) \circ r\left(u_{\varphi(\xi)}^{\varphi(\beta)}\right)$ holds for every $\xi < \beta$. For fix $\xi < \beta$ and let $p_\alpha := r(f_\xi) \circ r(u_{\varphi(\xi)})$ for $\alpha \geq \varphi(\xi)$. If $\alpha < \alpha'$ then

$$p_{\alpha'} \circ e(u_{\alpha'}^\xi) = r(f_\xi) \circ r(u_{\varphi(\xi)}^{\varphi(\beta)}) \circ e(u_{\varphi(\xi)}) \circ e(u_{\alpha'}^\xi) = r(f_\xi) \circ r(u_{\varphi(\xi)}) \circ e(u_{\alpha'}^\xi) = p_\alpha.$$ 

Thus, by semicontinuity, there exists a unique arrow $q: u_{\varphi(\beta)} \to x_\beta$ in $\mathcal{K}$ satisfying $p_\alpha \circ e(u_{\varphi(\beta)}^{\varphi(\beta)}) = q$ for $\alpha \in [\varphi(\xi), \varphi(\beta)]$. On the other hand, $q := r(x_\xi^\beta) \circ r(f_\beta)$ satisfies this. In particular, setting $\alpha := \varphi(\xi)$, we get $r(x_\xi^\beta) \circ r(f_\beta) = r(f_\xi) \circ r\left(u_{\varphi(\xi)}^{\varphi(\beta)}\right)$. This shows that the construction can be carried out until we reach the length of the sequence $\bar{x}$, which is assumed to be not greater than $\kappa$. This completes the proof. \hfill \square

In order to apply the above theorem, we shall need the following fact about sequences of left-invertible arrows.

**Proposition 5.4.** Let $\bar{x}$ be a continuous sequence in a category $\mathcal{K}$ and assume that each bonding arrow $x_\alpha^\beta$ is left-invertible in $\mathcal{K}$. Then there exists a sequence $\bar{y}$ in $\mathcal{K}$ such that $\bar{x} = e[\bar{y}]$.

**Proof.** [...] \hfill \square

We now turn to the question of homogeneity and uniqueness.

**Theorem 5.5.** Let $\mathcal{K}$ be a category. Every two semicontinuous Fraïssé sequences of the same regular length in $\mathcal{K}$ satisfy the back-and-forth principle.

**Proof.** Let $\bar{u}, \bar{v}$ be semicontinuous Fraïssé sequences of length $\kappa = \text{cf} \kappa$ in $\mathcal{K}$ and let $f: u_0 \to \bar{v}$ be given. We define inductively strictly increasing functions $\varphi: \kappa \to \kappa$, $\psi: \kappa \to \kappa$ and arrows $f_\alpha: u_{\varphi(\alpha)} \to v_{\varphi(\alpha)}$, $g_\alpha: v_{\varphi(\alpha)} \to u_{\varphi(\alpha+1)}$ in $\mathcal{K}$, so that

(i) $\alpha \leq \psi(\alpha) < \varphi(\alpha) \leq \psi(\alpha + 1)$. 

14
(ii) \( g_\alpha \circ f_\alpha = u^{\psi(\alpha+1)}_\psi \) and \( f_{\alpha+1} \circ g_\alpha = v^{\psi(\alpha+1)}_\varphi \).

(iii) \( \xi < \eta \implies v^{\psi(\eta)}_\varphi \circ f_\xi = f_\eta \circ v^{\psi(\eta)}_\psi \).

We start with \( \psi(0) = 0, \varphi(0) = \alpha \) and \( f_0 = f \), where \( 0 < \alpha < \kappa \) is such that \( f \) is equivalent to \( f_0 : u_0 \to v_\alpha \).

Fix \( \beta > 0 \) and assume that \( \varphi \upharpoonright \beta, \psi \upharpoonright \beta \), \( \{f_\alpha\}_{\alpha < \beta} \) and \( \{g_\alpha\}_{\alpha < \beta} \) have already been defined. In case where \( \beta \) is a successor ordinal, we proceed like in the proof of Theorem 3.10 using the fact that both sequences are Fraïssé, we first define \( f_\beta \) and later \( g_\beta \), so that (i)–(iii) hold. So assume \( \beta \) is a limit ordinal.

Let \( \rho := \sup_{\alpha < \beta} \varphi(\alpha) = \sup_{\alpha < \beta} \psi(\alpha) \). Define \( \varphi(\beta) := \psi(\beta) := \rho \). Semicontinuity says that \( u_\rho \) with the cocone of arrows \( \{e(u^e_{\psi(\alpha)})\}_{\alpha < \beta} \) is the colimit of the sequence \( e[u \circ (\varphi \upharpoonright \beta)] \) in \( \mathcal{F} \). Similarly, \( v_\rho \) is the colimit of \( e[v \circ (\varphi \upharpoonright \beta)] \). Thus, there exist unique arrows \( p : u_\rho \to v_\rho, q : v_\rho \to u_\rho \) in \( \mathcal{F} \) such that the diagrams

\[
\begin{array}{ccc}
  u_\rho & \xrightarrow{p} & v_\rho \\
  e(u^e_{\psi(\alpha)}) \downarrow & & \downarrow e(v^e_{\varphi(\alpha)}) \\
  u_{\psi(\alpha)} & \xrightarrow{f_\alpha} & v_{\varphi(\alpha)} \\
\end{array}
\quad
\begin{array}{ccc}
  u_\rho & \xleftarrow{q} & v_\rho \\
  e(u^e_{\psi(\alpha+1)}) \downarrow & & \downarrow e(v^e_{\varphi(\alpha)}) \\
  u_{\psi(\alpha+1)} & \xleftarrow{g_\alpha} & v_{\varphi(\alpha)} \\
\end{array}
\]

commute for every \( \alpha < \beta \). In particular, \( q \circ p = \text{id}_{u_\rho} \) and \( p \circ q = \text{id}_{v_\rho} \), so \( f_\beta := \langle p, q \rangle \) is an arrow of \( \mathcal{F} \) (in fact, it is an isomorphism between \( u_\rho \) and \( v_\rho \)). We need to check (iii) with \( \eta := \beta \). For fix \( \xi < \beta \). The first diagram in (3) says that

\[
\begin{equation}
1 \quad e\left(v^e_{\psi(\xi)}\right) \circ e(f_\xi) = e(f_\beta) \circ e\left(u^e_{\psi(\xi)}\right).
\end{equation}
\]

We need to show that

\[
2 \quad r(f_\xi) \circ r\left(v^e_{\psi(\xi)}\right) = r\left(u^e_{\psi(\xi)}\right) \circ r(f_\beta).
\]

We shall use the fact that \( v_\rho \) is the colimit of \( e[v \circ \varphi] \) restricted to the ordinal interval \( [\xi, \beta) \). Given \( \alpha \in [\xi, \beta) \), define

\[
k_\alpha := r(f_\xi) \circ r\left(v^e_{\psi(\alpha)}\right) = r(f_\alpha) \circ r\left(v^e_{\psi(\xi)}\right).
\]

Then \( k_\alpha : v_{\varphi(\alpha)} \to u_{\psi(\xi)} \) is an arrow in \( \mathcal{F} \). Observe that for \( \alpha < \alpha' \) we have

\[
\begin{equation}
k_{\alpha'} \circ e\left(v^e_{\varphi(\alpha)}\right) = r(f_\xi) \circ r\left(v^e_{\varphi(\alpha') \circ v_{\varphi(\alpha)}\varphi(\xi)}\right) \circ e\left(v^e_{\varphi(\alpha')}\right) = r(f_\xi) \circ r\left(v^e_{\varphi(\xi)}\right) = k_\alpha.
\end{equation}
\]
By the definition of a colimit, there exists a unique arrow \( k : u_\varphi(\xi) \to u_\psi(\xi) \) in \( \mathcal{K} \) satisfying

\[
(3) \quad k \circ e \left( v_\varphi^{(\alpha)} \right) = k_\alpha
\]

for every \( \alpha \in [\xi, \beta] \). Observe that, given \( \alpha \in [\xi, \beta] \), we have

\[
r(f_\xi) \circ r \left( v_\varphi^{\psi(\xi)} \right) \circ e \left( v_\varphi^{\psi(\alpha)} \right) = r(f_\xi) \circ r \left( v_\varphi^{\psi(\alpha)} \right) \circ e \left( v_\varphi^{\psi(\alpha)} \right) = k_\alpha.
\]

By uniqueness, it follows that

\[
k = r(f_\xi) \circ r \left( v_\varphi^{\psi(\xi)} \right).
\]

Now let

\[
\ell := r \left( u_{\psi(\xi)}^e \right) \circ q.
\]

For each \( \alpha \in [\xi, \beta] \), using the second diagram in (2), the first part of (ii) and (iii), we obtain

\[
\ell \circ e \left( v_\varphi^{\psi(\alpha)} \right) = r \left( u_{\psi(\xi)}^e \right) \circ e \left( u_{\psi(\alpha+1)}^e \right) \circ e(g_\alpha)
\]

\[
= r \left( u_{\psi(\xi)}^e \right) \circ r \left( u_{\psi(\alpha+1)}^e \right) \circ e(g_\alpha)
\]

\[
= r \left( u_{\psi(\xi)}^e \right) \circ e(g_\alpha) = r \left( u_{\psi(\xi)}^e \right) \circ r \left( u_{\psi(\alpha+1)}^e \right) \circ e(g_\alpha)
\]

\[
= r \left( u_{\psi(\xi)}^e \right) \circ r(f_\alpha) \circ r(g_\alpha)
\]

\[
= r \left( f_\alpha \circ u_{\psi(\xi)}^e \right) = r \left( v_\varphi^{\psi(\alpha)} \circ f_\xi \right)
\]

\[
= r(f_\xi) \circ r \left( v_\varphi^{\psi(\alpha)} \right) = k_\alpha.
\]

It follows that \( \ell \) satisfies (2), therefore \( k = \ell \), by uniqueness. This means that (2) is true. Recalling that \( \varrho = \varphi(\beta) = \psi(\beta) \), we have proved that \( f_\beta \circ u_{\psi(\xi)}^{\psi(\beta)} = v_{\varphi(\xi)}^{\psi(\beta)} \circ f_\xi \), i.e. condition (iii) is satisfied with \( \eta := \beta \). We still need to define \( \psi(\beta+1) \) and \( g_\beta \). Since \( f_\beta \) is invertible in \( \mathcal{K} \) (its inverse is \( (g, p) \)), we may set \( \psi(\beta+1) := \varrho + 1 \) and \( g_\beta := u_{\varphi(\xi)}^{\varrho+1} \circ f_\beta^{-1} \).

Clearly, condition (i) and the first part of (ii) are fulfilled. The second part of (ii) with \( \alpha \) replaced by \( \beta \) is taken care of in the successor step \( \beta + 1 \), which we have already justified.

Finally, like in the proof of Theorem 3.10, we deduce from conditions (i)–(iii) that \( \vec{f} = \{ f_\alpha \}_{\alpha < \kappa} \) and \( \vec{g} = \{ g_\alpha \}_{\alpha < \kappa} \) are arrows of sequences in \( \mathcal{K} \) such that \( \vec{f} \) extends \( f \) and, by condition (ii), these arrows are isomorphisms in the category of sequences.

**Corollary 5.6.** Let \( \kappa \) be a regular cardinal and let \( \mathcal{K} \) be a \( \kappa \)-continuous category. Assume that \( \mathcal{K} \) has a Fraïssé sequence of length \( \kappa \). Then \( \mathcal{K} \) also has a semicontinuous Fraïssé sequence of length \( \kappa \).
6 Applications

In this section we collect few applications of our results – mainly of those from Section 5 – to Banach spaces, Valdivia compacta and linearly ordered sets. We also describe a natural category of binary trees which has many pairwise incomparable Fraïssé sequences of length $\omega_1$.

6.1 Universal Banach spaces

[History ...]

We first discuss a Banach space which is universal for linear isometric embeddings. Its existence, under the continuum hypothesis, actually follows from the results of Droste & Göbel [2].

Let $\mathcal{B}_0^{iso}$ denote the category whose objects are separable Banach spaces and arrows are linear isometries.

Lemma 6.1. $\mathcal{B}_0^{iso}$ has the amalgamation property.

Proof. Fix $X, Y, Z \in \mathcal{B}_0^{iso}$ and fix linear isometric embeddings $f: Z \to X$ and $g: Z \to Y$. Without loss of generality, we may assume that $f$ and $g$ are inclusions, i.e. $Z \subseteq X$ and $Z \subseteq Y$. We may also assume that $X \cap Y = Z$. Now let $W$ be the formal algebraic sum of $X$ and $Y$, i.e. $W = \{x + y : x \in X, y \in Y\}$ and $x + y = x' + y'$ whenever $x - x' = y' - y$. Let $G$ be the convex hull of $B_X$ and $B_Y$, where $B_X, B_Y$ denote the closed unit balls of $X$ and $Y$ respectively. Let $\| \cdot \|$ be the norm induced by the Minkowski functional of $G$. Then the completion of $\langle W, \| \cdot \| \rangle$ is a separable Banach space which provides an amalgamation of $f$ and $g$ in $\mathcal{B}_0^{iso}$.

Clearly, $\mathcal{B}_0^{iso}$ has an initial object, the zero space. Thus, the joint embedding property follows from amalgamation. The next statement is rather clear.

Lemma 6.2. $\mathcal{B}_0^{iso}$ is $\sigma$-continuous.

Theorem 6.3. Assume CH. There exists a Banach space $V$ of density $\aleph_1$ such that every Banach space of density $\leq \aleph_1$ is linearly isometric to a subspace of $V$ and every linear isometry $T: X \to Y$ between separable subspaces of $V$ can be extended to a linear isometry of $V$. Moreover, the space $V$ is unique, up to a linear isometry.
Proof. Assuming the continuum hypothesis, there are only \(\aleph_1\) many isometric types of separable Banach spaces and there are only \(\aleph_1\) many types of linear isometries. Thus, by Lemmas 6.1 and 6.2 and by Theorem 3.3 \(B_{\aleph_0}\) has a Fraïssé sequence \(\bar{u}\) of length \(\omega_1\). We may further assume that this sequence is continuous. Let \(V\) be the colimit of \(\bar{u}\) in the category of all Banach spaces.

Fix a Banach space \(X\) of density \(\leq \aleph_1\). We can write \(X = \bigcup_{\alpha<\omega_1} X_\alpha\), where \(\{X_\alpha\}_{\alpha<\omega_1}\) is an increasing chain of closed separable subspaces of \(X\) such that \(X_\delta = \text{cl}(\bigcup_{\xi<\delta} X_\xi)\) for every limit ordinal \(\xi < \omega_1\). Translating it to the language of category theory, we obtain a continuous \(\omega_1\)-sequence in \(B_{\aleph_0}\) whose colimit, in the category of all Banach spaces, is \(X\). By Theorem 3.7 there is an arrow of sequences \(F : \bar{x} \to \bar{u}\). This arrow has a colimit in the category of all Banach spaces, which is just a linear isometric embedding of \(X\) into \(V\).

The second statement is obtained by the back-and-forth principle, using the continuity of \(\bar{u}\).

We do not know much about the space \(V\) from the above theorem, although we remark below that it cannot be isometric to any \(C(K)\) space.

**Proposition 6.4.** Let \(K\) be a compact space which contains at least two points. Then there exists a linear isometry \(T : X \to Y\) between 1-dimensional subspaces of \(C(K)\), which cannot be extended to a linear isometry of \(C(K)\).

**Proof.** Fix \(a \neq b\) in \(K\). Let \(X\) consist of all constant functions on \(K\). Let \(R : C(\{a, b\}) \to C(K)\) be a regular extension operator for the inclusion \(\{a, b\} \subseteq K\). That is, \(R\) is a linear operator which assigns to each \(f \in C(\{a, b\})\) its extension \(Rf \in C(K)\) so that \(R1 = 1\) and \(Rf \geq 0\) whenever \(f \geq 0\). For example, let \((Rf)(t) = \varphi(t)f(a) + (1 - \varphi(t))f(b)\), where \(\varphi : K \to [0, 1]\) is a continuous function such that \(\varphi(a) = 1\) and \(\varphi(b) = 0\) (which exists by Urysohn’s Lemma). Note that \(R\) is an isometric embedding of \(C(\{a, b\})\) into \(C(K)\).

Now define \(T : X \to C(K)\) by \(T1 = R1_{\{a\}}\), where \(1_{\{a\}}\) is the function which takes value 1 at \(a\) and value 0 at \(b\). Let \(Y = T[X] = \{\lambda R1_{\{a\}} : \lambda \in \mathbb{R}\}\). Then \(T\) is an isometry.

Suppose \(T : C(K) \to C(K)\) is a linear isometry extending \(T\). Let \(v = (T)^{-1}[R1_{\{b\}}]\). Then \(\|v\| = 1\). By compactness, there exists \(t \in K\) such that \(\|v(t)\| = 1\). Let \(a = v(t)\) and consider \(u = R(\alpha 1_{\{a\}} + 1_{\{b\}}) = \alpha R1_{\{a\}} + R1_{\{b\}}\). Notice that \(\|u\| = 1\), while \(\|(T)^{-1}(u)\| = \|\alpha 1_{K} + v\| \geq |\alpha + v(t)| = 2\). Hence \(\|(T)^{-1}\| \geq 2\), a contradiction.

It is well known that, under the continuum hypothesis, the compact space \(\omega^* := \beta\omega \setminus \omega\) is the universal continuous preimage for compact spaces of weight \(\leq \aleph_1\). Moreover, \(\omega^*\) is homogeneous with respect to quotients on metrizable compacta. In fact, these statements hold without any extra set-theoretic assumptions, see \([\ldots]\). Now let \(W = \ell_\infty/c_0 = C(\omega^*)\). Then \(W\) contains a linear isometric copy of any Banach space of density \(\leq \aleph_1\). Indeed, if \(X\) is such a space then \(X \subseteq C(K)\), where \(K\) is the unit ball of the

18
dual space $X^*$. Now, a quotient map $f: \omega^* \to K$ induces a linear isometric embedding of $\mathcal{C}(K)$ into $\mathcal{C}(\omega^*) = \ell_\infty/c_0$. Let $V$ be the space from Theorem 6.3. By the above proposition, $V$ is not linearly isometric to any space of the form $\mathcal{C}(K)$, where $K$ is a compact space. In particular, $V \neq \ell_\infty/c_0$. We do not know whether $V$ could be just isomorphic to $\mathcal{C}(K)$ for some compact space $K$ of weight $\aleph_1 = 2^{\aleph_0}$. Motivated by topological properties of the compact space $\beta \omega \setminus \omega$, we ask:

**Question 6.5.** Does there exist in ZFC a Banach space $V$ of density $2^{\aleph_0}$ which has properties like the space in Theorem 6.3?

We now turn to a more special class of Banach spaces, namely Banach spaces with projectional resolutions. From this point on, we consider the category $\mathcal{B}_{\aleph_0}$ whose objects are again all separable Banach spaces and arrows are linear operators of norm $\leq 1$. We shall apply the results of Section 5. Our aim is to obtain a universal Banach space with a projectional resolution of the identity or, equivalently, with a countably norming Markushevich basis.

**Lemma 6.6.** Let $f: Z \to X$, $g: Z \to Y$ be left-invertible arrows in $\mathcal{B}_{\aleph_0}$. Then $f, g$ have a pushout in $\mathcal{B}_{\aleph_0}$.

**Proof.** We may assume that $X = Z \oplus X_1$, $Y = Z \oplus Y_1$ and $f$, $g$ are the canonical embeddings. Let $W = Z \oplus X_1 \oplus Y_1$ and let $G$ be the convex hull of $B_X \cup B_Y$, where $B_X$, $B_Y$ denote the closed unit balls of $X$ and $Y$ respectively. Let $\| \cdot \|$ be the Minkowski functional of $G$. Then $W$ becomes a Banach space and there are natural embeddings $f': X \to W$, $g': Y \to W$. It is easy to see that $f'$, $g'$ is a pushout of $f$, $g$. Indeed, let $p: X \to U$, $q: Y \to U$ be arrows of $\mathcal{B}_{\aleph_0}$ such that $p \circ f = q \circ g$. There is a unique linear transformation $h: W \to U$ such that $h \circ f' = p$ and $h \circ g' = q$. Observe that $p[B_X] \subseteq B_U$ and $q[B_Y] \subseteq B_U$, therefore $h[G] \subseteq B_U$. It follows that $\|h\| \leq 1$. \hfill $\square$

**Lemma 6.7.** Let $\bar{a}: \omega \to \mathcal{B}_{\aleph_0}$ be such that $a^m_n$ is left-invertible in $\mathcal{B}_{\aleph_0}$ for every $m < n < \omega$. Then $\bar{a}$ has the colimit in $\mathcal{B}_{\aleph_0}$.

**Proof.** The assumption that $a^m_n$ is left-invertible is superfluous: it suffices to assume that each $a^m_n$ is a linear isometric embedding. We may assume that $a_m \subseteq a_n$ for $m < n$ and that $a^m_n$ is just the inclusion for $m < n$. Let $X_0 = \bigcup_{n<\omega} a_n$. Then $X_0$ is a normed linear space, endowed with the obvious norm. Let $X$ be the completion of $X_0$. We claim that $X$ together with the cocone of inclusions $a^\infty_n: a_n \to X$ is the colimit of $\bar{a}$. For fix an object $y \in \mathcal{B}_{\aleph_0}$ and a family of arrows $\{f_n\}_{n<\omega}$ such that $f_n: a_n \to y$ and $f_n = f_m \circ a^m_n$ for every $n < m < \omega$. There is a unique function $h_0: X_0 \to y$ satisfying $h_0 \circ a^\infty_n = f_n$. It is clear that $h_0$ is linear and $\|h_0\| \leq 1$. Thus, $h_0$ has a unique extension to a continuous linear transformation $h: X \to y$. Clearly, $\|h\| \leq 1$ and $h \circ a^\infty_n = f_n$ holds for every $n < \omega$. \hfill $\square$

We now have all ingredients needed to construct a universal Banach space with a PRI.
Theorem 6.8. Assume CH. There exists a Banach space $E$ with a PRI and of density $\aleph_1$, which has the following properties:

(a) The family $\{X \subseteq E: X$ is 1-complemented in $E\}$ is, modulo linear isometries, the class of all Banach spaces of density $\leq \aleph_1$ with a PRI.

(b) Given separable subspaces $X, Y \subseteq E$, norm one projections $P: E \to X$, $Q: E \to Y$ and a linear isometry $T: X \to Y$, there exist a linear isometry $S: E \to E$ extending $T$ and satisfying $P \circ S^{-1} = T^{-1} \circ Q$.

Moreover, the above properties describe the space $E$ uniquely, up to a linear isometry.

Proof. [...] \qed

6.2 Binary trees

In this subsection we describe the announced example of a category of trees which has many pairwise non-equivalent $\omega_1$-Fraïssé sequences.

By a tree we mean a partially ordered set $(T, \leq)$ which is a meet semilattice, i.e. every two elements of $T$ have the greatest lower bound, and for every $t \in T$ the interval $\{x \in T: x < t\}$ is well ordered. Every tree $T$ has a single minimal element $0_T$, called the root of $T$. An immediate successor of $t \in T$ is an element $s > t$ such that no $x \in T$ satisfies $t < x < s$. A subtree of a tree $T$ is a subset $S \subseteq T$ which is closed under the meet operation. A tree $T$ is binary if every $t \in T$ has at most two immediate successors.

We shall denote by $\text{max } T$ the set of all maximal elements of $T$. A tree $T$ is bounded if for every $x \in T$ there is $t \in \text{max } T$ such that $x \leq t$. Recall that an initial segment of a poset $(T, \leq)$ is a subset $A$ of $T$ satisfying $\{x \in T: x \leq t\} \subseteq A$ for every $t \in A$. A subset $A$ of $T$ is closed if $\sup C \in A$ for every chain $C \subseteq A$. This is equivalent to saying that $A$ is closed with respect to the interval topology on $T$ generated by intervals of the form $[0, t]$ and $(s, t]$, where $s < t$.

We define the category $\mathfrak{T}_2$ as follows. The objects of $\mathfrak{T}_2$ are nonempty countable bounded binary trees. An arrow from $T \in \mathfrak{T}_2$ into $S \in \mathfrak{T}_2$ is a semilattice embedding $f: T \to S$ such that $f[T]$ is a closed initial segment of $S$.

A tree $T$ is healthy if every element of $T \setminus \text{max}(T)$ has at least two immediate successors and for every $t \in T$ and $\alpha < \text{ht}(T)$ there exists $s \geq t$ such that $\text{Lev}_T(s) \geq \alpha$. An example of a healthy tree of height $\omega_1$ is

$$T = \{x \in 2^{<\omega_1}: |\{\alpha: x(\alpha) = 1\}| < \aleph_0\}.$$

Note that all levels of $T$ are countable. Setting $T_\alpha = \{x \in T: \text{dom}(x) \subseteq \alpha + 1\}$, we obtain an inductive sequence $\{T_\alpha\}_{\alpha < \omega_1}$ in the category $\mathfrak{T}_2$. More generally, if $S$ is any binary tree of height $\omega_1$ whose all levels are countable then, setting

$$S_\alpha = \{x \in S: \text{the order type of } [0, x) \text{ is } \leq \alpha\}$$

20
we obtain an inductive sequence $\vec{S}$ in $\mathfrak{T}_2$, where each $S_\alpha^0$ is the inclusion, which is an arrow in $\mathfrak{T}_2$. We shall say that $\vec{S}$ is the natural decomposition of $S$.

**Lemma 6.9.** Let $V \in \mathfrak{T}_2$ be a healthy tree of height $\alpha + 1$. Then every $T \in \mathfrak{T}_2$ with $\text{ht}(T) \leq \alpha + 1$ is isomorphic to a closed initial segment of $V$.

**Proof.** Denote by $\mathfrak{M}$ the class of all nonempty bounded binary trees $T$ of height $\leq \alpha + 1$ such that $\text{max}(T)$ is finite. Given such a tree $T$, write $\text{max}(T) = \{w_0, \ldots, w_{m-1}\}$ and define inductively $T_0 := [0, w_0]$ and $T_k := [0, w_k] \setminus (T_0 \cup \cdots \cup T_{k-1})$. Then $\mathcal{D} = \{T_0, \ldots, T_{m-1}\}$ is a natural decomposition of $T$ into connected chains, induced by the enumeration of $\text{max}(T)$.

Let $\mathcal{L}$ be a category whose objects are pairs $\langle T, \mathcal{D} \rangle$, where $T \in \mathfrak{M}$ and $\mathcal{D}$ is a natural decomposition into connected chains induced by an enumeration of $\text{max}(T)$, as described above. Given $\langle T, \mathcal{D} \rangle, \langle S, \mathcal{F} \rangle \in \mathcal{L}$, an arrow in $\mathcal{L}$ is a tree embedding $f: T \to S$ with the following properties:

(a) For each $D \in \mathcal{D}$ there is $F \in \mathcal{F}$ such that $f[D]$ is an initial segment of $F$.

(b) For each $F \in \mathcal{F}$ there exists at most one $D$ such that $f[D] \subseteq F$.

It is clear that these properties are preserved under the usual composition, so $\mathcal{L}$ is indeed a category. Now consider the given healthy tree $V$ with $\text{max}(V) = \{e_n\}_{n \in \omega}$ and define $V^n = \bigcup_{i \leq n} [0, e_i]$. Let $\mathcal{D}^n$ be the decomposition of $V^n$ induced by the enumeration $\{e_0, \ldots, e_n\}$ of $\text{max}(V^n)$. Then $\langle V^n, \mathcal{D}^n \rangle \in \mathcal{L}$ and $\vec{V} = \{\langle V^n, \mathcal{D}^n \rangle\}_{n \in \omega}$ is an inductive sequence in $\mathcal{L}$. We claim that:

1. $\vec{V}$ is a Fraïssé sequence in $\mathcal{L}$ which has property (E).

2. If $\vec{T}$ is an inductive sequence in $\mathcal{L}$ and $\vec{f} = \{f_n\}_{n \in \omega}$ is an embedding of $\vec{T}$ into $\vec{V}$ then the embedding $f: T \to V$ induced by $\vec{f}$ has the property that $f[T]$ is a closed initial segment of $V$.

We first show (2): Fix $y \in V \setminus f[T]$. Find $m$ and $D \in \mathcal{D}_m$ such that $y \in D$. Let $S_n$ be the natural decomposition of $T_n$, $S = \bigcup_{n \in \omega} S_n$. Then there is at most one $S \in S$ such that $f[S] \subseteq D$. Moreover $f[S]$ is closed in $D$, so $D \setminus f[S]$ is a neighborhood of $y$ disjoint from $f[T]$.

For the proof of (1), fix $\langle T, \mathcal{F} \rangle \in \mathcal{L}$ and assume $f: T \to V_n$ is an arrow in $\mathcal{L}$. Let $T \subseteq T'$ and let $\mathcal{F}' \supseteq \mathcal{F}$ so that the inclusion $T \subseteq T'$ is an arrow between $\langle T, \mathcal{F} \rangle$ and $\langle T', \mathcal{F}' \rangle$. Without loss of generality, we may assume that $\mathcal{F}' = \mathcal{F} \cup \{A\}$, i.e. $T'$ differs from $T$ by only one new branch. Let $a = \min A$. Then, by the definition of the natural decomposition, $a$ has an immediate predecessor $c \in T$ (note that $0 \in T$ so $a > 0$).

Find $F \in \mathcal{F}$ such that $c \in F$. Then $f(c)$ has exactly two immediate successors in $V$ and at most one belongs to $f[T]$, since $c$ has only two immediate successors in $T$. Let
Let \( d \in V \setminus f[T] \) be an immediate of \( f(c) \). Find a big enough \( m > n \) so that there exists \( D \in D_m \) with \( d \in D \). Since each maximal element of \( V \) has height \( \alpha \), \( D \) is a cofinal branch in \( V \) and hence \( A \) can be (uniquely) embedded into \( D \) as an initial segment. This embedding defines an extension \( \overrightarrow{T}: T' \to V_m \) of \( f \). Since \( \mathfrak{L} \) has a minimal object, this shows that \( \overrightarrow{V} \) is Fra"{i}ssé and satisfies (E).

Finally, fix \( T \in \mathfrak{S}_2 \) with \( \text{ht}(T) \leq \alpha + 1 \). Decompose \( T \) into an inductive \( \omega \)-sequence, according to a fixed enumeration of \( \text{max}(T) \). Claims (1) and (2) say that \( T \) can be embedded into \( V \) as a closed initial subtree. This completes the proof. \( \square \)

**Theorem 6.10.** Assume \( U \) is a healthy binary tree of height \( \omega_1 \), whose all levels are countable. Let \( \overrightarrow{U} \) be the natural \( \mathfrak{S}_2 \)-decomposition of \( U \). Then \( \overrightarrow{U} \) is a Fra"{i}ssé sequence in \( \mathfrak{S}_2 \) which has the extension property. In particular, \( \mathfrak{S}_2 \) has both the amalgamation and the joint embedding property.

**Proof.** Note that \( \mathfrak{S}_2 \) has a minimal object, namely the one-element tree. Clearly, such a tree embeds into \( U_0 \). It suffices to show that \( \overrightarrow{U} \) satisfies (E) – it will then follow that \( \overrightarrow{U} \) is a Fra"{i}ssé sequence. Since there exists a healthy binary tree of height \( \omega_1 \) with countable levels, we shall be able to conclude that \( \mathfrak{S}_2 \) has a Fra"{i}ssé sequence satisfying (E) and consequently \( \mathfrak{S}_2 \) has both the amalgamation and the joint embedding property. Thus, it remains to show that \( \overrightarrow{U} \) has the extension property.

Fix \( \alpha < \omega_1 \) and fix an arrow \( f: T \to U_\alpha \) in \( \mathfrak{S}_2 \) and assume that \( T \) is a closed initial subtree of \( S \), i.e. the inclusion \( T \subseteq S \) is an arrow of \( \mathfrak{S}_2 \). Fix \( \alpha < \omega_1 \) so that \( \text{ht}(S) < \alpha \). Let \( \{s_n: n \in \omega \} \) enumerate all minimal elements of \( S \setminus T \). Let \( t_n \) be the immediate predecessor of \( s_n \). Then \( t_n \in T \). Recall that \( f(t_n) \) has exactly two immediate successors in \( U_\alpha \) and at least one of them does not belong to \( f[T] \), since otherwise \( t_n \) would already have two immediate successors in \( T \). Let \( y_n \) be an immediate successor of \( f(t_n) \) which does not belong to \( f[T] \). Let \( V_n = \{y \in U_\alpha: y \geq y_n\} \). Then \( V_n \) is a healthy binary tree of countable height. By Lemma 6.9 we can embed \( G_n = \{s \in S: s \geq s_n\} \) onto a closed initial segment of \( V_n \). Combining all these embeddings, we obtain an extension \( \overrightarrow{f}: S \to U_\alpha \) of \( f \). We claim that \( \overrightarrow{f}[S] \) is closed in \( U_\alpha \). Indeed, if \( C \subseteq S \) is a chain and \( C \nsubseteq T \) then \( c \geq s_n \) for some \( c \in C \) and for some \( n \). Thus, \( \sup C \) exists in \( S \) and hence \( \sup \overrightarrow{f}[C] \) exists in \( \overrightarrow{f}[G_n] \subseteq \overrightarrow{f}[S] \). \( \square \)

**Theorem 6.11.** Assume \( \overrightarrow{U} \) and \( \overrightarrow{V} \) are \( \omega_1 \)-Fra"{i}ssé sequences in \( \mathfrak{S}_2 \) inducing healthy trees \( U \) and \( V \) respectively. Assume further that \( F: \overrightarrow{U} \to \overrightarrow{V} \) is an arrow of sequences. Then the trees \( U, V \) are isomorphic.

**Proof.** Let \( f: U \to V \) be the embedding induced by \( F \), i.e. assuming both \( \overrightarrow{U}, \overrightarrow{V} \) are chains of trees, \( f \) is the union of arrows \( f_\alpha: U_\alpha \to V_{\varphi(\alpha)} \) in \( \mathfrak{S}_2 \), where \( \varphi: \omega_1 \to \omega_1 \) is an increasing function. We claim that \( f[U] \) is closed in \( V_{\varphi(\alpha)} \). Suppose otherwise and fix a sequence \( x_0 < x_1 < \ldots \) in \( U \) such that \( y = \sup_{n \in \omega} f(x_n) \notin f[U] \). Find \( \beta < \omega_1 \) such that \( \{x_n: n \in \omega\} \subseteq U_\beta \) and \( y \in V_{\varphi(\beta)} \). Then \( f(x_n) = f_\beta(x_n) \) and \( y \notin f_\beta[U_\beta] \), which shows that \( f_\beta \) is not an arrow in \( \mathfrak{S}_2 \), a contradiction.
We finally claim that $V = f[U]$, which of course shows that $f$ is an isomorphism. Suppose $V \neq f[U]$ and fix a minimal element $y \in V \setminus f[U]$. Find $\alpha < \omega_1$ such that $y \in V_{\varphi(\alpha)}$. Since $f[U]$ is closed in $V_{\varphi(\alpha)}$, $y$ has an immediate predecessor, say $v = f(u)$. Let $a, b$ be the two immediate successors of $u$ in $U$ (which exist, because $U$ is healthy). Then either $y = f(a)$ or $y = f(b)$, because $V$ is binary and $f[U]$ is an initial segment of $U$. This is a contradiction.

It can be easily shown that every tree induced by an $\omega_1$-Fraïssé sequence in $\mathfrak{T}_2$ is healthy, therefore this assumption can be removed from the above statement.

Recall that sequences $\vec{a}$ and $\vec{b}$ of the same length are comparable if there exists an arrow of sequences $\vec{f}$ such that either $\vec{f}: \vec{a} \rightarrow \vec{b}$ or $\vec{f}: \vec{b} \rightarrow \vec{a}$. Otherwise, we say that $\vec{a}$ and $\vec{b}$ are incomparable.

**Corollary 6.12.** There exist two incomparable $\omega_1$-Fraïssé sequences in $\mathfrak{T}_2$. 

**Proof.** Let $U = \{x \in 2^{<\omega_1} : |x^{-1}(1)| < \aleph_0 \}$ and let $V$ be a healthy binary Aronszajn tree. Clearly, $U$ and $V$ are not isomorphic. Both trees can be naturally decomposed into $\omega_1$-sequences $\vec{U}$ and $\vec{V}$ respectively. By Theorem 6.10, $\vec{U}$ and $\vec{V}$ are Fraïssé sequences. By Theorem 6.11, these sequences are incomparable. 

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TO DO:

- A universal Valdivia compact of weight $\aleph_1$, under CH.
- A universal linearly ordered Valdivia compact; retractive linearly ordered sets.

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