Integrals with respect to the Euler characteristic over spaces of functions and the Alexander polynomial.

A.Campillo F.Delgado * S.M.Gusein-Zade †

Recently there were obtained some formulae which express the Alexander polynomial (and thus the zeta-function of the classical monodromy transformation) of a plane curve singularity in terms of the ring of functions on the curve: [10]. One of them describes the coefficients of the Alexander polynomial or of zeta-function of the monodromy transformation as Euler characteristics of some explicitly constructed spaces. For the Alexander polynomial these spaces are complements to arrangements of projective hyperplanes in projective spaces. Thus for the zeta-function they are disjoint unions of such spaces. A little bit later J.Denef and F.Loeser ([3]) described the Lefschetz numbers of iterates of the monodromy transformation of a hypersurface singularity (of any dimension) as Euler characteristics of some subspaces of the space of (truncated) arcs. After that it was understood that the results of [10] are connected with the notion of the motivic integration or rather with its version (in some sense a dual one) where the space of arcs is substituted by the space of functions. The final aim of this paper is to discuss the notion of the integral with respect to the Euler characteristics (or with respect to the generalized Euler characteristic) over the space of functions (or over its projectivization) and its connection with the formulae for the coefficients of the Alexander polynomial and of the zeta-function of the monodromy transformation as Euler characteristics of some spaces.

There were two results which preceded the formulae for the Alexander polynomial of a plane curve singularity from [10]. At first there was seen

*First two authors were partially supported by DGICYT PB97-0471. Address: University of Valladolid, Dept. of Algebra, Geometry and Topology, 47005 Valladolid, Spain. E-mail: campillo@cpd.uva.es, fdelgado@agt.uva.es
†Partially supported by the grants RFBR, INTAS–97–1644 and NWO 047.008.005. Address: Moscow State University, Dept. of Mathematics and Mechanics, Moscow, 119899, Russia. E-mail: sabir@mccme.ru
no real connection between them. We discuss these results in the next two sections.

1 The Poincaré series of the ring of functions and the zeta-function of the monodromy transformation

One of the results (or rather an observation) identifies the Poincaré series of the ring of functions on an irreducible plane curve singularity with the zeta-function of the classical monodromy transformation of it ([8]).

Let \((C, 0) \subset (\mathbb{C}^2, 0)\) be an irreducible germ of a plane curve. Let \(f = 0\) be an equation of the curve \(C\), i.e., a germ of an analytic map such that \(\text{Im } \varphi = C\) and \(\varphi\) is an isomorphism between \(\mathbb{C}\) and \(C\) outside of the origin. For a germ \(g\) from the ring \(\mathcal{O}_{C, 0}\) of germs of holomorphic functions at the origin in the plane \(\mathbb{C}^2\), let \(v(g) \in \mathbb{Z}_{\geq 0}\) be the order of zero at the origin of the function \(g \circ \varphi : (\mathbb{C}, 0) \to \mathbb{C}\), i.e., \(g \circ \varphi(\tau) = a \cdot \tau^v(g) + \text{terms of higher degree, } a \neq 0\) (if \(g \circ \varphi \equiv 0\), i.e., if \(g \in (f)\), one assumes \(v(g) = +\infty\)). The set \(S_C\) of integers \(v = v(g)\) for all germs \(g \in \mathcal{O}_{C, 0}\) with \(v(g) < +\infty\) is a subsemigroup in \(\mathbb{Z}_{\geq 0}\) and is called the semigroup (of values) of the (irreducible) plane curve singularity \(C\).

Let \(\varphi^* : \mathcal{O}_{C, 0} \to \mathcal{O}_{C, 0}\) be the natural homomorphism induced by the map \(\varphi\). The ring \(\mathcal{O}_{C, 0}\) of germs of holomorphic functions in one variable has a natural filtration by powers of the maximal ideal \(m \subset \mathcal{O}_{C, 0}\): \(\mathcal{O}_{C, 0} = m^0 \supset m^1 \supset m^2 \supset \ldots\) Let \(W_i = (\varphi^*)^{-1}(m^i) \subset \mathcal{O}_{C, 0}\). The ideals \(W_i\) form a filtration of the ring \(\mathcal{O}_{C, 0}\) of functions on the curve \(C\): \(\mathcal{O}_{C, 0} = W_0 \supset W_1 \supset W_2 \supset \ldots\) Since \(\dim m^i/m^{i+1} = 1\), the dimension of each factor \(W_i/W_{i+1}\) is either 1 or 0. It is equal to 1 if and only if \(i\) is an element of the semigroup \(S_C\) of the curve \(C\).

Let \(P_C(t) = \sum_{i=0}^{\infty} \dim(W_i/W_{i+1}) \cdot t^i\) be the Poincaré series of the filtration \(W_i\). One can write \(P_C(t)\) as \(\sum_{v \in S_C} t^v\). Since all sufficiently large integers belong to the semigroup \(S_C\), \(P_C(t)\) is the power series decomposition of a rational function (which we also denote by \(P_C(t)\)), moreover of a polynomial divided by \((1 - t)\).

Combinatorial properties of the semigroup of an irreducible plane curve
singularity (see, e.g., [11]) permit to calculate the rational function $P_C(t)$ in the following terms. Let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of the plane curve singularity $C$. The exceptional divisor $\mathcal{D} = \pi^{-1}(0)$ of the resolution is the union of its irreducible components, each of them isomorphic to the projective line $\mathbb{CP}^1$. A resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ of a plane curve singularity $C = \{ f = 0 \}$ (not necessarily irreducible) can be described by its dual graph $\Gamma$. Vertices of the graph $\Gamma$ are in one-to-one correspondence with irreducible components of the total transform $\pi^{-1}(C)$ of the curve $C$, i.e., with components of the exceptional divisor $\mathcal{D}$ and of the strict transform $\pi^{-1}(C) \setminus \mathcal{D}$ of the curve $C$. The vertices corresponding to the components of the strict transform of the curve $C$ are depicted by arrows. Two vertices of the graph $\Gamma$ are connected by an edge iff the corresponding components of the total transform $\pi^{-1}(C)$ intersect.

For a vertex $\sigma$ corresponding to a component $E_\sigma$ of the exceptional divisor $\mathcal{D}$, let $m^\sigma$ be the multiplicity (order) of the lifting $f \circ \pi$ of the function $f$ (the equation of the curve $C$) along $E_\sigma$. The dual graph of the minimal resolution of an irreducible plane curve singularity $C$ is shown in Fig.1. Here $s$ is the number of Puiseux pairs of the curve $C = \{ f = 0 \}$. Let $\beta_i$ ($i = 0, 1, \ldots, s$) and $\alpha_i$ ($i = 1, 2, \ldots, s$) be the dead ends and the star points of the graph $\Gamma$ (see Fig.1), let $\beta_j = m^{\beta_j}$, $\alpha_j = m^{\alpha_j}$. The set of integers $\{ \beta_j | j = 0, 1, \ldots, s \}$ is the minimal set of generators of the semigroup $S_C$ of the curve $C$. It is known that the integers $\alpha_j$ are multiples of the integers $\beta_j$ for $j = 1, \ldots, s$: $\alpha_j = (n_j + 1)\beta_j$, and that each element $v \in S_C$ can be represented in a unique way in the form $v = k_0\beta_0 + \sum_{j=1}^{s} k_j\beta_j$ with $k_0 \geq 0$, $0 \leq k_j \leq n_j$ for $1 \leq j \leq s$. Using this fact one can obtain the following formula (see, e.g., [8]).
Statement 1

\[ P_C(t) = \frac{\prod_{j=1}^{s} (1 - t^{\beta_j})}{\prod_{j=0}^{s} (1 - t^{\alpha_j})}. \]  

For a plane curve singularity \( C \) (again not necessarily irreducible) defined by an equation \( \{f = 0\} \) let \( h_f : V_f \to V_f \) be the classical monodromy transformation of the germ \( f \). Here \( V_f \) is the Milnor fibre of the singularity \( f : V_f = \{z \in \mathbb{C}^2 : ||z|| \leq \varepsilon, f(z) = \delta\} \), where \( 0 < |\delta| \ll \varepsilon \), \( \varepsilon \) is small enough. The monodromy transformation \( h_f \) is a diffeomorphism of the Milnor fibre \( V_f \) well defined up to isotopy. The zeta-function \( \zeta_h(t) \) of a transformation \( h : X \to X \) is the rational function in the variable \( t \) defined as follows:

\[ \zeta_h(t) = \prod_{q \geq 0} \left[ \det \left( id - t \cdot h_* |_{H_q(X; \mathbb{R})} \right) \right]^{(-1)^{q+1}}. \]

Let \( \zeta_C(t) = \zeta_{h_f}(t) \) be the zeta-function of the classical monodromy transformation \( h_f \). By the formula of N.A’Campo \((\text{[1]}))\) the zeta-function \( \zeta_C(t) \) of a plane curve singularity \( C = \{f = 0\} \) can be expressed in terms of an embedded resolution \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) of the curve \( C \) as

\[ \zeta_C(t) = \prod_{E \subset D} \left( 1 - t^{\sigma} \right)^{-\chi(E)} \]  

where \( E \) is "the smooth part" of the component \( E \) of the exceptional divisor \( D \), i.e., \( E \) minus intersection points with all other components of the total transform of the curve \( C \).

For the irreducible plane curve singularity \( C \) (the dual graph \( \Gamma \) of the minimal resolution of which is shown in Fig.\( \text{[4]} \) \( \chi(E_{\beta_j}) = 1, \chi(E_{\alpha_j}) = -1 \), and \( \chi(E_{\sigma}) = 0 \) for all other components \( E \) of the exceptional divisor \( D \). Therefore

\[ \zeta_C(t) = \frac{\prod_{j=1}^{s} (1 - t^{\beta_j})}{\prod_{j=0}^{s} (1 - t^{\alpha_j})}. \]  

Comparing (1) and (3) and one has

**Theorem 1** The Poincaré series \( P_C(t) \) of the ring of functions of an irreducible plane curve singularity \( C \) (or rather the rational function represented by it) coincides with the zeta-function \( \zeta_C(t) \) of the singularity \( C \):

\[ P_C(t) = \zeta_C(t). \]
Up to now the authors do not know an explanation of this coincidence. We consider it as an exciting challenge.

2 The extended semigroup of a plane curve singularity and the Euler characteristic of its projectivization

It appears that one of possible generalizations of the result of Theorem 3 uses the notion of the extended semigroup of a curve singularity (4). According to this generalization one can say that, for an irreducible plane curve singularity \( C \), the coefficients of the Poincaré series \( P_C(t) \) which are equal to 0 or 1 are Euler characteristics of the empty set and of a point respectively.

Let \( (C, 0) \subset (\mathbb{C}^n, 0) \) be an arbitrary (reduced) curve singularity (not necessarily plane) and let \( C = \bigcup_{i=1}^{r} C_i \) be its representation as the union of irreducible components \( C_i \). Let \( \mathbb{C}_i \) be the complex line with the coordinate \( \tau_i \) \((i = 1, \ldots, r)\) and let \( \varphi_i : (\mathbb{C}_i, 0) \to (\mathbb{C}^n, 0) \) be parameterizations (uniformizations) of the branches \( C_i \) of the curve \( C \), i.e., germs of analytic maps such that \( \text{Im} \varphi_i = C_i \) and \( \varphi_i \) is an isomorphism between \( \mathbb{C}_i \) and \( C_i \) outside of the origin. Let \( \mathcal{O}_{\mathbb{C}^n, 0} \) be the ring of germs of holomorphic functions at the origin in \( \mathbb{C}^n \). For a germ \( g \in \mathcal{O}_{\mathbb{C}^n, 0} \), let \( v_i = v_i(g) \in \mathbb{Z}_{\geq 0} \) and \( a_i = a_i(g) \in \mathbb{C}^* \) be the degree of the leading term and the coefficient at it in the power series decomposition of the composition \( g \circ \varphi_i : (\mathbb{C}_i, 0) \to \mathbb{C} \): \( g \circ \varphi_i(\tau_i) = a_i \cdot \tau_i^{v_i} + \text{terms of higher degree}, \ a_i \neq 0 \) (if \( g \circ \varphi_i(t) \equiv 0 \), \( v_i(g) = +\infty \) and \( a_i(g) \) is not defined). The numbers \( v_i(g) \) and \( a_i(g) \) are defined for elements \( g \) of the ring \( \mathcal{O}_C \) of functions on the curve \( C \) as well. Let \( v(g) = ||g|| = v_1(g) + \ldots + v_r(g) \).

The semigroup \( S_C \) of the curve singularity \( C \) is the subsemigroup of \( \mathbb{Z}_{\geq 0}^r \) which consists of elements of the form \( \underline{v}(g) = (v_1(g), \ldots, v_r(g)) \) for all germs \( g \in \mathcal{O}_C \) with \( v_i(g) < \infty \); \( i = 1, \ldots, r \). The set \( \hat{S}_C \) of elements of the form \( (\underline{v}(g); \underline{a}(g)) = (v_1(g), \ldots, v_r(g); a_1(g), \ldots, a_r(g)) \in \mathbb{Z}_{\geq 0}^r \times (\mathbb{C}^*)^r \) for all germs \( g \in \mathcal{O}_C \) with \( v_i(g) < \infty \), \( i = 1, \ldots, r \), is a subsemigroup of \( \mathbb{Z}_{\geq 0}^r \times (\mathbb{C}^*)^r \) and is called the extended semigroup of the curve singularity \( C = \bigcup_{i=1}^{r} C_i \). The extended semigroup \( \hat{S}_C \) is well defined (does not depend on the choice of the parameterizations \( \varphi_i \)) up to a natural equivalence relation.

For plane curve singularities, the extended semigroup is not a topological invariant, but reflects some moduli of it (see 4). Namely, it determines the
exceptional divisor of the minimal embedded resolution of a plane curve singularity up to projective equivalence, i.e., up to projective equivalences of its irreducible components (each of them is isomorphic to the projective line \(\mathbb{C}P^1\)) which preserve all the intersection points with other components of the total transform \(\pi^{-1}(C)\) of the curve \(C\).

There is a natural semigroup homomorphism (projection) \( p : \mathbb{Z}_{\geq 0}^r \times (\mathbb{C}^*)^r \to \mathbb{Z}_{\geq 0}^r \) which sends \((\underline{v}, \underline{a})\) to \(\underline{v}\). It maps the extended semigroup \(\hat{S}_C\) of the curve \(C\) onto its (usual) semigroup \(S_C\). The preimages \(F_{\underline{v}} = p^{-1}(\underline{v}) \subset \{\underline{v}\} \times (\mathbb{C}^*)^r \subset \{\underline{v}\} \times \mathbb{C}^r\) of the map \(p : \hat{S}_C \to \mathbb{Z}_{\geq 0}^r\) are called fibres of the extended semigroup \(\hat{S}_C\). \(F_{\underline{v}}\) is not empty if and only if \(\underline{v} \in S_C\).

The fibres \(F_{\underline{v}}\) can be described in the following way. For \(\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r\) (not only for \(\underline{v} \in \mathbb{Z}_{\geq 0}^r\)), let \(J(\underline{v})\) be the ideal in the ring \(O_{\mathbb{C}^n, 0}\) of germs of functions in \(n\) variables which consists of germs \(g \in O_{\mathbb{C}^n, 0}\) such that \(v_i(g) \geq v_i\) (i.e., \(v_i(g) \geq v_i\) for \(i = 1, \ldots, r\)). There is a natural linear map \(j_{\underline{v}} : J(\underline{v}) \to \mathbb{C}^r\), which sends \(g \in J(\underline{v})\) to \(\underline{a} = (a_1, \ldots, a_r) \in \mathbb{C}^r\), where \(a_i\) is the coefficient in the power series decomposition \(g \circ \varphi_i(\tau_i) = a_i \tau_i + \text{terms of higher degree}\) (coefficients \(a_i\) can be equal to zero). The kernel of the map \(j_{\underline{v}}\) coincides with the ideal \(J(\underline{v} + \underline{1})\) \((\underline{1} = (1, \ldots, 1))\). Therefore the image \(\text{Im} \ j_{\underline{v}} \subset \mathbb{C}^r\) of it is isomorphic to the vector space \(C(\underline{v}) = J(\underline{v})/J(\underline{v} + \underline{1})\). One can easily see that \(F_{\underline{v}} = \text{Im} \ j_{\underline{v}} \cap (\mathbb{C}^*)^r\) (under the natural identification of \(\{\underline{v}\} \times (\mathbb{C}^*)^r\) and \((\mathbb{C}^*)^r\)). Therefore, for \(\underline{v} \in S_C\), the fibre \(F_{\underline{v}}\) is the complement to an arrangement of linear hyperplanes in a linear space. One can show that dimensions of the fibres \(F_{\underline{v}}\) (or of the spaces \(C(\underline{v})\)) and combinatorial types of the corresponding arrangements of hyperplanes \(\text{Im} \ j_{\underline{v}} \cap (\mathbb{C}^r \setminus (\mathbb{C}^*)^r) \subset \text{Im} \ j_{\underline{v}}\) are topological invariants of a plane curve singularity (i.e., do not change along the stratum \(\mu = \text{const}\)).

We shall be mostly interested in Euler characteristics of spaces under consideration. The usual definition of the Euler characteristic of a topological space (say, of a \(CW\)-complex) \(X\) is \(\chi(X) = \sum_{q \geq 0} (-1)^q \dim H_q(X; \mathbb{R})\).

If \(X = X_1 \cup X_2\) where the spaces \((CW\text{-complexes})\) \(X_1\) and \(X_2\) are compact, one has \(\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2)\). Therefore for compact spaces the Euler characteristic possesses the additivity property. This permits to consider it as a generalized (nonpositive) measure on the algebra of such spaces. However spaces we are interested in (e.g., fibres of the extended semigroup of a curve) are noncompact semialgebraic sets (complex or real). The Euler characteristic defined above does not possess the additivity property for such spaces. For example, let \(X\) be the circle \(S^1\), let \(X_1\) be a point of \(X\), and let \(X_2 = X \setminus X_1\) be a (real) line. Then one has \(\chi(X) = 0\), \(\chi(X_1) = \chi(X_2) = 1\), \(\chi(X_1 \cap X_2) = \chi(\emptyset) = 0\), and \(0 \neq 1 + 1 - 0\).
In order to have the desired additivity property one should define the Euler characteristic $\chi(X)$ of a space $X$ (say, semialgebraic) as

$$\sum_{q \geq 0} (-1)^q \dim H_q(X^*, *, \mathbb{R}),$$

where $X^*$ is the one-point compactification of the space $X$ (if $X$ is compact, the one-point compactification of it is the disjoint union of $X$ with a point), $*$ is the added ("infinite") point. We shall use this definition. One can show that defined this way the Euler characteristic does possess the additivity property for semialgebraic sets (in the example above $\chi(X) = 0$, $\chi(X_1) = 1$, $\chi(X_2) = -1$). Moreover, a semialgebraic set $X$ can be represented as a disjoint union of a finite number of open cells so that the boundary of a cell of some dimension (its closure minus it itself) lies in the union of cells of smaller dimensions. (This does not mean a representation of the space $X$ as a CW–complex since in general (for noncompact sets) one does not have maps of closed balls into $X$ which determine the cells. For example the real line $\mathbb{R}^1$ is simply one cell of dimension 1.) One can see that the Euler characteristic of the (semialgebraic) set $X$ is equal to the alternative sum of numbers of cells of different dimensions.

By a graded topological space $X$ (with a $\mathbb{Z}[[t_1, \ldots, t_r]]$–grading) we shall mean a collection of topological spaces $X_\mathbf{u}$ corresponding to elements $\mathbf{u} = (v_1, \ldots, v_r)$ of $\mathbb{Z}_{\geq 0}^r$. We shall (formally) write $X = \sum_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^r} X_\mathbf{u} \cdot t^\mathbf{u}$ where $t = (t_1, \ldots, t_r)$, $t^\mathbf{u} = t_1^{v_1} \cdot \ldots \cdot t_r^{v_r}$. The (disjoint) union and the product of topological spaces defines the sum and the product of graded spaces. By definition the Euler characteristic of the graded space $X = \sum X_\mathbf{u} \cdot t^\mathbf{u}$ is the power series $\chi(X) = \sum \chi(X_\mathbf{u}) \cdot t^\mathbf{u}$. One has $\chi(X + Y) = \chi(X) + \chi(Y)$, $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. A graded map from $X = \sum X_\mathbf{u} \cdot t^\mathbf{u}$ to $Y = \sum Y_\mathbf{u} \cdot t^\mathbf{u}$ is a collection of maps $X_\mathbf{u} \to Y_\mathbf{u} (\mathbf{u} \in \mathbb{Z}_{\geq 0}^r)$.

The extended semigroup $\widehat{S}_C$ of a curve singularity $C = \bigcup_{i=1}^r C_i$ is in a natural sense a graded space: $\widehat{S}_C = \sum_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^r} F_\mathbf{u} \cdot t^\mathbf{u}$. The extended semigroup $\widehat{S}_C$ (i.e., each fibre $F_\mathbf{u}$) has a natural (free) $\mathbb{C}^*$–action: multiplication of an element $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbb{C}^*)^r$ by a constant. Therefore $\chi(\widehat{S}_C) = 0$. In order to have a nontrivial Euler characteristic one can (or even should) "kill" this $\mathbb{C}^*$. One way to do that is to factorize the extended semigroup by this $\mathbb{C}^*$–action, i.e., to consider the projectivization of it. (Another (somewhat dual) way was used in [3]).

Thus, let $\mathbb{P}\widehat{S}_C = \sum \mathbb{P} F_\mathbf{u} \cdot t^\mathbf{u}$ be the projectivization of the extended semigroup $\widehat{S}_C (\mathbb{P}\widehat{S}_C \subset \mathbb{P}(\mathbb{C}^*)^r)$. This projectivization itself is a (graded)
semigroup. Its fibres $\mathbb{P}F_\mathbb{P}$ are complements to arrangements of projective hyperplanes in complex projective spaces. Let $X_C(t)$ be the Euler characteristics of the projectivization $\mathbb{P}\tilde{S}_C$ of the extended semigroup $\tilde{S}_C$, i.e.,

$$X_C(t) = \sum \chi(\mathbb{P}F_\mathbb{P}) \cdot t^\mathbb{P}.$$ 

It is not difficult to understand that, for a reducible plane curve singularity ($r > 1$), the Euler characteristics $\chi(\mathbb{P}F_\mathbb{P})$ of the fibres of the projectivization of the extended semigroup are equal to zero for $\|\mathbb{P}\|$ big enough and therefore $X_C(t)$ is in fact a polynomial.

For an irreducible curve singularity $C$ ($r = 1$), each fibre $F_\mathbb{P}$ ($\mathbb{P} \in \mathbb{Z}_{r=0}^\mathbb{P}$) is either empty (if $\mathbb{P} \not\in S_C$) or isomorphic to $\mathbb{C}^*$ (if $\mathbb{P} \in S_C$). Thus its projectivization $\mathbb{P}F_\mathbb{P}$ is empty for $\mathbb{P} \not\in S_C$ and is a point for $\mathbb{P} \in S_C$ and

$$X_C(t) = \sum_{\mathbb{P} \in S_C} t^\mathbb{P} = P_C(t).$$

Therefore, for a reducible curve singularity $C = \bigcup_{i=0}^r C_i$, the polynomial $X_C(t)$ to some extend can be considered as a possible generalization of the Poincaré series $P_C(t)$.

3 Poincaré series of the ring of functions of a reducible curve singularity

Another (by definition, not by essence: see Theorem 2) generalization of the Poincaré series of the ring of germs of functions for a reducible curve singularity $C = \bigcup_{i=0}^r C_i \subset (\mathbb{C}_o,0)$ can be constructed in the following way. For $\mathbb{P} \in \mathbb{Z}_r$ (not only for $\mathbb{P} \in \mathbb{Z}_{r=0}^\mathbb{P}$), let $c(\mathbb{P})$ be the (non-negative) integers $\dim J(\mathbb{P})/J(\mathbb{P} + 1)$. Let $\mathcal{L}(t_1, \ldots, t_r) = \sum_{\mathbb{P} \in \mathbb{Z}_r} c(\mathbb{P})t^\mathbb{P}$.

For $r > 1$, the integers $c(\mathbb{P})$ can be positive for $\mathbb{P}$ with (some) negative components $v_i$ as well. For example, if there exists a germ $g \in \mathcal{O}_{\mathbb{C}_o,0}$ with $v_1(g) = v_1^*$, then for any $v_2, \ldots, v_r$ such that $v_i \leq v_i(g)$ (including negative ones), the germ $g$ represents a non-trivial element in the factor $J(\mathbb{P})/J(\mathbb{P} + 1)$ where $\mathbb{P} = (v_1^*, v_2, \ldots, v_r)$, $J(\mathbb{P}) = \{ g \in \mathcal{O}_{\mathbb{C}_o,0} : v_i(g) \geq v \}$, and therefore $c(\mathbb{P}) > 0$. Therefore $\mathcal{L}(t_1, \ldots, t_r)$ is not a power series but an element of the set $\mathbb{Z}[[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}]]$ of Laurent series in $t_1, \ldots, t_r$ infinite in all the directions. The set $\mathbb{Z}[[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}]]$ is not a ring (generally speaking, the product of its elements is not defined), but it is a $\mathbb{Z}[t_1, \ldots, t_r] -$ or $\mathbb{Z}[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}]$-module. One can understand that along each line in the lattice $\mathbb{Z}_r$ parallel to a coordinate one the coefficients $c(\mathbb{P})$ stabilize in each direction. This is the reason why $P_C'(t_1, \ldots, t_r) = \mathcal{L}(t_1, \ldots, t_r) \cdot$
\( \prod_{i=1}^{r} (t_i - 1) \) is a polynomial. Moreover, one can show that, for \( r > 1 \), the polynomial \( P'_C(t_1, \ldots, t_r) \) is divisible by \( (t_1 \cdot \ldots \cdot t_r - 1) \). Let \( P_C(t_1, \ldots, t_r) = P'_C(t_1, \ldots, t_r)/(t_1 \cdot \ldots \cdot t_r - 1) \). Then \( P_C(t) \) is a polynomial for \( r > 1 \) and is a power series for \( r = 1 \) (i.e. for an irreducible curve singularity). In the last case \( P_C(t) \) coincides with the Poincaré series of the ring of functions on the curve discussed above and, thus, it can be considered as a generalization of it. We shall call \( P_C(t_1, \ldots, t_r) \) the Poincaré series of the curve singularity \( C = \cup_{i=1}^{r} C_i \).

Though the definitions of the polynomials (series for \( r = 1 \)) \( X_C(t) \) and \( P_C(t) \) look rather different, in fact they coincide (for any curve, not only for a plane one).

**Theorem 2** For a curve singularity \( C = \bigcup_{i=1}^{r} C_i \subset (\mathbb{C}^n, 0) \) one has

\[
X_C(t) = P_C(t).
\]

The proof consists in application of the inclusion–exclusion formula for computing the Euler characteristic of the projectivization \( \mathbb{P} F_v \) as the complement to an arrangement of projective hyperplanes in a projective space (taking into account that \( \chi(\mathbb{P}(\mathbb{C}^n)) = \chi(\mathbb{CP}^{n-1}) = n) \).

4 The Alexander polynomial of a plane curve singularity and the Poincaré polynomial

The Alexander polynomial (in \( r \) variables) is an invariant of a link with \( r \) (numbered) components in the sphere \( S^3 \). The general definition can be found, e.g., in [3]. To a plane curve singularity \( C = \bigcup_{i=1}^{r} C_i \subset (\mathbb{C}^2, 0) \) there corresponds the link \( C \cap S^3_\varepsilon \) in the 3-sphere \( S^3_\varepsilon \) of radius \( \varepsilon \) centred at the origin in the complex plane \( \mathbb{C}^2 \) with \( \varepsilon \) small enough. Let \( \Delta_C(t_1, \ldots, t_r) \) be the Alexander polynomial of this link (≡ the Alexander polynomial of the curve \( C \)). For such a link (an algebraic one) we rather use not the general definition of the Alexander polynomial, but a formula for it in terms of an embedded resolution \( \pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0) \) of the curve singularity \( C \).

Let the curve \( C = \bigcup_{i=1}^{r} C_i \) be given by an equation \( f = 0 \) \((f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)) \) and let \( f = \prod_{i=1}^{r} f_i \) where \( f_i = 0 \) are equations of the components \( C_i \) of the curve \( C \) \((i = 1, \ldots, r) \). For each component \( E_\sigma \) of the exceptional divisor \( \mathcal{D} = \pi^{-1}(0) \) let \( m_\sigma^i \) be the multiplicity of the lifting \( f_i \circ \pi \)
of the function \( f \) to the space \( X \) of the resolution along the component \( E_\sigma, m^\sigma = (m_1^\sigma, \ldots, m_r^\sigma) \). One has \( \sum_{i=1}^r m_i^\sigma = m^\sigma \). As above let \( \hat{E}_\sigma \) be the smooth part of the component \( E_\sigma \), i.e., \( E_\sigma \) minus its intersections with all other components of the total transform \( \pi^{-1}(C) \) of the curve \( C \). These intersection points on the component \( E_\sigma \) are in one-to-one correspondence with connected components of the complement \( (f \circ \pi)^{-1}(0) \setminus \hat{E}_\sigma \). We shall say that an intersection point is \textbf{essential} if the corresponding connected component contains the strict transform of a component of the curve \( C \).

Let \( s_\sigma \) be the number of essential points on the component \( E_\sigma \), and let \( \tilde{E}_\sigma \) be the complement to the set of essential points in \( E_\sigma \).

\textbf{Theorem 3} ([6]) \textit{For} \( r > 1 \),

\[ \Delta^C(t_1, \ldots, t_r) = \prod_{E_\sigma \subset D} \left( 1 - t^{m^\sigma} \right)^{-\chi(\hat{E}_\sigma)} \]  

(4)

One can see that this formula is an analogue of the formula (2) for the zeta–function \( \zeta_C(t) \) of the monodromy transformation of the curve \( C \). Moreover one has \( \zeta_C(t) = \Delta^C(t, \ldots, t) \) (for \( r > 1 \)).

\textbf{Remarks.} 1. For an irreducible curve singularity (i.e., for \( r = 1 \)) the right-hand side of the formula (3) is not a polynomial, but a polynomial (in fact the Alexander polynomial in one variable) divided by \( 1 - t \).

2. Generally speaking the Alexander polynomial is defined only up to multiplication by a monomial \( \pm t^{k_1} \), \( k = (k_1, \ldots, k_r) \in \mathbb{Z}^r \). For algebraic links the formula (3) fixes the choice of the Alexander polynomial in such a way that it is really a polynomial (i.e., does not contain monomials with negative powers) and its value at the origin \( (t = 0) \) is equal to 1.

\textbf{Theorem 4} \textit{For a plane curve singularity} \( C = \bigcup_{i=1}^r C_i \subset (\mathbb{C}^2, 0), r > 1 \),

\[ \Delta^C(t) = X_C(t) = P_C(t) \]

The main course of the \textbf{proof} goes as follows. First we construct a graded space (in fact a graded semigroup) \( Y \) such that its Euler characteristic is equal to the Alexander polynomial \( \Delta^C(t_1, \ldots, t_r) \) together with a map (a semigroup homomorphism) to the projectivization \( \mathbb{P}\hat{S}_C \) of the extended semigroup \( \hat{S}_C \). Let \( \pi : (X, D) \rightarrow (\mathbb{C}^2, 0) \) be an embedded resolution of the curve singularity \( C \subset (\mathbb{C}^2, 0) \), let the exceptional divisor \( D \) of the resolution \( \pi \) be the union of irreducible components \( E_\sigma \) (\( \sigma \in \Gamma \)). For a topological space \( X \), let \( S^kX = X^k/S_k \) (\( k \geq 0 \)) be the \( k \)th symmetric power
of the space $X$, i.e., the space of subsets of the space $X$ with $k$ elements ($S^0 X$ is a point).

Let
\[ Y = \sum_{\{k_\sigma\}} \left( \prod_{\sigma} S^{k_\sigma} \tilde{E}_\sigma \cdot t^{k_\sigma m_\sigma} \right), \]
where the union (sum) is over all sets of nonnegative integers $k_\sigma$, $\sigma \in \Gamma$, $S^{k_\sigma} \tilde{E}_\sigma$ is the $k$-th symmetric power of the nonsingular part $\tilde{E}_\sigma$ of the component $E_\sigma$ of the exceptional divisor. The graded topological space $Y$ is a (graded) semigroup with respect to the operation defined by the union of the subsets.

There is a natural map (a graded semigroup homomorphism) $\Pi : Y \to \mathbb{P} \hat{S}_C$ defined as follows. An element
\[ y \in Y = \prod_{\sigma} \left( \bullet + S^1 \tilde{E}_\sigma \cdot t^{m_\sigma} + S^2 \tilde{E}_\sigma \cdot t^{2m_\sigma} + \ldots \right), \]
where $\bullet$ is a point ($= S^0 \hat{E}_\sigma$), is represented by a subset of "nonsingular" points of the exceptional divisor $D$ (i.e., of $\hat{D} = \bigcup_{\sigma} \hat{E}_\sigma$) with $k_\sigma$ points $P_1^\sigma$, $\ldots$, $P_{k_\sigma}^\sigma$ on the component $\tilde{E}_\sigma$. For a point $A \in \hat{D}$, let $\tilde{L}_A$ be the germ of a nonsingular (complex analytic) curve transversal to the exceptional divisor $\hat{D}$ at the point $A$. Let the image $L_A = \pi(\tilde{L}_A) \subset (\mathbb{C}^2, 0)$ of the curve $\tilde{L}_A$ be given by an equation $g_A = 0$, $g_A \in \mathcal{O}_{\mathbb{C}^2, 0}$. Let $g_y = \prod_{\sigma} \prod_{j=1}^{k_\sigma} g_{P_j^\sigma}$. Then
\[ \Pi(y) := (v_1(g_y), \ldots, v_r(g_y); a_1(g_y) : \ldots : a_r(g_y)). \]

To prove that the map $\Pi$ is well defined, i.e., that $\Pi(y) \in \mathbb{P} \hat{S}_C$ does not depend on the choice of curves $\tilde{L}_A$ and of the equations $g_A = 0$, suppose that $\tilde{L}_A'$ is another germ of a nonsingular (complex analytic) curve transversal to the exceptional divisor $\hat{D}$ at the point $A \in \hat{D}$, $g_A' = 0$ is an equation of the curve $L_A' = \pi(\tilde{L}_A') \subset (\mathbb{C}^2, 0)$. Let $g_y' = \prod_{\sigma} \prod_{j=1}^{k_\sigma} g_{P_j^\sigma}'$, let $\tilde{g}_y = g_y \circ \pi$ and $\tilde{g}_y' = g_y' \circ \pi$ be the liftings of the germs $g_y$ and $g_y'$ to the space $\mathcal{X}$ of the resolution, and let $\psi = \tilde{g}_y'/\tilde{g}_y$ be their ratio. The meromorphic function $\psi$ on $\mathcal{X}$ has (simple) zeros along the curves $\tilde{L}_{P_j^\sigma}$ and (also simple) poles along the curves $\tilde{L}_{P_j^\sigma}$. Therefore the restriction of the function $\psi$ to the exceptional divisor $\hat{D}'$ has neither zeroes no poles, i.e., it is a regular (holomorphic) function without zeroes and thus it is a constant (say, $c$) on $\hat{D}$. It implies that $\underline{\psi}(g_y') = \underline{\psi}(g_y)$, $a_\sigma(g_y') = c \cdot a_\sigma(g_y)$ and therefore the
points \((g_1'; a_1(g_2') : \ldots : a_r(g_3'))\) and \((g_1; a_1(g_2) : \ldots : a_r(g_3'))\) in the projectivization \(\mathbb{P}\hat{S}_C\) coincide.

To compute the Euler characteristic \(\chi(Y)\) of the graded space \(Y\) one uses the following statement.

**Lemma 1** For a topological space \(X\),

\[
\chi(\bullet + S^1X \cdot t + S^2X \cdot t^2 + \cdots) = (1 - t)^{-\chi(C)}.
\]

**Corollary 1** \(\chi(Y) = \prod_{\sigma} (1 - t^{m_\sigma})^{-\chi(E_\sigma)} = \Delta^C(t_1, \ldots, t_r).\)

Let \(Y\) be an arbitrary point of the lattice \(\mathbb{Z}^r_\geq 0\) and suppose that the resolution \(\pi: (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)\) of the curve \(C\) is such that, for any germ \(g \in \mathcal{O}_{\mathbb{C}^2, 0}\) with \(v(g) \leq \varphi\), the strict transform \((g \circ \pi)^{-1}(0) \setminus \mathcal{D}\) of the curve \(\{g = 0\}\) intersects the exceptional divisor \(\mathcal{D}\) only at nonsingular points of it. One can get such a resolution from the minimal one making sufficiently many blow-ups at intersection points of components of the total transform \(\pi^{-1}(C)\) of the curve \(C\). It is not difficult to see that in this case, for \(\varphi \leq \varphi\), the image \(\Pi(Y_\varphi \cdot t^\varphi)\) of the part of grading \(\varphi\) of the space \(Y\) coincides with the projectivization \(\mathbb{P}\hat{S}_C\) of the extended semigroup \(\hat{S}_C\) (or rather with the corresponding summand \(\mathbb{P}F_{\varphi}\)). Therefore in order to prove that \(\chi(\mathbb{P}\hat{S}_C) = \Delta^C(t_1, \ldots, t_r)\) it is sufficient to show that \(\chi(\text{Im} \Pi) = \chi(Y)\). There would be no problem to show this if the map \(\Pi\) would be injective. This is not the case. Thus one has to analyse how non-injective is it.

One possibility to decrease the set where the map \(\Pi\) is non-injective is is to reduce the graded space \(Y\) in the following way. Let \(D'\) be the union of components \(E_\sigma\) of the exceptional divisor with at least two essential points, i.e., with \(s_\sigma \geq 2\), and let \(\Delta'\) be the set of the corresponding vertices of the dual graph of the resolution. Connected components of the complement \(D \setminus D'\), which do not contain the starting divisor \(1\), are tails of the dual graph \(\overline{\Gamma}\) of the resolution and correspond to (some) dead ends \(\delta\) of the graph \(\Gamma\). Let \(\Delta\) be the set of these dead ends. For \(\delta \in \Delta\), let \(st_\delta\) be the vertex of \(\Delta'\) such that \(E_{st_\delta}\) intersects the corresponding connected component of \(\overline{D \setminus D'}\) (see Fig.2). Let

\[
Y' = \prod_{\sigma \in \Delta'} (\bullet + S^1\overline{E_\sigma} \cdot t^{m_\sigma} + S^2\overline{E_\sigma} \cdot t^{2m_\sigma} + \ldots).
\]
Pay attention that the spaces \( \tilde{E}_\sigma \) ("nonsingular parts" of the components of the exceptional divisor) in the definition of the graded space \( Y \) are substituted here by the spaces \( \tilde{E}_\sigma \) (the components of the exceptional divisor minus essential points of them).

For a dead end \( \delta \in \Delta \), \( m^{st_\delta} \) is a multiple of \( m^\delta \): \( m^{st_\delta} = (n_\delta + 1) \cdot m^\delta \).

Let \( Y_\delta = \sum_{k=0}^{n_\delta} \cdot t^{km^\delta} \).

Let \( \tilde{Y} = Y' \times Y_1 \times \prod_{\delta \in \Delta} Y_\delta \). One can show that \( \chi(\tilde{Y}) = \chi(Y) \).

There exists a map \( \Pi : \tilde{Y} \to \mathbb{P}\tilde{S}_C \) such that \( \text{Im} \Pi = \text{Im} \Pi_1 \). To define it, one can say that \( \tilde{Y} \) is a subset of the graded semigroup

\[
\tilde{Y}^* = Y' \times \left( \sum_{m \in S_1} \cdot t^m \right) \times \prod_{\delta \in \Delta} \left( \sum_{k=0}^{\infty} \cdot t^{km^\delta} \right)
\]
(each factor of $\tilde{Y}$ is a graded semigroup) and the map $\tilde{\Pi}$ is the restriction of a graded semigroup homomorphism $\tilde{Y} \to \mathbb{P}S_C$. Because of that it should be defined for points of $\bigcup_{\sigma \in \Delta'} \tilde{E}_\sigma$ and also for "monomials" of the form $\bullet \cdot \ell^{m_{\sigma'}}$ for $\delta \in \Delta$ and $\bullet \cdot \ell^{m_{\beta_{\delta'}}}$ for $i = 0, 1, \ldots, q$.

For a point $A$ of $\tilde{E}_\sigma$, $\sigma \in \Delta'$, (or rather for the monomial $[A] \cdot \ell^{m_{\sigma'}}$) $\tilde{\Pi}$ coincides with $\Pi$. A point of $\tilde{E}_\sigma \setminus \tilde{E}_\sigma$, $\sigma \in \Delta'$, corresponds either to a dead end $\delta \in \Delta$ (and in this case $\sigma = st_\delta$) or to the initial divisor 1 (in this case $\sigma = st_1$). In the first case one puts $\tilde{\Pi}([A] \cdot \ell^{m_{\sigma'}}) = (n_\delta + 1) \cdot \Pi([A_\delta] \cdot \ell^{n_{\delta}})$ for any point $A_\delta \in \tilde{E}_\delta$ (see the definition of $n_\delta$ above); in the second case there exists $\ell_0, \ldots, \ell_q$ such that $m_{\sigma'} = m_{st_1} = \sum_{i=0}^q \ell_i m_{\beta_{\delta_i}}$ and one puts $\tilde{\Pi}([A] \cdot \ell^{m_{\sigma'}}) = \sum_{i=0}^q \ell_i \cdot \Pi([A_{\beta_i}] \cdot \ell^{m_{\beta_{\delta_i}}})$ for any points $A_{\beta_i} \in \tilde{E}_{\beta_i}$. One puts $\tilde{\Pi}(\bullet \cdot \ell^{m_{\delta}}) = \Pi([A_\delta] \cdot \ell^{n_{\delta}})$ for any point $A_\delta \in \tilde{E}_\delta$, $\delta \in \Delta$, $\tilde{\Pi}(\bullet \cdot \ell^{m_{\beta_{\delta_i}}}) = \Pi([A_{\beta_i}] \cdot \ell^{m_{\beta_{\delta_i}}})$ for any point $A_{\beta_i} \in \tilde{E}_{\beta_i}$, $i = 0, 1, \ldots, q$ (one can easily see that the result does not depend on the choice of the points $A_\delta, A_{\beta_i}$ in these cases).

It is not difficult to see that $\text{Im} \tilde{\Pi} = \text{Im} \Pi$. Now Theorem 3 follows from the statement that $\chi(\tilde{Y}) = \chi(\text{Im} \tilde{\Pi})$. To prove this, one analyses places where the map $\tilde{\Pi}$ is not injective. At such places we indicate some parts of the space $\tilde{Y}$ which are fibred into complex tori $(\mathbb{C}^*)^{s-1}$, $s \geq 2$, (and thus have zero Euler characteristic) and which can be removed without changing the image $\tilde{\Pi}(\tilde{Y})$. To obtain such a proof, the following three types of statements are used.

1. Connected components of the space $\tilde{Y}$ are labelled by multi-indices

$$k = \{\{k_\sigma\}_{\sigma \in \Delta'}, \overline{m}, \{k_\delta\}_{\delta \in \Delta}\}$$

with $k_\sigma \geq 0$ for each $\sigma \in \Delta'$, $\overline{m} \in S'_1$, and $0 \leq k_\delta \leq n_\delta$ for each $\delta \in \Delta$. A vertex $\sigma \in \Delta'$ is called a cut of a multi-index $k = \{k_\sigma, \overline{m}, k_\delta\}$ if $k_\sigma \geq s_\sigma$. One shows that if $\tilde{\Pi}(y_1) = \tilde{\Pi}(y_2)$ for two different elements $y_1$ and $y_2$ from $\tilde{Y}$ corresponding to multi-indices $k^i$, $i = 1, 2$ (i.e., at a place where the map $\tilde{\Pi}$ is not injective), then each multi-index $k^i$ ($i = 1, 2$) has a cut. Moreover one proves more fine statements about the distribution of the cuts of the multi-indices $k^1$ and $k^2$ on the tree $\Delta'$. Namely, up to the numbering of $y_1$ and $y_2$ there exists a cut $\sigma$ of the multi-index $k^1$ such that for each strict transform $\tilde{C}_i$ of a branch $C_i$ of the curve $C$ greater than $\sigma$ there is a cut of the multi-index $k^2$ on the geodesic from $\sigma$ to $\tilde{C}_i$. (For vertices $\sigma_1$ and $\sigma_2$ of the dual graph $\Gamma$ of the resolution, one says that $\sigma_1 \geq \sigma_2$ if $\sigma_2$ lies
on the geodesic between \( \sigma_1 \) and the starting divisor 1.) This is the most complicated (combinatorial) part of the proof.

2. It is not difficult to describe the difference between the images \( \Pi(y) \) and \( \Pi(y) \) for points \( y \) and \( y \) from the space \( Y \) corresponding to one and the same multi-index \( k \) and such that there exists only one vertex \( \sigma \in \Delta' \) for which the sets of points on \( \hat{E}_\sigma \) for the elements \( y \) and \( \tilde{y} \) (\( P_1^\sigma, \ldots, P_k^\sigma \) and \( \tilde{P}_1^\sigma, \ldots, \tilde{P}_k^\sigma \)) respectively) are different. Let \( Q_0^\sigma, Q_1^\sigma, \ldots, Q_{s-1}^\sigma \) be the essential points on the component \( E_\sigma \) of the exceptional divisor. Let \( \Psi \) be a meromorphic function on \( E_\sigma \) (\( \cong \mathbb{P}^1 \)) with zeroes at the points \( \tilde{P}_1^\sigma, \ldots, \tilde{P}_k^\sigma \) and poles at the points \( P_1^\sigma, \ldots, P_k^\sigma \). Such function is well-defined up to multiplication by a non-zero constant. For \( i = 1, \ldots, r \), let \( j(i) \) be defined by the condition that the connected component of \( \pi^{-1}(C) \setminus E_\sigma \) which contains the strict transform \( \tilde{C}_i \) of the branch \( C_i \) of the curve \( C \) intersects the component \( E_\sigma \) of the exceptional divisor at the point \( Q_j(i) \).

Then \( \Pi(y) \) is obtained from \( \Pi(y) \) by multiplying the coordinate \( a_i \) \( (i = 1, \ldots, r) \) by the constant \( \Psi(Q_j(i)) \) (the value of the described meromorphic function \( \Psi \) at the corresponding essential point). Therefore it is necessary to describe sets of possible values of the function \( \Psi \) at the essential points \( Q_0^\sigma, Q_1^\sigma, \ldots, Q_{s-1}^\sigma \) for different sets of points \( \tilde{P}_1^\sigma, \ldots, \tilde{P}_k^\sigma \).

3. Let \( Q_0, Q_1, \ldots, Q_{s-1} \) be (different) points of a projective line \( E \), \( \tilde{E} = E \setminus \{Q_\ell : \ell = 0, 1, \ldots, s-1\} \), let \( P_1^\sigma, \ldots, P_k^\sigma \) be \( k \) points (not necessarily different), different from \( Q_0, Q_1, \ldots, Q_{s-1} \). Let \( \Phi \) be the map from the symmetric power \( S^k \tilde{E} \) of the space \( \tilde{E} \) to \( \mathbb{P}((\mathbb{C}^*)^s) \) defined in the following way. For an element from \( S^k \tilde{E} \); i.e., for \( k \) points \( P_1, \ldots, P_k \) from \( \tilde{E} \), let \( \psi \) be a meromorphic function on \( E \) with zeroes at the points \( P_1, \ldots, P_k \) and poles at the points \( P_1^\sigma, \ldots, P_k^\sigma \); let \( \Phi(\{P_j\}) := (\psi(Q_0) : \psi(Q_1) : \ldots : \psi(Q_{s-1})). \)

Then, if \( k \geq s-1 \), one has \( \text{Im } \Phi = \mathbb{P}((\mathbb{C}^*)^s) \); if \( k \leq s-1 \), \( \Phi \) is an embedding. Moreover in both cases the map \( \Phi \) is a (smooth) locally trivial (in fact a trivial) fibration over its image the fibre of which is a (complex) affine space of dimension \( \max(0, k-s+1) \).

In order to prove this one can proceed as follows. Without loss of generality one can suppose that \( P_1^\sigma = P_2^\sigma = \cdots = P_k^\sigma = P^\sigma \). Let us choose an affine coordinate on the projective line \( E \) such that \( P^\sigma = \infty \). Then \( \psi \) is a polynomial of degree \( \leq k \) with zeroes at those points \( P_1, \ldots, P_k \) which are different from \( P^\sigma \). Let \( z_\ell \) be the coordinate of the point \( Q_\ell, \ell = 0, 1, \ldots, s-1 \).

For \( k \geq s-1 \), the statement that the map \( \Phi \) is onto can be reduced to the following obvious one: for an arbitrary prescribed set of values \( \{\psi_0, \psi_1, \ldots, \psi_{s-1}\} \), there exists a polynomial \( \psi \) of degree \( \leq k \) such that \( \psi(z_\ell) = \psi_\ell, \ell = 0, 1, \ldots, s-1 \). The statement that \( \Phi \) is a locally trivial fibration over its
image follows from the fact that if \( \psi_1 \) and \( \psi_2 \) are polynomials with coinciding values at the points \( Q_\ell, \ell = 0, 1, \ldots, s - 1 \), then \( \psi_1 = \psi_2 + q(z)(z - z_0)(z - z_1) \cdot \cdot \cdot (z - z_{s-1}) \) where \( q(z) \) is an arbitrary polynomial of degree \( k - s \).

For \( k \leq s - 1 \), the statement follows from the fact that such a polynomial of degree \( \leq s - 1 \) is unique.

This statement implies that at places where the map \( \tilde{\Pi} \) is not injective (and thus in the presence of cuts) one can find parts of the graded space \( \tilde{Y} \) which have zero Euler characteristic and which can be deleted without changing the image of the map \( \tilde{\Pi} \).

Combination of all these arguments (and/or of their versions) proves Theorem 4.

5 The Alexander polynomial as an integral with respect to the Euler characteristic

It appears that Theorem 4 can be reformulated so that the Alexander polynomial \( \Delta^C(t_1, \ldots, t_r) \) will be written as a certain integral with respect to the Euler characteristic over the space \( \mathcal{O}_{\mathbb{C}^2,0} \) of germs of functions on the plane \( \mathbb{C}^2 \) at the origin or rather over its projectivization \( \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} \). In order to do that one has to define the notion of the Euler characteristic for (some) subsets of the space \( \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} \). The idea is the same which is used to define the notion of the Euler characteristic (or of the generalized Euler characteristic with values in the Grothendieck ring of semialgebraic sets localized by the class of the complex line) for subsets of the space of arcs (in the framework of the theory of the motivic integration; see, e.g., [4]). The definitions of the Euler characteristic of (some) subsets and of the integral with respect to the Euler characteristic is practically the same for the space \( \mathcal{O}_{\mathbb{C}^n,0} \) of germs of functions at the origin in the space \( \mathbb{C}^n \) and for its projectivization \( \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \). Since here we shall use the last one, we shall give them in this setting.

Let \( J_{\mathbb{C}^n,0}^k \) be the space of \( k \)-jets of functions at the origin in \( (\mathbb{C}^n,0) \) \( (J_{\mathbb{C}^n,0}^k = \mathcal{O}_{\mathbb{C}^n,0}/m^{k+1} \cong \mathbb{C}^{\binom{n+k}{k}} ) \), where \( m \) is the maximal ideal in the ring \( \mathcal{O}_{\mathbb{C}^n,0} \). For a complex vector space \( L \) (finite or infinite dimensional) let \( \mathbb{P}L = (L \setminus \{0\})/\mathbb{C}^* \) be its projectivization, let \( \mathbb{P}^*L \) be the disjoint union of \( \mathbb{P}L \) with a point (in some sense \( \mathbb{P}^*L = L/\mathbb{C}^* \)). One has natural maps \( \pi_k : \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{P}^*J_{\mathbb{C}^n,0}^k \) and \( \pi_{k,\ell} : \mathbb{P}^*J_{\mathbb{C}^n,0}^k \to \mathbb{P}^*J_{\mathbb{C}^n,0}^\ell \) for \( k \geq \ell \). Over \( \mathbb{P}J_{\mathbb{C}^n,0} \subset \mathbb{P}^*J_{\mathbb{C}^n,0}^k \) the map \( \pi_{k,\ell} \) is a locally trivial fibration, the fibre of which is a complex vector space of dimension \( \binom{n+k}{k} - \binom{n+\ell}{\ell} \).

**Definition:** A subset \( X \subset \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \) is said to be cylindric if \( X = \pi_k^{-1}(Y) \) for
a semi-algebraic subset \( Y \subset \mathbb{P}^k_{\mathbb{C},0} \subset \mathbb{P}^*J^k_{\mathbb{C},0} \).

**Definition:** For a cylinder subset \( X \subset \mathbb{P}O_{\mathbb{C},0} \) \( (X = \pi_1^{-1}(Y), Y \subset \mathbb{P}^k_{\mathbb{C},0}) \) its Euler characteristic \( \chi(X) \) is defined as the Euler characteristic \( \chi(Y) \) of the set \( Y \).

Let \( \psi : \mathbb{P}O_{\mathbb{C},0} \to A \) be a function with values in an Abelian group \( A \).

**Definition:** We say that the function \( \psi \) is cylindric if, for each \( a \neq 0 \), the set \( \psi^{-1}(a) \subset \mathbb{P}O_{\mathbb{C},0} \) is cylindric.

**Definition:** The integral of a cylindric function \( \psi \) over the space \( \mathbb{P}O_{\mathbb{C},0} \) with respect to the Euler characteristic is

\[
\int_{\mathbb{P}O_{\mathbb{C},0}} \psi d\chi := \sum_{a \in A, a \neq 0} \chi(\psi^{-1}(a)) \cdot a
\]

if this sum has sense in \( A \). If the integral exists (has sense) the function \( \psi \) is said to be integrable.

**Remarks.** 1. In a similar way one can define the notion of the generalized Euler characteristic \([X]\) of a cylindric subset of the space \( \mathbb{P}O_{\mathbb{C},0} \) (or of the space \( \mathcal{O}_{\mathbb{C},0} \)) with values in the Grothendieck ring of complex algebraic varieties localized by the class \( \mathbb{L} \) of the complex line and thus the corresponding notion of integration (see, e.g., [4]). For that one should define \([X] = [Y] \cdot \mathbb{L}^{-\binom{n+k}{k}}\).

2. There are the same notions (of the Euler characteristic, of the generalized Euler characteristic, and of the integral with respect to the Euler characteristic) in the real setting as well. Since the Euler characteristic \( \chi(\mathbb{L}_\mathbb{R}) \) of the real line \( \mathbb{L}_\mathbb{R} \) is equal to \(-1\), one has to define the Euler characteristic \( \chi(X) \) of a cylinder subset \( X \subset \mathbb{P}E_{\mathbb{R},0} \) \( (X = \pi_1^{-1}(Y), Y \subset \mathbb{P}^k_{\mathbb{R},0}) \) as \((-1)^{-\binom{n+k}{k}} \cdot \chi(Y)\).

Let \( \mathbb{Z}[\![t]\!] \) (respectively \( \mathbb{Z}[\![t_1,\ldots,t_r]\!] \)) be the group (with respect to the addition) of formal power series in the variable \( t \) (respectively in the variables \( t_1,\ldots,t_r \)). As usual, for \( \mathbf{v} = (v_1,\ldots,v_r) \in \mathbb{Z}^r_{\geq 0} \), \( t^\mathbf{v} = t_1^{v_1} \cdots t_r^{v_r} \); we assume \( t^\infty = 0 \).

**Theorem 5** For each \( \mathbf{v} \in \mathbb{Z}^r_{\geq 0} \), the subset \( \{ g \in \mathbb{P}O_{\mathbb{C},0} : \varphi_i(g) = \mathbf{v} \} \) of \( \mathbb{P}O_{\mathbb{C},0} \) is cylindric. Therefore the functions \( \ell^\mathbf{v}(g) \) and \( t^v(g) \) \( (v(g) = \|\ell^\mathbf{v}(g)\| = v_1(g) + \ldots + v_r(g)) \) on \( \mathbb{P}O_{\mathbb{C},0} \) with values in \( \mathbb{Z}[\![t_1,\ldots,t_r]\!] \) and \( \mathbb{Z}[\![t]\!] \) respectively are cylindric.

**Proof** follows from the fact that, for \( g \in m^s \), \( v_i(g) \geq s \), i.e., the power series decomposition of \( g \circ \varphi_i(\tau_i) \) starts from terms of degree at least
Therefore the functions $t^v(g)$ and $tv(g)$ on $\mathbb{P}O_{C^2,0}$ are integrable (since $\sum_{v \in \mathbb{Z}_{\geq 0}} \ell(v)tv \in \mathbb{Z}[[t_1, \ldots, t_r]]$ for any integers $\ell(v)$).

**Theorem 6** For $r > 1$,

$$\int_{\mathbb{P}O_{C^2,0}} t^v(g) d\chi = \Delta^C(t_1, \ldots, t_r);$$

for $r \geq 1$,

$$\int_{\mathbb{P}O_{C^2,0}} tv(g) d\chi = \zeta_C(t).$$

**Proof** follows from Theorem 4. For $v = (v_1, \ldots, v_r) \in \mathbb{Z}_{\geq 0}^r$, let $k = 1 + \max_{1 \leq i \leq r} v_i$, and let $Z_v \subset \mathbb{P}J^{k}_{C^2,0}$ be the set $\{j^k g \in \mathbb{P}J^{k}_{C^2,0} : v(g) = v\}$ (we have mentioned that the condition $v(g) = v$ is determined by the $k$-jet $j^k g$ of the germ $g$). The natural map $Z_v \rightarrow \mathbb{P}F_v$ of the set $Z_v$ to the fibre $\mathbb{P}F_v$ of the projectivization $\mathbb{P}\hat{S}_C$ of the extended semigroup $\hat{S}_C$ is a locally trivial fibration the fibre of which is a complex vector space of some dimension. Now Theorem 6 follows from the facts that $\chi(\mathbb{P}\hat{S}_C) = \Delta^C(t_1, \ldots, t_r)$, $\zeta_C(t) = \Delta^C(t, \ldots, t)$. □

6 Polynomial functions in two variables and the zeta-function of the monodromy transformation at infinity

There exists a global analogue of the discussed results for a plane algebraic curve with one branch at infinity ([9]). Thus one can say that this is an analogue of the statement about the coincidence of the Poincaré series of the coordinate ring and the zeta-function of the classical monodromy transformation of an irreducible curve singularity (i.e., for $r = 1$).

Let $F(x, y)$ be a complex polynomial of degree $d$. It is well known that the map $F : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a locally trivial fibration over the complement to a finite set in the target $\mathbb{C}$. Its fibre is a generic level curve $C_{gen}$ of the polynomial $F$. Let $h_F^\infty : C_{gen} \rightarrow C_{gen}$ be the monodromy transformation of this fibration corresponding to the loop $\gamma(\tau) = R \cdot \exp(2\pi i \tau)$ with real $R$ large enough. The transformation $h_F^\infty$ is called the monodromy transformation of the polynomial $F$ at infinity. Let $\zeta_F^\infty(t)$ be the zeta–function of the monodromy transformation $h_F^\infty$.
Let $C_\lambda = (x, y) \in \mathbb{C}^2 : F(x, y) = \lambda$ be the level curve of the polynomial $F$. Assume that the curve $C_0$ has only one branch at infinity, i.e., that the projective curve $\overline{C_0}$ has one point $A_\infty$ at infinity and that it has only one branch at this point. This implies that the curve $\overline{C_\lambda}$ has only one branch at infinity (i.e., at the point $A_\infty$) for any $\lambda \in \mathbb{C}$.

Without loss of generality one can assume that $A_\infty = (0 : 1 : 0)$. The germ of the curve $\overline{C_0}$ at the point $A_\infty$ is determined by the local equation \{\(Q(u, v) = 0\}\}, where $Q(u, v) = u^d \cdot P(v/u, 1/u)$; the infinite line $L_\infty$ is given by the equation \{\(u = 0\)\}.

To the curve $C_0$ there corresponds a semigroup $\Gamma \in \mathbb{Z}_{\geq 0}$ — the semigroup of (orders of) poles along the curve $\overline{C_0}$ of polynomials in the affine plane $\mathbb{C}^2$ at the point $A_\infty$, i.e.,

$$\Gamma := \{-v(H(v/u, 1/u)) | H(x, y) \in \mathbb{C}[x, y], H \text{ is not divisible by } F\} \subset \mathbb{Z}.$$  

One can easily see that in fact $\Gamma \subset \mathbb{Z}_{\geq 0}$, since a polynomial not identically equal to zero on the curve $C_0$ cannot have a zero at the point $A_\infty$ of it. Let $P_\Gamma(t)$ be the Poincaré series for the semigroup $\Gamma$:

$$P_\Gamma(t) = \sum_{i \in \Gamma} t^i.$$  

**Theorem 7** $P_\Gamma(t) = \zeta_\infty^\infty(t)$.

The proof is essentially the same as of Theorem 4. Using known combinatorial properties of the semigroup $\Gamma$ (given by the Abhyankar–Moh theorem: [2], [3]) one can compute the Poincaré series $P_\Gamma(t)$ in terms of the minimal embedded resolution of the plane curve $C_0$. The zeta-function $\zeta_\infty^\infty(t)$ of the polynomial is given by an analogue of the A’Campo formula. The results of the computations show that they coincide. \(\Box\)

**Remark.** One can easily see that the statement of Theorem 7 can be rewritten in the form

$$\zeta_\infty^\infty(t) = \int_{\mathbb{P}\mathbb{C}[x, y]} t^{-v(H)} d\chi,$$

where $v(H) = v(H(v/u, 1/u))$. The integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathbb{C}[x, y]$ of the space $\mathbb{C}[x, y]$ of polynomials in two variables is defined in the same way as one in the section 5.
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