Remarks on the statistical origin of the geometrical formulation of quantum mechanics

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Abstract

A quantum system can be entirely described by the Kähler structure of the projective space \( \mathbb{P}(H) \) associated to the Hilbert space \( H \) of possible states; this is the so-called geometrical formulation of quantum mechanics.

In this paper, we give an explicit link between the geometrical formulation (of finite dimensional quantum systems) and statistics through the natural geometry of the space \( \mathbb{P} \times \times_{n} \) of non-vanishing probabilities \( p : E_{n} \to \mathbb{R} \) defined on a finite set \( E_{n} := \{x_{1}, ..., x_{n}\} \). More precisely, we use the Fisher metric \( g_{F} \) and the exponential connection \( \nabla^{(1)} \) (both being natural statistical objects living on \( \mathbb{P} \times \times_{n} \)) to construct, via the Dombrowski splitting Theorem, a Kähler structure on \( T\mathbb{P} \times \times_{n} \) which has the property that it induces the natural Kähler structure of a suitably chosen open dense subset of \( \mathbb{P}(\mathbb{C}^{n}) \).

As a direct physical consequence, a significant part of the quantum mechanical formalism (in finite dimension) is encoded in the triple \( (\mathbb{P} \times \times_{n}, g_{F}, \nabla^{(1)}) \).

1 Introduction

In quantum mechanics, it is well known that if \( \psi_{1}, \psi_{2} \) are two collinear vectors belonging to the Hilbert space \( H \) describing the possible states of a quantum system, then they are physically equivalent. This means that the true configuration space of quantum mechanics is the projective space \( \mathbb{P}(H) \), and this leads to a formulation of quantum mechanics uniquely based on the geometry of \( \mathbb{P}(H) \) (see [2] for a detailed discussion). In this formulation, the dynamics is governed by the Fubini-Study symplectic form, and the observables are no more operators, rather functions on the projective space having the particularity to preserve, in some sense, the Fubini-Study metric; eigenstates are critical points of the observable functions, and the corresponding critical values are the eigenvalues. In other words, quantum mechanics can be completely formulated in terms of the Kähler structure of the projective space \( \mathbb{P}(H) \). Subsequently, we shall refer to this formulation as the geometrical formulation of quantum mechanics.

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The geometrical formulation, although complete and very elegant, has a very disappointing feature: it doesn’t explain, nor even justify, why its probabilistic interpretation, based on the Fubini-Study metric, is consistent. Consistency is only something that one observes once the computations are done, and, somehow, this appears as “magical”. But of course, magic doesn’t exist, and this is clearly the sign that there is something in the geometrical formulation that we still don’t understand. This concerns the various formulas where probabilities are expected of course, but more generally, it concerns our understanding of the link between the Kähler structure of the projective space $\mathbb{P}(\mathcal{H})$, and statistics.

This link has, until now, attracted very little attention in the existing literature, and its nature is still quite mysterious. The only mention (to our knowledge) of a possible link between the Kähler structure of the projective space, and statistics, comes from information geometry, where it is argued that the Fubini-Study metric (at least for $\mathbb{P}(\mathbb{C}^n)$), is a kind of “quantum analogue” of the famous Fisher metric, the latter being a metric living on the space $\mathcal{P}_n$ of non-vanishing probabilities $p : E_n \to \mathbb{R}$ defined on a finite set $E_n = \{x_1, ..., x_n\}$. Such “link” is, however, only an analogy, and in particular, it has no precise mathematical formulation.

Nevertheless, the idea that the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$ may originate from the intrinsic geometry of a statistical manifold is an appealing one, and the main observation of this paper is that it is “almost” true. More precisely, we show that the restriction of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$ to the open dense subset $\mathcal{P}(\mathbb{C}^n)^\times := \{[z_1, ..., z_n] \in \mathbb{P}(\mathbb{C}^n) \mid z_k \neq 0 \ \forall \ k = 1, ..., n\}$ can be completely recovered from the statistical manifold $\mathcal{P}_n^\times$, supplemented with the Fisher metric $g_F$ and from another important statistical object living naturally on $\mathcal{P}_n^\times$, namely the exponential connection $\nabla^{(1)}$. Together, these geometrical objects form a triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ which allows the construction of an almost Hermitian structure $(T\mathcal{P}_n^\times, G, J)$ on $T\mathcal{P}_n^\times$; the construction, which is due to Dombrowski (see [3]), is straightforward. For $u_p \in T_p\mathcal{P}_n^\times$, the exponential connection $\nabla^{(1)}$ gives a splitting of $T_u_p T\mathcal{P}_n^\times$ into the direct sum of the spaces of horizontal and vertical tangent vectors that one may identify with $T_p\mathcal{P}_n^\times \oplus T_p\mathcal{P}_n^\times$. With this splitting, the metric $G$ at the point $u_p$ is simply the direct sum $(g_F)_p \oplus (g_F)_p$, while the almost complex structure $J$ is given, for $v_p, w_p \in T_p\mathcal{P}_n^\times$, by $J_{u_p}(v_p, w_p) := (-w_p, v_p)$. The resulting almost Hermitian structure will be referred to as the almost splitting Hermitian structure associated to the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$.

Now, the link between the almost splitting Hermitian structure associated to the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ and the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$ goes as follows. As we show, there exists a covering map $\tau : T\mathcal{P}_n^\times \to \mathbb{P}(\mathbb{C}^n)^\times$ having the property that the “pull back” of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$ is exactly the above almost splitting Hermitian structure on $T\mathcal{P}_n^\times$ (the latter being actually also a Kähler structure). Hence, we have a precise description of the statistical origin of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$.

A direct physical consequence—at least theoretically—is that a significant part
of the quantum mechanical formalism (in finite dimension) is encoded in the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ via its associated almost splitting Hermitian structure. This physical consequence is, however, not discussed in this paper for which we refer the reader to [4].

This paper is organized as follows. In §2, we recall the definition of the Fisher metric $g_F$ and the exponential connection $\nabla^{(1)}$ on the space of probabilities $\mathcal{P}_n^\times$, and we define the almost splitting Hermitian structure associated to $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$. In §3, we introduce a covering map $\tau : T\mathcal{P}_n^\times \to \mathbb{P}(\mathbb{C}^n)^\times$ and show that the “pull back” of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$ via $\tau$ is exactly the almost splitting Hermitian structure on $T\mathcal{P}_n^\times$ (Proposition 3.3).

2 Information geometry and the statistical model $\mathcal{P}_n^\times$

Our statistical model will be the space $\mathcal{P}_n^\times$ of non-vanishing probability distributions $p$ on a discrete set $E_n := \{x_1, \ldots, x_n\}$:

$$\mathcal{P}_n^\times := \{p : E_n \to \mathbb{R} \mid p(x_i) > 0 \text{ for all } x_i \in E_n \text{ and } \sum_{i=1}^{n} p(x_i) = 1\}. \quad (1)$$

This space is clearly a connected manifold of dimension $n - 1$. From a topological point of view, it is trivial since it can be realized as the interior of a simplex, but from a statistical point of view, it is naturally endowed with nontrivial geometric structures that we now want to describe (see [1]).

First of all, and for the rest of this paper, we will always use the so-called exponential representation for the tangent space:

$$T_p\mathcal{P}_n^\times \cong \{u = (u_1, \ldots, u_n) \in \mathbb{R}^n \mid u_1 p_1 + \ldots + u_n p_n = 0\}, \quad (2)$$

where $p \in \mathcal{P}_n^\times$, and where by definition, $p_i := p(x_i)$ for all $x_i \in E_n$.

If $u \in \mathbb{R}^n$ is a vector satisfying $u_1 p_1 + \ldots + u_n p_n = 0$ for a given probability $p$, then we shall denote by $[u]_p$ the unique tangent vector of $\mathcal{P}_n^\times$ at the point $p$ determined by the exponential representation. One easily sees that if $p(t)$ is a smooth curve in $\mathcal{P}_n^\times$, then

$$\left. \frac{d}{dt} \right|_0 p(t) = [u]_{p(0)} \iff \left. \frac{d}{dt} \right|_0 p_i(t) = p_i(0) u_i \text{ for all } i = 1, \ldots, n, \quad (3)$$

where $p_i(t) := (p(t))(x_i)$.

Equation (3) is actually one way to define the exponential representation.

The Fisher metric $g_F$ is now defined, for $[u]_p, [v]_p \in T_p\mathcal{P}_n^\times$, by\footnote{The Fisher metric is actually defined only up to a multiplicative constant, and thus we are free, for later convenience, to introduce the factor $1/4$.}

$$(g_F)_p([u]_p, [v]_p) := \frac{1}{4} \sum_{k=1}^{n} p_k u_k v_k = \frac{1}{4} E_p(uv), \quad (4)$$
where \( E_p(.) \) denotes the expectation with respect to the probability \( p \) and where \( uv \) denotes the vector in \( \mathbb{R}^n \) whose \( k \)-th component is \( u_k v_k \).

In most interesting statistical models, the Fisher metric is naturally and intrinsically defined, and is of central importance in information geometry (see [1]).

For a real parameter \( \alpha \), we also introduce the so-called \( \alpha \)-connection \( \nabla^{(\alpha)} \) on the space \( \mathcal{P}_n^\times \). One way to define it is to use its associated covariant derivative along curves \( D^{(\alpha)}/dt : \) if \( [V(t)]_{p(t)} \) is a vector field along a curve \( p(t) \) in \( \mathcal{P}_n^\times \) such that \( dp(t)/dt = [u(t)]_{p(t)} \), then, by definition,

\[
\frac{D^{(\alpha)}}{dt} [V(t)]_{p(t)} := \left[ \dot{V}(t) + \alpha/2 \cdot u(t) V(t) - E_p(t) \left( \dot{V}(t) + \alpha/2 \cdot u(t) V(t) \right) \cdot n \right]_{p(t)} ,
\]

where \( u \) and \( V \) are viewed as maps \( \mathbb{R} \to \mathbb{R}^n \), \( \dot{V}(t) \) is the usual derivative of \( V(t) \) with respect to \( t \) and where \( n := (1,...,1) \in \mathbb{R}^n \).

It may be shown that \( \nabla^{(0)} \) corresponds to the Levi-Civita connection associated to the Fisher metric \( g_F \) and also, if \( X,Y,Z \) are vector fields on \( \mathcal{P}_n^\times \), that

\[
X g_F(Y,Z) = g_F(\nabla^{(\alpha)}_X Y,Z) + g_F(Y,\nabla^{(-\alpha)}_X Z) .
\]

Because of (7), the triple \((g_F,\nabla^{(\alpha)},\nabla^{(-\alpha)})\) is called a dualistic structure on \( \mathcal{P}_n^\times \), and is also one of the major tools in information geometry (see [1]).

In the sequel, we will not use this dualistic structure, but we will restrict our attention to the 1-connection \( \nabla^{(1)} \), also called exponential connection. This connection is probably, in view of (3), the simplest and the most natural connection among the family of \( \alpha \)-connections since its expression reduces to

\[
\frac{D^{(1)}}{dt} [V(t)]_{p(t)} = \left[ \dot{V}(t) - E_p(t) \left( \dot{V}(t) \right) \cdot n \right]_{p(t)} ,
\]

We now want to describe the natural geometry of \( T\mathcal{P}_n^\times \). Recall that if \( M \) is a manifold endowed with an affine connection \( \nabla \), then the Dombrowski splitting Theorem holds (see [3]):

\[
T(TM) \cong TM \oplus TM \oplus TM ,
\]

this splitting being viewed as an isomorphism of vector bundles over \( M \), and the isomorphism, say \( \Phi_M \), being

\[
T_{u_x} TM \ni A_{u_x} \xrightarrow{\Phi_M} (u_x, (\pi^{TM})_{u_x} A_{u_x}, K A_{u_x}) ,
\]

where \( \pi^{TM} : TM \to M \) is the canonical projection and where \( K : T(TM) \to TM \) is the canonical connector associated to the connection \( \nabla \).

Applied to the couple \((\mathcal{P}_n^\times, \nabla^{(1)})\), the Dombrowski splitting Theorem yields

\[
T(T\mathcal{P}_n^\times) \cong T\mathcal{P}_n^\times \oplus T\mathcal{P}_n^\times \oplus T\mathcal{P}_n^\times .
\]
It turns out that the inverse of the vector bundle isomorphism \( \Phi_{p_n} : T(T^n P_n) \to T^n P_n \oplus T^n P_n \oplus T^n P_n \) can be expressed very explicitly:

**Lemma 2.1.** For \([u]_p, [v]_p, [w]_p \in T_p P_n^\times\), we have:

\[
\Phi_{p_n}^{-1}([u]_p, [v]_p, [w]_p) = \left. \frac{d}{dt} \right|_0 \left[ u + tw - E_p(t)(u + tw) \cdot n \right]_{p(t)},
\]

where \(p(t)\) is a smooth curve in \(P_n^\times\) satisfying \(p(0) = p\) and \(dp(t)/dt|_0 = [v]_p\).

**Proof.** It suffices to show that

\[
\Phi_{p_n} \left( \frac{d}{dt} \right|_0 \left[ u + tw - E_p(t)(u + tw) \cdot n \right]_{p(t)} \right) = ([u]_p, [v]_p, [w]_p).
\]

To this end, let us consider the curve \(\gamma\) in \(T P_n^\times\) which is defined by

\[
\gamma(t) := \left. \left[ u + tw - E_p(t)(u + tw) \cdot n \right] \right|_{p(t)}.
\]

We have

- \(\gamma(0) = [u]_p\),
- \((\pi T P_n^\times)_{\gamma(0)} \frac{d}{dt} \gamma(t) = \frac{d}{dt} (\pi T P_n^\times \circ \gamma)(t) = \frac{d}{dt} p(t) = [v]_p\),

and thus, we immediately see that

\[
\Phi_{p_n} \left( \frac{d}{dt} \gamma(t) \right) = ([u]_p, [v]_p, *),
\]

where “*” has to be determined. But, according to \([7], [9]\) and the fact that \(K^{(1)}\) is a connector (here \(K^{(1)}\) is the connector associated to the connection \(\nabla^{(1)}\)),

\[
* = K^{(1)} \left. \frac{d}{dt} \right|_0 \gamma(t) = D^{(1)} \left. \gamma(t) \right|_0 = \left. \frac{d}{dt} \gamma(t) \right|_0 - E_p \left. \left[ \frac{d}{dt} \gamma(t) \right] \cdot n \right|_p
\]

\[
= \left. \left[ w - \frac{d}{dt} \gamma(t) \right] \cdot E_p(t)(u + tw) \cdot n + \left( \frac{d}{dt} E_p(t)(u + tw) \right) \cdot E_p(n) \cdot n \right|_p = [w]_p.
\]

Hence \(\Phi_{p_n} \left( \frac{d}{dt} \gamma(t)/dt \right|_0 \right) = ([u]_p, [v]_p, [w]_p)\). The lemma follows. \(\square\)

In the sequel, we will identify a triple \(([u]_p, [v]_p, [w]_p)\) with the corresponding element of \(T_{[u]_p}(T P_n^\times)\) via the isomorphism \(\Phi_{p_n}\).
Having the decomposition \([10]\), it is a simple matter to define on \(T^n_P\) an almost Hermitian structure. Indeed, we define a metric \(G\), a 2-form \(\Omega\) and an almost complex structure \(J\) by setting

\[
G_{[u]_p} \left( ([v]_p, [w]_p) \right) := (g_F)_p ([v]_p, [w]_p), \\
\Omega_{[u]_p} \left( ([v]_p, [w]_p) \right) := (g_F)_p ([v]_p, [w]_p) - (g_F)_p ([w]_p, [v]_p), \\
J_{[u]_p} \left( ([v]_p, [w]_p) \right) := ([u]_p, -[w]_p, [v]_p),
\]

(18)

where \([u]_p, [v]_p, [w]_p, [\overline{v}]_p, [\overline{w}]_p \in T^n_P\).

Clearly, \(J^2 = -\text{Id}\) and \(G(J\cdot J\cdot) = G(\cdot\cdot\cdot)\), which means that \((T^n_P, G, J)\) is an almost Hermitian manifold, and one readily sees that \(G, J\) and \(\Omega\) are compatible, i.e., that \(\Omega = G(J\cdot\cdot\cdot)\); the 2-form \(\Omega\) is thus the fundamental 2-form of the almost Hermitian manifold \((T^n_P, G, J)\).

The above geometric construction is a particular case of a more general construction which is due to Dombrowski (see [3]).

3 The statistical nature of \(\mathbb{P}(\mathbb{C}^n)\)

Recall that the projective space \(\mathbb{P}(\mathbb{C}^n)\) is simply the quotient \((\mathbb{C}^n - \{0\})/\sim\), where the equivalence relation \(\sim\) is defined by

\[
(z_1, \ldots, z_n) \sim (w_1, \ldots, w_n) \iff \exists \lambda \in \mathbb{C} - \{0\} : (z_1, \ldots, z_n) = \lambda (w_1, \ldots, w_n).
\]

(19)

For \(z = (z_1, \ldots, z_n) \in \mathbb{C}^n - \{0\}\), we shall denote by \([z] = [z_1, \ldots, z_n]\) the corresponding element of \(\mathbb{P}(\mathbb{C}^n)\). One may identify \([z]\) with the complex line \(\mathbb{C} \cdot z\).

The manifold structure of \(\mathbb{P}(\mathbb{C}^n)\) may be defined as follows. For a vector \(u = (u_1, \ldots, u_n) \in \mathbb{C}^n\) such that \(|u|^2 = \langle u, u \rangle = \overline{u}_1 u_1 + \cdots + \overline{u}_n u_n = 1\) (our convention for the Hermitian product \(\langle , \rangle\) on \(\mathbb{C}^n\) is that \(\langle , \rangle\) is linear in the second argument), we define a chart \((U_u, \phi_u)\) of \(\mathbb{P}(\mathbb{C}^n)\) by letting

\[
\begin{align*}
U_u &:= \{[z] \in \mathbb{P}(\mathbb{C}^n) \mid \langle u, z \rangle = 0\}, \\
\phi_u : U_u &\to [u]^\perp \subseteq \mathbb{C}^n, [z] \mapsto \frac{1}{\langle u, z \rangle} z - u.
\end{align*}
\]

(20)

If \(u\) varies among all the unit vectors in \(\mathbb{C}^n\), then the corresponding charts \((U_u, \phi_u)\) form an atlas for \(\mathbb{P}(\mathbb{C}^n)\); the projective space is thus a real manifold of dimension \(2(n - 1)\), and, using the above charts, we have the identification

\[
T_{[u]} \mathbb{P}(\mathbb{C}^n) \cong [u]^\perp = \{w \in \mathbb{C}^n \mid \langle u, w \rangle = 0\}.
\]

(21)

The Fubini-Study metric \(g_{FS}\) and the Fubini-Study symplectic form \(\omega_{FS}\) are now defined at the point \([u]\) in \(\mathbb{P}(\mathbb{C}^n)\) via the formulas:

\[
\left( (\phi_u^{-1})^* g_{FS} \right)_0 (\xi_1, \xi_2) := \text{Re} \langle \xi_1, \xi_2 \rangle, \quad \left( (\phi_u^{-1})^* \omega_{FS} \right)_0 (\xi_1, \xi_2) := \text{Im} \langle \xi_1, \xi_2 \rangle,
\]

(22)
where \( \xi_1, \xi_2 \in [u]^\perp \cong T_u \mathbb{P}(\mathbb{C}^n) \).
One may show that \( g_{FS} \) and \( \omega_{FS} \) are globally well defined on \( \mathbb{P}(\mathbb{C}^n) \).

We now want to relate the Kähler structure of \( \mathbb{P}(\mathbb{C}^n) \) with the almost splitting Hermitian structure associated to the triple \((\mathcal{P}_n^\times, g_F, \nabla^{(1)})\) discussed in [2]. To this end, we set

\[
\mathbb{P}(\mathbb{C}^n)^\times := \{ [z_1, ..., z_n] \in \mathbb{P}(\mathbb{C}^n) \mid z_i \neq 0 \text{ for all } i = 1, ..., n \}
\tag{23}
\]
and introduce the following smooth map

\[
\tau: T\mathcal{P}_n^\times \rightarrow \mathbb{P}(\mathbb{C}^n)^\times, \quad [u]_p \rightarrow [\sqrt{p_1} e^{iu_1/2}, ..., \sqrt{p_n} e^{iu_n/2}].
\tag{24}
\]

The geometrical nature of the map \( \tau \) is given by the following lemma.

**Lemma 3.1.** The map \( \tau: T\mathcal{P}_n^\times \rightarrow \mathbb{P}(\mathbb{C}^n)^\times \) is a universal covering map whose deck transformation group is a copy of \( \mathbb{Z}^{n-1} \).

**Proof.** Let \( T^{n-1} \) denotes the \((n-1)\)-dimensional torus, and let us consider the following diagram:

\[
\begin{array}{ccc}
T\mathcal{P}_n^\times & \xrightarrow{\tau} & \mathbb{P}(\mathbb{C}^n)^\times \\
j_1 \downarrow & & \downarrow j_2 \\
\mathcal{P}_n^\times \times \mathbb{R}^{n-1} & \xrightarrow{\tau} & \mathcal{P}_n^\times \times T^{n-1}
\end{array}
\tag{25}
\]

where

- \( j_1([u]_p) := (p, (u_1 - u_2, ..., u_{n-1} - u_n)) \),
- \( j_2([\sqrt{p_1} e^{iu_1/2}, ..., \sqrt{p_n} e^{iu_n/2}]) := (p, (e^{i(u_1-u_2)/2}, ..., e^{i(u_{n-1}-u_n)/2})) \),
- \( \tau(p, u) := (p, (e^{iu_1/2}, ..., e^{iu_{n-1}/2})) \).

Clearly, \( j_1 \) and \( j_2 \) are diffeomorphisms, and one easily sees that \( \tau \) is nothing but the quotient map associated to the (free and proper) action of the group \( \mathbb{Z}^{n-1} \) on \( \mathcal{P}_n^\times \times \mathbb{R}^{n-1} \) given by \( (k_1, ..., k_{n-1}) \cdot (p, u) := (p, (u_1 + 4k_1 \pi, ..., u_{n-1} + 4k_{n-1} \pi)) \). As the diagram is manifestly commutative, the lemma follows. \( \square \)

**Lemma 3.2.** For \( ([u]_p, [v]_p, [w]_p) \in T_{([u]_p)} T\mathcal{P}_n^\times \), we have

\[
(\phi_z \circ \tau)_{([u]_p, [v]_p, [w]_p)} = \left(1/2 \sqrt{p_1} e^{iu_1/2}(v_1 + iw_1), ..., 1/2 \sqrt{p_n} e^{iu_n/2}(v_n + iw_n)\right),
\tag{26}
\]
where \( z := (\sqrt{p_1} e^{iu_1/2}, ..., \sqrt{p_n} e^{iu_n/2}) \in \mathbb{C}^n \) and where \( \phi_z : U_z \subseteq \mathbb{P}(\mathbb{C}^n) \rightarrow [z]^\perp \subseteq \mathbb{C}^n \) is the chart on \( \mathbb{P}(\mathbb{C}^n) \) introduced in [20].
Proof. Let us fix \([u]_p, [v]_p, [w]_p \in \mathcal{T}^\times P^\times_n\) and set
\[
z(t) := \left(\sqrt{p_1(t)} e^{i(\lambda u_1 + tw_1)/2}, ..., \sqrt{p_n(t)} e^{i(\lambda u_n + tw_n)/2}\right) \in \mathbb{C}^n - \{0\},
\]  
where \(p(t)\) is a smooth curve in \(\mathcal{P}^\times_n\) satisfying \(dp(t)/dt|_0 = [v]_p\). Observe that \([z] = [z(0)] = \tau([u]_p), (z(t), \dot{z}(t))^2 = 1\) and that \((z(t), \dot{z}(t)) = 0\) since \(z(t)\) is normalized. Using the identification \([11],\) Lemma 2.1 as well as the formula \([\lambda\cdot z] = [z] (\lambda \in \mathbb{C} - \{0\})\) which holds on \(\mathbb{P}(\mathbb{C}^n)\), we see that
\[
(\phi_z(0) \cdot \tau)_{[u]_p} ([u]_p, [v]_p, [w]_p) =\frac{d}{dt}|_0 (\phi_z(0) \cdot \tau) \left( [u + tw - E_p(t)(u + tw) \cdot n]_{p(t)} \right)
\]
\[
= \frac{d}{dt}|_0 \phi_z(0) \left( [\sqrt{p_1(t)} e^{i(\lambda u_1 + tw_1)/2} e^{-iE_p(t)(u + tw)/2}, ..., \sqrt{p_n(t)} e^{i(\lambda u_n + tw_n)/2} e^{-iE_p(t)(u + tw)/2}] \right)
\]
\[
= \frac{d}{dt}|_0 \phi_z(0) ([z(t)]) = \frac{d}{dt}|_0 \left( \frac{1}{\langle z(0), z(t) \rangle} \cdot z(t) - z(0) \right)
\]
\[
= \frac{-\langle z(0), \dot{z}(0) \rangle}{\langle z(0), z(0) \rangle^2} z(0) + \frac{1}{\langle z(0), z(0) \rangle} \cdot \dot{z}(0) = -\langle z(0), \dot{z}(0) \rangle z(0) + \dot{z}(0) = \dot{z}(0) .
\]

The lemma is now a direct consequence of
\[
(\dot{z}(0))_j = \frac{d}{dt}|_0 \sqrt{p_j(t)} e^{i(\lambda u_j + tw_j)/2} = \frac{1}{2\sqrt{p_j}} p_j v_j e^{i\lambda u_j/2} + \sqrt{p_j} \frac{i}{2} w_j e^{i\lambda u_j/2}
\]
\[
= 1/2 \sqrt{p_j} e^{i\lambda u_j/2} (v_j + iw_j)
\]  
(of course, in the above computations we use extensively the exponential representation \([30]\).)

For the next proposition (which is the main observation of this paper), recall that \(\mathcal{T}^\times P^\times_n\) is endowed with its associated splitting Hermitian structure \((G, J, \Omega)\) introduced at the end of \([2]\) and that \(\mathbb{P}(\mathbb{C}^n)\) possesses its natural Kähler structure \((g_{FS}, J_{FS}, \omega_{FS})\).

**Proposition 3.3.** The covering map \(\tau : \mathcal{T}^\times P^\times_n \to \mathbb{P}(\mathbb{C}^n)^\times\) defined in \([2]\) has the following properties:
\[
\tau^* g_{FS} = G , \quad \tau^* \omega_{FS} = \Omega , \quad \tau_* J = J_{FS} \tau_* .
\]

**Proof.** Let us fix \([u]_p, [v]_p, [w]_p \in \mathcal{T}_p \mathcal{P}^\times_n\) and define the normalized vector
\[
z := \left(\sqrt{p_1} e^{i\lambda u_1/2}, ..., \sqrt{p_n} e^{i\lambda u_n/2}\right) \in \mathbb{C}^n .
\]
According to Lemma 3.2 we have:

\[
\left\langle (\phi_z \circ \tau) \star_{[u]_p} ([u]_p, [v]_p, [w]_p), (\phi_z \circ \tau) \star_{[u]_p} ([u]_p, [\overline{v}]_p, [\overline{w}]_p) \right\rangle = \left\langle \left(1/2 \sqrt{p_1} e^{i u_1/2} (v_1 + i w_1), \ldots, 1/2 \sqrt{p_n} e^{i u_n/2} (v_n + i w_n) \right)\right. \\
\left. , \left(1/2 \sqrt{p_1} e^{i u_1/2} (\overline{v}_1 + i \overline{w}_1), \ldots, 1/2 \sqrt{p_n} e^{i u_n/2} (\overline{v}_n + i \overline{w}_n) \right) \right\rangle \\
= \sum_{j=1}^n \frac{1}{4} p_j (v_j - i w_j)(\overline{v}_j + i \overline{w}_j) = \sum_{j=1}^n \frac{1}{4} p_j \left( v_j \overline{v}_j + w_j \overline{w}_j + i(v_j \overline{w}_j - \overline{v}_j w_j) \right) \\
= \frac{1}{4} \sum_{j=1}^n p_j (v_j \overline{v}_j + w_j \overline{w}_j) + \frac{i}{4} \sum_{j=1}^n p_j (v_j \overline{w}_j - \overline{v}_j w_j) \\
= G_{[u]_p} \left( ([u]_p, [v]_p, [w]_p), ([u]_p, [\overline{v}]_p, [\overline{w}]_p) \right) \\
+ i \Omega_{[u]_p} \left( ([u]_p, [v]_p, [w]_p), ([u]_p, [\overline{v}]_p, [\overline{w}]_p) \right) \right. \\
\left. \right). \tag{32} \]

Comparing (32) with the definition of $g_{FS}$ and $\omega_{FS}$ given in (22) gives the first two relations in (30), and these two relations, together with the fact that both triples $(G, J, \Omega)$ and $(g_{FS}, J_{FS}, \omega_{FS})$ are compatible, imply the last relation in (30). The proposition follows.

Acknowledgments

It is a pleasure to thank Hsiung Tze who pointed out to me the existence of the geometrical formulation of quantum mechanics, and who, by his enthusiasm, gave me the necessary motivation to study quantum mechanics outside of its usual presentation. This work was done with the financial support of the Japan Society for the Promotion of Science.

References

[1] S. Amari and H. Nagaoka, Methods of information geometry (American Mathematical Society, Providence, RI, 2000).

[2] A. Ashtekar and T. A. Schilling, Geometrical formulation of quantum mechanics, in On Einstein’s path (New York, 1996) (Springer, New York, 1999) pp. 23–65.

[3] P. Dombrowski, On the geometry of the tangent bundle, J. Reine Angew. Math. 210 (1962), 73–88.

[4] M. Molitor, Exponential families, Kähler geometry and quantum mechanics, in preparation.