GMsFEM on unstructured grids for single-phase flow in fractured porous media

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Abstract. In this work, we consider an unstructured Generalized Multiscale Finite Element Method (GMsFEM) for solution of the filtration problem in a fractured media. The basic idea is that coarse grid blocks are formed as sets of fine grid triangular cells and, thus, can be of an almost arbitrary polygonal shape. In this approach, when constructing the fine grid, the faces of coarse blocks are not considered. Discrete fractured model is used to describe a flows in a fractured media.

1. Introduction

In numerical solution of the applied problems using multiscale methods, a fine-scale grid is constructed by taking into account coarse blocks, which is a rather complicated process. In order to avoid difficulties with coarse grid definition, various modifications of the multiscale methods are developed. For example, in [1] and [2] a dynamic a grid coarsening strategy was developed for a mixed flow, where the grid should separate high-flow regions from low-flow regions. In [3], an unstructured coarse grids are used in Multiscale Finite Volume Method (MSFVM) for simulation of the flow processes in the fractured porous media. Generalazed Multiscale Finite Element Method with discontinuous Galerkin coupling on the unstructured coarse grid was presented in [11] for solution of the problems in perforated domains. GMsFEM for flow in fractured porous media was presented in [10] for structured coarse grids.

In this work, we consider a Generalized Multiscale Finite Element Method on unstructured coarse grids for solution of the filtration problem in fractured porous media. In Section 2, we present a problem formulation and fine grid approximation using discrete fracture model. In Section 3, we present a multiscale algorithm for solution of the problem on unstructured coarse grids based on the GMsFEM. Numerical results are presented in Section 4.
2. Model problem and fine grid approximation

In this work we consider a single-phase fluid flow in fractured porous media [10]

\[
\begin{align*}
  c_m \frac{\partial p_m}{\partial t} - \text{div} \left( \frac{k_m}{\mu} \text{grad} p_m \right) + \sigma_{mf}(p_m - p_f) &= 0, \quad x \in \Omega, \\
  c_f \frac{\partial p_f}{\partial t} - \text{div} \left( \frac{k_f}{\mu} \text{grad} p_f \right) - \sigma_{fm}(p_m - p_f) &= 0, \quad x \in \gamma,
\end{align*}
\]

(1)

where \( \Omega \) is the porous media domain, \( \gamma \) is the lower-dimensional domain for fractures, \( p_m \) and \( p_f \) are the porous matrix and fractures pressures. Here \( \mu \) is the fluid viscosity, \( c_\alpha = \phi_\alpha c_{R,\alpha} \), \( \phi_\alpha \), \( c_{R,\alpha} \) and \( k_\alpha \) are the porosity, compressibility and permeability for continuum \( \alpha \) (\( \alpha = m, f \)). Note that the flow of an incompressible fluid in an elastically deformable porous medium is considered and \( k_f \) is much larger than \( k_m \) (high - permeable fractures).

System of equations (1) is considered with following initial and boundary conditions

\[
\begin{align*}
  p_\alpha(x,0) &= p_0(x), \quad x \in \Omega, \\
  p_\alpha &= p_1, \quad x \in \Gamma_L, \quad -\frac{k_\alpha \partial p_\alpha}{\mu} = 0, \quad x \in \partial\Omega/\Gamma_L.
\end{align*}
\]

(2)

for \( \alpha = m, f, n \) is the outward unit normal to the boundary \( \partial\Omega \), \( \Gamma_L \) is left boundary.

The finite element method and discrete fracture model are used to discretize the system on a fine grid, where fractures are represented on the fine grid as the facets of finite element mesh ([4] – [6]). Using superposition principle and suppose that \( p = p_f = p_m \) on the fractures \( p = p_m \) on porous matrix domain, we obtain following approximation in the matrix form

\[
M^{n+1}p - Ap^n = 0,
\]

(3)

where

\[
A = [a_{i,j}], \quad a_{i,j} = \int_\Omega \left( \frac{k_m}{\mu} \text{grad} \phi_i \cdot \text{grad} \phi_j \right) dx + \int_\gamma \left( \frac{k_f}{\mu} \text{grad} \phi_i^f \cdot \text{grad} \phi_j^f \right) dx,
\]

\[
M = [m_{i,j}], \quad m_{i,j} = \int_\Omega c_m \phi_i \phi_j dx + \int_\gamma c_f \phi_i^f \phi_j^f dx.
\]

and \( \phi_i, \phi_i^f \) are the linear basis functions in domain \( \Omega \) and \( \gamma \). Here for approximation by time, we used an implicit scheme with time step size \( \tau \), superscripts \( n \) and \( n+1 \) denote previous and current time levels, respectively.

3. Coarse grid approximation using GMsFEM

In this section, we give a brief description of the unstructured GMsFEM for flow problems in fractured media [7] – [11]. We start with the construction of the coarse grids. We define unstructured grid as a partition of the fine grid by the mesh partitioning library SCOTCH [12]. We emphasize the use of \( \omega_i \) to denote a coarse neighborhood, and \( K \) to denote a coarse block \((i = 1, ..., N, N \) is the number of coarse neighborhoods).

To construct the multiscale space for pressure approximation, we solve following spectral problem

\[
A^{\omega_i} \Psi_j^i = \lambda_j^i S^{\omega_i} \Psi_j^i,
\]

(4)

where \( A^{\omega_i} \) is the restriction of the global matrix \( A \) in local domain \( \omega_i \) and

\[
S^{\omega_i} = [s_{i,j}^{\omega_i}], \quad s_{i,j}^{\omega_i} = \int_{\omega_i} \frac{k_m}{\mu} \phi_i \phi_j dx + \int_\gamma \frac{k_f}{\mu} \phi_i^f \phi_j^f dx.
\]
Next, we choose the eigenvectors corresponded to the smallest $M_i$ eigenvalues.

For construction of the conforming multiscale space, we define a multiscale partition of unity functions $\chi_i$ as follows

$$A^K \chi_i = 0, \quad \forall K \in \omega_i,$$

where $A^K$ is the restriction of the global matrix $A$ in coarse partitioning $K$. We solve eq. (5) with linear boundary conditions $\chi_i = g_i$ on $\partial K$, where $g_i$ is a continuous function on $\partial K$ and is linear on each edge of $\partial K$.

To form a multiscale space and projection matrix, we multiply the partition of unity functions $\chi_i$ by the eigenfunctions $\Psi^i$

$$V_{ms} = \text{span}\{\psi_{i,j} = \chi_i \Psi^i_j : 1 \leq i \leq N, 1 \leq j \leq M_i\}, \quad R = [\psi_1, \ldots, \psi_N, N \times N]^T,$$

and define a projection matrix $R^T = [\psi_1, \ldots, \psi_N]$, where $\psi_i$ are used to denote the nodal values of each basis function defined on the fine grid.

Finally, we have following coarse grid system

$$M_c p_{c, n+1} - p_c^n + A_c p_{c, n+1} = 0,$$

where $A_c = RAR^T$, $M_c = RMR^T$ and $p_{ms}^{n+1} = R^T p_{c, n+1}$.

4. Numerical results

In this section, we present numerical results to show the performance of the presented multiscale method on unstructured coarse grids. For coefficients representing matrix and fracture properties, we set $\phi_m = 0.4$, $c_{R,m} = 10^{-9}\text{Pa}^{-1}$, $k_m = 10^{-15}\text{m}^2$, $\phi_f = 1$, $c_{R,f} = 10^{-9}\text{Pa}^{-1}$, $k_f = 10^{-10}\text{m}^2$, $\mu = 2 \cdot 10^{-3}\text{Pa} \cdot \text{s}$, $p_0 = 10\text{MPa}$, $p_1 = 20\text{MPa}$. Final time is 30 months with 30 time steps. We construct three coarse grids (Figure 4). Fine grid contains 70297 elements and 35471 vertices. Coarse grid 1 contains 25 elements and 52 vertices. Coarse grid 2 contains 50 elements and 102 vertices. Coarse grid 3 contains 100 elements and 202 vertices. To compare the results, we use relative errors in $L^2_a$ and $H^1_a$ norms.

![Figure 1](image_url). Fine grid and three unstructured coarse grids. First: Fine grid with 70297 elements and 35471 vertices. Second: Coarse grid 1 with 25 elements and 52 vertices. Third: Coarse grid 2 with 50 elements and 102 vertices. Fourth: Coarse grid 3 with 100 elements and 202 vertices.

The fracture geometry with several connected fractures is illustrated in Figure 4. The fine grid (reference) and multiscale solutions on coarse grid 2 with $M = 4$ at different times are shown in Figure 4. We observe that the coarse-scale solution provide a good approximation of the reference solution. The relative $L^2_a$ and $H^1_a$ relative errors are shown in Table 4. We see that the $H^1_a$ relative error and $L^2_a$ relative error on coarse grid 3 are 0.04 % and 2.3 % for $M = 32$ on the coarse grid 1. We obtain a smaller error for finer coarse grid and error reduce with increasing number of the multiscale basis functions.
Figure 2. Fine-scale solution (top) and multiscale solutions on coarse grid 2 with $M_{\text{off}} = 4$ (bottom) for three time steps $t_m$, $m = 4, 13, 30$.

Table 1. Relative errors in % at the final time for different number of multiscale basis functions $M = M_i$, $i = 1, ..., N$.

| $M$ | Coarse grid 1 | Coarse grid 2 | Coarse grid 3 |
|-----|---------------|---------------|---------------|
|     | $L^2_\alpha$  | $H^1_\alpha$  | $L^2_\alpha$  | $H^1_\alpha$  | $L^2_\alpha$  | $H^1_\alpha$  |
| 1   | 12.6          | 81.7          | 10.9          | 76.6          | 5.9           | 63.8          |
| 2   | 5.1           | 52.1          | 4.1           | 46.5          | 2.45          | 40.1          |
| 4   | 1.6           | 23.8          | 1.5           | 22.9          | 0.78          | 19.6          |
| 8   | 0.8           | 15.1          | 0.3           | 10.2          | 0.16          | 7.8           |
| 16  | 0.1           | 5.2           | 0.08          | 4.3           | 0.04          | 3.7           |
| 32  | 0.04          | 2.3           | 0.02          | 1.9           | 0.01          | 1.9           |

5. Conclusion
In this paper, we presented an unstructured multiscale method for single phase flow problem with discrete fracture model. Multiscale basis functions are constructed in the offline stage via local spectral problems. To represent our approach we solve the flow problem on three different coarse grids. Numerical results are presented and demonstrate that GMsFEM can provide accurate solution on the unstructured coarse grids.

6. Acknowledgements
Work is supported by the mega-grant of the Russian Federation Government N14.Y26.31.0013. MV’s work is supported by the grant RSF N17-71-20055.
Figure 3. Relative errors $L_2^a$ and $H_1^a$ in % by time (top and bottom rows). Coarse grid 1, 2 and 3 (from left to right)

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