A NEW STATISTIC ON DYCK PATHS FOR COUNTING 3-DIMENSIONAL CATALAN WORDS

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Abstract. A 3-dimensional Catalan word is a word on three letters so that the subword on any two letters is a Dyck path. For a given Dyck path $D$, a recently defined statistic counts the number of Catalan words with the property that any subword on two letters is exactly $D$. In this paper, we enumerate Dyck paths with this statistic equal to certain values, including all primes. The formulas obtained are in terms of Motzkin numbers and Motzkin ballot numbers.

1. Introduction

Dyck paths of semilength $n$ are paths from the origin $(0,0)$ to the point $(2n,0)$ that consist of steps $u = (1,1)$ and $d = (1,-1)$ and do not pass below the $x$-axis. Let us denote by $D_n$ the set of Dyck paths of semilength $n$. It is a well-known fact that $D_n$ is enumerated by the Catalan numbers.

A 3-dimensional Catalan path (or just Catalan path) is a higher-dimensional analog of a Dyck path. It is a path from $(0,0,0)$ to $(n,n,n)$ with steps $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, so at each lattice point $(x,y,z)$ along the path, we have $x \geq y \geq z$. A 3-dimensional Catalan word (or just Catalan word) is the word on the letters $\{x,y,z\}$ associated to a Catalan path where $x$ corresponds to the step in the $x$-direction $(1,0,0)$, $y$ corresponds to the step in the $y$-direction $(0,1,0)$, and $z$ corresponds to a step in the $z$ direction $(0,0,1)$. As an example, the complete list of Catalan words with $n = 2$ is:

\[ xxxyyz \, xxyyzz \, xxyzzz \, xyxzzz \, xyzxzz \, xzxyzz \, yzxxzz \, yzxzzz \, yzzyxz \, zzxxyy \, zzyxxy \, zyxxyz. \]

Given a Catalan word $C$, the subword consisting only of $x$’s and $y$’s corresponds to a Dyck path by associating each $x$ to a $u$ and each $y$ to a $d$. Let us call this Dyck path $D_{xy}(C)$. Similarly, the subword consisting only of $y$’s and $z$’s is denoted by $D_{yz}(C)$ by relabeling each $y$ with a $u$ and each $z$ with a $d$. For example, if $C = xxxyyzzzxxyyzz$, then $D_{xy}(C) = uudududd$ and $D_{yz}(C) = uudduudd$.

Catalan words have been studied previously, see for example in [4, 5, 6, 7]. In [4] and [5], the authors study Catalan words $C$ of length $3n$ with $D_{xy}(C) = udud\ldots ud$ and determine that the number of such Catalan words is equal to $\frac{1}{2n+1}(3n)_n$. Notice that when $n = 2$, the three Catalan words with this property are those in the above list whose $x$’s and $y$’s alternate.

In [1], though it wasn’t stated explicitly, it was found that the number of Catalan words $C$ of length $3n$ with $D_{xy}(C) = D_{yz}(C)$ is also $\frac{1}{2n+1}(3n)_n$. Such Catalan words have the property that the subword consisting of $x$’s and $y$’s is the same pattern as the subword consisting of $y$’s and $z$’s. For $n = 2$, the three Catalan words with this property are:

\[ xxxyyz \, xxyyzz \, xyzzyz. \]

The authors further show that for any fixed Dyck path $D$, the number of Catalan words $C$ with $D_{xy}(C) = D_{yz}(C) = D$ is given by

\[ L(D) = \prod_{i=1}^{n-1} \frac{r_i(D) + s_i(D)}{r_i(D)} \]
Theorem 4.2

Sequence starting at 1

| D    | L(D) | Catalan word C with $D_{xy}(C) = D_{yz}(C) = D$ |
|------|------|------------------------------------------------|
| uuudd | 1    | xxyyzzz                                  |
| uudud | 1    | xxyyzzz                                  |
| uuddud | 3    | xyyxzxyyz, xyyxzxyz, xyyxzxyzz            |
| udud  | 3    | xyyxzxyzz, xyyxzxyzz, xyyxzxyzz           |
| udud   | 4    | xyyxzxyzz, xyyxzxyzz, xyyxzxyzz, xyyxzxyzz|

Figure 1. All Dyck words $D \in D_3$, and all corresponding Catalan words $C$ with $D_{xy}(C) = D_{yz}(C) = D$. There are $\frac{1}{7} \binom{9}{3} = 12$ total Catalan words $C$ of length 9 with $D_{xy}(C) = D_{yz}(C)$.

where $r_i(D)$ is the number of down steps between the $i^{th}$ and $(i + 1)^{st}$ up step in $D$, and $s_i(D)$ is the number of up steps between the $i^{th}$ and $(i + 1)^{st}$ down step in $D$. The table in Figure 1 shows all Dyck words $D \in D_3$ and all corresponding Catalan paths $C$ with $D_{xy}(C) = D_{yz}(C) = D$.

As an application of the statistic $L(D)$, in [1] it was found that the number of 321-avoiding permutations of length $3n$ composed only of 3-cycles is equal to the following sum over Dyck paths:

$$|S_{3n}^*(321)| = \sum_{D \in D_{3n}} L(D) \cdot 2^{h(D)},$$

where $h(D)$ is the number of returns, that is, the number of times a down step in the Dyck path $D$ touches the $x$-axis.

In this paper, we study this statistic more directly, asking the following question.

**Question 1.1.** For a fixed $k$, how many Dyck paths $D \in D_n$ have $L(D) = k$?

Equivalently, we could ask: how many Dyck paths $D \in D_n$ correspond to exactly $k$ Catalan words $C$ with $D_{xy}(C) = D_{yz}(C) = D$? We completely answer this question when $k = 1$, $k$ is a prime number, or $k = 4$. The number of Dyck paths with $L = 1$ is found to be the Motzkin numbers; see Theorem 3.1. When $k$ is prime, the number of Dyck paths with $L = k$ can be expressed in terms of the Motzkin numbers. These results are found in Theorem 4.1 and Theorem 4.2. Finally, when $k = 4$, the number of Dyck paths with $L = 4$ can also be expressed in terms of the Motzkin numbers; these results are found in Theorem 5.1. A summary of these values for $k \in \{1, 2, \ldots, 7\}$ can be found in the table in Figure 2.

| $|D_n^k|$ | Sequence starting at $n = k$ | OEIS          | Theorem    |
|-------|-----------------------------|---------------|------------|
| $|D_1^1|$ | 1, 1, 2, 4, 9, 21, 51, 127, 323, . . . | A001006       | Theorem 3.1 |
| $|D_1^2|$ | 1, 0, 1, 2, 6, 16, 45, 126, 357, . . . | A005717       | Theorem 4.1 |
| $|D_2^3|$ | 2, 2, 4, 10, 26, 70, 192, 534, . . . | 2 \cdot (A005773) | Theorem 4.2 |
| $|D_3^4|$ | 2, 5, 9, 25, 65, 181, 505, 1434, . . . | 2 \cdot (A025565) + A352916 | Theorem 5.1 |
| $|D_4^5|$ | 2, 6, 14, 36, 96, 262, 726, 2034, . . . | 2 \cdot (A225034) | Theorem 4.2 |
| $|D_5^6|$ | 14, 34, 92, 252, 710, 2026, 5844, . . . |                  | Section 6 |
| $|D_6^7|$ | 2, 10, 32, 94, 272, 784, 2260, 6524, . . . | 2 \cdot (A353133) | Theorem 4.2 |

Figure 2. The number of Dyck paths $D$ of semilength $n$ with $L(D) = k$. 2
2. Preliminaries

We begin by stating a few basic definitions and introducing relevant notation.

**Definition 2.1.** Let $D \in D_n$.

1. An ascent of $D$ is a maximal set of contiguous up steps; a descent of $D$ is a maximal set of contiguous down steps.
2. If $D$ has $k$ ascents, the ascent sequence of $D$ is given by $\text{Asc}(D) = (a_1, a_2, \ldots, a_k)$ where $a_1$ is the length of the first ascent and $a_i - a_{i-1}$ is the length of the $i$th ascent for $2 \leq i \leq k$.
3. Similarly, the descent sequence of $D$ is given by $\text{Des}(D) = (b_1, \ldots, b_k)$ where $b_1$ is the length of the first descent and $b_i - b_{i-1}$ is the length of the $i$th descent for $2 \leq i \leq k$. We also occasionally use the convention that $a_0 = b_0 = 0$.
4. The $r$-$s$ array of $D$ is the $2 \times n$ vector,
   \[
   \begin{pmatrix}
   r_1 & r_2 & \cdots & r_{n-1} \\
   s_1 & s_2 & \cdots & s_{n-1}
   \end{pmatrix}
   \]
   where $r_i$ is the number of down steps between the $i^{\text{th}}$ and $(i+1)^{\text{st}}$ up step, and $s_i$ is the number of up steps between the $i^{\text{th}}$ and $(i+1)^{\text{st}}$ down step.
5. The statistic $L(D)$ is defined by
   \[
   L(D) = \prod_{i=1}^{n-1} \left( \frac{r_i(D) + s_i(D)}{r_i(D)} \right).
   \]

We note that both the ascent sequence and the descent sequence are increasing, $a_i \geq b_i > 0$ for any $i$, and $a_k = b_k = n$ for any Dyck path with semilength $n$. Furthermore, it is clear that any pair of sequences satisfying these properties produces a unique Dyck path. There is also a relationship between the $r$-$s$ array of $D$ and the ascent and descent sequences as follows:

\[
\begin{align*}
  r_k &= \begin{cases} 
    0 & \text{if } k \notin \text{Asc}(D) \\
    b_i - b_{i-1} & \text{if } k = a_i \text{ for some } a_i \in \text{Asc}(D)
  \end{cases} \\
  s_k &= \begin{cases} 
    0 & \text{if } k \notin \text{Des}(D) \\
    a_{i+1} - a_i & \text{if } k = b_i \text{ for some } b_i \in \text{Des}(D)
  \end{cases}
\end{align*}
\]

The following example illustrates these definitions.

**Example 2.2.** Consider the Dyck path $D = uuudduuudduuudduudduudddd$, which is pictured in Figure 3. The ascent sequence and descent sequence of $D$ are

\[
\text{Asc}(D) = (2, 4, 7, 10, 11, 12, 15) \quad \text{and} \quad \text{Des}(D) = (2, 3, 5, 6, 8, 10, 15),
\]
Conversely, given $M$ thus, the positions in $M\mathcal{D}$.

Furthermore, given $D$ and the procedure described in the following definition.

Example 2.4.

For each modified Motzkin word $h, u, d, \ast$ to be the Dyck path in $\mathcal{D}_n$ by the procedure described in the following definition.

Definition 2.3. Let $M^* \in \mathcal{M}_{n-1}^*$. Define $D_{M^*}$ to be the Dyck path in $\mathcal{D}_n$ where Asc($D_{M^*}$) is the increasing sequence with elements from the set

$$\{j : m_j = d \text{ or } m_j = \ast\} \cup \{n\}$$

and Des($D_{M^*}$) is the increasing sequence with elements from the set

$$\{j : m_j = u \text{ or } m_j = \ast\} \cup \{n\}.$$ 

Furthermore, given $D \in \mathcal{D}_n$, define $M^*_D = m_1 m_2 \cdots m_{n-1} \in \mathcal{M}_{n-1}^*$ by

$$m_i = \begin{cases} 
\ast & \text{if } r_i > 0 \text{ and } s_i > 0 \\
u & \text{if } r_i = 0 \text{ and } s_i > 0 \\
d & \text{if } r_i > 0 \text{ and } s_i = 0 \\
h & \text{if } r_i = s_i = 0.
\end{cases}$$

Notice that this process defines a one-to-one correspondence between $\mathcal{M}_{n-1}^*$ and $\mathcal{D}_n$. That is, $D_{M^*_D} = D$ and $M^*_D$ is $M^*$. Because this is used extensively in future proofs, we provide the following example.

Example 2.4. Let $D$ be the Dyck path defined in Example 2.2, pictured in Figure 3 with $r$-s array:

$$\begin{pmatrix}
0 & 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 0 \\
0 & 2 & 3 & 0 & 3 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

The columns of the $r$-s array help us to easily find $M^*_D$:

- if column $i$ has two 0’s, the $i$th letter in $M^*_D$ is $h$;
- if column $i$ has a 0 on top and a nonzero number on bottom, the $i$th letter in $M^*_D$ is $u$;
- if column $i$ has a 0 on bottom and a nonzero number on top, the $i$th letter in $M^*_D$ is $d$; and
- if column $i$ has a two nonzero entries, the $i$th letter in $M^*_D$ is $\ast$.

Thus,

$$M^*_D = h \ast u \ast d \ast u \ast d \ast h.$$ 

Conversely, given $M^*_D$ as above, we find $D = D_{M^*_D}$ by first computing Asc($D$) and Des($D$). The sequence Asc($D$) contains all the positions in $M^*_D$ that are either $d$ or $\ast$ while Des($D$) contains all the positions in $M^*_D$ that are either $u$ or $\ast$. Thus,

$$\text{Asc}(D) = (2, 4, 7, 10, 11, 12, 15) \quad \text{and} \quad \text{Des}(D) = (2, 3, 5, 6, 8, 10, 15).$$
Notice that $L(D)$ is determined by the product of the binomial coefficients corresponding to the positions of $\ast$’s in $M_D^*$. One final notation we use is to let $\mathcal{D}_n^k$ be the set of Dyck paths $D$ with semilength $n$ and $L(D) = k$. With these definitions at hand, we are now ready to prove our main results.

3. Dyck paths with $L = 1$ or $L = \binom{r_k + s_k}{s_k}$ for some $k$

In this section, we enumerate Dyck paths $D \in \mathcal{D}_n$ where $M_D^*$ has at most one $\ast$. Because $L(D)$ is determined by the product of the binomial coefficients corresponding to the $\ast$ entries in $M_D^*$, Dyck paths with $L = 1$ correspond exactly to the cases where $M_D^*$ has no $\ast$’s and are thus Motzkin paths. Therefore, these Dyck paths will be enumerated by the well-studied Motzkin numbers.

**Theorem 3.1.** For $n \geq 1$, the number of Dyck paths $D$ with semilength $n$ and $L(D) = 1$ is

$$|\mathcal{D}_n^1| = M_{n-1},$$

where $M_{n-1}$ is the $(n-1)^{\text{st}}$ Motzkin number.

**Proof.** Let $D \in \mathcal{D}_n^1$. Since $L(D) = 1$, it must be the case that either $r_i(D) = 0$ or $s_i(D) = 0$ for all $i$. By Definition 2.3, $M_D^*$ consists only of elements in $\{h, u, d\}$ and is thus a Motzkin path in $\mathcal{M}_{n-1}$. This process is invertible, as given any Motzkin path $M \in \mathcal{M}_{n-1} \subseteq \mathcal{M}_n$, we have $D_M = D$. □

As an example, the table in Figure 4 shows the $M_9 = 9$ Dyck paths in $\mathcal{D}_9^1$ and their corresponding Motzkin paths.

We now consider Dyck paths $D \in \mathcal{D}_n$ where $D_M^*$ has exactly one $\ast$. Such Dyck paths have $L = \binom{r_k + s_k}{s_k}$ where $k$ is the position of $\ast$ in $D_M^*$. We call the set of Dyck paths of semilength $n$ with $L = \binom{r_k + s_k}{s_k}$ obtained in this way $\mathcal{D}_n^{r_k,s_k}$.

For ease of notation, if $D \in \mathcal{D}_n^{r_k,s_k}$, define

- $x(D)$ to be the number of ups before the $\ast$ in $M_D^*$, and
- $y(D)$ be the number of downs before the $\ast$ in $M_D^*$.

We can then easily compute the value of $L(D)$ based on $x(D)$ and $y(D)$ as stated in the following observation.

**Observation 3.2.** Suppose $D \in \mathcal{D}_n^{r_k,s_k}$ and write $x = x(D)$ and $y = y(D)$. Then in $M_D^*$, the following are true.

- The difference in positions of the $(y+1)^{\text{st}}$ occurrence of either $u$ or $\ast$ and the $y^{\text{th}}$ occurrence of $u$ is $r$; or, when $y = 0$, the first occurrence of $u$ is in position $r$.
- The difference in positions of the $(x+2)^{\text{nd}}$ occurrence of either $d$ or $\ast$ and the $(x+1)^{\text{st}}$ occurrence of either $d$ or $\ast$ is $s$; or, when $x$ is the number of downs in $M_D^*$, the last occurrence of $d$ is in position $n - s$.

**Example 3.3.** Consider the Dyck path

$$D = uuuudduuddduudduudddd.$$

The ascent sequence and descent sequence of $D$ are

$$\text{Asc}(D) = (5, 7, 9, 11) \quad \text{and} \quad \text{Des}(D) = (2, 6, 7, 11),$$

and the $r$-$s$ array of $D$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0
\end{pmatrix}. $$
There is only one column, column 7, where both entries are nonzero. Thus,

\[
L(D) = \binom{r_7 + s_7}{r_7} = \binom{4 + 2}{4} = 15,
\]

and \( D \in \mathcal{D}^{12}_{11} \). Note also that

\[
M_D^* = huhhdu * hdh
\]

has exactly one *. Now let’s compute \( L(D) \) more directly using Observation 3.2. Notice \( x(D) = 2 \) and \( y(D) = 1 \) since there are two \( u \)'s before the * in \( M_D^* \) and one \( d \) before the *. In this case, the position of the second occurrence of either \( u \) or * is 6 and the position of the first occurrence of \( u \) is 2, so \( r = 6 - 2 = 4 \). Since there are only two downs in \( M_D^* \), we note the last \( d \) occurs in position 9, so \( s = 11 - 9 = 2 \).

In order to proceed, we need to define the Motzkin ballot numbers. The Motzkin ballot numbers are the number of Motzkin paths that have their first down step in a fixed position. These numbers appear in [2] and are similar to the well-known Catalan ballot numbers (see [3]). If \( n \geq k \), we let \( T_{n,k} \) be the set of Motzkin paths of length \( n \) with the first down in position \( k \), and we define \( T_{k-1,k} \) to be the set containing the single Motzkin path consisting of \( k - 1 \) horizontal steps.
Given any Motzkin path $M$, define the reverse of $M$, denoted $M^R$, to be the Motzkin path found by reading $M$ in reverse and switching $u$’s and $d$’s. For example, if $M = huuhdhd$, $M^R = uhuhdhd$.

Given $M \in \mathcal{T}_{n,k}$, the Motzkin path $M^R$ has its last up in position $n - k + 1$.

The following lemma gives the generating function for the Motzkin ballot numbers $T_{n,k} = |\mathcal{T}_{n,k}|$.

**Lemma 3.4.** For positive integers $n \geq k$, let $T_{n,k} = |\mathcal{T}_{n,k}|$. Then for a fixed $k$, the generating function for $T_{n,k}$ is given by

$$
\sum_{n=k-1}^{\infty} T_{n,k}x^n = (1 + xm(x))^{k-1} x^{k-1}.
$$

**Proof.** Consider a Motzkin path of length $n$ with the first down in position $k$. It can be rewritten as $a_1a_2\cdots a_{k-1}a_1a_2\cdots a_{k-1}$ where either

- $a_i = f$ and $\alpha_i$ is the empty word, or
- $a_i = u$ and $\alpha_i$ is $dM_i$ for some Motzkin word $M_i$,

for any $1 \leq i \leq k - 1$. The generating function is therefore $(x + x^2m(x))^{k-1}$. \qed

In later proofs we decompose certain Motzkin paths as shown in the following definition.

**Definition 3.5.** Let $r, s$, and $n$ be positive integers with $n \geq r + s - 2$, and let $P \in \mathcal{T}_{n,r+s-1}$. Define $P_s$ to be the maximal Motzkin subpath in $P$ that begins at the $r$th entry, and define $P_r$ be the Motzkin path formed by removing $P_s$ from $P$.

Given $P \in \mathcal{T}_{n,r+s-1}$, notice that $P_r \in \mathcal{T}_{\ell,r}$ for some $r - 1 \leq \ell \leq n - s + 1$ and $P_s \in \mathcal{T}_{n-\ell,s}$. In other words, the first down in $P_s$ must be in position $s$ (or $P_s$ consists of $s - 1$ horizontal steps), and the first down in $P_r$ must be in position $r$ (or $P_r$ consists of $r - 1$ horizontal steps). This process is invertible as follows. Given $P_r \in \mathcal{T}_{\ell,r}$ and $P_s \in \mathcal{T}_{n-\ell,s}$, form a Motzkin path $P \in \mathcal{T}_{n,r+s-1}$ by inserting $P_s$ after the $(r - 1)$st element in $P_r$.

Because this process is used extensively in subsequent proofs, we illustrate this process with an example below.

**Example 3.6.** Let $r = 3$, $s = 4$, and $n = 13$. Suppose $P = uhuhhdhdudud \in \mathcal{T}_{13,6}$. By definition, $P_s$ is the maximal Motzkin path obtained from $P$ by starting at the $3$rd entry:

$$P = \text{uh} \text{uhhdhdudud}.$$  

Thus, $P_s = uhhdhd \in \mathcal{T}_{6,4}$ as seen in the boxed subword of $P$ above, and $P_r = uhdudud \in \mathcal{T}_{7,3}$. Conversely, given $P$ as shown above and $r = 3$, we note that the maximal Motzkin path in $P_s$ starting at position 3 is exactly the boxed part $P_s$.

Using the Motzkin ballot numbers and this decomposition of Motzkin paths, we can enumerate the set of Dyck paths in $\mathcal{D}_n^{r,s}$. These are enumerated by first considering the number of returns. Suppose a Dyck path $D \in \mathcal{D}_n^r$ has a return after $2k$ steps with $k < n$. Then $r_k(D)$ is the length of the ascent starting in position $2k + 1$, and $s_k(D)$ is the length of the descent ending where $D$ has a return. Thus, the binomial coefficient $\binom{r_k + s_k}{r_k}$ > 1. This implies that if $D \in \mathcal{D}_n^{r,s}$, it can have at most two returns (including the end). Dyck paths in $\mathcal{D}_n^{r,s}$ that have exactly two returns are counted in Lemma 3.7 and those that have a return only at the end are counted in Lemma 3.9.

**Lemma 3.7.** For $r \geq 1, s \geq 1$, and $n \geq r + s$, the number of Dyck paths $D \in \mathcal{D}_n^{r,s}$ that have two returns is $T_{n - 2, r + s - 1}$. 

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Lemma 3.9. For $D$ counts those Dyck paths in $\mathcal{D}_{n,r,s}^{r,s}$ that have exactly two returns and $T_{n-2,r+s-1}$. First, suppose $P \in T_{n-2,r+s-1}$. Thus, there is some $r - 1 \leq \ell \leq n- s + 1$ so that $P_r \in T_{\ell,r}$ and $P_s \in T_{n-2-\ell,s}$ where $P_r$ and $P_s$ are as defined in Definition 3.5.

Now create the modified Motzkin word $M^* \in \mathcal{M}_{n-1}^*$ by concatenating the reverse of $P_r$, the letter $*$, and the word $P_s$; that is, $M^* = P_r^R \ast P_s$. Because $P_r$ and $P_s$ have a combined total length of $n - 2$, the modified Motzkin word $M^*$ is length $n-1$. Let $D = D_{M^*}$ as defined in Definition 2.3 and let $x = x(D)$ and $y = y(D)$. Since $M^*$ has only the Motzkin word $P_r^R$ before $*$, we have $x = y$ and $D$ must have exactly two returns.

Using Observation 3.2, we can show that $D \in \mathcal{D}_{n,r,s}^{r,s}$ as follows. The $(y+1)$st occurrence of either a $u$ or $* \ast$ is the $* \ast$ and the $y$th occurrence of $u$ is the last $u$ in $P_r^R$; the difference in these positions is $r$. Also, the $(x+1)$st occurrence of either a $d$ or $* \ast$ is the $* \ast$ and the $(x+2)$nd occurrence of either a $d$ or $* \ast$ is the first $d$ in $P_s$; the difference in these positions is $s$.

To see that this process is invertible, consider any Dyck path $D \in \mathcal{D}_{n,r,s}^{r,s}$ that has exactly two returns. Since $D \in \mathcal{D}_{n,r,s}^{r,s}$, $M_D^n$ has exactly one $\ast$. Furthermore, since $D$ has a return after $2k$ steps for some $k < n$, it must be that $\ast$ decomposes $M_D^n$ into two Motzkin paths. That is, the subword of $M_D^n$ before the $\ast$ is a Motzkin path as well as the subword of $M_D^n$ after the $\ast$. We will call the subword of $M_D^n$ consisting of the first $k - 1$ entries $M_r$ and the subword of $M_D^n$ consisting of the last $n - 1 - k$ entries $M_s$.

Since $r_k = r$ and there are the same number of ups and downs before the $\ast$ in $M_D^n$, the last up before $\ast$ must be in position $k - r$. Similarly, since $s_k = s$, the first down after $\ast$ must be in position $k + s$. Thus, $M_r^R \in T_{k-1,r}$ and $M_s \in T_{n-1-k,s}$. Let $P$ be the Motzkin path formed by inserting $M_s$ after the $(r-1)$st element in $M_r^R$. Then $P \in T_{n-2,r+s-1}$ as desired.

The following example shows the correspondence.

Example 3.8. Let $r = 3$, $s = 4$, and $n = 15$. Suppose $P = uhuhddhdudud \in \mathcal{T}_{13,6}$. The corresponding Dyck path $D \in \mathcal{D}_{13,4}^{3,4}$ is found as follows. First, find $P_r = uhdudud$ and $P_s = uhhdhh$ as in Example 3.6. Then let $M^* = P_r^R \ast P_s$ or

$$M^* = ududuhd \ast uhddhh.$$  

Letting $D = D_{M^*}$, we see that $x(D) = y(D) = 3$. The fourth occurrence of either $u$ or $* \ast$ is the $* \ast$ in position 8, and the third occurrence of $u$ is in position 5, so $r = 8 - 5 = 3$. Similarly, the fourth occurrence of either $d$ or $* \ast$ is the $* \ast$ in position 8, and the fifth occurrence of $d$ is in position 12, so $s = 12 - 8 = 4$ as desired.

For completion, we write the actual Dyck path $D$ using Definition 2.3 by first seeing $\text{Asc}(D) = (2, 4, 7, 8, 12, 15)$ and $\text{Des}(D) = (1, 3, 5, 8, 9, 15)$. Thus

$$D = uuduududuududduddudduddudduddudu.$$  

Lemma 3.7 counted the Dyck paths in $\mathcal{D}_{n,r,s}^{r,s}$ that have exactly two returns; the ensuing lemma counts those Dyck paths in $\mathcal{D}_{n,r,s}^{r,s}$ that have only one return (at the end).

Lemma 3.9. For $r \geq 1, s \geq 1,$ and $n \geq r + s + 2,$ the number of Dyck paths $D \in \mathcal{D}_{n,r,s}^{r,s}$ that only have a return at the end is

$$\sum_{i=0}^{n-2-s-r} (i+1)M_i T_{n-4-i,r+s-1}. $$  

Proof. Consider a pair of Motzkin paths, $M$ and $P$, where $M$ is length $i$ with $0 \leq i \leq n - 2 - s - r$, and $P \in T_{n-4-i,r+s-1}$. For each such pair, we consider $1 \leq j \leq i+1$ and find a corresponding Dyck path $D \in \mathcal{D}_{n,r,s}^{r,s}$. Thus, there will be $i+1$ corresponding Dyck paths for each pair $M$ and $P$. Each Dyck path $D$ will have exactly one $\ast \ast$ in $M_D^n$. 

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We begin by letting $\overline{M}$ be the modified Motzkin path obtained by inserting $\ast$ before the $j$th entry in $M$ or at the end if $j = i + 1$. Let $\overline{x}$ be the number of ups before the $\ast$ in $\overline{M}$, and let $\overline{y}$ be the number of downs before the $\ast$ in $\overline{M}$.

Recall that by Definition 3.5, there is some $r - 1 \leq \ell \leq n - 3 - s - i$ so that $P$ can be decomposed into $P_r \in \mathcal{T}_\ell,r$ and $P_s \in \mathcal{T}_{n-4-\ell,s}$. We now create a modified Motzkin word, $M^* \in \mathcal{M}_{n-1}$ by inserting one $u$, one $d$, $P_r^R$, and $P_s$ into $\overline{M}$ as follows.

1. Insert a $d$ followed by $P_s$ immediately before the $(\overline{x} + 1)st$ $d$ in $\overline{M}$ or at the end if $\overline{x}$ is equal to the number of downs in $\overline{M}$.

2. Insert the reverse of $P_r$ followed by $u$ after the $\overline{y}th$ $u$ or at the beginning if $\overline{y} = 0$.

Call the resulting path $M^*$. We claim that $D_{M^*} \in \mathcal{D}_n^{r,s}$ and that $D_{M^*}$ only has one return at the end. For ease of notation, let $D = D_{M^*}$, $x = x(D)$, and $y = y(D)$. Notice that the number of downs (and thus the number of ups) in $P_r$ is $y - \overline{y}$. Then the $(y + 1)st$ $u$ or $\ast$ in $M^*$ is the inserted $u$ following $P_r^R$ from Step (2), and the $y$th $u$ is the last $u$ in $P_r^R$. The difference in these positions is $r$. Similarly, the $(x + 1)st$ $d$ or $\ast$ in $M^*$ is the inserted $d$ before the $P_s$ from Step (1), and the $(x + 2)nd$ $d$ or $\ast$ in $M^*$ is the first down in $P_s$. The difference in these positions is $s$, and thus by Observation 3.2, $D \in \mathcal{D}_n^{r,s}$.

To see that $D$ only has one return at the end, we note that the only other possible place $D$ can have a return is after $2k$ steps where $k = \ell + j + 1$, the position of $\ast$ in $M^*$. However, $x > y$ so $D$ only has one return at the end.

We now show that this process is invertible. Consider any Dyck path $D \in \mathcal{D}_n^{r,s}$ that has one return at the end. Since $D$ only has one return at the end, the $\ast$ does not decompose $M^*_D$ into two Motzkin paths, and we must have $x(D) > y(D)$.

Let $P_1$ be the maximal Motzkin word immediately following the $(x + 1)st$ occurrence of $d$ or $\ast$ in $M^*_D$. Note that $P_1$ must have its first down in position $s$ or $P_1$ consists of $s - 1$ horizontal steps. Let $P_2$ be the maximal Motzkin word preceding the $(y + 1)st$ up in $M^*$. Then either $P_2$ consists of $r - 1$ horizontal step or the last $u$ in $P_2$ is $r$ from the end; that is, the first $d$ in $P_2^R$ is in position $r$.

Since $x > y$, the $(y + 1)st$ $u$ comes before the $x$th $d$. Thus, deleting the $\ast$, the $(y + 1)st$ $u$, the $x$th $d$, $P_1$, and $P_2$ results in a Motzkin path we call $M$. Note that if $M$ is length $i$, then the combined lengths of $P_1$ and $P_2$ is length $n - 4 - i$. This inverts the process by letting $P_s = P_1$ and $P_r = P_2^R$. 

We again illustrate the correspondence from the above proof with an example.

**Example 3.10.** Let $r = 3$, $s = 4$, $n = 24$, and consider the following pair of Motzkin paths

$$M = uuhdhhhdudud \quad \text{and} \quad P = uuhdhhhdudud.$$ 

As in Example 3.6, $P_r = uuhdud$ and $P_s = uuhdhh$. Following the notation in the proof of Lemma 3.9 we have $i = 7$. Our goal is to find 8 corresponding Dyck paths for each $1 \leq j \leq 8$. If $j = 1$, we first create $\overline{M}$ by inserting $\ast$ before the 1st entry in $M$:

$$\overline{M} = *uuhdudd.$$

Now there are $\overline{x} = 0$ ups and $\overline{y} = 0$ downs before the $\ast$ in $\overline{M}$. Thus, we form $M^*$ by inserting $P_r^R u$ at the beginning of $\overline{M}$ and $dP_s$ immediately before the 1st down in $\overline{M}$ yielding

$$M^* = \boxed{uuhdud} \ u * \boxed{uuhdhh} \ dhudd.$$

The paths $P_r^R$ and $P_s$ are boxed in the above notation and the inserted $u$ and $d$ are in bold. If $D = D_{M^*}$, then $x(D) = 4$ and $y(D) = 3$ because there are four $u$’s and three $d$’s before $\ast$ in $M^*$. The $(y + 1)st$ (or fourth) occurrence of $u$ or $\ast$ in $M^*$ is the bolded $u$ in position 8, and the third occurrence of $u$ is the last $u$ in $P_r^R$ in position 5; thus $r = 3$. Similarly, the $(x + 2)nd$ (or
Figure 5. Given \( r = 3 \), \( s = 4 \), \( n = 24 \), and the pair of Motzkin paths 
\( M = uudhudd \in \mathcal{M}_7 \) and 
\( P = uuhhhddhudd \in \mathcal{T}_{136} \), the Dyck words formed 
by \( D_M^* \) are the 8 corresponding Dyck paths in \( D_{24}^{3,4} \) that only have one return.

### Proposition 3.11
For \( r \geq 1 \), \( s \geq 1 \), and \( n \geq r + s \), the number of Dyck paths \( D \in D_{n}^{r,s} \) is

\[
|D_{n}^{r,s}| = T_{n-2,r+s-1} + \sum_{i=0}^{n-2-s-r} (i + 1)M_iT_{n-4-i,r+s-1}.
\]

**Proof.** Dyck paths in \( D_{n}^{r,s} \) can have at most two returns. Thus, this is a direct consequence of Lemmas 3.7 and 3.9. \( \square \)

Interestingly, we remark that the formula for \( |D_{n}^{r,s}| \) only depends on the sum \( r + s \) and not the individual values of \( r \) and \( s \). For example, \( |D_{n}^{1,3}| = |D_{n}^{2,2}| \). Also, because the formula for \( |D_{n}^{r,s}| \) is given in terms of Motzkin paths, we can easily extract the generating function for these numbers using Lemma 3.4.

### Corollary 3.12
For \( r, s \geq 1 \), the generating function for \( |D_{n}^{r,s}| \) is

\[
x^{r+s}(1 + xm(x))^{r+s-2} \left(1 + x^2(xm(x))'\right).
\]

### 4. Dyck paths with \( L = p \) for prime \( p \)

When \( L = p \), for some prime \( p \), we must have that every term in the product \( \prod_{i=1}^{n-1} \binom{r_i+s_i}{r_i} \) is equal to 1 except for one term which must equal \( p \). In particular, we must have that there is exactly
one \(1 \leq k \leq n - 1\) with \(r_k \neq 0\) and \(s_k \neq 0\). Furthermore, we must have that either \(r_k = 1\) and \(s_k = p - 1\) or \(r_k = p - 1\) and \(s_k = 1\). Therefore, when \(L = 2\), we have

\[ |D_2^2| = |D_1^{1,1}|. \]

When \(L = p\) for an odd prime number, we have

\[ |D_p^n| = |D_1^{p-1}| + |D_2^{p-1}| = 2|D_1^{p-1}|. \]

Thus the results from the previous section can be used in the subsequent proofs.

**Theorem 4.1.** For \(n \geq 4\), the number of Dyck paths with semilength \(n\) and \(L = 2\) is

\[ |D_2^n| = (n - 3)M_{n-4}, \]

where \(M_{n-4}\) is the \((n - 4)\)th Motzkin number. Additionally, \(|D_2^2| = 1\) and \(|D_2^3| = 0\). Thus the generating function for \(|D_2^n|\) is given by

\[ L_2(x) = x^2 + x^4 (xm(x))' \]

where \(m(x)\) is the generating function for the Motzkin numbers.

**Proof.** By Proposition 3.11 for \(n \geq 2\),

\[ |D_1^{1,1}| = T_{n-2,1} + \sum_{i=0}^{n-4} (i + 1)M_i T_{n-4-i,1}. \]

In the case where \(n = 3\) or \(n = 4\), the summation is empty and thus \(|D_2^3| = T_{0,1} = 1\) and \(|D_2^4| = T_{1,1} = 0\). For \(n \geq 4\), the term \(T_{n-2,1} = 0\). Furthermore, the terms in the summation are all 0 except when \(i = n - 4\). Thus,

\[ |D_1^{1,1}| = (n - 3)M_{n-4}T_{0,1} \]

or

\[ |D_2^2| = (n - 3)M_{n-4}. \]

\[ \square \]

The sequence for the number of Dyck paths of semilength \(n\) with \(L = 2\) is given by:

\[ |D_2^n| = 1, 0, 1, 2, 6, 16, 45, 126, 357, \ldots \]

This can be found at OEIS A005717.

Because the formula for \(|D_2^n|\) is much simpler than the one found in Proposition 3.11, the correspondence between Dyck paths in \(D_2^n\) and Motzkin paths of length \(n - 4\) is actually fairly straightforward. For each Motzkin word of length \(n - 4\), there are \(n - 3\) corresponding Dyck paths of semilength \(n\) having \(L = 2\). These corresponding Dyck paths are found by modifying the original Motzkin word \(n - 3\) different ways. Each modification involves adding a \(u, d,\) and placeholder \(*\) to the original Motzkin word. The \(n - 3\) distinct modifications correspond to the \(n - 3\) possible positions of the placeholder \(*\) into the original Motzkin word.

As an example, in the case where \(n = 6\), there are \(M_2 = 2\) Motzkin words of length 2. For each word, we can insert a placeholder \(*\) in \(n - 3 = 3\) different positions, and thus there are a total of 6 corresponding Dyck paths of semilength 6 having \(L = 2\). Figure 5 provides the detailed process when \(n = 6\).

When \(L = p\) for an odd prime number, we can also enumerate \(D_p^n\) using Proposition 3.11 as seen in the following theorem.
Theorem 4.2. For a prime number $p \geq 3$ and $n \geq p$, the number of Dyck paths with semilength $n$ and $L = p$ is

$$|D_p^n| = 2 \left( T_{n-2,p-1} + \sum_{i=0}^{n-2-p} (i+1)M_iT_{n-i-4,p-1} \right).$$

Thus, the generating function for $|D_p^n|$ is

$$2x^p(1 + xm(x))^{p-2} (x^2(xm(x))' + 1).$$

Proof. This lemma is a direct corollary of Proposition 3.11 with $r = 1$ and $s = p - 1$. We multiply by two to account for the case where $r = p - 1$ and $s = 1$, and $r = 1$ and $s = p - 1$. □

5. Dyck paths with $L = 4$

When $L(D) = 4$, things are more complicated than in the cases for prime numbers. If $D \in D_4^n$, then one of the following is true:

- $D \in D_{n,1}^{1,3}$ or $D \in D_{n,1}^{3,1}$; or
- All but two terms in the product $\prod_{i=1}^{n-1} \frac{r_i + s_i}{r_i}$ are equal to 1, and those terms must both equal 2.

Because the first case is enumerated in Section 3, this section will be devoted to counting the Dyck paths $D \in D_4^n$ where $M_D$ has exactly two *'s in positions $k_1$ and $k_2$ and

$$L(D) = \binom{r_{k_1} + s_{k_1}}{r_{k_1}} \binom{r_{k_2} + s_{k_2}}{r_{k_2}}$$

with $r_{k_1} = s_{k_1} = r_{k_2} = s_{k_2} = 1$. 

```markdown
| Motzkin word | $M^*$ | Asc($D$) | Des($D$) | $r$-s array | Dyck path, $D$ |
|--------------|-------|----------|----------|-------------|----------------|
| $u \cdot hhd$ | (2 5 6) | (0 1 0 0 1) | (3 1 0 0 0) | | |
| $u \cdot hd$ | (3 5 6) | (0 0 1 0 2) | (2 0 1 0 0) | | |
| $uh \cdot d$ | (4 5 6) | (0 0 0 1 3) | (1 0 0 1 0) | | |
| $u \cdot udd$ | (2 4 5 6) | (0 1 0 1 1) | (2 1 1 0 0) | | |
| $uu \cdot dd$ | (3 4 5 6) | (0 0 1 1 1) | (1 1 1 0 0) | | |
| $uud \cdot d$ | (3 4 5 6) | (0 0 1 1 2) | (1 1 0 1 0) | | |
```
For ease of notation, let $\hat{D}_n$ be the set of Dyck paths $D \in \mathcal{D}_n^d$ with the property that $M^*_D$ has exactly two *'s. Also, given $D \in \hat{D}_n$, define $x_i(D)$ to be the number of ups before the $i$th * in $M^*_D$ and let $y_i(D)$ be the number of downs before the $i$th * for $i \in \{1, 2\}$.

**Example 5.1.** Let $D$ be the Dyck path with ascent sequence and descent sequence

$$\text{Asc}(D) = (3, 6, 7, 8, 10, 11) \quad \text{and} \quad \text{Des}(D) = (1, 3, 4, 5, 8, 11)$$

and thus $r$-$s$ array

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 3 \\
3 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$ 

By inspection of the $r$-$s$ array and noticing that only columns 3 and 8 have two nonzero entries, we see that $L(D) = \left(\begin{smallmatrix} 1+1 \\ 1 \end{smallmatrix}\right) = 4$ and thus $D \in \hat{D}_{11}$. Furthermore, we can compute

$$M^*_D = uh * wudd * hd.$$ 

Since there is one $u$ before the first * and no $d$'s, we have $x_1(D) = 1$ and $y_1(D) = 0$. Similarly, there are three $u$'s before the second * and two $d$'s so $x_2(D) = 3$ and $y_2(D) = 2$.

In this section, we will construct $M^*$ from smaller Motzkin paths. To this end, let us notice what $M^*_D$ should look like if $D \in \hat{D}_n$.

**Lemma 5.2.** Suppose $M^* \in \mathcal{M}^*_{n-1}$ has exactly two *'s. Then $D_{M^*} \in \hat{D}_n$ if, writing $x_i = x_i(D_{M^*})$ and $y_i = y_i(D_{M^*})$, we have:

- The $(x_1 + 1)$st occurrence of either $a$ or * is followed by another $d$ or *;
- The $(x_2 + 2)$nd occurrence of either $a$ or * is followed by another $d$ or *; or $x_2$ is equal to the number of $d$'s and $M^*$ ends in $d$ or *;
- The $(y_1)_\text{th}$ occurrence of either $u$ or * is followed by another $u$ or *; or $y_1 = 0$ and the $M^*$ begins with $u$ or *;
- The $(y_2 + 1)$st occurrence of either $u$ or * is followed by another $u$ or *.

**Proof.** Suppose $M^* \in \mathcal{M}^*_{n-1}$ has two stars in positions $k_1$ and $k_2$. Then it is clear that

$$L(D) = \begin{pmatrix}
\frac{r_{k_1} + s_{k_1}}{r_{k_1}} & \frac{r_{k_2} + s_{k_2}}{r_{k_2}}
\end{pmatrix}$$

so it suffices to show that $r_{k_1} = s_{k_1} = r_{k_2} = s_{k_2} = 1$. Recall that $\text{Asc}(D) = (a_1, a_2, \ldots, a_k)$ is the increasing sequence of positions $i$ in $M^*$ with $m_i = d$ or *. Similarly, $\text{Des}(D) = (b_1, b_2, \ldots, b_k)$ is the increasing sequence of positions $i$ in $M^*$ with $m_i = u$ or *

First notice that $r_{k_1} = 1$ only if $b_i - b_{i-1} = 1$ where $a_i = k_1$. However, $i = y_1 + 1$ since the first star must be the $(y_1 + 1)$st occurrence of $d$ or *. Therefore $b_{k_1}$ is the position of the $(y_1 + 1)$st $u$ or * and $b_{k_1} - 1$ is the position of the $(y_1)_\text{th}$ $u$ or *.

The difference in positions is 1 exactly when they are consecutive in $M^*$. The other three bullet points follow similarly.

Enumerating Dyck paths $D \in \hat{D}_n$ will be found based on the values of $x_1(D)$ and $y_2(D)$. The cases we consider are

- $x_1(D) \notin \{y_2(D), y_2(D) + 1\}$;
- $x_1(D) = 1$ and $y_2(D) = 0$;
- $x_1(D) = y_2(D) + 1 \geq 2$; and
- $x_1(D) = y_2(D)$.

The next four lemmas address each of these cases separately. Each lemma is followed by an example showing the correspondence the proof provides.

**Lemma 5.3.** For $n \geq 7$, the number of Dyck paths $D \in \hat{D}_n$ with $x_1(D) \notin \{y_2(D), y_2(D) + 1\}$ is $\binom{n-5}{2} M_{n-7}$. 

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Proof. We will show that for any $M \in \mathcal{M}_{n-7}$, there are \binom{n-5}{2} corresponding Dyck paths $D \in \hat{\mathcal{D}}_n$ with $x_1(D) \notin \{y_2(D), y_2(D) + 1\}$. To this end, let $M \in \mathcal{M}_{n-7}$ and let $1 \leq j_1 < j_2 \leq n - 5$. There are \binom{n-5}{2} choices for $j_1$ and $j_2$ each corresponding to a Dyck path with the desired properties. We create a modified Motzkin word $\overline{M}^* \in \mathcal{M}_{n-5}^*$ with *'s in position $j_1$ and $j_2$ and the subword of $\overline{M}^*$ with the *'s removed is equal to $M$. Let $\overline{x}_i$ be the number of ups before the $i$th * in $\overline{M}^*$ and let $\overline{y}_i$ be the number of downs before the $i$th * in $\overline{M}^*$ for $i \in \{1, 2\}$. We create the modified Motzkin word $M^* \in \mathcal{M}_{n-1}^*$ as follows:

1. Insert $d$ before the $(\overline{x}_2 + 1)$th down or at the very end of $\overline{x}_2$ is the number of downs in $\overline{M}^*$.
2. Insert $d$ before the $(\overline{x}_1 + 1)$th down or at the very end of $\overline{x}_1$ is the number of downs in $\overline{M}^*$.
3. Insert $u$ after the $\overline{y}_2$th up or at the beginning if $\overline{y}_2 = 0$.
4. Insert $u$ after the $\overline{y}_1$th up or at the beginning if $\overline{y}_1 = 0$.

Notice that in Step (1), the $d$ is inserted after the second * and in Step (4), the $u$ is inserted before the first *. Let $D = D_{M^*}$. We first show that $D \in \hat{\mathcal{D}}_n$ by showing $L(D) = 4$. We proceed by examining two cases.

In the first case, assume $\overline{x}_1 + 1 \leq \overline{y}_2$. In this case, the inserted $d$ in Step (2) must occur before the second * since there were $\overline{y}_2$ $d$'s before the second * in $M^*$. Similarly, the inserted $u$ in Step (3) must occur after the first *.

Thus, we have

$$x_1(D) = \overline{x}_1 + 1, \quad y_1(D) = \overline{y}_1, \quad x_2(D) = \overline{x}_2 + 2, \quad \text{and} \quad y_2(D) = \overline{y}_2 + 1.$$  

We now use the criteria of Lemma 5.2 to see that $L(D) = 4$:

- The $(x_1 + 1)$th occurrence of a $d$ or * is the inserted $d$ from Step (2) and is thus followed by $d$;
- The $(x_2 + 2)$th occurrence of a $d$ or * is the inserted $d$ from Step (1) and is thus followed by $d$;
- The $y_1$th occurrence of a $u$ is the inserted $u$ from Step (4) and is thus followed by $u$; and
- The $(y_2 + 1)$th occurrence of a $u$ is the inserted $u$ from Step (3) and is thus followed by $u$.

We also have

$$x_1(D) = \overline{x}_1 + 1 \leq \overline{y}_2 < \overline{y}_2 + 1 = y_2(D),$$

and thus $D \in \hat{\mathcal{D}}_n$ with $x_1(D) \notin \{y_2(D), y_2(D) + 1\}$ as desired.

In the second case where $\overline{x}_1 \geq \overline{y}_2$, the inserted $d$ in Step (2) occurs after the second * and the inserted $u$ in Step (3) occurs before the first *.

Here we have

$$x_1(D) = \overline{x}_1 + 2, \quad y_1(D) = \overline{y}_1, \quad x_2(D) = \overline{x}_2 + 2, \quad \text{and} \quad y_2(D) = \overline{y}_2.$$  

We can easily check that the criteria of Lemma 5.2 are satisfied to show that $L(D) = 4$. Also,

$$x_1(D) = \overline{x}_1 + 2 \geq \overline{y}_2 + 2 = y_2(D) + 2,$$

and thus $D$ has the desired properties.

To see that this process is invertible, consider any $D \in \hat{\mathcal{D}}_n$ with $x_1(D) \notin \{y_2(D), y_2(D) + 1\}$ and let $k_1 \leq k_2$ be the positions of the *'s in $M^*_D$. We consider the two cases where $x_1(D) < y_2(D)$ and where $x_1(D) \geq y_2(D) + 2$. Since for each case, we’ve established the relationship between $x_i$ and $\overline{x}_i$ and between $y_i$ and $\overline{y}_i$, it is straightforward to undo the process.

Begin with the case where $x_1(D) < y_2(D)$. In this case:

- Delete the $(x_2(D))$th $d$ and the $(x_1(D))$th $d$.
- Delete the $(y_2(D) + 1)$th $u$ and the $(y_1(D) + 1)$th $u$.
- Delete both *'s.

Now consider the case where $x_1(D) \geq y_2(D) + 2$. In this case:

- Delete the $(x_2(D) + 2)$th $d$ and the $(x_1(D) + 1)$th $d$. 

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• Delete the \((y_2(D)+1)\)th \(u\) and the \((y_1(D)+2)\)th \(u\).
• Delete both \(*\)'s.

\[\square\]

**Example 5.4.** Suppose \(n = 11\) and let \(M = hudh \in \mathcal{M}_4\). There are \(\binom{6}{2} = 15\) corresponding Dyck paths \(D \in \hat{\mathcal{D}}_n\) with \(x_1(D) \notin \{y_2(D), y_2(D)+1\}\), and we provide two of these in this example.

First, suppose \(j_1 = 2\) and \(j_2 = 5\) so that \(\hat{M} = h * ud * h\). We then count the number of ups and downs before each \(*\) to get
\[\bar{x}_1 = 0, \quad \bar{y}_1 = 0, \quad \bar{x}_2 = 1, \quad \text{and} \quad \bar{y}_2 = 1.\]

Following the steps in the proof, we insert two \(u\)'s and two \(d\)'s to get
\[M^* = uh * wuddh.\]

Let \(D = D_{M^*}\) and notice that the number of ups before the first \(*\) is \(x_1(D) = 1\) and the number of downs before the second \(*\) is \(y_2(D) = 2\) and thus \(x_1(D) < y_2(D)\). Since \(L(D) = 4\), \(D\) satisfies the desired criteria. To see that the process is invertible, we would delete the third and first \(d\), the third and first \(u\), and the two \(*\)'s.

Now, suppose \(j_1 = 2\) and \(j_2 = 4\) so that \(\hat{M}^* = h * u * dh\). We again count the number of ups and downs before each \(*\) to get
\[\bar{x}_1 = 0, \quad \bar{y}_1 = 0, \quad \bar{x}_2 = 1, \quad \text{and} \quad \bar{y}_2 = 0,\]

and insert two \(u\)'s and two \(d\)'s to get
\[M^* = uuh * u * ddhd.\]

Now, if \(D = D_{M^*}\) we have \(x_1(D) = 2\) and \(y_2(D) = 0\) and so \(x_1(D) \geq y_2(D) + 2\) as desired. We can also easily check the \(L(D) = 4\).

**Lemma 5.5.** For \(n \geq 5\), the number of Dyck paths \(D \in \hat{\mathcal{D}}_n\) with \(x_1(D) = 1\) and \(y_2(D) = 0\) is \(M_{n-5}\).

**Proof.** We find a bijection between the set of Dyck paths \(D \in \hat{\mathcal{D}}_n\) with \(x_1(D) = 1\) and \(y_2(D) = 0\) with the set \(\mathcal{M}_{n-5}\). First, suppose \(M \in \mathcal{M}_{n-5}\). Let \(\bar{x}_2\) be the number of ups before the first down. Now create the modified Motzkin word \(M^* \in \mathcal{M}_{n-1}\) as follows.

1. Insert \(d\) before the \((\bar{x}_2+1)\)st \(d\) in \(M\) or at the end if the number of downs in \(M\) is \(\bar{x}_2\).
2. Insert \(*\) before the first \(d\).
3. Insert \(u\) followed by \(*\) before the first entry.

Let \(D = D_{M^*}\). By construction, we have
\[x_1(D) = 1, \quad y_1(D) = 0, \quad x_2(D) = \bar{x}_2 + 1, \quad \text{and} \quad y_2(D) = 0.\]

In particular, Step (3) gives us \(x_1(D)\) and \(y_1(D)\), while Step (2) gives us \(x_2(D)\), and \(y_2(D)\). We also have the four criteria of Lemma 5.2:

- The second occurrence of a \(d\) or \(*\) is the second \(*\) which is followed by \(d\);
- The \((x_2 + 2)\)st occurrence of a \(d\) or \(*\) is the inserted \(d\) from Step (1) which is followed by \(d\) (or at the end);
- \(M^*\) begins with \(u\); and
- The first occurrence of a \(u\) or \(*\) is the first entry \(u\) which is followed by \(*\).

Let us now invert the process. Starting with a Dyck path \(D\) with \(x_1(D) = 1\) and \(y_2(D) = 0\), and its corresponding modified Motzkin word \(M^*_D\). Since \(y_2(D) = 0\), we also have \(y_1(D) = 0\), and thus by Lemma 5.2 we must have that the first two entries are either uu, u*, **, or *u. However, since we know \(x_1(D) = 1\), it must be the case that \(M^*_D\) starts with uu. As usual, let \(x_2(D)\) be the number of \(u\)'s before the second star. Obtain \(M \in \mathcal{M}_{n-5}\) by starting with \(M^*_D\) and then:
• Delete the \((x_2)\)th \(d\).
• Delete the first \(u\).
• Delete both \(*\)’s.

\[ \square \]

**Example 5.6.** Suppose \(n = 11\) and let \(M = huuuhd \in \mathcal{M}_6\). By Lemma 5.5, there is one corresponding Dyck path \(D \in \widehat{D}_{11}\) with \(x_1(D) = 1\) and \(y_2(D) = 0\). Following the notation in the proof, we have \(\pi_2 = 2\) and we get

\[ M^* = u * huu * dhdd. \]

Let \(D = D_{M^*}\). We can easily check that \(L(D) = 1\). Also, \(x_1(D) = 1\) and \(y_2(D) = 0\) as desired.

**Lemma 5.7.** For \(n \geq 7\), the number of Dyck paths \(D \in \widehat{D}_n\) with \(x_1(D) = y_2(D) + 1 \geq 2\) is

\[ \sum_{i=0}^{n-7} (i + 1)M_iM_{n-7-i}. \]

**Proof.** Consider a pair of Motzkin paths, \(M\) and \(P\), where \(M \in \mathcal{M}_i\) and \(P \in \mathcal{M}_{n-7-i}\) with \(0 \leq i \leq n - 7\). For each such pair, we consider \(1 \leq j \leq i + 1\) and find a corresponding Dyck path \(D \in \widehat{D}_n\) with \(x_1(D) = y_2(D) + 1 \geq 2\). Thus, there will be \(i + 1\) corresponding Dyck paths for each pair \(M\) and \(P\).

We begin by creating a modified Motzkin word \(\overline{M}\) ∈ \(\mathcal{M}^{**}_{n-4}\) by inserting \(u\), followed by \(*\), followed by the path \(P\), followed by \(d\) before the \(i\)th entry in \(M\). Then, let \(\overline{\pi}_1\) be the number of ups before \(*\) in \(\overline{M}\) and \(\overline{\pi}_1\) be the number of downs before the \(*\) in \(\overline{M}\). Notice that \(\overline{\pi}_1 \geq 1\). We now create a modified Motzkin word \(M^* \in \mathcal{M}^*\) as follows.

1. Insert \(*\) before the \((\overline{\pi}_1 + 1)\)st \(d\) or at the end if \(\overline{\pi}_1 + 1\) equals the number of downs in \(\overline{M}\).

   Now let \(\pi_2\) be the number of ups before this second \(*\).

2. Insert \(u\) after the \(\overline{\pi}_1\)th \(u\) in \(\overline{M}\) or at the beginning if \(\overline{\pi}_1 = 0\).

3. Insert \(d\) before the \((\pi_2 + 1)\)st \(d\) (or at the end).

Let \(D = D_{M^*}\). By construction, we have

\[ x_1(D) = \overline{\pi}_1 + 2, \quad y_1(D) = \overline{\pi}_1, \quad x_2(D) = \overline{\pi}_2 + 1, \quad \text{and} \quad y_2(D) = \overline{\pi}_1, \]

and thus \(x_1(D) = y_2(D) + 1 \geq 2\).

We also have the four criteria of Lemma 5.2:

• The \((x_1 + 1)\)st occurrence of a \(d\) or \(*\) is the second \(*\) from Step (1) which is followed by \(d\);
• The \((x_2 + 2)\)st occurrence of a \(d\) or \(*\) is the inserted \(d\) from Step (3) which is followed by \(d\) (or at the end);
• The \((y_1 + 1)\)st occurrence of a \(u\) or \(*\) is the inserted \(u\) from Step (2) and thus is preceded by \(u\); and
• The \((y_2 + 2)\)nd occurrence of a \(u\) or \(*\) is the first \(*\) which immediately follows a \(u\).

To see that this process is invertible, consider any Dyck path \(D \in \widehat{D}_n\) with \(x_1(D) = y_2(D) + 1 \geq 2\). To create \(\overline{M}\), start with \(M^*_D\) and then:

• Delete the \((x_2)\)th \(d\).
• Delete the second \(*\).
• Delete the \((y_1 + 1)\)st \(u\).

Because \(x_1 = y_2 + 1\), we have \(y_1 + 1 \leq x_2\) and so this process results in \(\overline{M} \in \mathcal{M}^{**}_{n-4}\). Now let \(P\) be the maximal subpath in \(\overline{M}\) beginning with the entry immediately following the \(*\). Deleting the \(u\) and the \(*\) preceding \(P\), all of \(P\), and the \(d\) following \(P\) inverts the process.  \(\square\)
Example 5.8. Suppose \( n = 11 \) and let \( M = ud \in \mathcal{M}_2 \) and \( P = hh \in \mathcal{M}_2 \). There are 3 corresponding Dyck paths with \( D \in \hat{\mathcal{D}}_n \) with \( x_1(D) = y_2(D) + 1 \geq 2 \) and we provide one example. First, let \( j = 1 \) and create the word \( M^* \in \mathcal{M}_7^* \) by inserting \( u \ast Pd \) before the first entry in \( M \):

\[
M^* = [u \ast hh] ud.
\]

Notice \( x_1 = 1 \) and \( y_1 = 0 \) since there is only one entry, \( u \), before the \( \ast \). Then, following the procedure in the proof of Lemma 5.7, we insert \( \ast \) before the second \( d \) and note that \( x_2 = 2 \). Then we insert \( u \) at the beginning and \( d \) at the end to get \( M^* = u[hhd u \ast dd] \). Let \( D = D_{M^*} \). By inspection, we note \( x_1(D) = 2 \) and \( y_2(D) = 1 \), and we can easily check that \( L(D) = 4 \).

Lemma 5.9. For \( n \geq 7 \), the number of Dyck paths \( D \in \hat{\mathcal{D}}_n \) with \( x_1(D) = y_2(D) \) is

\[
\sum_{i=0}^{n-7} (i+1)M_iM_{n-7-i}.
\]

Also, for \( n = 3 \), there is exactly 1 Dyck path \( D \in \hat{\mathcal{D}}_3 \) with \( x_1(D) = y_2(D) \).

Proof. Similar to the proof of Lemma 5.7, consider a pair of Motzkin paths, \( M \) and \( P \), where \( M \in \mathcal{M}_i \) and \( P \in \mathcal{M}_{n-7-i} \) with \( 0 \leq i \leq n - 7 \). For each such pair, we consider \( 1 \leq j \leq i+1 \) and find a corresponding Dyck path \( D \in \hat{\mathcal{D}}_n \) with \( x_1(D) = y_2(D) \). Thus, there will be \( i+1 \) corresponding Dyck paths for each pair \( M \) and \( P \).

We begin by creating a modified Motzkin word \( M^* \in \mathcal{M}_{n-4}^* \) by inserting \( \ast \), followed by \( u \), followed by the path \( P \), followed by \( d \) before the \( j \)th entry in \( M \). Then, let \( x_1 \) be the number of ups before \( \ast \) in \( M^* \) and \( y_1 \) be the number of downs before the \( \ast \) in \( M^* \). We now create a modified Motzkin word \( M^* \in \mathcal{M}^* \) as follows.

1. Insert \( \ast \) after the \((x_1 + 1)\)st \( d \) in \( M^* \). Let \( x_2 \) be the number of ups before this second \( \ast \).
2. Insert \( u \) after the \((x_2 + 1)\)th \( u \) in \( M^* \) or at the beginning if \( x_2 = 0 \).
3. Insert \( d \) before the \((x_2 + 1)\)st \( d \) (or at the end).

Let \( D = D_{M^*} \). By construction, we have

\[
x_1(D) = x_1 + 1, \quad y_1(D) = y_1 + 1, \quad x_2(D) = x_2 + 1, \quad \text{and} \quad y_2(D) = x_1 + 1.
\]

It is easy to verify that the criteria in Lemma 5.2 are satisfied and so \( D \in \hat{\mathcal{D}}_n \) with \( x_1(D) = y_2(D) \).

To see that this process is invertible, consider any Dyck path \( D \in \hat{\mathcal{D}}_n \) with \( x_1(D) = y_2(D) \). Since \( x_1 = y_2 \), there are \( y_2 \) ups before the first \( \ast \), in \( M_D^* \) and thus the first \( \ast \) in \( M_D^* \) is the \((y_2 + 1)\)th occurrence of a \( u \) or \( \ast \). By the fourth criterium in Lemma 5.2 the first \( \ast \) must be followed by another \( u \) or \( \ast \). Similarly, the \((x_1 + 2)\)th occurrence of either a \( d \) or \( \ast \) is the second \( \ast \). Thus, by the second criterium of Lemma 5.2 the second \( \ast \) must be immediately preceded by a \( d \) or \( \ast \).

We now show the cases where the first \( \ast \) is immediately followed by the second \( \ast \) results in only one Dyck path. In this case, \( x_1 = y_1 = x_2 = y_2 \), and thus \( M_D^* \) can be decomposed as a Motzkin path, followed by \( \ast \ast \), followed by another Motzkin path. By the second criterium in Lemma 5.2 the entry after the second \( \ast \) must be a \( d \) (which is not allowed) and thus \( M_D^* \) ends in \( \ast \). Similarly, the third criterium in Lemma 5.2 tells us \( M_D^* \) begins with \( \ast \) and so \( M_D^* = \ast \ast \). Thus, \( D \in \hat{\mathcal{D}}_n \) and is the path \( D = ududud \).

We now assume the first \( \ast \) is followed by \( u \) which implies the second \( \ast \) is preceded by \( d \). In this case, we must have at least \( y_1 + 1 \) downs before the second \( \ast \) and at least \( x_1 + 1 \) ups before the second \( \ast \) yielding

\[
y_1 + 1 \leq x_1 = y_2 \leq x_2 - 1.
\]
Thus the \((y_1 + 1)\)th \(u\) comes before the first * and the \(x_2\)th \(d\) comes after the second *. To find \(\overline{M}'\) from \(M'_7\):

- Delete the \(x_2\)th \(d\).
- Delete the second *.
- Delete the \((y_1 + 1)\)st \(u\);

which results in \(\overline{M}' \in M'_{n-4}\). Now let \(P\) be the maximal subpath in \(\overline{M}'\) beginning with the entry after the \(u\) that immediately follows the remaining *. (The entry after \(P\) must be \(d\) since \(P\) is maximal and \(\overline{M}'\) is a Motzkin path when ignoring the *.) Removing \(uPd\) and the remaining * from \(\overline{M}'\) results in a Motzkin path \(M\) as desired. \(\square\)

**Example 5.10.** Suppose \(n = 11\) and let \(M = ud \in M_2\) and \(P = hh \in M_2\). There are 3 corresponding Dyck paths with \(D \in \widehat{D}_n\) with \(x_1(D) = y_2(D)\) and we provide one example. First, let \(j = 1\) and create the word \(\overline{M}' \in M'_7\) by inserting *\(uPd\) before the first entry in \(M\):

\[\overline{M}' = [uhhd] ud.\]

Notice \(\overline{P}_1 = 0\) and \(\overline{P}_2 = 0\) since there are no entries before the *. Then, following the procedure in the proof of Lemma 5.9, we insert * after the first \(d\) and note that \(\overline{P}_2 = 1\). Then we insert \(u\) at the beginning and \(d\) before the second \(d\) to get

\[M' = u[uhhd]*udd.\]

Let \(D = D_{M'}\). By inspection, we note \(x_1(D) = y_2(D) = 1\), and we can easily check that \(L(D) = 4\).

**Theorem 5.11.** The number of Dyck paths with semilength \(n \geq 4\) and \(L = 4\) is

\[|D^4_n| = 2 \left( T_{n-2,3} + \sum_{i=0}^{n-6} (i + 1)M_iT_{n-4-i,3} \right) + \binom{n-5}{2} M_{n-7} + M_{n-5} + \sum_{i=0}^{n-7} (i + 1)M_iM_{n-7-i}.\]

Also, \(|D^4_3| = 1\).

**Proof.** This is a direct consequence of Proposition 3.11 along with Lemmas 5.3, 5.5, 5.7, and 5.9 \(\square\)

6. Further Remarks

As seen in Section 5, finding \(|D^k_n|\) is more complicated when \(k\) is not prime, as there could be many ways to write \(k\) as a product of binomial coefficients. For example, consider \(k = 6\). If \(D \in D^6_n\), then one of the following is true:

- \(D \in D^1_3\) or \(D \in D^5_1\),
- \(D \in D^2_3\), or
- All but two terms in the product \(\prod_{i=1}^{n-1} \left( \binom{r_i + s_i}{r_i} \right)\) are equal to 1, and those terms must equal 2 and 3.

The number of Dyck paths in the first two cases is given by Proposition 3.11

\[|D^1_3| = |D^5_1| = T_{n-2,5} + \sum_{i=0}^{n-8} (i + 1)M_iT_{n-4-i,5}\]

and

\[|D^2_3| = T_{n-2,3} + \sum_{i=0}^{n-6} (i + 1)M_iT_{n-4-i,3}.\]

In the final case, we have

\[L(D) = \binom{r_{k_1} + s_{k_1}}{r_{k_1}} \binom{r_{k_2} + s_{k_2}}{r_{k_2}}\]
where exactly one of \( \{r_{k_1}, r_{k_2}, s_{k_1}, s_{k_2}\} \) is equal to 2 and the other three values are 1. By symmetry, we need only to consider two cases: when \( r_{k_1} = 2 \) and when \( s_{k_1} = 2 \). We can appreciate that these cases can become quite involved; the proofs would involve similar techniques to those found in Section 5 along with the proof of Proposition 3.11. Although we do not provide a closed form, the number of Dyck paths \( D \in D_n^6 \) in this case are (starting at \( n = 4 \)):

\[
2, 4, 8, 16, 44, 122, 352, 1028, 3036, \ldots.
\]

Combining the first two cases with this case, we provide the first terms of the values of \( |D_n^6| \) (starting at \( n = 4 \)):

\[
3, 6, 14, 34, 92, 252, 710, 2026, 5844, \ldots.
\]

Further work in this area could involve finding formulas for \( |D_k| \) when \( k \) is a non-prime number greater than 4. It also still remains open to refine the enumeration of \( D_k^6 \) with respect to the number of returns. Having such a refinement in terms of number of returns would yield a new formula for the number of 321-avoiding permutations of length \( 3n \) composed only of 3-cycles as seen in Equation (1).

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