Nonparametric inference about mean functionals of nonignorable nonresponse data without identifying the joint distribution

Wei Li\textsuperscript{1}, Wang Miao\textsuperscript{2}, and Eric Tchetgen Tchetgen\textsuperscript{3}

\textsuperscript{1}Center for Applied Statistics and School of Statistics, Renmin University of China
\textsuperscript{2}School of Mathematical Sciences, Peking University
\textsuperscript{3}Department of Statistics, University of Pennsylvania

Abstract

We consider identification and inference about mean functionals of observed covariates and an outcome variable subject to nonignorable missingness. By leveraging a shadow variable, we establish a necessary and sufficient condition for identification of the mean functional even if the full data distribution is not identified. We further characterize a necessary condition for $\sqrt{n}$-estimability of the mean functional. This condition naturally strengthens the identifying condition, and it requires the existence of a function as a solution to a representer equation that connects the shadow variable to the mean functional. Solutions to the representer equation may not be unique, which presents substantial challenges for nonparametric estimation and standard theories for nonparametric sieve estimators are not applicable here. We construct a consistent estimator for the solution set and then adapt the theory of extremum estimators to find from the estimated set a consistent estimator for an appropriately chosen solution. The estimator is asymptotically normal, locally efficient and attains the semiparametric efficiency bound under certain regularity conditions. We illustrate the proposed approach via simulations and a real data application on home pricing.

Keywords: Identification; Model-free estimation; Nonignorable missingness; Shadow variable.

1 Introduction

Nonresponse is frequently encountered in social science and biomedical studies, due to such as reluctance to answer sensitive survey questions. Certain characteristics of the missing data mechanism is used to define a taxonomy to describe the missingness process (Rubin 1976; Little and Rubin 2002). It is called missing at random (MAR) if the propensity of missingness conditional on all study variables is unrelated to the missing values. Otherwise, it is called missing not at random (MNAR) or nonignorable. MAR has been commonly used for statistical analysis in the presence of missing data; however, in many fields of study, suspicion that the missing data mechanism may be nonignorable is often warranted (Scharfstein et al. 1999; Robins et al. 2000).
Rotnitzky and Robins (1997), Rotnitzky et al. (1998), Ibrahim et al. (1999). For example, nonresponse rates in surveys about income tend to be higher for low socio-economic groups (Kim and Yu, 2011). In another example, efforts to estimate HIV prevalence in developing countries via household HIV survey and testing such as the well-known Demographic and Health Survey, are likewise subject to nonignorable missing data on participants’ HIV status due to highly selective non-participation in the HIV testing component of the survey study (Tchetgen Tchetgen and Wirth, 2017). There currently exist a variety of methods for the analysis of MAR data, such as likelihood based inference, multiple imputation, inverse probability weighting, and doubly robust methods. However, these methods can result in severe bias and invalid inference in the presence of nonignorable missing data.

In this paper, we focus on identification and estimation of mean functionals of observed covariates and an outcome variable subject to nonignorable missingness. Estimating mean functionals is a goal common in many scientific areas, e.g., sampling survey and causal inference, and thus is of significant practical importance. However, there are several difficulties for analysis of nonignorable missing data. The first challenge is identification, which means that the parameter of interest is uniquely determined from observed data distribution. Identification is straightforward under MAR as the conditional outcome distribution in complete-cases equals that in incomplete cases given fully observed covariates, whereas it becomes difficult under MNAR because the selection bias due to missing values is no longer negligible. Even if stringent fully-parametric models are imposed on both the propensity score and the outcome regression, identification may not be achieved; for counterexamples, see Wang et al. (2014); Miao et al. (2016). To resolve the identification difficulty, previous researchers (Robins et al., 2000; Kim and Yu, 2011) have assumed that the selection bias is known or can be estimated from external studies, but this approach should be used rather as a sensitivity analysis, until the validity of the selection bias assumption is assessed. Without knowing the selection bias, identification can be achieved by leveraging fully observed auxiliary variables that are available in many empirical studies. For instance, instrumental variables, which are related to the non-response propensity but not related to the outcome given covariates, have been used in missing data analysis since Heckman (1979). The corresponding semiparametric theory and inference are recently established by Liu et al. (2020); Sun et al. (2018); Tchetgen Tchetgen and Wirth (2017); Das et al. (2003). Recently, an alternative approach called the shadow variable approach has grown in popularity in sampling survey and missing data analysis. In contrast to the instrumental variable, this approach entails a shadow variable that is associated with the outcome but independent of the missingness process given covariates and the outcome itself. Shadow variable is available in many applications (Kott, 2014). For example, Zahner et al. (1992) and Ibrahim et al. (2001) considered a study of the children’s mental health evaluated through their teachers’ assessments in Connecticut. However, the data for the teachers’ assessments are subject to nonignorable missingness. As a proxy of the teacher’s assessment, a separate parent report is available for all children in this study. The parent report is likely to be correlated with the teacher’s assessment, but is unlikely to be related to the teacher’s response rate given the teacher’s assessment and fully observed covariates. Hence, the parental assessment is regarded as a shadow variable in this study. The shadow variable design is quite general. In health and social sciences, an accurately measured outcome is routinely available only for a subset of patients, but one or more surrogates may be fully observed. For instance, Robins et al. (1994) considered a cardiovascular disease setting where, due to high cost of laboratory analyses, and to the small amount of stored serum per subject (about 2% of study subjects had stored serum
thawed and assayed for antioxidants serum vitamin A and vitamin E), error prone surrogate measurements of the biomarkers derived from self-reported dietary questionnaire were obtained for all subjects. Other important settings include the semi-supervised set up in comparative effectiveness research where the true outcome is measured only for a small fraction of the data, e.g., diagnosis requiring a costly panel of physicians while surrogates are obtained from databases including ICD-9 codes for certain comorbidities. Instead of assuming MAR conditional on the surrogates, the shadow variable assumption may be more appropriate in presence of informative selection bias. By leveraging a shadow variable, D’Haultfœuille (2010); Miao et al. (2019) established identification results for nonparametric models under completeness conditions. In related work, Wang et al. (2014); Shao and Wang (2016); Tang et al. (2003); Zhao and Shao (2015); Morikawa and Kim (2021) proposed identification conditions for a suite of parametric and semiparametric models that require either the propensity score or the outcome regression, or both to be parametric. All existing shadow variable approaches need to further impose sufficiently strong conditions to identify the full data distribution, although in practice one may only be interested in a parameter or a mean functional which may be identifiable even if the full data distribution is not.

The second challenge for MNAR data is the threat of bias due to model misspecification in estimation, after identification is established. Likelihood-based inference (Greenlees et al., 1982; Tang et al., 2003, 2014), imputation-based methods (Kim and Yu, 2011), inverse probability weighting (Wang et al., 2014) have been developed for analysis of MNAR data. These estimation methods require correct model specification of either the propensity score or the outcome regression, or both. However, bias may arise due to specification error of parametric models as they have limited flexibility, and moreover, model misspecification is more likely to appear in the presence of missing values. Vansteelandt et al. (2007); Miao and Tchetgen Tchetgen (2016); Miao et al. (2019); Liu et al. (2020) proposed doubly robust methods for estimation with MNAR data, which affords double protection against model misspecification; however, these previous proposals require an odds ratio model characterizing the degree of nonignorable missingness to be either completely known or correctly specified.

In this paper, we develop a novel strategy to nonparametrically identify and estimate a generic mean functional. In contrast to previous approaches in the literature, this work has the following distinctive features and makes several contributions to nonignorable missing data literature. First, given a shadow variable, we directly work on the identification of the mean functional without identifying the full data distribution, whereas previous proposals typically have to first ensure identification of the full data distribution. In particular, we establish a necessary and sufficient condition for identification of the mean functional, which is weaker than the commonly-used completeness condition for nonparametric identification. For estimation, we propose a representer assumption which is shown to be necessary for $\sqrt{n}$-estimability of the mean functional. The representer assumption involves the existence of a function as a solution to a representer equation that relates the mean functional and the shadow variable. Second, under the representer assumption, we propose nonparametric estimation that no longer involves parametrically modelling the propensity score or the outcome regression. Nonparametric estimation has been largely studied in the literature and has received broad acceptance with machine learning tools in many applications (Newey and Powell, 2003; Chen and Pouzo, 2012; Kennedy et al., 2017). Because the solution to the representer equation may not be unique, we first construct a consistent estimator of the solution set. We use the method of sieves to approximate unknown smooth functions as possible solutions and estimate corresponding coefficients.
by applying a minimum distance procedure, which has been routinely used in semiparametric
and nonparametric econometric literature (Newey and Powell 2003; Ai and Chen 2003; Santos
2011; Chen and Pouzo 2012). We then adapt the theory of extremum estimators to find from
the estimated set a consistent estimator of an appropriately chosen solution. Based on such
an estimator, we propose a representer-based estimator for the mean functional. Under certain
regularity conditions, we establish consistency and asymptotic normality for the proposed esti-
mator. The proposed estimator is shown to be locally efficient for the mean functional under our
shadow variable model. Besides, due to the nonuniqueness of solutions to the representer equa-
tion, the asymptotic results cannot be simply obtained by applying the standard nonparametric
sieve theory and additional techniques are required.

The remainder of this paper is organised as follows. In Section 2, we provide a necessary
and sufficient identifying condition for a generic mean functional within the shadow variable
framework, and also introduce a representer assumption that is shown to be necessary for \( \sqrt{n} \)
estimability of the mean functional. In Section 3, we develop a model-free estimator for the mean
functional, establish the asymptotic theory, discuss its semiparametric efficiency, and provide
computational details. In Section 4, we study the finite-sample performance of the proposed
approach via both simulation studies and a real data example about home pricing. We conclude
with a discussion in Section 5 and relegate proofs to the supporting information.

2 Identification

Let \( X \) denote a vector of fully observed covariates, \( Y \) the outcome variable that is subject
to missingness, and \( R \) the missingness indicator with \( R = 1 \) if \( Y \) is observed and \( R = 0 \)
otherwise. The missingness process may depend on the missing values. We let \( f(\cdot) \) denote
the probability density or mass function of a random variable (vector). The observed data
contain \( n \) independent and identically distributed realizations of \((R, X, Y, Z)\) with the values of
\( Y \) missing for \( R = 0 \). We are interested in identifying and making inference about the mean
functional \( \mu = E\{\tau(X, Y)\} \) for a generic function \( \tau(\cdot) \). In particular, when \( \tau(X, Y) = Y \), the
parameter \( \mu = E(Y) \) corresponds to the population outcome mean that is of particular interest
in sampling survey and causal inference. This setup also covers the ordinary least squares
regression problem if we choose \( \mu = E(X_j Y) \) for any component \( X_j \) of \( X \). Suppose we observe
an additional shadow variable \( Z \) that meets the following assumption.

**Assumption 1 (Shadow variable).** (i) \( Z \perp \perp R \mid (X, Y) \); (ii) \( Z \not\perp Y \mid X \).

Assumption 1 reveals that the shadow variable does not affect the missingness process given
the covariates and outcome, and it is associated with the outcome given the covariates. This
assumption has been used for adjustment of selection bias in sampling surveys (Kott 2014) and
in missing data literature (D’Haultfoeuille 2010; Wang et al. 2014; Miao and Tchetgen Tchet-
gen 2016). Examples and extensive discussions about the assumption can be found in Zahner
et al. (1992); Ibrahim et al. (2001); Miao and Tchetgen Tchetgen (2016); Miao et al. (2019).
Under Assumption 1, we have

\[
E\{\gamma(X, Y) \mid R = 1, X, Z\} = \beta(X, Z),
\]

where

\[
\gamma(X, Y) = \frac{f(R = 0 \mid X, Y)}{f(R = 1 \mid X, Y)}, \quad \text{and} \quad \beta(X, Z) = \frac{f(R = 0 \mid X, Z)}{f(R = 1 \mid X, Z)}.
\]
Without further assumptions such as completeness conditions, \( \gamma(X,Y) \) is generally not identifiable from \( \{1\} \). Nevertheless, by noting that \( \mu = E\{R\tau(X,Y) + R\tau(X,Y)\gamma(X,Y)\} \), the expectation functional \( \mu \) can be identified under conditions weaker than those required for identification of \( \gamma(X,Y) \) itself. The discussions below are based on operators on Hilbert spaces. Let \( L_2(X,Y,Z) \) denote the set of real valued functions of \( (X,Y,Z) \) that are square integrable with respect to the conditional distribution of \( (X,Y,Z) \) given \( R = 1 \). Let \( T : L_2(X,Y) \rightarrow L_2(X,Z) \) be the linear operator given by \( T(\xi) = E\{\xi(X,Y) \mid R = 1, X, Z\} \). Its adjoint \( T^\prime : L_2(X,Z) \rightarrow L_2(X,Y) \) is the linear map \( T^\prime(\eta) = E\{\eta(X,Z) \mid R = 1, X, Y\} \). The range and null space of \( T \) is denoted by \( \mathcal{R}(T) \) and \( \mathcal{N}(T) \), respectively. The orthogonal complement of a set \( \mathcal{A} \) is denoted by \( \mathcal{A}^\perp \) and its closure in the norm topology is cl(\( \mathcal{A} \)).

**Theorem 1.** Under Assumption \( \{1\} \), \( \mu \) is identifiable if and only if \( \tau(X,Y) \in \mathcal{N}(T)^\perp \).

Since the definition of the operator \( T \) involves only observed data, the identifying condition could be justified in principle without extra model assumptions on the missing data distribution. In contrast to previous approaches ([D’Haultfoeuille, 2010; Miao et al., 2019]) that have to identify the full data distribution under varying completeness conditions, our identification strategy allows for a larger class of models where only the parameter of interest is uniquely identified even though the full data law may not be. One of the commonly-used completeness conditions is that for any square-integrable function \( g \), \( E\{g(X,Y) \mid R = 1, X, Z\} = 0 \) almost surely if and only if \( g(X,Y) = 0 \) almost surely. Under such circumstances, the function \( \gamma(X,Y) \) defined after \( \{1\} \) is identified, i.e., \( \mathcal{N}(T) = \{0\} \), then the functional \( \mu \) is identifiable because \( \tau(X,Y) \in \mathcal{N}(T)^\perp = L_2(X,Y) \). However, the identifying condition \( \tau(X,Y) \in \mathcal{N}(T)^\perp \) does not suffice for \( \sqrt{n} \)-estimability of \( \mu \), particularly for functionals that lie in the boundary of \( \mathcal{N}(T)^\perp \).

In fact, following the proof of Lemma 4.1 in [Severini and Tripathi, 2012], we can show that Assumption \( \{2\} \) is necessary for \( \sqrt{n} \)-estimability of \( \mu \) under some regularity conditions.

**Assumption 2** (Representer). \( \tau(X,Y) \in \mathcal{R}(T^\prime) \), i.e., there exists a function \( \delta_0(X,Z) \) such that

\[
E\{\delta_0(X,Z) \mid R = 1, X, Y\} = \tau(X,Y).
\]  

(2)

Note that \( \mathcal{N}(T)^\perp = \text{cl}\{\mathcal{R}(T^\prime)\} \), the representer assumption naturally strengthens the identifying condition \( \tau(X,Y) \in \mathcal{N}(T)^\perp \). Assumption \( \{2\} \) is not only a sufficient condition for identifiability of \( \mu \), but also necessary for \( \sqrt{n} \)-estimability of \( \mu \). Equation (2) relates the function \( \tau(x,y) \) and the shadow variable via the representer function \( \delta_0(x,z) \). The equation is a Fredholm integral equation of the first kind, and the requirement for existence of solutions to (2) is mild. Assumption \( \{2\} \) is nearly necessary for the completeness condition. Specifically, if the completeness condition holds, then the solution to (2) exists under the technical conditions \( A_1 \sim A_3 \) given in the Appendix. Similar conditions for existence of solutions to the Fredholm integral equation of the first kind are discussed in [Miao et al., 2018; Carrasco et al., 2007a; Cui et al., 2020].

We give some specific examples that Assumption \( \{2\} \) holds. If there exists some transformation of \( Z \) such that \( E\{\lambda(Z) \mid X, Y\} = \alpha(X) + \beta(X)\tau(X,Y) \) and \( \beta(x) \neq 0 \), then Assumption \( \{2\} \) is met with \( \delta_0(X,Z) = \{\lambda(Z) - \alpha(X)\}/\beta(X) \). As a special case, \( \lambda(Z) = Z \) when \( E(Z \mid X,Y) \) is linear in \( \tau(X,Y) \). For simplicity, we may drop the arguments in \( \delta_0(X,Z) \) and directly use \( \delta_0 \) in what follows, and notation for other functions are treated in a similar way.

Note that Assumption \( \{2\} \) only requires the existence of solutions to equation (2), but not uniqueness. For instance, if both \( Z \) and \( Y \) are binary, then \( \delta_0 \) is unique and

\[
\delta_0(X,Z) = \frac{Z\{\tau(X,1) - \tau(X,0)\} - f_0(X)\tau(X,1) + f_1(X)\tau(X,0)}{f_1(X) - f_0(X)},
\]
where $f_0(X) = f(Z = 1 \mid R = 1, X, Y = 0)$ and $f_1(X) = f(Z = 1 \mid R = 1, X, Y = 1)$. However, if $Z$ has more levels than $Y$, $\delta_0$ may not be unique.

**Corollary 1.** Under Assumptions 2 and 3, $\mu$ is identifiable, and

$$
\mu = E\{R\tau(X,Y) + (1 - R)\delta_0(X,Z)\}.
$$

From Corollary 1 even if $\delta_0$ is not uniquely determined, all solutions to Assumption 2 must result in an identical value of $\mu$. Moreover, this identification result does not require identification of the full data distribution $f(R, X, Y, Z)$. In fact, identification of $f(R, X, Y, Z)$ is not ensured under Assumptions 1 and 2 only; see Example 1. To our knowledge, the identifying Assumptions 1–2 are so far the weakest for the shadow variable approach. Besides that, Assumption 2 is also necessary for $\sqrt{n}$-estimability of $\mu$ within the shadow variable framework.

We further illustrate Assumption 2 with the following example.

**Example 1.** Consider the following two models:

Model 1: $Y \sim U(0, 1)$, $Z \mid y \sim Bern(y)$, and $f(R = 1 \mid y, z) = 4y^2(1 - y)$, where $Bern(y)$ denotes Bernoulli distribution with probability $y$.

Model 2: $Y \sim Be(2, 2)$, $Z \mid y \sim Bern(y)$, and $f(R = 1 \mid y, z) = 2y/3$, where $Be(2, 2)$ denotes Beta distribution with parameters 2 and 2.

Suppose we are interested in estimating the outcome mean $\mu = E(Y)$. It is easy to verify that the above two models satisfy Assumption 3 by choosing $\delta_0(X, Z) = Z$. These two models imply the same outcome mean $E(Y) = 1/2$ and the same observed data distribution, because $f(R = 1, y, z) = 4y^2(1 - y)\{zy + (1 - z)(1 - y)\}$ and $f(z) = 1/2$ in these two models. However, the full data distributions of these two models are different.

## 3 Estimation, inference, and computation

In this section, we provide a novel estimation procedure without modeling the propensity score or outcome regression. Previous approaches often require fully or partially parametric models for at least one of them. For example, Qin et al. (2002) and Wang et al. (2014) assumed a fully parametric model for the propensity score; Kim and Yu (2011) and Shao and Wang (2016) relaxed their assumption and considered a semiparametric exponential tilting model for the propensity; Miao and Tchetgen Tchetgen (2016) proposed doubly robust estimation methods by either requiring a parametric propensity score or an outcome regression to be correctly specified. Our approach aims to be more robust than existing methods by avoiding (i) point identification of the full data law under more stringent conditions, and (ii) over-reliance on parametric assumptions either for identification or for estimation.

As implied by Corollary 1 any solution to (2) provides a valid $\delta_0$ for recovering the parameter $\mu$. Suppose that all such solutions belong to a set $\Delta$ of smooth functions, with specific requirements for smooth functions given in Definition 1. Then the set of solutions to (2) is denoted by

$$
\Delta_0 = \{\delta \in \Delta : E\{\delta(X, Z) \mid R = 1, X, Y\} = \tau(X,Y)\}.
$$

For estimation and inference about $\mu$, we need to construct a consistent estimator for some fixed $\delta_0 \in \Delta_0$. If $\Delta_0$ were known, then we would simply select one element $\delta_0$ from the set and use this element to estimate $\mu$. Unfortunately, the solution set $\Delta_0$ is unknown, and the lack of identification of $\delta_0$ presents important technical challenges. Directly solving (2) does not
generally yields a consistent estimator for some fixed \( \delta_0 \). Instead, by noting that the solution set \( \Delta_0 \) is identified, we aim to obtain an estimator \( \hat{\delta}_0 \) in the following two steps: first, construct a consistent estimator \( \hat{\Delta}_0 \) for the set \( \Delta_0 \); second, carefully select \( \hat{\delta}_0 \in \hat{\Delta}_0 \) such that it is a consistent estimator for a fixed element \( \delta_0 \in \Delta_0 \).

3.1 Estimation of the solution set \( \Delta_0 \)

Define the criterion function

\[
Q(\delta) = E\left[ R \left\{ E(\tau(X, Y) - \delta(X, Z) \mid R = 1, X, Y) \right\}^2 \right].
\]

Then the solution set \( \Delta_0 \) in (3) is equal to the set of zeros of \( Q(\delta) \), i.e.,

\[
\Delta_0 = \{ \delta \in \Delta : Q(\delta) = 0 \},
\]

and hence, estimation of \( \Delta_0 \) is equivalent to estimation of zeros of \( Q(\delta) \). This can be accomplished with the approximate minimizers of a sample analogue of \( Q(\delta) \) [Chernozhukov et al., 2007].

We adopt a method of sieves approach to construct a sample analogue function \( Q_n(\delta) \) for \( Q(\delta) \) and a corresponding approximation \( \Delta_n \) for \( \Delta \). Let \( \{\psi_q(x, z)\}_{q=1}^{\infty} \) denote a sequence of known approximating functions of \( x \) and \( z \), and

\[
\Delta_n = \left\{ \delta \in \Delta : \delta(x, z) = \sum_{q=1}^{q_n} \beta_q \psi_q(x, z) \right\}
\]

(4)

for some known \( q_n \) and unknown parameters \( \{\beta_q\}_{q=1}^{q_n} \). The construction of \( Q_n \) entails a non-parametric estimator of conditional expectations. Let \( \{\phi_k(x, y)\}_{k=1}^{\infty} \) be a sequence of known approximating functions of \( x \) and \( y \). Denote the vector of the first \( k_n \) terms of the basis functions by

\[
\phi(x, y) = \{ \phi_1(x, y), \ldots, \phi_{k_n}(x, y) \}^T,
\]

and let

\[
\Phi = \{ \phi(X_1, Y_1), \ldots, \phi(X_n, Y_n) \}^T, \quad \Lambda = \text{diag}(R_1, \ldots, R_n).
\]

For a generic random variable \( B = B(X, Y, Z) \) with realizations \( \{B_i = B(X_i, Y_i, Z_i)\}_{i=1}^{n} \), the nonparametric sieve estimator of \( E(B \mid R = 1, x, y) \) is obtained by the linear regression of \( B \) on the vector \( \phi(X, Y) \) with observed data, i.e.,

\[
\hat{E}(B \mid R = 1, X, Y) = \phi^T(X, Y) (\Phi^T \Lambda \Phi)^{-1} \sum_{i=1}^{n} R_i \phi(X_i, Y_i) B_i.
\]

Then the sample analogue \( Q_n \) of \( Q \) is

\[
Q_n(\delta) = \frac{1}{n} \sum_{i=1}^{n} R_i \hat{e}^2(X_i, Y_i, \delta),
\]

(6)

with

\[
\hat{e}(X_i, Y_i, \delta) = \hat{E}\{ \tau(X, Y) - \delta(X, Z) \mid R = 1, X_i, Y_i \},
\]

(7)
We establish the set consistency of $\hat{\delta}$. Finally, the proposed estimator of $\Delta_0$ is
\[
\hat{\Delta}_0 = \{ \delta \in \Delta_n : Q_n(\delta) \leq c_n \},
\]
where $\Delta_n$ and $Q_n(\delta)$ are given in \([4]\) and \([6]\), respectively, and $\{c_n\}_{n=1}^\infty$ is a sequence of small positive numbers converging to zero at an appropriate rate. The requirement on the rate of $c_n$ will be discussed later for theoretical analysis.

### 3.2 Set consistency

We establish the set consistency of $\hat{\Delta}_0$ for $\Delta_0$ in terms of Hausdorff distances. For a given norm $\| \cdot \|$, the Hausdorff distance between two sets $\Delta_1, \Delta_2 \subseteq \Delta$ is
\[
d_H(\Delta_1, \Delta_2, \| \cdot \|) = \max \{ d(\Delta_1, \Delta_2), d(\Delta_2, \Delta_1) \},
\]
where $d(\Delta_1, \Delta_2) = \sup_{\delta_1 \in \Delta_1} \inf_{\delta_2 \in \Delta_2} \| \delta_1 - \delta_2 \|$ and $d(\Delta_2, \Delta_1)$ is defined analogously. Thus, $\hat{\Delta}_0$ is consistent under the Hausdorff distance if both the maximal approximation error of $\hat{\Delta}_0$ by $\Delta_0$ and of $\Delta_0$ by $\hat{\Delta}_0$ converge to zero in probability.

We consider two different norms for the Hausdorff distance: the pseudo-norm $\| \cdot \|_{\text{w}}$ defined by
\[
\| \delta \|_{\text{w}}^2 = E \left[ R \{ E(\delta(X, Z) | R = 1, X, Y) \}^2 \right],
\]
and the supremum norm $\| \cdot \|_{\infty}$ defined by
\[
\| \delta \|_{\infty} = \sup_{x,z} |\delta(x,z)|.
\]
From the representer equation (2), we have that for any $\hat{\delta}_0 \in \hat{\Delta}_0$ and $\delta_0, \delta \in \Delta_0$, $\| \hat{\delta}_0 - \delta \|_{\text{w}} = \| \delta_0 - \delta \|_{\text{w}}$. Hence,
\[
\| \hat{\delta}_0 - \delta_0 \|_{\text{w}} = \inf_{\delta \in \Delta_0} \| \hat{\delta}_0 - \delta \|_{\text{w}} \leq d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_{\text{w}}).
\]
This result implies that we can obtain the convergence rate of $\| \hat{\delta}_0 - \delta_0 \|_{\text{w}}$ by deriving that of $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_{\text{w}})$. However, the identified set $\Delta_0$ is an equivalence class under the pseudo-norm, and the convergence under $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_{\text{w}})$ does not suffice to consistently estimate a given element $\delta_0 \in \Delta_0$. Whereas the supremum norm $\| \cdot \|_{\infty}$ is able to differentiate between elements in $\Delta_0$, and $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_{\infty}) = o_p(1)$ under certain regularity condition as we will show later.

We make the following assumptions to guarantee that $\hat{\Delta}_0$ is consistent under the metric $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_{\infty})$ and to obtain the rate of convergence for $\hat{\Delta}_0$ under the weaker metric $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_{\text{w}})$.

**Assumption 3.** The vector of covariates $X \in \mathbb{R}^d$ has support $[0, 1]^d$, and the outcome $Y \in \mathbb{R}$ and the shadow variable $Z \in \mathbb{R}$ have compact supports.

Assumption 3 requires $(X, Y, Z)$ to have compact supports, and without loss of generality, we assume that $X$ has been transformed such that the support is $[0, 1]^d$. These are standard conditions that are usually required in the semiparametric literature. Although $Y$ and $Z$ are also required to have compact support, the proposed approach may still be applicable if the supports are infinite with sufficiently thin tails. For instance, in our simulation studies where the variables $Y$ and $Z$ are drawn from a normal distribution in Section 4, the proposed approach continues to perform quite well.
We next impose restrictions on the smoothness of functions in the set $\Delta$. We use the following Sobolev norm to characterize the smoothness of functions.

**Definition 1.** For a generic function $\rho(w)$ defined on $w \in \mathbb{R}^d$, we define

$$\|\rho\|_{\infty, \alpha} = \max_{|\lambda| \leq \alpha} \sup_{w} |D^\lambda \rho(w)| + \max_{\lambda=\alpha} \sup_{w \neq w'} \frac{D^\lambda \rho(w) - D^\lambda \rho(w')}{\|w - w'\|^{\alpha - \alpha}} ,$$

where $\lambda$ be a $d$-dimensional vector of nonnegative integers, $|\lambda| = \sum_{i=1}^{d} \lambda_i$, $\alpha$ denotes the largest integer smaller than $\alpha$, $D^\lambda \rho(w) = \partial^{\lambda_1} \rho(w)/\partial w_1^{\lambda_1} \ldots \partial w_d^{\lambda_d}$, and $D^0 \rho(w) = \rho(w)$.

A function $\rho$ with $\|\rho\|_{\infty, \alpha} < \infty$ has uniformly bounded partial derivatives up to order $\alpha$; besides, the $\alpha$th partial derivative of this function is Lipschitz of order $\alpha - \alpha$.

**Assumption 4.** The following conditions hold:

(i) $\sup_{\delta \in \Delta} \|\delta\|_{\infty, \alpha} < \infty$ for some $\alpha > (d + 1)/2$; in addition, $\Delta_0 \neq \emptyset$, and both $\Delta_n$ and $\Delta$ are closed;

(ii) for every $\delta \in \Delta$, there is $\Pi_n \delta \in \Delta_n$ such that $\sup_{\delta \in \Delta} \|\delta - \Pi_n \delta\|_{\infty} = O(\eta_n)$ for some $\eta_n = o(1)$.

Assumption 4(i) requires that each function $\delta \in \Delta$ is sufficiently smooth and bounded. The closedness condition in this assumption and Assumption 3 together imply that $\Delta$ is compact under $\| \cdot \|_{\infty}$. It is well known that solving integral equations as in (2) is an ill-posed inverse problem. The ill-posedness due to noncontinuity of the solution and difficulty of computation can have a severe impact on the consistency and convergence rates of estimators. The compactness condition is imposed to ensure that the consistency of the proposed estimator under $\| \cdot \|_{\infty}$ is not affected by the ill-posedness. Such a compactness condition is commonly made in the nonparametric and semiparametric literature; see, e.g., Newey and Powell (2003), Ai and Chen (2003), and Chen and Pouzo (2012). Alternatively, it is possible to address the ill-posed problem by employing a regularization approach as in Horowitz (2009) and Darolles et al. (2011).

Assumption 4(ii) quantifies the approximation error of functions in $\Delta$ by the sieve space $\Delta_n$. This condition is satisfied by many commonly-used function spaces (e.g., Hölder space), whose elements are sufficiently smooth, and by popular sieves (e.g., power series, splines). For example, consider the function set $\Delta$ with $\sup_{\delta \in \Delta} \|\delta\|_{\infty, \alpha} < \infty$. If the sieve functions $\{v_q(x, z)\}_{q=1}^{\infty}$ are polynomials or tensor product univariate splines, then uniformly on $\delta \in \Delta$, the approximation error of $\delta$ by functions of the form $\sum_{q=1}^{Q_n} \beta_q v_q(x, z) \in \Delta_n$ under $\| \cdot \|_{\infty}$ is of the order $O\{q_n^{-\alpha/(d+1)}\}$. Thus, Assumption 4(ii) is met with $\eta_n = q_n^{-\alpha/(d+1)}$; see Chen (2007) for further discussion.

**Assumption 5.** The following conditions hold:

(i) the smallest and largest eigenvalues of $E\{R\phi(X, Y)\phi(X, Y)^\top\}$ are bounded above and away from zero for all $k_n$;

(ii) for every $\delta \in \Delta$, there is a $\pi_n(\delta) \in \mathbb{R}^{k_n}$ such that

$$\sup_{\delta \in \Delta} \|E\{\delta(X, Z) \mid r = 1, x, y\} - \phi^{\top}(x, y)\pi_n(\delta)\|_{\infty} = O\left(k_n^{-\frac{\alpha}{\alpha + 1}}\right).$$
(iii) $\xi_n^2 k_n = o(n)$, where $\xi_n = \sup_{x,y} \|\phi(x,y)\|_2$.

Assumption 5 bounds the second moment matrix of the approximating functions away from singularity, presents a uniform approximation error of the series estimator to the conditional mean function, and restricts the magnitude of the series terms. These conditions are standard for series estimation of conditional mean functions; see, e.g., [Newey (1997), Ai and Chen (2003), and Huang (2003)]. Primitive conditions are discussed below so that the rate requirements in this assumption hold. Consider any $f$ satisfying Assumption 4, i.e., $\sup_{d \in \Delta} \|d\|_{\infty, \alpha} < \infty$. If the partial derivatives of $f(z \mid r = 1, x, y)$ with respect to $(x, y)$ are continuously differentiable up to order $\alpha + 1$, then under Assumption 3 we have $\sup_{d} \|E(\delta(X, Z) \mid R = 1, x, y)\|_{\infty, \alpha} < \infty$.

In addition, if the sieve functions $\{\phi_k(x, y)\}_{k=1}^{\infty}$ are polynomials or tensor product univariate splines, then by similar arguments after Assumption 4 we conclude that the approximation error under $\| \cdot \|_{\infty}$ is of the order $O(k_n^{-\alpha/(d+1)})$ uniformly on $d \in \Delta$. Verifying Assumption 5(iii) depends on the relationship between $\xi_n$ and $k_n$. For example, if $\{\phi_k(x, y)\}_{k=1}^{\infty}$ are tensor product univariate splines, then $\xi_n = O(k_n^{d+1/2})$.

Write $c_n$ in (8) by $b_n/a_n$ with appropriate sequences $a_n$ and $b_n$, and define $\lambda_n = k_n/n + k_n^{-2\alpha/(d+1)} + \eta_n^2$.

**Theorem 2.** Suppose that Assumptions 3–5 hold. If $a_n = O(\lambda_n^{-1})$, $b_n \to \infty$ and $b_n = o(a_n)$, Then $d_H(\Delta_0, \Delta_0, \| \cdot \|_w) = o_p(1)$, and $d_H(\Delta_0, \Delta_0, \| \cdot \|_w) = O_p(c_n^{1/2})$.

Theorem 2 shows the consistency of $\Delta_0$ under the supremum-norm metric $d_H(\Delta_0, \Delta_0, \| \cdot \|_w)$ and establishes the rate of convergence of $\Delta_0$ under the weaker pseudo-norm metric $d_H(\Delta_0, \Delta_0, \| \cdot \|_w)$. In particular, if we let $k_n^2 = o(n)$, $k_n^{-3\alpha/(d+1)} = o(n^{-1})$, and $\eta_n = o(n^{-1/3})$, as imposed in Assumption 7 in the next subsection, then $\lambda_n = o(n^{-2/3})$ or $\lambda_n^{-1}n^{-2/3} \to \infty$. We take $a_n = \lambda_n^{-1/2}n^{1/3} \to \infty$ and $b_n = a_n^{1/2} / n^{1/3}$. Thus, $a_n = \lambda_n^{-1/2}n^{2/3} \to \infty$, $b_n = a_n - a_n^{-1}n^{-2/3} = o(n)$, and $b_n = a_n - a_n^{-2/3}n^{-1/3} = o(a_n)$. In fact, under such rate requirements, we have $n^{2/3}b_n = o(a_n)$ and $d_H(\Delta_0, \Delta_0, \| \cdot \|_w) = o_p(n^{-1/4})$, which are sufficient to establish the asymptotic normality of the proposed estimator given in subsection 3.4.

### 3.3 A representer-based estimator

After we have obtained a consistent estimator $\hat{\Delta}_0$ for $\Delta_0$, we remain to select an estimator from $\hat{\Delta}_0$ such that it converges to a unique element belonging to $\Delta_0$. We adopt the theory of extremum estimators to achieve this goal. Let $M : \Delta \to \mathbb{R}$ be a population criterion functional that attains a unique minimum $\delta_0$ on $\Delta_0$ and $M_n(\delta)$ be its sample analogue. We then choose the minimizer of $M_n(\delta)$ over the estimated solution set $\hat{\Delta}_0$, denoted by $\hat{\delta}_0 \in \arg\min_{\delta \in \Delta_0} M_n(\delta)$, (10)

which is expected to converge to the unique minimum $\delta_0$ of $M(\delta)$ on $\Delta_0$.

**Assumption 6.** The function set $\Delta$ is convex; the functional $M : \Delta \to \mathbb{R}$ is strictly convex and attains a unique minimum at $\delta_0$ on $\Delta_0$; its sample analogue $M_n : \Delta \to \mathbb{R}$ is continuous and $\sup_{\delta \in \Delta} |M_n(\delta) - M(\delta)| = o_p(1)$.
One example of particular interest is

\[ M(\delta) = E\left[ \{(1 - R)\delta(X, Z)\}^2 \right]. \]

This is a convex functional with respect to \( \delta \). In addition, since \( E\{(1 - R)\delta_0(X, Z)\} = E\{(1 - R)\tau(X, Y)\} \) for any \( \delta_0 \in \Delta_0 \), the minimizer of \( M(\delta) \) on \( \Delta_0 \) in fact minimizes the variance of \( (1 - R)\delta_0(X, Z) \) among \( \delta_0 \in \Delta_0 \). Its sample analogue is

\[ M_n(\delta) = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i)\delta^2(X_i, Z_i). \]

Under Assumptions 3–4, one can show that the function class \( \{(1 - R)\delta : \delta \in \Delta\} \) is a Glivenko-Cantelli class, and thus \( \sup_{\delta \in \Delta} |M_n(\delta) - M(\delta)| = o_p(1) \).

**Theorem 3.** Suppose that Assumptions 3–6 hold. Then

\[ \|\hat{\delta}_0 - \delta_0\|_\infty = o_p(1), \]

where \( \hat{\delta}_0 \) is defined through (10) and \( \delta_0 \) is defined in Assumption 6. In addition, if \( a_n = O(\lambda_n^{-1}) \), \( b_n \to \infty \) and \( b_n = o(a_n) \), we then have

\[ \|\hat{\delta}_0 - \delta_0\|_w = O_p(c_1^{1/2}/n). \]

Theorem 3 implies that by choosing an appropriate function \( M(\delta) \), it is possible to construct a consistent estimator \( \hat{\delta}_0 \) for some unique element \( \delta_0 \in \Delta_0 \) in terms of supremum norm \( \| \cdot \|_\infty \) and further obtain its rate of convergence under the weaker pseudo-norm \( \| \cdot \|_w \).

Based on the estimator \( \hat{\delta}_0 \) given in (10), we obtain the following representer-based estimator \( \hat{\mu}_{\text{rep}} \) of \( \mu \):

\[ \hat{\mu}_{\text{rep}} = \frac{1}{n} \sum_{i=1}^{n} \left\{ R_i \tau(X_i, Y_i) + (1 - R_i)\hat{\delta}_0(X_i, Z_i) \right\}. \] (11)

Below we discuss the asymptotic expansion of the estimator \( \hat{\mu}_{\text{rep}} \).

Let \( \overline{\Delta} \) be the closure of the linear span of \( \Delta \) under \( \| \cdot \|_w \), which is a Hilbert space with inner product:

\[ \langle \delta_1, \delta_2 \rangle_w = E\left[ RE\{\delta_1(X, Z) \mid R = 1, X, Y\} E\{\delta_2(X, Z) \mid R = 1, X, Y\} \right] \]

for any \( \delta_1, \delta_2 \in \overline{\Delta} \).

**Assumption 7.** The following conditions hold:

(i) there exists a function \( h_0 \in \Delta \) such that

\[ \langle h_0, \delta \rangle_w = E\{(1 - R)\delta(X, Z)\} \text{ for all } \delta \in \overline{\Delta}. \]

(ii) \( \eta_n = o(n^{-1/3}) \), \( k_n^{-3\alpha/(d+1)} = o(n^{-1}) \), \( k_n^3 = o(n) \), \( \xi_n^2k_n^2 = o(n) \), and \( \xi_n^2k_n^{-2\alpha/(d+1)} = o(1) \).
Note that the linear functional $\delta \mapsto E\{ (1 - R) \delta(X, Z) \}$ is continuous under $\| \cdot \|_w$. Hence, by the Riesz representation theorem, there exists a unique $h_0 \in \Delta$ (up to an equivalence class in $\| \cdot \|_w$) such that $\langle h_0, \delta \rangle_w = E\{ (1 - R) \delta(X, Z) \}$ for all $\delta \in \Delta$. However, Assumption 7(ii) further requires that this equivalence class must contain at least one element that falls in $\Delta$. A primitive condition for Assumption 7(i) is that the inverse probability weight also has a smooth representer: if

$$E \{ h_0(X, Z) + 1 \mid R = 1, X, Y \} = \frac{1}{f(R = 1 \mid X, Y)},$$

then $h_0$ satisfies Assumption 7(i).

Assumption 7(ii) imposes some rate requirements, which can be satisfied as long as the function classes being approximated in Assumptions 4 and 5 are sufficiently smooth.

**Theorem 4.** Suppose that Assumptions 3-7 hold. We have that

$$\sqrt{n} (\hat{\mu}_{\text{rep}} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ (1 - R_i) \delta_0(X_i, Z_i) + R_i \tau(X_i, Y_i) + R_i E \{ h_0(X, Z) \mid R = 1, X, Y \} \right]$$

$$\times \left\{ \tau(X_i, Y_i) - \delta_0(X_i, Z_i) \right\} - \mu - \sqrt{n} r_n(\delta_0) + o_P(1),$$

with

$$r_n(\delta_0) = \frac{1}{n} \sum_{i=1}^{n} R_i \hat{E} \{ \Pi_n h_0(X, Z) \mid R = 1, X, Y \} \hat{c}(X_i, Y_i, \hat{\delta}_0),$$

where $\Pi_n h_0 \in \Delta_n$ approximates $h_0$ as given in Assumption 4(ii), $\hat{E}(\cdot)$ and $\hat{c}(\cdot)$ are defined in (5) and (7), respectively.

Theorem 4 reveals an asymptotic expansion of $\hat{\mu}_{\text{rep}}$. However, the estimator $\hat{\mu}_{\text{rep}}$ is not necessarily asymptotically normal as the bias term $\sqrt{n} r_n(\delta_0)$ may not be asymptotically negligible. In the next subsection, we propose a debiased estimator which is regular and asymptotically normal. We further establish that the debiased estimator is semiparametric locally efficient under a shadow variable model at a given submodel where a key completeness condition holds.

### 3.4 A debiased semiparametric locally efficient estimator

Note that only $\Pi_n h_0$ is unknown in the bias term $r_n(\delta_0)$ in (13). We propose to construct an estimator of $\Pi_n h_0$ and then subtract the bias to obtain an estimator of $\mu$ that is asymptotically normal. We define the criterion function:

$$C(\delta) = E \left[ R \{ E(\delta(X, Z) \mid R = 1, X, Y) \}^2 \right] - 2 E \{ (1 - R) \delta(X, Z) \}, \quad \delta \in \Delta$$

and its sample analogue,

$$C_n(\delta) = \frac{1}{n} \sum_{i=1}^{n} R_i \left( \hat{E}(\delta(X, Z) \mid R = 1, X, Y) \right)^2 - \frac{2}{n} \sum_{i=1}^{n} (1 - R_i) \delta(X_i, Z_i), \quad \delta \in \Delta.$$ 

Since $E\{ (1 - R) \delta(X, Z) \} = \langle h_0, \delta \rangle_w$ by Assumption 7, it follows that $C(\delta) = \| \delta - h_0 \|_w^2 - \| h_0 \|_w^2$. Thus, $h_0$ is the unique minimizer of $\delta \mapsto C(\delta)$ up to the equivalence class in $\| \cdot \|_w$. In addition,
Since $h_0$ and $\Pi_n h_0$ are close under the metric $|| \cdot ||_\infty$ by Assumption 1(ii), we then define the estimator for $\Pi_n h_0$ by:

$$\hat{h} \in \arg\min_{\delta \in \Delta_n} C_n(\delta),$$

(14)

Given the estimator $\hat{h}$, the approximation to the bias term $r_n(\hat{h})$ is

$$\hat{r}_n(\hat{h}) = \frac{1}{n} \sum_{i=1}^{n} R_i \hat{E}\left\{ \hat{h}(X, Z) \mid R = 1, X_i, Y_i \right\} \hat{c}(X_i, Y_i, \hat{h}).$$

(15)

**Lemma 1.** Suppose that Assumptions 3–7 hold. Then it follows that

$$\sup_{\delta \in \Delta_0} |\hat{r}_n(\hat{h}) - r_n(\hat{h})| = O_p \left[ c_n^{1/2} \left\{ \left( \frac{k_n}{n} \right)^{1/4} + k_n^{-\alpha/(\alpha+1)} \right\} \right].$$

This lemma establishes the rate of convergence of $\hat{r}_n(\hat{h})$ to $r_n(\hat{h})$ uniformly on $\Delta_0$. If $c_n$ converges to zero sufficiently fast, then $\sup_{\delta \in \Delta_0} \sqrt{n} |\hat{r}_n(\hat{h}) - r_n(\hat{h})| = o_p(1)$. The rate conditions imposed in Assumption 7(ii) guarantee that such a choice of $c_n$ is feasible. As a result, Theorem 7 and Lemma 1 imply that it is possible to construct a debiased estimator that is $\sqrt{n}$-consistent and asymptotically normal by subtracting the estimated bias $\hat{r}_n(\hat{h})$ from $\hat{\mu}_{\text{rep}}$:

$$\hat{\mu}_{\text{rep}} = \hat{\mu}_{\text{rep}} - \hat{r}_n(\hat{h}).$$

(16)

**Theorem 5.** Suppose that Assumptions 3–7 hold. If $a_n = O(\lambda_n^{-1})$, $b_n \to \infty$ and $n^{2/3}b_n = o(a_n)$, then $\sqrt{n}(\hat{\mu}_{\text{rep}} - \mu)$ converges in distribution to $N(0, \sigma^2)$, where $\sigma^2$ is the variance of

$$(1 - R)\delta_0(X, Z) + R\tau(X, Y) + \hat{R}E\{h_0(X, Z) \mid R = 1, X, Y\}\{\tau(X, Y) - \delta_0(X, Z)\} - \mu. \quad (17)$$

Based on (17), one can easily obtain a consistent estimator of the asymptotic variance

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( (1 - R_i)\hat{\delta}_0^2(X_i, Z_i) + R_i\tau^2(X_i, Y_i) - \left\{ \frac{1}{n} \sum_{i=1}^{n} (1 - R_i)\hat{\delta}_0(X_i, Z_i) + R_i\tau(X_i, Y_i) \right\}^2 \right. \left. + R_i\left\{ \hat{E}\left( \hat{h}(X, Z) \mid R = 1, X_i, Y_i \right) \right\}^2 \{\tau(X_i, Y_i) - \hat{\delta}_0(X_i, Z_i)\} \right)^2.$$ 

Then given $\alpha \in (0, 1)$, an asymptotic $100(1 - \alpha)$% confidence interval is $[\hat{\mu}_{\text{rep}} - z_{\alpha}\hat{\sigma}/\sqrt{n}, \hat{\mu}_{\text{rep}} + z_{\alpha}\hat{\sigma}/\sqrt{n}]$, where $z_{\alpha} = \Phi^{-1}(1 - \alpha/2)$. The formula (17) presents the influence function for $\hat{\mu}_{\text{rep}}$. The influence function is locally efficient in the sense that it attains the semiparametric efficiency bound for the outcome mean under certain conditions in the semiparametric model $\mathcal{M}_{\text{np}}$ defined through (4).

**Assumption 8.** The following conditions hold:

(i) **Completeness:** (1) for any square-integrable function $\xi(x, y)$, $E\{\xi(X, Y) \mid R = 1, X, Z\} = 0$ almost surely if and only if $\xi(X, Y) = 0$ almost surely; (2) for any square-integrable function $\eta(x, z)$, $E\{\eta(X, Z) \mid R = 1, X, Y\} = 0$ almost surely if and only if $\eta(X, Z) = 0$ almost surely.

(ii) **Denote** $\Omega(x, z) = E\{[\gamma(X, Y) - \beta(X, Z)]^2 \mid R = 1, X = x, Z = z\}$. **Suppose that** $0 < \inf_{x, z}\Omega(x, z) \leq \sup_{x, z}\Omega(x, z) < \infty$. 

13
(iii) The operator $T$ and its adjoint $T'$ defined through conditional expectations in Section 2 are both bounded.

Under the completeness condition in Assumption 8(i), $\gamma(X, Y)$ is identifiable, $\delta_0(X, Z)$ and $h_0(X, Z)$ that respectively solve (2) and (12) are also uniquely identified. Assumption 8(ii) bounds $\Omega(x, z)$ away from zero and infinity. Note that conditional expectation operators can be shown to be bounded under weak conditions on the joint density [Carrasco et al. 2007b].

**Corollary 2.** The influence function (17) attains the efficiency bound of $\mu$ in $M_{np}$ at the submodel where $h_0(x, z)$ solves (12) and Assumptions 1–8 hold.

### 3.5 Computation

In this section, we discuss the computations of $\hat{\delta}_0$ in (10) and $\hat{h}$ in (14) that are both required for $\hat{\mu}_{np}$. For the computation of $\delta_0$, we aim to solve the following constrained optimization problem:

$$\hat{\delta}_0 \in \arg \min \limits_{\delta \in \Delta_n} M_n(\delta), \quad \text{s.t.} \quad Q_n(\delta) \leq c_n. \quad (18)$$

The constrained function $Q_n$ in (8) is quadratic in $\delta$ and the objective function

$$M_n(\delta) = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) \delta^2(x_i, z_i)$$

is also a quadratic function. It remains to impose some constraints on $\Delta_n$ so that we could have a tractable optimization problem.

A computationally simple choice for $\Delta_n$ are linear sieves as defined in (4); that is, for any $\delta \in \Delta_n$, we have $\delta = \psi^T \theta$, where $\theta = (\theta_1, \ldots, \theta_m)^T$ and $\psi(x, z) = \{\psi_1(x, z), \ldots, \psi_q(x, z)\}^T$.

For this choice of $\Delta_n$, if we define $\Delta = \{\delta : \|\delta\|_{\infty, \alpha} \leq K\}$ for some $K > 0$ as required by Assumption 4, then the constraint $\|\psi^T \theta\|_{\infty, \alpha} \leq K$ can be highly nonlinear in $\theta$. We thus follow Newey and Powell (2003) by defining $\Delta$ as the closure of the set $\{\delta : \|\delta\|_{2, \alpha_0} \leq K\}$ under $\|\cdot\|_{2, \alpha_0}$, where $\alpha_0 > \alpha + (d + 1)/2$ and

$$\|\delta\|_{2, \alpha_0}^2 = \sum_{|\lambda| \leq \alpha_0} \int_{[0,1]^d \times Z} D^\lambda \delta(x, z)^2 \; dx \; dz,$$

In the above equation, $Z$ denotes the support of $Z$, $\lambda$ is a $(d + 1)$-dimensional vector of non-negative integers, $|\lambda| = \sum_{i=1}^{d+1} \lambda_i$, and the operator $D^\lambda(\cdot)$ is defined in Definition 4. Then the constraint $\psi^T \theta \in \Delta_n$ now turns to $\theta^T H_n \theta \leq K^2$, where

$$H_n = \sum_{|\lambda| \leq \alpha_0} \int_{[0,1]^d \times Z} \left\{D^\lambda \psi(x, z) D^\lambda \psi^T(x, z)\right\} \; dx \; dz.$$
and then let $\hat{h} = \psi^\top \hat{\theta}_h$. In the above optimization problems, $c_n$, $K$, and the basis function dimensions $k_n, q_n$ are all tuning parameters. Several data-driven methods for selecting tuning parameters in series estimation have been discussed in [Li and Racine (2007)]. Here we suggest using 5-fold cross-validation to select the tuning parameters.

4 Numerical studies

4.1 Simulation

In this subsection, we conduct simulation studies to evaluate the performance of the proposed estimators in finite samples. We consider two different cases. In case (I), the data are generated under models where the full data distribution is identified. In case (II), the full data distribution is not identified but Assumption 2 holds.

For case (I), we generate four covariates $X = (X_1, X_2, X_3, X_4)^\top$ according to $X_j \sim U(0, 1)$ for $j = 1, \ldots, 4$. We consider four data generating settings, including combinations of two choices of outcome models and two choices of propensity score models.

The missing data proportion in each of these settings is about 50%. For each setting, we replicate 1000 simulations at sample sizes 500 and 1000. We apply the proposed estimators $\hat{\mu}_\text{repA} (\text{REP-DB})$ and $\hat{\mu}_\text{rep} (\text{REP})$ to estimate the population outcome mean $\mu = E(Y)$. For comparison, we also use an inverse probability weighted estimator ($\text{IPW}$) with a linear-logistic propensity score model assuming MNAR and a regression-based estimator ($\text{marREG}$) assuming MAR to estimate $\mu$.

Simulation results are reported in Figure 1. In all four settings, the proposed estimators $\text{REP-DB}$ and $\text{REP}$ have negligible bias. In contrast, the $\text{IPW}$ estimator can have comparable bias with ours only when the propensity score model is correctly specified; see settings (a) and (c). If the propensity score model is incorrectly specified as in settings (b) and (d), the $\text{IPW}$ estimator exhibits an obvious downward bias and does not vanish when the sample size increases. As expected, the $\text{marREG}$ estimator has non-negligible bias in all settings.

We also calculate the 95% confidence interval based on the proposed estimator $\text{REP-DB}$ and the $\text{IPW}$ estimator. Coverage probabilities of these two approaches are shown in Table 1. The $\text{REP-DB}$ estimator based confidence intervals have coverage probabilities close to the nominal level of 0.95 in all scenarios even under small sample size $n = 500$. In contrast, the $\text{IPW}$ estimator based confidence intervals have coverage probabilities well below the nominal value if the propensity score model is incorrectly specified.

For case (II), we generate data according to Model 1 in Example 1. As with case (I), we consider two different sample sizes $n = 500$ and $n = 1000$. We calculate the bias (Bias), Monte Carlo standard deviation (SD) and 95% coverage probabilities (CP) based on 1000 replications in each setting. For comparison, we also apply the $\text{IPW}$ estimator with a correct propensity score model to estimate $\mu$. Since the full data distribution is not identified, the performance of $\text{IPW}$ estimator depends on initial values during the optimization process. We consider two
Figure 1: Comparisons in case (I) between the proposed two estimators (REP−DB and REP) and existing estimators (IPW and marREG) under sample sizes \( n = 500 \) and \( n = 1000 \). The abbreviation LL stands for Linear-logistic propensity score model with Linear outcome model, and the other three scenarios are analogously defined. The horizontal line marks the true value of the outcome mean.
Table 1: Coverage probability of the 95% confidence interval of the REP-DB and IPW estimators.

| n    | Methods | LL      | NL      | LN      | NN      |
|------|---------|---------|---------|---------|---------|
| 500  | REP-DB  | 0.940   | 0.932   | 0.942   | 0.939   |
|      | IPW     | 0.930   | 0.635   | 0.928   | 0.491   |
| 1000 | REP-DB  | 0.945   | 0.933   | 0.948   | 0.951   |
|      | IPW     | 0.943   | 0.381   | 0.951   | 0.177   |

Table 2: Comparisons in case (II) between REP-DB and IPW under n = 500 and n = 1000.

| n    | REP-DB Bias | SD | CP | IPW-true Bias | SD | CP | IPW-uniform Bias | SD | CP |
|------|-------------|----|----|---------------|----|----|------------------|----|----|
| 500  | 0.008       | 0.033 | 0.917 | -0.003       | 0.050 | 0.923 | -0.113       | 0.206 | 0.709 |
| 1000 | 0.003       | 0.024 | 0.942 | -0.004       | 0.035 | 0.933 | -0.134       | 0.216 | 0.667 |

different settings of initial values for optimization parameters: true values and random values from the uniform distribution \(U(0, 1)\). The results are summarized in Table 2.

We observe from Table 2 that the proposed estimator REP-DB has negligible bias, small standard deviation and satisfactory coverage probability even under sample size \(n = 500\). As sample size increases to \(n = 1000\), the 95% coverage probability is close to the nominal level. For the IPW estimator, only when the initial values for optimization parameters are set to be true values, it has comparable performance with REP-DB. However, if the initial values are randomly drawn from \(U(0, 1)\), the IPW estimator has non-negligible bias, large standard deviation and low coverage probability. As sample size increases, the situation becomes worse. We also calculate the IPW estimator when initial values are drawn from other distributions, e.g., standard normal distribution. The performance is even worse and we do not report the results here. The simulations in this case demonstrate the superiority of the proposed estimator over existing estimators which require identifiability of the full data distribution.

4.2 Empirical example

We apply the proposed methods to the China Family Panel Studies, which was previously analyzed in Miao et al. (2019). The dataset includes 3126 households in China. The outcome \(Y\) is the log of current home price (in \(10^4\) RMB yuan), and it has missing values due to the nonresponse of house owner and the non-availability from the real estate market. The missingness process of home price is likely to be not at random, because subjects having expensive houses may be less likely to disclose their home prices. The missing data rate of current home price is 21.8%. The completely observed covariates \(X\) includes 5 continuous variables: travel time to the nearest business center, house building area, family size, house story height, log of family income, and 3 discrete variables: province, urban (1 for urban household, 0 rural), refurbish status. The shadow variable \(Z\) is chosen as the construction price of a house, which is also completely observed. The construction price is related to the current price of a house, and it the shadow variable assumption that nonresponse is independent of the construction price conditional on the current price and fully observed covariates is a reasonable assumption as the
Table 3: Point estimates and 95% confidence intervals of the outcome mean for the home pricing example

| Methods   | Estimate | 95% confidence interval       |
|-----------|----------|-------------------------------|
| REP-DB    | 2.591    | (2.520, 2.661)                |
| IPW       | 2.611    | (2.544, 2.678)                |
| marREG    | 2.714    | (2.661, 2.766)                |
| marIPW    | 2.715    | (2.659, 2.772)                |

construction price can be viewed as error prone proxy for the current home value, and as such is no longer predictive of the missingness mechanism once the current home value has been accounted for.

We apply the proposed estimator REP-DB to estimate the outcome mean and the 95% confidence interval. We also use the competing IPW estimator and two estimators assuming MAR (marREG and marIPW) for comparison. The results are shown in Table 3. We observe that the results from the proposed estimator are similar to those from the IPW estimator, both yielding slightly lower estimates of home price on the log scale than those obtained from the standard MAR estimators. However, when the data are transformed back to the original scale, the deviations are notable and amount to approximately $1.13 \times 10^4$ RMB yuan. These analysis results are generally consistent with those in Miao et al. (2019).

5 Discussion

With the aid of a shadow variable, we have established a necessary and sufficient condition for nonparametric identification of mean functionals of nonignorable missing data even if the joint distribution is not identified. Then we strengthen this condition by imposing a representer assumption that is necessary for $\sqrt{n}$-estimability of the mean functional. The assumption involves the existence of solutions to a representer equation, which is a Fredholm integral equation of the first kind and can be satisfied under mild requirements. Based on the representer equation, we propose a sieve-based estimator for the mean functional, which bypasses the difficulties of correctly specifying and estimating the unknown missingness mechanism and the outcome regression. Although the joint distribution is not identifiable, the proposed estimator is shown to be consistent for the mean functional. In addition, we establish conditions under which the proposed estimator is asymptotically normal. We would like to point out that since the solution to the representer equation is not uniquely determined, one cannot simply apply standard theories for nonparametric sieve estimators to derive the above asymptotic results. In fact, we need to first construct a consistent estimator for the solution set, and then find from the estimated set a consistent estimator for an appropriately chosen solution. We finally show that the proposed estimator attains the semiparametric efficiency bound for the shadow variable model at a key submodel where the representer is uniquely identified.

The availability of a valid shadow variable is crucial for the proposed approach. Although it is generally not possible to test the shadow variable assumption via observed data without making another untestable assumption, the existence of such a variable is practically reasonable in the empirical example presented in this paper and similar situations where one or more proxies
or surrogates of a variable prone to missing data may be available. In fact, it is not uncommon in survey studies and/or cohort studies in the health and social sciences, that certain outcomes may be sensitive and/or expensive to measure accurately, so that a gold standard measurement is obtained only for a select subset of the sample, while one or more proxies or surrogate measures may be available for the remaining sample. Instead of a standard measurement error model often used in such settings which requires stringent identifying conditions, the more flexible shadow variable approach proposed in this paper provides a more robust alternative to incorporate surrogate measurement in a nonparametric framework, under minimal identification conditions. Still, the validity of the shadow variable assumptions generally requires domain-specific knowledge of experts and needs to be investigated on a case-by-case basis. As advocated by Robins et al. (2000), in principle, one can also conduct sensitivity analysis to assess how results would change if the shadow variable assumption were violated by some pre-specified amount.

The proposed methods may be improved or extended in several directions. Firstly, the proposed identification and estimation framework may be extended to handle nonignorable missing outcome regression or missing covariate problems. Secondly, one can use modern machine learning techniques to solve the representer equation so that an improved estimator may be achieved that adapts to sparsity structures in the data. Thirdly, it is of great interest to extend our results to handling other problems of coarsened data, for instance, unmeasured confounding problems in causal inference. We plan to pursue these and other related issues in future research.

Appendix

Existence of solutions to representer equation (2) under completeness conditions

We adopt the singular value decomposition (Carrasco et al. (2007b), Theorem 2.41) of compact operators to characterize conditions for existence of a solution to (2). Let $L^2\{F(t)\}$ denote the space of all square-integrable functions of $t$ with respect to a cumulative distribution function $F(t)$, which is a Hilbert space with inner product $\langle g,h \rangle = \int g(t)h(t)dF(t)$. Let $K_x$ denote the conditional expectation operator $L^2\{F(z \mid x, r = 1)\} \rightarrow L^2\{F(y \mid x, r = 1)\}$, $K_x h = E\{h(Z) \mid x, y, r = 1\}$ for $h \in L^2\{F(z \mid x, r = 1)\}$, and let $(\lambda_n, \varphi_n, \psi_n)_{n=1}^{+\infty}$ denote a singular value decomposition of $K_x$. We assume the following regularity conditions:

**Condition A1.** $\int \int f(z \mid x, y, r = 1)f(y \mid x, z, r = 1)dydz < +\infty$

**Condition A2.** $\int \tau^2(x, y)f(y \mid x, r = 1)dy < +\infty$.

**Condition A3.** $\sum_{n=1}^{+\infty} \lambda_n^{-2} |\langle \tau(x, y), \psi_n \rangle|^2 < +\infty$.

Given $f(z \mid x, y, r = 1)$, the solution to (2) must exist if the completeness condition and Conditions A1, A3 all hold. The proof follows immediately from Picard’s theorem (Kress (1989), Theorem 15.18) and Lemma 2 of Miao et al. (2018).

Supporting information

Supporting information includes additional lemmas and proofs of all the theoretical results.
References

Ai, C. and Chen, X. (2003). Efficient estimation of models with conditional moment restrictions containing unknown functions. *Econometrica*, 71(6):1795–1843.

Carrasco, M., Florens, J. P., and Renault, E. (2007a). Linear inverse problems in structural econometrics estimation based on spectral decomposition and regularization. In Heckman, J. J. and Leamer, E., editors, *Handbook of Econometrics*, volume 6B, pages 5633–5751. Elsevier, Amsterdam.

Carrasco, M., Florens, J.-P., and Renault, E. (2007b). Linear inverse problems in structural econometrics estimation based on spectral decomposition and regularization. *Handbook of Econometrics*, 6:5633–5751.

Chen, X. (2007). Large sample sieve estimation of semi-nonparametric models. *Handbook of Econometrics.*

Chen, X. and Pouzo, D. (2012). Estimation of nonparametric conditional moment models with possibly nonsmooth generalized residuals. *Econometrica*, 80(1):277–321.

Chernozhukov, V., Hong, H., and Tamer, E. (2007). Estimation and confidence regions for parameter sets in econometric models. *Econometrica*, 75(5):1243–1284.

Cui, Y., Pu, H., Shi, X., Miao, W., and Tchetgen Tchetgen, E. (2020). Semiparametric proximal causal inference. *arXiv preprint arXiv:2011.08411.*

Darolles, S., Fan, Y., Florens, J.-P., and Renault, E. (2011). Nonparametric instrumental regression. *Econometrica*, 79(5):1541–1565.

Das, M., Newey, W. K., and Vella, F. (2003). Nonparametric estimation of sample selection models. *The Review of Economic Studies*, 70:33–58.

D’Haultfoeuille, X. (2010). A new instrumental method for dealing with endogenous selection. *Journal of Econometrics*, 154(1):1–15.

Greenlees, J. S., Reece, W. S., and Zieschang, K. D. (1982). Imputation of missing values when the probability of response depends on the variable being imputed. *Journal of the American Statistical Association*, 77(378):251–261.

Heckman, J. J. (1979). Sample selection bias as a specification error. *Econometrica*, 47:153–161.

Horowitz, J. L. (2009). *Semiparametric and Nonparametric Methods in Econometrics*, volume 12. New York: Springer.

Huang, J. Z. (2003). Local asymptotics for polynomial spline regression. *Annals of Statistics*, 31(5):1600–1635.

Ibrahim, J. G., Lipsitz, S. R., and Chen, M.-H. (1999). Missing covariates in generalized linear models when the missing data mechanism is non-ignorable. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 61(1):173–190.
Ibrahim, J. G., Lipsitz, S. R., and Horton, N. (2001). Using auxiliary data for parameter estimation with non-ignorably missing outcomes. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 50(3):361–373.

Kennedy, E. H., Ma, Z., McHugh, M. D., and Small, D. S. (2017). Non-parametric methods for doubly robust estimation of continuous treatment effects. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79(4):1229–1245.

Kim, J. K. and Yu, C. L. (2011). A semiparametric estimation of mean functionals with nonignorable missing data. *Journal of the American Statistical Association*, 106(493):157–165.

Kott, P. S. (2014). Calibration weighting when model and calibration variables can differ. In Mecatti, F., Conti, L. P., and Ranalli, G. M., editors, *Contributions to Sampling Statistics*, pages 1–18. Springer, Cham.

Kress, R. (1989). *Linear Integral Equations*. Springer, Berlin.

Li, Q. and Racine, J. S. (2007). *Nonparametric Econometrics: Theory and Practice*. Princeton University Press.

Little, R. J. and Rubin, D. B. (2002). *Statistical Analysis With Missing Data*. New York, NY: Wiley-Interscience.

Liu, L., Miao, W., Sun, B., Robins, J., and Tchetgen Tchetgen, E. (2020). Identification and inference for marginal average treatment effect on the treated with an instrumental variable. *Statistica Sinica*, 30:1517–1541.

Miao, W., Ding, P., and Geng, Z. (2016). Identifiability of normal and normal mixture models with nonignorable missing data. *Journal of the American Statistical Association*, 111(516):1673–1683.

Miao, W., Geng, Z., and Tchetgen Tchetgen, E. J. (2018). Identifying causal effects with proxy variables of an unmeasured confounder. *Biometrika*, 105(4):987–993.

Miao, W., Liu, L., Tchetgen Tchetgen, E., and Geng, Z. (2019). Identification, doubly robust estimation, and semiparametric efficiency theory of nonignorable missing data with a shadow variable. *arXiv preprint arXiv:1509.02556v3*.

Miao, W. and Tchetgen Tchetgen, E. J. (2016). On varieties of doubly robust estimators under missingness not at random with a shadow variable. *Biometrika*, 103:475–482.

Morikawa, K. and Kim, J. K. (2021). Semiparametric optimal estimation with nonignorable nonresponse data. *The Annals of Statistics*, 49(5):2991–3014.

Newey, W. K. (1997). Convergence rates and asymptotic normality for series estimators. *Journal of Econometrics*, 79(1):147–168.

Newey, W. K. and Powell, J. L. (2003). Instrumental variable estimation of nonparametric models. *Econometrica*, 71(5):1565–1578.
Qin, J., Leung, D., and Shao, J. (2002). Estimation with survey data under nonignorable nonresponse or informative sampling. *Journal of the American Statistical Association*, 97(457):193–200.

Robins, J. M., Rotnitzky, A., and Scharfstein, D. O. (2000). Sensitivity analysis for selection bias and unmeasured confounding in missing data and causal inference models. In *Statistical models in Epidemiology, the Environment, and Clinical Trials*, pages 1–94. Springer.

Robins, J. M., Rotnitzky, A., and Zhao, L. P. (1994). Estimation of regression coefficients when some regressors are not always observed. *Journal of the American Statistical Association*, 89(427):846–866.

Rotnitzky, A. and Robins, J. (1997). Analysis of semi-parametric regression models with non-ignorable non-response. *Statistics in Medicine*, 16(1):81–102.

Rotnitzky, A., Robins, J. M., and Scharfstein, D. O. (1998). Semiparametric regression for repeated outcomes with nonignorable nonresponse. *Journal of the American Statistical Association*, 93(444):1321–1339.

Rubin, D. B. (1976). Inference and missing data (with discussion). *Biometrika*, 63(3):581–592.

Santos, A. (2011). Instrumental variable methods for recovering continuous linear functionals. *Journal of Econometrics*, 161(2):129–146.

Scharfstein, D. O., Rotnitzky, A., and Robins, J. M. (1999). Adjusting for nonignorable drop-out using semiparametric nonresponse models. *Journal of the American Statistical Association*, 94(448):1096–1120.

Severini, T. A. and Tripathi, G. (2012). Efficiency bounds for estimating linear functionals of nonparametric regression models with endogenous regressors. *Journal of Econometrics*, 170(2):491–498.

Shao, J. and Wang, L. (2016). Semiparametric inverse propensity weighting for nonignorable missing data. *Biometrika*, 103(1):175–187.

Sun, B., Liu, L., Miao, W., Wirth, K., Robins, J., and Tchetgen Tchetgen, E. J. (2018). Semiparametric estimation with data missing not at random using an instrumental variable. *Statistica Sinica*, 28(4):1965–1983.

Tang, G., Little, R. J., and Raghunathan, T. E. (2003). Analysis of multivariate missing data with nonignorable nonresponse. *Biometrika*, 90(4):747–764.

Tang, N., Zhao, P., and Zhu, H. (2014). Empirical likelihood for estimating equations with nonignorably missing data. *Statistica Sinica*, 24(2):723.

Tchetgen Tchetgen, E. J. and Wirth, K. E. (2017). A general instrumental variable framework for regression analysis with outcome missing not at random. *Biometrics*, 73(4):1123–1131.

Vansteelandt, S., Rotnitzky, A., and Robins, J. (2007). Estimation of regression models for the mean of repeated outcomes under nonignorable nonmonotone nonresponse. *Biometrika*, 94:841–860.
Wang, S., Shao, J., and Kim, J. K. (2014). An instrumental variable approach for identification and estimation with nonignorable nonresponse. *Statistica Sinica*, 24:1097–1116.

Zahner, G. E., Pawelkiewicz, W., DeFrancesco, J. J., and Adnopoz, J. (1992). Children’s mental health service needs and utilization patterns in an urban community: An epidemiological assessment. *Journal of the American Academy of Child & Adolescent Psychiatry*, 31(5):951–960.

Zhao, J. and Shao, J. (2015). Semiparametric pseudo-likelihoods in generalized linear models with nonignorable missing data. *Journal of the American Statistical Association*, 110(512):1577–1590.