SPECTRAL INCLUSION AND POLLUTION FOR A CLASS OF NON-SELF-ADJOINT PERTURBATIONS

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Abstract. Spectral inclusion and spectral pollution results are proved for sequences of linear operators of the form $T_0 + i\gamma s_n$ on a Hilbert space, where $s_n$ is strongly convergent and $\gamma > 0$. We work in both an abstract setting and a more concrete Sturm-Liouville framework. The results rigorously justify a method for computing eigenvalues in spectral gaps.

1. Introduction

It is well known that the numerical approximation of the spectra of linear operators is often complicated by the possible presence of spectral pollution [2, 12, 23, 31]. In this article, we study the eigenvalues of linear operators under a certain class of perturbations. We are primarily motivated by a method, called the dissipative barrier method (described below), designed to circumvent the issues caused by spectral pollution. In addition, these types of perturbations, often referred to as complex absorbing potentials in the context of Schrödinger operators, arise in the study of the damped wave equation [9, 10, 18], in the computation of resonances in quantum chemistry [32, 33, 37], as well as in the study of resonances in quantum chaos [11, 29, 30].

Suppose that we are interested in approximating the spectrum of a (linear) operator $H$ on a Hilbert space $\mathcal{H}$ with domain $D(H)$. Let $(H_n)$ be a sequence of operators on $\mathcal{H}$ whose spectra we hope will approximate the spectrum $\sigma(H)$ of $H$ as $n \to \infty$. The limiting spectrum of $(H_n)$ is defined by

$$\sigma((H_n)) = \{\lambda \in \mathbb{C} : \exists I \subset \mathbb{N} \text{ infinite}, \exists \lambda_n \in \sigma(H_n), n \in I \text{ with } \lambda_n \to \lambda\}.$$  

$(H_n)$ is said to be spectrally inclusive for $H$ in some $\Omega \subset \mathbb{C}$ if

$$\sigma(H) \cap \Omega \subset \sigma((H_n)).$$

The set of spectral pollution for $(H_n)$ with respect to $H$ is defined by

$$\sigma_{\text{poll}}((H_n)) = \{\lambda \in \sigma((H_n)) : \lambda \notin \sigma(H)\}.$$  

In order to reliably approximate the spectrum of $H$ in $\Omega \subset \mathbb{C}$ using $(H_n)$, we require that there is no spectral pollution in $\Omega$, $\sigma_{\text{poll}}((H_n)) \cap \Omega = \emptyset$, and that $(H_n)$ is spectrally inclusive for $H$ in $\Omega$. If this holds, we say that $(H_n)$ is spectrally exact for $H$ in $\Omega$.

A typical scenario in which the set of spectral pollution may be non-empty is one in which the essential spectrum $\sigma_e(H)$ of $H$ has a band-gap structure and

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the operators $H_n$ have compact resolvents. For this reason, spectral pollution often causes issues for the numerical computation of eigenvalues in spectral gaps. Various methods have been proposed to deal with such issues, we mention, for instance [5, 12, 20, 24, 25, 36]. We focus on one such method, which involves perturbing the operator of interest such as to move the spectrum, in a predictable way, away from the set of spectral pollution caused by numerical discretisation [28].

Let $T_0$ be an operator on Hilbert space $\mathcal{H}$; suppose we are interested in numerically computing the spectrum of $T_0$. Suppose that $T_0$ is self-adjoint, so that $\sigma(T_0) \subset \mathbb{R}$. Let $(s_n)$ be a sequence of bounded, $T_0$-compact, self-adjoint operators tending strongly to the identity operator on $\mathcal{H}$. If $\mathcal{H} = L^2(0, \infty)$, for instance, a typical choice for $s_n$ would be $\chi_{[0,n]}$, where $\chi$ is the characteristic function. Define the perturbed operators by

$$T_n = T_0 + i\gamma s_n \quad (n \in \mathbb{N})$$

where $\gamma > 0$. The limit operator $T$ is defined by

$$T = T_0 + i\gamma.$$  

The spectrum of $T_0$ is exactly encoded in the spectrum of $T$ since $\sigma(T) = \sigma(T_0) + i\gamma$.

Under appropriate conditions on $T_0$ and $s_n$, it can be proved that the spectrum of $T_n$, for fixed $n$, can be reliably numerically computed [1, 26, 27, 28, 35]. In particular, spectral pollution due to numerical discretisation of $T_n$ lies on the real-line, away from $\sigma(T)$, which lies along the line $i\gamma + \mathbb{R}$ (see [4] and the discussion in Section 2.2). Because of this, if $(T_n)$ can be shown to be spectrally exact for $T$ in an open neighbourhood in $\mathbb{C}$ of $i\gamma + I \subset i\gamma + \mathbb{R}$, then in principle one can reliably numerically compute the spectrum of $T_0$ in $I$. Such spectral exactness results are currently limited. We mention, however, the results [28, Theorem 10] and [21, Theorem 4.6] regarding spectral inclusion for isolated eigenvalues and the spectral enclosure result [21, Theorem 3.3].

The aim of the present article is to provide spectral inclusion and spectral pollution results for sequences of operators of the form (1.4). Our results can be summarised in a nutshell as:

(A) (Theorem 2.12) Enclosures for the spectral pollution and for the regions where spectral inclusion fails, for abstract operators of the form (1.4).

(B) (Theorems 3.6 and 3.7) A spectral inclusion result for isolated eigenvalues (with convergence rate) and a spectral pollution result, for Sturm-Liouville operators.

(C) (Theorem 4.5) A spectral inclusion result for the essential spectrum (with convergence rate), for Sturm-Liouville operators.

Our results are illustrated by numerical examples in Section 5. Note that in result (A) (Section 2) we do not require that the operators $(s_n)$ are $T_0$-compact and that in results (B) and (C) (Sections 3 and 4) we do not require that $T_0$ is self-adjoint.

1.1. Summary of Results. The definitions of the essential spectrum $\sigma_e(H)$ and the discrete spectrum $\sigma_d(H)$ for an operator $H$ are given by equations (1.7) and (1.8) below.

1.1.1. Limiting Essential Spectrum and Spectral Pollution. In Section 2, we consider a self-adjoint operator $T_0$ on Hilbert space $\mathcal{H}$. We assume that the operators $s_n$ $(n \in \mathbb{N})$ on $\mathcal{H}$ are self-adjoint, tend strongly to the identity operator as $n \to \infty$ and
are bounded independently of \( n \). For \( \gamma > 0 \), we define the perturbed operators \( T_n \) (\( n \in \mathbb{N} \)) by (1.4) and the limit operator \( T \) by (1.5).

The main tool in this section is the notion of *limiting essential spectrum* \( \sigma_e((T_n)) \) (see Definition 2.1), introduced by Bögli (2018) [3]. The results of [3] show that (Corollary 2.7)

\[
(T_n) \text{ is spectrally exact for } T \in \mathbb{C}\setminus[\sigma_e((T_n)) \cup \sigma_e((T_n^*)^\ast) \cup \sigma_e(T)].
\]

The *limiting essential numerical range* \( W_e((T_n)) \) of \( (T_n) \) (see Definition 2.5), introduced by Bögli, Marletta and Tretter (2020) is a convex set which, in our set-up, satisfies (Propositions 2.6 and 2.9)

\[
\sigma_e((T_n)) \cup \sigma_e((T_n^*)^\ast) \subset W_e((T_n)) \subset \text{conv}(\sigma_e(T_0)) \setminus \{\pm \infty\} \times i\gamma[s_-, s_+]
\]

where \( \sigma_e(T_0) \) denotes the extended essential spectrum of \( T_0 \) (see Definition 2.8) and \( s_\pm \in \mathbb{R} \) (defined by (2.7)) satisfy \( s_- - \varepsilon \leq s_n \leq s_+ + \varepsilon \) for all \( \varepsilon > 0 \) and all large enough \( n \).

The main results of Section 2 are non-convex enclosures for \( \sigma_e((T_n)) \) improving the enclosure provided by \( W_e((T_n)) \).

(A) (Theorem 2.12) If \( s_n \) is a projection operator for all \( n \), that is \( s_n^2 = s_n \), then \( \sigma_e((T_n)) \cup \sigma_e((T_n^*)^\ast) \subset \Gamma_a = \Gamma_a(\sigma_e(T_0), \gamma) \), where

\[
\Gamma_a := \{ \lambda \in \mathbb{C} : \Im(\lambda) \in [0, \gamma] \text{, \ dist}(\Re(\lambda), \sigma_e(T_0)) \leq \sqrt{\Im(\lambda)(\gamma - \Im(\lambda))} \}.
\]

If for any sequence \( (u_n) \subset D(T_0) \) bounded in \( \mathcal{H} \) with \( (T_0u_n) \) bounded in \( \mathcal{H} \) we have

\[
\langle s_n u_n, T_0 u_n \rangle - \langle T_0u_n, s_n u_n \rangle \to 0 \text{ as } n \to \infty
\]

(Assumption 1) then \( \sigma_e((T_n)) \cup \sigma_e((T_n^*)^\ast) \subset \Gamma_b = \Gamma_b(\sigma_e(T_0), \gamma, s_\pm) \), where

\[
\Gamma_b := \sigma_e(T_0) \times i\gamma[s_-, s_+].
\]

In particular, if \( s_n \) are projection operators or if Assumption 1 is satisfied then

\( (T_n) \) is spectrally exact for \( T \) in some open neighbourhood of any \( \lambda \in \sigma_d(T) \).

We clarify that by open neighbourhood we mean open neighbourhood in \( \mathbb{C} \). The enclosures \( \Gamma_a \) and \( \Gamma_b \) are illustrated in Figure 1. Assumption 1 is verified for a class of perturbations for Schrödinger operators on Euclidean domains. The proof of Theorem 2.12 uses an adaptation (Lemma 2.11) of a result of Kato (1949) [22, Lemma 1].

1.1.2. Second Order Operators on the Half-Line. In Section 3, we consider the consider a case in which \( T_0 \) is a Sturm-Liouville operator on \( L^2(0, \infty) \) and provide a more precise analysis compared to Section 2. The Sturm-Liouville operator \( T_0 \) is allowed to have complex coefficients and is endowed with a complex mixed boundary condition at 0.

We assume that for any \( \lambda \in \mathbb{C}\setminus\sigma_e(T_0) \), the solution space of the equation \( \tilde{T}_0 u = \lambda u \) (here, \( \tilde{T}_0 \) is the differential expression corresponding to \( T_0 \)) is spanned by solutions \( \psi_\pm(\cdot, \lambda) \) admitting the decomposition

\[
\psi_\pm(x, \lambda) = e^{\pm ik(\lambda)x} \tilde{\psi}_\pm(x, \lambda).
\]

Here, \( k \) and \( \tilde{\psi}_\pm(\cdot, \lambda) \) are analytic functions on \( \mathbb{C}\setminus\sigma_e(T_0) \) with \( \Im k > 0 \) and with \( \tilde{\psi}_\pm(\cdot, \lambda) \) bounded. A similar decomposition is required for \( \psi'_\pm \) - see Assumption 2 for the precise statement.
The perturbed operators in Section 3 are defined by

\[(1.6) \quad T_R = T_0 + i\gamma \chi_{[0,R]} \quad (R \in \mathbb{R}_+)\]

where \(\gamma \in \mathbb{C} \setminus \{0\}\). The limit operator \(T\) is defined by the expression (1.5). Under these assumptions, for any \((R_n)\) with \(R_n \to \infty\), we construct a set \(S_p((R_n))\) (equation (3.21)) and prove the following:

(B) (Theorems 3.6 and 3.7) For any eigenvalue \(\lambda \in \sigma_d(T) \setminus \sigma_e(T_0)\) of \(T\) with \(\lambda \notin S_p((R_n))\), there exists eigenvalues \(\lambda_n\) of \(T_{R_n}\) \((n \in \mathbb{N})\) such that

\[|\lambda - \lambda_n| = O(e^{-\beta R_n}) \quad \text{as} \quad n \to \infty\]

for some \(\beta > 0\) independent of \(n\). Furthermore, the set of spectral pollution for \((T_{R_n})\) with respect to \(T\) satisfies

\[\sigma_{\text{poli}}((T_{R_n})) \subset \sigma_e(T_0) \cup S_p((R_n)).\]

The proofs utilise Rouché’s theorem applied to an analytic function (Lemma 3.4) whose zeros are the eigenvalues of \(T_R\). (B) implies that \((T_{R_n})\) is spectrally exact for \(T\) in \(\mathbb{C} \setminus (\sigma_e(T_0) \cup \sigma_e(T) \cup S_p((R_n)))\).

Assumption 2 is verified in two cases:

- (Examples 3.2 and 3.8) \(T_0\) is a Schrödinger equation with an \(L^1\) potential. In this case, \(S_p((R_n)) = \emptyset\).
- (Examples 3.3 and 3.9) \(T_0\) is a Schrödinger operator with an eventually real \(a\)-periodic potential, \(\gamma > 0\) and \(R_n - R_{n-1} = a\) for all \(n\). In this case, \(S_p((R_n))\) is expressed as the zeros of a certain analytic function (equation (3.27)).

1.1.3. Inclusion for the Essential Spectrum. In Section 4, we let \(T_0\) be a Sturm-Liouville operator satisfying Assumption 2, as described above. In addition, we require that \(\sigma_e(T_0) \subset \mathbb{R}\) and that \(k\) and \(\psi_\pm(x,\cdot)\), hence the solutions \(\psi_\pm(x,\cdot)\), admit analytic continuations into an open neighbourhood of any point in the interior of \(\sigma_e(T_0)\). See Assumption 3 for the precise statement.

The perturbed operators \(T_R\) and the limit operator \(T\) in Section 4 are defined by (1.6) and (1.5) respectively, as in Section 3. We construct a set \(S_T \subset i\gamma + \mathbb{R}\) (equation (4.6)) and prove that:

(C) (Theorem 4.5) For any \(\mu\) in the interior of \(\sigma_e(T_0)\) with \(\mu + i\gamma \notin S_T\), there exists eigenvalues \(\lambda_R\) of \(T_R\) \((R \in \mathbb{R}_+)\) such that

\[|\lambda_R - (\mu + i\gamma)| = O\left(\frac{1}{R}\right) \quad \text{as} \quad R \to \infty.\]

The proof utilises Rouché’s theorem applied to an analytic function (Lemma 4.4) whose zeros are the eigenvalues of \(T_R\). In the case that

- (Examples 4.2 and 4.6) \(T_0\) is a Schrödinger operator with an integrable potential satisfying the Naimark condition or a dilation analyticity condition, or,
- (Examples 4.3 and 4.7) \(\gamma > 0\) and \(T_0\) is a Schrödinger operator with a real, eventually periodic potential, endowed with a real mixed boundary condition at 0,
it is proven that Assumption 3 is satisfied and that
\[ \mu + i\gamma \in S_r \text{ if and only if } \mu \text{ is a resonance of } T_0 \text{ embedded in } \sigma_e(T_0). \]

See equation (4.20) for the precise definition of a resonance used here. For these cases, since resonances on the real line form a set of measure zero, we can combine Theorem 4.5 with Theorem 3.7 and the characterisation of \( S_p((R_n)) \) to conclude that

\( (T_n) \) is spectrally exact for \( T \) in some open neighbourhood of any \( \mu \in \text{int}(\sigma_e(T)) \).

1.2. Notation and Conventions. Let \( H \) be a separable Hilbert space with corresponding inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( B_n \xrightarrow{s} B \) as \( n \to \infty \) denote strong convergence in \( H \) for bounded operators \( B_n \) and \( B \) on \( H \). Let \( f_n \xrightarrow{w} f \) as \( n \to \infty \) denote weak convergence in \( H \) for \( f_n, f \in H \). In this article, we define the essential spectrum of an operator \( H \) on \( H \) as

\[
\sigma_e(H) = \left\{ \lambda \in \mathbb{C} : \exists (u_n) \subset D(H) \text{ with } \|u_n\| = 1, u_n \to 0, \| (H - \lambda)u_n \| \to 0 \right\}
\]

which corresponds to \( \sigma_{e2} \) in [16]. The sequence \( (u_n) \) appearing in (1.7) is referred to as a singular sequence. The discrete spectrum is defined by

\[
\sigma_d(H) = \sigma(H) \setminus \sigma_e(H).
\]

The convention we take with regards to the square-root function is to make the branch-cut along the positive semi-axis, so that \( \Im \sqrt{z} \geq 0 \) for all \( z \in \mathbb{C} \). We let \( B_r(z) \) denote an open ball of radius \( r > 0 \) around a point \( z \in \mathbb{C} \). In Sections 3 and 4, \( \psi'(x,z) := \frac{d}{dx} \psi(x,z) \).

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2. Limiting Essential Spectrum and Spectral Pollution

In this section, we study spectral exactness for sequences of abstract operators of the form (1.4). In Section 2.1, we briefly review the notions of limiting essential spectrum and essential numerical range. We refer the reader to [3] and [4] for a more detailed exposition. In Section 2.2, we discuss the application of limiting essential spectrum and essential numerical range to operators of the form (1.4). In Section 2.3, we prove enclosures for the limiting essential spectrum.

2.1. Limiting Essential Spectrum and Numerical Range. Let \( \mathcal{H}_n \subset \mathcal{H} \) (\( n \in \mathbb{N} \)) be closed subspaces and let \( P_n : \mathcal{H} \to \mathcal{H}_n \) be the corresponding orthogonal projections. Assume that \( P_n \xrightarrow{s} I \). Let \( H \) and \( H_n \) (\( n \in \mathbb{N} \)) be closed, densely-defined operators acting on \( \mathcal{H} \) and \( \mathcal{H}_n \) respectively.

Definition 2.1. The limiting essential spectrum of \( (H_n) \) is defined by

\[
\sigma_e((H_n)) = \left\{ \lambda \in \mathbb{C} : \exists J \subset \mathbb{N} \text{ infinite, } \exists u_n \in D(H_n), n \in J \text{ with } \|u_n\| = 1, u_n \to 0, \| (H_n - \lambda)u_n \| \to 0 \right\}.
\]
Definition 2.2. \((H_n)\) converges to \(H\) in the \textit{generalised strong resolvent sense}, denoted by \(H_n \xrightarrow{\text{gsr}} H\), if
\[
\exists n_0 \in \mathbb{N} : \exists \lambda_0 \in \bigcap_{n \geq n_0} \rho(H_n) \cap \rho(H) : (H_n - \lambda_0)^{-1} P_n \xrightarrow{\sigma} (H - \lambda_0)^{-1}.
\]

In the case that \(H_n = H\) for all \(n\), generalised resolvent convergence is equivalent to strong resolvent convergence and denoted by \(H_n \xrightarrow{\text{sr}} H\).

Theorem 2.3 \([3, \text{Theorem 2.3}]\). If \(H_n \xrightarrow{\text{gsr}} H\) and \(H_n^* \xrightarrow{\text{gsr}} H^*\) then
\[
(2.3)
\sigma_{\text{poll}}((H_n)) \subset \sigma_e((H_n)) \cup \sigma_e((H_n^*))^*.
\]
and every isolated \(\lambda \in \sigma(H)\) outside \(\sigma_e((H_n)) \cup \sigma_e((H_n^*))^*\) is approximated by \((H_n)\), that is,
\[
\{\lambda \in \sigma(H) : \lambda \text{ isolated, } \lambda \notin \sigma_e((H_n)) \cup \sigma_e((H_n^*))^*\} \subset \sigma((H_n)).
\]

Definition 2.4. The \textit{essential numerical range} of \(H\) is defined by
\[
W_e(H) = \{\lambda \in \mathbb{C} : \exists (u_n) \subset D(H) \text{ with } \|u_n\| = 1, u_n \to 0, \langle Hu_n, u_n \rangle \to \lambda\}.
\]

Definition 2.5. The \textit{limiting essential numerical range} of \((H_n)\) is defined by
\[
W_e((H_n)) = \left\{\lambda \in \mathbb{C} : \exists I \subset \mathbb{N} \text{ infinite, } \exists u_n \in D(H_n), n \in I \text{ with } \|u_n\| = 1, u_n \to 0, \langle (H_n - \lambda)u_n, u_n \rangle \to 0\right\}.
\]

Proposition 2.6 \([4, \text{Proposition 5.6}]\). The \textit{limiting essential numerical range} of \((H_n)\) is closed and convex with
\[
\text{conv}(\sigma_e((T_n))) \subset W_e((H_n)).
\]
Furthermore, if \(D(H_n) \cap D(H_n^*)\) is a core of \(H_n^*\) for all \(n\) then
\[
\text{conv}(\sigma_e((T_n)) \cup \sigma_e((T_n^*))^*) \subset W_e((H_n)).
\]

2.2. Enclosures for the Limiting Essential Spectrum. Throughout the remainder of the section, let \(T_0\) and \(s_n (n \in \mathbb{N})\) be self-adjoint operators on \(\mathcal{H}\). Let \(\gamma > 0\) and define the perturbed operators, as in the introduction, by
\[
T_n = T_0 + i\gamma s_n, \quad (n \in \mathbb{N})
\]
Assume that \(s_n \xrightarrow{\sigma} I\) and that \(\|s_n\| \leq C\) for some \(C > 0\) independent of \(n\). Define the limit operator by \(T = T_0 + i\gamma\) as in the introduction - \(T_n\) converges strongly to \(T\).

Corollary 2.7. \((T_n)\) is spectrally exact for \(T\) in \(\mathbb{C}\setminus[\sigma_e((T_n)) \cup \sigma_e((T_n^*))^*] \cup \sigma_e(T)\)

Proof. The fact that \(T_n \xrightarrow{\sigma} T\) and
\[
T_n^* = T_0 - i\gamma s_n \xrightarrow{\sigma} T_0 - i\gamma = T^*
\]
follows from an application of the resolvent identity, using \(s_n \xrightarrow{\sigma} I\), the self-adjointness of \(T_0\) and the uniform boundedness of the sequence of operators \((s_n)\). By Theorem 2.3, \(\sigma_{\text{poll}}((T_n)) \subset \sigma_e((T_n)) \cup \sigma_e((T_n^*))^*\) and
\[
\{\lambda \in \sigma(T) : \lambda \text{ isolated, } \lambda \notin \sigma_e((T_n)) \cup \sigma_e((T_n^*))^*\} \subset \sigma((T_n)).
\]
The corollary follows from the fact that every element of \(\sigma_d(T) = \sigma_d(T_0) + i\gamma\) is isolated since \(T_0\) is self-adjoint \([16]\). \qed
In this section, we do not require that $T_0$ is semi-bounded or that $s_n$ are $T_0$-compact. However, in applications to the numerical computation of eigenvalues in spectral gaps, we typically require that at least the latter condition holds. If both conditions hold, then the spectral pollution due to the numerical discretisation of $T_n$, for fixed $n$, lies on the real-line. This can be proved using the notion of essential numerical range.

If $T_0$ is semi-bounded and $s_n$ is $T_0$-compact for all $n$, then $W_e(T_n) = W_e(T_0) \subset \mathbb{R}$ for any $n$ [4, Theorem 4.5]. The spectral pollution due to projection or domain truncation methods is contained inside the essential numerical range hence the real line [4, Theorems 6.1 and 7.1]. In addition, any eigenvalue of $T_n$ off the real line, hence away from the essential numerical range, is approximated.

If $s_n$ are $T_0$-compact then, by Weyl’s theorem, we have

\begin{equation}
\forall n \in \mathbb{N} : \sigma_e(T_0) = \sigma_e(T_n) \subset \sigma(T_n).
\end{equation}

Therefore, since $\sigma_e(T_0) \cap \sigma(T) = \emptyset$, there will be a contribution by $\sigma_e(T_0)$ to the spectral pollution for $(T_n)$ with respect to $T$,

\begin{equation}
\sigma_e(T_0) \subset \sigma_{\text{poll}}((T_n)).
\end{equation}

This is in stark contrast to the examples of spectral pollution that are typically studied, due to Galerkin, projection or domain truncation approximations, where spectral pollution is caused by spurious eigenvalues.

Since $D(T_n) = D(T_n^*) = D(T_0)$, Proposition 2.6 implies that the quantity $\sigma_e((T_n)) \cup \sigma_e((T_n^*))^*$ is contained in the limiting essential numerical range $W_e((T_n))$ and so $(T_n)$ is spectrally exact for $T$ in $\mathbb{C} \setminus [W_e((T_n)) \cup \sigma_e(T)]$. The limiting essential numerical range is typically easier to study than the limiting essential spectrum. For sequences of operators of the form (2.4), the limiting essential numerical range $W_e((T_n))$ is contained in a strip. To state and prove this fact, we shall require the notion of extended essential spectrum.

**Definition 2.8.** The extended essential spectrum $\hat{\sigma}_e(H) \subset \sigma_e(H) \cup \{\pm \infty\}$ of a self-adjoint operator $H$ on $\mathcal{H}$ is defined as the union of $\sigma_e(H)$ with $+\infty$ and/or $-\infty$ if $H$ is unbounded from above and/or below respectively.

Throughout the remainder of the section, let

\begin{equation}
s_- := \liminf_{n \to \infty} \inf_{u \in \mathcal{H}, \|u\|=1} \langle s_n u, u \rangle \quad \text{and} \quad s_+ := \limsup_{n \to \infty} \sup_{u \in \mathcal{H}, \|u\|=1} \langle s_n u, u \rangle.
\end{equation}

Then, for any $\varepsilon > 0$, $s_- - \varepsilon \leq s_n \leq s_+ + \varepsilon$ for large enough $n$.

**Proposition 2.9.** $W_e((T_n)) \subset \text{conv}(\hat{\sigma}_e(T_0)) \setminus \{\pm \infty\} \times i[\varepsilon, \varepsilon]$

**Proof.** Let $\lambda \in W_e((T_n))$. Then there exists $(u_n) \subset D(T_0)$ such that $\|u_n\| = 1$ for all $n$, $u_n \to 0$ and $(T_n - \lambda)u_n, u_n \to 0$. Taking the real part of the inner product, we have $(\langle T_0 - \Re(\lambda) \rangle)u_n, u_n \to 0$ which implies that

$\Re(\lambda) \in W_e(T_0) = \text{conv}(\hat{\sigma}_e(T_0)) \setminus \{\pm \infty\}$

where we used [4, Theorem 3.8]. Also, $\Im((T_n - \lambda)u_n, u_n) \to 0$ so

$\Im(\lambda) = \gamma \langle s_n u_n, u_n \rangle + o(1) \in \varepsilon[s_-, s_+]$.

□

The next example shows that the inclusion in Proposition 2.9 is sharp.
Example 2.10. Suppose that $T_0 = -\Delta + q$ is a self-adjoint Schrödinger operator on $\mathcal{H} = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is some open set and $q$ is a real function on $\Omega$. Assume that $T_0$ is endowed with Dirichlet or Neumann boundary conditions on $\partial \Omega$ and that $q$ is bounded below. For each $R \in \mathbb{R}_+$, let $s_R$ be a bounded multiplication operator on $L^2(\Omega)$ defined by

$$(s_R u)(x) = \chi_{\Omega \cap B_R(0)}(x)u(x). \quad (u \in L^2(\Omega), x \in \Omega)$$

There exists $(R_n) \subset \mathbb{R}_+$ with $R_n \to \infty$, such that

$$W_e((T_{R_n})) = [\text{conv}(\hat{\sigma}_e(T_0)) \setminus \{\pm \infty\}] \times i[0, \gamma]$$

$$= [\inf \sigma_e(T_0), \infty) \times i[0, \gamma],$$

where $T_R := T_0 + i\gamma s_R$.

Proof. By Proposition 2.9, it suffices to show that there exists $(R_n) \subset \mathbb{R}_+$ such that $R_n \to \infty$ and

$$[\text{conv}(\hat{\sigma}_e(T_0)) \setminus \{\pm \infty\}] \times i[0, \gamma] \subset W_e((T_{R_n})).$$

Let $\lambda \in \mathbb{C}$ be such that

$$\Re(\lambda) \in \text{conv}(\hat{\sigma}_e(T_0)) \setminus \{\pm \infty\} = W_e(T_0)$$

and $\Im(\lambda) \in [0, \gamma]$. Then there exists $(u_n) \subset D(T_0)$ with $\|u_n\| = 1$, $u_n \to 0$ such that $\langle (T_0 - \Re(\lambda))u_n, u_n \rangle \to 0$. For each $n \in \mathbb{N}$, the map

$$f_n : [0, \infty) \to [0, \gamma]; R \mapsto \gamma \langle s_R u_n, u_n \rangle$$

is continuous and monotonically increasing with $f_n(0) = 0$ and $f_n(R) \to \gamma$ as $R \to \infty$. Consequently, we can choose $R_n$ such that $f_n(R_n) = \Im(\lambda)$ for all $n$. Then,

$$\langle (T_{R_n} - \lambda)u_n, u_n \rangle = \langle (T_0 - \Re(\lambda))u_n, u_n \rangle \to 0$$

so $\lambda \in W_e((T_{R_n}))$.

It remains to show that $R_n \to \infty$ necessarily. Suppose for contradiction that there exists $R > 0$ independent of $n$ such that $R_n \leq R$. Since the boundary conditions for $T_0$ are Dirichlet or Neumann, an integration by parts yields

$$\langle T_0 u_n, u_n \rangle = \int_\Omega |
abla u_n|^2 + \int_\Omega q|u_n|^2 = \Re(\lambda) + o(1) \text{ as } n \to \infty. \quad (2.8)$$

Since $q$ is bounded below, $(||\nabla u_n||)$ is bounded hence so is $(||u_n||_{H^1(\Omega \cap B_R(0))})$. By the Rellich–Kondrachov theorem, $H^1(\Omega \cap B_R(0))$ is compactly embedded in $L^2(\Omega \cap B_R(0))$ so there exists a subsequence $(u_{n_k})$ converging strongly in $L^2(\Omega \cap B_R(0))$. Since $u_n \to 0$ this subsequence converges strongly to zero and so

$$\frac{\Im(\lambda)}{\gamma} \leq \langle s_{R_{n_k}} u_{n_k}, u_{n_k} \rangle \leq \langle s_R u_{n_k}, u_{n_k} \rangle \leq \|s_R u_{n_k}\| = \|u_{n_k}\|_{L^2(\Omega \cap B_R(0))} \to 0$$

which is the desired contradiction since $\Im(\lambda) \neq 0$ in general.

\[ \square \]
2.3. **Main Abstract Results.** In the main result of this section, Theorem 2.12, we shall prove enclosures for the limiting essential spectrum that offer improvements over the enclosure provided by the limiting essential numerical range. Unlike the limiting essential numerical range, these enclosures are not convex in general. An interesting feature of the enclosure $\Gamma_a$ of Theorem 2.12 (a) is that it is independent of the perturbing operators $(s_n)$, depending only on $\sigma_e(T_0)$ and $\gamma$. The enclosures are illustrated in Figure 1.

We will require an adaptation of [22, Lemma 1]. A corollary of [22, Lemma 1] is that if $H$ is a self-adjoint operator on $\mathcal{H}$ and there exists $u \in D(H)$ with $\|u\| = 1$ such that $\langle Hu, u \rangle = \eta$ and $\|(H - \eta)u\| = \varepsilon$ for some $\eta \in \mathbb{R}$ and $\varepsilon > 0$, then $\text{dist}(\eta, \sigma(H)) \leq \varepsilon$. Our adaptation allows one to get an estimate for the distance of a point to the essential spectrum of $H$, provided a certain sequence in $D(H)$ is given.

**Lemma 2.11.** Let $H$ be a self-adjoint operator on $\mathcal{H}$. Suppose that there exists a sequence $(u_n) \subset D(H)$ with $\|u_n\| = 1$ for all $n$ and $u_n \to 0$ such that
\[
\langle Hu_n, u_n \rangle \to \eta \quad \text{and} \quad \|(H - \eta)u_n\| \to \varepsilon
\]
for some $\eta \in \mathbb{R}$ and $\varepsilon > 0$. Then,
\[
\text{dist}(\eta, \sigma_e(H)) \leq \varepsilon.
\]

**Proof.** Suppose for contradiction that
\[
[\eta - \varepsilon, \eta + \varepsilon] \cap \sigma_e(H) = \emptyset.
\]
Then,
\[
[\eta - \varepsilon, \eta + \varepsilon] \cap \sigma(H) = \{\lambda_1, \ldots, \lambda_m\} =: I
\]
where $\lambda_1, \ldots, \lambda_m$ are eigenvalues of $H$ of finite multiplicity and $m \geq 0$ is an integer. For any $\lambda \in \sigma(H) \setminus I$,
\[
(\lambda - (\eta - \varepsilon))(\lambda - (\eta + \varepsilon)) \geq C
\]
for some $C > 0$ independent of $n$, hence, expanding the polynomial,

$$
(2.10) \quad \lambda^2 - 2\eta\lambda + \eta^2 - \varepsilon^2 \geq C.
$$

Let $E(\lambda)$ denote the resolution of identity corresponding to the self-adjoint operator $H$. Integrating both sides of the inequality (2.10) by the measure $d\langle E(\lambda)u_n, u_n\rangle$, for any $n \in \mathbb{N}$ we have

$$
(2.11) \quad \int_{\sigma(H) \setminus I} \lambda^2 d\langle E(\lambda)u_n, u_n\rangle - 2\eta \int_{\sigma(H) \setminus I} \lambda d\langle E(\lambda)u_n, u_n\rangle + (\eta^2 - \varepsilon^2 - C) \int_{\sigma(H) \setminus I} d\langle E(\lambda)u_n, u_n\rangle \geq 0
$$

For all $j \in \{1, \ldots, m\}$, $\lambda_j$ is of finite multiplicity so $\langle E(\lambda_j) - E(\lambda_j - 0)\rangle u_n = 0$ for all large enough $n$. Consequently,

$$
\lim_{n \to \infty} \int_{\sigma(H) \setminus I} \lambda^2 d\langle E(\lambda)u_n, u_n\rangle = \lim_{n \to \infty} \int_{\sigma(H)} \lambda^2 d\langle E(\lambda)u_n, u_n\rangle = \lim_{n \to \infty} \|Hu_n\|^2 = \eta^2 + \varepsilon^2.
$$

Here, we used (2.9) and the fact that

$$
\|(H - \eta)u_n\|^2 = \|Hu_n\|^2 - 2\langle Hu_n, u_n\rangle \eta + \eta^2 = \|Hu_n\|^2 - \eta^2 + o(1)
$$

as $n \to \infty$. Similarly,

$$
\lim_{n \to \infty} \int_{\sigma(H) \setminus I} \lambda d\langle E(\lambda)u_n, u_n\rangle = \eta \quad \text{and} \quad \lim_{n \to \infty} \int_{\sigma(H) \setminus I} d\langle E(\lambda)u_n, u_n\rangle = 1.
$$

Therefore, taking the limit $n \to \infty$ in (2.11) yields

$$
(2.12) \quad (\eta^2 + \varepsilon^2) - 2\eta^2 + (\eta^2 - \varepsilon^2) \geq C
$$

which is the desired contradiction since the left hand side of (2.12) is zero.

Finally, we prove enclosures for the limiting essential spectrum of the sequences of operators $(T_n)$ defined by (2.4). We shall require additional assumptions on the perturbing operators $(s_n)$.

In part (a) of the theorem, we simply require that $s_n$ are projection operators. An indicative example for this case in the setting $\mathcal{H} = L^2(\mathcal{M})$, where $\mathcal{M}$ is a Riemannian manifold, is $s_n := \chi_{f^{-1}(-\infty, n)}$, where $f$ is a Morse function on $\mathcal{M}$.

The hypothesis for part (b) of the theorem, Assumption 1, is given below. An example of a class of perturbations for Schrödinger operators satisfying this assumption is provided in Example 2.14. We expect that this assumption can be verified in many other interesting settings, provided the perturbations $(s_n)$ are properly constructed.

**Assumption 1.** If $(u_n) \subset D(T_0)$ is bounded in $\mathcal{H}$ with $(T_0 u_n)$ is bounded in $\mathcal{H}$ then

$$
\langle s_n u_n, T_0 u_n \rangle - \langle T_0 u_n, s_n u_n \rangle \to 0 \text{ as } n \to \infty.
$$

**Theorem 2.12.** (a) Assume that $s_n$ is a projection operator for all $n$, that is, $s_n^2 = s_n$. Then $\sigma_e(T_n) \cup \sigma_e(T_n)^* \subset \Gamma_a = \Gamma_a(\sigma_e(T_0), \gamma)$ where

$$
(2.13) \quad \Gamma_a := \left\{ \lambda \in \mathbb{C} : \Im(\lambda) \in [0, \gamma], \dist(\Re(\lambda), \sigma_e(T_0)) \leq \sqrt{\Im(\lambda)(\gamma - \Im(\lambda))} \right\},
$$
Suppose that Assumption 1 holds. Then
\[ \sigma_e((T_n)) \cup \sigma_e((T_n^*)) \subset \Gamma_b = \Gamma_b(\sigma_e(T_0), \gamma, s_{\pm}) \]
where
\[ (2.14) \quad \Gamma_b := \sigma_e(T_0) \times i\gamma [s_-, s_+] \].

Proof. We will only prove that \( \sigma_e((T_n)) \subset \Gamma_a, \Gamma_b \) - the proof that \( \sigma_e((T_n^*)) \subset \Gamma_a, \Gamma_b \) is similar since \( T_n^* = T_0 - i\gamma s_n \).

Let \( \lambda \in \sigma_e((T_n)) \). Then there exists \( (u_n) \subset D(T_0) \) with \( ||u_n|| = 1 \), \( u_n \to 0 \) and \( ||(T_n - \lambda)u_n|| = o(1) \).

By Cauchy-Schwarz,
\[ (2.15) \quad \langle (T_n - \lambda)u_n, u_n \rangle = o(1). \]
Taking the real part of (2.15) yields
\[ (2.16) \quad (T_0u_n, u_n) = \Re(\lambda) + o(1). \]
Taking the imaginary part of (2.15) yields
\[ (2.17) \quad \gamma \langle s_n u_n, u_n \rangle = \Im(\lambda) + o(1). \]

Since \( (||s_n u_n||) \) is bounded, \( (||T_0 u_n||) \) is bounded, hence by Cauchy-Schwarz,
\[ (2.18) \quad \langle (T_n - \lambda)u_n, T_0 u_n \rangle = o(1). \]
Taking the real part of (2.18),
\[ (2.19) \quad ||T_0 u_n||^2 - \Re(\lambda) \langle T_0 u_n, u_n \rangle - \gamma \Im\langle s_n u_n, T_0 u_n \rangle = o(1). \]
By (2.16),
\[ \langle (T_0 - \Re(\lambda))u_n \rangle^2 = ||T_0 u_n||^2 - \Re(\lambda) \langle T_0 u_n, u_n \rangle + o(1), \]
hence (2.19) yields,
\[ (2.20) \quad ||(T_0 - \Re(\lambda))u_n||^2 = \gamma \Im\langle s_n u_n, T_0 u_n \rangle + o(1). \]
(a) In this case, \( \sigma(s_n) = \{0, 1\} \) so \( 0 \leq s_n \leq 1 \) for all \( n \), and so by (2.17),
\[ (2.21) \quad \forall n \in \mathbb{N} : \langle s_n u_n, u_n \rangle \in [0, 1] \quad \Rightarrow \quad \Im(\lambda) \in [0, \gamma]. \]

By Cauchy-Schwarz,
\[ (2.22) \quad \langle (T_n - \lambda)u_n, s_n u_n \rangle = o(1). \]
Taking the imaginary part of (2.22) and using the hypothesis \( s_n^2 = s_n \) as well as (2.17),
\[ \Im\langle s_n u_n, T_0 u_n \rangle = \gamma ||s_n u_n||^2 - \Im(\lambda) \langle s_n u_n, u_n \rangle + o(1) \]
\[ = (\gamma - \Im(\lambda)) \langle s_n u_n, u_n \rangle + o(1) \]
\[ = (\gamma - \Im(\lambda)) \frac{\Im(\lambda)}{\gamma} + o(1). \]
Combining (2.20) and (2.23), we have
\[ (2.24) \quad ||(T_0 - \Re(\lambda))u_n|| = \sqrt{(\gamma - \Im(\lambda)) \Im(\lambda) + o(1)}. \]
Using Lemma 2.11, (2.16) and (2.24) imply
\[ \text{dist}(\Re(\lambda), \sigma_e(T_0)) \leq \sqrt{\Im(\lambda)(\gamma - \Im(\lambda))} \]
as required for this case.
(b) In this case, Assumption 1 (i) and a similar reasoning as in (2.21) implies that
\[ \Im(s_n u_n, T_0 u_n) = o(1) \Rightarrow \|(T_0 - \Re(\lambda))u_n\| = o(1) \]
so \((u_n)\) is a singular sequence proving that \(\Re(\lambda) \in \sigma_c(T_0)\), as required.  
\[
\square
\]

Remark 2.13. It is interesting to note that Lemma 2.11 is not required in case (b)
of Theorem 2.12. This is because Assumption 1 ensures that the following holds:
\[ (u_n) \subset D(T_n) = D(T_0), \|u_n\| = 1, u_n \to 0, \|(T_n - \lambda)u_n\| \to 0 \]
\[ \Rightarrow (u_n) \subset D(T_0), \|u_n\| = 1, u_n \to 0, \|(T_0 - \Re(\lambda))u_n\| \to 0, \]
that is, if \((u_n)\) is a singular-type sequence for a point \(\lambda\) in the limiting essential
spectrum then \((u_n)\) is also a singular sequence for \(\Re(\lambda) \in \sigma_c(T_0)\).

Example 2.14. Suppose that \(T_0 = -\Delta + q\) is the Schrödinger operator on \(\mathcal{H} = L^2(\Omega)\) introduced in Example 2.10. Let \(\varphi \in W^{1,\infty}(0, \infty)\) be such that \(\varphi(0) = 1\) and let \((R_n) \subset \mathbb{R}_+\) be any sequence such that \(R_n \to \infty\). For any \(n \in \mathbb{N}\), define
multiplication operator \(s_n\) on \(L^2(\Omega)\) by
\[
(2.25) \quad (s_n u)(x) = \varphi \left( \frac{\langle x \rangle}{R_n} \right) u(x) \quad (u \in L^2(\Omega), x \in \Omega)
\]
where \(\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}\). Then \(s_n\) is uniformly bounded, \(s_n \xrightarrow{\ast} I\) and Assumption
1 is satisfied.

Proof. Define \(\varphi_n : \Omega \to \mathbb{R}\) by
\[
\varphi_n(x) = \varphi \left( \frac{\langle x \rangle}{R_n} \right), \quad (x \in \Omega)
\]
Step 1 (Uniform boundedness). The uniform boundedness of the sequence of operators \((s_n)\)
follows from the fact that, for all \(u \in L^2(\Omega)\) and all \(n \in \mathbb{N}\),
\[
\langle s_n u, u \rangle = \int_{\Omega} \varphi_n(x)|u|^2 \leq \text{ess sup}_{t \in (0, \infty)} \varphi(t)\|u\|^2
\]
and, similarly,
\[
\langle s_n u, u \rangle \geq \text{ess inf}_{t \in (0, \infty)} \varphi(t)\|u\|^2.
\]
Step 2 \((s_n \xrightarrow{\ast} I)\). Let \(u \in L^2(\Omega)\) and let \((X_n) \subset \mathbb{R}_+\) be any sequence such that
\(X_n \to \infty\) and \(X_n = o(R_n)\). For any \(n \in \mathbb{N}\),
\[
(2.26) \quad \|(s_n - I)u\| \leq \|\varphi(\langle \cdot \rangle/R_n)-1\|_{L^\infty(\Omega \cap B_{X_n}(0))}\|u\|+(\|s_n\|+1)\|u\|_{L^2(\Omega \setminus B_{X_n}(0))}.
\]
By Morrey’s inequality, \(\varphi\) is continuous, so, since \(\varphi(0) = 1\), the first term on the
right hand side of (2.26) tends to zero as \(n \to \infty\). The second term tends to zero
because \(u \in L^2(\Omega)\) and \(\|s_n\|\) is bounded.
Step 3 (Assumption 1). Let \((u_n) \subset D(T_0)\) be any sequence which is bounded in \(H\) such that \((T_0 u_n)\) is bounded in \(H\). Then,

\[
\langle s_n u_n, T_0 u_n \rangle - \langle T_0 u_n, s_n u_n \rangle = - \int_\Omega \varphi_n u_n \Delta(\varphi_n) + \int_\Omega \varphi_n \overline{\varphi_n} \Delta(u_n)
\]

\[
= \int_\Omega \nabla (\varphi_n u_n) \cdot \nabla (\varphi_n) - \int_\Omega \nabla (\varphi_n \overline{\varphi_n}) \cdot \nabla (u_n)
\]

\[
= \int_\Omega u_n \nabla (\varphi_n) \cdot \nabla (\varphi_n) - \int_\Omega \overline{\varphi_n} \nabla (\varphi_n) \cdot \nabla (u_n).
\]

Note that the second equality holds by an integration by parts since \(T_0\) is endowed with Dirichlet or Neumann boundary conditions. Hence we have,

\[
(2.27) \quad \|\langle s_n u_n, T_0 u_n \rangle - \langle T_0 u_n, s_n u_n \rangle\| \leq 2\|\nabla \varphi_n\|_{L^\infty(\Omega)} \|\nabla u_n\|\|u_n\|.
\]

By the chain rule and the fact that \(\varphi \in W^{1,\infty}(0, \infty), \|\nabla \varphi_n\|_{L^\infty(\Omega)} \to 0\) as \(n \to \infty\), \((u_n)\) is bounded by hypothesis and \((\nabla u_n)\) can be seen to be bounded by performing an integration by parts as in (2.8), using the hypotheses that \(|\langle \langle T_0 u_n \rangle \rangle|\) is bounded and that \(q\) is bounded below. The right hand side of (2.27) tends to zero as \(n \to \infty\) hence Assumption 1 is satisfied.

\[
\square
\]

3. Second order operators on the half-line

Consider the differential expression

\[
\dot{T}_0 u = \frac{1}{r} (-pu')' + qu \quad \text{on} \quad [0, \infty)
\]

where \(p, q\) and \(r\) are functions on \([0, \infty)\) satisfying the minimal hypotheses: \(p\) and \(q\) are complex in general, \(r > 0\), \(p \neq 0\) and \(q, 1/p, r \in L^1_{\text{loc}}(0, \infty)\). These assumptions on \(p, q\) and \(r\) ensure that for any \(\lambda, u_1, u_2 \in \mathbb{C}\) there exists a unique solution \(u\) to the initial value problem

\[
\dot{T}_0 u = \lambda u \quad \text{on} \quad [0, \infty), \quad u(0) = u_1, \quad pu'(0) = u_2
\]

such that \(u, pu' \in AC_{\text{loc}}[0, \infty)\). The solution space of \(\dot{T}_0 u = \lambda u\) on \([0, \infty)\) is therefore two-dimensional.

Consider a Sturm-Liouville operator \(T_0\) on the weighted Lebesgue space \(L^2_r(0, \infty)\), endowed with a complex mixed boundary condition at 0,

\[
(3.1) \quad BC[u] := \cos(\eta)u(0) - \sin(\eta)pu'(0) = 0
\]

for some \(\eta \in \mathbb{C}\). \(T_0\) is defined by

\[
(3.2) \quad D(T_0) = \{u \in L^2_r(0, \infty) : u, pu' \in AC_{\text{loc}}[0, \infty), \dot{T}_0 u \in L^2_r(0, \infty), BC[u] = 0\}.
\]

Fix \(\gamma \in \mathbb{C}\setminus\{0\}\). Define the perturbed operators by

\[
(3.3) \quad T_R u = T_0 u + i\gamma \chi_{[0, R]} u, \quad D(T_R) = D(T_0) \quad (R \in \mathbb{R}_+)
\]

where \(\chi\) is the characteristic function. Define the limit operator by

\[
(3.4) \quad T = T_0 + i\gamma.
\]

Next, we introduce the main hypotheses of this section, which we will later assume holds throughout the section. The assumption ensures that for any \(\lambda \in \mathbb{C}\)
Consequently, Assumption 2 is satisfied in this case.

Remark 3.1 (See [7]). The conditions of Assumption 2 do not exclude a situation in which $\sigma(T_0) = \sigma_c(T_0) = \mathbb{C}$. A sufficient condition to ensure that this does not occur is that
\[ \overline{co} \left\{ \frac{q(x)}{r(x)} + yp(x) : x, y \in [0, \infty) \right\} \neq \mathbb{C}, \]
where $\overline{co}$ denotes the closed convex hull, and that $\tilde{T}_0$ is in Sims case I. In this case, the operator $T_0$ defined by (3.2) is quasi-$m$-accretive and $J$-self-adjoint with respect to the complex conjugation operator [7, Theorem 4.5].

Example 3.2 (Schrödinger operators with $L^1$ potentials). Consider the case $p = r = 1$ with $q \in L^1(0, \infty)$. Then,
\[ \sigma_c(T_0) = [0, \infty). \]

By the Levinson asymptotic theorem [15, Theorem 1.3.1], for any $z \in \mathbb{C}\setminus\{0\}$, the solution space of $T_0u = zu$ is spanned by $\psi_\pm(\cdot, z)$, where
\begin{align*}
\psi_\pm(x, z) &= e^{\pm ik(z)x} \tilde{\psi}_\pm(x, z) \\
\psi'_\pm(x, z) &= e^{\pm ik(z)x} \tilde{\psi}'_\pm(x, z).
\end{align*}

Consider the case $\sigma(T_0) = \sigma_c(T_0) = \mathbb{C}$. A sufficient condition to ensure that this does not occur is that
\[ \overline{co} \left\{ \frac{q(x)}{r(x)} + yp(x) : x, y \in [0, \infty) \right\} \neq \mathbb{C}, \]
where $\overline{co}$ denotes the closed convex hull, and that $\tilde{T}_0$ is in Sims case I. In this case, the operator $T_0$ defined by (3.2) is quasi-$m$-accretive and $J$-self-adjoint with respect to the complex conjugation operator [7, Theorem 4.5].
The discriminant is defined by
\[ D(z) = \phi_1(X + a, z) + \phi_2'(X + a, z). \]

The essential spectrum of \( T_0 \) is
\[ \sigma_e(T_0) = \{ z \in \mathbb{R} : |D(z)| \leq 2 \}. \]

The Floquet multipliers \( \rho_\pm \) are defined by
\[ \rho_\pm(z) = \frac{1}{2} \left( D(z) \pm i \sqrt{4 - D(z)^2} \right). \]

Note that \( \rho_\pm \) have branch cuts along \( \sigma_e(T_0) \), \( |\rho_+(z)| < 1 \) for all \( z \in \mathbb{C} \setminus \sigma_e(T_0) \) and \( \rho_+(z)\rho_-(z) = 1 \). Define \( k \) by
\[ k(z) = -\frac{i}{a} \ln(\rho_+(z)). \]

In this setting, \( k \) is referred to as the Floquet exponent. \( k \) is analytic and satisfies \( \Im k > 0 \) on \( \mathbb{C} \setminus \sigma_e(T_0) \) hence satisfies Assumption (i).

Define the Floquet solutions \( \psi_\pm \) by
\[ \psi_\pm(x, z) = -\phi_2(X + a, z)\phi_1(x, z) + (\phi_1(X + a, z) - \rho_\pm(z))\phi_2(x, z) \]
for any \( x \in [0, \infty) \) and \( z \in \mathbb{C} \). \( \psi_\pm(\cdot, z) \) span the solution space of \( \tilde{T}_0u = zu \) and satisfy
\[ \psi_\pm(x_0 + na, z) = e^{+ik(z)na}\psi_\pm(x_0, z) \]
\[ \psi'_\pm(x_0 + na, z) = e^{+ik(z)na}\psi'_\pm(x_0, z) \]
for any \( x_0 \in [X, X + a) \) and \( n \in \mathbb{N} \). For any \( x \), the Floquet solutions \( \psi_\pm(x, \cdot) \) and \( \psi'_\pm(x, \cdot) \) are analytic on \( \mathbb{C} \setminus \sigma_e(T_0) \). Define the band-ends \( B_{\text{ends}} \) by
\[ B_{\text{ends}} = \{ \lambda \in \mathbb{C} : |D(\lambda)| = 2 \}. \]

For any \( z_0 \in \sigma_e(T_0) \setminus B_{\text{ends}} \), \( \rho_\pm \) and \( k \) can be analytically continued into an open neighbourhood of \( z_0 \) in \( \mathbb{C} \), hence for any \( x \in [0, \infty) \), \( \psi_\pm(x, \cdot) \) and \( \psi'_\pm(x, \cdot) \) can be analytically continued into an open neighbourhood of \( z_0 \).

Finally, Assumption 2 can be satisfied by setting
\[ \tilde{\psi}_\pm(x, \lambda) = \begin{cases} e^{+ik\lambda}x\psi_\pm(x, \lambda) & \text{if } x \in [0, X) \\ e^{+ik\lambda}x_0(x)\psi_\pm(x_0(x), \lambda) & \text{if } x \in [X, \infty) \end{cases} \]
and
\[ \tilde{\psi}'_\pm(x, \lambda) = \begin{cases} e^{+ik\lambda}x\psi'_\pm(x, \lambda) & \text{if } x \in [0, X) \\ e^{+ik\lambda}x_0(x)\psi'_\pm(x_0(x), \lambda) & \text{if } x \in [X, \infty) \end{cases} \]
where \( x_0(x) := X + (x - X) \mod a \).

Throughout the remainder of the section, let
\[ S = \sigma_e(T_0) \cup (i\gamma + \sigma_e(T_0)) \]
and suppose that the conditions of Assumption 2 are satisfied. Recall that \( BC \) denotes the boundary condition functional defined by equation (3.1).
Lemma 3.4. \( \lambda \in \mathbb{C}\backslash S \) is an eigenvalue of \( T_R \) if and only if
\[
f_R(\lambda) = \alpha_+(R, \lambda)e^{ik(\lambda-i\gamma)R} + \alpha_-(R, \lambda)e^{-ik(\lambda-i\gamma)R} = 0
\]
where
\[
\alpha_+(R, \lambda) := BC[\psi_-(\cdot, \lambda-i\gamma)]\left(\tilde{\psi}_+(R, \lambda-i\gamma)\tilde{\psi}_d^d(R, \lambda) - \tilde{\psi}_+(R, \lambda-i\gamma)\tilde{\psi}_+(R, \lambda)\right)
\]
and
\[
\alpha_-(R, \lambda) := BC[\psi_+(\cdot, \lambda-i\gamma)]\left(\tilde{\psi}_+(R, \lambda)\tilde{\psi}_d^d(R, \lambda-i\gamma) - \tilde{\psi}_+(R, \lambda)\tilde{\psi}_-(R, \lambda-i\gamma)\right).
\]
Furthermore, \( f_R \) is analytic on \( \mathbb{C}\backslash S \).

Proof. Let \( \lambda \in \mathbb{C}\backslash S \). Any solution of the boundary value problem
\[
(\tilde{T}_0 + i\gamma \chi_{[0,R]})u = \lambda u \text{ on } [0,R], \ BC[u] = 0
\]
must be of the form \( C_1u_1(\cdot, \lambda) \), where \( u_1 \) is defined by
\[
\begin{align*}
    u_1(x, \lambda) &= BC[\psi_-(\cdot, \lambda-i\gamma)]\psi_+(x, \lambda-i\gamma) - BC[\psi_+(\cdot, \lambda-i\gamma)]\psi_-(x, \lambda-i\gamma)
\end{align*}
\]
and \( C_1 \in \mathbb{C} \) is independent of \( x \). Also, any solution of the boundary value problem
\[
(\tilde{T}_0 + i\gamma \chi_{[0,R]})u = \lambda u \text{ on } [R, \infty), \ u \in L^2_+(R, \infty)
\]
must be of the form \( C_2\psi_+(x, \lambda) \), where \( C_2 \in \mathbb{C} \) is independent of \( x \). \( \lambda \) is an eigenvalue if and only if there exists \( C_1, C_2 \in \mathbb{C}\backslash\{0\} \) independent of \( x \) such that
\[
\begin{align*}
    \begin{cases}
        C_1u_1(x, \lambda) & \text{if } x \in [0,R) \\
        C_2\psi_+(x, \lambda) & \text{if } x \in [R, \infty)
    \end{cases}
\end{align*}
\]
is absolutely continuous. This holds if and only if
\[
    u_1(R, \lambda)\psi_+(R, \lambda) - u_1'(R, \lambda)\psi_+(R, \lambda) = 0
\]
which holds if and only if the following quantity is zero
\[
(BC[\psi_-(\cdot, \lambda-i\gamma)]\psi_+(x, \lambda-i\gamma) - BC[\psi_+(\cdot, \lambda-i\gamma)]\psi_-(x, \lambda-i\gamma))\tilde{\psi}_d^d(R, \lambda)
\]
\[
- (BC[\psi_-(\cdot, \lambda-i\gamma)]\psi_+(x, \lambda-i\gamma) - BC[\psi_+(\cdot, \lambda-i\gamma)]\psi_-(x, \lambda-i\gamma))\tilde{\psi}_+(R, \lambda)
\]
which in turn is equivalent to
\[
\alpha_+(R, \lambda)e^{ik(\lambda-i\gamma)R} + \alpha_-(R, \lambda)e^{-ik(\lambda-i\gamma)R} = 0.
\]
The analyticity claim follows from Assumptions 2 (i) and (ii). \( \square \)

In the regions of the complex plane for which \( \alpha_-(R, \cdot) \) becomes small for large \( R \), we are unable to prove the spectral pollution and spectral inclusion results of Theorems 3.6 and 3.7. We shall now define a subset of the complex plane capturing such regions.

Define function \( \Lambda : [0, \infty) \times \mathbb{C}\backslash S \rightarrow \mathbb{C} \) by
\[
\Lambda(R, \lambda) = \tilde{\psi}_+(R, \lambda)\tilde{\psi}_d^d(R, \lambda-i\gamma) - \tilde{\psi}_+(R, \lambda)\tilde{\psi}_-(R, \lambda-i\gamma),
\]
so that \( \alpha_-(R, \lambda) = BC[\psi_+(\cdot, \lambda-i\gamma)]\Lambda(R, \lambda) \). Note that the zeros of \( \lambda \rightarrow BC[\psi_+(\cdot, \lambda-i\gamma)] \) are exactly the eigenvalues of the limit operator \( T = T_0 + i\gamma \).
Let \((R_n) \subseteq \mathbb{R}_+\) be any sequence such that \(R_n \to \infty\) as \(n \to \infty\). Define subset \(S_p((R_n))\) of \(\mathbb{C}\) by

\[
S_p((R_n)) = \left\{ z \in \mathbb{C} \setminus S : \liminf_{n \to \infty} |\Lambda(R_n, z)| = 0 \right\}.
\]

The set \(S \cup S_p((R_n))\) plays a similar role in this section as the limiting essential spectrum did in section 2. We shall show in Theorems 3.6 and 3.7 that there is no spectral pollution for \((T_{R_n})\) with respect to \(T\) outside of \(S \cup S_p((R_n))\) and that eigenvalues of \(T\) lying outside of \(S \cup S_p((R_n))\) are approximated (with exponentially small error) by the eigenvalues of \(T_{R_n}\).

**Proposition 3.5.** Let \((R_n) \subseteq \mathbb{R}_+\) be any sequence such that \(R_n \to \infty\) as \(n \to \infty\).

(a) \(S \cup S_p((R_n))\) is a closed subset of \(\mathbb{C}\).

(b) For any \(\lambda \in \mathbb{C} \setminus (S \cup S_p((R_n)))\) there exists a bounded, open neighbourhood \(U\) of \(\lambda\) with \(\overline{U} \subseteq \mathbb{C} \setminus (S \cup S_p((R_n)))\). For such \(U\), we have \(|\Lambda(R_n, z)| \geq C\) for all large \(n\) and all \(z \in U\), where \(C > 0\) is some constant independent of \(n\) and \(z\).

**Proof.** (a) By Assumption 2 (ii), \(\Lambda(R_n, \cdot)\) is analytic for all \(n\) and \(\Lambda(\cdot, z)\) is bounded for all \(z\). Let \(\lambda\) be a limit point of \(S \cup S_p((R_n))\). The desired lemma holds if and only if \(\lambda\) lies in either \(S\) or in \(S_p((R_n))\). If \(\lambda\) is a limit point of \(S\) then \(\lambda \in S\) since \(S\) is closed.

In the only other case, \(\lambda\) is a limit point of \(S_p((R_n))\) so there exists \((\lambda_k) \subseteq S_p((R_n))\) such that \(\lambda_k \to \lambda\) as \(k \to \infty\). Since \(\liminf_{n \to \infty} |\Lambda(R_n, \lambda_k)| = 0\) for all \(k\), there exists a subsequence \((R_{n_k})\) such that \(|\Lambda(R_{n_k}, \lambda_k)| \to 0\) as \(k \to \infty\). Let \(\varepsilon > 0\) be small enough so that \(\overline{B}_\varepsilon(\lambda) \subseteq \mathbb{C} \setminus S\). Since the magnitude of \(\Lambda(R, z)\) is bounded above uniformly for all \(R > 0\) and all \(z \in \overline{B}_\varepsilon(\lambda)\), by Cauchy’s integral formula,

\[
|\Lambda(R_{n_k}, z)| \leq |\Lambda(R_{n_k}, \lambda_k)| + |\Lambda(R_{n_k}, \lambda) - \Lambda(R_{n_k}, \lambda_k)| \to 0\quad\text{as}\quad k \to \infty.
\]

Finally,

\[
|\Lambda(R_{n_k}, \lambda)| \leq |\Lambda(R_{n_k}, \lambda_k)| + |\Lambda(R_{n_k}, \lambda) - \Lambda(R_{n_k}, \lambda_k)| \to 0\quad\text{as}\quad k \to \infty.
\]

so \(\lambda \in S_p((R_n))\), completing the proof.

(b) Let \(\lambda \in \mathbb{C} \setminus (S \cup S_p((R_n)))\). \(\mathbb{C} \setminus (S \cup S_p((R_n)))\) is an open subset of \(\mathbb{C}\) so there exists a bounded open neighbourhood \(U\) of \(\lambda\) such that \(\overline{U} \subseteq \mathbb{C} \setminus (S \cup S_p((R_n)))\). Suppose that the desired result does not hold with this choice for \(U\). Then there exists a subsequence \((R_{n_k})\) and a sequence \((z_k) \subseteq U\) such that \(|\Lambda(R_{n_k}, z_k)| \to 0\) as \(k \to \infty\). Since \(\overline{U}\) is compact, there exists \(z \in \mathbb{C} \setminus (S \cup S_p((R_n)))\) such that \(z_k \to z\). By the arguments in (a), \(\liminf_{n \to \infty} |\Lambda(R_n, z)| = 0\), which is the desired contradiction.

Next, we prove the main results of this section, regarding spectral inclusion and pollution for the operators \(T_{R_n}\) defined by equation (3.3) such that \(T_0\) satisfies Assumption 2. Recall, also, that \(S\) is defined by equation (3.19) and \(S_p((R_n))\) is defined by (3.21).

**Theorem 3.6.** Let \((R_n) \subseteq \mathbb{R}_+\) be any sequence such that \(R_n \to \infty\) as \(n \to \infty\). Let \(\mu\) be an eigenvalue of \(T_0\). Assume that \(\mu + i\gamma \notin S \cup S_p((R_n))\). Then there
exists eigenvalues $\lambda_n$ of $T_{R_n}$ \((n \in \mathbb{N})\) and constants $C_0 = C_0(T_0, \gamma, \mu) > 0$ and \(\beta = \beta(T_0, \gamma, \mu) > 0\) such that
\[ |\lambda_n - (\mu + i\gamma)| \leq C_0 e^{-\beta R_n} \]
for all large enough $n$.

Proof. Throughout the proof, let $C > 0$ be an arbitrary constant independent of $\lambda$ and $n$. Assume that $C_1, C_2, C_3 > 0$ are independent of $\lambda$ and $n$.

Since $\mu$ is an eigenvalue of $T_0$, $\mu + i\gamma$ is a zero of the analytic function
\[ \lambda \mapsto \tilde{f}(\lambda) := BC[\psi_{\lambda}(\cdot, \lambda - i\gamma)]. \]

Since it is assumed that $\mu + i\gamma \notin S \cup S_p((R_n))$, Lemma 3.5 (b) guarantees the existence of an open neighbourhood $U$ of $\mu + i\gamma$ with $U \subset \mathbb{C} \setminus (S \cup S_p((R_n)))$ and constant $N_0 \in \mathbb{N}$ independent of $n$, such that $|\Lambda(R_n, \lambda)| \geq C$ for all $\lambda \in U$ and all $n \geq N_0$. For $n \geq N_0$, $\lambda$ is an eigenvalue of $T_{R_n}$ if and only if
\[ \tilde{f}_n(\lambda) := e^{i k(\lambda - i\gamma) R_n} \frac{f_{R_n}(\lambda)}{\Lambda(R_n, \lambda)} = 0. \]

It suffices to show that there exists $R_0 > 0$ independent of $n$ and zeros $\lambda_n$ of $\tilde{f}_n$ such that inequality (3.23) holds for all $n \geq N_0$ with $R_n \geq R_0$.

For all $\lambda \in U$,
\[ |\tilde{f}_n(\lambda) - \tilde{f}(\lambda)| = \left| e^{i k(\lambda - i\gamma) R_n} \frac{\alpha_{+}(R_n, \lambda)}{\Lambda(R_n, \lambda)} \right| \leq C_1 e^{-C_2 R_n} \]
for some $C_1, C_2 > 0$. Here, we used the uniform boundedness of $\alpha_{+}(R_n, \lambda)$ for all $\lambda \in U$ and all $n$ which is guaranteed by Assumption 2 (ii), the inequality for $\Lambda$ and the fact the $\Im(\cdot - i\gamma) \geq C$ on $U$. Since $\tilde{f}$ is analytic at $\mu + i\gamma$, there exists $\varepsilon > 0$ small enough such that for all $\lambda \in B_{\varepsilon}(\mu + i\gamma)$ we have
\[ |\tilde{f}(\lambda)| \geq C_3 |\lambda - (\mu + i\gamma)|^\nu \]
for some $C_3 > 0$. Here, $\nu$ is the algebraic multiplicity of the eigenvalue $\mu$ of $T_0$, that is, the multiplicity of the zero $\mu$ of the analytic function $z \mapsto BC[\psi_{\lambda}(\cdot, z)]$.

Let $C_0 = (2C_1/C_3)^{1/\nu}$ and $\beta = C_2/\nu$. Let $R_0 > 0$ be large enough such that $C_0 e^{-\beta R_0} < \varepsilon$. Combining (3.24) and (3.25), for all $n \geq N_0$ with $R_n \geq R_0$ and all $\lambda \in \mathbb{C}$ with
\[ |\lambda - (\mu + i\gamma)| = C_0 e^{-\beta R_n} \]
we have
\[ |\tilde{f}_n(\lambda) - \tilde{f}(\lambda)| \leq \frac{1}{2} |\tilde{f}(\lambda)| < |\tilde{f}(\lambda)|. \]

By Rouche’s theorem, for all $n \geq N_0$ with $R_n \geq R_0$, there exists a zero $\lambda_n$ of $\tilde{f}_n(\lambda)$ satisfying $|\lambda_n - (\mu + i\gamma)| < C_0 e^{-\beta R_n}$, as required.

The next result concerns spectral pollution - the set of spectral pollution is defined by equation (1.3).

**Theorem 3.7.** Let $(R_n) \subset \mathbb{R}_+$ be any sequence such that $R_n \to \infty$ as $n \to \infty$. The set of spectral pollution of the sequence of operators $(T_{R_n})$ with respect to the limit operator $T = T_0 + i\gamma$ satisfies
\[ \sigma_{\text{poll}}((T_{R_n})) \subset \sigma_e(T_0) \cup S_p((R_n)). \]
Proof: Throughout the proof, let $C > 0$ be an arbitrary constant independent of $\lambda$ and $n$.

Let $\mu \in \mathbb{C}\backslash(S \cup S_\psi((R_n)))$ and assume that $\mu$ is not an eigenvalue of $T$. Then $\mu$ is an arbitrary element of $\rho(T)\backslash(\sigma_c(T_0) \cup S_p((R_n)))$. We aim to show that $\mu \notin \sigma_{\text{poll}}(T_{R_n}))$, for which it suffices to show that there exists a neighbourhood $U$ of $\mu$ and $N_0 \in \mathbb{N}$ independent of $n$ such that $T_{R_n}$ has no eigenvalues in $U$ for any $n \geq N_0$.

Since $BC[\psi_+(\cdot, \mu - i\gamma)] \neq 0$ and by the definition of $S_p((R_n))$, there exists $N_0 \in \mathbb{N}$ independent of $n$ such that

$$|\alpha_-(R_n, \mu)| = |BC[\psi_+(\cdot, \mu - i\gamma)]\Lambda(R_n, \mu)| \geq C$$

for all $n \geq N_0$. By Assumption 2, $\alpha_{\pm}(R_n, \cdot)$ is analytic on $\mathbb{C}\backslash S$ for all $n$ and $\alpha_{\pm}(\cdot, \lambda)$ is in $L^\infty(0, \infty)$ for all $\lambda \in \mathbb{C}\backslash S$. Consequently, letting $\varepsilon > 0$ be small enough so that $B_\varepsilon(\mu) \subset \mathbb{C}\backslash S$, we have $|\alpha_{\pm}(R_n, \lambda)| \leq C$ for all $n$ and all $\lambda \in B_\varepsilon(\mu)$, so, using Cauchy’s integral formula as in (3.22), and making $\varepsilon > 0$ small enough, we have

$$|\alpha_{\pm}(R_n, \lambda) - \alpha_{\pm}(R_n, \mu)| \leq C|\lambda - \mu|$$

for all $n$ and all $\lambda \in B_\varepsilon(\mu)$.

Define approximation $f_n^{(\mu)}$ to $f_{R_n}$ by

$$f_n^{(\mu)}(\lambda) = \alpha_+(R_n, \mu)e^{i(k(\lambda - i\gamma)R_n)} + \alpha_-(R_n, \mu)e^{-i(k(\lambda - i\gamma)R_n)}.$$ 

For all $\lambda \in B_\varepsilon(\mu)$, we have by (3.26),

$$|f_{R_n}(\lambda) - f_n^{(\mu)}(\lambda)| \leq C|\lambda - \mu|e^{3k(\lambda - i\gamma)R_n}.$$ 

Since $3k(\lambda - i\gamma) \geq C$ for all $\lambda \in B_\varepsilon(\mu)$ and by the boundedness of $\alpha_{\pm}(\cdot, \mu)$, we can make $N_0$ large enough and independent of $n$ such that for all $n \geq N_0$ and all $\lambda \in B_\varepsilon(\mu)$,

$$|e^{ik(\lambda - i\gamma)R_n}f_n^{(\mu)}(\lambda)| = |\alpha_+(R_n, \mu)e^{2ik(\lambda - i\gamma)R_n} + \alpha_-(R_n, \mu)|$$

$$\geq \left|\alpha_-(R_n, \mu)\right| - |\alpha_+(R_n, \mu)|e^{-23k(\lambda - i\gamma)R_n}$$

$$\geq \frac{|\alpha_-(R_n, \mu)|}{2} \geq C.$$

Finally, making $\varepsilon > 0$ small enough if necessary, we have $|f_{R_n}(\lambda) - f_n^{(\mu)}(\lambda)| < |f_n^{(\mu)}(\lambda)|$ for all $n \geq N_0$ and all $\lambda \in B_\varepsilon(\mu)$. $f_{R_n}$ therefore has no zeros in $U := B_\varepsilon(\mu)$ for $n \geq N_0$ and so $\mu \notin \sigma_{\text{poll}}((T_n))$, completing the proof.

In the case of Schrödinger operators on $L^2(0, \infty)$ with $L^1$ potentials, described in Example 3.2, $S_p((R_n))$ can be easily computed to be the empty set.

**Example 3.8** (Schrödinger operators with $L^1$ potentials, continued). Consider again the case $p = r = 1$ with $q \in L^1(0, \infty)$. Then, using expression (3.20) for $\Lambda$ and the expressions for $\psi_\pm$, $\tilde{\psi}_\pm$ in Example 3.2 (ii), $\Lambda$ satisfies

$$\Lambda(R, \lambda) \to -i\left(\sqrt{\lambda - i\gamma} + \sqrt{\lambda}\right) \text{ as } R \to \infty$$

for any $\lambda \in \mathbb{C}\backslash S$. Since $\sqrt{\lambda - i\gamma} \neq -\sqrt{\lambda}$ for all $\lambda \in \mathbb{C}$ we have

$$S_p((R_n)) = \emptyset$$

for any $(R_n) \subset \mathbb{R}_+$ with $R_n \to \infty$ as $n \to \infty$. 


For Schrödinger operators with eventually real periodic potentials, described in Example 3.3, the computation of $S_p((R_n))$ is more involved. We shall require the notion of essential numerical range (see Definition 2.4).

**Example 3.9** (Eventually periodic Schrödinger operators, continued). Consider again the case $p = r = 1$ with $q|_{X,\infty}$ real-valued and a-periodic for some $X \geq 0$ and $a > 0$. Assume that $\gamma > 0$ and let $R_n = x_0 + na$ $(n \in \mathbb{N})$ for any fixed $x_0 \in [X, X + a]$.

Using the expressions (3.17) and (3.18) for $\tilde{\psi}_\pm$ and $\tilde{\psi}^d_\pm$ as well as the definition of $S_p((R_n))$ in equation (3.21), we infer that $\lambda \in S_p((R_n))$ if and only if

$$\psi_+(x_0, \lambda)\psi'_-(x_0, \lambda - i\gamma) - \psi'_+(x_0, \lambda)\psi_-(x_0, \lambda - i\gamma) = 0.$$  
(3.27)

$\psi_\pm(x_0, \cdot)$ and $\psi'_\pm(x_0, \cdot)$ are analytic on $\mathbb{C}\setminus\sigma_e(T_0)$ and can be analytically continued into an open neighbourhood in $\mathbb{C}$ of any point in $\sigma_e(T_0)\setminus\text{Bends}$ (recall that $\text{Bends}$ denotes the set of band-ends for the essential spectrum of the operator $T_0$). Consequently, $S_p((R_n))$ consists of isolated points in the complex plane that can only accumulate to the band-ends of either $T_0$ or $T$, that is, to the set $\text{Bends} \cup (i\gamma + \text{Bends})$. Furthermore, $S_p((R_n))$ satisfies the inclusions

$$S_p((R_n)) \subset \sigma_e((T_{R_n}))$$  
(3.28)

and

$$S_p((R_n)) \subset [\inf \sigma_e(T_0), \infty) \times i(0, \gamma).$$  
(3.29)

**Proof of inclusions (3.28) and (3.29).** By $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$, we mean $\|\cdot\|_{L^2(0,\infty)}$ and $\|\cdot\|_{L^\infty(0,\infty)}$ respectively.

Let $\lambda \in S_p((R_n))$. Then, using the property (3.15) of the Floquet solutions, (3.27) implies that,

$$\psi_+(R_n, \lambda)\psi'_-(R_n, \lambda - i\gamma) - \psi'_+(R_n, \lambda)\psi_-(R_n, \lambda - i\gamma) = 0$$  
(3.30)

for all $n$.

**Step 1** ($\lambda \in \sigma_e((T_{R_n}))$). (3.30) ensures that there exists $C_{1,n}, C_{2,n} \in \mathbb{C}\setminus\{0\}$ independent of $x$ such that

$$u_n(x) := \begin{cases} C_{1,n}\psi_-(x, \lambda - i\gamma) & \text{if } x \in [0, R_n) \\ C_{2,n}\psi_+(x, \lambda) & \text{if } x \in [R_n, \infty) \end{cases}$$  
(3.31)

is absolutely continuous and solves the Schrödinger equation $\tilde{T}_{R_n}u = \lambda u$, where $\tilde{T}_{R_n}$ denotes the differential expression on $[0, \infty)$ corresponding to $T_{R_n}$. Define

$$v_n = \frac{\tilde{\chi}_n u_n}{\|\tilde{\chi}_n u_n\|_{L^2}}.$$  

where $\tilde{\chi}_n(x) := \tilde{\chi}(x/R_n)$ and $\tilde{\chi} : [0, \infty) \to [0, 1]$ is any smooth function such that $\tilde{\chi} = 0$ on $[0, \frac{1}{n}]$ and $\tilde{\chi} = 1$ on $[\frac{1}{n}, \infty)$. Then $v_n \in D(T_0) = D(T_{R_n})$, $\|v_n\|_{L^2} = 1$ and, since $\langle v_n, \varphi \rangle_{L^2} = 0$ for any $\varphi \in C_c^\infty(0, \infty)$ and any large enough $n$, $v_n \to 0$ in $L^2(0, \infty)$. 


Using the expression (3.31) for \( u_n \) and the property (3.15) of the Floquet solutions,
\[
\|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, R_n)}^2
= \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, X)}^2 + \sum_{m=0}^{n-1} e^{2\Im k(\lambda - i\gamma)ma} \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, X+a)}^2
\]
\[
+ e^{2\Im k(\lambda - i\gamma)na} \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(X, x_0)}^2.
\]
A similar equation holds for \( \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, R_n)}^2 \). By unique continuation, we have
\[
\|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(I)} \leq C \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(I)}
\]
for \( I = [0, X] \), \([X, X+a]\) or \([X, x_0] \) where \( C > 0 \) is an arbitrary constant independent of \( n \) so we have
\[
\tag{3.32}
\|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, R_n)}^2 \leq C \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, R_n)}^2
\]
for all \( n \). Also, using the property (3.15) of the Floquet solutions and noting that
\[
\|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, x)} \text{ is exponentially growing in } x,
\]
we deduce that,
\[
\|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, \frac{X}{2} R_n)}^2 \leq C \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(\frac{X}{2} R_n, R_n)}^2
\]
for all large enough \( n \) hence
\[
\tag{3.33}
\|u_n\|_{L^2} \leq C \|u_n\|_{L^2(\frac{X}{2} R_n, \infty)} \leq C \|
\tilde{\chi} u_n\|_{L^2}.
\]
for all large enough \( n \).

We have
\[
\|(T_{R_n} - \lambda) v_n\|_{L^2} \leq \frac{1}{\|\tilde{\chi} u_n\|_{L^2}} \left[ \|\tilde{\chi} (T_{R_n} - \lambda) u_n\|_{L^2} + 2\|\tilde{\chi} u_n\|_{L^2} + \|
\tilde{\chi} u_n\|_{L^2} \right].
\]

The first term in the square brackets above vanishes and \( \tilde{\chi}^{(k)} \) are supported in \([0, R_n] \) with \( \|\tilde{\chi}^{(k)}\|_{L^\infty} \leq C/R_n^k \) so
\[
\|(T_{R_n} - \lambda) v_n\|_{L^2} \leq C \left[ \frac{\|u_n\|_{L^2}}{\|\tilde{\chi} u_n\|_{L^2}} \left[ \frac{1}{R_n} \|\psi_-^\prime(\cdot, \lambda - i\gamma)\|_{L^2(0, R_n)} + \frac{1}{R_n^2} \right] \right] \to 0 \text{ as } n \to \infty.
\]

Here, we used estimates (3.32) and (3.33). Consequently, by the definition of limiting essential spectrum (see Definition 2.1), we have \( \lambda \in \sigma_e((T_{R_n})) \).

**Step 2** \( (\Re(\lambda) \in [\inf \sigma_e(T_0), \infty)) \). \( v_n \) also satisfies
\[
\langle (T_{R_n} - \lambda) v_n, v_n \rangle_{L^2} \to 0 \text{ as } n \to \infty
\]
since \( v_n(0) = 0 \) and \( \gamma > 0 \), integrating by parts and taking the real part of the inner product yields
\[
\langle (-\frac{d^2}{dx^2} + \Re(q) - \Re(\lambda)) v_n, v_n \rangle_{L^2} \to 0 \text{ as } n \to \infty.
\]

Consequently, \( \Re(\lambda) \in W_e(T_{0,\text{real}}) \) where
\[
T_{0,\text{real}} := -\frac{d^2}{dx^2} + \Re(q)
\]
acts on \( L^2(0, \infty) \) and is endowed with a Dirichlet boundary condition at 0. \( T_{0,\text{real}} \) is self-adjoint so by [4, Theorem 3.8]
\[
\Re(\lambda) \in W_e(T_{0,\text{real}}) = \text{conv}(\hat{\sigma}_e(T_{0,\text{real}})) \setminus \{\pm \infty\} = [\inf \sigma_e(T_0), \infty)
\]
Recall here that $\bar{\sigma}_e$ denotes the extended essential spectrum, defined by Definition 2.8.

**Step 2 (3(\lambda) \in (0, \gamma))**. Without loss of generality, take

\[(3.34) \quad C_{1,n} = C_{1,n}(\lambda) := \psi_{+}(R_n, \lambda) = e^{ik(\lambda)na} \psi_{+}(x_0, \lambda)\]

and

\[C_{2,n} = C_{2,n}(\lambda) := \psi_{-}(R_n, \lambda - i\gamma) = e^{-ik(\lambda - i\gamma)na} \psi_{-}(x_0, \lambda - i\gamma).\]

Note that if $\psi_{+}(x_0, \lambda) = 0$ or $\psi_{-}(x_0, \lambda - i\gamma) = 0$ then we can instead take $C_{1,n} = \psi_{+}(R_n, \lambda)$ and $C_{2,n} = \psi_{-}(R_n, \lambda - i\gamma)$, without affecting the proof. Integrating by parts from $X$ to $\infty$ we have,

\[
\lambda \int_{X}^{\infty} |u_n|^2 = \int_{X}^{\infty} |u'_n|^2 + \int_{X}^{\infty} q |u_n|^2 + i\gamma \int_{X}^{R_n} |u_n|^2
- |C_{1,n}|^2 \psi_{-}(X, \lambda - i\gamma) \overline{\psi}_{-}(X, \lambda - i\gamma)
\]

Since $q|_{X,\infty}$ is real-valued and $\gamma > 0$, we can take the imaginary part of the above equation to see that $\lambda$ must satisfy

\[
\Im(\lambda) = \frac{1}{\int_{X}^{\infty} |u_n|^2} \left( \gamma \int_{X}^{R_n} |u_n|^2 - |C_{1,n}|^2 \Im(\psi_{-}(X, \lambda - i\gamma) \overline{\psi}_{-}(X, \lambda - i\gamma)) \right) =: F_n
\]

for all $n$. Using the property (3.15) of the Floquet solutions, we compute

\[
\frac{1}{|C_{1,n}|^2} \int_{X}^{R_n} |u_n|^2 = \int_{X}^{R_n} |\psi_{-}(t, \lambda - i\gamma)|^2 \, dt
= \sum_{m=0}^{n-1} e^{2\Im(k(\lambda - i\gamma)ma)} |\psi_{-}(\cdot, \lambda - i\gamma)|^2_{L^2(X, X + a)} + e^{2\Im(k(\lambda - i\gamma)na)} |\psi_{-}(\cdot, \lambda - i\gamma)|^2_{L^2(X, x_0)}.
\]

The sum in the above equality satisfies

\[
\sum_{m=0}^{n-1} e^{2\Im(k(\lambda - i\gamma)na)} = \frac{1 - e^{2\Im(k(\lambda - i\gamma)na)}}{1 - e^{2\Im(k(\lambda - i\gamma)a}} \sim \frac{e^{2\Im(k(\lambda - i\gamma)na)}}{e^{2\Im(k(\lambda - i\gamma)a}} - 1 \text{ as } n \to \infty
\]

which, using the expression (3.34) for $C_{1,n}$, implies that

\[
\int_{X}^{R_n} |u_n|^2 \sim C_3 e^{2\Im(k(\lambda - i\gamma) - k(\lambda))na} \text{ as } n \to \infty
\]

for some $C_3 = C_3(\lambda) > 0$ independent of $n$. Similarly, we compute

\[
\frac{1}{|C_{2,n}|^2} \int_{R_n}^{\infty} |u_n|^2 = \int_{R_n}^{\infty} |\psi_{+}(t, \lambda)|^2 \, dt
= e^{-2\Im(k(\lambda)na)} |\psi_{+}(\cdot, \lambda)|^2_{L^2(x_0, X + a)} + \sum_{m=n+1}^{\infty} e^{-2\Im(k(\lambda)ma)} |\psi_{+}(\cdot, \lambda)|^2_{L^2(X, X + a)}/a
\]

which implies that

\[
\int_{R_n}^{\infty} |u_n|^2 = C_4 e^{2\Im(k(\lambda - i\gamma) - k(\lambda))na}
\]

for some $C_4 = C_4(\lambda) > 0$ independent of $n$. Furthermore, using the expression (3.34) for $C_{1,n}$ and denoting $\lambda$ dependence explicitly now,

\[-|C_{1,n}(\lambda)|^2 \Im(\psi_{-}'(X, \lambda - \gamma) \overline{\psi}_{-}(X, \lambda - \gamma)) = C_5(\lambda) e^{-2\Im(k(\lambda)na)}
\]
for some $C_5(\lambda) \in \mathbb{R}$ independent of $n$ so

$$F_n(\lambda) \sim \gamma \frac{C_3(\lambda)}{C_3(\lambda) + C_4(\lambda)} \frac{C_5(\lambda)}{C_3(\lambda) + C_4(\lambda)} \text{ as } n \to \infty.$$ 

Since $\lambda$ does not depend on $n$ and $\Im(\lambda) = F_n$ for all $n$, we must have

$$\Im(\lambda) = \gamma \frac{C_3(\lambda)}{C_3(\lambda) + C_4(\lambda)} \in (0, \gamma).$$

$\square$

4. Inclusion for the essential spectrum

Consider the Sturm-Liouville operator $T_0$ introduced in Section 3. Suppose that the conditions of Assumption 2 are met. As before, fix $\gamma \in \mathbb{C}\{0\}$, define the perturbed operators by

$$T_{R,n} = T_0 u + i \gamma \chi_{[0,R]} u, \quad D(T_{R,n}) = D(T_0) \quad (R \in \mathbb{R}_+)$$

and define the limit operator by $T = T_0 + i \gamma$.

In this section, we prove that the essential spectrum of the limit operator $T$ is approximated by the eigenvalues of $(T_{R,n})_{R \in \mathbb{R}_+}$ as $R \to \infty$. To achieve this, we require an additional assumption which ensures that the solutions $\psi_\pm$ of $T_0 u = \lambda u$ introduced in Assumption 2 can be analytically continued, with respect to the spectral parameter $\lambda$, into an open neighbourhood in $\mathbb{C}$ of any point in the interior of $\sigma_e(T_0)$. The interior of the essential spectrum is denoted by $\text{int}(\sigma_e(T_0))$ and defined with respect to the subspace topology.

**Assumption 3.** $T_0$ is such that $\sigma_e(T_0) \subset \mathbb{R}$. For any $\mu \in \text{int}(\sigma_e(T_0))$, there exists an open neighbourhood $V_\mu$ of $\mu$ such that:

(i) $k$ admits analytic continuations $\kappa_u$ ($\kappa_l$) from the half-planes $\mathbb{C}_+ (\mathbb{C}_-$) respectively into $V_\mu$, with

$$\Im \kappa_u(z), -\Im \kappa_l(z) \begin{cases} > 0 & \text{if } z \in \mathbb{C}_+ \cap V_\mu \\ = 0 & \text{if } z \in \mathbb{R} \cap V_\mu \\ < 0 & \text{if } z \in \mathbb{C}_- \cap V_\mu \end{cases} \quad (4.1)$$

(ii) For any $R > 0$, $\tilde{\psi}_u(R, \cdot)$ admits analytic continuations $\tilde{\phi}_u(R, \cdot) (\tilde{\phi}_l(R, \cdot))$ from $\mathbb{C}_+ (\mathbb{C}_-$) respectively into $V_\mu$ and $\tilde{\psi}_u^d(R, \cdot)$ admits analytic continuations $\tilde{\phi}_u^d(R, \cdot) (\tilde{\phi}_l^d(R, \cdot))$ from $\mathbb{C}_+ (\mathbb{C}_-$) respectively into $V_\mu$. For $j = u$ and $l$, $\tilde{\phi}_j$ and $\tilde{\phi}_j^d$ satisfy

$$\|\tilde{\phi}_j(\cdot, z)\|_{L^\infty(0, \infty)}, \|\tilde{\phi}_j^d(\cdot, z)\|_{L^\infty(0, \infty)} < \infty \quad (4.2)$$

for all $z \in V_\mu$.

Define

$$\phi_j(x, z) = e^{i \kappa_j(x) x} \tilde{\phi}_j(x, z). \quad (j = u \text{ or } l)$$

$\phi_u$ and $\phi_l$ are analytic continuations of $\psi_\pm$ so we have:

**Lemma 4.1.** Let $j = u$ or $l$. For all $x \in [0, \infty)$ and $z \in V_\mu$, we have:

(a) $T_0 \phi_j(x, z) = z \phi_j(x, z)$

(b) $\phi'_j(x, z) = e^{i \kappa_j(x) x} \tilde{\phi}'_j(x, z)$
Proof. We only consider the case \( j = u \), the case \( j = l \) is similar. By Morera’s theorem and Cauchy’s theorem, the function \( z \mapsto (\tilde{T}_0 - z)\varphi_u(x, z) \) is analytic on \( V_{\mu} \) for any fixed \( x \in [0, \infty) \). In addition it vanishes on the open subset \( C_+ \cap V_{\mu} \) so, by unique continuation, it must vanish on \( V_{\mu} \), proving (a). (b) can be proved similarly by considering a function \( z \mapsto \varphi_u'(x, z) - e^{i\kappa_u(z)x} \varphi_d(x, z) \).

Example 4.2 (Schrödinger operators with \( L^1 \) potentials, continued). Consider again the case \( p = r = 1 \) with \( q \in L^1(0, \infty) \), introduced in Example 3.2. Recall that \( k(\lambda) = \sqrt{\lambda} \) so Assumption 3 (i) is satisfied in this case. Recall that \( \tilde{\psi}_\pm(x, z) = 1 + E_\pm(x, z) \) and \( \tilde{\psi}_d^d(x, z) = \pm i\sqrt{2}(1 + E_\pm^d(x, z)) \). In order to show that Assumption 3 (ii) holds in this case it suffices to show that, for any \( \mu \in \text{int}(\sigma_c(T_0)) \) and any \( x, E_\pm(x, \cdot) \) and \( E_\pm^d(x, \cdot) \) admit analytic continuations into an open neighbourhood \( V_{\mu} \) of \( \mu \) with
\[
|E_\pm(x, z)|, |E_\pm^d(x, z)| \to 0 \text{ as } x \to \infty
\]
for all \( z \in V_{\mu} \). Additional conditions on the potential \( q \) are required to ensure that this holds. Two such conditions are:

(a) (Naimark condition [34, Lemma 1]) There exists \( a > 0 \) such that
\[
\int_0^{\infty} e^{\alpha x}|q(x)|\,dx < \infty.
\]

(b) (Dilation analyticity [6]) \( q \) is real-valued and can be analytically continued into some open, convex region \( U \subset \mathbb{C} \) containing a sector \( \{ z \in \mathbb{C} : \arg(z) \in [-\theta, \theta] \} \) for some \( \theta \in (0, \frac{\pi}{2}] \). Furthermore, there exists \( C_0 > 0 \) and \( \beta > 1 \) independent of \( z \) such that
\[
|q(z)| \leq C_0|z|^{-\beta}
\]
for all \( z \in U \).

Example 4.3 (Eventually periodic Schrödinger operators, continued). Consider again the case \( p = r = 1 \) with \( q \) eventually real periodic, introduced in Example 3.3. As mentioned in example 3.3, for any \( \mu \in \text{int}(\sigma_c(T_0)) = \sigma_c(T_0) \backslash B_{\text{ends}} \) and any \( x, k, \tilde{\psi}_\pm(x, \cdot) \) and \( \tilde{\psi}_d^d(x, \cdot) \) admit analytic continuations into an open neighbourhood \( V_{\mu} \) of \( \mu \). By the expression (3.12) for \( \tilde{\rho}_+ \), the analytic continuation \( \tilde{\rho}_+ \) for \( \rho_+ \), from \( \mathbb{C}_\pm \) into \( V_{\mu} \), satisfies
\[
|\tilde{\rho}_+| = \begin{cases} 
1 & \text{if } z \in \mathbb{C}_+ \cap V_{\mu} \\
1 & \text{if } z \in \mathbb{R} \cap V_{\mu} \\
> 1 & \text{if } z \in \mathbb{C}_\mp \cap V_{\mu}
\end{cases}
\]
Hence, the analytic continuations of \( k \) satisfy equation (4.1) of Assumption 3 (ii). By the expressions (3.17) and (3.18), the analytic continuations of \( \psi_+(x, \cdot) \) and \( \tilde{\psi}_d^d(x, \cdot) \) satisfy the \( L^\infty \) estimates (4.2) of Assumption 3 (ii).

Throughout the remainder of the section, let \( \mu \in \text{int}(\sigma_c(T_0)) \) and suppose that the conditions of Assumption 3 are satisfied. Also, assume without loss of generality that \( (i\gamma + V_{\mu}) \cap \mathbb{R} = \emptyset \).

Lemma 4.4. \( \lambda \in i\gamma + V_{\mu} \) is an eigenvalue of \( T_R \) if and only if
\[
g_R(\lambda) = \beta_u(R, \lambda)e^{i\kappa_u(\lambda - i\gamma)}R + \beta_l(R, \lambda)e^{i\kappa_l(\lambda - i\gamma)}R = 0
\]
The proof is similar to the proof of Lemma 3.4.

Proof. Furthermore, $g$ and $\mu$ must lie in $\text{span}_C \{u_1(\cdot, \cdot)\}$, where $u_1$ is defined by

$$u_1(x, \lambda) = BC[\varphi_u(\cdot, \lambda - i\gamma)]\varphi_u(x, \lambda - i\gamma) - BC[\varphi_u(\cdot, \lambda - i\gamma)]\varphi_u(x, \lambda - i\gamma).$$

Since $(i\gamma + V_\mu) \cap \mathbb{R} = \emptyset$, any $L^2$ solution of $(\tilde{T}_0 + i\gamma \chi_{[0, \tilde{R}]}\varphi_u(x, \lambda + \lambda_i)$ must lie in $\text{span}_C \{\psi_u(\cdot, \cdot)\}$. $\lambda$ is an eigenvalue if and only if

$$u_1(R, \lambda)\psi_u(R, \lambda) - u_1'(R, \lambda)\psi_u(R, \lambda) = 0$$

which in turn holds if and only if

$$\beta_u(R, \lambda)e^{i\kappa_u(\lambda - i\gamma)}R + \beta_l(R, \lambda)e^{i\kappa_l(\lambda - i\gamma)}R = 0.$$

$$\Box$$

We proceed on to the proof of inclusion for the essential spectrum of $T_R$, which consists of proving that there exists eigenvalues of $T_R$ accumulating to $\mu + i\gamma$ as $R \to \infty$. We can only achieve this with the additional assumption that $\mu + i\gamma$ does not lie in a region of the complex plane in which either $\beta_u(R, \cdot)$ or $\beta_l(R, \cdot)$ become small as $R \to \infty$. Define a subset $S_\varepsilon$ of $\mathbb{C}$, capturing such regions, by

$$S_\varepsilon = \{\lambda \in i\gamma + V_\mu \cap \mathbb{R} : \liminf_{R \to \infty} |\beta_j(R, \lambda)| = 0, j = u or l\}. \quad (4.6)$$

The strategy of the proof is to first introduce an approximation $g_R^\infty(\lambda)$ to $g_R(\lambda)$ which is valid for $\lambda$ near $\mu + i\gamma$. It is then shown that there exists zeros $\lambda_R^\infty$ of $g_R^\infty$ converging to $\mu + i\gamma$ as $R \to \infty$. A family of simple closed contours $\ell_R$ surrounding $\lambda_R^\infty$ are constructed such that $\text{dist}(\ell_R, \mu + i\gamma) \to 0$ as $R \to \infty$. We estimate $|g_R^\infty|$ from below and $|g_R - g_R^\infty|$ from above on $\ell_R$ to conclude, using Rouche’s Theorem, that there exists a zero $\lambda_R$ of $g_R$ inside $\ell_R$ for all large enough $R$. Such $(\lambda_R)$ would be eigenvalues of $T_R$ and would converge to $\mu + i\gamma$ as $R \to \infty$, giving the result.

**Theorem 4.5.** Assume that $\mu \in \text{int}(\sigma_e(T_0))$ is such that $\mu + i\gamma$ does not lie in $S_\varepsilon$. There exists eigenvalues $\lambda_R$ of $T_R$ ($R \in \mathbb{R}_+$) and a constant $C_0 = C_0(T_0, \gamma, \mu) > 0$ such that

$$|\lambda_R - (\mu + i\gamma)| \leq \frac{C_0}{R}$$

for all large enough $R$.

**Proof.** Let $C > 0$ be an arbitrary constant independent of $R$.

Define approximation $g_R^\infty$ to $g_R$ by

$$g_R^\infty(\lambda) = \beta_u, R e^{i\kappa_u(\lambda - i\gamma)}R - \beta_l, R e^{i\kappa_l(\lambda - i\gamma)}R$$
where $\beta_{u,R} := \beta_u(R, \mu + i\gamma)$ and $\beta_{l,R} := -\beta_l(R, \mu + i\gamma)$. By the definition of $S_\epsilon$, the $L^\infty$ estimates (3.5) of Assumption 2 (ii) and the $L^\infty$ estimates (4.2) of Assumption 3 (ii), there exists $C_1, C_2 > 0$ independent of $R$ such that $\beta_{u,R}$ and $\beta_{l,R}$ satisfy

\begin{equation}
C_1 \leq |\beta_{j,R}| \leq C_2 \quad (j = u \text{ or } l)
\end{equation}

for all large enough $R$. $g_R^\infty(\lambda) = 0$ holds if and only if

\begin{equation}
(\kappa_u - \kappa_l)(\lambda - i\gamma) = -i R \left( \ln \left( \frac{\beta_{l,R}}{\beta_{u,R}} \right) + 2\pi i n \right) =: \tilde{\kappa}(n)
\end{equation}

for some $n \in \mathbb{Z}$. The branch cut $L_R$ of the logarithm function $\ln R$ is defined as the ray from the origin through the point $-\beta_{l,R}/\beta_{u,R}$.

Let $w_0 = (\kappa_u - \kappa_l)(\mu)$, $\tilde{n}(R) = Rw_0/(2\pi)$, $n(R) = \lceil \tilde{n}(R) \rceil$,

\[ A(R) = \ln \left| \frac{\beta_{l,R}}{\beta_{u,R}} \right| \text{ and } B(R) = \text{arg}_R \left( \frac{\beta_{l,R}}{\beta_{u,R}} \right) \]

where $\text{arg}_R$ is the $[0, 2\pi)$-valued argument function that takes value zero on $L_R$. Since $L_R$ includes the point $-\beta_{l,R}/\beta_{u,R}$, we have $B(R) = \pi$. Note that $\mu$ is real since $\sigma_u(T_0) \subset \mathbb{R}$, therefore, by Assumption 3 (i), $\exists w_0 = 0$ and $\tilde{n}(R) \in \mathbb{R}$. We have

\[ \tilde{\kappa}(n(R)) = \frac{B(R) + 2\pi n(R)}{R} - i \frac{A(R)}{R} \]

and so, using estimates (4.7) to bound $A(R)$,

\[ |\tilde{\kappa}(n(R)) - w_0| \leq |\Re \tilde{\kappa}(n(R)) - w_0| + |\Im \tilde{\kappa}(n(R))| \]

\[ \leq \left| \frac{B(R)}{R} \right| + \frac{2\pi |n(R) - \tilde{n}(R)|}{R} + \frac{|2\pi \tilde{n}(R) - w_0|}{R} + \frac{|A(R)|}{R} \]

\[ \leq \frac{C}{R}. \]

Let $h = \kappa_u - \kappa_l - w_0$. Let $\varepsilon > 0$ be small enough so that $|h| > 0$ on $\partial B_\varepsilon(\mu)$. Assumption 3 (i) implies that any $z \in \partial B_\varepsilon(\mu)$ satisfies

\begin{equation}
\text{arg} \left( \frac{h}{|h|} (z) \right) = \text{arg}(h(z)) \in \begin{cases} (0, \pi) & \text{if } z \in \mathbb{C}_+ \cap \mathcal{V}_\mu \\ (0, \pi) & \text{if } z \in \mathbb{R} \cap \mathcal{V}_\mu \\ (\pi, 2\pi) & \text{if } z \in \mathbb{C}_- \cap \mathcal{V}_\mu \end{cases}.
\end{equation}

Note that arg is set so that $\text{arg}(z) = 0$ if $z \in \mathbb{R}_+$. The topological degree (i.e. the winding number) of the map

\[ \frac{h}{|h|} : \partial B_\varepsilon(0) \to \partial B_1(0) \]

is equal to the number of zeros for $h$ in $B_\varepsilon(0)$, counted with multiplicity [19, pg. 110]. (4.9) implies that the topological degree of $h/|h|$ can only be 1, hence $\mu$ is a simple zero of $\kappa_u - \kappa_l$. Consequently, there exists an analytic inverse $(\kappa_u - \kappa_l)^{-1} : B_{2\delta}(w_0) \to U$ for some small enough $\delta > 0$ and some neighbourhood $U$ of $\mu$.

Let $R_0 > 0$ be large enough such that $\tilde{\kappa}(n(R)) \in B_\delta(w_0)$ for all $R \geq R_0$. Define

\begin{equation}
\lambda_R^\infty = (\kappa_u - \kappa_l)^{-1}(\tilde{\kappa}(n(R)) + i\gamma) \quad (R \geq R_0)
\end{equation}

Then $g_R^\infty(\lambda_R^\infty) = 0$ and, by the analyticity of $(\kappa_u - \kappa_l)^{-1}$,

\begin{equation}
|\lambda_R^\infty - (\mu + i\gamma)| \leq C |\tilde{\kappa}(n(R)) - w_0| \leq \frac{C}{R}
\end{equation}
for large enough $R$. For $R \geq R_0$, define family $\ell_R$ of simple closed contours around $\lambda_R^\infty$ by
\begin{equation}
\ell_R = \{ \ell_R(\theta) : \theta \in [0, 2\pi) \},
\end{equation}
\begin{equation}
\ell_R(\theta) = (\kappa_u - \kappa_l)^{-1}(\hat{\kappa}(n(R)) + \frac{\delta}{R} e^{i\theta}) + i\gamma.
\end{equation}
By the analyticity of $(\kappa_u - \kappa_l)^{-1}$ and estimate (4.11), we have that
\begin{equation}
|\ell_R(\theta) - (\mu + i\gamma)| \leq |\ell_R(\theta) - \lambda_R^\infty| + |\lambda_R^\infty - (\mu + i\gamma)| \leq C
\end{equation}
for large enough $R$.

By a direct computation, we have
\begin{equation}
e^{i\kappa_l(\ell_R(\theta) - i\gamma)R} = \frac{\beta_l(R) e^{i\delta e^{i\theta}}}{\beta_u(R)} e^{i\kappa_l(\ell_R(\theta) - i\gamma)R}.
\end{equation}
By Assumption 3 (ii), $\beta_u(R, \cdot)$ and $\beta_l(R, \cdot)$ are analytic and bounded in $R$ uniformly in a small enough neighbourhood of $\mu + i\gamma$, so, using the Cauchy integral formula as in (3.22) and noting that $\text{dist}(\ell_R(\theta), \mu + i\gamma) = O(1/R)$, for $j = u$ or $l$ we have
\begin{equation}
|\beta_j(R, \ell_R(\theta)) - \beta_j(R, \mu + i\gamma)| \leq C|\ell_R(\theta) - (\mu + i\gamma)|
\end{equation}
for large enough $R$. Using (4.7), (4.14), (4.15) and (4.16)
\begin{align*}
|g_R(\ell_R(\theta)) - g_R^\infty(\ell_R(\theta))| &\leq \left| \beta_u(R, \ell_R(\theta)) - \beta_u(R) \right| \left| \frac{\beta_l(R)}{\beta_u(R)} e^{i\delta e^{i\theta}} \right| + |\beta_l(R, \ell_R(\theta)) - \beta_l(R)| e^{-\lambda_R^\kappa(\ell_R(\theta) - i\gamma)R} \\
&\leq C|\ell_R(\theta) - (\mu + i\gamma)| e^{-\lambda_R^\kappa(\ell_R(\theta) - i\gamma)R} \leq \frac{C}{R} e^{-\lambda_R^\kappa(\ell_R(\theta) - i\gamma)R}
\end{align*}
for large enough $R$. Similarly,
\begin{align*}
|g_R^\infty(\ell_R(\theta))| &\leq |\beta_l(R)| e^{i\delta e^{i\theta}} - 1 | e^{-\lambda_R^\kappa(\ell_R(\theta) - i\gamma)R} \geq C e^{-\lambda_R^\kappa(\ell_R(\theta) - i\gamma)R}.
\end{align*}
For large enough $R$, Rouché’s condition
\begin{align*}
|g_R(\ell_R(\theta)) - g_R^\infty(\ell_R(\theta))| < |g_R^\infty(\ell_R(\theta))|
\end{align*}
is satisfied so there exists a zero $\lambda_R$ of $g_R$ in the interior of $\ell_R$ such that
\begin{align*}
|\lambda_R - (\mu + i\gamma)| \leq |\lambda_R - \lambda_R^\infty| + |\lambda_R^\infty - (\mu + i\gamma)| \leq C_0
\end{align*}
for some $C_0 > 0$ independent of $R$. \qed

We finish this section with a characterisation of the set $S_\varepsilon$ in the case that $T_0$ is a Schrödinger operator with an $L^1$ or an eventually real periodic potential.

Define function $\Lambda_u : [0, \infty) \times (i\gamma + V_\mu) \to \mathbb{C}$ by
\begin{equation}
\Lambda_u(R, \lambda) = \tilde{\varphi}_+(R, \lambda - i\gamma) \tilde{\psi}_+(R, \lambda) - \tilde{\varphi}_-(R, \lambda - i\gamma) \tilde{\psi}_-(R, \lambda)
\end{equation}
so that, by the definition of $\beta_u$ in Theorem 4.4, $\beta_u(R, \lambda) = BC[\varphi_u(\cdot, \lambda - i\gamma)]\Lambda_u(R, \lambda)$. Similarly, define function $\Lambda_l : [0, \infty) \times (i\gamma + V_\mu) \to \mathbb{C}$ by
\begin{equation}
\Lambda_l(R, \lambda) = \tilde{\psi}_+(R, \lambda) \tilde{\varphi}_+^d(R, \lambda - i\gamma) - \tilde{\psi}_-(R, \lambda) \tilde{\varphi}_-(R, \lambda - i\gamma)
\end{equation}
so that $\beta_l(R, \lambda) = BC[\varphi_u(\cdot, \lambda - i\gamma)]\Lambda_l(R, \lambda)$.

$S_\varepsilon$ can be decomposed as
\begin{equation}
S_\varepsilon = (i\gamma + S_{\varepsilon, 0}) \cup S_{\varepsilon, u} \cup S_{\varepsilon, l}
\end{equation}
where
\begin{equation}
S_{\tau,0} := \{ z \in V_\mu \cap \mathbb{R} : BC[\varphi_u(\cdot, z)] = 0 \text{ or } BC[\varphi_l(\cdot, z)] = 0 \}
\end{equation}
and
\begin{equation}
S_{\tau,j} := \left\{ \lambda \in i\gamma + V_\mu \cap \mathbb{R} : \liminf_{R \to \infty} \left| \Lambda_j(R, \lambda) \right| = 0 \right\} \quad (j = u \text{ or } l).
\end{equation}
The elements of $S_{\tau,0}$ are precisely the resonances of $T_0$ in $V_\mu \cap \mathbb{R}$, by definition.

**Example 4.6** (Schrödinger operators with $L^1$ potentials, continued). Consider again the case $p = r = 1$ with $q \in L^1(0, \infty)$ satisfying the necessary conditions ensuring that Assumption 3 holds, as discussed in Example 4.2. In this case, since the functions $E^\pm(R, \lambda)$ and $E'^\pm(R, \lambda)$ tend to zero as $R \to \infty$ for any $\lambda$, $\Lambda_u$ and $\Lambda_l$ satisfy
\[
|\Lambda_j(R, \lambda)| \to |\sqrt{\lambda - i\gamma} - \sqrt{\lambda}| \quad \text{as } R \to \infty \quad (j = u \text{ or } l)
\]
for all $\lambda \in i\gamma + V_\mu$, where the square-root is understood to have been analytically continued from $\mathbb{C}_+$ ($\mathbb{C}_-$) into $V_\mu$ in the case $j = u$ ($j = l$) respectively. Since $\sqrt{\lambda - i\gamma} \neq \sqrt{\lambda}$ for all $\lambda \in i\gamma + V_\mu$, regardless of which branch-cut for the square-root is chosen, we have
\[
S_{\tau,u} = S_{\tau,l} = \emptyset.
\]
Consequently,
\[
S_{\tau} = i\gamma + S_{\tau,0},
\]
that is, $\mu + i\gamma \in S_{\tau}$ if and only if $\mu$ is a resonance of $T_0$

**Example 4.7** (Eventually periodic Schrödinger operators, continued). Consider the case $p = r = 1$ with $q$ real-valued and $q_{||X,\infty||} a$-periodic for some $X \geq 0$ and $a > 0$. Assume that $\eta \in [0, \pi)$, so that $T_0$ is equipped with a real mixed boundary condition at 0. Note that $T_0$ is self-adjoint in this case. $q$ is eventually real periodic so by Example 4.3, Assumption 3 is satisfied. The sets $S_{\tau,u}$ and $S_{\tau,l}$ satisfy
\begin{equation}
S_{\tau,u} = S_{\tau,l} = \emptyset.
\end{equation}
Consequently,
\[
S_{\tau} = i\gamma + S_{\tau,0},
\]
that is, $\mu + i\gamma \in S_{\tau}$ if and only if $\mu$ is a resonance of $T_0$

**Proof of (4.22).** We will only prove (4.22) for $j = u$, the proof for $j = l$ is similar.

Assume for contradiction that $S_{\tau,u}$ is non-empty and let $\lambda \in S_{\tau,u}$. By unique continuation, expressions analogous to (3.17) and (3.18) hold for $\check{\varphi}_u$ and $\check{\varphi}_u^L$. By these expressions, there exists a sequence $(x_{0,n}) \subset [X, X + a)$ such that $\Lambda_u(x_{0,n}, \lambda) \to 0$ as $n \to \infty$. Let $x_0$ be any accumulation point of $(x_{0,n})$. Then, since $\Lambda_u(\cdot, \lambda)$ is absolutely continuous, it holds that $\Lambda_u(x_0, \lambda) = 0$, so,
\begin{equation}
\varphi_u(x_0, \lambda - i\gamma)\psi_+(x_0, \lambda) - \varphi_u^L(x_0, \lambda - i\gamma)\psi_+(x_0, \lambda) = 0.
\end{equation}
Noting that the solutions $\phi_1(\cdot, \lambda - i\gamma)$ and $\phi_2(\cdot, \lambda - i\gamma)$ defined by (3.9) are real since $\lambda - i\gamma \in \mathbb{R}$ and that the analytic continuations $\rho_u(\rho_1)$ for $\rho_+$ from $\mathbb{C}_+(\mathbb{C}_-)$ respectively satisfy $\rho_u(\lambda - i\gamma) = \rho(\lambda - i\gamma)$, the expression analogous to (3.14) for the Floquet solution $\check{\varphi}_u$ implies that
\[
\check{\varphi}_u(x, z) = -\phi_2(X + a, z)\phi_1(x, z) + (\phi_1(X + a, z) - \check{\varphi}_u(z))\phi_2(x, z)
\]
\[
= -\phi_2(X + a, z)\phi_1(x, z) + (\phi_1(X + a, z) - \rho(z))\phi_2(x, z) = \varphi_l(x, z)
\]
which is the desired contradiction. □

Consider perturbed operators of the form

Example 5.1.

\begin{align}
\psi(x, 0, \lambda - i\gamma) \overline{\psi_+(x, 0, \lambda)} - \overline{\psi_+(x, 0, \lambda - i\gamma)} \psi_+(x, 0, \lambda) = 0.
\end{align}

By (4.23) and (4.24), there exists \( C_{1, u}, C_{2, u}, C_{1, t}, C_{2, t} \in \mathbb{C} \) independent of \( x \) such that the functions

\[
u_u(x, \lambda) := \begin{cases} 
C_{1, u} \varphi_u(x, \lambda - i\gamma) & \text{if } x \in [0, x_0) \\
C_{2, u} \psi_+(x, \lambda) & \text{if } x \in [x_0, \infty)
\end{cases}
\]

\[
u_l(x, \lambda) := \begin{cases} 
C_{1, l} \varphi_l(x, \lambda - i\gamma) & \text{if } x \in [0, x_0) \\
C_{2, l} \psi_+(x, \lambda) & \text{if } x \in [x_0, \infty)
\end{cases}
\]

are absolutely continuous and solve the Schrödinger equation \( \hat{T}_{x_0} u = \lambda u \). Note that \( \psi_+ \) solves the Schrödinger equation \( \hat{T}_{x_0} \psi = \lambda \psi \) on \( [x_0, \infty) \) because \( q \) is real-valued. By orthogonality, there exists \( (a_u, a_t) \in \mathbb{C}^2 \setminus \{(0, 0)\} \) such that

\[
BC[a_u u_u(\cdot, \lambda) + a_t u_l(\cdot, \lambda)] = a_u C_{1, u} BC[\varphi_u(\cdot, \lambda - i\gamma)] + a_t C_{1, t} BC[\varphi_l(\cdot, \lambda - i\gamma)] = 0
\]

This implies that \( \lambda \) is an eigenvalue of \( T_{x_0} \) with corresponding eigenfunction \( u := a_u u_u + a_t u_l \). By a standard integration by parts,

\[
\Im(\lambda) = \gamma \frac{\int_0^{x_0} |u|^2}{\int_0^{\infty} |u|^2} < \gamma
\]

which is the desired contradiction. □

5. Numerical examples

In this section, we illustrate the results from sections 3 and 4 with numerical examples.

Example 5.1. Consider perturbed operators of the form

\begin{align}
T_R = -\frac{d^2}{dx^2} + i\chi_{[0, \eta]}(x) \quad (R \in \mathbb{R}_+)
\end{align}

endowed with Dirichlet boundary conditions at 0. This corresponds to the case \( p = r = 1, q = 0, \eta = 0, \gamma = 1 \) in Sections 3 and 4.

By an explicit computation, \( \lambda \in \mathbb{C} \setminus [0, \infty) \) is an eigenvalue of \( T_R \) if and only if

\begin{align}
f_R(\lambda) = i\sqrt{\lambda} \sin(\sqrt{\lambda - iR}) - \sqrt{\lambda - iR} \cos(\sqrt{\lambda - iR}) = 0.
\end{align}

Note that our convention is that the branch cut of the square-root is along \([0, \infty)\). By suitably analytically continuing the square root function in (5.2), any \( \lambda \) in the lower right quadrant of the complex plane is a resonance of \( T_R \) if and only if \( f_R(\lambda) = 0 \).

To numerically compute the zeros of \( f_R \), hence the eigenvalues and resonances of \( T_R \), in a fixed bounded region, we use a Python implementation of an algorithm utilising the argument principle [13]. The results are illustrated in Figure 2.

For small enough \( R > 0 \), \( T_R \) has no eigenvalues [17]. As \( R \) increased, we observe resonances in the lower half plane emerging out of \( \sigma_e(T_R) = [0, \infty) \), to become eigenvalues in

\[
\Gamma_\gamma := (0, \infty) \times i(0, \gamma)
\]

accumulating to \( \sigma_e(T) = i\gamma + [0, \infty) \), as expected by Theorem 4.5.
Figure 2. Plot of the eigenvalues and resonances of the operator $T_R$ defined by (5.1).

Example 5.2. Consider perturbed operator of the form

$$T_R = T_0 + i\chi_{[0,R]}(x) = -\frac{d^2}{dx^2} + i\chi_{[0,R_0]}(x) + i\chi_{[0,R]}(x) \quad (R \in \mathbb{R}_+),$$

endowed with Dirichlet boundary conditions at 0. This corresponds to the case $p = r = 1$, $\eta = 0$, $q = i\chi_{[0,R_0]}$ for some $R_0 > 0$ and $\gamma = 1$ in Sections 3 and 4.

By an explicit computation, $\lambda \in \mathbb{C}\setminus[0, \infty)$ is an eigenvalue of $T_R$ if and only if

$$f_R(\lambda) = i\sqrt{\lambda-i} \left[ e^{-2i\sqrt{\lambda-i}(R-R_0)} - \frac{\sqrt{\lambda-i} - \sqrt{\lambda}}{\sqrt{\lambda-i} + \sqrt{\lambda}} \right] \sin(\sqrt{\lambda-2i}R_0)$$

$$- \sqrt{\lambda-2i} \left[ e^{-2i\sqrt{\lambda-i}(R-R_0)} + \frac{\sqrt{\lambda-i} - \sqrt{\lambda}}{\sqrt{\lambda-i} + \sqrt{\lambda}} \right] \cos(\sqrt{\lambda-2i}R_0) = 0$$

As before, by suitably analytically continuing the square root function in (5.4), any $\lambda$ in the lower right quadrant of the complex plane is a resonance of $T_R$ if and only if $f_R(\lambda) = 0$.

A numerical computation of the zeros of $f_R$, hence the eigenvalues and resonances of $T_R$ is shown in Figure 3. We observe that there are eigenvalues of $T_R$ converging
Figure 3. Plot of eigenvalues and resonances of the operators $T_R$ and $T = T_0 + i \gamma$ defined by (5.3), with $R_0 = 4.7$.

rapidly to the eigenvalues of $T$ and that eigenvalues of $T_R$ accumulate to $\sigma_e(T) = i\gamma + [0, \infty)$, as expected by Theorems 3.6 and 4.5.

Recall that Example 4.7 guarantees that the rate of convergence of eigenvalues of $T_R$ to $\mu \in \text{int}(\sigma_e(T)) = i\gamma + (0, \infty)$ is $O(1/R)$, unless $\mu$ is a resonance of $T$. The limit operator $T$ for our choice of parameters has a resonance embedded in $\sigma_e(T)$. We seem to observe a distinction between the way the eigenvalues of $T_R$ accumulate to the resonance compared to other points in the interior of $\sigma_e(T)$. It seems reasonable to conjecture that the rate of convergence to embedded resonances is indeed slower that $O(1/R)$.

Example 5.3. Consider perturbed operators of the form

(5.5) \[ T_R = T_0 + \frac{i}{4} \chi_{[0,R]}(x) = -\frac{d^2}{dx^2} + \sin(x) + \frac{i}{4} \chi_{[0,R]}(x) \ (R \in \mathbb{R}_+) \]

endowed with a Dirichlet boundary condition at 0. This corresponds to the case $p = r = 1$, $\eta = 0$, $q(x) = \sin(x)$ and $\gamma = \frac{1}{4}$ in Section 3 and 4. The essential
spectrum of $T_0$ has a band gap structure - the first spectral band, which we denote by $B$, is approximately $[-0.3785, -0.3477]$ [28, Example 15].

To numerically compute the eigenvalues of $T_R$, we first perform a domain truncation onto an interval $[0, X]$, imposing a Dirichlet boundary condition at $X$. Applying a finite difference method with step-size $h$, we obtain a finite matrix $T_{R,X,h}$. For fixed $R$, the eigenvalues of $T_{R,X,h}$ accumulate to every point in $\sigma(T_R)$ as $X \to \infty$ and $h \to 0$. Moreover, any point of accumulation that does not lie in $\sigma(T_R)$ must lie on the real-line (see [8] and [28]).

For a fixed small value of $h$, a fixed large value of $X - R$, the eigenvalues of $T_{R,X,h}$ for increasing $R$ are plotted in Figure 4. We first observe an accumulation of eigenvalues of $T_{R,X,h}$ to the interval $B$ in $\mathbb{R}$. These eigenvalues of $T_{R,X,h}$ are due to the domain truncation method approximating $\sigma_e(T_R)$ and should not be interpreted as approximations of the eigenvalues of $T_R$. All other points in the plots are approximations of the eigenvalues of $T_R$.

In Figure 4, we observe that as $R$ increases, eigenvalues of $T_R$ emerge out of the spectral band $B$ and tend to the shifted spectral band $i\gamma + B$, which is a subset of $\sigma_e(T)$. For large $R$, we observe an accumulation of eigenvalues to $i\gamma + B$, as predicted by Theorem 4.5. We refer the reader to Example 3.9 for a characterisation of the possible points of spectral pollution for this system.
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