NONLINEAR DIRAC EQUATIONS and NONLINEAR GAUGE TRANSFORMATIONS

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Abstract
Nonlinear Dirac equations (NLDE) are derived through a group $\mathfrak{g}^2$ of nonlinear (gauge) transformation acting in the corresponding state space. The construction generalises a construction for nonlinear Schrödinger equations. To relate $\mathfrak{g}^2$ with physically motivated principles we assume: locality (i.e. it contains no explicit derivative and no derivatives of the wave function), separability (i.e. it acts on product states componentwise) and Poincaré invariance. Furthermore we want that a positional density is invariant under $\mathfrak{g}^2$. Such nonlinear transformations yield NLDE which describe physically equivalent systems. To get 'new' systems, we extend this NLDE (gauge extension) and present a family of NLDE which is a slight nonlinear generalisation of the Dirac equation. We discuss and comment the fact that nonlinear evolutions are not consistent with the usual framework of quantum theory. To develop a corresponding extended framework one needs models for nonlinear evolutions which also indicate possible physical consequences of nonlinearities.
1 INTRODUCTION

There is some recent interest in fundamental nonlinear quantum mechanical evolution equations. Different proposals for (non relativistic) nonlinear Schrödinger equations (NLSE) are known: Some of them are physically plausible modifications of the linear equation (LSE) [1–4]; others are based on some first principles (non relativistic quantum mechanics) [5–8]. It is known [7–11] (see [12] for a short review) that nonlinear evolutions are not consistent with the usual (linear) quantum mechanical framework. For an interpretation and application of NLSE a corresponding extension of this framework is needed [11, 13]. If this extension allows to show in approximation, e.g., with a nonlinear evolution equation, that a nonlinearity yields measurable effects, one can decide whether a deviation from the linear prediction can be viewed as an information on a nonlinear extension of quantum mechanics [16–17].

The recent results refer to non relativistic Schrödinger equations. A generalisation to relativistic Dirac equations is possible. One can use the method [7, 8] of nonlinear transformations in the state space; they are assumed to be consistent with first principles: Locality and separability condition, Poincaré invariance and a notion for an equivalent description of quantum systems. We present such a generalisation with a step-by-step construction (see a detailed review [13]). A family of nonlinear Dirac equations (NLDE) is obtained which is ‘near’ or ‘similar’ equivalent to those families which are equivalent to linear ones.

We explain our construction in section 2. Families of nonlinear transformations are constructed in section 3 with the resulting NLDE and their extensions. Section 4 contains concluding remarks.

2 A METHOD TO CONSTRUCT NONLINEAR DIRAC EQUATIONS (NLDE)

Consider the function space \( \mathcal{G}_4 = C^1(\mathbb{C}^4, \mathbb{C}) \) on Minkowski space-time, i.e. complex (column) vector valued functions \( \psi \) with components \( \psi_j = \psi_j(x_0, \ldots, x_3), \) \( j = 0, 1, 2, 3; \) define \( \bar{\psi} = \psi^* \gamma_0. \) The Hilbert space of a 1-particle Dirac system is denoted as \( \mathcal{H}_D; \) use \( \tilde{\mathcal{G}}_4 = \mathcal{G}_4 \cap \mathcal{H}_D \) for an interpretation of the later results. The evolution is given through a family \( \mathcal{F}_0 \) of linear Dirac operators and equations (parameter \( m \geq 0 \))

\[
\mathcal{D}_D = (\gamma_\mu p^\mu - m), \quad \mathcal{D}_D \psi = 0 \text{ with } p^\mu = i \frac{\partial}{\partial x^\mu} \text{ on } \tilde{\mathcal{G}}_4. \tag{1}
\]

Consider a group \( \mathcal{N} \) of invertible (in general nonlinear) transformations \( N \) acting on \( \mathcal{G}_4 \) or \( \tilde{\mathcal{G}}_4 \)

\[
N : \psi \mapsto N \psi.
\]

The \( N \) may depend on \( \psi \) (and \( \bar{\psi} \)), derivatives of \( \psi, \bar{\psi} \) and explicitly on \( \partial_\mu \) and \( x_\mu. \)
If $\psi$ is a solution of $\mathbb{D}_D \psi = 0$ then $\psi' = N^{-1} \psi$ is a solution of
$$\mathbb{D}_D^N \psi' = 0 \quad \text{with} \quad \mathbb{D}_D^N = \mathbb{D}_D N.$$  

$\mathbb{D}_D^N$ is a (linearisable) nonlinear Dirac operator.

Our construction of a physical acceptable nonlinear Dirac operator starts with the group $N$. As mentioned (section 1) we select subgroups of $N$ which are consistent with plausible additional requirements ('first principles') in step 1 - 4. In step 5 we extend the obtained families of NLDE to families not equivalent to $\mathcal{F}_0$.

**STEP 1 (Locality):**

The Dirac operator $\mathbb{D}_S$ is a first order PDO. We want the same for $\mathbb{D}_S^N$. This implies a locality condition: $N$ does not depend on derivatives of $\psi$, $\bar{\psi}$ and not explicitly on $\partial_\mu$. For later invariance properties (step 4) it is reasonable to assume that also an explicit dependence of $x_\mu$ (and functions of $x_\mu$) will not appear. Hence one has

$$N\psi = N(\psi) \psi = (N_0(\psi) \psi_0, \ldots, N_3(\psi) \psi_3)^\top. \quad (2)$$

It is convenient to use also $N\psi \equiv N[\psi] \equiv N(\psi) \psi$. The set of transformations (2) is a group $N^l \subset N$; the $N(\psi)$ act on $\mathcal{G}_4$ as matrix valued multiplication operators.

**STEP 2 (Separability)**

Quantum theory describes not only 1-particle systems but necessarily also systems build from $n$ particles. The corresponding $n$-particle Hilbert space is - following first principles - a product space of 1-particle spaces; it is spanned through the linear completion of a set of product wave functions

$$\mathcal{P}^{(n)} = \{ \psi^1 \otimes \cdots \otimes \psi^n \mid \psi^i \in \mathcal{H}^1_D, i = 1, \ldots, n \}. $$

To extend linear operators $A \equiv A^{(1)}$ on $\mathcal{H}^1_D$ to (linear) operators $A^{(n)}$ on $\mathcal{H}^n_D$ one defines first an action on $\mathcal{P}^{(n)}$,

$$A^{(n)}(\psi^1 \otimes \cdots \otimes \psi^n) = A\psi^1 \otimes \cdots \otimes A\psi^n. $$

This definition extends uniquely by linear completion from $\mathcal{P}^{(n)}$ to $\mathcal{H}^n_D$ (or a dense set in $\mathcal{H}^n_D$). For the nonlinear operators - like those considered here - this construction for $n$-particle operators from 1-particle ones is not possible. To have a property of $N^{(n)}$ which is at least partly consistent with the linear theory we assume that $N$ acting on $\mathcal{H}^1_D$ extend to the set $\mathcal{P}^{(n)}$ as

$$N^{(n)}(\psi^1 \otimes \cdots \otimes \psi^n) = N\psi^1 \otimes \cdots \otimes N\psi^n, \quad (3)$$

which is a (weak) separation property for $N$. The set of $N$ with (3) forms a subgroup $N^s$ of $N$. Note that one needs additional information (see e.g. II) to extend $N^{(n)}$ from $\mathcal{P}^{(n)}$ to $\mathcal{H}^n_D$. This above construction yields evolution equations for 1-particle systems only.
STEP 3 (Poincaré Invariance)

The Dirac Operator $\mathcal{D}_D$ in $\mathcal{H}_D^1$ behaves under the inhomogeneous Lorentz group with a spin $\frac{1}{2}$ representation $U$ in $\mathcal{H}_D^1$. This invariance is a principal property of the Dirac system. However, for the transformed operator $\mathcal{D}_D^N$ this property may be lost. Therefore one has to guarantee it through an assumption on $N$. For this we use the following information: The generators of $U$ are

$$P_\mu = p_\mu; \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).$$

$N$ is a function of $\psi$ (step 1). This leads easily to a condition

$$N(U\psi) = U N(\psi) \text{ with } U = \exp \sum_{\mu,\nu=0,\mu \neq \nu}^3 \alpha_{\mu\nu} \gamma_\mu \gamma_\nu,$$

which implies Poincaré invariance of $\mathcal{D}_D^N$. The set of local transformations in $\mathcal{N}^l$ with condition (4) form a group $\mathcal{N}^l,P$.

STEP 4 (Equivalence)

In non relativistic quantum mechanics (e.g., in $\mathbb{R}^3_x$) one can argue that the positional density $\rho(x,t)$ for all $x$ and $t$ is a fundamental observable of a system in the sense that all observables can be calculated from the information encoded in $\rho(x,t)$. This density is connected to (pure) states $\phi$ through $\phi^* \phi$, and was called utility function in [18]. For transformations $N$ with an invariant utility function

$$N[\phi]^* N[\phi] = \phi^* \phi,$$

the positional densities (5) for $\phi$ and for $N^{-1}\phi$ are equal. This implies a notion of equivalence. (5) is an equivalence condition. Wave functions $\psi$ which evolve with a Schrödinger operator

$$\mathcal{D}_S = i\partial_t - H$$

($H$ a Hamiltonian) and a transformed wave function $N^{-1}\phi$ with Schrödinger operator (which may be nonlinear) $\mathcal{D}_SN$ describe the same physics; they are equivalent. Linear $N[\phi]$, i.e. $N[\phi] = \exp ia(x, t)\phi$ are usual gauge transformations. It is reasonable to denote general $N$ in the following as nonlinear gauge transformations $[7, 8]$.

In the relativistic Dirac case a corresponding argument is used. Here the utility function with a physically reasonable density $\rho$ is

$$F_\rho = \psi^\dagger \psi.$$

with $\psi^\dagger = (\psi^*_0, \ldots, \psi^*_3)$. Dirac systems related through $N$ are equivalent if $N$ leaves $F_\rho$ invariant, i.e.

$$N[\psi]^\dagger N[\psi] = \psi^\dagger \psi.$$

Transformations $N \in \mathcal{N}^l$ with (7) form a group $\mathcal{N}^l,e$. 

4
STEP 5 (Extension)

The intersection
\[ \mathcal{N}^{l,s} \cap \mathcal{N}^{l,P} \cap \mathcal{N}^{l,e} = \mathcal{N}^2 \] (8)
yields a family \( \mathcal{F}_1 \) of NLDE which is - because of the invariance of the utility function - equivalent to the linear family \( \mathcal{F}_0 \). With steps 1.-4. one obtains interesting nonlinear reformulations of the linear Dirac equation. However, we are looking for 'new' systems which are 'near' to 'physically equivalent' ones, i.e. in systems which are local, separable and Poincaré invariant but not equivalent. The following construction leads to such systems through a simple 'extension' of \( \mathcal{F}_1 \): It turns out that \( \mathcal{N}^2 \) is characterised by a real number and through complex functions which are related among themselves but otherwise arbitrary. The evolution equations depend on the relation between these functions. Hence one can 'extend' the family \( \mathcal{N}^2 \) if one breaks this relation, i.e. if one chooses the functions and the number independently. Such a family describes 'new' systems. We remark already here that a general framework to treat spin 1/2 particles with nonlinear evolutions is not (yet) known and a NLDE is not immediately applicable.

3 RESULTS OF THE CONSTRUCTION

We explain details of the above step-by-step method and present some obtained NLDE families.

3.1 LOCALITY AND SEPARABILITY

In step 1 we motivated the subgroup \( \mathcal{N}^l \) of nonlinear transformations which are functions of \( \psi \) and which act as \( 4 \times 4 \) matrices on \( \mathcal{G}_4 \).

To fulfil the separation property (3) in step 2 it is sufficient to discuss the two particle case \( n=2 \),
\[ N^{(2)}(\psi^1 \otimes \psi^2) = N\psi^1 \otimes N\psi^2, \quad \psi^i \in \mathcal{H}^1_D, \quad i = 1, 2, \] (9)
i.e. we demand the existence of an \( n \) component function \( N^{(2)} \) on such that for \( N \) and for any two \( \psi^1 \) and \( \psi^2 \) relation (9) holds. For \( n > 2 \) we get corresponding results.

To calculate the resulting form of \( N \) use the (non unique) polar decomposition of the components \( \psi^k_i = R^k_i \exp iS^k_i, \quad k = 1, 2, \quad i = 0, \ldots, 3 \), express the components of \( N(\psi^k)\psi^k \) in terms of \( R^k, S^k \) with \( R^k = \{ R^k_1, \ldots, R^k_n \}, \quad S^k = \{ S^k_1, \ldots, S^k_n \} \) and write
\[ N(\psi^k)\psi^k = F(R^k, S^k), \quad k = 1, 2. \]

To ensure (9) we have to prove the existence of functions \( G_{ij} \) such that
\[ F_i(R^1, S^1)F_j(R^2, S^2) = G_{ij}(\mathcal{R}, \mathcal{S}), \] (10)
where \( R^1, S^1, R^2, S^2 \) have 4 components, \( \mathcal{R}, \mathcal{S} \) have the components \( R^1_i R^2_j \) and \( S^1_i + S^2_j \) and \( i, j = 0, \ldots, 3 \). The \( G_{ij} \) are invariant under the following one-parameter groups with real parameters \( \tau, \theta \)

1. The group of scale transformations on \( \mathcal{R} \)
\[
R'^{\tau} = R^1 \exp(\tau), \quad R'^{\tau} = R^2 \exp(-\tau),
\]
2. The group of translations on \( \mathcal{S} \)
\[
S'^{\tau} = S^1 + \theta, \quad S'^{\tau} = S^2 - \theta
\]

which are generated by Lie vector fields
\[
D = \sum_{j=0}^{3} \left( R^1_j \frac{\partial}{\partial R^1_j} - R^2_j \frac{\partial}{\partial R^2_j} \right), \quad \text{and} \quad P = \sum_{j=0}^{3} \left( \frac{\partial}{\partial S^1_j} - \frac{\partial}{\partial S^2_j} \right).
\]

Take the functional equation (9) and apply \( D \) and \( P \); the right hand side vanishes and yields the following system of differential-functional equations:
\[
(DF_i)F_j + (DF_j)F_i = 0, \quad (PF_i)F_j + (PF_j)F_i = 0,
\]
\( i, j = 0, \ldots, 3 \). Dividing these equations by (non-zero) functions \( F_iF_j \) we represent them as follows:
\[
\frac{1}{F_i} \sum_{k=0}^{3} R^1_k \frac{\partial F_i}{\partial R^1_k} = \frac{1}{F_j} \sum_{k=0}^{3} R^2_k \frac{\partial F_j}{\partial R^2_k}, \quad (11)
\]
\[
\frac{1}{F_i} \sum_{k=0}^{3} \frac{\partial F_i}{\partial S^1_k} = \frac{1}{F_j} \sum_{k=0}^{3} \frac{\partial F_j}{\partial S^2_k}, \quad (12)
\]
for \( i, j = 0, \ldots, 3 \). Both sides of the above equations depend on different variables, hence there exist complex parameters \( a \) and \( b \) such that
\[
\sum_{k=0}^{3} R_k \frac{\partial F_i}{\partial R_k} = a F_i, \quad (13)
\]
\[
\sum_{k=0}^{3} \frac{\partial F_i}{\partial S_k} = b F_i, \quad (14)
\]
holds for any \( i = 0, \ldots, 3 \).

For \( n > 1 \) the general solution of (13) can be written in the form
\[
F_i = (R_i)^a H_i \left( \frac{R_1}{R_0}, \ldots, \frac{R_3}{R_0}, S_0, \ldots, S_3 \right), \quad i = 0, \ldots, 3.
\]
with arbitrary smooth complex-valued functions \( H_i \); instead of \( \frac{R_j}{R_0} \), \( j = 1, 2, 3 \), one can also use \( \frac{R_j}{R_k} \), \( j \neq k \), \( k \) fixed, \( k, j = 0, \ldots, 3 \). With (13)
\[
\sum_{k=0}^{3} \frac{\partial H_i}{\partial S_k} = b, \quad i = 0, \ldots, 3
\]
holds. Integrating the equations we arrive at the final form\(^1\) for \(F_i\)

\[
F_i = (R_i)^a \exp(bS_i) G_i \left( \frac{R_1}{R_0}, \ldots, \frac{R_3}{R_0}, S_1 - S_0, \ldots, S_3 - S_0 \right). 
\]

(15)

Here \(G_i\) are arbitrary smooth functions of the indicated variables and \(i = 0, \ldots, 3\). We suppose that the functions \(H_i\) are well-defined under \(R_3 \to 0\). This form of \(N\) is necessary for the separation condition of \(N^{(n)}\) on \(P^{(n)}\) for any \(n \geq 2\). It is also sufficient. For a straightforward proof use the identities for \(n = 2\)

\[
\frac{R_1^1}{R_1^n} = \frac{R_1^1 R_2^2}{R_1^n R_2^j}, \quad \frac{R_2^2}{R_2^j} = \frac{R_2^2 R_1^1}{R_2^j}, \\
S_1^1 - S_1^n = (S_1^1 + S_2^2) - (S_1^n + S_2^1), \\
S_2^2 - S_2^n = (S_2^1 + S_1^1) - (S_2^n + S_1^1),
\]

Analogous identities are used for \(n > 2\). We summarize the result:

The general form of a transformation \(N\):

\[
N : G_4 \to G_4
\]

which satisfies locality and separation conditions is given through

\[
N_{j[a,b,G]}(\psi) = (R_j)^{a-1} \exp((b - i)S_j) \\
\times G_j \left( \frac{R_1}{R_0}, \ldots, \frac{R_3}{R_0}, S_1 - S_0, \ldots, S_3 - S_0 \right),
\]

(16)

\(N\) is labelled by two arbitrary complex parameter \(a, b\) and 4 functions \((G_0, \ldots, G_3) \equiv G\). The \(G_j\) can be written also in terms of \(\psi, \bar{\psi}\)

\[
G_j = G_j \left( \frac{\psi_1}{\psi_0}, \ldots, \frac{\psi_3}{\psi_0}, \frac{\bar{\psi}_1}{\bar{\psi}_0}, \ldots, \frac{\bar{\psi}_3}{\bar{\psi}_0} \right)
\]

(17)

We expect that the transformations \(N\) form a group. Arrange the labelling complex parameters \(a\) and \(b\) in matrix form

\[
K = K(a, b) = \begin{pmatrix} \tilde{a} & \hat{a} \\ \tilde{b} & \hat{b} \end{pmatrix}, \quad a = \tilde{a} + i\tilde{b}, \quad b = \tilde{b} + i\hat{b}
\]

(18)

and compute the product

\[
N_{[K_1,G_1]} \circ N_{[K_2,G_2]} = N_{[K_3,G_3]},
\]

or, more detailed,

\[
N_{[K_2,G_2]}(N_{[K_1,G_1]}(\psi) \psi) N_{[K_1,G_1]}(\psi) = N_{[K_3,G_3]}(\psi) \psi.
\]

(19)

\(^1\)Instead of \(R_k/R_3, k = 0, \ldots, 2\) one can use also \(R_k/R_l, l\) fixed, \(k = 0, \ldots, 3, k \neq l, \text{e.g.,} l = 0\).
The result for $K_3$ is the matrix product

$$K_3 = K_2 K_1.$$ 

For the components of $G_3$ we find

$$G_{3j} = |G_{1j}|^{a_2} \exp(b_2 \arg G_{1j}) G_{2j}(u_1, \ldots, u_3, v_1, \ldots, v_3)$$

(20)

the variables of $G_{1j}$ were given in (16); the $u_l$, $v_l$ in $G_{2j}$ are $(l = 1, 2, 3)$

$$u_l = \left(\frac{R_l}{R_0}\right)^{a_1} \exp \left(b_1 (S_l - S_0)|G_{1l}||G_{10}|^{-1}\right),$$

$$v_l = \hat{a}_1 \ln \left(\frac{R_l}{R_0}\right) + \hat{b}_1 (S_l - S_0) + \arg(G_{1l} - G_{10}).$$

Hence the transformations (16) build a local (infinite parameter) group $\mathfrak{M}^{l,s}$ for $|K| \neq 0$ and for appropriate $G$. The element $N_{[1,1]}$ is the identity; $N_{[K,G]}$ is locally invertible in a neighbourhood of the identity; the associativity is respected.

### 3.2 POINCARÉ INVARIANCE

We gave in (4) a condition for $N \subset \mathfrak{M}^l$ which yields a Poincaré invariant nonlinear Dirac operator. A straightforward evaluation of (4) (see [19], Theorem 1.2.1; [20]), yields the most general form for $N[\psi]$,

$$N(\psi)\psi = (f_1(\bar{\psi}\psi, \bar{\psi}\gamma_5\psi) + f_2(\bar{\psi}\psi, \bar{\psi}\gamma_5\psi)\gamma_5)\psi.$$ (21)

$f_1$, $f_2$ are independent complex functions depending on the invariant quantities $\bar{\psi}\psi \equiv X$, $\bar{\psi}\gamma_5\psi \equiv Y$ ($X$ depends on $R^2_k$, $Y$ on $R_1 R_k$ and $S_i - S_k$). The result reflects that $N$ transforms under the Poincaré group like the corresponding scalar and pseudoscalar invariants. The invariant transformations $N \in \mathfrak{M}^l$ form a group $\mathfrak{M}^{l,P}$.

We are interested now in local, separable and Poincaré invariant $N$, i.e., in

$$\mathfrak{M}^{l,s,P} = \mathfrak{M}^{l,s} \cap \mathfrak{M}^{l,P}.$$ 

The intersection is given (see (16) and (21)) through

$$N_{j[a,b,G]}(\psi)\psi_j = R_j a^{-1} \exp(b - i)S_j G_j(\ldots)\psi_j = f_1(X,Y)\psi_j + f_2(X,Y)(\gamma_5\psi)_j$$

(22)

e.g. for $j = 0$ as

$$R_0 a^{-1} \exp(b - i)S_0 G_0(\ldots) = f_1(X,Y)R_0 + f_2(X,Y)R_2 \exp i(S_2 - S_0).$$

(23)

This relates $G_j(\ldots)$ and $f_1$, $f_2$, which depend on $R_i$, $S_i$. Similarly as in section 3.1 we use groups of translations in $\mathcal{S}$ and of scale transformation on $\mathcal{R}$,

$$S_j \mapsto S_j + \theta, \quad R_j \mapsto \lambda R_j.$$ 

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The variables in $G_j(\ldots)$ are invariant under both groups; the variables in $f_1, f_2$, i.e. $X$ and $Y$, are translation invariant and behave under scale transformation as $X \mapsto \lambda^2 X, Y \mapsto \lambda^2 Y$. The translations imply for $j = 0$ (and any $j$) in (28)

$$b = i, \text{ i.e. } \bar{b} = 0, \quad \hat{b} = 1. \quad (24)$$

For scale transformations we use in $f_1, f_2$ instead of $X$ and $Y$ the variables $X$ and $Z = \frac{X}{Y}$ ($XY \neq 0$) and get

$$R_0^0 G_0(\ldots) = \lambda^{-\alpha+1} F(\lambda^2 X, \ldots), \quad F(\lambda^2 X, \ldots) = f_1(\lambda^2 X, Z) R_0 + f_2(\lambda^2 X, Z) R_2 \exp i(S_2 - S_0).$$

Differentiate this in respect to $\lambda$ and obtain a differential equation; its solution specify for $\lambda = 1$ the $X$ and $Y$ dependence of $f_1$ and $f_2$;

$$f_i(X, Y) = g(X) h_i(Z), \quad g(X) = (\bar{\psi} \psi)^{a-1}, \quad h_i(Z) \text{ arbitrary }, i = 1, 2. \quad (25)$$

or with $X = \bar{\psi} \psi = R_0^2 + R_1^2 - R_2^2 - R_3^2$

$$g(X) = (R_0^2 + R_1^2 - R_2^2 - R_3^2)^{a-1}. \quad (26)$$

For the relation between $G_j(\ldots)$ and $h_1, h_2$ we get

$$G_j(\ldots) = \left( \frac{R_0^2 + R_1^2 - R_2^2 - R_3^2}{R_j} \right)^{a-1} (h_1(Z) - i h_2(Z) \frac{R_j}{R_j} \exp i(S_{j(j)} - S_j)) \quad (27)$$

($J$ maps $0 \mapsto 2, 1 \mapsto 3, 2 \mapsto 0, 3 \mapsto 1.)$ $Z$ depends on $R_j, S_j$ as $R_j R_k, S_j - S_k$.

Nonlinear transformations (22) with (25) leads to NLDE which are local, separable and Poincaré invariant. They build a group $\mathfrak{N}^1 \equiv \mathfrak{N}^{l,s,P}$; their elements are labelled through $a, h_1(Z), h_2(Z)$ and act on $G_4$ as

$$N_{[a, h_1, h_2]}(\psi) \equiv (\bar{\psi} \psi)\left( \frac{a-1}{2} \cdot (h_1(Z) + h_2(Z)\gamma_5) \right). \quad (28)$$

### 3.3 Invariance of the Utility Function

The utility function $F_\rho$ was defined in (1) - step 4 - and those $N$ with invariant $F_\rho$ define equivalent systems. This equivalence condition is an additional restriction to the different subgroups of $\mathfrak{N}$. We discuss first the case $\mathfrak{N}^{l,s}$ and specialise the result to $\mathfrak{N}^1 = \mathfrak{N}^{l,s,P}$.

**CASE $\mathfrak{N}^{l,s}$:**

Insert $N_{[a,b,c]}(\psi)\psi$ from (16) in the equivalence condition (7)

$$\sum_{k=0}^3 R_k^{2\tilde{b}} \exp \tilde{b} S_k |G_k|^2 = \sum_{k=0}^3 R_k^2 \quad (29)$$

As in the last section use translations in $S$ and scale transformations in $R$. Because (29) is invariant under both types one gets

$$\exp 2\tilde{b} \theta \cdot \sum_{k=0}^3 R_k^{2\tilde{b}} \exp (2\tilde{b} S_k) |G_k|^2 = \sum_{k=0}^3 R_k^2 \quad (30)$$

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\[ \lambda^{2\tilde{a} - 2} \cdot \sum_{k=0}^{3} R_k^2 \exp(2\tilde{b} S_k) |G_k|^2 = \sum_{k=0}^{3} R_k^2. \]

From this we have immediately
\[ \tilde{a} = 1, \; \tilde{b} = 0; \; \tilde{a} \text{ and } \tilde{b} \text{ arbitrary}, \]

and the \( G_k \) are restricted to those \( G'_k \) which fulfil
\[ \sum_{k=0}^{3} R_k^2 (|G'_k|^2 - 1) = 0 \] (31)
i.e. the \( G'_k \) with \( k = 0, 1, 2, 3 \) are not independent.
Hence, a transformation \( N \) which satisfies locality, separation and equivalence conditions is given through
\[ N_{j[K',G']} \psi_j = R_{j}^{1+ia} \exp i\hat{b} S_j G'_j \left( \frac{R_1}{R_0}, \ldots, \frac{R_3}{R_0}, S_1 - S_0, \ldots, S_3 - S_0 \right) \]
with
\[ K' = \begin{pmatrix} 1, & \hat{a} \\ 0, & \hat{b} \end{pmatrix}, \; \hat{a}, \hat{b} \in \mathbb{R} \text{ arbitrary, } \hat{b} \neq 0, \; \sum_{k=0}^{3} R_k^2 (|G'_k|^2 - 1) = 0. \] (32)
The restriction in (32) from \( K \) to \( K' \) and from \( G_k \) to \( G'_k \) is enforced through the equivalence condition.
The transformations \( N_{[K',G']} \) form the group \( \mathcal{N}^{l,s,e} \). For \( N_{[K'_3,G'_3]} = N_{[K'_1,G'_1]} \circ N_{[K'_2,G'_2]} \) we find for \( K'_3 \)
\[ K'_3 = K'_2 K'_1 = \begin{pmatrix} 1, & \hat{a}_1 + \hat{a}_2 \hat{b}_1 \\ 0, & \hat{b}_1 \hat{b}_2 \end{pmatrix} \] (33)
and for the \( G'_{3j}, j = 0, 1, 2, 3 \) the following condition (from (20), (26), (27):
\[ |G'_{3j}|^2 = |G'_{1j}|^2 |G'_{2j}|^2, \; \sum_{k=0}^{3} R_k^2 (|G'_{1k}|^2 |G'_{2k}|^2 - 1) = 0 \] (34)
with \( G'_j \) and its variables from (16), (20). An explicit form of \( G'_{3,j} \) is given in the next example.
Case $\mathfrak{N}_1 = \mathfrak{N}^{l,s,P}$:
Local, separable and Poincaré invariant $N_{[a,b_1,b_2]}$ in (28) are special cases of $N_{[a,b,G]}$ in (16). Therefore the equivalence condition (7) leads to (29) and we have together with the invariance condition (24)

$$a = 1 + i\hat{a}, \quad (b = 0).$$

If we insert $G_j(., .)$ from (27) in (29) we have with (33)

$$3 \sum_{k=0}^3 R_k^2(|h_1(Z) - ih_2(Z)|^2 \frac{R_{J(k)}}{R_k} \exp |S_{J(k)} - S_k|^2 - 1) = 0.$$ 

This is a condition for $h_i(Z), i = 1, 2,$ which are functions of $Z = \frac{X}{Y}$ and implies

$$|h_1(Z)|^2 + |h_2(Z)|^2 = 1.$$

Hence we have

$$h_1(Z) = \exp i\phi(Z) \cos \rho(Z), \quad h_2(Z) = \exp i\phi(Z) \sin \rho(Z),$$

with two real functions $\phi(Z), \rho(Z)$ which label $N_{[a,\phi,\rho]}$ together with a real number $\hat{a}$; they act on $G_4$ as

$$\begin{align*}
N_{[\hat{a},\phi,\rho]}(\psi) \psi &= (\bar{\psi}\psi)^{\frac{1}{2}} \exp i(\cos \rho(Z) + i \sin \rho(Z)\gamma_5)\psi \\
&= \exp(i\frac{\hat{a}}{2} \ln(\bar{\psi}\psi) + i\phi(Z) + \rho(Z)\gamma_5)\psi.
\end{align*}$$

Again, the $N_{[\hat{a},\phi,\rho]}$ form a group $\mathfrak{N}^2 = \mathfrak{N}^{l,s,P,e}$. The group relation is

$$N_{[\hat{a}_1,\phi_1,\rho_1]} \circ N_{[\hat{a}_2,\phi_2,\rho_2]} = N_{[\hat{a}_3,\phi_3,\rho_3]}$$

with

$$\hat{a}_3 = \hat{a}_1 + \hat{a}_2, \quad \rho_3(z) = \rho_1(z) + \rho_2(\bar{z}), \quad \bar{z}(z) = \frac{z \cos 2\rho_1(z) - \sin 2\rho_1(z)}{\cos 2\rho_1(z) + \sin 2\rho_1(z)}.$$ 

$$\phi_3(z) = \frac{\hat{a}_2}{2} \ln \left(\cos 2\rho_1(z) - \frac{1}{z} \sin 2\rho_1(z)\right) + \phi_1(z) + \phi_2(\bar{z}).$$ 

$|G_{j,3}|^2$ is in this example independent of $j$ and $|G_{j,3}|^2 = 1$ holds; condition (31) is fulfilled.

### 3.4 THE RESULTING NLDE AND THEIR EXTENSIONS

Part of our construction is that invertible $N$ imply for $N^{-1}\psi$ a nonlinear Dirac operator and equation

$$(\gamma_\mu p^\mu - m)N(\psi)\psi = (\gamma_\mu p^\mu + H(\psi))\psi = 0$$ (38)
with $H(\psi)$ as nonlinear term depending on $m$ and the labels of $N$. We calculate the NLDE for $N \in \mathcal{N}^1, \mathcal{N}^2$.

In case of $N \in \mathcal{N}^1$ we find for $H_1 \left( \frac{dh_i(Z)}{dz} = h_i'(Z), i = 1, 2 \right)$ from (28)

$$H_1(X, Y) = (\gamma_\mu p^\mu X) \frac{a - 1}{2X} + (\gamma_\mu p^\mu Z) \frac{h_1(Z)h_1'(Z) + h_2(Z)h_2'(Z) + (h_1(Z)h_2'(Z) - h_1'(Z)h_2(Z))\gamma_5}{h_1(Z)^2 + h_2(Z)^2} + m \frac{h_1(Z)^2 - h_2(Z)^2 + 2h_1(Z)h_2(Z)\gamma_5}{h_1(Z)^2 + h_2(Z)^2}.$$  (39)

This is a family $\mathcal{F}_1$ of Poincaré invariant NLDE which respect the separability condition.

For the interesting subfamily $\mathcal{F}_2 \subset \mathcal{F}_1$ which describes in addition physically equivalent systems we find with $h_i(X)$ in (36) and with (35)

$$H_2(X, Y) = (\gamma_\mu p^\mu X) i \frac{\hat{a}}{2X} + (\gamma_\mu p^\mu Z) (i\phi'(Z) + \rho'(Z)\gamma_5)) - m \exp 2\rho(Z)\gamma_5.$$  (40)

$\mathcal{F}_2$ is invariant under $\mathcal{N}^2$, i.e. if one calculates $D_D^N \circ N_2, N_{1,2} \in \mathcal{N}^2$ one gets a nonlinear term of the form (40).

The structure in (39), (40) stems from the general form of local, Poincaré invariant $N$ in (21). The operator $\gamma_\mu p^\mu$ acts on the invariants $X, Z$ and the coefficient functions have a scalar and pseudoscalar part. Solutions $\psi'$ for $\mathcal{F}_1$ are available from solutions $\psi$ of $D_S\psi = 0$ through $\psi' = N^{-1}\psi$, $N \in \mathcal{N}^1$ or $\mathcal{N}^2$. By construction, the family $\mathcal{F}_2$ describes equivalent systems.

To get ‘new’ systems we use the method explained in Section 2, step 5. We generalize $H_2(X, Y)$ with real functions $g(X), k(Z), l(Z), n(Z)$ to

$$H_{2,ex}(X, Y) = (\gamma_\mu p^\mu X)ig(X) + (\gamma_\mu p^\mu Z) (ik(Z) + l(Z)\gamma_5)) - m \exp 2n(Z)\gamma_5.$$  (41)

The resulting ‘extended’ family $\mathcal{F}_{2,ex}$ is local and Poincaré invariant; furthermore it is not equivalent to $\mathcal{F}_0$ but it has a ‘similar’ structure. $\mathcal{F}_2$ is a subfamily of $\mathcal{F}_{2,ex}$ with

$$k(Z) = \phi'(Z), \quad l(Z) = \rho'(Z), \quad n(Z) = 2\rho(Z).$$  (42)

To obtain solutions of (41) the above mentioned technique could be useful. Because $\mathcal{F}_2$ is constructed from $\mathcal{N}^2$, i.e. from a family of nonlinear gauge functions (cf. Section 2, step 5), $\mathcal{F}_{2,ex}$ is denoted as gauge extension of $\mathcal{F}_0$.

## 4 CONCLUDING REMARKS

We derived from physically reasonable ‘principles’ a family $\mathcal{F}_{2,ex}$ of NLDE for one-particle systems which is a ‘mild’ (gauge) extension of the family $\mathcal{F}_2$ which describes
the linear Dirac family $\mathcal{F}_0$ through a nonlinear Dirac operator. With ‘mild’ we understand that the deviation of $\mathcal{F}^{ex}_2$ from $\mathcal{F}_2$ or the effect of a nonlinearity $H_{2,ex}$ compared to $H_2$ is ‘very small’; this assumes that we work in a region in which very small nonlinear corrections behave well and that some information is known on a solution variety of $\mathcal{F}^{ex}_2$.

Concerning the physical relevance of a family of NLDE we mention the following arguments:

1. Because of the successes of the (linear) formulation and interpretation of quantum theory and the fact that a linear framework does not allow nonlinear evolution of one particle systems like $\mathcal{F}^{ex}_2$ one has to develop a new framework together with an interpretation if one wants to incorporate such evolutions. Furthermore one has to show that nonlinearities yields experimental effects which are measurable through precision experiments. We have no corresponding results for a nonlinear Dirac theory (and a connected quantum field theory). However we showed, using parts of the linear framework and interpretations - denoted as ‘first principles’ - that one can construct physically equivalent NLDE which yield after gauge-extensions a family of physically motivated one particle NLDE.

For the generalisation $\mathcal{F}^{ex}_2$ to $n$-particle systems one has to extend the evolution operator, which is defined only on product states, to a dense set in $\mathcal{H}^*_D$ (operator-extension). In the nonrelativistic case such a method is known [1].

2. A reasonable attempt to develop a framework for a nonlinear quantum theory should start with the dynamic of the system, i.e. with a physical justification of a nonlinearity, e.g. from ‘first principles’ or from geometrical properties [5], [6], [7], [8]. Such a framework depends on the structure of the nonlinearity. Therefore some information on a method to derive a NLDE with a special class of nonlinear terms like $H_2$ could be useful.

3. We mentioned in section 1 some recent interest in nonrelativistic nonlinear quantum mechanical evolutions equations, e.g. nonlinear one-particle Schrödinger equation (NLSE). It would be interesting to see whether NLSE appear as nonrelativistic limits from a NLDE, as it is the case for linear Schrödinger and Dirac equations. One can realise the nonrelativistic case with the program from section 1: take a Hilbert space of scalar functions, the (nonrelativistic) utility function [9] and use the central extension of the inhomogeneous Galilei group (with time translations) instead of the Poincaré group as space-time invariance. The result is known (see [9], section 4); the nonlinear terms depend on second order terms $\partial^2_k \varphi$, $\partial_k \varphi \partial_l \varphi$, $k, l = 1, 2, 3$ because the Schrödinger equation is of second order. The usual procedure to derive a nonrelativistic limit of the Dirac equation leads to the Pauli equation (or the Schrödinger equation for vanishing electromagnetic potentials). If one tries an analogous procedure for the NLDE family $\mathcal{F}^{ex}_2$, second order terms in
\( \psi, \bar{\psi} \) are absent. Hence for a NLDE with a nonrelativistic limit given through a NLSE of the type given in \[5\] - \[8\] one should use a nonlinear transformation \( N \) which depends also on derivatives of \( \psi \). Such transformations were discussed in the nonrelativistic case in \[21\].

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