PHOTON "MASS" AND ATOMIC LEVELS IN A SUPERSTRONG MAGNETIC FIELD

M. I. VYSOTSKY*
A.I. Alikhanov Institute of Theoretical and Experimental Physics,
Moscow 117218 Russia
*E-mail: vysotsky@itep.ru

The structure of atomic levels originating from the lowest Landau level in a superstrong magnetic field is analyzed. The influence of the screening of the Coulomb potential on the values of critical nuclear charge is studied.

Keywords: Atomic levels, superstrong magnetic field, critical charge.

It is a great pleasure for me to present this paper to Sergei Gaikovich Matinyan on his 80th birthday.

1. Introduction
We will discuss the modification of the Coulomb law and atomic spectra in superstrong magnetic field. The talk is based on papers,1–3 see also.4

2. $D = 2$ QED
Let us consider two dimensional QED with massive charged fermions. The electric potential of the external point-like charge equals:

$$
\Phi(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)},
$$

(1)

where $\Pi(k^2)$ is the one-loop expression for the photon polarization operator:

$$
\Pi(k^2) = 4g^2 \left[ \frac{1}{\sqrt{t(1 + t)}} \ln(\sqrt{1 + t + \sqrt{t}}) - 1 \right] \equiv -4g^2 P(t),
$$

(2)

and $t \equiv -k^2/4m^2$, $[g]$ = mass.

In the coordinate representation for $k = (0, k_\parallel)$ we obtain:

$$
\Phi(z) = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_\parallel z}dk_\parallel/2\pi}{k_\parallel^2 + 4g^2 P(k_\parallel^2/4m^2)}.
$$

(3)
With the help of the interpolating formula
\[ \mathcal{T}(t) = \frac{2t}{3 + 2t} \] (4)
the accuracy of which is better than 10% for \( 0 < t < \infty \) we obtain:
\[ \Phi = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ikz}dk}{2\pi k^2 + g^2(k_0^2/2m^2)} = \]
\[ = \frac{4\pi g}{1 + 2g^2/3m^2} \left[ -\frac{1}{2}z^2 + \frac{g^2/3m^2}{\sqrt{6m^2 + 4g^2}} \exp(-\sqrt{6m^2 + 4g^2}|z|) \right]. \] (5)

In the case of heavy fermions (\( m \gg g \)) the potential is given by the tree level expression; the corrections are suppressed as \( g^2/m^2 \).

In the case of light fermions (\( m \ll g \)):
\[ \Phi(z) \bigg|_{m \ll g} = \begin{cases} \pi e^{-2g|z|}, & z \ll \frac{1}{g} \ln \left( \frac{g}{m} \right) \\ -2\pi g \left( \frac{3m^2}{2g^2} \right) |z|, & z \gg \frac{1}{g} \ln \left( \frac{g}{m} \right) \end{cases} \] (6)

\( m = 0 \) corresponds to the Schwinger model; photon gets a mass due to a photon polarization operator with massless fermions.

3. Electric potential of the point-like charge in \( D = 4 \) in superstrong \( B \)

We need an expression for the polarization operator in the external magnetic field \( B \). It simplifies greatly for \( B \gg B_0 = m_\alpha^2/e \), where \( m_\alpha \) is the electron mass and we use Gauss units, \( e^2 = \alpha = 1/137 \). The following results were obtained in:
\[ \Phi(k) = \frac{4\pi e}{k_0^2 + k_+^2 + \frac{e^3B}{m^2} \exp \left( -\frac{k_+^2}{2eB} \right) \sqrt{\frac{2}{3\pi e^2}}} \] (7)
\[ \Phi(z) = 4\pi e \int \frac{e^{ikz}dk}{k_0^2 + k_+^2 + \frac{e^3B}{m^2} \exp(-k_+^2/(2eB))(k_0^2/2m^2)/(3 + k_0^2/2m^2)} = \]
\[ = \frac{e}{|z|} \left[ 1 - e^{-\sqrt{6m^2}|z|} + e^{-\sqrt{2/(2\pi)}e^3B+6m^2}|z| \right] \] (8)

For \( B \ll 3\pi m^2/e^3 \) the potential is Coulomb up to small corrections:
\[ \Phi(z) \bigg|_{e^3B \ll m_\alpha^2} = \frac{e}{|z|} \left[ 1 + O \left( \frac{e^3B}{m_\alpha^2} \right) \right], \] (9)

analogously to \( D = 2 \) case with substitution \( e^3B \rightarrow g^2 \).

For \( B \gg 3\pi m^2/e^3 \) we obtain:
\[ \Phi(z) = \begin{cases} \frac{e}{|z|} \exp(-\sqrt{(2/\pi)e^3B}|z|), & \sqrt{(2/\pi)e^3B} > |z| > \frac{1}{\sqrt{eB}} \\ \frac{e}{|z|} \left( 1 - e^{-\sqrt{6m^2}|z|} \right), & \frac{1}{m_\alpha} > |z| > \frac{1}{\sqrt{(2/\pi)e^3B}} \\ \frac{e}{|z|}, & |z| > \frac{1}{m_\alpha} \end{cases} \] (10)
The close relation of the radiative corrections at $B >> B_0$ in $D = 4$ to the radiative corrections in $D = 2$ QED allows to prove that just like in $D = 2$ case higher loops are not essential (see, for example,\cite{5}).

4. Hydrogen atom in the magnetic field

For $B > B_0 = m_e^2/e$ the spectrum of Dirac equation consists of ultrarelativistic electrons with only one exception: the electrons from the lowest Landau level (LLL, $n = 0$, $\sigma_z = -1$) are nonrelativistic. So we will find the spectrum of electrons from LLL in the screened Coulomb field of the proton.

The wave function of electron from LLL is:

$$R_{0m}(\rho) = \left[\pi(2a_H^2)^{1+|m|(|m|!)}\right]^{-1/2} \rho^{m} e^{i(m\varphi - \rho^2/(4a_H^2))},$$

where $m = 0, -1, -2$ is the projection of the electron orbital momentum on the direction of the magnetic field.

For $a_B \equiv 1/\sqrt{eB} \ll a_H = 1/(m_e e^2)$ the adiabatic approximation is applicable and the wave function looks like:

$$\Psi_{n0m-1} = R_{0m}(\rho)\chi_n(z),$$

where $\chi_n(z)$ satisfy the one-dimensional Schrödinger equation:

$$\left[-\frac{1}{2m_e} \frac{d^2}{dz^2} + U_{eff}(z)\right]\chi_n(z) = E_n\chi_n(z).$$

Since screening occurs at very short distances it is not important for odd states, for which the effective potential is:

$$U_{eff}(z) = -e^2 \int |R_{0m}(\rho)|^2 \sqrt{\rho^2 + z^2} d^2\rho \left[1 - e^{-\sqrt{6m_e^2} z} + e^{-\sqrt{(2/\pi)}e^3B+6m_e^2} z \right].$$

It equals the Coulomb potential for $|z| \gg a_H$ and is regular at $z = 0$.

Thus the energies of the odd states are:

$$E_{odd} = -\frac{m_e e^4}{2n^2} + O \left(\frac{m_e^2 e^3}{B}\right), \quad n = 1, 2, \ldots,$$

and for the superstrong magnetic fields $B > m_e^2/e^3$ they coincide with the Balmer series with high accuracy.

For even states the effective potential looks like:

$$\tilde{U}_{eff}(z) = -e^2 \int |R_{0m}(\rho)|^2 \sqrt{\rho^2 + z^2} d^2\rho \left[1 - e^{-\sqrt{6m_e^2} z} + e^{-\sqrt{(2/\pi)}e^3B+6m_e^2} z \right].$$
Integrating the Schrödinger equation with the effective potential from $x = 0$ till $x = z$, where $a_H \ll z \ll a_B$, and equating the obtained expression for $\chi'(z)$ to the logarithmic derivative of Whittaker function – the solution of the Schrödinger equation with Coulomb potential – we obtain the following equation for the energies.
of even states:

$$\ln \left( \frac{H}{1 + \frac{e^6}{3\pi H}} \right) = \lambda + 2\ln \lambda + 2\psi \left( 1 - \frac{1}{\lambda} \right) + \ln 2 + 4\gamma + \psi(1 + |m|) ,$$  \hspace{1cm} (18)

where $H \equiv B/(m_e^2 e^3)$, $\psi(x)$ is the logarithmic derivative of the gamma-function and

$$E = -(m_e e^4/2)\lambda^2 .$$  \hspace{1cm} (19)

The spectrum of the hydrogen atom in the limit $B \gg m_e^2 e^3$ is shown in Fig. 1.

5. Screening versus critical nucleus charge

Hydrogen-like ion becomes critical at $Z \approx 170$: the ground level reaches lower continuum, $\varepsilon_0 = -m_e$, and two $e^+e^-$ pairs are produced from vacuum. Electrons with the opposite spins occupy the ground level, while positrons are emitted to infinity. According to the strong magnetic field $Z_{cr}$ diminishes: it equals approximately 90 at $B = 100B_0$; at $B = 3 \cdot 10^4B_0$ it equals approximately 40. Screening of the Coulomb potential by the magnetic field acts in the opposite direction and with account of it larger magnetic fields are needed for a nucleus to become critical.

The bispinor which describes an electron on LLL is:

$$\psi_e = \begin{pmatrix} \varphi_e \\ \chi_e \end{pmatrix} ,$$

$$\varphi_e = \begin{pmatrix} 0 \\ g(z) \exp \left( -\frac{\rho^2}{4a_H^2} \right) \end{pmatrix} ; \chi_e = \begin{pmatrix} 0 \\ if(z) \exp \left( -\frac{\rho^2}{4a_H^2} \right) \end{pmatrix} .$$  \hspace{1cm} (20)

Dirac equations for functions $f(z)$ and $g(z)$ look like:

$$g_z - (\varepsilon + m_e - \bar{V})f = 0 ,$$

$$f_z + (\varepsilon - m_e - \bar{V})g = 0 ,$$  \hspace{1cm} (21)

where $g_z \equiv dg/dz$, $f_z \equiv df/dz$. They describe the electron motion in the effective potential $\bar{V}(z)$:

$$\bar{V}(z) = \frac{Ze^2}{a_H^2} \left[ 1 - e^{-\frac{\sqrt{6m_e^2|z|}}{e^3}} + e^{-\sqrt{\frac{(2/\pi)e^3B+6m_e^2}2}z} \right] \times$$

$$\times \int_0^\infty \frac{e^{-\rho^2/2a_H^2}}{\sqrt{\rho^2 + z^2}} \rho \, d\rho .$$  \hspace{1cm} (22)

Integrating numerically we find the dependence of $Z_{cr}$ on the magnetic field with the account of screening. The results are shown in Fig. 2. For the given nucleus to become critical larger magnetic fields are needed and the nuclei with $Z < 52$ do not become critical.
Fig. 2. The values of $B_Z^{cr}$: a) without screening according to, dashed (green) line; b) numerical results with screening, solid (blue) line. The dotted (black) line corresponds to the field at which $a_H$ becomes smaller than the size of the nucleus.

Acknowledgments

I am grateful to the organizers for wonderful time in Nor-Amberd and Tbilisi and to my coauthors Sergei Godunov and Bruno Machet for helpful collaboration. I was partly supported by the grants RFBR 11-02-00441, 12-02-00193, by the grant of the Russian Federation government 11.G34.31.0047, and by the grant NSh-3172.2012.2.

References

1. M.I. Vysotsky, JETP Lett. 92, 15(2010).
2. B. Machet and M.I. Vysotsky, Phys. Rev. D 83, 025022 (2011).
3. S.I. Godunov, B. Machet, and M.I. Vysotsky, Phys. Rev. D 85, 044058 (2012).
4. A.E. Shabad, V.V. Usov, Phys. Rev. Lett. 98, 180403 (2009); Phys. Rev. D 77, 025001 (2008).
5. V.B. Beresteckii, Proceedings of LIYaF Winter School 9, part 3, 95 (1974).
6. Ya.B. Zeldovich, V.S. Popov, UFN 105, 403 (1971); W. Greiner, J. Reinhardt, Quantum Electrodynamics (Springer-Verlag, Berlin, Heidelberg, 1992); W. Greiner, B. Müller, and J. Rafelski, Quantum Electrodynamics of Strong Fields (Springer-Verlag, Berlin, Heidelberg, 1985).
7. V.N. Oraevskii, A.I. Rez, and V.B. Semikoz, Zh. Eksp. Teor. Fiz. 72, 820 (1977) [Sov. Phys. JETP 45, 428 (1977)].