THE EDGE METRIC DIMENSION OF THE GENERALIZED PETERSEN
GRAPH $P(n, 3)$ IS AT MOST 4

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Abstract. It is known that the edge metric dimension of the generalized Petersen graph $P(n, 3)$ is at least 3. We give a formula for the distance between any two vertices in $P(n, 3)$, and a formula for the distance between any vertex and any edge in $P(n, 3)$. Then we show by construction that the edge metric dimension of $P(n, 3)$ is at most 4, and conjecture that the dimension is 4 for $n \geq 11$.

1. Introduction

For any sequence $w = (v_1, \ldots, v_k)$ of vertices and any vertex $v$ in a connected graph $G$, the representation of $v$ with respect to $w$ is the $k$-tuple $(d(v, v_1), \ldots, d(v, v_k))$, where $d(x, y)$ is the distance between the vertices $x$ and $y$. The set $\{v_1, \ldots, v_k\}$ is said to be a resolving set for $G$ if every two vertices of $G$ have distinct representations. It was Slater [17] who firstly considered the minimum cardinality of a resolving set for $G$, called the metric dimension of $G$. Kelenc, Tratnik, and Yero [13] introduced the analogous concept of edge metric dimension, defined to be the minimum cardinality of a vertex set $(v_1, \ldots, v_k)$ such that the vectors $(d(e, v_1), \ldots, d(e, v_k))$ for all edges $e$ are distinct, where $d(e, v_i)$ is the distance between the edge $e$ and the vertex $v_i$.

According to Kelenc et al. [13], the edge metric dimension has applications in network security surveillance. For instance, an intruder accesses a network through edges can be identified by an edge resolving set. They proved the NP-hardness of the edge metric dimension problem. The exact values for the edge metric dimension of some classes of graphs were known, while bounds are given to some other graph classes. The upper bound of the edge metric dimension of an $n$-vertex graph is $n - 1$, around which some research can be found from [20, 21]. For a rich resource of kinds of resolving sets of graphs with applications, see Kelenc, Kuziak, Taranenko and Yero [12].

In this paper, we consider the edge metric dimension of the generalized Petersen graph $P(n, 3)$. It was Coxeter [4] who firstly studied the generalized Petersen graphs $P(n, k)$ with coprime parameters $n$ and $k$. Many graph theoretic and algorithmic properties of the generalized Petersen graphs have been investigated; see [1, 3, 5, 7, 10, 11, 14–16, 18, 19]. Some of these properties are quite difficult to show.

Filipović, Kartelj and Kratica [8] showed that the edge metric dimension of any $r$-regular graph is no less than $1 + \log_2 r$, from which it follows immediately that the edge metric dimension of a generalized Petersen graph $P(n, k)$ is at least 3. They showed that the edge metric dimension of the graphs $P(n, 1)$ and $P(n, 2)$ are both 3 for $n \geq 10$. We find it challenging to confirm the edge metric dimension of the graph $P(n, 3)$. Here is the main result of this paper.

Theorem 1.1. The edge metric dimension of the generalized Petersen graph $P(n, 3)$ is at most 4.

Though it is complicated considerably, to our surprise, it is computationally more difficult to show that no 3 vertices in $P(n, 3)$ form an edge resolving set. We pose the following conjecture.

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Conjecture 1.1. For \( n \geq 11 \), the edge metric dimension of the generalized Petersen graph \( P(n, 3) \) is 4.

2. Preliminaries

Let \( G = (V, E) \) be a connected graph and \( x \in V \). The distance between \( x \) and \( v \in V \) is the length of a shortest \( xv \)-path, denoted \( d(x, v) \). The distance between \( x \) and an edge \( e = uv \) is \( d(x, e) = \min\{d(x, u), d(x, v)\} \). We say that \( x \) resolves two vertices \( v \) and \( v' \) if \( d(x, v) \neq d(x, v') \), and that \( x \) resolves two edges \( e \) and \( e' \) if \( d(x, e) \neq d(x, e') \). A set \( R = \{x_1, \ldots, x_k\} \) of vertices is said to be a vertex resolving set (resp., edge resolving set) if every two distinct vertices (resp., edges) in \( G \) are resolved by a member of \( R \). The minimum cardinality of a vertex (resp., edge) resolving set is the vertex metric dimension (resp., edge metric dimension) of \( G \). A vertex (resp., edge) resolving set of the minimum cardinality is said to be a vertex (resp., edge) metric basis of \( G \). For any edge \( e \in E \), the sequence \( (d(e, x_1), \ldots, d(e, x_k)) \) is called the edge metric representation of \( e \) with respect to \( R \).

Let \( n \geq 3 \) and \( 1 \leq k < n/2 \). The generalized Petersen graph, denoted \( P(n, k) \), is the graph with vertex set \( \{u_j, v_j : j \in \mathbb{Z}_n\} \) and edge set \( \{u_ju_{j+1}, v_jv_{j+k}, u_jv_j : j \in \mathbb{Z}_n\} \), where \( \mathbb{Z}_n \) is the additive group of integers modulo \( n \). The Petersen graph is \( P(5, 2) \). The induced subgraphs \( P(n, k)[u_0, u_1, \ldots, u_{n-1}] \) and \( P(n, k)[v_0, v_1, \ldots, v_{n-1}] \), are called the outer cycle and the graph of inner cycle(s), respectively. Frucht, Graver, and Watkins [9] characterized automorphisms of generalized Petersen graphs, which implies a characterization for \( P(n, k) \) to be vertex-transitive, and a characterization for \( P(n, k) \) to be edge-transitive. As a result,

\[
P(n, k) \cong P(n, k) \iff hk \equiv \pm 1 \pmod{n};
\]

see also [2]. In particular, the graphs \( P(7, 3) \) and \( P(7, 2) \) are isomorphic to each other. It is routine to check that the Möbius-Kantor graph \( P(8, 3) \) has an edge resolving set \( \{u_0, u_1, v_0, v_2\} \), \( P(9, 3) \) has an edge resolving set \( \{u_0, v_3, v_1\} \), and \( P(10, 3) \) has an edge resolving set \( \{u_0, u_2, v_3\} \). Henceforth we suppose that \( n \geq 11 \).

For any integers \( a \) and \( b \), we denote

\[
[a, b] = \begin{cases} 
\{a, a + 1, \ldots, b\}, & \text{if } a \leq b; \\
\emptyset, & \text{if } a > b.
\end{cases}
\]

When a symbol \( w \in \{u, v\} \) is used, we mean \( w_i = u_i \) if \( w = u \), and \( w_i = v_i \) if \( w = v \).

Lemma 2.1. Consider the graph \( P(n, k) \). Let \( i \in \mathbb{Z} \), \( p_0 = \{u_0, v_0\} \) and \( w \in \{u, v\} \). Then

\[
d(p_0, w_i) = d(p_0, w_{i-1}) \quad \text{and} \quad d(u_0, v_i) = d(v_0, u_i).
\]

Moreover, for any \( i, j \in \mathbb{Z} \), we have

\[
d(u_i, e^u_{ij}) = d(u_0, e^u_{i-j-1}) = d(u_0, e^u_{i-j-1}),
\]

\[
d(u_i, e^e_{ij}) = d(u_0, e^e_{i-j-1}) = d(u_0, e^e_{i-j}),
\]

\[
d(u_i, e^v_{ij}) = d(u_0, e^v_{i-j-1}) = d(u_0, e^v_{i-j-k}).
\]

Proof. All the results follow from the symmetry of the graph \( P(n, k) \). \( \square \)

Let \( p = w_0w_1 \cdots w_i = a_1a_2 \cdots a_1 \) be a path in \( P(n, k) \), where \( w_i \) are vertices of \( P(n, k) \), and \( a_i \) is the arc \( w_{i-1}w_i \). Denote the length \( l \) of \( p \) by \( \ell_p \). Let \( S_p = \{a_j : 1 \leq j \leq s\} \) be the set of spokes in \( p \). In particular, \( a_i = w_{i+1}w_i \) is the first spoke. Then \( p \) can be written alternatively as \( p = p_0p_1 \cdots p_s \), where for \( 0 \leq j \leq s \),

\[
p_j = w_{i_j}w_{i_j+1} \cdots w_{i_{j+1}-1}, \quad \text{with} \quad w_{i_0} = w_0 \text{ and } w_{i_{s+1}} = w_i,
\]

is a subpath of \( p \), called a section of \( p \). Note that the vertex set \( V(p_j) \) is contained either in the outer cycle entirely or in an inner cycle entirely. We call \( p_j \) an outer section in the former case and an
inner section in the latter. We define the direction of a non-spoke arc to be clockwise if it is of the form \( u_j u_{j+1} \) or \( v_j v_{j+k} \), and to be counterclockwise otherwise. Note that the length of a section \( p_j \) is positive except either (i) \( j = 0 \) and \( w_0 w_1 \) is a spoke, or (ii) \( j = s \) and \( w_{l-1} w_l \) is a spoke. A section \( p_j \) of positive length is said to be clockwise (resp., counterclockwise) if any arc in it is so. We call \( p \) clockwise (resp., counterclockwise) if every section \( p_j \) of positive length is so. We call \( p \) undeviating if it is either clockwise or counterclockwise. For any \( 1 \leq j \leq s \), we call the subpath \( p_{j-1} p_j \) a turn of \( p \) if the sections \( p_{j-1} \) and \( p_j \) have different directions.

We define a reflection \( f \) on the vertex set of \( P(n, k) \) by \( f(w_i) = w_{-i} \), where \( w \in \{ u, v \} \). It extends naturally to act on paths.

**Lemma 2.2.** Let \( x \) and \( y \) be vertices of \( P(n, 3) \). Then there is an undeviating path \( xp_1 p_2 \cdots p_{d(x,y)-1} y \) which contains at most 2 spokes.

**Proof.** Without loss of generality we can suppose that \( x \in \{ u_0, v_0 \} \) and \( y = w_t \), where \( w \in \{ u, v \} \) and \( 0 \leq t \leq n - 1 \). Let \( p \) be a path from \( x \) to \( y \) of length \( \ell_p = d(x, y) \), with the minimum number of turns. If there is no undeviating path from \( x \) to \( y \), then \( p \) has at least one turn. We will show that this is impossible by contradiction. Let \( qq' \) be the first turn in \( p \), where \( q \) and \( q' \) are sections of \( p \) with different directions.

**Case 1.** \( q \) is a clockwise outer section. Then we can suppose that \( q = u_{j-s} u_{j-s+1} \cdots u_j \) for some \( s \geq 1 \) and \( p = \alpha q v_j \beta \), where \( \alpha \) is a clockwise subpath and \( \beta \) is a path starting from \( v_{j-3} \).

If \( s \geq 2 \), then the path

\[
p' = \begin{cases} \alpha u_{j-s} u_{j-s+1} \cdots u_{j-3} \beta, & \text{if } s \geq 3 \\ \alpha u_{j-3} \beta, & \text{if } s = 2 \end{cases}
\]

is shorter than \( p \), contradicting the choice of \( p \). When \( s = 1 \), the path \( p \) reduces to \( p = \alpha u_{j-1} u_j v_j \beta \).

- If \( \alpha \) contains a non-spoke arc, then \( \alpha = \alpha' v_{j-4} v_{j-1} \) for some clockwise path \( \alpha' \), and the path \( \alpha' v_{j-4} u_{j-4} u_{j-3} \beta \) is shorter than \( p \), the same contradiction.
- If \( \alpha \) has no non-spoke arcs, then the path \( \alpha u_{j-1} u_j u_{j-3} \beta \) has the same length as \( p \) and a less number of turns than \( p \).

This proves that Case 1 is impossible.

**Case 2.** \( q \) is a counterclockwise outer section. The path \( f(p) \), which is from \( f(x) \) to \( f(y) \), has length \( \ell_{f(p)} = d(f(x), f(y)) \) and the minimum number of turns. Since \( f(q) f(q') \) is the first turn in \( f(p) \), and \( f(q) \) is a clockwise outer section, we know that Case 2 is impossible by the impossibility of Case 1.

**Case 3.** \( q' \) is a counterclockwise outer section. Then we can suppose that \( q' = u_j u_{j-1} \cdots u_{j-s} \) and \( p = \alpha v_j q' \beta \), where \( s \geq 1 \), \( \alpha \) is a clockwise subpath ending at \( v_{j-3} \), and \( \beta \) is a subpath.

If \( s \geq 2 \), then the path

\[
p' = \begin{cases} \alpha u_{j-3} u_{j-4} \cdots u_{j-s} \beta, & \text{if } s \geq 3 \\ \alpha u_{j-3} u_{j-2} \beta, & \text{if } s = 2 \end{cases}
\]

is shorter than \( p \), contradicting the choice of \( p \). When \( s = 1 \), \( p \) reduces to \( p = \alpha v_j u_j u_{j-1} \beta \).

- If \( \beta \) contains no non-spoke arcs, then the path \( \alpha u_{j-3} u_{j-2} u_{j-1} \beta \) has the same length as \( p \) and a less number of turns than \( p \).
- If \( \beta \) contains a non-spoke arc, then \( \beta = v_{j-1} \beta' \) for some path \( \beta' \) starting from \( v_{j-4} \) or from \( v_{j+2} \). In the former case, the path \( \alpha u_{j-3} u_{j-4} \beta' \) is shorter than \( p \); in the latter case, the path \( \alpha v_j u_{j-1} u_{j+2} \beta' \) has the same length as \( p \) and a less number of turns than \( p \).

This proves that Case 3 is impossible.
Case 4. \( q' \) is a clockwise outer section. Since \( f(q)f(q') \) is the first turn in the path \( f(p) \), and \( f(q') \) is a counterclockwise outer section, we know that Case 4 is impossible by the impossibility of Case 3.

This proves the existence of an undeviating path from \( x \) to \( y \) of length \( d(x,y) \). Let \( p \) be such a path. It remains to show that the number of spokes in \( p \) is at most 2. By symmetry, we can suppose that \( p \) is clockwise. Assume that \( p \) has at least 3 spokes.

If \( x = v_0 \), then there exists \( h \geq 0 \) and \( i, j \geq 1 \) such that
\[
p = v_0v_3 \cdots v_{3h}u_{3h+1} \cdots u_{3h+i}v_{3h+i+1} \cdots v_{3h+i+3j+3}.
\]
where \( \beta \) is a clockwise path starting from \( u_{3h+i+3j} \). Then
\[
\ell_p = h + 1 + i + 1 + j + 1 + \ell_\beta = h + i + j + 1 + \ell_\beta + 3.
\]
Suppose that \( 3h + i + 3j = 3q + r \), where \( q \geq 1 \) and \( r \in \{0, 1, 2\} \). The path
\[
p' = v_0v_3 \cdots v_{3q}u_{3q+1} \cdots u_{3q+r-1}.
\]
has length \( \ell_p - \ell_p' = q + r + 1 + \ell_\beta \), and
\[
\frac{\ell_p}{3} = 2 + (i - r) + (h + j - q) = 2\left(\frac{i}{3} + 1 - \frac{r}{3}\right) \geq 2\left(\frac{1}{3} + 1 - \frac{2}{3}\right) = \frac{4}{3},
\]
contradicting the choice of \( p \).

Otherwise \( x = u_0 \). Then there exists \( h \geq 0 \) and \( i, j \geq 1 \) such that
\[
p = u_0u_1 \cdots u_hv_hv_{h+3} \cdots v_{h+3i}u_{h+3i+1} \cdots u_{h+3i+j},
\]
where \( \beta \) is a clockwise path starting from \( v_{h+3i+j} \). Then
\[
\ell_p = h + 1 + i + 1 + j + 1 + \ell_\beta = h + i + j + 1 + \ell_\beta + 3.
\]
Suppose that \( h + 3i + j = 3q + r \), where \( q \geq 1 \) and \( r \in \{0, 1, 2\} \). The path
\[
p' = v_0v_3 \cdots v_{3q}u_{3q+1} \cdots u_{3q+r-1}.
\]
has length \( \ell_p - \ell_p' = q + r + 1 + \ell_\beta \), and
\[
\frac{\ell_p}{3} = 2 + (i - q) + (h + j - r) = 2\left(\frac{h + j}{3} + 1 - \frac{r}{3}\right) \geq 2\left(\frac{1}{3} + 1 - \frac{2}{3}\right) = \frac{4}{3},
\]
contradicting the choice of \( p \). This completes the proof.

Let \( x \) be a vertex in \( P(n,3) \) and let \( y \) be a vertex or an edge in \( P(n,3) \). Denote by \( P(x,y) \) the set of shortest undeviating paths from \( x \) to \( y \). Let \( P_-(x,y) \) be the set of clockwise paths in \( P(x,y) \), and \( P_-(x,y) \) the length of any path in \( P_-(x,y) \), called the clockwise distance from \( x \) to \( y \). Let \( P_+(x,y) \) be the set of counterclockwise paths in \( P(x,y) \), and \( P_+(x,y) \) the length of any path in \( P_+(x,y) \), called the counterclockwise distance from \( x \) to \( y \). For any \( n \in \mathbb{Z} \), define \( q_n, r_n \in \mathbb{Z} \) by \( n = 3q_n + r_n \), where \( r_n \in \{0, 1, 2\} \). Define
\[
r = \begin{cases} 0, & \text{if } (r_n, r_i) = (0,2); \\ |r_n - r_i|, & \text{otherwise.} \end{cases}
\]

Lemma 2.3. Let \( 3 \leq i \leq n - 3 \). Then we have the following in the graph \( P(n,3) \).

(1) \( \ell_-(u_0, u_i) = q_i + r_i + 2 \) and \( \ell_-(u_0, u_i) = q_n - q_i + r + 2 \).
(2) \( \ell_-(u_0, v_i) = q_i + r_i + 1 \) and \( \ell_-(u_0, v_i) = q_n - q_i + r + 1 \). Furthermore, \( |P_-(u_0, v_i)| = |P_+(u_0, v_i)| = 1 \).
(3) The clockwise and counterclockwise distances from \( v_0 \) to \( v_i \) are respectively
\[
\ell_-(v_0, v_i) = \begin{cases} q_i, & \text{if } r_i = 0; \\ q_i + r_i + 2, & \text{if } r_i \neq 0 \end{cases} \quad \text{and} \quad \ell_+(v_0, v_i) = \begin{cases} q_n - i, & \text{if } r_n = r_i; \\ q_n - i + r_n - i + 2, & \text{if } r_n \neq r_i. \end{cases}
\]
Proof. It is elementary to compute that
\[(q_n - q_i, r_n - r_i), \quad \text{if } r_n \geq r_i\]
\[(q_n - q_i - 1, 1), \quad \text{if } (r_n, r_i) = (0, 2)\]
\[(q_n - q_i - 1, 2), \quad \text{otherwise}\]
\[(2.1) \quad (q_{n-i}, r_{n-i}) = \begin{cases} \end{cases}\]

First, consider \(p \in P_-(u_0, u_i)\). By Lemma 2.2, the path \(p\) has either 0 or 2 spokes. In the former case, \(\ell_p = i\); in the latter case, \(\ell_p = q_i + r_i + 2\) for \(p\) has \(q_i\) steps on an inner cycle between the two spokes and \(r_i\) steps on the outer cycle. Since \(i \geq 3\), we can deduce that
\[\ell_-(u_0, u_i) = \min(i, q_i + r_i + 2) = q_i + r_i + 2.\]
By symmetry and by Eq. (2.1), \(\ell_+(u_0, u_i) = q_{n-i} + r_{n-i} + 2 = q_n - q_i + r + 2\).

Second, consider \(p \in P_-(u_0, v_i)\). By Lemma 2.2, the path \(p\) has exactly one spoke. Then
\[p = u_0u_1 \cdots u_{r_i}v_{r_i}v_{r_i+1} \cdots v_i\]
is unique, with length \(\ell_p = q_i + r_i + 1\). Indeed, the unique spoke in \(p\) must be immediately after the first \(r_i\) steps on the outer cycle. By symmetry, \(|P_+(u_0, v_i)| = 1\). By Eq. (2.1),
\[\ell_+(u_0, v_i) = q_{n-i} + r_{n-i} + 1 = q_n - q_i + r + 1.\]

Third, consider \(p \in P_-(v_0, v_i)\). By Lemma 2.2, the path \(p\) has either 0 or 2 spokes. If \(r_i = 0\), then \(p = v_0v_1 \cdots v_i\) is unique and \(\ell_p = q_i\). Otherwise \(r_i \in \{1, 2\}\). If \(p\) has no spokes, then \(\ell_p = i\); if \(p\) has exactly two spokes, then \(\ell_p = q_i + r_i + 2\) for \(p\) has \(q_i\) steps on an inner cycle between the two spokes and \(r_i\) steps on the outer cycle. This proves the first desired formula. It is direct to obtain the second one by symmetry. \(\square\)

Note that \(\ell_\pm(u_0, u_i) = \ell_\pm(u_0, u_i) + 1\), and the path obtained by adding the arc \(v_iu_i\) to the unique path in \(P_\pm(u_0, u_i)\) belongs to \(P_\pm(u_0, u_i)\).

For any \(n \in \mathbb{Z}\), define \(q'_n, r'_n \in \mathbb{Z}\) by \(n = 6q'_n + r'_n\), where \(0 \leq r'_n \leq 5\). Define
\[M_n = \max\{j < \lfloor n/2 \rfloor : r_j = r_n\}.

Theorem 2.1. Let \(0 \leq i \leq \lfloor n/2 \rfloor\). Then we have the following in the graph \(P(n, 3)\).

\[d(u_0, u_i) = \begin{cases} \end{cases}\]
\[d(u_0, v_i) = \begin{cases} \end{cases}\]
\[d(v_0, v_i) = \begin{cases} \end{cases}\]

Proof. Let \(0 \leq i \leq \lfloor n/2 \rfloor\). We show them individually.

Consider \(d(u_0, u_i)\). If \(i \leq 2\), it is easy to check that \(d = q_i + r_i\). Let \(3 \leq i \leq \lfloor n/2 \rfloor\). Suppose that \(\ell_+(u_0, u_i) < \ell_-(u_0, u_i)\). By Lemma 2.3, \(q_n - q_i + r + 2 < q_i + r_i + 2\). It is elementary to show that \((r'_n, i) = (5, \lfloor n/2 \rfloor)\). In this case,
\[d(u_0, u_i) = q_n - q_i + r + 2 = 2q'_n + 1 - q'_n + 0 + 2 = q_i + r_i + 1.\]
Since \(r_{n-i} = 0\), the set \(P\) consists of a unique path, which has no edge in the outer cycle.
Consider \( d(u_0, v_i) \). Suppose that \( \ell_+(u_0, v_i) < \ell_-(u_0, v_i) \). By Lemma 2.3 and the previous case, we find \((r'_n, i) = (5, \lfloor n/2 \rfloor)\) and \(d(u_0, u_i) = q_i + r_i\).

Let \( d = d(v_0, v_i) \). Suppose that \( \ell_+(v_0, v_i) < \ell_-(v_0, v_i) \). By Lemma 2.3, it is elementary to show that \( r_n = r_i \neq 0, d = q_n - i \), and

\[
q_n - q_i \leq q_i + r_i + 1.
\]

We proceed according to the value of \( r'_n \).

- If \( n = 6q'_n + 1 \), then \( i \leq 3q'_n - 2 \) since \( r_i = 1 \). By Ineq. (2.2), we find \( i = 3q'_n - 2 = M_n \) and \( d = q'_n + 1 = q_i + r_i + 1 \).
- If \( n = 6q'_n + 2 \), then \( i \leq 3q'_n - 1 \) since \( r_i = 2 \). By Ineq. (2.2), we find \( i = 3q'_n - 1 = M_n \) and \( d = q'_n + 1 = q_i + r_i \).
- If \( n = 6q'_n + 4 \), then \( i \leq 3q'_n + 1 \) since \( r_i = 1 \). By Ineq. (2.2), we find \( i = 3q'_n + 1 = M_n \) and \( d = q'_n + 1 = q_i + r_i \).
- If \( n = 6q'_n + 5 \), then \( i \leq 3q'_n + 2 \) since \( r_i = 2 \). By Ineq. (2.2), we find that either \( i = 3q'_n + 2 = \lfloor n/2 \rfloor \) or \( i = 3q'_n - 1 = M_n \). In the former case, \( d = q'_n + 1 = q_i + r_i + 1 \); in the latter case, \( d = q'_n + 2 = q_i + r_i + 1 \).

This completes the proof.

### Lemma 2.4

Let \( i, j \in \mathbb{Z} \) and \( j > i \). We have the following equivalence:

\[
q_j + r_j < q_i + r_i \iff j = i + 1 \quad \text{and} \quad r_i = 2.
\]

In this case, \( q_{i+1} + r_{i+1} + 1 = q_i + r_i \).

**Proof.** It is elementary and we omit the proof.

### Corollary 2.2

Let \( 0 \leq i \leq \lfloor n/2 \rfloor - 1 \). Then the following equivalences hold:

\[
\begin{align*}
d(u_0, u_{i+1}) < d(u_0, u_i) & \iff i \in \{5, 8, 11, \ldots, 3q'_n - 1\}, \\
d(v_0, u_{i+1}) < d(v_0, u_i) & \iff i \in \{2, 5, 8, \ldots, 3q'_n - 1\}.
\end{align*}
\]

**Proof.** We show the two equivalences individually.

Let \( d_1 = d(u_0, u_i) \) and \( d_2 = d(u_0, u_{i+1}) \). Suppose that \( d_2 < d_1 \). We proceed by contradiction.

**Case 1.** \( i \leq 2 \). Then \((d_1, d_2) = (i, i + 1)\), contradicting the premise \( d_2 < d_1 \).

**Case 2.** \( r'_n = 5 \) and \( i = \lfloor n/2 \rfloor \). Then \( n \) is odd and \( d_2 = d(u_0, u_{n-i-1}) = d_1 \).

**Case 3.** \((r'_n, i) \neq (5, \lfloor n/2 \rfloor - 1)\). Then \((n, i) = (6q'_n + 5, 3q'_n + 1)\). By Theorem 2.1, \( d_1 = q'_n + 3 = d_2 \).

**Case 4.** None of the above. By Theorem 2.1, the premise \( d_2 < d_1 \) reduces to the inequality in Lemma 2.4, which is equivalent to \( r_i = 2 \); in this case, \( d_2 = d_1 - 1 \) by Lemma 2.4.

Rearranging the above results, we obtain the first desired equivalence.

Now, let \( d_1 = d(v_0, u_i) \) and \( d_2 = d(v_0, u_{i+1}) \). Suppose that \( d_2 < d_1 \). We treat 3 cases.

**Case 1.** \( i = \lfloor n/2 \rfloor - 1 \). If \( n \) is odd, then \( d_1 = d_2 \) by symmetry, a contradiction. Suppose that \( n \) is even. By Theorem 2.1, the assumption \( d_2 < d_1 \) reduces to the inequality in Lemma 2.4, which is equivalent to \( r_i = 2 \). Since \( i = n/2 - 1 \), we find \( r'_n = 0 \).

**Case 2.** \((r'_n, i) = (5, \lfloor n/2 \rfloor - 2)\). Then \( i = 3q'_n + 1 \) and \( d_1 = q'_n + 2 = d_2 \), a contradiction.

**Case 3.** None of the above. Since \( i + 1 \leq \lfloor n/2 \rfloor \), by Theorem 2.1 and Lemma 2.4, the assumption \( d_2 < d_1 \) reduces to \( r_i = 2 \).
Rearranging the above results, we obtain the second desired equivalence. □

**Corollary 2.3.** Let \(0 \leq i \leq \lfloor n/2 \rfloor\). Then the following equivalences hold:
\[
\begin{align*}
d(u_0, u_i) < d(u_0, v_i) &\iff i \leq 2, \\
d(v_0, v_i) < d(v_0, u_i) &\iff r_i = 0 \text{ or } (r_n', i) \in \{(5, \lfloor n/2 \rfloor), (2, M_n), (4, M_n)\}.
\end{align*}
\]

**Proof.** Direct from Theorem 2.1. □

**Corollary 2.4.** Let \(0 \leq i \leq \lfloor n/2 \rfloor - 2\), \(d_1 = d(u_0, v_i)\) and \(d_2 = d(u_0, v_{i+3})\).

1. If \((r_n', i) = (2, n/2 - 2)\), then \(d_1 > d_2 = q_i + r_i\).
2. Otherwise, \(d_2 \geq d_1 = q_i + r_i + 1\).

**Proof.** Let \(0 \leq i \leq \lfloor n/2 \rfloor - 2\) and \(j = n - i - 3\). Then \(d_2 = d(u_0, v_j)\) by symmetry. Suppose that \(d_2 < d_1\). We treat 3 cases.

**Case 1.** \(i = \lfloor n/2 \rfloor - 2\). Then \(j = \lfloor n/2 \rfloor - 1\). If \(n\) is odd, then \(j = i\) and \(d_1 = d_2\), a contradiction. Otherwise \(n\) is even, then \(j = i + 1\). By Theorem 2.1, the assumption \(d_2 < d_1\) reduces to the inequality in Lemma 2.4, which is equivalent to \(r_i = 2\). It follows that \(r_n' = 2\) and \(d_2 = q_i + r_i + 1\).

**Case 2.** \(i = \lfloor n/2 \rfloor - 3\) and \(n\) is odd. Then \(j = \lfloor n/2 \rfloor = i + 2\). If \(r_n' = 5\), then \((r_i, r_j) = (0, 2)\). On the other hand, by Theorem 2.1, the assumption \(d_2 < d_1\) reduces to \(q_j + r_j < q_i + r_i + 1\), i.e., \(q_j + 1 < q_i\), which is impossible since \(j = i + 2\). Otherwise \(r_n' \in \{1, 3\}\). Then \((r_i, r_j) = \{(1, 0), (2, 1)\}\) and \(r_j = r_i - 1\). By Theorem 2.1, the assumption \(d_2 < d_1\) reduces to \(q_j + r_j + 1 < q_i + r_i + 1\), i.e., \(q_j < q_i\). Since \(j = i + 2\), we find \(q_j = q_i\). It follows that \(r_j \geq r_i\), a contradiction.

**Case 3. None of the above.** Then \(i + 3 \leq \lfloor n/2 \rfloor\). By Theorem 2.1,
\[d_2 - d_1 = q_i + r_i - q_i = 1,
\]
contradicting the assumption \(d_2 < d_1\).

Therefore, by Theorem 2.1, as if \((r_n', i) \neq (2, n/2 - 2)\), \(d_2 \geq d_1 = q_i + r_i + 1\). □

**Corollary 2.5.** Let \(0 \leq i \leq \lfloor n/2 \rfloor - 2\), \(d_1 = d(v_0, v_i)\) and \(d_2 = d(v_0, v_{i+3})\). Suppose that \(d_2 < d_1\). Then one of the following is true.

1. \(r_n' = 2\) and \((i, d_2) = (M_n, q_r + r_i - 1)\).
2. \(r_n' \in \{1, 5\}\) and \((i, d_2) = (M_n, q_r + r_i)\).
3. \(r_n' \in \{2, 4\}\) and \((i, d_2) = (M_n - 3, q_r + r_i + 1)\).

**Proof.** Suppose that \(r_n = 0\). When \(i + 3 \leq n/2\), we can deduce Eq. (2.3) by Theorem 2.1, contradicting the premise \(d_2 < d_1\). Consider the other case \(n/2 - 3 < i \leq \lfloor n/2 \rfloor - 2\). If \(n = 6q_n'\), then \(i = 3q_n' - 2\) and \(d_1 = q_n' + 2 < q_n' + 3 = d_2\), the same contradiction. Otherwise \(n = 6q_n' + 3\). Then \(i = 3q_n' - 1\) and \(d_1 = d_2 = q_n' + 3\), the same contradiction.

Below we can suppose that \(r_n \neq 0\). When \(i + 3 < n/2\), we obtain the same contradiction Eq. (2.3) to the premise \(d_2 < d_1\). It remains to compute the pair \((d_1, d_2)\) for \(M_n - 3 \leq i \leq \lfloor n/2 \rfloor - 2\). We observe that the upper bound can be further improved to \(\lfloor n/2 \rfloor - 2\) when \(n\) is odd since \(d_1 = d_2\) by symmetry when \(i = (n + 1)/2 - 2\). We treat 4 cases according to the residue of \(n\) modulo 6.

**Case 1.** \(n = 6q_n' + 1\). Then \(M_n = 3q_n' - 2 = \lfloor n/2 \rfloor - 2\). By Theorem 2.1, we find Table 2.1, from which we see that \((i, d_2) = (M_n, q_i + r_i)\).

**Case 2.** \(n = 6q_n' + 2\). Then \(M_n = 3q_n' - 1 = n/2 - 2\). By Theorem 2.1, we find Table 2.2, from which we see that \((i, d_2) = (M_n, q_i + r_i - 1)\) or \((i, d_2) = (M_n - 3, q_i + r_i + 1)\).
Table 2.1. The pair \((d_1, d_2)\) when \(n = 6q_n^i + 1\) and \(M_n - 3 \leq i \leq \lfloor n/2 \rfloor - 2\).

| \(i\)   | \(d_1\) | \(d_2\) |
|---------|---------|---------|
| 3q_n^i - 2 | \(q_n^i + 1\) | \(q_n^i\) |
| 3q_n^i - 3 | \(q_n^i - 1\) | \(q_n^i\) |
| 3q_n^i - 4 | \(q_n^i + 2\) | \(q_n^i + 3\) |
| 3q_n^i - 5 | \(q_n^i + 1\) | \(q_n^i + 1\) |

Table 2.2. The pair \((d_1, d_2)\) when \(n = 6q_n^i + 2\) and \(M_n - 3 \leq i \leq \lfloor n/2 \rfloor - 2\).

| \(i\)   | \(d_1\) | \(d_2\) |
|---------|---------|---------|
| 3q_n^i - 1 | \(q_n^i + 1\) | \(q_n^i\) |
| 3q_n^i - 2 | \(q_n^i + 2\) | \(q_n^i + 3\) |
| 3q_n^i - 3 | \(q_n^i - 1\) | \(q_n^i\) |
| 3q_n^i - 4 | \(q_n^i + 2\) | \(q_n^i + 1\) |

**Case 3.** \(n = 6q_n^i + 4\). Then \(M_n = 3q_n^i + 1\) and \(n/2 - 2 = 3q_n^i\). By Theorem 2.1, we find Table 2.3, from which we see that \((i, d_2) = (M_n - 3, q_i + r_i + 1)\).

Table 2.3. The pair \((d_1, d_2)\) when \(n = 6q_n^i + 4\) and \(M_n - 3 \leq i \leq \lfloor n/2 \rfloor - 2\).

| \(i\)   | \(d_1\) | \(d_2\) |
|---------|---------|---------|
| 3q_n^i | \(q_n^i + 3\) | \(q_n^i + 3\) |
| 3q_n^i - 1 | \(q_n^i + 3\) | \(q_n^i + 4\) |
| 3q_n^i - 2 | \(q_n^i + 2\) | \(q_n^i + 1\) |

**Case 4.** \(n = 6q_n^i + 5\). Then \(M_n = 3q_n^i + 1\) and \(\lfloor n/2 \rfloor - 2 = 3q_n^i\). By Theorem 2.1, we find Table 2.4, from which we see that \((i, d_2) = (M_n, q_i + r_i)\).

Table 2.4. The pair \((d_1, d_2)\) when \(n = 6q_n^i + 5\) and \(M_n - 3 \leq i \leq \lfloor n/2 \rfloor - 2\).

| \(i\)   | \(d_1\) | \(d_2\) |
|---------|---------|---------|
| 3q_n^i | \(q_n^i\) | \(q_n^i + 1\) |
| 3q_n^i - 1 | \(q_n^i + 2\) | \(q_n^i + 1\) |
| 3q_n^i - 2 | \(q_n^i + 2\) | \(q_n^i + 3\) |
| 3q_n^i - 3 | \(q_n^i - 1\) | \(q_n^i\) |
| 3q_n^i - 4 | \(q_n^i + 2\) | \(q_n^i + 2\) |

Rearranging the above results we obtain the desired statement. \(\square\)

By using Corollaries 2.2 to 2.5, we can compute the distance between any vertex and any edge in \(P(n, 3)\). With the aid of the notation \(r\), we can write up a formula for such a distance. Denote \(e_{n}^{u} = u_{i}u_{i+1}\), \(e_{n}^{v} = v_{i}v_{i+3}\), and \(e_{n}^{s} = u_{i}v_{i}\).

**Corollary 2.6.** Let \(n \geq 13\) and \(0 \leq i \leq n - 1\). Then

\[
\begin{align*}
    d(u_0, e_{i}^{v}) &= \begin{cases}
        \min(i, n-1-i), & \text{if } i \leq 2 \text{ or } i \geq n-3, \\
        \min\left(\lfloor i/3 \rfloor + 2, \lceil (n-i-1)/3 \rceil + 2 \right), & \text{otherwise};
    \end{cases}
\end{align*}
\]
The edge dimension of $P(n, 3)$ is at most 4

\[ d(u_0, e_{i}^n) = \begin{cases} \min(i, n - i), & \text{if } i \leq 2 \text{ or } i \geq n - 2, \\ \min(q_i + r_i + 1, q_n - q_i + r + 1), & \text{otherwise;} \end{cases} \]

\[ d(u_0, e_{i}^n) = \begin{cases} \min(i + 1, n - i + 1), & \text{if } i \leq 1 \text{ or } i = n - 1, \\ \min(q_i + r_i + 1, q_n - q_i + r), & \text{otherwise;} \end{cases} \]

\[ d(v_0, e_{i}^u) = \begin{cases} \min(q_i + \lfloor r_i/2 \rfloor + 1, q_n - q_i + \lfloor r_n/2 \rfloor - \lfloor r_i/2 \rfloor + 1), & \text{if } r_i = 0 \text{ or } (r_n, r_i) = (0, 2), \\ \min(q_i + 2, q_n - q_i + 1), & \text{otherwise;} \end{cases} \]

\[ d(v_0, e_{i}^v) = \begin{cases} \min(q_i, q_n - q_i), & \text{if } r_i = r_n = 0, \\ \min(q_i + r_n + 1, q_n - q_i), & \text{if } r_i = r_n \neq 0, \\ \min(q_i + 2, q_n - q_i + 2), & \text{if } r_i = 1 \neq r_n, \\ \min(q_i + 3, q_n - q_i + r_n), & \text{if } r_i = 2 \neq r_n; \end{cases} \]

\[ d(v_0, e_{i}^v) = \begin{cases} \min(q_i, q_n - q_i - 1), & \text{if } r_i = r_n = 0, \\ \min(q_i + r_i + 2, q_n - q_i + r_n - r_i - 1), & \text{if } r_i = r_n \neq 0, \\ \min(q_i + 3, q_n - q_i + 2), & \text{if } r_i = 1 \neq r_n, \\ \min(q_i + 4, q_n - q_i + r_n + 1), & \text{if } r_i = 2 \neq r_n. \end{cases} \]

**Proof.** One may show them by using Corollaries 2.2 to 2.5. We omit the details. \[ \square \]

## 3. Edge resolving sets for $P(n, 3)$

**Lemma 3.1.** Suppose that $r_n = 0$. For $n \geq 18$, the set $(u_0, u_1, v_2, u_{\lfloor n/2 \rfloor - 1})$ is a resolving one.

**Proof.** Since $r_n = 0$, the parameter $r$ is 1 if $r_i = 1$ and 0 otherwise. First of all, the edges whose distances from $u_0$ are less than 3 are distinguishable; see Table 3.1 for their metric representations with respect to the vector $(u_0, u_1, v_2, u_{\lfloor n/2 \rfloor - 1})$, where an asterisk * entry means that it is unnecessary to be computed for the purpose of distinguishing the edge indicated by the row the entry lies in.

While most entries in Table 3.1 can be obtained directly from Corollary 2.6, we explain the $u_{\lfloor n/2 \rfloor - 1}$-coordinate of the row $e_{3}^{v}$, which is derived in the following way. Consider

\[ i = \lfloor n/2 \rfloor + 4 = \begin{cases} 3q_n' + 4, & \text{if } n = 6q_n'; \\ 3q_n' + 6, & \text{if } n = 6q_n' + 3. \end{cases} \]

It follows that

\[ (q_i, r_i, r) = \begin{cases} (q_n' + 1, 1, 1), & \text{if } n = 6q_n'; \\ (q_n' + 2, 0, 0), & \text{if } n = 6q_n' + 3. \end{cases} \]

Therefore,

\[ d(u_{\lfloor n/2 \rfloor - 1}, e_{3}^{v}) = d(u_0, e_{\lfloor n/2 \rfloor + 4}^{v}) = \min(q_i + r_i + 1, q_n - q_i + r) = \begin{cases} q_n', & \text{if } n = 6q_n'; \\ q_n' - 1, & \text{if } n = 6q_n' + 3 \end{cases} \]

is at most $\lfloor q_n/2 \rfloor$.

Let $d \geq 3$ be an integer. By using Corollary 2.6, one may solve out the set of edges whose distance from $u_0$ is $d$, under some lower bound conditions on $q_n$ and $d$; see Table 3.2. The lower bound of $d$ works for all integers $n \geq 18$, since it is derived by requiring that distance from the edge $e_{3}^{v}$ to $u_0$ is computed by using the second expression in each of the first three formulas in Corollary 2.6. It remains to show that the edge metric representations are distinct.
Table 3.1. The edge metric representations of edges $e_i$ whose distances from $u_0$ are less than 3, when $n = 3q_n$.

| $l$ | $i$ | $d(u_0, e_i)$ | $d(u_1, e_i)$ | $d(v_2, e_i)$ | $d(u_{\lfloor n/2 \rfloor - 1}, e_i)$ |
|-----|-----|---------------|---------------|---------------|-------------------------------|
| $u$ | 0   | 0             | 0             | *             | *                              |
|     | 1   | 1             | 0             | 1             | *                              |
|     | 2   | 2             | 1             | 1             | *                              |
|     | −3  | 2             | 3             | 3             | $\lceil q_n/2 \rceil + 2$     |
|     | −2  | 1             | 2             | 2             | *                              |
|     | −1  | 0             | 1             | 2             | *                              |
| $s$ | 0   | 0             | 1             | 3             | *                              |
|     | 1   | 1             | 0             | 2             | *                              |
|     | 2   | 2             | 1             | 0             | *                              |
|     | 3   | 2             | 2             | 2             | *                              |
|     | −3  | 2             | 3             | 4             | $\lceil q_n/2 \rceil + 2$     |
|     | −2  | 2             | 2             | 3             | *                              |
|     | −1  | 1             | 2             | 1             | *                              |
| $v$ | 0   | 1             | 2             | 3             | *                              |
|     | 1   | 2             | 1             | $\lceil q_n/2 \rceil$ | $\lceil q_n/2 \rceil$         |
|     | 3   | 2             | 3             | 3             | $\leq \lceil q_n/2 \rceil$    |
|     | −6  | 2             | 3             | 4             | $\lceil q_n/2 \rceil$         |
|     | −4  | 2             | 3             | 1             | *                              |
|     | −3  | 1             | 2             | 4             | *                              |
|     | −2  | 2             | 1             | 3             | $\lceil q_n/2 \rceil + 1$     |
|     | −1  | 2             | 2             | 0             | *                              |

Table 3.2. The edges $e_i$ whose distances from $u_0$ are $d$, with the corresponding lower bounds of $q_n$ and $d$, when $n = 3q_n$.

| $l$ | $i$ | lower bound of $q_n$ | lower bound of $d$ |
|-----|-----|-----------------------|---------------------|
| $u$ | 3$d$ − 6 | 2$d$ − 4             | 3                   |
|     | 3$d$ − 7 | 2$d$ − 4             | 4                   |
|     | 3$d$ − 8 | 2$d$ − 5             | 4                   |
|     | 5 − 3$d$ | 2$d$ − 4             | 3                   |
|     | 6 − 3$d$ | 2$d$ − 4             | 4                   |
|     | 7 − 3$d$ | 2$d$ − 5             | 4                   |
| $s$ | ±(3$d$ − 3) | 2$d$ − 2             | 3                   |
|     | ±(3$d$ − 5) | 2$d$ − 4             | 3                   |
|     | ±(3$d$ − 7) | 2$d$ − 4             | 4                   |
| $v$ | 3$d$ − 3 | 2$d$ − 1             | 3                   |
|     | 3$d$ − 5 | 2$d$ − 3             | 3                   |
|     | 3$d$ − 7 | 2$d$ − 3             | 3                   |
|     | −3$d$  | 2$d$ − 1             | 3                   |
|     | 2 − 3$d$ | 2$d$ − 3             | 3                   |
|     | 4 − 3$d$ | 2$d$ − 3             | 3                   |
From Table 3.2, we see that the minimum lower bound of \( q_n \) is \( 2d - 5 \), which is attained by the edges \( e_{3d-8}^u \) and \( e_{7-3d}^v \). In fact, these two edges coincide with each other, because

\[
(3d - 8) - (7 - 3d) = 6d - 15 = 3(2d - 5) = 3q_n = n.
\]

Using the same idea, we see 4 edges for which the lower bound of \( q_n \) is \( 2d - 4 \):

\[
e_{3d-6}^u = e_{6-3d}^u, \quad e_{3d-7}^u = e_{5-3d}^u, \quad e_{3d-5}^u = e_{7-3d}^v, \quad \text{and} \quad e_{3d-7}^v = e_{5-3d}^v.
\]

They have distances \( d - 1 \), \( d - 2 \), \( d \), and \( d - 3 \) from \( v_2 \) respectively, and thus distinguishable.

Below we can suppose that \( q_n \geq 2d - 3 \). By Corollary 2.6, we compute the metric representations of edges in Table 3.2 with respect to the remaining vertex triple \((u_1, v_2, u_{(n/2)-1})\); see Table 3.3, where \( I_{even}(q_n) \) equals 1 if \( q_n \) is odd and 0 if \( q_n \) is even. Note that

| \( l \) | \( i \) | \( q_n \) | \( d(u_1, e_i^1) \) | \( d(v_2, e_i^1) \) | \( d(u_{(n/2)-1}, e_i^1) \) |
|------|------|------|----------------|----------------|----------------|
| \( u \) | 3d - 6 | \( \geq 2d - 3 \) | \( 2 (d = 3) \) | \( d - 1 \) | \( \leq \lceil q_n/2 \rceil - d + 4 \) |
| \( 3d - 7 \) | \( 2d - 3 \) | \( d \) | \( d - 2 \) | 0 |
| \( 2d - 2 \) | 2 |
| \( \geq 2d - 1 \) | * |
| \( 3d - 8 \) | \( \geq 2d - 3 \) | \( d - 1 \) | \( d - 2 \) | * |
| \( 5 - 3d \) | \( 2d - 3 \) | \( d + 1 \) | \( d - 1 \) | * |
| \( \geq 2d - 2 \) | \( d \) | \( \lceil q_n/2 \rceil - d + 4 \) |
| \( 6 - 3d \) | \( \geq 2d - 3 \) | \( d \) | * |
| \( 7 - 3d \) | \( \geq 2d - 3 \) | \( d \) | \( \lceil q_n/2 \rceil - d + 5 \) |
| \( s \) | \( 3d - 3 \) | \( \geq 2d - 2 \) | \( d + 1 \) | \( d \) | \( \leq \lceil q_n/2 \rceil - d + 3 \) |
| \( 3d - 5 \) | \( \geq 2d - 3 \) | \( d - 1 \) | \( d \) | \( \leq \lceil q_n/2 \rceil - d + 3 \) |
| \( 3d - 7 \) | \( 2d - 3 \) | \( d - 1 \) | \( d - 3 \) | 1 |
| \( \geq 2d - 2 \) | \( \lceil q_n/2 \rceil - d + 3 \) |
| \( 3 - 3d \) | \( 2d - 2 \) | \( d + 1 \) | \( d \) | * |
| \( 2d - 1 \) | \( d + 1 \) | 2 |
| \( \geq 2d \) | \( d + 2 \) | \( \lceil q_n/2 \rceil - d + 3 \) |
| \( 5 - 3d \) | \( 2d - 3 \) | \( d \) | \( d - 2 \) | 2 |
| \( \geq 2d - 2 \) | \( d + 1 \) | \( d - 1 \) | \( \lceil q_n/2 \rceil - d + 3 + 2I_{even}(q_n) \) |
| \( 7 - 3d \) | \( \geq 2d - 3 \) | \( d - 1 \) | \( d \) | \( \lceil q_n/2 \rceil - d + 5 \) |
| \( v \) | \( 3d - 3 \) | \( 2d - 1 \) | \( d + 1 \) | \( d + 1 \) | 1 |
| \( \geq 2d \) | * |
| \( 3d - 5 \) | \( 2d - 3 \) | \( d - 1 \) | \( d + 1 \) | * |
| \( 2d - 2 \) | 2 |
| \( \geq 2d - 1 \) | \( \lceil q_n/2 \rceil - d + 2 \) |
| \( 3d - 7 \) | \( 2d - 3 \) | \( d - 1 \) | \( d - 3 \) | 2 |
| \( \geq 2d - 2 \) | \( \lceil q_n/2 \rceil - d + 2 \) |
| \( -3d \) | \( 2d - 1 \) | \( d + 1 \) | \( d + 1 \) | * |
| \( 2d \) | \( d + 2 \) | 2 |
| \( \geq 2d + 1 \) | \( d + 3 \) | * |
| \( 2 - 3d \) | \( 2d - 3 \) | \( d - 1 \) | \( d - 3 \) | * |
| \( 2d - 2 \) | \( d \) | \( d - 2 \) | 1 |
| \( \geq 2d - 1 \) | \( d + 1 \) | \( d - 1 \) | \( \lceil q_n/2 \rceil - d + 2 + 2I_{even}(q_n) \) |
| \( 4 - 3d \) | \( \geq 2d - 3 \) | \( d - 1 \) | \( d + 1 \) | \( \lceil q_n/2 \rceil - d + 4 \) |

Table 3.3. The metric representations of the edges whose distances from \( u_0 \) are the same \( d \), when \( n = 3q_n \geq 3(2d - 3) \).
Table 3.4. The edge metric representations for \( d \leq 2 \) that need the vertex \( u_{\lfloor n/2 \rfloor - 1} \) as a resolving set, when \( n = 6h + 1 \).

| \( l \) | \( i \) | \( d(u_0, e^l_i) \) | \( d(u_1, e^l_i) \) | \( d(v_2, e^l_i) \) | \( d(u_{\lfloor n/2 \rfloor - 1}, e^l_i) \) |
|-------|-------|-----------------|-----------------|-----------------|-----------------|
| \( u \) | \(-3\) | 2 | 3 | 3 | \( h + 2 \) |
| \( s \) | \(-3\) | 2 | 3 | 4 | \( h + 2 \) |
| \( v \) | 1 | 2 | 1 | 3 | \( h \) |
| \( 3 \) | 2 | 3 | 3 | \( h \) |
| \(-6\) | 2 | 3 | 4 | \( h + 1 \) |
| \(-2\) | 2 | 1 | 3 | \( h + 1 \) |

\[ 3d - 5 \equiv 4 - 3d \pmod{n} \] when \( q_n = 2d - 3 \),
\[ 3d - 7 \equiv 2 - 3d \pmod{n} \] when \( q_n = 2d - 3 \),
\[ 3d - 3 \equiv 3 - 3d \pmod{n} \] when \( q_n = 2d - 2 \),
\[ 3d - 3 \equiv -3d \pmod{n} \] when \( q_n = 2d - 1 \).

From Table 3.3, we see that every edge is recognizable, as desired. \( \square \)

Lemma 3.2. Suppose that \( n = 6q'_n + 1 \), where \( q'_n \geq 3 \). Then the set \( (u_0, u_1, v_2, u_{\lfloor n/2 \rfloor - 1}) \) is a resolving one.

Proof. We write \( h = q'_n \) for short. Let \( d \geq 0 \) be an integer. For \( d \leq 2 \), all entries except the non-asterisks in the last column in Table 3.1 keep invariant. The non-asterisks are listed in Table 3.4. From which we see that the edges whose distance \( d \) from \( u_0 \) is at most 2 are distinguishable by the vector \( (u_0, u_1, v_2, u_{\lfloor n/2 \rfloor - 1}) \).

Let \( d \geq 3 \). Note that
\[ 3d - 3 \equiv 2 - 3d \quad \text{and} \quad 3d - 5 \equiv -3d \pmod{n} \] when \( d = q'_n + 1 \).

We list the edges whose distance from \( u_0 \) is \( d \), with their metric representations with respect to the remaining vertex triple \( (u_1, v_2, u_{\lfloor n/2 \rfloor - 1}) \) as in Table 3.5, From which we see that all edges are distinguishable by \( (u_0, u_1, v_2, u_{\lfloor n/2 \rfloor - 1}) \), as desired. \( \square \)

Lemma 3.3. Suppose that \( n = 6q'_n + 2 \), where \( q'_n \geq 3 \). Then the set \( (u_0, u_1, u_{n/2 - 3}, v_{n/2 - 2}) \) is a resolving one.

Proof. Write \( h = q'_n \) for short. For \( d \leq 2 \), we compute the edge metric representations as in Table 3.6, in which the columns for \( d(u_0, e^l_i) \) and \( d(u_1, e^l_i) \) are same to those in Table 3.1.

Let \( d \geq 3 \). Note that
\[ 3d - 5 \equiv 5 - 3d \quad \text{and} \quad 3d - 7 \equiv 3 - 3d \pmod{n} \] when \( d = h + 2 \).

We list the edges whose distance from \( u_0 \) is \( d \), with their metric representations with respect to the remaining vertex triple in Table 3.7, from which we see that all edges are distinguishable, as desired. \( \square \)

Lemma 3.4. Suppose that \( n = 6q'_n + 4 \), where \( q'_n \geq 3 \). Then the set \( (u_0, u_1, v_2, u_{n/2 + 3}) \) is a resolving one.

Proof. We write \( h = q'_n \) for short. For \( d \leq 2 \), all entries except the non-asterisks in the last column in Table 3.1 keep invariant. The non-asterisks are listed in Table 3.8, from which we see that the edges whose distances from \( u_0 \) are at most 2 are distinguishable by \( (u_0, u_1, v_2, u_{n/2 + 3}) \).

Let \( d \geq 3 \). Note that
\[ 3d - 7 \equiv 7 - 3d \quad \text{and} \quad 3d - 8 \equiv 6 - 3d \pmod{n} \] when \( d = h + 3 \),
Table 3.5. The metric representations of the edges $e_i^l$ whose distances from $u_0$ are $d$, when $n = 6h + 1$.

| l  | i  | range of $d$ | $d(u_1, e_i^l)$ | $d(v_2, e_i^l)$ | $d(u_{[n/2]-1}, e_i^l)$ |
|----|----|--------------|-----------------|-----------------|-----------------|
| u  | $3d - 6$ | $h + 2$ | $d \geq 4$ | $d - 1$ | $h - d + 3$ |
|    |       | $h + 1$ |                     | $h - d + 2$ |                  |
|    |       | $[3, h]$ | $2 (d = 3)$ | $h - d + 4$ |                  |
|    |       |          | $d \geq 4$ |                    |                  |
|    | $3d - 7$ | $[4, h + 2]$ | $d$ | $d - 2$ | $*$ |
|    | $3d - 8$ | $h + 2$ | $d - 1$ | $d - 2$ | $h - d + 2$ |
|    |       | $[4, h + 1]$ |                      | * |                  |
|    | $5 - 3d$ | $h + 2$ | $d$ | $d - 1$ | $*$ |
|    |       | $[3, h + 1]$ | $d + 1$ | $h - d + 5$ |                  |
|    | $6 - 3d$ | $h + 2$ | $d$ | $d - 1$ | $h - d + 4$ |
|    |       | $[4, h + 1]$ |                      | * |                  |
|    | $7 - 3d$ | $[4, h + 2]$ | $d$ | $d - 1$ | $h - d + 5$ |
| s  | $3d - 3$ | $[3, h + 1]$ | $d + 1$ | $d$ | $\leq h - d + 3$ |
|    | $3d - 5$ | $[3, h + 1]$ | $d - 1$ | $d$ | $\leq h - d + 3$ |
|    | $3d - 7$ | $h + 2$ | $d - 1$ | $d - 3$ | $h - d + 2$ |
|    |       | $[4, h + 1]$ |                      | $h - d + 3$ |                  |
|    | $3 - 3d$ | $h + 1$ | $d$ | $d + 1$ | $*$ |
|    |       | $[3, h]$ | $d + 1$ | $d + 2$ | $*$ |
|    | $5 - 3d$ | $[3, h + 1]$ | $d + 1$ | $d - 1$ | $h - d + 4$ |
|    | $7 - 3d$ | $h + 2$ | $d - 1$ | $d - 2$ | $h - d + 4$ |
|    |       | $[4, h + 1]$ |                      | * |                  |
| v  | $3d - 3$ | $h + 1$ | $d + 1$ | $d - 1$ | $*$ |
|    |       | $[3, h]$ | $d + 1$ | $*$ |                  |
|    | $3d - 5$ | $h + 1$ | $d - 1$ | $d + 1$ | $h - d + 3$ |
|    |       | $[3, h]$ |                     | $h - d + 2$ |                  |
|    | $3d - 7$ | $h + 2$ | $d - 1$ | $d - 3$ | $h - d + 3$ |
|    |       | $[3, h + 1]$ |                      | $h - d + 2$ |                  |
|    | $-3d$ | $h + 1$ | $d - 1$ | $d + 1$ | $*$ |
|    |       | $[3, h]$ | $d + 1$ | $d + 3$ | $*$ |
|    | $2 - 3d$ | $[3, h + 1]$ | $d + 1$ | $d - 1$ | $h - d + 3$ |
|    | $4 - 3d$ | $h + 2$ | $d - 1$ | $d - 3$ | $*$ |
|    |       | $h + 1$ |                         | * |                  |
|    |       | $[3, h]$ | $d + 1$ | $h - d + 3$ |                  |

$3d - 3 \equiv 5 - 3d \pmod{n}$ and $3d - 5 \equiv 3 - 3d \pmod{n}$ when $d = h + 2$.

We list the edges whose distance from $u_0$ is $d$, with their metric representations with respect to the remaining vertex triple as in Table 3.9, from which we see that all edges are distinguishable, as desired.

\[ \square \]

**Lemma 3.5.** Suppose that $n = 6q_n' + 5$, where $q_n' \geq 3$. Then the set $(u_0, u_1, u_{(n-3)/2}, v_{(n-1)/2})$ is a resolving one.

**Proof.** For $d \leq 2$, we compute the edge metric representations as in Table 3.10. All entries for the first four columns in Table 3.1 keep invariant.
Table 3.6. The metric representations of edges $e^l_i$ whose distances from $u_0$ are at most 2, when $n = 6h + 2$.

| $l$ | $i$ | $d(u_0, e^l_i)$ | $d(u_1, e^l_i)$ | $d(u_{n/2-3}, e^l_i)$ | $d(v_{n/2-2}, e^l_i)$ |
|-----|-----|-----------------|-----------------|---------------------|---------------------|
| $u$ | 0   | 0               | 0               | *                   | *                   |
|     | 1   | 1               | 0               | $h+1$               | *                   |
|     | 2   | 2               | 1               | $h+1$               | $h$                 |
|     | $-3$| 2               | 3               | $h+2$               | $h+1$               |
|     | $-2$| 1               | 2               | $h+2$               | $h+1$               |
|     | $-1$| 0               | 1               | $h+2$               | *                   |
| $s$ | 0   | 0               | 1               | $h+1$               | *                   |
|     | 1   | 1               | 0               | $h$                 | *                   |
|     | 2   | 2               | 1               | $h+1$               | $h-1$               |
|     | 3   | 2               | 2               | $h$                 | *                   |
|     | $-3$| 2               | 3               | $h+2$               | $h$                 |
|     | $-2$| 2               | 2               | $h+1$               | $h+2$               |
|     | $-1$| 1               | 2               | $h+2$               | $h$                 |
| $v$ | 0   | 1               | 2               | $h$                 | *                   |
|     | 1   | 2               | 1               | $h-1$               | *                   |
|     | 3   | 2               | 3               | $h-1$               | *                   |
|     | $-6$| 2               | 3               | $h+1$               | $h-1$               |
|     | $-4$| 2               | 3               | $h+1$               | $h$                 |
|     | $-3$| 1               | 2               | $h+1$               | *                   |
|     | $-2$| 2               | 1               | $h$                 | *                   |
|     | $-1$| 2               | 2               | $h+1$               | $h-1$               |

Let $d \geq 3$. We note that $d = h+3$ happens only when $l = u$. Since $n = 6h + 5 = 6d - 13$, among the 18 edges there are only three distinct edges have the same distance $d$ from $u_0$, that is,

$$e^u_{3d-6} = e^u_{7-3d}, \quad e^u_{3d-7} = e^u_{6-3d}, \quad \text{and} \quad e^u_{3d-8} = e^u_{5-3d}.$$ 

Their distances from $u_{(n-3)/2}$ are respectively 2, 1, and 0. Thus these three edges are distinguishable. For $d \leq h + 2$, we compute out Table 3.11, from which we see that all edges are distinguishable, as desired. □

Now we are in a position to prove Theorem 1.1.

Proof. By Lemmas 3.1 to 3.5, we know that for any $n \geq 18$, the graph $P(n, 3)$ has an edge resolving set of order 4. For each $11 \leq n \leq 17$, an edge resolving set of order 4 is listed below:

- When $n = 11$, an edge resolving set is $(u_0, u_1, v_0, v_5)$;
- When $n = 12$, an edge resolving set is $(u_0, u_1, v_2, v_3)$;
- When $n = 13$, an edge resolving set is $(u_0, u_1, v_3, v_7)$;
- When $n = 14$, an edge resolving set is $(u_0, u_3, v_4, v_5)$;
- When $n = 15$, an edge resolving set is $(u_0, u_1, v_2, v_4)$;
- When $n = 16$, an edge resolving set is $(u_0, u_2, v_3, v_4)$;
- When $n = 17$, an edge resolving set is $(u_0, u_2, v_3, v_{15})$.

We omit the verification for they can be done by computer easily. □
Table 3.7. The metric representations of the edges $e_i^t$ whose distances from $u_0$ are $d$, when $n = 6h + 2$.

| $l$ | $i$ | range of $d$ | $d(u_1, e_i^t)$ | $d(u_{n/2-3}, e_i^t)$ | $d(v_{n/2-2}, e_i^t)$ |
|-----|-----|--------------|-----------------|---------------------|---------------------|
| $u$ | $3d - 6$ | $h + 2$ | $d (d \geq 4)$ | $h - d + 4$ | * |
|     |       | $h + 1$ | $h - d + 1$ | * |
|     | $[3, h]$ | $2 (d = 3)$ | $h - d + 3$ | * |
|     |       | $d (d \geq 4)$ | * |
|     | $3d - 7$ | $h + 2$ | $d$ | $h - d + 3$ | * |
|     |       | $h + 1$ | $h - d + 2$ | * |
|     | $[4, h]$ | $h - d + 4$ | * |
|     | $3d - 8$ | $h + 2$ | $d - 1$ | $h - d + 2$ | * |
|     |       | $h + 1$ | $h - d + 3$ | $h - d + 3$ |
|     | $[4, h]$ | $h - d + 4$ | |
|     | $5 - 3d$ | $[3, h + 2]$ | $\geq d$ | $h - d + 5$ | * |
|     | $6 - 3d$ | $[4, h + 2]$ | $d$ | $h - d + 6$ | $h - d + 4$ |
|     | $7 - 3d$ | $[4, h + 2]$ | $d$ | $h - d + 6$ | $h - d + 5$ |
| $s$ | $3d - 3$ | $h + 2$ | $d - 1$ | $h - d + 6$ | $h - d + 3$ |
|     |       | $h + 1$ | $h - d + 3$ | * |
|     |       | $h$ | $h - d + 1$ | * |
|     | $3d - 5$ | $h + 2$ | $d - 1$ | $h - d + 4$ | * |
|     |       | $h + 1$ | $h - d + 1$ | * |
|     | $[3, h]$ | $h - d + 2$ | * |
|     | $3d - 7$ | $h + 2$ | $d - 1$ | $h - d + 3$ | * |
|     |       | $h + 1$ | $h - d + 3$ | $h - d + 2$ |
|     | $[4, h]$ | $h - d + 4$ | $h - d + 2$ |
|     | $3 - 3d$ | $h + 2$ | $d - 1$ | $h - d + 3$ | * |
|     | $[3, h + 1]$ | $d + 1$ | $h - d + 4$ | $h - d + 2$ |
|     | $5 - 3d$ | $h + 2$ | $d - 1$ | $h - d + 4$ | * |
|     | $[3, h + 1]$ | $d + 1$ | $h - d + 4$ | $(d = 3)$ |
|     |       | $h - d + 5 (d \geq 4)$ |
|     | $7 - 3d$ | $[4, h + 2]$ | $d - 1$ | $h - d + 6$ | $h - d + 5$ |
| $v$ | $3d - 3$ | $h + 1$ | $d$ | $h - d + 4$ | * |
|     |       | $h$ | $h - d + 2$ | * |
|     | $3d - 5$ | $h + 1$ | $d - 1$ | $h - d + 2$ | * |
|     | $[3, h]$ | $h - d + 1$ | * |
|     | $3d - 7$ | $[3, h + 1]$ | $d - 1$ | $h - d + 3$ | $h - d + 1$ |
|     | $-3d$ | $h + 1$ | $d$ | $h - d + 3$ | $h - d + 1$ |
|     | $[3, h]$ | $d + 1$ | |
|     | $2 - 3d$ | $h + 1$ | $d$ | $h - d + 3$ | $\geq h - d + 4$ |
|     | $[3, h]$ | $d + 1$ | |
|     | $4 - 3d$ | $[3, h + 1]$ | $d - 1$ | $h - d + 4 (d = 3)$ | * |
|     |       | $h - d + 5 (d \geq 4)$ | * |

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Table 3.8. The metric representations for edges with $d \leq 2$ that need the vertex $u_{n/2+3}$ as a resolving set, when $n = 6h + 4$.

| $l$ | $i$ | $d(u_0, e_i^1)$ | $d(u_1, e_i^1)$ | $d(v_2, e_i^1)$ | $d(u_{n/2+3}, e_i^1)$ |
|-----|-----|-----------------|-----------------|-----------------|------------------|
| $u$ | $-3$| 2               | 3               | 3               | $h + 1$          |
| $s$ | $-3$| 2               | 3               | 4               | $h + 1$          |
| $v$ | 1   | 2               | 1               | 3               | $h + 1$          |
|     | 3   | 2               | 3               | 3               | $h + 2$          |
|     | $-6$| 2               | 3               | 4               | $h$              |
|     | $-2$| 2               | 1               | 3               | $h$              |

Table 3.9. The metric representations of the edges $e_i^1$ whose distances from $u_0$ are $d$, when $n = 6h + 4$.

| $l$ | $i$ | range of $d$ | $d(u_1, e_i^1)$ | $d(v_2, e_i^1)$ | $d(u_{n/2+3}, e_i^1)$ |
|-----|-----|-------------|-----------------|-----------------|------------------|
| $u$ | $3d - 6$ | $[3, h + 2]$ | 2 ($d = 3$) | $d - 1$ | $h - d + 6$ |
|     | $3d - 7$ | $[4, h + 3]$ | $d$ | $d - 2$ | * |
|     | $3d - 8$ | $[4, h + 3]$ | $d - 1$ | $d - 2$ | $h - d + 6$ |
|     | $5 - 3d$ | $h + 2$ | $d + 1$ | $d$ | $h - d + 3$ |
|     | $3, h$ | $h + 1$ | $h - d + 2$ |
|     | $6 - 3d$ | $h + 3$ | $d - 1$ | $d - 2$ | * |
|     | $[4, h + 2]$ | $d$ | $d$ | * |
|     | $7 - 3d$ | $h + 3$ | $d$ | $d - 2$ | * |
|     | $h + 2$ | $d - 1$ | $h - d + 2$ |
|     | $[4, h + 1]$ | | $h - d + 4$ |
| $s$ | $3d - 3$ | $h + 2$ | $d + 1$ | $d - 1$ | * |
|     | $[3, h + 1]$ | $d$ | $h - d + 5$ |
|     | $3d - 5$ | $[3, h + 2]$ | $d - 1$ | $d$ | $h - d + 5$ |
|     | $3d - 7$ | $[4, h + 3]$ | $d - 1$ | $d - 3$ | $h - d + 5$ |
|     | $3 - 3d$ | $h + 2$ | $d - 1$ | $d$ | * |
|     | $[3, h]$ | $h + 1$ | $h - d + 2$ |
|     | $5 - 3d$ | $h + 2$ | $d + 1$ | $d - 1$ | * |
|     | $[3, h]$ | $h + 1$ | $h - d + 3$ |
|     | $7 - 3d$ | $h + 3$ | $d - 1$ | $d - 3$ | * |
|     | $h + 2$ | $d - 1$ | * |
|     | $[4, h + 1]$ | $d$ | $h - d + 3$ |
| $v$ | $3d - 3$ | $h + 1$ | $d + 1$ | $d$ | $h - d + 4$ |
|     | $[3, h]$ | $d + 1$ | * |
|     | $3d - 5$ | $[3, h + 1]$ | $d - 1$ | $d + 1$ | $h - d + 4$ |
|     | $3d - 7$ | $[3, h + 2]$ | $d - 1$ | $d - 3$ | $h - d + 4$ |
|     | $-3d$ | $h + 1$ | $d$ | $d + 2$ | * |
|     | $[3, h]$ | $d + 1$ | $d + 3$ | * |
|     | $2 - 3d$ | $h + 1$ | $d + 1$ | $d - 1$ | $h - d + 3$ |
|     | $[3, h]$ | | $h - d + 2$ |
|     | $4 - 3d$ | $h + 2$ | $d - 1$ | $d - 2$ | $h - d + 3$ |
|     | $h + 1$ | $d$ | $h - d + 2$ |
|     | $[4, h]$ | $d + 1$ | $h - d + 2$ |
THE EDGE DIMENSION OF $P(n, 3)$ IS AT MOST 4

Table 3.10. The metric representations of edges $e_i^l$ whose distances from $u_0$ are at most 2, when $n = 6h + 5$.

| $i$ | $d(u_0, e_i^1)$ | $d(u_1, e_i^1)$ | $d(u_{(n-3)/2}, e_i^1)$ | $d(v_{(n-1)/2}, e_i^1)$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| $u$ | 0               | 0               | *               | *               |
|     | 1               | 1               | 0               | $h+2$           |
|     | 2               | 2               | 1               | $h+2$           |
|     | $-3$            | 2               | 3               | $h+3$           |
|     | $-2$            | 1               | 2               | $h+3$           |
|     | $-1$            | 0               | 1               | $h+3$           |
| $s$ | 0               | 0               | 1               | $h+2$           |
|     | 1               | 1               | 0               | $h+1$           |
|     | 2               | 2               | 1               | $h+2$           |
|     | 3               | 2               | 2               | $h+1$           |
|     | $-3$            | 2               | 3               | $h+2$           |
|     | $-2$            | 2               | 2               | $h+2$           |
|     | $-1$            | 1               | 2               | $h+1$           |
| $v$ | 0               | 1               | 2               | $h+1$           |
|     | 1               | 2               | 1               | $h$             |
|     | 3               | 2               | 3               | $h$             |
|     | $-6$            | 2               | 3               | $h+1$           |
|     | $-4$            | 2               | 3               | $h+1$           |
|     | $-3$            | 1               | 2               | $h+2$           |
|     | $-2$            | 2               | 1               | $h+1$           |
|     | $-1$            | 2               | 2               | $h+2$           |

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Table 3.11. The metric representations of the edges $e_i^1$ whose distance from $u_0$ are $d$, when $n = 6h + 5$ and $d \leq h + 2$.

| $l$ | $i$ | range of $d$ | $d(u_1, e_i^1)$ | $d(u_{(n-3)/2}, e_i^1)$ | $d(v_{(n-1)/2}, e_i^1)$ |
|-----|-----|--------------|------------------|------------------|------------------|
| $u$ | $3d - 6$ | $h + 2$ | $d$ ($d \geq 4$) | $h - d + 2 = 0$ | * |
|     |     | $[3, h + 1]$ | $2$ ($d = 3$) | $h - d + 4$ | $h - d + 4$ |
|     |     | $d$ ($d \geq 4$) |               |                 | * |
|     | $3d - 7$ | $h + 2$ | $d$ | $h - d + 3 = 1$ | $h - d + 4$ |
|     |     | $[4, h + 1]$ |               |                 | * |
|     | $3d - 8$ | $h + 2$ | $d - 1$ | $h - d + 4 = 2$ | $h - d + 4$ |
|     |     | $[4, h + 1]$ |               |                 | * |
| $s$ | $3d - 3$ | $h + 2$ | $d$ | $h - d + 4 = 2$ | $h - d + 4$ |
|     |     | $[3, h]$ |               |                 | * |
|     | $3d - 5$ | $h + 2$ | $d - 1$ | $h - d + 2 = 0$ | * |
|     |     | $[3, h + 1]$ |               |                 | * |
|     | $3d - 7$ | $h + 2$ | $d - 1$ | $h - d + 4 = 2$ | $h - d + 3$ |
|     |     | $[4, h + 1]$ |               |                 | * |
|     | $3 - 3d$ | $h + 2$ | $d$ | $h - d + 3 = 1$ | $h - d + 2$ |
|     |     | $[3, h + 1]$ |               |                 | * |
|     | $5 - 3d$ | $h + 2$ | $d$ | $h - d + 4$ | $h - d + 4$ |
|     |     | $[3, h + 1]$ |               |                 | * |
|     | $7 - 3d$ | $[3, h + 1]$ | $d + 1$ | $h - d + 5 (d = 3)$ | $h - d + 5$ |
|     |     |               |               | $h - d + 6 (d \geq 4)$ | * |
| $v$ | $3d - 3$ | $h + 2$ | $d - 1$ | $h - d + 6 = 4$ | * |
|     |     | $[3, h]$ |               |                 | * |
|     | $3d - 5$ | $h + 2$ | $d - 1$ | $h - d + 4 = 3$ | $h - d + 4$ |
|     |     | $[3, h + 1]$ |               |                 | * |
|     | $3d - 7$ | $[3, h + 2]$ | $2$ ($d = 3$) | $h - d + 4$ | $h - d + 2$ |
|     |     | $d - 1$ ($d \geq 4$) |               |                 | * |
|     | $-3d$ | $h + 2$ | $d$ | $h - d + 5 = 3$ | * |
|     |     | $[3, h + 1]$ |               |                 | * |
|     | $2 - 3d$ | $[3, h + 2]$ | $d$ | $h - d + 3$ | $h - d + 5$ |
|     |     | $[3, h + 2]$ | $d - 2$ | $h - d + 5$ | * |

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THE EDGE DIMENSION OF $P(n,3)$ IS AT MOST 4

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