Electromagnetic nonlinear X-waves

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Nonlinear optical media that are normally dispersive, support a new type of localized (non-diffractive and nondispersive) wavepackets that are X-shaped in space and time and have slower than exponential decay. High-intensity X-waves, unlike linear ones, can be formed spontaneously through a trigger mechanism of conical emission, thus playing an important role in experiments.

The nonlinear response of condensed matter can compensate for the diffractive spreading of optical beams, or the dispersive broadening of pulses due to group-velocity dispersion (GVD), forming spatial [1-3] or temporal solitons [4-8], respectively. Recent experimental results concerning the self-focusing behavior of intense ultrashort pulses [4] indicate, however, that the spatial and temporal degrees of freedom cannot be treated separately. When the three length scales naturally associated with diffraction, GVD, and nonlinearity become comparable, the most intriguing consequence of space-time coupling is the possibility to form a nondiffractive and nondispersive localized wavepacket (LWP), namely a spatio-temporal soliton or light bullet [9] characterized by exponentially decaying tails. A strict constraint for the formation of light bullets is that the nonlinear phase changes counteract both the linear wavefront curvature and the GVD-induced chirp, leading to space-time focusing, as occurring in Kerr-like focusing media with anomalous GVD [10].

Vice versa, a normal GVD rules out the possibility to achieve bullet-type LWPs. In this regime, the field evolution is known to be qualitatively different, and involves complex phenomena such as temporal splitting and spectral breaking [3,11]. This is the reason why no attempts have been made to answer the fundamental question as to whether any form of nonlinearity-induced localization could still take place in normally dispersive media. In this letter, we show that LWPs do exist also with normal GVD in the form of nonlinear X-waves (NLXWs) or X-wave solitons. To date X-shaped waves are known only in the context of linear acoustic [12], or electromagnetic [13] propagation, and constitute the polychromatic generalization of diffraction-free Bessel (or Durnin [12]) beams. They have been observed in both acoustical [13] and optical [4] experiments, both requiring beam-shaping techniques. Here, we find propagation-invariant NLXWs that can be naturally regarded as the continuation of linear X-waves into the nonlinear regime. Yet, we find one fundamental difference between linear and NLXWs. At high intensity, the formation of X-shaped LWPs occurs spontaneously from conventional bell-shaped (in space and time) beams through self-induced spectral reshaping triggered by a mechanism of conic emission. Thus, NLXWs are expected to have stronger impact on experiments than linear X waves.

To support the generality of NLXW concept, we choose two different phenomena, whose spatio-temporal dynamics have been widely investigated experimentally. Specifically, we consider self-action of a scalar wavepacket \( u_1 \) (carrier \( \omega_0 \)) due to a pure (cubic) focusing Kerr effect, or generation of a \( u_2 \) wavepacket at second-harmonic (SH, \( 2\omega_0 \)) in non-centrosymmetric (quadratic) media. In the paraxial regime, the evolution in Kerr media is ruled by the scalar 1+3 nonlinear Schrödinger (NLS) equation,

\[
id\partial_t u_1 + \nabla^2 u_1 - d_1 \partial_{\tau\tau} u_1 + \Gamma|u_1|^2 u_1 = 0, \quad (1)
\]

whereas SH generation is ruled by the vector NLS model

\[
(i\partial_{\tau} + \sigma_1 \nabla^2 u_1 - d_1 \partial_{\tau\tau} u_1 + \Gamma u_2 u_1^* e^{i\delta k \zeta} = 0,
\]

\[
(i\partial_{\tau} + \sigma_2 \nabla^2 u_2 + i\nu \partial_{\tau} - d_2 \partial_{\tau\tau} u_2 + \Gamma u_1^* \nabla \equiv 0, \quad (2)
\]

In Eqs. (1) and (2), the link with real-world variables \( X, Y, Z, T \) is as follows: \( \zeta \equiv Z/Z_{\text{eff}} \) is the propagation distance in units of diffraction length \( Z_{\text{eff}} = 2k_0W_0^2 \) associated with the beam waist \( W_0 \), \( \nabla^2 u_1 \equiv \partial^2 + \partial_{\tau}^2 \) is the transverse Laplacian where \( (X, Y) = W_0(\xi, \eta) \), and \( t = (T - Z/V_g)/T_0 \) is time in a frame traveling at group-velocity \( V_g \) (of \( u_1 \)) in units of \( T_0 = (|k|^2|Z_{\text{eff}}|/2)^{1/2}, k_m^0 (m = 1, 2) \) being the GVD at \( n\omega_0 \). The coefficients are \( d_1 = k_m^0/|k|^4 \), \( \sigma_1 = k_m^0/k_m (\sigma \approx 1/2) \), the wavevector mismatch \( \delta k = (k_2 - 2k_1)Z_{\text{eff}} \), and \( \Gamma = Z_{\text{eff}}/Z_{\text{nl}} \) (not rescaled out to quantify the impact of nonlinearities). Here \( Z_{\text{nl}} = (\chi_3 I_p)^{-1} \) in Eq. (1) and \( Z_{\text{nl}} = (\chi_2 W)^{-1} \) in Eqs. (2) are nonlinear length scales associated with the input peak intensity \( I_p \) at \( \omega_0 \) (\( |u_1|^2 |_{\text{max}} = 1 \)), and \( \chi_3 [m/W] \) and \( \chi_2 [W^{-1/2}] \) are standard nonlinear coefficients. Finally, \( v = Z_{\text{eff}} \delta \nu/T_0 \) accounts for the walk-off due to group-velocity mismatch (GVM) \( \delta \nu = V_g^{-1} - V_1^{-1} \), though we analyse (where not stated otherwise) the GVM-matched case \( v = 0 \).
We seek for propagation-invariant, radially-symmetric LWPs of the form $u_1 = f_1(r,t) \exp(-i\beta t)$, accompanied in Eqs. (2) by a symbiotic SH $u_2 = f_2(r,t) \exp(-i(2\beta + \delta k) t)$. Here $f_{1,2}$ are real, and $\beta$ is a nonlinear phase-shift (in Eqs. (4) it makes the two waves nonlinearly phase-matched). In the cubic case, $f_1$ obeys the equation
\[
\ddot{f}_1 + r^{-1} \dot{f}_1 - D_1 \partial_t f_1 + b f_1 + \gamma f_1^3 = 0,
\]
while Eqs. (2) yield (after setting $f_1/\sqrt{\sigma_2} \to f_1$)
\[
\ddot{f}_1 + r^{-1} \dot{f}_1 - D_1 \partial_t f_1 + b f_1 + \gamma f_2 f_1 = 0,
\]
\[
\ddot{f}_2 + r^{-1} \dot{f}_2 - D_2 \partial_t f_2 + \alpha f_2 + \gamma f_2^3 = 0,
\]
where we have set $D_m = d_m/\sigma_m$, $f = \partial f/\partial r$, and $r = |\beta| \rho \equiv (x^2 + y^2)^{1/2}/(\rho^2 \equiv \xi^2 + \eta^2)$, $t = |\beta| \tau$, $\gamma = \Gamma/|\beta|$, $b = |\delta| |\beta|$ and $\alpha = (2b + |\delta| |\beta|)/\sigma_2$ in the nondegenerate case ($\beta \neq 0$), while ($r,t) = (\rho,\tau)$, $\alpha = \delta k/\sigma_2$, $\gamma = \Gamma$ and $b = 0$ in the degenerate case ($\beta = 0$). Eqs. (3) must be integrated along with the boundary conditions $f_{1,2}(r,\pm \infty) = f_{1,2}(\pm \infty, t) = 0$, and $f_{1,2}(0,t) = 0$. While the anomalous GVD regime ($D_m < 0$) guarantees that, for $\beta < 0$ ($\alpha < 0$), nearly separable LWPs (light bullets) exist with exponentially decaying tails, in the normal GVD regime ($D_m > 0$) the nature of the LWP solutions (if any) must change dramatically, because the low-intensity exponential damping no longer takes place. Pseudo-spectral numerical techniques (i.e., solve Eqs. (4) as a dynamical evolution problem in $r$ with appropriate discretization in $t$) are well suited to search for strongly nonseparable objects with slow spatio-temporal decay. We have implemented such methods (using two different algorithms), and found LWPs when $\beta \geq 0$ ($\alpha \geq 0$). Efficient convergence occurs by employing as a trial function (of time, at $r = 0$) the real part of the waveform,
\[
u_m = \frac{1}{\sqrt{(\Delta - i(\tau/d_m))^2 + \xi^2 + \eta^2}}; \quad m = 1,2,
\]
which represent X-shaped LWP solutions of Eqs. (2) in the linear limit ($\Gamma = 0$). Here $\Delta$ is a free parameter: the smaller $\Delta$, the stronger the localization. The LWP solutions, obtained from Eq. (3) with $\Delta = 1$ in Eq. (4) are shown in Fig. 1. For $b = 0$ and moderate nonlinearities [$\gamma = 1$, Fig. 1(a)], the ”ground-state” LWP mode has a clear X-shape (in $x-t$, or V-shape in $r-t$ variables), encompassing a space-time tightly confined structure, with slow axisymmetric spatial decay ($\sim 1/r$) accompanied by (radially increasing) temporal pulse-splitting. In the case $b = 1$, while the field maintains its basic X-shape, it develops radial (damped) oscillations as shown in Fig. 1(b), e.g. for the strong nonlinear case ($\gamma = 10$). From Eqs. (4) we obtain similar NLXWs, where the two symbiotic LWPs $f_{1,2}$ can be both of ground-state type ($b = \alpha = 0$), ground-oscillatory type ($b = 0, \alpha \neq 0$), or both of oscillatory type ($b = 1, \alpha \neq 0$, with out-of-phase coherent oscillation), an example of the latter case being shown in Fig. 2.

In both nonlinear processes the oscillations stem from the fact that the spatial behavior of the low-intensity portion of the LWPs is governed by a zero-th order Bessel equation, which can be easily obtained by Fourier-transforming the linear ($\gamma = 0$) limit of Eq. (4) or (de-coupled) Eqs. (4). Therefore NLXWs exhibit the characteristic damped spatial oscillations of $J_0$ Bessel functions, and can be regarded as the natural generalization of monochromatic nondiffractive conical $J_0$ beams, whose physical realization was demonstrated both in the linear and nonlinear regimes. More precisely, once the NLXW solutions $f_m = f_m(r,t)$ are obtained, they can be represented in the spectral domain of transverse wavevector $K = (K^2 + K^2)^{1/2}$ and frequency detuning $\Omega$, through the Fourier-Bessel transform $S_m = S_m(K,\Omega)$, as
\[
f_m = \int_0^{+\infty} \int_{-\infty}^{+\infty} S_m(\Omega, K) J_0(\Omega r) e^{i\Omega t} K dKd\Omega,
\]
showing their nature of a weighted superposition of conical $J_0$ beams with different frequency.
LWP (e.g., by means of lensaons or axicons). Vice versa, we have found that the interplay of the nonlinearity and normal GVD is responsible for a universal mechanism, namely colored conical emission (CE), that allows for the self-induced spectral (in $\mathbf{K}$-K) reshaping necessary to turn conventional, e.g. gaussian, pulsed beams into X-waves. In fact, it is known that in Kerr media the (modulational) stability analysis of the cw plane-wave solution $u = e^{i\mathbf{K} \cdot \mathbf{r}}$ of Eq. (1) yields exponential amplification of conical plane-wave (or Bessel) perturbations with wavevector $\mathbf{K}$ and frequency detuning $\Omega$, such to yield real values of gain $g = \left\{ |\mathbf{D}|^2 - |\mathbf{K}|^2 + i \right\}^{1/2}$, where $\mathbf{D} = (\Omega^2 - c^2)^{1/2}$. This analysis can be readily generalized for the cw plane-wave eigensolutions $u_1 = u_{10} \exp(-i\mathbf{K} \cdot \mathbf{r})$, $u_2 = u_{20} \exp(-i(\mathbf{K} + \delta \mathbf{k}) \cdot \mathbf{r})$ of Eqs. (2-3) to obtain the gain $g = \left| b \pm (b^2 - c)^{1/2} \right|^{1/2}$, where $\beta = (\Omega_1 + \Omega_2 - \Gamma u^{2}_{10} - \Gamma u^{2}_{20})^2 - \Omega_2$, $\Omega_m = \Theta_{m} K_0 - \Omega_2 \Omega^2 + m \beta - (m - 1) \delta k$, $m = 1, 2$. In the normal GVD regime ($m > 0$) these expressions entail CE, i.e. preferential amplification of waves at $\mathbf{K}$ (angles) linearly increasing with frequency detuning. By comparing (see Fig. 3) the CE gain with the NLXW spectrum, obtained by inverting Eq. (1), it is clear that the instability provides amplification at $\mathbf{K} - \Omega$ pairs that favour the formation of NLXWs. Although the stability analysis is carried out for cw plane-wave pumping beams, CE occurs also from tightly-focused short-pulse input beams [17]. In this case, we expect CE to amplify frequencies $\mathbf{K} - \Omega$ contained in the broad input spectrum, while preserving the phase coherence between different spectral components necessary for the formation of a NLXW. In order to support this conjecture and prove that NLXWs are the key to understand the dynamics of experiments carried out with narrow beams and short pulses in the normal GVD regime, we have conducted numerical simulations of the propagation. This is crucial also to assess the observability of such type of LWP with finite-energy, which are the nonlinear counterpart of monochromatic finite-aperture $J_0$ or Bessel-Gauss beams, and nonmonochromatic finite-energy linear X-waves [3,4], observable in the real world. While extensive results will be reported elsewhere, we focus specifically on quadratic media [Eqs. (1)] in the large negative mismatch limit, where the field $u_1$ plays a leading role, and experiences an effective focusing Kerr effect [1,3]. This case has twofold relevance: (i) higher-order effects not included in our models (Raman, self-steepening, space-time coupling, saturation, etc.) have lesser impact on the propagation as compared with true Kerr media; (ii) experimental results in Lithium Triborate (LBO) indicates the occurrence of pulse compression in spite of the fact that the medium is normally dispersive [19]. We model the latter case assuming $\chi_2 \simeq 7 \times 10^{-5} \text{W}^{-2}$, $k_0^2 = 0.015 \text{ps}^2/\text{m}$, $k_0^2 = 0.07 \text{ps}^2/\text{m}$, a mismatch $\Delta k = -30 \text{ cm}^{-1}$ ($\delta k = -180$), and a spatio-temporal input gaussian beam $u_1(Z = 0)$ with FWHM 170 fsec duration and 65 $\mu$m beam width. Figure 2(top panel) shows the output intensity in a 4 cm crystal, in the GVM-matched case ($\lambda = 1.3\mu$m, $T = 35 \text{ C in LBO}$) for an intensity $I_p = 50 \text{ GW/cm}^2$ ($Z_{nl} = 0.6 \text{ mm}$). As shown, while a moderate fraction of the energy lags behind and in front in the form of pulse satellites, the main portion of the beam develops the characteristic structure of a NLXW. Importantly, we observe that the formation of NLXW-type of LWP in Fig. 4 is accompanied by strong pulse compression. In fact, it can be shown that the peculiar spatio-temporal structure of a NLXW leads to an effective GVD, which we obtain in the form $k''_{eff} = \frac{d^2 k}{d \omega^2} \simeq \frac{d^2 k}{d \omega^2} \left[ k(\omega) - k^2 \right] \simeq k'' - k_0(\delta \omega)^2$, where the angle $\theta \simeq \sin \theta = k_t/k_0$, and $k_t = (k_0^2 - k_2)^{1/2}$, $K = |\Omega| \|\beta|$. Therefore the dispersive contribution which stems from the angular dispersion $\theta = \theta(\omega)$ counteracts the material (normal) GVD, leading to an effective anomalous GVD, which in turn explains the compression.

NLXWs are robust also against strong GVM. First, when $\nu \neq 0$ in Eqs. (2), we find CE as well as NLXW solutions of Eqs. (2) characterized by an additional complex phase profile (details will be given elsewhere). On the other hand, our simulations show that, at sufficiently high intensity, after a short distance, the launched ($u_1$) and generated ($u_2$) fields tend to develop spontaneously NLXW shapes, meanwhile leading to nonlinear walk-off compensation ($u_{1,2}$ travel locked together). For instance in Fig. 4 (bottom panel) we show the profile $|u_1|^2$ obtained for an input intensity $I_p = 70 \text{ GW/cm}^2$ after 1.5 mm propagation in LBO at $\lambda = 1.006 \mu$m, where a GVM as large as $\delta V = 45 \text{ ps/m}$ ($\nu = 75$) is compensated.

Our results show that, contrary to common belief, nonlinear space-time localization takes place also in normally dispersive media. NLXWs are the eigenmodes of 1+3D paraxial wave propagation models, and will be central to interpret correctly numerical and experimental results. Having restricted ourselves to NLXWs that travel at the natural group-velocity of light, further work will be devoted to search for NLXW solutions with finite energy (of which we have given numerical evidence), and/or sub-or super-luminal nature, as well as the role played by nonparaxiality and higher-order nonlinear terms.

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