Abstract: We show split property of gapped ground states for Fermion systems on a one-dimensional lattice and clarify mathematical meaning of string order of fermions.

Keywords: operator algebra, CAR algebra, split property, gapped ground state, string order, Jordan-Wigner transformation.

AMS subject classification: 82B10
1 Introduction

Analysis of gapped ground states is a subject of intensive study in mathematical physics. Matrix product states are typical examples for investigation of various aspects of gapped ground states. When we proceed to study of more general gapped ground states for one-dimensional systems, we arrive at states of quantum spin chains with split property. Split property is statistical independence of two subsystems in an infinite quantum spin chain. Historically the notion of split property was introduced in research of local quantum field theories in the last century and it is relatively recent to apply it to analysis of ground states of quantum spin chains. In [13] and in [14] we have shown that split property is valid for gapped ground states of infinite quantum spin chains and if split property is valid, the state has a matrix product state representation with an infinite dimensional auxiliary boundary space. As the matrix product representation is constructed canonically on the basis of the physical Hilbert space of the system, symmetry property of a gapped ground state is inherited to its auxiliary boundary space. Using matrix product representations, we have a Lieb-Mattis-Schultz type theorem for general SU(2) symmetric ground states. Our result is extended to certain discrete symmetry including space reflection by H.Tasaki and Y.Ogata [16]. Split property is a standing base of $Z_2$ invariant of Y.Ogata in [17]) which is a generalization of various invariant of symmetry protected topological phases.

The aim of the present paper is to investigate some basic aspects of split property for Fermion systems and to clarify mathematical meaning of string order for quasi-one dimensional Fermion systems.

Though a part of results stated below can be proved in a more general setting of unital $Z_2$ graded infinite dimensional $C^*$-algebras, for concreteness, we focus on CAR algebras (algebras generated by creation and annihilation operators of fermions).

Let $\mathfrak{A}(K)$ be the (self-dual) CAR(canonical anticommutation relations) algebra $\mathfrak{A}(K)$ over a one-particle Hilbert space $K$ equipped with a conjugate unitary involution $J$. (c.f. [1].) Hereafter we assume that $K$ is separable. By the conjugate unitary involution, we mean $J$ is a conjugate linear operator acting on the complex Hilbert space $K$ satisfying

$$J^2 = 1, \quad (Jh, Jk)_K = (k, h)_K \quad (h, k \in K) \tag{1.1}$$

where $(k, h)_K$ is the inner product of vectors $k, h$ in $K$. The CAR algebra $\mathfrak{A}(K)$ is the UHF $C^*$ algebra generated by $B(f), f \in K$ where $B(f)$ depends linearly on $f \in K$ and satisfies the following canonical anticommutation relation and $^*$ relations:

$$B(h)^* = B(Jh), \quad \{B(h), B(k)^*\} = (k, h)_K 1. \tag{1.2}$$

$\mathfrak{A}(K)$ is $Z_2$ graded where the $Z_2$ grading is specified with a parity automorphism $\Theta$ determined via the following equations: $\Theta(B(f)) = -B(f)$. Obviously, $\Theta^2(Q) = (Q)$ for any $Q$ in $\mathfrak{A}(K)$. The even (odd ) part of $\mathfrak{A}(K)$ is denoted by
We extend $\Theta$ via the equation $\Theta(T) = T$.

\[ \Theta(T) = T \]

Let $\hat{A}$ be the crossed product of the $\mathbb{Z}_2$ action induced by $\Theta_-$, $\hat{A} = A \times \mathbb{Z}_2$, namely $\hat{A}$ is generated by $A$ and a self-adjoint unitary $T$ implementing $\Theta_-$.

\[ T^2 = 1, \quad T = T^*, \quad TQT^{-1} = \Theta_-(Q), \quad Q \in \hat{A} \]

We extend $\Theta$ via the equation $\Theta(T) = T$ to an automorphism of $\hat{A}$. For any $Q \in \hat{A}$ we set

\[ Q_\pm = 1/2(Q + \Theta(Q)). \]

Let $\mathfrak{A}^{S}$ be the $C^*$-subalgebra of $\hat{A}$ generated by $\mathfrak{A}_L$ and $T\mathfrak{A}_R^{(-)}$ and let $\mathfrak{A}_L^{S}$ be the $C^*$-subalgebra of $\mathfrak{A}^{S}$ generated by $T\mathfrak{A}_R^{(-)}$ and we set

\[ \mathfrak{A}_L^{S}(\pm) = \{ Q \in \mathfrak{A}_L^{S} \mid \Theta(Q) = \pm Q \}, \quad \mathfrak{A}_L^{S}(\pm) \cap \mathfrak{A}_R^{S} = \mathfrak{A}_L^{S}(\pm) \cap \mathfrak{A}_R^{S} \cap \mathfrak{A}_L^{S}/R. \]

Obviously, $\mathfrak{A}_L^{S}(\pm) = \mathfrak{A}_L^{S}(\pm)$, $\mathfrak{A}_L$ and $T\mathfrak{A}_R^{(-)}$ commute and $\mathfrak{A}_L/R$, and $\mathfrak{A}_L^{S}/R$ are isomorphic to the UHF $C^*$-algebra of the type $2^\infty$, as a consequence, we can identify $\mathfrak{A}^{S}$ with $\mathfrak{A}_L \otimes \mathfrak{A}_R$:

\[ \mathfrak{A}^{S} = \mathfrak{A}_L \otimes \mathfrak{A}_R, \quad \mathfrak{A}_L^{S} = \mathfrak{A}_L \otimes 1, \quad \mathfrak{A}_R^{S} = 1 \otimes \mathfrak{A}_R. \]

Let us recall definition of (Z$_2$ graded) product states. Note that our definition of $\mathfrak{A}^{S}$ is different from that of the Pauli spin system via Jordan Wigner transformation in [1]. The definition is convenient for proof of split property in Section 2.

**Lemma 1.1** Let $\psi_L$ (resp. $\psi_R$) be a state of $\mathfrak{A}_L$ (resp. $\mathfrak{A}_R$) and suppose that $\psi_L$ is $\Theta$ invariant. There exists a state $\psi_L \otimes_{\mathbb{Z}_2} \psi_R$ of $\mathfrak{A}$ satisfying

\[ \psi_L \otimes_{\mathbb{Z}_2} \psi_R(QP) = \psi_L(Q+)\psi_R(P) \]

for any $Q \in \mathfrak{A}_L$ and any $P \in \mathfrak{A}_R$. 

For proof of this lemma, it suffices to present the GNS representation associated with \( \psi_L \otimes \mathbb{Z}_2 \psi_R \). Let \( \{ \pi_L(\mathfrak{A}_L), \Omega_L, \mathcal{H}_L \} \) be the GNS triple of \( \mathfrak{A}_L \) associated with the state \( \psi_L \) where \( \pi_L(\cdot) \) is the representation of \( \mathfrak{A}_L \) on the Hilbert space \( \mathcal{H}_L \) with the GNS cyclic vector \( \Omega_L \), and let \( \{ \pi_R(\cdot), \Omega_R, \mathcal{H}_R \} \) be the GNS triple of \( \mathfrak{A}_R \) associated with the state \( \psi_R \). Since \( \psi_L \) is \( \Theta \) invariant, there exists a self-adjoint unitary \( \Gamma_L \) acting on \( \mathcal{H}_L \) satisfying

\[
\Gamma_L \Omega_L = \Omega_L, \quad \Gamma_L \pi_L(Q) \Gamma_L^{-1} = \pi_L(\Theta(Q)), \quad Q \in \mathfrak{A}_L.
\]  

(1.11)

Let \( P_L \) be the orthogonal projection from \( K \) to \( K_L \) and \( P_R \) be that to \( K_L \) and set

\[
\pi(B(h)) = \pi_L(B(P_L h) \otimes 1 + \Gamma_L \otimes \pi_R(B(P_R h) h \in K.
\]  

(1.12)

Then, \( (1.12) \) gives rise to the GNS representation of \( \mathfrak{A} \) associated with \( \psi_L \otimes \psi_R \).

If, in stead, \( \psi_R \) is \( \Theta \) invariant, we have a state \( \psi_L \otimes \mathbb{Z}_2 \psi_R \) determined by

\[
\psi_L \otimes \mathbb{Z}_2 \psi_R(\pi_P) = \psi_L(\pi_P)\psi_R(\pi_P)
\]  

(1.13)

for any \( Q \in \mathfrak{A}_L \) and any \( P \in \mathfrak{A}_R \). We employ the same notation \( \psi_L \otimes \mathbb{Z}_2 \psi_R \) for the state determined by \( (1.10) \) or by \( (1.11) \) and \( \psi_L \otimes \mathbb{Z}_2 \psi_R \) will be referred to as the graded product state of \( \psi_L \) and \( \psi_R \) or simply the product state if there is no risk of confusion.

**Definition 1.2** Let \( \psi \) be a state of \( \mathfrak{A} \). We say split property is valid for \( \psi \) if \( \psi \) is quasi-equivalent to a product state \( \psi_L \otimes \mathbb{Z}_2 \psi_R \).

In the above Definition 1.2, we are assuming, at least, one of \( \psi_L \) and \( \psi_R \) is \( \Theta \) invariant. If a state \( \psi \) of \( \mathfrak{A} \) is \( \Theta \) invariant, satisfying split property, we may suppose that both \( \psi_L \) and \( \psi_R \) are \( \Theta \) invariant, for example, if \( \psi_L \) is \( \Theta \) invariant and \( \psi \) is quasi-equivalent to a product state \( \psi_L \otimes \mathbb{Z}_2 \psi_R \) of the form \( \psi \otimes \mathbb{Z}_2 \psi_R \circ \Theta \), hence, \( \psi \) is quasi-equivalent to \( \psi_L \otimes \mathbb{Z}_2 \psi_R 1/2(\psi_R + \psi_R \circ \Theta) \) as well.

**Theorem 1.3** Let \( \psi \) be a pure state of \( \mathfrak{A} \) which is \( \Theta \) invariant.

Suppose that split property is valid for \( \psi \). Then, \( \psi \) restricted to \( \mathfrak{A}_R \) (resp. \( \mathfrak{A}_L \)) gives rise to a type I representation of \( \mathfrak{A}_R \) (resp. of \( \mathfrak{A}_L \)).

Conversely, if \( \mathfrak{A}_R \) (resp. \( \mathfrak{A}_L \)) gives rise to a type I representation, the split property is valid for \( \psi \).

Next we turn to gapped ground states. We will see that gapped ground states with string order cannot be connected to the Fock vacuum state in Section 3.

Let \( \mathfrak{A} \) be the CAR algebra generated by creation and annihilation operators \( c_j^+, c_i \) \( i, j \in \mathbb{Z} \) satisfying the standard canonical anticommutation relations:

\[
\{c_j, c_i\} = 0, \quad \{c_j^+, c_i^+\} = 0, \quad \{c_j^+, c_i\} = \delta_{i,j} \mathbb{1}.
\]  

(1.14)

The relation to the selfdual formalism of . \( [1] \) will be explained later in Section 3.
For integers \( n < m \), let \( \mathcal{A}_{[n,m]} \) be the subalgebra of \( \mathcal{A} \) generated by \( c_j^* \), \( c_i \) \((n \leq i, j \leq m)\) and \( \mathcal{A}^{(+)}_{[n,m]} \) be the even part of \( \mathcal{A}_{[-n,n]} \). Let \( \mathcal{A}_{\text{loc}} \) be the algebra of strictly local observables:

\[
\mathcal{A}_{\text{loc}} = \bigcup_{-\infty < n < m < \infty} \mathcal{A}_{[n,m]}.
\]

By \( \tau_k \ (k \in \mathbb{Z}) \) we denote the lattice translation which is an automorphism of \( \mathcal{A}(K) \) satisfying

\[
\tau_k(c_i) = c_{i+k}, \quad \tau_k(c_i^*) = c_i^*. \quad \text{(1.16)}
\]

Fix \( h = h^* \in \mathcal{A}^{(+)}_{[-n,n]} \) for some \( n > 1 \) and we consider a finite volume Hamiltonian \( H_M \) defined by

\[
H_M = \sum_{-M+n \leq k \leq M-n} \tau_k(h) \in \mathcal{A}^{(+)}_{[-M,M]}.
\]

The formal infinite volume Hamiltonian

\[
H = \lim_{M \to \infty} H_M
\]

generates a time evolution of our Fermion system, more precisely, an infinite volume time evolution \( \alpha^k_h(Q) \) of an observable \( Q \) is determined via the following equation:

\[
\lim_{M \to \infty} e^{itH_M} Q e^{-itH_M} = \alpha^h_t(Q) \quad Q \in \mathcal{A}(K).
\]

**Definition 1.4** Let \( \psi \) be a translationally invariant state.

(i) \( \psi \) is a ground state of \( H \) if \( \psi \) minimizes the energy density:

\[
\psi(h) = \inf \varphi(h)
\]

where \( \inf \) is taken among the set of all translationally invariant states.

(ii) A translationally invariant ground state \( \psi \) is gapped if the following inequality is valid for some positive constant \( m \):

\[
\psi(Q^*[H,Q]) \geq m(\psi(Q^*Q) - |\psi(Q)|^2) \quad Q \in \mathcal{A}_{\text{loc}}.
\]

**Remark 1.5** (i) Any translationally invariant pure state \( \psi \) is \( \Theta \) invariant. This is a folklore among specialists, and we present a brief sketch of proof. Suppose \( \psi \) is pure, translationally invariant but not \( \Theta \) invariant. There exists \( Q \in \mathcal{A}(-) \) and an increasing sequence of even numbers, \( j_k \in 2\mathbb{N} \) such that \( j_1 < j_2 < j_k < j_{k+1} \ldots \) such that

\[
\psi(Q) \neq 0, \quad \lim_{k \to \infty} \pi_\psi(\tau_{j_k}(Q)) = W \neq 0.
\]

As a consequence, we have

\[
W \pi_\psi(Q) = \pi_\psi(\Theta(Q))W, \quad W^* \pi_\psi(Q) = \pi_\psi(\Theta(Q))W^*, \quad W^2 = c_1,
\]
for some $c \in \mathbb{C}$. By multiplying a suitable constant, we may assume that $W$ is a self-adjoint unitary $W = W^*, W^2 = 1$ implementing $\Theta$. If we denote the normal extension of $\tau_1$ to $\pi_\psi(\mathfrak{A})$ by the same symbol, we have $\tau_1(W) = \pm W$. This is because $\tau_1(W)$ implements $\Theta$ as well. Set $\xi_\pm = \frac{1}{2}(1 \pm W)\Omega_\psi$. The positive normal functional $\psi_\pm(Q) = (\xi_\pm, \pi_\psi(Q)\xi_\pm)$ are $\Theta$ and $\tau_2$ invariant because

$$\psi_\pm(\tau_2(Q)) = \frac{1}{4}((1 \pm \tau_{-2}(W))\Omega_\psi, \pi_\psi(Q)(1 \pm \tau_{-2}(W))\Omega_\psi)$$

$$\psi_\pm(\Theta(Q)) = \frac{1}{4}(W(1 \pm W)\Omega_\psi, \pi_\psi(Q)W(1 \pm W)\Omega_\psi).$$

By definition, $\psi_\pm(W) = \pm 1$, however, due to $\Theta$ and $\tau_2$ invariance of $\psi_\pm$

$$\psi_\pm(\tau_{jk}(Q)) = 0$$

because $Q \in \mathfrak{A}^$. $W = \lim_{k \to \infty} \pi_\psi(\tau_{jk}(Q))$ implies a contradiction.

**Remark 1.6** If a translationally invariant state $\psi$ satisfies \[1.20\], $\psi$ is $\alpha_t^h$ invariant. On the GNS Hilbert space $\mathcal{H}_\psi$ with the GNS cyclic vector $\Omega_\psi$, there exists a positive self-adjoint operator $H_\psi$ on $\mathcal{H}_\psi$ satisfying

$$e^{itH_\psi}\pi_\psi(Q)\Omega_\psi = \pi_\psi(\alpha_t(Q))\Omega_\psi.$$ $H_\psi$ is regarded as the regularized Hamiltonian.

By $\text{spec } H_\psi$ we denote the spectrum of $H_\psi$. The condition \[1.21\] is equivalent to the spectrum gap condition, namely, the infimum of the spectrum of $H_\psi$ is non-degenerate and $\text{spec } H_\psi \subset \{0\} \cup [m, \infty)$.

**Definition 1.7** Let $\psi$ be a translationally invariant pure state. $\psi$ has string order if there exist $Q_1$ in $\mathfrak{A}^(-) \cap \mathfrak{A}_{[n, -1]}$ and $Q_2$ in $\mathfrak{A}_{[m]}^(-) \cap \mathfrak{A}_{\mathbb{Z}}[0,m]$ with $n < 0 \leq m$ satisfying

$$\lim_{k \to \infty} \psi(Q_1 S[0, 2k - 1] Q_2) \neq 0$$

where

$$S[0, k - 1] = \prod_{j=0}^{k-1} (2c_j^* c_j - 1).$$

By the standard Fock state $\psi_F$ we mean a state of $\mathfrak{A}(K)$ uniquely determined by the equation $\psi_F(c_j^* c_j) = 0$ for any $j \in \mathbb{Z}$.

**Theorem 1.8** Let $\psi$ be a translationally invariant pure gapped ground state for $H$. If $\psi$ has string order, $\psi$ cannot be connected to the standard Fock state $\psi_F$ by any one-parameter group of automorphisms $\alpha_t^{h'}$,

$$\psi \circ \alpha_t^{h'} \neq \psi_F$$

for any $t$ and any $h' \in \mathfrak{A}_{\text{loc}}$. 

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Remark 1.9 Historically, in [9], M.den Nijs, and K.Rommelse introduced the string order of gapped ground states for quantum spin chains to characterize the Haldane phase.

In their pioneer work [12] of the Haldane phase, T. Kennedy and H. Tasaki discovered a relation between the string order of M.den Nijs, and K.Rommelse and hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking. We recommend a monograph [20] by H.Tasaki for detail and further extension of research.

Turning to string order of Fermion systems discussed here, among various physics literature, and in the present context, one relevant is a paper [4] of Y. Bahri and A. Vishwanath. Y. Bahri and A. Vishwanath examined Fermion systems associated with the XY model. According to Y. Bahri and A. Vishwanath, string order plays a role of detecting majorana fermions. The string order we consider here is same as that of Y. Bahri and A. Vishwanath and it characterizes the hidden $\mathbb{Z}_2$ symmetry breaking where the hidden system is a Pauli spin chain obtained by Jordan Wigner transformation. Our results are valid for general short range periodic Hamiltonians with gapped ground states, in particular, quasi one-dimensional tight binding models are within reach of our results.

Remark 1.10 Theorem 1.8 is valid for periodic gapped ground states. Theorem 1.8 is valid not only for finite range Hamiltonians but for long range Hamiltonians as long as the interaction across $(-\infty, -1]$ and $[0\infty)$ is bounded.

We expect the stability of the Fermion string order can be proven for states change under one-parameter family of automorphisms considered in [15], however, in this paper we concentrate on general mathematical parts.

In Theorem 1.8 we consider translationally invariant states. We believe a similar result is valid for non-translationally invariant setting. In that case, the bottom of spectrum can be infinitely degenerate, and some additional work is needed.

After completing our first manuscript, we noticed a work [5] of C.Bourne and H. Schulz-Baldes. There are overlaps both in methods and in contents between C.Bourne and H. Schulz-Baldes ours. We mention here difference between the two.

(i) Theorem 1.3 was remarked without proof in our previous paper [14]. Consider the translationally invariant Hamiltonian for $\mathfrak{A}$ defined by

\[ H^F = -\sum_{k=-\infty}^{\infty} (c_k^* - c_k)(c_{k+1}^* + c_{k+1}). \]  

(1.24)

After the Jordan Wigner transformation, the corresponding Hamiltonian $H^P$ for our Pauli spin algebra $\mathfrak{A}^P$ is

\[ H^P = -\sum_{k=-\infty}^{\infty} \sigma_x^{(k)} \sigma_x^{(k+1)}. \]  

(1.25)
A unique translationally invariant, $\Theta$ invariant ground state $\psi^P$ of $H^P$ is an average of two product states $\varphi$ and $\varphi \circ \Theta$:

$$\psi^P = \frac{\varphi + \varphi \circ \Theta}{2}, \quad \varphi(\sigma_x^{(k)}) = 1, \quad k \in \mathbb{Z}. \quad (1.26)$$

(1.26) shows that restriction of $\psi$ for $H^F$ to $A^+_L \otimes A^+_R$ is not a factor state, not a product state. We have to take into account this possibility in our proof of Theorem 1.3.

(ii) We proved that any translationally invariant pure gapped ground states satisfy split property and the $\mathbb{Z}_2$ index is well-defined for those states. To prove our claim we employ results of [2] and [3].

(iii) The definition of $\mathbb{Z}_2$ index introduced in [5] by C.Bourne and H. Schulz-Baldes and that by ourself are different but equivalent. We claim that the $\mathbb{Z}_2$ index is determined by presence or absence of string order.

We believe both [5] of C.Bourne and H. Schulz-Baldes and ours are complementary and both papers shed a new scope of string order of Fermion chains and the $\mathbb{Z}_2$ index.

We will prove results within the framework of the theory of operator algebras. Basic references of this approach are [7] and [8]. For more physics oriented readers, the monograph [20] of H.Tasaki is a very good introduction to various mathematical aspects of quantum many body problems.

## 2 Proof of Theorem 1.3

For later use, we collect results of $\Theta$ invariance of pure states. Let $\mathfrak{B}$ be a unital $C^*$-algebra and $\Theta$ be an involutive automorphism of $\mathfrak{B}$, $\Theta^2(Q) = Q$ for $Q \in \mathfrak{B}$. Set

$$\mathfrak{B}^{(\pm)} = \{ Q \in \mathfrak{B} \mid \Theta(Q) = \pm Q \}.$$ 

We assume that $\mathfrak{B}^{(-)}$ is generated by a single self-adjoint unitary $Z$ in $\mathfrak{B}^{(-)}$ and $\mathfrak{B}^{(+)}$,

$$\mathfrak{B}^{(-)} = Z \mathfrak{B}^{(+)}$$

Let $\psi$ be a $\Theta$ invariant state of $\mathfrak{B}$. We denote the GNS triple of $\mathfrak{B}$ associated with $\psi$ by $\{ \pi_\psi(\mathfrak{B}), \Omega_\psi, \mathcal{H}_\psi \}$ where $\pi_\psi(\mathfrak{B})$ is the representation of $\mathfrak{B}$ on the GNS Hilbert space $\mathcal{H}_\psi$ and $\Omega_\psi \in H_\psi$ is the GNS cyclic vector. As $\psi$ is $\Theta$ invariant, there exists a unique self-adjoint unitary operator $\Gamma(\Theta)$ on $\mathcal{H}_\psi$ satisfying

$$\Gamma(\Theta) \pi_\psi(Q) \Omega_\psi = \pi_\psi(\Theta(Q)) \Omega_\psi \quad (Q \in \mathfrak{B}), \quad \Gamma(\Theta)^* = \Gamma(\Theta), \quad \Gamma(\Theta)^2 = 1.$$ 

We set $\mathcal{H}_\psi^{\pm} = \pi_\psi(\mathfrak{B}^{(\pm)}) \Omega_\psi$ and the representation of $\mathfrak{B}^{(+)}$ restricted to $\mathcal{H}_\psi^{\pm}$ is denoted by $\pi^{(\pm)}_\psi(\cdot)$. 

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Lemma 2.1 (Lemma 4.1 of [3]) Let $\mathcal{B}$, $\Theta$ be as above and let $\psi$ be a $\Theta$-invariant state of $\mathcal{B}$.

(i) Suppose that $\psi$ is a pure state of $\mathcal{B}$. Then, the restriction of $\psi$ to $\mathcal{B}^{(+)}$ is a pure state of $\mathcal{B}^{(+)}$.

(ii) Suppose that the GNS representations $\pi_{\psi}^{(+)}(\cdot)$ of $\mathcal{B}^{(+)}$ is irreducible. Then, $\psi$ is a pure state of $\mathcal{B}$ if and only if $\pi_{\psi}^{(\pm)}(\cdot)$ of $\mathcal{B}^{(+)}$ are mutually disjoint (= not unitarily equivalent).

(iii) Suppose that the GNS representations $\pi_{\psi}^{(+)}(\cdot)$ of $\mathcal{B}^{(+)}$ is irreducible and that $\psi$ is not a pure state of $\mathcal{B}$. Then, there exists a pure state $\varphi$ of $\mathcal{B}$ satisfying $\psi = \frac{\varphi + \varphi \circ \Theta}{2}$. $\varphi$ and $\varphi \circ \Theta$ are mutually disjoint.

Lemma 2.2 Let $\mathcal{B}$ and $\Theta$ be as above. Suppose that a state $\varphi$ of $\mathcal{B}$ is pure and that $\varphi$ and $\varphi \circ \Theta$ are disjoint.

Then, $\varphi$ restricted to $\mathcal{B}^{(+)}$ is pure, and $\pi_{\varphi}(\mathcal{B})'' = \pi_{\varphi}(\mathcal{B}^{(-)})''$.

Proof. Set

$$\psi = \frac{1}{2}(\varphi + \varphi \circ \Theta).$$

We employ the GNS triple $\{\pi_{\psi}(\mathcal{B}), \Omega_{\psi}, \mathcal{H}_{\psi}\}$ associated with $\psi$ is realized in terms of representations associated with $\varphi$.

$$\mathcal{H}_{\psi} = \mathcal{H}_{\varphi} \oplus \mathcal{H}_{\varphi}, \quad \Omega_{\psi} = \frac{1}{\sqrt{2}}(\Omega_{\varphi} \oplus \Omega_{\varphi}), \quad \pi_{\psi}(Q) = \pi_{\varphi}(Q) \oplus \pi_{\varphi}(\Theta(Q))$$

We now denote $h = (f, g) \in \mathcal{H}_{\psi}$ by a column vector and any bounded operator on $\mathcal{H}_{\psi}$ by a 2 by 2 matrix as follows.

$$h = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}_{\psi}, \quad \pi_{\psi}(Q) = \begin{pmatrix} \pi_{\varphi}(Q) & 0 \\ 0 & \pi_{\varphi}(\Theta(Q)) \end{pmatrix}, \quad Q \in \mathcal{B}.$$ 

Let $V(\Theta)$ and $X$ be self-adjoint unitaries on $\mathcal{H}_{\psi}$ defined via the following identities:

$$V(\Theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}_{\psi}), \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.1)$$

Due to disjointness of $\varphi$ and $\varphi \circ \Theta$, $X$ is an element of $\pi_{\varphi}(\mathcal{B})''$ and by a direct calculation, we see

$$V(\Theta)XV^{-1}(\Theta) = -X, \quad V(\Theta)\pi_{\varphi}(Q)V^{-1}(\Theta) = \pi_{\varphi}(\Theta(Q)), \quad Q \in \mathcal{B}. \quad (2.2)$$

The first identity in (2.2) suggests that $X$ is in the weak closure of $\pi_{\varphi}(\mathcal{B}^{(-)})$. The center of the von Neumann algebra $\pi_{\varphi}(\mathcal{B})''$ is two dimensional, generated by $X$ and

$$\varphi(Q) = \begin{pmatrix} \Omega_{\psi}, \pi_{\varphi}(Q) & (1 + X) \Omega_{\psi} \\ 0 & \pi_{\varphi}(Q) \end{pmatrix}, \quad \varphi \circ \Theta(Q) = \begin{pmatrix} \Omega_{\psi}, \pi_{\varphi}(Q) & (1 - X) \Omega_{\psi} \\ 0 & \pi_{\varphi}(Q) \end{pmatrix}, \quad Q \in \mathcal{B}.$$
We claim that
\[ \pi_\psi(\mathcal{B}^{(+)}\Omega_\varphi) = \mathcal{H}_\varphi = \pi_\psi(\mathcal{B})\Omega_\varphi. \]  
(2.3)

As the self-adjoint unitary \( X \) is in in the weak closure of \( \pi_\psi(\mathcal{B}^{(-)}) \), any element \( R \) in the weak closure of \( \pi_\psi(\mathcal{B}^{(-)}) \) is a product
\[ R = R^+ X, \quad R^+ = RX \in \pi_\psi(\mathcal{B}^{(+)})''. \]  
(2.4)

We obtain
\[ X \begin{pmatrix} \Omega_\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \Omega_\varphi \\ 0 \end{pmatrix}, \quad R \begin{pmatrix} \Omega_\varphi \\ 0 \end{pmatrix} = R^+ \begin{pmatrix} \Omega_\varphi \\ 0 \end{pmatrix} \]
which implies (2.3).

Next we show that \( \varphi \) restricted to \( \mathcal{B}^{(+)} \) is pure. If this is not the case, there exists a non-trivial projection \( E \) in the commutant \( \pi_\psi(\mathcal{B}^{(+)}') \). As \( \Omega_\varphi \) is a cyclic vector for \( \pi_\psi(\mathcal{B}^{(+)})'' \), \( E\Omega_\varphi \neq 0 \) in \( \mathcal{H}_\varphi \). Set
\[ \tilde{E} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \in \mathcal{B}(\mathcal{H}_\varphi). \]

Then, \( \tilde{E} \) commutes with \( X \), and with any element of \( \pi_\psi(\mathcal{B}^{(-)}) \) due to (2.4). However, this contradicts with irreducibility of \( \pi_\psi(\mathcal{B}) \) on \( \mathcal{H}_\varphi \). It turns out that \( \varphi \) restricted to \( \mathcal{B}^{(+)} \) is pure. As \( \pi_\psi(\mathcal{B}^{(+)})'' \) is trivial,
\[ \pi_\psi(\mathcal{B}^{(+)})'' = \pi_\psi(\mathcal{B})'' = \mathcal{B}(\mathcal{H}_\varphi). \]

End of Proof.

We will borrow the following lemma in [19] due to S.Strâtil˘a and D.Voiculescu as well. (c.f. 2.7 of [19])

**Lemma 2.3** Let \( \mathcal{B} \) and \( \Theta \) be as in the previous lemma \( \ref{lem:b**} \) and \( \psi_1 \) and \( \psi_2 \) be \( \Theta \) invariant states of \( \mathcal{B} \). If \( \psi_1 \) and \( \psi_2 \) restricted to \( \mathcal{B}^{(+)} \) are quasi-equivalent, so are \( \psi_1 \) and \( \psi_2 \).

**Lemma 2.4** (i) Suppose that \( \varphi \) is a pure state of \( \mathcal{A}^S \). Restriction of \( \varphi \) to \( \mathcal{A}_L^S \) and that to \( \mathcal{A}_R^S \) are factor states.

(ii-a) If \( \psi \) is a \( \Theta \) invariant factor state of \( \mathcal{A} \).

(ii-b) If \( \psi \) is not quasi-equivalent to \( \psi \circ \Theta_- \), restriction of \( \psi \) to \( \mathcal{A}_L \) and that to \( \mathcal{A}_R \) are factor states.

(ii-c) If \( \psi \) is pure , equivalent to \( \psi \circ \Theta_- \), and the restriction of \( \psi \) to \( \mathcal{A}^{(+)} \) and that of \( \psi \circ \Theta_- \) to \( \mathcal{A}^{(+)} \) are equivalent, then, the restriction of \( \psi \) to \( \mathcal{A}_L \) and \( \psi \) restricted to \( \mathcal{A}_R \) are factor states.

(ii-d) If \( \psi \) is equivalent to \( \psi \circ \Theta_- \), and \( \psi \) restricted to \( \mathcal{A}^{(+)} \) and \( \psi \circ \Theta_- \) restricted to \( \mathcal{A}^{(+)} \) are not equivalent, then, \( \psi \) restricted to \( \mathcal{A}_L \) and to \( \mathcal{A}_R \) are not factor states. \( \psi \) restricted to \( \mathcal{A}_L \) is an average of two mutually non quasi-equivalent factor states.
Examples of Lemma 2.4 (ii) are provided by Fock states associated with ground states of one-dimensional XY model. (c.f. [3])

**Proof of Lemma 2.4**

The claim of Lemma 2.4 (i) follows from identification of the center of $\pi_\phi(A_S''')$ and the algebra at infinity. (c.f. Theorem 2.6.10 of [7].) The claim of Lemma 2.4 (ii) is not mentioned in this form in [7] and we present our proof here.

First we recall proof of Lemma 2.4 (i) briefly. As $C^*$-algebras, $A_L$ and $A_R$ are an infinite tensor product of $M_2(C)$. We assign a non-negative integer to specify each tensor product component of $A_R$ and a negative integer to specify each tensor product component of $A_L$. $A_R$ is generated by Pauli spin matrix $\sigma^{(j)}_\alpha$ where $j \in \mathbb{Z}, 0 \leq j$, $\alpha = x, y, z$ and $A_L$ is generated by Pauli spin matrix $\sigma^{(j)}_\alpha$ where $j \in \mathbb{Z}, j < 0$, $\alpha = x, y, z$.

Let $\Lambda$ be a subset of $\mathbb{Z}$ and $A_{\Lambda}^{S}$ be the $C^*$-subalgebra of $A^{S}$ generated by $\sigma^{(j)}_\alpha$ where $j \in \Lambda, \alpha = x, y, z$. $A^{S}(\Lambda)$ is the set of observables localized in $\Lambda$. We set

$$A_{\infty}^{S} = \bigcup_{\Lambda \subset \mathbb{Z}, |\Lambda| < \infty} A^{S}(\Lambda).$$

If $\Lambda$ is a finite set, and let $tr_{\Lambda}$ be the partial trace $tr_{\Lambda}$ satisfying

$$tr_{\Lambda}(QR) = Qtr(R) \in C \cdot 1 \quad (Q \in A^{S}(\Lambda^c), \ R \in A^{S}(\Lambda)) \quad (2.5)$$

$tr_{\Lambda}$ is a projection from $A^{S}$ to $A^{S}(\Lambda^c)$ constructed in the following manner:

$$tr_{\Lambda}(Q) = c \sum_{k,l} e_{kl}Qe_{lk} \quad (2.6)$$

where $\{e_{kl} \in A^{S}(\Lambda)\}$ is a matrix unit system of $A^{S}(\Lambda)$, i.e. the linear hull of $\{e_{kl}\}$ is $A^{S}(\Lambda)$ and $e_{kl}$ satisfy

$$e_{kl}e_{ab} = \delta_{la}e_{kb}, \quad e_{kl}^* = e_{lk}, \quad \sum_{k} e_{kk} = 1,$$

and $c$ is the normalization constant determined by $tr_{\Lambda}(1) = 1$.

Due to realization (2.6), $tr_{\Lambda}$ can be extended to a normal projection from $\pi(A^{S})''$ to $A^{S}(\Lambda^c)''$ for any representation $\pi(\cdot)$ of $A^{S}$.

We turn to our proof of Lemma 2.4 (i). Let $\{\pi_{\varphi^{s}}(A^{S}), \Omega_{\varphi^{s}}, \mathcal{H}_{\varphi^{s}}\}$ be the GNS triple associated with $A^{S}(\Lambda)$.

$$\varphi(Q^*R^*RQ) = \varphi(R^*Q^*RQ) \leq ||Q||^2\varphi(R^*R) \quad (2.7)$$

for $Q \in A^{S}_L$, $R \in A^{S}_R$, and the representation of $\pi_{\varphi^{s}}(A^{S}_R)$ on $\mathcal{H}_{\varphi^{s}}$ and that on the closure of $\pi_{\varphi^{s}}(A^{S}_R)^{''}$ are quasi-equivalent and we can identify the center of $\pi_{\varphi^{s}}(A^{S}_R)^{''}$ on both spaces.

We introduce the algebra at infinity via the following equation:

$$\mathfrak{M}_{\infty} = \cap_{n=1,2,3,\ldots} \pi_{\varphi^{s}}(A^{S}([-n,n])^{''}),$$

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and we claim that the center of $\pi_\varphi^\prime(\mathfrak{A}^S)''$ coincides with $\mathfrak{M}_\infty$. By definition, inclusion $\mathfrak{M}_\infty \subset \pi_\varphi^\prime(\mathfrak{A}^S)'' \cap \pi_\varphi^\prime(\mathfrak{A}^S)'$. If $C$ is an element of the center of $\pi_\varphi^\prime(\mathfrak{A}^S)''$, there exists $C_\alpha \in \mathfrak{M}_\infty$ such that $C = w - \lim_{\alpha \to \infty} C_\alpha$. Then, for each $n > 0$,

$$w - \lim_{\alpha \to \infty} tr_{[-n,n]}(C_\alpha) = tr_{[-n,n]}(C) = C$$

which tells us $\mathfrak{M}_\infty \subset \pi_\varphi^\prime(\mathfrak{A}^S)'' \cap \pi_\varphi^\prime(\mathfrak{A}^S)'$. In the same line of reasoning, the algebra at infinity of $\pi_\varphi^\prime(\mathfrak{A}^R)''$ is the center of $\pi_\varphi^\prime(\mathfrak{A}^R)''$. As any element in $\pi_\varphi^\prime(\mathfrak{A}^R)''$ satisfies the condition of elements of the algebra at infinity of $\pi_\varphi^\prime(\mathfrak{A}^S)''$ and the algebra at infinity of $\pi_\varphi^\prime(\mathfrak{A}^R)''$ is contained in that of $\pi_\varphi^\prime(\mathfrak{A}^S)''$. If $\psi$ is a factor state of $\mathfrak{A}^S(\Lambda)$, $\pi_\varphi^\prime(\mathfrak{A}^R)''$ is factor.

Next, we consider the fermionic case (ii) of Lemma 2.4. We assume that $\psi$ is $\Theta$ invariant, $\Theta$ can be extended to an automorphism of $\pi_\varphi^\prime(\mathfrak{A}^L)$ and any central element $C$ of $\pi_\varphi^\prime(\mathfrak{A}^R)''$ is a sum of even and odd elements $C_{\pm}$, $\Theta(C_{\pm}) = \pm C_{\pm}$, $C = C_{\pm} + C_{\mp}$.

If we identify the one-particle space $K$ with $l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$, $K_L$ with $l^2(\mathbb{Z}_{<0}) \oplus l^2(\mathbb{Z}_{<0})$ and $K_R$ with $l^2(\mathbb{N} \cup \{0\}) \oplus l^2(\mathbb{N} \cup \{0\})$ where $\mathbb{Z}_{<0} = \{k \in \mathbb{Z} \mid k < 0\}$, $\mathfrak{A}^S_{\Lambda/R}$ is isomorphic to $\mathfrak{A}^L_{\Lambda/R}$. However, as odd elements of $\mathfrak{A}^S_{\Lambda}$ and $\mathfrak{A}^R_{\Lambda}$ are anticommuting, we can only apply our argument for (i) of Lemma 2.4 for the even part $C_{\pm}$. (c.f. Notes and Remark of Section 2.6.1 of [11] If $C \in \pi_\varphi^\prime(\mathfrak{A}^R)'' \cap \pi_\varphi^\prime(\mathfrak{A}^R)'$, and $\Theta(C) = C$, triviality $C = d1$ holds for some $d \in C$.

If $C = -\Theta(C) \neq 0$ is self-adjoint, belonging to the center of $\pi_\varphi^\prime(\mathfrak{A}^R)''$, then, $\Theta(C^2) = d1$ with $d > 0$, and $X = d^{-1/2}C$ is a self-adjoint unitary. $X$ anti-commutes with odd elements of $\pi_\varphi^\prime(\mathfrak{A}^L)'$. Thus, $X \in \pi_\varphi^\prime(\mathfrak{A}^R)''$ is a self-adjoint unitary satisfying

$$X \pi_\varphi^\prime(Q) X^{-1} = \pi_\varphi^\prime(\Theta_{-}(Q)) \quad Q \in \mathfrak{A}. \quad (2.8)$$

In another word, $X$ implements $\Theta_{-}$. We arrive at our claim of (ii) of Lemma 2.4. For any factor state $\psi$, $\psi$ and $\psi \circ \Theta_{-}$ are quasi-equivalent or disjoint. If $\psi$ and $\psi \circ \Theta_{-}$ are disjoint, $X$ satisfying (2.8) cannot exist and $\pi_\varphi^\prime(\mathfrak{A}^R)''$ is a factor. This implies (ii-a) of Lemma 2.4.

If $\psi$ is pure, $\psi$ restricted to $\mathfrak{A}^{(+)}$ and $\psi \circ \Theta_{-}$ restricted to $\mathfrak{A}^{(+)}$ are either unitarily equivalent or disjoint. If they are unitarily equivalent, $\pi_\varphi^\prime(\mathfrak{A}^R)''$ is a factor because if $X$ satisfying (2.8) exists, $\psi$ cannot be pure due to Lemma 2.4. This shows (ii-b) of Lemma 2.4.

If $\psi$ is pure, and if $\psi$ and $\psi \circ \Theta_{-}$ are unitarily equivalent, and $\psi$ restricted to $\mathfrak{A}^{(+)}$ and $\psi \circ \Theta_{-}$ restricted to $\mathfrak{A}^{(+)}$ are disjoint, there exists a self-adjoint unitary $X$ satisfying satisfying (2.8) again due to Lemma 2.4. Let $\psi_R$ be the normal extension of $\psi$ to $\pi_\varphi^\prime(\mathfrak{A}^R)''$ and Set

$$\psi_1(Q) = \psi_R((1 + X)Q), \quad \psi_2(Q) = \psi_R((1 - X)Q) \quad Q \in \pi_\varphi^\prime(\mathfrak{A}^R)''.$$
In Lemma 2.5 we consider equivalence of projections in a von Neumann algebra. All the knowledge we need here is explained in Chapter 6 of [11]. Let $M$ be a von Neumann algebra and $p, q$ be projections in $M$. $p, q$ are equivalent if and only if there exists a partial isometry $u \in M$ satisfying $uu^* = p$ and $u^*u = q$. If $M$ is a factor, any two projections are comparable, i.e. either (a) $p$ is equivalent to a sub-projection of $q$ or (b) $q$ is equivalent to a sub-projection of $p$. If $M$ is a type III factor, any two projections are equivalent. If $M$ is a type II factor with a normal semi-finite trace $\Theta$, two projections $p, q$ are equivalent if and only if $tr(p) = tr(q)$.

**Lemma 2.5** Let $M$ be a von Neumann algebra with a finite center acting on a Hilbert space $\mathcal{H}$ and $\Gamma$ be a self-adjoint unitary on $\mathcal{H}$ giving rise to a $Z_2$ action $\Theta$ on $M$, and set $\mathcal{N} = \{Q \in M \mid \Gamma Q = Q\Gamma\}$. If $M$ is of type I with a finite center, $M$ is a type I von Neumann algebra.

**Proof of Lemma 2.5**

Due to our assumption of Lemma 2.5, $M$ cannot be a finite factor, we proceed to our proof assuming that $M$ is either of type $II_1$ or of type $III$.

Suppose that $M$ is of type $II_1$ and let $tr(\cdot)$ be a $\Theta$ invariant semi-finite normal trace of $M$. Let $P$ be a minimal projection in a factor component of $M$. For any small positive number $c$ satisfying $0 < c < 1/2d = 1/2tr(P)$ there exists a projection $Q \in M$ such that $Q \leq P, c = tr(Q)$. Set $Q_{\pm} = 1/2(Q \pm \Theta(Q))$. As $tr(\cdot)$ is $\Theta$ invariant, $c = tr(Q_+)$.

Suppose $d = \infty$. As $Q_+ \leq P, Q_+ \in M$ and $P$ is minimal, $Q_+ = aP$ for some $a > 0$ which leads to a contradiction, $c = tr(Q) = atr(P) = \infty$. Suppose $d$ is finite. As $Q$ is a projection, $Q^2 = Q_+^2 + (Q_ Q_+ + Q_ Q_+) + Q_2 = Q \leq P$.

Applying $\Theta$, we obtain

$$0 \leq Q_+ = Q_+^2 + Q_2 \leq P, \quad Q_- = Q_+ Q_- + Q_- Q_+.$$  \hfill (2.9)

As $Q_+$ and $Q_2$ are positive elements in $\mathcal{N}$ and $P$ is a minimal projection, we obtain $Q_+ = cP, Q_2 = (c - c^2)P$. Then,

$$PQ_- P = PQ_+ Q_- P + PQ_- Q_+ P = 2cPQ_- P$$

and as a consequence, $PQ_+ P = 0$ and Due to the second identity of (2.9),

$$PQ_- = PQ_+ Q_- + PQ_- Q_+ = cPQ_- + cPQ_+ = cPQ_-.$$

This shows that $PQ_- = 0$ and that $Q_- = Q_+ Q_- + Q_- Q_+ = cPQ_- + cQ_- P = 0$. It turns out $Q = Q_+ + Q_- = Q_+ \in \mathcal{N}$, and $P = Q$ which leads to a contradiction.

Next we assume that $M$ is a type $III$ von Neumann algebra. We can take a $\Theta$ invariant central projection $P \neq 0$ of $M$ and a projection $Q$ of a rank, at most 2, in in one factor component of $M$. (If $P_1$ is a minimal central projection of $M$ such that $\Theta(P_1)P_1 = 0$ we set $P = P_1 + \Theta(P_1)$ and we take a minimal projection.
Q_1 of \mathfrak{M} equivalent to P_1 and a minimal projection Q_2 of \mathfrak{M} equivalent to P_2 orthogonal to Q_1. Then, \rho is equivalent to Q = Q_1 + Q_2.) We now consider \mathfrak{M}P and by abuse of notation P = 1, \mathfrak{M} = \mathfrak{M}P.

Let U be a partial isometry satisfying Q = UU^*, 1 = U^*U. We can write \( U = U_+ + U_-X \) where \( X = X^* = -\Theta(X), X^2 = 1, U_\pm \in \mathfrak{M}. \) As \( Q, 1 \in \mathfrak{M}, \)

\[ Q = U_+ U_+^* + U_- U_-^* \quad 1 = U_+^* U_+ + X U_-^* U_- X \]

\[ U_+^* U_- X + X U_-^* U_+ = 0, \quad U_- X U_+^* + U_+ X U_-^* = 0, \]

As \( U_\pm U_\pm^* \leq Q, U_\pm = Q U_\pm \) which means the rank( the dimension of the range) of \( U_\pm \) is at most two. As a consequence, the rank of \( U_+^* U_+ \) and that of \( X U_-^* U_- X \) are at most two and \( U_2^* U_+ + X U_-^* U_- \) is of finite rank, which is a contradiction.

**Proof of Theorem 1.6**

Let \( \psi \) be a \( \Theta \) invariant pure state of \( \mathfrak{A} \) and we focus on the restriction of \( \psi \) to \( \mathfrak{A}_L. \) We fix \( h \in \mathcal{K}_L \) such that \( Jh = h \) and \( ||h|| = 1. \) Then, \( \mathcal{B}(h) \) is a self-adjoint unitary in \( \mathfrak{A}_L. \) Let \( \{ \pi_\psi(\mathfrak{A}), \mathfrak{H}_\psi, \Omega_\psi \} \) be the GNS triple associated with \( \psi. \) Set \( \mathfrak{H}_\psi^{(\pm)} = \pi(\mathfrak{A}^{(\pm)})\Omega_\psi \) and by \( \pi_{\pm}(\mathfrak{A}^{(\pm)}) \) we denote the representation \( \pi_\psi(\mathfrak{A}^{(\pm)}) \) of \( \mathfrak{A}^{(\pm)} \) restricted to \( \mathfrak{H}_\psi \). As we mentioned before, \( \pi_\psi(\mathfrak{A}^{(\pm)}) \) are disjoint. By definition \( \pi_\psi(\mathfrak{A}^{(\pm)}) \) of \( \mathfrak{H}_\psi \) is a direct sum of \( \pi_\psi(\mathfrak{A}^{(\pm)}) \).

As \( \mathfrak{H}_\psi^{(\pm)} = \pi(\mathfrak{A}^{(\pm)} B(h))\Omega_\psi, \pi_\psi(\mathfrak{A}^{(\pm)}) \) of \( \mathfrak{A}^{(\pm)} \) on \( \mathfrak{H}_\psi \) is unitarily equivalent to the GNS representation associated with \( 1/2\{ \psi + \psi \circ \mathcal{A}(B(h)) \}. \)

Suppose that \( \psi \) is quasi-equivalent to \( \psi_L \otimes \psi_R \) and that both \( \psi_R \) and \( \psi_L \) are \( \Theta \) invariant. As to the restriction to \( \mathfrak{A}^{(\pm)} \) the we have three possible case.

(Case-1) \( \psi \) restricted to \( \mathfrak{A}^{(\pm)} \) is quasi-equivalent to \( \psi_L \otimes \psi_R \) restricted to \( \mathfrak{A}^{(\pm)} \).
(Case-2) \( \psi \circ \mathcal{A}(B(h)) \) restricted to \( \mathfrak{A}^{(\pm)} \) is quasi-equivalent to \( \psi_L \otimes \psi_R \) restricted to \( \mathfrak{A}^{(\pm)} \).
(Case-3) \( 1/2\{ \psi + \psi \circ \mathcal{A}(B(h)) \} \) restricted to \( \mathfrak{A}^{(\pm)} \) is quasi-equivalent to \( \psi_L \otimes \psi_R \) restricted to \( \mathfrak{A}^{(\pm)} \).

Note that in all cases, the state \( \psi, \psi \circ \mathcal{A}(B(h)) \) or \( 1/2\{ \psi + \psi \circ \mathcal{A}(B(h)) \} \) restricted to \( \mathfrak{A}^{(\pm)} \) extended to a \( \Theta \) invariant state of \( \mathfrak{A}^S \) is of type I.

Let \( \varphi \) be the \( \Theta \) invariant extension of \( \mathfrak{A}^{(\pm)} \) to \( \mathfrak{A}^S \) and let \( \varphi_L \) be the \( \Theta \) invariant extension of \( \psi_L \) restricted to \( \mathfrak{A}^{(\pm)}_L \) to \( \mathfrak{A}^S_L \) and \( \varphi_R \) be the \( \Theta \) invariant extension of \( \psi_R \) restricted to \( \mathfrak{A}^{(\pm)}_R \) to \( \mathfrak{A}^S_R. \)

If \( Q \in \mathfrak{A}_L R \in \mathfrak{A}_R \) satisfying \( QR \in \mathfrak{A}^{(\pm)} \) we have \( QR = Q_+ R_+ + Q_- R_- \) and

\[ \psi_L \otimes \psi_R(QR) = \psi_L(Q_+ \psi_R R_+) = \varphi_L(Q_+) \varphi_R(R_+) \]
\[ = \varphi_L \otimes \varphi_R((Q_+ + Q_- T)(R_+ + R_- T)). \]  \hspace{1cm} (2.10)

(2.10) shows that if we extend \( \psi_L \otimes \psi_R \) restricted to \( \mathfrak{A}^{(\pm)} \) to \( \Theta \) invariant state of \( \mathfrak{A}^S \), we obtain a product state \( \varphi_L \otimes \varphi_R. \) If (Case-1) is valid, \( \varphi \) is quasi-equivalent to \( \varphi_L \otimes \varphi_R \) due to Lemma 2.3 and \( \varphi \) restricted to \( \mathfrak{A}^S_L \) which shows \( \psi \) restricted to \( \mathfrak{A}_L \) is of type I. The same argument is valid for (Case-2) and (Case-3).
3 Fermionic String Order

First we begin with Fock states of our CAR algebra $\mathfrak{A}$. A Bogoliubov automorphism $\beta_u$ is an automorphism of $\mathfrak{A}$ defined by the following equation:

$$\beta_u(B(f)) = B(uf), \quad f \in K$$

where $u$ is a unitary on $K$ satisfying $JuJ = u$. On the one-particle $K = K_R \oplus K_L$, we introduce self-adjoint unitaries $\theta$ and $\Theta_-$ satisfying the following equations:

$$\theta(f) = -f, \quad f \in K, \quad \Theta_-(f_L \oplus f_R) = (-f_L \oplus f_R), \quad f_L \in K_L, \quad f_R \in K_R. \quad (3.1)$$

Obviously, $\Theta(Q) = \beta\theta(Q), \Theta_-(Q) = \beta_\Theta_-(Q)$ for $Q \in \mathfrak{A}$.

**Definition 3.1** (i) A projection $E$ acting on $K$ is a basis projection if and only if $JEJ = 1 - E$.

(ii) Let $E$ be a basis projection. Let $\psi_E$ be a pure state of $\mathfrak{A}(K)$ uniquely determined by the following equation:

$$\psi_E(B^*(f)B(g)) = (f, Eg)_K, \quad f, g \in K. \quad (3.2)$$

We call $\psi_E$ a Fock state associated with $E$.

In the above Fock state, $\psi_E(B^*((1 - E)g)B((1 - E)g)) = 0$ implies $B((1 - E)g)$ plays a role of an annihilation operator and $B((1 - E)g)^* = B(J(1 - E)g) = B(EJg)$ plays a role of a creation operator. The GNS Hilbert space is an antisymmetric Fock space with a one-particle space $EK$.

It is known that two Fock states $\psi_{E_1}$ and $\psi_{E_2}$ are unitarily equivalent if and only if $E_1 - E_2$ is a Hilbert-Schmidt operator on $K$. (See [1].) In particular, for a unitary $u$ satisfying $JuJ = u$ and for a basis projection, if $uEu - E$ is a Hilbert-Schmidt operator, there exists a unitary $\Gamma(u)$ on the GNS Hilbert space implementing $u$:

$$\Gamma(u)\pi_{\psi_E}(Q)\Gamma(u)^* = \pi_{\psi_{\beta_u(Q)}}, \quad Q \in \mathfrak{A}.$$ 

If $E_1 - E_2$ is a compact operator, the dimension $\dim E_1 \wedge (1 - E_2)$ of the range of $E_1 \wedge (1 - E_2)$ is finite.

**Theorem 3.2** (i) Let $E$ be a basis projection. The split property is valid for $\psi_E$ if and only if $\theta_+ E_\theta_+ - E$ is a Hilbert-Schmidt operator on $K$.

(ii) If $\psi_E$ has the split property, the restriction of $\psi_E$ to $\mathfrak{A}_L$ is factor if and only if $\dim E \wedge (1 - \theta E_\theta)$ is even.

As all elements for proof of Theorem 3.2 are contained in [1] and [3], we sketch our proof briefly. Suppose that $\psi_E$ is quasi-equivalent to $\psi^1 \otimes_{\mathbb{Z}_2} \psi^2$. As quasi-equivalence classes of representations matters we may replace $\psi^1$ with another state of $\mathfrak{A}_L$ quasi-equivalent to $\psi^1$ in $\psi^1 \otimes \psi^2$ and in view of the fermion version
of (2.7), we may assume that \( \psi^1 \) is restriction of \( \psi_E \) to \( \mathfrak{A}_L \) and that \( \psi^2 \) is restriction of \( \psi_E \) to \( \mathfrak{A}_R \). As a consequence, \( \psi^1 \otimes_{Z_2} \psi^2 \) is a quasifree state defined in Definition 3.1 of [1] where the covariance operator \( S \) in (3.3) of [1] is written as follows:

\[
S = qE + (1 - q)E, \quad q = \frac{1}{2}(1 - \theta). 
\]  

(3.3)

Due to Theorem 1 and Lemma 5.1 of [1], \( \psi^1 \otimes \psi^2 \) is quasi-equivalent to \( \psi_E \) if and only if \( P_E - P_S \) is in the Hilbert Schmidt class where \( P_S \) is defined in (4.4) of [1]. This condition is equivalent to the conditions that \( \theta - E\theta - (1 - E)\theta \) is in the Hilbert Schmidt class and that \( \theta - E\theta \) is in the trace class, however, the former condition implies the latter. (ii) of Theorem 3.2 is same as Theorem 4 of [2].

\((-1)^{dim E_1 \wedge (1 - E_2)}\) is nothing but the classical \( \mathbb{Z}_2 \) index of real Fredholm operators and based on this observation, we introduce the following \( \mathbb{Z}_2 \) index.

**Definition 3.3** Let \( \psi \) be a \( \Theta \) invariant pure state of \( \mathfrak{A} \) with split property.

(i) \( \text{ind}_{Z_2} \psi = 1 \) if the restriction of \( \psi \) to \( \mathfrak{A}_L \) is a factor state.

(ii) \( \text{ind}_{Z_2} \psi = -1 \) if the restriction of \( \psi \) to \( \mathfrak{A}_L \) is not a factor state.

**Remark 3.4** \( \psi \) and \( \psi \circ \Theta_{-} \) are unitarily equivalent if split property holds for any \( \Theta \) invariant pure state \( \psi \) of \( \mathfrak{A} \). We are wondering if the converse holds. Namely, unitary equivalence of \( \psi \) and \( \psi \circ \Theta_{-} \) implies split property or not. We are unable to prove the claim, unable to provide counter any example.

For automorphisms \( \alpha_L \) of \( \mathfrak{A}_L \) and \( \alpha_R \) of \( \mathfrak{A}_R \) commuting with \( \Theta \), we can introduce a \( \mathbb{Z}_2 \) graded product automorphism \( \alpha_L \otimes_{Z_2} \alpha_R \) of \( \mathfrak{A} \) satisfying

\[
\alpha_L \otimes_{Z_2} \alpha_R(Q_L) = \alpha_L(Q_L), \quad \alpha_L \otimes_{Z_2} \alpha_R(Q_R) = \alpha_L(Q_R) 
\]  

(3.4)

for \( Q_L \in \mathfrak{A}_L \), \( Q_R \in \mathfrak{A}_R \) and

\[
(\varphi_L \otimes_{Z_2} \varphi_R) \circ (\alpha_L \otimes_{Z_2} \alpha_R) = (\varphi_L \circ \alpha_L) \otimes_{Z_2} (\varphi_R \circ \alpha_R) 
\]  

(3.5)

for any \( \mathbb{Z}_2 \) graded product state \( \varphi_L \otimes_{Z_2} \varphi_R \).

**Definition 3.5** An automorphism \( \alpha \) of \( \mathfrak{A} \) is an almost product automorphism if there exists a \( \mathbb{Z}_2 \) graded product automorphism \( \alpha_L \otimes_{Z_2} \alpha_R \) and a unitary \( U \) of \( \mathfrak{A} \) such that

\[
\alpha = (\alpha_L \otimes \alpha_R) \circ Ad(U). 
\]  

(3.6)

As an inner perturbation \( Ad(U) \) preserves quasi-equivalence classes of representations, the following proposition is obvious.

**Proposition 3.6** Let \( \psi \) be a \( \Theta \) invariant pure state of \( \mathfrak{A} \) and let \( \alpha \) be an almost product automorphism of \( \mathfrak{A} \). Then,

\[
\text{ind}_{Z_2} \psi = \text{ind}_{Z_2} (\psi \circ \alpha) 
\]  

(3.7)
Next we look into gapped ground states of one-dimensional Fermion systems. The one-particle space we deal with is \( K = l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}) \). We assume that systems are translationally invariant without loss of generality. Our argument below is valid for periodic systems or for one-particle subspace with an internal degree of freedom \( V \), \( K = l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}) \) provided the dimension of \( V \) is finite. The correspondence of creation and annihilation operators \( c^*_k \) and \( c_k \) and \( B(f \oplus g) \) is as follows.

Set \( \mathbb{Z}_< = \{ j \in \mathbb{Z} \mid j < 0 \} \), \( K = l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}), K_L = l^2(\mathbb{Z}_\cup \{0\}) \oplus l^2(\mathbb{Z}_\cup \{0\}) \)

\[
B(f \oplus g) = c^*(f) + c(g) = \sum_j (c^* f_j + c g_j)
\]

\( J(f \oplus g) = (f \oplus \overline{g}) \) (3.8)

for \( f = \{ f_j \} \in K_0, g = \{ g_j \} \in K_0 \), \( \overline{f} \) is the complex conjugation \( \overline{f} = \{ f_j \} \).

**Proposition 3.7** Let \( \psi \) be a translationally invariant pure ground state of a finite range translationally invariant Hamiltonian of (1.18). If \( \psi \) is a gapped ground state, the split property holds, i.e. \( \psi \) restricted to \( A_L \) is type I.

We may prove Proposition 3.7 in several ways. As repeating proof of [10], [6] and [14] for fermion systems is somewhat lengthy, here we employ the Jordan-Wigner transformation a la mani`ere de [3] and reduce the problem to the Pauli spin chain. We work in the algebra \( \hat{A} \) and \( \mathcal{T} \) to define our Jordan-Wigner transformation. For any \( l \in \mathbb{Z} \), set

\[
\sigma^{(l)}_{x} = S_l(c^*_l + c_l) \\
\sigma^{(l)}_{y} = i S_l(c_l - c^*_l) \\
\sigma^{(l)}_{z} = (2c^*_l c_l - 1)
\]

where

\[
S_l = \begin{cases} 
T \prod_{k=0}^{l-1} \sigma^{(k)}_{z} & (1 \leq j) \\
T & (j = 0) \\
T \prod_{k=j}^{-1} \sigma^{(k)}_{z} & (j < 0).
\end{cases}
\]

We obtain relations for Pauli spin matrices:

\[
\sigma^{(l)}_{x} \sigma^{(l)}_{y} = i \sigma^{(l)}_{z}, \quad \{ \sigma^{(l)}_{x}, \sigma^{(l)}_{y} \} = 0, \quad (\sigma^{(l)}_{x})^2 = 1 \quad [\sigma^{(l)}_{x}, \sigma^{(k)}_{z}] = 0
\]

for \( l, k \in \mathbb{Z}, l \neq k \) and \( \alpha, \beta = x, y, z \).

Let \( \mathfrak{A}^P \) be a \( C^* \)-subalgebra of \( \hat{A} \) generated by \( \sigma^{(l)}_{\alpha} \ l \in \mathbb{Z}, \alpha = x, y, z \) where \( P \) stands for Pauli spin algebra. Set

\[
\mathfrak{A}^{P} (\pm) = \{ Q \in \mathfrak{A}^P \mid \Theta(Q) = \pm Q \}, \quad Q_{\pm} = \frac{Q \pm \Theta(Q)}{2} \in \mathfrak{A}^{P} (\pm)
\]

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and let $\mathfrak{A}^P_{[n,m]}$ be the $C^*$-subalgebra of $\mathfrak{A}^P$ generated by $\sigma^{(l)}_{\alpha}$, $n \leq l \leq m$, $\alpha = x, y, z$. Set

\[
\mathfrak{A}^P_{[n,m]} = \mathfrak{A}^P_{[n,m]} \cap \mathfrak{A}^P (\pm),
\]

\[
\mathfrak{A}^P_{loc} = \bigcup_{-\infty < n \leq m < \infty} \mathfrak{A}^P_{[n,m]}.
\]

, i.e. $\mathfrak{A}^P_{loc}$ is the set of all elements strictly localized in finite sets. Finally we introduce $\mathfrak{A}^P_L = \mathfrak{A}^P_{(-\infty,-1]}$, $\mathfrak{A}^P_R = \mathfrak{A}^P_{[1,\infty)}$. By split property of a state $\varphi$ of $\mathfrak{A}^P$ we mean split property with respect to $\mathfrak{A}^P_L$ and $\mathfrak{A}^P_R$.

Due to our definition, it easy to see that $\mathfrak{A}(+)$ and $\mathfrak{A}^P (+)$ coincide: $\mathfrak{A}(+) = \mathfrak{A}^P (+)$. The lattice translation $\tau_j$ on $\mathfrak{A}(+)$ can be extended to $\mathfrak{A}^P$.

For any $\Theta$ invariant pure state $\psi$ of $\mathfrak{A}$, we extend $\psi$ to a state $\hat{\psi}$ of $\hat{\mathfrak{A}}$ via the following equation:

\[
\hat{\psi}(Q_1 + Q_2 T) = \psi(Q_1, Q_2) \quad Q_1, Q_2 \in \mathfrak{A}.
\]

$\hat{\psi}$ is invariant under $\Theta$.

We recall results proved in [3]. See Section 5 of [3]. In our proof Section 5 of [3], quasifree property of states is not used but a crucial point is (non-)existence of $U(\Theta_-)$ where $U(\Theta_-)$ is the self-adjoint unitary implementing $\Theta_-$ on $\mathfrak{A}$ on the GNS space of $\psi$. Note that if $\psi$ be a $\Theta$ invariant pure state of $\mathfrak{A}$ and if $\psi$ and $\psi \circ \Theta_-$ are unitarily equivalent, there exists a self-adjoint unitary $U(\Theta_-)$ on the GNS space $\mathfrak{H}_\psi$ satisfying

\[
U(\Theta_-)\pi_\psi((Q))U(\Theta_-)^* = \pi_\psi(\Theta_-(Q)), \quad Q \in \mathfrak{A}.
\]

If we assume further that $\psi$ and $\psi \circ \Theta_-$ restricted to $\mathfrak{A}(+)$ are unitarily equivalent, $U(\Theta_-) \in \pi_\psi(\mathfrak{A}(+)')''$. If we assume, instead, that $\psi$ and $\psi \circ \Theta_-$ restricted to $\mathfrak{A}(+)$ are disjoint, $U(\Theta_-)$ is in the closure of $\pi_\psi(\mathfrak{A}(+)')''$ by weak operator topology.

**Proposition 3.8** Let $\psi$ be a $\Theta$ invariant pure state of $\mathfrak{A}$ and $\psi^P$ be the state of $\mathfrak{A}^P$ uniquely determined by the following equation :

\[
\psi^P(Q) = \psi(Q_+), \quad Q \in \mathfrak{A}^P. \tag{3.10}
\]

(i) If $\psi$ and $\psi \circ \Theta_-$ are disjoint, $\psi^P$ is a pure state of $\mathfrak{A}^P$. The split property does not hold.

(ii) Suppose $\psi$ and $\psi \circ \Theta_-$ are unitarily equivalent.

(ii-a) Assume further that $\psi$ and $\psi \circ \Theta_-$ restricted to $\mathfrak{A}(+)$ are unitarily equivalent, $\psi^P$ is a pure state of $\mathfrak{A}^P$.

(ii-b) Assume further that $\psi$ and $\psi \circ \Theta_-$ restricted to $\mathfrak{A}(+)$ are disjoint, $\psi^S$ is not a pure state of $\mathfrak{A}^P$. There exists a pure state $\varphi$ of $\mathfrak{A}^P$ satisfying

\[
\varphi^S = 1/2 (\varphi + \varphi \circ \Theta), \quad \varphi \neq \varphi \circ \Theta. \tag{3.11}
\]

$\varphi$ and $\varphi \circ \Theta$ are disjoint and $\varphi$ is described as

\[
\varphi(Q) = \psi((1 + U(\Theta_-)T)Q), \quad Q \in \mathfrak{A}^P. \tag{3.12}
\]
where by abuse of notations, we denote the normal extension of $\varphi$ to $\pi_P(\mathfrak{A}^P)'$ by the same symbol $\varphi$. $U(\Theta_-)$ is the self-adjoint unitary implementing $\Theta_-$ on $\pi_P(\mathfrak{A})'$ and $U(\Theta_-)$ is in the weak limit of elements in $\pi_P(\mathfrak{A}^-)$. The existence of $U(\Theta_-)$ follows from disjointness of $\psi$ and $\psi \circ \Theta_-$ restricted to $\mathfrak{A}^P$.

As any $\mathbb{Z}_2$ product state is $\Theta_-$ invariant, for a $\Theta$ invariant pure state $\psi$ of $\mathfrak{A}$ with split property $\psi$ and $\psi \circ \Theta_-$ are unitarily equivalent. Proposition 3.8 and (ii) of Lemma 2.4 imply the following.

**Corollary 3.9** Let $\psi$ be a $\Theta$ invariant pure state of $\mathfrak{A}$ with split property.

(i) if $\text{ind}_{\mathbb{Z}_2} \psi = 1$, $\psi^P$ is a pure state.

(ii) if $\text{ind}_{\mathbb{Z}_2} \psi = -1$, $\psi^P$ is not a pure state and a pure state $\varphi$ satisfying (3.11) and (3.12) exists.

Furthermore, if $\psi$ is periodic with a period $p$, $\psi \circ \tau_p = \psi$, $\psi^P$ is periodic with a period $p$. In the case of (ii), one of the following two possibility (a), (b) is valid.

(a) $\varphi$ is periodic with a period $p$.

(b) $\varphi \circ \tau_p = \varphi \circ \Theta$.

It is now time to talk about gapped ground states of $\mathfrak{A}$. The finite volume Hamiltonian $H_M$ of (1.17) belongs to $\mathfrak{A}_P^P(+) = \mathfrak{A}(+)$ and generates an infinite volume time evolution of $\mathfrak{A}^P$ via the same limit as in (1.19). We denote the time evolution of $\mathfrak{A}^P$ by the same symbol $\alpha_{ih}$:

$$
\lim_{M \to \infty} e^{itH_M} Q e^{-itH_M} = \alpha_{ih}^P(Q) \quad Q \in \mathfrak{A}^P.
$$

(3.13)

Suppose that $\psi$ is a translationally invariant pure ground state of $\mathfrak{A}$. In view of (1.19) for $\mathfrak{A}^P$ the $\Theta$ invariant extension $\psi^P$ in (3.10) to $\mathfrak{A}^P$ is a translationally invariant $\Theta$ invariant ground state for $\alpha_{ih}^P$. Due to Proposition 3.7, either (i) $\psi^P$ is pure or (ii) $\psi^P$ is an average of two pure states $\varphi$ and $\varphi \circ \Theta$.

**Proposition 3.10** Suppose that $\psi$ is a translationally invariant pure gapped ground state of $\mathfrak{A}$.

(i) if $\psi^P$ is a pure state of $\mathfrak{A}^P$, $\psi^P$ is a translationally invariant gapped ground state.

(ii) If $\psi^P$ is not pure, $\varphi$ in (3.11) is a gapped ground state which is periodic with period 2.

**Proof of Proposition 3.10**

Let $\{\pi(\mathfrak{A}^P), \Omega, \mathfrak{F}\}$ be the GNS triple associated with $\psi^P$ and set

$$
\mathfrak{F}^\pm = \pi(\mathfrak{A}^P(\pm)) \Omega.
$$

Let $H_{\psi^P}$ be a positive self-adjoint operator on $\mathfrak{F}$ implementing $\alpha_{ih}^P$ on $\pi(\mathfrak{A}^P)$.

Now we show (i). As restriction of $\psi^P$ to $\mathfrak{A}^P(+) = \mathfrak{A}(+)$ is a gapped ground state, $H_{\psi^P}$ has positive spectrum on $\mathfrak{F}^+$

$$
\text{spec } H_{\psi^P}|_{\mathfrak{F}^+} \subset \{0\} \cup [m, \infty)
$$
Suppose that there exists a gapless excitation on $\mathcal{F}^-$, in another word, for any small $\delta > 0$ and some $\eta > 0$

$$\text{spec } H_{\psi^p}|_{\mathcal{F}^-} \cap (-\eta, \delta) \neq \emptyset.$$ 

there exists $Q \in \mathfrak{U}^{P(-)}$ and a smooth function $f(t)$ such that its fourier transform $\hat{f}(\xi)$ is supported in $(-\delta, \eta)$ and that

$$\pi(\alpha^h_j(Q)) = \hat{f}(-H_{\psi^p})\pi(Q)\Omega \neq 0 \quad (3.14)$$

where $\alpha^h_j(Q)$ is an operator creating small energy excitation:

$$\alpha^h_j(Q) = \int_{\mathbb{R}} \alpha^h_j(t) f(t) \, dt \in \mathfrak{U}^{P(-)}.$$ 

Due to cluster property of any pure state,

$$\lim_{k \to \infty} \psi^P(\alpha^h_j(Q)^* \tau_k(\alpha^h_j(Q)^* \alpha^h_j(Q)) \alpha^h_j(Q)) = 0 \quad (3.15)$$

As a consequence we see that, for a large $k$,

$$\xi = \pi(\alpha^P_{\psi^h}(Q))\tau_k(\pi(\alpha^P_{\psi^h}(Q)))\Omega \neq 0. \quad (3.16)$$

By the proof of Lemma 3.2.42 (iii) in [1], $\xi$ has the spectral support in $(0, 2\delta)$ in the special decomposition of $H_{\psi^p}$. $\xi$ belongs to $\mathcal{F}^+$ and we obtain a contradiction.

(ii) If $\psi^P$ is not pure, representation of $\mathfrak{U}^{P(+)}$ on $\mathcal{F}^+$ are unitary equivalent the spectrum of $H_{\psi^p}$ on each space is same with doubled multiplicity. If we restrict $H_{\psi^p}$ to irreducible representations of $\mathfrak{U}^P$, the degeneracy of the ground state energy spectrum is removed.

Proof of Proposition 3.7

(i) If $\psi^P$ is pure, $\psi^P$ is a gapped ground state for the Pauli spin system and split property is valid due to a result in [10] and [14]. Thus $\psi^P(\mathfrak{U}^P_L)^\nu$ is type I and $\Theta$ invariance of $\psi^P$ restricted to $\mathfrak{U}^P_L$ implies $\pi_P(\mathfrak{U}^{P(+)}_L)^\nu$ is type I as the commutant of a single self-adjoint unitary in $B(\mathcal{F})$ is type I. Since $\pi_P(\mathfrak{U}^{P(+)}_L)^\nu = \pi_\psi(\mathfrak{U}^{(+)}_L)^\nu$, $\pi_\psi(\mathfrak{U}^P_L)^\nu$ is type I.

(ii) If $\psi^P$ is not pure, $\varphi$ is a gapped ground state for the Pauli spin system and split property is valid for $\varphi$ and $\varphi \circ \Theta$. As $\varphi$ is pure, $\varphi$ is equivalent to a product pure state $\varphi_1 \otimes \varphi_2$ where both $\varphi_1$ and $\varphi_2$ are pure. As $\varphi$ is not equivalent to $\varphi \circ \Theta$ at least $\varphi_1$ or $\varphi_2$ is not equivalent to $\varphi_1 \circ \Theta$ and $\varphi_2 \circ \Theta$.

If $\varphi_1$ and $\varphi_1 \circ \Theta$ are equivalent, there exists a $\Theta$ invariant pure state $\tilde{\varphi_1}$ equivalent to $\varphi_1$ and the argument of (i) applies to show our claim.

If $\varphi_1$ and $\varphi_1 \circ \Theta$ are disjoint, due to Lemma 2.2, $\pi_{\varphi_1}(\mathfrak{U}^{P(+)}_L)^\nu = \pi_{\varphi_1}(\mathfrak{U}^P_L)^\nu$ is type I and hence $\pi_\psi(\mathfrak{U}^{(+)}_L)^\nu$ is type I and so is $\pi_\psi(\mathfrak{U}^P_L)^\nu$.

End of Proof
All the staffs being ready, we summarize what we have shown in this section, which automatically clarifies the relationship between string order and the index we defined.

We started with a gapped ground state $\psi$ which is pure, and translationally invariant (and hence $\Theta$ invariant). Via Jordan-Wigner transformation, we obtain a translationally invariant $\Theta$ invariant ground state $\psi^P$.

If $\text{ind}_{\mathbb{Z}_2} \psi = 1$, $\psi^P$ is a gapped pure ground state, has no long range order. For any $Q_1, Q_2 \in \mathfrak{X}_{\text{loc}}(-)$

$$\lim_{k \to \infty} \psi^P(Q_1 \tau_k(Q_2)) = 0$$

which implies

$$\lim_{k \to \infty} \psi^P(Q_1 S[a, b + k] \tau_k(Q_2)) = 0$$

for some $a$ and $b$ (No String Order).

If $\text{ind}_{\mathbb{Z}_2} \psi = -1$, $\psi^P$ is still a ground state but not pure. We have two possibilities:

(a) $\varphi$ is a translationally invariant gapped pure ground state $\varphi$ of a Pauli spin system such that $\varphi$ and $\varphi \circ \Theta$ are disjoint. There exist $Q_1, Q_2 \in \mathfrak{X}_{\text{loc}}(-)$ such that

$$\lim_{k \to \infty} \psi^P(Q_1 \tau_k(Q_2)) \neq 0$$

Hence going back to $\mathfrak{X}$ string order exists

$$\lim_{k \to \infty} \psi^P(Q_1 S[a, b + k] \tau_k(Q_2)) \neq 0$$

(b) $\varphi$ is a periodic gapped ground state of a Pauli spin system such that

$$\varphi \circ \tau_1 = \varphi \circ \Theta, \quad \varphi \circ \tau_1 \neq \varphi.$$  

There exist $Q_1, Q_2 \in \mathfrak{X}_{\text{loc}}(-)$ such that

$$\lim_{k \to \infty} \psi^P(Q_1 \tau_{2k}(Q_2)) \neq 0$$

String order exists

$$\lim_{k \to \infty} \psi^P(Q_1 S[a, b + 2k] \tau_k(Q_2)) \neq 0$$

Apparently the converse is correct. For example, if string order exists, $\psi^P$ is not pure and $\text{ind}_{\mathbb{Z}_2} \psi = -1$. If no string order exists, there is no long range order for $\psi^P$ and $\text{ind}_{\mathbb{Z}_2} \psi = 1$. This shows Theorem 1.8.

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References

[1] H. Araki. *On Quasifree States of CAR and Bogoliubov Automorphisms.* Publ. RIMS Kyoto Univ. **6**, 385-442 (1970/71).

[2] H. Araki and D.E. Evans: On a $C^*$-algebraic Approach to Phase Transition in the Two-Dimensional Ising Model. Commun.Math.Phys. **91**, 489-503 (1983).

[3] H. Araki and Taku Matsui: Ground States of the XY Models. Commun.Math.Phys. **101**, 213-245 (1985).

[4] Y. Bahri and A. Vishwanath: *Detecting Majorana fermions in quasi-one-dimensional topological phases using nonlocal order parameters.* Phys. Rev. B **89**, 155135 (2014).

[5] C. Bourne and H. Schulz-Baldes: *On $Z_2$-indices for ground states of fermionic chains.* preprint, arXiv:1905.11556v2

[6] F. Brandão and M. Horodecki: *Exponential Decay of Correlations Implies Area Law.* Commun. Math. Phys. **333**, 761-798 (2015).

[7] O. Bratteli and D. Robinson: *Operator algebras and quantum statistical mechanics I.*, 2nd edition Springer (1987).

[8] O. Bratteli and D. Robinson: *Operator algebras and quantum statistical mechanics II.*, 2nd edition Springer (1997).

[9] M. den Nijs, and K. Rommelse: *Preroughning transitions in crystal surfaces and valence-bond phases in quantum spin chains.* Phys. Rev. B **40**, 4709 (1989).

[10] M. Hastings: *An area law for one dimensional quantum systems.* JSTAT, P08024 (2007).

[11] R.V. Kadison and J. Ringrose: *Fundamentals of the Theory of Operator Algebras II.* Graduate Studies in Mathematics **16**, AMS(1997).

[12] T. Kennedy and H. Tasaki: *Hidden symmetry breaking and the Haldane phase in S = 1 quantum spin chains.* Comm. Math. Phys. **147**, 431-484 (1992).

[13] Taku Matsui: *The split property and the symmetry breaking of the quantum spin chain.* Commun.Math.Phys. **218**, 393-416, (2001).

[14] Taku Matsui: *Boundedness of entanglement entropy and split property of quantum spin chains.* Rev. Math. Phys. **13**, 0017, (2013).

[15] A. Moon, and Y. Ogata: *Automorphic equivalence within gapped phases in the bulk.* Journ. Funct. Anal. **278**, Issue 8, 108422(2020)
[16] H.Tasaki and Y.Ogata: Lieb-Schultz-Mattis type theorems for quantum spin chains without continuous symmetry. Commun. Math. Phys. 372, 951 - 96(2019)

[17] Y.Ogata: A $\mathbb{Z}_2$-index of symmetry protected topological phases with time reversal symmetry for quantum spin chains. Commun.Math.Phys. To appear

[18] Y.Ogata: A $\mathbb{Z}_2$-index of symmetry protected topological phases with reflection symmetry for quantum spin chains. preprint

[19] S.Strătilă and D.Voiculescu On a class of KMS states for the unitary group $U(\infty)$. Math.Ann.235, 87-110(1978)

[20] H. Tasaki: Physics and mathematics of quantum many-body systems. Springer (to appear)