Qualitative properties of a nonlinear system involving the $p$-Laplacian operator.

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Abstract

In this article we consider the nonlinear system involving the $p$-Laplacian

\[
\begin{align*}
|u'|^{p-2}u'' &= u^{p-1}v^p \\
|v'|^{p-2}v'' &= v^{p-1}u^p \\
u &\geq 0, v \geq 0
\end{align*}
\]

for which we prove symmetry, asymptotic behavior and non degeneracy properties. This can help to a better understanding to what happens in the $N$ dimensional case, for which several authors prove a De Giorgi Type result under some additional growth and monotonicity assumptions.

1 Introduction

In this article we extend some of the results obtained in [5] in the case of the Laplacian, to the $p$-Laplacian case. More precisely we consider the system in $\mathbb{R}^N$:

\[
\begin{align*}
\text{div}(|\nabla u|^{p-2}\nabla u) &= u^{p-1}v^p \\
\text{div}(|\nabla v|^{p-2}\nabla v) &= v^{p-1}u^p,
\end{align*}
\]

where $u$ and $v$ are supposed to be positive.

In the case $p = 2$ this problem comes from a phase separation model. As an example the Gross Pitaevskii, [9] model can be described by the non linear elliptic system

\[
\begin{align*}
-\Delta u + \alpha u^3 + \Lambda u^2 v &= \lambda_{1,\Lambda} u & \text{in } \Omega \\
-\Delta v + \alpha v^3 + \Lambda u^2 v &= \lambda_{2,\Lambda} v & \text{in } \Omega \\
u > 0, v > 0, & \text{in } \Omega, u = v = 0 & \text{on } \partial\Omega \\
\int_{\Omega} u^2 &= \int_{\Omega} v^2 = 1
\end{align*}
\] (1.1)
where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ and $\alpha, \beta$ are positive parameters, $\Lambda$ will become large. Assuming that $\sup(\lambda_{1,\Lambda}, \lambda_{2,\Lambda}) \leq C$ for some constant independent of $\Lambda$, formally and up to subsequences $(u_{\Lambda}, v_{\Lambda})$ converges to some pair $(u, v)$ which satisfies $uv = 0$ and the equations

\[
\begin{cases}
-\Delta u + \alpha u^3 = \lambda_{1,\Lambda} u & \text{in } \Omega_u = \{x, u(x) > 0\} \\
-\Delta v + \beta v^3 = \lambda_{2,\Lambda} v & \text{in } \Omega_v = \{x, v(x) > 0\}
\end{cases}
\] (1.2)

Several papers treat the convergence of $(u_{\Lambda}, v_{\Lambda})$ away the interface $\gamma = \{x, u(x) = 0 = v(x)\}$, see for example [29] and [10], [24] for the uniform equicontinuity of $(u_{\Lambda}, v_{\Lambda})$.

Near the interface, the profile of bounded solutions of (1.1) of the blow up equation is a system, which is completely classified in the one dimensional case, [5], [6]. This system is the following

\[
\begin{cases}
U'' = UV^2 \\
V'' = VU^2 \\
U > 0, V > 0
\end{cases}
\]

In [6], the authors expect that the same system occurs in the $N$ dimensional case, say

\[
\begin{cases}
\Delta U = UV^2 \\
\Delta V = VU^2 \\
U > 0, V > 0
\end{cases}
\]

in $\mathbb{R}^N$.

Furthermore they conjecture that for any dimension $N \leq 8$, the system is in fact one dimensional. They obtain this result under some additional assumption on the growth of the solution, in dimension 2. The assumption $N \leq 8$ is motivated by the case of scalar equations for which it is known that the scalar equation is not necessary one dimensional for $N \geq 9$, [13]. The condition on the growth in the two dimensional case is satisfied in particular if the solution of the system is at most linear at infinity, as it is proved in the case $N = 1$. When $N > 2$, this is not sufficient, and up to now, even in the case $N = 2$, this increasing behaviour is not proved.

In [18] the author improved the result by establishing that for $N = 2$, as soon as $u$ and $v$ have at most algebraic increasing behavior and satisfy for some component $\partial_N u > 0$, $\partial_N v < 0$, then the solution is one dimensional. This result is recently improved by Farina and Soave by replacing the condition $\partial_N u > 0$, $\partial_N v < 0$ by the weaker condition $\lim_{x_N \to \pm \infty} u(x', x_N) - v(x', x_N) = \pm \infty$, $2$
uniformly with respect to $x'$, conserving the assumption of algebraic growth. Let us cite also the recent result of K. Wang \[28\] which replaces the monotonicity condition by the fact that $(u, v)$ is a local minimizer.

The present paper is motivated by the asymptotic study of (1.1), say

$$
\begin{align*}
\begin{cases}
-d\text{div}(\abs{\nabla u}^{p-2}\nabla u) + \alpha u^{p+1} + \Lambda u^{p}u^{p-1} = \lambda_{1,\Lambda}u^{p-1} & \text{in } \Omega \\
-d\text{div}(\abs{\nabla v}^{p-2}\nabla v) + \beta v^{p+1} + \Lambda u^{p}v^{p-1} = \lambda_{2,\Lambda}v^{p-1} & \text{in } \Omega \\
u > 0, \ v > 0 \text{ in } \Omega, \ u = v = 0 \text{ on } \partial\Omega
\end{cases}
\end{align*}
$$

(1.3)

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^{N}$ and $\alpha, \beta$ are positive parameters, $\Lambda$ will become large. Such a pair of solutions is a critical point for the functional $E_{\Lambda}(u, v) = \frac{1}{p}\int_{\Omega}((\abs{\nabla u}^{p} + \abs{\nabla v}^{p}) + \frac{\alpha}{p+2}\int_{\Omega}u^{p+2} + \frac{\beta}{p+2}\int_{\Omega}v^{p+2} + \frac{\Lambda}{p}\int_{\Omega}u^{p}v^{p})$ under the constraint $\int_{\Omega}u^{p} = \int_{\Omega}v^{p} = 1$.

Assume that there exists some constant $C$ independent on $\Lambda$ with

$$
\sup_{\Lambda}(\lambda_{1,\Lambda}, \lambda_{2,\Lambda}) \leq C, \text{ for } \Lambda \text{ large }.
$$

As $\Lambda$ goes to infinity, and up to subsequence $(u_{\Lambda}, v_{\Lambda})$ tends formally to some pair $(u, v)$ which satisfies

$$
\begin{align*}
\begin{cases}
-d\text{div}(\abs{\nabla u}^{p-2}\nabla u) + \alpha u^{p+1} = \lambda_{1,\Lambda}u^{p-1} & \text{in } \Omega_{u} \\
-d\text{div}(\abs{\nabla v}^{p-2}\nabla v) + \beta v^{p+1} = \lambda_{2,\Lambda}v^{p-1} & \text{in } \Omega_{v}
\end{cases}
\end{align*}
$$

(1.4)

where $\Omega_{u} = \{x, u(x) > 0\}$, $\Omega_{v} = \{x, v(x) > 0\}$.

It is not our purpose here to follow this way. We are interested in the one dimensional case and especialy in the behavior of the limit pair of solutions $(u, v)$ near the interface $\gamma = \{x \in \Omega, u(x) = v(x) = 0\}$.

When $N = 1$ and $\Omega = ]a, b[\text{ one has the result}

**Theorem 1.1.** Assume that $\Omega = ]a, b[$, that $(u_{\Lambda}, v_{\Lambda})$ solves (1.3). There exists $x_{\Lambda} \in \Omega$ such that $u_{\Lambda}(x_{\Lambda}) = m_{\Lambda} = v(x_{\Lambda})$ goes to zero, and $x_{\Lambda}$ tends to some point in $\gamma$. Furthermore if $\bar{u}_{\Lambda} = \frac{1}{m_{\Lambda}}u_{\Lambda}(m_{\Lambda}y + x_{\Lambda})$, $\bar{v}_{\Lambda} = \frac{1}{m_{\Lambda}}v_{\Lambda}(m_{\Lambda}y + x_{\Lambda})$ and $y \in ]\frac{a-x_{\Lambda}}{m_{\Lambda}}, \frac{b-x_{\Lambda}}{m_{\Lambda}}[$, $(\bar{u}_{\Lambda}, \bar{v}_{\Lambda})$ converges locally uniformly to some pair $U, V$ which satisfies

$$
\begin{align*}
\begin{cases}
(U''|^{p-2}U')' = V^{p}U^{p-1} & \text{on } \mathbb{R} \\
(V''|^{p-2}V')' = U^{p}V^{p-1} \\
U(0) = V(0) = 1.
\end{cases}
\end{align*}
$$

(1.5)

Furthermore there exists some positive constant $T_{\infty}$ such that

$$
|U'|^{p} + |V'|^{p} - U^{p}V^{p} = T_{\infty}.
$$

3
Next we are interested in the existence and in the properties of the solutions of (1.5). The existence of a non trivial solution is given in Theorem 3.1. In a second time we prove the following

**Theorem 1.2.** Let $N = 1$ and $(U, V)$ be a non negative solution of

$$
\begin{align*}
\frac{(|U'|^p - 2U')'}{p} &= V^pU^{p-1}, \\
\frac{(|V'|^p - 2V')'}{p} &= U^pV^{p-1}
\end{align*}
$$

on $\mathbb{R}$. Then up to exchanging $U$ and $V$,

1) Up to translation $V(y) = U(-y)$.

2) $U' > 0$ everywhere, $U'(+\infty) = (T_\infty)^{\frac{1}{p}}$, and there exist some positive constants, $m, M, k, K, c, C$ such that near $-\infty$, $me^{-Kx^2} \leq U(x) \leq Me^{-kx^2}$, $c|x|U \leq U' \leq C|x|U$.

Symmetric estimates hold for $V$ exchanging $-\infty$ and $+\infty$.

3) Suppose that $\phi, \psi$ is a bounded solution of the linearized system

$$
\begin{align*}
\frac{(|U'|^p - 2\phi')'}{p} &= (p-1)U^{p-2}V^p\phi + pV^{p-1}U^{p-1}\psi, \\
\frac{(|V'|^p - 2\psi')'}{p} &= (p-1)V^{p-2}U^p\psi + pU^{p-1}V^{p-1}\phi
\end{align*}
$$

in $\mathbb{R}$, then there exists some constant $c$ such that $(\phi, \psi) = c(U', V')$.

We end this introduction by some reflexions about the De Giorgi’s conjecture, which, even if we do not treat it here, is after all, at the origin of the present paper.

As we said in the abstract, the previous classification is a first step if one want to prove a De Giorgi type result on the system

$$
\begin{align*}
\text{div}(|\nabla u|^{p-2}\nabla u) &= u^{p-1}v^p, \\
\text{div}(|\nabla v|^{p-2}\nabla v) &= u^pv^{p-1}
\end{align*}
$$

in $\mathbb{R}^N$.

In [15], [16] the authors consider a more general system than the present one, the quasilinear operator she studies includes the $p$-Laplacian operator, and the right hand side included the case studied here. She proves that under some condition of growth on the solution, together with some stability assumption on the couple of solutions, then the solution is in fact one dimensional. The stability is in particular implied when the solution is ”half monotone ”, i.e. if there exists one direction say $e_1$ such that $\partial_1 u > 0$ and $\partial_1 v < 0$. It is an exercise to prove that the growth condition (1.14) in [15] is satisfied when the
solution has at most linear growth at infinity, in dimension 2, hence we recover a generalization of the results in [6] in the case $p \neq 2$.

Several questions are of interest: As we said above, in [18], and [19] the authors suppose that the solution has at most algebraic growth. Can we have the same result in the $p$-Laplacian case? The answer is not immediate, since the proof of Farina and Soave relies on some properties of the Almgren frequency function which do not extend to the $p$-Laplacian case.

Another interesting question is the following: Suppose that one replaces the Laplacian system by a Fully Nonlinear system. Of course for Pucci’s operators the one dimensional system is reduced, up to constant, to the Laplacian case. But due to the non differentiability of the Pucci’s operators, the definition of stable solutions must be precised. In the same order of ideas, one can imagine to treat the case of Fully Nonlinear degenerate or singular systems, based on the model of the $p$-Laplacian type treated here, but not under divergence form, as the following

\[
\begin{cases}
|\nabla u|^{\alpha} F(D^2 u) = u^{\alpha+1} v^{\alpha+2} \\
|\nabla v|^{\alpha} F(D^2 v) = v^{\alpha+1} u^{\alpha+2} 
\end{cases}
\text{in } \mathbb{R}^N
\]

where $\alpha$ is some number $> -1$ and $F$ is fully nonlinear elliptic. The reader may consult [7] for properties of such operators and a convenient definition of viscosity solutions.

It would be far too long to cite all the papers written about the De Giorgi type result in the case of one equation in place of a system. Let us cite for the $p$-Laplacian case the recent paper of A. Farina and Valdinoci [21], and the very complete paper of Savin et al. [26]. For variations on the subject on De Giorgi’s conjecture in the case of a single equation the reader may consult [12], [17], [22], [20], [2], [3], [23], [1], [25], [8].

The paper is organized as follows: In Section 2, we consider the one dimensional system defining $(u_\Lambda, v_\Lambda)$ and prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. Most of the technical details of this section are postponed to the appendix in section 4.

Acknowledgment The author wishes to thank Alberno Farina and Enrico Valdinoci for having pointed out several papers on the question.
2 The original problem: Proof of theorem 1.1

In all that section we will frequently use in place of sequences, subsequences, without mentioning it. Let us consider for $\alpha, \beta$ and $\Lambda, \lambda_1, \lambda_2$ some given positive constants

$$\begin{aligned}
\begin{cases}
-u_A'' + \alpha u_A^{p+1} + \Lambda u_A^{p-1} = \lambda_1 u_A^{-1} \\
v_A'' + \beta v_A^{p+1} + \Lambda v_A^{p-1} = \lambda_2 v_A^{-1}
\end{cases}
\end{aligned} \quad \text{in } [a, b].
$$

(2.1)

$$u, v > 0, u(a) = u(b) = v(a) = v(b) = 0, \quad \int_a^b |u|^p = \int_a^b |v|^p = 1.$$ (2.2)

$(u_A, v_A)$ is then a solution of the minimizing eigenvalue problem

$$\inf_{|u|_p = |v|_p = 1, (u, v) \in W^{1, p}([a, b])} E_\Lambda(u, v),$$

where

$$E_\Lambda(u, v) = \frac{1}{p} \int_a^b |u'|^p + \frac{1}{p} \int_a^b |v'|^p + \alpha \int_a^b \frac{|u|^{p+2}}{p+2} + \beta \int_a^b \frac{|v|^{p+2}}{p+2} + \Lambda \int_a^b |u|^p |v|^p.$$ (2.3)

Assume that

$$\max_{\lambda} (\lambda_1, \lambda_2) \leq C,$$

Due to the first equation in (2.1) multiplied by $u_A$, and integrated over $[a, b]$, one gets

$$\int_a^b |u'_A|^p + \alpha \int_a^b u_A^{p+2} + \Lambda \int_a^b u_A v_A^p = \lambda_1.$$ (2.2)

Doing the same for $v_A$, one gets that $(u_A, v_A)$ is bounded in $W^{1, p}([a, b])^2$.

Furthermore

**Lemma 2.1.** There exist $T_\Lambda$, $C_1$ and $C_2$ independent on $\Lambda$ such that

$$\frac{|u'_A|^p}{p} + \frac{|v'_A|^p}{p} - \Lambda \frac{u_A^p v_A^p}{p} - \alpha \frac{u_A^{p+2}}{p+2} - \beta \frac{v_A^{p+2}}{p+2} + \lambda_1 \frac{u_A^p}{p} + \lambda_2 \frac{v_A^p}{p} = T_\Lambda$$ (2.3)

and

$$0 < C_1 \leq T_\Lambda \leq C_2 < \infty$$

6
Proof

Multiply the first equation in (2.1) by \( u'_\Lambda \), the second one by \( v'_\Lambda \), and add the two equations, we obtain that

\[
-\left( \frac{|u'_\Lambda|^p}{p} \right)' - \left( \frac{|v'_\Lambda|^p}{p} \right)' + \alpha \left( \frac{|u''_\Lambda|^p}{p + 2} \right)' + \beta \left( \frac{|v''_\Lambda|^p}{p + 2} \right)' + \Lambda \left( \frac{|u''_\Lambda|^p}{p} \right)' = \lambda_{1,\Lambda} \left( \frac{|u'_\Lambda|^p}{p} \right)' + \lambda_{2,\Lambda} \left( \frac{|v'_\Lambda|^p}{p} \right)'
\]

Hence there exists some constant \( T_\Lambda \) such that (2.3) is satisfied. Integrating on \([a, b]\) the equation defining \( T_\Lambda \), one gets

\[
\int_a^b \frac{|u'_\Lambda|^p}{p} + \int_a^b \frac{|v'_\Lambda|^p}{p} - \Lambda \int_a^b \frac{|u''_\Lambda|^p}{p + 2} - \alpha \int_a^b \frac{|u''_\Lambda|^p}{p + 2} - \beta \int_a^b \frac{|v''_\Lambda|^p}{p + 2} + \lambda_{1,\Lambda} + \lambda_{2,\Lambda} = \frac{T_\Lambda (b - a)}{p}.
\]

On the other hand, using (2.2) for \( u_\Lambda \) and its analogous for \( v_\Lambda \), one has

\[
\int_a^b |u'_\Lambda|^p + \int_a^b |v'_\Lambda|^p + \alpha \int_a^b |u''_\Lambda|^p + \beta \int_a^b |v''_\Lambda|^p + 2\Lambda \int_a^b u''_\Lambda v''_\Lambda = \lambda_{1,\Lambda} + \lambda_{2,\Lambda}.
\]

Combining the two equations one gets

\[
T_\Lambda (b - a) = 2 \int_a^b |u'_\Lambda|^p + 2 \int_a^b |v'_\Lambda|^p + \frac{2\alpha}{p} \int_a^b |u''_\Lambda|^p + \frac{2\beta}{p + 2} \int_a^b |v''_\Lambda|^p + \lambda_{1,\Lambda} \int_a^b u''_\Lambda v''_\Lambda
\]

and using Poincaré’s inequality \( \int_a^b |u'_\Lambda|^p \geq C \int_a^b |u_\Lambda|^p \geq C \), one obtains that \( T_\Lambda \geq C > 0 \).

On the other hand since \((u_\Lambda)\) and \((v_\Lambda)\) are bounded in \( W^{1, p}([a, b]) \) and since \( \lambda_{1,\Lambda} \) and \( \lambda_{2,\Lambda} \) are bounded, one gets that \( T_\Lambda \) is bounded from above.

Furthermore using the equation defining \( T_\Lambda \), and the fact that \( u_\Lambda \) and \( v_\Lambda \) vanish on the end points, one gets that \( u'_\Lambda (a) \), \( u'_\Lambda (b) \), \( v'_\Lambda (a) \), \( v'_\Lambda (b) \) are bounded independently on \( \Lambda \). Integrating the first equation in (2.1) between \( a \) and \( x \) one gets

\[
|u'_\Lambda|^{p-2} u'_\Lambda (x) - |u'_\Lambda|^{p-2} u'_\Lambda (a) = \alpha \int_a^x u''_\Lambda + \Lambda \int_a^x u''\Lambda u''_\Lambda + \lambda_{1,\Lambda} \int_a^x u''_\Lambda
\]

using this on \( \{ x = b \} \) one gets \( \Lambda \int_a^b u''_\Lambda v''_\Lambda \leq C \), and by the positivity that \( |u'_\Lambda|^{p-2} u'_\Lambda (x) - |u'_\Lambda|^{p-2} u'_\Lambda (a) \leq C \). Doing the same between \( x \) and \( b \) one obtains that \( |u'_\Lambda| \leq C \) for some constant independent on \( \Lambda \). In the same manner, \( |v'_\Lambda| \leq C \).
Theorem 2.2. Assume that \((u_\Lambda, v_\Lambda)\) solves the system (2.1) in \([a, b]\). There exists \(x_\Lambda\) such that \(m_\Lambda = u(x_\Lambda) = v(x_\Lambda) \rightarrow 0\) as \(\Lambda\) goes to infinity, and \(x_\Lambda \rightarrow x_\infty \in ]a, b[\). Furthermore \(m_\Lambda^{2p} \rightarrow C_0 > 0\) and \(\Lambda^{\frac{1}{2p}} \min(x_\Lambda - a, b - x_\Lambda) \rightarrow +\infty\).

Proof of Theorem 2.2

By the previous estimates \((u_\Lambda)\) and \((v_\Lambda)\) are relatively compact in \(C([a, b])\). In particular up to subsequence, \(u_\Lambda\) and \(v_\Lambda\) are uniformly convergent. Let \((u_\infty, v_\infty)\) be the limit of such subsequence. By the identity \(\int_a^b |u_\infty|^p = \int_a^b |v_\infty|^p\), and by the uniform convergence, there exists \(x_\infty\) and \(x_\Lambda\) which tends to \(x_\infty\), such that \(m_\Lambda = u_\Lambda(x_\Lambda) = v_\Lambda(x_\Lambda)\) and \(x_\infty \in \{u_\infty(x) = v_\infty(x)\}\) = \(\gamma\).

To prove that

\[ \lim \sup \Lambda m_\Lambda^{2p} < \infty, \]

we argue by contradiction and define \(\tilde{u}_\Lambda = \frac{1}{m_\Lambda} u\left(\frac{y}{m_\Lambda^{\frac{1}{p}}} + x_\Lambda\right)\), \(\tilde{v}_\Lambda = \frac{1}{m_\Lambda} v\left(\frac{y}{m_\Lambda^{\frac{1}{p}}} + x_\Lambda\right)\) where \(y \in ]-x_\Lambda + am_\Lambda^{\frac{1}{p}}, -x_\Lambda + bm_\Lambda^{\frac{1}{p}}[\), interval which tends to \(\mathbb{R}\).

Then \((\tilde{u}_\Lambda, \tilde{v}_\Lambda)\) satisfy the equation

\[ \frac{\tilde{u}'_{\Lambda}}{p} + \frac{\tilde{v}'_{\Lambda}}{p} - \frac{\tilde{u}^p_{\Lambda} \tilde{v}^p_{\Lambda}}{p} - \alpha \frac{\tilde{u}^{p+2}_{\Lambda}}{(p+2)m^{p-2}_{\Lambda}} - \beta \frac{\tilde{v}^{p+2}_{\Lambda}}{(p+2)m^{p-2}_{\Lambda}} - \frac{T_\Lambda}{pm^{2p}_{\Lambda}} = 0, \]

where

\[ \frac{\tilde{u}^p_{\Lambda}}{p m^{p}_{\Lambda}} + \frac{\tilde{v}^p_{\Lambda}}{p m^{p}_{\Lambda}} = \frac{T_\Lambda}{pm^{2p}_{\Lambda}}. \]

Using the fact that \((u'_\Lambda)\) is bounded independently on \(\Lambda\), by the mean value’s theorem

\[ |\tilde{u}_\Lambda(y) - \frac{1}{m_\Lambda} u(x_\Lambda)| \leq \frac{1}{m_\Lambda^{\frac{1}{p}}} |u'_\Lambda|_\infty \]

and we have analogous estimates for \(\tilde{v}_\Lambda\), so using \(u_\Lambda(x_\Lambda) = m_\Lambda = v(x_\Lambda)\) one obtains that \(\tilde{u}_\Lambda\) goes to 1 uniformly, \(\tilde{v}_\Lambda\) goes to 1, finally passing to the limit in the equation (2.3) one gets that

\[ -\frac{1}{p} = 0, \]

a contradiction. We have obtained that \(\Lambda m_\Lambda^{2p}\) is bounded.

To end the proof, suppose by contradiction that \(\Lambda m_\Lambda^{2p} \rightarrow 0\) for a subsequence and let \(\tilde{u}_\Lambda(y) = \frac{1}{m_\Lambda} u_\Lambda(m_\Lambda y + x_\Lambda)\), then \(\tilde{u}_\Lambda\) and \(\tilde{v}_\Lambda\) satisfy the identity
constant before. We have obtained that \( \tilde{m} \) these constant are zero, which yields a contradiction with the identity (the sense, and so, also does its derivative. By passing to the limit one obtains that \( |\tilde{u}| \) converges strongly in \( L^p \), in particular \( |u'|^{p-2}u'' \) converges in the distributional sense, and so, also does its derivative. By passing to the limit one obtains that \((\tilde{u}'', \tilde{v}')\) tends locally uniformly to \((u, v)\) which satisfies \( |u'|^{p-2}u'' = 0 \) and \( |v'|^{p-2}v'' = 0 \), hence \( u = cte \), \( v' = cte \). Since \( u\) and \( v\) are bounded, these constant are zero, which yields a contradiction with the identity

\[
\frac{|u|}{p} + \frac{|v|}{p} = T_{\infty},
\]

where we have used the fact that \( T_{\infty} := \lim_{\Lambda \to \infty} T_{\Lambda}, \) by the estimates on \( T_{\Lambda} \) proved before. We have obtained that \( m^2p\Lambda \) is bounded from above by some constant \( > 0 \).

We finally prove that \( \Lambda \frac{2}{p} \min(x_{\Lambda} - a, b - x_{\Lambda}) \to +\infty \). Suppose for example and by contradiction that up to a subsequence \( \Lambda \frac{1}{p} (x_{\Lambda} - a) \to C_1. \) Define \( \tilde{u}_{\Lambda} = \Lambda \frac{1}{p} u(x_{\Lambda} - a) \). Then \( \tilde{u}_{\Lambda}, \tilde{v}_{\Lambda} \) satisfy

\[
|\tilde{u}|^{p-2}u' - \frac{\alpha}{\Lambda^{p+2}} \tilde{u}^{p-1}_{\Lambda} + \lambda_{1, \Lambda} \tilde{u}^{p-1}_{\Lambda} = 0,
\]

\[
|\tilde{v}|^{p-2}v' - \frac{\beta}{\Lambda^{p+2}} \tilde{v}^{p-1}_{\Lambda} + \lambda_{2, \Lambda} \tilde{v}^{p-1}_{\Lambda} = 0.
\]

We also get from the energy estimate

\[
\frac{|\tilde{u}'|}{p} + \frac{|\tilde{v}'|}{p} = \frac{\tilde{u}^{p+2}}{(p + 2)\Lambda^{p+2}} - \frac{\beta \tilde{v}^{p+2}}{(p + 2)\Lambda^{p+2}} + \lambda_{1, \Lambda} \tilde{u}^{p}_{\Lambda} + \lambda_{2, \Lambda} \tilde{v}^{p}_{\Lambda} = T_{\Lambda}.
\]

Remark as before that when \( \Lambda \) goes to infinity, \((\tilde{u}, \tilde{v})\) tends locally uniformly to some \((U, V)\), which satisfies
\[ |U'|^{p-2}U'' = V^p U^{p-1} \]
\[ |V'|^{p-2}V'' = U^p V^{p-1} \]

and \( U(-C_1) = V(-C_1) = 0 \). Note that if \((b - x_A) \Lambda \frac{1}{x} \rightarrow C_2 < \infty\) one has \( U(C_2) = 0 \) and then \( U'' = 0 \) and \( U \geq 0 \) implies \( U \equiv 0 \). We then assume that \( C_2 = +\infty \).

Furthermore using Fatou’s lemma one gets

\[
\int_{-C_1}^{\infty} V^p U^{p-1} \leq \liminf \int_{(a-x_A) \Lambda \frac{1}{x}}^{(b-x_A) \Lambda \frac{1}{x}} \tilde{v}^p \tilde{u}^{p-1} = \Lambda \int_{a}^{b} v^p u^{p-1} \leq C
\]

as stated in the proof of Lemma 2.1. Since \( U' \) is increasing and \( U \geq 0 \), if \( U \) is not identically zero, there exists \( C'_1 \geq C_1 \), such that \( U'(C'_1) > 0 \), and \( U = 0 \) on \([-C_1, -C'_1]\) then \( U(x) \geq U(-C'_1) + U'(-C'_1)(x - C'_1) = U'(-C'_1) + (x - C'_1) \). In the same manner there exists \( C''_1 \geq C_1 \), such that \( V(x) \geq V(-C''_1) + V'(-C''_1)(x - C''_1) \).

We have obtained that near \(+\infty\), \( U^{p-1}V^p \geq C x^{2p-1} \), which contradicts the fact that \( U^{p-1}V^p \) is integrable on \([-C_1, +\infty[\). Finally \( U = V = 0 \), a contradiction with the identity defining \( T_{\infty} \), when passing to the limit. We have obtained that \( \Lambda \min(x_A - a, b - x_A) \rightarrow +\infty \).

This ends the proof of Theorem 1.1.

3 Qualitative properties of the \( p \)-system in the one dimensional case : Proof of Theorem 1.2

In this section we want to prove the existence of non trivial solutions to the limit system (1.6). Note that the previous existence’s result is obtained under the assumption \((\lambda_{1, A}, \lambda_{2, A}) \leq C\). Theorem 1.2 is a consequence of several Theorems and propositions:

**Theorem 3.1.** There exists an entire solution for (1.6) such that \( U(x) = V(-x) \).

**Proof**
We argue as in [5], up to technical arguments due to the non linearity of the \( p \)-Laplacian, and due to the singularity (\( p < 2 \)) or the degeneracy (\( p > 2 \)).

Let us consider for \( R \) large the variational problem

\[
\inf_{\{(U,V) \in H^1([-R,R],[U(x)=V(-x),U(-R)=0, U(R)=R]\}} \frac{1}{p} \int_{-R}^{R} |U'|^p + \frac{1}{p} \int_{-R}^{R} |V'|^p + \frac{1}{p} \int |U|^p |V|^p.
\]

This problem admits a unique solution \((U_R, V_R)\).

We prove that \( U_R \) is non negative. Indeed one has

\[
|U'_R|^{p-2}U''_R = |U_R|^{p-2}U_R V_R|^p,
\]

and a symmetric equation for \( V_R \). Multiplying by \( U_R^- \) and integrating by parts, using the fact that \( U_R(-R) \) and \( U_R(R) \) are nonnegative, one gets that \( U_R^- = 0 \) and then \( U_R \geq 0 \). The same is valid for \( V_R \).

By the strong maximum principle of Vasquez [27], \( U_R > 0 \), on \([0, R], U'_R (-R) > 0 \) and \( |U'_R|^{p-2}U''_R \) is increasing implies that \( U'_R > 0 \) everywhere, finally \( U''_R \geq 0 \). Analogously \( V'_R < 0 \), then \( U_R - V_R \) vanishes only on zero. \( U_R - V_R > 0 \) on \([0, R] \) implies that \( |U'_R|^{p-2}U''_R - |V'_R|^{p-2}V''_R = U''_R - V''_R \leq 0 \) on \([0, R] \).

**First case** \( p \geq 2 \).

Using \( V'_R(x) = -U'_R(-x) \) one has for \( x > 0 \) \( |V'_R|(x) = U'_R(-x) < U'_R(x) \) and then \( |U'_R|^{p-2}U''_R \leq |V'_R|^{p-2}V''_R \) which implies since \( U'_R > 0 \), that \( U''_R - V''_R \leq 0 \).

From this one gets that \( U_R \geq V_R + x \) for \( x > 0 \). Indeed on 0, \( U_R(0) = V_R(0) \geq V_R(0) + x \) and on \( R, U_R(R) = R \geq V_R(0) + R = R \), in particular \( U_R(x) > x \) for \( x > 0 \).

We have obtained that

\[
|V'_R|^{p-2}V''_R \geq (x^+)^{p-1} \quad \text{since } V''_R \geq 0 \text{ anywhere else, in particular on } \mathbb{R}^-.
\]

Let \( \tilde{V} \) be the solution, (given by Lemma (4.3)), of

\[
|\tilde{V}'|^{p-2}\tilde{V}'' = x^p |\tilde{V}|^{p-2} \tilde{V},
\]

on \( \mathbb{R}^+ \), which is positive and satisfies \( \tilde{V}'(0) = -2 \). Let us extend \( \tilde{V} \) on \( \mathbb{R}^- \) by the linear function \(-2x + \tilde{V}(0)\). Since \( \tilde{V} \) hence defined is \( C^2 \) and is a solution of

\[
|\tilde{V}'|^{p-2}\tilde{V}'' = (x^+)^{p} |\tilde{V}|^{p-2} \tilde{V}
\]

on both \( \mathbb{R}^+ \) and \( \mathbb{R}^- \), one gets that it is a solution on \([0, R] \). For \( R \) large enough, \( V_R(-R) = R \leq -2(-R) + \tilde{V}(0) = \tilde{V}(-R) \), while \( V_R(R) = 0 \leq \tilde{V}(R) \) since \( \tilde{V} \) is positive. Using the comparison principle, since \( \tilde{V} \) and \( V_R \) are
respectively solution and sub-solution of the same equation, one gets that $V_R \leq \tilde{V}$ on $[-R, R]$. Using Harnack’s inequality, one gets that $(U_R, V_R)$ tends to a non trivial solution (since $U_R \geq x^+$), $(U, V)$ which satisfies $V(x) = U(-x)$.

**Second case** $p < 2$:

We begin to prove that $U_R'(x) \geq \frac{1}{1+2^{\frac{1}{p-1}}}$ for $x > 0$. Note that for $x > 0$

$$(U_R')^{p-1} \leq |V_R'|^{p-2}V_R'(x) + 2(U_R'(0))^{p-1}$$

which implies that $U_R'(x) \leq 2^{\frac{1}{p-1}}U_R'(0)$. Hence $U_R(R) = R \leq U_R(0) + 2^{\frac{1}{p-1}}RU_R'(0)$

On the other hand when $x \in \mathbb{R}^-$

$$U_R'(x) \leq U_R'(0)$$

which implies by the mean value’s theorem that $U_R(0) \leq U_R(-R) + U_R'(0)R$.

We have obtained that

$$U_R'(0) \geq \frac{1}{1+2^{\frac{1}{p-1}}}$$

as soon as $R$ is large enough. We derive from this that on $\mathbb{R}^+$ $|V_R'|^{p-2}V_R'' \geq \left(\frac{1}{1+2^{\frac{1}{p-1}}}\right)^p (x^+)pV_R^{p-1}$.

We now consider the solution $\tilde{V}$ of $|\tilde{V}'|^{p-2}\tilde{V}'' = \left(\frac{1}{1+2^{\frac{1}{p-1}}}\right)^p x^p \tilde{V}^{p-1}$ on $\mathbb{R}^+$, $\tilde{V} > 0$ given by Proposition 4.1 which satisfies $\tilde{V}'(0) = -2$, extended by $-2x + \tilde{V}(0)$ on $\mathbb{R}^-$.

One obtains as in the case $p \geq 2$ that $V_R \leq \tilde{V}$ and by Harnack’s inequality, one gets that $(U_R, V_R)$ tends locally uniformly to $(U, V)$ which is not identically zero.

**Lemma 3.2.** Suppose that $(U, V)$ satisfies (1.6). Then either $U' > 0$ and $V' < 0$ or $U' < 0$ and $V' > 0$. Furthermore there exists some constant $C$ such that $|U'|^p + |V'|^p \leq C$.

**Proof**

Clearly the identity

$$|U'|^p + |V'|^p - U^pV^p = T_\infty$$

holds for some finite constant $T_\infty$. Since $U'' \geq 0$, either $U' > 0$ or $U' < 0$ or there exists $x_1$ such that $U''(x) > 0$ for $x > x_1$ and conversely for $x < x_1$, and the same for $V'$.  

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Suppose that \( U' \) and \( V' \) have the same sign somewhere, then if this sign is positif, by the increasing behavior of \( U' \) and \( V' \), it is true also for \( x \) large.

In particular \( U(+\infty) = V(+\infty) = +\infty \).

**The case** \( p \geq 2 \)

Let \( \varphi = U + V \). For \( x \) large enough, \(|\varphi'|^{p-2} = (|U'| + |V'|)^{p-2} \geq c(|U'|^{p-2} + |V'|^{p-2})\) and then \(|\varphi'|^{p-2}\varphi'' \geq c(|U'|^{p-2} + |V'|^{p-2})(U'' + V'') \geq c(|U'|^{p-2}U'' + |V'|^{p-2}V'') \geq 2^{p-1}(V^p + U^p) \geq (U + V)^p\) as soon as \( x \) is large enough since \( U \) and \( V \) go to infinity.

Then \(|\varphi'|^{p-2}\varphi'' \geq \varphi^p\), for \( x \) large enough.

Multiplying by \( \varphi' \geq 0 \), one gets that \( \frac{d^2}{dx^2}(|\varphi'|^p) \geq c\frac{d^1}{dx}(|\varphi'|^{p+1}) \). Then \((\varphi')^p \geq c(\varphi^{p+1} + 1)\), and since \( \varphi \) tends to infinity, for \( x \) large enough, \((\varphi')^p(x) \geq \frac{p(p+1)}{2} \).

This implies \( \varphi' \geq c\varphi^{\frac{p+1}{p}} \), and then for \( x \) large enough, \( \frac{d}{dx}(-\varphi^-\frac{1}{p}) \geq c_p \), which would imply that \( \varphi \) becomes negative for \( x \) large enough.

**The case** \( p < 2 \)

Let \( \varphi = ((U')^{p-1} + (V')^{p-1})^{\frac{1}{p-1}} \), note that \( \varphi^{p-1} = (U')^{p-1} + (V')^{p-1} \), that \( \varphi \geq U' + V' \) and \( \varphi \leq 2^{\frac{1}{p-1}}(U' + V')^p \).

One has

\[
(\varphi^{p-1})' = \frac{d}{dx}((U')^{p-1} + (V')^{p-1}) \\
= (p-1)(U'^{p-1}V + V'^{p-1}U) \\
\geq (V + U)^p,
\]

for \( x \) large by the behavior at infinity of \( U \) and \( V \).

Multiplying by \( U' + V' > 0 \) one obtains that

\[
\left( \frac{d}{dx}(\varphi^{p-1}) \right)(U' + V') \geq \frac{1}{p + 1} \frac{d}{dx}((V + U)^{p+1}).
\]

On the other hand, by the positivity of \( \frac{d}{dx}(\varphi^{p-1}) \), and \( U' + V' \) one has

\[
\frac{d}{dx}(\varphi^{p-1})(U' + V') \leq \frac{p-1}{p} \frac{d}{dx}(\varphi^p)
\]

hence integrating and using the fact that \( U + V \) goes to infinity when \( x \) goes to \( +\infty \), one gets that there exists some constant \( c_p \) such that for \( x \) large enough

\[
(U' + V')^p \geq c_p \varphi^p \geq c_p(U + V)^{p+1}
\]

We end as in the case \( p \geq 2 \) and get an absurdity.
If the sign of $U'$ and $V'$ are both negative somewhere they are both negative for $x < -x_1$. By considering the invariance of the equation by changing $x$ in $-x$, and reasoning as above one gets a contradiction.

We have obtained that up to exchanging $U$ and $V$, $U' > 0$ and $V' < 0$.

Suppose that $U' \to +\infty$ somewhere, then it occurs at $+\infty$ since $U'$ is increasing, in particular $U$ goes to $+\infty$ at $+\infty$, and using \[3.1\] so does $UV$. Then $(V'|V'|^{-2})' = (p-1)(UV)^{p-1}U \to +\infty$, which implies that $V'$ goes to $+\infty$ at $+\infty$, a contradiction with $V' < 0$. We have obtained that $U'$ is bounded. If $V' \to -\infty$ somewhere, it occurs at $-\infty$, then $V$ goes to $+\infty$ at $-\infty$, by \[3.1\] $UV$ goes to $+\infty$ at $-\infty$ and $|U'|^{-2}U' = (UV)^{p-1}V \to +\infty$ at $-\infty$, hence $U'$ becomes $< 0$ for $x$ large negative, a contradiction.

We have obtained that $|U'| + |V'| \leq C$.

**Proposition 3.3.** let $(U, V)$ be a solution such that $U' > 0$ and $V' < 0$. Then $V^pU^{p-1} \to 0$, $U^pV^{p-1} \to 0$ at $\pm\infty$. Furthermore the following assertions hold:

$$
U(-\infty) = 0, U'(-\infty) = 0, U'(+\infty) = (T_\infty)^{1\over p}, \quad (3.2)
$$

$$
V(+\infty) = 0, V'(+\infty) = 0, V'(-\infty) = -(T_\infty)^{1\over p} \quad (3.3)
$$

Proof

Since $d\over dx(|U'|^{-2}U') \geq 0$, and $d\over dx(|V'|^{-2}V') \geq 0$, $U'$ and $V'$ have a limit at infinity. Furthermore $V' \leq 0$ and is increasing so it converges at $+\infty$. Its limit must be zero since if not for $x$ large enough, $V' \leq -m < 0$ and $V$ would become negative for $x$ large.

By Lemma 1.2, $U'$ is bounded. Furthermore it has a positive finite limit at infinity, and by \[3.1\] it-s so does $U^pV^p$. Then $U$ goes to infinity, more precisely $U$ behaves like an increasing linear function. Then $U^{p-1}V^p = {1\over b}(U^pV^p) \to 0$. Furthermore for $x > x_o$ and $x_o$ large, $|V'|^{-2}V'' - V^{p-1} \geq 0$.

Let us consider $W = V(x_o)e^{-x+x_o}$ which satisfies $|W'|^{-2}W'' - W^{p-1} \leq 0$. Using lemma 4.1 in the appendix one gets that $V \leq W$ on $[x_o, \infty]$ and then $\lim_{x \to +\infty} UV = 0$ as well as $\lim_{x \to +\infty} U^pV^{p-1} = 0$. Since $V' \to 0$ at infinity, \[3.1\] implies that $|U'|p(+\infty) = T_\infty$. In particular $T_\infty > 0$.

A symmetric result holds near $-\infty$ exchanging $U$ and $V$.

**Lemma 3.4.** Let $U$ and $V$ be as in Proposition 3.3. There exist some positive constants $m, M, k, K$ which depend on $T_\infty$, such that

$$
me^{-Kx^2} \leq U(x) \leq Me^{-kx^2}
$$

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and some positive constants $c$, $C$ such that $c|x|U(x) \leq U'(x) \leq C|x|U(x)$, for $x$ large negative, and analogous inequalities for $v$ near $+\infty$.

Furthermore $U$ has two asymptotic lines $y = 0$ at $-\infty$ and $y = (T_\infty)^{\frac{1}{p}}x + b_1$ for some $b_1 \in \mathbb{R}$, at $+\infty$. Similarly $V$ has asymptotic lines $0$ at $+\infty$ and $y = -(T_\infty)^{\frac{1}{p}}x + b_2$ for some $b_2$ at $-\infty$.

**Remark 3.5.** The constant $k$ and $K$ can be explicitly determined in function of $T_\infty$.

Proof:

Let $k$ be such that $V(x) \geq 2k|x|$ near $-\infty$, define $W(x) = C e^{-k|x|^2}$ which satisfies $|W'|^{p-2}W'' \leq V^pW^{-1}$ for $x$ large enough negative, where the constant has been chosen in order that $W(-M) = U(-M)$, and $M$ is large enough $> 0$. By Lemma 3.1, using the fact that $W'$ and $U'$ are bounded, one gets that $U \leq W$.

The lower bound can be obtained by considering some $K$ such that near $-\infty V(x) \leq K|x|$. Consider $W(x) = C e^{-K|x|^2}$ where $C$ is chosen so that $W(-M) = U(-M)$ and $M$ is large. $|W'|^{p-2}W'' \geq V^pW^{-1}$ as soon as $x$ is large enough negative. From this one derives that $W \leq U$.

We prove the assertions concerning $U$ and $U'$. We begin to prove that $U' \geq U$ for $x$ large negative. Indeed let us observe that for $|x|$ large negative $|U'|^{p-2}U'' \geq U^{-1}$ and then multiplying by $U'$ and integrating, using $U(-\infty) = U'(-\infty) = 0$, one obtains $(U')^p \geq U^p$.

To prove a better estimate, observe that by the behavior of $V$ at $-\infty |U'|^{p-2}U'' \geq C|x|^pU^p$.

We multiply by $U'$ and prove that $\int_{-\infty}^{-x} |t|^p U^{-1}(t)U'(t)dt \geq C|x|^pU^p$

Indeed $\int_{-\infty}^{-x} |t|^p U^{-1}(t)U'(t)dt = \int |t|^p \frac{U(t)}{p} \frac{-x}{p} dt + \int_{-\infty}^{-x} |t|^{p-1} U(t)dt = |x|^{\frac{p}{p}(-x)} + \int_{-\infty}^{-x} |t|^{p-1} U(t)dt \geq |x|^{\frac{p}{p}(-x)}$, from this one yields

$$\int_{-\infty}^{-x} |t|^p U^{-1}(t)U'(t)dt \geq |x|^{\frac{p}{p}(-x)},$$

and then using the fact that near $-\infty$, $U'$ and $|x|^pU^p$ tend to zero, one gets that $(U')^p - C|x|^pU^p \geq 0$.

In the same manner by the behaviour of $V$ at $-\infty$ there exists $C$ such that $|U'|^{p-2}U'' \leq C|x|^pU^{p-1}$, here we use $\int_{-\infty}^{-x} |t|^p U^{p-1}(t)U'(t)dt = \int |t|^p \frac{U(t)}{p} \frac{-x}{p} dt + \int_{-\infty}^{-x} |t|^{p-1} U(t)dt \leq |x|^{\frac{p}{p}(-x)} + \frac{1}{|x|} \int_{-\infty}^{-x} |t|^p U^{p-1}(t)U'(t)dt$. From this as soon
as $|x| > 2$, 
\[
\int_{-\infty}^{-x} |t|^p U^{p-1}(t) U'(t) dt \leq 2|x|^p \frac{U^p}{p}(-x).
\]

We have obtained the estimate on the right $U' \leq CU|x|$.

To deduce from this the asymptotic of $U$ and $V$, we use the previous estimates on $U$ for $V$ near $+\infty$. So we have $|V'| \geq C|x| e^{-kx^2}$ and then by (3.1) one derives that $(U')^p - T_\infty \geq -C|x| e^{-p k x^2}$, hence $U'(x) - T_\infty \geq -C|x| e^{-kx^2}$ which implies by integrating that $U(x) - T_\infty x \geq -C$ for $x$ large positive.

Since $U$ is convex, $(U' - T_\infty \frac{1}{p})$ is increasing and since it tends to $0$ at infinity, it is negative, hence $U - T_\infty \frac{1}{p} x$ is decreasing. Finally $U - (T_\infty \frac{1}{p}) x$ is decreasing and minorated, hence defining $b_1 = \lim_{x \to +\infty} (U - T_\infty \frac{1}{p} x)$, one has $(U - T_\infty \frac{1}{p} x) \geq b_1$. Of course a symmetric result holds for $V$.

**Proposition 3.6.** Let $(U, V)$ be a solution of (1.1), with $U(0) = V(0) = 1$. Then $V(y) = U(-y)$.

We assume that $U' > 0$, hence we are in the hypothesis of the previous propositions. We can assume that $b_1 \geq b_2$, since if not one can replace $(U(x), V(x))$ by $(V(-x), U(-x))$ which exchanges $b_1$ and $b_2$. We use the sliding method of Berestycki and Nirenberg, [2].

Let $I_\lambda = \{ x, x > \lambda \}$ and $U_\lambda(x) = U(2\lambda - x)$, $V_\lambda(x) = V(2\lambda - x)$ Let $w_1 = U - V_\lambda$, $w_2 = U_\lambda - V$. We prove in what follows that for $\lambda$ large enough and $x \in I_\lambda$, $w_1(x) > 0$ as well as $w_2 > 0$. From the asymptotic behaviour of $U$ and $V$, and since $U$ is convex, $U(x) \geq T_\infty \frac{1}{p} x + b_1$, and by the asymptotic behavior of $V$ there exists $K$ such that $V(x) \leq -T_\infty \frac{1}{p} x + K$, this implies that $w_1(x) \geq T_\infty \frac{1}{p} (x + (2\lambda - x)) + b_1 - K$ for $x \in I_\lambda$. So by taking $\lambda$ such that $\lambda T_\infty \frac{1}{p} > K - b_1$ one gets that if $x \in \lambda, 2\lambda$, $w_1(x) \geq T_\infty \frac{1}{p} (\lambda + 0) + b_1 - K > 0$ and for $x > 2\lambda$, $w_1(x) \geq T_\infty \frac{1}{p} (x + x - 2\lambda) + b_1 - K \geq 2\lambda T_\infty \frac{1}{p} + b_1 - K > 0$.

We now derive from this that $w_2$ is also $> 0$ in the same $I_\lambda$ for large values of $\lambda$. Indeed, we have
\[
|U_\lambda'|^{p-2} U_\lambda'' - |V'|^{p-2} V'' = U^p(U_\lambda^{p-1} - V^{p-1}) + U_\lambda^{p-1}(V^p - U_p).
\]

Multiplying this by $w_2^\gamma$, integrating between $\lambda$ and $x$ and using $(U_\lambda - V)(\lambda) = (U - V)(\lambda) = w_1(\lambda) > 0$ one gets
implies that \( \lambda \) behaviour, there exists \( w \) strong maximum principle \( \lambda < \) has if 

\[
- \int_\lambda^x (|U'_\lambda|^{p-2}U'_\lambda - |V'|^{p-2}V')(w_2^-)' + \left[ \frac{1}{p-1}(|U'_\lambda|^{p-2}U'_\lambda - |V'|^{p-2}V')(w_2^-) \right]_\lambda^x \\
= \int_\lambda^x U^p(U_{-1}^p - V^{p-1})w_2^- \\
+ \int_\lambda^x U_{-1}^p(V - U)p)w_2^- \\
\leq 0
\]

Using \(- \int_\lambda^x (|U'_\lambda|^{p-2}U'_\lambda - |V'|^{p-2}V')(w_2^-) \geq 0, w_2^- (\lambda) = 0 \) and \( w_2(\infty) = 0 \), as well as the fact that \( U' \) and \( V' \) are bounded, letting \( x \) go to infinity, one gets that \( w_2^- = 0 \) and then \( w_2 > 0 \) for \( x \in I_\lambda \) and \( \lambda \) large enough.

We now define \( \lambda^* = \inf\{\lambda > 0, w^\mu_1(x) > 0 \text{ in } I_\mu, \text{ for all } \mu > \lambda\} \). By the previous observations \( w^\mu_2 > 0 \) in \( I_\mu \) for all \( \mu > \lambda^* \). Since \( U(0) = V(0) = 1 \), one has if \( \lambda < 0 \), by the increasing behaviour of \( U - V \), \( (U - V)(\lambda) < 0 \), which implies that \( \lambda^* \geq 0 \). We want to prove that \( \lambda^* = 0 \).

Let us observe that by continuity \( w^\lambda_1^* \geq 0 \) and \( w^\lambda_2^* \geq 0 \) on \( I_\lambda^* \). By the strong maximum principle \( w^\lambda_1^* > 0 \) and \( w^\lambda_2^* > 0 \) in \( I_\lambda^* \). By the asymptotic behaviour, there exists \( B_1 \) such that for \( x < B_1 < 0, V(x) + \frac{1}{\lambda_1^*}x - b_2 < \frac{b_1 - b_2}{2} \).

Take \( A = \sup(2\lambda^* - B_1, \lambda^*) \) then for \( x > A \), and for \( 0 < \lambda < \lambda^* \),

\[
V(2\lambda - x) + \frac{1}{\lambda_1^*}(2\lambda - x) - b_2 < \frac{b_1 - b_2}{2},
\]
hence for \( x > A \) and \( \lambda \in [0, \lambda^*], w^\lambda_1^*(x) - \frac{1}{\lambda_1^*}2\lambda \geq \frac{b_1 - b_2}{2}.
\]

We now observe that \( \inf_{[\lambda^*, A]} w^\lambda_1^* = m > 0 \), indeed \( w^\lambda_1^*(\lambda^*) = U(\lambda^*) - V(\lambda^*) > U(0) - V(0) \) since \( U' > 0, V' < 0, U(0) = V(0) \) and \( \lambda^* > 0 \). By the uniform continuity of \( V \) in a compact set, there exists \( \eta < \lambda^* \) such that for \( |\lambda - \lambda^*| < \eta, \) for all \( x \in [\lambda^* - \eta, A], \) one has \( |V(2\lambda^* - x) - V(2\lambda - x)| \leq \frac{m}{2} \) and then for \( x > \lambda > \lambda^* - \eta \) and \( x < A, U(x) - V(2\lambda - x) \geq \frac{m}{2} \). Finally \( \inf_{[\lambda, \lambda^*]} w^\lambda_1^* > 0 \) for \( \lambda^* - \eta < \lambda < \lambda^* \), and then \( w^\lambda_1^* > 0 \) on a neighborhood on the left of \( \lambda^* \). This contradicts the definition of \( \lambda^* \). We have obtained \( \lambda^* = 0 \) and then \( U(x) \geq V(-x) \) for \( x \geq 0 \).

Since we have seen before that \( w^0_1 \geq 0 \) implies \( w^0_2 \geq 0 \), \( U(-x) \geq V(x) \) for \( x > 0 \). We have obtained that \( U(x) \geq V(-x) \) for \( x \in \mathbb{R} \).

Since \( U(0) = V(0), U(x) - V(-x) \) reaches its minimum at zero. This implies that \( (w^0_1)''(0) = U'(0) + V'(0) = 0 \). By the strong comparison principle one gets that \( U(x) = V(-x) \). We have obtained in the same time that \( b_1 = b_2 \).

Part 3) in Theorem 12 is contained in the
Proposition 3.7. Suppose that \( \phi, \psi \) are bounded solutions of
\[
\begin{align*}
\{ & (|U'|^{p-2} \phi')' = (p-1)U^{p-2}V'\phi + pU^{p-1}V'\psi \\
& (|V'|^{p-2} \psi')' = (p-1)V^{p-2}U'\phi + pU^{p-1}V'\phi.
\end{align*}
\]
Then there exists some constant \( c \) such that \( (\phi, \psi) = c(U', V') \).

Proof

For personal convenience we use minuscule letters \( (u, v) \) in place of \( (U, V) \).

We do not distinguish the case \( p > 2 \) or \( p < 2 \) for the moment. Let \( \bar{\phi}, \bar{\psi} \) be defined as \( \phi = u'\bar{\phi}, \psi = v'\bar{\psi} \). One has
\[
\begin{align*}
(|u'|^{p-2} \phi')' &= (|u'|^{p-2} u'') \bar{\phi} + p|u'|^{p-2} u'' \bar{\phi}' \\
& \quad + |u'|^{p-2} u'' \bar{\phi} \\
& = pu^{p-1} v' \bar{\phi} + pu^{p-1} v' \bar{\phi} + (p-1)u^{p-2} v' u' \bar{\phi} \\
& \quad + |u'|^{p-2} u'' \bar{\phi}.
\end{align*}
\]
On the other hand using the equation satisfied by \( \phi \) one gets
\[
p|u'|^{p-2} u'' \bar{\phi}' + |u'|^{p-2} u' \bar{\phi}'' = pu^{p-1} v^{p-1} v' (\bar{\psi} - \bar{\phi}).
\]
In the same manner for \( v \)
\[
p|v'|^{p-2} v'' \bar{\psi}' + |v'|^{p-2} v' \bar{\psi}'' = pu^{p-1} v^{p-1} v' (\bar{\phi} - \bar{\psi}).
\]
Multiplying the first equation by \( u' \bar{\phi} \) and the second one by \( v' \bar{\psi} \), one gets
\[
p|u'|^{p-2} u' u' \bar{\phi} \bar{\phi}' + |u'|^{p} \bar{\phi} \bar{\phi}'' + p|v'|^{p-2} v' v' \bar{\psi} \bar{\psi}' + |v'|^{p} \bar{\psi} \bar{\psi}'' = -p(uv)^{p-1} u' v'(\bar{\psi} - \bar{\phi})^2.
\]
Let us now observe that
\[
p|u'|^{p-2} u' u'' \bar{\phi} \bar{\phi}' + |u'|^{p} \bar{\phi} \bar{\phi}'' = (|u'|^{p} \bar{\phi} \bar{\phi}')' - |u'|^{p} (\bar{\phi}')^2.
\]
Claim: \( |u'|^{p} \bar{\phi} \bar{\phi}' \) and \( |v'|^{p} \bar{\psi} \bar{\psi}' \) go to zero at \( +\infty \) and \( -\infty \)

This claim will end the proof since then we will have
\[
0 = \lim_{\infty} [(|u'|^{p} \bar{\phi} \bar{\phi}') + (|v'|^{p} \bar{\psi} \bar{\psi}')] = \int_{\mathbb{R}} |u'|^{p}(\bar{\phi}')^2 + |v'|^{p}(\bar{\psi}')^2 \\
- p \int_{\mathbb{R}} (uv)^{p-1} u' v' (\bar{\psi} - \bar{\phi})^2 \geq 0.
\]
and since $u'v' < 0$ this will imply $\bar{\phi}' = \bar{\psi}' = 0$ and $\bar{\phi} = \bar{\psi}$.

In the sequel we prove the claim for $u$ and $\phi$. The result for $v$ and $\psi$ can be done by obvious symmetric arguments.

**Proof of the claim for $u$ and $\bar{\phi}$**

\[ |u'|^p \bar{\phi} \bar{\phi}' = |u'|^{p-2}(u' \bar{\phi})(u' \bar{\phi}') \]
\[ = |u'|^{p-2} \phi u' \left( \frac{\phi'}{u'} - \frac{\phi u''}{u'^2} \right) \]
\[ = |u'|^{p-2} \phi \phi' - |\phi|^2 \frac{|u'|^{p-2} u''}{u'} \]
\[ = |u'|^{p-2} \phi \phi' - \phi^2 \frac{u^{p-1}v^p}{u'} \]

We consider separately the cases $+\infty$ and $-\infty$.

**The case $+\infty$.**

Since $u$ increases like a linear function, $u'$ is minorated by some positive constant and $v$ goes exponentially towards zero, the term $\phi^2 \frac{u^{p-1}v^p}{u'}$ on the right goes to zero.

On the other hand $|u'|^{p-2} \phi'$ tends to zero. Indeed, its derivative is integrable for $x$ large by the asymptotic behavior of $u$ and $v$, and the fact that $\phi$ and $\psi$ are bounded. So it has a limit. Suppose that the limit is $l \neq 0$, then $\phi' \sim \frac{l}{T_\infty^p}$, this contradicts $\phi$ bounded. Finally $\lim_{x \to +\infty} |u'|^p \bar{\phi} \bar{\phi}'(x) = 0$.

**The case $-\infty$.**

We now distinguish the case $p \geq 2$ and $p < 2$

**The case $p \geq 2$.**

Let us recall that there exist some positive constants $M, k, c$ such that near $-\infty$, $u \leq Me^{-kx^2}$, and $u' \geq c|x|u$ for $x$ large enough negative. In particular since $v$ is linear at infinity $\frac{|u^{p-1}v^p|}{u'} \leq c|x|^{p-1}e^{-(p-2)kx^2}$.

Using $\phi$ bounded, one gets that $\phi^2 \frac{u^{p-1}v^p}{u'}$ goes to zero at $-\infty$. Furthermore $|u'|^{p-2} \phi'$ has a limit at $-\infty$ by the equation, if this limit was $\neq 0$, this would imply that $\phi'$ goes to $\pm \infty$ and would contradict $\phi$ bounded. All this implies that $|u'|^p \bar{\phi} \bar{\phi}'$ tends to zero at $-\infty$.

**The case $p < 2$**

This case is much more involved and requires several steps.

**Step 1:** There exists $t_p$ which goes to $-\infty$ such that $(|u'|^{p-2} \phi')(t_p) \to 0$. 

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Suppose for a while that there does not exist \( t_p \) which goes to \(-\infty\) such that 

\[ |u'|^{p-2} \phi'(t_p) \to 0. \]

Then there exists \( C > 0 \) such that for all \( t \) large negative either 

\[ (|u'|^{p-2} \phi')(t) \geq C \] or 

\[ (|u'|^{p-2} \phi')(t) \leq -C. \]

One assumes that we are in the first case and will give at the end the arguments in the other case. Then \( \phi' > 0 \) near \(-\infty\), hence \( \phi \) has a finite limit since \( \phi \) is bounded. We begin to prove that \( \phi \) tends to zero at \(-\infty\). Suppose \( \phi \) does not tend to zero, then there exists \( m > 0 \) such that either \( \phi > m \) or \( \phi \leq -m \) for \( x \) large negative. In the first case for some constant \( c > 0 \) which can vary from one line to another, \( (|u'|^{p-2} \phi') \geq cu^{p-2} \geq cu^{p-2} |x|^p \) since the last term \( pu^{p-1}v^{p-1} \psi \) tends to zero. Integrating between \(-x \) and \(-x_o \) large negative, one gets for \( x > x_o \), 

\[ |u'|^{p-2} \phi'(-x_o) - |u'|^{p-2} \phi'(-x) \geq c \int_{-x_o}^{-x} u^{p-3}(t)u'(t)|t|^{p-1}dt \geq c|x|^{p-1}u^{-2}(-x). \]

Indeed

\[ \int_{-x}^{-x_o} u^{p-3}(t)u'(t)|t|^{p-1}dt = \frac{u^{p-2}|t|^{p-1}}{p-2} |_{-x}^{-x_o} + \frac{p-1}{p-2} \int_{-x}^{-x_o} u^{p-2}(t)|t|^{p-2}dt \]

\[ \geq \frac{u^{p-2}(-x)|x|^{p-1}}{p-2} - \frac{p-1}{(2-p)|x|^2} \int_{-x}^{-x_o} u^{p-3}u'|t|^{p-1}dt \]

which implies that 

\[ |u'|^{p-2} \phi'(-x) \leq C_1 - C_2 |x|^{p-1}u^{p-2} \]

and then in particular 

\[ \phi' \leq -C_3x^{2-p}x^{p-1} = -C_3x. \]

This contradicts \( \phi \) bounded. In the same manner if \( \phi \leq -m < 0 \) one gets 

\[ |u'|^{p-2} \phi'(-x_o) - |u'|^{p-2} \phi'(-x) \leq -c|x|^{p-1}u^{p-2} \]

which implies \( \phi' \geq C_3x \), this still contradicts \( \phi \) bounded. So we are in the hypothesis that \( \phi \) tends to zero and 

\[ |u'|^{p-2} \phi' \geq C > 0. \]

Then \( \phi(x) \geq \int_{-\infty}^{-x} C(u')^{2-p}(t)dt \geq \int_{-\infty}^{-x} |t|^{1-p}u^{1-p}(t)u'(t)dt. \]

Observe that 

\[ \int_{-\infty}^{-x} |t|^{1-p}u^{1-p}(t)u'(t)dt \geq c|x|^{1-p}u^{2-p}. \]

Indeed

\[ \int_{-\infty}^{-x} |t|^{1-p}u^{1-p}u'(t)dt = \frac{1}{2-p} \int_{-\infty}^{-x} |t|^{1-p} \frac{d}{dt}(u^{2-p})dt \]

\[ = \frac{1}{2-p} \left[ |t|^{1-p}u^{2-p} \right]^{-x}_{-\infty} - \frac{p-1}{(2-p)|x|^2} \int_{-\infty}^{-x} |t|^{1-p}u^{1-p}u'(t)dt. \]

This ends the proof by taking \( |x| \) large enough.

We have obtained that 

\[ \phi(x) \geq C|x|^{1-p}u^{2-p} \]

and replacing in the equation satisfied by \( \phi \) one gets 

\[ |u'|^{p-2} \phi'(-x_o) - |u'|^{p-2} \phi'(-x) \geq C \int_{-x_o}^{-x} |t|^{1-p}|t|^{p}dt = C|x|^2. \]
From this one derives that \(|u'|^{p-2}\phi'(-x) \leq -Cx^2\), a contradiction with the assumption.

The case where \(|u'|^{p-2}\phi(-x) \leq -C < 0\) for \(x\) large enough can be recovered by changing \(\phi\) in \(-\phi\), noting the fact that the previous computations do not use the sign of \(\psi\).

We have obtained that there exists \(t_p\) which goes to \(-\infty\), such that \((|u'|^{p-2}\phi')(t_p) \to 0\).

**Step 2:** \((u')^{p-2}\phi\phi'\) and \(u^{p-1}\phi' v\) both tend to zero at \(-\infty\).

We multiply the equation satisfied by \(\phi\), by \(\phi\) and integrate between \(t_p\) and \(t_{p+1}\), where \(t_p\) is some subsequence decreasing to \(-\infty\), given by step 1. One obtains

\[
[u'|^{p-2}\phi\phi']_{t_{p+1}}^{t_p} = \int_{t_{p+1}}^{t_p} ((p-1)u^{p-2}v^p\phi^2 + |u'|^{p-2}(\phi')^2) + \int_{t_{p+1}}^{t_p} pu^{p-1}v^{p-1}\phi\psi,
\]

and since \(u^{p-1}v^{p-1}\phi\psi\) is absolutely integrable by the estimates on \(u\) and \(v\) one gets with the positivity of \((p-1)u^{p-2}v^p\phi^2 + |u'|^{p-2}(\phi')^2\) and summing on \(p\) that \(u^{p-2}v^p\phi^2\) and \(|u'|^{p-2}(\phi')^2\) are integrable. Finally for all \(s\) and \(t\) going to \(-\infty\) \([u'|^{p-2}\phi\phi']_{t}^{s}\) tends to zero, hence \(|u'|^{p-2}\phi\phi'\) has a limit, and since it possesses a subsequence which tends to zero, this limit is zero.

We now prove that \(\phi^2 u^{p-1}v^p\) has a finite limit. For this it is enough to prove that its derivative is integrable at \(-\infty\).

\[
(\phi^2 u^{p-1}v^p)' = (\phi^2) \left( \frac{u^{p-1}v^p}{u'} \right)' + 2\phi\phi' u^{p-1}v^p \]

\+

\[ (p-1)u^{p-2}v^p + \frac{u^{p-1}v^p - (u')^2}{u'} \phi^2 + 2\phi\phi' u^{p-1}v^p \]

\=\n
\[ (p-1)u^{p-2}v^p + \frac{u^{p-1}v^p - (u')^2}{u'} - u''(u')^{p-2} \left( \frac{u}{u'} \right) v^p \phi^2 + 2\phi\phi' u^{p-1}v^p \]

\=\n
\[ (p-1)u^{p-2}v^p + \frac{u^{p-1}v^p - (u')^2}{u'} - u^{p-2}v^p \left( \frac{u}{u'} \right)^{p-2} \phi^2 + 2\phi\phi' u^{p-1}v^p. \]

Each of the first three terms above can be majorized near \(-\infty\) by \(Cu^{p-2}v^p\phi^2\) and then are integrable near \(-\infty\).
Lemma 4.1. We begin with the comparison principle used in section 3.

For \((\phi^2)' \frac{u^{p-2}}{\partial u} \) we use Cauchy Schwarz’s inequality as follows

\[
\left| \phi' \frac{u^{p-1}}{u'} \right| = |\phi'| u' \left| \frac{u^{p-2}}{\partial u} u^{p-1} \phi \right| \\
\leq C |\phi'| (u')^{\frac{p-2}{2}} |xu|^{\frac{p}{2}} u^{p-1} \phi | \\
\leq C |\phi'| (u')^{\frac{p-2}{2}} |x|^{\frac{p}{2}} |\phi| \\
\leq C (\phi')^2 (u')^{p-2} + C v^p u^{p-2} \phi^2
\]

We deduce that since \(\phi^2 u^{p-2} v^p\) is integrable near \(-\infty\), so is \(\phi^2 u^{p-1} v^p\) and then it tends to zero.

Of course we would obtain symmetric properties for \(\psi\) and \(v\) near \(+\infty\). This ends the proof.

4 Appendix : Global existence uniqueness and qualitative results for solutions of \( |y'|^{p-2} y'' = x^p |y|^{p-2} \) on \( \mathbb{R}^+ \)

We begin with the comparison principle used in section 3.

Lemma 4.1. Suppose that \(a\) is some continuous and bounded function such that \(a(x) > 0\) for \(x > x_o\). Suppose that \(W'\) and \(V'\) are bounded at infinity, that \(W(x_o) = V(x_o)\), or \(W'(x_o) = V'(x_o)\), \(\lim_{x \to +\infty} W(x) = \lim_{x \to +\infty} V(x) = 0\), and \(|W'|^{p-2} W'' - a(x)|W|^{p-2} W \leq 0\) for \(x > x_o\),

\( |V'|^{p-2} V'' - a(x)|V|^{p-2} V \geq 0 \). Then \(V \leq W\) for \(x > x_o\).

Proof : Let us multiply the difference of the equations satisfied by \(V\) and \(W\), by \((V-W)^+\) and integrate by parts, one gets \(\int_{x_o}^x (|V'|^{p-2} V'' - |W'|^{p-2} W'')((V-W)^+) + \int_{x_o}^x a(t)(|V|^{p-2} V - |W|^{p-2} W)(V-W)^+ (t) dt \leq 0\). Passing to the limit when \(x\) goes to infinity and using \((|V'|^{p-2} V'' - |W'|^{p-2} W'')((V-W)^+) > 0\) one gets in particular that \(\int_{x_o}^x a(t)(|V|^{p-2} V - |W|^{p-2} W)(V-W)^+ (t) dt = 0\) and then \(V \leq W\) on \([x_o, +\infty]\).

Proposition 4.2. For \((y_o, y_1)\) given there exists a unique global solution on \(\mathbb{R}^+\) of

\[ |y'|^{p-2} y'' = x^p |y|^{p-2} y, \ y(0) = y_o, \ y'(0) = y_1. \]
Proof of Proposition 4.2

We begin to prove local existence and uniqueness of solutions. Suppose that \( x_0 \geq 0 \).

Let \( y_0 = y(x_0) \), \( y_1 = y'(x_0) \). If \( y'(x_0) \neq 0 \), Cauchy Lipschitz theorem can be applied and provides local existence and uniqueness of the solution.

Suppose that \( y_0 = y_1 = 0 \). Then we use some strict maximum principle to get that \( y \equiv 0 \) on the right and the left of \( x_0 \).

Suppose indeed that \( y \) is not identically zero. We begin to prove that if \( y(x_0 + h) > 0 \) for some \( h > 0 \), then \( y \geq 0 \) on \( [x_0, x_0 + h] \). We multiply the equation by \( y^- \) and integrate between \( x_0 \) and \( x_0 + h \), we get

\[
\int_{x_0}^{x_0 + h} \gamma |y'|^{p-2}y'(-y') - p \int_{x_0}^{x_0 + h} x^p |y|^{p-2}yy^- + ||y'|^{p-2}y'(-y')|_{x_0}^{x_0 + h} = 0 \]

and since \( y^- = 0 \) on \( x_0 \) and \( x_0 + h \) one obtains

\[
\int_{x_0}^{x_0 + h} |(y^-)'|^{p-2} + p \int_{x_0}^{x_0 + h} x^p |y|^{p-2} |y^-| = 0 \quad \text{and} \quad y^- = 0 \quad \text{on} \quad [x_0, x_0 + h].
\]

So we are in the situation where \( y \geq 0 \) on \( [x_0, x_0 + h] \). We prove that if \( y \) is not identically zero on the right, there is a contradiction with \( y'(x_0) = 0 \). Let \( \gamma \) be such that \( \gamma > x \) on \( [x_0, x_0 + \delta] \), \( \beta \) such that \( \beta(e^{\gamma \delta} - 1) < y(x_0 + \delta) \), and consider \( w = \beta(e^{\gamma (x-x_0)} - 1) \). Then \( w \leq y \) on \( \{x_0\} \) and on \( \{x = x_0 + \delta\} \), and

\[
|w'|^{p-2}w'' > x^p |w|^{p-2}w.
\]

By the classical comparison principle one gets that \( u \geq w \) on \( [x_0, x_0 + \delta] \), which implies that

\[
\liminf \frac{y(x_0 + h) - y(x_0)}{h} \geq \liminf \frac{w(x_0 + h) - w(x_0)}{h} = \gamma \beta > 0
\]

and contradicts \( y'(x_0) = 0 \). Doing the same on the left, one gets that \( y \equiv 0 \).

We now suppose that \( y_1 = 0 \) and \( y_0 \neq 0 \).

We use the fixed point theorem to obtain existence and uniqueness of solution. One can suppose without loss of generality that \( y_0 = 1 \).

In the following we suppose first that \( x_0 \neq 0 \), and will give at the end the changes to bring when \( x_0 = 0 \), considering only the right hand side.

Define for \( y \in B_{x_0,\delta}(1, \frac{1}{2}) := \{y \in C([x_0-\delta, x_0+\delta]), |y-1|_{C([x_0-\delta, x_0+\delta])} \leq \frac{1}{2}\} \).

Let us define the function \( \phi \) as \( \phi(Z) = |Z|^{p-1} \), and the operator \( T \) as

\[
T(y)(x) = 1 + \int_{x_0}^{x} \phi(\int_{x_0}^{t} (p-1) s^p |y|^{p-2} y(s) ds) dt.
\]

We prove in what follows that one can choose \( \delta \) small enough in order that \( T \) sends \( B_{x_0,\delta}(1, \frac{1}{2}) \), into itself and is contracting in that ball.

Indeed we use for \( y \) in \( B_{x_0,\delta}(1, \frac{1}{2}) \)

\[
(p-1)(|x_0-\delta|)^p |t-x_0| \left( \frac{1}{2} \right)^{p-1} \leq \int_{x_0}^{t} (p-1) s^p |y|^{p-2} y(s) ds \leq (p-1)(|x_0+\delta|)^p |t-x_0| \left( \frac{3}{2} \right)^{p-1}.
\]

Using the mean value’s theorem, denoting \( Y_i(t) = \left( \int_{x_0}^{t} (p-1) s^p |y_i|^{p-2} y_i(s) ds \right) \) for \( i = 1, 2 \), one gets
that for some $C$ independent on $\delta$

$$|\phi(Y_1(t)) - \phi(Y_2(t))| \leq |Y_1(t) - Y_2(t)||t - x_o|\frac{1}{p^{-1}}.$$

We now observe that $|Y_1(t) - Y_2(t)| \leq (p-1)(|x_o| + \delta)^p \left(\frac{3}{2}\right)^{p-2} ||y_1 - y_2||_{\infty}|t - x_o|$, from this one derives that

$$|\phi(Y_1(t)) - \phi(Y_2(t))| \leq C||y_1 - y_2||_{\infty}|t - x_o|\frac{1}{p^{-1}},$$

and then for $x > x_o$

$$\int_{x_o}^{x} |\phi(Y_1(t)) - \phi(Y_2(t))|dt \leq C||y_1 - y_2||_{\infty}|x - x_o|\frac{1}{p^{-1}}.$$

In particular choosing $\delta$ such that $C\delta\frac{1}{p^{-1}} < \frac{1}{2}$, the map $T$ is contracting. Under the same condition $T$ maps $B_{x_o,\delta}(1, \frac{1}{2})$ into itself.

Then it possesses a unique fixed point. Since any solution is around $x_o$ a fixed point of $T$ we have obtained the local existence and uniqueness.

We now give the changes to bring when $x_o = 0$.

Define for $y \in B_{0,\delta}(1, \frac{1}{2}) := \{y \in C([0, \delta]), \ |y - 1|_{C[0, \delta]} \leq \frac{1}{2}\}$, $T(y) = 1 + \int_0^t \phi(\int_0^t (p-1)s^p|y|^{p-2}y(s)ds)dt$. We prove in what follows that one can choose $\delta$ small enough in order that $T$ send $B_{0,\delta}(1, \frac{1}{2})$, into itself and is contracting in this ball.

Indeed we use for $y$ in that ball $\frac{p-1}{p+1}\left(\frac{1}{2}\right)^{p-1} t^{p+1} \leq |\int_0^t (p-1)s^p|y|^{p-2}y(s)ds| \leq \frac{p-1}{p+1}\left(\frac{3}{2}\right)^{p-1}$ and by the mean value’s theorem, denoting $(\int_0^t (p-1)s^p|y|^{p-2}y(s)ds) = Y_i(t)$, one gets

$$|\phi(\int_0^t (p-1)s^p|y|^{p-2}y_1(s)ds) - \phi(\int_0^t (p-1)s^p|y|^{p-2}y_2(s)ds)| $$

$$\leq |Y_1(t) - Y_2(t)|Ct^{\frac{p+1}{p+1}-(p+1)}$$

We now observe that $|Y_1(t) - Y_2(t)| \leq (p-1)(|x_o| + \delta)^p \left(\frac{3}{2}\right)^{p-1} ||y_1 - y_2||_{\infty}|t|^{p+1}$, from this one derives that

$$|\phi(Y_1(t)) - \phi(Y_2(t))| \leq C||y_1 - y_2||_{\infty}|t|^{\frac{p+1}{p+1}}$$

and then for $x > 0$

$$\int_0^x |\phi(Y_1(t)) - \phi(Y_2(t))|dt \leq C||y_1 - y_2||_{\infty}|x|^{\frac{2p}{p+1}}.$$

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In particular choosing δ such that $C\delta^{\frac{2p}{p-1}} < \frac{1}{2}$, the map $T$ is contracting. Under the same condition $T$ maps $B_{0, \delta}(1, \frac{1}{2})$ into itself.

We want to prove global existence. For that aim, suppose that there exists $\bar{x}$ such that either $y'(\bar{x}) = +\infty$ or $y(\bar{x}) = +\infty$. If $y(\bar{x}) < \infty$, $|y'|^{p-2}y' = |y'|^{p-2}y'(\bar{x} - h) + \int_{\bar{x} - h}^{\bar{x}} pt^{p-1}|y'|^{p-2}y(t)\,dt$ then $\lim_{x \to \bar{x}} y'(x)$ is finite and we get local existence after $\bar{x}$, so we can assume that $y(\bar{x}) = +\infty$. We begin to observe that by continuity $y > 0$ in a neighborhood on the left of $\bar{x}$, and then by the equation $|y'|^{p-2}y'$ is increasing on the left of $\bar{x}$, hence has a limit for $x \to \bar{x}$, $x < \bar{x}$. Suppose for a while that this limit is $L \leq 0$. Then one would have for $x < \bar{x}$

$$|y'|^{p-2}y'(x) = L + (p - 1) \int_{\bar{x}}^{x} t^{p-1}|y'|^{p-1}\,dt < 0.$$ 

Then $y(x) > y(\bar{x}) = +\infty$ a contradiction. We have obtained that $\lim_{x \to \bar{x}} y'(x) = L > 0$.

We now write using the equation and the increasing behaviour of $y$ on $[x_o, \bar{x}]$, $(y')^{p-1}(x) \leq (y')^{p-1}(x_o) + (p - 1) \int_{x_o}^{x} t^{p-1}|y'|^{p-1}\,dt \leq (y')^{p-1}(x_o) + (p - 1)y^{p-1}(x)(\bar{x})^{p+1}$. This implies that $y'(x) \leq \sup(2^{\frac{2-p}{p-1}}, 1)(y'(x_o)+(p-1)(\bar{x})^{\frac{p+1}{p-1}}y(x))$, and by integrating between $x_o$ and any $x < \bar{x}$, $\int_{x_o}^{x} \frac{y'(t)\,dt}{y'(x_o)+(p-1)(\bar{x})^{\frac{p+1}{p-1}}y(t)} \leq \sup(2^{\frac{2-p}{p-1}}, 1)(x-x_o)$, which implies that $\log(y'(x_o)+(\bar{x})^{\frac{p+1}{p-1}}y(x)) \leq C(x-x_o)$ and this contradicts the fact that $y(\bar{x}) = +\infty$. We have obtained that $y$ is defined on $\mathbb{R}^+$. 

We now consider the equation

$$V''|V'|^{p-2} = x^p|V'|^{p-2}V$$

and suppose that $V(0) < 0$. Then either $V \leq 0$, or there exists $\bar{x}$ such that for $x > \bar{x}$, $V > 0$.

Indeed, if we contradict this fact, there exists $\bar{x}_1$ which is such that $V(\bar{x}_1) > 0$ and it is a local maximum for $V$. Then $V'(\bar{x}_1) = 0$. Since $V'$ is increasing around $\bar{x}_1$ by the equation, $V'(x) < 0$ for $x < x_1$, $V'(x) > 0$ for $x > x_1$, which contradicts the fact that $x_1$ is a local maximum. So we are in the hypothesis that $V(x) \geq 0$ for $x$ large and by the strict maximum principle $V > 0$, hence $V'$ is increasing in particular either it is negative and in that case necessarily tends to zero, or $V' > 0$ somewhere and then it remains $> 0$, which implies that $V$ goes to infinity at $+\infty$.

We want to prove that it is possible to choose $V(0) > 0$ in order that for $V'(0) = -2$, the solution satisfy $V > 0$ on $[0, \infty]$, and $V$ and $V'$, tend to zero at infinity).
Lemma 4.3. There exists a solution which satisfies on $\mathbb{R}^+$

$$|y'|^{p-2}y'' = t^p y^{p-1},$$

which is positive and satisfies $y'(\infty) = y(\infty) = 0$. Furthermore, for $y_1 < 0$ given, there exists a unique positive solution as above with the initial condition $y'(0) = y_1$.

We use the existence of a sub- and a supersolution on $[0, \infty]$ which satisfies $y'(\infty) = y(\infty) = 0$. Note that any positive constant is a supersolution.

Let us exhibit a sub-solution.

Let $w_2(t) = e^{-(x^2+2x)}$. Then $w_2'(t) = -(2x + 2) w_2$, $w_2''(t) = ((2x + 2)^2 - 2)w_2 \geq 4x^2w$, then $|w_2'|^{p-2}w_2'' \geq 2^p x^pw_2^{p-1} \geq x^pw_2^{p-1}$.

Now we use Perron's method on every compact set $[0, R]$, ie we define $y_R = \sup\{y, w_2(t) \leq y(t) \leq 1, \text{ on } [0, R]\}$, $y$ is a sub - solution on $[0, R]$.

$y_R$ is classically a solution on $[0, R]$.

The sequence $y_R$ is locally uniformly bounded and then by Harnack’s inequality, it converges locally uniformly to a solution $y$ which satisfies $w_2 \leq y \leq 1$. Since $y > 0$ and is bounded we know by the analysis made previously that $y$ goes to zero at infinity, as well as $y'$.

Let us observe that for $V'(0) < 0$ given, there exists some $V(0) > 0$ such that $V$ is a solution for such initial conditions, which satisfies $V(+\infty) = V'(+\infty) = 0$.

Indeed, let $y$ be the positive solution obtained above, let $V = \frac{V'(0)}{y'(0)} y$, then $V$ is a positive solution which satisfies the required condition. Let us prove the uniqueness of solutions $V$ such that $\lim V = 0$, $V'$ is bounded and $V'(0)$ given. Suppose for that aim that $V_i, i = 1, 2$ are two such solutions.

Then substracting the equations satisfied by $V_1$ and $V_2$, multiplying by $(V_1 - V_2)$ and integrating on $[0, R]$ with $R$ large, one gets using $(V_1 - V_2)'(0) = 0$ and $(|V_1'|^{p-2}V_1' - |V_2'|^{p-2}V_2')(V_1 - V_2)(R) \to 0$ that $\int_0^R (|V_1'|^{p-2}V_1' - |V_2'|^{p-2}V_2')(V_1 - V_2) \to 0$ and then $V_1 = V_2$.

Proposition 4.4. Let $\beta, \gamma > 0$ be given. There exists a unique solution $W$ which satisfies

$$\begin{cases}
|W'|^{p-2}W'' = \beta^p x^p W^{p-1} \text{ on } \mathbb{R}^+ \\
W'(0) = -\gamma, \lim_{x \to +\infty} W(x) = 0
\end{cases}$$

Furthermore $W$ and $W'$ are bounded.
Proof Let $W$ be a solution of the previous equation with $W'(0) = -1$, and consider $\bar{W}(x) = \frac{2}{\beta} W(\beta x)$, then $\bar{W}$ satisfies

$$|\bar{W}|^{p-2}\bar{W}'' = \left(\frac{2}{\beta}\right)^{p-1} \beta^p |W'|^{p-2}W''(\beta x)$$

$$= \beta^p x^p \bar{W}^{p-1}(x)$$

$W$ is convenient.

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