Decentralized State-Dependent Markov Chain Synthesis for Swarm Guidance

Samet Uzun, Nazım Kemal Üre

Abstract—This paper introduces a decentralized state-dependent Markov chain synthesis method for probabilistic swarm guidance of a large number of autonomous agents to a desired steady-state distribution. The probabilistic swarm guidance approach is based on using a Markov chain that determines the transition probabilities of agents to transition from one state to another while satisfying prescribed transition constraints and converging to a desired steady-state distribution. Our main contribution is to develop a decentralized approach to the Markov chain synthesis that updates the underlying column stochastic Markov matrix as a function of the state, i.e., the current swarm probability distribution. Having a decentralized synthesis method eliminates the need to have complex communication architecture. Furthermore, the proposed method aims to cause a minimal number of state transitions to minimize resource usage while guaranteeing convergence to the desired distribution. It is also shown that the convergence rate is faster when compared with previously proposed methodologies.

Index Terms—Decentralized control, Markov chains, Swarm robotics, Probabilistic swarm guidance.

I. INTRODUCTION

Swarm behavior is common in nature; animal colonies like bees or ants aggregate together and exhibit collective behaviors, such as gathering food or avoiding a threat. This behavior allows these animals to tackle challenging tasks and improve long term survival chances. Many scientists took inspiration from these natural display of swarm behaviors and attempted to adapt them to control of engineered multi-agent systems. Instead of using a few large agents, using a swarm that has too many small agents can be more robust and efficient to complete some challenging tasks like area exploration, surveillance and coverage. One of the most essential parts that must be developed in order to implement such systems is swarm guidance. Swarm guidance methods should generate optimal trajectories for each agents of the swarm to perform their task.

A. Related Works

Deterministic methods, which are useful for swarms containing up to 10 – 20 agents are developed, but they are computationally infeasible for the systems that contain agents ranging from hundreds to tens of thousands [11–16]. Density based deterministic and probabilistic guidance methods are developed for both discrete and continuous state spaces to handle this scalability issue. Both deterministic and probabilistic density based swarm guidance methods are developed for continuous state spaces.

Deterministic density control method that is inspired by Smoothed Particle Hydrodynamic (SPH) for group motion and segregation is presented in [7]. In [8], a deterministic velocity field method is illustrated to drive the swarm of robots smoothly to the desired density distribution. A probabilistic notion of density control is presented in [9] to converge a predetermined distribution by a diffusing swarm of robots that take local measurements of an underlying scalar field.

There are also probabilistic methods for the guidance of swarm in a discrete state space. These probabilistic methods consider the swarm as a statistical ensemble and treat the guidance problem as convergence to a desired swarm density distribution. In [10], a Markov chain is designed using the Metropolis-Hastings algorithm to guide swarm agents to the desired swarm density distribution in discrete state space. Convex optimization techniques are used by modeling objective function and constraints as linear matrix inequalities (LMI) for designing of the corresponding Markov chain [11]. This work is extended to control of swarm agents using density feedback [12] to increase the convergence rate and some safety constraints are also considered in designing of the Markov chain in [13]. However, these methods do not consider the number of transitions of the agents. Swarm guidance is formulated as an optimal transport problem to minimize the number of transitions of the agents [14]; however, the computation time of the algorithm increases rapidly with increasing the dimension of the state space, also the performance of the algorithm drops significantly with estimation errors of the current swarm distribution. Time-inhomogeneous Markov chain approach to probabilistic swarm guidance (PSG-IMC algorithm) is developed to minimize the number of transitions of the agents [15]. This method is computationally efficient and gives reasonable results with estimation errors. However, all these mentioned feedback based methods require global current distribution feedback. Communication between all bins has to be established to estimate density distribution for all bins and some estimation errors may also occur. An alternative approach that requires only local-information is developed for the probabilistic guidance problem [16]. In this work, time-inhomogeneous Markov chain approach presented in [15] is modified to work with local information and the method is used to minimize number of transitions of agents. However, in both global and local-information based time-inhomogeneous Markov chain approaches, number of transitions of agents is minimized but convergence rate of the swarm distribution decreases sharply as transition constraints of the agents increases.

1Istanbul Technical University, Department of Aeronautics and Astronautics, Istanbul, 34469, Turkey \{uzunsame, ure\}@itu.edu.tr
B. Main Contributions

In this paper, we propose a decentralized state-dependent Markov chain synthesis method for probabilistic guidance of a large number of autonomous swarm agents to a desired steady-state distribution. Transitions probabilities of the agents from one state to another is determined in a decentralized manner while satisfying prescribed transition constraints. Hence, any complex communication architecture is not required. Markov matrix of the swarm distribution becomes an identity matrix as swarm distribution convergence to desired distribution, so proposed method cause minimal number of state transitions. We show that convergence rate of the swarm distribution to desired steady-state distribution is much more faster than previous methodologies especially in the operational regions that have dense transition constraints. We provide theoretical guarantees on the convergence of the proposed algorithm by using concepts from spectral graph theory and Lyapunov stability.

C. Organization

The paper is organized as follows. Section II introduces the probabilistic guidance problem formulation and the Probabilistic Guidance Algorithm. Section III introduces our Markov matrix synthesis methods to converge a stationary desired density distribution. Section IV presents several numerical simulations of swarms converging to desired distributions. This paper is concluded in Section V.

Note: In this paper, the time index is donated by a right subscript and the agent index are donated as lower-case right superscript. 0 and 1 are zero matrix and matrix of ones in appropriate dimensions. \( V \setminus W \) is the elements in set \( V \) that are not in set \( W \). \( P \) denotes probability of a random variable. \( M > (\geq 0) \) implies that \( M \) is a positive (non-negative) matrix. \( \sigma(A) \) is the set of eigenvalues \( \lambda \) of \( A \). \( \lambda_{\text{min}}(A) \) and \( \lambda_{\text{max}}(A) \) are minimum and maximum eigenvalues of the matrix \( A \). \( ||x||_2 = \sum_i \{x[i]\}^2 \) denotes the L2 norm of vector \( x \). \((v_1, v_2, ..., v_n)\) represents a vector, such that \((v_1, v_2, ..., v_n) = [v_1^T, v_2^T, ..., v_n^T]^T \). \( P = P^T \geq (\geq 0) \) implies that \( P \) is a symmetric (semi-) definite matrix.

II. BACKGROUND

Most of the underlying definitions and baseline algorithms are based on [10], in this section, we briefly review this material for completeness.

A. Swarm Distribution Guidance Problem

Definition 1. (Bins) The operational region, which the swarm agents are distributed, is denoted as \( \mathcal{R} \). The region is assumed to be partitioned as the union of \( m \) disjoint regions, which are referred to as bins \( R_i \), \( i = 1, ..., m \), such that \( \mathcal{R} = \bigcup_{i=1}^m R_i \), and \( R_i \cap R_j = \emptyset \) for \( i \neq j \).

Definition 2. (Density distribution of the swarm) Let an agent have position \( r(k) \) at time index \( k \in \mathbb{Z}^+ \). Let \( x(k) \) be a vector of probabilities, \( 1^T x(k) = 1 \), such that the \( i \)-th element \( x[i](k) \) is the probability of the event that this agent will be in bin \( R_i \) at time \( k \). Consider a swarm comprised of \( N \) agents. Each agent is assumed to act independently of the other agents, so that the following equation holds for \( N \) separate events,

\[
x[i](k) := P(r_i(k) \in R_i), \quad l = 1, ..., N,
\]

where \( r_i(k) \) denotes the position of the \( l \)-th agent at time index \( k \), and the probabilities of these \( N \) events are jointly statistically independent. We refer to \( x(k) \) as the density distribution of the swarm.

Definition 3. (Desired steady-state distribution) It is desired to guide the agents to a specified steady-state distribution described by a probability vector \( v \in \mathbb{R}^m \).

\[
\lim_{k \to \infty} x(k) = v.
\]

States of the desired distribution can be classified as in the following definition.

Definition 4. (Recurrent and transient states) The states that have non-zero elements in the desired distribution \( v \) are called recurrent states. The other states with zero elements in the desired distribution \( v \) are called transient states.

The main idea of the probabilistic guidance is to drive the propagation of probability vector \( x \), instead of individual agent positions \( \{r_i(k)\}_{i=1}^N \). Swarms are formed as a statistical ensemble of agents to facilitate the guidance of swarm problem. Although the distribution of swarm agent positions \( n/N \) is usually different from \( x \) numerically, it is equal to \( x \) on the average. Using the law of large numbers, \( x \) can be made arbitrarily close to \( n/N \) as the number of agents increases.

B. Decentralized Probabilistic Swarm Guidance

1) Probabilistic Guidance Algorithm:

Assumption 1. (Agent’s capability) All agents can determine their current bins to use their stochastic policy for the transition.

Definition 5. (Stochastic policy) All swarm agents are propagated at time \( k \) with a column stochastic matrix \( M(k) \in \mathbb{R}^{m \times m} \) that is called as Markov matrix [17]. Then, \( M(k) \) has to satisfy

\[
1^T M(k) = 1^T, \quad M(k) \geq 0.
\]

The entries of matrix \( M(k) \) are defined as transition probabilities. Specifically, for any \( k \in N^+ \) and \( i, j = 1, ..., m \),

\[
M[i,j](k) = P(r(k+1) \in R_i|r(k) \in R_j).
\]

i.e., an agent in bin \( j \) transitions to bin \( i \) with probability \( M[i,j](k) \).

The constraints \( M(k) \geq 0 \) and \( 1^T M(k) = 1^T \) simply implies that the probability of moving from one bin to another bin is nonnegative and the sum of probabilities of transition from a given bin to another bin is equal to 1. The Markov matrix is supplied to each of the agents to propagate their position, which only depends on their local states.
The matrix $M(k)$ determines the time evolution of the probability vector $x$ as
\[
x(k + 1) = M(k)x(k), \quad k = 0, 1, 2, \ldots
\]
with $x(0) \geq 0$ and $1^T x(0) = 1$. \hfill (5)

**Algorithm 1 Probabilistic Guidance Algorithm**

1. Each agent determines its current bin, e.g., $r_I(k) \in R_e$.
2. Each agent generates a random number $z$ that is uniformly distributed in $[0, 1]$.
3. Each agent transitions to bin $j$, i.e., $r_I(k + 1) \in R_j$, if
\[
\sum_{s=1}^{j-1} M(s, i)(k) \leq z \leq \sum_{s=1}^{j} M(s, i)(k).
\]

The probabilistic guidance algorithm is given in the Algorithm \[1\]. The first step of the algorithm is to determine the agent’s current bin. In the following steps, each agent samples from a discrete distribution and transitions to another bin depending on the column of the Markov matrix, which is determined by the agent’s current bin.

2) **Convergence to Desired Steady-State Distribution:** Desired distribution of the swarm is donated by the vector $v$. The main idea of the probabilistic guidance law is to synthesize a Markov matrix that satisfies the condition given in the Definition \[3\] The requirement for a Markov chain to converge a desired density distribution is given in Eq. \[4\]. Let define the difference between swarm distribution at time-step $k$ and the desired distribution with an error vector as $e(k) = x(k) - v$. Then, Eq. \[4\] implies that,
\[
\lim_{k \to \infty} e(k) = \lim_{k \to \infty} x(k) - v = 0. \hfill (6)
\]

3) **Transition Constraints:** The transition between two bins is represented by an edge of a directed graph, where the adjacency matrix of the directed graph is defined similar to the second section of \[18\].

**Definition 6.** (Transition Constraints) Adjacency matrix is used to restrict the allowable transitions of the agents. $A_a[i, j] = 1$ if the transition from bin $i$ to bin $j$ is allowable, and is zero otherwise. Transition constraint of the agents are restricted with the following inequality:
\[
(11^T - A_a^T) \odot M = 0. \hfill (7)
\]

**Assumption 2.** (Strongly Connected) It is assumed that all bins is strongly connected by the adjacency matrix which means there exists a directed path between all bins, or equivalently, $(I + A_a)^{m-1} > 0$ for $A_a \in \mathbb{R}^{m \times m}$ \[17\], section 6.2.19. Then, there exist a directed path between all bins.

**Definition 7.** (Neighbor Bins) All bins $j$ that satisfy the condition $A_a[j, i] = 1$ are neighbor bins of the bin $i$.

III. **Synthesis of the Markov Matrix**

In this section, two different methods are presented for the synthesis of the Markov matrix for a swarm to converge recurrent and transient states of the desired stationary distribution, respectively. We propose a decentralized state-dependent Markov chain synthesis method to converge the recurrent states of the desired distribution. For the transient states, a shortest-path algorithm is proposed that is presented in modified Metropolis-Hastings algorithm \[10\]. In \[15\], \[16\], a similar shortest-path algorithm is used for the transient states. In these algorithms, agents that are in any transient state are propagated to the recurrent states using shortest-path. We briefly review this algorithm for completeness.

Assume that there are $m_r$ recurrent and $m_t = m - m_r$ transient states in the desired distribution. Markov matrix and desired distribution are split as Eq. \[8\] to synthesize the Markov matrix for the recurrent and transient states, separately.
\[
v = (v_t, v_r), \quad M = \begin{bmatrix} M_1 & 0 \\ M_2 & M_3 \end{bmatrix}_{m \times m}, \hfill (8)
\]
where $v_t \in \mathbb{R}^{m_r}$, $v_t = 0$, $v_r > 0$, $M_1 \in \mathbb{R}^{m_r \times m_t}$, and $M_3 \in \mathbb{R}^{m_t \times m_r}$. Since desired distribution and Markov matrix are partitioned with renumbering the bins, current distribution and adjacency matrix are also partitioned using the same bin renumbering as follows,
\[
x = (x_{tr}, x_{rc}), \quad A_a = \begin{bmatrix} A_{a_1} & 0 \\ A_{a_2} & A_{a_3} \end{bmatrix}_{m \times m}. \hfill (9)
\]

In Section III-A it is assumed that $1^T x_{rc}(k) = 1$ for any $k \in \mathbb{Z}^+$ which means all agents in the recurrent states. Thereafter, in Section III-B it is proved that, after a certain time all agents will be in the recurrent states.

A. **Synthesis of the Markov Matrix for the Recurrent States**

We propose a decentralized state-dependent Markov chain method to converge recurrent states of the desired distribution via minimal number of transition and rapid convergence rate. Since we use the current density distribution as feedback, the following assumption is required.

**Assumption 3.** (Agent’s capability) Besides Assumption \[7\] it is assumed that all agents know the density values of their own bins an their neighbor bins.

Since the algorithm is decentralized, complex communication architecture is not required. Communication only with neighbor bins is sufficient for an agent to determine its transition probabilities.

As described in Section I-A probabilistic swarm guidance time-inhomogeneous Markov chain (PSG-IMC) algorithm, which is presented in \[15\], uses global density distribution feedback to minimize the number of transitions of the agents. There is also a version of this algorithm that uses local density distribution feedback to minimize the number of transitions of the agents \[16\]. These algorithms become impractical for Markov chain synthesis problems that have dense transition constraints since the convergence rate of these algorithms decreases dramatically as the adjacency matrix becomes sparse. As can be seen in the Corollary 1 or Corollary 3 in \[15\] or \[16\], the transition probability for an agent is the desired density value of the target bin at maximum. Hence, if any
bin has limited neighbor bins that have small desired density values, the total transition probability to these bins is very small for an agent. These algorithms are effective for cases that have dense adjacency matrices. Also, the performance of these algorithms changes dramatically with the selection of some pre-determined constants. The selection of these constants should be made carefully for each experiment.

In our method, synthesis of the Markov matrix occurs in a decentralized manner and number of transitions of agents are much more lower respect to homogeneous Markov chain methods that are presented in [10], [11]. We show that convergence rate of our algorithm is much faster than both Metropolis-Hastings algorithm and PSG-IMC algorithm, which are homogeneous and time-inhomogeneous Markov chain methods, respectively. For all discussed Markov matrix synthesis methods, recurrent states have to be strongly connected amongst themselves, so the following assumption is required for convergence of swarm to desired distribution.

**Assumption 4.** It is assumed that all recurrent bins of the desired distribution is strongly connected by the adjacency matrix which means there exists a directed path between all recurrent bins, or equivalently, \((I + A_{chsn})^{m_r - 1} > 0\) for \(A_{chsn} \in R^{m_r \times m_r}\). Then, there exist a directed path between all recurrent bins.

In our algorithm, difference between the desired distribution and current distribution is calculated as error vector donated as \(e\). Transitions of agents occur from the bins that have lower error values to the bins that have high error values to equalize error values of all bins. Since sum of these error values is equal to 0 (i.e., \(1^T e(k) = 1^T v_r - 1^T x_{rc} = 1 - 1 = 0\)), after a certain time, error value of the all bins are balanced at zero. Hence, distribution of swarm agents convergence to the desired stationary distribution. The synthesis of the Markov matrix is given in Algorithm 2.

**Algorithm 2** Decentralized state-dependent Markov chain synthesis

\[
e[i](k) = v_r[i] - x_{rc}[i](k),
\]

\[
d_{chsn} > \max_{j} \left( \sum_{i \neq j} A_{chsn}[i,j] \right),
\]

\[
T[i,j](k) = \max \left( 0, \frac{e[i](k) - e[j](k)}{d_{chsn}} \right) \odot A_{chsn}[i,j],
\]

\[
R[i,j](k) = \begin{cases} T[i,j](k) \quad & \text{if } x_{rc}[j](k) > 0 \\ 0 \quad & \text{if } x_{rc}[j](k) = 0 \end{cases}
\]

\[
R[j,i](k) = 1 - \sum_i R[i,j] \quad \text{if } \sum_i R[i,j] < 1
\]

\[
M_3[i,j](k) = \frac{R[i,j]}{\sum_i R[i,j]}
\]

In the Algorithm 2, the error vector between the desired and the current distributions is calculated as in Eq. (10). Then, a \(d_{chsn}\) value which is higher than half of maximum number of neighbor for any bin is determined in Eq. (11). Agents are propagated from their bins to the neighbor bins that have higher error value than the error value of their bins until both bins reach their average error value. In the Eq. (12), the density value that must be passed from a bin to another bin to reach their average error value is determined. If there are just two bins and they have only one neighbor bin, they can reach to their average error value in one time step when \(d_{chsn}\) is neglected (i.e., \(d_{chsn} = 1\)). Since each bin can have number of neighbors, determined density value for transition should be divided a \(d_{chsn}\) value for stabilization, which is proved in the Lemma 2.

After the density values are determined for transition, these numbers are divided by the density values of the bins as in the Eq. (13) to decide what percentage of their agents will be propagated. Since sums of the some columns of the \(R\) matrix may be lower than value of 1, remaining probabilities are added to diagonal elements of the corresponding columns for these columns as in the Eq. (14). Since division by the density vector \(x\) in Eq. (13) may cause sums of some columns to exceed the number 1, values of the corresponding columns are scaled so that their sums of the columns are 1 in Eq. (15). Hence, the Markov matrix is synthesized.

The following example is given in order to internalize the algorithm. Transition of density values are represented on a graph for two time-steps in the Figure 1. Desired density values, current density values and error values of the vertices are given for each time-step in the Table 1. Since degree of all vertices is 2, \(d_{chsn}\) value is chosen as 3. Density transition values between vertices are determined using Eq. (12), and represented with red arrows in the Figure 1. It should be noted that the density transition value between the vertex 3 and vertex 4 should be 0.1 at time-step 0.1. However, this density value is determined as 0, since current density value of the vertex 3 is 0 at time-step 0. The following theorem is given to represent the Markov matrix that synthesized by the Algorithm 2 is a column stochastic matrix.

**Theorem 1.** The Markov matrix synthesized by the Algorithm 2 satisfies the \(M_3(k) \geq 0\) and \(1^T M_3(k) = 1^T\) constraints.

**Proof.** (Proof of \(M_3(k) \geq 0\) \(T[i,j](k) \geq 0\) and \(x_{rc}[j](k) \geq 0\) imply that \(R[i,j](k) \geq 0\) in Eq. (13). \(R[j,i](k)\) that is diagonal of the matrix \(R\) is determined in the Eq. (14) as \(1 - \sum_i R[i,j]\) only if \(\sum_i R[i,j] < 1\), so \(R[i,j] \geq 0\). Since \(\sum_i R[i,j] \geq 1\), \(M_3[i,j](k) \geq 0\) for all \(i\) and \(j\).
Definition 8. Let \( \mathcal{G}(\mathcal{V}, \mathcal{E}) \) be a strongly connected graph, consist of \( \mathcal{V} \), a nonempty set of vertices, and \( \mathcal{E} \), a set of unordered pairs of distinct elements of \( \mathcal{V} \) called edges. \( \mathcal{V}(\mathcal{G}) = \{v_1, ..., v_m\} \) is called vertex set with \( m = |\mathcal{V}| \). Two vertices \( v_i \) and \( v_j \) are called adjacent in \( \mathcal{G} \), if \( \{v_i,v_j\} \) is an edge of the graph \( \mathcal{G} \). Adjacency matrix of the graph \( \mathcal{G} \) is represented as \( A_{\mathcal{G}} \). Degree matrix \( D(\mathcal{G}) \) is a diagonal matrix and represents the number of edges that is connected to each vertex such that \( D(\mathcal{G})[i,i] = \sum_j A_{\mathcal{G}}[i,j] \). \( L(\mathcal{G}) \) is the Laplacian matrix for the graph \( \mathcal{G} \) such that \( L(\mathcal{G}) = D(\mathcal{G}) - A_{\mathcal{G}} \). \( \lambda \) is the maximum degree of \( \mathcal{G} \) such that \( \lambda = \max_{0 \leq i \leq m} (D(\mathcal{G})[i,i]) \).

Definition 9. Let \( e \in \mathbb{R}^m \) be a vector which represent the values of each vertices of the graph defined in Definition 8 such that summation of the all values are zero, \( 1^T e = 0 \).

Definition 10. Let \( \mathcal{G}^o \) is the largest subgraph of \( \mathcal{G} \) with self-loops removed. Degree and maximum degree of the graph \( \mathcal{G}^o \) are represented as \( D(\mathcal{G}^o) \) and \( d(\mathcal{G}^o) \). Adjacency matrix and Laplacian matrix of the graph \( \mathcal{G}^o \) are represented as \( A_{\mathcal{G}_o}(\mathcal{G}^o) \) and \( L(\mathcal{G}^o) \), respectively.

The following lemma is given to represent value stabilization of vertices of defined graph at 0 under a given transition rule.

Lemma 2. Given a graph \( \mathcal{G}(\mathcal{V}, \mathcal{E}) \) that satisfies the properties in Definition 3. Values of these nodes for time-step \( k \) are represented with \( c(k) \). When value of the each node is evolving as

\[
e[i](k + 1) = e[i](k) + \sum_{j \neq i} A_{\mathcal{G}}[i,j] \left( \frac{e[j](k) - e[i](k)}{d_{chsn}} \right)
\]

where \( d_{chsn} > d(\mathcal{G}^o) \).

Then, \( e(k) \) vector asymptotically convergences to 0 (i.e. \( \lim_{k \to \infty} e(k) = 0 \)).

Proof. Eq. (16) is equivalent to following equation,

\[
e(k + 1) = Fe(k)
\]

where

\[
F = I - \frac{L(\mathcal{G}^o)}{d_{chsn}}.
\]

From spectrum of Laplacians for graphs [19], maximum eigenvalue of the Laplacian matrix for a connected undirected graph without any self-loops satisfies the following inequality,

\[
\max \left( \sigma \left( L(\mathcal{G}^o) \right) \right) \leq 2d(\mathcal{G}^o).
\]

Since Laplacian matrix is a positive semi-definite matrix and \( d_{chsn} > d(\mathcal{G}^o) \) is given in Eq. (16), eigenvalues of the Laplacian matrix satisfies the following inequality,

\[
0 \leq \lambda_i < 2d_{chsn} \quad \text{for} \quad \lambda_i \in \sigma \left( L(\mathcal{G}^o) \right).
\]

Since \( d_{chsn} \in \mathbb{R} \), the following inequality can be written for the eigenvalues of the matrix \( \frac{L(\mathcal{G}^o)}{d_{chsn}} \),

\[
0 \leq \lambda_i < 2d_{chsn} \quad \text{for} \quad \lambda_i \in \sigma \left( \frac{L(\mathcal{G}^o)}{d_{chsn}} \right).
\]

Since all eigenvalues of the identity matrix is 1 and both the identity matrix and the matrix \( \frac{L(\mathcal{G}^o)}{d_{chsn}} \) are symmetric matrices, the following inequality can be written for the eigenvalues of the matrix \( F \),

\[
-1 < \lambda_i \leq 1 \quad \text{for} \quad \lambda_i \in \sigma(F).
\]

When corresponding graph is strongly connected, null space of the Laplacian matrix \( L(\mathcal{G}^o) \) is spanned by the vector 1. Hence, eigenvalue of the matrix \( F \) is 1 only if corresponding eigenvector is 1.

Since \( 1^T e(k) = 0 \), Eq. (17) can be re-written as follows,

\[
e(k + 1) = \left( F - \frac{L(\mathcal{G}^o)}{d_{chsn}} \right) e(k)
\]

where

\[
F = I - \frac{L(\mathcal{G}^o)}{d_{chsn}}.
\]
The corresponding eigenvector of the matrix $\mathbf{11}^T/m$ is $\mathbf{1}$ with an eigenvalue of 1 and other eigenvalues are 0. Since both matrix $F$ and the matrix $\mathbf{11}^T/m$ are symmetric, the following inequality can be written for the eigenvalues of the matrix $(F - \mathbf{11}^T/m)$,

$$-1 < \lambda_1 < 1 \quad \text{for} \quad \lambda_i \in \sigma \left( F - \mathbf{11}^T/m \right).$$

Since the matrix $(F - \mathbf{11}^T/m)$ is symmetric, also eigenvalues of the matrix $(F - \mathbf{11}^T/m)^T (F - \mathbf{11}^T/m)$ satisfies the following inequality,

$$0 < \lambda_i < 1 \quad \text{for} \quad \lambda_i \in \sigma \left( (F - \mathbf{11}^T/m)^T (F - \mathbf{11}^T/m) \right).$$

Then, $I \succ (F - \mathbf{11}^T/m)^T (F - \mathbf{11}^T/m)$. The linear discrete-time system defined in Eq. (22) asymptotically convergent if and only if there exist a Lyapunov matrix $P = P^T > 0$ such that $P \succ (F - \mathbf{11}^T/m)^T P (F - \mathbf{11}^T/m)$ [20, Chapter 3.1]. This condition is satisfied for the system defined in Eq. (22) when $P = I$. Then, the system given in Eq. (22) is a convergent system. Then, the following equality is satisfied,

$$\lim_{k \to \infty} e(k) = \mathbf{0}.$$

We prove that the system given in Lemma 2 is a convergent system. The following lemma is given to put a bound for the convergent rate of the system.

**Lemma 3.** Convergence rate of the system given in Lemma 2 is bounded as in the following equation,

$$\lambda_{\min} \left( \frac{2d_{chsn} \mathcal{L}(G^o) - \mathcal{L}(G^o)^2 + \mathbf{11}^T/m}{d_{chsn}^2} \right) \leq \frac{\|e(k)\|^2 - \|e(k + 1)\|^2}{\|e(k)\|^2} \leq \lambda_{\max} \left( \frac{2d_{chsn} \mathcal{L}(G^o) - \mathcal{L}(G^o)^2}{d_{chsn}^2} \right).$$

**Proof.** As discussed in the proof of Lemma 2, the system can be represented as in the Eq. (17). Then,

$$\|e(k)\|^2 - \|e(k + 1)\|^2 = e(k)^T e(k) - e(k + 1)^T e(k + 1)$$

$$= e(k)^T e(k) - e(k + 1)^T F e(k)$$

$$= e(k)^T (I - F)^T e(k)$$

$$= e(k)^T \left( I - \left( I - \frac{\mathcal{L}(G^o)}{d_{chsn}} \right)^T \left( I - \frac{\mathcal{L}(G^o)}{d_{chsn}} \right) \right) e(k)$$

$$= e(k)^T \left( \frac{2d_{chsn} \mathcal{L}(G^o) - \mathcal{L}(G^o)^2}{d_{chsn}^2} \right) e(k).$$

Let express the matrix $\left( \frac{2d_{chsn} \mathcal{L}(G^o) - \mathcal{L}(G^o)^2}{d_{chsn}^2} \right)$ as the $S$ matrix.

Since $1^T e(k) = 0$ is given in Definition 9, the following equation is also satisfied,

$$\|e(k)\|^2 - \|e(k + 1)\|^2 = e(k)^T \left( S + \mathbf{11}^T/m \right) e(k).$$

Since the matrix $\mathcal{L}(G^o)$ is symmetric and real, the matrix $\left( S + \mathbf{11}^T/m \right)$ is also symmetric and real. Then, we can bound the quadratic form of the matrix as follows,

$$\lambda_{\min} \left( S + \mathbf{11}^T/m \right) e(k)^T e(k) \leq e(k)^T \left( S + \mathbf{11}^T/m \right) e(k) \leq \lambda_{\max} \left( S + \mathbf{11}^T/m \right) e(k)^T e(k).$$

Then Eq. (23) is satisfied which completes the proof.

**Theorem 2.** Any swarm distribution $x_{rc}(k)$ controlled by the Markov matrix $M_3(k)$ synthesized by the Algorithm 2 converges to desired distribution $v_r$ (i.e., $\lim_{k \to \infty} x_{rc}(k) = v_r$).

**Proof.** As described in Section II-B2, $\lim_{k \to \infty} x_{rc}(k) = v_r$ if and only if $\lim_{k \to \infty} e(k) = 0$ where $e[i](k) = v_r[i] - x_{rc}[i](k)$. In Lemma 2 it is proved that $\lim_{k \to \infty} e(k) = 0$ if there is a transition between nodes as in Eq. (16). The differences of the Lemma 2 with the Algorithm 2 are in the Eq. (13) and (15). In Eq. (12), density values are determined for the transition. In Eq. (13), it is determined what percentage of agents is propagated. For the bins that have enough agents for the determined transition, remaining percentages is left to diagonal of the corresponding columns as in Eq. (15). Since the probability of the events of these columns are 1, Eq. (15) does not affect them. If some bins have not enough agents for the determined transition, sums of the entities in Eq. (12) exceed the value of 1. These columns are scaled to 1 so that their sums of the columns are 1 in Eq. (15). This scaling has the same effect as increasing the $d_{chsn}$ value in the Lemma 2. Since increasing of $d_{chsn}$ value does not affect the proof, not enough agents in some bins only causes the convergence rate to decrease. Then, any swarm distribution $x_{rc}(k)$ can converge to desired distribution $v_r$ using the Markov matrix synthesized by the Algorithm 2.

Major drawbacks of the our algorithm respect to homogeneous Markov chain methods are that the current density value
of their own and neighbor bins have to be known by all agents. Also, the Markov matrix should be synthesized for each time-step since the current density distribution is changing for each time-step.

B. Synthesis of the Markov Matrix for the Transient States

A shortest-path algorithm is used to synthesize a Markov matrix for the transient states of the desired distribution. This method is proposed in modified Metropolis-Hastings algorithm that is presented in [21] and similar shortest-path algorithm is used in [15], [16]. We briefly review this algorithm here for completeness. We should define the following index sets for the algorithm.

Definition 11. (Index sets respect to recurrent states) The index set $I_r$ and $I_t$ contains the recurrent and transient states, respectively. Also, transient states can be split subsets as $I_s$, $I_{s-1},...$ and so on. The index set $I_s$ consists of the bins that are directly connected by adjacency matrix to the bins $i \in I_r$ and $I_s \cap I_r = \emptyset$, the index set $I_{s-1}$ consists of the bins that are directly connected by adjacency matrix to bins $i \in I_s$ and $I_{s-1} \cap (I_r \cup I_s) = \emptyset$, and so on.

$$M[i,j] = \begin{cases} 
\text{as in Algorithm 2} & \text{if } i \in I_r, j \in I_r \\
0 & \text{if } i \in I_{s-n}, j \in I_r, \\
\frac{1}{(\sum_{l \in I_t} A_s[j,l])} & \text{if } i \in I_r, j \in I_s, A_s[j,i] = 1 \\
\frac{1}{(\sum_{l \in I_{s-n}} A_s[j,l])} & \text{if } i \in I_{s-n}, j \in I_{s-n-1}, A_s[j,i] = 1, \text{ for } n = \{0, 1, \ldots\} 
\end{cases}$$

In this algorithm, agents in transient states set which are neighbor of the recurrent states set, are propagated to the recurrent states directly. Agents in other transient states set are propagated to transient states set which are closer to the recurrent states set. Hence, all agents can be propagated to the recurrent states in finite time for the finite operational region. The synthesis of a Markov matrix using a combination of our algorithm and the shortest-path algorithm is given in the Eq. (24).

In [10], it is showed that the condition $\rho(M_1) < 1$ is satisfied using the properties of M-matrices, which are shown in theorem 2.5.3 (parts 2.5.3.2 and 2.5.3.12) of [21]. Then, it is proved that all agents will be in the recurrent states after a certain time that means there is a finite $k \in \mathbb{Z}^+$ that satisfy $1^T x_{rc}(k) = 1$.

IV. NUMERICAL SIMULATIONS

Convergence performance of our algorithm is represented on the numerical example given in Figure 2 which is the same with the example given in the [10]. Agents converge to the ‘E’ letter until 250. time-step. Then, approximately 1/3 agents are removed from the state space and remaining agents convergence to desired distribution again until 750 time-steps. Comparison of changing of the total variation and the total number of transitions with both the M-H algorithm and the PSG-IMC algorithm is given in Figure 3 and 4 for two different cases. In the first case, adjacency matrix only allows the transition to one step away bins which is at the top, bottom, right or left bins. In the second case, adjacency matrix allows the transition to ten step away bins. In both cases, the convergence rate is much better than the M-H and PSG-IMC algorithms for our algorithm. PSG-IMC algorithm cannot perform good results in the first case because of the sparse adjacency matrices problem discussed in Section III-A. In the second case which has dense adjacency matrix, PSG-IMC algorithm gives better results than M-H algorithm but our algorithm is still much better than PSG-IMC algorithm. The number of transitions of our algorithm and PSG-IMC algorithm is much lower than the M-H algorithm since as the convergence increases, the error value decreases, the Markov matrix turns into an identity matrix and unnecessary transitions are avoided. Since the transition of agents rarely occurred in PSG-IMC algorithm, the number of transitions of PSG-IMC algorithm is slightly better than the our algorithm.

V. CONCLUSION AND FUTURE WORKS

This paper introduces a probabilistic approach to guide the distribution of large-scale swarm of autonomous agents to a desired stationary distribution in a decentralized and scalable manner. The probabilistic approach is based on designing a Markov chain for the distribution of the swarm to converge a desired stationary distribution while satisfying some transition constraints. In this paper, a decentralized state-dependent Markov chain method is introduced to converge the desired distribution with high convergence rate and minimal number of state transitions. Difference between desired and current density values of the bins is considered as error value and our algorithm is based on to equalize the error values of the all neighbor bins. Since sum of the error values of the bins is zero, these error values go to zero for each bins and agents can converge to the desired distribution. In earlier literature, convergence rate of the global or local-information based time-inhomogeneous Markov chain synthesis methods is very slow for operational regions that have dense transition constraints that means sparse adjacency matrices. We showed that number of transitions of agents is lower than homogeneous Markov chain synthesis methods like the proposed time-inhomogeneous Markov chain method and we showed that convergence rate of our algorithm is much more faster than both proposed homogeneous or time-inhomogeneous Markov chain synthesis methods for both operational regions that have sparse and dense adjacency matrices. For the future works,

- A new Markov matrix synthesis method will be developed for the transient states to increase the convergence rate of the algorithm.
- Probabilistic swarm guidance algorithms and proposed Markov matrix synthesis methods will be used as a baseline and some strategies will be developed on these algorithms to solve the swarm to swarm engagement problem.
Fig. 2: Representation of the distribution of the swarm for the time-steps 0, 250, 251, and 750, respectively. There are 400 (20 × 20) bins and 5000 agents in the operational region at the beginning of the simulation. The agents converge to the 'E' letter until the 250. time-step and approximately 1/3 are removed from the operational space. Then, remaining agents converge to the same desired distribution until 750. time-steps.

Fig. 3: Comparison of changing of the total variation and the number of transitions with time for the M-H, PSG-IMC and DSMC algorithms. In this case, adjacency matrix only allows the transition to one step away bins for the agents.

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