Conformally Flat Minimal $C$-totally Real Submanifolds of $(\kappa, \mu)$-Nullity Space Forms

Ahmet YILDIZ$^1$*

$^1$İnönü University, Education Faculty, Department of Mathematics, Malatya, Turkey
a.yildiz@inonu.edu.tr, ORCID: 0000-0002-9799-1781

Received: 28.05.2020  Accepted: 08.10.2020  Published: 30.12.2020

Abstract

In this paper we study conformally flat minimal $C$-totally real submanifolds of $(\kappa, \mu)$-nullity space forms.

Keywords: Contact metric manifold; $(\kappa, \mu)$-space form; Conformally flat manifold; Second fundamental form; Totally geodesic.

$(\kappa, \mu)$-Nullity Uzay Formlarının Konformal Flat Minimal $C$-total Reel Altmanifoldları

Özet

Bu çalışmada $(\kappa, \mu)$-nullity uzay formlarının konformal flat minimal $C$-total reel altmanifoldlarını çalıştık.

Anahtar Kelimeler: Değme metrik manifold; $(\kappa, \mu)$-uzay formu; Konformal flat manifold; İkinci temel form; Total jeodezik.

1. Introduction

Let $M^m$ be a minimal $C$-totally real submanifold of dimension $m$, having constant $\bar{\varphi}$-sectional curvature $c$ in a $(2m + 1)$-dimensional Sasakian space form $\bar{M}$ of constant $\bar{\varphi}$-sectional curvature $\bar{c}$.
curvature $\tilde{c}$. B.Y. Chen and K. Ogiue [1] studied totally real submanifolds and proved that if a such a submanifold is totally geodesic, then it is of constant curvature $c = \frac{1}{4} \tilde{c}$. Then D. Blair [2] showed that such a submanifold is totally geodesic if and only if it is of constant curvature $c = \frac{1}{4} (\tilde{c} + 3)$. Also S. Yamaguchi, M. Kon and T. Ikawa [3] stated that if such a submanifold is compact and has constant scalar curvature, then it is totally geodesic and has constant sectional curvature $c$ satisfying $c = \frac{1}{4} (\tilde{c} + 3)$ or $c \leq 0$. Later D. E. Blair and K. Ogiue [4] proved that if $M$ is compact and $c > \frac{m-2}{4(2m-1)} (\tilde{c} + 3)$, then $M$ is totally geodesic. Also P. Verheyen and L. Verstraelen [5] obtained that if $M^m \ (m \geq 4)$ is a compact conformally flat submanifold admitting constant scalar curvature $\text{scal} > \frac{(m-1)^3 (m+2)}{4(m^2+m-4)} (\tilde{c} + 3)$ and $\tilde{\phi}$-sectional curvature $c$ satisfying $c > \frac{(m-1)^2}{4m(m^2+m-4)} (\tilde{c} + 3)$, then it is totally geodesic.

In the present paper, we study the results indicated above for a conformally flat minimal $\mathcal{C}$-totally real submanifold $M$ in a $(\kappa, \mu)$-nullity space form $\tilde{M}^{2m+1}$ with constant $\tilde{\phi}$-sectional curvature $\tilde{c}$. We prove the followings:

**Theorem 1.** Let $\tilde{M}^{2m+1}$ be a $(\kappa, \mu)$-nullity space form of constant $\tilde{\phi}$-sectional curvature $\tilde{c}$ and $M^m$ be an $m \geq 4$-dimensional compact conformally flat minimal $\mathcal{C}$-totally real submanifold of a $\tilde{M}^{2m+1}$. Then

$$\text{scal} > \frac{(m-1)^3 (m+2)}{4(m^2+m-4)} (\tilde{c} + 3) + \frac{2(m-1)[m(m-2)\lambda(\lambda+2)+(m-2)\lambda]}{4(m^2+m-4)},$$

implies that $M^m$ is totally geodesic, where $\lambda = \sqrt{1 - \kappa}$.

**Theorem 2.** Let $M^m$ be a minimal $\mathcal{C}$-totally real submanifold of a $(\kappa, \mu)$-nullity space form $\tilde{M}^{2m+1}$. If $M^m$ has constant curvature $c$, then either

$$c = \frac{1}{4} [(\tilde{c} + 3) + 2\lambda^2 + 8\lambda],$$

in which case $M^m$ is totally geodesic, or $c \leq 0$.

2. Preliminaries

Let $\tilde{M}^{2m+1}$ be a contact metric manifold with the $(\tilde{\phi}, \xi, \tilde{\eta}, \tilde{g})$ satisfying

$$\tilde{\phi}^2 = -I + \tilde{\eta} \otimes \xi,$$

$$\tilde{\eta}(\xi) = 1, \tilde{\phi} \xi = 0, \tilde{\eta}(U) = \tilde{g}(U, \xi),$$

(1)
\[ g(\phi U, \phi V) = \bar{g}(U, V) - \bar{\eta}(U)\bar{\eta}(V), \quad g(\phi U, V) = d\bar{\eta}(U, V), \]

for vector fields \( U \) and \( V \) on \( \tilde{M} \). The operator \( h \) defined by \( h = -\frac{1}{2}L_{\xi} \phi \), vanishes iff \( \xi \) is Killing. Also we have

\[
\phi h + h\phi = 0, \quad h\xi = 0, \quad \bar{\eta}oh = 0, \quad tr \ h = tr \ \phi h = 0.
\] (2)

Due to anti-commuting \( h \) with \( \phi \), if \( U \) is an eigenvector of \( h \) with the eigenvalue \( \lambda \) then \( \phi U \) is also an eigenvector of \( h \) with the eigenvalue \(-\lambda \) [6]. Moreover, for the Riemannian connection \( \tilde{\nabla} \) of \( \bar{g} \), we have

\[
\tilde{\nabla}_U \xi = -\phi U - \phi hU.
\] (3)

If \( \xi \) is Killing then contact metric manifold \( \tilde{M} \) is said to be a \( K\)-contact Riemannian manifold. On a \( K\)-contact Riemannian manifold, we have

\[
\tilde{R}(U, \xi)\xi = U - \bar{\eta}(U)\xi.
\]

A Sasakian manifold is known as a normal contact metric manifold. A contact metric manifold to be Sasakian if and only if \( \tilde{R}(U, V)\xi = \bar{\eta}(V)U - \bar{\eta}(U)V, \) where \( \tilde{R} \) is the curvature tensor on \( \tilde{M} \). Moreover, every Sasakian manifold is a \( K\)-contact manifold [2].

The \((\kappa, \mu)\)-nullity distribution for a contact metric manifold \( \tilde{M} \) is a distribution

\[
Null(\kappa, \mu): p \rightarrow Null_p(\kappa, \mu) = \left\{ W \in T_p\tilde{M} | \tilde{R}(U, V)W = \kappa[\bar{g}(V, W)U - \bar{g}(W, U)V] + \mu[\bar{g}(V, W)hU - \bar{g}(U, W)hV] \right\}
\]

for any \( U, V \in T_p(\tilde{M}) \), where \( \kappa, \mu \in \mathbb{R} \) and \( \kappa \leq 1 \). We consider that \( \tilde{M} \) is a contact metric manifold with \( \xi \) concerning to the \((\kappa, \mu)\)-nullity distribution, i.e.,

\[
R(U, V)\xi = \kappa[\bar{\eta}(V)U - \bar{\eta}(U)V] + \mu[\bar{\eta}(V)hU - \bar{\eta}(U)hV].
\] (4)

The necessary and sufficient condition for the manifold \( \tilde{M} \) to be a Sasakian manifold is that \( \kappa = 1 \) and \( \mu = 0 \) [7]. Also, for more details, one can see [8] and [9]. For \( \kappa < 1 \), \((\kappa, \mu)\)-contact metric manifolds have constant scalar curvature. Also, the sectional curvature \( \tilde{R}(U, \phi U) \) according to a \( \phi \)-section determined by a vector \( U \) is called a \( \phi \)-sectional curvature. A \((\kappa, \mu)\)-contact metric manifold with constant \( \phi \)-sectional curvature \( \bar{c} \) is a \((\kappa, \mu)\)-nullity space form. The curvature tensor of a \((\kappa, \mu)\)-nullity space form \( \tilde{M} \) is given by [10]

\[
\tilde{R}(U, V)W = \frac{1}{4}(\bar{c} + 3)[g(V, W)U - g(U, W)V]
\]
For any orthonormal basis \( \{ \mathbf{w}_i \} \) of \( T_pM \), the mean curvature vector \( \mathbf{H}(p) \) is given by

\[
\mathbf{H}(p) = \frac{1}{m} \sum_{i=1}^{m} B(\mathbf{w}_i, \mathbf{w}_i).
\]
The submanifold $M$ is totally geodesic in $\tilde{M}$ if $B = 0$, and minimal if $H = 0$. If $B(U, V) = g(U, V)H$ for all $U, V \in TM$, then $M$ is totally umbilical. For the second fundamental form $B$, with respect to the covariant derivation $\nabla$ is defined by

$$ (\nabla_u B)(V, W) = D_u(B(V, W)) - B(\nabla_v V, W) - B(V, \nabla_v W), \quad (9) $$

for all $U, V$ and $W$ on $M$ [11], where $\nabla$ is the covariant differentiation operator of van der Waerden-Bortolotti.

Also the equations of Gauss, Codazzi and Ricci are given by

$$ g(R(U, V)W, T) = g(\tilde{R}(U, V)W, T) $$

$$ + g(B(U, W), B(V, T)) - g(B(V, W), B(U, T)), \quad (10) $$

$$ (\tilde{R}(U, V)W) = (\nabla_u B)(V, W) - (\nabla_v B)(U, W), \quad (11) $$

$$ g(\tilde{R}(U, V)W, N) = g(R(U, V)W, N) + g([A_N, A_W]U, V), \quad (12) $$

where $R$ and $\tilde{R}$ are the Riemannian curvature tensor of $M$ and $\tilde{M}$ and $(\tilde{R}(U, V)W) = \nabla^2 B$ denotes the normal component of $\tilde{R}(U, V)W$ [11]. The second covariant derivative $\nabla^2 B$ of $B$ is defined by

$$ (\nabla^2 B)(W, T, U, V) = (\nabla_u \nabla_v B)(W, T) $$

$$ = \nabla^1_u(\nabla_v B)(W, T)) - \nabla^1_v(\nabla_u W, T) $$

$$ - (\nabla_u B)(W, \nabla_v T) - (\nabla_v B)(W, \nabla_u T). \quad (13) $$

Then, we have

$$ (\nabla_u \nabla_v B)(W, T) - (\nabla_v \nabla_u B)(W, T) = (\tilde{R}(U, V)B)(W, T) $$

$$ = R(U, V)B(W, T) - B(R(U, V)W, T) - B(W, R(U, V)T), \quad (14) $$

where $\tilde{R}$ is the curvature tensor belonging to the connection $\nabla$. The Laplacian of the square of the length of the second fundamental form is defined

$$ \frac{1}{2} \Delta \|B\|^2 = g(\nabla^2 B, B) + \|\nabla B\|^2, \quad (15) $$

where $\|B\|$ is the length of the second fundamental form $B$, so that
\[ \|B\|^2 = \sum_{i,j} g(B(w_i, w_j), B(w_i, w_j)) , \]

and using (3.8), we can write

\[ \|\nabla B\|^2 = \sum_{i,j,k} g((\nabla_{w_i} \nabla_{w_j} B)(w_k), (\nabla_{w_i} \nabla_{w_j} B)(w_k)) , \]

and

\[ g(\nabla^2 B, B) = \sum_{i,j,k} g((\nabla_{w_i} \nabla_{w_j} B)(w_k), B(w_j, w_k)) . \]

A submanifold \( M \) in a contact metric manifold is called a \( C\)-totally real submanifold [12] if every tangent vector of \( M \) belongs to the contact distribution. Hence, a submanifold \( M \) in a contact metric manifold is a \( C\)-totally real submanifold if \( \xi \) is normal to \( M \). A submanifold \( M \) in an almost contact metric manifold is called a \( C\)-totally real submanifold if \( \bar{\varphi}(TM) \subset T^\perp(M) \) [13].

4. Conformally Flat Minimal \( C\)-totally Real Submanifolds of \( (\kappa, \mu)\)-Nullity Space Forms

Let \( M^m \) be a \( C\)-totally real submanifold of a \( (\kappa, \mu)\)-nullity space form \( \bar{M}^{2m+1} \) with \( \bar{\varphi} \)-sectional curvature \( \bar{\xi} \) and structure tensors \( (\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{\mathcal{G}}) \), with \( \xi \) normal to \( M \). The conformal curvature tensor field of \( M^m \) is defined by

\[ C(U, V)W = R(U, V)W + \frac{1}{m-2} [Ric(U, W)V - Ric(V, W)U] \]

\[ - \frac{\text{scat}}{(m-1)(m-2)} [g(U, W)V - g(V, W)U] , \]

for all vector fields \( U, V, \) and \( W \), where \( Q \) denotes the Ricci operator defined by \( g(QU, V) = Ric(U, V) \). For \( m \geq 4 \), the manifold \( M \) is conformally flat manifold if and only if \( C = 0 \) [11].

**Lemma 3.** Let \( M \) be an \( m \)-dimensional \( C\)-totally real submanifold on \( (\kappa, \mu)\)-contact metric manifold \( \bar{M}^{2m+1} \). Then, we have

i) \( A_{\varphi w_i w_j} = A_{\varphi w_j w_i} \),

ii) \( tr(\sum_i A_i^2) = \sum_{i,j} (tr A_i A_j)^2 \).

**Lemma 4.** A \( C\)-totally real submanifold \( M \) of dimension \( m \geq 4 \) in a \( (\kappa, \mu)\)-nullity space form \( \bar{M}^{2m+1} \) conformally flat if and only if
\[(m - 1)(m - 2) \left\{ \sum_{\alpha} \left\{ g(A_{\alpha} w_j, w_k) g(A_{\alpha} w_l, w_l) - g(A_{\alpha} w_l, w_k) g(A_{\alpha} w_j, w_l) \right\} \right\} \\
+ \left\{ \sum_{\alpha} \left( tr(A_{\alpha})^2 - \|B\|^2 \right) \right\} \left\{ g(w_j, w_k) g(w_l, w_l) - g(w_l, w_k) g(w_j, w_l) \right\} \\
+ (m - 1) \left\{ \sum_{\alpha} tr(A_{\alpha}) \left\{ g(A_{\alpha} w_j, w_k) g(w_l, w_l) - g(A_{\alpha} w_l, w_k) g(w_j, w_l) \right\} \right\} \\
+ g(A_{\alpha} w_l, w_l) g(A_{\alpha} w_j, w_l) \\
- \left\{ \sum_{\alpha, \beta} \left\{ g(A_{\alpha} w_l, w_l) g(A_{\alpha} w_k, w_k) g(w_j, w_l) \\
- g(A_{\alpha} w_l, w_k) g(A_{\alpha} w_k, w_l) g(w_j, w_l) \\
+ g(A_{\alpha} w_l, w_l) g(A_{\alpha} w_l, w_k) g(w_j, w_k) \\
- g(A_{\alpha} w_l, w_k) g(A_{\alpha} w_l, w_l) g(w_j, w_k) \right\} \right\} = 0,
\]

where

\[\|B\|^2 = \sum_{\alpha, \beta, \gamma} g(A_{\alpha} w_i, w_j)^2 = trA^*, \]

and

\[A^* = \sum_{\alpha} (A_{\alpha})^2. \]

**Proof.** Let \(M\) be a conformally flat manifold. Then, from Eqn. (5) and Eqn. (19), we have

\[(m - 1)(m - 2) g(R(w_i, w_j)w_k, w_l) \]

\[+ (m - 1) \left\{ Ric(w_i, w_k) g(w_j, w_l) - Ric(w_j, w_k) g(w_i, w_l) \right\} \]

\[- scal\{g(w_i, w_k) g(w_j, w_l) - g(w_j, w_k) g(w_i, w_l)\} = 0. \]

Using Eqn. (10) in Eqn. (23), we get

\[(m - 1)(m - 2) \sum_{\alpha} \left\{ g(A_{\alpha} w_j, w_k) g(A_{\alpha} w_l, w_l) - g(A_{\alpha} w_l, w_k) g(A_{\alpha} w_j, w_l) \right\} \]

\[+ \frac{(m-1)(m-2)}{4} \left\{ (c + 3) + 2\lambda^2 + 8\lambda \right\} + scal\left\{ g(w_j, w_k) g(w_i, w_l) \right\} \]

\[- \left\{ g(w_j, w_l) g(w_i, w_k) \right\} = 0, \]

\[+ (m - 1) \left\{ Ric(w_i, w_k) g(w_j, w_l) - Ric(w_j, w_k) g(w_i, w_l) \right\} \]

\[+ Ric(w_j, w_l) g(w_i, w_k) - Ric(w_i, w_l) g(w_j, w_k) \right\} = 0, \]

\[= 0. \]
where $Ric$ and $scal$, respectively, the Ricci tensor and scalar curvature of $M$, defined by

$$Ric(w_j, w_k) = \frac{(m-1)}{4} \left\{ (c + 3) + 2\lambda^2 + 8\lambda \right\} g(w_j, w_k)$$

$$+ \sum_a \text{tr}(A_a) g(A_a w_j, w_k) - g(A_a w_j, A_a w_k),$$

and

$$scal = \frac{m(m-1)}{4} \left\{ (c + 3) + 2\lambda^2 + 8\lambda \right\} + \sum_a (\text{tr}(A_a))^2 - \|B\|^2.$$  

(25)

(26)

From Eqn. (24)-Eqn. (26), we have Eqn. (20).

**Lemma 5.** Let $M$ be an $m$-dimensional $C$-totally real submanifold on $(\kappa, \mu)$-contact metric manifold $\tilde{M}^{2m+1}$. If $M$ is minimal, then Eqn. (20) becomes

$$(m - 1)(m - 2)g([A_i, A_j]w_k, w_l)$$

$$-\|B\|^2 \{ g(w_j, w_k)g(w_l, w_i) - g(w_i, w_k)g(w_j, w_l) \}$$

$$- (m - 1) \{ g(w_j, w_l)\text{tr}(A_i A_k) - g(w_i, w_l)\text{tr}(A_j A_k)$$

$$+ g(w_i, w_k)\text{tr}(A_j A_l) - g(w_j, w_k)\text{tr}(A_i A_l) \} = 0.$$  

(27)

**Lemma 6.** Let $M$ be a conformally flat minimal $C$-totally real submanifold of dimension $m \geq 4$ in a $(\kappa, \mu)$-nullity space form $\tilde{M}^{2m+1}$, then

$$(m - 1)(m - 2) \sum_{i,j} \text{tr}(A_i A_j)^2 = \|B\|^4 + (m - 1)(m - 4)\text{tr}(A^*)^2.$$  

(28)

Also we have the following:

**Lemma 7.** In any $(\kappa, \mu)$-contact metric manifold, we have

$$i) \|\nabla B\|^2 \geq \|B\|^2,$$

$$ii) \text{tr}(A^*)^2 \leq \|B\|^4.$$  

(29)

(30)

Now using Lemma 7, we get the following:

**Lemma 8.** Let $\tilde{M}^{2m+1}$ be a $(\kappa, \mu)$-nullity space form of constant $\tilde{\varphi}$-sectional curvature $\tilde{\varphi}$ and $M$ be an $m \geq 4$-dimensional minimal $C$-totally real submanifold of $\tilde{M}$. The Laplacian of the square of the length of the second fundamental form $B$ of $M$
\[
\frac{1}{2} \Delta \|B\|^2 = \|\nabla B\|^2 + \left( (\bar{c} - 1) + \frac{m(\bar{c} + 3)}{4} + \frac{\lambda}{2} (m(\lambda + 4) - \lambda) \right) \|B\|^2 \\
+ 2 \sum_{\alpha, \beta} tr(A_\alpha A_\beta)^2 - 3tr(A^*)_2,
\]

where \(\lambda = \sqrt{1 - \kappa}\).

**Proof.** If \(M\) is minimal then, from [11], we have

\[
(\nabla^2 B)(U, V) = \sum_l (R(w_l, U)B)(w_l, V).
\]

For an orthonormal base \(w_i\), from Eqn. (12), we have

\[
(R(w_k, w_l)B)(w_k, w_j) = R(\bar{w}_k, w_l)B(\bar{w}_k, w_j) - B(R(w_k, w_l)w_k, w_j)
\]

\[
= -B(w_k, R(w_k, w_l)w_j).
\]

Using Eqn. (10) in Eqn. (33), we get

\[
g((R(w_k, w_l)B)(w_k, w_j), B(w_i, w_j)) = g(R(\bar{w}_k, w_l)B(\bar{w}_k, w_j), B(w_i, w_j)) \\
- g(B(\bar{R}(w_k, w_l)w_k, w_j), B(w_i, w_j)) - \sum_{\alpha, \beta} g(A_\alpha A_\beta w_k, A_\alpha A_\beta w_k) \\
+ \sum_{\alpha, \beta} tr(A_\alpha) tr(A_\beta A_\alpha) - g(B(w_k, \bar{R}(w_k, w_l)w_j), B(w_i, w_j)) \\
- \sum_{\alpha, \beta} (tr(A_\alpha A_\beta))^2 + \sum_{\alpha, \beta} tr(A_\beta A_\alpha)^2.
\]

Again using Eqn. (11) in Eqn. (34), we have

\[
g((R(w_k, w_l)B)(w_k, w_j), B(w_i, w_j)) = g(\bar{R}(w_k, w_l)B(\bar{w}_k, w_j), B(w_i, w_j)) \\
- g(B(\bar{R}(w_k, w_l)w_k, w_j), B(w_i, w_j)) - g(B(w_k, \bar{R}(w_k, w_l)w_j), B(w_i, w_j)) \\
+ \sum_{\alpha, \beta} \left[ tr(A_\alpha A_\beta - A_\beta A_\alpha)^2 - (tr(A_\beta A_\alpha))^2 \right].
\]

After some calculations, we have

\[
g(\bar{R}(w_k, w_l)B(w_k, w_j), B(w_i, w_j)) = \left( \frac{\bar{c} - 1}{4} - \frac{\lambda^2}{2} \right) \|B\|^2,
\]

\[
g(B(\bar{R}(w_k, w_l)w_k, w_j), B(w_i, w_j)) = \frac{(1 - m)(\bar{c} + 3) + 2\lambda(\lambda + 4)}{4} \|B\|^2,
\]
\[ g(B(w_k, \bar{R}(w_k, w_l), B(w_l, w_l))) = \left(\frac{-(\bar{c}+3)-2\lambda(\lambda+4)}{4}\right) \|B\|^2, \]  
(38)

\[ \sum_{\alpha, \beta} [tr(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha})^2 - (tr(A_{\beta}A_{\alpha}))^2] = \sum_{\alpha, \beta} 2tr(A_{\beta}A_{\alpha}) - 3tr(A^*)^2. \]  
(39)

Thus, using Eqn. (36)-(39) in Eqn. (35), we get Eqn. (31).

5. Proofs of the Main Results

For a conformally flat submanifold \( M \) of dimension \( m \geq 4 \) we use equation Eqn. (28) to replace \( \sum_{\alpha, \beta} tr(A_{\alpha}A_{\beta})^2 \) in Eqn. (31), we have

\[
\frac{1}{2} (m-1)(m-2)\Delta \|B\|^2 = (m-1)(m-2)\|\nabla B\|^2 \\
+ (m-1)(m-2) \left( \frac{(\bar{c}+1)+m(\bar{c}+3)}{4} + \frac{\lambda}{2} (m(\lambda+4) - \lambda) \right) \|B\|^2 \\
- (m-1)(m+2) tr(A^*)^2 + 2\|B\|^4. 
\]  
(40)

So from Lemma 7, we get

\[
\frac{1}{2} (m-1)(m-2)\Delta \|B\|^2 \\
\geq (m-1)(m-2)\|B\|^2 + 2\|B\|^4 - (m-1)(m+2)\|B\|^4 \\
+ \frac{1}{4} (m-1)(m-2)[(\bar{c}-1) + m(\bar{c}+3) + 2\lambda(m(\lambda+4) - \lambda)] \|B\|^2 \\
= \|B\|^2 \left[ \frac{(m-1)(m-2)(m+1)(\bar{c}+3)}{4} + (m-1)(m-2) \frac{\lambda(m(\lambda+4) - \lambda)}{2} \right]. 
\]  
(41)

If \( \bar{c} > -3 \), then

\[
\|B\|^2 \leq \frac{(m-1)(m-2)(\bar{c}+3)}{4(m^2+m-4)} + (m-1)(m-2) \frac{\lambda(m(\lambda+4) - \lambda)}{2(m^2+m-4)}, 
\]  
(42)

which implies that \( \Delta \|B\|^2 \geq 0 \). For a compact submanifold \( M \), Hopf’s lemma states that \( \Delta \|B\|^2 = 0 \) and from Eqn. (41) and Eqn. (42), we conclude that \( \|B\|^2 = 0 \). Hence, we have

\[
scal = \frac{m(m-1)}{4} \left[ (\bar{c}+3) + 2\lambda^2 + 8\lambda \right] - \|B\|^2, 
\]  
(43)

for every compact minimal \( C \)-totally real submanifold in a \((\kappa, \mu)\)-nullity space form \( \overline{M} \). Thus, the proof of Theorem 1 is completed.
On the other hand, since $M^m$ has constant curvature $c$ and $\text{scal} = m(m - 1)\tilde{c}$, from Eqn. (26), we have

$$
\|B\|^2 = m(m - 1)\left(\frac{\tilde{c}(c + 3) + 2\lambda^2 + 8\lambda}{4} - c\right),
$$

and

$$
c \leq \frac{\tilde{c}(c + 3) + 2\lambda^2 + 8\lambda}{4}.
$$

Also, Eqn. (10) becomes

$$
\left(c - \frac{1}{4}(c + 3 + 2\lambda^2 + 8\lambda)\right)\{g(w_j, w_k)g(w_i, w_l) - g(w_i, w_k)g(w_j, w_l)\}
= g([A_i, A_j]w_k, w_i). 
$$

Multiplying this equation by $\sum_{\bar{k}} g(A_{\bar{k}}w_i, w_l)g(A_{\bar{k}}w_j, w_k)$, we obtain

$$
\left(c - \frac{1}{4}(c + 3 + 2\lambda^2 + 8\lambda)\right)\|B\|^2 = \sum_{i, j} \text{tr}(A_i A_j)^2 - \sum_{i, j} (\text{tr}(A_i A_j))^2. 
$$

Since $\text{Ric} = \frac{\text{scal}}{m} g$, from Eqn. (25) and Lemma 3, we have

$$
\text{tr}(A_i A_l) = g(A_{\alpha l} w_j, A_{\alpha l} w_i) = \frac{\text{scal}}{m} g(w_j, w_i) = \frac{\|B\|^2}{m} g(w_j, w_i),
$$

and

$$
\text{tr}(A_i A_j)^2 = \left(c - \frac{1}{4}(c + 3 + 2\lambda^2 + 8\lambda)\right)\|B\|^2 + \frac{\|B\|^4}{m}.
$$

Substituting the last equation into Eqn. (31), we obtain

$$
\|\nabla B\|^2 = \left[\frac{m + 1}{m(m - 1)}\|B\|^2 - \frac{m(c + 3) + (c - 1)}{4} + \frac{\lambda(m(\lambda + 4) - \lambda)}{2}\right]\|B\|^2.
$$

Now using

$$
\|B\|^2 = m(m - 1)\left[\frac{1}{4}(c + 3 + 2\lambda^2 + 8\lambda)\right],
$$

and Lemma 7, we get

$$
\|\nabla B\|^2 = m(m^2 - 1)\left(c - \frac{(c + 3) + 2\lambda^2 + 8\lambda}{4}\right)\left(c - \frac{1}{m + 1}\right).
$$
\[ \geq m(m-1)\left\{ \frac{(\ell+3)+2\ell^2+1}{4} - \ell \right\}. \]

Thus, the proof of Theorem 2 is completed.

**Acknowledgement**

The authors are thankful to the referees for their valuable comments and suggestions towards the improvement of the paper.

**References**

[1] Chen B.Y., Ogiue K., *On totally real submanifolds*, Transactions of the American Mathematical Society, 193, 257-266, 1974.

[2] Blair D.E., *Contact manifolds in Riemannian geometry*, Lectures Notes in Mathematics 509, Springer-Verlag, Berlin, 146p, 1976.

[3] Yamaguchi S., Kon M., Ikawa T., *C-totally real submanifolds*, Journal of Differential Geometry, 11, 59-64, 1976.

[4] Blair D.E., Ogiue K., *Geometry of integral submanifolds of a contact distribution*, Illinois Journal of Mathematics, 19, 269-275, 1975.

[5] Verheyen P., Verstraelen L., *Conformally flat C-totally real submanifolds of Sasakian space forms*, Geometriae Dedicata, 12, 163-169, 1982.

[6] Tanno S., *Ricci Curvatures of Contact Riemannian manifolds*, Tôhoku Mathematical Journal, 40, 441-448, 1988.

[7] Blair D.E., Ogiue K., *Positively curved integral submanifolds of a contact distribution*, Illinois Journal of Mathematics, 19, 628-631, 1975.

[8] Blair D.E., Koufogiorgos T., Papantoniou, B.J., *Contact metric manifolds satisfying a nullity condition*, Israel Journal of Mathematics, 91, 189-214, 1995.

[9] Verstraelen L., Vrancken L., *Pinching Theorems for C-Totally Real Submanifolds of Sasakian Space Forms*, Journal of Geometry, 33, 172-184, 1988.

[10] Koufogiorgos T., *Contact Riemannian manifolds with constant \( \phi \)-sectional curvature*, Geometry and Topology of Submanifolds VIII, World Scientific, 1996, ISBN 981-02-776-0.

[11] Yano K., Kon M., *Structures on manifolds*, World Scientific, 508p, 1984.

[12] Yano K., Kon M., *Anti-invariant submanifolds of a Sasakian Space Forms*, Tôhoku Mathematical Journal, 29, 9-23, 1976.

[13] Yano K., Kon M., *Anti-Invariant submanifolds*, Marcel Dekker, New York. 185p, 1978.