Generalized Integral Inequalities of Chebyshev Type

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Abstract: In this paper, we present a number of Chebyshev type inequalities involving generalized integral operators, essentially motivated by the earlier works and their applications in diverse research subjects.

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1. Introduction

Many integral inequalities of various types have been presented in the literature. Among them, we choose to recall the following Chebyshev inequality (see [1]):

\[
\frac{1}{b-a} \int_{a}^{b} f(x)g(x)dx \geq \left( \frac{1}{b-a} \int_{a}^{b} f(x)dx \right) \left( \frac{1}{b-a} \int_{a}^{b} g(x)dx \right),
\]

where \( f \) and \( g \) are two integrable and synchronous functions on \([a, b] \), \( a < b \), \( a, b \in \mathbb{R} \). Here, two functions \( f \) and \( g \) are called synchronous on \([a, b]\) if

\[
(f(x) - f(y))(g(x) - g(y)) \geq 0 \quad (x, y \in [a, b]).
\]

In the case that we have \( -f \) and \( g \) (or similarly \( f \) and \( -g \)) the sense of the previous inequality is the opposite.

Inequality (1) has many applications in diverse research subjects such as numerical quadrature, transform theory, probability, existence of solutions of differential equations and statistical problems. Many authors have investigated generalizations of the Chebyshev inequality (1), these are called Chebyshev type inequalities (see, e.g., [2,3] or [4]).

We give the definition of a general fractional integral. We assume that the reader is familiar with the classic definition of the Riemann integral, so we will not present it. Throughout the paper we will suppose that the positive integral operator kernel \( T : I \to (0, \infty) \) defined below is an absolutely continuous function on interval \( I \subseteq \mathbb{R} \).

**Definition 1.** Let \( I \) be an interval \( I \subseteq \mathbb{R} \) and \( a, b \in I \). The generalized integral operators \( \int_{I,a}^{b} \) and \( \int_{I,b}^{a} \), called respectively, right and left, are defined for every locally integrable function \( f \) on \( I \) as follows:
\[ J_{T,a^+}(f)(x) = \int_a^x \frac{f(t)}{T(t-a)} \, dt, \quad x > a. \]
\[ J_{T,b^-}(f)(x) = \int_x^b \frac{f(t)}{T(b-t)} \, dt, \quad x < b. \]

Note that in special cases, \( J_{T,a^+} \) and \( J_{T,b^-} \) are equal to the following integrals:
\[ J_{T,0^+}(f)(1) = \int_0^1 \frac{f(t)}{T(t)} \, dt \]
and
\[ J_{T,1^-}(f)(0) = \int_0^1 \frac{f(t)}{T(1-t)} \, dt = \int_0^1 \frac{1-t}{T(t)} \, dt. \]

We say that \( f \) belongs to the function space \( L^+_T[a,b] \) if
\[ J_{T,a^+}(f)(b) < \infty, \]
similarly \( f \) belongs to \( L^-_T[a,b] \) if
\[ J_{T,b^-}(f)(a) < \infty, \]
and \( f \in L^-_T[a,b] \) if \( f \in L^+_T[a,b] \cap L^-_T[a,b] \).

It is easy to see that the case of the \( J_T \) operators defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. For details about the Riemann–Liouville fractional integrals (left-sided) of a function \( f \) of order \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \) the reader can consult [5,6]. In [7], Belarbi and Dahmani established some theorems related to the Chebyshev inequality involving Riemann–Liouville fractional integral operator. Recently, some new integral inequalities involving this fractional integral operator have appeared in the literature, see, e.g., [8–19].

Taking into account the previous research results and the generalized integral operator, we will obtain some Chebyshev type inequalities, which contain many of the inequalities reported in the literature as particular cases.

2. Main Results

**Theorem 1.** Let \( f \) and \( g \) be two functions from \( L^+_T[a,b] \) which are synchronous on \( [a,b] \). Then
\[ J_{T,a^+}(fg)(b) \geq [\tau(b-a)]^{-1} J_{T,a^+}(f)(b) J_{T,a^+}(g)(b) \]
where
\[ \tau(x) = \int_0^x \frac{ds}{T(s)}. \]

**Proof.** Since \( f \) and \( g \) are synchronous on \( [a,b] \), we have
\[ (f(u) - f(v))(g(u) - g(v)) \geq 0; \quad u,v \in [a,b] \]
or equivalently
\[ f(u)g(u) + f(v)g(v) \geq f(u)g(v) + f(v)g(u). \]
Multiplying both sides by \( \frac{1}{T(u-a)} \) yields
\[ \frac{f(u)g(u)}{T(u-a)} + \frac{f(v)g(v)}{T(u-a)} \geq \frac{f(u)g(v)}{T(u-a)} + \frac{f(v)g(u)}{T(u-a)}. \]
Integrating both sides of the resulting inequality with respect to the variable \( u \) from \( a \) to \( b \), gives us
\[
\int_a^b \frac{f(u)g(u)}{T(u-a)} \, du + \int_a^b \frac{f(v)g(v)}{T(u-a)} \, du \geq \int_a^b \frac{f(v)g(u)}{T(u-a)} \, du + \int_a^b \frac{f(u)g(v)}{T(u-a)} \, dv.
\]
From this, we have
\[
I_{T,a^+}(fg)(b) + f(v)g(v)\tau(b-a) \geq g(v) I_{T,a^+}(f)(b) + f(v) I_{T,a^+}(g)(b). \tag{3}
\]
After multiplying the inequality by \( \frac{1}{T(v-a)} \) and integrating with respect to \( v \) between \( a \) and \( b \), we get
\[
I_{T,a^+}(fg)(b) \tau(b-a) + \tau(b-a) \int_a^b \frac{f(v)g(v)}{T(v-a)} \, dv \geq I_{T,a^+}(f)(b) \int_a^b \frac{g(v)}{T(v-a)} \, dv + I_{T,a^+}(g)(b) \int_a^b \frac{f(v)}{T(v-a)} \, dv,
\]
that is
\[
2 I_{T,a^+}(fg)(b) \tau(b-a) \geq 2 I_{T,a^+}(f)(b) I_{T,a^+}(g)(b)
\]
and we have got (2). \( \square \)

**Remark 1.** Similar calculations as above shows that for any \( f, g \in L_T[a,b] \) synchronous on \( [a,b] \), we have
\[
I_{T,b^-}(fg)(a) \geq (\tau(b-a))^{-1} I_{T,b^-}(f)(a) I_{T,b^-}(g)(a). \tag{4}
\]

**Remark 2.** If we take \( T \equiv 1 \) in Theorem 1 (or in Remark 1), then inequality (2) (or (4)) reduces to the classic inequality (1) of Chebyshev.

**Remark 3.** If we consider the kernel \( \alpha, \beta > 0 \)
\[
T(x-t) = T(x-t, \alpha, \beta) = \frac{\Gamma(\beta)x^{\alpha}}{\Gamma(\beta+\alpha)}b^{-\beta}, \tag{5}
\]
we obtain ([16], Theorem 5) that contains ([7], Theorem 3.1) as a particular case.

**Theorem 2.** Let \( f \) and \( g \) be two functions from \( L_T^+[a,b] \cap L_T^-[a,b] \) which are synchronous on \( [a,b] \). Then
\[
\tau_2(b-a) I_{T_1,a^+}(fg)(b) + \tau_1(b-a) I_{T_2,a^+}(fg)(b) \geq I_{T_1,a^+}(f)(b) I_{T_2,a^+}(g)(b) + I_{T_1,a^+}(g)(b) I_{T_2,a^+}(f)(b). \tag{6}
\]
where
\[
\tau_1(x) = \int_0^x \frac{ds}{T_1(s)} \text{ and } \tau_2(x) = \int_0^x \frac{ds}{T_2(s)}.
\]

**Proof.** Writing \( T_1 \) in place of \( T \) and \( \tau_1 \) in place of \( \tau \) in (3) and then multiplying both sides by \( \frac{1}{T_2(v-a)} \) yields
\[
\frac{I_{T_1,a^+}(fg)(b)}{T_2(v-a)} + \tau_1(b-a) \frac{f(v)g(v)}{T_2(v-a)} \geq I_{T_1,a^+}(f)(b) \frac{g(v)}{T_2(v-a)} + I_{T_1,a^+}(g)(b) \frac{f(v)}{T_2(v-a)}.
\]
Integrating both sides of the resulting inequality with respect to the variable \( v \) between \( a \) and \( b \) gives us (6). \( \square \)
**Remark 4.** In case of $T_1 = T_2$, we obtain Theorem 1.

**Remark 5.** By taking the kernels $(\alpha, \beta, \tau > 0)$

\[
T_1(x - t) = \frac{\Gamma(\beta)\Gamma^{1-\alpha}}{\Gamma(\alpha - \tau/\beta)\Gamma^{1-\tau}} \quad \text{and} \quad T_2(x - t) = \frac{\Gamma(\beta)\Gamma^{1-\tau}}{\Gamma(\tau/\beta)\Gamma^{1-\tau}},
\]

we obtain ([16], Theorem 6) and hence ([7], Theorem 3.2) as a particular case.

**Theorem 3.** Let $\{f_i\}_{i=1,2,...,n}$ be positive increasing functions from $L^+_+\{a,b\}$. We have

\[
\left[ I_{T,a^+} \left( \prod_{i=1}^{n} f_i \right) \right](b) \geq \left[ \tau(b-a) \right]^{-1} \left[ I_{T,a^+} \left( \prod_{i=1}^{n} f_i \right) \right](b) I_{T,a^+} \left( f_{n+1} \right)(b) \geq \left[ \tau(b-a) \right]^{-n} \prod_{i=1}^{n} I_{T,a^+} \left( f_i \right)(b).
\]

**Proof.** We prove this theorem by induction on $n \in \mathbb{N}$. For $n = 1$, (7) trivially holds. For $n = 2$, (7) immediately comes from (2), since $f_1$ and $f_2$ are synchronous on $[a,b]$. Now assume that the inequality (7) is true for some $n \in \mathbb{N}$. Let $f := \prod_{i=1}^{n} f_i$ and $g := f_{n+1}$. Observe that $f$ and $g$ are increasing on $[a,b]$, therefore (2) and the induction hypothesis for $n$ yields

\[
I_{T,a^+} \left( \prod_{i=1}^{n} f_i f_{n+1} \right)(b) \geq \left[ \tau(b-a) \right]^{-1} I_{T,a^+} \left( \prod_{i=1}^{n} f_i \right) I_{T,a^+} \left( f_{n+1} \right)(b) \geq \left[ \tau(b-a) \right]^{-n} \prod_{i=1}^{n+1} I_{T,a^+} \left( f_i \right)(b).
\]

This completes the induction and the proof. \(\square\)

**Remark 6.** Taking kernel (5), we obtain ([16], Theorem 7), which is a generalization of ([7], Theorem 3.3).

**Theorem 4.** Let $f, g : \{0, \infty\} \to \mathbb{R}$, $f, g \in L^+_+\{a,b\}$ such that $f$ is increasing and $g$ is differentiable with $g'$ bounded below by $m = \inf_{t \in [0,\infty)} g'(t)$. Then we have

\[
I_{T,a^+} \left( fg \right)(b) \geq \left[ \tau(b-a) \right]^{-1} I_{T,a^+} \left( f \right)(b) I_{T,a^+} \left( g \right)(b) - \frac{m}{\tau(b-a)} I_{T,a^+} \left( f \right)(b) I_{T,a^+} \left( t \right)(b) + m I_{T,a^+} \left( tf \right)(b),
\]

where $t(x) = x$ is the identity function.

**Proof.** Let $p(x) = mx$ and $h(x) = g(x) - p(x)$. Note that $h$ is differentiable and increasing on $[0,\infty)$. Hence we can apply (2), and we obtain

\[
I_{T,a^+} \left( fh \right)(b) \geq \left[ \tau(b-a) \right]^{-1} I_{T,a^+} \left( f \right)(b) I_{T,a^+} \left( h \right)(b) = \left[ \tau(b-a) \right]^{-1} I_{T,a^+} \left( f \right)(b) I_{T,a^+} \left( g \right)(b) - \left[ \tau(b-a) \right]^{-1} I_{T,a^+} \left( f \right)(b) I_{T,a^+} \left( p \right)(b). \quad (8)
\]

Since

\[
I_{T,a^+} \left( p \right)(b) = m I_{T,a^+} \left( t \right)(b)
\]

and

\[
I_{T,a^+} \left( fp \right)(b) = m I_{T,a^+} \left( tf \right)(b),
\]

we obtain (16) and hence ([7], Theorem 3.2) as a particular case.
where the desired result is obtained.

\[ \text{Remark 7. Using kernel (5), we obtain (16), Theorem 8).} \]

\[ \text{Remark 8. Our results contain those of [20] with the right choice of kernel } T. \]

**Theorem 5.** Let \( f, g : [0, \infty) \to \mathbb{R} \) such that \( f \) and \( g \) are differentiable with \( f' \) bounded below by \( m_1 = \inf_{t \in [0, \infty)} f'(t) \) and \( g' \) bounded below by \( m_2 = \inf_{t \in [0, \infty)} g'(t) \). Then we have

\[
J_{T,a^+}(fh)(b) \geq \tau(b-a)^{-1} J_{T,a^+}(f)(b) J_{T,a^+}(g)(b) - m_2 \frac{b-a}{(b-a)^2} J_{T,a^+}(f)(b) J_{T,a^+}(t)(b) + \frac{m_1 m_2}{\tau(b-a)} J_{T,a^+}^2(t)(b) + m_2 J_{T,a^+}(tf)(b) + m_1 J_{T,a^+}(tg)(b) - m_1 m_2 J_{T,a^+}^2(t)(b),
\]

where \( t(x) = x \) is the identity function.

**Proof.** Let \( p_1(x) = m_1 x \) and \( h_1(x) = f(x) - p_1(x) \), similarly, \( p_2(x) = m_2 x \) and \( h_2(x) = g(x) - p_2(x) \). Since \( h_1 \) and \( h_2 \) is differentiable and increasing on \([0, \infty)\), applying (2) gives us

\[
J_{T,a^+}(h_1h_2)(b) \geq \tau(b-a)^{-1} J_{T,a^+}^2(h_1)(b) J_{T,a^+}^2(h_2)(b) + \frac{m_1 m_2}{\tau(b-a)} J_{T,a^+}^2(t)(b) + m_2 J_{T,a^+}(tf)(b) + m_1 J_{T,a^+}(tg)(b) - m_1 m_2 J_{T,a^+}^2(t)(b).
\]

Moreover,

\[
J_{T,a^+}(h_1p_2)(b) = m_2 J_{T,a^+}(th_1)(b) = m_2 J_{T,a^+}(tf)(b) - m_1 m_2 J_{T,a^+}^2(t)(b) \tag{10}
\]

similarly,

\[
J_{T,a^+}(h_2p_1)(b) = m_1 J_{T,a^+}(tg)(b) - m_1 m_2 J_{T,a^+}^2(t)(b) \tag{11}
\]

and

\[
J_{T,a^+}(p_1p_2)(b) = m_1 m_2 J_{T,a^+}^2(t)(b). \tag{12}
\]
From the equality
\[ f g = (h_1 + p_1)(h_2 + p_2) = h_1 h_2 + h_1 p_2 + h_2 p_1 + p_1 p_2 \]
we have
\[ J_{T,a^+}(fg)(b) = J_{T,a^+}(h_1 h_2)(b) + J_{T,a^+}(h_1 p_2)(b) + J_{T,a^+}(h_2 p_1)(b) + J_{T,a^+}(p_1 p_2)(b), \]
and this equality together with (9)–(12) implies the required result. \( \square \)

**Remark 9.** In case of \( m_1 = 0 \), we obtain Theorem 4.

**Remark 10.** The results obtained in this work can be extended if we consider instead of \( f \) and \( g \), \( -f \) and \( g \) or \( f \) and \( -g \), in the notion of synchronous functions, in which case the direction of the inequalities changes.

### 3. Conclusions

In this work, we have obtained the Chebyshev inequality from Theorem 1 within the framework of generalized integrals. In addition to the observations made, which prove the strength of our results, we would like to present a couple of variants of the classic Chebyshev inequality.

If we take kernel \( T = t^\alpha \), \( \alpha < 1 \), then we get
\[
J_{T,a^+}(fg)(b) \geq \frac{1 - \alpha}{(b-a)^{1-\alpha}} J_{T,a^+}(f)(b) J_{T,a^+}(g)(b).
\]

In case of taking kernel \( T = e^{at} \), \( \alpha \neq 0 \), then we have the following variant of the Chebyshev inequality:
\[
J_{T,a^+}(fg)(b) \geq \alpha \frac{1}{1 - e^{-\alpha(b-a)}} J_{T,a^+}(f)(b) J_{T,a^+}(g)(b).
\]

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