Periodic Hamiltonian systems in shape optimization problems with Neumann boundary conditions

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Abstract

The recent approach based on Hamiltonian systems and the implicit parametrization theorem, provides a general fixed domain approximation method in shape optimization problems, using optimal control theory. In previous works, we have examined Dirichlet boundary conditions with distributed or boundary observation. Here, we discuss the case of Neumann boundary conditions, with a combined cost functional, including both distributed and boundary observation. Extensions to nonlinear state systems are possible. This new technique allows simultaneous boundary and topological variations and we also report numerical experiments confirming the theoretical results.

Key Words: Hamiltonian systems, implicit parametrizations, shape optimization, optimal control, Neumann boundary conditions, boundary and topological variations

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1 Introduction

Shape optimization has started its development especially in the last quarter of the previous century and we just quote several monographs devoted to this subject Pironneau [24], Haslinger and Neittaanmäki [12], Sokolowski and Zolesio [27], Delfour and Zolesio [7], Neittaanmäki, Sprekels and Tiba [20], Bucur and Buttazzo [4], Henrot and Pierre
where more details on the history of the subject and comprehensive references can be found. It is to be noted that, in general, just certain variants of boundary variations are taken into account, while topological variations of the unknown domains are frequently not investigated.

A typical example of shape optimization problem, defined on a given family of domains $\Omega \in \mathcal{O}$ (in general, it is assumed that $\Omega \subset D$, a prescribed bounded domain), has the following structure:

$$\min_{\Omega \in \mathcal{O}} \int_{\Lambda} j(x, y_\Omega(x)) \, dx,$$

$$Ay_\Omega = f \text{ in } \Omega,$$

$$By_\Omega = 0 \text{ on } \partial \Omega$$

(1.1) (1.2) (1.3)

where $\Lambda$ may be $\Omega$ or some fixed given subdomain $E \subset \Omega$, or $\partial \Omega$; and $B$ is some boundary operator expressing the boundary condition, $A$ is some differential operator, $f \in L^p(D)$, $p > 2$ is given and $j(\cdot, \cdot)$ is a Carathéodory function. More constraints on the unknown domains $\Omega$, or on the state $y_\Omega$, more general cost functionals may be taken into account. Regularity assumptions on $\Omega \in \mathcal{O}$, on $j(\cdot, \cdot)$, other hypotheses, will be imposed as necessity appears.

Many geometric optimization problems arise in mechanics: minimize the thickness, the volume, the stresses, etc., in a plate, a beam, a curved rod in dimension three, an arch, a shell. Due to the formulation of the mechanical models, the geometric characteristics of the object (thickness, curvature) enter as coefficients in the governing differential system. Consequently, such geometric optimization problems take the form of an optimal control problem in a given domain, with the control acting in the coefficients. See [3], [2], [20] Ch VI, where detailed presentations, including numerical examples, may be found.

In fact, general shape optimization problems (1.1)-(1.3) have a similar structure with optimal control problems, the difference being that the minimization parameter is the unknown geometry itself, $\Omega \in \mathcal{O}$. It is a natural question to find a method that reduces/approximates general optimal design problems to/via optimal control theory, and some examples already appear in the classical monograph of Pironneau [24]. In the case of Dirichlet boundary conditions several approaches have been developed [19], [18], [16], [17] allowing both shape and topology optimization. Essential ingredients are functional variations that combine both aspects and the recent implicit parametrization method based on the representation of the geometry via iterated Hamiltonian systems [28], [21], [29], [30]. It turns out that this approach is very general and we show here that it works in the case of Neumann boundary conditions as well. This remains true for the Robin boundary conditions, nonlinear equations, etc., but we do not examine such questions. The methodology is of fixed domain type and it has important advantages at the numerical level: it avoids remeshing and recomputing the mass matrix in each
iteration of the algorithm. Related ideas are also applicable in free boundary problems, see [9], [10], optimization and control [31].

Concerning topological variations, we underline that the well known level set method [22], [23], [1], [15] is essentially different from our approach. In our method, while we also use level functions, no Hamilton-Jacobi equation is needed and simple ordinary differential Hamiltonian systems can handle the unknown geometry and its variations. We work in dimension two, $D \subset \mathbb{R}^2$, since the important periodicity argument is based on the Poincare-Bendixson theorem [14], [25], and certain related developments. This is a case of interest in shape optimization.

The paper is organized as follows. In the next Section, we collect some preliminaries and we give the precise formulation of the problem. Both distributed and boundary observations are taken into account. In Section 3 we introduce the fixed domain approximation process as an optimal control problem, we prove a general approximation property under very weak conditions and we also obtain some error estimates. As a corollary of the employed methods, an existence result is proved as well. Section 4 is devoted to the differentiability properties of our approach, that give the basis for numerical algorithms of gradient type. A key technical development is the proof of the differentiability of the period in Hamiltonian systems, with respect to functional variations. Discretization and numerical examples are discussed in the last two Sections.

2 Problem formulation and preliminaries

Let $\mathcal{O}$ be a given family of open, connected sets, $\Omega \subset D$, not necessarily simply connected, where $D \subset \mathbb{R}^2$ is a bounded domain and $\Omega, D$ have both $C^{1,1}$ boundaries.

In each $\Omega \in \mathcal{O}$, we consider the Neumann boundary value problem

\begin{align}
-\Delta y_\Omega + y_\Omega &= f \text{ in } \Omega, \quad (2.1) \\
\frac{\partial y_\Omega}{\partial n} &= 0 \text{ on } \partial \Omega, \quad (2.2)
\end{align}

where $f \in L^p(D)$, $p > 2$ is given. It is known that (2.1), (2.2) has a unique solution $y_\Omega \in W^{2,p}(\Omega)$, more general elliptic operators may be taken into account in (2.1) or the regularity conditions on the boundary may be relaxed, Grisvard [8]. Here, it is important to work in $\mathbb{R}^2$ since Poincaré-Bendixson type arguments are essential in the proof of the global existence result for the Hamilton system (2.10)-(2.12) that are introduced in the sequel for the description of the unknown geometries. In fact, all the other arguments to be used in this work are valid in arbitrary dimension, where iterated Hamiltonian systems are necessary for the description of the geometry and their solution is local [29].

We associate to the system (2.1), (2.2) a cost functional that combines distributed and boundary observation (the necessary regularity conditions are detailed in the se-
where $E \subset \subset D$ is a given subdomain such that $E \subset \Omega$ for any $\Omega \in \mathcal{O}$ and $J(\cdot, \cdot), j(\cdot, \cdot)$ are Carathéodory functions. More restrictions (for instance, on the state $y_\Omega$) may be added to the shape optimization problem (2.1)-(2.3), denoted by $(\mathcal{P})$. More assumptions will be formulated as necessity appears.

The approach based on functional variations [18], [19], [30] assumes that the family of admissible domain $\mathcal{O}$ is obtained starting from a family $\mathcal{F} \subset C(D)$ of level functions via the relation:

$$\Omega = \Omega_g = \text{int} \{ x \in D; \ g(x) \leq 0 \}, \quad g \in \mathcal{F}. \quad (2.4)$$

While $\Omega_g$ defined in (2.4) is an open set and may have many connected components, the domain $\Omega_g$ that we use in the sequel is the component that contains $E$. This is possible if we assume

$$g(x) \leq 0, \quad \forall x \in E, \quad \forall g \in \mathcal{F}. \quad (2.5)$$

Another variant, possible to be used in the definition of the domain $\Omega_g$, is to assume that

$$x_0 \in \partial \Omega_g, \quad \forall g \in \mathcal{F} \quad (2.6)$$

for some $x_0 \in D \setminus E$, given. One has to impose on the family $\mathcal{F}$ the simple constraint

$$g(x_0) = 0, \quad \forall g \in \mathcal{F}. \quad (2.7)$$

In this context, it is important to consider the closed bounded set:

$$G = \{ x \in D; \ g(x) = 0 \} \quad (2.8)$$

associated to any $g \in \mathcal{F}$. If $\mathcal{F} \subset C(D)$ without further conditions, then $\text{meas}(G) > 0$ is possible. We further assume, see [30], that $\mathcal{F} \subset C^1(D)$ and

$$|\nabla g(x)| > 0, \quad \forall x \in G, \quad \forall g \in \mathcal{F}. \quad (2.9)$$

Then, by (2.6)-(2.9) and the implicit functions theorem, we get $G = \partial \Omega_g$ and the Hamiltonian system

$$z_1'(t) = -\frac{\partial g}{\partial x_2} (z_1(t), z_2(t)), \quad t \in I_g, \quad (2.10)$$

$$z_2'(t) = \frac{\partial g}{\partial x_1} (z_1(t), z_2(t)), \quad t \in I_g, \quad (2.11)$$

$$(z_1(0), z_2(0)) = x_0 \in \partial \Omega_g \quad (2.12)$$
where \(I_g\) is the local existence interval for (2.10)-(2.12), gives a local parametrization of \(\partial\Omega_g\) around \(x_0\), [28]. The solution is unique due to the Hamiltonian structure [29]. We also assume that
\[
g(x) > 0, \quad \forall x \in \partial D, \quad \forall g \in F
\]
which ensures that \(G \cap \partial D = \emptyset\) for \(g \in F\).

Notice that the family \(O\) of domains defined by (2.4)-(2.5) is very rich, they may be multiply connected and this is one reason why the above approach combines boundary and topological variations in shape optimization.

Moreover, under hypothesis (2.9), we get \(\partial\Omega_g\) of class \(C^1\) and more regularity can be obtained if more regularity is imposed on \(F\). This ensures the previously mentioned regularity properties for the solution of (2.1), (2.2) and the cost (2.3) and its approximation (in the next section), are well defined.

It is proved in [30], that hypotheses (2.9) and (2.13) are sufficient for the global existence in (2.10)-(2.12).

**Theorem 2.1** For any \(x_0 \in D \setminus E\), the solution of (2.10)-(2.12) is periodic and \(I_g\) may be chosen as its period, \(I_g = [0, T_g]\).

Namely, the limit cycle situation from the Poincaré-Bendixson theory is not possible here. If \(\partial\Omega_g\) is not connected, its complete description may be obtained via (2.10)-(2.12), by choosing an initial condition on each component. Another crucial property proved in [30] is

**Theorem 2.2** Under the above hypotheses, the compact set \(G\) has a finite number of connected components, for any fixed \(g \in F\).

Clearly, the number of the connected components may be unbounded over the whole \(F\).

### 3 Approximation and existence

The approximation of shape optimization problems via cost penalization was introduced in [30] and further developed in [16]. The idea is to penalize the boundary condition on the unknown domains. This is possible due to the Hamiltonian representation of the unknown geometries, Thm. 2.1 and Thm. 2.2. We use here a penalization variant that has good differentiability properties and is formulated as an optimal control problem (\(\epsilon > 0\)):

\[
\min_{g,u} \left\{ \int_E J(x, y(x)) \, dx + \int_{I_g} j(z(t), y(z(t))) \sqrt{(z_1'(t))^2 + (z_2'(t))^2} \, dt \right. \\
+ \left. \frac{1}{\epsilon} \int_{I_g} \left[ \nabla y(z_1(t), z_2(t)) \cdot \frac{\nabla g(z_1(t), z_2(t))}{\nabla g(z_1(t), z_2(t))} \right]^2 \sqrt{(z_1'(t))^2 + (z_2'(t))^2} \, dt \right\} \quad (3.1)
\]
subject to

\[-\Delta y + y = f + g_m^2 u, \quad \text{in } D,\]
\[y = 0, \quad \text{on } \partial D,\]

and (2.5). Above \(z(t) = (z_1(t), z_2(t))\) is the solution of (2.10)-(2.12), the state \(y \in W^{2,p}(D) \cap H^1_0(D)\) from (3.2), (3.3) clearly depends on \(g \in F\) and \(u\) is measurable such that \(g_m^2 u \in L^p(D), \ p > 2\). In dimension 2, we have \(y \in C^1(D)\) by the Sobolev theorem and all the terms in (3.1) make sense. The penalization term in (3.1) is a detailed formula for

\[\int_{\partial \Omega_g} |\frac{\partial y}{\partial n}|^2 \, d\sigma\]

based on the Hamiltonian representation (2.10)-(2.12) of \(\partial \Omega_g\) and the fact that the unit normal to \(\partial \Omega_g = G\) is given by \(\frac{\nabla g(z_1(t), z_2(t))}{|\nabla g(z_1(t), z_2(t))|}\) in \((z_1(t), z_2(t)) \in \partial \Omega_g\) and it is well defined due to (2.9). In case \(\partial \Omega_g\) has several connected components (their number is finite by Thm. 2.2) then the penalization term is replaced by a finite sum of similar terms, with some initial condition in (2.10)-(2.12) fixed on each component. It is to be noticed that, in the “extended” equation (3.2), (3.3), we have Dirichlet boundary conditions, while the original state system (2.1), (2.2) is a Neumann boundary value problem. It turns out that the approximation properties of (3.1)-(3.3) remain valid even with this change of boundary conditions and we want to stress this property. In fact, it is also easier to work with (3.3) in the finite element discretization, in the next sections.

**Proposition 3.1** Let \(J(\cdot, \cdot)\) and \(j(\cdot, \cdot)\) be Carathéodory functions on \(D \times \mathbb{R}\), bounded from below by a constant and let \(F \subset C^2(D)\) satisfy (2.9), (2.13). Denote by \([y_n^m, g_n^m, u_n^m]\) a minimizing sequence in the penalized problem (3.1)-(3.3), (2.5). Then, on a subsequence denoted by \(m(n)\) the pairs \([\Omega_{g_{\text{n}}(m)}, y_{g_{\text{n}}(m)}]\) (not necessarily admissible) give a minimizing cost in (3.3), satisfy (2.1) and (2.2) is valid with a perturbation of order \(\epsilon^{1/2}\).

**Proof.** The proof follows the ideas from [30], [16]. Let \([y_{g_{m}}, g_{m}] \in W^{2,p}(\Omega_{g_{m}}) \times F\) be a minimizing sequence for the problem (2.1)-(2.5). Here, \(\partial \Omega_{g_{m}}\) is \(C^2\) and this ensures the regularity \(y_{g_{m}} \in W^{2,p}(\Omega_{g_{m}})\) due to \(f \in L^p(D)\). There is \(\bar{y}_{g_{m}} \in W^{2,p}(D \setminus \Omega_{g_{m}}), \) not unique, such that \(\bar{y}_{g_{m}} = y_{g_{m}}\) on \(\partial \Omega_{g_{m}}, \ \frac{\partial \bar{y}_{g_{m}}}{\partial n} = \frac{\partial y_{g_{m}}}{\partial n} = 0\) on \(\partial \Omega_{g_{m}}, \ \bar{y}_{g_{m}} = 0\) on \(\partial D\). We define an admissible control in (3.2) by

\[u_{g_{m}} = -\frac{\Delta \bar{y}_{g_{m}} + f - \bar{y}_{g_{m}}}{(g_{m})^2_+}, \quad \text{in } D \setminus \Omega_{g_{m}},\]

and zero otherwise. We infer by (3.4) that \((g_{m})^2 u_{g_{m}}\) is in \(L^p(D)\) and \(g_{m}, u_{g_{m}}\) is an admissible control pair for the penalized problem (3.1)-(3.3), (2.5). Moreover, the corresponding state in (3.2) is obtained by concatenation of \(y_{g_{m}}\) and \(\bar{y}_{g_{m}}\) and the corresponding penalization term in (3.1) is null. That is the corresponding costs in (3.1)
and in (2.3) are the same. This construction is also valid in the case $\Omega_{g_{m}}$ is not simply connected.

We obtain

$$
\int_{E} J(x, y_{n(m)}^{\epsilon}(x)) \, dx + \int_{I_{g_{n}(m)}} j(z_{n(m)}(t); y_{n(m)}^{\epsilon}(z_{n(m)}(t))) \, |z_{n(m)}^{\epsilon}(t)| \, dt \\
+ \frac{1}{\epsilon} \int_{I_{g_{n}(m)}} \left[ \nabla y_{n(m)}^{\epsilon}(z_{n(m)}(t)) \cdot \frac{\nabla g_{n(m)}^{\epsilon}(z_{n(m)}(t))}{|\nabla g_{n(m)}^{\epsilon}(z_{n(m)}(t))|} \right]^{2} |z_{n(m)}^{\epsilon}(t)| \, dt \\
\leq \int_{E} J(x, y_{m}(x)) \, dx + \int_{\partial \Omega_{g_{m}}} j(x, y_{m}(x)) \, d\sigma \rightarrow \inf(\mathcal{P})
$$

(3.5)

for $m \to \infty$. In (3.5), the index $n(m)$ is big enough in order to have the inequality valid and $z_{n}$ is the solution of (2.10)-(2.12) associated to $g_{n}^{\epsilon}$ (for simplicity, we don’t write $z_{n}^{\epsilon}$).

Since $J(\cdot, \cdot)$ and $j(\cdot, \cdot)$ are bounded from below by constants, from (3.5), we get the boundedness of the penalization term on the subsequence $n(m)$. This yields the last statement of Proposition 3.1 on $\partial \Omega_{g_{n}(m)}$. As $\left(g_{n}^{\epsilon}(m)\right)_{n}$ is null in $\Omega_{g_{n}(m)}$, we see that (2.1) is satisfied in $\Omega_{g_{n}(m)}$, due to (3.2). The minimizing property of the sequence $\left[\Omega_{g_{n}(m)}, y_{n}^{\epsilon}\right]$ in the original cost (2.3) is again an obvious consequence of (3.5), by the positivity of the penalization term(s).

By the Weierstrass theorem, there is $m_{g} > 0$ such that (2.9) becomes

$$
|\nabla g(x)| \geq m_{g}, \quad \forall x \in G, \quad \forall g \in \mathcal{F}.
$$

(3.6)

In order to strengthen the approximation property in Proposition 3.1, we impose that $\mathcal{F}$ is bounded in $C^{2}(\overline{D})$ and we require uniformity in (2.9), (3.6), where $m > 0$ is some given constant:

$$
|\nabla g(x)| \geq m, \quad \forall x \in G, \quad \forall g \in \mathcal{F}.
$$

(3.7)

Notice that (3.7) or the boundedness of $\mathcal{F}$ don’t modify the topological characteristics of the family of admissible domains $\Omega_{g}$, $g \in \mathcal{F}$. We denote by $y_{n,\epsilon}$ the solution of (2.1), (2.2) in $\Omega_{g_{n}}$.

**Proposition 3.2** Under the above assumptions, there is an absolute constant $C > 0$ such that

$$
|y_{n,\epsilon} - y_{n}^{\epsilon}|_{H^{1}(\Omega_{g_{n}})} \leq C\epsilon^{1/4}.
$$

**Proof.** We take the difference of the equations (2.1) in $\Omega_{g_{n}}$ corresponding to $y_{n,\epsilon}$, $y_{n}^{\epsilon}$ and we multiply by $y_{n,\epsilon} - y_{n}^{\epsilon}$. Then, we get:
\[ |y_{n,e} - y_{n}^\varepsilon|_{H^2(\Omega_{g_n}^\varepsilon)}^2 = -\int_{\partial\Omega_{g_n}^\varepsilon} (\frac{\partial y_{n,e}^\varepsilon}{\partial n})(y_{n,e} - y_{n}^\varepsilon) d\sigma \leq c\varepsilon^{1/2} |y_{n,e} - y_{n}^\varepsilon|_{L^2(\partial\Omega_{g_n}^\varepsilon)}, \]

where \( c > 0 \) is an absolute constant corresponding to the evaluation of the penalization term in (3.1), from the last statement in Proposition 3.1.

By (3.7) and Green’s formula, we have:

\[
m|y_{n,e} - y_{n}^\varepsilon|_{L^2(\Omega_{g_n}^\varepsilon)} \leq \int_{\Omega_{g_n}^\varepsilon} |y_{n,e} - y_{n}^\varepsilon|^2 |\nabla g_{n}^\varepsilon| d\sigma = \int_{\partial\Omega_{g_n}^\varepsilon} |y_{n,e} - y_{n}^\varepsilon|^2 \nabla y_{n}^\varepsilon \cdot \nu_{\varepsilon} d\sigma
\]

\[
\leq \int_{\Omega_{g_n}^\varepsilon} |y_{n,e} - y_{n}^\varepsilon|^2 |\Delta g_{n}^\varepsilon| dx + 2 \int_{\Omega_{g_n}^\varepsilon} |y_{n,e} - y_{n}^\varepsilon| |\nabla (y_{n,e} - y_{n}^\varepsilon)| \nabla g_{n}^\varepsilon| dx
\]

\[
\leq M [ |y_{n,e} - y_{n}^\varepsilon|_{L^2(\Omega_{g_n}^\varepsilon)}^2 + |y_{n,e} - y_{n}^\varepsilon|_{L^2(\Omega_{g_n}^\varepsilon)}^2 |\nabla (y_{n,e} - y_{n}^\varepsilon)|_{L^2(\Omega_{g_n}^\varepsilon)}^2 ] \leq M [ |y_{n,e} - y_{n}^\varepsilon|_{L^2(\Omega_{g_n}^\varepsilon)}^2 + \varepsilon^{1/2} |y_{n,e} - y_{n}^\varepsilon|_{L^2(\Omega_{g_n}^\varepsilon)}^2 + \varepsilon^{-1/2} |y_{n,e} - y_{n}^\varepsilon|_{L^2(\Omega_{g_n}^\varepsilon)}^2 ],
\]

where we also use the binomial inequality (with the same \( \varepsilon \) as in Proposition 3.1) together with the boundedness of \( F \) in \( C^2(D) \). The notation \( \nu_{\varepsilon} \) is the normal to the domain \( \Omega_{g_n}^\varepsilon \). Combining the above two inequalities, we end the proof. \( \square \)

**Remark 3.1** We note the very weak hypotheses on the cost functional in Proposition 3.1. Together with Proposition 3.2, the justification for the use of the control problem (3.1)-(3.3), (2.5) in the approximation of (P), is obtained. A detailed study of the convergence properties when \( \varepsilon \to 0 \), for a distributed cost functional, is performed in [30].

**Corollary 3.1** Under assumption (3.7) and the boundedness of \( F \) in \( C^1(D) \), the shape optimization problem has at least one optimal solution \( \Omega^* \).

**Proof.** Condition (3.7) allows to apply the implicit function theorem around any point \((x, y) \in G\) and to obtain the local representation of \( G \) via some function \( y = y(x) \). In particular, also taking into account the boundedness of \( F \) in \( C^1(D) \), it yields that \( y'(x) = -\frac{g_x(x, y(x))}{g_y(x, y(x))} \) is bounded, uniformly with respect to the family of admissible domains, under appropriate choices of the local axes. This allows the application of well known existence results due to Chenais (see [24], Ch. 3.3) and to end the proof. \( \square \)

### 4 Directional derivative

We consider now functional variations \( g + \lambda r, u + \lambda v, r \in F, \lambda \in \mathbb{R}, v \in L^p(D) \). In the sequel, we shall take into account the condition (2.6), (2.7) for \( g, r \) in the identification of the corresponding domains from (2.4). This is also necessary in (2.10)-(2.12) and at the numerical level it is very easy to implement (finding some \( x_0 \) arises to solve \( g(x) = 0 \),
which is a standard routine, and to use (2.10)-(2.12) to identify such initial conditions on each connected component of \( G \) by elimination; see [16] for other details). Notice that the perturbations of \( u \) are always admissible since we have no constraints on \( u \) and the perturbations of \( g \) satisfy (2.7), (2.9), (2.13) for \(|\lambda|\) small enough (depending on \( g \)).

We denote by \( y_\lambda \in W^{2,p}(D) \), \( z_\lambda \in C^1(\mathbb{R}) \) the solutions of (3.2), (3.3) and (2.10)-(2.12) corresponding to the above variations, respectively. From the previous section, we know that \( z_\lambda \) is periodic with some period \( T_\lambda > 0 \) and we take its definition interval to be \([0, T_\lambda]\). In [16], it is proved under conditions (2.9), (2.13), that \( T_\lambda \to T \) as \( \lambda \to 0 \), where \( T \) is the period of \( z \), i.e. \( I_g = [0, T] \).

**Proposition 4.1** The system in variations corresponding to (3.2), (3.3), (2.10)-(2.12) is:

\[
-\Delta q + q = g_\lambda^2 v + 2g_\lambda u r, \quad \text{in } D, \quad (4.1)
\]
\[
q = 0, \quad \text{on } \partial D, \quad (4.2)
\]
\[
w'_1 = -\nabla \partial_2 g(z) \cdot w - \partial_2 r(z), \quad \text{in } [0, T], \quad (4.3)
\]
\[
w'_2 = \nabla \partial_1 g(z) \cdot w + \partial_1 r(z), \quad \text{in } [0, T], \quad (4.4)
\]
\[
w_1(0) = 0, \quad w_2(0) = 0, \quad (4.5)
\]

where \( q = \lim_{\lambda \to 0} \frac{y_\lambda - y}{\lambda} \), \( w = [w_1, w_2] = \lim_{\lambda \to 0} \frac{z_\lambda - z}{\lambda} \) and the limits exist in \( W^{2,p}(D) \), respectively \( C^1([0, T]) \).

**Proof.** This is based on standard techniques in the calculus of variations and we quote [16] where relevant arguments can be found. \( \square \)

**Proposition 4.2** Under the above assumptions, we have:

\[
\lim_{\lambda \to 0} \frac{T_\lambda - T}{\lambda} = -\frac{w_2(T)}{z_2'(T)}
\]

if \( z_2'(T) \neq 0 \).

**Proof.** Clearly \( \nabla (g + \lambda r) \neq 0 \) on \( G_\lambda \) if \(|\lambda|\) small. Then, by the perturbed variant of (2.10)-(2.12) it yields \(|z_2'(T)\lambda + |z_2'(T)\lambda| > 0 \) and, similarly \(|z_1'(T)\lambda + |z_2'(T)\lambda| > 0 \), due to (2.9). We choose here \( z_2'(T) \neq 0 \) and, consequently, \( z_2'(T) \neq 0 \), for \( \lambda \) “small”. Then \( z_2^\lambda \) is invertible on some interval \([T - \alpha, T + \beta]\) with \( \alpha, \beta > 0 \), small, not depending on \( \lambda \), (and similarly around 0 due to the periodicity property).

This is due to \( z_\lambda \to z \in C^1([0,2T])^2 \) and \( T_\lambda \to T \). We have \( z_\lambda(T_\lambda) = x_0 \) and it yields:

\[
T_\lambda = (z_2^\lambda)^{-1}(x_0^2). \quad (4.6)
\]

We denote \( x_0^0 = z_2(T_0) \to x_0^2 \) as \( \lambda \to 0 \). We may write

\[
\frac{T_\lambda - T}{\lambda} = \frac{(z_2^\lambda)^{-1}(x_0^2) - (z_2)^{-1}(x_0^2)}{\lambda} = \frac{(z_2)^{-1}(x_0^2) - (z_2)^{-1}(x_0^2)}{\lambda}. \quad (4.7)
\]
By (4.6), (4.7) we get
\[
\frac{T_\lambda - T}{\lambda} = \frac{(z_2)^{-1}(x_0^\lambda) - (z_2)^{-1}(x_0^2)}{x_0^\lambda - x_0^2} z_2(T_\lambda) - z_2^2(T_\lambda).
\]

Passing to the limit in the above relation and using Proposition 4.1 we end the proof.
\[\square\]

**Remark 4.1** If \(z_1'(T) \neq 0\), the limit is \(-\frac{w_1(T)}{z_1'(T)}\). In general, we denote by \(\theta(g, r)\) this limit. The last condition in Proposition 4.2 is a consequence of (2.9).

To study the differentiability properties of the penalized cost function (3.1), we also assume \(f \in W^{1,p}(D)\), \(\partial D\) is in \(C^{2,1}\) and \(F \subseteq C^2(D)\). We get that \(g_2^2 \in W^{1,\infty}(D)\) and \(g_2^2 u \in W^{1,p}(D)\) if \(u \in W^{1,p}(D)\) and the solution of (3.2), (3.3) satisfies \(y \in W^{3,p}(D) \subset C^2(D)\).

**Proposition 4.3** Under the above conditions, assume that \(J(x, \cdot)\) is in \(C^1(\mathbb{R})\) and \(j(\cdot, \cdot)\) is in \(C^1(\mathbb{R}^3)\). Then, the directional derivative of (3.1), in the direction \([v, r] \in W^{1,p}(D) \times F\), is given by:
\[
\theta(g, r) \left[ j(x_0, y(x_0)) + \left| \frac{\partial y}{\partial t}(x_0) \right|^2 \right] |\nabla g(x_0)| + \int_E \partial_2 J(x, y(x)) q(x) dx \\
+ \int_0^T \nabla_1 j(z(t), y(z(t))) \cdot w(t) |z'(t)| dt \\
+ \int_0^T \partial_2 j(z(t), y(z(t))) [\nabla y(z(t)) \cdot w(t) + q(z(t))] |z'(t)| dt \\
+ \int_0^T j(z(t), y(z(t))) \frac{z'(t) \cdot w'(t)}{|z'(t)|} dt \\
+ \frac{2}{\varepsilon} \int_0^T \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|^2} \nabla r(z(t)) \cdot \nabla y(z(t)) |z'(t)| dt \\
+ \frac{2}{\varepsilon} \int_0^T \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \left[ (H y(z(t))) w(t) + \nabla q(z(t)) \right] \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} |z'(t)| dt \\
+ \frac{2}{\varepsilon} \int_0^T \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \nabla y(z(t)) \cdot \left[ \frac{(H g(z(t))) w(t)}{|\nabla g(z(t))|} \right] |z'(t)| dt \\
- \frac{\nabla g(z(t))}{|\nabla g(z(t))|^3} \nabla r(z(t)) \cdot \nabla g(z(t)) + \nabla g(z(t)) \cdot (H g(z(t))) w(t)) |z'(t)| dt \\
+ \frac{1}{\varepsilon} \int_0^T \left[ \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right]^2 \frac{z'(t) \cdot w'(t)}{|z'(t)|} dt.
\]

(4.8)
The notations are explained in the proof.

**Proof.** We compute

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ \int_E J(x, y_\lambda(x)) dx + \int_0^{T_\lambda} j(z_\lambda(t), y_\lambda(z_\lambda(t))) |z'_\lambda(t)| dt \right. \\
\left. + \frac{1}{\epsilon} \int_0^{T_\lambda} \left[ \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} \right]^2 |z'_\lambda(t)| dt - \int_E J(x, y(x)) dx \right. \\
- \int_0^T j(z(t), y(z(t))) |z'(t)| dt - \frac{1}{\epsilon} \int_0^T \left[ \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right]^2 |z'(t)| dt \right\}.
\]

Applying Proposition 4.1, (4.1), (4.2), and the differentiability hypotheses on \( J, j \), we get:

\[
\frac{1}{\lambda} \left[ \int_E J(x, y_\lambda(x)) dx - \int_E J(x, y(x)) dx \right] \to \int_E \partial_z J(x, y(x)) q(x) dx. \tag{4.9}
\]

We discuss now the term:

\[
\frac{1}{\lambda} \int_T^{T_\lambda} j(z_\lambda(t), y_\lambda(z_\lambda(t))) |z'_\lambda(t)| dt = \frac{T_\lambda - T}{\lambda} j(z_\lambda(\tau_\lambda), y_\lambda(z_\lambda(\tau_\lambda))) |z'_\lambda(\tau_\lambda)| \\
\to \theta(g, r) j(x_0, y(x_0)) |z'(T)| = \theta(g, r) j(x_0, y(x_0)) |\nabla g(x_0)|, \tag{4.10}
\]

due to (2.10)–(2.12) and Remark 4.1. Here \( \tau_\lambda \) is some intermediary point in the interval \([T, T_\lambda]\), depending on \( \lambda, g, r, j, \) etc. We also use Thm. 2.1 and \( T_\lambda \to T \).

Similarly, we consider the term:

\[
\frac{1}{\lambda} \int_T^{T_\lambda} \left[ \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} \right]^2 |z'_\lambda(t)| dt \\
\to \theta(g, r) \left[ \nabla y(x_0) \cdot \frac{\nabla g(x_0)}{|\nabla g(x_0)|} \right]^2 |\nabla g(x_0)| = \theta(g, r) |\frac{\partial y}{\partial n}(x_0)|^2 |\nabla g(x_0)|. \tag{4.11}
\]

In the last two limits, the regularity properties of \( y, z, y_\lambda, z_\lambda \) also play a key role.

Next, we investigate the last term:

\[
\frac{1}{\lambda} \left\{ \int_0^T j(z_\lambda(t), y_\lambda(z_\lambda(t))) |z'_\lambda(t)| dt \\
+ \frac{1}{\epsilon} \int_0^T \left[ \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} \right]^2 |z'_\lambda(t)| dt \\
- \int_0^T j(z(t), y(z(t))) |z'(t)| dt - \frac{1}{\epsilon} \int_0^T \left[ \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right]^2 |z'(t)| dt \right\}
\]
Clearly, the terms containing \(j(\cdot, \cdot)\) give the limit:

\[
\int_0^T \left[ \nabla_h, \nabla w(t) + \partial_j (z(t), y(z(t)) \nabla g(z(t)) \cdot w(t) \right] dt
+ \int_0^T \left[ \partial_j (z(t), y(z(t))) q(z(t)) |z'(t)| + j(z(t), y(z(t))) \frac{z'(t) \cdot w'(t)}{|z'(t)|} \right] dt \tag{4.12}
\]

where \(\nabla_h\) is the gradient of \(j(\cdot, \cdot)\) with respect to the two components of \(z\), and \(\partial_j\) is the partial derivative with respect to \(y\), other quantities are defined in (4.1)-(4.5).

Let us consider now the two terms corresponding to the penalization of Neumann boundary condition. We intercalate advantageous terms and we compute step by step:

\[
\frac{1}{\lambda} \int_0^T \left\{ \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} - \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right\} |z'_\lambda(t)| dt
= \frac{1}{\lambda} \int_0^T \left[ \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} - \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right] |z'_\lambda(t)| dt
= \int_0^T S \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} |z'_\lambda(t)| dt
+ \int_0^T S \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} |z'_\lambda(t)| dt
+ \frac{1}{\lambda} \int_0^T S \left[ \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} - \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right] |z'_\lambda(t)| dt
= I + II + III \tag{4.13}
\]

where \(S\) is the sum

\[
\nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla (g + \lambda r)(z_\lambda(t))}{|\nabla (g + \lambda r)(z_\lambda(t))|} + \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|}.
\]

We have:

\[
\lim_{\lambda \to 0} I = 2 \int_0^T \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \nabla r(z(t)) \cdot \nabla y_\lambda(z_\lambda(t)) |z'(t)| dt
\]

\[
\lim_{\lambda \to 0} II = 2 \int_0^T \nabla y_\lambda(z_\lambda(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} [H y(z(t)) + \nabla q(z(t))] \cdot \nabla g(z(t)) |z'(t)| dt,
\]

where \(H y\) is the Hessian matrix of \(y \in C^2(\mathcal{D})\).
Concerning part III, we get:

\[
\lim_{\lambda \to 0} III = 2 \int_0^T \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} |z'(t)| |\nabla y(z(t))| \left[ \frac{(H g(z(t))) w(t)}{|\nabla g(z(t))|} \right] dt \\
- \frac{\nabla g(z(t))}{|\nabla g(z(t))|^2} \left( \frac{\nabla g(z(t)) \cdot \nabla r(z(t))}{|\nabla g(z(t))|} + \frac{\nabla g(z(t)) \cdot (H g(z(t))) w(t)}{|\nabla g(z(t))|} \right) [z'(t)] dt \\
= 2 \int_0^T \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \nabla y(z(t)) \cdot \left[ \frac{(H g(z(t))) w(t)}{|\nabla g(z(t))|} \right] |z'(t)| dt \\
- \frac{\nabla g(z(t))}{|\nabla g(z(t))|^3} (\nabla g(z(t)) \cdot \nabla r(z(t)) + \nabla g(z(t)) (H g(z(t))) w(t)) [z'(t)] dt \\
\]

Finally, the term

\[
\int_0^T \left[ \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right]^2 \frac{|z'_\lambda(t)| - |z'(t)|}{\lambda} dt \\
\rightarrow \int_0^T \left[ \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} \right]^2 \frac{z'(t) \cdot w'(t)}{|z'(t)|} dt \tag{4.14}
\]

Summing up relations (4.9)-(4.14), we finish the proof of (4.8). \(\Box\)

5 Finite element descent directions

We use the piecewise cubic finite element \(P_3\) in \(\mathcal{T}_h\) a triangulation of \(D\). We define

\[
\mathbb{W}_h = \{ \varphi_h \in \mathcal{C}(\overline{D}); \varphi_h|_T \in P_3(T), \forall T \in \mathcal{T}_h \}
\]

of dimension \(n = \text{card}(I)\) (\(I\) the set of nodes in \(\mathcal{T}_h\)) and

\[
\mathbb{V}_h = \{ \varphi_h \in \mathbb{W}_h; \varphi_h = 0 \text{ on } \partial D \},
\]

of dimension \(n_0 = \text{card}(I_0)\) (\(I_0\) the set of nodes in \(\mathcal{T}_h\), outside \(\partial D\)) which are finite element approximations of Hilbert spaces \(\mathbb{W} = H^1(D), \mathbb{V} = H^1_0(D)\), respectively.

The parametrization function \(g\) is approached by the finite element function \(g_h \in \mathbb{W}_h\), \(g_h(x) = \sum_{i \in I} g_i \phi_i(x)\) where \(G = (G_i)_{i \in I} \in \mathbb{R}^n\) is a real vector and \(\phi_i\) is the basis in \(\mathbb{W}_h\). Similarly, we denote \(u_h \in \mathbb{W}_h, y_h \in \mathbb{V}_h\) and the associated vectors \(U = (U_i)_{i \in I} \in \mathbb{R}^n\) and \(Y = (Y_j)_{j \in I_0} \in \mathbb{R}^{n_0}\) for the discretization of the control, respectively the state. For the control term \(u_h\), one can also employ lower order finite elements, like continuous piecewise linear \(P_1\) or piecewise constant \(P_0\). See [3], [20] for a discussion of finite element spaces.
Here, we consider (2.1) with non homogeneous boundary condition \( \frac{\partial y}{\partial n} = \delta \) on \( \partial \Omega \), with \( \delta \) some given function in \( H^1(D) \). The objective function (3.1) is taken of the form

\[
\min_{g,u} J(g,u) = \left\{ \int_E J(x,y(x)) \, dx + \int_{I_g} j(z(t),y(z(t))) |z'(t)| \, dt \right. \\
+ \frac{1}{\epsilon} \int_{I_g} \left[ \nabla y(z(t)) \cdot \frac{\nabla g(z(t))}{|\nabla g(z(t))|} - \delta(z(t)) \right]^2 |z'(t)| \, dt \right\}. \tag{5.1}
\]

We denote the first term of (5.1) by

\[ t_1 = \int_E J(x,y(x)) \, dx. \]

The second and the third terms of (5.1) can be rewritten as integrals on \( \partial \Omega_g \), more precisely

\[ t_2 = \int_{\partial \Omega_g} j(s,y(s)) \, ds \]
\[ t_3 = \frac{1}{\epsilon} \int_{\partial \Omega_g} \left[ \nabla y(s) \cdot \frac{\nabla g(s)}{|\nabla g(s)|} - \delta(s) \right]^2 \, ds. \]

We employ the software FreeFem++, and these terms can be computed with the command `int1d(Th, levelset=gh)(...)`.

We use the general descent direction method

\( (G^{k+1}, U^{k+1}) = (G^k, U^k) + \lambda_k (R^k, V^k), \)

where \( \lambda_k > 0 \) is obtained via some line search

\[ \lambda_k \in \arg \min_{\lambda > 0} J((G^k, U^k) + \lambda (R^k, V^k)) \]

and \( (R^k, V^k) \) is a descent direction, i.e. \( dJ(G^k, U^k)(R^k, V^k) < 0 \). For \( E \neq \emptyset \), a projection is necessary in order to get (2.4). The algorithm stops if \( |J(G^{k+1}, U^{k+1}) - J(G^k, U^k)| < \text{tol} \) or \( dJ(G^k, U^k)(R^k, V^k) = 0 \). Other choices are possible, see [6] for details on such algorithms.

Since the approximating state system (3.2), (3.3) is similar to [16], we apply here a similar discretization technique of the gradient (4.8). In the following, we shall use descent directions based on the discrete simplified adjoint system: find \( p_h \in V_h \) such that

\[
\int_D \nabla \varphi_h \cdot \nabla p_h \, dx + \int_D \varphi_h p_h \, dx = \int_E \partial_2 J(x,y_h(x)) \varphi_h(x) \, dx \\
+ \int_{\partial \Omega_h} \partial_2 j(s,y_h(s)) \varphi_h(s) \, ds \\
+ \frac{2}{\epsilon} \int_{\partial \Omega_h} \left( \nabla y_h(s) \cdot \frac{\nabla g_h(s)}{|\nabla g_h(s)|} - \delta_h(s) \right) \nabla \varphi_h(s) \cdot \frac{\nabla g_h(s)}{|\nabla g_h(s)|} \, ds \tag{5.2}
\]
for all \( \varphi_h \in V_h \). In the right hand side of (5.2) appear just the terms multiplying \( q \) in the gradient (4.8) and \( \delta_h(s) \) is a continuous piecewise linear \( P_1 \) discretization of \( \delta(s) \) in \( D \).

**Proposition 5.1** Given \( g_h, u_h \in W_h \) and the variations \( r_h, v_h \in W_h \), let \( y_h \in V_h \) be the finite element solution of (3.2), (3.3), let \( q_h \in V_h \) be the finite element solution of (4.1), (4.2) depending in \( r_h, v_h \) and let \( p_h \in V_h \) be the solution of (5.2). Then

\[
\int_E \partial_2 J(x, y_h(x)) q_h(x) dx + \int_{\partial \Omega} \partial_2 j(s, y_h(s)) q_h(s) ds
+ \frac{2}{\epsilon} \int_{\partial \Omega} \left( \nabla y_h(s) \cdot \frac{\nabla g_h(s)}{|\nabla g_h(s)|} - \delta_h(s) \right) \nabla q_h(s) \cdot \frac{\nabla g_h(s)}{|\nabla g_h(s)|} ds \leq 0
\]

if we choose:

i) \( r_h = -p_h u_h \) and \( v_h = -p_h \) or

ii) \( r_h = -\tilde{d}_h \) and \( v_h = -p_h \) where \( \tilde{d}_h \in W_h \) is the solution of

\[
\int_D \nabla \tilde{d}_h \cdot \nabla \varphi_h dx + \int_D \tilde{d}_h \varphi_h dx = \int_D 2(g_h)_+ u_h p_h \varphi_h dx
\]

for all \( \varphi_h \in W_h \).

**Proof.** Putting \( \varphi_h = q_h \) in (5.2) and multiplying (4.1) by \( p_h \), integrating by parts over \( D \) and using (4.2), we get that the left hand side of (5.3) is equal to:

\[
\int_D (g_h)_+^2 v_h p_h dx + \int_D 2(g_h)_+ u_h r_h p_h dx.
\]

For \( v_h = -p_h \), we have

\[
\int_D (g_h)_+^2 v_h p_h dx = - \int_D (g_h)_+^2 p_h^2 dx \leq 0.
\]

If \( (g_h)_+ p_h \) is not null, then the above inequality is strict.

Case i). For \( r_h = -p_h u_h \), we have

\[
\int_D 2(g_h)_+ u_h r_h p_h dx = - \int_D 2(g_h)_+ (u_h p_h)^2 dx \leq 0.
\]

Case ii). For \( r_h = -\tilde{d}_h \), we have

\[
\int_D 2(g_h)_+ u_h r_h p_h dx = - \int_D 2(g_h)_+ u_h p_h \tilde{d}_h dx
= - \int_D \nabla \tilde{d}_h \cdot \nabla \tilde{d}_h dx - \int_D \tilde{d}_h \tilde{d}_h dx \leq 0.
\]

The second equality is obtained by putting \( \varphi_h = \tilde{d}_h \) in (5.4). This ends the proof. If \( (g_h)_+ p_h \) is not null, then the inequality (5.3) is strict. \( \Box \)
Remark 5.1 Due to the strong non convex character of the shape optimization problems, the descent algorithms find just a local minimum point of the penalized problem, in general. The penalization term may remain not null, that is the constraint \(2.2\) may be violated. However, the above methodology offers a systematic and general approximation procedure that can be applied in many examples and produces relevant results. Both topological and boundary variations are performed simultaneously.

6 Numerical tests

Example 1.

We choose \(D = [3, 3[x] - 3, 3[y] = x_1^2 + x_2^2 - 1^2, \quad f(x) = -4 + y_d(x)\) and the tracking type cost \(j(x) = \frac{1}{2} (y(x) - y_d(x))^2\). We fix \(\delta = 2\) for the non homogeneous Neumann boundary condition. We consider first the case \(E = \emptyset\) and \(J = 0\), with the numerical parameters: \(\epsilon = 0.5\), the mesh of \(D\) has 73786 triangles and 37254 vertices and the tolerance parameter for the stopping test is \(tol = 10^{-6}\).

The initial domain is the disk of center \((0, 0)\) and radius 2.5 with a circular hole of center \((-1, -1)\) and radius 0.5. The corresponding \(g_0(x_1, x_2)\) is given by

\[
\max \left( (x_1)^2 + (x_2)^2 - 2.5^2, -(x_1 + 1)^2 - (x_2 + 1)^2 + 0.5^2 \right).
\]

The initial guess for the control is \(u_0 = 0\).

We use the descent direction given by the Proposition \ref{prop:desc_dir} case ii) and the algorithm stops after 3 iterations. For the stopping test, we have computed just the left hand side of \(5.3\) and we replaced \(dJ(G^k, U^k)(R^k, V^k) = 0\) by: there are no smaller values than \(J(G^k, U^k)\) in the direction \((R^k, V^k)\) for \(\lambda \in \{ \rho^i; i \in \mathbb{N}, 0 \leq i < 30\}\), with \(\rho = 0.8\).

We can observe in Figure 1 the evolution of the domain (both boundary and topological changes) and in Table 1 the corresponding values of the objective function. For \(u_0 = 0\), we get \(g_1 = g_0\), but we have, for the cost functional, \(J_1 < J_0\), since there is minimization with respect to the control \(u\). We do not plot in Figure 1 the domain for \(k = 1\) because it is the same as for \(k = 0\), but there is a column in Table 1 corresponding to \(k = 1\), showing the evolution of the penalized cost.

For the solution of the elliptic problem \(2.1\)-\(2.2\) in the computed domains \(\Omega_g\), we obtain in fact the best value \(t_2 = 46.59\) (see Table 2), which is consistently better than \(t_2 = 67.60\) obtained for the solution of \(3.2\)-\(3.3\) in \(D\), in the corresponding iteration of the algorithm. This is due to the value of the penalization term \(t_3\), which remains “far” from zero. Such situations are frequent in penalization approaches for nonconvex minimization problems.
Figure 1: Example 1. Initial domain $k = 0$ (top, left), intermediate domains during the line-search after $k = 1$ and the final domain $k = 2$ (bottom, right).

| iteration | $k=0$ | $k=1$ | $k=2$ |
|-----------|-------|-------|-------|
| $t_2$     | 220.87| 171.13| 67.60 |
| $t_3$     | 35.50 | 34.63 | 54.75 |
| $J$       | 291.89| 240.39| 177.12|

Table 1: Example 1. The computed objective function $J = t_2 + \frac{1}{2} t_3$. The columns 4, 5, 6, 7 correspond to the intermediate configurations obtained during the line-search after $k = 1$. The descent property is valid just for the total cost, on the last line.

| iteration | $k=0$ | $k=1$ | $k=2$ |
|-----------|-------|-------|-------|
| $t_2$     | 96.39 | 74.76 | 46.59 |

Table 2: Example 1. The values of $t_2$ for the finite element solution of (2.1)-(2.2) in the domains presented in Figure 1.
Example 2.

We study now a case with $E \neq \emptyset$. The $D$, $y_d$, $f$, $\delta$ are the same as in Example 1. The observation domain $E$ is the disk of center $(0, 0)$ and radius 0.5 and we take $J(x) = \frac{1}{2} (y(x) - y_d(x))^2$ and $j = 0$. We fix $\epsilon = 0.9$ and the other numerical parameters are the same as in Example 1. Such a choice of a “big” penalization parameter (similar with the previous example) has the consequence that the constraint (2.2) is consistently relaxed and allows a large choice of descent directions.

For $g_0(x_1, x_2)$, given by

$$\max \left( (x_1 + 0.8)^2 + (x_2 + 0.8)^2 - 1.8^2, -(x_1 + 0.8)^2 - (x_2 + 0.8)^2 + 0.6^2 \right)$$

we obtain as initial domain the ring of center $(-0.8, -0.8)$, exterior radius 1.8 and interior radius 0.6.

In order to observe during the algorithm the restriction (2.5), we use the descent direction method with projection, see [6]. The descent direction is given by the Proposition 5.1, case ii) and the projection is computed as follows: $\Pi(g) = g_E$ in $E$ and $\Pi(g) = g$ outside $E$, where $g_E \in F$ is such that $g_E(x) < 0$ if and only if $x \in E$. In our test, $g_E(x_1, x_2) = (x_1)^2 + (x_2)^2 - 0.5^2$. The line search, with projection only for the parametrization function, is

$$\lambda_k \in \arg \min_{\lambda > 0} \mathcal{J}(\Pi(G^k + \lambda R^k), U^k + \lambda V^k)$$

and the next iteration is defined by

$$G^{k+1} = \Pi(G^k + \lambda_k R^k), \quad U^{k+1} = U^k + \lambda_k V^k.$$  

The initial guess for the control is $u_0 = 1$.

| iteration | k=0 | k=1 | k=2 |
|-----------|-----|-----|-----|
| $t_1$     | 8.03 | 4.00 | 0.35 |
| $t_3$     | 234.91 | 198.08 | 193.56 |
| $\mathcal{J}$ | 269.05 | 223.46 | 215.42 |

Table 3: Example 2. The computed objective function $\mathcal{J} = t_1 + \frac{1}{\epsilon}t_3$. The columns 3, 4, 5 correspond to the intermediate configurations obtained during the line-search after $k = 0$.

The domain evolution is presented in Figure 2 and the corresponding values of the objective function are in Table 3.

For the finite element solution of (2.1)-(2.2) in the domains presented in Figure 2, we have reported $t_1$ in Table 4. Due to the low value of the initial cost, we notice the oscillations around this value and the minimal cost is attained already in the first step of the line search. The interpretation of the penalization term is similar as in the previous example.
Figure 2: Example 2. Domain for $k = 0$ (top, left), intermediate domains during the line-search after $k = 0$, domain for $k = 1$ (bottom, middle) and the final domain for $k = 2$ (bottom, right).

Table 4: Example 2. The values of $t_1$ for the finite element solution of (2.1)-(2.2) in the domains presented in Figure 2.
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