Abstract

In this episode, it is shown how the octonion X-product is related to $E_8$ lattices, integral domains, sphere fibrations, and other neat stuff.
1. Introduction.

Let $\mathbf{O}$ be the octonion algebra [1], an 8-dimensional real division algebra, both noncommutative and nonassociative, and the last in the finite sequence of real division algebras including the reals, $\mathbb{R}$, complexes, $\mathbb{C}$, and quaternions, $\mathbb{Q}$. Let $e_a$, $a = 0, 1, \ldots, 7$, be a basis for $\mathbf{O}$, with

$$e_0 = 1,$$

the identity, and

$$(e_a)^2 = -1, \ a \in \{1, \ldots, 7\}. \quad (2)$$

These latter elements also anticommute:

$$e_a e_b = -e_b e_a, \ a \neq b \in \{1, \ldots, 7\}. \quad (3)$$

Finally, we choose an octonion multiplication whose quaternionic triples are determined by the cyclic product rule,

$$e_a e_{a+1} = e_{a+5}, \ a \in \{1, \ldots, 7\}, \quad (4)$$

where the indices in (4) are from 1 to 7, modulo 7 (and in particular we will set 7=7 mod 7 to avoid confusing $e_0$ with $e_7$). Given (4), the following useful property is also satisfied:

$$e_a e_b = \pm e_c \implies e_{2a} e_{2b} = \pm e_{2c}, \ a, b, c \in \{1, \ldots, 7\} \quad (5)$$

(this is the index doubling property; because of (4) and (5), proofs of many octonion properties can be done very generally by proving the property in one example only).

Let $X \in \mathbf{O}$ be a unit element. That is,

$$\|X\|^2 = XX^\dagger = (X^0 + \sum_{a=1}^7 X^ae_a)(X^0 - \sum_{a=1}^7 X^ae_a) = \sum_{a=0}^7 (X^a)^2 = 1. \quad (6)$$

So,

$$X \in S^7, \quad (7)$$

the 7-sphere. Because $\mathbf{O}$ is nonassociative, if $X \in S^7$, and $A, B \in \mathbf{O}$ are two other elements, then

$$A \circ_X B \equiv (AX)(X^\dagger B) \neq AB \quad (8)$$

in general. However, if we fix $X$, then $\mathbf{O}_X$, which denotes $\mathbf{O}$ with its original product replaced by the so-called X-product (8) (see [2], and also [1][3]), is yet another copy of the octonions, isomorphic to the starting copy (or any other copy).
In [3] I showed that if $X \in \Xi_0 \cup \Xi_1 \cup \Xi_2 \cup \Xi_3$, where

\[
\begin{align*}
\Xi_0 &= \{\pm e_a\}, \\
\Xi_1 &= \{(\pm e_a \pm e_b) / \sqrt{2} : a, b \text{ distinct}\}, \\
\Xi_2 &= \{(\pm e_a \pm e_b \pm e_c \pm e_d) / 2 : a, b, c, d \text{ distinct}, e_a(e_b(e_c(e_d))) = \pm 1\}, \\
\Xi_3 &= \{(\sum_{a=0}^7 \pm e_a) / \sqrt{8} : \text{odd number of '+'}s\}, \\
& \quad a, b, c, d \in \{0,\ldots,7\},
\end{align*}
\]

then for all $a, b \in \{0,\ldots,7\}$, there is some $c \in \{0,\ldots,7\}$ such that

\[
e_a \circ_X e_b = \pm e_c
\]

(in [3] the superscript +5 was affixed to the sets $\Xi_m$ to indicate that the starting multiplication was that determined by (4), but as I am using only this one starting multiplication here, I will dispense with the superscripts). That is, in this case $O_X$ can be obtained from $O$ by a rearrangement of the indices in $\{1,\ldots,7\}$ [3].

2. $E_8$ Lattices and Integral Domains.

The 240 elements of $\Xi_0 \cup \Xi_2$ are the nearest neighbors (first shell) to the origin of an $E_8$ lattice (so are the 240 elements of $\Xi_1 \cup \Xi_3$ (see [4][5])). Define

\[
\varepsilon^h_8 = G^h[\Xi_0 \cup \Xi_2], \quad h = 1, \ldots, 7,
\]

where $G^h$ is the $O(8)$ reflection taking $e_0 \leftrightarrow e_h$. These 7 sets are nearest neighbor points for 7 different $E_8$ lattices, but in this case it is well known [4][5] that the 240 points of $\varepsilon^h_8$, for each $h = 1, \ldots, 7$, close under multiplication (however, because of nonassociativity they do not form a finite group). One may also think of $\varepsilon^h_8$ as being the unital elements of a noncommutative and nonassociative integral domain.

It should be fairly obvious that if

\[
X \in \varepsilon^h_8,
\]

then $\varepsilon^h_{8X}$, the X-product variant of $\varepsilon^h_8$, is also closed under its multiplication, since for all $A, B \in \varepsilon^h_8$,

\[
AX \in \varepsilon^h_8 \quad \text{and} \quad X^\dagger B \in \varepsilon^h_8 \implies (AX)(X^\dagger B) \in \varepsilon^h_8.
\]
(Clearly if \( X \in \mathcal{E}_8^h \), then \( X^\dagger \in \mathcal{E}_8^h \).) However, only if \( X \) is also an element of \( \Xi_0 \cup \Xi_2 \) will the resulting X-product also satisfy (10). For example,

\[
\mathcal{E}_8^7 \cap \Xi_0 = \Xi_0,
\]

and

\[
\mathcal{E}_8^7 \cap \Xi_2 = \{(\pm 1 \pm e_1 \pm e_5 \pm e_7)/2, (\pm e_2 \pm e_3 \pm e_4 \pm e_6)/2, (\pm e_4 \pm e_6 \pm e_1 \pm e_5)/2, (\pm 1 \pm e_4 \pm e_6 \pm e_7)/2, (\pm e_1 \pm e_5 \pm e_1 \pm e_5)/2\}.
\]

Since \( \pm X \) results in the same X-product, there are 8 X-product variants \( \mathcal{E}_8^7 \) arising from \( \mathcal{E}_8^7 \cap \Xi_0 \), and \((6 \times 16)/2 = 48\) arising from \( \mathcal{E}_8^7 \cap \Xi_2 \). So there are 56 X-product variants of \( \mathcal{E}_8^7 \) (and by virtue of index cycling, all the \( \mathcal{E}_8^h \)) that satisfy (10), and close under the new X-product. (There are clearly other X-product variants that close but do not satisfy (10), that is, for which the resulting multiplication is not the result of a simple index rearrangement [3].)

3. \( \mathcal{E}_8^0 \equiv \Xi_0 \cup \Xi_2 \) is Closed.

Actually,

\[
\mathcal{E}_8^0 \equiv \Xi_0 \cup \Xi_2
\]

is not closed under the multiplication we started with. For example,

\[
(1 + e_1 + e_2 + e_6)/2, \quad (1 + e_1 + e_3 + e_4)/2 \in \mathcal{E}_8^0,
\]

but

\[
(1 + e_1 + e_2 + e_6)(1 + e_1 + e_3 + e_4)/4 = (e_1 + e_2 + e_4 + e_5)/2 \notin \mathcal{E}_8^0,
\]

because

\[
e_1(e_2(e_4e_5)) = e_1(e_2e_2) = -e_1 \neq \pm 1.
\]

However, take a look at the \( \mathcal{E}_8^h, \ h = 1, \ldots, 7 \). These satisfy

\[
\mathcal{E}_8^h = \Xi_0 \cup \Xi_2^h,
\]

where

\[
\Xi_2^h = \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct},
\quad e_a(e_b(e_c,e_d)) = \pm e_h \text{ if exactly one of } a, b, c, d = 0 \text{ or } h,
\quad e_a(e_b(e_c,e_d)) = \pm 1 \text{ otherwise } \}.
\]

As it turns out, we can achieve very much the same thing on \( \mathcal{E}_8^0 \) using the X-product. Consider

\[
X = (1 + e_7)/\sqrt{2} \in \Xi_1.
\]
Let
\[ A = (\pm e_a \pm e_b \pm e_c \pm e_d)/2 \in \Xi_2, \] (15)
so
\[ e_a(e_b(e_c e_d)) = \pm 1. \] (16)
If, however, we modify the product, using the X-product (8) with \( X \) given in (14), then the bits of \( A \) in (15) satisfying (16) also satisfy
\[ e_a \circ_X (e_b \circ_X (e_c \circ_X e_d)) = \pm e_7 \]
if exactly one of \( a, b, c, d = 0 \) or \( h = 7 \),
\[ e_a(e_b(e_c e_d)) = \pm 1 \] otherwise. (17)
In other words (see (13)), \( \mathcal{E}_8^0 \) is closed under this particular X-product. Note, for example, that
\[ (1 + e_1 + e_2 + e_6) \circ_X (1 + e_1 + e_3 + e_4)/4 = (e_2 + e_3 + e_4 + e_6)/2 \in \Xi_2, \] (18)
since
\[ e_2(e_3 e_4 e_6) = e_2(e_3 e_7) = e_2(e_2) = -1 \]
(don’t forget, \( \Xi_2 \) is defined in terms of the beginning product, not the X-product variant). Our principal result is then the following:
\[ \mathcal{E}_8^0 \text{ is X-product closed if } X \in \Xi_1. \] (19)
Given that modulo sign change \( \Xi_1 \) has 56 elements, there are also therefore 56 X-products variants \( \mathcal{E}_{8X}^0 \) of \( \mathcal{E}_8^0 \) that are closed under multiplication, and from which we may define integral domains.

4. Sphere Fibrations to Lattice Fibrations.

Let
\[ X = X^0 + X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7 \in S^7. \]
Then
\[ e_1 \circ_X e_2 = \]
\[ ((X^0)^2 + (X^1)^2 + (X^2)^2 + (X^6)^2 - (X^3)^2 - (X^4)^2 - (X^7)^2)e_6 + 2(X^0X^5 + X^1X^7 - X^2X^4 + X^3X^6)e_3 + 2(-X^0X^7 + X^1X^5 + X^2X^3 + X^4X^6)e_4 + 2(-X^0X^3 - X^1X^4 - X^2X^7 + X^5X^6)e_5 + 2(X^0X^4 - X^1X^3 + X^2X^5 + X^7X^6)e_7 = Y = Y^6e_6 + Y^3e_3 + Y^4e_4 + Y^5e_5 + Y^7e_7 \]
(20)
(see [1][3]) defines a map from \( S^7 \rightarrow S^4 \). That is,
\[ (Y^6)^2 + (Y^3)^2 + (Y^4)^2 + (Y^5)^2 + (Y^7)^2 = 1. \]
(21)
Therefore, relative to this X-product, the set \( \{ e_1, e_2, Y \} \) is a quaternionic triple. This implies that the set of all
\[ U = \exp(\theta^1e_1 + \theta^2e_2 + \theta^3Y) \]
(22)
is just \( SU(2) \simeq S^3 \), and
\[ e_1 \circ_{(U_X)} e_2 = (e_1 \circ_X U) \circ_X (U^\dagger \circ_X e_2) = e_1 \circ_X e_2 = Y \]
(23)
(see [3]), since the \( U \) and \( U^\dagger \) cancel each other out because these two elements are part of the quaternionic subalgebra of \( O_X \) generated by \( e_1 \) and \( e_2 \). Therefore,
\[ \{ UX : U = \exp(\theta^1e_1 + \theta^2e_2 + \theta^3Y) \} \simeq S^3 \]
(24)
is the \( S^3 \) fibre over \( Y \in S^4 \) in the exact sequence
\[ S^3 \rightarrow S^7 \rightarrow S^4, \]
(25)
implicit in (20-24). (Clearly the map (20) could be replaced by
\[ X \rightarrow e_a \circ_X e_b, \ a \neq b \in \{ 1, ..., 7 \}; \]
many other possibilities exist, which I will leave to the reader to explore.)

All of this translates to lattices, the shells of which are discrete versions of spheres. In particular, let
\[ X \in \mathcal{E}_8^0 \equiv \Xi_0 \cup \Xi_2 \subset S^7. \]
(26)
Consider the map

\[ X \longrightarrow e_1 \circ X \circ e_2. \]  

(27)

Because of (10), the image of this map is the ten element set

\[ Z^5 \equiv \{ \pm e_6, \pm e_3, \pm e_4, \pm e_5, \pm e_7 \} \subset S^4, \] 

(28)

which is the inner shell of the 5-dimensional cubic lattice, \( Z^5 \) (see [5]). Consider the fibre of elements of \( E_0^8 \) mapping to \( e_6 \in Z^5 \), which is

\[ D_4 \equiv \{ \pm 1, \pm e_1, \pm e_2, \pm e_6 \} \cup \{ (\pm 1 \pm e_1 \pm e_2 \pm e_6)/2 \} \subset S^3 \longrightarrow \{ e_6 \}, \] 

(29)

which is the inner shell of a 24-dimensional \( D_4 \) lattice and integral domain (see [4][5]). Generalizing further from (25), we have an exact sequence

\[ D_4 \subset S^3 \longrightarrow E_0^8 \subset S^7 \longrightarrow Z^5 \subset S^4. \] 

(30)

• NOTE: 10 \( \times \) 24 = 240, which I leave to the reader to prove.

5. Conclusion.

The motivation for this work is curiousity fired by the beauty of the mathematics. I have only scratched the surface of this vast interconnected mathematical realm, and many of its connections I will never see. But I share a profound belief that the design of our physical reality is intimately linked to this web of mathematical notions [1], and this is my way of gaining a better understanding of the web and its power.
References

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