An answer to a question of Coleman on scattered sets

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Abstract
The aim of this paper is to show that every scattered subset of a dense-in-itself semi-$T_D$-space is nowhere dense. We are thus able to answer a recent question of Coleman [1] in the affirmative. In terms of Digital Topology, we prove that in semi-$T_D$-spaces with no open screen, trace spaces have no consolidations.

1 Introduction

It is well-known that in dense-in-themselves $T_1$-spaces, all scattered subsets are nowhere dense. This result was established by Kuratowski in the proof that in $T_1$-spaces the finite union of scattered subsets is scattered.

In a recent paper Coleman asked the following question [1, Question 4]: Is it true that in dense-in-themselves, $T_D$-spaces all scattered sets are nowhere dense? In what follows, we will show that even in dense-in-themselves semi-$T_D$-spaces all $\alpha$-scattered sets are nowhere dense.

The question of Coleman is in fact very well motivated not only because it is interesting to know how low one can go on the separations below $T_1$ and still have the scattered sets being

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nowhere dense but also from a ‘digital point of view’. In terms of Digital Topology, we will prove that in semi-$T_D$-spaces with empty open screen, trace spaces have no consolidations.

In Digital Topology several spaces that fail to be $T_1$ are very often important in the study of the geometric and topological properties of digital images [6, 7]. Such is in fact the case with the major building block of the digital n-space – the digital line or the so called Khalimsky line. This is the set of the integers, $\mathbb{Z}$, equipped with the topology $\mathcal{K}$, generated by $\mathcal{G}_K = \{\{2n - 1, 2n, 2n + 1\} : n \in \mathbb{Z}\}$.

A fenestration [7] of a space $X$ is a collection of disjoint nonempty open sets whose union is dense. The consolidation $A^+$ [7] of a set $A$ is the interior of its closure. When there is a fenestration of a space $(X, \tau)$ by singletons, the space $(X, \tau)$ is called $\alpha$-scattered [4] or a trace space [7]. For example, in the digital line the collection $\{\{n\} : n \in \mathbb{Z} \text{ and } n \text{ is odd}\}$ is a fenestration of $(\mathbb{Z}, \mathcal{K})$. All scattered sets are $\alpha$-scattered by not vice versa [4]. In $T_0$-spaces without isolated points, we may encounter a trace space which fails to be nowhere dense [1]. Nevertheless, as we will show, with the presence of the very weak separation 'semi-$T_D$', in spaces with no isolated points we have all $\alpha$-scattered sets being nowhere dense.

A topological space $X$ is a called a $T_D$-space if every singleton is locally closed or equivalently if the derived set $d(x)$ is closed for every $x \in X$. Recall that $X$ is a semi-$T_D$-space if every singleton is open or semi-closed [3]. Recall that a subset $A$ of a space $(X, \tau)$ is called locally dense [2] if $A \subseteq A^+$. Note that every open and every dense set is locally dense.

2 When is $\mathcal{N}$ finer than $S$?

Recall that a topological ideal $\mathcal{I}$ is a nonempty collection of sets of a space $(X, \tau)$ closed under heredity and finite additivity. For example, the families $\mathcal{N}$ (of all nowhere dense sets) and $\mathcal{F}$ (of all finite sets) always form ideals while the family $S$ of all scattered sets is an ideal if and only if the space is $T_0$.

**Proposition 2.1** For a topological space $(X, \tau)$ the following conditions are equivalent:

1. $X$ is a dense-in-itself semi-$T_D$-space.
2. Every singleton is nowhere dense.
There are no locally dense singletons in $X$.

Proof. (1) ⇒ (2) Let $x \in X$. Since $X$ is a semi-$T_D$-space, $\{x\}$ is open or semi-closed. Since $X$ is dense-in-self, $\{x\}$ is semi-closed. On the other hand, in any topological space every singleton is locally dense (= preopen) or nowhere dense. If $\{x\}$ is preopen, then it must be (due to semi-closedness) regular open. As $X$ has no isolated points, we conclude that $\{x\}$ is nowhere dense.

(2) ⇒ (3) Obvious, since the ideal of nowhere dense sets is closed under finite additivity.

(3) ⇒ (4) Follows easily from the fact that singletons are either locally dense or nowhere dense.

(4) ⇒ (1) If some point $x \in X$ were isolated, then it would be locally dense. This shows that $X$ is dense-in-itself. That $X$ is a semi-$T_D$-space follows easily from the fact that nowhere dense sets are semi-closed.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\beta$-open if $A$ is dense in some regular closed subspace of $X$. Note that every locally dense set is $\beta$-open.

Observation 2.2 (i) Every $\beta$-open subset of a dense-in-itself semi-$T_D$-space is also dense-in-itself and semi-$T_D$.

(ii) Let $(X_i, \tau_i)_{i \in I}$ be a family of topological spaces such that at least one of them is a dense-in-itself semi-$T_D$-space. Then the product space $X = \prod_{i \in I} X_i$ is also dense-in-itself and semi-$T_D$.

Theorem 2.3 If a topological space $(X, \tau)$ is dense-in-itself and semi-$T_D$, then every $\alpha$-scattered subset of $X$ is nowhere dense.

Proof. Let $A \subseteq X$ be $\alpha$-scattered. Assume that $A^+\neq \emptyset$ is nonempty, i.e., there exists a nonempty $U \in \tau$ such that $U \subseteq \overline{A}$. Since $(A, \tau|A)$ is $\alpha$-scattered, $U$ meets $I(A)$, the set of all isolated points of $(A, \tau|A)$. Let $x \in U \cap I(A)$ and let $V$ be an open subset of $(X, \tau)$ such that $V \cap A = \{x\}$. Set $W = U \cap V$. Note that $W \subseteq V \cap \overline{A} \subseteq \overline{V \cap A} = \overline{x}$ and so $\{x\}$
has nonempty consolidation, i.e. it is not nowhere dense in $X$. By Proposition 2.1, we have a contradiction. Hence, $A$ is nowhere dense. \[\square\]

The digital interpretation of Theorem 2.3 is as follows: In semi-$T_D$-spaces with no isolated points, i.e., semi-$T_D$-spaces with empty open screens, the trace spaces have empty consolidations.

Now, we can apply the result above in order to show that the $\alpha$-scattered subsets of the density topology are in fact its Lebesgue null set.

**Definition 1** A measurable set $E \subseteq \mathbb{R}$ has density $d$ at $x \in \mathbb{R}$ if

$$
\lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{2h}
$$

exists and is equal to $d$. Set $\phi(E) = \{x \in \mathbb{R}: d(x, E) = 1\}$. The open sets of the density topology $\mathcal{T}$ are those measurable sets $E$ that satisfy $E \subseteq \phi(E)$. Note that the density topology $\mathcal{T}$ is finer than the usual topology on the real line.

**Corollary 2.4** The trace spaces (i.e., the $\alpha$-scattered subsets) of the density topology are precisely its Lebesgue null set.

**Proof.** Follows from Theorem 2.3 and the fact that a subset $A$ of the density topology is nowhere dense if and only if it is a Lebesgue null set \[\square\]

**References**

[1] J.P. Coleman, On weaker notions of scatteredness, *Questions Answers Gen. Topology*, 16 (1998), 11–15.

[2] H.H. Corson and E. Michael, Metrizability of certain countable unions, *Illinois J. Math.*, 8 (1964), 351–360.

[3] J. Dontchev, On point generated spaces, *Questions Answers Gen. Topology*, 13 (1) (1995), 63–69.

[4] J. Dontchev, M. Ganster and D. Rose, $\alpha$-scattered spaces II, *Houston J. Math.*, 23 (2) (1997), 231–246.
[5] D. Janković and I. Reilly, On semiseparation properties, *Indian J. Pure Appl. Math.*, 16 (9) (1985), 957–964.

[6] T.Y. Kong and A. Rosenfeld, Digital topology: Introductions and survey, *Computer Vision, Graphics and Image Processing*, 48 (1989), 357–393.

[7] E.H. Kronheimer, The topology of digital images, *Topology Appl.*, 46 (3) (1992), 279–303.

[8] F.D. Tall, The density topology, *Pacific J. Math.*, 62 (1976), 275–284.

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