SEMIGROUPS OF WEIGHTED COMPOSITION OPERATORS IN SPACES OF ANALYTIC FUNCTIONS

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Abstract. We study the strong continuity of weighted composition semigroups of the form $T_tf = \varphi'_t(f \circ \varphi_t)$ in several spaces of analytic functions. First we give a general result on separable spaces and use it to prove that these semigroups are always strongly continuous in the Hardy and Bergman spaces. Then we focus on two non-separable family of spaces, the mixed norm and the weighted Banach spaces. We characterize the maximal subspace in which a semigroup of analytic functions induces a strongly continuous semigroup of weighted composition operators depending on its Denjoy-Wolff point, via the study of an integral-type operator.

1. Introduction

Let $\{T_t : t \geq 0\}$ be a family of bounded operators on a Banach space $X$ of analytic functions on the unit disk. The family is a semigroup of bounded operators if it satisfies the following conditions,

1. $T_0$ is the identity in the space of bounded operators on $X$,
2. $T_{t+s} = T_t T_s$, for all $t, s \geq 0$.

Moreover, the semigroup is called strongly continuous if

$$\lim_{t \to 0^+} \|T_tf - f\|_X = 0.$$ 

One of the most commonly studied semigroups of bounded operators is the semigroup of composition operators, where, for $t \geq 0$, $T_t = C_{\varphi_t}$, and $\{\varphi_t : t \geq 0\}$ is a family of analytic self-maps of the disk. Such a family has to satisfy some conditions in order to generate a semigroup of composition operators, namely

1'. $\varphi_0$ is the identity in $D$,
2'. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
3'. $\varphi_t \to \varphi_0$ as $t \to 0$ uniformly on compact sets of $D$.

The family is then called a (one-parameter) semigroup of analytic functions.

Given a semigroup of analytic functions, the induced family of weighted composition operators $T_t$ given by $T_t f = \varphi'_t(f \circ \varphi_t)$ is a semigroup of weighted composition operators, as long as they are bounded on $X$. Note that, from now on, we will write $\varphi'_t(z) = \frac{d\varphi_t}{dz}(z)$ to distinguish it from $\frac{d\varphi_t}{dt}(z)$. We want to understand operator theory properties of the semigroup of operators,
such as spectrum, ideals or dynamics, in terms of geometric function theory of the semigroup of analytic functions, and the first step is to characterize the strong continuity.

The weighted composition operators of this type are related to the isometries of the Hardy space (see [26]) and the semigroups were studied, in the context of the BMOA space, by Stilosgiannis in [38]. Other semigroups of weighted composition operators, with more general multiplication symbols, were studied by Siskakis in [33].

In this paper we are interested in the strong continuity of these weighted composition semigroups in several spaces of analytic functions, including the Hardy and Bergman spaces, and non-separable spaces such as the mixed norm spaces and weighted Banach spaces. We will see that there are fundamental differences in the characterization of the strongly continuous weighted composition semigroups depending on the separability properties of the space.

In [13] the authors prove the following basic properties of semigroups of analytic functions that will be useful for us:

- If \( \{ \varphi_t \} \) is a semigroup, then each map \( \varphi_t \) is univalent.
- The **infinitesimal generator** of \( \{ \varphi_t \} \) is the function
  \[
  G(z) := \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}, \quad z \in \mathbb{D}.
  \]
  This convergence holds uniformly on compact subsets of \( \mathbb{D} \), so \( G \in \mathcal{H}(\mathbb{D}) \). The generator satisfies
  \[
  G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z) \frac{\partial \varphi_t(z)}{\partial z}
  \]
  and characterizes the semigroup uniquely.
- The function \( G \) has a unique representation
  \[
  G(z) = (bz - 1)(z - b)P(z), \quad z \in \mathbb{D},
  \]
  where \( P \in \mathcal{H}(\mathbb{D}) \) with \( \text{Re} P \geq 0 \) in \( \mathbb{D} \) and \( b \in \mathbb{T} \) is the **Denjoy-Wolff point** of the semigroup, that is, all self-maps in the semigroup share a common Denjoy-Wolff point \( b \).
- If \( \{ \varphi_t \} \) is non-trivial, there exists a unique univalent function \( h : \mathbb{D} \to \mathbb{C} \), called the **Koenigs function** of \( \{ \varphi_t \} \) such that:
  - If \( b \in \mathbb{D} \) then \( h(b) = 0 \), \( h'(b) = 1 \),
    \[
    h(\varphi_t(z)) = e^{G(b)t}h(z)
    \]
    for \( t \geq 0 \), \( z \in \mathbb{D} \) and
    \[
    h'(z)G(z) = G'(b)h(z),
    \]
    \( z \in \mathbb{D} \).
  - If \( b \in \mathbb{T} \) then \( h(0) = 0 \),
    \[
    h(\varphi_t(z)) = h(z) + t
    \]
    for \( t \geq 0 \), \( z \in \mathbb{D} \) and
    \[
    h'(z)G(z) = 1,
    \]
    \( z \in \mathbb{D} \).

See also [34] for a review on semigroups of analytic functions and composition operators.

The structure of the paper is as follows. Section 2 is an introduction on the spaces of analytic functions that will appear later on. Section 3 will be devoted to the study of general semigroups of weighted composition operators. We first give a result for separable spaces, and prove that in
the Hardy and Bergman spaces every semigroup of weighted composition operators is strongly continuous. To study the non-separable case we define the maximal closed linear subspace of a space $X$ such that the semigroup $\{\varphi_t\}$ generates a semigroup of weighted composition operators on it,

$$[\varphi'_t, X] = \{ f \in X : \lim_{t \to 0^+} \| \varphi'_t (f \circ \varphi_t) - f \|_X = 0 \}.$$  

In Sections 4 and 5 we characterize this subspace for the mixed norm spaces and the weighted Banach spaces. We will use the characterizations from Section 3 and, thus, we will need to study the boundedness and compactness of an integral-type operator on such spaces.

Throughout the paper, we will understand $1/\infty$ as zero, the letters $A, B, C, C', K, m$ will denote positive constants, and we will say that two quantities are comparable, denoted by $\alpha \approx \beta$, if there exist a positive constant $C$ such that $C^{-1} \alpha \leq \beta \leq C \alpha$.

2. Spaces of analytic functions

2.1. Hardy and Bergman spaces. The Hardy space $H^p$ is the space of analytic functions on the unit disk such that its integral means

$$M_p(r, f) = \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

for $0 < p < \infty$ and

$$M_\infty(r, f) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$$

are bounded as $r \to 1$. For every $p$, $0 < p \leq \infty$, the polynomials are dense in $H^p$, and if $1 \leq p < \infty$, the space $H^p$ is a Banach space with the norm

$$\|f\|_{H^p} = \lim_{r \to 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \text{ when } 1 \leq p < \infty,$$

and

$$\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)|.$$

The Bergman space $A^p$, $0 < p < \infty$, is the space of analytic functions on the unit disk such that

$$\int_\mathbb{D} |f(z)|^p dA(z) < \infty,$$

where $dA$ is the normalized Lebesgue area measure, that is, the subspace of $L^p(\mathbb{D}, dA)$ whose elements are analytic functions. For any $0 < p < \infty$, the Bergman space $A^p$ is a complete space of analytic functions on the unit disk where polynomials are dense. It becomes a Banach space for $1 \leq p < \infty$ with the norm

$$\|f\|_{A^p} = \left( \int_\mathbb{D} |f(z)|^p dz \right)^{1/p}.$$

More generally, we also define the weighted Bergman space $A^p_\alpha$, $0 < p < \infty$, $-1 < \alpha < \infty$, of analytic functions on the unit disk such that

$$\int_\mathbb{D} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty.$$
The Bergman spaces are closely related to the Hardy spaces. Rewriting the integral condition as
\[
\frac{1}{\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p r^q d\theta dr = 2 \int_0^1 M_p^q(r, f) r dr
\]
it is clear that every function in the Hardy space \( H^p \) belongs also to the Bergman space \( A^p \). Moreover, Hardy and Littlewood proved \( H^p \subseteq A^{2p} \).

2.2. Mixed norm spaces. The mixed norm spaces \( H(p, q, \alpha) \), \( 0 < p, q \leq \infty \), \( 0 < \alpha < \infty \), are the spaces of analytic functions on \( \mathbb{D} \) such that
\[
\int_0^1 (1 - r)^{\alpha q - 1} M_p(q, f) dr < \infty,
\]
for \( q < \infty \), and
\[
\sup_{0 < r < 1} (1 - r)^{\alpha p} M_p(r, f) < \infty
\]
for \( q = \infty \). We also define the "little-oh version", \( H_0(p, \infty, \alpha) \), as the subspace of functions in \( H(p, \infty, \alpha) \) such that
\[
\lim_{r \to 1^-} (1 - r)^{\alpha p} M_p(r, f) = 0.
\]

These spaces are closely related to the Hardy and Bergman spaces, since we can identify the weighted Bergman space \( A^p_\alpha \), \( 0 < p < \infty \), \( -1 < \alpha < \infty \), with the space \( H\left(p, p, \frac{\alpha + 1}{p}\right) \) and the Hardy space \( H^p \) with the limit case \( H(p, \infty, 0) \). The mixed norm spaces are also related to other spaces of analytic functions, such as Besov and Lipschitz spaces, via fractional derivatives (see [28, Chapter 7]).

They were explicitly defined in Flett’s works [21, 25]. Since then, these spaces have been studied by many authors (see [1, 16, 19, 27, 37]). Recently, the mixed norm spaces are mentioned in the works [7, 8, 5], and the monograph [28].

These spaces are Banach for \( p, q \geq 1 \), and we will denote its norm by \( ||f||_{p,q,\alpha} \). We have the following results for these spaces [28, Proposition 7.1.3.]:

**Proposition 1.** For \( 0 \leq r \leq 1 \), let \( f_r(z) = f(rz) \), \( z \in \mathbb{D} \).

- If \( f \in H(p, q, \alpha) \), \( 0 < p \leq \infty \), \( 0 < q, \alpha < \infty \), then \( \|f_r - f\|_{p,q,\alpha} \to 0 \), as \( r \to 1 \).
- If \( f \in H_0(p, \infty, \alpha) \), \( 0 < p \leq \infty \), \( 0 < \alpha < \infty \), then \( \|f_r - f\|_{p,\infty,\alpha} \to 0 \), as \( r \to 1 \).

Moreover, if \( f \in H(p, \infty, \alpha) \) and \( \|f_r - f\|_{p,\infty,\alpha} \to 0 \), as \( r \to 1 \), then \( f \in H_0(p, \infty, \alpha) \).

Notice that, in the language of semigroups of composition operators on mixed norm spaces, Proposition 1 proves that the semigroup of dilations of the unit disk, defined as \( \{\varphi_t\} \) with \( \varphi_t(z) = e^{-t}z \), for all \( t \geq 0 \) and \( z \in \mathbb{D} \), induces a strongly continuous semigroup of composition operators on \( H(p, q, \alpha) \) for \( q < \infty \) and on \( H_0(p, \infty, \alpha) \), but not on \( H(p, \infty, \alpha) \). In a similar way we can prove that the semigroup of weighted composition operators induced by the same semigroup of analytic functions is strongly continuous on \( H(p, q, \alpha) \) for \( q < \infty \) and on \( H_0(p, \infty, \alpha) \), but not on \( H(p, \infty, \alpha) \). Notice first that the operators
\[
T_t f(z) = \varphi_t(z)f(\varphi_t(z)) = e^{-t}f(e^{-t}z)
\]
are bounded on every mixed norm space.

**Theorem 2.** Let \( \varphi_t(z) = e^{-t}z \), \( t \geq 0 \) and \( z \in \mathbb{D} \).

- If \( f \in H(p, q, \alpha) \), \( 0 < p \leq \infty \), \( 0 < q, \alpha < \infty \), then \( \|T_t f - f\|_{p,q,\alpha} \to 0 \), as \( t \to 0 \).
- If \( f \in H_0(p, \infty, \alpha) \), \( 0 < p \leq \infty \), \( 0 < \alpha < \infty \), then \( \|T_t f - f\|_{p,\infty,\alpha} \to 0 \), as \( t \to 0 \).
Moreover, if \( f \in H(p, \infty, \alpha) \) and \( \| T_t f - f \|_{p, \infty, \alpha} \to 0 \), as \( t \to 0 \), then \( f \in H_0(p, \infty, \alpha) \).

**Proof.** Denote by \( f_t \) the function \( f_t(z) = f(e^{-t}z) \), \( z \in \mathbb{D} \). Since \( e^{-t}f_t \) converges uniformly to \( f \) as \( t \to 0 \) on the compact subset \( |z| \leq r \), then

\[
M_p(r, e^{-t}f_t - f) \to 0
\]
as \( t \to 0 \). On the other hand, since

\[
M_p(r, e^{-t}f_t - f) \leq 4 \left( M_p(r, e^{-t}f_t) + M_p(r, f) \right) \leq 4 \left( M_p(e^{-t}r, f) + M_p(r, f) \right) \leq 8M_p(r, f),
\]

the Lebesgue’s Dominated Convergence theorem yields

\[
\| T_t f - f \|_{p,q,\alpha}^q = \alpha q \int_0^1 (1 - r)^{\alpha q - 1} M_p^q(r, e^{-t}f_t - f) \, dr \to 0
\]
as \( t \to 0 \).

If \( f \in H_0(p, \infty, \alpha) \), fix \( \varepsilon > 0 \) and let \( r_0 > 0 \) be such that \( \sup_{r \geq r_0} (1 - r)^\alpha M_p(r, f) < \varepsilon/16 \). Let \( t_0 \) be such that, for every \( 0 < t < t_0 \) and \( r < r_0 \),

\[
|e^{-t}(re^{i\theta}) - f(re^{i\theta})| < \varepsilon/2.
\]

Then

\[
\| T_t f - f \|_{p,\infty,\alpha} \leq \sup_{r < r_0} (1 - r)^\alpha M_p(r, e^{-t}f_t - f) + \sup_{r \geq r_0} (1 - r)^\alpha M_p(r, e^{-t}f_t - f)
\]

\[
\leq \frac{\varepsilon}{2} \sup_{r < r_0} (1 - r)^\alpha + 8 \sup_{r \geq r_0} (1 - r)^\alpha M_p(r, f) \leq \varepsilon.
\]

Finally, let \( f \in H(p, \infty, \alpha) \) such that \( \| T_t f - f \|_{p,\infty,\alpha} \to 0 \), as \( t \to 0 \). Fix \( \varepsilon > 0 \). Let \( t_0 > 0 \) such that, for every \( t < t_0 \),

\[
\sup_{0 < r < 1} (1 - r)^\alpha M_p(r, e^{-t}f_t - f) < \varepsilon.
\]

Then

\[
\lim_{r \to 1} (1 - r)^\alpha M_p(r, f) \leq 4 \left( \lim_{r \to 1} (1 - r)^\alpha M_p(r, e^{-t}f_t) + \lim_{r \to 1} (1 - r)^\alpha M_p(r, e^{-t}f_t - f) \right)
\]

\[
\leq 4 \left( \| f_t \|_{H_p} \lim_{r \to 1} (1 - r)^\alpha + \varepsilon \right) = 4\varepsilon.
\]

Notice that, in the proof that if \( f \in H(p, \infty, \alpha) \) and \( \| T_t f - f \|_{p,\infty,\alpha} \to 0 \), as \( t \to 0 \), then \( f \in H_0(p, \infty, \alpha) \) we have used that \( f_t \) is bounded, and therefore the argument above is not valid if the semigroup of analytic functions \( \{ \varphi_t \} \) has radial limits of modulus one.

A consequence of Proposition 3 is that polynomials are dense in \( H(p, q, \alpha) \), \( 0 < p < \infty \), \( 0 < q, \alpha < \infty \) and \( H_0(p, \infty, \alpha) \), \( 0 < p \leq \infty \), \( 0 < \alpha < \infty \). The closure in \( H(p, \infty, \alpha) \) of the set of all analytic polynomials is \( H_0(p, \infty, \alpha) \).

This result was also proved by Lusky in \[39\] in a more general setting. He also proved the following theorem.

**Theorem 3.** The space \( H(p, q, \alpha) \) is reflexive for \( 1 < q < \infty \) and

\[
H_0(p, \infty, \alpha)^{**} = H(p, \infty, \alpha).
\]

We list some properties of the functions in these spaces that will be needed later. They can be found in \[5\].
Proposition 4. Let $0 < p, q \leq \infty$ and $0 < \alpha < \infty$. There exists $C > 0$ such that for every $f \in H(p, q, \alpha)$ and $z \in \mathbb{D}$,
\[ |f(z)| \leq \frac{C\|f\|_{p, q, \alpha}}{1 - |z|^{\alpha + \frac{1}{p}}}. \]

Moreover, for every $z \in \mathbb{D}$ the function
\[ f_z(w) = \frac{(1 - |z|^2)^{\alpha + \frac{1}{p}}}{(1 - zw)^{-2(\alpha + \frac{1}{p})}} \]
belongs to $H_0(p, \infty, \alpha)$ (and therefore $f_z \in H(p, q, \alpha)$ for $0 < q \leq \infty$),
\[ \|f_z\|_{p, q, \alpha} < 1 \quad \text{and} \quad |f_z(z)| = \frac{1}{(1 - |z|^2)^{\alpha + \frac{1}{p}}}. \]

In the subsequent work, the following result will be very useful for us, since it relates the membership of derivatives of functions belonging to a mixed norm space to another mixed norm space. It is based on a theorem by Hardy and Littlewood (see [23 Thm. 5.5]).

Lemma 5. For $0 < p \leq \infty$ and $\alpha > 0$,

1. $M_p(r, f) = O(1 - r)^{-\alpha} \Leftrightarrow M_p(r, f') = O(1 - r)^{-(\alpha + 1)}$ (that is, $f \in H(p, \infty, \alpha)$ if and only if $f' \in H(p, \infty, \alpha + 1)$).
2. $M_p(r, f) = o(1 - r)^{-\alpha} \Leftrightarrow M_p(r, f') = o(1 - r)^{-(\alpha + 1)}$ (that is, $f \in H_0(p, \infty, \alpha)$ if and only if $f' \in H_0(p, \infty, \alpha + 1)$).

Notice that this means that if we denote by $D$ the differentiation operator, $Df(z) = f'(z)$, $z \in \mathbb{D}$, then it is bounded from $H(p, \infty, \alpha)$ to $H(p, \infty, \alpha + 1)$ and from $H_0(p, \infty, \alpha)$ to $H_0(p, \infty, \alpha + 1)$.

2.3. Weighted Banach spaces. A function $v : \mathbb{D} \to \mathbb{R}_+$ is a weight if it is a bounded, continuous, positive, and radial function. The weighted Banach spaces with weight $v$ are the spaces
\[ H_v^\infty = \{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty \} \]
and
\[ H_v^0 = \{ f \in H_v^\infty : \lim_{|z| \to 1} v(z) |f(z)| = 0 \}. \]

These spaces appear naturally in the study of the growth of analytic functions, see, for instance, [32, 33, 39, 1]. They are Banach with respect to the norm
\[ \|f\|_v = \sup_{z \in \mathbb{D}} v(z) |f(z)|. \]

If $\limsup_{|z| \to 1} v(z) > 0$ we have that $H_v^\infty = H_\infty$ and $H_v^0 = \{0\}$. Therefore, we will only be interested in what is called a typical weight, that is, a weight with $\lim|z| \to 1 v(z) = 0$. The spaces induced by these typical weights satisfy that $(H_v^0)^{**} = H_v^\infty$, and that polynomials are dense in $H_v^0$.

To each weight there is a weight defined via an associated growth condition, called the associated weight $\tilde{v}$ (see [15]),
\[ \tilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\}} \]
for $z \in \mathbb{D}$. In [15] the authors prove that for each $z \in \mathbb{D}$ there exists a $f_z \in H_v^\infty$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$ and $\|f_z\| \leq 1$. Moreover, $H_v^\infty = H_{\tilde{v}}^\infty$ and $H_v^0 = H_{\tilde{v}}^0$. 
As in the case of the mixed norm spaces, we will be interested in the behavior of the derivatives of functions of $H_v^\infty$. We will require several definitions in order to study whether the derivative of a function in a weighted Banach space belongs to a space of the same family.

Firstly, the weighted Bloch space $B_v$ is the space of analytic functions $f$ on the unit disk $D$ such that
\[
\sup_{z \in D} v(z)|f'(z)| < \infty.
\]
The corresponding "little-oh" space is the little Bloch space $B_v^0$ of functions in $B_v$ satisfying
\[
\lim_{|z| \to 1} v(z)|f'(z)| = 0.
\]
The weighted Bloch spaces are clearly related to the weighted Banach spaces, since $f \in B_v$ if and only if $f' \in H_v^\infty$.

Following [30], a weight $v$ has the property (U) if there exists a positive number $\alpha$ such that the function $r \to v(r)/(1-r)^\alpha$ is almost increasing (that is, there exists a constant $C$ such that, for $r_1 < r_2$ we have $v(r_1)/(1-r_1)^\alpha \leq C v(r_2)/(1-r_2)^\alpha$), and it satisfies the property (L) if there exists a positive number $\beta$ such that the function $r \to v(r)/(1-r)^\beta$ is almost decreasing.

A weight is called normal if it satisfies both properties (U) and (L).

Theorem 6. Assume that the weight $v$ has property (U). Then $v$ has property (L) if and only if $H_v^0 = B_{(1-r^2)v(r)}^0$. In particular, if $v$ is a normal weight then $H_v^0 = B_{(1-r^2)v(r)}^0$, and by duality, $H_v^\infty = B_{(1-r^2)v(r)}^{\infty}$.

In other words, $f \in H_v^\infty$ if and only if $f' \in H_{(1-r^2)v(r)}^\infty$. In [9], the authors study the Volterra operators on weighted Banach spaces and then apply the results to the semigroups of composition operators on those spaces. Like in our work, they use the fact that, for some weights, derivatives of functions on a weighted Banach space belongs to another weighted Banach space. Following this reference, we say that a weight $v$ is quasi-normal if $H_v^\infty = B_{(1-r^2)v(r)}^{\infty}$. Discussion about different weights, including characterizations of whether a weight is quasi-normal, can be found in [9].

3. SEMIGROUPS OF WEIGHTED COMPOSITION OPERATORS IN GENERAL SPACES OF ANALYTIC FUNCTIONS

Our first goal is to find the semigroups of analytic functions that induce strongly continuous semigroups of weighted composition operators on Banach spaces of analytic functions for which polynomials are dense. To do this we have the next theorem:

Theorem 7. Let $\{\varphi_t\}$ be a semigroup of analytic functions and $\{T_t\}$ be the induced bounded operator semigroup on a Banach space of analytic functions $X$ satisfying:

(1) polynomials are dense in $X$,
(2) for every $f, g \in X$ such that $|f| \leq |g|$ in $D$ then $\|f\|_X \leq C\|g\|_X$ for some constant $C$,
(3) $\sup_{0 \leq t \leq 1} \|T_t\| < \infty$, and
(4) $\|\varphi_t - id\|_X \to 0$ and $\|\varphi'_t - 1\|_X \to 0$

Then $\{T_t\}$ is strongly continuous on $X$. 
Therefore, we have that $p$ and hence, $n$ and from here it is enough to show that, for every $t$ as $t \to 0$, suppose $z$ for $z \in \mathbb{D}$, and $z \in \mathbb{D}$,
\[\varphi^n_t(z) - z^n = \left(\sum_{k=0}^{n-1} \varphi_k(z)z^{n-1-k}\right)(\varphi_t(z) - z)\]
and hence,
\[|\varphi^n_t(z) - z^n| \leq n|\varphi_t(z) - z|\]

This theorem allows us to prove that any bounded weighted composition semigroup induces a strongly continuous semigroup on a Banach space where polynomials are dense, for instance the classical Hardy and Bergman spaces and the Mixed Norm spaces.

**Proposition 8.** Every semigroup of analytic functions for which the family of weighted composition operators $\{T_t\}$ are uniformly bounded induces a strongly continuous semigroup of weighted composition operators on the Hardy spaces $H^p$ and on the mixed norm spaces $H(p, q, \alpha)$ for $1 \leq q < \infty$ and $H_0(p, \infty, \alpha)$ (and therefore for every weighted Bergman space $A^n_p$).

**Proof.** Clearly both $H^p$ and $H(p, q, \alpha)$ satisfy properties (1) and (2), while (3) is given by hypothesis, so we only need to check (4). Since $\varphi_t$ tends to $\varphi_0$ uniformly on compact subsets of the disk we have that, likewise, $\varphi'_t \to 1$ uniformly on compact subsets. Therefore, for $0 < r < 1$, $M_p(r, \varphi_t - \varphi_0) \to 0$ and $M_p(r, \varphi'_t - 1) \to 0$, and we get (4) applying Lebesgue’s Dominated Convergence Theorem.

Once we have studied the separable case, we are interested in the spaces where polynomials are not dense, such as the mixed norm space $H(p, \infty, \alpha)$ and the weighted Banach spaces $H^\infty_p$. The semigroups of composition operators on these spaces were studied in [26, 17, 18] and [9].
Theorem 10. Let $W$ be a maximal subspace of $X$ where the semigroup of weighted composition operators is strongly continuous.

Proposition 9. Let $X$ be a Banach space of analytic functions and $\{\varphi_t\}$ a semigroup of analytic functions such that $\sup_{0 \leq t \leq 1} \|T_t\| < \infty$. Then there exists a closed subspace $Y$ of $X$ such that:

- The induced semigroup $\{T_t\}$ is strongly continuous on $Y$.
- $T_t(Y) \subset Y$ for every $t \geq 0$.
- Every other subspace of $X$ with the above properties is contained in $Y$.

The proof is analogous to [38 Prop. 5.1]. Following this reference, we will denote the maximal subspace $Y$ of the above Proposition as $[\varphi^\prime_t, X]$, that is, 

$$[\varphi^\prime_t, X] = \{f \in X : \lim_{t \to 0^+} \|T_t f - f\|_X = 0\}.$$ 

We can relate this subspace with the generator of the semigroup of analytic functions.

Theorem 10. Let $\{\varphi_t\}$ be a semigroup of analytic functions with generator $G$. Let $\{T_t\}$ be the induced operator semigroup on $X$ and suppose that $\sup_{0 \leq t \leq 1} \|T_t\| = M < \infty$. Then

$$[\varphi^\prime_t, X] = \{f \in X : (Gf)' \in X\}.$$

Proof. The proof of $[\varphi^\prime_t, X] \subseteq \{f \in X : (Gf)' \in X\}$ is similar to the proofs of [17 Thm. 1] and [38 Thm. 5.3], so we omit it here.

To prove $\{f \in X : (Gf)' \in X\} \subseteq [\varphi^\prime_t, X]$, let $f \in X$ with $(Gf)' \in X$. If we call $Gf = h$, then $h' = (Gf)' \in X$. Note that, by the properties of $G$ seen in the introduction,

$$(h(\varphi_s(z))')' = \varphi'^\prime_s(z)h'(\varphi_s(z)) = \frac{1}{G(z)} \frac{\partial \varphi_s(z)}{\partial s} h'(\varphi_s(z)) = \frac{1}{G(z)} \frac{\partial (h(\varphi_s(z)))}{\partial s},$$

and from here, for $t \geq 0$ and $z \in \mathbb{D}$,

$$T_t f(z) - f(z) = \varphi'^\prime_t(z)f(\varphi_t(z)) - f(z) = \frac{G(\varphi_t(z))}{G(z)} f(\varphi_t(z)) - f(z)$$

$$= \frac{1}{G(z)} (h(\varphi_t(z)) - h(z)) = \frac{1}{G(z)} \int_0^t \frac{\partial h(\varphi_s(z))}{\partial s} ds = \int_0^t (h(\varphi_s(z)))' ds.$$ 

Therefore, for $t < 1$,

$$\|T_t f - f\|_X = \|\varphi'^\prime_t(f \circ \varphi_t) - f\|_X \leq \int_0^t \| (h \circ \varphi_s)' \|_X ds \leq \sup_{0 \leq s \leq 1} \| (h \circ \varphi_s)' \|_X \cdot t$$

$$= \sup_{0 \leq s \leq 1} \| T_s (h') \|_X \cdot t \leq \sup_{0 \leq s \leq 1} \| T_s \| \| h' \|_X \cdot t \leq M \| h' \|_X \cdot t,$$

and $\|T_t f - f\|_X \to 0$ as $t \to 0^+$. Taking closures, we have 

$$\overline{\{f \in X : (Gf)' \in X\}} \subseteq [\varphi^\prime_t, X].$$

Another useful rewriting of this subspace is given by the following operator: For an analytic function $g$ let $W_g : X \to X$ be the operator

$$W_g f(z) = g(z) \int_0^z f(\zeta) \, d\zeta$$

for $f \in \mathcal{H}(\mathbb{D})$. Clearly, $W_g f$ is an analytic function, and the operator can be rewritten as

$$W_g = M_g V_{id},$$
Lemma 12. Let \( \{ \phi_t \} \) be a semigroup with associated generator \( G \). Let \( X \) be a Banach space of analytic functions with the properties:

(i) \( X \) contains the constant functions,

(ii) If \( \{ T_t \} \) is the induced semigroup on \( X \) then \( \sup_{t \in [0,1]} \| T_t \| < \infty \).

Then

\[
[\phi'_t, X] = X \cap (W_\gamma(X) \oplus \mathbb{C}),
\]

Moreover,

\[
[\phi'_t, X] = \overline{X \cap (W_\gamma(X) \oplus \mathbb{C})},
\]

where

\[
W_\gamma f(z) = \frac{1}{1 - z^2} \int_0^z f(\zeta) d\zeta
\]

if the Denjoy-Wolff point of \( \{ \phi_t \} \) is \( b = 1 \) and, if \( X \) also satisfies \( f \in X \iff \frac{f(z) - f(b)}{z - b} \in X \) then

\[
W_\gamma f(z) = \frac{1}{P(z)} \int_0^z f(\zeta) d\zeta
\]

if \( b = 0 \).

Proof. By Theorem 10 we are interested in the closure of the set \( \{ f \in X : (Gf)' \in X \} \), that is,

\[
f(z) = \frac{1}{c(T)} \int_0^z h(\zeta) d\zeta + C
\]

for some \( h \in X \) and constant \( C \). Thus

\[
\{ f \in X : (Gf)' \in X \} = X \cap (W_\gamma(X) \oplus \mathbb{C}),
\]

and we just only need to take closures.

Now, if the Denjoy-Wolff point of \( \{ \phi_t \} \) is \( b \in \mathbb{T} \), we can compose the semigroup by a rotation and assume \( b = 1 \). Then, by the representation of \( G \) discussed in the introduction, \( G(z) = (1 - z)^2 P(z) \). If \( b \in \mathbb{D} \), without lack of generality we can assume \( b = 0 \) and then \( G(z) = -z P(z) \), where \( \text{Re} P(z) > 0 \). If the space \( X \) also satisfies that for each \( b \in \mathbb{D} \), \( f \in X \iff \frac{f(z) - f(b)}{z - b} \in X \), then we will need only study \( W_\gamma \).

In the next sections we will use Theorem 10 and Proposition 11 to characterize the semigroups of weighted composition operators that are strongly continuous on the mixed norm spaces and the weighted Banach spaces.

4. SEMIGROUPS OF WEIGHTED COMPOSITION OPERATORS ON MIXED NORM SPACES

Suppose that the semigroup of analytic functions \( \{ \phi_t \} \) induces a family of bounded weighted composition operators \( \{ T_t \} \) on \( H(p, \infty, \alpha) \). By Proposition 11 we are interested on the boundedness of the operator \( W_g = M_g V_d \) on \( H(p, \infty, \alpha) \). Since it is the product of an integral operator and a multiplier, first we will need the following lemma on multipliers on \( H(p, \infty, \alpha) \).

Lemma 12. Let \( g \) be an analytic function in the unit disk and \( M_g \) the pointwise multiplier with symbol \( g \). The following are equivalent:

(a) \( M_g : H(p, \infty, \alpha - 1) \to H(p, \infty, \alpha) \);
(b) \( M_g : H_0(p, \infty, \alpha - 1) \to H_0(p, \infty, \alpha) \);
(c) \( g \in H(\infty, \infty, 1) \).

Proof. First, assume \( g \in H(\infty, \infty, 1) \). Therefore
\[
|g(z)| \leq \frac{\|g\|_{\infty, \infty, 1}}{1 - |z|}
\]
for every \( z \in \mathbb{D} \). Let \( f \in H(p, \infty, \alpha - 1) \), then
\[
\|gf\|_{p, \infty, \alpha} = \sup_{0 \leq r < 1} (1 - r)^{\alpha} M_p(r, gf)
\]
\[
\leq \|g\|_{\infty, \infty, 1} \sup_{0 \leq r < 1} \frac{(1 - r)^{\alpha}}{1 - r} M_p(r, f) = \|g\|_{\infty, \infty, 1} \|f\|_{p, \infty, \alpha - 1}.
\]
The same inequalities show that if \( f \in H_0(p, \infty, \alpha - 1) \) then \( gf \in H_0(p, \infty, \alpha) \).

On the other hand, suppose \( gf \in H(p, \infty, \alpha) \) for every \( f \in H(p, \infty, \alpha - 1) \). This means that the operator \( M_g \) is bounded from \( H(p, \infty, \alpha - 1) \) into \( H(p, \infty, \alpha) \). Let us denote by \( M \) the norm of this operator. Then, by Proposition \( \text{II} \)
\[
|g(z)f(z)| \leq \frac{CM\|gf\|_{p, \infty, \alpha}}{(1 - |z|)^{\alpha + \frac{1}{p}}} \leq \frac{CM\|f\|_{p, \infty, \alpha - 1}}{(1 - |z|)^{\alpha + \frac{1}{p}}},
\]
Choosing for every \( z \in \mathbb{D} \) \( f_z \in H(p, \infty, \alpha - 1) \) as
\[
f_z(w) = \frac{(1 - |z|^2)^{\alpha - 1 + \frac{1}{p}}}{(1 - \overline{w}z)^{2(\alpha - 1 + \frac{1}{p})}},
\]
a function that satisfies \( |f_z(z)| = (1 - |z|^2)^{-(\alpha - 1 + \frac{1}{p})} \) and \( \|f_z\|_{p, \infty, \alpha - 1} \approx 1 \), we get, for every \( z \in \mathbb{D} \),
\[
\frac{|g(z)|}{(1 - |z|^2)^{\alpha - 1 + \frac{1}{p}}} = \frac{|g(z)f_z(z)|}{CM} \leq \frac{CM}{(1 - |z|)^{\alpha + \frac{1}{p}}}
\]
From here it is clear that \( g \in H(\infty, \infty, 1) \). The same argument shows that if \( W_g \) is bounded from \( H_0(p, \infty, \alpha - 1) \) to \( H_0(p, \infty, \alpha) \) then \( g \in H(\infty, \infty, 1) \).

Once we understand the behavior of the multiplier, we can prove the boundedness of the operator \( W_g \) on the "big-Oh" space and from the "little-oh" space to itself or to the bigger space. The key in this proposition is the fact that, thanks to Lemma \( \text{V} \) the Volterra operator \( V_g \) maps \( H(p, \infty, \alpha) \) into \( H(p, \infty, \alpha - 1) \), and the previous lemma.

**Proposition 13.** Let \( g \) be an analytic function in the unit disk. The following are equivalent:

\begin{itemize}
  \item[(a)] \( W_g : H(p, \infty, \alpha) \rightarrow H(p, \infty, \alpha) \);
  \item[(b)] \( W_g : H_0(p, \infty, \alpha) \rightarrow H_0(p, \infty, \alpha) \);
  \item[(c)] \( W_g : H_0(p, \infty, \alpha) \rightarrow H(p, \infty, \alpha) \);
  \item[(d)] \( g \in H(\infty, \infty, 1) \).
\end{itemize}

**Proof.** Recall that \( W_g = M_g V_{id} \). By Lemma \( \text{V} \) it is clear that
\[
V_{id} : H(\infty, \infty, 1) \rightarrow H(p, \infty, \alpha - 1).
\]
Moreover, by the previous Lemma, if \( g \in H(\infty, \infty, 1) \) then
\[
M_g : H(p, \infty, \alpha - 1) \rightarrow H(p, \infty, \alpha)
\]
and
\[
M_g : H_0(p, \infty, \alpha - 1) \rightarrow H_0(p, \infty, \alpha),
\]
and from here \( W_g \) is bounded on \( H(p, \infty, \alpha) \) and on \( H_0(p, \infty, \alpha) \). This proves \( (d) \Rightarrow (a) \) and \( (d) \Rightarrow (b) \). Since \( (a) \Rightarrow (c) \) and \( (b) \Rightarrow (c) \) are clear, we only need to prove \( (c) \Rightarrow (d) \).
Now, suppose $W_g$ is a bounded operator from $H_0(p, \infty, \alpha)$ to $H(p, \infty, \alpha)$. Since the differentiation operator $D$ is bounded from $H_0(p, \infty, \alpha - 1)$ to $H_0(p, \infty, \alpha)$ by Lemma 5, then the operator $M_g = W_g \circ D$ is bounded from $H_0(p, \infty, \alpha - 1)$ to $H(p, \infty, \alpha)$. By Lemma 12, this means $g \in H(\infty, \infty, 1)$.

The most useful result to study the semigroups of weighted composition operators will be the next one, that characterizes the boundedness of the operator from the bigger space $H(p, \infty, \alpha)$ to $H_0(p, \infty, \alpha)$. We also find that it is equivalent to the compactness and weakly compactness on $H(p, \infty, \alpha)$.

**Theorem 14.** Let $g \in H(\infty, \infty, 1)$. The following statements are equivalent:

(a) $W_g : H(p, \infty, \alpha) \to H(p, \infty, \alpha)$ is compact;

(b) $W_g : H(p, \infty, \alpha) \to H(p, \infty, \alpha)$ is weakly compact;

(c) $W_g : H(p, \infty, \alpha) \to H_0(p, \infty, \alpha)$;

(d) $g \in H_0(\infty, \infty, 1)$.

To prove this theorem we will need the following result, that can be found in [22, p. 482].

**Theorem 15.** Let $T : X \to Y$ be a bounded linear operator between two Banach spaces $X$ and $Y$. Then, $T$ is weakly compact if and only if $T^{**}(X^{**}) \subset Y$.

**Proof (Theorem 15).** We first prove that $(a) \iff (d)$. Let $g \in H_0(\infty, \infty, 1)$ and $\{f_n\}$ a sequence in the unit ball of $H(p, \infty, \alpha)$ that converges uniformly on compact subsets of the unit disk. Therefore, for fixed $\varepsilon > 0$ there is $R < 1$ such that $|g(z)|(1 - |z|) < \varepsilon/\|V_{id}\|$ for $|z| \geq R$. Moreover, if $r \leq R$, note that $(1 - r)^{\alpha - 1} \leq 1$ if $\alpha \geq 1$ and $(1 - r)^{\alpha - 1} \leq (1 - R)^{\alpha - 1}$ if $\alpha < 1$. Since $f_n \to 0$ uniformly on compact subsets, there is $N_0 \in \mathbb{N}$ such that

$$|f_n(z)| \leq \frac{\varepsilon}{((1 - R)^{\alpha - 1} + 1)\|g\|_{\infty, \infty, 1}}$$

for $n \geq N_0$ and for all $|z| \leq R$. Then, for $r < R$,

$$M_p(r, V_{id}f_n) \leq \left( \int_0^{2\pi} \left( \int_0^r |f_n(\zeta)|d\zeta \right)^p \frac{d\theta}{2\pi} \right)^{1/p} \leq \frac{\varepsilon r}{((1 - R)^{\alpha - 1} + 1)\|g\|_{\infty, \infty, 1}}.$$ 

From here,

$$(1 - r)^\alpha M_p(r, gV_{id}f_n) \leq (1 - r)^\alpha \sup_{\theta \in [0, 2\pi]} |g(re^{i\theta})|M_p(r, V_{id}f_n)$$

$$\leq \|g\|_{\infty, \infty, 1}(1 - r)^{\alpha - 1} M_p(r, V_{id}f_n) \leq \frac{\varepsilon r(1 - r)^{\alpha - 1}}{(1 - R)^{\alpha - 1} + 1} \leq \varepsilon$$

for $r < R$.

On the other hand, if $r > R$, then

$$(1 - r)^\alpha M_p(r, gV_{id}f_n) \leq (1 - r)^\alpha \sup_{\theta \in [0, 2\pi]} |g(re^{i\theta})|M_p(r, V_{id}f_n)$$

$$\leq \|V_{id}f_n\|_{p, \infty, \alpha - 1}(1 - r)^\alpha \sup_{\theta \in [0, 2\pi]} |g(re^{i\theta})| \leq \varepsilon,$$

since the operator $V_{id}$ is bounded. Thus, $\lim \|W_gf_n\|_{p, \infty, \alpha} = 0$ and the operator is compact.
Now we assume $W_g$ is compact on $H(p, \infty, \alpha)$. Since the differentiation operator $D$ is bounded from $H(p, \infty, \alpha - 1)$ to $H(p, \infty, \alpha)$ by Lemma 3, we have that the multiplier
\[ M_g = W_g \circ D : H(p, \infty, \alpha - 1) \to H(p, \infty, \alpha) \]
is compact. Therefore, if $\{f_n\}$ is a sequence in the unit ball of $H(p, \infty, \alpha - 1)$ such that $f_n \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{D}$, then $\|M_g f_n\|_{p, \infty, \alpha} \to 0$ as $n \to \infty$.

Suppose that $\|g\|_{H_0(\infty, \alpha, 1)} \neq 0$, then there exist a sequence $\{z_n\}$ and a constant $C > 0$ such that $|z_n| \to 1$ as $n \to \infty$, and
\[ |g(z_n)| \geq \frac{C}{1 - |z_n|}. \]

Given the sequence $\{z_n\}$ we will take the functions $\{f_{z_n}\}$, where $f_{z_n} \in H(p, \infty, \alpha - 1)$ are the functions in Prop. 4
\[ f_{z_n}(w) = \frac{(1 - |z_n|^2)^{\alpha + \frac{1}{p}}}{(1 - z_n w)^2(\alpha + \frac{1}{p})}, \]
$n \in \mathbb{N}$, $w \in \mathbb{D}$. These functions satisfy $|f_{z_n}(z_n)| = (1 - |z_n|^2)^{-(\alpha + \frac{1}{p})}$ and $\|f_{z_n}\|_{p, \infty, \alpha - 1} \approx 1$.

Moreover, $f_{z_n} \to 0$ uniformly on compact subsets of the disk as $n \to \infty$ and thus, since $M_g$ is compact, $\|M_g f_{z_n}\| \to 0$ as $n \to \infty$. Nevertheless,
\[ \frac{\|M_g f_{z_n}\|}{(1 - |z_n|)^{\alpha + \frac{1}{p}}} \geq |g(z_n)||f_{z_n}(z_n)| \geq \frac{C|f_{z_n}(z_n)|}{1 - |z_n|} = \frac{C}{(1 - |z_n|)^{\alpha + \frac{1}{p}}}, \]
that is,
\[ M_g f_{z_n} \geq C > 0, \]
getting a contradiction.

Suppose now that $W_g : H(p, \infty, \alpha) \to H_0(p, \infty, \alpha)$, or equivalently, $W_g^{**} : H^{**}(p, \infty, \alpha) \to H_0^{**}(p, \infty, \alpha)$. By Theorem 3, $H_0^{**}(p, \infty, \alpha) = H(p, \infty, \alpha)$, and therefore, $W_g^{**} : H^{**}(p, \infty, \alpha) \to H(p, \infty, \alpha)$. By Theorem 13, this is equivalent to $W_g : H(p, \infty, \alpha) \to H(p, \infty, \alpha)$ being weakly compact. Therefore we have proved (c) $\Leftrightarrow$ (b).

Now we see (d) $\Rightarrow$ (c). Let $g \in H_0(\infty, \alpha, 1)$ and $f \in H(p, \infty, \alpha)$, then, since $W_g = M_g \circ V_{id}$ and $V_{id}$ is bounded from $H(p, \infty, \alpha)$ to $H(p, \infty, \alpha - 1)$, we have that
\[ \lim_{r \to 1} (1 - r)^\alpha M_p(r, g V_{id} f) \leq \lim_{r \to 1} (1 - r)^\alpha \sup_{\theta \in [0, 2\pi]} |g(re^{i\theta})| M_p(r, V_{id} f) \leq \|V_{id} f\|_{p, \infty, \alpha - 1} \lim_{r \to 1} (1 - r)^\alpha \sup_{\theta \in [0, 2\pi]} |g(re^{i\theta})| = 0. \]

Therefore, $W_g$ is bounded from $H(p, \infty, \alpha)$ to $H_0(p, \infty, \alpha)$.

Finally, to prove (c) $\Rightarrow$ (d), suppose that $W_g$ is bounded from $H(p, \infty, \alpha)$ to $H_0(p, \infty, \alpha)$. Notice that, since the differentiation operator $D$ is bounded from $H(p, \infty, \alpha - 1)$ to $H(p, \infty, \alpha)$ by Lemma 3, then the operator $M_g = W_g \circ D$ is bounded from $H(p, \infty, \alpha - 1)$ to $H_0(p, \infty, \alpha)$.

Now, suppose $g \notin H_0(\infty, \alpha, 1)$, then there exist a sequence $\{z_n\}$ and a constant $C > 0$ such that $|z_n| \to 1$ and
\[ |g(z_n)| \geq \frac{C}{1 - |z_n|}. \]

Let
\[ f(z) = \frac{1}{(1 - z)^{\alpha + \frac{1}{p} - 1}}, \]
Theorem 17. Let \( f \) be a semigroup with Denjoy-Wolff point \( b \in \mathbb{D} \). Then
\[
H_0(p, \infty, \alpha) = [\varphi_t, H(p, \infty, \alpha)] \equiv \frac{1}{P} \in H_0(\infty, \infty, 1).
\]
Proof. By Proposition \[11\]
\[
[\varphi'_t, H(p, \infty, \alpha)] = H(p, \infty, \alpha) \cap (W_\gamma(H(p, \infty, \alpha)) \oplus \mathbb{C}),
\]
where \( W_\gamma \) is the operator
\[
W_\gamma f(z) = \frac{1}{P(z)} \int_0^z f(\zeta) d\zeta,
\]
z \in \mathbb{D}. The function \( P \) has positive real part, and so does \( \frac{1}{P(z)} \). Therefore, \( \frac{1}{|P(z)|} \leq \frac{C}{1-|z|} \) and \( \frac{1}{P} \in H(\infty, \infty, 1) \). Thus, by Proposition \[13\] the operator \( W_\gamma \) is bounded on \( H(p, \infty, \alpha) \) and
\[
W_\gamma(H(p, \infty, \alpha)) \oplus \mathbb{C} \subset H(p, \infty, \alpha).
\]
From here,
\[
H(p, \infty, \alpha) \cap (W_\gamma(H(p, \infty, \alpha)) \oplus \mathbb{C}) = W_\gamma(H(p, \infty, \alpha)) \oplus \mathbb{C}
\]
and
\[
H_0(p, \infty, \alpha) = [\varphi'_t, H(p, \infty, \alpha)] = W_\gamma(H(p, \infty, \alpha)) \oplus \mathbb{C}
\]
if and only if \( W_\gamma(H(p, \infty, \alpha)) \subseteq H_0(p, \infty, \alpha) \). By Theorem \[14\] this is equivalent to \( \frac{1}{P} \in H_0(\infty, \infty, 1) \). □

Now, for \( b = 1 \), recall that
\[
[\varphi'_t, H(p, \infty, \alpha)] = \{ f \in H(p, \infty, \alpha) : (1 - z)^2 P f \} \in H(p, \infty, \alpha)
\]
or equivalently, by Lemma \[5\]
\[
[\varphi'_t, H(p, \infty, \alpha)] = \{ f \in H(p, \infty, \alpha) : (1 - z)^2 P f \in H(p, \infty, \alpha - 1) \}.
\]

Theorem 18. For every nontrivial semigroup of analytic functions with Denjoy-Wolff point \( b \in \mathbb{T} \) we have
\[
H_0(p, \infty, \alpha) \not\subset [\varphi'_t, H(p, \infty, \alpha)].
\]
Proof. Let \( P \) be an arbitrary real-part function associated with a generator of a nontrivial semigroup. Since \( P \in H(\infty, \infty, 1) \), if \( f \) is such that \( (1 - z)^2 f \in H(p, \infty, \alpha - 2) \) then \( f \in [\varphi'_t, H(p, \infty, \alpha)] \). Now let \( f(z) = \frac{1}{(1-z)^{\alpha+1}} \), then \( f \in H(p, \infty, \alpha) \setminus H_0(p, \infty, \alpha) \), and \( (1 - z)^2 f \in H(p, \infty, \alpha - 2) \). Therefore, \( H_0(p, \infty, \alpha) \not\subset [\varphi'_t, H(p, \infty, \alpha)] \). □

5. Semigroups of Weighted Composition Operators on Weighted Banach Spaces

As stated in Section 2, we are interested in the weighted Banach spaces \( H^\infty_v \) with quasi-normal weights, that is, spaces satisfying
\[
H^\infty_v = B^\infty_v(1-r)^{v(r)}.
\]

Thanks to the weight being quasi-normal, several results in this section are similar to the analogous in the previous section. First, we study the boundedness of the multiplier.

Lemma 19. Let \( v \) be a weight and \( g \) an analytic function in the unit disk and \( M_g \) the pointwise multiplier with symbol \( g \). The following are equivalent:
(a) \( M_g : H^\infty_v(1-r) \to H^\infty_v \);
(b) \( M_g : H^0_v(1-r) \to H^0_v \);
(c) \( g \in H(\infty, \infty, 1) \).

Proof. The proof is analogous to the proof of Lemma \[12\], using the definition of the norm of \( H^\infty_v \), and the fact that there is a \( f_z \) in \( H^\infty_v \) such that \( |f_z(z)| = 1/\tilde{v}(z) \) and \( \|f_z\| \leq 1 \), and that \( H^\infty_v = H^\infty_v \). □
The boundedness of $W_g = M_g \circ V_{id}$ is now clear on $H_0^\infty$ if the weight is quasi-normal, since we can use Theorem 6.

**Proposition 20.** Let $v$ be a quasi-normal weight and $g$ an analytic function in the unit disk. The following are equivalent:

(a) $W_g : H_0^\infty \rightarrow H_0^\infty$;
(b) $W_g : H_0^0 \rightarrow H_0^0$;
(c) $W_g : H_0^0 \rightarrow H_0^\infty$;
(d) $g \in H(\infty, \infty, 1)$.

**Proof.** In this case Theorem 6 gives us the boundedness of

$$V_{id} : H_0^\infty v \rightarrow H_0^\infty v (r)/(1-r),$$

that, with Lemma 19, gives $(d) \implies (a)$. It also proves that the function $f_z'$, where $f_z \in H_0^0 (r)/(1-r)$ is a function satisfying

$$|f_z'(z)| = \frac{1 - |z|}{\tilde{v}(z)}$$

and $\|f_z\|_{(r)/(1-r)} = 1$, belongs to $H_0^0$. Therefore, if the operator $W_g$ is bounded from $H_0^0$ to $H_0^\infty$ then

$$|g(z)| \left| \int_0^z f_0(\zeta) \, d\zeta \right| = |g(z)||f_z(z)| = \frac{|g(z)|(1 - |z|)}{\tilde{v}(z)} \leq CM \tilde{v}(z),$$

and therefore $g \in H(\infty, \infty, 1)$.

\[ \Box \]

We will also use the analogue of Theorem 14.

**Theorem 21.** Let $v$ be a quasi-normal weight and $g \in H(\infty, \infty, 1)$. The following statements are equivalent:

(a) $W_g : H_0^\infty \rightarrow H_0^\infty$ is compact;
(b) $W_g : H_0^\infty \rightarrow H_0^\infty$ is weakly compact;
(c) $W_g : H_0^0 \rightarrow H_0^0$;
(d) $g \in H_0(\infty, \infty, 1)$.

**Proof.** The proof of (a) $\iff$ (d) $\implies$ (c) follows the same steps as in the mixed norm spaces, using the ideas of the last proof.

To prove (c) $\iff$ (b) we use that $(H_0^0)^{**} = H_0^\infty$ and Theorem 15.

Finally, the proof of (b) $\implies$ (a), that is, that a weakly compact multiplier is compact on $H_0^\infty$ is Theorem 5.2 of [20].

\[ \Box \]

As in the case of mixed norm spaces, we find that there are not nontrivial strongly continuous semigroups of weighted composition operators on $H_0^\infty$.

**Theorem 22.** No nontrivial semigroup of analytic functions induces a strongly continuous semigroup of weighted composition operators on $H_0^\infty$. In other words,

$$[\varphi_t, H_0^\infty] \not\subset H_0^\infty.$$

**Proof.** By Theorem 6 our problem is to find the semigroups with generator $G$ such that

$$H_0^\infty \subset \{ f \in H_0^\infty : Gf \in H_0^\infty (r)/(1-r) \}.$$
To find the analytic functions $G$ such that $Gf \in H^\infty_{v(r)/(1-r)}$ for every $f \in H_v^\infty$, we proceed as in the proof of Lemma 19. Suppose the multiplier $M_G$ is bounded from $H_v^\infty$ to $H^\infty_{v(r)/(1-r)}$, then

$$|G(z)f(z)| \leq \frac{\|M_G\|\|f\|_v(1-|z|)}{\tilde{v}(z)}.$$ 

Taking $f_z$ in $H^\infty_{v(z)}$ as a function with $|f_z(z)| = 1/\tilde{v}(z)$ and $\|f_z\| = 1$, we get that

$$|G(z)f_z(z)| = \frac{|G(z)|}{\tilde{v}(z)} \leq \frac{\|M_G\|(1-|z|)}{\tilde{v}(z)}$$

for any $z \in \mathbb{D}$, and therefore $G \equiv 0$.

Finally we study whether $H_v^0 = [\varphi'_t, H_v^0]$, in the case where the Denjoy-Wolff point is $b \in \mathbb{D}$ and in the case where $b \in \mathbb{T}$.

**Theorem 23.** Let $\{\varphi_t\}$ be a semigroup with Denjoy-Wolff point $b \in \mathbb{D}$. Then

$$H_v^0 = [\varphi'_t, H_v^\infty] \iff \frac{1}{P} \in H_0(\infty, \infty, 1).$$

**Proof.** Following the discussion of Theorem 17 we know that, if $b = 0$, the operator $W_\gamma$ is bounded on $H_v^\infty$ and, by Proposition 20,

$$W_\gamma(H_v^\infty) + \mathbb{C} \subset H_v^\infty.$$ 

Thus

$$H_v^\infty \cap (W_\gamma(H_v^\infty) + \mathbb{C}) = W_\gamma(H_v^\infty) + \mathbb{C}$$

and

$$H_v^0 = [\varphi_t, H_v^\infty] = W_\gamma(H_v^\infty) + \mathbb{C}$$

if and only if $W_\gamma(H_v^\infty) \subseteq H_v^0$. By Proposition 21 this is equivalent to $\frac{1}{P} \in H_0(\infty, \infty, 1)$. \hfill \Box

**Theorem 24.** For every nontrivial semigroup of analytic functions with Denjoy-Wolff point $b \in \mathbb{T}$ we have

$$H_v^0 \subsetneq [\varphi'_t, H_v^\infty].$$

**Proof.** Suppose $H_v^0 = [\varphi'_t, H_v^\infty]$. Since

$$[\varphi'_t, H_v^\infty] = \left\{ f \in H_v^\infty : (1-z)^2 P f \in H_v^\infty_{v(r)/(1-r)} \right\}$$

and $P \in H(\infty, \infty, 1)$, then if $f$ is such that $(1-z)^2 P f \in H_v^\infty_{v(r)/(1-r)^2}$ then $f \in [\varphi'_t, H_v^\infty]$. If we take $f \in H_v^\infty$ such that $|f_z(z)| = 1/\tilde{v}(z)$, we have that $f_z \not\in H_v^0$ and $(1-z)^2 f_z \in H_v^\infty_{v(r)/(1-r)^2}$, thus $f_z \in [\varphi'_t, H_v^\infty]$. Therefore, $H_v^0 \neq [\varphi'_t, H_v^\infty]$. \hfill \Box

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