On Density of State of Quantized Willmore Surface
—A Way to Quantized Extrinsic String in $\mathbb{R}^3$—

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Abstract.
Recently I quantized an elastica with Bernoulli-Euler functional in two-dimensional space using the modified KdV hierarchy. In this article, I will quantize a Willmore surface, or equivalently a surface with the Polyakov extrinsic curvature action, using the modified Novikov-Veselov (MNV) equation. In other words, I show that the density of state of the partition function for the quantized Willmore surface is expressed by volume of a subspace of the moduli of the MNV equation.

§1. Introduction

In series of works [1-7], I have considered the correspondence between an immersed object and the Dirac operator confined there. The Dirac operator confined in an immersed object is uniquely determined by the procedure which I proposed [1-4] and can be regarded as the representation matrix of the symmetry of the immersed object [1-7]. I had been studying it mainly on an elastica in a plane [1-6] and showed that the Dirac operator confined in an elastica is identified with the Lax operator of the modified KdV equation while the mathematical deformation of the elastica obeys the modified KdV hierarchy [6]. By investigating other quantum equations [8-9], I conjectured that such correspondence between the Dirac operator and geometry can be extended to higher dimensional immersed objects [2,3,4].

Couple years ago Konopelchenko [10,11] discovered that a conformal surface $S$ immersed in three dimensional flat space $\mathbb{R}^3$ obeys the Dirac equation, which I will call Konopelchenko-Kenmotsu-Weierstrass-Enneper (KKWE) [10-15] equation here,

$$\partial f_1 = V f_2, \quad \partial f_2 = -V f_1,$$

(1-1)

where

$$V := \frac{1}{2} \sqrt{\rho H},$$

(1-2)

$H$ is the mean curvature of the surface $S$ parameterized by complex $z$ and $\rho$ is the conformal metric induced from $\mathbb{R}^3$. The KKWE equation completely exhibits the immersed geometry as the old Weierstrass-Enneper equation expresses the minimal surface [10-15]. In ref.[16] I showed that it is identified with the Dirac operator confined in the surface $S$ and by quantizing the Dirac field I found that the quantized symmetry of the Dirac operator is also in agreement with the symmetry of the surface itself [17]. In other words, this KKWE equation is the equation which I conjectured before [2,3] and I had been searching for. Even though for more general surface, which is not conformal, the KKWE type equation was discovered by Burgess and Jensen [18] following my prescriptions [1], their equation is not easy to deal with and I could not find meaningful results. However the KKWE equation is very useful to investigate the immersed object and in terms of (1-1), Konopelchenko, Taimanov and other Russian group found non-trivial results related to the immersed surface [19-22].
By physical investigating the KKWE equation and its quantized version, the Willmore functional [21,22] and the modified Novikov-Veselov (MNV) equation naturally appears [10-15,17,19,20]. The Willmore functional is given as [21,22],

\[ W = \int_S \text{dvol} \; H^2, \tag{1-3} \]

where "dvol" is a volume form of the surface \( S \). The harmonic map associated with this functional has been studied in the differential geometry [21,22].

On the other hand, Polyakov introduced an extrinsic curvature action in the string theory and the theory of two-dimensional gravity from renomalizability [23]. However his action is just the Willmore functional (1-3). Thus his program recently was investigated by Carroll and Konopelchenko [19] and Grinevich and Schmidt [20] using KKWE equation (1-1). The main theme of this article is to quantize the Willmore surface but I emphasis that it means the study on the quantization of the Polyakov extrinsic curvature action.

It should be noted that the elastica problem of \( \mathbb{R}^2 \) has very similar structure of the Willmore surface problem of \( \mathbb{R}^3 \) [10-14]. Corresponding to the Willmore functional (2-20), there is Bernoulli-Euler functional for an elastica [24],

\[ E = \int dq^1 k^2, \tag{1-4} \]

where \( k \) is a curvature of the elastica [24]. While the Willmore surface is related to the modified Novikov-Veselov (MNV) equation, the elastica is related to the modified KdV equation [1-7,25-27].

Recently I exactly quantized the elastica of the Bernoulli-Euler functional (1-4) preserving its local length [25]. Then I found that its moduli is completely represented by the MKdV equation and closely related to the two-dimensional quantum gravity [28-30]. The quantized elastica obeys the MKdV equation and at a critical point, the Painlevé equation of the first kind appears [25] while in the quantized two-dimensional gravity which is defined at a critical point of the discrete tiling model, there appears the Painlevé equation of the first kind with the KdV hierarchy [28-30].

In this article instead of the local length preserving, I will impose that the surface preserves its complex structure and will quantize the Willmore functional. Then I will show that the MNV hierarchy appears as the quantized motion of a Willmore surface in the path integral.

The organization of this article is as follows. Section 2 reviews the argument of the quantized elastica following to that in ref.[25]. In §3, I will quantize the Willmore surface and then the density of states of the Willmore functional is given as the volume of the MNV equation. Section 4 gives the discussion for the results.

\section{2. Quantization of Elastica}

I will denote by \( \mathcal{C} \) a shape of the elastica embedded in a complex plane \( \mathbb{C} \) and by \( X(s) \) its affine vector [6]:

\[ S^1 \ni s \mapsto X(s) \in \mathcal{C} \subset \mathbb{C}, \quad X(s + L) = X(s), \tag{2-1} \]

where \( L \) is the length of the elastica. I will fix the metric of the curve \( \mathcal{C} \) induced from the natural metric of \( \mathbb{C} \); \[ ds = \sqrt{dX \overline{dX}}. \] The Frenet-Serret relations are expressed as [6,25-27]

\[ \psi_0 := \exp(i\phi/2) = \sqrt{\partial_s X}, \tag{2-2} \]

\[ \left( \begin{array}{cc} \partial_s & v \\ v & -\partial_s \end{array} \right) \left( \begin{array}{c} \psi_0 \\ iv_0 \end{array} \right) = 0, \quad v := \frac{1}{2} k := \frac{1}{2} \partial_s \phi, \tag{2-3} \]

where \( \phi \) is a real valued function of \( s \) and \( k \) is the curvature of the curve \( \mathcal{C} \), \( \phi(s + L) = \phi(s) \) and \( k(s + L) = k(s) \).
The energy functional of the elastica, which I will call Bernoulli-Euler functional here [24], is given as

$$E = \int_0^L ds \ k^2 = 4 \int_0^L ds \ v^2,$$  

(2-4)

and shape of a static elastica is realized as its stationary point satisfied with the boundary conditions. I assume that the elastica does not stretch and preserves its local infinitesimal length.

I will consider quantization of a closed elastica whose local length preserves in the quantization process. The partition function of the elastica is given as [25],

$$Z = \int DX \exp \left( -\beta \int_0^L ds \ [k^2] \right).$$  

(2-5)

Since there is trivial affine symmetry of the centroid of the elastica and the partition function diverges, I will regularize it,

$$Z_{\text{reg}} = \frac{Z}{\text{Vol}(\text{Aff})},$$  

(2-6)

where Vol(Aff) is the volume of the affine transformation.

Next I will consider the condition of local length preserving. In the path integral, I must pay attentions upon the higher perturbations of $\epsilon$ to gain an exact result. Hence I will assume that $X$ is parameterized by a parameter $t$ and the difference between perturbed affine vector $X_\epsilon$ and unperturbed one $X$ can be expressed by [6,25-27],

$$X_\epsilon(s,t) := e^{\epsilon \partial _t} X(s,t), \quad \epsilon \partial _t X = X_\epsilon - X + O(\epsilon^2).$$  

(2-7)

with the relation

$$\partial _t X = (u_1 + iu_2) \exp(i\phi), \quad u_1(L) = u_1(0), \quad u_2(L) = u_2(0),$$  

(2-8)

where $u$'s are real function of $s$ and $t$. This is virtual dynamics of the curve [6]. As well as the argument in refs.[6,25-27], due to the isometry condition, I require $[\partial _t, \partial _s] = 0$ for $X$. Then the isometry condition exactly preserves, $ds \equiv ds_\epsilon$ for $ds_\epsilon := \sqrt{\partial _s X_\epsilon \partial _s X_\epsilon} ds$. Even though $\epsilon$ is constant, dependence of the variation upon the position $s$ comes from the "equation of motion" (2-8) and $u_a(s), \ a = 1, 2$. Hence the deformation (2-7) contains non-trivial ones.

From $[\partial _t, \partial _s] = 0$, I have the relation [26,27],

$$-\partial _t \exp(i\phi) = ((u_{1s} - u_2k) + i(u_{2s} + u_1k)) \exp(i\phi).$$  

(2-9)

Noting that $\phi$ and $u$'s are real valued, (3-9) is reduced to two differential equations and by partially solving one of them, I obtain the "equation of motion" of the deformation,

$$\partial _s u_1 = ku_2, \quad u_1 = \int _s ^L ds \ u_2k =: \partial _s ^{-1} u_2k,$$

$$\partial _t k = \Omega u_2.$$

(2-10)

Here $\partial _s ^{-1}$ is the pseudo-differential operator with a parameter $c \in \mathbb{R}$ as an integral constant and

$$\Omega := \partial _s ^2 + \partial _s k \partial _s ^{-1} k.$$  

(2-11)
In ref. [6, 25], instead of the single deformation parameter, I used the infinite dimensional parameters $t = (t_1, t_3, \cdots)$ and investigated the moduli space of the partition function. Then the minimal set of the virtual equations of motion, which satisfies the physical requirements, is given as

$$
\partial_{t_{2n+1}}k = -\Omega^n \partial_t k, \quad \partial_{t_{2n+3}}k = \Omega \partial_{t_{2n+1}}k, \quad (n = 1, 2, \cdots).
$$

(2-12)

They are the MKdV hierarchy [27, 28]. As in ref. [6], I stated that these relations (2-12) should be regarded as the Neither currents for the immersed object and $t$'s should be considered as the Schwinger proper times, in ref. [25] I showed that (2-12) means the quantum fluctuations and a kind of currents of the quantized Neither theorem or the Ward-Takahashi identities.

However by the studying the moduli of the quantize elastica, the nontrivial deformation obeys the MKdV equation

$$
\partial_t v + 6v^2 \partial_s v + \partial_s^3 v = 0,
$$

(2-13)

because the solutions of the higher order equations belonging to the MKdV hierarchy are also satisfied with the MKdV equation.

Here it is a very remarkable fact that for the variation of $t$ obeying the MKdV equation, the Bernoulli-Euler functional is invariant,

$$
\partial_t \int ds \, v(s, t)^2 = \frac{1}{4} \partial_t E = 0,
$$

(2-14)

because

$$
\partial_t \int ds v^2 = -\int ds \partial_s (\frac{3}{2} v^4 + \frac{1}{2} (v \partial_s^2 v - (\partial_s v)^2)) = 0.
$$

(2-15)

Since the MKdV problem is an initial value problem, for any regular shape of elastica satisfied with the boundary conditions, the "time" $t$ development of the curvature can be expressed. In other words, for given any regular curve, there exists family of the solutions of the MKdV equation (2-13) which contains the given curve. Due to the integrability and (2-15), during the motion of $t$, the energy functional does not change its value. Hence the trajectory of the deformation parameter $t$ draws the functional space which has the same of value of the energy functional. This remind me of the fact that in the group theory the character of a group is invariant among the elements belonging the same conjugate class. In fact in the Sato theory, the solutions space of the MKdV equation is acted by the affine Lie algebra $A^{(1)}$ [31].

Thus I can estimate the functional space for each functional value. In other words by investigating the moduli of the MKdV equation which is satisfied with the boundary conditions,

$$
v(0) = v(L), \quad X(0) = X(L), \quad (2-16)
$$

the measure of the functional integral $d\mu$ can be decomposed,

$$
d\mu = \sum_E d\mu_E. \quad (2-17)
$$

So I let the set of these trajectories which occupy the same energy $E$ be denoted as $\Xi_E$.

Hence the partition function can be represented as

$$
Z_{\text{reg}} = \int d\mu \exp(-\beta E) = \sum_E \exp(-\beta E) \int \Xi_E d\mu_E = \sum_E \exp(-\beta E) \text{Vol}(\Xi_E) \quad (2-18)
$$

where

$$
\text{Vol}(\Xi_E) = \int_{\Xi_E} d\mu_E \quad (2-19)
$$

is the volume of the trajectories $\Xi_E$. 

In ref.[25], I explicitly expressed \( d \mu \) in terms of the moduli of the MKdV equation. According to the arguments in ref.[25], for a case of the solution represented by the hyperelliptic function of genus \( g \), \( d \mu_E \) is roughly expressed as \( dt_3 \wedge dt_5 \wedge \cdots \wedge dt_{2g-1} \) where \( t_g := (t_1, t_3, \cdots, t_{2g-1}) \) is a subset of the infinite dimensional deformed parameters such as (2-12). Even though I introduced the infinite dimensional coordinates \( t \) in ref.[25], they are often reduced to finite dimensional space, as the Jacobi variety with finite dimension is embedded in the universal grassmannian manifold in the Sato theory [25,31,32]. I showed that \( \Xi_E \) is given as the real subspace of the Jacobi variety corresponding to the hyperelliptic curve, which is the trajectory space of the solution [25].

I will note that the volume of \( \Xi_E \) is estimated by the unit of the elastica length \( L \). Due to the complex structure of the moduli of the MKdV equation, which is expressed as the Jacobi variety of the hyperelliptic curve and can be performed the coordinate transformation such as rotation, the volume can be evaluated in terms of the elastica length \( L \) [25,32]. However since the dimension of the trajectory space \( \Xi_E \) differs depending upon the energy \( E \), the sum of terms with different dimensional volume appear. It seems to be fancy but noting the facts that the dimension of the energy functional \( E \) is the inverse of the length and that \( \beta/[\text{length}] \) is order unit, the multiple of the length can be interpreted as the multiple of the quantizing parameter \( \beta^{-1} \). Hence such summation has physical meanings.

(2-18) means that the density of state of the Bernoulli-Euler functional system is completely represented by the moduli and solutions spaces of the MKdV equation. These space is acted by the infinite dimensional Lie group [31]. Then the measure \( d \mu \) can be regarded as the Haar measure for the subalgebra of infinite dimensional Lie algebra \( A^{(1)} \) [25].

### §3. Quantization of Willmore surface

I will denote by \( S \) a shape of a compact surface immersed in the three dimensional space \( \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R} \), and by \((Z(z, \bar{z}) := X^1 + iX^2, X^3(z, \bar{z})) \) its affine vector :

\[
\Sigma \ni z \mapsto (Z, X^3) \in S \subset \mathbb{C} \times \mathbb{R}.
\]  
(3-1)

Here \( \Sigma \) can be expressed as \( \Sigma = \mathbb{C}/\Gamma \) where \( \Gamma \) is a Fuchian group and then \( \Sigma \) is a complex analytic object [33]. The volume element and the infinitesimal length the surface \( S \) are given by

\[
d \text{vol} = \frac{1}{2} \rho d z \wedge d \bar{z} =: \frac{1}{2} \rho d^2 z, \quad d s^2 = \rho d z d \bar{z}.
\]  
(3-2)

The Konopelchenko-Kennmotsu-Weierstrass-Enneper (KKWE) relation will be expressed as

\[
\psi_+ = i \sqrt{\partial_z \bar{Z}}, \quad \psi_- = -i \sqrt{\partial_z Z}, \quad \partial X^3 = -\bar{\psi}_+ \bar{\psi}_-,
\]  
(3-3)

\[
\mathcal{D} \psi = \left( \frac{\partial}{V} - \frac{\partial V}{\partial \psi} \right) \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) = 0, \quad V = \frac{1}{2} H \rho,
\]  
(3-4)

where \( V \) is a real valued function of \( z \) and \( \bar{z} \) and \( H \) is the mean curvature of \( S \). The general form might be expressed by the Dirac operator which was calculated by Burgess and Jensen following my prescription of the confinement Dirac operator. However due to the conformal structure of the surface \( S \), the equation becomes simpler and given by (3-4) [16] while due to the isometry condition, the Frenet-Serret relation becomes simple (2-2).

As the energy integral of the Willmore surface or the Polyakov extrinsic curvature action is given as

\[
E = \int \rho d^2 z \ H^2 = 4 \int d^2 z \ V^2.
\]  
(3-5)

The Willmore surface realized by the solution \( \psi_+ \psi_- \neq 0 \) exists.
As I quantized the elastica using the MKdV equation [25], I will consider quantization of such a surface. The partition function of the surface is also given as [25],

$$
\tilde{Z}_{\text{reg}} = \frac{\int DX \exp (-\beta \int \rho d^2 z H^2)}{\text{Vol}(\text{Aff})},
$$

where Vol(Aff) means the volume of the affine transformation in $\mathbb{R}^3$.

I will search for the deformation flow of the surface which preserves the Willmore function or the Polyakov extrinsic curvature action and complex structure. My question is what equation the deformation flow obeys. By Taimanov and Konopelchenko, the answer of the question was founded that the modified Novikov-Veselov (MNV) equation preserves the complex structure and the functional (3-5) [10-14].

The MNV equation is given as

$$
V_t = V_{t^+} + V_{t^-},
$$

$$
V_{t^+} = \partial^3 V + 3\partial V U + \frac{3}{2} V \partial U, \quad V_{t^-} = \bar{\partial}^3 V + 3\bar{\partial} \bar{V} \bar{U} + \frac{3}{2} V \bar{\partial} \bar{U},
$$

$$\bar{\partial} U = \partial V^2, \quad \partial \bar{U} = \bar{\partial} V^2. \tag{3-7}
$$

Along the line $z = \bar{z}$, the MNV equation (3-7) is reduced to the MKdV equation (2-13).

As the Frenet-Serret relation can be regarded as the inverse scattering system of the MKdV equation, the KKWE equation can be also regarded as the inverse scattering system of the MNV equation.

$$
(\partial_t \pm B^\pm \bar{\partial} + [\bar{\partial}, A^\pm]) = 0, \tag{3-8}
$$

recovers (3-7) for

$$
A^+ = \begin{pmatrix}
\partial^3 & -3(\partial V) \partial + 3 V U \\
0 & \partial^3 + 3 U \partial + 3(\partial U)/2
\end{pmatrix},
$$

$$
B^+ = 3 \begin{pmatrix}
0 & (\partial V) \partial - V U \\
-(\partial V) \partial - (\partial^2 V) - UV & 0
\end{pmatrix},
$$

$$
A^- = \begin{pmatrix}
\partial^3 + \bar{U} \bar{\partial} + 3 \bar{\partial} \bar{U}/2 & 0 \\
3 \bar{\partial} V \bar{\partial} - 3 \bar{V} \bar{U} & \bar{\partial}^3
\end{pmatrix},
$$

$$
B^- = 3 \begin{pmatrix}
0 & (\bar{\partial} V) \bar{\partial} + (\bar{\partial}^2 V) - V \bar{U} \\
-(\bar{\partial} V) \bar{\partial} + V \bar{U} & 0
\end{pmatrix}. \tag{3-9}
$$

The variation of the Dirac field is given as

$$
\partial_t \psi = \partial_{t^+} \psi + \partial_{t^-} \psi, \quad \partial_t \bar{\psi} = A^\pm \bar{\psi}. \tag{3-10}
$$

For the variation of $t$ obeying the MNV equation, the Willmore functional is invariant,

$$
4\partial_t \int d^2z \ V^2 = \partial_t E = 0. \tag{3-11}
$$

because the integrand can be expressed by the boundary quantities [13],

$$
V_t^2 = \partial(V \partial^2 V - \frac{1}{2}(\partial V)^2 + \frac{3}{2} V^2 U) + \bar{\partial}(V \bar{\partial}^2 V - \frac{1}{2}(\bar{\partial} V)^2 + \frac{3}{2} \bar{V}^2 \bar{U}). \tag{3-12}
$$

Next I will check the preserving the complex structure of surface for the MNV flows following the argument of Taimanov.
First I will remark that the metric is represented by the Dirac field as
\[ \rho = (|\psi_1|^2 + |\psi_2|^2)^2, \] (3-13)
owing to the relation (3-3). Thus if the relation (3-3) is covariant or preserves for a point of the MNV flows, the conformal structure (3-13) maintains.

Thus I will evaluate \( \partial_t Z = \partial_+ Z + \partial_- Z \) and \( \partial_t X^3 \). By straightforward computations, these values calculated as \[ \partial_t Z = 2i \int z^{(z)} d(f_+ + g_+), \] (3-14)
and
\[ \partial_t X^3 = - \int z^{(z)} d(h_1 + h_2) \] (3-15)
where
\[ f_+ := \frac{3}{2} U \psi_-^2, \quad g_+ := \psi_- \partial^2 \psi_- - \frac{1}{2} (\partial \psi_-)^2, \]
\[ f_- := \frac{3}{2} U \bar{\psi}_+^2, \quad g_- := \psi_+ \bar{\partial}^2 \bar{\psi}_+ - \frac{1}{2} (\bar{\partial} \bar{\psi}_+)^2, \] (3-16)
\[ h_1 = \psi_+ \bar{\partial}^2 \psi_- + \psi_- \partial^2 \bar{\psi}_+ - \partial \psi_- \partial \bar{\psi}_+ + 3U \bar{\psi}_+ \psi_-,, \]
\[ h_2 = \psi_+ \bar{\partial}^2 \bar{\psi}_- + \bar{\psi}_- \partial^2 \psi_+ - \partial \bar{\psi}_- \partial \psi_+ + 3 \bar{\psi}_+ \psi_-. \] (3-17)

\( U^+(+) = U \quad \text{and} \quad U^(-) = \bar{U} \). Here \( df = \partial f dz + \bar{\partial} f d\bar{z} \).

Let the infinitesimal flows obeying the MNV equation (3-7) module \( \epsilon^2 \) be denoted as
\[ (Z_\epsilon, X_\epsilon^3) := (Z, X^3) + \epsilon \partial_t (Z, X^3) + O(\epsilon^2). \] (3-18)

(3-14)-(3-17) means the infinitesimal variation is given as the integral of the closed form defined over \( \Sigma \) [34] and can be regarded as a single function of \( \Sigma \). Since \( (Z, X^3) \) is also a periodic function of \( \Sigma \), \( (Z_\epsilon, X_\epsilon^3) \) is globally defined over \( \Sigma \) as a function of \( \Sigma \).

On the other hand, (3-14)-(3-17) guarantee that \( [\partial, \bar{\partial}] X_\epsilon^i = 0 \), which means that I can locally define the independent coordinates \( z \) and \( \bar{z} \) for \( X_\epsilon^i \) surface; I can locally find a conformal coordinate system of an open set of \( \Sigma \). Furthermore due to the global properties, their coordinate system can be extended to the global coordinate and the connection of each open set are trivial due to \( [\partial, \bar{\partial}] X_\epsilon^i = 0 \) for any point of \( \Sigma \).

Hence the MNV flows preserves the complex structure of the surface \( \mathcal{S} \).

I will emphasis that the MNV problem is also an initial value problem, for any shape of compact conformal surface, the "time" \( t \) development of the surface can be expressed and these conserves the energy functional and complex structure. Hence the trajectory of the deformation parameter \( t \) means the states of the same energy and its volume is the density of states of each energy \( E \).

As I did for the quantization elastica, the measure of the functional integral \( d\mu \) can be decomposed,
\[ d\tilde{\mu} = \sum_E d\tilde{\mu}_E \] (3-19)

and moduli of \( \tilde{Z}_{\text{reg}} \) restricted by \( E \) is denoted as \( \tilde{\mathcal{Z}}_E \)
\[ \tilde{Z}_{\text{reg}} = \int d\tilde{\mu} \exp(-\beta E) = \sum_E \exp(-\beta E) \int d\tilde{\mu}_E = \sum_E \exp(-\beta E) \text{Vol}(\tilde{\mathcal{Z}}_E) \] (3-20)
As I did for the Bernoulli-Euler functional, the density of state of the Willmore functional system might be completely represented by the moduli of the MNV equation.

However the Willmore surface has no natural length because for a global scale transformation $z \rightarrow \lambda z \ (\lambda > 0)$, the mean curvature changes as $H \rightarrow H/\lambda$ and the Willmore surface is invariant. Hence for given energy $E$, there are infinite degenerate states regarding to the global scaling parameter $\lambda \in (0, \infty)$ and the regularized partition function $\tilde{Z}_{\text{reg}}$ also diverges.

However along the line of $z = \bar{z}$, the deformation of MNV flows obeys the MKdV equation which conserves local length of the line. In other words, on the MNV flows, the length of the line is a conserved quantity and is well-defined. Hence in terms of this length, I can redefine the partition function by fixing the length of the line,

$$Z_{\text{reg}} := \tilde{Z}_{\text{reg}}|_{(\text{the length of } z = \bar{z}) = L}. \quad (3-21)$$

Due to the compactness of the surface $\mathcal{S}$, $L$ must be finite. By fixing the scale of the surface, I will define decomposed measure and the space of the trajectories, $d\mu_{E,L} := d\tilde{\mu}|_{L}$, $\Xi_{E,L} := \tilde{\Xi}|_{L}$,

$$Z_{\text{reg}} = \sum_E \int_{\Xi_{E,L}} d\mu_{E,L} \exp(-\beta E) = \sum_E \exp(-\beta E) \text{Vol}(\Xi_{E,L}) \quad (3-22)$$

The physical meaning of the summation in (3-22) is justified similar to (2-19).

§4. Discussion

Carroll and Konopelchenko also proved that the MNV flows conserves the extrinsic string action for the case that $\rho H$ is constant; here the extrinsic string action consists of the Nambu-Goto action, the Wess-Zumino-Witten type geometrical action, and the Polyakov extrinsic curvature action (3-3). Hence my result (3-2) can be extended to such a case and then it means that the algorithm of the calculation of the partition function of the extrinsic string in $\mathbb{R}^3$ is essentially the same as above arguments. In other words, the quantization of the string immersed in $\mathbb{R}^3$ can be partially performed even though only the string in $\mathbb{R}^n \ n < 3$ had been studied as the two dimensional gravity [28-30].

In ref. [2] and [3], I stated that as the self-dual Yang-Mills equation can be expressed by the integrable equation and be represented by the Dirac operator and as the MKdV equation governs the "virtual" motion of the elastica and can be written by the Dirac operator, the higher dimensional soliton surface might be expressed by the Dirac operator and has a physical meaning.

This conjecture was proved by the discoveries and studies of Konopelchenko and Taimanov. Using the Dirac operator, they investigated the surface itself and derived non-trivial results [10-14]. Categories of the linear analytic system of Dirac operator, the geometry and integrable system are closely connected with each other. Their connections should be interpreted as functors among them and the exact sequences and the associated cohomorogy in individual categories should be expressed by common language. Thus the expression of these systems should be unified and then using the relations their common hidden symmetries might be revealed. I believe that this quantization of surface and quantization of elastica [25] contributes such studies.

Finally I will comment upon open problems related to this system. In ref.[25], I found that at the critical point of the quantized elastica, a certain expectation value obeys the Painlevé equation of the first kinds. I have a question what equation appears at a critical point in the quantized Willmore surface system. If exists, it might be related to the higher dimensional analogue of the Painlevé equation.

Furthermore since the MNV equation is an initial value problem, more general Riemannian surfaces can be allowed, at least, an initial condition even though the energy manifold of the inverse scattering system is given as only hyperelliptic curves [6]. Hence I have another question whether there is an analytical connection between the general Riemannian surface or the general Fuchian group and the hyperelliptic function of the MKdV equation. If there is, this system should be holomorphically studied.
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