HOLOMORPHIC GEOMETRIC MODELS FOR REPRESENTATIONS OF C*-ALGEBRAS

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Abstract. Representations of C*-algebras are realized on section spaces of holomorphic homogeneous vector bundles. The corresponding section spaces are investigated by means of a new notion of reproducing kernel, suitable for dealing with involutive diffeomorphisms defined on the base spaces of the bundles. Applications of this technique to dilation theory of completely positive maps are explored and the critical role of complexified homogeneous spaces in connection with the Stinespring dilations is pointed out. The general results are further illustrated by a discussion of several specific topics, including similarity orbits of representations of amenable Banach algebras, similarity orbits of conditional expectations, geometric models of representations of Cuntz algebras, the relationship to endomorphisms of B(H), and non-commutative stochastic analysis.

1. Introduction

Originally, the interest in the study of representations of algebras and groups of operators on infinite-dimensional Hilbert or Banach spaces is to be found, as one of the main motivations, in problems arising from Quantum Physics. In this setting, unitary groups of operators can be interpreted as symmetry groups while the self-adjoint operators are thought of as observable objects, hence the direct approach to such questions leads naturally to representations both involving algebras generated by commutative or non-commutative canonical relations, and groups of unitaries on Hilbert spaces; see for instance [GW54], [Sh62], or [Se57]. Over the years, there have been important developments of this initial approach, in papers devoted to analyze or classify a wide variety of representations, and yet many questions remain open in the subject. It is certainly desirable to transfer to this field methods, or at least ideas, of the rich representation theory of finite-dimensional Lie groups.

In this respect, recall that geometric representation theory is a classical topic in finite dimensions. Its purpose is to shed light on certain classes of representations by means of their geometric realizations (see for instance [Ne00]). Thus the construction of geometric models of representations lies at the heart of that topic, and one of the classical results obtained in this direction is the Bott-Borel-Weil theorem concerning realizations of irreducible representations of compact Lie groups in spaces of sections (or higher cohomology groups) of holomorphic vector bundles over flag manifolds; see [Bo57]. Section spaces of vector bundles also appear in methods of induction of representations, of Lie groups, from representations initially defined on appropriate subgroups. Induced representations are required for instance by the so-called orbit method, consisting of establishing a neat link between general representations of a Lie group and the symplectic geometry of its coadjoint orbits; see [Ki04] or [Fo95]. Such sections are quite often obtained out of suitable square-integrable functions on the base space of the bundle.

Sometimes these ideas work well in the setting of infinite-dimensional Lie groups, in special situations or for particular aims; see for example [Bo80], [Ki04], and [Ne04]. However, several difficult points are encountered when one tries to extend these ideas in general, and perhaps the most important one is related to the lack of an algebraic structure theory for representations of these groups. Also, it is not a minor question the fact that, in infinite dimensions, there is no sufficiently well-suited theory of integration. The most reasonable way to deal with these problems seems to be to restrict both the class of groups and the
class of representations one is working with. Moreover, one is led quite frequently to employ methods of operator algebras. See for example [SV02], where the study of factor representations of the group $U(\infty)$ and AF-algebras is undertaken. There, a key role is played by the Gelfand-Naimark-Segal (or GNS, for short) representations constructed out of states of suitable maximal abelian self-adjoint subalgebras.

The importance of GNS representations as well as that of the geometric properties of state spaces in operator theory are well known. In [BR07], geometric realizations of restrictions of GNS representations to groups of unitaries in $C^*$-algebras are investigated. Suitable versions of reproducing kernels on vector bundles are considered, in order to build representation spaces formed by sections. This technique has already a well-established place in representation theory of finite-dimensional Lie groups; see for instance the monograph [Ne00]. In some more detail, let $1 \in B \subseteq A$ be unital $C^*$-algebras such that there exists a conditional expectation $E: A \to B$. Let $U_A$ and $U_B$ be the unitary groups of $A$ and $B$ respectively, and $\varphi$ a state of $A$ such that $\varphi \circ E = \varphi$. A reproducing kernel Hilbert space $\mathcal{H}_{\varphi,E}$ can be constructed out of $\varphi$ and $E$, consisting of $C^\infty$ sections of a certain Hermitian vector bundle with base the homogeneous space $U_A/U_B$, and the restriction to $U_A$ of the GNS representation associated with $\varphi$ can be realized (by means of a certain intertwining operator) as the natural multiplication of $U_A$ on $\mathcal{H}_{\varphi,E}$; see Theorem 5.4 in [BR07]. This theorem relates the GNS representations to the geometric representation theory, in the spirit of the Bott-Borel-Weil theorem. In view of this result and of the powerful method of induction developed in [Bo80], it is most natural to ask about similar results for more general representations of infinite-dimensional Lie groups.

Another circle of ideas is connected with holomorphy. Recall that this is the classical setting of the Bott-Borel-Weil theorem of [Bo57] involving the flag manifolds, and it reinforces the strength of applications. On the other hand, the idea of complexification plays a central role in this area, inasmuch as one of the ways to describe the complex structure of the flag manifolds is to view the latter as homogeneous spaces of complexifications of compact Lie groups. (See [LS91], [Bo93], [Bo04], and [Sz04] for recent advances in understanding the differential geometric flavor of the process of complexification.) In some cases involving finiteness properties of spectra and traces of elements in a $C^*$-algebra, it is possible to prove that the aforementioned infinite-dimensional homogeneous space $U_A/U_B$ is a complex manifold as well and the Hilbert space $\mathcal{H}_{\varphi,E}$ is formed by holomorphic sections (see Theorem 5.8 in [BR07]).

One can find in Section 2 of the present paper some related results of holomorphy in the important special case of tautological bundles over Grassmann manifolds associated with involutive algebras. For the reader’s convenience, these results are exposed in some detail since they illustrate the main ideas underlying the present investigation. (A complementary perspective on these manifolds can be found in [BN05].) Apart from the above two examples, the holomorphic character of the manifolds $U_A/U_B$ (and associated bundles) is far from being clear in general. On the other hand, the aforementioned conditional expectation $E: A \to B$ has a strong geometric meaning as a connection form defining a reductive structure in the homogeneous space $G_A/G_B$; see [ACS95] and [CG99]. Since $X$ is the Lie algebra of the complex Banach-Lie group $G_X$ for $X = A$ and $X = B$, it is important to incorporate full groups of involutives to the framework established in [BR07]. Note also that $G_X$ is the universal complexification of $U_X$, according to the discussion of [Ne02].

**Brief description of the present paper.** One of our aims in the present paper is to extend the geometric representation theory of unitary groups of operator algebras to the complex setting of full groups of involutive elements. For this purpose we need a method to realize the representation spaces as Hilbert spaces of sections in holomorphic vector bundles. If one tries to mimic the arguments of [BR07] then one runs into troubles very soon (regarding the construction of appropriate reproducing kernels), due to the fact that general involutive elements of a $C^*$-algebra lack, when considered in an inner product, helpful cancellative properties (that unitaries have). This can be overcome by using certain involutions $z \mapsto z^{-*}$ (that come from the involutions of $C^*$-algebras) on the bases of the bundles, but then the problem is that our bundles lose their Hermitian character.

So we are naturally led toward developing a special theory of reproducing kernels on vector bundles. Section 3 includes a discussion of a version of Hermitian vector bundles suitable for our purposes. We call them *like-Hermitian*. The bases of such vector bundles are equipped with involutive diffeomorphisms $z \mapsto z^{-*}$, so that we need to find out a class of reproducing kernels, compatible in a suitable sense with
the corresponding diffeomorphisms, which we call here reproducing \((-\ast)-\)kernels. The very basic elements for the theory of reproducing \((-\ast)-\)kernels are presented in Section 4 (it is our intention to develop such a theory more systematically in forthcoming papers). In Section 5 we discuss examples of the above notions which arise in relation to homogeneous manifolds obtained by (smooth) actions of complex Banach-Lie groups (see Definition 5.10). These examples play a critical role for our main constructions of geometric models of representations; see Theorem 5.2 and Theorem 5.4. In particular, Theorem 5.4 provides the holomorphic versions of such realizations. In order to include the homogeneous spaces of unitary groups \(U_A/U_B\) in the theory and to avoid the fact that they are not necessarily complex manifolds, we had to view them as embedded into their natural complexifications \(G_A/G_B\).

It is remarkable that, using a significant polar decomposition of \(G_A\) found by Porta and Rech, relative to a prescribed conditional expectation (see \([PR94]\)), it is possible to interpret the manifold \(G_A/G_B\) as (diffeomorphic to) the tangent bundle of \(U_A\) to a prescribed conditional expectation (see \([PR94]\)). These examples play a critical role for our main constructions of geometric models of representations; see Theorem 5.2 and Theorem 5.4. In particular, Theorem 5.4 provides the holomorphic versions of such realizations. In order to include the homogeneous spaces of unitary groups \(U_A/U_B\) in the theory and to avoid the fact that they are not necessarily complex manifolds, we had to view them as embedded into their natural complexifications \(G_A/G_B\).

We begin with several elementary considerations about idempotents in complex associative algebras. Notation 2.1. We are going to use the following notation: \(A\) is a unital associative algebra over \(\mathbb{C}\) with unit \(1\) and set of idempotents \(\mathcal{P}(A) = \{p \in A \mid p^2 = p\}\); for \(p_1, p_2 \in \mathcal{P}\) the notation \(p_1 \sim p_2\) means that we have both \(p_1 p_2 = p_2\) and \(p_2 p_1 = p_1\). For each \(p \in \mathcal{P}(A)\) we denote its equivalence class by

2. Grassmannians and homogeneous Hermitian vector bundles

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- a theory of reproducing kernels on vector bundles that takes into account prescribed involutions of the bundle bases (Section 4);
- in the case of homogeneous vector bundles we investigate a circle of ideas centered on the relationship between reproducing kernels and complexifications of homogeneous spaces (Theorems 5.4 and 5.6);
- by using the previous items we model the representation spaces of Stinespring dilations as spaces of holomorphic sections in certain homogeneous vector bundles; thereby we set forth a rich panel of differential geometric structures accompanying the dilations of completely positive maps (Section 6); for one thing, we provide a geometric perspective on induced representations of \(C^\ast\)-algebras (cf. \([Ri74]\)). We should point out here that there exist earlier approaches to questions in dilation theory with a geometric flavor—see for instance \([ALRS97], [Ar00], [Po01],\) or \([MS03]\)—however they are different from the present line of investigation.

The last section of the paper, Section 7, is devoted to showing, by means of several specific examples, that the theory established here has interesting links with quite a number of different subjects in operator theory and related areas.

For the sake of better explanation, we conclude this introduction by a summary of the main points considered in the paper. These are:

- an illustration of our results we describe in Section 7 a number of geometric properties of orbits of representations of nuclear \(C^\ast\)-algebras and injective von Neumann algebras (Corollary 7.2), similarity orbits of conditional expectations, and some relationships with representations of Cuntz algebras and endomorphisms of \(B(\mathcal{H})\), as well as with non-commutative stochastic analysis.
\[ \{ q \in \mathcal{P}(A) \mid q \sim p \}. \]

The quotient set is denoted by \( \text{Gr}(A) = \mathcal{P}(A)/\sim \) (the Grassmannian of \( A \)), and the quotient map by \( \pi : p \mapsto [p], \mathcal{P} \to \text{Gr}(A) \).

The group of invertible elements of \( A \) is denoted by \( G_A \), and it has a natural action on \( \mathcal{P}(A) \) by
\[
\alpha : (u, q) \mapsto upu^{-1}, \quad G_A \times \mathcal{P}(A) \to \mathcal{P}(A).
\]

The corresponding isotropy group at \( p \in \mathcal{P}(A) \) is \( \{ u \in G_A \mid \alpha(u, p) = p \} = G_A \cap \{ p \}' = \{ p \}' = : G(p) \)
where we denote by \( \{ p \}' \) the commutant subalgebra of \( p \) in \( A \) (see page 484 in [DG02]). \( \square \)

**Lemma 2.2.** There exists a well-defined action of the group \( G_A \) upon \( \text{Gr}(A) \) like this:
\[
\beta : (u, [p]) \mapsto [upu^{-1}], \quad G_A \times \text{Gr}(A) \to \text{Gr}(A),
\]
and the diagram
\[
\begin{array}{ccc}
G_A \times \mathcal{P}(A) & \xrightarrow{\alpha} & \mathcal{P}(A) \\
\text{id}_G \times \pi \downarrow & & \downarrow \pi \\
G_A \times \text{Gr}(A) & \xrightarrow{\beta} & \text{Gr}(A)
\end{array}
\]
is commutative.

**Proof.** See for instance the end of Section 3 in [DG01]. \( \square \)

**Definition 2.3.** For every idempotent \( p \in \mathcal{P}(A) \) we denote by \( G_A([p]) \) the isotropy group of the action \( \beta : G_A \times \text{Gr}(A) \to \text{Gr}(A) \) at the point \( [p] \in \text{Gr}(A) \), that is, \( G_A([p]) = \{ u \in G_A \mid [upu^{-1}] = [p] \} \).

The following statement concerns the relationship between the isotropy groups of the actions \( \alpha \) and \( \beta \) of \( G_A \) upon \( \mathcal{P}(A) \) and \( \text{Gr}(A) \), respectively.

**Proposition 2.4.** The following assertions hold.

(i) For every \( p \in \mathcal{P}(A) \) we have \( G_A([p]) \cap G_A([1 - p]) = G(p) \).

(ii) If \( U \) is a subgroup of \( G_A \) and \( p \in \mathcal{P}(A) \) is such that \( U \cap G_A([p]) = U \cap G_A([1 - p]) \), then \( U \cap G_A([p]) = U \cap \{ p \}' = : U(p) \).

**Proof.** (i) We have
\[
G_A([p]) = \{ u \in G_A \mid [upu^{-1}] = [p] \} \quad \text{and} \quad G_A([1 - p]) = \{ u \in G_A \mid [u(1 - p)u^{-1}] = [1 - p] \},
\]
so that clearly \( G_A([p]) \cap G_A([1 - p]) \supseteq G_A \cap \{ p \}' \). For the converse inclusion let \( u \in G_A([p]) \cap G_A([1 - p]) \) arbitrary. In particular \( u \in G_A([p]) \), whence \( upu^{-1} \sim p \), which is equivalent to the fact that \( (upu^{-1})p = p \) and \( p(upu^{-1}) = upu^{-1} \). Consequently we have both
\[
(2.1) \quad pu^{-1}p = u^{-1}p
\]
and
\[
(2.2) \quad pup = up.
\]
On the other hand, since \( u \in G_A([1 - p]) \) as well, it follows that \( (1 - p)u^{-1}(1 - p) = u^{-1}(1 - p) \) and \( (1 - p)u(1 - p) = u(1 - p) \). The later equation is equivalent to \( u - up - pu + pup = u - up \), that is, \( pup = pu \). Then (2.2) implies that \( up = pu \), that is, \( u \in G(p) \).

(ii) This follows at once from part (i). \( \square \)

**Remark 2.5.** For instance, Proposition 2.4(ii) can be applied if the algebra \( A \) is equipped with an involution \( a \mapsto a^* \) such that \( p = p^* \), and \( U = U_A := \{ u \in G_A \mid u^{-1} = u^* \} \) is the corresponding unitary group. In this case, it follows by (2.1) and (2.2) that \( up = pu \) whenever \( u \in U_A \cap G_A([p]) \), hence \( U_A \cap G_A([p]) = U_A \cap \{ p \}' = : U_A(p) \).

For \( q \in \mathcal{P}(A) \), put \( \hat{q} := 1 - q \) and \( A^q := \{ a \in A \mid \hat{q}aq = 0 \} \). The following result is partly a counterpart, for algebras, of Proposition 2.4.

**Proposition 2.6.** Assume that \( A \) is equipped with an involution and let \( p \in \mathcal{P}(A) \) such that \( p = p^* \). Then the following assertions hold:
(i) \( uA^pu^{-1} = A^p\), for every \( u \in U_A(p)\);
(ii) \( A^p \cap A^q = \{ p \} \); 
(iii) \( A^p + A^q = A \);
(iv) \((A^p)^* = A^\hat{p}\).

Proof. (i) This is readily seen. 
(ii) Firstly, note that, for \( a \in A \), we have \( \hat{pa}p = p\hat{a}\) if and only if \( ap = pa\). Moreover, if \( ap = pa\) then \( \hat{pa}p = ap - pap = ap - ap = 0 \) and analogously \( p\hat{a}a = 0 \). From this, the equality of the statement follows.
(iii) For every \( a \in A \) and \( q \in \mathcal{P}(A) \) we have \( qu \in A^q \). Hence \( a = pa + \hat{pa} \in A^p + A^\hat{p} \), as we wanted to show.
(iv) Take \( a \in A^p \). Then \( pap = ap \), that is, \( pa^p = pa^* \). Hence, \( \hat{pa}^*p = (a^* - pa^*)(1 - p) = a^* - pa^* - a^*p + a^*p = a^* - a^*p = a^*p \). This means that \( a^* \in A^p \). Conversely, if \( a \in A^p \) then, as above, \( pa^*p = a^*p \); that is, \( a = (a^*)^* \) with \( a^* \in A^p \). \(\square\)

Assume from now on that \( A \) is a unital \( C^* \)-algebra. Then \( G_A \) is a Banach-Lie group whose Lie algebra coincides with \( A \). The \( G_A \)-orbits in \( Gr(A) \), obtained by the action \( \beta \) and equipped with the topology inherited from \( Gr(A) \), are holomorphic Banach manifolds diffeomorphic to \( G_A/G_A([p]) \) (endowed with its quotient topology), see Theorem 2.2 in [DC02]. Also, the Grassmannian \( Gr(A) \) can be described as the discrete union of these \( G_A \)-orbits, see [DC01] and Theorem 2.3 in [DC02]. Moreover, \( U_A \) is a Banach-Lie subgroup of \( G_A \) with the Lie algebra \( u_A := \{ a \in A \mid a^* = -a \} \). As it is well known, the complexification of \( u_A \) is \( A \), via the decomposition \( a = \{(a - a^*)/2\} + i\{(a + a^*)/2\} \), \( (a \in A) \). Thus the conjugation of \( A \) is given by \( a \mapsto \pi := \{(a - a^*)/2\} - i\{(a + a^*)/2\} = -a^* \). We seek for possible topological and/or differentiable relationships between the \( G_A \)-orbits and the \( U_A \)-orbits \( U_A/U_A(p) \) in \( \mathcal{G}(A) \).

Let \( p = p^* \in \mathcal{P}(A) \) and \( U_A(p) := U_A \cap \{ p \} \). It is clear that \( U_A(p) + iU_A(p) = \{ p \} \). Also, there is a natural identification between \( U_A/U_A(p) \) and the tangent space \( T_{[p]}(U_A/U_A(p)) \).

The above observations and Proposition 2.6 yield immediately the following result. Let \( \text{Ad}_U \) denote the adjoint representation of \( U_A \).

**Proposition 2.7.** With the above notations,
\[
\text{Ad}_U(u)A^p \subset A^p, \quad (u \in U_A(p)); \quad A^p \cap \overline{A^p} = u_A(p) + iu_A(p); \quad A^p + \overline{A^p} = A.
\]
In particular, \( u_A/u_A(p) \simeq A/A^p \) whence we obtain that \( U_A/U_A(p) \) and \( G_A/G_A([p]) \) are locally diffeomorphic, and so \( U_A/U_A(p) \) inherits the complex structure induced by \( G(A)/G_A([p]) \).

Proof. The first part of the statement is just a rewriting of Proposition 2.6. Then the result follows from Theorem 6.1 in [Be06]. \(\square\)

**Remark 2.8.** Since \( G_A(p) \subset G_A([p]) \), there exists the canonical projection \( G_A/G_A(p) \to G_A/G_A([p]) \). It is clear that its restriction to \( U_A/U_A(p) \) becomes the identity map \( U_A/U_A(p) \to U_A/U_A([p]) \). We have seen that \( U_A/U_A(p) \) enjoys a holomorphic structure inherited from that of \( G_A/G_A([p]) \). Moreover, \( G_A/G_A(p) \) is a complexification of \( U_A/U_A(p) \), in the sense that there exists an anti-holomorphic diffeomorphism in \( G_A/G_A(p) \) whose set of fixed points coincides with \( U_A/U_A(p) \):

The mapping \( aG_A(p) \to (a^*)^{-1}G_A(p), \ G_A/G_A(p) \to G_A/G_A(p) \) is an anti-holomorphic diffeomorphism (which corresponds to the mapping \( apa^{-1} \to (a^*)^{-1}pa^* \) in terms of orbits). Then \( aG_A(p) = (a^*)^{-1}G_A(p) \) if and only if \( (a^*a)G_A(p) = G_A(p) \), that is, \( (a^*a)p = p(a^*a) \). Using the functional calculus for \( C^* \)-algebras, we can pick \( b = \sqrt{\alpha^*a} \) in \( A \) and obtain \( bp = pb \). Since \( a^*a = b^2 = b^*b \) we have \( (ab^{-1})^* = (b^{-1})^*a^* = (b^*)^{-1}a^* = ba^{-1} = (ab^{-1})^{-1} \) and therefore \( u := ab^{-1} \in U_A \). Finally, \( aG_A(p) = ubG_A(p) = uG_A(p) \equiv uU_A(p) \in U_A/U_A(p) \).

According to Proposition 2.3 (i), idempotents like \( apa^{-1} \equiv aG_A(p) \), for \( a \in G_A \), can be alternatively represented as pairs \( (a[p]a^{-1}, (a^*)^{-1}[p]a^*) \) so that the “orbit” \( G_A/G_A(p) \) becomes a subset of the Cartesian product \( G_A([p]) \times G_A([p]) \). In this perspective, the preceding projection and diffeomorphism are given, respectively, by
\[
(a[p]a^{-1}, (a^*)^{-1}[p]a^*) \mapsto a[p]a^{-1} \equiv (a[p]a^{-1}, a[p]a^{-1}), \ G_A/G_A(p) \to G_A/G_A([p])
\]
(so \((u[p]u^{-1}, u[p]u^{-1}) \mapsto upu^{-1} \equiv (u[p]u^{-1}, u[p]u^{-1})\), when \(u \in U_A\)) and
\[
(a[p]a^{-1}, (a^*)^{-1}[p]a^*) \mapsto ((a^*)^{-1}[p]a^*, a[p]a^{-1}),
\]
for every \(a \in G_A\).

**Remark 2.9.** Proposition [27] relates to the setting of [BR07]. Namely, assume that \(B\) is a \(C^*\)-subalgebra of \(A\), with \(1 \in B \subseteq A\), for which there exist a conditional expectation \(E: A \rightarrow B\) and a state \(\varphi: A \rightarrow \mathbb{C}\) such that \(\varphi \circ E = \varphi\). For \(X \in \{A, B\}\), we denote by \(\varphi_X\) the state \(\varphi\) restricted to \(X\). Let \(\mathcal{H}_X\) be the Hilbert space, and let \(\pi_X: X \rightarrow \mathcal{B}(\mathcal{H}_X)\) be the corresponding cyclic representation obtained by the Gelfand-Naimark-Segal (GNS, for short) construction applied to the state \(\varphi_X: X \rightarrow \mathbb{C}\). Thus, \(\mathcal{H}_X\) is the completion of \(X/N_X\) with respect to the norm \(\|y + N_X\|_\varphi := \varphi(y^*y)\), where \(N_X := \{y \in X \mid \varphi(y^*y) = 0\}\). The representation \(\pi_X\) is then defined as the extension to \(\mathcal{H}_X\) of the left multiplication of \(X\) on \(X/N_X\).

Let \(P\) denote the orthogonal projection \(P: \mathcal{H}_A \rightarrow \mathcal{H}_B\).

An equivalence relation can be defined in \(G_A \times \mathcal{H}_B\) by setting that \((g_1, h_1) \sim (g_2, h_2)\) (with \(g_1, g_2 \in G_A, h_1, h_2 \in \mathcal{H}_B\)) if and only if there exists \(w \in G_B\) such that \(g_2 = g_1w\) and \(h_2 = \pi_B(w^{-1})h_1\). The corresponding quotient space will be denoted by \(G_A \times_{G_B} \mathcal{H}_B\), and the equivalence class in \(G_A \times_{G_B} \mathcal{H}_B\) of the element \((g, h)\) \(\in G_A \times \mathcal{H}_B\) will be denoted by \(\{(g, h)\}\). Define \(U_A \times_{U_B} \mathcal{H}_B\) in an analogous fashion. Then the mappings \(\Pi_L: \{g, h\} \mapsto g \mathcal{B}_B, G_A \times_{G_B} \mathcal{H}_B \rightarrow G_A / \mathcal{B}_B\) and \(\Pi_U: \{u, h\} \mapsto u \mathcal{B}_B, U_A \times_{U_B} \mathcal{H}_B \rightarrow U_A / \mathcal{B}_B\) are vector bundles, \(\Pi_L\) being Hermitian, in fact. Moreover, \(\Pi_U\) admits a reproducing kernel \(K\) with the associated Hilbert space \(\mathcal{H}^K\), formed by continuous sections of \(\Pi_U\), such that the restriction of the GNS representation \(\pi_A\) to \(U_A\) can be realized on \(\mathcal{H}^K\), see [BR07].

Let us apply the above theory to the case when, for a given unital \(C^*\)-algebra \(A\), we take \(B = \{p\}^* \in P(A)\). Then \(E_p: a \mapsto \text{pop} + a \text{pop} \downarrow\text{pa}, A \rightarrow B\) is a conditional expectation from \(A\) onto \(B\). Let \(\mathcal{H}\) be a Hilbert space such that \(A \hookrightarrow \mathcal{B}(\mathcal{H})\). Pick \(x_0 \in \mathcal{H}\) such that \(\|x_0\| = 1\). Then \(\varphi_0: A \rightarrow \mathbb{C}, \text{given by } \varphi_0(a) := \langle ax_0 \mid x_0 \rangle\mathcal{H}\) for all \(a \in A\), is a state of \(A\) such that \(\varphi_0 \circ E_p = \varphi_0\). The GNS representation of \(A\) associated with \(\varphi_0\) is as follows. Set \((a_1 | a_2) := \varphi_0(a_2^*a_1) = \langle a_2^*a_1x_0 \mid x_0 \rangle\mathcal{H} = \langle a_1x_0 \mid a_2x_0 \rangle\mathcal{H}\) for every \(a_1, a_2 \in A\). So \(\varphi_0(a^*a) = \|a(x_0)\|^2\) for all \(a \in A\), whence the null space of \(\cdot | \cdot\rangle_0\) is \(N_0 := \{a \in A \mid \langle a | \cdot \rangle_0 = 0\} = \{a \in A \mid a(x_0) = 0\}\). The norm \(\| \cdot \|_\mathcal{H}\) induced by \(\cdot | \cdot\rangle_0\) on \(A/N_0\) is given by \(\|h\|_\mathcal{H} := \|a + N_0\|_\mathcal{H} = \varphi_0(a^*a)^{1/2} = \|a(x_0)\|_\mathcal{H} = \|h\|_\mathcal{H}\) for every \(h \in A(x_0) \subset \mathcal{H}\), where \(a(x_0) = h \mapsto a + N_0\). Hence \(\mathcal{H}_A\) is a closed subspace of \(\mathcal{H}\) such that \(a\mathcal{H}_A \subset \mathcal{H}_A\) for every \(a \in A\). Note that \(\mathcal{H}_A\) coincides with \(\mathcal{H}\) provided that we can choose \(x_0\) in \(\mathcal{H}\) such that \(A(x_0)\) is dense in \(\mathcal{H}\). This will be of interest in Remark [2.13] below.

Analogously, we can consider the restriction of \(\cdot | \cdot\rangle_0\) to \(B\) and proceed in the same way as above. Thus we obtain that the corresponding null space is \(B \cap N_0\), that the norm in \(B/(B \cap N_0)\) is that of \(p\mathcal{H}\) (so that one of \(\mathcal{H}\)), and that \(\mathcal{H}_B\) is a closed subspace of \(p\mathcal{H}\) such that \(b\mathcal{H}_B \subset \mathcal{H}_B\) for every \(b \in B\). Also, \(\mathcal{H}_B\) is \(p\mathcal{H}\) if \(x_0\) can be chosen in \(p\mathcal{H}\) and such that \(B(x_0)\) is dense in \(\mathcal{H}\).

The representation \(\pi_A: a \mapsto \pi(a), A \rightarrow \mathcal{B}(\mathcal{H}_A)\) is the extension to \(\mathcal{H}_A\) of the left multiplication \(\pi_A(a): a' + N_0 \mapsto \langle a'a \rangle + N_0, A/N_0 \rightarrow A/N_0\). Thus it satisfies \(\pi_A(a' + N_0) = \langle a'a \rangle + N_0 = a(a'x_0) = a(h),\) if \(a' + N_0 \mapsto a(x_0) = h\). In other words, \(\pi_A\) is the inclusion operator (by restriction) from \(A\) into \(\mathcal{B}(\mathcal{H}_A)\). Also, \(\pi_B\) is in turn the inclusion operator from \(B\) into \(\mathcal{B}(\mathcal{H}_B)\).

Since \(E_p(N_0) \subseteq N_0\), the expectation \(E_p\) induces a well-defined projection \(P: A/N_0 \rightarrow B/(N_0 \cap B)\). On the other hand, \(E_p(a^*a) - E_p(a)^*E_p(a) = \text{pa}^2 \text{pop} + \text{pa} \text{pop} \geq 0\) since \(p, \overline{p} \geq 0\). Hence \(P\) extends once again as a bounded projection \(P: H_A \rightarrow H_B\). Indeed, if \(h = a(x_0)\) with \(a \in A\), we have
\[
P(h) \equiv P(a + N_0) = E(a) + (B \cap N_0) = E(a)(x_0) = (pa)(x_0) = ph,
\]
that is, \(P = p|\mathcal{H}_A\).

In the above setting, note that \(U_B = U(p)\). Let \(\Gamma(U_A / U(p), U_A \times U(p) \mathcal{H}_B)\) be the section space of the bundle \(\Pi_U\). The reproducing kernel associated with \(\Pi_U\) is given by \(K_p(u_1U(p), u_2U(p))[(u_2, f_2)] := \langle [u_1, pu_1^{-1}u_2]f_2 \rangle, \) for every \(u_1, u_2 \in U_A\) and \(f_2 \in \mathcal{H}_B\). The kernel \(K_p\) generates a Hilbert subspace \(\mathcal{H}_B^{K_p}\) of sections in \(\Gamma(U_A / U(p), U_A \times U(p) \mathcal{H}_B)\). Let \(\gamma_p: \mathcal{H}_A \rightarrow \Gamma(U_A / U(p), U_A \times U(p) \mathcal{H}_B)\) be the mapping defined by \(\gamma_p(h)(uU(p)) := \langle [u, pu^{-1}h] \rangle \) for every \(h \in \mathcal{H}_A\) and \(u \in U_A\). Then \(\gamma_p\) is injective and it intertwines
the representation \( \pi_A \) of \( U_A \) on \( \mathcal{H}_A \) and the natural action of \( U_A \) on \( \mathcal{H}^{K_F} \); that is, the diagram

\[
\begin{array}{c}
\mathcal{H}_A \\
\gamma_p
\end{array}
\quad \xrightarrow{u} \quad
\begin{array}{c}
\mathcal{H}_A \\
\gamma_p
\end{array}
\]

is commutative for all \( u \in U_A \), where \( \mu(u)F := uF(u^{-1} \cdot \cdot \cdot) \) for every \( F \in \Gamma(U_A/U(p), U_A \times U(p) \mathcal{H}_B) \).

Remark 2.12.

Notation 2.11.

Let us recall the specific definition and some properties of the Grassmannian manifold in this case.

In fact \( \gamma(uh)(vU(p)) := [(v, pv^{-1}uh)] = u[(u^{-1}v, pv^{-1}uh)] =: u\{\gamma(h)(u^{-1}vU(p))\} \) for all \( u, v \in U_A \).

See Theorem 5.4 of [BR07] for details in the general case. We next show that \( \mathcal{H}^{K_F} \) in fact consists of holomorphic sections.

**Proposition 2.10.** Let \( A \) be a unital \( C^* \)-algebra, \( p = p^* \in \mathcal{P}(A) \), and \( B := \{p\}' \). In the above notation, the homogeneous Hermitian vector bundle \( \Pi_U : U_A \times U(p)H_B \to U_A/U(p) \) is holomorphic, and the image of \( \gamma_p \) consists of holomorphic sections. Thus \( \mathcal{H}^{K_F} \) is a Hilbert space of holomorphic sections of \( \Pi_U \).

**Proof.** Let \( u_0 \in U_A \). Then \( \Omega_G := \{u_0g \mid g \in G_A, \|1 - g^{-1}\| < 1\} \) is open in \( G_A \) and contains \( u_0 \), and similarly with \( \Omega_U := \Omega_G \cap U_A \) in \( U_A \).

It is readily seen that the mapping \( \psi_0 : \{[u, f] \} \to (uU(p), E_p(u^{-1}u_0^{-1}u)^{-1}f) \), \( \Pi_\Omega^{-1}(\Omega_U) \to \Omega_U \times \mathcal{H}_B \) is a diffeomorphism, with inverse map \( \gamma_p : \Pi_U \to \Omega_U \times \mathcal{H}_B \) (this shows the local triviality of \( \Pi_U \)). Thus every point in the manifold \( U_A \times U(p) \mathcal{H}_B \) has an open neighborhood which is diffeomorphic to the manifold product \( W \times \mathcal{H}_B \), where \( W \) is an open subset of \( U_A/U(p) \). By Proposition 2.7, \( U_A/U(p) \) is a complex homogeneous manifold and therefore the manifold \( U_A \times U(p) \mathcal{H}_B \) is locally complex, i.e., holomorphic. Also the bundle map \( \Pi_U \) is holomorphic.

On the other hand, for fixed \( h \in \mathcal{H}_A \), the mapping \( \sigma_0 : gG_A([p]) \to E_p(g^{-1}u_0^{-1})^{-1}pg^{-1}h \), \( \Omega_G \to \mathcal{H}_B \) is holomorphic on \( \Omega_G \), so it defines a holomorphic function \( \sigma_0 : \Omega_GG_A([p]) \to \mathcal{H}_B \). By Proposition 2.7, the injection \( j : U_A/U(p) \to G_AG_A([p]) \) is holomorphic, and so the restriction map \( r := \sigma_0 \circ j \) is holomorphic around \( u_0U(p) \). Since \( \gamma(h) = \psi_0^{-1}\circ (I_{\Omega_U} \times r) \) around \( u_0U(p) \), it follows that \( \gamma(h) \) is (locally) holomorphic.

Finally, by applying Theorem 4.2 in [BR07] we obtain that \( K_F \) is holomorphic. \( \square \)

The starting point for the holomorphic picture given in Proposition 2.10 has been the fact that \( U_A/U(p) \) enjoys a holomorphic structure induced by the one of \( G_A/G([p]) \), see Proposition 2.7. Such a picture can be made even more explicit if we have a global diffeomorphism \( U_A/U_A(p) \cong G_A/G_A([p]) \). The prototypical example is to be found when \( A \) is the algebra of bounded operators on a complex Hilbert space. Let us recall the specific definition and some properties of the Grassmannian manifold in this case.

**Notation 2.11.** We shall use the standard notation \( B(\mathcal{H}) \) for the \( C^* \)-algebra of bounded linear operators on the complex Hilbert space \( \mathcal{H} \) with the involution \( T \mapsto T^* \). Let \( GL(\mathcal{H}) \) be the Banach-Lie group of all invertible elements of \( B(\mathcal{H}) \), and \( U(\mathcal{H}) \) its Banach-Lie subgroup of all unitary operators on \( \mathcal{H} \). Also,

- \( Gr(\mathcal{H}) := \{S \mid S \text{ closed linear subspace of } \mathcal{H}\}; \)
- \( T(\mathcal{H}) := \{(S, x) \in Gr(\mathcal{H}) \times \mathcal{H} \mid x \in S\} \subseteq Gr(\mathcal{H}) \times \mathcal{H}; \)
- \( \Pi_T : (S, x) \mapsto S, T(\mathcal{H}) \to Gr(\mathcal{H}); \)
- for every \( S \in Gr(\mathcal{H}) \) we denote by \( ps : \mathcal{H} \to S \) the corresponding orthogonal projection. \( \square \)

**Remark 2.12.** The objects introduced in Notation 2.11 have the following well known properties:

(a) Both \( Gr(\mathcal{H}) \) and \( T(\mathcal{H}) \) have structures of complex Banach manifolds, and \( Gr(\mathcal{H}) \) carries a natural (non-transitive) action of \( U(\mathcal{H}) \). (See Examples 3.11 and 6.20 in [Up85], or Chapter 2 in [Do66].)

(b) For every \( S_0 \in Gr(\mathcal{H}) \) the corresponding connected component of \( Gr(\mathcal{H}) \) is the \( GL(\mathcal{H}) \)-orbit and is also the \( U(\mathcal{H}) \)-orbit of \( S_0 \), that is,

\[
Gr_{S_0}(\mathcal{H}) = \{gS_0 \mid g \in GL(\mathcal{H})\} = \{uS_0 \mid u \in U(\mathcal{H})\}
\]

\[
= \{S \in Gr(\mathcal{H}) \mid \dim S = \dim S_0 \text{ and } \dim S^\perp = \dim S_0^\perp \} \simeq U(\mathcal{H})/(U(S_0) \times U(S_0^\perp)).
\]

(See Proposition 23.1 in [Up85] or Lemma 2.13 below, alternatively.)
(c) The mapping \( \Pi_H : T(H) \to \text{Gr}(H) \) is a holomorphic Hermitian vector bundle, and we call it the universal (tautological) vector bundle associated with the Hilbert space \( H \). Set \( T_{\text{Gr}}(H) := \{(S, x) \in T(H) \mid S \in \text{Gr}_{\text{Gr}}(H)\} \). The vector bundle \( T_{\text{Gr}}(H) \to \text{Gr}_{\text{Gr}}(H) \) obtained by restriction of \( \Pi_H \) to \( T_{\text{Gr}}(H) \) will be called here the universal vector bundle at \( S_0 \). It is also Hermitian and holomorphic.

\[
\text{dim } S \text{ as said in Remark 2.12(b), every } \text{Gr}(H) \text{ unital } C^* \text{-algebra. Every element } g \in G_A \text{ has a unique polar decomposition } g = u a \text{ with } u \in U_A \text{ and }
\]

Property (b) in Remark 2.12 means that \( U_A / U_A(p_{S_0}) \simeq G_A / G_A(p_{S_0}) \) for \( A = B(H) \). For the sake of clarification we now connect Notation 2.14 and Notation 2.11 in more detail. For \( A = B(H) \) we have \( \text{Gr}(A) = \text{Gr}(H) \), and with this identification the action \( \beta \) of Lemma 2.2 corresponds to the natural action (so-called collineation action) of the group of invertible operators on \( H \) upon the set of all closed linear subspaces of \( H \). The following lemma gives us the collineation orbits of \( \text{Gr}(H) \) in terms of orbits of projections, and serves in particular to explain the property stated in Remark 2.12(b).

For short, denote \( G = \text{GL}(H) \) and \( U = U(H) \).

**Lemma 2.13.** Let \( S_0 \in \text{Gr}(H) \). Then the following assertions hold.

(i) \( \text{Gr}(p_{S_0}) = \{ g \in G \mid g S_0 = S_0 \} \) and \( \text{U}(p_{S_0}) = \{ u \in U \mid u S_0 = S_0 \} \).

(ii) For every \( g \in G \) and \( S = g S_0 \) we have \( S^\perp = (g^*)^{-1}(S_0^\perp) \).

(iii) We have \( \text{Gr}_{\text{Gr}}(H) = \{ g S_0 \mid g \in G \} = \{ [g S_0, g^{-1}] \mid g \in G \} \).

(iv) We have \( \text{U}/\text{U}(p_{S_0}) \simeq G/\text{G}(p_{S_0}) \simeq \text{Gr}_{\text{Gr}}(H) \), where the symbol \( \simeq \) means diffeomorphism between the respective differentiable structures, and that the differentiable structure of the quotient spaces is the one associated with the corresponding quotient topologies.

(v) \( G/\text{G}(p_{S_0}) \simeq \{(a S_0, (a^*)^{-1} S_0) \mid a \in G \} \) and the map \((a S_0, (a^*)^{-1} S_0) \mapsto ((a^*)^{-1} S_0, a S_0) \) is an involutive diffeomorphism on \( G/\text{G}(p_{S_0}) \). Its set of fixed points is \( \text{Gr}_{\text{Gr}}(H) \equiv \{ (u S_0, u S_0) \mid u \in U \} \).

**Proof.** (i) As shown in Proposition 2.4, an element \( g \) of \( G \) belongs to \( \text{G}(p_{S_0}) \) if and only if \( p_{S_0} g^{-1} p_{S_0} = g^{-1} p_{S_0} \) and \( p_{S_0} g p_{S_0} = g p_{S_0} \). From this, it follows easily that \( g(S_0) \subset S_0 \) and \( g^{-1}(S_0) \subset S_0 \), that is, \( g(S_0) = S_0 \). Conversely, if \( g(S_0) \subset S_0 \) then \( g(p_{S_0}) \subset p_{S_0} \subset (p_{S_0}) \) whence \( g p_{S_0} = p_{S_0} \); similarly, \( g^{-1}(S_0) \subset S_0 \) implies that \( p_{S_0} g^{-1} p_{S_0} = g^{-1} p_{S_0} \). In conclusion, \( \text{G}(p_{S_0}) = \{ g \in G \mid g S_0 = S_0 \} \).

Now, the above equality and Remark 2.2 imply that \( \text{U}(p_{S_0}) = \{ u \in U \mid u S_0 = S_0 \} \).

(ii) Let \( x \in S_0^\perp \), \( y \in S \). Then \((g^*)^{-1}(x) \mid y\} = ((g^{-1})^*(x) \mid y\} = (x \mid g^{-1}(y)) = 0 \), so \((g^*)^{-1}(S_0^\perp) \subset S^\perp \). Take now \( y \in S^\perp \), \( x = g^*(y) \) and \( z \in S_0 \). Then \((x \mid z) = (g^*(y) \mid z) = (y \mid g(z)) = 0 \), whence \( x \in S_0^\perp \) and therefore \( y = (g^*)^{-1}(g^* y) \in (g^*)^{-1}(S_0^\perp) \). In conclusion, \( S^\perp = (g^*)^{-1}(S_0^\perp) \).

(iii) By (ii), we have \( \text{U}(S_0^\perp) = u(S_0^\perp) \) for \( u \in U \). Thus \( S = u(S_0) \) if and only if \( \dim S = \dim S_0 \) and \( \dim S = \dim S_0 = u(S_0^\perp) \). Also, if \( S = u(S_0) \) and \( S = u(S_0^\perp) \), then \( p_{S_0} u = p_{S_0} u \), that is, \( p_{S_0} u = p_{S_0} u \). Hence \( \text{Gr}_{\text{Gr}}(H) = \{ u S_0 \mid u \in U \} = \{ S \in \text{Gr}(H) \mid \dim S = \dim S_0 \} \).

Suppose now that \( S = g S_0 \) with \( g \in G \). Then \( \dim S = \dim S_0 \). By (ii) again, \( S^\perp \) and \( (g^*)^{-1}(S_0^\perp) \) and so \( \dim S^\perp = \dim S_0^\perp \). Hence \( S \in \text{Gr}_{\text{Gr}}(H) \). Finally, the bijective correspondence between \( g S_0 \) and \( g[p_{S_0}]^{-1} \) is straightforward.

(iv) This is clearly a consequence of parts (iii) and (i) from above, and Theorem 2.2 in [DG02].

(v) For every \( a \in G \), the pairs \((a S_0, (a^*)^{-1} S_0) \) and \((a[p] a^{-1}, (a^*)^{-1} [p] a^* \) are in a one-to-one correspondence, by part (iii) from above. Hence, this part (v) is a consequence of Remark 2.5.

Parts (iv) and (v) of Lemma 2.13 tell us that the Grassmannian orbit \( \text{Gr}_{\text{Gr}}(H) \) is a complex manifold which in turn admits a complexification, namely the orbit \( \text{G}/\text{G}(p_{S_0}) \).

**Remark 2.14.** As said in Remark 2.12(b), every \( \text{GL}(H) \)-orbit (and so every \( U(H) \)-orbit) is a connected component of \( \text{Gr}(H) \). Let us briefly discuss the connected components of \( \text{Gr}(A) \) when \( A \) is an arbitrary unital \( C^* \)-algebra. Every element \( g \in G_A \) has a unique polar decomposition \( g = u a \) with \( u \in U_A \) and
$0 \leq a \in G_A$, hence there exists a continuous path $t \mapsto u \cdot ((1-t)1 + ta)$ in $G_A$ that connects $u = u \cdot 1$ to $g = u \cdot a$. Thus every connected component of the $G_A$-orbit of $[p] \in Gr(A)$ contains at least one connected component of the $U_A$-orbit of $[p] \in Gr(A)$ for any idempotent $p \in P(A)$. (Loosely speaking, the $U_A$-orbit of $[p]$ has more connected components than the $G_A$-orbit of $[p]$.) Example 7.13 in [PR87] shows that the $C^*$-algebra $A$ of the continuous functions $S^3 \to M_2(C)$ has the property that there indeed exist $G_A$-orbits of elements $[p] \in P(A)$ which are nonconnected.

If the unitary group $U_A$ is connected (so that the invertible group $G_A$ is connected), then all the $U_A$-orbits and the $G_A$-orbits in $Gr(A)$ are connected since continuous images of connected sets are always connected. On the other hand, as said formerly, the Grassmannian $Gr(A)$ is the discrete union of these $G_A$-orbits. Thus if the unitary group $U_A$ is connected, then the connected components of $Gr(A)$ are precisely the $G_A$-orbits in $Gr(A)$. One important case of connected unitary group $U_A$ is when $A$ is a $W^*$-algebra (since every $u \in U_A$ can be written as $u = \exp(ia)$ for some $a = a^* \in A$ by the Borel functional calculus, hence the continuous path $t \mapsto \exp(ita)$ connects $1$ to $u$ in $U_A$). For $W^*$-algebras such that $Gr(A)$ is the discrete union of $U_A$-orbits, it is then clear that the $G_A$-orbits and the $U_A$-orbits coincide. This is the case if $A$ is the algebra of bounded operators on a complex Hilbert space, as we have seen before.

The universal bundle $T_{S_0}(H) \to GrS_0(H)$ can be expressed as a vector bundle obtained from the so-called (principal) Stiefel bundle associated to $p_{S_0} \to S_0$, see [DC02]. A similar result holds, by replacing the Stiefel bundle with certain, suitable, of its sub-bundles. To see this, let us now introduce several mappings.

Put $p := p_{S_0}$. We consider $G \times G([p]) S_0$ and $U \times U([p]) S_0$ as in Remark 2.9. Note that $g_1 S_0 = g_2 S_0$ and $g_1(h_1) = g_2(h_2)$ if and only if $(g_1, h_1) \sim (g_2, h_2)$, or equivalently, if $w = g_1^{-1} g_2 \in G([p])$, in $G \times S_0$. Hence, the mapping $v_G : G \times S_0 \to T_{S_0}(H)$ defined by $v_G((g, h)) = (gS_0, g(h))$ for every $(g, h) \in G \times S_0$, induces the usual (canonical) quotient map $\tilde{v}_G : G \times G([p]) S_0 \to T_{S_0}(H)$. We denote by $v_U$ the restriction map of $v_G$ on $G \times S_0$. As above, the quotient mapping $\tilde{v}_U : U \times U([p]) S_0 \to T_{S_0}(H)$ is well defined.

Since $U([p]) = U \cap G([p])$, the inclusion mapping $j : U \times U([p]) S_0 \to G \times G([p]) S_0$ is well defined. Note that $j = (\tilde{v}_G)^{-1} \circ \tilde{v}_U$.

Finally, let $P_G : G \times G([p]) S_0 \to G/G([p])$ and $P_U : U \times U([p]) S_0 \to U/U([p])$ denote the vector bundles built in the standard way from the Stiefel sub-bundles $g \to gG([p]) \simeq gS_0$ and $g \to g/G([p]) \simeq GrS_0(H)$ and $u \mapsto u/U([p]) \simeq GrS_0(H)$ respectively.

**Proposition 2.15.** The following diagram is commutative in both sides, and the horizontal arrows are diffeomorphisms between the corresponding differentiable structures

$$
\begin{array}{ccc}
T_{S_0}(H) & \xrightarrow{(\tilde{v}_U)^{-1}} & U \times U([p]) S_0 \xrightarrow{j} G \times G([p]) S_0 \\
\downarrow \pi_U & & \downarrow \pi_U \\
GrS_0(H) & \xrightarrow{\sim} & U/U([p]) \xrightarrow{\sim} G/G([p])
\end{array}
$$

**Proof.** By construction, the mapping $\tilde{v}_U$ is clearly one-to-one. Now we show that it is onto. Let $(S, h) \in T_{S_0}(H)$. This means that $h \in S$ and that $S = uS_0$ for some $u \in U$. Then $f := u^{-1}(h) \in S_0$ and $h = u(f)$, whence $\tilde{v}_U([(u, f)]) = (S, h)$, where $[(u, f)]$ is the equivalence class of $(u, f)$ in $U \times U([p]) S_0$. Hence $\tilde{v}_U$ is a bijective map.

Analogously, we have that $\tilde{v}_G$ is bijective from $G \times G([p]) S_0$ onto $T_{S_0}(H)$ as well. As a consequence, $j = (\tilde{v}_G)^{-1} \circ \tilde{v}_U$ is also bijective. It is straightforward to check that all the maps involved in the diagram above are smooth.

**Example 2.16.** By Proposition 2.15, one can show that the universal, tautological bundle $\Pi_H : T_{S_0}(H) \to GrS_0(H)$ enters, as a canonical example, the framework outlined in Theorem 5.4 and Theorem 5.8 of [BR07].

To see this in terms of the bundle $\Pi_H$ itself, first note that the commutant algebra $\{p_{S_0}\}'$ of $p_{S_0}$ coincides with the Banach subalgebra $B$ of $A$ formed by the operators $T$ such that $T(S_0) \subset S_0$, $T(S_0^\perp) \subset S_0^\perp$. 


Remark 2.18. Assume again the situation where $\phi$ is a $C^*$-subalgebra of $A$, as it had to be.) Let $p = p_{B_0}$. From Lemma 2.13 $u \in U([p])$ if and only if $uB_0 = B$. Hence $u \in U(p) = U([p]) \cap U([1 - p])$ if and only if $uB_0 = B$ and $uB_0^+ = B_0^+$, that is, $U(p) = U_A \cap B = U_B$.

Similarly to what has been done in Remark 2.9, let $T \mapsto T(x_0)$, $B(H) \mapsto H$ and $T \mapsto T(x_0)$, $B \mapsto B_0$ are surjective, we obtain that $H_A = H$ and $H_B = B_0$ in the GNS construction associated with $A = B(H)$, $B$ and $\phi_0$. Moreover, in this case, $\pi_A$ coincides with the identity operator and the extension $P : H_A \mapsto H_B$ of $p$ is $P = p$. Denote by $p_1, p_2 : Gr(H) \times Gr(H) \mapsto Gr(H)$ the natural projections and define

$$Q_H : Gr(H) \times Gr(H) \mapsto \text{Hom}(p_2^*(\Pi_H), p_1^*(\Pi_H))$$

by

$$Q_H(S_1, S_2) = (p_{S_1})|_{S_2} : S_2 \mapsto S_1$$

whenever $S_1, S_2 \in \text{Gr}(H)$. This mapping $Q_H$ is called the universal reproducing kernel associated with the Hilbert space $H$. In fact, for $S_1, \ldots, S_n \in \text{Gr}(H)$ and $x_j \in S_j$ ($j = 1, \ldots, n$),

$$\sum_{j,l=1}^n (Q_H(S_1, S_j)x_j | x_l)_H = \sum_{j,l=1}^n (p_{S_j}x_j | x_l)_H = \sum_{j,l=1}^n (x_j | x_l)_H = \left(\sum_{j=1}^n x_j | \sum_{l=1}^n x_l\right)_H \geq 0,$$

so $Q_H$ is certainly a reproducing kernel in the sense of [BR07].

Using Example 2.16 we get the following special case of Theorem 5.8 in [BR07].

**Corollary 2.17.** For a complex Hilbert space $H$, the action of $\mathcal{U}$ on $\mathcal{H}$ can be realized as the natural action of $\mathcal{U}$ on a Hilbert space of holomorphic sections from $Gr_{S_0}(H)$ into $H$, such a realization being implemented by $\gamma(uh) = u \gamma(h)u^{-1}$, for every $h \in H$, $u \in U$.

**Proof.** If $S \in Gr_{S_0}(H)$, there exists $u \in U$ such that $uS_0 = S$ and then $p_S = up_{S_0}u^{-1}$. Thus for all $u_1, u_2 \in U$ and $x_1, x_2 \in S_0$ we have $Q_H(u_1S_0, u_2S_0)(u_1x_2) = p_{u_1S_0}(u_2x_2) = u_1p_{S_0}(u_1^{-1}u_2x_2)$. This formula shows that for every connected component $Gr_{S_0}(H)$ the restriction of $Q_H$ to $Gr_{S_0}(H) \times Gr_{S_0}(H)$ is indeed a special case of the reproducing kernels considered in Remark 2.9. For every $h \in H$, the mapping $\gamma_{p_S}(h) : Gr_{S_0}(H) \mapsto T_S(H)$ which corresponds to $Q_H$ can be identified to the holomorphic map $uS_0 \mapsto upu^{-1}h$, $Gr_{S_0}(H) \mapsto H$. Then the conclusion follows by using the diffeomorphism $\mathcal{U}/\mathcal{U}(p) \simeq G/\hat{G}(\{p\}) \simeq Gr_{S_0}(H)$ of Lemma 2.13 together with Proposition 2.15. \hfill \square

**Remark 2.18.** Assume again the situation where $A$ and $B$ are arbitrary $C^*$-algebras, $B$ is a $C^*$-subalgebra of $A$, with unit $1 \in B \subseteq A$. $A \mapsto B$ is a conditional expectation, and $\phi : A \mapsto C$ is a state such that $\phi \circ E = \phi$. With the same notations as in Remark 2.9 take $x_0 := 1 + N_B \in B/N_B \subseteq A/N_A$. It is well known that $x_0$ is a cyclic vector of $\pi_X$, for $X \in \{A; B\}$: let $h \in \mathcal{H}_X$ such that $0 = (\pi(x_0 | h))_{\mathcal{H}_X} = (e + N_X | h)_{\mathcal{H}_X}$ for all $e \in X$; since $X/N_X$ is dense in $\mathcal{H}_X$ we get $0 = (e | h)_{\mathcal{H}_X} = \|h\|^2$, that is, $h = 0$. Thus $\pi_X(x_0)$ is dense in $\mathcal{H}_X$.

Inspired by [AS94], we now consider the $C^*$-subalgebra $\mathfrak{A}$ of $B(\mathcal{H}_A)$ generated by $\pi_A(A)$ and $p$, where $p$ is the orthogonal projection from $\mathcal{H}_A$ onto $\mathcal{H}_B$. Set $\mathfrak{B} := \mathfrak{A} \cap \{p\}'$. Clearly, the GNS procedure is applicable to $\mathfrak{B} \subseteq \mathfrak{A} \subseteq B(\mathcal{H}_A)$, for the expectation $E_p : \mathfrak{A} \mapsto \mathfrak{B}$ and state $\phi_0$ defined by $x_0$, as we have done in Remark 2.9. Then $\pi_A(A)(x_0) \subseteq \mathfrak{A}(x_0) \subseteq \mathcal{H}_A$ and $\pi_A(B)(x_0) \subseteq \mathfrak{B}(x_0) \subseteq \mathcal{H}_B$, whence, by the choice of $x_0$, we obtain that $\mathfrak{A}(x_0) = \mathcal{H}_A$ and $\mathfrak{B}(x_0) = \mathcal{H}_B$. Thus we have that $\mathcal{H}_A = \mathcal{H}_A$ and $\mathcal{H}_B = \mathcal{H}_B$.

According to former discussions there are two (composed) commutative diagrams, namely

$$\begin{align*}
\begin{array}{cccccc}
G_A \times_{G_B} \mathcal{H}_B & \xrightarrow{\pi_A \times I} & G_{\mathfrak{A}} \times_{G_{\mathfrak{A}}(p)} \mathcal{H}_B & \xrightarrow{I \times \pi_{\mathfrak{A}}} & G_{\mathfrak{A}} \times_{G_{\mathfrak{A}}([p])} \mathcal{H}_B & \xrightarrow{\mathfrak{j} \times I} & \mathcal{G} \times_{\mathcal{G}([p])} \mathcal{H}_B \\
\downarrow \Pi_{G_{\mathfrak{A}}} & & \downarrow \Pi_{G_{\mathfrak{A}}} & & \downarrow \Pi_{G_{\mathfrak{A}}([p])} & & \downarrow \Pi_{\mathcal{G}([p])} \\
G_A/G_B & \xrightarrow{\pi_A} & G_{\mathfrak{A}}/G_{\mathfrak{A}}(p) & \xrightarrow{I} & G_{\mathfrak{A}}/G_{\mathfrak{A}}([p]) & \xrightarrow{\mathfrak{j}} & \mathcal{G}/\mathcal{G}([p])
\end{array}
\end{align*}$$

(2.4)
and

\[
\begin{array}{c}
U_A \times U_B \xrightarrow{\pi_A \times I} U_A \times U_B(p) \xrightarrow{\jmath \times I} \mathcal{U} \times \mathcal{U}(p) \xrightarrow{\sim} T_{\mathcal{H}_B}(\mathcal{H}_A) \\
U_A/U_B \xrightarrow{\hat{\pi}_A} U_A/U_B(p) \xrightarrow{j} \mathcal{U}/\mathcal{U}(p) \xrightarrow{\sim} \text{Gr}_{\mathcal{H}_B}(\mathcal{H}_A)
\end{array}
\]

(where the meaning of the arrows is clear). We suggest to call \( \Pi_G: G_A \times G_B \to G_A/G_B \) and \( \Pi_U: U_A \times U_B \to U_A/U_B \) the GNS vector bundle and the unitary GNS vector bundle, respectively, for data \( E: A \to B \) and \( \varphi: A \to \mathbb{C} \). Following the terminology used in \cite{AS94, ALRS97} for the maps \( G_A/G_B \to G_A/G_A(p) \), \( U_A/U_B \to U_A/U_A(p) \), we could refer to the left sub-diagrams of (2.4) and (2.5) as the basic vector bundle representations of \( \Pi_G \) and \( \Pi_U \), respectively. Since \( \mathcal{H}_A = \mathcal{H}_A \) and \( \mathcal{H}_B = \mathcal{H}_B \), the process to construct such “basic” objects, of Grassmannian type, is stationary. Also, since there is another way to associate Grassmannians to the GNS and unitary GNS bundles, which is that one of considering the tautological bundle of \( \mathcal{H}_A \) (see the right diagrams in (2.4), (2.5)), we might call \( G_A \times G_B \) \( \mathcal{H}_B \to G_A/G_B(p) \) the minimal Grassmannian vector bundle, and call \( T_{\mathcal{H}_B}(\mathcal{H}_A) \to \text{Gr}_{\mathcal{H}_B}(\mathcal{H}_A) \) the universal Grassmannian vector bundle, associated with data \( E: A \to B \) and \( \varphi: A \to \mathbb{C} \). In the unitary case, we should add the adjective “unitary” to both bundles.

Note that the vector bundles \( G \times G(p) \to G/G(p) \) and \( T_{\mathcal{H}_B}(\mathcal{H}_A) \to \text{Gr}_{\mathcal{H}_B}(\mathcal{H}_A) \) are isomorphic. In this sense, both diagrams (2.4) and (2.5) converge towards the tautological bundle for \( \mathcal{H}_A \).

Let us remark that (2.4) is holomorphic, and everything in (2.5) is holomorphic with the only possible exception of the bundle \( \Pi_U \). On the other hand, we have that \( G_A/G_A(p) \) and \( G/G(p) \) are complexifications of \( U_A/U_A(p) \) and \( \mathcal{U}/\mathcal{U}(p) \) respectively, on account of Remark 2.3 and Lemma 2.18. We shall see in Corollary 5.8 that \( G_A/G_B \) is also a complexification of \( U_A/U_B \) in general. Note in passing that the fact that \( G_A/G_B \) is such a complexification implies interesting properties of metric nature in the differential geometry of \( U_A/U_B \), see \cite{ALRS97}.

The above considerations strongly suggest to investigate the relationships between (2.4) and (2.5) in terms of holomorphy and geometric realizations. In this respect, note that the commutativity of (2.5) corresponds, at the level of reproducing kernels, with the equality

\[
(\pi_A \times I) \circ (u_1 U_B, u_2 U_B) = Q_{\mathcal{H}_B}(\pi_A(u_1) \mathcal{U}(p), \pi_A(u_2) \mathcal{U}(p)) \circ (\pi_A \times I)
\]

for all \( u_1, u_2 \in U_A \) (where the holomorphy supplied by \( Q_{\mathcal{H}_B} \) appears explicitly). From this, a first candidate to reproducing kernel on \( G_A/G_B \), in order to obtain a geometric realization of \( \pi_A \) on \( G_A \), would be defined by

\[
K(g_1 G_B, g_2 G_B)((g_2, f)) := [(g_1, p(\pi_A(g_1^{-1}) \pi_A(g_2)) f)]
\]

for every \( g_1, g_2 \in G_A \) and \( f \in \mathcal{H}_B \). Nevertheless, since the elements \( g_1, g_2 \) are not necessarily unitary, it is readily seen that the kernel \( K \) so defined need not be definite-positive in general. There is also the problem of the existence of a suitable structure of Hermitian type in \( \Pi_G \).

In the present paper, we propose a theory on bundles \( G_A \times G_B \to G_A/G_B \) and kernels \( K \text{ ad hoc} \), based on the existence of suitable involutive diffeomorphisms in \( G_A/G_B \), which allows us to incorporate those bundles to a framework that contains as a special case the one established in \cite{BR07}.

\[ \Box \]

3. LIKE-HERMITIAN STRUCTURES

We are going to introduce a variation of the notion of Hermitian vector bundle, which will turn out to provide the appropriate setting for the geometric representation theory of involutive Banach-Lie groups as developed in Section 5.

**Definition 3.1.** Assume that \( Z \) is a real Banach manifold equipped with a diffeomorphism \( z \mapsto z^{-*} \), \( Z \to Z \), which is involutive in the sense that \( (z^{-*})^{-*} = z \) for all \( z \in Z \). Denote by \( p_1, p_2: Z \times Z \to Z \) the natural projection maps. Let \( \Pi: D \to Z \) be a smooth vector bundle whose fibers are complex Banach spaces (see for instance \cite{AMR88} or \cite{Lu01} for details on infinite-dimensional vector bundles).

We define a like-Hermitian structure on the bundle \( \Pi \) (with typical fiber the Banach space \( \mathcal{E} \)) as a family \( \{ \cdot | . \} : Z \to Z \) with the following properties:

1. **Positivity:** \( \langle z, z^{-*} \rangle > 0 \) for all \( z \neq 0 \) in \( Z \).
2. **Symmetry:** \( \langle z, z^{-*} \rangle = \langle z^{-*}, z \rangle \) for all \( z \in Z \).
3. **Linearity:** For all \( z, w \in Z \) and scalars \( a, b \), \( \langle az + bw, z^{-*} \rangle = a \langle z, z^{-*} \rangle + b \langle w, z^{-*} \rangle \).
4. **Admissibility:** For all \( z \in Z \), \( \langle z, z^{-*} \rangle = 0 \) if and only if \( z = 0 \).

**Proposition 3.2.** Let \( \Pi: D \to Z \) be a like-Hermitian structure as above. Then for any \( z \in Z \), \( \langle z, z^{-*} \rangle \) is a positive definite inner product on \( D_z \) with respect to the norm \( \| \cdot \| := \langle \cdot, \cdot \rangle^{1/2} \).

**Proof.** (Sketch) The proof follows from the properties of like-Hermitian structures and the fact that \( \Pi \) is a smooth vector bundle. Details are provided in Section 5.

\[ \Box \]
(a) For every \( z \in Z \), \((\cdot | \cdot)_{z,z} : D_z \times D_{z^{-}} \to \mathbb{C} \) is a sesquilinear strong duality pairing.

(b) For all \( z \in Z, \xi \in D_z, \) and \( \eta \in D_{z^{-}} \), we have \((\xi | \eta)_{z,z} = (\eta | \xi)_{z^{-},z} \).

(c) If \( V \) is an arbitrary open subset of \( Z \), and \( \Psi_0 : V \times \mathcal{E} \to \Pi^{-1}(V) \) and \( \Psi_1 : V^{-*} \times \mathcal{E} \to \Pi^{-1}(V^{-*}) \) are trivializations of the vector bundle \( \Pi \) over \( V \) and \( V^{-*} := \{ z^{-*} | z \in V \} \), respectively, then the function \((z, x, y) \mapsto \langle \Psi_0(z, x) | \Psi_1(z^{-*}, y) \rangle_{z,z}, V \times \mathcal{E} \times \mathcal{E} \to \mathbb{C} \) is smooth.

We call like-Hermitian vector bundle any vector bundle equipped with a like-Hermitian structure. \( \square \)

**Remark 3.2.** Here we explain the meaning of condition (a) in Definition 3.1. To this end let \( \mathcal{X} \) and \( \mathcal{Y} \) be two complex Banach spaces. A functional \((\cdot | \cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C} \) is said to be a sesquilinear strong duality pairing if it is continuous, is linear in the first variable and antilinear in the second variable, and both:

\[
(\forall h \in \mathcal{H}, y \in \mathcal{Y}) \quad (Th | y) = (h | Sy)_{\mathcal{H}}.
\]

Conversely, for every bounded linear operator \( S : \mathcal{Y} \to \mathcal{H} \) there exists a unique bounded linear operator \( T : \mathcal{H} \to \mathcal{X} \) satisfying (3.1), and we denote \( S^{-*} := T \) and \( T^{-*} := S \). \( \square \)

**Remark 3.3.** For later use we now record the following fact: Assume that \( \mathcal{X} \) and \( \mathcal{Y} \) are two Banach spaces over \( \mathbb{C} \), and let \((\cdot | \cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C} \) be a sesquilinear strong duality pairing. Now let \( \mathcal{H} \) be a Hilbert space over \( \mathbb{C} \) and let \( T : \mathcal{H} \to \mathcal{X} \) be a continuous linear operator. Then there exists a unique linear operator \( S : \mathcal{Y} \to \mathcal{H} \) such that:

\[
(3.1) \quad (Th | y) = (h | Sy)_{\mathcal{H}}.
\]

**Example 3.5.** Let \( \Pi : D \to Z \) be a smooth vector bundle whose fibers are complex Banach spaces. Assume that there exist a complex Hilbert space \( \mathcal{H} \) and a smooth map \( \Theta : D \to \mathcal{H} \) with the property that \( \Theta|_{D_z} : D_z \to \mathcal{H} \) is a bounded linear operator for all \( z \in Z \). Then \( \Theta \) determines a family of continuous sesquilinear functionals:

\[
(\cdot | \cdot)_{z,z} : D_z \times D_{z^{-}} \to \mathbb{C}, \quad (\eta_1 | \eta_2)_{z,z} = (\Theta(\eta_1) | \Theta(\eta_2))_{\mathcal{H}}.
\]

If in addition \( \Theta|_{D_z} : D_z \to \mathcal{H} \) is injective and has closed range, and the scalar product of \( \mathcal{H} \) determines a sesquilinear strong duality pairing between the subspaces \( \Theta(D_z) \) and \( \Theta(D_{z^{-}}) \) whenever \( z \in Z \), then it is easy to see that we get a like-Hermitian structure on the vector bundle \( \Pi \). \( \square \)

**Definition 3.6.** An involutive Banach-Lie group is a (real or complex) Banach-Lie group \( G \) equipped with a diffeomorphism \( u \mapsto u^* \) satisfying \((uv)^* = v^*u^* \) and \((u^*)^* = u \) for all \( u, v \in G \). In this case we denote:

\[
(\forall u \in G) \quad u^{-*} := (u^{-1})^*
\]

and

\[
G^+ := \{ u^* u | u \in G \}
\]

and the elements of \( G^+ \) are called the positive elements of \( G \).

If in addition \( H \) is a Banach-Lie-subgroup of \( G \), then we say that \( H \) is an involutive Banach-Lie subgroup if \( u^* \in H \) whenever \( u \in H \). \( \square \)

**Remark 3.7.** If \( G \) is an involutive Banach-Lie group then for every \( u \in G \) we have \((u^{-1})^* = (u^*)^{-1} \) and moreover \( 1^* = 1 \). To see this, just note that the mapping \( u \mapsto (u^*)^{-1} \) is an automorphism of \( G \), hence it commutes with the inversion mapping and leaves \( 1 \) fixed. \( \square \)
Example 3.8. Every Banach-Lie group G has a trivial structure of involutive Banach-Lie group defined by \( u^* := u^{-1} \) for all \( u \in G \). In this case the set of positive elements is \( G^+ = \{1\} \). □

Example 3.9. Let \( A \) be a unital \( C^* \)-algebra with the group of invertible elements denoted by \( G_A \). Then \( G_A \) has a natural structure of involutive complex Banach-Lie group defined by the involution of \( A \). If \( B \) is any \( C^* \)-subalgebra of \( A \) such that there exists a conditional expectation \( E : A \to B \), then \( G_B \) is an involutive complex Banach-Lie subgroup of \( G_A \). □

Definition 3.10. Assume that we have the following data:

- \( G_A \) is an involutive real (respectively, complex) Banach-Lie group and \( G_B \) is an involutive real (respectively, complex) Banach-Lie subgroup of \( G_A \).
- For \( X = A \) or \( X = B \), assume \( \mathcal{H}_X \) is a complex Hilbert space with \( \mathcal{H}_B \) closed subspace in \( \mathcal{H}_A \), and \( \pi_X : G_X \to B(\mathcal{H}_X) \) is a uniformly continuous (respectively, holomorphic) \(*\)-representation such that \( \pi_B(u) = \pi_A(u)|_{\mathcal{H}_B} \) for all \( u \in G_B \). By \(*\)-representation we mean that \( \pi_A(u^*) = \pi_A(u)^* \)
  for all \( u \in G_A \).
- We denote by \( P : \mathcal{H}_A \to \mathcal{H}_B \) the orthogonal projection.

We define an equivalence relation on \( G_A \times \mathcal{H}_B \) by

\[
(u, f) \sim (u', f') \quad \text{whenever there exists } w \in G_B \quad \text{such that } u' = uw \quad \text{and } f' = \pi_B(w^{-1})f.
\]

For every pair \((u, f) \in G_A \times \mathcal{H}_B\) we define its equivalence class by \([u, f]\) and let \( D = G_A \times_{G_B} \mathcal{H}_B \) denote the corresponding set of equivalence classes. Then there exists a natural onto map

\[
\Pi : [(u, f)] \mapsto s := u G_B, \quad D \to G_A / G_B.
\]

For \( s \in G_A / G_B \), let \( D_s := \Pi^{-1}(s) \) denote the fiber on \( s \). Note that \((u, f) \sim (u', f')\) implies that \( \pi_A(u)f = \pi_A(u')f' \) so that the correspondence \([u, f] \mapsto \pi_A(u)f, \ D_s \to \pi_A(u)\mathcal{H}_B, \) gives rise to a complex linear structure on \( D_s \). Moreover,

\[
\|[u, f]\|_{D_s} := \|\pi_A(u)f\|_{\mathcal{H}_A}
\]

where \([u, f] \in D_s\), defines on \( D_s \) a Hilbertian norm.

Clearly, this structure does not depend on the choice of \( u \). Nevertheless, note that the natural bijection from \( \mathcal{H}_B \) onto the fiber \( \Pi^{-1}(s) \) defined by

\[
\Theta_u : f \mapsto [(u, f)], \quad \mathcal{H}_B \to \Pi^{-1}(s),
\]

is a topological isomorphism but it need not be an isometry. In other words, the fiberwise maps

\[
\Theta_u \Theta_u^{-1} : [(u, f)] \mapsto f \mapsto [(v, f)], \quad D_s \to \mathcal{H}_B \to D_t,
\]

where \( s = u G_B, \ t = v G_B \), are topological isomorphisms but they are not unitary transformations in general. As a complex Hilbert space, \( D_s \) has so many realizations of the topological dual or predual. We next consider the following ones. For \( \xi = [(u, f)], \ \eta = [(v, g)] \) in \( D \), and \( s = u G_B, \ t = v G_B \), we set as in Example 3.6

\[
(\xi \mid \eta)_D \equiv (\xi \mid \eta)_{s,t} := (\pi_A(u)f \mid \pi_A(v)g)_{\mathcal{H}_A}.
\]

where \((\cdot \mid \cdot)_{\mathcal{H}_A}\) is the inner product which defines the complex Hilbert structure on \( \mathcal{H}_A \) and, by restriction, on \( \mathcal{H}_B \). This is a well-defined, non-negative sesquilinear form on \( D \). In particular \((\cdot \mid \cdot)_{s,t} = (\cdot \mid \cdot)_{t,s} \). We are mainly interested in forms \((\cdot \mid \cdot)_{s,t}\) with \( t = s^{-1} \in G_A / G_B \). In this case

\[
[(u, f)] \mid [(u^{-1}, g)]_{s,s^{-1}} = (\pi_A(u)f \mid \pi_A(u^{-1})g)_{\mathcal{H}_A} = (\pi_A(u^{-1})\pi_A(u)f \mid g)_{\mathcal{H}_A} = (f \mid g)_{\mathcal{H}_B},
\]

whenever \([u, f] \in D_s\) and \([u^{-1}, g] \in D_{s^{-1}}\). Thus Example 3.3 shows that the basic mapping

\[
\Theta : [(u, f)] \mapsto \pi_A(u)f, \quad D \to \mathcal{H}_A,
\]

gives rise to a like-Hermitian structure on the vector bundle \( \Pi \).

We shall say that \( \Pi : D \to G_A / G_B \) is the (holomorphic) homogeneous like-Hermitian vector bundle associated with the data \((\pi_A, \pi_B, P)\). □
Remark 3.11. Let us see that Definition 3.10 is correct, that is, condition (a) of Definition 3.1 is satisfied. In fact, let \( u \in G_A \) arbitrary, denote \( z = uG_B \in G_A / G_B \), and let \( \varphi \) be any bounded linear functional on \( \overline{D}_z \). Then the mapping \( \tilde{\varphi}: g \mapsto [(u^{-*}, g)] \mapsto \varphi([(u^{-*}, g)]) \), \( H_B \to \overline{D}_z \to \mathbb{C} \), is antilinear and bounded. By the Riesz’ theorem there exists \( f \in H_B \) such that
\[
\varphi([(u^{-*}, g)]) = \tilde{\varphi}(g) = (f | g)_{H_B} \equiv \left( [(u^{-*}, g)] \right)_{z, z^{-*}}
\]
and so \( \langle \cdot, \cdot \rangle_{z, z^{-*}} \) is a sesquilinear strong duality pairing between \( D_z \) and \( D_{z^{-*}} \). \( \square \)

In order to get a better understanding of the structures introduced in Definition 3.10, we shall need the following notion.

Definition 3.12. Assume that we have the following objects: a complex involutive Banach-Lie group \( G \), a complex Banach manifold \( Z \) equipped with an involutive diffeomorphism \( z \mapsto z^{-*} \), and a holomorphic like-Hermitian vector bundle \( \Pi: D \to Z \), such that \( \Pi \circ \alpha = \beta \circ (\text{id}_G \times \Pi) \), where \( \alpha \) and \( \beta \) are holomorphic actions of \( G \) on \( D \) and \( Z \) and for all \( u \in G \) and \( z \in Z \) the mapping \( \alpha(u, \cdot) |_{D_z}: D_z \to D_{\beta(u, z)} \) is a bounded linear operator. In addition we assume that \( \beta(u^{-*}, z^{-*}) = \beta(u, z)^{-*} \) whenever \( u \in G \) and \( z \in Z \) and we let \( \pi: G \to B(\mathcal{H}) \) be a holomorphic \(*\)-representation.

We say that a holomorphic mapping \( \Theta: D \to \mathcal{H} \) relates \( \Pi \) to \( \pi \) if it has the following properties:

(i) for each \( z \in Z \) the mapping \( \Theta_z := \Theta |_{D_z}: D_z \to \mathcal{H} \) is an injective bounded linear operator and in addition we have \( (\Theta(\xi) | \Theta(\eta))_{\mathcal{H}} = \theta(\xi, \eta) \) whenever \( \xi, \eta \in D_z \);
(ii) for every \( u \in G \) and \( z \in Z \) we have \( \Theta_{\beta(u, z)} \circ \alpha(u, \cdot) |_{D_z} = \pi(u) \circ \Theta_z: D_z \to \mathcal{H} \).
\( \square \)

Now we turn to a result (Theorem 3.13) which points out that the basic mapping \( \Theta \) introduced in Definition 3.10 indeed plays a central role in the whole picture. In this statement we denote by \( \iota_0(\cdot) \) the pull-back of a bundle by the mapping \( \iota_0 \); see for instance [Ln01] for some details.

Theorem 3.13. In the setting of Definition 3.12 let \( z_0 \in Z \) such that \( z_0^{-*} = z_0 \), and assume that the isotropy group \( G_0 := \{ u \in G | \beta(u, z_0) = z_0 \} \) is a Banach-Lie subgroup of \( G \). In addition assume that the orbit of \( z_0 \), that is, \( \mathcal{O}_{z_0} = \{ \beta(u, z_0) | u \in G \} \), is a submanifold of \( Z \), and denote by \( \iota_0: \mathcal{O}_{z_0} \to Z \) the corresponding embedding map. Then there exists a closed subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \) such that the following assertions hold:

(i) For every \( u \in G_0 \) we have \( \pi(u) |_{\mathcal{H}_0} \leq \mathcal{H}_0 \).
(ii) Denote by \( \pi_0: u \mapsto \pi(u) |_{\mathcal{H}_0}, G_0 \to B(\mathcal{H}_0) \), the corresponding representation of \( G_0 \) on \( \mathcal{H}_0 \), by \( \Pi_0: D_0 \to G/G_0 \) the like-Hermitian vector bundle associated with the data \( (\pi, \pi_0, P_{\mathcal{H}_0}) \), and by \( \Theta_0: D_0 \to \mathcal{H} \) the basic mapping associated with the data \( (\pi, \pi_0, P_{\mathcal{H}_0}) \). Then there exists a biholomorphic bijective \( G \)-equivariant map \( \theta: D_0 \to \iota_0^*(D) \) such that \( \theta \) sets up an isometric isomorphism of like-Hermitian vector bundles over \( G/G_0 \simeq \mathcal{O}_{z_0} \) and the diagram

\[
\begin{array}{c}
D_0 \xrightarrow{\theta} \iota_0^*(D) \\
\downarrow \Theta_0 \downarrow \Theta_0 |_{\iota_0^*(D)} \\
\mathcal{H} \xrightarrow{\text{id}_{\iota_0^*(D)}} \mathcal{H}
\end{array}
\]

is commutative.

Proof. For \( \eta, \xi \in D_{z_0} \) we have
\[
(\Theta_{z_0}^{−1}(\Theta_{z_0}^\dagger \xi) | \eta)_{z_0, z_0} = (\Theta_{z_0}^\dagger \xi | \Theta_{z_0} \eta)_{\mathcal{H}} = (\xi | \eta)_{z_0, z_0},
\]
where the first equality is derived from Remark 3.11 and the second one is the hypothesis of Definition 3.12 (i), since \( z_0 = z_0^{-*} \). As \( (\cdot | \cdot)_{z_0, z_0} \) is a strong duality pairing we have \( \Theta_{z_0}^{−1}(\Theta_{z_0}^\dagger \xi) = \xi \). This implies that \( \text{Ran} (\Theta_{z_0}) \) is closed in \( \mathcal{H} \) since \( \Theta_{z_0}^{−1} \) is bounded. Put \( \mathcal{H}_0 := \text{Ran} (\Theta_{z_0}) \).
For arbitrary \( u \in G_0 \) we have \( \beta(u, z_0) = z_0 \). Then property (ii) in Definition 3.12 shows that we have a commutative diagram

\[
\begin{array}{ccc}
D_{z_0} & \xrightarrow{\alpha(u, \cdot)D_{z_0}} & D_{z_0} \\
\downarrow \Theta_{z_0} & & \downarrow \Theta_{z_0} \\
\mathcal{H} & \xrightarrow{\pi(u)} & \mathcal{H}
\end{array}
\]

whence \( \pi(u)(\Theta_{z_0}(D_{z_0})) \subseteq \Theta_{z_0}(D_{z_0}) \), that is, \( \pi(u)H_0 \subseteq H_0 \). Thus \( H_0 \) has the desired property (i).

To prove (ii) we first note that, since \( G_0 \) is a Banach-Lie subgroup of \( Z \), it follows that the \( G \)-orbit \( \mathcal{O}_{z_0} \simeq G/G_0 \) has a natural structure of Banach homogeneous space of \( G \) (in the sense of \([Ra77]\)) such that the inclusion map \( i_0 : \mathcal{O}_{z_0} \hookrightarrow Z \) is an embedding.

Next define

\[
(3.3) \quad \tilde{\theta} : G \times H_0 \rightarrow D, \quad \tilde{\theta}(u, f) := \alpha(u, \Theta_{z_0}^{-1}(f)) = \Theta_{\beta(u, z_0)}^{-1}(\pi(u)f) \in D_{\beta(u, z_0)} \subseteq D,
\]

where the equality follows by property (ii) in Definition 3.12. Then for all \( u \in G \), \( u_0 \in G_0 \), and \( f \in H_0 \) we have \( \beta(uu_0^{-1}, z_0) = \beta(u, z_0) \) and

\[
\tilde{\theta}(uu_0^{-1}, \pi(u_0)f) = \Theta_{\beta(uu_0^{-1}, z_0)}^{-1}(\pi(uu_0^{-1})\pi(u_0)f) = \Theta_{\beta(u, z_0)}^{-1}(\pi(u)f) = \tilde{\theta}(u, f).
\]

In particular there exists a well defined map

\[
\theta : [(u, f)] \mapsto \Theta_{\beta(u, z_0)}^{-1}(\pi(u)f), \quad G \times_{G_0} H_0 \rightarrow D.
\]

This mapping is \( G \)-equivariant with respect to the actions of \( G \) on \( G \times_{G_0} H \) and on \( D \) since \( \tilde{\theta} \) is \( G \)-equivariant: for all \( u, v \in G \) and \( f \in H_0 \) we have

\[
\tilde{\theta}(uv, f) = \Theta_{\beta(uv, z_0)}^{-1}(\pi(uv)f) = \Theta_{\beta(u, \beta(v, z_0))}^{-1}(\pi(u)v)f) = \alpha(u, \Theta_{\beta(v, z_0)}^{-1}(\pi(v)f)) = \alpha(u, \tilde{\theta}(v, f)),
\]

where the second equality follows since \( \beta : G \times Z \rightarrow Z \) is a group action, while the third equality is a consequence of property (ii) in Definition 3.12. Besides, it is clear that \( \theta \) is a bijection onto \( i_0^*(D) \) and a fiberwise isomorphism. Also it is clear from the above construction of \( \theta \) and from the definition of the basic mapping \( \Theta_0 : D_0 \rightarrow \mathcal{H} \) associated with the data \((\pi, \pi_0, P_{H_0})\) (see Definition 3.10) that \( \Theta \circ \theta = \Theta_0 \), that is, the diagram in the statement is indeed commutative. In addition, since both mappings \( \Theta \) and \( \Theta_0 \) are fiberwise “isometric” (see property (i) in Definition 3.12 above and Definition 3.10), it follows by \( \Theta \circ \theta = \Theta_0 \) that \( \theta \) gives us an isometric morphism of like-Hermitian bundles over \( G/G_0 \simeq \mathcal{O}_{z_0} \).

Now we still have to prove that the map \( \theta : D_0 = G \times_{G_0} H_0 \rightarrow i_0^*(D) \subseteq D \) is biholomorphic. We first show that it is holomorphic. Since \( \mathcal{O}_{z_0} \) is a submanifold of \( Z \), it follows that \( i_0^*(D) \) is a submanifold of \( D \) (see for instance the comments after Proposition 1.4 in Chapter III of \([Lin01]\)).

Thus it will be enough to show that \( \theta : G \times_{G_0} H_0 \rightarrow D \) is holomorphic. And this property is equivalent (by Corollary 8.3(ii) in \([Up85]\)) to the fact that the mapping \( \bar{\theta} : G \times H_0 \rightarrow D \) is holomorphic, since the natural projection \( G \times H_0 \rightarrow G \times_{G_0} H_0 \) is a holomorphic submersion. Now the fact that \( \bar{\theta} : G \times H_0 \rightarrow D \) is a holomorphic map follows by the first formula in its definition \((3.3)\), since the group action \( \alpha : G \times D \rightarrow D \) is holomorphic.

Consequently the mapping \( \theta : G \times_{G_0} H_0 \rightarrow i_0^*(D) \) is holomorphic. Then the fact that the inverse \( \theta^{-1} : i_0^*(D) \rightarrow G \times_{G_0} H_0 \) is also holomorphic follows by general arguments in view of the following facts (the first one and the second of them have been already established, and the third one is well-known): Both \( G \times_{G_0} H_0 \) and \( i_0^*(D) \) are locally trivial holomorphic vector bundles; we have a commutative diagram

\[
\begin{array}{ccc}
G \times_{G_0} H_0 & \xrightarrow{\theta} & i_0^*(D) \\
\downarrow & & \downarrow \\
G/G_0 & \xrightarrow{} & \mathcal{O}_{z_0}
\end{array}
\]

where the bottom arrow is the biholomorphic map \( G/G_0 \simeq \mathcal{O}_{z_0} \) induced by the action \( \beta : G \times Z \rightarrow Z \), and the vertical arrows are the projections of the corresponding holomorphic like-Hermitian vector bundles; the inversion mapping is holomorphic on the open set of invertible operators on a complex Hilbert space.
The proof is completed. \qed

**Remark 3.14.** The significance of Theorem 3.13 is the following one: In the setting of Definition 3.12 the special situation of Definition 3.10 is met precisely when the action $\beta: G \times Z \to Z$ is transitive, and in this case the basic mapping is essentially the unique mapping that relates the bundle $\Pi$ to the representation of the bigger group $G$.

On the other hand, by considering direct products of homogeneous Hermitian vector bundles, we can construct obvious examples of other maps relating bundles to representations as in Definition 3.12 \qed

4. Reproducing ($-\ast$)-kernels

**Definition 4.1.** Let $\Pi: D \to Z$ be a like-Hermitian bundle, with involution $-\ast$ in $Z$. A reproducing ($-\ast$)-kernel on $\Pi$ is a section $K \in \Gamma(Z \times Z, \operatorname{Hom}(p_s^1 \Pi, p_t^1 \Pi))$ (whence $K(s, t): D_t \to D_s$ for all $s, t \in Z$) which is ($-\ast$)-positive definite in the following sense: For every $n \geq 1$ and $t_j \in Z$, $\eta_j^{-\ast} \in D_{t_j^{-\ast}}$ $(j = 1, \ldots, n)$,

$$\sum_{j=1}^{n} (\eta_j^{-\ast} | K(t_j, t_j^{-\ast}) \eta_j^{-\ast})_{t_j^{-\ast}, t_j} = \sum_{j=1}^{n} (K(t_j, t_j^{-\ast}) \eta_j^{-\ast} | \eta_j^{-\ast})_{t_j^{-\ast}, t_j} \geq 0.$$ 

Here $p_1, p_2: Z \times Z \to Z$ are the natural projection mappings. If in addition $\Pi: D \to Z$ is a holomorphic like-Hermitian vector bundle and $K(\cdot, t)\eta \in \mathcal{O}(Z, D)$ for all $\eta \in D_t$ and $t \in Z$, then we say that $K$ is a holomorphic reproducing ($-\ast$)-kernel. \qed

**Remark 4.2.** In Definition 4.1 the symbol $\eta_j^{-\ast}$ is just a way to refer to elements of $D_{t_j^{-\ast}}$, that is, $\eta_j^{-\ast}$ is not associated to any element $\eta_j$ of $D_t$, necessarily. From the definition we have that $K(s, s^{-\ast}) \geq 0$ in the sense that $(K(s, s^{-\ast})\xi^{-\ast} | \xi^{-\ast})_{s, s^{-\ast}} \geq 0$ for all $\xi^{-\ast} \in D_{s^{-\ast}}$. \qed

The following results are related to the extension of Theorem 4.2 in [BR07] to reproducing kernels on like-Hermitian vector bundles.

**Proposition 4.3.** Let $\Pi: D \to Z$ be a smooth like-Hermitian vector bundle and, as usually, denote by $p_1, p_2: Z \times Z \to Z$ the projections. Next consider a section $K \in \Gamma(Z \times Z, \operatorname{Hom}(p_s^1 \Pi, p_t^1 \Pi))$ and for all $s \in Z$ and $\xi \in D_s$ denote $K_\xi = K(\cdot, s)\xi \in \Gamma(Z, D)$. Also denote

$$\mathcal{H}_0^K := \text{span}_\mathbb{C}\{K_\xi | \xi \in D\} \subseteq \Gamma(Z, D).$$

Then $K$ is a reproducing ($-\ast$)-kernel on $\Pi$ if and only if there exists a complex Hilbert space $\mathcal{H}$ such that $\mathcal{H}_0^K$ is a dense linear subspace of $\mathcal{H}$ and

$$K_\eta | K_\xi)_\mathcal{H} = (K(s^{-\ast}, t)\eta | \xi)_{s^{-\ast}, s} \tag{4.1}$$

whenever $s, t \in Z$, $\xi \in D_s$, and $\eta \in D_t$.

**Proof.** First assume that there exists a Hilbert space as in the statement. Then for all $n \geq 1$ and $t_j \in Z$, $\eta_j \in D_{t_j^{-\ast}}$ $(j = 1, \ldots, n)$, it follows by (4.1) that

$$\sum_{j=1}^{n} (K(t_j, t_j^{-\ast}) \eta_j | \eta_j)_{t_j^{-\ast}, t_j} = \sum_{j=1}^{n} (K_\eta | K_\eta)_\mathcal{H} = \left(\sum_{j=1}^{n} K_{\eta_j} | \sum_{j=1}^{n} K_{\eta_j}\right)_\mathcal{H} \geq 0.$$ 

In addition, for arbitrary $s, t \in Z$, $\xi \in D_s$ and $\eta \in D_t$, we have

$$(\eta | K(t^{-\ast}, s)\xi)_{t^{-\ast}, s} = (K(t^{-\ast}, s)\xi | \eta)_{t^{-\ast}, s} = (K_\xi | K_\eta)_\mathcal{H} = (K_\eta | K_\xi)_\mathcal{H} = (K(s^{-\ast}, t)\eta | \xi)_{s^{-\ast}, s},$$

where the second equality and the fourth one follow by (4.1).

Conversely, let us assume that $K$ is a reproducing ($-\ast$)-kernel. We are going to define a positive Hermitian sesquilinear form on $\mathcal{H}_0^K$ by

$$\theta(\Theta | \Delta)_\mathcal{H} = \sum_{j=1}^{n} (K(s_j^{-\ast}, t_j)\eta_j | \xi)_{s_j^{-\ast}, s_j} \tag{4.2}$$
for \( \Theta, \Delta \in \mathcal{H}_0^K \) of the form \( \Theta = \sum_{j=1}^{n} K_{\eta_j} \) and \( \Delta = \sum_{l=1}^{n} K_{\xi_l} \), where \( \eta_j \in D_{t_j}, \xi_l \in D_{s_l}, \) and \( t_j, s_l \in Z \) for \( j, l = 1, \ldots, n \). The assumption that \( K \) is a reproducing \((-*)\)-kernel implies at once that for all \( \Theta, \Delta \in \mathcal{H}_0^K \) we have \( (\Theta | \Theta)_{\mathcal{H}} \geq 0 \) and \( (\Theta | \Delta)_{\mathcal{H}} = (\Delta | \Theta)_{\mathcal{H}} = 0 \). To see that \((\cdot | \cdot)_{\mathcal{H}}\) is well defined, note that it is clearly sesquilinear and \((4.2)\) implies

\[
(\Delta | \Theta)_{\mathcal{H}} = \Theta(j,l)_{\mathcal{H}} \leq \sum_{l=1}^{n} (\Theta(s_l^{-*}) | \xi_l)_{s_l^{-*},s_l}.\]

hence \((\Theta | \Theta)_{\mathcal{H}} = (\Delta | \Delta)_{\mathcal{H}} = 0 \) if it happens that \( \Theta = 0 \). This implies that \((\cdot | \cdot)_{\mathcal{H}}\) is well defined, and the above remarks show that this is a nonnegative Hermitian sesquilinear form on \( \mathcal{H}_0^K \).

To check that \((\cdot | \cdot)_{\mathcal{H}}\) is also non-degenerate, let \( \Theta \in \mathcal{H}_0^K \) such that \((\Theta | \Theta)_{\mathcal{H}} = 0 \). Since \((\cdot | \cdot)_{\mathcal{H}}\) is a nonnegative Hermitian sesquilinear form, it follows that it satisfies the Cauchy-Schwarz inequality, hence for all \( \Delta \in \mathcal{H}_0^K \) we have \( |(\Theta | \Delta)_{\mathcal{H}}|^2 \leq (\Theta | \Theta)_{\mathcal{H}}(\Delta | \Delta)_{\mathcal{H}} = 0 \), whence \((\Theta | \Delta)_{\mathcal{H}} = 0 \). It follows by this property along with the formula \((4.3)\) that for arbitrary \( s \in Z \) and \( \xi \in D_{s^{-*}} \) we have

\[
(\Theta(s) | \xi_{s,s^{-*}}) = (\Theta | K\xi)_{\mathcal{H}} = 0.
\]

Since \( \{(\cdot | \cdot)_{z,z^{-*}}\}_{z \in Z} \) is a like-Hermitian structure, it then follows that \( \Theta(s) = 0 \) for all \( s \in Z \), hence \( \Theta = 0 \). Consequently \((\cdot | \cdot)_{\mathcal{H}}\) is a scalar product on \( \mathcal{H}_0^K \), and then the completion of \( \mathcal{H}_0^K \) with respect to this scalar product is a complex Hilbert space with the asserted properties.

**Definition 4.4.** Let \( \Pi: D \to Z \) be a smooth like-Hermitian vector bundle, \( p_1, p_2: Z \times Z \to Z \) the projections, and let \( K \in \Gamma(Z \times Z, \text{Hom}(p_2^*\Pi, p_1^*\Pi)) \) be a reproducing \((-*)\)-kernel. As above, for all \( s \in Z \) and \( \xi \in D_{s} \), put \( K_\xi = K(\cdot, s) \xi \in \Gamma(Z, D) \). It is clear that the Hilbert space \( \mathcal{H} \) given by Proposition 4.3 is uniquely determined. We shall denote it by \( \mathcal{H}^K \) and we shall call it the reproducing \((-*)\)-kernel Hilbert space associated with \( K \).

In the same framework we also define the mapping

\[
K: D \to \mathcal{H}^K, \quad K(\xi) = K_\xi.
\]

It follows by Lemma 4.3 below that for every \( s \in Z \) there exists a bounded linear operator \( \theta_s: \mathcal{H}^K \to D_{s^{-*}} \), such that

\[
(\forall \xi \in D_s, h \in \mathcal{H}^K) \quad (K(\xi) | h)_{\mathcal{H}^K} = (\xi | \theta_s h)_{s,s^{-*}}.
\]

Note that the operator \( \theta_s \) is uniquely determined since \( \{(\cdot | \cdot)_{z,z^{-*}}\}_{z \in Z} \) is a like-Hermitian structure, and in the notation of Remark 3.3 we have

\[
(\theta_s)^{-*} = K|_{D_{s^{-*}}}.
\]

**Lemma 4.5.** Assume the setting of Definition 4.4. Then for every \( s \in Z \) the operator \( K|_{D_s}: D_s \to \mathcal{H}^K \) is bounded, linear and adjointable, in the sense that there exists a bounded linear operator \( \theta_s: \mathcal{H}^K \to D_{s^{-*}} \), such that \((4.3)\) is satisfied.

**Proof.** Since at every point of \( Z \) we have a sesquilinear strong duality pairing, it will be enough to show that for arbitrary \( s \) the linear operator \( K|_{D_s}: D_s \to \mathcal{H}^K \) is continuous. (See Remark 3.3) To this end, let us denote by \( \|\cdot\|_{D_s} \) any norm that defines the topology of the fiber \( D_s \). Then for every \( \xi \in D_s \) we have

\[
\|K(\xi)\|_{\mathcal{H}^K} = \|K_\xi\|_{\mathcal{H}^K} = (K_\xi | K_\xi)_{\mathcal{H}^K}^{1/2} \leq (K(s^{-*}, s) \xi | \xi)_{s^{-*},s}^{1/2} \leq M_1^{1/2}\|\xi\|_{D_s},
\]

where \( M_1 > 0 \) denotes the norm of the continuous sesquilinear functional \( D_s \times D_s \to \mathbb{C} \) defined by \( (\xi, \eta) \mapsto (K(s^{-*}, s) \xi | \eta)_{s^{-*},s} \).

So the operator \( K|_{D_s}: D_s \to \mathcal{H}^K \) is indeed bounded and \( \|K|_{D_s}\| \leq M_1^{1/2} \).

**Example 4.6.** Every reproducing kernel on a Hermitian vector bundle (see e.g., Section 4 in [BR07]) provides an illustration for Definition 4.4. In fact, this follows since every Hermitian vector bundle is like-Hermitian.
Proposition 4.7. Let \( \Pi : D \to Z \) be a like-Hermitian bundle, and denote by \( p_1, p_2 : Z \times Z \to Z \) the natural projections. Then for every reproducing \((\ast)\)-kernel \( K \in \Gamma(Z \times Z, \text{Hom}(p_1^2 \Pi, p_1^1 \Pi)) \) there exists a unique linear mapping \( \iota : \mathcal{H}^K \to \Gamma(Z, D) \) with the following properties:

(a) The restriction of \( \iota \) to the dense subspace \( \mathcal{H}^K_0 \) is the identity mapping.
(b) The mapping \( \iota \) is injective.
(c) The evaluation operator \( \text{ev}_s^\iota : h \mapsto (\iota(h))(s) \), \( \mathcal{H}^K \to D_s \), is continuous linear for arbitrary \( s \in Z \), and we have

\[
(\forall s,t \in Z) \quad K(s,t^{-\ast}) = \text{ev}_s^\iota \circ (\text{ev}_t^\iota)^{-\ast}.
\]

Definition 4.8. In the setting of Proposition 4.7 we shall say that \( \iota \) is the realization operator associated with the reproducing \((\ast)\)-kernel \( K \).

Proof of Proposition 4.7. The uniqueness of \( \iota \) is clear. To prove the existence of \( \iota \), note that for every \( s \in Z \) there exists a bounded linear operator \( \theta_s : \mathcal{H}^K \to D_{s^{-\ast}} \) such that

\[
(\forall \xi \in D_s, h \in \mathcal{H}^K) \quad (K_\xi \mid h)_{\mathcal{H}^K} = (\xi \mid \theta_s h)_{s,s^{-\ast}}
\]

(see Lemma 4.5). We shall define the wished-for mapping \( \iota \) by

\[
(\forall h \in \mathcal{H}^K, s \in Z) \quad (\iota(h))(s) := \theta_{s^{-\ast}} h
\]

whenever \( h \in \mathcal{H}^K \) and \( s \in Z \). In particular we have

\[
(\forall s \in Z) \quad \text{ev}_s^\iota = \theta_{s^{-\ast}},
\]

and in addition equation (4.8) holds.

It is also clear that the mapping \( \iota \) defined by (4.8) is linear. To prove that it is injective, let \( h \in \mathcal{H}^K \) with \( \iota(h) = 0 \). Then \((\iota(h))(s)^{-\ast} = 0 \) for all \( s \in Z \), so that \( \theta_s h = 0 \) for all \( s \in Z \), according to (4.8).

Now (4.7) shows that \( (K_\xi \mid h)_{\mathcal{H}^K} = 0 \) for all \( \xi \in D \), whence \( h \perp \mathcal{H}^K_0 \) in \( \mathcal{H}^K \). Since \( \mathcal{H}^K_0 \) is a dense subspace of \( \mathcal{H}^K \), it then follows that \( h = 0 \).

We shall check that the restriction of \( \iota \) to \( \mathcal{H}^K_0 \) is the identity mapping. To this end it will be enough to see that for all \( t \in Z \) and \( \eta \in D_t \) we have \( \iota(K_\eta) = K_\eta \). In fact, at any point \( s \in Z \) we have \( (\iota(K_\eta))(s) = \theta_{s^{-\ast}}(K_\eta) \) by (4.8). Hence for all \( \xi \in D_{s^{-\ast}} \) we get

\[
(\xi \mid (\iota(K_\eta))(s))_{s^{-\ast},s} = (\xi \mid \theta_{s^{-\ast}}(K_\eta))_{s^{-\ast},s} = (K_\xi \mid K_\eta)_{\mathcal{H}^K} = (K(t^{-\ast}, s^{-\ast})\xi \mid \eta)_{t^{-\ast},t} = (\xi \mid K(s,t)\eta)_{s^{-\ast},s} = (\xi \mid K_\eta(s))_{s^{-\ast},s}.
\]

Since \( \xi \in D_{s^{-\ast}} \) is arbitrary and \( \{\cdot \mid \cdot\}_{z,z^{-\ast}}^{z,z^{-\ast}} \in Z \) is a like-Hermitian structure, it then follows that \( (\iota(K_\eta))(s) = K_\eta(s) \) for all \( s \in Z \), whence \( \iota(K_\eta) = K_\eta \) as desired.

Next we shall prove that \( \iota \) has the asserted property (c). To this end, let \( s, t \in Z \), \( \eta \in D_{t^{-\ast}} \), and \( \xi \in D_{s^{-\ast}} \) arbitrary. Then

\[
((\text{ev}_s^\iota \circ (\text{ev}_t^\iota)^{-\ast})\eta)(s,s^{-\ast}) = ((\theta_{s^{-\ast}} \circ (\theta_{t^{-\ast}})^{-\ast})\eta)(s,s^{-\ast}) = (((\theta_{t^{-\ast}} \circ (\theta_{s^{-\ast}})^{-\ast})\eta)(K_\xi)_{\mathcal{H}^K} = (K_\eta \mid K_\xi)_{\mathcal{H}^K} = (K(s,t^{-\ast})\eta)(s,s^{-\ast}).
\]

Since \( \eta \in D_{t^{-\ast}} \) and \( \xi \in D_{s^{-\ast}} \) are arbitrary and \( \{\cdot \mid \cdot\}_{z,z^{-\ast}}^{z,z^{-\ast}} \in Z \) is a like-Hermitian structure, it then follows that \( \text{ev}_s^\iota \circ (\text{ev}_t^\iota)^{-\ast} = K(s, t^{-\ast}) \) for arbitrary \( s, t \in Z \), as desired.

We now extend to our framework some basic properties of the classical reproducing kernels (see for instance the first chapter of [Nao]).

Proposition 4.9. Assume that \( \Pi : D \to Z \) is a like-Hermitian vector bundle, and \( K \) is a continuous reproducing \((\ast)\)-kernel on \( \Pi \) with the realization operator \( \iota : \mathcal{H}^K \to \Gamma(Z, D) \). Then the following assertions hold:

(a) We have \( \text{Ran} \ \iota \subseteq C(Z, D) \) and the mapping \( \iota \) is continuous with respect to the topology of \( C(Z, D) \) defined by the uniform convergence on the compact subsets of \( Z \).
(b) If \( \Pi \) is a holomorphic bundle and \( K \) is a holomorphic reproducing \((\ast)\)-kernel then we have \( \text{Ran} \ \iota \subseteq \mathcal{O}(Z, D) \).
Proof. The proof has two stages.

1° At this stage we prove that every \( s \in Z \) has an open neighborhood \( V_s \) such that for every sequence \( \{h_n\}_{n \in \mathbb{N}} \) in \( \mathcal{H}^K \) convergent to some \( h \in \mathcal{H}^K \) we have \( \lim_{n \in \mathbb{N}} (\iota(h_n))(z) = (\iota(h))(z) \) uniformly for \( z \in V_s \).

In fact, since the vector bundle \( \Pi \) is locally trivial, there exists an open neighborhood \( V \) of \( s \) such that \( \Pi \) is trivial over both \( V \) and \( V^{-*} := \{z^{-*} \ | \ z \in V\} \). Let \( \Psi_0 : V \times \mathcal{E} \to \Pi^{-1}(V) \) and \( \Psi_1 : V^{-*} \times \mathcal{E} \to \Pi^{-1}(V^{-*}) \) be trivializations of the vector bundle \( \Pi \) over \( V \) and \( V^{-*} \) respectively, where the Banach space \( \mathcal{E} \) is the typical fiber of \( \Pi \). In particular, these trivializations allow us to endow each fiber \( D_z \) with a norm (constructed out of the norm of \( \mathcal{E} \)) for \( z \in V \cup V^{-*} \). On the other hand, property (c) in Definition 3.1 shows that the function

\[
B : (z, x, y) \mapsto (\Psi_0(z, x) \mid \Psi_1(z^{-*}, y)), \quad V \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}
\]

is smooth. Then by property (c) in Definition 3.1 we get a well-defined mapping

\[
\tilde{B} : V \to \text{Iso}(\mathcal{E}, \mathcal{E}^\ast), \quad \tilde{B}(z) x := B(z, x, \cdot) \quad \text{for} \quad z \in V \quad \text{and} \quad x \in \mathcal{E},
\]

and it is straightforward to prove that \( \tilde{B} \) is continuous since \( B \) is so. Here \( \text{Iso}(\mathcal{E}, \mathcal{E}^\ast) \) stands for the set of all topological isomorphisms \( \mathcal{E} \to \mathcal{E}^\ast \), which is an open subset of the complex Banach space \( B(\mathcal{E}, \mathcal{E}^\ast) \). Then by shrinking the open neighborhood \( V \) of \( s \) we may assume that there exists \( M > 0 \) such that \( \max\{\|\tilde{B}(z)\|, \|\tilde{B}(z)^{-1}\|\} < M \) whenever \( z \in V \). In particular, for such \( z \) and \( x \in \mathcal{E} \) we have \( \|x\| \leq M\|\tilde{B}(z)x\| \), and then the definition of the norm of \( \tilde{B}(z) \in \mathcal{E}^\ast \) implies the following fact:

\[
(\forall z \in V)(\forall x \in \mathcal{E})(\exists y \in \mathcal{E}, \ |y| = 1) \quad \|x\| \leq M\|B(z, x, y)\|.
\]

In view of the fact that the norms of the fibers \( D_z \) and \( D_{z^{-*}} \) are defined such that the operators \( \Psi_0(z, \cdot) : \mathcal{E} \to D_z \) and \( \Psi_1(z^{-*}, \cdot) : \mathcal{E} \to D_{z^{-*}} \) are isometries whenever \( z \in V \), it then follows that

\[
(\forall z \in V)(\forall \eta \in D_z)(\exists \xi \in D_{z^{-*}}), \ |\xi| = 1 \quad \|\eta\| \leq M\|\xi\| \quad \text{and} \quad \|\xi\| = 1
\]

On the other hand, it follows by (4.10) that \( \|\iota(h)(z)\|_{D_z} = \|\theta_{z^{-*}}(h)\|_{D_z} \) for arbitrary \( z \in V \) and \( h \in \mathcal{H}^K \). Then by (4.10) there exists \( \xi \in D_{z^{-*}} \) such that \( \|\xi\| = 1 \) and

\[
\|\iota(h)(z)\|_{D_z} \leq M\|\xi\| \quad \text{and} \quad \|\xi\| = 1
\]

On the other hand, since \( K : Z \times Z \to \text{Hom}(pZ\Pi, pZ\Pi) \) is continuous, it follows that after shrinking again the neighborhood \( V \) of \( s \) we may suppose that \( m := \sup M_z < \infty \), where \( M_z \) denotes the norm of the bounded sesquilinear functional \( D_z \times D_z \to \mathbb{C} \) defined by \( (\eta_1, \eta_2) \mapsto (K(z^{-*}, z)\eta_1 \mid \eta_2)_{z^{-*}, z} \) whenever \( z \in V \). Then the computation from the proof of Lemma 4.5 shows that \( K \in \mathcal{H}^K \) and

\[
\|K\|_{\mathcal{H}^K} \leq m^{1/2}\|\|D_s\|_{m^{1/2}}
\]

It then follows by the above inequalities that we end up with an open neighborhood \( V \) of \( s \) with the following property:

\[
(\forall h \in \mathcal{H}^K)(\forall z \in V) \quad \|\iota(h)(z)\|_{D_z} \leq m^{1/2}M\|h\|_{\mathcal{H}^K},
\]

which clearly implies the claim from the beginning of the present stage of the proof.

2° At this stage we come back to the proof of the assertions (a)–(b). Assertion (a) follows by what we proved at stage 1° by means of a straightforward compactness reasoning and by what we proved at stage 1°, since \( K_z \in \mathcal{C}(Z, D) \) whenever \( \xi \in D \) and \( \text{span}_C\{K_z \mid \xi \in D\} = \mathcal{H}^K \). Finally, assertion (b) follows by the assertion (a) in a similar manner, since \( \mathcal{O}(Z, D) \) is a closed subspace of \( \mathcal{C}(Z, D) \) with respect to the topology of uniform convergence on the compact subsets of \( Z \) (see Corollary 1.14 in [Up85]).

Remark 4.10. It follows by Proposition 4.9 (a) that every reproducing \((\ast\ast)\)-kernel Hilbert space \( \mathcal{H}^K \) is a Hilbert subspace of \( \mathcal{C}(Z, D) \) in the sense of [Sc64]. Thus the theory of reproducing \((\ast\ast)\)-kernels developed in the present section provides a new class of examples of reproducing kernels in the sense of Laurent Schwartz.
5. Homogeneous Like-Hermitian Vector Bundles and Kernels

We develop here some aspects of the theory of kernels introduced in the previous section, when the manifold \( Z \) is assumed to be a homogeneous manifold arising from the (smooth) action of a Banach-Lie group, as in Definition 3.10. Specifically, we shall construct realizations of \(*\)-representations, of Banach-Lie groups, on spaces of analytic sections in like-Hermitian vector bundles. A critical role in this connection will be played by the following class of examples of reproducing \((-*)\)-kernels (compare the special case discussed in Example 2.16).

**Example 5.1.** Assume the setting of Definition 3.10: Let \( G_A \) be an involutive real (respectively, complex) Banach-Lie group and \( G_B \) an involutive real (respectively, complex) Banach-Lie subgroup of \( G_A \). For \( X = A \) or \( X = B \), let \( \mathcal{H}_X \) be a complex Hilbert space with \( \mathcal{H}_B \) closed subspace in \( \mathcal{H}_A \) and \( P : \mathcal{H}_A \to \mathcal{H}_B \) the corresponding orthogonal projection, and let \( \pi_X : G_X \to \mathcal{B}(\mathcal{H}_X) \) be a uniformly continuous (respectively, holomorphic) \(*\)-representations such that \( \pi_B(u) = \pi_A(u)|_{\mathcal{H}_B} \) for all \( u \in G_B \). In addition, denote by \( \Pi : D = G_A \times_{G_B} \mathcal{H}_B \to G_A/G_B \) the homogeneous like-Hermitian vector bundle associated with the data \((\pi_A,\pi_B,P)\), and let \( p_1, p_2 : G_A/G_B \to G_A/G_B \) be the natural projections. Set

\[
K(s,t\eta) = [(u,P(\pi_A(u^{-1})\pi_A(v)f))] 
\]

for \( s,t \in G_A/G_B \) and \( \eta = [(v,f)] \in D_\eta \subset D \). Then \( K \) is a reproducing \((-*)\)-kernel for which the corresponding reproducing \((-*)\)-kernel Hilbert space \( \mathcal{H}^K \subset \mathcal{C}(G_A/G_B,D) \) (respectively \( \mathcal{H}^K \subset \mathcal{O}(G_A/G_B,D) \)) consists of sections of the form \( F_h := [(\cdot,P(\pi_A(\cdot)^{-1})h)] \), \( h \in \text{span}_B(\pi_A(G_A)\mathcal{H}_B) \) in \( \mathcal{H}_A \).

To see this, let \( s_j = u_j G_B \in G_A/G_B \) and \( \xi_j = [(u_j^{-*,f_j})] \in D_{s_j} \) for \( j = 1,\ldots,n \). We have

\[
\sum_{j,l=1}^{n} (K(s_l,s_j^{-*})\xi_j | \xi_l)_{s_l,s_j^{-*}} = \sum_{j,l=1}^{n} (P(\pi_A(u_l^{-1})\pi_A(u_j^{-*})f_j) | f_l)_{\mathcal{H}_B} = \sum_{j,l=1}^{n} (\pi_A(u_l^{-1})\pi_A(u_j^{-*})f_j | f_l)_{\mathcal{H}_A}
\]

\[
= \sum_{j=1}^{n} \pi_A(u_j^{-*})f_j | \sum_{l=1}^{n} \pi_A(u_l^{-*})f_l \geq 0.
\]

On the other hand, by the above calculation we get

\[
(K(s_l,s_j^{-*})\xi_j | \xi_l)_{s_l,s_j^{-*}} = (\pi_A(u_j^{-*})f_j | \pi_A(u_l^{-*})f_l)_{\mathcal{H}_A} = (\pi_A(u_l^{-1})f_l | \pi_A(u_j^{-*})f_j)_{\mathcal{H}_A}
\]

\[
= (K(s_j,s_l^{-*})\xi_l | \xi_j)_{s_j,s_l^{-*}} = (\xi_j | K(s_j,s_l^{-*})\xi_l)_{s_j,s_l^{-*}}.
\]

Thus \( K \) is a reproducing \((-*)\)-kernel. Again by the above calculation it follows that

\[(K_{\xi_j} | K_{\xi_j})_{\mathcal{H}^K} = (K(s_j,s_j^{-*})\xi_j | \xi_j)_{s_j,s_j^{-*}} = (\pi_A(u_j^{-*})f_j | \pi_A(u_j^{-*})f_l)_{\mathcal{H}_A} \]

Now, by Proposition 4.9, \( \mathcal{H}^K \subset \mathcal{C}(G_A/G_B,D) \). Let \( F \) be a section in \( \mathcal{H}^K \). By definition \( F \) is a limit, in the norm of \( \mathcal{H}^K \), of a sequence of sections of the form \( \sum_{j=1}^{n(m)} K_{\xi_j}^{m} \), where \( \xi_j^{m} = [(v_j^m,f_j^m)] \in D \), \( j = 1,\ldots,n(m) \), \( m = 1,2,\ldots \). By Proposition 4.9, \( \sum_{j=1}^{n(m)} \pi_A(v_j^m) f_j^m \) is a Cauchy sequence in \( \mathcal{H}_A \), so that there exists \( h := \lim_{m \to \infty} \sum_{j=1}^{n(m)} \pi_A(v_j^m) f_j^m \) in \( \mathcal{H}_A \). Now, by the proof of Proposition 4.9, convergence in \( \mathcal{H}^K \) implies (locally uniform) convergence in \( \mathcal{C}(G_A/G_B,D) \) whence, for every \( s = u G_B \in G_A/G_B \), we get

\[
F(s) = \lim_{m \to \infty} \sum_{j=1}^{n(m)} K_{\xi_j}^m(s) = \lim_{m \to \infty} \sum_{j=1}^{n(m)} [(u,P(\pi_A(u)^{-1}\pi_A(v_j^m) f_j^m))]
\]

\[
= \lim_{m \to \infty} [(u,P(\pi_A(u)^{-1}\sum_{j=1}^{n(m)} \pi_A(v_j^m) f_j^m))]
\]

in \( D_\ast \). On the other hand, since the norm in \( D_\ast \) is the copy of the norm in \( \mathcal{H}_A \), through the action of the basic mapping \( \Phi \) associated with data \((\pi_A,\pi_B,P)\) (see Example 3.5 and the bottom of Definition 3.10).
we also have \( \lim_{m \to \infty} [(u, P(\pi_A(u)^{-1} \sum_{j=1}^m \pi_A(u_j^m)f_j^m))] = [(u, P(\pi_A(u)^{-1}h))] \). Thus we have shown that \( F = F_h \). Also, for arbitrary \( h \in \mathcal{H}_A \),

\[
F_h = 0 \iff (\forall u \in G_A) \quad P(\pi_A(u^{-1})h) = 0 \iff (\forall u \in G_A) \quad \pi_A(u^{-1})h \perp B = 0.
\]

Since \( \pi_A \) is a \( * \)-representation, it then follows that \( F_h = 0 \) if and only if \( h \perp \text{span}_C(\pi_A(G_A)\mathcal{H}_B) \). Hence \( \mathcal{H}_A/(\text{span}_C(\pi_A(G_A)\mathcal{H}_B))^\perp \equiv \mathcal{H}_K = \{F_h \mid h \in \text{span}_C(\pi_A(G_A)\mathcal{H}_B)\} \).

Finally, note that \( \mathcal{H}_K \subset C^\infty(G_A/G_B, D) \) indeed, by definition of \( F_h \) (\( h \in \mathcal{H}_A \)). In the case where \( G_A \) and \( G_B \) are complex Banach-Lie groups then \( \mathcal{H}_K \subset O(G_A/G_B, D) \), by the definition of \( F_h \) as well.

Clearly, the mapping \( F_h \mapsto h, \mathcal{H}_K \mapsto \text{span}_C(\pi_A(G_A)\mathcal{H}_B) \subset \mathcal{H}_A \) is an isometry, which we denote by \( W \), such that \( W(K_h) = \pi_A(v)f \) for \( \eta = [(v, f)] \in D \). In addition, if \( \text{span}_C(\pi_A(G_A)\mathcal{H}_B) = \mathcal{H}_A \) then the operator \( W \) is unitary. Recall the mapping \( \tilde{K} : D \to \mathcal{H}_K \) given by \( \tilde{K}(\xi) = K\xi \) if \( \xi \in D \), as in \( (4.4) \).

Clearly \( W \circ \tilde{K} = \Theta \), where \( \Theta \) is the basic mapping for the data \( (\pi_A, \pi_B, P) \) (see Definition \( 3.10 \)).

The following result is an extension of Theorem 5.4 in \( [BR07] \) and provides geometric realizations for \( * \)-representations of involutive Banach-Lie groups.

**Theorem 5.2.** In the preceding setting, the following assertions hold:

(a) The linear operator

\[
\gamma : \mathcal{H}_A \to \mathcal{H}_K \subset C^\infty(G_A/G_B, D), \quad (\gamma(h))(uG_B) = [(u, P(\pi_A(u^{-1})h))],
\]

satisfies \( \text{Ker}\gamma = (\text{span}_C(\pi_A(G_B)\mathcal{H}_B))^\perp \) and the operator \( \iota := \gamma \circ W \) is the canonical inclusion \( \mathcal{H}_K \to C^\infty(G_A/G_B, D) \). Moreover, \( \gamma \circ \Theta = \tilde{K} \).

(b) For every point \( t \in G_A/G_B \) the evaluation map \( \text{ev}_t^* = \iota(t) : \mathcal{H}_K \to D_t \) is a continuous linear operator such that

\[
(\forall s, t \in G_A/G_B) \quad K(s, t^{-*}) = \text{ev}_s^* \circ (\text{ev}_t^*)^{-*}.
\]

(c) The mapping \( \gamma \) is a realization operator in the sense that it is an intertwiner between the \( * \)-representation \( \pi_A : G_A \to \mathcal{B}(\mathcal{H}_A) \) and the natural representation of \( G_A \) on the space of cross sections \( C^\infty(G_A/G_B, D) \).

**Proof.** (a) This part is just a reformulation of what has been shown prior to the statement of the theorem. The equality \( \gamma \circ \Theta = \tilde{K} \) is obvious.

(b) Let \( t \in G_A/G_B \) arbitrary and then pick \( u \in G_A \) such that \( t = uG_B \). In particular, once the element \( u \) is chosen, we get a norm on the fiber \( D_t \) (see Definition \( 3.10 \)) and then for every \( F \in F_h \in \mathcal{H}_K \), where \( h \in \text{span}_C(\pi_A(G_A)\mathcal{H}_B) \), we have

\[
||\text{ev}_t(F_h)||_{D_t} = ||F_h(t)||_{D_t} = ||(u, P(\pi_A(u^{-1})h))||_{D_t} = ||\pi_A(u)P(\pi_A(u^{-1})h)||_{\mathcal{H}_A} \leq ||\pi_A(u)|| \cdot ||\pi_A(u^{-1})|| \cdot ||h||_{\mathcal{H}_A} = C_u||F_h||_{\mathcal{H}_K},
\]

so that the evaluation map \( \text{ev}_t^* : \mathcal{H}_A \to D_t \) is continuous.

Let us keep \( s = uG_B \) fixed for the moment. We first prove that

\[
(\tilde{K}|_{D_s})^{-*} = \text{ev}_{s^{-*}}^* : \mathcal{H}_K \to D_{s^{-*}}.
\]

To this end we check that condition \( (4.3) \) in Definition \( 1.4 \) is satisfied with \( \theta_s = \text{ev}_{s^{-*}}^* : \mathcal{H}_K \to D_{s^{-*}} \). In fact, let \( \xi = [(u, f)] \in D_s \) arbitrary.

Then for all \( h \in \text{span}_C(\pi_A(G_A)\mathcal{H}_B) \) we have

\[
(\xi | \theta_s F_h)_{s^{-*}} = \left( [(u, f)] \mid [(u^{-*}, P(\pi_A((u^{-*})^{-1})h))] \right)_{s^{-*}}
= (\pi_A(u)f \mid \pi_A(u^{-*})P(\pi_A((u^{-*})^{-1})h))_{\mathcal{H}_A}
= (f \mid P(\pi_A(u^*)h))_{\mathcal{H}_A} = (\pi_A(u)f \mid h)_{\mathcal{H}_A}
= (W(\gamma \circ \Theta)(\xi) \mid W(\gamma(h))|_{\mathcal{H}_A} = ((\gamma \Theta)(\xi) \mid F_h)_{\mathcal{H}_K}
= (\tilde{K}(\xi) \mid F_h)_{\mathcal{H}_K} = (K_\xi \mid F_h)_{\mathcal{H}_K}.
\]
Now let \( s, t \in G_A/G_B \) arbitrary and \( u, v \in G_A \) such that \( s = uG_B \) and \( t = vG_B \). It follows by (5.2) that \((\ev_1^*)^{-*} \circ K_{D_{-t}} \), hence for every \( \eta = [(v^{*-*}f)] \in D_{-t} \) we have

\[
\ev_1^* \circ (\ev_1^*)^{-*} \eta = \ev_1^*(K_{D_{-t}}(\eta)) = (\iota(K_{\eta}))(s) = [(u, P(\pi_A(u^{-1})\pi_A(v^{*-*}f)))] = K(s, t^{*-*})\eta.
\]

(c) Let \( h \in \mathcal{H}_A \) and \( v \in G_A \) arbitrary. Then at every point \( t = uG_B \in G_A/G_B \) we have

\[
(\gamma(\pi_A(v)h))(t) = [(u, P(\pi_A(u^{-1})\pi_A(v)h))] = [(u, P(\pi_A((v^{-1}u^{-1})h)))] = v \cdot [(v^{-1}u, P(\pi_A((v^{-1}u)^{-1}h)))] = v \cdot (\gamma(h))(v^{-1}t)
\]

and the proof ends. \( \square \)

Part (c) of the above theorem tells us that it is possible to realize representations like \( \pi_A : G_A \to \mathcal{B}(\mathcal{H}_A) \) as natural actions on spaces of analytic sections. We next take advantage of these geometric realizations to point out some phenomena of holomorphic extension in bundle vectors and sections of them. Firstly, we record some auxiliary facts in the form of a lemma.

**Lemma 5.3.** Let \( G_A \) be an involutive Banach-Lie group and \( G_B \) an involutive Banach-Lie subgroup of \( G_A \), and denote by \( \beta : \{v, uG_B\} \mapsto vuG_B, G_A \times G_A/G_B \to G_A/G_B \) the corresponding transitive action. Also denote \( U_X = \{u \in G_X \mid u^{-*} = u\} \) for \( X \in \{A, B\} \). Then the following assertions hold:

(a) There exists a correctly defined involutive diffeomorphism

\[
z \mapsto z^{*-*}, \quad G_A/G_B \to G_A/G_B,
\]

defined by \( uG_B \mapsto u^{-*}G_B \). This diffeomorphism has the property \( \beta(v^{*-*}, z^{*-*}) = \beta(v, z)^{-*} \) whenever \( v \in G_A \) and \( z \in G_A/G_B \).

(b) The group \( U_X \) is a Banach-Lie subgroup of \( G_X \) for \( X \in \{A, B\} \) and \( U_B \) is a Banach-Lie subgroup of \( U_A \).

(c) If \( G_B^+ = G_A^+ \cap G_B \), then the mapping

\[
\lambda : uU_B \mapsto uG_B, \quad U_A/U_B \to G_A/G_B,
\]

is a diffeomorphism of \( U_A/U_B \) onto the fixed-point submanifold of the involutive diffeomorphism of \( G_A/G_B \) introduced above in assertion (a).

**Proof.** Assertion (a) follows since the mapping \( u \mapsto u^{-*} \) is an automorphism of \( G \) (Remark 5.7). The proof of assertion (b) is straightforward.

As regards (c), what we really have to prove is the equality \( \lambda(U_A/U_B) = \{z \in G_A/G_B \mid z^{*-*} = z\} \). The inclusion \( \subseteq \) is obvious. Conversely, let \( z \in G_A/G_B \) with \( z^{*-*} = z \). Pick \( u \in G_A \) arbitrary such that \( z = uG_B \). Since \( z^{*-*} = z \), it follows that \( u^{-1}u^{*-*} \in G_B \). On the other hand, \( u^{-1}u^{*-*} \in G_A^+ \), hence the hypothesis \( G_B^+ = G_A^+ \cap G_B \) implies that \( u^{-1}u^{*-*} \in G_B^+ \). That is, there exists \( w \in G_B \) such that \( u^{-1}u^{*-*} = uw^* \). Hence \( uw = (uw)^{-*} \), so that \( uw \in U_A \), and in addition \( z = uG_B = uwG_B = \lambda(uwU_B) \). \( \square \)

The next theorem gives a holomorphic extension of the Hermitian vector bundles and kernels introduced in [BR07].

**Theorem 5.4.** For \( X \in \{A, B\} \), let \( G_X \) be a complex Banach-Lie group and \( G_B \) a Banach-Lie subgroup of \( G_A \). As above, set \( U_X = \{u \in G_X \mid u^{*-*} = u\} \). Let \( \pi_X : X \to \mathcal{B}(\mathcal{H}_X) \) be a holomorphic *-representation such that \( \pi_B(u) = \pi_A(u)|_{\mathcal{H}_B} \) for all \( u \in G_B \). Denote by \( \Pi : D \to G_A/G_B \) the like-Hermitian vector bundle, \( K \) the reproducing \((-*)\)-kernel, and \( W : \mathcal{H}^K \to \mathcal{H}_A \) the isometry and \( \gamma : \mathcal{H}_A \to C^\infty(G_A/G_B, D) \) the realization operator associated with the data \( (\pi_A, \pi_B, P) \), where \( P : \mathcal{H}_A \to \mathcal{H}_B \) is the orthogonal projection.

Also denote by \( \Pi^U : D^U \to U_A/U_B \) the like-Hermitian vector bundle, \( K^U \) the reproducing \((-*)\)-kernel, and \( W^U : \mathcal{H}^{K^U} \to \mathcal{H}_A \) the isometry and \( \gamma^U : \mathcal{H}_A \to C^\infty(U_A/U_B, D) \) the operators associated with the data \( (\pi_A|_{U_A}, \pi_B|_{U_B}, P) \).

Let assume in addition that \( G_B^+ = G_A^+ \cap G_B \). Then the following assertions hold:
(a) The inclusion \( \iota := \gamma \circ W : \mathcal{H}^K \to \mathcal{O}(G_A/G_B, D) \) is the realization operator associated with the reproducing \((-\ast\)-)kernel \( K \). Moreover, \( \gamma \) intertwines the \(*\)-representation \( \pi_A : G_A \to \mathcal{B}(\mathcal{H}_A) \) and the natural representation of \( G_A \) on the space of cross sections \( \mathcal{O}(G_A/G_B, D) \).

(b) The like-Hermitian vector bundle \( IV^U : D^U \to U_A/U_B \) is actually a Hermitian vector bundle. The mapping \( \lambda : uU_B \mapsto uG_B, U_A/U_B \mapsto G_A/G_B \), is a diffeomorphism of \( U_A/U_B \) onto a submanifold of \( G_A/G_B \) and we have

\[
\lambda(U_A/U_B) = \{ z \in G_A/G_B \mid z^{-\ast} = z \}.
\]

In addition, there exists an \( U_A \)-equivariant real analytic embedding \( \Lambda : D^U \to D \) such that the diagrams

\[
\begin{array}{ccc}
D^U & \xrightarrow{\Lambda} & D \\
\Pi^U \downarrow & & \downarrow \Pi \\
U_A/U_B & \xrightarrow{\lambda} & G_A/G_B
\end{array}
\]

\[
\begin{array}{ccc}
D^U & \xrightarrow{\Lambda} & D \\
\gamma(\Pi(h)) \uparrow & & \uparrow \gamma(h) \\
U_A/U_B & \xrightarrow{\lambda} & G_A/G_B
\end{array}
\]

for arbitrary \( h \in \overline{\text{span}}_C(\pi_A(G_A)\mathcal{H}_B) \) are commutative, the mapping \( \Lambda \) is a fiberwise isomorphism, and \( \Lambda(D^U) = \Pi^{-1}(\lambda(U_A/U_B)) \).

(c) The inclusion \( I^U := \gamma^U \circ W : \mathcal{H}^K \to \mathcal{C}^\gamma(U_A/U_B, D^U) \) is the realization operator associated with the reproducing kernel \( K^U \), where \( \mathcal{C}^\gamma(U_A/U_B, D^U) \) is the subspace of \( \mathcal{C}^\infty(U_A/U_B, D^U) \) of real analytic sections. In addition, \( \gamma^U \) is an intertwiner between the unitary representation \( \pi_A : U_A \to \mathcal{B}(\mathcal{H}_A) \) and the natural representation of \( \mathcal{C}^\infty(U_A/U_B, D^U) \).

Proof. (a) This follows by Theorem 5.2 applied to the data \((\pi_A, \pi_B, P)\). The fact that the range of the realization operator \( \iota \) consists only of holomorphic sections follows either by Proposition 4.9(c) or directly by the definition of \( \gamma \) (see Theorem 5.2(a)).

d) The fact that \( \Pi^U \) is a Hermitian vector bundle (Remark 4.4) follows by 8.2. Moreover, the asserted properties of \( \lambda \) follow by Lemma 5.4(c) as we are assuming that \( G_B^+ = G_B \cap G_A^+ \). As regards the \( U_A \)-equivariant embedding \( \Lambda : D^U \to D \), it can be defined as the mapping that takes every equivalence class \([[(u, f)]]) \in D^U \) into the equivalence class \([[(u, f)]]) \in D \). Then \( \Lambda \) clearly has the wished-for properties.

c) Use again Theorem 5.2 for the data \((\pi_A|_{U_A}, \pi_B|_{U_B}, P)\). It is clear from definitions (see also [BR71]) that \( K^U \) is a reproducing kernel indeed. The fact that the range of \( \gamma^U \), or \( \iota^U \), consists of real analytic sections follows by the definition of \( \gamma^U \) (see Theorem 5.2(a) again). Alternatively, one can use assertions (a) and (b) above to see that for arbitrary \( h \in \mathcal{H}_A \) the mapping \( \lambda \circ \gamma^U(h) \circ \lambda^{-1} : \lambda(U_A/U_B) \to D \) is real analytic since it is a section of \( \Pi \) over \( \lambda(U_A/U_B) \) which extends to the holomorphic section \( \gamma(h) : G_A/G_B \to D \).

\]

\[\square\]
Let \( p := (\text{Ker} \, E) \cap u_A \), which is a real Banach space acted on by \( U_B \) by means of the adjoint action \( (u, X) \mapsto u X u^{-1} \). Consider the corresponding quotient map \( \kappa: (u, X) \mapsto [(u, X)] \), \( U_A \times p \rightarrow U_A \times U_B \), \( p \), and define the mapping \( \Psi^E_p: (u, X) \mapsto u \exp(iX) \), \( U_A \times p \rightarrow G_A \). Then there is a unique \( U_A \)-equivariant, real analytic diffeomorphism \( \Psi^E: U_A \times U_B \rightarrow G_A/G_B \) such that the diagram

\[
\begin{array}{ccc}
U_A \times p & \xrightarrow{\Psi^E_p} & G_A \\
\kappa \downarrow & & \downarrow \sigma \\
U_A \times U_B & \xrightarrow{\Psi^E} & G_A/G_B
\end{array}
\]

is commutative. Thus the complex homogeneous space \( G_A/G_B \) has the structure of a \( U_A \)-equivariant real vector bundle over its real form \( U_A/U_B \), the corresponding projection being given by the composition (depending on the conditional expectation \( E \))

\[
G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times U_B \xrightarrow{\Xi} U_A/U_B,
\]

where the typical fiber of the vector bundle \( \Xi \circ (\Psi^E)^{-1} \) is the real Banach space \( p = (\text{Ker} \, E) \cap u_A \).

Proof. The uniqueness of \( \Psi^E \) follows since the mapping \( \kappa \) is surjective. For the existence of \( \Psi^E \), note that for all \( u \in U_A \), \( v \in U_B \), and \( X \in p \) we have

\[
q(\Psi^E_0(u v, v^{-1} X v)) = q(u v \cdot \exp(i v^{-1} X v)) = q(u v \cdot \exp(i X) v) = q(u \exp(i X) v) = u \exp(i X) v G_B = u \exp(i X) G_B = q(\Psi^E_0(u, X)).
\]

This shows that the mapping

\[
(5.4) \quad \Psi^E: [(u, X)] \mapsto u \exp(iX) G_B, \quad U_A \times U_B \rightarrow G_A/G_B,
\]

is well defined, and it is clearly \( U_A \)-equivariant. Moreover, since \( \kappa \) is a submersion and \( \Psi^E \circ \kappa = q \circ \Psi^E_0 \) is a real analytic mapping, it follows by Corollary 8.4(i) in [Up85] that \( \Psi^E \) is real analytic.

Now we prove that \( \Psi^E \) is bijective. To this end we need the following fact:

\[
(5.5) \quad \text{for all } a \in G_A \text{ there exist a unique } (u, X, b) \in U_A \times p \times G_B^1 \text{ such that } a = u \exp(i X) \cdot b
\]

(see Theorem 8 in [PR94]). It follows by (5.4) and (5.5) that the mapping \( \Psi^E: U_A \times U_B \rightarrow G_A/G_B \) is surjective. To see that it is also injective, assume that \( u_1 \exp(i X_1) G_B = u_2 \exp(i X_2) G_B \), where \( (u_j, X_j) \in U_A \times p \) for \( j = 1, 2 \). Then there exists \( b_1 \in G_B \) such that \( u_1 \exp(i X_1) b_1 = u_2 \exp(i X_2) \). Let \( b_1 = v b \) be the polar decomposition of \( b_1 \in G_B \), where \( v \in U_B \) and \( b \in G_B^+ \). Then

\[
u_1 \exp(i X_1) b_1 = u_1 \exp(i X_1) v b = u_1 v \exp(i v^{-1} X_1) v b.
\]

Note that \( u_1 v \in U_A \) and \( v^{-1} X_1 v \in p \) since \( E(v^{-1} X_1 v) = v^{-1} E(X_1) v = 0 \). Since \( u_1 \exp(i X_1) b_1 = u_2 \exp(i X_2) \), it then follows by the uniqueness assertion in (5.5) that \( u_2 = u_1 v \) and \( X_2 = v^{-1} X_1 v \). Hence \( [(u_1, X_1)] = [(u_2, X_2)] \), and thus the mapping \( \Psi^E: U_A \times U_B \rightarrow G_A/G_B \) is injective as well.

Finally, we show that the inverse function

\[
(\Psi^E)^{-1}: a G_B = u \exp(iX) G_B \mapsto [(u, X)], \quad G_A/G_B \rightarrow U_A \times U_B \]

is also smooth. For this, note that \( u \) and \( X \) in (5.5) depend on \( a \) in a real analytic fashion (see [PR94]). Hence, the mapping \( \sigma: a \mapsto [(u, X)] \), \( G_A \rightarrow U_A \times U_B \) is smooth. Since \( \sigma = (\Psi^E)^{-1} \circ q \) and \( q \) is a submersion, it follows again from Corollary 8.4(i) in [Up85] that \( (\Psi^E)^{-1} \) is smooth. In conclusion, \( \Psi^E \) is a real analytic diffeomorphism (see page 268 in [Bo04]), as we wanted to show. \( \square \)

Remark 5.7. From the observation in the second part of the above statement, it follows that the mapping \( \Xi \circ (\Psi^E)^{-1} \) can be thought of as an infinite-dimensional version of Mostow fibration; see [Mo55, Mo05] and Section 3 in [Bo04] for more details on the finite-dimensional setting. See also Theorem 1 in Section 3 of [La78] for a related property of complexifications of compact symmetric spaces.
In fact the construction of the diffeomorphism \( \Psi^E \) in Theorem 5.6 relies on the representation \([5.5]\), and so it depends on the decomposition of \( A \) obtained in terms of the expectation \( E \), see [PR94]. It is interesting to see how \( \Psi^E \) depends explicitly on \( E \) at the level of tangent maps: We have

\[
T_{(1,0)}: (Z,Y) \mapsto ((1-E)Z,Y), \quad u_A \times p \to T_{(1,0)}((U_A \times U_B) p) \simeq p \times p,
\]

\[
T_{(1,0)}(\Psi^E): (Z,Y) \mapsto Z + iY, \quad u_A \times p \to A,
\]

\[
T_1: Z \mapsto (1-E)Z, \quad A \to \text{Ker} E,
\]

hence \( T_{[1,0]}(\Psi^E)((1-E)Z,Y) = (1-E)(Z+iY) = (1-E)Z+iY \) whenever \( Z \in u_A \) and \( Y \in p \). Thus

\[
T_{[1,0]}(\Psi^E): (Y_1,Y_2) \mapsto Y_1 + iY_2, \quad p \times p \to \text{Ker} E,
\]

which is an isomorphism of real Banach spaces since \( \text{Ker} E = p + ip \).

\[\square\]

**Corollary 5.8.** Let \( A \) and \( B \) two \( C^* \)-algebras as in the preceding theorem. Then \( G_A/G_B \simeq U_A \times U_B \) \( p \) is a complexification of \( U_A/U_B \) with respect to the anti-holomorphic involutive diffeomorphism

\[
u \exp(iX)G_B \mapsto u \exp(-iX)G_B, \quad G_A/G_B \to G_A/G_B
\]

where \( u \in U_A, X \in p \) (alternatively, \( [(u,X)] \mapsto [(u,-X)] \)).

**Proof.** First, note that \( G_B^+ = G_B \cap G_A^+ \). This is a direct consequence of the fact that the \( C^* \)-algebras are closed under taking square roots of positive elements. So Theorem 5.6 applies to get \( U_A/\text{U}_B \) as the set of fixed points of the mapping \( aG_B \mapsto a^{-*}G_B \) on \( G_A/G_B \), where \( a^{-*} := (a^*)^{-1} \) for \( a \in A \), and \( * \) is the involution in \( A \). By \([5.8]\), every element \( aG_B \) in \( G_A/G_B \) is of the form \( aG_B = uG_B \) with \( u \in U_A \) and \( X \in p \), and the correspondence \( uG_B \mapsto [(u,X)] \) is a bijection. But then \( uG_B \mapsto u \exp(-iX) \) since \( X^* = -X \), and the proof ends. \[\square\]

To put Theorem 5.6 and Corollary 5.8 in a proper perspective, we recall that for \( X \in \{A,B\} \) the Banach-Lie group \( G_X \) is the universal complexification of \( U_X \) (see Example VI.9 in [Ne02], and also \([GN03]\)). Besides this, we have seen in Theorem 5.4 that the homogeneous space \( G_A/G_B \) is a complexification of \( U_A/U_B \). Now Corollary 5.8 implements such a complexification in the explicit terms of a sort of polar decomposition (if \( X \in p \) then \( \exp(iX)^* = \exp(-i(-X)) = \exp(iX) \) whence \( \exp(iX) = \exp(iX/2) \exp(-iX/2)^* \) is positive). For the group case, see \([GN03]\).

**Remark 5.9.** It is to be noticed that there is an alternative way to express the involution mapping considered in this section as multiplication by positive elements. This representation was suggested by Axiom 4 for involutions of homogeneous reductive spaces as studied in the paper [MR92].

Specifically, under the conditions assumed above the following condition is satisfied:

\[
(a^*)^{-1} a = a_+ ab_+.
\]

To see this first note that we can assume \( |a^*| < \sqrt{2} \). Then, if \( H \) is a Hilbert space such that \( A \) is canonically embedded in \( B(H) \) and \( x \in H \),

\[
\|a^* a\|^2 < 2 \|a^* a\| \Leftrightarrow (x - a^* a x | x - a^* a x)_{\mathcal{H}} < (x | x)_{\mathcal{H}} \Rightarrow \|1 - a^* a\|^2 \leq \|x\|^2.
\]

Thus \( \|1 - a^* a\| < 1 \) and so \( \|1 - E(a^* a)\| = \|E(1) - E(a^* a)\| < 1 \), whence \( b_+ := E(a^* a) \in G_B^+ \). Now it is clear that (5.6) holds with \( a_+ := (a^*)^{-1}a^{-1} \in G_A^+ \).

As a consequence of (5.6), we have that \( a^{-*}G_B = a_+ aG_B \) for every \( a \in G_B \). Let us see the correspondence of such an identity with the decomposition of \( a^{-*}G_B \) given in Theorem 5.6. Since \( a = u e^{iX} b \) in (5.5), we have \( a^* a = (b e^{iX} u^{-1}) (u e^{iX} b) = b e^{2iX} b \) and so \( E(a^* a) = b E(e^{2iX}) b \). It follows that \( a^{-*} = a_+ ab_+ \) where \( b_+ = b E(e^{2iX}) b \) and \( a_+ = u e^{-iX} b e^{-2iX} b^{-1} e^{-iX} u^{-1} \).

\[\square\]

There is a natural identification between the vector bundle \( \Xi: U_A \times U_B \to U_A/U_B \) and the tangent bundle \( T(U_A/U_B) \to U_A/U_B \). In view of Theorem 5.6, we get an interesting interpretation of the homogeneous space \( G_A/G_B \) as the tangent bundle of \( U_A/U_B \).
Corollary 5.10. In the above notation, the vector bundle $\Xi: U_A \times U_B \to U_A/U_B$ is $U_A$-equivariantly isomorphic to the tangent bundle $T(U_A/U_B) \to U_A/U_B$. Hence, the composition

$$G_A/G_B \xrightarrow{(\psi^E)^{-1}} U_A \times U_B \xrightarrow{\sim} T(U_A/U_B)$$

defines a $U_A$-equivariant diffeomorphism between the complexification $G_A/G_B$ and the tangent bundle $T(U_A/U_B)$ of the homogeneous space $U_A/U_B$.

Proof. Let $\alpha: (u,vU_B) \mapsto u\lambda vU_B$, $U_A \times U_B \to U_A/U_B$. Then let $p_0 = 1U_B \in U_A/U_B$ and $\partial_2 \alpha: U_A \times T(U_A/U_B) \to T(U_A/U_B)$ the partial derivative of $\alpha$ with respect to the second variable. Since $T_{p_0}(U_A/U_B) \cong p$, by restricting $\partial_2 \alpha$ to $U_A \times T_{p_0}(U_A/U_B)$ we get a mapping $\alpha^E: U_A \times p \to T(U_A/U_B)$. Then it is straightforward to show that there exists a unique $U_A$-equivariant diffeomorphism $\alpha^E: U_A \times U_B \to T(U_A/U_B)$ such that $\alpha^E \circ \kappa = \alpha^E$.

Now it follows by Theorem 5.7 that the composition $G_A/G_B \xrightarrow{(\psi^E)^{-1}} U_A \times U_B \xrightarrow{\alpha^E} T(U_A/U_B)$ defines a $U_A$-equivariant diffeomorphism between the complexification $G_A/G_B$ and the tangent bundle $T(U_A/U_B)$ of the homogeneous space $U_A/U_B$. □

Remark 5.11. It is known that conditional expectations can be regarded as connection forms of principal bundles, see [ACS93], [CG99], and [Gal06]. Thus Corollary 5.10 leads to numerous examples of real analytic Banach manifolds whose tangent bundles have complex structures associated with certain connections. See for instance [LS01], [Bo03], and [Sz04] for the case of finite-dimensional manifolds. □

6. Stinespring representations

In this section we are going to apply the preceding theory of reproducing ($-\ast$)-kernels, for homogeneous-like-Hermitian bundles, to explore the differential geometric background of completely positive maps. Thus we shall find geometric realizations of the Stinespring representations which will entail an unexpected bearing on the Stinespring dilation theory. Specifically, it will follow that the classical constructions of extensions of representations and induced representations of $C^*$-algebras (see [Di64] and [Rt74], respectively), which seemed to pass beyond the realm of geometric structures, actually have geometric interpretations in terms of reproducing kernels on vector bundles. See Remark 6.9 below for some more details.

Notation 6.1. For every linear map $\Phi: X \to Y$ between two vector spaces and every integer $n \geq 1$ we denote $\Phi_n = \Phi \otimes \text{id}_{M_n(\mathbb{C})}: M_n(X) \to M_n(Y)$, that is, $\Phi_n((A_{ij}))_{1 \leq i,j \leq n} = (\Phi(A_{ij}))_{1 \leq i,j \leq n}$ for every matrix $(A_{ij})_{1 \leq i,j \leq n} \in M_n(X)$.

Definition 6.2. Let $A_1$ and $A_2$ be two unital $C^*$-algebras and $\Phi: A_1 \to A_2$ a linear map. We say that $\Phi$ is completely positive if for every integer $n \geq 1$ the map $\Phi_n: M_n(A_1) \to M_n(A_2)$ is positive in the sense that it takes positive elements in the $C^*$-algebra $M_n(A_1)$ to positive ones in $M_n(A_2)$.

If moreover $\Phi(1) = 1$ then we say that $\Phi$ is unital and in this case we have $\|\Phi_n\| = 1$ for every $n \geq 1$ by the Russo-Dye theorem (see e.g., Corollary 2.9 in [Paul02]). □

Definition 6.3. Let $A$ be a unital $C^*$-algebra, $\mathcal{H}_0$ a complex Hilbert space and $\Phi: A \to \mathcal{B}(\mathcal{H}_0)$ a unital completely positive map. Define a nonnegative sesquilinear form on $A \otimes \mathcal{H}_0$ by the formula

$$\left( \sum_{j=1}^n b_j \otimes \eta_j, \sum_{i=1}^n a_i \otimes \xi_i \right) = \sum_{i,j=1}^n (\Phi(a_i^* b_j) \eta_j | \xi_i)$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}_0$ and $n \geq 1$. In particular

\begin{equation}
\left( \sum_{j=1}^n b_j \otimes \eta_j, \sum_{j=1}^n b_j \otimes \eta_j \right) = (\Phi_n((b_j^* b_j)_{1 \leq i,j \leq n}) \eta | \eta),
\end{equation}
where $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in M_{n,1}(\mathbb{C}) \otimes \mathcal{H}_0$. Consider the linear space $N = \{ x \in A \otimes \mathcal{H}_0 \mid (x \mid x) = 0 \}$ and denote by $\mathcal{K}_0$ the Hilbert space obtained as the completion of $(A \otimes \mathcal{H}_0)/N$ with respect to the scalar product defined by $\langle \cdot \mid \cdot \rangle$ on this quotient space.

On the other hand define a representation $\tilde{\pi}$ of $A$ by linear maps on $A \otimes \mathcal{H}_0$ by

$$(\forall a, b \in A)(\forall \eta \in \mathcal{H}_0) \quad \tilde{\pi}(a)(b \otimes \eta) = ab \otimes \eta.$$ 

Then every linear map $\tilde{\pi}(a) : A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0$ induces a continuous map $(A \otimes \mathcal{H}_0)/N \to (A \otimes \mathcal{H}_0)/N$, whose extension by continuity will be denoted by $\pi_{\Phi}(a) \in \mathcal{B}(\mathcal{K}_0)$. We thus obtain a unital $*$-representation $\pi_{\Phi} : A \to \mathcal{B}(\mathcal{K}_0)$ which is called the Stinespring representation associated with $\Phi$.

Additionally, denote by $V : \mathcal{H}_0 \to \mathcal{K}_0$ the bounded linear map obtained as the composition

$$V : \mathcal{H}_0 \to A \otimes \mathcal{H}_0 \to (A \otimes \mathcal{H}_0)/N \to \mathcal{K}_0,$$

where the first map is defined by $A \ni h \mapsto 1 \otimes h \in A \otimes \mathcal{H}_0$ and the second map is the natural quotient map. Then $V : \mathcal{H}_0 \to \mathcal{K}_0$ is an isometry satisfying $\Phi(a) = V^* \pi(a)V$ for all $a \in A$.

\textbf{Remark 6.4.} The construction sketched in Definition 6.3 essentially coincides with the proof of the Stinespring theorem on dilations of completely positive maps (\cite{St55}); see for instance Theorem 5.2.1 in \cite{ER00} or Theorem 4.1 in \cite{Pau02}. Minimal Stinespring representations are uniquely determined up to a unitary equivalence; see Proposition 4.2 in \cite{Pau02}.

We also note that in the case when $\dim \mathcal{H}_0 = 1$, that is, $\Phi$ is a state of $A$, the Stinespring representation associated with $\Phi$ coincides with the Gelfand-Naimark-Segal (GNS) representation associated with $\Phi$. Thus in this case the isometry $V$ identifies with an element $h$ in $\mathcal{K}_0$ such that $\Phi(a) = (\pi(a)h \mid h)_{\mathcal{H}}$ for all $a \in A$.

We now start the preparations necessary for obtaining the realization theorem for Stinespring representations (Theorem 6.10).

\textbf{Lemma 6.5.} Let $\Phi : A \to B$ be a unital completely positive map between two $C^*$-algebras. Then for every $n \geq 1$ and every $a \in M_n(A)$ we have $\Phi_n(a)^*\Phi_n(a) \leq \Phi_n(a^*a)$.

\textbf{Proof.} Note that $\Phi_n : M_n(A) \to M_n(B)$ is in turn a unital completely positive map, hence after replacing $A$ by $M_n(A)$, $B$ by $M_n(B)$, and $\Phi$ by $\Phi_n$, we may assume that $n = 1$. In this case we may assume $B \subseteq \mathcal{B}(\mathcal{H}_0)$ for some complex Hilbert space $\mathcal{H}_0$ and then, using the notation in Definition 6.3 we have

$$\Phi(a^*a) = V^*\pi_{\Phi}(a^*a)V = V^*\pi_{\Phi}(a)^*\text{id}_{\mathcal{K}_0}\pi_{\Phi}(a)V \geq V^*\pi_{\Phi}(a)^*VV^*\pi_{\Phi}(a)V = \Phi(a)^*\Phi(a),$$

where the second equality follows since the Stinespring representation $\pi_{\Phi} : A \to \mathcal{B}(\mathcal{K}_0)$ is in particular a $*$-homomorphism. See for instance Corollary 5.2.2 in \cite{ER00} for more details.

For later use we now recall the theorem of Tomiyama on conditional expectations.

\textbf{Remark 6.6.} Let $1 \in B \subseteq A$ be two $C^*$-algebras and such that there exists a conditional expectation $E : A \to B$, that is, $E$ is a linear map satisfying $E^2 = E$, $\|E\| = 1$ and $\text{Ran} \ E = B$. Then for every $a \in A$ and $b_1, b_2 \in B$ we have $E(a^*a) = E(a^*a)^*$, $0 \leq E(a)^*E(a) \leq E(a^*a)$, and $E(b_1ab_2) = b_1E(a)b_2$. (See for instance \cite{D57} or \cite{S71}.) Additionally, $E$ is completely positive and $E(1) = 1$, and this explains why $E$ has the Schwarz property stated in the previous Lemma 6.5.

\textbf{Lemma 6.7.} Assume that $1 \in B \subseteq A$ are $C^*$-algebras with a conditional expectation $E : A \to B$ and a unital completely positive map $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ satisfying $\Phi \circ E = \Phi$, where $\mathcal{H}_0$ is a complex Hilbert space. Denote by $\pi_A : A \to \mathcal{B}(\mathcal{H}_A)$ and $\pi_B : B \to \mathcal{B}(\mathcal{H}_B)$ the Stinespring representations associated with the unital completely positive maps $\Phi$ and $\Phi|_B$, respectively. Then $\mathcal{H}_B \subseteq \mathcal{H}_A$, and for every $h_0 \in \mathcal{H}_0$ and
b ∈ B we have the commutative diagrams
\[
\begin{array}{c}
A \xrightarrow{1_{h_0}} \mathcal{H}_A \\
E \downarrow \quad P \downarrow \quad P \\
B \xrightarrow{1_{h_0}} \mathcal{H}_B \\
\end{array}
\]
where \(P : \mathcal{H}_A \to \mathcal{H}_B\) is the orthogonal projection, and \(\iota_{h_0} : A \to \mathcal{H}_A\) is the map induced by \(a \mapsto a \otimes h_0\).

**Proof.** We first check that the right-hand square is a commutative diagram. In fact, it is clear from the construction in Definition 6.3 that \(\mathcal{H}_B \subseteq \mathcal{H}_A\) and for every \(b \in B\) we have \(\pi_A(b^*)|_{\mathcal{H}_B} = \pi_B(b^*)\). In other words, if we denote by \(I : \mathcal{H}_B \hookrightarrow \mathcal{H}_A\) the inclusion map, then \(\pi_A(b^*) \circ I = I \circ \pi_B(b^*)\). Now note that \(I^* = P\) and take the adjoints in the previous equation to get \(P \circ \pi_A(b) = \pi_B(b) \circ P\).

To check that the left-hand square is commutative, first note that \(E \otimes \text{id}_{\mathcal{H}_0} : A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0\) is an idempotent mapping. To investigate the continuity of this map, let \(x = \sum_{i=1}^n a_i \otimes \xi_i \in A \otimes \mathcal{H}_0\) and note that \((E \otimes \text{id}_{\mathcal{H}_0})x = (E \otimes \text{id}_{\mathcal{H}_0})x = \Phi_n((E(a_i^*)E(a_j))_{1 \leq i,j \leq n}) \xi_i\), where \(\xi = \left(\begin{array}{c} \xi_1 \\ \vdots \\ \xi_n \end{array}\right) \in M_n(C) \otimes \mathcal{H}_0\). On the other hand \(E(a_i^*)E(a_j)_{1 \leq i,j \leq n} = E_n(a^*a) = E_n((a_i^*)_{1 \leq i,j \leq n})\), where
\[
a = \begin{pmatrix}
a_1 & \cdots & a_n \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix} \in M_n(A)
\]
and the above inequality follows by Lemma 6.3. Now, since \(\Phi_n : M_n(A) \to M_n(B)\) is a positive map, we get \(\Phi_n((E(a_i^*)E(a_j))_{1 \leq i,j \leq n}) \leq \Phi_n(E_n((a_i^*)_{1 \leq i,j \leq n}))\). Furthermore we have \(\Phi_n \circ E_n = (\Phi \circ E)_n = \Phi_n\) by hypothesis, hence \((E \otimes \text{id}_{\mathcal{H}_0})x \leq \Phi_n((a_i^*)_{1 \leq i,j \leq n}) \xi_0 = (x \mid x)\). Thus the linear map \(E \otimes \text{id}_{\mathcal{H}_0} : A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0\) is continuous (actually contractive) with respect to the semi-scalar product \((\cdot \mid \cdot)\) and thus induces a bounded linear operator \(E : \mathcal{H}_A \to \mathcal{H}_A\). Moreover, since \(E^2 = E\) and \(E(A) = B\), it follows that \(E^2 = E\) and \(E(\mathcal{H}_A) = \mathcal{H}_B\). On the other hand, it is obvious that for every \(h_0 \in \mathcal{H}\) we have \(E \circ \iota_{h_0} = \iota_{h_0} \circ E\). Hence it will be enough to prove that \(E = P\).

To this end let \(x = \sum_{i=1}^n a_i \otimes \xi_i \in A \otimes \mathcal{H}_0\) and \(y = \sum_{j=1}^m b_j \otimes \eta_j \in B \otimes \mathcal{H}_0\) arbitrary. We have
\[
\begin{aligned}
((E \otimes \text{id}_{\mathcal{H}_0})x \mid y) &= \sum_{i=1}^n (E(a_i) \otimes \xi_i) \mid \sum_{j=1}^m (b_j \otimes \eta_j) \\
&= \sum_{i,j=1}^n (\Phi(b_j^*E(a_i)) \mid \eta_j) \\
&= \sum_{i,j=1}^n (\Phi(b_j^*a_i) \mid \eta_j) = (x \mid y),
\end{aligned}
\]
where the third equality follows since \(E(ba) = bE(a)\) for all \(a \in A\) and \(b \in B\), while the next-to-last equality follows by the hypothesis \(\Phi \circ E = \Phi\). Since \(y \in B \otimes \mathcal{H}_0\) is arbitrary, the above equality shows that \((E \otimes \text{id}_{\mathcal{H}_0})x \in B \otimes \mathcal{H}_0\). This implies that \(E(\mathcal{H}_A) \subseteq B \otimes \mathcal{H}_0\), whence \(E(\mathcal{H}_A) = \mathcal{H}_B\) for all \(x \in A \otimes \mathcal{H}_0\), where \(x \mapsto \tilde{x}, A \otimes \mathcal{H}_0 \to \mathcal{H}_A\), is the canonical map obtained as the composition \(A \otimes \mathcal{H}_0 \to (A \otimes \mathcal{H}_0)/N \hookrightarrow A\).

(See Definition 6.3.) Since \(\{x \mid x \in A \otimes \mathcal{H}_0\}\) is a dense linear subspace of \(\mathcal{H}_A\), it follows that \(E = P\) throughout \(\mathcal{H}_A\), and we are done. \(\square\)

**Remark 6.8.** Under the assumptions of the previous lemma, we also obtain that \(\mathcal{H}_A = \text{span}_{\text{c}} \pi_A(U_A)\mathcal{H}_B\): By standard arguments in \(C^*\)-algebras, we have that \(A = \text{span}_{\text{c}} U_A\) or, equivalently, \(A = \text{span} U_A \cdot B\) since we have \(1 \in B\). So \(A \otimes \mathcal{H}_0 = \text{span}_{\text{c}} U_A \cdot (B \otimes \mathcal{H}_0)\) whence by quotienting and then by passing to the completion we get \(\mathcal{H}_A = \text{span}_{\text{c}} \pi_A(U_A)\mathcal{H}_B\).
Hence the mapping $\gamma$ is an isometry from $H_A$ onto $H^K$ and the inverse mapping $\gamma^{-1}$ coincides with $W$, see the remark prior to Theorem 3.2.

Remark 6.9. In the setting of Lemma 6.7 if the restriction of $\Phi$ to $B$ happens to be a nondegenerate $*$-representation of $B$ on $H_0$, then $H_B = H_0$ and $\pi_B = \Phi|_B$ by the uniqueness property of the minimal Stinespring dilation (see Remark 6.8). In this special case our Lemma 6.7 is related to the constructions of extensions of representations (see Proposition 2.10.2 in [Di64]) and induced representations of $C^*$-algebras (see Lemma 1.7, Theorem 1.8, and Definition 1.9 in [Riz74]).

In the following theorem we are using notation of Section 5.

Theorem 6.10. Assume that $B \subseteq A$ are two unital $C^*$-algebras such that there exists a conditional expectation $E: A \to B$ from $A$ onto $B$, and let $\Phi: A \to B(H_0)$ be a unital completely positive map satisfying $\Phi \circ E = \Phi$, where $H_0$ is a complex Hilbert space. Let $(\pi_A|_G_A, \pi_B|_G_B, P)$ be the Stinespring data associated with $E$ and $\Phi$. Set $\lambda: uU_B \mapsto uGB, U_A/U_B \mapsto G_A/G_B$.

Then the following assertions hold:

(a) There exists a real analytic diffeomorphism $a \mapsto (u(a), X(a), b(a))$, $G_A \to U_A \times p \times G_B^+$ so that

$$aGB = u(a) \exp(iX(a))b(a)$$

for all $a \in A$, which induces the polar decomposition in $G_A/G_B$,

$$aG_B = u(a) \exp(iX(a))G_B, \quad a \in G_A.$$

(b) The mapping $-\ast: u \exp(iX)GB \mapsto u \exp(-iX)GB$, $G_A/G_B \to G_A/G_B$ is an anti-holomorphic involutive diffeomorphism of $G_A/G_B$ such that

$$\lambda(U_A/U_B) = \{s \in G_A/G_B \mid s = s^{-\ast}\}.$$

(c) The projection

$$\exp(iX)GB \mapsto uU_B, \quad G_A/G_B \to U_A/U_B$$

has the structure of a vector bundle isomorphic to the tangent bundle $U_A \times U_B \mapsto U_A/U_B$ of the manifold $U_A/U_B$. The corresponding isomorphism is given by $u \exp(iX)GB \mapsto [(u, X)]$ for all $u \in U_A, X \in p$.

(d) Set $H(E, \Phi) := \{\gamma(h) \mid h \in H_A\} \subset \mathcal{O}(G_A/G_B, D)$ where $\gamma: H_A \to \mathcal{O}(G_A/G_B, D)$ is the realization operator defined by $\gamma(h)(aGB) = [a, P(\pi_A(a)^{-1}h)]$ for $a \in G_A$ and $h \in H_A$. Put $\gamma^U := \gamma|_{U_A/U_B}: H_A \to C^*(U_A/U_B, D^U)$ and $H^U(E, \Phi) := \{\gamma^U(h) \mid h \in H_A\}$. Denote by $\mu(a)$ the operator on the spaces $C^*(U_A/U_B, D^U)$ and $\mathcal{O}(G_A/G_B, D)$ defined by natural multiplication by $a \in G_A$. Then $H(E, \Phi)$ and $H^U(E, \Phi)$ are Hilbert spaces isometric with $H_A$. Moreover, for every $a \in G_A$ the following diagram is commutative

$$
\begin{array}{ccc}
H_A & \xrightarrow{\gamma^U} & H^U(E, \Phi) \\
\pi(a) \downarrow & & \downarrow \mu(a) \\
H_A & \xrightarrow{\gamma^U} & H^U(E, \Phi)
\end{array}
$$

that is, $\gamma \circ \pi(a) = \mu(a) \circ \gamma$.

(e) There exists an isometry $V_{E, \Phi}: H_0 \to H(E, \Phi)$ such that

$$\Phi(a) = V_{E, \Phi}(T_1\mu(a))V_{E, \Phi}^*, \quad a \in A,$$

where $T_1\mu$ is the tangent map of $\mu(\cdot)|_{H(E, \Phi)}$ at $1 \in G_A$. In fact, $T_1\mu$ is a Banach algebra representation of $A$ which extends $\mu$.

Proof. (a) Let $(\pi_A|_G_A, \pi_B|_G_B, P)$ be the Stinespring data introduced in Lemma 6.7, so that $H_A = \pi_A(G_A)H_B$ according to Remark 6.8. We have that $G_B^+ = G_B \cap G_A^+$ as a direct consequence of the fact that the $C^*$-algebras are closed under taking square roots of positive elements. Then parts (a)-(d) of the theorem follows immediately by application of Theorem 6.6, Corollary 5.8, Corollary 5.10 and Theorem 6.4.

As regards (e) note that for every $a \in A$ and $h \in H_A$,

$$T_1\mu(a)\gamma(h) = (d/dt)|_{t=0}\mu(e^{ta})\gamma(h) = (d/dt)|_{t=0}e^{ta}[\pi_A(e^{ta}\gamma(h))] = \gamma(\pi_A(a)(h)).$$
Since $\gamma$ is bijective (and isometric) we have that $T_\lambda \mu(a) = \gamma^{-1} \pi_A(a) \gamma$ for all $a \in A$, whence it is clear that $T_\lambda \mu$ becomes a Banach algebra representation (and not only a Banach-Lie algebra representation).

Now take $V_{E,\phi} := \gamma \circ V$ where $V$ is the isometry $V : H_0 \rightarrow H_A$ given in Definition 6.3. It is clear that $\gamma^* = \gamma^{-1}$ and then that $V_{E,\phi}$ is the isometry we wanted to find.

\[ \Box \]

Remark 6.11. Theorem 6.10 extends to the holomorphic setting, and for Stinespring representations, the geometric realization framework given in Theorem 5.4 of [BR07] for GNS representations. As part of such an extension we have found that the real analytic sections obtained in [BR07] are always restrictions of holomorphic sections of suitable (like-Hermitian) vector bundles on fairly natural complexifications.

Part (e) of the theorem provides us with a strong geometric view of the completely positive mappings on $C^*$-algebras $A$: such a map is the compression of the “natural action of $A^*$ (in the sense that it is obtained by differentiating the non-ambiguous natural action of $G_A$) on a Hilbert space formed by holomorphic sections of a vector bundle of the formerly referred to type.

\[ \Box \]

7. Further applications and examples

1) Banach algebra amenability

Example 7.1. Let $\mathfrak{A}$ be a Banach algebra. A virtual diagonal of $\mathfrak{A}$ is by definition an element $M$ in the bidual $\mathfrak{A}^{\ast\ast}$ such that

$$ y \cdot M = M \cdot y \quad \text{and} \quad m(M) \cdot y = y \quad (y \in \mathfrak{A}) $$

where $m$ is the extension to $(\mathfrak{A} \otimes \mathfrak{A})^{\ast\ast}$ of the multiplication map in $\mathfrak{A}$, $y \otimes y' \mapsto yy'$. The algebra $\mathfrak{A}$ is called amenable when it possesses a virtual diagonal as above. When $\mathfrak{A}$ is a $C^*$-algebra, then $\mathfrak{A}$ is amenable if and only if it is nuclear. Analogously, a dual Banach algebra $\mathfrak{M}$ is called Connes-amenable if $\mathfrak{A}$ has a virtual diagonal which in addition is normal. Then a von Neumann algebra $\mathfrak{A}$ is Connes-amenable if and only it is injective. For all these concepts and results, see [Ru02].

Let $\mathfrak{A}$ be a $C^*$-algebra and let $\mathfrak{B}$ be a von Neumann algebra. By $\text{Rep}(\mathfrak{A}, \mathfrak{B})$ we denote the set of bounded representations $\rho : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\rho(\mathfrak{A})\mathfrak{B}_s = \mathfrak{B}_s$, where $\mathfrak{B}_s$ is the (unique) predual of $\mathfrak{B}$ (recall that $\mathfrak{B}_s$ is a left Banach $\mathfrak{B}$-module). In the case $\mathfrak{B} = \mathcal{B}(\mathcal{H})$, for a complex Hilbert space $\mathcal{H}$, the property that $\rho(\mathfrak{A})\mathfrak{B}_s = \mathfrak{B}_s$ is equivalent to have $\rho(\mathfrak{A})\mathcal{H} = \mathcal{H}$, that is, $\rho$ is nondegenerate. Let $\text{Rep}_s(\mathfrak{A}, \mathfrak{B})$ denote the subset of $*$-representations in $\text{Rep}(\mathfrak{A}, \mathfrak{B})$. For a von Neumann algebra $\mathfrak{M}$, we denote by $\text{Rep}_s(\mathfrak{M}, \mathfrak{B})$ the subset of homomorphisms in $\text{Rep}(\mathfrak{M}, \mathfrak{B})$ which are ultraweakly continuous, or normal for short. As above, the set of $*$-representations of $\text{Rep}^s(\mathfrak{M}, \mathfrak{B})$ is denoted by $\text{Rep}^s(\mathfrak{M}, \mathfrak{B})$.

From now on, $\mathfrak{A}, \mathfrak{M}$ will denote a nuclear $C^*$-algebra and an injective von Neumann algebra respectively. Fix $\rho \in \text{Rep}(\mathfrak{A}, \mathfrak{B})$. The existence of a virtual diagonal $M$ for $\mathfrak{A}$ allows us to define an operator $E_{\rho} : \mathfrak{B} \rightarrow \mathfrak{B}$ by

$$ (E_{\rho}(T)x \mid x') := M(y \otimes y' \mapsto (\rho(y)T)(\rho(y')x \mid x')) \equiv \int_{\mathfrak{A} \otimes \mathfrak{A}} (\rho(y)T)\rho(y')x \mid x' \ dM(y,y') $$

where $x, x'$ belong to a Hilbert space $\mathcal{H}$ such that $\mathfrak{B} \hookrightarrow \mathcal{B}(\mathcal{H})$ canonically, and $T \in \mathfrak{B}$. In the formula, the “integral” corresponds to the Effros notation, see [CG98]. The operator $E_{\rho}$ is a bounded projection such that

$$ E_{\rho}(\mathfrak{B}) = \rho(\mathfrak{A})' := \{ T \in \mathfrak{B} \mid T \rho(y) = \rho(y)T, \ y \in \mathfrak{A} \}. $$

In fact, it is readily seen that $\|E_{\rho}\| \leq \|M\| \|\rho\|^2$, so that $E_{\rho}$ becomes a conditional expectation provided that $\|M\| = \|\rho\| = 1$. For instance, if $\rho$ is a $*$-homomorphism then its norm is one, see page 7 in [Pau02]. The existence of (normal) virtual diagonals of norm one in (dual) Banach algebras is not a clear fact in general, but it is true, and not simple, that such (normal) virtual diagonals exist for (injective von Neumann) nuclear $C^*$-algebras, see page 188 in [Riu02].

For $\rho \in \text{Rep}(\mathfrak{A}, \mathfrak{B})$ and $T \in \mathfrak{B}$, let $T \rho T^{-1} \in \text{Rep}(\mathfrak{A}, \mathfrak{B})$ defined as $(T \rho T^{-1})(y) := T \rho(y)T^{-1} \ (y \in \mathfrak{A})$. Put

$$ G(\rho) := \{ T \rho T^{-1} \mid T \in G_\mathfrak{B} \} \quad \text{and} \quad U(\rho) := \{ T \rho T^{-1} \mid T \in U_\mathfrak{B} \}. $$
The set $\mathcal{S}(\rho)$ is called the similarity orbit of $\rho$, and $\mathcal{U}(\tau)$ is called the unitary orbit of $\tau \in \text{Rep}_s(\mathfrak{A}, \mathfrak{B})$. It is known that $\text{Rep}(\mathfrak{A}, \mathfrak{B})$, endowed with the norm topology, is the direct union of orbits $\mathcal{S}(\rho)$. Moreover, each orbit $\mathcal{S}(\rho)$ is a homogeneous Banach manifold with a reductive structure induced by the connection form $E_\rho$. In the same way, $\text{Rep}_s(\mathfrak{A}, \mathfrak{B})$ is the disjoint union of orbits $\mathcal{U}(\tau)$, and the restriction of $E_\rho$ on $u_B$ is a connection form which induces a homogeneous reductive structure on $\mathcal{U}(\tau)$ — see [ACS95], [CG98], and [Ga06]. We next compile some more information, about the similarity and unitary orbits, which is obtained on the basis of results of the present section.

Let $\rho \in \text{Rep}(\mathfrak{A}, \mathfrak{B})$. As it was proved in [Bu81] and independently in [Ch81] (see also [Ha83]), there exists $\tau \in \text{Rep}_s(\mathfrak{A}, \mathfrak{B}) \cap \mathcal{S}(\rho)$, whence $\mathcal{S}(\rho) = \mathcal{S}(\tau)$. Hence, without loss of generality, $\rho$ can be assumed to be a $*$-representation, so that $\|\rho\| = 1$. Moreover, since we are assuming that $\mathfrak{A}$ is nuclear, we can choose a virtual diagonal $M$ of $\mathfrak{A}$ of norm one. Thus the operator $E_\rho$ is a conditional expectation. Set $A := \mathfrak{B}$, $B := \rho(\mathfrak{A})'$. With this notation, $\mathcal{S}(\rho) = G_A/G_B$ and $\mathcal{U}(\rho) = U_A/U_B$ diffeomorphically.

For $X \in p_p := \text{Ker} E_\rho \cap u_A$, let $[X]$ denote the equivalence class of $X$ under the adjoint action of $U_B$ on $p_p$, considered in Theorem 6.10. Also, set $e^{iX} := \exp(\rho(X))$.

**Corollary 7.2.** Let $\mathfrak{A}$ be a nuclear C$^*$-algebra and let $\mathfrak{B}$ be a von Neumann algebra. The following assertions hold:

(a) Each connected component of $\text{Rep}(\mathfrak{A}, \mathfrak{B})$ is a similarity orbit $\mathcal{S}(\rho)$, for some $\rho \in \text{Rep}_s(\mathfrak{A}, \mathfrak{B})$. Moreover, each orbit $\mathcal{S}(\rho)$ is the disjoint union

$$\mathcal{S}(\rho) = \bigcup_{[X] \in p_p/U_B} \mathcal{U}(e^{iX} \rho e^{-iX})$$

where $\mathcal{U}(e^{iX} \rho e^{-iX})$ is connected, for all $[X] \in p_p/U_B$.

(b) The similarity orbit $\mathcal{S}(\rho)$ is a complexification of the unitary orbit $\mathcal{U}(\rho)$ with respect to the involutive diffeomorphism $ue^{iX} \rho e^{-iX}u^{-1} \mapsto ue^{-iX} \rho e^{iX}u^{-1}$ ($u \in U_B$).

(c) The mapping $ue^{iX} \rho e^{-iX}u^{-1} \mapsto upu^{-1}$, $\mathcal{S}(\rho) \rightarrow \mathcal{U}(\rho)$ is a continuous retraction which defines a vector bundle diffeomorphic to the tangent bundle $U_A \times_{U_B} p_p \rightarrow \mathcal{U}(\rho)$ of $\mathcal{U}(\rho)$.

(d) Let $\mathcal{H}_0$ be a Hilbert space such that $\mathfrak{B} \hookrightarrow \mathcal{B}(\mathcal{H}_0)$. For every $\rho \in \text{Rep}(\mathfrak{A}, \mathfrak{B})$ there exists a Hilbert space $\mathcal{H}_0(\rho)$ isometric with $\mathcal{H}_0$, which is formed by holomorphic sections of a like-Hermitian vector bundle with base $\mathcal{S}(\rho)$. Moreover, $\mathfrak{B}$ acts continuously by natural multiplication on $\mathcal{H}_0(\rho)$, and the representation $R$ obtained by transferring $\rho$ on $\mathcal{H}_0(\rho)$ coincides with multiplication by $\rho$; that is, $R(y)F = \rho(y) \cdot F$ for all $y \in \mathfrak{A}$ and section $F \in \mathcal{H}_0(\rho)$.

**Proof.** (a) As said before, every similarity orbit of $\text{Rep}(\mathfrak{A}, \mathfrak{B})$ is of the form $\mathcal{S}(\rho)$ for some $\rho \in \text{Rep}_s(\mathfrak{A}, \mathfrak{B})$. Since $\mathfrak{A}$ is a von Neumann algebra, the set of unitaries $U_A = U_B$ is connected whence it follows (as in Remark 2.14) that the orbits $\mathcal{S}(\rho)$ and $\mathcal{U}(e^{iX} \rho e^{-iX})$ are connected for all $X \in p_p$. For $X,Y \in p_p$, we have $\mathcal{U}(e^{iX} \rho e^{-iX}) = \mathcal{U}(e^{iY} \rho e^{-iY})$ if and only if there exists $u \in U_A$ such that $e^{iY} \rho e^{-iY} = ue^{iX} \rho e^{-iX}$, which means that $u \in U_B$ and $Y = uXu^{-1}$ (see Theorem 5.6). Hence $[X] = [Y]$. Finally, by Theorem 5.6 again we have $\mathcal{S}(\rho) = \bigcup_{[X] \in p_p/U_B} \mathcal{U}(e^{iX} \rho e^{-iX})$.

(b) This is Theorem 6.10 (b).

(c) This follows by Theorem 6.10 (c).

(d) Given $\rho$ in $\text{Rep}(\mathfrak{A}, \mathfrak{B})$, there is $\tau = \tau(\rho)$ in $\text{Rep}_s(\mathfrak{A}, \mathfrak{B})$ such that $\mathcal{S}(\rho) = \mathcal{S}(\tau)$. Now we fix a virtual diagonal of $\mathfrak{A}$ of norm one and then define the conditional expectation $E_\rho \equiv E_{\tau(\rho)}$ as prior to this corollary. So $E_\rho: \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_0)$ is a completely positive mapping and one can apply Theorem 6.10 (d). As a result, one gets a Hilbert space $\mathcal{H}(\rho) := \mathcal{H}(E_\rho, E_\rho)$ of holomorphic sections of a like-Hermitian bundle on $\mathcal{S}(\rho) = G_B/G_B$ (where $B = \rho(\mathfrak{A})'$), and an isometry $V_\rho := V_{E_\rho}: \mathcal{H}(\rho) \rightarrow \mathcal{H}(\rho)$, satisfying $E_\rho(\rho(y)) = V_\rho^* T_{11} \mu(\rho(y))V_\rho$ for all $y \in \mathfrak{A}$, in the notations of Theorem 6.10. Note that $V_{E_\rho}^* V_\rho = 1$ and therefore the correspondence $y \mapsto V_\rho \rho(y) V_\rho^*$ defines a (bounded) representation of $\mathcal{H}(\rho)$. Now take $\mathcal{H}_0(\rho) := V(\mathcal{H}_0)$ and define $R(y)$ as the restriction of $V_\rho \rho(y) V_\rho^*$ on $\mathcal{H}_0(\rho)$ for every $y \in \mathfrak{A}$. Clearly, $R$ is the transferred representation of $\rho$ from $\mathcal{H}_0$ to $\mathcal{H}_0(\rho)$. 


Also, for every \( F \in \mathcal{H}_0(\rho) \) there exists \( h_0 \in \mathcal{H}_0 \) such that \( F = V_\rho(1 \otimes h_0) \), that is, \( F(aG_B) = [a, P(a^{-1} \otimes h_0)] \) for all \( a \in G_\mathfrak{A} \), where \( P \) is as in Lemma 6.7. Then, for \( y \in \mathfrak{A} \),
\[
R(y)F = R(y)V_\rho(1 \otimes h_0) = V_\rho \rho(y)V_\rho^*V_\rho(1 \otimes h_0) = V_\rho \rho(y)(1 \otimes h_0)
\]
\[
= V_\rho(\rho(y) \otimes h_0) = T_1\mu(\rho(y)) \quad V_\rho(1 \otimes h_0) = \rho(y) : F,
\]
as we wanted to show.  

\[\square\]

**Remark 7.3.**
(i) The first part of Corollary 7.2 (a) was already well known (see for example [ACS95]). In the decomposition of the second part, the orbit \( \Omega(e^{iX}pe^{-iX}) \) for \( X = 0 \) corresponds to the unitary orbit of \( \rho \). So the disjoint union supplies a sort of configuration of the similarity orbit \( \mathfrak{S}(\rho) \) by relation with the unitary orbit \( \mathfrak{U}(\rho) \).

(ii) Parts (a), (b), (c) of Corollary 7.2 are consequences of the Porta-Recht decomposition given in [PR94], see (5.3). Such a decomposition has been considered previously in relation with similarity orbits of nuclear \( C^* \)-algebras, through in a different perspective, see Theorem 5.7 in [ACS95], for example.

(iii) Corollary 7.2 admits a version entirely analogous for injective von Neumann algebras \( \mathfrak{M} \) (replacing the nuclear \( C^* \)-algebra \( \mathfrak{A} \) of the statement) and representations in Rep\(^*\)\((\mathfrak{A}, \mathfrak{B})\) and Rep\(^*\)\((\mathfrak{A}, \mathfrak{B})\).

Proofs are similar to the nuclear, \( C^* \), case. For the analog of (d) one needs to take a normal virtual diagonal of \( \mathfrak{M} \) of norm one.

(iv) Corollary 7.2 applies in particular to locally compact groups \( G \) for which the group \( C^* \)-algebra \( C^*(G) \) is amenable, see [ACS95] and [CG94]. When the group is compact the method to define the expectation \( E_\rho \) works for every representation \( \rho \) taking values in any Banach algebra \( A \). We shall see a particular example of this below, involving Cuntz algebras.

\[\square\]

2) **Completely positive mappings**

Let \( A \) be a complex unital \( C^* \)-algebra, with unit \( 1 \), included in the algebra \( B(\mathcal{H}) \) of bounded operators on a Hilbert space \( \mathcal{H} \). Assume that \( \Phi : A \to B(\mathcal{H}) \) is a unital, completely bounded mapping. (In the following we shall assume freely that \( \mathcal{H} \) is separable, when necessary.)

**Lemma 7.4.** Given \( \Phi \) as above and \( u \in G_A \), let \( \Phi_u \) denote the mapping \( \Phi_u := u\Phi(u^{-1} \cdot u)u^{-1} \). Then

(i) For every \( v \in G_A \), \( \Phi_u \) is completely bounded and \( \|\Phi_u\|_{cb} \leq \|\Phi\|_{cb} \cdot \|u\| \cdot \|u^{-1}\|^2 \).

(ii) If \( \Phi \) is completely positive then \( \Phi_u \) is completely positive for every \( u \in U_A \).

**Proof.** (i) Let \( u \) be a natural number. Take \( f = (f_1, \cdots , f_n), h = (h_1, \cdots , h_n) \in \mathcal{H}^n \) and \( (a_{ij})_{ij} \in M_n(A) \) all of them of respective norms less than or equal to 1. In the following we shall think of \( f \) and \( g \) in their column version. Then, for \( u \in G_A \), we have
\[
\|\Phi_u^n(\Phi_{a_{ij}}f) h \|_{\mathcal{H}^n} = |\sum_{i,j} \Phi_u^n(\Phi_{a_{ij}}f) h_i |_{\mathcal{H}}| \\
\leq \|\Phi_u^n(\Phi_{a_{ij}}f) h_i |_{\mathcal{H}}|_{\mathcal{H}^n} \| (u^{-1} f_j) h_i |_{\mathcal{H}^n} \\
= \|\Phi_u^n(\Phi_{a_{ij}}f) h_i |_{\mathcal{H}}|_{\mathcal{H}^n} \| u^{-1} \| \|u\| \|u^{-1}\|^2 \|u\| \|u^*\|.
\]

(ii) Assume now that \( \Phi \) is completely positive. For natural \( n \), take \( (a_{ij})_{ij} \geq 0 \) in \( M_n(A) \) and \( h = (h_1, \cdots , h_n) \in \mathcal{H}^n \). Then
\[
\Phi_u^n(\Phi_{a_{ij}}f) h \|_{\mathcal{H}^n} = ((\sum_{j=1}^n \Phi_u(a_{ij})h_j, \cdots , \sum_{j=1}^n \Phi_u(a_{nj})h_j) \| h \}_{\mathcal{H}^n} \\
= \sum_{i,j=1}^n \Phi_u(a_{ij})h_i \| h_i \}_{\mathcal{H}} = \sum_{i,j=1}^n (\Phi_u^n(b_{ij}f) \| f \}_{\mathcal{H}^n}
\]
where \( f = u^{-1}h, b_{ij} = u^{-1}a_{ij}u, u \in U_A \). So \( (b_{ij})_{ij} \geq 0 \) in \( M_n(A) \) and, since \( \Phi \) is completely positive, we conclude that \( \Phi_u^n(b_{ij}) \geq 0 \) as we wanted to show.  

\[\square\]
Now let \( \Phi : A \to \mathcal{B}(\mathcal{H}) \) be a fixed, unital completely positive mapping. By Proposition 3.5 in [Pau02], \( \Phi \) is completely bounded. According to the preceding Lemma 7.4, if \( \mathcal{U}(\Phi) \) and \( S(\Phi) \) denote, respectively, the unitary orbit \( \mathcal{U}(\Phi) := \{ \Phi_u \mid u \in U_A \} \) and the similarity orbit \( S(\Phi) := \{ \Phi_u \mid u \in G_A \} \) of \( \Phi \), there are natural actions of \( G_A \) on \( S(\Phi) \) and of \( U_A \) on \( \mathcal{U}(\Phi) \), under usual conjugation. Note that the elements of the orbit \( S(\Phi) \) are completely bounded maps but they do not need to be completely positive.

Let \( G(\Phi) := \{ u \in G_A \mid \Phi_u = \Phi \} \) and \( U(\Phi) := G(\Phi) \cap U_A \).

**Corollary 7.5.** In the above notation, \( S(\Phi) = G_A/G(\Phi) \) and \( \mathcal{U}(\Phi) = U_A/U(\Phi) \).

**Proof.** It is enough to observe that \( G(\Phi) \) and \( U(\Phi) \) are the isotropy subgroups of the actions of \( G_A \) on \( S(\Phi) \) and of \( U_A \) on \( \mathcal{U}(\Phi) \), respectively.

Note that \( G(\Phi) \) is defined by the family of polynomial equations

\[
\varphi(\Phi(axa^{-1}) - a\Phi(x)a^{-1}) = 0, \quad x \in A, \varphi \in \mathcal{B}(\mathcal{H})_*, \ a \in G_A
\]

on \( G_A \times G_A \), so \( G(\Phi) \) is algebraic and a Banach-Lie group with respect to the relative topology of \( A \) (see for instance the Harris-Kaup theorem in [Up85]). To see when the isotropy groups \( G(\Phi) \) and \( U(\Phi) \) are Banach-Lie subgroups of \( G_A \), we need to compute their Lie algebras \( g(\Phi) \) and \( u(\Phi) \), respectively, and to see whether they are complemented subspaces of \( A \).

**Lemma 7.6.** In the above notation we have \( g(\Phi) = \{ X \in A \mid (\forall a \in A) \quad \Phi([a,X]) = [\Phi(a),X]\} \), and therefore \( u(\Phi) = \{ X \in A \mid (\forall a \in A) \quad \Phi([a,X]) = [\Phi(a),X]\} \).

**Proof.** To prove the inclusion “\( \subseteq \)” just note that if \( X \in A \) and \( e^iX := \exp(tX) \in G(\Phi) \), then for every \( a \in A \) we get \( \Phi(e^{iX}ae^{-iX}) = e^{iX}\Phi(a)e^{-iX} \) for all \( t \in \mathbb{R} \). Hence by differentiating in \( t \) and taking values at \( t = 0 \) we obtain \( \Phi(axa) = \Phi(a)X - X\Phi(a) \); that is, \( \Phi([a,X]) = [\Phi(a),X] \).

Now let \( X \) in the right-hand side of the first equality from the statement. Then \( A \) is an invariant subspace for the mapping \( adX : [\cdot,Z] : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \), since \( X \in A \). In addition, \( \Phi \circ (adX)|_A = (adX) \circ \Phi \). Hence for every \( t \in \mathbb{R} \) and \( n \geq 0 \) we get \( \Phi \circ (tadX)^n|_A = (tadX)^n \circ \Phi \), whence \( \Phi \circ \exp(t(adX)|_A) = \exp(tadX) \circ \Phi \). Since \( \exp(tadX)b = e^{tX}be^{-tX} \) for all \( b \in \mathcal{B}(\mathcal{H}) \), it then follows that \( e^{tX} \in G(\Phi) \) for all \( t \in \mathbb{R} \), whence \( X \in g(\Phi) \). The remainder of the proof is now clear.

As regards the description of the isotropy Lie algebra \( g(\Phi) \) in Lemma 7.6, let us note the following fact:

**Proposition 7.7.** The isotropy Lie algebra \( g(\Phi) \) is a closed involutive Lie subalgebra of \( A \). If the range of \( \Phi \) is contained in the commutant of \( A \) then \( g(\Phi) \) is actually a unital \( C^* \)-subalgebra of \( A \), given by \( g(\Phi) = \{ X \in A \mid \Phi(ax) = \Phi(Xa) \quad \text{for all} \ a \in A \} \). In this case, \( G_{g(\Phi)} = G(\Phi) \).

**Proof.** It is clear from Lemma 7.4 that \( g(\Phi) \) is a closed linear subspace of \( A \) which contains the unit \( 1 \). Moreover, since \( \Phi(a^*) = \Phi(a)^* \) for all \( a \in A \) (this is automatic by the Stinespring’s dilation theorem, for instance), then for \( X \in g(\Phi) \) and \( a \in A \) we have \( \Phi([X^*,a]) = \Phi([a^*,X])^* = \Phi([a,X])^* = [\Phi(a^*),X]^* = [X^*,\Phi(a)] \) whence \( g(\Phi) \) is stable under involution as well. The fact that \( g(\Phi) \) is a Lie subalgebra of \( A \) follows by Theorem 4.13 in [Be00] (see the proof there).

If the range of \( \Phi \) is contained in the commutant of \( A \), then \( g(\Phi) = \{ X \in A \mid \Phi(ax) = \Phi(Xa) \quad \text{for all} \ a \in A \} \), and so \( g(\Phi) \) is a \( C^* \)-subalgebra of \( A \). Finally, note that \( u \in G_{g(\Phi)} \) if and only if \( u \in G_A \) and \( \Phi(uau^{-1}) = \Phi(a) = a\Phi(a)u^{-1} \) (since \( \Phi(1) \subseteq A' \)), if and only if \( u \in G(\Phi) \).

The condition in the above mapping for \( \Phi \) to be contained in the commutant of \( A \) holds if for instance, \( A \) is a state of \( A \). Next, we give another example suggested by Example 7.1. For a \( C^* \)-algebra \( \mathfrak{A} \) and von Neumann algebra \( A \) with predual \( A_* \), let \( \rho : \mathfrak{A} \to A \) be a bounded \( * \)-homomorphism such that \( \rho(\mathfrak{A})A_* = A_* \). Denote \( \rho^* \) the (generic) weak operator topology in a von Neumann algebra.

**Corollary 7.8.** Assume that \( \mathfrak{A} \) is a nuclear \( C^* \)-algebra or an injective von Neumann algebra (in the second case we assume in addition that \( \rho \) is normal), and that \( A = \rho(\mathfrak{A})w^* \). Let \( \Phi = \rho : \mathfrak{A} \to A \) be a conditional expectation associated with \( \rho \) as in Example 7.7. Then \( B := \Phi(A) \subseteq A' \) and therefore \( g(\Phi) \) is a von Neumann subalgebra of \( A \). Also, \( B \) is commutative and so it is isomorphic to an algebra of \( L^\infty \) type.
Proof. From \( \Phi(A) = \rho(\mathfrak{A})' \) and \( A = \overline{\rho(\mathfrak{A})}'^* \) it is readily seen (recall that \( A_\ast \) is an \( A \)-bimodule for the natural module operations) that \( \Phi(a) \) commutes with every element of \( A \) for all \( a \in A \). The remainder of the corollary is clear. \( \square \)

It is not difficult to find representations as those of the preceding corollary. If \( \pi: \mathfrak{A} \to \mathfrak{B} \) is a representation as in Example 7.1 then it is enough to take \( A := \pi(\mathfrak{A})' \) in \( \mathfrak{B} \), and \( \rho: \mathfrak{A} \to A \) defined by \( \rho(y) := \pi E(y) \) \( \,(y \in \mathfrak{A})\), to obtain a representation satisfying the hypotheses of Corollary 7.8. It is straightforward to check that \( A \) is a \( C^* \)-algebra and, moreover, that \( A \) is a dual Banach space. In effect, if \( \mathfrak{B}_s \) is the predual of \( \mathfrak{B} \) and \( A \) is the pre-annihilator subspace of \( A \) in \( \mathfrak{B}_s \), then the quotient \( \mathfrak{B}_s / A \) is a predual of \( A \), and an \( A \)-submodule of \( A^* \), such that \( \rho(\mathfrak{A})(\mathfrak{B}_s / A) = \mathfrak{B}_s / A \). Note that in the case when \( \mathfrak{B} = \mathcal{B}(\mathcal{H}) \) the von Neumann’s bicommutant theorems says that \( A = \pi(\mathfrak{A})'' \).

3) Conditional expectations

As regards the isotropy group \( G(\Phi) \) of a completely positive map \( \Phi: A \to \mathcal{B}(\mathcal{H}) \), we are going to see that it is actually a Banach-Lie subgroup of \( G_A \) (in the sense that the isotropy Lie algebra \( \mathfrak{g}(\Phi) \) is a complemented subspace of \( A \)) in the important special case when \( \Phi \) is a faithful normal conditional expectation.

Thus, let assume in this subsection that \( \Phi = E \) is a faithful, normal, conditional expectation \( E: A \to B \), where \( A \) and \( B \) are von Neumann algebras with \( B \subseteq A \subseteq \mathcal{B}(\mathcal{H}) \). In this case all of the elements in the unitary orbit \( U(E) \) are conditional expectations, whereas all we can say about the elements in the similarity orbit \( S(E) \) is that they are completely bounded quasi-expectations. We would like to present \( S(E) \) and \( U(E) \) as examples of the theory given in the previous Theorems 5.3 and/or 5.6 or even Section 6 of this paper.

Denote \( A_E := \{ x \in \mathcal{B}' \cap A \mid E(ax) = E(xa), a \in A \} \) and fix a faithful, normal state \( \varphi \) on \( B \). (Such a faithful state exists if the Hilbert space \( \mathcal{H} \) is separable.) The set \( A_E \) is a von Neumann subalgebra of \( A \) and, using the modular group of \( A \) induced by the gauge state \( \psi := \varphi \circ E \), it can be proven that there exists a faithful, normal, conditional expectation \( F: A \to A_E \) such that \( E \circ F = F \circ E \) and \( \psi \circ F = \psi \) (see Proposition 4.5 in [AS01]). Set \( \Delta = E + F - EF \). Then \( \Delta \) is a bounded projection from \( A \) onto \( \Delta(A) = A_E + B = (A_E \cap \ker E) \oplus B \).

By considering the connected 1-component \( G(E)^0 = G_{A_E} \cdot G_B \) of the isotropy group \( G(E) \) (see Proposition 3.3 in [AS01]), the existence of \( \Delta \) implies that \( G(E) \) is in fact a Banach-Lie subgroup of \( G_A \), the orbits \( S(E) \) and \( U(E) \) are homogeneous Banach manifolds, and the quotient map \( G_A \to S(E) \cong G_A / G(E) \) is an analytic submersion, see Corollary 4.7 and Theorem 4.8 in [AS01]. Also, the following assertions hold:

Proposition 7.9. In the notations from above and from the first subsection, \( \Delta(A) = \mathfrak{g}(E) \). In particular, \( A \) splits trough \( \mathfrak{g}(E) \) and \( \mathfrak{g}(E) \) is a \( w^* \)-closed Lie subalgebra of \( A \).

Proof. By Theorem 4.8 in [AS01] the quotient mapping \( G_A \to S(E) = G_A / G(E) \) is an analytic submersion. In fact the kernel of its differential is \( \mathfrak{g}(E) \) (see Theorem 8.19 in [Up85]). Also, \( \mathfrak{g}(E) := T_1(G(E)) = \Delta(A) \) by Proposition 4.6 in [AS01]. \( \square \)

Now let \( \Phi: A \to \mathcal{B}(\mathcal{H}_0) \) be any unital completely positive map such that \( \Phi \circ E = \Phi \) and apply Stinespring’s dilation procedure to the mapping \( \Phi \) and the conditional expectation \( E: A \to B \). Thus, for \( J = A, B \) there are the Hilbert spaces \( \mathcal{H}_J(\Phi) \) and (Stinespring) representations \( \pi_J: J \to \mathcal{B}(\mathcal{H}_J(\Phi)) \) such that \( \mathcal{H}_B(\Phi) \subseteq \mathcal{H}_A(\Phi) \) and \( \pi_B(u) = \pi_A(u)|_{\mathcal{H}_B(\Phi)} \) for each \( u \in B \), as given in Lemma 6.7. Denote by \( P: \mathcal{H}_A(\Phi) \to \mathcal{H}_B(\Phi) \) the corresponding orthogonal projection.

We are going to construct representations of the intermediate groups in the sequence

\[ G_B \subseteq G(E)^0 \subseteq G(E) \subseteq G_A. \]

For this purpose set \( \mathcal{H}_E(\Phi) := \overline{\operatorname{span}(\pi_A(G(E)|_{\mathcal{H}_B(\Phi)}) \cap \mathcal{H}_E(\Phi)} \) and \( P_E \) the orthogonal projection from \( \mathcal{H}_A(\Phi) \) onto \( \mathcal{H}_E(\Phi) \). We have that \( \overline{\operatorname{span}(\pi_A(G_A)|_{\mathcal{H}_E(\Phi)}) = \mathcal{H}_A(\Phi) \) since \( \overline{\operatorname{span}(\pi_A(G_A)|_{\mathcal{H}_B(\Phi)}) = \mathcal{H}_A(\Phi) \) by Remark 6.8. For every \( u \in G(E) \), put \( \pi_E(u) := \pi_A(u)|_{\mathcal{H}_E(\Phi)} \). Then \( \pi_E(u)(\mathcal{H}_E(\Phi)) \subseteq \mathcal{H}_E(\Phi) \) and so \( (\pi_A, \pi_E, P_E) \) is a data in the sense of Definition 8.10 (with holomorphic \( \pi_A \) and \( \pi_E \)). Similarly, set
Remark 7.12. In connection with the commutative diagram of Corollary 7.10, note that since \( G(\Phi) \) is the connected 1-component of \( G(E) \), it follows that the arrow \( G_A/G(E)^0 \to G_A/G(E) = S(E) \) is actually a covering map whose fiber is the Weyl group \( G(E)/G(E)^0 \) of the conditional expectation \( E \) (cf. [AS01] and the references therein).

Remark 7.13. It is interesting to observe how Corollary 7.10 looks in the case when \( g(E) \) is an associative algebra, as in the second part of Proposition 7.7.

Thus let assume that for a conditional expectation \( E: A \to A' \) as in former situations we have that \( B := E(A) \subseteq A' \). Then \( B \) is commutative and \( A \subseteq B' \) (note that \( B' \) need not be commutative; in other words, \( B \) is not maximal abelian). Hence, by Proposition 7.7,

\[
g(E) = \{ X \in A \mid E(aX) = E(Xa), a \in A \} = \{ X \in B' \cap A \mid E(aX) = E(Xa), a \in A \} := A_E.
\]
By Proposition 7.9, $A_E + B = \Delta(A) = g(E) = A_E$ whence $B \subseteq A_E$. Also, as regards to groups, Proposition 7.9 applies to give $G_{g(E)} = G(E)$ whence we have

$$G(E) = G_{g(E)} = G_{A_E} \subseteq G_{A_E} \cdot GB = G(E)^0 \subseteq G(E),$$

and we obtain that $G(E)^0 = G(E)$. This implies that the bundles $D_E^0 \to G_A/G(E)^0$ and $D_E \to S(E)$ of Corollary 7.10 coincide.

Moreover, from the fact that $B \subseteq A_E = g(E)$ it follows that $F \circ E = E$ where $F$ is the conditional expectation given prior to Proposition 7.9. In fact, for $a \in A$, $E(a) \in A_E = F(A)$ so there is some $a' \in A$ such that $E(a) = F(a')$. Then $(FE)(a) = F(F(a')) = (FF)(a') = F(a') = E(a)$ as required. Since $FE = EF$ we have eventually that $FE = EF = E$.

Suppose now that $\Phi : A \to B(H_0)$ is a completely positive mapping such that $\Phi \circ E = \Phi$. Then $\Phi \circ F = (\Phi \circ E) \circ F = \Phi \circ (EF) = \Phi \circ E = \Phi$, and one can use again the argument preceding Corollary 7.10 to find vector bundles with corresponding Hilbert spaces (fibers) and representations $\pi_A : A \to B(H_A(\Phi))$, $\pi_{A_E} : A_E \to B(H_{A_E}(\Phi))$ and so on. In particular, from $E|_{A_E} : A_E \to B$ one gets $H_{A_E}(\Phi) = \text{span} \{ \pi_{A_E}(g(E))H_B(\Phi) \} = \text{span} \pi_{A}(g(E))H_B(\Phi) = H_E(\Phi)$. Hence, in this case, the bundle $D_E \to S(E)$ is a Stinespring bundle with respect to the data $\pi_{A_E}|_{g(E)}$, $\pi_B|_{G_B}$ (and the corresponding projection) to which Theorem 6.10 can be applied.

More precisely, part (c) of that theorem implies that $S(E)$ is diffeomorphic to the tangent bundle $U_A \times_{U(E)} p^E$ of $U(E)$, where $p^E = \text{ker} E \cap u_A$, in the same way as $G_A/G_B$ is diffeomorphic to $U_A \times_{U_B} p^E$. $p^E = \text{ker} E \cap u_A$.

\[\square\]

4) Representations of Cuntz algebras

We wish to illustrate the theorem on geometric realizations of Stinespring representations by an application to representations of Cuntz algebras. For the sake of simplicity we shall be working in the classical setting ([Cu77]), although a part of what we are going to do can be extended to more general versions of these C*-algebras (see [CK80], [Pi97], and also [DPZ98]) or to more general C*-dynamical systems.

Example 7.14. Let $N \in \{2, 3, \ldots \} \cup \{\infty\}$ and denote by $O_N$ the C*-algebra generated by a family of isometries $\{v_j\}_{0 \leq j < N}$ that act on the same Hilbert space and satisfy the condition that their ranges are mutually orthogonal, and in addition $v_0v_0^* + \cdots + v_{N-1}v_{N-1}^* = 1$ in the case $N \neq \infty$. The Cuntz algebra $O_N$ has a canonical uniqueness property with respect to the choice of the generators $\{v_j\}_{1 \leq j < N}$ subject to the above conditions (see [Cu77]). In particular, this implies that there exists a pointwise continuous gauge *-automorphism group parameterized by the unit circle, $\lambda \mapsto \tau(\lambda) = \tau_\lambda$, $\mathbb{T} \to \text{Aut}(O_N)$, such that $\tau_\lambda(v_j) = \lambda v_j$ if $0 < j < N$ and $\lambda \in \mathbb{T}$. For every $m \in \mathbb{Z}$ we denote

\begin{equation}
O_N^{(m)} = \{ x \in O_N \mid (\forall \lambda \in \mathbb{T}) \quad \tau_\lambda(x) = \lambda^m x\}
\end{equation}

the spectral subspace associated with $m$, and then $F_N := O_N^{(0)}$ (the fixed-point algebra of the gauge group). For every $m \in \mathbb{Z}$ we have a contractive surjective linear idempotent mapping $E^{(m)} : O_N \to O_N^{(m)}$ defined by

\begin{equation}
(\forall x \in O_N) \quad E^{(m)}(x) = \int_{\mathbb{T}} \lambda^{-m} \tau_\lambda(x) d\lambda,
\end{equation}

which is a faithful conditional expectation in the case $m = 0$ (see for instance Theorem V.4.3 in [Da96]). We shall denote $E^{(0)} = E$ for the sake of simplicity.\[\square\]

The following statement is inspired by some remarks from Section 2 in [La93b]. It shows that, under specific hypothesis on the C*-algebras from Theorem 6.10, the corresponding reproducing kernel Hilbert space has a circular symmetry that resembles the one of the classical spaces of holomorphic functions on the unit disk. Thus we get series expansions and the natural setting of harmonic analysis in the spaces of bundle sections associated with completely positive maps.

Corollary 7.15. Let $N \in \{2, 3, \ldots \} \cup \{\infty\}$, a completely positive unital map $\Phi : O_N \to B(H_0)$, and the corresponding Stinespring representation $\pi_\Phi : O_N \to B(K_0)$, and isometry $V : H_0 \to K_0$ such that
\[ \Phi = \text{V}^*\pi\text{V}. \]  
Put \( \tilde{\mathcal{H}}_0 := \text{V}(\mathcal{H}_0) \). Then the condition \( \Phi \circ E = \Phi \) is satisfied if and only if \( \Phi \) is gauge invariant, in the sense that for each \( \lambda \in \mathbb{T} \) we have \( \Phi \circ \tau_\lambda = \Phi \). In addition, if this is the case, then the following assertions hold:

(a) Consider the geometric realization \( \gamma: K_0 \to \mathcal{H}(E, \Phi) \) of the Stinespring representation \( \pi_\Phi \) and let \( \Pi: D \to G_{O_N}/G_{F_N} \) be the corresponding homogeneous vector bundle. Then the gauge automorphism group of \( O_N \) induces smooth actions \( \tau \) and \( \tau \) of the circle group \( \mathbb{T} \) on the total space \( D \) and the base \( G_{O_N}/G_{F_N} \), respectively, such that the diagram

\[
\begin{array}{ccc}
\mathbb{T} \times D & \xrightarrow{\tau} & D \\
\downarrow \text{id} \times \Pi & & \downarrow \Pi \\
\mathbb{T} \times (G_{O_N}/G_{F_N}) & \xrightarrow{\tau} & G_{O_N}/G_{F_N}
\end{array}
\]

is commutative. The action on the base of the vector bundle also commutes with the natural involutive diffeomorphism thereof.

(b) If for all \( m \in \mathbb{Z} \) we denote by \( \mathcal{H}(E, \Phi)^{(m)} \) the closed linear subspace generated by \( \gamma(\pi_\Phi(O_N^{(m)}))\tilde{\mathcal{H}}_0 \), then we have the orthogonal direct sum decomposition \( \mathcal{H}(E, \Phi) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}(E, \Phi)^{(m)} \) and each term of this decomposition is \( G_{F_N} \)-invariant.

(c) For each \( m \in \mathbb{Z} \), the orthogonal projection \( P_{E, \Phi}^{(m)}: \mathcal{H}(E, \Phi) \to \mathcal{H}(E, \Phi)^{(m)} \) is given by the formula

\[
(P_{E, \Phi}^{(m)})(z) = \int_{\mathbb{T}} \lambda^{-m}(\tau_\lambda \circ \Delta \circ \tau_\lambda^{-1})(z) \, d\lambda
\]

whenever \( z \in G_{O_N}/G_{F_N} \) and \( \Delta \in \mathcal{H}(E, \Phi) \).

Proof. Firstly note that \( (7.2) \) implies that \( E \circ \tau_\lambda = \tau_\lambda \circ E = E \) for all \( \lambda \in \mathbb{T} \), because of the invariance property of the Haar measure \( d\lambda \) on the unit circle \( \mathbb{T} \). Consequently, if we assume that \( \Phi \) is gauge invariant, then for all \( x \in O_N \) we have \( \Phi(E(x)) = \Phi(\int_{\mathbb{T}} \tau_\lambda(x) \, d\lambda) = \int_{\mathbb{T}} \Phi(\tau_\lambda(x)) \, d\lambda = \int_{\mathbb{T}} \Phi(x) \, d\lambda = \Phi(x) \).

Conversely, if \( \Phi \circ E = \Phi \), then for every \( \lambda \in \mathbb{T} \) we have \( \Phi \circ \tau_\lambda = \Phi \circ \tau_\lambda \circ E = \Phi = \Phi \).

(a) To define the action of \( \tau \) upon the base \( G_{O_N}/G_{F_N} \) we use the fact that each gauge automorphism \( \tau_\lambda \) leaves \( F_N \) pointwise invariant and therefore induces a mapping of \( G_{O_N}/G_{F_N} \) onto itself. It is straightforward to show that in this way we get an action \( (\lambda, aG_{F_N}) \mapsto \tau_\lambda(a)G_{F_N} \) of \( \mathbb{T} \) as claimed. The action of the circle group upon the total space \( D \) can be defined by the formula \([([a, f]) \mapsto ([\tau_\lambda(a), f])] \) for all \([([a, f]) \in D \) and \( \lambda \in \mathbb{T} \).

(b) The realization operator \( \gamma: K_0 \to \mathcal{H}(E, \Phi) \) is unitary, hence it will be enough to prove that 

\[ K_0 = \bigoplus_{m \in \mathbb{Z}} K_0^{(m)} \]

and that each term of this decomposition is invariant under all of the operators in the \( C^* \)-algebra \( \pi_\Phi(F_N) \), where \( K_0^{(m)} \) is the closed linear subspace of \( K_0 \) spanned by \( \pi_\Phi(O_N^{(m)})\tilde{\mathcal{H}}_0 \) for all \( m \in \mathbb{Z} \).

(Note that \( K_0 = \text{sp} \pi_\Phi(F_N)\tilde{\mathcal{H}}_0 \) by construction.)

The proof of this assertion follows the lines of Section 1 in [La93] and relies on the fact that, as an easy consequence of \( (7.1) \), we have \( O_N^{(m)}O_N^{(n)} \subseteq O_N^{(m+n)} \) and \( (O_N^{(m)})^* \subseteq O_N^{(m-n)} \) for all \( m, n \in \mathbb{Z} \) (where \( (\cdot)^* \) stands for the image under the involution of \( O_N \)). Note that \( \text{V}^*V : K_0 \to \tilde{\mathcal{H}}_0 \) is the orthogonal projection from \( K_0 \) onto \( \tilde{\mathcal{H}}_0 \). It follows that for all \( m, n \in \mathbb{Z} \) with \( m \neq n \), and \( x \in O_N^{(m)} \), \( y \in O_N^{(n)} \), \( \xi, \eta \in \mathcal{H}_0 \) we have

\[
(\pi_\Phi(x)V \xi | \pi_\Phi(y)V \eta) = (\pi_\Phi(y^*x)V \xi | V \eta) = (\text{V}^*V(\pi_\Phi(y^*x)V \xi | V \eta)
= (\Phi(\text{V}(y^*x))\xi | \eta)_{\tilde{\mathcal{H}}_0} = (\Phi(\text{V}(y^*x))\xi | \eta)_{\tilde{\mathcal{H}}_0} = 0,
\]

where the latter equality follows since \( y^*x \in O_N^{(m-n)} \) with \( m - n \neq 0 \), so that \( E(y^*x) = 0 \) as an easy consequence of \( (7.2) \). The above computation shows that \( K_0^{(m)} \perp K_0^{(n)} \) whenever \( m \neq n \).

To see that \( \bigcup_{m \in \mathbb{Z}} K_0^{(m)} \) spans the whole \( K_0 \), just recall from [Ni74] that the set \( \bigcup_{m \in \mathbb{Z}} O_N^{(m)} \) spans a dense linear subspace of \( O_N \), and use the image of this set under the unital \( * \)-homomorphism \( \pi_\Phi : O_N \to \mathcal{B}(K_0) \).
Consequently the asserted orthogonal direct sum decomposition of $K_{0}$ is proved. On the other hand, since $F_{N} = O_{N}^{(0)}$, it follows that for all $m \in \mathbb{Z}$ we have $F_{N}O_{N}^{(m)} = O_{N}^{(0)}O_{N}^{(m)} \subseteq O_{N}^{(n)}$, so $\pi_{\Phi}(F_{N})K_{0}^{(m)} \subseteq K_{0}^{(m)}$ according to the construction of $K_{0}^{(m)}$.

(c) For every $\lambda \in \mathbb{T}$ denote by $a \otimes f \mapsto a \otimes f$, $O_{N} \otimes \mathcal{H}_{0} \to K_{0}$, the canonical map. Since $\Phi \circ \tau_{\lambda} = \Phi$, it follows that the mapping $a \otimes f \mapsto \tau_{\lambda}(a) \otimes f$, $O_{N} \otimes \mathcal{H}_{0} \to O_{N} \otimes \mathcal{H}_{0}$, induces a unitary operator $V_{\lambda}: a \otimes f \mapsto \tau_{\lambda}(a) \otimes f$, $K_{0} \to K_{0}$. As the closure of the image of $O_{N}^{(m)} \otimes \mathcal{H}_{0}$ in $K_{0}$ is equal to $K_{0}^{(m)}$, it follows that for all $m \in \mathbb{Z}$ we have

\[(7.3) \quad K_{0}^{(m)} = \{ h \in K_{0} \mid (\forall \lambda \in \mathbb{T}) \quad V_{\lambda}h = \lambda^{m}h \}.\]

On the other hand, for all $a, c \in O_{N}$ and $f \in \mathcal{H}_{0}$ we have

\[
\pi_{\Phi}(\tau_{\lambda}(c))a \otimes f = \tau_{\lambda}(c) \circ \tau_{\lambda}(a) \otimes f = V_{\lambda}c \tau_{\lambda-1}(a) \otimes f = V_{\lambda}c \pi_{\Phi}(c) \tau_{\lambda-1}(a) \otimes f
\]

whence

\[(7.4) \quad (\forall \lambda \in \mathbb{T}, c \in O_{N}) \quad \pi_{\Phi}(\tau_{\lambda}(c)) = V_{\lambda} \pi_{\Phi}(c) V_{\lambda}^{-1}.\]

It follows by (7.3) and (7.4) that

\[(7.5) \quad (\forall \lambda \in \mathbb{T}, n \in \mathbb{Z}, c \in O_{N}, h \in K_{0}^{(n)}) \quad \pi_{\Phi}(\tau_{\lambda}(c))h = \lambda^{-n}V_{\lambda} \pi_{\Phi}(c)h.\]

Now for $\lambda \in \mathbb{T}$, $\eta \in K_{0}$, and $c \in G_{O_{N}}$, we get

\[
(\tilde{\tau}_{\lambda} \circ \gamma(h) \circ \tau_{\lambda}^{-1})(cG_{F_{N}}) = \tilde{\tau}_{\lambda}(\gamma(h)(\tau_{\lambda}(c)G_{F_{N}})) = \tilde{\tau}_{\lambda}((\tau_{\lambda}(c), P(\pi_{\Phi}(\tau_{\lambda}(c^{(-1)}\eta))))
\]

whence by (7.6) it follows that for $h \in K_{0}^{(n)}$ we have

\[
(\tilde{\tau}_{\lambda} \circ \gamma(h) \circ \tau_{\lambda}^{-1})(cG_{F_{N}}) = \lambda^{n}[(c, P(V_{\lambda}^{-1} \pi_{\Phi}(c^{-1})h))]
\]

where for $m \in \mathbb{Z}$, $P^{(m)}: K_{0} \to K_{0}^{(m)}$ is the orthogonal projection and $P = P^{(0)}$.

On the other hand, it follows by (7.6) that

\[
\int_{\mathbb{T}} \lambda^{-m}(V_{\lambda}h) d\lambda = P^{(m-n)}h \quad \text{for all } h \in K_{0}, \text{ and thus by}
\]

the above equality we get for $h \in K_{0}^{(n)}$,

\[
\int_{\mathbb{T}} \lambda^{-m}(\tilde{\tau}_{\lambda} \circ \gamma(h) \circ \tau_{\lambda}^{-1})(cG_{F_{N}}) d\lambda = \int_{\mathbb{T}} \lambda^{-m}[(c, P(V_{\lambda}^{-1} \pi_{\Phi}(c^{-1})h))] d\lambda
\]

\[
= [(c, P^{(n)}(P^{(m-n)}(\pi_{\Phi}(c^{-1})h)))] = \begin{cases} 
\gamma(h)(cG_{F_{N}}) & \text{if } n = m, \\
0 & \text{if } n \neq m. 
\end{cases}
\]

Thus we have proved the asserted formula for $\Delta = \gamma(h)$ with $h \in \bigcup_{n\in\mathbb{Z}} K_{0}^{(n)}$. On the other hand, the reproducing ($-\epsilon$)-kernel associated with the Stinespring representation $\pi_{\Phi}$ is continuous, hence the realization operator $\gamma: K_{0} \to O(G_{O_{N}}/G_{F_{N}}, D)$ is continuous with respect to the uniform convergence on the compact subsets of the base $G_{O_{N}}/G_{F_{N}}$. Since $K_{0} = \bigoplus_{n\in\mathbb{Z}} K_{0}^{(n)}$ and the right hand side of the asserted formula is linear and continuous with respect to the latter topology, it follows that the corresponding equality extends by linearity and continuity to the whole space $H(\mathcal{E}, \Phi) = \gamma(K_{0})$. \hfill \qed

**Remark 7.16.** It is noteworthy that orthogonal decompositions similar to the one of Corollary (7.10) also show up in connection with representations of Cuntz algebras that do not necessarily occur as Stinespring dilations of gauge invariant maps; see for instance the representations studied in [339]. \hfill \qed

There is a close relationship between *-endomorphisms of algebras $B(\mathcal{H})$ and representations of Cuntz algebras, see for instance [333a, 333b]. In the remainder of this subsection, we point out that some of the notions underlying this relationship provide us with more examples of the theory proposed in the present paper. Thus let $\mathcal{H}$ be a separable Hilbert space, let $A := B(\mathcal{H})$ and let $\alpha: A \to A$ a unital *-endomorphism; then $\alpha$ is normal, as noted for instance in [333a]. By a celebrated result of
W. Arveson there exist $N \in \{1, \ldots, +\infty\}$ and a $*$-representation $\rho: \mathcal{O}_N \to A$, where $\mathcal{O}_N$ is the Cuntz algebra generated by $N$ isometries $v_j$, on $\mathcal{H}$, such that

$$\alpha(T) = \sum_{j=1}^{N} \rho(v_j) T \rho(v_j^*), \quad T \in A. \quad (7.6)$$

(See Proposition 2.1 in [At89].) Since $\alpha$ is unital we have that $\sum_{j=1}^{N} \rho(v_j) \rho(v_j^*) = 1$ even for $N = \infty$ (in the strong operator topology, in the latter case). In the sequel we assume that $N \geq 2$.

We do observe that, in order to make a link between the geometry of $\rho$ and the one of $\alpha$, the endomorphism $\alpha$ can be regarded either as a $*$-representation of $A$ of the injective von Neumann algebra $A$ on $\mathcal{H}$ or as a completely positive mapping from $A$ into $\mathcal{B}(\mathcal{H})$. In fact, it seems natural at first glance to take into consideration the second option, since $\alpha$ induces the correspondence $u \rho u^{-1} \mapsto u \alpha(u^{-1} \cdot u) u^{-1}$, $(u \in G_A)$; that is, the canonical map $\mathcal{S}(\rho) \to \mathcal{S}(\alpha)$. Nevertheless, in such a case we cannot be sure that the algebraic isomorphism $\mathcal{S}(\alpha) \cong G_A/G(\alpha)$ entails a structure of smooth homogeneous manifold. Namely, the Lie algebra of $G(\alpha)$ is $\mathfrak{g}(\alpha) = \{ X \in A \mid X - \alpha(X) \in \alpha(A)' \}$, see Lemma [La93a] and it is not clear that $\mathfrak{g}(\alpha)$ is topologically complemented in $A$.

On the other hand, by looking at $\alpha$ as a $*$-representation and by using a bit of the structure of von Neumann algebras, it is possible to relate the orbits $\mathcal{S}(\rho)$ and $\mathcal{S}(\alpha)$ to each other, as we are going to see.

Since $A$ is Connes-amenable (that is, $A$ is injective, see [Ru02]) an appropriate virtual diagonal $M$ for $A$ can be fixed, so that the mapping

$$E_\alpha(T) := \int_{A \otimes A} \alpha(a) T \alpha(b) \, dM(a,b), \quad (T \in A),$$

is a conditional expectation from $A$ onto $B_\alpha := E_\alpha(A)$. This endows $\mathcal{S}(\alpha)$ with the corresponding smooth homogeneous manifold structure.

Similarly, $\mathcal{O}_N$ is a nuclear (amenable) $C^*$-algebra, see [Cu77], and therefore we can fix a contractive virtual diagonal $M_N$ for $\mathcal{O}_N$ so that the mapping

$$E_\rho(T) := \int_{\mathcal{O}_N \otimes \mathcal{O}_N} \rho(s) T \rho(t) \, dM_N(s,t), \quad (T \in A),$$

is a conditional expectation from $A$ onto $B_\rho := E_\rho(A)$. In this way, we regard $\alpha: A \to \mathcal{B}(\mathcal{H})$ and $\rho: \mathcal{O}_N \to A$ as special cases of Example [7.11].

There are two $C^*$-subalgebras of $A$ which are important in the study of the endomorphism $\alpha$. These are $\{ a \in A \mid \alpha(a) = a \}$ and $\bigcap_{n > 0} \alpha^n(A)$. Recall that $B_\alpha = \{ a \in A \mid E_\alpha(a) = a \} = \alpha(A)'$ and $B_\rho = \{ a \in A \mid E_\rho(a) = a \} = \rho(\mathcal{O}_N)'$, see [CG98]. (In this notation, $\mathfrak{g}(\alpha) = (I - \alpha)^{-1}(E_\alpha(A))$.) By Proposition 3.1(i) in [La93a] we have $B_\rho = \rho(\mathcal{O}_N)' = \{ a \in \mathcal{B}(\mathcal{H}) \mid \alpha(a) = a \}$ (which means in particular that $B_\rho = \mathcal{C}1$ if the representation $\rho$ is irreducible or, equivalently, if $\alpha$ is an ergodic endomorphism of $A = \mathcal{B}(\mathcal{H})$). Hence, for every $b \in B_\rho$ we get $b = \alpha(b) \in E_\alpha(A)'$. Shortly,

$$B_\rho \subseteq \alpha(A)' = \alpha(A). \quad (7.7)$$

Note that $\alpha(A)'$ is $w^*$-closed in $A$ and so it is a von Neumann subalgebra of $A$. Moreover $\alpha(A)'$ is the range of the conditional expectation $E_\alpha$ and then it turns out to be injective. Let $\Delta$ be the anti-unitary operator ($\Delta$ is an antilinear isometry with $\Delta^2 = \text{id}$) which appears in the standard form of $\alpha(A)'$, see [Ta83], volume II. Then the mapping $E_{\Delta \alpha \Delta}: A \to A$ given by

$$E_{\Delta \alpha \Delta}(T) := \Delta E_\alpha(\Delta T \Delta) \Delta \quad (T \in A),$$

is a conditional expectation such that $\alpha(A)'' = E_{\Delta \alpha \Delta}(A)$. It follows that $\alpha(A)''$ is also injective, see [Ta83], volume III. The above notation is not by chance since $E_{\Delta \alpha \Delta}$ corresponds exactly to the expectation defined by the virtual manifold $M$ and representation $\Delta \alpha \Delta$. Thus we have

$$B_\rho \subseteq B_{\Delta \alpha \Delta}. \quad (7.8)$$

Passing to quotients, (7.8) implies that we obtain a canonical surjection $\mathcal{S}(\rho) \to \mathcal{S}(\Delta \alpha \Delta)$. In fact, it is a submersion mapping: The above map is $G_A$-equivariant and its tangent map at $\rho$ is implemented by
the bounded projection 

\[ \text{id} - E_{\Delta \alpha \Delta}: (\text{id} - E_{\rho})(a) \mapsto (\text{id} - E_{\Delta \alpha \Delta})(a). \]

$$
from A/B_\rho \text{ onto } A/B_{\Delta \alpha \Delta}.
$$

On the other hand, the involutive transformation \( \beta \mapsto \Delta \beta(\cdot) \Delta \) carries diffeomorphically similarity (unitary) orbits into similarity (unitary) orbits of the space of representations \( \text{Rep}^\omega (A) \). In particular \( \mathcal{G}(\alpha) \) and \( \mathcal{G}(\Delta \alpha \Delta) \) are diffeomorphic through the map

\[ a \alpha a^{-1} \mapsto (\Delta \alpha \Delta)(\Delta \alpha \Delta)(\Delta a^{-1} \Delta), \quad \mathcal{G}(\alpha) \rightarrow \mathcal{G}(\Delta \alpha \Delta). \]

(note that \( u a = \alpha u \) if and only if \( (\Delta u \Delta)(\Delta \alpha \Delta) = (\Delta \alpha \Delta)(\Delta u \Delta) \)). Putting all the above facts together we obtain the analytic submersion \( \mathcal{G}(\rho) \rightarrow \mathcal{G}(\alpha) \) given by

\[ a \rho a^{-1} \mapsto (\Delta \alpha \Delta)\alpha(\Delta a^{-1} \Delta), \quad (a \in G_A). \]

At this point one can return to (7.9), which also tells us that

\[ B_{\alpha} = \alpha(A)' \subseteq B_{\rho}'. \]

Thus we can proceed as formerly, just replacing \( \alpha \) with \( \rho \). In particular, the von Neumann algebra \( B_\rho = \rho(O_N)' \) is injective and then its commutant subalgebra \( B_{\rho}' \) in \( A \) is also injective. In effect, it is the image \( B_{\rho}' = E_{R_{\rho} R}(A) \) of the conditional expectation \( E_{R_{\rho} R} \) defined by the virtual diagonal \( M_N \) and representation \( R_{\rho} R \), where \( R \) is the anti-unitary operator involved in the standard form of \( \rho(O_N)' \).

Hence, there exists an analytic submersion \( \mathcal{G}(\alpha) \rightarrow \mathcal{G}(\rho) \) given by

\[ u \rho u^{-1} \mapsto (R \rho)(Ru^{-1} \rho), \quad (u \in G_A). \]

In summary, we have proved the following result.

**Theorem 7.17.** Let \( \Delta, R \) be the anti-unitary operators associated with the standard forms of the injective von Neumann algebras \( \alpha(A)', \rho(O_N)' \) respectively. Then \( B_\rho \subseteq B_{\Delta \alpha \Delta} \) and \( B_\alpha \subseteq B_{R_{\rho} R} \), and the mappings

\[ q_\rho: a \rho a^{-1} \mapsto (\Delta \alpha \Delta)\alpha(\Delta a^{-1} \Delta), \quad \mathcal{G}(\rho) \rightarrow \mathcal{G}(\alpha), \]

\[ q_\alpha: u \omega u^{-1} \mapsto (R \rho)(Ru^{-1} \rho), \quad \mathcal{G}(\alpha) \rightarrow \mathcal{G}(\rho) \]

are analytic submersions.

**Remark 7.18.** The joint action of suitably chosen mappings in the proposition yields new submersions

\[ a \rho a^{-1} \mapsto (V a V^{-1}) \rho(V a^{-1} V^{-1}), \quad \mathcal{G}(\rho) \rightarrow \mathcal{G}(\rho) \]

and

\[ u \omega u^{-1} \mapsto (V^{-1} u V) \alpha(V^{-1} u^{-1} V), \quad \mathcal{G}(\alpha) \rightarrow \mathcal{G}(\alpha), \]

where \( V \) is the unitary operator \( V = R \Delta \). Such submersions need not be diffeomorphisms.

Because of the inclusion \( B_\rho \subseteq B_{\Delta \alpha \Delta} \) we have \( E_{\Delta \alpha \Delta} \circ E_{\rho} = E_{\rho} \). Set \( F_\rho := E_{\rho} E_{\Delta \alpha \Delta} \). Then \( F_\rho \) is also a conditional expectation and \( F_\rho \) and \( E_{\rho} \) are equivalent: \( F_\rho E_{\rho} = E_{\rho} \) and \( E_{\rho} F_\rho = F_\rho \), so that \( F_\rho(A) = B_\rho \).

In addition, \( F_\rho \) and \( E_{\Delta \alpha \Delta} \) commute: \( E_{\Delta \alpha \Delta} F_\rho = F_\rho = E_{\rho} E_{\Delta \alpha \Delta} \).

Let \( \Phi: A \rightarrow B(H_0) \) be a completely positive map, for some Hilbert space \( H_0 \). Put \( \Phi_{\rho} := \Phi \circ F_\rho \). Then \( \Phi_{\rho} F_\rho = \Phi_{\rho} F_{\rho} = \Phi_{\rho} E_{\Delta \alpha \Delta} = \Phi_{\rho} \). Applying Stinepring’s dilation theorem we find Hilbert spaces \( H_J(\Phi_{\rho}) \), for \( J = A, B_{\Delta \alpha \Delta}, B_\rho \) with \( H_J(\Phi_{\rho}) \subseteq H_{B_{\Delta \alpha \Delta}}(\Phi_{\rho}) \subseteq H_A(\Phi_{\rho}) \), and \( \ast \)-representations \( \pi_J: J \rightarrow B(H_J(\Phi_{\rho})) \) satisfying \( \pi_{B_{\Delta \alpha \Delta}}(u) = \pi_A(u)|_{H_{B_{\Delta \alpha \Delta}}(\Phi_{\rho})} \) for each \( u \in B_{\Delta \alpha \Delta} \), and \( \pi_{B_\rho}(u) = \pi_{B_{\Delta \alpha \Delta}}(u)|_{H_{B_\rho}(\Phi_{\rho})} \) for each \( u \in B_\rho \).

**Corollary 7.19.** In the above setting, the following commutative diagram and \( G_A \)-equivariant diagram:

\[
\begin{array}{ccc}
G_A \times_{G_{B_\rho}} H_{B_\rho}(\Phi_{\rho}) & \longrightarrow & G_A \times_{G_{B_\rho}} H_{B_\rho}(\Phi_{\rho}) \\
\downarrow & & \downarrow \\
\mathcal{G}(\rho) & \xrightarrow{q_\rho} & \mathcal{G}(\alpha)
\end{array}
\]

whose arrows are \( G_A \)-equivariant and compatible with the involutive diffeomorphisms \( \ast \) on both \( \mathcal{G}(\rho) \) and \( \mathcal{G}(\alpha) \). Moreover, the representation \( \pi_A \) of \( G_A \) on \( H_A(\Phi_{\rho}) \) can be extended to \( A \) and realized as
multiplication on a reproducing kernel Hilbert space formed by holomorphic sections of the left-side vector bundle in the diagram.

Proof. Firstly, a diagram similar to that of the statement, and concerning the algebras \( B_\rho \subseteq B_{\Delta_0 \Delta} \), is immediately obtained by mimicking the proof of Corollary 7.10. Then using the diffeomorphism \( \Delta(\cdot) \Delta \) one gets the wanted result or diagram, where the action of \( B_\rho \) on \( \mathcal{H}_B(\Phi_\rho) \) is given just by transferring the action of \( B_{\Delta_0 \Delta} \) through \( \Delta(\cdot) \Delta \) (note that \( B_{\Delta_0 \Delta} = \Delta B_{\Delta_0 \Delta} \)).

A diagram entirely analogous to the previous one of the corollary can be obtained by interchanging roles of representations \( \rho \) and \( \alpha \).

Remark 7.20. Using the representation \( \rho \), we can make a link between Corollary 7.15 and the preceding setting. Let \( \tau \) be the gauge automorphism group of \( \mathcal{O}_N \), and let \( E_\tau \) be the expectation defined by (7.2) for \( m = 0 \). Corollary 7.15 applies to completely positive mappings \( \Phi: \mathcal{O}_N \to \mathcal{B}(\mathcal{H}_0) \) such that \( \Phi \circ E_\tau = \Phi \).

Assume that \( \rho \) is a \( * \)-representation of \( \mathcal{O}_N \) on a von Neumann algebra \( A \), and \( E_\rho: A \to A \) the conditional expectation associated with some fixed, virtual diagonal \( M \) of norm one for \( A \). Let \( \Phi: E_\rho(A) \to \mathcal{B}(\mathcal{H}_0) \) be completely positive, and let consider \( \Phi_{\rho,\tau} := \Phi \circ E_\rho \circ \rho \circ E_\tau \). Then \( \Phi_{\rho,\tau} E_\tau = \Phi_{\rho}\tau \) and we obtain Hilbert spaces and their decompositions like those of Corollary 7.15 associated with the representation \( \rho \) and algebra \( A \).

Finally, the subalgebra \( \rho(\mathcal{F}_N)' = \bigcap_{n \geq 0} \alpha^n(A) \) (see Proposition 3.1(ii) in [La93a]) suggested us to form sequences of vector bundles in the following manner. Let \( \alpha: A \to A \) be a normal, \( * \)-representation where \( A = \mathcal{B}(\mathcal{H}) \) as above. Then \( \alpha^*(A_n) \subseteq A_n \) where \( A_n \) denotes the predual of \( A \) formed by the trace-class operators on \( \mathcal{H} \), and \( \alpha^* \) is the transpose mapping of \( \alpha \). For \( n \in \mathbb{N} \) we are going to consider the iterative mappings \( \beta_n := \alpha^* \circ \rho \) and corresponding expectations denoted by \( E_n := E_{\beta_n} \) and put \( E_0 = E_\rho \). Then, for \( \xi \in A_n \) and \( T \in A \),

\[
(E_n \circ \alpha)(T)(\xi) = \int_{\mathcal{O}_N \otimes \mathcal{O}_N} (\alpha^n \rho)(s) (\alpha(T)(\alpha^n \rho)(t)(\xi)) \, dM_N(s,t)
\]

whence, by a reiterated process and since \( \alpha E_\rho = E_\rho \), we get

\[
E_n \circ \alpha^n = E_\rho, \quad n \in \mathbb{N}
\]

Hence we get \( E_n E_\rho = E_\rho \) and therefore \( B_\rho \subseteq B_n \), where \( B_n := E_n(A) \), for all \( n \). Further, we have \( \alpha E_n(A) = E_{n+1} \alpha(A) \subseteq E_{n+1}(A) \) by (7.10), that is, \( \alpha(B_n) \subseteq B_{n+1} \), \( n \in \mathbb{N} \).

Now consider a countable family \( (\Phi_n)_{n \geq 0} \) of completely positive mappings \( \Phi_n: A \to \mathcal{B}(\mathcal{H}_0) \), for some Hilbert space \( \mathcal{H}_0 \), such that

\[
\Phi_{n+1} \circ \alpha = \Phi_n, \quad \Phi_n \circ E_n = \Phi_n, \quad n \in \mathbb{N}
\]

Such a family exists. Take for instance \( \phi_n := E_\rho E_1 \cdots E_n \), and a completely positive map \( \Phi: A \to \mathcal{B}(\mathcal{H}_0) \). Then the family \( \Phi_n := \Phi \circ \phi_n, n \geq 0 \), satisfies (7.12). In these conditions the diagram

\[
\begin{array}{cccccccc}
A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \Phi_{n+1} \mathcal{B}(\mathcal{H}_0) \\
\downarrow E_0 & & \downarrow E_1 & & \downarrow E_n & & \downarrow E_{n+1} & & \downarrow \text{id} \\
B_0 & \xrightarrow{\alpha} & B_1 & \xrightarrow{\alpha} & \cdots & \xrightarrow{\alpha} & B_n & \xrightarrow{\alpha} & \Phi_{n+1} \mathcal{B}(\mathcal{H}_0)
\end{array}
\]

is commutative in each of its subdiagrams.
For every \( n \geq 0 \), by applying Theorem 6.10 to the conditional expectation \( E_n : A \to B_n \) and mapping \( \Phi_n \), one finds the corresponding Hilbert space \( \mathcal{H}_{B_n}(\Phi_n) \) for the representation which is the Stinespring dilation of \( \Phi_n \). Take a finite set of elements \( b_j \) in \( B_n \). As \( \Phi_n(b_i^* b_j) = \Phi_{n+1}(\alpha(b_i^*) \alpha(b_j)) \) it follows that

\[
\| \sum_j b_j \otimes f_j \|_{\Phi_n} = \| \sum_j \alpha(b_j) \otimes f_j \|_{\Phi_{n+1}}
\]

for all \( \sum_j b_j \otimes f_j \in B_n \otimes \mathcal{H}_n \), see Section 6. Hence, \( \alpha(\mathcal{H}_{B_n}(\Phi_n)) \subseteq \mathcal{H}_{B_{n+1}}(\Phi_{n+1}) \). This implies that we have found the (countable) system of vector bundle homomorphisms

\[
\begin{array}{ccccccc}
G_A \times_{G_{B_n}} \mathcal{H}_{B_n}(\Phi_0) & \xrightarrow{id \otimes \alpha} & G_A \times_{G_{B_1}} \mathcal{H}_{B_1}(\Phi_1) & \cdots & & & & \cdots \\
\downarrow & & \downarrow & & & & \cdots \\
\mathcal{G}(\rho) & \xrightarrow{\tilde{\alpha}_0} & \mathcal{G}(\alpha^1 \rho) & \cdots & & & & \cdots \\
\downarrow & & \downarrow & & & & \cdots \\
\mathcal{G}(\alpha^n \rho) & \xrightarrow{\tilde{\alpha}_{n-1}} & \cdots & & & & \cdots \\
\downarrow & & & & & & \cdots \\
\mathcal{G}(\alpha^n \rho) & \xrightarrow{\tilde{\alpha}_n} & \cdots & & & & \cdots 
\end{array}
\]

where \( \tilde{\alpha}_n \) is the canonical submersion induced by \( \alpha|_{B_n} : B_n \to B_{n+1}, n \geq 0 \). Of course the above sequence of diagrams gives rise to the corresponding statements about complexifications, and realizations of representations on spaces of holomorphic sections.

5) Non-commutative stochastic analysis

We have just shown a sample of how to find sequences of homogeneous vector bundles of the type dealt with in this paper. As a matter of fact, continuous families of such bundles are also available, which could hopefully be of interest in other fields. More precisely, the geometric models developed in the present paper might prove useful in order to get a better understanding of the phenomena described by the various theories of non-commutative probabilities. By way of illustrating this remark, we shall briefly discuss from our geometric perspective a few basic ideas related to the stochastic calculus on full Fock spaces as developed in [BV00] and [BV02]. (See also [VDN92] and [Ex80] for a complementary perspective that highlights the role of the Cuntz algebras in connection with full Fock spaces.)

In the paper [BV00], a family of conditional expectations \( \{E_t\}_{t \geq 0} \) is built on the von Neumann algebra \( A \) of bounded operators on the full Fock space, generated by the annihilation, creation, and gauge operators. Set \( A_t := E_t(A) \) for \( t > 0 \). It is readily seen that \( A_t \subseteq A_s \) and that \( E_t E_s = E_t \) whenever \( 0 < s \leq t \) (check first for the so-called in [BV00] basic elements). Applying the Stinespring dilation procedure to the conditional expectation \( E_t \) and completely positive mapping \( E_t \) one gets Hilbert spaces \( \mathcal{H}_{A_j}(E_t) \subseteq \mathcal{H}(A_j(E_t)) \) and the consequent Stinespring representations \( \pi_{A_j} : A_j \to \mathcal{B}(\mathcal{H}_{A_j}(E_t)) \), where \( j = 0, t, \) and \( A_0 = A \). This entails the commutative diagram

\[
\begin{array}{ccccccc}
G_A \times_{G_s} \mathcal{H}_{A_s}(E_t) & \longrightarrow & G_A \times_{G_s} \mathcal{H}_{A_s}(E_t) & \longrightarrow & G_A \times_{G_s} \mathcal{H}_{A_s}(E_t) \\
\downarrow & & \downarrow & & \downarrow \\
G_{A/G_t} & \longrightarrow & G_{A/G_s} & \longrightarrow & G_{A/G_r},
\end{array}
\]

for \( r < s < t \), where \( G_j = G_{A_j} \) for \( j = r, s, t \). Moreover, as usual, the geometrical framework of the present paper works to produce a Hilbert space \( \mathcal{H}_{A_s}(E_t) \), formed by holomorphic sections on \( G_{A/G_s} \), which is isometric to \( \mathcal{H}(A_t) \) and enables us to realize \( \pi_{A_j} \) as natural multiplication.

On the other hand, from the point of view of the quantum stochastic analysis (see for instance [Par90] and [BP95]), it is worth considering unital completely positive mappings \( \Phi : A \to \mathcal{B}(\mathcal{H}_0) \) with the following filtration property: There exists a family \( \{\Phi_t : A \to \mathcal{B}(\mathcal{H}_t)\}_{t \geq 0} \) of completely positive mappings which approximate \( \Phi \) in some sense and satisfy \( \Phi_t \circ E_t = \Phi_t \) for all \( t > 0 \). Then we get commutative diagrams

\[
\begin{array}{ccccccc}
G_A \times_{G_s} \mathcal{H}_{A_s}(\Phi_t) & \longrightarrow & G_A \times_{G_s} \mathcal{H}_{A_s}(\Phi_s) \\
\downarrow & & \downarrow \\
G_{A/G_t} & \longrightarrow & G_{A/G_s},
\end{array}
\]

whenever \( s < t \). By means of the realizations of the full Fock space as reproducing kernel Hilbert spaces of sections in appropriate holomorphic vector bundles we find geometric interpretations for most concepts.
tradtionally related to the Fock spaces (for instance, annihilation, creation, and gauge operators). We thus arrive at the challenging perspective of a relationship between the non-commutative stochastic analysis and the infinite-dimensional complex geometry, which certainly deserves to be understood in more detail. For one thing, this might provide useful geometric insights in areas like the theory of quantum Markov processes.

Acknowledgment. We wish to thank Professor Karl-Hermann Neeb for pointing out to us some pertinent references. The present research was begun during the first-named author’s visit at the Department of Mathematics of the University of Zaragoza with financial support from Proyecto MTM2004-03036, DGI-FEDER, of the MCYT, Spain. Our research was further developed during the visit of the second-named author’s visit at the Institute of Mathematics ”Simion Stoilow” of the Romanian Academy with support from the aforementioned grant and from the Softwin Group. Partial financial support from Proyecto E-64 of the DG Aragón, Spain, and from Grant 2-CEX06-11-22/2006 of the Romanian Government is also acknowledged.

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