Colombeau’s Generalized Functions on Arbitrary Manifolds

Herbert BALASIN

Institute for Theoretical Physics, University of Alberta
Edmonton, T6G 2J1, CANADA

Abstract

We extend the Colombeau algebra of generalized functions to arbitrary (paracompact) \( C^\infty \) \( n \)-manifolds \( M \). Embedding of continuous functions and distributions is achieved with the help of a family of \( n \)-forms defined on the tangent bundle \( TM \), which form a partition of unity upon integration over the fibres.

PACS numbers: 9760L, 0250

Alberta-Thy-35-96
TUW 96 – 20
August 1996
Introduction

Colombeau theory [1, 2] set out to give a mathematically consistent way of multiplying distributions. From a physical point of view the Colombeau algebra provides a sound framework to accommodate calculations involving regularization methods employed to deal with singular quantities that actually arise as products of distributions. The most prominent example in this regard is provided by the renormalization procedure of perturbative quantum field theory. Recently, there have been some attempts [3, 4, 5, 6, 7, 8] to apply a similar formalism to spacetime singularities that arise in general relativity. Unless one is willing to restrict to manifolds that are topologically $\mathbb{R}^n$ the application of the Colombeau formalism requires its generalization to arbitrary manifolds.

Although it is a pretty recent development the definition of the Colombeau algebra has undergone various changes, starting from a functional analytic motivation, involving the delicate subject of differential calculus in locally convex vector spaces [9].

The definition [10] we are going to generalize in this work considers the elements of the Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$ as moderate one-parameter families $(f_\epsilon)$ of $C^\infty$ functions denoted by $C^\infty_M(\mathbb{R}^n)$ up to negligible families denoted by $C^\infty_N(\mathbb{R}^n)$, where the adjectives refer to certain growth conditions in the parameter $\epsilon$ of the family. The embedding of continuous functions and distributions into algebra is achieved with the help of a smoothing-kernel, which has to obey certain properties. The above quotient is then such that $C^\infty(\mathbb{R}^n)$ becomes a subalgebra of $\mathcal{G}(\mathbb{R}^n)$ thereby reconciling the two different embeddings as constant sequences or with the aid of the smoothing kernel.

Although the definition of the Colombeau algebra used above is fairly straightforwardly generalized to an arbitrary manifold [11], the notion of the smoothing kernel presents some difficulties since it draws heavily upon concepts specific to $\mathbb{R}^n$. On the other hand the smoothing kernels play an important role for the embedding of distributions as linear subspaces into the Colombeau algebra. Although there have been proposals [12] which weaken the condition on the moments, we will stick to the original condition by taking the smoothing kernel to be an $n$-form on the tangent-bundle $TM$ of the manifold $M$ (more precisely a family of $n$-forms of this type, which becomes a partition of unity upon integration along the fibre). Since diffeomorphisms of $M$ act linearly on the fibres of $TM$ this approach gives (invariant) meaning to the conditions on the moments familiar from $\mathbb{R}^n$. The embedding
of smooth and continuous functions with the aid of the above smoothing kernel produces locally defined moderate functions, whose sum defines the corresponding Colombeau-object.

Our work will be organized as follows. In chapter one we recall the basic notions of the Colombeau algebra. Section two will be devoted to a brief survey of distribution theory on arbitrary manifolds, thereby giving a manifestly covariant formulation of the latter. Finally in chapter three we present our generally covariant formulation of the Colombeau algebra motivated by the extension of smoothing kernels which we will consider to be a suitable family of $n$-forms on the $2n$-dimensional tangent-bundle.

1) Moderate and negligible functions, association and all that

The basic idea of Colombeau’s approach for the multiplication of distributions is to find a (differential) algebra large enough to contain all the usual $C^\infty$ functions as a subalgebra and the distributions as a linear subspace. The construction starts by considering one-parameter families $(f_\epsilon)$ of $C^\infty$ functions subject to the condition

\[
C_M^\infty = \{(f_\epsilon) | f_\epsilon \in C^\infty(\mathbb{R}^n) \quad \forall K \subset \mathbb{R}^n \text{compact}, \forall \alpha \in \mathbb{N}^n
\]

\[
\exists N \in \mathbb{N}, \exists \eta > 0, \exists c > 0 \quad \text{s.t.} \sup_{x \in K} |D^\alpha f_\epsilon(x)| \leq \frac{c}{\epsilon^N} \quad \forall 0 < \epsilon < \eta, \}
\]

where \( D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}. \)

One way of getting some intuition for these objects is by considering them as regularizations of functions that (possibly) become singular in the limit $\epsilon \to 0$. All operations like addition and multiplication are defined pointwise, and one easily proves that (1) is an algebra, which will be denoted $C_M^\infty$.

$C^\infty$-functions are canonically embedded into $C_M^\infty$ as constant sequences, whereas continuous functions (and distributions) of at most polynomial growth require a smoothing kernel $\varphi \in \mathcal{S}$. Since it represents an approximate $\delta$-
function one requires that
\[ \int d^n x \varphi(x) = 1 \quad \int d^n x \, x^\alpha \varphi(x) = 0 \quad |\alpha| \geq 1 \]  
(2)

The embedding is done by convoluting the (rescaled and shifted) smoothing kernel \( \varphi \) with \( f \), i.e.
\[ f_\varepsilon(x) = \int d^n y \frac{1}{\varepsilon^n} \varphi\left(\frac{y-x}{\varepsilon}\right) f(y). \]

In order to reconcile the different embeddings of \( C^\infty \) functions one identifies them by employing a suitable ideal \( C_N^\infty(\mathbb{R}^n) \). Its members will be addressed as negligible functions in the following.

\[ C_N^\infty(\mathbb{R}^n) = \{ (f_\varepsilon) | (f_\varepsilon) \in C_M^\infty(\mathbb{R}^n) \quad \forall K \subset \mathbb{R}^n \text{ compact}, \forall \alpha \in \mathbb{N}^n, \forall N \in \mathbb{N} \quad \exists \eta > 0, \exists c > 0, \quad \sup_{x \in K} |D^\alpha f_\varepsilon(x)| \leq c \varepsilon^N \quad \forall 0 < \varepsilon < \eta \} \]  
(3)

Considering the difference \( f(x) - \int d^n y (1/\varepsilon^n) \varphi(\frac{y-x}{\varepsilon}) f(y) \), where \( f \) denotes an arbitrary \( C^\infty \) function of at most polynomial growth, one easily checks that it is a negligible function. The Colombeau algebra \( G(\mathbb{R}^n) \) is therefore defined to be the quotient of \( C_M^\infty(\mathbb{R}^n) \) with respect to \( C_N^\infty(\mathbb{R}^n) \). A Colombeau object is thus a moderate family \( (f_\varepsilon(x)) \) of \( C^\infty \) functions modulo negligible families.

The usual distribution theory arises from coarse graining the Colombeau algebra employing an equivalence relation called association. Two Colombeau objects \( (f_\varepsilon) \) and \( (g_\varepsilon) \) will be considered associated if
\[ \lim_{\varepsilon \to 0} \int d^n x (f_\varepsilon(x) - g_\varepsilon(x)) \varphi(x) = 0 \quad \forall \varphi \in \mathcal{D}. \]  
(4)

This equivalence relation is much coarser than the one used for the definition of \( G \). It is compatible with addition, differentiation and multiplication by \( C^\infty \) functions. It is, however, not compatible with multiplication. Intuitively speaking different Colombeau objects are packaged together into one association-class. One might think of such a class as containing different regularizations of the same (possibly singular) non-smooth function.

Let us give a simple (by now classical \[10\]) example showing the power of the association calculus. Consider \( \theta^n(x) \) which as piecewise continuous function gives rise to the same (regular) distribution as \( \theta \). Upon naive differentiation one obtains
\[ \theta^n(x) = \theta(x) \Rightarrow n\theta(x)\theta'(x) = \theta'(x) \Rightarrow \theta(0) = \frac{1}{n}. \]
which is a contradiction since $\theta(0)$ is independent of $n$. With regard to the Colombeau algebra $\theta^n$ is no longer equal to $\theta$, they are however associated. Since association respects differentiation we have

$$\theta^n(x) \approx \theta(x) \Rightarrow n\theta^{n-1}(x)\theta'(x) \approx \theta'(x) \Rightarrow \theta^n(x)\theta'(x) \approx \frac{1}{n+1}\delta(x),$$

which only tells us that we are allowed to replace $\theta^n\theta'$ by $\delta/(n + 1)$. Since multiplication breaks association we do not encounter any ambiguities.

### 2) Distributions on arbitrary manifolds

The main goal of this section is to give a manifestly covariant formulation of distribution theory, which shows that the distribution concept relies only on the differentiable structure of the underlying manifold $M$ and does not require any additional notions such as the existence of a metric or a volume-form \cite{12, 4}. Usually distributions on $\mathbb{R}^n$ are defined as elements of the (topological) dual of test-function space $D$, which in the simplest case is considered to consist of all $C^\infty$ functions with compact support. Locally integrable functions $f(x)$ are embedded into distribution space as so-called regular functionals

$$(f, \varphi) = \int d^n x f(x)\varphi(x). \tag{5}$$

Operations like differentiation and multiplication by arbitrary $C^\infty$ functions are defined via the corresponding operations on $D$, namely

$$(D^\alpha f, \varphi) := (-)^{\alpha}(f, D^\alpha \varphi) \quad \alpha \in \mathbb{N}^n$$

$$(g f, \varphi) := (f, g \varphi),$$

which reduce to standard integral relations if one restricts to regular distributions generated by differentiable functions. Both of the above operations are well-defined since they map $D$ onto itself.

In order to generalize the above concepts to arbitrary manifolds $M$, we first have to decide what to do about test-function space. Although $C^\infty$ functions with compact support are a concept that makes sense on arbitrary manifolds this would not allow us to embed locally integrable functions in the same way as we did in $\mathbb{R}^n$, where we made use of the natural volume form.
$d^nx$, unless we are willing to single out a volume form. Thinking, however, of $\varphi$ and $d^nx$ as parts of one object, we are immediately lead to consider $C^\infty$ $n$-forms $\tilde{\varphi}$ with compact support as the natural generalization of test-functions. Taking into account that the latter are sections of a vector-bundle and by employing a partition of unity (which requires the underlying manifold $M$ to be paracompact) we basically may construct the locally convex vector-space topology in very much the same way as in $\mathbb{R}^n$. Distributions are then once again defined as the elements of the topological dual of this space.

The concepts of multiplication by $C^\infty(M)$ functions is without problems. In order to generalize differentiation we make use of the Lie-derivative along an arbitrary $C^\infty$ vector-field and the classical identity

$$(L_X f, \tilde{\varphi}) = \int_M L_X f \tilde{\varphi} = \int_M d(i_X (f \tilde{\varphi})) - \int_M fL_X \tilde{\varphi} = -(f, L_X \tilde{\varphi})$$

where the last equality is taken to be true for arbitrary distributions. Let us once again emphasize that the embedding of locally integrable functions led us to generalizing test-function space to $C^\infty$ $n$-forms with compact support.

3) Generally covariant formulation of the Colombeau algebra

This chapter is devoted to a formulation of the Colombeau algebra suitable for arbitrary manifolds $M$. Considering the definition of moderate and negligible functions (13) one sees that they immediately generalize to $M$, since the conditions required for the definition remain invariant under coordinate transformations. A manifestly covariant definition may be given with the aid of the Lie-derivative

$$C^\infty_M(M) = \{(f_\epsilon) | f_\epsilon \in C^\infty(M) \forall K \subset M compact, \forall \{X_1, \ldots, X_p\}$$

$$X_i \in \Gamma(TM), [X_i, X_j] = 0, \exists N \in \mathbb{N}, \exists \eta > 0, \exists c > 0$$

$$s.t. \sup_{x \in K} |L_{X_1} \ldots L_{X_p} f_\epsilon(x)| \leq \frac{c}{\epsilon^N} \ 0 < \epsilon < \eta \},$$

$$C^\infty_N(M) = \{(f_\epsilon) \in C^\infty_M(M) \forall K \subset M compact, \forall \{X_1, \ldots, X_p\}$$

$$X_i \in \Gamma(TM), [X_i, X_j] = 0, \forall q \in \mathbb{N}, \exists \eta > 0, \exists c > 0$$

$$s.t. \sup_{x \in K} |L_{X_1} \ldots L_{X_p} f_\epsilon(x)| \leq c\epsilon^q \ 0 < \epsilon < \eta \}.$$
However, looking at the smoothing kernel $\varphi$ required for the embedding of continuous functions (and distributions) one realizes that the condition on the moments does not remain invariant under coordinate transformations. One way of remedying this situation was presented in [11], where instead of a fixed smoothing kernel $\varphi$ a whole family $\varphi_\epsilon$ was considered, which allowed a weakening of the conditions on the moments.

We will try to follow a different path, which allows us to keep the conditions on the moments in a coordinate invariant manner. This seemingly paradoxical statement is easily understood in terms of the tangent bundle $T M$ of $M$. Let us consider a bundle-atlas of $T M$ induced by an atlas $M$ and let $(x, \xi)$ denote the respective coordinates of an arbitrary chart. Now fix a differential $n$-form
\[ \tilde{\varphi} = \varphi(x, \xi)(d\xi^1 + N_1 dx^1) \wedge \cdots \wedge (d\xi^n + N_n dx^n) \] (6)
which we require to obey
\[ \int_{T_x M} i^*_x \tilde{\varphi} = \int \varphi(x, \xi) d^n \xi = 1 \]
\[ \int_{T_x M} \xi^\alpha i^*_x \tilde{\varphi} = \int \xi^\alpha \varphi(x, \xi) d^n \xi = 0 \] (7)
where $i_x : T_x M \to TM$ and $i_x^* \tilde{\varphi} = \varphi(x, \xi) d^n \xi$.

The advantage of the tangent-bundle formulation comes from the fact that diffeomorphisms in $M$ induce a specific type of diffeomorphism on $T M$, namely fibre-preserving bundle morphisms, which act linearly on the fibres thereby leaving (7) invariant. Moreover, the rescaling and shift operations are now naturally interpreted as action of the structure group on $T M$, given by
\[ \phi_\epsilon : TM \to TM \quad (x, \xi) \mapsto (x, \frac{1}{\epsilon} \xi), \]
\[ \phi_a : TM \to TM \quad (x, \xi) \mapsto (x, \xi + a) \] (8)
which are specific $IGL(n, \mathbb{R})$ transformations. Let us now use the smoothing-form $\tilde{\varphi}$ to embed a continuous function $f$
\[ f_\epsilon(x) := \int_{T_x M} \phi_\epsilon^* \phi_{-\epsilon}^* i^*_x \tilde{\varphi} f = \int d^n \xi \frac{1}{\epsilon^n} \varphi(x, \frac{\xi - x}{\epsilon}) f(\xi) = \int d^n \xi \varphi(x, \xi) f(x + \epsilon \xi). \]

\[^2\text{we actually consider the fibres } T_x M \text{ to be affine spaces} \]
The last relation makes explicit use of the coordinate representation of the function $f$ with respect to a given coordinate chart (which we assume to be $\mathbb{R}^n$) in order to lift $f$ to a function on $T_xM$. This entails that $f_\epsilon$ is also only defined locally. However, using a $C^\infty$ partition of unity ($\rho_i$) subordinate to the cover $(U_i)$ allows us to patch the local objects together to a global one

$$f_\epsilon(x) := \sum_i \rho_i(x) f_{i,\epsilon}(x),$$

where the sum is always finite due to the local finiteness of the cover $(U_i)$. Note that actually every term in the sum is a well-defined $C^\infty$–object on $M$.

The above construction may be completely absorbed into the $n$-form $\tilde{\varphi}$ by taking a family of $n$-forms $\tilde{\varphi}_i$ subordinate to the cover $(U_i)$ such that

$$\sum_i \int_{T_xM} i_x^* \tilde{\varphi}_i = 1$$

That is to say $\rho_i(x) := \int_{T_xM} i_x^* \tilde{\varphi}_i$ defines a partition of unity subordinate to the (locally finite) cover $(U_i)$. It is now easy to show that the two different embeddings of $C^\infty$ functions differ only by elements belonging to $C^\infty_N(M)$.

$$f_\epsilon(x) - f(x) = \sum_i \int d^n\xi \varphi_i(x, \xi) (f(x + \epsilon \xi) - f(x)) = O(\epsilon^q),$$

where the last equality is derived from Taylor-expanding $f(x + \epsilon \xi)$. Although the embedding (9) depends on the atlas employed, since we made specific use of the local representation of the function in order to lift it to the tangent-bundle, this dependence disappears for $C^\infty$ functions

$$\tilde{f}_\epsilon(\bar{x}) = \int d^n\bar{\xi} \tilde{\varphi}(\bar{x}, \bar{\xi}) \tilde{f}(\bar{x} + \epsilon \bar{\xi}) = \int d^n\xi \varphi(x, \xi) f(\mu^{-1}(\mu(x) + \epsilon \partial x \xi))$$

$$= \int d^n\xi \varphi(x, \xi) f(x + \epsilon \xi) + O(\epsilon^q) = f_\epsilon(x) + O(\epsilon^q),$$

where once again the result was obtained by Taylor-expanding up to order $q$.

*Acknowledgement:* The author wants to thank Michael Oberguggenberger and Michael Kunzinger for introducing him to the subject of Colombeau theory, during his stay in Innsbruck. Furthermore the author wants to thank the relativity and cosmology group at the University of Alberta and especially Werner Israel, for their kind hospitality, during the final stage of this work.
Conclusion

In this paper we generalized the Colombeau algebra to arbitrary manifolds. The main problem that had to be overcome was the embedding of continuous functions and distributions with the help of a smoothing kernel. The definition of the latter used in the standard $\mathbb{R}^n$-approach required all its moments to vanish. Unfortunately, this concept does not remain invariant under coordinate transformations and does therefore not generalize to arbitrary manifolds. Taking advantage of the tangent bundle we were, however, able to maintain the condition on the moments by taking the smoothing kernel to be a differential $n$-form defined on the $2n$-dimensional tangent bundle. The coordinate invariance of this construction is guaranteed by linear action of $M$-diffeomorphisms on the fibres of $TM$. 
References

[1] Colombeau J, New Generalized Functions and Multiplication of Distributions Mathematics Studies 84, North Holland (1984).

[2] Aragona J and Biagioni H, Analysis Mathematica 17, 75 (1991).

[3] Parker P, J. Math. Phys. 20, 1423 (1979).

[4] Balasin H and Nachbagauer H, Class. Quantum Grav. 10, 2271 (1993).

[5] Balasin H and Nachbagauer H, Class. Quantum Grav. 11, 1453 (1994).

[6] Balasin H, Geodesics of impulsive pp-waves and the multiplication of distributions gr-qc/9607076.

[7] Loustó C and Sánchez N Nucl. Phys. B383 377 (1992).

[8] Clark C, Vickers J and Wilson J, Class. Quantum Grav. 13, 2485 (1996).

[9] Colombeau J, Differential calculus and holomorphy Mathematics Studies 64, North Holland (1982).

[10] Colombeau J, Multiplication of Distributions LNM 1532, Springer (1992).

[11] Colombeau J and Meril A, Journal of Mathematical Analysis and Applications 186, 357-364 (1994).

[12] Choquet-Bruhat Y, Morette-DeWitt C and Dillard-Bleick M, Analysis Manifolds and Physics, North-Holland (1982).