A DOUBLY GENERATED UNIFORM ALGEBRA
WITH A ONE-POINT GLEASON PART
OFF ITS SHILOV BOUNDARY

ALEXANDER J. IZZO

ABSTRACT. It is shown that there exists a compact set $X$ in $\mathbb{C}^2$
with a nontrivial polynomial hull $\hat{X}$ such that some point of $\hat{X} \setminus X$
is a one-point Gleason part for $P(X)$. Furthermore, $X$ can chosen
so that $P(X)$ has a dense set of invertible elements.

1. Introduction

Examples of uniform algebras $A$ with a one-point Gleason part lying
off the Shilov boundary for $A$ have long been known. See for instance
[11, p. 187] for a well-known example defined on the 2-torus and known
as the big disc algebra. Brian Cole’s method of root extensions [2] yields
uniform algebras with nontrivial Shilov boundary such that every point
is a one-point Gleason part. Recently, Cole, Ghosh, and the present
author [3] gave a triply generated example, that is, they constructed
a compact set $X$ in $\mathbb{C}^3$ with a nontrivial polynomial hull $\hat{X}$ such that
every point of $\hat{X}$ is a one-point Gleason part for $P(X)$. This suggests
the question of whether for $X$ a compact set in $\mathbb{C}^2$, the set $\hat{X} \setminus X$ can
contain a one-point Gleason part. The main purpose of the present
paper is to show that this can indeed happen. Furthermore, $X$ can
be chosen so that $P(X)$ has a dense set of invertible elements, and
hence in particular, so that $\hat{X}$ contains no analytic discs. (As noted by
Garth Dales and Joel Feinstein in [4], the condition that $P(X)$ has a
dense set of invertible elements is strictly stronger than the condition
that $\hat{X}$ contains no analytic disc.) Since for $X$ a compact set in $\mathbb{C}^1$,
there can never be a one-point Gleason part for $P(X)$ lying in $\hat{X} \setminus X$,
this result is, in a certain sense, optimal. Whether the result can be
strengthened to obtain a compact set $X$ in $\mathbb{C}^2$ with nontrivial polynomial
hull such that every point of $\hat{X}$ is a one-point Gleason part for
$P(X)$ remains open.

2000 Mathematics Subject Classification. 32E20, 46J10, 46J15.

Key words and phrases. doubly generated uniform algebra, one-point Gleason
part, Shilov boundary, polynomial hull, dense invertibles, analytic disc, Cantor set.
We will denote the open unit ball in $\mathbb{C}^N$ by $B$ and the boundary of an open set $\Omega$ in $\mathbb{C}^N$ by $\partial \Omega$.

**Theorem 1.1.** There exists a compact set $X \subset \partial B \subset \mathbb{C}^2$ such that the origin is in $\hat{X} \setminus X$ and is a one-point Gleason part for $P(X)$.

Following Dales and Feinstein, we will say that a uniform algebra $A$ has dense invertibles if the invertible elements of $A$ are dense in $A$. As alluded to earlier, the set $X$ in the above theorem can be chosen so that $P(X)$ has a dense invertibles. Using extensions of the notions of polynomial and rational hulls introduced by the author in [8], we will establish the following more general result. The definitions of these new notions, the $k$-polynomial hull $\hat{X}^k$ of $X$ and the $k$-rational hull $h_r^{k-1}(X)$ of $X$, will be recalled below in the next section.

**Theorem 1.2.** Let $\Sigma \subset \mathbb{C}^N$ be any compact set with nontrivial $k$-polynomial hull with $k \geq 2$, and let $x_0 \in \hat{\Sigma}^k$. Then $\Sigma$ contains a compact set $X$ such that $x_0$ is in $h_r^{k-1}(X) \setminus X \subset \hat{X} \setminus X$ and is a one-point Gleason part for $P(X)$, and $P(X)$ has dense invertibles.

The following corollary is an almost immediate consequence.

**Corollary 1.3.** Let $\Omega \subset \mathbb{C}^N$ ($N \geq 2$) be any bounded open set, and let $x_0 \in \Omega$. Then there exists a compact set $X \subset \partial \Omega$ such that $x_0$ is in $\hat{X} \setminus X$ and is a one-point Gleason part for $P(X)$, and $P(X)$ has dense invertibles.

Using results from the author’s paper [8], we will obtain the following as another corollary.

**Corollary 1.4.** There exists a Cantor set $X$ in $\mathbb{C}^3$ such that $\hat{X} \setminus X$ is nonempty, some point of $\hat{X} \setminus X$ is a one-point Gleason part for $P(X)$, and $P(X)$ has dense invertibles.

The existence of a Cantor set $X$ in $\mathbb{C}^3$ such that $\hat{X} \setminus X$ is nonempty and $P(X)$ has dense invertibles was proved by the author as [8, Theorem 1.4]. What is new in Corollary 1.4 is that the Cantor set $X$ can be chosen so that some point of $\hat{X} \setminus X$ is a one-point Gleason part for $P(X)$. Whether there exists a Cantor set $X$ in $\mathbb{C}^3$ with nontrivial polynomial hull so that every point of $\hat{X}$ is a one-point Gleason part remains open. However, the author showed in [8, Theorem 7.2] that a Cantor set with these properties does exist in $\mathbb{C}^4$.

In connection with the above results, we mention that, as noted by Dales and Feinstein [4] (and implicitly noted by Stolzenberg [10]), the polynomial and rational hulls of a compact set $X$ coincide whenever $P(X)$ has dense invertibles.
In the next section, in addition to giving the definitions of the \( k \)-polynomial and \( k \)-rational hulls, we make explicit some standard definitions and notations already used above, and we recall some known results we will need. The proofs of the results stated above are given in Section 3.

2. Preliminaries

For \( X \) a compact Hausdorff space, we denote by \( C(X) \) the algebra of all continuous complex-valued functions on \( X \) with the supremum norm \( \| f \|_X = \sup \{|f(x)| : x \in X\} \). A uniform algebra on \( X \) is a closed subalgebra of \( C(X) \) that contains the constant functions and separates the points of \( X \).

For a compact set \( X \) in \( \mathbb{C}^N \), the polynomial hull \( \hat{X} \) of \( X \) is defined by
\[
\hat{X} = \{ z \in \mathbb{C}^N : |p(z)| \leq \max_{x \in X} |p(x)| \text{ for all polynomials } p \},
\]
and the rational hull \( h_r(X) \) of \( X \) is defined by
\[
h_r(X) = \{ z \in \mathbb{C}^N : p(z) \in p(X) \text{ for all polynomials } p \}.
\]
An equivalent formulation of the definition of \( h_r(X) \) is that \( h_r(X) \) consists precisely of those points \( z \in \mathbb{C}^N \) such that every polynomial that vanishes at \( z \) also has a zero on \( X \).

Observe that when \( N = 2 \), the statement \( z \in h_r(X) \) means that every analytic subvariety of \( \mathbb{C}^2 \) (of pure positive dimension) passing through \( z \) intersects \( X \), whereas when \( N > 2 \), the statement \( z \in h_r(X) \) means only that every pure codimension 1 analytic subvariety of \( \mathbb{C}^N \) passing through \( z \) intersects \( X \). This observation suggests the following extensions of the notions of polynomial and rational hulls introduced by the author in [8, Definitions 3.1 and 3.2]. (Here and throughout the paper, by an analytic subvariety of \( \mathbb{C}^N \) of pure codimension \( m \), we mean a pure dimensional analytic subvariety of \( \mathbb{C}^N \) of pure dimension \( N - m \).) For \( 1 \leq k \leq N \), the \( k \)-rational hull \( h_r^k(X) \) of a compact set \( X \subset \mathbb{C}^N \) is defined by
\[
h_r^k(X) = \{ z \in \mathbb{C}^N : \text{every analytic subvariety of } \mathbb{C}^N \text{ of pure codimension } \leq k \text{ that passes through } z \text{ intersects } X \}.
\]
For \( 2 \leq k \leq N \), the \( k \)-polynomial hull \( \hat{X}^k \) of \( X \subset \mathbb{C}^N \) is defined by
\[
\hat{X}^k = \{ z \in \mathbb{C}^N : z \in h_r^{k-1}(X) \text{ and } z \in \hat{X}^{k-1} \cap V \text{ for every analytic subvariety } V \text{ of } \mathbb{C}^N \text{ of pure codimension } \leq k - 1 \text{ that passes through } z \}.
\]
The 1-polynomial hull \( \hat{X}^1 \) of \( X \) is defined to be the usual polynomial hull \( \hat{X} \).

It is immediate from the definitions that
\[
\hat{X} = \hat{X}^1 \supset h_r(X) = h_1^r(X) \supset \hat{X}^2 \supset h_2^r(X) \supset \cdots \supset \hat{X}^N \supset h_r^N(X) = X.
\]

The set \( X \) is said to be \( k \)-polynomially convex if \( \hat{X}^k = X \) and \( k \)-rational convex if \( h_r^k(X) = X \). It is easily verified that the \( k \)-polynomial hull of a compact set is \( k \)-polynomially convex and the \( k \)-rational hull is \( k \)-rationally convex.

One could also consider modifications of the above definitions. For \( 1 \leq k \leq N \), the quasi-\( k \)-rational hull \( h_r^k(X) \) of \( X \) is defined by
\[
h_r^k(X) = \{ z \in \mathbb{C}^N : p_1, \ldots, p_k \text{ are polynomials such that } p_1(z) = 0, \ldots, p_k(z) = 0, \text{ then } p_1, \ldots, p_k \text{ have a common zero on } X \}.
\]

For \( 2 \leq k \leq N \), the quasi-\( k \)-polynomial hull \( \hat{X}^k \) of \( X \) is defined by
\[
\hat{X}^k = \{ z \in \mathbb{C}^N : z \in \hat{X}^k \text{ and if } p_1, \ldots, p_{k-1} \text{ are polynomials such that } p_1(z) = 0, \ldots, p_{k-1}(z) = 0, \text{ then } z \in (X \cap \{ p_1 = 0, \ldots, p_{k-1} = 0 \})^c \}.
\]

The quasi-1-polynomial hull \( \hat{X}^1 \) of \( X \) is defined to be the usual polynomial hull \( \hat{X} \).

Note that the \( k \)-hulls are ostensibly smaller than the corresponding quasi-\( k \)-hulls. We will use exclusively the \( k \)-hulls. However, the quasi-\( k \)-hulls are perhaps more intuitive than the \( k \)-hulls, and the reader who wishes to do so, may replace the \( k \)-hulls by the quasi-\( k \)-hulls throughout.

We say that a hull \( X_H \) is nontrivial if the set \( X_H \setminus X \) is nonempty.

We denote by \( P(X) \) the uniform closure on \( X \subset \mathbb{C}^N \) of the polynomials in the complex coordinate functions \( z_1, \ldots, z_N \), and we denote by \( R(X) \) the uniform closure of the rational functions holomorphic on (a neighborhood of) \( X \). Both \( P(X) \) and \( R(X) \) are uniform algebras, and it is well known that the maximal ideal space of \( P(X) \) can be naturally identified with \( \hat{X} \), and the maximal ideal space of \( R(X) \) can be naturally identified with \( h_r(X) \).

By an analytic disc in \( \mathbb{C}^N \), we mean an injective holomorphic map \( \sigma : \{ z \in \mathbb{C} : |z| < 1 \} \to \mathbb{C}^N \). By the statement that a subset \( S \) of \( \mathbb{C}^N \) contains no analytic discs, we mean that there is no analytic disc in \( \mathbb{C}^N \) whose image is contained in \( S \).

Let \( A \) be a uniform algebra on a compact space \( X \). The Gleason parts for the uniform algebra \( A \) are the equivalence classes in the maximal ideal space of \( A \) under the equivalence relation \( \varphi \sim \psi \) if \( \| \varphi - \psi \| < 2 \) in the norm on the dual space \( A^* \). (That this really is an equivalence
relation is well-known but not obvious! An alternative formulation of
the definition of Gleason part is that two points $\varphi$ and $\psi$ lie in the same
Gleason part if and only if there exists a constant $c > 0$ such that
$$1/c < u(\varphi)/u(\psi) < c$$
for every function $u > 0$ that is the real part of a function in $A$. See
[6, Theorem VI.2.1] for the equivalence of these two formulations. It
is easily seen that in the second formulation, the condition $u > 0$ can
be replaced by $u \geq 0$, and it is obvious that $u \geq 0$ can be replaced by
$u \leq 0$. We say that a Gleason part is nontrivial if it contains more
than one point. It is immediate that the presence of an analytic disc
in the maximal ideal space of $A$ implies the existence of a nontrivial
Gleason part.

The real part of a complex number (or function) $z$ will be denoted
by $\Re z$.

For the reader’s convenience we recall here some lemmas that we will
use. The first of these is standard, and a short proof can be found in
[12, Lemma 1.7.4]. The others are proven in [8].

**Lemma 2.1.** If $X \subset \mathbb{C}^N$ is a polynomially convex set, and if $E \subset X$ is
polynomially convex, then for every holomorphic function $f$ defined on
a neighborhood of $X$, the set $E \cup (X \cap f^{-1}(0))$ is polynomially convex.

**Lemma 2.2.** [8, Lemma 4.1] Let $\mathcal{K}$ be a collection of compact sets
in $\mathbb{C}^N$ totally ordered by inclusion. Let $K_\infty = \bigcap_{K \in \mathcal{K}} K$. Then
$\hat{K}_k^\infty = \bigcap_{K \in \mathcal{K}} \hat{K}_k^k$ and $h_k^k(K_\infty) = \bigcap_{K \in \mathcal{K}} h_k^k(K)$.

Recall that a subset of a space is called perfect if it is closed and has
no isolated points. Every space contains a unique largest perfect
subset (which can be empty), namely the closure of the union of all
perfect subsets of the space.

**Lemma 2.3.** [8, Lemma 4.2] Let $X \subset \mathbb{C}^N$ be a compact set and let
$E$ be the largest perfect subset of $X$. Then $\hat{E}_k \\supset \hat{X}_k \\supset \hat{E}_k(X) \\supset \hat{X}_k(X)$.

The next lemma plays a key role in our proofs.

**Lemma 2.4.** [8, Lemma 5.9] Let $\Sigma \subset \mathbb{C}^N$ be a compact set, let $p$ be a
polynomial on $\mathbb{C}^N$, and let $X = \{\Re p \leq 0\} \cap \Sigma$. Let $k \geq 2$ be an integer.
Then $\{\Re p \leq 0\} \cap \hat{\Sigma}^{k-1} \supset \hat{X}^{k-1} \supset h_k^{k-1}(X) \supset \{\Re p \leq 0\} \cap \hat{\Sigma}^k$.

Our final lemma is a special case of the previous one because, as is
shown in the proof of Corollary [13] below, $\partial B = \partial B_N = \overline{B}$. The reader
who wishes can, however, supply a direct proof of the lemma similar
to the proof of [7, Lemma 3.2].
Lemma 2.5. [8, Lemma 6.3] Let $N \geq 2$. Let $p$ be a polynomial on $\mathbb{C}^N$, and let $X = \{\Re p \leq 0\} \cap \partial B$. Then $\hat{X} = \mathcal{H}_p^{-1}(X) = \{\Re p \leq 0\} \cap \overline{B}$.

3. The Proofs

In this section we prove the results stated in the introduction. Theorem 1.1 is of course contained in Corollary 1.3. However, we will give a direct proof so as to present the construction giving the one-point Gleason part in its simplest form without the additional complications involved in obtaining dense invertibles.

The proof of Theorem 1.1 has some similarities with the construction of a polynomial hull without analytic discs given by Julien Duval and Norman Levenberg [5] (or see [12, Lemma 1.7.5]). The proof of Theorem 1.2 involves combining the ingredients in the proof of Theorem 1.1 with ingredients in the proof of [8, Theorem 5.2]. (Theorem 5.2 of [8] generalizes [7, Theorem 4.1] whose proof was inspired by the constructions of Duval and Levenberg [5] and of Dales and Feinstein [4].)

Proof of Theorem 1.1. Choose a sequence $\{E_j\}_{j=1}^{\infty}$ of compact polynomially convex subsets of $\mathbb{B} \setminus \{0\}$ such that every point of $\mathbb{B} \setminus \{0\}$ lies in $E_j$ for infinitely many values of $j$. We will construct a sequence of polynomials $\{f_j\}_{j=1}^{\infty}$ such that the sets

$$X_j = \{\Re f_j \leq 0\} \cap \partial B$$

form a decreasing sequence and such that for each $j$ we have

$$f_j(0) = -1 \quad \text{and} \quad \Re f_j > -1/j \text{ on } E_j \cap \hat{X}_{j+1}.$$ 

Then letting $X = \bigcap_{j=1}^{\infty} X_j$ we have $\hat{X} = \bigcap_{j=1}^{\infty} \hat{X}_j = \bigcap_{j=1}^{\infty}(\{\Re f_j \leq 0\} \cap \overline{B})$ by Lemmas 2.2 and 2.5. In particular, $0 \in \hat{X}$. Furthermore, since each point of $\hat{X} \setminus \{0\}$ lies in $E_j$ for infinitely many $j$, for each point $x \in \hat{X} \setminus \{0\}$ there are infinitely many $j$ such that $\Re f_j(x) > -1/j$, and hence such that $0 \leq \Re f_j(x)/\Re f_j(0) < 1/j$. Since $\Re f_j \leq 0$ on $\hat{X}$ for each $j$, this gives that $\{0\}$ is a one-point Gleason part for $P(X)$ by the remarks in Section 2.

We construct the $f_j$ inductively. To begin, set $f_1$ identically equal to $-1$. Then assume for the purpose of induction that polynomials $f_1, \ldots, f_n$ have been chosen such that the sets $X_1, \ldots, X_n$ defined as in (1) for $j = 1, \ldots, n$ form a decreasing sequence, for each $j = 1, \ldots, n$ we have $f_j(0) = -1$, and for each $j = 1, \ldots, n-1$ we have $\Re f_j > -1/j$ on $E_j \cap \hat{X}_{j+1}$. Let $L_n = \{\Re f_n \geq 0\} \cap \overline{B}$ and $C_n = E_n \cap \{\Re f_n \leq -1/n\}$. The sets $L_n$ and $C_n$ are each polynomially convex. Because the sets $f_n(L_n)$ and $f_n(C_n)$ lie in disjoint half-planes, their polynomial hulls
are disjoint. Therefore, $L_n \cup C_n$ is polynomially convex by Kallin’s lemma [9] (or see [12, Theorem 1.6.19]). Adjoining a single point to a polynomially convex set yields another polynomially convex set [1, Lemma 2.2], so $L_n \cup C_n \cup \{0\}$ is also polynomially convex. Therefore, there exists a polynomial $f_{n+1}$ such that $f_{n+1}(0) = -1$ and $\Re f_{n+1} > 0$ on $L_n \cup C_n$. Now define $X_{n+1}$ as in (11) with $j = n + 1$ and observe that then $X_n \supset X_{n+1}$ and $\Re f_n > -1/n$ on $E_n \cap \hat{X}_{n+1}$, so the induction can continue.

**Proof of Theorem 1.2.** Choose a sequence $\{E_j\}_{j=1}^\infty$ of compact polynomially convex subsets of $\hat{\Sigma} \setminus \{x_0\}$ such that every point of $\hat{\Sigma} \setminus \{x_0\}$ lies in $E_j$ for infinitely many values of $j$. Also choose a sequence of polynomials $\{p_j\}_{j=1}^\infty$ that is dense in $P(\Sigma)$ and such that $p_j(x_0) \neq 0$ for each $j$. Let $Z_j = \hat{\Sigma} \cap p_j^{-1}(0)$ for each $j$. We will construct a sequence of polynomials $\{f_j\}_{j=1}^\infty$ such that the sets

(2) \[ X_j = \{\Re f_j \leq 0\} \cap \Sigma \]

form a decreasing sequence and such that for each $j$ we have

\[ f_j(x_0) = -1, \quad \Re f_j > 0 \quad \text{on} \quad Z_j, \quad \text{and} \quad \Re f_j > -1/j \quad \text{on} \quad E_j \cap \hat{X}_{j+1}. \]

Then letting $X = \bigcap_{j=1}^\infty X_j$, we have by Lemmas 2.2 and 2.4 that $h_r^{k-1}(X) = \bigcap_{j=1}^\infty h_r^{k-1}(X_j) \supset \bigcap_{j=1}^\infty (\{\Re f_j \leq 0\} \cap \hat{\Sigma}^k)$. In particular, $x_0 \in h_r^{k-1}(X)$. Also $\hat{X} = \bigcap_{j=1}^\infty \hat{X}_j \subset \bigcap_{j=1}^\infty (\{\Re f_j \leq 0\} \cap \hat{\Sigma})$, again by Lemmas 2.2 and 2.4. Thus $\hat{X}$ is disjoint from each $Z_j$. Consequently, each $p_j$ is invertible in $P(X)$, and hence $P(X)$ has dense invertibles. Furthermore, since each point of $\hat{\Sigma} \setminus \{x_0\}$ lies in $E_j$ for infinitely many $j$, for each point $x \in \hat{\Sigma} \setminus \{x_0\}$ there are infinitely many $j$ such that $\Re f_j(x) > -1/j$, and hence such that $0 \leq \Re f_j(x)/\Re f_j(x_0) < 1/j$. Since $\Re f_j \leq 0$ on $\hat{X}$ for each $j$, this gives that $\{x_0\}$ is a one-point Gleason part for $P(X)$ by the remarks in Section 2.

We construct the $f_j$ inductively. The set $Z_1$ is polynomially convex, and hence $Z_1 \cup \{x_0\}$ is also polynomially convex (by [1, Lemma 2.2]), so there is a polynomial $f_1$ such that

\[ f_1(x_0) = -1 \quad \text{and} \quad \Re f_1 > 0 \quad \text{on} \quad Z_1. \]

Set $X_1 = \{\Re f_1 \leq 0\} \cap \Sigma$. For the inductive step, assume that polynomials $f_1, \ldots, f_n$ have been chosen such that the sets $X_1, \ldots, X_n$ defined as in (2) for $j = 1, \ldots, n$ form a decreasing sequence, for each $j = 1, \ldots, n$ we have $f_j(x_0) = -1$ and $\Re f_j > 0$ on $Z_j$, and for each $j = 1, \ldots, n-1$ we have $\Re f_j > -1/j$ on $E_j \cap \hat{X}_{j+1}$. Let $L_n = \{\Re f_n \geq 0\} \cap \hat{\Sigma}$ and $C_n = E_n \cap \{\Re f_n \leq -1/n\}$. The sets $L_n$ and $C_n$ are each polynomially
convex. Because the sets \( f_n(L_n) \) and \( f_n(C_n) \) lie in disjoint half-planes, their polynomial hulls are disjoint. Therefore, \( L_n \cup C_n \) is polynomially convex by Kallin’s lemma \([9]\) (or see \([12, \text{Theorem 1.6.19}]\)). Applying Lemma 2.1 now gives that \( L_n \cup C_n \cup Z_{n+1} \) is polynomially convex. Consequently, \( L_n \cup C_n \cup Z_{n+1} \uplus \{x_0\} \) is also polynomially convex (by \([1, \text{Lemma 2.2}]\)). Therefore, there exists a polynomial \( f_{n+1} \) such that \( f_{n+1}(x_0) = -1 \) and \( \Re f_{n+1} > 0 \) on \( L_n \cup C_n \cup Z_{n+1} \). Now set \( X_{n+1} = \{\Re f_{n+1} \leq 0\} \cap \Sigma \). Because \( \Re f_{n+1} > 0 \) on \( L_n \), we have \( X_n \supset X_{n+1} \), and because \( \Re f_{n+1} > 0 \) on \( C_n \), we have \( \Re f_n > -1/n \) on \( E_n \cap \hat{X}_{n+1} \). Thus the induction can continue. □

Proof of Corollary 1.3. Let \( x \in \Omega \) be arbitrary. Then every analytic subvariety \( V \) of \( \mathbb{C}^N \) of pure positive dimension passing through \( x \) intersects \( \partial \Omega \). Consequently, \( x \) lies in \( h^{r-1}_n(\partial \Omega) \), and applying the maximum principle to the irreducible component of \( V \) through \( x \) shows that \( x \) is in \( \partial \Omega \cap \hat{V} \). Thus \( \Omega \subset \partial \Omega \cap \mathbb{C}^N \). The corollary now follows immediately from Theorem 1.2. □

Proof of Corollary 1.4. By \([8, \text{Theorem 6.1}]\) there exists a Cantor set \( \Sigma \) in \( \mathbb{C}^3 \) such that \( \hat{\Sigma}^2 \) contains the closed unit ball of \( \mathbb{C}^3 \). Let \( x_0 \) be a point of \( \hat{\Sigma}^2 \setminus \Sigma \). Theorem 1.2 gives that \( \Sigma \) contains a compact set \( K \) such that \( x_0 \) is in \( \hat{K} \setminus K \) and is a one point Gleason part for \( P(K) \), and \( P(K) \) has dense invertibles. Let \( X \) be the largest perfect subset of \( K \). Then \( \hat{X} \setminus X \supset \hat{K} \setminus K \) by Lemma 2.3, so \( x_0 \) is in \( \hat{X} \setminus X \), and the condition that \( x_0 \) is a one-point Gleason part for \( P(K) \) implies that \( x_0 \) is also a one-point Gleason part for \( P(X) \). The set \( X \) is a Cantor set since it is a perfect subset of the Cantor set \( \Sigma \). Finally density of the invertibles in \( P(K) \) implies density of the invertibles in \( P(X) \). □

References

[1] J. T. Anderson, A. Izzo, and J. Wermer, “Polynomial approximation on three-dimensional real-analytic submanifolds of \( \mathbb{C}^n \),” \textit{Proc. Amer. Math. Soc.} \textbf{129} (2001) no. 8, 2395–2402.
[2] B. J. Cole, \textit{One-point parts and the peak point conjecture}, Ph.D. dissertation, Yale University, 1968.
[3] B. J. Cole, S. N. Ghosh, and A. J. Izzo, \textit{A hull with no nontrivial Gleason parts}, \textit{Indiana Univ. Math. J.} \textbf{67} (2018), 739–752.
[4] H. G. Dales and J. F. Feinstein, \textit{Banach function algebras with dense invertible group}, \textit{Proc. Amer. Math. Soc.}, \textbf{136} (2008), 1295 - 1304.
[5] J. Duval and N. Levenberg, \textit{Large polynomial hulls with no analytic structure}, Complex Analysis and Geometry (Trento, 1995), Longman, Harlow, 1997, 119-122.
[6] T. W. Gamelin, \textit{Uniform Algebras}, 2nd ed., Chelsea Publishing Company, New York, NY, 1984.
[7] A. J. Izzo, *Gleason parts and point derivations for uniform algebras with dense invertible group*, Trans. Amer. Math. Soc. **370** (2018), 4299-4321.

[8] A. J. Izzo, *Spaces with polynomial hulls that contain no analytic discs* (submitted).

[9] E. Kallin, *Polynomial convexity: The three spheres problem*, Conference on Complex Analysis Held in Minneapolis, 1964, Springer-Verlag, Berlin, 1965, 301–304.

[10] G. Stolzenberg, *A hull with no analytic structure*, J. Math. Mech. **12** (1963), 103–111.

[11] E. L. Stout, *The Theory of Uniform Algebras*, Bogden & Quigley, New York, 1971.

[12] E. L. Stout, *Polynomial Convexity*, Birkhäuser, Boston, 2007.

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403

*E-mail address*: aizzo@bgsu.edu