NEW METHODS FOR BOUNDING THE NUMBER OF POINTS ON CURVES OVER FINITE FIELDS

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Abstract. We provide new upper bounds on \( N_q(g) \), the maximum number of rational points on a smooth absolutely irreducible genus-\( g \) curve over \( \mathbb{F}_q \), for many values of \( q \) and \( g \). Among other results, we find that \( N_4(7) = 21 \) and \( N_8(5) = 29 \), and we show that a genus-12 curve over \( \mathbb{F}_2 \) having 15 rational points must have characteristic polynomial of Frobenius equal to one of three explicitly given possibilities.

We also provide sharp upper bounds for the lengths of the shortest vectors in Hermitian lattices of small rank and determinant over the maximal orders of small imaginary quadratic fields of class number 1.

Some of our intermediate results can be interpreted in terms of Mordell–Weil lattices of constant elliptic curves over one-dimensional function fields over finite fields. Using the Birch and Swinnerton-Dyer conjecture for such elliptic curves, we deduce lower bounds on the orders of certain Shafarevich–Tate groups.

1. Introduction

The last three decades have seen increasing interest in the calculation of the value of \( N_q(g) \), the maximum number of rational points on a smooth, absolutely irreducible curve \( C \) of genus \( g \) over a finite field \( \mathbb{F}_q \). Initially this increased interest was motivated in part by new constructions of error-correcting codes exceeding the Gilbert–Varshamov bound, but now there are many problems related to the computation of \( N_q(g) \) that are mathematically attractive in their own right, independent of possible applications in coding theory.

In the 1940s, André Weil [46–49] showed that if \( C \) is a genus-\( g \) curve over \( \mathbb{F}_q \), then

\[
q + 1 - 2g\sqrt{q} \leq \#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q},
\]

so that \( N_q(g) \leq q + 1 + 2g\sqrt{q} \). In the 1980s this upper bound was improved in a number of ways. Serre [35] showed that

\[
N_q(g) \leq q + 1 + g[2\sqrt{q}],
\]

and Manin [25] and Ihara [16] showed that the Weil bound could be improved even further when \( g \) is large with respect to \( q \). Generalizing these ideas, Drinfel’d and Vlăduţ showed that for fixed \( q \),

\[
N_q(g) \leq (\sqrt{q} - 1 + o(1))g \quad \text{as } g \to \infty,
\]
and Serre developed the “explicit formulae” method (optimized by Oesterlé), which gives the best bound on \( N_q(g) \) that can be obtained formally using only Weil’s “Riemann hypothesis” for curves and the fact that for every \( d \geq 0 \) the number of degree-\( d \) places on a curve is non-negative. For general \( q \) and \( g \) the Oesterlé bound has not been improved upon, but for certain families and special cases improvements can be made [6, 13, 17, 20–24, 33, 35–38, 41, 42, 50].

In 2000, van der Geer and van der Vlugt published a table of the best upper and lower bounds on \( N_q(g) \) known at the time, for \( g \leq 50 \) and for \( q \) ranging over small powers of 2 and 3. They updated their paper twice a year after its publication, and the revised versions were made available on van der Geer’s website. In 2010, van der Geer, Ritzenthaler, and the authors, with technical assistance from Geerit Oomens, incorporated the updated tables from [8] into the online tables now available at manypoints.org. These new online tables display results for many more prime powers \( q \) than were in [8]: the primes less than 100, the prime powers \( p^i \) for \( p < 20 \) and \( i \leq 5 \), and the powers of 2 up to \( 2^7 \). The original tables of van der Geer and van der Vlugt inspired us to do the work that appeared in our earlier paper [14]: afterwards, we continued to work on the problem of improving the known upper bounds on \( N_q(g) \), and the work we present in this paper was used to help populate the manypoints tables when the site was created.

In our 2003 paper we used a number of techniques to show that certain isogeny classes of abelian varieties over finite fields do not contain Jacobians of curves; by enumerating the isogeny classes of a given dimension \( g \) over a given field \( F_q \) that could possibly contain a Jacobian of a curve with \( N \) points, and then applying our techniques, we were able to show that some values of \( N \) could not occur. We were thus able to improve the known upper bounds on \( N_q(g) \) for many pairs \((q, g)\). In this paper, we introduce four new techniques that can sometimes be used to show that an isogeny class of abelian varieties does not contain a Jacobian. These new techniques were responsible for improving more than 16% of the upper bounds in the 2009 version of [8] when those results were integrated into the manypoints tables.

The first of our new techniques concerns isogeny classes containing product varieties. In our earlier paper, we showed that one can sometimes deduce arithmetic and geometric properties of curves whose Jacobians are isogenous to a product \( A \times B \) when the resultant of the radicals of the “real Weil polynomials” (see Section 2) of \( A \) and \( B \) is small. The first improvement we introduce here is to show that we can replace the resultant by the reduced resultant in these arguments. The reduced resultant is defined, and the new results are explained, in Section 2. We also explain how in certain circumstances we can replace the reduced resultant by an even smaller quantity that depends more delicately on the varieties \( A \) and \( B \).

In our earlier paper, we showed that if \( E \) is a supersingular elliptic curve over \( F_q \) with all endomorphisms defined over \( F_q \), and if \( A \) is an ordinary elliptic curve such that the resultant of the real Weil polynomials of \( E \) and \( A \) is squarefree, then there is no Jacobian isogenous to \( E^n \times A \) for any \( n > 0 \). Our second improvement is to show that the same statement holds when \( A \) is an arbitrary ordinary variety. This is explained in Section 3.

Our third new technique concerns isogeny classes that contain a variety of the form \( A \times E^n \), where \( E \) is an ordinary elliptic curve over \( F_q \). We show in Section 4 that if a curve \( C \) has Jacobian isogenous to \( A \times E^n \), then there is a map from \( C \)
to $E$ whose degree is bounded above by an explicit function of the discriminant of $\text{End} E$, the reduced resultant of the real Weil polynomials of $E$ and $A$, and the exponent $n$. In order to produce the sharpest bounds possible, we give an algorithm for computing the length of the shortest nonzero vectors in small Hermitian lattices over imaginary quadratic fields of class number 1. We provide tables of some of these sharp upper bounds in Section 4.

Our fourth technique is a theorem that gives an easy-to-check necessary and sufficient condition for the entire category of abelian varieties in a given ordinary isogeny class over a finite field to be definable over a subfield. We present this result and explain its significance in Section 5.

We have implemented all of our calculations in the computer algebra package Magma \[3\]. The programs we use are found in the package \texttt{IsogenyClasses.magma}, which is available on the first author’s website: Go to the bibliography page \url{http://alumni.caltech.edu/~however/biblio.html} and follow the link associated to this paper. We outline the structure of these Magma routines in Section 6 and in Section 7 we present a sampling of the computational results we have obtained. These include two new values of $N_q(g)$ and an analysis of the possible Weil polynomials of genus-12 curves over $\mathbb{F}_2$ meeting the Oesterlé bound.

As we have mentioned, some of our arguments give upper bounds for the degrees of maps from a curve $C$ to an elliptic curve $E$. Such upper bounds give restrictions on the determinant of the Mordell–Weil lattice of the base extension of $E$ to the function field $K$ of $C$. In Section 8 we indicate how some of our results, when combined with proven cases of the conjecture of Birch and Swinnerton–Dyer, allow us to give lower bounds (and sometimes even exact formulas) for the size of the Shafarevich–Tate group of $E$ over $K$.

2. Reduced resultants and the gluing exponent

In our previous paper \[14\] we analyzed non-simple isogeny classes of abelian varieties over finite fields by bounding the ‘distance’ between a variety $A$ in the isogeny class and a product variety, measured essentially by the degree of the smallest isogeny from $A$ to a product. We continue to use this same strategy, but we will improve upon our earlier bounds.

In the following definition, we use the convention that the greatest common divisor of the set $\{0\}$ is $\infty$.

\textbf{Definition 2.1.} Let $A_1$ and $A_2$ be abelian varieties over a finite field $k$. Let $E$ be the set of integers $e$ with the following property: If $\Delta$ is a finite group scheme over $k$ that can be embedded in a variety isogenous to $A_1$ and in a variety isogenous to $A_2$, then $e\Delta = 0$. We define the gluing exponent $e(A_1, A_2)$ of $A_1$ and $A_2$ to be the greatest common divisor of the set $E$.

If $A_1$ and $A_2$ have no isogeny factor in common, there exist \textit{nonzero} integers $e$ with the property mentioned in the definition — for example, the proof of \[14\] Lem. 7, p. 1684] shows that the quantity $s(A_1, A_2)$ defined in \[14\] §1] has the desired property — so $e(A_1, A_2)$ is finite in this case. Clearly $e(A_1, A_2)$ depends only on the isogeny classes of $A_1$ and $A_2$.

We make this definition because many of the results in \[14\] remain true if their statements are modified by replacing $s(A_1, A_2)$ with $e(A_1, A_2)$; this is so because
the only property of \(s(A_1, A_2)\) used in the proofs of these results is that it lies in the set \(E\). In particular, [14] Thm. 1, p. 1678] becomes the following.

**Theorem 2.2.** Let \(A_1\) and \(A_2\) be nonzero abelian varieties over a finite field \(k\).

(a) If \(e(A_1, A_2) = 1\) then there is no curve \(C\) over \(k\) whose Jacobian is isogenous to \(A_1 \times A_2\).

(b) Suppose \(e(A_1, A_2) = 2\). If \(C\) is a curve over \(k\) whose Jacobian is isogenous to \(A_1 \times A_2\), then there is a degree-2 map from \(C\) to another curve \(D\) over \(k\) whose Jacobian is isogenous to either \(A_1\) or \(A_2\). \(\Box\)

Also, [14] Lem. 7, p. 1684] becomes:

**Lemma 2.3.** Suppose \(B\) is an abelian variety over a finite field \(k\) isogenous to a product \(A_1 \times A_2\), where \(e(A_1, A_2) < \infty\). Then there exist abelian varieties \(A'_1\) and \(A'_2\), isogenous to \(A_1\) and \(A_2\), respectively, and an exact sequence

\[
0 \to \Delta \to A'_1 \times A'_2 \to B \to 0
\]

such that the projection maps \(A'_1 \times A'_2 \to A'_1\) and \(A'_1 \times A'_2 \to A'_2\) give monomorphisms from \(\Delta\) to \(A'_1[e]\) and to \(A'_2[e]\), where \(e = e(A_1, A_2)\).

Suppose in addition that \(B\) has a principal polarization \(\mu\). Then the pullback of \(\mu\) to \(A'_1 \times A'_2\) is a product polarization \(\lambda_1 \times \lambda_2\), and the projection maps \(A'_1 \times A'_2 \to A'_1\) and \(A'_1 \times A'_2 \to A'_2\) give isomorphisms of \(\Delta\) with \(\ker \lambda_1\) and \(\ker \lambda_2\). In particular, \(\Delta\) is isomorphic to its own Cartier dual. \(\Box\)

Furthermore, [14] Prop. 11, p. 1688] becomes:

**Proposition 2.4.** Let \(A_1\) and \(A_2\) be abelian varieties over a finite field \(k\), and let \(e = e(A_1, A_2)\). Suppose that \(e < \infty\) and that for every \(A'_1\) isogenous to \(A_1\) and every \(A'_2\) isogenous to \(A_2\), the only self-dual finite group scheme that can be embedded in both \(A'_1[e]\) and \(A'_2[e]\) as the kernel of a polarization is the trivial group scheme. Then there is no curve over \(k\) with Jacobian isogenous to \(A_1 \times A_2\). \(\Box\)

And finally, [14] Prop. 13, p. 1689] becomes:

**Proposition 2.5.** Suppose \(C\) is a curve over a finite field \(k\) whose Jacobian is isogenous to the product \(A \times E\) of an abelian variety \(A\) with an elliptic curve \(E\), where \(e(A, E) < \infty\). Then there is an elliptic curve \(E'\) isogenous to \(E\) for which there is map from \(C\) to \(E'\) of degree dividing \(e(A, E)\), and we have \(\#C(k) \leq e(A, E) \cdot \#E(k)\). \(\Box\)

These results will only be more useful than their predecessors from [14] if we can produce bounds on \(e(A_1, A_2)\) that are better than the bound \(e(A_1, A_2) \mid s(A_1, A_2)\) that follows from the proof of [14] Lem. 7, p. 1684]. The main result of this section, Proposition 2.8 below, provides such improved bounds. To state the proposition, we must introduce the idea of the reduced resultant of two polynomials, together with a result about its computation.

**Definition 2.6** (Polst [32], p. 179)]). The reduced resultant of two polynomials \(f\) and \(g\) in \(Z[x]\) is the non-negative generator of the ideal \(Z \cap (f, g)\) of \(Z\), where \((f, g)\) is the ideal of \(Z[x]\) generated by \(f\) and \(g\).

Alternatively (but equivalently), one can define the reduced resultant of \(f\) and \(g\) to be the characteristic of the quotient ring \(Z[x]/(f, g)\). The reduced resultant of \(f\) and \(g\) is 0 if and only if \(f\) and \(g\) are both divisible by a nonconstant polynomial;
also, the reduced resultant divides the usual resultant, and is divisible by all of the prime divisors of the usual resultant.

Computing reduced resultants of monic elements of \( \mathbb{Z}[x] \) is straightforward, as the following lemma shows.

**Lemma 2.7.** Let \( f_1 \) and \( f_2 \) be coprime elements of \( \mathbb{Z}[x] \), not both constant, whose leading coefficients are coprime to one another, and let \( n \) be the reduced resultant of \( f_1 \) and \( f_2 \).

1. There are unique polynomials \( b_1 \) and \( b_2 \) in \( \mathbb{Z}[x] \) such that
   - \( \deg b_1 < \deg f_2 \)
   - \( \deg b_2 < \deg f_1 \)
   - \( n = b_1 f_1 + b_2 f_2 \).

2. Let \( a_1 \) and \( a_2 \) be the unique elements of \( \mathbb{Q}[x] \) such that
   - \( \deg a_1 < \deg f_2 \)
   - \( \deg a_2 < \deg f_1 \)
   - \( 1 = a_1 f_1 + a_2 f_2 \).

Then the reduced resultant \( n \) of \( f_1 \) and \( f_2 \) is the least common multiple of the denominators of the coefficients of \( a_1 \) and \( a_2 \), and the polynomials \( b_1 \) and \( b_2 \) from statement (1) satisfy \( b_1 = na_1 \) and \( b_2 = na_2 \).

**Proof.** First we prove that there exist polynomials \( b_1 \) and \( b_2 \) with the properties listed in statement (1). The unicity of these polynomials will follow from statement (2).

Let \( b_1 \) and \( b_2 \) be arbitrary elements of \( \mathbb{Z}[x] \) such that \( n = b_1 f_1 + b_2 f_2 \). We will show that if \( \deg b_1 \geq \deg f_2 \) or \( \deg b_2 \geq \deg f_1 \) then there are elements \( \overline{b}_1 \) and \( \overline{b}_2 \) of \( \mathbb{Z}[x] \), whose degrees are smaller than those of \( b_1 \) and \( b_2 \), such that \( n = \overline{b}_1 f_1 + \overline{b}_2 f_2 \). By successively replacing the \( b \)'s with the \( \overline{b} \)'s, we will find a pair \( (b_1, b_2) \) of polynomials that satisfy the conditions in statement (1).

Let \( d_1 \) and \( d_2 \) be the degrees of \( b_1 \) and \( b_2 \). Since \( b_1 f_1 + b_2 f_2 \) has degree 0, we see that

\[
d_1 + \deg f_1 = d_2 + \deg f_2,
\]

so that \( d_1 \geq \deg f_2 \) if and only if \( d_2 \geq \deg f_1 \). For each \( i \) let \( u_i \) be the leading coefficient of \( b_i \) and let \( v_i \) be the leading coefficient of \( f_i \). Then we must have \( u_1 v_1 + u_2 v_2 = 0 \), and since \( v_1 \) and \( v_2 \) are coprime to one another, \( v_1 \) divides \( u_2 \) and \( v_2 \) divides \( u_1 \). If we set

\[
\overline{b}_1 = b_1 - (u_1/v_2)x^{d_1-\deg f_2}f_2,
\]
 \[
\overline{b}_2 = b_2 - (u_2/v_1)x^{d_2-\deg f_1}f_1,
\]

then \( \overline{b}_1 \) and \( \overline{b}_2 \) have the desired properties.

Let \( a_1 \) and \( a_2 \) be as in statement (2). The unicity of the \( a \)'s shows that we must have \( b_1 = na_1 \) and \( b_2 = na_2 \), and it follows that \( n \) is a multiple of all of the denominators of the coefficients of \( a_1 \) and \( a_2 \).

On the other hand, if \( m \) is the least common multiple of the denominators of \( a_1 \) and \( a_2 \), then \( ma_1 \) and \( ma_2 \) are elements of \( \mathbb{Z}[x] \), and the equality \( m = (ma_1)f_1 + (ma_2)f_2 \) shows that \( m \) is a multiple of \( n \). Therefore \( n = m \), and the lemma is proved. \( \square \)

We are almost ready to state Proposition 2.8. Recall that the Weil polynomial of a \( d \)-dimensional abelian variety \( A \) over a finite field \( \mathbb{F}_q \) is the characteristic polynomial \( f \) of the Frobenius endomorphism of \( A \), and that the real Weil polynomial
of $A$ is the unique polynomial $h \in \mathbb{Z}[x]$ such that $f(x) = x^d h(x + q/x)$. Recall also that the \textit{radical} of a polynomial is the product of its irreducible factors, each taken once. The radical of the real Weil polynomial of an abelian variety over a finite field is the minimal polynomial of the endomorphism $F + V$, where $F$ is Frobenius and $V$ is Verschiebung.

\textbf{Proposition 2.8.} Let $A_1$ and $A_2$ be nonzero abelian varieties over a finite field $k$ with no isogeny factor in common. Let $g_1$ and $g_2$ be the radicals of their real Weil polynomials, let $n$ be the reduced resultant of $g_1$ and $g_2$, and let $b_1$ and $b_2$ be the unique elements of $\mathbb{Z}[x]$ such that $n = b_1 g_1 + b_2 g_2$ and $\deg b_1 < \deg g_2$ and $\deg b_2 < \deg g_1$.

1. The gluing exponent $e(A_1, A_2)$ divides $n$.
2. Let $g = g_1 g_2$, and suppose that $g$ is divisible by $x^2 - 4q$, where $q = \#k$. If the coefficients of the polynomial $b_1 g_1 + xg/(x^2 - 4q)$ are all even, then $n$ is even, and $e(A_1, A_2)$ divides $n/2$.

We prove Proposition 2.8 at the end of this section, after we state and prove two lemmas. Throughout the rest of this section, $A_1$ and $A_2$ will be abelian varieties as in the statement of the proposition, and $F$ and $V$ will be the Frobenius and Verschiebung endomorphisms of $A_1 \times A_2$.

Our first lemma shows that we can find bounds on $e(A_1, A_2)$ by understanding the endomorphism ring of $A_1 \times A_2$.

\textbf{Lemma 2.9.} For each $i$ let $\varphi_i$ be the projection map $\text{End}(A_1 \times A_2) \to \text{End} A_i$. Suppose $\beta$ is an element of the subring $\mathbb{Z}[F, V]$ of the center of $\text{End}(A_1 \times A_2)$ with the property that $\varphi_1(\beta) = 0$ and $\varphi_2(\beta)$ is an integer $n$. Then the gluing exponent $e(A_1, A_2)$ is a divisor of $n$.

\textbf{Proof.} Suppose $A'_1$ and $A'_2$ are abelian varieties over $k$ that are isogenous to $A_1$ and $A_2$, respectively. Note that $F$ and $V$ are endomorphisms of $A'_1 \times A'_2$, and that every isogeny from $A'_1 \times A'_2$ to $A_1 \times A_2$ respects the actions of $F$ and $V$. Therefore $\beta$ can be viewed as an endomorphism of $A'_1 \times A'_2$, and the projections of $\beta$ to $\text{End} A'_1$ and to $\text{End} A'_2$ are 0 and $n$, respectively.

Suppose $\Delta$ is a finite group scheme over $k$ for which there are monomorphisms $\Delta \to A'_1$ and $\Delta \to A'_2$ for some $A'_1$ and $A'_2$ isogenous to $A_1$ and $A_2$. Frobenius and Verschiebung also act on $\Delta$, and the existence of a monomorphism from $\Delta$ to $A'_1$ shows that $\beta$ acts as 0 on $\Delta$. But the existence of a monomorphism from $\Delta$ to $A'_2$ shows that $\beta$ acts as $n$ on $\Delta$. Therefore $\Delta$ is killed by $n$, and $e(A_1, A_2)$ is a divisor of $n$. \hfill $\square$

\textbf{Lemma 2.10.} Let $\alpha$ be an element of the subring $\mathbb{Z}[F, V]$ of $\text{End}(A_1 \times A_2)$ and let $g_1$ and $g_2$ be the minimal polynomials of $\alpha$ restricted to $A_1$ and $A_2$, respectively. Then the gluing exponent $e(A_1, A_2)$ is a divisor of the reduced resultant of $g_1$ and $g_2$.

\textbf{Proof.} Let $n$ be the reduced resultant of $g_1$ and $g_2$, so that there are elements $b_1$ and $b_2$ of $\mathbb{Z}[x]$ such that $n = b_1 g_1 + b_2 g_2$. Let $\beta = (b_1 g_1)/\alpha$. Then $\beta$ acts as 0 on $A_1$, because it is a multiple of $g_1(\alpha)$, and it acts as $n$ on $A_2$, because $n - \beta = (b_2 g_2)/\alpha$ is a multiple of $g_2(\alpha)$. It follows from Lemma 2.9 that $e(A_1, A_2)$ divides $n$. \hfill $\square$

\textbf{Proof of Proposition 2.8.} Statement (1) follows from Lemma 2.10, and the fact that $g_1$ and $g_2$ are the minimal polynomials of $F + V$ restricted to $A_1$ and $A_2$, respectively.
Suppose $g$ is divisible by $x^2 - 4q$, say $g = (x^2 - 4q)h$. Let $S$ be the subring $\mathbb{Z}[F, V]$ of the center of $\text{End}(A_1 \times A_2)$, and let $R$ be the subring $\mathbb{Z}[F + V]$ of $S$. The tensor product $R_Q = R \otimes Q$ is a product of fields $K_i$, with each $K_i$ corresponding to an irreducible factor of $g$, and the tensor product $S_Q = S \otimes Q$ is a product of fields $L_i$, with each $L_i$ being an extension of $K_i$, of degree 1 if the corresponding factor of $g$ divides $x^2 - 4q$, and of degree 2 otherwise. Let $K$ be the product of the $K_i$ corresponding to factors of $g$ that do not divide $x^2 - 4q$, let $K'$ be the product of the remaining $K_i$, and let $L$ and $L'$ be the products of the corresponding $L_i$. Note that on each factor $L_i$ of $L'$ we have $F = V$.

Suppose the coefficients of the polynomials $r = b_1g_1 + xh$ are all even. Consider the polynomial in $\mathbb{Z}[u, v]$ obtained by evaluating $r$ at $u + v$. Its coefficients are also all even, so the same is true of the polynomial

$$b_1(u + v)g_1(u + v) + (u - v)h(u + v).$$

Let $s$ be 1/2 times this polynomial, so that $s$ lies in $\mathbb{Z}[u, v]$, and $s(F, V)$ lies in $S$.

Consider the element $(F - V)h(F + V)$ of $S_Q$. On each factor $L_i$ of $L$, this element is 0, because $h(F + V) = 0$ on each factor $K_i$ of $K$. But on each factor $L_i$ of $L'$, we have $F - V = 0$. Thus, $(F - V)h(F + V) = 0$.

We see that $2s(F, V)$ differs from $b_1(F + V)g_1(F + V)$ by 0, so $2s(F, V)$ is equal to 0 in $\text{End} A_1$ and is equal to $n$ in $\text{End} A_2$. Since $s(F, V)$ lies in $\text{End}(A_1 \times A_2)$, it follows that $n$ is even, and from Lemma 4.11 we see that the gluing exponent of $A_1$ and $A_2$ divides $n/2$. \hfill \square

3. Supersingular factors in the Jacobian

We say that a Weil polynomial or a real Weil polynomial is ordinary if the corresponding isogeny class consists of ordinary abelian varieties. In this section we will prove the following theorem, which generalizes [14, Cor. 12, p. 1689].

**Theorem 3.1.** Suppose $h \in \mathbb{Z}[x]$ is the real Weil polynomial of an isogeny class of abelian varieties over a finite field $k$, where $q = \# k$ is a square. Suppose further that $h$ can be written $h = (x - 2s)^n h_0$, where $s^2 = q$, where $n > 0$, and where $h_0$ is a nonconstant ordinary real Weil polynomial such that the integer $h_0(2s)$ is squarefree. Then there is no curve over $k$ with real Weil polynomial equal to $h$.

The heart of the proof is a lemma about finite group schemes over finite fields.

**Lemma 3.2.** Suppose $\Delta$ is a finite $\ell$-torsion group scheme over a finite field $k$ with $\# \Delta > \ell$, and suppose the Frobenius and Verschiebung endomorphisms of $\Delta$ act as multiplication by an integer $s$ with $s^2 = q$. If $\Delta$ can be embedded in an ordinary abelian variety $A$ over $k$ with real Weil polynomial $h_0$, then the integer $h_0(2s)$ is divisible by $\ell^2$.

**Proof.** First we prove the lemma under the additional assumption that $A$ is simple. Let $R$ be the subring $\mathbb{Z}[F, V]$ of $\text{End} A$, where $F$ and $V$ are the Frobenius and Verschiebung endomorphisms. Since $A$ is ordinary and simple, the ring $R$ is an order in a CM-field $K$ whose degree over $\mathbb{Q}$ is twice the dimension $d$ of $A$. If we define $R^+$ to be the subring $\mathbb{Z}[F + V]$ of $R$, then $R^+$ is an order in the maximal real subfield $K^+$ of $K$. (These facts follow, for instance, from the Honda–Tate theorem [13, Théorème 1, p. 96].) It is easy to see that the elements

$$1, F, V, F^2, V^2, \ldots, F^{d-1}, V^{d-1}, F^d$$

are the elements of $R^+$. Since $\Delta$ is finite, the Frobenius and Verschiebung endomorphisms of $\Delta$ act as multiplication by $s$, and since $s^2 = q$, we have $h_0(2s)$ is divisible by $\ell^2$. \hfill \square
form a basis for \( R \) as a \( \mathbb{Z} \)-module.

Let \( \mathfrak{p} \) be the ideal \( (\ell, F - s, V - s) \) of \( R \). We will show that \( \mathfrak{p} \) is a prime ideal, and that if \( h_0(2s) \) is not divisible by \( \ell^2 \) then the localization \( R_\mathfrak{p} \) is a regular local ring. We will accomplish this by analyzing the rings \( R/\mathfrak{p} \) and \( R/\mathfrak{p}^2 \).

Let \( F' \) and \( V' \) be the Frobenius and Verschiebung endomorphisms of \( \Delta \), and choose an embedding \( \iota \) of \( \Delta \) into \( A \). The embedding \( \iota \) gives us a homomorphism from \( R \) to \( \text{End} \Delta \) that sends \( F \) to \( F' = s \) and \( V \) to \( V' = s \), and clearly \( \mathfrak{p} \) is contained in the kernel of this homomorphism. Therefore \( \mathfrak{p} \) is not the unit ideal. (We note that the fact that \( \mathfrak{p} \) is not the unit ideal tells us that \( s \) is coprime to \( \ell \); for otherwise \( \mathfrak{p} \) would contain \( F \) and \( V \), and \( F \) and \( V \) are coprime in \( R \) because \( A \) is ordinary — see [12, Lem. 4.12, p. 2372].)

On the other hand, since \( F \equiv V \equiv s \mod \mathfrak{p} \), we see that every power of \( F \) and \( V \) is congruent to an integer modulo \( \mathfrak{p} \), so every element of \( R \) is congruent to an integer modulo \( \mathfrak{p} \). Furthermore, since \( \ell \in \mathfrak{p} \), we find that every element of \( R \) is congruent modulo \( \mathfrak{p} \) to a nonnegative integer less than \( \ell \). This shows that \( R/\mathfrak{p} \cong F_\ell \), so \( \mathfrak{p} \) is prime.

Now we analyze the ring \( R/\mathfrak{p}^2 \). Note that \( \mathfrak{p}^2 \) contains \( (F - s)^2 = F^2 - 2sF + q \) and \( (V - s)^2 = V^2 - 2sV + q \). Using multiples of these elements to eliminate higher powers of \( F \) and \( V \), we see that every element of \( R/\mathfrak{p}^2 \) can be represented by an element of the form \( aF + bV + c \). Using the fact that \( \mathfrak{p}^2 \) contains \( \ell^2 \), \( \ell(F - s) \), and \( \ell(V - s) \), we see that every element of \( R/\mathfrak{p}^2 \) can be represented by an element of this form with the further restriction that \( a, b, \) and \( c \) are nonnegative integers with \( a, b < \ell \) and \( c < \ell^2 \).

Now, \( \mathfrak{p}^2 \) also contains \( (F - s)(V - s) = 2q - (F + V)s = s(2s - (F + V)) \), and since \( s \) is coprime to \( \ell \) and hence not in \( \mathfrak{p} \), we see that \( \mathfrak{p}^2 \) contains \( F + V - 2s \). This shows that we can find representatives as above for which \( b = 0 \).

Suppose, to obtain a contradiction, that \( h_0(2s) \) is not divisible by \( \ell^2 \). Since \( F + V \equiv 2s \mod \mathfrak{p}^2 \), we have \( 0 = h_0(F + V) \equiv h_0(2s) \mod \mathfrak{p}^2 \). This can only happen if \( h_0(2s) \) is divisible by \( \ell \), and then our assumption that \( h_0(2s) \) is not divisible by \( \ell^2 \) implies that \( \mathfrak{p}^2 \) contains \( \ell \). This means that every element of \( R/\mathfrak{p}^2 \) has a representative of the form \( aF + c \) where \( a \) and \( c \) are both non-negative integers less than \( \ell \). In particular, \( \mathfrak{p}/\mathfrak{p}^2 \) is a 1-dimensional \( R/\mathfrak{p} \)-vector space, and so \( R_\mathfrak{p} \) is a regular local ring.

Deligne [5] proved a result that implies that there is an equivalence of categories between the category of abelian varieties over \( k \) that are isogenous to \( A \) and the category of nonzero finitely-generated \( R \)-submodules of \( K \). This equivalence of categories is fleshed out in [12]; the only result we will use in our argument is that if our variety \( A \) corresponds to the isomorphism class of an \( R \)-module \( \mathfrak{A} \subset K \), then the \( R \)-modules \( A[\ell] \) and \( \mathfrak{A}/\ell\mathfrak{A} \) are isomorphic to one another.

The image of \( \Delta \) in \( A \) must sit inside the largest subgroup of \( A[\ell] \) on which \( F \) and \( V \) acts as \( s \), so we would like to analyze the \( \mathfrak{p} \)-primary part of \( A[\ell] \). This \( \mathfrak{p} \)-primary part is simply \( \mathfrak{A}_p/\ell\mathfrak{A}_p \), and since \( R_p \) is regular, this last module is isomorphic to \( R_p/\mathfrak{p}^a \), where \( \mathfrak{p}^a \) is the largest power of \( \mathfrak{p} \) dividing \( \ell \). The submodule of \( R_p/\mathfrak{p}^a \) on which \( \mathfrak{p} \) acts trivially is \( \mathfrak{p}^{a-1}/\mathfrak{p}^a \), which has order \( \ell \). Since \( \Delta \) has order greater than \( \ell \), we see that \( \Delta \) cannot be embedded in \( A[\ell] \), a contradiction.

This proves the lemma in the case where \( A \) is simple. Now we turn to the general case. The decomposition of \( A \) up to isogeny corresponds to the factorization of \( h_0 \).
Suppose 

\[ h_0 = h_1^{e_1} h_2^{e_2} \cdots h_r^{e_r} \]

is the factorization of \( h_0 \) into powers of distinct irreducibles, and suppose, to obtain a contradiction, that \( h_0(2s) \) is not divisible by \( \ell^2 \). As before, choose an embedding \( \iota \) of \( \Delta \) into \( A \).

The radical of \( h_0 \) is the minimal polynomial of \( F + V \), viewed as an element of \( \text{End} A \), so \( A \) is killed by \( h_0(F + V) \). Since \( \Delta \) can be embedded in \( A \), it is also killed by this polynomial in \( F + V \). But since \( F + V = 2s \) on \( \Delta \), we find that \( h_0(2s) \) kills \( \Delta \), and since \( \Delta \) is \( \ell \)-torsion, we see that \( h_0(2s) \) is divisible by \( \ell \).

This means that exactly one prime factor \( h_i \) of \( h_0 \) has the property that \( h_i(2s) \) is divisible by \( \ell \), and for this \( i \) we must have \( e_i = 1 \). By renumbering, we may assume that \( h_1(2s) \) is divisible by \( \ell \) and that \( e_1 = 1 \).

Let \( H = h_2^{e_2} \cdots h_r^{e_r} \), so that \( H(2s) \) is coprime to \( \ell \). If we apply \( H(F + V) \) to \( A \), we obtain a subvariety \( A_1 \) of \( A \) on which \( F + V \) satisfies \( h_1 \). Since \( H(F + V) \) acts as \( H(2s) \) on \( \Delta \), we see that the image of \( \iota(\Delta) \) under \( H(F + V) \) is simply \( \iota(\Delta) \). Thus, \( \iota \) provides an embedding of \( \Delta \) into the simple variety \( A_1 \). As we have shown, the existence of this embedding is inconsistent with the fact that \( h_1(2s) \) is not divisible by \( \ell^2 \), and this contradiction proves the lemma. \( \square \)

Proof of Theorem 3.1. Let \( E \) be an elliptic curve over \( k \) with real Weil polynomial equal to \( x - 2s \) and let \( A \) be an abelian variety over \( k \) with real Weil polynomial equal to \( h_0 \). Let \( e \) be the gluing exponent of \( E^n \) and \( A \). Proposition 2.8 says that \( e \) divides the reduced resultant of \( x - 2s \) and the radical of \( h_0 \); this reduced resultant divides the resultant of \( x - 2s \) and \( h_0 \), which is the squarefree integer \( h_0(2s) \), so \( e \) is squarefree.

Suppose \( \Delta \) is a nontrivial self-dual group scheme that can be embedded in a variety isogenous to \( E^n \) as the kernel of a polarization, and that can also be embedded a variety isogenous to \( A \) as the kernel of a polarization. Let \( \ell \) be a prime divisor of the order of \( \Delta \); then the \( \ell \)-primary part \( \Delta_\ell \) of \( \Delta \) is a nontrivial group scheme that can be embedded in a variety isogenous to \( E^n \) and in a variety isogenous to \( A \). Furthermore, since \( e \Delta_\ell = 0 \), and \( e \) is squarefree, we see that \( \Delta_\ell \) is \( \ell \)-torsion. And finally, since \( \Delta \) is isomorphic to the kernel of a polarization and hence has square order, we see that the order of \( \Delta_\ell \) is divisible by \( \ell^2 \).

Frobenius and Verschiebung act on \( E \) as multiplication by the integer \( s \), so they act on \( E^n \) and on all varieties isogenous to \( E^n \) in the same way. Since \( \Delta_\ell \) can be embedded in a variety isogenous to \( E^n \), we see that Frobenius and Verschiebung act as multiplication by \( s \) on \( \Delta_\ell \) as well.

But then \( \Delta_\ell \) satisfies the hypotheses of Lemma 3.2, so we find that \( h_0(2s) \) is divisible by \( \ell^2 \), a contradiction. Thus, no nontrivial self-dual group scheme can be embedded as the kernel of a polarization both in a variety isogenous to \( E^n \) and in a variety isogenous to \( A \). It follows from Proposition 2.4 that there is no curve over \( k \) with real Weil polynomial equal to \( h \). \( \square \)

4. Hermitian lattices

We saw in Proposition 2.4 that if the Jacobian \( J \) of a curve \( C \) over a finite field is isogenous to a product \( A \times E \), where \( A \) is an abelian variety and \( E \) is an elliptic curve with \( \text{Hom}(E, A) = \{0\} \), then we can derive an upper bound on the degree of the smallest-degree map from \( C \) to an elliptic curve isogenous to \( E \). Our goal in
this section is to prove a similar result when \( J \) is isogenous to \( A \times E^n \), for \( n > 0 \) and \( E \) ordinary.

**Proposition 4.1.** Suppose \( C \) is a curve over \( \mathbf{F}_q \) whose Jacobian is isogenous to a product \( A \times E^n \), where \( n > 0 \), where \( E \) is an ordinary elliptic curve of trace \( t \), and where \( A \) is an abelian variety such that the gluing exponent \( e = e(A, E) \) is finite. Let \( h \) be the real Weil polynomial of \( A \) and let \( b = \gcd(e^n, b(t)) \). Let \( E' \) be any elliptic curve isogenous to \( E \) whose endomorphism ring is generated over \( \mathbf{Z} \) by the Frobenius. Then there is a map from \( C \) to \( E' \) of degree at most

\[
\gamma_{2n} b^{1/n} \sqrt{|t^2 - 4b|}/4,
\]

where \( \gamma_{2n} \) is the Hermite constant for dimension \( 2n \).

**Remark.** We have

\[
\gamma_2^2 = 4/3, \quad \gamma_4^2 = 4, \quad \gamma_6^8 = 64/3, \quad \gamma_8^8 = 256, \quad \text{and} \quad \gamma_{10}^{10} < 5669.
\]

The value of \( \gamma_2 \) was given by Hermite \([11]\), the value of \( \gamma_4 \) by Korkine and Zolotareff \([15]\), and the values of \( \gamma_6 \) and \( \gamma_8 \) by Blichfeldt \([2]\). The upper bound for \( \gamma_{10} \) follows from an estimate of Blichfeldt \([1]\). General upper bounds for \( \gamma_n \) can be found in \([10]\, \S38]\).

In any specific instance, it may be possible to improve the bound from Proposition \([4.1]\) by using refinements of the individual lemmas from which the proof of the proposition is built. We turn now to these lemmas. After presenting the proof of Proposition \([4.1]\), we will explore some cases in which the proposition can be improved.

**Lemma 4.2.** Let \( E \) be an ordinary elliptic curve over \( \mathbf{F}_q \), let \( R \) be the endomorphism ring of \( E \), let \( A \) be an abelian variety isogenous to \( E^n \) such that \( R \) is contained in the center of \( \text{End} A \), and let \( \Lambda \) be a polarization of \( A \). Let \( Q : \text{Hom}(E, A) \to \mathbf{Z} \) be the map that sends 0 to 0 and that sends a nonzero morphism \( \psi \) to the square root of the degree of the pullback polarization \( \psi^* \Lambda \). Then \( Q \) is a positive definite quadratic form on \( \text{Hom}(E, A) \), the determinant of \( Q \) is equal to \( |(\text{disc } R)/4|^n \deg \Lambda \), and there is a nonzero element \( \psi \) of \( \text{Hom}(E, A) \) such that

\[
Q(\psi) \leq \gamma_{2n} (\deg \Lambda)^{1/(2n)} \sqrt{|\text{disc } R|/4}.
\]

**Remark.** By the determinant of a positive definite quadratic form \( Q : L \to \mathbf{Z} \) on a free \( \mathbf{Z} \)-module \( L \), we mean the following: Let \( B \) be the unique symmetric bilinear form \( L \times L \to \mathbf{Q} \) such that \( Q(x) = B(x, x) \) for all \( x \). Then we define the determinant of \( Q \) to be the determinant of the Gram matrix for \( B \).

If we let \( L^* = \text{Hom}(L, \mathbf{Z}) \) be the dual of \( L \), then \( 2B \) defines a homomorphism \( b : L \to L^* \), and we have

\[
\det Q = (1/2)^{\text{rank } L} |L^*:b(L)|.
\]

**Proof of Lemma 4.2.** Let \( p \) be the characteristic of \( \mathbf{F}_q \), let \( \pi \) and \( \overline{\pi} \) be the Frobenius and Verschiebung endomorphisms of \( E \), and let \( R_0 \) be the subring \( \mathbf{Z}[\pi, \overline{\pi}] \) of \( R \). The theory of Deligne modules \([5,12]\) shows that the category of abelian varieties over \( \mathbf{F}_q \) that are isogenous to some power of \( E \) is equivalent to the category of torsion-free finitely generated \( R_0 \)-modules. The equivalence depends on a choice: we must specify an embedding \( \varepsilon : W \to \mathbf{C} \) of the Witt vectors \( W \) over \( \mathbf{F}_q \) into the complex numbers. The equivalence sends an abelian variety to the first integral
homology group of the complex abelian variety obtained by base-extension (via $\varepsilon$) from the canonical lift of the variety. It follows from the irreducibility of Hilbert class polynomials that we can choose the embedding $\varepsilon$ so that the equivalence takes the elliptic curve $E$ to the $R_0$-module $R$.

We recall from \cite[§4]{12} how the concept of a polarization translates to the category of Deligne modules, at least in the special case we are considering. Let $K$ be the quotient field of $R_0$. The embedding $\varepsilon$ determines a $p$-adic valuation $\nu$ on the field of algebraic numbers sitting inside the complex numbers; we let $\varphi : K \to \mathbb{C}$ be the complex embedding of $K$ such that $\nu(\varphi(\pi)) > 0$. A polarization on a finitely-generated torsion-free $R_0$-module $M$ is a skew-Hermitian form

$$S : (M \otimes \mathbb{Q}) \times (M \otimes \mathbb{Q}) \to K$$

such that $\text{Tr}_{K/\mathbb{Q}} S(M, M) \subseteq \mathbb{Z}$ and such that $\varphi(S(x, x))$ lies in the lower half-plane for all nonzero $x \in M$. (Note that $\varphi(S(x, x))$ must be pure imaginary, since $S$ is skew-Hermitian.) The composition $\text{Tr}_{K/\mathbb{Q}} \circ S$ gives a map $M \to \text{Hom}(M, \mathbb{Z})$; the degree of the polarization is the size of the cokernel of this map.

Now let $M$ be the $R_0$-module corresponding to the $n$-dimensional variety $A$. Since $R$ lies in the center of the endomorphism ring of $M$, we see that the $R_0$-module structure of $M$ extends to an $R$-module structure. Thus, every element $x$ of $M$ determines a map $\alpha_x : R \to M$ defined by $\alpha_x(r) = rx$, and every map from $R$ to $M$ is of this form. We find that $\text{Hom}(E, A) \cong M$.

Let $S$ be the polarization on $M$ corresponding to the polarization $\Lambda$ of $A$, and let $\psi$ be a nonzero map from $E$ to $A$, corresponding to a map $\alpha : R \to M$, say $\alpha(r) = rx$ for a nonzero $x \in M$. The polarization $\psi^* \Lambda$ then corresponds to the skew-Hermitian form

$$S_x : K \times K \to K$$

defined by

$$S_x(u, v) = S(ux, vx) = u\overline{\psi}S(x, x).$$

Our map $R \to \text{Hom}(R, \mathbb{Z})$ is then

$$v \mapsto \left( u \mapsto \text{Tr}_{K/\mathbb{Q}} u\overline{\psi}S(x, x) \right),$$

and the size of the cokernel of this map is the index of the $R$-module $S(x, x)R$ inside the trace dual of $R$. Let $\delta$ be a generator of the different of $R$, chosen so that $\varphi(\delta)$ is pure imaginary and in the upper half plane. Then the size of the cokernel is the norm of $\delta S(x, x)$. Since $S(x, x)$ and $\delta$ are both pure imaginary, and their images under $\varphi$ lie in opposite half-planes, the product $\delta S(x, x)$ is a positive rational number, so its norm is just its square. Thus, under the identification $\text{Hom}(E, A) \cong M$, the function $Q$ in the statement of the lemma is the map $M \to \mathbb{Z}$ defined by $Q(x) = \delta S(x, x)$. Therefore $Q$ is a quadratic form.

We compute that the symmetric bilinear form $B$ on $M$ such that $Q(x) = B(x, x)$ is given by

$$B(x, y) = (1/2) \text{Tr}_{K/\mathbb{Q}} \delta S(x, y).$$

The map $M \to \text{Hom}(M, \mathbb{Z})$ determined by $\text{Tr}_{K/\mathbb{Q}} \circ S$ has cokernel of size $\text{deg} \Lambda$; replacing $S$ with $\delta S$ increases the size of the cokernel by $\text{Norm}(\delta)^n = |\text{disc } R|^n$. Therefore

$$\det Q = (1/2)^\text{rank}_K M |\text{disc } R|^n \text{ deg } \Lambda$$

$$= |\text{(disc } R/2|^n \text{ deg } \Lambda.$$
The final statement of the lemma follows from [10, Thm. 38.1, p. 386] or [4, Thm. 12.2.1, p. 260].

**Lemma 4.3.** Let $C$ be a curve over a field $k$, let $E$ be an elliptic curve over $k$, and let $\mu: E \to \hat{E}$ and $\lambda: \text{Jac} C \to \text{Jac} \hat{C}$ be the canonical polarizations of $E$ and of the Jacobian of $C$, respectively. Suppose $C$ has a $k$-rational divisor $D$ of degree 1, and let $\varepsilon: C \to \text{Jac} C$ be the embedding that sends a point $P$ to the class of the divisor $P - D$.

Suppose $\psi$ is a nonzero homomorphism from $E$ to $\text{Jac} C$, so that $\hat{\psi} \lambda \psi = d\mu$ for some positive integer $d$. Let $\varphi: C \to E$ be the map $\mu^{-1} \hat{\psi} \lambda \varepsilon$. Then $\psi = \varphi^*$, and $\varphi$ has degree $d$.

**Proof.** The maps in the lemma can be arranged into the following diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{d} & E \\
\psi & & \sim \\
C & \xrightarrow{\varepsilon} & \text{Jac} C \\
\downarrow & & \downarrow \\
\text{Jac} C & \xrightarrow{\lambda} & \text{Jac} C \\
\end{array}
\]

Inserting another copy of $\text{Jac} C$ into the bottom row we obtain

\[
\begin{array}{ccc}
E & \xrightarrow{d} & E \\
\psi & & \sim \\
C & \xrightarrow{\varepsilon} & \text{Jac} C \\
\downarrow & & \downarrow \\
\text{Jac} C & \xrightarrow{\lambda} & \text{Jac} C \\
\end{array}
\]

and we see that the middle vertical map from $\text{Jac} C$ to $E$ is equal to $\varphi_*$. Lemma 4.4 below, shows that then we must have $\psi = \varphi^*$, which is the first part of the conclusion of the lemma. Since $\varphi_* \varphi^*$ is equal to multiplication by the degree of $\varphi$, we find that $\deg \varphi = d$. \hfill $\Box$

**Lemma 4.4.** Let $f: C \to D$ be a nonconstant morphism of curves over a field $k$, and let $f_*: \text{Jac} C \to \text{Jac} D$ and $f^*: \text{Jac} D \to \text{Jac} C$ be the associated push-forward and pullback maps between the Jacobians of $C$ and $D$. Under the natural isomorphisms between $\text{Jac} C$ and $\text{Jac} D$ and their dual varieties, the isogenies $f_*$ and $f^*$ are dual to one another.

**Remark.** This statement is proven by Mumford [29, §1]. Mumford assumes that $f$ has degree 2 because his paper is concerned with double covers; however, the proof does not use this assumption. For the convenience of the reader, we include a version of his proof here.

**Proof of Lemma.** Let $\lambda_C$ and $\lambda_D$ be the canonical principal polarizations of $\text{Jac} C$ and $\text{Jac} D$. It suffices to prove the lemma in the case where $k$ is algebraically closed, so we may assume that $C$ has a $k$-rational point $P$. Let $g_C$ be the embedding of $C$ into $\text{Jac} C$ that sends a point $Q$ to the class of the divisor $Q - P$, and let $g_D$ be the embedding of $D$ into $\text{Jac} D$ that sends a point $Q$ to the class of the divisor $Q - P$.
$Q - f(P)$. Then we have a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{gC} & \text{Jac } C \\
\downarrow f & & \downarrow f_* \\
D & \xrightarrow{gD} & \text{Jac } D.
\end{array}
$$

Applying the functor Pic$^0$, we obtain the diagram

$$
\begin{array}{ccc}
\text{Jac } C & \xrightarrow{\lambda_C^{-1}} & \text{Jac } C \\
\downarrow f^* & & \downarrow f_* \\
\text{Jac } D & \xrightarrow{\lambda_D^{-1}} & \text{Jac } D,
\end{array}
$$

which expresses the fact that $f^*$ is isomorphic to the dual of $f_*$; that is, $f^*$ and $f_*$ are dual to one another. \qed

**Proof of Proposition 4.1.** Suppose $C$ is a curve over a finite field $\mathbb{F}_q$ whose Jacobian is isogenous to a product $A \times E^n$, where $n > 0$, where $E$ is an ordinary elliptic curve with trace $t$, and where $A$ is an abelian variety such that the gluing exponent $e = E(A, E)$ is finite. Let $E'$ be an ordinary elliptic curve over $\mathbb{F}_q$ isogenous to $E$ such that the endomorphism ring $R$ of $E'$ is generated by the Frobenius; this means that the discriminant of $R$ is equal to $t^2 - 4q$.

Lemma 2.3 says that there is a variety $A'$ isogenous to $A$, a variety $B$ isogenous to $E^n$, and an exact sequence

$$0 \to \Delta \to A' \times B \to \text{Jac } C \to 0$$

where $\Delta$ is a finite group scheme and the induced maps $\Delta \to A'$ and $\Delta \to B$ are monomorphisms. Let $\lambda$ be the canonical principal polarization of $\text{Jac } C$. Again by Lemma 2.3, pulling $\lambda$ back to $B$ gives us a polarization $\Lambda$ of $B$ with kernel isomorphic to $\Delta$. The lemma also says that $\Delta$ can be embedded into the $e$-torsion of $B$, so the order of $\Delta$ is a divisor of $e^{2n}$.

Let $\eta_\Delta \in \text{End } \Delta$ be the sum of the Frobenius and Verschiebung endomorphisms of $\Delta$, and let $\eta \in \text{End } A'$ be the sum of the Frobenius and Verschiebung endomorphisms of $A'$. Since $\Delta$ embeds into $B$, we have $\eta_\Delta = t$ on $\Delta$. Thus, the image of $\Delta$ in $A'$ must lie in the kernel of the endomorphism $\eta - t$. The degree of this endomorphism is the constant term of its characteristic polynomial, and since the characteristic polynomial of $\eta$ is $h^2$, the characteristic polynomial of $\eta - t$ is $h(x+t)^2$, whose constant term is $h(t)^2$. Thus, $\Delta$ embeds into a group scheme of order $h(t)^2$, so the order of $\Delta$ is a divisor of $b^2$, where $b = \gcd(e^n, h(t))$.

Let $Q$ be the map that sends a nonzero homomorphism $\psi: E' \to B$ to the square root of the degree of the pullback polarization $\psi^* \Lambda$. Lemma 4.2 says that there is a nonzero element $\psi$ of $\text{Hom}(E', B)$ such that

$$Q(\psi) \leq \gamma_{2n} (\# \Delta)^{1/(2n)} \sqrt{\text{disc } R} / 4 \leq \gamma_{2n} b^{1/n} \sqrt{\text{disc } R} / 4.$$
Thus we have a diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\psi^*} & \tilde{E} \\
\downarrow & & \downarrow \\
B & \xrightarrow{\lambda} & \tilde{B} \\
\downarrow & & \downarrow \\
\text{Jac } C & \xrightarrow{\lambda} & \text{Jac } C
\end{array}
\]

where the vertical arrows on the right are the dual morphisms of the vertical arrows on the left. Using Lemma 4.3, we obtain a map from $C$ to $E$ whose degree $d$ is equal to $Q(\psi)$, so that $d \leq \gamma n b^{1/n} \sqrt{|t^2 - 4q|/4}$. □

As we mentioned earlier, the bound in Proposition 4.1 can sometimes be improved. There are two places in the proof of the proposition where improvements can be made: First, when one has specific varieties $A$ and $E$ in hand, the estimate for the size of the group scheme $\Delta$ can often be sharpened by a more thorough analysis of the $e(A, E)$-torsion group schemes that can be embedded in a variety isogenous to $E^n$ and in a variety isogenous to $A$. Second, we obtain upper bounds on short vectors for the quadratic form $Q$ by using general bounds on short vectors in lattices. But the lattices we are considering are quite special — they come provided with an action of an imaginary quadratic order — so there is no reason to suspect that the bounds for general lattices will be sharp in our situation. Improving upper bounds on the lengths of short vectors in such lattices is helpful enough in practice that we will devote the remainder of this section to doing so.

We will start by studying pullbacks of polarizations on powers of elliptic curves. Then we will focus on the very special case of ordinary elliptic curves over finite fields that are isogenous to no other curves.

**Lemma 4.5.** Let $E$ be an elliptic curve over a field $k$ and let $R = \text{End } E$, so that $R$ is a ring with a positive involution. Let $\lambda_0$ be the canonical principal polarization of $E$. Fix an integer $n > 0$, and let $\Lambda_0$ denote the product polarization $\lambda_0^n$ of $E^n$. Let $\Phi_n$ denote the map from polarizations of $E^n$ to $\text{End } E^n \cong M_n(R)$ that sends a polarization $\Lambda$ of $E^n$ to the element $\Lambda_0^{-1} \Lambda$ of $\text{End } E^n$. Then:

1. The image of $\Phi_n$ is the set of positive definite Hermitian matrices in $M_n(R)$.
2. The degree of a polarization $\Lambda$ of $E^n$ is the square of the determinant of $\Phi(\Lambda)$.
3. Let $\Lambda$ be a polarization of $E^n$ and let $H$ be the Hermitian form on $R^n$ determined by $\Phi_n(\Lambda)$. If $\alpha : E \to E^n$ is a nonzero map corresponding to a vector $v$ of elements of $R$, then the polarization $\alpha^* \Lambda$ of $E$ is equal to $d\lambda_0$, where $d = H(v, v)$.

**Proof.** In general, if $A$ is an abelian variety with a principal polarization $\mu_0$, the map $\mu \to \mu_0^{-1} \mu$ identifies the set of polarizations of $A$ with the set of elements of $\text{End } A$ that are fixed by the Rosati involution associated to $\mu_0$ and whose minimal polynomials have only positive real roots. (See the final paragraph of §21 of [28].) The Rosati involution on $\text{End } E^n$ associated to the product polarization $\Lambda_0$ is the conjugate transpose, and the roots of the minimal polynomial of a Hermitian matrix are all positive precisely when the matrix is positive definite. This proves (1).
The degree of an element of \( \text{End} \, E^n \) is equal to the norm (from \( \text{End} \, E \to \mathbb{Z} \)) of its determinant. Since the determinant of a Hermitian matrix already lies in \( \mathbb{Z} \), its norm is just its square. This proves (2).

Item (3) follows from noting that upon identifying \( E \) with its dual via \( \lambda_0 \), the dual map \( \tilde{\alpha} : E^n \to E \) is given by the conjugate transpose \( v^* \) of the vector \( v \). The pullback of \( \Lambda \) to \( E \) is then given by \( v^* \Phi_n(\Lambda) \nu \lambda_0 \), and this is \( H(v, v) \lambda_0 \).

Given an imaginary quadratic order \( R \), an integer \( n > 0 \), and an integer \( D > 0 \), let \( d(R, n, D) \) be the smallest integer \( d \) with the following property: For every positive definite Hermitian matrix \( M \in M_n(R) \) of determinant \( D \), the associated Hermitian form over \( R^n \) has a short vector of length at most \( d \). The next lemma shows that in a very special case, the function \( d(R, n, D) \) gives a bound on the minimum nonzero value of the function \( Q \) from Lemma 4.2.

**Lemma 4.6.** Let \( E \) be an elliptic curve over \( \mathbb{F}_q \), let \( t \) be the trace of \( E \), and suppose \( t^2 - 4q \) is the discriminant of the maximal order \( R \) of an imaginary quadratic field of class number 1. Let \( \Lambda \) be a polarization of a variety \( A \) isogenous to \( E^n \), and let \( Q : \text{End}(E, A) \to \mathbb{Z} \) be as in Lemma 4.2. Then there is a nonzero element \( \psi \in \text{End}(E, A) \) such that \( Q(\psi) \leq d(R, n, \sqrt{\deg \Lambda}) \).

**Proof.** The theory of Hermitian modules [24, Appendix], or of Deligne modules [5, 12], shows that the varieties isogenous to \( E^n \) correspond to rank-\( n \) modules over \( R \). There is only one such module up to isomorphism, because \( R \) has class number 1. Therefore \( A \) is isomorphic to \( E^n \).

Let \( \Phi_n \) be as in Lemma 4.3 and let \( M \) be the Hermitian matrix \( \Phi_n(\Lambda) \), so that part (2) of the lemma shows that \( \det M = \sqrt{\deg \Lambda} \). Let \( H \) be the Hermitian form on \( R^n \) determined by \( M \). If \( \psi : E \to E^n \) corresponds to a vector \( v \in R^n \), then part (3) of Lemma 4.3 shows that \( Q(\psi) = H(v, v) \leq d(R, n, \sqrt{\deg \Lambda}) \).

**Lemma 4.7.** The integer entries in Tables 1–5 give correct values of \( d(R, n, D) \).

**Proof.** Our proof is computational. We will outline two algorithms for computing \( d(R, n, D) \) when \( R \) is a maximal order of class number 1. We have implemented these algorithms in Magma, and the resulting programs are available at the URL mentioned in the introduction — follow the links related to this paper, and download the file HermitianForms.magma. The entries in Tables 1–5 reflect the output of these programs.

Let \( R \) be the maximal order of an imaginary quadratic field \( K \) of class number 1. Our first algorithm will compute, for any rank \( n \) and determinant \( D \), the value of \( d(R, n, D) \).

Let \( M \) be an \( n \)-by-\( n \) Hermitian matrix with entries in \( R \) and let \( L \) be the corresponding Hermitian \( R \)-lattice. For each positive integer \( i \leq n \) we define the \( i \)‘th successive \( R \)-minimum of \( L \) to be the smallest integer \( N_i \) such that the elements of \( L \) of length \( N_i \) or less span a \( K \)-vector space of dimension at least \( i \).

Let the successive \( R \)-minima of \( L \) be \( N_1, \ldots, N_n \). Let \( L_2 \) be the \( R \)-lattice \( L \) viewed as a rank-\( 2n \) lattice over \( \mathbb{Z} \), and let \( M_1, \ldots, M_{2n} \) be the successive minima of \( L_2 \). Then

\[
N_1 = M_1 \\
N_2 \leq M_3 \\
N_3 \leq M_5
\]
and so on, so that
\[
(N_1 \cdots N_n)^2 = N_1^2 N_2^2 \cdots N_n^2 \\
\leq (M_1 M_2)(M_3 M_4) \cdots (M_{2n-1} M_{2n}).
\]

Arguing as in the proof of Lemma 4.2, we find that
\[
\det L_Z = (\det M)^2 (|\text{disc } R|/4)^n,
\]
and combining this with [4, Thm 12.2.2, p. 262] we find that
\[
N_1 \cdots N_n \leq \gamma_{2n}^n \left( \det M \right) (|\text{disc } R|/4)^{n/2}.
\]

From this, we obtain an upper bound on \(N_1\).

Let this initial upper bound be \(B\). We let a variable \(s\) take on successive values \(B, B-1, \ldots\), down to 1. For a given value of \(s\), we try to construct \(R\)-lattices whose successive \(R\)-minima are all greater than or equal to \(s\). The first \(s\) for which we succeed in constructing such a lattice will be the value of \(d(R, n, D)\).

We attempt to construct an \(R\)-lattice with \(N_1 = s\) as follows:

Suppose the successive \(R\)-minima of \(L\) are all \(s\) or larger. The product bound above gives us a finite set of values of the \(N_i\) to consider. For each possible set of \(N_i\), suppose we have an \(R\)-lattice \(L\) with those minima. Consider the sublattice \(L'\) of \(L\) generated by vectors giving those minima. The Gram matrix for \(L'\) will have the \(N_i\) on its diagonal, and we get bounds for the other entries from the fact that each rank-2 sublattice of \(L'\) is positive definite and has no vectors of length less than \(s\). So we can enumerate all of the \(L'\), and then see whether any of the \(L'\) have superlattices with discriminant \(D\) and with no vectors of length less than \(s\).

Even without a formal complexity analysis, it is not hard to see that the work required to run the algorithm outlined above grows at least on the order of
\[
(\gamma_{2n}/2)(n^2-n)/2 \ D^{(n-1)/2} \ |\text{disc } R|^{(n^2-n)/4}.
\]

We have implemented the algorithm for \(n = 2, 3, 4, \) and 5 in Magma, in the routines \FindMinimum2, FindMinimum3, FindMinimum4, and FindMinimum5, respectively. In practice, for \(n = 5\) our implementation took more time to run than we were willing to wait, and for \(n = 4\) we only ran the algorithm for the orders of discriminants \(-3\)

| \(D\) | Rank 1 | Rank 2 | Rank 3 | Rank 4 | Rank 5 | \(D\) | Rank 1 | Rank 2 | Rank 3 | Rank 4 | Rank 5 |
|-------|--------|--------|--------|--------|--------|-------|--------|--------|--------|--------|--------|
| 1     | 1 1 1 1 1 | 11 3 2 2 | 21 5 3 3 2 |
| 2     | 1 1 1 —   | 12 4 3 2 2 | 22 5 2 3 — |
| 3     | 2 2 2 2 2 | 13 4 2 2 2 | 23 5 3 2 — |
| 4     | 2 2 2 2 2 | 14 3 2 2 — | 24 6 4 3 — |
| 5     | 2 1 2 —   | 15 4 3 2 — | 25 5 3 3 2 |
| 6     | 3 2 2 —   | 16 4 3 3 3 | 26 5 3 2 — |
| 7     | 2 2 2 2 2 | 17 4 3 2 — | 27 6 3 3 3 |
| 8     | 3 2 2 —   | 18 5 3 3 — | 28 5 3 3 3 |
| 9     | 3 3 3 3 2 | 19 4 3 2 2 | 29 6 3 3 — |
| 10    | 2 2 2 —   | 20 4 3 2 — | 30 6 4 3 — |

Table 1. Values of \(d(R, n, D)\) for the quadratic order \(R\) of discriminant \(-3\). Dashes indicate values that we have not computed.
Table 2. Values of \(d(R, n, D)\) for the quadratic order \(R\) of discriminant \(-4\). Dashes indicate values that we have not computed.

| \(D\) | \(\text{Rank} n\) |
|---|---|
| 1 | 1 1 2 1 |
| 2 | 2 2 2 2 |
| 3 | 2 1 2 2 |
| 4 | 2 2 2 2 |
| 5 | 2 2 2 2 |
| 6 | 2 2 2 2 |
| 7 | 3 2 2 2 |
| 8 | 4 2 2 2 |
| 9 | 3 2 2 2 |
| 10 | 3 2 2 2 |
| 11 | 4 2 2 2 |
| 12 | 4 3 2 2 |
| 13 | 3 3 2 2 |
| 14 | 4 2 2 2 |
| 15 | 4 3 2 2 |
| 16 | 4 4 4 3 |
| 17 | 5 3 3 2 |
| 18 | 6 3 3 2 |
| 19 | 4 3 2 2 |
| 20 | 5 4 3 3 |
| 21 | 5 3 3 |
| 22 | 5 3 3 |
| 23 | 6 3 3 |
| 24 | 6 4 3 |
| 25 | 5 3 4 3 |
| 26 | 6 4 3 |
| 27 | 6 4 3 |
| 28 | 6 4 3 |
| 29 | 6 3 3 |
| 30 | 6 4 |

and \(-4\). All of the values in the tables for \(n = 5\), and most of the values for \(n = 4\), came from running our second algorithm, which computes \(d(R, n, D)\) only in the case where \(D\) is the norm of an element of \(R\).

We must introduce some additional notation before outlining the second algorithm. Throughout, \(R\) will continue to denote the maximal order of an imaginary quadratic field \(K\) with class number 1.

Let \(L\) be the lattice \(R^n\), viewed as a subset of \(K^n\). For every prime ideal \(p\) of \(R\), we fix a finite set \(S_p\) of matrices in \(M_n(R)\) such that

\[
\{P^{-1}L : P \in S_p\}
\]

is the complete set of the superlattices \(M \supset L\) in \(K^n\) such that \(M/L \cong R/p\) as \(R\)-modules. For example, if \(\pi\) is a generator of the principal ideal \(p\), and if \(X \subset R\) is a set of representatives for the residue classes of \(p\), then one choice for \(S_p\) would be

\[
\left\{ \begin{array}{c} \pi \ 0 \ 0 \ \cdots \ 0 \\ x_2 \ 1 \ 0 \ \cdots \ 0 \\ x_3 \ 0 \ 1 \ \cdots \ 0 \\ \vdots \ \vdots \ \vdots \ \ddots \vdots \\ x_n \ 0 \ 0 \ \cdots \ 1 \end{array} : x_i \in X \right\} \cup \left\{ \begin{array}{c} 1 \ 0 \ 0 \ \cdots \ 0 \\ 0 \ \pi \ 0 \ \cdots \ 0 \\ 0 \ x_3 \ 1 \ \cdots \ 0 \\ \vdots \ \vdots \ \vdots \ \ddots \vdots \\ 0 \ x_n \ 0 \ \cdots \ 1 \end{array} : x_i \in X \right\} \cup \cdots
\]

We say that two Hermitian matrices \(A\) and \(B\) in \(M_n(R)\) are isomorphic to one another if there is an invertible \(C \in M_n(R)\) such that \(A = C^*BC\), where \(C^*\) is the conjugate transpose of \(C\). We also fix a finite set \(U\) of representatives of the isomorphism classes of unimodular Hermitian matrices in \(M_n(R)\). For the \(R\) and \(n\) we will be considering, Schiemann \([34]\) has computed such sets \(U\).

**Lemma 4.8.** If \(A\) is a matrix in \(M_n(R)\) whose determinant generates a prime ideal \(p\), then there is an element \(P\) of \(S_p\) and an invertible \(C \in M_n(R)\) such that \(A = CP\).

**Proof.** The lattice \(M = A^{-1}L\) is a superlattice of \(L\) such that \(M/L \cong R/p\), so there is a \(P \in S_p\) so that \(A^{-1}L = P^{-1}L\). If we set \(C = AP^{-1}\) then \(CL = L\), so \(C\) is an invertible element of \(M_n(R)\). \(\square\)
Lemma 4.9. Suppose $A$ is a Hermitian matrix in $M_n(R)$ whose determinant is equal to $x\tau$ for some $x$ in $R$. Write the ideal $xR$ as a product $p_1\cdots p_r$ of prime ideals. Then $A$ is isomorphic to a product

$$P_1P_2\cdots P_r U P_r^* \cdots P_2^* P_1^*$$

where $U \in \mathcal{U}$ and $P_i \in \mathcal{S}_{p_i}$.

**Proof.** We prove this statement by induction on the number $r$ of prime factors of $xR$. The statement is certainly true when $r = 0$, because in that case $\det A = 1$ and $A$ is isomorphic to one of the matrices in $\mathcal{U}$.

Suppose $r > 0$, and let $q = \overline{p_1}$. Let $M = A^{-1}L$ and let $G$ be the finite $R$-module $M/L$, whose cardinality is $(\det A)^2$. Pick $\delta \in R$ with $\delta^2 = \text{disc} R$, so that $\delta$ is a purely imaginary generator of the different of $R$. We define a pairing $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ by setting

$$b(x, y) = \text{Tr}(x^*Ay/\delta) \mod \mathbb{Z};$$

it is easy to check that this pairing is well-defined, and by using the fact that $(1/\delta)L$ is the trace dual of $L$, we see that $b$ is nondegenerate. Note also that $b$ is alternating, and that for all $r \in R$ and $x, y \in G$ we have $b(rx, y) = b(x, yr)$, so that in the terminology of [12], $b$ is semi-balanced.

The ideal $q$ occurs in the Jordan–Hölder decomposition for the $R$-module $G$, so the $q$-torsion $T$ of $G$ is nontrivial. We claim that we can find a 1-dimensional $R/q$-vector subspace of $T$ on which the pairing $b$ is identically 0. If $q \neq \overline{q}$ this follows from [12] Lem. 7.2, p. 2378. If $q = \overline{q}$ and the dimension of $T$ as a $R/q$-vector space is at least 2, then this follows from [12] Lem. 7.3, p. 2378. If $q = \overline{q}$ and $T$ is 1-dimensional, we let $U$ be the $q$-power torsion of the $R$-module $G$. Since $q^2$ divides $\det A$, the $R$-module $U$ is strictly larger than $T$, and the annihilator of $T$ in $U$ is nontrivial. Thus, the $q$-torsion of the annihilator of $T$ must be $T$ itself, so $b$ restricted to $T$ is trivial. This proves the claim.

Let $N$ be the sublattice of $M$ consisting of elements that reduce modulo $L$ to elements of $T$. Then $N$ is a superlattice of $L$ with the property that $N/L \cong R/q$, so there is an element $Q$ of $\mathcal{S}_q$ such that $N = Q^{-1}L$.  

| $D$ | $\text{Rank } n$ | $D$ | $\text{Rank } n$ | $D$ | $\text{Rank } n$ |
|-----|-----------------|-----|-----------------|-----|-----------------|
| $1$ | 1               | $11$ | 3               | $21$ | 7               |
| $2$ | 2               | $12$ | 4               | $22$ | 5               |
| $3$ | 2               | $13$ | 4               | $23$ | 5               |
| $4$ | 2               | $14$ | 5               | $24$ | 6               |
| $5$ | 3               | $15$ | 4               | $25$ | 6               |
| $6$ | 2               | $16$ | 4               | $26$ | 6               |
| $7$ | 3               | $17$ | 5               | $27$ | 6               |
| $8$ | 4               | $18$ | 6               | $28$ | 7               |
| $9$ | 3               | $19$ | 5               | $29$ | 6               |
| $10$ | 3              | $20$ | 6               | $30$ | 6               |

Table 3. Values of $d(R, n, D)$ for the quadratic order $R$ of discriminant $-7$. Dashes indicate values that we have not computed.
Table 4. Values of $d(R, n, D)$ for the quadratic order $R$ of discriminant $-8$. Dashes indicate values that we have not computed.

Let $x$ and $y$ be arbitrary elements of $L$. Since $b$ is trivial on $N/L$, we have

$$\text{Tr}((Q^{-1}x)^*A(Q^{-1}y)/\delta) \in \mathbb{Z},$$

so that

$$\text{Tr}(x^*Q^{-1}z^*A^{-1}y)/\delta) \in \mathbb{Z} \text{ for all } x, y \in L.$$

If we set $B = Q^{-1}z^*A^{-1}$, we see that $By/\delta$ must lie in the trace dual of $L$, which is $(1/\delta)L$, so $B$ must send $L$ to $L$. In other words, the entries of $B$ must all be elements of $R$. This shows that $A = Q^*BQ$ for a Hermitian matrix $B$ in $M_n(R)$ whose determinant can be written $y\mathfrak{p}$ for an element $y$ of $R$ with $yR = \mathfrak{p}_2 \cdots \mathfrak{p}_r$.

Applying the induction hypothesis, we find that

$$B = C^*P_2P_3 \cdots P_rU P_r^* \cdots P_3P_2^*C$$

where each $P_i$ lies in $S_{\mathfrak{p}_i}$, where $U$ lies in $\mathcal{U}$, and where $C$ is an invertible element of $M_n(R)$. Thus

$$A = Q^*C^*P_2P_3 \cdots P_rU P_r^* \cdots P_3P_2^*CQ.$$

Note that $Q^*C^*$ is a matrix whose determinant generates the prime ideal $\mathfrak{p}_1$, so by Lemma 4.8 there is an element $P_1$ of $S_{\mathfrak{p}_1}$ and an invertible $D \in M_n(R)$ such that $Q^*C^* = DP_1$. Thus, we find that

$$A = DP_1P_2P_3 \cdots P_rU P_r^* \cdots P_3P_2^*P_1^*D^*.$$

In other words, $A$ is isomorphic to a product as in the statement of the lemma. □

Lemma 4.9 thus gives us an easy way to enumerate all of the isomorphism classes of Hermitian matrices in $M_n(R)$ of a given determinant, provided that the determinant is a norm. For each isomorphism class, we can compute the shortest vector using standard techniques that are built into Magma, and in this way we can compute an upper bound on the lengths of the short vectors of such matrices. This completes the proof of Lemma 4.7. □

Remark. The proof of Lemma 4.9 is very similar to the proof of [12, Prop. 7.1, p. 2378], and indeed we initially thought of Lemma 4.9 not as a statement about the decomposition of Hermitian matrices but rather as a statement about the decomposition of non-principal polarizations of abelian varieties.
Table 5. Values of $d(R, n, D)$ for the quadratic order $R$ of discriminant $-11$. Dashes indicate values that we have not computed.

As an application of the sharp values of $d(R, n, D)$, we prove a generalization of [24, Thm. 4, pp. 95–96].

**Proposition 4.10.** Let $q$ be a prime power, let $m = \lfloor 2\sqrt{q} \rfloor$, and suppose there is an elliptic curve $E$ over $\mathbb{F}_q$ with trace $-m$. Let $F$ be an arbitrary elliptic curve over $\mathbb{F}_q$. Then there is no Jacobian over $\mathbb{F}_q$ isogenous to $E^{g-1} \times F$ if $m^2 - 4q$, $g$, and the trace of $F$ lie in the following table:

| $m^2 - 4q$ | $g$ | trace $F$ |
|------------|-----|-----------|
| $-3$       | $3$ | $-m + 2$ |
| $4$        | $-m + 2$ |
| $4$        | $-m + 5$ |
| $5$        | $-m + 2$ |
| $-4$       | $3$ | $-m + 3$ |
| $-11$      | $3$ | $-m + 2$ |

**Proof.** Suppose $m^2 - 4q$, $g$, and the trace of $F$ lie in the table, and suppose $C$ is a curve of genus $g$ over $\mathbb{F}_q$ whose Jacobian is isogenous to $E^{g-1} \times F$. Write the trace of $F$ as $-m + f$. Then the reduced resultant of the real Weil polynomials of $E$ and $F$ is $f$, and the largest group scheme that can be embedded in the $f$-torsion of an elliptic curve isogenous to $F$ has order $f^2$. Pulling the principal polarization of $\text{Jac} C$ back to $E^{g-1}$ gives us a polarization of degree $f^2$, corresponding to a positive definite Hermitian matrix, with determinant $f$, over the quadratic order $R$ of discriminant $m^2 - 4q$. Looking in Tables 1, 2, and 5, we find that for all of the cases listed in the proposition, the value of $d(R, g-1, f)$ is 1, so that the Hermitian form associated to this matrix has a vector of length 1. This vector gives us an embedding of $E$ into $E^{g-1}$; let $\psi$ be the composition of this embedding with the map $E^{g-1} \to \text{Jac} C$. Then Lemma 1.5 shows that $\psi$ pulls the principal polarization of $C$ back to the principal polarization of $E$, and Lemma 4.3 shows that there is a map of degree 1 from $C$ to $E$. This is clearly impossible, so there must not be a $C$ with Jacobian isogenous to $E^{g-1} \times F$. \qed
5. Galois descent

For some isogeny classes \( C \) of abelian varieties over a finite field \( k \), one can show that every principally-polarized variety in \( C \) can be defined over a subfield \( k_0 \) of \( k \); it follows that every Jacobian in \( C \) comes from a curve that can be defined over \( k_0 \). This reduces the problem of determining whether there are Jacobians in \( C \) to the problem of determining whether there are Jacobians in a collection of isogeny classes over a smaller field. This idea was used in [38, pp. Se42–Se43] and in [23]; the appendix to the latter paper describes some methods for determining whether principally-polarized varieties can be defined over subfields.

In this section we give a simple necessary and sufficient condition for determining whether the entire category of varieties in an ordinary isogeny class \( C \) can be descended in this way. To begin, we set some notation and make a formal definition.

Let \( k_0 \) be a finite field, \( C_0 \) an isogeny class of abelian varieties over \( k_0 \), and \( k \) a finite extension of \( k_0 \), say of degree \( e \) over \( k_0 \). Base extension by \( k/k_0 \) takes the isogeny class \( C_0 \) to an isogeny class \( C \) over \( k \), and the base extension functor respects properties such as the degrees of isogenies, the duality of varieties, and whether or not an isogeny is a polarization.

**Definition 5.1.** We say that \( C \) descends to \( C_0 \) if base extension induces an equivalence between the category of abelian varieties in \( C_0 \) and the category of abelian varieties in \( C \).

Let \( C \) be an arbitrary isogeny class of ordinary abelian varieties over a finite field \( k \), and let \( A \) be any variety in \( C \). Let \( \pi \) and \( \overline{\pi} \) be the Frobenius and Verschiebung endomorphisms of \( A \), respectively; then the subring \( R := \mathbb{Z}[\pi, \overline{\pi}] \) of \( \text{End} A \) is contained in the center \( K \) of the ring \( E := (\text{End} A) \otimes \mathbb{Q} \). Up to isomorphism, the ring \( R \) and the \( \mathbb{Q} \)-algebras \( K \) and \( E \) do not depend on the choice of \( A \); we denote them by \( R_C, K_C, \text{ and } E_C \). The algebra \( K \) is a product of CM fields, and \( R \) is an order in \( K \). Furthermore, complex conjugation on \( K \) sends \( \pi \) to \( \overline{\pi} \).

**Theorem 5.2.** Let \( C \) be an isogeny class of ordinary abelian varieties over a finite field \( k \) that contains an index-\( e \) subfield \( k_0 \).

If there is an element \( \pi_0 \) of \( R_C \) such that \( \pi = \pi_0^e \), then the characteristic polynomial of \( \pi_0 \) (as an element of \( E_C \)) is the Weil polynomial for an isogeny class \( C_0 \) of abelian varieties over \( k_0 \), and \( C \) descends to \( C_0 \).

Conversely, if \( C \) descends to an isogeny class \( C_0 \) over \( k_0 \), then there is a \( \pi_0 \) in \( R_C \) whose characteristic polynomial (as an element of \( E_C \)) is equal to the Weil polynomial for \( C_0 \) and such that \( \pi = \pi_0^e \).

**Proof.** Let \( q = \#k \) and \( q_0 = \#k_0 \), so that \( q = q_0^e \), and let \( p \) be the characteristic of \( k \). Let \( R = R_C \) and \( K = K_C \).

Suppose there is an element \( \pi_0 \) of \( R \) such that \( \pi = \pi_0^e \), and let \( g \) be its characteristic polynomial. The product \( \pi_0 \pi_0^{-1} \) of \( \pi_0 \) with its complex conjugate is totally positive and real, and since

\[
(\pi_0 \overline{\pi}_0)^e = \pi \pi = q = q_0^e
\]

we see that \( \pi_0 \overline{\pi}_0 = q_0 \). This shows that all of the complex roots of \( g \) have magnitude \( \sqrt{q_0} \), so all of the roots of \( g \) are \( q_0 \)-Weil numbers. To show that the corresponding isogeny class of varieties is ordinary, we must show that for every homomorphism \( \varphi \) of \( K_C \) to \( \mathbb{Q}_p \), one of the numbers \( \varphi(\pi_0) \) and \( \varphi(\overline{\pi}_0) \) is a unit and the other is not.
But this follows from the fact that for each \( \varphi \), one of the numbers \( \varphi(\pi_0) \) and \( \varphi(\pi'_0) \) is a unit and the other is not, which is true because \( \mathcal{C} \) is ordinary. The Honda–Tate theorem then shows that \( g \) is the Weil polynomial of an isogeny class \( \mathcal{C}_0 \) of ordinary abelian varieties over \( k_0 \).

Let \( f \) be the Weil polynomial of \( \mathcal{C} \), and let \( f = f_1^{e_1} \cdots f_r^{e_r} \) be its factorization into powers of distinct irreducibles. Each \( f_i \) defines a CM-field \( K_i \), and \( K \) is the product of these \( K_i \). Likewise, we can write the factorization of \( g \) as \( g = g_1^{e_1} \cdots g_r^{e_r} \), where each \( g_i \) also defines \( K_i \).

Let \( R_0 \) be the ring \( \mathbb{Z}[\pi_0, \pi'_0] \), so that \( R_0 = R_{\mathcal{C}_0} \). Deligne’s theorem on ordinary abelian varieties \([5]\) shows that the category of abelian varieties in \( \mathcal{C}_0 \) is equivalent to the category of finitely generated \( R_0 \)-modules that can be embedded in \( V := K_1^{e_1} \times \cdots \times K_r^{e_r} \) as submodules whose images span \( V \) as a \( \mathbb{Q} \)-vector space. (The first author \([12]\) has shown how dual varieties and polarizations can be interpreted in this category of \( R_0 \)-modules.) Likewise, the category of abelian varieties in \( \mathcal{C} \) is equivalent to the category of finitely generated \( R \)-modules that can be embedded in \( V \) as submodules whose images span \( V \) as a \( \mathbb{Q} \)-vector space. The base extension functor sends an \( R_0 \)-module \( M \) to the same module, viewed as a module over the subring \( \mathbb{Z}[\pi_0, \pi'_0] = R \) of \( R_0 \). But since \( \pi_0 \) and \( \pi'_0 \) lie in \( R \), we have \( R = R_0 \), so base extension gives an equivalence of categories. This shows that \( \mathcal{C} \) descends to \( \mathcal{C}_0 \), and proves the first statement of the theorem.

Now assume that \( \mathcal{C} \) is an ordinary isogeny class that descends to an isogeny class \( \mathcal{C}_0 \) over \( k_0 \). Clearly \( \mathcal{C}_0 \) must also be ordinary. Let \( R_0 = R_{\mathcal{C}_0} \), and let \( \pi_0 \in R_0 \) be the Frobenius for \( \mathcal{C}_0 \). Then the Frobenius \( \pi \) for the isogeny class \( \mathcal{C} \) is \( \pi_0 \), and the ring \( R = R_{\mathcal{C}} \) is isomorphic to the subring \( \mathbb{Z}[\pi_0, \pi'_0] \) of \( R_0 \).

We know that \( R_0 \) is contained in the center of the endomorphism ring of every variety in \( \mathcal{C}_0 \), and it follows from \([15\) Thm. 7.4, p. 554)] or from \([5\) that there exist varieties in \( \mathcal{C}_0 \) whose endomorphism rings have centers equal to \( R_0 \). Thus, \( R_0 \) can be characterized as the smallest ring that occurs as the center of the endomorphism ring of a variety in \( \mathcal{C}_0 \). Likewise, \( R \) is the smallest ring that occurs as the center of the endomorphism ring of a variety in \( \mathcal{C} \). Since we are assuming that base extension gives an equivalence of categories from \( \mathcal{C}_0 \) to \( \mathcal{C} \), we find that we must have \( R_0 \cong R \). It follows that the natural inclusion \( R = \mathbb{Z}[\pi_0, \pi'_0] \subset R_0 \) is an isomorphism, so \( R \) contains an element \( \pi_0 \) whose characteristic polynomial is the Weil polynomial for \( \mathcal{C}_0 \) and with \( \pi = \pi_0 \).

\[ \square \]

**Remark.** We could also have proven Theorem 5.2 by using Théorèmes 6 and 7 from \([23\) §§4.5)] to show that each variety in \( \mathcal{C} \), and each polarization of each variety in \( \mathcal{C} \), descends to \( k_0 \). However, we felt that the argument above, which gives us an entire equivalence of categories between the isogeny classes \( \mathcal{C} \) and \( \mathcal{C}_0 \) all at once, was worth the small additional effort of introducing Deligne modules into the proof.

If \( A \) is an abelian variety over a finite field \( \mathbb{F}_q \), the **standard quadratic twist** \( A' \) of \( A \) is the twist of \( A \) corresponding to the element of the cohomology set \( H^1(\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \operatorname{Aut} A) \) represented by the cocycle that sends the \( q \)-th power Frobenius automorphism of \( \overline{\mathbb{F}}_q \) to the automorphism \( -1 \) of \( A \). Suppose \( \mathcal{C} \) is an isogeny class of abelian varieties over a finite field. The **quadratic twist** \( \mathcal{C}' \) of \( \mathcal{C} \) is the isogeny class consisting of the standard quadratic twists of the elements of \( \mathcal{C} \). If the Weil polynomial of \( \mathcal{C} \) is \( f(x) \), then the Weil polynomial of \( \mathcal{C}' \) is \( f(-x) \).
The next result shows how Theorem 5.2 can help us show there are no Jacobians in an isogeny class.

**Theorem 5.3.** Suppose \( C \) is an isogeny class of ordinary abelian varieties over a finite field \( k \) that descends to an isogeny class \( C_0 \) over a subfield \( k_0 \) of \( k \) of index \( e \), and suppose \( C \) is a curve over \( k \) whose Jacobian lies in \( C \).

1. If \( e \) is odd, then \( C \) has a model over \( k_0 \) whose Jacobian lies in \( C_0 \).
2. If \( e \) is even, then \( C \) has a model over \( k_0 \) whose Jacobian lies either in \( C_0 \) or in the quadratic twist of \( C_0 \).

Thus, to show that there are no Jacobians in \( C \), it suffices to show there are no Jacobians in \( C_0 \) and, if \( e \) is even, in the quadratic twist of \( C_0 \).

**Proof.** If \( C \) is hyperelliptic, Théorème 4 of the appendix to [23] shows that \( C \) has a model over \( k_0 \) whose Jacobian lies in \( C_0 \). If \( C \) is not hyperelliptic, Théorème 5 of the same appendix shows that \( C \) has a model \( C_0 \) over \( k_0 \) whose Jacobian has an \( \varepsilon \)-twist that lies in \( C_0 \), where \( \varepsilon \) is a homomorphism from \( \text{Gal} k/k_0 \) to \( \{\pm 1\} \). If \( e \) is odd \( \varepsilon \) must be trivial, so \( \text{Jac} C_0 \) lies in \( C_0 \), and we get statement (1) of the theorem. If \( e \) is even, then the \( \varepsilon \)-twist is either trivial or the standard quadratic twist, and we get statement (2). \( \square \)

6. **Magma Implementation**

As we indicated in the introduction, we have implemented our various tests in Magma. The main program is `isogeny_classes(q, g, N)`, which takes as input a prime power \( q \), a genus \( g \), and a desired number of points \( N \). Using the algorithm outlined in [14], we enumerate all of the monic degree-\( g \) polynomials in \( \mathbb{Z}[x] \) whose leading terms are \( x^g + (N - q - 1)x^{g-1} \) and all of whose roots are real numbers of absolute value at most \( 2\sqrt{q} \). This set of polynomials includes the set of real Weil polynomials of Jacobians of curves with \( N \) points. For each such polynomial \( f \), the program runs the subroutine `process_isogeny_class`, which answers ‘no’, ‘maybe’, or ‘yes’ to the question “Is there a Jacobian whose real Weil polynomial is equal to \( f \)?”

The procedure `process_isogeny_class`, when supplied with a polynomial, performs the following steps:

1. The procedure checks whether the polynomial corresponds to an isogeny class of abelian varieties; that is, it checks whether the polynomial satisfies the conditions of the Honda–Tate theorem [43, Théorème 1, p. 96]. If not, the answer to the question is ‘no’.
2. If the dimension of the isogeny class is 2, the procedure checks whether it meets the conditions of the Howe/Nart/Ritzenthaler classification of 2-dimensional isogeny classes that contain Jacobians [15]. The answer to the question is ‘yes’ or ‘no’, accordingly.
3. The procedure checks whether the Weil polynomial predicts a non-negative number of degree-\( d \) places for all \( d \) less than or equal to the genus. (The isogeny classes returned by `isogeny_classes()` have this property, but for isogeny classes that arise recursively in some of the following steps, this condition must be checked.) If not, the answer to the question is ‘no’.
4. If the isogeny class is maximal (that is, if \( N \) is equal to the Weil bound for genus-\( g \) curves over \( \mathbb{F}_q \)), the procedure checks whether the results of
Korchmaros and Torres [18] forbid the existence of a curve with Jacobian in the isogeny class. If so, the answer to the question is ‘no’.

(5) The procedure checks whether the isogeny class factors as an ordinary isogeny class times the class of a power of a supersingular elliptic curve with all endomorphisms defined. If so, it checks to see whether Theorem 5.1 shows that the isogeny class does not contain a Jacobian. If so, the answer to the question is ‘no’.

(6) The procedure uses Theorem 5.2 to check whether the isogeny class can be descended to an isogeny class over a subfield. If so, the procedure uses Theorem 5.3 to recurse, and checks whether the associated isogeny classes over the subfield contain Jacobians. If they do not, then the answer to the question is ‘no’.

(7) The procedure checks whether the real Weil polynomial can be split into two factors whose resultant is 1. A result of Serre (see [14, Thm. 1(a), p. 1678]) says that no Jacobian can lie in such a class, so if there is such a splitting, the answer to the question is ‘no’.

(8) Using Proposition 2.8, the procedure checks whether the real Weil polynomial can be split into two factors whose gluing exponent is 2. In this case, any curve whose Jacobian lies in the isogeny class must have an involution (Theorem 2.2), and so must be a double cover of a curve \( D \) whose real Weil polynomial \( g \) can be determined up to at most two possibilities. If a contradiction can be deduced from this, either using Lemma 6.1 (below) or by showing recursively that there is no curve with real Weil polynomial equal to \( g \), the answer to the question is ‘no’.

(9) The procedure checks to see whether Proposition 4.1, or a refinement using our tables of maximal lengths of short vectors of Hermitian forms, can be used to deduce the existence of a map of known degree \( n \) from any curve \( C \) with real Weil polynomial \( f \) to an elliptic curve \( E \) with a known trace. If such a map can be shown to exist, and if its existence leads to a contradiction (either by using Lemma 6.1 if \( n = 2 \), or by noting that \( \#C(F_q) > n\#E(F_q) \)), the answer to the question is ‘no’.

(10) If at this point the question has not yet been answered, the answer defaults to ‘maybe’, because we have no proof that the answer is ‘no’, and we do not know that the answer is ‘yes’.

To decide whether there is a problem with there being a double cover from a curve \( C \) whose Jacobian lies in an isogeny class \( C_1 \) to a curve \( D \) whose Jacobian lies in an isogeny class \( C_2 \), we use the following lemma:

**Lemma 6.1.** Suppose \( C \) and \( D \) are curves over \( \mathbf{F}_q \) of genus \( g_C \) and \( g_D \), respectively, and for each \( i \) let \( a_i \) and \( b_i \) denote the number of places of degree \( i \) on \( C \) and on \( D \), respectively. Suppose \( \varphi : C \to D \) is a map of degree 2. Let \( r \) denote the number of geometric points of \( D \) that ramify in the double cover, and let \( r_1 \) denote the number of \( \mathbf{F}_q \)-rational points of \( D \) that ramify in the double cover.

1. We have \( 2b_1 - 2a_2 - a_1 \leq r_1 \leq 2b_1 - a_1 \).
2. We have \( r_1 \equiv a_1 \mod 2 \) and \( r_1 \geq 0 \).
3. We have \( r \geq r_1 + \sum_{1 < d \leq g_C, d \text{ odd}, a_d \text{ odd}, d} \).
4. If \( q \) is even, then \( r = 2\text{-rank } C - 2(2\text{-rank } D) + 1 \) and \( r \leq g_C - 2g_D + 1 \).
5. If \( q \) is odd, then \( r = 2g_C - 4g_D + 2 \).
Proof. Let $s_1$ and $i_1$ be the number of rational points of $D$ that split and are inert (respectively) in $\varphi$. Then we have

$$b_1 = s_1 + i_1 + r_1, \quad a_1 = 2s_1 + r_1, \quad \text{and} \quad a_2 \geq i_1.$$  

These relations lead to statements (1) and (2).

If $d$ is odd, then the number of degree-$d$ places on $C$ is equal to twice the number of splitting degree-$d$ places on $D$, plus the number of ramifying degree-$d$ places on $D$. If $d$ and $a_d$ are both odd, then there must be at least one degree-$d$ place of $D$ that ramifies. This leads to statement (3).

If $q$ is even, then the Deuring–Shafarevich formula and the Riemann–Hurwitz formula show that

$$r = 2 \text{rank } C - 2(2 \text{rank } D) + 1 \quad \text{and} \quad r \leq g_C - 2g_D + 1,$$

respectively. This is statement (4). If $q$ is odd, then the Riemann–Hurwitz formula shows that $r = 2g_C - 4g_D + 2$, which is (5). \hfill \square

Lemma 6.1 gives a simple way of testing whether there is a problem with the existence of a degree-2 map $C \to D$, where Jac $C$ and Jac $D$ lie in known isogeny classes. If we find contradictory statements about $r$ and $r_1$, we know there is a problem with there being such a double cover. We note that in the Magma routines from our earlier paper, we did not make use of all of the inequalities listed in Lemma 6.1 so this is another place where the new program improves upon the old.

Finally, in some circumstances we can deduce that a curve with $N$ points must be a degree-$n$ cover of an elliptic curve $E$, for some $n > 2$. This will lead to a contradiction if $N > n\#E(\mathbb{F}_q)$.

Using these criteria, the procedure process_isogeny_class decides whether it can deduce a contradiction from the existence of a Jacobian in a given isogeny class.

7. New results and applications

Our new program improved the best known upper bound on $N_q(g)$ for more than 16% of the $(q, g)$-pairs in the 2009 version of the tables of van der Geer and van der Vlugt; this improvement is in addition to the improvements that came from our earlier paper. In this section we present a sample of some of these new results to indicate how the theorems from earlier in the paper come into play. We also give some examples that show that the information obtained from our program, combined with further analysis, can be used to restrict the possible Weil polynomials of curves with a given number of points.

7.1. Proof that $N_9(12) \leq 61$. Consider the case $q = 9$ and $g = 12$. The Ihara bound says that $N_9(12) \leq 63$, and the Magma program we wrote for our earlier paper [14] showed that in fact $N_9(12) \leq 62$. Our new program shows that $N_9(12) \leq 61$; in this subsection we explain how our new techniques eliminate cases that our old techniques could not.

Our old program could show that if a genus-12 curve over $\mathbb{F}_9$ has 62 points, then its real Weil polynomial is either

$$(x + 2)(x + 4)^6(x + 5)^4(x + 6) \quad \text{or} \quad (x + 4)^8(x + 6)^2(x^2 + 8x + 14).$$

Here is how our new program shows that the first of these polynomials is not the real Weil polynomial of a curve.
Let $E$ be the unique elliptic curve over $\mathbb{F}_9$ with 15 points, so that $E$ has trace $-5$. Using the gluing exponent, or even just using the resultant, one can show that if $A$ is a principally polarized abelian variety with real Weil polynomial equal to $(x + 2)(x + 4)^6(x + 5)^4(x + 6)$, then the principal polarization of $A$ pulls back to a polarization of $E^4$ of degree 1 or degree 32. Using Table 5 we find that this polarization of $E^4$ pulls back to give a polarization of degree at most 32 on $E$, and Lemma 4.3 shows that if $A$ is the Jacobian of a curve $C$, then $C$ has a map of degree at most 3 to $E$. But then $C$ could only have at most 3 times the number of rational points that $E$ has, so that $\#C(\mathbb{F}_9) \leq 45$, and in particular $C$ does not have 62 points. This eliminates the first of the two polynomials above.

We turn now to the second polynomial. The smallest resultant that we can obtain between two complementary factors of the radical of the second real Weil polynomial is 4, and in fact we get this resultant from each of the three possible splittings. However, the reduced resultant of $(x + 4)(x + 6)$ and $(x^2 + 8x + 14)$ is 2, so using Theorem 2.2 we find that a genus-12 curve $C$ with the given real Weil polynomial must be a double cover of a genus-2 curve $D$ with real Weil polynomial equal to $x^2 + 8x + 14$. From its real Weil polynomial we see that the curve $D$ has only 18 rational points, so a double cover of $D$ can have at most 36 rational points. This eliminates the second real Weil polynomial, and shows that $N_9(12) \leq 61$. \[\square\]

The best lower bound on $N_9(12)$ that we know at this time is 56, as shown by Gebhardt [7, Tbl. 2, p. 96], so there is still a gap between our current lower and upper bounds.

7.2. **New values of** $N_q(g)$. Running our new program, we find that $N_4(7) \leq 21$ and $N_8(5) \leq 29$. Niederreiter and Xing [31] showed that there is a genus-7 curve over $\mathbb{F}_4$ with 21 points, and van der Geer and van der Vlugt [9] showed that there is a genus-5 curve over $\mathbb{F}_8$ with 29 points, so we see that $N_4(7) = 21$ and $N_8(5) = 29$.

Let us sketch how our new program was able to improve upon the earlier program to show that $N_4(7) \leq 21$. The earlier program showed that any genus-7 curve over $\mathbb{F}_4$ with 22 points must have one of the following five real Weil polynomials:

- $x(x + 2)^2(x + 3)^3(x + 4)$,
- $(x + 3)^3(x^4 + 8x^3 + 20x^2 + 16x + 1)$,
- $(x + 1)(x + 3)^4(x^2 + 4x + 1)$,
- $(x + 1)(x + 3)^3(x + 4)(x^2 + 3x + 1)$,
- $(x + 2)^3(x + 4)^2(x^2 + 3x + 1)$.

Our new program eliminates these possibilities. The first real Weil polynomial is forbidden by an argument on Hermitian forms. The second and fifth are eliminated by Theorem 2.2(b); one can show that curves with these real Weil polynomials must be double covers of other curves, and we obtain contradictions from Lemma 6.1. The third is eliminated for the same reason; however, for this polynomial we need to use the gluing exponent and not just the resultant in order to show that the curve is a double cover. And finally, the fourth polynomial can be eliminated by using the supersingular factor method from Section 3.

7.3. **Correcting an error.** In [14, §7] we attempted to show two particular polynomials could not occur as real Weil polynomials of curves, but we made an error,
as is documented in the Corrigendum to [14]. We sketched corrected arguments in the second appendix of the arXiv version of [14]; here we provide all the details.

First, we would like to show that
\[ f = (x + 2)^2(x + 3)(x^3 + 4x^2 + x - 3) \]
cannot be the real Weil polynomial of a genus-6 curve \( C \) over \( \mathbb{F}_3 \). Using Proposition 4.1 and Table 4 we find that any curve with real Weil polynomial equal to \( f \) must be a double cover of the unique elliptic curve \( E \) over \( \mathbb{F}_3 \) with trace \(-2\). But \( E \) has 6 rational points, so a double cover of \( E \) can have at most 12 points. Since \( C \) is supposed to have 15 points, we see that no such curve \( C \) can exist. Our new program makes these deductions automatically.

The other argument in [14, §7] that we must correct concerns genus-4 curves over \( \mathbb{F}_{27} \); we would like to show that no such curve can have 65 points. Our new program shows that if there were a genus-4 curve over \( \mathbb{F}_{27} \) with 65 points, it would have to be a double cover of the unique elliptic curve \( E \) over \( \mathbb{F}_{27} \) having 38 points. It is not hard to enumerate all of the genus-4 double covers of this elliptic curve; Magma code for doing so can be found on the first author’s web site, in the section associated to the paper [13]. The largest number of points we find on genus-4 double cover of \( E \) is 64.

7.4. Proof that \( N_{32}(4) \leq 72 \). In our earlier paper [14, §6.2] we showed that \( N_{32}(4) < 75 \). Our new program shows that any genus-4 curve over \( \mathbb{F}_{32} \) having more than 72 points must be a double cover of the unique elliptic curve with trace \(-11\). Enumerating the genus-4 double covers of this curve is a feasible computational problem which can be solved by a simple modification of the method outlined in [14, §6.2] (in general we must consider three possible arrangements of ramification points, but in the specific situation we faced in [14, §6.2] we could eliminate one of these arrangements). We have implemented an algorithm to enumerate these double covers in Magma, and the resulting program can be found in the file 32-4.magma, available at the URL mentioned in the Introduction. We find that no such double cover has more than 71 points, so \( N_{32}(4) \leq 72 \). This fact, combined with information gleaned from IsogenyClasses.magma, shows that a genus-4 curve over \( \mathbb{F}_{32} \) having 72 points must have real Weil polynomial equal to \((x + 11)^2(x^2 + 17x + 71)\).

In 1999 Mike Zieve found a genus-4 curve over \( \mathbb{F}_{32} \) with 71 points, and while searching for double covers of the trace \(-11\) curve we also found a number of curves with 71 points. For example, if \( r \in \mathbb{F}_{32} \) satisfies \( r^5 + r^2 + 1 = 0 \), then the genus-4 curve
\[
\begin{align*}
g^2 + xy &= x^3 + x \\
z^2 + z &= (r^{14}x^2 + r^{24}x + r^{18})/(x + r)
\end{align*}
\]
has 71 points. Thus, \( 71 \leq N_{32}(4) \leq 72 \).

7.5. Genus 12 curves over \( \mathbb{F}_2 \). The smallest genus \( g \) for which the exact value of \( N_2(g) \) is unknown is \( g = 12 \); the Oesterlé bound is 15, and a genus-12 curve with 14 points is known. Our program, plus some additional work, allows us to show that a genus-12 curve over \( \mathbb{F}_2 \) with 15 points must have one of three real Weil polynomials.

**Theorem 7.1.** If \( C \) is a genus-12 curve over \( \mathbb{F}_2 \) with 15 rational points, then the real Weil polynomial of \( C \) is equal to one of the following polynomials:
I. $(x + 1)^2(x + 2)^2(x^2 - 2)(x^2 + 2x - 2)^3$
II. $(x - 1)(x + 2)^2(x^3 + 2x^2 - 3x - 2)(x^3 + 3x^2 - 3)(x^3 + 4x^2 + 3x - 1)$
III. $(x^2 + x - 3)(x^3 + 3x^2 - 3)(x^3 + 4x^2 + 3x - 1)(x^4 + 4x^3 + 2x^2 - 5x - 3)$

Proof. Our program shows that the real Weil polynomial of such a curve is either one of the three listed in the theorem, or one of the following two:
IV. $(x^2 + x - 1)^2(x^4 + 5x^3 + 4x^2 - 10x - 11)(x^4 + 5x^3 + 5x^2 - 5x - 5)$
V. $(x^3 + 3x^2 - 3)(x^4 + 5x^3 + 5x^2 - 5x - 5)(x^5 + 4x^4 + 4x^3 - 9x^2 - 5x + 3)$

So to prove the theorem, all we must do is show that possibilities (IV) and (V) cannot occur. First we analyze possibility (IV).

Let
\[
\begin{align*}
h_1 &= x^2 + x - 1 \\
h_2 &= x^4 + 5x^3 + 4x^2 - 10x - 11 \\
h_3 &= x^4 + 5x^3 + 5x^2 - 5x - 5
\end{align*}
\]
be the irreducible factors of the real Weil polynomial $h$ listed as item (IV). Since the constant terms of these polynomials are odd, they correspond to isogeny classes of ordinary abelian varieties. The corresponding Weil polynomials are
\[
\begin{align*}
f_1 &= x^4 + x^3 + 3x^2 + 2x + 4 \\
f_2 &= x^8 + 5x^7 + 12x^6 + 20x^5 + 29x^4 + 40x^3 + 48x^2 + 40x + 16 \\
f_3 &= x^8 + 5x^7 + 13x^6 + 25x^5 + 39x^4 + 50x^3 + 52x^2 + 40x + 16
\end{align*}
\]

Let $K_1$, $K_2$, and $K_3$ be the CM fields defined by these three Weil polynomials, and for each $i$ let $\pi_i$ be a root of $f_i$ in $K_i$. For each $i$ we use Magma to check that $K_i$ has class number 1, and we compute the discriminant of the maximal order $O_i$ of $K_i$. Using [12] Prop. 9.4, p. 2384 we find that for each $i$, the discriminant of the subring $R_i := \Z[\pi_i, \mathfrak{p}_i]$ of $O_i$ is equal to the discriminant of $O_i$, so we have $R_i = O_i$. Using the theory of Deligne modules [12], we find for each $i$ the isogeny class with real Weil polynomial $h_i$ contains a unique abelian variety $A_i$.

Suppose there were a curve $C$ with real Weil polynomial equal to $h_1^2h_2h_3$. Factoring this real Weil polynomial as $h_1^2h_2$ times $h_3$, noting that the reduced resultant of $h_1h_2$ and $h_3$ is 3, and applying Lemma 2.3 we find that we have an exact sequence
\[
0 \rightarrow \Delta \rightarrow B \times A_3 \rightarrow \text{Jac} C \rightarrow 0,
\]
where $B$ is isogenous to $A_1^2 \times A_2$ and where $\Delta$ is a self-dual group scheme that can be embedded into both $B[3]$ and $A_3[3]$.

Since the reduced resultant of $h_1$ and $h_2$ is 19, there is an isogeny $A_1^2 \times A_2 \rightarrow B$ whose degree is a power of 19; in particular, the group scheme $B[3]$ is isomorphic to $A_1[3]^2 \times A_2[3]$.

The group scheme structure of each $A_i[3]$ is determined by the $R_i$-module structure of $O_i/3O_i$ (see [12] Lem. 4.13, p. 2372)). Since 3 is unramified in $K_1$ and $K_2$, we find that $B[3]$ is a direct sum of simple group schemes. However, $A_3[3]$ is not semisimple; the prime 3 is ramified in $K_3$, and in fact in $K_3$ we have $3 = -\zeta^2(1 - \zeta)^2$, where
\[
\zeta = -(98 + 69 \pi_3 + 40 \pi_3^2 + 18 \pi_3^3 + 5 \pi_3^4 + 56 \pi_3 + 25 \pi_3^2 + 7 \pi_3^3)\]
is a cube root of unity. From this we see that the unique semi-simple subgroup-scheme of $A_3[3]$ is the kernel of $1 - \zeta$, so the image of $\Delta$ under the projection from $B \times A_3$ to $A_3$ lies in $A_3[1 - \zeta]$.

Now consider the automorphism $\alpha := (1, \zeta)$ of $B \times A_3$. Clearly $\alpha$ acts trivially on $B \times A_3[1 - \zeta]$, so it acts trivially on the image of $\Delta$, and therefore it descends to give an automorphism $\beta$ (of order 3) on $\text{Jac} C$. Furthermore, since $\zeta \overline{\zeta} = 1$, the automorphism $\alpha$ respects the pullback to $B \times A_3$ of the principal polarization of $\text{Jac} C$, so $\beta$ is an automorphism of $\text{Jac} C$ as a polarized variety. The strong form of Torelli’s theorem [27, Thm. 12.1, p. 202] shows that $C$ must therefore have an automorphism $\gamma$ of order 3. Let $D$ be the quotient $C/\langle \gamma \rangle$, so that there is a degree-3 Galois cover $C \to D$.

We calculate from its Weil polynomial that $C$ has 41 places of degree 8. Since 8 is not a multiple of 3, every degree-8 place of $C$ lies over a degree-8 place of $D$, and since 41 is congruent to 2 mod 3, we see that at least 2 degree-8 places of $D$ ramify in the triple cover $C \to D$. Thus, the degree of the different of the cover is at least 32. The Riemann–Hurwitz formula then gives

$$22 = 2g_C - 2 = 3(2g_D - 2) + \text{(degree of different)} \geq 6g_D + 26,$$

so that the genus of $D$ must be negative. This contradiction shows that no curve over $\mathbb{F}_2$ can have real Weil polynomial equal to possibility (IV) above.

Next we consider the polynomial $h$ from item (V) above. As in the preceding case, $h$ corresponds to an ordinary isogeny class. We will use the results of [12] to show that there are no principally polarized abelian varieties in this isogeny class. Let us sketch what these results are and how they are used.

A CM-order is a ring $R$ that is isomorphic to an order in a product of CM fields and that is stable under complex conjugation. Section 5 of [12] defines a contravariant functor $B$ from the category of CM-orders to the category of finite 2-torsion groups. If $C$ is an isogeny class of ordinary abelian varieties corresponding to a Weil polynomial $f$, then each irreducible factor $f_i$ of $f$ defines a CM field $K_i$ with maximal order $O_i$. Frobenius and Verschiebung generate an order $R$ in the product $K$ of the $K_i$. Section 5 of [12] uses properties of the polynomial $f_i$ to define an element $l_i$ of $B(O_i)$. The map $R \subset \prod O_i \to O_i$ gives a homomorphism $B(O_i) \to B(R)$, and we define $l_C$ to be the sum of the images of the $l_i$ in $B(R)$. Theorem 5.6 (p. 2375) of [12] says that there is a principally polarized variety in $C$ if and only if $l_C = 0$.

Proposition 10.1 (p. 2385) of [12] shows how to compute $B(O_i)$, and Proposition 10.5 (p. 2387) shows how to compute $B(R)$ and the map $B(O_i) \to B(R)$. Proposition 11.3 (p. 2390) and Proposition 11.5 (pp. 2391–2392) show how to compute the elements $l_i$ of $B(O_i)$. We will apply these results to show that for the isogeny class $C$ defined by the ordinary real Weil polynomial $h$ in item (V), the element $l_C$ is nonzero, so that there are no principally polarized varieties in $C$, and in particular no Jacobians.

Let

$$h_1 = x^3 + 3x^2 - 3$$

$$h_2 = x^4 + 5x^3 + 5x^2 - 5x - 5$$

$$h_3 = x^5 + 4x^4 + x^3 - 9x^2 - 5x + 3$$
so that \( h = h_1 h_2 h_3 \). The Weil polynomials corresponding to the \( h_i \) are

\[
\begin{align*}
  f_1 &= x^6 + 3x^5 + 6x^4 + 9x^3 + 12x^2 + 12x + 8 \\
  f_2 &= x^8 + 5x^7 + 13x^6 + 25x^5 + 39x^4 + 50x^3 + 52x^2 + 40x + 16 \\
  f_3 &= x^{10} + 4x^9 + 11x^8 + 23x^7 + 41x^6 + 63x^5 + 82x^4 + 92x^3 + 88x^2 + 64x + 32.
\end{align*}
\]

Let \( K_1, K_2, \) and \( K_3 \) be the CM fields defined by these three Weil polynomials, and for each \( i \) let \( \pi_i \) be a root of \( f_i \) in \( K_i \). Let \( \mathcal{O}_i \) be the maximal order of \( K_i \), and let \( R \) be the subring of \( \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \) generated by \( \pi := (\pi_1, \pi_2, \pi_3) \) and \( \bar{\pi} := (\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3) \).

From [12] Prop 10.1, p. 2385 we see that \( \mathcal{B}(\mathcal{O}_1) \cong \mathcal{B}(\mathcal{O}_3) \cong 0 \) and \( \mathcal{B}(\mathcal{O}_2) \cong \mathbb{Z}/2\mathbb{Z} \). According to [12] Props. 11.3 and 11.5, the element \( I_2 \) of \( \mathcal{B}(\mathcal{O}_2) \) will be zero if and only if the positive square root of \( \text{Norm}_{K_2/\mathbb{Q}}(\pi_2 - \bar{\pi}_2) \) is congruent to the middle coefficient of \( f_2 \) modulo 4. The norm of \( \pi_2 - \bar{\pi}_2 \) is 1, and the middle coefficient of \( f_2 \) is 39, so we find that \( I_2 \neq 0 \).

Let \( S = \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \). To calculate \( \mathcal{B}(R) \) and the map \( i^* : \mathcal{B}(S) \to \mathcal{B}(R) \) obtained from the inclusion \( i : R \to S \), we apply [12] Prop. 10.5, p. 2387. That proposition shows that there is a push-out diagram

\[
\begin{array}{ccc}
  D_s & \longrightarrow & B(S) \\
  \downarrow N & & \downarrow i^* \\
  C_s & \longrightarrow & B(R)
\end{array}
\]

where \( D_s \) and \( C_s \) are certain finite 2-torsion groups. Let \( R^+ \) and \( S^+ \) be the subrings of \( R \) and \( S \) consisting of elements fixed by complex conjugation, so that \( S^+ = \mathcal{O}_1^+ \times \mathcal{O}_2^+ \times \mathcal{O}_3^+ \) and \( R^+ = \mathbb{Z}[\pi + \bar{\pi}] \). Then \( C_s \) has a basis (as an \( \mathbb{F}_2 \)-vector space) consisting of elements indexed by the set

\[
\{ \text{maximal ideals } p \text{ of } R^+ \mid p \text{ is singular and is inert in } R/R^+ \}
\]

and \( D_s \) has a basis consisting of elements indexed by the set

\[
\left\{ \text{maximal ideals } q \text{ of } S^+ \mid \begin{array}{l}
  q \text{ is inert in } S/S^+, \\
  q \cap R^+ \text{ is singular in } R^+, \text{ and } \\
  q \cap R^+ \text{ is inert in } R/R^+
\end{array} \right\}.
\]

Let us compute the maximal ideals of \( R^+ \) that are singular and that are inert in \( R/R^+ \). Since \( R^+ \cong \mathbb{Z}[x]/(h) \) and \( R = R^+[x]/(x^2 - (\pi + \bar{\pi})x + q) \), this is a straightforward matter. We leave the details to the reader, but the only prime we find that is singular and inert is \( p = (3, \pi + \bar{\pi}) \).

There are two maximal ideals of \( S^+ \) lying over \( p \): the ideal \( q_1 = (3, \pi_1 + \bar{\pi}_1) \) of \( \mathcal{O}_1^+ \) and the ideal \( q_3 = (3, \pi_3 + \bar{\pi}_3) \) of \( \mathcal{O}_3^+ \). We compute that \( q_1 \) splits in \( \mathcal{O}_1 \) and that \( q_3 \) splits in \( \mathcal{O}_3 \). Therefore the group \( D_s \) is trivial. It follows from the push-out diagram that \( i^* \) is injective, so \( i^*(I_2) \) is nonzero, and there is no principally-polarized variety in the isogeny class associated to \( h \).

8. Bounds on Shafarevich–Tate groups

Propositions 4.3 and 4.4 both give upper bounds on the degrees of the smallest map from a curve \( C \) to an elliptic curve \( E \), and therefore say something about the Mordell–Weil lattice of maps from \( C \) to \( E \). The Birch and Swinnerton-Dyer conjecture for constant elliptic curves over one-dimensional function fields over finite fields, proven by Milne [26], relates the determinants of such Mordell–Weil lattices.
to certain Shafarevich–Tate groups. In this section we make some comparisons between our results and the conjecture of Birch and Swinnerton-Dyer, and deduce some results about Shafarevich–Tate groups.

Let $C$ be a curve over a finite field $\mathbb{F}_q$ of characteristic $p$, and let $K$ be its function field. Suppose there is an embedding $\psi : E \to \text{Jac} C$ of an elliptic curve $E$ into the Jacobian of $C$. Pick a degree-1 divisor $X$ on $C$, and let $\varphi : C \to E$ be the map associated to $\psi$ and $X$ as in Lemma 4.3. The assumption that $\psi$ is an embedding implies that $\varphi$ is minimal; that is, $\varphi$ does not factor through an isogeny from another elliptic curve to $E$. (Conversely, a minimal map from $C$ to an elliptic curve gives rise to an embedding of the elliptic curve into the Jacobian of $C$.)

Suppose further that there is only one factor of $E$ in the Jacobian of $C$, up to isogeny; that is, assume that $\text{Jac} C$ is isogenous to $A \times E$ for an abelian variety $A$ such that the gluing exponent $e := e(A, E)$ is finite. Let $h$ be the real Weil polynomial of $A$, let $g$ be the radical of $h$, and let $t$ be the trace of $E$. Note that Proposition 2.5 says that $\deg \varphi$ divides $e$, and that Proposition 2.8 says that $e$ divides the reduced resultant of $g$ with $x - t$, which is equal to $g(t)$.

**Theorem 8.1.** Let $R$ be the endomorphism ring of $E$, let $E$ be the base extension of $E$ from $\mathbb{F}_q$ to $K$, and let $\text{III}$ be the Shafarevich–Tate group of $E$.

(a) Suppose $R$ is an order in a quadratic field, so that we may write $t^2 - 4q = F^2 \Delta_0$ for some fundamental discriminant $\Delta_0$ and conductor $F$, and so that the discriminant of $\text{End} E$ is equal to $f^2 \Delta_0$ for some divisor $f$ of $F$. Then

$$\sqrt{\# \text{III}} = \frac{F \mid h(t)}{f \deg \varphi},$$

and $\sqrt{\# \text{III}}$ is divisible by

$$\frac{F \mid h(t)}{f \mid g(t)}.$$

(b) Suppose $R$ is an order in a quaternion algebra. Then $q$ is a square, we have

$$\sqrt{\# \text{III}} = \frac{\sqrt{q} \mid h(t)}{p \deg \varphi},$$

and $\sqrt{\# \text{III}}$ is divisible by

$$\frac{\sqrt{q} \mid h(t)}{p \mid g(t)}.$$

**Proof.** Since every map from an elliptic curve isogenous to $E$ to $\text{Jac} C$ factors through the embedding $\psi : E \to \text{Jac} C$, it follows that any map from $C$ to an elliptic curve isogenous to $E$ factors through $\varphi$. In particular, the set $L$ of maps from $C$ to $E$ that take $X$ to a divisor on $E$ that sums to the identity is equal to $R \varphi$.

The *Mordell–Weil lattice* of $E$ over $K$ is the group $E(K)/E(\mathbb{F}_q)$ provided with the pairing coming from the canonical height (see [33]). The natural map $L \to E(K)/E(\mathbb{F}_q)$ is a bijection, and the quadratic form on $L$ obtained from the height pairing on $E(K)$ is twice the quadratic form given by the degree map (see [40], Thm. III.4.3, pp. 217–218).

Suppose that $R$ is an order in a quadratic field. Using the fact that $L = R \varphi$, we find that the determinant $D$ of the Mordell–Weil lattice for $E$ satisfies

$$D = (\deg \varphi)^2 |\text{disc} R| = (\deg \varphi)^2 f^2 |\Delta_0|.$$
Let the eigenvalues of Frobenius for $\text{Jac } C$ be 

$$\pi_1, \pi_1, \pi_2, \pi_2, \ldots, \pi_g, \pi_g,$$

indexed so that $\pi_1$ and $\pi_1$ are the (distinct) eigenvalues of Frobenius for $E$. The Birch and Swinnerton-Dyer conjecture for constant elliptic curves over function fields \cite[Thm. 3, pp. 100–101]{26} says that the product $D\#\text{III}$ is equal to

$$q^g \left( 1 - \frac{\pi_1}{\pi_1} \right) \left( 1 - \frac{\pi_i}{\pi_1} \right) \prod_{i>1} \left( 1 - \frac{\pi_1}{\pi_i} \right) \left( 1 - \frac{\pi_i}{\pi_1} \right) \left( 1 - \frac{\pi_i}{\pi_1} \right).$$

Combining this with the relations $\pi_1\pi_1 = q$, and using the facts that $\pi_1 + \pi_1 = t$ and $(\pi_1 - \pi_1)^2 = F^2 \Delta_0$, we find that

$$D\#\text{III} = -q^{1-g}(\pi_1 - \pi_1)^2 \prod_{i>1} ((\pi_i - \pi_1)(\pi_i - \pi_1)(\pi_i - \pi_1)(\pi_i - \pi_1))$$

$$= -q^{1-g}(\pi_1 - \pi_1)^2 \prod_{i>1} q(\pi_1 + \pi_1 - \pi_i - \pi_1)^2$$

$$= F^2 |\Delta_0| \prod_{i>1} (t - \pi_i - \pi_i)^2$$

$$= F^2 |\Delta_0| h^2(t).$$

Using the equation for $D$ given above, we find that

$$\sqrt{\#\text{III}} = \frac{F}{f} h(t) \deg \varphi.$$

As we noted earlier, $\deg \varphi$ is divisor of $q(t)$. This proves statement (a).

Suppose $R$ is a (necessarily maximal) order in a quaternion algebra. This implies that $q$ is a square and the Frobenius eigenvalues of $E$ are both equal to $\sqrt{q}$ or to $-\sqrt{q}$. Again calculating the determinant $D$ of the Mordell–Weil lattice of $E$ by using the identification $L = R\varphi$, we find that $D = (\deg \varphi)^2 p^2$; using \cite[Thm. 3, pp. 100–101]{26} we find that

$$D\#\text{III} = qh^2(t).$$

From these equalities we obtain

$$\sqrt{\#\text{III}} = h(t) \frac{\sqrt{q}}{p} \deg \varphi,$$

and as above we find that $\sqrt{\#\text{III}}$ is a multiple of

$$\frac{\sqrt{q} h(t)}{p} \frac{\deg \varphi}{g(t)}.$$

Our results tell us something about Shafarevich–Tate groups in other situations as well. Suppose $C$ is a curve over a finite field $\mathbb{F}_q$ whose Jacobian is isogenous to $E^g$, for some ordinary elliptic curve $E$ over $\mathbb{F}_q$ with trace $t$. Let $R$ be the endomorphism ring of $E$. There is a universal isogeny $\Psi : A \to \text{Jac } C$, unique up to isomorphism, with the property that every map $\psi : E \to \text{Jac } C$ factors through $\Psi$; the Deligne module for $A$ is the largest submodule of the Deligne module for $\text{Jac } C$ that is also an $R$-module.

Write $t^2 - 4q = F^2 \Delta_0$ for a fundamental discriminant $\Delta_0$ and a conductor $F$, and let $f$ be the conductor of $\text{End } E$, so that $f \mid F$. 

Theorem 8.2. Let $K$ be the function field of $C$, let $E$ be the base extension of $E$ from $\mathbb{F}_q$ to $K$, and let $\text{III}$ be the Shafarevich–Tate group of $E$. Then
\[
\sqrt{\#\text{III}} = (F/f)^9 \deg \Psi.
\]

Proof. Let $\mu$ be the canonical principal polarization of $\text{Jac} C$, and let $\lambda$ be the pullback of $\mu$ to $A$ via the isogeny $\Psi$. Consider the lattice of homomorphisms $E \to \text{Jac} C$, with the quadratic form $Q$ provided by the square root of the degree of the pullback of $\mu$ to $E$. Since every homomorphism $E \to \text{Jac} C$ factors through $\Psi$, this lattice is isomorphic to to the lattice of homomorphisms $E \to A$, with the quadratic form given by the square root of the degree of the pullback of $\mu$ to $E$. Applying Lemma 4.2, we see that the determinant of this lattice is
\[
|\text{disc } R|/4^9 (\deg \Psi)^2 = (f/2)^{2g} |\Delta_0|^g (\deg \Psi)^2.
\]

As we have noted earlier, the Mordell–Weil lattice of $E$ is isomorphic to the lattice of homomorphisms $E \to \text{Jac} C$, with the quadratic form $2Q$. Therefore the determinant $D$ of the Mordell–Weil lattice is
\[
D = f^{2g} |\Delta_0|^g (\deg \Psi)^2.
\]

On the other hand, the Birch and Swinnerton-Dyer conjecture says that in this case we have
\[
D\#\text{III} = q^g \left(1 - \frac{\pi}{\pi_1}\right)^g \left(1 - \frac{\pi}{\pi_2}\right)^g
\begin{align*}
&= (\pi - \pi_1)^g (\pi - \pi_2)^g \\
&= (4q - t^2)^g \\
&= F^{2g} |\Delta_0|^g.
\end{align*}
\]

Thus we find that
\[
\sqrt{\#\text{III}} = (F/f)^9 \deg \Psi.
\]

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