REDUCTION PROCEDURES IN CLASSICAL AND QUANTUM MECHANICS

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Abstract. We present, in a pedagogical style, many instances of reduction procedures appearing in a variety of physical situations, both classical and quantum. We concentrate on the essential aspects of any reduction procedure, both in the algebraic and geometrical setting, elucidating the analogies and the differences between the classical and the quantum situations.

Keywords: Generalized reduction procedure, symplectic reduction Poisson reduction, Quantum systems

1. Introduction

Reduction procedures, the way we understand them today (i.e. in terms of Poisson reduction) can be traced back to Sophus Lie in terms of function groups, reciprocal function groups and indicial functions [30, 41, 47]. Function groups provide an algebraic description of the cotangent bundle of a Lie group but are slightly more general because can arise from Lie group actions which do not admit a momentum map [15]. Recently they have reappeared as “dual pairs” [35].

Physicists have used reduction procedures as an aid in trying to integrate the dynamics “by quadratures”. Dealing, as usual, with a coordinate formulation, reduction and coordinate separability have overlapped a good deal. From the point of view of the integration of the equations of motion, the so called decomposition into independent motions may be formalized as follows. Consider a dynamical vector field $\Gamma$ on a carrier manifold $M$ and a decomposition

$$\Gamma = \sum_i \Gamma_i,$$

with the requirement that:

- $[\Gamma_i, \Gamma_j] = 0$
- $\text{span} \{\Gamma_i(p)\} = T_pM$, $\forall p \in X \subset M$, where $X$ is open and dense submanifold in $M$.

When such a decomposition exists, the evolution is given by the product of the one parameter groups associated with each $\Gamma_j$.

Looking for superposition rules which would generalize the usual superposition rule of linear systems, Lie [41] introduced dynamical systems admitting a decomposition

$$\Gamma = \sum_i a^j(t)\Gamma_j$$

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with $[\Gamma_i,\Gamma_j] = \sum_k c^k_{ij} \Gamma_k$ and $c^k_{ij} \in \mathbb{R}$ (i.e. the vector fields $\Gamma_k$ span a finite-dimensional Lie algebra) and still $\{\text{span} \Gamma_i(p)\} = T_p M$, $\forall p \in X \subset M$, where $X$ is open and dense in $M$. The integration of these systems may be achieved by finding a fundamental set of solutions: they admit a superposition rule even if the system is nonlinear. These systems have been called Lie Scheffers systems (see e.g. [11] and references therein) and have an important representative given by the Riccati equation. It is worth illustrating this example because it is an instance of a nonlinear equation which is obtained as reduction of a linear one.

Example 1. Let us consider $\mathbb{R}^2$ and the following system of first-order differential equation

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ (1)

where $A$ is a $2 \times 2$ matrix with real entries, maybe depending on time. By performing a reduction with respect to the dilation group, or its infinitesimal generator $\Delta = x_1 \partial x_1 + x_2 \partial x_2$, i.e. by introducing the variable $\xi = x_1/x_2$, the linear equation (1) becomes:

$$\dot{\xi} = b_0 + b_1 \xi + b_2 \xi^2,$$ (2)

with $b_0 = a_{12}$, $b_1 = a_{11} - a_{22}$, $b_2 = -a_{21}$.

This is an instance of Riccati equation and is associated with a “free” motion on the group $\text{SL}(2, \mathbb{R})$: $\dot{g}^{-1} = -b^i(t) A_j$, where $\{A_j\}$ is a basis of the Lie algebra of $\text{SL}(2, \mathbb{R})$. Associated to it we find a nonlinear superposition rule for the solutions: if $x_1, x_2, x_3$ are independent solutions, every other solution $x$ is obtained from the following ratio:

$$\frac{(x - x_1)(x_2 - x_3)}{(x - x_2)(x_1 - x_3)} = K$$

Riccati type equations arise also in the reduction of the Schrödinger equation from the Hilbert space of states to the space of pure states [13].

Another example, but for partial differential equations, is provided by the following variant of the Burgers equation.

Example 2. To illustrate the procedure for partial differential equations in one space and one time, we consider the following variant of the Burgers equation

$$\frac{\partial w}{\partial t} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{k}{2} \left( \frac{\partial^2 w}{\partial x^2} \right) = 0.$$ 

This equation admits a superposition rule of the following kind: for any two solutions, $w_1$ and $w_2$,

$$w = -k \log \left( \exp \left( -\frac{w_1 + \ell_1}{k} \right) + \exp \left( -\frac{w_2 + \ell_2}{k} \right) \right)$$

is again a solution with $\ell_1$ and $\ell_2$ and $k$ real constants.

The existence of a superposition rule might suggest that the equation may be related to a linear one. This is indeed the case and we find that the heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{2} \frac{\partial^2 u}{\partial x^2}$$

is indeed related to the nonlinear equation by the replacement $u = \exp \left( -\frac{w}{k} \right)$. 
Out of this experience one may consider the possibility of integrating more general evolution systems of differential equations by looking for a simpler system (simple here meaning that it is a system explicitly integrable) whose reduction gives the system that we would like to integrate. In some sense, with a sentence, we could say that the reduction procedure provides us with interacting systems out of free (or Harmonic) ones.

The great interest for new completely integrable systems boosted the research in this direction in the past twenty five years and many interesting physical systems, both in finite and infinite dimensions were shown to arise in this way [50].

In the same ideology one may also put the attempts for the unification of all the fundamental interactions in Nature by means of Kaluza-Klein theories. In addition the attempt to quantize theories described by degenerate Lagrangians called for a detailed analysis of reduction procedures connected with constraints. These techniques came up again when considering geometric quantization as a procedure to construct unitary irreducible representations for Lie groups by means of the orbit method [37].

The simplest example to show how “nonlinearities” arise from reduction of a free system is the three dimensional free particle. Of course if our concern is primarily with the equations of motion we have to distinguish the various available descriptions: Newtonian, Lagrangian, Hamiltonian. Each description carries additional structures with respect to the equations of motion and one has to decide whether the reduction should be performed within the chosen category or if the reduced dynamics will be allowed to belong to another one.

The present paper is a substantially revised version of a talk delivered at a workshop. We have decided to keep the colloquial and friendly style aimed at exhibiting the many instances of reduction procedures appearing in a variety of physical situations, both classical and quantum. This choice may give the impression of an episodic paper, however it contains an illustration of the essential aspects of any reduction procedure, both in the algebraic and geometrical setting, pointing out the analogies and the differences between the classical and the quantum situation. Moreover it shows a basic philosophical principle: The unmanifest world is simple and linear, it is the manifest world which is “folded” and nonlinear.

1.1. Interacting systems from free ones. In what follows, we are going to consider few examples where “nonlinearities” obtained from reduction of linear systems are carefully examined.

Example 3. On $\mathbb{R}^3$ we consider the equations of motion of a free particle of unit mass in Newtonian form:

$$\ddot{r} = 0.$$ (3)

This system is associated to the second order vector field in $T\mathbb{R}^3$, $\Gamma = \dot{r} \frac{\partial}{\partial \dot{r}}$ and has constants of the motion

$$\frac{d}{dt}(\dot{r} \wedge \dot{r}) = 0, \quad \frac{d}{dt} \dot{r} = 0.$$

By introducing spherical polar coordinates

$$\vec{r} = r \vec{n} \quad \vec{n} \cdot \vec{n} = 1, \quad r > 0$$
where \( \vec{n} = \vec{r}/||\vec{r}|| = \vec{r}/r \) is the unit vector in the direction of \( \vec{r} \), and taking derivatives we find
\[
\dot{\vec{r}} = \vec{r} \hat{n} + r \dot{\vec{n}}, \quad \ddot{\vec{r}} = \vec{r} \ddot{n} + 2 \dot{r} \dot{\vec{n}} + r \ddot{\vec{n}}.
\]

Moreover, from the identities
\[
\vec{n} \cdot \vec{n} = 1, \quad \vec{n} \cdot \dot{\vec{n}} = 0, \quad \ddot{\vec{n}} = -\vec{n} \dddot{n},
\]
we see that \( \dot{r} \cdot \vec{n} = \dot{r} \), and \( \ddot{r} \cdot \vec{n} = r \dddot{n} \); using \( \dddot{r} = 0 \) we obtain
\[
\dddot{\vec{r}} = -r \vec{n} \cdot \dddot{\vec{n}} = r \dddot{n}^2, \tag{4}
\]
and, of course,
\[
\vec{r} \wedge \dot{\vec{r}} = r^2 \vec{n} \wedge \dot{\vec{n}}.
\]

The equations of motion (4) are not equations in the variable \( r \) only, because of the term \( \dot{\vec{n}}^2 \). However by making use of constants of the motion, we can choose invariant submanifolds \( \Sigma \) for \( \Gamma \) such that taking the restrictions on such submanifolds, we can associate with this equation an equation of motion involving only \( r, \dot{r} \) and some “coupling constants” related to the values of the constants of motion. So, we can restrict ourselves to initial conditions with a fixed value of the angular momentum, say, for instance,
\[
l^2 = r^4 (\dot{\vec{n}})^2,
\]
in order to get
\[
\dddot{\vec{r}} = \frac{l^2}{r^3}.
\]

If, on the other hand, we restrict ourselves to initial conditions satisfying
\[
(\dot{\vec{r}})^2 = 2E,
\]
we get
\[
\dddot{\vec{r}} = \frac{2E}{r} - \frac{\dot{\vec{r}}^2}{r}.
\]

By selecting an invariant submanifold of \( \mathbb{R}^3 \) by means of a convex combination of energy and angular momentum, i.e. \( \alpha (\vec{r} \wedge \dot{\vec{r}})^2 + (1 - \alpha) \dot{\vec{r}}^2 = k \), we would find
\[
\ddot{\vec{r}} = \left( \frac{\alpha l^2 + (1 - \alpha)(2E - \dot{\vec{r}}^2)r^2}{r^3} \right)
\]

We might even select a time dependent constant of the motion, for instance
\[
r^2 + \dot{r}^2 t^2 - 2r \cdot \dot{r} t = k^2,
\]
to get rid of \( (\dot{\vec{n}})^2 \),
\[
(\dot{\vec{n}})^2 = \frac{1}{r^2} \left[ (k^2 + 2r \cdot \dot{r} t - r^2) t^{-2} - \dot{r}^2 \right]
\]
and thus we would get a time-dependent reduced dynamics:
\[
\dddot{\vec{r}} = \frac{k^2}{r} t^{-2} + 2 \dot{r} t^{-1} - \frac{t^{-2}}{r} - \frac{\dot{r}^2}{r}.
\]

The geometrical interpretation of what we have done is rather simple: we have selected an invariant submanifold \( \Sigma \subset \mathbb{R}^3 \) (the level set of a constant of the motion), we have restricted the dynamics to it, and then we have used the rotation group to foliate \( \Sigma \) into orbits. The reduced dynamics is a vector field acting on the space of orbits \( \tilde{\Sigma} = \Sigma / SO(3) \). It should be remarked that even if \( \Sigma \) is selected in various ways, the choice we have made is compatible with the action of the rotation group.
It should be clear now that our presentation goes beyond the standard reduction in terms of the momentum map, which involves additional structures. Indeed this reduction, when carried out with the canonical symplectic structure, would give us only the first solution in the example above.

There is another way to undertake the reduction. On $T^* \mathbb{R}^3$ with coordinates $(\vec{r}, \vec{p})$, we can consider the functions

$$
\xi_1 = \frac{1}{2} \langle \vec{r}, \vec{r} \rangle, \quad \xi_2 = \langle \vec{p}, \vec{p} \rangle, \quad \xi_3 = \langle \vec{r}, \vec{p} \rangle.
$$

Here $\langle \vec{a}, \vec{b} \rangle$ denotes the scalar product $\vec{a} \cdot \vec{b}$, but it can be extended to a non definite positive scalar product.

The equation of motion (3) on these coordinate functions becomes

$$
\frac{d}{dt} \xi_1 = \xi_3, \quad \frac{d}{dt} \xi_2 = 0, \quad \frac{d}{dt} \xi_3 = \xi_2.
$$

Note that any constant of the motion of this system is then a function of $\xi_2$ and $(2\xi_1 \xi_2 - \xi_3^2)$. Consider first the invariant submanifold $\xi_2 = k \in \mathbb{R}$. Then we find,

$$
\frac{d}{dt} \xi_1 = \xi_3, \quad \frac{d}{dt} \xi_3 = k,
$$

i.e. a uniformly accelerated motion in the variable $\xi_1$. It may be described by the Lagrangian $L = \frac{1}{2} v^2 + k x$, where $x = \xi_1$, $v = \dot{\xi}_1 = \xi_3$.

Had we selected a different invariant submanifold, for instance,

$$
2\xi_1 \xi_2 - \xi_3^2 = l^2,
$$

the restricted dynamics would have been:

$$
\frac{d}{dt} \xi_1 = \xi_3, \quad \frac{d}{dt} \xi_3 = \frac{\xi_3^2 + l^2}{2 \xi_1}.
$$

A corresponding Lagrangian description is provided by the function $L = \frac{1}{2} \dot{x}^2 - \frac{2 l^2}{\xi_1}$, with $x = \xi_1$ and $v = \xi_3$.

If we start with the dynamics of the isotropic harmonic oscillator, say $\dot{\vec{r}} = \vec{p}$ and $\dot{\vec{p}} = -\vec{r}$, on functions $\eta_1 = \xi_1 - \frac{1}{2} \xi_2$, $\xi_3$ and $\eta_2 = \xi_1 + \xi_2$, we would get $\dot{\eta}_1 = 2 \xi_3$, $\dot{\xi}_3 = 2 \xi_1 - \xi_2$ and $\dot{\eta}_2 = 0$, i.e. $\dot{\eta}_1 = 2 \xi_3$ and $\dot{\xi}_3 = -2 \eta_1$, i.e. we get a one dimensional oscillator. We would like to stress that the “position” of this reduced system, say $\eta_1$ is not a function depending only on the initial position variables.

**Remark 1.** Let us point out a general aspect of the example we just considered. We first notice that the functions $\xi_1 = \frac{1}{2} x_a x^a$, $\xi_2 = p_a p^a$ and $\xi_3 = x_a p^a$ may be defined on any phase space $\mathbb{R}^{2n} = T^* \mathbb{R}^n$, with $\mathbb{R}^n$ an Euclidean space. If we consider the standard Poisson bracket, say

$$
\{ p_a, x^b \} = \delta^b_a, \quad \{ p_a, p_b \} = 0 = \{ x^a, x^b \},
$$

we find that for the new variables

$$
\{ \xi_3, \xi_1 \} = 2 \xi_1, \quad \{ \xi_2, \xi_3 \} = 2 \xi_2, \quad \{ \xi_2, \xi_1 \} = 2 \xi_3.
$$

Thus the functions we are considering close on the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The infinitesimal generators $\{ \xi_i, \cdot \}$ are complete vector fields and integrate to a symplectic action of $SL(2, \mathbb{R})$ on $\mathbb{R}^{2n}$.

Then, in the stated conditions there is always a symplectic action of $SL(2, \mathbb{R})$ on $T^* \mathbb{R}^n \simeq \mathbb{R}^{2n}$ with a corresponding momentum map $\mu : T\mathbb{R}^n \rightarrow \mathfrak{sl}^* (2, \mathbb{R})$. If
we denote again the coordinate functions on this three dimensional vector space by 
\{\xi_1, \xi_2, \xi_3\}, we have the Poisson bracket \([3]\) and the momentum map provides a symplectic realization of the Poisson manifold \(\mathfrak{sl}^\ast(2, \mathbb{R})\). In the language of Lie, the coordinate functions \{\xi_1, \xi_2, \xi_3\} along with all the smooth functions of them \(\{f(\xi_1, \xi_2, \xi_3)\}\) define a function group. The Poisson subalgebra of functions of \(\mathcal{F}(\mathbb{R}^3)\) commuting with all the functions \(f(\xi_1, \xi_2, \xi_3)\), constitute the reciprocal function group, and all functions in the intersection of both sets, say functions of the form \(F(\xi_1 \xi_2 - \frac{1}{2} \xi_3^2)\), constitute the indicial functions.

By setting \(\xi_1 = \frac{1}{2}, \xi_3 = 0\) we identify a submanifold in \(T\mathbb{R}^n\) diffeomorphic with \(T\mathbb{S}^{n-1}\), the tangent bundle of the \((n-1)\)-dimensional sphere. It is clear that the reciprocal function group is generated by functions \(J_{ab} = p_a x^b - p_b x^a\). Thus, the reduced dynamics which we usually associate with the Hamiltonian \(H = \frac{1}{2}p_r^2 + \frac{l^2}{2r^2}\) is actually a dynamics on \(\mathfrak{sl}^\ast(2, \mathbb{R})\) and therefore it has the same form independently of the dimension of the space \(T\mathbb{R}^n\) we start with. Symplectic leaves in \(\mathfrak{sl}^\ast(2, \mathbb{R})\) are diffeomorphic to \(\mathbb{R}^2\) and pairs of conjugated variables may be introduced as

\[
\begin{align*}
\left\{\frac{\xi_1}{\xi_3}, \frac{1}{2} \xi_1 \right\} &= 1 \\
\left\{\frac{\xi_2}{\sqrt{2\xi_3}}, \sqrt{2\xi_3} \right\} &= 1
\end{align*}
\]

We see in all these examples that the chosen invariant submanifold appears eventually as a “coupling constant” in the reduced dynamics. Moreover, the final “second order description” may be completely unrelated to the original one.

Another remark is in order. We have not specified the signature of our scalar product on \(\mathbb{R}^3\). It is important to notice that the final result does not depend on it. However, because in the reduced dynamics \(\xi_1\) appears in the denominator, when the scalar product is not positive definite we have to remove the full algebraic variety \((\vec{r}, \vec{r}) = 0\) to get a smooth vector field. If the signature is \((+, +, -)\), the relevant group will not be \(\text{SO}(3)\) anymore but will be replaced by \(\text{SO}(2, 1)\).

We can summarize by saying that the reduction of the various examples that we have considered are based on the selection of an invariant submanifold and the selection of an invariant subalgebra of functions.

A few more remarks are necessary:

**Remark 2.** If we consider the Lagrangian description of the free particle

\[ L = \frac{1}{2} (\dot{r}^2 + \dot{\vec{r}}^2), \]

in polar coordinates it becomes

\[ L = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\vec{r}}^2 \right), \]

which restricted to the submanifold \(l^2 = r^4 \dot{\vec{r}}^2\) would give

\[ L = \frac{1}{2} \left( \dot{r}^2 + \frac{l^2}{r^2} \right), \]

which is not the Lagrangian giving rise to the dynamics \(\ddot{r} = l^2/r^3\). Therefore, we must conclude that the reduction, if done in the Lagrangian formalism, must be considered as a symplectic reduction in terms of the symplectic structure of Lagrangian systems (i.e. in terms of the symplectic form \(\omega_L\) and the energy function \(E_L\)).
Remark 3. The free particle admits many alternative Lagrangians, therefore once an invariant submanifold $\Sigma$ has been selected, we have many alternative symplectic structures to pull-back to $\Sigma$ and define alternative involutive distributions to quotient $\Sigma$. The possibility of endowing the quotient manifold with a tangent bundle structure has to be investigated separately because the invariant submanifold $\Sigma$ does not need to have a particular behaviour with respect to the tangent bundle structure. A recent generalization consists of considering that the quotient space may not have a tangent bundle structure but may have a Lie algebroid structure. Further examples and additional comments on previous examples may be found in [39, 44].

We shall close now these preliminaries by commenting on the generalization of this procedure to free systems on higher dimensional spaces.

1.2. Generalized polar coordinates. In the existing literature, examples have been already considered to get Calogero-Moser potentials, Toda and other well-known systems, starting with free or harmonic motions on the space of $n \times n$ Hermitian matrices, free motions on $U(n)$, and free motions on the coset space $GL(n, \mathbb{C})/U(n, \mathbb{C})$ [50].

The main idea is to start with a space of diagonalizable matrices $\{X\}$ and to consider a diagonalizing matrix $G$ in such a way that

$$X = GQG^{-1}.$$  

The diagonal matrix will play the role of “radial coordinates” while $G$ plays the role of angular coordinates. Some care is needed when the parametrization of $X$, $X = X(Q,G)$, is not unique.

Example 4. Let us study again the three dimensional example discussed above. Consider matrices

$$X = \begin{pmatrix} x_1 & \frac{x_2}{\sqrt{2}} \\ \frac{x_2}{\sqrt{2}} & x_3 \end{pmatrix},$$

satisfying the evolution equation

$$\ddot{X} = 0.$$

Therefore, the matrix $M = [X, \dot{X}]$ is such that $dM/dt = 0$, because

$$\dot{M} = [\dot{X}, X] + [X, \dot{X}] = 0.$$

We can introduce new coordinates for the symmetric matrix $X$ by using the rotation group: such a matrix $X$ can be diagonalized by means of an orthogonal transformation $G$, thus, $X$ can be written as $X = GQG^{-1}$ with

$$Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad G = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$$

and therefore, as

$$GQG^{-1} = \begin{pmatrix} q_1 \cos^2 \phi + q_2 \sin^2 \phi & (q_2 - q_1) \sin \phi \cos \phi \\ (q_2 - q_1) \sin \phi \cos \phi & q_1 \sin^2 \phi + q_2 \cos^2 \phi \end{pmatrix}$$

and $x_1 = q_1 \cos^2 \phi + q_2 \sin^2 \phi$, $x_3 = q_1 \sin^2 \phi + q_2 \cos^2 \phi$, we get the relation

$$x_1 + x_3 = q_1 + q_2, \quad x_2 = \frac{1}{\sqrt{2}}(q_2 - q_1) \sin 2\phi, \quad x_1 - x_3 = (q_1 - q_2) \cos 2\phi.$$
Note that $G^{-1} \dot{G} = \dot{G} G^{-1} = \dot{\phi} \sigma$ and $G \sigma = \sigma G$, where
\[
\sigma = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and $M = l \sigma$, where $l$ is the sum of the first and third components of the angular momentum.

Then, using
\[
\frac{d}{dt} (G^{-1} \dot{G}) = -G^{-1} \dot{G} G^{-1},
\]
we see that
\[
\dot{X} = G \dot{Q} G^{-1} - G Q G^{-1} \dot{G} G^{-1} + G \dot{Q} G^{-1} = G \left([G^{-1} \dot{G}, Q] + \dot{Q}\right) G^{-1},
\]
i.e.
\[
\dot{X} = G(\dot{Q} + \dot{\phi} \left[\sigma, Q\right] G^{-1}).
\]

Notice that $[\sigma, Q] = (q_2 - q_1) \sigma_1$ and $[Q, \dot{Q}] = 0$. Consequently,
\[
M = [X, \dot{X}] = G [Q, \dot{\phi} (\sigma + [Q, \dot{Q}])] G^{-1} = \dot{\phi} (q_2 - q_1) G [Q, \sigma] G^{-1} = -\dot{\phi} (q_2 - q_1)^2 \sigma,
\]
and then $l$ is given by
\[
l = \dot{\phi} (q_2 - q_1)^2.
\]
The equations of motion along the radial variables become
\[
\ddot{Q} - \dot{\phi}^2 [\sigma, [\sigma, Q]] = 0
\]
Restricting to the submanifold $\Sigma_l$ given by
\[
\Sigma_l = \left\{ l = -\frac{1}{2} \text{Tr} M \sigma \right\},
\]
we find
\[
\ddot{Q} = \frac{l^2}{(q_2 - q_1)^2} [\sigma, [\sigma, Q]]
\]
or, more explicitly
\[
\ddot{q}_1 = -\frac{2l^2}{(q_2 - q_1)^3}, \quad \ddot{q}_2 = \frac{2l^2}{(q_2 - q_1)^3}
\]
They provide us with Calogero equations for two interacting particles on a line and are the Euler–Lagrange equations associated with the Lagrangian function,
\[
L = \frac{1}{2} \left( q_1^2 - q_2^2 \right) - \frac{\dot{q}_1^2}{(q_2 - q_1)^2}.
\]

1.3. A Lagrangian description and solutions of the Hamilton-Jacobi equation. On the space of symmetric matrices $\{X\}$ we define the Lagrangian function $L = \frac{1}{2} \text{Tr}(\dot{X} \dot{X})$. This Lagrangian gives rise to the Euler-Lagrange equations of motion $\ddot{X} = 0$. Moreover, the symplectic structure associated to it is defined by $\omega_L = \text{Tr}(d\dot{X} \wedge dX)$ and $E_L = L$. The invariance of the Lagrangian under translations and rotations implies the conservation of linear momentum $P = \dot{X}$ and angular momentum $M = [X, \dot{X}]$. The corresponding explicit solutions of the dynamics are thus given by
\[
X(t) = X_0 + t P_0.
\]
It is possible to find easily a solution of the corresponding Hamilton-Jacobi equation. Indeed, by integrating the Lagrangian along the solutions or by solving

\[ P_t \frac{dX}{dt} - P_0 \frac{dX_0}{dt} = dS(X_t, X_0; t) \]

we find that the action is written as

\[ S = \frac{1}{2} \text{Tr}(X_t - X_0)^2. \]

By fixing a value \( \ell^2 = \frac{1}{2} \text{Tr}M^2 \) we select an invariant manifold \( \Sigma \). The corresponding reduced dynamics gives the Calogero equations. Therefore we restrict \( S \) to those solutions which satisfy

\[ \frac{1}{2} \text{Tr}(X_t^2X_0^2 - (X_tX_0)^2) = \ell^2 \]

and we find a solution for the Hamilton-Jacobi equation associated with the reduced dynamics.

**Remark 4.** For any invertible symmetric matrix \( K \), the Lagrangian function

\[ L_K = \frac{1}{2} \text{Tr} \dot{X}K\dot{X} \]

would describe again the free motion. More generally, for any monotonic function \( f \), the composition \( f(L_K) \) would be a possible alternative Lagrangian. The corresponding Lagrangian symplectic structure could be used to find alternative Hamilton-Jacobi equations. For those aspects we refer to [12].

### 2. Summarizing and formalizing

To prepare the ground for the Poisson reduction we emphasize that the reduction procedure that we have considered so far uses two basic ingredients:

- An invariant subalgebra (of functions) \( R \).
- An invariant submanifold of the carrier space \( \Sigma \subset M \).

#### 2.1. The geometrical description.

Let us try to identify the basic aspects of the reduction procedures we shall consider. We denote by \( M \) the manifold containing the states of our system. The equations of motion will be represented by a vector field \( \Gamma \), and we suppose that it gives rise to a one parameter group of diffeomorphisms

\[ \Phi : \mathbb{R} \times M \to M. \]

Occasionally, when we want our map to keep track of the infinitesimal generator we will write \( \Phi_\Gamma \) or \( \Phi_\Gamma : \mathbb{R} \times M \to M \).

To apply the general reduction procedure we need:

- a submanifold \( \Sigma \), invariant under the \( \Phi \) evolution, i.e.

\[ \Phi(\mathbb{R} \times \Sigma) \subset \Sigma, \quad \text{or} \quad \Phi(t, m) \in \Sigma, \quad \forall t \in \mathbb{R}, \quad m \in \Sigma. \]

- An invariant equivalence relation among points of \( \Sigma \), i.e. we consider equivalence relations for which

\[ m \sim m' \Rightarrow \Phi(\mathbb{R}, m) \sim \Phi(\mathbb{R}, m'). \]

The reduced dynamics or “reduced evolution” is defined on the manifold of equivalence classes (assumed to be endowed with a differentiable structure).

One may also start the other way around: we could first consider an invariant equivalence relation on the whole manifold \( M \) and then select an invariant submanifold for the reduced dynamics, to further reduce the dynamical evolution.
2.1.1. Some remarks. In real physical situations the invariant submanifolds arise as level set of functions. These level sets were called invariant relations by Levi-Civita [3] to distinguish them from level sets of constants of the motion. Usually, equivalence classes will be generated by orbits of Lie groups or leaves of involutive distributions. “Closed subgroup” theorems are often employed to guarantee the regularity of the quotient manifold [51].

When additional structures are present, like Poisson or symplectic structures, it is possible to get involutive distributions out of a family of invariant relations. The so called “symplectic reduction” is an example of this particular situation.

When the space is endowed with additional structures, say a tangent or a cotangent bundle, with the starting dynamics being, for instance, second order (in the tangent case), we may also ask for the reduced one to be second order, once we ask the reduced space to be also endowed with a tangent space structure. This raises natural questions on how to find appropriate tangent or cotangent bundle structures on a given manifold obtained as a reduced carrier space. Similarly, we may start with a linear dynamics, perform a reduction procedure (perhaps by means of quadratic invariant relations) and enquire on possible linear structures on the reduced carrier space. A simple example of this situation is provided by the Maxwell equations. These equations may be written in terms of the Faraday 2–form \( F \) encoding the electric field \( E \) and the magnetic field \( B \), as:

\[
dF = 0 \quad d^* F = 0,
\]

when considered in the vacuum [40]. We may restrict these equations to the invariant submanifold

\[
F \wedge F = 0, F \wedge * F = 0.
\]

Even though these relations are quadratic the reduced Maxwell equations provide as solutions the radiation fields and are still linear.

In conclusion, when additional structures are brought into the picture, we may end up with extremely rich mathematical structures and quite difficult mathematical problems.

Example 5. A charged non-relativistic particle in a magnetic monopole field

This system was considered by Dirac [21] and a variant of it, earlier by Poincaré [52]. To describe it in terms of a Lagrangian Dirac introduced a “Dirac string”. The presence of this unphysical singularity leads to technical difficulties in the quantization of this system. Several proposals have been made to deal with these problems.

Here we would like to show how our reduction procedure allows to deal with this system and provides a clear way for its quantization. In doing this we shall follow mainly [7, 8, 9].

The main idea is to replace \( \mathbb{R}^3_0 \) with \( \mathbb{R}^4_0 \) described as the product \( \mathbb{R}^4_0 = S^3 \times \mathbb{R}_+ \), and to get back our space of relative coordinates for the charge-monopole by means of a reduction procedure.

We set first \( \vec{x} \cdot \vec{\sigma} = rs\sigma_3 s^{-1} \), where \( r^2 = x_1^2 + x_2^2 + x_3^2 \) and \( s \in SU(2) \) (realized as \( 2 \times 2 \) matrices of the defining representation; while \( \{\sigma_1, \sigma_2, \sigma_3\} \) are the Pauli matrices. We write the Lagrangian function on \( \mathbb{R}^4_0 \) as

\[
L = \frac{1}{2} m \text{Tr} \left( \frac{d}{dt} (rs\sigma_3 s^{-1}) \right)^2 - k (\text{Tr} \sigma_3 s^{-1} \frac{1}{s} \frac{d}{dt} s)^2.
\]
This expression for the Lagrangian shows clearly the invariance under the left action of SU(2) on itself and an additional invariance under the right action $s \mapsto se^{i\omega}0$ for $\theta \in [0, 2\pi)$. It is convenient to introduce left invariant one forms $\theta^a$ by means of $i\sigma_a\theta^a = s^{-1}ds$ and related left invariant vector fields $X_a$ which are dual to them $\theta^a(X_b) = \delta^a_b$. If $\Gamma$ denotes any second order vector field on $\mathbb{R}^4_0$ we set $\hat{\theta}^a = \theta^a(\Gamma)$, where, with some abuse of notation, we are using the same symbol for $\theta^a$ on $\mathbb{R}^4_0$ and its pull-back to $T\mathbb{R}^4_0$. It is also convenient to use the unit vector $\hat{n}$ defined by $\hat{x} = \hat{n}r$, i.e $\hat{n}\sigma = s\sigma_3s^{-1}$.

After some computations, the Lagrangian becomes

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{4}mr^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + k\theta_3^2.$$ 

It is not difficult to find the canonical 1- and 2-forms for the Lagrangian symplectic structure. For instance $\theta_L = m\dot{r}dr + \frac{1}{2}mr^2(\dot{\theta}_1\dot{\theta}_1 + \dot{\theta}_2\dot{\theta}_2) + 2k\dot{\theta}_3\theta_3$; and of course $\omega_L = d\theta_L$. The energy function $E_L$ coincides with $L$.

If we fix the submanifold $\Sigma_c$ by setting

$$\Sigma_c = \{(r, v) \in T\mathbb{R}^4_0 | \theta_3 = c\},$$

the submanifold turns out to be invariant because $\theta_3$ is a constant of the motion.

On $\Sigma_c$, $\theta_L = m\dot{r}dr + \frac{1}{2}mr^2(\dot{\theta}_1\dot{\theta}_1 + \dot{\theta}_2\dot{\theta}_2) + 2k\theta_3\theta_3$. If we then use the foliation associated with $X_3^i$ (the tangent lift of $X_3$ to $T\mathbb{R}^4_0$), we find that $\omega_L$ is the pull-back of a 2-form on the quotient because $d\theta_3 = \theta_1 \wedge \theta_2$, and hence contains $X^3_4$ in its kernel. The term $d\theta_3$ is exactly proportional to the magnetic field of the magnetic monopole sitting at the origin. Thus on the quotient space of $\Sigma_c$ by the action of the left flow of $X_3^i$ we recover the dynamics of the electron-monopole system on the (quotient) space $T(S^2 \times \mathbb{R}) = T\mathbb{R}^3_0$. It is not difficult to show that

$$\frac{d}{dt} \left( \frac{i}{2}[\hat{n}\sigma, mr^2\hat{n}\sigma] + k\hat{n}\sigma \right) = 0; \quad k = \frac{eg}{4\pi}.$$ 

These constants of the motion are associated with the rotational invariance and replace the usual angular momentum functions.

This example shows that the reduction of the Lagrangian system of Kaluza-Klein type on $T\mathbb{R}^4$ does not reduce to a Lagrangian system on $T\mathbb{R}^3$ but just to a symplectic system.

2.2. The algebraic description. The evaluation map $ev : M \times \mathcal{F} \rightarrow \mathbb{R}$ defined as $(m, f) \mapsto f(m)$, allows to dualize the basic ingredients from the manifold to the algebra of functions on $M$, the observables.

We first notice that to any submanifold $\Sigma \subset M$ we can associate a short exact sequence of associative algebras

$$0 \longrightarrow \mathcal{I}_\Sigma \longrightarrow \mathcal{F} \xrightarrow{\pi_\Sigma} \mathcal{F}_\Sigma \longrightarrow 0$$

defined in terms of the identification map $i_\Sigma : \Sigma \hookrightarrow M$, $\Sigma \ni m \mapsto m \in M$. We have thus:

$$\mathcal{I}_\Sigma = \{ f \in \mathcal{F} | i_\Sigma^*(f) = 0 \}$$

From the property $i_\Sigma^*(fg) = i_\Sigma^*(f)i_\Sigma^*(g)$ we find that $\mathcal{I}_\Sigma$ is a bilateral ideal in $\mathcal{F}$. The algebra $\mathcal{F}_\Sigma$ is then the quotient algebra $\mathcal{F}/\mathcal{I}_\Sigma$. 
Any derivation $\Gamma$ acting on the set of functions of $\mathcal{F}(M)$ will define a derivation on the set of functions $\mathcal{F}_\Sigma$ if and only if $L_\Gamma I_\Sigma \subset I_\Sigma$, so that $\Gamma$ acting on equivalence classes will define a derivation on the reduced carrier space.

A simple example illustrates the procedure. On $T\mathbb{R}^3$ we consider the bilateral ideal $I_\Sigma$, when $\Sigma$ is defined from

\begin{equation}
\begin{aligned}
f_1 &= \vec{r} \cdot \vec{r} - 1 \\
f_2 &= \vec{r} \cdot \vec{v},
\end{aligned}
\end{equation}

and we set $f_1 = 0 = f_2$.

We get the submanifold $\Sigma$ as $TS^2$. The algebra of functions $\mathcal{F}_\Sigma$ is obtained from $\mathcal{F}(T\mathbb{R}^3)$ simply by using in the argument of $f(\vec{r}, \vec{v})$ the constraints (6), i.e.

\begin{equation}
L_\Gamma (\vec{r} \cdot \vec{r} - 1) = \alpha (\vec{r} \cdot \vec{r} - 1) + \beta \vec{r} \cdot \vec{v},
\end{equation}

for arbitrary functions $\alpha, \beta$ and also

\begin{equation}
L_\Gamma (\vec{r} \cdot \vec{v}) = \alpha' (\vec{r} \cdot \vec{r} - 1) + \beta' \vec{r} \cdot \vec{v}.
\end{equation}

It is not difficult to show that the module of such derivations is generated by $R_l = \epsilon_{jkl} (x_j \frac{\partial}{\partial x_k} + v_j \frac{\partial}{\partial v_k})$; $\nu_l = \epsilon_{ijl} x_j \frac{\partial}{\partial v_i}$.

An invariant subalgebra in $\mathcal{F}$, say $\bar{\mathcal{F}}$, for which $I_\Sigma$ is an ideal, defines an invariant equivalence relation by setting

\begin{equation}
m' \sim m'' \text{ iff } f(m') = f(m''), \quad \forall f \in \bar{\mathcal{F}}
\end{equation}

It follows that $\bar{\mathcal{F}}$ defines a subalgebra in $\mathcal{F}_\Sigma$ and corresponds to a possible quotient manifold of $\Sigma$ by the equivalence relation defined by $\bar{\mathcal{F}}$.

In general, a subalgebra in $\mathcal{F}$, say $\mathcal{F}_Q$, defines a short exact sequence of Lie algebras

\begin{equation}
0 \rightarrow \mathfrak{X}^v \rightarrow \mathfrak{X}^N \rightarrow \mathfrak{X}_Q \rightarrow 0
\end{equation}

where $\mathfrak{X}^v$ is the Lie algebra of vector fields annihilating $\mathcal{F}_Q$, $\mathfrak{X}^N$ is the normalizer of $\mathfrak{X}^v$ in $\mathfrak{X}(M)$, and $\mathfrak{X}_Q$ is the quotient Lie algebra. This sequence of Lie algebras may be considered a sequence of Lie modules with coefficients in $\mathfrak{F}_Q$. In the previous case, $\mathfrak{F}_Q$ would be the invariant subalgebra in $\mathcal{F}_\Sigma$ and the equivalence relation would be defined by the leaves of the involutive distribution $\mathfrak{X}^v$ (regularity requirements should be then imposed on $\mathfrak{F}_Q$). See [38] for details.

From the dual point of view it is now clear that reducible evolutions will be defined by one-parameter groups of transformations which are automorphisms of the corresponding short exact sequences. The corresponding infinitesimal versions will be defined in terms of derivations of the appropriate short exact sequence of algebras.

To illustrate this aspect, we consider the associative subalgebra of $\mathcal{F}(T\mathbb{R}^3)$ generated by $\{\vec{r} \cdot \vec{r}, \vec{v} \cdot \vec{v}, \vec{r} \cdot \vec{v}\}$. For this algebra it is not difficult to see that the vector fields

\begin{equation}
X_c = \epsilon_{abc} \left( x^a \frac{\partial}{\partial x_b} + v^a \frac{\partial}{\partial v_b} \right)
\end{equation}

generate $\mathfrak{X}^v$, while $\mathfrak{X}^N$ is generated by $\mathfrak{X}^v$ and

\begin{align*}
&\vec{r} \frac{\partial}{\partial \vec{r}}, \quad \vec{v} \frac{\partial}{\partial \vec{v}}, \quad \vec{r} \frac{\partial}{\partial \vec{r}}, \quad \vec{v} \frac{\partial}{\partial \vec{v}}.
\end{align*}
The quotient \( \mathfrak{X}_Q \), with a slight abuse of notation, can also be considered to be generated by the vector fields

\[
\tilde{r} \frac{\partial}{\partial \tilde{v}}, \quad \tilde{v} \frac{\partial}{\partial \tilde{r}}, \quad \tilde{r} \frac{\partial}{\partial \tilde{r}}, \quad \tilde{v} \frac{\partial}{\partial \tilde{v}},
\]

which however are not all independent. Any combination of them with coefficients in the subalgebra may be considered a “reduced dynamics”.

2.3. Additional structures: Poisson reduction. When a Poisson structure is available, we can further qualify the previous picture. We can consider associated short exact sequences of Hamiltonian derivations. Hence, a Poisson reduction can be formulated in the following way: we start with \( \mathcal{I}_\Sigma \), again an ideal in the commutative and associative algebra \( \mathcal{F} \). We consider then the Hamiltonian derivations which map \( \mathcal{I}_\Sigma \) into itself:

\[
W(\mathcal{I}_\Sigma) = \{ f \in \mathcal{F} \mid \{ f, \mathcal{I}_\Sigma \} \subset \mathcal{I}_\Sigma \}.
\]

Then we consider \( \mathcal{I}_\Sigma' = \mathcal{I}_\Sigma \cap W(\mathcal{I}_\Sigma) \) and get the exact sequence of Poisson algebras

\[
0 \to \mathcal{I}_\Sigma' \to W(\mathcal{I}_\Sigma) \to Q_\Sigma \to 0.
\]

When the ideal \( \mathcal{I}_\Sigma \) is given by constraint functions as in the Dirac approach, \( W(\mathcal{I}_\Sigma) \) are first class functions and \( \mathcal{I}_\Sigma' \) are the first class constraints.

Example 6. We give here an example of an iterated reduction. We consider a parametrization of \( T\mathbb{R}^4 \) in terms of the identity matrix in dimension 2 \( (\sigma_0) \) and the \( 2 \times 2 \) Pauli matrices as follows: \( \pi = p_0\sigma_0 + p_a\sigma_a \) and \( g = y_0\sigma_0 + y_a\sigma_a \).

A preliminary “constraint” manifold is selected by requiring that\n
\[
\mathrm{Tr} g^+ g = 1 \quad \mathrm{Tr} \pi^+ = 0.
\]

This manifold is diffeomorphic to the tangent bundle of \( S^3 \), i.e. \( TS^3 \). The Hamiltonian \( H = (p_\mu p^\mu)(y_\mu y^\mu) \) defines a vector field tangent to the constraint manifold. Similarly for the “potential” function \( V = \frac{1}{2}(y_0^2 + y_1^2 - y_2^2 - y_3^2)/y_\mu y^\mu \).

The Hamiltonian function \( \frac{1}{2}H + V \), when restricted to \( TS^3 \) with a slight abuse of notation acquires the suggestive form

\[
H = \frac{1}{2}(p_0^2 + p_1^2 + y_0^2 + y_3^2) + \frac{1}{2}(p_1^2 + p_2^2 - y_1^2 - y_2^2).
\]

By using the relation \( y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1 \) we may also write it in the form

\[
H = \frac{1}{2}(p_0^2 + p_1^2 + 2(y_0^2 + y_3^2)) + \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{2}.
\]

Starting now with \( TS^3 \) we may consider the further reduction by fixing

\[
\Sigma_K = \{ (y_\mu, p^\nu) \in TS^3 \mid y_0p_3 - p_0y_3 + y_1p_2 - y_2p_1 = K \},
\]

and quotienting by the vector field

\[
X = y_0 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_0} + y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1} + p_0 \frac{\partial}{\partial p_3} - p_3 \frac{\partial}{\partial p_0} + p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1}.
\]

The final reduced manifold will be \( TS^2 \subset T\mathbb{R}^3 \), with projection \( TS^3 \to TS^2 \) provided by the tangent of the Hopf fibration \( \pi : S^3 \to S^2 \), defined as

\[
x_1 = 2(y_1y_3 - y_0y_2) \quad x_2 = 2(y_2y_3 - y_0y_1) \quad x_3 = y_0^2 + y_3^2 - y_1^2 - y_2^2.
\]

The final reduced dynamics will be associated with the Hamiltonian function of the spherical pendulum.
The spherical pendulum is thus identified by
\[ S^2 \subset \mathbb{R}^3 = \{ x \in \mathbb{R}^3 \mid \langle x, x \rangle = x_1^2 + x_2^2 + x_3^2 = 1 \} \]
\[ TS^2 \subset T\mathbb{R}^3 = \{ (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle x, x \rangle = 1, \langle x, v \rangle = 0 \} \]

The dynamics is given by means of \( \omega = \sum dx_i \wedge dv_i \) when restricted to \( TS^2 \), in terms of \( E = \frac{1}{2} \langle v, v \rangle + x_3 \). The angular momentum is a constant of the motion corresponding to the rotation around the \( Ox_3 \) axis. The energy momentum map
\[ \mu : TS^2 \to \mathbb{R}^2 : (x, v) \mapsto (E(x, v), L(x, v)) \]
has quite interesting properties as shown by [20, 24].

3. Formulations of Quantum Mechanics

Having defined Poisson reduction we are now on the good track to define a reduction procedure for quantum systems. After all, according to deformation quantization the Poisson bracket provides us with a first order approximation to Quantum Mechanics. However, before entering a general discussion of the reduction procedure for quantum systems, let us recall very briefly the various formalisms to describe quantum dynamical evolution.

The description of quantum systems is done basically by means of either the Hilbert space of states, where we define dynamics by means of the Schrödinger equation, or by means of the algebra of observables, where dynamics is defined by means of the Heisenberg equation. We may also consider other pictures like the Ehrenfest picture, the phase-space picture and the \( C^* \)-algebra approach.

3.1. The Schrödinger equation in Wave Mechanics.

3.1.1. The framework. Let us consider first the usual description of Schrödinger formulation of Quantum Mechanics. We consider the set of states of our quantum system to be the space of square integrable functions on some domain \( D \) (which, for simplicity can be assumed to be some open subset of \( \mathbb{R}^n \) but that can also be considered to be a general differential manifold, possibly with boundary). Thus the Hilbert space describing the set of states will be \( L^2(D, d\mu) \), where we denote by \( d\mu \) the measure associated to a volume form. The states themselves will be denoted as \( \psi \) or as \( |\psi\rangle \), in the standard bra-ket notation. Observables are required to be symmetric operators on this space, and usually realized as differential operators. In this setting, dynamics is introduced through the Schrödinger equation:
\[ i\hbar \frac{d}{dt} \psi = H \psi \quad \psi \in \mathcal{H} \]  
(9)

where the Hamiltonian operator \( H \) corresponds to an essentially self-adjoint differential operator acting on \( L^2(D, d\mu) \) and written as
\[ H\psi = \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x), \]  
(10)

where \( V(x) \) represents the potential energy of the system usually assumed to act as a multiplicative operator.

Dynamics can also be encoded in a unitary operator \( U(t, t_0) \) such that
\[ |\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle. \]
It is possible to write a differential equation to encode Schrödinger equation on $U$ by setting

$$i\hbar \frac{d}{dt}U(t, t_0) = HU(t, t_0).$$

By using eigenstates of the position operator $Q|x\rangle = x|x\rangle$, we can write the identity in the form

$$\mathbb{I} = \int_D |x\rangle dx.$$

Then we can give the operator $U$ an integral form:

$$\psi(x, t) = \langle x, \psi(t) \rangle = \int_D G(x, t; x_0, t_0)\psi(x_0, t_0)dx_0$$

where $G(x, t; x_0, t_0) = \langle x, U(t, t_0)x_0 \rangle$. The function $G$ is known as the propagator or the Green function of the system.

Schrödinger equation exhibits interesting properties but we would like to focus now on the fact that it can be given a Hamiltonian form with respect to the symplectic structure that can be associated to the imaginary part of the Hermitian structure of the Hilbert space, when considered as a real manifold. We will elaborate a little further on this statement in the next sections.

Keeping this in mind we can think now of the analogue of the reduction procedures that we have seen in the classical setting. The idea is quite the same for simple examples such as the free motion, but the quantum nature of the system provides us with some new features:

3.1.2. Example: The reduction of free motion in the quantum case. The description of the free quantum evolution is rather simple because the semi-classical treatment is actually exact [26]. In what follows we are setting $\hbar = 1$ for simplicity.

The Hamiltonian operator for free motions in two dimensions, written in polar coordinates is

$$H = -\frac{1}{2} \frac{Q}{Q} \frac{\partial}{\partial Q} Q \frac{\partial}{\partial Q} - \frac{1}{Q^2} \frac{\partial^2}{\partial \phi^2}.$$

By a similarity transformation $H' = Q^{\frac{1}{2}} HQ^{-\frac{1}{2}}$ we get rid of the linear term and obtain

$$H' = -\frac{1}{2} \left( \frac{\partial^2}{\partial Q^2} + \frac{1}{Q^2} \left( \frac{1}{4} + \frac{\partial^2}{\partial \phi^2} \right) \right).$$

Restricting $H'$ to the subspace of square integrable functions of the form $S_m = \{ \psi = e^{im\phi} f(Q) \}$, we find that on this particular subspace

$$H'\psi = -\frac{1}{2} \left( \frac{\partial^2}{\partial Q^2} - \frac{1}{Q^2} \left( m^2 - \frac{1}{4} \right) \right) \psi.$$

Going back to the Hamiltonian operator, we find

$$Q^{\frac{1}{2}} H Q^{-\frac{1}{2}}(Q^{\frac{1}{2}}\psi) = -\frac{1}{2} \left( \frac{\partial^2}{\partial Q^2} - \frac{1}{Q^2} \left( m^2 - \frac{1}{4} \right) \right) Q^{\frac{1}{2}}\psi = EQ^{\frac{1}{2}}\psi.$$  

This determines a Hamiltonian operator along the radial coordinate and setting $g^2 = m^2 - \frac{1}{4}$ we have

$$\tilde{H} = -\frac{1}{2} \frac{\partial^2}{\partial Q^2} + \frac{1}{2} \frac{g^2}{Q^2}. $$
If we parametrize the Euclidean space with matrices $X$, solutions of the free problem, in generic coordinates $\{X\}$, are given, of course, by wave-packets formed out of “plane-waves”
\[
\psi_p(X) = Ae^{i \text{Tr}XP},
\]
where $A$ is a normalization constant, chosen in such a way as to give a delta function normalization.

By decomposing $X$ into a “radial” part $Q$ and an “angular” part $G$, say
\[
X = G - 1^2 QG,
\]
we can write the wave function in the form
\[
\psi(Q, G) = Ae^{i \text{Tr}(G^{-1}QGP)} = \psi_p(X).
\]

In this particular case it is not difficult to show that $I_j(X, P) = \text{Tr}(P^j)$ are constants of the motion in involution and give rise to the operators $(-i)^j \text{Tr} \left( \frac{\partial}{\partial X} \right)^j$.

To perform specific computations let us go back to the two-dimensional situation.

We consider $\psi_p = Ae^{i \text{Tr}PX}$ and project it along the eigenspace $S_m$ of the angular momentum corresponding to the fixed value $m$.

We recall that (in connection with the unitary representations of the Euclidean group)
\[
\int_0^{2\pi} d\phi e^{im\phi} e^{iPQ \cos \phi} = 2\pi J_m(PQ),
\]
where $J_m$ is the Bessel function of order $m$. Thus we conclude
\[
\psi_p(Q) = 2\pi \sqrt{PQ} J_m(PQ).
\]

In the particular case we are considering free motion is described by a quadratic Hamiltonian in $\mathbb{R}^2$. Therefore the Green function becomes
\[
G(X_t - X_0, 0; t) = \frac{C}{2t} e^{\text{Tr}(X_t - X_0)^2 t}.
\]
The Green function can be written in terms of the action (the solution of the Hamilton-Jacobi equation, see Section 13) and the Van Vleck determinant ([26], appendix 4.B).

By using polar coordinates the kernel of the propagator is
\[
G(Q_t, Q_0; t) = \sqrt{Q_t Q_0} \int_0^{2\pi} d\phi e^{im\phi} K(X_t, X_0; t) = \sqrt{Q_t Q_0} e^{i \frac{(Q_t^2 + Q_0^2)^2 - \frac{(Q_t Q_0)^2}{x^2}}{2it}} J_m \left( \frac{Q_t Q_0}{t} \right),
\]
where the angle is coming from the scalar product of $X_t$ with $X_0$.

3.2. Reduction in terms of Differential operators. With this simple example we have discovered that in wave mechanics the reduction procedure involves differential operators and their eigenspaces. Let us therefore consider some general aspects of reduction procedures for differential operators.

In general, the Hamiltonian operator defining the Schrödinger equation on $L^2(D, d\mu)$ is a differential operator, which may exhibit a complicated dependence in the potential. It makes sense thus to study a general framework for the reduction of differential operators acting on some domain $D$, when we assume that the reduction procedure consists in the suitable choice of some “quotient” domain $D'$.  

3.2.1. Abstract definition of Differential operators. We consider \( \mathcal{F} = C^\infty(\mathbb{R}^n) \), the algebra of smooth functions on \( \mathbb{R}^n \). A differential operator of degree at most \( k \) is defined as a linear map \( D^k: \mathcal{F} \to \mathcal{F} \) of the form
\[
D^k = \sum_{|\sigma| \leq k} g_{\sigma} \frac{\partial^{|\sigma|}}{\partial x^\sigma}, \quad g_{\sigma} \in \mathcal{F}
\]
(12)
where \( \sigma = (i_1, \cdots, i_n) \), \( |\sigma| = \sum_k i_k \) and
\[
\frac{\partial^{|\sigma|}}{\partial x^\sigma} = \frac{\partial^{|\sigma|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}
\]

This particular way of expressing differential operators relies on the generator of “translations”, \( \frac{\partial}{\partial x} \). Therefore, when the reduced space does not carry an action of the translation group this way of writing differential operators is not very convenient. There is an intrinsic way to define differential operators which does not depend on coordinates [2, 32, 33]. One starts from the following observation

\[
\left[ \frac{\partial}{\partial x_j}, \hat{f} \right] = \frac{\partial f}{\partial x_j},
\]

where \( \hat{f} \) is the multiplication operation by \( f \), i.e. an operation of degree zero \( \hat{f}: g \mapsto fg \), with \( f, g \in \mathcal{F} \).

It follows that
\[
[D^k, \hat{f}] = \sum_{|\sigma| \leq k} g_{\sigma} \left[ \frac{\partial^{|\sigma|}}{\partial x^\sigma}, \hat{f} \right],
\]
is of degree at most \( k - 1 \). Iterating for a set of \( k + 1 \) functions \( f_0, \cdots, f_k \in \mathcal{F} \), one finds that
\[
[\cdots, [D^k, \hat{f}_0], \hat{f}_1], \cdots, \hat{f}_k] = 0;
\]

This algebraic characterization allows for a definition of differential operators on any manifold.

The algebra of differential operators of degree 1 is a Lie subalgebra with respect to the commutator and splits into a direct sum
\[
D^1 = \mathcal{F} \oplus D^1_c
\]
where \( D^1_c \) are derivations, i.e. differential operators of degree one which give zero on constants. We can endow the set with a Lie algebra structure by setting
\[
[(f_1, X_1), (f_2, X_2)] = (X_1 f_2 - X_2 f_1, [X_1, X_2])
\]

If we consider \( \mathcal{F} \) as an Abelian Lie algebra, \( D^1_c \) is the algebra of its derivations and then \( D^1 \) becomes what is known in the literature as the “holomorph” of \( \mathcal{F} \). In this way the algebra of differential operators becomes the enveloping algebra of the holomorph of \( \mathcal{F} \).

The set of differential operators on \( M \), denoted as \( \mathcal{D}(M) \), can be given the structure of a graded associative algebra and it is also a module over \( \mathcal{F} \). Notice that this property would not make sense at the level of abstract operator algebra. To consider the problem of reduction of differential operators we consider the problem of reduction of first order differential operators. Because the zeroth order ones are just functions, we restrict our attention to vector fields, i.e. the set \( D^1_c \).
Given a projection $\pi : M \to N$ between smooth manifolds, we say that a vector field $X_M$ projects onto a vector field $X_N$ if

$$L_{X_M} \pi^* f = \pi^*(L_{X_N} f) \quad \forall f \in \mathcal{F}(N).$$

We say thus that $X_M$ and $X_N$ are $\pi$–related.

Thus if we consider the subalgebra $\pi^*(\mathcal{F}(N)) \subset \mathcal{F}(M)$, a vector field is projectable if it defines a derivation of the subalgebra $\pi^*(\mathcal{F}(N))$. More generally, for a differential operator $D^k$, we shall say that it is projectable if

$$D^k \pi^*(\mathcal{F}(N)) \subset \pi^*(\mathcal{F}(N)).$$

It follows that projectable differential operators of degree zero are elements in $\pi^*(\mathcal{F}(N))$. Therefore projectable differential operators are given by the enveloping algebra of the holomorph of $\pi^*(\mathcal{F}(N))$, when the corresponding derivations are considered as belonging to $\mathfrak{X}(M)$.

**Remark 5.** Given a subalgebra of differential operators in $\mathcal{D}(M)$ it is not said that it is the enveloping algebra of the first order differential operators it contains. When this happens, we cannot associate a corresponding quotient manifold with an identifiable subalgebra of differential operators. An example of this situation arises with $J_z$ and $J^2$. It is clear that this commuting subalgebra of differential operators can not be generated by its “first order content”.

In the quantization procedure, this situation gives rise to anomalies [1].

### 3.2.2. Example: differential operators and the Kustainheiro-Stiefel (KS) fibration.

In this section we would like to consider the reduction of differential operators associated with the KS projection $\pi_{KS} : \mathbb{R}^4_0 \to \mathbb{R}^3_0$, where $\mathbb{R}^0 = \mathbb{R}^1 - \{0\}$, and show that the hydrogen atom operator may be obtained as a reduction of the operators associated with a family of harmonic oscillators.

Let us recall first how this map is defined. We first notice that $\mathbb{R}^3_0 = S^3 \times \mathbb{R}^+ \sim SU(2) \times \mathbb{R}^+$ and $\mathbb{R}^3_0 = S^2 \times \mathbb{R}^+$. By introducing polar coordinates

$$g = Rs \quad s \in SU(2), \quad R \in \mathbb{R}^+,$$

we define $\pi_{KS} : \mathbb{R}^4_0 \to \mathbb{R}^3_0$ as

$$\pi_{KS} : g \mapsto g\sigma_3 g^+ = R^2 s\sigma_3 s^{-1} = x^k \sigma_k,$$

where $\{\sigma_k\}$ are the Pauli matrices. In a Cartesian coordinate system one has

$$x_1 = 2(y_1 y_3 + y_2 y_0) \quad x_2 = 2(y_2 y_3 - y_1 y_0) \quad x_3 = y_1^2 + y_2^2 - y_3^2 - y_0^2,$$

where $g = \sum_i y_i \sigma_i$. Moreover, $\sqrt{x^i x_j} = r = R^2 = y^k y_k$.

The KS projection defines a principal fibration with structure group $U(1)$.

By the definition of $\pi_{KS}$ it is easy to see that acting with $e^{i\lambda \sigma_3}$ on $SU(2)$ does not change the projected point on $\mathbb{R}^3_0$. The associated fundamental vector field is the left invariant infinitesimal generator associated with $\sigma_3$, i.e. $i_{X_3} s^{-1} ds = i \sigma_3$. In coordinates it reads

$$X_3 = y^3 \frac{\partial}{\partial y^3} - y^0 \frac{\partial}{\partial y^0} + y^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial y^1}.$$

We consider the Lie algebra of differential operators generated by $X_3$ and $\pi_{KS}(\mathcal{F}(\mathbb{R}^3_0))$. Projectable differential operators with respect to $\pi_{KS}$ are given by the normalizer of this algebra in the algebra of differential operators $\mathcal{D}(\mathbb{R}^4_0)$. As we already remarked
this means that this subalgebra must map \( \pi^* (\mathcal{F}(\mathbb{R}^3_0)) \) into itself. If we denote this subalgebra by \( D^\pi \) we may also restrict our attention to the operators in \( D^\pi \) commuting with \( X_3 \). To explicitly construct this algebra of differential operators we use the fact that \( SU(2) \times \mathbb{R}_+ \) is a Lie group and therefore it is parallelizable. Because the KS map has been constructed with left invariant vector field \( X_3 \), we consider the generators of the left action of \( SU(2) \), say right invariant vector fields \( Y_1, Y_2, Y_3 \), and a central vector field along the radial coordinate, say \( R \). All these vector fields are projectable and therefore along with \( \pi^*_K (\mathcal{F}(\mathbb{R}^3_0)) \) generate a projectable sub-algebra of differential operators which covers the algebra of differential operators on \( \mathbb{R}^3_0 \). This map is surjective and we can ask to find the “inverse image” of the operator \( \hat{H} = -\Delta_3 - \frac{k}{R} \), which is the operator associated with the Schrödinger equation of the hydrogen atom (\( \Delta_3 \) denotes the Laplacian in the three dimensional space). As this operator is invariant under the action of \( so(4) \sim su(2) \oplus su(2) \), associated with the angular momentum and the Runge-Lenz vector, we may look for a representative in the inverse image which shares the same symmetries. As the pull-back of the potential \( \frac{k}{R} \) creates no problems, we may concentrate our attention on the Laplacian. Because of the invariance requirements, our candidate for the inverse image will have the form

\[
D = f(R) \frac{\partial^2}{\partial R^2} + g(R) \frac{\partial}{\partial R} + h(R) \Delta_3^s + c(R),
\]

where \( R \) is the radial coordinate in \( \mathbb{R}^3_0 \), and \( f, g, h \) are functions to be determined.

We recall that in polar coordinates the Laplacian \( \Delta_3 \) has the expression

\[
\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_2^s,
\]

where we denote by \( \Delta_2^s \) the Laplacian on the \( n \)-dimensional sphere.

By imposing \( D \pi^*_K f = \pi^*_K (\hat{H} f) \) for any \( f \in \mathcal{F}(\mathbb{R}^3_0) \) we find that the representative in the inverse image has the expression

\[
H' = -\frac{1}{2} \frac{1}{4R^2} \Delta_4 - \frac{k}{R^2}.
\]

This operator is usually referred to as the conformal Kepler Hamiltonian \([5]\).

Now, with this operator we may try to solve the eigenvalue problem

\[
\left( -\frac{1}{2} \frac{1}{4R^2} \Delta_4 - \frac{k}{R^2} \right) \psi - E \psi = 0.
\]

It defines a subspace in \( \mathcal{F}(\mathbb{R}^4_0) \) which coincides with the one determined by the equation

\[
\left( -\frac{1}{2} \Delta_4 - 4ER^2 - 4k \right) \psi = 0.
\]

This implies that the subspace is given by the eigenfunctions of the Harmonic oscillator with frequency \( \omega(E) = \sqrt{-8E} \). We notice then that a family of oscillators is required to solve the eigenvalue problem associated with the hydrogen atom. To find the final wave functions on \( \mathbb{R}^3 \) we must require that \( L_{X_3} \psi = 0 \) in order to find eigenfunctions for the three dimensional problem. Eventually we find the correct relations for the corresponding eigenvalues

\[
E_m = -\frac{k^2}{2(m + 1)^2}, \quad m \in \mathbb{N}.
\]
Of course, dealing with Quantum Mechanics we should ensure that the operator \( H' = -\frac{1}{2}\Delta_4 + \frac{k}{R^2} \) is essentially self-adjoint to be able to associate with it a unitary dynamics. One finds that the Hilbert space should be constructed as a space of square integrable functions on \( \mathbb{R}^4 \) with respect to the measure \( 4R^2d^4y \) instead of the Euclidean measure on \( \mathbb{R}^4 \). We shall not go into the details of this, but the problem of a different scalar product is strictly related to the reparametrization of the classical vector field, required to turn it into a complete one. This would be a good example for J. Klauder’s saying: “these are classical symptoms of a quantum illness” (see [56]). Further details can be found in [5, 6]. As for the reduction of the Laplacian in Quantum Mechanics see also [27, 28, 29]. For a geometrical approach to the problem of self-adjoint extensions see [4].

3.3. Heisenberg formalism. A different approach to Quantum Mechanics is given by what is known as the Heisenberg picture. Here dynamics is encoded in the algebra of observables, considered as the real part of an abstract \( \mathbb{C}^* \)-algebra.

First, we have to consider observables as associated with Hermitian operators (finite dimensional matrices if the system is finite dimensional). These matrices do not define an associative algebra because the product of two Hermitian matrices is not Hermitian. However we may complexify this space by writing a generic matrix as the sum of a real part \( A \) and an imaginary part \( iB \), \( A \) and \( B \) being Hermitian. In this way we find that:

**Proposition 1.** The complexification of the algebra of observables allows us to write an associative product of operators \( A = A_1 + iA_2 \), where \( A_1 \) and \( A_2 \) are real Hermitian. We shall denote by \( \mathcal{A} \) the corresponding associative algebra.

Finally we can proceed to define the equations of motion on this complexified algebra of observables. It is introduced by means of the **Heisenberg equation:**

\[
i\hbar \frac{d}{dt} A = [A, H], \quad A \in \mathcal{A}, \quad (13)
\]

where \( H \) is called the Hamiltonian of the system we are describing. To take into account an explicit time-dependence of the observable we may also write the equation of motion in the form

\[
\frac{d}{dt} A = -\frac{i}{\hbar} [A, H] + \frac{\partial A}{\partial t} \quad A \in \mathcal{A}. \quad (14)
\]

From a formal point of view, this expression is similar to the expression of Hamilton equation written on the Poisson algebra of classical observables (i.e. on the algebra of functions representing the classical quantities with the structure provided by the Poisson bracket we assume our classical manifold is endowed with). This similarity is not casual and turns out to be very useful in the study of the quantum-classical transition. We shall come back to this point later on.

**Remark 6.** The equations of motion written in this form are necessarily derivations of the associative product and can therefore be considered as “intrinsically Hamiltonian”. In the Schrödinger picture, however, if the vector field is not anti-Hermitian, the equation still makes sense, but the dynamics need not be Kählerian. To recover a similar treatment, one has to give up the requirement that the evolution preserves the product structure on the space of observables.
This approach to Quantum Mechanics relies on the non-commutative algebra of observables, therefore it is instructive to consider a reduction procedure for non-commutative algebras.

3.3.1. An example of reduction in a non-commutative setting. The example of reduction procedure in a non-commutative setting that we are going to discuss reproduces the Poisson reduction in the “quantum-classical” transition and goes back to the celebrated example of the quantum $SU(2)$ written by Woronowicz [55] and is adapted from [31].

We consider the space $S^3 \subset \mathbb{R}^4$, identified with the group $SU(2)$ represented in terms of matrices. The $\star$-algebra $A$ generated by matrix elements is dense in the algebra of continuous functions on $SU(2)$ and can be characterized as the “maximal” unital commutative $\star$-algebra $A$, generated by elements which we can denote as $\alpha, \nu, \alpha^*, \nu^*$ satisfying $\alpha^*\alpha + \nu^*\nu = 1$. This algebra can be generalized and deformed into a non-commutative one by replacing some relations with the following ones:

$$\alpha\alpha^* - \alpha^*\alpha = (2q - q^2)\nu^*\nu$$

and

$$\nu\alpha - \alpha\nu = q\nu\alpha$$

$$\nu^*\alpha - \alpha\nu^* = q\nu^*\alpha.$$

This algebra reduces to the previous commutative one when $q = 0$. In this respect this situation resembles the one on the phase-space where we consider “deformation quantization” and the role of the parameter $q$ is played by the Planck constant. Pursuing this analogy we may consider the formal product depending on the parameter $q$:

$$u \star_q v = uv + \sum_n q^n P_n(u, v),$$

where $P_n$ are such that the product $\star_q$ is associative.

Since the commutator bracket

$$[u, v]_q = u \star_q v - v \star_q u$$

is a biderivation (as for any associative algebra) and satisfies the Jacobi identity we find that the “quantum Poisson bracket” gives a Poisson bracket when restricted to “first order elements”

$$\{u, v\} = P_1(u, v) - P_1(v, u).$$

In general, we can write

$$\lim_{q \to 0} \frac{1}{q} [u, v]_q = \{u, v\}$$

(15)

From the defining commutation relations written by Woronowicz we get the corresponding quadratic Poisson brackets on the matrix elements of $SU(2)$:

$$\{\alpha, \bar{\alpha}\} = 2\bar{\nu}\nu, \quad \{\nu, \bar{\nu}\} = 0, \quad \{\nu, \alpha\} = \nu\alpha, \quad \{\bar{\nu}, \alpha\} = \bar{\nu}\alpha.$$

Passing to real coordinates, $\alpha = q_2 + ip_2$ and $\nu = q_1 + ip_1$, we get a purely imaginary bracket whose imaginary part is the following quadratic Poisson bracket

$$\{p_1, q_1\} = 0, \quad \{p_1, p_2\} = q_1q_2, \quad \{p_1, q_1\} = -p_1p_2,$$

$$\{q_1, p_2\} = q_1q_2, \quad \{q_1, q_2\} = -q_1p_2, \quad \{p_2, q_2\} = q_2^2 + p_1^2.$$

The functions $q_1^2 + q_2^2 + p_1^2 + p_2^2$ is a Casimir function for this Lie algebra.
By performing a standard Poisson bracket reduction we find a bracket on $S^3$. If we identify this space with the group $SU(2)$ we get the Lie-Poisson structure on $SU(2)$.

The vector field

$$X = -q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q_2}$$

selects a subalgebra of functions $\mathcal{F}$ by imposing the condition $L_X \mathcal{F} = 0$. This reduced algebra can be regarded as the algebra generated by

$$u = -p_1^2 - q_1^2 + p_2^2 + q_2^2, \quad \nu = 2(p_1 p_2 + q_1 q_2), \quad z = 2(p_1 q_2 - q_1 p_2),$$

with brackets

$$\{v, u\} = 2(1 - u)z, \quad \{u, z\} = 2(1 - u)v, \quad \{z, v\} = 2(1 - u)u.$$

One finds that $u^2 + \nu^2 + z^2 = 1$ so that the reduced space of $SU(2)$ is the unit sphere $S^2$ and the reduced bracket vanishes at the North Pole ($u = 1, v = z = 0$).

It may be interesting to notice that the stereographic projection from the North Pole pulls-back the standard symplectic structure on $\mathbb{R}^2$ onto the one associated with this one on $S^2 - \{\text{North Pole}\}$.

It is now possible to carry on the reduction at the non-commutative level. We identify the subalgebra $\mathcal{A}'_q \subset \mathcal{A}_q$ generated by the elements $u = -2\nu^*\nu = \alpha^*\alpha - \nu^*\nu$, $w = 2\nu^*\alpha$ and $w^* = 2\alpha^*\nu$. We have $uw^* + w^*w = I$ and the algebra $\mathcal{A}'_q$ admits a limit given by $\mathcal{A}'_0$ generated by the two dimensional sphere $S^2$. The subalgebra $\mathcal{A}'_q$ can be considered as a quantum sphere.

The quantum Poisson bracket on $S^2$ is given by

$$[w, u] = (q^2 - 2q)(1 - u)w, \quad [w^*, u] = -(q^2 - 2q)(1 - u)w^*,$$

and

$$[w, w^*] = -(2q^2 - 2q)(1 - u) + (4q - 6q^2 + 4q^3 - q^4)(1 - u)^2.$$

Passing to the classical limit we find, by setting $v = \text{Re}(w)$, $z = -\text{Im}(w)$:

$$\{v, u\} = 2(1 - u)z, \quad \{u, z\} = 2(1 - u)v, \quad \{z, v\} = 2(1 - u)u,$$

which coincides with the previous reduced Poisson bracket associated with the vector field $X$. In this case, the reduction procedure commutes with the “quantum-classical” limit.

In this same setting it is now possible to consider a “quantum dynamics” and the corresponding “classical” one to see how they behave with respect to the reduction procedure.

On the algebra $\mathcal{A}_q$ we consider the dynamics defined by the Hamiltonian

$$H = \frac{1}{2}u = \frac{1}{2}(1 - 2\nu^*\nu) = \frac{1}{2}(\alpha^*\alpha - \nu^*\nu).$$

This choice ensures that our Hamiltonian defines a dynamics on $\mathcal{A}'_q$. The resulting equations of motion are

$$[H, \nu] = 0, \quad [H, \nu^*] = 0, \quad [H, \alpha] = (q^2 - 2q)\nu^*\nu\alpha, \quad [H, \alpha^*] = -(q^2 - 2q)\nu^*\nu\alpha^*,$$

so that the dynamics written in the exponential form is

$$U(t) = e^{itH}$$

and gives

$$\nu(t) = \nu_0, \quad \nu^*(t) = \nu^*(0)$$
\[ \alpha(t) = e^{it(q^2 - 2q)v^* \nu} \alpha_0, \quad \alpha^*(t) = e^{-it(q^2 - 2q)v^* \nu} \alpha_0^* . \]

Going to the “classical limit” we find
\[ H = \frac{1}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2), \]
with the associated vector field on \( S^3 \) given \([42]\) by
\[ \Gamma = 2(q_1^2 + p_1^2) \left( q_2 \frac{\partial}{\partial q_2} - p_2 \frac{\partial}{\partial q_2} \right), \]
the corresponding solutions are given by
\[ q_1(t) = q_1(0), \quad p_1(t) = p_1(0) \]
\[ p_2(t) = \cos(2t(q_1^2 + p_1^2)) p_2(0) + \sin(2t(q_1^2 + p_1^2)) q_2(0), \]
\[ q_2(t) = -\sin(2t(q_1^2 + p_1^2)) p_2(0) + \cos(2t(q_1^2 + p_1^2)) q_2(0). \]

If we remember \([15]\), this flow is actually the limit of the quantum flow when we take the limit of the deformation parameter \( q \to 0 \) and hence \( q/q \to 1 \).

Indeed in this case \( \nu^* \nu = q_1^2 + p_1^2 \) and \( \alpha = q_2 + ip_2 \). As the Hamiltonian was chosen to be an element of \( A'_q \) we get a reduced dynamics given by
\[ [H, w] = -\frac{1}{2}(q^2 - 2q)(1 - u)w, \quad [H, w^*] = -\frac{1}{2}(q^2 - 2q)(1 - u)w^*. \]

The corresponding solutions for the endomorphism \( e^{it\text{ad}_H} \) become
\[ w(t) = e^{-it\frac{1}{2}(q^2 - 2q)(1 - u)} w(0), \quad w^*(t) = e^{it\frac{1}{2}(q^2 - 2q)(1 - u)} w^*(0). \]

Passing to the classical limit we find the corresponding vector field on \( \mathbb{R}^3 \) tangent to \( S^2 \)
\[ \tilde{\Gamma} = (1 - u) \left( 2 \frac{\partial}{\partial v} - \nu \frac{\partial}{\partial z} \right), \]
which is the reduced dynamics
\[ \frac{du}{dt} = 0 \]
\[ \frac{dv}{dt} = 2(q_1^2 + p_1^2)(q_2 p_1 - p_2 q_1) = (1 - u)z, \]
\[ \frac{dz}{dt} = -2(q_1^2 + p_1^2)(p_1 p_2 + q_1 q_2) = -(1 - u)v. \quad (16) \]

By using the stereographic projection \( S^2 \to \mathbb{R}^2 \) given by \( (x, y) = \frac{1}{1-u}(v, z) \) we find the associated vector field on \( \mathbb{R}^2 \)
\[ \Gamma(x, y) = \frac{2}{x^2 + y^2 + 1} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). \]

This example is very instructive because provides us with an example of reduced quantum dynamics that goes onto the corresponding reduced classical dynamics, i.e. reduction “commutes” with “dequantization”. Further details can be found in \([31]\).
3.3.2. Example: deformed oscillators. Another instance of a non-commutative algebra reduction is provided by the case of the deformed harmonic oscillator. Let us start thus by analyzing the case of deformed harmonic oscillators described in the Heisenberg picture. By including the deformation parameter in the picture we can deal with several situations at the same time, as we are going to see.

We consider a complex vector space \( V \) generated by \( a, a^+ \). Out of \( V \) we construct the associative tensorial algebra \( A = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots \). A dynamics on \( V \), say

\[
\frac{d}{dt} a = -i\omega a, \quad \frac{d}{dt} a^+ = i\omega a^+
\]

defines a dynamics on \( A \) by extending it by using the Leibniz rule with respect to the tensor product.

A bilateral ideal \( I_{r,q} \) of \( A \), generated by the relation \( a^+ a - qaa^+ + r = 0 \), i.e. the most general element of \( I_{r,q} \) has the form \( A(a^+ a - qaa^+ + r)B \), with \( A, B \in A \), is also invariant under the previously defined equations of motion. It follows then that the dynamics defines a derivation, a “reduced dynamics” on the quotient algebra \( A_{r,q} = A/I_{r,q} \). When \( q = 1 \) and \( r = 0 \) the dynamics becomes a dynamics on a commuting algebra and therefore can be considered to be a classical dynamics. When \( q = 1 \) and \( r = \hbar \) we get back the standard quantum dynamics of the harmonic oscillator. If we consider \( r \) to be a function of the “number operator” defined as \( n = a^+ a \) we obtain many of the proposed deformations of the harmonic oscillator existing in the literature. In particular, these deformations have been applied to the description of the magnetic dipole [43]. It is clear now that this reduction procedure may be carried over to any realization or representation of the abstract algebra and the corresponding ideal \( I_{r,q} \). In this example it is important that the starting dynamics is linear. The extension to the universal tensorial algebra gives a kind of abstract universal harmonic oscillator. The bilateral ideal we choose to quotient the tensor algebra is responsible for the physical identification of variables and may arise from a specific realization of the tensor algebra in terms of functions or operators.

3.4. Ehrenfest formalism.

3.4.1. The formalism. This picture of Quantum Mechanics is not widely known but it arises in connection with the so called Ehrenfest theorem which may be seen from the point of view of \( \star \)-products on phase space (see [26]). Some aspects of this picture have been considered by Weinberg [54] and more generally appear in the geometrical formulation of Quantum Mechanics [16, 17, 18, 19].

We saw above how Schrödinger picture assumes as a starting point the Hilbert space of states and derive the observable as real operators acting on this space of states. The Heisenberg picture starts from the observables, enlarged by means of complexification into a \( \mathbb{C}^* \)-algebra and derives the states as positive normalized linear functionals on the algebra of observables. In the Ehrenfest picture both spaces are considered jointly to define quadratic functions as

\[
f_A(\psi) = \frac{1}{2}\langle \psi, A\psi \rangle.
\]  

(17)

In this way all operators are transformed into quadratic functions which are real valued when the operators are Hermitian. The main advantage of this picture relies
on the fact that we can define a Poisson bracket on the space of quadratic functions by setting
\[
\{ f_A, f_B \} := i f_{[A,B]},
\]
where \([A, B]\) stands for the commutator on the space of operators. By introducing an orthonormal basis in \(H\), say \(\{\psi_k\}\), we may write the function \(f_A\) as
\[
f_A(\psi) = \frac{1}{2} \sum_{jk} c_j c^*_k \langle \psi_j, A \psi_k \rangle, \quad \psi = \sum_k c_k \psi_k
\]
and the Poisson bracket then becomes
\[
\{ f_A, f_B \} = i \sum_k \left( \frac{\partial f_A}{\partial c_k} \frac{\partial f_B}{\partial c^*_k} - \frac{\partial f_A}{\partial c^*_k} \frac{\partial f_B}{\partial c_k} \right).
\]

This bracket can be used to write the equations of motion in the form
\[
if_A \frac{df_A}{dt} = \{ f_H, f_A \},
\]
where \(f_H\) is the function associated to the Hamiltonian operator.

While this way of writing the dynamics is very satisfactory because allows us to write the equations of motion in a “classical way”, one has lost the associative product of operators. Indeed, the point-wise product (somehow a natural one for the functions defined on a real differential manifold) of two quadratic functions will not be quadratic but a quartic function. To recover the associative product we can, however, get inspiration from the definition of the Poisson bracket (18) and introduce
\[
(f_A \ast f_B)(\psi) := f_{AB}(\psi) = \frac{1}{2} \langle \psi, AB \psi \rangle.
\]

By inserting a resolution of the identity \(\sum_j |\psi_j\rangle\langle \psi_j| = I\) (since there is a numerable basis for \(H\)) in between the two operators in \(AB\), say
\[
\langle \psi, A \sum_j |\psi_j\rangle\langle \psi_j| B \psi \rangle,
\]
and writing the expression of \(\psi\) in terms of the basis elements \(\psi = \sum_k c_k \psi_k\) we find a product
\[
(f_A \ast f_B)(\psi) = \sum_{jkl} c_j c^*_l \langle \psi_j, A \psi_k \rangle \langle \psi_k, B \psi_l \rangle,
\]
which reproduces the associative product of operators but now it is not point-wise anymore.

As a matter of fact the Poisson bracket defines derivations for this product, i.e.
\[
\{ f_A, f_B \ast f_C \} = \{ f_A, f_B \} \ast f_C + f_B \ast \{ f_A, f_C \} \quad \forall f_A, f_B, f_C.
\]

Therefore it is an instance of what Dirac calls a quantum Poisson bracket [22]. In the literature it is known as a Lie-Jordan bracket [25, 40].

Using both products, the Ehrenfest picture becomes equivalent to Schrödinger and Heisenberg ones.

Let us consider now how the expressions of the products are written in terms of a different basis, namely the basis of eigenstates of the position operator \(Q\) or the momentum operator \(P\). We have thus two basis \(\{|q\}\) and \(\{|p\}\) satisfying \(Q|q\rangle = q|q\rangle\) and \(P|p\rangle = p|p\rangle\) and
\[
\int_{-\infty}^{\infty} |q\rangle dq |q\rangle = I = \int_{-\infty}^{\infty} |p\rangle dp |p\rangle.
\]
Now the matrix elements $A_{kj} = \langle \psi_j, A \psi_k \rangle$ of the operators in the definition of the $\star$ product above become

$$A(q', q) = \langle q', Aq \rangle$$

or

$$A(p', p) = \langle p', Ap \rangle,$$

and the sum is replaced by an integral:

$$(f_A \star f_B)(\psi) = \int dq dq' dq'' c(q'') c^*(q') A(q'', q) B(q, q').$$

Thus this is a product of functions defined on $\mathbb{R}^n \times \mathbb{R}^n$ or $(\mathbb{R}^n)^* \times (\mathbb{R}^n)^*$, i.e. two copies of the configuration space or two copies of the momentum space. Following an idea of Dirac [23] one may get functions on $\mathbb{R}^n \times (\mathbb{R}^n)^*$ by using eigenstates of the position operator on the left and eigenstates of the momentum operator on the right:

$$A_l(q, p) = \langle q, Ap \rangle e^{-\frac{i \hbar}{\sqrt{2}} qp},$$

or also interchanging the roles of position and momentum:

$$A_l(p, q) = \langle p, Aq \rangle e^{\frac{i \hbar}{\sqrt{2}} qp}.$$

Without elaborating much on these aspects (we refer to [14] for details) we simply state that the $\star$–product we have defined, when considered on phase space, becomes the standard Moyal product.

It is now clear that we may consider the reduction procedure in terms of non-commutative algebras when we consider the $\star$–product. We shall give a simple example where from a $\star$–product on $\mathbb{R}^4$ we get by means of a reduction procedure a $\star$–product on the dual of the Lie algebra of $SU(2)$. Further details connected with their use in non-commutative geometry can be found in [34].

### 3.4.2. Example: Star products on $su(2)$.

We are going to show how it is possible to define star products on spaces such as $su(2)$ by using the reduction of the Moyal star product defined on a larger space ($\mathbb{R}^4$ in this case).

Let us then consider the coordinates $\{q_1, q_2, p_1, p_2\}$ for $\mathbb{R}^4$, $\{x, y, w\}$ for $su(2)$ and the mapping $\pi : \mathbb{R}^4 \to \mathbb{R}^3 \sim su(2)$ defined as:

$$f_1(q_1, q_2, p_1, p_2) = \pi^*(x) = \frac{1}{2}(q_1 q_2 + p_1 p_2)$$

$$f_2(q_1, q_2, p_1, p_2) = \pi^*(y) = \frac{1}{2}(q_1 p_2 - q_2 p_1)$$

$$f_3(q_1, q_2, p_1, p_2) = \pi^* w = \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2).$$

It is useful to consider also the pull-back of the Casimir function of $su(2)$, $C = \frac{1}{2}(x^2 + y^2 + w^2)$, which becomes

$$\pi^* C = \frac{1}{32} (p_1^2 + q_1^2 + p_2^2 + q_2^2)^2.$$
identifies a star product on functions defined on the quotient. We consider thus a vector field $H$ on $\mathbb{R}^4$ satisfying

$$L_H \pi^* x = 0 = L_H \pi^* y = L_H \pi^* w.$$ 

This condition characterizes the point-wise subalgebra of functions of $\mathbb{R}^4$ which are projectable on functions of $\mathbb{R}^3$. Such a vector field can be taken to be the Hamiltonian vector field associated to the Casimir function $\pi^* C$. It is simple to see that the Poisson subalgebra generated by the functions $\{ \pi^* x, \pi^* y, \pi^* w, f_H \}$ where $f_H = q_1^2 + q_2^2 + p_1^2 + p_2^2$ is the Poisson commutant of the function $f_H$ (see [34]). And this set is an involutive Moyal subalgebra when we consider the Moyal product on them, i.e. for any functions $F, G$

$$\{ f_H, F \} = 0 = \{ f_H, G \} \Rightarrow \{ f_H, F \star G \} = 0.$$

The star product on $\mathfrak{su}(2)$ is then defined as:

$$\pi^* (F \star_{\mathfrak{su}(2)} G) = \pi^* F \star \pi^* G.$$

As an example we can consider the product:

$$x_j \star_{\mathfrak{su}(2)} f(x_i) = \left( x_j - \frac{i\theta}{8} \epsilon_{jklm} x_k \frac{\partial}{\partial x_m} - \frac{\theta^2}{32} \left( \left( 1 + x_k \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_k} - \frac{1}{2} x_j \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} \right) \right) f(x_i).$$

The same procedure may be applied to obtain a reduced star product for all three dimensional Lie algebras (see [34]) and to deal with a non-commutative differential calculus [48].

4. The complex projective space as a reduction of the Hilbert space

4.1. Geometric Quantum Mechanics. It is possible to show that the various pictures we have presented so far can be given an unified treatment. To this aim it is convenient to consider a realification of the Hilbert space and to deal with our different pictures from a geometric perspective.

Let us start by considering again the complex Hilbert space $\mathcal{H}$ which contains the set of states of our quantum system. Originally it is considered to be a complex vector space, but we can also look at it as a real vector space by considering the real and imaginary parts of the vectors $\mathcal{H} \ni |\psi\rangle = (\psi_R, \psi_I)$. The Hermitian structure of $\mathcal{H}$ is then encoded in two real tensors, one symmetric and one skew-symmetric; which together with the complex structure provide us with a Kähler structure. Let us first discuss this point in some detail.

We consider therefore $\mathcal{H}_R$ with the structure of a Kähler manifold $(\mathcal{H}_R, J, g, \omega)$, i.e. a complex structure $J : T\mathcal{H}_R \to T\mathcal{H}_R$, a Riemannian metric $g$ and a symplectic form $\omega$. First of all, we are going to make use of the linear structure of the Hilbert space (encoded in the dilation vector field $\Delta$) to identify the tangent vectors at any point of $\mathcal{H}_R$. In this way we can consider the Hermitian structure on $\mathcal{H}_R$ as an Hermitian tensor on $T\mathcal{H}_R$. With every vector we can associate a vector field

$$X_\psi : \phi \to (\phi, \psi).$$

Therefore, the Hermitian tensor, denoted in the same way as the scalar product is

$$\langle X_{\psi_1}, X_{\psi_2} \rangle = \langle \psi_1, \psi_2 \rangle.$$
Fixing an orthonormal basis $\{e_k\}$ of the Hilbert space allows us to identify this product with the canonical Hermitian product of $\mathbb{C}^n$:

$$\langle \psi_1, \psi_2 \rangle = \sum_k \langle \psi_1, e_k \rangle \langle e_k, \psi_2 \rangle.$$ 

The Hilbert space becomes then identified with $\mathbb{C}^n$. As a result, the group of unitary transformations on $\mathcal{H}$ becomes identified as the group $U(n, \mathbb{C})$, its Lie algebra $\mathfrak{u}(n, \mathbb{C})$ and so on.

The choice of the basis also allows us to introduce coordinates for the realified structure:

$$\langle e_k, \psi \rangle = (q_k + ip_k) (\psi),$$ 

and write the geometrical objects introduced above as:

$$J = \frac{\partial}{\partial p_k} \otimes dq_k - \frac{\partial}{\partial q_k} \otimes dp_k, \quad g = dq_k \otimes dq_k + dp_k \otimes dp_k \quad \omega = dq_k \wedge dp_k.$$ 

If we combine them in complex coordinates $z_k = q_k + ip_k$ we can write the Hermitian structure in a simple way

$$h = dz_k \otimes d\bar{z}_k.$$ 

The space of observables (i.e. of Hermitian operators acting on $\mathcal{H}$) is identified with the dual $\mathfrak{u}^*(\mathcal{H})$ of the real Lie algebra $\mathfrak{u}(\mathcal{H})$, by means of the scalar product existing on the Lie algebra and using the fact that the multiplication of an Hermitian matrix by the imaginary unit gives an element in the Lie algebra. Then we get

$$A(T) = \frac{i}{2} \text{Tr} AT \quad A \in \mathfrak{u}^*, \quad T \in \mathfrak{u}.$$ 

The product given by the trace allows us to establish an isomorphism between $\mathfrak{u}(\mathcal{H})$ and $\mathfrak{u}^*(\mathcal{H})$ and identifies the adjoint and the coadjoint action of the Lie group $U(\mathcal{H})$.

Under the previous isomorphism, $\mathfrak{u}^*(\mathcal{H})$ becomes a Lie algebra with Lie bracket defined by

$$[A, B]_\text{c} = i[A, B] = i(AB - BA).$$ 

Moreover, we can also define a scalar product on $\mathfrak{u}^*$, given by:

$$\langle A, B \rangle = \frac{1}{2} \text{Tr} AB,$$

which turns the vector space into a real Hilbert space.

The identification of vectors and covectors allows to write the isomorphism from $\mathfrak{u}^*$ to $\mathfrak{u}$. The metric becomes then

$$\langle \hat{A}, \hat{B} \rangle_\mathfrak{u} = \frac{1}{2} \text{Tr} AB.$$ 

We can also associate complex valued functions to linear operators $A \in \mathfrak{gl}(\mathcal{H})$ as we have seen in the Ehrenfest picture,

$$\mathfrak{gl}(\mathcal{H}) \ni A \mapsto f_A = \frac{1}{2} \langle \psi, A\psi \rangle_{\mathcal{H}}.$$ 

We can use this mapping in the dual $\mathfrak{u}(\mathcal{H})$. The way to do it is to consider the complexification of $\mathfrak{u}(\mathcal{H})$ and then consider general linear transformations (i.e. elements of $\mathfrak{gl}(\mathcal{H})$) and associate with them the complex valued functions we saw above.
Now, by using the contravariant form $G + i\Omega$ of the Hermitian tensor given by:

$$G + i\Omega = \frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial q_k} + \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial p_k} + i \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial p_k},$$

it is possible to define a bracket (see for instance [10])

$$\{f, h\}_H = \{f, h\}_g + i\{f, h\}_\omega$$

In particular, for quadratic functions we have

$$\{f_A, f_B\} = f_{AB} + f_{BA} = 2f_{AB}$$

Thus in this way we can define a tensorial version of the symmetric product on the space of Hermitian matrices which defines a Jordan algebra, along with the Lie product given by the commutator.

For Hermitian operators we find:

$$\text{grad} f_A = \tilde{A} \quad \text{Ham} f_A = \tilde{iA},$$

where the vector fields associated with operators are defined by:

$$\tilde{A} : \mathcal{H}_R \to T\mathcal{H}_R \quad \psi \mapsto (\psi, A\psi),$$

$$\tilde{iA} : \mathcal{H}_R \to T\mathcal{H}_R \quad \psi \mapsto (\psi, JA\psi).$$

The action of $U(\mathcal{H})$ on $\mathcal{H}$ defines a momentum map

$$\mu : \mathcal{H} \to u^*(\mathcal{H}).$$

The fundamental vector fields associated with the operator $A$ is given by $\tilde{iA}$ and the momentum map is such that

$$\mu(\psi)(\tilde{iA}) = \frac{1}{2}\langle \psi, A\psi \rangle_{\mathcal{H}}$$

Thus we can write the momentum map from $\mathcal{H}_R$ to $u^*(\mathcal{H})$ as

$$\mu(\psi) = |\psi\rangle\langle \psi|$$

For Hermitian operators, linear functions on $u^*(\mathcal{H})$ (i.e. elements of the unitary algebra) are pulled-back to quadratic functions. Therefore $\mu$ provides a symplectic realization of the Poisson manifold $u^*(\mathcal{H})$ and the Ehrenfest picture is nothing but the “pullback” of the Heisenberg picture to the symplectic manifold $\mathcal{H}_R$. If we denote by $\hat{A}$ the linear function on $u^*(\mathcal{H})$ associated with the element $-iA \in u(\mathcal{H})$, we see immediately that the momentum map relates the contravariant tensors $G$ and $\Omega$ (defined on $\mathcal{H}_R$) with the linear contravariant tensors $R$ and $\Lambda$ on $u^*(\mathcal{H})$ corresponding to its Lie-Jordan brackets.

We have then the obvious definitions:

$$R(\xi)(\hat{A}, \hat{B}) = \langle \xi, [A, B]_+ \rangle_{u^*}, \quad \Lambda(\xi)(\hat{A}, \hat{B}) = \langle \xi, [A, B]_- \rangle_{u^*},$$

and together they form the complex tensor

$$(R + i\Lambda)(\xi)(\hat{A}, \hat{B}) = 2\langle \xi, AB \rangle_{u^*} = \text{Tr} \xi AB.$$ 

Clearly,

$$G(\mu^* \hat{A}, \mu^* \hat{B}) + i\Omega(\mu^* \hat{A}, \mu^* \hat{B}) = \mu^*(R(\hat{A}, \hat{B}) + i\Lambda(\hat{A}, \hat{B})).$$
Thus the momentum map provides also an unified view of the Schrödinger, Ehrenfest and Heisenberg pictures. Clearly the Schrödinger vector field being associated with the Hamiltonian function $\mu^{*}(\hat{A})$ is just the usual equation written as

$$i\frac{d}{dt}\psi = A\psi.$$ 

The geometrical formulation of Quantum Mechanics we have presented shows that the reduction procedure in the quantum setting may use most of the procedures available from the classical setting. Of course now care must be used to deal with the reduction of the nonlocal product. Again we may find that a reduced $\ast$–algebra need not be associated with a product defined on functions defined on some “quotient” manifold. Thus whether or not the reduction procedure commutes with the quantum-classical transition has to be considered an open problem.

### 4.2. Pure states: the complex projective space.

The consideration that the probabilistic interpretation of Quantum Mechanics requires state vectors to be normalized to one, i.e. $\langle \psi, \psi \rangle = 1$, and that the probability density $\psi^\ast(x,t)\psi(x,t)$ is invariant under multiplication by a phase, i.e. replacing $\psi$ with $e^{i\phi}\psi$ does not alter the probabilistic interpretation, imply that the carrier space of “physical states” is really the complex projective space $\mathbb{P}\mathcal{H}$ or the ray space $\mathcal{R}\mathcal{H}$. If one considers the natural projection from $\mathcal{H} - \{0\}$ to $\mathcal{R}\mathcal{H}$:

$$\mathcal{H} - \{0\} \ni \psi \mapsto \pi(\psi) = \rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi,\psi\rangle},$$

one discovers that $\mathcal{H} - \{0\}$ can be considered as a principal bundle over $\mathcal{R}\mathcal{H}$ with group structure $\mathbb{C}_0 = S^1 \times \mathbb{R}_+$. The infinitesimal generators of this group action, i.e. the corresponding fundamental vector fields are $\Delta$ and $J(\Delta)$ (remember that $\Delta$ was the dilation vector field and $J$ the tensor representing the complex structure of $\mathcal{H}$).

From the action of $U(\mathcal{H})$ on $\mathcal{H}$, that we can write as

$$\psi \mapsto g\psi \quad g \in U(\mathcal{H}),$$

we can introduce a “projected” action on $\mathcal{R}\mathcal{H}$ given by

$$\rho \mapsto g\rho g^{-1}.$$ 

This action is transitive. In the particular case of the unitary evolution operator (i.e. the one-parameter group of unitary transformations associated with the Schrödinger equation (9), where we assume for simplicity that the Hamiltonian does not depend on time), the evolution written in terms of the elements of $\mathcal{R}\mathcal{H}$ is written as

$$\rho(t) = \exp\left(-\frac{iHt}{\hbar}\right)\rho(0)\exp\left(\frac{iHt}{\hbar}\right).$$

This expression provides a solution of the von Neumann equation

$$i\hbar\frac{dp}{dt} = [H, p]. \quad (21)$$

Thus this equation becomes another instance of the quantum equations of motion, in this case defined on $\mathcal{R}\mathcal{H}$. 
According to our previous treatment of the momentum map of the unitary group, the ray space \( \mathcal{R}\mathcal{H} \) can be identified with a symplectic leaf of \( u^*(\mathcal{H}) \) passing through a rank-one projector

\[
\frac{|\psi\rangle\langle\psi|}{\langle\psi,\psi\rangle} \in u^*(\mathcal{H}).
\]

Here we are interested in considering it as the complex projective space obtained as a reduction of \( \mathcal{H} - \{0\} \).

Now we would like to transfer the geometric objects we introduced on \( u^*(\mathcal{H}) \) onto the ray space \( \mathcal{R}\mathcal{H} \). In particular the structures we defined on the set of quadratic functions. As we are interested now in functions which are projectable with respect to \( \Delta \) and \( J(\Delta) \), we are going to consider the functions associated with the expectation values of the observables, i.e.

\[
e_A(\psi) = \frac{\langle\psi,A\psi\rangle}{\langle\psi,\psi\rangle}.
\]

**Lemma 1.** These functions are invariant with respect the actions of \( \Delta \) and \( J(\Delta) \).

**Proof.** The invariance with respect to dilations is obvious. To prove the invariance under \( J(\Delta) \), it is useful to notice that this vector field is the Hamiltonian vector field (with respect the canonical symplectic structure) associated with the function

\[
f_I(\psi) = \frac{1}{2}\langle\psi,I\psi\rangle.
\]

Hence it commutes with any quadratic function associated to an operator, because any operator commutes with the identity. \( \square \)

This observation also shows that this example may be considered to be the Poisson reduction associated to the ideal generated by the functions \( \langle\psi,\psi\rangle - 1 \). The associated first class functions in the family of quadratic ones are exactly given by \( f_A(\psi) = \langle\psi,A\psi\rangle \), with \( A \) a generic operator. In this way the complex projective space provides an instance of Poisson reduction.

It is important to remark that in this construction we are not passing through the submanifold \( \mathcal{H} - \{0\} \supset \Sigma = \{\psi \in \mathcal{H} - \{0\}|\langle\psi,\psi\rangle = 1\} \), as it is usually done in the definition of the geometric description of the projective space. The reason is that if we want to have the freedom to consider alternative Hermitian structures on \( \mathcal{H} \) (this would be the analog of the bi-Hamiltonian structures for classical mechanical systems) we can not privilege a given one with respect to others. If we change the Hermitian structure, the submanifold \( \Sigma \) would be different while the corresponding projective space, as a manifold, would not change.

It is now simple to understand why the tensors \( G \) and \( \Omega \), associated with the Hermitian structure, will not be projectable objects. In spite of this, we can turn them into projectable objects by introducing a conformal factor:

\[
\widetilde{G} = \langle\psi,\psi\rangle G, \quad \widetilde{\Omega} = \langle\psi,\psi\rangle \Omega.
\]

But with this change, \( \widetilde{\Omega} \) is no longer representing a Poisson structure, but a Jacobi one, whose defining vector field is the Hamiltonian vector field associated (with respect to the symplectic structure) with the function \( \frac{1}{2}\langle\psi,\psi\rangle \), via \( G \). The reduction of this Jacobi algebra gives rise to the expected Poisson structure on the ray space \( \mathcal{R}\mathcal{H} \).

It is interesting to look at the particular form of these tensors when we introduce adapted coordinates:
In complex coordinates the expression is
\[ \sum_k (z_k^* z_k) \sum_l \frac{\partial}{\partial z_l} \otimes \frac{\partial}{\partial \bar{z}_l}. \]

If we use real coordinates, the principal bundle we have mentioned on the space of pure states will admit a connection one-form (which is Hermitian) given by
\[ \theta = \frac{\langle \psi, d\psi \rangle}{\langle \psi, \psi \rangle}, \]
which satisfies \( \theta(J(\Delta)) = i \) and \( \theta(\Delta) = 1 \); and takes the form
\[ \frac{(q - ip)d(q + ip)}{q^2 + p^2} = \frac{q dq + pdp}{q^2 + p^2} + i \frac{q dp - pdq}{q^2 + p^2} \]
with
\[ \Delta = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p}, \quad J(\Delta) = q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}. \]

Once we have found symmetric and skew-symmetric tensors on the ray space we can invert them and find the associated covariant form. When pulled-back to \( \mathcal{H} - \{0\} \) these tensors may be represented by
\[ \frac{\langle d\psi, d\bar{\psi} \rangle - \langle \psi, d\bar{\psi} \rangle \langle d\psi, \psi \rangle}{\langle \psi, \psi \rangle^2}. \quad (22) \]

This presentation shows very clearly that by changing the Hermitian structure we also change the connection one form \( \theta \), while \( \Delta \) and \( J \) remain unchanged if we do not change the complex structure. The curvature two form of this connection represents a symplectic structure on the ray space \( RH \) and is usually considered as a starting point to deal with geometric phases.

We consider the differential \( d_J \) associated with the \((1, 1)\)-tensor \( J \), defined (see [49, 53]) on functions and one–forms as
\[ (d_J f)(X) = df(JX) \quad d_J \beta(X, Y) = L_{JX} \beta(Y) - L_{JY} \beta(X) - \beta([J[X, Y]]); \]
for \( \beta \) a one-form and \( X \) and \( Y \) vector fields and extended naturally to higher order forms. With this we find that the Kählerian two form \( \omega \) may be written as
\[ \omega = dd_J \log(\langle \psi, \psi \rangle), \]
while the translational invariant two form on \( \mathcal{H} \) would be \( dd_J(\langle \psi, \psi \rangle). \)

In this expression we see that \( \log(\langle \psi, \psi \rangle) \) represents the Kähler potential on \( \mathcal{H}_R \) and depends on the chosen Hermitian structure. It is not projectable on the ray-space, while the two form we associate with it will be the pull-back of a two form on \( RH \).

Before closing this section we notice that by taking convex combinations of our pure states, rank-one projectors, we can generate the whole set of density states. If, on the other hand, we consider real combinations, we generate the full \( u^*(\mathcal{H}) \) space. Therefore it is possible to derive Heisenberg picture from the von Neumann description.

From our description in terms of geometrical Quantum Mechanics it should be clear that the equivalence of the various pictures is naturally presented in our generalized reduction procedure.

Another comment is in order. The reduction procedures within Quantum Mechanics are most effective when they are formulated in a way such that the classical
limit may be naturally considered in the chosen formalism. We believe that this may be considered as an indication that Quantum Mechanics should be formulated in a way that in some form it incorporates the so called “correspondence principle”.

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