Girsanov theorem for \( G \)-Brownian motion: the degenerate case

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Abstract. In this paper, we prove the Girsanov theorem for \( G \)-Brownian motion without the non-degenerate condition. The proof is based on the perturbation method in the nonlinear setting by constructing a product space of the \( G \)-expectation space and a linear space that contains a standard Brownian motion. The estimates for exponential martingales of \( G \)-Brownian motion are important for our arguments.

Key words: \( G \)-expectation, \( G \)-Brownian motion, Girsanov theorem

AMS 2010 subject classifications: 60H10, 60H30

1 Introduction

Motivated by financial problems with model uncertainty, Peng \[6, 7, 8\] systematically introduced the nonlinear \( G \)-expectation theory. Under the \( G \)-expectation framework, the \( G \)-Brownian motion and related Itô’s stochastic calculus were constructed. Moreover, the existence and uniqueness theorem of (forward and backward) stochastic differential equations driven by \( G \)-Brownian motion were obtained in Gao \[2\], Peng \[8\] and Hu, Ji, Peng and Song \[3\].

\( G \)-Brownian motion \( B = (B_t)_{t \geq 0} \) is a continuous process with independent and stationary increments under \( G \)-expectation \( \hat{\mathbb{E}} \). It is characterized by a function \( G(A) = \hat{\mathbb{E}}[\langle AB_1, B_1 \rangle] \), for \( A \in \mathcal{S}(d) \), where \( \langle \cdot, \cdot \rangle \) is the inner product for vectors and \( \mathcal{S}(d) \) is the sets of symmetric \( d \times d \) matrices. We say that the function \( G \) (or \( G \)-Brownian motion \( B \)) is non-degenerate if there exist a constant \( \sigma^2 > 0 \) such that

\[
G(A) - G(A') \geq \frac{1}{2} \sigma^2 \text{tr}[A - A'], \quad \text{for } A \geq A'.
\]

(1.1)

Under this non-degenerate condition, Osuka \[5\] and Xu, Shang and Zhang \[9\] proved the Girsanov theorem for \( G \)-Brownian motion. Their arguments used the so-called PDE method which usually applies Taylor’s expansion or Itô’s formula to the solutions of \( G \)-heat equations that corresponding to \( G \)-Brownian motion. So this method relies heavily on the non-degenerate condition since the later guarantees the regularity of the solutions.

The aim of this paper is to generalize the Girsanov theorem to the case that the non-degenerate condition \[(1.1)\] for \( B \) may not hold. Using the product space theory in the nonlinear expectation setting, we obtain a non-degenerate \( G \)-Brownian motion perturbation by adding a small linear Brownian motion term to \( G \)-Brownian motion \( B \). Then the Girsanov theorem for \( G \)-Brownian motion under the non-degenerate condition applies. To get the results for \( B \), we consider a limit procedure, and the main difficulty is that the dominated convergence theorem does not hold under the nonlinear framework. We overcome this problem by utilizing the exponential martingale property of \( G \)-Brownian motion and proving some useful estimates.

The paper is organized as follows. In Section 2, we recall some basic notions and results of \( G \)-expectation, \( G \)-Brownian motion and Girsanov theorem for \( G \)-Brownian motion in the non-degenerate case. In Section 3, we give the main results on Girsanov theorem for possibly degenerate \( G \)-Brownian motion.

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2 Preliminaries

In this section, we review some basic notions and results of $G$-expectation, $G$-Brownian motion and the corresponding Girsanov theorem. More relevant details can be found in [5, 6, 7, 8, 9].

2.1 $G$-expectation space

Let $\Omega$ be a given nonempty set and $\mathcal{H}$ be a linear space of real-valued functions on $\Omega$ such that if $X_1, \ldots, X_d \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_d) \in \mathcal{H}$ for each $\varphi \in \mathcal{C}_{b, \text{Lip}}(\mathbb{R}^d)$, where $\mathcal{C}_{b, \text{Lip}}(\mathbb{R}^d)$ is the space of bounded, Lipschitz functions on $\mathbb{R}^d$. $\mathcal{H}$ is considered as the space of random variables.

**Definition 2.1** A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for each $X, Y, \in \mathcal{H}$,

(i) Monotonicity: $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ if $X \geq Y$;

(ii) Constant preserving: $\hat{\mathbb{E}}[c] = c$ for $c \in \mathbb{R}$;

(iii) Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

(iv) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

**Definition 2.2** Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. A $d$-dimensional random vector $Y$ is said to be independent from another $m$-dimensional random vector $X$ under $\hat{\mathbb{E}}[\cdot]$ if, for each test function $\varphi \in \mathcal{C}_{b, \text{Lip}}(\mathbb{R}^{m+d})$, we have

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\varphi(x, y)|_{x=X}]$$

A family of $d$-dimensional random vectors $(X_t)_{t \geq 0}$ on the same sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a $d$-dimensional stochastic process.

**Definition 2.3** Two $d$-dimensional processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ defined respectively on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ are called identically distributed, denoted by $(X_t)_{t \geq 0} \overset{d}{=} (Y_t)_{t \geq 0}$, if for each $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$, $(X_{t_1}, \ldots, X_{t_n}) \overset{d}{=} (Y_{t_1}, \ldots, Y_{t_n})$, i.e.,

$$\hat{\mathbb{E}}_1[\varphi(X_{t_1}, \ldots, X_{t_n})] = \hat{\mathbb{E}}_2[\varphi(Y_{t_1}, \ldots, Y_{t_n})] \text{ for each } \varphi \in \mathcal{C}_{b, \text{Lip}}(\mathbb{R}^{n \times d})$$

**Definition 2.4** A $d$-dimensional process $(X_t)_{t \geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is said to have independent increments if, for each $0 \leq t_1 < \cdots < t_n$, $X_{t_n} - X_{t_{n-1}}$ is independent from $(X_{t_1}, \ldots, X_{t_{n-1}})$. A $d$-dimensional process $(X_t)_{t \geq 0}$ is said to have stationary increments if, for each $t, s \geq 0$, $X_{t+s} - X_s \overset{d}{=} X_t$.

**Definition 2.5** A $d$-dimensional process $(B_t)_{t \geq 0}$ on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a $G$-Brownian motion if the following properties are satisfied:

1. $B_0 = 0$;
2. It is a process with stationary and independent increments;
(3) For each $t \geq 0$, $\hat{E}[\varphi(B_t)] = u^2(t, 0)$ for each $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$, where $u^2$ is the viscosity solution of the following $G$-heat equation:

$$\begin{cases}
\partial_t u(t, x) - G(D^2_{xx} u(t, x)) = 0, \\
u(0, x) = \varphi(x).
\end{cases}$$

Here $G(A) := \hat{E}[\langle AB_1, B_1 \rangle]$, for $A \in \mathbb{S}(d)$, where $\langle \cdot, \cdot \rangle$ is the inner product for vectors and $\mathbb{S}(d)$ is the sets of symmetric $d \times d$ matrices.

Remark 2.6 If $G(A) = \frac{1}{2}[A][A]$, for $A \in \mathbb{S}(d)$, then $B$ is a standard Brownian motion.

Now we recall the construction of $G$-Brownian motion on the path space. We denote by $\Omega := C_0([0, \infty); \mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \geq 0}$ started from the origin and equipped with the distance $d(\omega, \eta) := \sum_{n=1}^{\infty} 2^{-n} |(\max_{t \in [0, N]} |\omega^1_t - \eta^1_t|) \wedge 1|.$

Let $B_t(\omega) := \omega_t$ for $\omega \in \Omega$, $t \geq 0$ be the canonical process. We set $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}) : n \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_n \leq T, \varphi \in C_{b,\text{Lip}}(\mathbb{R}^{d \times n})\}$ as well as $L_{ip}(\Omega) := \bigcup_{m=1}^{\infty} L_{ip}(\Omega_m).$ (2.1)

Let $G : \mathbb{S}(d) \rightarrow \mathbb{R}$ be a monotonic and sublinear function. We define the $G$-expectation $\hat{E} : L_{ip}(\Omega) \rightarrow \mathbb{R}$ by two steps.

Step 1. For $X = \varphi(B_{t+s} - B_s)$ with $t, s \geq 0$ and $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$, we define $\hat{E}[X] = u(t, 0),$ where $u$ is the solution of the following $G$-heat equation:

$$\partial_t u - G(D^2_{xx} u) = 0, \quad u(0, x) = \varphi(x).$$

Step 2. For $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ with $0 \leq t_0 < \cdots < t_n$ and $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^{d \times n})$, we define $\hat{E}[X] = \varphi_n,$ where $\varphi_n$ is obtained via the following procedure:

$$\begin{align*}
\varphi_1(x_1, \cdots, x_{n-1}) &= \hat{E}[\varphi(x_1, \cdots, x_{n-1}, B_{t_n} - B_{t_{n-1}})], \\
\varphi_2(x_1, \cdots, x_{n-2}) &= \hat{E}[\varphi_1(x_2, \cdots, x_{n-2}, B_{t_{n-1}} - B_{t_{n-2}})], \\
&\vdots \\
\varphi_n &= \hat{E}[\varphi_{n-1}(B_{t_1} - B_{t_0})].
\end{align*}$$

The corresponding conditional expectation $\hat{E}_t$ of $X$ with $t = t_i$ is defined by $\hat{E}_t[X] = \varphi_{n-i}(B_{t_i} - B_{t_0}, \cdots, B_{t_i} - B_{t_{i-1}}).$

For each $p \geq 1$, we denote by $L^p_G(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under the norm $||X||_p := (\hat{E}[|X|^p])^{1/p}$. The $G$-expectation $\hat{E}[\cdot]$ and conditional $G$-expectation $\hat{E}_t[\cdot]$ can be extended continuously to $L^p_G(\Omega)$ and $(\Omega, L^p_G(\Omega), \hat{E})$ forms a sublinear expectation space. Moreover, it is easy to check that the canonical process $B$ is a $G$-Brownian motion on $(\Omega, L^p_G(\Omega), \hat{E})$ and $G(A) = \hat{E}[\langle AB_1, B_1 \rangle]$ for $A \in \mathbb{S}(d)$.

Indeed, the $G$-expectation can be regarded as an upper expectation on $L^p_G(\Omega)$.
Theorem 2.7 ([1, 4]) There exists a weakly compact set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \quad \text{for all } \xi \in L^1_G(\Omega).$$

For this $\mathcal{P}$, we define the following capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

A set $A \subset \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$ q.s.

We set

$$L(\Omega) := \{X \in \mathcal{B}(\Omega) : \hat{E}[P[X]] \text{ exists for each } P \in \mathcal{P}\}.$$ 

Then the $G$-expectation can be extended to the space $L(\Omega)$ and we still denote it by $\hat{E}$, i.e.,

$$\hat{E}[X] := \sup_{P \in \mathcal{P}} E_P[X], \quad \text{for each } X \in L(\Omega).$$

Definition 2.8 A real function $X$ on $\Omega$ is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon$ such that $X|_O$ is continuous.

Definition 2.9 We say that $X : \Omega \mapsto \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \mapsto \mathbb{R}$ such that $X = Y$, q.s.

Then we have the following characterization of the space $L^p_G(\Omega)$, which can be seen as a counterpart of Lusin’s theorem in the nonlinear expectation theory.

Theorem 2.10 ([1]) For each $p \geq 1$, we have

$$L^p_G(\Omega) = \{X \in \mathcal{B}(\Omega) : \lim_{N \to \infty} \hat{E}[|X|^p I_{|X| \geq N}] = 0 \text{ and } X \text{ has a quasi-continuous version}\}.$$ 

Note that the monotone convergence theorem is different from the classical case due to the nonlinearity.

Proposition 2.11 Suppose $X_n$, $n \geq 1$ and $X$ are $\mathcal{B}(\Omega)$-measurable.

1. Assume $X_n \uparrow X$ q.s. and $E_P[X_n^+] < \infty$ for all $P \in \mathcal{P}$. Then $\hat{E}[X_n] \uparrow \hat{E}[X]$.

2. If $\{X_n\}_{n=1}^{\infty}$ in $L^1_G(\Omega)$ satisfies that $X_n \downarrow X$, q.s., then $\hat{E}[X_n] \downarrow \hat{E}[X]$.

Proposition 2.12 (Jensen’s inequality) Let $X \in L(\Omega)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Assume that $\hat{E}[|X|] < \infty$ and $\varphi(X) \in L(\Omega)$. Then

$$\hat{E}[\varphi(X)] \geq \varphi(\hat{E}[X]).$$

Proof. We can take $P_k \in \mathcal{P}$ such that $E_{P_k}[X] \to \hat{E}[X]$. Note that convex function is continuous, then by the classical Jensen’s inequality,

$$\varphi(\hat{E}[X]) = \lim_{k \to \infty} \varphi(E_{P_k}[X]) \leq \lim_{k \to \infty} E_{P_k}[\varphi(X)] \leq \hat{E}[\varphi(X)].$$
Remark 2.13 Since $\hat{E}$ is the upper-expectation, we cannot expect that Jensen’s inequality $\hat{E}[\varphi(X)] \leq \varphi(\hat{E}[X])$ holds for concave functions. For example, we take $d = 1$, $\varphi = -x$, $X = |B_t|^2$ with $-\hat{E}[-|B_t|^2] < \hat{E}[|B_t|^2]$. Then

$$\hat{E}[-X] > -\hat{E}[X].$$

For each $1 \leq i, j \leq d$, we denote by $\langle B^i, B^j \rangle$ the mutual quadratic variation process. Then for two processes $\eta \in M^1_G(0, T)$ and $\xi \in M^1_G(0, T)$, the $G$-Itô integrals $\int_0^T \eta_t dB^i_t$ and $\int_0^T \xi_t dB^{i,j}_t$ are well defined. Moreover, we have $\int_0^T \eta_t dB^i_t \in L^2_G(\Omega)$ and $\int_0^T \eta_t dB^{i,j}_t \in L^2_G(\Omega)$.

**Definition 2.14** A process $\{M_t\}$ with values in $L^2_G(\Omega)$ is called a $G$-martingale if $M_t \in L^1_G(\Omega)$ and $\hat{E}_t(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both $G$-martingales, we call $\{M_t\}$ a symmetric $G$-martingale.

We say that the function $G$ is non-degenerate if there exists a constant $\sigma^2 > 0$ such that

$$G(A) - G(A') \geq \frac{1}{2} \sigma^2 \text{tr}[A - A'], \quad A \geq A'.$$

**(2.2)**

**Remark 2.15** By the Hahn-Banach theorem, one can check that there exists a bounded, convex and closed subset $\hat{\mathcal{S}}_+(d)$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A], \quad A \in \mathcal{S}(d),$$

**(2.3)**

where $\mathcal{S}_+(d)$ denotes the collection of nonnegative elements in $\mathcal{S}(d)$. Then (2.3) is equivalent to the condition that

$$\gamma \geq \sigma^2 I_{d \times d}, \quad \text{for each } \gamma \in \Gamma.$$

In the one-dimensional case, the non-degenerate condition reduces to the condition that the lower variance of $B$ is strictly positive, i.e., $-\hat{E}[-|B_t|^2] > 0$.

Now we give the Girsanov theorem under the non-degenerate condition. Given $T > 0$ and $h \in M^2_G(0, T; \mathbb{R}^d)$. We define, for $0 \leq t \leq T$,

$$\mathcal{E}(h)_t := \exp \left( \int_0^t \langle h_s, dB_s \rangle - \frac{1}{2} \int_0^t \langle h_s, h^T_s, dB_s \rangle \right),$$

$$\hat{B}_t := B_t - \int_0^t dB_s h_s,$$

where $\langle \cdot, \cdot \rangle$ is the Euclid inner product for vectors and matrices. We set

$$\mathcal{H} := \{ \varphi(\hat{B}_t_1, \hat{B}_t_2, \cdots, \hat{B}_t_n) : n \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_n \leq T, \varphi \in C_b Lip(\mathbb{R}^{n \times d}) \}.$$

We define a sublinear expectation $\hat{E}$ by

$$\hat{E}[\xi] := E[\xi \mathcal{E}(h)_T], \quad \text{for } \xi \in \mathcal{H}.$$

We shall assume the following $G$-Novikov’s condition:

**(H)** There exists some constant $\delta > 0$ such that

$$\hat{E} \left[ \exp \left( \frac{1}{2}(1 + \delta) \int_0^T \langle h_t h^T_t, dB_t \rangle \right) \right] < \infty.$$ 

**(2.4)**
Girsanov theorem for $G$-Brownian motion is stated as follows.

**Theorem 2.16** [6, 7] If $G$ is non-degenerate and $h$ satisfies the G-Novikov’s condition (H), then the process $(\tilde{B}_{t})_{0 \leq t \leq T}$ is a $G$-Brownian motion on the sublinear expectation space $(\Omega, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$.

The G-Novikov’s condition guarantees that $(\mathcal{E}(h_{t}))_{0 \leq t \leq T}$ is a symmetric $G$-martingale. It worth noting that the non-degenerate assumption is not needed here.

**Proposition 2.17** [6, 7] If the G-Novikov’s condition holds, then $(\mathcal{E}(h_{t}))_{0 \leq t \leq T}$ is a symmetric $G$-martingale on $(\Omega, L_{G}^{1}(\Omega), \tilde{\mathbb{E}})$.

### 3 Main results

We first present a convergence theorem for sequences of random variables in the following exponential form.

**Proposition 3.1** Let $h \in M_{d}^{2}(0, T; \mathbb{R})$ such that $\tilde{\mathbb{E}}[\exp(\delta_{0} \int_{0}^{T} \langle h_{t}^{2}, d(B)_{t} \rangle)] < \infty$ for some $\delta_{0} > 0$. For any fixed $\alpha, \beta \in \mathbb{R}$, we denote

$$J_{\varepsilon} := \exp\left(\alpha \varepsilon \int_{0}^{T} \langle h_{t}, dB_{t} \rangle - \frac{\beta \varepsilon^{2}}{2} \int_{0}^{T} \langle h_{t}h_{t}^{T}, d(B)_{t} \rangle \right), \text{ for } \varepsilon > 0.$$  

Then

$$\hat{\mathbb{E}}[J_{\varepsilon}] \to 1, \text{ as } \varepsilon \downarrow 0,$$

and

$$\hat{\mathbb{E}}[-J_{\varepsilon}] \to -1, \text{ as } \varepsilon \downarrow 0. \tag{3.1} \tag{3.2}$$

**Proof.** Part I: Proof of (3.1).

We first show that $\lim_{\varepsilon \to 0} \hat{\mathbb{E}}[J_{\varepsilon}] \leq 1$. Let any $q > 1$ be given and $q'$ be the corresponding Hölder conjugate. Denote $\alpha_{\varepsilon} = \frac{\alpha \varepsilon^{2} q'}{2}$. Then by Hölder’s inequality, we have

$$\hat{\mathbb{E}}[J_{\varepsilon}] = \hat{\mathbb{E}}\left[\exp\left(\alpha_{\varepsilon} \int_{0}^{T} \langle h_{t}, dB_{t} \rangle - \frac{\beta \varepsilon^{2}}{2} \int_{0}^{T} \langle h_{t}h_{t}^{T}, d(B)_{t} \rangle \right)\right] \leq \hat{\mathbb{E}}\left[\left(\exp\left(\alpha_{\varepsilon} \int_{0}^{T} \langle h_{t}, dB_{t} \rangle \right) - \alpha_{\varepsilon} \int_{0}^{T} \langle h_{t}h_{t}^{T}, d(B)_{t} \rangle \right)\right]^{rac{q'}{2}} \hat{\mathbb{E}}\left[\left(\exp\left(\alpha_{\varepsilon} \int_{0}^{T} \langle h_{t}h_{t}^{T}, d(B)_{t} \rangle \right)\right)^{q'}\right].$$

Note that $\hat{\mathbb{E}}[\exp(\delta_{0} \int_{0}^{T} \langle h_{t}^{2}, d(B)_{t} \rangle)] < \infty$ for some $\delta_{0} > 0$ implies $\hat{\mathbb{E}}[\exp(\delta' \int_{0}^{T} \langle h_{t}^{2}, d(B)_{t} \rangle)] < \infty$ for each $\delta' \leq \delta_{0}$. Then applying Proposition 2.17, we get

$$\hat{\mathbb{E}}\left[\left(\exp\left(\alpha_{\varepsilon} \int_{0}^{T} \langle h_{t}, dB_{t} \rangle \right) - \alpha_{\varepsilon} \int_{0}^{T} \langle h_{t}h_{t}^{T}, d(B)_{t} \rangle \right)\right]^{rac{q'}{2}} = \mathbb{E}\left[\exp\left(\alpha_{\varepsilon}q \int_{0}^{T} \langle h_{t}, dB_{t} \rangle - \frac{\alpha_{\varepsilon}^{2} q^{2}}{2} \int_{0}^{T} \langle h_{t}h_{t}^{T}, d(B)_{t} \rangle \right)\right] = 1, \text{ when } \varepsilon > 0 \text{ is small.}$$

Thus,

$$\hat{\mathbb{E}}[J_{\varepsilon}] \leq \hat{\mathbb{E}}\left[\left(\exp\left(\alpha_{\varepsilon} \int_{0}^{T} \langle h_{t}h_{t}^{T}, d(B)_{t} \rangle \right)\right)^{q'}\right]^{rac{1}{q'}} \text{, when } \varepsilon > 0 \text{ is small.}$$

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It remains to show that
\[
\mathbb{E} \left[ \left( \exp \left( \left( \frac{1}{2} \beta \varepsilon^2 - \frac{1}{2} (\alpha \varepsilon^2 + \beta \varepsilon^2) \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right) \right)^q \right] = \mathbb{E} \left[ \left( \exp \left( \frac{1}{2} \beta \varepsilon^2 \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right)^q \right] \rightarrow 1, \text{ as } \varepsilon \downarrow 0.
\]

If \( q\alpha^2 - \beta \leq 0 \), since
\[
\left( \exp \left( \frac{1}{2} \beta \varepsilon^2 \left( \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right) \right)^q \uparrow 1, \text{ as } \varepsilon \downarrow 0,
\]
we have, by Proposition 2.11(1),
\[
\mathbb{E} \left[ \left( \exp \left( \frac{1}{2} \beta \varepsilon^2 \left( \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right) \right)^q \right] \uparrow 1, \text{ as } \varepsilon \downarrow 0.
\]

If \( q\alpha^2 - \beta \geq 0 \), from Theorem 2.10 and the assumption that \( \mathbb{E} \left[ \exp \left( \delta \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right] < \infty \) for some \( \delta > 0 \), it is easy to see that
\[
\left( \exp \left( \frac{1}{2} \beta \varepsilon^2 \left( \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right) \right)^q \in L_1^G(\Omega), \text{ for } \varepsilon > 0 \text{ small.}
\]

Note that
\[
\left( \exp \left( \frac{1}{2} \beta \varepsilon^2 \left( \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right) \right)^q \downarrow 1, \text{ as } \varepsilon \downarrow 0.
\]

Applying Proposition 2.11(2), we then get
\[
\mathbb{E} \left[ \left( \exp \left( \frac{1}{2} \beta \varepsilon^2 \left( \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right) \right)^q \right] \downarrow 1, \text{ as } \varepsilon \downarrow 0. \tag{3.3}
\]

Now we prove that \( \lim_{\varepsilon \to 0} \mathbb{E} \left[ J_\varepsilon \right] \geq 1 \). From Jensen’s inequality, we get
\[
\mathbb{E} \left[ J_\varepsilon \right] = \mathbb{E} \left[ \exp \left( \alpha \varepsilon \int_0^T \langle h_t, dB_t \rangle - \frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t h_t^T, dB_t \rangle \right) \right] \geq \exp \left( \mathbb{E} \left[ \alpha \varepsilon \int_0^T \langle h_t, dB_t \rangle - \frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t h_t^T, dB_t \rangle \right] \right) = \exp \left( \mathbb{E} \left[ -\frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t h_t^T, dB_t \rangle \right] \right) = \exp \left( \frac{\varepsilon^2}{2} \mathbb{E} \left[ -\beta \int_0^T \langle h_t h_t^T, dB_t \rangle \right] \right) \rightarrow 1, \text{ as } \varepsilon \downarrow 0.
\]

**Part II: Proof of (3.2).**
Applying the classical Jensen’s inequality under each \( P \in \mathcal{P} \), we have

\[
\hat{E}[-J_\varepsilon] = \hat{E}\left[- \exp\left(\alpha \int_0^T \langle h_t, dB_t \rangle - \frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t^T, d(B_t) \rangle \right)\right]
\]

\[
= \sup_{P \in \mathcal{P}} \exp\left(\frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t^T, d(B_t) \rangle \right)
\]

\[
= \sup_{P \in \mathcal{P}} \exp\left(\frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t, dB_t \rangle \right)
\]

\[
= \sup_{P \in \mathcal{P}} \exp\left(- \frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t^T, d(B_t) \rangle \right)
\]

Since \( y \rightarrow \exp(-y) \) is increasing, we further get

\[
\hat{E}[-J_\varepsilon] \leq \sup_{P \in \mathcal{P}} \exp\left(- \frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t^T, d(B_t) \rangle \right)
\]

\[
= - \exp\left(- \frac{\beta \varepsilon^2}{2} \int_0^T \langle h_t^T, d(B_t) \rangle \right)
\]

\[
= - \exp\left(- \frac{\varepsilon^2}{2} \beta \int_0^T \langle h_t^T, d(B_t) \rangle \right)
\]

\[
\rightarrow -1, \text{ as } \varepsilon \downarrow 0.
\]

Moreover, from Part I, we know that

\[
\hat{E}[-J_\varepsilon] \geq -\hat{E}[J_\varepsilon] \rightarrow -1, \text{ as } \varepsilon \downarrow 0.
\]

□

The main result of our paper is the following Girsanov theorem for G-Brownian motion in the degenerate case.

Let \( h \in \mathcal{F}_G^2(0, T; \mathbb{R}^d) \). We define

\[
\tilde{B}_t := B_t - \int_0^t d\langle B \rangle_s h_s, \text{ for } 0 \leq t \leq T,
\]

and

\[
\tilde{H} := \{ \varphi(\tilde{B}_{t_1}, \tilde{B}_{t_2}, \cdots, \tilde{B}_{t_n}) : n \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_n \leq T, \varphi \in \mathcal{C}_b(Lip(\mathbb{R}^{n \times d})) \}.
\]

**Theorem 3.2** Assume that for \( h \in \mathcal{F}_G^2(0, T; \mathbb{R}^d) \), the G-Novikov’s condition (H) holds for some \( \delta > 0 \) and

\[
\hat{E}\left[\exp\left(\int_0^T \delta_0 |h_t|^2 dt\right)\right] < \infty \text{ for some } \delta_0 > 0.
\]

Define a sublinear expectation \( \hat{E} \) by

\[
\hat{E}[\xi] := \hat{E}[\hat{E}(\xi|h)_T], \text{ for } \xi \in \tilde{H}.
\]

Then the process \((\tilde{B}_t)_{t \geq 0}\) is a G-Brownian motion on the sublinear expectation space \((\Omega, \tilde{H}, \hat{E})\).
Remark 3.3 (i) Compared with Theorem 2.10, we have imposed in Theorem 3.2 an additional assumption that \( \mathbb{E} \left[ \exp \left( \int_0^T \delta_0 |B_s|^2 ds \right) \right] < \infty \) for some \( \delta_0 > 0 \). In the non-degenerate case, this assumption is implied by the G-Novikov’s condition by noting that, from Corollary 5.7 in Chapter III of [3] and Remark 2.15, \( \frac{d|B_t|^2}{dt} \geq \epsilon^2 I_{d \times d} \). But in the degenerate case, it is needed for our arguments.

(ii) According to the proofs of Lemma 2.2 in [3] and Proposition 5.10 in [5], the G-Novikov’s implies that \( \mathbb{E}[|E(B)|^p] < \infty \) for some \( p > 1 \). This property will be used in the proof of the main theorem.

To prove Theorem 3.2, it suffices to show for \( t_1 \leq t_2 \leq \cdots \leq t_n \leq T \) and \( \varphi \in C_{b, \text{Lip}}(\mathbb{R}^{n \times d}) \), it holds that
\[
\mathbb{E} \left[ \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}) \right] = \tilde{\mathbb{E}} \left[ \varphi(\tilde{B}_{t_1}, \tilde{B}_{t_2}, \cdots, \tilde{B}_{t_n}) \right].
\] (3.4)

Since \( B \) is possibly degenerate, we use the following product space method in the nonlinear expectation setting to add a small linear Brownian motion term to \( B \), so to get a non-degenerate perturbation \( B^\epsilon \).

Let
\[
\tilde{G}(A') = G(A) + \frac{1}{2} \text{tr}[C], \quad \text{for } A' = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \in S(2d), \quad \text{where } A, B, C \in S(d).
\]
Following the method in Section 2, we can construct an auxiliary \( G \)-expectation space \((\tilde{\Omega}, L^1_G(\tilde{\Omega}), \tilde{\mathbb{E}})\) such that

(i) \( \tilde{\Omega} = \Omega \times C_0([0, \infty); \mathbb{R}^d) \);

(ii) \( \tilde{B}_t := (B_t, W_t)_{t \geq 0} \) is a \( 2d \)-dimensional \( G \)-Brownian motion, where \( W \) is the canonical process on \( C_0([0, \infty); \mathbb{R}^d) \).

Moreover, by the definition of \( \tilde{\mathbb{E}} \), we also have:

Lemma 3.4 Let \((\tilde{\Omega}, L^1_G(\tilde{\Omega}), \tilde{\mathbb{E}})\) be defined as above. Then

(iii) \( \tilde{\mathbb{E}} = \mathbb{E} \) on \( L^1_G(\tilde{\Omega}) \) and \( (B_t)_{t \geq 0} \) is a \( d \)-dimensional \( G \)-Brownian motion under \( \tilde{\mathbb{E}} \);

(iv) \( (W_t)_{t \geq 0} \) is a \( d \)-dimensional standard Brownian motion under \( \tilde{\mathbb{E}} \).

Proof. We only prove that \( \tilde{\mathbb{E}} = \mathbb{E} \) on \( L^1_G(\tilde{\Omega}) \), which implies \( (B_t)_{t \geq 0} \) is a \( G \)-Brownian motion under \( \tilde{\mathbb{E}} \), and the proof for (iv) is similar. By Step 2 in the definition of \( G \)-expectation in Section 2, we only need to show that, for any given \( X = \varphi(B_{t+s} - B_s) \), where \( \varphi \in C_{b, \text{Lip}}(\mathbb{R}^d) \), we have
\[
\tilde{\mathbb{E}}[X] = \mathbb{E}[X].
\] (3.5)
From Step 1 in the definition of \( G \)-expectation, we know that
\[
\tilde{\mathbb{E}}[X] = \mathbb{E}[X] = \hat{u}(t, 0, 0).
\]
Here \( \hat{u}(r, x_1, x_2) \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}^d) \) is the solution of the following \( G \)-heat equation:
\[
\partial_t \hat{u} - G(D^2_{xx} \hat{u}) = 0, \quad \hat{u}(0, x_1, x_2) = \varphi(x_1), \quad \text{where } x = (x_1, x_2).
\] (3.6)
Similarly,
\[
\tilde{\mathbb{E}}[X] = u(t, 0),
\]
where \( u(r, x_1) \in C([0, T] \times \mathbb{R}^d) \) is the solution of the following \( G \)-heat equation:
\[
\partial_t u - G(D^2_{x_1x_1} u) = 0, \quad u(0, x_1) = \varphi(x_1).
\]
It is easy to check that $u(r, x_1)$ is also a solution of (3.6). Then, from the uniqueness theorem of viscosity solutions, we get

$$u(r, x_1) = \bar{u}(r, x_1, x_2), \text{ for } (r, x_1, x_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d,$$

which implies the desired (3.5). □

For each fixed $\varepsilon \in (0, 1)$, we define $B_t^\varepsilon = B_t + \varepsilon W_t$. Following Proposition 1.4 in Chapter III of [8], we deduce that $(B_t^\varepsilon)_{t \geq 0}$ is a $d$-dimensional $G_\varepsilon$-Brownian motion under $\bar{\mathbb{E}}$, where

$$G_\varepsilon(A) = \mathbb{E}[\langle AB_t^\varepsilon, B_t^\varepsilon \rangle] = G \left( \begin{bmatrix} A & \varepsilon A \\ \varepsilon A & \varepsilon^2 A \end{bmatrix} \right) = G(A) + \frac{\varepsilon^2}{2} \text{tr}[A], \text{ for } A \in S(d).$$

We claim that the $G_\varepsilon$ is non-degenerate. Indeed, for $A \geq B$, we have

$$G_\varepsilon(A) - G_\varepsilon(B) = G(A) - G(B) + \frac{\varepsilon^2}{2} \text{tr}[A - B].$$

The following two lemmas concern respectively the quadratic variation and the stochastic exponential of $B^\varepsilon$.

**Lemma 3.5** We have

$$\langle B_t^\varepsilon \rangle_t = \langle B_t \rangle_t + \varepsilon^2 t I_{d \times d}, \quad (3.7)$$

where $I_{d \times d}$ is the $d \times d$ identity matrix.

**Proof.** We can find a set $\Gamma \subset S_+(d)$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A], \text{ for } A \in S(d). \quad (3.8)$$

Then it is easy to check that

$$G(A') = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr} \left[ A' \left[ \begin{array}{cc} \gamma & 0 \\ 0 & I_{d \times d} \end{array} \right] \right], \text{ for } A' \in S(2d).$$

By Corollary 5.7 in Chapter III of [8], we have

$$\langle B_t \rangle_t = \left[ \begin{array}{c} \langle B_t \rangle_t \\ \langle B_t, W_t \rangle_t \\ \langle W_t \rangle_t \end{array} \right] \in \left\{ t \left[ \begin{array}{cc} \gamma & 0 \\ 0 & I_{d \times d} \end{array} \right] : \gamma \in \Gamma \right\}.$$ From this we deduce that $\langle B_t, W_t \rangle_t = 0$, and thus,

$$\langle B_t^\varepsilon \rangle_t = \langle B_t \rangle_t + 2\varepsilon \langle B_t, W_t \rangle_t + \varepsilon^2 \langle W_t \rangle_t = \langle B_t \rangle_t + \varepsilon^2 t I_{d \times d}.$$ This completes the proof. □

**Lemma 3.6** Under the assumptions of Theorem 3.2, the $G$-Novikov’s condition holds for $B^\varepsilon$: For any given $0 < \delta' < \delta$, there exists some $\varepsilon_{\delta'} > 0$ such that for each $0 < \varepsilon \leq \varepsilon_{\delta'}$,

$$\bar{\mathbb{E}} \left[ \exp \left( \frac{1}{2}(1 + \delta') \int_0^T \langle h_t h_t^T, d\langle B_t^\varepsilon \rangle_t \rangle \right) \right] < \infty. \quad (3.9)$$

**Proof.** We first take $p > 1$ so small such that

$$p(1 + \delta') \leq 1 + \delta.$$
Let \( p' \) be the Hölder conjugate of \( p \). Then after taking \( \varepsilon > 0 \) small, we have
\[
\frac{\varepsilon^2 p'}{2} (1 + \delta') \leq \delta_0.
\]
Applying the Hölder’s inequality, we obtain
\[
\hat{E} \left[ \exp \left( \frac{1}{2} (1 + \delta') \int_0^T \langle h_t h_t^T, d\langle B^c \rangle_t \rangle \right) \right] 
= \hat{E} \left[ \exp \left( \frac{1}{2} (1 + \delta') \int_0^T \langle h_t h_t^T, d\langle B \rangle_t \rangle \right) \exp \left( \frac{\varepsilon^2}{2} (1 + \delta') \int_0^T |h_t|^2 dt \right) \right] 
\leq \hat{E} \left[ \exp \left( \frac{p}{2} (1 + \delta') \int_0^T \langle h_t h_t^T, d\langle B \rangle_t \rangle \right) \right]^{\frac{1}{p'}} \hat{E} \left[ \exp \left( \frac{\varepsilon^2 p'}{2} (1 + \delta') \int_0^T |h_t|^2 dt \right) \right]^{\frac{1}{p'}} < \infty,
\]
where in the last inequality we have applied the G-Novikov’s condition for \( B \) and the assumption that
\( \hat{E} \left[ \exp \left( \int_0^T |h_t|^2 dt \right) \right] \) < \( \infty \). \( \square \)

Now we are ready to state the proof of Theorem 3.2.

**Proof.** We define, for \( 0 \leq t \leq T \),
\[
N_t^\varepsilon := \exp \left( \int_0^t \langle h_s, d\langle B^c \rangle_s \rangle - \frac{1}{2} \int_0^t \langle h_s h_s^T, d\langle B \rangle_s \rangle \right) = \mathcal{E}(h_t) \exp \left( \int_0^t \varepsilon \langle h_s, dW_s \rangle - \frac{1}{2} \varepsilon^2 \int_0^t |h_s|^2 ds \right), \tag{3.10}
\]
\[
\tilde{B}_t^\varepsilon := B_t^\varepsilon - \int_0^t d\langle B^c \rangle_s h_s = B_t^\varepsilon - \int_0^t d\langle B \rangle_s h_s - \varepsilon^2 \int_0^t h_s ds, \tag{3.11}
\]
where we have used Lemma 3.6 in the second equalities in (3.10) and (3.11). We also define
\[
\hat{E}^\varepsilon[\xi] := \hat{E}[\xi N_0^\varepsilon], \text{ for } \xi \in \tilde{\mathcal{H}}.
\]

Since \( (B_t^\varepsilon)_{t \geq 0} \) is non-degenerate and from Lemma 3.6 it satisfies the G-Novikov’s condition for small enough \( \varepsilon > 0 \), then we can apply Theorem 2.16 to obtain that, for \( \varphi \in C_{b, Lip}(\mathbb{R}^{n \times d}) \),
\[
\hat{E}[\varphi(B_t^\varepsilon, B_{t_2}^\varepsilon, \ldots, B_{t_n}^\varepsilon)] = \hat{E}^\varepsilon[\varphi(B_0^\varepsilon, B_{t_2}^\varepsilon, \ldots, B_{t_n}^\varepsilon)], \text{ for } \varepsilon > 0 \text{ small}. \tag{3.12}
\]
To completes the proof, we shall show that the left-hand side (right-hand side resp.) of (3.12) converges to the left-hand side (right-hand side resp.) of (3.2) by the following two steps.

*Step 1. The left-hand side.* By the Lipschitz continuity assumption of \( \varphi \), we have
\[
|\hat{E}[\varphi(B_t^\varepsilon, B_{t_2}^\varepsilon, \ldots, B_{t_n}^\varepsilon)] - \hat{E}[\varphi(B_t, B_{t_2}, \ldots, B_{t_n})]| 
\leq L_{\varphi} \hat{E}[|B_t^\varepsilon - B_t| + |B_{t_2}^\varepsilon - B_{t_2}| + \cdots + |B_{t_n}^\varepsilon - B_{t_n}|] \to 0, \text{ as } \varepsilon \to 0,
\]
where \( L_{\varphi} \) is the Lipschitz constant of \( \varphi \).

*Step 2. The right-hand side.* Let \( p > 1 \) be the constant in Remark 3.3 (ii). Then from the definition of
where $C_\varphi$ is the bound of $\varphi$ and $p'$ is the Hölder conjugate of $p$.

Now we show that $I_1, I_2 \to 0$, as $\varepsilon \to 0$. The proof of $I_2 \to 0$ is similar to that of the left-hand side in Step 1, so we omit it, and we only need to consider the $I_1$ term. By Hölder’s inequality, we get

$$
\mathbb{E}[N_T^\varepsilon - \mathcal{E}(h)_T] = \mathbb{E}[ \mathcal{E}(h)_T | \exp \left( \int_0^T \varepsilon \langle h_s, dW_s \rangle - \frac{1}{2} \varepsilon^2 \int_0^T |h_s|^2 ds \right) - 1 ]
$$

$$
\leq \mathbb{E}[\mathcal{E}(h)_T | p' ]^{\frac{1}{p'}} \mathbb{E} \left[ \exp \left( \int_0^T \varepsilon \langle h_s, dW_s \rangle - \frac{1}{2} \varepsilon^2 \int_0^T |h_s|^2 ds \right) - 1 \right]^{\frac{1}{p'}}
$$

(3.13)

Let any $r \geq 0$ be fixed. From the assumption that $\mathbb{E} \left[ \exp \left( \int_0^T \delta_0 |h_s|^2 ds \right) \right] = \mathbb{E} \left[ \exp \left( \int_0^T \delta_0 |h_s|^2 ds \right) \right] < \infty$ for some $\delta_0 > 0$ and Proposition 3.13 we have

$$
\mathbb{E} \left[ \pm \left( \exp \left( \int_0^T \varepsilon \langle h_s, dW_s \rangle - \frac{1}{2} \varepsilon^2 \int_0^T |h_s|^2 ds \right) \right)^r \right]
$$

$$
= \mathbb{E} \left[ \pm \exp \left( r \varepsilon \int_0^T \langle h_s, dW_s \rangle - \frac{1}{2} r \varepsilon^2 \int_0^T |h_s|^2 ds \right) \right] \to \pm 1, \text{ as } \varepsilon \downarrow 0.
$$

Then applying the binomial theorem, we get

$$
\mathbb{E} \left[ \exp \left( \int_0^T \varepsilon \langle h_s, dW_s \rangle - \frac{1}{2} \varepsilon^2 \int_0^T |h_s|^2 ds \right) - 1 \right]^{\frac{1}{p'}}
$$

$$
\leq \mathbb{E} \left[ \left( \exp \left( \int_0^T \varepsilon \langle h_s, dW_s \rangle - \frac{1}{2} \varepsilon^2 \int_0^T |h_s|^2 ds \right) - 1 \right)^{N/k} \right]^{\frac{1}{p'}}
$$

$$
\leq \left\{ \frac{N}{k} \mathbb{E} \left[ \left( \exp \left( \int_0^T \varepsilon \langle h_s, dW_s \rangle - \frac{1}{2} \varepsilon^2 \int_0^T |h_s|^2 ds \right) \right)^{N-k} (-1)^k \right] \right\}^{\frac{1}{p'}}
$$

$$
= \left\{ \left( 1 - 1 \right)^N \right\}^{\frac{1}{p'}}
$$

$$
= 0, \text{ as } \varepsilon \downarrow 0,
$$

where $N$ is an even number not smaller than $p'$. Therefore, combining this with (3.13), we obtain

$$
\mathbb{E}[N_T^\varepsilon - \mathcal{E}(h)_T] \to 0, \text{ as } \varepsilon \downarrow 0,
$$

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which implies \( I_1 \to 0, \) as \( \varepsilon \downarrow 0, \)

as desired. \( \Box \)

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**References**

[1] L. Denis, M. Hu and S. Peng, Function spaces and capacity related to a sublinear expectation: application to \( G \)-Brownian motion paths, Potential Anal. 34 (2011) 139–161.

[2] F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by \( G \)-Brownian motion. Stochastic Process. Appl. 119 (2009), no. 10, 3356-3382.

[3] M. Hu, S. Ji, S. Peng and Y. Song, Backward stochastic differential equations driven by \( G \)-Brownian motion. Stochastic Process. Appl. 124, 759-784, 2014.

[4] M. Hu and S. Peng, On representation theorem of \( G \)-expectations and paths of \( G \)-Brownian motion, Acta Math. Appl. Sin. Engl. Ser. 25 (2009) 539–546.

[5] E. Osuka, Girsanov's formula for \( G \)-Brownian motion. Stochastic Process. Appl. 123 (2013), no. 4, 1301-1318.

[6] S. Peng, \( G \)-expectation, \( G \)-Brownian motion and related stochastic calculus of Itô type, in: Stochastic Analysis and Applications, in: Abel Symp., vol. 2, 2007, pp. 541–567.

[7] S. Peng, Multi-dimensional \( G \)-Brownian motion and related stochastic calculus under \( G \)-expectation, Stochastic Process. Appl. 118 (2008) 2223–2253.

[8] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty, [arXiv:1002.4546], 2010.

[9] J. Xu, H. Shang and B. Zhang, A Girsanov type theorem under \( G \)-framework, Stoch. Anal. Appl. 29 (2011) 386-406.