Lower bound of the Dirichlet function $L(s, \chi_q)$ at $s = 1$

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Abstract. In this paper, we use the convolution method to give a good lower bound of the Dirichlet function at $L(s, \chi_q)$ at $s = 1$.

Keyword. Dirichlet L function, Lower bound, convolution.

MR(2000) Subject Classification 11M20

Lower bound of the Dirichlet function $L(s, \chi_q)$ at $s = 1$ be an important subject in the analytic number theory. This subject has important theoretical value and wide application.

Let $\chi_q$ denote the primitive real character mod $q$, $L(s, \chi_q)$ be its Dirichlet $L$ function, $\beta$ be an exceptional real zero of the function.

In this paper, we give the following theorem and the corollary.

Theorem. There is a positive constant $c_1$, we have

$$L(1, \chi_q) \geq \frac{c_1}{\log^6 q}$$

Corollary. There is a positive constant $c_2$, we have

$$\beta \leq 1 - \frac{c_2}{\log^8 q}$$

We use three parts to prove the theorem and the corollary.

1. Some Lemma

Lemma 1.1. When $x \geq 2$, 

1
\[ \pi(x) = \sum_{2 \leq p \leq x} 1 = \int_{2}^{x} \frac{dt}{\log t} + R_1(x) \]

where \( p \) be the prime number, \( R_1(x) \ll x \exp \left( -c_3 \sqrt{\log x} \right) \) and \( c_3 \) be a positive constant.

see the page 179 of references[1].

**Lemma 1.2** Let \( \chi_q \) be the primitive real character mod \( q \), \( \beta \) be an exceptional real zero. If \( q \leq \exp \left( 2c_4 \sqrt{\log x} \right) \), then

\[ \vartheta(x, \chi_q) = \sum_{2 \leq p \leq x} \chi_q(p) \log p = -\frac{x^{\beta}}{\beta} + R_2(x) \]

where \( p \) be the prime number, \( R_2(x) \ll x \exp \left( -c_4 \sqrt{\log x} \right) \) and \( c_4 \) be a positive constant.

see the page 383 of references[1].

**Lemma 1.3** Let \( \chi_q \) be a non-principal character modulo \( q \). Then for any integers \( M \) and \( N \) with \( N > 0 \),

\[ \sum_{n=M+1}^{M+N} \chi_q(n) \ll q^{\frac{1}{2}} \log q \]

see the page 307 of references[1].

**Lemma 1.4** Let \( \beta \) be an exceptional real zero of the function \( L(s, \chi_q) \), then

1. \( 1 \ll \frac{L(1, \chi_q)}{1 - \beta} \ll \log^2 q \)

see the page 363 of references[1].

2. \( L(1, \chi_q)) \gg q^{-\frac{1}{2}} \)

see the page 370 of references[1].

3. \( 1 - \beta \gg q^{-\frac{1}{2}} \log^{-2} q \)
Lemma 1.5  Let integer $n > 0$, we have

$$\mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{if } n \neq 1 
\end{cases}$$

where $\mu(n)$ be M"obius function.

Let $\chi_q$ be the primitive real character mod q, we write $a(n) = \sum_{d|n} \chi_q(d)$, this is a multiplicative function and $a(n) \geq 0$.

Lemma 1.6  When $x \geq q$, we have

$$\sum_{1 \leq n \leq x} \frac{a(n)}{n} = (\log x + \gamma) L(1, \chi_q) + L'(1, \chi_q) + O \left( x^{-\frac{1}{2}} q^\frac{3}{4} (\log x)^\frac{3}{2} \right)$$

where $\gamma$ be Euler’s constant.

2. Preparatory Work

Lemma 2.1  Let $\chi_q$ be the primitive real character mod q, when $x \geq q^4$, we have

$$\sum_{1 \leq n \leq x} a(n) \leq 2x L(1, \chi_q)$$

where $a(n) = \sum_{d|n} \chi_q(d)$.

Proof.  Let’s start with the definition of $a(n)$,

$$\sum_{1 \leq n \leq x} a(n) = \sum_{1 \leq n \leq x} \sum_{d|n} \chi_q(d) = \sum_{1 \leq dm \leq x} \chi_q(d) = \sum_{1 \leq d \leq x} \chi_q(d) \sum_{1 \leq m \leq x/d} 1$$

$$= \sum_{1 \leq d \leq \sqrt{x}} \chi_q(d) \sum_{1 \leq m \leq x/d} 1 + \sum_{\sqrt{x} < d \leq x} \chi_q(d) \sum_{1 \leq m \leq x/d} 1$$
\[
= \sum_{1 \leq d \leq \sqrt{x}} \chi_q(d) \left( \frac{x}{d} + O(1) \right) + \sum_{1 \leq m \leq \sqrt{x}} \sum_{\sqrt{x} < d \leq x/m} \chi_q(d)
\]

By lemma 1.3, the above formula
\[
= x \sum_{1 \leq d \leq \sqrt{x}} \frac{\chi_q(d)}{d} + O \left( x^{\frac{1}{2}} q^{\frac{1}{2}} \log q \right)
\]
because
\[
\sum_{1 \leq d \leq \sqrt{x}} \frac{\chi_q(d)}{d} = L(1, \chi_q) + O \left( x^{-\frac{1}{2}} q^{\frac{1}{2}} \log q \right)
\]
therefore
\[
\sum_{1 \leq n \leq x} a(n) = xL(1, \chi_q) + O \left( x^{\frac{1}{2}} q^{\frac{1}{2}} \log q \right)
\]
By (2) of lemma 1.4, we completed the proof of lemma.

**Lemma 2.2** Let \( \mu(n) \) be Möbius function, we have
\[-\mu(n) = b_1(n) + b_2(n) + b_3(n)\]
where
\[
b_1(n) = \begin{cases} 
-2\mu(n) & \text{if } n < y \\
0 & \text{if } n \geq y 
\end{cases}
\]
\[
b_2(n) = \sum_{mdr = n} \mu(m)\mu(d)
\]
\[
b_3(n) = \sum_{mk = n} \left( \sum_{d|k} \mu(d) \right)
\]
Proof. When $\text{Re}s > 1$, Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\zeta^{-1}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$, we write $M(s) = \sum_{1 \leq n < y} \frac{\mu(n)}{n^s}$. Then, we have the following identity

$$-\zeta^{-1}(s) = -2M(s) + M^2(s)\zeta(s) + (\zeta^{-1}(s) - M(s)) (\zeta(s)M(s) - 1)$$

Compare the coefficients on both sides of this identity, we proved the lemma. For this lemma, the reader can refer to the page 114 of references [3].

From now on, let $y = \exp \left( c_5 \log^2 q \right)$, $c_5 = \max \left( 6(c_3)^{-2}, 6(c_4)^{-2}, 6 \right)$ and $x = y^7$.

Lemma 2.3 Let $\chi_q$ be the primitive real character mod $q$, then

$$\sum_{2y < p \leq x} (1 + \chi_q(p)) = (1 - \beta)x + O \left( (1 - \beta)L(1, \chi_q)x \log^2 q + xq^{-2} \right)$$

where $p$ be the prime number, $\beta$ be an exceptional real zero.

Proof. by lemma 1.2,

$$\sum_{2y < p \leq x} \chi_q(p) = \int_{2y}^{x} \frac{1}{\log t} d \left( \sum_{p \leq t} \chi_q(p) \log p \right)$$

$$= \int_{2y}^{x} \frac{1}{\log t} d \left( -\frac{t^\beta}{\beta} + R_2(t) \right) = -\int_{2y}^{x} \frac{t^{\beta-1}}{\log t} dt + \int_{2y}^{x} \frac{1}{\log t} d (R_2(t))$$

Let’s assume that $(1 - \beta) \log x \leq \frac{1}{4}$, Otherwise the theorem already holds, so we have

$$-\int_{2y}^{x} \frac{t^{\beta-1}}{\log t} dt = -\int_{2y}^{x} \frac{e^{(\beta-1)\log t}}{\log t} dt$$

$$= -\int_{2y}^{x} \frac{dt}{\log t} - (\beta - 1) \int_{2y}^{x} dt + O \left( (1 - \beta)^2 \int_{2y}^{x} \log t dt \right)$$

$$= -\int_{2y}^{x} \frac{dt}{\log t} + (1 - \beta)(x - 2y) + O \left( (1 - \beta)^2 x \log x \right)$$

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By (1) of lemma 1.4

\[(1 - \beta)^2 x \log x \ll (1 - \beta)L(1, \chi_q)x \log x \ll (1 - \beta)L(1, \chi_q)x \log^2 q\]

Clearly \(-(1 - \beta)2y \ll xq^{-2}\)

In addition, we have

\[
\int_{2y}^{x} \frac{1}{\log t} d(\frac{R_2(t)}{t}) = \frac{R_2(x)}{\log x} - \frac{R_2(2y)}{\log 2y} + \int_{2y}^{x} \frac{R_2(t)}{t \log^2 t} dt
\]

\[\ll \frac{|R_2(x)|}{\log x} + \frac{|R_2(2y)|}{\log 2y} + \int_{2y}^{x} \frac{|R_2(t)|}{t \log^2 t} dt
\]

\[\ll R_2(x) + R_2(2y) \int_{2y}^{x} \frac{1}{t \log^2 t} dt \ll x \exp \left(-c_4 \sqrt{\log x}\right) \ll xq^{-2}\]

By lemma 1.1, we have

\[\sum_{2y < \rho \leq x} \frac{1}{\log \rho} + O \left(x \exp(-c_3 \sqrt{\log x})\right) = \int_{2y}^{x} \frac{dt}{\log t} + O(xq^{-2})\]

this completes the proof of the lemma.

From now on, let’s assume \(L(1, \chi_q) \log^6 q \leq 1\), otherwise the theorem already holds.

**Lemma 2.4** Let \(\chi_q\) be the primitive real character mod \(q\), then

\[\sum_{1 \leq r \leq \sqrt{x}} a(r) \sum_{\sqrt{r} \leq mk \leq x/r} a(m)a(k) \ll L^2(1, \chi_q)x \log^4 q + xq^{-2}\]

where \(a(n) = \sum_{d \mid n} \chi_q(d)\).

**Proof.** For the convenience of writing, let’s denote \(L(1, \chi_q)\) by \(L(1)\).

\[\sum_{\sqrt{x} < mk \leq x/r} a(m)a(k) \leq \sum_{mk \leq x/r} a(m)a(k)\]
\[
= \sum_{y \leq k \leq x/ry} a(k) \sum_{y \leq m \leq x/rk} a(m) \leq \sum_{y \leq k \leq x/ry} a(k) \sum_{1 \leq m \leq x/rk} a(m)
\]

By lemma 2.1, the above formula
\[
\ll \sum_{y \leq k \leq x/ry} a(k) \left( L(1) \frac{x}{rk} \right) \ll L(1) \frac{x}{r} \ll \sum_{y \leq k \leq x/ry} \frac{a(k)}{k}
\]
by lemma 1.6, the above formula
\[
\ll L(1) \frac{x}{r} \left( L(1) \log x + y^{-\frac{1}{2}} q^{\frac{1}{4}} \log^2 x \right)
\]
\[
\ll \frac{x}{r} L^2(1) \log^2 q + \frac{x}{r} L(1) y^{-\frac{1}{2}} q^{\frac{1}{4}} \log^3 q
\]
by lemma 1.6 and \( L(1) \log^6 q \leq 1 \), we have
\[
\sum_{1 \leq r \leq \sqrt{x}} a(r) \sum_{\sqrt{x} < mk \leq x/r \atop m \geq y, k \geq y} a(m)a(k)
\]
\[
\ll (L'(1) + 1) L^2(1) x \log^2 q + (L'(1) + 1) L(1) x y^{-\frac{1}{2}} q^{\frac{1}{4}} \log^3 q
\]
because \( L'(1) \ll \log^2 q \), the above formula
\[
\ll L^2(1) x \log^4 q + L(1) x y^{-\frac{1}{2}} q^{\frac{1}{4}} \log^5 q \ll L^2(1) x \log^4 q + x y^{-\frac{1}{2}} q^{\frac{1}{4}}
\]
\[
\ll L^2(1) x \log^4 q + xq^{-2}
\]
this completes the proof of the lemma.

**Lemma 2.5** Let \( \chi_q \) be the primitive real character mod \( q \), then
\[
\sum_{\frac{mk}{m \geq y, k \geq y}} a(m)a(k) \sum_{1 \leq r \leq x/mk} a(r) \ll L^2(1, \chi_q) x + xq^{-2}
\]
where \( a(n) = \sum_{d \mid n} \chi_q(d) \)
Proof. For the convenience of writing, let’s denote $L(1, \chi_q)$ by $L(1)$. By lemma 2.1, we have

$$\sum_{mk \leq \sqrt{x}} a(m)a(k) \sum_{1 \leq r \leq x/mk} a(r) \ll \sum_{mk \leq \sqrt{x}} a(m)a(k) \left( L(1) \frac{x}{mk} \right)$$

$$\ll L(1)x \sum_{mk \leq \sqrt{x}} a(m) a(k) \frac{m}{k}$$

Clearly

$$\sum_{mk \leq \sqrt{x}} \frac{a(m) a(k)}{m} \frac{a(k)}{k} = \sum_{y \leq m \leq \sqrt{x}/y} \frac{a(m)}{m} \sum_{y \leq k \leq \sqrt{x}/m} \frac{a(k)}{k}$$

$$\leq \sum_{y \leq m \leq \sqrt{x}} \frac{a(m)}{m} \sum_{y \leq k \leq \sqrt{x}} \frac{a(k)}{k} = \left( \sum_{y \leq m \leq \sqrt{x}} \frac{a(m)}{m} \right)^2$$

by lemma 1.6, the above formula

$$= \left( \log \frac{\sqrt{x}}{y} + O \left( y^{-\frac{1}{2}} q^{\frac{3}{4}} \log^{\frac{3}{2}} x \right) \right)^2$$

$$\ll L^2(1) \log^2 x + y^{-1}q^{\frac{1}{2}} \log^3 x \ll L^2(1) \log^4 q + y^{-1}q^{\frac{1}{2}} \log^6 q$$

therefore

$$\sum_{mk \leq \sqrt{x}} a(m)a(k) \sum_{1 \leq r \leq x/mk} a(r) \ll L^3(1)x \log^4 q + L(1)xy^{-1}q^{\frac{1}{2}} \log^6 q$$

$$\ll L^2(1)x + xy^{-1}q^{\frac{1}{2}} \ll L^2(1)x + xq^{-2}$$

this completes the proof of the lemma.

3. Proof of Theorem and Corollary

We write
\[ P(2y) = \prod_{2 \leq p \leq 2y} p \]

where \( p \) be the prime number.

Let \( \chi_q \) be the primitive real character mod \( q \), \( a(n) = \sum_{d|n} \chi_q(d) \). \( \mu(n) \) be Möbius function.

By lemma 2.2, we have

\[
- \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} \mu(n) = \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} b_1(n)
\]

\[
+ \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} b_2(n) + \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} b_3(n)
\]

namely

\[
\sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} b_2(n) = - \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} b_3(n)
\]

\[
- \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} \mu(n) - \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} b_1(n)
\]

Next, we prove the theorems and the corollary in three steps.

First step. By lemma 1.5, we have

\[
- \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} \mu(n) = -1
\]

Because \( (u, P(2y)) = 1 \), therefore \( (n, P(2y)) = 1 \), but when \( 1 \leq n < y \), the only possibility is \( n = 1 \). by lemma 2.2,

\[
- \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n|u} b_1(n) = 2
\]
The second step. By lemma 2.2,

\[ \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n \mid u} b_2(n) = \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n \mid u} \sum_{1 \leq m < y, 1 \leq d < y} \mu(m)\mu(d) \]

Because \((u, P(2y)) = 1\), therefore \((n, P(2y)) = 1\), \((d, P(2y)) = 1\) and \((m, P(2y)) = 1\), but \(1 \leq d < y, 1 \leq m < y\), then the only possibility is \(d = 1, m = 1\). Therefore the above formula

\[ = \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n \mid u} \sum_{r \mid u} 1 = \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{r \mid u} 1 \]

\[ = \sum_{1 \leq u \leq x} \mu^2(u)a(u)\tau(u) \]

where \(\tau(u)\) be the divisor function. Let \(p\) be the prime number, the above formula

\[ \geq \sum_{2y < p \leq x} \mu^2(p)a(p)\tau(p) \geq \sum_{2y < p \leq x} a(p) = \sum_{2y < p \leq x} (1 + \chi_4(p)) \]

By lemma 2.3, we have

\[ \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n \mid u} b_2(n) \geq (1 - \beta)x + O \left( (1 - \beta)L(1, \chi_4)x \log^2 q + xq^{-2} \right) \]

The third step. By lemma 2.2,

\[ - \sum_{1 \leq u \leq x} \mu^2(u)a(u) \sum_{n \mid u} b_3(n) \]
\[ - \sum_{1 \leq u \leq x} \mu^2(u) a(u) \sum_{n \mid u} \sum_{\text{m.k.n.m} = n, m \geq y, k \geq y} \mu(m) \left( \sum_{1 \leq d < y} \mu(d) \right) \]

Because \((u, P(2y)) = 1\), therefore \((n, P(2y)) = 1\), \((d, P(2y)) = 1\), but \(1 \leq d < y\), then the only possibility is \(d = 1\), therefore the above formula

\[ = - \sum_{1 \leq u \leq x} \mu^2(u) a(u) \sum_{n \mid u} \sum_{\text{m.k.n.m} = n, m \geq y, k \geq y} \mu(m) \]

\[ \ll \sum_{1 \leq u \leq x} \mu^2(u) a(u) \sum_{m.k} 1 \]

\[ = \sum_{1 \leq u \leq x} \mu^2(u) a(u) \sum_{m.k} 1 = \sum_{1 \leq m.k.r \leq x} \mu^2(m.k.r) a(m.k.r) \]

When \(\mu^2(m.k.r) \neq 0\), we have \((m, k) = 1, (m, r) = 1, (k, r) = 1\), therefore \(a(m.k.r) = a(m)a(k)a(r)\). Then, the above formula

\[ = \sum_{1 \leq m.k.r \leq x} a(m)a(k) \mu^2(k)a(k) \mu^2(r)a(r) \leq \sum_{1 \leq m.k.r \leq x} a(m)a(k)a(r) \]

\[ = \sum_{m.k \leq x} a(m)a(k) \sum_{1 \leq r \leq x/mk} a(r) = \sum_{m.k \leq \sqrt{x}} a(m)a(k) \sum_{1 \leq r \leq x/mk} a(r) \]

\[ + \sum_{\sqrt{x} < m.k \leq x} a(m)a(k) \sum_{1 \leq r \leq x/mk} a(r) \]

\[ = \sum_{m.k \leq \sqrt{x}} a(m)a(k) \sum_{1 \leq r \leq x/mk} a(r) + \sum_{1 \leq r \leq \sqrt{x}} a(r) \sum_{\sqrt{x} < m.k \leq x} a(m)a(k) \]
By Lemma 2.4 and Lemma 2.5, we have

\[- \sum_{1 \leq u \leq x \atop (u, P(2y)) = 1} \mu^2(u) a(u) \sum_{n \mid u} b_3(n) \ll L^2(1, \chi_q) x \log^4 q + x q^{-2}\]

Combining the above calculations, we have

\[(1 - \beta)x \ll L^2(1, \chi_q) x \log^4 q + (1 - \beta) L(1, \chi_q) x \log^2 q + x q^{-2}\]

Divide both sides of this inequality by \((1 - \beta)x\), then

\[1 \ll L(1, \chi_q) \frac{L(1, \chi_q)}{1 - \beta} \log^4 q + L(1, \chi_q) \log^2 q + \frac{1}{(1 - \beta)q^2}\]

By (1) and (3) of lemma 1.4,

\[1 \ll L(1, \chi_q) \log^6 q + q^{-1}\]

when \(q\) is sufficiently large, we have

\[c_1 \leq L(1, \chi_q) \log^6 q\]

therefore

\[\frac{c_1}{\log^6 q} \leq L(1, \chi_q)\]

By (1) of lemma 1.4, we have

\[\beta \leq 1 - \frac{c_2}{\log^6 q}\]

this completes the proof of the theorem and the corollary.

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