Transport equation in generalized Campanato spaces

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Abstract

In this paper we study the transport equation in \(\mathbb{R}^n \times (0, T)\), \(T > 0\),
\[
\partial_t f + v \cdot \nabla f = g, \quad f(\cdot, 0) = f_0 \quad \text{in} \quad \mathbb{R}^n
\]
in generalized Campanato spaces \(\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)\). The critical case is particularly
interesting, and is applied to the local well-posedness problem in a space close
to the Lipschitz space in our companion paper\[^6\]. More specifically, in the
critical case \(s = q = N = 1\) we have the embedding relations, \(B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \hookrightarrow C^{0,1}(\mathbb{R}^n)\), where \(B^1_{\infty,1}(\mathbb{R}^n)\) and \(C^{0,1}(\mathbb{R}^n)\) are the Besov space and
the Lipschitz space respectively. For \(f_0 \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n), v \in L^1(0, T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)),\) and \(g \in L^1(0, T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))\), we prove the existence and uniqueness of solutions
to the transport equation in \(L^\infty(0, T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))\) such that
\[
\|f\|_{L^\infty(0, T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))} \leq C\left(\|v\|_{L^1(0, T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))}, \|g\|_{L^1(0, T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))}\right).
\]

Similar results in the other cases are also proved.

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1 Introduction

Let $0 < T < +\infty$ and $Q = \mathbb{R}^n \times [0,T]$ with $n \in \mathbb{N}, n \geq 2$. We consider the transport equation

\begin{equation}
\begin{cases}
\partial_t f + (v \cdot \nabla)f = g & \text{in } Q, \\
v = v_0 & \text{on } \mathbb{R}^n \times \{0\},
\end{cases}
\end{equation}

where $f = f(x_1, \ldots, x_n)$ is unknown, while $v = (v_1, \ldots, v_n) = v(x,t)$ represents a given drift velocity and $g = g(x_1, \ldots, x_n)$ given function.

Our aim in this paper is to obtain estimates of solutions to (1.1) in generalized Campanato spaces. The proof relies on a key estimate in terms of local oscillation. As byproduct we get existence of solutions in Besov spaces and Tribel-Lizorkin spaces, which can be estimated by the data belonging to these spaces. One of the main motivations to study the transport equation in such generalized Campanato spaces is to apply it to prove local well-posedness of the incompressible Euler equations in function space embedded in the Lipschitz space, which includes linearly growing functions at spatial infinity. For recent developments of the local well-posedness/ill-posedness of the Euler equations in various critical function spaces embedded in $C^{0,1} (\mathbb{R}^n)$ we refer [3, 4, 11, 13, 16, 17, 1, 7, 12]. We would also like to refer [8] for the study of transport equation with drift velocity in less regular space. For the application of our new function spaces in the critical case to the Euler equations please see our companion paper [6].

Let us introduce the function spaces we will use throughout the paper. Let $N \in \mathbb{N} \cup \{0, -1\}$. By $\mathcal{P}_N$ ($\mathcal{P}_N$ respectively) we denote the space of all polynomial (all homogenous polynomials respectively) of degree less or equal $N$. We equip the space $\mathcal{P}_N$ with the norm $\|P\|_{(p)} = \|P\|_{L^p(B(1))}$. Note that since $\dim(\mathcal{P}_N) < +\infty$ all norms $\|\cdot\|_{(p)}, 1 \leq p \leq \infty$, are equivalent. For notational convenience, in case $N = -1$ we use the convention $\mathcal{P}_{-1} = \{0\}$, which consists of the trivial polynomial $P \equiv 0$. 
Let \( f \in L^p_{\text{loc}}(\mathbb{R}^n), 1 \leq p \leq +\infty \). For \( x_0 \in \mathbb{R}^n \) and \( 0 < r < \infty \) we define the oscillation

\[
\text{osc}_{p,N}(f; x_0, r) := |B(r)|^{-\frac{1}{p}} \inf_{P \in \mathcal{P}_N} \| f - P \|_{L^p(B(x_0, r))}.
\]

We note that from our convention above in the case \( N = -1 \) we have

\[
\text{osc}_{p,-1}(f; x_0, r) := |B(r)|^{-\frac{1}{p}} \| f \|_{L^p(B(x_0, r))}.
\]

Then, we define for \( 1 \leq q, p \leq +\infty \) and \( s \in (-\infty, N + 1] \) the spaces

\[
\mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) \mid \| f \|_{\mathcal{L}^s_{q(p,N)}} := \left\| \left( \sum_{j \in \mathbb{Z}} \left( 2^{-sj} \text{osc}_{p,N}(f; \cdot, 2^j) \right)^q \right)^{\frac{1}{q}} \right\|_{L^\infty} < +\infty \right\}.
\]

Furthermore, by \( \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \), \( k \in \mathbb{N} \), we denote the space of all \( f \in W^{k,p}_{\text{loc}}(\mathbb{R}^n) \) such that \( D^k f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \). The space \( \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \) will be equipped with the norm

\[
\| f \|_{\mathcal{L}^{k,s}_{q(p,N)}} = \| D^k f \|_{\mathcal{L}^s_{q(p,N)}} + \| f \|_{L^p(B(1))}, \quad f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n).
\]

According to the characterization theorem of the Triebel-Lizorkin spaces in terms of oscillation, we have

\[
\left\{ \begin{array}{l}
\begin{aligned}
f \in F^s_{r,q}(\mathbb{R}^n) &\iff \| f \|_{L^\infty(B_1)} + \left\| \left( \sum_{j = -\infty}^0 \left( 2^{-sj} \text{osc}_{p,N}(f; \cdot, 2^j) \right)^q \right)^{\frac{1}{q}} \right\|_{L^r} < +\infty.
\end{aligned}
\end{array} \right.
\]

(cf. \cite[Theorem, Chap. 1.7.3]{L}), and we could regard the spaces \( \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \) as an extension of the limit case of \( F^s_{r,q}(\mathbb{R}^n) \) as \( r \to +\infty \).

In fact in case \( q = +\infty \) and \( s > 0 \) we get the usual Campanato spaces with the isomorphism relation (cf. \cite{M, T})

\[
\mathcal{L}^{n+ps,p}_{\infty(p,N)}(\mathbb{R}^n) \cong \mathcal{L}^s_{\infty(p,N)}(\mathbb{R}^n).
\]

Furthermore, in the case \( N = 0, s = 0 \) and \( q = \infty \) we get the space of bounded mean oscillation, i.e.,

\[
\mathcal{L}^0_{\infty(0,0)}(\mathbb{R}^n) \cong BMO.
\]

In case \( N = -1 \) and \( s \in (-\frac{n}{p}, 0) \) the above space coincides with the usual Morrey space \( \mathcal{M}^{n+ps}(\mathbb{R}^n) \).

We note that the oscillation introduced in (1.2) is attained by a unique polynomial \( P_\ast \in \mathcal{P}_N \).

According to Theorem \cite[3.6]{M} (see Section 3 below), for the spaces \( \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \) we have the following embedding properties

\[
\text{(1.3)} \quad B^1_{r,1} \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{0,1}_{1(p,1)}(\mathbb{R}^n) \hookrightarrow C^{0,1}(\mathbb{R}^n).
\]
Accordingly, 

(1.4) \[ \| \nabla u \|_\infty \leq c \| u \|_{\mathcal{L}^1_{1(p,1)}}. \]

Furthermore, for every \( f \in \mathcal{L}^k_{1(p,k)}(\mathbb{R}^n) \), \( k \in \{0, 1\} \), there exists a unique \( \dot{P}^k_\infty f \in \dot{P}_1 \), such that for all \( x_0 \in \mathbb{R}^n \)

\[ f \text{ converge asymptotically to } \dot{P}^k_\infty f \text{ as } |x| \to +\infty. \]

The precise meaning of this asymptotic limit will be given in Section 3 below.

We are now in a position to present our first main result.

**Theorem 1.1** (The case \( N = 0 \)). Let \( 0 < T < +\infty \). Let \( s \in (-\frac{n}{q}, 0) \), \( 1 < p < +\infty \), \( 1 \leq q \leq +\infty \). Let \( v \in L^1(0, T; L^p_{\text{loc}}(\mathbb{R}^n)) \), with

(1.5) \[ \int_0^T \| \nabla v(\tau) \|_\infty d\tau < +\infty. \]

Then for every \( f_0 \in \mathcal{L}^s_{p(0)}(\mathbb{R}^n) \) and \( g \in L^1(0, T; \mathcal{L}^s_{p(0)}(\mathbb{R}^n)) \) there exists a unique solution \( f \in L^\infty(0, T; \mathcal{L}^s_{p(0)}(\mathbb{R}^n)) \) to the transport equation (1.1). Furthermore, it holds for almost all \( t \in (0, T) \)

(1.6) \[ |f(t)|_{\mathcal{L}^s_{p(0)}} \leq c \left( |f_0|_{\mathcal{L}^s_{p(0)}} + \int_0^T |g(\tau)|_{\mathcal{L}^s_{p(0)}} d\tau \right) \exp \left( c \int_0^T \| \nabla v(\tau) \|_\infty d\tau \right). \]

In case \( N = 1 \) we get

**Theorem 1.2** (The case \( N = 1 \) and \( s = 1 \)). Let \( 0 < T < +\infty \) and \( 1 < p < +\infty \), \( 1 \leq q \leq +\infty \). Let \( v \in L^1(0, T; \mathcal{L}^1_{q(1)}(\mathbb{R}^n)) \) with (1.5) and

(1.7) \[ \int_0^T \sup_{x_0 \in \mathbb{R}^n} \left( \sum_{j=-\infty}^0 (-j)^{q-1}2^{-jq} \text{osc}_{p,1}(v(\tau); x_0, 2^j) \right) \frac{1}{q} d\tau < +\infty \]

Let \( f_0 \in \mathcal{L}^1_{q(1)}(\mathbb{R}^n) \) and \( g \in L^1(0, T; \mathcal{L}^1_{q(1)}(\mathbb{R}^n)) \) satisfying the condition

(1.8) \[ \sup_{x_0 \in \mathbb{R}^n} \text{osc}(f_0; x_0, 1) + \int_0^T \sup_{x_0 \in \mathbb{R}^n} \text{osc}(g(\tau); x_0, 1) d\tau < +\infty. \]

Then, there exists a unique solution \( f \in L^\infty(0, T; \mathcal{L}^1_{q(1)}(\mathbb{R}^n)) \) to the transport equation (1.1). Furthermore, it holds for all \( t \in (0, T) \)

(1.9) \[ |f(t)|_{\mathcal{L}^1_{q(1)}} \leq c \left( |f_0|_{\mathcal{L}^1_{q(1)}} + \int_0^T |g(\tau)|_{\mathcal{L}^1_{q(1)}} d\tau \right) \exp \left( c \int_0^T C(\tau) d\tau \right), \]
where we set
\[ C(\tau) = \| \nabla v(\tau) \|_\infty + \sup_{x_0 \in \mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} \left( j^n - 2^{j-1} \text{osc}_p(v(\tau); x_0, 2^j) \right)^{\frac{1}{q}} \right), \]
\[ j^- = -\min\{j, 0\}, \text{ and } |z|_{\mathcal{Z}^s_{q(p,0)}} \text{ stands for the semi norm} \]
\[ |z|_{\mathcal{Z}^s_{q(p,0)}} = |z|_{\mathcal{Z}^s_{q(p,1)}} + \sup_{x_0 \in \mathbb{R}^n} |\nabla P_{x_0,1}(z)|. \]

Our third main result concerns the case \( s > 1 \).

**Theorem 1.3** (The case \( N \geq 1 \) and \( s > 1 \)). Let \( 0 < T < +\infty \), \( N \in \mathbb{N} \), \( 1 < s < +\infty \), \( 1 < p < +\infty \), and \( 1 \leq q < +\infty \). Let \( v \in L^1(0,T; \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)) \) with (1.3) and Let \( f_0 \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \) and \( g \in L^1(0,T; \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)) \) satisfying the condition

\[ (1.10) \quad \| \nabla f_0 \|_\infty + \int_0^T \| \nabla g(\tau) \|_\infty d\tau < +\infty. \]

Then, there exists a unique solution \( f \in L^\infty(0,T; \mathcal{L}^1_{q(p,1)}(\mathbb{R}^n)) \) to the transport equation (1.1) together with the estimate

\[ (1.11) \quad |f(t)|_{\mathcal{Z}^s_{q(p,0)}} \leq c \left\{ |f_0|_{\mathcal{Z}^s_{q(p,0)}} + \int_0^T |g(\tau)|_{\mathcal{Z}^s_{q(p,0)}} \right\} \exp \left( c \int_0^T \| v(\tau) \|_{\mathcal{Z}^s_{q(p,0)}} d\tau \right), \]

where \( |z|_{\mathcal{Z}^s_{q(p,0)}} \) stands for the semi norm defined by
\[ |z|_{\mathcal{Z}^s_{q(p,0)}} = |z|_{\mathcal{Z}^s_{q(p,1)}} + \| \nabla z \|_\infty. \]

From Theorem 1.2 we get the following corollary for the special case \( s = q = N = 1 \), which will be useful for our future application to the Euler equations in the critical spaces.

**Corollary 1.4.** Let \( 0 < T < +\infty \), \( 1 < p < +\infty \). Let \( v \in L^1(0,T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)) \), \( f_0 \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \) and \( g \in L^1(0,T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)) \). Then there exists a unique solution \( f \in L^\infty(0,T; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)) \) to the transport equation (1.1). Furthermore, it holds for all \( t \in (0,T) \)

\[ (1.12) \quad \| f(t) \|_{\mathcal{Z}^1_{1(p,1)}} \leq C \left\{ 1 + \int_0^T |v(\tau)|_{\mathcal{Z}^1_{1(p,1)}} d\tau \right\} \exp \left( c \int_0^T \| \nabla v(\tau) \|_\infty d\tau \right). \]

where
\[ C = c \left( \| f_0 \|_{\mathcal{Z}^1_{1(p,1)}} + \int_0^T \| g(\tau) \|_{\mathcal{Z}^1_{1(p,1)}} d\tau \right), \]

while \( c = \text{const} > 0 \) depending on \( n \) and \( p \).
**Remark 1.5.** Using the well-known characterization of $B^1_{\infty,1}(\mathbb{R}^n)$ in terms of oscillation, we easily verify the embeddings

\[(1.13) \quad B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^1_{(p,1)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow L^1_{(p,1)}(\mathbb{R}^n).\]

Indeed, referring to [15, Theorem, Chap.1.7.3]), we see that

\[v \in B^1_{\infty,1}(\mathbb{R}^n) \iff \sum_{j=-\infty}^{0} 2^{-j} \| \text{osc}(v; \cdot, 2^j) \|_{L^\infty} + \|v\|_{L^\infty} < +\infty.\]

This shows that for $x \in \mathbb{R}^n$ it holds

\[\sum_{j \in \mathbb{Z}} 2^{-j} \text{osc}(v; x, 2^j) \leq \sum_{j=-\infty}^{0} 2^{-j} \| \text{osc}(v; \cdot, 2^j) \|_{L^\infty} + \sum_{j=1}^{\infty} 2^{-j} \text{osc}(v; x, 2^j) + \|v\|_{L^\infty}.\]

On the other hand, it is readily seen that $\text{osc}(v; x, 2^j) \leq 2\|v\|_{L^\infty}$. Accordingly, the second sum on the right-hand side is bounded by $\|v\|_{L^\infty}$. This yields

\[\|v\|_{L^1_{(p,1)}} = \sum_{j \in \mathbb{Z}} 2^{-j} \text{osc}(v; x, 2^j) + \|v\|_{L^2(B(1))} \leq \sum_{j \in \mathbb{Z}} 2^{-j} \text{osc}(v; x, 2^j) + c\|v\|_{B^1_{\infty,1}} \leq c \sum_{j=-\infty}^{0} 2^{-j} \| \text{osc}(v; \cdot, 2^j) \|_{L^\infty} + c\|v\|_{L^\infty} \leq c\|v\|_{B^1_{\infty,1}}.\]

Secondly, according to [14, p. 85] (see also [1]) we have the embedding

\[B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).\]

On the other hand, there exists a function $f \in L^1_{(p,1)}(\mathbb{R}^n)$ which is not in $C^1(\mathbb{R}^n)$ (see Appendix B). This clearly shows that $L^1_{(p,1)}(\mathbb{R}^n)$ contains less regular functions than $B^1_{\infty,1}(\mathbb{R}^n)$.

Thirdly, since $L^1_{(p,1)}(\mathbb{R}^n)$ contains linearly growing function at infinity, in particular polynomials of of degree less or equal one, $L^1_{(p,1)}(\mathbb{R}^n)$ is strictly bigger than $B^1_{\infty,1}(\mathbb{R}^n)$ in terms of asymptotic behaviors as infinity. We also note that the use of our generalized Campanato spaces to handle the bounded domain problem is quite convenient as in the case of usual Campanato spaces.

### 2 Preliminariy lemmas

Let $X = \{X_j\}_{j \in \mathbb{Z}}$ be a sequence of non-negative real numbers. Given $s \in \mathbb{R}$ and $0 < q < +\infty$, we denote

\[
\{2^{js}\} \cdot X := \{2^{js}X_j\}_{j \in \mathbb{Z}}, \quad X^q := \{X_j^q\}_{j \in \mathbb{Z}}
\]
respectively. We define \( S_{\alpha,q} : X = \{X_j\}_{j \in \mathbb{Z}} \mapsto Y = \{Y_j\}_{j \in \mathbb{Z}} \), where
\[
Y_j = (S_{\alpha,q}(X))_j = 2^{i\alpha} \left( \sum_{l=j}^{\infty} (2^{-\alpha} X_l)^q \right)^{\frac{1}{q}}, \quad j \in \mathbb{Z}.
\]
From the above definition, in case of \( \alpha = 0 \), it follows that
\[
\|S_{0,q}(X)\|_{\ell^{\infty}} = \|X\|_{\ell^{\infty}} \leq \|X\|_{\ell^{\alpha}} \quad \forall X \in \ell^{q}.
\]
Clearly, for all \( \alpha, \beta \in \mathbb{R} \) it holds
\[
2^{\beta j}(S_{\alpha,q}(X))_j = S_{\alpha+\beta,q}(\{2^{\beta l} X_l\})_j, \quad j \in \mathbb{Z}.
\]
Given \( X = \{X_j\}_{j \in \mathbb{Z}} \), \( Y = \{Y_j\}_{j \in \mathbb{Z}} \), we denote \( X \leq Y \) if \( X_j \leq Y_j \) for all \( j \in \mathbb{Z} \). Throughout this paper, we frequently make use of the following lemma, which could be regarded as a generalization of the result in [2].

**Lemma 2.1.** For all \( \beta < \alpha \) and \( 0 < p \leq q \leq +\infty \) it holds
\[
S_{\beta,q}(S_{\alpha,p}(X)) \leq \frac{1}{1 - 2^{-(\alpha-\beta)}} S_{\beta,q}(X).
\]

**Proof:** We first observe
\[
(S_{\beta,q}(S_{\alpha,p}X))_j = 2^{i\beta} \left\{ \sum_{i=j}^{\infty} 2^{-i\beta q} (S_{\alpha,p}X)_i^q \right\}^{\frac{1}{q}}
\]
\[
= 2^{i\beta} \left\{ \sum_{i=j}^{\infty} 2^{-i\beta q} \left[ 2^{i\alpha} \left( \sum_{l=i}^{\infty} (2^{-\alpha l} X_l)^p \right)^{\frac{1}{p}} \right]^q \right\}^{\frac{1}{q}}
\]
\[
= 2^{i\beta} \left\{ \sum_{i=j}^{\infty} 2^{i(\alpha-\beta)q} \left( \sum_{l=i}^{\infty} (2^{-\alpha p l} 2^{\beta p l} X_l^p)^{q} \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}}
\]
\[
= (S_{0,q}(S_{\alpha-\beta,p}(\{2^{\beta p l} X_l^p\})))_j.
\]

1. The case \( p = 1, \beta = 0 \). Let \( X \) be sequence with \( X_j = 0 \) except finite \( j \in \{m, m+1, \ldots\} \). By the aid of Hölder’s inequality, we get
\[
(S_{0,q}(S_{\alpha,1}(X)))_j^q
\]
\[
= \sum_{i=j}^{\infty} \left( \sum_{l=0}^{i} 2^{-\alpha(l+i)} X_{l+i} \right)^q \sum_{i=j}^{\infty} \left( \sum_{l=0}^{i} 2^{-\alpha l} X_l \right)^{q-1}
\]
\[
= \sum_{i=j}^{\infty} 2^{i\alpha} \sum_{l=0}^{i} 2^{-\alpha(l+i)} X_{l+i} \left( \sum_{i=j}^{\infty} 2^{-\alpha l} X_l \right)^{q-1}
\]
\[
= \sum_{i=j}^{\infty} 2^{-\alpha} \sum_{l=0}^{i} X_{l+i} S_{\alpha,1}(X)^{q-1} \leq \sum_{l=0}^{\infty} 2^{-\alpha} \left( \sum_{i=j}^{\infty} X_{l+i}^q \right)^{q-1} \left( \sum_{i=j}^{\infty} (S_{\alpha,1}(X))^q \right)^{\frac{q-1}{q}}
\]
\[
\leq \frac{1}{1 - 2^{-\alpha}} (S_{0,q}(X))_j (S_{0,q}(S_{\alpha,1}(X)))_j^{q-1},
\]

where we used the fact \((\sum_{i=j}^{\infty} X_{l+i}^q)^{\frac{1}{q}} \leq (\sum_{i=j}^{\infty} X_i^q)^{\frac{1}{q}} = (S_{0,q}X)_j\) for all \(l \geq 0\). Dividing both sides by \((S_{0,q}(S_{\alpha,1}(X)))_{j}^{q-1}\), we get \((2.3)\).

In the general case \(S_{0,q}(X)_j < +\infty\) we obtain from \((2.3)\) for the truncated sequence the property \(S_{0,q}(S_{\alpha,1}(X))_j < +\infty\). This shows \((2.3)\) for the general case.

2. The case \(0 < p \leq q \leq +\infty, \beta < \alpha\). Recalling the definition of \(S_{\alpha,p}(X)\), we find

\[
(2.5) \quad S_{\alpha,p}(X)_j = \left(2^{j\alpha p} \sum_{i=j}^{\infty} 2^{-i\alpha p} X_i^p\right)^{\frac{1}{p}} = (S_{\alpha,p,1}(\{X_i^p\})_j^{\frac{1}{p}}, \ j \in \mathbb{Z}.
\]

Using \((2.5)\) with \(\alpha - \beta\) in place of \(\alpha\) together with \((2.5)\) with \(\beta = 0\) and \(p = q\), we obtain the following two identities for \(j \in \mathbb{Z}\)

\[
(2.6) \quad (S_{\alpha-\beta,p}(\{2^{-\beta i} X_i\}))_j = (S_{(\alpha-\beta)p,1}(\{2^{-\beta i} X_i^p\})_j^{\frac{1}{p}}.
\]

\[
(2.7) \quad [S_{0,q}(S_{(\alpha-\beta)p,1}(\{2^{-\beta i} X_i^p\}))_j^{\frac{1}{q}} = [S_{0,1}(\{S_{(\alpha-\beta)p,1}(\{2^{-\beta i} X_i^p\})_j^{\frac{2}{q}})]_j^{\frac{1}{q}}.
\]

Applying, \(S_{0,q}\) to both sides of and using first \((2.6), (2.7)\) together \((2.2)\), and applying the inequality from the first part of the proof, we arrive at

\[
\left(S_{0,q}(S_{\alpha-\beta,p}(\{2^{-\beta i} X_i\}))_j\right)^{\frac{1}{q}} = \left[S_{0,q}(S_{(\alpha-\beta)p,1}(\{2^{-\beta i} X_i^p\}))_j^{\frac{1}{q}}
\right.

= \left[S_{0,1}(\{S_{(\alpha-\beta)p,1}(\{2^{-\beta i} X_i^p\})_j^{\frac{2}{q}})]_j^{\frac{1}{q}}
\right.

= \left(S_{0,\frac{2}{p}}(S_{(\alpha-\beta)p,1}(\{2^{-\beta i} X_i^p\}))_j^{\frac{p}{q}}
\right)

\leq \frac{1}{(1 - 2 - (\alpha - \beta)p)^{\frac{1}{p}}}(S_{0,\frac{2}{p}}(\{2^{-\beta i} X_i^p\})_j^{\frac{1}{p}}
\right)

\leq \frac{1}{1 - 2 - (\alpha - \beta)^{-\beta j}}(S_{\beta,q}(X))_j,
\]

where we used the fact \((1 - x^a)^{\frac{1}{a}} \geq 1 - x\) for all \(0 < x < 1\) and \(a > 1\). Combining this with \((2.4)\), we have \((2.3)\).

\section{Properties of the spaces \(\mathcal{L}^s_{q,p,N}(\mathbb{R}^n)\)}

In this section our objective is to provide important properties of the space \(\mathcal{L}^{k,s}_{q,p,N}(\mathbb{R}^n)\) such as embedding properties, equivalent norms, interpolations properties and product estimates. First, let us recall the definition of the generalized mean for distributions \(f \in \mathcal{S}'\), where \(\mathcal{S}\) denotes the usual Schwarz class of rapidly decaying functions. For \(f \in \mathcal{S}'\) and \(\varphi \in \mathcal{S}\) we define the convolution

\[
f * \varphi(x) = \langle f, \varphi(x - \cdot)\rangle, \quad x \in \mathbb{R}^n.
\]
where $< \cdot, \cdot >$ denotes the dual pairing. Below we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, $f \ast \varphi \in C^\infty(\mathbb{R}^n)$ and for every multi index $\alpha \in \mathbb{N}_0^n$ it holds

$$D^\alpha(f \ast \varphi) = f \ast (D^\alpha \varphi) = (D^\alpha f) \ast \varphi.$$  

Given $x_0 \in \mathbb{R}^n, 0 < r < +\infty$ and $f \in \mathcal{S}'$ we define the mean

$$[f]_{r,x}^\alpha = f \ast D^\alpha \varphi_r(x).$$

where $\varphi_r(y) = r^{-n} \varphi(r^{-1}(y))$, and $\varphi \in C^\infty_c(B(1))$ stands for the standard mollifier, such that $\int \varphi dx = 1$. Note that in case $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we get

$$[f]_{r,x}^0 = \int_{\mathbb{R}^n} f(x-y) \varphi_r(y) dy = \int_{B(x,r)} f(y) \varphi_{x,r}(-y) dy,$$

where $\varphi_{x,r} = \varphi_r(\cdot + x)$. Furthermore, from the above definition it follows that

(3.1) $$[f]_{r,x}^\alpha = (D^\alpha f) \ast \varphi_r(x) = [D^\alpha f]_{x,r}^0.$$  

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ we immediately get

(3.2) $$[f]_{x,r}^\alpha \leq cr^{-|\alpha|-n} \|f\|_{L^1(B(x,r))} \quad \forall x \in \mathbb{R}^n, r > 0.$$

**Lemma 3.1.** Let $x_0 \in \mathbb{R}^n, 0 < r < +\infty$ and $N \in \mathbb{N}_0$. For every $f \in \mathcal{S}'$ there exists a unique polynomial $P_{x_0,r}^N(f) \in \mathcal{P}_N$ such that

(3.3) $$[f - P_{x_0,r}^N(f)]_{x_0,r}^\alpha = 0 \quad \forall |\alpha| \leq N.$$

**Proof:** Set $L = \binom{n+N}{N}$. Clearly, $\dim \mathcal{P}_N = L$. We define the mapping $T_N: \mathcal{P}_N \to \mathbb{R}^L$, by

$$(T_N Q)_\alpha = [Q]_{x_0,r}^\alpha, \quad |\alpha| \leq N, \quad Q \in \mathcal{P}_N.$$  

In order to prove the assertion of the lemma it will be sufficient to show that $T_N$ is injective, since by $\mathcal{P}_N = L$ this implies, $T_N$ is also surjective. In fact, this can be proved by induction over $N$. In case $N = 0$ we see this by the fact that

$$(T_0 1)_0 = [1]_{x_0,r}^0 = 1.$$

This $T_0$ stands for the identity in $\mathcal{P}_0 \cong \mathbb{R}$. Assume $T_{N-1}$ is injective. Let $Q = \sum_{|\alpha| \leq N} a_\alpha x^\alpha \in \mathcal{P}_N$ such that $T_N(Q) = 0$. Using (3.1), this implies for $|\alpha| = N$

$$0 = [Q]_{x_0,r}^\alpha = \left[ \sum_{|\beta| \leq N} a_\beta D^\beta x^\beta \right]_{x_0,r}^0 = [\alpha! a_\alpha]_{x_0,r}^0 = \alpha! a_\alpha.$$

Here, we used the formula $D^\alpha x^\beta = \alpha! \delta_{\alpha \beta}$ for all $|\beta| \leq N$. Accordingly, $Q \in \mathcal{P}_{N-1}$, and it holds $T_{N-1}(Q) = T_N(Q) = 0$. By our assumption it follows $Q = 0$. This proves that $T_N$ is injective and thus surjective. $ \blacksquare $
Lemma 3.2. 1. Let \( f \in \mathcal{S}' \). Then for all \( |\beta| \leq N \) it holds

\[
P_{x_0,r}^{N-|\beta|}(D^\beta f) = D^\beta P_{x_0,r}^N(f).
\]

2. The mapping \( P_{x_0,r}^N : L^p(B(x_0,r)) \rightarrow \mathcal{P}_N, 1 \leq p \leq +\infty \), defines a projection, i.e.

\[
P_{x_0,r}^N(Q) = Q \quad \forall Q \in \mathcal{P}_N,
\]

\[
\|P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))} \leq c\|P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))} \leq c\|P_{0,1}\|_p \|f\|_{L^p(B(x_0,r))}.
\]

where

\[
\|P_{0,1}\|_p = \sup_{g \in L^p(B(1)) \neq 0} \frac{\|P_{0,1}(g)\|_{L^p(B(1))}}{\|g\|_{L^p(B(1))}} = \frac{\|P_{x_0,r}^N(g)\|_{L^p(B(x_0,r))}}{\|g\|_{L^p(B(x_0,r))}}.
\]

3. For all \( f \in W^{p,j}(B(x_0,r)), 1 \leq p < +\infty, 1 \leq j \leq N + 1 \), it holds

\[
\|f - P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))} \leq c r^j \sum_{|\alpha| = j} \|D^\alpha f - D^\alpha P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))}.
\]

Proof: 1. Let \( \gamma \in \mathbb{N}_0^n \) be a multi index with \( |\gamma| \leq N - |\beta| \). Obviously, \( |\beta + \gamma| \leq N \). From the definition of \( P_{x_0,r}^N \), observing (3.5), and employing (3.1) we find

\[
[P_{x_0,r}^{N-|\beta|}(D^\beta f)]^\gamma_{x_0,r} = [D^\beta f]^\gamma_{x_0,r}
\]

\[
= D^\beta f * D^\gamma \varphi_r(x_0) = f * D^{\beta + \gamma} \varphi_r(x_0)
\]

\[
= [f]_{x_0,r}^{\beta + \gamma} = [P_{x_0,r}^N(f)]_{x_0,r}^{\beta + \gamma} = [D^\beta P_{x_0,r}^N(f)]_{x_0,r}^\gamma.
\]

As we have seen in the proof of Lemma 3.1 the mapping \( T_{N-|\beta|} : \mathcal{P}_{N-|\beta|} \rightarrow \mathcal{P}_{N-|\beta|} \) is injective. This yields (3.4).

2. We show that \( P_{x_0,r}^N \) is a projection, i.e. \( P_{x_0,r}^N(Q) = Q \) for all \( Q \in \mathcal{P}_N \). Indeed, given \( Q \in \mathcal{P}_N \), by the definition of \( P_{x_0,r}^N \) (3.3) it follows that

\[
[Q - P_{x_0,r}^N(Q)]_{x_0,r}^\alpha = 0 \quad \forall |\alpha| \leq N.
\]

Consequently, \( T_N(Q - P_{x_0,r}^N(Q)) = 0 \). Since \( T_N \) is injective we get \( P_{x_0,r}^N(Q) = Q \). The inequality (3.6) can be verified by a standard scaling and translation argument.

3. We prove (3.8) by induction over \( j \). For \( j = 1 \) (3.8) follows from the usual Poincaré inequality, since \( [f - P_{x_0,r}^N(f)]_{x_0,r}^0 = 0 \). Assume (3.8) holds for \( j - 1 \). Thus,

\[
\|f - P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))} \leq c r^{j-1} \sum_{|\alpha| = j-1} \|D^\alpha f - D^\alpha P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))}.
\]

Thanks to (3.3) for all \( |\alpha| = j - 1 \) it holds,

\[
D^\alpha P_{x_0,r}^N(f) = P_{x_0,r}^{N-j+1}(D^\alpha f).
\]

Hence, \( [D^\alpha f - D^\alpha P_{x_0,r}^N(f)]_{x_0,r}^0 = 0 \). An application of the Poincaré inequality gives

\[
\|D^\alpha f - D^\alpha P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))} \leq c r \|DD^\alpha f - DD^\alpha P_{x_0,r}^N(f)\|_{L^p(B(x_0,r))}.
\]

Combining (3.9) and (3.10), we get (3.8).
Remark 3.3. From (3.8) with \( j = N + 1 \) we get the generalized Poincaré inequality

\[
(3.11) \quad \begin{cases}
\| f - P^N_{x_0,r}(f) \|_{L^p(B(x_0,r))} \leq c r^{N+1} \| D^{N+1} f \|_{L^p(B(x_0,r))} \\
\forall f \in W^{N+1,p}(B(x_0,r)).
\end{cases}
\]

Corollary 3.4. For all \( x_0 \in \mathbb{R}^n, 0 < r < +\infty, N \in \mathbb{N}_0, \) and \( 1 \leq p < +\infty \) it holds

\[
(3.12) \quad \| f - P^N_{x_0,r}(f) \|_{L^p(B(x_0,r))} \leq c \inf_{Q \in \mathcal{P}_N} \| f - Q \|_{L^p(B(x_0,r))} = c r^{\frac{N}{p}} \text{osc}(f; x_0, r).
\]

Proof: Let \( Q \in \mathcal{P}_N \) be arbitrarily chosen. In view of (3.9) we find

\[
f - P^N_{x_0,r}(f) = f - Q - P^N_{x_0,r}(f - Q).
\]

Hence, applying triangle inequality, along with (3.6) we get

\[
\| f - P^N_{x_0,r}(f) \|_{L^p(B(x_0,r))} \leq \| f - Q \|_{L^p(B(x_0,r))} + \| P^N_{x_0,r}(f - Q) \|_{L^p(B(x_0,r))}
\]

\[
\leq c \| f - Q \|_{L^p(B(x_0,r))}.
\]

This shows the validity of (3.12). \( \blacksquare \)

In our discussion below and in the sequel of the paper it will be convenient to work with smooth functions. Using the standard mollifier we get the following estimate in \( \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \) for the mollification.

Lemma 3.5. Let \( \varepsilon > 0. \) Given \( f \in \mathcal{S} \), we define the mollification

\[
f_\varepsilon(x) = [f]_{x,\varepsilon}^0 = f \ast \varphi_\varepsilon(x), \quad x \in \mathbb{R}^n.
\]

1. For all \( f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \), and all \( \varepsilon > 0 \) it holds

\[
(3.13) \quad \| f_\varepsilon \|_{\mathcal{L}^{k,s}_{q(p,N)}} \leq c \| f \|_{\mathcal{L}^{k,s}_{q(p,N)}}.
\]

2. Let \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) such that for all \( 0 < \varepsilon < 1, \)

\[
(3.14) \quad \| f_\varepsilon \|_{\mathcal{L}^{k,s}_{q(p,N)}} \leq c_0,
\]

then \( f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \) and it holds \( \| f \|_{\mathcal{L}^{k,s}_{q(p,N)}} \leq c_0. \)

Proof: 1. We may restrict ourselves to the case \( k = 0 \). Let \( x_0 \in \mathbb{R}^n \) and \( j \in \mathbb{Z} \). Set \( 0 < r < +\infty. \) By the definition of \( P^N_{x_0,r}(f) \) (cf. (3.3) ) together with (3.11) it follows that for all \( |\alpha| \leq N \) and for almost all \( y \in \mathbb{R}^n, \)

\[
f \ast D^\alpha \varphi_r(x_0 - y) = [f]_{x_0 - y, r}^\alpha = [P^N_{x_0-y,r}(f)]_{x_0-y, r}^\alpha = P^N_{x_0-y,r}(f) \ast D^\alpha \varphi_r(x_0 - y).
\]

11
Multiplying both sides by \( \varphi_{0,\varepsilon}(y) \), integrate the result over \( \mathbb{R}^n \) and apply Fubini’s theorem, we get for all \( |\alpha| \leq N \)

\[
[f_\varepsilon]_{x_0,r}^\alpha = (f * \varphi_\varepsilon * D^\alpha \varphi_r)(x_0) = (f * D^\alpha \varphi_r * \varphi_\varepsilon)(x_0)
\]

\[
= \int \limits_{\mathbb{R}^n} (f * D^\alpha \varphi_r)(x_0 - y)\varphi_\varepsilon(y)dy \\
= \int \limits_{\mathbb{R}^n} P_{x_0-y,r}^N(f) * D^\alpha \varphi_r(x_0 - y)\varphi_\varepsilon(y)dy \\
= \int \limits_{\mathbb{R}^n} \int \limits_{\mathbb{R}^n} P_{x_0-y,r}^N(f)(x)D^\alpha \varphi_r(x_0 - y - x)\varphi_\varepsilon(y)dxdy \\
= \int \limits_{\mathbb{R}^n} \int \limits_{\mathbb{R}^n} P_{x_0-y,r}^N(f)(x-y)D^\alpha \varphi_r(x_0 - x)\varphi_\varepsilon(y)dxdy \\
= \int \limits_{\mathbb{R}^n} \int \limits_{\mathbb{R}^n} P_{x_0-y,r}^N(f)(x-y)\varphi_\varepsilon(y)d(\varphi)(x_0 - x)d \\
= \left[ \int \limits_{\mathbb{R}^n} P_{x_0-y,r}^N(f)(x-y)\varphi_\varepsilon(y)dy \right]_{x_0,r}^\alpha.
\]

This shows that

\[
(3.15) \quad P_{x_0,r}^N(f_\varepsilon)(x) = \int \limits_{\mathbb{R}^n} P_{x_0-y,r}^N(f)(x-y)\varphi_\varepsilon(y)dy, \quad x \in \mathbb{R}^n,
\]

\[
(3.16) \quad = \int \limits_{\mathbb{R}^n} P_{x_0-\varepsilon y,r}^N(f)(x-\varepsilon y)\varphi(y)dy, \quad x \in \mathbb{R}^n.
\]

Accordingly,

\[
|f_\varepsilon(x) - P_{x_0,2^j}^N(f_\varepsilon)(x)|^p \leq \left( \int \limits_{\mathbb{R}^n} |f(x-\varepsilon y) - P_{x_0-\varepsilon y,2^j}^N(f)(x-\varepsilon y)|\varphi(y)dy \right)^p.
\]

Integration of both sides over \( B(x_0,2^j) \) and multiplication with \( \frac{1}{|B(2^j)|} \), using Jensen’s inequality with respect to the probability measure \( \varphi dy \), we find

\[
\operatorname{osc}_{p,N}^\alpha(f_\varepsilon; x_0, 2^j) \leq \left( \int \limits_{B(x_0,2^j)} \left( \int \limits_{\mathbb{R}^n} |f(x-\varepsilon y) - P_{x_0-\varepsilon y,2^j}^N(f)(x-\varepsilon y)|\varphi(y)dy \right)^p dx \right)^\frac{1}{p}
\]

\[
= \int \limits_{\mathbb{R}^n} \left( \int \limits_{B(x_0-\varepsilon y,2^j)} |f(x) - P_{x_0-\varepsilon y,2^j}(f)(x)|^p dx \right)^\frac{1}{p} \varphi(y)dy
\]

\[
\leq c \int \limits_{B(1)} \operatorname{osc}_{p,N}^\alpha(f; x_0 - \varepsilon y; 2^j) \varphi(y)dy.
\]
Multiplying both sides by $2^{-js}$ applying the $\ell^q$ norm to both sides of the resultant inequality, and using Minkowski’s inequality, we are led to
\[
\left( \sum_{j \in \mathbb{Z}} (2^{-js} \text{osc}(f; x_0, 2^j))^q \right)^{\frac{1}{q}} \leq c \int_{B(1)} \left( \sum_{j \in \mathbb{Z}} (2^{-js} \text{osc}(f; x_0 - \varepsilon y, 2^j))^q \right)^{\frac{1}{q}} \varphi(y) dy
\]
\[
\leq c |f|_{L^q_{\Phi(p,N)}}.
\]
Taking the supremum over all $x_0 \in \mathbb{R}^n$ in the above inequality shows (3.13).

2. Let $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satifying (3.14). This implies that $f \in W^{k,p}_{\text{loc}}(\mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$ and $l, m \in \mathbb{Z}$, $l < m$. According to the absolutely continuity of the Lebesgue measure together with (3.14) it follows
\[
\sum_{j=l}^{m} (2^{-js} \text{osc}(D^k f; x_0, 2^j))^q = \lim_{\varepsilon \downarrow 0} \sum_{j=l}^{m} (2^{-js} \text{osc}(D^k f; x_0, 2^j))^q \leq c_0^q.
\]
This shows that $\{2^{-js} \text{osc}(D^k f; x_0, 2^j)\}_{j \in \mathbb{Z}} \in \ell^q$, and its sum is bounded by $c_0$. Accordingly, $f \in L^q_{\Phi(p,N)}(\mathbb{R}^n)$, and it holds $|f|_{L^q_{\Phi(p,N)}} \leq c_0$.

We are now in a position to prove the following embedding properties. First, let us introduce the definition of the projection to the space of homogeneous polynomial $\hat{P}^N_{x_0,r} : \mathcal{S} \rightarrow \mathcal{P}_N$ defined by means of
\[
\hat{P}^N_{x_0,r}(f)(x) = \sum_{|\alpha|=N} \frac{1}{\alpha!} [f]_{x_0,r}^\alpha, \quad x \in \mathbb{R}^n.
\]
Clearly, for all $f \in \mathcal{S}$ it holds
\[
D^\alpha \hat{P}^N_{x_0,r}(f) = \hat{P}^N_{x_0,r-|\alpha|} (D^\alpha f) \quad \forall \, |\alpha| \leq k.
\]

**Theorem 3.6.** 1. For every $N \in \mathbb{N}_0$ the following embedding holds true
\[
\mathcal{L}^N_{1(p,N)}(\mathbb{R}^n) \hookrightarrow C^{N-1,1}(\mathbb{R}^n) \quad \text{if} \quad N \geq 1
\]
\[
\mathcal{L}^0_{1(p,0)}(\mathbb{R}^n) \hookrightarrow L^{\infty}(\mathbb{R}^n) \quad \text{if} \quad N = 0.
\]
2. For every $f \in \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n)$ there exists a unique $\hat{P}^N_{\infty} \in \mathcal{P}_N$, such that for all $x_0 \in \mathbb{R}^n$
\[
\lim_{r \rightarrow \infty} \hat{P}^N_{x_0,r}(f) = \hat{P}^N_{\infty}(f) \quad \text{in} \quad \mathcal{P}_N.
\]
Furthermore, $\hat{P}^N_{\infty} : \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n) \rightarrow \mathcal{P}_N$ is a projection, with the property
\[
D^\alpha \hat{P}^N_{\infty}(f) = \hat{P}^N_{\infty-|\alpha|} (D^\alpha f) \quad \forall \, |\alpha| \leq N.
\]
3. For all $g, f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ it holds
\[
\hat{P}^1_{\infty}(g \partial_k f) = \hat{P}^1_{\infty}(g) \partial_k \hat{P}^1_{\infty}(f) = \hat{P}^1_{\infty}(g) \hat{P}^0_{\infty}(\partial_k f), \quad k = 1, \ldots, n.
\]
In addition, for \( g \in C^{0,1}(\mathbb{R}^n; \mathbb{R}^n) \), and for all \( f \in \mathcal{L}_{1(p,0)}^0(\mathbb{R}^n) \) it holds

\[
(3.21) \quad \dot{P}_\infty^0(g \partial_k f) := \lim_{r \to \infty} P_{0,r}^0(g \partial_k f) = 0, \quad k = 1, \ldots, n,
\]

where \( g \partial_k f = \partial_k(g f) - \partial_k g f \in \mathcal{S}' \).

4. For all \( v \in \mathcal{L}_{1(p,1)}^1(\mathbb{R}^n; \mathbb{R}^n) \) with \( \nabla \cdot v = 0 \) almost everywhere in \( \mathbb{R}^n \) and \( f \in \mathcal{L}_{1(p,1)}^1(\mathbb{R}^n) \) it holds

\[
(3.22) \quad \dot{P}_\infty^0(\nabla v \cdot \nabla f) = \dot{P}_\infty^0(\nabla v) \cdot \dot{P}_\infty^0(\nabla f).
\]

**Proof:** 1. Let \( \varepsilon > 0 \) be arbitrarily chosen. Let \( f \in \mathcal{L}_{1(p,N)}^N(\mathbb{R}^n) \). Set \( f_\varepsilon = f \ast \varphi_\varepsilon \). By Lemma 3.5 we get \( f_\varepsilon \in \mathcal{L}_{1(p,N)}^N(\mathbb{R}^n) \) and it holds

\[
(3.23) \quad |f_\varepsilon|_{\mathcal{L}_{1(p,N)}^N(\mathbb{R}^n)} \leq c|f|_{\mathcal{L}_{1(p,N)}^N(\mathbb{R}^n)}.
\]

Let \( x_0 \in \mathbb{R}^n \) be fixed. Let \( j \in \mathbb{Z} \). Clearly, \( f_\varepsilon \in C^\infty(\mathbb{R}^n) \). Let \( \alpha \in \mathbb{N}_0^n \) be a multi index with \( |\alpha| = N \). Then

\[
D^\alpha P_{x_0,2^j}^0(f_\varepsilon) = [D^\alpha(f_\varepsilon)]_{x_0,2^j} = D^\alpha \dot{P}_{x_0,2^j}^0(f_\varepsilon).
\]

Let \( m \in \mathbb{Z} \). Since \( D^\alpha f_\varepsilon \) is continuous we have

\[
D^\alpha f_\varepsilon(x) = \lim_{j \to +\infty} [D^\alpha f_\varepsilon]_{x,2^j} \quad \forall x \in \mathbb{R}^n.
\]

Using triangle inequality along with (3.5) and (3.13), and using (3.2), we get

\[
|D^\alpha f_\varepsilon(x) - [D^\alpha f_\varepsilon]_{x,2^{m-1}}| = \sum_{j=-\infty}^{m} |[D^\alpha f_\varepsilon]_{x,2^{j-1}} - [D^\alpha f_\varepsilon]_{x,2^{j}}| \\
\leq \sum_{j=-\infty}^{m} |[D^\alpha f_\varepsilon]_{x,2^{j-1}} - [D^\alpha f_\varepsilon]_{x,2^{j}}| = \sum_{j=-\infty}^{m} |f_\varepsilon^{\alpha}_{x,2^{j-1}} - f_\varepsilon^{\alpha}_{x,2^{j}}| \\
= \sum_{j=-\infty}^{m} |f_\varepsilon - P_{x,2^{j}}^N(f_\varepsilon)|_{x,2^{j-1}} \\
\leq c \sum_{j=-\infty}^{m} 2^{-jN} \text{osc}_{p,N}(f_\varepsilon; x, 2^j) \leq c |f_\varepsilon|_{\mathcal{L}_{1(p,N)}^N(\mathbb{R}^n)} \leq c |f|_{\mathcal{L}_{1(p,N)}^N(\mathbb{R}^n)}.
\]

Thus, \( \{D^N f_\varepsilon\} \) is bounded in \( L^\infty(B(r)) \) for all \( 0 < r < +\infty \). By means of Banach-Alaoglu’s theorem and Cantor’s diagonalization principle we get a sequence \( \varepsilon_k \searrow 0 \) as \( k \to +\infty \) and \( f \in \text{W}^{N,\infty}_{\text{loc}}(\mathbb{R}^n) \), such that for all \( 0 < r < +\infty \)

\[
D^N f_{\varepsilon_k} \to D^N f \text{ weakly-}* \text{ in } L^\infty(B(r)) \text{ as } k \to +\infty.
\]
Furthermore, from (3.24) we get for almost all \( x \in \mathbb{R}^n \) and all \( m \in \mathbb{Z} 
abla
\leq \alpha \right\} x, 2^m \right| \leq c \left| f \right|_{\mathcal{L}^1_{1(p, N)}} + \sum_{|\alpha|=N} \left| f \right|_{x, 2^m}^\alpha. 
\]

Let \( x_0 \in \mathbb{R}^n \) be fixed. We now choose \( m \in \mathbb{Z} \) such that \( 2^{m-1} \leq |x_0| < 2^m \). Then noting \( B(x_0, 2^m) \subset B(2^{m+1}) \), employing (3.5) and (3.2), we get

\[
\left| \left| [f]_{x_0, 2^m}^\alpha - [f]_{L_0, 2^m}^\alpha \right| \right| \leq \left| \left| f \right|_{x_0, 2^m}^\alpha \right| - \left| \left| f \right|_{L_0, 2^m}^\alpha \right| = \left| \left| f - P_{0, 2^m+1}^N \right|_{x_0, 2^m}^\alpha \right| - \left| \left| f - P_{0, 2^m+1}^N \right|_{L_0, 2^m}^\alpha \right| 
\leq c 2^{-mN} \text{osc}(f; 0, 2^{m+1}) \leq c \left| f \right|_{\mathcal{L}^1_{1(p, N)}}.
\]

Similarly, we get for all \( j \in \mathbb{Z} \)

\[
\left| \left| [f]_{L_0, 2^m}^\alpha - [f]_{L_0, 2^j}^\alpha \right| \right| \leq c \left| f \right|_{\mathcal{L}^1_{1(p, N)}}.
\]

Thus, combining the two inequalities we have just obtained, using triangle inequality, we find for all \( j \in \mathbb{Z} \)

\[
\sum_{|\alpha|=N} \left| f \right|_{x, 2^m}^\alpha \leq c \left| f \right|_{\mathcal{L}^1_{1(p, N)}} + \sum_{|\alpha|=N} \left| f \right|_{L_0, 2^j}^\alpha.
\]

This together with (3.25), we infer for all \( j \in \mathbb{Z} \)

\[
\| D^N f \|_\infty \leq c \left| f \right|_{\mathcal{L}^1_{1(p, N)}} + c \sum_{|\alpha|=N} \left| f \right|_{L_0, 2^j}^\alpha \leq c \left| f \right|_{\mathcal{L}^1_{1(p, N)}} + c \| D^N \|_{0, 2/j}^N (f).
\]

This completes the proof of (3.18).

2. Let \( x_0 \in \mathbb{R}^n \). Let \( m, l \in \mathbb{Z}, l < m \). Noting that \( \hat{P}_{x_0, 2^l}^N (Q) = Q \) for all \( Q \in \hat{P}_N \) and \( \hat{P}_{x_0, 2^l}^N (Q) = 0 \) for all \( Q \in \mathcal{P}_{N-1} \), we get the following identity for all \( j, k \in \mathbb{Z} \)

\[
\hat{P}_{x_0, 2^l}^N (P_{x_0, 2^k}^N (f)) = \hat{P}_{x_0, 2^l}^N (f).
\]

Using triangle inequality together with the above identity, (3.2) and (3.12) we estimate

\[
\| \hat{P}_{x_0, 2^l}^N (f) - \hat{P}_{x_0, 2^m}^N (f) \| \leq \sum_{j=l+1}^m \| \hat{P}_{x_0, 2^{j-l}}^N (f) \| - \hat{P}_{x_0, 2^l}^N (f) \| 
\leq c \sum_{j=l+1}^m 2^{-jN-j} \| \hat{P}_{x_0, 2^{j-l}}^N (f) - \hat{P}_{x_0, 2^l}^N (f) \| \| f - P_{x_0, 2^l}^N (f) \|_{L^p(B(x_0, 2^l))} 
\leq c \sum_{j=l+1}^m 2^{-jN} \text{osc}(f; x_0, 2^l). 
\]

15
Owing to $f \in \mathcal{S}^{N}_{1(0,N)}(\mathbb{R}^n)$ the right-hand side of the above inequality tends to zero as $m, l \to +\infty$. This shows that $\{\hat{P}^{N}_{0,2m}(f)\}$ is a Cauchy sequence in $\hat{P}_{N}$ and converges to a unique limit $\hat{P}^{N}_{\infty,x_0}$. We claim that

$$\hat{P}^{N}_{\infty,x_0} = \hat{P}^{N}_{\infty,0} =: \hat{P}^{N}_{\infty}(f).$$

In fact, for $m \in \mathbb{Z}$ such that $|x_0| \leq 2^m$, we obtain

$$\|\hat{P}^{N}_{0,2m}(f) - \hat{P}^{N}_{0,2m+1}(f)\| \leq c2^{-m} \|\hat{P}^{N}_{0,2m}(f) - \hat{P}^{N}_{0,2m+1}(f)\|_{L^p(B(x_0,2^m))} \leq c2^{-m} \text{osc}(f;0,2^{m+1}) \to 0 \quad \text{as} \quad m \to +\infty.$$ 

Consequently, (3.27) must hold. The identity (3.19) is an immediate consequence of (3.17).

3. Now, let $g, f \in \mathcal{S}^{1}_{1(p,1)}(\mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$. We first show that $\{[\partial_\alpha f]_{x_0,2^j}\}_{j \in \mathbb{N}}$, $k \in \{1, \ldots, n\}$, is a Cauchy sequence. Let $j \in \mathbb{N}$ be fixed. We easily calculate,

$$[g\partial_\alpha f]_{x_0,2^{j-1}} - [g\partial_\alpha f]_{x_0,2^j} = \left[ g\partial_\alpha f - P_{x_0,2^j}(g)[\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^{j-1}}^\alpha - \left[ g\partial_\alpha f - P_{x_0,2^j}(g)[\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^j}^\alpha.$$ 

Furthermore, applying integration by parts, we get,

$$\left[ g\partial_\alpha f - P_{x_0,2^j}(g)[\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^{j-1}}^\alpha = \left[ g\partial_\alpha f - P_{x_0,2^j}(g)[\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^{j-1}}^\alpha + \left[ g \cdot (\partial_\alpha f - [\partial_\alpha f]_{x_0,2^j}) \right]_{x_0,2^{j-1}}^\alpha - \int_{\mathbb{R}^n} \left[ g \cdot (\partial_\alpha f - [\partial_\alpha f]_{x_0,2^j}) \right]_{x_0,2^{j-1}}^\alpha d\varphi_{x_0,2^{j-1}} dx$$

$$= - \int_{\mathbb{R}^n} \left[ g\partial_\alpha f - [\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^{j-1}}^\alpha d\varphi_{x_0,2^{j-1}} dx$$

$$= - \int_{\mathbb{R}^n} \left[ g\partial_\alpha f - [\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^{j-1}}^\alpha d\varphi_{x_0,2^{j-1}} dx$$

$$= - \int_{\mathbb{R}^n} \left[ g\partial_\alpha f - [\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^{j-1}}^\alpha d\varphi_{x_0,2^{j-1}} dx$$

This together with (3.12) yields

$$\left[ g\partial_\alpha f - P_{x_0,2^j}(g)[\partial_\alpha f]_{x_0,2^j} \right]_{x_0,2^{j-1}}^\alpha \leq c \|\nabla f\|_{L^\infty} 2^{-j} \text{osc}(v; x_0, 2^j) + c \|\nabla v\|_{L^\infty} 2^{-j} \text{osc}(f; x_0, 2^j).$$
By an analogous reasoning we find
\[
\left[ g\partial_k f - P_{x_0,2^j}^1 (g) [\partial_k f]_{x_0,2^j}^0 \right] \alpha_{x_0,2^j} \leq c\|\nabla f\|_\infty 2^{-j} \text{osc}(v; x_0, 2^j) + c\|\nabla v\|_\infty 2^{-j} \text{osc}(f; x_0, 2^j).
\]

Let \(l, m \in \mathbb{Z}\) with \(l < m\) be arbitrarily chosen. Using triangle inequality together with the two estimates we have just obtained, we estimate
\[
\left| [g\partial_k f]_{x_0,2^l}^\alpha - [g\partial_k f]_{x_0,2^m}^\alpha \right| = \sum_{j=l+1}^m \left| [g\partial_k f]_{x_0,2^{j-1}}^\alpha - [g\partial_k f]_{x_0,2^j}^\alpha \right| \leq c\|\nabla f\|_\infty \sum_{j=l+1}^m 2^{-j} \text{osc}(g; x_0, 2^j) + c\|\nabla g\|_\infty \sum_{j=l+1}^m 2^{-j} \text{osc}(f; x_0, 2^j).
\]

Since \(g, f \in \mathcal{L}_{1(p,1)}(\mathbb{R}^n)\) the right-hand side converges to zero as \(l, m \to +\infty\). Thus, \([g\partial_k f]_{x_0,2^j}^\alpha\) is a Cauchy sequence, and has a unique limit say \(a_{x_0}\). Let \(j \in \mathbb{N}\) such that \(2^j \geq |x_0|\). Thus, \(B(x_0, 2^j) \subset B(2^{j+1})\). By the same reasoning as above we estimate
\[
\left| [g\partial_k f]_{x_0,2^j}^0 - [g\partial_k f]_{x_0,2^{j+1}}^0 \right| = c\|\nabla f\|_\infty 2^{-j} \text{osc}(g; 0, 2^{j+1}) + c\|\nabla g\|_\infty 2^{-j} \text{osc}(f; 0, 2^{j+1}).
\]

Since the right-hand side converges to zero as \(j \to +\infty\) we get \(a_{x_0} = a_0\). Setting \([g\partial_k f]_{x_0,2^j}^\alpha = a_0\), we complete the proof of (3.20).

Next, we prove (3.21). Let \(g \in C^{0,1}(\mathbb{R}^n)\) and \(f \in \mathcal{L}_{1(p,0)}^0(\mathbb{R}^n)\). Applying integration by parts and product rule, we calculate
\[
[g\partial_k f]_{x_0,r}^0 = - \int_{B(x_0,r)} \partial_k g(y)(f(y) - [f]_{x_0,r}^0) \varphi_r(x_0 - y) dy + \int_{B(x_0,r)} g(y)(f(y) - [f]_{x_0,r}^0) \partial_k \varphi_r(x_0 - y) dy.
\]

Applying Hölder’s inequality, we easily get
\[
[g\partial_k f]_{x_0,r}^0 \leq c\|\nabla g\|_\infty \text{osc}(f; x_0, r) + cr^{-1}\|g\|_{L^\infty(B(x_0,r))} \text{osc}(f; x_0, r).
\]

Noting that \(r^{-1}\|g\|_{L^\infty(B(x_0,r))} \leq c\|g(x_0)| + c\|\nabla g\|_\infty\), and using the fact that \(\text{osc}_{p,0}(f; x_0, r) \to 0\) as \(r \to r + \infty\), we obtain (3.21).

It remains to show the identity (3.22). Let \(v \in \mathcal{L}_{1(p,1)}^1(\mathbb{R}^n; \mathbb{R}^n)\) with \(\nabla \cdot v = 0\) and \(f \in \mathcal{L}_{1(p,1)}^1(\mathbb{R}^n)\). Using (3.19) together with \(\nabla \cdot v = 0\) and (3.20), we obtain
\[
[\partial_k v \cdot \nabla f]_{x_0,r}^0 = \partial_j P_{x_0,r}(v_j) f = \partial_j \hat{P}_{x_0,r}^1 ((\partial_k v_j) f) \to \partial_j \hat{P}_{x_0}^1 ((\partial_k v_j) f) = \hat{P}_{x_0}^0(\partial_k v) \cdot \hat{P}_{x_0}^0(\nabla f) \quad \text{as} \quad r \to +\infty.
\]
This shows that
\[
\hat{P}^0(\partial_k v \cdot \nabla f) = \lim_{r \to \infty} [\partial_k v \cdot \nabla f]_{x_0,r}^0 = \hat{P}^0(\partial_k v) \cdot \hat{P}^0(\nabla f).
\]

This completes the proof of the Lemma.

Next, we prove the following norm equivalence which is similar to the properties of the known Campanato space.

**Lemma 3.7.** Let \(1 \leq p < +\infty, 1 \leq q \leq +\infty, \) and \(N, N' \in \mathbb{N}_0, N < N', s \in [-\frac{n}{p}, N+1). \)
If \(f \in L^q_{p,N'}(\mathbb{R}^n), \) and satisfies
\[
(3.28) \quad \lim_{m \to \infty} \hat{P}^L_{0,2m}(D^k f) = 0 \quad \forall L = N + 1, \ldots, N'.
\]
then \(f \in L^q_{p,N}(\mathbb{R}^n)\) and it holds,
\[
(3.29) \quad |f|_{L^q_{p,N}(\mathbb{R}^n)} \leq |f|_{L^q_{p,N'}(\mathbb{R}^n)} \leq c|f|_{L^q_{p,N}(\mathbb{R}^n)}.
\]

**Proof:** We may restrict ourself to the case \(k = 0. \) First, let us prove that for all \(s \in [-\frac{n}{p}, N) \) and for all \(f \in L^q_{p,N}(\mathbb{R}^n)\) such that
\[
(3.30) \quad \lim_{m \to \infty} \hat{P}^N_{0,2m}(f) = 0.
\]

it follows that \(f \in L^q_{p,N-1}(\mathbb{R}^n), \) together with the estimate
\[
(3.31) \quad |f|_{L^q_{p,N-1}(\mathbb{R}^n)} \leq c|f|_{L^q_{p,N}(\mathbb{R}^n)}.
\]

Let \(x_0 \in \mathbb{R}^n, 0 < r < +\infty. \) Noting that \(P^{N}_{x_0,2r}(f) - \hat{P}^{N}_{x_0,2r}(f) \in P_{N-1}, \) we see that
\[
\hat{P}^{N}_{x_0,r}(P^{N}_{x_0,2r}(f)) = \hat{P}^{N}_{x_0,r}(P^{N}_{x_0,2r}(f) - \hat{P}^{N}_{x_0,2r}(f)) + \hat{P}^{N}_{x_0,r}(\hat{P}^{N}_{x_0,2r}(f)) = \hat{P}^{N}_{x_0,2r}(f).
\]

By a scaling argument and triangle inequality we infer
\[
\begin{align*}
r^{-\frac{n}{p}} \|\hat{P}^{N}_{x_0,r}(f)\|_{L^p(B(x_0,r))} & - (2r)^{-\frac{n}{p}} \|\hat{P}^{N}_{x_0,2r}(f)\|_{L^p(B(x_0,2r))} \\
& = \|\hat{P}^{N}_{x_0,r}(f)\| - \|\hat{P}^{N}_{x_0,2r}(f)\| \\
& \leq \|\hat{P}^{N}_{x_0,r}(f) - \hat{P}^{N}_{x_0,2r}(f)\| \\
& \leq c(2r)^{-\frac{n}{p}} \|\hat{P}^{N}_{x_0,2r}(f)\|_{L^p(B(x_0,2r))} \\
& \leq cr^{-N} \text{osc}(f; x_0, 2r).
\end{align*}
\]

Let \(j, m \in \mathbb{Z}, j < m. \) Using the above estimate we deduce that
\[
\begin{align*}
& \left|2^{-jN-j\frac{n}{p}} \|\hat{P}^{N}_{x_0,2^{j}}(f)\|_{L^p(B(x_0,2^{j+1}))} - 2^{-mN-m\frac{n}{p}} \|\hat{P}^{N}_{x_0,2^{m}}(f)\|_{L^p(B(x_0,2^{m}))}\right| \\
& \leq c \sum_{i=j}^{m-1} 2^{-iN} \text{osc}(f; x_0, 2^{i+1}) \\
& \leq c 2^{N} \sum_{i=j}^{m-1} 2^{-iN} \text{osc}(f; x_0, 2^{i}).
\end{align*}
\]
Observing (3.36), we see that
\[
\lim_{m \to \infty} \| 2^{-mN-m} \hat{P}_{x_0,2^m}(f) \|_{L^p(B(x_0,2^m))} = \lim_{m \to \infty} \| \hat{P}_{x_0,2^m}(f) \| = 0.
\]
Thus, letting \( m \to +\infty \) in the above estimate, we arrive at
\[
2^{2jN} \| \hat{P}_{x_0,2^j}(f) \| = 2^{-jN} \| \hat{P}_{x_0,2^j}(f) \|_{L^p(B(x_0,2^j))} \leq c2^{jN} \sum_{i=j}^{\infty} 2^{-iN} \text{osc}(f; x_0, 2^i)
\]
(3.32)
\[
= c \left( S_{N,1}(\text{osc}(f; x_0))^j \right)
\]
where \( \text{osc}_{p,N}(f; x_0) \) stands for a sequence defined as
\[
\text{osc}_{p,N}(f; x_0)_i = \text{osc}_{p,N}(f; x_0, 2^i), \quad i \in \mathbb{Z}.
\]
Using triangle inequality together with (3.32), we obtain
\[
\text{osc}_{p,N-1}(f; x_0, 2^j)
\]
\[
= 2^{-jN} \inf_{P \in \mathcal{P}_{N-1}} \| f - P \|_{L^p(B(x_0,2^j))}
\]
\[
\leq c2^{-jN} \| f - P_{x_0,2^j}(f) + \hat{P}_{x_0,2^j}(f) \|_{L^p(B(x_0,2^j))}
\]
\[
\leq c2^{-jN} \| f - P_{x_0,2^j}(f) \|_{L^p(B(x_0,2^j))} + c2^{-jN} \| \hat{P}_{x_0,2^j}(f) \|_{L^p(B(x_0,2^j))}
\]
\[
\leq c \text{osc}_{p,N}(f; x_0, 2^j) + 2^{iN} \| \hat{P}_{x_0,2^j}(f) \|
\]
(3.33)
\[
\leq c \text{osc}_{p,N}(f; x_0, 2^j) + c \left( S_{N,1}(\text{osc}(f; x_0))^j \right)
\]
Noting that \( \text{osc}_{p,N}(f; x_0, 2^j) \leq S_{N,1}(\text{osc}_{p,N}(f; x_0, 2^j)) \), we infer from (3.33)
\[
\text{osc}_{p,N-1}(f; x_0)_j = \text{osc}_{p,N-1}(f; x_0, 2^j) \leq c \left( S_{N,1}(\text{osc}(f; x_0))^j \right), \quad j \in \mathbb{Z}.
\]
Applying \( S_{s,q} \) to both sides of (3.33), and using Lemma 2.1, we get the inequality
\[
|f|_{\mathcal{L}^{s}_{q,p,N-1}} = \sup_{x_0 \in \mathbb{R}^n} S_{s,q}(\text{osc}_{p,N-1}(f; x_0)) \leq c \sup_{x_0 \in \mathbb{R}^n} S_{s,q}(\text{osc}_{p,N}(f; x_0)) = |f|_{\mathcal{L}^{s}_{q,p,N}},
\]
which implies (3.31). We are now in a position to apply (3.31) iteratively, replacing \( N \) by \( N + 1 \) to get
\[
|f|_{\mathcal{L}^{s}_{q,p,N}} \leq c |f|_{\mathcal{L}^{s}_{q,p,N+1}} \leq \ldots \leq c |f|_{\mathcal{L}^{s}_{q,p,N'}}.
\]
This completes the proof of the lemma.

**Remark 3.8.** For all \( f \in \mathcal{L}^{s}_{q,p,N} (\mathbb{R}^n) \), \( 1 \leq p < +\infty, 1 \leq q \leq +\infty, s \in [-\frac{n}{p}, N + 1) \), the condition (3.28) is fulfilled, and therefore (3.29) holds for all \( f \in \mathcal{L}^{s}_{q,p,N} (\mathbb{R}^n) \) under
the assumptions on $p, q, s, N$ and $N'$ of Lemma 3.7. To verify this fact we observe for $f \in \mathcal{L}^{s}_{q(p,N)}(\mathbb{R}^{n})$ that

$$\text{(3.35)} \quad \sup_{m \in \mathbb{Z}} 2^{-Nm} \text{osc}(f, 0, 2^m) \leq |f|_{\mathcal{L}^{s}_{q(p,N)}}. \quad (3.35)$$

Then for $L \in \mathbb{N}, L > N$, we estimate for multi index $\alpha$ with $|\alpha| = L$

$$|D^{\alpha} P_{0,2^m}^{L}(f)| = |D^{\alpha} P_{0,2^m}^{L}((f - P_{0,2^m}^{N})| \leq c2^{-Lm} \text{osc}(f, 0, 2^m)$$

$$\leq c2^{m(N-L)} |f|_{\mathcal{L}^{s}_{q(p,N)}} \to 0 \quad \text{as} \quad m \to +\infty.$$ 

Hence, (3.28) is fulfilled.

**Remark 3.9.** In case $q = \infty$, since $\mathcal{L}^{s}_{\infty(p,N)}(\mathbb{R}^{n})$ coincides with the usual Campanato space, and Lemma 3.7 is well known (cf. [10, p. 75]).

A careful inspection of the proof of Lemma 3.7 gives the following.

**Corollary 3.10.** Let $N, N' \in \mathbb{N}_{0}, N < N'$. Let $f \in L^{p}_{joc}(\mathbb{R}^{n})$ satisfy (3.28) with $k = 0$. Then, for all $x_{0} \in \mathbb{R}^{n}$ and $j \in \mathbb{Z}$ it holds,

$$\text{(3.36)} \quad \text{osc}_{p,N}(f; x_{0}, 2^{j}) \leq c(S_{N+1,1}(\text{osc}(f; x_{0})))_{j}. \quad (3.36)$$

**Proof:** Set $k = N' - N$. Using (3.34) with $N'$ in place of $N$, we find

$$\text{(3.37)} \quad \text{osc}_{p,N'-1}(f; x_{0}, 2^{j}) \leq c(S_{N',1}(\text{osc}(f; x_{0})))_{j}, \quad j \in \mathbb{Z}. \quad (3.37)$$

Iterating this inequality $k$-times and applying Lemma 2.1, we arrive at

$$\text{osc}_{p,N}(f; x_{0}) = \text{osc}_{p,N'-k}(f; x_{0}) \leq cS_{N+1,1}(S_{N+2,1} \ldots S_{N',1}(\text{osc}(f; x_{0})))$$

$$\leq cS_{N+1,1}(\text{osc}(f; x_{0})). \quad (3.37)$$

Whence, (3.36).

We also have the following growth properties of functions in $\mathcal{L}^{s}_{q(p,N)}(\mathbb{R}^{n})$ as $|x| \to +\infty$

**Lemma 3.11.** Let $N \in \mathbb{N}_{0}$. Let $f \in \mathcal{L}^{s}_{q(p,N)}(\mathbb{R}^{n}), 1 \leq q \leq +\infty, 1 \leq p < +\infty, s \in \left[N, N + 1\right)$.

1. In case $s \in (N, N + 1)$ it holds

$$\text{(3.38)} \quad |f(x)| \leq c(1 + |x|^{n}) ||f||_{\mathcal{L}^{s}_{q(p,N)}} \quad \forall x \in \mathbb{R}^{n}. \quad (3.38)$$

2. In case $s = N$ it holds

$$\text{(3.39)} \quad |f(x)| \leq c(1 + \log(1 + |x|)^{\frac{1}{p'}} |x|^{N}) ||f||_{\mathcal{L}^{N}_{q(p,N)}} \quad \forall x \in \mathbb{R}^{n}. \quad (3.39)$$

Here $q' = \frac{q}{q-1}, c = \text{const} > 0$, depending on $q, p, s, N$ and $n$.\[20\]
Proof: 1. The case $s \in (N, N + 1)$. Let $x_0 \in \mathbb{R}^n$. Let $j \in \mathbb{N}_0$ such that $2^j \leq 1 + |x_0| \leq 2^{j+1}$. Let $\alpha$ be a multi index with $|\alpha| = N$. Verifying that $D^{\alpha} f(x_0) = \lim_{i \to -\infty} D^{\alpha} \hat{P}^N_{x_0,2^i}(f)$, using triangle inequality we find

$$|D^{\alpha} f(x_0)| \leq \sum_{i=-\infty}^{j} |D^{\alpha} \hat{P}^N_{x_0,2^i}(f) - D^{\alpha} \hat{P}^N_{x_0,2^{i-1}}(f)| + |D^{\alpha} \hat{P}^N_{x_0,2^j}(f)|$$

$$\leq c \sum_{i=-\infty}^{j} 2^{-iN} \text{osc}_p,f,0,2^i + |D^{\alpha} \hat{P}^N_{x_0,2^j}(f)|.$$

By the aid of Hölder’s inequality we find

$$\sum_{i=-\infty}^{j} 2^{-iN} \text{osc}_p,f,0,2^i = \sum_{i=-\infty}^{j} 2^{-i(N-s)} 2^{-is} \text{osc}_p,f,0,2^i$$

$$\leq c 2^{j(s-N)} |f| \mathcal{X}^{s,p}_{q,(p,N)} \leq c (1 + |x_0|^{s-N}) |f| \mathcal{X}^{s,p}_{q,(p,N)}.$$

On the other hand,

$$|D^{\alpha} \hat{P}^N_{x_0,2^j}(f)| = |D^{\alpha} \hat{P}^N_{x_0,2^j}(f - P^N_{0,2^{j+1}}(f))| + |D^{\alpha} (P^N_{0,2^{j+1}}(f) - P^N_{0,1}(f))| + |D^{\alpha} P^N_{0,1}(f)|$$

$$\leq 2^{-jN - \frac{n}{p}} \|f - P^N_{0,2^{j+1}}(f)\|_{L^p(x_0,2^{j+1})} + c \sum_{i=0}^{j} 2^{-i(N-s)} 2^{-is} \text{osc}_p,f,0,2^i$$

$$+ c \|f\|_{L^p(B(1))}$$

$$\leq \text{osc}_p,f,0,2^{j+1} + c \sum_{i=0}^{j} 2^{-i(N-s)} 2^{-is} \text{osc}_p,f,0,2^i + c \|f\|_{L^p(B(1))}$$

$$\leq c (1 + |x_0|^{s-N}) \|f\| \mathcal{X}^{s,p}_{q,(p,N)}.$$

Accordingly,

$$\|D^N f(x)\| \leq c (1 + |x|^{s-N}) \|f\| \mathcal{X}^{s,p}_{q,(p,N)}.$$  \hspace{1cm} (3.40)

This implies \((3.38)\).

2. The case $s = N$. Let $x_0 \in \mathbb{R}^n$. As above we choose $j \in \mathbb{N}_0$ such that $2^j \leq 1 + |x_0| < 2^{j+1}$.

In this case we first claim

$$\|D^N \hat{P}^N_{x_0,1}(f)\| \leq (\log (1 + |x_0|))^{\frac{1}{2}} \|f\| \mathcal{X}^{s,p}_{q,(p,N)}.$$  \hspace{1cm} (3.41)

Indeed, arguing as above using triangle inequality along with Hölder’s inequality, we
Let
\[\|D^N \dot{P}^N_{x_0,1}(f)\| \leq \sum_{i=1}^j \|D^N \dot{P}^N_{x_0,2^i}(f)\| - \|D^N \dot{P}^N_{x_0,2^{i-1}}(f)\| + \|D^N \dot{P}^N_{x_0,2^i}(f)\|\]
\[\leq \sum_{i=1}^j 2^{-Ni} \text{osc}_{p,N}(f; x_0, 2^i) + \|D^N \dot{P}^N_{x_0,2^i}(f)\|\]
\[\leq \sum_{i=1}^{j+1} 2^{-Ni} \text{osc}_{p,N}(f; x_0, 2^i) + \|D^N \dot{P}^N_{0,2^{i+1}}(f)\|\]
\[\leq c_2 \frac{1}{2} |f|_\mathcal{L}^N_{q(p,N)} + \|D^N \dot{P}^N_{0,2^{i+1}}(f)\|.
\]

Similarly,
\[\|D^N \dot{P}^N_{0,2^{i+1}}(f)\| \leq c_2 \frac{1}{2} |f|_\mathcal{L}^N_{q(p,N)} + \|D^N \dot{P}^N_{0,2^{i+1}}(f)\|.
\]

Combining the two inequalities we have just obtained, we get (3.41).

Let \(i \in \mathbb{Z}\). Then by triangle inequality together with (3.41) we find
\[
2^{-\frac{n}{p} - iN} \|\dot{P}^N_{x_0,2^i}(f)\|_{L^p(x_0,2^i)} \leq c \|D^N \dot{P}^N_{x_0,2^i}(f)\|
\]
\[
\leq c \sum_{i=l}^1 \left(\|D^N \dot{P}^N_{x_0,2^i}(f)\| - \|D^N \dot{P}^N_{x_0,2^{i-1}}(f)\| + c_2 \|D^N \dot{P}^N_{x_0,1}(f)\|\right)
\]
\[
\leq c \sum_{i=l}^1 \left(\|D^N \dot{P}^N_{x_0,2^i}(f) - D^N \dot{P}^N_{x_0,2^{i-1}}(f)\| + c_2 \|D^N \dot{P}^N_{x_0,1}(f)\|\right)
\]
\[
\leq c \sum_{i=l}^1 2^{-Ni} \text{osc}_{p,N}(f; x_0, 2^i) + c_2 \|D^N \dot{P}^N_{x_0,1}(f)\|
\]
\[
\leq c|i|\frac{1}{2} |f|_\mathcal{L}^N_{q(p,N)} + c(\log(1 + |x_0|))\frac{1}{2} \|f\|_{\mathcal{L}^N_{q(p,N)}}.
\]

This shows that
\[
2^{-i(N-1)} \text{osc}_{p,N^{-1}}(f; x_0, 2^i)
\]
\[
\leq 2^{-i(N-1)} \text{osc}_{p,N}(f; x_0, 2^i) + 2^{-\frac{n}{p}} \|\dot{P}^N_{x_0,2^i}(f)\|_{L^p(x_0,2^i)}
\]
\[
(3.42) \quad \leq 2^{-i(N-1)} \text{osc}_{p,N}(f; x_0, 2^i) + c_2^i \left(i\frac{1}{2} + (\log(1 + |x_0|))\frac{1}{2}\right) \|f\|_{\mathcal{L}^N_{q(p,N)}}.
\]

Summing both sides over \(i = -\infty\) to \(i = 1\) and applying Hölder’s inequality, we get
\[
(3.43) \quad \sum_{i=-\infty}^1 2^{-i(N-1)} \text{osc}_{p,N^{-1}}(f; x_0, 2^i) \leq c \left(1 + (\log(1 + |x_0|))\frac{1}{2}\right) \|f\|_{\mathcal{L}^N_{q(p,N)}}.
\]

Let \(\alpha\) be a multi index with \(|\alpha| = N - 1\). Noting that \(D^\alpha f(x_0) = \lim_{i \to -\infty} D^\alpha \dot{P}^N_{x_0,2^i}(f),\)
Lemma 3.12. Using the Poincaré’s inequality and Lemma 3.7, we get the following embedding.

\[ |D^\alpha f(x_0)| \leq |D^\alpha \hat{P}_{x_0,2^i}^{N-1}(f)| + c \sum_{i=-\infty}^{1} |D^\alpha \hat{P}_{x_0,2^{i-1}}^{N-1}(f)| \]

\[ \leq |D^\alpha \hat{P}_{x_0,2^i}^{N-1}(f)| + c \sum_{i=-\infty}^{1} 2^{-(N-1)i} \text{ osc}_{p,N-1} (f; x_0, 2^i) \]

\[ \leq \|D^{N-1}\hat{P}_{x_0,2^i}^{N-1}(f)\| + c \left( 1 + (\log(1 + |x_0|)^{\frac{1}{p}}) \right) \|f\|_{L^{q(p,N)}}. \]

Arguing as above using triangle inequality, using (3.42), we find

\[ \|D^{N-1}\hat{P}_{x_0,2^i}^{N-1}(f)\| \leq c \sum_{i=0}^{j} 2^{-(N-1)i} \text{ osc}_{p,N-1} (f, x_0, 2^i) + \|D^{N-1}\hat{P}_{x_0,2^i}^{N-1}(f)\| \]

\[ \leq c \sum_{i=0}^{j} 2^{-(N-1)i} \text{ osc}_{p,N-1} (f, x_0, 2^i) + \sum_{i=0}^{j+1} 2^{-(N-1)i} \text{ osc}_{p,N-1} (f, 0, 2^i) \]

\[ + \|D^{N-1}\hat{P}_{x_0,2^i}^{N-1}(f)\| \]

\[ \leq c2^j \|f\|_{L^{q(p,N)}} \leq c(1 + \log(1 + |x_0|)^{\frac{1}{p}}|x_0|) \|f\|_{L^{q(p,N)}}. \]

Combining the above inequalities we obtain

\[ |D^{N-1}f(x_0)| \leq (1 + \log(1 + |x_0|)^{\frac{1}{p}}|x_0|) \|f\|_{L^{q(p,N)}}. \]

This yields (3.39).

Using the Poincaré’s inequality and Lemma 3.7, we get the following embedding.

**Lemma 3.12.** Let \( N \in \mathbb{N}_0, k \in \mathbb{N}_0, 1 < p < +\infty, 1 \leq q \leq +\infty, s \in [N, N+1) \).

1. In case \( q = \infty \) and \( s \notin \mathbb{N} \) it holds

\[ \mathcal{L}^{k,s}_{\infty(p,N)}(\mathbb{R}^n) \cong C^{k+N, s-N}(\mathbb{R}^n). \]  

2. In case \( q = \infty \) and \( s \in \mathbb{N} \) it holds

\[ \mathcal{L}^{k,s}_{\infty(p,N)}(\mathbb{R}^n) \cong BMO_{k+s}(\mathbb{R}^n). \]

where

\[ BMO_N = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) \left| \sup_{j \in \mathbb{Z}} 2^{-Nj} \text{ osc}_{1,N} (f; x_0, 2^j) < +\infty \right. \right\}. \]

3. In case \( 1 \leq q < \infty \) it holds

\[ \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{k+s}_{q(p,N+k)}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{k,s}_{\infty(p,N)}(\mathbb{R}^n). \]
**Proof:** 1. In case $k = 0$ the space $L^s_{\infty(p,N)}(\mathbb{R}^N)$ coincides with the Campanato space $L^s_{\infty,N}(\mathbb{R}^N)$ which is isomorphic to $C^{N,s-N}(\mathbb{R}^N)$ (cf. [10, Chap. III 1.]). In case, $k \geq 1$. For $f \in L^s_{\infty(p,N)}(\mathbb{R}^N)$ we get $D^k f \in L^s_{\infty(p,N)}(\mathbb{R}^N) \cong C^{N,s-N}(\mathbb{R}^N)$, which shows \((3.44)\).

2. In case $k = 0$, and $s = N$ the space $L^s_{\infty(p,N)}(\mathbb{R}^N)$ coincides with the Campanato space $L^p_{\infty,N}(\mathbb{R}^N)$. According to [10, Chap. III 1.] this space coincides with the space $BMO_N$. In case $k \geq 1$ we argue as above to verify \((3.45)\).

3. Let $L^k_{q(p,N)}(\mathbb{R}^N)$. Using Poincaré inequality \((3.8)\), with $j = k$, we find $\text{osc}_{p,N+k}(f; x_0, 2^j) \leq c2^k \text{osc}_{p,N}(D^k f; x_0, 2^j)$. Accordingly,

$$\|\{2^{-(s+k)j} \text{osc}_{p,N+k}(f; x_0, 2^j)\}_{j \in \mathbb{Z}}\|_{l^q} \leq c\|\{2^{-sj} \text{osc}_{p,N}(D^k f; x_0, 2^j)\}_{j \in \mathbb{Z}}\|_{l^q},$$

where

$$\text{osc}_{p,N}(f; x_0) = \{\text{osc}_{p,N}(f; x_0, 2^j)\}_{j \in \mathbb{Z}}.$$

Taking the supremum over all $x_0 \in \mathbb{R}^n$ on both sides of the above estimate, we get the first embedding.

It remains to show the second embedding. To see this we first notice that $L^k_{q(p,N+k)}(\mathbb{R}^n) \hookrightarrow L^k_{\infty(p,N+k)}(\mathbb{R}^n)$. Indeed,

$$2^{-(s+k)j} \text{osc}_{p,N+k}(f; x_0, 2^j) \leq 2^{-sj} \text{osc}_{p,N}(D^k f; x_0, 2^j) \leq |f|_{L^k_{q(p,N+k)}}.$$

Taking the supremum over all $j \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^n$, we get the embedding

$$L^k_{q(p,N+k)}(\mathbb{R}^n) \hookrightarrow L^k_{\infty(p,N+k)}(\mathbb{R}^n).$$

On the other hand, in case $s \in (N, N + 1)$, from \((3.44)\) it follows $L^k_{\infty(p,N+k)}(\mathbb{R}^n) \cong C^{k+N,s-N}(\mathbb{R}^n)$ (cf. [10, Chap. III 1.]). Using Gagliardo-Nirenberg’s inequalities, we can get the interpolation properties. First let us recall the Gagliardo-Nirenberg inequalities.

**Lemma 3.13.** Let $j, N \in \mathbb{N}_0, 0 \leq j < k$. Let $1 \leq p, p_0, p_1 \leq +\infty$, and $\theta \in \left[\frac{j}{n}, 1\right]$, satisfying

$$\frac{1}{p} = \frac{j}{n} + \frac{1 - \theta}{p_0} + \left(\frac{1}{p_1} - \frac{k}{n}\right)\theta.$$ \((3.47)\)

Then, for all $f \in L^{p_0}(B(1)) \cap W^{k,p_1}(B(1))$ it holds

$$\|D^j f\|_{L^p(B(1))} \leq c\|f\|^{1-\theta}_{L^{p_0}(B(1))}\|f\|^{\theta}_{W^{k,p_1}(B(1))}.$$ \((3.48)\)
Notice that, using the generalized Poincaré inequality, under the assumption of Lemma 3.13 for all \( f \in L^{p_0}(B(1)) \cap W^{k,p_1}(B(1)) \), and \( N \in \mathbb{N}_0, N \geq k - 1 \) the following inequality holds

\[ (3.49) \quad \| D^j (f - P_{0,1}^N(f)) \|_{L^p(B(1))} \leq c \| f - P_{0,1}^N(f) \|_{L^{p_0}(B(1))}^{1-\theta} \| D^k f - D^k P_{0,1}^N(f) \|_{L^p(B(1))}^\theta. \]

By a standard scaling and translation argument, we deduce from \((3.49)\) that for all \( x_0 \in \mathbb{R}^n, 0 < r < +\infty, N \in \mathbb{N}_0, N \geq k - 1 \), and for all \( f \in L^{p_0}(B(x_0, r)) \cap W^{k,p_1}(B(x_0, r)) \) the following inequality holds

\[ (3.50) \quad \| D^j f - P_{x_0,r}^N(D^j f) \|_{L^p(B(x_0, r))} = \| D^j (f - P_{x_0,r}^N(f)) \|_{L^p(B(x_0, r))} \leq c \| f - P_{x_0,r}^N(f) \|_{L^{p_0}(B(x_0, r))}^{1-\theta} \| D^k f - P_{x_0,r}^{N-k}(D^k f) \|_{L^{p_1}(B(x_0, r))}^\theta. \]

**Theorem 3.14.** Let \( j, k, N \in \mathbb{N}_0, 0 \leq j < k \leq N + 1 \). Let \( 1 \leq p, p_0, p_1 < +\infty, 1 \leq q, q_0, q_1 \leq +\infty, -\infty < s, s_0, s_1 < N + 1 \), and \( \theta \in \left[ \frac{j}{N}, 1 \right] \), satisfying

\[ (3.51) \quad \frac{1}{p} = \frac{j}{n} + \frac{1-\theta}{p_0} + \frac{1}{p_1} - \frac{k}{n}, \]

\[ (3.52) \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \]

\[ (3.53) \quad s + j = (1-\theta)s_0 + \theta(s_1 + k). \]

Then, for all \( L^{q_0(p_0,N)}(\mathbb{R}^n) \cap L^{k,s_1}(p_1,N) \) \( \mathbb{R}^n \) it holds

\[ (3.54) \quad \| f \|_{L^{q_0(p_0,N)}(\mathbb{R}^n)} \leq c \| f \|_{L^{k,s_1}(p_1,N)}^{1-\theta} \| f \|_{L^{q_0(p_0,N)}}^{\theta}. \]

**Proof:** Observing \((3.51)\) and \((3.52)\), thanks to \((3.50)\) we find

\[
2^{-s l} \text{osc}_{p,N-j} (D^j f; x_0, 2^l) \leq c 2^{-s l (1-\theta)-l s_1 \theta} \text{osc}_{p_0,N} (f; x_0, 2^l)^{1-\theta} \text{osc}_{p_1,N-k} (D^k f; x_0, 2^l)^\theta \\
= c [2^{-s l \theta} \text{osc}_{p_0,N} (f; x_0, 2^l)]^{1-\theta} [2^{-s l \theta} \text{osc}_{p_1,N-k} (D^k f; x_0, 2^l)]^\theta.
\]

According to \((3.53)\), we may apply \( \ell^q \) norm to both sides of the above inequality and use Hölder’s inequality. This gives

\[
\left( \sum_{l \in \mathbb{Z}} (2^{-s l \theta} \text{osc}_{p_0,N} (f; x_0, 2^l))^{q_0} \right)^{\frac{1}{q_0}} \leq c \left( \sum_{l \in \mathbb{Z}} (2^{-s l \theta} \text{osc}_{p_0,N} (f; x_0, 2^l))^{q_0} \right)^{\frac{1}{q_0}} \left( \sum_{l \in \mathbb{Z}} (2^{-s l \theta} \text{osc}_{p_1,N-k} (D^k f; x_0, 2^l))^{q_1} \right)^{\frac{\theta}{q_1}}.
\]

Taking the supremum over all \( x_0 \in \mathbb{R}^n \), we get the assertion \((3.54)\).
Remark 3.15. Consider the special case

\[
\begin{aligned}
N = k, p = p_0 = p_1, \theta = \frac{q}{J}, s = s_0 = s_1 = 0, \\
1 \leq q < +\infty, q_0 = +\infty, q_1 = \frac{qk}{J}.
\end{aligned}
\]

Then, (3.54) reads

\[
\|f\|_{\mathcal{L}^{j,0}_{q(p,k-j)}(\mathbb{R}^n)} \leq c\|f\|_{\mathcal{L}^{j,0}_{\infty(p,k)}}^{1-\frac{q}{J}} \|f\|_{\mathcal{L}^{j,0}_{q(p,k)}}^{\frac{q}{J}} \leq c\|f\|_{BMO}^{-1} \|f\|_{\mathcal{L}^{j,0}_{q(p,k)}}.
\]

Under the assumption that

\[
\lim_{m \to \infty} \hat{P}_{0,2m}^L(D^j u) = 0 \quad \forall \ L = 1, \ldots, k - j,
\]

we estimate the term on the left hand side by the aid of (3.29) with \(N = 0\) and \(N' = k - j\). This yields

\[
\|f\|_{\mathcal{L}^{j,0}_{q(p,k-j)}(\mathbb{R}^n)} \leq c\|f\|_{BMO}^{-1} \|f\|_{\mathcal{L}^{j,0}_{q(p,k)}}.
\]

We are now in a position to prove the following product estimate.

Theorem 3.16. Let \(1 < p < +\infty\). Let \(N \in \mathbb{N}_0\) and \(s \in (-\infty, N + 1)\). Then for all \(f, g \in \mathcal{L}^{p,\infty}_{q(p,N)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), it holds

\[
\|fg\|_{\mathcal{L}^{p,\infty}_{q(p,N)}} \leq c\left(\|f\|_{L^\infty} \|g\|_{\mathcal{L}^{p,\infty}_{q(p,N)}} + \|g\|_{L^\infty} \|f\|_{\mathcal{L}^{p,\infty}_{q(p,N)}}\right).
\]

Proof: Let \(\alpha, \beta \in \mathbb{N}_0^n\) two multi index both are not zero with \(|\alpha + \beta| = k\). Set \(|\alpha| = j\). Using triangle inequality, we see that

\[
\|D^\alpha f D^\beta g - P^{N+k-j}(D^\alpha f)P^{N+j}(D^\beta g)\|_{L^p(B(x_0,r))} \\
\leq c\|((D^\alpha f - P^{N+k-j}(D^\alpha f))(D^\beta g - P^{N+j}(D^\beta g))\|_{L^p(B(x_0,r))} \\
+ c\|((D^\alpha f - P^{N+k-j}(D^\alpha f))P^{N+j}(D^\beta g)\|_{L^p(B(x_0,r))} \\
+ c\|P^{N+k-j}(D^\alpha f)(D^\beta g - P^{N+j}(D^\beta g))\|_{L^p(B(x_0,r))} \\
= I + II + III.
\]

Using Hölder’s inequality together with Gagliardo-Nirenberg’s inequality (3.50), we estimate

\[
I \leq c\|D^\alpha f - P^{N+k-j}(D^\alpha f)\|_{L^p(B(x_0,r))} \|D^\beta g - P^{N+j}(D^\beta g)\|_{L^{\frac{J}{J-1}}(B(x_0,r))} \\
= c\|D^\alpha f - D^\alpha P^{N+k}(f)\|_{L^p(B(x_0,r))} \|D^\beta g - D^\beta P^{N+k}(g)\|_{L^{\frac{J}{J-1}}(B(x_0,r))} \\
\leq c\|D^\beta (f - P^{N+k}(f))\|_{L^p(B(x_0,r))} \|D^{k-j}(g - P^{N+k}(g))\|_{L^{\frac{J}{J-1}}(B(x_0,r))} \\
\leq c\|f - P^{N+k}(f)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \|D^k (f - P^{N+k}(f))\|_{L^p(B(x_0,r))}^{\frac{q}{J}} \times \|g - P^{N+k}(g)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \|D^k (g - P^{N+k}(g))\|_{L^p(B(x_0,r))}^{\frac{q}{J}} \times \\
\leq c\|f\|_{L^\infty(B(x_0,r))} \|D^k f - P^{N+k}(D^k f)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \|g\|_{L^\infty(B(x_0,r))} \|D^k g - P^{N+k}(D^k g)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \times \\
\times \|g\|_{L^\infty(B(x_0,r))} \|D^k g - P^{N+k}(D^k g)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \times \\
\times \|g\|_{L^\infty(B(x_0,r))} \|D^k g - P^{N+k}(D^k g)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \times \\
\times \|g\|_{L^\infty(B(x_0,r))} \|D^k g - P^{N+k}(D^k g)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \times \\
\times \|g\|_{L^\infty(B(x_0,r))} \|D^k g - P^{N+k}(D^k g)\|_{L^p(B(x_0,r))}^{1-\frac{q}{J}} \times \]
Applying Young’s inequality, we obtain
\[
I \leq c \|f\|_{L^\infty(B(x_0,r))} \|D^k g - P_{x_0,r}^N(D^k g)\|_{L^p(B(x_0,r))} + c \|g\|_{L^\infty(B(x_0,r))} \|D^k f - P_{x_0,r}^N(D^k f)\|_{L^p(B(x_0,r))}.
\]

In order to estimate \(II\) we make use of the inequality
\[
\|P_{x_0,r}^{N+j}(D^\beta g)\|_{L^\infty(B(x_0,r))} \leq c r^{-(k-j)} \|g\|_{L^\infty(B(x_0,r))},
\]

which can be proved by a standard scaling argument. Together with Poincaré’s inequality we find
\[
II \leq c r^{k-j} \|D^{k-j}(D^\alpha f - P_{x_0,r}^{N+k-j}(D^\alpha f))\|_{L^p(B(x_0,r))} \|g\|_{L^\infty(B(x_0,r))}
\leq c \|g\|_{L^\infty(B(x_0,r))} \|D^k f - P_{x_0,r}^N(D^k f)\|_{L^p(B(x_0,r))}.
\]

By an analogous reasoning we get
\[
III \leq c \|f\|_{L^\infty(B(x_0,r))} \|D^k g - P_{x_0,r}^N(D^k g)\|_{L^p(B(x_0,r))}.
\]

Inserting the estimates of \(I, II\) and \(III\) into the right-hand side of (3.60), we arrive at
\[
\|D^\alpha f D^\beta g - P_{x_0,r}^{N+k-j}(D^\alpha f) P_{x_0,r}^{N+j}(D^\beta g)\|_{L^p(B(x_0,r))}
\leq c \|f\|_{L^\infty(B(x_0,r))} \|D^k g - P_{x_0,r}^N(D^k g)\|_{L^p(B(x_0,r))}
+ c \|g\|_{L^\infty(B(x_0,r))} \|D^k f - P_{x_0,r}^N(D^k f)\|_{L^p(B(x_0,r))}.
\]

(3.61)

Let \(\gamma \in \mathbb{N}_0\) be a multi index with \(|\gamma| = k\). Using Leibniz formula, we compute
\[
D^\gamma(f g) = \sum_{\alpha + \beta = \gamma} \binom{\gamma}{\alpha! \beta!} D^\alpha f D^\beta g.
\]

Thus, employing Corollary\(3.4\), using triangle inequality together with (3.61), we obtain
\[
\|D^\gamma(f g) - P_{x_0,r}^{2N+k}(D^\gamma(f g))\|_{L^p(B(x_0,r))}
\leq c \inf_{Q \in P_{x_0,r}^{2N+k}} \|D^\gamma(f g) - Q\|_{L^p(B(x_0,r))}
\leq c \|D^\gamma(f g) - \sum_{\alpha + \beta = \gamma} \binom{\gamma}{\alpha! \beta!} P_{x_0,r}^{N+k-j}(D^\alpha f) P_{x_0,r}^{N+j}(D^\beta g)\|_{L^p(B(x_0,r))}
\leq c \|f\|_\infty \|D^k g - P_{x_0,r}^N(D^k g)\|_{L^p(B(x_0,r))}
+ c \|g\|_\infty \|D^k f - P_{x_0,r}^N(D^k f)\|_{L^p(B(x_0,r))}.
\]

This yields the product estimate
\[
(3.62) \quad \text{osc}_{p,2N+k}(D^k (f g); x_0, r) \leq c \|f\|_\infty \text{osc}_{p,N}(D^k g; x_0, r) + c \|g\|_\infty \text{osc}_{p,N}(D^k f; x_0, r).
\]

Into (3.62) we insert \(r = 2^j, j \in \mathbb{Z}\), and multiply this by \(2^{-sj}\). Then, applying the \(\ell^q\) norm to both sides of (3.62), we are led to
\[
(3.63) \quad \|fg\|_{\mathcal{X}_{q(p,2N+k)}^{k,s}} \leq c \left( \|f\|_\infty \|g\|_{\mathcal{X}_{q(p,N)}^{k,s}} + \|g\|_\infty \|f\|_{\mathcal{X}_{q(p,N)}^{k,s}} \right).
\]

Verifying (3.28) holds for \(N' = 2N + k\), we are in a position to apply Lemma\(3.7\) with \(N' = 2N + k\). This gives (3.59).
4 Proof of the main theorems

We start with the following energy identity for solutions to the transport equation. Let $1 < p < +\infty$, $x_0 \in \mathbb{R}$ and $0 < r < +\infty$. We denote $\varphi_{x_0,r} = \varphi(r^{-1}(x_0 - \cdot))$. We define the following minimal polynomial $P_{x_0,r}^N(f), f \in L^p(B(x_0,r))$, by

$$\| (f - P_{x_0,r}^N(f)) \varphi_{x_0,r} \|_p = \min_{Q \in \mathbb{P}_N} \| (f - Q) \varphi_{x_0,r} \|_p. \quad (4.1)$$

The existence and uniqueness of such polynomial is shown in appendix of the paper. We recall the notation $\varphi_{x_0,r} = r^{-n}\varphi(r^{-1}(x_0 - \cdot))$. We have the following.

**Lemma 4.1.** Given $v \in L^1(0,T;C^{0,1}(\mathbb{R}^n;\mathbb{R}^n))$, and $g \in L^1(0,T;L^p_{loc}(\mathbb{R}^n))$, let $f \in L^\infty(0,T;C^{0,1}(\mathbb{R}^n)) \cap C([0,T];L^p_{loc}(\mathbb{R}^n))$ be a weak solution to the transport equation

$$\partial_t f + (v \cdot \nabla)f = g \quad \text{in} \quad Q_T. \quad (4.2)$$

Let $N \in \mathbb{N}_0$. Define,

$$L = \begin{cases} 
2N - 1 & \text{if } N \geq 1 \\
0 & \text{if } N = 0.
\end{cases}$$

Then for all $t \in [0,T]$ it holds

$$e(t) = e(0) + t \int_0^t v \cdot \nabla \varphi_{x_0,r} |f - P_{x_0,r}^L(f)|^p \varphi_{x_0,r}^{p-1} e(\tau)^{1-p} \, dx \, d\tau$$

$$+ \frac{1}{p} \int_0^t \int_{B(x_0,r)} \nabla \cdot v |f - P_{x_0,r}^L(f)|^p \varphi_{x_0,r}^p e(\tau)^{1-p} \, dx \, d\tau$$

$$+ \int_0^t \int_{B(x_0,r)} v \cdot \nabla P_{x_0,r}^L(f) : |f - P_{x_0,r}^L(f)|^{p-2}(f - P_{x_0,r}^L(f)) \varphi_{x_0,r}^{p-2} \, dx \, d\tau$$

$$+ \int_0^t \int_{B(x_0,r)} (g - P_{x_0,r}^N(g)) |f - P_{x_0,r}^L(f)|^{p-2}(f - P_{x_0,r}^L(f)) \varphi_{x_0,r}^{p-2} \, dx \, d\tau$$

$$= e(0) + \int + II + III + IV, \quad (4.3)$$

where

$$e(\tau) = \| (f(\tau) - P_{x_0,r}^L(f(\tau))) \varphi_{x_0,r} \|_p, \quad \tau \in [0,T].$$
In addition, the following inequality holds for all $t \in [0, T]$

$$
\text{osc}_{p,L} \left( f(t); x_0, \frac{r}{2} \right) \leq c \text{osc}_{p,L}(f(0); x_0, r) + c r^{-1} \int_0^t \| v(\tau) \|_{L^\infty(B(x_0,r))} \text{osc}_{p,N}(f(\tau); x_0, 2r) d\tau
$$

$$
+ c \int_0^t \| \nabla \cdot v(\tau) \|_{L^\infty(B(x_0,r))} \text{osc}_{p,N}(f(\tau); x_0, 2r) d\tau
$$

$$
+ \delta_{N_0} c \int_0^t \text{osc}_{p,N}(v(\tau); x_0, r) \| \nabla P^N_{x_0,r}(f(\tau)) \|_{L^\infty(B(x_0,r))} d\tau
$$

$$
(4.4) + c \int_0^t \text{osc}_{p,N}(g(\tau); x_0, r) d\tau,
$$

where $\delta_{N_0} = 0$ if $N = 0$ and $1$ otherwise.

**Proof:** Let $x_0 \in \mathbb{R}^n, 0 < r < +\infty$ be fixed. Let $\delta \geq 0$ we define

$$
F_\delta(z) = (\delta + |z|^2)^{\frac{n-2}{2}} z, \quad z \in \mathbb{R}^n.
$$

Let $N \in \mathbb{N}_0$. Set $L = 0$ if $N = 0$ and $L = 2N - 1$ if $L \geq 1$. For $\delta > 0$ by $P^L_{x_0,r}(f(\tau)) \in \mathcal{P}_L, 0 \leq \tau \leq T,$ we denote the minimal polynomial, defined in the Appendix A. (cf. Lemma A.1), such that

$$
(4.5) \int_{B(x_0,r)} F_\delta(f(\tau) - P^L_{x_0,r}(f(\tau)) \cdot Q_{x_0,r}^p, dx = 0 \quad \forall \tau \in [0, T], \quad \forall Q \in \mathcal{P}_L.
$$

Furthermore, for all $\tau \in [0, T]$ it holds

$$
(4.6) P^L_{x_0,r}(f(\tau)) \to P^L_{x_0,r}(f(\tau)) \text{ in } L^p(B(x_0,r)) \text{ as } \delta \searrow 0.
$$

According to (A.1) the function $s \mapsto P^L_{x_0,r}(f(s))$ is differentiable for $\delta > 0$, and from (4.2) we get

$$
\partial_t(f - P^L_{x_0,r}(f)) + (v \cdot \nabla)(f - P^L_{x_0,r}(f)) + (v \cdot \nabla)P^L_{x_0,r}(f)
$$

$$
(4.7) = g - \partial_t P^L_{x_0,r}(f) \text{ in } Q_T.
$$

First let us verify that $\partial_t P^L_{x_0,r}(f(\tau)) \in \mathcal{P}_L$ for all $\tau \in [0, T]$. In fact, for any multi index $\alpha \in \mathbb{N}_0$ with $|\alpha| = L + 1$, recalling $P^L_{x_0,r}(f) \in \mathcal{P}_L$, we get $D^\alpha \partial_t P^L_{x_0,r}(f) = \partial_t D^\alpha P^L_{x_0,r}(f) = 0$. This shows the claim.

We multiply (4.7) by $F_\delta(f(\tau) - P^L_{x_0,r}(f(\tau))) \gamma^p_{x_0,r}$, integrate over $B(x_0, r)$ and apply
integration by parts. This together with (4.5) yields

$$\partial_t \| \delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p \| (\delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p$$

$$= \frac{1}{p} \partial_t \| (\delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p$$

$$= \int_{B(x_0,r)} v(\tau) \cdot \nabla \varphi_{x_0,r}(\delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p$$

$$+ \int_{B(x_0,r)} \nabla \cdot v(\tau)(\delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p$$

$$+ \int_{B(x_0,r)} v(\tau) \cdot P_{x_0,r}^\delta(f(\tau))F_\delta(f(\tau) - P_{x_0,r}^\delta(f(\tau))) \varphi_{x_0,r} \|_p$$

$$+ \int_{B(x_0,r)} (g(\tau) - P_{x_0,r}^N(g(\tau)))F_\delta(f(\tau) - P_{x_0,r}^\delta(f(\tau))) \varphi_{x_0,r} \|_p.$$

In the last line we used identity (4.5) for $Q = P_{x_0,r}^N(g(\tau))$.

Multiplying both sides of the above identity by $e_\delta(\tau)^1 - p$, where $e_\delta(\tau) := \| (\delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p$, integrating the result over $(0, t), t \in [0, T]$, with respect to $\tau$, and applying integration by parts, we find

$$e_\delta(t) = e_\delta(0) + \int_{0}^{t} \int_{B(x_0,r)} v(\tau) \cdot \nabla \varphi_{x_0,r}(\delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p$$

$$+ \int_{0}^{t} \int_{B(x_0,r)} \nabla \cdot v(\tau)(\delta + |f(\tau) - P_{x_0,r}^\delta(f(\tau))|^2 \frac{2^p}{p} \varphi_{x_0,r} \|_p$$

$$+ \int_{0}^{t} \int_{B(x_0,r)} v(\tau) \cdot P_{x_0,r}^\delta(f(\tau))F_\delta(f(\tau) - P_{x_0,r}^\delta(f(\tau))) \varphi_{x_0,r} \|_p$$

$$+ \int_{0}^{t} \int_{B(x_0,r)} (g(\tau) - P_{x_0,r}^N(g(\tau)))F_\delta(f(\tau) - P_{x_0,r}^\delta(f(\tau))) \varphi_{x_0,r} \|_p.$$

In the above identity, letting $\delta \to 0$ and making use of (4.6), we obtain (4.4).
2. Using the triangle inequality, we estimate

\[
I \leq c \int_0^t \| \nabla \varphi_{x_0,r} \cdot v(\tau) \|_\infty \| f(\tau) - P_{L_{x_0,r}}^* (f(\tau)) \|_{L^p(B(x_0,r))} e(\tau)^{p-1} \| e(\tau) \|^{1-p} d\tau
\]

\[
\leq c \int_0^t \| \nabla \varphi_{x_0,r} \cdot v(\tau) \|_\infty \| f(\tau) - P_{L_{x_0,r}}^* (f(\tau)) \|_{L^p(B(x_0,r))} d\tau
\]

\[
\leq c \int_0^t \| \nabla \varphi_{x_0,r} \cdot v(\tau) \|_\infty \| (f(\tau) - P_{L_{x_0,r}}^* (f(\tau))) \varphi_{x_0,2r} \|_p d\tau
\]

\[
+ c \int_0^t \| \nabla \varphi_{x_0,r} \cdot v(\tau) \|_\infty \| P_{L_{x_0,r}}^* (f(\tau)) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,r))} d\tau = I_1 + I_2.
\]

Thanks to the minimizing property (4.1) we get

\[
I_1 \leq c r^{-1} \int_0^t \| v(\tau) \|_{L^\infty(B(x_0,r))} \| f(\tau) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,2r))} d\tau.
\]

On the other hand, for estimating \( I_2 \), making use of (A.12), we see that for all \( \tau \in [0, T] \),

\[
P_{L_{x_0,r}}^* (f(\tau)) - P_{L_{x_0,2r}}^* (f(\tau)) = P_{L_{x_0,r}}^* (f(\tau) - P_{L_{x_0,r}}^* (f(\tau))) - P_{L_{x_0,2r}}^* (f(\tau) - P_{L_{x_0,2r}}^* (f(\tau))).
\]

This, together with (A.8) and (A.1), yields

\[
\| P_{L_{x_0,r}}^* (f(\tau)) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,r))}
\]

\[
\leq \| P_{L_{x_0,r}}^* (f(\tau) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,2r))} + \| P_{L_{x_0,2r}}^* (f(\tau) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,2r))}
\]

\[
\leq c \| f(\tau) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,2r))}.
\]

Consequently, \( I_2 \) enjoys the same estimate as \( I_1 \), which gives

\[
I \leq c r^{-1} \int_0^t \| v(\tau) \|_{L^\infty(B(x_0,r))} \| f(\tau) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,2r))} d\tau.
\]

Using (A.1), we immediately get

\[
II \leq c \int_0^t \| \nabla \cdot v(\tau) \|_{L^\infty(B(x_0,r))} \| (f(\tau) - P_{L_{x_0,r}}^* (f(\tau))) \varphi_{x_0,r} \|_{L^p(B(x_0,r))} d\tau
\]

\[
\leq c \int_0^t \| \nabla \cdot v(\tau) \|_{L^\infty(B(x_0,r))} \| f(\tau) - P_{L_{x_0,2r}}^* (f(\tau)) \|_{L^p(B(x_0,2r))} d\tau.
\]

31
We proceed with the estimation of $III$. Clearly, in case $N = 0$, since $P_{x, r}^L(f(\tau)) = \text{const}$ for all $\tau \in [0, T]$, the integral $III$ vanishes. Thus, it only remains the case $N > 0$. Let $\tau \in [0, T]$ be fixed. Making use of (4.5) with $\delta = 0$, we find

$$
\int_{B(x, r)} v(\tau) \cdot \nabla P_{x, r}^L(f(\tau)) \cdot |f(\tau) - P_{x, r}^L(f(\tau))|^{p-2}(f(\tau) - P_{x, r}^L(f(\tau))) \varphi_{x, r}^p dx
$$

$$
= \int_{B(x, r)} v(\tau) \cdot \nabla (P_{x, r}^L(f(\tau)) - P_{x, r}^N(f(\tau))) \cdot F_0 f(\tau) - P_{x, r}^L(f(\tau))) \varphi_{x, r}^p dx
$$

$$
+ \int_{B(x, r)} v(\tau) \cdot \nabla P_{x, r}^N(f(\tau)) \cdot F_0 (f(\tau) - P_{x, r}^L(f(\tau))) \varphi_{x, r}^p dx
$$

$$
= \int_{B(x, r)} v(\tau) \cdot \nabla (P_{x, r}^L(f(\tau)) - P_{x, r}^N(f(\tau))) \cdot F_0 (f(\tau) - P_{x, r}^L(f(\tau))) \varphi_{x, r}^p dx
$$

$$
+ \int_{B(x, r)} (v(\tau) - P_{x, r}^N(v(\tau))) \cdot \nabla P_{x, r}^N(f(\tau)) \cdot F_0 (f(\tau) - P_{x, r}^L(f(\tau))) \varphi_{x, r}^p dx
$$

$$
= J_1 + J_2.
$$

Using the fact that $P_{x, r}^L(Q) = P_{x, r}^N(Q) = Q$ for all $Q \in P_N$, we get with $Q = P_{x, r}^N(f(\tau))$ for all $\tau \in (0, t)$

$$
\|\nabla (P_{x, r}^L(f(\tau)) - P_{x, r}^N(f(\tau)))\|_{L^p(B(x, r))} \leq c r^{-1}\|f(\tau) - P_{x, r}^N(f(\tau))\|_{L^p(B(x, r))}. 
$$

Then Hölder’s inequality yields

$$
J_1 \leq c r^{-1}\|v(\tau)\|_{L^\infty(B(x, r))} \|f(\tau) - P_{x, r}^N(f(\tau))\|_{L^p(B(x, r))} \epsilon(\tau)^{p-1}.
$$

Similarly,

$$
J_2 \leq c \|v(\tau) - P_{x, r}^N(v(\tau))\|_{L^p(B(x, r))} \|\nabla P_{x, r}^N(f(\tau))\|_{L^\infty(B(x, r))} \epsilon(\tau)^{p-1}.
$$

Inserting the estimates of $J_1$ and $J_2$ into the integral of $III$, we obtain

$$
III \leq c r^{-1} \int_0^t \|v(\tau)\|_{L^\infty(B(x, r))} \|f(\tau) - P_{x, r}^N(f(\tau))\|_{L^p(B(x, r, 2r))} d\tau
$$

$$
+ c \int_0^t \|v(\tau) - P_{x, r}^N(v(\tau))\|_{L^p(B(x, r))} \|\nabla P_{x, r}^N(f(\tau))\|_{L^\infty(B(x, r))} d\tau.
$$

To estimate $IV$, we use Hölder’s inequality. This leads to

$$
IV \leq \int_0^t \|g(\tau) - P_{x, r}^N(g(\tau))\|_{L^p(B(x, r))} d\tau.
$$
Inserting the estimates of $I, II, III$ and $IV$ into the right-hand side of (4.3), we find

$$e(t) \leq e(0) + cr^{-1} \int_0^t \|v(\tau)\|_{L^\infty(B(x_0,r))} \|f(\tau) - P_{x_0,2r}^N f(\tau)\|_{L^p(B(x_0,2r))} d\tau$$

$$+ c \int_0^t \|\nabla \cdot v(\tau)\|_{L^\infty(B(x_0,r))} \|f(\tau) - P_{x_0,2r}^N f(\tau)\|_{L^p(B(x_0,2r))} d\tau$$

$$+ c \int_0^t \|v(\tau) - P_{x_0,r}^N (v(\tau))\|_{L^p(B(x_0,r))} \|\nabla P_{x_0,r}^N (f(\tau))\|_{L^\infty(B(x_0,r))} d\tau$$

$$+ c \int_0^t \|g(\tau) - P_{x_0,r}^N (g(\tau))\|_{L^p(B(x_0,r))} d\tau.$$  
(4.8)

Noting that

$$\|f(t) - P_{x_0,\frac{t}{2}}^L (f(t))\|_{L^p(B(x_0,\frac{t}{2}))} \leq c \|f(t) - P_{x_0,r}^{L,*} (f(t))\|_{L^p(B(x_0,r))} = c e(t),$$

and using (A.1), recalling that $L = 2N - 1$, the inequality (4.4) follows from (4.8).  
\[\blacksquare\]

**Remark 4.2.** Given $v \in L^1(0,T; C^{0,1}(\mathbb{R}^n; \mathbb{R}^n))$, and $\pi \in L^1(0,T; W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n))$, let $f \in L^\infty(0,T; C^{0,1}(\mathbb{R}^n; \mathbb{R}^n))$ with $\nabla \cdot f = 0$ be a weak solution to the system

$$\partial_t f + (v \cdot \nabla) f = -\nabla \pi \quad \text{in} \quad Q_T.$$  
(4.9)

Then, repeating the proof of Lemma 4.1 for the case $p = 2$ and $N = 1$ in the vector valued case, we find

$$e(t) = e(0) + \int_0^t v \cdot \nabla \varphi_{x_0,r} |f - P_{x_0,r}^{1,*} (f)|^2 \varphi_{x_0,r} e(\tau)^{-1} dx d\tau$$

$$+ \frac{1}{2} \int_0^t \int_{B(x_0,r)} \nabla \cdot v (f - P_{x_0,r}^{1,*} (f) \varphi_{x_0,r}^2 e(\tau)^{-1} dx d\tau$$

$$+ \int_0^t \int_{B(x_0,r)} v \cdot \nabla P_{x_0,r}^{1,*} (f) \cdot (f - P_{x_0,r}^{1,*} (f)) \varphi_{x_0,r}^2 e(\tau)^{-1} dx d\tau$$

$$+ \int_0^t \int_{B(x_0,r)} (\nabla \pi - P_{x_0,r}^1 (\nabla \pi)) (f - P_{x_0,r}^{1,*} (f)) \varphi_{x_0,r}^2 e(\tau)^{-1} dx d\tau$$

$$= e(0) + I + II + III + IV,$$

where

$$e(\tau) = \|(f(\tau) - P_{x_0,r}^{1,*} (f(\tau))) \varphi_{x_0,r}\|_2, \quad \tau \in [0, T].$$

33
The integrals $I, II$ and $III$ can be estimated as in the proof of Lemma 4.1. For the estimation of $IV$ we proceed as follows.

Assume that the mollifier $\varphi \in C^\infty_c(B(1))$ is radial symmetric. Let $u \in L^1(B(x_0, r))$. It can be checked easily that the minimal polynomial $P_{x_0,r}^1(u)$ is given by

$$P_{x_0,r}^1(u)(x) = \frac{1}{\int_{\mathbb{R}^n} \varphi_{x_0,r}^2 dy} \int_{\mathbb{R}^n} u \varphi_{x_0,r}^2 dy + \int_{\mathbb{R}^n} \varphi_{x_0,r}^2 \left| x_0 - y \right|^2 dy \int_{\mathbb{R}^n} u \varphi_{x_0,r}^2 (y_i - x_0,i) dy(x_i - x_0,i).$$

In case $u = (u_1, \ldots, u_n)$ with $\nabla \cdot u = 0$ almost everywhere in $B(x_0, r)$, recalling that $\varphi$ is radialsymmetric, by Gauss’ theorem we get

$$\nabla \cdot P_{x_0,r}^1(u)(x) = \frac{n}{\varphi_{x_0,r}^2} \int_{B(x_0,r)} u \cdot (y - x_0) \varphi_{x_0,r}^2 dy = 0.$$

Using integration by parts together with $\nabla \cdot P_{x_0,r}^1(f(\tau)) = 0$, and applying Sobolev-Poincaré inequality, we get

$$\int_{B(x_0, r)} (\nabla \pi(\tau) - P_{x_0,r}^1(\nabla \pi(\tau))(f(\tau) - P_{x_0,r}^1(f(\tau)))\varphi_{x_0,r}^2 e(\tau)^{-1} dx$$

$$= -2 \int_{B(x_0, r)} (\pi(\tau) - P_{x_0,r}^2(\pi(\tau))(f(\tau) - P_{x_0,r}^1(f(\tau)))\varphi_{x_0,r} \cdot \nabla \varphi_{x_0,r} e(\tau)^{-1} dx$$

$$\leq cr^{-1} \left( \int_{B(x_0, r)} \left| \nabla \pi(\tau) - P_{x_0,r}^1(\nabla \pi(\tau)) \right|^{2n} \frac{\pi^2}{n+2} dx \right)^{\frac{1}{2n}} \leq c r^\frac{2}{2n+2} \frac{\text{osc} (\nabla \pi(\tau); x_0, r)}{\pi^{n+2}}.$$

This yields

$$IV \leq cr^\frac{2}{2n+2} \int_0^t \frac{\text{osc} (\nabla \pi(\tau); x_0, r)}{\pi^{n+2}} d\tau.$$

Inserting the estimates of $I, II, III$ and $IV$ into the right-hand side of (4.10), and arguing as in the proof of Lemma 4.1, we arrive at

$$\text{osc}_{2,1} \left( f(t); x_0, \frac{r}{2} \right) \leq c \text{osc}_{2,1}(f(0); x_0, r) + cr^{-1} \int_0^t \left\| v(\tau) \right\|_{L^\infty(B(x_0,r))} \text{osc}_{2,1}(f(\tau); x_0, 2r) d\tau$$

$$+ c \int_0^t \left\| \nabla \cdot v(\tau) \right\|_{L^\infty(B(x_0,r))} \text{osc}_{2,1}(f(\tau); x_0, 2r) d\tau$$

$$+ c \int_0^t \text{osc}_{2,1}(v(\tau); x_0, r) \left\| \nabla P_{x_0,r}^1(f(\tau)) \right\| d\tau$$

(4.11)$$+ c \int_0^t \frac{\text{osc}_{2,1} (\nabla \pi(\tau); x_0, r)}{\pi^{n+2}} d\tau.$$
Proof of the main theorems

1. Existence and uniqueness in terms of particle trajectories. Assume \( f_0 \in L^{s}_{q(p,N)}(\mathbb{R}^n) \), \( g \in L^{1}(0,T;L^{s}_{q(p,N)}(\mathbb{R}^n)) \), and \( \nabla v \in L^{1}(0,T;L^\infty(\mathbb{R}^n)) \). Let \( (x,t) \in Q_{T} \) be fixed. By \( X_t(x,\cdot) \) we denote the unique solution to the ODE

\[
\frac{d}{d\tau}X_t(x,\tau) = v(X_t(x,\tau),\tau), \quad \tau \in [0,T], \quad X_t(x,t) = x,
\]

which is ensured by Carathéodory’s theorem. We define the flow map \( \Phi_{t,\tau} : \mathbb{R}^n \to \mathbb{R}^n \) by means of

\[
\Phi_{t,\tau}(x) = X_t(x,\tau), \quad x \in \mathbb{R}^n, \quad \tau, t \in [0,T].
\]

By the uniqueness of this flow we get the inverse formula

\[
\Phi_{t,\tau}^{-1}(x) = \Phi_{\tau,t}(x).
\]

Furthermore, from (4.12) we deduce that

\[
\frac{d}{d\tau}\Phi_{t,\tau}(x) = v(\Phi_{t,\tau}(x),\tau), \quad \tau \in [0,T], \quad \Phi_{t,t}(x) = x.
\]

Let \( (x,t) \in Q_{T} \). We set \( y = \Phi_{t,0}(x) \), which is equivalent to \( x = \Phi_{0,t}(y) \). We define \( f \) by means of

\[
f(x,t) = f_0(y) + \int_{0}^{t} g(\Phi_{0,s}(y),s)ds.
\]

Recalling that \( f(t) \) is Lipschitz for almost all \( t \in (0,T) \), we see that \( f \) is differentiable with respect to time almost everywhere in \( (0,T) \). Recalling the inverse formula, it holds \( x = \Phi_{0,t}(y) \). Consequently, for \( y \in \mathbb{R}^n \) fixed we get from (4.14)

\[
f(\Phi_{0,t}(y),t) = f_0(y) + \int_{0}^{t} g(\Phi_{0,s}(y),s)ds \quad \forall t \in (0,T).
\]

Differentiating (4.15) with respect to \( t \), and observing (4.13), we obtain

\[
\partial_{t}f(\Phi_{0,t}(y),t) + (v(\Phi_{0,t}(y),t) \cdot \nabla)f(\Phi_{0,t}(y),t) = g(\Phi_{0,t}(y),t).
\]

This shows that \( f \) solves (1.1) in \( Q_{T} \). In addition, verifying that \( \Phi_{0,0}(x) = x \), we get from (4.15)

\[
f(x,0) = f_0(x) \quad \forall x \in \mathbb{R}^n.
\]

This solution is also unique. In fact, assume there is another solution \( \tilde{f} \) solves (1.1). Setting \( w = f - \tilde{f} \), then \( w \) solves (1.1) with homogenous data. In other words for every \( y \in \mathbb{R}^n \) the function \( Y(t) = w(\Phi_{0,t}(y),t) \) solves the ODE

\[
\dot{Y} = 0, \quad Y(0) = 0,
\]

35
which implies \( Y \equiv 0 \), and thus \( w(\Phi_{0,t}(y),t) = 0 \). With \( y = \Phi_{t,0}(x) \) we get \( w(x,t) = 0 \) for all \((x,t) \in Q_T\).

2. **Growth of the solution as \(|x| \to +\infty\).** Applying \( \nabla_x \) to both sides of (4.13), and using the chain rule, we find that

\[
(4.17) \quad \frac{d}{d\tau} \nabla \Phi_{s,\tau}(x) = \nabla v(\Phi_{s,\tau}(x), \tau) \cdot \nabla \Phi_{s,\tau}(x).
\]

Integration with respect to \( \tau \) over \((s,t)\) yields

\[
\nabla \Phi_{s,t}(x) = I + \int_s^t \nabla v(\Phi_{s,\tau}(x), \tau) \cdot \nabla \Phi_{s,\tau}(x) d\tau,
\]

where \( I \) stands for the unit matrix. Thus, for all \( s,t \in (0,T) \),

\[
|\nabla \Phi_{s,t}(x)| \leq 1 + \int_s^t \|\nabla v(\tau)\|_{\infty} |\nabla \Phi_{s,\tau}(x)| d\tau.
\]

By means of Gronwall’s lemma it follows that for all \( s,t \in (0,T) \)

\[
(4.18) \quad |\nabla \Phi_{s,t}(x)| \leq \exp \left( \int_s^t \|\nabla v(\tau)\|_{\infty} d\tau \right).
\]

From the definition (4.12) we deduce that

\[
\nabla f(x,t) = \nabla f_0(\Phi_{t,0}(x)) \cdot \nabla \Phi_{t,0}(x) + \int_0^t \nabla g(\Phi_{0,\tau}(\Phi_{t,0}(x)), \tau) d\tau
\]

\[
= \nabla f_0(\Phi_{t,0}(x)) \cdot \nabla \Phi_{t,0}(x)
\]

\[
+ \int_0^t \nabla g(\Phi_{0,\tau}(\Phi_{t,0}(x)), \tau) \cdot \nabla \Phi_{0,\tau}(\Phi_{t,0}(x)) \cdot \nabla \Phi_{t,0}(x) d\tau.
\]

Thus, in case \( \nabla f_0 \in L^\infty(\mathbb{R}^n) \) and \( g \in L^1(0,T; L^\infty(\mathbb{R}^n)) \), in view of (4.18) we get for all \( t \in (0,T) \)

\[
(4.19) \quad \|\nabla f(t)\|_{\infty} \leq \left( \|\nabla f_0\|_{\infty} + \int_0^T \|\nabla g(\tau)\|_{\infty} \right) \exp \left( 2 \int_0^T \|\nabla v(\tau)\|_{\infty} d\tau \right).
\]

Using integration by parts, from (4.13) we get for all \( s,t \in (0,T) \)

\[
\Phi_{s,t}(x) - x = \Phi_{s,t} - \Phi_{s,s}(x) = \int_s^t v(\Phi_{s,\tau}(y), \tau) - v(0,\tau) d\tau + \int_s^t v(0,\tau) d\tau.
\]
This leads to the inequality
\[
|\Phi_{s,t}(x)| \leq |x| + \int_0^T |v(0, \tau)| d\tau + \int_{s}^{t} \|\nabla v(\tau)\|_{\infty} |\Phi_{s,\tau}(y)| d\tau.
\]

By means of Gronwall’s lemma we find for all \(s, t \in (0, T)\)
\[
|\Phi_{s,t}(x)| \leq \left(|x| + \int_0^T |v(0, \tau)| d\tau\right) \exp\left(\int_0^T \|\nabla v(\tau)\|_{\infty} d\tau\right) \leq c(1 + |x|).
\]

Let \(x \in \mathbb{R}^n\) and \(t \in (0, T)\). In case \(N = 0, s \in [0, 1)\), using Lemma 3.11, we get
\[
|f_0(x)| \leq c(1 + |x^s|) \|f_0\|_{L^s_{q(p,N)}},
\]
\[
|g(x, \tau)| \leq c(1 + |x^s|) \|g(\tau)\|_{L^s_{q(p,N)}}.
\]

In case \(N = 1, s = 1\) and \(1 < q \leq \infty\) we get by Lemma 3.11
\[
|f_0(x)| \leq c(1 + \log(1 + |x^s|) \frac{1}{q'} |x^s|) \|f_0\|_{L^s_{q(p,N)}},
\]
\[
|g(x, \tau)| \leq c(1 + \log(1 + |x^s|) \frac{1}{q'} |x^s|) \|g(\tau)\|_{L^s_{q(p,N)}}
\]

with \(q' = \frac{q}{q-1}\). In the remaining cases having \(\nabla f_0 \in L^\infty(\mathbb{R}^n)\) and \(\nabla g \in L^1(0, T; L^\infty(\mathbb{R}^n))\), we find,
\[
|f_0(x)| \leq c(1 + |x^s|) (\|f_0\|_{L^s_{q(p,N)}} + \|\nabla f_0\|_{\infty}),
\]
\[
|g(x, \tau)| \leq c(1 + |x^s|) (\|g(\tau)\|_{L^s_{q(p,N)}} + \|\nabla g(\tau)\|_{\infty}).
\]

Setting \(y = \Phi_{t,0}(x)\), we get from (4.15)
\[
|f(x, t)| \leq |f_0(y)| + \int_0^t |g(\Phi_{0,s}(y), s)| ds
\]

Employing (4.21)–(4.26) together with (4.20), we see that for all \((x, t) \in Q_T\)
\[
|f(x, t)| \leq c \begin{cases} 
(1 + |x|^{\min\{s, 1\}}) & \text{if } s \neq 1, \\
(1 + \log(1 + |x|) \frac{1}{q'} |x^s|) & \text{if } s = 1,
\end{cases}
\]

where \(c\) stands for a constant depending on \(s, q, p, N, n\) and \(f_0, g\) and \(v\).

3. Local energy estimation. Let \(x_0 \in \mathbb{R}^n\). Let \(\xi \in C^2([0, T]; \mathbb{R}^n)\) be a solution to the ODE
\[
\dot{\xi}(\tau) = v(x_0 + \xi(\tau), \tau) \quad \tau \in [0, T].
\]
We set
\[ F(x, \tau) = f(x + \xi(\tau), \tau), \quad V(x, \tau) = v(x + \xi(\tau), \tau) - \dot{\xi}(\tau), \quad G(x, s) = g(x + \xi(\tau), \tau), \quad (x, s) \in Q_T. \]

It is readily seen that \( V \) solves the transport equation
\[ \partial_t F + (V \cdot \nabla) F = G \quad \text{in} \quad Q_T. \]

In particular, from (4.28) we infer
\[ V(x_0, \tau) = 0 \quad \forall \tau \in [0, T]. \]

Set \( L = 2N - 1 \) if \( N > 0 \) and \( L = 0 \) if \( N = 0 \). According to (4.4) of Lemma 4.1 with \( r = 2^{j+1}, j \in \mathbb{Z} \), noting that in view of (4.30) it holds \( 2^{-j}\|V(\tau)\|_{L^\infty(B(x_0, 2^{j+1}))} \leq c\|\nabla v(\tau)\|_\infty \), we find
\[
\text{osc}_{p,L}^{p,L}(F(t); x_0, 2^j) \leq c \text{osc}_{p,L}^{p,L}(f_0(\cdot + \xi(0)); x_0, 2^{j+1})
\]
\[
+ c \int_0^t \|\nabla v(\tau)\|_\infty \text{osc}_{p,N}^{p,N}(F(\tau); x_0, 2^{j+2}) d\tau
\]
\[
+ \delta_{N_0} c \int_0^t \text{osc}_{p,N}^{p,N}(V(\tau); x_0, 2^{j+1}) \|\nabla P^{2}_{x_0,\tau}(F(\tau))\|_{L^\infty(B(x_0, 2^{j+1}))} d\tau
\]
\[
+ c \int_0^t \text{osc}_{p,N}^{p,N}(G(\tau); x_0, 2^{j+1}) d\tau,
\]
where \( \delta_{N_0} = 0 \) if \( N = 0 \) and 1 otherwise.

**Proof of (1.6) in Theorem 1.1** Inequality (4.31) gives
\[
\text{osc}_{p,0}^{p,0}(F(t); x_0, 2^j) \leq c \text{osc}_{p,0}^{p,0}(F(0); x_0, 2^{j+1}) + c \int_0^t \|\nabla v(\tau)\|_\infty \text{osc}_{p,0}^{p,0}(F(\tau); x_0, 2^{j+2}) d\tau
\]
\[
+ c \int_0^t \text{osc}_{p,0}^{p,0}(G(\tau); x_0, 2^{j+1}) d\tau.
\]

Observing (4.27), since \( s < 1 \), we get \( S_{1,1}(\text{osc}_{p,0}(f(\tau); x_0)) < +\infty \). Thus, applying \( S_{1,1} \)
to both sides of (4.37), we obtain

\[
S_{1,1}(\text{osc}(F(t); x_0))
\]

\[
\leq c S_{1,1}(\text{osc}(F(0); x_0)) + c \int_0^t \|\nabla v(\tau)\|_\infty S_{1,1}(\text{osc}(F(\tau); x_0)) d\tau
\]

\[
(4.33)
\]

\[
+ c \int_0^t S_{1,1}(\text{osc}(G(\tau); x_0)) d\tau.
\]

Applying Gronwall’s lemma, we deduce from (4.40)

\[
\text{osc}(F(t); x_0)
\]

\[
\leq S_{1,1}(\text{osc}(F(t); x_0))
\]

\[
\leq c \left\{ S_{1,1}(\text{osc}(F(0); x_0)) + \int_0^t S_{1,1}(\text{osc}(G(\tau); x_0)) d\tau \right\} \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty d\tau \right).
\]

(4.34)

Let \( t \in [0, T] \). Clearly, the constant in (4.34) is independent of the choice of the characteristic for \( \xi \). Therefore, we may choose \( \xi \) such that \( \xi(t) = 0 \), which implies \( F(t) = f(t) \). Hence, we may replace \( F(t) \) by \( f(t) \) on the left-hand side of (4.34). Afterwards, with the help of Lemma 2.1 we are in a position to operate \( S_{s,q} \) to both sides of (4.34), verifying \( F(0) = f_0(\cdot - \xi(0)) \), that yields

\[
(S_{s,q}(\text{osc}(f(t); x_0)))_j
\]

\[
\leq c \left\{ (S_{s,q}(\text{osc}(f_0(\cdot - \xi(0)); x_0)))_j
\right.
\]

\[
+ \int_0^t (S_{s,q}(\text{osc}(G(\tau); x_0))))_j d\tau \right\} \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty d\tau \right).
\]

(4.35)

Multiplying both sides by \( 2^{-js} \), we get

\[
\left( \sum_{i=j}^{\infty} (2^{-si} \text{osc}(f(t); x_0; 2^j))_q \right)^{\frac{1}{q}}
\]

\[
\leq c \left\{ |f_0|_{L^q(f_0)} + \int_0^t |G(\tau)|_{L^q(f_0)} d\tau \right\} \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty d\tau \right).
\]

(4.36)

Passing \( j \to -\infty \) and taking the supremum over \( x_0 \in \mathbb{R}^n \) in (4.36), we get (1.6). 

**Proof of (1.9) in Theorem 1.2** Recalling that \( V(x_0, \tau) = 0 \) for all \( \tau \in [0, T] \), we see that \( 2^{-j} \|V(\tau)\|_{L^\infty(B(x_0,2^{j+1}))} \leq c \|\nabla v(\tau)\|_\infty \) and \( 2^{-j} \text{osc}(V(\tau); x_0, 2^{j+1}) \leq c \|\nabla v(\tau)\|_\infty \).
Thus, (4.15) leads to
\[
\begin{align*}
\text{osc}_{p,1}(F(t); x_0, 2^j) \\
\leq c \text{osc}_{p,1}(F(0); x_0, 2^{j+1}) + c \int_0^t \|\nabla v(\tau)\|_{\infty} \text{osc}_{p,1}(F(\tau); x_0, 2^{j+2})d\tau \\
+ c \int_0^t \text{osc}_{p,1}(V(\tau); x_0, 2^{j+1})|\nabla \dot{P}_{x_0,2^{j+1}}^1(F(\tau))|d\tau \\
+ c \int_0^t \text{osc}_{p,1}(G(\tau); x_0, 2^{j+1})d\tau.
\end{align*}
\]
(4.37)

In case \(j \geq 0\), using triangle inequality, we get
\[
|\nabla \dot{P}_{x_0,2^j}^1(F(\tau))| \leq c \sum_{i=0}^j 2^{-i} \text{osc}_{p,1}(F(\tau); x_0, 2^i) + |\nabla \dot{P}_{x_0,1}^1(F(\tau))|
\leq c 2^{-j} (S_{3,1}(\text{osc}_{p,1}(F(\tau); x_0)))_j + |\nabla \dot{P}_{x_0,1}^1(F(\tau))|.
\]

In case \(j < 0\), using triangle inequality along with Hölder’s inequality, we find
\[
|\nabla \dot{P}_{x_0,2^j}^1(F(\tau))| \leq c \sum_{i=0}^j 2^{-i} \text{osc}_{p,1}(F(\tau); x_0, 2^i) + |\nabla \dot{P}_{x_0,1}^1(F(\tau))|
\leq \left(-j \right)^{\frac{\alpha}{p}} \left(\sum_{i=j}^0 2^{-iq} (\text{osc}_{p,1}(F(\tau); x_0, 2^i))^q\right)^{\frac{1}{q}} + |\nabla \dot{P}_{x_0,1}^1(F(\tau))|.
\]

Summing up the above estimates, we arrive at
\[
\begin{align*}
\text{osc}_{p,1}(V(\tau); x_0, 2^{j+1})|\nabla \dot{P}_{x_0,2^{j+1}}^1(F(\tau))| \\
\leq 2^{-j} \text{osc}_{p,1}(V(\tau); x_0, 2^{j+1}) (S_{3,1}(\text{osc}_{p,1}(F(\tau); x_0)))_j \\
+ c(j^-)^{\frac{\alpha}{p}} \text{osc}_{p,1}(V(\tau); x_0, 2^{j+1}) \left\{ \left(\sum_{i=-\infty}^0 2^{-iq} (\text{osc}_{p,1}(F(\tau); x_0, 2^i))^q\right)^{\frac{1}{q}} + |\nabla \dot{P}_{x_0,1}^1(F(\tau))| \right\},
\end{align*}
\]
(4.38)
where \(j^- = - \min\{j, 0\}\). Applying the operator \(S_{2,1}\) to the both sides of the above inequality, and making use of Lemma 2.1 with \(p = q = 1, \alpha = 3\) and \(\beta = 2\), we obtain
\[
\begin{align*}
S_{2,1}\left\{ \left\{ \text{osc}_{p,1}(V(\tau); x_0, 2^{i+1})|\nabla \dot{P}_{x_0,2^{i+1}}^1(F(\tau))| \right\} \right\} \\
\leq c|v(\tau)|_{\mathcal{X}_{\beta q(p,1)}^1} S_{2,1}(\text{osc}_{p,1}(F(\tau); x_0)) \\
+ cS_{2,1}\left\{ \left\{ \left(-i\right)^{\frac{\alpha}{q}} \text{osc}_{p,1}(V(\tau); x_0, 2^i) \right\} \left\{ \left(\sum_{i=-\infty}^0 2^{-iq} (\text{osc}_{p,1}(F(\tau); x_0, 2^i))^q\right)^{\frac{1}{q}} + |\nabla \dot{P}_{x_0,1}^1(F(\tau))| \right\} \right\},
\end{align*}
\]
(4.39)
Observing (1.27), all sum in the above estimates are finite. Again appealing to (1.20), we are in a position to apply $S_{2,1}$ to both sides of (1.37) to get

$$S_{2,1}(\text{osc}(F(t); x_0))$$

$$\leq cS_{2,1}(\text{osc}(F(0); x_0)) + c \int_0^t (\| \nabla v(\tau) \|_\infty + |v(\tau)|_{L^q_{\infty}})S_{2,1}(\text{osc}(F(\tau); x_0))d\tau$$

$$+ c \int_0^t S_{2,1}\left(\left\{ (i^-)^{\frac{q}{q-1}} \text{osc}(V(\tau); x_0, 2^i) \right\}\right) \left( \sum_{i = -\infty}^0 2^{-iq}(\text{osc}(F(\tau); x_0, 2^i))^q \right)^{\frac{1}{q}} + |\nabla \dot{F}_{x_0,1}(F(\tau))| d\tau$$

(4.40)\hspace{1cm} + c \int_0^t S_{2,1}(\text{osc}(G(\tau); x_0))d\tau.

Applying Gronwall’s lemma, we are led to

$$\text{osc}(F(t); x_0)$$

$$\leq S_{2,1}(\text{osc}(F(t); x_0))$$

$$\leq \left\{ cS_{2,1}(\text{osc}(f_0(\cdot + \xi(0); x_0))$$

$$+ c \int_0^t S_{2,1}\left(\left\{ (i^-)^{\frac{q}{q-1}} \text{osc}(V(\tau); x_0, 2^i) \right\}\right) \left( \sum_{i = -\infty}^0 2^{-iq}(\text{osc}(F(\tau); x_0, 2^i))^q \right)^{\frac{1}{q}} + c|\nabla \dot{F}_{x_0,1}(F(\tau))| d\tau$$

(4.41)\hspace{1cm} + c \int_0^t S_{2,1}(\text{osc}(G(\tau); x_0))d\tau \right\} \exp \int_0^T (\| \nabla v(\tau) \|_\infty + |v(\tau)|_{L^q_{\infty}})d\tau.$$

Observing (1.7), using Lemma 2.1, we may apply $S_{1,q}$ to both sides of (4.41). Accordingly,

$$\sup_{t \in [0, T]} S_{1,q}(\text{osc}(F(t); x_0)) < +\infty.$$

For given $t \in [0, T]$ we may choose $\xi$ such $\xi(t) = 0$. Thus, the same holds for $f(t)$ in place of $F(t)$. Now, we are able to apply $S_{1,q}$ to both sides of (4.38), which yields

$$S_{1,q}\left(\left\{ \text{osc}(V(\tau); x_0, 2^{i+1})|\nabla \dot{F}_{x_0,2^{i+1}}(F(\tau))| \right\}\right)$$

$$\leq c|v(\tau)|_{L^q_{\infty}}S_{1,q}(\text{osc}(F(\tau); x_0))$$

$$+ cS_{1,q}\left(\left\{ (i^-)^{\frac{q}{q-1}} \text{osc}(V(\tau); x_0, 2^i) \right\}\right) \left( \sum_{i = -\infty}^0 2^{-iq}(\text{osc}(F(\tau); x_0, 2^i))^q \right)^{\frac{1}{q}} + |\nabla \dot{F}_{x_0,1}(F(\tau))|.$$
we infer
\[
\left( \sum_{i=-\infty}^{\infty} 2^{-i q} (\text{osc}(F(t); x_0, 2^i))^q \right)^{\frac{1}{q}} \leq c |f_0|_{\mathcal{Q}^{1}_{q(p,1)}} + c \int_{0}^{t} \|\nabla v(\tau)\|_{\infty} \left( \sum_{i=-\infty}^{\infty} 2^{-i q} (\text{osc}(F(t); x_0, 2^i))^q \right)^{\frac{1}{q}} d\tau + c \int_{0}^{t} \left( \sum_{i=-\infty}^{\infty} (i^{-1} 2^{-i} \text{osc}(V(\tau); x_0, 2^i))^q \right)^{\frac{1}{q}} \left( \sum_{i=-\infty}^{\infty} 2^{-i q} (\text{osc}(F(\tau); x_0, 2^i))^q \right)^{\frac{1}{q}} + |\nabla \hat{P}_{x_0,1}^1(F(\tau))| d\tau.
\]

(4.43)

Next, we require to estimate $|\nabla \hat{P}_{x_0,1}^1(F(\tau))|$ by the initial data $f_0$ and $g$. We apply $\hat{P}_{x_0,1}^1$ to both sides (4.29). This gives

\[
\left( \begin{array}{c}
\partial_t \hat{P}_{x_0,1}^1(F) + \hat{P}_{x_0,1}^1(V \cdot \nabla F)
\end{array} \right) = \hat{P}_{x_0,1}^1(G) \quad \text{in} \quad Q_T.
\]

Noting that $\hat{P}_{x_0,1}^1(P_{x_0,1}^1(V) \cdot \nabla \hat{P}_{x_0,1}^1(F)) = P_{x_0,1}^1(V) \cdot \nabla \hat{P}_{x_0,1}^1(F)$, and applying $\nabla$ to both sides of (4.44), we infer

\[
\frac{d}{dt} \nabla \hat{P}_{x_0,1}^1(F) + (\nabla \hat{P}_{x_0,1}^1(V)) \cdot \nabla \hat{P}_{x_0,1}^1(F)
\]
\[
= \nabla \hat{P}_{x_0,1}^1 \left( P_{x_0,1}^1(V) \cdot \nabla \hat{P}_{x_0,1}^1(F) - V \cdot \nabla F \right) + \nabla \hat{P}_{x_0,1}^1(G) \quad \text{in} \quad [0, T].
\]

(4.45)

On the other hand,

\[
\nabla \hat{P}_{x_0,1}^1 \left( P_{x_0,1}^1(V) \cdot \nabla \hat{P}_{x_0,1}^1(F) - V \cdot \nabla F \right)
\]
\[
= \nabla \hat{P}_{x_0,1}^1 \left( (P_{x_0,1}^1(V) - V) \cdot \nabla \hat{P}_{x_0,1}^1(F) \right) + \nabla \hat{P}_{x_0,1}^1 \left( V \cdot \nabla (\hat{P}_{x_0,1}^1(F) - F) \right)
\]
\[
= \nabla \hat{P}_{x_0,1}^1 \left( (P_{x_0,1}^1(V) - V) \cdot \nabla \hat{P}_{x_0,1}^1(F) \right) + \nabla \cdot V (P_{x_0,1}^1(F) - F).
\]

Inserting this identity into the right-hand side of (4.45), multiplying the result by $\frac{\nabla \hat{P}_{x_0,1}^1(F)}{|\nabla \hat{P}_{x_0,1}^1(F)|}$, we get the following differential inequality

\[
\frac{d}{dt} |\nabla \hat{P}_{x_0,1}^1(F)| \leq c \|\nabla v\|_{\infty} |\nabla \hat{P}_{x_0,1}^1(F)| + c \|\nabla v\|_{\infty} \text{osc}(F, x_0; 1) + |\nabla \hat{P}_{x_0,1}^1(G)|.
\]
Integrating this inequality over \((0, t)\) and applying integration by parts, we obtain

\[
|\nabla \hat{P}_{x_0,1}(F(t))| \leq |\nabla \hat{P}_{x_0,1}(F(0))| + c \int_0^t |\nabla v(\tau)| \|\nabla \hat{P}_{x_0,1}(F(\tau))| d\tau
\]

\[
+ \int_0^t |\nabla v(\tau)| \|\text{osc}(F(\tau), x_0; 1)\| d\tau + \int_0^t |\nabla \hat{P}_{x_0,1}(G(\tau))| d\tau
\]

\[
\leq \|f_0\|_{\overline{\mathcal{L}}_q(p, 1)} + c \int_0^t |\nabla v(\tau)| \|\text{osc}(F(\tau), x_0; 1)\| d\tau
\]

\[
+ \int_0^t \|g(\tau)\|_{\overline{\mathcal{L}}_q(p, 1)} d\tau; \tag{4.46}
\]

where \(|z|_{\overline{\mathcal{L}}_q(p, 1)}\) stands for the semi norm

\[
|z|_{\overline{\mathcal{L}}_q(p, 1)} = |z|_{\overline{\mathcal{L}}_q(p, 1)} + \sup_{x_0 \in \mathbb{R}^n} |\nabla \hat{P}_{x_0,1}(z)|.
\]

Combining (4.43) and (4.46), we arrive at

\[
\left( \sum_{i=\infty}^{-\infty} 2^{-iq(\text{osc}(F(t); x_0, 2^i))} \right)^{\frac{1}{q}} + |\nabla \hat{P}_{x_0,1}(F(t))|
\]

\[
\leq c |f_0|_{\overline{\mathcal{L}}_q(p, 1)} + c \int_0^t |\nabla v(\tau)| \left( \sum_{i=\infty}^{-\infty} 2^{-iq(\text{osc}(F(t); x_0, 2^i))} \right)^{\frac{1}{q}} d\tau
\]

\[
+ c \int_0^t \left( \sum_{i=\infty}^{-\infty} (i^{-q})^{q-1}(2^{-iq(\text{osc}(V(\tau); x_0, 2^i))})^{\frac{q}{q}} \{ \left( \sum_{i=\infty}^{-\infty} 2^{-iq(\text{osc}(F(\tau); x_0, 2^i))} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} d\tau
\]

\[
+ |\nabla \hat{P}_{x_0,1}(F(\tau))| d\tau + c \int_0^t |g(\tau)|_{\overline{\mathcal{L}}_q(p, 1)} d\tau. \tag{4.47}
\]

Applying Gronwall’s lemma and for given \(t \in [0, T]\) choosing \(\xi\) such that \(\xi(t) = 0\), and taking the supremum over \(x_0 \in \mathbb{R}^n\), we obtain the desired estimate \((1.9)\). \(\blacksquare\)

**Proof of (1.11) in Theorem 1.3** We first define

\[
\chi(x_0, t) = \sup_{j \in \mathbb{Z}} 2^{-j} \text{osc}(F(t); x_0, 2^j), \quad (x_0, t) \in \mathbb{R}^n \times [0, T].
\]

Clearly, thanks to (1.27) \(\chi(x_0, t)\) is finite. Noting that \(\|\nabla P_{x_0,2^{j+1}}^N(F(\tau))\|_{L^\infty(B(x_0,2^{j+1}))} \leq 43\)
Thus, we apply (4.31) with $L = 2N - 1$
\[
osc_{p,2N-1} (F(t); x_0, 2^j) \leq c osc_{p,N} (F(0); x_0, 2^{j+1})
\]
\[
+ c \int_0^t \| \nabla v(\tau) \|_{\infty} osc_{p,N} (F(\tau); x_0, 2^{j+2}) d\tau
\]
\[
(4.48)
\]
\[
+ c 2^{-j} \int_{p,N} osc(V(\tau); x_0, 2^{j+1}) \chi(x_0, \tau) d\tau + c \int_{p,N} osc(G(\tau); x_0, 2^{j+1}) d\tau.
\]

First let us estimate the term $osc_{p,0} (F(t); x_0, 2^{j+1})$. In view of (4.37) with $j + 1$ in place of $j$, and recalling that $\nabla f_0 \in L^{\infty}(\mathbb{R}^n), g \in L^1(0, T; L^\infty(\mathbb{R}^n))$, we see that
\[
osc_{p,0} (F(t); x_0, 2^{j+1})
\]
\[
(4.49)
\]
\[
\leq c 2^j \| \nabla f_0 \|_{\infty} + c \int_0^t \| \nabla v(\tau) \|_{\infty} osc_{p,0} (F(\tau); x_0, 2^{j+3}) d\tau + c \int_0^t 2^j \| \nabla g(\tau) \|_{\infty} d\tau.
\]

Multiplying both sides of (4.49) by $2^{-j}$ and taking the supremum over all $j \in \mathbb{Z}$, using the triangle inequality, we obtain
\[
\chi(x_0, t) \leq c \| \nabla f_0 \|_{\infty} + c \int_0^t \| \nabla v(\tau) \|_{\infty} \chi(x_0, \tau) d\tau + c \int_0^t \| \nabla g(\tau) \|_{\infty} d\tau,
\]
(4.50)

Thanks to (4.27) we have $S_{N+1,1}(osc_{p,N} F(t); x_0) < +\infty$ for all $t \in [0, T]$. Applying $S_{N+1,1}$ to both sides of (4.48), and using Corollary 3.10 with $N' = 2N - 1$, we get
\[
osc_{p,N} (F(t); x_0) \leq S_{N+1,1}(osc_{p,2N-1} (F(t); x_0))
\]
\[
\leq c S_{N+1,1}(osc_{p,N} (F(0); x_0)) + c \int_0^t \| \nabla v(\tau) \|_{\infty} S_{N+1,1}(osc_{p,N} (F(\tau); x_0)) d\tau
\]
\[
+ c \int_0^t S_{N+1,1}(osc_{p,N} (V(\tau); x_0)) \chi(x_0, \tau) d\tau
\]
\[
(4.51)
\]
\[
+ c \int_0^t S_{N+1,1}(osc_{p,N} (G(\tau); x_0)) d\tau.
\]

Next, once more using (4.27) we see that $S_{s,q}(osc_{p,N} (F(t); x_0)) < +\infty$, for all $t \in [0, T]$. Thus, we apply $S_{s,q}$ to both sides of (4.51) and use Lemma 2.4. This combined with
\[ (4.50) \text{ gives } \]
\[ 2^{-j^s}(S_{s,q}(\text{osc}(F(t); x_0)))_j + \chi(x_0, t) \]
\[ \leq c|f_0|_{L^{s}_{q(p,1)}} + \chi(x_0, 0) \]
\[ + c \int_0^t (|v(\tau)|_{L^{s}_{q(p,1)}} + \|\nabla v(\tau)\|_\infty) \left[ 2^{-j^s}(S_{s,q}(\text{osc}(F(t); x_0)))_j + \chi(x_0, \tau) \right] d\tau \]
\[ + c \int_0^t (|g(\tau)|_{L^{s}_{q(p,1)}} + \|\nabla g(\tau)\|_\infty) d\tau. \]
\[ (4.52) \]

By virtue of Gronwall’s lemma we deduce from (4.52)
\[ 2^{-j^s}(S_{s,q}(\text{osc}(F(t); x_0)))_j + \chi(x_0, t) \]
\[ \leq c \left\{ |f_0|_{L^{s}_{q(p,1)}} + \|\nabla f_0\|_\infty \right\} \exp \left( \int_0^t (|v(\tau)|_{L^{s}_{q(p,1)}} + \|\nabla v(\tau)\|_\infty) d\tau \right). \]
\[ (4.53) \]

Whence, (1.11).\[\square\]

**Proof of (1.12) in Corollary 1.4**  In view of Theorem 3.6 we have \( \nabla f_0 \in L^\infty(\mathbb{R}^n) \), \( \nabla g \in L^1(0, T; L^\infty(\mathbb{R}^n)) \). More precisely, (3.26) yields
\[ \|\nabla f_0\|_\infty \leq c \|f_0\|_{L^1_{q(p,1)}}, \quad \int_0^T \|\nabla g(\tau)\|_\infty d\tau \leq c \|g\|_{L^1(0,T;L^1_{q(p,1)})}. \]

In particular, this shows that condition (1.8) of Theorem 1.2 is fulfilled. Furthermore, since \( v \in L^1(0, T; L^1_{p,1}(\mathbb{R}^n)) \), condition of Theorem 1.2 (1.7) is also satisfied. Now, we are in a position to apply of Theorem 1.2 which yields \( f \in L^\infty(0, T; L^1_{p,1}(\mathbb{R}^n)) \). This allows to apply \( S_{1,1} \) to both sides of (4.31). This together with Gronwall’s Lemma and the inequality \( \|\nabla P^1_{x_0,2t+1}(F(\tau))\| \leq c 2^{-j}\|\nabla f(\tau)\|_\infty \) yields
\[ (S_{1,1}(\text{osc}(F(t); x_0)))_j \]
\[ \leq c \left\{ (S_{1,1}(\text{osc}(F(0); x_0)))_j + \int_0^t (S_{1,1}(\text{osc}(G(\tau); x_0)))_j d\tau \right. \]
\[ + \int_0^t (S_{1,1}(\text{osc}(V(\tau); x_0)))_j \|\nabla f(\tau)\|_\infty d\tau \] \[ \left. \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty d\tau \right) \right\}. \]
\[ (4.54) \]
Choosing $\xi$ so that $\xi(t) = 0$, multiplying both sides by $2^{-j}$ and letting $j \to -\infty$ taking the supremum over $x_0 \in \mathbb{R}^n$, we deduce from (4.54)

$$
|f(t)|_{L^1_{1(p,1)}} 
\leq c \left\{ \|f_0\|_{L^1_{1(p,1)}} + \int_0^t \|g(\tau)\|_{L^1_{1(p,1)}} d\tau 
+ \int_0^t \|v(\tau)\|_{L^1_{1(p,1)}} \|\nabla f(\tau)\|_\infty d\tau \right\} \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty d\tau \right).
$$

(4.55)

Combining (4.55) and (4.19) along with (4.27) in order to estimate $\|f(t)\|_{L^p(B(1))}$, we get the desired estimate (1.12).

Below we prove the uniqueness parts of Theorem 1.1, Theorem 1.2, Theorem 1.3 and Corollary 1.4. In fact we prove the stronger version of it, namely the strong-weak uniqueness.

**Strong-weak uniqueness.** Let $\overline{f} \in L^2_{loc}(\mathbb{R}^n)$ be a weak solutions to (1.1). Then $w = f - \overline{f}$ solves the transport equation with homogenous data

$$
\partial_t w + (v \cdot \nabla) w = 0 \quad \text{in} \; Q_T, \quad w = 0 \quad \text{on} \; \mathbb{R}^n \times \{0\}
$$

in a weak sense, i.e. for all $t \in (0, T)$, and for all $\varphi \in L^\infty(0, t; W^{1,2}(\mathbb{R}^n)) \cap W^{1,1}(0, t; L^2(\mathbb{R}^n))$ with supp$(\varphi) \subset \mathbb{R}^n \times [0, t]$, it holds

$$
- \int_0^t \int_{\mathbb{R}^n} w \partial_t \varphi + (v \cdot \nabla) \varphi w + \nabla \cdot v \varphi w dxds = - \int_{\mathbb{R}^n} w(t) \varphi(t) dx.
$$

(4.57)

Let $\psi \in C^\infty_c(\mathbb{R}^n)$ be a given function. Using the method of characteristics, for every $\varepsilon > 0$ we get a solution $\varphi^\varepsilon \in L^\infty(0, t; W^{1,2}(\mathbb{R}^n)) \cap W^{1,1}(0, t; L^2(\mathbb{R}^n))$ of the following dual problem

$$
\partial_t \varphi^\varepsilon + v \cdot \nabla \varphi^\varepsilon + \nabla \cdot v \varphi^\varepsilon = 0 \quad \text{in} \; Q_t, \quad \varphi^\varepsilon(t) = \psi \quad \text{in} \; \mathbb{R}^n.
$$

(4.58)

Noting that $\|\nabla v(\tau)\|_\infty \leq \|\nabla v(\tau)\|_\infty$, using Gronwall’s lemma we see that $\|\varphi^\varepsilon\|_1 + \|\varphi^\varepsilon\|_\infty \leq c$ with a constant $c > 0$ independent of $\varepsilon > 0$. Since $v(0, \cdot), \|\nabla v(\cdot)\|_\infty \in L^1(0, T)$ using (1.20), we get a number $0 < R < +\infty$ such that supp$(\varphi^\varepsilon) \subset B(R) \times [0, t]$. In (4.57) putting $\varphi = \varphi^\varepsilon$, and using (4.58), we infer

$$
\int_{\mathbb{R}^n} w(t) \psi dx = \int_0^t \int_{\mathbb{R}^n} w \partial_t \varphi^\varepsilon + (v \cdot \nabla) \varphi^\varepsilon w + \nabla \cdot v \varphi^\varepsilon w dxds
$$

$$
= \int_0^t \int_{B(R)} \nabla \cdot (v - v^\varepsilon) \varphi^\varepsilon w dxds.
$$

(4.59)
Noting that $\nabla \cdot (v(s) - v_\varepsilon(s)) \to 0$ in $L^2(B(R))$ as $\varepsilon \searrow 0$ for almost all $s \in (0,t)$, by the aid of Vitali’s convergence theorem ([9, p. 180]) it follows that

$$\int_0^t \int_{B(R)} \nabla \cdot (v - v_\varepsilon)\varphi^\varepsilon wdxds \to 0 \quad \text{as} \quad \varepsilon \searrow 0.$$

Letting $\varepsilon \searrow 0$ in (4.59), we deduce that

$$\int_{\mathbb{R}^n} w(t)\psi dx = 0. \quad \text{Whence,} \quad w \equiv 0. \quad \text{This shows the uniqueness.}$$

\section{A Minimal polynomials}

Let $< p < +\infty$. Let $x_0 \in \mathbb{R}^n$ and $0 < r < +\infty$ be fixed. Set $\phi = \varphi(r^{-1}(x_0 - \cdot))$, where $\varphi \in C_c^\infty(B(1))$, being radial symmetric, stands for the standard mollifier. For $\delta \geq 0$ we define the following functional $J_\delta : L^p(B(x_0,r)) \to \mathbb{R}$ by

$$J_\delta(f) = \int_{B(x_0,r)} (\delta + |f|^2)^{\frac{p}{2}}\phi^p dx, \quad f \in L^p(B(x_0,r)).$$

Recall $\mathcal{P}_N$, $N \in \mathbb{N}_0$, denotes the space of all polynomial of degree less or equal $N$. Since $J_\delta$ is strict convex and lower semi continuous with $J_\delta(f) \to +\infty$ as $\|f\|_{L^p(B(x_0,r))} \to +\infty$. For each $f \in L^p(B(x_0,r))$ there exists a unique $P^{x_0,\delta}_{x_0,\varepsilon}(f) \in \mathcal{P}_N$ with

(A.1) \quad $J_\delta(P^{x_0,\delta}_{x_0,\varepsilon}(f) - f) = \min_{P \in \mathcal{P}_N} J_\delta(P - f)$

Clearly, the mapping $J_{\delta,f} : P \mapsto J_\delta(P - f)$ is differentiable as a function from $\mathcal{P}_N$ into $\mathbb{R}$. Since the first variation must vanish at each minimizer, we get

(A.2) \quad $\langle DJ_{\delta,f}(P^{x_0,\delta}_{x_0,\varepsilon}(f), P) \rangle = 0 \quad \forall \ P \in \mathcal{P}_N.$

This shows that

(A.3) \quad $\int_{B(x_0,r)} F_\delta(P^{x_0,\delta}_{x_0,\varepsilon}(f) - f) \cdot P\phi^p dx = 0 \quad \forall \ P \in \mathcal{P}_N.$

where

$$F_\delta(u) = (\delta + |u|^2)^{\frac{p-2}{2}}u, \quad u \in \mathbb{R}^n.$$

It is well known that $F_\delta$ is monotone and continuously differentiable for each $\delta > 0$. Furthermore, there exists a constant $c > 0$ independent of $\delta$ such that for all $u, v \in \mathbb{R}^m$,

(A.4) \quad $(F_\delta(u) - F_\delta(v))(u - v) \geq c(p-1)(\delta + |u| + |u - v|)^{p-2}|u - v|^2,$

(A.5) \quad $|F_\delta(u) - F_\delta(v)| \leq cp(\delta + |u| + |u - v|)^{\frac{p-2}{2}}|u - v|.$

47
We now define the mapping $G_\delta : L^p(B(x_0, r)) \times \mathcal{P}_N \to (\mathcal{P}_N)'$ by

$$
\langle G_\delta(f, P), Q \rangle = \int_{B(x_0, r)} F_\delta(f(x) - P) \cdot Q \phi^2(x) dx, \quad f \in L^p(B(x_0, r), P, Q \in \mathcal{P}_N.
$$

Clearly, (A.3) is equivalent to

(A.6) \quad G_\delta(f, P_{x_0, r}^{N, \delta}(f)) = 0.

We obtain the following properties of $G_\delta$.

**Lemma A.1.**

1. For every $f \in L^p(B(x_0, r))$ the mapping $G_\delta(f, \cdot) : \mathcal{P}_N \to (\mathcal{P}_N)'$ is strictly monotone, bijective, and in case $\delta > 0$ strongly monotone and is a $C^1$ diffeomorphism.

2. In case $\delta > 0$, the mapping $f \mapsto P_{x_0, r}^{N, \delta}(f) : L^p(B(x_0, r)) \to \mathcal{P}_N$ is Fréchet differentiable, and its derivative is given by

(A.7) \quad DP_{x_0, r}^{N, \delta}(f) = -[D_2G_\delta(f, P_{x_0, r}^{N, \delta}(f))]^{-1} \circ D_1G_\delta(f, P_{x_0, r}^{N, \delta}(f)), \quad f \in L^p(B(x_0, r)),

where $D_1G_\delta(f, P) \in \mathcal{L}(L^p(B(x_0, r)), (\mathcal{P}_N)')$ stands for derivative with respect to the first variable, while $D_2G_\delta(f, P) \in \mathcal{L}(\mathcal{P}_N, (\mathcal{P}_N)')$ stands for derivative with respect to the second variable.

Furthermore it holds for every $f \in L^p(B(x_0, r))$

(A.8) \quad \|P_{x_0, r}^{N, \delta}(f)\|_{L^p(B(x_0, r))}^p \leq 2^p \int_{B(x_0, r)} (\delta + |P - f|)^{\frac{p}{2}} \phi^p dx.

3. For all $f \in L^p(B(x_0, r))$ it holds

(A.9) \quad P_{x_0, r}^{N, \delta}(f) \to P_{x_0, r}^{N, \delta}(f) \text{ in } \mathcal{P}_N \text{ as } \delta \searrow 0,

where $P_{x_0, r}^{N, \delta}(f) = P_{x_0, r}^{N, 0}(f)$.

**Proof:** 1. Observing (A.4), we get for all $f \in L^p(B(x_0, r))$, and $P, Q \in \mathcal{P}_N$

\[
\langle (G_\delta(f, P) - G_\delta(f, Q)), (P - Q) \rangle \geq c(p - 1) \int_{B(x_0, r)} (\delta + |P - f| + |P - Q|)^{p-2} |P - Q|^2 \phi^2 dx.
\]

This immediately shows that $G_\delta(f, \cdot)$ is strictly monotone and in case $\delta > 0$ strongly monotone. Here we have used the fact that $\|P\|_{L^2(B(x_0, r))}$ defines an equivalent norm on $\mathcal{P}_N$. Furthermore, if $\delta > 0$ we see that $G_\delta(f, \cdot) : \mathcal{P}_N \to (\mathcal{P}_N)'$ is continuously differentiable and coercive, i.e.

$$
\frac{\langle G_\delta(f, P), P \rangle}{\|P\|} \to 0 \quad \text{as} \quad \|P\| \to +\infty.
$$
Applying the theory of monotone operators, we see that \( G_\delta(f, \cdot) \) is bijective, and is a \( C^1 \) diffeomorphism.

2. Let \( \delta > 0 \) and \( f \in L^p(B(x_0, r)) \). Let \( P_{x_0,r}^{N,\delta}(f) \in \mathcal{P}_N \) denote the minimizer of the functional \( J_\delta(\cdot - f) \) in \( \mathcal{P}_N \). In view of (A.6) we have \( G_\delta(f, P_{x_0,r}^{N,\delta}(f)) = 0 \). Since \( D_2G_\delta \) is an isomorphism from \( \mathcal{P}_N \) into \( (\mathcal{P}_N)' \), by the implicit function theorem we infer that the mapping \( P_{x_0,r}^{N,\delta} : L^p(B(x_0, r)) \to \mathcal{P}_N \) is Fréchet, differentiable, and it holds (A.7).

**Proof of (A.8).** Since \( J_\delta \) is convex and recalling the minimizing property of \( P_{x_0,r}^{N,\delta}(f) \), we get
\[
J_\delta(P_{x_0,r}^{N,\delta}(f)) \leq \frac{1}{2} (J_\delta(P_{x_0,r}^{N,\delta}(f) - f) + J_\delta(f)) \leq J_\delta(f).
\]

This shows that
\[
2^{-p} \int_{B(x_0, r)} |P_{x_0,r}^{N,\delta}(f)|^p dx \leq J_\delta(f).
\]
Whence, (A.8)

3. Now, let \( \delta_k \downarrow 0 \) as \( k \to +\infty \). By (A.8) we see that \( \{P_{x_0,r}^{N,\delta_k}(f)\} \) is bounded. Thus, there exists a subsequence, and \( P_{x_0,r}^{N,\delta_k}(f) \in \mathcal{P}_N \) such that \( P_{x_0,r}^{N,\delta_k}(f) \to P_{x_0,r}^{N,\delta}(f) \) in \( \mathcal{P}_N \) as \( j \to +\infty \). Since \( F_{\delta,j}(f(x) - P_{x_0,r}^{N,\delta_k}(f)) \to F_0(f(x) - P_{x_0,r}^{N,\delta}(f)) \) as \( j \to +\infty \) for all \( x \in B(x_0, r) \) by Lebesgue’s theorem of dominated convergence it follows \( 0 = G_{\delta_k}(P_{x_0,r}^{N,\delta_k}(f)) \to G_0(f, P_{x_0,r}^{N,\delta}(f)) \). Since \( G_0(f, \cdot) \) is strictly monotone, the zero is unique, and thus \( P_{x_0,r}^{N,\delta}(f) = P_{x_0,r}^{N,0}(f) \). Thus, convergence property (A.9) is verified.

Furthermore, in (A.8) letting \( \delta \downarrow 0 \), we see that
\[
\|P_{x_0,r}^{N,\delta}(f)\|_{L^p(B(x_0, r))} \leq 2\|f\phi\|_{L^p(B(x_0, r))}.
\]
This completes the proof of the lemma.

**Remark A.2.** The mapping \( P_{x_0,r}^{N,\delta} : L^p(B(x_0, r)) \to \mathcal{P}_N \) fulfills the projection property
\[
P_{x_0,r}^{N,\delta}(Q) = Q \quad \forall \, Q \in \mathcal{P}_N.
\]
In fact, this follows immediately from (A.1) by setting \( f = Q \) therein.

**B Example of a function in \( \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \setminus C^1(\mathbb{R}^n) \)**

The following example shows that \( \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \) is not in \( C^1(\mathbb{R}^n) \). For simplicity we only consider the case \( n = 1 \) since general case \( n \in \mathbb{N} \) can be reduced to \( n = 1 \). We define
\[
f(x) = \int_0^x u(y)dy, \quad x \in \mathbb{R},
\]
where
\[ u(x) = \begin{cases} 
1 - 2^{2m}|x - 2^{-m}| & \text{if } x \in I_m, \ m \in \mathbb{N}, \\
0 & \text{elsewhere,}
\end{cases} \]
and \( I_m = [2^{-m} - 2^{-2m}, 2^{-m} + 2^{-2m}] \).

**Proof of** \( f \in L^1_{1(p,1)}(\mathbb{R}) \): Thanks to (3.46) it will be sufficient to show that \( u \in L^0_{1(p,0)}(\mathbb{R}) \). In what follows we estimate \( \text{osc}_{p,0}(u; x, r) \) for \( x \in [0, 1] \) and \( 0 < r < +\infty \).

We start with the case \( x = 0 \). For \( 2^{-m - 1} < r \leq 2^{-m} \) we get
\[
\text{osc}_{p,0}(u; 0, r) \leq 2 \left( \frac{1}{2r} \int_{-r}^{r} |u(y)|^p \, dy \right)^{\frac{1}{p}} \leq 2 \left( \frac{1}{2r} \int_{-r}^{r} \sum_{j=m}^{\infty} \chi_{I_j} \, dy \right)^{\frac{1}{p}} \leq cr^{-\frac{1}{p}} \left( \sum_{j=m}^{\infty} 2^{-2j} \right)^{\frac{1}{p}} \leq c2^{-\frac{m}{p}}.
\]
This yields,
\[
\sum_{j=-\infty}^{+\infty} \text{osc}_{p,0}(u; 0, 2^j) = \sum_{j=-\infty}^{-1} \text{osc}_{p,0}(u; 0, 2^j) + \sum_{j=0}^{\infty} \text{osc}_{p,0}(u; 0, 2^j) \leq c \sum_{j=-\infty}^{-1} 2^{\frac{j}{p}} + c \sum_{j=0}^{\infty} 2^{-\frac{j}{p}} < +\infty.
\]

(B.1)

Let \( x \in (0, 1] \). Then there exists \( m \in \mathbb{N} \) such that \( 2^{-m} < x \leq 2^{-m+1} \). Let \( 0 < r < +\infty \). We consider the following three cases.

1. First, in case \( 2^{-m-1} < r < +\infty \) by triangle inequality we get
\[
\text{osc}_{p,0}(u; x, r) \leq c \text{osc}_{p,0}(u; 0, 8r).
\]

2. In case \( 2^{2m} < r \leq 2^{-m-1} \), again by triangle inequality we find
\[
\text{osc}_{p,0}(u; x, r) \leq 2 \left( \frac{1}{2r} \int_{-r}^{r} |u'(y)|^p \, dy \right)^{\frac{1}{p}} \leq 2 \left( \frac{1}{2r} \int_{-r}^{r} (\chi_{I_{m+1}} + \chi_{I_m} + \chi_{I_{m-1}}) \, dy \right)^{\frac{1}{p}} \leq cr^{-\frac{1}{p}} 2^{-\frac{2m}{p}}.
\]

3. In case \( 0 < r \leq 2^{-2m} \), using Poincaré’s inequality, we obtain
\[
\text{osc}_{p,0}(u; x, r) \leq cr \left( \frac{1}{2r} \int_{-r}^{r} |u'(y)|^p \, dy \right)^{\frac{1}{p}} \leq cr^{-\frac{1}{p}} \left( \int_{-r}^{r} (2^{2(m+1)} \chi_{I_{m+1}} + 2^{2m} \chi_{I_m} + 2^{2(m-1)} \chi_{I_{m-1}}) \, dy \right)^{\frac{1}{p}} \leq c r^{-\frac{1}{p}} 2^{\frac{2m}{p}},
\]
where \( p' = \frac{p}{p-1} \).
Using the estimates above together with (B.1), we obtain

\[
\sum_{j \in \mathbb{Z}} \text{osc}_{p,0}(u; x, 2^j) = \sum_{j = -m+1}^{\infty} \text{osc}_{p,0}(u; x, 2^j) + \sum_{j = -\infty}^{-m} \text{osc}_{p,0}(u; x, 2^j) + \sum_{j = -\infty}^{-2m} \text{osc}_{p,0}(u; x, 2^j) \\
\leq c \sum_{j = -m+1}^{\infty} \text{osc}_{p,0}(u; 0, 2^j) + c2^{-m/p} \sum_{j = -2m+1}^{-m} 2^{-j/p} \\
+ 2^{2m/p} \sum_{j = -\infty}^{-2m} 2^{j/p} \leq c,
\]

where the \(c\) stands for an absolute constant. Accordingly,

(B.2) \[\sup_{x \in [0, 1]} \sum_{j \in \mathbb{Z}} \text{osc}_{p,0}(u; x, 2^j) < +\infty.\]

In case \(x < 0\) there exists \(m \in \mathbb{Z}\) such that \(-2^{m+1} < x \leq -2^m\). Using triangle inequality together with (B.1), we easily see that

\[
\sum_{j \in \mathbb{Z}} \text{osc}_{p,0}(u; x, 2^j) \leq \sum_{j = m}^{\infty} \text{osc}_{p,0}(u; x, 2^j) \leq c \sum_{j \in \mathbb{Z}} \text{osc}_{p,0}(u; x, 2^j) \leq c.
\]

Similarly by the aid of (B.2) we get \(\sum_{j \in \mathbb{Z}} \text{osc}_{p,0}(u; x, 2^j) \leq c \sum_{j \in \mathbb{Z}} \text{osc}_{p,0}(u; 1, 2^j) \leq c\) for all \(x \geq 1\). This shows that \(u \in L^0_{1(p,0)}(\mathbb{R}^n)\), and thus \(f \in L^1_{1(p,1)}(\mathbb{R}^n)\) but \(u \notin C^1(\mathbb{R})\).

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