Thermodynamics of modified Schwarzschild black hole in a non–commutative gauge theory

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Abstract

In this work, we study the thermodynamic properties on a non–commutative background via gravitational gauge field potentials. Particularly, we focus on the static spherically symmetric black hole. Such a procedure is accomplished after contracting de Sitter (dS) group, SO(4, 1), with the Poincaré group, ISO(3, 1). After that, we calculate the surface area $\kappa$ and the modified Hawking temperature $T_\Theta$. Additionally, we obtain the following deformed thermal state quantities: the entropy, the Helmholtz free energy, the pressure and the heat capacity. Finally, we check that the second law of thermodynamics holds in this context.

Keywords: Thermodynamics; black hole; non–commutative gauge theory.

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1 Introduction

The formalism of classical thermodynamics can effectively be employed to investigate the physical aspects of black holes based on semiclassical methods of general relativity. The most famous studies in this direction were proposed by Hawking [1–3] and Bekenstein [4, 5] with the purpose of solving the well-known information paradox [6]. Remarkably, they found that the event horizon area $A$ of a black hole, in the context of asymptotically–flat spacetimes, turns out to be mathematically related to its corresponding entropy, which is supported by quantum field theory in curved spacetimes [7–9].

With the knowledge of the black hole thermodynamics, we can certainly address to the problem of thermodynamic stability. From the crucial work of Hawking and Gibbons [10], it is well–known that such an issue may be extended to the black holes in non–asymptotically flat spacetimes [11]. In that Ref., it is seen that the thermodynamic information of de Sitter black holes shows a significant discrepancy if compared to that ones in the context of an asymptotically flat spacetimes. Also, the authors found that, for a particular reference frame, the latter ones have a perfect black body radiation spectra, having their temperature calculated through the surface gravity $\kappa$. More so, within de Sitter spacetime, a cosmological event horizon is present and the emission of particle modes can also occur for a very particular situation in which both surface gravities are equal. This configuration only happens when there exists a degenerate case of extreme solutions [11, 12].

Such a formalism does not properly possess a bound for the precision with which measurements of distances are made. In addition, it is believed that the bounds should exist, being given by the Planck length. In this sense, one of the most common approaches to accomplish such a feature is via non–commutativity of spacetimes. In an overall context, the main motivation to study non–commutative geometry stems from string/M-theory [13–15]. Other important applications arise in the context of supersymmetric Yang–Mills theories within the superfield approach [16–18]. Particularly, in the context of gravity, this technique is typically inserted by using the Seiberg–Witten map, which gauges an appropriate group [19]. In the non–commutative approach, many works have been made in the context of black holes [20–23], their evaporating aspects [24] and their thermodynamic properties [21, 25–28]. Nevertheless, up to now, there is a gap in the literature concerning a concise thermal study of the Schwarzschild black hole coming from the deformed gravitational gauge potentials.

In this work, based on the notable works of Chaichian et al. [29] and Zet et al. [30], we are motivated to investigate the thermodynamic properties of a Schwarzschild black hole arising from the deformed gravitational gauge potentials (or tetrad fields), namely, $\hat{\mathcal{e}}^{\mu}_{\nu}(x, \Theta)$. Such a formalism is achieved by contracting the non–commutative de Sitter (dS) group, SO(4, 1), with the Poincaré (inhomogeneous Lorentz) group, ISO(3, 1), through the Seiberg–Witten (SW) map approach [15, 31, 32]. After that, we calculate the surface area $\kappa$ and the modified Hawking temperature $T_{\Theta}$. Next, we obtain the following deformed state quantities: the entropy, the Helmholtz free energy, the pressure and the heat capacity.

2 Deformed gravitational gauge potentials

In this section, we shall provide the main tools to address the non–commutative gauge theory of gravity. As it is said before, the gauge group corresponds to the dS one, SO(4, 1). Let us briefly describe the SO(4, 1) gauge theory on a commutative $(3 + 1)$–dimensional Minkowski spacetime whose metric in spherical coordinates is given by

$$ds^2 = dr^2 + r^2 d\Omega_2^2 - c^2 dt^2,$$

(2.1)

with $d\Omega_2^2 = \left(d\theta^2 + \sin^2 \varphi d\varphi^2\right)$. It is worth mentioning that the SO(4, 1) group is a ten–dimensional group whose infinitesimal generators are indicated by $\mathcal{M}_{AB} = -\mathcal{M}_{BA}$, in which $A = a, 5$ and $B = b, 5$, where $a, b = 1, 2, 3, 0$. The generators $\mathcal{M}_{AB}$ may properly be distinguished from rotations $\mathcal{M}_{ab} = -\mathcal{M}_{ba}$ and translations $\mathcal{P}_a = \mathcal{M}_{a5}$. The non–deformed gauge potentials, $\omega^{AB}_\mu(x) =$
$-\omega_{BA}(x)$, are distinguished from the spin connection, $\omega_{\mu}^{ab}(x) = -\omega_{\mu}^{ba}(x)$, and from the tetrad fields, $e_{\mu}^{a}(x)$, so that $\omega_{\mu}^{a5}(x) = K e_{\mu}^{a}(x)$, in which $K$ is a contraction parameter.

Moreover, there is another gauge field via $\tilde{\phi}_{\mu}(x) = K \tilde{\phi}_{\mu}(x, \Theta)$, which suffices to eliminate $\tilde{\phi}_{\mu}(x, \Theta)$ by considering the limit $K \to 0$. In other words, we deal with the Poincarè gauge group ISO(3, 1), if we regard such a limit $K \to 0$ [29, 30]. The field strength related to $\omega_{\mu}^{AB}(x)$ reads as follows

$$F_{\mu}^{AB} = \partial_{\mu} \omega_{\nu}^{AB} - \partial_{\nu} \omega_{\mu}^{AB} + (\omega_{\mu}^{AC} \omega_{\nu}^{DB} - \omega_{\nu}^{AC} \omega_{\mu}^{DB}) \eta_{CD}$$

(2.2)

where $\mu, \nu = 1, 2, 3, 0$, and $\eta_{AB} = \text{diag}(+++-)$. More so, we can write

$$F_{\mu}^{a5} = K [\partial_{\mu} e_{\nu}^{a} - \partial_{\nu} e_{\mu}^{a} + (\omega_{\mu}^{ab} e_{\nu}^{a} - \omega_{\nu}^{ab} e_{\mu}^{a}) \eta_{bc}] = K T_{\mu\nu}^{a},$$

(2.3a)

$$F_{\mu}^{ab} = \partial_{\mu} \omega_{\nu}^{ab} - \partial_{\nu} \omega_{\mu}^{ab} + (\omega_{\mu}^{ac} \omega_{\nu}^{db} - \omega_{\nu}^{ac} \omega_{\mu}^{db}) \eta_{cd} + K (e_{\mu}^{a} e_{\nu}^{b} - e_{\nu}^{a} e_{\mu}^{b}) = R_{\mu}^{ab},$$

(2.3b)

in which $\eta_{ab} = \text{diag}(+++-)$. Here, it is worthy to be mentioned that the Poincarè gauge group under consideration possesses geometric structures coming from the Riemann–Cartan spacetime, including curvature and torsion fields [29, 33]. The torsion tensor $T_{\mu\nu}^{a} \equiv F_{\mu\nu}^{a5}/K$ and the curvature tensor $R_{\mu}^{ab} \equiv F_{\mu}^{ab}$ associated with Riemann–Cartan spacetimes are fundamentally described by $e_{\mu}^{a}(x)$ and $\omega_{\mu}^{a5}(x)$. Given Eq. (2.3a), in the absence of torsion fields, one can obtain the spin connections in terms of tetrad fields.

Now, let us start presenting a possible scenario of spherically gauge fields of the SO(4,1) group [29, 33]:

$$e_{\mu}^{1} = \left( \frac{1}{A}, 0, 0, 0 \right), \quad e_{\mu}^{2} = (0, r, 0, 0), \quad e_{\mu}^{3} = (0, 0, r \sin \theta, 0), \quad e_{\mu}^{0} = (0, 0, 0, \mathcal{A}),$$

(2.4)

and

$$\omega_{\mu}^{12} = (0, \mathcal{W}, 0, 0), \quad \omega_{\mu}^{13} = (0, 0, \mathcal{Z} \sin \theta, 0), \quad \omega_{\mu}^{10} = (0, 0, 0, \mathcal{U}),$$

$$\omega_{\mu}^{23} = (0, 0, -\cos \theta, \mathcal{V}), \quad \omega_{\mu}^{20} = \omega_{\mu}^{30} = (0, 0, 0, 0),$$

(2.5)

where $\mathcal{A}, \mathcal{U}, \mathcal{V}, \mathcal{W}$ and $\mathcal{Z}$ are defined as a function of three–dimensional radius only. Furthermore, the non–null components of the torsion tensor are given as follows [30]

$$T_{01}^{0} = -\frac{A A' + U}{A}, \quad T_{03}^{0} = r \mathcal{V} \sin \theta T_{12}^{0} = \frac{A + \mathcal{W}}{A},$$

$$T_{02}^{3} = -r \mathcal{V}, \quad T_{13}^{3} = \frac{(A + \mathcal{Z}) \sin \theta}{A},$$

(2.6)

and curvature tensor is [30]

$$R_{01}^{01} = \mathcal{U}', \quad R_{01}^{23} = -\mathcal{V}', \quad R_{23}^{23} = (\mathcal{Z} - \mathcal{W}) \cos \theta, \quad R_{01}^{01} = -\mathcal{U} \mathcal{W}, \quad R_{01}^{13} = -\mathcal{V} \mathcal{W},$$

$$R_{03}^{03} = -\mathcal{U} \mathcal{Z} \sin \theta, \quad R_{13}^{12} = \mathcal{V} \mathcal{Z} \sin \theta R_{12}^{12} = \mathcal{W}', \quad R_{23}^{23} = (1 - \mathcal{Z} \mathcal{W}) \sin \theta, \quad R_{13}^{13} = \mathcal{Z}' \sin \theta.$$  

(2.7)

Prime symbols $\mathcal{A}', \mathcal{U}', \mathcal{V}', \mathcal{W}'$, and $\mathcal{Z}'$ denote derivatives with respect to radial components. Taking into account Eq. (2.6), we propose the following constraints to guarantee the absence of the torsion field

$$\mathcal{V} = 0, \quad \mathcal{U} = -A A', \quad \mathcal{W} = -A = \mathcal{Z}.$$  

(2.8)

With respect to the field equation

$$R_{\mu}^{a} - \frac{1}{2} R e_{\mu}^{a} = 0,$$

(2.9)

written in terms of the tetrad fields $e_{\mu}^{a}(x)$ with $R_{\mu}^{a} = R_{\mu\nu}^{ab} e_{\nu}^{b}$ and $R = R_{\mu\nu}^{ab} e_{\nu}^{a} e_{\mu}^{b}$, we obtain the following solution

$$\mathcal{A}(r) = \sqrt{1 - \frac{\alpha}{r}},$$

(2.10)
where \( \alpha \) is supposed to be an arbitrary constant given by \( \alpha = 2GM/c^2 \), where the speed of light, the black hole mass, and the gravitational constant are represented by \( c \), \( M \), and \( G \), respectively. To find the relevant deformed metric \( ds^2 = g_{\mu\nu}(x, \Theta)dx^\mu dx^\nu \), with \( x^\mu = r, \theta, \varphi, t \), related to the Schwarzschild solution \((3 + 1)\)-dimensional non–commutative corrections stemming from constraint \( s \) (\ref{constraint}) following non–commutative corrections to \( \hat{g}_{\mu\nu}(x, \Theta) \) arising from the contraction of the non–commutative gauge dS group \( SO(4,1) \), with the Poincaré group, \( ISO(3,1) \), based on the SW map approach \([15,31,32]\). The non–commutative spacetime structure may be constructed to satisfy

\[
[x^\mu, x^\nu] = i\Theta^{\mu\nu},
\]  
(2.11)

where parameters \( \Theta^{\mu\nu} \) are regarded to be real constants in such a way that \( \Theta^{\mu\nu} = -\Theta^{\nu\mu} \). Therefore, the gravitational fields, i.e., \( \hat{e}_\mu^a(x, \Theta) \), and \( \hat{\omega}_{\mu AB}(x, \Theta) \), subjected to a non–commutative spacetime structure, are written in the power series of the parameter \( \Theta \) \([15,29–31]\)

\[
\hat{e}_\mu^a(x, \Theta) = e_\mu^a(x) - i\Theta^\nu{}^\rho e_\mu^a(x) + \Theta^\nu{}^\rho \Theta^{\lambda\tau} e_\mu^a(x) \ldots,
\]  
(2.12a)

\[
\hat{\omega}_{\mu AB}(x, \Theta) = \omega_{\mu AB}(x) - i\Theta^\nu{}^\rho \omega_{\mu AB}(x) + \Theta^\nu{}^\rho \Theta^{\lambda\tau} \omega_{\mu AB}(x) \ldots .
\]  
(2.12b)

It should be noted that \( \hat{e}_\mu^a(x, \Theta) \), in Eq. (2.12a), is obtained by using an expanded form of the following non–commutative corrections to \( \hat{\omega}_{\mu AB}(x, \Theta) \), given by Eq. (2.12b), up to the second order

\[
\omega_{\mu\rho\lambda\tau}^{AB}(x) = \frac{1}{4} \{ \omega_\rho, \partial_\mu \omega_\lambda + R_{\mu\lambda} \}^{AB},
\]  
(2.13a)

\[
\omega_{\mu\rho\lambda\tau}^{AB}(x) = \frac{1}{32} \left( - \{ \omega_\lambda, \partial_\tau \{ \omega_\rho, \partial_\mu \omega_\mu + R_{\mu\rho} \} \} \right) + 2 \{ \omega_\lambda, \{ \omega_\rho, \partial_\mu \omega_\mu + R_{\mu\rho} \} \}
\]  
(2.13b)

written after applying the SW map. Eqs. (2.13a) and (2.13b) obey the following relations

\[
[\alpha, \beta]^{AB} = \alpha^{AC} \beta^B_C - \beta^{AC} \alpha^B_C, \quad \{ \alpha, \beta \}^{AB} = \alpha^{AC} \beta^B_C + \beta^{AC} \alpha^B_C,
\]  
(2.14)

and also

\[
D_\mu R_{\rho\sigma}^{AB} = \partial_\mu R_{\rho\sigma}^{AB} + (\omega_\mu^{AC} R_{\rho\sigma}^{DB} + \omega_\mu^{BC} R_{\rho\sigma}^{DA}) \eta_{CD}.
\]  
(2.15)

It is important to note that we highlight some constraints related to \( \hat{\omega}_{\mu AB}(x, \Theta) \):

\[
\hat{\omega}_{\mu AB}^*(x, \Theta) = -\hat{\omega}_{\mu AB}(x, \Theta), \quad \hat{\omega}_{\mu AB}(x, \Theta) \equiv \hat{\omega}_{\mu AB}^*(x, -\Theta) = -\hat{\omega}_{\mu AB}^*(x, \Theta),
\]  
(2.16)

where the * superscript is a symbol which refers to the complex conjugate. Moreover, the corresponding non–commutative corrections stemming from constraints (2.16) can be expressed as

\[
\omega_{\mu AB}(x) = -\omega_{\mu AB}^*(x), \quad \omega_{\mu\rho\lambda\tau}^{AB}(x) = \omega_{\mu\rho\lambda\tau}^{AB}(x), \quad \omega_{\mu\rho\lambda\tau}^{AB}(x) = -\omega_{\mu\rho\lambda\tau}^{AB}(x).
\]  
(2.17)

We realize that Eq. (2.12a) is provided by using Eqs. (2.13a) and (2.13b) with the absence of torsion field \( T_{\mu\nu}^a \), and the limit \( K \to 0 \). For Eq. (2.12a), the complex conjugate of deformed tetrad fields can be written as

\[
\hat{e}_\mu^a(x, \Theta) = e_\mu^a(x) + i\Theta^\nu{}^\rho e_\mu^a(x) + \Theta^\nu{}^\rho \Theta^{\lambda\tau} e_\mu^a(x) \ldots ,
\]  
(2.18)

where

\[
e_{\mu\nu\rho}^a = \frac{1}{4} \left[ \omega_{\mu\nu\rho}^a + \left( \partial_\mu \omega_{\nu\rho}^a + R_{\mu\rho}^{ac} \right) \eta_{ad} \right],
\]  
(2.19)
and
\[ e^a_{\mu
u
\rho\lambda\tau}(x) = \frac{1}{32} \left[ 2 \{ R_{\tau\nu}, R_{\mu\rho}\} \right]^{ab} e^c_\lambda - \omega^a_{\chi} (D_\rho + \partial_\rho) R^b_{\mu\chi} \eta^{dm} \\
- \{ \omega_{\nu}, (D_\rho + \partial_\rho) R^b_{\mu\tau}\}^{ab} e^c_\lambda - \partial_\nu \{ \omega_{\tau}, (\partial_\rho \omega_{\mu} + R^b_{\mu\rho})\}^{ab} e^c_\lambda \\
- \omega^a_{\chi} \partial_\rho \{ \omega^b_{\chi} \partial_\rho e^m_{\mu} + (\partial_\rho \omega^b_{\chi} + R^m_{\rho\chi}) e^m_{\mu}\} \eta^{dm} + 2 \partial_\rho \omega^a_{\chi} \partial_\rho \partial_\tau e^c_\mu \\
- 2 \partial_\rho (\partial_\tau \omega^a_{\rho} + R^a_{\tau\rho}) \partial_\nu e^c_\lambda - \{ \omega_{\nu}, (\partial_\rho \omega_{\lambda} + R^a_{\rho\lambda})\}^{ab} \partial_\tau e^c_\mu \\
- (\partial_\tau \omega^a_{\rho} + R^a_{\tau\rho}) \{ \omega^b_{\chi} \partial_\rho \partial_\nu \omega_{\mu} + (\partial_\rho \omega^b_{\chi} + R^m_{\rho\chi}) e^m_{\mu}\} \eta^{bc}. \tag{2.20} \]

Thus, a deformed metric tensor can be written as
\[ \hat{g}_{\mu\nu}(x, \Theta) = \frac{1}{2} h_{ab} \left( \hat{e}^a_{\mu}(x, \Theta) \ast \hat{e}^b_{\nu}(x, \Theta) + \hat{e}^b_{\mu}(x, \Theta) \ast \hat{e}^a_{\nu}(x, \Theta) \right), \tag{2.21} \]

where symbol $\ast$ denotes the well–known star product.

### 3 Deformed Schwarzschild black hole

After taking into account Eqs. (2.4) (2.5), (2.21), (2.12a) and (2.18), we are properly able to introduce non–null components of the deformed metric tensor $\hat{g}_{\mu\nu}(x, \Theta)$, up to the second order of parameter $\Theta$. With this, we may address to the Schwarzschild solution of a non–commutative $(3 + 1)$–dimensional spherically symmetric spacetime

\[ \hat{g}_{11}(r, \Theta) = \frac{1}{\mathcal{A}(r)^2} + \frac{1}{4} \frac{\mathcal{A}''(r)}{\mathcal{A}(r)} \Theta^2 + O \left( \Theta^4 \right), \tag{3.1a} \]
\[ \hat{g}_{22}(r, \Theta) = r^2 + \frac{1}{16} \left[ \mathcal{A}(r)^2 + 11r \mathcal{A}(r) \mathcal{A}'(r) + 16r^2 \mathcal{A}'(r)^2 \right. \\
\left. + 12r^2 \mathcal{A}(r) \mathcal{A}''(r)^2 \right] \Theta^2 + O \left( \Theta^4 \right), \tag{3.1b} \]
\[ \hat{g}_{33}(r, \Theta) = r^2 \sin^2\theta + \frac{1}{16} \left[ 4 \left( 2r \mathcal{A}(r) \mathcal{A}'(r) - r \frac{\mathcal{A}'(r)}{\mathcal{A}(r)} + r^2 \mathcal{A}(r) \mathcal{A}''(r) \right) \right. \\
\left. + 2r^2 \mathcal{A}'(r)^2 \right] \sin^2\theta + \cos^2\theta \Theta^2 + O \left( \Theta^4 \right), \tag{3.1c} \]
\[ \hat{g}_{00}(r, \Theta) = -\mathcal{A}(r)^2 - \frac{1}{4} \left[ 2r \mathcal{A}(r) \mathcal{A}'(r)^3 + r \mathcal{A}(r)^3 \mathcal{A}''(r) + \mathcal{A}(r)^3 \mathcal{A}''(r) \right. \\
\left. + 2r^2 \mathcal{A}'(r)^2 + 5r \mathcal{A}(r)^2 \mathcal{A}'(r)^3 \right] \Theta^2 + O \left( \Theta^4 \right). \tag{3.1d} \]

As it can be seen, the deformed metric tensor $\hat{g}_{\mu\nu}(x, \Theta)$ is diagonal. It should be noted that the non–commutative structure of spacetime is specified by Eq. (2.11). In this scenario, the deformed diagonal metric $\hat{g}_{\mu\nu}(x, \Theta)$ is provided by considering the non–null components of the $\Theta^{\mu\nu}$ tensor
\[ \Theta^{12} = \Theta = -\Theta^{21}, \tag{3.2} \]

where $\Theta$ specifies the non–commutativity of the spacetime coordinates. If we consider the first non–null component of Eq. (3.1d) being equal to zero, $\hat{g}_{00}(r, \Theta)|_{r=r_s} = 0$, we can acquire the mass of the deformed Schwarzschild black hole as
\[ M_\Theta = \frac{r_s c^2}{2G} \left( \frac{4}{11} - \frac{8}{11} \left( \frac{r_s}{\Theta} \right)^2 + \frac{4}{\sqrt{1 + 7 \left( \frac{r_s}{\Theta} \right)^2 + 4 \left( \frac{r_s}{\Theta} \right)^4}} \right), \tag{3.3} \]

where the event horizon of the usual Schwarzschild radius is denoted by $r_s$. The mass of the Schwarzschild black hole in the framework of the non–commutative gauge theory of gravitation, i.e., a special case
of non–commutative gravity, is obtained up to the second order of parameter \((r_s/\Theta)\). Thereby, we recover the usual mass of the Schwarzschild black hole if we consider the limit below:

\[
\lim_{r_s\Theta \to 0} M_\Theta = \frac{r_s c^2}{2G} = M, \tag{3.4}
\]

where \(r_s \Theta = r_s/\Theta\). For the sake of providing an application to this scenario, we exhibit the deformed Schwarzschild black hole temperature in the semiclassical framework:

\[
T_\Theta = \frac{\hbar \kappa}{2\pi}, \quad \kappa = -\frac{1}{2\sqrt{-g_{00}(r, \Theta)g_{11}(r, \Theta)}} \frac{d\hat{g}_{00}(r, \Theta)}{dr} \bigg|_{r=r_s}, \tag{3.5}
\]

where \(\kappa\) is the respective surface gravity. Now, considering the non–commutativity corrections up to the second order in \(\Theta\), Eq. (3.5) can be calculated as

\[
T_\Theta = \frac{\alpha \hbar (11\alpha \Theta^2 + 4r_s^3 - 6\Theta^2 r_s)}{16\pi r_s^5 \left(\frac{\alpha \Theta^2 (8r_s - 11\alpha)}{16r_s^2} - \frac{\alpha}{r_s} + 1\right) \left(\frac{r_s}{4(r_s - \alpha)} - \frac{\alpha \Theta^2 (4r_s - 3\alpha)}{16r_s^2 (r_s - \alpha)^2}\right)}. \tag{3.6}
\]

One can see that the corrected temperature in Eq. (3.6) depends on the non–commutativity parameter \(\Theta\), the usual Schwarzschild radius \(r_s\), and the parameter \(\alpha\), which includes the usual Schwarzschild black hole mass \(M\). Furthermore, we can write the limit of \(T_\Theta\) when \(\Theta \to 0\) as follows

\[
\lim_{\Theta \to 0} T_\Theta = \frac{\alpha \hbar}{4\pi r_s^2} = T. \tag{3.7}
\]

Additionally, we can provide a correspondence between usual Schwarzschild black hole and the deformed one:

\[
\frac{T_\Theta}{T} = \frac{11\alpha \Theta^2 + 4r_s^3 - 6\Theta^2 r_s}{4r_s^3 \left(\frac{\alpha \Theta^2 (8r_s - 11\alpha)}{16r_s^2} - \frac{\alpha}{r_s} + 1\right) \left(\frac{r_s}{4(r_s - \alpha)} - \frac{\alpha \Theta^2 (4r_s - 3\alpha)}{16r_s^2 (r_s - \alpha)^2}\right)}. \tag{3.8}
\]

As a matter of fact, the black hole entropy is derived from a surface area term of the gravitational action at the event horizon [34]. In the literature, there exist some methods to obtain the black hole entropy, \(S\). Particularly, we apply a method in which the black hole, like any thermodynamic system, needs to satisfy the first law of thermodynamics [35]

\[
dM = TdS + \sum_{i=1} X_i dy_i, \tag{3.9}
\]

where the generalized force associated with variable \(y_i\) is denoted by \(X_i\). Indeed, \(y_i\) is supposed to be one of the variables of a black hole, such as, the electric charge. Here, it worth mentioning that the thermodynamic state quantities were also calculated in other different cosmological scenarios [36–42].

Here, if we consider the first law of thermodynamics, i.e., \(dM_\Theta = T_\Theta dS_\Theta\), we can determine the non–commutative entropy in an analytical manner

\[
S_\Theta = \int T_\Theta^{-1} \left(\frac{\partial M_\Theta}{\partial r_\Theta}\right)_{Q_i, P_i} dr_\Theta. \tag{3.10}
\]

Thereby, with the usage of Eqs. (3.3), (3.6) and (3.10), it is written as

\[
S_\Theta = \frac{\pi G \sqrt{-\frac{3c^4}{G^2} - \frac{16c^2}{G} + 44 \left[12r_s^2 (3c^2 + 22G) + 11\Theta^2 (6c^2 + 23G) \ln (r_s)\right]}}{4\sqrt{11} (c^2 - 2G) (3c^2 + 22G)}. \tag{3.11}
\]
From the Legendre transform of internal energy $U(y_i)$ and enthalpy $H$, we arrive at

$$U = H - PV.$$  \tag{3.12}

In the context of thermodynamics, we know that relation $U = M$ makes sense. In addition, based on the claim of Ref. [43], i.e., that $H = M$, as well as considering Eqs. (3.13) and (3.12), we can rewrite the first law of thermodynamics as

$$dU = TdS + \sum_{i=1}^{N} X_i dy_i - d(PV),$$  \tag{3.13}

where $X_i$ can mimic the $V$ volume, the angular velocity $\Omega$ and the electrostatic potential $\Phi$. In our case, the horizon area and the volume of the black hole are geometrically associated; for instance, the volume is given by $V = \frac{4}{3} \pi r^3$, where the radius of the non–commutative Schwarzschild black hole is denoted by $r_{\Theta}$. With these explanations, the non–commutative Helmholtz free energy $F_{\Theta}$ can be obtained by $F_{\Theta} = M_{\Theta} - T_{\Theta} S_{\Theta}$. In this sense, with respect to Eqs. (3.3), (3.6) and (3.11), it reads

$$F_{\Theta} = \frac{r_{\Theta} c^2}{2G} \left( \frac{4}{11} - \frac{8}{11} \left( \frac{r_{\Theta}}{6} \right)^2 + \frac{4}{11} \sqrt{1 + 7 \left( \frac{r_{\Theta}}{6} \right)^2 + 4 \left( \frac{r_{\Theta}}{6} \right)^4} \right)$$

$$- \frac{\alpha}{16 \pi r_{\Theta}^5} \left[ \alpha \left( \frac{11 \alpha^2 + 4r_{\Theta}^2 - 6 \Theta^2 r_{\Theta}}{10r_{\Theta}^3} - \frac{1}{r_{\Theta}^2} + 1 \right) \left( \frac{r_{\Theta}}{10r_{\Theta}^3 - \alpha} - \frac{\alpha \Theta^2 (4r_{\Theta} - 3\Theta)^2}{10r_{\Theta}^3 - \alpha} \right)^2 \right]$$

$$\times \left[ \frac{\pi G (\frac{3\pi^2}{G} - \frac{16 \pi^2}{G} + 44 \left[ 12r_{\Theta}^2 (3c^2 + 22G) + 11\Theta^2 (6c^2 + 23G) \ln (r_{\Theta}) \right] )}{4\sqrt{11^5} (c^2 - 2G) (3c^2 + 22G)} \right],$$

and we keep the simplified form of Eq. (3.14) up to the second order of $(r_{\Theta}/\Theta)$. Now, let us apply a well–known standard expression to derive the pressure of the black hole:

$$P_{\Theta} = -\frac{1}{4 \pi r_{\Theta}^2} \left( \frac{\partial F_{\Theta}}{\partial r_{\Theta}} \right)_T,$$  \tag{3.15}

which after substituting Eq. (3.14) into Eq. (3.15), we get

$$P_{\Theta} = \frac{T_{\Theta} G \left( \frac{3\pi^2}{G} - \frac{16 \pi^2}{G} + 44 \left( \frac{11 \Theta^2 (6c^2 + 22G)}{r_{\Theta}} + 24r_{\Theta} (3c^2 + 22G) \right) \right)}{16 \sqrt{11^5} r_{\Theta}^2 (c^2 - 2G) (3c^2 + 22G)}$$

$$- 2c^2 r_{\Theta} \left( \frac{\Theta^2}{r_{\Theta}^2} + \sqrt{\frac{11\Theta^2}{r_{\Theta}^2} + \left( \frac{\Theta^2}{r_{\Theta}^2} - 1 \right)^2 - 1} - \frac{\Theta^2 (6c^2 + 22G)}{11\pi G \Theta^2} \right)$$

$$- \frac{c^2 \left( \frac{-3\Theta^2 + 3r_{\Theta}^3 - 22r_{\Theta}^2}{\sqrt{\Theta^2 r_{\Theta}^2 + \Theta^2 r_{\Theta}^2 + 11\Theta^2 r_{\Theta}^2}} - 3r_{\Theta}^3 \right)}{22\pi G r_{\Theta}^5}.$$

Furthermore, the non–commutative heat capacity for such a deformed black hole can be obtained by

$$C_{V_{\Theta}} = T_{\Theta} \frac{\partial S_{\Theta}}{\partial T_{\Theta}} = T_{\Theta} \frac{\partial S_{\Theta}}{\partial r_{\Theta}} \frac{\partial r_{\Theta}}{\partial T_{\Theta}},$$  \tag{3.17}
where it can properly be written as

\[ C_{V\Theta} = - \left( \sqrt{44 - \frac{3c^4}{G^2} - \frac{16c^2}{G} G\pi (r_s - \alpha) (4r_s^3 - 6r_s\Theta^2 + 11\alpha\Theta^2)} (16r_s^4 - 16r_s^3\alpha + 3c^2 (72r_s^2 + 66\Theta^2)) / \right. \\
\left. \quad + 4\sqrt{11} (c^2 - 2G) (3c^2 + 22G) (2048r_s^{12} - 6144r_s^{11}\alpha - 61056r_s^8\alpha^2\Theta^2 - 14520r_s^5\alpha^3\Theta^4 + 2790r_s^2\alpha^4\Theta^6 - 2431r_s\alpha^5\Theta^6 + 726\alpha^6\Theta^6 + 6144r_s^{10}\Theta^2 - 128r_s^9 (16\alpha^3 - 255\alpha^2\Theta^2) + 64r_s^7 (764\alpha^3\Theta^2 - 15\alpha^4) - 32r_s^6 (4484\alpha^2\Theta^2 - 195\alpha^2\Theta^4) + 16r_s^4 (893\alpha^4\Theta^4 + 12\alpha^2\Theta^6) - 4r_s^3 (1256\alpha^5\Theta^4 + 323\alpha^3\Theta^6)) \right) \quad (3.18) \]

In addition, above expression may be simplified if we take into account small \( \Theta \) as follows:

\[ C_{V\Theta} = - \left( G\pi\Theta^2 \sqrt{4 - \frac{3c^4 + 16c^2G}{11G^2}} (256r_s^6 (174c^2 + 1045G) - 384r_s^5\alpha (537c^2 + 3476G) - 33r_s^4\alpha^2 (4479c^2 + 3822G) + 256\alpha^3 (-573 (3c^2 + 22G) + 11r_s^2 (66c^2 + 505G)) + 2304r_s^2\alpha^4 (3c^2 + 22G)) (c^2 - 2G) (3c^2 + 22G) (r_s - \alpha)^3) \right) \\
\left. \quad + \frac{3G\pi r_s^2}{c^2 - 2G} \sqrt{4 - \frac{3c^4 + 16c^2G}{11G^2}} + \mathcal{O}(\Theta)^4. \right) \quad (3.19) \]

### 4 Conclusion

Designing general relativity on a non–commutative spacetime, we presented the modified Einstein’s gravity arising from deformed tetrad fields by contracting the non–commutative gauge dS group, SO(4, 1), with the Poincarè group, ISO(3, 1), based on the SW map approach. According to the non–null components of the \( \Theta^{\mu\nu} \) tensor, we obtained an exact spherically symmetric solution to the modified Einstein’s equation. The deformation of gravitational gauge potentials, achieved in \((3 + 1)\)-dimensional non–commutative spacetime, established the deformed Schwarzschild black hole from the prominent deformed metric tensor \( \tilde{g}_{\mu\nu}(x, \Theta) \). For the first time, the thermodynamic properties of such a Schwarzschild–like black hole were investigated. In this way, the surface gravity, the modified Hawking temperature, the entropy, the Helmholtz free energy, the pressure and the heat capacity were calculated within this non–commutative scenario. It must be noted that, the second law of the thermodynamics was maintained and, when the non-commutative parameter \( r_s/\Theta \) went to zero, we obtained the thermodynamic properties of the standard Schwarzschild black hole, as one could expect. As a further perspective, we intend to study the main aspects of the evaporation of such a black hole similar to what was proposed by Ref. [24].

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### 5 Data Availability Statement

Data Availability Statement: No Data associated in the manuscript
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