Scalar curvature of systems with fractal distribution functions

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Abstract

Starting with the relative entropy for two close statistical states we define the metric and calculate the scalar curvature $R$ for systems with classical, boson and fermion fractal distribution functions with moment order parameter $q$. In particular, we find that for $q \neq 1$ the scalar curvature is closer to zero implying that the fractal bosonic and fermionic systems are more stable than the standard ones.

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1 Introduction

From the theory of fractals [1] we learned that given a statistical weight $\Omega(q, \delta)$ of a system with order parameter $q$ and resolution $\delta$, the fractal dimension is defined as the exponent $d = D_q$ which will make the product $\lim_{\delta \to 0} \Omega(q, \delta) \delta^d$ finite. With use of the definition of the Boltzmann entropy $S(q, \delta) = \ln \Omega(q, \delta)$, the relation between the entropy and the fractal dimension $D_q$ is given by

$$D_q = - \lim_{\delta \to 0} \frac{S(q, \delta)}{\ln \delta}$$

(1)

Based on these definitions and with use of the Boltzmann’s H theorem, the generalized entropy and distribution functions for classical and quantum gases

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were calculated in Ref. [2]. The average number of particles with energy \( \epsilon \) was shown to be given by
\[
<n(\epsilon)> = \frac{1}{[1 + \beta(q - 1)(\epsilon - \mu)]^{1/(q-1)} + a},
\]
(2)
where \( a = 0 \) for the classical case, and the values \( a = -1 \) and \( a = 1 \) correspond to Bose-Einstein and Fermi-Dirac cases, respectively. For \( q = 1 \), Equation (2) becomes the standard textbook result for classical and quantum ideal gases. The distribution functions in Equation (2) were also obtained in Ref. [3] by considering a dilute gas approximation to the partition function of a non-extensive statistical mechanics originally proposed in Ref. [4]. It is our purpose to study some of the geometric properties of systems with average particle number according to Equation (2). The idea of using geometry in thermodynamics is not new [5]-[9], and several authors developed formalisms to measure the distance between equilibrium states through the definition of a metric and the calculation of the corresponding scalar curvature as a measure of the interactions [10]-[20]. Some of the applications include classical and quantum gases [11],[16],[21],[22], magnetic systems [23]-[26], non-extensive statistical thermodynamics [27],[28],[29], anyon gas [30],[31], fractional statistics [32] and deformed boson and fermion systems [33]. Some of the basic results of these approaches include the relationships between the metric with the correlations of the stochastic variables, and the scalar curvature \( R \) with the stability of the system, and the facts that the scalar curvature \( R \) vanishes for the classical ideal gas, \( R > 0 (R < 0) \) for a boson (fermion) ideal gas, and it is singular at a critical point. Here, we wish to study systems with an average particle number given in Equation (2). In Section 2 we briefly describe the formalism of systems with fractal distribution functions as reported in Ref. [34]. In Section 3 we obtain the metric from the second order term in the expansion of the relative entropy between two close statistical states, and in Section 4 we use the metric to compute the scalar curvature for the classical, Bose-Einstein and Fermi-Dirac cases for \( q \neq 1 \). In Section 5 we summarize our results.

2 Fractal models

2.1 Classical case

We will use the short notation
\[
\rho_l = [1 + (q - 1)\beta(\epsilon_l - \mu)]^{1/(q-1)}.
\]
(3)
The probability density is defined
\[
\rho = \frac{1}{Z_{MB}} \prod_{l=0} \frac{1}{n_l!} \rho_l^{-n_l},
\]
(4)
where the partition function
\[ Z_{MB} = \prod_{l=0}^{\infty} \sum_{n_l=0}^{\infty} \frac{1}{n_l!} \rho_l^{-n_l} \]
\[ = \prod_{l=0}^{\infty} e^{\rho_l - 1} \]  
(5)

From the definition of the average number of particles with energy \( \epsilon_l \)
\[ < n_l > = \frac{\sum_{n_l=0}^{\infty} \frac{1}{n_l!} \rho_l^{-n_l}}{\sum_{n_l=0}^{\infty} \frac{1}{n_l!} \rho_l^{-n_l}}, \]
we find after summing the series that
\[ < n_l > = \rho_l^{-1}. \]  
(6)

In the thermodynamic limit we write for the average total number of particles
\[ < N > = 4\pi V \hbar^3 \left( \frac{2m}{\beta(q-1)} \right)^{3/2} \int_0^\infty \frac{x^2 dx}{[1 + x^2 - (q-1)\beta \mu]^{1/(q-1)}} \]  
(8)

leading to the expression
\[ < N > = -2\pi V \hbar^3 \left( \frac{2m}{\beta(q-1)} \right)^{3/2} \frac{1}{[1 - (q-1)\beta \mu]^{1/(q-1)-(3/2)}} S, \]  
(9)

where \( S = C_0 + \sum_{i=1}^\infty \frac{(-1)^m}{m!} \left( \frac{1}{2} \right) \ldots \left( \frac{3}{2} - m \right) C_m, \)  
(10)

with \( C_m = \frac{1}{1/(q-1)-m+(3/2)}. \) Solving Equation (8) we find that the fugacity \( z = e^{\beta \mu} \) has a temperature dependence given by
\[ \ln z = \frac{1}{q-1} \left\{ 1 - \left[ -2\pi V \hbar^3 \left( \frac{2m}{(q-1)\beta} \right)^{3/2} S \right]^{1/\omega} \right\}, \]  
(11)

where \( \omega = \frac{5-3q}{2(q-1)}. \) From Equation (11) we see that the fugacity is restricted to the interval \( 0 < z < e^{1/(q-1)}, \) which serves as a cut-off that avoids a negative average occupation number. The correct definition of the average energy is given by
\[ < \epsilon > = \frac{4\pi V}{\hbar^3} \int_0^\infty \frac{p^2}{2m} < n(p) >^q p^2 dp, \]  
(12)

leading, with Equation (8), to the required classical result \( < \epsilon > = \frac{3}{4} < N > kT \) \[ 33 \]. We should remark, that Equations (5) and (12) are related by the standard definition
\[ < \epsilon > = -\frac{\partial \ln Z_{MB}}{\partial \beta}. \]  
(13)
2.2 Boson case

Similarly to the classical case, the definition

\[
\langle N \rangle = \sum_{j=0}^{\infty} \sum_{n_j=0}^{\infty} \frac{n_j \rho_j^{n_j}}{\rho_j^{n_j}},
\]

leads to the average occupation number

\[
\langle n_j \rangle = \frac{1}{\rho_j - 1}
\]

with the probability density and the partition function

\[
\rho = \frac{1}{Z_{BE}} \prod_{j=0}^{\infty} \rho_j^{n_j},
\]

\[
Z_{BE} = \prod_{j=0}^{\infty} \sum_{n_j=0}^{\infty} \rho_j^{n_j}
= \prod_{j=0}^{\infty} \frac{1}{1 - \rho_j^{-1}}.
\]

As in the standard, \( q = 1 \), Bose-Einstein case the chemical potential is negative.

2.3 Fermion case

For the Fermi-Dirac case the average occupation number

\[
\langle n_j \rangle = \frac{1}{\rho_j + 1}
\]

is obtained by defining

\[
\rho = \frac{1}{Z_{FD}} \prod_{j=0}^{\infty} \rho_j^{n_j},
\]

\[
Z_{FD} = \prod_{j=0}^{\infty} \sum_{n_j=0}^{\infty} \rho_j^{n_j}
= \prod_{j=0}^{\infty} (1 + \rho_j^{-1}),
\]

and the requirement that the average occupation number \( \langle n_j \rangle \in [0, 1] \) leads to restrict the fugacity to the interval \( 0 < z < e^{1/(q-1)} \). For Bose-Einstein and Fermi-Dirac cases we obtain

\[
-\frac{\partial \ln Z}{\partial \beta} = \langle \epsilon \rangle + \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \binom{1 - q}{k} < n_l >^{q+k} \epsilon_l (-a)^k.
\]

\(^1\)We should remark that our partition function differs from that reported in Ref. [3].
3 The metric

For the three cases discussed in the previous section we cannot adopt any of the standard definitions for the metric like for example \( g_{\alpha\gamma} = \frac{\partial^2 \ln Z}{\partial \beta^\alpha \partial \beta^\gamma} \); \( \beta^1 = \beta; \beta^2 = -\beta \mu \) \hspace{1cm} (22)

which is valid for exponential distributions.

The relative entropy, \( H(p||P) = \sum_i p_i \left(-\ln \left(\frac{p_i}{P_i}\right)\right)^\mu \), is a very useful concept.

For example, for two close distribution functions \( p(x) \) and \( p(x + \Delta) \) we can obtain a Fisher’s information measure

\[
I_\mu = \int dx \left( \frac{dp/dx}{p(x)} \right)^\mu \frac{dp}{dx}
\] \hspace{1cm} (23)

as part of the second order term in \( \Delta \) for the entropic form, with \( \mu = 1 \), \( S = -\sum_i p_i \ln p_i \) \hspace{1cm} [35], and \( S = \sum_i p_i (-\ln p_i)^\mu \) \hspace{1cm} [36] where \( \mu \) is a fractional parameter. By defining \( \phi(x) = p^{1+\mu} \) and considering the Fisher information as a lagrangian density leads to linear and nonlinear differential equations for \( \mu = 1 \) and \( \mu \neq 1 \), respectively.

Here, based on work in Ref.[10] we expand the relative entropy for \( \mu = 1 \) between two close densities \( \rho(\beta) \) and \( \rho(\beta + d\beta) \) up to second order in \( d\beta^\alpha \). Therefore, the information distance \( I(\rho(\beta), \rho(\beta + d\beta)) \) between the two close states is written

\[
I(\rho(\beta^\alpha), \rho(\beta^\alpha + d\beta^\alpha)) = Tr \rho \left( \ln \rho(\beta^\alpha) - \ln \rho(\beta^\alpha + d\beta^\alpha) \right),
\] \hspace{1cm} (24)

such that expanding the second order term gives for the metric

\[
g_{\alpha\gamma} = \frac{\partial^2 \ln \rho}{\partial \beta^\alpha \partial \beta^\gamma} + Tr \rho \sum_{l=0} <n_l> \frac{\partial^2 \ln \rho_l}{\partial \beta^\alpha \partial \beta^\gamma}. \hspace{1cm} (25)
\]

A simple inspection shows that in the \( q \rightarrow 1 \) limit Equation (26) reduces to Equation (22). Equation (25) can be simplified leading to the three corresponding metrics:

\[
g_{\alpha\gamma}^{MB} = \sum_{l=0} \frac{1}{\rho_l} \frac{\partial \rho_l}{\partial \beta^\alpha} \frac{\partial \rho_l}{\partial \beta^\gamma} \hspace{1cm} (26)
\]

\[
g_{\alpha\gamma}^{BE} = \sum_{l=0} \frac{1}{\rho_l (\rho_l - 1)^2} \frac{\partial \rho_l}{\partial \beta^\alpha} \frac{\partial \rho_l}{\partial \beta^\gamma} \hspace{1cm} (27)
\]

\[
g_{\alpha\gamma}^{FD} = \sum_{l=0} \frac{1}{\rho_l (\rho_l + 1)^2} \frac{\partial \rho_l}{\partial \beta^\alpha} \frac{\partial \rho_l}{\partial \beta^\gamma} \hspace{1cm} (28)
\]

which can be summarized in the general formula

\[
g_{\alpha\gamma} = \sum_{l=0} \frac{<n_l>^2}{\rho_l} \frac{\partial \rho_l}{\partial \beta^\alpha} \frac{\partial \rho_l}{\partial \beta^\gamma}. \hspace{1cm} (30)
\]
Writing in general \( \rho_l = [1 + (q - 1) \sum \beta^\alpha F_\lambda^\alpha]^{1/(q-1)} \) we find

\[
\frac{\partial}{\partial \beta^\alpha} \frac{\partial}{\partial \beta^\lambda} \ln Z = \sum_{l=0}^{\rho_l} \rho_l^{2q} F_\alpha^\alpha F_\lambda^\lambda (q < n_l > -a < n_l >^2) \quad (31)
\]

\[
g_{\alpha\lambda} = \sum_{l=0}^{\rho_l} \rho_l^{2q} F_\alpha^\alpha F_\lambda^\lambda (q < n_l > -a < n_l >^2) \quad (32)
\]

4 Scalar curvature

4.1 Classical case

It has been shown that the scalar curvature [11][16] vanishes for the standard case, but it is tempting to speculate whether that is also the case for \( q \neq 1 \). In the thermodynamic limit, with \( x = \beta \epsilon \), we write for example

\[
g_{11} = \frac{2}{\sqrt{\pi}} V \beta^{-2} \lambda^{-3} \int_0^\infty \frac{x^{5/2} dx}{[1 + (q - 1)(x + \gamma)]^{1+1}} \quad (33)
\]

where hereafter \( \gamma = -\beta \mu \). This integral converges for \( \frac{5-3q}{2(q-1)} > 0 \), restricting the values of \( q \) to the interval \( q \in [1, 5/3) \). With use of the integral representation of the \( \Gamma \)-function

\[
\Gamma(y) = w^y \int_0^\infty t^{y-1} e^{-wt} dt, \quad y > 0 ; \quad w > 0,
\]

we obtain for the components of the metric tensor

\[
g_{11} = V \beta^{-2} \lambda^{-3} h_{5/2},
\]

\[
g_{12} = V \beta^{-1} \lambda^{-3} h_{3/2},
\]

\[
g_{22} = V \lambda^{-3} h_{1/2},
\]

where the function

\[
h_\lambda = \frac{2}{\sqrt{\pi}(q-1)^{(\lambda+1)}} \frac{\Gamma(\lambda + 1)\Gamma(\frac{2q-1}{q-1} - \lambda)}{\Gamma(\frac{2q-1}{q-1})} \frac{1}{[1 + (q - 1)(x + \gamma)]^{1+1}},
\]

satisfies

\[
\frac{\partial h_\lambda}{\partial \gamma} = -\lambda h_{\lambda-1}. \quad (37)
\]

As is well known [17], the scalar curvature is given by

\[
R = \frac{2}{detg} R_{1212}, \quad (38)
\]

where \( detg = g_{11}g_{22} - g_{12}g_{12} \) and the non-vanishing part of the curvature tensor \( R_{\alpha\beta\gamma\lambda} \) is given in terms of the Christoffel symbols.
Figure 1: The scalar curvature $R$, in units of $\lambda^3 V^{-1}$, as a function of the fugacity $z$ for bosons at constant $\beta$ for the cases of $q = 1$ (solid line), $q = 1.1$ (dashed line) and $q = 1.2$ (dotted line).

\[ R_{\alpha\beta\gamma\lambda} = g^{\eta\theta} \left( \Gamma_{\eta\alpha\lambda} \Gamma_{\theta\beta\gamma} - \Gamma_{\eta\alpha\gamma} \Gamma_{\theta\beta\lambda} \right). \]  

(39)

A simple calculation leads to the result

\[ R = \frac{V^{-1} \lambda^3}{4 (\det g)^2} \left( 5 h_{1/2}^2 h_{3/2}^2 - 6 h_{1/2}^2 h_{5/2} + h_{3/2} h_{-1/2} h_{5/2} \right), \]  

(40)

such that after replacement of the definition of the function $h_\lambda$ we get that the scalar curvature for the classical fractal case is identically equal to zero. Therefore, in this case the parameter $q$ does not play any role as far as correlations are concerned.

### 4.2 Boson and fermion cases

Here, in order to evaluate the corresponding scalar curvatures we need to replace the summations in Equations (27) and (28) by integrals

\[ G^\pm_\lambda = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^\lambda \Omega^{\frac{3-2\lambda}{2}}}{(\Omega^{1/\gamma} \pm 1)^2} dx, \]  

(41)

where the $+$ sign is for fermions and the $-$ sign for bosons, and the function $\Omega = 1 + (q - 1)(x + \gamma)$. In particular, the metric component $g_{11}$ is written

\[ g_{11} = V \lambda^{-3} \beta^{-2} G^+_{5/2}, \]  

(42)

and its integral converges for $1 \leq q < 5/3$.

The functions $G^\pm_\lambda$ also satisfy

\[ \frac{\partial G^\pm_\lambda}{\partial \gamma} = -\lambda G^\pm_{\lambda-1}, \]  

(43)
and thus the corresponding equations for $R$ are equivalent to Equation (40) with the replacement of the function $h_\lambda$ by the functions $G^{\pm}_\lambda$. Figures 1 and 2 show the results of a numerical calculation of the scalar curvature $R$ as a function of the fugacity $z$ for the parameter values $q = 1, 1.1, 1.2$ for boson and fermions respectively.

5 Conclusions

In this paper, starting from the relative entropy for two close statistical states we defined the metric for systems with fractal distribution functions with order parameter $q$. We calculated the scalar curvature $R$ and found that it vanishes for the classical ideal gas, as in the standard case. Numerical calculations for the boson and fermion systems show that the corresponding values of $R$ as a function of the fugacity $z$ are closer to zero than those in the $q = 1$ case, implying that the departure from the value $q = 1$ makes the systems more stable. Therefore, for $q \neq 1$ bosons will be less attractive and fermions less repulsive than their standard counterparts. Our results are in agreement with those obtained in a previous work [37] wherein we showed that long-range correlations for the fractal Bose case decrease when the parameter $q$ departs from the standard value $q = 1$. On the other hand, if one wishes to consider the order parameter $q$ as a non-extensive parameter it has to be within the context of considering these fractal systems as a dilute approximation to non-extensive statistical mechanics, which consists in replacing the Tsallis partition function by a factorized one. This type of approximation has been shown [38] to be good outside a temperature interval that shifts to higher values of $T$ when the number of energy levels increases. Our results also show that the sign of $R$ remains unchanged as a function of $z$ implying that these systems do not exhibit anyonic behavior, a fact that looks impossible to check by performing an expansion for $z \approx 0$ to obtain the second
virial coefficient because the partition function is a function of \( \ln z \). In addition, our results contrast with the cases of systems with quantum group symmetry where the parameter \( q \) interpolates between bosons and fermions in two and three dimensions [39].

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