OPTIMAL SWEEPOUTS OF A RIEMANNIAN 2 SPHERE

GREGORY R. CHAMBERS AND YEVGENY LIOKUMOVICH

Abstract. We prove the following conjecture of R. Rotman. Suppose we are given an \( \epsilon > 0 \) and a sweepout of a Riemannian 2-sphere which is composed of curves of length at most \( L \). We can then find a second sweepout which is composed of curves of length at most \( L + \epsilon \), which are pairwise disjoint, and which are either constant curves or simple curves.

We use the techniques involved in proving this statement to partly answer a question due to N. Hingston and H.-B. Rachemacher, and we also use these methods to extend the results of [CL] concerning converting homotopies to isotopies in an effective way.

1. Introduction

Let \((S^2, g)\) be a Riemannian 2-sphere and let \( \Lambda \) denote the space of all smooth closed curves on \((S^2, g)\). A sweepout \( \gamma \) is a smooth map from \( S^1 \times S^1 \) to \( S^2 \) of degree 1. We may consider \( \gamma \) to be a map from \( S^1 \) to \( \Lambda \), and we use the notation \( \gamma_t(s) \) to denote \( \gamma(t, s) \). We say that a family of curves \( \gamma_t \) for \( t \in [-1, 1] \) is a slicing of \((S^2, g)\) if there exists a diffeomorphism \( f \) from the standard 2-sphere to \((S^2, g)\) such that \( f \) sends each longitudinal circle \( \{(x, y, z) \in S^2 | z = t\} \) to \( \gamma_t \).

In this paper we prove the following conjecture of R. Rotman.

**Theorem 1.1.** If there exists a sweepout \( \gamma_t \) of \((S^2, g)\) consisting of curves of length at most \( L \), then for any \( \epsilon > 0 \) there exists a slicing of \((S^2, g)\) consisting of curves of length at most \( L + \epsilon \).

Our construction of a slicing proceeds in three steps. The first step is to modify our original sweepout \( \gamma \) so that it begins and ends on a constant curve. This is accomplished by means of a certain surgery along a self-intersection point. In Section 3 we define a procedure that involves cutting some of the curves in \( \gamma \) at their self-intersection points and assembling the resulting subcurves into a new noncontractible family of curves that starts and ends at a point. This is illustrated on Figure 1.

For a general family of curves such a surgery may not be possible (see Figure 5 and the discussion in the end of 3.1). We show, however, that it is possible whenever the family \( \gamma \) is noncontractible. To prove this we derive a certain formula (Proposition 2.1) that relates the degree of the map \( \gamma \) to the number of times a small loop (oriented
Figure 1. Producing a sweepout with constant curves as endpoints

appropriately) is created or destroyed in this family. This formula is a consequence of Whitney’s theorem that the turning number of a curve on the plane does not change under regular homotopies.

The second step is to remove the self-intersections of each curve in the family. To accomplish this, we apply Theorem 1.1’ from [CL]. This theorem takes a homotopy between two simple closed curves $\gamma_0$ and $\gamma_1$ through curves of length less than $L$, and either constructs an isotopy between $\gamma_0$ and $\gamma_1$ or an isotopy between $\gamma_0$ and $-\gamma_1$ through curves of length at most $L + \epsilon$. Here, $-\gamma_1$ is $\gamma_1$ with the opposite orientation. We give a brief description of this construction as we will need to use some of its properties in this paper.

Consider a closed curve $C$ as a graph with vertices corresponding to self-intersection points and edges corresponding to arcs of the curve. A redrawing of $C$ is a connected closed curve obtained by traversing the edges of this graph exactly once in such a way that the result is simple after a small perturbation. In [CL] we showed that, given a homotopy $\gamma$, the space of all redrawings of all curves in $\gamma$ is homeomorphic to a graph and that this graph contains a path that connects a redrawing of the initial curve to a redrawing of the final curve. We apply the same argument to our family of curves that starts and ends at a point. As a result we obtain a family that starts and ends at a point and consists of simple closed curves. A priori this new family may be contractible in $\Lambda$. This indeed happens if the degree of the map $\gamma$ is even.
In Section 4 we show that if we start with a family of curves that corresponds to a map of the torus of odd degree then this construction will produce a sweepout.

Finally, we want to remove pairwise intersections between curves. To accomplish this, we use a theorem in [CR] that says that any homotopy of simple curves on a Riemannian surface can be turned into a homotopy of mutually disjoint simple curves with the same length bound. The theorem also indicates how this new homotopy is related to the original initial and final curves. After the first two steps each curve in our modified family consists of arcs of some curve in the original family. After the application of this last step the resulting slicing will consist of curves made out of arcs of curves in the original homotopy as well as certain geodesic arcs.

Our argument works in the same way if we start not from a sweepout of the sphere, but from any family of closed curves which corresponds to a map from the torus of odd degree. In particular, the methods above prove the following result, which partially answers a question of N. Hingston and H.-B. Rademacher which appeared in [HR] and [BM].

Let $M$ be a Riemannian manifold. Given a homology class $X \in H_m(\Lambda M, \mathbb{Z})$ define a critical level of $X$ to be the following min-max quantity

$$cr(X) = \inf_{\gamma \in X} \sup_{t \in S^1} \text{length}(\gamma_t)$$
where the infimum runs over all families of curves in the homology class $X$. It is a standard result in Morse theory that every critical level corresponds to a closed geodesic on $M$ of length equal to $cr(X)$. N. Hingston and H.-B. Rademacher asked how $cr(X)$ and $cr(kX)$ are related for some integer $k$ and some homology class $X$ of the free loop space of $M = (S^n, g)$ of infinite order. We give a negative answer to this question in the following case.

**Corollary 1.2.** If $X$ is the generator of $H_1(\Lambda, \mathbb{Z})$ and $k$ is odd, then $cr(X) = cr(kX)$.

In the last section we use the methods developed in this article to prove a conjecture from [CL] about isotopies of curves on a Riemannian 2-surface.

**Theorem 1.3.** Let $M$ be a 2-dimensional Riemannian manifold (with or without boundary) and let $\gamma_0$ and $\gamma_1$ be two simple closed curves which are homotopic through curves of length $\leq L$. For any $\epsilon > 0$ one of the following statements holds:

1. $\gamma_0$ and $\gamma_1$ are homotopic through simple closed curves of length at most $L + \epsilon$.
2. $\gamma_0$ and $\gamma_1$ are each contractible through simple closed curves of length $L + \epsilon$.

Here, we mean that all curves except for the final curve are simple.

To illustrate this theorem, let $\gamma_0$ be a small contractible loop on a surface $M$ and let $\gamma_1$ be the same loop with the opposite orientation and suppose they are homotopic through short curves. If $M$ is a torus then $\gamma_0$ and $\gamma_1$ are homotopic, but not isotopic. If $M$ is a sphere then $\gamma_0$ and $\gamma_1$ are isotopic, but possibly only through curves of much larger length.

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2. REIDEMEISTER MOVES AND DEGREES OF MAPS

2.1. **Generic sweepouts.** We begin by defining three types of Reidemeister moves, shown in Figure 2. These are ways in which a generic one parameter family of smooth curves may locally self-interact. As in this figure, we categorize them as Type 1, Type 2, and Type 3 moves.

Now, fix a map $\gamma : S^1 \to \Lambda$. For any $\epsilon > 0$, Thom’s Multijet Transversality theorem (see [GG], [CL, Proposition 2.1]) implies that we can perturb $\gamma$ to $\tilde{\gamma}$ so that $\tilde{\gamma}$ has the following properties:

1. For every $t \in S^1$, $\text{length}(\tilde{\gamma}_t) \leq \text{length}(\gamma(t)) + \epsilon$.
2. $\tilde{\gamma}$ is a sweepout.
(3) For each \( t \in S^1 \), \( \tilde{\gamma}_t \) contains only isolated self-intersections. That is, for each self-intersection of each \( \tilde{\gamma}_t \), there is an open ball centered at that self-intersection that contains no other self-intersection.

(4) \( \tilde{\gamma} \) is composed of a finite set of Reidemeister moves. That is, there is a finite sequence of points \( t_1, \ldots, t_n \in S^1 \) such that exactly one Reidemeister move occurs between \( t_i \) and \( t_{i+1} \) for each \( i \in \{1, \ldots, n-1\} \), and exactly one Reidemeister move occurs between \( t_n \) and \( t_1 \). At each time \( t_i \), the curve has transverse intersections only.

Since any smooth map can be perturbed to have this property, we will assume that \( \gamma \) has already been put into this form with \( \epsilon \) sufficiently small. Additionally, for any sweepout \( \gamma \), we say that it is generic if it has the above properties. We define a generic homotopy and a generic family \( S^1 \to \Lambda \) of closed curves in analogous ways.

2.2. Degree and Type 1 Reidemeister moves. We will say that a continuous family of closed curves \( \{\gamma_t\}_{t \in S^1} \) has degree \( d \) if the corresponding map from the torus to \( S^2 \) has degree \( d \). In this section, we will describe a procedure that assigns either a \( +1 \) or a \( -1 \) to each Type 1 move in a sweepout. This will be called the sign of the corresponding Type 1 move. We will then prove a formula relating the sum of these signs to the degree of \( \gamma \). This formula is a consequence of Whitney’s theorem that connected components of the space of immersed curves in \( \mathbb{R}^2 \) are classified by their turning numbers.

The turning number \( T(\gamma) \) of an immersed curve \( \gamma \in \mathbb{R}^2 \) is defined as the degree of the Gauss map sending a point \( t \in S^1 \) to \( \frac{d\gamma(t)}{|d\gamma(t)|} \).

Fix an orientation on \( S^2 \). Let \( \gamma \) be a generic homotopy of closed curves on \( S^2 \). Suppose a Type 1 move happens at time \( t' \). Note that for a small \( \delta > 0 \), the curves \( \gamma_t \) with \( t' - \delta \leq t \leq t' + \delta \) are immersed everywhere except for a small open disc where a small loop is created or destroyed. Fix a point \( p \in S^2 \) such that \( p \) does not intersect \( \gamma_t \) for all \( t \in [t' - \delta, t' + \delta] \). Let \( St_p : S^2 \setminus \{p\} \to \mathbb{R}^2 \) denote the stereographic projection with respect to \( p \). We say that the Type 1 move is positive if the turning number of the curve \( St_p(\gamma_t) \) in \( S^2 \setminus \{p\} \) increases by one and we say that it is negative if it decreases by 1. Observe that this definition is independent of the choice of \( p \) as long as \( \gamma_t \) does not intersect \( p \) for \( t' - \delta \leq t \leq t' + \delta \).

**Proposition 2.1.** Let \( \gamma : S^1 \to \Lambda \) be a generic family of closed curves of degree \( d \). The sum of the signs of all Type 1 moves is equal to \( 2d \).

**Proof.** Let \( \{\gamma_t\}_{t \in S^1} \) be a generic homotopy and let \( C \) be the union of self-intersection points of \( \gamma_t \) for all \( t \). Observe that, for a generic homotopy \( \gamma \), the set \( C \) has Hausdorff dimension 1.

Define \( U \subset S^2 \setminus C \) to be the set of points such that, for each \( x \in U \), there are only finitely many times \( t \in \{t_1, \ldots, t_k\} \) when \( \gamma_t \) intersects \( x \). Moreover, for each \( t_i \)
Figure 3. The degree of $St_x(\gamma_t)$ changes by 2 when $\gamma_t$ passes through the point at infinity.

and $s_i$ with $\gamma_{t_i}(s_i) = x$, we require that $\frac{\partial \gamma}{\partial t}(t_i, s_i) \neq 0$. Since $\gamma$ is generic, $U$ is a set of full measure in $S^2$. Fix $x \in U$. For each moment of time when $\gamma_t$ passes through $x$, define the local degree $d_i(x)$ to be the sign of the frame $(\frac{\partial \gamma}{\partial t}(t_i, s_i), \frac{\partial \gamma}{\partial s}(t_i, s))$. The sum of all $d_x(i)$'s is then equal to the degree of $\gamma_t$.

Consider a segment of the homotopy $\gamma_t$ between $t_i$ and $t_{i+1}$. By a result of Whitney [W] the turning number of the composition of $\gamma_t$ with the stereographic projection $St_x(\gamma_t)$ does not change unless $\gamma_t$ undergoes a Type 1 move. In this case, the turning number changes by 1 (respectively $-1$) under a positive (respectively negative) Type 1 move.

When $t$ approaches $t_i$, the curve $St_x(\gamma_t)$ is contained in some large disc $D$ except for a simple arc $a$ which stretches to infinity. As $\gamma_t$ passes through $x$ at time $t_i$, the arc $a$ is replaced by an arc $b$ as shown in Figure 3. We observe that the turning number of $St_x(\gamma_t)$ decreases by 2 if $d_x(i)$ is positive and increases by 2 if $d_x(i)$ is negative.

Since the homotopy is defined on $S^1$, it starts and ends on the same curve with the same turning number. It follows that the sum of signs of Type 1 moves equals $2 \sum d_x(i)$.

When $\gamma$ has no Type 1 moves this result simplified to the following statement.

**Corollary 2.2.** If $\gamma : S^1 \to \Lambda$ is a generic family that contains no Type 1 Reidemeister moves, then $\gamma$ is contractible.
Remark 2.3. Corollary 2.2 also follows from the fact that the fundamental group of each connected component of the space of immersed curves is isomorphic to \( \mathbb{Z}_2 \) (see [S], [I], and [T]). Here’s a sketch of the proof of this fact. Let \( \gamma_0 \) be an immersed curve and let \( F_{\gamma_0} \) be the connected component of the space of immersed curves containing \( F_{\gamma_0} \). It follows from the parametric h-principle [Gr] that the space of immersed curves is weak homotopy equivalent to \( \Lambda SM \), the space of free loops on the spherical tangent bundle \( SM \) of \( M \). It can be shown (see [H]) that the group \( \pi_1(F_{\gamma_0}) \) is isomorphic to the centralizer of the class represented by \( (\gamma_0(s), \frac{d\gamma_0}{ds}(s)) \) in \( \pi_1(SM, (\gamma_0(0), \frac{d\gamma_0}{ds}(0))) \). Since \( S S^2 \cong SO_3 \cong \mathbb{R}P^3 \) we obtain that \( \pi_1(F_{\gamma_0}) = \mathbb{Z}_2 \).

The inclusion map \( \iota : F_{\gamma_0} \to \Lambda M \) induces a homomorphism \( \iota_* : \pi_1(F_{\gamma_0}) \to \pi_1(\Lambda M) \). Since \( \pi_1(F_{\gamma_0}) = \mathbb{Z}_2 \) and \( \pi_1(\Lambda M) = \mathbb{Z} \), \( \iota_* \) must be trivial.

2.3. Degree of a family containing constant curves. From Corollary 2.2 we obtain the following characterization of the degree of a family of curves that consists of immersed curves and constant curves. This characterization will be important in the proof of our main theorem. Let \( \gamma : S^1 \to \Lambda \) be a family of closed curves such that, for some closed interval \( I \subset S^1 \), we have that \( \gamma_t \) is a constant curve for all \( t \in I \) and \( \gamma_{it} \) is a generic homotopy with no moves of Type 1 for \( t \in S^1 \setminus I \). We can show then that the family \( \gamma \) has degree 0 or \( \pm 1 \). If we consider \( \gamma \) on \( S^1 \setminus I \), we get a map from an open interval of \( S^1 \) to \( \Lambda \). After a small perturbation, we may assume that this map is generic, and so if we choose points close to the endpoints of the open interval, the corresponding curves are simple. The degree of \( \gamma \) then depends on the orientations of these curves. This dependence is described as follows. We assume that \( \gamma \) has been perturbed slightly so the above property is true.

Fix an orientation on the sphere. As above, for a small \( \delta > 0 \), let \( t_1 \) and \( t_2 \) be two points at the distance \( \delta \) from the two endpoints of the open interval \( S^1 \setminus I \). If \( \delta \) is chosen to be sufficiently small, then for each \( i = 1, 2 \), the curve \( \gamma_{ti} \) is a simple closed curve bounding a small disc \( D_i \). The orientation of the sphere then induces an orientation of \( D_i \), which in turn induces an orientation on \( \partial D_i \). If this orientation coincides with the orientation of \( \gamma_{ti} \), we say that \( \gamma_{ti} \) is positively oriented and we say that it is negatively oriented otherwise. We say that \( \gamma \) has the same orientation at the endpoints of \( S^1 \setminus I \) if for all sufficiently small \( \epsilon > 0 \) both \( \gamma_{t_1} \) and \( \gamma_{t_2} \) are oriented positively or both are oriented negatively. Otherwise, we say that \( \gamma \) has different orientations at the endpoints of \( S^1 \setminus I \).

Corollary 2.4. Let \( \gamma \) be as described above. If \( \gamma \) has the same orientation at the endpoints of \( S^1 \setminus I \), then \( \gamma \) has degree 0. If \( \gamma \) has different orientations at the endpoints of \( S^1 \setminus I \), then \( \gamma \) has degree \( \pm 1 \).

Proof. Suppose first that \( \gamma \) has the same orientation at the endpoints. Define a new homotopy \( \tilde{\gamma} \) as follows. For a sufficiently small \( \delta > 0 \), choose \( t_1 \) and \( t_2 \) as above. On
the interval $[t_1, t_2] \subset S^1 \setminus I$, we set $\tilde{\gamma}$ to be equal to $\gamma$. On the complement of this interval we define a homotopy from $\gamma_{t_1}$ to $\gamma_{t_2}$ through short simple closed curves as depicted in Figure 4. Observe that $\tilde{\gamma}$ will have the same degree as $\gamma$, since two maps coincide on $[t_1, t_2]$ and on the complement of $[t_1, t_2]$ the images of both maps have very small measure. By Lemma 2.2 the degree of $\tilde{\gamma}$ is 0.

Suppose $\gamma$ has different orientations at the endpoints. As in the first case we will replace a portion of $\gamma$ on the interval $I$ with a different family of curves. Instead of a family of short curves on Figure 4, we will use a family that comes from a slicing of the sphere. Let $\beta$ be a slicing of $S^2$ with the point $\beta(-1)$ contained in the small disc $D_1$ and $\beta(1)$ contained in the small disc $D_2$. Note that we do not require any control over the lengths of curves in $\beta$ for this lemma, as the statement that we are proving is purely topological.

After a small perturbation we may assume that, for some small $\epsilon > 0$, the image of $\beta(-1 + \epsilon)$ coincides with the image of $\gamma_{t_1}$ and the image of $\beta(1 - \epsilon)$ coincides with the image of $\gamma_{t_2}$. Observe that $\beta$ has different orientation at the endpoints, and so after a reparametrization we may assume that $\gamma_{t_1} = \beta(-1 + \epsilon)$ and $\gamma_{t_1} = \beta(1 - \epsilon)$. We define a new family $\gamma'$ by setting it to be equal to $\gamma$ on the interval $[t_1, t_2]$, and to coincide with $\beta[-1 + \epsilon, 1 - \epsilon]$ on the rest of $S^1$. Since $\gamma'$ consists of immersed curves, it must have degree 0. For any point $x \in S^2$ that does not lie in one of the small discs $D_1$ or $D_2$, there is exactly one preimage of $x$ under $\beta$. Therefore, if $x$ is also a regular point of $\gamma'$, then the total sum of signed preimages of $x$ under $\gamma$ restricted to $[t_1, t_2]$ must be equal to $\pm 1$. We conclude that the degree of $\gamma$ is $\pm 1$. \qed
Surgery on Sweepouts

The purpose of this section is to perform surgery on sweepouts to prove the following proposition:

**Proposition 3.1.** Given a generic sweepout \( \gamma : S^1 \to \Lambda \) through curves of length at most \( L \) and an \( \epsilon > 0 \), we can find a sweepout \( \tilde{\gamma} \) such that \( \tilde{\gamma}_0 \) is a constant curve and \( \tilde{\gamma} \) consists of curves of length no more than \( L + \epsilon \).

To prove this theorem, we will require methods from the article [CL] by the two authors. This method begins with defining a certain graph. Fix a map \( \gamma : S^1 \to \Lambda \), and choose \( t_1, \ldots, t_{n+1} \in S^1 \) such that \( t_i \neq t_j \) if \( i \neq j \) and \( 1 \leq i, j \leq n \), exactly one Reidemeister move occurs between each \( t_i \) and \( t_{i+1} \) for \( i \in \{1, \ldots, n\} \), and \( t_1 = t_{n+1} \). Note that at each time \( t_i \), \( \gamma_{t_i} \) is an immersed curve with transverse self-intersections, each of which consists of exactly two arcs meeting at a point.

### 3.1. Graph of self-intersections.

Form the graph \( \Gamma \) as follows. We will first describe how to add vertices, and then describe how to connect them with edges.

**Vertices** The vertices of \( \Gamma \) will fall into \( n + 1 \) sets \( V_1, \ldots, V_{n+1} \) and are defined as follows. First of all, \( V_{n+1} \) will simply be a copy of \( V_1 \). Next, to construct \( V_i \) for \( i \in \{1, \ldots, n\} \), we do the following. If \( \gamma_{t_i} \) is simple, then \( V_i \) is empty. If \( \gamma_{t_i} \) is not simple, then every vertex in \( V_i \) corresponds to a self-intersection of \( \gamma_{t_i} \).

**Edges** To add edges to the graph, we do the following. First, add an edge between each vertex in \( V_1 \) and each corresponding vertex in \( V_{n+1} \). Next, for each \( i \in \{1, \ldots, n\} \), consider the Reidemeister move \( R \) in \( \gamma \) between \( \gamma_{t_i} \) and \( \gamma_{t_{i+1}} \). We add edges on a case-by-case basis:

1. If \( R \) is of Type 1, then we have two cases. The first case is if \( V_i \) or \( V_{i+1} \) is empty; in this case we do not add any vertices to \( \Gamma \). The second case is if either \( V_i \) or \( V_{i+1} \) has more than 1 vertex. In this case, we do the following. For every vertex in \( V_i \) that corresponds to a self-intersection \( z \) of \( \gamma_{t_i} \), either \( z \) is deleted by the Reidemeister move \( R \), in which case we don’t add an edge to \( z \), or we can follow \( z \) forward to a self-intersection \( z' \) of \( \gamma_{t_{i+1}} \), in which case we join the vertex that corresponds to \( z \) to the vertex that corresponds to \( z' \) with an edge.
2. If \( R \) is of Type 2, then one of two things are true. One possibility is that we can find two vertices in \( V_i \) that correspond to two distinct self-intersection points \( x \) and \( y \) in \( \gamma_{t_i} \) that are deleted by \( R \). We then join these two vertices by an edge. For every other vertex in \( V_i \), that vertex corresponds to a self-intersection \( z \) of \( \gamma_{t_i} \). We can follow this self-intersection forward in time to a self-intersection \( z' \) of \( \gamma_{t_{i+1}} \). We join the vertex that corresponds to \( z \) to the vertex that corresponds to \( z' \) with an edge.
The other possibility is that we can find two vertices in $V_{i+1}$ that correspond to self-intersection points $x$ and $y$ that were created by $R$. In this case, we join these two vertices by an edge. Additionally, for every other vertex in $V_{i+1}$, that vertex corresponds to a self-intersection point $z$ of $\gamma_{t_i+1}$. There is a self-intersection point $z'$ of $\gamma_{t_i}$ such that $z'$ can be followed forward to $z$. We join the vertex that corresponds to $z'$ to the vertex that corresponds to $z$ with an edge.

(3) If $R$ is of Type 3, then every vertex in $V_i$ corresponds to a self-intersection point $z$ of $\gamma_{t_i}$. This self-intersection point can be followed forward to a self-intersection point $z'$ of $\gamma_{t_{i+1}}$. We join the vertex that corresponds to $z$ to the vertex that corresponds to $z'$ with an edge.

The following lemma characterizes the degree of each vertex of the graph $\Gamma$.

**Lemma 3.2.** Each vertex in $\Gamma$ has degree 0, 1 or 2. Furthermore, using the above notation, consider any vertex $v \in V_i$ and the self-intersection $s$ of $\gamma_{t_i}$ that corresponds to $v$. Let $x, y \in \{0, 1\}$ be defined as follows. If $s$ is destroyed in a Type 1 deletion between $t_i$ and $t_{i+1}$, then $x = 1$, otherwise $x = 0$. If $s$ is created in a Type 1 creation between $t_{i-1}$ and $t_i$, then $y = 1$, otherwise $y = 0$. We then have that the degree of $v$ is $2 - x - y$. In the above statements, if $i = 1$, then $y = 1$, and if $i = n + 1$, then $x = 1$.

**Proof.** For each vertex $v \in V_i$, consider the Reidemeister move $R_1$ that occurs between $t_{i-1}$ and $t_i$, and let $R_2$ be the Reidemeister move that occurs between $t_i$ and $t_{i+1}$. Again, if $i = 1$, then $R_1$ is not defined, and if $i = n + 1$, then $R_2$ is not defined.

From the definition of the edges of $\Gamma$, we see that the edges added to $v$ (with corresponding self-intersection $s$ of $\gamma_{t_i}$) work exactly as follows. If $R_1$ is not defined or does not create $s$ in a Type 1 creation, then an edge is added to $\Gamma$ at $v$. If $R_2$ is not defined or does not destroy $s$ in a Type 1 deletion, then a separate edge is added to $\Gamma$ at $v$. This coincides with the degree computations in the statement of the lemma. \qed

We can look at the set of all vertices $V$ such that a vertex $v \in V$ corresponds to a vertex right after it was created by a Type 1 move, or just before it is destroyed by a Type 1 move. A corollary of this lemma is that $V$ can be decomposed in a particular way.

**Corollary 3.3.** The set $V$ is equal to the union of a number of disjoint pairs of vertices such that, for each pair $(v, v')$, there is a path in the graph from $v$ to $v'$. Furthermore, the path between a pair $(v, v')$ and the path between a different pair $(w, w')$ are completely disjoint (they do not share any edges). Note that we may have that $v = v'$ for a given pair $(v, v')$. 
We will use such paths to generate our new homotopy. We first require a definition concerning how to cut a curve at a self-intersection.

**Definition 3.4.** Given a smooth curve \( \alpha : S^1 \to S^2 \) with isolated self-intersections, we say that a curve \( \beta : S^1 \to S^2 \) is a *subcurve* of \( \alpha \) if there is some closed interval \([a, b] \subset S^1\) (possibly with \(a = b\)) such that \( \beta \) is simply \( \alpha \) restricted to \([a, b]\). We have that \( \alpha(a) = \alpha(b) \), so \( \beta \) is all of \( \alpha \), is a point, or \( \alpha(a) \) is a self-intersection of \( \alpha \).

For each pair of self-intersections \((v, v')\) from Corollary 3.3, the self-intersection \(s\) which corresponds to \(v\) produces two subcurves, \(C_{v,1}\) and \(C_{v,2}\). To form \(C_{v,1}\), begin at \(s\) and follow the loop around according to its orientation until we get back to \(s\). If we continue along the loop according to its orientation, we will encounter \(s\) another time. This forms the second subcurve \(C_{v,2}\). Similarly, the self-intersection which corresponds to \(v'\) produces two subcurves \(C_{v',1}\) and \(C_{v',2}\). Since each edge in the graph corresponds to a continuous path between self-intersections, the path between \(v\) and \(v'\) produces a continuous path between the self-intersection that corresponds to \(v\) and the self-intersection that corresponds to \(v'\). This path then induces a homotopy from \(C_{v,1}\) to \(C_{v',i}\), for some \(i \in \{1, 2\}\), and it induces a homotopy from \(C_{v,2}\) to \(C_{v',j}\), where \(j \neq i\). For consistency, assume that \(C_{v,1}\) and \(C_{v',1}\) correspond to the loops that were just created or are about to be destroyed by the appropriate Type 1 moves. Let these two homotopies be denoted by \(h_{v,v',1}\) and \(h_{v,v',2}\), respectively.

**Definition 3.5.** Given a pair \((v, v')\) as above, we say that it is *good* if \(h_{v,v',1}\) ends at \(C_{v',2}\) (\(h_{v,v',1}\) starts at \(C_{v,1}\) by definition).

If a good pair \((v, v')\) exists, we can contract any curve in the sweepout to a point through curves of controlled length.

**Lemma 3.6.** If the graph \(\Gamma\) contains a good pair, then for any curve \(\gamma_t\) in the sweepout, there is a contraction of \(\gamma_t\) to a point through curves of length at most \(L + \epsilon\).

**Proof.** Observe that we can use the Type 1 destruction or creation of a small loop which corresponds to the terminal vertex \(v'\) to homotope \(C_{v',2}\) to a curve in the sweepout \(\gamma\). Denote this curve by \(\gamma_{t*}\). Similarly, we can use the Type 1 move corresponding to the initial vertex \(v\) to homotope the curve \(C_{v,1}\) to a point. The fact that \((v, v')\) is a good pair means that \(C_{v,1}\) and \(C_{v',2}\) are homotopic via the homotopy \(h_{v,v',1}\) through subcurves of curves in \(\gamma\). Hence, \(\gamma_{t*}\) (and every other curve in \(\gamma\)) is homotopic to a point through subcurves of curves in \(\gamma\). \(\square\)

We now have that, if there is a good pair \((v, v')\), then our proof of Proposition 3.1 is true. To see this let the homotopy \(\alpha_t\) be a contraction of a curve in \(\gamma\). Define \(\tilde{\gamma} = (-\alpha) * \gamma * \alpha\), where \(-\alpha\) signifies \(\alpha\) in the reverse direction. This family of curves has the same degree as \(\gamma\) and starts at a constant curve. By Lemma 3.6 we can...
Figure 5. We cannot remove self-intersections by repeatedly applying the cutting procedure.

choose $\alpha$ so that the lengths of curves are bounded by $L + \epsilon$. The next subsection proves the existence of such a good pair for any sweepout $\gamma$.

If $(v, v')$ is not a good pair, then the “cutting” procedure as above produces two homotopies, at least one of which is nontrivial. One is tempted to apply both of these cutting procedures repeatedly in the hope of constructing a homotopy with the desired properties. However, even for a simple case of homotopies of curves with at most 2 self-intersections it may happen that the maximal number of self-intersections of curves in the new homotopy does not decrease, no matter how many times we apply the cutting procedure. In fact, there are situations in which a portion of the homotopy is replicated every time we apply the cutting procedure. This is illustrated in Figure 5.

The top picture describes the original homotopy. After we apply the cutting procedure we obtain two new homotopies. One of these homotopies will start at a point and end at a point, but it may happen that it has degree 0. Then we have to consider the other homotopy, which is shown on the bottom of Figure 5. This homotopy has two Type 1 creations exactly like the original homotopy, so we cannot get rid of these intersections by applying the cutting procedure again.

3.2. **Existence of a good pair** $(v, v')$. Assigning to each Type 1 move a sign as we did when defining the degree formula in Proposition 2.1 (using some appropriate $x \in S^2$), we see that since the degree of $\gamma$ is odd, the sum of all of the signs of all of the Type 1 moves is $2 \mod 4$. Hence, we can find a pair $(v, v')$ such that the sign of the Type 1 move associated to $v$ is positive, as is the sign of the Type 1 move...
associated with \( v' \). Note that if \( v = v' \), then there is some ambiguity as to which Type 1 moves we are referring to. In this case, there is a self-intersection that is created by a positive Type 1 move, and then which is immediately destroyed by a positive Type 1 move. In this case, clearly \((v,v')\) is a good pair, and these are the two Type 1 moves to which we are referring. For the remainder of this section, we may thus assume that \( v \neq v' \). We then have the following.

**Lemma 3.7.** If \( v \) and \( v' \) correspond to Type 1 moves of the same sign then the pair \((v,v')\) is a good pair.

**Proof.** Note that a positive Type 1 move can be either a creation of a small positively oriented loop or a destruction of a small negatively oriented loop. Each point on the graph \( \Gamma \) corresponds to a transverse self-intersection point of \( \gamma_t \) for some \( t \). By cutting \( \gamma_t \) at the self-intersection point we obtain two connected curves \( \gamma_{t}^{A} \) and \( \gamma_{t}^{B} \). To each point on the graph and a choice of \( \gamma_{t}^{A} \) or \( \gamma_{t}^{B} \), we will associate a binary invariant \( L \) which we will call the local orientation. This invariant is based only on local data in the neighbourhood of the self-intersection. We will show that the invariant changes sign every time the path in the graph between \( v \) and \( v' \) changes direction.

Let \( p \) be a self-intersection point of \( \gamma_t \). Without any loss of generality we may assume that two arcs intersect at \( p \) perpendicularly. We cut \( \gamma_t \) at \( p \) and smooth out the intersection in such a way that we obtain two connected curves that inherit their orientations from \( \gamma_t \). Let \( Q \) be a small disc in the neighbourhood of \( p \). After smoothing, the curve separates \( D \) into three connected components (see Figure 6). Let \( A \) and \( B \) denote two components that do not share a boundary and let \( \gamma_{t}^{A} \) and \( \gamma_{t}^{B} \) denote curves adjacent to \( A \) and \( B \) correspondingly. Let \( v_1 \) be a tangent vector of \( \gamma_{t}^{A} \) at \( p \) (solid line on Figure 6) and let \( v_2 \) be a vector pointing from \( p \) to a point on \( \partial A \cap \partial D \) (dashed line on Figure 6). We define \( L(\gamma_{t}^{A}) = 1 \) if the ordered pair \((v_1,v_2)\) is positively oriented and \( L(\gamma_{t}^{A}) = -1 \) otherwise. We define \( L(\gamma_{t}^{B}) \) in the same manner. Observe that \( L(\gamma_{t}^{A}) = -L(\gamma_{t}^{B}) \).
For each point $q$ in the path $P$ between $v$ and $v'$, let $L_q$ denote $L(\gamma^A_t)$, where $\gamma^A_t$ corresponds to the appropriate curve in $h_{v,v',1}$.

We observe that $L_q$ does not change when $\gamma$ undergoes a Type 3 move or a move that does not involve the self-intersection that we follow to form the path $P$. Consider a segment of the path $P$ corresponding to a part of the homotopy where $\gamma_t$ goes through a Type 2 move involving the self-intersection that we follow to form $P$. On the graph $\Gamma$, this looks like a change of direction of the path $P$ (see Figure 8). Let $q_1$ be a point on the path just before the change of direction and let $q_2$ be a point just after it. We claim that $L_{q_1} = -L_{q_2}$. This follows by considering Figure 7.

With this result we can now prove the lemma. Suppose $(v,v')$ is not a good pair. Consider two cases. Suppose first that the path $P$ changes direction an odd number of times as in Figure 8(a). It follows that the Type 1 move corresponding to $v$ and the Type 1 move corresponding to $v'$ are either both creations, or are both destructions. In either case, the orientation of the loop that is being created or destroyed at $v$ is different than the orientation of the loop that is being created or destroyed at $v'$ since $L$ changes sign from the beginning of $P$ to the end of $P$ and $(v,v')$ is not a good pair. Thus, the sign of the Type 1 move associated with $v$ is opposite to the sign of the Type 1 move associated with $v'$, which is a contradiction.

Suppose $P$ changes direction an even number of times as in Figure 8(b). Then either the Type 1 move associated with $v$ is a creation and the Type 1 move associated with $v'$ is a destruction, or the move associated with $v$ is a destruction and the move associated with $v'$ is a creation. In either case, the orientation of the small loop that is created or destroyed at $v$ is the same as the orientation of the small loop that is being created or destroyed at $v'$, since the $L$ invariant has the same value at the beginning and at the end of $P$, and since $(v,v')$ is not a good pair. This implies that the sign of the Type 1 move at $v$ is opposite to that of the Type 1 move at $v'$, which is a contradiction.
As described above, this implies that Proposition 3.1 is true.

4. Proof of Theorem 1.1

Given a sweepout $\gamma$, we apply Proposition 3.1 to it to obtain a sweepout $\tilde{\gamma}$ that starts and ends on a constant curve. For simplicity we will think of $\tilde{\gamma}$ as a family of curves defined on $[0, 1]$ with $\tilde{\gamma}_0 = \tilde{\gamma}_1$ being the constant curve. It follows from the construction that for all $t$ sufficiently close to the endpoints $\tilde{\gamma}_t$ is a simple closed curve, and $\tilde{\gamma}_t$ is not constant for $t \in (0, 1)$.

Our goal now is to modify $\tilde{\gamma}$ so that it consists of curves without self-intersections, satisfies the desired length bound and has an odd degree (in fact, we will show that the new family has degree $\pm 1$). To do this we use the results of [CL]. Let $t_1 = \delta$ and $t_2 = 1 - \delta$ be two points near the endpoints of $[0, 1]$ so that $\tilde{\gamma}_{t_1}$ and $\tilde{\gamma}_{t_2}$ are simple closed curves each bounding a small closed disc, denoted by $D_1$ and $D_2$ respectively. We perturb the homotopy slightly so that discs $D_1$ and $D_2$ are disjoint. If $\delta$ is sufficiently small we can always do this without increasing the lengths of curves by more than $\epsilon$.

We can now apply Theorem 1.1’ from [CL] to the homotopy $\tilde{\gamma}[t_1, t_2]$. This produces an isotopy $\alpha$ which starts at $\tilde{\gamma}_{t_1}$ and ends at $\pm \tilde{\gamma}_{t_2}$, where $-\tilde{\gamma}_{t_2}$ denotes $\tilde{\gamma}_{t_2}$ with the opposite orientation. Curves $\alpha_0$ and $\alpha_1$ are simple curves of some very small length. We can contract each of them to a point in the corresponding small disc $D_i$ through short simple closed curves. Hence, we can turn $\alpha$ into a map $\beta : S^2 \to S^2$.

**Lemma 4.1.** Map $\beta$ has degree $\pm 1$.

**Proof.** First, we show that the degree of $\beta$ must be odd.
Let \( x \in S^2 \) be a regular point of \( \tilde{\gamma} \), which does not intersect any of the curves \( \tilde{\gamma}_t \) for \( t \) in one of the small intervals \([0, t_1]\) and \([t_2, 1]\), and which also does not lie on any of the self-intersection points of curves in \( \tilde{\gamma} \). By selecting \( t_1 \) and \( t_2 \) sufficiently close to 0 and 1, we can ensure that the set of such points has nearly full measure in \( S^2 \). In addition we require that \( x \) does not lie on any curve of \( \tilde{\gamma}_{[t_1, t_2]} \) at which a Reidemeister move occurs, that is, any curve which has a singularity, a tangential self-intersection or an intersection which involves three arcs. There will be only finitely many such curves in the homotopy (see Subsection 2.1).

For each \( t \in [t_1, t_2] \), the curve \( \alpha_t \) is a redrawing of some curve \( \tilde{\gamma}_{t'} \), where \( t' \in [t_1, t_2] \), but may differ from \( t \). We can modify the original homotopy \( \tilde{\gamma}_t \) so that \( \alpha_t \) is a redrawing of \( \tilde{\gamma}_t \) for each \( t \). This is done simply by making the homotopy move back and forth through the same curves multiple times for certain subintervals of the parameter space. This does not change the sum of the signs of the signed preimages of \( x \) with respect to \( \tilde{\gamma} \), and does not affect the initial and final curves. Since the total number of preimages of \( x \) under \( \tilde{\gamma} \) is odd (as the sum of the signed preimages is 1), the same must be true about \( \alpha \), and so the sum of all of the signed preimages of \( \alpha \) must be odd as well. Since \( \beta \) consists of constant curves and simple closed curves, by Corollary 2.4, \( \beta \) has degree \( \pm 1 \).

We now have that \( \alpha : [0, 1] \to S^2 \) consists of simple curves (except for \( \alpha_0 \) and \( \alpha_1 \) which are constant), but different curves in \( \alpha \) may intersect. We will use methods from [CR] to modify curves on the interval \([a, b]\) so that they have no pairwise intersections.

Fix an orientation on \( S^2 \). Let \( \gamma_t \) be a simple closed curve on \( S^2 \). \( \gamma_t \) separates \( S^2 \) into two discs. Let \( D_{\gamma_t} \) denote the connected component of \( S^2 \setminus \gamma_t \), such that the orientation inherited by \( D_{\gamma_t} \) from \( S^2 \) coincides with the orientation of \( \gamma_t = \partial D_{\gamma_t} \). We will say that \( D_{\gamma_t} \) is the interior of \( \gamma_t \) and \( S^2 \setminus D_{\gamma_t} \) is the exterior of \( \gamma_t \).

In [CR] R. Rotman and the first author introduced a procedure that takes a homotopy of simple closed curves on a surface and produces a monotone homotopy of curves of controlled length. By a monotone homotopy, we mean a homotopy through simple closed curves which are pairwise disjoint. Depending on the initial homotopy, there are two possibilities for the kind of homotopy this procedure will output. We describe these two possibilities below.

**Theorem 4.2.** ([CR]) If \( \gamma_0 \) and \( \gamma_1 \) are isotopic through simple closed curves of length at most \( L \), then at least one of the following two options holds

1. There exists an embedding \( f : [0, 1] \times S^1 \to (S^2, g) \), such that \( f(0 \times S^1) \) is in the interior of \( \gamma_0 \), \( f(1 \times S^1) \) is in the exterior of \( \gamma_1 \), and the length of \( f(t \times S^1) \) is bounded above by \( L + \epsilon \) for all \( 0 \leq t \leq 1 \).
(2) There exists a embedding $f : \mathbb{D} \to (S^2, g)$ from the Euclidean disc $\mathbb{D} = \{x^2 + y^2 \leq 1 \}$ to $S^2$ such that $f(0)$ is contained in the closure of one of the connected components of $S^2 \setminus \gamma_0$, while $f(\partial \mathbb{D})$ is contained in the closure of the other connected component of $S^2 \setminus \gamma_0$ Moreover, the length of $f(\{x^2+y^2 = t^2\})$ is bounded above by $L + \epsilon$ for all $0 \leq t \leq 1$.

We will also need the following lemma from [CR].

**Lemma 4.3.** ([CR]) Suppose there exist two embeddings $f_1 : [0, 1] \times S^1 \to (S^2, g)$ and $f_2 : [0, 1] \times S^1 \to (S^2, g)$, such that $f_1(1 \times S^1)$ is contained in the interior of $f_2(0 \times S^1)$. Moreover, assume the length of $f_i(t \times S^1)$ is bounded above by $L$ for $i = 1, 2$ and $0 \leq t \leq 1$. Then there exists an embedding $f_3 : [0, 1] \times S^1 \to (S^2, g)$ such that $f_3(0 \times S^1)$ is contained in the exterior of $f_1(0 \times S^1)$, $f_3(1 \times S^1)$ is contained in the interior of $f_2(1 \times S^1)$, and the length of $f_3(t \times S^1)$ is bounded above by $L + \epsilon$ for all $0 \leq t \leq 1$.

We want to apply Theorem 4.2 to the isotopy $\alpha$ restricted to $[t_1, t_2]$. Observe that the interior of $\alpha_{t_1}$ is the small disc $D_1$, while the interior of $\alpha_{t_2}$ is the closure of the complement of the small disc $D_2$; see Figure 9. This follows from Corollary 2.4.

If the first case of Theorem 4.2 is true, then we can contract $f(0 \times S^1)$ and $f(1 \times S^1)$ in the corresponding small discs in a monotone way, obtaining the desired slicing. If the second case of Theorem 4.2 is true and $f(0)$ is contained in the exterior of $\alpha_{t_1}$, then again we can contract $f(\partial \mathbb{D})$ in $D_1$ in a monotone way. See Figure 9 for more details.

The problematic case is when the second case of Theorem 4.2 is true, but $f(0)$ is contained in the interior of $\alpha_{t_1}$. In this case we do the following. Define $t^*$ to be the supremum over all $t \in [t_1, t_2]$ such that, if we apply Theorem 4.2 to $\alpha$ on $[t_1, t]$, then either the first case is true, or the second case is true and $f(0)$ lies in the interior of $\alpha_t$. Choosing $\delta > 0$ to be sufficiently small, we can consider the curves $\alpha_{t^*+\delta}$ and $\alpha_{t^*-\delta}$. We can then apply Theorem 4.2 to $\alpha$ on $[t_1, t^*-\delta]$, and to $\alpha$ on $[t_1, t^*+\delta]$. Let these monotone homotopies be $\rho_1$ and $\rho_2$. Due to the definition of $t^*$ and the choice of $\delta > 0$, $\rho_1$ ends at a curve in the exterior of $\alpha_{t^*-\delta}$ and begins at a constant curve, or at a curve in the interior of $D_1$. We can thus modify $\rho_1$ if necessary to start at a short simple closed curve whose interior is a small disc, and to end at a curve in the exterior of $\alpha_{t^*}$.

$\rho_2$ starts at a curve in the interior of $\alpha_{t^*+\delta}$, and ends at a constant curve in the exterior of $\alpha_{t^*+\delta}$. Again, we can modify $\rho_2$ if necessary to start in the interior of $\alpha_{t^*}$ and end at a short simple closed curve whose exterior is a small disc. $\rho_1$ and $\rho_2$ (after modification) are depicted in Figure 9.

We now apply Lemma 4.3 to $\rho_1$ and $\rho_2$. The result can easily be deformed if necessary into a slicing by contracting the endpoint curves (if they are not constant)
Figure 9. Removing intersections between different curves
in the appropriate discs in a monotone way. This completes the proof of Theorem 1.1.

5. Applications to converting homotopies to isotopies in an effective way

In this section we use the methods of this paper to prove Theorem 1.3. This result was conjectured to be true by the authors in ([CL], remark in the end of Section 3).

We wish to show that if two simple curves $\gamma_0$ and $\gamma_1$ on a Riemannian manifold $(M,g)$ are homotopic through curves of length $L$, then for any $\epsilon > 0$,

1. $\gamma_0$ and $\gamma_1$ are homotopic through simple curves of length at most $L + \epsilon$, or
2. $\gamma_0$ and $\gamma_1$ are each contractible through simple closed curves of length at most $L + \epsilon$. Here, by simple curves, we mean that all curves except for the final constant curve are simple.

We break the proof into two cases. If $\gamma_0$ is non-contractible, then since $\gamma_1$ is homotopic to $\gamma_0$, it is also non-contractible. In [CL] it was shown that $\gamma_0$ and $\gamma_1$ are isotopic through curves of length at most $L + \epsilon$.

The case that we deal with now is if $\gamma_0$ (and $\gamma_1$) are contractible. We can assume that the images of $\gamma_0$ and $\gamma_1$ do not coincide by perturbing our homotopy slightly. If we can show that the result holds for the perturbed curves, then we can perturb the curves back to the original curves, proving the theorem.

Let $M$ be any surface and consider the universal cover $\tilde{M}$ of $M$. The universal cover is either $S^2$ or a contractible subset of $\mathbb{R}^2$. Since the curves $\gamma_0$ and $\gamma_1$ are contractible we can lift the homotopy between them to a homotopy in the universal cover. Hence, without any loss of generality, we may assume that $M$ is either a plane or a sphere.

Suppose $M$ is a contractible subset of $\mathbb{R}^2$. In this case, we can define the turning number $T_0$ of $\gamma_0$ and the turning number $T_1$ of $\gamma_1$ as before. Since both $\gamma_0$ and $\gamma_1$ are simple, $T_0 = \pm 1$, and $T_1 = \pm 1$. We can apply the procedure described in [CL] to produce a homotopy $H : [0, 1] \times S^1 \to \mathbb{R}^2$ through simple curves of length at most $L + \epsilon$ from $\gamma_0$ to $\pm \gamma_1$. Since this map goes through simple closed curves, the turning number of $\gamma_0$ and the turning number of $H(1 \times S^1)$ agree. Hence, if $T_0 = T_1$, we are done.

If $T_0 \neq T_1$, then by an argument analogous to the one used to prove Proposition 2.1, the total sum of signs of Type 1 moves in $\gamma$ is $\pm 2$. Hence, as in Section 3, we can construct a graph $\Gamma$ and find a good pair of vertices $(v, v')$ connected by a path $P$ in the graph. We can cut homotopy $\gamma$ along $P$ to obtain a contraction of $\gamma_0$ through curves of length at most $L + \epsilon$. Using Theorem 1.1’ from [CL], we can turn this into a contraction through simple closed curves of length at most $L + \epsilon$. We can do the same with $\gamma_1$. This completes the proof of our theorem in the case when $M = \mathbb{R}^2$. 
Now, suppose that $M = S^2$. We begin with a lemma:

**Lemma 5.1.** If we consider the map $f : [0, 1] \times S^2$ defined by $\gamma(t, s)$, then there is a regular point $p \in S^2$ such that $f^{-1}(p)$ contains an even number of points.

**Proof.** To see this we can find a smooth map $g : [-1, 2] \times S^1$ such that

1. $f(t, s) = g(t, s)$ for $t \in [0, 1]$.
2. $g(t, s)$ on $[-1, 0]$ goes from a constant curve to $\gamma_0$.
3. $g(t, s)$ on $[1, 2]$ goes from $\gamma_1$ to a constant curve.
4. $S^2 \setminus (g([-1, 0] \times S^1) \cup g([1, 2] \times S^1))$ contains an open set $U \subset S^2$.
5. There exists an open set $V \subset S^2$ such that $V \subset (g([-1, 0] \times S^1) \cup g([1, 2] \times S^1))$ and for every $x \in V$, there is exactly one pair $(t, s) \in [-1, 0] \times S^1 \cup [1, 2] \times S^1$ that maps to $x$.

$g$ is constructed as follows. To construct $g$ on $[-1, 0]$, we choose one of the discs bounded by $\gamma_0$, and contract $\gamma_0$ to a point in the disc in a monotone way. We repeat the same procedure with $\gamma_1$. Since $\gamma_0$ and $\gamma_1$ do not have the same image, all of the above properties are satisfied.

Since $g$ maps $2 \times S^1$ to a point and $-1 \times S^1$ to a point, we can define a corresponding map from $S^2$ to $S^2$. If this degree is even, then for every regular value $p$ in $U$, the total number of points in $f^{-1}(p)$ is even, as the sum of all of the signed preimages is equal to the degree of $g$, which is even. If this degree is odd, then for every regular value in $p \in V$, the total number of preimages of $p$ ($f^{-1}(p)$) is even, since it is equal to the degree of $g \pm 1$. This completes the proof of our claim. \hfill $\Box$

We can now prove our theorem. From the above lemma, choose a regular point $p \in S^2$ whose preimage has an even number of points. Consider the stereographic projection $St_p$ from $S^2 \setminus \{p\}$ to $\mathbb{R}^2$. We can additionally assume that $p$ is not in the image of any curve in $\gamma$ at which a Reidemeister move occurs (that is, any curve in $\gamma$ that has a singularity, non-traverse intersection or an intersection which involved three arcs) and that $p$ is not in the image of $\gamma_0$ or $\gamma_1$. Let $\tilde{\gamma}$ denote the isotopy from Theorem 1.1' of [CL] that starts on $\gamma_0$ and ends on $\pm \gamma_1$. As remarked in the proof of Lemma 4.1 we can assume that $\tilde{\gamma}$ is a redrawing of the curve $\gamma_t$ for each $t$. Let $S_t$ denote the difference between the turning number of $St_p(\gamma_t)$ and the turning number of $St_p(\tilde{\gamma}_t)$. Note that $S_t$ is undefined for finitely many times $t$ when the curve $\gamma_t$ (and, as a result, $\tilde{\gamma}_t$) intersects $p$. Since both homotopies start on the same curve we have $S_0 = 0$. When $\gamma$ goes through a positive Type 1 move its turning number increases by 1, while the turning number of the corresponding curve in $\tilde{\gamma}$ remains the same. Hence $S_t$ increases by one. Similarly, it decreases by one whenever $\gamma$ goes through a negative Type 1 move. If $\gamma$ passes through the point $p$ then, as in the proof of Proposition 2.1 (see Figure 3), we have that the turning number of $\gamma$ changes by
±2, as does the turning number of $\tilde{\gamma}$. Hence, every time the curve goes through the point $p$, we have that $S_t$ changes by 0 or ±4.

In the end we have two possibilities. If $S_1 = 0$, then $\tilde{\gamma}$ defines the desired homotopy and we are done. If $S_1 = ±2$, then it follows that the total sum of signed Type 1 moves in homotopy $\gamma$ is ±2. Hence, there is a path between a good pair of vertices $(v, v')$ in the appropriate graph $\Gamma$, and so using the methods from Section 4, we can contract $\gamma_0$ to a point through simple curves of length at most $L + \epsilon$. Similarly, we can do this for $\gamma_1$. This completes the proof.

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