Directing Words of Fuzzy Finite Automata

Magnus Steinby
Department of Mathematics and Statistics
University of Turku
FI-20014 Turku, Finland

March 2, 2021

Abstract

A deterministic finite automaton is directable if it has a directing word which takes the automaton from every state to the same state. These notions have been extended also to other kinds of automata. Thus, B. Imreh and M. Steinby (1999) identified three natural types of directing words, called D1–, D2– and D3–directing words, for non-deterministic finite automata (NFAs). Here we adapt these notions for fuzzy finite automata (FFAs). The D3–directing words obtained this way are precisely the directing words introduced by V. Karthikeyan and M. Rajasekar (2015). With any FFA $F$ we associate an NFA $F_{nd}$ which has the same D$i$–directing words as $F$. Thus, if these definitions are used, the theory of directable FFAs reduces to that of NFAs.

We also introduce three new kinds of directing words of fuzzy automata that we call DD1–, DD2– and DD3–directing words, respectively which depend more on the fuzzy transition degrees between states. We establish some basic properties of the sets $DD_i(F)$ of DD$i$–directing words of any given FFA $F$. In particular, it is shown that these languages are regular, and that DD$i$–directability is decidable. For so-called normal FFAs the languages $DD_i(F)$ are shown to have some special properties. Several relationships between the families of the corresponding sets $DD_i(F)$ of DD$i$–directing words are presented. We also determine the complete ∩-semilattice of the various classes of DD$i$–directable FFAs and normal FFAs and their intersections.

Keywords: fuzzy automaton, directing word, directable automaton, nondeterministic automaton

2010 Mathematics Subject Classification: 68Q45
1 Introduction

An input word \(w\) of a deterministic finite automaton (DFA) \(A\) is said to be \textit{directing}, or \textit{synchronizing}, if it takes \(A\) from every state to a fixed state \(c\), and \(A\) is \textit{directable} if it has directing words. The topic has been extensively studied ever since it was introduced by Černý [4] in 1964. Directability has also been defined for other kinds of automata. In particular, Imreh and Steinby [12] identify three types of directing words of nondeterministic finite automata (NFAs). In [16] Karthikeyan and Rajasekar define directing words for fuzzy finite automata (FFAs), and using this notion they attempt to fuzzify the contents of [11] (without mentioning this paper).

The three types of directing words of NFAs, the D1–, D2– and D3–directing words, considered in [12], can be defined in a natural way for fuzzy automata, too. Then the directing words of [16] are the same as the D3–directing words. We show that for every FFA there is an NFA such that the two automata have exactly the same directability properties. Hence, all the results aimed at in [16], and much more, follow from known facts about NFAs (cf. [3, 5, 6, 7, 10, 12, 13, 14, 25], for example).

Then we introduce three new types of directing words of fuzzy automata that depend more on the transition degrees between states. The study of the sets of such directing words, the families of languages formed by them, and the classes of the corresponding directable FFAs form the main contents of this paper.

Section 2 introduces some basic terminology, notation and the automata to be considered. In Section 3 we first recall the definitions of directing words for DFAs as well as the D1–, D2– and D3–directing words of NFAs. Then we define D1–, D2– and D3–directing words of FFAs. With each FFA \(F\) we associate an NFA \(F^{\text{nd}}\) that faithfully reflects some important properties of the original automaton. In particular, for each \(i = 1, 2, 3\), the FFA \(F\) and the NFA \(F^{\text{nd}}\) have the same \(Di\)–directing words, which means that the properties of \(Di\)–directable FFAs and their directing words can be derived directly from the theory of \(Di\)–directable NFAs. We also note that for any FFA \(F\), the languages \(D_1(F)\), \(D_2(F)\) and \(D_3(F)\) of D1–, D2– and D3–directing words of \(F\) are regular, and that \(Di\)–directability is therefore decidable. Finally, it is shown how an algorithm for testing the directability of a DFA presented in [11] can be modified for deciding whether a given (complete) FFA is D3–directable.

In Section 4 we introduce DD1–, DD2– and DD3–directing words of FFAs, and DD1–, DD2– and DD3–directable FFAs, and present some properties of the sets \(DD_1(F)\), \(DD_2(F)\) and \(DD_3(F)\) of the respective sets of
directing words of an FFA $\mathcal{F}$. Among other matters, we exhibit relationships between the languages $DD_i(\mathcal{F})$ and $D_i(\mathcal{F})$. Also the languages $DD_i(\mathcal{F})$ are shown to be regular.

In Section 5, we study the relationships between the families of languages formed by the sets $DD_i(\mathcal{F})$ of DDi-directing words of such FFAs. In Section 6 we establish the inclusion relationships between the classes of DDi-directable FFAs and the classes of DDi-directable so-called normal FFAs.

Some familiarity with the basic theory of finite automata and regular languages is assumed (cf. [1], [8], [18], [23] or [24], for example). For the general theory of fuzzy automata and fuzzy languages, the reader may consult [20] or [22]. The papers [2] and [26] survey parts of the theory of directable automata. Much of our terminology and notation stems from [11, 12].

2 Some basic notions

We may write $A := B$ to emphasize that $A$ is defined to be equal to $B$. The cardinality of a set $A$ is denoted by $|A|$, and the set of all subsets of $A$ by $\mathcal{P}(A)$. For any integer $n \geq 0$, let $n := \{1, \ldots, n\}$.

In what follows, $X$ is always a finite nonempty alphabet. The set of all (finite) words over $X$ is denoted by $X^*$ and the empty word by $\varepsilon$. Furthermore, let $X^+ := X^* \setminus \{\varepsilon\}$. The length of a word $w \in X^*$ is denoted by $\text{lg}(w)$. Subsets of $X^*$ are called languages. A family of languages $\mathcal{L}$ assigns to each finite nonempty alphabet $X$ a set $\mathcal{L}(X)$ of languages over $X$. We write $\mathcal{L} = \{\mathcal{L}(X)\}_X$.

A fuzzy language over $X$ is a mapping $\lambda : X^* \to [0, 1]$, where the set of degrees of membership is the real unit interval $[0, 1]$ ordered by the usual $\leq$-relation and equipped with the lattice operations $r \lor s = \max(r, s)$ and $r \land s = \min(r, s)$. Thus we follow Zadeh’s [27] original definition. The support of $\lambda$, is the language $\text{supp}(\lambda) := \{w \in X^* \mid \lambda(w) > 0\}$, and its range is the set $\text{ran}(\lambda) := \{\lambda(w) \mid w \in X^*\}$. If $\text{supp}(\lambda)$ is a finite set $\{w_1, \ldots, w_n\}$, we may write

$$\lambda = \{w_1/\lambda(w_1), \ldots, w_n/\lambda(w_n)\}.$$  

If $\text{ran}(\lambda) \subseteq \{0, 1\}$, then $\lambda$ is said to be crisp.

A deterministic finite automaton, a DFA for short, $A = (A, X, \delta)$ consists of a finite nonempty set $A$ of states, the input alphabet $X$, and a transition function $\delta : A \times X \to A$ which is extended in the usual way to a mapping $\delta^* : A \times X^* \to A$. 

3
A deterministic finite recognizer (DFR) $A = (A, X, \delta, a_0, F)$ is a DFA $A = (A, X, \delta)$ equipped with an initial state $a_0 \in A$ and a set $F \subseteq A$ of final states. It recognizes the language $L(A) := \{ w \in X^* \mid \delta^*(a_0, w) \in F \}$, and a language $L \subseteq X^*$ is said to be recognizable, or regular, if $L = L(A)$ for such a recognizer $A$. The set of regular languages over $X$ is denoted by $\text{Rec}(X)$.

A nondeterministic finite automaton (NFA) $N = (A, X, \alpha)$ consists of a finite nonempty set $A$ of states, the input alphabet $X$, and a nondeterministic transition function $\alpha : A \times X \to \wp(A)$. The transition function is extended to a mapping $\alpha^* : \wp(A) \times X^* \to \wp(A)$ as follows: for every $H \subseteq A$,

1. $\alpha^*(H, \varepsilon) = H$, and
2. $\alpha^*(H, ux) = \bigcup\{\alpha(b, x) \mid b \in \alpha^*(H, u)\}$ for any $u \in X^*$ and $x \in X$.

For any $H \subseteq A$ and $w \in X^*$, $\alpha^*(H, w)$ is the set of states reachable by the input word $w$ from some state in $H$. For any $a \in A$ and $w \in X^*$, we write $\alpha^*(a, w)$ for $\alpha^*(\{a\}, w)$. The NFA $N$ is complete, if $\alpha(a, x) \neq \emptyset$ for all $a \in A$ and $x \in X$. Obviously, then $\alpha^*(a, w) \neq \emptyset$ for all $a \in A$ and $w \in X^*$.

A fuzzy finite automaton (FFA) $F = (A, X, f)$ consists of a finite nonempty set $A$ of states, the input alphabet $X$, and a fuzzy transition function

$$f : A \times X \times A \to [0, 1].$$

In concrete examples of an FFA $F = (A, X, f)$, we specify the transition function $f$ by giving the nonzero values $f(a, x, b)$.

The transition function is extended to $f^* : A \times X^* \times A \to [0, 1]$ by setting

1. $f^*(a, \varepsilon, a) = 1$, and $f^*(a, \varepsilon, b) = 0$ if $b \neq a$, and
2. $f^*(a, w, b) = \max\{f^*(a, v, c) \land f(c, x, b) \mid c \in A\}$ for $w = vx$ with $v \in X^*$ and $x \in X$,

for all $a, b \in A$. For any $a \in A$ and $w \in X^*$, let

$$\mathcal{F}(a, w) := \{ b \in A \mid f^*(a, w, b) > 0 \}. $$

Thus $\mathcal{F}(a, w)$ is the set of states reachable (to a nonzero degree) from state $a$ by the input word $w$. With any pair $a \in A$ and $w \in X^*$, we also associate a fuzzy subset $\mathcal{F}_{a, w} : A \to [0, 1], b \mapsto f^*(a, w, b)$, of $A$. Obviously, $\text{supp}(\mathcal{F}_{a, w}) = \mathcal{F}(a, w)$.

For any $H \subseteq A$ and $w \in X^*$, let $\mathcal{F}(H, w) := \bigcup\{\mathcal{F}(a, w) \mid a \in H\}$.

The fuzzy automaton $F$ is complete if $\mathcal{F}(a, x) \neq \emptyset$ for all $a \in A$ and $x \in X$. If $F$ is complete, then clearly $\mathcal{F}(a, w) \neq \emptyset$ for all $a \in A$ and $w \in X^*$. 

4
3 Directing words and directable automata

A word \( w \in X^* \) is a directing word of a DFA \( A = (A, X, \delta) \) if there is a state \( c \in A \) such that \( \delta^*(a, w) = c \) for every \( a \in A \). The set of directing words of \( A \) is denoted by \( DW(A) \), and \( A \) is directable if \( DW(A) \neq \emptyset \).

In [12], three kinds of directing words of an NFA \( F = (A, X, \alpha) \) were identified. A word \( w \in X^* \) was said to be:

(D1) \( D1\text{-directing} \) if \( (\exists c \in A)(\forall a \in A)(\alpha^*(a, w) = \{c\}) \),

(D2) \( D2\text{-directing} \) if \( (\forall a, b \in A)(\alpha^*(a, w) = \alpha^*(b, w)) \), and

(D3) \( D3\text{-directing} \) if \( (\exists c \in A)(\forall a \in A)(c \in \alpha^*(a, w)) \).

For each \( i = 1, 2, 3 \), the set of \( Di\text{-directing} \) words of \( F \) is denoted by \( D_i(F) \), and \( F \) is called Di-directable if \( D_i(F) \neq \emptyset \).

If we regard a DFA as a special NFA, all three conditions \( (Di) \) yield the usual directing words. To state this more precisely, we associate with each DFA \( A = (A, X, \delta) \) an NFA \( A^{\text{nd}} = (A, X, \alpha) \), where \( \alpha : A \times X \to \wp(A) \) is defined by \( \alpha(a, x) := \{\delta(a, x)\} \) (\( a \in A, x \in X \)). It is clear that \( \alpha^*(a, w) = \{\delta^*(a, w)\} \) for every \( a \in A \) and \( w \in X^* \), which implies the following facts.

**Remark 3.1** For any DFA \( A \), \( DW(A) = D_1(A^{\text{nd}}) = D_2(A^{\text{nd}}) = D_3(A^{\text{nd}}) \).

Clearly, the directability of fuzzy automata may also be defined in several ways. The definition suggested by Karthikeyan and Rajasekar [16] may be stated as follows: \( w \in X^* \) is a directing word of an FFA \( F = (A, X, f) \) if there is a state \( c \in A \) such that \( f^*(a, w, c) > 0 \) (i.e. \( c \in F(a, w) \)) for every \( a \in A \), and \( F \) is said to be directable if it has a directing word. Using these notions the authors attempt to present some ideas and results appearing in [11] in a ‘fuzzified’ form. The discussion of these and related matters can be simplified by the following reduction to nondeterministic automata.

Consider an FFA \( F = (A, X, f) \). A word \( w = x_1 \ldots x_k (x_1, \ldots, x_k \in X) \) is directing for \( F \) if and only if there is a state \( c \in A \) such that for every \( a \in A \) there is a chain of states \( a = a_0, a_1, \ldots, a_k = c \) such that \( f(a_{i-1}, x_i, a_i) > 0 \) for every \( i \in k \). The actual values of the degrees \( f(a_{i-1}, x_i, a_i) \) do not matter as long as they are positive. Hence, we could assign every positive transition the value 1 without changing the set of directing words. This means that when we are interested just in directing words, \( F \) may be replaced by the associated NFA \( F^{\text{nd}} = (A, X, \alpha) \), where \( \alpha \) is defined by

\[
\alpha(a, x) := \{b \in A \mid f(a, x, b) > 0\} \quad (x \in X, a \in A).
\]
Example 3.2 Let $\mathcal{F} = (\{a, b, c\}, \{x, y\}, f)$ be the fuzzy automaton, where the non-zero transitions are $f(a, x, b) = 0.3$, $f(b, x, c) = 0.4$, $f(c, x, b) = 0.2$, $f(c, x, c) = 0.6$, $f(b, y, b) = 0.5$ and $f(b, y, c) = 0.1$. Then the transitions of $\mathcal{F}^{\text{nd}}$ are given by $\alpha(a, x) = \{b\}$, $\alpha(b, x) = \{c\}$, $\alpha(c, x) = \{b, c\}$, $\alpha(a, y) = \emptyset$, $\alpha(b, y) = \{b, c\}$ and $\alpha(c, y) = \emptyset$.

It is easy to verify the following lemma by induction on $\lg(w)$.

Lemma 3.3 Let $\mathcal{F} = (A, X, f)$ be any FFA and $\mathcal{F}^{\text{nd}} = (A, X, \alpha)$ be the associated NFA. Then $\alpha^*(a, w) = \mathcal{F}(a, w)$ for all $a \in A$ and $w \in X^*$.

Let us also note the following obvious fact.

Corollary 3.4 A fuzzy finite automaton $\mathcal{F}$ is complete if and only if the associated NFA $\mathcal{F}^{\text{nd}}$ is complete.

Remark 3.5 A converse construction is possible, too, i.e., with any NFA $\mathcal{N} = (A, X, \alpha)$, we may associate a related FFA $\mathcal{N}^{\text{fz}} = (A, X, f)$, where $f : A \times X \times A \to [0, 1]$ is defined by

$$f(a, x, b) = \begin{cases} 1, & \text{if } b \in \alpha(a, x) \\ 0, & \text{if } b \notin \alpha(a, x). \end{cases}$$

It is easy to see that $\mathcal{N}^{\text{fz}}$ is a crisp FFA, and that

(a) $\mathcal{N}^{\text{fz}}$ is complete if and only if $\mathcal{N}$ is complete,
(b) $\mathcal{N}^{\text{fz}}(a, w) = \alpha^*(a, w)$ for all $a \in A$ and $w \in X^*$, and
(c) $(\mathcal{N}^{\text{fz}})^{\text{nd}} = \mathcal{N}$.

However, $(\mathcal{F}^{\text{nd}})^{\text{fz}} = \mathcal{F}$ holds for an FFA $\mathcal{F}$ only in case $\mathcal{F}$ is crisp.

The above definitions of D1–, D2– and D3–directing words can be applied in a natural way to fuzzy automata, too. For an FFA $\mathcal{F} = (A, X, f)$, let us call a word $w \in X^*$

(D1') $D1$–directing if $(\exists c \in A)(\forall a \in A)(\mathcal{F}(a, w) = \{c\})$,

(D2') $D2$–directing if $(\forall a, b \in A)(\mathcal{F}(a, w) = \mathcal{F}(b, w))$, and

(D3') $D3$–directing if $(\exists c \in A)(\forall a \in A)(c \in \mathcal{F}(a, w))$.

For each $i = 1, 2, 3$, the set of $Di$–directing words of $\mathcal{F}$ is denoted by $D_i(\mathcal{F})$, and $\mathcal{F}$ is said to be $D_i$–directable if $D_i(\mathcal{F}) \neq \emptyset$.

The D3–directing words of a fuzzy automaton $\mathcal{F}$ are exactly the directing words as defined in [16]. A pairwise comparison of conditions (Di) and (Di') ($i = 1, 2, 3$) yields by Lemma 3.3 the following facts.
Proposition 3.6 For any FFA $\mathcal{F}$ and each $i = 1, 2, 3$, $D_i(\mathcal{F}) = D_i(\mathcal{F}^{\text{nd}})$, and hence $\mathcal{F}$ is Di-directable if and only if $\mathcal{F}^{\text{nd}}$ is Di-directable.

Proposition 3.6 implies that all the known results concerning Di–directing words of NFAs (cf. [3, 6, 7, 12, 14, 25], for example) apply to fuzzy automata, too. For example, the following relations between the sets $D_1(\mathcal{F})$, $D_2(\mathcal{F})$, and $D_3(\mathcal{F})$, for any fuzzy automaton $\mathcal{F}$, hold by Remark 3.2 of [12].

Remark 3.7 $D_1(\mathcal{F}) \subseteq D_2(\mathcal{F}) \cap D_3(\mathcal{F})$ for any fuzzy automaton $\mathcal{F}$, and if $\mathcal{F}$ is complete, then $D_1(\mathcal{F}) \subseteq D_2(\mathcal{F}) \subseteq D_3(\mathcal{F})$. Moreover, any one of the inclusions may be proper.

As noted in [11], for example, $X^*DW(A)X^* = DW(A)$ for any DFA $A$, but the corresponding fact does not always hold for the D3–directing words of a fuzzy automaton $\mathcal{F}$ as claimed in [16]. In fact, neither $X^*D_3(\mathcal{F}) = D_3(\mathcal{F})$ nor $D_3(\mathcal{F})X^* = D_3(\mathcal{F})$ holds generally. For example, the fuzzy automaton $\mathcal{F}$ of Example 3.2 has the D3–directing word $xx$ while neither $yxx$ nor $xxy$ is in $D_3(\mathcal{F})$. Remark 3.3 of [12] yields the following facts.

Remark 3.8 $D_2(\mathcal{F})X^* = D_2(\mathcal{F})$ for any FFA $\mathcal{F}$. If $\mathcal{F}$ is complete, then $X^*D_1(\mathcal{F}) = D_1(\mathcal{F})$, $X^*D_2(\mathcal{F})X^* = D_2(\mathcal{F})$ and $X^*D_3(\mathcal{F})X^* = D_3(\mathcal{F})$.

It is rather obvious that $D_1(\mathcal{F})$, $D_2(\mathcal{F})$ and $D_3(\mathcal{F})$ are regular languages for every FFA $\mathcal{F}$. Although this also follows directly from Proposition 3.4 of [12], let us present a simple proof formulated for FFAs.

Proposition 3.9 For any FFA $\mathcal{F}$, the languages $D_1(\mathcal{F})$, $D_2(\mathcal{F})$ and $D_3(\mathcal{F})$ are regular.

Proof. If $\mathcal{F} = (A, X, f)$ with $A = \{a_1, \ldots, a_n\}$, let $(B, X, \delta)$ be the DFA with $B = \{\mathcal{F}(a_1, u), \ldots, \mathcal{F}(a_n, u) | u \in X^*\}$ and $\delta$ defined by

$$\delta(\{C_1, \ldots, C_k\}, x) = \{\mathcal{F}(C_1, x), \ldots, \mathcal{F}(C_k, x)\}$$

for all $\{C_1, \ldots, C_k\} \in B$ and $x \in X$. For $b_0 = \{\{a_1\}, \ldots, \{a_n\}\}$, we get $\delta(b_0, u) = \{\mathcal{F}(a_1, u), \ldots, \mathcal{F}(a_n, u)\}$ for every $u \in X^*$. It is easy to see that, for each $i = 1, 2, 3$, $L(B_i) = D_i(\mathcal{F})$ for $B_i = (B, X, \delta, b_0, F_i)$, when we choose $F_1 = \{\{c\} | c \in A\} \cap B$, $F_2 = \{\{C\} | C \subseteq A\} \cap B$, and $F_3 = \{\{C_1, \ldots, C_k\} \in B | C_1 \cap \cdots \cap C_k \neq \emptyset\}$. □

The construction used in the above proof is clearly effective if $\mathcal{F}$ is effectively given. This means that we may decide whether $\mathcal{F}$ is D1–, D2– or
D3-directable. However, the decision method suggested by this observation is feasible for very small FFAs only. We shall now present a more effective way to decide the D3-directability of a complete FFA.

Two states $a$ and $b$ of a DFA $A = (A, X, \delta)$ are merged by a word $w \in X^*$ if $\delta^*(a, w) = \delta^*(b, w)$. The algorithm for testing the directability of a DFA $A$ presented in [11] is based on the obvious fact that $A$ is directable if and only if every pair of states of $A$ can be merged.

By the definition suggested in [16], two states $a$ and $b$ of an FFA $F = (A, X, f)$ are merged by a word $w \in X^*$ if $f^*(a, w, c) > 0$ and $f^*(b, w, c) > 0$ for some $c \in A$. Theorem 4.1 of [16] claims that $F$ is directable if and only if every pair of states can be merged. The condition is trivially necessary, but the converse may fail if the automaton is incomplete, and then the algorithm of [16], obtained from that of [11] by replacing the Boolean matrix by a fuzzy matrix, may incorrectly proclaim the automaton to be directable.

**Example 3.10** Let $F = (\{a, b, c\}, \{x, y, z\}, f)$ be the FFA in which $f(a, x, a)$, $f(b, x, a)$, $f(b, y, b)$, $f(c, y, b)$, $f(a, z, c)$ and $f(c, z, c)$ are the transitions with non-zero degrees. Clearly, $a$ and $b$ are merged by $x$, $b$ and $c$ by $y$, and $a$ and $c$ by $z$, but $D_3(F) = \emptyset$ since $f^*(a, yw, d) = f^*(b, zw, d) = f^*(c, xw, d) = 0$ for all $w \in X^*$ and $d \in A$.

However, the algorithm of [11] can be used, almost without any change, for testing a complete FFA for D3-directability.

In [12] a word $w \in X^*$ was defined to D3-merge two states $a, b \in A$ of an NFA $N = (A, X, \alpha)$ if $\alpha^*(a, w) \cap \alpha^*(b, w) \neq \emptyset$. Let us say that $w$ D3-merges two states $a, b \in A$ of an FFA $F$ if $F(a, w) \cap F(b, w) \neq \emptyset$. A word merges, in the sense of [16], two states if and only if it D3-merges them.

**Lemma 3.11** A word D3-merges two states of an FFA $F$ if and only if it D3-merges them in the associated NFA $F^{nd}$. Hence a complete FFA is D3-directable if and only if every pair of its states has a D3-merging word.

**Proof.** The first assertion is valid by Lemma 3.3. The rest of the lemma follows then from Lemma 5.4 of [12] which expresses the corresponding fact for NFAs.

For any $k \geq 0$, two states $a, b \in A$ of an FFA $F = (A, X, f)$ are said to be $D_3 k$-mergeable if they are D3-merged by a word of length $\leq k$. We denote the D3 $k$-mergeability relation of $F$ by $\mu(k)$, and let $\mu := \bigcup_{k \geq 0} \mu(k)$.

It is easy to verify the following proposition that justifies the algorithm to be presented. It corresponds exactly to Proposition 4.1 of [11].
Proposition 3.12 Let $\mathcal{F} = (A, X, f)$ be a complete FFA.

(a) $\mathcal{F}$ is $D3$–directable if and only if $\mu = A \times A$.
(b) The relations $\mu(k)$ and $\mu$ are reflexive and symmetric.
(c) $\mu(0) = \Delta_A$, and $\mu(k) = \mu(k - 1) \cup \{(a, b) \mid (\exists x \in X)[(f(a, x) \times f(b, x)) \cap \mu(k - 1) \neq \emptyset]\}$ for $k \geq 1$.
(d) If $\mu(k) = \mu(k - 1)$ for some $k \geq 1$, then $\mu(k) = \mu$.
(e) $\Delta_A = \mu(0) \subset \mu(1) \subset \ldots \subset \mu(k) = \mu$ for some $k \leq \binom{n}{2}$, where $n = |A|$.

Let $\mathcal{F} = (A, X, f)$ be an $n$-state complete FFA with $A = \{1, \ldots, n\}$. The algorithm implicitly computes the relations $\mu(0), \mu(1), \ldots$ until $\mu(k - 1) = \mu(k)$ for some $k \geq 1$. It employs a Boolean $n \times n$-matrix $M$ and a list $NewPair$ of pairs of states. The meaning of $M[i, j] = 1$ is that states $i$ and $j$ are known to be $D3$-mergeable. It suffices to consider the entries $M[i, j]$ with $1 \leq i < j \leq n$. A pair $(i, j)$ appears on the list $NewPair$ if it is known to be $D3$-mergeable but this fact has not yet been utilized for finding new $D3$-mergeable pairs. We also use the inverted transition table $I = (I[a, x])_{a \in A, x \in X}$, where $I[a, x] := \{i \in A \mid a \in f(i, x)\}$ for all $a \in A$ and $x \in X$. The steps of the algorithm are as follows:

1. Set $M[i, j] := 0$ for $1 \leq i < j \leq n$, $NewPair := \epsilon$ (empty list), and compute $I$.
2. Find all pairs $(a, x) \in A \times X$ for which $|I[a, x]| > 1$. For each such pair $(a, x)$ consider every pair $i, j \in I[a, x]$ with $i < j$. If $M[i, j] = 0$, let $M[i, j] := 1$ and append $(i, j)$ to $NewPair$.
3. Until $NewPair = \epsilon$ do the following. Delete the first pair, say $(a, b)$, from $NewPair$. For each $x \in X$, find all pairs $(i, j)$ with $i < j$ such that $i \in I[a, x]$ and $j \in I[b, x]$, or $j \in I[a, x]$ and $i \in I[b, x])$. If $M[i, j] = 0$, let $M[i, j] := 1$ and append $(i, j)$ to $NewPair$.
4. $\mathcal{F}$ is $D3$–directable if and only if $M[i, j] = 1$ whenever $1 \leq i < j \leq n$.

Quite the same way as in [11], one can show that the time bound for the algorithm is $O(m \cdot n^2)$, where $m = |X|$ and $n = |A|$.

It appears unlikely that anything like the above algorithm exists for $D1$– or $D2$–directability as there does not seem to be any useful notions of $D1$– or $D2$–merging words. Furthermore, while the shortest $D3$–directing word of a complete $D3$–directable $n$-state NFA is at most of length of order $O(n^3)$ (cf. Proposition 5.3 of [12]), no polynomial upper bounds exist for shortest $D1$– and $D2$–directing words (cf. [6, 8, 9, 12]).
4 Directability with degrees

We shall now introduce three new types of directing words for FFAs that are analogous to the D1–, D2– and D3–directing words but depend more on the transition degrees $f(a, x, b)$ of the automaton.

For a given FFA $\mathcal{F} = (A, X, f)$, we say that a word $w \in X^*$ is

(DD1) $DD_1$–directing if $(\exists c \in A)(\exists r \in (0, 1])(\forall a \in A)(\mathcal{F}_{a, w} = \{c/r\})$,

(DD2) $DD_2$–directing if $(\forall a, b \in A)(\mathcal{F}_{a, w} = \mathcal{F}_{b, w})$, and

(DD3) $DD_3$–directing if $(\exists c \in A)(\forall a \in A)(\mathcal{F}_{a, w}(c) = \max\{\mathcal{F}_{a, w}(b) \mid b \in A\} > 0)$.

For each $i = 1, 2, 3$, the set of $DD_i$–directing words of $\mathcal{F}$ is denoted by $DD_i(\mathcal{F})$, and $\mathcal{F}$ is said to be $DD_i$–directable if $DD_i(\mathcal{F}) \neq \emptyset$.

If condition (DD1) holds, then for every state $a \in A$, $c$ is the only state reachable from $a$ by the input word $w$, and its reachability degree $\mathcal{F}_{a, w}(c)$ has the same value $r > 0$ for every $a$. That $w$ satisfies (DD2) means that the fuzzy set $\mathcal{F}_{a, w}$ of states reached from a given state $a$ by input $w$ is the same for every $a$, i.e., $f^*(a, w, c) = f^*(b, w, c)$ for all $a, b, c \in A$. Finally, condition (DD3) means that there is a state $c$ which has the maximal reachability degree $\mathcal{F}_{a, w}(c) > 0$ by the input $w$ from any state $a \in A$.

Let us see how these notions are related to each other and the previously considered types of directing words of fuzzy automata.

**Proposition 4.1** Let $\mathcal{F} = (A, X, f)$ be any FFA.

(a) $DD_1(\mathcal{F}) \subseteq D_1(\mathcal{F})$,

(b) $DD_1(\mathcal{F}) \subseteq DD_2(\mathcal{F}) \subseteq D_2(\mathcal{F})$,

(c) $DD_1(\mathcal{F}) \subseteq DD_3(\mathcal{F}) \subseteq D_3(\mathcal{F})$, and

(d) $DD_1(\mathcal{F}) \subseteq DD_2(\mathcal{F}) \cap DD_3(\mathcal{F})$.

Any one of the inclusions may be proper.

**Proof.** The inclusions follow rather easily from the definitions. Let us prove statements (b) and (d).

If $w \in DD_1(\mathcal{F})$, and (DD1) holds for $c \in A$ and $r \in (0, 1]$, then $\mathcal{F}_{a, w} = \{c/r\} = \mathcal{F}_{b, w}$ for all $a, b \in A$, and thus $w \in DD_2(\mathcal{F})$.

If $w \in DD_2(\mathcal{F})$, then $\mathcal{F}(a, w) = \text{supp}(\mathcal{F}_{a, w}) = \text{supp}(\mathcal{F}_{w, b}) = \mathcal{F}(b, w)$ for all $a, b \in A$, and hence $w \in D_2(\mathcal{F})$.

To show that the inclusion $DD_1(\mathcal{F}) \subseteq DD_2(\mathcal{F})$ may be proper, we consider the FFA $\mathcal{F} = (A, X, f)$, where $A = \{a, b\}$, $X = \{x\}$, and $f$ is defined by
Proof. Let us verify the first pair of equalities. For any word \( w \) in \( X^* \), we associate its fuzzy \( m \times n \)-matrix representations. For any \( a, b \in A \) and \( x \in X \), their matrix representations. For any \( f(a, x) = f(a, x, b) = f(b, x, b) = 1 \). The word \( x \) does not belong to \( DD_1(F) \) because \( F_{a,x} = \{a/1, b/1\} \). On the other hand, \( x \in DD_2(F) \) because \( F_{a,x} = \{a/1, b/1\} = F_{b,x} \).

The inclusion \( DD_2(F) \subseteq D_2(F) \) is proper for \( F = (\{a, b\}, \{x\}, f) \), if \( f \) is defined by \( f(a, x, b) = 0.1 \) and \( f(b, x, b) = 0.2 \). Now \( x \notin DD_2(F) \) because \( F_{a,x}(b) = 0.1 \) while \( F_{b,x}(b) = 0.2 \). On the other hand, \( x \in D_2(F) \) since \( F(a, x) = \{b\} = F(b, x) \).

The inclusion relation in (d) follows from (b) and (c). That it may be proper, is seen by considering the first FFA defined above in this proof; the word \( x \) belongs to \( DD_2(F) \cap DD_3(F) \), but not to \( DD_1(F) \).

Let us say that an FFA \( F = (A, X, f) \) is crisp if \( f(a, x, b) \in \{0, 1\} \) for all \( a, b \in A \) and \( x \in X \). It is easy to see that if \( F \) is crisp, then for all \( a, b \in A \) and \( w \in X^* \),

1. \( f^*(a, w, b) \in \{0, 1\} \), and
2. \( F_{a,w} = \{a_1/1, \ldots, a_k/1\} \), where \( \{a_1, \ldots, a_k\} = F(a, w) \).

From (2) we get the following facts.

**Remark 4.2** If \( F = (A, X, f) \) is a crisp FFA, then \( DD_1(F) = D_1(F^{\text{nd}}) = D_1(F) \), \( DD_2(F) = D_2(F^{\text{nd}}) = D_2(F) \) and \( DD_3(F) = D_3(F^{\text{nd}}) = D_3(F) \).

**Proof.** Let us verify the first pair of equalities. For any \( w \in X^* \),

\[
\begin{align*}
w \in DD_1(F) & \iff (\exists c \in A)(\exists r \in (0, 1])(\forall a \in A) F_{a,w} = \{c/r\} \\
& \iff (\exists c \in A)(\forall a \in A) F_{a,w} = \{c/1\} \\
& \iff (\exists c \in A)(\forall a \in A) F(a, w) = \{c\} \\
& \iff w \in D_1(F^{\text{nd}}).
\end{align*}
\]

Hence, \( DD_1(F) = D_1(F^{\text{nd}}) \), and \( D_1(F^{\text{nd}}) = D_1(F) \) by Proposition 3.6.

The other two pairs of equalities have similar straightforward proofs. \( \square \)

When dealing with fuzzy finite automata it may be convenient to use their matrix representations. For any \( m, n \geq 1 \), a fuzzy \( m \times n \)-matrix is an \( m \times n \)-array \( (r_{ij}) \) in which \( r_{ij} \in [0, 1] \) for all \( i \in \mathbf{m} \) and \( j \in \mathbf{n} \). The product of a fuzzy \( m \times k \)-matrix \( R = (r_{ij}) \) and a fuzzy \( k \times n \)-matrix \( S = (s_{ij}) \) is the fuzzy \( m \times n \)-matrix \( RS = (t_{ij}) \), where

\[
t_{ij} = (r_{i1} \wedge s_{1j}) \vee \ldots \vee (r_{ik} \wedge s_{kj}) \quad (i \in \mathbf{m}, j \in \mathbf{n}).
\]

Let \( F = (A, X, f) \) be an \( n \)-state FFA with \( A = \{a_1, \ldots, a_n\} \). With any word \( w \in X^* \), we associate its fuzzy \( n \times n \) transition matrix
\[ M^F = M_w = (r_{ij}), \text{ where } r_{ij} = f^*(a_i, w, a_j) \quad (i, j \in n). \]

For any words \( u, v \in X^* \), \( M_{uv} = M_uM_v \) because
\[
f(a_i, uv, a_j) = \max\{f^*(a_i, u, a_k) \land f^*(a_k, v, a_j) \mid k \in n\} \quad (i, j \in n).
\]

For any \( w \in X^* \) and \( i \in n \), the \( i^{th} \) row \((r_{i1} r_{i2} \ldots r_{in})\) of \( M_w \) represents the fuzzy set \( F_{a_i,w} \), i.e.,
\[
r_{i1} = F_{a_i,w}(a_1), \ r_{i2} = F_{a_i,w}(a_2), \ldots, r_{in} = F_{a_i,w}(a_n) \quad (i \in n)
\]

By this observation, one may deduce from the matrix \( M_w \) whether the word \( w \) is DD1–, DD2– or DD3–directing, and this will be used in the proof of the following proposition.

**Proposition 4.3** For any FFA \( F = (A, X, f) \), the languages DD1\((F)\), DD2\((F)\) and DD3\((F)\) are recognizable.

**Proof.** Let \( A = \{a_1, \ldots, a_n\} \). The set \( B := \{M_w \mid w \in X^*\} \) is finite because all the matrix entries belong to the finite set \{\( f(a_i, x, a_j) \mid i, j \in n, x \in X \}\}. To see this, it suffices to notice that no new elements are introduced when matrices are multiplied as the only operations used are the join \( \max(r, s) \) and meet \( \min(r, s) \) in the chain \(( [0, 1], \leq )\).

Let \((B, X, \delta)\) be the DFA for which \( \delta : B \times X \rightarrow B \) is defined by
\[
\delta(M_w, x) = M_{wx} \quad (w \in X^*, x \in X).
\]
It is easy to show that \( \delta^*(M_\varepsilon, w) = M_w \) for every \( w \in X^* \). From this observation, it follows that for each \( i = 1, 2, 3 \), we get a DFR \( B_i = (B, X, \delta, M_\varepsilon, F_i) \) recognizing DD1\((F)\) by the following choices of the sets \( F_i \) of final states.

1. \( M_w \in F_1 \) if and only if for some \( j \in n \) and \( r \in (0, 1] \), \( r_{1j} = \ldots = r_{nj} = r \) and \( r_{ik} = 0 \) for all \( i, k \in n \) with \( k \neq j \).
2. \( M_w \in F_2 \) if and only if all the rows in \( M_w \) are identical.
3. \( M_w \in F_3 \) if and only if for some \( j \in n \), in every row, the \( j^{th} \) element is maximal and positive.

In Remark 3.8 we noted some equalities satisfied by the \( D_i(F) \)-languages, but not all of them hold for the DD1\((F)\)-languages. For example, let \( F = (\{a, b\}, \{x, y\}, f) \) be the complete FFA in which the nonzero transitions are
\[
f(a, x, b) = f(b, x, b) = f(b, y, b) = 1 \text{ and } f(a, y, a) = 0.5.
\]
Then \( x \in DD1(F) \) because \( F_{a,x} = F_{b,x} = \{b/1\} \), but \( yx \notin DD1(F) \) since \( F_{a,yx} = \{b/0.5\} \) while \( F_{b,yx} = \{b/1\} \). Hence, \( X^*DD1(F) \neq DD1(F) \). Similarly, \( X^*DD2(F) \neq DD2(F) \) because \( x \) is a D2–directing word of \( F \), but \( yx \) is not. However,
the equality $X^*DD_1(\mathcal{F}) = DD_1(\mathcal{F})$, for example, holds if we replace the completeness of an FFA by the stronger property of “normality”.

Let us say that an FFA $\mathcal{F} = (A, X, f)$ is normal if for every pair $a \in A$, $x \in X$, there is a state $b \in A$ for which $f(a, x, b) = 1$, i.e., $\mathcal{F}_{a, x}$ is a normal fuzzy set.

**Lemma 4.4** If $\mathcal{F} = (A, X, f)$ is a normal FFA, then for all $a \in A$ and $w \in X^*$, there is a state $b \in A$ such that $f^*(a, w, b) = 1$. Hence, if $\mathcal{F}$ is normal and $w \in DD_3(\mathcal{F})$, then there is a $c \in A$ such that $f^*(a, w, c) = 1$ for every $a \in A$.

**Proposition 4.5** For any FFA $\mathcal{F} = (A, X, f)$,

(a) $DD_2(\mathcal{F})X^* = DD_2(\mathcal{F})$.

If $\mathcal{F}$ is normal, then we also have

(b) $X^*DD_1(\mathcal{F}) = DD_1(\mathcal{F})$,

(c) $X^*DD_2(\mathcal{F})X^* = DD_2(\mathcal{F})$, and

(d) $X^*DD_3(\mathcal{F})X^* = DD_3(\mathcal{F})$.

**Proof.** To prove (a), consider any words $w \in DD_2(\mathcal{F})$ and $u \in X^*$. For all $a, b, c \in A$,

\[
\mathcal{F}_{a,wu}(c) = \max \{ f^*(a, w, d) \land f^*(d, u, c) \mid d \in A \} = \max \{ f^*(b, w, d) \land f^*(d, u, c) \mid d \in A \} = \mathcal{F}_{b,wu}(c),
\]

which shows that $DD_2(\mathcal{F})X^* \subseteq DD_2(\mathcal{F})$. The converse inclusion is obvious.

Assume now that $DD_2(\mathcal{F})X^* \subseteq DD_2(\mathcal{F})$. To prove (b), consider any $w \in DD_1(\mathcal{F})$, and let $c \in A$ and $r \in (0, 1]$ be such that $\mathcal{F}_{a,w} = \{c/r\}$ for every $a \in A$. Take any $u \in X^*$ and $a \in A$. By Lemma 4.4, there is a $b \in A$ such that $f^*(a, u, b) = 1$, and therefore

\[
f^*(a, uw, c) = \max \{ f^*(a, u, d) \land f^*(d, w, c) \mid d \in A \} = 1 \land r = r,
\]

where we used the fact that $f^*(d, w, c) = r$ for every $d \in A$. On the other hand, if $c' \in A$, $c' \neq c$, then for every $a \in A$

\[
f^*(a, uw, c') = \max \{ f^*(a, u, d) \land f^*(d, w, c') \mid d \in A \} = \max \{ f^*(a, u, d) \land 0 \mid d \in A \} = 0.
\]
This shows that $uw \in DD_1(\mathcal{F})$, and hence $X^*DD_1(\mathcal{F}) \subseteq DD_1(\mathcal{F})$.

As we already have $DD_2(\mathcal{F})X^* = DD_2(\mathcal{F})$, we get (c) by showing that $X^*DD_2(\mathcal{F}) \subseteq DD_2(\mathcal{F})$. Let $u \in X^*$ and $w \in DD_2(\mathcal{F})$, and consider any $a, b, c \in A$. Let $a', b' \in A$ be such that $f^*(a, u, a') = f^*(b, u, b') = 1$. Then

$$f^*(a, uw, c) = \max\{f^*(a, u, d) \land f^*(d, w, c) \mid d \in A\}$$

$$= f^*(a, u, a') \land f^*(a', w, c)$$

$$= 1 \land f^*(a', w, c) = 1 \land f^*(b', w, c)$$

$$= \max\{f^*(b, u, d) \land f^*(d, w, c) \mid d \in A\}$$

$$= f^*(b, uw, c),$$

where we also used the fact that $f^*(a', w, c) = f^*(d, w, c) = f^*(b', w, c)$ for every $d \in A$. Thus we have shown that $uw \in DD_2(\mathcal{F})$.

Finally, to prove (d), consider any $u \in X^*$ and $w \in DD_3(\mathcal{F})$. Let $c \in A$ be a state such that, for every $a \in A$, $f^*(a, w, c) > 0$ and $f^*(a, w, b) \leq f^*(a, w, c)$ for every $b \in A$. If $c' \in A$ is a state such that $f^*(c, u, c') = 1$, then for any $a \in A$,

$$f^*(a, wu, c') = \max\{f^*(a, w, d) \land f^*(d, u, c') \mid d \in A\}$$

$$\geq f^*(a, w, c) \land f^*(c, u, c')$$

$$= f^*(a, w, c) > 0.$$

Moreover, for any $b \in A$,

$$f^*(a, wu, b) = \max\{f^*(a, w, d) \land f^*(d, u, b) \mid d \in A\}$$

$$\leq f^*(a, w, c) \land \max\{f^*(d, u, b) \mid d \in A\}$$

$$\leq f^*(a, w, c) = f^*(a, w, c) \land f^*(c, u, c')$$

$$\leq f^*(a, wu, c'),$$

where we used the fact that $f(a, w, d) \leq f(a, w, c)$ for every $d \in A$, and thus $wu \in DD_3(\mathcal{F})$. The converse inclusion $DD_3(\mathcal{F}) \subseteq DD_3(\mathcal{F})X^*$ holds trivially. To complete the proof of (d), we show that $X^*DD_3(\mathcal{F}) \subseteq DD_3(\mathcal{F})$.

Let $w \in DD_3(\mathcal{F})$, $c \in A$ and $u \in X^*$ be as above. Consider any $a \in A$ and let $a' \in A$ be a state for which $f^*(a, u, a') = 1$. Then

$$f^*(a, uw, c) = \max\{f^*(a, u, d) \land f^*(d, w, c) \mid d \in A\}$$

$$\geq f^*(a, u, a') \land f^*(a', w, c) = f^*(a', w, c) > 0$$

Moreover, for any $b \in A$,

$$f^*(a, uw, b) = \max\{f^*(a, u, d) \land f^*(d, w, b) \mid d \in A\}$$

$$\leq \max\{f^*(a, u, d) \land f^*(d, w, c) \mid d \in A\} = f^*(a, uw, c),$$

$$\leq f^*(a, wu, c'),$$

where we also used the fact that $f(a, w, d) \leq f(a, w, c)$ for every $d \in A$, and thus $wu \in DD_3(\mathcal{F})$. The converse inclusion $DD_3(\mathcal{F}) \subseteq DD_3(\mathcal{F})X^*$ holds trivially. To complete the proof of (d), we show that $X^*DD_3(\mathcal{F}) \subseteq DD_3(\mathcal{F})$.
and hence $uw \in DD_3(\mathcal{F})$. \hfill \Box

It is easy to see that if $L = RX^*(\subseteq X^*)$, where $R$ is a nonempty regular language, then any minimal DFR $A = (A, X, \delta, a_0, F)$ of $L$ has exactly one final state, and this state $d$ is a trap state, i.e., $\delta(d, x) = d$ for every $x \in X$. Hence, Proposition 4.5 yields the following facts.

**Corollary 4.6** If an FFA $\mathcal{F} = (A, X, f)$ is $DD_2$-directable, then any minimal DFR recognizing $DD_2(\mathcal{F})$ has exactly one final state, and this state $d$ is a trap state. If $\mathcal{F}$ is normal and $DD_3$-directable, then any minimal DFR recognizing $DD_3(\mathcal{F})$ has exactly one final state, and this is a trap state.

### 5 Families of sets of $DD_i$-directing words

Let us now consider the families of languages $DD_i(\mathcal{F})$ formed by the $DD_i$-directing words of $DD_i$-directable FFAs. In [9] Imreh and Ito studied the families of the sets $D_i(\mathcal{N})$ of $D_i$-directing words of $D_i$-directable NFAs. Their results, also presented in [13], apply directly to fuzzy automata, too.

Let $\mathcal{L} = \{\mathcal{L}(X)\}_{X}$ and $\mathcal{K} = \{\mathcal{K}(X)\}_{X}$ be families of languages. $\mathcal{K}$ is a subfamily of $\mathcal{L}$, $\mathcal{K} \subseteq \mathcal{L}$ in symbols, if $\mathcal{K}(X) \subseteq \mathcal{L}(X)$ for every alphabet $X$. If $\mathcal{K} \subseteq \mathcal{L}$, but $\mathcal{K} \neq \mathcal{L}$, then $\mathcal{K}$ is a proper subfamily of $\mathcal{L}$ and we write $\mathcal{K} \subset \mathcal{L}$. The intersection of $\mathcal{K}$ and $\mathcal{L}$ is the family $\mathcal{K} \cap \mathcal{L} := \{\mathcal{K}(X) \cap \mathcal{L}(X)\}_{X}$. The families $\mathcal{K}$ and $\mathcal{L}$ are incomparable if neither $\mathcal{K} \subseteq \mathcal{L}$ nor $\mathcal{L} \subseteq \mathcal{K}$ holds. This is expressed by $\mathcal{K} \parallel \mathcal{L}$.

For each $i \in 3$, let $DD_i = \{DD_i(X)\}_{X}$ be the family of the sets $DD_i(\mathcal{F})$ of $DD_i$-directing words of $DD_i$-directable FFAs. Similarly, let $nDD_i = \{nDD_i(X)\}_{X}$ be the family of the sets $DD_i(\mathcal{F})$, where $\mathcal{F}$ is a $DD_i$-directable normal FFA. Furthermore, let $DW = \{DW(X)\}_{X}$ be the family of the sets $DW(A)$ of directing words of directable DFAs. Note that any language belonging to $DD_1(X)$, $DD_2(X)$, $DD_3(X)$ or $DW(X)$ for some $X$ is, by definition, nonempty.

The following facts are easily verified.

**Lemma 5.1** For any DFA $A = (A, X, \delta)$, the FFA $A^{fz} = (A, X, f)$ is normal, and for all $a, b \in A$ and $w \in X^*$,

$$ f^*(a, w, b) = \begin{cases} 
1, & \text{if } \delta^*(a, w) = b; \\
0, & \text{otherwise.} 
\end{cases} $$

Furthermore, $DD_1(A^{fz}) = DD_2(A^{fz}) = DD_3(A^{fz}) = DW(A)$.

**Corollary 5.2** $DW \subseteq nDD_1 \cap nDD_2 \cap nDD_3$.  

15
It is clear that $X^*DW(A)X^* = DW(A)$ for any DFA $A$, and in [9] it was shown that this is a characteristic property of the languages $DW(A)$:

**Proposition 5.3** [9] A language $L \subseteq X^*$ belongs to $\mathcal{D}W(X)$ if and only if $L \in \text{Rec}(X)$, $L \neq \emptyset$ and $X^*LX^* = L$.

**Proposition 5.4** $\mathcal{D}W \subset \mathcal{D}D_1 \cap \mathcal{D}D_2 \cap \mathcal{D}D_3$.

*Proof.* The inclusion holds by Corollary 5.2 (naturally $n\mathcal{D}D_1 \subseteq \mathcal{D}D_i$ for each $i$). Let $\mathcal{F} = (A, X, f)$ be the FFA in which $A = \{a, b\}$, $X = \{x, y\}$, and $f$ is given by $f(a, x, b) = f(b, x, b) = f(b, y, b) = 1$. Clearly, $F_{a,xu} = \mathcal{F}_{b,xu} = \{b/1\}$ and $F_{a,uy} = \emptyset$ for every $u \in X^*$. Thus $\mathcal{D}D_1(\mathcal{F}) = \mathcal{D}D_2(\mathcal{F}) = \mathcal{D}D_3(\mathcal{F}) = xX^*$. Since $xX^* \notin \mathcal{D}W(X)$, the claimed inclusion is proper. \[\square\]

**Proposition 5.5** (a) $\mathcal{D}W \subset n\mathcal{D}D_1 \subset \mathcal{D}D_1$.

(b) $\mathcal{D}W = n\mathcal{D}D_2 \subset \mathcal{D}D_2$.

(c) $\mathcal{D}W = n\mathcal{D}D_3 \subset \mathcal{D}D_3$.

*Proof.* The relations $\mathcal{D}W \subseteq n\mathcal{D}D_i$ $(i = 1, 2, 3)$ hold by Corollary 5.2. Let us prove separately the remaining parts of the three statements.

To prove $\mathcal{D}W \subset n\mathcal{D}D_1$, let us consider the normal FFA $\mathcal{F} = (A, X, f)$, where $A = \{a, b\}$, $X = \{x, y\}$, and $f$ is given by $f(a, x, b) = f(a, y, a) = f(b, x, b) = f(b, y, a) = f(b, y, b) = 1$. Clearly, $\mathcal{D}D_1(\mathcal{F})$ is the language $X^*x$ which does not belong to $\mathcal{D}W(X)$.

That the inclusion $n\mathcal{D}D_1 \subset \mathcal{D}D_1$ is proper, is seen by considering the FFA $\mathcal{F} = (A, X, f)$, where $A = \{a, b\}$, $X = \{x, y\}$, and $f$ is given by $f(a, x, b) = f(b, x, b) = f(b, y, b) = 1$ and $f(a, y, a) = 0.5$. Obviously, $\mathcal{D}D_1(\mathcal{F}) = xX^*$, but $xX^* \notin n\mathcal{D}D_1(X)$ by Proposition 4.5(b).

To prove that $\mathcal{D}W = n\mathcal{D}D_2$, it suffices to show that $n\mathcal{D}D_2 \subseteq \mathcal{D}W$. If $L \in n\mathcal{D}D_2(X)$, then $L \neq \emptyset$ and $L \in \text{Rec}(X)$. Moreover, $X^*LX^* = L$ by Proposition 4.5(c), and thus $L \in \mathcal{D}W(X)$.

Let $\mathcal{F} = (A, X, f)$ be the FFA, where $A = \{a, b\}$, $X = \{x, y\}$ and $f$ is given by $f(a, x, b) = f(b, x, b) = f(b, y, b) = 1$. By Proposition 4.5(c), $\mathcal{D}D_2(\mathcal{F}) = xX^*$ is not in $n\mathcal{D}D_2(X)$. Hence inclusion $n\mathcal{D}D_2 \subseteq \mathcal{D}D_2$ is proper.

(c) That $\mathcal{D}W = n\mathcal{D}D_3$ can be shown similarly as $\mathcal{D}W = n\mathcal{D}D_2$ above. To prove $n\mathcal{D}D_3 \subset \mathcal{D}D_3$, we consider again the FFA $\mathcal{F}$ defined in part (b) above: $\mathcal{D}D_3(\mathcal{F}^*) = xX^*$ is not in $n\mathcal{D}D_3(X)$. \[\square\]
Proposition 5.6

\[ DW = nDD1 \cap nDD2 \cap nDD3 = nDD1 \cap DD2 = nDD1 \cap DD3. \]

Proof. The first equality follows from Corollary 5.2 and Proposition 5.5.

Also the inclusions \( DW \subseteq nDD1 \cap DD2 \) and \( DW \subseteq nDD1 \cap DD3 \) hold by Corollary 5.2 and Proposition 5.5. On the other hand, if \( L \in nDD1 \cap DD2 \), then \( L \neq \emptyset \), \( L \in Rec(X) \), and \( X^* LX^* = L \) by Proposition 4.5(a,b). Thus \( L \in DW(X) \) by Proposition 5.3. Hence, \( DW = nDD1 \cap DD2 \). The equality \( DW = nDD1 \cap DD3 \) has a similar proof. \( \square \)

Proposition 5.7 \( DD1 \parallel DD3 \).

Proof. There are two noninclusions to be shown.

If \( DD1 \subseteq DD3 \), then naturally also \( nDD1 \subseteq DD3 \). By Proposition 5.6 this would imply that \( nDD1 = nDD1 \cap DD3 = DW \), which contradicts Proposition 5.3(a). Thus \( DD1 \nsubseteq DD3 \).

To prove \( DD3 \nsubseteq DD1 \), let \( F = (A, X, f) \) be the FFA, where \( A = \{p, q\} \), \( X = \{x, y\} \) and \( f \) is defined by

\[ f(p, x, q) = f(q, x, p) = f(q, x, q) = f(p, y, p) = 1. \]

Then \( x, xxy \in DD3(F) \) as \( F_{p,x} = \{q/1\} \), \( F_{q,x} = \{p/1, q/1\} \), and \( F_{p,xx} = \{p/1\} = F_{q,xx} \). Assume that there exists an FFA \( G = (B, X, g) \) for which \( DD1(G) = DD3(F) \). Then there would exist \( c, d \in B \) and \( r, s \in (0,1] \) such that for every \( a \in B \), \( G_{a,x} = \{c/r\} \) and \( G_{a,xx} = \{d/s\} \). Thus, \( G(a, x) = \{c\} \) for every \( a \in B \), and \( G(c, x) = \{c\} \) in particular. Since \( G(a, xx) = \{d\} \) for every \( a \in B \), this must mean also that \( G(c, y) = \{d\} \). Moreover, for every \( a \in B \), \( g(a, x, c) = c \) and \( g(a, x, b) = 0 \) if \( b \neq c \) \((b \in B) \). Thus we may conclude that for every \( a \in B \),

\[ g^*(a, xy, d) = \max\{g(a, x, b) \land g(b, y, d) \mid b \in B\} = g(a, x, c) \land g(c, y, d) = g(a, x, c) \land g(c, x, c) \land g(c, y, d) = g^*(a, xxy, d) = s > 0. \]

On the other hand, for any \( d' \neq d \),

\[ g^*(a, xy, d') = \max\{g(a, x, b) \land g(b, y, d') \mid b \in B\} = g(a, x, c) \land g(c, y, d') = g(a, x, c) \land g(c, x, c) \land g(c, y, d') \leq g^*(a, xxy, d') = 0. \]

Thus \( G_{a,xy} = \{d/s\} \), for every \( a \in B \). This would mean that \( xy \in DD1(G) \) although \( xy \notin DD3(F) \) (as \( F_{p,xy} = \emptyset \)). \( \square \)
Proposition 5.8  \( \mathcal{D}D_1 \not\subset \mathcal{D}D_2 \) and \( \mathcal{D}D_3 \not\subset \mathcal{D}D_2 \).

Proof. The noninclusion \( \mathcal{D}D_1 \not\subset \mathcal{D}D_2 \) must hold because \( \mathcal{D}D_1 \subset \mathcal{D}D_2 \) would imply that \( n\mathcal{D}D_1 = n\mathcal{D}D_1 \cap \mathcal{D}D_1 \subset n\mathcal{D}D_1 \cap \mathcal{D}D_2 = \mathcal{D}W \subset n\mathcal{D}D_1 \), a contradiction.

To prove \( \mathcal{D}D_3 \not\subset \mathcal{D}D_2 \), consider the FFA \( \mathcal{F} = (\{a, b\}, \{x, y\}, f) \), where \( f \) is defined by \( f(a, x, a) = f(a, x, b) = f(a, y, a) = f(b, x, b) = 1 \). Assume then that \( \mathcal{D}D_3(\mathcal{F}) = \mathcal{D}D_2(\mathcal{G}) \) for some FFA \( \mathcal{G} \). Now \( x \in \mathcal{D}D_2(\mathcal{F}) \) because \( \mathcal{F}_{a,x} = \{a/1, b/1\} \) and \( \mathcal{F}_{b,x} = \{b/1\} \). However, \( x \in \mathcal{D}D_2(\mathcal{G}) \) would imply \( xy \in \mathcal{D}D_2(\mathcal{G}) \) although \( xy \not\in \mathcal{D}D_3(\mathcal{F}) \). □

Proposition 5.9  \( n\mathcal{D}D_1 \parallel \mathcal{D}D_1 \cap \mathcal{D}D_2 \cap \mathcal{D}D_3 \).

Proof. If \( n\mathcal{D}D_1 \subset \mathcal{D}D_1 \cap \mathcal{D}D_2 \cap \mathcal{D}D_3 \), then \( n\mathcal{D}D_1 \subset \mathcal{D}D_2 \). By Proposition 5.6 this would imply that \( \mathcal{D}W = n\mathcal{D}D_1 \cap \mathcal{D}D_2 = n\mathcal{D}D_1 \), which contradicts Proposition 5.5(a). On the other hand, if \( \mathcal{D}D_1 \cap \mathcal{D}D_2 \cap \mathcal{D}D_3 \subset n\mathcal{D}D_1 \), then

\[
\mathcal{D}D_1 \cap \mathcal{D}D_2 \cap \mathcal{D}D_3 = n\mathcal{D}D_1 \cap \mathcal{D}D_2 \cap \mathcal{D}D_3 = n\mathcal{D}D_1 \cap \mathcal{D}D_3 \cap \mathcal{D}D_3 = \mathcal{D}W \cap \mathcal{D}D_3 = \mathcal{D}W,
\]

by Propositions 5.5(a) and 5.6. This would contradict Proposition 5.4. □

6 Classes of DD\(i\)-directable FFAs

In this section we shall consider the classes of FFAs and normal FFAs with the various DD\(i\)-directability properties. For each \( i = 1, 2, 3 \), let

(1) \( \mathcal{D}D(i) \) denote the class of all DD\(i\)-directable FFAs, and
(2) \( n\mathcal{D}D(i) \) denote the class of all DD\(i\)-directable normal FFAs.

Furthermore, let \( \text{Dir} \) be the class of all directable DFAs; recall that in Section 5 we interpreted with each DFA \( \mathcal{A} \) a crisp FFA \( \mathcal{A}^c \).

The following proposition justifies the Hasse diagram of Figure 11 of the \( \cap \)-semilattice of the classes \( \mathcal{D}D(i) \), \( n\mathcal{D}D(i) \), \( \text{Dir} \) and their intersections.

Proposition 6.1  (a) \( n\mathcal{D}D(i) \subset \mathcal{D}D(i) \) for \( i = 1, 2, 3 \).

(b) \( \mathcal{D}D(1) \subset \mathcal{D}D(2) \cap \mathcal{D}D(3) \).

(c) \( \mathcal{D}D(2) \parallel \mathcal{D}D(3) \).

(d) \( \mathcal{D}D(2) \cap \mathcal{D}D(3) \subset \mathcal{D}D(2) \) and \( \mathcal{D}D(2) \cap \mathcal{D}D(3) \subset \mathcal{D}D(3) \).
Figure 1: Classes of DD-directable FFAs

(e) $\text{nDD}(1) \subset \text{nDD}(2)$.

(f) $\text{nDD}(2) \subset \text{nDD}(3)$.

(g) $\text{DD}(1) \parallel \text{nDD}(2)$.

(h) $\text{DD}(2) \cap \text{DD}(3) \| \text{nDD}(3)$.

(i) $\text{nDD}(2) \subset \text{DD}(2) \cap \text{DD}(3)$.

(j) $\text{DD}(1) \cap \text{nDD}(2) = \text{nDD}(1)$.

(k) $(\text{DD}(2) \cap \text{DD}(3)) \cap \text{nDD}(3) = \text{nDD}(2)$.

(l) $\text{Dir} \subset \text{nDD}(1)$.

Proof. (a) $\text{nDD}(i) \subset \text{DD}(i)$ for $i = 1, 2, 3$:
The inclusions are obvious. It is easy to show that they are proper by finding
for each $i = 1, 2, 3$ an FFA that is DD-$i$–directable but not normal.

(b) $\text{DD}(1) \subset \text{DD}(2) \cap \text{DD}(3)$:
The inclusion follows from Proposition 4.1(d). That it is proper, can be
seen by considering the FFA $F = (A, X, f)$, where $A = \{a, b\}$, $X = \{x\}$, and
$f(a, x, a) = f(a, x, b) = f(b, x, a) = f(b, x, b) = 1$. For any $w = x^n$, where
$n \geq 1$, $F_{a,w} = \{a/1, b/1\} = F_{b,w}$. Thus $DD_2(F) = DD_3(F) = \{x^n \mid n \geq 1\}$,
while $DD_1(F) = \emptyset$. 

19
(c) **DD(2) || DD(3):**
Consider the FFA \( \mathcal{F} = (A, X, f) \), where \( A = \{a, b\} \), \( X = \{x, y\} \), \( f(a, x, a) = 0.2 \), \( f(a, x, b) = f(a, y, a) = f(b, x, b) = f(b, y, b) = 1 \). If \( w = y^n \) with \( n \geq 1 \), then \( \mathcal{F}_{a,w} = \{a/1\} \) and \( \mathcal{F}_{b,w} = \{b/1\} \), which means that \( w \) is not DD2-directing. If the letter \( x \) appears in \( w \) \( \in X^* \) at least once, then \( \mathcal{F}_{a,x} = \{a/0.2, b/1\} \) and \( \mathcal{F}_{b,x} = \{b/1\} \), and thus \( w \in DD_3(\mathcal{F}) \setminus DD_2(\mathcal{F}) \). This means that \( \mathcal{F} \in DD(3) \setminus DD(2). \)

Next, let \( \mathcal{G} = (\{a, b\}, \{x\}, g) \) be the FFA, where the only nonzero transition is \( g(a, x, b) = 1 \). Obviously, \( \mathcal{G}_{a,x} = \{b/1\} \), \( \mathcal{G}_{b,x} = \emptyset \), and \( \mathcal{G}_{a,w} = \mathcal{G}_{b,w} = \emptyset \) for any \( w = x^n \) with \( n \geq 2 \). This shows that \( \mathcal{G} \in DD(2) \setminus DD(3). \)

(d) **DD(2) \cap DD(3) \subset DD(2) and DD(2) \cap DD(3) \subset DD(3):**
The properness of these inclusions follows from (c).

(e) **nDD(1) \subset nDD(2):**
By (b), any DD1-directable FFA is also DD2-directable, and naturally this applies to normal FFAs, too. The properness of the inclusion is shown by the FFA \( \mathcal{F} \) used in the proof of (b) as it is also normal.

(f) **nDD(2) \subset nDD(3):**
Let \( \mathcal{F} = (A, X, f) \) be a DD2-directable normal FFA, and consider any \( w \in DD_2(\mathcal{F}) \) and \( b \in A \). Since \( \mathcal{F} \) is normal, \( f^*(b, w, c) = 1 \) for some \( c \in A \). As \( w \) is DD2-directing, this implies that \( f^*(a, w, c) = 1 \) for every \( a \in A \), and thus \( w \) is also DD3-directing. To show that the inclusion \( nDD(2) \subseteq nDD(3) \) is proper, consider the FFA \( \mathcal{F} \in DD(3) \setminus DD(2) \) used in the proof of (c). Since \( \mathcal{F} \) is normal, also \( \mathcal{F} \in nDD(3) \setminus nDD(2) \) holds.

(g) **DD(1) || nDD(2):**
Consider the FFA \( \mathcal{F} = (A, X, f) \), where \( A = \{a, b\} \), \( X = \{x, y\} \), and \( f \) is given by \( f(a, x, b) = f(b, x, b) = f(b, y, b) = 1 \). It is in DD(1) because \( \mathcal{F}_{a,x} = \mathcal{F}_{b,x} = \{b/1\} \), but it is not in nDD(2) as it is not normal. On the other hand, \( nDD(2) \subseteq DD(1) \) cannot hold either because \( nDD(2) \setminus nDD(1) \neq \emptyset \) by statement (e).

(h) **DD(2) \cap DD(3) || nDD(3):**
The FFA \( \mathcal{F} = (\{a, b\}, \{x\}, f) \) with \( f \) defined by \( f(a, x, b) = f(b, x, b) = 0.5 \) is DD2- and DD3-directable, but it is not normal. The FFA \( \mathcal{G} = (\{a, b, c\}, \{x\}, g) \), in which \( g \) is given by \( g(a, x, b) = 0.5 \) and \( g(a, x, a) = g(b, x, c) = g(c, x, c) = 1 \), is normal and DD3-directable. For example, \( xx \in DD_3(\mathcal{G}) \). On the other hand, there are no DD2-directing words as \( \mathcal{G}_{a,x} = \{a/1, b/0.5\} \), \( \mathcal{G}_{b,x} = \mathcal{G}_{c,x} = \{c/1\} \), and \( \mathcal{G}_{a,w} = \mathcal{G}_{a,w} = \mathcal{G}_{c,w} = \{c/1\} \) for any \( w = x^n \) with \( n \geq 2 \).

(i) **nDD(2) \subset DD(2) \cap DD(3):**
Of course, \( nDD(2) \subseteq DD(2) \), and by using statements (f) and (a), we get
\[ \text{nDD}(2) \subseteq \text{nDD}(3) \subseteq \text{DD}(3). \] Thus \( \text{nDD}(2) \subseteq \text{DD}(2) \cap \text{DD}(3) \). The FFA \( \mathcal{F} \) used in the proof of (h) is DD2– and DD3–directable, but it is not normal. Hence, the inclusion is proper.

(j) \( \text{DD}(1) \cap \text{nDD}(2) = \text{nDD}(1) \):

The inclusion \( \text{nDD}(1) \subseteq \text{DD}(1) \cap \text{nDD}(2) \) holds because \( \text{nDD}(1) \subseteq \text{DD}(1) \) by (a), and \( \text{nDD}(1) \subseteq \text{nDD}(2) \) by (e). On the other hand, if \( F \in \text{DD}(1) \cap \text{nDD}(2) \), then \( F \) is both DD1–directable and normal, i.e., \( F \in \text{nDD}(1) \).

(k) \( (\text{DD}(2) \cap \text{DD}(3)) \cap \text{nDD}(3) = \text{nDD}(2) \):

Obviously, (k) is equivalent to the equality \( \text{DD}(2) \cap \text{nDD}(3) = \text{nDD}(2) \) which has a similar proof as (j).

(l) \( \text{Dir} \subseteq \text{nDD}(1) \):

Any DFA is clearly normal when viewed as an FFA, and by Lemma 5.1 is DD1–directable. On the other hand, the FFA \( \mathcal{F} = (A, X, f) \), where \( A = \{a, b, c\}, X = \{x\} \), and \( f \) is given by \( f(a, x, b) = f(a, x, c) = f(b, x, c) = f(c, x, c) = 1 \), is DD1–directable and normal, but not a DFA. \( \square \)

Let us now consider the subautomata, epimorphic images and finite direct products of DD1–, DD2– and DD3–directable FFAs. In what follows, \( \mathcal{F} = (A, X, f) \) and \( \mathcal{G} = (B, X, g) \) are any two FFAs.

We call \( \mathcal{G} \) a subautomaton of \( \mathcal{F} \) if (1) \( B \subseteq A \), (2) \( \mathcal{F}(b, x) \subseteq B \) for all \( b \in B \) and \( x \in X \), and (3) \( g(b, x, b') = f(b, x, b') \) for all \( b, b' \in B \) and \( x \in X \).

**Lemma 6.2** Let \( \mathcal{G} \) be a subautomaton of \( \mathcal{F} \). For any \( b \in B \) and \( w \in X^* \),

(a) \( \mathcal{F}(b, w) \subseteq B \),

(b) \( \mathcal{G}(b, w) = \mathcal{F}(b, w) \), and

(c) \( \mathcal{G}_{b, w} = \mathcal{F}_{b, w}|_B \) (\( \mathcal{G}_{b, w} \) is the restriction of \( \mathcal{F}_{b, w} \) to \( B \)).

**Proof.** All three statements have easy proofs by induction on \( \text{lg}(w) \). \( \square \)

**Proposition 6.3** Let \( \mathcal{G} \) be a subautomaton of an FFA \( \mathcal{F} \). For each \( i = 1, 2, 3 \), \( \text{DD}_i(\mathcal{F}) \subseteq \text{DD}_i(\mathcal{G}) \), and thus any subautomaton of a DD\( i \)--directable FFA is also DD\( i \)--directable. Furthermore, if \( \mathcal{F} \) is normal, then so is \( \mathcal{G} \).

**Proof.** If \( w \in \text{DD}_1(\mathcal{F}) \), then there exist \( c \in A \) and \( r \in (0, 1] \) such that \( \mathcal{F}_{a, w} = \{c/r\} \) for every \( a \in A \). In particular, for any \( b \in B \), \( c \in \mathcal{F}(b, w) \subseteq B \), and hence \( \mathcal{G}_{b, w} = \mathcal{F}_{b, w}|_B = \{c/r\} \). This means that \( w \in \text{DD}_1(\mathcal{G}) \).

If \( w \in \text{DD}_2(\mathcal{F}) \), then \( \mathcal{G}_{a, w} = \mathcal{F}_{a, w}|_B = \mathcal{F}_{b, w}|_B = \mathcal{G}_{b, w} \) for all \( a, b \in B \), i.e., \( w \in \text{DD}_2(\mathcal{G}) \).
Let \( w \in DD_3(\mathcal{F}) \), and let \( c \in A \) be such that for every \( a \in A \), \( \mathcal{F}_{a,w}(c) > 0 \) and \( \mathcal{F}_{a,w} \) is maximal among all the values of \( \mathcal{F}_{a,w} \). Since this holds also for every \( a \in B \), we have \( c \in B \), and therefore \( w \in DD_3(\mathcal{G}) \).

Assume now that \( \mathcal{F} \) is normal, and consider any \( a \in A \) and \( b \in B \). Since \( \mathcal{F} \) is normal, there is a state \( c \in A \) such that \( f(b,x,c) = 1 \). By Lemma 6.2, \( c \in B \) and \( g(b,x,c) = f(b,x,c) = 1 \), which shows that \( \mathcal{G} \) is normal. \( \square \)

A mapping \( \phi : A \to B \) defines a homomorphism \( \phi : \mathcal{F} \to \mathcal{G} \) of FFAs (cf. [19, 20, 21], for example) if

\[
g(a\phi, x, b) = \max\{f(a, x, a') \mid a' \in b\phi^{-1}\},
\]

for all \( a \in A \), \( b \in B \) and \( x \in X \). A surjective homomorphism is called an epimorphism.

Lemma 6.4 If \( \phi : \mathcal{F} \to \mathcal{G} \) is an epimorphism of FFAs, then for all \( a \in A \), \( b \in B \) and \( w \in X^* \),

(a) \( g^*(a\varphi, x, b) = \max\{f^*(a, w, a') \mid a' \in b\varphi^{-1}\} \), and

(b) \( \mathcal{F}(a, w)\varphi = \mathcal{G}(a\varphi, w) \).

Proof. Statement (a) has a straightforward proof by induction on \( \lg(w) \).

Let us consider (b). Also here we proceed by induction on \( \lg(w) \). If \( a' \in \mathcal{F}(a, w) \), then

\[
g^*(a\varphi, w, a'\varphi) = \max\{f^*(a, w, a'\varphi) \mid a'' \in A, a''\varphi = a'\varphi\} = f(a, w, a') > 0
\]

implies that \( a'\varphi \in \mathcal{G}(a\varphi, w) \), and hence that \( \mathcal{F}(a, w)\varphi \subseteq \mathcal{G}(a\varphi, w) \). Conversely, if \( b \in \mathcal{G}(a\varphi, w) \), then

\[
\max\{f^*(a, w, a') \mid a' \in b\varphi^{-1}\} = g^*(a\varphi, w, b) > 0
\]

means that \( f^*(a, w, a') > 0 \), i.e., \( a' \in \mathcal{F}(a, w) \), for some \( a' \in b\varphi^{-1} \), and thus \( b \in \mathcal{F}(a, w)\varphi \). \( \square \)

Proposition 6.5 If \( \phi : \mathcal{F} \to \mathcal{G} \) is an epimorphism of FFAs, then \( DD_i(\mathcal{F}) \subseteq DD_i(\mathcal{G}) \) for \( i = 1, 2, 3 \), and thus all epimorphic images of a \( DDi \)-directable FFA are \( DDi \)-directable. Moreover, if \( \mathcal{F} \) is normal, then so is \( \mathcal{G} \).

Proof. Let us prove \( DD_i(\mathcal{F}) \subseteq DD_i(\mathcal{G}) \) for each value of \( i \).

If \( w \in DD_1(\mathcal{F}) \), then there exist a \( c \in A \) and an \( r \in (0,1] \) satisfying \( \mathcal{F}_{a,w} = \{c/r\} \) for every \( a \in A \). Consider any \( b \in B \). Since \( \varphi \) is surjective, \( b = a\varphi \) for some \( a \in A \). For any \( d \in B \),
\[ g^*(b, w, d) = \max\{f^*(a, w, a') \mid a' \in d\varphi^{-1}\} = \begin{cases} r, & \text{if } d = c\varphi \\ 0, & \text{if } d \neq c\varphi. \end{cases} \]

This means that \( G_{b, w} = \{c\varphi/r\} \) for every \( b \in B \), and thus \( w \in DD_1(G) \).

If \( w \in DD_2(F) \), then \( f^*(a_1, w, a) = f^*(a_2, w, a) \) for all \( a_1, a_2, a \in A \). Consider any states \( b_1, b_2 \in B \) of \( G \), and let \( a_1, a_2 \in A \) be states of \( F \) for which \( b_1 = a_1\varphi \) and \( b_2 = a_2\varphi \). For any \( b \in B \),

\[ g^*(b_1, w, b) = \max\{f^*(a_1, w, a) \mid a \in b\varphi^{-1}\} = \max\{f^*(a_2, w, a) \mid a \in b\varphi^{-1}\} = g^*(b_2, w, b), \]

which shows that \( w \in DD_2(G) \).

Let \( w \in DD_3(F) \). Then there is a \( c \in A \) such that, for every \( a \in A \), \( f^*(a, w, c) > 0 \) and \( f^*(a, w, d) \leq f^*(a, w, c) \) for every \( d \in A \). Consider any \( b, b' \in B \), and let \( a, a' \in A \) be states for which \( b = a\varphi \) and \( b' = a'\varphi \). Then

\[ g^*(b, w, c\varphi) = \max\{f^*(a, w, a'') \mid a''\varphi = c\varphi\} \geq f^*(a, w, c) > 0, \]

and

\[ g^*(b, w, b') = \max\{f^*(a, w, a'') \mid a'' \in b'\varphi^{-1}\} \leq f^*(a, w, c) \leq g^*(b, w, c\varphi), \]

which means that \( w \in DD_3(G) \).

Assume now that \( F \) is normal, and consider any \( b \in B \) and \( x \in X \). If \( a \in A \) and \( c \in A \) are states for which \( a\varphi = b \) and \( f(a, x, c) = 1 \), then

\[ g(b, x, c\varphi) = g(a\varphi, x, c\varphi) = \max\{f(a, x, d) \mid d\varphi = c\varphi\} \geq f(a, x, c) = 1, \]

i.e., \( g(b, x, c\varphi) = 1 \).

The direct product (cf. [19, 20], for example) of two FFAs \( F \) and \( G \) is the FFA \( F \times G = (A \times B, X, h) \), where \( h \) is defined by

\[ h((a, b), x, (a', b')) = f(a, x, a') \land g(b, x, b') \quad (a, a' \in A, b, b' \in B, x \in X). \]

The following example shows that DD\(i\)–directability \( (i = 1, 2, 3) \) is not preserved under this product.

**Example 6.6** Let \( F = \{a, b\}, \{x, y\}, f \) and \( G = \{1, 2\}, \{x, y\}, g \) be FFAs, where \( f \) and \( g \) are given by \( f(a, x, b) = f(b, x, b) = f(b, y, b) = 1, \) and \( g(1, y, 2) = g(2, y, 2) = g(2, x, 2) = 1 \). These FFAs belong to all three classes \( DD(1), DD(2) \) and \( DD(3) \) because \( x \in DD_1(F) \cap DD_2(F) \cap DD_3(F) \) and \( y \in DD_1(G) \cap DD_2(G) \cap DD_3(G) \).

As \( (F \times G)((a, 1), x) = (F \times G)((a, 1), y) = \emptyset \) while \( (F \times G)((b, 2), x) = (F \times G)((b, 2), y) = \{ (b, 2) \} \neq \emptyset \), it is clear that \( F \times G \) has no DD1–, DD2– or DD3–directing words, and thus \( F \times G \notin DD(1) \cup DD(2) \cup DD(3) \).
References

[1] M.A. Arbib, *Theories of Abstract Automata*, Prentice-Hall, Englewood Cliffs, N.J., 1969.

[2] S. Bogdanović, B. Imreh, M. Ćirić and T. Petković, Directable automata and their generalizations: a survey, *Novi Sad Journal of Mathematics* **29**, No. 2 (1999), 19-69.

[3] H.V. Burkhard, Zum Längenproblem homogener Experimente an determinierten und nicht-determinierten Automaten, *Elektronische Informationsverarbeitung und Kybernetik, EIK* **12** (1976), 301-306.

[4] J. Černý, Poznámka k homogénnym experimentom s konečnými automatmi. *Matematicko-fysikalný Časopis SAV* **14**(4) (1964), 208-215.

[5] H. Don and H. Zantema, Synchronizing non-deterministic finite automata, *Journal of Automata, Languages and Combinatorics* **23**, No. 4 (2018), 307-328.

[6] Zs. Gazdag, Sz. Iván and J. Nagy-Görgy: Improved upper bounds on synchronizing nondeterministic automata, *Information Processing Letters* **109** (2009), 986-990.

[7] P. Goralčík, Z. Hedrlín, V. Koubek and J. Ryšlinková, A game of composing binary relations, *R.A.I.R.O. Informatique théorique/Theoretical Informatics* **16**(4) (1982), 365-369.

[8] J.E. Hopcroft, R. Motwani and J.D. Ullman, *Automata, Languages, and Computation* (3. ed.), Pearson Boston 2007.

[9] B. Imreh and M. Ito, On some special classes of regular languages, in: J. Karhumäki, H. Maurer, G. Paun, G. Rozenberg (eds.), *Jewels are Forever*, Springer, Berlin 1999, pp. 25-34.

[10] B. Imreh, M. Ito and M. Steinby, On commutative directable nondeterministic automata, in: C. Martin-Vide and V. Mitrana (eds.), *Grammars and Automata for String Processing. From Mathematics and Computer Science to Biology, and Back*, Taylor and Francis, London 2003, pp. 141-150.

[11] B. Imreh and M. Steinby, Some Remarks on Directable Automata, *Acta Cybernetica* **12** (1995), 23-35.

[12] B. Imreh and M. Steinby, Directable nondeterministic automata, *Acta Cybernetica* **14** (1999), 105-115.

[13] M. Ito, *Algebraic Theory of Automata and Languages*, World Scientific Publishing Co., Singapore 2004.
[14] M. Ito and K. Shikishima-Tsuji, Shortest directing words of nondeterministic directable automata, *Discrete Mathematics* **308** (2008), 4900-4905.

[15] Sz. Iván, Synchronizing weighted automata, in: Z. Ésik and Z. Fülöp (eds.), *Automata and Formal Languages* (AFL 2014), EPTCS **151** (2014), 301-313.

[16] V. Karthikeyan and M. Rajasekar, Directable Fuzzy Automata, *International Journal of Computer Applications* (0975-8887) **125**, No. 8 (2015).

[17] D.J. Kfoury, Synchronizing sequences of probabilistic automata, *Studies in Applied Mathematics* **49** (1970), 101-103.

[18] D.C. Kozen, *Automata and Computability*, Springer-Verlag, New York 1997.

[19] D.S. Malik, J.N. Mordeson and M.K. Sen, Products of fuzzy finite state machines, *Fuzzy Sets and Systems* **92** (1997), 95-102.

[20] J. Mordeson and D. Malik, *Fuzzy Automata and Languages: Theory and Applications*, Chapman & Hall (CRC), London 2002.

[21] T. Petković, Congruences and homomorphisms of fuzzy automata, *Fuzzy Sets and Systems* **157** (2006), 444-458.

[22] G. Rahonis, Fuzzy languages, in: M. Droste, W. Kuich, and H. Vogler (eds.), *Handbook of Weighted Automata*, Springer-Verlag, Berlin 2009. pp. 481-517.

[23] J. Sakarovich, *Elements of Automata Theory*, Cambridge University Press, Cambridge 2009.

[24] A. Salomaa, *Theory of Automata*, Pergamon Press, Oxford, 1969.

[25] H. Shabana, $D_2$-synchronization in nondeterministic automata, *Ural Mathematical Journal* **4**, No. 2 (2018), 99-110.

[26] M.V. Volkov, Synchronizing automata and the Černý conjecture, in: C. Martín-Vide, F. Otto and H. Fernau (eds.), *LATA 2008*, Lecture Notes in Computer Science **5196**, Springer-Verlag, Berlin 2008, pp. 11-27.

[27] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338-353.