Demonstration of an optical-coherence converter

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Optical coherence is evaluated by assessing the correlations between field fluctuations at different points in space and time. When multiple degrees of freedom (DoFs) of an optical field – spatial, temporal, and polarization – are relevant, the coherence of each DoF is typically studied separately. For example, spatial coherence is evaluated via double-slit interference, temporal coherence through two-path (e.g., Michelson) interference, and polarization coherence by measuring the Stokes parameters. Although this traditional approach succeeds when the DoFs are uncoupled, it fails at capturing key features of the field’s coherence if the DoFs are indeed correlated – a situation that arises often. By viewing coherence as a ‘resource’ that can be shared among the DoFs, we show that the field can be reconfigured to allocate the entropy from one DoF to another without loss of energy. Starting from a linearly polarized spatially incoherent field – one that produces no spatial interference fringes – we obtain a spatially coherent field that is unpolarized. By reallocating the entropy to polarization, the field becomes invariant with regard to the action of a polarization scrambler, thus suggesting a strategy for avoiding the deleterious effects of a randomizing system on a DoF of the optical field.
state $S_p = 0$ corresponds to a fully polarized field (no statistical fluctuations), whereas the maximal-entropy state $S_p = 1$ corresponds to an unpolarized field (maximal fluctuations)\cite{12}; similarly for the spatial DoF based on $G_s$. Entropy so defined is a \textit{unitary invariant} of the field DoF; it cannot be changed by applying lossless deterministic optical transformations.

B. Joint polarization and spatial coherence formalism

Evaluating $G_p$ and $G_s$ is not sufficient to completely identify the coherence of a vector field in which the polarization and spatial DoFs are potentially correlated. A $4 \times 4$ coherency matrix $G$ is necessary to capture the full vector-field coherence\cite{13,6},

$$G = \begin{pmatrix} 
C_{aa}^{HH} & C_{ab}^{HV} & C_{ba}^{VH} & C_{bb}^{VV} \\
C_{ab}^{HV} & C_{dd}^{VV} & C_{db}^{VH} & C_{bb}^{VV} \\
C_{ba}^{VH} & C_{db}^{VH} & C_{dd}^{VV} & C_{bb}^{VV} \\
C_{bb}^{VV} & C_{bb}^{VV} & C_{bb}^{VV} & C_{bb}^{VV} 
\end{pmatrix},$$

where $G_{ii} = \langle E_{i}^\dagger (E_i)^* \rangle$, $i,j = H,V$, and $k,l = a,b$. The matrix $G$ is Hermitian positive semi-definite and normalized such that $\text{Tr} \{ G \} = 1$ ($\text{Tr}$ is the matrix trace). The diagonal elements are the power-fractions from the mutually exclusive contributions: $C_{aa}^{HH}$ and $C_{aa}^{VV}$ are the H and V components at $\vec{a}$, respectively, and $C_{bb}^{HH}$ and $C_{bb}^{VV}$ are those at $\vec{b}$. The off-diagonal elements are normalized correlations between field components. The double-slit visibility observed when overlapping the fields from $\vec{a}$ and $\vec{b}$ is $V = 2 | \langle G_{bb}^{HH} + G_{bb}^{VV} \rangle |^{1/2}$. Crucially, $V$ is not a unitary invariant of the field\cite{15,16}, and reversible optical transformations that span the spatial and polarization DoFs can increase $V^{17,18}$.

Each \textit{physically independent} DoF (spatial and polarization) carries one bit of entropy, so the vector field now carries 2 bits of entropy: $S = -\lambda_{ab1} \log_2 \lambda_{ab1} - \lambda_{ab2} \log_2 \lambda_{ab2} - \lambda_{ab3} \log_2 \lambda_{ab3} - \lambda_{ab4} \log_2 \lambda_{ab4}$, where $0 \leq S \leq 2$ and $\{ \lambda \} = \{ \lambda_{ab1}, \lambda_{ab2}, \lambda_{ab3}, \lambda_{ab4} \}$ are the real positive eigenvalues of $G$. The zero-entropy state $S = 0$ corresponds to a fully polarized and spatially coherent field (no statistical fluctuations in either DoF and $\{ \lambda \} = \{ 1,1,0,0 \}$), whereas the maximal-entropy state $S = 2$ corresponds to an unpolarized spatially incoherent field (maximal fluctuations in both DoFs and $\{ \lambda \} = \{ 1,1,1,1 \}$).

II. CONCEPT OF OPTICAL COHERENCY CONVERSION

In general $S \leq S_p + S_s$, with equality achieved only when the two DoFs are independent, in which case $G$ can be written in separable form $G = G_s \otimes G_p$. In general, $S_s$ and $S_p$ are obtained from the $2 \times 2$ ‘reduced’ spatial and polarization coherency matrices

$$G_s^{(r)} = \begin{pmatrix} 
C_{aa}^{HH} + C_{bb}^{VV} & C_{ab}^{HV} + C_{bb}^{VV} \\
C_{ab}^{HV} + C_{bb}^{VV} & C_{bb}^{VV} + C_{bb}^{VV} 
\end{pmatrix},$$

$$G_p^{(r)} = \begin{pmatrix} 
C_{aa}^{HH} + C_{bb}^{HH} & C_{ab}^{VH} + C_{bb}^{VV} \\
C_{ab}^{VH} + C_{bb}^{VV} & C_{bb}^{VV} + C_{bb}^{VV} 
\end{pmatrix},$$

which are obtained from $G$ by a ‘partial trace’\cite{19}, that is, by tracing over one DoF\cite{6,7}.

The concept of an optical-coherence converter is illustrated in Fig. 1(a). Consider the case when the field carries one bit of entropy ($S = 1$) and the DoFs are independent ($S = S_p + S_s$), in which case a single DoF can accommodate this entropy. The field may be maximally \textit{incoherent} but polarized ($S_p = 1$ and $S_s = 0$), whereupon no interference fringes can be observed [Fig. 1(b)]. Alternatively, the field may be spatially coherent but unpolarized ($S_s = 0$ and $S_p = 1$), in which case full-visibility fringes can be observed [Fig. 1(c)]. We demonstrate here that an optical field can be \textit{reversibly} transformed from the former configuration to the latter without loss of energy, thus \textit{converting} coherence from one DoF (polarization) to the other (spatial). Throughout the procedure $S$ remains constant; that is, no uncertainty is added or removed from the field, only an internal reorganization of the field engendered by a unitary transformation confines the statistical fluctuations to one DoF while freeing the other from uncertainty. We call such a system a ‘coherence converter’.

The optical arrangement we propose to convert coherence between the spatial and polarization DoFs is depicted in Fig. 2. We start from two points $\vec{a}$ and $\vec{b}$ of equal intensity in a spatially \textit{incoherent} H-polarized field (the fields are mutually incoherent or statistically independent), which thus produce no interference fringes. The polarization at $\vec{b}$ is rotated to become orthogonal to that at $\vec{a}$ (H $\rightarrow$ V) before combining the fields at a polarizing beam splitter (PBS), which yields an unpolarized field. We then split the field into two points $\vec{a}$ and $\vec{b}$ using a nonpolarizing beam splitter, which creates two copies of the field that can demonstrate high-visibility interference fringes. We proceed now to present the measurements at each step of this coherence-conversion process.

Figure 1 | Concept of an optical-coherence converter. (a) Starting with a polarized but spatially incoherent field ($S_p = 0$ and $S_s = 1$), coherence is converted from polarization to the spatial DoF, thereby yielding an unpolarized but spatially coherent field ($S_p = 1$ and $S_s = 0$) but without introducing further fluctuations (fixed total entropy $S = 1$). The device thus converts the statistical fluctuations (and the attendant entropy) from one DoF to the other. (b) When a polarized but spatially incoherent field is incident on a double-slit, no interference fringes are observed. (c) After converting coherence from polarization to the spatial DoF, high-visibility (but unpolarized) interference fringes appear.

III. SOURCE CHARACTERIZATION

The optical field we study is extracted from a broadband, unpolarized LED (center wavelength 850 nm, 30-nm-FWHM bandwidth; Thorlabs M850L3 IR). The field is spectrally filtered (10-nm-FWHM), polarized along H, and spatially filtered through a 100-μm-wide slit placed at a distance of 180 mm from the source. The ‘input’ plane that includes the points $\vec{a}$ and $\vec{b}$ (each defined by a 100-μm-wide slit) is located 420 mm away from the slit [the source in Fig. 2(a)]. We first confirm that the field is spatially coherent within $\vec{a}$ and $\vec{b}$ separately (i.e., the spatial coherence width of the field, estimated to be $\approx 1$ mm, is larger than the slit width). This is accomplished using a narrow pair of slits (50-μm-wide separated by $\Delta = 150$ μm) at either $\vec{a}$ or $\vec{b}$ and observing the double-slit interference on a CCD camera (Hamamatsu 1394) at a
We have thus confirmed the relationship between the two length where the subscripts ‘s’ and ‘p’ refer to the spatial and polarization DoFs, we measure $G_p$ polarized (scalar) but spatially incoherent, thus the fields from $G_s \otimes G_p$ have the form

\[ G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

where the subscripts ‘s’ and ‘p’ refer to the spatial and polarization DoFs, respectively, and the notation $\otimes$ identifies a diagonal matrix with the entries along the diagonal listed between the curly brackets.

To fully characterize the field coherence across the spatial and polarization DoFs, we measure $G_1$ via OCmT\(^{27,28}\), which requires 16 measurements to reconstruct $G_1$. Since 4 polarization projections are required to identify $G_p$ and 4 spatial projections are required to determine $G_s$, for a scalar field, $4 \times 4$ linearly independent combinations of these spatial and polarization projections are necessary to reconstruct $G$ subject to the constraints of Hermiticity, semi-positiveness, and unity-trace. These measurements are in one-to-one correspondence to those required to reconstruct a two-qubit density matrix in quantum mechanics, a process known as ‘quantum state tomography’\(^{20-22}\). Carrying out these optical measurements (see\(^8\) for details), $G_1$ is reconstructed [Fig. 3(b)] and is found to be in good agreement with the theoretical expectation [Fig. 3(c)], with the remaining slight deviations attributable to unequal powers at $\vec{a}$ and $\vec{b}$.

The measured coherency matrix $G_1$ in the $(\vec{a}, \vec{b})$-plane yields $S = 1.001$, and the reduced spatial and polarization coherency matrices $G_{s(p)}$ obtained from $G$ yield $S_s = 0.991$, $S_p = 0.037$, respectively. The field entropy is thus associated with the spatial DoF and not polarization, resulting in an absence of interference fringes [Fig. 3(a)]. The lack of interference fringes is consistent with the fact that all measures of spatial coherence or double-slit interference fringes for a vector field rely on the cross-correlation matrix\(^{14}\) or beam-coherence matrix\(^{23}\), $G_{ab} = \begin{pmatrix} G_{HH} & G_{HV} \\ G_{VH} & G_{VV} \end{pmatrix}$, which is the top right $2 \times 2$ block of the coherency matrix $G$.

Figure 3 | (a) The four measurements required to reconstruct the spatial coherence matrix $G_s$ for a scalar field at $\vec{a}$ and $\vec{b}$. The intensity pattern is recorded with both slits open (left), and two measurements are made: the intensity on the optical axis (red dot) and at the location mid-way along the first expected fringe location calculated from the slit separation (green dot). No fringes are observed here since the field is spatially incoherent. Next, the intensity on the optical axis is recorded when $\vec{a}$ (left) and $\vec{b}$ (right) are blocked (the red dots; see Refs.\(^7,8\) for details). (b) Plot depicting graphically the real parts of the elements of the spatial-polarization coherency matrix $G_1$ for the source plane as reconstructed from OCmT that utilizes the measurements in (a) when carried out in conjunction with polarization measurements. (c) Plot depicting graphically the elements of the theoretically expected coherency matrix $G = \frac{1}{2} \text{diag}(1,0,1,0)$, corresponding to a scalar H-polarized field that is spatially incoherent (Eq. 1).
wave plate (HWP) placed after ports of a PBS (Thorlabs CM1-PBS252), where the V component loss of power or increase in total entropy becomes unpolarized but spatially coherent (G-field on the PBS is linearly polarized, whereas their superposition unpolarized, which we confirm by registering a flat Malus curve to the PBS, the input fields are reconstituted. The field is now V = \{1, 0\}, whereas its superposition is V-polarized light, respectively. (c) Graphical depiction of the field at the output port highlights its random polarization. The dashed interference fringes, the randomly polarized field at \(\vec{a}''\) is split symmetrically into two halves by a 50:50 non-polarizing beam splitter to points \(\vec{a}\) and \(\vec{b}\) [Fig. 4(a)], which can then be overlapped to produce high-visibility fringes. This step does not change the values of \(S, S_\nu,\) or \(S_p\). The \((\vec{a}, \vec{b})\)-plane is the image plane relayed from the \((\vec{a}', \vec{b}')\)-plane by a lens [Fig. 2(c)]. The coherency matrix \(G_3\) in this plane:

\[
G_3 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \frac{1}{2} \left( \frac{1}{s} \right) \left( \frac{1}{p} \right). \tag{3}
\]

The measured \(G_3\) reconstructed via OCmT [Fig. 5(b)] is in good agreement with the theoretical expectation [Fig. 5(c)].

Now that we have \(G_{1b}^{ab} + G_{bb}^{ab} = \frac{1}{2}\), the predicted visibility is \(V = 1^{14}\). Furthermore, because the two DoFs are independent and the field is unpolarized (as is clear from the separable form of \(G_2\) in Eq. 2), projecting the polarization on any direction will not affect the high visibility [Fig. 5(d)]. Note that the coherence time of the field is determined by its spectral bandwidth, and observing the fringes [Fig. 5(d)] requires introducing a relative delay in the path of the fields from \(\vec{a}\) or \(\vec{b}\) before overlapping them [delay line 2 in Fig. 2(c)]. The variation in the measured visibility with introduced relative time delay corresponds to the expected field coherence time \(\approx 100\, \text{fs} \) (coherence length \(\approx 30\, \mu\text{m}\) [Fig. 5(e)].

**V. SURVIVING RANDOMIZATION THROUGH ENTROPY REALLOCATION**

The ability to redistribute or reallocate the field entropy \(S\) between the DoFs can be exploited in protecting a DoF from the deleterious impact of a randomizing medium. Consider a depolarizing medium represented by a Mueller matrix \(\hat{M} = \text{diag}\{1, 0, 0, 0\}\) that converts any state of polarization into a completely unpolarized state. The initial field \(G_1\) [Fig. 3(b,c)] would be converted into the incoherent unpolarized field \(G_1' = \frac{1}{2} \text{diag}\{1, 1, 1, 1\}\) upon traversing this medium with \(S_p \rightarrow 1\), such that \(S \rightarrow 2\). If, however, coherence is first reversibly converted from the spatial DoF to polarization (\(S_p \rightarrow 1\) and \(S_\nu \rightarrow 0\)), then traversing a depolarizing medium cannot increase \(S_p\), and the field is thus left unchanged. Subsequently, the coherence-conversion can be reversed and a polarized field retrieved after emerging from the depolarizing medium without loss of energy.

We have carried out the proof-of-concept experiment depicted in Fig. 6(a) where we place a depolarizer or polarization scrambler in the path of the field \(G_2\) in the \((\vec{a}', \vec{b}')\)-plane. The polarization scrambler is implemented by rotating a HWP. The CCD camera exposure time is increased to 10 s, corresponding to the rotation.
Figure 5 | (a) Schematic for the coherence converter that transforms two linearly polarized, spatially incoherent fields (at \(\mathbf{a}'\) and \(\mathbf{b}'\)) into two randomly polarized mutually coherent fields (at \(\mathbf{a}\) and \(\mathbf{b}\)). (b) Graphical depiction of the real part of the entries of the experimentally reconstructed \(G_3\) via OCmT. (c) The theoretical expectation for \(G_3\). (d) Interference patterns obtained by overlapping the fields from \(\mathbf{a}\) and \(\mathbf{b}\) after a polarization projection, with high-visibility fringes observed in all cases. The top panels are CCD camera images and the lower panels are obtained by integrating the fringes vertically. (e) Visibility as a function of a relative delay inserted between the fields at \(\mathbf{a}\) and \(\mathbf{b}\) before overlapping them at the CCD camera [Fig. 2(c)] for the diagonal polarization projection case in (d).

Time of the waveplate, to capture the averaged interference pattern in a single shot. The Mueller matrix associated with a polarization scrambler is \(\hat{M} = \text{diag}\{1, 0, 0, 0\}\). The visibility observed after the \((\mathbf{a}, \mathbf{b})\)-plane remains high. This result can be modeled theoretically by first noting that a unitary transformation \(\hat{U}\) transforms the field according to \(G_2 \rightarrow G'_2 = \hat{U}G_2\hat{U}^\dagger\), where \(\hat{U}\) is a \(4 \times 4\) unitary transformation that spans the spatial and polarization DoFs. If the device in question impacts the two DoFs independently, then \(\hat{U} = \hat{U}_s \otimes \hat{U}_p\), where \(\hat{U}_s\) and \(\hat{U}_p\) are \(2 \times 2\) unitary transformations for the spatial and polarization DoFs, respectively. The impact of a randomizing but energy-conserving transformation can be modeled as a statistical ensemble of unitary transformations\(^{26-28}\). The transformation of \(G_2\) upon traversing a polarization scrambler can be expressed as

\[
G'_2 = \sum_{j=1}^{4} p_j (\hat{I} \otimes \hat{U}_p^{(j)}) G_2 (\hat{I} \otimes \hat{U}_p^{(j)})^\dagger,
\]

where \(\hat{I}\) is the \(2 \times 2\) identity matrix and the ensemble \(\{\hat{U}_p^{(j)}\}\) comprises with equal probabilities \(p_j = \frac{1}{4}\) the Jones matrices:

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]

which correspond to polarization rotations of \(\pm 45^\circ\) in the H-V basis, and a HWP in the H-V basis and rotated by \(45^\circ\). Substituting the ensemble \(\{\hat{U}_p^{(j)}\}\) into Eq. 4 yields \(G'_2 = G_2\), which entails that the high-visibility seen in Fig. 5(d) should be retained, as confirmed in Fig. 6(a). In other words, \(G_2\) is invariant with regards to any polarization randomization.

If the polarization scrambler is position-dependent, the coherency matrix \(G_2\) will no longer be invariant under randomization (because the spatial and polarization DoFs become coupled). In the experiment illustrated in Fig. 6(b), the polarization scram-
bler is placed at $\vec{d}$ in the plane of $G_3$. The spatial-polarization transformation of $G_3$ takes the form
\[
G_3 = \{\hat{A}_b \otimes 1\} G_3 \{\hat{A}_b \otimes 1\} + \sum_{j=1}^{4} p_j \{\hat{A}_d \otimes \hat{U}_{j}^{(j)}\} G_3 \{\hat{A}_d \otimes \hat{U}_{j}^{(j)\dagger}\},
\]
where we have $\hat{A}_d = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ and $\hat{A}_b = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$, resulting in $G_3' = \frac{1}{2} \mathbb{1} \otimes \frac{1}{2} \mathbb{1} = \frac{1}{2} \text{diag}(1,1,1,1)$; that is, an unpolarized and spatially incoherent field with maximal entropy $S = 2$ is produced. No fringes will appear in this case [Fig. 6(b)].

### VI. DISCUSSION AND CONCLUSION

To facilitate the analysis of the coherence of optical fields encompassing multiple DoFs, it is becoming increasingly clear that the mathematical formalism of multi-partite quantum mechanical states is most useful. The underlying foundation for this utility is the analogy between the mathematical description used in these domains. The Hilbert space of a multi-partite system is the tensor product of the Hilbert spaces of the single-particle subsystems. In classical optics, the multiple DoFs of a beam are described in a linear vector space having formally the structure of a tensor product of the linear vector spaces of the individual DoFs. In the quantum setting, pure multi-partite states that cannot be separated into products of single-particle states are said to be entangled; whereas in the classical setting, fields in which the DoFs cannot be separated are now being called ‘classically entangled’, coherence can be viewed as a ‘resource’ that may be reversibly converted from one DoF of the beam to another, just as entanglement is a resource shared among multiple quantum particles. There is a critical difference though between the quantum and classical settings. Entanglement between initially separable particles requires nonlocal operations of particle-particle interactions (which are particularly challenging for photons); on the other hand, entangling operations that couple different DoFs of a beam are readily available in classical optics. Adopting this information-driven standpoint has led to a host of novel opportunities and applications. For example, Bell’s measure, originally developed for ascertaining nonlocality, becomes a useful in quantifying the coherence of a multi-DoF beam and assessing the resources required to synthesize a beam of given characteristics; beams in which spatial modes and polarization are classically entangled have been shown to enable fast particle tracking and full Mueller characterization of a sample; and introducing spatio-temporal spectral correlations leads to propagation-invariant pulsed optical beams.

We have demonstrated – for the first time to the best of our knowledge – the ‘conversion’ or transformation of coherence from one DoF of an optical field to another; namely, from polarization to the spatial DoF. Starting from a field that is spatially incoherent but polarized, we redirect the statistical fluctuations from space towards polarization, resulting in an unpolarized field that is spatially coherent – reversibly, without losing optical energy along the way. Specifically, the 1 bit of entropy characterizing the spatial DoF was removed and added instead to the initially zero-entropy polarization. Entropy-engineering of partially coherent optical fields can open the path to a variety of possible future extensions and applications with regards to optimizing the interaction of optical fields with disordered media.

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