All you need is N: Baryon spectroscopy in two large N limits

Aleksey Cherman†
Center for Fundamental Physics, Department of Physics,
University of Maryland, College Park, MD 20742-4111

Thomas D. Cohen‡
Maryland Center for Fundamental Physics, Department of Physics,
University of Maryland, College Park, MD 20742-4111

Richard F. Lebed§
Department of Physics, Arizona State University, Tempe, AZ 85287-1504

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The generalization of QCD to many colors is not unique; each distinct choice corresponds to a distinct $1/N_c$ expansion. The familiar ’t Hooft $N_c \to \infty$ limit places quarks in the fundamental representation of SU($N_c$), while an alternative approach places quarks in its two-index antisymmetric representation. At $N_c = 3$ these two $1/N_c$ expansions coincide. We compare their predictions for certain observables in baryon spectroscopy, particularly mass combinations organized according to SU(3) flavor breaking. Each large $N_c$ limit generates an emergent spin-flavor symmetry that leads to the vanishing of particular linear combinations of baryon masses at specific orders in the expansions. Experimental evidence shows that these relations hold at the expected orders regardless of which large $N_c$ limit one uses, suggesting the validity of either limit in the study of baryons. We also consider a hybrid large $N_c$ limit in which one flavor is taken to transform in the two-index antisymmetric representation and the rest of the flavors are in the fundamental representation. While this hybrid large $N_c$ limit is theoretically attractive, we show that for a wide class of observables it faces some phenomenological difficulties.

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I. INTRODUCTION

The study of large $N_c$ QCD and the $1/N_c$ expansion is nearly as old as QCD itself [1]. While much of its significance lies in the formal theoretical domain, the $1/N_c$ expansion has proven to be an important source of qualitative and semi-quantitative insights into hadronic physics. Underlying the approach is the premise that a world with a very large number of colors is not too different from a world with $N_c = 3$. Thus it is sensible to compute some hadronic properties of interest in the $N_c \to \infty$ world and then to improve predictions systematically by taking $1/N_c$ corrections into account. Implicit in this formulation is the assumption of a unique and well-defined large $N_c$ world. However, the extrapolation from $N_c = 3$ to $N_c \to \infty$ has long been known not to be unique [2]: one can construct distinct theories that agree at $N_c = 3$ but differ at $N_c = \infty$. Each distinct extrapolation corresponds to a distinct $1/N_c$ expansion for physical quantities. It is thus important to uncover which large $N_c$ limits are appropriate for various quantities of phenomenological interest.

The large $N_c$ limit is usually taken by placing quarks in the fundamental representation F of SU($N_c$), as occurs for the physical case $N_c = 3$. But $N_c = 3$ is exceptional because the (anti)fundamental representation and two-index antisymmetric representation AS [$N_c(N_c - 1)/2$ dimensional in general] are isomorphic: we can associate each AS quark $q^i$ with an F quark $q_i$ via $q_i = \frac{1}{2} \epsilon_{ijk} q^j k$. For example, an $N_c = 3$ red-blue quark in the AS representation is equivalent to an anti-green quark in F. Exchanging quarks with antiquarks, one sees that this “QCD$_{AS}$” at $N_c = 3$ is just ordinary “QCD$_F$”. For $N_c > 3$ however, QCD$_{AS}$ is clearly different from QCD$_F$. Each of them has a distinct large $N_c$ limit, which we refer to respectively as the large $N_c^{AS}$ limit and the large $N_c^F$ limit. When generically referring to either large limit, we continue to use $N_c$ without a superscript.

Since these two limits describe different large $N_c$ worlds, our $N_c = 3$ world may in effect be described more accurately by one of them, in the sense that one of the $1/N_c$ expansions may turn out to be more useful for describing some sets of observables. Indeed, it is not a priori clear if either limit is viable for a given set of observables.

Compelling theoretical reasons underlie the study of AS Dirac quarks and the large $N_c^{AS}$ limit. A key aspect is the large $N_c$ connection of QCD$_{AS}$ to QCD$_{Adj}$, which is Yang-Mills theory with Majorana quarks in the adjoint (Adj) representation. Armoni, Shifman, and Veneziano [3] have found an orientifold equivalence between the respective large $N_c^{AS}$ and $N_c^{Adj}$ limits. More precisely, for a certain set of observables, the differences between QCD$_{AS}$ and QCD$_{Adj}$ as $N_c \to \infty$ are sublead-

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*Electronic address: aleksey@physics.umd.edu
†Electronic address: cohen@physics.umd.edu
‡Electronic address: alekseyc@physics.umd.edu
§Electronic address: Richard.Lebed@asu.edu
ing in $1/N_c$, a fact that is interesting in its own right. Moreover, QCD$_{Adj}$ with a single massless flavor is supersymmetric, allowing one to use SUSY techniques to compute some observables explicitly in one-flavor QCD$_{Adj}$, and equate them at large $N_c$ to corresponding ones in one-flavor QCD$_{AS}$ [3].

While the large $N_c^{AS}$ limit is clearly theoretically interesting, determining its phenomenological relevance in describing the real $N_c = 3$ world is equally important. This paper compares the phenomenological viability of the large $N_c^{AS}$ limit to that of the standard large $N_c^{F}$ limit. We focus on testing the two large $N_c$ limits with observables from baryon spectroscopy, which can be used to systematically probe the differences between the two limits. Such a study is possible because a spin-flavor symmetry emerges in the baryon sector at large $N_c$ [1,5,6], and when combined with the hierarchy associated with SU(3)$_{flavor}$ breaking, this symmetry makes quantitative predictions for particular combinations of baryon masses.

Each of these baryon mass relations has been previously shown to hold to the expected accuracy for the usual large $N_c^{F}$ limit of QCD [7]. In particular, Ref. [7] showed that these relations hold far better than if one had merely used SU(3)$_{flavor}$ relations. The large $N_c^{F}$ world is therefore sufficiently close to the $N_c = 3$ world that the $1/N_c$ expansion has predictive power for the mass observables.

In this paper, we use the large $N_c$ spin-flavor symmetry together with SU(3)$_{flavor}$ breaking to show that the predictions of the large $N_c^{AS}$ expansion match the experimental data for the mass observables much better than those using SU(3)$_{flavor}$ breaking only. Furthermore, we show that the large $N_c^{AS}$ expansion explains the patterns in the data approximately as well as the large $N_c^{F}$ expansion. Thus one sees that, while the data require some large $N_c$ limit, it does not determine which large $N_c$ expansion is the correct one. The large $N_c^{F}$ world and the large $N_c^{AS}$ world are about equally close to the $N_c = 3$ world, at least as determined by the baryon mass spectrum.

As defined above, the large $N_c^{AS}$ and $N_c^{F}$ limits of QCD both treat all quark flavors symmetrically, and we focus on these two flavor-symmetric large $N_c$ limits for most of this paper. We also briefly discuss a hybrid large $N_c$ limit in which one flavor is singled out to transform in the AS representation, while the rest remain in the F representation. This hybrid “Corrigan-Ramond” expansion has a number of theoretical attractions, but as shown in Sec. [V] the lack of flavor symmetry for $N_c \neq 3$ makes its phenomenological utility questionable for a large class of observables that includes (for instance) baryon mass spectra.

The organization of the paper is as follows. In Sec. [II] we discuss some general phenomenological features of the large $N_c^{AS}$ limit and define the phenomenologically appropriate large $N_c$ generalization of baryons in QCD$_{AS}$. Next, in Sec. [III] we briefly review the emergent spin-flavor symmetry for baryons in the large $N_c$ limit, and discuss its application to the $N_c^{AS}$ limit, and in Sec. [IV] we confront the predictions of the two large $N_c$ limits with experimental data. We then discuss the phenomenology of a hybrid large $N_c$ limit in Sec. [V] and summarize in Sec. [VI].

II. BARYONS IN THE LARGE $N_c^{AS}$ LIMIT

In many ways the $1/N_c^{F}$ and $1/N_c^{AS}$ expansions are similar: Both large $N_c$ limits are planar, contain an infinite number of narrow mesons, and so on. However, internal F quark loops are suppressed in the large $N_c^{F}$ limit but not in the large $N_c^{AS}$ limit. This feature results from gluons and AS quarks sharing one important property: Summing over their internal quantum numbers means including $O(N_c^2)$ color index values, whereas sums over F quarks include only $N_c$ index values.

Thus the OZI rule and all of its phenomenological implications (e.g., a small $\rho-\omega$ mass splitting) naturally emerge in the $N_c^{F}$ limit, but not in the $N_c^{AS}$ limit. However, this particular phenomenological virtue of the $N_c^{F}$ limit induces a corresponding vice, as indicated by cases like the $\pi-$$\eta'$ mass splitting in which Zweig’s rule fails badly. Which of the two expansions is preferable depends on the observable.

In this paper we focus on mass observables associated with baryon spectroscopy. We now define what we mean by “baryons” in the two large $N_c$ limits, and discuss some of their properties at large $N_c$. In the large $N_c^{F}$ limit, the construction of color-singlet baryons $B_F$ with $N_c$ quarks is straightforward [8]:

$$B_F \sim \epsilon^{i_1,i_2,\cdots,i_{N_c}} q_{i_1} q_{i_2} \cdots q_{i_{N_c}}. \quad (1)$$

The situation is somewhat more complicated in the large $N_c^{AS}$ limit, since more than one way exists to construct large $N_c^{AS}$ color-singlet “baryons” [9]. One may couple $N_c$ quarks with color indices contracted as follows:

$$B_{\phi} \sim \epsilon^{i_1,i_2,\cdots,i_{N_c},i_{N_c-1},i_{N_c}} q_{j_1} q_{j_2} q_{j_3} \cdots q_{j_{N_c}}, \quad (2)$$

which represents a sum of terms, each one containing $N_c/2$ quark fields. A natural alternative construction uses sums involving two epsilon tensors and $N_c(N_c-1)/2$ quarks, and is fully antisymmetric under the interchange of any two quarks [9]. For $N_c = 3$, this can be written simply as

$$B_{\phi} \sim \epsilon_{i_2,j_2,i_1} \epsilon_{i_3,j_3,j_1} q_{i_1} q_{j_1} q_{i_2} q_{j_2} q_{i_3} q_{j_3}, \quad (3)$$

and one must keep in mind that the indices of the two-index quarks are to be antisymmetrized. The methods of Ref. [9] can be used to straightforwardly write similar expressions for general $N_c$, but the resulting expressions are somewhat cumbersome and we do not show them here.

Each of these combinations of quarks can reasonably be described as a baryon since each carries nonzero baryon
number. At the same time, these two classes of baryon have dramatically different properties. For example, denoting by $B_\phi$ and $B_\psi$ the baryon number of $B_\phi$ and $B_\psi$ as determined by quark number, one has
\[
\frac{B_\phi}{B_\psi} = \frac{1}{N_c - 1}.
\] (4)

It might appear that an ambiguity has arisen as to which of these types of baryons should be taken as the large $N_c^{\text{AS}}$ analog of our usual $N_c = 3$ baryons. In fact, this is not the case: By construction, $B_\phi$ baryons only exist for $N_c$ even (and have other questionable physical properties [9]). Thus the $B_\psi$ baryon is the better generalization of the $N_c = 3$ case. Accordingly, we take the baryon number per quark to be $[N_c(N_c - 1)/2]^{-1}$. Extrapolating solely to odd values of large $N_c$ eliminates $B_\phi$ baryons from further consideration. When discussing QCD$_{\text{AS}}$ “baryons” in the remainder of this paper, we mean $B_\psi$ baryons.

Baryons at large $N_c^{\text{AS}}$ behave as one might expect: As the number of quarks in a baryon scales like $N_c^2$, the baryon mass also scales like $N_c^2$ [9]. However, showing that the interactions between AS quarks are consistent with this counting involves somewhat more subtle combinatorics than for F quarks [10]. Witten noted long ago [11] that baryon properties in the conventional large $N_c^F$ limit scale as in a generic soliton model with the mesonic coupling constant scaling as $g \sim 1/N_c^{1/2}$. Similarly, one expects that baryon properties in the large $N_c^{\text{AS}}$ limit scale as in a generic soliton model with the mesonic coupling constant scaling as $g \sim 1/N_c$. It has been shown that observables in QCD$_{\text{AS}}$ Skyrme-type models self-consistently obey the correct $1/N_c$ scaling [10].

III. SPIN-FLAVOR SYMMETRY AS A PROBE

The central question of this paper is the extent to which the large $N_c^{\text{AS}}$ limit is viable for phenomenology, specifically for baryon spectroscopy. Qualitatively, the large $N_c^{\text{AS}}$ limit differs from the large $N_c^F$ limit in three ways. First, the generic scaling of many quantities associated with meson and baryon observables is different, and many quantities behave as if the expansion parameter in the meson and baryon sector is $1/N_c^2$ rather than $1/N_c$. For instance, meson decay constants scale as $f_m^2 \sim N_c^2$ rather than $f_m^2 \sim N_c$, with analogous changes for many other quantities [10]. Second, the detailed dynamics are different in the two theories: One cannot simply compute quantities in QCD$_F$ at large $N_c^F$ and replace $N_c$ by $N_c(N_c - 1)/2$ to obtain the large $N_c^{\text{AS}}$ limit of QCD$_{\text{AS}}$. At a diagrammatic level, this distinction is reflected by rather different combinatorics. It is largely for this reason that we take the expansion parameter to be $1/N_c^2$ and not $[N_c(N_c - 1)/2]^{-1}$. Third, observables associated with internal quark loops, e.g., strangeness form factors of the proton [12], are suppressed in the large $N_c^F$ limit but not in the large $N_c^{\text{AS}}$ limit.

How to study all three of these distinctions systematically is not obvious. For example, identifying loop effects in baryons is not generally straightforward, with the exception of using the strangeness content of the nucleon as a probe. Unfortunately, rather limited data of this sort exist, making the identification of patterns difficult. Evidence suggests that strange-quark matrix elements are typically quite small (for some recent reviews, see Refs. [13, 14, 15]), in contrast to expectations from the large $N_c^{\text{AS}}$ limit. However, it has been argued that the strange scalar matrix elements are large [12], as expected in the large $N_c^{\text{AS}}$ but not the large $N_c^F$ limit. Given these results, real conclusions remain elusive. Distinguishing between the limits based upon detailed dynamical differences is even more difficult; short of dedicated lattice studies of baryons built around either limit, no obvious scheme suggests itself. We therefore focus on the first difference, the distinct $1/N_c$ scaling behaviors of certain observables in the two expansions.

Fortunately, the scaling is easily probed by considering the spectroscopy of low-lying baryons: the octet and decuplet. The existence of an emergent spin-flavor symmetry SU(2$f$) at large $N_c$ [4, 5, 6], which places all these baryons in a single multiplet (the old-fashioned 56 when $N_c = 3$ and $N_f = 3$), provides a tool for comparing the two large $N_c$ limits. This symmetry, which implies mass relations that hold to various orders in a $1/N_c$ expansion, can be uncovered by studying pion-nucleon scattering. The Born and cross-Born graphs in that process are each proportional to $(g_{A1}/f_{\pi})^2$, which is large in either large $N_c$ limit, scaling as either $N_c$ or $N_c^2$ in QCD$_F$ and QCD$_{\text{AS}}$, respectively. In either case, these graphs individually would violate unitarity at large $N_c$, which necessitates cancellations between them in order to maintain a consistent theory. The consistency conditions implicit in these cancellations yield the spin-flavor symmetry at large $N_c$. At finite $N_c$ the cancellations are only partial, but a remarkable feature survives: If the underlying $N_c \to \infty$ symmetry is exact, the cancellations hold at next-to-leading order in the $1/N_c$ expansion [5, 6]. For the standard large $N_c^F$ limit this means that the corrections first appear at order $1/N_c^2$; for the large $N_c^{\text{AS}}$ limit, the order is $1/N_c^4$. Moreover, one can use explicit breaking of SU(3)$_{\text{flavor}}$ as a lever to probe the $1/N_c$ expansion. When SU(3)$_{\text{flavor}}$ is broken at first order (due to the strange quark mass, say), then the spin-flavor symmetry is also broken—but in a controlled way. By studying both the SU(3)$_{\text{flavor}}$ and $1/N_c$ scalings of operators giving rise to baryon masses, one can show that different linear combinations of masses that vanish in the SU(3)$_{\text{flavor}}$ limit can have different $N_c$ scalings: Some are leading order (LO), some are next-to-leading order (NLO), and some are next-to-next-to-leading order (NNLO). Since the phenomenological analysis sets $N_c = 3$ at the end of the calculation, the series stops at NNLO. Some details of this operator analysis appear in the Appendix.

Using these methods, Ref. [7] studied the baryon mass
relations in the standard large $N_c^F$ limit. The patterns seen in [7] are quite instructive, and give some of the strongest evidence to date for the validity of the $1/N_c$ expansion in hadronic physics. The expansion has real predictive power: Relations dependent on SU(3)$_{\text{flavor}}$ breaking that hold at LO are not satisfied as well as those that hold at NLO, which in turn are not satisfied as well as those at NNLO. Even more impressively, if one parameterizes these violations by constants times appropriate powers of $1/N_c$ and inserts $N_c = 3$, one finds that the coefficients are of natural size. What would one expect upon redoing this analysis using the large $N_c^{AS}$ limit? It is not hard to see that the general pattern determining which relations are well satisfied and which ones are not is unaltered, because the emergent large $N_c$ spin-flavor symmetry—a direct consequence of the nontrivial $N_c$ scaling of meson-baryon coupling constants—is common to both large $N_c$ limits. One might worry that the lack of suppression of quark loops in the large $N_c^{AS}$ limit could change the scaling of the suppressed baryon mass combinations compared to the large $N_c^F$ limit. To see that this does not happen, recall that the emergent SU(2)$_N$ symmetry mandates the degeneracy of all ground-state baryons in both large $N_c$ limits. While quark loops certainly contribute to baryon masses, they do not change the scaling of baryon mass differences at leading order. Incorporating explicit SU(3)$_{\text{flavor}}$ breaking does not change these conclusions, because each possible operator has a leading-order contribution in $1/N_c$ that does not include a quark loop. Thus the spin-flavor symmetry predicts that the same baryon mass combinations are suppressed at large $N_c$ for both large $N_c$ limits. However, the _degree_ of $1/N_c$ suppressions is different for the large $N_c^{AS}$ and $N_c^F$ limits.

Since the relevant expansion is in powers of $1/N_c^2 = 1/9$ for the large $N_c^{AS}$ limit versus $1/N_c = 1/3$ for large $N_c^F$, one might expect to find coefficients that are no longer natural. Were this true, one would conclude that the large $N_c^{AS}$ limit is less useful for describing baryons than the large $N_c^F$ limit. Remarkably, as shown in the remainder of this paper, the data show that _both_ expansions give natural coefficients. Such a surprising conclusion is possible because spin-flavor symmetry for $N_c = 3$ gives access to only two subleading orders in the expansion. Based upon this evidence, _both_ expansions appear to be viable.

**IV. CONFRONTING THE TWO LARGE $N_c$ LIMITS WITH DATA**

The suppressed baryon mass combinations are classified according to the isospin $I$ transformation properties of operators that contribute to them. References [7, 10] examine the $I = 0, 1, 2$ mass combinations that are suppressed in the combined $1/N_c$ and flavor-breaking analysis. Unfortunately, the experimental errors in the isospin-breaking $I = 1, 2$ mass relations are too large to allow a meaningful comparison that could discriminate between the two large $N_c$ limits. Thus in this paper, we focus on the $I = 0$ mass relations.

The specific $I = 0$ mass combinations analyzed here are defined as

$$
N_0 = \frac{1}{2}(p + n),
$$
$$
\Sigma_0 = \frac{1}{3}(\Sigma^+ + \Sigma^0 + \Sigma^-),
$$
$$
\Xi_0 = \frac{1}{2}(\Xi^0 + \Xi^-),
$$
$$
\Delta_0 = \frac{1}{4}(\Delta^{++} + \Delta^+ + \Delta^0 + \Delta^-),
$$
$$
\Sigma_+^* = \frac{1}{3}(\Sigma^{++} + \Sigma^0 + \Sigma^{*-}),
$$
$$
\Xi_+^* = \frac{1}{2}(\Xi^{*0} + \Xi^{*-}).
$$

As shown in Ref. [7] (and briefly reviewed in the Appendix), these mass combinations satisfy certain relations summarized in Table 1. Each large $N_c$ limit predicts that these relations are suppressed by specific powers of $1/N_c$ and powers of the SU(3)$_{\text{flavor}}$ symmetry-breaking parameter $\epsilon$.

To compare the mass relations to the theoretical predictions from the $1/N_c$ and SU(3)$_{\text{flavor}}$-breaking analysis, one forms dimensionless ratios $R_i$ from the mass combinations $M_i$. Given an $M_i$ from Table 1 ($i = 1, 2, \ldots, 7$), the ratios are defined by $R_i \equiv M_i / (M_i^* / 2)$, where $M_i^*$ is the same combination of masses as in $M_i$, except that each numerical coefficient is replaced by its absolute magnitude. The ratios $R_i$ defined in this way measure the experimental size of each mass combination $M_i$ relative to the weighted average $M_i^*$ of all the masses appearing in $M_i$.

We then compute the expected theoretical suppression $S_i$ for each $R_i$, which consists of the powers of $\epsilon$ and $1/N_c$ appropriate to $M_i$, as compiled in Table 1. Each $M_i^*$, being a baryon average mass, is $O(N_c^4)$ in the large $N_c^F$ limit and $O(N_c^2)$ in the large $N_c^{AS}$ limit. If the combined SU(3)$_{\text{flavor}}$-breaking and $1/N_c$ expansions are effective, in the sense that a fit to data produces coefficients of natural size, then one expects each $R_i / S_i$ to be of order unity. A more telling metric for testing this hypothesis is the set of _accuracy factors_ $A_i \equiv \log(R_i / S_i)$, since this expression encodes the fact that, as far as naturalness is concerned, deviations from unity like $R_i / S_i \approx 2$ and $R_i / S_i \approx 1/2$ are equally significant. If the theoretical and experimental suppressions agree, one expects that the $A_i$ should lie in a range of about $-1 \leq A_i \leq 1$, since $1 \approx \log 3$ (the amount that would be caused by an inappropriate factor of $N_c$ in the large $N_c^F$ limit).

To compare the data with the theoretical predictions, one must also estimate the size of the SU(3)$_{\text{flavor}}$-breaking parameter $\epsilon$. To obtain an estimate independent of large $N_c$ effects and sensitive only to SU(3)$_{\text{flavor}}$-breaking effects, we take $\epsilon$ to be the average of the mass splittings

between $N_0$ and $\Lambda$, $\Sigma_0$, and $\Xi_0$. Specifically, we define

$$\epsilon \equiv \frac{1}{3} \sum_{i=1}^{3} \frac{B_i - N_0}{(B_i + N_0)/2},$$

(6)

where $B_i = \Lambda, \Sigma_0, \Xi_0$. This particular choice, among many other possible options, gives $\epsilon \approx 0.25$, which is the same value used in Ref. [7].

| Mass Combination | Large $N_c^F$ suppression | Large $N_c^{AS}$ suppression |
|------------------|---------------------------|-------------------------------|
| $M_1$ | $5(2N_0 + 3\Sigma_0 + \Lambda + 2\Xi_0) - 4(4\Delta_0 + 3\Sigma_0^* + 2\Xi_0^* + \Omega)$ | $1/N_c$ | $1/N_c^2$ |
| $M_2$ | $5(6N_0 - 3\Sigma_0 + \Lambda - 4\Xi_0) - 2(2\Delta_0 - \Xi_0^* - \Omega)$ | $\epsilon$ | $\epsilon$ |
| $M_3$ | $N_0 - 3\Sigma_0 + \Lambda + \Xi_0$ | $\epsilon/N_c$ | $\epsilon/N_c^2$ |
| $M_4$ | $-2N_0 - 9\Sigma_0 + 3\Lambda + 8\Xi_0 + 2(2\Delta_0 - \Xi_0^* - \Omega)$ | $\epsilon^2/N_c$ | $\epsilon^2/N_c^2$ |
| $M_5$ | $35(2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0) - 4(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega)$ | $\epsilon^3/N_c$ | $\epsilon^3/N_c^2$ |
| $M_6$ | $7(2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0) - 2(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega)$ | $\epsilon^2/N_c^2$ | $\epsilon^2/N_c^4$ |
| $M_7$ | $\Delta_0 - 3\Sigma_0^* + 3\Xi_0^* - \Omega$ | $\epsilon^3/N_c^2$ | $\epsilon^3/N_c^4$ |

TABLE I: The $I = 0$ mass relations and their theoretical suppression factors.

In Fig. 1 we plot the accuracy factors $A_i$ for $I = 0$ mass ratios $R_i$ when only SU(3)$_{\text{flavor}}$-breaking suppressions are included.

FIG. 1: Plot of the accuracy factors $A_i$ for $I = 0$ mass ratios $R_i$ when only SU(3)$_{\text{flavor}}$-breaking suppressions are included.

In Fig. 1 we plot the accuracy factors $A_i$ for $i = 1, 2, \ldots, 7$ when one includes only SU(3)$_{\text{flavor}}$-breaking in the theoretical suppression factors $S_i$. In Fig. 2 we plot the $A_i$ using theoretical suppression factors $S_i$ obtained from the large $N_c^F$ limit, while in Fig. 3 we plot the $A_i$ using $S_i$ obtained from the large $N_c^{AS}$ limit. In all three plots, the error bars are due to experimental uncertainties in the baryon mass combinations $M_i$ (and only visible for $M_7$).

Figure 1 clearly shows that the accuracy factors $A_i$ do not assume natural values when one uses only SU(3)$_{\text{flavor}}$-breaking effects for the suppression factors $S_i$. It is also clear from Figs. 2 and 3 that the $A_i$ assume natural values for both large $N_c$ limits. Different definitions for the value of $\epsilon$ do not change this result.

Thus, from the plots in Figs. 1-3 one sees that a large $N_c$ limit explains the global pattern of suppressions in the baryon mass splittings $M_i$, and the suppressions cannot be explained purely by SU(3)$_{\text{flavor}}$-breaking effects. The combined large $N_c$ (from either large $N_c$ limit) and SU(3)$_{\text{flavor}}$-breaking suppression factors give results much more consistent with the data than do SU(3)$_{\text{flavor}}$-breaking factors alone.

Surprisingly, however, it is also clear that both the large $N_c^F$ and large $N_c^{AS}$ expansions are qualitatively similar to the data: the spread of $A_i$ values in each case is only about 1.5 units. Our results suggest that baryon spectroscopy data are consistent with two different large $N_c$ limits. The analysis shows that large $N_c$ physics is necessary to explain baryon mass splittings, but it does not definitively show which large $N_c$ limit is closest to our $N_c = 3$ world.

V. A HYBRID LARGE $N_c$ LIMIT

In the last section we compared the phenomenology of two distinct flavor-symmetric $1/N_c$ expansions extrapo-
lated to $N_c = 3$. In this section we discuss a hybrid large $N_c$ limit. Specifically, suppose one extrapolates to the large $N_c$ limit with one quark flavor in the AS representation, but keeps the rest of the flavors in the F representation. This extrapolation has a number of theoretical virtues connected to the large $N_c$ orientifold equivalence with QCD$_{Adj}$ described in the Introduction, and it has been explored by Ref. [3] [17] [18] [19] [20] [21] [22]. We refer to this large $N_c$ extrapolation, first suggested in 1979 by Corrigan and Ramond (CR) [2], as the large $N_c^{CR}$ limit.

Reference [3] observed that the F quarks decouple from the dynamics of the glue and the AS quarks in the large $N_c^{CR}$ limit, so that QCD in the CR large $N_c$ limit is equivalent to $N_f = 1$ QCD$_{AS}$ at large $N_c$. Since the latter theory is orientifold-equivalent to $N = 1$ SUSY for large $N_c^{Adj}$, one can use SUSY methods to make predictions for the behavior of the theory in the large $N_c^{CR}$ limit. Thus, to the extent that large $N_c$ limit and the chiral limit of this flavor-asymmetric theory are relatively close to the real world, one obtains an ab initio method of predicting certain QCD observables. For instance, Ref. [18], showed that the $N_f = 3$ chiral condensate $\langle q\bar{q} \rangle$ in the large $N_c^{CR}$ limit can be calculated using the orientifold equivalence.

This theory has another virtue, noted long ago in Ref. [2]. The CR theory has states with the color structure $Q^{[i,j]} q_i q_j$ at all $N_c$. At $N_c = 3$, using the AS isomorphism described above, this state simply becomes the usual three-quark baryon. If one assigns each fundamental quark $q_i$, $q_j$ as well as $Q^{[i,j]}$ baryon number 1/3 at general $N_c$, then three-quark baryons $Q^{[i,j]} q_i q_j$ exist at any $N_c > 1$. Thus, in contrast to Witten’s construction in Eq. (1)], CR baryon masses do not grow with $N_c$, which is an attractive feature since physical $N_c = 3$ baryon and meson masses are not significantly different.

Moreover, CR baryons and mesons have recently been shown to exhibit a very interesting property: Their Regge slopes coincide [21]. Since this feature approximately holds in the physical $N_c = 3$ hadronic spectrum, one might conclude that the large $N_c^{CR}$ limit is phenomenologically useful.

In the rest of this section we explore whether a $1/N_c$ expansion around the large $N_c^{CR}$ limit is actually phenomenologically viable. Clearly, one can construct a class of theories at any $N_c$ with one flavor of AS quark and one or more F quark flavors. Such theories are certainly sensible at any $N_c$; at $N_c = 3$ they correspond to real QCD. However, one may ask whether the large $N_c^{CR}$ world is “close” to the $N_c = 3$ world. In certain crucial aspects the answer is no.

For example, the approximate $SU(2N_f) \times SU(2N_f)$ chiral symmetry is an essential feature of $N_c = 3$ hadronic physics; let us consider the fate of this symmetry in the large $N_c^{CR}$ limit. For simplicity we restrict to two massless flavors $u$ and $d$, and without loss of generality we take $u$-flavor quarks to be AS quarks, denoted by $u_{AS}$, while $d$-flavor quarks transform in the F representation, and are denoted by $d_F$. For $N_c = 3$, the exact $SU(2)_L \times SU(2)_R$ chiral symmetry is spontaneously broken to $SU(2)_V$, yielding three pseudoscalar Goldstone pions and a light $\eta'$ associated with the anomalous U(1) transformation $\exp(i\theta\gamma_5)$. For any finite $N_c \neq 3$ in the CR approach, no symmetry connects $u_{AS}$ and $d_F$: The chiral symmetry and its associated Goldstone bosons are present only in the $N_c = 3$ CR theory.

Indeed, one cannot construct color-singlet operators with the quantum numbers of the pion for $N_c \neq 3$. Consider an operator with the quantum numbers of the $\pi^+$; for $N_c = 3$ this is the color singlet $ie^{ijkl} \pi_{jk} d_l$. For $N_c > 3$, $e^{ijkl}$ is no longer an invariant tensor, and three color indices cannot be combined into a color singlet. Thus for any finite $N_c > 3$, no Goldstone bosons occur. As $N_c \rightarrow \infty$, one light boson associated with the U(1) rotation $d_F \rightarrow \exp(i\theta\gamma_5) d_F$, call it $\eta_d$, emerges. While this chiral current is anomalous, the anomaly is suppressed by $1/N_c$. However, the anomaly for the analogous current for $u_{AS}$ is unsuppressed at large $N_c$, since AS quark loops are unsuppressed. Thus the large $N_c^{CR}$ world has one $\eta_d$ boson, while the $N_c = 3$ world has three $\pi$ Goldstone bosons and an $\eta'$. This is an essential difference.

The problem with the large $N_c^{CR}$ limit is not that chiral symmetry is explicitly broken at $N_c > 3$, but rather that the transformations associated with the symmetry do not even exist away from $N_c = 3$. Moreover, isospin transformations also do not exist for $N_c > 3$. Hadrons do not form isospin multiplets for $N_c > 3$; for example, while protons and neutrons both exist at $N_c = 3$, only conventional neutrons ($u_{AS} d_F d_F$) exist for any $N_c > 3$. The large $N_c^{CR}$ world is qualitatively very different from the $N_c = 3$ world. A phenomenologically viable $1/N_c$ CR expansion seems unlikely for observables sensitive to flavor physics.

The essential nature of the problem is the absence of a smooth limit. Had the issue merely been that isospin and chiral symmetry were badly broken at large $N_c^{CR}$, then one might hope that the symmetries at $N_c = 3$ emerge when $1/N_c$ corrections are included to sufficiently higher order. If, for example, isospin-violating quantities such as the proton-neutron mass difference were substantial at large $N_c$, but $1/N_c$ corrections reduced the difference
in such a way that their sum added to zero at $N_c = 3$, one might be able to construct a sensible $1/N_c$ expansion that is useful down to $N_c = 3$. However, since the proton simply does not exist for $N_c > 3$, the proton-neutron mass difference cannot be studied in a $1/N_c$ expansion.

Given these considerations, we conclude that the large $N_c^{CR}$ limit, while exhibiting interesting Regge behavior, does not appear to admit a usable $1/N_c$ expansion that connects smoothly to the $N_c = 3$ world for a wide class of observables.

VI. SUMMARY

In this paper, we have investigated the phenomenology of two large $N_c$ limits of QCD: the large $N_c^F$ and large $N_c^{AS}$ limits. Using the emergent spin-flavor symmetry of baryons in the large $N_c$ limit, we have compared the predictions of both these large $N_c$ limits for certain baryon mass relations to experimental data. We have found that these baryon mass relations hold to the expected order for both large $N_c$ limits. Thus at least for the class of observables we have considered, both large $N_c$ limits appear to be phenomenologically viable.

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APPENDIX A: OPERATOR ANALYSIS AND MASS RELATIONS

In this appendix we briefly sketch the arguments leading to the mass relations given in Table I.

The phenomenological analysis of large $N_c$ baryons relies upon three features: the large number of quarks in the baryon wave function [$N_c$ in the $N_c^F$ counting, $N_c(N_c - 1)/2$ in the $N_c^{AS}$ counting] and their combinatorics [3], the ‘t Hooft scaling $\frac{G_{YM}}{N_c} \sim N_c^{-1/2}$ of the QCD coupling constant, and the existence of a ground-state multiplet completely symmetric under the combined spin-flavor symmetry, which is the generalization of the old SU(6) 56-plet [6, 23, 24]. Of course, the baryon contains not only valence quark but gluon and sea quark degrees of freedom as well. However, the assumption that physical baryons fill specific spin and flavor representations based upon the quantum numbers of quarks implies that one may describe the baryon wave function as comprised entirely of quark interpolating fields that exhaust the baryon wave function [25], and it is these fields that one calls “quarks” in the large $N_c$ analysis.

Interactions among the quarks in the the baryon may be described in terms of operators classified [26] by their transformation properties under the spin-flavor symmetry (singlet, adjoint under spin but not flavor, three-index symmetric tensor under flavor, etc.) as well as the number $n$ of quarks that participate in the interaction, which defines an $n$-body operator. The quark combinatorics combined with the ‘t Hooft scaling shows that $n$-body operators are generically suppressed by a factor $1/N_c^n$ in the $N_c^F$ counting. However, to obtain the true suppression factor one must also take into account cases in which the contributions from the quarks add coherently in the baryon matrix elements, as well as eliminate operators whose matrix elements form a linearly dependent set when evaluated on the ground-state baryon multiplet [26]. Once this is accomplished, one is left with a linearly independent set of operators, each one with a well-defined power counting in $1/N_c$, that carries precisely the same dimension as the space of independent baryon observables. In essence, the operators and observables form equivalent bases for the baryons, and the operators form a hierarchy in powers of $1/N_c$ such that their assembly forms an effective baryon Hamiltonian.

An explicit example clarifies the comments made above. In either the large $N_c^F$ or $N_c^{AS}$ limit, the color indices of the quarks are combined to give a color wave function completely antisymmetric under quark exchange, as in Eqs. (1) or (3), respectively. Since the baryons are fermions and therefore completely antisymmetric under the exchange of all quark indices, the spin-flavor-space part of the baryon wave function must be completely symmetric under quark exchange; and since the spatial wave functions of ground-state baryons are $L = 0$ and hence symmetric, one requires complete symmetry of the spin-flavor wave function. This is the reasoning that, for $N_c = 3$, leads to the SU(6) 56-plet.

Operators that break the spin-flavor symmetry carry spin, flavor, or both indices, and are labeled $J^S$, $T^a$, and $G^{\alpha a}$, respectively in Refs. [6, 7, 15]. For example,

$$T^a \equiv \sum_{\alpha} q^\dagger_\alpha \left( \mathbb{1} \otimes \frac{\lambda^\alpha}{2} \right) q_\alpha , \quad (A1)$$

where $\mathbb{1}$ is the identity matrix in spin space, $\lambda^\alpha$ is the usual Gell-Mann matrix in flavor space, and $\alpha$ sums over all the quarks in the baryon. Note that color indices do not appear explicitly in this expression, so the operator $T^a$ is well defined in both large $N_c^F$ and $N_c^{AS}$. All operators that have nonvanishing matrix elements on baryon states may be formed from polynomials of $J^S$, $T^a$, and $G^{\alpha a}$ (with suitable contractions of spin-flavor indices), and such a polynomial of nth order gives an n-body operator. Since the physical baryons have $N_c = 3$ quarks, such polynomials beyond cubic order applied to these baryons give matrix elements linearly dependent upon those of lower-order operators, which means that
the $1/N_c$ series for any finite given value of $N_c$ terminates after providing a complete set of independent operators.

An $n$-body operator requires an $n$-quark interaction, which in turn implies $2n$ factors of $g_{YM}$, for a suppression of $1/N_c^n$, as discussed above. Therefore, in the effective large $N_c$ baryon Hamiltonian, the operator $T^a$ appears multiplied by an explicit factor of $1/N_c$, as compared to the spin-flavor symmetric operator $I$ that has $O(N_c^2)$ matrix elements. Consider, however, the mass operator $T^8$; its matrix elements, which naively merely count strange quarks, are actually given by

$$\langle T^8 \rangle = \frac{1}{2\sqrt{3}}(N_c - 3N_s). \quad (A2)$$

Here we see a coherent $O(N_c^2)$ contribution that seems to upset the large $N_c$ counting; however, note that it is the same for all baryons and therefore simply provides an additional contribution to the leading-order spin-flavor symmetric mass operator $I$. The operator $T^8$ also breaks SU(3)$_{flavor}$ and therefore requires an explicit prefactor of $c$. When one repeats this analysis for a complete set of linearly independent operators (e.g., $cT^8, c^2(T^8, T^8)/N_c$, etc.), one finds that each operator contributes to a unique baryon mass combination, which defines the combinations $M_i$ in Table I. For example, the combination $M_2$ is associated with the operator $cT^8$ considered above. The analysis for $N_c^{as}$ proceeds exactly the same way, with the exception that each $1/N_c$ in $N_c^{as}$ is replaced by $1/N_c^2$.

Such an analysis is not unique to the $1/N_c$ expansion. All that is required is a finite multiplet of states under some symmetry, and a perturbative parameter that suppresses some of the independent operators that act upon the multiplet. For example, since the fundamental operators distinguishing strangeness break SU(3)$_{flavor}$ symmetry by transforming as an 8, flavor symmetry-breaking transforming as a 27 ($\subset 8 \otimes 8$) does not occur until second order in SU(3)$_{flavor}$ breaking, and this operator must be associated with a doubly-suppressed SU(3)$_{flavor}$-breaking mass combination. Indeed, when evaluated for the $N_c=3$ baryon octet, this mass combination turns out to be just the one that defines the Gell-Mann–Okubo relation, $2N_0 - \Sigma_0 - 3\Lambda_0 + 2\Xi_0$, using the notation of Table I.

This analysis, applied to the baryons in large $N_c$ limit, was first performed in Ref. [7], and was later improved as better measurements for baryon isospin splittings were obtained [10].