ON THE ZERO-VISCOSITY LIMIT OF THE NAVIER-STOKES EQUATIONS IN THE HALF-SPACE

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Abstract. We consider the zero viscosity limit of the incompressible Navier-Stokes equations with non-slip boundary condition in the half-space for the initial vorticity located away from the boundary. By using the vorticity formulation and Cauchy-Kowaleskaya theorem, Maekawa proved the local in time convergence of the Navier-Stokes equations in the half-plane to the Euler equations outside a boundary layer and to the Prandtl equations in the boundary layer. In this paper, we develop the direct energy method to generalize Maekawa’s result to the half-space.

1. Introduction

In this paper, we are concerned with the zero-viscosity limit of the incompressible Navier-Stokes equations in the half-space $\mathbb{R}^3_+$:

$$\begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon + \nabla_x p^\varepsilon &= \varepsilon^2 \Delta u^\varepsilon, \\
\partial_t v^\varepsilon + u^\varepsilon \cdot \nabla_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon + \partial_y p^\varepsilon &= \varepsilon^2 \Delta v^\varepsilon, \\
\nabla_x \cdot u^\varepsilon + \partial_y v^\varepsilon &= 0, \\
(u^\varepsilon, v^\varepsilon)(t, x, 0) &= (0, 0).
\end{align*}$$

(1.1)

Here and in what follows, $(x, y) \in \mathbb{R}^2 \times \mathbb{R}_+$ and $\nabla_x = (\partial_{x_1}, \partial_{x_2})$, $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$, $\varepsilon^2$ is the viscosity coefficient, $(u^\varepsilon, v^\varepsilon)$ and $p^\varepsilon$ denote the velocity field and the pressure respectively.

In the absence of the boundary, the Navier-Stokes equations indeed converge to the Euler equations:

$$\begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla_x u^\varepsilon + \nabla_x p^\varepsilon &= 0, \\
\partial_t v^\varepsilon + u^\varepsilon \cdot \nabla_x v^\varepsilon + \partial_y v^\varepsilon + \partial_y p^\varepsilon &= 0, \\
\nabla_x \cdot u^\varepsilon &= 0.
\end{align*}$$

(1.2)

This problem has been well studied in various functional settings [11, 29, 5, 18, 19].

In the presence of the boundary, the zero-viscosity limit will become very complicated due to the possible appearance of boundary layer. For the Navier slip boundary condition

$$v^\varepsilon = 0, \quad \partial_y u^\varepsilon = 0 \quad \text{on} \quad y = 0,$$

the boundary layer is weak. In such case, the limit from the Navier-Stokes equations to the Euler equations was justified by Xiao and Xin for the half-space [33] and by Roussiet and Masmoudi [21] for general domain, see [9, 10, 32] and references therein for more relevant results. For the non-slip boundary condition, the boundary layer is strong. In 1904, Prandtl introduced the boundary layer theory in [26]. Using a formal boundary layer expansion

$$\begin{align*}
u^\varepsilon(t, x, y) &= u^\varepsilon(t, x, y) + u^p(t, x, \frac{y}{\varepsilon}) + O(\varepsilon), \\
v^\varepsilon(t, x, y) &= v^\varepsilon(t, x, y) + \varepsilon v^p(t, x, \frac{y}{\varepsilon}) + O(\varepsilon),
\end{align*}$$

(1.3)
he derived the Prandtl boundary layer equation
\[
\begin{align*}
\partial_t u + u \cdot \nabla_x u + v \partial_y u + \partial_x p &= \partial_y^2 u, \\
\nabla_x \cdot u + \partial_y v &= 0,
\end{align*}
\]
(1.4)

Up to now, the justification of this formal boundary expansion is still a challenging problem.

The first step toward this problem is to deal with the well-posedness of the Prandtl equation. Initiated by Oleinik and Samokhin [21], the well-posed problem was well understood for the monotonic data in Sobolev spaces [34, 1, 22, 6] and general analytic data [27, 16, 7, 35].

The first rigorous verification of the Prandtl boundary layer theory was achieved in the analytic setting by Sammartino and Caffiisch [28] (see also [31] for a proof based on direct energy method). In the case when the domain and the initial data have a circular symmetry, the convergence was justified in [17, 23]. Guo and Nguyen [8] justify the zero-viscosity limit of steady Navier-Stokes equations over a moving plate. Initiated by Kato [12], there are many works devoted to the conditional convergence [13, 14, 30, 4].

Recently, Maekawa [20] justified the zero-viscosity limit for the initial vorticity located away from the boundary in the half-plane. A very interesting point is that this kind of data is only analytic near the boundary. Intuitively, this seems enough to exclude the instability of boundary layer. However, the proof in [20] used another mechanism in a crucial way: weak interaction between the outer vorticity and the inner vorticity.

The goal of this paper is two fold. The first one is to generalize Maekawa’s result to the half space \( \mathbb{R}^3_+ \). In \( \mathbb{R}^3_+ \), the data with vorticity located away from the boundary is not analytic near the boundary. However, we find that the data is still analytic in the tangential direction near the boundary. Indeed, the tangential analyticity is enough to ensure the well-posedness of the Prandtl equation [16, 35]. The second one is to develop a direct energy method for the zero-viscosity limit problem. The proof in [28, 20] is based on the Cauchy-Kowaleskaya theorem, where the representation formula of the solution was used in a crucial way. In particular, the representation formula of the vorticity is used in [20]. So, this method seems difficult to apply to the zero-viscosity limit problem in general physical domain. While, energy method may be applicable for the case of general domain.

For the simplicity, we consider the initial data of the form
\[
\begin{align*}
u^\varepsilon (0, x, y) &= u_0(x, y), \quad v^\varepsilon (0, x, y) = v_0(x, y),
\end{align*}
\]
which satisfies
\[
\nabla_x \cdot u_0 + \partial_y v_0 = 0, \quad u_0(x, 0) = 0, \quad v_0(x, 0) = 0,
\]
and the initial vorticity \( \omega_0 = \text{curl}(u_0, v_0) \) satisfies
\[
2d_0 \triangleq \text{dist}(\text{supp}\omega_0, \{y = 0\}) > 0.
\]
Without loss of generality, we take \( d_0 = 1 \).

To state our main result, we introduce the following Prandtl system
\[
\begin{align*}
\partial_t u^p - \partial_z u^p + u^p \cdot \nabla_x u^\varepsilon (t, x, 0) + \left( u^\varepsilon (t, x, 0) + u^p \right) \cdot \nabla_x u^p \\
+ \left( v^p - \int_0^\infty \partial_x u^p (t, x, z) dz + z \partial_y v^\varepsilon (t, x, 0) \right) \partial_z u^p &= 0,
\end{align*}
\]
\[
\nabla_x \cdot u^p + \partial_z v^p = 0,
\]
\[
u^p (0, x, y) = 0,
\]
\[
\lim_{z \to \infty} (u^p, v^p) (t, x, z) = 0, \quad u^p (t, x, 0) = -u^\varepsilon (t, x, 0),
\]
(1.7)
where \((u^\varepsilon, v^\varepsilon, p^\varepsilon)\) is the solution of the Euler equations (1.2).

Now, our main result is stated as follows.

**Theorem 1.1.** There exist \(T > 0\) and \(C > 0\) independent of \(\varepsilon\) such that for any \((u_0, v_0) \in H^{30}(\mathbb{R}^3)\) satisfying (1.5) and (1.6), there exists a unique solution \((u^\varepsilon, v^\varepsilon)\) of the Navier-Stokes equations (1.1) in \([0, T]\), which satisfies

\[
\sup_{0 \leq t \leq T} \left\| u^\varepsilon(t, x, y) - u^\varepsilon(t, x, y) - u^\varepsilon(t, x, y) \right\|_{L^2(\mathbb{R}^3)} \leq C \varepsilon,
\]

\[
\sup_{0 \leq t \leq T} \left\| v^\varepsilon(t, x, y) - v^\varepsilon(t, x, y) - \varepsilon p(t, x, y) \right\|_{L^2(\mathbb{R}^3)} \leq C \varepsilon.
\]

Let us present a sketch of the proof and ideas.

1. Construction of the approximate solution \(U^a = (u^a, v^a)\) of the system (1.1) by using the asymptotic matched expansion method.
2. The error \(U^R\) between the solution and the approximate solution satisfies
   \[
   \partial_t U^R - \varepsilon^2 \Delta U^R + U^R \cdot \nabla U^a + U^a \cdot \nabla U^R + U^R \cdot \nabla U^R + \nabla P^R = R.
   \]
   In this equation, the main trouble is to control the linear terms \(U^R \cdot \nabla U^a\) and \(U^a \cdot \nabla U^R\), while the nonlinear term \(U^R \cdot \nabla U^R\) is in fact easy to control. One of the most singular terms is
   \[
   v^R \partial_y (u^R(t, x, y)) = v^R \frac{y}{\varepsilon} \partial_z u^R(t, x, y) \sim -\nabla_x \cdot u^R \partial_z u^R(t, x, y).
   \]
   So, this term will lead to the loss of one horizontal derivative in the process of energy estimates. To remedy the loss of the derivative, it is natural to work in the analytic setting. In our case, we will use the tangential analyticity to recover one derivative loss near the boundary, and use the exponential decay in \(z\) of \(u^\varepsilon(t, x, z)\) away from the boundary.
3. In order to avoid the singularity like \(\partial_y w^\varepsilon(t, x, y) \sim \frac{1}{\varepsilon}\), it is better to work in the conormal Sobolev spaces with \(\partial_y\) replaced by conormal derivative \(y \partial_y\). The disadvantage is that we have no control on the regularity in \(y\) variable near the boundary.
4. To gain one derivative in \(y\) variable, we need to use the vorticity formulation of the error equation, which takes the form
   \[
   \partial_t w - \varepsilon^2 \Delta w + \tilde{U}_a \cdot \nabla w + \tilde{U} \cdot \nabla w_a + \tilde{U} \cdot \nabla w - w_a \cdot \nabla U - w \cdot \nabla U_a - w \cdot \nabla U = -\text{curl}(R_h, R_e) - M,
   \]
   where the boundary condition of the vorticity can be determined by introducing the Dirichlet-Neumann operator.
5. In the vorticity formulation, one of trouble terms is \(\tilde{v}_a \partial_y w\). To handle it, we need to decompose the vorticity into two parts: Euler part \(w_e\) and Prandtl part \(w_p\). The Euler part has the exponential decay in \(\varepsilon\) near the boundary, and the Prandtl part has the exponential decay in \(\frac{y}{\varepsilon}\). Some ideas are motivated by [3, 25].
6. In our case, we will use the tangential analyticity to recover one derivative loss near the boundary.
7. The third component of the vorticity has better behaviour. This observation is crucial to close our estimate.
8. The most subtle task is to construct a suitable energy functional to reveal all mechanism such as the analyticity near the boundary, the exponential decay in \(\varepsilon\) of \(w_e\) near the boundary and the exponential decay in \(\frac{y}{\varepsilon}\) of \(w_p\). Some ideas are motivated by [3, 25].

The paper is organized as follows. In Section 2, we construct the approximate solution by using the matched asymptotic expansion method. In Section 3, we derive the error equation and give a decomposition of vorticity formulation of the error equation. Section 4 is devoted to the functional framework and some product estimates. In Section 5, we construct the energy functional and prove Theorem 1.1 under some assumptions. Section 6-Section 12 is devoted to the key energy estimates.
in the analytic setting and Sobolev setting for the velocity and the vorticity. Finally, we present the well-posedness of the Euler system and the Prandtl system in the appendix.

2. Construction of the approximate solution

In this section, we use the asymptotic matched method to construct the approximate solution.

2.1. Outer(Euler) expansions. Away from the boundary, we construct the approximate solution by the following expansions

\[ u^e(t, x, y) = u_e^{(0)}(t, x, y) + \varepsilon u_e^{(1)}(t, x, y) + \cdots, \]
\[ v^e(t, x, y) = v_e^{(0)}(t, x, y) + \varepsilon v_e^{(1)}(t, x, y) + \cdots, \]
\[ p^e(t, x, y) = p_e^{(0)}(t, x, y) + \varepsilon p_e^{(1)}(t, x, y) + \cdots. \]

By substituting the above expansions into (1.1) and matching the (leading) zeroth order terms, we find that \((u_e^{(0)}, v_e^{(0)}, p_e^{(0)})\) should satisfy the Euler equations

\[
\begin{aligned}
\partial_t u_e^{(0)} + u_e^{(0)} \cdot \nabla_x u_e^{(0)} + v_e^{(0)} \partial_y u_e^{(0)} + \nabla_x p_e^{(0)} &= 0, \\
\partial_t v_e^{(0)} + u_e^{(0)} \cdot \nabla_x v_e^{(0)} + v_e^{(0)} \partial_y v_e^{(0)} + \partial_y p_e^{(0)} &= 0, \\
\nabla_x \cdot u_e^{(0)} + \partial_y v_e^{(0)} &= 0,
\end{aligned}
\]

which will be equipped with the boundary condition

\[ v_e^{(0)}(t, x, 0) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^2, \]

and the initial condition

\[ u_e^{(0)}(0, x, y) = u_0(x, y), \quad v_e^{(0)}(0, x, y) = v_0(x, y), \quad (x, y) \in \mathbb{R}^2 \times \mathbb{R}_+. \]

By matching the \(\varepsilon\)-order terms, we find that \((u_e^{(1)}, v_e^{(1)}, p_e^{(1)})\) should satisfy the linearized Euler equations

\[
\begin{aligned}
\partial_t u_e^{(1)} + u_e^{(1)} \cdot \nabla_x u_e^{(1)} + v_e^{(1)} \partial_y u_e^{(1)} + u_e^{(0)} \cdot \nabla_x u_e^{(1)} + v_e^{(0)} \partial_y u_e^{(1)} + \nabla_x p_e^{(1)} &= 0, \\
\partial_t v_e^{(1)} + u_e^{(1)} \cdot \nabla_x v_e^{(1)} + v_e^{(1)} \partial_y v_e^{(1)} + u_e^{(0)} \cdot \nabla_x v_e^{(1)} + v_e^{(0)} \partial_y v_e^{(1)} + \partial_y p_e^{(1)} &= 0, \\
\nabla_x \cdot u_e^{(1)} + \partial_y v_e^{(1)} &= 0, \\
(u_e^{(1)}, v_e^{(1)})(0, x, y) &= (0, 0).
\end{aligned}
\]

Here the boundary condition on \(v_e^{(1)}\) is determined by \(v_p^{(1)}\)

\[ v_e^{(1)}|_{g=0} = -v_p^{(1)}|_{y=0}. \]

2.2. Boundary(Prandtl) layer expansions. Near the boundary, we will make the boundary layer(Prandtl layer) expansion. For this, we introduce the scaled(Prandtl) variable \(z = \frac{y}{\varepsilon} \in [0, +\infty)\) and write

\[ u^e(t, x, y) = u_p^{(0)}(t, x, y, z) + \varepsilon u_p^{(1)}(t, x, y, y, z) + \cdots, \]
\[ v^e(t, x, y) = v_p^{(0)}(t, x, y, z) + \varepsilon v_p^{(1)}(t, x, y, y, z) + \cdots, \]
\[ p^e(t, x, y) = p_p^{(0)}(t, x, y, z) + \varepsilon p_p^{(1)}(t, x, y, y, z) + \cdots, \]

where for every \(i \in \{1, 2, \cdots\}, \)

\[ u_p^{(i)}(t, x, y, z) = u_p^{(i)}(t, x, y) + u_p^{(i)}(t, x, z), \]
\[ v_p^{(i)}(t, x, y, z) = v_p^{(i)}(t, x, y) + v_p^{(i)}(t, x, z), \]
\[ p_p^{(i)}(t, x, y, z) = p_p^{(i)}(t, x, y) + p_p^{(i)}(t, x, z). \]
The matched boundary condition requires that
\[ u_p^{(i)}(t, x, z) \to 0, \quad v_p^{(i)}(t, x, z) \to 0, \quad p_p^{(i)}(t, x, z) \to 0, \quad \text{as } z \to +\infty. \] (2.3)

While, the boundary condition of \((u^\varepsilon, v^\varepsilon)\) on \(y = 0\) requires that
\[ u_p^{(i)}(t, x, 0) = -u_e^{(i)}(t, x, 0), \quad v_p^{(i)}(t, x, 0) = -v_e^{(i)}(t, x, 0), \quad i = 0, 1, \ldots. \] (2.4)

To derive the equation of \((u_p^{(i)}, v_p^{(i)}, p_p^{(i)})\), we put the expansions into (1.1) and then put the terms with the same order in \(\varepsilon\) together.

First of all, we deduce from the \(\varepsilon^{-1}\)-order terms and boundary condition (2.3) that
\[ v_p^{(0)} = 0, \quad p_p^{(0)} = 0. \]

Then, collecting \(\varepsilon^0\)-th term of the \(v^\varepsilon\) equation, divergence free condition and boundary condition, we obtain
\[
\begin{aligned}
\begin{cases}
\partial_t u_p^{(0)} - \partial_z u_p^{(0)} + u_p^{(0)} \cdot \nabla_x u_p^{(0)}(t, x, 0) + (u_p^{(0)} + u_e^{(0)}(t, x, 0)) \cdot \nabla_x u_p^{(0)} \\
+ (v_p^{(1)} + v_e^{(1)}(t, x, 0) + z\partial_y v_e^{(0)}(t, x, 0)) \partial_z u_p^{(0)} = 0, \\
\nabla_x \cdot u_p^{(0)} + \partial_z v_p^{(1)} = 0, \\
u_p^{(0)}(0, x, y) = 0, \\
limit_{z \to +\infty} (u_p^{(0)}, v_p^{(1)})(t, x, z) = 0, \\
u_p^{(0)}(t, x, 0) = -u_e^{(0)}(t, x, 0).
\end{cases}
\end{aligned}
\] (2.5)

While, using \(v_p^{(0)} = 0\) and boundary condition (2.3), collecting \(\varepsilon^0\)-order term of the \(v^\varepsilon\) equation, we obtain
\[ p_p^{(1)} = 0. \]

**Remark 2.1.** We set
\[ \tilde{u}_p^{(0)}(t, x, z) = u_p^{(0)}(t, x, z) + u_e^{(0)}(t, x, 0), \quad \tilde{v}_p^{(1)}(t, x, z) = v_p^{(1)}(t, x, z) + v_e^{(1)}(t, x, 0) + z\partial_y v_e^{(0)}(t, x, 0). \]

Then, by Bernoulli law,
\[ \partial_t u_e^{(0)}(t, x, 0) + u_e^{(0)}(t, x, 0) \cdot \nabla_x u_e^{(0)}(t, x, 0) + \nabla_x p_e^{(0)}(t, x, 0) = 0, \]
we arrive at
\[
\begin{aligned}
\begin{cases}
\partial_t \tilde{u}_p^{(0)} - \partial_z \tilde{u}_p^{(0)} + \tilde{u}_p^{(0)} \cdot \nabla_x \tilde{u}_p^{(0)} + \tilde{v}_p^{(1)} \partial_z \tilde{u}_p^{(0)} + \nabla_x p_e^{(0)}(t, x, 0) = 0, \\
\nabla_x \cdot \tilde{u}_p^{(0)} + \partial_z \tilde{v}_p^{(1)} = 0, \\
u_p^{(0)}(0, x, z) = u_e^{(0)}(0, x, 0), \\
limit_{z \to +\infty} \tilde{u}_p^{(0)}(t, x, z) = u_e^{(0)}(t, x, 0), \\
\tilde{u}_p^{(0)}(t, x, 0) = 0, \quad \tilde{v}_p^{(1)}(t, x, 0) = 0.
\end{cases}
\end{aligned}
\]
This is just the Prandtl equation.

Finally, we collect the \(\varepsilon\)-order terms that
\[
\begin{aligned}
&\partial_t u_p^{(1)} - \partial_z u_p^{(1)} + (u_p^{(0)} + u_e^{(0)}(t, x, 0)) \cdot \nabla_x u_p^{(1)} + (v_p^{(1)} + v_e^{(1)}(t, x, 0) + z\partial_y v_e^{(0)}(t, x, 0)) \partial_z u_p^{(1)} \\
&+ u_p^{(1)} \cdot \nabla_x (u_p^{(0)} + u_e^{(0)}(t, x, 0)) + (u^{(1)}(t, x, 0) + z\partial_y u_e^{(0)}(t, x, 0)) \cdot \nabla_x u_p^{(0)} \\
&+ (v_p^{(2)} - v_p^{(2)}(t, x, 0) + z\partial_y v_e^{(0)}(t, x, 0) + \frac{1}{2}z^2\partial_y v_e^{(0)}(t, x, 0)) \partial_z u_p^{(0)} \\
&= -\left( u_p^{(0)} \cdot \nabla_x \partial_y u_e^{(0)}(t, x, 0) + u_p^{(0)} \cdot \nabla_x u_e^{(1)}(t, x, 0) + v_p^{(1)} \partial_y u_e^{(0)}(t, x, 0) \right),
\end{aligned}
\] (2.6)
with the initial condition \( u_p^{(1)}(0, x, z) = 0 \) and the boundary condition
\[
v_p^{(1)}(t, x, 0) = -u_e^{(1)}(t, x, 0), \quad \lim_{z \to +\infty} u_p^{(1)}(t, x, z) = 0.
\]

Here \( v_p^{(2)} \) is determined by
\[
v_p^{(2)}(t, x, z) = \int_{z}^{+\infty} \nabla_x \cdot u_p^{(1)}(t, x, z') dz'.
\]

Moreover, the pressure \( p_p^{(2)} \) is determined by
\[
p_p^{(2)}(t, x, z) = -\int_{z}^{+\infty} \mathcal{P}_2(t, x, z') dz',
\]
where
\[
\mathcal{P}_2 = \partial_{zz} v_p^{(1)} - \partial_t v_p^{(1)} - u_e^{(0)}(t, x, 0) \cdot \nabla_x v_p^{(1)} - u_p^{(0)} : \left( \nabla_x v_e^{(1)}(t, x, 0) + \nabla_x v_p^{(1)} \right) - v_p^{(1)} \partial_y v_e^{(0)}(t, x, 0)
- (v_e^{(1)}(t, x, 0) + v_p^{(1)}) \partial_z v_p^{(1)} - z \partial_y \nabla_x v_p^{(0)}(t, x, 0) \cdot v_p^{(0)} - z \partial_y v_p^{(0)}(t, x, 0) \partial_z v_p^{(1)}.
\]

**Remark 2.2.** These equations can be solved in the following way
\[
(u_e^{(0)}, v_e^{(0)}, p_e^{(0)}) \to (u_e^{(1)}, v_e^{(1)}) \to (u_p^{(1)}, v_e^{(1)}, p_e^{(1)}) \to (u_p^{(1)}, v_p^{(2)}, p_p^{(2)}).
\]

### 2.3. Construction of the approximate solution.

Let us define the approximate solution \( (u_a^\varepsilon, v_a^\varepsilon, p_a^\varepsilon) \) as follows
\[
\begin{align*}
u_a^\varepsilon(t, x, y) & \equiv \sum_{i=0}^{1} \varepsilon^i u_e^{(i)}(t, x, y) + \sum_{i=0}^{1} \varepsilon^i u_p^{(i)}(t, x, \frac{y}{\varepsilon}), \\
v_a^\varepsilon(t, x, y) & \equiv \sum_{i=0}^{1} \varepsilon^i v_e^{(i)}(t, x, y) + \varepsilon v_p^{(1)}(t, x, \frac{y}{\varepsilon}), \\
p_a^\varepsilon(t, x, y) & \equiv \sum_{i=0}^{1} \varepsilon^i p_e^{(i)}(t, x, y) + \varepsilon^2 p_p^{(2)}(t, x, \frac{y}{\varepsilon}).
\end{align*}
\]

We set
\[
f(t, x) \equiv \int_{0}^{\infty} \partial_z u_p^{(1)}(t, x, z) dz. \tag{2.7}
\]

In view of the process of the asymptotic expansions, a straightforward computation gives
\[
\begin{align*}
\begin{cases}
\partial_t u_a^\varepsilon + u_a^\varepsilon \cdot \nabla_x u_a^\varepsilon + (v_a^\varepsilon - \varepsilon^2 f(t, x) e^{-y}) \partial_y u_a^\varepsilon + \nabla_x p_a^\varepsilon - \varepsilon^2 \Delta u_a^\varepsilon = -R_{e,h} - R_{p,h}, \\
\partial_t v_a^\varepsilon + u_a^\varepsilon \cdot \nabla_x v_a^\varepsilon + (v_a^\varepsilon - \varepsilon^2 f(t, x) e^{-y}) \partial_y v_a^\varepsilon + \partial_y p_a^\varepsilon - \varepsilon^2 \Delta v_a^\varepsilon = -R_{e,v} - R_{p,v}, \\
\nabla_x \cdot u_a^\varepsilon + \partial_y v_a^\varepsilon = 0,
\end{cases}
\end{align*}
\]
\[
(u_a^\varepsilon, v_a^\varepsilon)(0, x, y) = (u_0(x, y), v_0(x, y)), \\
(u_a^\varepsilon, v_a^\varepsilon)(t, 0, x) = (0, \varepsilon^2 f(t, x)),
\tag{2.8}
\]

where
\[
\begin{align*}
-R_{e,h} & = \varepsilon^2 \left( u_e^{(1)} \cdot \nabla_x u_e^{(1)} + v_e^{(1)} \partial_y u_e^{(1)} - f e^{-y} (\partial_y u_e^{(0)} + \varepsilon \partial_y u_e^{(1)}) \right) - \varepsilon^2 \Delta (u_e^{(0)} + \varepsilon u_e^{(1)}), \\
-R_{e,v} & = \varepsilon^2 \left( u_e^{(1)} \cdot \nabla_x v_e^{(1)} + v_e^{(1)} \partial_y v_e^{(1)} - f e^{-y} (\partial_y v_e^{(0)} + \varepsilon \partial_y v_e^{(1)}) \right) - \varepsilon^2 \Delta (v_e^{(0)} + \varepsilon v_e^{(1)}),
\end{align*}
\]
and

\[-R_{p,h} = \varepsilon^2 \left( u_e^{(1)} \cdot \nabla x u_p^{(1)} + u_p^{(1)} \cdot \nabla x u_e^{(1)} + u_e^{(1)} \cdot \nabla x u_p^{(1)} + v_p^{(1)} \partial_y u_e^{(1)} + v_p^{(1)} ( \partial_y u_e^{(1)} + \varepsilon \partial_y u_e^{(1)} ) \right) \]

\[- f e^{-y} \partial_z u_p^{(1)} + v_p^{(2)} \partial_z u_p^{(1)} + \varepsilon \left( (u_e^{(0)} - u_e^{(0)}(t, x, 0)) \cdot \nabla x u_p^{(1)} + (\nabla x u_e^{(0)} - \nabla x u_e^{(0)}(t, x, 0)) \cdot u_p^{(0)} \right) \]

\[+ (u_e^{(1)} - u_e^{(1)}(t, x, 0)) \cdot \nabla x u_p^{(0)} + (\nabla x u_e^{(0)} - \nabla x u_e^{(0)}(t, x, 0)) u_p^{(1)} + (v_e^{(1)} - v_e^{(1)}(t, x, 0)) \partial_z u_p^{(1)} \]

\[+ (\partial_y u_p^{(0)} - \partial_y u_e^{(0)}(t, x, 0)) v_p^{(1)} + f (1 - e^{-y}) \partial_z u_p^{(0)} \]

\[+ \frac{1}{\varepsilon} \left( v_e^{(0)} - y \partial_y v_e^{(0)}(t, x, 0) - \frac{y^2}{2} \partial_y v_e^{(0)}(t, x, 0) \right) \partial_z u_p^{(0)} + (v_e^{(0)} - y \partial_y v_e^{(0)}(t, x, 0)) \partial_z u_p^{(1)} \]

\[+ (v_e^{(1)} - v_e^{(1)}(t, x, 0) - y \partial_y v_e^{(1)}(t, x, 0)) \partial_z u_p^{(0)} + (u_e^{(0)} - u_e^{(0)}(t, x, 0) - y \partial_y u_e^{(0)}(t, x, 0)) \cdot \nabla x u_p^{(0)} \]

\[+ (\nabla x u_e^{(0)} - \nabla x u_e^{(0)}(t, x, 0) - y \partial_y \nabla x u_e^{(0)}(t, x, 0)) \cdot u_p^{(0)} - \varepsilon^2 \Delta_x u_{a,p} + \varepsilon^2 \partial_x P_{e,p}^{(2)}, \]

\[-R_{p,v} = \varepsilon^2 \left( \partial_t v_p^{(2)} + (u_e^{(0)} + v_p^{(0)}) \cdot \nabla x v_p^{(2)} + (v_e^{(1)} + v_p^{(1)}) \partial_z v_p^{(2)} + (u_e^{(1)} + v_p^{(1)}) \cdot \nabla x v_p^{(1)} \right) \]

\[+ v_p^{(2)} \partial_y v_{a,e} + (v_e^{(2)} - f e^{-y}) \partial_y v_{a,p} + (v_e^{(1)} + v_p^{(1)}) \cdot \nabla x v_{e}^{(1)} + v_p^{(1)} \partial_y v_{e}^{(1)} - \partial_z v_p^{(2)} \]

\[+ \varepsilon \left( (u_e^{(0)} - u_e^{(0)}(t, x, 0)) \cdot \nabla x u_p^{(1)} + (\nabla x u_e^{(1)} - \nabla x u_e^{(0)}(t, x, 0)) \cdot u_p^{(0)} \right) \]

\[+ (v_e^{(1)} - v_e^{(1)}(t, x, 0)) \partial_z v_p^{(1)} + (\partial_y v_e^{(0)} - \partial_y v_e^{(0)}(t, x, 0)) v_p^{(1)} + (u_e^{(1)} \cdot \nabla x v_e^{(0)} + v_e^{(0)} \cdot \partial_z v_p^{(2)} - \varepsilon^3 \Delta_x v_{a,p}. \]

Here and in what follows, \( u_{a,e}(t, x, y) = u_e^{(0)}(t, x, y) + \varepsilon u_e^{(1)}(t, x, y), \ u_{e,a}(t, x, y) = u_e^{(0)}(t, x, y) + \varepsilon u_e^{(1)}(t, x, y), \ u_{a,p}(t, x, z) = u_p^{(0)}(t, x, z) + \varepsilon u_p^{(1)}(t, x, z), \ u_{p,a}(t, x, z) = u_p^{(1)}(t, x, z) + \varepsilon u_p^{(2)}(t, x, z). \)

Formally, it holds that

\[ R_{e,h} = O(\varepsilon^2), \ R_{p,h} = O(\varepsilon^2), \ R_{e,v} = O(\varepsilon^2), \ R_{p,v} = O(\varepsilon^2). \]

Later on, we will make them precise.

3. The error system and vorticity formulation

To justify the boundary layer expansion, the most key ingredient is to show that the remainder is uniformly small in a suitable sense.

3.1. The error system. We introduce the error between the solution and the approximate solution

\[ u_e^{\varepsilon} \overset{\text{def}}{=} u_e - u_a^{\varepsilon}, \ v_p^{\varepsilon} \overset{\text{def}}{=} v_p - v_a^{\varepsilon}, \ p_p^{\varepsilon} \overset{\text{def}}{=} p_p - p_a^{\varepsilon}. \]

From (1.1) and (2.8), we deduce that \((u_e^{\varepsilon}, v_p^{\varepsilon})\) satisfies the following system

\[
\begin{align*}
\partial_t u_e^{\varepsilon} + \nabla x u_e^{\varepsilon} + v_e^{\varepsilon} \cdot \partial_y u_e^{\varepsilon} + (v_e^{\varepsilon} - \varepsilon^2 f(t, x) e^{-y}) \partial_y u_e^{\varepsilon} + u_e^{\varepsilon} \cdot \nabla x u_e^{\varepsilon} + (v_e^{\varepsilon} + \varepsilon^2 f(t, x) e^{-y}) \partial_y u_e^{\varepsilon} &+ u_e^{\varepsilon} \cdot \nabla x u_{a}^{\varepsilon} + (v_e^{\varepsilon} + \varepsilon^2 f(t, x) e^{-y}) \partial_y u_e^{\varepsilon} + u_e^{\varepsilon} \cdot \nabla x u_{a}^{\varepsilon} \\
\partial_t v_p^{\varepsilon} + u_p^{\varepsilon} \cdot \nabla x v_p^{\varepsilon} + (v_p^{\varepsilon} - \varepsilon^2 f(t, x) e^{-y}) \partial_y v_p^{\varepsilon} + u_p^{\varepsilon} \cdot \nabla x v_p^{\varepsilon} + (v_p^{\varepsilon} + \varepsilon^2 f(t, x) e^{-y}) \partial_y v_p^{\varepsilon} + u_p^{\varepsilon} \cdot \nabla x v_{a}^{\varepsilon} &+ (v_p^{\varepsilon} + \varepsilon^2 f(t, x) e^{-y}) \partial_y v_p^{\varepsilon} + u_p^{\varepsilon} \cdot \nabla x v_{a}^{\varepsilon} \\
\nabla x \cdot u_e^{\varepsilon} + \partial_y v_p^{\varepsilon} &- \varepsilon^2 \Delta x v_{a}^{\varepsilon} = 0,
\end{align*}
\]

\[(u_e^{\varepsilon}, v_p^{\varepsilon})(t, x, 0) = (0, -\varepsilon^2 f(t, x)), \]

\[(u_e^{\varepsilon}, v_p^{\varepsilon})(0, x, y) = (0, 0). \]
For simplicity of notations, we will omit the superscript $\varepsilon$ and set $u := v_R^\varepsilon$, $v := v_R^\varepsilon$, $p := p_R^\varepsilon$, $u_a := u_a^\varepsilon$, $v_a := v_a^\varepsilon$ and introduce

\[
U_a = (u_a, v_a), \quad \tilde{U}_a = (u_a, v_a - \varepsilon^2 f e^{-y}),
\]

\[
U = (u, v), \quad \tilde{U} = (u, v - \varepsilon^2 f e^{-y}), \quad R = (R_h, R_v),
\]

then the error system (3.1) reads

\[
\begin{aligned}
\partial_t U - \varepsilon^2 \Delta U + \tilde{U}_a \cdot \nabla u + \tilde{U} \cdot \nabla U_a + \tilde{U} \cdot \nabla U + \nabla p &= R, \\
\nabla_{x,y} \cdot U &= 0, \\
U(t, x, 0) &= (0, -\varepsilon^2 f(t, x)), \\
U(0, x, y) &= (0, 0).
\end{aligned}
\]

(3.2)

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\[3.2. \text{ The vorticity formulation of error equation.} \text{ To gain the derivative in } y \text{ variable, we need to use the vorticity formulation. One can check [13] for the derivation in 3-D.}\]

\[\text{Let us first introduce the Dirichlet-Neumann map and Neumann-Dirichlet map.}\]

\[\text{Definition 3.1. Let } f \in C_0^\infty(\mathbb{R}^2). \text{ We denote by } E_D f \text{ and } E_N f, \text{ respectively, the solution to the Dirichlet problem}\]

\[
\begin{aligned}
\{ & \Delta E_D f = 0, \\
& E_D f|_{\partial \Omega_3^1} = f,
\end{aligned}
\]

and the solution to the Neumann problem

\[
\begin{aligned}
\{ & \Delta E_N f = 0, \\
& -\partial_3 E_N f|_{\partial \Omega_3^1} = f.
\end{aligned}
\]

\[\text{Then the Dirichlet-Neumann map } \Lambda_{DN} \text{ and Neumann-Dirichlet map } \Lambda_{ND} \text{ are respectively defined by}\]

\[
\Lambda_{DN} f = -\gamma \partial_3 E_D f, \quad \Lambda_{ND} f = \gamma E_N f,
\]

where $\gamma$ is the trace operator.

Next, we introduce

\[w = (w_h, w_3) = \text{curl}(u, v), \quad w_a = (w_{a,h}, w_{a,3}) = \text{curl}(u_a, v_a).\]

Then taking curl on both sides of (3.2), we arrive at

\[
\begin{aligned}
\partial_t w - \varepsilon^2 \Delta w + \tilde{U}_a \cdot \nabla w + \tilde{U} \cdot \nabla w_a + \tilde{U} \cdot \nabla w - w_a \cdot \nabla U - w \cdot \nabla U_a - w \cdot \nabla U &= \text{curl}(R_h, R_v) - M, \\
w(0, x, z) &= 0, \\
-\varepsilon^2 (\partial_y + |D_x|) w_h(t, x, 0) - \varepsilon^2 \partial_x \Lambda_{ND}(\gamma \nabla x \cdot w_h)(t, x, 0) &= -\partial_y (-\Delta D)^{-1} J_h + \partial_x (-\Delta N)^{-1} J_3, \\
w_3(t, x, 0) &= 0.
\end{aligned}
\]

Here

\[M = \begin{pmatrix}
\varepsilon^2 \partial_{x_2} f e^{-y} \partial_y v_a + \varepsilon^2 f e^{-y} \partial_y u_{a,2} \\
-\varepsilon^2 \partial_{x_1} f e^{-y} \partial_y v_a - \varepsilon^2 f e^{-y} \partial_y u_{a,1} \\
\varepsilon^2 \partial_{x_2} f e^{-y} \partial_y u_{a,2} - \varepsilon^2 \partial_{x_1} f e^{-y} \partial_y u_{a,1}
\end{pmatrix},
\]

and

\[J = \text{curl} \begin{pmatrix}
-\tilde{U}_a \cdot \nabla u - \tilde{U} \cdot \nabla u_a - \tilde{U} \cdot \nabla u + R_h \\
-\tilde{U}_a \cdot \nabla v - \tilde{U} \cdot \nabla v_a - \tilde{U} \cdot \nabla v + R_v
\end{pmatrix} \equiv (J_h, J_3).\]
In the following, let us explain the derivation of boundary condition of the vorticity. First, we have the following Biot-Savart law

\[ u = \text{curl}(\Psi(w)), \quad \Psi(w) = \left( \begin{array}{c} (-\Delta_D)^{-1}w_h \\ (-\Delta_N)^{-1}w_3 \end{array} \right). \]

Therefore,

\[
0 = \partial_t u_1|_{y=0} = \partial_t(\partial_{x_2}(-\Delta_N)^{-1}w_3 - \partial_y(-\Delta_D)^{-1}w_2)|_{y=0}
\]

\[
= \partial_{x_2}(-\Delta_N)^{-1}(\varepsilon^2 \Delta w_3 + J_3)|_{y=0} - \partial_y(-\Delta_D)^{-1}(\varepsilon^2 \Delta w_2 + J_2)|_{y=0}
\]

\[
= -\varepsilon^2(\gamma \partial_{x_2} w_3 + \partial_{x_2} \Lambda_{ND} \gamma \partial_y w_3) + \partial_{x_2}(-\Delta_N)^{-1}J_3|_{y=0}
\]

\[
+ \varepsilon^2(\gamma \partial_y w_2 + \Lambda_{DN} \gamma w_2) - \partial_y(-\Delta_D)^{-1}J_2|_{y=0},
\]

which gives, by \( w_3|_{y=0} = 0 \) and divergence free of vorticity, that

\[
-\varepsilon^2(\gamma \partial_y w_2 + \Lambda_{DN} \gamma w_2) - \varepsilon^2 \partial_{x_2} \Lambda_{ND} \gamma (\nabla x \cdot w_h) = \partial_{x_2}(-\Delta_N)^{-1}J_3|_{y=0} - \partial_y(-\Delta_D)^{-1}J_2|_{y=0}.
\]

Similarly, there holds

\[
0 = \partial_t u_2|_{y=0} = \partial_t(-\partial_{x_1}(-\Delta_N)^{-1}w_3 + \partial_y(-\Delta_D)^{-1}w_1)|_{y=0}
\]

\[
= -\partial_{x_1}(-\Delta_N)^{-1}(\varepsilon^2 \Delta w_3 + J_3)|_{y=0} + \partial_y(-\Delta_D)^{-1}(\varepsilon^2 \Delta w_1 + J_1)|_{y=0}
\]

\[
= \varepsilon^2(\gamma \partial_{x_1} w_3 + \partial_{x_1} \Lambda_{ND} \gamma \partial_y w_3) - \partial_{x_1}(-\Delta_N)^{-1}J_3|_{y=0}
\]

\[
- \varepsilon^2(\gamma \partial_y w_1 + \Lambda_{DN} \gamma w_1) + \partial_y(-\Delta_D)^{-1}J_1|_{y=0},
\]

thus

\[
-\varepsilon^2(\gamma \partial_y w_1 + \Lambda_{DN} \gamma w_1) - \varepsilon^2 \partial_{x_1} \Lambda_{ND} \gamma (\nabla x \cdot w_h) = \partial_{x_1}(-\Delta_N)^{-1}J_3|_{y=0} - \partial_y(-\Delta_D)^{-1}J_1|_{y=0}.
\]

Using the fact that

\[ \Lambda_{DN} f = |D_x| f, \]

we deduce the boundary condition of the vorticity.

### 3.3. Decomposition of the vorticity

Motivated by [20], we decompose the vorticity into two parts: Euler part \( w_e \) and Prandtl part \( w_p \), which are respectively defined by the following system

\[
\begin{cases}
\partial_t w_e - \varepsilon^2 \Delta w_e + \tilde{U}_a \cdot \nabla w_e + \tilde{U} \cdot \nabla w_{a,e} + \tilde{U} \cdot \nabla w_e - w_{a,e} \cdot \nabla U - w_e \cdot \nabla U_a - w_e \cdot \nabla U \\
= \text{curl}(R_{e,h}, R_{e,v}) - M_e,
\end{cases}
\]

\[
\begin{array}{ll}
w_e(0, x, y) = 0, & w_{e,3}(t, x, 0) = 0, \\
-\varepsilon^2(\partial_y + |D_x|)w_{e,h}(t, x, 0) - \varepsilon^2 \partial_x \Lambda_{ND}(\gamma \nabla x \cdot w_{e,h})(t, x, 0) = 0,
\end{array}
\]

and

\[
\begin{cases}
\partial_t w_p - \varepsilon^2 \Delta w_p + \tilde{U}_a \cdot \nabla w_p + \tilde{U} \cdot \nabla w_{a,p} + \tilde{U} \cdot \nabla w_p - w_{a,p} \cdot \nabla U - w_p \cdot \nabla U_a - w_p \cdot \nabla U \\
= \text{curl}(R_{p,h}, R_{p,v}) - M_p,
\end{cases}
\]

\[
\begin{array}{ll}
w_p(0, x, y) = 0, & w_{p,3}(t, x, 0) = 0, \\
-\varepsilon^2(\partial_y + |D_x|)w_{p,h}(t, x, 0) - \varepsilon^2 \partial_x \Lambda_{ND}(\gamma \nabla x \cdot w_{p,h})(t, x, 0) = 0,
\end{array}
\]

Here we denote

\[
w_{a,e} = (w_{a,e,h}, w_{a,e,3}) = \text{curl}(u_{a,e}, v_{a,e}), \quad w_{a,p} = (w_{a,p,h}, w_{a,p,3}) = \text{curl}(u_{a,p}, v_{a,p}),
\]

and

\[
M_e = \begin{pmatrix}
\varepsilon^2 \partial_{x_2} f e^{-y} \partial_y v_{a,e} + \varepsilon^2 f e^{-y} \partial_y u_{a,e,2} \\
-\varepsilon^2 \partial_{x_1} f e^{-y} \partial_y v_{a,e} - \varepsilon^2 f e^{-y} \partial_y u_{a,e,1} \\
\varepsilon^2 \partial_{x_2} f e^{-y} \partial_y u_{a,e,2} - \varepsilon^2 \partial_{x_2} f e^{-y} \partial_y u_{a,e,1}
\end{pmatrix},
\]

\[
M_p = \begin{pmatrix}
\end{pmatrix}.
and
\[
M_p = \begin{pmatrix}
\varepsilon^2 \partial_{x_1} f e^{-y} \partial_y v_{a,p} + \varepsilon^2 f e^{-y} \partial_y u_{a,p,2} \\
-\varepsilon^2 \partial_{x_1} f e^{-y} \partial_y v_{a,p} - \varepsilon^2 f e^{-y} \partial_y u_{a,p,1} \\
\varepsilon^2 \partial_{x_2} f e^{-y} \partial_y u_{a,p,2} - \varepsilon^2 f e^{-y} \partial_y u_{a,p,1}
\end{pmatrix}.
\]

It is easy to find that \( w \) and \( w_e + w_p \) satisfy the same equation and initial-boundary conditions. Therefore, we get
\[
w = w_e + w_p.
\]

4. Functional framework and product estimates

In this section, we introduce the functional spaces we are working on and some product estimates, which will be used in the energy estimate.

Throughout this paper, let \( \delta > 0 \) be a small constant and \( \lambda > 0 \) be a large constant, which are determined later. We denote by \( C_0 \) a constant independent of \( \delta \), which may change from line to line.

4.1. Functional framework. Let \( \varphi \) be a smooth function such that
\[
\varphi(y) = \begin{cases}
\delta y, & y \leq 1, \\
\frac{\delta y}{1 + y}, & y \geq 2.
\end{cases}
\]

We introduce the conormal operator \( Z = \varphi(y) \partial_y \) and denote
\[
Z^k \triangleq \varphi(y)^k \partial_y^k, \quad Z^k \triangleq (\delta z)^k \partial_z^k.
\]

Let us introduce the following Sobolev type spaces
\[
H^m_{\text{co}}(\mathbb{R}_+^3) \triangleq \left\{ u \in L^2(\mathbb{R}_+^3) : \| u \|_{H^m_{\text{co}}} = \sum_{|i| + j \leq m} \| \partial_x^i Z^j u \|_{L^2(\mathbb{R}_+^3)} < \infty \right\},
\]
\[
H^m_{\text{tan}}(\mathbb{R}_+^3) \triangleq \left\{ u \in L^2(\mathbb{R}_+^3) : \| u \|_{H^m_{\text{tan}}} = \sum_{|i| \leq m} \| \partial_x^i u \|_{L^2(\mathbb{R}_+^3)} < \infty \right\}.
\]

Here and in what follows, \( \partial_x^i u \) means \( \partial_x^{i_1} \partial_{x_2}^{i_2} u \) for \( i = (i_1, i_2) \in \mathbb{N}^2 \). \( H^m_{\text{co}}(\mathbb{R}_+^3) \) is the so called conormal Sobolev space.

We also introduce the norms
\[
\| u \|^2_{H^m_{\text{co}}(0,a)} = \sum_{|i| + j \leq m} \int_0^a \int_{\mathbb{R}^2} | \partial_x^i Z^j u(x,y) |^2 \, dx \, dy,
\]
\[
\| u \|^2_{H^m_{\text{tan}}(0,a)} = \sum_{|i| \leq m} \int_0^a \int_{\mathbb{R}^2} | \partial_x^i u(x,y) |^2 \, dx \, dy,
\]
\[
\| u \|^2_{H^m_{\text{co}}(0,a)} = \sum_{|i| + j \leq m} \int_0^a \int_{\mathbb{R}^2} | \langle D_x \rangle^{\frac{j}{2}} \partial_x^i Z^j u(x,y) |^2 \, dx \, dy,
\]
\[
\| u \|^2_{H^m_{\text{tan}}(0,a)} = \sum_{|i| \leq m} \int_0^a \int_{\mathbb{R}^2} | \langle D_x \rangle^{\frac{j}{2}} \partial_x^i u(x,y) |^2 \, dx \, dy
\]

for some \( a \in \mathbb{R}_+ \) and the inner product
\[
\langle u, v \rangle_{H^m_{\text{co}}} = \sum_{|i| + j \leq m} \int_0^{+\infty} \int_{\mathbb{R}^2} \partial_x^i Z^j u \partial_x^i Z^j v(x,y) \, dx \, dy,
\]
\[
\langle u, v \rangle_{H^m_{\text{tan}}} = \sum_{|i| \leq m} \int_0^{+\infty} \int_{\mathbb{R}^2} \partial_x^i u \partial_x^i v(x,y) \, dx \, dy.
\]
For the vorticity, we need to make the estimates in the weighted type spaces. Let $\theta(y)$ be an increasing function satisfying

$$
|\theta'(y)| + |\theta''(y)| \leq C_0 \delta, \quad \theta(0) = 0, \quad \theta\left(\frac{1}{2}\right) = \delta, \quad \theta'(0) = 0, \quad \theta'(y) = 0 \quad \text{for} \quad y \geq \frac{1}{2}.
$$

(4.2)

We define

$$\phi(t, y) \overset{\text{def}}{=} \delta - \theta(y) - \lambda t.$$

Let $y(t) \in (0, \frac{1}{2})$ be such that $\phi(t, y(t)) = 0$ for small $t \leq \frac{\delta}{\lambda}$. Let $T_0 = \frac{\delta}{\lambda}$. Then for $t \in [0, T_0]$, there exists $c_0 > 0$ such that

$$y(t) \geq c_0 > 0.$$

(4.3)

We introduce two weights

$$\Psi_e(t, y) \overset{\text{def}}{=} e^{\frac{1}{2} \phi(t, y)}, \quad \Psi_p(t, y) \overset{\text{def}}{=} e^{\frac{1}{2} \phi(t, y) - \lambda t},$$

where $\Psi_e$ is the weight for Euler part $w_e$ and $\Psi_p$ is the weight for Prandtl part $w_p$. We denote

$$\|u\|_{H^{m}(\mathbb{R}^3_+)}^2 = \sum_{|i|+j \leq m} \left\| e^{\Psi_e \partial_x^i \partial_y^j} u \right\|_{L^2(\mathbb{R}^3_+)}^2, \quad \|u\|_{H^{m}_p(\mathbb{R}^3_+)}^2 = \sum_{|i|+j \leq m} \left\| e^{\Psi_p \partial_x^i \partial_y^j} u \right\|_{L^2(\mathbb{R}^3_+)}^2.$$

Moreover, we denote

$$\|u\|_{H^{m}(0, a)}^2 = \sum_{|i|+j \leq m} \left\| e^{\Psi_e \partial_x^i \partial_y^j} u \right\|_{L^2(\mathbb{R}^{2} \times (0, a))}^2,$$

$$\|u\|_{H^{m}_p(0, a)}^2 = \sum_{|i|+j \leq m} \left\| e^{\Psi_p \partial_x^i \partial_y^j} u \right\|_{L^2(\mathbb{R}^{2} \times (0, a))}^2,$$

$$\|u\|_{H^{m}_{e, p}(0, a)}^2 = \sum_{|i|+j \leq m} \left\| e^{\Psi_e \langle D_x \rangle^\frac{1}{2} \partial_x^i \partial_y^j} u \right\|_{L^2(\mathbb{R}^{2} \times (0, a))}^2,$$

$$\|u\|_{H^{m}_{p, e}(0, a)}^2 = \sum_{|i|+j \leq m} \left\| e^{\Psi_p \langle D_x \rangle^\frac{1}{2} \partial_x^i \partial_y^j} u \right\|_{L^2(\mathbb{R}^{2} \times (0, a))}^2.$$

For function $u$ compactly supported in Fourier space in $x$ variable, we define

$$u_\Phi(t, x, y) \overset{\text{def}}{=} \mathcal{F}^{-1}_{\xi \to x} \left( e^{\Phi(t, \xi, y)} \mathcal{F}_{x \to \xi} u \right)(t, x, y),$$

where $\Phi(t, \xi, y) = \phi(t, y)\langle \xi \rangle$.

4.2. Product estimates. Let $a \in [0, \frac{1}{2}]$ and $I = [0, a]$.

**Lemma 4.1.** Let $m \geq 8$ and $\sigma \in [0, 1]$. It holds that

$$\|\langle D_x \rangle^{-\sigma} uv \|_{H^{m}_e(I)} \leq C \left( \sum_{|i|+j \leq m} \| \partial_x^i \partial_y^j \langle D_x \rangle^{-\sigma} u f \|_{L^\infty(I; L^2(\mathbb{R}^3_+))}^2 \| v f \|_{H^{m-2}_e(I)}^2 \right),$$

$$\|\langle D_x \rangle^{-\sigma} u f \|_{H^{m}_e(I)} \leq C \left( \sum_{|i|+j \leq m-2} \| \partial_x^i \partial_y^j u f \|_{L^\infty(I; L^2(\mathbb{R}^3_+))}^2 \| \langle D_x \rangle^{-\sigma} v f \|_{H^{m}_e(I)}^2 \right).$$

$$\|\langle D_x \rangle^{-\sigma} uv \|_{H^{m}_p(I)} \leq C \left( \sum_{|i|+j \leq m-2} \| \partial_x^i \partial_y^j u f \|_{L^\infty(I; L^2(\mathbb{R}^3_+))}^2 \| \langle D_x \rangle^{-\sigma} v f \|_{H^{m}_p(I)}^2 \right),$$

$$\|\langle D_x \rangle^{-\sigma} u f \|_{H^{m}_p(I)} \leq C \left( \sum_{|i|+j \leq m-2} \| \partial_x^i \partial_y^j v f \|_{L^\infty(I; L^2(\mathbb{R}^3_+))}^2 \| \langle D_x \rangle^{-\sigma} u f \|_{H^{m}_p(I)}^2 \right).$$
In particular, we have
\[
\|(uv)\Phi\|_{H^m_{\text{co}}(I)}^2 \leq C \left( \|u\Phi\|_{H^{m-1}_{\text{co}}(I)}^2 + \|\partial_y u\Phi\|_{H^{m-2}_{\text{co}}(I)}^2 \right) \|v\Phi\|_{H^m_{\text{co}}(I)}^2 + \left( \|v\Phi\|_{H^{m-1}_{\text{co}}(I)}^2 + \|\partial_y v\Phi\|_{H^{m-2}_{\text{co}}(I)}^2 \right) \|u\Phi\|_{H^m_{\text{co}}(I)}^2.
\]

Similar estimates also hold in the space $H^m_{\text{tan}}(I)$.

**Proof.** We deduce from the definition of $H^m_{\text{co}}(I)$ that
\[
\|\langle D_x \rangle^{-\sigma} (uv)\Phi\|_{H^m_{\text{co}}(I)}^2 = \sum_{|i|+j \leq m} \int_0^a \int_{\mathbb{R}^2} |\langle D_x \rangle^{-\sigma} \partial_x^i Z^j (uv)\Phi|^2(x,y) dxdy.
\]

We only show the case of $j = m$, other cases are similar. By Leibniz’s rule, we have
\[
\int_0^a \int_{\mathbb{R}^2} |\langle D_x \rangle^{-\sigma} Z^m (uv)\Phi|^2 dxdy \\
\leq C \sum_{m_2 + m_3 = m - m_1} \int_0^a \int_{\mathbb{R}^2} \|\langle D_x \rangle^{-\sigma} (Z^{m_2} u Z^{m_3} v)\Phi\|^2_{H^{1/2}(\mathbb{R}^2)} dy \triangleq I.
\]

We split it into two cases according to the value of $m_1$.

**Case 1.** $m_1 \geq 4$.

By classical product estimate(2), we have
\[
I \leq C \sum_{m_2 + m_3 = m - m_1} \int_0^a \|\langle D_x \rangle^{-\sigma} (Z^{m_2} u)\Phi\|^2_{H^2(\mathbb{R}^2)} \|\langle D_x \rangle^{-\sigma} (Z^{m_3} v)\Phi\|^2_{H^2(\mathbb{R}^2)} dy \\
+ \sum_{m_2 + m_3 = m - m_1} \int_0^a \|\langle D_x \rangle^{-\sigma} (Z^{m_2} u)\Phi\|^2_{H^2(\mathbb{R}^2)} \|\langle D_x \rangle^{-\sigma} (Z^{m_3} v)\Phi\|^2_{H^2(\mathbb{R}^2)} dy \\
\leq C \left( \sum_{|i|+j \leq m} \|\partial_x^i Z^j (D_x)^{-\sigma} u\Phi\|^2_{L^\infty(I; L^2(\mathbb{R}^2))} \|v\Phi\|^2_{H^{m-2}_{\text{co}}(I)} \\
+ \sum_{|i|+j \leq m-2} \|\partial_x^i Z^j u\Phi\|^2_{L^\infty(I; L^2(\mathbb{R}^2))} \|\langle D_x \rangle^{-\sigma} v\Phi\|^2_{H^m_{\text{co}}(I)} \right).
\]

**Case 2.** $m_1 \leq 3$.

First of all, we have the following classical product estimate: for any $s \geq 0$,
\[
\|\langle D_x \rangle^{-\sigma} (uv)\|_{H^s(\mathbb{R}^2)} \leq C \|\langle D_x \rangle^{-\sigma} u\|_{H^s(\mathbb{R}^2)} \|v\|_{H^{s+2}(\mathbb{R}^2)}.
\]

We decompose $I$ into three parts as following
\[
I \leq C \sum_{m_2 \leq \left[ \frac{m-m_1}{2} \right] - 1} \int_0^a \|\langle D_x \rangle^{-\sigma} (Z^{m_2} u Z^{m_3} v)\Phi\|^2_{H^{m_1}(\mathbb{R}^2)} dy \\
+ C \sum_{m_3 \leq \left[ \frac{m-m_1}{2} \right] - 1} \int_0^a \|\langle D_x \rangle^{-\sigma} (Z^{m_2} u Z^{m_3} v)\Phi\|^2_{H^{m_1}(\mathbb{R}^2)} dy \\
+ C \int_0^a \|\langle D_x \rangle^{-\sigma} (Z^{\left[ \frac{m-m_1}{2} \right]} u Z^{m-m_1-\left[ \frac{m-m_1}{2} \right]} v)\Phi\|^2_{H^{m_1}(\mathbb{R}^2)} dy \triangleq I_1 + I_2 + I_3.
\]
It follows from (22) that
\[ I_1 \leq C \sum_{m_2 \leq \left( \frac{m+1}{2} \right)} \int_0^a \left\| (Z^{m_2} u) \|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \right\| (D_x)_{-\sigma} (Z^{m_3} v) \|_{H^{m_1}(\mathbb{R}^2)}^2 \, dy \]
\[ \leq C \sum_{|i|+j \leq m-2} \left\| \partial_x^i Z^j u \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \left\| (D_x)_{-\sigma} v \Phi \right\|_{H^{m_0}(I)}^2. \]

Similar argument gives
\[ I_2 \leq C \sum_{|i|+j \leq m} \left\| \partial_x^i Z^j (D_x)_{-\sigma} u \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \left\| v \Phi \right\|_{H^{m_0-2}(I)}^2. \]

For \( I_3 \), we have
\[ I_3 \leq C \int_0^a \left\| (Z^{m_2} u) \|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \right\| (D_x)_{-\sigma} (Z^{m_3} v) \|_{H^{m_1}(\mathbb{R}^2)}^2 \, dy \]
\[ + C \int_0^a \left\| (D_x)_{-\sigma} (Z^{m_3} v) \|_{H^{m_1}(\mathbb{R}^2)}^2 \right\| (Z^{m_3} v) \|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \, dy \]
\[ \leq C \left( \sum_{|i|+j \leq m} \left\| \partial_x^i Z^j u \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \left\| v \Phi \right\|_{H^{m_0-2}(I)}^2 \right) \]
\[ + \sum_{|i|+j \leq m-2} \left\| \partial_x^i Z^j u \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \left\| (D_x)_{-\sigma} v \Phi \right\|_{H^{m_0}(I)}^2. \]

The first inequality follows by collecting these estimates. The proof of the second inequality is similar. The third inequality can be deduced from Sobolev embedding and the second one directly.

To deal with the term like \( u \partial_x v \), we need the following lemma.

**Lemma 4.2.** Let \( m \geq 8 \). It holds that
\[ \left\| (u \partial_x v) \|_{H^{m_0}(I)} \right\| \leq C \left( \sum_{|i|+j \leq m} \left\| \partial_x^i Z^j (D_x)_{-\sigma} u \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \right) \]
\[ + \sum_{|i|+j \leq m-2} \left\| \partial_x^i Z^j u \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \left\| v \Phi \right\|_{H^{m_0-2}(I)}^2 \]
\[ + \sum_{|i|+j \leq m-1} \left\| \partial_x^i Z^j v \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \left\| (D_x)_{-\sigma} u \Phi \right\|_{H^{m_0}(I)}^2 \]
\[ + \sum_{|i|+j \leq m-1} \left\| \partial_x^i Z^j v \Phi \right\|_{L^\infty(I;L^2(\mathbb{R}^2))}^2 \left\| (D_x)_{-\sigma} v \Phi \right\|_{H^{m_0-2}(I)}^2 \]
\[ + C \left\| v \Phi \right\|_{H^{m_0-1}(I)}^2 \left\| u \Phi \right\|_{H^{m_0-1}(I)}^2. \]

Similar estimates also hold in the space \( H^m_{\text{tan}}(I) \).
Proof. Using the first inequality of Lemma 4.1, we deduce that
\[
\left| \langle (u \partial_x v) \phi, w \phi \rangle_{H^m(I)} \right| = \left| \langle (D_x) - \frac{1}{2} (u \partial_x v) \phi, (D_x)^{-\frac{1}{2}} w \phi \rangle_{H^m(I)} \right|
\leq C \| (D_x)^{-\frac{1}{2}} (u \partial_x v) \phi \|_{L^2(I; L^2(\mathbb{R}^2))}^2 + C \| w \phi \|_{H^m(I)}^2
\]
\[
\leq C \sum_{|i| + j \leq m} \| \partial_x^i Z^j (D_x)^{-\frac{1}{2}} u \phi \|_{L^2(I; L^2(\mathbb{R}^2))}^2 \| v \phi \|_{H^m(I)}^2,
\]
which shows the first inequality. The second inequality can be proved in a similar way. By the second inequality of Lemma 4.1 and Sobolev embedding, we can deduce the third inequality. □

The following two lemmas are the analogous of Lemma 4.1 and Lemma 4.2 in the weighted spaces. Since the proof is the almost same, we omit the details.

Lemma 4.3. Let \( m \geq 8 \) and \( \sigma \in [0, 1] \). It holds that
\[
\| (D_x)^{-\sigma} (uv) \phi \|_{H^m(I)}^2 \leq C \left( \sum_{|i| + j \leq m} \| \partial_x^i Z^j (D_x)^{-\sigma} u \phi \|_{L^2(I; L^2(\mathbb{R}^2))} \| v \phi \|_{H^m(I)}^2, \right.
\]
\[\left. + \sum_{|i| + j \leq m-2} \| \partial_x^i Z^j u \phi \|_{L^2(I; L^2(\mathbb{R}^2))} \| (D_x)^{-\sigma} v \phi \|_{H^m(I)} \right),
\]
\[
\| (D_x)^{-\sigma} (uv) \phi \|_{H^m(I)}^2 \leq C \left( \sum_{|i| + j \leq m} \| \partial_x^i Z^j (D_x)^{-\sigma} u \phi \|_{L^2(I; L^2(\mathbb{R}^2))}, \right.
\]
\[\left. + \sum_{|i| + j \leq m-2} \| \partial_x^i Z^j u \phi \|_{L^2(I; L^2(\mathbb{R}^2))} \| (D_x)^{-\sigma} v \phi \|_{H^m(I)} \right) + \sum_{|i| + j \leq m-2} \| e^{\Psi} \partial_x^i Z^j v \phi \|_{L^2(I; L^2(\mathbb{R}^2))} \| (D_x)^{-\sigma} u \phi \|_{H^m(I)}^2.
\]

Similar estimates also hold in the weighted space \( H^m_p(I) \).

Lemma 4.4. Let \( m \geq 8 \). It holds that
\[
\left| \langle (u \partial_x v) \phi, w \phi \rangle_{H^m(I)} \right| \leq C \left( \sum_{|i| + j \leq m} \| \partial_x^i Z^j (D_x)^{-\frac{1}{2}} u \phi \|_{L^2(I; L^2(\mathbb{R}^2))}, \right.
\]
\[\left. + \sum_{|i| + j \leq m-2} \| \partial_x^i Z^j u \phi \|_{L^2(I; L^2(\mathbb{R}^2))} \| v \phi \|_{H^m(I)} + C \| w \phi \|_{H^m(I)}^2 \right),
\]
\[
\left| \langle (u \partial_x v) \phi, w \phi \rangle_{H^m(I)} \right| \leq C \| u \phi \|_{H^m(I)}^2 \left( \sum_{|i| + j \leq m} \| \partial_x^i Z^j (D_x)^{\frac{1}{2}} v \phi \|_{L^2(I; L^2(\mathbb{R}^2))}, \right.
\]
\[\left. + \sum_{|i| + j \leq m-1} \| \partial_x^i Z^j v \phi \|_{L^2(I; L^2(\mathbb{R}^2))} + C \| w \phi \|_{H^m(I)}^2 \right),
\]

Similar estimates also hold in the weighted space \( H^m_p(I) \).
5. Energy functional and Proof of Theorem 1.1

5.1. Construction of energy functional. To control the error, let us introduce the following energy functional

\[
E_v(t) \overset{\text{def}}{=} \varepsilon^{-2} \left( \| U_\Phi \|_{H^1_{tan}}^2 + \| U \|_{H^1_{tan}}^2 \right),
\]

\[
K_v(t) \overset{\text{def}}{=} \varepsilon^{-2} \| U_\Phi \|_{H_{tan}^2(0,y(t))}^2,
\]

\[
E_w(t) \overset{\text{def}}{=} \varepsilon^{-2} \left( \| (\varphi w_\epsilon) \|_{H^1_\Phi}^2 + \| (\varphi w_p) \|_{H^1_\Phi}^2 + \| (w_{e,3}) \|_{H^1_\Phi}^2 + \| (w_{p,3}) \|_{H^1_\Phi}^2 + \right.
\]

\[
\left. + \| (w_\epsilon) \|_{H^1_{co}}^2 + \| (w_p) \|_{H^1_{co}}^2 + \| w_{e,3} \|_{H^1_{co}}^2 + \| w_{p,3} \|_{H^1_{co}}^2 \right).
\]

\[
K_w(t) \overset{\text{def}}{=} \varepsilon^{-2} \left( \| (\varphi w_\epsilon) \|_{H^1_{\Phi} \frac{1}{2}(0,y(t))}^2 + \| (\varphi w_p) \|_{H^1_{\Phi} \frac{1}{2}(0,y(t))}^2 + \| (w_{e,3}) \|_{H^1_{\Phi} \frac{1}{2}(0,y(t))}^2 + \right.
\]

\[
\left. + \| (w_{p,3}) \|_{H^1_{\Phi} \frac{1}{2}(0,y(t))}^2 + \| (w_\epsilon) \|_{H^1_{\Phi} \frac{1}{2}(0,y(t))}^2 + \| (w_p) \|_{H^1_{\Phi} \frac{1}{2}(0,y(t))}^2 \right).
\]

We denote

\[
E(t) \triangleq E_v(t) + E_w(t), \quad K(t) \triangleq K_v(t) + K_w(t).
\]

In the following Section 7-Section 12, we will show that if \( E(t) \leq C_1 \varepsilon^2 \), then it holds that

\[
\frac{d}{dt} E(t) + (\lambda - C) K(t) \leq C \varepsilon^2
\]

under the following uniform estimates for the approximate solutions and the remainders \( R_{e,h}, R_{e,v}, R_{p,h}, R_{p,v} \). The proof will be presented in appendix.

Lemma 5.1. There exist \( T_a > 0 \) such that for any \( t \in [0, T_a] \), there holds

\[
\left\| (u^{(i)}_\epsilon, v^{(i)}_\epsilon) \Phi \right\|_{H^{15}_G(R^3)} \leq C,
\]

\[
\sum_{|m| + n \leq 15} \left( \sum_{|l| \leq 1} \right) \left\| e^{z^2} \partial_x^n \partial_t^m \Phi \right\|_{L^2(R^3)} \leq C
\]

for \( i = 0, 1 \), where \( \Phi_e = (1 - y) \langle \xi \rangle \) and \( \Phi_p = \left( \frac{1}{z} - \lambda p \right) \langle \xi \rangle \).

Lemma 5.2. There exist \( T_a > 0 \) such that for any \( t \in [0, T_a] \), there holds

\[
\left\| (R_{e,h}, R_{e,v}) \Phi_e \right\|_{H^{11}_G(R^3)} \leq C \varepsilon^2,
\]

\[
\sum_{|m| + n \leq 10} \left\| e^{z^2} \partial_x^n \partial_t^m (R_{p,h}, R_{p,v}) \Phi_p \right\|_{L^2(R^3)} \leq C \varepsilon^2.
\]

5.2. Proof of Theorem 1.1. Let us prove Theorem 1.1 under the energy inequality (5.1).

For fixed \( \varepsilon > 0 \), the local well-posedness of the Navier-Stokes equations can be easily proved in the energy space like

\[
\mathcal{E}(t) = \| U_\Phi \|_{H^1_{tan}}^2 + \| U \|_{H^1_{tan}}^2 + \| (w_\epsilon) \Phi \|_{H^1_{co}}^2 + \| (w_p) \Phi \|_{H^1_{co}}^2 + \| w_\epsilon \|_{H^1_{co}}^2 + \| w_p \|_{H^1_{co}}^2,
\]

where \( w_\epsilon = \text{curl} U_\epsilon + w_p \) with

\[
\begin{aligned}
&\partial_t w_\epsilon - \varepsilon^2 \Delta w_\epsilon + U_\epsilon \cdot \nabla w_\epsilon - w_\epsilon \cdot \nabla U_\epsilon = 0, \\
w_\epsilon(0, x, y) = w_0, \quad w_{e,3}(t, x, 0) = 0, \\
-\varepsilon^2 (\partial_y + D_x) w_{e,h}(t, x, 0) - \varepsilon^2 \nabla_x A_N D (\gamma \nabla_x \cdot w_{e,h})(t, x, 0) = 0,
\end{aligned}
\]
and
\[
\begin{aligned}
\partial_t w_\varepsilon - \varepsilon^2 \Delta w_\varepsilon + U_\varepsilon \cdot \nabla w_\varepsilon - w_\varepsilon \cdot \nabla U_\varepsilon &= 0, \\
|w^\varepsilon|_p(0, x, y) &= 0, \\
-\varepsilon^2 (\partial_y + |D_x|) w^\varepsilon_{p,h}(t, x, 0) &= -\varepsilon^2 \nabla_x \Lambda_{ND}(\gamma \nabla_x \cdot w^\varepsilon_{p,h})(t, x, 0) \\
= -\partial_y (-\Delta D_h)^{-1} J^\varepsilon_h + \partial_x (-\Delta N)^{-1} J^\varepsilon_x.
\end{aligned}
\]

Here \((J^\varepsilon_h, J^\varepsilon_x) = \text{curl}(U_\varepsilon \cdot \nabla U_\varepsilon)\).

Let \(T^*_{\varepsilon}\) be the maximal existence time of the solution \(U_\varepsilon\). If we take \(\lambda \geq C \) and \(T_1 \leq T_\alpha\) so that \(T_1 C \leq \frac{C}{2}\), we deduce from \([5.1]\) that
\[
E(t) \leq \frac{C_1}{2} \varepsilon^2 \quad \text{for} \quad t \in [0, \min(T^*_{\varepsilon}, T_1)],
\]
which in turn implies \(T^*_{\varepsilon} = T_1\) by a continuous argument. Therefore, there holds
\[
\sup_{0 \leq t \leq T_0} \left( \varepsilon^{-2} \|U_\varepsilon(t) - U_\alpha(t)\|^2_{H^{10}_{\text{loc}}} + \|w_\varepsilon(t) - w_{\varepsilon,\alpha}(t)\|^2_{H^s_{\text{co}}} + \|w_{p}\varepsilon(t) - w_{p,\alpha}(t)\|^2_{H^s_{\text{co}}} \right) \leq C \varepsilon^2.
\]

Then Sobolev embedding gives
\[
\|U_\varepsilon(t) - U_\alpha(t)\|_{L^\infty(\mathbb{R}_1)} \leq C \varepsilon^{\frac{3}{2}} \quad \text{for} \quad t \in [0, T_1]
\]
with \(C\) independent of \(\varepsilon\). The proof of Theorem \([1.1]\) is completed.

Let us conclude this section by the following lemma, which will be used in the energy estimate of the vorticity.

**Lemma 5.3.** Let \(w\) solve the equation
\[
\begin{aligned}
\partial_y w + |\partial_y w| &= f, \\
\lim_{y \to +\infty} w(x, y) &= 0.
\end{aligned}
\]

Then we have
\[
\|D_x w\|^2_{L^2_{x,y}} + \|\partial_y w\|^2_{L^2_{x,y}} \leq C \|f\|^2_{L^2_{x,y}}.
\]

**Proof.** Taking Fourier transform in \(x\) variable, we get
\[
\partial_y \hat{w} + |\xi| \hat{w} = \hat{f}.
\]

Solving this ODE, we get
\[
\hat{w}(\xi, y) = -\int_y^{+\infty} e^{-|\xi|(y-y')} \hat{f}(\xi, y') dy',
\]
from which, it follows that
\[
\|D_x w\|_{L^2_{x,y}} \leq \|\xi e^{-|\xi|y} \ast \hat{f}(\xi, \cdot)\|_{L^2_{x,y}} \leq \sup_{\xi} \left( \|\xi\|_{L^1}\|e^{-|\xi|y}\|_{L^1} \right) \|f\|_{L^2_{x,y}} \leq \|f\|_{L^2_{x,y}}.
\]

Using the equation, we can obtain the estimate of \(\|\partial_y w\|^2_{L^2_{x,y}}\). □
6. Tangential analytic limit and Sobolev estimates of the pressure

6.1. Elliptic equation of the pressure. Taking \( \text{div} \) on both sides of the system (6.2), we obtain the following elliptic equation on the pressure \( p \) with Neumann boundary condition

\[
\begin{cases}
-\Delta p = (\text{div} \, F + \partial_y G) - (\text{div} \, R_h + \partial_y R_v), \\
\partial_y p(x, 0) = \varepsilon^2 \partial_{yy} v(t, x, 0) + R_v(x, 0) + \varepsilon^2 \partial_t f(t, x) - \varepsilon^4 \Delta_x f(t, x),
\end{cases}
\]

(6.1)

where

\[ F \triangleq \tilde{U}_a \cdot \nabla u + \bar{U} \cdot \nabla u_a + \bar{U} \cdot \nabla u, \quad G \triangleq \tilde{U}_a \cdot \nabla v + \bar{U} \cdot \nabla v_a + \bar{U} \cdot \nabla v. \]

(6.2)

It is easy to see that \( F(t, x, 0) = 0 \) and \( G(t, x, 0) = 0 \).

To proceed, let us present the estimate of \((F, G)\).

Lemma 6.1. It holds that

\[ \left\| (F, G) \right\|_{H^s_{tan}(0, y(t))}^2 \leq C \varepsilon^2 (1 + E(t)) \left( E(t) + K(t) + \varepsilon^2 \right). \]

Proof. We first handle \( F \) and estimate it term by term. By Lemma 4.1, Lemma 5.1 \( \tilde{v}_a|_{y=0} = 0 \) and \( \partial_y u = (w_2, -w_1) + \nabla_x v \equiv w_1^+ + \nabla_x v \), we deduce that

\[
\left\| (\tilde{U}_a \cdot \nabla u) \right\|_{H^s_{tan}(0, y(t))}^2 \leq \left\| (u_a \partial_x u) \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| (\tilde{v}_a \partial_y u) \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| (\partial_y u_a) \right\|_{H^s_{tan}(0, y(t))}^2
\]

\[
\leq C \left\| \varphi \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| \left( \tilde{v}_a \varphi (\varphi w_1^+ + \varphi \nabla_x v) \right) \right\|_{H^s_{tan}(0, y(t))}^2
\]

\[
\leq C \left( \left\| \varphi \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| (\varphi w_1^+) \right\|_{H^s_{tan}(0, y(t))}^2 + \varepsilon^4 \right)
\]

\[ \leq C \varepsilon^2 (K(t) + \varepsilon^2). \]

Similarly, using \( \tilde{v}|_{y=0} = 0 \) and \( -\partial_y v = \nabla_x u \), we get

\[
\left\| (\bar{U} \cdot \nabla a) \right\|_{H^s_{tan}(0, y(t))}^2 \leq \left\| (u \partial_x u_a) \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| (\tilde{v} \partial_y u_a) \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| (\partial_y u_a) \right\|_{H^s_{tan}(0, y(t))}^2
\]

\[
\leq C \left\| \varphi \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| \left( \tilde{v} \varphi \right) \right\|_{H^s_{tan}(0, y(t))}^2 + \left\| (\varphi Z u_a) \right\|_{H^s_{tan}(0, y(t))}^2
\]

\[ \leq C \left( \left\| \varphi \right\|_{H^s_{tan}(0, y(t))}^2 + \varepsilon^4 \right) \leq C \varepsilon^2 (K(t) + \varepsilon^2). \]

To deal with the nonlinear term, we need the following product estimate

\[ \left\| (D_x)^{\frac{s}{2}}(uv) \varphi (\cdot, y) \right\|_{L^2(R^2)}^2 \leq C \left\| (D_x)^{\frac{s}{2}} u \varphi (\cdot, y) \right\|_{L^2(R^2)}^2 \left\| v \varphi (\cdot, y) \right\|_{H^2(R^2)}^2 \quad \text{for} \quad y \in (0, y(t)). \]

Then by Sobolev embedding, \( \partial_y u = w_1^+ + \nabla_x v \) and \( -\partial_y v = \nabla_x u \), we get

\[
\left\| (u \partial_x u) \right\|_{H^s_{tan}(0, y(t))}^2 \leq C \sum_{|i| \leq s, |j| \leq 4} \int_0^{y(t)} \left\| (D_x)^{\frac{s}{2}} (\partial_x^i u \partial_x^j \partial_x^j u) \varphi (\cdot, y) \right\|_{L^2(R^2)}^2 dy
\]

\[
+ C \sum_{|i| \leq s, |j| > 4} \int_0^{y(t)} \left\| (D_x)^{\frac{s}{2}} (\partial_x^i u \partial_x^j \partial_x^j u) \varphi (\cdot, y) \right\|_{L^2(R^2)}^2 dy
\]

\[ \leq C \left\| u \varphi \right\|_{H^s_{tan}} \left( \left\| (\partial_y u) \varphi \right\|_{H^s_{tan}} + \left\| u \varphi \right\|_{H^s_{tan}} \right) \left\| u \varphi \right\|_{H^s_{tan}(0, y(t))}^2
\]

\[ \leq C \left( \left\| w \varphi \right\|_{H^s_{tan}}^2 + \left\| U \varphi \right\|_{H^s_{tan}}^2 \right) \left\| u \varphi \right\|_{H^s_{tan}(0, y(t))}^2 \leq C \varepsilon^2 E(t) K(t). \]
Proof. We only prove the case of Lemma 6.2. It holds that for $G$ and $18$ MINGWEN FEI, TAO TAO, AND ZHIFEI ZHANG

Similarly, we have

Summing up, we deduce that

The estimate for $G$ is similar. 

**Lemma 6.2.** It holds that for $k = 7, 8, 9,$

Proof. We only prove the case of $k = 9$ for $F$. By Lemma 4.1 and Lemma 5.1, $\tilde{v}_a|_{y = 0} = 0$, $\partial_y u = u_h^+ + \nabla_x v$ and $-\partial_y v = \nabla_x \cdot u$, we deduce that

and

Similarly, we have

and

and

Similarly, we have

and

and

and
Putting these estimates together, we deduce the estimate of $F$. □

6.2. Tangential analytic estimate of the pressure.

**Lemma 6.3.** There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there holds

$$\delta \left\| (\nabla p) \phi \right\|_{H_{tan}^7}^2 + \left\| \theta' (\nabla p) \phi \right\|_{H_{tan}^4}^2 \leq C \varepsilon^2 \left( E(t) + K(t) + \varepsilon^2 \right) \left( 1 + E(t) \right) + C_0 \delta \varepsilon^4 \left\| (\partial_y + |D_x|) w \right\|_{H_{tan}^2}^2 + C_0 \delta \varepsilon^4 \left\| (\partial_y + |D_x|) w \right\|_{H_{tan}^2}^2 .$$

**Proof.** A straightforward computation gives

$$\begin{cases}
-D_x p_{\phi} - \partial_y (\partial_y p_{\phi}) - \langle D_x \rangle \theta' (\partial_y p_{\phi}) = \nabla_x \cdot F_{\phi} + \partial_y G_{\phi} + \theta' \langle D_x \rangle G_{\phi} \\
-\nabla_x \cdot (R_h)_{\phi} - \partial_y (R_v)_{\phi} - \langle D_x \rangle \theta' (R_v)_{\phi} , \\
(\partial_y p_{\phi})(x,0) = -\varepsilon^2 (\partial_y \nabla_x \cdot u)_{\phi}(x,0) + (R_v)_{\phi}(x,0) + \varepsilon^2 (\partial_t f)_{\phi}(t,x) - \varepsilon^4 \Delta_x f_{\phi}(t,x). 
\end{cases} \tag{6.3}
$$

Acting $\langle D_x \rangle \frac{1}{2} \partial_x^i$ on both sides of (6.3), then taking $L^2(\mathbb{R}_+^3)$ inner product with $(\theta')^2 \langle D_x \rangle \frac{1}{2} \partial_x^i p_{\phi}$ and summing over $1 \leq |i| \leq 8$, we obtain

$$\sum_{1 \leq |i| \leq 8} \left\{ -\langle D_x \rangle \frac{1}{2} \partial_x^i D_x p_{\phi} - \langle D_x \rangle \frac{1}{2} \partial_x^i \partial_y (\partial_y p_{\phi}) - \langle D_x \rangle \theta' \langle D_x \rangle \frac{1}{2} \partial_x^i (\partial_y p_{\phi}), (\theta')^2 \langle D_x \rangle \frac{1}{2} \partial_x^i p_{\phi} \right\}$$

$$= \sum_{1 \leq |i| \leq 8} \left\{ -\langle D_x \rangle \frac{1}{2} \partial_x^i \left[ \nabla_x \cdot F_{\phi} + \partial_y G_{\phi} + \langle D_x \rangle \theta' G_{\phi} \right], (\theta')^2 \langle D_x \rangle \frac{1}{2} \partial_x^i p_{\phi} \right\}$$

$$+ \sum_{1 \leq |i| \leq 8} \left\{ -\langle D_x \rangle \frac{1}{2} \partial_x^i \left[ -\nabla_x \cdot (R_h)_{\phi} - \partial_y (R_v)_{\phi} - \langle D_x \rangle \theta' (R_v)_{\phi} \right], (\theta')^2 \langle D_x \rangle \frac{1}{2} \partial_x^i p_{\phi} \right\}$$

$$\triangleq I_1 + I_2. \tag{6.4}$$

First, integrating by parts and using (4.2), the left hand side of (6.4) is bigger than

$$\left( \frac{1}{2} - C_0 \delta \right) \sum_{1 \leq |i| \leq 8} \left\| \theta' \langle D_x \rangle \frac{1}{2} \partial_x^i (\nabla p)_{\phi} \right\|_{L^2}^2 - C_0 \delta \left\| (\partial_x p)_{\phi} \right\|_{H_{tan}^7}^2 .$$

We get by integration by parts and Lemma 5.2 that

$$I_1 \leq C \left( \left\| F_{\phi} \right\|_{H_{tan}^4}^2 + \left\| G_{\phi} \right\|_{H_{tan}^4}^2 \right) + \frac{1}{10} \sum_{1 \leq |i| \leq 8} \left\| \theta' \langle D_x \rangle \frac{1}{2} \partial_x^i (\nabla p)_{\phi} \right\|_{L^2}^2 ,$$

$$I_2 \leq C \varepsilon^4 + \frac{1}{10} \sum_{1 \leq |i| \leq 8} \left\| \theta' \langle D_x \rangle \frac{1}{2} \partial_x^i (\nabla p)_{\phi} \right\|_{L^2}^2 .$$

Thus, collecting the above estimates and fixing $\delta$ small, we arrive at

$$\sum_{1 \leq |i| \leq 8} \left\| \theta' \langle D_x \rangle \frac{1}{2} \partial_x^i (\nabla p)_{\phi} \right\|_{L^2}^2 \leq C \left( \left\| F_{\phi} \right\|_{H_{tan}^4}^2 + \left\| G_{\phi} \right\|_{H_{tan}^4}^2 \right) + C_0 \delta \left\| (\partial_x p)_{\phi} \right\|_{H_{tan}^7}^2 + C \varepsilon^4 .$$

Moreover, we get by (4.2) that

$$\left\| \theta' \langle D_x \rangle \frac{1}{2} (\nabla p)_{\phi} \right\|_{L^2(\mathbb{R}_+^3)}^2 \leq C_0 \delta \left\| (\nabla p)_{\phi} \right\|_{L^2}^2 \leq C_0 \delta \left\| (\nabla p)_{\phi} \right\|_{H_{tan}^7}^2 .$$

Thus, we arrive at

$$\left\| \theta' (\nabla p)_{\phi} \right\|_{H_{tan}^7}^2 \leq C \left\| (F,G)_{\phi} \right\|_{H_{tan}^4}^2 + C_0 \delta \left\| (\nabla p)_{\phi} \right\|_{H_{tan}^7}^2 + C \varepsilon^4 . \tag{6.5}$$
Acting $|D_x|^\frac{1}{2} \partial_x^i$ on both sides of (6.3), then taking $L^2$ inner with $|D_x|^\frac{1}{2} \partial_x^i \phi$ and summing over $|i| \leq 7$, we obtain

$$
\sum_{|i| \leq 7} \left< |D_x|^\frac{1}{2} \partial_x^i \Delta_x p - |D_x|^\frac{1}{2} \partial_x^i \partial_y (\partial_y p) \frac{1}{2} \partial_x^i (\partial_y p) - \langle D_x \rangle \theta' \langle D_x \rangle \frac{1}{2} \partial_x^i (\partial_y p) \phi, |D_x|^\frac{1}{2} \partial_x^i p \phi \right>
$$

$$
= \sum_{|i| \leq 7} \left< |D_x|^\frac{1}{2} \partial_x^i \left[ \nabla_x \cdot F_\Phi + \partial_y G_\Phi + \langle D_x \rangle \theta' G_\Phi \right], |D_x|^\frac{1}{2} \partial_x^i p \phi \right>
$$

$$
+ \sum_{|i| \leq 7} \left< |D_x|^\frac{1}{2} \partial_x^i \left[ - \nabla_x \cdot (R_h) \phi - \partial_y (R_v) \phi - \langle D_x \rangle \theta' (R_v) \phi \right], |D_x|^\frac{1}{2} \partial_x^i p \phi \right>.
$$

Integrating by parts and using $\partial_y v = -\nabla_x \cdot u$, the left hand side of (6.6) is bigger than

$$
(1 - C_0 \delta) \| (\nabla p) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} + C \| \nabla p \|^2_{L^2(\mathbb{R}^2)} + \sum_{|i| \leq 7} \int_{\mathbb{R}^2} |D_x|^\frac{1}{2} \partial_x^i p \phi |D_x|^\frac{1}{2} \partial_x^i (R_v) \phi(t, x, 0) dx
$$

$$
+ \sum_{|i| \leq 7} \int_{\mathbb{R}^2} |D_x|^\frac{1}{2} \partial_x^i p \phi |D_x|^\frac{1}{2} \partial_x^i (-\varepsilon^2 (\partial_y \nabla_x \cdot u) \Phi) + \varepsilon^2 (\partial_t f) - \varepsilon^4 \Delta_x f \phi)(t, x, 0) dx.
$$

Recalling that $f = \partial_x \int_{-\infty}^0 u_1(t, x, y) dy$, by Lemma 5.1 we get

$$
\sum_{|i| \leq 7} \left| \int_{\mathbb{R}^2} |D_x|^\frac{1}{2} \partial_x^i p \phi (|D_x|^\frac{1}{2} \partial_x^i (\varepsilon^2 (\partial_t f) - \varepsilon^4 \Delta_x f \phi))(t, x, 0) dx \right|
$$

$$
\leq \frac{1}{4} \left( \| (\nabla p) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} + C \varepsilon^4 \right),
$$

and using $\partial_y u = w^i + \nabla_x v,$

$$
\sum_{|i| \leq 7} \left| \int_{\mathbb{R}^2} |D_x|^\frac{1}{2} \partial_x^i p \phi (|D_x|^\frac{1}{2} \partial_x^i (\varepsilon^2 (\partial_t f) - \varepsilon^4 \Delta_x f \phi))(t, x, 0) dx \right|
$$

$$
= \frac{1}{4} \left( \| (\nabla p) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} + C \varepsilon^4 \right),
$$

Thus, fixing $\delta$ small, the left hand side of (6.6) is bigger than

$$
\frac{1}{2} \left( \| (\nabla p) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} - C_0 \| \nabla p \|^2_{L^2(\mathbb{R}^2)} + \sum_{|i| \leq 7} \int_{\mathbb{R}^2} |D_x|^\frac{1}{2} \partial_x^i p \phi |D_x|^\frac{1}{2} \partial_x^i R_v(t, x, 0) dx \right.
$$

$$
- \frac{1}{4} \left( \| (\nabla p) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} + C \varepsilon^4 \right).
$$

Integrating by parts and using Lemma 5.2, the right hand side of (6.6) can be bounded by

$$
\frac{1}{4} \left( \| (\nabla p) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} + \sum_{|i| \leq 7} \int_{\mathbb{R}^2} |D_x|^\frac{1}{2} \partial_x^i p \phi |D_x|^\frac{1}{2} \partial_x^i R_v(t, x, 0) dx + C \varepsilon^4 + C \| (F, G) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}}.
$$

Therefore, we arrive at

$$
\| (\nabla p) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} \leq C \| (F, G) \phi \|^2_{H^{\frac{7}{4}}_{\text{tan}}} + C_0 \varepsilon^4 \| (\partial_y + |D_x|) w \phi \|^2_{H^{\frac{7}{2}}_{\text{tan}}}
$$

$$
+ C \varepsilon^4 \left( \| w \phi \|^2_{H^{\frac{7}{2}}_{\text{tan}}} + \| U \phi \|^2_{H^{\frac{7}{2}}_{\text{tan}}} + 1 \right) + C_0 \| \nabla p \|^2_{L^2(\mathbb{R}^2)}.
$$

(6.8)
Following the proof of (6.8), we have
\[ \|\nabla p\|_{L^2(\mathbb{R}^3)}^2 \leq C \| (F, G) \|_{L^2}^2 + C_0 \varepsilon^4 \| (\partial_y + |D_x|)w \|_{L^2}^2 + C_0 \varepsilon^4 \| w \|_{H^1_{co}}^2 + \| U \|_{H^2_{tan}}^2 + 1 , \] 
(6.9)
Putting (6.8) and (6.9) into (6.5), we obtain
\[ \| \partial_x (\nabla p) \phi \|_{H^1_{tan}}^2 \leq C \left( \left\| (F, G) \phi \right\|_{H^s_{tan}(0, \varepsilon^2(t))}^2 + \left\| (F, G) \right\|_{H^s_{tan}}^2 \right) + C_0 \varepsilon^4 \| (\partial_y + |D_x|)w \|_{L^2}^2 + C_0 \varepsilon^4 \| (\partial_y + |D_x|)w \|_{H^1_{co}}^2 + C \varepsilon^4 \| (\partial_y + |D_x|)w \|_{H^1_{co}}^2 , \] 
which along with Lemma 6.1 and Lemma 6.2 gives our result.

6.3. Tangential Sobolev estimate of the pressure.

Lemma 6.4. It holds that
\[ \| \nabla p \|_{H^1_{co}}^2 \leq C \varepsilon^2 (E(t) + \varepsilon^2) (1 + E(t)) + C_0 \varepsilon^4 \| (\partial_y + |D_x|)w \|_{H^1_{co}}^2 . \]

Proof. Acting \( \partial_x \) on both sides of (6.11), and then taking \( L^2 \) inner product with \( \partial_x p \), summing over all \( |i| \leq 7 \), we arrive at
\[ - \sum_{|i| \leq 7} \langle \partial_x^i \Delta p, \partial_x^i p \rangle = \sum_{|i| \leq 7} \langle \partial_x^i (\nabla_x \cdot F + \partial_y G) - \partial_x^i (\nabla_x \cdot R_b + \partial_y R_v), \partial_x^i p \rangle . \] 
(6.10)
By integrating by parts, the left hand side of (6.10) is bigger than
\[ \| \nabla p \|_{H^1_{tan}}^2 + \sum_{|i| \leq 7} \int_{\mathbb{R}^2} \partial_x^i p \partial_x^i R_v (t, x, 0) dx + \sum_{|i| \leq 7} \int_{\mathbb{R}^2} \partial_x^i p (\varepsilon^2 \partial_x^i \partial_y v + \varepsilon^2 \partial_x^i f - \varepsilon^4 \Delta_x \partial_x^i f) (t, x, 0) dx . \]
Recalling that \( f(t, x) = \partial_x \int_0^{+\infty} u_p^{(1)} (t, x, y) dy \) and by Lemma 5.1, we have
\[ \sum_{|i| \leq 7} \left| \int_{\mathbb{R}^2} \partial_x^i p (\varepsilon^2 \partial_x^i \partial_y v + \varepsilon^2 \partial_x^i f - \varepsilon^4 \Delta_x \partial_x^i f) (t, x, 0) dx \right| \leq \frac{1}{4} \| \nabla p \|_{H^1_{tan}}^2 + C \varepsilon^4 . \]
Using \( \partial_y v = -\nabla_x \cdot u \) and \( \partial_y w = w^1_{b} + \nabla_x v \), we get
\[ \sum_{|i| \leq 7} \left| \int_{\mathbb{R}^2} \partial_x^i p (\varepsilon^2 \partial_x^i \partial_y u) (t, x, 0) dx \right| \leq \sum_{|i| \leq 7} \left| \int_{\mathbb{R}^3} \partial_x^i p (\varepsilon^2 \partial_x^i \partial_y u) (t, x, y) dx dy \right| \leq \frac{1}{4} \| \nabla p \|_{H^1_{tan}}^2 + C_0 \varepsilon^4 \left( \| \partial_y w \|_{H^1_{co}}^2 + \| U \|_{H^2_{co}}^2 \right) . \]
So, the left hand side of (6.11) is bigger than
\[ \frac{1}{2} \| \nabla p \|_{H^1_{tan}}^2 + \sum_{|i| \leq 7} \int_{\mathbb{R}^2} \partial_x^i p \partial_x^i R_v (t, x, 0) dx - C_0 \varepsilon^4 \left( 1 + \| \partial_y w \|_{H^1_{co}}^2 + \| U \|_{H^2_{co}}^2 \right) . \]
By integrating by parts, the right hand side of (6.11) can be bounded by
\[ \frac{1}{4} \| \nabla p \|_{H^1_{tan}}^2 + C \| (F, G) \|_{H^1_{tan}}^2 + C \varepsilon^4 + \sum_{|i| \leq 7} \int_{\mathbb{R}^2} \partial_x^i p \partial_x^i R_v (t, x, 0) dx . \]
Thus, we arrive at
\[ \| \nabla p \|_{H^1_{tan}}^2 \leq C \| (F, G) \|_{H^1_{tan}}^2 + C \varepsilon^4 \left( 1 + \| w \|_{H^1_{co}}^2 + \| U \|_{H^2_{co}}^2 \right) + C_0 \varepsilon^4 \left( \| \partial_y + |D| \right) \| \|_{H^1_{co}}^2 , \]
which along with Lemma 6.2 gives our result.
7. Tangential Analytic Type Estimate of the Velocity

In this section, we make tangential analytic type estimates for the velocity. In what follows, we always assume \( t \in [0, \min(T_0, T_\alpha)] \).

**Proposition 7.1.** There exists \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), there holds

\[
\frac{1}{2} \frac{d}{dt} \| U_\Phi \|_{H^1_{\text{tan}}}^2 + \lambda \| U_\Phi \|_{H^{1/2}_{\text{tan}}(0, y(t))}^2 + \frac{\varepsilon^2}{2} \| (\nabla U)\Phi \|_{H^1_{\text{tan}}}^2 \\
\leq C \varepsilon^2 (E(t) + K(t) + \varepsilon^2) (1 + E(t)) \\
+ \frac{\varepsilon^4}{100} \left( \| (\partial_y + |D_x|) w \|_{H^1_{\text{tan}}}^2 + \| (\partial_y + |D_x|) w \|_{H^1_{\text{tan}}}^2 \right).
\]

**Proof.** Acting \( \partial^i_x e^\Phi \) on both sides of (3.2), taking \( L^2 \) inner product with \( \partial^i_x U_\Phi \), and then summing over all \( |i| \leq 9 \), we arrive at

\[
\sum_{|i| \leq 9} \langle \partial^i_x (\partial_i U)_\Phi, \partial^i_x U_\Phi \rangle - \varepsilon^2 \sum_{|i| \leq 9} \langle \partial^i_x (\Delta U)_\Phi, \partial^i_x U_\Phi \rangle \triangleq I_0 \\
\leq \left| \sum_{|i| \leq 9} \langle \partial^i_x (\bar{U}_a \cdot \nabla U)_\Phi, \partial^i_x U_\Phi \rangle \right| + \left| \sum_{|i| \leq 9} \langle \partial^i_x (\bar{U} \cdot \nabla U_a)_\Phi, \partial^i_x U_\Phi \rangle \right| \\
+ \left| \sum_{|i| \leq 9} \langle \partial^i_x (\bar{U} \cdot \nabla U_a)_\Phi, \partial^i_x U_\Phi \rangle \right| + \left| \sum_{|i| \leq 9} \langle \partial^i_x (\nabla p)_\Phi, \partial^i_x U_\Phi \rangle \right| \\
+ \left| \sum_{|i| \leq 9} \langle \partial^i_x (R_h, R_v)_\Phi, \partial^i_x U_\Phi \rangle \right| \triangleq \sum_{i=1}^5 I_i.
\]

Let us now handle them term by term.

**Step 1. Estimate of \( I_0 \)**

We get by integration by parts that

\[
\sum_{|i| \leq 9} \langle \partial^i_x (\partial_i U)_\Phi, \partial^i_x U_\Phi \rangle = \sum_{|i| \leq 9} \langle \partial^i_x (\partial_i U_\Phi) - \partial^i_x \partial_i \Phi U_\Phi, \partial^i_x U_\Phi \rangle \\
\geq \frac{1}{2} \frac{d}{dt} \| U_\Phi \|_{H^1_{\text{tan}}}^2 + \lambda \| U_\Phi \|_{H^{1/2}_{\text{tan}}(0, y(t))}^2.
\]

Using \( u(t, x, 0) = 0, \partial_y v|_{y=0} = -\nabla_x \cdot u|_{y=0} = 0 \) and \( |\theta'(y)| \leq C_0 \delta \), we deduce that

\[
-\varepsilon^2 \sum_{|i| \leq 9} \langle \partial^i_x (\Delta U)_\Phi, \partial^i_x U_\Phi \rangle = -2\varepsilon^2 \langle \theta'(\partial_y U)_\Phi, (D_x U)\Phi \rangle_{H^1_{\text{tan}}} + \varepsilon^2 \| (\nabla U)\Phi \|_{H^1_{\text{tan}}}^2 \\
\geq \frac{\varepsilon^2}{2} \| (\nabla U)\Phi \|_{H^1_{\text{tan}}}^2 - C \varepsilon^2 \| U_\Phi \|_{H^1_{\text{tan}}}^2
\]

for small \( \delta \). Thus, we obtain

\[
I_0 \geq \frac{1}{2} \frac{d}{dt} \| U_\Phi \|_{H^1_{\text{tan}}}^2 + \lambda \| U_\Phi \|_{H^{1/2}_{\text{tan}}(0, y(t))}^2 + \frac{\varepsilon^2}{2} \| (\nabla U)\Phi \|_{H^1_{\text{tan}}}^2 - C \varepsilon^2 \| U_\Phi \|_{H^1_{\text{tan}}}^2.
\]

**Step 2. Estimate of \( I_1 \)**

First of all, we deal with \( \left| \langle \bar{U}_a \cdot \nabla u \rangle \Phi, u_\Phi \right|_{H^1_{\text{tan}}} \), which can be controlled by

\[
\sum_{|i| \leq 9} \left| \int_{y(t)}^{y(t)} \langle \partial^i_x (\bar{U}_a \cdot \nabla u)_\Phi, \partial^i_x u_\Phi \rangle_{L^2} dy \right| + \sum_{|i| \leq 9} \left| \int_{y(t)}^{y(t)} \langle \partial^i_x (\bar{U}_a \cdot \nabla u)_\Phi, \partial^i_x u_\Phi \rangle_{L^2} dy \right| \triangleq I_{11} + I_{12}.
\]
It follows from the first inequality of Lemma 4.2 and Lemma 5.1 that
\[ \sum_{|i| \leq 9} \left| \int_0^{y(t)} \langle \partial_x^i (u_a \partial_x u \phi), \partial_x^i u \phi \rangle \right| dy \leq C \left\| u \phi \right\|_{H^2_{tan}((0,y(t)))}^2. \]

Using \( \partial_y u = w_h^+ + \nabla_x v \) and \( (v_a - \varepsilon^2 f(t,x)e^{-y})|_{y=0} = 0 \), the same argument as above gives
\[ \sum_{|i| \leq 9} \left| \int_0^{y(t)} \langle \partial_x^i ((v_a - \varepsilon^2 f(t,x)e^{-y}) \partial_y u \phi), \partial_x^i u \phi \rangle \right| dy \]
\[ = \sum_{|i| \leq 9} \left| \int_0^{y(t)} \langle \partial_x^i \left( \frac{(v_a - \varepsilon^2 f(t,x)e^{-y})}{\varphi} (\varphi w_h^+ + \varphi \partial_y v) \phi, \partial_x^i u \phi \right) \right| dy \]
\[ \leq C \left( \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| \varphi w \right\|_{H^2_{tan}((0,y(t)))}^2 \right). \]

So, we get
\[ I_{11} \leq C \left( \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| \varphi w \right\|_{H^2_{tan}((0,y(t)))}^2 \right). \]

Thanks to \( \phi(t,y) \leq 0 \) for \( y \geq y(t) \), \( I_{12} \) can be controlled by
\[ C \sum_{|i| \leq 9} \left| \int_0^{y(t)} \langle \partial_x^i (u_a \partial_x u + (v_a - \varepsilon^2 f(t,x)e^{-y}) \partial_y u), \partial_x^i u \phi \rangle \right| dy \]
\[ \leq C \left( \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| \varphi w \right\|_{H^2_{tan}((0,y(t)))}^2 \right). \]

Here we used \( \partial_y u = w_h^+ + \nabla_x v \) again.

Collecting the above estimates, we obtain
\[ \left| \langle \tilde{U}_a \cdot \nabla u \phi, u \phi \rangle \right| \]
\[ \leq C \left( \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| \varphi w \right\|_{H^2_{tan}((0,y(t)))}^2 \right) \]
\[ \leq C \left( \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| \varphi w \right\|_{H^2_{tan}((0,y(t)))}^2 \right). \]

Using the first inequality of Lemma 4.2, \( \partial_y v = -\nabla_x \cdot u \) and Lemma 5.1, similar argument as above gives
\[ \left| \langle \tilde{U}_a \cdot \nabla v \phi, v \phi \rangle \right| \leq C \left( \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 \right). \]

This shows that
\[ I_1 \leq C \left( \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| \varphi w \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| U \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| \varphi w \right\|_{H^2_{tan}((0,y(t)))}^2 \right) \]
\[ \text{Step 3. Estimate of } I_2. \]

Using Lemma 4.1 and Lemma 5.1, it is easy to deduce that
\[ \left| \langle (u \partial_x u_a) \phi, u \phi \rangle \right| \leq C \left( \left\| u \phi \right\|_{H^2_{tan}((0,y(t)))}^2 + \left\| u \right\|_{H^2_{tan}((0,y(t)))}^2 \right) \]

On the other hand, we have
\[ \sum_{|i| \leq 9} \left| \int_0^{y(t)} \langle \partial_x^i ((v + \varepsilon^2 f(t,x)e^{-y}) \partial_y u_a \phi), \partial_x^i u \phi \rangle \right| dy \]
\[ \leq \sum_{|i| \leq 9} \left| \int_0^{y(t)} \langle \partial_x^i ((v + \varepsilon^2 f(t,x)e^{-y}) \partial_y u_a) \phi, \partial_x^i u \phi \rangle \right| L^2 dy \]
\[ + \sum_{|i| \leq 9} \left| \int_{y(t)}^{+\infty} \langle \partial_x^i ((v + \varepsilon^2 f(t,x)e^{-y}) \partial_y u_a) \phi, \partial_x^i u \phi \rangle \right| L^2 dy \triangleq I_{21} + I_{22}. \]
Using the fact that \( \|\partial_x^i \partial_y^j u_a\|_{L^\infty(\mathbb{R}^2 \times (y(t), \infty))} \leq C \) due to \( y(t) \geq c_0 \), it is easy to get
\[
I_{22} \leq C(\|U\|_{H_{tan}^2}^2 + \varepsilon^4).
\]
Thanks to \( (v + \varepsilon^2 f(t, x)e^{-y})|_{y=0} = 0 \), \( \partial_y v = -\nabla_x \cdot u \), Hardy's inequality and Lemma 5.1, we arrive at
\[
I_{21} \leq \sum_{|i| \leq 9} \int_0^{y(t)} \left\langle \partial_x^i \left( \frac{1}{y} \int_0^y \left( -\text{div}_x u(y, t) - \varepsilon^2 f(t, x)e^{-y} \right) dy' \right) \partial_x^j u \right\rangle_{L^2_y} \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \varepsilon^4\right).
\]
This shows that
\[
\left\langle \left( \frac{U_t}{\Phi} \right) \Phi, u \right\rangle_{H_{tan}^2} \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \|U\|_{H_{tan}^2} \right).
\]
Using the fact that \( \|\partial_x^i \partial_y^j u_a\|_{L^\infty} \leq C \), it is easy to deduce that
\[
\left\langle \left( \frac{U_t}{\Phi} \right) \Phi, u \right\rangle_{H_{tan}^2} \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \|U\|_{H_{tan}^2} \right).
\]
Thus, we obtain
\[
I_2 \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \|U\|_{H_{tan}^2} \right).
\]
**Step 4. Estimate of \( I_3 \).**

We first consider \( \left\langle \left( \frac{U}{\Phi} \right) \Phi, u \right\rangle_{H_{tan}^2} \), which can be bounded by
\[
\sum_{|i| \leq 9} \int_0^{y(t)} \left\langle \partial_x^i \Phi, \partial_x^j u \right\rangle_{L^2_y} \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \|U\|_{H_{tan}^2} \right).
\]
Using the interpolation inequality \( \|g\|_{L^\infty} \leq C\|g\|_{L^2_y}^{\beta} \|\partial_y g\|_{L^2_y}^{\frac{1}{\beta}} \) and \( \partial_y u = w^1 + \nabla_x v \), we deduce that
\[
\sum_{|i| \leq 9} \int_0^{y(t)} \left\langle \left( \frac{\partial_x^i (u \cdot \nabla_x u) \Phi, \partial_x^j u \right\rangle_{L^2_y} \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \|U\|_{H_{tan}^2} \right)
\]
and
\[
\sum_{|i| \leq 9} \int_0^{y(t)} \left\langle \left( \frac{\partial_x^i (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y u \Phi, \partial_x^j u \right\rangle_{L^2_y} \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \|U\|_{H_{tan}^2} \right)
\]
from which, we infer that
\[
I_{32} \leq C\left(\|U\|_{H_{tan}^2}^2 + \varepsilon^4\right)(\|w\|_{H_{tan}^2} + \|U\|_{H_{tan}^2}^2).
\]
For \( I_{31} \), using the third inequality of Lemma 4.2 and \( \partial_y u = w^1 + \nabla_x v \), we obtain
\[
\sum_{|i| \leq 9} \int_0^{y(t)} \left\langle \partial_x^i (u \partial_x u) \Phi, \partial_x^j u \right\rangle_{L^2_y} \leq C\left(\|\Phi\|_{H_{tan}^2}^2 + \|w\|_{H_{tan}^2}^2\right),
\]
and by Lemma 5.2 and \( \partial_y u = w_t^+ + \nabla_x v \) and \( \partial_y v = -\nabla_x \cdot u \),

\[
\sum_{|i| \leq 9} \left| \int_0^{y(t)} \langle \partial_i^x ((v + \varepsilon^2 f(t,x)e^{-y})\partial_y u)_{\Phi}, \partial_i^x u_{\Phi} \rangle_{L^2(y)} dy \right|
\]

\[
\leq C \left\langle (D_x)^{-\frac{1}{2}}((v + \varepsilon^2 f(t,x)e^{-y})\partial_y u)_{\Phi}, \partial_x u_{\Phi} \right\rangle_{L^2(y)}^2 + C\|u_{\Phi}\|_{H^\frac{9}{2\alpha}(0,y(t))}^2
\]

\[
\leq C \left( \|U_{\Phi}\|_{H^\frac{9}{2\alpha}(0,y(t))}^2 + \varepsilon^4 \right) \left( \|w_{\Phi}\|_{H^{s_9}(0,y(t))}^2 + \|v_{\Phi}\|_{H^{n_9}(0,y(t))}^2 \right)
\]

\[
+ C \left( \|U_{\Phi}\|_{H_{\alpha}(0,y(t))}^2 + \varepsilon^4 \right) \left( \|w_{\Phi}\|_{H^{s_9}(0,y(t))}^2 + \|v_{\Phi}\|_{H^{n_9}(0,y(t))}^2 \right) + C\|u_{\Phi}\|_{H^\frac{9}{2\alpha}(0,y(t))}^2.
\]

Similarly, \( \left\langle (\tilde{U} \cdot \nabla v)_{\Phi}, v_{\Phi} \right\rangle_{H^\frac{9}{2\alpha}(0,y(t))} \) can be controlled by

\[
\sum_{|i| \leq 9} \int_0^{y(t)} \langle \partial_i^x (\tilde{U} \cdot \nabla v)_{\Phi}, \partial_i^x v_{\Phi} \rangle_{L^2} dy + \sum_{|i| \leq 9} \int_{y(t)}^{+\infty} \langle \partial_i^x (\tilde{U} \cdot \nabla v)_{\Phi}, \partial_i^x v_{\Phi} \rangle dy \triangleq I_{33} + I_{34}.
\]

Similar arguments as \( I_{32} \) give

\[
I_{34} \leq C \left( \|U\|_{H^1_{\alpha}(0,y(t))}^2 + \varepsilon^4 \right) \left( \|U\|_{H^2_{\alpha}(0,y(t))} + \|w\|_{H^2_{\alpha}(0,y(t))} \right).
\]

Using \( \partial_y v = -\nabla_x \cdot u \) and the third inequality of Lemma 5.2, we obtain

\[
I_{33} \leq C \|U_{\Phi}\|_{H^\frac{9}{2\alpha}(0,y(t))}^2 \left( 1 + \|U_{\Phi}\|_{H^{n_9}(0,y(t))}^2 + \|w_{\Phi}\|_{H^{s_9}(0,y(t))}^2 \right).
\]

Summing up the estimates of \( I_{31} - I_{34} \), we deduce that

\[
I_3 \leq C\varepsilon^2 (1 + E(t)) \left( \varepsilon^2 + E(t) + K(t) \right).
\]

**Step 5. Estimate of \( I_4 \).**

Using \( \partial_y v = -\nabla_x \cdot u \) and integration by parts, we arrive at

\[
I_4 \leq \sum_{1 \leq |i| \leq 9} \left| \int_{\mathbb{R}^d} \langle D_x \rangle^\frac{1}{2} \partial_i^x p_{\Phi} \theta' \langle D_x \rangle^\frac{1}{2} \partial_i^x v_{\Phi} dx \right|
\]

\[
+ \sum_{1 \leq |i| \leq 9} \left| \int_{\mathbb{R}^d} \partial_i^x p_{\Phi} \partial_i^x v_{\Phi}(x,0) dx \right| + C_0 \delta \|\nabla p\|_{L^2} + C \|U_{\Phi}\|_{L^2}^2
\]

\[
\leq C_0 \left( \|\theta'(\partial_x p)_{\Phi}\|_{H^{n_9}(0,y(t))}^2 + \|U_{\Phi}\|_{H^{n_9}(0,y(t))}^2 \right)
\]

\[
+ C \varepsilon^4 + C_0 \delta \|\nabla p\|_{H^{n_9}(0,y(t))}^2 + C \|U_{\Phi}\|_{H^1_{\alpha}(0,y(t))}^2.
\]

Here we used \( v(t,x,0) = -\varepsilon^2 f(t,x) \).

**Step 6. Estimate of \( I_5 \).**

It follows from Lemma 5.2 that

\[
I_5 \leq C \|U_{\Phi}\|_{H^1_{\alpha}(0,y(t))}^2 + C\varepsilon^4.
\]

Putting the estimates of \( I_1 - I_5 \) together, we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|U_{\Phi}\|_{H^1_{\alpha}(0,y(t))}^2 + \lambda \|U_{\Phi}\|_{H^{n_9}(0,y(t))}^2 + \varepsilon^2 \|U_{\Phi}\|_{H^{n_9}(0,y(t))}^2 \leq C\varepsilon^2 (E(t) + K(t) + \varepsilon^2) (1 + E(t))
\]

\[
+ C_0 \delta \|\nabla p\|_{H^{n_9}(0,y(t))}^2 + C_0 \|\theta'(\partial_x p)_{\Phi}\|_{H^{n_9}(0,y(t))}^2,
\]

which along with Lemma 6.3 gives our result.
8. Tangential Sobolev estimates of velocity

In this section, we make the energy estimate of the velocity in tangential Sobolev space, which will be used to control the regularity of the velocity away from the boundary.

**Proposition 8.1.** There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there holds

$$\frac{1}{2} \frac{d}{dt} \|U\|_{H_{tan}^0}^2 + \varepsilon^2 \|\nabla U\|_{H_{tan}^0}^2 \leq C\varepsilon^4 (E(t) + \varepsilon^2)^{\frac{7}{2}}$$

$$+ C\varepsilon^2 (E(t) + 1)(E(t) + \varepsilon^2) + \frac{\varepsilon^4}{100} \|(\partial_y + \|D_x\|)w\|_{H_{co}^0}^2.$$ 

Proof. Acting $\partial_x^i$ on both sides of (E.22), and taking $L^2$ inner product with $\partial_x^i U$, then summing over $|i| \leq 10$, we arrive at

$$\frac{1}{2} \frac{d}{dt} \|U\|_{H_{tan}^0}^2 + \varepsilon^2 \|\nabla U\|_{H_{tan}^0}^2 \leq \sum_{|i|\leq 10} \langle \partial_x^i (\tilde{U}_a \cdot \nabla U), \partial_x^i U \rangle \bigg| + \sum_{|i|\leq 10} \langle \partial_x^i (\tilde{U} \cdot \nabla U), \partial_x^i U \rangle \bigg| \bigg| + \sum_{|i|\leq 10} \langle \partial_x^i (\nabla p), \partial_x^i U \rangle \bigg|$$

$$+ \sum_{|i|\leq 10} \langle \partial_x^i (R_h, R_v), \partial_x^i U \rangle \bigg| \triangleq \sum_{i=1}^5 I_i.$$ 

Let us now handle them term by term.

**Step 1. Estimate of $I_1$.**

The term $I_1$ is bounded by

$$\bigg| \sum_{|i|\leq 10} \langle \tilde{U}_a \cdot \nabla \partial_x^i U, \partial_x^i U \rangle \bigg| + \bigg| \sum_{|i|\leq 10, j<i} \langle \partial_x^{-i-j} \tilde{U}_a \cdot \nabla \partial_x^j U, \partial_x^j U \rangle \bigg|.$$

Due to $\tilde{U}_a|_{y=0} = 0$, integrating by parts and using Lemma 5.1, the first term can be controlled by $C\|U\|_{H_{tan}^0}^2.$ Using $\partial_y u = w_d^\perp + \nabla_x v, \partial_y v = -\nabla_x u \text{ and } (v_a - \varepsilon^2 f(t,x)e^{-y})|_{y=0} = 0$ and Lemma 5.1, the second term can be bounded by

$$C\bigg( \|U\|_{H_{tan}^0}^2 + \|\varphi w\|_{H_{co}^0}^2 \bigg),$$

where we used the fact that

$$\bigg| \sum_{|i|\leq 10, j<i} \langle \partial_x^{-i-j}(v_a - \varepsilon^2 f(t,x)e^{-y})\partial_x^j w, \partial_x^j U \rangle \bigg|$$

$$= \bigg| \sum_{|i|\leq 10, j<i} \frac{\langle \partial_x^{-i-j}(v_a - \varepsilon^2 f(t,x)e^{-y})\varphi, \partial_x^j U \rangle}{\varphi} \bigg| \leq C\|\varphi w\|_{H_{co}^0}^2.$$ 

This shows that

$$I_1 \leq C\bigg( \|U\|_{H_{tan}^0}^2 + \|\varphi w\|_{H_{co}^0}^2 \bigg).$$

**Step 2. Estimate of $I_2$.**

It is easy to deduce from Lemma 5.1 that

$$\bigg| \sum_{|i|\leq 10} \langle \partial_x^i (u \partial_x U_a + (v + \varepsilon^2 f(t,x)e^{-y})\partial_y (U_{a,e} + \varepsilon v_{a,p} e_3), \partial_x^i U \rangle \bigg| \leq C\|U\|_{H_{tan}^0}^2,$$
where \( e_3 = (0, 0, 1) \). On the other hand, we have

\[
\left| \sum_{|i| \leq 10} \left( \partial_x^i ((v + \varepsilon^2 f(t, x)e^{-y}) \partial_y u_{a,p}) \right), \partial_x^i u \right|
\]

\[
= \left| \sum_{|i| \leq 10} \int^{+\infty}_{0} \left( \partial_x^i ((v + \varepsilon^2 f(t, x)e^{-y}) \partial_y u_{a,p}) \right), \partial_x^i u \right|
\]

\[
+ \left| \sum_{|i| \leq 10} \int^{y(t)}_{0} \left( \partial_x^i ((v + \varepsilon^2 f(t, x)e^{-y}) \partial_y u_{a,p}) \right), \partial_x^i u \right|
\]

Using the fact that \( \|\partial_x^i \partial_y u_{a,p}\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \infty)} \leq C \) due to \( y(t) \geq c_0 \), the first term can be bounded by

\[
C \left( \|U\|^2_{H^{10}_{\text{tan}}} + \varepsilon^4 \right).
\]

Thanks to \( (v + \varepsilon^2 f(t, x)e^{-y})|_{y=0} = 0, \partial_y v = -\nabla_x \cdot u \), Hardy’s inequality and Lemma 5.11 the second term can be bounded by

\[
\sum_{|i| \leq 10} \int^{y(t)}_{0} \left| \partial_x^i \left( \frac{1}{y} \int^{y}_{0} \left(-\text{div}_x u(x, y') - \varepsilon^2 f(t, x)e^{-y'} \right) dy' \left(z\partial_z u_{a,p}\right) \right), \partial_x^i U \right|_{L^2_y}(y) dy
\]

\[
\leq C \left( \|U\|^2_{H^{9}_{\text{tan}}} + \varepsilon^4 \right),
\]

where we used the fact that

\[
\sum_{|i| \leq 11} \int_{\mathbb{R}^2} \int^{y(t)}_{0} \left| \partial_x^i U \right|^2 dxdy \leq \sum_{|i| \leq 9} \int_{\mathbb{R}^2} \int^{y(t)}_{0} \left| \partial_x^i U \right|^2 dxdy \leq C \left( \|U\|^2_{H^{9}_{\text{tan}}} \right),
\]

due to \( \phi(t, y) \geq c_0 > 0 \) for \( y \leq \frac{y(t)}{2} \).

Summing up, we arrive at

\[
I_2 \leq C \left( \|U\|^2_{H^{9}_{\text{tan}}} + \|U\|^2_{H^{10}_{\text{tan}}} + \varepsilon^4 \right).
\]

**Step 3. Estimate of \( I_3 \).**

By Sobolev embedding and \( \partial_y u = w^+_{h} + \nabla_x v \), we deduce that

\[
\sum_{|i| \leq 10, j < 1} \int^{+\infty}_{0} \left| \left( \partial_x^{i-j} u \partial_y \partial_x^j u, \partial_x^i u \right)_{L^2_x} \right| dy
\]

\[
\leq C \|u\|^2_{H^{10}_{\text{tan}}} \|u\|_{L^\infty H^5_{\text{top}}(\mathbb{R}^2)} \leq C \|u\|^2_{H^{10}_{\text{tan}}} \left( \|w^+_{h}\|_{H^8_{\text{tan}}} + \partial_x v \right)^{\frac{3}{4}}_{H^8_{\text{tan}}}
\]

\[
\leq C \|u\|^2_{H^{10}_{\text{tan}}} \left( \|U\|_{H^{10}_{\text{tan}}} + \|w\|_{H^9_{\text{tan}}} + \|\partial_y u\|_{H^9_{\text{tan}}} \right),
\]

and

\[
\sum_{|i| \leq 10, j < 1} \int^{+\infty}_{0} \left| \left( \left( \partial_x^{i-j} (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y \partial_x^j u, \partial_x^i u \right)_{L^2_x} \right| dy
\]

\[
\leq C \|u\|_{H^{10}_{\text{tan}}} \left( \|v\|_{H^{10}_{\text{tan}}} + \varepsilon^2 \|\partial_y u\|_{L^\infty H^2_{\text{top}}} \right) + C \|u\|_{H^{10}_{\text{tan}}} \|\partial_y u\|_{H^9_{\text{tan}}} \|v + \varepsilon^2 f(t, x)e^{-y} \|_{L^\infty H^2_{\text{top}}}
\]

\[
\leq C \left( \|U\|^2_{H^{10}_{\text{tan}}} + \varepsilon^4 \right) \left( \|U\|_{H^{10}_{\text{tan}}} + \|w\|_{H^9_{\text{tan}}} + \|\partial_y u\|_{H^9_{\text{tan}}} \right).
\]
On the other hand, we get by integration by parts and Lemma 5.1 that
\[
\left| \sum_{|i| \leq 10} \langle \bar{u} \partial_x \partial_x^i U + (v + \varepsilon^2 f(t,x)e^{-y}) \partial_y \partial_y^i U, \partial_x^i U \rangle \right| \leq C(\|U\|_{H_{tan}^{10}}^2 + \varepsilon^4).
\]

Thus, we obtain
\[
I_3 \leq C(\|U\|_{H_{tan}^{10}}^2 + \varepsilon^4)(1 + \|U\|_{H_{tan}^{10}}^2 + \|w\|_{H_{co}^2}^2 + \|\partial_y w\|_{H_{co}^2}^2 + \|w\|_{H_{co}^2}^2).
\]

**Step 4. Estimate of \(I_4\) and \(I_5\).**

Using the divergence free condition and Lemma 5.1, integrating by parts, we obtain
\[
I_4 \leq \sum_{|i| \leq 10} \left| \int_{\mathbb{R}^2} \partial_x^i p \partial_x^i v(x,0) dx \right| \leq C \varepsilon^4 + \delta \|\nabla p\|_{H_{tan}^{10}}^2.
\]

Here we used \(v(t,x,0) = -\varepsilon^2 f(t,x)\).

It follows from Lemma 5.2 that
\[
I_5 \leq C(\|U\|_{H_{tan}^{10}}^2 + \varepsilon^4).
\]

Summing up the estimates of \(I_1 - I_5\), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \|U\|_{H_{tan}^{10}}^2 + \frac{\varepsilon^2}{2} \|\nabla U\|_{H_{tan}^{10}}^2 \leq C \varepsilon^4 (E_v(t) + \varepsilon^2 + E_w(t) + 1)(E(t) + \varepsilon^2) + \delta \|\nabla p\|_{H_{tan}^{10}}^2,
\]
which along with Lemma 5.3 gives our result. \(\Box\)

9. Tangential analytic estimate of the vorticity: Euler part

In this section, we make tangential analytic estimates for the Euler part \(w_e\) of the vorticity.

9.1. Tangential analytic estimate of \(w_e\). Using (3.3), we first observe that \((w_e)_\Phi\) satisfies
\[
\begin{align*}
\partial_t (w_e)_\Phi + \lambda(D_e)(w_e)_\Phi - \varepsilon^2(\Delta w_e)_\Phi + (\bar{U}_e \cdot \nabla w_e)_\Phi + (\bar{U}_e \cdot \nabla w_{a,e})_\Phi + (\bar{U}_e \cdot \nabla w_e)_\Phi \\
- (w_{a,e} \cdot \nabla U)_\Phi - (w_e \cdot \nabla U_e)_\Phi - (w_e \cdot \nabla U)_\Phi = (\text{curl}(R_{e,h}, R_{e,e}) - M_e)_\Phi
\end{align*}
\]

(9.1)

together with the following initial-boundary conditions
\[
\begin{cases}
-\varepsilon^2((\partial_y + |D_e|)w_{e,h})_\Phi(t,x,0) - \varepsilon^2 \partial_x (\Lambda ND(\gamma \nabla x \cdot w_{e,h}))_\Phi(t,x,0) = 0, \\
(w_{e,3})_\Phi(t,x,0) = 0, \\
(w_e)_\Phi(0,x,y) = 0.
\end{cases}
\]

**Proposition 9.1.** There exists \(\delta_0 > 0\) such that for any \(\delta \in (0, \delta_0)\), there holds
\[
\frac{1}{2} \frac{d}{dt} \|w_e\|_{H^2_e}^2 + \lambda \|w_e\|_{H^2_e}^2 + \frac{1}{4} \|\partial_y |D_e| w_e\|_{H^2_e(0,y(t))}^2 + \frac{\varepsilon^2}{10} \|((\partial_y + |D_e|)w_e)_\Phi\|_{H^2_e}^2
\]
\[
\leq C(E(t) + K(t) + \varepsilon^2)(1 + E(t)) + C \varepsilon^{-2} E(t)^{\frac{5}{2}} + \frac{\varepsilon^2}{100} \|((\partial_y + |D_e|)w_e)_\Phi\|_{H^2_e}^2.
\]
Proof. Acting $\partial_x^j Z^i$ on both sides of (9.1), taking $L^2$ inner product with $e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi$, and then summing over all $|i| + j \leq 8$, we arrive at

$$\frac{1}{2} \frac{d}{dt} \left\| (w_e)_\Phi \right\|_{H^0_e}^2 + \lambda \left\| (w_e)_\Phi \right\|_{H^{3/2}_e(0,y(t))}^2 \geq -\frac{2}{\epsilon^2} \sum_{|i| + j \leq 8} \langle \partial_x^j Z^i (\Delta w_e)_\Phi, e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \rangle$$

$$\leq \left| \sum_{|i| + j \leq 8} \langle \partial_x^j Z^i (\tilde{U}_a \cdot \nabla w_e)_\Phi, e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \rangle \right| + \left| \sum_{|i| + j \leq 8} \langle \partial_x^j Z^i (\tilde{U} \cdot \nabla w_{a,e})_\Phi, e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \rangle \right|$$

$$+ \left| \sum_{|i| + j \leq 8} \langle \partial_x^j Z^i (\tilde{U}_a \cdot \nabla U_{a,e})_\Phi, e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \rangle \right| + \left| \sum_{|i| + j \leq 8} \langle \partial_x^j Z^i (\tilde{U}_e \cdot \nabla U_{e,n})_\Phi, e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \rangle \right|$$

Hence, we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| (w_e)_\Phi \right\|_{H^0_e}^2 + \lambda \left\| (w_e)_\Phi \right\|_{H^{3/2}_e(0,y(t))}^2 \geq -\frac{2}{\epsilon^2} \sum_{|i| + j \leq 8} \langle \partial_x^j Z^i (\Delta w_e)_\Phi, e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \rangle$$

Step 1. Estimate of $I_1$.

For any $|i| + j \leq 8$, there holds

$$\langle \partial_x^j Z^i (\tilde{U}_a \cdot \nabla w_e)_\Phi, e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \rangle = \int_0^{y(t)} \int_{R^2} \partial_x^j Z^i (\tilde{U}_a \cdot \nabla w_e)_\Phi e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \, dx \, dy$$

$$+ \int_{y(t)}^{+\infty} \int_{R^2} \partial_x^j Z^i (\tilde{U}_a \cdot \nabla w_e)_\Phi e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \, dx \, dy$$

Thanks to $\phi(t, y) \leq 0$ and $\varphi(y) \geq c\delta$ for $y \geq y(t)$, we get by Lemma 5.1 that

$$I_{12} \leq C \left\| w_e \right\|_{H^{3/2}_e}^2.$$

For $I_{11}$, by Lemma 5.1 and the first inequality of Lemma 4.3, we have

$$\left| \int_0^{y(t)} \int_{R^2} \partial_x^j Z^i (u_{a} \partial_x w_e - (v - \varepsilon^2 f(t, x)e^{-y}) |D_x| w_e)_\Phi e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \, dx \, dy \right| \leq C \left\| (w_e)_\Phi \right\|_{H^{3/2}_e(0,y(t))}^2,$$

and by Lemma 5.1, and the first inequality of Lemma 4.3 we get

$$\left| \int_0^{y(t)} \int_{R^2} \partial_x^j Z^i (\tilde{U}_e (\partial_y + |D_x|) w_e)_\Phi e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \, dx \, dy \right|$$

Hence, we obtain

$$I_{11} \leq C \left\| (w_e)_\Phi \right\|_{H^{3/2}_e(0,y(t))}^2 + C \left\| (w_e)_\Phi \right\|_{H^{3/2}_e(0,y(t))}^2 + \frac{\varepsilon^2}{100} \left\| \partial_y + |D_x| w_e \right\|_{H^{3/2}_e(0,y(t))}^2.$$

This shows that

$$I_1 \leq C \left( \left\| w_e \right\|_{H^2_e}^2 + \left\| (w_e)_\Phi \right\|_{H^{3/2}_e(0,y(t))}^2 \right) + \frac{\varepsilon^2}{100} \left( \left\| \partial_y + |D_x| w_e \right\|_{H^{3/2}_e(0,y(t))}^2 \right).$$

Step 2. Estimate of $I_2$ and $I_7$.

Using the fact that $w_{a,e} = 0$ for $y \leq y(t)$, it is easy to deduce that

$$I_2 = \left| \sum_{|i| + j \leq 8} \int_{y(t)}^{+\infty} \int_{R^2} \partial_x^j Z^i (u \cdot \nabla x w_{a,e} + (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y w_{a,e})_\Phi e^{2\Psi_e} \partial_x^j Z^i(w_e)_\Phi \, dx \, dy \right|$$

$$\leq C \left( \left\| w_e \right\|_{H^2_e}^2 + \left\| U \right\|_{H_{tan}^2}^2 + \left\| \varphi w \right\|_{H^{3/2}_e}^2 + \varepsilon^4 \right).$$
Here we used
\[ \|U\|_{H^0_{co}} \leq C (\|U\|_{H^0_{tan}} + \|\varphi w\|_{H^8_{co}}), \] (9.2)
which follows from \( Zu = \varphi w^\perp + \varphi \nabla_x v \) and \(-\partial_y v = \nabla_x \cdot u\).

It follows from the fact that \( w_{a,e} = 0 \) for \( y \leq y(t) \) and Lemma 5.2 that
\[ I_7 \leq C (\|w_e\|_{H^8_{co}}^2 + \epsilon^4). \]

**Step 3. Estimate of \( I_3 \) and \( I_6 \).**

Firstly, \( I_3 \) can be controlled by
\[
\left| \sum_{|i+j| \leq 8} \int_0^{y(t)} \int_{\mathbb{R}^2} \partial_x^i \partial_y^j (\bar{U} \cdot \nabla w_e) \varphi e^{2\varphi} \partial_x^i \partial_y^j (w_e) \Phi dx dy \right|
\]
\[+ \left| \sum_{|i+j| \leq 8} \int_{y(t)}^{+\infty} \int_{\mathbb{R}^2} \partial_x^i \partial_y^j (\bar{U} \cdot \nabla w_e) \varphi e^{2\varphi} \partial_x^i \partial_y^j (w_e) \Phi dx dy \right| \leq I_{31} + I_{32}. \]

As \( \phi(t, y) \leq 0 \) and \( \varphi(y) \geq c\delta \) for \( y \geq y(t) \), by (9.2), we get
\[ I_{32} \leq C \|w_e\|_{H^8_{co}}^2 \left( \epsilon^2 + \|U\|_{H^{10}_{tan}}^2 + \|\varphi w\|_{H^8_{co}}^2 \right). \]

Using the first inequality of Lemma 1.4, we obtain
\[
\left| \sum_{|i+j| \leq 8} \int_0^{y(t)} \int_{\mathbb{R}^2} \partial_x^i \partial_y^j (u \cdot \nabla w_e - (v + \epsilon^2 f(t,x)e^{-y})|D_x| w_e) \varphi e^{2\varphi} \partial_x^i \partial_y^j (w_e) \Phi dx dy \right|
\]
\[\leq C \left( \epsilon^2 + \|U\|_{H^0_{co}}^2 + \|w_e\|_{H^8_{co}}^2 \right) \|w_e\|_{H^{\frac{8}{2}}(0,y(t))}. \]

Here we used
\[ \|\partial_y U\|_{H^8_{co}} \leq C (\|w\|_{H^8_{co}} + \|U\|_{H^{10}_{tan}}), \] (9.3)
which can be deduced from \( Zu = \varphi w^\perp + \varphi \nabla_x v \) and \(-\partial_y v = \nabla_x \cdot u\).

On the other hand, we have
\[
\left| \sum_{|i+j| \leq 8} \int_0^{y(t)} \int_{\mathbb{R}^2} \partial_x^i \partial_y^j ((v + \epsilon^2 f(t,x)e^{-y})|\partial_y + |D_x|| w_e) \varphi e^{2\varphi} \partial_x^i \partial_y^j (w_e) \Phi dx dy \right|
\]
\[\leq \|w_e\|_{H^8_{co}} \|((v + \epsilon^2 f(t,x)e^{-y})|\partial_y + |D_x|| w_e) \varphi\|_{H^8_{co}} \leq \frac{\epsilon^2}{100} \|((\partial_y + |D_x|| w_e) \varphi\|_{H^8_{co}}^2 + C \epsilon^{-2} \left( \epsilon^4 + \|U\|_{H^{10}_{tan}}^2 + \|\varphi w\|_{H^8_{co}}^2 \right) \|w_e\|_{H^{8}_{co}}^2. \]

Summing up, we obtain
\[
I_3 \leq \frac{\epsilon^2}{100} \|((\partial_y + |D_x|| w_e) \varphi\|_{H^8_{co}}^2 + C \epsilon^{-2} \|w_e\|_{H^8_{co}}^2 \|U\|_{H^{10}_{tan}}^2 + \|\varphi w\|_{H^8_{co}}^2 \epsilon^4
\]
\[+ C \left( \epsilon^2 + \|U\|_{H^{10}_{tan}}^2 + \|w\|_{H^{8}_{co}}^2 \right) \|w_e\|_{H^{8}_{co}}^2 \|U\|_{H^{10}_{tan}}^2 \|\varphi w\|_{H^8_{co}}^2 \epsilon^4
\]
\[+ C \|w_e\|_{H^{8}_{co}} \left( \epsilon^2 + \|U\|_{H^{10}_{tan}}^2 + \|\varphi w\|_{H^8_{co}}^2 \right), \]
Similarly, $I_6$ can be controlled by

$$
\left| \sum_{|i|+j\leq 8} \int_{t_0}^{t(t)} \int_{\mathbb{R}^2} \partial_x^i Z^j (w_{e,h} \cdot \nabla_x U + w_{e,3} \partial_y U) \Phi e^{2\Psi} \partial_x^i Z^j (w_e) \Phi \, dx \, dy \right|
$$

$$
+ \left| \sum_{|i|+j\leq 8} \int_{t_0}^{+\infty} \int_{\mathbb{R}^2} \partial_x^i Z^j (w_{e,h} \cdot \nabla_x U + w_{e,3} \partial_y U) \Phi e^{2\Psi} \partial_x^i Z^j (w_e) \Phi \, dx \, dy \right| \triangleq I_{61} + I_{62}.
$$

Similar to $I_{32}$, we have

$$
I_{62} \leq (\| U \|_{H_{10}^{t\downarrow}} + \| w \|_{H_{10}^{t\uparrow}}) \| w_e \|_{H_{10}^{t\uparrow}}^2.
$$

By the second inequality of Lemma 4.3, Sobolev embedding and (9.3), we obtain

$$
I_{61} \leq \| (w_{e,h} \cdot \nabla_x U + w_{e,3} \partial_y U) \Phi \|_{H_8^{t\downarrow}(y(t))} \| (w_e) \Phi \|_{H_8^{t\uparrow}}
$$

$$
\leq C \left( \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right)^{\frac{1}{2}} + \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}(y(t))} \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right) \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}} \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right) \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}} \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right) \right)
$$

$$
+ C \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}} \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right) \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}} \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right) \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}} \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right).
$$

**Step 4. Estimate of $I_4$ and $I_5$.**

By Lemma 5.1 (9.2) and (9.3), we get

$$
I_4 \leq \left| \sum_{|i|+j\leq 8} \langle \partial_x^i Z^j (w_{e,h} \cdot \nabla_x U + w_{e,3} \partial_y U) \Phi, e^{2\Psi} \partial_x^i Z^j (w_e) \Phi \rangle \right|
$$

$$
\leq C \left( \| U \|_{H_{10}^{t\downarrow}}^2 + \| w \|_{H_{10}^{t\uparrow}}^2 + \| (w_e) \Phi \|_{H_8^{t\downarrow}}^2 + \| U \|_{H_{10}^{t\downarrow}}^2 \right).
$$

And $I_5$ can be controlled by

$$
\left| \sum_{|i|+j\leq 8} \langle \partial_x^i Z^j (w_{e,h} \cdot \nabla_x Ua) \Phi, e^{2\Psi} \partial_x^i Z^j (w_e) \Phi \rangle \right| + \left| \sum_{|i|+j\leq 8} \langle \partial_x^i Z^j (w_{e,h} \cdot \nabla_x Ua) \Phi, e^{2\Psi} \partial_x^i Z^j (w_e) \Phi \rangle \right|
$$

$$
\triangleq I_{51} + I_{52}.
$$

By Lemma 5.1 it is obvious that

$$
I_{52} \leq C \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}}^2 + \| w_e \|_{H_{10}^{t\downarrow}}^2 \right).
$$

We decompose $I_{51}$ into two parts

$$
I_{51} \leq \left| \sum_{|i|+j\leq 8} \int_{t_0}^{t(t)} \int_{\mathbb{R}^2} \partial_x^i Z^j (w_{e,h} \partial_y Ua) \Phi e^{2\Psi} \partial_x^i Z^j (w_e) \Phi \, dx \, dy \right|
$$

$$
+ \left| \sum_{|i|+j\leq 8} \int_{t_0}^{+\infty} \int_{\mathbb{R}^2} \partial_x^i Z^j (w_{e,h} \partial_y Ua) \Phi e^{2\Psi} \partial_x^i Z^j (w_e) \Phi \, dx \, dy \right| \triangleq I_{511} + I_{512}.
$$

Using the fact $\| \partial_x^i Z^j \partial_y Ua \|_{L^\infty (\mathbb{R}^2 \times (y(t),+\infty))} \leq C$, we deduce that

$$
I_{511}^2 \leq C \| w_e \|_{H_8^{t\downarrow}}^2.
$$

And we get by Lemma 5.1 that

$$
I_{51} \leq C \left( \| (w_e) \Phi \|_{H_8^{t\downarrow}} \right)^2.
$$

Thus, we arrive at

$$I_5 \leq C\|w_e\|^2_{H^8} + C\|\frac{w_e}{\varepsilon}\|^2_{H^8(0,y(t))}.$$  

**Step 5. Estimate of dissipative term.**

Let $A \triangleq \partial_y + |D_x|$. Using the boundary conditions of $w_e$, we get by integration by parts that

$$-\varepsilon^2 \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j(\Delta w_e)\Phi, e^{2\Psi_e} \partial_x^i Z^j(w_e)\Phi \rangle$$

$$= -\varepsilon^2 \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j [\partial_y (Aw_e)\Phi - |D_x|(Aw_e)\Phi + 0'(D_x)(Aw_e)\Phi], e^{2\Psi_e} \partial_x^i Z^j(w_e)\Phi \rangle$$

$$= \varepsilon^2 \|(Aw_e)\Phi\|^2_{H^8} - \varepsilon^2 \sum_{|i|+j \leq 8, j \geq 1} \langle [\partial_x^i Z^j, \partial_y] (Aw_e)\Phi, e^{2\Psi_e} \partial_x^i Z^j(w_e)\Phi \rangle$$

$$- \varepsilon^2 \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \partial_x^i \nabla_x (\Lambda_{ND}(\gamma \nabla_x \cdot w_{e,h}))\Phi \cdot e^{2\Psi_e} \partial_x^i (w_{e,h})\Phi(t, x, 0)dx$$

$$+ \varepsilon^2 \sum_{|i|+j \leq 8, j \geq 1} \langle \partial_x^i Z^j (Aw_e)\Phi, e^{2\Psi_e} [\partial_y, \partial_x^i Z^j](w_e)\Phi - 2 \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j (Aw_e)\Phi, e^{2\Psi_e} \partial' \partial_x^i Z^j(w_e)\Phi \rangle$$

$$- 2\varepsilon^2 \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j (Aw_e)\Phi, e^{2\Psi_e} \partial' (D_x) \partial_x^i Z^j(w_e)\Phi \rangle$$

$$\geq \frac{\varepsilon^2}{2} \|(Aw_e)\Phi\|^2_{H^8} - C_0\delta^2 \varepsilon^2 \|(D_x)(w_e)\Phi\|^2_{H^8} - C_0\delta^2 \left\| \frac{w_e}{\varepsilon} \right\|^2_{H^8}$$

$$- \varepsilon^2 \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \partial_x^i \nabla_x (\Lambda_{ND}(\gamma \nabla_x w_{e,h}))\Phi \cdot e^{2\Psi_e} \partial_x^i (w_{e,h})\Phi(t, x, 0)dx.$$

(9.4)

By Lemma 5.3 and (12), we have

$$\varepsilon \|(D_x)(w_e)\Phi\|^2_{H^8} = \varepsilon \sum_{|i|+j \leq 8} \left\| (D_x)(e^{\Psi_e} \partial_x^i Z^j(w_e)\Phi) \right\|^2_{L^2(\mathbb{R}^2)}$$

$$\leq C_0 \varepsilon \sum_{|i|+j \leq 8} \left\| (\partial_y + |D_x|)(e^{\Psi_e} \partial_x^i Z^j(w_e)\Phi) \right\|^2_{L^2(\mathbb{R}^2)}$$

$$\leq C_0 \delta \|(w_e)\Phi\|_{H^8} + C_0 \varepsilon \left\| (\partial_y + |D_x|)w_e\Phi \right\|_{H^8} + C_0 \delta \|(D_x)(w_e)\Phi\|^2_{H^8} + C \left\| \frac{w_e}{\varepsilon} \right\|^2_{H^8},$$

which implies that

$$\varepsilon \|(D_x)(w_e)\Phi\|^2_{H^8} \leq C_0 \delta \|(w_e)\Phi\|_{H^8} + C_0 \varepsilon \left\| (\partial_y + |D_x|)w_e\Phi \right\|_{H^8} + C \left\| \frac{w_e}{\varepsilon} \right\|^2_{H^8}.$$

(9.5)

Using the fact that $F(\Lambda_{ND} f) = \int_{|k|} \frac{\hat{f}(k)}{|k|}$, we find that

$$- \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \partial_x^i \nabla_x (\Lambda_{ND}(\gamma \nabla_x w_{e,h}))\Phi \cdot e^{2\Psi_e} \partial_x^i (w_{e,h})\Phi(t, x, 0)dx$$

$$= \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \partial_x^i (\Lambda_{ND}(\gamma \nabla_x w_{e,h}))\Phi e^{2\Psi_e} \partial_x^i (\nabla_x w_{e,h})\Phi(t, x, 0) \geq 0.$$  

(9.6)
Putting (9.5)-(9.6) into (9.4) and fixing \( \delta \) small, we arrive at
\[
- \varepsilon^2 \sum_{i+j \leq 8} \langle \partial_i^\varepsilon Z^j (\Delta w_e), \varepsilon^2 \partial_i^\varepsilon Z^j (w_e) \rangle_F \\
\geq \frac{\varepsilon^2}{4} \left\| (\partial_y + |D_x|) w_e \right\|_{H^8}^2 - C \left\| (w_e) \right\|_{H^8}^2 - C \frac{\left( (w_e) \right)^2}{\varepsilon^2}.
\]

Summing up the estimates in Step 1-Step 5, we conclude the proposition. \( \square \)

### 9.2. Improved tangential analytic estimate of \( w_{e,3} \).

Recall that \( (w_{e,3}) \) satisfies
\[
\partial_t (w_{e,3}) + \lambda (D_x) (w_{e,3}) - \varepsilon^2 (\Delta w_{e,3}) + (\bar{U}_a \cdot \nabla w_{e,3})_\Phi + (\bar{U} \cdot \nabla w_{e,3})_\Phi + (\bar{U} \cdot \nabla w_{e,3})_\Phi \\
- (w_{a,e} \cdot \nabla) w_{e,3} = (\text{curl} (R_{e,h}, R_{e,v})_3 - M_{e,3}) w_{e,3}.
\]

**Proposition 9.2.** There exists \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), there holds
\[
\frac{1}{2} \frac{d}{dt} \left\| (w_{e,3})_\Phi \right\|_{H^8}^2 - C \varepsilon^2 (E(t) + 1) (\varepsilon^2 + E(t) + K(t)) + C \varepsilon^4 E(t)^\frac{5}{3} \\
+ \varepsilon^2 \left\| (\partial_y + |D_x|) w_e \right\|_{H^8}^2 + \frac{\varepsilon^2}{2} \left\| (w_e) \right\|_{H^8}^2 + \varepsilon^2 \left\| (\partial_y + |D_x| w_e) \right\|_{H^8}^2.
\]

**Proof.** Compared with the case in Proposition 9.1, we have more decay in \( \varepsilon \) for the third component \( w_{e,3} \) of \( w_e \). The key reason is that the following terms
\[
(w_{a,e} \cdot \nabla) w_{e,3} + (w_e \cdot \nabla v_a)_\Phi + (w_e \cdot \nabla v)_\Phi
\]
behave better than the corresponding terms in the equation of \( w_e \) due to \( \partial_y v = -\nabla_x \cdot u \). Let us just show the following estimates:
\[
\left| \left\{ (w_{e,h} \cdot \nabla_x v_a)_\Phi, (w_{e,3})_\Phi \right\|_{L^2} \right| \\
\leq \left| \left\{ (w_{e,h} \cdot \nabla_x (v_a - \varepsilon^2 f(t, x)e^{-y}))_\Phi, (w_{e,3})_\Phi \right\|_{H^8} \right| + \left| (w_{e,h} \cdot \nabla_x (v + \varepsilon^2 f(t, x)e^{-y}))_\Phi, (w_{e,3})_\Phi \right\|_{H^8}.
\]

Using \( (v_a - \varepsilon^2 f(t, x)e^{-y})(t, x, 0) = 0 \) and \( \| \partial_x^2 \|_{L^\infty} \leq C \), the first term can be bounded by
\[
\left| \left\{ (\varphi w_{e,h} \cdot \nabla_x v_a - \varepsilon^2 f(t, x)e^{-y})_\Phi, (w_{e,3})_\Phi \right\|_{H^8} \right| \\
\leq C \left( \| (w_{e,3})_\Phi \|_{H^8} + \| w_{e,3} \|_{H^8}^2 + \| (\varphi w_e) \|_{H^8}^2 + \| (\varphi w_e) \|_{H^8}^2 \right).
\]

Using the second inequality of Lemma 4.3 and Sobolev embedding, the second term in \( (0, y(t)) \) can be controlled by
\[
C \left( \| (w_{e,3})_\Phi \|_{H^8} \| (w_{e,h} \partial_x (v + \varepsilon^2 f(t, x)e^{-y}))_\Phi \|_{H^8(0, y(t))} \right) \\
\leq C \left( \| (w_{e,3})_\Phi \|_{H^8} \right) \left( \| U \|_{H^4} + \varepsilon^2 + \| (\varphi w) \|_{H^8} \right) \\
+ \varepsilon^4 \left( \| (w_{e,3})_\Phi \|_{H^8} \right) \left( \| U \|_{H^4} + \varepsilon^2 + \| w \|_{H^8} \right).
\]

and using Sobolev embedding, the second term in \( (y(t), \infty) \) can be controlled by
\[
\frac{\varepsilon^4}{100} \left\| (\partial_y + |D_x| w_e) \right\|_{H^8}^2 + C \left( \| w_{e,3} \|_{H^8} \right) \left( \| U \|_{H^{10}} + \varepsilon^2 + \| w \|_{H^8} \right).
\]

The estimates of the other terms can follow the proof of Proposition 9.1 line by line. \( \square \)
9.3. Improved tangential analytic estimate of $\varphi w_e$. It is easy to verify that $\varphi w_e$ satisfies

$$\partial_t(\varphi w_e) - \varepsilon^2 \Delta (\varphi w_e) + \varphi \tilde{U}_a \cdot \nabla w_e + \varphi \tilde{U} \cdot \nabla w_{a,e} + \varphi \tilde{U}_a \cdot \nabla w_{a,e} - \varphi w_{a,e} \cdot \nabla U$$

$$- (\varphi w_e) \cdot \nabla U_a - (\varphi w_e) \cdot \nabla U = \varphi (\text{curl}(R_{e,h}, R_{e,v}) - M_e) - \varepsilon^2 \varphi'' w_e - 2\varepsilon^2 \varphi' \partial_y w_e.$$

**Proposition 9.3.** There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there holds

$$\frac{1}{2} \frac{d}{dt} \| (\varphi w_e) \|_{H^s_\varepsilon}^2 + \lambda \| (\varphi w_e) \|_{H^{s+\frac{1}{2}}(0,y(t))}^2 + (\lambda - C) \| \frac{(\varphi w_e) \|}{\varepsilon} \|_{H^s_\varepsilon(0,y(t))}^2 + \frac{\varepsilon^2}{2} \| (\nabla (\varphi w_e)) \|_{H^s_\varepsilon}^2$$

$$\leq C\varepsilon^2 (1 + E(t)) \left( \varepsilon^2 + E(t) + K(t) \right) + C \varepsilon \| E(t) \| \epsilon + \varepsilon^4 \| (\partial_y + |D_x|) w \|_{H^s_{e,\varepsilon}}^2 + \varepsilon^2 \| (w_e) \|_{H^s_\varepsilon}^2.$$

**Proof.** The reason why $\varphi w_e$ has more decay is very similar to the case $w_{3,e}$. Now the weight $\varphi$ cancels one singularity from the derivative $\partial_y$ due to $\varphi \partial_y = Z$. Let us just deal with the following terms. First, it is easy to get

$$\langle \varepsilon^2 \varphi'' (w_e) \phi, (\varphi w_e) \phi \rangle_{H^s_\varepsilon} \leq C \| (\varphi w_e) \phi \|_{H^s_\varepsilon}^2 + C \varepsilon \| (w_e) \phi \|_{H^s_\varepsilon}^2.$$

We get by integration by parts that

$$\left| \left( \varepsilon^2 \varphi' (\partial_y w_e) \phi, (\varphi w_e) \phi \right)_{H^s_\varepsilon} \right|$$

$$\leq \frac{\varepsilon^2}{100} \| (\partial_y (\varphi w_e) \phi \|_{H^s_\varepsilon}^2 + C \varepsilon \| (w_e) \phi \|_{H^{s+\frac{1}{2}}_\varepsilon}^2 + \varepsilon^2 \| (w_e) \phi \|_{H^s_\varepsilon}^2 + C \| (\varphi w_e) \phi \|_{H^s_\varepsilon}^2.$$

The estimates of the other terms can follow the proof of Proposition [0.1] line by line. □

10. Tangential analytic estimate of the vorticity: Prandtl part

Let us observe that $(w_p)\phi$ satisfies

$$\partial_t (w_p)\phi + \lambda (D_x) (w_p)\phi - \varepsilon^2 (\Delta w_p)\phi + (\tilde{U}_a \cdot \nabla w_p)\phi + (\tilde{U} \cdot \nabla w_{a,p})\phi + (\tilde{U} \cdot \nabla w_p)\phi$$

$$- (w_{a,p} \cdot \nabla U)\phi - (w_p \cdot \nabla U_a)\phi - (w_p \cdot \nabla U)\phi = (\text{curl}(R_{p,h}, R_{p,v}) - M_p)\phi$$

(10.1)

together with the following initial-boundary conditions

$$\left\{ \begin{array}{l}
-\varepsilon^2 (\partial_y + |D_x|) w_{p,h})\phi |_{t,x,0} - \varepsilon^2 \partial_x (\Delta_N \varphi (\nabla x \cdot w_{p,h})) \phi |_{t,x,0} \\
= - (\partial_y (-\Delta)^{-1} J_h) \phi |_{y=0} + (\partial_x (-\Delta)^{-1} J_3) \phi |_{y=0}, \\
(w_{p,3})\phi |_{t,x,0} = 0, \\
(w_p)\phi |_{t,0,y} = 0.
\end{array} \right.$$

**Proposition 10.1.** There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there holds

$$\frac{1}{2} \frac{d}{dt} \| (w_p)\phi \|_{H^s_\varepsilon}^2 + \lambda \| (w_p)\phi \|_{H^{s+\frac{1}{2}}(0,y(t))}^2 + (\lambda - C) \| \frac{y(w_p)\phi}{\varepsilon} \|_{H^s_\varepsilon(0,y(t))}^2$$

$$+ \frac{\varepsilon^2}{100} \| (\partial_y + |D_x|) w)\phi \|_{H^s_{e,\varepsilon}}^2 \leq C (1 + E(t)) (K(t) + E(t) + \varepsilon^2) + C \varepsilon^{-2} E(t)^2$$

$$+ \frac{\varepsilon^2}{100} \| (\partial_y + |D_x|) w)\phi \|_{H^s_{e,\varepsilon}}^2.$$
Proof. Acting $\partial_x^i Z^j$ on both sides of (10.11) and then taking $L^2$ inner product with $e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi$, summing over all $|i| + j \leq 8$, we arrive at

$$\frac{1}{2} \frac{d}{dt} \left\| (w_p)_\Phi \right\|_{H^8_p}^2 + \frac{\lambda}{2} \left\| (w_p)_\Phi \right\|^2_{H^8_p(0, y(t))} + \frac{\lambda}{2} \left\| \frac{y(w_p)_\Phi}{\varepsilon} \right\|^2_{H^8_p} - \varepsilon^2 \sum_{|i| + j \leq 8} \langle \partial_x^i Z^j (\Delta w_p)_\Phi, e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi \rangle$$

$$\leq \sum_{|i| + j \leq 8} \langle \partial_x^i Z^j (\tilde{U}_\Phi \cdot \nabla w_p)_\Phi, e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi \rangle + \sum_{|i| + j \leq 8} \langle \partial_x^i Z^j (\tilde{U} \cdot \nabla w_{a,p})_\Phi, e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi \rangle$$

$$+ \sum_{|i| + j \leq 8} \langle \partial_x^i Z^j (w_p \cdot \nabla U)_\Phi, e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi \rangle + \sum_{|i| + j \leq 8} \langle \partial_x^i Z^j (w_p \cdot \nabla U)_\Phi, e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi \rangle$$

$$+ \sum_{|i| + j \leq 8} \langle \partial_x^i Z^j (\text{curl}(R_{p,h}, R_{p,v}) - M_p)_\Phi, e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi \rangle \equiv \sum_{i=1}^7 I_i.$$

Step 1. Estimate of $I_1$.

Following the argument of $I_1$ in Proposition 9.1, we can deduce that

$$I_1 \leq C \left( \left\| w_p \right\|^2_{H^8_p} + \left\| (w_p)_\Phi \right\|^2_{H^8_p(0, y(t))} + \left\| \frac{y(w_p)_\Phi}{\varepsilon} \right\|^2_{H^8_p(0, y(t))} \right) + \frac{\varepsilon^2}{100} \left\| ((\partial_y + |D_x| w_p)_\Phi \right\|^2_{H^8_p}.$$

Step 2. Estimate of $I_2$.

First, $I_2$ can be controlled by

$$\left| \sum_{|i| + j \leq 8} \int_0^{y(t)} \int_{\mathbb{R}^2} \partial_x^i Z^j (u \partial_x w_{a,p} + (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y w_{a,p}) \Phi e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi dx dy \right|$$

$$\leq \sum_{|i| + j \leq 8} \int_0^{y(t)} \int_{\mathbb{R}^2} \partial_x^i Z^j (u \partial_x w_{a,p} + (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y w_{a,p}) \Phi e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi dx dy \right| \equiv I_{21} + I_{22}.$$

Using the fact that $\|e^{\Psi r} \partial_x^i Z^j \partial_{x,y} w_{a,p}\|_{L^\infty(\mathbb{R}^2 \times (y(t), \infty))} \leq C$, it is easy to get by (9.2) that

$$I_{21} \leq C \left( \|w_p\|^2_{H^8_p} + \|U\|^2_{H^8_p} + \|\varphi w\|^2_{H^8_p} + \varepsilon^4 \right).$$

As $w_{a,p} = \partial_y (u_{a,p}) - \partial_x v_{a,p}$, we get by (9.2) that

$$\left| \sum_{|i| + j \leq 8} \int_0^{y(t)} \int_{\mathbb{R}^2} \partial_x^i Z^j ((v + \varepsilon^2 f(t, x)e^{-y}) \partial_y w_{a,p}) \Phi e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi dx dy \right|$$

$$\leq C \left( \|w_p\|^2_{H^8_p} + \varepsilon^2 \|U\|^2_{H^8_p} + \varepsilon^2 \|\varphi w\|^2_{H^8_p} \right).$$

Using $\partial_y v = -\nabla_x \cdot u$ and $(v + \varepsilon^2 f(t, x)e^{-y})|_{y=0} = 0$ and Lemma 5.1, we deduce that

$$\left| \sum_{|i| + j \leq 8} \int_0^{y(t)} \int_{\mathbb{R}^2} \partial_x^i Z^j ((v + \varepsilon^2 f(t, x)e^{-y}) \partial_y w_{a,p}) \Phi e^{2\Psi r} \partial_x^i Z^j (w_p)_\Phi dx dy \right|$$

$$\leq C \left( \|w_p\|^2_{H^8_p} + C \varepsilon^2 \left( \|\varphi w\|^2_{H^8_p} + \|U\|^2_{H^8_p} + \varepsilon^4 \right) \right).$$
Thus, we obtain
\[
I_{22} \leq C \|(w_p)_\Phi\|_{H^p_{2\Phi}}^2 + C \varepsilon^{-2} \left( \|(\varphi w)\Phi\|_{H^s_{2\Phi}}^2 + \|U\Phi\|_{H^p_{\kappa_1}}^2 + \varepsilon^4 \right).
\]
This shows that
\[
I_2 \leq C \left( \|w_p\|_{H^p_{2\Phi}}^2 + \|U\|_{H^p_{\kappa_1}}^2 + \|\varphi w\|_{H^s_{2\Phi}}^2 + \|(w_p)\Phi\|_{H^p_{2\Phi}}^2 \right)
+ C \varepsilon^{-2} \left( \|(\varphi w)\Phi\|_{H^s_{2\Phi}}^2 + \|U\Phi\|_{H^p_{\kappa_1}}^2 + \varepsilon^4 \right).
\]

**Step 3. Estimate of $I_3$ and $I_6$.**
Following the arguments of $I_3$ and $I_6$ in Proposition 9.1, we obtain
\[
I_3 \leq \frac{\varepsilon^2}{100} \left( \|((\partial_y + |D_x|)w_p)\Phi\|_{H^p_{2\Phi}}^2 + C \varepsilon^{-2} \left( \|(w_p)_\Phi\|_{H^p_{2\Phi}}^2 + \|(\varphi w)\Phi\|_{H^s_{2\Phi}}^2 + \varepsilon^4 \right) \right)
+ C \left( \varepsilon^2 + \|U\Phi\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 \right) \left( \|(w_p)_\Phi\|_{H^p_{2\Phi}}^2 \right)
+ C \|w_p\|_{H^p_{2\Phi}}^2 \left( \varepsilon^2 + \|U\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 \right),
\]
and
\[
I_6 \leq \frac{\varepsilon^2}{100} \left( \|((\partial_y + |D_x|)w_p)\Phi\|_{H^p_{2\Phi}}^2 + \|\frac{y(w_p)_\Phi}{\varepsilon}\|_{H^p_{\kappa_1}(0,y(t))}^2 + C \varepsilon^{-\frac{\delta}{2}} \|(w_p)_\Phi\|_{H^p_{\kappa_1}}^2 \left( \|U\Phi\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 \right) \right)
+ C \left( \|U\Phi\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 \right) \left( \|(w_p)_\Phi\|_{H^p_{\kappa_1}}^2 \right) \|w_p\|_{H^p_{2\Phi}}^2.
\]

**Step 4. Estimate of $I_4$.**
We split $I_4$ into two parts
\[
I_4 \leq \sum_{|i|+j \leq 8} \left| \langle \partial^i_x Z^j (w_{a,\rho,\Phi} \partial_x U)_\Phi, e^{2\varepsilon^2 \partial^i_x Z^j (w_p)\Phi} \rangle \right|
+ \sum_{|i|+j \leq 8} \left| \langle \partial^i_x Z^j (w_{a,\rho,\Phi} \partial_y U)_\Phi, e^{2\varepsilon^2 \partial^i_x Z^j (w_p)\Phi} \rangle \right| \triangleq I_{41} + I_{42}.
\]
Using $\|e^{\varepsilon^2 \partial^i_x Z^j (w_{a,\rho,\Phi})}_{L^\infty} \leq C$, we get by (9.2) and (9.3) that
\[
I_{42} \leq C \left( \|U\Phi\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 + \|(w_p)_\Phi\|_{H^p_{\kappa_1}}^2 + \|U\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 + \|w_p\|_{H^p_{2\Phi}}^2 \right),
\]
and using $\|e^{\varepsilon^2 \partial^i_x Z^j (w_{a,\rho,\Phi})}_{L^\infty(\mathbb{R} \times (y(t),+\infty))} \leq C$, we deduce that
\[
I_{41} \leq C \left( \|U\Phi\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 + \|w_p\|_{H^p_{2\Phi}}^2 \right)
+ \sum_{|i|+j \leq 8} \int_{\mathbb{R}^2} \left| \partial^i_x Z^j (w_{a,\rho,\Phi} \partial_x U)_\Phi e^{2\varepsilon^2 \partial^i_x Z^j (w_p)\Phi} dx dy \right|
\leq C \left( \|U\Phi\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 + \|w_p\|_{H^p_{2\Phi}}^2 + \|(w_p)_\Phi\|_{H^p_{\kappa_1}}^2 \right)
+ C \varepsilon^{-2} \left( \|(\varphi w)\Phi\|_{H^s_{2\Phi}(0,y(t))}^2 + \|U\Phi\|_{H^p_{\kappa_1}(0,y(t))}^2 \right).
\]
Thus, we obtain
\[
I_4 \leq C \left( \|U\Phi\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 + \|(w_p)_\Phi\|_{H^p_{\kappa_1}}^2 + \|U\|_{H^p_{\kappa_1}}^2 + \|w\|_{H^s_{2\Phi}}^2 + \|w_p\|_{H^p_{2\Phi}}^2 \right)
+ C \varepsilon^{-2} \left( \|(\varphi w)\Phi\|_{H^s_{2\Phi}}^2 + \|U\Phi\|_{H^p_{\kappa_1}}^2 \right).
\]

**Step 5. Estimate of $I_5$.**
We split $I_5$ into two parts

$$I_5 \leq \left| \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j(w_{p,0}\partial_x U_a)\Phi, e^{2\Psi \cdot} \partial_x^i Z^j(w_p)\Phi \rangle \right|$$

$$+ \left| \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j(w_{p,3}\partial_y U_a)\Phi, e^{2\Psi \cdot} \partial_x^i Z^j(w_p)\Phi \rangle \right| \equiv I_{51} + I_{52}.$$ 

By Lemma 5.1, we have

$$I_{51} \leq C \left\| (w_p)_\Phi \right\|_H^2 + \left\| w_p \right\|_H^2,$$

and

$$I_{52} \leq C \left\| (w_p)_\Phi \right\|_H^2 + \left\| (w_p)_\Phi \right\|_H^2 + C\varepsilon^{-2} \left( \left\| (w_{p,3})_\Phi \right\|_H^2 + \left\| (w_{p,3})_\Phi \right\|_H^2 \right).$$

This shows that

$$I_5 \leq C \left\| (w_p)_\Phi \right\|_H^2 + \left\| (w_p)_\Phi \right\|_H^2 + C\varepsilon^{-2} \left( \left\| (w_{p,3})_\Phi \right\|_H^2 + \left\| (w_{p,3})_\Phi \right\|_H^2 \right).$$

**Step 6. Estimate of $I_7$.**

Using Lemma 5.2, it is easy to get

$$I_7 \leq C \| (w_p)_\Phi \|_H^2 + C\varepsilon^2.$$

**Step 7. Estimate of dissipative term.**

Following the arguments in Proposition 9.1, we can deduce that

$$-\varepsilon^2 \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j(\Delta w_p)_\Phi, e^{2\Psi \cdot} \partial_x^i Z^j(w_p)_\Phi \rangle$$

$$\geq \frac{\varepsilon^2}{2} \| (Aw_p)_\Phi \|_H^2 - C_0\delta^2\varepsilon^2 \| (D_x)(w_p)\Phi \|_H^2 - C_0\delta^2 \| y(w_p)\Phi \|_H^2$$

$$+ \varepsilon^2 \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \partial_x^i (Aw_p)_\Phi \partial_x^i (w_p)_\Phi (x,0) dx,$$ 

(10.2)

Following the argument of (9.5), we have

$$\varepsilon \| D_x((w_p)_\Phi) \|_H \leq C\varepsilon \| (w_p)_\Phi \|_H + C\varepsilon \left\| (Aw_p)_\Phi \right\|_H^2 + C \left\| y(w_p)\Phi \right\|_H^2.$$

Putting this estimate into (10.2) and fixing $\delta$ small, we arrive at

$$-\varepsilon^2 \sum_{|i|+j \leq 8} \langle \partial_x^i Z^j(\Delta w_p)_\Phi, e^{2\Psi \cdot} \partial_x^i Z^j(w_p)_\Phi \rangle$$

$$\geq \frac{\varepsilon^2}{4} \| (Aw_p)_\Phi \|_H^2 - C \| y(w_p)\Phi \|_H^2$$

$$+ \varepsilon^2 \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \partial_x^i ((\partial_y + |D_x|)w_p)_\Phi \partial_x^i (w_p)_\Phi (t, x, 0) dx.$$ 

(10.3)
Now, using the boundary condition of \(w_p\), we arrive at
\[
\varepsilon^2 \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \partial_x^i ((\partial_y + |D_x|)w_p) \Phi \partial_x^i (w_p) \Phi (t, x, 0) dx
\]
\[
= - \sum_{|i| \leq 8} \varepsilon^2 \int_{\mathbb{R}^2} \partial_x^i \nabla_x (\Lambda_N D(\gamma \nabla_x \cdot w_{p,h})) \Phi \cdot \partial_x^i (w_{p,h}) \Phi (t, x, 0) dx
\]
\[
+ \sum_{|i| \leq 8} \int_{\mathbb{R}^2} \varepsilon^2 (\delta - \lambda t) D_x \partial_x^i w_{p,h} \partial_x^i (-\partial_y (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3)(t, x, 0) dx.
\]

The same argument as (9.19) gives
\[
- \varepsilon^2 \int_{\mathbb{R}^2} \partial_x^i \nabla_x (\Lambda_N D(\gamma \nabla_x \cdot w_{p,h})) \Phi \cdot \partial_x^i (w_{p,h}) \Phi (t, x, 0) dx \geq 0.
\]

To deal with another term, we introduce a smooth cut-off function \(\zeta(t, y)\) satisfies \(\zeta(t, 0) = 1\) and \(\zeta(t, y) = 0\) for \(y > y(t)\). Then we have
\[
\int_{\mathbb{R}^2} \varepsilon^2 (\delta - \lambda t) (D_x) \partial_x^i w_{p,h} (x, 0) \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3)(t, x, 0) dx
\]
\[
= \int_{\mathbb{R}^2} \int_0^{y(t)} \partial_y \left\{ \zeta(t, y) e^\Phi \partial_x^i w_{p,h} \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3 \right\} dy dx
\]
\[
= \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^\Phi \partial_x^i w_{p,h} \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3 \right\} dy dx
\]
\[
- \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^\Phi \partial_x^i w_{p,h} \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3 \right\} dy dx
\]
\[
+ \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^\Phi \partial_x^i w_{p,h} \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3 \right\} dy dx
\]
\[
+ \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^\Phi \partial_x^i w_{p,h} \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3 \right\} dy dx
\]
\[
= \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^\Phi \partial_x^i w_{p,h} \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3 \right\} dy dx
\]
\[
= \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^\Phi \partial_x^i w_{p,h} \partial_x^i (\Delta_D)^{-1} J_h + \partial_x (\Delta_N)^{-1} J_3 \right\} dy dx
\]
\[
\geq \sum_{k=1}^5 I_k^b.
\]

Recall that \(J = (J_h, J_3) = -\text{curl}(F, G) + \text{curl}R\), where
\[
(F, G) = \bar{U}_a \cdot \nabla U + \bar{U} \cdot \nabla U_a + \bar{U} \cdot \nabla U.
\]

Using \(L^2\) boundness of operators \(\partial_y (\Delta_D)^{-1} \partial_x\), \(\partial_y (\Delta_D)^{-1} \partial_y\), \(\partial_y (\Delta_D)^{-1}\), \(\nabla (\Delta_N)^{-1} \partial_x\) and \(\partial_y (\Delta_N)^{-1}\), we can deduce from Lemma 6.11 that
\[
|I_1^b| \leq C \|(w_p)\|_{H^p} + C \sum_{|i| \leq 8} \|\partial_x^i (F, G)\|_{L^2(\mathbb{R}^2 \times (0, y(t)))} + C \varepsilon^4
\]
\[
\leq C \|(w_p)\|_{H^p} + C \varepsilon^2 (1 + E(t))(E(t) + K(t) + \varepsilon^2),
\]
and
\[
|I_2^b| \leq C \|(w_p)\|_{H^p} + C \sum_{|i| \leq 8} \|D_x \frac{1}{\varepsilon} \partial_x^i (F, G)\|_{L^2(\mathbb{R}^2 \times (0, y(t)))} + C \varepsilon^4
\]
\[
\leq C (1 + E(t))(\varepsilon^2 + E(t) + K(t)).
\]
Similarly, we have
\[ |I_0^b| \leq C \| (w_p)_t \|_{H_p^s}^2 + C \sum_{|i| \leq 8} \| \langle D_x \rangle^{-\frac{1}{2}} \partial_x^i (J_3)_t \|_{L^2(\mathbb{R}^2 \times (0, y(t)))}^2 \]
\[ \leq C(1 + E(t)) (\varepsilon^2 + E(t) + K(t)). \]

For \( I_3^b \), we have
\[ I_3^b \leq C(1 + E(t)) (\varepsilon^2 + E(t) + K(t)) + \frac{\varepsilon^2}{100} \| (Aw_p)_t \|_{H_p^s}^2 + C\varepsilon^{-2} E(t)^2. \]

We write
\[ I_4^b = \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^{\Phi} \partial_x^i w_{p, h} \partial_x^j (J_{h, \Phi}) dy dx + \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) e^{\Phi} \partial_x^i w_{p, h} \partial_x^j (-\partial_{xx} (\Delta D)^{-1} J_{h, \Phi}) dy dx. \]

Now, the second term can be handled as \( I_3^b \), while the first term can be handled as \( I_1^b, I_2^b, I_3^b \) by integrating by parts and \((F, G)(t, x, 0) = 0\).

Thus, we arrive at
\[ -\varepsilon^2 \sum_{|i| + j \leq 8} \langle \partial_x^i Z^j (\Delta w_p)_t \rangle, e^{2} \partial_x^i Z^j (w_p, \Phi) \rangle_2 \]
\[ \geq \frac{\varepsilon^2}{8} \| (Aw_p)_t \|_{H_p^s}^2 - C \| (w_p)_t \|_{H_p^s}^2 - C(1 + E(t)) (E(t) + K(t)) \]
\[ + \varepsilon^2 \left( 1 + E(t) \right) (E^2 + E(t) + K(t)) + C\frac{\varepsilon}{100} \| (\Delta w_{p, h})_t \|_{H_p^s}^2 + \varepsilon^2 \| (\partial_x^i w_{p, h})_t \|_{H_p^s}^2 + \varepsilon^2 \| (\partial_x^i w_{p, h})_t \|_{H_p^s}^2 + \varepsilon^2 \| (\partial_x^i w_{p, h})_t \|_{H_p^s}^2. \]

The proposition follows by combing the estimates in Step 1-Step 7. \( \square \)

As in \( w_{e, 3}, \varphi w_{e, 3}, w_{p, 3} \) and \( \varphi w_{p} \) have more decay in \( \varepsilon \). Let us just state the following results without proof.

**Proposition 10.2.** There exists \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), there holds
\[ \frac{1}{2} \frac{d}{dt} \| (w_{p, 3})_t \|_{H_p^s}^2 + \lambda \| (w_{p, 3})_t \|_{H_p^s}^2 + \left( \lambda - C \right) \| \frac{y(w_{p, 3})_t}{\varepsilon} \|_{H_p^s}^2 + \varepsilon^2 \| (\nabla w_{p, 3})_t \|_{H_p^s}^2 \]
\[ \leq C\varepsilon^2 \left( 1 + E(t) \right) (E^2 + E(t) + K(t)) + C\varepsilon^2 E(t)^2 \]
\[ + \frac{\varepsilon^2}{100} \| (\partial_x^i + |D_x|) w_{p, 3} \|_{H_p^s}^2 \]
\[ + \frac{\varepsilon^2}{100} \| (\partial_x^i + |D_x|) w_{p, 3} \|_{H_p^s}^2 \]
\[ + \frac{\varepsilon^2}{100} \| (\partial_x^i + |D_x|) w_{p, 3} \|_{H_p^s}^2. \]

**Proposition 10.3.** There exists \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), there holds
\[ \frac{1}{2} \frac{d}{dt} \| (\varphi w_p)_t \|_{H_p^s}^2 + \lambda \| (\varphi w_p)_t \|_{H_p^s}^2 + \left( \lambda - C \right) \| \frac{y(\varphi w_p)_t}{\varepsilon} \|_{H_p^s}^2 + \varepsilon^2 \| (\nabla (\varphi w_p))_t \|_{H_p^s}^2 \]
\[ \leq C\varepsilon^2 \left( 1 + E(t) \right) (E^2 + E(t) + K(t)) + C\varepsilon^2 E(t)^2 \]
\[ + \frac{\varepsilon^2}{100} \| (\partial_x^i + |D_x|) w_{p} \|_{H_p^s}^2 + \varepsilon^2 \| (\partial_x^i + |D_x|) w_{p} \|_{H_p^s}^2 \]
\[ + \frac{\varepsilon^2}{100} \| (\partial_x^i + |D_x|) w_{p} \|_{H_p^s}^2. \]

11. **Conormal Sobolev estimate of the vorticity: Euler part**

**Proposition 11.1.** There exists \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), there holds
\[ \frac{1}{2} \frac{d}{dt} \| w_e \|_{H_P^s}^2 + \frac{\varepsilon^2}{2} \| (\partial_y + |D_x|) w_e \|_{H_P^s}^2 \]
\[ \leq C \left( E(t) + \varepsilon^2 \right) + \frac{\varepsilon^2}{100} \| (\partial_y + |D_x|) w \|_{H_P^s}^2 + C\varepsilon^{-2} E(t)^2. \]
Proof. First, acting $\partial_i^j Z^j$ on both sides of (9.1), and then taking $L^2$ inner product with $\partial_x^j Z^j w_e$, summing over all $|i| + j \leq 9$, we arrive at

$$
\frac{1}{2} \frac{d}{dt} \| w_e \|^2_{H^{29}_e} - \epsilon^2 \sum_{|i| + j \leq 9} \langle \partial_x^i Z^j (\Delta w_e), \partial_x^j Z^j w_e \rangle 
\leq \sum_{|i| + j \leq 9} \langle \partial_x^i Z^j (\bar{U} \cdot \nabla w_e), \partial_x^j Z^j w_e \rangle + \sum_{|i| + j \leq 9} \langle \partial_x^i Z^j (\bar{U} \cdot \nabla w_{a,e}), \partial_x^j Z^j w_e \rangle 
+ \sum_{|i| + j \leq 9} \langle \partial_x^i Z^j (w_e \cdot \nabla U), \partial_x^j Z^j w_e \rangle + \sum_{|i| + j \leq 9} \langle \partial_x^i Z^j (w_{a,e} \cdot \nabla U), \partial_x^j Z^j w_e \rangle 
+ \sum_{|i| + j \leq 9} \langle \partial_x^i Z^j (\text{curl}(R_{e,h}, R_{e,v}) - M_e), \partial_x^j Z^j w_e \rangle \approx \sum_{i=1}^7 I_i.
$$

Step 1. Estimate of $I_1$.

We bound $I_1$ as

$$
I_1 \leq C \sum_{|i| + j \leq 9} \sum_{(m,n) < (i,j)} \left| \int_{\mathbb{R}^3_+} \left( \partial_x^{i-m} Z^{j-n} u_\alpha \partial_x^m Z^n w_e 
+ \partial_x^{i-m} Z^{j-n} (v_a - \epsilon^2 f(t,x) e^{-y}) \partial_x^m Z^n \partial_y w_e \partial_x^j Z^j w_e dx dy \right) 
+ \sum_{|i| + j \leq 9} \left| \int_{\mathbb{R}^3_+} (u_\alpha \partial_x Z^j w_e + (v_a - \epsilon^2 f(t,x) e^{-y}) \partial_x Z^j \partial_y w_e) \partial_x^j Z^j w_e dx dy \right|.
$$

By Lemma 5.1, the first term can be controlled by $C \| w_e \|^2_{H^{29}_e}$, where we used $(v_a - \epsilon^2 f(t,x) e^{-y}) |_{y=0} = 0$ and the fact

$$
\| \partial_x^{i-m} Z^{j-n} (v_a - \epsilon^2 f(t,x) e^{-y}) \varphi \|_{L^\infty} \leq C.
$$

Notice that $|\partial_y, \partial_x^j Z^j| = j \varphi \partial_x Z^{j-1} \partial_y$. By Lemma 5.1 and integrating by parts, we deduce that the second term can be bounded by

$$
\sum_{|i| + j \leq 9, j \geq 1} \left| \int_{\mathbb{R}^3_+} (v_a - \epsilon^2 f(t,x) e^{-y}) \varphi \partial_x Z^{j-1} \partial_y w_e \partial_x^j Z^j w_e dx dy \right| + C \| w_e \|^2_{H^{29}_e}.
$$

Thus, by (11.1), we deduce that the second term is bounded by $C \| w_e \|^2_{H^{29}_e}$. Thus, we arrive at

$$
I_1 \leq C \| w_e \|^2_{H^{29}_e}.
$$

Step 2. Estimate of $I_2$ and $I_7$.

By Lemma 5.1 $Zu = \varphi w_0^h + \varphi \nabla x v$ and $\partial_y v = -\nabla_x \cdot u$, it is easy to obtain

$$
I_2 \leq C \left( \| w_e \|^2_{H^{29}_e} + \| \varphi w \|^2_{H^{29}_e} + \| U \|^2_{H^{29}_e} + \epsilon^4 \right).
$$

We infer from Lemma 5.2 that

$$
I_7 \leq C (\| w_e \|^2_{H^{29}_e} + \epsilon^4).
$$

Step 3. Estimate of $I_3$ and $I_6$. 

First, $I_3$ can be controlled by
\[
\sum_{|i|+j \leq 9} \sum_{(m,n) \leq (i,j)} \left| \langle \partial_x^{i-m} Z^{j-n} \widetilde{U} \cdot \partial_x^m Z^n \nabla \varphi, \partial_x^{i} Z^j \varphi \rangle \right| + \sum_{|i|+j \leq 9} \left| \langle \widetilde{U} \cdot \partial_x^{i} Z^j \nabla \varphi, \partial_x^{i} Z^j \varphi \rangle \right|.
\]

Integrating by parts, the second term can be bounded by
\[
C \varepsilon^2 \|w\|_{H^2}^2 + \sum_{|i|+j \leq 9, j \geq 1} \left| \langle (v + \varepsilon^2 f(t, x)e^{-y}) \partial_x^i Z^j \varphi, \partial_x^{i} Z^j \varphi \rangle \right|.
\]

Using $(v + \varepsilon^2 f(t, x)e^{-y})|_{y=0} = 0$, we infer that
\[
\sum_{|i|+j \leq 9, j \geq 1} \left| \langle (v + \varepsilon^2 f(t, x)e^{-y}) \partial_x^i Z^j \varphi, \partial_x^{i} Z^j \varphi \rangle \right|
\]
\[
= \sum_{|i|+j \leq 9, j \geq 1} \left| \langle (v + \varepsilon^2 f(t, x)e^{-y}) \varphi \partial_x^i Z^j \varphi, \partial_x^{i} Z^j \varphi \rangle \right|
\]
\[
\leq C \left( \|U\|_{H^2}^2 + \varepsilon^2 \right) \|w\|_{H^2}^2.
\]

Therefore, the second term can be bounded by
\[
C \left( \varepsilon^2 + \|U\|_{H^2}^2 \right) \|w\|_{H^2}^2.
\]

Using $Z = \varphi w^1 + \varphi \nabla_x v$ and $\partial_y v = -\nabla_x \cdot u$, the first term is bounded by
\[
\frac{\varepsilon^2}{100} \|\partial_y + |D_x|\|w\|_{H^2}^2 + C \varepsilon^2 \left( \varepsilon^2 + \|U\|_{H^2}^2 + \|w\|_{H^2}^2 \right) \|w\|_{H^2}^2.
\]

Thus, we obtain
\[
I_3 \leq \frac{\varepsilon^2}{100} \|\partial_y + |D_x|\|w\|_{H^2}^2 + C \varepsilon^2 \left( \varepsilon^2 + \|U\|_{H^2}^2 + \|w\|_{H^2}^2 \right) \|w\|_{H^2}^2.
\]

Now, $I_6$ can be controlled by
\[
\left| \sum_{|i|+j \geq 9, |m| + n \geq 5} \int_{\mathbb{R}^3} \partial_x^{i-m} Z^{j-n} \varphi \partial_x^m Z^n \nabla \varphi, \partial_x^{i} Z^j \varphi dxdy \right|
\]
\[
+ \sum_{|i|+j \leq 9, |m| + n \geq 6} \int_{\mathbb{R}^3} \partial_x^{i-m} Z^{j-n} \varphi \partial_x^m Z^n \nabla \varphi, \partial_x^{i} Z^j \varphi dxdy \right| \triangleq I_{61} + I_{62}.
\]

By Sobolev inequality and $\partial_y u = w^1 + \nabla_x v$, we deduce that
\[
I_{61} \leq C \|w\|_{H^3}^2 \left( \|w\|_{H^2}^2 + \|\partial_y w\|_{H^2}^2 + \|w\|_{H^2}^2 + \|U\|_{H^2}^2 + \|w\|_{H^2}^2 \right),
\]
\[
I_{62} \leq C \|w\|_{H^3} \left( \|w\|_{H^3} + \|U\|_{H^2} \right) \|w\|_{H^3} \|\nabla w\|_{H^3}.
\]

This shows that
\[
I_6 \leq C \varepsilon^2 \left( \varepsilon^2 + \varepsilon^2 \right) + C \varepsilon \left( \varepsilon^2 + \varepsilon^2 \right) + \frac{\varepsilon^2}{100} \|\partial_y + |D_x|\|w\|_{H^2}^2.
\]

Step 4. Estimate of $I_4$ and $I_5$.

By Lemma 5.1 and $\partial_y u = w^1 + \partial_x v$, we deduce that
\[
I_4 \leq \sum_{|i|+j \leq 9} \left| \langle \partial_x^{i} Z^j (w_{a,e,h} \cdot \nabla_x U + w_{a,e,3} \partial_y U), \partial_x^{i} Z^j \varphi \rangle \right| \leq C \left( \|U\|_{H^2}^2 + \|w\|_{H^2}^2 \right).
\]
And $I_5$ can be controlled by

$$
\left| \sum_{|i|+j \leq 9} \langle \partial^i_x Z^j (w_{e,3} \partial_y u_{a,p}), \partial^i_x Z^j w_{e,h} \rangle \right| + \sum_{|i|+j \leq 9} \left| \langle \partial^i_x Z^j (w_{e,3} \partial_y u_{a,p}), \partial^i_x Z^j w_{e,3} \rangle + \langle \partial^i_x Z^j (w_{e,3} \partial_y U_{a,e} + w_{e,h} \partial_x U_{a}), \partial^i_x Z^j w_{e} \rangle \right|.
$$

By Lemma $5.1$, the second term is bounded by $C \| w_e \|_{H^9_o}^2$, and the first term can be controlled by $C \left( \| w_e \|_{H^9_o}^2 + \left\| \frac{w_{e,3}}{\varepsilon} \right\|_{H^9_o}^2 \right)$. Hence,

$$
I_5 \leq C \left( \| w_e \|_{H^9_o}^2 + \left\| \frac{w_{e,3}}{\varepsilon} \right\|_{H^9_o}^2 \right).
$$

**Step 5. Estimate of dissipative term.**

We get, by integrating by parts, that

$$
-\varepsilon^2 \sum_{|i|+j \leq 9} \langle \partial^i_x Z^j (\Delta w_e), \partial^i_x Z^j w_e \rangle = \varepsilon^2 \left\| A w_e \right\|_{H^9_o}^2 - \varepsilon^2 \sum_{|i|+j \leq 9, j \geq 1} \langle \partial^i_x Z^j, \partial_y \partial^i_x Z^j w_e \rangle + \varepsilon^2 \sum_{|i|+j \leq 9, j \geq 1} \langle \partial^i_x Z^j w_e, \partial_y \partial^i_x Z^j w_e \rangle + \sum_{|i| \leq 9} \int_{R^2} \partial^i_x \nabla \Lambda_{ND}(\gamma \nabla (x \cdot w_{e,h})) \cdot \partial^i_x w_{e,h}(t, x, 0)dx 
$$

where we used

$$
-\varepsilon^2 \sum_{|i| \leq 9} \int_{R^2} \partial^i_x \nabla \Lambda_{ND}(\gamma \nabla (x \cdot w_{e,h})) \cdot \partial^i_x w_{e,h}(t, x, 0)dx \geq 0.
$$

Combining the estimates in Step 1-Step 5, we complete the proof of the proposition. \qed

Similarly, we can prove the following improved decay estimate in $\varepsilon$ for $w_{e,3}$ and $\varphi w_e$. Here we omit the details.

**Proposition 11.2.** There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there holds

$$
\frac{1}{2} \frac{d}{dt} \left\| w_{e,3} \right\|_{H^9_o}^2 + \frac{\varepsilon^2}{2} \left\| \nabla w_{e,3} \right\|_{H^9_o}^2 \leq C \varepsilon^2 \left( E(t) + \varepsilon^2 \right) \left( 1 + E(t) \right) + \frac{\varepsilon^4}{100} \left\| (\partial_y + |D_x|) w \right\|_{H^9_o}^2 + C \varepsilon^4 E(t)^{\frac{5}{2}}.
$$

**Proposition 11.3.** There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there holds

$$
\frac{1}{2} \frac{d}{dt} \left\| \varphi w_e \right\|_{H^9_o}^2 + \frac{\varepsilon^2}{2} \left\| \nabla (\varphi w_e) \right\|_{H^9_o}^2 \leq C \varepsilon^2 \left( E(t) + \varepsilon^2 \right) \left( 1 + E(t) \right) + \frac{\varepsilon^4}{100} \left\| (\partial_y + |D_x|) w \right\|_{H^9_o}^2 + C \varepsilon^4 E(t)^{\frac{5}{2}}.
$$
12. Conormal Sobolev estimates of the vorticity: Prandtl part

Proposition 12.1. There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, there holds

$$\frac{1}{2} \frac{d}{dt} \| w_p \|^2_{H^2_p} + (\lambda - C) \left\| \frac{yw_p}{\varepsilon} \right\|_{H^2_p}^2 + \frac{\varepsilon^2}{10} \left\| (\partial_y + |D_x|)w_p \right\|_{H^2_p}^2 \leq C(E(t) + \varepsilon^2) + \frac{\varepsilon^2}{100} \left\| (\partial_y + |D_x|)w \right\|_{H^2_p}^2 + C\varepsilon^{-2} E(t).$$

Proof. First, acting $\partial^j_x Z^j$ on both sides of (3.3), and then taking $L^2$ inner product with $e^{2\Psi} \partial^j_x Z^j w_p$, summing over all $|i| + j \leq 9$, we arrive at

$$\frac{1}{2} \frac{d}{dt} \| w_p \|^2_{H^2_p} + \lambda \left\| \frac{yw_p}{\varepsilon} \right\|_{H^2_p}^2 - \varepsilon^2 \sum_{|i| + j \leq 9} \langle \partial^j_x Z^j (\Delta w_p), e^{2\Psi} \partial^j_x Z^j w_p \rangle \leq \sum_{|i| + j \leq 9} \langle \partial^j_x Z^j (\tilde{U}_a \cdot \nabla w_p), e^{2\Psi} \partial^j_x Z^j w_p \rangle + \sum_{|i| + j \leq 9} \langle \partial^j_x Z^j (\tilde{U} \cdot \nabla v_{a,p}), e^{2\Psi} \partial^j_x Z^j w_p \rangle + \sum_{|i| + j \leq 9} \langle \partial^j_x Z^j (\nabla U), e^{2\Psi} \partial^j_x Z^j w_p \rangle + \sum_{|i| + j \leq 9} \langle \partial^j_x Z^j (\text{curl}(R_{p,h} - M_p)), e^{2\Psi} \partial^j_x Z^j w_p \rangle \triangleq \sum_{i=1}^7 I_i.$$

Step 1. Estimate of $I_1$

We bound $I_1$ as follows

$$I_1 \leq \sum_{|i| + j \leq 9} \sum_{(m,n) < (i,j)} \left| \int_{\mathbb{R}^+} \left[ \partial^{-m}_x Z^{j-n} u_{a} \partial_x \partial^n_x Z^m w_p \right. \right.$$

$$+ \partial^{-m}_x Z^{j-n} (v_a - \varepsilon^2 f(t,x) e^{-y}) \partial^n_x Z^m \partial_y w_p \left] e^{2\Psi} \partial^j_x Z^j w_p \right. \right. dx dy \left. \right. \left. \right. + \sum_{|i| + j \leq 9} \left| \int_{\mathbb{R}^+} \left[ u_{a} \partial_x \partial^i_x Z^i w_p + (v_a - \varepsilon^2 f(t,x) e^{-y}) \partial^i_x Z^i \partial_y w_p \right] e^{2\Psi} \partial^j_x Z^j w_p dx dy \right|.$$

By Lemma 5.1, the first term can be bounded by $C\| w_p \|^2_{H^2_p}$, where we used $(v_a - \varepsilon^2 f(t,x)e^{-y})|_{y=0} = 0$ and (11.1). Integrating by parts, the second term can be bounded by

$$\sum_{|i| + j \leq 9} \left| \int_{R^3} (v_a - \varepsilon^2 f(t,x)e^{-y}) \partial^i_x Z^i \partial_y w_p e^{2\Psi} \partial^j_x Z^j w_p dx dy \right| + C\| w_p \|^2_{H^2_p},$$

$$\sum_{|i| + j \leq 9} \left| \int_{R^3} (v_a - \varepsilon^2 f(t,x)e^{-y}) \partial^i_x Z^i \partial_y w_p \right| \frac{y}{\varepsilon^2} e^{2\Psi} dx dy \right|.$$

Using $[\partial^i_x Z^i, \partial_y] = \psi \partial^i_x Z^{-1} \partial_y$ and (11.1), we have

$$\sum_{|i| + j \leq 9} \left| \int_{R^3} (v_a - \varepsilon^2 f(t,x)e^{-y}) \partial^i_x Z^i \partial_y w_p e^{2\Psi} \partial^j_x Z^j w_p dx dy \right| \leq C\| w_p \|^2_{H^2_p}.$$
and
\[
\sum_{|i|+j\leq 9} \left| \int_{\mathbb{R}_+^2} (v_a - \varepsilon^2 f(t, x)e^{-y}) \left( \frac{w}{y} \right)^{1/2} \frac{\partial_z Z^j w_p}{\epsilon^2} dxdy \right| \leq C \frac{\|w_p\|}{\epsilon}^{2} H_p^2.
\]

Thus, we obtain
\[
I_1 \leq C \left( \frac{\|w_p\|}{\epsilon}^{2} H_p^2 + \|w_p\|^{2} H_p^2 \right).
\]

Step 2. Estimate of $I_2$.

We bound $I_2$ as follows
\[
\left| \sum_{|i|+j\leq 9} \int_{y(t)}_{y(t)+\infty} \int_{\mathbb{R}_+^2} \frac{\partial_z Z^j}{\epsilon^2} (u \cdot \nabla_x w_{a,p} + (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y w_{a,p}) e^{2\Psi_r} \partial_z Z^j w_p dxdy \right|
\]
\[
+ \left| \sum_{|i|+j\leq 9} \int_{y(t)}_{y(t)+\infty} \int_{\mathbb{R}_+^2} (\frac{\partial_z Z^j}{\epsilon^2} (u \cdot \nabla_x w_{a,p} + (v + \varepsilon^2 f(t, x)e^{-y}) \partial_y w_{a,p}) e^{2\Psi_r} \partial_z Z^j w_p dxdy \right|
\]

Using the fact that $\|e^{\Psi_r} \partial_z Z^j \partial_x y w_{a,p}\|_{L^\infty(\mathbb{R}_x \times (y(t),\infty))} \leq C$, $Z u = \varphi w^1_H + \varphi \nabla_x v$ and $\partial_y v = -\nabla_x \cdot u$, the first term can be bounded by
\[
C \left( \|w_p\|_H^{2} + \|U\|_H^{2} + \|\varphi v\|_H^{2} \right).
\]

Notice that $w_{a,p} = \partial_y (u_{a,p}) - \partial x v_{a,p}$, $Z u = \varphi w^1_H + \varphi \nabla_x v$, thus, by Lemma 5.1 we have
\[
\left| \sum_{|i|+j\leq 9} \int_{y(t)}_{y(t)+\infty} \int_{\mathbb{R}_+^2} \frac{\partial_z Z^j}{\epsilon^2} (u \cdot \nabla_x w_{a,p} + \varphi \nabla_x v) e^{2\Psi_r} \partial_z Z^j w_p dxdy \right|
\]
\[
\leq C \|w_p\|_H^{2} + C\varepsilon^{-2} \left( \|U\|_H^{2} + \|\varphi v\|_H^{2} \right).
\]

Using $\partial_y v = -\nabla_x \cdot u$, $(v + \varepsilon^2 f(t, x)e^{-y})|_{y=0} = 0$ and Lemma 5.1, we get
\[
\left| \sum_{|i|+j\leq 9} \int_{y(t)}_{y(t)+\infty} \int_{\mathbb{R}_+^2} \frac{\partial_z Z^j}{\epsilon^2} (u \cdot \nabla_x w_{a,p} + \varphi \nabla_x v) e^{2\Psi_r} \partial_z Z^j w_p dxdy \right|
\]
\[
\leq C \|w_p\|_H^{2} + C\varepsilon^{-2} \left( \|U\|_H^{2} + \|\varphi v\|_H^{2} + \epsilon^4 \right).
\]

Thus, we arrive at
\[
I_2 \leq C \|w_p\|_H^{2} + C\varepsilon^{-2} \left( \|U\|_H^{2} + \|\varphi v\|_H^{2} + \epsilon^4 \right).
\]

Step 3. Estimate of $I_3$ and $I_0$.

Following the arguments of Step 3 in Proposition 11.1, we obtain
\[
I_3 \leq \frac{\varepsilon^2}{100} \|\partial_y + |D_x|\|w_p\|_H^{2} + \|w_p\|_H^{2} + C\varepsilon^{-2} \left( \epsilon^4 + \|U\|_H^{2} + \|w\|_H^{2} \right) \|w_p\|_H^{2}
\]
\[
+ C \left( \varepsilon^2 + \|U\|_H^{2} + \|w\|_H^{2} \right) \|w_p\|_H^{2},
\]
and

\[
I_6 \leq C \varepsilon^{-\frac{4}{9}} E(t)^{\frac{8}{9}} + CE(t)^{\frac{2}{9}} + \frac{\varepsilon^2}{100} \left\| (\partial_y + |D_y|)w \right\|_{H^{10}_p}^2 + \frac{\varepsilon^2}{100} \left\| (\partial_y + |D_y|)w_p \right\|_{H^9_p}^2 + \frac{\|w_p\|_p^2}{\varepsilon} \left\| w_p \right\|_{H^9_p}^2.
\]

**Step 4. Estimate of \( I_4 \).**

We split \( I_4 \) into two parts

\[
I_4 \leq \left| \sum_{|i|+j \leq 9} \left\langle \partial_x^i Z^j (w_{a,p,h} \cdot \nabla_x U), e^{2\Phi_p} \partial_x^j Z^j w_p \right\rangle \right|
\]

\[
+ \left| \sum_{|i|+j \leq 9} \left\langle \partial_x^i Z^j (w_{a,p,3} \partial_y U), e^{2\Phi_p} \partial_x^j Z^j w_p \right\rangle \right| \triangleq I_{41} + I_{42}.
\]

Using that \( \|e^{2\Phi_p} \partial_x^i Z^j w_{a,p,3}\|_{L^\infty} \leq C \), \( \partial_y u = w_{h,1} + \nabla_x v \) and \( \partial_y v = -\nabla_x \cdot u \), we obtain

\[
I_{42} \leq C \left( \|w\|_{H^{10}_p}^2 + \|U\|_{H^{10}_{tan}}^2 + \|w_p\|_{H^9_p}^2 \right).
\]

While, using \( \|e^{2\Phi_p} \partial_x^i Z^j w_{a,p,h}\|_{L^\infty} \leq \frac{C}{\varepsilon} \), we obtain

\[
I_{41} \leq C \varepsilon^{-2} \left( \|\varphi w\|_{H^{10}_p}^2 + \|U\|_{H^{10}_{tan}}^2 \right) + C \|w_p\|_{H^9_p}^2.
\]

This shows that

\[
I_4 \leq C \varepsilon^{-2} \left( \|\varphi w\|_{H^{10}_p}^2 + \|U\|_{H^{10}_{tan}}^2 \right) + C \|w_p\|_{H^9_p}^2.
\]

**Step 5. Estimate of \( I_5 \).**

First, we split \( I_5 \) into two parts

\[
I^p_{51} \leq \left| \sum_{|i|+j \leq 9} \left\langle \partial_x^i Z^j (w_{p,h} \partial_x U_a), e^{2\Phi_p} \partial_x^j Z^j w_p \right\rangle \right|
\]

\[
+ \left| \sum_{|i|+j \leq 9} \left\langle \partial_x^i Z^j (w_{p,3} \partial_y U_a), e^{2\Phi_p} \partial_x^j Z^j w_p \right\rangle \right| \triangleq I_{51} + I_{52}.
\]

We get by Lemma 5.1 that

\[
I_{51} \leq C \|w_p\|_{H^9_p}^2.
\]

Using \( \|\partial_x^i Z^j \partial_y U_a\|_{L^\infty} \leq \frac{C}{\varepsilon} \), we have

\[
I_{52} \leq C \left( \|w_p\|_{H^9_p}^2 + \left\| \frac{w_p}{\varepsilon} \right\|_{H^9_p}^2 \right).
\]

Thus, we get

\[
I_5 \leq C \left( \|w_p\|_{H^9_p}^2 + \left\| \frac{w_p}{\varepsilon} \right\|_{H^9_p}^2 \right).
\]

**Step 6. Estimate of \( I_7 \).**

It follows from Lemma 5.2 that

\[
I_7 \leq C \|w_p\|_{H^9_p}^2 + \varepsilon^2.
\]

**Step 7. Estimate of dissipative term.**
We get, by integrating by parts, that
\[ -\varepsilon^2 \sum_{|i|+j\leq 9} \langle \partial^i_x Z^j (\Delta w_p), e^{2\Psi_N} \partial^i_x Z^j w_p \rangle \]
\[ = \varepsilon^2 \| Aw_p \|_{H^p}^2 - \varepsilon^2 \sum_{|i|+j\leq 9} \langle [\partial^i_x Z^j, \partial_y] (Aw_p), e^{2\Psi_N} \partial^i_x Z^j w_p \rangle + \varepsilon^2 \sum_{|i|\leq 9} \int_{\mathbb{R}^2} \partial^i_x (Aw_p) \partial^i_x w_p(x, 0)dx \]
\[ + \varepsilon^2 \sum_{|i|+j\leq 9} \langle \partial^i_x Z^j (Aw_p), e^{2\Psi_N} [\partial_y, \partial^i_x Z^j] w_p \rangle + 4(\delta - \lambda t) \sum_{|i|+j\leq 9} \langle \partial^i_x Z^j (Aw_p), e^{2\Psi_N} \partial^i_x Z^j w_p \rangle \]
\[ \geq \frac{\varepsilon^2}{2} \| Aw_p \|_{H^p}^2 - C \delta \varepsilon^2 \| w_p \|_{H^p} - C \delta^2 \left\| \frac{\partial w_p}{\varepsilon} \right\|_{H^p}^2 + \varepsilon^2 \sum_{|i|\leq 9} \int_{\mathbb{R}^2} \partial^i_x (Aw_p) \partial^i_x w_p(x, 0)dx. \] (12.1)

Using the boundary condition of \((\partial_y + |D_x|)w_p\), we arrive at
\[ \varepsilon^2 \sum_{|i|\leq 9} \int_{\mathbb{R}^2} \partial^i_x ((\partial_y + |D_x|)w_p) \partial^i_x w_p(t, x, 0)dx \]
\[ = -\varepsilon^2 \sum_{|i|\leq 9} \int_{\mathbb{R}^2} \partial^i_x \nabla_x (\Lambda_N \partial_y \cdot w_p h) \cdot \partial^i_x w_p(t, x, 0)dx \]
\[ + \sum_{|i|\leq 9} \int_{\mathbb{R}^2} \partial^i_x w_p h \partial^i_x (\partial_y (-\Delta)^{-1} J_h + \partial_x (-\Delta_N)^{-1} J_3)(t, x, 0)dx \]
\[ \geq \sum_{|i|\leq 9} \int_{\mathbb{R}^2} \partial^i_x w_p h \partial^i_x (\partial_y (-\Delta)^{-1} J_h + \partial_x (-\Delta_N)^{-1} J_3)(t, x, 0)dx. \]

Thus, we only need to deal with
\[ \sum_{|i|\leq 9} \int_{\mathbb{R}^2} \partial^i_x w_p h \partial^i_x (\partial_y (-\Delta)^{-1} J_h + \partial_x (-\Delta_N)^{-1} J_3)(t, x, 0)dx, \]
which can be controlled by
\[ \sum_{|i|\leq 9} \left| \int_{\mathbb{R}^2} \int_0^{y(t)} 2 \zeta(t, y) \partial^i_x w_p h \partial^i_x (\partial_y (-\Delta)^{-1} J_h + \partial_x (-\Delta_N)^{-1} J_3)dydx \right| \]
\[ + \sum_{|i|\leq 9} \left| \int_{\mathbb{R}^2} \int_0^{y(t)} 2 \zeta(t, y) \partial^i_x \partial_y w_p h \partial^i_x (\partial_y (-\Delta)^{-1} J_h + \partial_x (-\Delta_N)^{-1} J_3)dydx \right| \]
\[ + \sum_{|i|\leq 9} \left| \int_{\mathbb{R}^2} \int_0^{y(t)} 2 \zeta(t, y) \partial^i_x w_p h \partial^i_x \partial_y (\partial_y (-\Delta)^{-1} J_h)dydx \right| \]
\[ + \sum_{|i|\leq 9} \left| \int_{\mathbb{R}^2} \int_0^{y(t)} 2 \zeta(t, y) \partial^i_x w_p h \partial^i_x \partial_{xy} (-\Delta_N)^{-1} J_3)dydx \right| \triangleq \sum_{k=1}^4 I^k_b, \]

where \(\zeta(t, y)\) is a smooth cut-off function which satisfies \(\zeta(t, 0) = 1\) and \(\zeta(t, y) = 0\) for \(y \geq y(t)/2\). Using \(L^2\) boundness of operators \(\partial_y (-\Delta_D)^{-1} \partial_x, \partial_y (-\Delta_D)^{-1} \partial_y, \partial_{yy} (-\Delta_N)^{-1}, \nabla (-\Delta_N)^{-1} \partial_x\) and \(\partial_{xy} (-\Delta_N)^{-1}\), we can deduce from Lemma \(6.2\) that
\[ I^1_b \leq C \| w_p \|_{H^p}^2 + C \| \tilde{U} a \cdot \nabla U + \tilde{U} \cdot \nabla U_a + \tilde{U} \cdot \nabla U + R \|_{L^2_{H^p}}^2 \]
\[ \leq C \| w_p \|_{H^p}^2 + C \varepsilon^2 (\varepsilon^2 + E(t)) (E(t) + 1), \]
and by Lemma 5.3
\[
I_b^2 \leq \kappa \varepsilon^2 \|\partial_y w_p\|_{H^2_{tan}}^2 + C\varepsilon^{-2} \|\tilde{U}_a \cdot \nabla U + \tilde{U} \cdot \nabla U_a + \tilde{U} \cdot \nabla U + R\|_{H^2_{tan}}^2
\]
\[
\leq \kappa \varepsilon^2 \|(\partial_y + |D_x|)w_p\|_{H^2_p}^2 + C(\varepsilon^2 + E(t))(E(t) + 1).
\]

On the other hand, using the fact \(\phi(t, y) \geq c\delta > 0\) for \(y \leq \frac{y(t)}{2}\), we deduce that
\[
I_b^4 \leq C\|(w_p)\|_{H^2_p}^2 + C\|\tilde{U}_a \cdot \nabla U + \tilde{U} \cdot \nabla U_a + \tilde{U} \cdot \nabla U + R\|_{H^2_{tan}}^2
\]
\[
\leq C\|(w_p)\|_{H^2_p}^2 + C\varepsilon^2(\varepsilon^2 + E(t))(E(t) + 1).
\]

We write
\[
I_3^b \leq \sum_{|i| \leq 9} \left| \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) \partial_x^i w_{p,h} \partial_2 J_h dydx \right|
\]
\[
+ \sum_{|i| \leq 9} \left| \int_{\mathbb{R}^2} \int_0^{y(t)} \zeta(t, y) \partial_x^i w_{p,h} \partial_x^9 (\Delta_D)^{-1} J_h dydx \right|.
\]

Noticing that \((F, G)(t, x, 0) = 0\), the first term can be handled as \(I_1^b, I_2^b\) by integrating by parts, while the second term can be handled as \(I_4^b\).

Finally, fixing \(\kappa\) small, we obtain
\[
-\varepsilon^2 \sum_{|i|+j \leq 9} \langle \partial_x^i Z^j(\Delta w_p), e^{2\Psi_p} \partial_x^i Z^j w_p \rangle
\]
\[
\geq \frac{\varepsilon^2}{4} \|Aw_p\|_{H^2_p}^2 - C\left(\frac{y_{w_p}}{\varepsilon}\right)^2 - C(\varepsilon^2 + E(t))(1 + E(t)).
\]

The proposition follows by combing the estimates in Step 1-Step 7. \(\square\)

Similarly, we can prove the following improved decay estimate in \(\varepsilon\) for \(w_{p,3}\) and \(\varphi w_p\). Again we omit the details.

**Proposition 12.2.** There exists \(\delta_0 > 0\) such that for any \(\delta \in (0, \delta_0)\), there holds
\[
\frac{1}{2} \frac{d}{dt} \|w_{p,3}\|_{H^2_p}^2 + (\lambda - C) \left(\frac{y_{w_{p,3}}}{\varepsilon}\right)^2 + \frac{\varepsilon^2}{2} \|\nabla w_{p,3}\|_{H^2_p}^2
\]
\[
\leq C\varepsilon^2(\varepsilon^2 + E(t))(1 + E(t)) + \frac{\varepsilon^4}{100} \|(\partial_y + |D_x|)w\|_{H_{co}^2}^2 + C\varepsilon^2 E(t)^\frac{3}{2}.
\]

**Proposition 12.3.** There exists \(\delta_0 > 0\) such that for any \(\delta \in (0, \delta_0)\), there holds
\[
\frac{1}{2} \frac{d}{dt} \|\varphi w_p\|_{H^2_p}^2 + (\lambda - C) \left(\frac{y_{\varphi w_p}}{\varepsilon}\right)^2 + \frac{\varepsilon^2}{2} \|\nabla (\varphi w_p)\|_{H^2_p}^2
\]
\[
\leq C\varepsilon^2(\varepsilon^2 + E(t))(1 + E(t)) + \frac{\varepsilon^4}{100} \|(\partial_y + |D_x|)w\|_{H_{co}^2}^2 + C\varepsilon^2 E(t)^\frac{3}{2}.
\]

13. **Appendix**

In this appendix, we prove the well-posedness of the Euler system (2.1) and Prandtl system (2.5). The proof of well-posedness of the linearized equations (2.2) and (2.6) is similar, thus we omit the details.
13.1. Well-posedness of the Euler system.

Proposition 13.1. Let \((u_0, v_0) \in H^3(\mathbb{R}^3)\) with \(\text{div}(u_0, v_0) = 0\) and \((u_0, v_0)(x, 0) = 0\). Moreover, assume that \(\text{curl}(u_0, v_0) = 0\) in the domain \(\{(x, y) \in \mathbb{R}^3 : y \leq 2\}\). Then there exists \(T_e > 0\) such that the system (2.1) has a unique solution \(\{U^e, \Phi^e\}\) on \([0, T_e]\), which satisfies

\[
\sup_{0 \leq t \leq T_e} \|U^e_t\|_{H^3}^2 \leq C, \quad \sup_{0 \leq t \leq T_e} \|\Phi^e_t\|_{H^3}^2 \leq C,
\]

where \(\Phi^e = (1 - y)\eta(x)\) and \(H^m\) is the usual Sobolev space.

Proof. Here we only present a priori estimate of the solution. We consider the vorticity equation of the system (2.1)

\[
\begin{aligned}
\partial_t w^e + U^e \cdot \nabla w^e - w^e \cdot \nabla U^e &= 0, \\
w^e(0, x, y) &= \text{curl}(u_0, v_0) \triangleq w_0^e.
\end{aligned}
\]

First of all, the standard energy estimate ensures that

\[
\sup_{t \in [0, T_e]} \|w^e(t)\|_{H^m} \leq C
\]

for some \(T_e > 0\). Because of \(w_0^e = 0\) in \(\{(x, y) \in \mathbb{R}^3 : y \leq 2\}\), we can deduce that \(w^e(t, x, y) = 0\) in \(\{(x, y) \in \mathbb{R}^3 : y \leq \frac{3}{2}\}\) for any \(t \in [0, T_e]\) (take \(T_e\) smaller if necessary).

Thanks to

\[
\Delta w^e = \partial_{x^2} w_1^e - \partial_{x^1} \partial_{x^2} w_2^e, \quad w^e|_{y=0} = 0,
\]

we deduce that

\[
\|w^e\|_{H^3}^2 \leq C\left(\|w^e\|_{H^3}^2 + \|\Phi^e\|_{H^3}^2\right).
\]

On the other hand,

\[
\partial_y u^e = u^e_{y^1} + \partial_x v^e, \quad \nabla_x \cdot u^e = -\partial_y v^e, \quad \text{curl}_x u^e = w_3^e,
\]

therefore,

\[
\begin{aligned}
\Delta_x u_1^e &= -\partial_{y^1} v^e + \partial_{x^2} w_2^e, \quad \partial_y u_1^e = w_2^e - \partial_{x^1} v^e, \\
\Delta_x u_2^e &= -\partial_{y^2} v^e - \partial_{x^1} w_3^e, \quad \partial_y u_2^e = -w_1^e + \partial_{x^2} v^e,
\end{aligned}
\]

which along with (13.1) imply that

\[
\|w^e\|_{H^3}^2 \leq C\left(\|w^e\|_{H^3}^2 + \|\Phi^e\|_{H^3}^2\right).
\]

Thus, we arrive at

\[
\|w^e\|_{H^3}^2 \leq C\left(\|w^e\|_{H^3}^2 + \|\Phi^e\|_{H^3}^2\right),
\]

which implies

\[
\|U^e\|_{H^3}^2 \leq C\left(\|w^e\|_{H^3}^2 + \|\Phi^e\|_{H^3}^2\right).
\]

by using the fact that for any \(\varepsilon \in (0, 1)\),

\[
\|U^e\|_{H^3}^2 \leq \varepsilon \|w^e\|_{H^3}^2 + C(\varepsilon)\|U^e\|_{H^3}^2.
\]

A similar argument as above gives

\[
\|\partial_t w^e\|_{H^3}^2 \leq C\left(\|\partial_t w^e\|_{H^3}^2 + \|\partial_t U^e\|_{H^3}^2\right).
\]

Now, we deduce \(\|\partial_t w^e\|_{H^3}^2 \leq C\) from the vorticity equation, and \(\|\partial_t U^e\|_{H^3}^2 \leq C\) from the velocity equation. Thus,

\[
\|\partial_t U^e\|_{H^3}^2 \leq C.
\]

This completes the proof. \(\square\)
13.2. **Well-posedness of the Prandtl system.** To prove the well-posedness of the Prandtl system, we first introduce some weighted norms

\[
\|U^p\|_{H^m_p}^2 = \sum_{|j|+k\leq m} \int_{\mathbb{R}^3_+} |e^{\phi_p(t,z)} \partial^j_x \tilde{Z}^k U^p|^2 \, dx \, dz,
\]

\[
\|U^p\|_{H^m_p}^{2,1} = \sum_{|j|+k\leq m} \int_{\mathbb{R}^3_+} |e^{\phi_p(t,z)} \langle D_x \rangle^j \partial^k_x \tilde{Z}^k U^p|^2 \, dx \, dz + \sum_{|j|+k\leq m} \int_{\mathbb{R}^3_+} |ze^{\phi_p(t,z)} \partial^j_x \tilde{Z}^k U^p|^2 \, dx \, dz.
\]

where \(\phi_p(t,z) = \rho_p(t)z^2\) with \(\rho_p(t) = 1 - \lambda_p t \geq 1\) and \(\lambda_p\) defined later.

**Proposition 13.2.** Let \((u^e, v^e)\) be given as above proposition. There exists \(T_p > 0\) such that the system (2.5) has a unique solution \(U^p = (u^p, v^p)\) on \([0, T_p]\), which satisfies

\[
\sup_{0 \leq t \leq T_p} \|U^p \|_{H^m_p}^2 \leq C_0, \quad \sup_{0 \leq t \leq T_p} (\| (\partial_t U^p) \|_{H^{m+1}_p}^2 + \| (\partial_x^2 U^p) \|_{H^{m+2}_p}^2) \leq C,
\]

where \(\Phi_p = \rho_p(t)\langle \xi \rangle\).

**Proof.** As in Proposition 13.1, we only give a priori estimate. Recall that \(u^p\) satisfies

\[
\begin{cases}
\partial_t u^p - \partial_{zz} u^p + u^p \cdot \nabla_x u^e (t, x, 0) + (u^e(t, x, 0) + u^p) \cdot \nabla_x u^p
+ \left( - \int_0^z \nabla_x \cdot u^p(t, x, \xi) d\xi + z \partial_y v^e(t, x, 0) \right) \partial_z u^p = 0, \\
u^p(0, x, z) = 0, \\
limit_{z \to \infty} u^p(t, x, z) = 0, \quad u_p(t, x, 0) = -u^e(t, x, 0).
\end{cases}
\]

We set

\[
\overline{\mu}^p = u^p + e^{-2\phi_p(t,z)} u^e(t, x, 0) \triangleq u^p + g.
\]

Thus, it’s easy to verify that \(\overline{\mu}^p\) satisfies

\[
\begin{cases}
\partial_t \overline{\mu}^p - \partial_{zz} \overline{\mu}^p + F^p = 0, \\
\overline{\mu}^p(0, x, y) = 0, \\
limit_{z \to \infty} \overline{\mu}^p(t, x, z) = 0, \quad \overline{\mu}^p(t, x, 0) = 0,
\end{cases}
\]

where

\[
F^p = - \partial_t g + \partial_{zz} g + (\overline{\mu}^p - g) \cdot \nabla_x u^e(t, x, 0) + (u^e(t, x, 0) + \overline{\mu}^p - g) \cdot \nabla_x (\overline{\mu}^p - g)
+ \left( - \int_0^z \nabla_x \cdot (\overline{\mu}^p - g)(t, x, \xi) d\xi + z \partial_y v^e(t, x, 0) \right) \partial_z (\overline{\mu}^p - g).
\]

Acting \(\partial_x^j \tilde{Z}^k e^{\phi_p(t,z)} \delta(D_x)\) on both sides of (13.2), then taking \(L^2\) inner product with \(e^{2\phi_p(t,z)} \partial_x^j \tilde{Z}^k \overline{\mu}^p\), integrating by parts, summing over \(|j| + k \leq 27\) and fixing \(\delta\) small, we arrive at

\[
\frac{1}{2} \frac{d}{dt} \| \overline{\mu}^p \|_{H^m_p}^2 + \frac{\lambda_p}{2} \| \overline{\mu}^p \|_{H^m_p}^{2,1} + \frac{1}{2} \| (\partial_x^2 \overline{\mu}^p) \|_{H^{m+2}_p}^2 \leq C \sum_{|j| + k \leq 27} \left| \left\langle \partial_x^j \tilde{Z}^k F^p, e^{2\phi_p(t,z)} \partial_x^j \tilde{Z}^k \overline{\mu}^p \right\rangle \right|.
\]

Notice that \(g = e^{-2\phi_p(t,z)} u^e(t, x, 0)\). It is easy to get by Proposition 13.1 that

\[
\sum_{|j| + k \leq 27} \left| \left\langle \partial_x^j \tilde{Z}^k (\partial_t g + \nabla g + g \partial_x u^e(t, x, 0) + (u^e(t, x, 0) - g) \partial_x g) \phi_p, e^{2\phi_p(t,z)} \partial_x^j \tilde{Z}^k \overline{\mu}^p \right\rangle \right| \leq C + \| \overline{\mu}^p \|_{H^m_p}^{2,1}.
\]
and
\[
\sum_{|j|+k \leq 27} \left| \partial_x^j \tilde{Z}^k \left( \int_0^z \partial_z g(t, x, z') dz' \partial_z g + z \partial_y v^e(t, x, 0) \partial_z g \right) \right|_{\Phi_p} e^{2\phi_p(t, z)} \partial_x^j \tilde{Z}^k \left| \right|_{\Phi_p} \leq C + \left| \right|_{\Phi_p} \leq 27.
\]

By Proposition 13.1 again, we have
\[
\sum_{|j|+k \leq 27} \left| \partial_x^j \tilde{Z}^k \left( \partial_x^j \partial_x^k u^e(t, x, 0) \right) \right|_{\Phi_p} e^{2\phi_p(t, z)} \partial_x^j \tilde{Z}^k \left| \right|_{\Phi_p} \leq C \left| \right|_{\Phi_p} \leq 27.
\]

Finally, we consider the transport term
\[
\sum_{|j|+k \leq 27} \left| \partial_x^j \tilde{Z}^k \left( \partial_x^j \partial_x^k - \int_0^z \partial_z \partial_x^j(t, x, z') dz' \partial_z \partial_x^k \right) \right|_{\Phi_p} e^{2\phi_p(t, z)} \partial_x^j \tilde{Z}^k \left| \right|_{\Phi_p}.
\]

First, there holds
\[
\sum_{|j|+k \leq 27} \left| \int_{\mathbb{R}_+^3} \partial_x^j \tilde{Z}^k \left( \partial_x^j \partial_x^k \right) \partial_x^j \tilde{Z}^k \partial_x^j \left| \right|_{\Phi_p} \right| \leq C \left( \left| \right|_{\Phi_p} \right) + 1 \left| \right|_{\Phi_p} \leq 27 + 1 \left| \right|_{\Phi_p} \leq 27.
\]

Then, a direct computation gives
\[
\sum_{|j|+k \leq 27} \left| \int_{\mathbb{R}_+^3} \partial_x^j \tilde{Z}^k \left( \partial_x^j \partial_x^k \right) \partial_x^j \tilde{Z}^k \partial_x^j \left| \right|_{\Phi_p} \right| \leq C \left( \left| \right|_{\Phi_p} \right) \left| \right|_{\Phi_p} \left| \right|_{\Phi_p} \leq 27 + 1 \left| \right|_{\Phi_p} \leq 27.
\]

Thus, collecting these estimates, we arrive at
\[
\sum_{t \leq T_p} \int_{\mathbb{R}_+^3} \partial_x^j \tilde{Z}^k \left( \partial_x^j \partial_x^k \right) \partial_x^j \tilde{Z}^k \partial_x^j \left| \right|_{\Phi_p} \left| \right|_{\Phi_p} \leq C \left( \left| \right|_{\Phi_p} \right) \left| \right|_{\Phi_p} \leq 27 + 1 \left| \right|_{\Phi_p} \leq 27.
\]

With this, a continuous argument ensures that there exists \( T_p > 0 \) so that
\[
\sup_{0 \leq t \leq T_p} \left| \right|_{\Phi_p} \leq C,
\]
from which and Proposition 13.1 we infer that
\[
\sup_{0 \leq t \leq T_p} \left| \right|_{\Phi_p} \leq C.
\]

For the second estimate, we can first show that
\[
\sup_{0 \leq t \leq T_p} \left| \right|_{\Phi_p} \leq C,
\]
then the desired estimate can be deduced by using the equation. \( \square \)
13.3. Proof of Lemma [5.1] and Lemma [5.2] By the same arguments as in Proposition [13.1] and Proposition [13.2] we can prove the well-posedness of the linearized Euler equation (2.2) and hold the following uniform bounds for the approximate solution:

\[
\left\| (u_e^{(0)}, v_e^{(0)})_{\phi_e} \right\|_{H^{20}} \leq C, \quad \left\| (u_e^{(1)}, v_e^{(1)})_{\phi_e} \right\|_{H^{20}} \leq C, \quad \left\| \partial_t (u_e^{(0)}, v_e^{(0)})_{\phi_e} \right\|_{H^{20}} \leq C,
\]

\[
\left\| \partial_t (u_e^{(1)}, v_e^{(1)})_{\phi_e} \right\|_{H^{17}} \leq C, \quad \left\| (u_p^{(0)}, v_p^{(1)})_{\phi_p} \right\|_{H^{27}_{p}} \leq C, \quad \left\| (u_p^{(1)}, v_p^{(2)})_{\phi_p} \right\|_{H^{20}_{p}} \leq C,
\]

\[
\left\| \partial_t (u_p^{(0)}, v_p^{(1)})_{\phi_p} \right\|_{H^{24}_{p}} + \left\| \partial_{zz} (u_p^{(0)}, v_p^{(1)})_{\phi_p} \right\|_{H^{24}_{p}} \leq C,
\]

\[
\left\| \partial_t (u_p^{(1)}, v_p^{(2)})_{\phi_p} \right\|_{H^{17}_{p}} + \left\| \partial_{zz} (u_p^{(1)}, v_p^{(2)})_{\phi_p} \right\|_{H^{17}_{p}} \leq C.
\]

With these bounds, Lemma [5.1] and Lemma [5.2] follow easily.

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