Abstract

We extend Thomason’s homotopy colimit construction in the category of permutative categories to categories of algebras over an arbitrary $Cat$ operad and analyze its properties. We then use this homotopy colimit to prove that the classifying space functor induces an equivalence between the category of $n$-fold monoidal categories and the category of $C_n$-spaces after formally inverting certain classes of weak equivalences, where $C_n$ is the little $n$-cubes operad. As a consequence we obtain an equivalence of the categories of $n$-fold monoidal categories and the category of $n$-fold loop spaces and loop maps after localization with respect to some other class of weak equivalences. We recover Thomason’s corresponding result about infinite loop spaces and obtain related results about braided monoidal categories and 2-fold loop spaces.

Keywords:

2010 MSC: 55P35, 55P47, 55P48, 18D10, 18D50, 18C20

1. Introduction

In the last two decades there has been an increasing interest in algebraic structures on a category such as a monoidal structure, a symmetric monoidal one, a
braiding or a ribbon structure, motivated by questions arising from knot theory or mathematical physics. This paper deals with structures encoded by an operad.

Our motivation is the problem to determine those structures on a category which correspond to \( n \)-fold loop spaces: it has been known for quite some time that the classifying space of a monoidal category is an \( A_\infty \)-monoid, whose group completion is weakly equivalent to a loop space. In the same way, a symmetric monoidal category gives rise to an infinite loop space \([19]\), and a braided monoidal category to a double loop space \([8]\). In \([1]\) we introduced the notion of an \( n \)-fold monoidal category. The structure of such a category is encoded by a \( \Sigma \)-free operad \( \mathcal{M}_n \) in \( \text{Cat} \), the category of small categories. The classifying space functor turns \( \mathcal{M}_n \) into a \( \Sigma \)-free topological operad \( B\mathcal{M}_n \), and we could show that there is a sequence of weak equivalences of operads between \( B\mathcal{M}_n \) and the little \( n \)-cubes operad \( \mathcal{C}_n \). Hence each \( n \)-fold monoidal category gives rise to an \( n \)-fold loop space.

Now one would like to know whether each \( n \)-fold loop space can be obtained in this way up to weak equivalence. There has been evidence for this: it has been known for about forty years that each topological space is weakly equivalent to the classifying space of a category, and in 1995 Thomason proved that each infinite loop space comes from a symmetric monoidal category \([29]\). The result is obtained in two major steps. Using categorical coherence theory extensively, Thomason first shows that each infinite loop space arises from a simplicial symmetric monoidal category. He then applies the homotopy colimit construction in the category of symmetric monoidal categories to get rid of the simplicial parameter. The construction and analysis of this homotopy colimit is an essential part of \([27]\).

Refining Thomason’s argument and combining categorical with homotopical coherence theory we were able to achieve the first step for \( n \)-fold monoidal categories: in \([10]\) we proved that each \( n \)-fold loop space comes from a simplicial \( n \)-fold monoidal category and it remains to bridge the gap from simplicial \( n \)-fold monoidal categories to \( n \)-fold monoidal categories.

For this we extend Thomason’s homotopy colimit construction for symmetric monoidal categories to algebras over an arbitrary \( \text{Cat} \)-operad. Thomason constructed a Quillen model category structure on \( \text{Cat} \) whose weak equivalences are those functors \( F : \mathcal{C} \to \mathcal{D} \) for which \( BF : B\mathcal{C} \to B\mathcal{D} \) is a weak homotopy equivalence \([27]\). Unfortunately this model structure is neither monoidal nor are the operads \( \mathcal{M}_n \) cofibrant so that the rich literature on algebras in model categories cannot be applied to \( \text{Cat} \), and, as a consequence, the results on homotopy colimits in model categories are not available either. It is our intention to close this gap.

We define and investigate homotopy colimits \( \text{hocolim}^\mathcal{M}X \) of diagrams \( X \) of algebras over an arbitrary \( \text{Cat} \)-operad \( \mathcal{M} \) and prove that there is a natural weak equivalence

\[
\text{hocolim}^{BM} BX \to B(\text{hocolim}^\mathcal{M}X)
\]
if $\mathcal{M}$ is $\Sigma$-free (Theorem 6.5). In the process we correct a minor flaw in Thomason’s argument (see Remark 8.8). Although we only use the homotopy colimit construction to prove results about iterated loop spaces, we believe that it is of separate interest. In particular, it sheds some light into Thomason’s original construction, where an operad cannot be seen explicitly.

We use this result to prove that in the $\Sigma$-free case the classifying space functor induces an equivalence of categories

$$B : \text{Cat}^M[\text{we}^{-1}] \to \text{Top}^B[\text{we}^{-1}]$$

where $\text{Cat}^M$ and $\text{Top}^B$ are the categories of $\mathcal{M}$-algebras in $\text{Cat}$, respectively $\mathcal{B}$-algebras in $\text{Top}$, and the weak equivalences are those morphisms of algebras whose underlying morphisms are weak equivalences in $\text{Cat}$, respectively weak homotopy equivalences in $\text{Top}$. Since we do not have a model type structure on $\text{Cat}^M$ it is not clear that the localization $\text{Cat}^M[\text{we}^{-1}]$ exists. To make the above equivalence hold we can use Grothendieck’s language of universes, where $\text{Cat}^M[\text{we}^{-1}]$ exists in some higher universe. Alternatively we can embed $\text{Cat}^M$ with the nerve functor $N$ into the category of $NM$-algebras in the category $\text{SSets}$ of simplicial sets and consider $\text{Cat}^M[\text{we}^{-1}]$ as the full subcategory of $\text{SSets}^{NM}[\text{we}^{-1}]$ of objects $NA$ with $A$ in $\text{Cat}^M$.

Combining this equivalence with the equivalence induced by the change of operad functors, we obtain an equivalence of categories

$$\text{Cat}^{M^n}[\text{we}^{-1}] \simeq \text{Top}^{BM^n}[\text{we}^{-1}] \simeq \text{Top}^{C_n}[\text{we}^{-1}].$$

In particular, each $n$-fold loop space arises up to weak equivalence from an $n$-fold monoidal category. Specializing to the operad encoding symmetric monoidal categories (or their equivalent more rigid formulations, the permutative categories) we obtain a version of Thomason’s result before group completion. In a similar way, we relate double loop spaces to braided monoidal categories.

The paper is organized as follows: In the following two sections we introduce the category theoretical tools we need for this paper. Sections 4 and 5 contain the construction of the homotopy colimit and compare it with known constructions in model categories. In Section 6 we determine the homotopy type of our homotopy colimit in the case of a $\Sigma$-free $\text{Cat}$-operad. In Section 7 we establish the equivalences of categories listed above. The last section contains the applications of our results to iterated loop spaces.

2. Categorical prerequisites

Throughout this paper we will work with categories enriched over a symmetric monoidal category $\mathcal{V}$. We will start with a general $\mathcal{V}$, but from Definition 2.2 onwards $\mathcal{V}$ will always be one of the categories $\text{Cat}$ of small categories, $\text{Sets}$ of sets, $\text{SSets}$ of simplicial sets, and $\text{Top}$ of $k$-spaces in the sense of [30, 5(ii)]. For basic concepts we refer to the books of Kelly [16] and Borceux [2].
Recall that \((\mathcal{V}, \Box, \Phi)\) is a \textit{symmetric monoidal category} if we are given a bifunctor 
\[ \Box : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \]
which is associative and commutative up to coherent natural isomorphisms and has a coherent unit object \(\Phi\) [2, 6.1.2]. If the associativity and unit isomorphisms are the identities we call \((\mathcal{V}, \Box, \Phi)\) a \textit{permutative category}. \(\mathcal{V}\) is \textit{closed} if it has an internal Hom-functor, i.e. a bifunctor \(\mathcal{V}(-, -) : \mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}\) together with a natural isomorphism
\[ \mathcal{V}(X \Box Y, Z) \cong \mathcal{V}(X, \mathcal{V}(Y, Z)). \]

A functor \(F : \mathcal{V} \to \mathcal{V}'\) between symmetric monoidal is called \textit{symmetric monoidal} if there are natural transformations
\[ \Phi_{\mathcal{V}}' \to F(\Phi_{\mathcal{V}}) \quad \text{and} \quad F(-) \Box_{\mathcal{V}}' F(-) \to F(-) \Box_{\mathcal{V}} F(-) \]
compatible with the coherence isomorphisms. If these maps are isomorphisms \(F\) is called \textit{strong}.

A \(\mathcal{V}\)-\textit{category} or \(\mathcal{V}\)-\textit{enriched category} is a category \(\mathcal{C}\) equipped with an enrichment functor
\[ \mathcal{C}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V} \]
and composition morphisms
\[ \mathcal{C}(B, C) \Box \mathcal{C}(A, B) \to \mathcal{C}(A, C) \]
in \(\mathcal{V}\) with identities \(j_A : \Phi \to \mathcal{C}(A, A)\), subject to the obvious conditions. \(\mathcal{V}\)-functors and \(\mathcal{V}\)-natural transformations are defined in the obvious way.

A \(\mathcal{V}\)-category \(\mathcal{C}\) is \textit{tensored} over \(\mathcal{V}\) if there is a functor
\[ \mathcal{V} \times \mathcal{C} \to \mathcal{C}, \quad (K, X) \mapsto K \otimes X \]
such that
1. there are natural isomorphisms
\[ K \otimes (L \otimes X) \cong (K \Box L) \otimes X \quad \text{and} \quad \Phi \otimes X \cong X, \]
   satisfying the standard associativity and unit coherence conditions,
2. there are natural isomorphisms
\[ \mathcal{C}(K \otimes X, Y) \cong \mathcal{V}(K, \mathcal{C}(X, Y)). \]

Setting \(K = \Phi\), we obtain \(\mathcal{C}(X, Y) \cong \mathcal{V}(\Phi, \mathcal{C}(X, Y))\), thus recovering the underlying category \(\mathcal{C}\) from its enriched version. It is easy to show that condition (2) implies an enriched version: there are natural isomorphisms
\[ \mathcal{C}(K \otimes X, Y) \cong \mathcal{V}(K, \mathcal{C}(X, Y)) \]
in \(\mathcal{V}\).

Dually, a \(\mathcal{V}\)-category \(\mathcal{C}\) is \textit{cotensored} over \(\mathcal{V}\) if there is a functor
\[ \mathcal{V}^{op} \times \mathcal{C} \to \mathcal{C}, \quad (K, X) \mapsto X^K \]
such that
(1) there are natural isomorphisms
\[ X^{K \Box L} \cong (X^L)^K \quad \text{and} \quad X^\otimes \cong X, \]
satisfying the standard associativity and unit coherence conditions,

(2) there are natural isomorphisms
\[ \mathcal{C}(X, Y^K) \cong \mathcal{V}(K, \mathcal{L}(X, Y)). \]

Again, the last condition implies an enriched version: there are natural isomorphisms
\[ \mathcal{C}(X, Y^K) \cong \mathcal{V}(K, \mathcal{L}(X, Y)) \]
in \( \mathcal{V} \).

2.1 Let \( \mathcal{L} \) be a small category, \( \mathcal{C} \) a \( \mathcal{V} \)-category tensored over \( \mathcal{V} \) and let
\[ D : \mathcal{L}^{\text{op}} \to \mathcal{V} \quad \text{and} \quad E : \mathcal{L} \to \mathcal{C} \]
be functors. We define \( D \otimes_{\mathcal{L}} E \) be the coend (see \[14\], IX.6) of the functor
\[ \mathcal{L}^{\text{op}} \times \mathcal{L} \xrightarrow{D \times E} \mathcal{V} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C} \]
A tensor is a special case of an indexed colimit and a cotensor a special case of an indexed limit (for a definition see \[16\], pp. 71-73). We call a \( \mathcal{V} \)-category \( \mathcal{V} \)-complete if it has all small indexed limits and \( \mathcal{V} \)-cocomplete if it has all small indexed colimits. Recall that \( \mathcal{C} \) is \( \mathcal{V} \)-complete if it is complete in the ordinary sense and cotensored, and \( \mathcal{V} \)-cocomplete if it is cocomplete in the ordinary sense and tensored \[16\], Thm. 3.7.3.

As mentioned in the beginning of this section, henceforth \( \mathcal{V} \) will be one of the categories \( \mathbf{Cat}, \mathbf{Sets}, \mathbf{SSets}, \) or \( \mathbf{Top} \). The symmetric monoidal structure of each of these categories is given by the product functor. The categories are closed, their internal Hom-functors are the functor categories, the sets of all maps, the simplicial Hom-functor, and the \( k \)-function spaces respectively. They are tensored and cotensored over themselves by the product and the internal Hom-functors.

Observe that \( \mathbf{Sets} \) can be considered as a symmetric monoidal subcategory of each of the others if we interpret a set as a discrete category, discrete simplicial set, or discrete space respectively. Hence the following definition makes sense.

2.2 Definition: Let \( \mathcal{V} \) be one of the categories \( \mathbf{Cat}, \mathbf{Sets}, \mathbf{SSets}, \) or \( \mathbf{Top} \). A \( \mathcal{V} \)-operad is a permutative \( \mathcal{V} \)-enriched category \( (\mathcal{M}, \oplus, 0) \) such that \( \text{ob} \mathcal{M} = \mathbb{N}, m \oplus n = m + n \), and the maps
\[ \prod_{r_1 + \ldots + r_n = m} \mathcal{M}(r_1, 1) \times \ldots \times \mathcal{M}(r_n, 1) \times \Sigma_{r_1} \times \ldots \times \Sigma_{r_n} \Sigma_m \to \mathcal{M}(m, n) \]
sending \((f_1, \ldots, f_n; \sigma)\) to \((f_1 \oplus \ldots \oplus f_n) \circ \sigma\) are isomorphisms in \(V\) (note that \(\Sigma_n\) operates on \(n = 1 \oplus \ldots \oplus 1\)).

An \(\mathcal{M}\)-algebra in \(V\) is a strong symmetric monoidal functor \(A : \mathcal{M} \to V\). A map of \(\mathcal{M}\)-algebras is a natural transformation \(\alpha : A \to B\) compatible with the symmetric monoidal structure; in particular, it is determined by \(\alpha(1) : A(1) \to B(1)\).

**Notation:** The morphism objects \(\mathcal{M}(m, n)\) of a \(V\)-operad \(\mathcal{M}\) are uniquely determined by the objects \(\mathcal{M}(r, 1)\). As is common usage we denote \(\mathcal{M}(r, 1)\) by \(\mathcal{M}(r)\).

Let \(V^\mathcal{M}\) be the category of \(\mathcal{M}\)-algebras in \(V\).

Recall that a monad on a category \(C\) is a functor \(\mathcal{M} : C \to C\) together with natural transformations
\[
\mu : \mathcal{M} \circ \mathcal{M} \to \mathcal{M} \quad \text{and} \quad \iota : \text{Id}_C \to \mathcal{M}
\]
satisfying the associativity condition \(\mu \circ \mu \mathcal{M} = \mu \circ \mathcal{M} \mu\) and the unit conditions \(\mu \circ \iota \mathcal{M} = \mu \circ \mathcal{M} \iota\).

An algebra over \(\mathcal{M}\) or \(\mathcal{M}\)-algebra in \(C\) is a pair \((A, \xi)\) consisting of an object \(A\) of \(C\) and a map \(\xi : \mathcal{M}A \to A\) satisfying
\[
\xi \circ \iota(A) = \text{id}_A \quad \text{and} \quad \xi \circ \mu(A) = \xi \circ \mathcal{M}\xi.
\]

A map of \(\mathcal{M}\)-algebras \(f : (A, \xi_A) \to (B, \xi_B)\) is a map \(f : A \to B\) in \(C\) such that
\[
\xi_B \circ \mathcal{M}f = f \circ \xi_A.
\]

We denote the category of \(\mathcal{M}\)-algebras in \(C\) by \(C^\mathcal{M}\).

**2.3** For a \(V\)-operad \(\mathcal{M}\) we have an adjoint pair of \(V\)-functors
\[
F : V \Rightarrow \mathcal{M}^V : U
\]
consisting of the free algebra functor \(F\) and the underlying object functor \(U\) given by \(U(A) = A(1)\). The former is defined by
\[
F(X) = \prod_{n \geq 0} \mathcal{M}(n) \times_{\Sigma_n} X^n.
\]

Let \(\iota : \text{Id} \to UF\) and \(\varepsilon : FU \to \text{Id}\) be the unit and counit of this adjunction. Then \((\mathcal{M}, \mu, \iota) = (UF, U\varepsilon F, \iota)\) defines a monad on \(V\).

**2.4 Proposition:** Let \(\mathcal{M}\) be a \(V\)-operad. Then \(\mathcal{M}^V\) and \(V^\mathcal{M}\) are isomorphic.
Proof If $A$ is an $\mathcal{M}$-algebra in $\mathcal{V}$ the adjoint of the structure maps

$$A : \mathcal{M}(n) \to \mathcal{V}(A(1)^n, A(1)) \quad (\ast)$$

factor as

$$\mathcal{M}(n) \times A(1)^n \xrightarrow{\text{proj}} \mathcal{M}(n) \times \Sigma_n \xrightarrow{\xi_n} A(1) \quad (\ast\ast)$$

and the $\xi_n$ define an $\mathcal{M}$-algebra structure $\xi$ on $A(1)$. The correspondence $A \mapsto (A(1),\xi)$ extends to a functor $\mathcal{V}^{\mathcal{M}} \to \mathcal{V}^{\mathcal{M}}$. Conversely, given an $\mathcal{M}$-algebra $(A(1),\xi)$, the adjoints $(\ast)$ of the maps $(\ast\ast)$ define an $\mathcal{M}$-algebra $A$ in $\mathcal{V}$. This correspondence defines the inverse functor. □

If $\mathcal{M}$ is a $\mathcal{V}$-operad, we work in either of the categories $\mathcal{V}^{\mathcal{M}}$ and $\mathcal{V}^{\mathcal{M}}$ with a preference for $\mathcal{V}^{\mathcal{M}}$.

2.5 Let $A$ be an $\mathcal{M}$-algebra in $\mathcal{V}$ and $K \in \mathcal{V}$. Then the cotensor $A(1)^K$ in $\mathcal{V}$ has a canonical $\mathcal{M}$-algebra structure whose structure maps are the adjoints of $\mathcal{M}(n) \times (A(1)^K)^n \times K \cong \mathcal{M}(n) \times (A(1)^n)^K \times K \xrightarrow{id \times F} \mathcal{M}(n) \times A(1)^n \xrightarrow{a} A(1)$

where $e$ is the evaluation and $a$ is the structure map of $A$.

2.6 If $\mathbb{M}$ is a monad on $\mathcal{V}$ we enrich $\mathcal{V}^{\mathcal{M}}$ and hence $\mathcal{V}^{\mathcal{M}}$ over $\mathcal{V}$ as follows:

If $\mathcal{V} = \text{Cat}$ and $f,g : (A,\xi_A) \to (B,\xi_B)$ are maps of $\mathbb{M}$-algebras, a 2-morphism $s : f \Rightarrow g$ in $\text{Cat}^{\mathbb{M}}$ is a natural transformation satisfying

$$\xi_B \circ \mathbb{M}s = s \circ \xi_A$$

If $\mathcal{V} = \text{Sets}$ and $(A,\xi_A)$ and $(B,\xi_B)$ are $\mathbb{M}$-algebras in $\mathcal{V}$, then $B^{\Delta^n}$ is an $\mathbb{M}$-algebra by 2.3 and we define $\text{Sets}^{\mathbb{M}}(A,B)$ to be the set of all algebra maps $f : A \to B^{\Delta^n}$.

If $\mathcal{V} = \text{Top}$, we define $\text{Top}^{\mathbb{M}}(A,B)$ to be the subspace of $\text{Top}(A,B)$ of all maps of $\mathbb{M}$-algebras.

Let $\mathcal{M}$ be $\mathcal{V}$-operad and $\mathbb{M}$ its associated monad. The forgetful functor $U : \mathcal{V}^{\mathcal{M}} \to \mathcal{M}$ is faithful. If we consider $\mathcal{V}^{\mathcal{M}}(A,B)$ as a subobject of $\mathcal{V}(A(1),B(1))$ we obtain the $\mathcal{V}$-enrichment of $\mathcal{V}^{\mathcal{M}}$ corresponding to the $\mathcal{V}$-structure of $\mathcal{V}^{\mathcal{M}}$ under the isomorphism of Proposition 2.4.

2.7 Proposition: Let $\mathcal{M}$ be a $\mathcal{V}$-operad and $\mathbb{M}$ its associated monad. Then $\mathcal{V}^{\mathcal{M}}$ and $\mathcal{V}^{\mathbb{M}}$ are $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete.

Proof We first show that $U : \mathcal{V}^{\mathbb{M}} \to \mathcal{V}$ creates all limits and cotensors. Let $D : \mathcal{L} \to \mathcal{V}^{\mathbb{M}}$ be a diagram. Then

$$\lim U \circ D \subset \prod_{L \in \mathcal{L}} D(L)(1).$$
Coordinatewise operation gives the product an $\mathcal{M}$-structure which is inherited by $\lim U \circ D$. Hence $\lim D$ exists in $\mathcal{V}^\mathcal{M}$.

Let $A$ be an $\mathcal{M}$-algebra and $K \in \mathcal{V}$. Then $A(1)^K$ is the underlying object of the $\mathcal{M}$-algebra by \cite{2} which defines the cotensor in $\mathcal{V}^\mathcal{M}$.

Next we prove that $\mathcal{V}^\mathcal{M}$ is $\mathcal{V}$-complete. By \cite[VII.2.10]{7} it suffices to show that $\mathcal{M}$ preserves reflexive coequalizers. The statement of \cite[VII.2.10]{7} is proved for categories enriched over the category of weak Hausdorff $k$-spaces, but the proof holds verbatim also for categories enriched over $\mathcal{Cat}$, $\mathcal{SSets}$, or $\mathcal{Top}$.

A coequalizer diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\text{ } & \searrow{g} & \downarrow{h} \\
\text{ } & \text{ } & Z
\end{array}
$$

is called reflexive if there is a map $s : Y \to X$ such that $f \circ s = \text{id}_Y = g \circ s$.

We show that $\mathcal{M} = U \circ F : \mathcal{Cat} \to \mathcal{Cat}$ preserves reflexive coequalizers. The proofs for $\mathcal{SSets}$ and $\mathcal{Top}$ are analogous.

Since $F$ has a right adjoint it suffices to prove that $U$ preserves reflexive coequalizers. So let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\text{ } & \text{ } & \downarrow{h} \\
\text{ } & \text{ } & Z
\end{array}
$$

be a diagram in $\mathcal{Cat}^\mathcal{M}$ such that $f \circ s = \text{id}_Y = g \circ s$ and let

$$
\begin{array}{ccc}
UX & \xrightarrow{Uf} & UY \\
\text{ } & \searrow{Uh} & \downarrow{h} \\
\text{ } & \text{ } & Z
\end{array}
$$

be a coequalizer diagram in $\mathcal{Cat}$. We have to show that $Z$ has a unique $\mathcal{M}$-structure making $h$ a map of $\mathcal{M}$-algebras. The maps

$$
\xi : \mathcal{M}(n) \times Z^n \to Z
$$

are defined on objects $(A; z_1, \ldots, z_n)$ by choosing $y_1, \ldots, y_n \in UY$ such that $h(y_i) = z_i$ and setting

$$
\xi(A; z_1, \ldots, z_n) = h(\beta(A; y_1, \ldots, y_n))
$$

where $\beta$ is the $\mathcal{M}$-structure map of $Y$. This definition is independent of the choice of the $y_i$: for let $x$ be an object in $UX$ such that $f(x) = y_1$; let $\overline{y}_1 = g(x)$ and let $\overline{\alpha}(A; x, s(y_2), \ldots, s(y_n))$, where $\alpha$ is the $\mathcal{M}$-structure map of $X$. Since $f, g,$ and $s$ are maps of $\mathcal{M}$-algebras

$$
f(\overline{\alpha}) = \beta(A; f(x), f \circ s(y_2), \ldots, f \circ s(y_n)) = \beta(A; y_1, y_2, \ldots, y_n)
$$

$$
g(\overline{\alpha}) = \beta(A; g(x), g \circ s(y_2), \ldots, g \circ s(y_n)) = \beta(A; \overline{y}_1, y_2, \ldots, y_n)
$$

Hence

$$
h(\beta(A; y_1, y_2, \ldots, y_n)) = h(\beta(A; \overline{y}_1, y_2, \ldots, y_n))
$$
On morphisms we proceed analogously: if suffices to define $\xi$ for the generating morphisms

$$(w : z \to z') = h(v : y \to y')$$

which can be done in the same way as for objects. 

\[\square\]

2.8 Corollary: Let $\mathcal{M}$ be a $\mathcal{V}$-operad and $\mathcal{M}$ its associated monad. Then $\mathcal{V}^\mathcal{M}$ and $\mathcal{V}^{\mathcal{M}^e}$ are tensored and cotensored over $\mathcal{V}$.

2.9 We have a sequence of adjunctions where $N$ is the nerve functor, Sing is the singular, $|-|$ the topological realization, and $\text{cat}$ the categorification functor. The lower arrows are the left adjoints.

\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{N} & \text{SSets} \\
\text{cat} & \xleftarrow{\text{Sing}} & \text{Top} \\
\end{array}
\]

Since all these functors preserve products, they are strong monoidal functors. Hence any $\text{Cat}$-enriched category $A$ is $\text{SSets}$-enriched via the nerve functor $N$ and each $\text{Top}$-enriched category $B$ is $\text{SSets}$-enriched via the singular functor Sing [2, 6.4.3]. Moreover, $A$ as a $\text{SSets}$-category is tensored over $\text{SSets}$ by

$$K \otimes^\ast A = \text{cat}(K) \otimes A$$

and $B$ as a $\text{SSets}$-category is tensored over $\text{SSets}$ by

$$K \otimes^\ast B = |K| \otimes B,$$

where $\otimes^\ast$ stands for the tensor over $\text{SSets}$ while $\otimes$ stands for the tensor over $\text{Cat}$ respectively $\text{Top}$.

2.10 In the proof of the homotopy colimit theorem we will make use of lax functors and their rectifications. Let $\mathcal{L}$ be a small category. A lax functor $F : \mathcal{L} \to \text{Cat}$ is a pair of functions assigning a category $F(L)$ to each $L \in \text{ob}\mathcal{L}$ and a functor $F(f) : F(K) \to F(L)$ to each morphism $f : K \to L$ of $\mathcal{L}$ together with natural transformations

$$\rho(L) : F(id_L) \to id_{F(L)} \quad \sigma(f,g) : F(f \circ g) \to F(f) \circ F(g)$$

such that the following diagrams commute:

\[
\begin{array}{ccc}
F(f) \circ F(id_K) & \xrightarrow{\sigma(f,\text{id}_K)} & F(f) \\
\text{F(f)F(id_K)} & \xleftarrow{\rho(id_L,F(f))} & F(id_L) \circ F(f) \\
\end{array}
\]

\[
\begin{array}{ccc}
F(f \circ g \circ h) & \xrightarrow{\sigma(f \circ g, h)} & F(f \circ g) \circ F(h) \\
\text{F(f)(g)h} & \xleftarrow{\text{F}(f)\sigma(g,h)} & F(f) \circ F(g) \circ F(h) \\
\end{array}
\]
Let \( F, G : \mathcal{L} \to \mathcal{Cat} \) be lax functors. A (left) lax natural transformation \( d : F \to G \) is a pair of functions assigning a functor \( d(L) : F(L) \to G(L) \) to each \( L \in \text{ob} \mathcal{L} \) and a natural transformation \( d(f) : G(f) \circ d(K) \to d(L) \circ F(f) \) to each morphism \( f : K \to L \) of \( \mathcal{L} \) such that the following diagrams of natural transformations commute:

\[
\begin{array}{ccc}
G(id_L) \circ d(L) & \xrightarrow{d(id_L)} & d(L) \circ F(id_L) \\
\rho(L)d(L) & \downarrow & \downarrow d(L)\rho(L) \\
& & d(L)
\end{array}
\]

and for \( g \circ f : K \to L \to M \):

\[
\begin{array}{ccc}
G(g \circ f) \circ d(K) & \xrightarrow{d(g \circ f)} & d(M) \circ F(g \circ f) \\
\sigma(g,f)d(K) & \downarrow & \downarrow d(M)\sigma(g,f) \\
& & d(M) \circ F(g) \circ F(f) \\
G(g) \circ G(f) \circ d(K) & \xrightarrow{G(g)d(f)} & G(g) \circ d(L) \circ F(f)
\end{array}
\]

We can compose lax natural transformations in the obvious way and obtain a category of lax functors and lax natural transformations.

Lax functors can be rectified by Street’s rectification constructions \[25\]. We will use his first one:

**2.11 Proposition:** Let \( \mathcal{L} \) be a small category. There is a functor \( F \mapsto SF \) from the category of of lax functors \( F : \mathcal{L} \to \mathcal{Cat} \) and lax natural transformations to the category of strict functors \( \mathcal{L} \to \mathcal{Cat} \) and strict natural transformations with the following properties:

1. For each \( L \in \text{ob} \mathcal{L} \) there is a pair of adjoint functors

\[
\varepsilon(L) : SF(L) \Rightarrow F(L) : \eta(L)
\]

The \( \eta(L) \) combine to a lax natural transformation \( \eta : F \to SF \) such that for a lax natural transformation \( d : F \to G \) the diagram

\[
\begin{array}{ccc}
F(L) & \xrightarrow{d(L)} & G(L) \\
\eta(L) \downarrow & & \downarrow \eta(L) \\
SF(L) & \xrightarrow{Sd(L)} & SG(L)
\end{array}
\]

commutes. If \( F \) is a genuine functor then the \( \varepsilon(L) \) combine to a strict natural transformation \( \varepsilon : SF \to F \).
(2) If \( k : \mathcal{K} \to \mathcal{L} \) is a functor, then \( F \circ k : \mathcal{K} \to \text{Cat} \) is a lax functor and there is a natural transformation \( \xi : S(F \circ k) \to SF \circ k \) such that

\[
\begin{array}{ccc}
S(F \circ k)(K) & \xrightarrow{\eta_F(k(K))} & SF \circ k(K) \\
\downarrow & & \downarrow \\
F \circ k(K) & \xrightarrow{\xi(k)} & F \circ k(K)
\end{array}
\]

commutes for all \( K \in \text{ob}\mathcal{K} \). Moreover, \( \xi \) is natural with respect to lax natural transformations \( F \to G \).

3. Lax morphisms of \( M \)-algebras

3.1 Notation: Let

\[
\begin{array}{ccc}
f_1 & \xrightarrow{f_2} & g_1 \\
\sigma_1 \downarrow & & \tau \downarrow \\
f_3 & \xrightarrow{f_2} & g_2
\end{array}
\]

be a diagram of 1- and 2-cells in a 2-category \( \mathcal{K} \). We denote by \( \sigma_2 \circ \sigma_1 \) and \( \tau \circ_1 \sigma_1 \) the composite 2-cells

\[
\sigma_2 \circ_2 \sigma_1 : f_1 \xRightarrow{\sigma} f_2 \xRightarrow{\sigma_2} f_3
\]

\[
\tau \circ_1 \sigma_1 : g_1 \circ f_1 \Rightarrow g_2 \circ f_2
\]

We use the convention that \( g_1 \circ_1 \sigma_1 = \text{id}_{g_1} \circ_1 \sigma_1 \).

We are interested in a lax version of \( \text{Cat}^M \).

3.2 Definition: Let \( (M, \mu, \iota) \) be a monad on \( \text{Cat} \). A lax morphism

\[
(f, f) : (A, \xi_A) \to (B, \xi_B)
\]
of $M$-algebras is a pair consisting of a functor $f : A \to B$ and a natural transformation $\xi : \xi_B \circ Mf \Rightarrow f \circ \xi_A$

\[
\begin{array}{cccc}
M_A & \xrightarrow{Mf} & M_B \\
\downarrow \xi_A & & \downarrow \xi_B \\
A & \xrightarrow{f} & B
\end{array}
\]

satisfying

1. $\xi \circ 1 = (\xi_B \circ Mf) \circ 1 = \xi_B \circ Mf$

\[
\begin{array}{cccc}
M^2 A & \xrightarrow{M^2f} & M^2 B \\
\downarrow \mu_A & & \downarrow \mu_B \\
M A & \xrightarrow{Mf} & M B \\
\downarrow \xi_A & & \downarrow \xi_B \\
A & \xrightarrow{f} & B
\end{array}
= 
\begin{array}{cccc}
M^2 A & \xrightarrow{M^2f} & M^2 B \\
\downarrow \mu_A & & \downarrow \mu_B \\
M A & \xrightarrow{Mf} & M B \\
\downarrow \xi_A & & \downarrow \xi_B \\
A & \xrightarrow{f} & B
\end{array}

(2) $\xi \circ 1 = \mu^{-1}(A)$

2. $\xi_B \circ Mf = \mu \circ (\xi_B \circ Mf)$

\[
\begin{array}{cccc}
M^2 A & \xrightarrow{M^2f} & M^2 B \\
\downarrow \mu_A & & \downarrow \mu_B \\
M A & \xrightarrow{Mf} & M B \\
\downarrow \xi_A & & \downarrow \xi_B \\
A & \xrightarrow{f} & B
\end{array}
\]

(recall that $\xi_B \circ M \xi_A = \xi_B \circ \mu(A)$)

(2) $\xi_B \circ \mu(A) = \text{id}_f$
A 2-cell \( s : (f, \overline{g}) \Rightarrow (g, \overline{g}) \) between two lax morphisms \( (f, \overline{g}), (g, \overline{g}) : A \to B \) is a natural transformation \( s : f \Rightarrow g \) satisfying

\[ \overline{g} \circ \alpha \circ (\xi_B \circ 1_M s) = (s \circ 1_{\xi_A}) \circ \overline{f} \]

We can compose lax morphisms and 2-cells between lax morphisms in the obvious way and obtain a category \( \mathcal{Cat}_{\mathcal{M}}^{\text{lax}} \).

For later use it is convenient to have a more explicit description of a lax morphism.

### 3.3

A lax morphism \( (f, \overline{f}) : A \to B \) of \( \mathcal{M} \)-algebras is explicitly given by a functor \( f : A \to B \) and morphisms

\[ \overline{f}(A; K_1, \ldots, K_n) : f(K_1, \ldots, fK_n) \to f(A(K_1, \ldots, K_n)) \]

in \( B \), where \( A \in \mathcal{M}(n), K_1, \ldots, K_n \in A \), such that

1. for \( \alpha : A_1 \to A_2 \) in \( \mathcal{M}(n) \) and \( k_i : K_i \to K_i' \) in \( A \), the diagram

   \[
   \begin{array}{c}
   A_1(fK_1, \ldots, fK_n) \xrightarrow{\overline{f}(A_1; K_1, \ldots, K_n)} f(A_1(K_1, \ldots, K_n)) \\
   \downarrow_{\alpha(f(k_1), \ldots, f(k_n))} \quad \downarrow_{f(\alpha(k_1, \ldots, k_n))} \\
   A_2(fK'_1, \ldots, fK'_n) \xrightarrow{\overline{f}(A_2; K'_1, \ldots, K'_n)} f(A_2(K'_1, \ldots, K'_n))
   \end{array}
   \]

   commutes.

2. for \( A \in \mathcal{M}(k), B_i \in \mathcal{M}(r_i), K_{ij} \in A \)

   \[
   \overline{f}(A \ast (B_1 \oplus \ldots \oplus B_k); K_{11}, \ldots, K_{1r_1}, \ldots, K_{kr_k}) = \overline{f}(A; B_1(K_{11}, \ldots, K_{1r_1}), \ldots, B_k(K_{kr_1}, \ldots, K_{kr_k}))
   \]

   where \( \ast \) is the composition in the operad \( \mathcal{M} \).
Inputs to our homotopy colimit construction will be strict diagrams in $CAt^{MLax}$, but lax maps between such diagrams. We now introduce the relevant functor 2-category $Func(L, CAt^{MLax})$.

3.4 Let $L$ be a small category and $K$ be an arbitrary 2-category. The objects of $Func(L, K)$ are functors $D : L \to K$ of 1-categories, i.e. $L$-diagrams in $K$. A morphism or 1-cell $j : D \to F$ of $L$-diagrams is a homotopy morphism (or an op-lax natural transformation in Thomason’s terminology [28,29]). It assigns to each object $L$ in $L$ a functor

$$j_L : DL \to FL$$

and to each morphism $\lambda : L_0 \to L_1$ a natural transformation

$$j_\lambda : F\lambda \circ j_{L_0} \Rightarrow j_{L_1} \circ D\lambda$$

such that $j_{id} = id$ and for $L_0 \xrightarrow{\lambda_0} L_1 \xrightarrow{\lambda_2} L_2$

$$j_{\lambda_1 \circ \lambda_2} = (j_{\lambda_1} \circ_1 D\lambda_0) \circ_2 (F\lambda_1 \circ_1 j_{\lambda_0})$$

A 2-cell $D \xleftarrow{k} F$ in $Func(L, K)$ assigns to each object $L$ in $L$ a 2-cell

$$j_L : DL \to FL$$

in $K$ such that for $\lambda : L_1 \to L_2$ in $L$

$$k_\lambda \circ_2 (F\lambda \circ_1 s_{L_1}) = (s_{L_2} \circ_1 D\lambda) \circ_2 j_\lambda$$

Composition in $Func(L, K)$ is the canonical one.

If $K$ is $CAt^{MLax}$ the explicit coherence conditions of homotopy morphisms in $Func(L, K)$ are quite involved because the morphisms in each diagram are lax. For the reader’s convenience we formulate them in an appendix.
Let $\mathcal{K}^L$ be the subcategory of $\mathcal{F}unc(L, \mathcal{K})$ where the 1-cells are strict morphisms $j : D \to F$, i.e. for $\lambda : L_0 \to L_1$

\[
F\lambda \circ j_{L_0} = j_{L_1} \circ D\lambda \quad \text{and} \quad j_{\lambda} = \text{id}.
\]

If $D, F : L \to \mathcal{K}$ are $L$-diagrams in $\mathcal{K}$ we have an induced diagram

\[
\mathcal{K}(D, F) : L^{op} \times L \to \mathcal{C}at, \quad (L, L') \mapsto \mathcal{K}(DL, FL').
\]

Let $L/\mathcal{L}/L'$ denote the category whose object are diagrams

\[
(L_0, f_0, g_0) : L \xrightarrow{j_{L_0}} L_0 \xrightarrow{g_0} L'
\]

and whose morphisms $\lambda_1 : (L_0, f_0, g_0) \to (L_1, f_1, g_1)$ are morphisms $\lambda : L_0 \to L_1$ in $\mathcal{L}$ making

\[
\begin{array}{ccc}
L_0 & \xrightarrow{\lambda_1} & L' \\
\downarrow f_0 & & \downarrow g_1 \\
L_1 & \xrightarrow{\lambda} & L
\end{array}
\]

commute. We define

\[
- / \mathcal{L}/- : \mathcal{L}^{op} \times \mathcal{L} \to \mathcal{C}at, \quad (L, L') \mapsto L/\mathcal{L}/L'
\]

The following result will imply a rectification result if $\mathcal{K} = \mathcal{C}at^M$.

3.5 Proposition: There is a natural isomorphism of categories

\[
\alpha : \mathcal{F}unc(L, \mathcal{K})(D, F) \cong \mathcal{C}at^{\mathcal{L}^{op} \times \mathcal{L}}((- / \mathcal{L}/-), \mathcal{K}(D, F))
\]

Proof Let $j : D \to F$ be an object in $\mathcal{F}unc(L, \mathcal{K})(D, F)$, i.e. for each $\lambda : L_0 \to L_1$ we are given

\[
\begin{array}{ccc}
DL_0 & \xrightarrow{D\lambda_1} & DL_1 \\
\downarrow j_{L_0} & & \downarrow j_{L_1} \\
FL_0 & \xrightarrow{D\lambda_1} & FL_1
\end{array}
\]

If $\lambda_1 : (L_0, f_0, g_0) \to (L_1, f_1, g_1)$ is a morphism in $L/\mathcal{L}/L'$ we have a diagram

\[
\begin{array}{ccc}
DL_0 & \xrightarrow{D\lambda_1} & DL_1 \\
\downarrow j_{L_0} & & \downarrow j_{L_1} \\
FL_0 & \xrightarrow{F\lambda_1} & FL_1
\end{array}
\]

(A)
with commutative triangles. We define the functor
\[ \alpha(j)_{L,L'} : L / \mathcal{L} / L' \to \mathcal{K}(DL, FL') \]
on objects by \[ \alpha(j)_{L,L'}(L_0, f_0, g_0) = Fg_0 \circ j_{L_0} \circ Df_0, \] and on the morphism \( \lambda_1 \) to be the 2-cell depicted by diagram (A). The conditions on \( j \) make this a functor. By construction, the collection of the \( \alpha(j)_{L,L'} \) define a morphism \[ \alpha(j) : (-/\mathcal{L}/-) \to \mathcal{K}(D, F) \] of \((\mathcal{L}^{op} \times \mathcal{L})\)-diagrams in \( \text{Cat} \).

Now let \( s : i \to j \) be a morphism in \( \text{Func}(\mathcal{L}, \mathcal{K})(D, F) \). By definition, \( s \) consists of 2-cells
\[
\begin{array}{c}
\xymatrix{DL \ar@<2ex>[d]^{i_L} \ar@<1ex>[d]_{j_L} & FL \ar@<1ex>[d]^{s_L} \\
}
\end{array}
\]
satisfying
\[ j_{\lambda} \circ (F \circ_1 s_{L_0}) = (s_{L_1} \circ_1 D\lambda) \circ_2 i_{\lambda} \quad \text{for } \lambda : L_0 \to L_1. \quad (B1) \]

Then \( \alpha(s) \) is required to be a modification
\[
\begin{array}{c}
\xymatrix{L / \mathcal{L} / L' \ar@<2ex>[d]^{\alpha(j)_{L,L'}} \ar@<1ex>[d]_{\alpha(s)_{L,L'}} & \mathcal{K}(DL, FL') \ar@<1ex>[d]^{\alpha(i)_{L,L'}} \\
}
\end{array}
\]
satisfying
\[ ((D\lambda)^* \circ (F\mu)_*) \circ_1 \alpha(s)_{L,L'} = \alpha(s)_{L,L'} \circ_1 (\lambda^* \circ \mu_*) \quad (B2) \]
for \( \lambda : L \to \overline{L} \) and \( \mu : \overline{L} \to L' \) in \( \mathcal{L} \).

For an object \((L_0, f_0, g_0)\) in \( L/\mathcal{L}/L' \) we define the natural transformation \( \alpha(s)_{L,L'} \) by

\[
\alpha(s)_{L,L'}(L_0, f_0, g_0) = F(g_0) \circ_1 s_{L_0} \circ_1 D(f_0)
\]

which satisfies condition (B2).

This defines the functor \( \alpha \).

We now define the inverse functor

\[
\beta : \text{Cat}^{\mathcal{L}^{op} \times \mathcal{L}}((-/\mathcal{L}/-), \mathcal{K}(D, F)) \to \mathcal{F}\text{unc}(\mathcal{L}, \mathcal{K})(D, F)
\]

Let \( G, H : (-/\mathcal{L}/-) \to \mathcal{K}(D, F) \) be strict morphisms of \( (\mathcal{L}^{op} \times \mathcal{L}) \)-diagrams in \( \text{Cat} \). For each pair \((L, L') \in \mathcal{L}^{op} \times \mathcal{L} \) we are given a functor

\[
G_{L,L'} : L/\mathcal{L}/L' \to \mathcal{K}(DL, FL')
\]

and for each pair of morphisms \( \lambda : L \to \overline{L} \) and \( \mu : \overline{L} \to L' \) in \( \mathcal{L} \) a commutative diagram

\[
\begin{array}{ccc}
L/\mathcal{L}/L' & \xrightarrow{\lambda^* \circ \mu_*} & \mathcal{K}(DL, FL') \\
\downarrow \lambda^* & & \downarrow (D\lambda)^* \circ (F\mu) \\
L/\mathcal{L}/L' & \xrightarrow{G_{L,L'}} & \mathcal{K}(DL, FL)
\end{array}
\]

so that

\[
G_{L,L'}(L_0, f_0 \circ_0 \lambda, \mu \circ_0 g_0) = F\mu \circ G_{\overline{L}, \overline{L}'}(L_0, f_0, g_0) \circ D\lambda,
\]

and for a morphism \( \lambda_1 : (L_0, f_0, g_0) \to (L_1, f_1, g_1) \) in \( L/\mathcal{L}/L' \)

\[
F\mu \circ G_{\overline{L}, \overline{L}'}(\lambda_1 : (L_0, f_0, g_0) \to (L_1, f_1, g_1)) \circ D\lambda
= G_{L,L'}(\lambda_1 : (L_0, f_0 \circ_0 \lambda, \mu \circ_0 g_0) \to (L_1, f_1 \circ_0 \lambda, \mu \circ_0 g_1))
\]

We define

\[
\beta(G)_{L} = G_{L,L}(L, \text{id}_L, \text{id}_L) : DL \to FL
\]

and for \( \lambda : L \to \overline{L} \)

\[
\beta(G)_{\lambda} = G_{L,\overline{L}}(\lambda : (L, \text{id}, \lambda) \to (\overline{L}, \lambda, \text{id}))
\]

Since \( G_{L,\overline{L}}(L, \text{id}, \lambda) = F\lambda \circ G_{L,L}(L, \text{id}, \text{id}) \) and \( G_{L,\overline{L}}(\overline{L}, \lambda, \text{id}) = G_{\overline{L},\overline{L}}(\lambda, \text{id}, \text{id}) \circ D\lambda \), we have

\[
\beta(G)_{\lambda} : F\lambda \circ \beta(G)_{L} \Rightarrow \beta(G)_{\overline{L}} \circ D\lambda
\]

Given \( L_0 \xrightarrow{\lambda_1} L_1 \xrightarrow{\lambda_2} L_2 \), then

\[
F\lambda_2 \circ_1 G_{L_0,L_1}(\lambda_1 : (L_0, \text{id}, \lambda_1) \to (L_1, \lambda_1, \text{id})) = G_{L_0,L_2}(\lambda_1 : (L_0, \text{id}, \lambda_2 \circ_1 \lambda_1) \to (L_1, \lambda_1, \lambda_2))
\]

and

\[
G_{L_1,L_2}(\lambda_2 : (L_1, \text{id}, \lambda_2) \to (L_2, \lambda_2, \text{id})) \circ_1 D\lambda_1 = G_{L_0,L_2}(\lambda_2 : (L_1, \lambda_1, \lambda_2) \to (L_2, \lambda_2 \circ_1 \lambda_1, \text{id}))
\]
so that the coherence conditions for the $\beta(G)_\lambda$ hold.

Now let $s : G \Rightarrow H$ be a modification satisfying condition (B2), i.e. for $\lambda : L \rightarrow L'$ and $\mu : L' \rightarrow L''$

$$F\mu \circ_1 s_{L,L''}(L_0, f_0, g_0) \circ_1 D\lambda = s_{L,L''}(L_0, f_0 \circ \lambda, \mu \circ g_0).$$

We have to find a 2-cell

$$\beta(s) : \beta(G) \Rightarrow \beta(H)$$

in $Func(\mathcal{L}, \mathcal{K})(D, F)$. We define

$$\beta(s)_L = s_{L,L}(L, \text{id}, \text{id}) : \beta(G)_L \Rightarrow \beta(H)_L.$$

Since $s$ is a modification the following diagram commutes for $\lambda : L_0 \rightarrow L_1$

$$\begin{array}{ccc}
G_{L_0,L_1}(L_0, \text{id}, \lambda) & \xymatrix{\ar[rr]^{s_{L_0,L_1}(L_0, \text{id}, \lambda)} & & H_{L_0,L_1}(L_0, \text{id}, \lambda)} \\
G_{L_0,L_1}(L_1, \lambda, \text{id}) & \xymatrix{\ar[rr]^{s_{L_0,L_1}(L_1, \lambda, \text{id})} & & H_{L_0,L_1}(L_1, \lambda, \text{id})}
\end{array}$$

and we obtain for $\lambda : (L_0, \text{id}, \lambda) \rightarrow (L_1, \lambda, \text{id})$

$$\beta(H)_\lambda \circ_2 (F\lambda \circ_1 \beta(s)_L) = H_{L_0,L_1}(\lambda) \circ_2 (F\lambda \circ_1 s_{L_0,L_1}(L_0, \text{id}, \text{id})) = H_{L_0,L_1}(\lambda) \circ_2 (s_{L_0,L_1}(L_0, \text{id}, \lambda)) = s_{L_0,L_1}(L_1, \lambda, \text{id}) \circ_2 G_{L_0,L_1}(\lambda) = (s_{L_1,L_1}(L_1, \text{id}, \text{id}) \circ_1 D\lambda) \circ_2 \beta(G)_\lambda = (\beta(s)_L \circ_1 D\lambda) \circ_2 \beta(G)_\lambda$$

Hence $\beta(s)$ satisfies (B1). \qed

We are interested in the case $\mathcal{K} = \text{Cat}^\mathcal{M}$ where $\mathcal{M}$ is the monad associated with a $\Sigma$-free $\text{Cat}$-operad. In Section 5 we will deal with the case $\mathcal{K} = \text{Cat}^\mathcal{M}Lax$.

Since $\text{Cat}^\mathcal{M}$ is tensored over $\text{Cat}$ Proposition 3.5 and the properties of the tensor provide natural isomorphisms

3.6

$$Func(\mathcal{L}, \text{Cat}^\mathcal{M})(D, F) \cong \text{Cat}^{\mathcal{L}^\text{op} \times \mathcal{L}}(\mathcal{L} / - , \text{Cat}^\mathcal{M}(D, F))$$

$$\cong (\text{Cat}^\mathcal{M})^\mathcal{L}(\mathcal{L} / - \otimes_\mathcal{L} D, F)$$

We obtain

3.7 Proposition: There is a $\text{Cat}$-enriched adjunction

$$\begin{array}{c}
R : Func(\mathcal{L}, \text{Cat}^\mathcal{M}) \rightleftharpoons (\text{Cat}^\mathcal{M})^\mathcal{L} : i
\end{array}$$

with $R(D) = (- / \mathcal{L} / -) \otimes_\mathcal{L} D$ and $i$ the inclusion functor. \qed
We next analyze this adjunction.

3.8 Definition: A functor $F : \mathcal{A} \to \mathcal{B}$ of small categories is called a weak equivalence if the induced map of classifying spaces $BF : BA \to BB$ is a weak homotopy equivalence. A homomorphism of $\mathbb{M}$-algebras $f : X \to Y$ is called a weak equivalence if the underlying morphism of categories is a weak equivalence.

We will need the following lemma.

3.9 Lemma: Let $\mathcal{M}$ be a $\Sigma$-free $\text{Cat}$-operad and $\mathbb{M}$ its associated monoid. Let $F, G : \mathcal{L}^{op} \to \text{Cat}$ and $X : \mathcal{L} \to \text{Cat}^{BM}$ be diagrams. Let $f, g : F \to G$ be strict maps of diagrams and suppose we are given a natural transformation $\tau : f(L) \Rightarrow g(L)$ for each $L$ in $\mathcal{L}$ such that

$$G(\lambda) \circ \tau_{L_0} = \tau_{L_1} \circ F(\lambda)$$

for each morphism $\lambda : L_0 \to L_1$ in $\mathcal{L}$. Then the induced maps of classifying spaces

$$B(f \otimes \text{id}), B(g \otimes \text{id}) : B(F \otimes \mathcal{L} X) \to B(G \otimes \mathcal{L} X)$$

are homotopic as homomorphisms of $\mathcal{B}M$-algebras.

Proof Let $[1]$ denote the category $0 \to 1$, and let $i_0, i_1 : * \to [1]$ be the inclusion of the 0 and the 1 respectively. Each $\tau_L$ defines a functor $\tau_L : [1] \times F(L) \to G(L)$ such that $\tau_L \circ (i_0 \times \text{id}) = f(L)$ and $\tau_L(i_1 \times \text{id}) = g(L)$. These functors define a strict morphism $T : [1] \times F \to G$ of $\mathcal{L}^{op}$-diagrams. By the universal properties of the tensor we have a natural map

$$(0, 1) \otimes B(F \otimes \mathcal{L} X) = B([1]) \otimes B(F \otimes \mathcal{L} X) \to B(1) \otimes (F \otimes \mathcal{L} X)$$

$$= B(1 \times F) \otimes \mathcal{L} X) \xrightarrow{B(T \otimes \text{id})} B(G \otimes \mathcal{L} X)$$

which defines the required homotopy in $\mathcal{L}^{BM}$.

The unit $\alpha(D) : D \to R(D)$ of the adjunction $\mathcal{L} \dashv \text{Cat}$ is a homotopy morphism and the counit $\beta(D) : R(D) \to D$ a strict morphism of $\mathcal{L}$-diagrams.

We compare $(-/\mathcal{L}/-)$ with the functor

$$\mathcal{L} : \mathcal{L}^{op} \times \mathcal{L} \to \text{Cat}, \quad (L_0, L_1) \mapsto \mathcal{L}(L_0, L_1)$$

where we consider the set $\mathcal{L}(L_0, L_1)$ as a discrete category. Let

$$p^{L_0, L_1} : L_0/\mathcal{L}/L_1 \to \mathcal{L}(L_0, L_1) \quad \text{and} \quad q^{L_0, L_1} : \mathcal{L}(L_0, L_1) \to L_0/\mathcal{L}/L_1$$

be the functors defined by

$$p^{L_0, L_1}(L_2, f, g) = g \circ f \quad \text{and} \quad q^{L_0, L_1}(h) = (L_1, h, \text{id}).$$

Then $p^{L_0, L_1} \circ q^{L_0, L_1} = \text{id}$, and there is a natural transformation

$$\tau^{L_0, L_1} : \text{Id}_{L_0/\mathcal{L}/L_1} \to q^{L_0, L_1} \circ p^{L_0, L_1}$$

19
defined by
\[ \tau^{L_0, L_1}(L_2, f, g) = g : (L_2, f, g) \to (L_1, g \circ f, \text{id}). \]
The \( p^{L_0, L_1} \) induce a morphism of \( \mathcal{L} \)-diagrams
\[ p : RD \to D \]
and tracing \( \text{id}_D \) through the two natural isomorphisms of \[3.6\] shows that \( p = \beta(D) \).

The \( q^{L_0, L_1} \) induce a morphism
\[ q^{L_1} : D(L_1) = \mathcal{L}(\_ , L_1) \otimes_{\mathcal{L}} D \to (\_ / L_1) \otimes_{\mathcal{L}} D = RD(L_1) \]
and for \( \lambda : L_1 \to L_2 \) we have a natural transformation \( \rho_\lambda : RD(\lambda) \circ q^{L_1} \Rightarrow q^{L_2} \circ D(\lambda) \) induced by the natural transformation

\[
\begin{array}{ccc}
\mathcal{L}(L_0, L_1) & \xrightarrow{\lambda^*} & \mathcal{L}(L_0, L_2) \\
q^{L_0, L_1} \downarrow & & \downarrow q^{L_0, L_2} \\
L_0 / \mathcal{L} / L_1 & \xrightarrow{\lambda^*} & L_0 / \mathcal{L} / L_2
\end{array}
\]
where
\[ \rho_\lambda^{L_0}(f) = \lambda : (L_1, f, \lambda) \to (L_2, \lambda \circ f, \text{id}) \]
The \( q^\mathcal{L} \) together with the \( \rho_\lambda \) define a morphism
\[ q : (-/ \mathcal{L}/-) \to \text{Cat}^\mathcal{M}(D, RD) \]
of \( (\mathcal{L}^\mathcal{Op} \times \mathcal{L}) \)-diagrams defined by the functors
\[ q(L_0, L_1) : (L_0 / \mathcal{L} / L_1) \to \text{Cat}^\mathcal{M}(D(L_0), RD(L_1)) \]
mapping an object \((L_2, f, g)\) to \( DL_0 \xrightarrow{Df} DL_2 \xrightarrow{q^{L_2}} RD(L_2) \xrightarrow{RD(g)} RD(L_1) \)
and a morphism \( \lambda : (L_2, f, h \circ \lambda) \to (L_3, \lambda \circ f, h) \) to the 2-cell \( RD(h) \circ_1 \rho_\lambda \circ_1 D(f) \).
The map of \( \mathcal{L} \)-diagrams \( RD \to RD \) induced by \( q \) is the identity, because \( q \) takes values in the canonical maps \( D(L_0) \to RD(L_1) \). Hence \( q \) induces the unit \( \alpha(D) : D \to R(D) \).

If we fix \( L_1 \) the natural transformations \( \tau^{L_0, L_1} : \text{Id}_{L_0 / \mathcal{L} / L_1} \to q^{L_0, L_1} \circ p^{L_0, L_1} \)
satisfy the requirements of Lemma \[3.9\] so that
\[ q : (-/ \mathcal{L}/-) \otimes_{\mathcal{L}} D \to D \]
is objectwise a weak equivalence for each diagram \( D : \mathcal{L} \to \text{Cat}^\mathcal{M} \).

We summarize
3.10 Rectification Theorem: Let $M$ be the monad associated with a $\Sigma$-free $\mathsf{Cat}$-operad $\mathcal{M}$. There is a $\mathsf{Cat}$-enriched adjunction

$$R : \mathcal{F}\mathcal{u}\mathcal{n}\mathcal{c}(\mathcal{L}, \mathcal{C}\mathcal{a}\mathcal{t}^M) \rightleftarrows (\mathcal{C}\mathcal{a}\mathcal{t}^M)\mathcal{L} : i$$

with $R(D) = (-/\mathcal{L}/-) \otimes_{\mathcal{L}} D$ and $i$ the inclusion. The unit $\alpha(D) : D \to R(D)$ is induced by the functors $q^{L_0, L_1} : \mathcal{L}(L_0, L_1) \to L_0/\mathcal{L}/L_1$, the counit $\beta(D) : RD \to D$ by the functors $p^{L_0, L_1} : L_0/\mathcal{L}/L_1 \to \mathcal{L}(L_0, L_1)$. The morphism $\beta(D) : R(D) \to D$ of diagrams is objectwise a weak equivalence; an inverse of $\beta(D)(L) : RD(L) \to D(L)$ is given by $\alpha(D)(L) : D(L) \to RD(L)$, which satisfies $\beta(D)(L) \circ \alpha(D)(L) = \text{id}_{D(L)}$.

4. The homotopy colimit functor

Let $\mathcal{M}$ be an $\Sigma$-free operad in $\mathsf{Cat}$ and

$$X : \mathcal{L} \to \mathcal{C}\mathcal{a}\mathcal{t}^M\mathcal{L}\mathcal{a}\mathcal{x}$$

an $\mathcal{L}$-diagram in $\mathcal{C}\mathcal{a}\mathcal{t}^M\mathcal{L}\mathcal{a}\mathcal{x}$.

4.1 The homotopy colimit construction

An object of $\text{hocolim} X$ is an equivalence class $[A; (K_1, L_1), \ldots, (K_n, L_n)]$ of tuples consisting of an object $A$ in $\mathcal{M}(n)$, objects $L_i$ in $\mathcal{L}$ and objects $K_i$ in $X(L_i)$, subject to the relation

$$\left( A \cdot \sigma ; (K_1, L_1), \ldots, (K_n, L_n) \right) \sim \left( A; (K_{\sigma^{-1}(1)}, L_{\sigma^{-1}(1)}), \ldots, (K_{\sigma^{-1}(n)}, L_{\sigma^{-1}(n)}) \right)$$

We visualize objects as equivalence classes of rooted trees

```
   (K_1, L_1)  \cdots  (K_n, L_n)
\downarrow
   A
```

with one node, $n$ input edges, and labeled inputs. The relation reads

4.2

```
(K_1, L_1)  \cdots  (K_n, L_n)  (K_{\sigma^{-1}(1)}, L_{\sigma^{-1}(1)})  \cdots  (K_{\sigma^{-1}(n)}, L_{\sigma^{-1}(n)})
\downarrow  \sim  \downarrow
   A \cdot \sigma  \sim  A
```

Before we define morphisms we introduce some notation:
We already made the convention that

![Diagram](attachment:image.png)

stands for the object \( A \cdot \sigma \in \mathcal{M}(n) \) with \( \sigma \in \Sigma_n \) defined by \( \sigma^{-1}(k) = i_k \).

Hence a morphism

![Diagram](attachment:image2.png)

is a morphism \( A \cdot \sigma \to B \cdot \tau \) in \( \mathcal{M}(n) \) with \( \tau^{-1}(k) = j_k \).

A morphism \([A_1; (K_1, L_1), \ldots, (K_n, L_n)] \to [A_2; (K'_1, L'_1), \ldots, (K'_p, L'_p)]\) in \( \text{hocolim} X \) consists of

1. a map \( \varphi : \{1, \ldots, n\} \to \{1, \ldots, p\} \) (if \( n = 0 \), then \( \varphi : \emptyset \to \{1, \ldots, p\} \))
2. a \( p \)-tuple of objects \((C_1, \ldots, C_p)\) with \( C_k \in \mathcal{M}(|\varphi^{-1}(k)|) \), where \(|M|\) denotes the cardinality of the set \( M \),
3. a morphism \( \gamma : A_1 \to A_2 \ast (C_1 \oplus \cdots \oplus C_p) \cdot \sigma \).

Here \( \ast \) is the operad composition and \( \sigma \) is determined as above by

![Diagram](attachment:image3.png)

where the nodes are thought of composed into one node \( A_2 \ast (C_1 \oplus \cdots \oplus C_p) \) and the input labels to \( C_j \) are the elements of \( \varphi^{-1}(j) \) in natural order.

4. morphisms \( \lambda_i : L_i \to L'_{\varphi(i)} \) in \( \mathcal{L} \)
5. for each \( k = 1, \ldots, p \) a map

\[
\begin{align*}
f_k : C_k(\lambda_{i_1} K_{i_1}, \ldots, \lambda_{i_r} K_{i_r}) & \to K'_k
\end{align*}
\]

in \( X(L'_k) \) where \( \lambda K \) stands for \( X(\lambda)(K) \), where \( \{i_1 < \ldots < i_r\} = \varphi^{-1}(k) \), and \( C_k(\lambda_{i_1} K_{i_1}, \ldots, \lambda_{i_r} K_{i_r}) \in X(L'_k) \) is obtained by the action of \( C_k \in \mathcal{M}(r) \) on the tuple \((\lambda_{i_1} K_{i_1}, \ldots, \lambda_{i_r} K_{i_r})\). If \( \varphi^{-1}(k) = \emptyset \), this is a map \( C_k(\ast) \to K'_k \) in \( X(L'_k) \).
Again we will use a tree description and combine the data (1), (2), (4), (5) to
\[
\begin{array}{c}
\lambda_{i_{11}} \\
\downarrow & & \downarrow \\
C_1, f_1 & & \ldots & & \ldots & & \ldots & & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\lambda_{i_{r_1}} & & \lambda_{i_{p_1}} & & \lambda_{i_{p_{r_p}}} \\
\end{array}
\]

where \( \{i_{j_1} < \ldots < i_{j_r}\} = \varphi^{-1}(j) \). Relation \( \text{I}\text{I}\text{I}\text{I}\text{I}\text{I}\text{I} \) applies to these trees. The remaining data is the morphism \( \gamma : A_1 \to A_2 \ast (C_1 \oplus \ldots \oplus C_p) \cdot \sigma \), where \( \sigma^{-1} \) can be read off the input labels of \( C_1, \ldots, C_p \), given by the indices of the \( \lambda \)'s (see also example below).

Composition can best be explained by an example. The general case is accordingly.

**4.3 Example:** Morphism 1:
\[
[A_1; (K_1, L_1), \ldots, (K_7, L_7)] \to [A_2; (K'_1, L'_1), \ldots, (K'_4, L'_4)] \text{ is given by}
\]
(1)
\[
\begin{array}{c}
\lambda_2 & \lambda_4 & \lambda_6 \\
\downarrow & \downarrow & \downarrow \\
C_1, f_1 & C_2, f_2 & C_3, f_3 \\
\uparrow & \uparrow & \uparrow \\
\rho_2 & \rho_1 & \rho_4 \\
D_1, g_1 & D_2, g_2 & D_3, g_3 \\
\end{array}
\]

(2) \( \gamma_1 : A_1 \to A_2 \ast (C_1 \oplus \ldots \oplus C_4) \cdot \sigma \) with \( \sigma^{-1} = \left( \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\
2 & 4 & 6 & 1 & 5 \\
3 & 1 & 4 & 2 & 6 \\
4 & 3 & 2 & 1 & 7 \\
7 & 7 & 7 & 7 & 7 \end{array} \right) \)

Morphism 2:
\[
[A_2; (K'_1, L'_1), \ldots, (K'_4, L'_4)] \to [A_3; (K''_1, L''_1), \ldots, (K''_3, L''_3)] \text{ is given by}
\]
(1)
\[
\begin{array}{c}
\rho_2 & \rho_3 & \rho_1 & \rho_4 \\
D_1, g_1 & D_2, g_2 & D_3, g_3 \\
\end{array}
\]

(2) \( \gamma_2 : A_2 \to A_3 \ast (D_1 \oplus D_2 \oplus D_3) \cdot \tau \) with \( \tau^{-1} = \left( \begin{array}{cc} 1 & 2 \\
2 & 3 \\
3 & 4 \\
4 & 1 \end{array} \right) \)

The composite of these two morphisms can be read off the following "aiding picture":

\[
\begin{array}{c}
\lambda_1 & \lambda_5 \\
\downarrow & \downarrow \\
C_2 & C_3 \\
\rho_2 & \rho_3 \\
D_1 & D_1 \\
\end{array}
\] \hspace{1cm}
\begin{array}{c}
\lambda_2 & \lambda_4 & \lambda_6 & \lambda_3 & \lambda_7 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
C_1 & C_6 & C_4 & C_4 & C_4 \\
\rho_1 & \rho_4 & \rho_1 & \rho_4 & \rho_4 \\
D_2 & D_2 & D_2 & D_2 & D_3 \\
\end{array}
\]
The composite morphism
\[ [A_1; (K_1, L_1), \ldots, (K_7, L_7)] \rightarrow [A_3; (K'_1, L'_1), \ldots, (K'_3, L'_3)] \]
consists of
\[ (1) \]

\[ \begin{array}{c}
\mu_1 & \mu_5 & \mu_2 & \mu_4 & \mu_6 & \mu_3 & \mu_7 \\
E_1, h_1 & E_2, h_2 & E_3, h_3
\end{array} \]

(2) \[ \gamma_3 : A_1 \rightarrow A_3 \ast (E_1 \oplus E_2 \oplus E_3) \cdot \sigma_3 \]
with \( \sigma_3^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 4 & 6 & 3 & 7 \end{pmatrix} \) (use the "aiding picture" above)
and
\[
\begin{align*}
\mu_1 &= \rho_2 \circ \lambda_1, & \mu_5 &= \rho_2 \circ \lambda_5, & \mu_2 &= \rho_1 \circ \lambda_2, & \mu_4 &= \rho_1 \circ \lambda_4, \\
\mu_6 &= \rho_1 \circ \lambda_6, & \mu_3 &= \rho_4 \circ \lambda_3, & \mu_7 &= \rho_4 \circ \lambda_7
\end{align*}
\]
\[
E_1 = D_1 \ast (C_2 \oplus C_3), \quad E_2 = D_2 \ast (C_1 \oplus C_4), \quad E_3 = D_3
\]

\[ \gamma_3 : A_1 \xrightarrow{\gamma_3} A_2 \ast (C_1 \oplus \ldots \oplus C_4) \cdot \sigma \xrightarrow{\gamma_3 \ast \text{id}} A_3 \ast (D_1 \oplus D_2 \oplus D_3) \cdot \tau \ast (C_1 \oplus \ldots \oplus C_4) \cdot \sigma \]

We check that \( \gamma_3 \) has the correct target:
\[
(D_1 \oplus D_2 \oplus D_3) \cdot \tau \ast (C_1 \oplus \ldots \oplus C_4) \cdot \sigma
= (D_1 \oplus D_2 \oplus D_3) \ast (C_2 \oplus C_3 \oplus C_1 \oplus C_4) \cdot (\tau \ast \sigma)
\]
where \( \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 1 & 2 & 6 & 7 \end{pmatrix} \)
is the block permutation given by \( \tau \). Then \( (\tau \ast \sigma) = \sigma_3 \).
Finally, to specify the \( h_k \) we again consult the "aiding picture":
\[
\begin{align*}
h_1 : E_1(\mu_1 K_1, \mu_5 K_5) &= D_1(C_2(\rho_2 \lambda_1 K_1, \rho_2 \lambda_3 K_5)), C_3(\rho_3(\ast))) \\
&\xrightarrow{D_1(\rho_2(f_2), \rho_3(f_3))} D_1(\rho_2 K_2', \rho_3 K_3') \xrightarrow{\eta_1} K_1''
\end{align*}
\]
\( h_2 \) is defined analogously while
\[ h_3 : E_3(\ast) = D_3(\ast) \xrightarrow{\eta_3} K_3'' \]

Here recall that we have a diagram in \( \text{Cat}^{\text{bt}} \text{Lax} \). So for a morphism \( \rho : L \rightarrow L' \) in \( \mathcal{L} \) we have a lax morphism, which in abuse of notation we denote by \( (\rho, \overline{\rho}) : X(L) \rightarrow X(L') \) rather than by \( X(\rho) \). The definition of \( h_1 \) makes use of \( \overline{\rho}_2 \) and \( \overline{\rho}_3 \) and their explicit description in [3.3].
With the help of the “aiding picture” and the explicit description of lax morphisms in 3.3 it is straightforward to verify that composition in $\text{hocolim} X$ is associative.

The identity of $[A; (K_1, L_1), \ldots, (K_n, L_n)]$ is given by the trees

\[
\begin{array}{c}
\text{id}_{L_1} \\
\vdots \\
\text{id}_{L_k} \\
\text{id}
\end{array}
\quad \rightarrow 
\begin{array}{c}
\text{id} \\
\vdots \\
\text{id}
\end{array}
\]

and $\gamma = \text{id} : A \to A * (\text{id} \oplus \ldots \oplus \text{id}) = A$.

The $\mathcal{M}$-structure on $\text{hocolim} X$ is defined by

\[
B * ([A_1; (K_{1_1}, L_{1_1}), \ldots, (K_{1r_1}, L_{1r_1})], \ldots, [A_n; (K_{nr_n}, L_{nr_n})]) = [B * (A_1 \oplus \ldots \oplus A_n); (K_{11}, L_{11}), \ldots, (K_{nr_n}, L_{nr_n})]
\]

4.4 As an $\mathcal{M}$-algebra $\text{hocolim} X$ is generated by morphisms of the following types, called atoms:

1. $\gamma : [A; (K_1, L_1), \ldots, (K_n, L_n)] \to [B; (K_1, L_1), \ldots, (K_n, L_n)]$

\[
\begin{array}{c}
(K_1, L_1) \\
\downarrow \\
A
\end{array}
\quad \gamma 
\quad \begin{array}{c}
(K_n, L_n) \\
\downarrow \\
B
\end{array}
\]

with $\gamma : A \to B$ in $\mathcal{M}(n)$. The other data of the morphism are identities.

2. $\lambda : [\text{id}; (K, L)] \to [\text{id}; (\lambda K, L')]$ for a morphism $\lambda : L \to L'$ in $\mathcal{L}$.

\[
\begin{array}{c}
(K, L) \\
\downarrow \\
\text{id}
\end{array}
\quad \lambda 
\quad \begin{array}{c}
(\lambda K, L') \\
\downarrow \\
\text{id}
\end{array}
\]

The other data are identities.

Using the algebra structure we can combine a collection of such atoms $\lambda_i : [\text{id}; (K_i, L_i)] \to [\text{id}; (\lambda_i K_i, L_i')]$ to the morphism $A(\lambda_1, \ldots, \lambda_n)$:

\[
\begin{array}{c}
(K_1, L_1) \\
\downarrow \\
A
\end{array}
\quad \rightarrow 
\begin{array}{c}
(K_n, L_n) \\
\downarrow \\
A
\end{array}
\quad \begin{array}{c}
(\lambda_1 K_1, L_1') \\
\downarrow \\
A
\end{array}
\quad \begin{array}{c}
(\lambda_n K_n, L_n') \\
\downarrow \\
A
\end{array}
\]
(3) \( \text{ev}(C) : [C; (K_1, L), \ldots, (K_n, L)] \to \text{id}; (C(K_1, \ldots, K_n), L) \), called evaluation, for objects \( C \) in \( \mathcal{M}(n) \).

\[
\begin{array}{c}
(K_1, L) \\
\downarrow \\
C \\
\downarrow \\
(K_n, L) \\
\downarrow \\
(C(K_1, \ldots, K_n), L)
\end{array}
\xrightarrow{\text{ev}(C)} \downarrow \\
\text{id}
\]

(4) \( f : \text{id}; (K, L) \to \text{id}; (K', L) \) for morphisms \( f : K \to K' \) in \( X(L) \).

\[
\begin{array}{c}
(K, L) \\
\downarrow \\
\text{id}
\end{array}
\xrightarrow{f} \\
\begin{array}{c}
(K', L) \\
\downarrow \\
\text{id}
\end{array}
\]

4.5 Each morphism in \( \text{hocolim} X \) decomposes canonically into atoms.

We demonstrate this with an example.

4.6 Example: The second morphism of Example 4.3 decomposes as follows

\[
\begin{array}{c}
(K_1', L_1') \\
\downarrow \\
A_2 \\
\downarrow \\
(K_2', L_2') \\
\downarrow \\
A_3 \ast (D_1 \oplus D_2 \oplus D_3)
\end{array}
\xrightarrow{?_2} \\
\begin{array}{c}
(K_3', L_3') \\
\downarrow \\
A_3 \ast (D_1 \oplus D_2 \oplus D_3)
\end{array}
\]

\[
\begin{array}{c}
\langle \rho_2, K_2', L_2' \rangle \\
\langle \rho_3, K_3', L_3' \rangle \\
\downarrow \\
A_3 \ast (D_1 \oplus D_2 \oplus D_3)
\end{array}
\xrightarrow{?_2} \\
\begin{array}{c}
\langle \rho_1, K_1', L_1'' \rangle \\
\langle \rho_4, K_4', L_4'' \rangle \\
\downarrow \\
A_3 \ast (D_1 \oplus D_2 \oplus D_3)
\end{array}
\]

\[
\begin{array}{c}
\langle D_1 \rho_2, K_2', L_2' \rangle \\
\downarrow \\
A_3
\end{array}
\xrightarrow{?_2} \\
\begin{array}{c}
\langle D_2 \rho_3, K_3', L_3' \rangle \\
\downarrow \\
A_3
\end{array}
\]

We now establish the universal property of our homotopy colimit construction.

4.7 The universal property: Let \( X : \mathcal{L} \to \text{Cat}^Lax \) be a diagram and let

\( c(\text{hocolim} X) : \mathcal{L} \to \text{Cat}^Lax \)

be the constant diagram on \( \text{hocolim} X \). The universal map

\( j : X \to c(\text{hocolim} X) \)
in \( F\text{unc}(\mathcal{L}, \text{Cat}^M_{\text{Lax}}) \) is given by lax morphisms
\[
j_L = (j_L, j_L) : X(L) \to \text{hocolim}X
\]
and 2-cells \( j_\lambda : j_{L_0} \Rightarrow j_{L_1} \circ X \lambda \) for \( \lambda : L_0 \to L_1 \) defined as follows. The functor \( j_L : X(L) \to \text{hocolim}X \) sends an object \( K \) to \([\id; (K, L)]\) and \( f : K_1 \to K_2 \) to the atom \( f : [(\id; (K_1, L)) \to (\id; (K_2, L))]\).

These functors do not define a strict morphism of \( M \)-algebras: Given \( A \in M(n) \), we have
\[
j_L(A(K_1, \ldots, K_n)) = A(K_1, \ldots, K_2)
\]
and
\[
A(j_L(K_1), \ldots, j_L(K_n))
\]

The natural transformation
\[
\overrightarrow{j_L}(A; K_1, \ldots, K_n) : A(j_L(K_1), \ldots, j_L(K_n)) \to j_L(A(K_1, \ldots, K_n))
\]
is given by the evaluation \( \text{ev}(A) \).

Given a morphism \( \lambda : L_0 \to L_1 \) in \( \mathcal{L} \) we have to define the 2-cell \( j_\lambda \)

\[
\begin{array}{ccc}
X(L_0) & \xrightarrow{X\lambda} & X(L_1) \\
\downarrow j_{L_0} & \searrow j_{L_1} & \downarrow \text{hocolim}X \\
\end{array}
\]

For \( K \) in \( X(L_0) \) we get \( j_{L_1} \circ X \lambda (K) = [\id; (\lambda K, L_1)] \) and \( j_{L_0} (K) = [\id; (K, L_0)] \).

We define \( j_\lambda (K) : j_{L_0} (K) \to j_{L_0} \circ X(\lambda)(K) \) to be the atom \( \lambda \) in \( \text{hocolim}X \).

We note that \( j_\lambda \) has to be compatible with the natural transformations \( \overrightarrow{X\lambda}, \overrightarrow{j_{L_0}}, \) and \( \overrightarrow{j_{L_1}} \), which are part of the homotopy morphisms \( (X\lambda, \overrightarrow{X\lambda}), (j_{L_0}, \overrightarrow{j_{L_0}}) \) and \( (j_{L_1}, \overrightarrow{j_{L_1}}) \). The verification of this is left to the reader and refer to the appendix \( \text{Func}(\mathcal{L}, \text{Cat}^M_{\text{Lax}}) \).

**4.8 Theorem:** Let \( S \) be an \( M \)-algebra and \( cS : \mathcal{L} \to \text{Cat}^M_{\text{Lax}} \) the constant \( \mathcal{L} \)-diagram on \( S \). Let \( k : X \to cS \) be a morphism in \( \text{Func}(\mathcal{L}, \text{Cat}^M_{\text{Lax}}) \). Then there is a unique strict homomorphism
\[
r : \text{hocolim}X \to S
\]
of $M$-algebras such that $(r, \tau) \circ (j_L, j_L) = (k_L, \overline{k}_L)$ in $\text{Cat}^M_{\text{Lax}}$ and $r \circ_1 j_\lambda = k_\lambda$, where $\tau$ is the identity.

**Proof** On objects, $r$ has to satisfy
\[ r([\text{id}; (K, L)]) = k_L(K) \]
Since $r$ is supposed to be a strict homomorphism of $M$-algebras, this condition determines $r$ on all objects.

For morphisms it suffices to define $r$ on atoms

(1) Since $r$ is a strict homomorphism of $M$-algebras it is determined on $\gamma : [A; (K_1, L_1), \ldots, (K_n, L_n)] \to [B; (K_1, L_1), \ldots, (K_n, L_n)]$ for $\gamma : A \to B$ in $\mathcal{M}(n)$.

(2) For $\lambda : L \to L'$ we need
\[ r(\lambda) : r([\text{id}; (K, L)]) = k_L(K) \to k_{L'}(\lambda K) = r([\text{id}; (\lambda K, L')]) \]
which is determined by the condition $r \circ_1 j_\lambda = k_\lambda$, so that $r(\lambda) = k_\lambda(K)$.

(3) Since $r$ is a strict homomorphism, an evaluation $ev(C) : [C; (K_1, L)，\ldots, (K_n, L)] \to [\text{id}; (C(K_1,\ldots,K_n), L)]$ has to be mapped to $\overline{k}_L(C; K_1,\ldots,K_n) : C(k_LK_1,\ldots,k_LK_n) \to k_L(C(K_1,\ldots,K_n))$

(4) A map $f : [\text{id}; (K, L)] \to [\text{id}; (K', L')]$ for $f : K \to K'$ in $X(L)$ has to be mapped to $r(f) = k_L(f)$ because of $r \circ j_L = k_L$.

So $r$ is uniquely determined, and we leave it to the reader to check that it is a functor.

\[ \square \]

4.9 **Addendum:** Let $k' : X \to cS$ be another morphism in $\mathcal{F}unc(L, \text{Cat}^M_{\text{Lax}})$ and $s : k \Rightarrow k'$ a 2-cell. Let $r' : \text{hocolim}X \to S$ be the homomorphism induced by $k'$. Then there is a unique 2-cell $t : r \Rightarrow r'$ in $\text{Cat}^M_{\text{Lax}}$ such that
\[ t \circ_1 j_L = s_L. \]

**Proof** Since $t$ is a 2-cell in $\text{Cat}^M$ it has to satisfy
\[
\begin{align*}
    t([A; (K_1, L_1), \ldots, (K_n, L_n)]) &= t(A([\text{id}; (K_1, L_1)], \ldots, [\text{id}; (K_n, L_n)])) \\
    &= A(t(j_{L_1}(K_1)), \ldots, t(j_{L_n}(K_n))) \\
    &= A(s_{L_1}(K_1), \ldots, s_{L_n}(K_n))
\end{align*}
\]

Hence $t$ is uniquely determined, and it is easy to check that it is a natural transformation.  \[ \square \]

The universal property and its addendum imply

28
4.10 Theorem: \( \text{hocolim} : \mathcal{F}\text{unc}(\mathcal{L}, \text{Cat}^{\text{MLax}}) \to \text{Cat}^{\text{ML}} \) is a 2-functor which is left adjoint to the constant diagram functor. \( \square \)

4.11 Let \( c : \text{Cat}^{\text{ML}} \to \mathcal{F}\text{unc}(\mathcal{L}, \text{Cat}^{\text{MLax}}) \) be the constant diagram functor. The unit of the adjunction 4.10 is given by the universal map \( j : X \to c(\text{hocolim} X) \).

The counit \( r : \text{hocolim} c S \to S \) is obtained from Theorem 4.8 by taking \( k = \text{id}_{c S} \).

It is explicitly given by the evaluation, which sends an object \([A; (K_1, L_1), \ldots, (K_n, L_n)]\) to \( A(K_1, \ldots, K_n) \) and which is the identity on the atoms 4.4(2) and (3), while the atoms \( \gamma : A \to B \) and \( f : [\text{id}; (K, L)] \to [\text{id}; (K', L)] \) are send to \( \gamma : A(K_1, \ldots, K_n) \to B(K_1, \ldots, K_n) \), respectively \( f : K \to K' \).

4.12 Change of indexing category: Let \( X : \mathcal{L} \to \text{Cat}^{\text{MLax}} \) be a diagram and let \( F : \mathcal{N} \to \mathcal{L} \) be a functor of small categories. Then \( F \) induces a functor

\( F_* : \text{hocolim} X \circ F \to \text{hocolim} X \)

given on objects by

\[ F_*([A; (K_1, N_1), \ldots, (K_n, N_n)]) = [A; (K_1, F(N_1)), \ldots, (K_n, F(N_n))] \]

and on atoms by

\[ F_*(\mu) = F(\mu) \quad \text{for } \mu : N \to N' \text{ in } \mathcal{N} \]

while all the other atoms of \( \text{hocolim} X \circ F \) are mapped identically to the corresponding ones in \( \text{hocolim} X \).

The construction implies:

4.13 Proposition: Let \( \xrightarrow{G} \mathcal{N} \xrightarrow{F} \mathcal{L} \) be functors of small categories. Then

\( (F \circ G)_* = F_* \circ G_* : \text{hocolim} X \circ F \circ G \to \text{hocolim} X \)

\( \square \)

4.14 Suppose we are given two functors \( F, H : \mathcal{N} \to \mathcal{L} \) and a natural transformation \( \tau : F \Rightarrow H \), then \( \tau \) induces a natural transformation

\( \tau_* : F_* \Rightarrow H_* \)

defined by

\[ \tau_*([A; (K_1, N_1), \ldots, (K_n, N_n)]) = A(\tau(N_1), \ldots, \tau(N_n)) \]

with the atoms \( \tau(N_i) : [\text{id}; (K_i, F(N_i))] \to [\text{id}; (K_i, G(N_i))] \) corresponding to the morphism \( \tau(N_i) : F(N_i) \to G(N_i) \) in \( \mathcal{L} \).

We will prove the following two results in Section 6:
4.15 Cofinality Theorem: Let \( \mathcal{M} \) be a \( \Sigma \)-free operad. Let \( X : \mathcal{L} \to \mathcal{Cat}^{\mathcal{M} \text{Lax}} \) be a diagram and let \( F : \mathcal{N} \to \mathcal{L} \) be a functor of small categories. Suppose that for each object \( L \) in \( \mathcal{L} \) the space \( B(L \downarrow F) \) is contractible. Then

\[
F_* : \hocolim X \circ F \to \hocolim X
\]

is a weak equivalence.

4.16 Proposition: Let \( \mathcal{M} \) be a \( \Sigma \)-free operad. Let \( f : X \to Y \) be a strict morphism of \( \mathcal{L} \)-diagrams in \( \mathcal{Cat}^{\mathcal{M} \text{Lax}} \) such that for each object \( L \in \mathcal{L} \) the underlying map \( f(L) \) of the lax morphism \( (f(L), f(L)) : X(L) \to Y(L) \) is a weak equivalence. Then

\[
\hocolim f : \hocolim X \to \hocolim Y
\]

is a weak equivalence.

4.17 Comparison with Thomason’s homotopy colimit in \( \mathcal{Cat} \): Let \( \mathcal{M} \) be the initial operad in \( \mathcal{Cat} \), i.e. \( \mathcal{M}(1) = \{\text{id}\} \) and \( \mathcal{M}(n) = \emptyset \) for \( n \neq 1 \). Then \( \mathcal{Cat}^{\mathcal{M}} = \mathcal{Cat}^{\mathcal{M} \text{Lax}} = \mathcal{Cat} \) and our homotopy colimit of a diagram \( X : \mathcal{L} \to \mathcal{Cat}^{\mathcal{M} \text{Lax}} = \mathcal{Cat} \) coincides with the Grothendieck construction \( \mathcal{L} \int X \), which is Thomason’s homotopy colimit of \( X \) in \( \mathcal{Cat} \).

The universal property of \( \mathcal{L} \int X \) [26, 1.3.1] is a special case of our Theorem 4.8. Thomason does not introduce the category \( \mathcal{F} \text{unc}(\mathcal{L}, \mathcal{Cat}) \) of strict functors \( \mathcal{L} \to \mathcal{Cat} \) and homotopy morphisms in [26], so he has no analogue of Theorem 4.10; instead he considers lax functors \( \mathcal{L} \to \mathcal{Cat} \) and homotopy morphisms of such in [26, Section 3].

5. Strictification and consequences

If \( \mathcal{L} \) is the trivial category consisting of one object and its identity morphism, then \( \mathcal{F} \text{unc}(\mathcal{L}, \mathcal{Cat}^{\mathcal{M} \text{Lax}}) = \mathcal{Cat}^{\mathcal{M} \text{Lax}} \), and Theorem 4.10 translates to

5.1 Theorem: There is a strictification 2-functor

\[
\text{str} : \mathcal{Cat}^{\mathcal{M} \text{Lax}} \to \mathcal{Cat}^{\mathcal{M}}
\]

which is left adjoint to the inclusion functor. □

By Theorem 4.8 we have a lax morphism \( j : X \to \text{str}X \) for an \( \mathcal{M} \)-algebra \( X \), and a strict homomorphism

\[
r : \text{str}X \to X
\]

such that \( r \circ j = \text{id} \). Both are natural in \( X \).
5.2 Explicit description of $r$: (See 4.11)

Objects: $r$ maps $([A; K_1, \ldots, K_n])$ to $A(K_1, \ldots, K_n)$

Atoms: $\gamma : [A; K_1, \ldots, K_n] \to [B; K_1, \ldots, K_n]$ is mapped to $\gamma(K_1, \ldots, K_n)$, the evaluation $ev(C)$ is mapped to the identity, and $f : [id, K] \to [id; \gamma]$ is mapped to $f : K \to K'$.

The composite $j \circ r : strX \to strX$ maps an object $[A; K_1, \ldots, K_n]$ to $[id; A(K_1, \ldots, K_n)]$ and we have a natural transformation

$$s : \text{id}_{str} \Rightarrow j \circ r$$

defined by

$$s([A; K_1, \ldots, K_n]) = ev(A) : [A; K_1, \ldots, K_n] \to [id; A(K_1, \ldots, K_n)].$$

It follows

5.3 Proposition: $r$ is a weak equivalence with section $j$.

If $X$ is a free $M$-algebra we can do better.

5.4 Proposition: Let $F_C : \text{Cat} \to \text{Cat}^M$ be the free algebra functor 2.3 and let $X = F_C Y$. Then the natural homomorphism $r : strX \to X$ has a section $k : X \to strX$ in $\text{Cat}^M$, natural in $Y$, and there is a natural transformation $\tau : k \circ r \Rightarrow \text{id}_{str}$.

Proof On objects we define $k(A; y_1, \ldots, y_n) = [A; (id, y_1), \ldots, (id, y_n)]$ and extend this definition to morphisms in the canonical way. Then $k : F_C Y \to strF_CY$ is a homomorphism such that $r \circ k = \text{id}_{F_C Y}$. Let $(B_i, y_i)$ with $B_i \in \mathcal{M}(r_i)$ and $y_i = (y_{i1}, \ldots, y_{ir_i})$ be an object in $F_C Y$ and $A$ an object in $\mathcal{M}(n)$. Then

$$k \circ r([A; (B_1, y_1), \ldots, (B_n, y_n)]) = [A \circ (B_1 \oplus \ldots \oplus B_n); (id, y_{i1}), \ldots, (id, y_{n,r_n})]$$

and the natural transformation $\tau([A; (B_1, y_1), \ldots, (B_n, y_n)])$ is the map

$$[A \circ (B_1 \oplus \ldots \oplus B_n); (id, y_{i1}), \ldots, (id, y_{n,r_n}) \to [A; (B_1, y_1), \ldots, (B_n, y_n)]$$

obtained by applying $A$ to the atoms $ev(B_i) : [B_i; (id, y_{i1}), \ldots, (id, y_{i,r_i})] \to [id; (B_i; y_{i1}, \ldots, y_{i,r_i})].$

$\text{Cat}^M$ is $\text{Cat}$-enriched, tensored and cotensored 2.7. We now make use of this additional structure.

5.5 Let $X : \mathcal{L} \to \text{Cat}^M\text{Lax}$ be an $\mathcal{L}$-diagram in $\text{Cat}^M\text{Lax}$ and $Y : \mathcal{L} \to \text{Cat}^M$ an $\mathcal{L}$-diagram in $\text{Cat}^M$. Let $i : \text{Cat}^M \to \text{Cat}^M\text{Lax}$ be the inclusion functor. Then
Proposition 3.5, Theorem 5.1, and Theorem 3.10 provide the following chain of isomorphisms of categories

\[ \mathcal{F}unc(\mathcal{L}, \text{Cat}^{\text{Lax}})_{(X, iY)} \cong \mathcal{C}at^{\mathcal{L} \times \mathcal{L}^{op}}_{((-\mathcal{L}/-), \text{Cat}^{\text{Lax}}(X, iY))} \cong \mathcal{C}at^{\mathcal{L} \times \mathcal{L}^{op}}_{((-\mathcal{L}/-), \text{Cat}^{\text{M}}(\text{str}X, Y))} \cong (\text{Cat}^{\text{M}})^{\mathcal{L}}_{((-\mathcal{L}/-) \otimes_{\mathcal{L}} \text{str}X, Y)} \cong (\text{Cat}^{\text{M}})^{\mathcal{L}}(R(\text{str}X), Y) \]

and, if \( Y \) is the constant diagram \( cS \) on \( S \in \text{Cat}^{\text{M}} \), Theorem 4.10 gives

\[ \text{Cat}^{\text{M}}(\text{hocolim}X, S) \cong \mathcal{F}unc(\mathcal{L}, \text{Cat}^{\text{Lax}})_{(X, cS)} \cong (\text{Cat}^{\text{M}})^{\mathcal{L}}_{(R(\text{str}X), cS)} \cong \text{Cat}^{\text{M}}((-\mathcal{L}) \otimes_{\mathcal{L}} \text{str}X, S). \]

The last isomorphism is induced by the isomorphisms

\[ \text{colim}R(Z) = \text{colim}((-\mathcal{L}/-) \otimes_{\mathcal{L}} Z) = * \otimes_{\mathcal{L}} ((-\mathcal{L}/-) \otimes_{\mathcal{L}} Z) \cong (* \times_{\mathcal{L}} (-\mathcal{L}/-)) \otimes_{\mathcal{L}} Z \cong (-\mathcal{L}) \otimes_{\mathcal{L}} Z \]

for a diagram \( Z : \mathcal{L} \to \text{Cat}^{\text{M}} \).

We obtain

5.6 Theorem: The three functors hocolim and \((-\mathcal{L}) \otimes_{\mathcal{L}} \text{str}\) and \(\text{colim}R\text{str}\)

\[ \mathcal{F}unc(\mathcal{L}, \text{Cat}^{\text{Lax}}) \to \text{Cat}^{\text{M}} \]

sending \( X \) to hocolim\( X \), to \((-\mathcal{L}) \otimes_{\mathcal{L}} \text{str}X\), and to \(\text{colim}R\text{str}X\) respectively are naturally isomorphic.

If \( \mathcal{M} \) is the initial operad in \( \text{Cat} \) the strictification functor

\[ \text{str} : \text{Cat}^{\text{M}}_{\text{Lax}} = \text{Cat}^{\text{M}} = \text{Cat} \to \text{Cat}^{\text{M}} = \text{Cat} \]

is the identity functor. Since the tensor in \( \text{Cat} \) is just the product Theorem 5.6 and 4.11 imply

5.7 Proposition: Thomason’s homotopy colimit functor \( \mathcal{L} \int - \) and the functor \((-\mathcal{L}) \times_{\mathcal{L}} - \) are naturally isomorphic functors from the category \( \mathcal{F}unc(\mathcal{L}, \text{Cat}) \) of functors \( X : \mathcal{L} \to \text{Cat} \) and homotopy morphisms to the category \( \text{Cat} \).

5.8 Proposition: The free functor \( F_C : \text{Cat} \to \text{Cat}^{\text{M}} \) preserves homotopy colimits up to weak equivalences.

Proof Let \( X : \mathcal{L} \to \text{Cat} \) be a diagram in \( \text{Cat} \). By 5.6 and 5.7 we have to show that the canonical map

\[ (-\mathcal{L}) \otimes_{\mathcal{L}} \text{str}F_C X \to F_C((-\mathcal{L}) \times_{\mathcal{L}} X) \]
is a weak equivalence. Since $F_C$ is a left adjoint $\mathcal{C}$-functor it preserves coends and tensors, so that $F_C((-/\mathcal{L}) \times X) \cong (-/\mathcal{L}) \otimes \mathcal{C}X$ and we are left to show that the homomorphism $r : \text{str}F_CX \to F_CX$ induces a weak equivalence

$$(-/\mathcal{L}) \otimes \text{str}F_CX \to (-/\mathcal{L}) \otimes F_CX.$$ 

But this follows from 5.4. 

5.9 Remark: We have seen that Thomason’s homotopy colimit of a functor $X : \mathcal{L} \to \mathcal{C}$ can be identified with $(-/\mathcal{L}) \times \mathcal{L}X$. In comparison to this, the Bousfield-Kan homotopy colimit $\text{hocolim} Z$ of a diagram $Z : \mathcal{L} \to \text{SSets}$ is $N(-/\mathcal{L}) \otimes \mathcal{Z}$ where $N$ is the nerve functor. Both $\mathcal{C}$ and $\text{SSets}$ are self-enriched, and the tensor is given by the product. So both constructions are tensoring constructions involving $(-/\mathcal{L})$ respectively the nerve of $(-/\mathcal{L})$. Theorem 5.6 shows that our homotopy colimit construction fits into this set-up, the only difference being the objectwise strictification of our diagrams, which can be interpreted as a kind of “cofibrant replacement”. The latter is part of homotopy invariant homotopy colimit constructions in Quillen model categories (e.g. see [14, 18.5.3].

5.10 Proposition: Let $S$ be an $M$-algebra and $cS : \mathcal{L} \to \mathcal{C}^{M\text{Lax}}$ the constant diagram on $S$. Then in $\mathcal{C}^{M}$

$$\text{hocolim} cS \cong \mathcal{L} \otimes \text{str}S,$$

and the counit of the adjunction 4.10 corresponds to the composite

$$\mathcal{L} \otimes \text{str}S \to * \otimes \text{str}S \cong \text{str}S \to S.$$

Proof Let $F : \mathcal{L}^{op} \to \mathcal{C}$ be defined by $F(L) = L/\mathcal{L}$. Then

$$\text{hocolim} cS \cong (-/\mathcal{L}) \otimes \text{str}S \cong \text{colim} F \otimes \text{str}S \cong \mathcal{L} \otimes \text{str}S.$$ 

The second statement holds because the counit factors through the change of indexing category functor $F*$ for $F : \mathcal{L} \to *$ (see 4.11). 

6. The homotopy colimit theorem

In this section we compare our homotopy colimit with the homotopy colimit constructions of algebras in the categories $\text{SSets}$ and $\text{Top}$. Let

$$N : \mathcal{C} \to \text{SSets}, \quad | - | : \text{SSets} \to \text{Top}, \quad \text{Sing} : \text{Top} \to \text{SSets}$$

be the nerve, the topological realization functor, and the singular functor respectively, let $B = |N|$ be the classifying space functor. We recall

6.1 Definition: A map of topological spaces is called a weak equivalence if it is a weak homotopy equivalence. A functor $F : \mathcal{A} \to \mathcal{B}$ of categories and a map $f : K \to L$ of simplicial sets is called a weak equivalence if $BF : B\mathcal{A} \to B\mathcal{B}$ respectively $|f| : |K| \to |L|$ is a weak equivalence.
The categories $SSets$ and $Top$ are cofibrantly generated simplicial model categories with these weak equivalences; in particular, they are simplicially enriched \(\{2.9\}\). The realization and the singular functor are simplicial functors and we have a $SSets$-enriched Quillen equivalence \(\{22\}\)

\[ | - | : SSets \rightleftharpoons Top : Sing. \]

Let $\mathcal{M}$ be a $\Sigma$-free operad in $\mathbf{Cat}$. Then $N\mathcal{M}$ is a simplicial and $BM$ a topological operad. Let $SSets^{NM}$ and $Top^{BM}$ denote the categories of $NM$-algebras in $SSets$ respectively $BM$-algebras in $Top$, where $NM$ and $BM$ are the monads associated with the operads $N\mathcal{M}$ and $BM$ respectively. It is well-known that the Quillen model structures of $SSets$ and $Top$ lift to cofibrantly generated simplicial Quillen model structure on $SSets^{NM}$ and $Top^{BM}$, whose weak equivalences are those homomorphisms of algebras whose underlying maps are weak equivalences in $SSets$ respectively $Top$ (e.g. see \(\{22\}\); for the simplicial enrichment and the simplicial tensor recall \(\{2.9\}\)). Since the singular functor $Sing : Top \rightarrow SSets$ and the realization functor preserve products, the classical Quillen equivalence $| - | : SSets \rightleftharpoons Top : Sing$ lifts to a $SSets$-enriched Quillen equivalence $| - |^{\text{alg}} : SSets^{NM} \rightleftharpoons Top^{BM} : Sing^{\text{alg}}$.

6.2 Definition: Let $X : \mathcal{L} \rightarrow SSets^{NM}$ be an $\mathcal{L}$-diagram in $SSets^{NM}$, then its homotopy colimit $\text{hocolim}^{NM}X$ is defined by

\[ \text{hocolim}^{NM}X = N(-/\mathcal{L}) \otimes_{\mathcal{L}} QX \]

where $QX \rightarrow X$ is a fixed functorial objectwise cofibrant replacement.

If $X : \mathcal{L} \rightarrow Top^{BM}$ is an $\mathcal{L}$-diagram in $Top^{BM}$ its homotopy colimit is defined by

\[ \text{hocolim}^{BM}X = B(*, \mathcal{L}, \mathcal{L}) \otimes_{\mathcal{L}} QtX \]

where $QtX \rightarrow X$ again is a fixed functorial objectwise cofibrant replacement and $B(*, \mathcal{L}, \mathcal{L})$ is the 2-sided bar construction (see \(\{15\}\)).

6.3 Remark: Hirschhorn defines the homotopy colimit as $N(-/\mathcal{L}) \otimes_{\mathcal{L}} X$ without the cofibrant replacement \(\{14\}, 18.1.2\). Our definition has the advantage of being homotopy invariant (see \(\{14\}, 18.5.3\)). Up to weak equivalences our definition is independent of the choice of $Q$ respectively $Q_t$. For let $Q' \leftrightarrow$ another objectwise cofibrant replacement functor, the morphisms $Q'X \leftarrow Q'QX \rightarrow QX$ are objectwise weak equivalences between objectwise cofibrant diagrams and hence induce weak equivalences

\[ N(-/\mathcal{L}) \otimes_{\mathcal{L}} Q'X \leftarrow N(-/\mathcal{L}) \otimes_{\mathcal{L}} Q'QX \rightarrow N(-/\mathcal{L}) \otimes_{\mathcal{L}} QX. \]

by \(\{14\}, 18.5.3\]

6.4 Proposition: For a diagram $X : \mathcal{L} \rightarrow SSets^{NM}$ there are natural weak equivalences

\[ \text{hocolim}^{BM}|X| \leftarrow \text{hocolim}^{BM}|QX| \rightarrow \text{hocolim}^{NM}|X| \]
Proof Since $|QX| \to |X|$ is an objectwise weak equivalence the left map is a weak equivalence by homotopy invariance.

Note that $B(*, L, L) = B(-/L)$. Since realization is a simplicial left Quillen functor it preserves coends, tensors, and maps cofibrant objects to cofibrant objects. So we have a natural isomorphism

$$|N(-/L) \otimes L QX| \cong N(-/L) \otimes_L |QX| = |N(-/L)| \otimes_L |QX|$$

and each $|QX(L)|$ is cofibrant. Since $Q_i|QX| \to |QX|$ is an objectwise weak equivalence

$$\text{hocolim}^{BM}|QX| = |N(-/L)| \otimes_L Q_i|QX| \to |N(-/L)| \otimes_L |QX|$$

is a weak equivalence by [14, 18.5.3].

Now let $X : L \to \mathcal{C}at^M Lax$ be a $L$-diagram in $\mathcal{C}at^M Lax$. Composing with the strictification and the nerve functor we have

$$L \xrightarrow{X} \mathcal{C}at^M Lax \xrightarrow{\text{str}} \mathcal{C}at^M \xrightarrow{N} \mathbb{S}ets^{NM}$$

By definition of the tensor there is a map

$$\alpha : N(-/L) \otimes_L QN\text{str}X \xrightarrow{\text{hocolim}^{NM}\text{str}X} N(-/L) \otimes_L N\text{str}X \xrightarrow{N((-/L) \otimes_L \text{str}X)} N(\text{hocolim}^M X)$$

which is natural with respect to strict morphisms of $L$-diagrams in $\mathcal{C}at^M Lax$.

6.5 Theorem: For each diagram $X : L \to \mathcal{C}at^M Lax$ the map

$$\alpha : \text{hocolim}^{NM}\text{Nstr}X \to N(\text{hocolim}^M X)$$

is a weak equivalence, natural with respect to strict morphisms of $L$-diagrams in $\mathcal{C}at^M Lax$.

We prove the theorem in steps. We first reduce it to the case that $X$ is a diagram in $\mathcal{C}at^M$.

6.6 Lemma: It suffices to prove the theorem for diagrams in $\mathcal{C}at^M$.

Proof Let $X : L \to \mathcal{C}at^M Lax$ be a diagram in $\mathcal{C}at^M Lax$. For each $\mathcal{M}$-algebra $Z$ the natural lax morphism $j_Z : Z \to \text{str}Z$ and the natural homomorphism $r_Z : \text{str}Z \to Z$ induce morphisms $j : X \to \text{str}X$ and $r : \text{str}X \to X$ of diagrams in $\mathcal{C}at^M Lax$ such that $r \circ j = \text{id}$. Hence

$$\alpha_X : \text{hocolim}^{NM}\text{Nstr}X \to N(\text{hocolim}^M X)$$

35
is a retract of
\[ \alpha_{\text{str} X} : \text{hocolim}^\text{NM} N\text{str}(\text{str} X) \longrightarrow N(\text{hocolim}^\text{M} \text{str} X). \]

If the latter is a weak equivalence so is the former, because retracts of weak equivalences are weak equivalences. But \( \text{str} X \) is a diagram in \( \text{Cat}^\text{M} \).

We now deal with the free case. Consider the diagram

\[
\begin{array}{ccc}
\text{Cat}^\text{M} & \xrightarrow{N^\text{alg}} & \mathcal{SSets}^\text{NM} \\
\downarrow U_C & & \downarrow \uparrow U_S \\
\text{Cat} & \xrightarrow{N} & \mathcal{SSets}
\end{array}
\]

where \( U_C \) and \( U_S \) are the forgetful and \( F_C \) and \( F_S \) the free algebra functors. Clearly, \( U_S \circ N^\text{alg} = N \circ U_C \), and since \( \mathcal{M} \) is \( \Sigma \)-free we have

6.7 \( N^\text{alg} \circ F_C = F_S \circ N \).

The functors \( U_C \) and \( F_C \) are \( \text{Cat} \)-enriched, while \( U_S \) and \( F_S \) are simplicial functors. Since \( F_C \) and \( F_S \) are enriched left adjoints they preserve tensors and coends.

Since \( N^\text{alg} \) is just \( N \) applied to some algebra we usually drop \( \text{alg} \) from the notation. By [5,4] \( -/\mathcal{L} \times_{\mathcal{L}} Z \) is isomorphic to Thomason’s homotopy colimit of the diagram \( Z : \mathcal{L} \to \text{Cat} \), and by [20, Thm. 1.2] there is a weak equivalence

\[ N(-/\mathcal{L}) \times_{\mathcal{L}} N Z \xrightarrow{\sim} N(-/\mathcal{L} \times_{\mathcal{L}} Z) \]

in \( \mathcal{SSets} \). Since \( \mathcal{M} \) is \( \Sigma \)-free the free functor \( F_S \) preserves weak equivalences. So we obtain a weak equivalence

\[ F_S(N(-/\mathcal{L}) \times_{\mathcal{L}} N Z) \xrightarrow{\sim} F_S \circ N(-/\mathcal{L} \times_{\mathcal{L}} Z) \]

Since

\[ F_S(N(-/\mathcal{L}) \times_{\mathcal{L}} N Z) \cong N(-/\mathcal{L}) \otimes_{\mathcal{L}} (F_S \circ N)Z = N(-/\mathcal{L}) \otimes_{\mathcal{L}} (N \circ F_C)Z \]

and

\[ F_S(N(-/\mathcal{L} \times_{\mathcal{L}} Z)) = N \circ F_C(-/\mathcal{L} \otimes_{\mathcal{L}} Z) \cong N(-/\mathcal{L} \otimes_{\mathcal{L}} F_C Z) \]

the lower horizontal map in following commutative diagram

\[
\begin{array}{ccc}
N(-/\mathcal{L}) \otimes_{\mathcal{L}} Q\text{str} F_C Z & \xrightarrow{\alpha} & N(-/\mathcal{L} \otimes_{\mathcal{L}} \text{str} F_C Z) \\
\downarrow & & \downarrow \\
N(-/\mathcal{L}) \otimes_{\mathcal{L}} Q\text{F}_C Z \\
\downarrow & & \\
N(-/\mathcal{L} \otimes_{\mathcal{L}} F_S N Z & \longrightarrow & N(-/\mathcal{L} \otimes_{\mathcal{L}} F_C Z)
\end{array}
\]

36
is a weak equivalence. The two left vertical maps are induced by \( r : \text{str} F_C Z \rightarrow F_C Z \) and the functorial cofibrant replacement map \( QNF_C Z \rightarrow NF_C Z = F_S NZ \).

Since these maps are objectwise weak equivalences between objectwise cofibrant diagrams (for any \( Y \) in \( S\text{Sets} \) the algebra \( F_S Y \) is cofibrant), the two vertical maps of the diagram are weak equivalences by \([14, 18.5.3]\). The right vertical map is a weak equivalence by the proof of \([15, \S 3]\). Hence \( \alpha \) is a weak equivalence.

We summarize:

**6.8 Lemma:** Given a diagram \( X : L \rightarrow \text{Cat} \xrightarrow{F_C} \text{Cat}^M \), the natural map

\[
\alpha : \text{hocolim}^M N \text{str} X \rightarrow N(\text{hocolim}^M X)
\]

is a weak equivalence. \( \square \)

Now let \( X : L \rightarrow \text{Cat}^M \) be an arbitrary diagram. We will resolve \( X \) to analyze its homotopy colimit:

**6.9** Let \( \Delta_+ \) denote the category of ordered sets \( \underline{n} = \{-1 < 0 < \ldots < n\} \) and monotone maps preserving \(-1\). We have an obvious inclusion of the simplicial indexing category \( \Delta \subset \Delta_+ \). Giving a functor \( X : \Delta_+ \rightarrow \mathcal{C} \) amounts to giving a simplicial object \( X_\bullet \) in \( \mathcal{C} \) together with an object \( X_{-1} \), a morphism \( \varepsilon = d_0 : X_0 \rightarrow X_{-1} \), and additional degeneracy morphisms \( s_{-1} : X_n \rightarrow X_{n+1} \) for \( n = -1, 0, \ldots \) satisfying the extra simplicial identities

\[
\begin{align*}
\varepsilon d_0 &= \varepsilon d_1 \quad \text{i.e.} \quad d_0 d_0 = d_0 d_1 \\
 s_{-1} s_i &= s_{i+1} s_{-1} \\
 d_0 s_{-1} &= \text{id} \\
 d_i s_{-1} &= s_{-1} d_{i-1} \quad \text{for} \quad i > 0
\end{align*}
\]

Equivalently, a functor \( X : \Delta_+^{op} \rightarrow \mathcal{C} \) into any category \( \mathcal{C} \) consists of a simplicial object \( X_\bullet \) in \( \mathcal{C} \) together with a simplicial map

\[
\varepsilon : X_\bullet \rightarrow X_{-1}
\]

to the constant simplicial object on \( X_{-1} \), which in degree \( n \) is defined by \( \varepsilon_n = (d_0)^{n+1} \), and which we also denote by \( \varepsilon \), a simplicial section

\[
s : X_{-1} \rightarrow X_\bullet
\]
given in degree \( n \) by \( (s_{-1})^{n+1} \), and a simplicial homotopy \( s \circ \varepsilon \simeq \text{id} \).

The Godement resolution of \( M = U_C \circ F_C \) \([11, \text{App.}]\) is a functor

\[
M^+ : \text{Cat}^M \rightarrow \text{Cat}^{\Delta_+^{op}}
\]

better known as the functorial 2-sided bar construction

\[
B\bullet(M, M, X) \xrightarrow{\varepsilon} X
\]

37
with its augmentation \( \varepsilon \) and section \( s \) (see [13, Chap. 9]). Recall that \( B_\bullet(\mathcal{M}, \mathcal{M}, X) \) is a simplicial object in \( \text{Cat}^\mathcal{M} \) and \( \varepsilon \) is a simplicial map in \( \text{Cat}^\mathcal{M} \) to the constant simplicial object on \( X \). The morphism \( s \) is a section of \( \varepsilon \) in \( \text{Cat}^\Delta^\text{op} \) but not in \( (\text{Cat}^\mathcal{M})^\Delta^\text{op} \). For further details see [20, 2.2.2].

Let \( \mathcal{M}_\bullet = B_\bullet(\mathcal{M}, \mathcal{M}, -) \) denote the restriction of \( \mathcal{M}_\bullet^+ \) to \( \Delta^\text{op} \). Applying the natural map \( \alpha \) dimensionwise we obtain a map of bisimplicial sets

\[
\alpha_\bullet : N(-/\mathcal{L}) \otimes \mathcal{L} \text{QNstr}\mathcal{M}_\bullet X \rightarrow N(-/\mathcal{L}) \otimes \mathcal{L} \text{str}\mathcal{M}_\bullet X
\]

Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{diag}(N(-/\mathcal{L}) \otimes \mathcal{L} \text{QNstr}\mathcal{M}_\bullet X) & \xrightarrow{\text{diag}(\alpha_\bullet)} & \text{diag}N(-/\mathcal{L}) \otimes \mathcal{L} \text{str}\mathcal{M}_\bullet X \\
\varepsilon_1 & & \varepsilon_2 \\
N(-/\mathcal{L}) \otimes \mathcal{L} \text{QNstrX} & \xrightarrow{\alpha} & N(-/\mathcal{L}) \otimes \mathcal{L} \text{strX}
\end{array}
\]

where the vertical maps are induced by the augmentation \( \varepsilon : \mathcal{M}_\bullet X \rightarrow X \).

Recall that \( \mathcal{M}_n = \mathcal{M}_{n+1} \), so that \( \mathcal{M}_n X \) is a free \( \mathcal{M} \)-algebra for each \( n \geq 0 \). Hence the natural maps \( \alpha_n \) are weak equivalences by \([6, \text{IV.1.7}].\) To prove Theorem 6.5 we show that the two vertical maps of the diagram are weak equivalences.

6.10 Lemma: For any diagram \( X : \mathcal{L} \rightarrow \text{Cat}^\mathcal{M} \) the map

\[
\varepsilon_1 : \text{diag}(N(-/\mathcal{L}) \otimes \mathcal{L} \text{QNstr}\mathcal{M}_\bullet X) \rightarrow N(-/\mathcal{L}) \otimes \mathcal{L} \text{QNstrX}
\]

is a weak equivalence.

Proof If \( K_\bullet \) is a simplicial object in \( \text{SSets} \) and \( \nabla : \Delta \rightarrow \text{SSets} \) is the functor which maps \([n]\) to the standard simplicial \( n \)-simplex then

\[
\text{diag}K_\bullet \cong \nabla \times_{\Delta^\text{op}} K_\bullet
\]

By the argument of [21, 4.4] the “internal realization” in \( \text{SSets}^\mathcal{NM} \) coincides with the “external realization” in \( \text{SSets} \); in other words,

\[
\text{diag}A_\bullet \cong \nabla \otimes_{\Delta^\text{op}} A_\bullet
\]

in \( \text{SSets}^\mathcal{NM} \) for any simplicial object \( A_\bullet \) in \( \text{SSets}^\mathcal{NM} \). Since two tensors commute we obtain

\[
\text{diag}(N(-/\mathcal{L}) \otimes \mathcal{L} \text{QNstr}\mathcal{M}_\bullet X) \cong \nabla \otimes_{\Delta^\text{op}} (N(-/\mathcal{L}) \otimes \mathcal{L} \text{QNstr}\mathcal{M}_\bullet X) \cong N(-/\mathcal{L}) \otimes \mathcal{L} (\nabla \otimes_{\Delta^\text{op}} \text{QNstr}\mathcal{M}_\bullet X) \cong N(-/\mathcal{L}) \otimes \mathcal{L} \text{diag}(\text{QNstr}\mathcal{M}_\bullet X)
\]

in \( \text{SSets}^\mathcal{NM} \). For an object \( L \) in \( \mathcal{L} \) we consider the commutative diagram

\[
\begin{array}{ccc}
\text{diag}(\text{QNstr}\mathcal{M}_\bullet X(L)) & \xrightarrow{\text{diag}(\text{QNitr})} & \text{diag}(\text{QNM}_\bullet X(L)) \\
\varepsilon \downarrow & & \varepsilon \downarrow \\
\text{QNstrX}(L) & \xrightarrow{\text{QNr}} & \text{QN}_X(L)
\end{array}
\]

38
where \( r \) is the homomorphism of \( 5.2 \). By \( 5.3 \) the maps \( N r \) and \( N r_n : N \text{str} X(L) \to NM_n X(L) \) are weak equivalences. Hence so are \( Q N r \) and, by \( [13, IV.1.7.] \), also \( \text{diag}(Q N r \bullet) \).

From \( 6.7 \) we deduce that \( N \circ M^{n+1} = (NM)^{n+1} \circ N \).

It follows that \( NM_\bullet X(L) = (NM)_\bullet X(L) \) is a resolution of \( NX(L) \), so that \( QN \varepsilon : \text{diag}(QM_\bullet X(L)) \to QNX(L) \) is a weak equivalence. Since \( \text{diag}(QN \text{str} \bullet X(L)) \to QN \text{str} X(L) \) is a weak equivalence for each object \( L \) in \( L \), \( [14, 18.5.3] \) implies that \( \varepsilon_1 : \text{diag}(N(-/L) \otimes L \text{str} \bullet X(L)) \to N(-/L) \otimes L \text{str} X(L) \) is a weak equivalence.

Finally we investigate the map

\[
\varepsilon_2 : \text{diag}N(-/L \otimes L \text{str} \bullet X(L)) \to N(-/L \otimes L \text{str} X(L)).
\]

Since \( |\text{diag}(N(-/L \otimes L \text{str} \bullet X(L)))| \cong |B(-/L \otimes L \text{str} \bullet X(L))| \) we may investigate

\[
|B(-/L \otimes L \text{str} \bullet X(L))| \to B(-/L \otimes L \text{str} X(L)).
\]

Since \( B(-/L \otimes L \text{str} \bullet X(L)) \) is a simplicial CW-complex, the inclusions of the spaces of degenerate elements are closed cofibrations. Hence its realization is equivalent to its fat realization, which ignores degeneracies and which is known to be equivalent to its homotopy colimit along \( \Delta^\text{op} \) in \( \text{Top} \):

6.11

\[
|B(-/L \otimes L \text{str} \bullet X(L))| \xrightarrow{\sim} \text{hocolim}_{\Delta^\text{op}} \text{Top} BH
\]

where \( H \) is the functor

\[
H : \Delta^\text{op} \to \text{Cat}, \quad [n] \mapsto (-/L) \otimes \text{str} M_n X(L).
\]

If \( \mathcal{X} = (-/L) \otimes L \text{str} X \) we have to show that the augmentation \( \varepsilon \) induces a weak equivalence

\[
\text{hocolim}_{\Delta^\text{op}} \text{Top} BH \to BX.
\]

We now follow in part Thomason’s argument of \( [28, p 1641ff] \):

Let \( F : \mathcal{L} \to \text{Cat} \) be a functor. The dual Grothendieck construction \( F \backslash \mathcal{L} \) is the category whose objects are pairs \( (L, X) \) with an object \( L \) of \( \mathcal{L} \) and an object \( X \) of \( F(L) \). A morphism

\[
(l, x) : (L, X) \to (L', X')
\]
consists of a morphism \( l : L' \to L \) in \( \mathcal{L} \) and a morphism \( x : X \to F(l)X' \) in \( F(L) \). Composition is defined by \( (l_2, y) \circ (l_1, x) = (l_1 \circ l_2, F(l_1)y \circ x) \). It is related to Thomason’s Grothendieck construction \( \mathcal{L} \int F \) by

\[
F \int \mathcal{L} = (\mathcal{L} \int F^{op})^{op}
\]

where \( F^{op} : \mathcal{L} \to \text{Cat} \) sends \( L \) to \( F(L)^{op} \).

We have a commutative diagram

\[
\begin{array}{ccc}
B(\Delta^{op} \int H^{op}) & \xrightarrow{\tau} & B(\Delta^{op} \int H^{op}) \\
\downarrow \tau & & \downarrow h \\
B(\Delta^{op} \int H^{op}) & \xrightarrow{\tau} & \text{hocolim}_{\Delta^{op}} B(H^{op}) \\
\downarrow \tau & & \downarrow \text{hocolim}_{\Delta^{op}} BH \\
B(\Delta^{op} \int H^{op}) & \xrightarrow{\tau} & B(\Delta^{op} \int H^{op}) \\
\downarrow \tau & & \downarrow \tau \\
B(\Delta^{op} \int H^{op}) & \xrightarrow{\tau} & B(\Delta^{op} \int H^{op})
\end{array}
\]

where \( \tau \) is the weak equivalence of Thomason’s homotopy colimit construction in \( \text{Cat} \) [20], the unspecified maps are induced by the augmentation, and \( \tau : B(C^{op}) \cong BC \) is the well-known natural homeomorphism. Hence we are left to prove that the augmentation induces a weak equivalence

\[
\varepsilon : H \int \Delta^{op} \to \mathcal{X} = \text{hocolim}_{\Delta^{op}} X.
\]

By Quillen’s Theorem A [22, §1] it suffices to show that the comma category \( \varepsilon / K \) is contractible for each object \( K \) of \( \mathcal{X} \).

Let \( F : \mathcal{L} \to \text{Cat} \) be a functor and \( \varepsilon : F \int \mathcal{L} \to \mathcal{C} \) be any functor. Let \( j(L) : F(L) \to F \int \mathcal{L} \) be the functor sending \( X \) to \( (L, X) \) and let \( \varepsilon(L) = \varepsilon \circ j(L) \). Then for any object \( C \) of \( \mathcal{L} \) the comma category \( \varepsilon / C \) is isomorphic to \( (\varepsilon(\_)/C) \int \mathcal{L} \) where \( \varepsilon(\_)/C : \mathcal{L} \to \text{Cat} \) sends \( L \) to \( \varepsilon(L)/C \) [28, Lemma 4.6].

We apply this to our functor \( H \): Define

\[
\varepsilon_n = \varepsilon \circ j([n]) : H([n]) \to H \int \Delta^{op} \to \text{hocolim}_{\Delta^{op}} \mathcal{X},
\]

so that \( \varepsilon_n : (-/\mathcal{L}) \otimes_{\mathcal{L}} \text{str} \mathbb{M}_n X \to (-/\mathcal{L}) \otimes_{\mathcal{L}} \text{str} X \). Let \( A[K, L] \) stand for the object \( [A; (K_1, L_1), \ldots, (K_r, L_r)] \) in \( \text{hocolim}_{\Delta^{op}} \mathcal{X} \), and let \( G \) be the functor

\[
G : \Delta^{op} \to \text{Cat}, \quad [n] \to \varepsilon_n / A[K, L].
\]

We have to show that \( G \int \Delta^{op} \) is contractible. By [26, Thm.1.2] and [6.11] we have homotopy equivalences

\[
B(G \int \Delta^{op}) \simeq \text{hocolim}_{\Delta^{op}} BG \simeq [n] \mapsto BG[n] \|
\]

6.12 Lemma: The functor \( G \) extends to a lax functor \( G_+ : \Delta^{op} \to \text{Cat} \) such that \( G_+([-1]) = \varepsilon_{-1} / A[K, L] \), where \( \varepsilon_{-1} : (-/\mathcal{L}) \otimes_{\mathcal{L}} \text{str} X \to (-/\mathcal{L}) \otimes_{\mathcal{L}} \text{str} X \) is the identity (recall that \( \mathbb{M}_n X = \mathbb{M}^{n+1} X \)).
We postpone the proof. We apply Street’s first rectification construction with $k : \Delta^\text{op} \subset \Delta^\text{op}_+$. Since $G_+ \circ k = G$ is a strict functor we obtain maps of simplicial spaces

$$B((SG_+)|\Delta^\text{op}) \xrightarrow{B\xi} B(SG) \xrightarrow{B\varepsilon} BG$$

which are dimensionwise homotopy equivalences, because an adjunction induces a homotopy equivalence of classifying spaces. Hence

$$\parallel BG \parallel \simeq \parallel B((SG_+)|\Delta^\text{op}) \parallel \simeq B(SG_+([-1]) \simeq \ast.$$ 

The latter holds by definition of $\Delta^\text{op}_+$ and the fact that $B(SG_+([-1]) \simeq B(G_+([-1])) \simeq \ast$, because $G_+([-1])$ has a terminal object. It remains to prove the lemma.

**Proof of Lemma 6.12** Throughout this proof we denote various maps induced by the augmentation $M_0 X \to X$ by $\varepsilon$.

To extend $G : \Delta^\text{op} \to \mathcal{C}$ at $\varepsilon$ together with the augmentation $\varepsilon : G([0]) \to G([-1])$ to a lax functor $G_+ : \Delta^\text{op}_+ \to \mathcal{C}$ we have to define functors

$$s_{-1} : \varepsilon_n/A[K, L] \to \varepsilon_{n+1}/A[K, L] \quad n = -1, 0, 1, \ldots$$

satisfying the simplicial identities up to coherent natural transformations. An object $Z$ of $\varepsilon_n/A[K, L]$ is a pair consisting of an object $[B; (K_1', L_1'), \ldots, (K_n', L_n')]$ of $\text{hocolim}^{M_n} X$ and a morphism

$$\alpha : [B(\varepsilon K_1', L_1'), \ldots, (\varepsilon K_n', L_n')] \to A[K, L]$$

in $\text{hocolim}^{M_n} X$, where this time $\varepsilon$ is the augmentation $M_n X \to X$. Suppose $\alpha$ is given by the data

$$\begin{array}{c}
\lambda_{i_{11}} & \cdots & \lambda_{i_{1s_1}} & \lambda_{i_{1r_1}} & \cdots & \lambda_{i_{r_s_1}} \\
C_1, f_1 & & & & & C_r, f_r
\end{array}$$

and $\gamma : B \to A \ast (C_1 \oplus \ldots \oplus C_r) \cdot \sigma$, where $\sigma^{-1}$ can be read off the labels of the $\lambda$'s. Let

$$\overline{K}_j = C_j (\lambda_{i_{1j_1} K_{1j_1}', \ldots, \lambda_{i_{rj_r} K_{rj_r}'}}) \in M_n X(L_j).$$

Then the data

$$\begin{array}{c}
\lambda_{i_{11}} & \cdots & \lambda_{i_{1s_1}} & \lambda_{i_{1r_1}} & \cdots & \lambda_{i_{r_s_1}} \\
C_1, \text{id} & & & & & C_r, \text{id}
\end{array}$$

and $\gamma$ define a morphism

$$\overline{\pi} : [B(K_1', L_1'), \ldots, (K_n', L_n')] \to [A; (\overline{K}_1, L_1), \ldots, (\overline{K}_r, L_r)]$$

41
in hocolim$^\mathbb{M}\mathbb{M}_n X$ such that $\alpha = (f_1, \ldots, f_r) \circ \varepsilon(\omega)$.

We define

$$s_{-1}(Z) = ([A; (i\overline{K}_1, L_1), \ldots, (i\overline{K}_r, L_r)], (f_1, \ldots, f_r))$$

where $\varepsilon : \text{Id} \to \mathbb{M}$ is the unit of the monad $\mathbb{M}$.

Since multiple indices obscure the definition of $s_{-1}$ on morphisms we give a typical example. We refer the reader to Example 4.3.

Let $A[K, L]$ be the object $[A; (K'_1, L'_1), \ldots, (K'_n, L'_n)]$ in hocolim$^\mathbb{M} X$. Let $[A_2; (K'_1, L'_1), \ldots, (K'_4, L'_4)]$ be an object in hocolim$^\mathbb{M}\mathbb{M}_n X$ and

$$\alpha : [A_2; (\varepsilon K'_1, L'_1), \ldots, (\varepsilon K'_4, L'_4)] \to [A_3; (K''_1, L''_1), \ldots, (K''_3, L''_3)]$$

be given by

and $\gamma_2 : A_2 \to A_3 \ast (D_1 \oplus D_2 \oplus D_3) \cdot \tau$ where $\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$.

Let

$$\beta : [A_1; (K_1, L_1), \ldots, (K_7, L_7)] \to [A_2; (K'_1, L'_1), \ldots, (K'_4, L'_4)]$$

be the morphism in hocolim$^\mathbb{M}\mathbb{M}_n X$ given by

and $\gamma_1 : A_1 \to A_2 \ast (C_1 \oplus \ldots \oplus C_4) \cdot \sigma$ with $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 5 & 3 & 7 \end{pmatrix}$.

This $\beta$ defines a morphism

$$([A_1; (K_1, L_1), \ldots, (K_7, L_7)], \alpha \circ \varepsilon(\beta)) \to ([A_2; (K'_1, L'_1), \ldots, (K'_4, L'_4)], \alpha)$$

in $\varepsilon_n/A[K, L]$. Recall from Example 4.3 that the composite $\alpha \circ \varepsilon(\beta)$ is given by

and $\gamma_3 : A_1 \to A_3 \ast (E_1 \oplus E_2 \oplus E_3) \cdot \sigma_3$, where

$$\begin{align*}
h_1 &= g_1 \circ D_1(\rho_2(\varepsilon f_2), \rho_3(\varepsilon f_3)) : E_1(\mu_1 \varepsilon K_1, \mu_5 \varepsilon K_5) \to K''_1 \\
h_2 &= g_2 \circ D_2(\rho_1(\varepsilon f_1), \rho_4(\varepsilon f_4)) : E_2(\mu_2 \varepsilon K_2, \mu_7 \varepsilon K_7) \to K''_2 \\
h_3 &= g_3 : E_3(*) \to K''_3
\end{align*}$$
Here note that $X$ is a diagram in $\text{Cat}^M$ so that $\overline{\varphi} = \text{id}$ in the notation of Example 4.13.

We define

$$s_{-1}(\beta) = (\iota D_1(\rho_2(f_2), \rho_3(f_3)), \iota D_2(\rho_1(f_1), \rho_4(f_1)), \text{id})$$

It is straightforward to show that $s_{-1} : \varepsilon_n/A[K, L] \to \varepsilon_{n+1}/A[K, L]$ is a functor satisfying all simplicial identities 6.9 except for $d_0 \circ s_{-1} = \text{id}$. But there is a natural transformation $\tau : \text{id} \to d_0 \circ s_{-1}$ given by

$$\tau([B; (K'_1, L'_1), \ldots, (K'_n, L'_n)], \alpha) = \overline{\varphi}$$

(see (*)). By inspection, $\tau$ has the following properties

$$\tau \circ d_i = d_i \circ \tau \quad \text{for all } i$$

$$\tau \circ s_i = s_i \circ \tau \quad \text{for } i \geq 0$$

(***)

\[ \tau \circ s_{-1} = s_{-1} \circ \tau \]

Given morphisms $g$ and $h$ in $\Delta^\text{op}_+$ we present them in the generators in standard form

$$g = s_{k_r} \circ \ldots \circ s_{k_1} \circ d_{l_p} \circ \ldots \circ d_{l_1}$$

$$h = s_{j_r} \circ \ldots \circ s_{j_1} \circ d_{l_p} \circ \ldots \circ d_{l_1}$$

with $k_r > \ldots > k_1 \geq -1$, $l_p > \ldots > l_1 \geq 0$, $j_r > \ldots > j_1 \geq -1$, $i_r > \ldots > i_1 \geq 0$. In the notation of 2.10 we define

$$\sigma(h, g) : G_+(h \circ g) \to G_+(h) \circ G_+(g)$$

by

$$\sigma(h, g) = G_+(s_{j_r} \circ \ldots \circ s_{j_1} \circ d_{l_p} \circ \ldots \circ d_{l_1}) \circ \tau \circ 1 \circ G_+(s_{k_r} \circ \ldots \circ s_{k_1} \circ d_{l_p} \circ \ldots \circ d_{l_1})$$

if $i_1 = 0$ and $k_1 = -1$, and to be the identity otherwise. Here we use the relation

$$s_{k_r} \circ \ldots \circ s_{k_1} \circ s_{-1} = s_{-1} \circ s_{k_r} \circ \ldots \circ s_{k_1}.$$

Since $G_+$ preserves identities, we define $\rho([n]) = \text{id}_{G_+([n])}$. The equations (***), for $\tau$ ensure that the coherence diagrams 2.10 commute.

We now prove the statements 4.15 and 4.16.

**Proof of the Cofinality Theorem:** By the Homotopy Colimit Theorem the natural horizontal maps in the commutative diagram

$$\begin{array}{ccc}
\text{hocolim}^B_M B(\text{str}(X \circ F)) & \xrightarrow{F_*} & B(\text{hocolim}^M_N X \circ F) \\
\downarrow & & \downarrow \quad F_* \\
\text{hocolim}^B_M B(\text{str}X) & \xrightarrow{F_*} & B(\text{hocolim}^M_N X)
\end{array}$$

are weak equivalences. By [14, 19.6.7] the left vertical map is a weak equivalence, hence the result follows.

**Proof of Proposition 4.16:** By 3.3, the morphism $\text{str} f : \text{str} X \to \text{str} Y$ of diagrams is an objectwise weak equivalence. Hence $N\text{str} f$ is objectwise a weak equivalence, which in turn implies that $\text{hocolim}^M(N\text{str} f)$ is a weak equivalence. 

\[ \square \]
7. Equivalences of categories

Throughout this section let $\mathcal{M}$ be a $\Sigma$-free $\mathbf{Cat}$-operad. Then $N \mathcal{M}$ is a simplicial and $BM\mathcal{M}$ a topological operad. Let $M$, $N\mathcal{M}$ and $BM\mathcal{M}$ be their associated monads. For a category $\mathcal{C}$ let $\mathcal{S}\mathcal{C}$ denote the category of simplicial objects in $\mathcal{C}$. In particular, $S^2\mathbf{Sets}$ is the category of bisimplicial sets. We work with the diagram

$$\begin{array}{cccccc}
\mathcal{S}\mathcal{C} \mathcal{M} & \xrightarrow{S\mathcal{N}\mathcal{alg}} & S^2\mathbf{Sets}^{N\mathcal{M}} & \xrightarrow{S|-\mathcal{alg}} & cw\mathbf{St}\mathcal{op}^{BM} & \subset \mathbf{St}\mathcal{op}^{BM} \\
\text{hocolim} & & I & & II & \\
\mathcal{C} \mathcal{M} \quad N\mathcal{alg} & \xrightarrow{\text{diag}} & SS\mathbf{ets}^{N\mathcal{M}} & \xrightarrow{\text{alg}} & \mathbf{Top}^{BM} \\
U_C & \quad III & \quad U_S & IV & \quad U_T \\
\mathcal{C} \quad N & \xrightarrow{\quad} & SS\mathbf{ets} & \quad \xrightarrow{\quad} & \mathbf{Top} \\
\end{array}$$

where $cw\mathbf{St}\mathcal{op}^{BM}$ is the full subcategory of $\mathbf{St}\mathcal{op}^{BM}$ of those simplicial $BM\mathcal{M}$-algebras whose underlying spaces are simplicial CW-complexes with cellular structure maps. Again, $|-\mathcal{alg}$ is the usual topological realization functor, which lifts to algebras because it is a product preserving left adjoint. The squares $III$ and $IV$ commute while the square $II$ commutes up to natural isomorphisms. We will deal with square $I$ later.

We define weak equivalences in each of these categories to be morphisms which are mapped by the functors of the diagram to weak equivalences in $\mathbf{Top}$. For $\mathcal{S}\mathcal{C}\mathcal{M}$ we here chose the composition through $\mathbf{St}\mathcal{op}^{BM}$. The weak equivalences in all categories except for $\mathcal{S}\mathcal{C}\mathcal{M}$, $\mathcal{C}\mathcal{M}$ and $cw\mathbf{St}\mathcal{op}^{BM}$ are part of a Quillen model structure so that the localizations of these categories with respect to their classes of weak equivalences exist. It is easy to see that the localization $cw\mathbf{St}\mathcal{op}^{BM}[we^{-1}]$ exists and that it can be considered as a full subcategory of $\mathbf{St}\mathcal{op}^{BM}[we^{-1}]$. We cannot show that the localizations of $\mathcal{S}\mathcal{C}\mathcal{M}$ and $\mathcal{C}\mathcal{M}$ exist in the G"odel-Bernay set theory setting. We offer two remedies:

7.2 The reader may choose one of the following conventions. Our results hold for either of them:

- We work in the setting of Grothendieck universes where the localizations exist in a possibly higher universe.
In abuse of notation we define $\mathcal{S}Cat^M[\mathrm{we}^{-1}]$ and $\mathcal{C}at^M[\mathrm{we}^{-1}]$ to be the full subcategories of $\mathcal{S}^2\mathcal{S}ets^{NM}[\mathrm{we}^{-1}]$ respectively $\mathcal{S}ets^{NM}[\mathrm{we}^{-1}]$ of objects of the form $SN^{alg}(A_\bullet)$ respectively $N^{alg}(A_\bullet)$.

7.3 Lemma: Let $A_\bullet$ be a simplicial $M$-algebra in $\mathcal{C}at$ and $Q^{\text{Reedy}}N\text{str}A_\bullet$ the functorial Reedy-cofibrant replacement of $QN\text{str}A_\bullet$. Then there are natural weak equivalences

$$\text{hocolim}^{NM}N\text{str}A_\bullet \leftarrow N(-/\Delta^{op}) \otimes_{\Delta^{op}} Q^{\text{Reedy}}N\text{str}A_\bullet \rightarrow \text{diag}NA_\bullet$$

In particular, square $I$ of diagram 7.1 commutes up to a broken arrow of natural weak equivalences.

Proof $Q^{\text{Reedy}}X_\bullet$ is degreewise cofibrant for any $X_\bullet$ in $S^2\mathcal{S}ets^{NM}$ by [14, 15.3.11]. Hence the left morphism is a weak equivalence by the argument of Remark 6.3.

By [14, 18.7.4] the Bousfield-Kan map defines a natural weak equivalence

$$N(-/\Delta^{op}) \otimes_{\Delta^{op}} Q^{\text{Reedy}}N\text{str}A_\bullet \rightarrow \text{diag}Q^{\text{Reedy}}N\text{str}A_\bullet.$$ 

Since the natural map $Q^{\text{Reedy}}N\text{str}A_\bullet \rightarrow NA_\bullet$ is degreewise a weak equivalence $\text{diag}Q^{\text{Reedy}}N\text{str}A_\bullet \rightarrow \text{diag}NA_\bullet$ is a weak equivalence by [13, IV.1.7].

7.4 Corollary: A morphism $f_\bullet : A_\bullet \rightarrow B_\bullet$ in $\mathcal{S}Cat^M$ is a weak equivalence if and only if $\text{hocolim}f_\bullet$ is a weak equivalence.

The main theorem of this section is a consequence of the Homotopy Colimit Theorem and the following result:

7.5 Proposition: [14, 2.7] If $\mathcal{M}$ is a $\Sigma$-free $\mathcal{C}at$-operad, then the functors of diagram 7.1 induce equivalences of categories

$$\mathcal{S}Cat^M[\mathrm{we}^{-1}] \simeq \text{cwSTop}^{BM}[\mathrm{we}^{-1}] \simeq \text{STop}^{BM}[\mathrm{we}^{-1}]$$

7.6 Theorem: $\text{hocolim}^M : \mathcal{S}Cat^M \rightarrow \mathcal{C}at^M$ and the constant diagram functor $c : \mathcal{C}at^M \rightarrow \mathcal{S}Cat^M$ induce an equivalence of categories

$$\mathcal{S}Cat^M[\mathrm{we}^{-1}] \simeq \mathcal{C}at^M[\mathrm{we}^{-1}]$$

For the proof consider the adjunction

(A) $\text{hocolim}^M : \mathcal{F}unc(\Delta^{op}, \mathcal{C}at^M_{Lax}) \Rightarrow \mathcal{C}at^M : c$

The functor $c$ factors through the inclusion $\mathcal{S}Cat^M \subset \mathcal{F}unc(\Delta^{op}, \mathcal{C}at^M_{Lax})$. Theorem 7.6 is a consequence of the next two lemmas.

7.7 Lemma: The counit $\varepsilon(A) : \text{hocolim}^M cA \rightarrow A$ of adjunction (A) is a weak equivalence for all $A$ in $\mathcal{C}at^M$. 

45
Proof By 5.10 the counit \( \varepsilon(A) \) can be identified with the composition

\[
\Delta^{op} \otimes \text{str}A \xrightarrow{F \otimes \text{id}} \text{str}A \xrightarrow{\varepsilon} A
\]

where \( F : \Delta^{op} \to * \) is the constant functor. Since \( \Delta^{op} \) has an initial object \( F \otimes \text{id} \) is a weak equivalence by Lemma 3.9, and \( r \) is a weak equivalence by 5.3.

For the next lemma we will need the following additional adjunctions:

(B) \( \text{Cat}^{bM}\text{Lax}(A, i_1B) \cong \text{Cat}^{bM}(\text{str}A, B) \)

(C) \( \text{Func}(\Delta^{op}, \text{Cat}^{bM})(Y, i_2Z) \cong (\text{Cat}^{bM})^{\Delta^{op}}(RY, Z) \)

(D) \( (\text{Cat}^{bM})^{\Delta^{op}}(U, \bar{c}S) \cong \text{Cat}^{bM}(\text{colim}U, S) \)

where \( i_1 : \text{Cat}^{bM} \to \text{Cat}^{bM}\text{Lax} \) and \( i_2 : (\text{Cat}^{bM})^{\Delta^{op}} \to \text{Func}(\Delta^{op}, \text{Cat}^{bM}) \) are the inclusions, \( RY = (-/\Delta^{op}/- \otimes \Delta^{op} Y \), and \( \bar{c} : \text{Cat}^{bM} \to (\text{Cat}^{bM})^{\Delta^{op}} \) is the constant diagram functor.

For an \( X \) in \( \mathcal{SCat}^{bM} \) consider the commutative diagram

\[
\begin{array}{cccccc}
X & \xrightarrow{j} & \text{str}X & \xrightarrow{\alpha(\text{str}X)} & R\text{str}X & \xrightarrow{\beta(\text{str}X)} & \text{str}X & \xrightarrow{r} & X \\
\downarrow j & & \downarrow \mu(\text{str}X) & & \downarrow \beta(\text{str}X) & & \downarrow \beta(\text{str}X) & & \downarrow r \\
\text{colim}R\text{str}X = \text{hocolim}^{bM}X
\end{array}
\]

where \( j, \bar{j}, \alpha(\text{str}X), \) and \( \mu(\text{str}X) \) are the units of the adjunctions (A),..., (D), and \( \beta(\text{str}X) \) and \( r \) are the counits of the adjunctions (C) and (D) (\( \bar{j} \) and \( r \) are applied degreewise).

7.8 Lemma: For \( X \) in \( \mathcal{SCat}^{bM} \) we have natural weak equivalences

\[
c(\text{hocolim}^{bM}X) \xleftarrow{\mu(\text{str}X)} \text{Rstr}X \xrightarrow{r \circ \beta(\text{str}X)} X
\]

in \( \mathcal{SCat}^{bM} \).

Proof By 7.4 it suffices to show that the homotopy colimits of these morphisms are weak equivalences.

By 5.10 and 5.3 the homotopy colimits of \( r \) and \( \beta(\text{str}X) \) are weak equivalences. It remains to prove that \( \mu(\text{str}X) \) is a weak equivalence. By construction, \( r \circ \beta(\text{str}X) \circ \alpha(\text{str}X) \circ \bar{j} = \text{id}_X \). Hence \( \text{hocolim}(\alpha(\text{str}X) \circ \bar{j}) \) is a weak equivalence. Since \( j \) is the unit of the adjunction (A)

\[
\text{hocolim}^{bM}X \xrightarrow{\text{hocolim}^{bM}j} \text{hocolim}^{bM}c(\text{hocolim}^{bM}X) \xrightarrow{\varepsilon(\text{hocolim}^{bM}X)} \text{hocolim}^{bM}X
\]

is the identity. Since \( \varepsilon(\text{hocolim}^{bM}X) \) is a weak equivalence by 7.7, so is \( \text{hocolim}j \) and hence \( \mu(\text{str}X) \). □
8. Applications

Since we will consider various different $\text{Cat}$-operads $\mathcal{M}$ we will shift from the categories $\text{Cat}^{\mathcal{M}}$ of algebras over the associated monads $\mathcal{M}$ back to the isomorphic categories $\text{Cat}^{\mathcal{M}}$ of $\mathcal{M}$-algebras.

The operads $\mathcal{M}$ in this section will be reduced, i.e. $\mathcal{M}_0$ consists of a single element. Hence $\mathcal{M}$-algebras are naturally based by the images of the nullary operation.

For the comparison of categories of algebras recall the following result:

8.1 Proposition: Let $\varepsilon : \mathcal{P} \to \mathcal{Q}$ be a morphism of $\Sigma$-free topological operads such that $\varepsilon(n) : \mathcal{P}(n) \to \mathcal{Q}(n)$ is a weak equivalence for each $n \in \mathbb{N}$. Then the forgetful functor and its left adjoint define a Quillen equivalence

$$L : \text{Top}^{\mathcal{P}} \rightleftarrows \text{Top}^{\mathcal{Q}} : R$$

This is the reformulation in the language of model category structures of the well-known change of operads functor of [20], defined by a functorial two-sided bar construction. In fact, the two-sided bar construction is just the derived functor of the left adjoint (see also [10, Prop. 2.8]).

In connection with Theorem 7.6 this implies

8.2 Proposition: Let $\varepsilon : \mathcal{P} \to \mathcal{Q}$ be a morphism of $\Sigma$-free $\text{Cat}$-operads such that each $\varepsilon(n) : \mathcal{P}(n) \to \mathcal{Q}(n)$ is a weak equivalence. Then the forgetful functor and its left adjoint induce an equivalence of categories

$$\text{Cat}^{\mathcal{Q}[\text{we}^{-1}]} \rightleftarrows \text{Cat}^{\mathcal{P}[\text{we}^{-1}]}$$

Iterated monoidal categories and iterated loop spaces

We will rely heavily on the results of [1].

Let $\mathcal{M}_k$ denote the $\text{Cat}$-operad constructed in [1], which encodes $k$-fold monoidal categories, and let $\mathcal{C}_k$ denote the little $n$-cubes operad of [3]. In [1] there is constructed a $\Sigma$-free topological operad $\mathcal{D}_k$ and maps of operads

$$BM_k \xleftarrow{\varepsilon_k} \mathcal{D}_k \xrightarrow{\eta_k} \mathcal{C}_k$$

such that each $\varepsilon_k(n)$ and each $\eta_k(n)$ is a homotopy equivalence. Let $\iota_k : \mathcal{M}_k \to \mathcal{M}_{k+1}$ denote the canonical inclusion functor and $\rho_k : \mathcal{C}_k \to \mathcal{C}_{k+1}$ the canonical embedding. Although not stated explicitly, a quick check of [1, Chapt. 6] shows that there is also an embedding of operads $\delta_k : \mathcal{D}_k \to \mathcal{D}_{k+1}$ making the diagram

$$BM_k \xleftarrow{\varepsilon_k} \mathcal{D}_k \xrightarrow{\eta_k} \mathcal{C}_k \xleftarrow{\delta_k} \mathcal{D}_{k+1} \xrightarrow{\rho_k} \mathcal{C}_{k+1}$$
commute. If we define \( M_\infty = \text{colim}_k M_k \) and \( D_\infty = \text{colim}_k D_k \), we obtain induced morphisms of topological operads

\[
B M_\infty \xrightarrow{\epsilon_\infty} D_\infty \xrightarrow{\eta_\infty} C_\infty
\]

Since \( \iota_k \) is an inclusion, each \( B \iota_k(n) \) is a cofibration. Since each \( \rho_k(n) \) is a cofibration by [4, Proof of Lemma 2.50] and each \( \delta_k(n) \) is one by inspection, each \( \varepsilon_\infty(n) \) and each \( \eta_\infty(n) \) is a homotopy equivalence.

Hence Theorem 7.6 and Proposition 8.1 imply

8.3 Theorem: For \( 1 \leq k \leq \infty \) the classifying space functor \( B \) and the change of operads functors induced by \( \varepsilon_k \) and \( \eta_k \) determine equivalences of categories

\[
\text{Cat}^{M_k}[\text{we}^{-1}] \simeq \text{Top}^{B M_k}[\text{we}^{-1}] \simeq \text{Top}^{E_k}[\text{we}^{-1}]
\]

It is well-known that the group completion of a \( C_k \)-algebra is a \( k \)-fold loop space. Hence we obtain

8.4 Theorem: The group completion of the classifying space of a \( k \)-fold monoidal category is homotopy equivalent to a \( k \)-fold loop space. Conversely, up to weak homotopy equivalences and group completion, each \( k \)-fold loop space comes from a \( k \)-fold monoidal category this way.

8.5 Remark: The statement of the theorem can be put into the following form: let \( \text{we}_{gp} \subset \text{mor}(\text{Cat}^{M_k}) \) denote the class of those morphisms which are mapped to weak equivalences by the classifying space functor composed with the group completion functor, and let \( \Omega^k \text{Top} \) denote the category of \( k \)-fold loops spaces and loop maps. Then

\[
\text{Cat}^{M_k}[\text{we}_{gp}^{-1}] \simeq \Omega^k \text{Top}[\text{we}^{-1}].
\]

For the existence of \( \text{Cat}^{M_k}[\text{we}_{gp}^{-1}] \) see [2]. In [24] the second author constructs a group completion functor

\[
Q : \text{SSets}^{E_k} \to \text{SSets}^{E_k},
\]

where \( E_k \) is any cofibrant reduced \( E_k \)-operad, and a model structure on \( \text{SSets}^{E_k} \) whose weak equivalences are those morphisms \( f \) for which \( Q(f) \) is a weak equivalence. The realization functor transports this model structure to a structure on \( \text{cuTop}^{\text{we}_{gp}} \), the full subcategory of \( \text{Top}^{\text{we}_{gp}} \) of those algebras whose underlying space is a CW-complex. This structure is weaker than a model structure but strong enough to guarantee the existence of the localization with respect to the induced weak equivalences. So both alternatives of Remark 7.2 are available.
Symmetric monoidal categories

Recall from [1, Remark 1.9] that each permutative category is an infinite monoidal category of a special kind. Let \( \mathcal{P} \text{erm} \) be the category of permutative categories, and let \( \tilde{\Sigma} \) denote the \( \mathcal{C} \text{at} \)-operad given by the translation categories of the symmetric groups (see [20, 1.1]). Then \( \mathcal{P} \text{erm} \cong \mathcal{C} \text{at}^{\tilde{\Sigma}} \) (see [20, Section 4]). The projections

\[
B(\tilde{\Sigma}) \leftarrow B(\tilde{\Sigma}) \times \mathcal{C}_\infty \rightarrow \mathcal{C}_\infty
\]

are weak equivalences of \( \Sigma \)-free operads. Hence, by Theorem 7.6 and Proposition 8.1 we obtain

8.6 Theorem: The classifying space functor and the change of operads functors induce equivalences of categories

\[
\mathcal{P} \text{erm}[\text{we}^{-1}] = \mathcal{C} \text{at}^{\tilde{\Sigma}}[\text{we}^{-1}] \cong \mathcal{T} \text{op}^{B(\tilde{\Sigma})}[\text{we}^{-1}] \cong \mathcal{T} \text{op}^{\mathcal{C}_\infty}[\text{we}^{-1}].
\]

Since each symmetric monoidal category is equivalent to a permutative one [20], Theorem 8.6 induces the following result of Thomason [29]:

8.7 Theorem: Let \( \text{SymMon} \) denote the category of symmetric monoidal categories and \( \Omega^\infty \text{Top} \) the category of infinite loop spaces. The classifying space functor composed with the group completion functor induces a functor

\[
L : \text{SymMon} \rightarrow \Omega^\infty \text{Top}
\]

which is an equivalence after formally inverting weak equivalences in \( \Omega^\infty \text{Top} \) and those symmetric monoidal functors which are mapped to weak equivalences.

8.8 Remark: Theorem 8.6 is a stronger version of Thomason’s result, since it explains what happens before group completion. Mandell has obtained a similar result by completely different means [18].

There is a minor flaw in Thomason’s proof of the homotopy colimit theorem [28, Theorem 4.1] which is an essential part in his proof of Theorem 8.7. Using a spectral sequence argument he shows that the map corresponding to our

\[
\varepsilon_2 : \text{diag}N(-/\mathcal{L} \otimes_{\mathcal{L}} \text{strM}_*X) \rightarrow N(-/\mathcal{L} \otimes_{\mathcal{L}} \text{strX})
\]

is a homology equivalence and concludes that it is a weak equivalence because the spaces involved are \( H \)-spaces [28, pp.1641-1644]. This is not true because he considers only non-unital permutative categories at that point. Since he applies a group completion functor to his construction this flaw does not affect his result.

There is an operadic description of [1, Remark 1.9] in terms of a morphism of operads \( \lambda : \mathcal{M}_\infty \rightarrow \tilde{\Sigma} \), which is determined by sending an object \( i_1 \square_{r_1} i_2 \square_{r_2} \ldots \square_{r_{n-1}} i_n \) of \( \mathcal{M}_\infty(n) \) with any form of bracketing to the permutation in \( \Sigma_n \) sending \( k \) to \( i_k \). As a corollary to Theorem 8.6 we obtain

49
8.9 Corollary: The forgetful functor $\mathcal{P}erm \to \mathcal{C}at^{M\infty}$ induced by $\lambda$ defines an equivalence of categories

$$\mathcal{P}erm[we^{-1}] \simeq \mathcal{C}at^{M\infty}[we^{-1}]$$

Braided monoidal categories

8.10 Definition: A weak braided monoidal category is a category $\mathcal{C}$ together with a functor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is strictly associative and has a strict 2-sided unit object $0$ and with a natural commutativity morphisms $c_{A,B} : A \Box B \to B \Box A$ such that $c_{A,0} = c_{0,A} = id_A$ and the diagrams commute.

If the $c_{A,B}$ are isomorphisms, we call $\mathcal{C}$ a braided monoidal category.

8.11 Remark: Our terminology regarding braided monoidal categories differs from standard usage, as for example in [17]. The usual notion of braided monoidal categories has $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ only associative up to coherent natural isomorphisms and similarly with the 2-sided unit $0$. Braided monoidal categories with a strictly monoidal structure are called strict braided monoidal. Essentially the same proof as for symmetric monoidal categories, given in [20], shows that any braided monoidal category, in this sense, is equivalent to a strict one. Weak braided monoidal categories, in our sense, where the braidings are not required to be isomorphisms, have not been much studied.

By [1, Remark 1.5] any weak braided monoidal category is a special kind of a 2-fold monoidal category. Let $B_n$ denote the braid group on $n$ strings and
$p : B_n \to \Sigma_n$ its projection onto the symmetric group, whose kernel is the pure braid group $P_n$. Recall the standard presentation of $B_n$:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

Let $B_n^+$ denote the braid monoid on $n$ strings defined as a monoid by this presentation.

We define $\text{Cat-operads} \ B_r$ and $\text{Br}^+$ as follows: $ob B_r(n) = \Sigma_n$ and the morphisms from $\sigma$ to $\pi$ in $B_r(n)$ are the elements $b \in B_n$ such that $p(b) \circ \sigma = \pi$. Composition is the obvious one. The action of $\Sigma_n$ on $B_r(n)$ is by composition on the right, and the operad composition is defined as in $\tilde{\Sigma}$. The definition of $\text{Br}^+$ is the same with $B_n$ replaced by $B_n^+$ throughout and $p$ replaced by the composition with the natural map $u : B_n^+ \to B_n$. It is easy to check that $\text{Cat}^{\text{Br}}$ is the category of braided monoidal categories and $\text{Cat}^{\text{Br}^+}$ the category of weak braided monoidal categories. By [1, Remark 1.5] there are morphisms of operads

$$\mathcal{M}_2 \xrightarrow{\beta} \text{Br}^+ \xrightarrow{\mu} \text{Br}.$$ 

**8.12 Proposition:** There are topological operads $\mathcal{D}^+$ and $\mathcal{D}$ and a commutative diagram of morphisms of operads

\[
\begin{array}{ccc}
B(B_r^+) & \xleftarrow{B} & \mathcal{D}^+ \\
\downarrow B_{\mu} & & \downarrow \mathcal{C}_2 \\
B(B_r) & \xleftarrow{B} & \mathcal{D}
\end{array}
\]

which are equivalences.

We postpone the proof.

Like in the symmetric monoidal category case we get

**8.13 Theorem:** The classifying space functor and the change of operads functor induce equivalences of categories

$$\text{Cat}^{\text{Br}}[\text{we}^{-1}] \simeq T_{\text{op}}^{B(B_r)}[\text{we}^{-1}] \simeq T_{\text{op}}^{\mathcal{C}_2}[\text{we}^{-1}]$$

$$\text{Cat}^{\text{Br}^+}[\text{we}^{-1}] \simeq T_{\text{op}}^{B(B_r^+)}[\text{we}^{-1}] \simeq T_{\text{op}}^{\mathcal{C}_2}[\text{we}^{-1}]$$

The proof that $B_{\mu}$ is an equivalence uses an observation which might be of separate interest.
8.14 Definition: Let $S$ be a finite set. A Coxeter matrix $M = (m_{s,t})$ is a symmetric $S \times S$-matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ and $m_{s,t} \geq 2$ for $s \neq t$. A Coxeter matrix has an associated Artin monoid $G^+$ and an associated Artin group $G$ both given by the presentation

$$< S | \text{prod}(m_{s,t}; s, t) = \text{prod}(m_{s,t}; t, s) \text{ for } s \neq t, m_{s,t} < \infty >,$$

where $\text{prod}(n; x, y) = xxyy \ldots (n \text{ factors})$. [So $\text{prod}(2n; x, y) = (xy)^n$ and $\text{prod}(2n + 1; x, y) = (xy)^n x$.]

We call $G^+$ and $G$ spherical, if the associated Coxeter group, obtained from the Artin group by taking the quotient by the additional relations $\{s^2 = 1 | s \in S\}$, is finite.

Note that the braid monoid is a spherical Artin monoid.

8.15 Proposition: Let $G^+$ be a spherical Artin monoid with Artin group $G$. Then the canonical homomorphism $u: G^+ \to G$ is injective (see [6, (4.14)]) and induces a homotopy equivalence

$$Bu : B(G^+) \simeq B(G)$$

Proof For a discrete monoid $M$ and its algebraic group completion $UM$ the canonical map $u: M \to UM$ always induces an isomorphism $u_*: \pi_1(BM) \to \pi_1(BUM)$. Now let $G^+$ be a spherical Artin monoid with associated Artin group $G = UG^+$. By [6, (4.17)], we obtain $G$ from $G^+$ by inverting a certain central element. Since $u$ is also injective [6, (4.14)], it induces an isomorphism

$$H_*(G^+, A) \to H_*(G, A)$$

for any $\mathbb{Z}[G]$-module $A$ by [5, (X.4.1)]. Consequently, $B(G^+) \simeq B(G)$. \hfill $\square$

It remains to prove Proposition 8.12.

A braided operad is defined in the same way as an operad with the symmetric group replaced by the braid group. To define composition, we use the homomorphism $p : B_n \to \Sigma_n$. Examples of braided operads in $\text{Cat}$ are the translation operad $\overline{B}$ of the braid groups and $\overline{B}^+$, for which $\overline{B}^+(n)$ has the same objects as $\overline{B}(n)$, but the morphisms are elements of the braid monoid $B_n^+$ rather than the braid group $B_n$. The inclusions $u: B_n^+ \to B_n$ define a morphism of braided $\text{Cat}$-operads

$$\overline{B}^+ \to \overline{B}$$

The operad $\overline{C}_2$, where $\overline{C}_2(n)$, is the universal cover of $C_2(n)$ is a braided topological operad, for details see [5]. We have a commutative diagram of braided
topological operads

\[
\begin{array}{ccc}
B(\tilde{B}^+) & \leftarrow & B(\tilde{B}^+) \times \tilde{C}_2 \\
\downarrow & & \downarrow \\
B(\tilde{B}) & \leftarrow & B(\tilde{B}) \times \tilde{C}_2
\end{array}
\]

The morphisms are all equivalences, because the \( n \)-th space of each operad is contractible for all \( n \): this is clear for \( B(\tilde{B})(n) \), for \( B(\tilde{B}^+)(n) \) this follows from (8.13), because \( B(\tilde{B}^+)(n) \) is the universal cover of \( B(B^+_n) \) (see proof of [4, (4.4)]), for \( \tilde{C}_2(n) \) this follows from the well-known fact that \( C_2(n) \simeq K(P_n, 1) \), where \( P_n \) is the pure braid group.

Given a braided operad \( \mathcal{P} \), we can “debraid” it to obtain a genuine operad by replacing \( \mathcal{P}(n) \) by \( \mathcal{P}(n) \times_{B_n} \Sigma_n \). Note that \( B(Br^+) \), \( B(Br) \), and \( C_2 \) are the debraidings of \( B(\tilde{B}^+) \), \( B(\tilde{B}) \), and \( \tilde{C}_2 \).

The debraiding of the above diagram gives us the diagram required in 8.12 where \( \mathcal{D}^+ \) and \( \mathcal{D} \) are the debraidings of \( B(\tilde{B}^+) \times \tilde{C}_2 \) and \( B(\tilde{B}) \times \tilde{C}_2 \) respectively.

**Appendix**

We give an explicit description of \( \mathcal{F}unc(\mathcal{L}, \text{Cat}^{BLax}) \).

**Objects:** Objects are strict functors \( X\mathcal{L} \rightarrow \text{Cat}^{BLax} \).

So for \( \lambda : L_0 \rightarrow L_1 \) in \( \mathcal{L} \) we have a morphism

\[
(X\lambda, \overline{X\lambda}) : XL_0 \rightarrow XL_1
\]

depicted in the square

\[
\begin{array}{ccc}
\mathbb{M}XL_0 & \overset{MX\lambda}{\rightarrow} & \mathbb{M}XL_1 \\
\downarrow_{\xi_{XL}} & & \downarrow_{\xi_{XL_1}} \\
XL_0 & \overset{X\lambda}{\rightarrow} & XL_1
\end{array}
\]

satisfying

1. \( \overline{X\lambda} \circ_1 \mu(XL_0) = (\overline{XX} \circ_1 \mathbb{M}\xi_{XL_0}) \circ_2 (\xi_{XL_1} \circ_1 \mathbb{M}\overline{X\lambda}) \)
2. \( \overline{X\lambda} \circ_1 \iota(XL_0) = id_{X\lambda} \)

53
where $\xi_{XL_0}$ and $\xi_{XL_1}$ are the structure maps of the $\mathcal{M}$-algebras $XL_0$ and $XL_1$ and $\mu$ and $\iota$ are the multiplication and unit of the monad $\mathcal{M}$.

**Morphisms:** A morphism $j : X \to Y$ is a lax natural transformation. Given $L_0 \xrightarrow{\lambda_0} L_1 \xrightarrow{\lambda_1} L_2$, we have a diagram

\[
\begin{array}{ccc}
XL_0 & \xrightarrow{X\lambda_0} & XL_1 & \xrightarrow{X\lambda_1} & XL_2 \\
\downarrow{j_{L_0}} & \cong & \downarrow{j_{L_1}} & \cong & \downarrow{j_{L_2}} \\
YL_0 & \xrightarrow{Y\lambda_0} & YL_1 & \xrightarrow{Y\lambda_1} & YL_2
\end{array}
\]

such that

\[
j\lambda_1 \circ \lambda_2 = (j\lambda_1 \circ_1 X\lambda_0) \circ_2 (Y\lambda_1 \circ_1 j\lambda_0)
\]

Moreover, $j\lambda_0 : Y\lambda_0 \circ_1 j_{L_0} \Rightarrow j_{L_1} \circ_1 X\lambda_0$ (and accordingly $j\lambda_1$) has to satisfy

\[
(j\lambda_0 \circ_1 \xi_{XL_0}) \circ_2 (Y\lambda_0 \circ_1 j_{L_0}) = (jL_1 \circ_1 X\lambda_0) \circ_2 (\xi_{YL_1} \circ_1 Mj\lambda_0)
\]

where

\[
\begin{align*}
(Y\lambda_0 \circ j_{L_0}) & = (Y\lambda_0 \circ 1 j_{L_0}) \circ_2 (Y\lambda_0 \circ 1 Mj\lambda_0) \\
(j_{L_1} \circ X\lambda_0) & = (j_{L_1} \circ 1 X\lambda_0) \circ_2 (j_{L_1} \circ 1 M_X\lambda_0)
\end{align*}
\]

**2-cells:** $X \xrightarrow{j} Y$ consists of 2-cells $s_L : j_L \Rightarrow k_L$ in $\text{Cat}^{\mathcal{M}}\text{Lax}$, one for each object $L$ of $\mathcal{L}$ such that

\[
k\lambda \circ_2 (Y\lambda \circ_1 s_{L_0}) = (s_{L_1} \circ_1 X\lambda) \circ_2 j\lambda
\]

for each morphism $\lambda : L_0 \to L_1$ in $\mathcal{L}$. As a 2-cell in $\text{Cat}^{\mathcal{M}}\text{Lax}$ the natural transformation $s_L$ has to satisfy

\[
\xi_{YL} \circ_2 (\xi_{YL} \circ_1 Ms_L) = (s_L \circ_1 \xi_{XL}) \circ_2 \xi_{YL}.
\]

**References**

[1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R.M. Vogt, Iterated monoidal categories, Adv. in Math. 176 (2003), 277-349.

[2] F. Borceux, Handbook of categorical algebra 2, Encyclopedia of Mathematics and its Applications 51, Cambridge University Press 1994.

[3] J.M. Boardman, R.M. Vogt, Homotopy-everything H-spaces, Bull. Amer. Math. Soc. 74 (1968), 1117-1122.
[4] J.M. Boardman, R.M. Vogt, Homotopy invariant algebraic structures on topological spaces, Springer Lecture Notes in Mathematics 347 (1973).

[5] H. Cartan, S. Eilenberg, Homological algebra, Princeton University Press, 1956.

[6] P. Deligne, Les immeubles des groupes de tresses généralisés, Inventiones math. 17 (1972), 273-302.

[7] A.D. Elmendorf, I. Kriz, M.A. Mandell, J.P. May, Rings, modules, and algebras in stable homotopy theory, Mathematical Surveys Monograms 47, Amer. Math. Soc., Providence, RI, 1996.

[8] Z. Fiedorowicz, The symmetric bar construction, preprint.

[9] Z. Fiedorowicz, Classifying spaces of topological monoids and categories, Amer. J. Math. 106 (1984), 301-350.

[10] Z. Fiedorowicz, R.M. Vogt, Simplicial $n$-fold monoidal categories model all loop spaces, Cahier Topologie Géom. Differentielle 44 (2003), 105-148.

[11] M. Artin, A. Grothendieck, J.-L. Verdier, Théorie des topos et cohomologie étale des schémas, Springer Lecture Notes in Mathematics 269 (1972).

[12] R. Godement, Topologie algébrique et théorie des faisceaux, Paris: Hermann, 1973.

[13] P.G. Goers, J.F. Jardine, Simplicial homotopy theory, Progress in Mathematics 174, Birkhäuser Verlag, Basel (1999).

[14] P.S. Hirschhorn, Model categories and their localizations, Mathematical Surveys and Monographs 99, Amer. Math. Soc. (2002).

[15] J. Hollender and R.M. Vogt, Modules of topological spaces, applications to homotopy limits and $E_\infty$ structures, Archiv der Mathematik 59 (1992), 115-129.

[16] G.M. Kelly, Basic concepts of enriched category theory, London Math. Soc. Lecture Note Series 64, Cambridge University Press (1982).

[17] S. Mac Lane, Categories for the working mathematician, 2nd edition, Springer Verlag 1998.

[18] M.A. Mandell, An inverse $K$-theory functor, Documenta Math. 15 (2010), 765-791.

[19] J.P. May, The geometry of iterated loop spaces, Springer Lecture Notes in Math. 171 (1972).
[20] J.P. May, $E_\infty$ spaces, group completion, and permutative categories, London Lecture Notes in Math. 11 (1974), 61-93.

[21] J.E. McClure, R. Schwänzl, R.M. Vogt, $THH(R) \cong R \otimes S^1$ for $E_\infty$ ring spectra, J. Pure Appl. Algebra 140 (1990), 23-32.

[22] D.G. Quillen, Homotopical algebra, Springer Lecture Notes in Mathematics 43 (1967).

[23] R. Schwänzl, R.M. Vogt, The categories of $A_\infty$ and $E_\infty$ monoids and ring spaces as closed simplicial and topological model categories, Arch. Math. 56 (1991), 405-411

[24] M. Stelzer, A model categorical approach to group completion of $E_n$-algebras, Preprint 2010.

[25] R. Street, Two constructions on lax functors, Cahier Topologie Géom. Différentielle 13 (1972), 217-264.

[26] R. W. Thomason, Homotopy colimits in the category of small categories, Math. Proc. Cambridge Phil. Soc. 85 (1979), 91-109.

[27] R.W. Thomason, Cat as a closed model category, Cahiers Topol. Géom. Diff. 21 (1980), 305-324.

[28] R. W. Thomason, First quadrant spectral sequences in algebraic $K$-theory via homotopy colimits, Communications in Algebra 10 (1982), 1589-1668.

[29] R. W. Thomason, Symmetric monoidal categories model all connective spectra, Theory and Appl. of Categories 1 (1995), 78-11.

[30] R.M. Vogt, Convenient categories of topological spaces for homotopy theory, Arch. Math. 22 (1971), 545 - 555.