Bessel Identities in the Waldspurger Correspondence over the Complex Numbers

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Abstract. We prove certain identities between relative Bessel functions attached to irreducible unitary representations of $\text{PGL}_2(\mathbb{C})$ and Bessel functions attached to irreducible unitary representations of $\text{SL}_2(\mathbb{C})$. These identities reflect the Waldspurger correspondence over $\mathbb{C}$. We also prove several regularity theorems for Bessel and relative Bessel distributions which appear in the relative trace formula. This paper constitutes the local spectral theory of Jacquet’s relative trace formula over $\mathbb{C}$.

1. Introduction

1.1. Motivations. There is a pair of exponential integral formulae of Weber and Hardy on the Fourier transform of Bessel functions on the real numbers. Let $e(x) = e^{2\pi ix}$.

Weber’s formula is as follows,

$$
\int_0^\infty \frac{1}{\sqrt{x}} J_\nu (4\pi \sqrt{x}) e(\pm xy) \, dx = \frac{1}{\sqrt{2y}} e\left( \pm \left( \frac{1}{2y} - \frac{1}{8} \nu - \frac{1}{8} \right) \right) J_{\frac{1}{2}\nu} \left( \frac{\pi}{y} \right),
$$

with $y \in (0, \infty)$, valid when $\text{Re} \, \nu > -1$. Hardy’s formula is in a similar fashion,

$$
\int_0^\infty \frac{1}{\sqrt{x}} K_\nu (4\pi \sqrt{x}) e(\pm xy) \, dx = - \frac{\pi}{2 \sin(\pi \nu)} \frac{1}{\sqrt{2y}} e\left( \pm \left( \frac{1}{2y} + \frac{1}{8} \right) \right)
\left( e \left( \frac{\pi}{8} \right) J_{\frac{1}{2}\nu} \left( \frac{\pi}{y} \right) - e \left( \frac{1}{8} \nu \right) J_{-\frac{1}{2}\nu} \left( \frac{\pi}{y} \right) \right),
$$

when $|\text{Re} \, \nu| < 1$. Here $J_\nu$ and $K_\nu$ are Bessel functions (see [Waa]).

In the work [BM2] of Baruch and Mao, the formulae (1.1) and (1.2) are used to establish an identity between the relative Bessel functions for $\text{PGL}_2(\mathbb{R})$ and the Bessel functions for $\text{SL}_2(\mathbb{R})$ and hence a correspondence from irreducible unitary representations of $\text{PGL}_2(\mathbb{R})$ to irreducible genuine unitary representations of $\text{SL}_2(\mathbb{R})$. This correspondence is exactly the Shimura-Waldspurger correspondence over $\mathbb{R}$! Completely analogous results for the non-Archimedean case were obtained in [BMT]. These results fit into the theory.
of the relative trace formula developed by Jacquet and constitute the local (real and non-Archimedean) spectral theory that complements the global theory in [Jac]. Ultimately, the Waldspurger formula over a totally real field was obtained and used to study the central value of PGL$_2$ automorphic $L$-functions in [BM3].

Recently, it is proven in [Qi1, Qi2] the following complex analogue of the classical formulae of Weber and Hardy,

$$\left(1.3\right)\int_0^{2\pi} \int_0^\infty J_{\mu,m}(xe^{i\phi}) e(-2xy\cos(\phi + \theta))dx\,d\phi = \frac{1}{4y} e^{i\frac{\cos\theta}{y}} J_{\frac{1}{2}+\mu,\frac{1}{2}+m}\left(\frac{1}{16y^2}e^{2i\theta}\right),$$

for $y \in (0, \infty)$ and $\theta \in [0, 2\pi)$, provided that $|\text{Re}\mu| < \frac{1}{2}$ and $m$ is even. Here $J_{\mu,m}(z)$ is the Bessel function over the complex numbers defined as

$$J_{\mu,m}(z) = \begin{cases} \frac{2\pi^2}{\sin(2\pi\mu)} (J_{\mu,m}(4\pi \sqrt{z}) - J_{-\mu,-m}(4\pi \sqrt{z})) , & \text{if } m \text{ is even}, \\ \frac{2\pi^2i}{\cos(2\pi\mu)} (J_{\mu,m}(4\pi \sqrt{z}) + J_{-\mu,-m}(4\pi \sqrt{z})) , & \text{if } m \text{ is odd}, \end{cases}$$

with

$$J_{\mu,m}(z) = J_{-2\mu,-\frac{1}{2}m}(z) J_{2\mu+\frac{1}{2}m}(\overline{z}).$$

See [4,1] for more discussions on the definition of $J_{\mu,m}(z)$.

In this paper, we shall use the formula (1.3) to establish the Bessel identity for the Shimura-Waldspurger correspondence over $\mathbb{C}$. This completes the local spectral theory of the relative trace formula of Jacquet, complementary to [BM1] and [BM2]. As application of this paper, we wish to further generalize the Waldspurger formula onto an arbitrary number field.

1.2. Main theorem. We now give a sample of the Bessel identities that we obtain. Let

$$N = \left\{ \begin{pmatrix} 1 \\ z \\ 1 \end{pmatrix} : z \in \mathbb{C} \right\}, \quad A = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{C} \setminus \{0\} \right\}.$$

Let $\psi_1(z) = e(\text{Tr} z)$, viewed as a character on $N$. Let $\pi$ be an infinite-dimensional irreducible unitary representation of GL$_2(\mathbb{C})$ with trivial central character (that is, a representation of PGL$_2(\mathbb{C})$). We attach to $\pi$ the relative Bessel function $i_{\pi,\psi_1}$ on GL$_2(\mathbb{C})$ which is left $A$-invariant and right $(\psi_1,N)$-equivariant. $i_{\pi,\psi_1}$ is real analytic on an open subset of the Bruhat cell in GL$_2(\mathbb{C})$. Let $\sigma$ be an irreducible unitary representation of SL$_2(\mathbb{C})$. We attach to $\sigma$ the Bessel function $j_{\sigma,\psi_1}$ on SL$_2(\mathbb{C})$ which is both left and right $(\psi_1,N)$-equivariant. $j_{\sigma,\psi_1}$ is real analytic on the open Bruhat cell in SL$_2(\mathbb{C})$. We stress that $\pi$ or $\sigma$ is determined by $i_{\pi,\psi_1}$ or $j_{\sigma,\psi_1}$ respectively. Our main theorem of this paper (see Theorem 8.2) is as follows.

\footnote{This was done very recently while the present paper was under peer review. See [CQ].}
Theorem 1.1. Let $\pi$ be as above. There exists $\sigma$ as above such that for any $z \in \mathbb{C} \setminus \{0\}$ we have

$$i_{\pi, \psi_1} \left( \frac{z}{4}, \frac{1}{2} \right) = \frac{2e(\pi, 1/2) \psi_1(2/z) |z|}{L(\pi, 1/2)} j_{\sigma, \psi_1} \left( \frac{-z^{-1}}{2} \right),$$

in which $L(\pi, 1/2)$ and $e(\pi, 1/2)$ are the central values of the $L$-factor and the $e$-factor associated with $\pi$.

The correspondence in Theorem 1.1 is given by

$$\pi_{\mu, m} \longrightarrow \sigma_{\frac{1}{2} \mu, \frac{1}{2} m},$$

reflecting the index correspondence $(\mu, m) \longrightarrow \left( \frac{1}{2} \mu, \frac{1}{2} m \right)$ between the Bessel functions on the two sides of (1.3). Here $\pi_{\mu, m}$ and $\sigma_{\mu, m}$ are the unitary principal series or complementary series of $GL_2(\mathbb{C})$ and $SL_2(\mathbb{C})$ parameterized by $(\mu, m)$ respectively (see §2.2 for the definitions). To be precise, we have

$$j_{\pi, \psi_1} \left( \frac{z}{2} \right) = \frac{z}{|z|} J_{\mu, m} (-z), \quad \pi = \pi_{\mu, m},$$

$$j_{\sigma, \psi_1} \left( \frac{-z^{-1}}{2} \right) = (-1)^{\frac{1}{2} m} |z|^{-2} J_{\frac{1}{2} \mu, \frac{1}{2} m} (z^{-2}), \quad \sigma = \sigma_{\frac{1}{2} \mu, \frac{1}{2} m},$$

and we shall prove in the sense of distributions that

$$i_{\pi, \psi_1} \left( \frac{\mu}{2}, \frac{1}{2} \right) = \frac{1}{L(\pi, 1/2)} \int_{\mathbb{C} \setminus \{0\}} j_{\pi, \psi_1} \left( \frac{z}{2} \right) \psi_1(uz) \frac{dz}{|z|^2},$$

where $d_{1z}$ denotes twice of the Lebesgue measure on $\mathbb{C}$. Moreover, note that $e(\pi, 1/2) = i^{\frac{1}{2} m}$ if $\pi = \pi_{\mu, m}$. Thus the identity (1.4) follows from the integral formula (1.3).

Actually, we shall prove a more general identity between $i_{\pi, \phi}$ and $j_{\sigma, \psi'}$ for any two nontrivial characters $\psi$ and $\psi'$. The correspondence $\pi \rightarrow \sigma$ turns out to be exactly the Waldspurger correspondence $\pi \rightarrow \Theta(\pi)$. However, unlike the real case as in [BM2], the correspondence is now independent on $\psi'$.

1.3. Remarks. Admittedly, the Waldspurger correspondence over $\mathbb{C}$ is tremendously simpler than that over $\mathbb{R}$, because all the double covers of $SL_2(\mathbb{C})$ are isomorphic to the trivial product $SL_2(\mathbb{C}) \times \{\pm 1\}$. Moreover, the representation theory of $GL_2(\mathbb{C})$ or $SL_2(\mathbb{C})$ is simpler as discrete series do not exist.

Our expositions would be simplified if we view $PGL_2(\mathbb{C})$ as $PSL_2(\mathbb{C})$ and work only on $SL_2(\mathbb{C})$. Nevertheless, we choose to work in the framework of the representation theory of $GL_2(\mathbb{C})$ in order to preserve the analogy between this work and [BM2].

As the formula (1.3) is the foundation of this paper, we now make some remarks on its analytic perspective and applications.

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2It is preferred here to view $j_{\mu, \phi}$ as function on $PGL_2(\mathbb{C}) (= PSL_2(\mathbb{C}))$. 
The proof of (1.3) in [Q2] is considerably harder than that of (1.1) or (1.2). Interestingly, besides an incorporation of stationary phase and differential equations, also arise in the course of proof certain complicated combinatorial formulae.

It is the distributional variant of (1.3) (see (6.1)) that we shall use to deduce the formula of the relative Bessel function $i_{\nu, \psi}$. Critical is that the test function in (6.1) only needs to be rapidly decaying at infinity but not zero. By using this variant, we may completely avoid the analysis of differential operators as in [BM2]. This idea would also work in the real context as in [BM2].

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2. Notations and preliminaries

2.1. Basic notations. Let $G = \text{GL}_2(\mathbb{C})$ and $S = \text{SL}_2(\mathbb{C})$. Let $B, A$ and $Z$ denote the Borel subgroup of upper triangular matrices, the diagonal subgroup and the center of $G$ respectively. Set

$$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{C} \right\},$$

and

$$w = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad w_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$s(a) = \begin{pmatrix} a & a^{-1} \\ 1 & 1 \end{pmatrix}, \quad t(a) = \begin{pmatrix} a \\ 1 \end{pmatrix}, \quad z(c) = \begin{pmatrix} c \\ c \end{pmatrix}.$$}

Recall that $e(x) = e^{2\pi i x}$. For $\lambda \in \mathbb{C}$, let $\psi = \psi_\lambda$ be the additive character of $\mathbb{C}$ defined by

$$\psi_\lambda(z) = e(\text{Tr}(\lambda z)) = e(\lambda z + \bar{\lambda} z).$$

We also view $\psi$ as a character of $N$ by $\psi(n(z)) = \psi(z)$. Let $\|z\| = |z|_1 = |z|^2$. Take $dz = dz_1$ to be $2\sqrt{|\lambda|}$ times of the Lebesgue measure on $\mathbb{C}$, which is self-dual with respect to $\psi$. Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Set $d^\times z = dz/|z|.$

For $f \in L^1(\mathbb{C})$, we define the $\psi$-Fourier transform of $f$ as

$$\hat{f}(u) = \int_\mathbb{C} f(z)\psi(uz)dz.$$

(2.1)

With our choice of measure $dz$, the Fourier transform is self-dual, namely, $\hat{f}(u) = f(-u)$.

2.2. Representations of $\text{GL}_2(\mathbb{C})$ and $\text{SL}_2(\mathbb{C})$. Denote by $\chi_{\nu, l}$ the character of $\mathbb{C}^\times$ given by

$$\chi_{\nu, l} : z \mapsto |z|^{\nu}(z/|z|)^l,$$

with $\nu \in \mathbb{C}$ and $l \in \mathbb{Z}$. According to Langlands’ classification for $G$, any irreducible admissible representation of $G$ may be parametrized by $(\nu_1, \nu_2, l_1, l_2) \in \mathbb{C}^2 \times \mathbb{Z}^2$. First, we introduce the principal series representation $\pi(\chi_{\nu_1, l_1}, \chi_{\nu_2, l_2})$. It is known that $\text{Ind}_B^G(\chi_{\nu_1, l_1}, \chi_{\nu_2, l_2})$, 
defined by the unitary induction, has a unique irreducible quotient provided that \( \Re \nu_1 \geq \Re \nu_2 \). We denote this quotient by \( \pi(\chi_{\nu_1,l_1}, \chi_{\nu_2,l_2}) \) according to [JL] § 6. Second, these principal series \( \pi(\chi_{\nu_1,l_1}, \chi_{\nu_2,l_2}) \) exhaust all the irreducible admissible representations of \( G \) up to infinitesimal equivalence that occurs when we permute \( \chi_{\nu_1,l_1} \) and \( \chi_{\nu_2,l_2} \). Note that we can always permute \( (\nu_1, l_1) \) and \( (\nu_2, l_2) \) if necessary so that \( \Re \nu_1 \geq \Re \nu_2 \) is satisfied. See [Kna2] § 4 and [JL] § 6 for more details.

For \( \mu \in \mathbb{C}, m \in \mathbb{Z} \), we let \( \pi_{\mu,m} \) denote the principal series of \( G \) with parameter \( (\mu, -\mu, \frac{1}{2}m, -\frac{1}{2}m) \) if \( m \) is even or \( (\mu, -\mu, \frac{1}{2}(m + 1), -\frac{1}{2}(m - 1)) \) if \( m \) is odd. Note that \( \pi_{\mu,m} \otimes \chi_{\nu,l}(\det) \) exhaust all the principal series of \( G \) as above. It is clear that \( \pi_{\mu,m} \) has trivial central character if and only if \( m \) is even.

When restricting \( \pi_{\mu,m} \) on \( S \), we obtain the principal series \( \sigma_{\mu,m} \) of \( S \) induced from the character \( s(a) \to \chi_{\nu,m}(a) \). The principal series \( \sigma_{\mu,m} \) exhaust all the irreducible admissible representations of \( S \).

Finally, the representation \( \pi_{\mu,m} \) or \( \sigma_{\mu,m} \) is unitary if

- (unitary principal series) \( \Re \mu = 0 \), or
- (complementary series) \( \mu \in \left( 0, \frac{1}{2} \right) \) and \( m = 0 \).

For more information, the reader may consult the book of Knapp [Kna1].

### 2.3. Whittaker functions.

Let \( \pi \) be an infinite dimensional irreducible unitary representation of \( G \) on a Hilbert space \( \mathcal{H} \), with \( \mathcal{H}_c \) its subspace of smooth vectors. It is well known that \( \pi \) is generic in the sense that there exists a nonzero continuous \( \psi \)-Whittaker functional \( L \) on \( \mathcal{H}_c \), unique up to scalars, satisfying

\[
L(\pi(n)v) = \psi(n)L(v), \quad n \in N, \ v \in \mathcal{H}_c.
\]

Let

\[
(2.2) \quad W_v(g) = L(\pi(g)v), \quad v \in H_c, \ g \in G,
\]

be the Whittaker function corresponding to \( v \). All these definitions are valid for \( S \).

First, we have the following lemma for the asymptotic of Whittaker functions on the torus.

**Lemma 2.1.** For \( v \in H_c \), the function \( W_v(t(a)) \) is rapidly decreasing at infinity and is of order \( |a|^\rho \) for certain \( \rho > 0 \) when \( a \) is in the vicinity of zero.

This lemma is a consequence of a much more general result of Jacquet and Shalika for \( \text{GL}_n \) over a local field in [JS] § 4.4 Proposition 3. Specialized to the case of \( \text{GL}_2(\mathbb{C}) \), their result may be phrased as follows. There is a finite set \( C \) of characters of \( \mathbb{C}^\times \) with positive real part, namely, characters \( \chi \) with \( |\chi(z)| = |z|^\rho \) for \( \rho > 0 \), and for each \( \chi \in C \) a nonnegative integer \( r_\chi \) with the following property: let \( X \) be the set of finite functions of the form \( \chi(a)/(\log |a|^\rho)' \), with \( \chi \in C \) and \( r \leq r_\chi \), then for any given \( v \in H_c \) there are

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3For \( \text{GL}_2 \), one should have \( r_\chi = 0 \) or 1.
functions $\phi_\xi$ in the Schwartz space on $\mathbb{C} \times \mathbb{U}_2(\mathbb{C})$ such that

$$W_\xi(t(a)) = \sum_{\xi \in \mathbb{X}} \phi_\xi(a,k) \xi(a).$$

The rapid decay of $W_\xi(t(a))$ as $\|a\| \to \infty$ is already proven in [God] §2.5 (and also [JL] §6), in particular (68)-(72). However, we do not find any concrete statement in either [JL] or [God] on the asymptotic of $W_\xi(t(a))$ for $a$ small. Nevertheless, we may prove the asymptotic $W_\xi(t(a)) = O(\|a\|^p)$ as $\|a\| \to 0$ if we examine and estimate the integrals in (69) and (70) in [God, §2.5] more carefully; it is important here that $\pi$ is unitary so that either $\text{Re} \mu = 0$ or $\mu \in (0, \frac{1}{2})$ (we may choose $0 < \rho < \frac{1}{2} - \text{Re} \mu$). It should be noted that only $\mathbb{U}_2(\mathbb{C})$-finite vectors are treated in [JL] §6 and [God] §2.5.

Moreover, we have the following analogue of [BM2 Lemma 2.1].

**Lemma 2.2.** Let $v \in H_\infty$, $f, g \in C^\infty(\mathbb{C}^\times)$. Assume that both $f(a)W_\xi(t(a))$ and $g(a)W_\xi(t(a))$ are in $L^1(\mathbb{C}^\times, d^\times a)$. If

$$\int_{\mathbb{C}^\times} f(a)W_{\pi(a)}(t(a))d^\times a = \int_{\mathbb{C}^\times} g(a)W_{\pi(a)}(t(a))d^\times a$$

for all $n \in \mathbb{N}$, then $f(a)W_\xi(t(a)) = g(a)W_\xi(t(a))$ for all $a \in \mathbb{C}^\times$.

**Proof.** The proof is literally the same as that of [BM2 Lemma 2.1]. We have

$$\int_{\mathbb{C}^\times} f(a)W_{\pi(a)}(t(a))d^\times a = \int_{\mathbb{C}^\times} f(a)W_\xi(t(a))d^\times a = \int_{\mathbb{C}^\times} f(a)W_\xi(t(a))\psi(ax)d^\times a.$$

Hence follows the integrability of the first integral for all $x \in \mathbb{C}$ and it is equal to the Fourier transform of the function $\|a\|^{-1}f(a)W_\xi(t(a))$. The identity in the lemma then yields the equality between the Fourier transform of $\|a\|^{-1}f(a)W_\xi(t(a))$ and that of $\|a\|^{-1}g(a)W_\xi(t(a))$. It follows that $f(a)W_\xi(t(a)) = g(a)W_\xi(t(a))$ for all $a \in \mathbb{C}^\times$. Q.E.D.

3. Bessel and relative Bessel distributions for $GL_2(\mathbb{C})$

Let $\langle \cdot, \cdot \rangle$ be a $G$-invariant nonzero inner product on $H$. We can normalize the Whittaker functional $L$ so that

$$\langle v_1, v_2 \rangle = \int_{\mathbb{C}^\times} W_\xi(t(a))\overline{W_\xi(t(a))} d^\times a. \tag{3.1}$$

**3.1. The normalized torus invariant functional.** We define

$$P(v) = \frac{1}{L(\pi, 1/2)} \int_{\mathbb{C}^\times} W_\xi(t(a))d^\times a, \quad v \in H_\infty, \tag{3.2}$$

in which the normalization factor $L(\pi, 1/2)$ is the central value of the $L$-function for $\pi$ (see for example [Kna2, §4]). In view of Lemma 2.2, the integral above is absolutely convergent. Moreover, it may be verified that $P$ is a nonzero continuous linear functional on $H_\infty$ satisfying

$$P(\pi(a)v) = P(v), \quad a \in A, v \in H_\infty,$$

if the central character of $\pi$ is trivial. Hence $P$ is the nonzero unique up to scalar continuous linear functional with the above invariance property.
3.2. Bessel and relative Bessel distributions. For every continuous linear functional \( \lambda \) on \( H_\infty \), and every \( f \in C_c^\infty(G) \), the map \( v \to \lambda(\pi(f)v) \) is continuous on \( H \) by [Sha Proposition 3.2]. By the Riesz representation theorem, there exists a unique vector \( v_{\lambda,f} \) such that

\[
\lambda(\pi(f)v) = \langle v, v_{\lambda,f} \rangle, \quad \text{all } v \in H_\infty.
\]

More concretely, for any orthonormal basis \( \{v_i\} \) in \( H_\infty \),

\[
v_{\lambda,f} = \sum \lambda(\pi(f)v_i)v_i.
\]

Then we define the Bessel distribution by

\[
J(f) = J_{\pi,\phi}(f) = \overline{L(v_{\lambda,f})}
\]

and the relative Bessel distribution by

\[
I(f) = I_{\pi,\phi}(f) = \overline{L(v_{\rho,f})}.
\]

In view of (3.4), for any orthonormal basis \( \{v_i\} \) in \( H_\infty \), we have

\[
J(f) = \sum L(\pi(f)v_i)L(v_i),
\]

and

\[
I(f) = \sum P(\pi(f)v_i)L(v_i).
\]

Letting \( \tilde{f}(g) = f(g^{-1}) \), it follows from [BM2 Corollary 23.7] that \( J(f) = L(v_{L,\tilde{f}}) \) and \( I(f) = P(v_{L,\tilde{f}}) \). Thus

\[
I(f) = P(v_{L,\tilde{f}}) = \frac{1}{L(\pi, 1/2)} \int_{C^\times} W_{\nu_L,\tilde{f}}(t(a))d^\times a = \frac{1}{L(\pi, 1/2)} \int_{C^\times} L(\pi(t(a)v_{L,\tilde{f}}))d^\times a.
\]

Let \( \rho_t \) denote the left translation of functions on \( G \), namely, \( (\rho_t(h)f)(g) = f(h^{-1}g) \). Note that for any \( v \in H_\infty \)

\[
\pi((\rho_t(t(a))f)v) = f(t(a^{-1})\pi(g)v) \quad \text{for all } v \in H_\infty.
\]

and therefore \( \pi(t(a)v_{L,\tilde{f}}) = v_{L,\tilde{f}(t(a))f} \). Hence

\[
I(f) = \frac{1}{L(\pi, 1/2)} \int_{C^\times} J(\rho_t(t(a))f)d^\times a.
\]

Finally, we define Bessel distributions for \( S \) in the same manner. Let \( (\sigma, H) \) be a unitary representation of \( S \). Let \( L \) be a nonzero continuous \( \psi \)-Whittaker functional on \( H_\infty \). Similar as above, for any \( h \in C_c^\infty(S) \), there exists a unique vector \( v_{L,h} \in H \) such that

\[
L(v_{L,h}) = \langle v, v_{L,h} \rangle, \quad \text{all } v \in H_\infty.
\]

Then one can define similarly the Bessel distribution \( J_{\sigma,\phi} \) on \( S \) associated with \( \sigma \) by

\[
J_{\sigma,\phi}(h) = \overline{L(v_{L,h})}.
\]
4. Bessel functions for $GL_2(\mathbb{C})$

Bessel functions for $PSL_2(\mathbb{C})$ were first discovered by Bruggeman and Motohashi [BM5], and later by Lokvenec-Guleska [LG] for $SL_2(\mathbb{C})$, arising in their Kuznetsov trace formulae (Bessel functions for spherical representations of $SL_2(\mathbb{C})$) however appeared much earlier in the work of Miatello and Wallach [MW]). Recently, the second author rediscovered Bessel functions for $GL_2(\mathbb{C})$ as an example of the Bessel functions for $GL_n(\mathbb{C})$ occurring in the Voronoï summation formula; see [Q13 §3, 15, 17, 18].

In the representation theoretic aspect, most important is a kernel formula in [Q13 §18] for the action of the Weyl element in the Kirillov or Whittaker model of an irreducible unitary representation of $GL_2(\mathbb{C})$. This action is given by the Hankel transform over $\mathbb{C}^\times$ with integral kernel the associated Bessel function. Such a kernel formula for $GL_2(\mathbb{R})$ and $GL_2(\mathbb{C})$ lies in the center of the representation theoretic approach to the Kuznetsov trace formula; see [CPS] and [Q4]. In the case of $GL_2(\mathbb{R})$ or $SL_2(\mathbb{R})$, there are three proofs of the kernel formula in [CPS §8], [Mot1] and [BM2 Appendix 2]. Methods in the latter two proofs were generalized onto $SL_2(\mathbb{C})$ in [BM4], [Mot2] and [BBA], but there are unpleasant restrictions on both of them due to some convergence issues. In [BM4], an integral representation of the Bessel function is used but it is valid only for $|\text{Re} \, \mu| < \frac{1}{2}$. In [BBA], it requires that $\text{Re} \, \mu \neq 0$ and that the functions in the Kirillov model are compactly supported. The approach in [Q13] is quite different. It is based on the sophisticated harmonic analysis by gamma factors and the Mellin transforms (see [Q13 §1-3]). Also the ideas in [CPS §8] are followed and generalized in [Q13 §17] to $GL_m(\mathbb{R})$ and $GL_m(\mathbb{C})$.

4.1. The definition of $J_{\mu,m}(z)$. Let $\mu \in \mathbb{C}$ and $m \in \mathbb{Z}$. We define

$$J_{\mu,m}(z) = J_{-2\mu - \frac{1}{2}m}(z) J_{-2\mu + \frac{1}{2}m}^\prime(z).$$

The function $J_{\mu,m}(z)$ is well defined in the sense that the expression on the right of (4.1) is independent on the choice of the argument of $z$ modulo $2\pi$. Next, we define

$$J_{\mu,m}(z) = \begin{cases} 
\frac{2\pi^2}{\sin(2\pi\mu)} (J_{\mu,m}(4\pi \sqrt{z}) - J_{-\mu,-m}(4\pi \sqrt{z})), & \text{if } m \text{ is even,} \\
\frac{2\pi^2 i}{\cos(2\pi\mu)} (J_{\mu,m}(4\pi \sqrt{z}) + J_{-\mu,-m}(4\pi \sqrt{z})), & \text{if } m \text{ is odd,}
\end{cases}$$

where $\sqrt{z}$ is the principal branch of the square root, and it is understood that in the non-generic case when $4\mu \in 2\mathbb{Z} + m$ the right hand side should be replaced by its limit. $J_{\mu,m}(z)$ is a well defined function on $\mathbb{C}^\times$ only when $m$ is even, but it becomes well defined after multiplying the factor $\sqrt{z}/|z|$ when $m$ is odd.

For later use, the following crude estimates for $J_{\mu,m}(z)$ are sufficient. See for example [Q12 (2.28)].

It should be noted that our parametrization is slightly different from theirs.
4.2. Bessel functions for \( GL_2(\mathbb{C}) \). As defined in \( \S 2 \), let \( \pi = \pi_{\mu,m} \) be a principal series representation of \( G = GL_2(\mathbb{C}) \), and, for \( \lambda \in \mathbb{C}^\times \), let \( \psi = \psi_\lambda \) be a nontrivial additive character on \( N \). The central character \( \omega_\pi(z(c)) = c/|c| \) if \( m \) is odd and \( \omega_\pi(z(c)) \equiv 1 \) if \( m \) is even. We define a function \( j = j_{\pi,\psi} \) supported on the open Bruhat cell \( X = Bw_0B \) such that
\[
j_{\pi,\psi}(t(a)w_0) = \begin{cases} |a| J_{\mu,m}(-\lambda^2 a), & \text{if } m \text{ is even,} \\ -i|a| \sqrt{a} |a| J_{\mu,m}(-\lambda^2 a), & \text{if } m \text{ is odd,} \end{cases}
\]
and that \( j_{\pi,\psi} \) is left and right \((\psi,N)\)-equivariant and also \((\omega_\pi,Z)\)-equivariant, namely,
\[
j_{\pi,\psi}(n(x)z(c)t(a)w_0n(y)) = \psi(x)\psi(y)\omega_\pi(c)j_{\pi,\psi}(t(a)w_0).
\]
According to \( \text{[Qi]} \S 18 \), we have
\[
W_e(t(b)w_0) = \int_{\mathbb{C}^\times} \omega_\pi(z(c))^{-1} j(t(ab)w_0) W_e(t(a)) \, d^\times a,
\]
for all \( v \in H_\infty \).

As an easy consequence of \( (4.4) \) and \( (4.5) \), we have the following formula.

**Theorem 4.2.** Let \( v \in H_\infty \). Then
\[
W_e(g) = \int_{\mathbb{C}^\times} j(gt(a^{-1})) W_e(t(a)) \, d^\times a,
\]
for all \( g \in Bw_0B \).

It readily follows from \( (4.3), (4.4) \) that the restriction of the Bessel function \( j_{\pi,\psi} \) on \( S = SL_2(\mathbb{C}) \) is given by
\[
j_{\pi,\psi}(s(a)w) = (-1)^m |a|^2 J_{\mu,m} \lambda^2 a^2 \)
and
\[
j_{\pi,\psi}(n(x)s(a)w \gamma) = \psi(x)\psi(y) j_{\pi,\psi}(s(a)w).
\]

**Remark 4.3.** Of course, the factor \( \omega_\pi(z(c))^{-1} \) in \( (4.5) \) disappears if \( \pi \) is a representation of \( PGL_2(\mathbb{C}) \) in the case when \( m \) is even. Otherwise, it will be gone if one uses \( s(a) \) instead of \( t(a) \) for torus elements, so the kernel formula looks simpler for \( SL_2(\mathbb{C}) \).

\[
W_e(s(b)w) = \int_{\mathbb{C}^\times} j(s(ab)w) W_e(s(a)) \, d^\times a,
\]
for all \( v \in H_\infty \). See \([\text{Mot1]} \) Theorem 2 and \([\text{BBA]} \) Theorem 2.3.

We extend the definition of \( j \) to the whole group \( G \) by setting \( j(g) = 0 \) for \( g \notin X \).

---

[Qi] as well as \([\text{CPS}] \), the measure is not normalized by the factor \( \sqrt{|a|} \) and the choice of Weyl element is \( w \) instead of \( w_0 \), so the Bessel functions in \([\text{Qi]} \) Proposition 18.5 are slightly different.
Proposition 4.4. Assume that \( \pi \) is unitary. Then \( j \) is locally integrable over \( G \). Namely, for any \( f \in C_c^\infty (G) \), we have
\[
\int_G |f(g)j(g)| \, dg < \infty.
\]

Proof. Recall that the measure on \( X = NAw_0N \) is given by \( dg = |a|^{-2} dx d^\infty c dy \) if \( g = n(x)\zeta(c)\tau(a)w_0n(y) \). We have
\[
\int_G |f(g)j(g)| \, dg = \int_X |f(g)j(g)| \, dg
\]
\[
= \int |j(t(a)w_0)| \left( \int |f(n(x)\zeta(c)\tau(a)w_0n(y))| dx d^\infty c \right) |a|^{-2} d^\infty a
\]
\[
= \int |j(t(a)w_0)| \left( \int J(\sqrt{a}, c, |f|) d^\infty c \right) |a|^{-2} d^\infty a,
\]
where \( J(\sqrt{a}, c, |f|) \) is the orbital integral defined as in (A.1). In view of Proposition A.1, the inner integral over \( c \) is zero for small \( |a| \) and of the order \( |a|^{1+\varepsilon} \) for large \( |a| \). In view of Lemma 4.1 and the expression of \( j = f_{\pi,\phi} \) as in (4.3), \( j(t(a)w_0) \) is bounded by \( |a|^2 \) when \( |a| \) is large. It is now clear that the integral above is convergent.

Q.E.D.

5. Bessel distributions for \( GL_2(\mathbb{C}) \)

In this section, we show that the Bessel distribution \( J = J_{\pi,\phi} \) is represented by the Bessel function \( j = f_{\pi,\phi} \).

Lemma 5.1. Let \( v \in H_x \), then
\[
\int_{\mathbb{C}^*} J(\rho_r(t(a))f) W_v(t(a)) d^\infty a = \int_G f(g) W_v(g) \, dg,
\]
in which \( \rho_r \) is the right translation, that is, \( (\rho_r(h)f)(g) = f(gh) \).

Proof. The proof is similar to that of [BM2, Lemma 7.1]. For any \( v \in H_x \), we have
\[
\pi(\rho_r(t(a))f) v = \int_G f(gt(a)) \pi(g) \, dg
\]
\[
= \int_G f(g) \pi(g) \pi(t(a^{-1})) v = \pi(\rho_r(t(a^{-1})) v).
\]
Thus it follow from (3.3), (3.5) and (2.2) that \( \nu_{L, \rho_r(t(a))} = \pi(t(a)) \nu_{L, f} \) and \( J(\rho_r(t(a))f) = W_{\nu_{L, f}}(t(a)) \). Hence, in view of (3.1), we have
\[
\int_{\mathbb{C}^*} J(\rho_r(t(a))f) W_v(t(a)) d^\infty a = \langle v, \nu_{L, f} \rangle.
\]
On the other hand, by (3.3) and (2.2),
\[
\langle v, \nu_{L, f} \rangle = L(\pi(f)v) = \int_G f(g) W_v(g) \, dg,
\]
which finishes the proof.

Q.E.D.
Recall the definition of the Bessel function \( j = j_{\pi, \phi} \) in the last section. We define a distribution \( \hat{J} = \hat{J}_{\pi, \phi} \) on \( C_c^\infty(G) \) by
\[
\hat{J}(f) = \int_G f(g) j(g) dg.
\]
This distribution is well defined as we have proven in Proposition 4.4 that \( j_{\pi, \phi}(g) \) is locally integrable on \( G \).

**Lemma 5.2.** Let \( v \in H_\infty \) and \( f \in C_c^\infty(G) \). Then
\[
\int_{C_c^\infty} \hat{J}(\rho_v(t(a))f) W_v(t(a)) d^\times a = \int_G f(g) W_v(g) dg.
\]

**Proof.** We have
\[
\begin{align*}
\int_{C_c^\infty} \hat{J}(\rho_v(t(a))f) W_v(t(a)) d^\times a &= \int \left( \int_X f(gt(a)) j(g) dg \right) W_v(t(a)) d^\times a \\
&= \int \left( \int_X f(g) j(gt(a^{-1})) dg \right) W_v(t(a)) d^\times a \\
&= \int_X f(g) \left( \int j(gt(a^{-1})) W_v(t(a)) d^\times a \right) dg \\
&= \int_X f(g) W_v(g) dg.
\end{align*}
\]
Here we have obtained the last equality from (4.2) in Theorem 4.2. It is however needed to justify the change of order of integrations in the second to the last equality. For this, it suffices to verify the absolute convergence of the integral in the third line. Note that
\[
\begin{align*}
\int_X |f(g) j(gt(a^{-1}))| dg &\leq \int |j(t(ab)w_0)| \left( \int |f(n(x)z(c)t(b)w_0n(y))|dxdydz \right) |b|^{-2} d^\times b \\
&\leq \int |j(t(ab)w_0)| \left( \int J(\sqrt{b}, c, |f|) d^\times c \right) |b|^{-2} d^\times b,
\end{align*}
\]
where \( J(\sqrt{b}, c, |f|) \) is the orbital integral defined as in (A.1). In view of Proposition A.1, the inner integral over \( c \) is dominated by \( |b|^{1+\varepsilon} \Theta_B(b) \) for some constant \( B \), with \( \Theta_B \) the characteristic function on \( \{b : |b| \geq B\} \). Since \( \pi = \pi_{\mu, m} \) is unitary, we have \( |\text{Re} \mu| < \frac{1}{2} \).

In view of Lemma 4.1 and the expression of \( j = j_{\pi, \phi} \) as in (4.3), \( j(t(ab)w_0) \) is bounded by \( 1 + |a|^{\frac{1}{2}} \). Consequently, we have the following estimations for the integral above
\[
\leq \int (1 + |ab|^{\frac{1}{2}}) |b|^{-3+\epsilon} \Theta_B(b) db \leq 1 + |a|^{\frac{1}{2}}.
\]
Finally, recall from Lemma 2.1 that \( W_v(t(a)) \) is rapidly decreasing at infinity and of the order \( |a|^{2p} \) near zero for certain \( \rho > 0 \), then follows the absolute convergence of the integral.

Q.E.D.
Corollary 5.3. The two distributions \( I \) and \( J \) are the same. That is, for any \( f \in C^\infty_c(G) \), we have

\[
J(f) = \int_G f(g) j(g) dg.
\]

Proof. Choose \( v \in H_{\infty} \) with \( W_v(1) = 1 \), say. By Lemma 5.1 and 5.2 we have

\[
\int J(\rho_v(t(a)) f) W_v(t(a)) d^\sigma a = \tilde{J}(\rho_v(t(a)) f) W_v(t(a)) d^\sigma a
\]

for all \( f \in C^\infty_c(G) \). We replace \( v \) by \( \pi(n)v \) for any \( n \in N \) and apply Lemma 2.2 it follows that

\[
J(\rho_v(t(a)) f) W_v(t(a)) = \tilde{J}(\rho_v(t(a)) f) W_v(t(a))
\]

for all \( a \in \mathbb{C}^\times \). Letting \( a = 1 \), the conclusion follows immediately.

Q.E.D.

Let \( \sigma = \sigma_{\mu, m} \) and \( \pi = \pi_{\mu, m} \) be unitary principal series of \( S \) and \( G \) respectively; see (2.2). Recall that \( \sigma \) is the restriction of \( \pi \) on \( S \). Let \( H \) be their common underlying space. Let \( L \) be a fixed common \( \psi \)-Whittaker functional. Let the Bessel distribution \( J_{\pi, \psi} \) be defined by (3.10) (3.11). We now prove that the Bessel distribution \( J_{\sigma, \psi} \) is represented by the restriction of the Bessel function \( J_{\pi, \psi} \) to \( S \).

Proposition 5.4. Let \( \sigma \) and \( \pi \) be as above. For any \( h \in C^\infty_c(S) \), we have

\[
J_{\sigma, \psi}(h) = \int_S h(s) j_{\pi, \psi}(s) ds.
\]

As such, we shall write \( j_{\sigma, \psi} \) the restriction of \( j_{\pi, \psi} \) on \( S \).

Proof. Fix \( h \in C^\infty_c(S) \). Let \( U \subset Z \) be a small open neighborhood of the identity 1 so that the map \( (s, z) \rightarrow sz \) is an injection from \( S \times U \) into \( G = S \cdot Z \). Choose a function \( w \in C^\infty_c(U) \) with

\[
\int_U w(z) dz = 1,
\]

where \( dz = d^v c \) if \( z = z(c) \). Set \( f(sz) = h(s) \omega_{\pi}^{-1}(z) w(z) \). Clearly, \( f \) is well defined and \( f \in C^\infty_c(G) \) (indeed, \( f \in C^\infty_c(S \cdot U) \)). By Corollary 5.3 along with the \((\omega_{\pi}, Z)\)-equivariance of \( j_{\pi, \psi} \) (see (4.4)), we have

\[
J_{\sigma, \psi}(f) = \int_G f(g) j_{\pi, \psi}(g) dg = \int_U w(z) ds \int_S h(s) j_{\pi, \psi}(s) ds = \int_S h(s) j_{\pi, \psi}(s) ds.
\]

On the other hand, since \( \sigma \) is the restriction of \( \pi \) to \( S \), we may prove in a similar fashion that \( \pi(f) v = \sigma(h) v \) for any \( v \in H \). Precisely,

\[
\int_G f(g) \pi(g) v dg = \int_U w(z) ds \int_S h(s) \pi(s) v ds = \int_S h(s) \sigma(s) v ds.
\]

In view of the definitions in (3) it follows that \( v_L, f = v_L, h \) and that \( J_{\sigma, \psi}(f) = J_{\sigma, \psi}(h) \). The proof is now completed.

Q.E.D.

For \( h \in C^\infty_c(S) \) we introduced the orbital integral

\[
O^N_{h, \psi}(g) = \int h(n(x)g n(y)) \psi(x) \psi(y) dx dy.
\]
This orbital integral was studied by Jacquet. In particular, it is proven in [Jac] that the integral in (5.1) converges absolutely for all \( g \in X \cap S = N(A \cap S)wN \). In view of Proposition 5.3 along with (4.8), for \( h \in \mathcal{C}_c(S) \) we have

\[
J_{\sigma, \varphi}(h) = \int \mathcal{O}_{h, \varphi}^{N, N}(s(a)w)f_{\sigma, \varphi}(s(a)w)\|a\|^{-2}\,d^\times a.
\]

6. Relative Bessel functions for \( GL_2(\mathbb{C}) \)

In this section, we prove that relative Bessel distributions can be represented by real analytic functions on \( U = A(N \setminus \{0\})w_0N \). This follows directly from (the distributional version of) an explicit formula in [Qi2] (see (1.3) and (6.1) below) for the Fourier transform of the Bessel function \( J_{\mu, m}(z) \).

Unlike [BM2], the distributional integral formula in [Qi2] would enable us to prove the regularity of relative Bessel distributions on the whole open Bruhat cell \( X = ANw_0N \) rather than its open dense subset \( U \). Furthermore, with this observation, our proof of the full regularity of relative Bessel distributions on \( G \) becomes tremendously easier compare to the approach in [BM2]. See §7 for the details.

6.1. A formula for the Fourier transform of Bessel functions. The Fourier transform \( \hat{f} \) of a Schwartz function \( f \) on \( \mathbb{C} \) is defined by

\[
\hat{f}(u) = \int_{\mathbb{C}} f(z)e(-\text{Tr}(uz)) \, idz \wedge d\overline{z}.
\]

According to [Qi2, Corollary 1.5], when \( |\text{Re}\mu| < \frac{1}{2} \) and \( m \) is even, we have

\[
\left( \int_{\mathbb{C}} J_{\mu, m}(z) \hat{f}(z) \frac{idz \wedge d\overline{z}}{|z|^2} \right) = \frac{1}{2} \left( \int_{\mathbb{C}} e \left( \text{Tr} \left( \frac{1}{2u} \right) \right) J_{\frac{1}{2} \mu, \frac{1}{2} m} \left( \frac{1}{16u^2} \right) f(u) \frac{idu \wedge d\overline{u}}{|u|^2} \right),
\]

if \( f \) is a Schwartz function on \( \mathbb{C} \). Note that there is abuse of the notation \( dz \), as \( dz \) in this article denotes \( 2|\lambda| \) times of the Lebesgue measure on \( \mathbb{C} \).

It is critical that the test function \( f \) in (6.2) only need to be Schwartz on \( \mathbb{C} \) and rapid decay or vanishing at 0 is not required. We also remark that the deduction from (1.3) to (6.1) is not so straightforward; see [Qi2] §6 for more details.

Recall from (2.1) that the Fourier transform with respect to the additive character \( \psi(z) = \psi_\lambda(z) = e(\text{Tr}(\lambda z)) \) is defined by

\[
\hat{f}(u) = \int_{\mathbb{C}} f(z)\psi(uz) \, dz.
\]

Thus, in view of (4.3), the identity (6.1) may be rephrased as

\[
\int_{\mathbb{C}^2} j_{\pi, \varphi}(t(a)w_0) \hat{f}(a) \, d^\times a = \int_{\mathbb{C}^2} e \left( \text{Tr}(\lambda/2x) \right) |\lambda/2x| J_{\frac{1}{2} \mu, \frac{1}{2} m} \left( \frac{\lambda^2}{16x^2} \right) f(x) \, dx,
\]

if \( f \) is a Schwartz function on \( \mathbb{C} \).

6.2. Relative Bessel functions for \( GL_2(\mathbb{C}) \). Let \( \pi \) be an infinite-dimensional irreducible unitary representation of \( G = GL_2(\mathbb{C}) \) with trivial central character. So if \( \pi = \pi_{\mu, m} \) then \( m \) is even. Let \( J = J_{\pi, \varphi} \) and \( I = I_{\pi, \varphi} \) be the normalized Bessel and relative Bessel functions.
distributions defined as in \[3.3.\] Recalling the formula \[3.9.,\]

\[I(f) = \frac{1}{L(\pi, 1/2)} \int_{C^\times} J_j(t(b)) f \, d^\times b.\]

Let \(X = ANw_0N.\) For \(f \in C_c^\infty(X),\) since \(J\) is represented by the Bessel function \(j = j_{\mu, \phi}\)
on \(X\) (Corollary \[5.3,\]), it follows that

\[(6.3) \quad I(f) = \frac{1}{L(\pi, 1/2)} \int_{C^\times} \left( \int_X f(t(b)) j(g) \, dg \right) \, d^\times b.\]

**Lemma 6.1.** Let \(f \in C_c^\infty(X)\) be of the form

\[(6.4) \quad f(t(z(c)n(x)w_0 n(y)) = f_1(a)f_2(c)f_3(x)f_4(y),\]

with \(f_1, f_2 \in C_c^\infty(C^\times)\) and \(f_3, f_4 \in C_c^\infty(C).\) Then

\[(6.5) \quad \int_{C^\times} \int_G f(t(b)) j(g) \, dg \, d^\times b \]

\[= \int_{C^\times} f_1(b) d^\times b \int_{C^\times} f_2(c) d^\times c \int_{C} f_3(y) \psi(y) \, dy \int_{C^\times} j(t(a)w_0) \tilde{f}_3(a) d^\times a,
\]

where \(\tilde{f}_2\) is the \(\psi\)-Fourier transform of \(f_2\) defined as in \[2.1.\]

**Proof.** Let \(dg = d^\times c dxdy\) be a Haar measure on \(X\) for \(g = t(z(c)n(x)w_0 n(y)).\)

In view of \(6.4.4,\) and \(6.4.\), the integral on the left hand side of \(6.5.\) splits into the product

\[
\int f_2(c) d^\times c \int f_1(y) \psi(y) \, dy \int \left( \int f_1(ab) j(t(a)w_0) \int f_3(x) \psi(x) \, dx \, d^\times a \right) \, d^\times b
\]

\[= \int f_2(c) d^\times c \int f_1(y) \psi(y) \, dy \int \int f_1(ab) j(t(a)w_0) \tilde{f}_3(a) d^\times c \, \tilde{f}_3(a) d^\times a \, d^\times b.
\]

We claim that the last double integral converges absolutely. Hence we may change the order of integrations and the variable \(b\) to \(b/a,\) getting

\[
\int \int f_1(ab) j(t(a)w_0) \tilde{f}_3(a) d^\times c \, d^\times a = \int j(t(a)w_0) \tilde{f}_3(a) \int f_1(ab) d^\times b d^\times a
\]

\[= \int f_1(b) d^\times b \int j(t(a)w_0) \tilde{f}_3(a) d^\times a.
\]

We now show the absolute convergence. Choose \(|\text{Re } \mu| < \rho < \frac{1}{2}.\) Lemma \[5.1.\] implies that \(j(t(a)w_0)\) may be bounded by \(|a|^{1-2\rho} + |a|^{-\frac{\mu}{2}}.\) We can find positive constants \(A\) and \(B\) such that \(f_1(a) \leq \Upsilon_{A,B}(a),\) with \(\Upsilon_{A,B}(a)\) defined to be the characteristic function on the annulus \(\{a : A \leq |a| \leq B\}.\)

Hence

\[
\int f_1(ab) j(t(a)w_0) \tilde{f}_3(a) |d^\times a| \leq \int \Upsilon_{A,B}(a) |d^\times a| |a|^{1-2\rho} + |a|^{-\frac{\mu}{2}} da.
\]

Since \(\tilde{f}_3(a)\) is rapidly decreasing when \(|a|\) is large, it follows that the value of the integral above is rapidly decreasing when \(|b|\) is small. When \(|b|\) is large the integral is bounded by \(|b|^{2\rho - 1} + |b|^{-\frac{\mu}{2}}.\) It is then clear that integrating further with \(d^\times b = db/|b|^{\frac{\mu}{2}}\) will converge absolutely at both 0 and \(\infty.\)

Q.E.D.
Combining (6.2), (6.3) and (6.5), it follows that if we define a function \( i = i_{\pi, \psi} \) supported on \( U \) such that
\[
(6.6) \quad i_{\pi, \psi}(n(x)u_0) = e \left( \text{Tr}(\lambda/2) \right) \frac{|\lambda|}{2} J_{\frac{1}{2}y}(\lambda^2/16x^2)/L(\pi, 1/2),
\]
and that \( i_{\pi, \psi} \) is right \((\psi, N)\)-equivariant and left \( A \)-invariant, namely,
\[
(6.7) \quad i_{\pi, \psi}((t(a)z(c)n(x)u_0n(y))) = \psi(y)i_{\pi, \psi}(n(x)u_0),
\]
then
\[
(6.8) \quad I_{\pi, \psi}(f) = \int_U f(g)i_{\pi, \psi}(g)dg.
\]
for all \( f \) of the form (6.4) in Lemma 6.1. Since such functions span a dense subspace of \( C^\infty_c(X) \) (the Stone-Weierstrass theorem), (6.8) is actually valid for all \( f \in C^\infty_c(X) \).

7. Regularity of relative Bessel distributions over \( \text{GL}_2(\mathbb{C}) \)

In this section, we prove that the relative Bessel distribution \( I = I_{\pi, \psi} \) is given by the integration against the relative Bessel function \( i = i_{\pi, \psi} \) on the full group \( G \).

Since we have already proven in \( \text{BM2} \) the regularity of \( I \) on \( X \), Proposition 2.10 in \( \text{Sha} \) is now directly applicable to deduce its full regularity on \( G \). In particular, we may avoid the introduction of differential operators and almost all the arguments in §5 of \( \text{BM2} \). This simplification of course works in the real context as in \( \text{BM2} \).

We begin with the local integrability of \( i \) on \( G \). Recall that \( i \) is set to be zero outside \( U = A(N \setminus \{0\})u_0N \).

**Proposition 7.1.** The Bessel function \( i \) is local integrable on \( G \), namely, for any \( f \in C^\infty_c(G) \),
\[
\int_G |f(g)i(g)| dg < \infty.
\]

**Proof.** The proof is similar to that of Proposition 4.4. We have
\[
\begin{align*}
\int_G |f(g)i(g)| dg &= \int_U |f(g)i(g)| dg \\
&= \int |i(n(x)u_0)| \left( \int \int f(s(a)z(c)n(x)u_0n(y))d^\varepsilon cd'y \right) dx \\
&= \int |i(n(x)u_0)| \int M(x, c, |f|)d^\varepsilon dx,
\end{align*}
\]
in which \( M(x, c, |f|) \) is the orbital integral defined as in (A.2). Applying Proposition (A.2), the inner integral over \( c \) is zero for \( |x| \) large and of the order \( 1/|x|^\varepsilon \) for small values of \( |x| \). On the other hand, by the expression of \( i = i_{\pi, \psi} \) in (6.6) and Lemma 4.1, we know that \( i(n(x)u_0) \) is bounded when \( |x| \) is small. It is now clear that the integral above is convergent.

Q.E.D.

By Proposition 7.1 we can define the distribution \( \tilde{I} = \tilde{I}_{\pi, \psi} \) by
\[
\tilde{I}(f) = \int_G f(g)i(g)dg.
\]
**Proposition 7.2.** For any \( f \in C_c^\infty(G) \), we have
\[
I(f) = \tilde{I}(f).
\]

**Proof.** Let \( T \) be the distribution on \( G \) defined by the difference \( T = I - \tilde{I} \). First, in view of (6.8), \( I \) and \( \tilde{I} \) coincide on \( X \), so \( T \) is supported on the Borel \( B = G \times X \). Second, it is clear that \( T \) satisfies \( \rho_t(n(y))I = \psi(y)I \). Third, we claim that \( T \) is an eigen-distribution of the Casimir element \( \Delta \) in the universal enveloping algebra of the Lie algebra of \( G \). To see this, we first note that \( I \) is an eigen-distribution of \( \Delta \) according to [Sha §3], that is, \( \Delta I = \kappa I \) \((\kappa \in \mathbb{C}) \). As aforementioned, \( I \) is represented by the (real analytic) function \( i \) when restricted on \( U \), so \( i \) is an eigen-function of \( \Delta \) on \( U \). Precisely, we have \( \Delta i = ki \), which further implies \( \Delta \tilde{I} = k\tilde{I} \). The third claim is now proven. Under the three conditions above, by [Sha Proposition 2.10], we must have \( T = 0 \). Q.E.D.

Finally, for \( f \in C_c^\infty(G) \) define the orbital integral
\[
O_{f,\phi}^{A,N}(g) = \int_A \int_{J_N} f(agn)\psi(n)d^x \text{ad}n.
\]
It is proven in [Jac] that the integral in (7.1) converges absolutely for any \( g \in U = A(N \setminus \{0\})w_0N \). In view of Proposition 7.1 and 7.2, along with (6.7), for \( f \in C_c^\infty(G) \) we have
\[
I_{S,\phi}(f) = \int_{\mathbb{C}^\times} O_{f,\phi}^{A,N}(n(x)w_0)j_{x,\phi}(n(x)w_0)dx.
\]

8. Bessel identities over \( \mathbb{C} \)

We are now ready to establish the Waldspurger correspondence between irreducible unitary representations of \( G = GL_2(\mathbb{C}) \) with trivial central character (that is, representations of \( G/Z = PGL_2(\mathbb{C}) \)) and irreducible unitary representations of \( S = SL_2(\mathbb{C}) \) from the identities between their Bessel functions.

Fix the non-trivial additive character \( \psi = \psi_1 \) of \( \mathbb{C} \) defined by \( \psi(z) = e(\text{Tr}(jz)) \). Let \( D \in \mathbb{C}^\times \) and define \( \psi^D(z) = \psi(Dz) \). Define the Weil factor \( \gamma(z,\psi^D) \) by
\[
\gamma(z,\psi^D) = 1/\sqrt{\|D\|}.
\]
Note that \( \gamma(z,\psi^D) \) does not depend on \( z \), but we would rather keep \( z \) in this conventional notation. Define a transfer factor
\[
\Delta_{D,\phi}(z) = \gamma(z,\psi^D)\psi(2D/z)\sqrt{|z|}.
\]
Let \( \sigma \) be an irreducible unitary representation of \( S \). When changing \( \psi = \psi_1 \) to \( \psi^D = \psi_{1D} \), we wish to keep the Haar measure fixed on \( S \) with respect to \( \psi \) and instead re-normalize the formulae of \( j_{\sigma,\phi} \) in §4.2 by an extra factor \( \sqrt{|D|^2} \).

**Definition 8.1.** Let \( \pi \) be an irreducible unitary representation of \( G/Z \). We say that an irreducible unitary representation \( \sigma \) of \( S \) corresponds to \( \pi \) if the following equality (Bessel

\[\text{In the paper [BM], the more reasonable re-normalizing factor should be } \sqrt{|D|} \text{ not } 1/\sqrt{|D|}, \text{ so their formulae (19.2) and (19.4) need small modifications on the factors involving } |D|.\]
identity) holds

\[
\pi_{\mu,m} \longrightarrow \sigma_{\frac{\mu}{2}, \frac{\mu}{2} + m}, \quad \text{for } \mu \in i\mathbb{R}, \text{even, or } \mu \in (0, \frac{1}{2}), m = 0.
\]

**Theorem 8.2.** For each irreducible unitary representation \( \pi \) of \( G/Z \), there exists a corresponding irreducible unitary representation \( \sigma \) of \( S \) satisfying the Bessel identity (8.2). The correspondence is given by

\[
\pi_{\mu,m} \longrightarrow \sigma_{\frac{\mu}{2}, \frac{\mu}{2} + m}, \quad \text{for } \mu \in i\mathbb{R}, \text{even, or } \mu \in (0, \frac{1}{2}), m = 0.
\]

**Proof.** The proof is a simple comparison between the formulae (4.7) and (6.6). To verify the equality, it should be noted that by [Kna2, (4.7)] and [Tat, §3] we have \( \epsilon(\pi, s, \psi_{\lambda}) = 2^{m} \omega_{\pi}(\lambda)|\lambda|^{2s-1} \) if \( \pi = \pi_{\mu,m} \), and hence \( \epsilon(\pi, 1/2, \psi_{\lambda}) = 2^{m} = (-1)^{m} \) when \( m \) is even so that \( \omega_{\pi}(\lambda) = 1 \). Note that \( \epsilon(\pi, 1/2, \psi) \) is actually independent on \( \psi \). Q.E.D.

We remark that \( \sigma = \Theta(\pi) = \Theta(\pi, \psi^D) \) according to the notation of the theta correspondence of Waldspurger [Wal]. Unlike the real case (see [BM2, Theorem 19.2]), here \( \sigma \) is independent on the additive character \( \psi^D \).

Finally, we would like to prove an identity in the level of distributions. It is proven in [Jac] that for each \( f \in C_c^\infty(G) \) there exists \( f' \in C_c^\infty(S) \) such that

\[
O_{f,\phi}^{N,X}(n(z/4D)w_0) = O_{f',\phi^D}^{N,X}(ws(z))\psi(-2D/z) \sqrt{\gamma(z, \psi^D)},
\]

for all \( z \in \mathbb{C}^\times \). See (5.1) and (7.1) for the definitions of these orbital integrals (the Haar measure in (5.1) however is now chosen with respect to \( \psi \) rather than \( \psi^D \)). By (5.2), (7.2) and Theorem 8.2, we have the following theorem.

**Theorem 8.3.** Assume that \( f \) and \( f' \) satisfy (8.3) and that \( \pi \) and \( \sigma \) correspond as in Theorem 8.2. Then

\[
I_{\pi,\phi}(f) = J_{\sigma,\phi^D}(f')\epsilon(\pi, 1/2, \psi)|2D|L(\pi, 1/2).
\]

**Appendix A. Orbital integrals**

In this appendix, we state some preliminary analytic results on the \((N,N)\) and \((A,N)\) orbital integrals that were needed for verifying the absolute convergence of certain integrals. While the volume estimates are slightly more complicated (they can still be done in the polar coordinates without much difficulty), the proofs are literally identical with those for the real case in [BM2, §4.1, 4.2] and will be omitted here.

**A.1. \((N,N)\) orbital integrals.** For \( f \in C_c(G), a, c \in \mathbb{C}^\times \), we define the following orbital integral

\[
J(a,c,f) = \iint f(n(x)z(c)s(a)w_0n(y))dx\,dy.
\]
Proposition A.1. Let \( f \in C_c(G) \). Then

1. \( J(a, c, f) \) converges absolutely for all \( a, c \in \mathbb{C}^\times \).
2. \( J(a, c, f) \) is compactly supported as a function of \( c \) in \( \mathbb{C}^\times \), independent on \( a \).
3. \( J(a, c, f) \) is zero when \( |a| \) is small, independent on \( c \).
4. \( J(a, c, f) = O(|a|^{2+\varepsilon}) \) for any \( \varepsilon > 0 \), when \( |a| \) is large, independent on \( c \).

A.2. \((A, N)\) orbital integrals. For \( f \in C_c(G), x \in \mathbb{C}, c \in \mathbb{C}^\times \), define orbital integral

\[
M(x, c, f) = \int \int f(s(a)z(c)n(x)n(y))|w_0(y)|d^\times ady.
\]

Proposition A.2. Let \( f \in C_c(G) \). Then

1. \( M(x, c, f) \) converges absolutely for all \( c, x \in \mathbb{C}^\times \).
2. \( M(x, c, f) \) is compactly supported as a function of \( c \) in \( \mathbb{C}^\times \), independent on \( x \).
3. \( M(x, c, f) \) is zero for large values of \( |x| \), independent on \( c \).
4. \( M(x, c, f) = O(1/|x|^\varepsilon) \) for any \( \varepsilon > 0 \), when \( |x| \) is small, independent on \( c \).

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