Open Quantum System Stochastic Dynamics and the Rotating Wave Approximation

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We study the stochastic dynamics of a two-level quantum system interacting with a stochastic magnetic field, and a single frequency electromagnetic field, with and without making the rotating wave approximation (RWA). The transformation to the rotating frame does not commute with the stochastic Hamiltonian if the stochastic field has nonvanishing components in the transverse direction. Hence, making the RWA modifies the stochastic terms in the Hamiltonian. Modification of the decay terms is also required in a master equation approach (i.e., the Liouville–von Neumann density matrix equation) for describing the dynamics. For isotropic Gaussian white noise, the RWA dynamics remains Markovian, although the Lindblad terms in the master equation for the density matrix become time-dependent when the non-commutation of the RWA transformation and the noise Hamiltonian is properly accounted for. We also treat Ornstein–Uhlenbeck noise, and find, in contra-distinction to the white noise case, a significant difference in the dynamics calculated with the RWA when the non-commutation of the RWA transformation and the noise Hamiltonian is taken into account. These findings are applicable to the modeling of any open quantum system coupled to an electromagnetic field.

I. INTRODUCTION

One of the most basic quantum processes studied in physics is a two-level system driven by an electromagnetic field. At least six Nobel prizes were awarded for work on such processes: I. I. Rabi, for the resonance method applied to molecules and NMR, F. Bloch and E. M. Purcell for their development of new methods for NMR, C. W. Townes, N. G. Basov, and A. M. Prokhorov for masers, lasers and quantum optics, A. Kastler for optical pumping, N. F. Ramsey for the separated oscillatory fields method and its use in atom clocks, and S. Haroche and D. J. Wineland for developing methods for observing individual quantum particles without destroying them. But quantum systems are never isolated; they interact with their environment, and this gives rise to perturbations that can strongly affect their behavior. Such interactions affect all the phenomena just enumerated, as well as specific phenomena such as dephasing in metals, nuclear-spin-dependent ground-state dephasing in diamond nitrogen-vacency centers, broadening and shift of atomic clock transitions, and decoherence in quantum information processes.

The dynamics of a quantum system coupled to an environment (a bath) is often treated in terms of the reduced density matrix of the system obtained by tracing out the bath degrees of freedom in the state of the system plus bath. Upon making the Born-Markov approximation, the resulting density matrix is the solution of a Lindblad master equation. An alternative treatment models the coupling of the system and the bath by introducing stochastic fields that interact with the system, where the stochastic fields are generated by a complex environment. In principle, the bath could be affected by the system (backaction). This backaction would modify the properties of the noise felt by the system and effectively appear as a self-interaction mediated by the environment. However, if the perturbation caused to the environment by the system is weak, which is similar to one of the approximations made in the Born-Markov approximation, the self-interaction correction can be neglected. Neglect of backaction is called the external noise approximation. The statistical properties of the stochastic variables are determined by the properties of the environment, and the environment itself is often modeled as a ensemble of approximately independent fluctuating fields in steady state (e.g., in thermal equilibrium). Hence, the resultant stochastic fields felt by the system are a superposition of a large number of components. Due to the central limit theorem, the stochastic fields can be represented by Gaussian, stationary stochastic processes which are completely specified by their first two moments. Moreover, if the timescales of the bath are small compared to those of the system, the stochastic processes can be taken to be Gaussian white noise. The averaged (over stochastic realizations) quantities obtained using Gaussian white noise is completely equivalent to those obtained using the Lindblad master equation approach. This stochastic process method is called the Schrödinger-Langevin stochastic differential equation formalism.

Let us explicitly consider a two-level system, e.g., a spin 1/2 particle. The system interacts with a constant magnetic field, whose direction can be taken, without loss of generality, to be along the z axis, an electromagnetic field with frequency ω, and a stochastic magnetic field, which can be viewed as being due to interaction with a bath of other particles having magnetic dipole moments. Taking Planck’s constant to be unity, ħ = 1, the deterministic Hamiltonian can be written as

\[ H(t) = \begin{pmatrix} \frac{\delta}{2} & \Omega \sin \omega t \\ \Omega \sin \omega t & -\frac{\delta}{2} \end{pmatrix}, \]  

(1)
where the energy difference of the two-level system is $\delta = -g\mu B_z$, where $B_z$ is the static magnetic field, and the Rabi frequency $\Omega$ is proportional to the electromagnetic field strength that oscillates at frequency $\omega$. Denoting the stochastic magnetic field $B_{st}(t)$, the stochastic Hamiltonian takes the form,

$$\mathcal{H}_{st}(t) = -\frac{g\mu}{2} B_{st}(t) \cdot \sigma \equiv \mathbf{b}(t) \cdot \sigma = \left( \begin{array}{cc} b_x(t) & b_z(t) - i b_y(t) \\ b_z(t) + i b_y(t) & -b_x(t) \end{array} \right),$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli spin matrices and $\mathbf{b}(t)$ is the dimensionless stochastic magnetic field. The mean of $\mathbf{b}(t)$ is taken to vanish, and its correlation function depends upon the type of noise:

$$\overline{\mathbf{b}(t) \cdot \sigma} = 0, \quad \overline{\mathbf{b}_i(t) \mathbf{b}_j(t')} = \kappa_{ij}(t - t'), \quad i, j = x, y, z.$$  

(3)

Here $\overline{(...)}$ denotes the stochastic average, i.e., the average over the noise fluctuations. The full Hamiltonian for the two-level system is $H(t) = \mathcal{H}(t) + \mathcal{H}_{st}(t)$.

Here we consider the stochastic dynamics of a quantum system exposed to random fluctuating fields and driven by an electromagnetic field, and explore the consequences of making the rotating wave approximation (RWA) for such systems. There is a considerable literature on the use of the RWA is such problems [12–25]. We also explore the resulting master equations obtained using the stochastic models. We shall explicitly consider white Gaussian noise (Wiener processes) and colored Gaussian noise (Ornstein–Uhlenbeck processes). We find that one must exercise care in applying the RWA because the transformation to the rotating frame does not always commute with the stochastic Hamiltonian. Comparison of the results obtained using these methods leads to surprising conclusions regarding the self-consistency of the RWA and stochastic dynamics, and the use of the RWA in master equation dynamics.

The outline of this paper is as follows. In Sec. II for comparison with the stochastic dynamics in the coming sections, we present results for the dynamics of the two-level system in an oscillating field without any stochasticity present, both without and with making the RWA. We discuss the stochastic dynamics in Sec. IIII first treating dephasing due to white noise in the transverse magnetic field ($b_z$) in Sec. III A, then isotropic white noise in Sec. III B. In Sec. IV V we present the master (Liouville–von Neumann) equation results for this problem. Section VI considers Ornstein–Uhlenbeck noise, and finally a summary and conclusions is presented in Sec. VI.

II. DYNAMICS IN AN OSCILLATING FIELD

The time-dependent Schrödinger equation for our two-level system is, $i\hbar \dot{\psi} = \mathcal{H}(t) \psi$, where $\psi(t) = (\psi_a(t) \psi_b(t))$ is the two-component solution and $\mathcal{H}(t)$ is the time-dependent Hamiltonian given by the sum of (1) and (2). In this section, for the sake of comparison with the stochastic dynamics to be presented in Secs. III and IV, we present the results of solving the Schrödinger equation without a stochastic Hamiltonian, both without and with making the RWA (i.e., transforming to the rotating frame wherein the Hamiltonian $\mathcal{H}_{RWA}$ is time-independent).

Figure I(a) shows the dynamics obtained without the presence of noise for the on-resonance case, starting with the initial condition, $\psi(0) \equiv (\psi_{a0}, \psi_{b0}) = (1, 0)$ at time $t = 0$. We use a set of units such that time is measured in units of $1/\omega$, and frequencies $\delta$ and $\Omega$ are in units of $\omega$ (i.e., we take $\omega = 1$). The probabilities $P_b(t) = |\psi_b(t)|^2$ and $P_a(t) = |\psi_a(t)|^2$ are plotted versus time for the on-resonance case, $\delta = 1$, and Rabi frequency $\Omega = 0.2$. Figure I(c) shows the off-resonance case with $\delta = 1.2$, and $\Omega = 0.2$ [Figs. I(b) and (d) are obtained using the RWA and will be discussed in the next section]. In both Figs. I(a) and I(c), the probabilities oscillate (Rabi-flop) with generalized Rabi frequency $\Omega_g = \sqrt{\Omega^2 + \Delta^2}$, where $\Delta = \omega - \delta$ is the detuning from resonance, moreover, there is a fast oscillation at frequency $\omega + \delta$, which is clearly evident. There is also a Bloch–Siegert shift of the resonance frequency by $\delta \omega_{BS} = \Omega^2/(4\omega)$ [20].

A. The Rotating Wave Approximation

If $\delta \approx \omega$, one often makes the rotating wave approximation (RWA), wherein one transforms to a rotating frame wherein the Hamiltonian, after neglecting a quickly oscillating component, is approximately time-independent. Letting the transformation to the rotating frame, $\mathcal{U}(t)$, be such that

$$\psi(t) = \mathcal{U}(t) \varphi(t), \quad \mathcal{U}(t) = \begin{pmatrix} e^{-i\delta t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix},$$

(4)
Fig. 1: (Color online) On and off resonance Rabi-flopping, calculated without and with the RWA. (a) Probabilities \( P_b(t) = |\psi_b(t)|^2 \) and \( P_a(t) = |\psi_a(t)|^2 \) versus time for the on-resonance case, for \( \omega = 1.0, \delta = 1.0 \) (on-resonance), and Rabi frequency \( \Omega = 0.2 \). (b) RWA probabilities \( P_b(t) \) and \( P_a(t) \) versus time for detuning \( \Delta = 0 \) and \( \Omega = 0.2 \). (c) Probabilities \( P_b(t) \) and \( P_a(t) \) versus time, with \( \omega = 1.0, \delta = 1.2 \) (non-resonant), and \( \Omega = 0.2 \). (d) RWA probabilities \( P_b(t) \) and \( P_a(t) \) versus time for detuning \( \Delta = 0.2 \) and \( \Omega = 0.2 \).

Taking \( \delta_a = -\delta/2 \) and \( \delta_b = \omega + \delta_a \), and noting that

\[
\frac{i\hbar}{\partial t} \frac{\partial \varphi(t)}{\partial t} = [\mathcal{U}(t)H\mathcal{U}(t) - i\hbar \mathcal{U}(t)\mathcal{U}^\dagger(t)]\varphi(t),
\]

and dropping quickly oscillatory terms, yields the following Schrödinger equation for the spinor \( \varphi(t) \):

\[
\frac{i}{\hbar} \frac{d}{dt} \begin{pmatrix} \varphi_b \\ \varphi_a \end{pmatrix} = \begin{pmatrix} -\Delta & i\Omega \\ -i\Omega & \omega \end{pmatrix} \begin{pmatrix} \varphi_b \\ \varphi_a \end{pmatrix}.
\]

(6)

Applying a further transformation, \( \varphi_a \rightarrow -i\varphi_a \), the full RWA transformation becomes

\[
\mathcal{U}(t) = e^{i\delta t/2} \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix}.
\]

(7)

This last transformation turns the complex Hermitian time-independent Hamiltonian matrix on the RHS of (6) into a real symmetric time-independent Hamiltonian, and the Schrödinger equation becomes [27],

\[
\frac{i}{\hbar} \frac{d}{dt} \begin{pmatrix} \varphi_b \\ \varphi_a \end{pmatrix} = \begin{pmatrix} -\frac{\Delta}{2} & \frac{\Omega}{2} \\ -\frac{\Omega}{2} & \omega \end{pmatrix} \begin{pmatrix} \varphi_b \\ \varphi_a \end{pmatrix}.
\]

(8)

Hence, the (constant) RWA Hamiltonian matrix is given by \( \mathcal{H}_{\text{RWA}} = \begin{pmatrix} -\Delta/2 & \Omega/2 \\ -\Omega/2 & \omega \end{pmatrix} \). Figure 1(b) shows the RWA dynamics obtained for the on-resonance case, \( \Delta = 0 \) (without the presence of noise). Comparison with Fig. 1(a)
shows that, aside from the additional oscillations due to the high frequency components and the small Bloch–Siegert shift (which is barely visible here, since \( \omega_{BS} = 0.01 \)), the nature of the dynamics is rather similar. Figure 1(d) shows the RWA dynamics for the off-resonance case with detuning \( \Delta = 0.2 \); here the magnitude of the Rabi oscillations of the probabilities is reduced from unity to \( \Omega/\Omega_g \).

### III. STOCHASTIC DYNAMICS

There are a number of ways of modeling stochastic processes, including a master equation method \([3]\), a Monte Carlo wave function method \([28]\), and a stochastic differential equations method \([7, 8, 29, 30]\). In this section, we use stochastic differential equations.

If the characteristic timescale of the fluctuations is much shorter than the timescale of free evolution of the system, the noise correlations can be well approximated by a Dirac \( \delta \) function to obtain the Gaussian white noise limit wherein the correlation functions \( \kappa_{ij}(\tau) \) appearing in Eq. 3 are proportional to Dirac \( \delta \) functions; moreover, if the noise in the different components of the magnetic field are uncorrelated,

\[
\overline{b_i(t)b_j(t')} = \kappa_{ij}(t - t') \approx w_{0,i}^2 \delta_{ij} \delta(t - t').
\]

The quantity \( w_{0,i} \) is the volatility of the \( i \)th component of the stochastic field \( b(t) \).

A Wiener process \( w(t) \) is the integral over time of white noise \( \xi(t) \), i.e., \( \xi(t) = dw(t)/dt \), with \( \overline{\xi(t)\xi(t')} = w_0^2 \delta(t - t') \) [compare with Eq. 3]. The Schrödinger–Langevin equation for a quantum system coupled to a Wiener stochastic process \( w(t) \) via operator \( \mathcal{V} \) is given by \([3]\),

\[
i \dot{\psi} = \mathcal{H} \psi + w_0 \xi(t) \mathcal{V} \psi - \frac{w_0^2}{2} \mathcal{V}^\dagger \mathcal{V} \psi.
\]

The \( w_0^2 \) term in Eq. 10 insures unitarity if \( \mathcal{V} \) is a Hermitian operator \([3]\). Equation 10 can be easily generalized to include sets of operators \( \mathcal{V}_i \), stochastic processes \( w_i(t) \), and volatilities \( w_{0,i} \), to obtain the general Schrödinger–Langevin equation,

\[
i \dot{\psi} = \mathcal{H} \psi + \sum_i \left( w_{0,i} \xi_i(t) \mathcal{V}_i \psi - \frac{w_{0,i}^2}{2} \mathcal{V}_i^\dagger \mathcal{V}_i \psi \right).
\]

Equation 10 is equivalent to a Markovian quantum master equation with a Lindblad operator \( \mathcal{V} \), and Eq. 11 is equivalent to a Markovian quantum master equation with set of Lindblad operators \( \mathcal{V}_i \).

For example, for the dephasing case to be studied in Sec. III A, the Lindblad operator is taken to be \( \mathcal{V} = \sigma_z \), and the wave function \( \psi \) is a two component spinor; hence, Eq. 10, written in stochastic process notation \([8, 29, 30]\), takes the form

\[
d\psi_1(t) = -i \left\{ [\mathcal{H}_{11} \psi_1(t) + \mathcal{H}_{12} \psi_2(t) - \frac{w_0^2}{2} \psi_b(t)] dt + w_0 \psi_b(t) dw \right\},
\]

\[
d\psi_2(t) = -i \left\{ [\mathcal{H}_{21} \psi_1(t) + \mathcal{H}_{22} \psi_2(t) - \frac{w_0^2}{2} \psi_a(t)] dt - w_0 \psi_a(t) dw \right\}.
\]

For any specific realization of the stochastic process, these equations are solved to yield the two component spinor \( \psi(t) = (\psi_1(t), \psi_2(t)) \) (which is itself a stochastic variable). The (survival) probability to be in state \( b \) at time \( t \) is \( P(t) = |\psi_b(t)|^2 \).

The distinction, as compared with the deterministic case \( (w_0 = 0) \), is that now \( P(t) \) is a random function with distribution \( D[P(t)] \). Equations 12 are easily generalizable to white noise in all three components of the magnetic field; the Lindblad operators appearing in 11 are then \( \mathcal{V}_i = \sigma_i \), and \( w_{0,i} \) are the volatilities for \( b_i(t) \). For the isotropic case (treated in Sec. III B), the numerical values of \( w_{0,i} \) are equal.

#### A. Dephasing due to transverse white noise

Dephasing of a system occurs due to interaction between the system and its environment which scrambles the phases of the wave function of the system without directly affecting probabilities. One of the methods for treating dephasing of a quantum system is to model the interaction with the environment in terms of a time-dependent random noise.
Such an approach enables the calculation of not only the averaged survival probability, $\bar{P}(t)$, but also its standard deviation, $\Delta P(t) = \left( \bar{P}(t)^2 - \left( \bar{P}(t) \right)^2 \right)^{1/2}$, and its statistical distribution function $D[P(T)]$. When the fluctuating magnetic field has a non-vanishing component only along $z$, Eq. (14) reduces to

$$\mathcal{H}_{st}(t) = \xi(t) \sigma_z,$$

i.e., $b(t) = \xi(t)\hat{z}$, where $\xi(t)$ can be taken as white noise if the correlation time of the bath is very fast in comparison to the timescales of the spin system, so

$$\overline{\xi(t)} = 0, \quad \overline{\xi(t)\xi(t')} = w_0^2 \delta(t - t').$$  \hspace{1cm} (14)

The white noise $\xi(t)$ can be written as the time derivative of a Wiener process $w(t)$, $\xi(t) = dw(t)/dt$, or more formally, the Wiener process $w(t)$ is the integral of the white noise. The stochastic Hamiltonian in Eq. (13) gives rise to dephasing of the wave function of the two-level system. In the case of dephasing due to collisions with particles, each collision can have a random duration and a random strength, and in the case of interactions with an environment, the many degrees of freedom of the environment can randomly affect the phase of the wave functions. In the case of dephasing due to collisions with particles, the RWA transformation and the transverse stochastic Hamiltonian do not commute. There is very little difference between the results in Figs. 4(b) and (c), which is not surprising, given that the time dependence of the plane waves $e^{i\omega t}$ is so slow compared to white noise. Figures 4(a)-c) are similar to those in Fig. 3 except that the off-resonance case, $\delta = 1.2$, $\omega = 1.0$, $\Omega = 0.2$, is presented. Again, there is here really is very little difference between
FIG. 2: (Color online) Dephasing of on-resonance transitions due to a stochastic field $b_z(t)$. (a) 100 stochastic realizations of the probability $P_b(t) = |\psi_b(t)|^2$ versus time for the on-resonance case, for $\delta = 1.0$, $\omega = 1.0$ and $\Omega = 0.2$ in the presence of a stochastic magnetic field along the $z$ direction with volatility $w_0 = 0.1$. (b) 100 stochastic realizations of the rotating wave approximation for the probability $P_b(t)$ versus time for the on-resonance case, for $\Delta = 0$, $\omega = 1.0$ and $\Omega = 0.2$ in the presence of a stochastic magnetic field along the $z$ direction with volatility $w_0 = 0.1$. (c) Average probability $P_b(t) = \psi_b^*(t)\psi_b(t)$ and the average plus and minus standard deviation of the probability versus time for $\delta = 1.0$, $\omega = 1.0$ and $\Omega = 0.2$ and a stochastic field in the $z$ direction with volatility $w_0 = 0.1$. (d) Average rotating wave approximation probability $P_b(t)$ and the average plus and minus standard deviation of the probability versus time for $\Delta = 0$, $\omega = 1.0$ and $\Omega = 0.2$ and a stochastic field in the $z$ direction with volatility $w_0 = 0.1$.

the results in Figs. 6(b) and (c). But for noise with a timescale comparable to $\omega^{-1}$ or smaller, large differences are expected. In Sec. IV we discuss the case of an Ornstein–Uhlenbeck process with mean reversion rates comparable to the frequency $\omega$; for such cases, we expect a big difference between the results of taking the non-commutation into account or not.

IV. MASTER (LIOUVILLE–VON NEUMANN) EQUATION RESULTS

As already mentioned, white noise produces a Markovian Liouville-von Neumann density matrix equation with Lindblad terms. Specifically, the isotropic white noise case in Eq. (15) yields the density matrix equation,

$$i\dot{\rho} = [H(t), \rho(t)] + w_0^2 \left(3\rho(t) - \sum_i \sigma_i \rho(t) \sigma_i \right),$$

whereas the case of white noise only in the $z$ component of the magnetic field given in Eq. (13) yields, $i\dot{\rho} = [H(t), \rho(t)] + w_0^2 (\rho(t) - \sigma_z \rho(t) \sigma_z)$. Figure 6 shows the results of such density matrix calculations, where Fig. 6(a) is for the on-resonance dephasing case with Lindblad operator $\sigma_z$, Fig. 6(b) is for the on-resonance isotropic white noise case, and
FIG. 3: (Color online) (a) Histogram with 100 paths (realizations) of the probability \( P_b(t) \) in the presence of a stochastic magnetic field along the \( z \) direction at the final computed time, \( t = 60 \) shown in Figs. 2(a) and (c). (b) Histogram with 100 paths (realizations) of the probability \( P_b(t) \) in the presence of an isotropic stochastic magnetic field at the final computed time, \( t = 60 \) to be shown in Fig. 4(a).

FIG. 4: (Color online) Decoherence and dephasing of on-resonance transitions. (a) Average probability \( \overline{P_b(t)} = \psi_b^*(t)\psi_b(t) \) and the standard deviation of the probability versus time for \( \delta = 1.0, \omega = 1.0, \Omega = 0.2 \) and volatilities \( w_{0,1} = w_{0,2} = w_{0,3} = 0.1 \). (b) Same as (a) (i.e., \( \Delta = 0 \)), except calculated using the RWA. (c) Same as (b), except that the non-commutativity of the RWA transformation and the transverse stochastic Hamiltonian not properly accounted for.

Fig. 4(c) is for the off-resonance case with \( \Delta = 1.2, \omega = 1.0 \). The results for \( \rho_{bb}(t) \) using the density matrix (master equation) treatment are identical, to within numerical accuracy, to the average probabilities \( \overline{P_b(t)} \) computed with the stochastic differential equation approach. However, the Liouville-von Neumann density matrix approach cannot determine the distribution of the probabilities or even the standard deviation of the probabilities.

V. ORNSTEIN–UHLENBECK PROCESS

There are many kinds of stochastic processes that have been studied. In order to see significant effects of the non-commutativity of the RWA transformation and stochastic terms in the total Hamiltonian \( H(t) = \mathcal{H}(t) + \mathcal{H}_{st}(t) \), we need a stochastic process with a finite bandwidth and with frequency components near or less than \( \omega \). A well-known such stochastic process is the Ornstein–Uhlenbeck process, also known as Gaussian colored noise and Brownian motion \[32\]. The mean and the autocorrelation function of a Ornstein–Uhlenbeck process are

\[
\overline{OU_i(t)} = OU_{0,i} e^{-\vartheta_i t} + \mu_i (1 - e^{-\vartheta_i t}), \quad \overline{OU_i(t) OU_j(t')} = \delta_{ij} \frac{w_{0,i}^2}{2\vartheta_i} e^{-\vartheta_i |t-t'|}[e^{\vartheta_i \tau_{corr}} - 1]. \tag{19}
\]

\( \vartheta_i \) is the mean reversion rate of the Ornstein–Uhlenbeck process \( OU_i(t) \), which is the inverse of the correlation time \( \tau_{corr} \) of the noise, \( w_{0,i} \) is its volatility, and \( \mu_i \) is the mean of the process, which we take to vanish, \( \mu_i = 0 \); we also take
FIG. 5: (Color online) Decoherence and dephasing of off-resonance transitions. (a) Average probability $P_b(t) = \psi_b^\ast(t)\psi_b(t)$ versus time for $\delta = 1.2$, $\omega = 1.0$, $\Omega = 0.2$ and volatilities $w_{0,1} = w_{0,2} = w_{0,3} = 0.1$. (b) Same as (a) (i.e., $\Delta = 0.2$), except calculated using the RWA. (c) Same as (b), except that the non-commutativity of the RWA transformation and the transverse stochastic Hamiltonian terms is not properly accounted for.

FIG. 6: (Color online) Master (Liouville-von Neumann density matrix) equation calculations of dephasing and decoherence. (a) Dephasing of the probability $\rho_{bb}(t)$ versus time for the on-resonance case, $\delta = 1.0$, $\omega = 1.0$, $\Omega = 0.2$, and white noise in the $z$ component of the magnetic field with volatility $w_{0} = 0.1$. (b) Decoherence and dephasing of the probability $\rho_{bb}(t)$ versus time for $\delta = 1.0$, $\omega = 1.0$, $\Omega = 0.2$, and isotropic white noise with volatilities $w_{0,1} = w_{0,2} = w_{0,3} = 0.1$. (c) Same as for (b), except for the off-resonance case with $\delta = 1.2$, $\omega = 1.0$.

$OU_{0,i} = 0$. The stochastic differential equations that we solve are,

$$d\psi(t) = -i \left[ H\psi + \sum_i OU_i(t)\sigma_i\psi(t) \right] dt,$$

$$dOU_i(t) = \partial_i [\mu_i - OU_i(t)] dt + w_{0,i} dw_i(t).$$

This yields a non-Markovian master equation for the density matrix of the system. For determining the effects of the non-commutativity of the RWA transformation and stochastic terms in the total Hamiltonian, it is sufficient to use only the term $i = x$ in the sum in Eq. (20a). Doing so simplifies the convergence of the calculation relative to using isotropic Ornstein–Uhlenbeck noise.

Figure 7 shows the calculated average probability $P_b(t)$ versus time and the average plus and minus the standard deviation of the probability calculated using Eq. (20) for Ornstein–Uhlenbeck noise in the $x$ component of the magnetic field for the on-resonance case, $\delta = 1.0$, $\omega = 1.0$, $\Omega = 0.2$. Figure 7(b) is calculated using the RWA, and (c) is as well, but ignoring the non-commutativity of the RWA transformation and the transverse stochastic Hamiltonian term, i.e., ignoring the factors $e^{\pm i\omega t}$ in the Hamiltonian

$$H_{\text{RWA}}(t) = H_{\text{RWA}} + H_{\text{st,RWA}}(t) = \left( \begin{array}{cc} -\Delta & \Omega \\ \Omega & 0 \end{array} \right) + \left( \begin{array}{c} b_x(t) \\ b_y(t) \end{array} \right) \left( \begin{array}{c} i e^{-i\omega t}[b_x(t) - ib_y(t)] \\ -b_z(t) \end{array} \right).$$

The calculations in Fig. 7 were hard to converge with respect to the time-step used, hence we only continued them out to a final time of $t = 20$. Note that the standard deviation in (c) is significantly reduced relative to (a) and (b), and the width becomes very close to zero at $t = 15.5$ where the average probability goes to zero, unlike the results in (a) and (b). Thus, the results of ignoring the non-commutativity of the RWA transformation and the transverse
stochastic Hamiltonian are very different from the RWA taking the non-commutativity into account. One can expect there to be a difference, with and without taking the non-commutativity into account, in a master equation approach as well, although we haven’t derived the master equation for this case and therefore haven’t carried out the calculation.

VI. SUMMARY AND CONCLUSIONS

Much of our practical experience with quantum mechanics comes from applying it to two-level systems driven by an electromagnetic field. But such systems are never truly isolated, and their interaction with their environment affect their mysterious quantum properties, i.e., their quantum coherence. Such interaction is at the heart of the fundamental limitations of quantum metrology and quantum information processing. Using the Schrödinger-Langevin stochastic differential equation formalism we studied the dynamics of a two-level quantum system driven by single frequency electromagnetic field, with and without making the rotating wave approximation (RWA). We found that, if the transformation to the rotating frame does not commute with the stochastic Hamiltonian, i.e., if the stochastic field has nonvanishing components in the transverse direction, making the RWA modifies the nature of the stochastic terms in the Hamiltonian. Hence, the decay terms in a master equation (i.e., Liouville–von Neumann density matrix equation) will also be affected. The RWA dynamics remains Markovian for Gaussian white noise, although the Lindblad terms in the stochastic Schrödinger-Langevin equation, and the master equation for the density matrix, become explicitly time-dependent when the non-commutation of the RWA transformation and the noise Hamiltonian is properly accounted for. For the non-Markovian Ornstein–Uhlenbeck noise case, we found significant difference in the dynamics calculated with the RWA when the non-commutation of the RWA transformation and the noise Hamiltonian is taken into account.

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