ON RECOVERY OF AN INHOMOGENEOUS CAVITY IN INVERSE ACOUSTIC SCATTERING

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ABSTRACT. Consider the time-harmonic acoustic scattering of an incident point source inside an inhomogeneous cavity. By constructing an equivalent integral equation, the well-posedness of the direct problem is proved in $L^p$ with using the classical Fredholm theory. Motivated by the previous work [10], a novel uniqueness result is then established for the inverse problem of recovering the refractive index of piecewise constant function from the wave fields measured on a closed surface inside the cavity.

1. Introduction. In this paper, we study an inverse scattering problem of determining an inhomogeneous cavity from many measurements inside the cavity. Precisely, let $D$ denote the inhomogeneous cavity, which is described by a bounded connected domain in $\mathbb{R}^3$ with the refractive index $n(x) \in L^\infty(D)$. Let $D_0 \subset D$ be a bounded connected part of the inhomogeneous cavity $D$ with the refractive index $n(x)$ equaling 1 in $D_0$. This shows that the medium is inhomogeneous in the subdomain $D_1 := D \setminus D_0$. Furthermore, we assume that both $\partial D_0$ and $\partial D$ are all of $C^2$-class.

Consider an incident field $u^i$ which is induced by a point source located at $y \in D_0$, i.e.,

$$u^i(x) := \Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi |x-y|}, \quad \text{in } \mathbb{R}^3 \setminus \{y\}.$$  

Then the scattering by the inhomogeneous cavity with the point source $u^i(x) = \Phi(x, y)$ is modelled by the Helmholtz equation:

$$\begin{cases}
\triangle u_y(x) + k^2 n(x) u_y(x) = 0 & \text{in } D \setminus \{y\}, \\
u_y(x) = 0 & \text{on } \partial D, \\
u_y(x) = u^i(x) + u^s_y(x) & \text{in } D,
\end{cases}$$

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where $u_s^x(\cdot)$ is the scattered field generated by $\Phi(\cdot, y)$, solving equation

\begin{equation}
\triangle u_s^x(x) + k^2 n(x)u_s^x(x) = k^2(1 - n(x))u^i(x) \quad \text{in } D
\end{equation}

with a Dirichlet boundary condition $u_s^x(x) = -u^i(x)$ on $\partial D$ from Problem (2). If 0 is assumed not to be a Dirichlet eigenvalue for the operator $\triangle + k^2 n(x)$ in $D$, we can easily show that equation (3) admits a unique solution $u_s^x(\cdot) \in H^1(D)$ which depends continuously on the data $\Phi(\cdot, y)$, using the variational approach.

In this paper we will consider the inverse cavity problem (ICP) of recovering the refractive index $n(x)$ described by a piecewise constant function, and the shape of the interior part $D_0$ of the inhomogeneous cavity $D$ from the knowledge of $u^x(x, y)$ for all $x, y \in \partial C$, corresponding to many point sources $\Phi(x, y)$. Here, $C$ denotes an open subregion in $D_0$ with a $C^2$-smooth boundary $\partial C$, i.e. $\overline{C} \subset D_0$, such that $k > 0$ is not a Dirichlet eigenvalue for the operator $-\triangle$ in $C$. And the boundary $\partial D$ of the inhomogeneous $D$ is assumed to be a priori known. Recently, this class of inverse problems have received a considerable interest and we here refer to [3, 5, 6, 7, 8, 9] for a detailed investigation, in which the central point related to the ICP is mainly focused on the study of qualitatively numerical methods of reconstructing the cavity in the homogeneous background medium, that is, $n(x) \equiv 1$ in $D$. For example, a factorization method was proposed in [6] with a Dirichlet or impedance condition on the boundary of the cavity and a sampling type method was proposed in [8] under the Dirichlet condition, which was later extended into the case of the Maxwell equation in [11]. However, to the best of the author’s knowledge, there is much little progress for the case when $D$ is an unknown inhomogeneous cavity in the literature. It is noticed that a related scattering problem was recently considered in [7] for a homogeneous cavity surrounded by a penetrable inhomogeneous media, where the uniqueness result was provided only for the shape of the cavity by means of the measurements inside.

Different from the previous works on the ICP, we will focus in the current paper on the uniqueness issue on the refractive index $n(x)$ and the cavity $D$ in Problem (2) by interior measurements generated by incident point sources (1). To this end, we first transfer Problem (2) into an equivalent integral equation by introducing a Dirichlet-Green function. With the help of this technique, we can prove that Problem (2) is uniquely solvable in the $L^p$-sense with $1 < p \leq 2$, using the classical Fredholm theory. This may be independently interesting in its own right direction. Meanwhile, we can also provide a generalized symmetry property of the solution to Problem (2) with incident fields located at two different points. Based on these analysis, we can propose a novel technique to deal with the ICP, especially for the unique recovery of the index of refraction in the inverse cavity scattering problem.

The remainder of this paper is planned as follows. In section 2, we study the cavity scattering problem with the incident field induced by a general point source wave. The uniqueness and existence of the solution to Problem (2) is obtained in the $L^p$-sense by employing an integral equation method. Section 3 is devoted to the inverse cavity scattering problem with the interior wave field measurements, corresponding to the incident point source wave (1). The uniqueness result is established for the refractive index and its support from many measurements inside the cavity.

2. Well-posedness in $L^p$ for $1 < p \leq 2$. In this section, we are devoted to study the solvability of Problem (2) with the data induced by a general point source wave.
$u_z^i$ located at $z \in D_0$ such that $u_z^i$ satisfies equation

$$\Delta u_z^i + k^2 u_z^i = 0 \quad \text{in } D \setminus \{z\}$$

and $u_z^i \in L^p(D)$ for $1 < p \leq 2$. The above condition induces a class of super-singular incident waves, such as $u_z^i(\cdot) = \Phi(\cdot, z)$ for $p = 2$ and $u_z^i(\cdot) = \partial_n \Phi(\cdot, z)$ for $1 < p < \frac{3}{2}$ and $l = 1, 2, 3$.

For a fixed $y \in D$, we introduce a related Dirichlet boundary value problem

$$\Delta G^y(\cdot, y) + k^2 G^y(\cdot, y) = 0 \quad \text{in } D, \quad G^y(\cdot, y) = -\Phi(\cdot, y) \quad \text{on } \partial D. \tag{4}$$

To ensure the uniqueness of solution to Problem (4), throughout the paper we assume that $k^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $D$. Under this assumption, it is known that there always exists a unique solution $G^y(\cdot, y) \in H^1(D)$ to Problem (4) for each $y \in D$, which continuously depends on the boundary data $\Phi(\cdot, y)|_{\partial D} \in H^\frac{1}{2}(\partial D)$.

Let $G(x, y) := \Phi(x, y) + G^y(x, y)$ for $x, y \in D$, which defines a Dirichlet-Green function for the operator $-\Delta + k^2$ in $D$, i.e., $\Delta G(x, y) + k^2 G(x, y) = -\delta_y(x)$ in $D$ and $G(x, y) = 0$ on $\partial D$. Using the function $G(\cdot, \cdot)$, we shall reduce Problem (1) to an equivalent integral equation defined in the subdomain $D_1$. To accomplish this, let $u_z(\cdot)$ denote the solution to Problem (2) with the data $w^i(x) = u_z^i(x)$. On one hand, we apply the Green theorem for $x \in D_1$ to obtain

$$\int_{D_1} k^2 (n(y) - 1) G(x, y) u_z(y) dy = -u_z(x) - \int_{\partial D_0} (\partial_n u_z(y) G(x, y) - u_z(y) \partial_n G(x, y)) ds(y), \tag{5}$$

where $\nu$ is the unit outward normal on $\partial D_0$ directed into $D_1$. On the other hand, we construct the function $P_z^i(\cdot)$ in $D$ by solving the Dirichlet problem

$$\begin{cases}
\Delta P_z^i + k^2 P_z^i = 0 & \text{in } D \setminus \{z\}, \\
P_z^i = 0 & \text{on } \partial D, \\
P_z^i = u_z^i + P_z^e & \text{in } D.
\end{cases} \tag{6}$$

It is easily seen that $P_z^e$ defines a $C^2$-smooth function in $D$. Therefore, $P_z^i$ has the same singularity with the incident field $u_z^i$ at $x = z$. We then have

$$\int_{\partial D_1} (\partial_n u_z(y) G(x, y) - u_z(y) \partial_n G(x, y)) ds(y) = \int_{\partial D_1} (\partial_n P_z^i(y) G(x, y) - P_z^i(y) \partial_n G(x, y)) ds(y) = -P_z^i(x). \tag{7}$$

This, combined with the equality (5), shows that $u_z$ solves the integral equation of the form

$$u_z(x) - \int_{D_1} k^2 (n(y) - 1) G(x, y) u_z(y) dy = P_z^i(x) \quad \text{for } x \in D_1. \tag{8}$$

**Theorem 2.1.** Let $u_z^e \in L^p(D)$ for $1 < p \leq 2$. If $u_z^e = u_z - u_z^i \in W^{2,p}(D)$ is the solution of (3), then $u_z|_{D_1}$ is a solution of (8). Conversely, if $u_z \in L^p(D_1)$ is the solution of (8), then $u_z - P_z^i \in W^{2,p}(D_1)$ and $u_z^e$ can be extended to be a solution of (3).

**Proof.** The first assertion follows from the argument from (5) and (8).
Conversely, assume that \( u_z \in L^p(D_1) \) is a solution of (8). We conclude from [4] that \( w_z := u_z - P_z^1 \in W^{2,p}(D_1) \) solves the equation
\[
\triangle w_z + k^2 w_z = -k^2(n(x) - 1)u_z \quad \text{in } D_1,
\]
which further means \( \triangle u_z + k^2 n(x)u_z = 0 \) in \( D_1 \). Notice that \( w_z \) can be extended into the inside of the domain \( D_0 \) by equation (8), as a new function in \( D \), such that \( w_z \in W^{2,p}(D) \) and \( \triangle w_z + k^2 w_z = 0 \) in \( D_0 \). The function \( w_z + P_z^1 \) is thus the solution to Problem (3). The proof is now completed.

Define the integral operator \( T : L^p(D_1) \to L^p(D_1) \) by
\[
(T\phi)(x) = \int_{D_1} k^2(n(y) - 1)G(x, y)\phi(y)\,dy, \quad x \in D_1,
\]
with the density \( \phi \in L^p(D_1) \). For \( 1 < p \leq 2 \), it follows from [4, Theorem 9.9] that \( T \) is bounded from \( L^p(D_1) \) into \( W^{2,p}(D) \) and then the compact embedding of \( W^{2,p}(D_1) \) into \( L^p(D_1) \) implies that \( T \) is compact on \( L^p(D_1) \). Equation (8) is now reduced to the second kind of operator equation
\[
(I - T)u_z = P_z^1 \quad \text{in } L^p(D_1).
\]

**Theorem 2.2.** For \( 1 < p \leq 2 \), there exists a unique solution to (10) such that
\[
\|u_z\|_{L^p(D_1)} \leq C\|P_z^1\|_{L^p(D_1)}.
\]

**Proof.** To prove (11), it is sufficient to show that \( (I - T) \) is injective from the Fredholm theory. Provided \( (I - T)\psi = 0 \) for some \( \psi \in L^p(D_1) \), we then have \( \triangle \psi + k^2 n(x)\psi = 0 \) in \( D \). Since 0 is not a Dirichlet eigenvalue for \( \triangle + k^2 n(x) \) in \( D \), we conclude that \( \psi = 0 \) in \( D \). Hence, the proof is completed.

**Remark 1.** By the well-posedness of the problem (6), it is clearly seen that \( P_z^1 \) has the same singularity as the incident field \( u_z^1(\cdot) \) at \( x = z \). Hence, we can obtain the estimate (11) for two different cases on the index \( p > 1 \):

(i) If \( u_z^1(\cdot) = \Phi(\cdot, z) \), then it holds that \( 1 < p \leq 2 \);

(ii) If \( u_z^1(\cdot) = \partial_{x_l}\Phi(\cdot, z) \) with \( l = 1, 2, 3 \), then it holds that \( 1 < p < \frac{3}{2} \).

Let \( z_1 \in \mathbb{R}^3 \) and \( z_2 \in \mathbb{R}^3 \) be in the interior of the domain \( D_0 \) with the assumption \( z_1 \neq z_2 \). Based on the above analysis, we now define two functions \( A_\varepsilon \) and \( B_\varepsilon \) in \( D_0 \):
\[
A_\varepsilon(z_1) := \int_{\partial\mathcal{O}_\varepsilon(z_1)} \left( \frac{\partial u_{z_1}^1(y)}{\partial \nu(y)} u_{z_2}^1(y) - u_{z_1}^1(y) \frac{\partial u_{z_2}^1(y)}{\partial \nu(y)} \right) \, ds(y),
\]
\[
B_\varepsilon(z_2) := \int_{\partial\mathcal{O}_\varepsilon(z_2)} \left( \frac{\partial u_{z_2}^1(y)}{\partial \nu(y)} u_{z_1}^1(y) - u_{z_2}^1(y) \frac{\partial u_{z_1}^1(y)}{\partial \nu(y)} \right) \, ds(y),
\]
by the solutions \( u_{z_l} \) to Problem (2) with the incident field \( u_l^1 = u_{z_l}^1 \) for \( l = 1, 2 \). Here, \( \mathcal{O}_\varepsilon(z_1) \) and \( \mathcal{O}_\varepsilon(z_2) \) are two small balls centered at \( z_1 \) and \( z_2 \), respectively, with the radii \( \varepsilon \) such that \( \mathcal{O}_\varepsilon(z_1) \subset D_0 \) and \( \mathcal{O}_\varepsilon(z_2) \subset D_0 \).

**Theorem 2.3.** Let \( A_\varepsilon \) and \( B_\varepsilon \) be defined by (12) and (13), respectively, for \( z_1 \neq z_2 \). Then
\[
\lim_{\varepsilon \to 0} A_\varepsilon(z_1) = \lim_{\varepsilon \to 0} B_\varepsilon(z_2) \quad \text{for } z_1, z_2 \in D_0,
\]
if either one of the limits in (14) exists.
Proof. The proof follows from a straightforward application of the second Green’s theorem. To facilitate the readers to understand the theorem, we here present a detailed proof.

Choosing \( \varepsilon > 0 \) sufficiently small such that \( \mathcal{O}_\varepsilon(z_1) \cap \mathcal{O}_\varepsilon(z_2) = \emptyset \), we have from the Green’s formula that

\[
0 = \int_{\partial\mathcal{O}_\varepsilon(z_1) \cup \mathcal{O}_\varepsilon(z_2)} (u_{z_2} \Delta u_{z_1} - u_{z_1} \Delta u_{z_2}) \, dy
\]

\[
(15) \quad = \int_{\partial D_0} - \int_{\partial \mathcal{O}_\varepsilon(z_1)} - \int_{\partial \mathcal{O}_\varepsilon(z_2)} \left( \frac{\partial u_{z_1}(y)}{\partial \nu(y)} u_{z_2}(y) - u_{z_1}(y) \frac{\partial u_{z_2}(y)}{\partial \nu(y)} \right) \, ds(y).
\]

The integral on \( \partial D_0 \) vanishes by applying the second Green’s theorem in \( D_1 \) with the homogeneous Dirichlet boundary conditions for both \( u_{z_1} \) and \( u_{z_2} \) on \( \partial D \).

Due to \( u(x) = 1 \) in \( D_0 \), it follows from equation (3) that the scattered solution \( u^s_{\varepsilon}(\cdot) = u_{z_1}(\cdot) - u^s_{\varepsilon}(\cdot) \) satisfies the homogeneous Helmholtz equation \((\Delta + k^2)u^s_{\varepsilon} = 0\) in \( D_0 \) for \( l = 1, 2 \), and is thus \( C^2 \)-smooth in \( D_0 \). Using the Green’s theorem again in \( \mathcal{O}_\varepsilon(z_1), l = 1, 2 \), yields that

\[
0 = \int_{\partial \mathcal{O}_\varepsilon(z_1)} \frac{\partial u^s_{z_1}(y)}{\partial \nu(y)} u^s_{z_2}(y) - u^s_{z_1}(y) \frac{\partial u^s_{z_2}(y)}{\partial \nu(y)} \, ds(y)
\]

\[
(16) \quad = \int_{\partial \mathcal{O}_\varepsilon(z_2)} \frac{\partial u^s_{z_2}(y)}{\partial \nu(y)} u^s_{z_2}(y) - u^s_{z_1}(y) \frac{\partial u^s_{z_2}(y)}{\partial \nu(y)} \, ds(y)
\]

since \( u^s_{z_1} \) and \( u^s_{z_2} \) are regular, respectively, in \( \mathcal{O}_\varepsilon(z_2) \) and \( \mathcal{O}_\varepsilon(z_1) \). Combining (16) with (17), equality (15) now becomes

\[
(18) \quad A_\varepsilon(z_1) = B_\varepsilon(z_2)
\]

for all sufficiently small \( \varepsilon > 0 \) from the definitions about \( A_\varepsilon \) and \( B_\varepsilon \). Finally, the required equality (14) is obtained by taking the limit from the left and right terms of (18). The proof of the theorem is thus accomplished.

Remark 2. (i) Suppose \( u^i_{z_1} (x) = \Phi(x, z_1) \) and \( u^i_{z_2} (x) = \Phi(x, z_2) \), i.e., the incident fields are induced by the fundamental solution to the Helmholtz equation. In this case, we have \( p = 2 \) in Theorem 2.2. Moreover, we can also conclude from the singularity of the fundamental solution that the limits exist for both \( A_\varepsilon \) and \( B_\varepsilon \) and

\[
(19) \quad u_{z_2}(z_1) = \lim_{\varepsilon \to 0} A_\varepsilon(z_1) = \lim_{\varepsilon \to 0} B_\varepsilon(z_2) = u_{z_1}(z_2)
\]

from Theorem 2.3.

(ii) Let \( u^i_{z_1}(x) = \Phi(x, z_1) \) and \( u^i_{z_2}(x) = \nabla_i \Phi(x, z_2) \cdot \vec{l} \) with \( \vec{l} \) a directional vector. In this case, it is checked that \( p = 2 \) in Theorem 2.2 since \( u^i_{z_1}(\cdot) \in L^2(D) \), and \( 1 < p < \frac{3}{2} \) since \( u^i_{z_2}(\cdot) \in L^p(D) \) \((1 < p < \frac{3}{2})\). Therefore, Theorem 2.3 remains valid to obtain that the limits exist for both \( A_\varepsilon \) and \( B_\varepsilon \) and

\[
(20) \quad u_{z_2}(z_1) = \lim_{\varepsilon \to 0} A_\varepsilon(z_1) = \lim_{\varepsilon \to 0} B_\varepsilon(z_2).
\]

By Theorem 2.3 and Remark 2, it is known that the solution to Problem (2) has a generalized symmetry property with two different incident sources in a general case for the index \( p > 1 \).
3. The inverse problem. In this section we aim to obtain the uniqueness on the refractive index \( n(x) \) and the support \( D_0 \) of the inhomogeneous cavity \( D \) under the assumption that the boundary \( \partial D \) is a priori known. In the inverse problem, we will assume that the refractive index \( n(x) \) is described as a piecewise constant function

\[
(21) \quad n(x) = \sum_{j \in \mathbb{Y}} c_j \cdot \chi_{\Omega_j}(x) \quad \text{for } x \in \overline{D} := \bigcup_{j \in \mathbb{Y}} \overline{\Omega_j}.
\]

Here, \( \mathbb{Y} = \{1, 2, 3, \ldots, N\} \) is an index set with the positive fixed integer \( N < +\infty \), \( c_j \) is a unknown constant for each \( j \in \mathbb{Y} \), and \( \chi_{\Omega_j}(\cdot) \) denotes the characteristic function of the unknown connected subdomains \( \Omega_j \), defined by 1 in \( \Omega_j \) and 0 otherwise. We also suppose that \( \Omega_{j_1} \cap \Omega_{j_2} = \emptyset \) for \( j_1 \neq j_2 \). Furthermore, for \( i \neq j \) if \( \partial \Omega_i \cap \partial \Omega_j \) has a non-empty open subset in the two-dimensional manifold of \( \mathbb{R}^3 \), we will naturally assume that \( c_i \neq c_j \). The mutual boundaries between the connected subdomains \( \Omega_j, j = 1, 2, \ldots, N \) in \( \mathbb{Y} \) are assumed to be of \( C^2 \)-class.

Due to the high nonlinearity and severely ill-posedness of the inverse problem, the ICP is very challenging in the general case. To the best of the authors’ knowledge, there are no results available in the literature, if \( D_0, D_1 \) and \( n(x) \) are all unknown.

For convenience, let the scattered solution denoted by \( u_j^{\delta}(\cdot, y), j = 1, 2, \) to Problem (2) be induced by an incident point source \( \Phi(\cdot, y) \) located at \( y \in \partial C \), corresponding to the domains \( D_j^{(1)}, D_j^{(2)} \) and the refractive index \( n_j(\cdot) \). Here, the domain \( C \) is assumed to be such that \( \overline{C} \subset (D_0^{(1)} \cap D_0^{(2)}) \).

We have the main result in the paper on the ICP.

**Theorem 3.1.** Assuming \( u_j^{\epsilon}(x, y) = u_2^{\epsilon}(x, y) \) for all \( x, y \in \partial C \), then \( D_0^{(1)} = D_0^{(2)} \) and \( n_1(x) = n_2(x) \).

**Proof.** **Step 1.** We shall first prove that \( D_0^{(1)} = D_0^{(2)} \) by contraction. Assuming that \( D_0^{(1)} \neq D_0^{(2)} \), then there exists some point \( z_\ast \in \partial(D_0^{(1)} \cap D_0^{(2)}) \) but \( z_\ast \notin \partial D_0^{(2)} \). Define the sequence

\[
z_j := z_\ast - \frac{\delta}{j} \nu(z_\ast), \quad j = 1, 2, 3, \ldots,
\]

where \( \nu(z_\ast) \) is the unit exterior normal at \( z_\ast \) and \( \delta > 0 \) is chosen such that \( z_j \in (D_0^{(1)} \cap D_0^{(2)}) \) for all \( j \in \mathbb{N} \). We now consider Problem (2) with the incident waves \( u_1^j(x) \) of the form

\[
u_j \cdot \Phi(x, z_j) \cdot \nu(z_\ast), \quad j = 1, 2, 3, \ldots,
\]

and let \( u_j^{\epsilon(1)}(x) \) and \( u_j^{\epsilon(2)}(x) \) be the solutions to Problem (2), generated by \( u_j^\epsilon(x) \). Using equality (20) in Remark 2 yields

\[
(23) \quad u_j^{\epsilon(p)}(y) = \lim_{\epsilon \to 0} \int_{\partial \sigma(z_j)} \left( \frac{\partial u_j^\epsilon(\tau)}{\partial \nu(\tau)} u_p(\tau, y) - u_j^\epsilon(\tau) \frac{\partial u_p(\tau, y)}{\partial \nu(\tau)} \right) \sigma \tau, \quad p = 1, 2,
\]

for \( y \in D_0^{(p)} \setminus \{z_j\}, j \in \mathbb{N} \).

For \( y \in \partial C \), since \( u_1^{\epsilon}(\cdot, y)|_{\partial C} = u_2^{\epsilon}(\cdot, y)|_{\partial C} \), it follows \( u_1^{\epsilon}(\cdot, y) = u_2^{\epsilon}(\cdot, y) \) in \( D_0^{(1)} \cap D_0^{(2)} \) from uniqueness of the Dirichlet problem in combination with the unique continuation principle. Therefore, we have \( u_1^{\epsilon}(\cdot, y) = u_2^{\epsilon}(\cdot, y) \) on \( \partial \sigma(z_j) \) for any sufficient small \( \epsilon > 0 \), which further gives

\[
u_j^{\epsilon(1)}(x) = u_j^{\epsilon(2)}(x) \quad \text{in} \quad (D_0^{(1)} \cap D_0^{(2)}) \setminus \{z_j\},
\]
from (23) for \( l = 1, 2, 3, \) and \( j \in \mathbb{N} \), using a similar discussion.

Due to \( z_* \in \partial D_0^{(1)} \), we can choose a Lipschitz domain \( D_{z_*} := O_{e_0}(z_*) \cap (D_0^{(2)} \setminus D_0^{(1)}) \) with a small \( \varepsilon_0 > 0 \) such that \( k^2 \) is not an eigenvalue of the interior transmission problem (ITP)

\[
\begin{aligned}
\Delta w_1 + k^2 n_1 w_1 &= 0 \quad \text{in } D_{z_*}, \\
\Delta w_2 + k^2 w_2 &= 0 \quad \text{in } D_{z_*}, \\
w_1 - w_2 &= f_1 \quad \text{on } \partial D_{z_*}, \\
\frac{\partial w_1}{\partial \nu} - \frac{\partial w_2}{\partial \nu} &= f_2 \quad \text{on } \partial D_{z_*},
\end{aligned}
\]

in the case when \( f_1 = 0 \) and \( f_2 = 0 \). For the ITP, it was shown in [1] that the smallest real eigenvalue \( \lambda_{\text{low}}(D_{z_*}) \) trends to \(+\infty\) as the parameter \( \varepsilon_0 \to 0 \). This means that for any fixed wavenumber \( k > 0 \) there exists \( \varepsilon_0' > 0 \) such that \( k > 0 \) is not an eigenvalue of the homogeneous ITP in \( D_{z_*} \) for any \( \varepsilon_0 \leq \varepsilon_0' \). Furthermore, we also conclude from [1] that the ITP is well posed with the \( L^2 \)-estimate

\[
\|w_1\|_{L^2(D_{z_*})} + \|w_2\|_{L^2(D_{z_*})} \leq C\|(f_2, f_1)\|_{L^2} \quad \text{on } \partial D_{z_*},
\]

assuming \( f_p, \ p = 1, 2, \) satisfy the condition: For \( f_1 \in H^\frac{1}{2}(\partial D_{z_*}) \) and \( f_2 \in H^{-\frac{1}{2}}(\partial D_{z_*}) \), there exists a function \( h_f \in H^1(D_{z_*}) \) satisfying \( h_f = f_1 \) on \( \partial D_{z_*} \) and \( \partial_n h_f = f_2 \) on \( \partial D_{z_*} \). Here, \( H^1(D_{z_*}) := \{ h \in H^1(D_{z_*}) : \Delta h \in L^2(D_{z_*}) \} \) and

\[
\inf_{h_f \text{satisfies condition}} \{ \|h_f\|_{H^1(D_{z_*})} + \|\Delta h_f\|_{L^2(D_{z_*})} \}.
\]

We now define \( w_{1,j} := u_j^{(1)} \) and \( w_{2,j} := u_j^{(2)} \) in \( D_{z_*} \). It is easily found that \((w_{1,j}, w_{2,j})\) is the unique solution of the ITP with the boundary data

\[
\begin{aligned}
f_1 &= f_{1,j} = u_j^{(1)} - u_j^{(2)} \quad \text{on } \partial D_{z_*}, \\
f_2 &= f_{2,j} = \frac{\partial u_j^{(1)}}{\partial \nu} - \frac{\partial u_j^{(2)}}{\partial \nu} \quad \text{on } \partial D_{z_*}.
\end{aligned}
\]

Next we will show that the functions \((f_{1,j}, f_{2,j})\) are uniformly bounded for all \( j \in \mathbb{N} \) with the norm defined in (27). It is first observed from (24) that

\[
\begin{aligned}
f_{1,j} &= 0 \quad \text{and } \quad f_{2,j} = 0, \quad \text{on } \partial D_{z_*} \cap O_{e_0}(z_*).
\end{aligned}
\]

Then we can look for the following function

\[
h_j(x) := (1 - \chi(x))(u_j^{(1)}(x) - u_j^{(2)}(x)) \quad \text{in } D_{z_*},
\]

where \( \chi(x) \in C^2(\mathbb{R}^3) \) is a cut-off function supported in \( O_{e_0}(z_*) \) with \( \varepsilon_1 < \varepsilon_0 \) and \( \chi(x) = 1 \) in \( O_{e_0}(z_*) \) with \( \varepsilon_2 < \varepsilon_1 \). Clearly, it holds by (30) that \( h_j = f_{1,j} \) and \( \partial_n h_j = f_{2,j} \) on \( \partial D_{z_*} \). Moreover, for any fixed \( j \in \mathbb{N} \), we have \( h_j \in H^1(D_{z_*}) \) since \( z_j \in D_0^{(1)} \cap D_0^{(2)} \). Therefore, \((f_{1,j}, f_{2,j})\) satisfies the above condition between (26) and (27). Using the definition of \((f_{2,j}, f_{1,j})\) yields that

\[
\begin{aligned}
\|(f_{2,j}, f_{1,j})\|_{L^2} &\leq (\|h_j\|_{H^1(D_{z_*})} + \|\Delta h_j\|_{L^2(D_{z_*})}) \\
&\leq C(\varepsilon_2) \sum_{p=1,2} (\|u_j^{(p)}\|_{H^1(D_{z_*} \setminus \partial O_{e_0}(z_*))} + \|\Delta u_j^{(p)}\|_{L^2(D_{z_*} \setminus \partial O_{e_0}(z_*))}) \\
&\leq C(\varepsilon_2) k^2 \max \{1, \sup_{x \in \Omega} n_1(x)\} \|u_j^{(p)}\|_{H^1(D_{z_*} \setminus \partial O_{e_0}(z_*))},
\end{aligned}
\]
where we have used the fact that $\chi = 1$ in $\mathcal{O}_{z_2}(z_*)$ and $u_j^{(p)}$ solves the Helmholtz equation in $D_{z_*} \setminus \overline{\mathcal{O}_{z_2}(z_*)}$ in the weak sense. Noticing that

$$\nabla_y \Phi(x, y) = \left( \frac{1}{|x - y|} - ik \right) \frac{e^{ik|x-y|}}{4\pi|x-y|} \nu(y)$$

it then follows from (22) that $\|u_j^i\|_{L^q(\mathcal{D}(p))} \leq C$ are uniformly bounded for all $j \in \mathbb{N}$ and $p = 1, 2$, with one fixed $1 < q < \frac{3}{2}$. This, together with Theorem 2.2, shows that $\|u_j^{(p)}\|_{L^q(\mathcal{D}(p))} \leq C$ is uniformly bounded for all $j \in \mathbb{N}$ and $p = 1, 2$, whence

$$\|u_j^{(p)}\|_{H^1(D_{z_*} \setminus \overline{\mathcal{O}_{z_2}(z_*))}} \leq C\|u_j^{(p)}\|_{W^{2,q}(D_{z_*} \setminus \overline{\mathcal{O}_{z_2}(z_*))}} \leq C\|u_j^{(p)}\|_{L^q(\mathcal{D}(p))} + \|P_{z_j}^{(p)}\|_{W^{2,q}(D_{z_*} \setminus \overline{\mathcal{O}_{z_2}(z_*))}} \leq C$$

(33)

follows from the continuous embedding of $W^{2,q}$ into $H^1$, equation (10) and the boundedness of the operator $T$ from $L^q$ into $W^{2,q}$ for $\frac{6}{5} < q < \frac{3}{2}$. Combining (26), (32) and (33), we now arrive at

$$\|w_{1,j}\|_{L^2(D_{z_*})} + \|w_{2,j}\|_{L^2(D_{z_*})} \leq \|(f_{2,j}, f_{1,j})\|_{L^2} \leq C,$$

where $C > 0$ is independent of $j \in \mathbb{N}$.

Further, it can be easily checked that

$$\int_{D_{z_*}} |\nabla_x \Phi(\cdot, z_j) \cdot \nu(z^*)|^2 dx \geq C \frac{1}{j^2} \int_{D_{z_*}} \frac{1}{|x - z_j|^6} dx = O(j),$$

which strikes with

$$+\infty \leftarrow \|u_j^{(p)}\|_{L^2(D_{z_*})} - \|w_{2,j} - u_j^{i}\|_{L^2(D_{z_*})} \leq \|w_{2,j}\|_{L^2(D_{z_*})} \leq C$$

as $j \to \infty$. Therefore, we have proved $D_0^{(1)} = D_0^{(2)}$.

**Step 2.** Let $D_0 := D_0^{(1)} = D_0^{(2)}$. We next show that $n_1(x) = n_2(x)$ when the refractive index $n_j(x), j = 1, 2$ satisfies the condition (21). Before going further, we first introduce some useful notations. For $j = 1, 2$, let $X_j := \{\Omega_{j,m} : m = 1, 2, \cdots, N_j\}$ denote the sets consisting of a finite number of subdomains of $D_1$ with respect to the refractive index $n_j$. For convenience, we may suppose that $\partial \Omega_{1,1} \cap \partial D_0 \neq \emptyset$, which means that $\partial \Omega_{1,1}$ and $\partial D_0$ share a non-empty open subset in the two-dimensional manifold of $\mathbb{R}^3$. Thus, we can choose some point $z_* \in \partial \Omega_{1,1} \cap \partial D_0$ and define $D_{\delta_0} := B_{\delta_0}(z_*) \cap \Omega_{1,1}$ with sufficiently small $\delta_0 > 0$, where $B_{\delta_0}(z_*)$ is a ball centered at $z_*$ with the radius $\delta_0 > 0$. Here, we remark that $z_*$ and $\delta_0$ can be chosen such that $n_1$ and $n_2$ are all fixed constants in $D_{\delta_0}$, due to the fact that both $n_1$ and $n_2$ are two piecewise constant functions. In what follows, we will first in this step prove $n_1 = n_2$ in $D_{\delta_0}$. To this end, we define $z_j := z_* - (\varepsilon/j)\nu(z_*)$, $j = 1, 2, 3, \cdots$ with $\varepsilon > 0$ small enough such that $z_j \in D_0$. Now let $u_j^{(1)}$ and $u_j^{(2)}$ be the solutions to the Problem (2) corresponding to the incident fields $u_j^i$ defined by (22). It is easily found that $(v_1, v_2) := (u_j^{(1)}, u_j^{(2)})$ solves the following interior transmission problem

$$\begin{cases}
\Delta v_1 + k^2 n_1 v_1 = 0 & \text{in } D_{\delta_0}, \\
\Delta v_2 + k^2 n_2 v_2 = 0 & \text{in } D_{\delta_0}, \\
v_1 - v_2 = f_1 & \text{on } \partial D_{\delta_0}, \\
\frac{\partial v_1}{\partial \nu} - \frac{\partial v_2}{\partial \nu} = f_2 & \text{on } \partial D_{\delta_0},
\end{cases}$$

(36)
where \( f_1 := f_{1,j} \) and \( f_2 := f_{2,j} \) are defined as in (28)-(29). Arguments similar to those between (25) and (35) lead to a contradiction. Then we obtain \( n_1 = n_2 \) in \( D_{\delta_0} \).

We continue to show that \( n_1 = n_2 \) in \( \Omega_{1,1} \). If \( \Omega_{1,1} \subseteq \Omega_{2,m_0} \) for some \( m_0 \leq N_2 \), we obviously have \( n_1 = n_2 \) in \( \Omega_{1,1} \). Otherwise, we will prove that

\[
\Omega_{1,1} = \bigcup_{1 \leq j \leq M} \Omega_{2,m_j} \quad \text{for} \quad M \leq N_2,
\]

that is, \( \Omega_{1,1} \) is just composed of a finite number of elements in \( X_2 \). We now prove (37) by contradiction. Assuming that (37) does not hold true, then there exists some subdomain \( \Omega_{2,m_{j_0}} \) in \( X_2 \) satisfying that \( \Omega_{1,1} \cap \Omega_{2,m_{j_0}} \neq \emptyset \), \( \Omega_{1,1} \not\subseteq \Omega_{2,m_{j_0}} \) and \( \Omega_{2,m_{j_0}} \not\subseteq \Omega_{1,1} \), see Figure 1.

![Figure 1. The inhomogeneous cavity](image-url)

So we can choose a connected part \( A_1 \subset (\Omega_{1,1} \cap \Omega_{2,m_{j_0}}) \) and another connected part \( A_2 \subset (\Omega_{2,m_{j_0}} \setminus \Omega_{1,1}) \) such that \( \partial A_2 \cap (\partial \Omega_{1,1} \cap \partial \Omega_{1,2}) \neq \emptyset \). This means that \( \partial A_2, \partial \Omega_{1,1} \) and \( \partial \Omega_{1,2} \) share a non-empty open subset in the two-dimensional manifold of \( \mathbb{R}^2 \), denoted by \( \Sigma \). For convenience, we can also assume that \( A_2 \subseteq \Omega_{1,2} \) in \( X_1 \) due to the fact that \( A_2 \not\subseteq \Omega_{1,1} \). Noticing that \( A_1 \subseteq \Omega_{1,1} \), it then follows from the arguments in the proof of the result that \( n_1 = n_2 \) in \( D_{\delta_0} \) (see (36)) in combination with the unique continuation principle that \( n_1 = n_2 \) in \( A_1 \). Further, from the assumptions on \( n_{j_0,j}, j = 1, 2 \) (see (21)), we may suppose that \( n_1 = c_{1,1} \) in \( A_1 \) with some constant \( c_{1,1} \), and consequently one has \( n_2 = n_1 = c_{1,1} \) in \( A_1 \). This further means \( n_2 = c_{1,1} \) in \( A_2 \) since \( n_2 \) is a fixed constant in \( \Omega_{2,m_{j_0}} \). On the other hand, recalling that \( A_2 \subseteq \Omega_{1,2} \) we can also have \( n_1 = c_{1,2} \) in \( A_2 \) with some constant \( c_{1,2} \). Next we will prove \( n_1 = n_2 \) in \( A_2 \).

Assume that \( n_1 \neq n_2 \) in \( A_2 \). We choose a point \( z_s \in \Sigma \) satisfying that \( B^s(z_s) \subset (\overline{\Omega_{1,1}} \cup \overline{\Omega_{1,2}}) \), where \( B^s(z_s) \) is a small ball centered at \( z_s \). Now we define \( z_j = z_s - (\delta/j) u(z_s) \) with a sufficiently small \( \delta > 0 \) such that \( z_j \in \Omega_{1,1} \), and consider the following boundary value problem

\[
\begin{aligned}
\Delta v_1 + k^2 n_1 v_1 &= 0 \quad \text{in} \quad D^{(1)} \setminus \overline{\Omega_{1,1}}, \\
\Delta v_1 + k^2 c_{1,1} v_1 &= 0 \quad \text{in} \quad \Omega_{1,1} \setminus \{z_j\}, \\
v_1 &= 0 \quad \text{on} \quad \partial D^{(1)}, \\
v_1 - v_1^j &= u^j \quad \text{on} \quad \partial \Omega_{1,1}, \\
\partial_n v_1 - \partial_n v_1^j &= \partial_n u^j \quad \text{on} \quad \partial \Omega_{1,1},
\end{aligned}
\]

(38)
where \( u_j^x, j \in \mathbb{N}, \) are defined by

\[
(39) \quad u_j^x(x) := \nabla_x \Phi(x, z_j; c_{1,1}) \cdot \nu(z_*) = \nabla_x \left( \frac{1}{4\pi} e^{\frac{i k c_{1,1}}{2}|x - z_j|} \right) \cdot \nu(z_*) .
\]

In (39), \( \Phi(\cdot, z_j; c_{1,1}) \) satisfies the Helmholtz equation \( \Delta \Phi(\cdot, z_j; c_{1,1}) + k^2 c_{1,1} \Phi(\cdot, z_j; c_{1,1}) = -\delta_{z_j} \) in \( \mathbb{R}^3 \), and then \( u_j^x \in L^q_{\text{loc}}(\mathbb{R}^3) \) for any fixed \( 1 < q < \frac{3}{2} \) and all \( j \in \mathbb{N} \). Moreover, we introduce another boundary value problem related to \( n \) \( \Omega \) and (35), we then derive that \( \phi(x) \in L^2(I) \). Then the assertion (37) holds true.

Step 3. We shall next show that \( n_1 = c_{1,1} \) in \( \Omega_{1,1} \). By (37), it is easily observed that one can choose a subdomain, denoted by \( \Omega_{2,m_1} \), satisfying that \( \Omega_{2,m_1} \subset \Omega_{1,1} \) and \( \partial \Omega_{2,m_1} \cap \partial D_0 \neq \emptyset \), that is, \( \partial \Omega_{2,m_1} \) and \( \partial D_0 \) share a non-empty open subset in the two-dimensional manifold of \( \mathbb{R}^3 \). Using the similar arguments between (25) and (35), we then derive that \( n_2 = c_{1,1} \) in \( \Omega_{2,m_1} \). If \( \Omega_{1,1} = \Omega_{2,m_1} \), the proof is ended. Otherwise, there must exist a subdomain, denoted by \( \Omega_{2,m_2} \), satisfying that \( \Omega_{2,m_2} \subset \Omega_{1,1} \) and \( \partial \Omega_{2,m_2} \cap \partial \Omega_{2,m_1} \neq \emptyset \). Again following the similar statements between (25) and (35), we deduce that \( n_2 = c_{1,1} \) in \( \Omega_{2,m_2} \). If \( \Omega_{1,1} = \Omega_{2,m_1} \cup \Omega_{2,m_2} \), the proof is ended. Otherwise, there must exist a subdomain, denoted by \( \Omega_{2,m_3} \), satisfying that \( \Omega_{2,m_3} \subset \Omega_{1,1} \) and \( \partial \Omega_{2,m_3} \cap \partial \Omega_{2,m_1} \neq \emptyset \) or \( \partial \Omega_{2,m_3} \cap \partial \Omega_{2,m_2} \neq \emptyset \) (see...
By the above analysis between (25) and (35) once again, we have that $n_2 = c_{1,1}$ in $\Omega_{2,m_3}$. Therefore, we can obtain in turn that $n_2 = c_{1,1}$ in $\Omega_{1,1}$.

**Step 4.** Finally, the assertion $n_2 = n_1$ in $\Omega_{1,j}$, $2 < j < N_1$ can be derived by using the similar arguments as **Step 3**. This completes the proof of the theorem. $\square$

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