Diffraction in time in terms of Wigner distributions and tomographic probabilities

Vladimir Man’ko, Marcos Moshinsky, Anju Sharma

\[ a \] Instituto de Ciencias Nucleares-UNAM.
Apdo. Postal 70-543, 04510 México, D. F. México

\[ b \] Instituto de Física-UNAM.
Apartado Postal 20-364, 01000 México, D. F. México

\[ * \text{Permanent address: P. N. Lebedev Physical Institute, Leninskiy Prospekt 53, 117924}
Moscow, Russian Federation, e-mail: manko@na.infn.it manko@sci.lebedev.ru
\]

\[ \dagger \text{Member of El Colegio Nacional} \]
Abstract

Long ago appeared a discussion in quantum mechanics of the problem of opening a completely absorbing shutter on which were impinging a stream of particles of definite velocity. The solution of the problem was obtained in a form entirely analogous to the optical one of diffraction by a straight edge. The argument of the Fresnel integrals was though time dependent and thus the first part in the title of this article. In section 1 we briefly review the original formulation of the problem of diffraction in time. In section 2 and 3 we reformulate respectively this problem in Wigner distributions and tomographical probabilities. In the former case the probability in phase space is very simple but, as it takes positive and negative values, the interpretation is ambiguous, but it gives a classical limit that agrees entirely with our intuition. In the latter case we can start with our initial conditions in a given reference frame but obtain our final solution in an arbitrary frame of reference.
1 Introduction

Long ago [1] one of us (M.M.) discussed in quantum mechanics the problem of opening at time $t = 0$ a completely absorbing shutter situated at $x = 0$, on which were impinging a stream of particles of definite velocity. In units in which $\hbar$ and the mass $m$ of the particles are unity, the problem reduces to finding a wave function that satisfies the free one dimensional time dependent Schrödinger equation i.e.,

$$i \frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi(x,t)}{\partial x^2},$$

with the initial condition

$$\psi(x,0) = \exp(ikx)\theta(-x),$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x < 0
\end{cases},$$

The solution of this problem was given in reference 1 and later Nussensveig [2] gave it the name $M(x,k,t)$ and it can be expressed as [1,2,3]

$$M(x,k,t) = \frac{1}{2} \exp \left[ i(kx - \frac{1}{2}k^2t) \right] \text{erfc}(e^{-i\pi/4}w)$$

$$= e^{-i\pi/4} \exp \left[ i(kx - \frac{1}{2}k^2t) \right] \frac{1}{\sqrt{2}} \left\{ \left[ \frac{1}{2} - C(w) \right] + i \left[ \frac{1}{2} - S(w) \right] \right\},$$

where

$$w = \frac{x - kt}{\sqrt{2}t},$$

and the error integral is

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-y^2}dy,$$

while the Fresnel integrals are defined by

$$C(w) = \sqrt{\frac{2}{\pi}} \int_0^w \cos y^2 dy, S(w) = \sqrt{\frac{2}{\pi}} \int_0^w \sin y^2 dy.$$
Although we have assumed $k$ real, as in the units we use it is the velocity or momentum of the impinging particles, all the above expressions remains valid for complex $k$ so long as $\text{Im} k < 0$. In that case we have the alternative representation \[2, 3\]

$$M(x, k, t) = i \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[p(i(\kappa x - \frac{1}{2}\kappa^2 t))] \frac{d\kappa}{\kappa - k},$$

(8)

which follows from the fact that both sides are solutions of (1) satisfying the initial condition (2). The Green function for the one dimensional free particle Schroedinger equation has the form

$$U(x - x', t) = \frac{\exp[i(x - x')^2/2t]}{\sqrt{2\pi it}},$$

(9)

as it satisfies Eq. (1) for any $t > 0$, but when $t = 0$ it becomes the delta function $\delta(x' - x)$. As our initial condition is (2) it is clear[1,2] that our function $M(x, k, t)$ can also be written as

$$M(x, t) = \int_{-\infty}^{0} U(x - x', t) \exp(ikx') dx'.$$

(10)

The expression

$$|M(x, k, t)|^2,$$

(11)

gives the probability density of finding the particle at the point $x$ at time $t$ when initially it was on the left side of the shutter i.e., with $x < 0$ and had a momentum $k$. From (4) we see that

$$|M(x, k, t)|^2 = \frac{1}{2} \left\{ \left[ \frac{1}{2} - C(w) \right]^2 + \left[ \frac{1}{2} - S(w) \right]^2 \right\},$$

(12)

and it is identical to the expression[4] for the intensity of light in the Fresnel diffraction by a straight edge. The variable $w$ has though a very different meaning from the optical problem as it is now a function of time given by (5). Thus the original paper [1] was given the name “Diffraction in time”.

All what we said above has been well known for a very long time, and had many applications among which we wish to mention those related to the time-energy uncertainty relations [5] and decay problems [6].

The reason that we return to this subject is that now we wish to see its behavior when formulated in terms of Wigner distributions functions [7] and also in relation with the tomographic probability developed recently by one of us (V.M.) and his collaborators [8].
2 Diffraction in time in Wigner distributions space

Normally quantum mechanics is discussed in configuration space or, in some cases, in momentum space, but not in both together. Wigner [7] found that this limitation interfered with the application of quantum mechanics to the statistical physics where the description is usually given in phase space. Thus he introduced his concept of Wigner distributions which allow us to discuss some features of quantum mechanics in phase space.

Our objective will be to formulate the diffraction in time problem, discussed in the previous section, in terms of Wigner distribution functions. In this way we can visualize the phenomena in phase space and more easily determine its classical limit, and compare it with our intuitive understanding of the behavior of a beam of particles of definite momentum impinging on a shutter, when the latter is opened.

In units in which $\hbar$ and the mass $m$ of the particle are unity, and where the configuration space wave function is denoted by $\psi(x)$, and the momentum by $p$, the Wigner distribution function is defined as [7]

$$W(x,p) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \psi^*(x+y)\psi(x-y) \exp(2ipy)dy,$$

which has the obvious property that

$$\int_{-\infty}^{\infty} W(x,p)dp = |\psi(x)|^2,$$

where the right hand side is the probability density at the point $x$, while an integration with respect $x$ gives us the usual probability density [7] at the momentum value $p$.

If we now wish to discuss the diffraction in time problem in terms of Wigner distributions we have to replace in (1) $\psi(x)$ by $M(x,k,t)$ of (4).

While for our analysis $k$ is real, we shall assume for the moment that $k$ is complex with a small negative imaginary part. In this way we can use the expression (8) for $M(x,k,t)$ and substituting in (1) we get

$$W(x,p; k, t) = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp \left\{ -i[\kappa(x+y) - \frac{1}{2}\kappa^2t] \right\} \exp \left\{ i[\kappa'(x-y) - \frac{1}{2}\kappa'^2t] \right\}}{(\kappa - k^*) (\kappa' - k)}.$$
\[ \times e^{i2py} \, dr\,dr'\,dy, \]  

where we now added the momentum \( k \) and time \( t \) to our Wigner function on the left hand side, as these parameters also appear in the \( M(x, k, t) \). We also indicate by \( k^* \) the complex conjugate of \( k \).

The evaluation of the triple integral (3) is done in Appendix A and it leads to the simple result

\[ W(x, p; k, t) = \frac{1}{2\pi(k - p)} \sin \left\{ 2(pt - x)(k - p) \right\} \theta(pt - x), \]  

where \( \theta \) is the step function defined in (3). Because of the presence of the sine function in (4) we see that the Wigner distribution for the diffraction in time problem oscillates between positive and negative values, where the physical significance of the latter is not clear. On the other hand, the presence of \( \theta \) indicates that the probability density in phase space vanishes when \( x > pt \). As in our units \( \hbar = m = 1 \), the momentum \( p \) is the same as the velocity, and this result is intuitively expected as the particles in our beam with momentum \( p \) could not yet have reached the point \( x \).

What is particularly interesting to us is the classical limit \( W(x, p; k, t) \) which is achieved when the Planck constant \( \hbar \to 0 \). We have then to abandon units in which \( \hbar \) and \( m \) were taken as 1 and instead use standard cgs ones. The modifications in the form of Eq.(3) are trivial and the resulting distribution function now has the form

\[ W(x, p; k, t) = \frac{1}{2} \frac{\sin \left[ g(k - p) \right]}{\pi(k - p)} \theta \left( \frac{pt}{m} - x \right), \]  

where

\[ g \equiv \frac{2}{\hbar} \left( \frac{pt}{m} - x \right). \]  

If we take the limit \( \hbar \to 0 \) then \( g \to +\infty \) as the step function takes the value 1 only if \( (pt/m) - x > 0 \). We can then use one of the definitions of the \( \delta \) function [9]

\[ \delta(k - p) = \lim_{g \to \infty} \frac{\sin [g(k - p)]}{\pi[k - p]}, \]  

to write the classical limit of our distribution function as

\[ W_{cl}(x, p; k, t) = \frac{1}{2} \delta(k - p) \theta \left( \frac{kt}{m} - x \right), \]  

where

\[ \theta(z) = \begin{cases} 
1 & \text{if } z > 0 \\
0 & \text{if } z < 0 
\end{cases} \]
where we used the presence of the $\delta(k - p)$ in (8) to replace in the step function the $p$ by $k$.

We now see that our classical limit is what we expect as the only value possible for the momentum of the particle is $p = k$ and besides this value is taken only when $x < (kt/m)$, as for $x > (kt/m)$ the particles would not have yet arrived at the point $x$. Thus the classical limit of the Wigner distribution function for our diffraction in time problem confirms our intuition.

### 3 Diffraction in time in terms of the tomographic probabilities

In ordinary quantum mechanics the essential concept is the wave function which in configuration space is denoted by $\psi(x)$. From this concept one derives the probability density $|\psi(x)|^2$ of finding the particle at point $x$ and also, through appropriate transforms of $\psi(x)$, the probabilities for given values of any other observables.

Recently a change of emphasis has been proposed in which the central concept is the probability itself, but defining it in a tomographic way [8, 10, 11]. This allows us to analyze thru a single concept the probability either in configuration or momentum space as well as for variables that are linear combinations of both. The tomographic probability density [8, 11] was given in terms of the Wigner distribution through the transform

$$w(X, \mu, \nu) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, p)e^{-iz(X-x\mu-p\nu)}dzdxdp,$$

where $X$ is the position considered in an ensemble of references frames [8, 11] which are rotated and scaled with respect to the initial ones through the parameters $\mu, \nu$. As an example we have that when $\mu = 1, \nu = 0$ the $X$ corresponds to the normal position coordinate, but when $\mu = 0, \nu = 1$, it is related with the momentum observable.

In (11) the $W(x, p)$ is the Wigner function defined in (11) and substituting it in (11) the tomographic probability density $w(X, \mu, \nu)$ is given in term of the configuration wave function $\psi(x)$ by

$$w(X, \mu, \nu) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x-y)\psi^*(x+y)e^{i2\mu y}e^{-iz(X-x\mu-p\nu)}dzdxdpdy$$

(2)
The integration with respect to \( p \) gives us the expression
\[
\int_{-\infty}^{\infty} dpe^{ip(2y + z\nu)} = \pi\delta\left(y + \frac{z\nu}{2}\right),
\]  
and substituting it in (2), and carrying out the integration with respect to \( y \), we obtain
\[
w(X, \mu, \nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x + \frac{z\nu}{2})\psi^*(x - \frac{z\nu}{2}) e^{-iz(x - \mu x)} dz dx
\]  
Introducing now the variables
\[
u = x + \frac{z\nu}{2}, \quad r = x - \frac{z\nu}{2},
\]  
we see the volume element \( dz dx \) in (4) becomes \( drdu/|\nu| \), so in terms of \( u, r, w(X, \mu, \nu) \) becomes
\[
w(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(u)\psi^*(r) \exp\left\{-i\frac{u-r}{\nu}\left[x - \mu \left(\frac{r + u}{2}\right)\right]\right\} dr du
\]  
where
\[
\chi(X, \mu, \nu) = \int_{-\infty}^{\infty} \psi(u) \exp\left[i\left(\frac{\mu}{2\nu}u^2 - u\frac{X}{\nu}\right)\right] du.
\]  
Thus, contrary to the Wigner distribution function, the tomographic probability density is always positive definite.

We now turn to the problem of diffraction in time which means replacing \( u \) by \( x \) in (3) and then \( \psi(x) \) by \( M(x, k, t) \) given in terms of its expression containing the Green function of the free particle motion. The expression \( \chi(x, \mu, \nu) \) takes then the form
\[
\chi(X, \mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\exp[i(x - x')^2/2t]}{\sqrt{2\pi(it)}} e^{ikx'} e^{-iX x'} e^{i\frac{ux^2}{2\nu}} dx dx'
\]  
This integral is evaluated in a straightforward but laborious way in Appendix B, where its value is given. As we are only interested in its absolute
value squared multiplied by \((2\pi|\nu|)^{-1}\) which, from (8) gives us the tomographical probability density, we see that it becomes

\[ w(X,\mu,\nu) = \frac{1}{2|\mu|}\left\{\left[\frac{1}{2} + C(\rho)\right]^2 + \left[\frac{1}{2} + S(\rho)\right]^2\right\}, \tag{9} \]

where

\[ \rho = \frac{k(\mu t + \nu) - X}{\sqrt{2\mu(\mu t + \nu)}}, \tag{10} \]

and \(C, S\) are the Fresnel integrals defined in Eq.(7).

We proceed now to discuss the meaning of the tomographical probability density given in (8). We mentioned above that \(\mu, \nu\) represent a rotation and scaling of an ensemble of reference frames in phase space with respect to the original one. Thus we can express them as

\[ \mu = e^\tau \cos \theta, \quad \nu = e^{-\tau} \sin \theta, \tag{11} \]

with \(\tau, \theta\) in the intervals \(-\infty \leq \tau \leq \infty, 0 \leq \theta \leq 2\pi\). These expressions of \(\mu, \nu\) imply that our new coordinate and momenta, which we designate by capital \(X, P\), are given in terms of the original ones, which we denote by lower case letters \(x, p\), through the relation

\[ \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} e^\tau \cos \theta & e^{-\tau} \sin \theta \\ -e^\tau \sin \theta & e^{-\tau} \cos \theta \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{12} \]

which is a linear canonical transformation as the determinant of the matrix is 1. The \(w(X, \mu, \nu)\) of (9), with \(\rho\) given by (11), gives then the probability density for the diffraction in time problem in the new configuration coordinate \(X\) defined in (12).

If we want to return to our original configuration space we see from (12) that we must take there \(\tau = \theta = 0\) which implies \(X = x\) and \(\mu = 1, \nu = 0\).

In that case \(\rho\) of (11) becomes

\[ \rho = \frac{kt - x}{\sqrt{2t}} = -w, \tag{13} \]

where \(w\) was defined in (5). As the Fresnel integrals are odd functions of the argument we have from (13) that

\[ C(\rho) = -C(w), \quad S(\rho) = -S(w), \tag{14} \]
and thus the particular tomographic density \( w(x, 1, 0) \) becomes

\[
w(x, 1, 0) = \frac{1}{2} \left\{ \left[ \frac{1}{2} - C(w) \right]^2 + \left[ \frac{1}{2} - S(w) \right]^2 \right\},
\]

which is identical to the expression (12), as we should expect.

Thus we see that the analysis of diffraction in time phenomena in terms of the tomographic probabilities, allows us to study the phenomena in a wide ensemble of reference frames in phase space as indicated in Eqs. (9–11).

This ensemble includes of course the original phase space \((x,p)\) in which the result is given by (14) agreeing exactly with the initial analysis of the problem [1].

### 4 Conclusion

In the present paper the problem of diffraction in time was visualized from three different viewpoints. The first was the original one [1], in which both the initial conditions and the solution of the problem were analyzed in this same frame of reference. The solution (5) was given in terms of the Fresnel integrals, and using Cornu spiral we showed that the usual diffraction pattern appeared as function of time.

In the second approach we translated our solution to the Wigner distribution space. The final expression for the probability density in phase space turned out to be very simple but, unfortunately, it could take both positive and negative values, which made its interpretation ambiguous.

Fortunately it was possible to consider its classical limit by taking \( \hbar \to 0 \), and the resulting expression (8) agreed entirely with our intuitive view, i.e., the probability in phase space was only different from zero when \( p = k \) and \( x < (pt/m) \), where all the observables are in cgs units.

The third approach implied formulating our solution in terms of tomographic probabilities. The latter have been introduced recently [8,10,11] to allow us to express the solutions in any reference frame that is rotated and scaled with respect to original one. In effect it implies carrying out a canonical transformation on the original solution of the diffraction in time problem. Our tomographic probability solutions (9) is again expressed in terms of Fresnel integrals but of an argument quite different from the one appearing (3).

If the canonical transformations is the unit one i.e., \( X = x, P = p \), then
our tomographic probability reduces to the solution (4), providing us with a
check of the analysis developed in section 3.

We finally wish to indicate that the diffraction in time phenomena de-

rived theoretically in 1952 was, in a somewhat changed form [13], measured
experimentally in 1996. Possibly a similar fate, in a distant future, awaits
the reformulation of the phenomena presented in this paper.
Appendix A: Determination the Wigner function $W(x, p; k, t)$.

We start with the expression (3) for $W(x, p; k, t)$ and rewrite it as

$$W(x, p; k, t) = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \exp\left[2ipy\left(p - \frac{\kappa + \kappa'}{2}\right)\right] dy \right\}$$

$$\times \left\{ \frac{\exp[-i(\kappa x - \frac{1}{2}\kappa^2 t)] \exp[i(\kappa' x + \frac{1}{2}\kappa'^2 t)]}{(\kappa - k^*)} \right\} d\kappa d\kappa'. \quad (A.1)$$

The first integral obviously gives the $\delta$ function $2\pi\delta(p - \frac{\kappa + \kappa'}{2})$ and so introducing it in (A.1) and integrating with respect to $\kappa'$ we obtain

$$W(x, p; k, t) = -\frac{1}{2\pi^2} \exp[2ip(x - pt)] \int_{-\infty}^{\infty} \frac{\exp[-2i\kappa(x - pt)]}{(\kappa - k^*)(\kappa + k - 2p)} d\kappa. \quad (A.2)$$

As we have that

$$\frac{1}{(\kappa - k^*)(\kappa + k - 2p)} = \frac{1}{(k + k^* - 2p)} \left[ \frac{1}{(\kappa - k^*)} - \frac{1}{(\kappa + k - 2p)} \right], \quad (A.3)$$

we see that by introducing it in (A.2) we get

$$W(x, p; k, t) = -\frac{\exp[2ip(x - pt)]}{(2\pi)^2(k + k^* - 2p)}$$

$$\times \left\{ \int_{-\infty}^{\infty} d\kappa \frac{\exp[-2i\kappa(x - pt)]}{(\kappa - k^*)} - \int_{-\infty}^{\infty} d\kappa \frac{\exp[-2i\kappa(x - pt)]}{(\kappa + k - 2p)} \right\}. \quad (A.4)$$

We now note, as indicated in the text after Eq(3), that we start by assuming that $k$ has as a small negative imaginary part so that

$$k \rightarrow k - i\epsilon; \quad k^* \rightarrow k + i\epsilon; \quad -k + 2p \rightarrow -k + 2p + i\epsilon \quad (A.5)$$

Thus the singularity in the integrals in (A.4) is in the upper half of the $\kappa$ plane.

We can close the contour in (A.4) by a large circle in the upper half of the complex $\kappa$ plane if $x - pt < 0$ thus getting the residues of the integrals at the points $k + i\epsilon, -k + 2p + i\epsilon$. On the other hand if $x - pt > 0$ we have to close the contour by a large circle in the lower half plane and, as the function
is analytic inside the contour, the integral vanishes. Then, passing to the limit when \( \epsilon \to 0 \), as required for our problem where \( k \) is real, we get, after carrying some of the multiplications, that

\[
W(x, p; k, t) = \frac{\theta(pt - x)}{4\pi i(k - p)} \left\{ \exp[2i(k - p)(x - pt)] - \exp[-2i(k - p)(x - pt)] \right\}.
\]

(A.6)

where \( \theta \) is the step function (3). As the curly bracket derived by \( 2i \) is a sine function we then obtain the expression (4).
Appendix B: Determination of the tomographic probability $w(X, \mu, \nu)$

The tomographic probability is proportional to the absolute square of $\chi(X, \mu, \nu)$ where the latter is given by (B.1) and we rewrite it in the form

$$w(X, \mu, \nu) = \int_{-\infty}^{0} \left\{ \frac{\exp[ikx' + i(x'^2/2t)]}{\sqrt{2\pi it}} \int_{-\infty}^{\infty} \exp[i(ax^2 - bx)]dx \right\}dx', \quad (B.1)$$

where

$$a \equiv \frac{\mu}{2\nu} + \frac{1}{2t}, \quad b \equiv \frac{X}{\nu} + \frac{x'}{t}, \quad (B.2)$$

We can rewrite the expressions in the last round bracket in (B.1) getting

$$ax^2 - bx = \left( \sqrt{ax} - \frac{b}{2\sqrt{a}} \right)^2 - \left( \frac{b^2}{4a} \right). \quad (B.3)$$

As $b^2/4a$ depends on $x'$ but not on $x$, we first evaluate the integral

$$\int_{-\infty}^{\infty} \exp \left[ i\left( \sqrt{ax} - \frac{b}{2\sqrt{a}} \right)^2 \right]dx = \sqrt{\frac{a}{\pi}} e^{i\pi/4}, \quad (B.4)$$

to obtain

$$\chi(X, \mu, \nu) = \frac{e^{i\pi/4}}{\sqrt{i\frac{\mu}{\nu} + 1}} \int_{-\infty}^{0} \exp[ikx' + i(x'^2/2t)] \exp \left[ \frac{-i(x'^2 + X^2)}{2(\frac{\mu}{\nu} + 1)} \right]dx'$$

$$= \frac{e^{i\pi/4}}{\sqrt{i\frac{\mu}{\nu} + 1}} \int_{-\infty}^{0} \exp[i(\alpha x'^2 + \beta x')]dx', \quad (B.5)$$

where

$$\alpha = \frac{(\mu/\nu)}{2(\frac{\mu}{\nu} + 1)}, \quad \beta = \frac{k(\frac{\mu}{\nu} + 1) - \frac{X}{\nu}}{\left(\frac{\mu}{\nu} + 1\right)}.$$

Using again the relation (B.3) we get

$$\alpha x'^2 + \beta x' = \left( \sqrt{\alpha} x' + \frac{\beta}{2\sqrt{\alpha}} \right)^2 - \frac{\beta^2}{4\alpha}, \quad (B.6)$$

and as $\beta^2/4\alpha$ is independent of $x'$ we need to consider first the integral

$$\int_{-\infty}^{0} e^{i(\sqrt{\alpha}x' + \frac{\beta}{2\sqrt{\alpha}})^2} dx' = \int_{-\infty}^{\frac{\beta}{2\sqrt{\alpha}}} e^{iy^2} \frac{dy}{\sqrt{\alpha}} = \int_{-\infty}^{0} e^{iy^2} \frac{dy}{\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha}} \int_{0}^{\frac{\beta}{2\sqrt{\alpha}}} (\cos y^2 + i \sin y^2) dy$$

14
\[ = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \frac{(1 + i)}{\sqrt{2}} + \frac{1}{\sqrt{\alpha}} \frac{\sqrt{\pi}}{\sqrt{2}} \left[ C \left( \frac{\beta}{2\sqrt{\alpha}} \right) + iS \left( \frac{\beta}{2\sqrt{\alpha}} \right) \right] \]  

(B.8)

where \( C, S \) are the Fresnel integrals of (1.7) and \( \alpha, \beta \) are given by (B.6).

Using (B.7) to introduce (B.8) in (B.5) we obtain

\[
\chi(X, \mu, \nu) = \frac{\sqrt{\pi} e^{i\pi/4}}{\sqrt{2(\mu/\nu) + 1}} \frac{1}{\alpha} \exp(-i\beta^2/4\alpha) \exp\left\{ -i(X^2/2\nu^2)\left( \frac{\mu}{\nu} + \frac{1}{t} \right)^{-1} \right\}
\times \left\{ \left[ \frac{1}{2} + C\left( \frac{\beta}{2\sqrt{\alpha}} \right) \right] + i \left[ \frac{1}{2} + S\left( \frac{\beta}{2\sqrt{\alpha}} \right) \right] \right\}.
\]  

(B.9)

Finally replacing \( \alpha, \beta \) by their values (B.6) we get.

\[
\chi(X, \mu, \nu) = \frac{\sqrt{\pi} e^{i\pi/4}}{\sqrt{(\mu/\nu)}} \exp \left\{ -i(X^2/2\nu^2)\left( \frac{\mu}{\nu} + \frac{1}{t} \right)^{-1} \right\}
\exp(-i\rho^2) \left\{ \left[ \frac{1}{2} + C(\rho) \right] + i \left[ \frac{1}{2} + S(\rho) \right] \right\}.
\]  

(B.10)

where

\[
\rho = \frac{k(\mu t + \nu) - X}{\sqrt{2\mu(\mu t + \nu)}}.
\]  

(B.11)

When taking the absolute square value of \( \chi(X, \mu, \nu) \) mainly the curly bracket remains and thus we get Eq.(8) whose properties are discussed in the main text.
References

[1] M. Moshinsky, Phys. Rev., 88, 625 (1952).

[2] H. M. Nussenzveig “Moshinsky Functions, Resonance and Tunneling” in Symmetries in Physics Ed. A. Frank and B. Wolf, (Springer Verlag, Berlin 1992) p. 294.

[3] M. Moshinsky, Phys. Rev., 84, 525 (1951).

[4] E. Jahnke and F. Emde, Tables of functions with formulas and curves Fourth Edition (Dover Publications, New York, 1945) pp. 34-37.

[5] M. Moshinsky, Am. J. Phys., 44, 1037 (1976).

[6] G. García Calderón, J. L. Mateos, M. Moshinsky, Phys. Rev. Lett., 74, 337 (1995); Ann. Phys., 249, 430 (1996).

[7] E. P. Wigner, Phys. Rev., 40, 749 (1932).

[8] S. Mancini, V. I. Man’ko, P. Tombesi, Phys. Lett. A, 213, 1 (1996).

[9] A. K. Ghatak and S. Loknathan, Quantum Mechanics: Theory and Applications, Third Edition, (Macmillan India Limited, 1994) pp. 601-603.

[10] Man’ko, V. I., and Mendes, R. V., “Non-commutative time-frequency tomography of analytic signals,” LANL physics/9712022 Data Analysis, Statistics, and Probability; IEEE Signal Processing (submitted 1998).

[11] V. I. Man’ko, L. Rosa, P. Vitale, Phys. Rev. A, 58, 3291 (1998); O. V. Man’ko, V. I. Man’ko, J. Russ. Laser Research (Plenum), 18, 407 (1997).

[12] M. Moshinsky and C. Quesne, “Oscillator systems” in 15th Solvay conference on Symmetry Properties in Nuclei (Gordon and Breach, New York 1974) pp. 239-242; J. Math. Phys. 12, 1772, (1971).

[13] P. Szriftigiser, D. Guéry-Odelin, M. Arndt, J. Dalibard, Phys. Rev. 77, 4 (1996).