Research Article

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The core inverse and constrained matrix approximation problem

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Abstract: In this article, we study the constrained matrix approximation problem in the Frobenius norm by using the core inverse:

\[ \|Mx - b\|_F = \min \{x \in \mathcal{R}(M) \}, \]

where \( M \in \mathbb{C}_n^m \). We get the unique solution to the problem, provide two Cramer’s rules for the unique solution and establish two new expressions for the core inverse.

Keywords: core inverse, Cramer’s rule, constrained matrix approximation problem

MSC 2010: 15A24, 15A29, 15A57

1 Introduction

Let \( M^*, \mathcal{R}(M) \) and \( \mathcal{N}(M) \) stand for the conjugate transpose, range space and null space of \( M \in \mathbb{C}_n^{m \times n} \), respectively. The symbol \( M(i \rightarrow b) \) denotes a matrix from \( M \) by replacing the \( i \)-th column of \( M \) by \( b \in \mathbb{C}^n \). The symbol \( e_i \) denotes the \( i \)-th column of \( L \) in which \( 1 \leq i \leq n \). The Moore-Penrose inverse of \( M \) is the unique matrix \( X \in \mathbb{C}_n^{m \times n} \) satisfying the relations: \( MXM = M \), \( XMX = X \), \( (MX)^* = MX \) and \( (XM)^* = XM \) and is denoted by \( X = M^* \) [1–3].

Let \( M \in \mathbb{C}_n^{m \times n} \) be singular. The smallest positive integer \( k \) for which \( \text{rk}(M^{k+1}) = \text{rk}(M^k) \) is called the index of \( M \) and is denoted by \( \text{Ind}(M) \). The index of a non-singular matrix is 0 and the index of a null matrix is 1. Furthermore,

\[ \mathbb{C}_n^m = \{ M | \text{Ind}(M) \leq 1, M \in \mathbb{C}_n^{m \times n} \}. \quad (1.1) \]

Let \( M \in \mathbb{C}_n^{m \times n} \) with \( \text{Ind}(M) = k \). A matrix \( X \) is the Drazin inverse of \( M \) if \( MXM^k = M^k \), \( XMX = X \) and \( MX = XM \). We write \( X = M^D \) for the Drazin inverse of \( M \). In particular, when \( M \in \mathbb{C}_n^m \), the matrix \( X \) is the group inverse of \( M \) and is denoted by \( X = M^g \) [1–3].

The core inverse of \( M \in \mathbb{C}_n^m \) is defined as the unique matrix \( X \in \mathbb{C}_n^{m \times n} \) satisfying the equations: \( MXM = M \), \( MX^2 = X \) and \( (MX)^* = MX \) and is denoted by \( X = M^\diamond \) [4,5]. It is noteworthy that the core inverse is a “least squares” inverse [6,7]. Moreover, it is proved that \( M^\diamond = M^g MM^D \) [4].

Recently, the relevant conclusions of the core inverse are very rich. In [7–10], generalizations of core inverse are introduced, for example, the core-EP inverse and the weak group inverse. In [11–15], their algebraic properties and calculating methods are studied. In [16,17], the studying of them is extended to
some new fields, for example, ring and operator. Moreover, those inverses are used to study partial orders in [4,5,10,18,19].

Consider the following equation:

\[ Mx = b. \]  (1.2)

Let \( M \in \mathbb{C}^{n \times n} \) with \( \text{Ind}(M) = k \) and \( b \in \mathcal{R}(M^k) \). Campbell and Meyer [20] show that \( x = M^n b \) is the unique solution of (1.2) with respect to \( x \in \mathcal{R}(M^k) \). Wei [21] gets the minimal \( P \)-norm solution of (1.2), where \( P \) is nonsingular, \( P^{-1}MP \) is the Jordan canonical form of \( M \) and \( \|x\|_p = \|P^{-1}x\|_2 \). Furthermore, let \( M \in \mathbb{C}^{m \times n} \). Wei [22] considered the unique solution of

\[ WMWx = b \quad \text{subject to} \quad x \in \mathcal{R}((WM)^k), \]

where \( W \in \mathbb{C}^{n \times m} \), \( k_1 = \text{Ind}(MW) \), \( k_2 = \text{Ind}(WM) \) and \( b \in \mathcal{R}((WM)^k) \). More results of (1.2) under some certain conditions can be found in [3,21,23–27].

It is well known that \( b \in \mathcal{R}(M) \) if and only if (1.2) is solvable. Let \( b \in \mathcal{R}(M) \) and the index of \( M \) is 1, then \( x = M^n b \) is the unique solution with \( x \in \mathcal{R}(M) \) [20]. It follows from \( M^n = M^n M^M \) that \( M^n b = M^n b \) [28]. Furthermore, the unique solution \( x = M^n b \) is given by Cramer’s rule [28, Theorem 3.3].

When \( b \notin \mathcal{R}(M) \), (1.2) is unsolvable, yet, it has least-squares solutions. Motivated by the aforementioned works, it is natural to consider the least-squares solutions of (1.2) under the certain condition \( x \in \mathcal{R}(M) \), i.e.,

\[ \|Mx - b\|_p = \min \quad \text{subject to} \quad x \in \mathcal{R}(M), \]  (1.3)

where \( M \in \mathbb{C}^{m \times n} \), \( \text{rk}(M) = r < n \) and \( b \in \mathbb{C}^n \).

2 Preliminaries

Lemma 2.1. [1] Let \( M \in \mathbb{C}^{n \times n} \) be idempotent. Then, \( M = P_{\mathcal{R}(M),\mathcal{N}(M)} \) with \( \mathcal{R}(M) \oplus \mathcal{N}(M) = \mathbb{C}^n \). In contrast, if \( F \oplus G = \mathbb{C}^n \), then there exists an idempotent \( P_{F,G} \) such that \( \mathcal{R}(P_{F,G}) = F \) and \( \mathcal{N}(P_{F,G}) = G \).

Furthermore, \( I_n - P_{F,G} = P_{G,F} \).

Lemma 2.2. [3] Let \( M \in \mathbb{C}^{m \times n} \). Then, \( \text{Ind}(M) = k \) if and only if

\[ \mathcal{R}(M^k) \oplus \mathcal{N}(M^k) = \mathbb{C}^n. \]  (2.1)

Lemma 2.3. [3] Let \( MXM = M \) and \( XMX = X \). Then,

\[ XM = P_{\mathcal{R}(X),\mathcal{N}(M)} \quad \text{and} \quad MX = P_{\mathcal{R}(M),\mathcal{N}(X)}. \]

Lemma 2.4. [3] Let \( F \oplus G = \mathbb{C}^n \). Then,

1. \( P_{F,G} M = M \Leftrightarrow \mathcal{R}(M) \subseteq F; \)
2. \( MP_{F,G} = M \Leftrightarrow \mathcal{N}(M) \supseteq G. \)

Lemma 2.5. [14] Let \( M \in \mathbb{C}^{n \times m}_n \) with \( \text{rk}(M) = r \). Then, there exists a unitary matrix \( V \) such that

\[ M = V \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} V^*, \]  (2.2)

where \( T \in \mathbb{C}^{r \times r} \) is nonsingular. Furthermore,

\[ M^* = V \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*. \]  (2.3)
3 Main results

3.1 Solution of (1.3)

Theorem 3.1. Let $M \in \mathbb{C}_n^{m \times n}$ and $b \in \mathbb{C}^n$. Then,

$$x = M^* b$$

is the unique solution of (1.3).

Proof. From $x \in \mathcal{R}(M)$, it follows that there exists $y \in \mathbb{C}^n$ for which $x = My$. Then, $x$ is the solution of (1.3) if and only if $y$ is the solution of

$$\|M^2y - b\|_F = \min.$$  

Let the decomposition of $M$ be as in (2.2). Denote

$$V^* y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad V^* b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{and} \quad M^* b = V \begin{bmatrix} T^{-1}b_1 \\ 0 \end{bmatrix},$$

where $y_1$, $b_1$ and $T^{-1}b_1 \in \mathbb{C}^{rk(M)}$. It follows that

$$\|Mx - b\|_F^2 = \left\| \begin{bmatrix} T^2y_1 + TSy_2 - b_1 \\ 0 \end{bmatrix} \right\|_F^2 = \|Ty_1 + Sy_2 - b_1\|_F^2 + \|b_2\|_F^2.$$  

Since $T$ is invertible, we have $\min_{y_1,y_2}\|T^2y_1 + TSy_2 - b_1\|_F^2 = 0$, that is, $\|M^2y - b\|_F = \min = \|b_2\|_F$, in which $y_2 \in \mathbb{C}^{n-rk(M)}$ is arbitrary, and $y_1 \cong T^{-2}b_1 - T^{-1}Sy_2$. It follows that

$$x = My = V \begin{bmatrix} T \\ 0 \end{bmatrix} S V^* y = V \begin{bmatrix} Ty_1 + Sy_2 \\ 0 \end{bmatrix} = V \begin{bmatrix} T^{-1}b_1 \\ 0 \end{bmatrix} = M^* b,$$

that is, (3.1) is the unique solution of (1.3). \hfill \Box

3.2 Determinantal formulas

When $M \in \mathbb{C}_n^{m \times n}$ is nonsingular, it is well known that the solution of (1.2) is unique and $x = M^{-1}b$. Let $x = (x_1, x_2, \ldots, x_n)^T$. Then,

$$x_i = \frac{\det(M(i \rightarrow b))}{\det(M)}, \quad i = 1, 2, \ldots, n,$$

is called Cramer’s rule for solving (1.2). In [29], Ben-Israel gets a Cramer’s rule for obtaining the least-norm solution of the consistent linear system (1.2),

$$x_i = \frac{\det\left[ \begin{bmatrix} M(i \rightarrow b) \\ V^* (i \rightarrow 0) \end{bmatrix} \right]}{\det\left[ \begin{bmatrix} U \\ V^* \end{bmatrix} \right]}, \quad i = 1, 2, \ldots, n,$$

where $U$ and $V$ are of full column rank, $\mathcal{R}(U) = \mathcal{N}(M^*)$ and $\mathcal{R}(V) = \mathcal{N}(M)$. In [26], Wang gives a Cramer’s rule for the unique solution $x \in \mathcal{R}(M^k)$ of (1.2), where $b \in \mathcal{R}(M^k)$ and $\text{Ind}(M) = k$. In [30], Ji proposes two new condensed Cramer’s rules for the unique solution $x \in \mathcal{R}(M^k)$ of (1.2), where $b \in \mathcal{R}(M^k)$ and...
Ind(M) = k. More details of Cramer’s rules for finding restricted solutions of (1.2) can be found in [1,3,31–36]. In Theorems 3.4 and 3.6, we will give two Cramer’s rules for the unique solution of (1.3).

First of all, we give the following two lemmas to prepare for a Cramer’s rule for core inverse in Theorem 3.4.

**Lemma 3.2.** Let $M \in C_n^{CM}$ with $\text{rk}(M) = r$, and let $L \in C_n^{n×(n-r)}$ with $\text{rk}(L) = n - r$ and $\mathcal{R}(L) = N(M^*)$. Then,

$$M^*M + (I_n - M^*M)L(L'L)^{-1}L^* = I_n.$$  \hspace{1cm} (3.5)

**Proof.** Let $M$ be as in (2.2), applying Lemma 2.2, we see that

$$\mathcal{R}(M) \oplus N(M) = C^n.$$  \hspace{1cm} (3.6)

Denote $M_1 = I_n - M^*M$ and $M_2 = L(L'L)^{-1}L^*$.

Applying Lemmas 2.1, 2.3 and $M^*M = M^*M$, we have

$$M^*M = P_{\mathcal{R}(M),N(M)};$$  \hspace{1cm} (3.7)

$$M_1 = I - M^*M = P_{N(M),\mathcal{R}(M)}.$$  \hspace{1cm} (3.8)

Since $(L(L'L)^{-1}L^*L(L'L)^{-1} = L(L'L)^{-1}$ and $L^*(L(L'L)^{-1}L^* = L^*$, applying Lemma 2.3, we obtain

$$M_2 = P_{\mathcal{R}(M),\mathcal{R}(M)}.$$  \hspace{1cm} (3.9)

Since $\mathcal{R}(L) = N(M^*)$, we obtain $M_2M_1 = M_2$ and

$$M_1M_2 = P_{N(M),\mathcal{R}(M)}.$$  \hspace{1cm} (3.10)

Therefore, applying Lemma 2.1, (3.7) and (3.10), we gain

$$M^*M + M_1M_2 = P_{\mathcal{R}(M),N(M)} + P_{N(M),\mathcal{R}(M)} = I_n,$$

i.e., (3.5). \hfill \Box

In [28, Theorems 3.2 and 3.3], let $M \in C_n^{CM}$, $b \in C^n$ and $b \in \mathcal{R}(M)$, and let $M_b$ and $M_c$ be of the full column ranks with $N(M^*) = \mathcal{R}(M_b)$ and $N(M_c^*) = \mathcal{R}(M)$. Then,

$$\begin{bmatrix} M & M_b \\ M_c & 0 \end{bmatrix}$$

is invertible and the unique solution $x = M^*b$ of (1.2) satisfying

$$x_i = \det \begin{bmatrix} M(i \rightarrow b) & M_b \\ M_c(i \rightarrow 0) & 0 \end{bmatrix} / \det \begin{bmatrix} M & M_b \\ M_c & 0 \end{bmatrix},$$

where $i = 1, 2, \ldots, n$. In Lemma 3.3 and Theorem 3.4, we give the unique least-squares solution of (1.3) in a similar way.

**Lemma 3.3.** Let $M$ and $L$ be as in Lemma 3.2. Then,

$$G = \begin{bmatrix} M & L \\ L^* & 0 \end{bmatrix}$$  \hspace{1cm} (3.11)

is invertible and

$$G^{-1} = \begin{bmatrix} M^* & (I_n - M^*M)L(L'L)^{-1} \\ (L^*L)^{-1}L^* & 0 \end{bmatrix}.$$  \hspace{1cm} (3.12)
Proof. Since $\mathcal{R}(L) = \mathcal{N}(M^*)$ and $M^* = M^TMM^T$, we have $M^*L = M^TMM^L = 0$ and $(L^*L)^{-1}L^*M = 0$. Furthermore, applying (3.5), we have

$$
\begin{bmatrix}
M^* & (I_n - M^*M)L(L^*L)^{-1}M^L \\
(L^*L)^{-1}L^* & 0
\end{bmatrix}
\begin{bmatrix}
M \\
L^*
\end{bmatrix}
= \begin{bmatrix}
M^*M + (I_n - M^*M)L(L^*L)^{-1}L^*M^*L \\
(L^*L)^{-1}L^*M
\end{bmatrix}
= I_{2n-r},
$$

that is, $G$ is invertible and $G^{-1}$ is of the form (3.12).

Based on Lemmas 3.2 and 3.3, we get a Cramer’s rule for the unique solution of (1.3).

**Theorem 3.4.** Let $M$ and $b$ be as in Lemma 3.2, and let $L$ be as in Lemma 3.2. Then, (1.3) has the unique solution $x = (x_1, x_2, \ldots, x_n)^T$ satisfying

$$
x_i = \frac{\det\left(\begin{bmatrix} M(i \rightarrow b) & L \end{bmatrix} \right)}{\det\left(\begin{bmatrix} M & L \end{bmatrix} \right)},
$$

where $i = 1, 2, \ldots, n$.

Proof. Since $G$ is invertible, applying Lemma 3.3, we get the unique solution $\hat{x} = G^{-1}\hat{b}$ of $G\hat{x} = \hat{b}$, in which $\hat{x} = \begin{bmatrix} x^* & y^* \end{bmatrix}^T$ and $\hat{b} = \begin{bmatrix} b^* & 0 \end{bmatrix}^T$. It follows from (3.12) that

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} M^* & (I_n - M^*M)L(L^*L)^{-1} \\
(L^*L)^{-1}L^* & 0
\end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} M^*b \\ (L^*L)^{-1}L^*b \end{bmatrix}.
$$

Applying (3.4) we obtain (3.13). □

In the following theorem, we give a characterization of the core inverse and prepare for a Cramer’s rule for the core inverse in Theorem 3.6.

**Theorem 3.5.** Let $M$ and $L$ be as in Lemma 3.2. Then,

$$
M^* = (MM^* + LL^*)^{-1}MM^*.
$$

Proof. Since $\mathcal{R}(L) = \mathcal{N}(M^*)$, $M \in \mathbb{C}_{\mathbb{N}}^{M \times M}$ and $\mathcal{R}(M)^\perp = \mathcal{N}(M^*)$, we obtain

$$
(LL^*)(LL^*)^\perp = P_{\mathcal{N}(M^*), \mathcal{R}(M)},
$$

and

$$
(MM^*M)(MM^*M)^\perp = P_{\mathcal{R}(M), \mathcal{N}(M^*)}
$$

and

$$
(MM^*M + LL^*)(MM^*M)^\perp + (LL^*)(LL^*)^\perp - (MM^*M)^\perp = P_{\mathcal{R}(M), \mathcal{N}(M^*)} + P_{\mathcal{N}(M^*), \mathcal{R}(M)} = I_n.
$$

Therefore, $(MM^*M + LL^*)$ is invertible.

Since $(LL^*)^\perp MM^* = 0$ and $(MM^*M)^\perp MM^* = M^*$, we have

$$
(MM^*M + LL^*)^{-1}MM^* = (MM^*M)^\perp MM^* + (LL^*)^\perp MM^* = M^*.
$$

It follows that we get (3.14). □

**Theorem 3.6.** Let $M$ and $L$ be as in Lemma 3.2. Then, (1.3) has the unique solution $x = (x_1, x_2, \ldots, x_n)^T$ satisfying

$$
x_j = \frac{\det(MM^*M + LL^*)(j \rightarrow MM^*b)}{\det(MM^*M + LL^*)},
$$

where $j = 1, 2, \ldots, n$. 

Proof. Applying Theorems 3.5 to 3.1, we have
\[ x = (MM^*M + LL^*)^{-1}MM^*b, \]
that is,
\[ (MM^*M + LL^*)x = MM^*b. \]
It follows from (3.4) that we get (3.15). \[ \square \]

In [30], Ji obtains the condensed determinantal expressions of $M^*$ and $M^T$. By using Theorem 3.5, we get a condensed determinantal expression of $M^*$.

**Theorem 3.7.** Let $M$ and $L$ be defined as in (3.11). Then, the core inverse $M^*$ is given by:
\[ M^*_{ij} = \frac{\det(MM^*M + LL^*)(i \to (MM^*)e_j)}{\det(MM^*M + LL^*)}, \quad (3.16) \]
where $1 \leq i, j \leq n$.

**Proof.** Since $MM^*M + LL^*$ is invertible, we consider
\[ (MM^*M + LL^*)x = (MM^*)e_j \]
and get the solution
\[ e_j^T x = \frac{\det(MM^*M + LL^*)(i \to (MM^*)e_j)}{\det(MM^*M + LL^*)}, \]
in which $i, j = 1, \ldots, n$.

It follows from (3.14) and $M^*_{ij} = e_i^T M^* e_j$ that we get (3.16). \[ \square \]

### 3.3 Examples

In the following examples, we show that our results are effective.

**Example 3.1.**
Let $M = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It is easy to check that $\mathcal{R}(L) = \mathcal{N}(M^*)$. By applying Lemma 3.3, we have $M^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then,
\[ (I_n - M^*M)L(L^*)^{-1} = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}, \quad (L^*)^{-1}L^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \]
\[ G = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \det(G) = 1 \text{ and } G^{-1} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \]

By applying Theorem 3.1, we get the solution of (1.3) is
\[ x = M^*b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \]
For $\det\begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = 1$ and $\det\begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 0$, by applying Theorem 3.4, we get $x_1 = \frac{1}{1}$, $x_2 = \frac{0}{0}$ and $x_3 = \frac{0}{0}$. Therefore, the solution of (1.3) is $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\det(MM'M + LL') = \det\begin{pmatrix} 9 & 18 & 18 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 9$, $MM'b = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$,

$\det(MM'M + LL')(1 \to MM'b) = \det\begin{pmatrix} 9 & 18 & 18 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 9$,

$\det(MM'M + LL')(2 \to MM'b) = 0$, $\det(MM'M + LL')(3 \to MM'b) = 0$, and by applying Theorem 3.6, we get $x_1 = \frac{2}{9}$, $x_2 = \frac{0}{9}$ and $x_3 = \frac{0}{9}$. Therefore, the solution of (1.3) is $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

**Example 3.2.**

Let

$$M = \begin{bmatrix} 1/4 & 1/8 & 1/8 \\ 1/4 & 1/8 & 1/8 \\ 1/4 & 0 & 1/4 \end{bmatrix}$$

Then,

$$M^# = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 3 & -2 \\ 1 & -5 & 6 \end{bmatrix} \quad M^# = \begin{bmatrix} 4/3 & 4/3 & 0 \\ 8/3 & 8/3 & -4 \\ -4/3 & -4/3 & 4 \end{bmatrix} \quad M^* = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

with $\text{rk}(L) = n - r$ and $\mathcal{R}(L) = \mathcal{N}(M^*)$. It is easy to check that

$$MM'M + LL' = \begin{bmatrix} 137/128 & -125/128 & 3/64 \\ -119/128 & 131/128 & 3/64 \\ 5/64 & 3/128 & 7/128 \end{bmatrix}.$$  

By applying Theorem 3.5, we get

$$(MM'M + LL')^{-1}MM' = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} = M^*.$$

For

$$\det(MM'M + LL') = 3/4096,$$

$$\det(MM'M + LL')(1 \to (MM')e_1) = 3/2048,$$

$$\det(MM'M + LL')(1 \to (MM')e_2) = 3/2048,$$

$$\det(MM'M + LL')(1 \to (MM')e_3) = -3/2048,$$

$$\det(MM'M + LL')(2 \to (MM')e_1) = 3/2048,$$
by applying Theorem 3.7, we get
\[
M_{11} = 2, \quad M_{12} = 2, \quad M_{13} = -2,
M_{21} = 2, \quad M_{22} = 2, \quad M_{23} = -2,
M_{31} = -2, \quad M_{32} = -2, \quad M_{33} = 6,
\]
that is,
\[
M^* = \begin{bmatrix}
2 & 2 & -2 \\
2 & 2 & -2 \\
-2 & -2 & 6
\end{bmatrix}.
\]

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References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd edn, Springer-Verlag, New York, 2003.
[2] D. S. Cvetković Ilić and Y. Wei, *Algebraic Properties of Generalized Inverses*, Springer, Singapore, 2017.
[3] G. Wang, Y. Wei, and S. Qiao, *Generalized Inverses: Theory and Computations*, 2nd edn, Springer, Singapore, 2018.
[4] O. M. Baksalary and G. Trenkler, *Core inverse of matrices*, Linear Multilinear Algebra 58 (2010), no. 5–6, 681–697, DOI: 10.1080/03081080902778222.
[5] H. Wang and X. Liu, *Characterizations of the core inverse and the core partial ordering*, Linear Multilinear Algebra 63 (2015), no. 9, 1829–1836, DOI: 10.1080/03081087.2014.975702.
[6] R. E. Cline, *Inverses of rank invariant powers of a matrix*, SIAM J. Numer. Anal. 5 (1968), 182–197, DOI: 10.1137/0705015.
[7] S. B. Malik and N. Thome, *On a new generalized inverse for matrices of an arbitrary index*, Appl. Math. Comput. 226 (2014), 575–580, DOI: 10.1016/j.amc.2013.10.060.
[8] O. M. Baksalary and G. Trenkler, *On a generalized core inverse*, Appl. Math. Comput. 236 (2014), 450–457, DOI: 10.1016/j.amc.2014.03.048.
[9] K. Manjunatha Prasad and K. S. Mohana, *Core-EP inverse*, Linear Multilinear Algebra 62 (2014), no. 6, 792–802, DOI: 10.1080/03081087.2013.791690.
[10] H. Wang and J. Chen, *Weak group inverse*, Open Math. 16 (2018), 1218–1232, DOI: 10.1515/math-2018-0100.
[11] Ivan Kyrchei, *Determinantal representations of the quaternion core inverse and its generalizations*, Adv. Appl. Clifford Algebr. 29 (2019), 104, DOI: 10.1007/s00006-019-1024-6.
[12] H. Ma, *Optimal perturbation bounds for the core inverse*, Appl. Math. Comput. 336 (2018), 176–181, DOI: 10.1016/j.amc.2018.04.059.
[13] K. Manjunatha Prasad and M. D. Raj, *Bordering method to compute core-EP inverse*, Spec. Matrices 6 (2018), 193–200, DOI: 10.1515/spma-2018-0016.
[14] Hongxing Wang, Core-EP decomposition and its applications, Linear Algebra Appl. 508 (2016), 289–300, DOI: 10.1016/j.laa.2016.08.008.

[15] H. Wang, J. Chen, and G. Yan, Generalized Cauchy-Hamilton theorem for core-EP inverse matrix and DMP inverse matrix, J. Southeast Univer. (Engl. Ed.) 1 (2018), no. 4, 135–138, DOI: 10.3969/j.issn.1003-7985.2018.01.019.

[16] Y. Gao and J. Chen, Pseudo core inverses in rings with involution, Comm. Algebra 46 (2018), no. 1, 38–50, DOI: 10.1080/00927872.2016.1260729.

[17] D. S. Rakić, N. Ć. Dinić, and D. S. Djordjević, Core inverse and core partial order of Hilbert space operators, Appl. Math. Comput. 244 (2014), 283–302, DOI: 10.1016/j.amc.2014.06.112.

[18] I. Kyrchei, Determinantal representations of the core inverse and its generalizations with applications, J. Math. 2019 (2019), 1–13, DOI: 10.1155/2019/1631979.

[19] H. Wang and X. Liu, A partial order on the set of complex matrices with index one, Linear Multilinear Algebra 66 (2018), no. 1, 206–216, DOI: 10.1080/03081087.2017.1292995.

[20] S. L. Campbell and C. D. Meyer, Generalized Inverses of Linear Transformations, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2009.

[21] Y. Wei, Index splitting for the Drazin inverse and the singular linear system, Appl. Math. Comput. 95 (1998), no. 2–3, 115–124, DOI: 10.1016/S0096-3003(97)10098-4.

[22] Y. Wei, A characterization for the W-weighted Drazin inverse and a Cramer rule for the W-weighted Drazin inverse solution, Appl. Math. Comput. 125 (2002), no. 2–3, 303–310, DOI: 10.1016/S0096-3003(01)00132-6.

[23] Y. L. Chen, Representations and Cramer rules for the solution of a restricted matrix equation, Linear and Multilinear Algebra 35 (1993), no. 3–4, 339–354, DOI: 10.1080/03081089308818266.

[24] K. Morikuni and M. Rozložník, On GMRES for singular EP and GP systems, SIAM J. Matrix Anal. Appl. 39 (2018), no. 2, 1033–1048, DOI: 10.1137/17M1128216.

[25] F. Toutounian and R. Buzhabadi, New methods for computing the Drazin-inverse solution of singular linear systems, Appl. Math. Comput. 294 (2017), 343–352, DOI: 10.1016/j.amc.2016.09.013.

[26] G. Wang, A Cramer rule for finding the solution of a class of singular linear equations, Linear Algebra Appl. 116 (1989), 27–34, DOI: 10.1016/0024-3795(89)90395-9.

[27] Y. Wei and H. Wu, Convergence properties of Krylov subspace methods for singular linear systems with arbitrary index, J. Comput. Appl. Math. 114 (2000), no. 2, 305–318, DOI: 10.1016/S0161-2007(99)00237-6.

[28] H. Ma and T. Li, Characterizations and representations of the core inverse and its applications, Linear and Multilinear Algebra (2019), DOI: 10.1080/03081087.2019.1588847.

[29] A. Ben-Israel, A Cramer rule for least-norm solutions of consistent linear equations, Linear Algebra Appl. 43 (1982), 223–226, DOI: 10.1016/0024-3795(82)90255-5.

[30] J. Ji, Explicit expressions of the generalized inverses and condensed Cramer rules, Linear Algebra Appl. 404 (2005), 183–192, DOI: 10.1016/j.laa.2005.02.025.

[31] I. Kyrchei, Analogs of Cramer’s rule for the minimum norm least squares solutions of some matrix equations, Appl. Math. Comput. 218 (2012), no. 11, 6375–6384, DOI: 10.1016/j.amc.2011.12.004.

[32] I. Kyrchei, Explicit formulas for determinantal representations of the Drazin inverse solutions of some matrix and differential matrix equations, Appl. Math. Comput. 219 (2013), no. 14, 7632–7644, DOI: 10.1016/j.amc.2013.01.050.

[33] I. Kyrchei, Cramer’s rule for generalized inverse solutions, in: I. Kyrchei (Ed.), Advances in Linear Algebra Research, pp. 79–132, Nova Science Publishers, New York, 2015.

[34] I. Kyrchei, Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse, Appl. Math. Comput. 309 (2017), 1–16, DOI: 10.1016/j.amc.2017.03.048.

[35] J. Ji, A condensed Cramer’s rule for the minimum-norm least-squares solution of linear equations, Linear Algebra Appl. 437 (2012), no. 9, 2173–2178, DOI: 10.1016/j.laa.2012.06.012.

[36] G. Wang and Z. Xu, Solving a kind of restricted matrix equations and Cramer rule, Appl. Math. Comput. 162 (2005), no. 1, 329–338, DOI: 10.1016/j.amc.2003.12.118.