ON THE UNIQUENESS FOR ONE-DIMENSIONAL
CONSTRAINED HAMILTON-JACOBI EQUATIONS

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Abstract. The goal of this paper is to study uniqueness of a one-dimensional
Hamilton-Jacobi equation
\[ \begin{cases} 
  u_t = |u_x|^2 + R(x, I(t)) & \text{in } \mathbb{R} \times (0, \infty), \\
  \max_{\mathbb{R}} u(\cdot, t) = 0 & \text{on } [0, \infty), 
\end{cases} \]
with an initial condition \( u_0(x, 0) = u_0(x) \) on \( \mathbb{R} \). A reaction term \( R(x, I(t)) \) is given
while \( I(t) \) is an unknown constraint (Lagrange multiplier) that forces maximum
of \( u \) to be always zero. In the paper, we prove uniqueness of a pair of unknowns
\((u, I)\) using dynamic programming principle in one dimensional space for some
particular class of nonseparable reaction \( R(x, I(t)) \).

1. Introduction

The non-local parabolic equations arising in adaptive dynamics (see [8, 9, 10, 11])
have an interesting feature so called Dirac concentration of density as a diffusion
coefficient vanishes. To illustrate this, we consider the following evolution equation
\[ \begin{align*}
  n^\varepsilon_t - \varepsilon \Delta n^\varepsilon &= \frac{n^\varepsilon}{\varepsilon} R(x, I^\varepsilon(t)) & \text{in } \mathbb{R}^n \times (0, \infty), \\
  n^\varepsilon(x, 0) &= n_0^\varepsilon \in L^1(\mathbb{R}^n) & \text{on } \mathbb{R}^n, \\
  I^\varepsilon(t) &= \int_{\mathbb{R}^n} \psi(x) n^\varepsilon(t, x) \, dx, 
\end{align*} \]
where the spatial variable \( x \) denotes ‘traits’ in the environment. Furthermore, \( n^\varepsilon \),
\( R(x, I^\varepsilon(t)) \), \( \varepsilon \) and \( \psi(x) \) describe density of the population, reproduction rate, mu-
tation rate and consumption rate by a trait \( x \). Here \( \psi \) assumed to be a non-
negative compactly supported function. We then take Hopf-Cole transformation
\[ n^\varepsilon(x, t) = e^{u^\varepsilon(x,t)/\varepsilon}. \]
It was shown in many literatures that as mutation rate \( \varepsilon \) van-
ishes, \( u^\varepsilon \) converges locally uniformly to \( u \) which is a viscosity solution to
\[ \begin{align*}
  u_t &= |Du|^2 + R(x, I(t)) & \text{in } \mathbb{R}^n \times (0, \infty), \\
  \max_{\mathbb{R}^n} u(\cdot, t) &= 0 & \text{on } [0, \infty), \\
  u(x, 0) &= u_0(x) & \text{on } \mathbb{R}^n. 
\end{align*} \]
The constraint of \( u \) is obtained from the property that \( I^\varepsilon \) is positive and uniformly
bounded. It was also shown that
\[ n^\varepsilon(x, t) n(x, t) \rightarrow \mathcal{P}(x)(x(t) - \mathcal{X}(t)) \text{ weakly in the sense of measure } \]

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where
\[ u(x,t) = \max_{\mathbb{R}} u(\cdot, t) = 0 \text{ and } \rho(t) = \frac{I(t)}{\psi(x)} \]
for the solution \( u^\varepsilon(x,t) \) to (1.2) (see [14, 6]). Despite the existence of solutions to (1.2) is quite well understood, the uniqueness is relatively less known. In the recent work by S. Mirrahimi, J. -M. Roquejoffre [13], the uniqueness of the solution is shown when the reaction and initial condition \( u_0(x) \) are strictly concave so that regularity of maximum point is obtained. However, the uniqueness for general initial data and a nonconcave reaction is still open. In this paper, the uniqueness property for constrained Hamilton-Jacobi equations in 1-D with some nonseparable reaction terms is obtained using dynamic programming principle.

1.1. Setting and main result. We need following assumptions on
\[ R(x,I) : \mathbb{R} \times [0, \infty) \to \mathbb{R} \text{ and } u_0(x) : \mathbb{R} \to \mathbb{R}. \]
where the reaction term is defined as
\[ R(x,I) = \begin{cases} b(x) - Q(I) & \text{for } x \geq 0, \\ R'(x,I) & \text{for } x < 0. \end{cases} \]

**Main Assumptions.** For a positive \( I_M \),
(A1) \( R \) is smooth and \( R'(\cdot,I) < 0 \) on \((-\infty,0)\) for any positive \( I \);
(A2) sup\( \{0 \leq I \leq I_M\} \|R(\cdot,I)\|_{W^{2,\infty}} < \infty \) and \( R \) is strictly decreasing in \( I \);
(A3) \( Q(I) \geq 0 \) is strictly increasing in \( I \) and \( Q(0) = 0 \)
(A4) sup\( R(\cdot,I_M) = 0 \);
(A5) min\( R(\cdot,0) = 0 \);
(A6) \( b(x) \) is strictly increasing on \([0, \infty)\) with \( b(0)=0 \);
(A7) \( b'(x) \) is Lipschitz continuous, hence, nonnegative;
(A8) \( u_0(x) \in C^2(\mathbb{R}) \) with \( \|u_0\|_{C^2(\mathbb{R})} < \infty \), max\( x \in \mathbb{R} \) \( u_0(\cdot) \) = \( u_0(0) = 0 \) and \( u_0(x) < 0 \) elsewhere.

Additionally, \( f \in W^{1,\infty}(\mathbb{R}^n) \), that is; \( \|f\|_{L^{\infty}(\mathbb{R}^n)} + \|Df\|_{L^{\infty}(\mathbb{R}^n)} < \infty \).

Now we are ready to state our main theorem. Under the assumptions above, we consider the following equation.
\[
\begin{cases}
    u_t = u_x^2 + R(x,I(t)) & \text{in } \mathbb{R} \times (0, \infty), \\
    \max_{\mathbb{R}} u(\cdot,t) = 0 & \text{on } [0, \infty), \\
    u(x,0) = u_0(x) & \text{on } \mathbb{R}.
\end{cases}
\]

**Theorem 1.1.** There exists at most one pair \((u,I)\) such that \( u(x,t) \in C(\mathbb{R} \times (0, \infty)) \)
solves (1.6) in viscosity sense and \( I(t) \in C([0, \infty)) \) is strictly increasing.

2. Preliminary

Throughout the section, let us assume \((u,I) \in C(\mathbb{R} \times (0, \infty)) \times C([0, \infty)) \) is a pair of solution to (1.6) in viscosity sense. By a Lipschitz estimate provided by the author in [15], one can assume further that \( u \) is Lipschitz continuous in \( \mathbb{R} \times [0, T] \) for
any positive $T$. Now we follow dynamic programming principle arguments presented in [13], which yields

$$u(x,t) = \sup_{\gamma(t) = x} \{ F(\gamma) : \gamma \in AC([0,t]; \mathbb{R}) \}$$

where

$$F(\gamma) := u_0(\gamma(0)) + \int_0^t \left( -\frac{\dot{\gamma}^2}{4} + R(\gamma(s), I(s)) \right) ds.$$ (2.1)

Furthermore, one can actually show that there exists a path $\gamma(s) \in C^1([0,t); \mathbb{R})$ such that

$$u(x,t) = u_0(\gamma(0)) + \int_0^t \left( -\frac{\dot{\gamma}^2}{4} + R(\gamma(s), I(s)) \right) ds$$ (2.3)

with $\gamma(t) = 0$ and it satisfies Euler-Lagrange equation

$$\begin{align*}
\ddot{\gamma}(s) + 2R_x(\gamma(s), I(s)) &= 0, \\
\dot{\gamma}(0) + 2u_0(\gamma(0)) &= 0, \\
\gamma(t) &= x.
\end{align*}$$ (2.4)

For the details, see [13] and references therein.

There could be more than one solution to the equation above. However, the Euler-Lagrange equation reduces to a simpler equation that results in the existence of a unique solution in our setting. We start with some generic properties.

**Proposition 2.1.** Assume that $\max_\mathbb{R} u(\cdot, t) = u(x', t) = 0$. Then $R(x', I(t)) = 0$.

**Proof.** By viscosity subsolution test, one can easily obtain $R(x', I(t)) \geq 0$. Now we assume that $R(x', I(t)) > 0$. Then there exists $t_0 > 0$ such that $R(x', I(s)) > 0$ on $[t, t + t_0]$ by the continuity of $I$ and $R$. Integrating (1.6) both sides over $\{x'\} \times [t, t + t_0]$ yields

$$u(x', t + t_0) - u(x', t) \geq \int_t^{t+t_0} R(x', I(s)) ds > 0$$

Hence, we get

$$u(x', t + t_0) > 0,$$

which violates the maximum constraint.

**Definition 1.** We define $x(t) \in \mathbb{R}$ to satisfy

$$R(x(t), I(t)) = 0$$

for $t > 0$ and a strictly increasing $I(t)$. Then, together with Proposition 2.1, we have

$$\max_\mathbb{R} u(\cdot, t) = u(x(t), t)) = 0.$$ (2.5)

**Proposition 2.2.** $I(0) = 0$ and $I(s) \leq I_M$ on $[0, \infty)$. 
Proof. Let us first prove $I(0) = 0$ when $(u, I)$ is a pair of solution. We may assume $I(0) > 0$. From the property (2.5), we deduce

$$0 = \lim_{t \to 0^+} u(x(t), t) = u(x(0^+), 0) < 0$$

where $x(0^+)$ is a right limit of $x(t)$, which yields contradiction. Therefore, $I(0) = 0$. The second part of the proposition, $I(s) \leq I_M$, is a straight consequence of Proposition 2.1 due to the assumption on $R$. \qed

We also need some regularity properties of the solution $u(x, t)$, which play crucial roles in analyzing the trajectory $\gamma(s)$.

**Definition 2.** For a real valued function $u(x)$ define for $x \in \mathbb{R}^n$, we define super differential and sub differential at $x$ as

$$D^+u(x) = \{ p \in \mathbb{R}^n : \lim_{y \to x} \inf \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \geq 0 \}$$

$$D^-u(x) = \{ p \in \mathbb{R}^n : \lim_{y \to x} \sup \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \}$$

**Lemma 2.3.** A solution $u(x, t)$ is semiconvex in $x \in \mathbb{R}$ for any fixed positive $T$.

Proof. Let us define $v(x, t) = -u(x, t)$ and prove $v(x, t)$ is semiconcave in $\mathbb{R} \times [0, T]$. Cleary, $v$ satisfies

$$\begin{cases}
  v_t + v_x^2 + R(x, I(t)) = 0 & \text{in } \mathbb{R}^n \times (0, T], \\
  v(x, 0) = -u(x, 0) & \text{on } \mathbb{R}.
\end{cases}$$

in viscosity sense. To prove semiconcavity of $v$, we first provide a priori estimate for $v^\varepsilon$ where $v^\varepsilon$ is a unique solution to

$$\begin{cases}
  v_t^\varepsilon + (v_x^\varepsilon)^2 + R(x, I(t)) = \varepsilon v_{xx}^\varepsilon & \text{in } \mathbb{R} \times (0, T], \\
  v^\varepsilon(x, 0) = -u_0(x) := v_0(x) & \text{on } \mathbb{R}.
\end{cases}$$

Differentiating (2.9) twice with respect to $x$ and substituting $w$ for $v_{xx}^\varepsilon$ yields

$$w_t + 2w^2 + 2v_xw_x + R_{xx} = \varepsilon w_{xx}.$$  \hspace{1cm} (2.10)

It is known that $w$ is bounded but the bound depends on $\varepsilon$. However, one can actually show that the bound is uniform in $\varepsilon$. To justify this, we first notice that $w$ is a subsolution to the following parabolic equation

$$\begin{cases}
  w_t + v_xw_x + R_{xx} = \varepsilon w_{xx} & \text{in } \mathbb{R} \times (0, T], \\
  w(x, 0) = v_0''(x) & \text{on } \mathbb{R}.
\end{cases}$$

(2.11)

On the other hand, $v_0 + Ct$ and $v_0 - Ct$ are supersolution and subsolution to (2.11) respectively where $C$ depends only on the bound for $R_{xx}$. Therefore, by comparison principle, one can obtain $|w| < C$ where $C$ does not depend on $\varepsilon$. As a last step, we need the following estimate.

**Claim.** There exists positive $C$ that depends only on $T$ such that

$$\|v_t^\varepsilon\|_{L^\infty(\mathbb{R} \times [0, T])} + \|v_x^\varepsilon\|_{L^\infty(\mathbb{R} \times [0, T])} < C$$
Proof of claim. Since $0 \leq I(t) \leq I_M$, $R(x,I)$ is bounded, for $C > 0$ large enough, we have
\begin{equation}
 v_0(x) - Ct \leq v^\varepsilon(x,t) \leq v_0(x) + Ct
\end{equation}
by the comparison principle. Using the comparison principle one more time yields
\begin{equation}
 v^\varepsilon(s + t) \geq v^\varepsilon(t) - Cs
\end{equation}
for $s, t \geq 0$. Therefore, $v^\varepsilon_t > -C$ in $\mathbb{R} \times [0, T]$. On the other hand, observing the original equation \[2.9\], we can derive $\|v^\varepsilon_x\|_{L^\infty(\mathbb{R} \times [0, T])} < C$ as $v^\varepsilon_{xx}$ is bounded above. Finally, an upper bound for $v^\varepsilon_t$ is obtained, and such bounds depend only on $T$.

As a consequence, $v^\varepsilon$ converges locally uniformly to $v$ as $\varepsilon$ goes to $0$ by Arzela-Ascoli and by the uniqueness and stability of a viscosity solution. Moreover, the semiconcavity of $v^\varepsilon$ in $x$ implies that
\begin{equation}
 v^\varepsilon(x,t) - K |x|^2
\end{equation}
is concave in $x$ for some positive $K$. Combining it with locally uniform convergence of $v^\varepsilon$, we get semiconcavity of $v$ in $x$. Therefore, $u$ is locally semiconvex in $x$.

**Lemma 2.4.** For each $t \in (0, \infty)$, $u(x,t)$ is differentiable at $(x(t), t)$ with respect to the space variable $x$ and it satisfies
\begin{equation}
 0 = u_x(x(t), t) = -\frac{\dot{\gamma}_x(t)}{2}.
\end{equation}
In addition to that, by the maximum constraint, we have $\dot{\gamma}_x(t) = 0$.

**Proof.** By Lemma 2.3, $v(x,t) = -u(x,t)$ is semiconcave in $x$. Hence, supper differential at $(x(t), t)$ is nonempty. On the other hand $p = 0$ is a subdifferential of for $v$ at $(x(t), t)$. Therefore, $u$ is differentiable with respect to the space variable at $(x(t), t)$. Moreover, the derivative is 0.

A classical result in \[3\] suggests that
\[\eta(t) \in \nabla^+ v(x(t), t)\]
where $\dot{\gamma}_x(s) = 2\eta(s)$ for $s \in [0, t]$ and $v$ is defined as above. Combining these two, we get the result using the differentiability of $v$ at $(x(t), t)$.

**Proposition 2.5.** Let $\gamma(s) \in C^1([0, t]; \mathbb{R})$ an optimizing path whose terminal point is $x(t)$ and $x(s) \in \mathbb{R}$ satisfy $R(x(s), I(s)) = 0$ for $s > 0$. Then we have $\gamma(s) > x(s)$ for $s \in (0, t)$.

**Proof.** We may assume first that $\gamma(s) \geq 0$ since $F(\gamma^+) \geq F(\gamma)$ where
\begin{equation}
 \gamma^+(s) = \begin{cases} 
 \gamma(s) & \text{if } \gamma(s) > 0 \\
 0 & \text{if } \gamma(s) \leq 0
\end{cases}
\end{equation}
Now we assume $\gamma(s) < x(s)$ on $(0, t)$. Then $R(\gamma(s), I(s)) < R(x(s), I(s)) = 0$ on $(0, t)$, which yields
\[0 = u(x(t), t) = \int_{t_0}^t \left( -\frac{\dot{\gamma}_x^2}{4} + R(\gamma(s), I(s)) \right) + u_0(\gamma(0)) < 0.\]
Hence, there exists $t' \in (t_0, t)$ such that $\gamma(t') = x(t')$. On the other hand, $\gamma(s)$ satisfies the Euler-Lagrange equation, which is,

$$\ddot{\gamma}(s) + R_x(\gamma(s), I(s)) = \dot{\gamma}(s) + b'(\gamma(s)) = 0. \quad (2.16)$$

Integrating the equation from $t'$ to $t$ gives

$$0 = \dot{\gamma}(t) - \dot{\gamma}(t_0) = \int_{t_0}^{t} b'(\gamma(s)) > 0,$$

by the lemma above. Therefore, $\gamma(s) > x(s)$ on $(0, t)$. □

3. Proof of the theorem 1.1

We assume that we have two pairs of solutions $(u_1, I_1)$ and $(u_2, I_2)$ to (1.6) for $n = 1$ and consider two cases. Let us fix the time $T$.

Case 1: $I_1(s)$ and $I_2(s)$ intersect only at the origin for $s \in [0, T]$. Without loss of generality, let us assume $I_1 < I_2$ except for the terminal point. Then $u_1$ is a viscosity supersolution to

$$\begin{cases} (u_2)_t = (u_2)_x^2 + R_x(I_2(t)) & \text{in } \mathbb{R} \times (0, t], \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \quad (3.1)$$

By the comparison principle and the maximum constraint, we have $x_1(s) = x_2(s)$ for all $s$, where $x_1, x_2$ are defined as above, which is a contradiction.

Case 2: $I_1(s)$ and $I_2(s)$ intersect at more than one point including the terminal point $t$. Let $t_0 < t_1 \in [0, t]$ be points such that

$$I_1(t_i) = I_2(t_i) \text{ for } i = 1, 2. \quad (3.2)$$

Hence, we have $x_1(t_0) = x_2(t_0) := \alpha$ and $x_1(t_1) = x_2(t_1) := \beta$. In addition to that, we may assume that

$$I_1 > I_2 \text{ for } i \in (t_0, t_1).$$

For the $t_i$'s above, we define $\gamma_1(s)$ and $\eta_1(s)$ as optimizing trajectories corresponding to $I_1$ whose terminal points are $\alpha$ and $\beta$ respectively. Similarly, one can define $\gamma_2(s)$ and $\eta_2(s)$ as optimizing trajectories corresponding to $I_2$ whose terminal points are $\alpha$ and $\beta$ respectively. By Proposition 2.5 and Lemma 2.4, for each $i = 1, 2$, $\gamma_i$ satisfies

$$\begin{cases} \ddot{\gamma}_i + 2b'(\gamma_i) = 0, \\ \dot{\gamma}_i(t) = 0, \\ \gamma(t) = \alpha. \end{cases}$$

Similarly, for each $i = 1, 2$, $\eta_i$ is a solution to

$$\begin{cases} \ddot{\eta}_i + 2b'(\eta_i) = 0, \\ \dot{\eta}_i(t) = 0, \\ \gamma(t) = \beta. \end{cases}$$
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Therefore, \( \gamma_1 = \gamma_2 := \gamma \) and \( \eta_1 = \eta_2 = \eta \). Applying this property to the relations

\[
0 = u_1(\beta, t_1) = \int_0^{t_1} \left( -\frac{\gamma^2}{4} + b(\gamma) - Q(I_1) \right) \, ds + u_0(\gamma(0), 0),
\]

\[
0 = u_2(\beta, t_0) = \int_0^{t_1} \left( -\frac{\gamma^2}{4} + b(\gamma) - Q(I_2) \right) \, ds + u_0(\gamma(0), 0),
\]

\[
0 = u_1(\alpha, t_1) = \int_0^{t_1} \left( -\frac{\eta^2}{4} + b(\gamma) - Q(I_1) \right) \, ds + u_0(\eta(0), 0),
\]

\[
0 = u_2(\alpha, t_0) = \int_0^{t_1} \left( -\frac{\eta^2}{4} + b(\gamma) - Q(I_2) \right) \, ds + u_0(\eta(0), 0),
\]

we end up getting

\[
0 = \int_{t_0}^{t_1} (Q(I_1) - Q(I_2)) \, ds,
\]

which contradicts \( I_1 > I_2 \) on \((t_0, t_1)\).

References

[1] S. Armstrong, H. V. Tran, *Viscosity solutions of general viscous Hamilton-Jacobi equations*, Mathematische Annalen. 361 (2014), 647-687.

[2] G. Barles, *Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit*, Nonlinear Analysis: Theory, Methods & Appl. 20 (1999), no. 9, 1123-1134.

[3] P. Cannarsa, C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and Their Applications.

[4] G. Barles, S. Mirrahimi, B. Perthame, *Concentration in Lotka-Volterra parabolic or integral equations: a general convergence result*, Methods Appl. Anal. 16 (2009), no. 3, pp.321-340.

[5] G. Barles, B. Perthame, *Concentrations and constrained Hamilton-Jacobi equations arising in adaptive dynamics*, Contemporary Math. 439 (2007), 57-68.

[6] O. Diekmann, P.-E. Jabin, S. Mischler, B. Perthame, *The dynamics of adaptation: an illuminating example and a Hamilton-Jacobi approach*, Th. Pop. Biol. 67 (2005), no. 4, 257-271.

[7] M. G. Crandall, L. C. Evans, P.-L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Transaction of American Mathematical Society, 282 (1984), no. 2, 487-502.

[8] O. Diekmann, *Beginner’s guide to adaptive dynamics*, Banach Center Publications 63 (2004), 47-86.

[9] S. A. H. Geritz, E. Kisdi, G. Mészáros, J. A. J. Metz, *Dynamics of adaptation and evolutionary branching*, Phy. Rev. Letters 78 (1997), 2024-2027.

[10] S. A. H. Geritz, E. Kisdi, G. Mészáros, J. A. J. Metz, *Evolutionary singular strategies and the adaptive growth and branching of the evolutionary tree*, Evolutionary Ecology 12 (1998), 35-57.

[11] S. A. H. Geritz, E. Kisdi, M. Gyllenberg, F. J. Jacobs, J. A. J. Metz *Link between population dynamics and dynamics of Darwinian evolution*, Phy. Rev. Letters 95 (2005), no. 7.

[12] N. Q. Le, H. Mitake, H. V. Tran, *Dynamical and Geometric Aspects of Hamilton-Jacobi and Linearized Monge-Ampere Equations*, Lecture notes in Mathematics 2183 (2016).

[13] S. Mirrahimi, J.-M. Roquejoffre, *A class of Hamilton-Jacobi equations with constraint: Uniqueness and constructive approach*, J. of Differential Equations 250.5 (2016), 4717-4738.

[14] B. Perthame, G. Barles, *Dirac concentrations in Lotka-Volterra parabolic PDEs*, Indiana Univ. Math., J. 57 (2008), no. 7, 3275-3301.

[15] Y. Kim, *Wellposedness for constrained Hamilton-Jacobi equations*, preprint
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