Statistical Einstein manifolds of exponential families with group-invariant potential functions

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Dedicated to Huafei Sun on the occasion of his sixtieth birthday.

Abstract

This paper mainly contributes to a classification of statistical Einstein manifolds, namely statistical manifolds at the same time are Einstein manifolds. Considering the Fisher information metric as a Riemannian metric, information geometry was developed to understand the intrinsic properties of statistical models, which play important roles in statistical inference, etc. Among all these models, exponential families is one of the most important kinds, whose geometric structures can be fully determined by their potential functions. Partial differential equation for potential functions is observed as the necessary and sufficient condition to amount to statistical Einstein manifolds; special solutions are obtained through the ansatz method as well as group-invariant solutions via reductions using Lie point symmetries.

Keywords: information geometry; Einstein manifold; symmetry reduction; group-invariant solutions

1 Introduction

Information geometry was founded based on the applications of differential geometry and Riemannian geometry into probability and statistics. The astonishing geometric structures are constructed based on the cheerful Fisher information metric, which is viewed as a Riemannian metric \(^{10,15,35}\). Decades later, Amari and his collaborators \(^{2,7}\) developed the dual geometric structures, which are considered as the core of parametrised information geometry; for more, see \(^{9,26,36,37}\). Beside statistics and entropic dynamical models \(^{3,8,14,15,24,25,27,29,31,33}\), information geometry has been successfully applied into other fields as well, such as neural networks \(^{5,6}\), decoding \(^{22}\), control systems \(^{3,39,41}\).

Meanwhile, there are scholars interested at the global properties of statistical manifolds themselves, rather than the applications of the theory of information geometry \(^{20,23,38}\). In this paper, we investigate statistical manifolds which are Einstein manifolds at the same time, that is, the following equation holds \(^{10}\)

\[
Ric = -\lambda g,
\]

where \(\lambda\) is a constant, \(Ric\) is the Ricci curvature tensor and \(g\) is the Riemannian or pseudo-Riemannian metric. We call such statistical manifolds statistical Einstein manifolds. In information geometry, \(g\) is the Fisher information metric, which is also called the Fisher information matrix. In particular, geometric quantities of exponential families, e.g. Fisher information metric, curvatures, can be fully determined by the so-called potential functions. Eq. \((1.1)\) for the metric \(g\) then becomes a differential equation of potential functions. A well-know example of exponential family is normal distribution.

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The complicity makes it extremely challenging to solve Eq. (1.1) or the equation for potential functions in general. In the current paper, we try to obtain special solutions in particular via reductions using Lie point symmetries; see for instance [11–13, 21, 30]. First introduced by Lie during 1880s, symmetry method has been greatly exploited and applied in various aspects. Continuous symmetries, in plain words, are local transformations unchanging the shape of an object; for differential equations, the object is the set of solutions. It is believed that every solvable differential equation has symmetries behind it.

The paper is organised as follows. In Section 2, we briefly review the theory of Lie point symmetries and group-invariant solutions. We will introduce the main concepts of information geometry in Section 3 and construct statistical Einstein manifolds in Section 4; some examples are provided. In Section 5, we obtain some two dimensional statistical Einstein manifolds by solving the corresponding equation (1.1) using an ansatz method and the symmetry method. The last section contributes to conclusions and open questions.

2 Lie point symmetries and group-invariant solutions of partial differential equations

This section is mostly based on [30]; see also [11–13, 21]. In general, let \( x = (x^1, x^2, \ldots, x^n) \) be the \( n \) independent variables and let \( u = (u^1, u^2, \ldots, u^m) \) be the \( m \) dependent variables. In the theory of information geometry below, they will be replaced by the parameters \( \theta \) and potential function \( \psi(\theta) \), respectively. Consider the following system of differential equations

\[
F_k(x, u, u_\alpha, \ldots) = 0, \quad k = 1, 2, \ldots, K, \tag{2.1}
\]

where each \( F_k \) is a function of finitely-many arguments. Here \( u_\alpha := \frac{\partial u^\alpha}{\partial x^i} \) denotes first order partial derivatives and we use the shorthand notation \( u_\alpha^J \) to denote partial derivatives:

\[
u_\alpha^J := \frac{\partial^{|J|}u^\alpha}{\partial(x^1)^{j_1}\partial(x^2)^{j_2}\cdots\partial(x^n)^{j_n}}.
\]

for \( J = (j_1, j_2, \ldots, j_n) \) with non-negative entries and \( |J| = j_1 + j_2 + \cdots + j_n \). For a one-parameter Lie group \( G \) of transformations:

\[
\tilde{x} = \tilde{x}(\varepsilon, x, u), \quad \tilde{u} = \tilde{u}(\varepsilon, x, u);
\]

\[
\tilde{x}|_{\varepsilon = 0} = x, \quad \tilde{u}|_{\varepsilon = 0} = u, \tag{2.2}
\]

the associated infinitesimal generator is

\[
X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \tag{2.3}
\]

where

\[
\xi^i := \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \tilde{x}^i, \quad \eta^\alpha = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \tilde{u}^\alpha. \tag{2.4}
\]

Here \( e \) is the identity element of \( G \). The Einstein summation convention is used here and all through the paper. This group action can be prolonged to partial derivatives of \( u \), leading to the prolongation of the vector field \( X \) [30]:

\[
\text{pr}X = \xi^i D_i + \cdots + D_J(\eta^\alpha - u^\alpha_\xi^i) \frac{\partial}{\partial u^\alpha_J} + \cdots. \tag{2.5}
\]

Here \( Q^\alpha := \eta^\alpha - u^\alpha_\xi^i \xi^i \) are called characteristics of the transformations; \( D_i \) is the total derivative with respect to \( x^i \). The multi-index notation \( D_J \) is a composition of total derivatives:

\[
D_J = D_{x_1}^{j_1} D_{x_2}^{j_2} \cdots D_{x_n}^{j_n}.
\]

This group of transformations is a symmetry group for the system of differential equations (2.1) if and only if the linearized symmetry condition (LSC) is satisfied

\[
\text{pr}X(F_k) = 0 \quad \text{on all solutions of (2.1)}. \tag{2.6}
\]
Such groups of symmetries with \( \xi \) and \( \eta \) being independent from partial derivatives are called groups of Lie point symmetries. Otherwise, they are called groups of higher symmetries.

**Example 2.1 (Heat equation).** Consider the heat equation

\[ u_t = u_{xx}. \] (2.7)

Let \( X = \xi^t(t, x, u) \partial_t + \xi^x(t, x, u) \partial_x + \eta(t, x, u) \partial_u \); its prolongation is

\[ \text{pr}X = \xi^t \partial_t + \xi^x \partial_x + \eta \partial_u + \eta^t \partial_u + \eta^{xx} \partial_{u_{xx}} + \cdots, \] (2.8)

where

\[ \eta^t = u_{tt} \xi^t + u_{tx} \xi^x + D_t Q, \quad \eta^{xx} = u_{xxx} \xi^x + u_{txx} \xi^t + D_x^2 Q. \] (2.9)

Here the characteristic is \( Q = \eta - \xi^t u_t - \xi^x u_x \). Then the LSC, that is \( \text{pr}X(u_t - u_{xx}) = 0 \) when \( u_t - u_{xx} = 0 \), gives

\[ \eta^t - \eta^{xx} = 0 \text{ when } u_t - u_{xx} = 0. \] (2.10)

Substitute \( u_t = u_{xx} \) (as well as \( u_{tt} = u_{xxxx}, \ u_{tx} = u_{txxx} \) and so on) to the equality \( \eta^t - \eta^{xx} = 0 \). We will get a polynomial with respect to partial derivatives \( u_x, \ u_{xx} \) and so on. All partial derivatives along the \( t \) direction will be replaced by partial derivatives about \( x \). By equating all independent coefficients of such a polynomial to zero, we get a system of differential equations about \( \xi^t, \ \xi^x \) and \( \eta \). Solve them we will obtain all infinitesimal generators. It is spanned by the six vector fields

\[ X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = u \partial_u, \quad X_4 = x \partial_x + 2t \partial_t, \quad X_5 = 2t \partial_x - xu \partial_u, \quad X_6 = 4tx \partial_x + 4t^2 \partial_t - (x^2 + 2t)u \partial_u, \]

and the infinite-dimensional subalgebra

\[ h(t, x) \partial_u, \]

where \( h(t, x) \) is an arbitrary solution of the heat equation.

Having the infinitesimal generators in hand, we are able to construct differential invariants and hence group-invariant solutions. A function \( f(x, u) \) is called group-invariant if for any section \( (x, u(x)) \), the equality holds identically

\[ X(f) = 0. \] (2.11)

Expanding this equality leads to a linear partial differential equation about \( f \),

\[ \eta^\alpha \frac{\partial f}{\partial u^\alpha} + \xi^t \frac{\partial f}{\partial x^t} = 0, \] (2.12)

which can often be solved using the method of characteristics, namely by solving

\[ \frac{dx^1}{\xi^t} = \cdots = \frac{dx^n}{\xi^t} = \frac{du^1}{\eta^t} = \cdots = \frac{du^m}{\eta^t}, \] (2.13)

if all coefficients \( \xi \) and \( \eta \) are nonzero. Solutions of (2.13) will provide us differential invariants, e.g. \( h^1, h^2, \ldots \) Any invariant function \( f(x, u) \) is hence rewritten as \( f(h^1, h^2, \ldots) \). Consider such invariants as new coordinates for the original system (2.1) and we obtain a reduced system. In specific cases, the reduced system can be solved exactly. Multi-reduction is also possible; see, for instance [21, 30]. Let us find out how it works in practice through the following example.
Example 2.2 (Heat equation cont.). For instance, consider the combination

\[ X = X_4 + aX_3 = x\partial_x + 2t\partial_t + au\partial_u, \quad (2.14) \]

where \( a \) is a nonzero constant. The equations for determining differential invariants (2.13) become

\[ \frac{dx}{x} = \frac{dt}{2t} = \frac{du}{au}. \quad (2.15) \]

Those equations can be solved exactly. For example consider the first equality and we get \( x = C\sqrt{t} \), which implies (by solving for the constant) that \( y := \frac{x}{\sqrt{t}} \)

is a differential invariant. Similarly we have another differential invariant \( v = t^{-a}u \).

The variable \( v \) is considered as a function of \( y \) only, namely \( v = v(y) \). Therefore, we can compute

\[ u_t = D_t(v(y)t^a) = v'yt^a + vat^{a-1} \quad (2.18) \]

and

\[ u_{xx} = D^2_x(v(y)t^a) = t^aD^2_x(v(y) = t^aD_x(v'y_x) = t^a(v''y_x^2 + v'y_{xx}). \quad (2.19) \]

Substitute these back to the heat equation \( u_t = u_{xx} \) and we obtain an ordinary differential equation

\[ v'' + \frac{1}{2}v'v' - av = 0. \quad (2.20) \]

Its general solution is

\[ v(y) = y \exp\left(-\frac{y^2}{4}\right) \left\{ C_1M\left(a + 1, \frac{3}{2}, \frac{y^2}{4}\right) + C_2U\left(a + 1, \frac{3}{2}, \frac{y^2}{4}\right) \right\}. \quad (2.21) \]

where \( M \) and \( U \) represent the Kummer’s and Tricomi’s confluent hypergeometric functions, respectively, e.g. [1]. Changing them back to the original coordinates, we get a group-invariant solution for the heat equation

\[ u(x,t) = xt^{a-\frac{1}{2}} \exp\left(-\frac{x^2}{4t}\right) \left\{ C_1M\left(a + 1, \frac{3}{2}, \frac{x^2}{4t}\right) + C_2U\left(a + 1, \frac{3}{2}, \frac{x^2}{4t}\right) \right\}. \]

In the next section, we will derive a system of differential equations for the potential functions of exponential families of probability density functions and investigate its group-invariant solutions. They contribute to statistical Einstein manifolds in information geometry.

3 Information geometry and exponential families

Although geometric analysis of any general finite parametric distribution can be developed, here we only consider regular distributions, whose probability density functions (pdfs) \( p(x;\theta) \) with parameters \( \theta = (\theta^1, \theta^2, \ldots, \theta^n) \in \Theta \) satisfy the following regularity conditions:

1. \( \Theta \) is a subset of \( \mathbb{R}^n \) such that for each \( x \), the mapping \( \theta \mapsto p(x;\theta) \) is smooth.

2. The order of integration and differentiation can be freely rearranged. For instance,

\[ \int \partial_i p(x;\theta) \, dx = \partial_i \int p(x;\theta) \, dx = \partial_i 1 = 0, \quad (3.1) \]

where \( \partial_i = \frac{\partial}{\partial \theta_i} \). For discrete distributions, we simply replace the integration by summation.
3. Different parameters stand for different pdfs, that is, \(\theta_1 \neq \theta_2\) implies that \(p(x; \theta_1)\) and \(p(x; \theta_2)\) are different pdfs. Moreover, every parameter \(\theta\) possesses a common support set where \(p(x; \theta) \geq 0\).

The family \(S\) of distribution represented by the pdf \(p(x; \theta)\) is called an \(n\)-dimensional statistical model

\[
S = \{p(x; \theta) \mid \theta \in \Theta \subset \mathbb{R}^n\}.
\]

There are many examples of statistical models, such as Poisson distribution [7], inverse Gamma distribution [25] and Pareto distribution [31]. The following examples are particularly interesting to us as both of them correspond to statistical Einstein manifolds.

**Example 3.1** (Normal distribution or Gaussian distribution). The dimension is \(n = 2\), \(x \in \mathbb{R}\), and the pdf is

\[
p(x; \theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\},
\]

where the parameter space is \(\Theta = \{\theta = (\mu, \sigma) \mid \mu \in \mathbb{R}, \sigma > 0\}\). The parameters \(\mu\) and \(\sigma\) are the mean and standard deviation.

**Example 3.2** (Weibull distribution). The dimension \(n = 2\), \(x \geq 0\), and the pdf is given by

\[
p(x; \theta) = \frac{b}{a} \left(\frac{x}{a}\right)^{b-1} \exp \left\{ -\left(\frac{x}{a}\right)^b \right\},
\]

where \(\Theta = \{\theta = (a, b) \mid a > 0, b > 0\}\). Here \(b\) is the shape parameter and \(a\) is the scale parameter of the distribution.

### 3.1 Fisher information metric and dual geometric structures

The Fisher information metric of a statistical model \(S\) is defined as

\[
g_{ij}(\theta) := E[\partial_i l_\theta \partial_j l_\theta] = \int \partial_i l(x; \theta) \partial_j l(x; \theta) p(x; \theta) \, dx,
\]

where \(l_\theta = l(x; \theta) = \ln p(x; \theta)\) and \(E\) means the expectation. For simplicity, we sometimes write \(g_{ij}\) instead of \(g_{ij}(\theta)\) and so forth. The positivity of \((g_{ij})\) makes \(S\) to be a Riemannian manifold; see e.g. [17, 36, 37]. There exists a one-parameter group of dual connections \(\nabla^{(\alpha)}\), the so called \(\alpha\)-connection, such that \(\nabla^{(0)}\) is the unique Riemannian connection corresponding to \(g\). The duality is illustrated by

\[
X g(Y, Z) = g \left(\nabla_X Y, Z\right) + g \left(Y, \nabla_X Z\right),
\]

where \(X, Y, Z\) are vector fields on the manifold. The coefficients of \(\alpha\)-connection are given by

\[
\Gamma^{(\alpha)}_{ij,k} := E[\partial_i \partial_j l_\theta \partial_k l_\theta] + \frac{1 - \alpha}{2} E[\partial_i l_\theta \partial_j l_\theta \partial_k l_\theta]
\]

\[
= \Gamma^{(0)}_{ij,k} + \frac{\alpha}{2} T_{ijk},
\]

with \(\Gamma^{(0)}_{ij,k}\) the coefficients of Riemannian connection \(\nabla^{(0)}\) and

\[
T_{ijk} := E[\partial_i l_\theta \partial_j l_\theta \partial_k l_\theta].
\]

Similarly to the Riemannian case, we have the \(\alpha\)-curvature tensor in components [17, 34]

\[
R^{(\alpha)}_{kl} = \partial_k \Gamma^{(\alpha)}_{lj} - \partial_j \Gamma^{(\alpha)}_{lk} + \Gamma^{(\alpha)}_{kj} \Gamma^{(\alpha)}_{lj} - \Gamma^{(\alpha)}_{kj} \Gamma^{(\alpha)}_{lj},
\]

which can also be written as \(R^{(\alpha)}_{kl} = R^{(\alpha)}_{klij} g_{si}\). The \(\alpha\)-Ricci curvature tensor and \(\alpha\)-scalar curvature are defined respectively as

\[
Ric^{(\alpha)}_{ij} = R^{(\alpha)}_{klij} g^{kl}
\]
and

\[ K^{(\alpha)}_i = \text{Ric}^{(\alpha)}_{ij} g^{ij}, \]  

(3.10)

where \((g^{ij})\) is the inverse of the Fisher information metric. The \(\alpha\)-sectional curvature tensor is given by

\[ \kappa^{(\alpha)}_{ij} = -\frac{R^{(\alpha)}_{ijkj} g^{ii} g^{jj} - g^{2}_{ij}}{g^{ij}}, \quad i \neq j. \]  

(3.11)

3.2 Exponential families

Pdfs of exponential families, one of the most important class of statistical models, can be written as

\[ p(x; \theta) = \exp \left\{ \frac{F_i(x) \theta^i + C(x)}{2} - \psi(\theta) \right\}, \]  

(3.12)

which are assumed to satisfy the regularity conditions. The functions \(F_i(x)\) are smooth with respect to \(x\). From the normalization condition \(\int p(x; \theta) \, dx = 1\), the potential function is given by

\[ \psi(\theta) = \ln \int \exp \left\{ \frac{F_i(x) \theta^i + C(x)}{2} \right\} \, dx. \]  

(3.13)

The normal distribution, e.g. Example 3.1, is a well-known example of exponential families however the Weibull distribution, e.g. Example 3.2, does not belong to exponential families. Geometric quantities of exponential families can be neatly written using the potential functions \(\psi(\theta)\).

**Theorem 3.3** ([3][7][87]). For exponential families:

1. The Fisher information metric is given by \(g_{ij} = \partial_i \partial_j \psi(\theta)\), which implies that potential functions are convex.

2. The coefficients of \(\alpha\)-connection are

\[ \Gamma^{(\alpha)}_{ij,k} = \frac{1}{2} T_{ijk}, \]  

(3.14)

where

\[ T_{ijk} = \partial_i \partial_j \partial_k \psi(\theta). \]  

(3.15)

3. The components of \(\alpha\)-curvature tensor are

\[ R^{(\alpha)}_{ijkl} = \frac{1}{4} \left( T_{kmi} T_{jln} - T_{kmj} T_{iln} \right) g^{m\ell}. \]  

(3.16)

4 Statistical Einstein manifolds of exponential families

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold which satisfies the equation (1.1), namely

\[ \text{Ric} = -\lambda g. \]

**Example 4.1** (Cont.). For Example 3.1, it is an exponential family as we can rewrite the pdf as

\[ p(x; \theta) = \exp \left\{ \frac{\mu x - \frac{1}{2} \sigma^2 x^2}{2} - \frac{\mu^2}{2 \sigma^2} - \ln \sigma - \frac{1}{2} \ln 2\pi \right\}. \]  

(4.1)

Introducing the so-called natural parameters \((\theta^1 = \frac{\mu}{\sigma}, \theta^2 = -\frac{1}{2 \sigma^2})\), and a 2-tuple \((F_1 = x, F_2 = x^2)\), we have

\[ p(x; \theta) = \exp \left\{ F_i \theta^i - \psi(\theta) \right\}, \]  

(4.2)

where

\[ \psi(\theta) = -\frac{(\theta^1)^2}{4 \theta^2} - \frac{1}{2} \ln (-\theta^2) + \frac{1}{2} \ln \pi. \]  

(4.3)
Therefore, we can get the Fisher information metric and the Ricci curvature tensor as

\[
g = \begin{pmatrix}
\sigma^2 & \frac{2\mu\sigma^2}{2\mu^2 + 2\sigma^4} \\
\frac{2\mu\sigma^2}{2\mu^2 + 2\sigma^4} & \frac{\mu^2 - \mu^2}{\sigma^2}
\end{pmatrix}
\] (4.4)

and

\[
Ric = \begin{pmatrix}
-\frac{\sigma^2}{2} & -\frac{\mu\sigma^2}{\sigma^2} \\
-\frac{\mu\sigma^2}{\sigma^2} & -\frac{2\mu^2 - \sigma^4}{\sigma^2}
\end{pmatrix}.
\] (4.5)

It is obvious that they satisfy the equation (1.1)

\[
Ric = -\frac{1}{2}g = -\kappa_{ij}g,
\] (4.6)

with \(-\lambda = \kappa_{12} = -\frac{1}{2}\) the sectional curvature.

Example 4.2 (Cont.). Example 3.2 does not belong to exponential families, however, it also corresponds to a statistical Einstein manifold. By viewing \(\theta = (a, b)\) as local coordinates, the Fisher information metric is

\[
g = \begin{pmatrix}
\frac{6b^2}{\pi^2a^2} & -\frac{6(1-\xi)}{\pi^2a} \\
-\frac{6(1-\xi)}{\pi^2a} & \frac{6\xi^2 - 2\xi + \frac{a^2}{\pi^2} + 1}{\pi^2}
\end{pmatrix},
\] (4.7)

where \(\xi\) is the Euler’s constant. The Ricci curvature is given by

\[
Ric = \begin{pmatrix}
\frac{6b^2}{\pi^2a^2} & -\frac{6(1-\xi)}{\pi^2a} \\
-\frac{6(1-\xi)}{\pi^2a} & \frac{6\xi^2 - 2\xi + \frac{a^2}{\pi^2} + 1}{\pi^2}
\end{pmatrix}.
\] (4.8)

Therefore,

\[
R = \frac{6}{\pi^2}g = -\kappa_{12}g,
\] (4.9)

where \(-\lambda = \kappa_{12} = -\frac{6}{\pi^2}\) is the Gaussian curvature. This fact implies that the Weibull distribution manifold is an Einstein manifold as well.

For exponential families, we are able to write down the equation (1.1) explicitly for the potential functions:

\[
R^{(0)}_{ijkl}g^{kl} = -\lambda g_{ij},
\] (4.10)

that is,

\[
\sum_{m,n,k,l} \frac{1}{4} \{\partial_i\partial_j\partial_m\psi(\theta)\partial_k\partial_j\partial_n\psi(\theta) - \partial_k\partial_l\partial_m\psi(\theta)\partial_i\partial_j\partial_n\psi(\theta)\}g^{mn}g^{kl} = -\lambda \partial_i\partial_j\psi(\theta).
\] (4.11)

For dimension two, in particular, we have the Fisher information metric

\[
g = \begin{pmatrix}
\partial_1\partial_1\psi & \partial_1\partial_2\psi \\
\partial_2\partial_1\psi & \partial_2\partial_2\psi
\end{pmatrix},
\] (4.12)

and therefore \(\det(g) = \partial_1\partial_1\psi\partial_2\partial_2\psi - (\partial_1\partial_2\psi)^2\). The Riemannian curvature tensor is given by

\[
R^{(0)}_{1212} = \frac{1}{4}\sum_{m,n} (\partial_1\partial_1\partial_1\partial_1\partial_2\partial_2\partial_n\psi - \partial_1\partial_2\partial_m\psi\partial_1\partial_2\partial_n\psi)g^{mn}
\]

\[
= \frac{1}{4\det(g)} \left\{ \partial_1\partial_1\partial_1\partial_1\partial_2\partial_2\partial_2\partial_2\psi - (\partial_1\partial_2\partial_2\partial_2\psi)^2 \\
- \partial_1\partial_2\partial_2\partial_2\partial_2\partial_2\partial_2\psi - \partial_1\partial_1\partial_2\partial_2\partial_2\partial_2\psi \\
+ \partial_2\partial_2\partial_2\partial_2\partial_2\partial_2\partial_2\psi - (\partial_1\partial_1\partial_2\partial_2\partial_2\psi)^2 \right\}.
\] (4.13)
The components of the Ricci curvature are
\[
Ric_{11}^{(0)} = -\frac{\partial_1 \partial_1 \psi}{\det(g)} R_{1212}^{(0)}, \quad Ric_{21}^{(0)} = Ric_{12}^{(0)} = -\frac{\partial_1 \partial_2 \psi}{\det(g)} R_{1212}^{(0)}, \quad Ric_{22}^{(0)} = -\frac{\partial_2 \partial_2 \psi}{\det(g)} R_{1212}^{(0)}.
\] (4.14)

Therefore, the equation has the following equivalent representation
\[
-\frac{R_{1212}^{(0)}}{\det(g)} = -\lambda,
\] (4.15)
the left hand side of that is the sectional curvature. So in this case, we can replace the constant $-\lambda$ by the curvature $\kappa_{12}$. In this situation, the Einstein manifold is equivalent to a manifold with constant curvature.

**Theorem 4.3** (Theorem 4.3). A 2 or 3-dimensional pseudo-Riemannian manifold is Einstein if and only if it has constant sectional curvature.

In the next section, some special solutions of the equation (4.15) are obtained, in particular group-invariant solutions via Lie point symmetries, that give us the potential functions for statistical Einstein manifolds of exponential families.

## 5 Two dimensional statistical Einstein manifolds of exponential families

In this section, we obtain some special solutions of the equation (4.15) for two dimensional exponential families. There is a prerequisite that the potential function is convex. For example, it is impossible to find any traveling wave solution of the form $\psi(\theta^1, \theta^2) = f(\theta^1 - c \theta^2)$ with $c$ a constant, since $\det(g) = 0$ for any $c$ and any smooth function $f$.

### 5.1 The ansatz method

First, we try the ansatz method, namely by pre-assuming the shapes of potential functions $\psi(\theta^1, \theta^2)$ and solving the relevant equation (4.15).

1. **Summation of two arbitrary functions**, i.e. $\psi(\theta^1, \theta^2) = f(\theta^1) + h(\theta^2)$.

   In this case, the Riemannian curvature tensor always vanishes by (4.13), and the Fisher information metric is $g = \text{diag}(f''+h'', h'')$. Therefore, the parameter must be $\lambda = 0$ and any functions $f$ and $h$ satisfying the convexity condition that $f'' > 0$ and $h'' > 0$ for all parameters $\theta^1$ and $\theta^2$ provide a potential function for a statistical Einstein manifold.

2. **Summation of an arbitrary function and a traveling wave function**, i.e. $\psi(\theta^1, \theta^2) = f(\theta^1) + h(\theta^1 - c \theta^2)$ with $c$ a nonzero constant.

   The Riemannian curvature tensor vanishes as well, and the Fisher information metric is given by
   \[
   g = \begin{pmatrix}
   f'' + h'' & -ch'' \\
   -ch'' & c^2 h''
   \end{pmatrix}.
   \] (5.1)

   Therefore $\lambda = 0$, and any set of functions $f$ and $h$ satisfying the convexity condition $f''+(1+c^2)h'' > 0$ and $f''h'' > 0$ provides a solution.

3. **Multiplication of two arbitrary functions**, i.e. $\psi(\theta^1, \theta^2) = f(\theta^1)h(\theta^2)$.

   In this case, we are only able to obtain some particular solutions. Now the Fisher information metric is
   \[
   g = \begin{pmatrix}
   f''h & f'h' \\
   f'h' & fh''
   \end{pmatrix},
   \] (5.2)

   which implies the convexity condition that $f''h + fh'' > 0$ and $\det(g) = fh(f''h'' - (f'h')^2) > 0$ for all $\theta^1$ and $\theta^2$ in some domain $\Theta \subset \mathbb{R}^2$. The equation (4.15) becomes
   \[
   4\lambda \det(g)^2 + f(f'^2 - f'f'')hh'h'' + f'(ff'' - f'f'')hh'' + f''(f'^2 - ff'')h^2h'' = 0.
   \] (5.3)
When $\lambda = 0$, the equation above is reduced to the zero-curvature equation

$$f(f'^2 - f''^2)hh'h'' + f'(f''^2 - f'f''')hh'^2 + f''(f'^2 - f''')h'^2h'' = 0,$$

(5.4)
and it can be solved explicitly as follows. Define $A_1(\theta^1) = f(f''^2 - f'f''')$, $A_2(\theta^1) = f'(f''^2 - f'f''')$ and $A_3(\theta^1) = f''(f'^2 - f''')$. As $h'' \neq 0$, rewrite the equation above as

$$A_1(\theta^1)hh'h'' + A_2(\theta^1)hh'^2 + A_3(\theta^1)h'^2h'' = 0,$$

(5.5)
and divide $hh'^2$ on both sides

$$A_1(\theta^1)\frac{h'h''}{h'^2} + A_2(\theta^1) + A_3(\theta^1)\frac{h'^2}{hh''} = 0.$$

(5.6)
Then we differentiate it with respect to $\theta^2$ to get

$$A_1(\theta^1)\left(\frac{h'h''}{h'^2}\right)' + A_2(\theta^1)\left(\frac{h'^2}{hh''}\right)' = 0.$$

(5.7)
Therefore, noting that $h' \neq 0$ from the convexity condition,

- If $\frac{h'^2}{hh''} = c_1 \neq 0$, that is $\left(\frac{h'^2}{hh''}\right)' = 0$, it can be solved as follows:
  - When $c_1 = 1$, we have
    $$h(\theta^2) = c_3 \exp(c_2 \theta^2)$$
    (5.8)
    and hence $\left(\frac{h'h''}{h'^2}\right)' = 0$. The equation (5.4) holds constantly for any function $f$. Therefore, a solution would be
    $$\psi(\theta^1, \theta^2) = f(\theta^1) \exp(c_2 \theta^2),$$
    (5.9)
    where $f$ is an arbitrary function such that the convexity condition is satisfied.
  - Otherwise when $c_1 \neq 1$, we have
    $$h(\theta^2) = (c_2 \theta^2 - c_3)^{\frac{1}{c_1 - 1}}$$
    (5.10)
    and again $\left(\frac{h'h''}{h'^2}\right)' = 0$ holds for the solution obtained above. Equation (5.14) gives an extra equation of $f$
    $$(2 - c_1)A_1(\theta^1) + A_2(\theta^1) + c_1 A_3(\theta^1) = 0,$$
    whose general solution is
    $$f(\theta^1) = c_6 (\theta^1 - c_5)^{c_4}.$$
    (5.11)

In general, the potential function is not convex. However, it is possible to find special convex solutions, e.g.

$$\psi(\theta^1, \theta^2) = (\theta^1 - c_5)^{c_4}(\theta^2 - c_3)^2, \quad \theta^1 > c_5, \theta^2 \neq -c_3, -1 < c_4 < 0.$$

(5.12)

- Let $\left(\frac{h'^2}{hh''}\right)' \neq 0$.
  - If $A_1(\theta^1) = 0$, we have $A_3(\theta^1) = 0$, namely $\frac{f'^2}{f'^2} = 1$. The solution is
    $$f(\theta^1) = c_3 \exp(c_2 \theta^1)$$
    (5.13)
    and $h$ is arbitrary.
If \( A_1(\theta^1) \neq 0 \), 
\[
\left( \frac{h' h''}{h'^2} \right)' = c_1 \left( \frac{h'^2}{h'^2} \right)'
\]
and 
\( A_3 = -c_1 A_1 \) where \( c_1 \neq 0 \). Simple calculation gives that
\[
\frac{h' h''}{h'^2} = c_1 \frac{h'^2}{h'^2} + c_2, \tag{5.14}
\]
and
\[
c_1 \frac{f' f''}{f'^2} = \frac{f'^2}{f'^2} + c_1 - 1. \tag{5.15}
\]
These two equations are similar that solution of (5.15) is simplify a change of parameter in the solution of (5.11). It is difficult to solve them but we can still find some special solutions. For instance, if we let \( c_1 = 1, c_2 = 0 \), both functions \( f \) and \( h \) are of the form
\[
c_4 \exp(c_3 \theta) + c_5 \exp(-c_3 \theta), \tag{5.16}
\]
where \( \theta \) is \( \theta^1 \) or \( \theta^2 \) respectively. For example, let us assume that all the constants are nonzero, then the function
\[
\psi(\theta^1, \theta^2) = (c_1^2 \exp(c_3 \theta^1) + c_2^2 \exp(-c_3 \theta^1)) (c_4^2 \exp(c_6 \theta^2) + c_5^2 \exp(-c_6 \theta^2)) \tag{5.17}
\]
is convex.

When \( \lambda \neq 0 \), the equation is even more complicated. It is still possible to obtain special solutions using the same algorithm above, however we do not provide details here.

In general, one may start with any ansatz and solve the resulting equation (4.15). For example, the ansatz \( \psi(\theta^1, \theta^2) = f(\theta^1) h(\theta^2) + g(\theta^1) \) will lead to the potential function of the normal distribution in Example 4.1. In principle, the solutions are potential functions for statistical Einstein manifolds once the convexity condition is satisfied.

### 5.2 Group-invariant solutions

In this part, we use the symmetry method to systematically consider group-invariant solutions of (5.14). We rewrite the equation using a time-space convention by letting \( \theta^1 = t \) and \( \theta^2 = x \),
\[
\psi_{tt} \psi_{xxx} - \psi^2_{txx} - \psi_{txx} \left( \psi_{tt} \psi_{xxx} - \psi_{txx} \right) + \psi_{xx} \left( \psi_{tt} \psi_{xxx} - \psi^2_{txx} \right) = 4 \lambda \left( \psi_{tt} \psi_{xxx} - \psi^2_{txx} \right)^2. \tag{5.18}
\]
Note that the \( x \) here is a space variable rather than the possible value of a random variable in a probability density function. The convexity condition is equivalent to the positive-definiteness of \( g \), namely
\[
\psi_{tt} + \psi_{xx} > 0 \text{ and } \det(g) = \psi_{tt} \psi_{xxx} - \psi^2_{txx} > 0. \tag{5.19}
\]

Using the LSC condition (2.3), we obtain a nine-dimensional Lie algebra of infinitesimal generators spanned by
\[
\begin{align*}
X_1 &= \partial_t, & X_2 &= \partial_x, & X_3 &= \partial_{\psi}, \\
X_4 &= t \partial_t, & X_5 &= x \partial_x, & X_6 &= x \partial_t, \\
X_7 &= t \partial_x, & X_8 &= t \partial_{\psi}, & X_9 &= x \partial_{\psi}. \tag{5.20}
\end{align*}
\]
Next, we use (linear combinations of) them to calculate group-invariant solutions of potential functions for the equation (5.18). The constant \( a \) is nonzero. The following list is not exhaustive; for instance, one may also consider a linear combination of multiple vector fields. Below we only present the group-invariant solutions which are potential to be convex.

- Using \( X_4 + a X_2 \):
\[
\psi(\theta^1, \theta^2) = -\frac{1}{4 \lambda} \ln \left\{ c_2 \exp \left( c_1 \theta^2 - c_1 a \ln \theta^1 \right) - 1 \right\} + c_3; \tag{5.21}
\]
• Using $X_5 + aX_1$:

$$\psi(\theta^1, \theta^2) = -\frac{1}{4\lambda} \ln \left\{ c_2 \exp \left( c_1 \theta^1 - c_1 a \ln \theta^2 \right) - 1 \right\} + c_3; \quad (5.22)$$

• Using $X_6 + aX_2$:

$$\psi(\theta^1, \theta^2) = -\frac{1}{4\lambda} \ln \left\{ c_2 \exp \left( c_1 (\theta^2)^2 - 2c_1 a \theta^1 \right) - 1 \right\} + c_3; \quad (5.23)$$

• Using $X_7 + aX_1$:

$$\psi(\theta^1, \theta^2) = -\frac{1}{4\lambda} \ln \left\{ c_2 \exp \left( c_1 (\theta^1)^2 - 2c_1 a \theta^2 \right) - 1 \right\} + c_3; \quad (5.24)$$

• Using $X_8 + aX_5$:

$$\psi(\theta^1, \theta^2) = \frac{(\theta^1 - c_1 a) \ln (\theta^1 - c_1 a) - (1 + 4c_1 \lambda) \theta^1 \ln \theta^1}{4ac_1 \lambda} + \frac{\theta^1 \ln \theta^2}{4a} + c_2 \theta^1 + c_3; \quad (5.25)$$

• Using $X_8 + aX_6$:

$$\psi(\theta^1, \theta^2) = \frac{(\theta^2 - c_1 a) \ln (\theta^2 - c_1 a) - \theta^2 \ln \theta^2}{4c_1 \lambda} + \frac{(\theta^1)^2}{2a \theta^2} + c_2 \theta^2 + c_3; \quad (5.26)$$

• Using $X_9 + aX_4$:

$$\psi(\theta^1, \theta^2) = \frac{(\theta^2 - c_1 a) \ln (\theta^2 - c_1 a) - (1 + 4c_1 \lambda) \theta^2 \ln \theta^2}{4ac_1 \lambda} + \frac{\theta^2 \ln \theta^1}{4a} + c_2 \theta^2 + c_3; \quad (5.27)$$

• Using $X_9 + aX_7$:

$$\psi(\theta^1, \theta^2) = \frac{(\theta^1 - c_1 a) \ln (\theta^1 - c_1 a) - \theta^1 \ln \theta^1}{4c_1 \lambda} + \frac{(\theta^2)^2}{2a \theta^1} + c_2 \theta^1 + c_1. \quad (5.28)$$

These group-invariant solutions are not necessary convex on the whole plane $\mathbb{R}^2$. It is always possible, however, to construct convexity domains for the invariant solutions. In statistics, the parameters are also often defined on a restricted domain rather than the whole plane, e.g. the normal distribution in Example 4.1.

6 Conclusion

In this paper, we have considered statistical Einstein manifolds of exponential families. Particularly in dimension two, this is equivalent to searching for constant curvature statistical manifolds. We used an ansatz method and symmetry method to construct potential functions of exponential families, whose constant curvature can be written using potential functions and their derivatives only. Nevertheless, there are challenging open questions to be investigated further.

One open question is to recover a probability density function from a given potential function, that we will call the inverse problem. As illustrated, we are always, at least in principle, able to get the potential function from a given pdf. It is not obvious if it is still possible and how we can do it vise versa. For instance, in dimension two, we will have to solve the following integral equation for unknown functions $F_1(x)$, $F_2(x)$ and $C(x)$,

$$\exp \left( \psi(\theta^1, \theta^2) \right) = \int \exp \left\{ F_1(x) \theta^1 + F_2(x) \theta^2 + C(x) \right\} \, dx,$$
where the potential function $\psi(\theta_1, \theta_2)$ is given.

Another one is the investigation of higher dimensional statistical Einstein manifolds of exponential families, especially for dimensions greater than 4 when the Einstein condition (1.1) is not equivalent to constant-curvature condition any more. Well-known higher dimensional exponential families include the multivariate normal distribution, the Dirichlet distribution, the Wishart distribution, the multinomial distribution and so on; see for instance, [19, 28].

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