Emission angle distribution and flavor transformation of supernova neutrinos

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Abstract

Using moment equations we analyze collective flavor transformation of supernova neutrinos. We study the convergence of moment equations and find that numerical results using a few moment converge quite fast. We study effects of emission angle distribution of neutrinos on neutrino sphere. We study scaling law of the amplitude of neutrino self-interaction Hamiltonian and find that it depends on model of emission angle distribution of neutrinos. Dependence of neutrino oscillation on different models of emission angle distribution is studied.

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1 Introduction

Flavor transformation of neutrinos in core-collapse supernova is one of the important remaining problems in neutrino physics. This problem has been investigated by many researchers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. It is realized that neutrino density above neutrino sphere in supernova can be so large that neutrino-neutrino refraction can dominate flavor transformation of neutrinos. Research on the effect of neutrino-neutrino refraction is difficult because it is caused by neutrino self-interaction and is of non-linear nature. Complete numerical analysis use discrete set of energy bins and angle bins of neutrinos. Evolutions of million equations have to be followed. It is very complicated.

In a recent work we derived a set of moment equations describing the transport and flavor transformation of neutrinos in supernova [23]. Distribution of neutrinos over angle $\theta_p$, the angle of neutrino direction intersecting with radial direction in supernova, is encoded in moments of density matrix. The equation of neutrino is expanded using these moments. Instead of using a large number of angle bins we just need a few moments to do numerical study. It is shown that numerical works can be simplified by about two orders of magnitude in comparison with multi-angle simulation. Moreover, this formulation of neutrino in supernova also offers us a way to study the effect of emission angle distribution of neutrinos on the transport and flavor transformation of neutrinos.
In this article we analyze the effect of emission angle distribution of neutrinos. In section 2 we make a quick review on the moment equations. In section 3 we analyze the scaling behavior of the strengths of moments in different models of neutrino emission. We check convergence property of moment equations in the analysis of the strengths of moments. In section 4 we analyze effect of different models of neutrino emission on collective neutrino oscillation. We summarize in section 5.

2 Moment equations

In Ref. [23] we introduced moments of $\rho_{\vec{p}}(t,r)$, density matrix for neutrinos at given time $t$ and radius $r$:

$$\rho_k(t,r,|\vec{p}|) = \int d\Omega_{\vec{p}} (1 - \cos \theta_p)^k \rho_{\vec{p}}(t,r), \ k = 0, 1, 2 \ldots ,$$  \hspace{1cm} (1)

where $\theta_p$ is the angle of neutrino direction intersecting with the radial direction, as shown in Fig 1. Similarly we introduced $\tilde{\rho}_k$ for anti-neutrinos. We also introduced re-scaled moments

$$\rho'_k = z^{2(k+1)} \rho_k,$$  \hspace{1cm} (2)

where

$$z = r/r_0,$$  \hspace{1cm} (3)

$r_0$ is the radius of neutrino sphere. Similarly we introduced $\tilde{\rho}'_k$ for anti-neutrinos. In Fig. 1 one can see clearly

$$\sin \theta_p = \frac{r_0}{r} \sin \theta_{p0}$$  \hspace{1cm} (4)

It is easy to see

$$1 - \cos \theta_p = \left(\frac{r_0^2}{r^2}\right)^{1/2} \left(1 + \sqrt{1 - \frac{r_0^2}{r^2} \sin^2 \theta_{p0}}\right).$$ \hspace{1cm} (5)

It scales approximately as $r^{-2}$. Together with the scaling behavior of the zeroth moment a geometric factor $z^{-2(k+1)}$ is found for moment $\rho_k$. The factor $z^{2(k+1)}$ is introduced in Eq. (2) to compensate this geometric scaling factor.

Using some approximations we arrive at the following set of moment equations

$$\frac{d\rho'_k}{dr} = -r_0^{-1} Q_k^1 - i[H_A, \rho'_k], \ k = 0, 1, \ldots , N$$  \hspace{1cm} (6)
Figure 1: Geometric picture of angles of the neutrino momentum intersect with \( \mathbf{r} \).

where \( N \geq 1 \) is an integer

\[
Q_k^1 = z^{2k} \sum_{l=k+1}^{N} (l + 1) z^{-(2l+1)} \rho'_l, \tag{7}
\]

\[
H_A = H_0 + \sqrt{2} G_F (L + z^{-4} D_1), \tag{8}
\]

\[
D_1 = \int \frac{dE}{(2\pi)^3} E^2 [\rho'_1(r, E) - \bar{\rho}'_1(r, E)]. \tag{9}
\]

Eq. (6) is a set of truncated moment equations in \( P_N \) approximation for which \( \rho'_k = 0 \) \((\bar{\rho}'_k = 0)\) has been set for \( k > N \). \( Q_k^1 = 0 \). \( H_0 \) is the Hamiltonian for vacuum oscillation, \( L = \text{diag}\{n_e, n_\mu, n_\tau\} \) in the flavor base is the matter term given by charged lepton number densities \( n_{e,\mu,\tau} \). \( G_F \) is the Fermi constant. Equation for \( \bar{\rho}_k \) is similar except replacing \( H_0 \) by \(-H_0\).

A few points concerning moment equations are as follows: a) Physical observables are described by \( \rho_0 \) and \( \rho_1 \). Integration of \( E^2 Tr[\rho_0] \) over energy gives the neutrino density and integration of \( E^2 Tr[\rho_0 - \rho_1] \) gives the neutrino flux; b) Emission angle distribution of neutrinos on neutrino sphere is described by moments \( \rho_k \) and their effect in the neutrino flavor transformation can be systematically studied; c) The strength of \( \rho'_k \), \( Tr[\rho'_k] \), is modified by \( Q_k^1 \) term and does not change if this term is neglected; d) The scaling law of the self-interaction Hamiltonian is no longer \( z^{-4} \) when \( N > 1 \) and is modified by higher moments. Precise scaling behavior should depend on the model of neutrino emission.
3 Emission angle distribution and scaling law of moments

In this section we analyze the strengths of zeroth and first moments, that is $Tr[\rho_0,1]$, in different models of neutrino emission. This analysis can tell us a lot on how strong neutrino self-interaction is. It can also tell us a lot on the convergence property of moment equations. This is because $\rho_0$ and $\rho_1$ are the most important quantities in our problem. Physical observables are given by $\rho_0$ and $\rho_1$. When neutrino self-interaction gives dominant contribution flavor transformation of neutrinos is controlled by $D_1$ which is directly related to $\rho_1$. Effects of higher moments on $Tr[\rho_0,1]$ tell us how large higher moments affect the flavor transformation of neutrinos.

We consider three models of neutrino emission on neutrino sphere.

Model I, neutrino is uniformly emitted with respect to the emission angle $\theta_{p_0}$ and

$$\rho_k(t,r_0) = \frac{1}{k+1} \rho_0(t,r_0) \tag{10}$$

Model II, emission angle distribution of neutrinos is proportional to $\cos \theta_{p_0}$ and

$$\rho_k(t,r_0) = \frac{2}{(k+1)(k+2)} \rho_0(t,r_0), \tag{11}$$

Model III, emission angle distribution of neutrinos is proportional to $(1 - \cos \theta_{p_0}) \cos \theta_{p_0}$
and
\[ \rho_k(t, r_0) = \frac{6}{(k + 2)(k + 3)} \rho_0(t, r_0). \] (12)

The evolution of \( \text{Tr}[\rho_k] \) is simple and is obtained by taking the trace of Eq. (6):
\[ \frac{d\text{Tr}[\rho'_k]}{dr} = -r_0^{-1}Q_k \] (13)
where
\[ Q_k = z^{2k} \sum_{l=k+1}^{N} (l + 1)z^{-(2l+1)}\text{Tr}[\rho'_l]. \] (14)

\( Q_N = 0 \). The second term in (6) does not contribute to \( \text{Tr}[\rho'_k] \).

We do numerical analysis for \( F_0 \) and \( F_1 \):
\[ F_0 = \text{Tr}[^0\rho'(r)]/\text{Tr}[^0\rho_0(r_0)], \quad F_1 = \text{Tr}[^1\rho'(r)]/\text{Tr}[^0\rho_0(r_0)] \] (15)
\( F_{0,1} \) are \( \text{Tr}[^0\rho_{0,1}] \) relative to \( \text{Tr}[^0\rho_0] \) at \( r = r_0 \). In our numerical analysis we work in two flavor system of \((\nu_e, \nu_x)\). We choose \( L_{\nu_e} = L_{\bar{\nu}_e} = L_{\nu_x} = L_{\bar{\nu}_x} = 3. \times 10^{51} \text{ erg/s}. \) The initial energy spectrum of neutrino is given by the Fermi-Dirac distribution
\[ f_\nu(E) = \frac{1}{N_2 T_\nu} \frac{x^2}{e^{x-\mu_\nu} + 1}, \] (16)
where \( x = E/T \) and \( N_2 \) is the normalization factor. Parameters of four types of neutrinos and anti-neutrinos are chosen as: \( T_{\nu_e} = 2.76 \text{ MeV, } T_{\bar{\nu}_e} = 4.01 \text{ MeV, } T_{\nu_x} = T_{\bar{\nu}_\mu} = 6.26 \text{ MeV. } \mu_{\nu_e} = \mu_{\bar{\nu}_e} = \mu_{\nu_x} = \mu_{\bar{\nu}_x} = 3. \)

In Fig. 2, 3 and 4 we show results in Model I, II and III separately. A number of characteristics can be read out in these figures:

i) In \( P_1 \) approximation \( F_1 \) keeps as a constant. This is because \( Q_1 \) is set to zero in this approximation.

ii) In \( P_2 \) approximation \( F_1 \) is modified. Results of \( F_0 \) do not agree with those in \( P_1 \) approximation.

iii) In \( P_3 \) approximation results of \( F_0 \) become close to those in \( P_2 \) approximations. Results of \( F_1 \) do not agree with those in \( P_2 \) approximation. This is because in \( P_2 \) approximation \( F_1 \) becomes corrected by \( Tr[\rho'_2] \) but \( Tr[\rho'_2] \) is still a constant. In \( P_3 \) approximation \( Tr[\rho'_2] \) is also corrected and its contribution to \( F_1 \) is modified.

iv) Results of Model II and III in \( P_4 \) and \( P_6 \) approximations agree perfectly for both \( F_0 \) and \( F_1 \).

v) Results of Model I in \( P_4 \) and \( P_6 \) approximations are in good agreement for \( F_0 \). For \( F_1 \) there are still some small differences.

A few comments are as follows:

a) Results in Model II and Model III converge faster than the results in Model I. This is in agreement with the observation that higher moments in Model II and Model III are more suppressed than those in Model I. Hence Model II and Model III should have better convergence properties.
b) Value of $F_0$ at large radius can be understood using flux conservation. The flux of neutrino is given by

$$Tr[\rho_0 - \rho_1] = z^{-2} Tr[\rho_0' - z^{-2} \rho_1']$$  \hspace{1cm} (17)$$

The flux, as it should be, scales as $z^{-2}$ (or $r^{-2}$) in stationary approximation. So $F_0 - z^{-2} F_1$ is a conserved quantity. At large $r$ this quantity approaches to $F_0$. On the other hand its initial value can be read out directly from the models of neutrino emission. Using Eqs. (10), (11) and (12) we find that at large $r$

$$F_0 \to \frac{1}{2}, \text{ in Model I}$$  \hspace{1cm} (18)$$
$$F_0 \to \frac{2}{3}, \text{ in Model II}$$  \hspace{1cm} (19)$$
$$F_0 \to \frac{1}{2}, \text{ in Model III}$$  \hspace{1cm} (20)$$

These values are in agreement with the plots in Figs. 2, 3 and 4.

c) The scaling behavior of $F_1$ tells us that in $P_N$ approximation with $N > 1$ the self-interaction Hamiltonian scales down faster than $r_0^4/r^4$.

d) Numerical study shows that $Tr[\rho_k']$ with $k > 1$ also drops down by $10^{-1} - 10^{-2}$ at large $r$. It is a further support to the point that moment equations converge quite fast.

4 Flavor transformation

In this section we do some analysis on flavor transformation of supernova neutrinos. We study the case of inverted mass hierarchy and for simplicity we neglect matter effect in the analysis.

In Fig. 5 we give plots of $\nu_e$ fraction versus radius. These plot are obtained by solving Eq. (6) numerically. For a small step we get

$$\rho_k'(r + \Delta r) = -\frac{\Delta r}{r_0}Q_{kk}[\rho_1'(r)] + e^{-iH_A \Delta r} \rho_k'(r)e^{iH_A \Delta r}$$  \hspace{1cm} (21)$$

In these plots one can see synchronized oscillation for which neutrinos of all energy point to the same direction in flavor space. Beyond the region of synchronized oscillation neutrino flavor vectors spin down which leads to neutrino flavor conversion.

We compare numerical results of $P_4$ approximation and of $P_6$ approximation in models I and III. We find nice agreements between these two approximations. This shows that $P_N$ approximation converge quite fast. This is in agreement with the discussion in the last section that the scaling law of the strength of the Hamiltonian converge quite fast.
In Fig. 5 one can see that result of $P_1$ approximation is quite different from that of $P_4$ and $P_6$ approximations. This is also consistent with discussion in the last section. Since the scaling law of the Hamiltonian in $P_1$ approximation is quite different from that in $P_4,6$ approximation we would expect to find difference in oscillation pattern.

In Fig. 6 we compare numerical results in models I, II and III. One can see that there are some differences in the oscillation pattern. This is consistent with the analysis on the strength of self-interaction Hamiltonian in these models. Numerical results show that at large $r$

$$F_1 \rightarrow 0.126, \text{ in Model I} \quad (22)$$

$$F_1 \rightarrow 0.134, \text{ in Model II} \quad (23)$$

$$F_1 \rightarrow 0.151, \text{ in Model III} \quad (24)$$

Since differences in Hamiltonian are not large at large radius and the differences in the oscillation pattern should not be large either.

We note that the scaling law of $F_1$, hence the amplitude of neutrino self-interaction Hamiltonian, is model dependent. This dependence on model is nicely described by the corrections given by higher moments in moment equations. Previous researches use fixed scaling function for the self-interaction Hamiltonian and do not take into account the dependence of the scaling law on the emission angle distribution of neutrinos. As a
comparison one can check the result using a fixed scaling function. For example, one can use \( \rho_1 = S^2 \rho_1(r_0) \) where \( S(r) = z^2/(1 + \sqrt{1 - z^2}) \). Hence \( F_1 = 0.5/(1 + \sqrt{1 - z^2})^2 \) in model I, \( F_1 = 0.33/(1 + \sqrt{1 - z^2})^2 \) in model II and \( F_1 = 0.5/(1 + \sqrt{1 - z^2})^2 \) in model III. At large radius this model independent scaling function gives \( F_1 \to 0.125 \) in model I, \( F_1 \to 0.083 \) in model II and \( F_1 \to 0.125 \) in model III. Only for model I this fixed scaling function gives a correct result at large radius. In model II this fixed scaling function gives a value quite different from the value obtained using moment equations.

5 Conclusion

In summary we have analyzed some properties of moment equations and the flavor transformation of supernova neutrinos. We have analyzed the scaling behavior of neutrino density and the amplitude of self-interaction Hamiltonian of neutrinos. They are related to quantities \( \rho_0 \) and \( \rho_1 \).

We analyzed the convergence of \( P_N \) approximation of moment equations. Numerical results show that the scaling behavior of \( Tr[\rho_{0,1}] \) converge for \( N < 10 \). We show that results of neutrino oscillation also converge quite fast. These analysis are consistent. Since the integration of \( E^2 \rho_1 \) give the self-interaction Hamiltonian the analysis on \( Tr[\rho_1] \) tell us how fast the amplitude of neutrino self-interaction converge.

We analyze neutrino flavor transformation. We find synchronized oscillation and bipolar oscillation in the oscillation pattern of supernova neutrinos. We find that oscillation

Figure 6: (color online) Fraction of \( \nu_e, n_{\nu_e}/(n_{\nu_e} + n_{\nu_x}) \), versus radius in different models. Neutrino parameters are the same as in Fig. 5
pattern of neutrinos converge quite fast for $N < 10$. The $P_1$ approximation can be used to make qualitative analysis but can not be used to do precise numerical study.

We study three models of emission angle distribution of neutrinos on the oscillation pattern and analyze model dependence of neutrino flavor transformation on the emission angle distribution of neutrinos. Different models of emission angle distribution can give different results in the scaling behavior of self-interaction Hamiltonian and in oscillation pattern of neutrinos. This model dependence is carefully taken into account in the correction given by higher moments in moment equations.

Previous works on oscillation of supernova neutrinos use fixed scaling function for the self-interaction Hamiltonian and do not take into the fact that the scaling law can be different in different models of neutrino emission. Analysis on the model dependent effect of emission angle distribution in neutrino oscillation is not presented in previous works.

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