STABILITY OF MINIMISING HARMONIC MAPS UNDER $W^{1,p}$ PERTURBATIONS OF BOUNDARY DATA: $p \geq 2$

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Abstract. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Consider a harmonic map $v : \Omega \rightarrow S^2$ with boundary data $v|\partial \Omega = \varphi$ which minimises the Dirichlet energy. For $p \geq 2$, we show that any energy minimiser $u$ whose boundary map $\psi$ has a small $W^{1,p}$-distance to $\varphi$ is close to $v$ in Hölder norm modulo bi-Lipschitz homeomorphisms, provided that $v$ is the unique minimiser attaining the boundary data. The index $p = 2$ is sharp: the above stability result fails for $p < 2$ due to the constructions by Almgren–Lieb [2] and Mazowiecka–Strzelecki [15].

1. Introduction

Let $u : \Omega \rightarrow S^2$, where $\Omega$ is a Lipschitz domain in $\mathbb{R}^3$ and $S^2$ is the unit 2-sphere. We are concerned with the boundary value problem for the harmonic map equation:

$$
\begin{cases}
-\Delta u = |\nabla u|^2 u & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
$$

(1.1)

This is the Euler–Lagrange equation for minimisers of the Dirichlet energy

$$E[u] := \int_{\Omega} |\nabla u(x)|^2 \, dx$$

(1.2)

over the space

$$W^{1,2}(\Omega, S^2) := \{ u \in W^{1,2}(\Omega, S^2) : u|\partial \Omega = \varphi \}.$$  

(1.3)

The existence of minimisers are well-known for $\varphi \in W^{1/2,2}(\partial \Omega, S^2)$ in the sense of trace, due to the lower semi-continuity of the functional $E$. Also, for $\varphi \in W^{1,2}(\partial \Omega, S^2)$ the space $W^{1,2}_\varphi(\Omega, S^2)$ is non-empty, as it contains the degree-0-homogeneous extension $\varphi(x/|x|)$.

The weak solutions to (1.1) are called (weakly) harmonic maps. Minimisers of the Dirichlet integral clearly satisfy (1.1), hence we call them minimising harmonic maps. The singular set of a harmonic map $u$, denoted by $\text{sing } u$, consists of the points that have an open neighbourhood in $\Omega$ in which $u$ is not Hölder continuous — equivalently, not real-analytic ([18], [3], [16]). We remark that there are non-minimising harmonic maps. As a prominent example, Rivière [17] constructed a harmonic map $v : B \rightarrow S^2$ with $\text{sing } v = \overline{B}$, but Schoen–Uhlenbeck [18] proved that minimising harmonic map $u : B \rightarrow S^2$ must have discrete singular sets ($\overline{B}$ is the unit 3-ball).

In this note, we study the stability of the minimising harmonic maps $u$ with respect to the boundary data $\varphi$. In a very interesting recent paper [15], by elaborating on Almgren–Lieb’s constructions in [2], Mazowiecka–Strzelecki proved that $u$ is highly non-stable under $W^{1,p}$-perturbations of $\varphi$ for $p < 2$ and $\Omega = B$.

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Proposition 1.1 (Theorem 1.1 in [15]). Let \( \varphi \in C^\infty(\partial B, S^2) \) be a degree-0 boundary map. Let \( 1 \leq p < 2 \) and \( N \in \mathbb{N} \) be arbitrary. Then, for each \( \epsilon > 0 \), there exists \( \psi \in C^\infty(\partial B, S^2) \) such that \( \deg \psi = 0 \), \( \|\varphi - \psi\|_{W^{1,p}} < \epsilon \), and the Dirichlet integral has a unique minimiser over \( W^{1,2} \) with at least \( N \) singularities in \( B \).

In contrast, R. Hardt and F.-H. Lin [10] proved that a minimising harmonic map is stable under Lipschitz perturbations of the boundary data, under an additional uniqueness assumption:

Proposition 1.2 (The Stability Theorem in [10]). Let \( \Omega \subset \mathbb{R}^3 \) be a smooth bounded domain and \( \varphi \in \text{Lip}(\partial \Omega, S^2) \). Suppose \( v \) is the unique energy-minimising map from \( \Omega \) to \( S^2 \) with \( v|\partial \Omega = \varphi \). Then there exist a positive number \( \beta > 0 \) and, for any \( \epsilon > 0 \), a positive number \( \delta > 0 \), such that for any \( \psi \in C^{1,\alpha}(\partial \Omega, S^2) \) with \( \|\varphi - \psi\|_{\text{Lip}} \leq \delta \) and any energy-minimising \( u \in W^{1,2}(\Omega, S^2) \) with \( u|\partial \Omega = \psi \), one has \( \|u - v \circ \eta\|_{C^{0,\beta}} \leq \epsilon \) for a bi-Lipschitz map \( \eta : \Omega \to \Omega \) with \( \|\eta - \text{id}_\Omega\|_{\text{Lip}} \leq \epsilon \).

Our main result shows that, under the same assumptions of [10], minimising harmonic maps are stable under \( W^{1,p} \)-perturbations of the boundary data for any \( p \geq 2 \). It demonstrates the sharpness of the index \( p = 2 \) in Proposition 1.1 Proposition 1.2 is the special case \( p = \infty \).

Theorem 1.3. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded Lipschitz domain and \( \varphi \in W^{1,p}(\partial \Omega, S^2) \) for \( p \geq 2 \). Suppose \( v \) is the unique energy-minimising map from \( \Omega \) to \( S^2 \) with \( v|\partial \Omega = \varphi \). Then there exist a positive number \( \beta > 0 \) and, for any \( \epsilon > 0 \), a positive number \( \delta > 0 \), such that for any \( \psi \in C^{1,\alpha}(\partial \Omega, S^2) \) with \( \|\varphi - \psi\|_{W^{1,p}} \leq \delta \) and any energy-minimising \( u \in W^{1,2}(\Omega, S^2) \) with \( u|\partial \Omega = \psi \), one has \( \|u - v \circ \eta\|_{C^{0,\beta}} \leq \epsilon \) for a bi-Lipschitz map \( \eta : \Omega \to \Omega \) with \( \|\eta - \text{id}_\Omega\|_{\text{Lip}} + \|\eta^{-1} - \text{id}_\Omega\|_{\text{Lip}} \leq \epsilon \).

The arguments essentially follow [10] by Hardt–Lin. We remark that the uniqueness of \( v \) is necessary: see §5, [10] for an example of a smooth boundary map that serves as boundary data for two minimisers from \( B \) to \( S^2 \), one with no singularity and the other with two singularities. Moreover, Almgren–Lieb [2] proved that the boundary data with unique minimisers are dense in the \( W^{1,2} \)-topology.

Notations. For embedded surfaces \( \Sigma \subset \mathbb{R}^3 \), we write \( dA \) for the surface measure on \( \Sigma \), and \( \nabla \) for the projection of the Euclidean gradient on \( \mathbb{R}^3 \) to \( T\Sigma \). In the spherical polar coordinates, we write \( x = r \omega \) for \( r = |x|, \omega = x/|x| \in S^2 \), the unit 2-sphere. For an \( m \)-dimensional submanifold \( M \) of \( \mathbb{R}^n \), \( |M| \) denotes the \( m \)-dimensional Hausdorff measure of \( M \). We write \( B(x, \rho) \) for an Euclidean 3-ball with centre \( x \) and radius \( \rho \); \( B_\rho := B(0, \rho) \) and \( B := B_1 \). For sets \( E \) and \( F \), we write \( E \sim F \) for the set difference, and \( 1_E \) for the indicator function on \( E \). The norms \( \|\bullet\|_{W^{1,p}}, \|\bullet\|_{\text{Lip}} \) and \( \|\bullet\|_{C^{0,\beta}} \) without explicitly indicating the domains are taken over the whole of \( \Omega, B \) or \( S^2 \), which will be clear from the context. \( O(3) \) is the group of \( 3 \times 3 \) orthogonal matrices.

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Note added. Upon completion of the paper, the author was informed of the very nice work [14] by Mazowiecka–Miskiewicz–Schikorra, in which a generalisation of Hardt–Lin’s stability theorem
is obtained independently. In [14] the stability in $W^{1,2}$-norm is proved for minimising harmonic maps with trace in $W^{s,p}$ for $s \in [1/2, 1]$, $p \in [2, \infty]$ such that $ps \geq 2$, provided that the traces are $W^{s,p}$-close. This may be compared with Theorem 1.3 above, in which we proved the stability in $C^{0,\beta}$-norm with traces in $C^{1,\alpha}$ being $W^{1,p}$-close. Additionally, in [14] Almgren–Lieb’s linear law on the number of singularities is also extended to the case of $W^{s,p}$-traces.

2. Uniform Boundary Regularity

In this section, we establish the following

Lemma 2.1. There exist constants $0 < e_0, \ell_0 \leq 1$ and $\rho_0 = \rho_0(\ell_0, e_0) > 0$ such that the following holds. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz map with $g(0) = 0 = |\nabla g(0)|$ and $\|g\|_{W^{1,\infty}} \leq \ell_0$. Denote by $\Omega_g := \{(x_1, x_2, x_3) \in \mathbb{B} : x_3 < g(x_1, x_2)\}$. Assume that $u \in W^{1,2}(\Omega_g, \mathbb{S}^2)$ is an energy-minimising map with $\|u\|_\mathbb{B} \cap \partial \Omega_g \leq e_0$; $2 \leq p \leq \infty$. Then $\|u\|_{W^p,\rho_0} \cap \Omega_g \leq e_0$ for some $0 < \beta < 1$.

The proof of Lemma 2.1 follows from an adaptation of §§5.4, 5.5, Hardt–Kinderlehrer–Lin [6] and §2, Hardt–Lin [10], in both of which the boundary data are assumed to be Lipschitz. On the other hand, if $\Omega_g$ is $C^\infty$ additionally, then we recover Corollary 2.5, Almgren–Lieb [2].

We need to modify the arguments in [6] [11] to deal with the lower regularity assumptions for the boundary map and the domain. One useful result is Theorem 5.7, Hardt–Lin [11]:

Lemma 2.2. Let $m$ be a positive integer, let $N$ be a smooth Riemannian manifold, and let $1 < p < \infty$. Denote by $\mathbb{B}^+ := \{(x^1, \ldots, x^m) \in \mathbb{R}^m : \sum_{i=1}^m |x^i|^2 < 1, x^m > 0\}$. If $u_0 \in W^{1,p}(\mathbb{B}^+, N)$ is a degree-0-homogeneous $p$-minimising harmonic map, and if $u_0$ is constant on $\partial \mathbb{B}^+ \cap \{x^m = 0\}$, then $u_0$ is a constant function.

We also recall the monotonicity formula: let $u$ be an energy-minimiser and $0 < \sigma < \rho < \rho_0$ such that $\mathbb{B}(y, \rho_0) \subseteq \mathbb{B}$. Then

$$
\frac{1}{\rho} \int_{\mathbb{B}(y, \rho)} |\nabla u|^2 \, dx - \frac{1}{\sigma} \int_{\mathbb{B}(y, \sigma)} |\nabla u|^2 \, dx = \int_{\mathbb{B}(y, \rho) \cap \mathbb{B}(y, \sigma)} \frac{2}{r} \left| \frac{\partial u}{\partial r} \right|^2 \, dx \geq 0. \tag{2.1}
$$

The proof follows by considering “squeeze deformations” of $u$; cf. Lemma 2.5, [18]; Lemma 1.3, [19] and §2.4, [21], among others.

Proof of Lemma 2.2. By a standard blowup argument — cf. §5 in Hardt–Kinderlehrer–Lin [6] — it suffices to prove a uniform bound on the rescaled energy: for $\rho_0$ sufficiently small, there exists $c_0 > 0$ such that

$$
\frac{1}{\rho_0} \int_{\mathbb{B}_{2\rho_0}(0) \cap \Omega_g} |\nabla u|^2 \, dx \leq c_0. \tag{2.2}
$$

(One may conclude by choosing $c_0$ depending on $e_0$, and then shrinking $\rho_0$ if necessary.)

As in [6], (2.2) will follow from an absolute bound

$$
\int_{\mathbb{B}_{1/2}(0) \cap \Omega_g} |\nabla u|^2 \, dx \leq c_1, \tag{2.3}
$$

where $c_1$ depends only on $p$ and $\ell_0$. In particular, the arguments for “energy decay/improvement” in §§5.4, 5.5, [19] can be directly adapted to the case of $W^{1,p}$-boundary data. In the sequel let us exhibit a $c_1$. 

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For $a.e. \ \sigma \in [1/2, 1]$, choose a bi-Lipschitz map $\Phi_\sigma : B_\sigma \cap \Omega_\sigma \to \Omega_\sigma$. The bi-Lipschitz constant of $\Phi_\sigma$ is universal; let us call it $\Lambda$. It depends only on $\|g\|_{W^{1, \infty}} \leq \ell_0$. We claim that there is $\omega_\sigma$, an extension of $(u \circ \Phi_\sigma^{-1})/\partial B_\sigma$, that satisfies the following inequality:

$$\int_{B_\sigma} |\nabla \omega_\sigma|^2 \, dx \leq c_2 \left\{ \int_{\partial B_\sigma} |\nabla (u \circ \Phi_\sigma^{-1})|^2 \, dA \right\}^{1/2}$$ \hspace{1cm} (2.4)

for $a.e. \ \sigma \in [1/2, 1]$.

To see this, we follow the arguments in §2.3, [10]. Let $\lambda = \lambda(\sigma)$ be the vector $|B_\sigma|^{-1} \int_{B_\sigma} (u \circ \Phi_\sigma^{-1}) \, dx$ in $\mathbb{R}^3$. By Fubini’s theorem, for $a.e. \ \sigma' \in [\sigma/2, \sigma]$ we have

$$\int_{\partial B_{\sigma'}} |\nabla (u \circ \Phi_\sigma^{-1})|^2 \, dA \leq 8 \int_{B_\sigma} |\nabla (u \circ \Phi_\sigma^{-1})|^2 \, dx,$$ \hspace{1cm} (2.5)

$$\int_{\partial B_{\sigma'}} |(u \circ \Phi_\sigma^{-1}) - \lambda|^2 \, dA \leq 8 \int_{B_\sigma} |(u \circ \Phi_\sigma^{-1}) - \lambda|^2 \, dx.$$ \hspace{1cm} (2.6)

The right-hand sides of (2.5) and (2.6) are finite, thanks to

$$\int_{B_\sigma} |\nabla (u \circ \Phi_\sigma^{-1})|^2 \, dx \leq \Lambda^2 \int_{\Omega_\sigma \cap B_\sigma} |\nabla u|^2 \, dx$$

and the Poincaré inequality.

Let $h : B_{\sigma'} \to \mathbb{R}^3$ be the harmonic function — i.e., $\Delta h = 0$ — with $h|\partial B_{\sigma'} = (u \circ \Phi_\sigma^{-1})|\partial B_{\sigma'}$. By an elementary computation, all harmonic functions fulfil the identity

$$\int_{\partial B_{\sigma'}} |\nabla h|^2 \, dA = \int_{B_{\sigma'}} |\nabla h|^2 \, dx + \sigma' \int_{\partial B_{\sigma'}} \left| \frac{\partial h}{\partial r} \right|^2 \, dA.$$ \hspace{1cm} (2.7)

Thus, using integration by parts, the Cauchy–Schwarz inequality, (2.7), and that $\nabla h = \nabla (u \circ \Phi_\sigma^{-1})$ on $\partial B_{\sigma'}$, we deduce

$$\int_{B_{\sigma'}} |\nabla h|^2 \, dx = \int_{B_{\sigma'}} (h - \lambda) \cdot \frac{\partial (h - \lambda)}{\partial r} \, dA$$

$$\leq \left\{ \int_{\partial B_{\sigma'}} |(u \circ \Phi_\sigma^{-1}) - \lambda|^2 \, dA \right\}^{1/2} \left\{ \int_{\partial B_{\sigma'}} |\nabla (u \circ \Phi_\sigma^{-1})|^2 \, dA \right\}^{1/2}.$$ \hspace{1cm} (2.8)

Now let us modify $h$ to a function with range in $S^2$ satisfying the same bound as in (2.8). Denote by $\Pi_a : \mathbb{R}^3 \to S^2$ the projection

$$\Pi_a(x) := \frac{x - a}{|x - a|}.$$  

By Sard’s theorem, $\Pi_a \circ h \in W^{1, 2}(B_{\sigma'}, S^2)$ for almost every $a \in B_{\sigma'/2}$. We have

$$|\nabla (\Pi_a \circ h)| = \left| \frac{\nabla h}{|h - a|} - \frac{\nabla h \cdot (h - a) \otimes (h - a)}{|h - a|^3} \right| \leq 2 \frac{|\nabla h|}{|h - a|}.$$  

Thus

$$\int_{B_{\sigma'/2}} \int_{B_{\sigma'}} \left| \nabla (\Pi_a \circ h(x)) \right|^2 \, dx \, da \leq 4 \int_{B_{\sigma'}} \left| \nabla h(x) \right|^2 \left\{ \int_{B_{\sigma'/2}} |h(x) - a|^{-2} \, da \right\} \, dx$$

$$\leq 4\pi \int_{B_{\sigma'}} \left| \nabla h(x) \right|^2 \, dx.$$
In particular, by Fubini we can choose $a \in B_{\sigma'/2}$ such that

$$
\int_{B_{\sigma'}} |\nabla (\Pi_a \circ h(x))|^2 \, dx \leq 8\pi \int_{B_{\sigma'}} |\nabla h(x)|^2 \, dx.
$$

One thus deduces from (2.8) that

$$
\int_{B_{\sigma'}} |\nabla (\Pi_a \circ h(x))|^2 \, dx \leq 8\pi \left\{ \int_{\partial B_{\sigma'}} \left| (u \circ \Phi^{-1}) - \lambda \right|^2 \, dA \right\}^{1/2} \left\{ \int_{\partial B_{\sigma'}} \left| \nabla (u \circ \Phi^{-1}) \right|^2 \, dA \right\}^{1/2}.
$$

But $u$ takes values in $S^2$; so

$$
\int_{\partial B_{\sigma'}} \left| (u \circ \Phi^{-1}) - \lambda \right|^2 \, dA
$$

$$
\leq 2 \int_{\partial B_{\sigma'}} \left| (u \circ \Phi^{-1}) \right|^2 \, dA + 2 \int_{\partial B_{\sigma'}} \lambda^2 \, dA \leq 4|\partial B_{\sigma'}| \leq 16\pi;
$$

hence

$$
\int_{B_{\sigma'}} |\nabla (\Pi_a \circ h(x))|^2 \, dx \leq 32\pi^{3/2} \left\{ \int_{\partial B_{\sigma'}} \left| \nabla (u \circ \Phi^{-1}) \right|^2 \, dA \right\}^{1/2}.
$$

(2.9)

Finally, set

$$
w_\sigma := (\Pi_a|\partial B_{\sigma'})^{-1} \circ \Pi_a \circ h.
$$

(2.10)

By construction $w_\sigma|\partial B_{\sigma'} = (u \circ \Phi^{-1})|\partial B_{\sigma'}$. The Lipschitz norm of $(\Pi_a|\partial B_{\sigma'})^{-1}$ can be bounded geometrically as follows. For $a \in B_{\sigma'/2}$ given, set up the polar coordinate centred at $a$. Then $\|(\Pi_a|\partial B_{\sigma'})^{-1}\|_{\text{Lip}}$ equals the maximal ratio $\ell_{a,\sigma'/\theta_a}$, where $\theta_a$ is the angle between two straight lines emanating from $a$, and $\ell_{a,\sigma'}$ is the length of the arc $A$ on $\partial B_{\sigma'}$ swept out by such straight lines opening at angle $\theta_a$. By elementary Euclidean geometry, the supremum over $a \in B_{\sigma'/2}$ of $\ell_{a,\sigma'/\theta_a}$ is attained only if $a \in \partial B_{\sigma'/2}$ and $\theta_a$ is bisected by the straight line through $a$ and 0. In this case, $\ell_{a,\sigma'/\theta_a} = \sigma'\alpha/\theta_a$, where $\alpha$ is the angle formed by arc $A$ and the origin. Clearly $\sigma'\alpha/\theta_a \leq 2\sigma' \leq 2$; hence

$$
\|(\Pi_a|\partial B_{\sigma'})^{-1}\|_{\text{Lip}} \leq 2.
$$

We can thus conclude (2.4) by choosing $c_2 = 128\pi^{3/2}$ (replacing $\sigma'$ with $\sigma$).

Now, define

$$
D(\sigma) := \int_{B_\sigma \cap \Omega_g} |\nabla u|^2 \, dx.
$$

(2.11)

By the minimality of $u$, we have

$$
D(\sigma) \leq \int_{B_{\sigma'} \cap \Omega_g} |\nabla (\omega_\sigma \circ \Phi_\sigma)|^2 \, dx
$$

$$
\leq \|\nabla \Phi_\sigma\|_{L^\infty}^2 \int_{B_{\sigma'}} |\nabla \omega_\sigma|^2 \, dx
$$

$$
\leq c_2 \|\nabla \Phi_\sigma\|_{L^\infty}^2 \left\{ \int_{\partial B_{\sigma'}} \left| \nabla (u \circ \Phi^{-1}) \right|^2 \, dA \right\}^{1/2}
$$

$$
\leq c_2 \|\nabla \Phi_\sigma\|_{L^\infty}^2 \|\nabla \Phi^{-1}\|_{L^\infty} \left\{ \int_{\partial B_{\sigma'} \cap \Omega_g} |\nabla u|^2 \, dA + \int_{B_{\sigma'} \cap \Omega_g} |\nabla u|^2 \, dA \right\}^{1/2}.
$$

Hölder’s inequality and the assumptions on $\|u|B \cap \partial \Omega_g\|_{W^{1,p}}$ and $g$ give us

$$
\int_{B_{\sigma'} \cap \partial \Omega_g} |\nabla u|^2 \, dA \leq \left\{ \int_{\partial \Omega_g} |\nabla u|^p \, dA \right\}^2 \|B_{\sigma'} \cap \partial \Omega_g\|^{p-2}_{\frac{p}{p}}
$$
\[
\leq 1 \times \left( \int_{\{z \in \mathbb{R}^2 : |z| \leq \sigma \}} \sqrt{1 + |\nabla g|^2} \, dz \right)^{\frac{p-2}{p}} \leq \left( \sqrt{2} \pi \sigma^2 \right)^{\frac{p-2}{p}}. \tag{2.12}
\]

Thus, for a.e. \( \sigma \in [1/2, 1] \), with the previously chosen value of \( c_2 \) we have
\[
D(\sigma) \leq 128 \pi^{3/2} \Lambda^3 \left( D'(\sigma) + \left( \sqrt{2} \pi \sigma^2 \right)^{\frac{p-2}{p}} \right)^{1/2}. \tag{2.13}
\]

To prove (2.13), it is enough to establish \( D(1/2) \leq c_1 \). Let us write \( c_1 = \theta^{-1} \) and assume for contradiction that \( D(1/2) > \theta^{-1} \) for each \( \theta > 0 \). Then
\[
\int_{1/2}^{1} \frac{D'(\sigma)}{D^2(\sigma)} \, d\sigma = \frac{1}{D(1)} - \frac{1}{D(1/2)} > -\theta.
\]

On the other hand, by (2.13) there holds
\[
\frac{D'(\sigma)}{D^2(\sigma)} \geq \left( \frac{1}{128 \pi^{3/2} \Lambda^3} \right)^2 - \left( \frac{1}{D(\sigma)} \right)^{\frac{p-2}{p}} \geq \left( \frac{1}{128 \pi^{3/2} \Lambda^3} \right)^2 - \left( \frac{1}{D(1/2)} \right)^{\frac{p-2}{p}} \theta^2.
\]

Integrating \( \sigma \) over \([1/2, 1]\), we get
\[
\varphi(\theta) := \left( \frac{1}{\sqrt{2} \pi} \right)^{\frac{p-2}{p}} \theta^2 + 2 \theta - \frac{1}{16384 \pi^3 \Lambda^6} > 0.
\]

However, \( \varphi \) has a positive root \( \theta_0 > 0 \), so any \( \theta \in (0, \theta_0] \) would violate the above inequality. To be concrete, we can take \( \theta = \theta_0/2 \), i.e.,
\[
c_1 = 2^{1+\frac{p-2}{p}} \frac{\pi^{\frac{p-2}{p}}}{\pi^{\frac{p-2}{p}}} \left( 1 + \left( \frac{16384 \pi^3 \Lambda^6}{\sqrt{2} \pi} - 1 \right)^{-1} \right),
\]

where \( \Lambda \) is the supremum of the bi-Lipschitz constant of \( \Phi_\sigma \) over \( \sigma \in [1/2, 1] \). This gives the desired contradiction and thus concludes (2.3).

Finally, let us establish the bound (2.2). If it were false, for some \( c > 0 \) there would exist sequences of positive numbers \( \{\rho_i\} \searrow 0 \), \( \{\epsilon_i\} \searrow 0 \), and \( \{\ell_i\} \searrow 0 \), Lipschitz maps \( \{g_i\} \) with \( \|g_i\|_{W^{1,\infty}} \leq \ell_i \), and minimisers \( \{u_i\} \subset W^{1,2}(\Omega_{g_i}, S^2) \), such that
\[
\|u_i\|_{\mathbb{B} \cap \partial \Omega_{g_i}} \leq \epsilon_i \quad \text{but} \quad \liminf_{i \to \infty} \frac{1}{\rho_i} \int_{\mathbb{B}_{2\rho_i} \cap \partial \Omega_{g_i}} |\nabla u_i|^2 \, dx \geq c. \tag{2.14}
\]

Denote by
\[
\tilde{u}_i(x) := u_i(2\rho_i x), \quad \tilde{g}_i(x) := g_i(2\rho_i x).
\]

Then \( \|	ilde{g}_i\|_{W^{1,\infty}} \leq 2 \rho_i \ell_i \) and
\[
\frac{1}{2 \rho_i} \int_{\mathbb{B}_{\rho_i} \cap \Omega_{\tilde{g}_i}} |\nabla u_i|^2 \, dx = \int_{\mathbb{B}_{1/2} \cap \Omega_{\tilde{g}_i}} |\nabla \tilde{u}_i|^2 \, dx \leq c_1,
\]

where \( c_1 \) is as in (2.3). As a result, a subsequence of \( \{\tilde{u}_i\} \) converges weakly to \( v \in W^{1,2}(\mathbb{B}^+, S^2) \). By monotonicity identity (2.1), \( v \) is degree-0-homogeneous. Thanks to Theorem 6.4 in Hardt–Lin [11], the convergence \( \tilde{u}_i \to v \) is indeed strong in the \( W^{1,2} \)-topology, and \( v \) is a minimising harmonic map. But the first inequality in (2.14) implies that the limiting map \( v \in W^{1,2}(\mathbb{B}^+, S^2) \) is constant on \( \mathbb{B} \cap \{x_3 = 0\} \), up to the choice of a representative in the Sobolev class. In view of Lemma 2.2, this contradicts the second inequality in (2.14).

Hence the assertion follows. \(\square\)
3. The Model Case: Stability of Hedgehog on $\Omega = \mathbb{B}$

In this section we prove Theorem 1.3 for the model case $\Omega = \mathbb{B}$, $\varphi = \text{id}_{S^2}$ with $p > 2$. The general case shall be obtained by gluing these building blocks together in §4, with modifications for the critical case $p = 2$. Recall from the hypotheses of Theorem 1.3 that the boundary map $\psi$ has $C^{1,\alpha}$-regularity; see [14] for results on $\psi$ with lower regularity.

We shall crucially rely on the result below due to L. Simon (see Theorem 1, [20] and the exposition [21]).

**Proposition 3.1.** Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $u \in W^{1,2}(\Omega, S^2)$ be an energy-minimising map. Assume that $\Theta(x/|x|)$ is a tangent map of $u$ at 0, where $\Theta \in \mathcal{O}(3)$ is a rotation. Then such $\Theta$ is unique. Moreover, there are uniform constants $\beta_0 \in [0, 1]$ and $c > 0$ depending only on $\Omega$ such that for all $r > 0$ sufficiently small, we have

$$A(r) := \left\| \frac{\partial}{\partial r} u(r \cdot) \right\|_{C^1(S^2)} + \| u(r \cdot) - \Theta \|_{C^2(S^2)} \leq cr^{\beta_0} \left\| u - \frac{x}{|x|} \right\|_{C^2(\mathbb{B}_2/\mathbb{B}_{1/1})}.$$

Let us recall the notion of tangent maps (see §3.1, [21] for details). In the setting of Proposition 3.1, take $\mathbb{B}(y, \rho_0) \Subset \Omega$, and for any $\rho \in [0, \rho_0]$ define the blowup maps

$$u_{y,\rho}(x) := u(y + \rho x).$$

Then, by the monotonicity formula (2.1), there holds $\int_{\mathbb{B}} |\nabla u_{y,\rho}|^2 \, dx \leq \rho_0^{-1} \int_{\mathbb{B}(y, \rho_0)} |\nabla u|^2 \, dx$. By [18] [11], for any $\{\rho_j\} \searrow 0$ we can select a subsequence (not relabelled) $\{u_{y,\rho_j}\}$ that converges strongly in $W^{1,2}_{\text{loc}}$ on $\mathbb{R}^n$ to an energy-minimiser $u_0$. Any $u_0$ thus obtained is called a tangent map of $u$ at $y$. The uniqueness of tangent maps remains a major open problem in the large.

Proposition 3.1 is a consequence of Simon’s asymptotic theory of nonlinear evolution equations developed in [20]. Indeed, Theorem 1 and Section 8 therein show that $A(r) \to 0$ as $r \searrow 0$, provided that the target manifold $N$ is real-analytic. When specialising to $N = S^2$, it follows from Brezis–Coron–Lieb [4] that the tangent map in the proposition must be of the form $\Theta(x/|x|)$ for a rotation $\Theta$. In this case, the integrability of Jacobi fields (see Simon, §6 in [21] and Gulliver–White [5]) yields the desired decay estimate of $A(r)$. Similar arguments were used in the proof of convergence to tangent cones of minimal submanifolds by Almgren–Allard in [1].

3.1. Singularity is Unique. Take $\Omega = \mathbb{B}$ and $\varphi = \text{id}_{S^2}$. Then $v : \mathbb{B} \to S^2$, the unique minimising map with $v|\partial \Omega = \text{id}_{S^2}$, is the “hedgehog”

$$v(x) = \frac{x}{|x|},$$

(see Brezis–Coron–Lieb [4]). Assume for contradiction that a sequence $\{u_i\} \subset W^{1,2}(\mathbb{B}, S^2)$ is energy-minimising with boundary data $\psi_i := u_i|\partial \mathbb{B} \in C^{1,\alpha}(\partial \mathbb{B}, S^2)$, so that

$$\delta_i := \| \psi_i - \text{id}_{S^2} \|_{W^{1,p}} \to 0$$

but $u_i$ has more than one singularity for large enough $i$.  

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First, by the minimality of \( u_i \), we get
\[
\int_{B} |\nabla u_i|^2 \, dx \leq \int_{B} \left| \nabla \left\{ \psi_i \left( \frac{x}{|x|} \right) \right\} \right|^2 \, dx
\]
\[
\leq \int_{S^2} |\nabla \psi_i(x)|^2 \, dA
\]
\[
\leq (1 + \kappa) \int_{S^2} |\nabla \text{id}_{S^2}|^2 \, dA + \left( 1 + \frac{1}{\kappa} \right) \int_{S^2} |\nabla (\psi_i - \text{id}_{S^2})|^2 \, dA
\]
for any small \( \kappa > 0 \). In the last line we used the simple inequality \((a+b)^2 \leq (1+\kappa)a^2+(1+\kappa^{-1})b^2\).

Moreover, it is well-known that \( x/|x| \) has the quantized energy \( 8\pi \):
\[
\int_{S^2} |\nabla \text{id}_{S^2}|^2 \, dA = \int_{B} \left| \nabla \left( \frac{x}{|x|} \right) \right|^2 \, dx = 8\pi.
\]

In addition,
\[
\int_{S^2} |\nabla (\psi_i - \text{id}_{S^2})|^2 \, dA \leq \|\psi_i - \text{id}_{S^2}\|^2_{W^{1,p}(B)} \leq (4\pi)^{\frac{p-2}{p}} \|\psi_i - \text{id}_{S^2}\|^2_{W^{1,p}}.
\]

Thus
\[
\int_{B} |\nabla u_i|^2 \, dx \leq (1 + \kappa) 8\pi + \left( 1 + \frac{1}{\kappa} \right) (4\pi)^{\frac{p-2}{p}} (\delta_i)^2. \tag{3.1}
\]

Thanks to the \( W^{1,2} \)-bound in (3.1), \( \{u_i\} \) has a subsequence (not relabelled) that converges weakly in \( W^{1,2} \). By sending first \( i \nearrow \infty \) and then \( \kappa \searrow 0 \), any such limit function has energy \( \leq 8\pi \) and boundary map \( \text{id}_{S^2} \). Again by Brezis–Coron–Lieb [1], it must be \( x/|x| \). Using the arguments by Schoen–Uhlenbeck [18], also see L. Simon [21] via Luckhaus’ lemma [13], we have
\[
u_i(x) \rightarrow \frac{x}{|x|} \quad \text{strongly in } W^{1,2}. \tag{3.2}
\]

Now, in view of Lemma 2.1, there exists a universal \( \rho_0 > 0 \) such that \( u_i \) are uniformly Hölder continuous with uniformly bounded energy on some neighbourhood of \( \partial(B \sim B_{1-\rho_0}) \). By the definition of \( \delta_i \), \( \deg(\psi_i) \) is equal to 1 for sufficiently large \( i \). So the singular set \( \text{sing} u_i \) is non-empty and lies in \( B_{1-\rho_0} \), i.e., away from the boundary \( \partial B \). As \( x/|x| \) is Hölder continuous away from 0, thanks to (3.2) and the interior regularity result in [18], we may conclude that the diameter of \( \text{sing} u_i \) tends to zero.

For any \( r \in [0, 1/20] \), there is \( i \) large enough such that \( B_{1-|a_i|} \subset B_{1-\rho_0} \), \( |a_i| < r/4 \) for every \( a_i \in \text{sing} u_i \). Consider \( \bar{u}_i(x) := u_i(x + a_i) \) defined on \( B_{1-|a_i|} \). Then we have
\[
\left\| \bar{u}_i \left( \frac{x}{|x|} \right) \right\|_{C^2(B(a_i, 1/2) \sim B_{r_0} \sim B_{r/2})} \leq \left\| u_i \left( \frac{x}{|x|} \right) \right\|_{C^2(B_1 \sim B_{2r_0} \sim B_{1/10})} + \left\| \frac{x - a_i}{|x|} - \frac{x}{|x|} \right\|_{C^2(B_{1-2r_0} \sim B_{1/5})} \rightarrow 0.
\]

The convergence of the first term follows from the interior regularity theory (see Schoen–Uhlenbeck [18]), and the convergence of the second term can be deduced from direct computation. Using the asymptotic theory of Simon (Proposition 5.1), we have \( \text{sing} \bar{u}_i = \{0\} \) for sufficiently large \( i \). This contradicts the assumption that \( u_i \) has more than one singularities.

Therefore, there exists \( \delta > 0 \) such that for any \( \psi \in C^{1,\alpha}(\partial B, S^2) \) with \( \|\psi - \text{id}_{S^2}\|_{W^{1,p}} \leq \delta \), any minimiser \( u \) with \( u|\partial B = \psi \) has a unique singular point.

In the sequel we say \( \text{sing} u = \{a\} \).
3.2. Modulus of Singularity. To estimate the modulus \(|a|\), we pick some \(\rho \in [0, 1]\) and define

\[
w(x) := \begin{cases} 
  u(\rho^{-1}x), & 0 \leq |x| < \rho, \\
  z(x)/|z(x)|, & \rho \leq |x| \leq 1,
\end{cases}
\]

where

\[
z(x) := \frac{1}{1 - \rho} \left\{ \left( 1 - |x| \right) \psi \left( \frac{x}{|x|} \right) + \left( |x| - \rho \right) \frac{x}{|x|} \right\}.
\]

In \(\mathbb{B}_\rho\) there holds

\[
\int_{\mathbb{B}_\rho} |\nabla w(x)|^2 \, dx = \rho \int_{\mathbb{B}} |\nabla u(y)|^2 \, dy.
\]

For \(x \in \mathbb{B} \sim \mathbb{B}_\rho\), we shall estimate by

\[
\int_{\mathbb{B} \sim \mathbb{B}_\rho} |\nabla w|^2 \, dx = \int_{\mathbb{B} \sim \mathbb{B}_\rho} \left\{ \frac{|z|^2|\nabla z|^2 - |z \cdot \nabla z|^2}{|z|^4} \right\} \, dx \leq \int_{\mathbb{B} \sim \mathbb{B}_\rho} \frac{|\nabla z|^2}{|z|^2} \, dx.
\]

Notice that

\[
z(x) - \frac{x}{|x|} = \frac{1 - |x|}{1 - \rho} \left| \psi - \text{id}_{S^2} \right| \left( \frac{x}{|x|} \right);
\]

so for \(\rho \leq |x| \leq 1\) one has

\[
|z(x)| \geq 1 - \left| \frac{1 - |x|}{1 - \rho} \right| \left| \psi - \text{id}_{S^2} \right|_{L^\infty} \geq 1 - c_5 \delta,
\]

where \(c_5 = c(\rho)\) is the Sobolev constant for \(W^{1,p}(\partial \mathbb{B}, S^2) \mapsto C^0(\partial \mathbb{B}, S^2)\) for \(p > 2\). Hence

\[
\int_{\mathbb{B} \sim \mathbb{B}_\rho} |\nabla w|^2 \, dx \leq \int_{\mathbb{B} \sim \mathbb{B}_\rho} \frac{|\nabla z(x)|^2}{(1 - c_5)^2} \, dx.
\]

But

\[
\nabla z(x) - \nabla \left( \frac{x}{|x|} \right) = \frac{1}{1 - \rho} \left\{ \frac{x}{|x|} \otimes \left( \text{id}_{S^2} - \psi \right) \left( \frac{x}{|x|} \right) + \left( 1 - |x| \right) \left[ \nabla \psi \left( \frac{x}{|x|} \right) - \nabla \left( \frac{x}{|x|} \right) \right] \right\};
\]

so, computing in spherical polar coordinates using \((a + b)^2 \leq (1 + \kappa)a^2 + (1 + \kappa^{-1})b^2\) and Hölder’s inequality, we get

\[
\int_{\mathbb{B} \sim \mathbb{B}_\rho} |\nabla z|^2 \, dx \leq (1 + \kappa) \int_{\mathbb{B} \sim \mathbb{B}_\rho} \left| \nabla \left( \frac{x}{|x|} \right) \right|^2 \, dx + (1 + \kappa^{-1}) \int_{\mathbb{B} \sim \mathbb{B}_\rho} \left\{ \frac{1 - |x|}{1 - \rho} \left| \nabla \left( \frac{x}{|x|} \right) \right| \right\}^2 \, dx
\]

\[
+ (1 + \kappa^{-1}) \int_{\mathbb{B} \sim \mathbb{B}_\rho} \left\{ \frac{1 - |x|}{1 - \rho} \left| \nabla \left( \frac{x}{|x|} \right) \right| \right\}^2 \, dx
\]

\[
\leq (1 + \kappa) \int_{\mathbb{B} \sim \mathbb{B}_\rho} \left| \nabla \left( \frac{x}{|x|} \right) \right|^2 \, dx + \frac{1 + \kappa^{-1}}{(1 - \rho)^2} \left\| \nabla \left( \frac{x}{|x|} \right) \right\|_{L^{p/2}(\mathbb{B} \sim \mathbb{B}_\rho)}^2
\]

\[
+ \frac{1 + \kappa^{-1}}{(1 - \rho)^2} \left\| \nabla \left( \frac{x}{|x|} \right) - \nabla \left( \frac{x}{|x|} \right) \right\|_{L^{p/2}(\mathbb{B} \sim \mathbb{B}_\rho)}^2 \left\| (1 - |x|)^2 \right\|_{L^{p/2}(\mathbb{B} \sim \mathbb{B}_\rho)}^2
\]

\[
\leq (1 + \kappa) \int_{\mathbb{B} \sim \mathbb{B}_\rho} \left| \nabla \left( \frac{x}{|x|} \right) \right|^2 \, dx + \frac{(1 + \kappa^{-1})(1 - \rho^2)}{3(1 - \rho^2)} \frac{4\pi}{r^2} \delta^2.
\]

Putting the above estimates together, we arrive at

\[
\int_{\mathbb{B}} |\nabla w|^2 \, dx \leq \rho \int_{\mathbb{B}} |\nabla u|^2 \, dx + \frac{1 + \kappa}{(1 - c_5 \delta)^2} \int_{\mathbb{B} \sim \mathbb{B}_\rho} \left| \nabla \left( \frac{x}{|x|} \right) \right|^2 \, dx + c_6 \frac{1 + \kappa^{-1}}{(1 - c_5 \delta)^2} \delta^2,
\]

where \(c_6\) depends only on \(p\) (via the Sobolev constant \(c_5\)) and \(\rho\).
by the area formula. Therefore, using \((3.5)(3.6)\) and the monotonicity formula \((2.1)\), one deduces
\[
C > \kappa > 0 \text{ for each } \delta \leq \pi.
\]
for each \(\delta \leq \pi\), respectively. Combining with \((3.7)\) and \((3.8)\), we get
\[
\int_B |\nabla u|^2 \, dx + 8\pi(1 - \rho) \left\{ \frac{1 + \kappa}{(1 - c_5\delta)^2} - 1 \right\} + c_6 \frac{1 + \kappa^{-1}}{(1 - c_5\delta)^2} \delta^2
\]
for each \(\rho > 2, 0 < \rho < 1, \kappa > 0\) and sufficiently small \(\delta > 0\).

On the other hand, as \(w|\partial B = id_{S^2}\) and \(\text{sing } w = \{a\}\), the estimates by Brezis–Coron–Lieb \([4]\); also see the last inequality on p.117, \([10]\) lead to
\[
\int_B |\nabla u|^2 \, dx \geq 8\pi + c_7|a|^2
\]
with a universal constant \(c_7\). Furthermore, the estimate \((3.11)\) holds with \(u_i, \delta_i\) replaced by \(u\) and \(\delta\), respectively. Combining with \((3.7)\) and \((3.8)\), we get
\[
c_7 |a|^2 \leq 8\pi \kappa + \left(1 + \frac{1}{\kappa}\right) \left(4\pi \right)^{\frac{p-2}{p}} \delta^2
+ 8\pi(1 - \rho) \left\{ \frac{1 + \kappa}{(1 - c_5\delta)^2} - 1 \right\} + c_6 \frac{1 + \kappa^{-1}}{(1 - c_5\delta)^2} \delta^2
\]
(3.9)

For each \(\rho \in ]0, 1[\) fixed, the penultimate term on the right-hand side of \((3.9)\) satisfies
\[
c_8 \left\{ \kappa + 2c_5\delta + \mathcal{O}(\delta^2) \right\}
\]
as \(\delta \searrow 0\),
where \(c_8 = 8\pi(1 - \rho)\). Also, for \(0 < \kappa, \delta \ll 1\), there exists \(c_9 = c(\rho, p)\) such that the final term of \((3.9)\) can be bounded as follows:
\[
c_6 \frac{1 + \kappa^{-1}}{(1 - c_5\delta)^2} \delta^2 \leq c_9 \kappa^{-1} \delta^2.
\]
The optimal \(\kappa > 0\) we may choose is of order \(\mathcal{O}(\delta)\). We thus conclude from \((3.9)\) that
\[
|a| \leq c_{10} \sqrt{\delta}
\]
for all \(\delta \leq \delta_0\), where \(\delta_0 = c(\rho, p) > 0\) is sufficiently small and \(c_{10} = c(\rho, p)\).

From now on, let us fix the parameter \(\rho \in ]0, 1[\).

3.3. \(W^{1,p}\)-Stability for \(x/|x|\) for \(p > 2\). As proved earlier in this section, \(u\) has a unique singularity \(a\), whose norm is controlled by \(\sqrt{\delta}\) with \(\|\psi - id_{S^2}\|_{W^{1,p}} \leq \delta\) and \(u|\partial B = \psi \in C^{1,\alpha}(\partial B, S^2)\). Here \(u\) satisfies the assumptions of Theorem \([1.3]\) with \(\Omega = B\) and \(\varphi = id_{S^2}\); in particular, it is a minimising harmonic map.

Several consequences can be deduced (see p.118, \([10]\) —
By §3.1 and [4] we have the quantisation of energy:

\[
\limsup_{r \to 0} \frac{1}{r} \int_{B(a,r)} |\nabla u|^2\,dx = 8\pi \tag{3.11}
\]

where \(a\) is the singularity of \(u\).

The tangent map of \(u\) at \(a\) is unique (by Proposition 6.1) and takes the form \(\Theta (x/|x|)\) with \(\Theta \in O(3)\) (by Corollary 7.12, Brezis–Coron–Lieb [4]).

By Proposition 3.1 there are universal constants \(\beta_0 \in [0,1]\) and \(c_{11} > 0\) such that for \(r > 0\) sufficiently small,

\[
\| \frac{\partial}{\partial r} \tilde{u}(r \cdot) \|_{C^1(S^2)} + \| \tilde{u}(r \cdot) - \Theta \|_{C^2(S^2)} \leq c_{11} E r^{\beta_0}. \tag{3.12}
\]

Specifically, for any \(\alpha \in [0,\beta_0]\) one has

\[
\| u - \Theta \left( \frac{x-a}{|x-a|} \right) \|_{C^{0,\alpha}(B_{1/2})} \leq c_{11} E. \tag{3.13}
\]

Here, for \(\tilde{u} : B_{1-|a|} \to S^2\) and \(\tilde{u}(x) := u(x + a)\) we set

\[
\mathcal{E} := \| \tilde{u} - \frac{x}{|x|} \|_{C^2(B_{2/3} \sim B_{1/3})}. \tag{3.14}
\]

By [5] 20 21 there is a universal constant \(c_{12}\) such that

\[
\| \Theta - \text{id}_{\mathbb{R}^3} \| \leq c_{12} \mathcal{E}; \tag{3.15}
\]

Here \(\| \cdot \|\) denotes the matrix norm.

Having summarised (i)–(iv) above, let us proceed as follows.

First, on the boundary \(\partial B\), there holds

\[
\| \psi - \Theta \left( \frac{x-a}{|x-a|} \right) \|_{W^{1,p}(\partial B)} \leq \| \psi - \text{id}_{S^2} \|_{W^{1,p}(\partial B)} + c_{12} \mathcal{E} + \| \frac{x-a}{|x-a|} - \frac{x}{|x|} \|_{W^{1,p}(\partial B)}.
\]

But

\[
\nabla \left( \frac{x-a}{|x-a|} \right) - \nabla \frac{x}{|x|} = \delta_{ij} \left( \frac{1}{|x-a|} - \frac{1}{|x|} \right) + \frac{(x-a) \otimes (x-a)}{|x-a|^3} - \frac{x \otimes x}{|x|^3}, \tag{3.16}
\]

thus a direct computation using \(|x| = 1, |a| \leq c_{10} \sqrt{\delta}\) yields

\[
\| \psi - \Theta \left( \frac{x-a}{|x-a|} \right) \|_{W^{1,p}(\partial B)} \leq \delta + c_{12} \mathcal{E} + c_{13} \sqrt{\delta} \tag{3.17}
\]

for \(c_{13} = c(p)\).

Next, thanks to (3.12) (3.15), we have

\[
\| \psi - \Theta \left( \frac{x-a}{|x-a|} \right) \|_{W^{1,p}(\partial B_{1/2})} \leq c_{14} \mathcal{E} + \| \frac{x-a}{|x-a|} - \frac{x}{|x|} \|_{W^{1,p}(\partial B_{1/2})},
\]

where \(c_{14} = c(\beta_0)\) with the universal constant \(\beta_0\) in (iii). Taking \(|x| = 1/2\) in (3.16), one obtains

\[
\| \psi - \Theta \left( \frac{x-a}{|x-a|} \right) \|_{W^{1,p}(\partial B_{1/2})} \leq c_{14} \mathcal{E} + c_{15} \sqrt{\delta} \tag{3.18}
\]

for a universal constant \(c_{15}\).

In what follows let us bound \(\mathcal{E}\) by a power of \(\delta\). Then, choosing \(\delta_0\) sufficiently small, for any \(\delta \in [0,\delta_0]\) we may apply the interior regularity theory (18) and Lemma 2.1 to deduce from
An application of Hölder’s inequality yields
\[ \| u - \Theta \left( \frac{x - a}{|x - a|} \right) \|_{C^{0, \alpha}(\mathbb{B} \sim \mathbb{B}_{1/2})} \leq c_{16}(\mathcal{E} + \sqrt{\delta}). \tag{3.19} \]

Here \( c_{16} = c(p) \) is determined from \( c_{12}, \ldots, c_{15} \) (one may shrink \( \alpha \in [0, \beta_0] \) if necessary to make it smaller than the universal constant \( \beta \) in Lemma 2.1). The desired bound for \( \mathcal{E} \) is achieved by adapting the arguments on pp.119–120, \([10]\).

To this end, we first notice that
\[ \mathcal{E} \leq J_1 + J_2 := \| u - \frac{x}{|x|} \|_{C^2(\mathbb{B}_{3/4} \sim \mathbb{B}_{1/4})} + \| \frac{x}{|x|} - \frac{x - a}{|x - a|} \|_{C^2(\mathbb{B}_{2/3} \sim \mathbb{B}_{1/3})}, \tag{3.20} \]
where
\[ J_2 \leq c_{17}|a|, \quad J_1 \leq c_{17}B. \tag{3.21} \]

By interior regularity, \( B \) can be chosen as an upper bound for the \( L^2 \)-norm of \( (u - x/|x|) \) in the larger annulus \( \mathbb{B} \sim \mathbb{B}_{1/8} \supset \mathbb{B}_{3/4} \sim \mathbb{B}_{1/4} \); the constant \( c_{17} = c(p) \).

Then, write \( x = r\omega \) for \( r = |x| \in [1/8, 1] \), \( \omega = x/|x| \in \mathbb{S}^2 \); we have
\[
\int_{\mathbb{B} \sim \mathbb{B}_{1/8}} |u(x) - \frac{x}{|x|}|^2 \, dx 
\leq 2 \int_{1/8}^1 \int_{\mathbb{S}^2} \left\{ |u(r\omega) - \psi(\omega)|^2 + |\psi(\omega) - \omega|^2 \right\} r^2 \, dA(\omega) \, dr =: J_{11} + J_{12}.
\]

An application of Hölder’s inequality yields
\[
J_{12} = 2 \left( \int_{1/8}^1 r^2 \, dr \right) \int_{\mathbb{S}^2} |\psi - i\mathbf{d}\mathbf{g}^2|^2 \, dA 
\leq \frac{2}{3} \left( 1 - \frac{1}{8^3} \right) \| \psi - i\mathbf{d}\mathbf{g}^2 \|_{L^p}^2 \| \mathbb{S}^2 \|_{L^\infty}^2 \leq c_{18} \delta^2,
\]
and a direct computation gives us
\[
J_{11} = 2 \int_{1/8}^1 \int_{\mathbb{S}^2} \left| \int_r^1 \frac{\partial u}{\partial r}(s\omega) \, ds \right|^2 \, r^2 \, dA(\omega) \, dr 
\leq 2 \int_{1/8}^1 \left\| \frac{\partial u}{\partial r} \right\|_{L^2(\mathbb{B} \sim \mathbb{B}_{1/8})}^2 (1 - r) \, dr \leq c_{19} \left\| \frac{\partial u}{\partial r} \right\|_{L^2(\mathbb{B} \sim \mathbb{B}_{1/8})}^2,
\]
where \( c_{18} = c(p) \) and \( c_{19} \) is a universal constant. But \( \| \partial u/\partial r \|_{L^2(\mathbb{B} \sim \mathbb{B}_{1/8})} \) can be controlled by the monotonicity formula \([2.1]\) and the quantisation of energy \([3.11]\):
\[
\left\| \frac{\partial u}{\partial r} \right\|_{L^2(\mathbb{B} \sim \mathbb{B}_{1/8})}^2 \leq \int_{\mathbb{B}} |\nabla u|^2 \, dx - (1 - 8|a|)\delta \pi.
\]
Furthermore, recall from \([3.1]\):
\[
\int_{\mathbb{B}} |\nabla u|^2 \, dx - 8\pi \leq 8\pi \kappa + \left( 1 + \frac{1}{\kappa} \right) (4\pi)^{\frac{p-2}{p}} \delta^2.
\]
Putting together the above estimates, one obtains
\[
\int_{\mathbb{B} \sim \mathbb{B}_{1/8}} \left( \frac{x}{|x|} \right)^2 \, dx \leq c_{18} \delta^2 + c_{19} \left\{ 64\pi |a| + 8\pi \kappa + \left( 1 + \frac{1}{\kappa} \right) (4\pi)^{\frac{p-2}{p}} \delta^2 \right\}. \tag{3.22}
\]
In view of \([3.10]\), the best decay rate of the right-hand side of \([3.22]\) is \( \mathcal{O}(\sqrt{\delta}) \) — e.g., by choosing \( \kappa = \mathcal{O}(\delta) \).
Therefore, taking the square root of (3.22) and utilising (3.20), (3.21), and (3.10), we can choose \( c_20 > 0 \) sufficiently small such that, for \( 0 < \delta \leq \delta_0 \), there holds
\[
\mathcal{E} \leq c_20 \delta^{4/4}.
\]
The constant \( c_20 = c(p) \). Moreover, by (3.19) and (3.13), for any sufficiently small \( \alpha > 0 \) we have
\[
\left\| u - \Theta \left( \frac{x - a}{|x - a|} \right) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{B})} \leq c_{21} \delta^{4/4} \quad \text{where} \quad c_{21} = c(p).
\] (3.23)

In summary, we obtain the following analogue of the Perturbation Lemma in [10]:

**Lemma 3.2.** Let \( \psi \in \mathcal{C}^{1,\alpha}(\partial \mathbb{B}, \mathbb{S}^2) \), \( 2 < p \leq \infty \) and \( \delta := \| \psi - \text{id}_{\mathbb{S}^2} \|_{W^{1,p}} \). There are positive constants \( \delta_0 \) and \( c \) (depending on \( p \)) and \( \alpha \in [0, 1] \), such that for any \( \delta \in [0, \delta_0] \) and \( u \in W^{1,2}(\mathbb{B}, \mathbb{S}^2) \) minimising the Dirichlet energy with \( u|\partial \mathbb{B} = \psi \), one has
\[
\text{sing} u = \{ a \}, \quad |a| \leq c\sqrt{\delta}, \quad \text{and} \quad \left\| u - \Theta \left( \frac{x - a}{|x - a|} \right) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{B})} \leq c\delta^{1/4},
\]
where \( \Theta \in \mathcal{O}(3) \) with \( \| \Theta - \text{id}_{\mathbb{S}^2} \| \leq c\delta^{1/4} \).

4. PROOF OF THEOREM 1.3

4.1. The case \( 2 < p \leq \infty \). With Lemma 3.2 at hand, Theorem 1.3 follows as in §3 of [10] for the case \( p > 2 \). To be self-contained we sketch the arguments below.

Assume \( u_i : \Omega \to \mathbb{S}^2 \) are energy-minimisers with \( u_i|\partial \Omega = \psi_i \in \mathcal{C}^{1,\alpha}(\partial \mathbb{B}, \mathbb{S}^2) \), such that \( \| \psi_i - \varphi \|_{W^{1,p}(\mathbb{S}^2)} \to 0 \) as \( i \to \infty \), and that \( v : \Omega \to \mathbb{S}^2 \) is the unique minimiser with \( v|\partial \Omega = \varphi \). Then \( \int_\Omega |\nabla u_i|^2 \, dx \) is bounded (e.g., by comparing with the harmonic extensions of \( \psi_i \) and the uniform bound on \( \| \psi_i \|_{W^{1,p}(\mathbb{S}^2)} \), \( p > 2 \)), \( u_i \to v \) strongly in \( W^{1,2} \) (by Theorem 6.4, [11]), and \( \text{sing} v \) is a finite set (by Theorem 2, [18]) — call it \( \{ a_j \}_{j=1}^k \subset \Omega \).

As before, the tangent map of \( v \) at each \( a_j \) is unique and equals \( \Theta_j(x/|x|) \) for \( \Theta_j \in \mathcal{O}(3) \). For \( 0 < \tau < \min\{ \text{dist}(a_j, \text{sing} v \cup \text{sing} \Omega) \}/2 \), we have
\[
\| u_i - v \|_{W^{1,p}(\partial \mathbb{B}(a_j, \tau))} \to 0
\] (4.1) thanks to Simon’s asymptotic theory (Proposition 3.1 also see [20, 21]) and a standard compactness argument.

Denote by \( \delta_i \) the larger of \( \| \psi_i - \varphi \|_{W^{1,p}(\mathbb{S}^2)} \) and \( \| u_i - v \|_{W^{1,p}(\partial \mathbb{B}(a_j, \tau))} \). Utilising the interior regularity theory ([18]) and the uniform boundary regularity Lemma 2.1 one may infer that
\[
\| u_i - v \|_{\mathcal{C}^{0,\alpha}(\Omega \setminus \bigcup_{1 \leq j \leq k} \mathbb{B}(a_j, \tau))} \leq c_{22} \delta_i.
\] (4.2)
This gives the desired stability of minimisers away from the singularities of the limiting map.

Now, apply the arguments in §3 to each \( \mathbb{B}(a_j, \tau) \), \( 1 \leq j \leq k \) and \( u_i \) for large enough \( i \). For each pair \( (i, j) \), there exists a unique point \( a_{ji} \in \mathbb{B}(a_j, \tau) \) such that \( \text{sing} u_i = \{ a_{ji} \} \). Moreover, there are rotations \( \Theta_{ji} \in \mathcal{O}(3) \) so that
\[
\sup_{1 \leq j \leq k} \left\{ |a_{ji} - a_j| + \| \Theta_{ji} - \Theta_j \| + \left\| u_i - \Theta_{ji} \left( \frac{x - a_{ji}}{|x - a_{ji}|} \right) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{B}(a_j, \tau))} \right\} \leq c_{23} \delta_i^{1/4}.
\] (4.3)
Also, set
\[
\tau_i := \max_{1 \leq j \leq k} |a_{ji} - a_j|^{1/2} \leq c_{24} \delta_i^{1/8}.
\]
Finally, we construct the bi-Lipschitz homeomorphism $\eta : \Omega \to \Omega$ such that some Hölder norm of $(u_i - v \circ \eta)$ and $\|\eta - \text{id}_\Omega\|_{\text{Lip}} + \|\eta^{-1} - \text{id}_\Omega\|_{\text{Lip}}$ are both made arbitrarily small. Define $\eta_i$ for each $i$, such that $\eta_i = \text{id}$ away from $\text{sing} v$, and near each $a_j$, $\eta_i$ maps $a_{ji}$ (the singularity of $u_i$) to $a_j$. In between, $\eta_i$ is connected by a smooth bump function. Then we take $\eta = \eta_i$ for large enough $i$. More precisely, as on p.121, we set

$$\eta_i := \begin{cases} \text{id} & \text{on } \Omega \sim \bigcup_{j=1}^k \mathbb{B}(a_j, \tau_i), \\ \lambda_{ji}x + (1 - \lambda_{ji})\text{id} & \text{on } \mathbb{B}(a_j, \tau_i) \text{ for each } 1 \leq j \leq k. \end{cases}$$

Here $\lambda_{ji}(x) = \Theta_{ji}(x - a_{ji}) + a_j$ and $\lambda_{ji} \in C^\infty(\Omega, [0, 1])$, $\lambda_{ji} \equiv 1$ on $\mathbb{B}(a_j, \tau_i/2)$, $\lambda_{ji} \equiv 0$ on $\Omega \sim \mathbb{B}(a_j, \tau_i)$, and $|\nabla \lambda_{ji}| \leq 2\tau_i$. Then, for sufficiently large $i$ and $\alpha' < \alpha/10$, we have

$$\|\eta_i^{-1} - \text{id}_\Omega\|_{\text{Lip}} + \|\eta_i - \text{id}_\Omega\|_{\text{Lip}} \leq c_{24}\delta_i^{1/8}, \quad \|u_i - v \circ \eta_i\|_{C^{0,\alpha'}} \leq c_{24}\delta_i^{1/4}. \quad (4.4)$$

This completes the proof of Theorem 1.3 for $p > 2$.

4.2. The case $p = 2$. Now let us modify the preceding arguments to deal with the critical case $p = 2$. The uniform boundary regularity Lemma 2.1 holds for $p = 2$, and the only place we used $p > 2$ is the Sobolev–Morrey embedding (3.4). So we just need to modify the arguments in §3.

Indeed, as the boundary maps $\psi, \text{id}_{\mathbb{S}^2} : \partial \mathbb{B} \to \mathbb{S}^2$ take values in the unit sphere, for $\psi \in C^{1,\alpha}(\partial \mathbb{B}, \mathbb{S}^2)$ we have $\|\psi - \text{id}_{\mathbb{S}^2}\|_{W^{1,\infty}(\mathbb{S}^2)} \leq c_{25}$, which depends only on the Lipschitz norm of $\psi$. Thus, applying the interpolation inequality

$$\|f\|_{L^q} \leq \|f\|_{L^2}^{2/q} \|f\|_{L^\infty}^{1-2/q}, \quad q > 2$$

to $f = \psi - \text{id}_{\mathbb{S}^2}$ and $f = \nabla \psi - \nabla \text{id}_{\mathbb{S}^2}$, we can find a constant $c_{26} = c(q, \|\psi\|_{\text{Lip}})$ such that

$$\|\psi - \text{id}_{\mathbb{S}^2}\|_{W^{1,q}(\mathbb{S}^2)} \leq c_{26}\delta^2_q =: \tilde{\delta}, \quad (4.5)$$

whenever $q \in [2, \infty]$ and

$$\|\psi - \text{id}_{\mathbb{S}^2}\|_{L^2(\mathbb{S}^2)} \leq \delta. \quad (4.6)$$

Now, one may repeat the arguments in §3.2, 3.3 with $\tilde{\delta}$ in place of $\delta$. In this way, equations (4.2) become, respectively,

$$\left\|u - \Theta\left(\frac{x - a}{|x - a|}\right)\right\|_{C^{0,\alpha}} \leq c_{27}\delta^{1/2q},$$

$$|a| \leq c_{27}\delta^{1/q},$$

$$\|\Theta - \text{id}_{\mathbb{R}^3}\| \leq c_{27}\delta^{1/2q},$$

where $c_{27} = c(q, \|\psi\|_{\text{Lip}})$. Therefore, a straightforward adaptation of the proof in §4.1 gives us

$$\|\eta_i^{-1} - \text{id}_\Omega\|_{\text{Lip}} + \|\eta_i - \text{id}_\Omega\|_{\text{Lip}} \leq c_{28}\delta_i^{1/2q},$$

$$\|u_i - v \circ \eta_i\|_{C^{0,\alpha'}} \leq c_{28}\delta_i^{1/2q}$$

with $c_{28} = c(q, \|\psi\|_{\text{Lip}})$.

We fix an arbitrary $q \in [2, \infty]$ to conclude the proof of Theorem 1.3 for $p = 2$.

5. Remarks and Prospective Questions

1. It is interesting to investigate the boundary stability of minimising harmonic maps with axial symmetry (cf. Hardt–Lin–Poon [12], Hardt–Kinderlehrer–Lin [13], and Hardt–Li [14]). That
is, the map \( u : \mathbb{B} \rightarrow \mathbb{S}^2 \) is determined by its value on the “orbit space” \( \{(r, z) : 0 \leq r \leq 1, r^2 + z^2 \leq 1\} \), where \( r = \sqrt{x^2 + y^2} \) for \( (x, y, z) \in \mathbb{B} \). In this case, the singularities are at most a discrete set on the \( z \)-axis, but the proof of the inequality (2.4) does not hold any more. The obstruction appears in an application of Fubini’s theorem for the projection \( \Pi_a \).

2. We proved the stability of energy-minimisers under \( W^{1,p} \)-perturbations of the boundary maps under suitable uniqueness conditions, \( p \geq 2 \); Hardt–Lin [10] proved for \( p = \infty \). This is in sharp contrast to the \( p < 2 \) case in [15] by Mazowiecka–Strzelecki; also see Almgren–Lieb [2]. In the nice recent work [14], Mazowiecka–Miśkiewicz–Schikorra proved (Theorem 7.1 therein):

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded smooth domain. Let \( s \in ]1/2, 1[ \) and \( p \in [2, \infty[ \). There are constants \( R, \gamma \) depending only on \( \Omega \) such that the following holds. Assume \( v \in W^{1,2}(\Omega; \mathbb{S}^2) \) is the unique minimising harmonic map with \( v|\partial \Omega = \psi \). For any \( \epsilon > 0 \), there is \( \delta = \delta(\Omega, \epsilon, \psi) > 0 \) such that if \( u \in W^{1,2}(\Omega, \mathbb{S}^2) \) is a minimising harmonic map with \( u|\partial \Omega = \varphi \) satisfying

\[
\sup_{\mathbb{B}(y, \rho) \in \Omega, \rho < R} \left\{ \rho^{p-2} |\psi|^{p} |\partial^{\gamma} \psi|_{W^{s,p}(\partial \Omega \cap \mathbb{B}(y, \rho))} \right\} < \gamma
\]

and

\[
|\psi - \varphi|_{W^{s,p}(\partial \Omega)} \leq \delta,
\]

then \( u \) has the same number of singularities as \( v \). Moreover, we have \( \|u - v\|_{W^{1,2}} \leq \epsilon \).

The above result in [14] by Mazowiecka–Miśkiewicz–Schikorra has weaker regularity assumption on the boundary map — \( \psi \in W^{s,p}(\partial \Omega, \mathbb{S}^2) \) — compared to \( \psi \in C^{1,\alpha}(\partial \Omega, \mathbb{S}^2) \) in Theorem [13] above. On the other hand, we bound the distance between \( u \) and \( v \) in a Hölder norm modulo bi-Lipschitz homeomorphisms, in comparison with the \( W^{1,2} \)-norm in [14].

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