INVERSE SOURCE PROBLEMS WITHOUT (PSEUDO) CONVEXITY ASSUMPTIONS†

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Abstract. We study the inverse source problem for the Helmholtz equation from boundary Cauchy data with multiple wave numbers. The main goal of this paper is to study the uniqueness and increasing stability when the (pseudo)convexity or non-trapping conditions for the related hyperbolic problem are not satisfied. We consider general elliptic equations of the second order and arbitrary observation sites. To show the uniqueness we use the analytic continuation, the Fourier transform with respect to the wave numbers and uniqueness in the lateral Cauchy problem for hyperbolic equations. Numerical examples in 2 spatial dimension support the analysis and indicate the increasing stability for large intervals of the wave numbers, while analytic proofs of the increasing stability are not available.

1. Introduction and main results. This paper studies the inverse source problem for the Helmholtz equation from boundary Cauchy data with multiple frequencies (wave numbers). The source term is supported in a bounded domain Ω and shall be recovered from full/partial boundary data on Γ ⊂ ∂Ω. Such inverse source problems have wide applications in antenna synthesis [2], biomedical imaging [1], and various kinds of tomography. It has been demonstrated that by boundary data of one single frequency, we could not identify the source function uniquely [11, Ch.4] while a family of equations (like the Helmholtz equation for various wave numbers in (0, K)) regains uniqueness. The crucial issue then is a stability estimate which is of importance in applications. Generally a stability estimate in inverse problems for elliptic equations is logarithmic which yields a robust recovery of only few source parameters and provides low resolution numerically. We mention that in [7] some uniqueness and numerical evidence were provided highlighting the improvement of multiple frequency measurement data. In [3], [4] the authors have proven the uniqueness and the first increasing stability results were obtained in more particular case by a different method of spatial Fourier analysis. In [6] we observed

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that the inverse source problem for Helmholtz equations, in odd dimensions, is highly connected with the identification of the initial data in the hyperbolic initial value problem by lateral Cauchy data (boundary observability in control theory) by means of the Fourier transform in time. As a consequence we also obtained most complete increasing (with growing $K$) stability results. In a recent paper [15] we further traced the exponential dependence on the attenuation term. We shall emphasize that in all papers mentioned above increasing stability bounds assume that the corresponding lateral Cauchy problem satisfies the so called non-trapping condition which is guaranteed by certain (pseudo)convexity of the domain and the surface with Cauchy data with respect to the hyperbolic equation. 

Other better investigated uniqueness and increasing stability by higher wave numbers measurement focus on the Cauchy problem of elliptic equations and the Schrödinger potential identification. Classical Carleman estimates imply some conditional Hölder type stability estimates for solutions of the elliptic Cauchy problem. Nevertheless, F. John in [16] showed that, in the continuation problem for the Helmholtz equation from the unit disk onto any larger disk, the stability estimate is still of a logarithmic type, uniformly with respect to the wave numbers. Increasing stability for the Schrödinger potential and conductivity coefficient has been demonstrated in [13], [14] respectively. In particular, in [13] we also traced the dependence of increasing stability on the attenuation. It has been demonstrated in [9] that the stability always is improving for larger $k$ under (pseudo) convexity conditions on the geometry of the domain and on the coefficients of the elliptic equation. On the other hand, [12] and [10] verify that in some cases increasing stability holds even when convexity and non-trapping conditions fail.

In this paper we rigorously demonstrate uniqueness in the inverse source problems without any convexity/non-trapping condition and give a strong numerical evidence of increasing stability for source identification. In absence of such conditions, increasing stability estimates for sources identification are not known.

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n, n = 2, 3$. We introduce another sub-domain $\Omega_0$ such that $\bar{\Omega}_0 \subset \Omega$, whose boundary $\partial \Omega_0$ is of $C^2$ class, and $\mathbb{R}^n \setminus \bar{\Omega}_0$ is connected.

Let $A$ be the second order elliptic operator

$$A := \sum_{j,m=1}^{n} \partial_j(a_{jm}\partial_m) + \text{i} \sum_{j=1}^{n} b_j \partial_j + c, \quad a_{jm}, b_j \in C^1(\mathbb{R}^n), \quad c \in L^\infty(\mathbb{R}^n),$$

and $A := \Delta$ in $\mathbb{R}^n \setminus \Omega$. Here we denote $b = (b_1, ..., b_n)$. We assume the ellipticity condition $\epsilon_0 |\xi|^2 \leq \sum_{j,m=1}^{n} a_{jm}(x)\xi_j \xi_m$ for some positive constant $\epsilon_0$ and all $x, \xi \in \mathbb{R}^n$ as well as the additional conditions

\begin{equation}
\text{the imaginary part } \Im b_j = 0, \quad b \cdot \nu \leq 0 \text{ on } \partial \Omega_0, \quad \nabla \cdot b \leq 0,
\end{equation}

\begin{equation}
b_0 \in L^\infty(\mathbb{R}^n), 0 \leq b_0, \text{ and } b_0 = 0 \text{ outside } \Omega,
\end{equation}

and

\begin{equation}
0 \leq \Im c \text{ in } \Omega.
\end{equation}

We consider the scattering problem

\begin{equation}
(A + b_0 k \text{i} + k^2)u = -f_1 + (i k - b_0) f_0 \quad \text{in } \mathbb{R}^n \setminus \bar{\Omega}_0,
\end{equation}
where $f_1 \in L^2(\mathbb{R}^n)$, $f_0 \in H^1(\mathbb{R}^n)$, are real valued, and $\text{supp} f_0 \cup \text{supp} f_1 \subset \Omega \setminus \Omega_0$ with the Neumann boundary condition

$$a \nabla u \cdot \nu = g_1(\cdot, k) \quad \text{on} \quad \partial \Omega_0,$$

where $a$ is the symmetric matrix $(a_{jm})$, and the Sommerfeld radiation condition

$$\lim_{r = |x| \to +\infty} r^{\frac{n-1}{2}} (\partial_r u - ik u) = 0.$$  

We will assume that

$$g_1(x, k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathcal{G}_1(x, t)e^{itk} dt$$

for some functions $\mathcal{G}_1, \partial_t \mathcal{G}_1 \in L^\infty((0, \infty); H^{\frac{1}{2}}(\partial \Omega_0))$ satisfying the inequality

$$\|\mathcal{G}_1(\cdot, t)\|_{L^\infty(\partial \Omega_0)} + \|\partial_t \mathcal{G}_1(\cdot, t)\|_{L^\infty(\partial \Omega_0)} + \|\mathcal{G}_1(\cdot, t)\|_{H^{\frac{1}{2}}(\partial \Omega_0)} \leq Ce^{-\delta t}$$

for some positive numbers $C, \delta$ depending on $\mathcal{G}_1$. In other words, $\mathcal{G}_1$ is a function of $(x, t)$ whose indicated norms with respect to $x \in \partial \Omega_0$ decay exponentially with respect to $t$. Then $g_1(\cdot, k)$ is (complex) analytic with respect to $k = k_1 + ik_2$ if $-\delta < k_2$.

The inverse source problem of interest is to find $f_0, f_1, g_1$ entering (3)-(5) from the additional Cauchy data

$$u(\cdot, k) = u_0, \quad \partial_r u(\cdot, k) = u_1 \text{ on } \Gamma, \quad K_0 < k < K,$$

where $\Gamma$ is a non(empt)y $C^2$ hyper surface in $\partial \Omega$, see Figure 1, and $K_0, K$ are some positive numbers satisfying $K_0 < K$.

**Figure 1.** Domain of the source problem. The source function is compactly supported in $\Omega \setminus \Omega_0$.

We present the main analytic result below:

**Theorem 1.1.** Let the assumptions (1), (2) hold. Then a solution $(f_0, f_1, g_1)$ to the inverse source problems (3)-(5) with the additional Cauchy data (7) is unique in each of the cases

- **a)**: $f_0 = 0, b_0 = 0$;
- **b)**: $g_1 = 0, \quad \mathbb{R}^n \setminus \Omega$ is connected.
This result will be proven in the next section by using analyticity of $u(\cdot,k)$ with respect to $k$ and an auxiliary hyperbolic initial boundary value problem obtained after the Fourier-Laplace transform of the scattering problem with respect to $k$. The source terms $f_0, f_1$ in the scattering problem will be the initial values for this hyperbolic problem and $G_1$ will be the boundary value data on $\partial \Omega_0 \times (0, +\infty)$. Using the John-Tataru's uniqueness of the continuation theorem one shows that the solution of the hyperbolic equation and hence the initial and boundary data are uniquely determined.

Observe that if $\Gamma = \partial \Omega$, then in Theorem 1.1 one can replace the Cauchy data (7) by the Dirichlet data $u(\cdot,k) = u_0$ on $\Gamma$, $K_0 < k < K$. Indeed, the radiating solution of the exterior Dirichlet problem is unique, so $u_0$ uniquely determines $u$ outside $\Omega$ and hence $u_1 = \partial_{\nu} u$ is attainable on $\partial \Omega$. Moreover, if $\partial \Omega$ is analytic and connected, and $A = \Delta$, $b_0 = 0$ near $\partial \Omega$, then in Theorem 1.1 instead of (7) it is sufficient to use $u(\cdot,k) = u_0$ on $\Gamma \subset \partial \Omega$. Indeed, then $u(\cdot,k)$ is (real) analytic near $\partial \Omega$, so $u_0$ uniquely determines $u$ on $\partial \Omega$, and as above, also $\partial_{\nu} u$ on $\Gamma$.

At any fixed $K$ one expects a logarithmic stability in this inverse source problem which is quite discouraging for applications. At present, there is no analytic results showing increasing stability for larger $K$ without convexity assumptions. In section 3 we present numerical examples of reconstruction of sources in the plane which either are characteristic functions of relatively complicated domains or smoothly spread point sources described by around 100 parameters. This reconstruction dramatically improves with growing $K$ definitely suggesting improving stability. While we consider the Helmholtz equation with constant coefficients the shape and location of the observation curve $\Gamma$ violates a non-trapping condition. Indeed, supports of sources are outside of the convex hull of $\Gamma$.

2. Proof of uniqueness. We first show solvability of the direct scattering problem and analyticity of its solution with respect to the wave number $k$ by the Lax-Phillips method [11]. Let $B$ be a ball containing $\bar{\Omega}$. We will use the following elliptic boundary value problem

$$
(A + b_0 ki + k^2) V = f^* \quad \text{in } \Omega_1 = B \setminus \bar{\Omega}_0,
$$

$$
a \nabla V \cdot \nu = g_1 \quad \text{on } \partial \Omega_1, \quad \partial_{\nu} V = iV \quad \text{on } \partial B.
$$

(8)

Observe that the boundary value problem (8) is elliptic and Fredholm. To show its unique solvability in the Sobolev space $H^2(\Omega_1)$, it suffices to demonstrate that the homogeneous problem ($f^* = 0$, $g_1 = 0$) has only the zero solution. Such argument follows from the energy integral. Indeed, the standard definition of a weak solution gives

$$
\int_{\partial \Omega_1} a \nabla V \cdot \nu \bar{V} - \int_{\Omega_1} \sum_{j,m=1}^n a_{jm} \partial_j V \partial_m \bar{V} + \int_{\Omega_1} \sum_{j=1}^n i b_j \partial_j V \bar{V} + \int_{\Omega_1} (c + i k b_0 + k^2) V \bar{V} = 0.
$$

(9)

Integrating by parts, we obtain

$$
\int_{\Omega_1} \sum_{j=1}^n i b_j \partial_j V \bar{V} = -i \int_{\partial \Omega_0} b \cdot \nu V \bar{V} - \int_{\Omega_1} \sum_{j=1}^n i b_j \partial_j V \bar{V} - i \int_{\Omega_1} \text{div} b V \bar{V}.
$$
Hence
\begin{equation}
2\text{Im} \int_{\Omega_1} \sum_{j=1}^{n} b_j \partial_j V \bar{V} = -\int_{\partial\Omega_0} b \cdot \nu \bar{V} - \int_{\Omega_1} \text{div} b \bar{V} \geq 0,
\end{equation}
due to the conditions (1), (2) on the coefficients $b$. If $k = k_1 + ik_2$, $0 < k_1, 0 \leq k_2$, then
\begin{equation}
\text{Im}(c + ikb_0 + k^2) = \text{Im} c + k_1 b_0 + 2k_1 k_2 \geq 0
\end{equation}
because of our assumptions on the coefficients $b_0, c$. Using the boundary conditions in (8) and taking the imaginary parts in (9) we conclude that
\[\int_{\partial B} V \bar{V} + 3\text{Im} \int_{\Omega_1} \sum_{j=1}^{n} b_j \partial_j V \bar{V} + \int_{\Omega_1} \text{Im}(c + ikb_0 + k^2) V \bar{V} = 0.\]
The above inequalities (10), (11) show that the boundary integral $\int_{\partial B} V \bar{V}$ is zero. Therefore $V = 0$ on $\partial B$ and from the boundary condition, $\partial_n V = 0$ on $\partial B$.

By uniqueness in the Cauchy problem for the elliptic equation (8) we have that $V = 0$ in $\Omega_1$. According to the theory of elliptic boundary value problems the operator mapping $V \in H^2(\Omega_1), \partial_n V = \bar{V}$ on $\partial B$ into $(f^*, g_1) \in H^0(\Omega_1) \times H^\frac{1}{2}(\partial\Omega_1)$ in (8) has the continuous inverse. Since this operator is continuous and analytic in these spaces, by known results ([18, pp.365]) its inverse $V(f^*, g_1; k)$ is analytic with respect to $k$ in an open set $S$ of the complex plane containing $k$ with $0 < k_1, 0 \leq k_2$.

We fix a smaller ball $B_3$ whose closure is in $B$ and contains $\Omega$. Let $\chi$ be a cut off $C^\infty$-function with $\chi = 1$ on $\Omega$ and $\chi = 0$ outside $B_3$. When showing solvability and analyticity of the scattering problem (3)-(5) we can assume that $g_1 = 0$. Indeed, $u = u_s + \chi V(0, g_1; k)$, where $u_s$ solves the scattering problem (3)-(5) with $g_1 = 0$ and $f$ is replaced by
\[f - \sum_{j,m=1}^{n} (\partial_j \chi a_{jm} \partial_m V_0 + \partial_j (a_{jm} \partial_m \chi V_0)) - i \sum_{j=1}^{n} b_j \partial_j \chi V_0, \quad V_0 = V(0, g_1; k),\]
which is a complex analytic function of $k \in S$ with the values in $H^0(\Omega_1)$. So it suffices to show existence and analyticity of $u_s$. For brevity, from now on we denote $u_s$ by $u$.

We look for a solution
\[u = w - \chi (w - V)\]
of the scattering problem (3)-(5) where $-f_1 + (ik - b_0) f_0$ is replaced by $f \in H^0(\mathbb{R}^n \setminus \Omega_0)$ with $V(f^*, 0; k)$ solving the elliptic boundary value problem (8) and
\[w(f^*; k)(x) = \int_{B} f^*(y) \Phi(x, y; k) dy\]
with the radiating fundamental solution $\Phi$ to the Helmholtz equation. Here $f^*$ is a function to be determined later. We extend $f, f^*$ onto $\Omega_0$ as zero. We recall that
\begin{equation}
\Phi(x, y; k) = -\frac{1}{4} H_n^{(1)}(k|x - y|) \quad \text{if} \quad n = 2, \quad \text{and} \quad \Phi(x, y; k) = \frac{e^{ik|x - y|}}{4\pi |x - y|} \quad \text{if} \quad n = 3,
\end{equation}
where $H_n^{(1)}$ is the Hankel function. Since $A = \Delta, b_0 = 0$ in $\mathbb{R}^n \setminus \Omega$, we have
\[
(A + ikb_0 + k^2)u = \Delta w + k^2 w - \Delta \chi (w - V) - 2\nabla \chi \cdot \nabla (w - V) - \chi (\Delta (w - V) + k^2 (w - V)) = f^* + E(f^*; k)
\]
on $B \setminus \Omega$ where $E(f^*; k) = -\Delta \chi(w - V) - 2\nabla \chi \cdot \nabla(w - V)$. Defining $E(f^*; k)$ as zero in $\Omega$ we conclude that $u$ solves our scattering problem if and only if

$$f = f^* + E(f^*; k) \text{ in } B.$$  \hspace{1cm} (13)

To show the unique solvability of (13) and analytic dependence of its solution on $k \in S$ it suffices to prove uniqueness of its solution. The operator $E(\cdot; k)$ is analytic and compact from $L^2(B)$ into itself. Indeed, the well known elliptic estimates show that both $w$ and $V$ are continuous linear operators from $L^2(B)$ into $H^2(B)$. Since the $E(f^*)$ is a linear first order partial differential operator with respect to $w - V$, it is continuous from $L^2(B)$ into $H^1(B)$ and hence compact from $L^2(B)$ into $L^2(B)$.

As a consequence, the equation (13) is Fredholm, so existence of its solution and the existence of the continuous inverse of $I + E$ follow from the uniqueness. Moreover, since the right hand side in (13) is an analytic continuous operator (from $L^2(B)$ into itself) with respect to $k \in S$, again from the known results about analytic continuous operators in Banach spaces ([18, pp.365]) it follows that its (continuous) inversion exists and is analytic with respect to $k \in S_1$ (an open subset of $S$ containing $k = k_1 + ik_2 : 0 < k_1, 0 \leq k_2$).

To show the uniqueness, we let $f = 0$ in (13) and we will prove that $f^* = 0$. From (13) we have that $f^* = 0$ in $\Omega$ and in $B \setminus B_1$. Then $u$ solves the homogeneous scattering problem (3), (4), (5), and, as we show next, $u = 0$ in $B \setminus \Omega_0$. Therefore, $V = u = 0$ in $\Omega \setminus \Omega_0$, we extend $V$ into $\Omega_0$ as zero. In addition, $w = u = 0$ in $B \setminus B_1$.

As a result, $(\Delta + k^2)(w - V) = (\Delta + k^2)u = f^* = 0$ in $\Omega$. Moreover, $(\Delta + k^2)(w - V) = f^* - f^* = 0$ in $B \setminus \Omega$. Since $w, V \in H^2(B)$ we have $(\Delta + k^2)(w - V) = 0$ in $B$. Using (8) we obtain $\partial_\nu(w - V) = -\partial_\nu V = -iV = i(w - V)$ on $\partial B$. Repeating the argument after (9) with $u, \Omega_0$ replaced by $w - V$, $B$ we conclude that $w - V = 0$ in $B$. Since $u = w - \chi(w - V)$ and $u = 0$ in $B \setminus \Omega_0$ it follows that $w = 0$ in $B \setminus \Omega_0$ and hence $f^* = (\Delta + k^2)w = 0$ in $B$.

In what follows we denote $B(R) = \{x : |x| < R\}$. To demonstrate uniqueness in the direct homogeneous scattering problem (3), (4), (5) as in (9), (10), (11) we have

$$\text{Im} \int_{\partial B(R)} \partial_\nu u \bar{u} + \text{Im} \int_{B(R) \setminus \Omega_0} \sum_{j=1}^n i b_j \partial_j u \bar{u} + \int_{B(R) \setminus \Omega_0} \text{Im}(c + ikb_0 + k^2)u \bar{u} = 0.$$ \hspace{1cm} (14)

We consider two cases: 1) $k_2 = 0$ and 2) $0 < k_2$.

In the case 1) using (10), (11), we have

$$\text{Im} \int_{\partial B(R)} \partial_\nu u \bar{u} \leq 0$$

and using the radiation condition (5) from the known results we derive that $u = 0$ outside $B$ and hence by the uniqueness of the continuation for the elliptic equations in $B \setminus \Omega_0$.

In the case 2) we will use the integral representation

$$u(x) = \int_B \Phi(x, y)(\Delta + k^2)u^*(y)dy, \; x \in \mathbb{R}^n \setminus B,$$ \hspace{1cm} (15)

where $u^* \in H^2(B)$ is an extension of the function $u$ from $\mathbb{R}^n \setminus B_1$ onto $B$. Indeed, both $u^*$ and the right hand side of (15) solve the same Helmholtz equation in $\mathbb{R}^n$ and satisfy the Sommerfeld radiation condition, so they are equal and we have (15). Since the fundamental solution (12) decays exponentially in $|x|$ when $0 < k_2$ (which
is obvious when $n = 3$ and follows from known inequality $|H_0^{(1)}(z)| + |d/dz H_0^{(1)}(z)| \leq C|e^z|$ when $1 < |z|$, [20, pp.210]), we have

$$\int_{\partial B(R)} \partial_x u \bar{u} \leq C(u)e^{-\delta(u)R}$$

(16)

and hence letting $R \to +\infty$ in (14) and using (10), (11) we conclude that

$$2k_1k_2 \int_{\mathbb{R}^n \setminus \Omega_0} u \bar{u} = 0$$

and hence $u = 0$ in $\mathbb{R}^n \setminus \Omega_0$.

Now we prove Theorem 1.1.

**Proof.** Due to the linearity it suffices to show that $u_0 = u_1 = 0$ on $\Gamma$, $K_0 < k < K$ implies that $f_0 = f_1 = 0$, $g_1 = 0$.

Let $U$ be a solution to the hyperbolic mixed initial boundary value problem

$$-\partial_t^2 U - b_0 \partial_t U + AU = 0 \quad \text{in } (\mathbb{R}^n \setminus \Omega_0) \times (0, \infty),$$

(17)

$$U = f_0, \quad \partial_t U = f_1 \quad \text{on } (\mathbb{R}^n \setminus \Omega_0) \times \{0\},$$

(18)

$$\partial U \cdot \nu = g_1 \quad \text{on } \partial \Omega_0 \times (0, \infty).$$

(19)

As known in [17] there is a unique solution $U \in L^\infty((0, T); H^1(\mathbb{R}^n \setminus \Omega_0))$ of this problem for any $T > 0$ and moreover by using the standard energy estimates, i.e. multiplying (17) by $\partial_t U e^{-\gamma_0 t}$ and integrating by parts over $(\mathbb{R}^n \setminus \Omega_0) \times (0, t)$ we have

$$\|\partial_t U(t, \cdot)\|_{(0)}(\mathbb{R}^n \setminus \Omega_0) + \|U(t, \cdot)\|_{(1)}(\mathbb{R}^n \setminus \Omega_0) \leq C(U)e^{\gamma_0 t}$$

for some positive $\gamma_0 = \gamma_0(U)$. Then the following Fourier-Laplace transform is well defined

$$u_+(x, k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty U(t, x)e^{ikt} dt, \quad k = k_1 + i\gamma, \quad \gamma_0 < \gamma.$$  

(20)

Approximating $f_0, f_1, g_1$ by smooth functions and integrating by parts we obtain

$$0 = \int_0^\infty (-\partial_t^2 U - b_0 \partial_t U + AU)(t, \cdot)e^{ikt} dt$$

$$= \partial_t U(0, \cdot) + (b_0 - ik)U(0, \cdot) + (k^2 + ikb_0) \int_0^\infty U(t, \cdot)e^{ikt} dt + A \int_0^\infty U(t, \cdot)e^{ikt} dt.$$  

Hence $u_+$ solves (3)-(4) with $k = k_1 + i\gamma$, $\gamma_0 < \gamma$. In addition, $u_+$ exponentially decays for large $|x|$.

Indeed, due to the finite speed of the propagation in the hyperbolic problems we have $U(x, t) = 0$ if $t < \theta R - \theta_0$ for some $\theta_0, \theta > 0$. Hence from (20)

$$\left| \frac{1}{2\pi} \int_{B(\mathbb{R}^n \setminus \Omega_0)} \int_{\theta R - \theta_0}^{\infty} U(t, x)e^{ikt} dt \right|^2$$

$$\leq \int_{B(\mathbb{R}^n \setminus \Omega_0)} \int_{\theta R - \theta_0}^{\infty} |U(t, x)|^2e^{-2(\gamma_0 - \epsilon)t} dt \int_{\theta R - \theta_0}^{\infty} e^{-2\epsilon t} dt dx$$

$$\leq C(U, \epsilon)e^{-2\epsilon \theta R} \int_{\theta R - \theta_0}^{\infty} \|U(t, \cdot)\|_{(0)}(\mathbb{R}^n \setminus \Omega_0)e^{-2(\gamma_0 - \epsilon)t} dt$$

$$\leq C(U, \epsilon)e^{-2\epsilon \theta R} \int_{\theta R - \theta_0}^{\infty} e^{-2(\gamma_0 - \epsilon)t} dt \leq C(U, \gamma, \gamma_0)e^{-2\epsilon \theta R},$$

(21)
if we choose $\varepsilon = \frac{2 - \alpha}{2}$. The same bound holds for $\partial_j u_*$. Let us choose a cut off function $\chi_1 \in C_c^\infty, \|\chi_1\|_{C^2(\mathbb{R}^2)} \leq C, \chi_1 = 1$ when $|x| < R + 1$, $\chi_1 = 0$ when $R + 3 < |x|$. From the Green’s formula,

$$\int_{\partial B(R)} \partial_\nu u_* \bar{u}_* + \int_{B(R+4) \setminus B(R)} (\nabla (\chi_1 u_*) \cdot \nabla (\chi_1 \bar{u}_*) - k^2 \chi_1^2 u_* \bar{u}_*)$$

$$= -\int_{B(R+4) \setminus B(R)} (\Delta (\chi_1 u_*) + k^2 \chi_1 u_*) \chi_1 \bar{u}_*$$

$$= -\int_{B(R+4) \setminus B(R)} (2(\nabla \chi_1 \cdot \nabla u_*) \chi_1 \bar{u}_* + \Delta \chi_1 u_* \chi_1 \bar{u}_*),$$

because $\Delta u_* + k^2 u_* = 0$ in $B(R+4) \setminus B(R)$. So

$$\int_{\partial B(R)} \partial_\nu u_* \bar{u}_*$$

$$= \int_{B(R+4) \setminus B(R)} (-\nabla (\chi_1 u_*) \cdot \nabla (\chi_1 \bar{u}_*) + k^2 \chi_1^2 u_* \bar{u}_* - 2(\nabla \chi_1 \cdot \nabla u_*) \chi_1 \bar{u}_*$$

$$- \Delta \chi_1 u_* \chi_1 \bar{u}_*).$$

From this equality and from the above exponential decay of $u_*, \nabla u_*$ for large $|x|$ it follows the bound

$$\left| \int_{\partial B(R)} \partial_\nu u_* \bar{u}_* \right| \leq C(u_*) e^{-\delta(u_*) R}. \tag{21}$$

The function $u(\cdot, k)$ solving (3)-(5) has a complex analytic extension from $(0, \infty)$ into a neighbourhood $S$ of the quarter plane $\{0 < \Re k, 0 \leq \Im k\}$, this extension satisfies (3)-(4) and decays exponentially as $|x| \to \infty$ when $0 < \Im k$ due to the above mentioned exponential decay of the Hankel function $H_0^{(1)}((k_1 + i\gamma)r)$ for large $r$. By uniqueness $u(\cdot, k) = u_*(\cdot, k)$ when $k = k_1 + i\gamma$, $0 < k_1, \gamma_0 < \gamma$.

By the uniqueness in the Cauchy problem for the second order elliptic equation $u(x, k) = 0$ when $x \in \Omega_T$, $K_0 < k < K_0$ for $\Omega_T$, where $\Omega_T$ is the connected component of $\mathbb{R}^n \setminus \Omega$ whose boundary contains $\Gamma$. Due to uniqueness of the analytic continuation with respect to $k$, $u(\cdot, k) = 0$ in $\Omega_T$ when $0 < k_1 < \infty$, $k = k_1 + i\gamma$, $\gamma_0 < \gamma$. Due to uniqueness in the Fourier-Laplace transform (20), we finally conclude that $U = 0$ in $\Omega_T \times (0, \infty)$.

We first handle the case a). Letting $U(-t, \cdot) = -U(t, \cdot), 0 < t$ (i.e. introducing odd reflection with respect to $t$) we conclude that the extended $U$ satisfies the hyperbolic equation (17) in $(\mathbb{R}^n \setminus \Omega_0) \times (-T, T)$ for any $T > 0$ and $U = 0$ in $\Omega_T \times (-T, T)$ by the John-Tataru’s Theorem [19, 11] we conclude that $U = 0$ in $(\mathbb{R}^n \setminus \Omega_0) \times (-T, T)$ for any $T > 0$. Hence $f_1 = 0$ and $g_1 = 0$.

Now we consider the case b). As above $U$ solves (17) in $(\mathbb{R}^n \setminus \Omega_0) \times (0, 2T)$ and $U = 0$ in $(\mathbb{R}^n \setminus \Omega_0) \times (0, 2T)$. Choosing $T$ sufficiently large and implementing the John-Tataru’s Theorem again, we conclude that $U = \partial_t U = 0$ in $(\Omega \setminus \Omega_0) \times \{T\}$. Since $\hat{U} = 0$ in $(\mathbb{R}^n \setminus \Omega) \times (0, T)$ and $\bar{\alpha} \nabla U \cdot v = \bar{g}_1 = 0$ on $\partial \Omega_0 \times (0, T)$ by uniqueness in the hyperbolic (backward) initial boundary value problem with the initial data $u(t) = 0$ in $(\Omega \setminus \Omega_0) \times (0, T)$. Hence $f_0 = U = 0$ and $f_1 = \partial_t U = 0$ on $(\Omega \setminus \Omega_0) \times \{0\}$. The proof is complete.

3. Numerical tests. In this section, we provide some examples numerically verifying the improving stability, when the wave number grows, without convexity.
assumptions. For simplicity’s sake we choose \( f_0 = 0 \) in (3) and only consider the recovery of a real source function \( f_1 \). We let \( \Omega_0 \) to be an empty set. The numerical tests for the inverse source problems are separated into two different cases. In the first case, we consider an interior inverse source problem such that \( \text{supp} f_1 \subset \Omega \) and the observation site \( \Gamma \subset \partial \Omega \). Whereas in the second case, we discuss an exterior inverse source problem where the compact support of \( f_1 \) is contained in the exterior domain of the convex hull of the observation site \( \Gamma \) which is not covered in the above discussion.

3.1. **Case one: Interior inverse source problems.** We firstly consider the interior inverse source problem with \( \text{supp} f_1 \subset \Omega \) which fits well the setting in Theorem 1.1.

![Figure 2. Annular domain of the interior inverse source problem.](image)

**3.1.1. Annular domain.** The first example concerns the annular domain \( \Omega = B_2(0.3, 0.3) \setminus \overline{B_{0.5}(0,0)} \) in Figure 2. Here we denote \( B_r(x_1, x_2) \) be the open ball with a radius \( r \) and centered at \((x_1, x_2)\). The radiating field \( u(x, k) \) satisfies the Helmholtz equation and the Sommerfeld radiation condition

\[
\begin{aligned}
\Delta u(x; k) + k^2 u(x; k) &= -f_1(x), \quad x \in \mathbb{R}^2, \\
\lim_{r = |x| \to \infty} r^{\frac{1}{2}} \left( \frac{\partial u(x, k)}{\partial r} - iku(x, k) \right) &= 0,
\end{aligned}
\]

where \( k \geq 0 \) is the wave number. We assume that \( f_1 \in L^2(\mathbb{R}^2) \) and is compactly supported in \( \Omega \). The Cauchy data is gathered at the observation site \( \Gamma = (0.3 + 2\cos \theta, 0.3 + 2\sin \theta) \) with \( \theta \in (0, \pi) \) which is half of the boundary of the sphere \( B_2(0.3, 0.3) \). For simplicity’s sake, we define the following two forward operators

\[
\begin{aligned}
(L_{1,k}f_1)(x) &= u(x; k)|_{\Gamma} = g_1(x; k) = \int_{\Omega} f_1(y)\Phi(x, y; k)dy, \quad x \in \Gamma, \\
(L_{2,k}f_1)(x) &= \frac{\partial u(x; k)}{\partial n}|_{\Gamma} = g_2(x; k) = \int_{\Omega} f_1(y)\frac{\partial \Phi(x, y; k)}{\partial n}dy, \quad x \in \Gamma,
\end{aligned}
\]

where the Hankel function \( \Phi(x, y; k) = -\frac{1}{4}H_0^{(1)}(k|x-y|) \) is the fundamental solution of (22).

In order to approximate the unknown source in \( \Omega \), we discretize a subset of the domain \( \Omega \) radically and spherically. More precisely, we discretize the radius equally in \([0.5, 1.5]\) with 30 points and the angle \([0, 2\pi]\) with 100 points. Mesh-points for
the source function are generated by the products of radical discretization and the angle discretization with 3000 unknowns. The observation data is gathered at the $\Gamma$ with equally distributed 50 angles in $(0, \pi)$. The collocation method is considered to generate the discretized matrices of $L_{1,k}$ and $L_{2,k}$. The same Kaczmarz-Landweber algorithm in [4, 5, 6, 15] is implemented to recover the unknown source. The wave number $k$ for the Kaczmarz iteration is chosen in the set $K = \{1, 2, \ldots, 200\}$ which is equally distributed in $[1, 200]$ and the inner Landweber iteration is fixed by 500 steps.

As shown in Figure 2 we consider a piecewise constant source function as follows

$$f_1(r, \theta) = \begin{cases} 
1 & \text{if } 0.8 \leq r \leq 0.9 \text{ and } \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]; \\
1 & \text{if } 0.9 \leq r \leq 1 \text{ and } \theta \in \left[-\frac{\pi}{4}, -\arctan\frac{1}{2}\right] \cup \left[\arctan\frac{1}{2}, \frac{\pi}{4}\right]; \\
1 & \text{if } 1 \leq r \leq 1.1 \text{ and } \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]; \\
0 & \text{elsewhere.}
\end{cases}$$

(23)

We present the reconstructed results in Figure 3. The left panel presents the approximate source function by the exact measurement data for the largest wave number $k = 200$. The middle panel shows the error between the exact source and the approximate one. In the right panel, we collect the relative error versus the increasing wave number $k$. As one can observe, the relative error decreases substantially when the wave number increases. The final iterates in the right panel has a relative error of 0.1856 where the piecewise constant structure is well captured. On the other hand, we shall emphasize that because the wave number interval $[1, 200]$ might not be sufficient in this inverse source problem and the error between both sources is still comparably large.

![Figure 3](image.png)

**Figure 3.** Annular domain with a piecewise constant source function for exact measurement data. Left: the approximate source; Middle: the error between both sources; Right: relative error versus the wavenumber $k$ with the minimal relative error 0.1856 at $k = 200$.

The next example considers a mixed-point source in the following form

$$f_1(x, y) = 0.5e^{-50((y-0.4)^2+(x-1)^2)} + 0.3e^{-150((y-0.01)^2+(x-0.75)^2)} + 0.2e^{-80((y-0.2)^2+(x-1.1)^2)} + 0.6e^{-120((y-0.6)^2+(x-0.6)^2)} + 0.5e^{-80((y+0.8)^2+(x-0.5)^2)} + 0.3e^{-150((y+0.4)^2+(x-0.7)^2)} + 0.2e^{-80((y+0.2)^2+(x-0.8)^2)} + 0.6e^{-120((y+0.6)^2+(x-0.6)^2)} + 0.5e^{-80((y-0.7)^2+(x+0.7)^2)} + 0.3e^{-150((y-0.4)^2+(x+0.5)^2)} + 0.2e^{-80((y-0.2)^2+(x+0.8)^2)} + 0.6e^{-120((y-0.6)^2+(x+0.6)^2)}$$
The approximate source function, error between both sources and the relative error are presented in Figure 4. Similar summary can be concluded referring to the piecewise constant source above.

The proposed Kaczmarz-Landweber iteration scheme is quite robust on noisy data. If the noise is small, i.e. less than 5%, the approximate source functions mimic both exact sources in good manners as those of [4, 6, 15]. For simplicity’s sake we skip all the discussion on the noise data but focus on different numerical tests if the observation site varies. More precisely, instead of the site $\Gamma = (0.3 + 2\cos \theta, 0.3 + 2\sin \theta)$, $\theta \in (0, \pi)$; $\Gamma_1 = (0.3 + 5\cos \theta, 0.3 + 5\sin \theta)$, $\theta \in (0, \frac{2}{5}\pi)$ and $\Gamma_2 = (0.3 + 10\cos \theta, 0.3 + 10\sin \theta)$, $\theta \in (0, \frac{1}{5}\pi)$. The values at the tail of each line are the minimal relative errors with $k = 200$ or $k = 400$.
domain is fixed as \( B_{1.5}(0,0) \setminus \bar{B}_{0.5}(0,0) \). For both the piecewise constant source (23) and the mixed-point source (24), the relative error slopes with increasing wave number are collected in Figure 5. Both panels show that the distance between the support of \( f_1 \) and the observation site is essential for the performance of the Kaczmarz-Landweber iteration scheme. When the observation site have certain distance to the compactly supported domain, the multi-frequency measurement data for wave numbers \( k \in [1, 200] \) can not improve the resolution essentially. To obtain better accuracy one has to increase the wave number interval as shown in Figure 5 where \( k \in [1, 400] \) is considered for large distance cases of \( \Gamma_1 \) and \( \Gamma_2 \).

3.1.2. Rectangular domain. In the second example we consider a recovery domain to be \((0.5, 2.5) \times (-1, 1)\) which is a subset of the rectangular domain \( \Omega = (0.45, 2.85) \times (-1.05, 1.05) \). The observation site is chosen by \( \Gamma = \{0.45\} \times [-0.45, 0.45] \), c.f. Figure 6. All the multi-frequency setting are the same as in the above annular domain where we choose the piecewise constant source as below

\[
f_1(x, y) = \begin{cases} 
1 & \text{if } 0.7 \leq x \leq 0.9 \text{ and } y \in [-0.4, 0.4]; \\
1 & \text{if } 0.9 \leq x \leq 1 \text{ and } y \in [-0.4, -0.2] \cup [0.2, 0.4]; \\
1 & \text{if } 1.0 \leq x \leq 1.2 \text{ and } y \in [-0.4, 0.4]; \\
0 & \text{elsewhere.}
\end{cases}
\]

For exact data, we present the approximate source, error with respect to the exact source and the relative error slope in Figure 7. It can be observed that with a short observation site, the approximate source does not mimic the exact one accurately. One can either extend the wave number set or the length of \( \Gamma \) to improve the resolution.

At the same time, we focus on the influence of the distance between \( \text{supp} f_1 \) and the observation site \( \Gamma \). Different from the annular domain discussed above, we fix the observation site \( \Gamma = \{0.45\} \times [-0.45, 0.45] \) but move the recovery domain and the source function \( f_1(x, y) \) among \((0.5, 2.5) \times (-1, 1)\), \((10.5, 12.5) \times (-1, 1)\), \((20.5, 22.5) \times (-1, 1)\) respectively. The relative error slopes with respect to the increasing wave number are presented in Figure 8. If the observation site is close to the recovery domain \((0.5, 2.5) \times (-1, 1)\), the wave number interval \( k \in [1, 200] \)
provides an approximate source with relative error 0.3149. Nevertheless, if the distance becomes large, i.e. the recovery domain $(10.5, 12.5) \times (-1, 1)$, we have to increase the wave number interval to $k \in [1, 400]$ to obtain similar accuracy. Concerning the largest distance of the recovery domain $(20.5, 22.5) \times (-1, 1)$, one need even larger wave number interval, which is hard to realize in real applications.

3.2. Case two: Exterior inverse source problems. In the second case, we consider an exterior inverse source problem where the support of $f_1$ is contained in the exterior domain of the convex hull of the observation site $\Gamma$ which also naturally appears in the inverse source problem.

3.2.1. Annular domain. We choose the same source function in Subsection 3.1.1 but modify the observation site $\Gamma = \partial B_{0.45}(0, 0)$ as shown in Figure 9. The length of the observation site is approximately half of the length of $\Gamma$ in Figure 2.

Figure 7. Rectangular domain with piecewise constant sources for exact measurement data. Left: the approximate source; Middle: the error between both sources; Right: relative error versus the wave number $k$ with the minimal relative error 0.3149 at $k = 200$.

Figure 8. Error slopes for fixed observation site $\Gamma$ but different recovery domains. The values at the tail of each line are the minimal relative errors with $k = 200$ or $k = 400$.  

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Figure 9. Annular domain of the exterior inverse source problem. Solid red line is the observation site $\Gamma$ and shadowed domain is $\Omega$.

We collect the numerical results for exact measurements in Figure 10. With shorter length, the approximate source has a comparably larger relative error with respect to the exact one. Referring to the right panel of Figure 10, the relative error for the final wave number $k = 200$ is 0.3923.

Figure 10. Annular domain with a piecewise constant source function for exact measurement data. Left: the approximate source; Middle: the error between both sources; Right: relative error versus the wave number $k$ with the minimal relative error 0.3923 at $k = 200$.

3.2.2. Rectangular domain. Similarly we can choose the same source function in Subsection 3.1.2 but adjust the observation site $\Gamma = \{ -0.45, 0.45 \} \times [ -0.45, 0.45 ] \cup [ -0.45, 0.45 ] \times [ -0.45, 0.45 ]$ and the source domain $(-1.5, 1.5) \times (-1.5, 1.5) \setminus [-0.5, 0.5] \times [-0.5, 0.5]$ in Figure 11.

For exact data, we present the approximate source and error between both sources and the relative error slope in Figure 12. Notice that we have four times more of the observation site length compared with that of Figure 6. With longer observation site $\Gamma$, the relative error for $k = 200$ is much smaller compared with those in Subsection 3.1.2.

4. Conclusion. Uniqueness can be extended onto isotropic Maxwell and linear elasticity systems by using uniqueness of the continuation results in [8] instead of [19]. The needed scattering theory is available, although not so transparent and explicit as for the Helmholtz equation.
The next really challenging issue is to obtain increasing stability bounds, similar to those in [6], [15]. At present, there is no decisive idea how to approach this question.

It is important to collect further numerical evidence of the increasing stability for more complicated geometries not satisfying non-trapping conditions.

An increasing (to optimal Lipschitz one) stability without convexity conditions has a fundamental significance, since it suggests that an arbitrary location of a measurement site \( \Gamma \) can still provide with a high resolution in many important applications.

One expects increasing stability in the inverse source problem when scattering is replaced by stationary waves in a bounded domain, i.e. when one replaces \( \mathbb{R}^n \) by a bounded reference domain \( \Omega_1 \) and the radiation condition by one of standard boundary conditions on \( \partial \Omega_1 \) (for example, by the Neumann condition which is most important in applications). In these cases there are additional difficulties due to eigenvalues of elliptic boundary value problems in bounded domains.

One can look at the different inverse source problem which is the linearized inverse problem for the Schrödinger potential: find \( f \) (supported in \( \Omega \subset \mathbb{R}^3 \)) from

\[
\int_{\Omega} f(y) e^{ik|x-y|} e^{ik|z-y|} |x-y| |z-y| dy
\]

given for \( x, z \in \Gamma \subset \partial \Omega \), where one expects quite explicit increasing stability bounds for \( f \) for larger \( k \).
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