Worst-case vs Average-case Design for Estimation from Fixed Pairwise Comparisons

Ashwin Pananjady†  Cheng Mao*  Vidya Muthukumar†
Martin J. Wainwright†,‡  Thomas A. Courtade†

Department of Electrical Engineering and Computer Sciences, UC Berkeley†
Department of Statistics, UC Berkeley‡
Department of Mathematics, MIT *

November 5, 2018

Abstract

Pairwise comparison data arises in many domains, including tournament rankings, web search, and preference elicitation. Given noisy comparisons of a fixed subset of pairs of items, we study the problem of estimating the underlying comparison probabilities under the assumption of strong stochastic transitivity (SST). We also consider the noisy sorting subclass of the SST model. We show that when the assignment of items to the topology is arbitrary, these permutation-based models, unlike their parametric counterparts, do not admit consistent estimation for most comparison topologies used in practice. We then demonstrate that consistent estimation is possible when the assignment of items to the topology is randomized, thus establishing a dichotomy between worst-case and average-case designs. We propose two estimators in the average-case setting and analyze their risk, showing that it depends on the comparison topology only through the degree sequence of the topology. The rates achieved by these estimators are shown to be optimal for a large class of graphs. Our results are corroborated by simulations on multiple comparison topologies.

1 Introduction

The problems of ranking and estimation from ordinal data arise in a variety of disciplines, including web search and information retrieval [DKNS01], crowdsourcing [CBCTH13], tournament play [HMG06], social choice theory [CN91] and recommender systems [BMR10]. The ubiquity of such datasets stems from the relative ease with which ordinal data can be obtained, and from the empirical observation that using pairwise comparisons as a means of data elicitation can lower the noise level in the observations [Bar03, SBC05].

Given that the number of items $n$ to be compared can be very large, it is often difficult or impossible to obtain comparisons between all $\binom{n}{2}$ pairs of items. A subset of pairs to compare, which defines the comparison topology, must therefore be chosen. For example, such topologies arise from tournament formats in sports, experimental designs in psychology set up to aid interpretability, or properties of the elicitation process. For instance, in rating movies, pairwise comparisons between items of the same genre are typically more abundant than comparisons between items of dissimilar genres. For these reasons, studying the performance of ranking algorithms based on fixed comparison topologies is of interest. Fixed comparison topologies are also important in rank breaking [HOX14, KO16], and more generally in matrix completion based on structured observations [KTT15, PABN16].
An important problem in ranking is the design of accurate models for capturing uncertainty in pairwise comparisons. Given a collection of $n$ items, the results of pairwise comparisons are completely characterized by the $n$-dimensional matrix of comparison probabilities and various models have been proposed for such matrices. The most classical models, among them the Bradley-Terry-Luce and Thurstone models, assign a quality vector to the set of items, and assign pairwise probabilities by applying a cumulative distribution function to the difference of qualities associated to the pair. There is now a relatively large body of work on methods for ranking in such parametric models (e.g., see the papers as well as references therein). In contrast, less attention has been paid to a richer class of models proposed decades ago in the sociology literature, which impose a milder set of constraints on pairwise comparison matrix. Rather than positing a quality vector, these models impose constraints that are typically given in terms of a latent permutation that rearranges the matrix into a specified form, and hence can be referred to as permutation-based models. Two such models that have been recently analyzed are those of strong stochastic transitivity, as well as the special case of noisy sorting. The strong stochastic transitivity (SST) model, in particular, has been shown to offer significant robustness guarantees and provide a good fit to many existing datasets, and this flexibility has driven recent interest in understanding its properties. Also, perhaps surprisingly, past work has shown that this additional flexibility comes at only a small price when one has access to all possible pairwise comparisons, or more generally, to comparisons chosen at random; in particular, the rates of estimation in these SST models differ from those in parametric models by only logarithmic factors in the number of items. On a related note, permutation-based models have also recently been shown to be useful in other settings like crowd-labeling, statistical seriation and linear regression.

Given pairwise comparison data from one of these models, the problem of estimating the comparison probabilities has applications in inferring customer preferences in recommender systems, advertisement placement, and sports, and is the main focus of this paper.

Our Contributions: Our goal is to estimate the matrix of comparison probabilities for fixed comparison topologies, studying both the noisy sorting and SST classes of matrices. Focusing first on the worst-case setting in which the assignment of items to the topology may be arbitrary, we show in Theorem that consistent estimation is impossible for many natural comparison topologies. This result stands in sharp contrast to parametric models, and may be interpreted as a “no free lunch” theorem: although it is possible to estimate SST models at rates comparable to parametric models when given a full set of observations, the setting of fixed comparison topologies is problematic for the SST class. This can be viewed as a price to be paid for the additional robustness afforded by the SST model.

Seeing as such a worst-case design may be too strong for permutation-based models, we turn to an average-case setting in which the items are assigned to a fixed graph topology in a randomized fashion. Under such an observation model, we propose and analyze two efficient estimators: Theorems and show that consistent estimation is possible under commonly used comparison topologies. Moreover, the error rates of these estimators depend only on the degree sequence of the comparison topology, and are shown to be unimprovable for a large class of graphs, in Theorem.

Our results therefore establish a sharp distinction between worst-case and average-case
designs when using fixed comparison topologies in permutation-based models. Such a phenomenon arises from the difference between minimax risk and Bayes risk under a uniform prior on the ranking, and may also be worth studying for other ranking models.

**Related Work:** The literature on ranking and estimation from pairwise comparisons is vast, and we refer the reader to some surveys [EV93, Mar96, Cat12] and references therein for a more detailed overview. Estimation from pairwise comparisons has been analyzed under various metrics like top-k ranking [CS15, SW15, JJSO16, CGMS17], comparison probability or parameter estimation [HOX14, SBB+16, SBGW17]. There have been studies of these problems under active [JN11, HSRW16, MG15], passive [NOS16, RA16], and collaborative settings [PNZ+15, NOTX17], and also for fixed as well as random comparison topologies [WJJ13, SBGW17]. Here we focus on the subset of papers that are most relevant to the work described here.

The problem of comparison probability estimation under a passively chosen fixed topology has been analyzed for parametric models by Hajek et al. [HOX14] and Shah et al. [SBB+16]. Both papers analyze the worst-case design setting in which the assignment of items to the graph is arbitrary, and derive bounds on the minimax risk of parameter (or equivalently, comparison probability) estimation. While their characterizations are not sharp in general, the rates are shown to depend on the spectrum of the Laplacian matrix of the topology. We point out an interesting consequence of both results: in the parametric model, the rates are shown to depend on the spectrum of the Laplacian matrix of the topology. We will see that this property no longer holds for the SST models considered in this paper: there are comparison topologies and SST matrices for which it is impossible to recover the full matrix even given an infinite amount of data per graph edge. It is also worth mentioning that the top-k ranking problem has been analyzed for parametric models by Hajek et al. [HOX14] and Shah et al. [SBB+16].

**Notation:** Here we summarize some notation used throughout the remainder of this paper. We use $n$ to denote the number of items, and adopt the shorthand $[n] := \{1, 2, \ldots, n\}$. We use $\text{Ber}(p)$ to denote a Bernoulli random variable with success probability $p$. For two sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, we write $a_n \lesssim b_n$ if there is a universal constant $C$ such that $a_n \leq Cb_n$ for all $n \geq 1$. The relation $a_n \gtrsim b_n$ is defined analogously, and we write $a_n \asymp b_n$ if the relations $a_n \lesssim b_n$ and $a_n \gtrsim b_n$ hold simultaneously. We use $c, c_1, c_2$ to denote universal constants that may change from line to line.

We use $e \in \mathbb{R}^n$ to denote the all-ones vector in $\mathbb{R}^n$. Given a matrix $M \in \mathbb{R}^{n \times n}$, its $i$-th row is denoted by $M_i$. For a graph $G$ with edge set $E$, let $M(G)$ denote the entries of the matrix $M$ restricted to the edge set of $G$, and let $\|M\|_E^2 = \sum_{(i,j) \in E} M_{ij}^2$. For a matrix $M \in \mathbb{R}^{n \times n}$ and a permutation $\pi : [n] \rightarrow [n]$, we use the shorthand $\pi(M) = \Pi M \Pi^\top$, where $\Pi$ represents the row permutation matrix corresponding to the permutation $\pi$. We let $\Pi$ denote the identity permutation. The Kendall tau distance $\text{KT}([\pi, \pi']) := \sum_{i,j \in [n]} 1\{\pi(i) < \pi(j), \pi'(i) > \pi'(j)\}$.

Let $\mathcal{G}(G)$ represent the set of all connected, vertex-induced subgraphs of a graph $G$, and let $V(S)$ and $E(S)$ represent the vertex and edge set of a subgraph $S$, respectively. We let
α(G) denote the size of the largest independent set of the graph G, which is a largest subset of vertices that have no edges among them. Define a biclique of a graph as two disjoint subsets of its vertices V₁ and V₂ such that (u, v) ∈ E(G) for all u ∈ V₁ and v ∈ V₂. Define the biclique number β(G) as the maximum number of edges in any such biclique, given by \( \max_{V₁ \cup V₂ \text{biclique}} |V₁||V₂| \). Let \( d_v \) denote the degree of vertex \( v \in V \).

2 Background and Problem Formulation

Consider a collection of \( n \geq 2 \) items that obey a total ordering or ranking determined by a permutation \( \pi^* : [n] \to [n] \). More precisely, item \( i \in [n] \) is preferred to item \( j \in [n] \) in the underlying ranking if and only if \( \pi^*(i) < \pi^*(j) \). We are interested in observations arising from stochastic pairwise comparisons between items. We denote the matrix of underlying comparison probabilities by \( M^* \in [0, 1]^{n \times n} \), with \( M^*_{ij} = \Pr\{i \succ j\} \) representing the probability that item \( i \) beats item \( j \) in a comparison.

Each item \( i \) is associated with a score, given by the probability that item \( i \) beats another item chosen uniformly at random. More precisely, the score \( \tau_i^* \) of item \( i \) is given by

\[
\tau_i^* := [\tau(M^*)]_i := \frac{1}{n-1} \sum_{j \neq i} M^*_{ij}.
\]

Arranging the scores in descending order naturally yields a ranking of items. In fact, for the models we define below, the ranking given by the scores is consistent with the ranking given by \( \pi^* \), i.e., \( \tau_i \geq \tau_j \) if \( \pi^*(i) < \pi^*(j) \). The converse also holds if the scores are distinct.

2.1 Pairwise comparison models

We consider a permutation-based model for the comparison matrix \( M^* \), one defined by the property of strong stochastic transitivity \cite{Fis73, ML65}, or the SST property for short. In particular, a matrix \( M^* \) of pairwise comparison probabilities is said to obey the SST property if for items \( i, j \) and \( k \) in the total ordering such that \( \pi^*(i) < \pi^*(j) < \pi^*(k) \), it holds that \( \Pr(i \succ k) \geq \Pr(i \succ j) \). Alternatively, recalling that \( \pi(M) \) denotes the matrix obtained from \( M \) by permuting its rows and columns according to the permutation \( \pi \), the SST matrix class can be defined in terms of permutations applied to the class \( \mathcal{C}_{\text{BISO}} \) of bivariate isotonic matrices as

\[
\mathcal{C}_{\text{SST}} : = \bigcup_{\pi} \pi(\mathcal{C}_{\text{BISO}}) = \bigcup_{\pi} \{\pi(M) : M \in \mathcal{C}_{\text{BISO}}\}.
\]

Here the class \( \mathcal{C}_{\text{BISO}} \) of bivariate isotonic matrices is given by

\[\{M \in [0, 1]^{n \times n} : M + M^T = e e^T \text{ and } M \text{ has non-decreasing rows and non-increasing columns}\},\]

where \( e \in \mathbb{R}^n \) denotes a vector of all ones.

As shown by Shah et al. \cite{SBGW17}, the SST class is substantially larger than commonly used class of parametric models, in which each item \( i \) is associated with a parameter \( w_i \in \mathbb{R} \), and the probability that item \( i \) beats item \( j \) is given by \( F(w_i - w_j) \), where \( F : \mathbb{R} \mapsto [0, 1] \) is a smooth monotone function of its argument.

\footnote{We set \( M^*_{ii} = 1/2 \) by convention.}
A special case of the SST model that we study in this paper is the *noisy sorting* model \[BM08\], in which the all underlying probabilities are described with a single parameter \(\lambda \in [0, 1/2]\). The matrix \(M_{NS}(\pi, \lambda) \in [0, 1]^{n \times n}\) has entries
\[
[M_{NS}(\pi, \lambda)]_{ij} = 1/2 + \lambda \cdot \text{sgn}(\pi(j) - \pi(i)),
\]
and the noisy sorting classes are given by
\[
C_{NS}(\lambda) := \bigcup_{\pi} \{M_{NS}(\pi, \lambda)\}, \quad \text{and} \quad C_{NS} := \bigcup_{\lambda \in [0, 1/2]} C_{NS}(\lambda). \tag{3}
\]
Here \(\text{sgn}(x)\) is the sign operator, with the convention that \(\text{sgn}(0) = 0\). In words, the noisy sorting class models the case where the probability \(\Pr\{i \succ j\}\) depends only on the parameter \(\lambda\) and whether \(\pi^*(i) < \pi^*(j)\). Although a noisy sorting model is a very special case of an SST model, apart from the degenerate case \(\lambda^* = 1/2\), it cannot be represented by any parametric model with a smooth function \(F\), and so captures the essential difficulty of learning in the SST class.

We now turn to describing the observation models that we consider in this paper.

### 2.2 Partial observation models

Our goal is to provide guarantees on estimating the underlying comparison matrix \(M^*\) when the comparison topology is fixed. Suppose that we are given data for comparisons in the form of a graph \(G = (V, E)\), where the vertices represent the \(n\) items and edges represent the comparisons made between items. We assume that the observations obey the probabilistic model
\[
Y_{ij} = \begin{cases} 
\text{Ber}(M^*_{ij}) & \text{for } (i, j) \in E, \text{ independently} \\
\ast & \text{otherwise,}
\end{cases} \tag{4}
\]
where \(\ast\) indicates a missing observation. We set the diagonal entries of \(Y\) equal to 1/2, and also specify that \(Y_{ji} = 1 - Y_{ij}\) for \(j > i\), so that \(Y + Y^\top = e e^\top\). We consider two different instantiations of the edge set given the graph.

#### 2.2.1 Worst-case setting

In this setting, we assume that the assignment of items to vertices of the comparison graph \(G\) is arbitrary. In other words, once the graph \(G\) and its edges \(E\) are fixed, we observe the entries of the matrix according to the observation model (4), and would like to provide uniform guarantees in the metric \(\|\hat{M} - M^*\|_F\) over all matrices \(M^*\) in our model class given this restricted set of observations.

This setting is of the worst-case type, since the adversary is allowed to choose the underlying matrix with knowledge of the edge set \(E\). Providing guarantees against such an adversary is known to be possible for parametric models \(HOX14\ [SBB^{+}16]\). However, as we show in Section 3.1, such a guarantee is impossible to obtain even over the the noisy sorting subclass of the full SST class. Consequently, the latter parts of our analysis apply to a less rigid, average-case setting.
2.2.2 Average-case setting

In this setting, we assume that the assignment of items to vertices of the comparison graph $G$ is random. Equivalently, given a fixed comparison graph $G$ having adjacency matrix $A$, the subset of the entries that we observe can be modeled by the operator $O = \sigma(A)$ for a permutation $\sigma : [n] \to [n]$ chosen uniformly at random. For a fixed comparison matrix $M^*$, our observations themselves consist of a random subset of the entries of the matrix $Y$ determined by the operator $O$: a location where $O_{ij} = 1$ (respectively $O_{ij} = 0$) indicates that entry $Y_{ij}$ is observed (respectively is not observed). Such a setting is reasonable when the graph topology is constrained, but we are still given the freedom to assign items to vertices of the comparison graph, e.g. in psychology experiments. A natural extension of such an observation model is the one of $k$ random designs, consisting of multiple random observation operators $\{O_i = \sigma_i(A)\}_{i=1}^k$, chosen with independent, random permutations $\{\sigma_i\}_{i=1}^k$.

Our guarantees in the one sample setting with the observation operator $O$ can be seen as a form of Bayes risk, where given a fixed observation pattern $E$ (consisting of the entries of the comparison matrix $Y$ determined by the adjacency matrix $A$ of the graph $G$, with $A_{ij}$ representing the indicator that entry $Y_{ij}$ is observed), we want to estimate a matrix $M^*$ under a uniform Bayesian prior on the ranking $\pi^*$. Studying this average-case setting is well-motivated, since given fixed comparisons between a set of items, there is no reason to assume a priori that the underlying ranking is generated adversarially.

We are now ready to state the goal of the paper. We address the problems of recovering the ranking $\pi^*$ and estimating the matrix $M^*$ in the Frobenius norm. More precisely, given the observation matrix $Y = Y(E)$ (where the set $E$ is random in the average-case observation model), we would like to output a matrix $\hat{M}$ that is function of $Y$, and for which good control on the Frobenius norm error $\|\hat{M} - M^*\|_F^2$ can be guaranteed.

3 Main results

In this section, we state our main results and discuss some of their consequences. Proofs are deferred to Section 5.

3.1 Worst-case design: minimax bounds

In the worst-case setting of Section 2.2.1, the performance of an estimator is measured in terms of the normalized minimax error

$$\mathcal{M}(G, \mathbb{C}) = \inf_{\hat{M} = f(Y(G))} \sup_{M^* \in \mathbb{C}} \mathbb{E}\left[\frac{1}{n^2} \|\hat{M} - M^*\|_F^2\right],$$

where the expectation is taken over the randomness in the observations $Y$ as well as any randomness in the estimator, and $\mathbb{C} \in \{\mathbb{C}_{\text{SST}}, \mathbb{C}_{\text{NS}}\}$ represents the model class. Our first result shows that for many comparison topologies, the minimax risk is prohibitively large even for the noisy sorting model.

**Theorem 1.** For any graph $G$, the diameter of the set consistent with observations on the edges of $G$ is lower bounded as

$$\sup_{M_1, M_2 \in \mathbb{C}_{\text{NS}}, M_1(G) = M_2(G)} \|M_1 - M_2\|_F^2 \geq \alpha(G)(\alpha(G) - 1) \vee \beta(G^*). \quad (5a)$$
Consequently, the minimax risk of the noisy sorting model is lower bounded as
\[
\mathcal{M}(G, \mathbb{C}_{\text{NS}}) \geq \frac{1}{4n^2} [\alpha(G)(\alpha(G) - 1) \lor \beta(G^*)]. \tag{5b}
\]

Note that via the inclusion \( \mathbb{C}_{\text{NS}} \subset \mathbb{C}_{\text{SST}} \), Theorem 1 also implies the same lower bound \((5b)\) on the risk \( \mathcal{M}(G, \mathbb{C}_{\text{SST}}) \). In addition to these bounds, the lower bounds for estimation in parametric models, known from past work \cite{SBB+16}, carry over directly to the SST model, since parametric models are subclasses of the SST class.

Theorem 1 is approximation-theoretic in nature: more precisely, inequality \((5a)\) is a statement purely about the size of the set of matrices consistent with observations on the graph. Consequently, it does not capture the uncertainty due to noise, and thus can be a loose characterization of the minimax risk for some graphs, with the complete graph being one example.

The bound \((5a)\) on the diameter of the set of consistent observations may be interpreted as the worst case error in the infinite sample limit of observations on \( G \). Hence, Theorem 1 stands in sharp contrast to analogous results for parametric models \cite{HOX14, SBB+16}, in which it suffices for the graph to be connected in order to obtain consistent estimation in the infinite sample limit. For example, connected graphs with large independent sets of order \( n \) do not admit consistent estimation over the noisy sorting and hence SST classes.

It is also worth mentioning that the connectivity properties of the graph that govern minimax estimation in the larger SST model are quite different from those appearing in parametric models. In particular, the minimax rates for parametric models are closely related (via the linear observation model) to the spectrum of the Laplacian matrix of the graph \( G \). In Theorem 1 however, we see other functions of the graph appearing that are not directly related to the Laplacian spectrum. In Section 4 we evaluate these functions for commonly used graph topologies, showing that for many of them, the risk is lower bounded by a constant even for graphs admitting consistent parametric estimation.

Seeing as the minimax error in the worst-case setting can be prohibitively large, we now turn to evaluating practical estimators in the random observation models of Section 2.2.2.

### 3.2 Average-case design: noisy sorting matrix estimation

In the average-case setting described in Section 2.2.2 we measure the performance of an estimator using the risk
\[
\sup_{M^* \in \mathbb{C}} \mathbb{E}_{\mathcal{O}, \mathcal{Y}} \frac{1}{n^2} \| \hat{M} - M^* \|_F^2.
\]
It is important to note that the expectation is taken over both the comparison noise, as well as the random observation pattern \( \mathcal{O} \) (or equivalently, the underlying random permutation \( \sigma \) assigning items to vertices). We propose the Average-Sort-Project estimator (ASP for short) for matrix estimation in this metric, which is a natural generalization of the Borda count estimator \cite{CM16, SBW16a}. It consists of three steps, described below for the noisy sorting model:

1. **Averaging step:** Compute the average \( \hat{\tau}_i = \frac{\sum_{j \neq i} Y_{ij} O_{ij}}{\sum_{j \neq i} O_{ij}} \), corresponding to the fraction of comparisons won by item \( i \).

2. **Sorting step:** Choose the permutation \( \hat{\pi}_{\text{ASP}} \) such that the sequence \( \{ \hat{\tau}_{\hat{\pi}_{\text{ASP}}(i)} \}_{i=1}^n \) is decreasing in \( i \), with ties broken arbitrarily.

7
(3) **Projection step**: Find the maximum likelihood estimate \( \hat{\lambda} \) by treating \( \hat{\pi}_{\text{ASP}} \) as the true permutation that sorts items in decreasing order. Output the matrix \( \hat{M}_{\text{ASP}} := M_{\text{NS}}(\hat{\pi}_{\text{ASP}}, \hat{\lambda}) \).

We now state an upper bound on the mean-squared Frobenius error achievable using the ASP estimator. It involves the degree sequence \( \{d_v\}_{v \in V} \) of a graph \( G \) without isolated vertices, meaning that \( d_v \geq 1 \) for all \( v \in V \).

**Theorem 2.** Let the observation process be given by \( O \). For any graph \( G = (V, E) \) without isolated vertices and any matrix \( M^* \in \mathbb{C}_{\text{NS}}(\lambda^*) \), we have

\[
\mathbb{E}_{O, Y} \left[ \frac{1}{n^2} \| \hat{M}_{\text{ASP}} - M^* \|^2_F \right] \lesssim \frac{1}{|E|} + \frac{n \log n}{|E|^2} + \frac{\lambda^*}{n} \sum_{v \in V} \frac{1}{\sqrt{d_v}}, \quad \text{and} \quad (6a)
\]

\[
\mathbb{E}_{O, Y} [\text{KT}(\pi^*, \hat{\pi}_{\text{ASP}})] \lesssim \frac{n}{\lambda^*} \sum_{v \in V} \frac{1}{\sqrt{d_v}}. \quad (6b)
\]

A few comments are in order. First, while the results are stated in expectation, a high probability bound can be proved for permutation estimation—namely

\[
\Pr_{O, Y} \left\{ \text{KT}(\pi^*, \hat{\pi}_{\text{ASP}}) \gtrsim \frac{n \sqrt{\log n}}{\lambda^*} \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right\} \leq n^{-10}.
\]

Second, it can be verified that \( \frac{1}{|E|} + \frac{n \log n}{|E|^2} \lesssim \frac{1}{n} \sum_{v \in V} \frac{1}{\sqrt{d_v}} \), so that taking a supremum over the parameter \( \lambda^* \in [0, 1/2] \) guarantees that the mean-squared Frobenius error is upper bounded as \( O \left( \frac{1}{n} \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right) \), uniformly over the entire noisy sorting class \( \mathbb{C}_{\text{NS}} \). Third, it is also interesting to note the dependence of the bounds on the noise parameter \( \lambda^* \) of the noisy sorting model. The “high-noise” regime \( \lambda^* \approx 0 \) is a good one for estimating the underlying matrix, since the true matrix \( M^* \) is largely unaffected by errors in estimating the true permutation. However, as captured by equation \( (6b) \), the permutation estimation problem is more challenging in this regime.

The bound \( (6a) \) can be specialized to the complete graph \( K_n \) and the Erdős-Rényi random graph with edge probability \( p \) to obtain the rates \( 1/\sqrt{n} \) and \( 1/\sqrt{np} \), respectively, for estimation in the mean-squared Frobenius norm. These rates are strictly sub-optimal for these graphs, since the minimax rates scale as \( 1/n \) and \( 1/(np) \), respectively; both are achieved by the global MLE [SBGW17]. Such a phenomenon is consistent with the gap observed between computationally constrained and unconstrained estimators in similar and related problems [SBGW17, FMR16, PWC17].

Interestingly, it turns out that the estimation rate \( (6a) \) is optimal in a certain sense, and we require some additional notions to state this precisely. Fix constants \( C_1 = 10^{-2} \) and \( C_2 = 10^2 \) and two sequences \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \) of (strictly) positive scalars. For each \( n \geq 1 \), define the family of graphs

\[
\mathcal{G}_n(a_n, b_n) := \left\{ G(V, E) \text{ is connected : } |V| = n, \quad C_1 a_n \leq |E| \leq C_2 a_n, \text{ and } C_1 b_n \leq \sum_{v \in V} \frac{1}{\sqrt{d_v}} \leq C_2 b_n \right\}.
\]

As noted in Section 2.2.2, the average-case design observation model is equivalent to choosing the matrix \( M^* \) from a random ensemble with the permutation \( \pi^* \) chosen uniformly at random, and observing fixed pairwise comparisons. Such a viewpoint is useful in order to state our lower bound. Expectations are taken over the randomness of both \( \pi^* \) and the Bernoulli observation noise.
Theorem 3. (a) Let $M^* = M_{NS}(\pi^*, 1/4)$, where the permutation $\pi^*$ is chosen uniformly at random on the set $[n]$. For any pair of sequences $(\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1})$ such that the set $G_n(a_n, b_n)$ is non-empty for every $n \geq 1$, and for any estimators $(\hat{M}, \hat{\pi})$ that are measurable functions of the observations on $G$, we have
\[
\sup_{G \in G_n(a_n, b_n)} \mathbb{E} \left[ \frac{1}{n^2} \| \hat{M} - M^* \|_F^2 \right] \gtrsim \frac{b_n}{n} \quad \text{and} \quad \sup_{G \in G_n(a_n, b_n)} \mathbb{E} \left[ KT(\pi^*, \hat{\pi}) \right] \gtrsim nb_n.
\]
(b) For any graph $G$, let $M^* = M_{NS}(\pi^*, c\sqrt{n/|E|})$, with the permutation $\pi^*$ chosen uniformly at random and the constant $c$ chosen sufficiently small. Then for any estimators $(\hat{M}, \hat{\pi})$ that are measurable functions of the observations on $G$, we have
\[
\mathbb{E} \left[ \frac{1}{n^2} \| \hat{M} - M^* \|_F^2 \right] \gtrsim \frac{n}{|E|}.
\]

Parts (a) and (b) of the lower bound may be interpreted respectively as the approximation error caused by having observations only on a subset of edges, and the estimation error arising from the Bernoulli observation noise. Note that part (b) applies to every graph, and is particularly noteworthy for sparse graphs. In particular, in the regime in which the graph has bounded average degree, it shows that the inconsistency exhibited by the ASP estimator is unavoidable for any estimator. A more detailed discussion for specific graphs may be found in Section 1.

Although part (a) of the theorem is stated for a supremum over graphs, we actually prove a stronger result that explicitly characterizes the class of graphs that attain these lower bounds. As an example, given the sequences $a_n = n^{2}$ and $b_n = \sqrt{n}$, we show that the ASP estimator is information-theoretically optimal for the sequence of graphs consisting of two disjoint cliques $K_n/2 \cup K_{n/2}$, which can be verified to lie within the class $G(a_n, b_n)$.

The ASP estimator for the SST model would replace step (iii), as stated, by a maximum likelihood estimate using the entries on the edges that we observe. However, analyzing such an estimator given only a single sample on the entries $O$ is a challenging problem due to dependencies between the different steps of the estimator, and the difficulty of solving the associated matrix completion problem. Consequently, we turn to an observation model consisting of two random designs, and design a different estimator that renders the matrix completion problem tractable.

### 3.3 Two random designs: SST matrix estimation

Recall the average-case setting with multiple random designs, as described in Section 2.2.2 in which the comparison topology is fixed ahead of time, but one can collect multiple observations by assigning items to the vertices of the underlying graph at random. In this section, we rely on two such independent observations $O_1$ and $O_2$ to design an estimator that is consistent over the SST class. In order to describe our estimator, we require some additional notation. For any matrix $X \in [0, 1]^{n \times n}$ such that $X + X^\top = ee^\top$, we use $r(X) := Xe$ to denote the vector of its row sums. Note that this vector is related to the vector of scores, as defined in equation (1), via $r(X) = (n - 1)\tau(X) + 1/2$.

Our estimator relies on the approximation of any matrix $M^* \in \mathbb{C}_{SST}$ by a block-wise constant matrix, and we require some more definitions to make this precise. For any vector $v \in \mathbb{R}_+^n$, fix some value $t \in (0, n)$ and define a block partition $bl_t(v)$ of $v$ as
\[
[bl_t(v)]_i = \{ j \in [n] : v_j \in [(i - 1)t], [it] \}.
\]
In particular, the blocking vector \( \hat{b}_l(\tau(X)) \) contains a partition of indices such that the row sums of the matrix within each block of the partition are within a gap \( t \) of each other. Denote the set of all possible partitions of the set \([n]\) by \( \chi_n \). For any partition \( C \in \chi_n \) of the indices \([n]\), define the set of blocks \( \mathcal{B}(C) = \{ S \times T : S, T \in C \} \).

By definition, given a partition \( C \in \chi_n \) of \([n]\), the set \( \mathcal{B}(C) \) is a partition of the set \([n] \times [n]\) into blocks. We are now ready to describe the blocking operation. For indices \( i, j \in [n] \), denote by \( B_C(i, j) \) the block in \( \mathcal{B}(C) \) that contains the tuple \((i, j)\). Given a matrix \( X \in [0, 1]^{n \times n} \) satisfying \( X + X^\top = \mathbf{e} \mathbf{e}^\top \), we define the blocked version \( X \) depending on observations in a set \( E \subseteq [n] \times [n] \) as

\[
[B(X, C, E)]_{ij} = \begin{cases} \frac{1}{|B_C(i,j) \cap E|} \sum_{(k, \ell) \in B_C(i,j) \cap E} X_{k\ell} & \text{if } B_C(i,j) \cap E \neq \emptyset \\ 1/2 & \text{otherwise.} \end{cases}
\]

(7)

In words, this defines a projection of the matrix \( X \) onto the set of block-wise constant matrices, by block-wise averaging the entries of \( X \) over the observed set of entries \( E \). We now turn to our estimator, called the Block-Average-Project estimator (BAP for short), of the underlying matrix \( M^* \in \mathbb{C}_{\text{SST}} \). Given the observation matrix \( Y_1 \), define

\[
[Y_1]_{ij} = \begin{cases} \frac{D_i}{n} [Y_1]_{ij} & \text{if entry } (i, j) \text{ is observed,} \\ 0 & \text{otherwise,} \end{cases}
\]

where \( D_i = \sum_{j=1}^n [O_1]_{ij} \) is the (random) degree of item \( i \). We now perform three steps:

1. **Blocking step:** Fix \( S = \sum_{v \in V} 1/\sqrt{d_v} \), and obtain the blocking vector \( \hat{b} = \text{bl}_S(\tau(Y_1')) \) and permutation \( \hat{\pi}_{\text{ASP}} \) as in step (2) of the ASP estimator.
2. **Averaging step:** Average the matrix \( Y_2 \) within each block to obtain the matrix \( \tilde{M} = \text{B}(Y_2, \hat{b}, E_2) \).
3. **Projection step:** Project onto the space \( \hat{\pi}_{\text{ASP}}(\mathbb{C}_{\text{BISO}}) = \{ \hat{\pi}_{\text{ASP}}(M) : M \in \mathbb{C}_{\text{BISO}} \} \), to obtain the estimator \( \tilde{M}_{\text{BAP}} \).

The blocking and averaging steps of the estimator are the main ingredients that we use to bound the error of the associated matrix completion problem. Also, the projection step of the estimator can be computed in polynomial time via bivariate isotonic regression [BDPR84].

**Theorem 4.** Let the observation process be given by \( O_1 \cup O_2 \). For any graph \( G \) without isolated vertices and any matrix \( M^* \in \mathbb{C}_{\text{SST}} \), we have

\[
\mathbb{E} \left[ \frac{1}{n^2} \| \tilde{M}_{\text{BAP}} - M^* \|_F^2 \right] \leq \frac{1}{n} \sum_{v \in V} \frac{1}{\sqrt{d_v}},
\]

where the expectation is taken over the noise, and observation patterns \( O_1 \) and \( O_2 \).

To be clear, the blocking estimate \( \tilde{M}_{\text{BAP}} \) is well-defined even when we have just one sample \( O_1 \) instead of two samples \( O_1 \) and \( O_2 \), where step (2) is replaced by the estimate \( \tilde{M} = \text{B}(Y_1, \hat{b}, E_1) \). In the simulations of Section 4, we see that for a large variety of graphs, using a single sample \( O_1 \) enjoys similar performance to using two independent samples \( O_1 \) and \( O_2 \). We require two independent samples of the observations in our theoretical analysis to decouple the randomness of the first step of the algorithm from the second. When using one sample \( O_1 \), the dependencies that are introduced between the different steps of the algorithm make the analysis challenging.
4 Dependence on graph topologies

In this section, we discuss implications of our results for some comparison topologies. Let us focus first on the worst-case design setting, and the lower bound of Theorem 1. For the star, path (or more generally, any graph with bounded average degree), and complete bipartite graphs, one can verify that we have $\alpha(G) \asymp n$, so $M(G, C_{NS}) \asymp 1$. If the graph is a union of disjoint cliques $K_{n/2} \cup K_{n/2}$ (or having a constant number of edges across the cliques, like a barbell graph), then we see that $\beta(G^c) \asymp n^2$, so $M(G, C_{NS}) \asymp 1$. Thus, our theory yields pessimistic results for many practically motivated comparison topologies under worst-case designs, even though all the connected graphs above admit consistent estimation for parametric models as the number of samples grows. In the average case-setting of Section 2.2.2, Theorems 2, 3 and 4 characterize the mean-squared Frobenius norm errors of the corresponding estimators (up to constants) as $D(G) = \frac{1}{n} \sum_{v \in V} \frac{1}{\sqrt{d_v}}$.

In order to illustrate our results for the average-case setting, we present the results of simulations on data generated synthetically from two special cases of the SST model. We fix $\pi^* = \text{id}$ without loss of generality, and generate the ground truth comparison matrix $M^*$ in one of two ways:

1. Noisy sorting with high SNR: We set $M^* = M_{NS}(\text{id}, 0.4)$.

2. SST with independent bands: We first set $M^*_i = 1/2$ for every $i$. Entries on the diagonal band immediately above the diagonal (i.e. $M^*_{i,i+1}$ for $i \in [n-1]$) are chosen i.i.d. and uniformly at random from the set $[1/2, 1]$. The band above is then chosen uniformly at random from the allowable set, where every entry is constrained to be upper bounded by 1 and lower bounded by the entries to its left and below. We also set $M^*_{ij} = 1 - M^*_{ji}$ to fill the rest of the matrix.

For each graph $G$ with adjacency matrix $A$, the data is generated from ground truth by observing independent Bernoulli comparisons under the observation process $O = \sigma(A)$, for a randomly generated permutation $\sigma$. For the SST model, we also generate data from two independent random observations $O_1$ and $O_2$ as required by the BAP estimator; however, we also simulate the behaviour of the estimator for one sample $O_1$ and show that it closely tracks that of the two-sample estimator.

Recall that the estimation error rate was dictated by the degree functional $D(G)$. While our graphs were chosen to illustrate scalings of $D(G)$, some variants of these graphs also naturally arise as comparison topologies.

1. **Two-disjoint-clique graph:** For this graph $K_{n/2} \cup K_{n/2}$, we have $d_v = \frac{n}{2} - 1$ for every $v \in V$, and simple calculations yield $D(G) \asymp \frac{1}{\sqrt{n}}$. It is interesting to note that this graph has unfavorable guarantees for parametric estimation under the adversarial model, because it is disconnected (and thus has a Laplacian with zero spectral gap.) We observe that this spectral property does not play a role in our analysis of the ASP or BAP estimator under the average-case observation model, and this behavior is corroborated by our simulations. Although we do not show it here, a similar behavior is observed for the stochastic block model, a practically motivated comparison topology when there are genres present among the items, which is a relaxation of the two-clique case allowing for sparser “communities” instead of cliques, and edges between the communities.

3. The complete bipartite graph, for instance, admits optimal rates of estimation.

4. Note that the SST model has been validated extensively on real data in past work (see, e.g. Ballinger and Wilcox [BW97]).
Figure 1. Normalized Frobenius norm error $\frac{1}{n^2}\|\hat{M}_{\text{ASP}} - M^*\|_F^2$ with data generated using the noisy sorting model $M^* = M_{\text{NS}}(\text{id}, 0.4)$, averaged over 10 trials.
Figure 2. Normalized Frobenius norm error $\frac{1}{n^2} \| \hat{M}_{BAP} - M^* \|_F^2$ with data generated using the SST model with independent bands, averaged over 10 trials, plotted for one and two samples.
(2) **Clique-plus-path graph:** The nodes are partitioned into two sets of \( n/2 \) nodes each. The graph contains an edge between every two nodes in the first set, and a path starting from one of the nodes in the first set and chaining the other \( n/2 \) nodes. This is an example of a graph construction that has many (\( \asymp n^2 \)) edges, but is unfavorable for noisy sorting or SST estimation. Simple calculations show that the degree functional is dominated by the constant degree terms and we obtain \( \mathcal{D}(G) \asymp 1 \).

(3) **Power law graph:** We consider the special power law graph \([BA99]\) with degree sequence \( d_i = i \) for \( 1 \leq i \leq n \), and construct it using the Havel-Hakimi algorithm \([Hav55, Hak62]\). For this graph, we have a disparate degree sequence, but \( \mathcal{D}(G) \asymp 1 \sqrt{n} \), and the simulated estimators are consistent.

(4) **\( [(n/2)^\alpha] \)-regular bipartite graphs:** A final powerful illustration of our theoretical guarantees is provided by a regular bipartite graph construction in which the nodes are partitioned into two sets of \( n/2 \) nodes each, and each node in one set is (deterministically) connected to \( [(n/2)^\alpha] \) nodes in the other set. This results in the degree sequence \( d_v = [(n/2)^\alpha] \) for all \( v \in V \), and the degree functional evaluates to \( \mathcal{D}(G) \asymp n^{-\alpha/2} \). The value of \( \alpha \) thus determines the scaling of the estimation error for the ASP estimator in the noisy sorting case, as well as the BAP estimator in the SST case, as seen from the slopes of the corresponding plots.

Some other graphs that were considered in parametric model environments \([SBB+16]\), such as the star, cycle, path and hypercube graphs, turn out to be unfavorable for permutation-based models even in the average-case setting, as corroborated by the lower bound of Theorem 3, part (b).

5 **Proofs**

In this section, we provide the proofs of our main results. We assume throughout that \( n \geq 2 \), and use \( c, c' \) to denote universal constants that may change from line to line.

5.1 **Proof of Theorem 1**

For each fixed graph \( G \), define the quantity

\[
\mathcal{A}(G) := \sup_{M, M' \in \mathcal{C}_{\text{NS}}} \frac{1}{n^2} \sum_{(i,j) \notin E} (M_{ij} - M'_{ij})^2
\]

corresponding to the diameter quantity that is lower bounded in equation (5a). Taking the lower bound (5a) as given for the moment, we first prove the lower bound (5b) on the minimax risk. It suffices to show that the minimax risk is lower bounded in terms of \( \mathcal{A}(G) \) as

\[
\inf_{\hat{M} = f(Y(G))} \sup_{M^* \in \mathcal{C}_{\text{NS}}} \mathbb{E} \left[ \frac{1}{n^2} \| \hat{M} - M^* \|^2_F \right] \geq \frac{1}{4} \mathcal{A}(G).
\]  

(8)

In order to verify this claim, consider the two matrices \( M^1, M^2 \in \mathcal{C}_{\text{SST}} \) that attain the supremum in the definition of \( \mathcal{A}(G) \); note that such matrices exist due to the compactness of the space and the continuity of the squared loss. By construction, these two matrices satisfy the properties

\[
M^1(G) = M^2(G), \quad \text{and} \quad \sum_{(i,j) \notin E} (M^1_{ij} - M^2_{ij})^2 = n^2 \mathcal{A}(G).
\]
We can now reduce the problem to one of testing between the two matrices $M^1$ and $M^2$, with the distribution of observations being identical for both alternatives. Consequently, any procedure can do no better than to make a random guess between the two, so we have

$$\inf M \sup M^* \in C_{NS} \mathbb{E} \left[ \| \hat{M} - M^* \|_F^2 \right] \geq \frac{1}{4} \sum_{(i,j) \notin E} (M_{ij}^1 - M_{ij}^2)^2,$$

which proves the claim (8).

It remains to prove the claimed lower bound (5a) on $A(G)$. This lower bound can be split into the following two claims:

$$A(G) \geq \frac{1}{n^2} \alpha(G) (\alpha(G) - 1), \quad (9a)$$
$$A(G) \geq \frac{1}{n^2} \beta(G^c). \quad (9b)$$

We use a different argument to establish each claim.

**Proof of claim (9a):** Recall from Section [1] the definition of the largest independent set. Without loss of generality, let the largest independent set be given by $I = \{v_1, \ldots, v_\alpha\}$. Assign item $i$ to vertex $v_i$ for $i \in [\alpha]$. Now we choose permutations $\pi$ and $\pi'$ so that

- $\pi(i) = i$ for $i \in [\alpha]$,
- $\pi'(i) = \alpha - i + 1$ for $i \in [\alpha]$,
- $\pi$ and $\pi'$ agree on $\{\alpha + 1, \ldots, n\}$.

Note that last step is possible because $\pi([\alpha]) = \pi'([\alpha])$. Moreover, define the matrices $M = M_{NS}(\pi, 1/2)$ and $M' = M_{NS}(\pi', 1/2)$. Note that by construction, we have ensured that $M(G) = M'(G)$. However, it holds that

$$\sum_{(i,j) \notin E} (M_{ij} - M'_{ij})^2 = \| M - M' \|_F^2 = 2 K_T(\pi, \pi') = \alpha(\alpha - 1),$$

which completes the proof.

**Proof of claim (9b):** Recall the definition of a maximum biclique from Section [1]. Since the complement graph $G^c$ has a biclique with $\beta(G^c)$ edges, the graph $G$ has two disjoint sets of vertices $V_1$ and $V_2$ with $|V_1||V_2| = \beta(G^c)$ that do not have edges connecting one to the other. We now pick the two permutations $\pi$ and $\pi'$ so that

- the permutation $\pi$ ranks items from $V_1$ as the top $|V_1|$ items, and ranks items from $V_2$ as the next $|V_2|$ items;
- the permutation $\pi'$ ranks items from $V_2$ as the top $|V_2|$ items, and ranks items from $V_1$ as the next $|V_2|$ items;
- the permutations $\pi$ and $\pi'$ agree with each other apart from the above constraints.

As before, we define $M = M_{NS}(\pi, 1/2)$ and $M' = M_{NS}(\pi', 1/2)$, and again, we have $M(G) = M'(G)$. The relative orders of items have been interchanged across the biclique, so it holds that $2 K_T(\pi, \pi') = \beta(G^c)$, which completes the proof. \qed
5.2 Some useful lemmas for average-case proofs

We now turn to proofs for the average-case setting. For convenience, we begin by stating two lemmas that are used in multiple proofs. The first lemma bounds the performance of the permutation estimator \( \hat{\pi}_{ASP} \) for a general SST matrix, and is thus of independent interest.

**Lemma 1.** For any matrix \( M^* \in C_{SST} \), the permutation estimator \( \hat{\pi}_{ASP} \) satisfies

\[
\| \hat{\pi}_{ASP}(M^*) - M^* \|_F^2 \leq 4(n - 1)\| \tau^* - \hat{\tau} \|_1, \tag{10a}
\]

and if additionally, \( M^* \in C_{NS}(\lambda^*) \), we have

\[
\| \hat{\pi}_{ASP}(M^*) - M^* \|_F^2 \leq 8\lambda^*(n - 1)\| \tau^* - \hat{\tau} \|_1. \tag{10b}
\]

In addition, the score estimates satisfy the bounds

\[
\mathbb{E}[\| \tau^* - \hat{\tau} \|_1] \leq c \sum_{v \in V} \frac{1}{\sqrt{d_v}}, \quad \text{and} \quad \Pr\left\{ \| \tau^* - \hat{\tau} \|_1 \geq c\sqrt{\log n} \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right\} \leq n^{-10}.
\]

Note that Lemma 1 implies the bound (6b), since for a matrix \( M^* \in C_{NS}(\lambda^*) \), we have

\[
8\lambda^2KT(\hat{\pi}_{ASP}, \tau^*) = \| \hat{\pi}_{ASP}(M^*) - M^* \|_F^2.
\]

Our second lemma is a type of rearrangement inequality.

**Lemma 2.** Let \( \{a_u\}_{u=1}^n \) be an increasing sequence of positive numbers and let \( \{b_u\}_{u=1}^n \) be a decreasing sequence of positive numbers. Then we have

\[
\left( \sum_{u=1}^n a_u \right) \left( \sum_{u=1}^n b_u \right) \geq n \sum_{u=1}^n a_u b_u.
\]

5.2.1 Proof of Lemma 1

Assume without loss of generality that \( \pi^* = \text{id} \). We begin by applying Hölder’s inequality to obtain

\[
\| \hat{\pi}_{ASP}(M^*) - M^* \|_F^2 \leq \| \hat{\pi}_{ASP}(M^*) - M^* \|_\infty \| \hat{\pi}_{ASP}(M^*) - M^* \|_1.
\]

In the case where \( M^* \in C_{NS}(\lambda^*) \), we have \( \| M^*_{\hat{\pi}_{ASP}(i)} - M^*_i \|_\infty \leq 2\lambda^* \); in the general case \( M^* \in C_{SST} \), we have \( \| M^*_{\hat{\pi}_{ASP}(i)} - M^*_i \|_\infty \leq 1 \). Next, if \( M^*_{\hat{\pi}_{ASP}} \) denotes the matrix obtained from permuting the rows of \( M^* \) by \( \hat{\pi}_{ASP} \), then it holds that

\[
\| \hat{\pi}_{ASP}(M^*) - M^* \|_1 \leq \| \hat{\pi}_{ASP}(M^*) - M^*_{\hat{\pi}_{ASP}} \|_1 + \| M^*_{\hat{\pi}_{ASP}} - M^* \|_1
\]

\[
= 2 \sum_{i=1}^n \| M^*_{\hat{\pi}_{ASP}(i)} - M^*_i \|_1,
\]

16
where the equality follows from the condition $M_{ij}^* + M_{ji}^* = 1$. We also have

$$\sum_{i=1}^n \| M_{\pi_i}^* - M_i^* \|_1 \overset{(i)}{=} (n - 1) \sum_{i=1}^n | \tau_{\pi_i}^* - \tau_i^* |$$

$$= (n - 1) \sum_{i=1}^n | \tau_i^* - \tau_{\pi_i}^* |$$

$$\leq (n - 1) \left[ \sum_{i=1}^n | \tau_i^* - \tau_{\pi_i}^* | + \sum_{i=1}^n | \tau_{\pi_i}^* - \tau_{\pi_i}^* | \right]$$

$$\overset{(ii)}{\leq} (n - 1) \left[ \sum_{i=1}^n | \tau_i^* - \tau_{\pi_i}^* | + \sum_{i=1}^n | \tau_i^* - \tau_i^* | \right]$$

$$= 2(n - 1) \| \tau^* - \hat{\tau} \|_1,$$

where step (i) is due to monotonicity along each column of $M^*$, and step (ii) follows from the $\ell_1$-rearrangement inequality (see, e.g., Example 2 in the paper [Vin90]), using the fact that both sequences $\{ \tau_i^* \}_{i=1}^n$ and $\{ \tau_{\pi_i}^* \}_{i=1}^n$ are sorted in decreasing order. Combining the last three displays yields the claimed bounds (10a) and (10b).

In order to prove the second part of the lemma, it suffices to show that the random variable $\| \tau^* - \hat{\tau} \|_1$ is sub-Gaussian with parameter $cS$, where $S := \sum_{v \in V} 1/\sqrt{d_v}$. Let $\sigma : [n] \to V$ be the uniform random assignment of items to vertices with $\sigma(A) = O$, and let $D_i$ denote the random degree $d_{\sigma(i)} = \sum_{j \neq i} O_{ij}$ of item $i$. Note that conditioned on the event $\sigma(i) = v$, the difference between a score and its empirical version can be written as

$$\hat{\tau}_i - \tau_i^* = \left( \frac{1}{d_v} \sum_{j: \sigma(j) \sim v} M_{ij}^* - \frac{1}{n - 1} \sum_{j \neq i} M_{ij}^* \right) + \frac{1}{d_v} \sum_{j: \sigma(j) \sim v} W_{ij},$$

where $\sim$ denotes the presence of an edge between two vertices. The term $\frac{1}{d_v} \sum_{j: \sigma(j) \sim v} M_{ij}^*$ is the empirical mean of $d_v$ numbers chosen uniformly at random without replacement from the set $\{ M_{ij}^* \}_{j \neq i}$, while $\frac{1}{n - 1} \sum_{j \neq i} M_{ij}^*$ is the true expectation. Moreover, $W_{ij}$ represents independent, zero-mean noise bounded within the interval $[-1, 1]$. Consequently, applying Hoeffding’s inequality for sampling without replacement [BM15, Proposition 1.2] and the standard Hoeffding bound [Hoe63] to the two parts respectively, we obtain

$$\Pr \left\{ | \hat{\tau}_i - \tau_i^* | \geq t \mid \sigma(i) = v \right\} \leq 4 \exp(-c d_v t^2). \quad (11)$$

Replacing $t$ by $t/\sqrt{d_v}$, we see that conditioned on the event $\sigma(i) = v$, the random variable $\sqrt{d_v} | \hat{\tau}_i - \tau_i^* |$ is sub-Gaussian with a constant parameter $c'$, or equivalently,

$$\mathbb{E} \left[ \exp \left( t \sqrt{D_i} | \hat{\tau}_i - \tau_i^* | \right) \mid \sigma(i) = v \right] \leq \exp(c t^2). \quad (12)$$
Since $S = \sum_{i=1}^{n} 1/\sqrt{D_i}$, Jensen’s inequality implies that

$$E\left[ \exp\left( t \sum_{i=1}^{n} |\tilde{\tau}_i - \tau^*_i| \right) \right] \leq E\left[ \sum_{i=1}^{n} \frac{1/\sqrt{D_i}}{S} \exp\left( tS \sqrt{D_i} |\tilde{\tau}_i - \tau^*_i| \right) \right]$$

$$= \sum_{i=1}^{n} \frac{1}{S} \sum_{v \in V} \Pr\{\sigma(i) = v\} E\left[ \frac{1}{\sqrt{D_i}} \exp\left( tS \sqrt{D_i} |\tilde{\tau}_i - \tau^*_i| \right) \right] | \sigma(i) = v$$

$$\leq \sum_{i=1}^{n} \frac{1}{S} \sum_{v \in V} \frac{1}{n} \frac{1}{\sqrt{d_v}} \exp(cS^2 t^2)$$

$$= \exp(cS^2 t^2),$$

where the last inequality follows from equation (12). Therefore, the random variable $\|\tilde{\tau} - \tau^*\|_1$ is sub-Gaussian with parameter $cS$, as claimed.

### 5.2.2 Proof of Lemma 2

For any increasing sequence $\{a_u\}$ and decreasing sequence $\{b_u\}$, the rearrangement inequality (see, e.g., Example 2 in the paper [Vin90]) guarantees that

$$\sum_{u=1}^{n} a_u b_u \leq \sum_{u=1}^{n} a_u b_{\pi(u)} \quad \text{for any permutation } \pi.$$

This inequality implies that

$$\frac{1}{n} \left( \sum_{u=1}^{n} a_u \right) \left( \sum_{u=1}^{n} b_u \right) = \frac{1}{n} \sum_{v=1}^{n} \sum_{u=1}^{n} a_u b_{\pi(v)(u)} \geq \frac{1}{n} \sum_{v=1}^{n} \sum_{u=1}^{n} a_u b_u$$

$$= \sum_{u=1}^{n} a_u b_u,$$

where $\pi^{(v)}(u) := (u + v) \mod n$ and we have used the rearrangement inequality for each of these permutations.

Equipped with these two lemmas, we are now ready to prove Theorem 2.

### 5.3 Proof of Theorem 2

Without loss of generality, reindexing as necessary, we may assume that the true permutation $\pi^*$ is the identity $\text{id}$, thereby ensuring that $M^* = M_{NS}(\text{id}, \lambda^*)$. We begin by applying the triangle inequality to upper bound the error as a sum of two terms:

$$\frac{1}{2} \|\hat{M}_\text{ASP} - M^*\|_F^2 \leq \|\hat{M}_\text{ASP} - \hat{\pi}_\text{ASP}(M^*)\|_F^2 + \|\hat{\pi}_\text{ASP}(M^*) - M^*\|_F^2.$$

Applying Lemma 1 yields bound on the approximation error. In particular, we have

$$E \left[ \|\hat{\pi}_\text{ASP}(M^*) - M^*\|_F^2 \right] \leq cn \sum_{v \in V} \frac{1}{\sqrt{d_v}}.$$
where we have written $Y_{ij} = M_{ij}^* + W_{ij}$, consequently, the error obeys

$$1/2 + \lambda^* = \frac{1}{|E|} \left( \sum_{(i,j) \in E \setminus I_{\pi_{ASP}}(E)} Y_{ij} + \sum_{(i,j) \in I_{\pi_{ASP}}(E)} (1 - Y_{ij}) \right)$$

$$= \frac{1}{|E|} \left( \sum_{(i,j) \in E} Y_{ij} + \sum_{(i,j) \in I_{\pi_{ASP}}(E)} (1 - 2Y_{ij}) \right)$$

$$= 1/2 + \lambda^* + \frac{1}{|E|} \left( \sum_{(i,j) \in E} W_{ij} \right) + \frac{1}{|E|} \left( \sum_{(i,j) \in I_{\pi_{ASP}}(E)} -2\lambda^* - 2W_{ij} \right),$$

where we have written $Y_{ij} = M_{ij}^* + W_{ij}$. Consequently, the error obeys

$$\left(\lambda - \lambda^*\right)^2 \leq \frac{3}{|E|^2} \left( \sum_{(i,j) \in E} W_{ij} \right)^2 + \frac{12}{|E|^2} (\lambda^*)^2 |I_{\pi_{ASP}}(E)|^2 + \frac{12}{|E|^2} \left( \sum_{(i,j) \in I_{\pi_{ASP}}(E)} W_{ij} \right)^2$$

$$\leq \frac{3}{|E|^2} \left( \sum_{(i,j) \in E} W_{ij} \right)^2 + \frac{12}{|E|^2} (\lambda^*)^2 |I_{\pi_{ASP}}(E)|^2 + \frac{12}{|E|^2} \left( \sum_{(i,j) \in I_{\pi_{ASP}}(E)} W_{ij} \right)^2,$$

where step (i) follows since $|I_{\pi_{ASP}}(E)| \leq |E|$ pointwise. We now bound each of the terms $T_1$, $T_2$ and $T_3$ separately. First, by standard sub-exponential tail bounds, and noting that $W_{ij} \in [-1, 1]$, we have

$$E[T_1] \leq \frac{3}{|E|}, \quad \text{and} \quad \Pr \left\{ T_1 \geq \frac{6}{|E|} \right\} \leq e^{-|E|}.$$

We also have

$$\frac{|E|}{12(\lambda^*)^2} E[T_2] = E \left[ |I_{\pi_{ASP}}(E)| \right]$$

$$= \sum_{i<j} \sum_{(u,v) \in E} \Pr[\sigma(i) = u, \sigma(j) = v] \Pr[\pi_{ASP}(i) > \pi_{ASP}(j)|\sigma(i) = u, \sigma(j) = v]$$

$$= \sum_{(u,v) \in E} \sum_{i<j} \frac{1}{n(n-1)} \Pr[\pi_{ASP}(i) > \pi_{ASP}(j)|\sigma(i) = u, \sigma(j) = v].$$

We now require the following lemma, which is proved at the end of this section.
Lemma 3. For any pair of vertices $u \neq v$, we have
\[
\sum_{i<j} \frac{1}{n(n-1)} \Pr[\hat{\pi}_{\text{ASP}}(i) > \hat{\pi}_{\text{ASP}}(j)|\sigma(i) = u, \sigma(j) = v] \leq \frac{c}{\lambda^*} \left( \frac{1}{\sqrt{d_u}} + \frac{1}{\sqrt{d_v}} \right). \tag{13}
\]

Using Lemma 3 in conjunction with our previous bounds yields
\[
\mathbb{E}[T_2] \leq c \frac{\lambda^*}{|E|} \sum_{(u,v) \in E} \left( \frac{1}{\sqrt{d_u}} + \frac{1}{\sqrt{d_v}} \right) = c \lambda^* \sum_{u \in V} \sqrt{d_u}, \tag{14}
\]
where the equality follows since each term $\frac{1}{\sqrt{d_u}}$ appears $d_u$ times in the sum over all edges, and $2|E| = \sum_{u \in V} d_u$. Let \(\{d_u\}_{u=1}^{n}\) represent the sequence of vertex degrees sorted in ascending order. An application of Lemma 2 with $a_u = d(u)$ and $b_u = \frac{1}{\sqrt{d(u)}}$ for $u \in [n]$ yields
\[
\sum_{u \in V} \sqrt{d_u} \leq \frac{1}{n} \left( \sum_{u \in V} d_u \right) \left( \sum_{u \in V} \frac{1}{\sqrt{d_u}} \right).
\]
Together with equation (14), we find that
\[
\mathbb{E}[T_2] \leq c \lambda^* \frac{1}{n} \sum_{u \in V} \frac{1}{\sqrt{d_u}}.
\]

In order to complete the proof, it remains to bound $\mathbb{E}[T_3]$. Note that this step is non-trivial, since the noise terms $W_{ij}$ for $(i, j) \in I_{\hat{\pi}_{\text{ASP}}}(E)$ depend on and are coupled through the data-dependent quantity $\hat{\pi}_{\text{ASP}}$. In order to circumvent this tricky dependency, consider some fixed permutation $\pi$, and let $T_3^\pi = \left( \sum_{(i,j) \in I_{\pi}(E)} W_{ij} \right)^2$. Note that $T_3^\pi$ has two sources of randomness: randomness in the edge set $E$ and randomness in observations. Since the observations \(\{W_{ij}\}\) are independent and bounded and $|I_{\pi}(E)| \leq |E|$, the term
\[
\sum_{(i,j) \in I_{\pi}(E)} W_{ij}
\]
is sub-Gaussian with parameter at most $\sqrt{|E|}$. We then have the uniform sub-exponential tail bound
\[
\Pr\{T_3^\pi \geq |E| + \delta\} \leq e^{-c\delta}. \tag{15}
\]

Notice that for any $\alpha \in \mathbb{R}$, the inequality $T_3 \geq \alpha$ implies that the inequality $\frac{12 |E|^2}{|E|^2} T_3^\pi \geq \alpha$ holds for some fixed permutation $\pi$. Taking a union bound over all $n! \leq e^{n \log n}$ fixed permutations, and setting $\delta = cn \log n$ for a constant $c > 1$ yields
\[
\Pr\left\{ T_3 \geq \frac{12 |E|}{|E|^2} + c \frac{n \log n}{|E|^2} \right\} \leq \exp \left\{ n \log n - cn \log n \right\} \leq \exp \left\{ -c'n \log n \right\}. \tag{16}
\]
Noticing that $T_3 \leq 1$, we obtain
\[
\mathbb{E}[T_3] \leq \Pr\left\{ T_3 \geq \frac{12 |E|}{|E|^2} + c \frac{n \log n}{|E|^2} \right\} + \left( 1 - \Pr\left\{ T_3 \geq \frac{12 |E|}{|E|^2} + c \frac{n \log n}{|E|^2} \right\} \right) \left( \frac{12 |E|}{|E|^2} + c \frac{n \log n}{|E|^2} \right)
\leq \exp \left\{ -c'n \log n \right\} + \frac{12 |E|}{|E|^2} + c \frac{n \log n}{|E|^2}
\leq c' \left( \frac{1}{|E|^2} + \frac{n \log n}{|E|^2} \right).
\]
Combining the pieces proves the claimed bound on the expectation. 

The only remaining detail is to prove Lemma 3.

5.3.1 Proof of Lemma 3

We fix $i, j \in [n]$ with $i < j$ and condition on the event that $\sigma(i) = u$ and $\sigma(j) = v$ throughout the proof. First, note that the bound stated is trivially true if one of the vertices $u$ or $v$ has degree 1, by adjusting the constant appropriately. Hence, we assume for the rest of the proof that $d_u, d_v \geq 2$. Define the quantity

$$\tilde{\Delta}_{ji} = 2\lambda^* \frac{j - i - 1}{n - 2}. \quad (17)$$

We divide the rest of our analysis into two cases.

Case 1, $(u, v) \notin E(G)$: When the vertices $u$ and $v$ are not connected, we have

$$\bar{\tau}_j := \mathbb{E}[\hat{\tau}_j] = \frac{1}{2} + \lambda^* \left( \frac{n - j}{n - 2} - \frac{j - 2}{n - 2} \right) \text{ and}$$

$$\bar{\tau}_i := \mathbb{E}[\hat{\tau}_i] = \frac{1}{2} + \lambda^* \left( \frac{n - i - 1}{n - 2} - \frac{i - 1}{n - 2} \right),$$

and it can be verified that $\bar{\tau}_i - \bar{\tau}_j = \tilde{\Delta}_{ji}$. Consequently, we have

$$\Pr \left\{ \tilde{\tau}_{\text{ASP}}(j) < \tilde{\tau}_{\text{ASP}}(i) \mid \sigma(i) = u, \sigma(j) = v \right\}$$

$$= \Pr \left\{ \tilde{\tau}_j > \tilde{\tau}_i \mid \sigma(i) = u, \sigma(j) = v \right\}$$

$$\leq \Pr \left\{ \frac{\bar{\tau}_j - \bar{\tau}_j}{\sqrt{d_u} + \sqrt{d_u}} > \tilde{\Delta}_{ji} \mid \sigma(i) = u, \sigma(j) = v \right\}$$

$$+ \Pr \left\{ \frac{\bar{\tau}_i - \bar{\tau}_i}{\sqrt{d_v} + \sqrt{d_u}} > \tilde{\Delta}_{ji} \mid \sigma(i) = u, \sigma(j) = v \right\}$$

$$\leq 4 \exp \left\{ -c \frac{d_u d_v}{(\sqrt{d_u} + \sqrt{d_v})^2} \tilde{\Delta}_{ji}^2 \right\}, \quad (18)$$

where the last step follows from the Hoeffding bound for sampling without replacement in conjunction with the standard Hoeffding bound for bounded independent noise, by an argument similar to that of equation (11).

Case 2, $(u, v) \in E(G)$: When the vertices $u$ and $v$ are connected, we have

$$\bar{\tau}_j := \mathbb{E}[\hat{\tau}_j] = \frac{1}{2} + \frac{d_v - 1}{d_v} \lambda^* \left( \frac{n - j}{n - 2} - \frac{j - 2}{n - 2} \right) - \frac{1}{d_v} \lambda^* \text{ and}$$

$$\bar{\tau}_i := \mathbb{E}[\hat{\tau}_i] = \frac{1}{2} + \frac{d_u - 1}{d_u} \lambda^* \left( \frac{n - i - 1}{n - 2} - \frac{i - 1}{n - 2} \right) + \frac{1}{d_u} \lambda^*,$$

and it can be verified that $\bar{\tau}_i - \bar{\tau}_j \geq \tilde{\Delta}_{ji}$.
Now, however, we must apply the Hoeffding bound for sampling without replacement to $d_u - 1$ and $d_v - 1$ random variables, respectively. Recalling that $d_u, d_v \geq 2$, we have

$$
\Pr \left\{ \tilde{\pi}_{\text{ASP}}(j) < \tilde{\pi}_{\text{ASP}}(i) \mid \sigma(i) = u, \sigma(j) = v \right\}
= \Pr \left\{ \tilde{\tau}_j > \tilde{\tau}_i \mid \sigma(i) = u, \sigma(j) = v \right\}
\leq \Pr \left\{ |\tilde{\tau}_j - \tilde{\tau}_i| > \frac{\sqrt{d_u}}{\sqrt{d_v} + \sqrt{d_u}} \Delta_{ji} \mid \sigma(i) = u, \sigma(j) = v \right\}
+ \Pr \left\{ |\tilde{\tau}_i - \tilde{\tau}_i| > \frac{\sqrt{d_v}}{\sqrt{d_v} + \sqrt{d_u}} \Delta_{ji} \mid \sigma(i) = u, \sigma(j) = v \right\}
\leq 4 \exp \left\{ -c \frac{(d_u - 1)(d_v - 1)}{(\sqrt{d_u} - 1 + \sqrt{d_v} - 1)^2} \Delta_{ji}^2 \right\}
\leq 4 \exp \left\{ -c' \frac{d_u d_v}{(\sqrt{d_u} + \sqrt{d_v})^2} \Delta_{ji}^2 \right\}.
$$

(19)

We use the shorthand $L_{uv}$ to denote the LHS of equation (13). Having established the bounds (18) and (19), we now combine them to derive that

$$
L_{uv} \leq \frac{1}{n(n-1)} \sum_{j=2}^{n} \sum_{i<j} 4 \exp \left\{ -c \frac{d_u d_v}{(\sqrt{d_u} + \sqrt{d_v})^2} (j - i - 1)^2 \frac{(\lambda^*)^2}{(n-2)^2} \right\}
\leq \frac{4}{n(n-1)} \sum_{m=1}^{n} \exp \left\{ -\frac{d_u d_v}{(\sqrt{d_u} + \sqrt{d_v})^2} m^2 \frac{(\lambda^*)^2}{(n-2)^2} \right\},
$$

where we have used $m = j - i$, and noted that there are at most $n - 1$ repetitions of each distinct value of $j - i$ in the sum over $j > i$.

Defining $\psi(q) = \sum_{m=1}^{\infty} q^m$, we recall the following theta function identity\footnote{For the rest of this subsection, $\pi$ denotes the universal constant.} for $ab = \pi$ (see, for instance, equation (2.3) in Yi [Yi04]):

$$
\sqrt{a} \left( 1 + 2 \psi(e^{-a^2}) \right) = \sqrt{b} \left( 1 + 2 \psi(e^{-b^2}) \right).
$$

Using the identity by setting $a^2 = c \frac{d_u d_v}{(\sqrt{d_u} + \sqrt{d_v})^2} \frac{(\lambda^*)^2}{n^2}$ yields

$$
L_{uv} \leq \frac{c}{\lambda^* n^2} \sqrt{d_u + \sqrt{d_v}} \left( 1 + 2 \sum_{m=1}^{\infty} \exp \left\{ -\frac{d_u d_v}{(\sqrt{d_u} + \sqrt{d_v})^2} m^2 \frac{n^2}{(\lambda^*)^2} \right\} \right)
\leq \frac{c}{\lambda^*} \sqrt{d_u + \sqrt{d_v}} \left( 1 + 2 \sum_{m=1}^{\infty} \exp \left\{ -\frac{d_u d_v}{(\sqrt{d_u} + \sqrt{d_v})^2} m^2 \frac{n^2}{(\lambda^*)^2} \right\} \right)
\leq \frac{c}{\lambda^*} \sqrt{d_u + \sqrt{d_v}} \left( 1 + \sum_{m=1}^{\infty} \exp \left\{ -16 \pi^2 n m \right\} \right),
$$

(20)

where in the last step, we have used the fact that $\lambda^* \leq 1/2$, and that $\frac{(\sqrt{d_u} + \sqrt{d_v})}{d_u d_v} \geq 4/n$. Bounding the geometric sum by a universal constant yields the required result.

### 5.4 Proof of Theorem 3

We prove the two parts of the theorem separately.
5.4.1 Proof of part (a)

The proof of part (a) is based on the following lemmas.

Lemma 4. Consider a matrix of the form \( M^* = M_{NS}(\pi^*, 1/4) \) where the permutation \( \pi^* \) is chosen uniformly at random. For any graph \( G = K_1 \cup K_2 \cup \ldots \) composed of multiple disjoint cliques with the number of vertices bounded as \( C \leq |K_i| \leq n/5 \) for all \( i \), and for any estimators \( (\hat{M}, \hat{\pi}) \) that are measurable functions of the observations on \( G \), we have

\[
\mathbb{E}\left[ \frac{1}{n^2} \| \hat{M} - M^* \|_F^2 \right] \geq \frac{c_2}{n} \sum_{v \in V} \frac{1}{\sqrt{d_v}}, \quad \text{and} \quad \mathbb{E}[KT(\pi^*, \hat{\pi})] \geq c_2 n \sum_{v \in V} \frac{1}{\sqrt{d_v}}. \tag{21}
\]

Lemma 5. Given any graph \( G \) with degree sequence \( \{d_v\}_{v \in V} \), there exists a graph \( G' \) consisting of multiple disjoint cliques with degree sequence \( \{d'_v\}_{v \in V} \) such that

\[
|E| \asymp |E'| \quad \text{and} \quad \sum_{v \in V} \frac{1}{\sqrt{d_v}} \asymp \sum_{v \in V} \frac{1}{\sqrt{d'_v}}. \tag{22}
\]

Part (a) follows by combining these two lemmas, so that it suffices to prove each of the lemmas individually.

Proof of Lemma 4. Our result is structural, and proved for permutation recovery. The bound for matrix recovery follows as a corollary. Assume we are given a graph on \( n \) vertices consisting of \( k \) disjoint cliques of sizes \( n_1, \ldots, n_k \). Let \( N_0 = 0 \) and \( N_j = \sum_{i=1}^{j} n_i \) for \( j \in [k] \). Without loss of generality, we let the \( j \)-th clique consist of the set of vertices \( V_j \) indexed by \( \{N_{j-1} + 1, \ldots, N_j\} \). By assumption, each \( n_j \) is upper bounded by \( n/5 \) and lower bounded by a universal constant.

Note that any estimator can only use the observations to construct the correct partial order within each clique, but not across cliques. We denote the induced partial order of a permutation \( \pi \) on the clique \( V_j \) by the permutation \( \pi_j : [n_j] \rightarrow [n_j] \). We will demonstrate that there exists a coupling of two marginally uniform random permutations \((\pi^*, \pi^\#)\) such that

\[
\mathbb{E}[KT(\pi^*, \pi^\#)] \geq c n \sum_{j=1}^{k} \sqrt{n_j} = c n \sum_{v \in V} \frac{1}{\sqrt{d_v}},
\]

and the partial order of \( \pi^* \) agrees with that of \( \pi^\# \) on each clique, that is, \( \pi^*_j = \pi^\#_j \) for all \( j \in [k] \). Another way of stating this is that for every clique \( V_j \) and every two vertices \( i_1, i_2 \in V_j \), we need that \( \pi^\#(i_1) < \pi^\#(i_2) \) if and only if \( \pi^*(i_1) < \pi^*(i_2) \).

Let \( \mathbb{E}[\cdot \mid \pi^*] \) denote the expectation over the observations conditional on \( \pi^* \). Given a pair of permutations \((\pi^*, \pi^\#)\) satisfying the above assumption, we view them as two hypotheses of the latent permutation. Then for any estimator \( \widehat{\pi} \), the Neyman-Pearson lemma \cite{NP66} guarantees that

\[
\mathbb{E}[KT(\widehat{\pi}, \pi^*) \mid \pi^*] + \mathbb{E}[KT(\widehat{\pi}, \pi^\#) \mid \pi^\#] \geq KT(\pi^\#, \pi^*)
\]

As an example, the identity permutation \( \pi = \text{id} \) would yield \( \pi_j = \text{id} \) on \( [n_j] \) for all \( j \in [k] \).
for each instance of \((\pi^*, \pi^\#)\), because the observations are identical for \(\pi^*\) and \(\pi^\#\). Taking expectation over \((\pi^*, \pi^\#)\), we obtain that

\[
2 \mathbb{E}[\text{KT}(\tilde{\pi}, \pi^*)] \geq \mathbb{E}[\text{KT}(\pi^*, \pi^\#)] \geq c n \sum_{v \in V} \frac{1}{\sqrt{d_v}}
\]
since both \(\pi^*\) and \(\pi^\#\) are marginally uniform.

To finish the proof, it remains to construct the required coupling \((\pi^*, \pi^\#)\). The construction is done as follows. First, permutations \(\pi^*\) and \(\tilde{\pi}\) are generated uniformly at random and independently. Second, we sort the permutation \(\tilde{\pi}\) on each clique according to \(\pi^*\), and denote the resulting permutation by \(\pi^\#\). Then the permutations \(\pi^*\) and \(\pi^\#\) are marginally uniform and have common induced partial orders on the cliques, which we denote by \(\{\pi_j^* : j \in [k]\}\).

With some extra notation, we can define the sorting step more formally for the interested reader. For a set of partial orders on the cliques \(\{\pi_j : j \in [k]\}\), we define a special permutation that effectively orders vertices within each clique \(V_j\) according to its corresponding partial order \(\pi_j\), but does not permute any vertices across cliques. We denote this special permutation by \(\pi_{\text{par}}(\{\pi_j : j \in [k]\})\). For every clique \(V_j\), we consider the permutation \(\pi_{\text{sort}, j} := \pi_j^* \circ (\pi_j)^{-1}\).

Now, we can formally define the sorting step to generate \(\pi^\#\) by

\[
\pi^\# := \pi_{\text{par}}(\{\pi_{\text{sort}, j} : j \in [k]\}) \circ \tilde{\pi}.
\]

Next, we need to evaluate the expected Kendall’s tau distance between these coupled permutations. By the tower property, we have

\[
\mathbb{E}[\text{KT}(\pi^*, \pi^\#)] = \mathbb{E}[\mathbb{E}[\text{KT}(\pi^*, \pi^\#) \mid \{\pi_j^* : j \in [k]\}]].
\]
The inner expectation can be simplified as follows. Pre-composing permutations \(\pi^*\) and \(\pi^\#\) with any permutation does not change the Kendall’s tau distance between them, so we have

\[
\mathbb{E}[\text{KT}(\pi^*, \pi^\#) \mid \{\pi_j^* : j \in [k]\}] = \mathbb{E}[\text{KT}(\pi, \pi')] \tag{23}
\]
where the permutations \(\pi\) and \(\pi'\) are drawn independently and uniformly at random from the set of permutations that are increasing on every clique. That is, for every clique \(V_j\) and every two vertices \(i_1, i_2 \in V_j\), we have \(\pi(i_1) < \pi(i_2)\) and \(\pi'(i_1) < \pi'(i_2)\).

We now turn to computing the quantity \(\mathbb{E}[\text{KT}(\pi, \pi')]\). It is well-known \cite{DG77} that \(2 \text{KT}(\pi, \pi') \geq \|\pi - \pi'\|_1\). This fact together with Jensen’s inequality implies that

\[
2 \mathbb{E}[\text{KT}(\pi, \pi')] \geq \sum_{i=1}^{n} \mathbb{E}[|\pi(i) - \pi'(i)|]
\]

\[
\geq \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{E}[|\pi(i) - \pi'(i) \mid \pi] \right]
\]

\[
= \sum_{i=1}^{n} \mathbb{E} \left[ |\pi(i) - \mathbb{E}[\pi'(i)]| \right]
\]

\[
= \mathbb{E} \left[ \|\pi - \mathbb{E}[\pi]\|_1 \right].
\]

\footnote{To understand why \(\pi\) and \(\pi'\) can be chosen independently, note that the only dependency between the original permutations \(\pi^*\) and \(\pi^\#\) is through the common induced partial orders \(\{\pi_j^* : j \in [k]\}\). By conditioning and pre-composing, we are able to remove that dependency.}
It therefore suffices to lower bound the quantity $\mathbb{E}[\|\pi - \mathbb{E}[\pi]\|_1]$.

Fix any $i \in [n]$. Then $i$ is $\ell$-th smallest index in the $j$-th clique for some $j \in [k]$ and $\ell \in [n_j]$, or succinctly, $i = N_{j-1} + \ell$. If we view $\pi^{-1}$ as random draws from the $n$ items, then $\pi(i)$ is equal to the the number of draws needed to get the $\ell$-th smallest element of $V_j$. Denoting $\mathbb{E}[\pi(i)]$ by $\mu$, we have

$$\mu = \ell + \mathbb{E} \left[ \sum_{r : \sigma(r) \notin V_j} 1\{r \text{ is drawn before } i\} \right] = \ell + (n - n_j) \frac{\ell}{n_j + 1} = \ell \frac{n + 1}{n_j + 1},$$

since the probability that an item not in $V_j$ is drawn before the $\ell$-th smallest element of $V_j$ is $\ell/(n_j + 1)$. Furthermore, $\pi(i) = s$ if and only if $\ell - 1$ elements of $V_j$ are selected in the first $s - 1$ draws and the $s$-th draw is from $V_j$, so

$$\Pr\{\pi(i) = s\} = \left( \frac{n_j}{\ell - 1} \right) \left( \frac{n - n_j}{s - \ell} \right) \left( \frac{n}{s - 1} \right) \frac{n_j - \ell + 1}{n - s + 1}. \quad (24)$$

We claim that for all $2n_j/5 \leq \ell \leq 3n_j/5$ and $|s - \mu| \leq n/\sqrt{n_j}$, it holds that

$$\Pr\{\pi(i) = s\} \leq c \sqrt{n_j}/n \quad (25)$$

where $c$ is a universal positive constant.

If the claim holds, then for any $0 \leq m \leq n/\sqrt{n_j}$, we have

$$\mathbb{E}[|\pi(i) - \mu|] \geq m \Pr\{|\pi(i) - \mu| \geq m\} \geq m \left[ 1 - c(2m + 1) \sqrt{n_j}/n \right]$$

by Markov’s inequality. Choosing $m = \frac{n}{6c\sqrt{n_j}}$ yields

$$\mathbb{E}[|\pi(i) - \mu|] \geq c_2 n/\sqrt{n_j}$$

for some positive constant $c_2$. Summing over $\ell$ in the given range, together with inequality (23), completes the proof.

**Proof of claim (25):** For $\ell \in [n_j]$ and $\ell \leq s \leq n - n_j + \ell$, define a bivariate function

$$p(\ell, s) := \left( \frac{n_j}{\ell - 1} \right) \left( \frac{n - n_j}{s - \ell} \right) \left( \frac{n}{s - 1} \right)^{-1}.$$  

Note that for any fixed $s$, the function $\ell \mapsto p(\ell, s)$ is the probability mass function of the hypergeometric distribution that describes the probability of $\ell - 1$ successes in $s - 1$ draws without replacement from a population of size $n$ with $n_j$ successes. Hence, its maximum is attained at $\ell = \lceil \frac{s n_j + 1}{n + 2} \rceil$. Now we consider the index set

$$\mathcal{I} = \left\{ (l, s) : \left[ \frac{n_j}{3} \right] \leq \ell \leq \left[ \frac{2n_j}{3} \right], \left[ \frac{n_j}{3} \right] \leq \left[ \frac{s n_j + 1}{n + 2} \right] \leq \left[ \frac{2n_j}{3} \right] \right\} \subset \left[ \frac{n_j}{3}, \frac{2n_j}{3} \right] \times \left[ \frac{n}{5}, \frac{4n}{5} \right].$$

In particular, the range of interest $2n_j/5 \leq \ell \leq 3n_j/5$ and $|s - \mu| \leq n/\sqrt{n_j}$, is contained within the set $\mathcal{I}$, since $\mu = \ell \frac{n + 1}{n_j + 1}$. Moreover, inequality (24) ensures that $\Pr\{\pi(i) = s\} \leq p(\ell, s)\frac{3n_j}{n}$ for $(\ell, s) \in \mathcal{I}$. Thus, in order to complete the proof, it suffices to prove that $p(\ell, s) \leq c/\sqrt{n_j}$ for $(\ell, s) \in \mathcal{I}$, and it suffices to consider $(\ell, s)$ such that $\ell = \left[ \frac{s n_j + 1}{n + 2} \right]$ since each function $\ell \mapsto p(\ell, s)$ attains its maximum at such a pair $(\ell, s)$.  

25
Toward this end, we use Stirling’s approximation [DM56] to obtain
\[
p(\ell, s) \leq c_2 \frac{\sqrt{n_j(n - n_j)(s - 1)(n - s + 1)}}{(\ell - 1)(n_j - \ell + 1)(s - \ell)(n - n_j - s + \ell)n} \cdot \frac{n_j^{n_j}(n - n_j)^{n - n_j}(s - 1)^{s - 1}(n - s + 1)^{n - s + 1}}{(\ell - 1)^{\ell - 1}(n_j - \ell + 1)(s - \ell)^{s - 1}(n - n_j - s + \ell)^{n - n_j - s + \ell}n^m}.
\] (26)

Since the factor in line (26) scales as \(1/c\) for \((\ell, s) \in \mathcal{I}\), it remains to bound the factor in line (27) by a universal constant. This follows from lengthy yet standard approximations which we briefly describe here. Assume that \(s \frac{n_j + 1}{n + 2}\) is an integer for simplicity, so that \(\ell\) is equal to this quantity and we have \(s = \ell \frac{n + 2}{n + 1}\); the extension to the general case is easy. We first group together
\[
\left[\frac{n_j(s - 1)}{(\ell - 1)n}\right]^{\ell - 1} = \left[\frac{n_j(n\ell + 2\ell - n_j - 1)/(n_j + 1)}{(\ell - 1)n}\right]^{\ell - 1} = \left[1 + \frac{1 + (2\ell n_j - n_j - \ell n)/(n_j n + n)}{\ell - 1}\right]^{\ell - 1},
\]
which is bounded by a constant for \((\ell, s) \in \mathcal{I}\) considering that \(\lim_{m \to \infty}(1 + \frac{m}{n})^m = e^a\). Then, we group together the terms
\[
\left[\frac{n_j(n - s + 1)}{(n_j - \ell + 1)n}\right]^{n_j - \ell + 1}, \left[\frac{(n - n_j)(s - 1)}{(s - \ell)n}\right]^{s - \ell} \text{ and } \left[\frac{(n - n_j)(n - s + 1)}{(n - n_j - s + \ell)n}\right]^{n - n_j - s + \ell}
\]
respectively, and a similar argument yields that each term is bounded by a constant. \(\square\)

**Proof of Lemma 5:** Fix a graph \(G\) with degree sequence \(\{d_v\}_{v \in V}\), and introduce the shorthand \(S = \sum_{v \in V} 1/\sqrt{d_v}\). For some parameter \(k\) to be chosen, define the graph \(G'\) on the same vertex set to be the disjoint union of one clique of size \(c_1 \sqrt{|E|}\), \(c_2 k\) cliques of size \([n/k]\) and \(c_3 S\) cliques of size \(2\), where \(c_1, c_2\) and \(c_3\) are constants to be determined such that the sizes of each clique are integers. The number of vertices remains the same, so that
\[
n = c_1 \sqrt{|E|} + c_2 k[n/k] + 2c_3 S.
\] (28)

The number of edges of \(G'\) is
\[
|E'| = \left(c_1 \sqrt{|E|} \right) + c_2 k \left[\frac{|n/k|}{2}\right] + c_3 S \times |E| + \frac{n^2}{k},
\]
where the last approximation holds because \(S \leq n \leq 2|E|\). Moreover, let
\[
S' = \sum_{v \in V} \frac{1}{\sqrt{d_v}} = \frac{c_1 \sqrt{|E|}}{c_1 \sqrt{|E|} - 1} + \frac{c_2 k [n/k]}{|n/k| - 1} + c_3 S \times \sqrt{n/k} + S,
\]
where the last approximation holds since \(|E|^{1/4} \leq \sqrt{n} \leq S\).

In order to guarantee that \(|E'| \asymp |E|\) and \(S' \asymp S\), we need to choose an integer \(k\) so that \(n^2/k \leq c|E|\) and \(\sqrt{n/k} \leq cS\), or equivalently
\[
\frac{n^2}{c|E|} \leq k \leq c^2 S^2 n.
\]
Such an integer \( k \) exists if \(|E|S^2 \geq n^3\). Indeed, applying Lemma 2 twice (with \( a_u = d(u) \) and \( b_u = 1/\sqrt{d(u)} \) the first time and \( a_u = \sqrt{d(u)} \) and \( b_u = 1/\sqrt{d(u)} \) the second time, where \( \{d(u)\}_{u=1}^n \) is the degree sequence in ascending order), we obtain that

\[
|E|S^2 = \left( \sum_{v \in V} d_v \right) \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^2 \geq n \left( \sum_{v \in V} \sqrt{d_v} \right) \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right) \geq n^3.
\]

With \( k \) selected, it is easy to choose \( c_1, c_2 \) and \( c_3 \) so that inequality (28) holds, since each of \( \sqrt{|E|}, k[n/k] \) and \( S \) is no larger than \( n \). The issue of integrality can be taken care of by constant-order adjustment of these numbers, so the proof is complete. \( \square \)

### 5.4.2 Proof of part (b)

Given a parameter space \( \Theta \), a set \( \mathcal{P} = \{\theta_1, \theta_2, \ldots, \theta_{|\mathcal{P}|}\} \) is said to be a \( \delta \)-packing in the metric \( \rho \) if \( \rho(\theta_i, \theta_j) > \delta \) for all \( i \neq j \). The lower bound of part (b) is based on the following packing lemma for the set of permutations in Kendall’s tau distance. We note that a similar lemma was proved by Barg and Mazumdar [BM10].

**Lemma 6.** For some positive constant \( c_1 \), there exists an \( c_1n^2 \)-packing \( \mathcal{P} \) of the set of permutations in the Kendall’s tau distance such that \( \log |\mathcal{P}| \geq n \).

Consider the random observation model with graph \( G = (V, E) \), where \( E \) denotes the random edge set of observations. We denote by \( \mathbb{Q}_M \) the law of the random observation noisy sorting model with underlying matrix \( M = M_{\text{NS}}(\pi, \lambda) \). We require the following lemma.

**Lemma 7.** Let \( \mathbb{P}_{M,G} \) denote the law of the noisy sorting model with underlying matrix \( M \in \mathcal{C}_{\text{NS}}(\lambda) \) for \( \lambda \in [0, 1/4] \) and comparison graph \( G \). Suppose that the entries of two matrices \( M, M' \in \mathcal{C}_{\text{NS}}(\lambda) \) differ in s edges of the graph \( G \). Then the KL divergence is bounded as

\[
\text{KL}(\mathbb{P}_{M,G}, \mathbb{P}_{M',G}) \leq 9\lambda^2s. \tag{29}
\]

Note that conditional on any instance of \( E \), Lemma 7 guarantees that

\[
\text{KL}(\mathbb{P}_{M,G}, \mathbb{P}_{M',G}) \leq 9\lambda^2\left|\{(i, j) \in E : i < j, M_{i,j} \neq M'_{i,j}\}\right|,
\]

where \( \mathbb{P}_{M,G} \) denotes the model for fixed graph \( G \). Hence taking expectation over the random edge set yields the upper bound

\[
\text{KL}(\mathbb{Q}_M, \mathbb{Q}_{M'}) \leq 9\lambda^2 \sum_{i<j, M_{i,j} \neq M'_{i,j}} \text{Pr}\{(i, j) \in E\} \leq 9\lambda^2 \sum_{i<j} \frac{2|E|}{n(n-1)} = 9\lambda^2|E|,
\]

valid for any \( M, M' \in \mathcal{C}_{\text{NS}}(\lambda) \).

Note that \( ||M - M'||_F^2 = 8\lambda^2K_T(\pi, \pi') \) for \( M = M_{\text{NS}}(\pi, \lambda) \) and \( M' = M_{\text{NS}}(\pi', \lambda) \). Hence Fano’s inequality applied to the packing given by Lemma 7 yields that

\[
\inf_M \sup_{M' \in \mathcal{C}_{\text{NS}}} \mathbb{E}\left[||M - M'||_F^2\right] \geq 8\lambda^2c_2n^2 \left(1 - \frac{9\lambda^2|E| + \log 2}{n}\right).
\]

The proof is completed by choosing \( \lambda^2 = c_2n/|E| \) for a sufficiently small constant \( c_2 \). \( \square \)

It remains to prove Lemmas 6 and 7.
Proof of Lemma 6: The inversion table $b = (b_1, \ldots, b_n)$ of a permutation $\pi$ has entries defined by

$$b_i = \sum_{j=i+1}^{n} 1\{\pi(i) > \pi(j)\} \text{ for each } i \in [n].$$

We refer the reader to Mahmoud [Mah00] and references therein for background on inversion tables. By definition, we have $b_i \in \{0, 1, \ldots, n-i\}$ and $\text{KT}(\pi, \text{id}) = \sum_{i=1}^{n} b_i$ where $\text{id}$ denotes the identity permutation. In fact, the set of tables $b$ satisfying $b_i \in \{0, 1, \ldots, n-i\}$ is bijective to the set of permutations via this relation [Mah00]. This bijection aids in counting permutations with constraints.

5.4.3 Proof of Lemma 7

The KL divergence between Bernoulli observations has the form

$$\text{KL}(\text{Ber}(1/2 + \lambda), \text{Ber}(1/2 - \lambda)) = \text{KL}(\text{Ber}(1/2 - \lambda), \text{Ber}(1/2 + \lambda))$$

$$= (1/2 + \lambda) \log \frac{1/2 + \lambda}{1/2 - \lambda} + (1/2 - \lambda) \log \frac{1/2 - \lambda}{1/2 + \lambda}$$

$$= 2\lambda \log \frac{1/2 + \lambda}{1/2 - \lambda}$$

$$\leq 9\lambda^2 \text{ for all } \lambda \in [0, 1/4],$$

where the last inequality follows by some simple algebra.

Note that the KL divergence between a pair of product distributions is equal to the sum of the KL divergences between individual pairs. Since $M$ and $M'$ differ in $s$ entries on the graph $G$ and the Bernoulli observations are independent for different edges, we see that $\text{KL}(\mathbb{P}_{M,G}, \mathbb{P}_{M',G}) \leq 9\lambda^2 s.$

5.5 Proof of Theorem 4

For the purpose of the proof, it is helpful to think of the observation model in its linearized form. In particular, we have two random edge sets $E_1$ and $E_2$ and the observation matrices

$$Y_i := M^* + W_i$$
for each \( i \in \{1, 2\} \). We also use the shorthand \( \mathcal{B}(X, C) := \mathcal{B}(X, C, [n] \times [n]) \), and recall the notation \( \|M\|_{F}^{2} := \sum_{(i,j) \in B} M_{ij} \).

By the triangle inequality, we have

\[
\|\hat{M}_{\text{BAP}} - M^{*}\|_{F}^{2} \leq 2\|\hat{M}_{\text{BAP}} - \hat{\pi}_{\text{ASP}}(M^{*})\|_{F}^{2} + 2\|M^{*} - \hat{\pi}_{\text{ASP}}(M^{*})\|_{F}^{2} \\
\leq 2\|\hat{M} - \hat{\pi}_{\text{ASP}}(M^{*})\|_{F}^{2} + 2\|M^{*} - \hat{\pi}_{\text{ASP}}(M^{*})\|_{F}^{2} \\
\leq 4\|\hat{M} - M^{*}\|_{F}^{2} + 6\|M^{*} - \hat{\pi}_{\text{ASP}}(M^{*})\|_{F}^{2},
\]

where step (i) follows from the non-expansiveness of the projection operator. We know from Lemma 1 that the second term in inequality (30) is bounded in expectation by the quantity \( nS = n \sum_{v \in V} 1/\sqrt{d_{v}} \) as desired, so it remains to bound the first term. Toward that end, again apply triangle inequality to write

\[
\|\hat{M} - M^{*}\|_{F}^{2} \leq 2\|\hat{M} - \mathcal{B}(M^{*}, \hat{b})\|_{F}^{2} + 2\|M^{*} - \mathcal{B}(M^{*}, \hat{b})\|_{F}^{2}.
\]

We now bound each of these terms separately. Starting with the first, let us define some notation. For a set \( S \subseteq [n] \times [n] \) and a matrix \( M \in \mathbb{R}^{n \times n} \), let \( \|M\|_{S}^{2} := \sum_{(i,j) \in S} M_{ij}^{2} \). We have

\[
\|\hat{M} - \mathcal{B}(M^{*}, \hat{b})\|_{F}^{2} = \sum_{B \in \mathcal{B}(\hat{b})} \|\hat{M} - \mathcal{B}(M^{*}, \hat{b})\|_{B}^{2}.
\]

Note that it is sufficient to consider off diagonal blocks in the sum, since both \( \hat{M} \) and \( \mathcal{B}(M^{*}, \hat{b}) \) are identically \( 1/2 \) in the diagonal blocks. Considering each block separately, we now split the analysis into two cases.

**Case 1, \( B \cap E_{2} = \phi \):** Because the entries of the error matrix are bounded within \([-1,1]\), we have

\[
\|\hat{M} - \mathcal{B}(M^{*}, \hat{b})\|_{B}^{2} \leq |B|.
\]

**Case 2, \( B \cap E_{2} \neq \phi \):** Since both \( \hat{M} \) and \( \mathcal{B}(M^{*}, \hat{b}) \) are constant on each block, we have

\[
\|\hat{M} - \mathcal{B}(M^{*}, \hat{b})\|_{B}^{2} = \frac{|B|}{|B \cap E_{2}|} \|\hat{M} - \mathcal{B}(M^{*}, \hat{b})\|_{B \cap E_{2}}^{2} \\
= \frac{|B|}{|B \cap E_{2}|} \|\mathcal{B}(M^{*} + W_{2}, \hat{b}, E_{2}) - \mathcal{B}(M^{*}, \hat{b})\|_{B \cap E_{2}}^{2} \\
\leq 2\frac{|B|}{|B \cap E_{2}|} \left( \|\mathcal{B}(M^{*} + W_{2}, \hat{b}, E_{2}) - \mathcal{B}(M^{*}, \hat{b})\|^{2}_{B \cap E_{2}} + \|\mathcal{B}(M^{*}, \hat{b}) + W_{2}, \hat{b}, E_{2}) - \mathcal{B}(M^{*}, \hat{b})\|^{2}_{B \cap E_{2}} \right). \tag{32}
\]

Let us handle each term on the RHS of the last inequality separately. First, by non-expansiveness of the projection operation defined by equation (7), we have

\[
\|\mathcal{B}(M^{*} + W_{2}, \hat{b}, E_{2}) - \mathcal{B}(M^{*}, \hat{b}) + W_{2}, \hat{b}, E_{2})\|_{B \cap E_{2}}^{2} \leq \|M^{*} - \mathcal{B}(M^{*}, \hat{b})\|_{B \cap E_{2}}^{2}.
\]

We also require the following technical lemma:
Lemma 8. For any block $B$ and tuple $(i, j) \in B$, we have
$$\Pr \left\{ (i, j) \in E_2 \mid |B \cap E_2| = k \right\} = \frac{k}{|B|}.$$  

See Section 5.5.1 for the proof of this claim.

Returning to equation (33) and taking expectation over the randomness in $E_2$ (which, crucially, is independent of the randomness in $\hat{b}$), we have
$$E_{E_2} \left[ \left\| M^* - B(M^*, \hat{b}) \right\|_{B \cap E_2}^2 \right] = \sum_{(i, j) \in B} \Pr \left\{ (i, j) \in E_2 \mid |B \cap E_2| = k \right\} \cdot \left[ M^* - B(M^*, \hat{b}) \right]_{ij}^2$$
$$= \frac{k}{|B|} \left\| M^* - B(M^*, \hat{b}) \right\|_B^2,$$

where step (ii) follows from Lemma 8.

Additionally, notice that $[W_{2}]_{ij}$ for $(i, j) \in E_2$ is independent and bounded within the interval $[-1, 1]$. Consequently, we have
$$E_{W_2} \left[ \left\| B(M^*, \hat{b}) + W_2, \hat{b}, E_2 \right\|_{B \cap E_2}^2 - B(M^*, \hat{b}) \right] \leq 1,$$

where we have used the fact that the entries of the matrix $B(M^*, \hat{b})$ are constant on the set of indices $B \cap E_2$.

It follows from equations (32), (33), (34) and (35) that
$$E \left[ \left\| \tilde{M} - B(M^*, \hat{b}) \right\|_B^2 \right] \leq 2E \left[ \frac{|B|}{|B \cap E_2|} \right] + 2E \left[ \left\| M^* - B(M^*, \hat{b}) \right\|_B^2 \right].$$

Combining the two cases and summing over the blocks, we obtain that
$$E \left[ \left\| \tilde{M} - B(M^*, \hat{b}) \right\|_B^2 \right] \leq 2 \sum_{B \in \mathcal{B}(\hat{b})} E \left[ \frac{|B|}{|B \cap E_2| \lor 1} \right] + 2E \left[ \left\| M^* - B(M^*, \hat{b}) \right\|_B^2 \right].$$

Note that the second term above is the same as the second term on the RHS of inequality (31).

We now require the following definition, and two lemmas to complete the proof. Given a matrix $M^*$ and a partition $C \in \chi_n$, define its row average as
$$[R(M^*, C)]_i = \frac{1}{|C(i)|} \sum_{j \in C(i)} M^*_j.$$  

Lemma 9. With $S = \sum_{v \in V} 1/\sqrt{d_v}$ and for the partition $\hat{b} = \text{bl}_t(r(Y_1'))$, we have
$$E_{E_2} \left[ \sum_{B \in \mathcal{B}(\hat{b})} \frac{|B|}{|B \cap E_2| \lor 1} \right] \leq nS.$$
Lemma 10. Given any matrix $X \in [0, 1]^{n \times n}$ with monotone columns, a score vector $\tilde{r} \in [0, n]^n$, and a value $t \in [0, n]$, we have

$$\|X - R(X, \text{bl}_t(\tilde{r}))\|_F^2 \leq nt + 2\|\tilde{r} - r(X)\|_1.$$ 

Applying Lemma 9 with the expectation taken over the edge set $E_2$ yields the desired bound on the first term of inequality (36).

In order to bound the second term of inequality (36), note that by definition, we have $B(M^*, C) = R(R(M^*, C)^\top)^\top$. Consequently, it holds that

$$\|M^* - B(M^*, C)\|_F^2 \leq 2\|M^* - R(M^*, C)\|_F^2 + 2\|R(M^*, C) - B(M^*, C)\|_F^2 = 2\|M^* - R(M^*, C)\|_F^2 + 2\|R(M^*, C)^\top - R(R(M^*, C)^\top, C)\|_F^2.$$ 

Setting $C = \text{bl}_S(\tilde{r})$ and applying Lemma 10 to both the terms, we obtain

$$\|M^* - B(M^*, \text{bl}_S(\tilde{r}))\|_F^2 \leq 2nS + 4\|\tilde{r} - r(M^*)\|_1.$$ 

Applying Lemma 1 yields a bound on the second term in expectation. This together with equations (31) and (36) completes the proof of Theorem 4 with the choice $t = \sum_{v \in V} 1/\sqrt{d_v}$. It remains to prove Lemmas 8, 9 and 10.

### 5.5.1 Proof of Lemma 8

Our proof relies crucially on the fact that one of the two sets is a block.

For a fixed integer $k$, we condition on the event $\{|B \cap E_2| = k\}$. Note that $E_2$ is the random edge set defined by

$$E_2 = \pi(E) = \{(i, j) : (\pi(i), \pi(j)) \in E\},$$

where $\pi$ is a uniform random permutation, and $E$ is a fixed instance of $E_2$. For any pair of tuples $(i, j), (k, \ell) \in B$, consider the permutation $\tilde{\pi}$ defined by

- $\tilde{\pi}(i) = k$, $\tilde{\pi}(k) = i$, $\tilde{\pi}(j) = \ell$ and $\tilde{\pi}(\ell) = j$;
- $\tilde{\pi}(m) = m$ for $m \neq i, j, k$ or $\ell$.

Note that right-composition by $\tilde{\pi}$ is clearly a bijection between the sets $\{\pi : (i, j) \in \pi(E)\}$ and $\{\pi : (k, \ell) \in \pi(E)\}$. Therefore, we have $|\{\pi : (i, j) \in E_2\}| = |\{\pi : (k, \ell) \in E_2\}|$. A counting argument then completes the proof. Indeed, conditioned on the event $\{|B \cap E_2| = k\}$, we have

$$\sum_{(i, j) \in B} \Pr\{(i, j) \in E_2\} = \mathbb{E}\left[\sum_{(i, j) \in B} 1\{(i, j) \in E_2\}\right] = k,$$

which implies that $\Pr\{(i, j) \in E_2\} = \frac{k}{|B|}$. 

31
5.5.2 Proof of Lemma 9

Fix an individual block $B$ of dimensions $h \times w$, and let $E = E_2$ for notational convenience. Define the random variable $Y = |B \cap E| + 1$ so that $(|B \cap E| \lor 1)^{-1} \leq 2/Y$. Hence we require a bound on the quantity $E[Y^{-1}]$. Toward this end, we write

$$Y = 1 + \sum_{(i,j) \in B} 1\{i,j\} \in E\}, \text{ and}$$

$$Y^2 = 1 + 2 \sum_{(i,j) \in B} 1\{i,j\} \in E\} + \sum_{(i,j) \neq (i',j') \in B} 1\{i,j\}, (i',j') \in E\}.$$

Note that for $(i,j), (i',j') \in B$ where $i \neq i'$ and $j \neq j'$, we have\]

$$\Pr\{i,j\} \in E\} = \frac{2|E|}{n(n-1)},$$

$$\Pr\{i,j\}, (i',j') \in E\} = \frac{\sum_{v \in V} d_v(d_v - 1)}{n(n-1)(n-2)}, \text{ and}$$

$$\Pr\{i,j\}, (i',j') \in E\} = \frac{4|E|^2 - 2\sum_{v \in V} d_v(d_v - 1) - 2|E|}{n(n-1)(n-2)(n-3)}.$$

Hence, we can compute the first two moments of $Y$ as

$$E[Y] = 1 + \sum_{(i,j) \in B} \Pr\{i,j\} \in E\} = 1 + \frac{2hw|E|}{n(n-1)}, \text{ and}$$

$$E[Y^2] = 1 + 2 \sum_{(i,j) \in B} \Pr\{i,j\} \in E\} + \sum_{(i,j), (i',j') \in B} \Pr\{i,j\}, (i',j') \in E\}$$

$$= 1 + \frac{4hw|E|}{n(n-1)} + \frac{2hw|E|}{n(n-1)} + \frac{hw(w-1) + wh(h-1)}{n(n-1)(n-2)} \sum_{v \in V} d_v(d_v - 1)$$

$$+ h(h-1)w(w-1) \frac{4|E|^2 - 2\sum_{v \in V} d_v(d_v - 1) - 2|E|}{n(n-1)(n-2)(n-3)}.$$

where for the last step we split into cases according to whether $i \neq i'$ or $j \neq j'$. Therefore, the variance $\text{var}(Y)$ is equal to

$$E[Y^2] - E[Y]^2 = \frac{2hw|E|}{n(n-1)} + \left[hw(w-1) + wh(h-1)\right] \frac{\sum_{v \in V} d_v(d_v - 1)}{n(n-1)(n-2)}$$

$$+ h(h-1)w(w-1) \frac{4|E|^2 - 2\sum_{v \in V} d_v(d_v - 1) - 2|E|}{n(n-1)(n-2)(n-3)} - \frac{4h^2w^2|E|^2}{n^2(n-1)^2}.$$

We note that

$$\frac{h(h-1)w(w-1)}{n(n-1)(n-2)(n-3)} - \frac{h^2w^2}{n^2(n-1)^2} = h\frac{hw(4n-6) - (h + w - 1)n(n-1)}{n(n-1)(n-2)(n-3)}$$

$$\leq \frac{2hw^2}{n(n-1)(n-2)(n-3)}.$$

where in the last step, we have used the fact that the quantity above is maximized when $h = w$, and that $2 \leq h + w \leq n$ by the construction of the blocks.
Combining the pieces, we conclude that \( \text{var}(Y) \) is bounded by

\[
\frac{hw|E|}{n^2} + c(hw^2 + wh^2) \sum_{v \in V} \frac{d_v^2}{n^3} + \frac{h^2 w^2 |E|^2}{n^6} \leq 2c \frac{hw|E|}{n^2} + c(hw^2 + wh^2) \sum_{v \in V} \frac{d_v^2}{n^3}
\]

where the inequality holds because \( h \leq n, \ w \leq n \) and \( |E| \leq n^2 \). Using the fact that \( Y \geq 1 \) and applying Chebyshev’s inequality, we obtain

\[
\mathbb{E}[Y^{-1}] \leq \Pr \left\{ Y \leq \frac{\mathbb{E}[Y]}{2} \right\} + \frac{2}{\mathbb{E}[Y]}
\]

\[
\leq \frac{4}{\mathbb{E}[Y]^2} \text{var}(Y) + \frac{2}{\mathbb{E}[Y]}
\]

\[
\leq \frac{n^4}{h^2 w^2 |E|^2} \left[ \frac{hw|E|}{n^2} + (hw^2 + wh^2) \sum_{v \in V} \frac{d_v^2}{n^3} \right] + \frac{n^2}{hw|E|}
\]

\[
= 2c \frac{n^2}{hw|E|} + cn \frac{h + w \sum_{v \in V} d_v^2}{|E|^2}.
\]

Now the above bound yields

\[
\mathbb{E} \frac{|B|}{Y} \leq 2c \frac{n^2}{|E|} + cn \frac{h + w \sum_{v \in V} d_v^2}{|E|^2}.
\]

Note that there are at most \( m^2 = (n/S)^2 \) blocks in total and the sum of \( h \) over \( m - 1 \) off-diagonal blocks vertically is bounded by \( n \) (similarly for \( w \)). Thus we conclude that

\[
\mathbb{E} \sum_{B \in B(b)} \frac{|B|}{|B \cap E| \vee 1} \leq c m \frac{n^2}{|E|} + c m n^2 \frac{\sum_{v \in V} d_v^2}{|E|^2}.
\]

In order to complete the proof, it suffices to show that

\[
\frac{n^2}{|E|} \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^{-2} + n \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^{-1} \frac{\sum_{v \in V} d_v^2}{|E|^2} \leq c \frac{\sum_{v \in V} \frac{1}{\sqrt{d_v}}}{n}.
\]

Note that Lemma \([2]\) implies that

\[
2|E| \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^2 = \left( \sum_{v \in V} d_v \right) \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^2 \geq n^3.
\]

It follows that

\[
\frac{n^2}{|E|} \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^{-2} \leq 2 - \frac{2}{n} \frac{1}{\sum_{v \in V} \frac{1}{\sqrt{d_v}}},
\]

and that

\[
n \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^{-1} \frac{\sum_{v \in V} d_v^2}{|E|^2} \leq 4 \frac{\sum_{v \in V} d_v^2}{n^2} \frac{\sum_{v \in V} \frac{1}{\sqrt{d_v}}}{ \left( \sum_{v \in V} \frac{1}{\sqrt{d_v}} \right)^2} \leq 4 \frac{\sum_{v \in V} \frac{1}{\sqrt{d_v}}}{n}.
\]

since \( d_v \leq n \).
5.5.3 Proof of Lemma \[10\]

This lemma is a generalization of an approximation theorem due to Chatterjee [Cha15] and Shah et al. [SBCW17] to the noisy and two-dimensional setting.

We use the shorthand $\hat{C}_t = \text{bl}_t(\hat{r})$ for the rest of the proof. Also define the set of placeholder elements in the partition $\hat{C}_t$ as

$$s(\hat{C}_t) = \{i : i \text{ is smallest index in some set } I \in \hat{C}_t\}.$$

We are now ready to prove the lemma. Begin by writing

$$\|X - R(X, \hat{C}_t)\|^2_F = \sum_{k=1}^n \left\| X_k - \frac{1}{|\hat{C}_t(k)|} \sum_{j \in \hat{C}_t(k)} X_j \right\|^2_2$$

\[\leq \sum_{k=1}^n \left\| X_k - \frac{1}{|\hat{C}_t(k)|} \sum_{j \in \hat{C}_t(k)} X_j \right\|^1_1 \]

\[\leq \sum_{k=1}^n \frac{1}{|\hat{C}_t(k)|} \sum_{j \in \hat{C}_t(k)} |r(X)_k - r(X)_j| \]

\[= \sum_{k \in s(\hat{C}_t)} \frac{1}{|\hat{C}_t(k)|} \sum_{i \in \hat{C}_t(k)} \sum_{j \in \hat{C}_t(k)} |r(X)_i - r(X)_j| \]

\[\leq \sum_{k \in s(\hat{C}_t)} \frac{1}{|\hat{C}_t(k)|} \sum_{i,j \in \hat{C}_t(k)} (|\hat{r}_i - r(X)_i| + |\hat{r}_j - r(X)_j| + |\hat{r}_i - \hat{r}_j|) \]

\[\leq 2\|\hat{r} - r(X)\|_1 + \|\hat{r} - r(X)\|_1 + \sum_{k \in s(\hat{C}_t)} t|\hat{C}_t(k)| \]

Step (i) follows from the fact that each entry of the difference matrix $X - R(X, \hat{C}_t)$ is bounded in the interval $[-1, 1]$; step (ii) follows from Jensen’s inequality and convexity of the $\ell_1$ norm; step (iii) uses the fact that for fixed $k$ and $j$, the quantity $X_{k\ell} - X_{j\ell}$ has the same sign for all $\ell \in [n]$ due to the monotonicity of columns of the matrix $X$; step (iv) uses the property of the blocking partition $\hat{C}_t$, which ensures that $|\hat{r}_i - \hat{r}_j| \leq t$ when the inclusion $i, j \in \hat{C}_t(k)$ is satisfied for some $k$. This completes the proof.

6 Discussion

In this paper, we studied the problem of estimating the comparison probabilities from noisy pairwise comparisons under worst-case and average-case design assumptions. We exhibited a dichotomy between worst-case and average-case models for permutation-based models, which suggests that a similar distinction may exist even for their parametric counterparts. Our bounds leave a few interesting questions unresolved: Is there a sharp characterization of the diameter $\mathcal{A}(G)$ quantifying the approximation error of a comparison topology $G$? The Borda
count estimator, a variant of which we analyzed, is known to achieve a sub-optimal rate in the case of full observations; the estimator of Braverman and Mossel [BM08] achieves the optimal rate over the noisy sorting class. What is the analog of such an estimator in the average-case setting with partial pairwise comparisons? Is there a computational lower bound to show that our estimators are the best possible polynomial-time algorithms for SST matrix estimation in the average-case setting?

Acknowledgements

This work was partially supported by National Science Foundation grants NSF-DMS-1612948, CCF-1528132, and CCF-0939370 (Science of Information), and DOD Advanced Research Projects Agency grant W911NF-16-1-0552. CM was supported in part by NSF CAREER DMS-1541099 and was visiting the Simons Institute for the Theory of Computing while this work was done.

A Bounds on the minimax denoising error

As we saw in Theorem 1, the minimax risk of Frobenius norm estimation is prohibitively large for many comparison topologies. In some applications, however, it may be of interest to control the denoising error, which is the error we make on the observations seen on the edges of the graph. Accordingly, we define the quantity

$$E(G, C) = \inf_{\hat{M} = f(Y(G))} \sup_{M^* \in C} \mathbb{E} \left[ \frac{1}{|E|} \| \hat{M} - M^* \|_F^2 \right],$$

where we have used a normalization of $|E|$ to provide an average entry-wise bound on the denoising error. The following theorem provides bounds on the minimax denoising error for fixed topologies.

**Theorem 5.** For any connected graph $G$, we have

$$E(G, C_{\text{NS}}) \geq c_1 \max_{S \subseteq G} \frac{|V(S)|^2}{|E(S)|}, \quad \text{and} \quad E(G, C_{\text{SST}}) \leq c_2 \frac{n \log^2 n}{|E|}. \quad (37)$$

Again, the lower bound on the error of the noisy sorting class provides a lower bound for the SST class. Conversely, the upper bound on the error for the SST class upper bounds the error for the noisy sorting class.

For many graphs used in practice, the lower bound can be evaluated to show that Theorem 5 provides a sharp characterization of the denoising error up to logarithmic factors.

The upper bound is obtained by the least squares estimator

$$\hat{M}_{\text{LS}} = \arg \min_{\hat{M} \in C_{\text{SST}}} \| Y - M^* \|_F^2.$$

While we do not know yet whether such an estimator is computable in polynomial time, analyzing it provides a notion of the fundamental limits of the problem. In particular, it is clear that the denoising problem is easier than Frobenius norm estimation, and we obtain consistent rates provided that the number of edges in the graph satisfies $|E| = \omega(n \log^2 n)$. 

35
A.1 Proof of Theorem 5

In this section, we prove Theorem 5 on the denoising error rate of the problem, splitting it into proofs of the lower and upper bounds.

A.1.1 Proof of lower bound

In order to prove the lower bound, we construct a suitable local packing $P$ of the parameter space $C_{NS}$, and then apply Fano’s inequality. For simpler presentation, we describe the packing $P$ by gradually putting constraints on its members. First, every matrix in $P$ is chosen to be $M_{NS}(\pi,\lambda)$ for a fixed $\lambda$ and some permutation $\pi$, so we focus on selecting the permutations $\pi$.

Consider any connected subgraph $S \in \mathcal{C}_G$ with at least two vertices. Let the vertices of $S$ form the top $|V(S)|$ items and choose the same ranking for the vertices of $S^c$ for each instance in the packing. Then all the matrices in the packing $P$ have the same $(i,j)$-th entry if $i \in S^c$ or $j \in S^c$. Hence the KL divergence between any two models with underlying matrices in the packing $P$ is bounded by $9\lambda^2|E(S)|$, by Lemma 7.

Next, fix a spanning tree $T(S)$ of $S$ which has $|V(S)|-1$ edges. Note that all the $2^{|V(S)|-1}$ assignments of values to these edges $\{M_{ij} : (i,j) \in T(S), i < j\} \in \{1/2 + \lambda, 1/2 - \lambda\}^{|V(S)|-1}$ are possible, since there are no cycle conflicts in the spanning tree. Using the Gilbert-Varshamov bound, we are guaranteed that there are constants $a$ and $b$ such that at least $2a|V(S)|$ of these assignments are separated pairwise by $b|V(S)|$ in the Hamming distance. We choose the packing $P$ consisting of matrices corresponding to these assignments, so that $\|M - M'\|^2_F \geq 8b\lambda^2|V(S)|$ for any distinct $M, M' \in P$.

Finally, Fano’s inequality implies that

$$|E| \mathcal{E}(G, C_{NS}) \geq 8b\lambda^2|V(S)| \left(1 - \frac{9\lambda^2|E(S)| + \log 2}{a|V(S)|}\right).$$

The proof then follows by choosing $\lambda^2 \approx \frac{|V(S)|}{|E(S)|}$, for a sufficiently small constant $c$.

A.1.2 Proof of upper bound

As mentioned before, we obtain the upper bound by considering the estimator $\hat{M}_{LS}$. The proof follows from previous results on the full observation case [SBGW17], but we provide it for completeness. Note that for each $(i,j) \in E$, the observation model takes the form

$$Y_{ij} = M_{ij}^* + W_{ij},$$

where $W_{ij}$ is a zero-mean noise variable lying in the interval $[-1,1]$.

The optimality of $\hat{M}_{LS}$ and feasibility of $M^*$ imply that we must have the basic inequality $\|Y - \hat{M}_{LS}\|^2_E \leq \|Y - M^*\|^2_E$, which after simplification, leads to

$$\frac{1}{2} \|\Delta\|^2_E \leq \langle \Delta, W \rangle_E,$$

where $\Delta = \hat{M}_{LS} - M^*$, and $\langle A, B \rangle_E = \sum_{(i,j) \in E} A_{ij} B_{ij}$ denotes the trace inner product restricted to the indices in $E$. 

36
In order to establish the upper bound, we first define the class of difference matrices $\mathbb{C}_{\text{DIFF}} := \{M - M' \mid M, M' \in \mathbb{C}_{\text{SST}}\}$, as well as the associated random variable $Z(t) := \sup_{D \in \mathbb{C}_{\text{DIFF}} \mid \|D\| \leq t} \langle D, W \rangle_E$.

With this notation, inequality (38) implies $\frac{1}{2} \|\Delta\|_E^2 \leq Z(\|\Delta\|_E)$. It follows from the star-shaped property of the set $\mathbb{C}_{\text{DIFF}}$ that the following critical inequality is satisfied for some $\delta > 0$:

$$E[Z(\delta)] \leq \frac{\delta^2}{2}.$$  

We are interested in the smallest such value $\delta$. In order to find it, we use Dudley’s entropy integral, for which we require a bound on the covering number of the class $\mathbb{C}_{\text{DIFF}}$. Such a bound was calculated for the Frobenius norm by Shah et al. [SBGW17] using the results of Gao and Wellner [GW07]. Clearly, since $\|M_i - M_j\|_E^2 \leq \|M_i - M_j\|_F^2$, a $\delta$-covering in the Frobenius norm automatically serves as a $\delta$-covering in the edge norm $\|\cdot\|_E$. Thus, we have the following lemma.

**Lemma 11.** [SBGW17] For every $\epsilon > 0$, we have the metric entropy bound

$$\log N(\epsilon, \mathbb{C}_{\text{DIFF}}, \|\cdot\|_E) \leq \log N(\epsilon, \mathbb{C}_{\text{DIFF}}, \|\cdot\|_F) \leq 9 \frac{n^2}{\epsilon^2} \left(\log \frac{n}{\epsilon}\right)^2 + 9n \log n.$$  

Dudley’s entropy integral then yields that for all $t > 0$, we have

$$E[Z(t)] \leq c \inf_{\delta \in [0, n]} \left\{ n\delta + \int_{\delta/2}^{t} \sqrt{\log N(\epsilon, \mathbb{C}_{\text{DIFF}} \cap B_E(\epsilon), \|\cdot\|_E)} d\epsilon \right\}$$  

$$\leq c \left\{ n^{-8} + \int_{n^{-9/2}}^{t} \sqrt{\log N(\epsilon, \mathbb{C}_{\text{DIFF}} \cap B_E(\epsilon), \|\cdot\|_E)} d\epsilon \right\}.$$  

After some algebra (for details, see Shah et al. [SBGW17]), we have

$$E[Z(t)] \leq c \left\{ n \log^2 n + t \sqrt{n \log n} \right\}.$$  

Setting $t = c \sqrt{n \log n}$ completes the proof.

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*A set $S$ is said to be star-shaped if $t \in S$ implies that $\alpha t \in S$ for all $\alpha \in [0, 1]$.*
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