HYPOELLIPTICITY OF THE $\bar{\partial}$-NEUMANN PROBLEM
BY MEANS OF SUBELLIPTIC MULTIPLIERS

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Abstract. We prove local hypoellipticity of the complex Laplacian $\Box$ and of the Kohn Laplacian $\Box_b$ in a pseudoconvex boundary when, for a system of cut-off $\eta$, the gradient $\partial_b \eta$ and the Levi form $\partial_b \partial_b \eta^2$ are subelliptic multipliers in the sense of [11].

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1. Introduction

For a pseudoconvex domain $\Omega \subset \subset \mathbb{C}^n$ with $C^\infty$-boundary $b\Omega$, we consider the problem of the local regularity of the canonical solution of $\partial_b$ and of the $\bar{\partial}$-Neumann problem at a point $z_o \in b\Omega$. We form the Kohn Laplacian $\Box_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*$ and the complex Laplacian $\Box = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$. The first problem can be restated in terms of the hypoellipticity of $\Box_b$

(Hypoellipticity) \quad $\Box_b u \in C^\infty_{z_o}$ implies $u \in C^\infty_{z_o}$.

In the same way the hypoellipticity of $\Box$ is defined. We search for general criteria of hypoellipticity. It was firstly noticed by Kohn that the presence in supporting complex hypersurfaces of propagators of boundary smoothness of holomorphic functions prevents from hypoellipticity. A related phenomenon is that of the propagation of holomorphic extendibility. According to [9], this takes place along complex curves. However, in the exponentially degenerate case, it was proved by [1] that real curves are also propagators. This is the case of the lines $\mathbb{R}_{\gamma_j}$ for the tube domain $2x_2 = e^{-\sum_{j=1}^{n-1} |x_j|^s}$ for $s \geq 1$. This propagation matches the non-hypoellipticity of $\Box_b$ proved by [6] in $\mathbb{C}^2$. Instead, if $s < 1$, the argument for propagation of [1] breaks down; again, this is in accordance with the hypoellipticity which occurs as a consequence of “superlogarithmic” estimates (cf. this section below). Thus propagation and hypoellipticity appear opposite one to another.

As for classical positive results on hypoellipticity, we recall that this is generally obtained through estimates on forms $v$ of degree $k \in [1, n-2]$ such as

(Subelliptic) \quad $||v||, \preceq ||\partial_b v|| + ||\partial_b^* v||$, 

(1.1)
or

\[(1.2) \quad \text{(Superlogarithmic) } \| \log(\Lambda)v \| \lesssim \delta(\| \bar{\partial}_b\partial v \| + \| \bar{\partial}_b^*v \|) + c_\delta \|v\|_{-1},\]

for any \( \delta \) and for suitable \( c_\delta \). Models are “decoupled” domains \( 2x_n = \sum_{j=1}^{n-1} h^j(z_j) \) with

\[
(\text{Subelliptic}) \quad h^j = |z_j|^{2m_j} \quad \text{or} \quad h^j = x_j^{2m_j},
\]

\[
(\text{Superlogarithmic}) \quad h^j = e^{-\frac{1}{|z_j|^{s}}} \quad \text{or} \quad h^j = e^{-\frac{1}{|x_j|^{s}}}, \quad s < 1,
\]

where we can replace the power \( |z_j|^s, \ s < 1 \) by \( |z_j|\log|z_j| \) and similarly for \( x_j \) (cf. [10]). To get hypoellipticity from (1.1) (cf. [14]), one substitutes \( \eta \Lambda^s u \) for \( v \) where \( \eta \) ranges in a system of cut-off and \( \Lambda^s \) is the standard tangential elliptic operator of order \( s \). The problem is to control the commutators \( [\bar{\partial}_b^*(\eta \Lambda^s), \partial \bar{\partial}_b u] \). First, these are estimated by \( |\partial^\eta \Lambda^s| \) by induction, and gets

\[
(1.3) \quad \|\eta u\|_{s} \lesssim \|\eta' \bar{\partial}_b u\|_{s} + \|\eta' \bar{\partial}_b^* u\|_{s} + \|u\|_{0} \quad \text{for } \eta' \succ \eta \quad \text{i.e. } \eta'|_{\text{supp } \eta} \equiv 1,
\]

which is sufficient for hypoellipticity. To get the same conclusion (1.3) starting from (1.2), one replaces \( \Lambda^s \) by the pseudodifferential operator \( R^s \) with symbol \( \sigma(R^s) = \Lambda^s \sigma(z) \) for \( \eta \prec \sigma \prec \eta' \) and notices that \( \eta \Lambda^s \prec \eta' R^s + O(\Lambda^{-\infty}), \ |\partial \eta|R^s = O(\Lambda^{-\infty}), \ ||\bar{\partial}_b^*(\eta' R^s)\| \leq c_s \log(\Lambda) R^s \)

and controls \( c_s \ll \delta^{-1} \) where \( \delta \) is the small constant in (1.2) (cf. [13]).

But hypoellipticity is not entirely ruled by estimates. In [12], Kohn proves hypoellipticity for boundaries defined by \( 2x_n = h(z', y_n) \) such that

(i) there are subelliptic estimates for \( |z'| \neq 0 \),
(ii) \( h_{z_j} \) are subelliptic multipliers.

In this situation, taking a cut-off \( \chi \) of one real variable and setting \( \zeta = \Pi_{j=1,\ldots,n-1}(|z_j|), \ \theta = \chi(|y_n|), \ \eta = \zeta \theta \), and denoting by \( \tilde{L}_j, \ j = 1, \ldots, n-1 \) a system of \((0,1)\) vector fields, we have

\[
[\tilde{L}_j, \eta] = \underbrace{\zeta_j \theta + h_j^j \zeta \theta}_{\text{controlled by (i)}}.
\]

The model is

\( 2x_n = e^{-\frac{1}{(\sum_{j=1}^{n-1} |z_j|^{s})^{s}}} \) for any \( s > 0 \).

When \( s < 1 \), hypoellipticity was already obtained from superlogarithmicity even with \( z_j \) replaced by \( x_j \); when \( s \geq 1 \), the conclusion is new and does not hold for \( x_j \) (cf. [1] and [6] already mentioned above).
It remained open the problem of the hypoellipticity of domains with model

\[ 2x_n = \sum_j e^{-\frac{1}{|z_j|^s}}, \quad s \geq 1, \]

in which summation is not taken at exponent. In this case (i) and (ii) do not hold at the points of the “cross” \( z_j = 0 \) for some \( j = 1, \ldots, n - 1 \). A first answer to this question has been given in [3] where hypoellipticity is stated on a class of domains which contains (1.4). This is obtained by modifying the localized “bad” vector field \( \zeta \theta T \) into

\[ (T)_{\zeta \theta} := \zeta \theta T - \left( \sum_j (h^j_{z_j \bar{z}_j})^{-1} L_j(\zeta \theta) \bar{L}_j + (h^j_{z_j \bar{z}_j})^{-1} L_j(\zeta \theta) L_j \right) \]

(cf. [7]). The class of domains in question is that for which the coefficients \( (h^j_{z_j \bar{z}_j})^{-1} L_j(\zeta \theta) \) are well defined, that is, the zeroe’s of \( (-)^j L_j(\zeta \theta) \) balance those of \( h^j_{z_j \bar{z}_j} \).

In the present paper, we give the geometric solution to the problem. Hypoellipticity holds whenever

\[ (1.5) \bar{\partial}_b \eta \text{ and } \partial_b \bar{\partial}_b \eta^2 \text{ are subelliptic multipliers,} \]

over “positively microlocalized” forms \( u^+ \). The model is

\[ 2x_n = \sum_j e^{-\frac{1}{|z_j|^s_j} x_j^{2m_j}} \quad \text{any } s_j > 0 \text{ and } m_j \geq 0. \]

The idea of the proof is to insert the cut-off \( \eta \) into the weight \( e^{-\varphi} \), \( \varphi = -\log \eta^2 + t|z|^2 \), \( z' \in T^\omega Cb^\Omega \), which occurs in the “basic estimate”. This dispenses from controlling \( \left[ \bar{\partial}_b^{(\omega)}, \eta \right] \) and reduces the problem only to the error in the Levi form and in the adjunction (in addition to the commutator \( \left[ \bar{\partial}_b^{(\omega)}, \Lambda^\omega \right] \), as usual):

\[
\begin{align*}
& e^{-\varphi} \left( \partial_b \bar{\partial}_b \varphi - \partial_b \bar{\partial}_b(t|z|^2) \right) \sim \partial_b \bar{\partial}_b \eta^2, \\
& e^{-\varphi} \left( \left( \bar{\partial}_b^* \right)_b - \left( \bar{\partial}_b^* \right)_b(t|z|^2) \right) \sim \partial_b \eta.
\end{align*}
\]

Thus, by the aid of (1.3), the basic estimate turns into a regularity estimate with cut-off. Note that the single entries of \( \partial_b \bar{\partial}_b \eta^2 \) and \( \partial_b \eta \) need not to be subelliptic multipliers for all components of \( u \) but just for those that they “pick up”.

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2. The main result

Let $\Omega \subset \subset \mathbb{C}^n$ be a smooth pseudoconvex domain and $z_o = 0$ a boundary point.

**Theorem 2.1.** Assume that there is a system of smooth cut-off $\eta$ in a neighborhood of $0$ such that

\begin{equation}
\partial_b \eta \text{ and } \partial_b \bar{\partial}_b \eta^2 \text{ are subelliptic multipliers in positive microlocalization in any degree } k \in [1, n-1] \text{ (cf. [11])}.
\end{equation}

Then $\Box_b$ and $\Box$ are $C^\infty$-hypoelliptic at $0$.

The main tool in the proof is the proposition below. Let $H_b = \ker \Box_b$ be the space of harmonic forms.

**Proposition 2.2.** Assume that for a system of cut-off $\eta$, (2.1) is satisfied. Then for any $\eta$ and for suitable $\eta' \succ \eta$, that is $\eta'|_{\text{supp } \eta} \equiv 1$, we have

\begin{equation}
||\eta u||_s \sim ||\eta' \bar{\partial}_b u||_s + ||\eta' \bar{\partial}_b^* u||_s + ||u||_0 \text{ for any } u \in H^+ \cap C^\infty(b\Omega)
\end{equation}

in any degree $k \in [0, n-1]$.

The same estimate holds for the $\bar{\partial}$-Neumann problem.

**Proof.** We choose the orientation $T^\pm$ of the purely imaginary vector field and consider the microlocal decomposition of the identity $\text{Id} = \Psi^+ + \Psi^- + \Psi^0$ and the corresponding decomposition of a form $u = u^+ + u^- + u^0$ (cf. [13] Section 2). We recall that $u^0$ enjoys elliptic estimates; we also observe that $[\bar{\partial}_b^*(\Psi^0)] \prec \Psi^0$ and hence it suffices to prove (2.2) separately for $u^+$ and $u^-$. We recall that the star-Hodge operator $u^- \mapsto \ast u^- = \ast \bar{u}^+$ settles up a correspondence between “negative” forms in degree $k$ and “positive” forms in complementary degree $n-1-k$. Thus it suffices to prove (2.2) for $u^+$.

We start from $k \geq 1$. We recall the weighted tangential estimates with weight $e^{-\varphi}$ for $\varphi = -\log \eta^2 + t|z'|^2$, $z' \in T_0^*b\Omega$; we point out that even though the weight $\varphi$ is not smooth, nevertheless $e^{-\varphi} \partial_b \varphi$ and $e^{-\varphi} \partial_b \bar{\partial}_b \varphi$ are bounded and hence all integrals below are well defined. Here is the estimate

\begin{equation}
\int e^{-\varphi} \partial_b \bar{\partial}_b \varphi(u^+, u^+)dV + ||\nabla u^+||_\varphi^2 \lesssim ||\bar{\partial}_b u^+||^2_\varphi + ||(\bar{\partial}_b^* u^+)||^2_\varphi + ||u^+||^2_\varphi.
\end{equation}
We first remove $e^{-t|z'|^2}$ from norms since it is uniformly bounded from above and below; thus the norms in (2.3) change into $|| \cdot ||^2_{\log \eta^2}$. We now describe $\bar{\partial}_b \partial_b \varphi$ and $(\bar{\partial}_b^* )_b u^+$. For the first

(2.4)

$$|\bar{\partial}_b \partial_b \varphi| = \bar{\partial}_b \partial_b (- \log \eta^2 + t|z'|^2) = - \frac{2}{\eta} \bar{\partial}_b \partial_b \eta + 2 \frac{\partial_b \eta \otimes \partial_b \eta}{\eta^2} + t \bar{\partial}_b z' \otimes \bar{\partial}_b z'. $$

For the second, we start from $(\bar{\partial}_b^*)_b = \bar{\partial}_b^* + \partial_b \log \eta^2 - t \bar{\partial}_b z' dz'$ and get

(2.5)

$$|| (\bar{\partial}_b^*)_b u^+ ||^2_{\varphi} \sim || (\bar{\partial}_b^*)_b u^+ ||^2_{\log \eta^2}$$

$$= \left| \left| \eta \bar{\partial}_b^* u^+ \right|_0^2 + 4 || \partial_b \eta \nabla \u_\varphi \bigg|_0^2 + || \eta \bar{\partial}_b z' dz' \bigg|_0^2 \right.$$  

$$+ 4 \Re \left( \eta \bar{\partial}_b^* u^+, \partial_b \eta \nabla u^+ + t \bar{\partial}_b z' dz' \bigg|_0^2 \right).$$

Taking supp $\eta$ in a small neighborhood of $z_o = 0$, $t \bar{\partial}_b z'$ is also small. By (2.1) and (2.3), equality (2.3) with $u^+$ replaced by $\Lambda^* u^+$ yields

(2.6)

$$\begin{align*}
  t || \eta u^+ ||_0^2 + || \eta \nabla u^+ ||_0^2 & \leq || \bar{\partial}_b \partial_b \varphi ||_0^2 + || \eta \bar{\partial}_b^* u^+ ||_0^2 \\
  & + || \partial_b \eta \nabla u^+ ||_0^2 + \int \partial_b \bar{\partial}_b (\eta^2)(u^+, u^+)dV.
\end{align*}$$

Note that, an alternative proof of (2.6) can be obtained from the boundary version of [15] Proposition 2.4 formula (2.24) with “twisting coefficient” $\sqrt{a} = \eta$ and weight $e^{-t|z'|^2}$. We apply (2.6) for $u^+$ replaced by $\Lambda^* u^+$ and wish to do two operations: to commutate $\bar{\partial}_b^*(\cdot)$ with $\Lambda^*$ in the right side of (2.6), and to estimate the two terms in the second line; (here $\bar{\partial}_b^*$ denotes either occurrence of $\bar{\partial}_b$ or $\bar{\partial}_b^*$). For this, we notice that, with the notation $c_s := \max_{z'} ||(c_s^j)_{|z'}||$, we have

(2.7) $\begin{cases}
  [\bar{\partial}_b^*(\cdot), \Lambda^*] = (c_s^j)_{|z'} \Lambda^*, \\
  || \partial \eta \nabla \Lambda^* u^+ ||^2 + \int \partial_b \bar{\partial}_b \eta^2 (\Lambda^* u^+, \Lambda^* u^+)dV \\
  \leq Q^b_{\eta, \Lambda^{*--}}(u^+, u^+) + c_s || \eta \Lambda^{*-} u^+ ||^2.
\end{cases}$

Here and in what follows, for an operator $\text{Op}$ such as $\eta' \Lambda^{*-}$, we write $Q^b_{\text{Op}}(u^+, u^+)$ for $|| \text{Op} \partial_b u ||^2 + || \text{Op} \bar{\partial}_b u ||^2$. We use (2.7) inside (2.6) in which $u^+$ is replaced by $\Lambda^* u^+$ and get

(2.8)

$$|| \eta \Lambda^* u^+ ||_0^2 \leq Q^b_{\eta, \Lambda^*}(u^+, u^+) + Q^b_{\eta', \Lambda^{*--}}(u^+, u^+) + c_s || \eta' \Lambda^{*-} u^+ ||_0^2 + \frac{c_s}{t} || \eta \Lambda^* u^+ ||_0^2.$$
We absorb the term in (2.8) with a factor of $\frac{c}{t}$ by taking $t$ large and restart (2.8) for $\eta$ replaced by $\eta'$ and $\Lambda^s$ by $\Lambda^{s-\epsilon}$ and, by induction on $j$ such that $j\epsilon > s$, get (2.2) for any form in degree $1 \leq k \leq n - 1$.

We have to show now that (2.2) also holds for forms in degree $k = 0$. In fact, given $u \in \mathcal{H}^+$, we use that $\bar{\partial}_b^*$ has closed range, and write

$$u = \bar{\partial}_b^* v$$

for some 1-form $v$ such that $\bar{\partial}_b v = 0$ and $||v||_0 \lesssim ||u||_0$.

We now observe that

$$\begin{cases}
\eta \bar{\partial}_b^*(v) = \bar{\partial}_b^*(\eta v) - \partial_b \eta \varpi v, \\
(\partial_b v)^+ = \bar{\partial}_b^* v^+ - [\bar{\partial}_b^*, \Psi^+] v
\end{cases}$$

for $v^0 \sim -\bar{\Psi}^+ v$. It follows

$$||\eta \Lambda^s u^+||^2 = \left(\Lambda^s u^+, \eta^2 \Lambda^s \bar{\partial}_b^* v^+\right) - \left(\Lambda^s u^+, \eta^2 \Lambda^s v^0\right)$$

$$= \left(\Lambda^s u^+, \bar{\partial}_b^*(\eta^2 \Lambda^s v^+\right) + \left(\eta \Lambda^s u^+, \sum_j (2L_j(\eta) + c_j^s \eta) \Lambda^s v^+_j\right) + \mathcal{E}$$

$$= \left(\eta \Lambda^s \bar{\partial}_b u^+, \eta \Lambda^s v^+\right) + 2\Re\left(\eta \Lambda^s u^+, \eta \sum_j c_j^s \Lambda^s v^+_j\right) + \left(\eta \Lambda^s u^+, 2 \sum_j L_j(\eta) \Lambda^s v^+_j\right) + \mathcal{E}$$

$$\leq ||\eta \Lambda^s \bar{\partial}_b u^+||^2 + sc||\eta \Lambda^s u^+||^2 + lc||\eta \Lambda^s v^+||^2 + lc||\partial_b(\eta) \Lambda^s v^+||^2 + \mathcal{E},$$

where, $sc$ and $lc$ denote a small and large constant respectively. Here and in the following, the notation $\mathcal{E}$ is used for an error subject to an elliptic gain which can therefore be disregarded. We have now to estimate $||\eta \Lambda^s v^+||^2$ and $||\partial_b(\eta) \Lambda^s v^+||^2$. For the second:

$$||\partial_b(\eta) \Lambda^s v^+||^2 \lesssim Q^b_{\Lambda^{s-\epsilon}}(v^+, v^0) + c_s||\eta' \Lambda^{s-\epsilon} v^0||^2$$

$$\lesssim ||\eta' \Lambda^{s-\epsilon} (\bar{\partial}_b v)^+||^2 + ||\eta' \Lambda^{s-\epsilon} v^0||^2 + c_s||\eta' \Lambda^{s-\epsilon} v^+||^2.$$

The central term in the last line above is of type $\mathcal{E}$. The first term is $||\eta' \Lambda^{s-\epsilon} u^+||^2$ which can be controlled by induction. Finally, to handle $||\eta' \Lambda^{s-\epsilon} v^+||^2$, we apply (2.8) and get an estimate by means of $||\eta' \Lambda^{s-\epsilon} u^+||^2 + ||\eta' \Lambda^{s-\epsilon} v^0||^2 + ||\eta'' \Lambda^{s-2\epsilon} u^+||^2 + ||\eta'' \Lambda^{s-2\epsilon} v^0||^2 + \frac{c}{t}||\eta' \Lambda^{s-\epsilon} v^+||^2$. In this way we control $||\partial_b(\eta) \Lambda^s v^+||^2$ (and similarly we can control $\int_{\partial \Omega} \partial_b \bar{\partial}_b \eta^2(\Lambda^s v^+, \Lambda^s v^+) dV$). We pass to $||\eta \Lambda^s v^+||^2$.
since $\bar{\partial}_b v = 0$, then
\begin{equation}
||\eta \Lambda^s v^+||^2 \leq \frac{1}{t} \left(||\eta \Lambda^s \bar{\partial}_b v^+||^2 + ||\partial_b(\eta) \bigwedge \Lambda^s v^+||^2 + \int_{\Omega} \bar{\partial}_b \partial_b \eta^2(\Lambda^s v^+, \Lambda^s v^+)dV\right)
\end{equation}

\leq \frac{1}{t} ||\eta \Lambda^s u^+||^2 + ||\eta \Lambda^s v^0||^2 + ||\partial_b(\eta) \bigwedge \Lambda^s v^+||^2 + \int_{\Omega} \bar{\partial}_b \partial_b \eta^2(\Lambda^s v^+, \Lambda^s v^+)dV.

In the last line of (2.12), the first term can be absorbed in the left of (2.10), the second is subject to an elliptic gain as $E$ above, the third has the estimate (2.11) and the last is similar. Altogether, $||\partial_b(\eta) \bigwedge \Lambda^s v^+||^2$ and $||\eta \Lambda^s v^+||^2$, in (2.10) are controlled. Thus induction works in (2.10) and has the effect of reducing the Sobolev index both of $u^+$ and $v^+$. At the last step, that is at $s = 0$, we use the closed range estimate $||v^+||_0 \leq ||v||_0 < ||u||_0$ and get rid of $v$.

This concludes the proof of estimate (2.2) of Proposition 2.2. The corresponding estimate for the $\bar{\partial}$-Neumann problem is obtained by the technique of [2].

Remark 2.3. We give an alternative proof of (2.2) for $u^−$ which avoids use of the star-Hodge operator. For this, we start, instead of (2.3) from
\begin{equation}
− \int e^\varphi \partial_b \bar{\partial}_b \varphi(u^−, u^−)dV + \sum_j \int e^\varphi \varphi_{, j j}|u^−|^2dV + ||\nabla u^−||_\varphi^2
\leq ||\bar{\partial}_b u^−||_\varphi^2 + ||(\bar{\partial}_b \varphi)_b u^−||_\varphi^2 + ||u^−||_\varphi^2.
\end{equation}

Using the analog of (2.4), (2.5) with $\varphi$ replaced by $−\varphi$, we end up with
\begin{equation}
t||\eta u^−||_0^2 + ||\eta \nabla u^−||_0^2 \leq Q_\eta(u^−, u^−) + ||\partial_b \eta \wedge u^−||_0^2
+ \int \partial_b \bar{\partial}_b \eta^2(u^−, u^−)dV - 2 \sum_j \int \eta \eta_{, j j}|u^−|^2dV.
\end{equation}

We then use the identities
\begin{align*}
\partial\bar{\partial} & \eta \wedge u^− = − * \bar{\partial}_b \eta \bigwedge * u^+, \\
\partial_b \bar{\partial}_b \eta^2(u^−, u^−) - 2\eta \sum_j \eta_{, j j}|u^−|^2 &= \partial_b \bar{\partial}_b \eta^2(*u^+, *u^+),
\end{align*}

which shows an action of subelliptic multiplier over $*u^+$. The rest of the proof goes through as before.

Proof of Theorem 2.4. By the $L^2$-theory of $\bar{\partial}_b$, there is well defined in $L^2$ the Green operator $G = \square_b^{-1}$. As an immediate consequence of
\(\bar\partial N\) and \(\partial N\) have exact local \(H^s\)-regularity at \(z_o\) over \(\ker \bar\partial\) and \(\ker \partial^*\) respectively. More precisely, we have

\[
\|\eta \bar\partial_b^* Gu\|_s \lesssim \|\eta' u\|_s + \|u\|_0.
\]

Let \(S\), resp. \(S^*\) be the Szegö, resp. anti-Szegö, projection. By Kohn’s formula \(S = \text{Id} - \bar\partial_b^* G \bar\partial_b\) and \(S^* = \text{Id} - \partial_b G \partial_b^*\), we have that the projections \(S^{(*)}\) are also regular, though a loss of one derivative may occur on account of the double application of \(\bar\partial_b^*\). In other words we have

\[
\|\eta S^{(*)} u\|_s \lesssim \|\eta' u\|_{s+1} + \|u\|_0.
\]

From this, we can get the (non-exact) regularity of \(G\) itself on account of

\[
\|\eta G\|_s = \|\eta \Box_b G^2 \alpha\|_s \\
\lesssim \|\eta \bar\partial_b \bar\partial_b^* G^2 S \alpha\|_s + \|\eta \bar\partial_b^* \bar\partial_b G^2 S^* \alpha\|_s \\
\lesssim \|\eta \bar\partial_b G \bar\partial_b^* G S \alpha\|_s + \|\eta \bar\partial_b^* G \bar\partial_b G S^* \alpha\|_s \\
\lesssim \|\eta' \bar\partial_b^* G S \alpha\|_s + \|\eta' \bar\partial_b G S^* \alpha\|_s \\
\lesssim \|\eta'' S \alpha\|_s + \|\eta'' S^* \alpha\|_s \\
\lesssim \|\eta''' \alpha\|_{s+1}.
\]

This estimate with loss of 1 derivative is an “a-priori” estimate. The method of the elliptic regularization makes it a “genuine” estimate; this clearly suffices for local \(C^\infty\)-regularity of the Green operator \(G\). The similar conclusion on the \(C^\infty\)-regularity of the Neumann operator \(N\) is obtained from the variant of (2.2) for the \(\bar\partial\)-Neumann problem.

\[\square\]

3. A class of Examples

A large class of domains to which Theorem 2.1 applies is provided by the following

**Theorem 3.1.** In \(\mathbb{C}^n\) we consider a “decoupled” pseudoconvex domain whose boundary \(b\Omega\) is defined in a neighborhood of 0 by

\[
2x_n = \sum_{j=1}^{n-1} h^j(z_j),
\]
for \( h^j \) real subharmonic, that is, satisfying \( h^j_{jj} \geq 0 \). We make the additional assumptions that each \( h^j \) has finite type \( 2m_j \) for \( z_j \neq 0 \) and that, up a harmonic term \( \Re F^j \), we have \( |h^j_{jj} + \Re F^j| \lesssim h^j_{jj} \).

Then, for a fundamental system of cut-off \( \eta \) at 0, \( \partial_b \eta \) and \( \partial_b \partial_b \eta^2 \) are \( \frac{1}{2m} \)-subelliptic multipliers for \( m = \sup_j m_j \geq 2 \) over forms \( u^+ \) in degree \( k \in [0, n - 1] \).

Proof. We choose a cut-off \( \chi \) in \( \mathbb{R} \) at 0, set \( \zeta = \Pi_j \chi(|z_j|) \), \( \theta = \chi(y_n) \), and define \( \eta = \zeta \theta \). We also write a general coefficient of \( u \) in degree \( k \) as \( u_{jK} \) for \( j = 1, \ldots, n - 1 \) and \( |K| = k - 1 \); we also use the notation \( r := 2x_n - \sum_j h^j \). The crucial point in the proof below is that, \( r \) being decoupled, we have

\[
\sum'_{|K|=k-1} \sum_{i=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} - \sum_{j=1}^{n-1} \sum'_{|K|=k-1} r_{jj} |u_{jK}|^2 = 0. \tag{3.1}
\]

Thus, the basic estimate not only yields

\[
\sum \sum'_{|J|=k} ||L_j u^+_j||^2_0 \lesssim Q^b(u^+, u^+) + ||u^+||^2_0, \tag{3.2}
\]

as usual, but also

\[
\sum \sum'_{|K|=k-1} ||L_j u^+_j||^2_0 \lesssim Q^b(u^+, u^+) + ||u^+||^2_0. \tag{3.3}
\]

We select an index \( j_0 \). Since, the iterated brackets \( \overbrace{(-L_{j_0}, [L_{j_0}, [\ldots]]}^{2m_{j_0}} \)

(\( \text{where } -L_{j_0} \) denotes either occurrence of \( L_{j_0} \) or \( \bar{L}_{j_0} \)) generate the purely imaginary vector field \( T = \partial_{y_n} \) over supp \( \zeta_{j_0} \subset \{ z_j : z_{j_0} \neq 0 \} \), then we have

\[
||T(\zeta_{j_0} u^+_{j_0,K})||_{-\frac{1}{2m_{j_0}}} \lesssim ||L_{j_0}(\zeta_{j_0} u^+_{j_0,K})||_0 + ||\bar{L}_{j_0}(\zeta_{j_0} u^+_{j_0,K})||_0 + ||u^+_{j_0,K}||_0 \lesssim Q^b(u^+, u^+) + ||u^+_{j_0,K}||^2_0. \tag{3.4}
\]

Thus \( Q^b + || \cdot ||^2_0 \) contains, over supp \( \zeta_{j_0} \), the norm of a fractional derivative \( ||T^{2m_{j_0}} u^+_{j_0,K}||^2_0 \) and of a full derivative \( ||\bar{L}_{j_0} u^+_{j_0,K}||^2_0 \) for any \( j = 1, \ldots, n - 1 \). As for \( L_j \), this is already contained in \( Q^b + || \cdot ||^2_0 \) for \( j = j_0 \) according to (3.3). For \( j \neq j_0 \), we have to change \( L_j \) into \( \bar{L}_j \).
For this, we use the identity
\[ ||L_j(\zeta_{j_0} u_{j_0 K})||^2_0 \lesssim ||\bar{L}_j(\zeta_{j_0} u_{j_0 K})||^2_0 + \left( [L_j, \bar{L}_j](\zeta_{j_0} u_{j_0 K}), \zeta_{j_0} u_{j_0 K} \right) + ||u_{j_0 K}||^2_0; \]

next, we express the commutator as \( [L_j, \bar{L}_j] = r_{jj} T + \sum_{h} a_h \bar{L}_h. \) The terms \( (\zeta_{j_0} a L_h u_{j_0 K}, \zeta_{j_0} u_{j_0 K}) \) can be estimated by \( sc \sum_j ||\zeta_{j_0} L_j u_{j_0 K}||^2_0 + lc ||u_{j_0 K}||^2_0 \) which yields
\[
\sum_j ||L_j(\zeta_{j_0} u_{j_0 K})||^2_0 \lesssim \sum_j ||\bar{L}_j(\zeta_{j_0} u_{j_0 K})||^2_0 + ||T(\zeta_{j_0} u_{j_0 K})||^2_0 + ||u_{j_0 K}||^2_0
\]
\[+ sc \sum_j ||L_j(\zeta_{j_0} u_{j_0 K})||^2_0 \lesssim \sum_j ||\bar{L}_j(\zeta_{j_0} u_{j_0 K})||^2_0 + \sum_j ||\bar{L}_j(\zeta_{j_0} u_{j_0 K})||^2_0 + lc ||u_{j_0 K}||^2_0.
\]

Taking summation over \( j_0 \) and \( K \), and the minimum \( \frac{1}{2m} \) of the \( \frac{1}{2m_{j_0}} \)'s, we get the estimate for \( \partial_b \zeta = \partial_b \Pi_j \zeta_j \)
\[ ||\partial_b \zeta \otimes u^+||^2_0 \lesssim Q^b(u^+, u^+) + ||u^+||^2_0 \]
\[ \lesssim Q^b(u^+, u^+), \]

where the second estimate follows from the closed range. Passing to a general \( \partial_b \eta = \partial_b \zeta \theta \), we notice that
\[
\partial_b \eta = (L_j \eta)_{j=1,\ldots,n-1} = (\zeta_j \theta)_{j=1,\ldots,n-1} + (\zeta_{j_0} \hat{h}^j \hat{\theta})_{j=1,\ldots,n-1}.
\]

Now, (a) has already been estimated in (3.6). As for (b), we observe that in new complex coordinates in which we get rid of harmonic terms in the \( h^j \)'s, we have by hypothesis \( |h_{j_0}^j|^2 < h_{j_0}^j \zeta_j \). It follows
\[
||T^\cdot u^+, T^\cdot u^+||^2_0 \lesssim \int \partial_b \bar{\partial}_b r(\bar{T}^\cdot u^+, T^\cdot u^+)dV + ||u^+||^2_0
\]
\[ \lesssim Q^b(u^+, u^+); \]
again, we have estimated \( ||u^+||^2_0 \lesssim Q^b \) by closed range. This, together with (3.6), shows that the gradient \( \partial_b \eta \) is a \( \frac{1}{2m} \)-subelliptic multiplier. We
pass to the Levi form. We start from the obvious equality $L_i \bar{L}_j(\eta^2) = 2L_i(\eta) \bar{L}_j(\eta) + 2\eta L_i \bar{L}_j(\eta)$ and

$$L_i \bar{L}_j(\eta) = (\zeta_i \zeta_j \theta + \zeta_i \bar{\zeta_j} \bar{\theta}) + (\zeta_i h^j_z \bar{\theta} + \zeta_j h^i_z \bar{\theta}) + \zeta_i h^i_z \bar{h}^j_z \bar{\theta}.$$

Now, the first and second terms in the right are controlled by (a) of (3.7) above. The third and fourth by

$$\left| \int \zeta_i u^+_i h^i_z \bar{u}^+_j \bar{\theta} dV \right| \leq \|\zeta_i u^+_i\|_0 \|h^i_z \bar{u}^+_j \bar{\theta}\|_0,$$

and then by (a) combined with (b). The fifth by

$$\sum_j \|T^\frac{1}{2}_j(h^j_z \zeta_j u^+_j)\|_0^2 \leq \int \partial_b \bar{\partial}_b r(T^\frac{1}{2}_j u^+, T^\frac{1}{2}_j u^+) dV + \|u^+\|_0^2 \leq Q^b(u^+, u^+).$$

Finally, the sixth by (b).

\[\square\]

Example 3.2. For the pseudoconvex domain with boundary defined, in a neighborhood of 0, by

$$2x_n = \sum_{j=1}^{n-1} e^{-s_j |x_j|^2} x_j^{2m_j} \quad \text{any } s_j > 0 \text{ and } m_j \geq 0,$$

we can readily verify that the hypotheses of Proposition 3.1 are satisfied. Hence, on account of Theorem 2.1, $\square_b$ and $\square$ are $C^\infty$-hypoelliptic at 0.

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