Time evolution of coupled multimode and multiresonator optomechanical systems

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Abstract. We study the time evolution of bosonic systems where multiple driven bosonic modes of light interact with multiple mechanical resonators through arbitrary, time-dependent, optomechanical-like interactions. We find the analytical expression for the full time evolution of the system and compute the expectation value of relevant quantities of interest. Among the most interesting ones, we are able to compute the first-order quantum bipartite coherence between pairs of subsystems, and the analytical expression for the mixedness induced by the nonlinear interaction in the reduced state of the mechanical oscillators. This result can be used to characterise the nonlinear nature of the system.
1. Introduction

Physical systems with many constituents are typically difficult to investigate in depth. Arguably, the most imposing challenge faced when studying such systems is the ability to obtain analytical insight into their dynamics, starting from fundamental equations of the theory that describes them. Classical and quantum many-body physics, statistical mechanics and thermodynamics have been developed to successfully tackle systems with large numbers of constituents. Approaches developed using tools and concepts from these areas of research rarely provide full analytical descriptions of the dynamics of complex systems, but are able to provide important coarse-grained information of key aspects of these systems. Regardless of the success of such approaches, a complete, analytical understanding of any system is highly desirable in order to explain existing features, and predict new ones.

In this work we study a quantum system composed of an arbitrary number of bosons, which we conveniently separate into field modes and mechanical oscillators. We assume that these two sets of bosonic modes interact through a nonlinear Hamiltonian, which can be used to model different physical implementations, such as Fabry–Pérot cavities with a moving-end mirror [1], levitated nano-diamonds [2] [3], membrane-in-the-middle configurations [4] and optomechanical crystals [5] [6].

We employ techniques developed to decouple the time-evolution operator analytically [7] [8] [9], and we provide a full analytical solution to the time evolution of the system. Our results applies to an arbitrary number of interacting systems with arbitrary time-dependent couplings, and is free from approximations of any kind. We employ the decoupling of the time-evolution operator to compute analytically the expectation value of physically interesting quantities, such as the average number of field- and resonator-excitations, and we compute the coherence that is induced between different bosonic modes. Furthermore, we are able to provide a full, analytical expression for the mixedness of the reduced state of any number of resonators, therefore contributing to the understanding of the nonlinear nature of the type of interaction considered here.

Finally, we apply our results to a simple cavity optomechanical scenario, where one cavity mode interacts with an arbitrary number of operators through a standard optomechanical Hamiltonian. Our analytical results simplify and provide us with an intuition of the quantum behavior of the resonators. In particular, we find a simple expression for the mixedness of the resonators which is always nonzero if the nonlinear coupling is present.

This work is organised as follows. In section 2 we introduce the necessary tools and the core Hamiltonian. In section 3 we decouple the time-evolution operator analytically. In section 4 we compute the time evolution of quantities of interest. In section 5 we specialise to specific initial states, which are of interest for modern and future applications. In section 6 we apply our techniques to a simple example, that of many resonators interacting with a single mode of the cavity. Finally, in section 7 we discuss the conclusions and outlook of this work.

‡ An alternative attempt to study analytically the time evolution of such systems was developed in parallel [10].
2. Tools

The tools that we develop here are not tied to a particular physical system. The only assumption is that the Hamiltonian of the system corresponds to the Hamiltonian presented below, which can be used to model, for example, a cavity optomechanical system \([11]\). Therefore, we emphasize that the approach and results of this work do not depend on the specific implementation chosen. For this reason, we choose to refer to the system as an optomechanical system for simplicity of presentations, and without loss of generality.

2.1. Optomechanics

Cavity optomechanics studies the interaction of light confined and matter \([11]\). A typical implementation is that of a cavity with a semitransparent mirror-wall, from which an electromagnetic beam can enter. The beam is in resonance with at least one cavity mode, therefore being “trapped”. On the other end, a mirror-membrane acts as the second wall of the cavity, fully reflecting light. The membrane can vibrate, which affects the fundamental frequency of the mode. The action of the membrane is then modelled effectively as the coupling of the position of a harmonic oscillator to one or more cavity modes. More membranes can be included, such as semitransparent membranes placed at the antinodes of the stationary cavity modes. When one such membrane is present, the system is known to be in a “membrane-in-the-middle” configuration \([4, 11]\).

In this work, we consider an optomechanical system with an arbitrary number of cavity field modes \(\{\hat{a}_n, \hat{a}_n^\dagger\}\) and an arbitrary number of mechanical modes \(\{\hat{b}_p, \hat{b}_p^\dagger\}\), where the creation and annihilation operators satisfy the canonical commutation relations \([\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{nn'}\) and \([\hat{b}_p, \hat{b}_{p'}^\dagger] = \delta_{pp'},\) and all other vanish. We then assume that these modes interact through the generalised optomechanical Hamiltonian

\[
\hat{H}_{\text{full}} = \hat{H}_0 + \sum_p \left[ \hbar \lambda_p^{(+)} \hat{B}_p^{(+)} + \hbar \lambda_p^{(-)} \hat{B}_p^{(-)} \right] + \sum_n \left[ \hbar \xi_n^{(+)} \hat{A}_n^{(+)} + \hbar \xi_n^{(-)} \hat{A}_n^{(-)} \right]
+ \sum_{n,p} \hbar g_{np}^{(+)} \hat{a}_n^\dagger \hat{a}_n \hat{B}_p^{(+)} + \sum_{n,p} \hbar g_{np}^{(-)} \hat{a}_n^\dagger \hat{a}_n \hat{B}_p^{(-)},
\]

where define the free Hamiltonian \(\hat{H}_0\) reads \(\hat{H}_0 := \sum_n \hbar \omega_c a_n^\dagger a_n + \sum_p \hbar \omega_m b_p^\dagger b_p\). We have also defined the cavity mode frequencies \(\omega_c,\) the mechanical resonator frequencies \(\omega_m,\) the time dependent couplings \(\lambda_p^{(\pm)}(t),\) \(\xi_p^{(\pm)}(t)\) and \(g_{np}^{(\pm)}(t)\) and the Hermitian operators

\begin{align*}
\hat{A}_p^{(+)} &= \hat{a}_p^\dagger + \hat{a}_p \\
\hat{A}_p^{(-)} &= i \left[ \hat{a}_p^\dagger - \hat{a}_p \right] \\
\hat{B}_p^{(+)} &= \hat{b}_p^\dagger + \hat{b}_p \\
\hat{B}_p^{(-)} &= i \left[ \hat{b}_p^\dagger - \hat{b}_p \right].
\end{align*}

The operators defined in \([2]\) can be cast in a more conventional form by noting that they are simply proportional to the quadrature operators \(\hat{x}_{c,p}, \hat{p}_{c,p}\) of the cavity modes and \(\hat{x}_{m,p}, \hat{p}_{m,p}\) of the mechanical resonators, i.e., \(\hat{A}_p^{(+)} \propto \hat{x}_{c,p}, \hat{A}_p^{(-)} \propto \hat{p}_{c,p}, \hat{A}_p^{(+)} \propto \hat{x}_{m,p}\) and \(\hat{A}_p^{(+)} \propto \hat{p}_{m,p}\). We will retain our convention because the decomposition in terms of creation and annihilation operators is more natural to this work.
The Hamiltonian (1) can be conveniently written as $\hat{H}_{\text{full}} = \hat{H}_{\text{NL}} + \hat{H}_{\text{Dr}}$, where we have decided to introduce the following contributions

$$\hat{H}_{\text{NL}} = \hat{H}_0 + \sum_p \left[ \hbar \lambda_p^{(+)} \hat{B}_p^{(+)} + \hbar \lambda_p^{(-)} \hat{B}_p^{(-)} \right] + \sum_{n,p} \hbar g_{np}^{(+)} \hat{a}_n^{\dagger} \hat{a}_n \hat{B}_p^{(+)} + \sum_{n,p} \hbar g_{np}^{(-)} \hat{a}_n^{\dagger} \hat{a}_n \hat{B}_p^{(-)}$$

$$\hat{H}_{\text{Dr}} = \sum_p \left[ \hbar \xi_p^{(+)} \hat{A}_p^{(+)} + \hbar \xi_p^{(-)} \hat{A}_p^{(-)} \right].$$

This splitting is important in the rest of our work, as we will see later.

2.2. Separation of drive

Our choice of the separation of the Hamiltonian (1) into the contributions (3) proves extremely convenient for the study of the time evolution of the system. First of all we note that the time-evolution operator $\hat{U}_{\text{full}}(t)$ induced by $\hat{H}_{\text{full}}$ has the following expression

$$\hat{U}_{\text{full}}(t) := \exp \left[ -\frac{i}{\hbar} \int_0^t dt' \hat{H}_{\text{full}}(t') \right].$$

We then manipulate the full expression (4) to obtain the equivalent expression

$$\hat{U}_{\text{full}}(t) := \hat{U}_{\text{NL}}(t) \exp \left[ -\frac{i}{\hbar} \int_0^t dt' \hat{H}_{\text{Dr}}(t') \hat{U}_{\text{NL}}(t') \right],$$

where we have defined $\hat{U}_{\text{NL}}(t) := \exp[ -\frac{i}{\hbar} \int_0^t dt' \hat{H}_{\text{NL}}(t') ]$. This separation facilitates our work in decoupling completely the time-evolution operator (4).

We proceed by noting that we can rearrange conveniently the contributing terms to $\hat{H}_{\text{NL}}(t)$ as

$$\hat{H}_{\text{NL}}(t) := \sum_p \hat{H}_p,$$

where we have introduced the multimode Hamiltonians $\hat{H}_p$ for fixed resonator $p$ as

$$\hat{H}_p = \sum_n \hbar \omega_{c,n} \hat{a}_n^{\dagger} \hat{a}_n + \hbar \omega_{m,p} \hat{b}_p^{\dagger} \hat{b}_p + \hbar \lambda_p^{(+)} \hat{B}_p^{(+)} + \hbar \lambda_p^{(-)} \hat{B}_p^{(-)} + \sum_n \hbar g_{np}^{(+)} \hat{a}_n^{\dagger} \hat{a}_n \hat{B}_p^{(+)}$$

$$+ \sum_n \hbar g_{np}^{(-)} \hat{a}_n^{\dagger} \hat{a}_n \hat{B}_p^{(-)}.$$

Finally, all of the Hamiltonians $\hat{H}_p$ commute with each other, that is $[\hat{H}_p, \hat{H}_{p'}] = 0$ for all $p, p'$. This is a paramount property for the next step.

2.3. Tackling time evolution of bosonic systems

We are now ready to discuss the last tool necessary for our work. Given a set of $N$ bosonic modes and an arbitrary, time dependent Hamiltonian $\hat{H}(t)$, the unitary time-evolution operator reads

$$\hat{U}(t) = \exp \left[ -\frac{i}{\hbar} \int_0^t dt' \hat{H}(t') \right].$$
where $\mathcal{T}$ is the time ordering operator [8]. This expression simplifies dramatically when the Hamiltonian $\hat{H}$ is time independent, in which case one simply has $\hat{U}(t) = \exp[-\frac{i}{\hbar} \hat{H} t]$. However, we are interested in general time evolution. The solution to the formal expression (8) can be found by decoupling techniques [8], as discussed below.

We note in passing that these techniques apply to any problem where the Lie algebra of the operators appearing in the Hamiltonian is closed, or has a particular structure. This is the case for the Lie algebra of the operators that appear in quadratic Hamiltonians, where there are $N(2N+1)$ independent elements of the Lie algebra, when considering systems of $N$ bosonic modes [8]. In this case, there are $N(2N+1)$ independent quadratic hermitian operators, which we can denote $\hat{G}_n$, that can be formed by arbitrary quadratic combinations of the creation and annihilation operators only (or, equivalently, of the quadrature operators) see [8]. For example, $\hat{G}_1 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_1 \hat{a}_1^\dagger$ or $\hat{G}_8 = \hat{a}_2^\dagger \hat{a}_5^\dagger + \hat{a}_5 \hat{a}_2$, where the numbering and ordering of the generators $\hat{G}_n$ is a matter of convenience. In these cases, it is convenient to choose to apply these techniques to solve problems where the states are Gaussian because linear transformations preserve the Gaussian character of the state. This, in turn, allows one to employ the covariance matrix formalism to compute any quantity of interest analytically [12].

3. Decoupling the time evolution of the system

In this section we will employ the techniques developed in [7, 8] to decouple the time-evolution operator $\hat{U}_{NL}$ induced the Hamiltonian $\hat{H}_{NL}$ in (3). This will allow us to subsequently compute an explicit form of the time-evolution operator $\hat{U}_{full}$. More details can be found in Appendix A.

3.1. Choice of core Hamiltonian

The first step is to employ the fact that $[\hat{H}_p, \hat{H}_{p'}] = 0$ for all $p, p'$. This stimulates us to look at an individual Hamiltonian $\hat{H}_p$ and we therefore drop the label $p$ for this first part of the work. This leaves us the Hamiltonian

$$\hat{H} := \sum_n \hbar \omega_{c,n} \hat{a}_n^\dagger \hat{a}_n + \hbar \omega_m \hat{b}^\dagger \hat{b} + \hbar \lambda^{(+)} \hat{B}^{(+)} + \hbar \lambda^{(-)} \hat{B}^{(-)} + \sum_n \hbar g^{(+)}_n \hat{a}_n^\dagger \hat{a}_n \hat{B}^{(+)} + \sum_n \hbar g^{(-)}_n \hat{a}_n^\dagger \hat{a}_n \hat{B}^{(-)}.$$

(9)

where we have re-defined the operators

$$\hat{A}^{(+)} = \hat{a}^\dagger + \hat{a} \quad \hat{A}^{(-)} = i [\hat{a}^\dagger - \hat{a}]$$

$$\hat{B}^{(+)} = \hat{b}^\dagger + \hat{b} \quad \hat{B}^{(-)} = i [\hat{b}^\dagger - \hat{b}].$$

(10)

The Hamiltonian (9) is the Hamiltonian we will be studying in detail in this work. It is an extension of the standard optomechanical Hamiltonian which, in its simplest form, is obtained from (9) by setting $g^{(-)}_n(t) = \lambda^{(\pm)}(t) = 0$ and considering only one cavity mode [11]. We will then use the results for this Hamiltonian to obtain the final result for the initial Hamiltonian.
Decoupling of the Hamiltonian requires considering the (formally infinite-dimensional) Lie algebra generated by the following set of Hermitian basis operators:

\[
\begin{align*}
\hat{N}_n &:= \hat{a}_n^\dagger \hat{a}_n \\
\hat{B}^{(+)} &:= \hat{b}^\dagger + \hat{b} \\
\hat{N}_m &:= \hat{a}_m^\dagger \hat{a}_m \\
\hat{B}^{(-)} &:= \mathbb{I} (\hat{b}^\dagger - \hat{b}) \\
\hat{N}_n \hat{B}^{(+)} &:= \hat{N}_n (\hat{b}^\dagger + \hat{b}) \\
\hat{N}_n \hat{B}^{(-)} &:= \hat{N}_n \mathbb{I} (\hat{b}^\dagger - \hat{b})
\end{align*}
\]

for all \( n, m \in \mathbb{N} \).

This allows us to find the following formal solution to the decoupling problem:

\[
\hat{U}(t) := e^{-i \sum_n \hat{N}_n e^{-i \int_0^t \hat{F}_n \hat{N}_n e^{-\frac{1}{2} \sum_{nm} \hat{F}_{nm} \hat{N}_n \hat{N}_m \hat{B}^{(+)} - i \int_0^t \hat{F}_n \hat{B}^{(+)} e^{-i \sum_{nm} \hat{F}_{nm} \hat{N}_n \hat{B}^{(-)}}} \right) \times e^{-i \int_0^t \hat{F}_n \hat{B}^{(-)} e^{-i \sum_{nm} \hat{F}_{nm} \hat{N}_n \hat{B}^{(-)}}}
\]

where the time dependent, real functions \( F \) depend on the coefficients of the Hamiltonian, and are related to these coefficients through coupled, nonlinear, first-order, ordinary differential equations. Notice that any change of the ordering of the operators in will only change these differential equations and therefore the functional form of the \( F \)-functions. This does not have an effect on the expectation value of any measurable quantity.

### 3.3. Solution to core differential equations

Here we present the results of the decoupling of the Hamiltonian. The full calculations can be found in Appendix A. The techniques developed for the decoupling of time-evolution operator \( \hat{U}_{\text{NL}}(t) \) lead to the differential equations, which have an analytical solution. After some algebra we find:

\[
\begin{align*}
F_b &= \omega_m t \\
F_n &= \omega_{c,n} t - 2 \int_0^t dt' \left[ \lambda^{(+)} s(t') + \lambda^{(-)} c(t') \right] \int_0^{t'} dt'' \left[ g_n^{(+)} c(t'') - g_n^{(-)} s(t'') \right] \\
&\quad - 2 \int_0^t dt' \left[ g_m^{(+)} s(t') + g_m^{(-)} c(t') \right] \int_0^{t'} dt'' \left[ \lambda^{(+)} c(t'') - \lambda^{(-)} s(t'') \right] \\
F_{nm} &= -4 \int_0^t dt' \left[ g_{nm}^{(+)} s(t') + g_{nm}^{(-)} c(t') \right] \int_0^{t'} dt'' \left[ g_n^{(+)} c(t'') - g_n^{(-)} s(t'') \right] \\
F_+ &= \int_0^t dt' \left[ \lambda^{(+)} c(t') - \lambda^{(-)} s(t') \right] \\
F_- &= - \int_0^t dt' \left[ \lambda^{(+)} s(t') + \lambda^{(-)} c(t') \right] \\
F_n^{(+)} &= \int_0^t dt' \left[ g_n^{(+)} c(t') - g_n^{(-)} s(t') \right] \\
F_n^{(-)} &= - \int_0^t dt' \left[ g_n^{(+)} s(t') + g_n^{(-)} c(t') \right]
\end{align*}
\]

where we have defined \( s(t) := \sin(\omega_m t) \) and \( c(t) := \cos(\omega_m t) \) for convenience of presentation.
This result is remarkable. Notwithstanding the fact that the Lie algebra is infinite-dimensional, and given an explicit expression for the time dependent couplings $\lambda_{\pm}$ and $g_{n}^{(\pm)}$, it is possible to obtain the full expression of the time evolution through (13).

3.4. Full nonlinear decoupled solution without drive

We have decoupled analytically the time-evolution operator induced by the Hamiltonian $\hat{H}$. Our result has been obtaining the analytical expressions (13). We recall that the Hamiltonian $\hat{H}$ was extracted from (6) for each resonator mode $b_{p}$. We can reconstruct the decoupled time-evolution operator $\hat{U}_{NL}$ induced by $\hat{H}_{NL}$ by noting again that $[\hat{H}_{p}, \hat{H}_{p}^{\dagger}] = 0$. We therefore have

$$\hat{U}_{NL}(t) = e^{-i\sum_{n} \omega_{n,t} t + \sum_{n,p} \hat{F}_{c,n}^{(p)} b_{p}^{\dagger} b_{p} e^{-\frac{i}{2} \sum_{n,m,p} F_{nm}^{(p)} \hat{N}_{n} \hat{N}_{m} e^{-i \sum_{p} F_{n}^{(p)} \hat{B}_{n}^{(p)}}} e^{-i \sum_{n,p} F_{n}^{(p)} \hat{B}_{n}^{(p)} e^{-i \sum_{n,p} F_{n}^{(p)} \hat{B}_{n}^{(p)}},$$

(14)

where the updated functions read

$$F_{n}^{(p)} = \omega_{n,p} t$$

$$\hat{F}_{c,n}^{(p)} = -2 \int_{0}^{t} dt' \left[ \lambda_{p}^{(+)} s_{p}(t') + \lambda_{p}^{(-)} c_{p}(t') \right] \int_{0}^{t'} dt'' \left[ g_{np}^{(+)} c_{p}(t'') - g_{np}^{(-)} s_{p}(t'') \right]$$

$$- 2 \int_{0}^{t} dt' \left[ g_{np}^{(+)} s_{p}(t') + g_{np}^{(-)} c_{p}(t') \right] \int_{0}^{t'} dt'' \left[ \lambda_{p}^{(+)} c_{p}(t'') - \lambda_{p}^{(-)} s_{p}(t'') \right]$$

$$F_{nm}^{(p)} = -4 \int_{0}^{t} dt' \left[ g_{mp}^{(+)} s_{p}(t') + g_{mp}^{(-)} c_{p}(t') \right] \int_{0}^{t'} dt'' \left[ g_{np}^{(+)} c_{p}(t'') - g_{np}^{(-)} s_{p}(t'') \right]$$

$$F_{+}^{(p)} = \int_{0}^{t} dt' \left[ \lambda_{p}^{(+)} c_{p}(t') - \lambda_{p}^{(-)} s_{p}(t') \right]$$

$$F_{-}^{(p)} = - \int_{0}^{t} dt' \left[ \lambda_{p}^{(+)} s_{p}(t') + \lambda_{p}^{(-)} c_{p}(t') \right]$$

$$F_{n}^{(p,+)} = \int_{0}^{t} dt' \left[ g_{np}^{(+)} c_{p}(t') - g_{np}^{(-)} s_{p}(t') \right]$$

$$F_{n}^{(p,-)} = - \int_{0}^{t} dt' \left[ g_{np}^{(+)} s_{p}(t') + g_{np}^{(-)} c_{p}(t') \right].$$

(15)

Above we have defined $s_{p}(t) := \sin(\omega_{m,p} t)$ and $c_{p}(t) := \cos(\omega_{m,p} t)$ for convenience of presentation.

Again, we would like to stress that this result is remarkable. It is a consequence of the structure of the full algebra, which allows for the use of many “tricks” to our advantage [7] [8].

4. Time evolution of quantities of interest

The decoupling achieved above allows us to obtain great analytical control on the time evolution of the system and is one of our main results. Here we employ this control to find the time evolution of different quantities of interest. We divide our analysis into two main parts. The first, where the linear drive of the modes is switched off (i.e., $\xi_{p}^{(\pm)} = 0$). This allows us two obtain analytical results in full detail. The second,
where we have defined $F_{\rho}$ of all possible combinations of excitations $I$ where we have introduced $\sum_{\rho} b$. has the form

4.1. Time evolution of quantities of interest: no external drive

We start by considering the drive to be switched off, i.e., $\xi^{(\pm)} = 0$. This is equivalent to studying an ideal cavity setup where the field is first fed in and then trapped inside (and does not leak, since we are not considering a master equation for the system). The Hamiltonian that describes the system is then $\hat{H}_{\text{NL}}$ and the induced time-evolution operator is $\hat{U}_{\text{NL}}$. From now on, until specified otherwise, time evolution will imply the use of $\hat{U}_{\text{NL}}$.

4.1.1. Mode operators

We start by computing the time evolution of the mode operators $\hat{a}_k$ and $\hat{b}_k$, through which one can compute the time evolution of all operators of interest. Their time dependence is induced by the nonlinear Hamiltonian $\hat{H}_{\text{NL}}$ which we discuss briefly, we will see that analytical results are unfortunately beyond reach.

Their time dependence is induced by the nonlinear Hamiltonian $\hat{H}_{\text{NL}}$ and is obtained through the standard Heisenberg equation as $\hat{a}_k(t) := \hat{U}_{\text{NL}} \hat{a}_k \hat{U}_{\text{NL}}^{-1}$ and $\hat{b}_k(t) := \hat{U}_{\text{NL}} \hat{b}_k \hat{U}_{\text{NL}}^{-1}$ respectively. After some algebra we find

$$\hat{a}_k(t) = \exp \left\{ -i \sum_{p} F_{\rho}^{(p)} U_{\text{NL}} \right\} \hat{a}_k \exp \left\{ i \sum_{p} F_{\rho}^{(p)} U_{\text{NL}} \right\}$$

$$\hat{b}_k(t) = \exp \left\{ -i \sum_{p} F_{\rho}^{(k)} U_{\text{NL}} \right\} \hat{b}_k \exp \left\{ i \sum_{p} F_{\rho}^{(k)} U_{\text{NL}} \right\}$$

where we have defined $F_{\rho}(t) := F_{\rho}^{(k)}(t) + i F_{\rho}^{(k)}(t)$ and $F_{\rho}^{(k)}(t) := F_{\rho}^{(k,+)}(t) + i F_{\rho}^{(k,-)}(t)$, and the notation $F_{\rho}^{(p)} := \frac{1}{2} \left[ F_{\rho}^{(p,+)} + F_{\rho}^{(p,-)} \right]$.

4.1.2. Final reduced state of the resonators

We continue by computing the final reduced state $\hat{\rho}_m$ of the mechanical resonators. The reduced state is defined by $\hat{\rho}_m := \text{Tr}_{\text{Phot}}(\hat{\rho}_{\text{NL}}(t))$, that is, by tracing over all of the cavity modes.

In Appendix C we provide all of the detailed computations for this part. We assume that the initial state $\hat{\rho}_c(0)$ of the cavity modes is separable from the initial state $\hat{\rho}_m(0)$ of the mechanical modes. This implies that the full initial state is $\hat{\rho}_0 = \hat{\rho}_c(0) \otimes \hat{\rho}_m(0)$. It is not difficult to show that the reduced state $\hat{\rho}_m(t)$ at time $t$ has the form

$$\hat{\rho}_m(t) = \sum_{\{n_k\}} \hat{D}_{\{n_k\}} \hat{\rho}_m(0) \hat{D}^\dagger_{\{n_k\}},$$

where we have introduced $\sum_{\{n_k\}} := \sum_{n_1, n_2, \ldots, n_N}$ for $N$ modes that belong to the set of all possible combinations of excitations $J$, while $\sum_{J} J_k := J_1 + J_2 + \ldots + J_N$ for
any \( k \)-dependent quantities \( J_k \). We have also introduced

\[
p(n_k) := \langle n_1, ..., n_N | \hat{\rho}_k(0) | n_1, ..., n_N \rangle
\]

\[
\hat{D}(n_k) := \exp \left[ -i \sum_p F_{m}^{(p)} \hat{b}_p^\dagger \hat{b}_p \right] \exp \left[ -i \sum_p \left( F_{+}^{(p)} + n_k F_{k}^{(p, +)} \right) \hat{b}_p \right] \times \exp \left[ -i \sum_p \left( F_{-}^{(p)} + \sum_{k \in \mathbb{Z}} n_k F_{k}^{(p, -)} \right) \hat{b}_p \right].
\]

\begin{equation}
(18)
\end{equation}

Note that we have \( \text{Tr}(\hat{\rho}_m(t)) = \sum_{n_k} p(n_k) = 1 \) as expected.

\subsection{4.1.3. Mode population}

The time evolution of the modes allows us to immediately compute the operators that “count” the number of excitations, namely \( \hat{a}_k^\dagger(t) \hat{a}_k(t) \) and \( \hat{b}_k^\dagger(t) \hat{b}_k(t) \). They read

\[
\hat{a}_k^\dagger(t) \hat{a}_k(t) = \hat{\alpha}_k^\dagger \hat{\alpha}_k,
\]

\[
\hat{b}_k^\dagger(t) \hat{b}_k(t) = \hat{\beta}_k^\dagger \hat{\beta}_k - \left[ F^{(k)} - \sum_n F_n^{(k)} \hat{N}_n \right] \hat{b}_k^\dagger - \left[ F^{(k)*} - \sum_n F_n^{(k)*} \hat{N}_n \right] \hat{b}_k
\]

\[
+ |F^{(k)}|^2 - F^{(k)} \sum_n F_n^{(k)*} \hat{N}_n - F^{(k)*} \sum_n F_n^{(k)} \hat{N}_n + \sum_{nm} F_n^{(k)} F_m^{(k)*} \hat{N}_n \hat{N}_m.
\]

\begin{equation}
(19)
\end{equation}

The number operator \( \hat{a}_k^\dagger(t) \hat{a}_k(t) \) of each field mode is a conserved quantity when the drive is switched off, as can be immediately seen from the Hamiltonian \( \text{H} \). However, the number operators \( \hat{b}_k^\dagger(t) \hat{b}_k(t) \) of the resonators are not, and they depend on the nonlinear coupling through the functions \( F^{(k)} \) and \( F_n^{(k)} \).

We note here that, if the nonlinear couplings are small, i.e., they are proportional to \( \epsilon \ll 1 \), we have that \( F^{(k)} \sim \epsilon \), \( F_n^{(k)} \sim \epsilon \) and \( F_m^{(k)*} \sim \epsilon^2 \). Therefore, the last term contributing to the number operator \( \hat{b}_k^\dagger(t) \hat{b}_k(t) \) in \((19)\) is a negligible contribution in this regime. When this occurs, it is easy to check that the result is equivalent to what would be obtained through first-order perturbation theory, as expected. In this sense, the impact of our techniques can already be seen here: the operators \((19)\) are exact and contain an evident signature of the full nonlinear character of the system.

\subsection{4.1.4. First-order bipartite quantum coherence}

Given two modes \( m \) and \( n \), we call the correlation \( \langle \hat{d}_{m}^{\dagger}(t) \hat{d}_{n}(t) \rangle \) the (first-order) bipartite coherence, sometimes denoted by \( G_{mn}^{(1)} \) in optics \([13, 14]\). This definition applies in general to any state. This measure corresponds to a simple interferometric setup, where we collect the photons in the modes \( m \) and \( n \), add a phase difference between their paths, and let them interfere.

We will witness the formation of an interference pattern only if the quantity \( \langle \hat{d}_{m}^{\dagger}(t) \hat{d}_{n}(t) \rangle \) is non-zero. This quantity can be normalized by the power in each mode, and in this case we recover the standard definition of first-order amplitude correlation function \( g_{mn}^{(1)} \) from quantum optics applied to modes \( m \) and \( n \), namely

\[
g_{mn}^{(1)}(t) := \frac{\langle |\hat{d}_{m}^{\dagger}(t)\hat{d}_{n}(t)| \rangle}{\sqrt{\langle \hat{d}_{m}^{\dagger}(t)\hat{d}_{m}(t) \rangle \langle \hat{d}_{n}^{\dagger}(t)\hat{d}_{n}(t) \rangle}}.
\]

\begin{equation}
(20)
\end{equation}
It is not difficult to employ our results and compute \( g_{nm} \) both for pairs of cavity modes \( \hat{a}_k \) and \( \hat{a}_{k'} \), for pairs of resonator modes \( \hat{b}_k \) and \( \hat{b}_{k'} \), or for pairs of cavity and resonator modes \( \hat{a}_k \) and \( \hat{b}_{k'} \).

The results can be obtained analytically but are not illuminating and we omit to print the general formulas. Instead, we will give a few explicit results later on, when we look at simple applications of our general results.

#### 4.1.5. Mixedness and linear entropy

Any pure state \( \hat{\rho} \) satisfies \( \hat{\rho}^2 = \hat{\rho} \). Since \( \text{Tr}(\hat{\rho}) = 1 \) this implies that \( \text{Tr}(\hat{\rho}^2) = 1 \). A measure of the mixedness of a state is the linear entropy \( S_N \), defined as \( S_N = 1 - \text{Tr}(\hat{\rho}^2) \) and vanishes for pure states \( [15] \).

The time evolution of the system induces coherence between the photonic part and the mechanical part, which we can quantify using the linear entropy \( S_N \) applies to the reduced state \( \hat{\rho}_m(t) \) found above \( [17] \).

In our case we have

\[
S_N = 1 - \text{Tr} \left( \hat{\rho}_m^2(t) \right)
\]

\[
= 1 - \sum_{(n_k, m_k) \in \mathcal{I}} p_{(n_k)} p_{(m_k)} \text{Tr} \left( \hat{D}_{(n_k)} \hat{D}^\dagger_{(m_k)} \hat{D}_{(m_k)} \hat{D}^\dagger_{(n_k)} \hat{\rho}_m(0) \hat{\rho}_m(0) \right)
\]

\[
= 1 - \sum_{(n_k, m_k) \in \mathcal{I}} p_{(n_k)} p_{(m_k)} \text{Tr} \left( \hat{D}^\dagger_{(m_k)} \hat{D}_{(n_k)} \hat{\rho}_m(0) \hat{D}^\dagger_{(n_k)} \hat{D}_{(m_k)} \hat{\rho}_m(0) \right),
\]

where it is easy to check that

\[
\hat{D}^\dagger_{(n_k)} \hat{D}_{(m_k)} = \prod_p e^{i \theta_p} \exp \left[ -i \sum_p F_{mp} \hat{b}^\dagger_p \hat{b}_p \right] \exp \left[ \Delta_{(k \in \mathcal{I})} (\hat{b}^\dagger - \text{h.c.}) \right],
\]

with \( F_{mp} := F_k^{(n_k m_k)} + i F_k^{(n_k m_k)} \) and \( \Delta_{(k \in \mathcal{I})} := \sum_{k \in \mathcal{I}} (n_k - m_k) F_k^{(n_k m_k)} \). The exact expression of the phase \( e^{i \theta_p} \) is irrelevant since it clearly cancels out in \( [21] \).

We know that when all the nonlinear couplings vanish, i.e., \( g_{np} = 0 \), we have \( F_k^{(n_k m_k)} = 0 \). As an immediate consequence of this is that the expression \( [21] \) reduces to \( S_N(t) = 1 - \text{Tr}(\hat{\rho}_m^2(0)) = S_N(0) \). Therefore, the mixedness introduced in the reduced state \( \hat{\rho}_m \) of the resonators is a direct and only consequence of the nonlinear interaction. This analytical insight is, of course, perfectly in line with what is expected.

#### 4.2. Time evolution of the driven, multimode, multiresonator systems

We would like to extend the results obtained above to tackle the full time evolution \( \hat{U}_{\text{full}} \) induced by \( H_{\text{full}} \). Since \( [\hat{H}_p, \hat{H}_p'] = 0 \), we can obtain

\[
\hat{U}_{\text{full}}(t) = U_{\text{NL}}(t) \mathcal{T} \exp \left[ -i \sum_n \int_0^t dt' \hat{U}_{\text{NL}}^\dagger(t') \left( \xi_n \hat{a}^\dagger_n + \xi^*_n \hat{a}_n \right) \hat{U}_{\text{NL}}(t') \right].
\]

\[\S\] We note that, contrary to the case of finite-dimensional systems with dimension \( d \), where the state \( \rho \) with maximal mixedness \( S_N = 1/d \) is the diagonal state with uniform eigenvalues \( \lambda = 1/d \), i.e., \( \rho = \sum_n \lambda |n\rangle \langle n| \) for an complete orthonormal basis \( |n\rangle \), the case of infinite dimensional systems is more subtle. Clearly, there cannot exist a diagonal state with uniform eigenvalues \( \lambda = 1/d \), since \( d \) is infinite. However, one could have diagonal states with finite amount of uniform, non-zero eigenvalues \( \lambda = 1/k \), were \( k \) can be arbitrary. In this case, one would still have \( S_N = 1/k \), which can be made arbitrarily small by increasing \( k \).
Here we have defined $\xi_n := \xi_n^{(+)} + i \xi_n^{(-)}$.

Unfortunately, we cannot proceed any further with simplifications of (23). The explicit expression in the time-ordered exponential leads to a time ordered exponential of exponential operators, which cannot be treated with the tools described here. We leave it to further work to study this case in more detail.

5. Choice of initial state: coherent state of the cavity modes and thermal state of the mechanical modes

Here we apply our results to a more concrete setup. We assume that there are a limited amount of modes $k \in \mathcal{I}$ that are initially in a coherent state $|\mu_k\rangle$ with parameter $\mu_k$ and defined by $\hat{a}_k |\mu_k\rangle = \mu_k |\mu_k\rangle$, while the cavity modes $s \notin \mathcal{I}$ are each in their respective vacuum state $|0\rangle_s$.

We also assume that the mechanical modes $\hat{b}_p$ are initially in a thermal state $\hat{\rho}_m(0) = \prod_{p \notin \mathcal{I}} \sum_j \frac{\tanh^{1/r_p}(r_p)}{\cosh^{1/r_p}(r_p)} |j_p\rangle\langle j_p|$, with temperature $T$ and parameter $r_p$ defined by $\tanh(r_p) := \exp[-\frac{\hbar \omega_{m,p}}{2k_B T}]$. This is the standard initial setup in most applications, such as those with mechanical oscillators [16] and with levitated nano-objects [17]. Note that the set $\mathcal{I}$ might include any number $N$ of modes with $N \geq 1$. More importantly, note that $N_{i,p} \equiv \sinh^2(r_p)$ is the initial population of thermal mechanical phonons.

The initial state of the system $\hat{\rho}_0$ is then separable in the mode/resonator bipartition, and has the expression

$$\hat{\rho}(0) = \prod_{k \in \mathcal{I}} |\mu_k\rangle\langle \mu_k| \otimes \prod_{s \notin \mathcal{I}} |0\rangle_s \otimes \hat{\rho}_m(0).$$

(24)

We already argued that the evolution driven by the Hamiltonian $\hat{H}_{\text{full}}$ cannot be treated analytically here. For the remainder of this work, we will consider evolution through the undriven Hamiltonian $\hat{H}_{\text{NL}}$.

5.1. Final reduced state of the mechanical resonators

The final reduced state $\hat{\rho}_m(t)$ of the mechanical modes has been computed for the general case and reads [17]. In the specific case we are studying here, we have to use $p_{(n_k)} = \prod_{k \in \mathcal{I}} \frac{\mu_k^{2n_k} e^{-|\mu_k|^2}}{n_k!}$ and we will obtain the final expression. This expression is cumbersome and we avoid printing it here.

5.2. First-order bipartite coherence

In this case, we can provide some explicit formulas given that we have specified the initial state of the system. We use the expression (20) and we provide expressions for the nominator and denominator separately.

∥ The temperatures can be lowered to values that allow to reduce the number $N_{i,p}$ of initial thermal phonons to $N_{i,p} \sim 0.34$, in the case of mechanical oscillators [16]. In the case of levitated nano-objects [17], one can reach temperatures that give rise to an average population of $N_{i,p} \sim 60$. 

\|

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∥
Time evolution of coupled multimode and multiresonator optomechanical systems

To compute the first-order coherence between two resonators \( \hat{a}_k \) and \( \hat{b}_{k'} \) or a cavity mode \( \hat{a}_k \) and resonator \( \hat{b}_{k'} \), we need the following on-diagonal expressions

\[
\langle \hat{a}_k^\dagger \hat{a}_k \rangle = |\mu_k|^2
\]

\[
\langle \hat{b}_{k'}^\dagger \hat{b}_{k'} \rangle = \sinh^2 r_p + |F^{(p)}|^2 - 2 \sum_n \Re(F^{(p)} F_n^{(p)*}) |\mu_n|^2 + \sum_n |F_n^{(p)}|^2 (|\mu_n|^2 + |\mu_n|^4)
\]

\[
+ 2 \sum_{n > m} \Re(F_n^{(p)} F_m^{(p)*}) |\mu_n|^2 |\mu_m|^2.
\] (25)

and off-diagonal expressions

\[
|\langle \hat{a}_k^\dagger \hat{a}_{k'} \rangle| = \exp \left[ -2 \sum m |\mu_m|^2 \sin^2 \left( \frac{1}{2} \sum_p \left( F(k_{km}) - F(k'_{km}) + 2 (F_k^{(p,+)} - F_{k'}^{(p,+)} F_m^{(p,-)}) \right) \right) \right]
\]

\[
\times |\mu_k| |\mu_k'| e^{-\frac{1}{2} \sum_p \cosh^2(2 r_p) |F(p)|^2}
\]

\[
|\langle \hat{a}_k^\dagger \hat{b}_{k'} \rangle| = \exp \left[ -2 \sum m |\mu_m|^2 \sin^2 \left( \frac{1}{2} \sum_p \left( F(k_{km}) + 2 F_k^{(p,+)} F_m^{(p,-)} \right) \right) \right]
\]

\[
\times \left| -F_k^{(p')} \sin^2 r_{p'} - F^{(p')} \right| + \sum_n F_n^{(p')} e^{-i \sum_p \left( F(k_{kn}) + 2 F_k^{(p,+)} F_n^{(p,-)} \right) |\mu_n|^2}
\]

\[
\times |\mu_k| e^{-\frac{1}{2} \sum_p \cosh^2(2 r_p) |F(p)|^2}
\]

\[
|\langle \hat{b}_{k'}^\dagger \hat{b}_{k'} \rangle| = |F^{(p')} F^{(p')} - \sum n |F^{(p)} F^{(p')}*| n |\mu_n|^2 - \sum n |F^{(p')} F_n^{(p)}| |\mu_n|^2
\]

\[
+ \sum n |F^{(p')} F_n^{(p')}| (|\mu_n|^2 + |\mu_n|^4) + \sum_{n \neq m} |F_n^{(p)} F_m^{(p')}| |\mu_n|^2 |\mu_m|^2 \right].
\] (26)

These expressions have been computed using the quantities in Appendix B. To obtain the expressions (25) and (26) we faced no conceptual hurdles but only lengthy algebra. We do not provide all the steps of the computations here for the sake of clarity of presentation.

5.3. Mixedness of the final reduced state of the mechanical resonators

Given our chosen initial state, and the final reduced state of the mechanical resonators, we can compute the mixedness induced by the nonlinear evolution \( \hat{U}_{NL} \). This can be obtained by first computing \( \text{Tr}(\{m_k\},\{n_k\}) := \text{Tr}(\hat{D}_{\{m_k\}}^\dagger \hat{D}_{\{n_k\}} \rho_m(0) \hat{D}_{\{n_k\}}^\dagger \hat{D}_{\{m_k\}} \rho_m(0)) \). This result is obtained in Appendix D and yields the surprisingly simple and compact expression

\[
\text{Tr}(\{m_k\},\{n_k\}) = \prod_p \exp \left[ -\frac{1}{\cosh(2 r_p)} \left| \Delta(p)_{k \in \mathcal{I}} \right|^2 \right],
\] (27)

where we have introduced \( \Delta(p)_{k \in \mathcal{I}} := \sum_{k \in \mathcal{I}} (n_k - m_k) F_k^{(p)} \) for simplicity of notation.
Therefore, we have that the linear entropy for finite temperature has a simple and analytical formula, which reads

\[
S_N = 1 - \sum_{\{n_k,m_k\} \in I} \prod_{k \in I} e^{-2|\mu_k|^2} \frac{|\mu_k|^2(n_k+m_k)}{n_k!m_k!} \prod_p \frac{1}{\cosh(2r_p)} \exp \left[ -\frac{1}{\cosh(2r_p)} \left| \Delta^{(p)}_{\{k \in I\}} \right|^2 \right]. \tag{28}
\]

We can now look at the contribution when the nonlinearity is switched off, i.e., \( \left| \Delta^{(p)}_{\{k \in I\}} \right| = 0 \). This implies that (28) yields the mixedness \( S^{in}_N \) of the initial state, which simply reads

\[
S^{in}_N = 1 - \sum_{\{n_k,m_k\} \in I} \prod_{k \in I} e^{-2|\mu_k|^2} \frac{|\mu_k|^2(n_k+m_k)}{n_k!m_k!} \prod_p \frac{1}{\cosh(2r_p)} = 1 - \prod_p \frac{1}{\cosh(2r_p)}. \tag{29}
\]

This allows us to express the full mixedness (28) as

\[
S_N = S^{in}_N + \sum_{\{n_k,m_k\} \in I} \prod_{k \in I} e^{-2|\mu_k|^2} \frac{|\mu_k|^2(n_k+m_k)}{n_k!m_k!} \prod_p \frac{1}{\cosh(2r_p)} \exp \left[ -\frac{1}{\cosh(2r_p)} \left| \Delta^{(p)}_{\{k \in I\}} \right|^2 \right].
\]

This is our main result.

The zero temperature \( T = 0 \) case is simply obtained by setting \( r_p = 0 \) for all \( p \). When the temperature becomes increasingly high, it tends to inhibit the generation of mixedness, i.e., correlations between the two systems. This is in line with previous results that studied the competition between initial mixedness (due to temperature) and the coherent generation of excitations [18].

It is now clear from (30) that the last fraction in the expression is the direct and full contribution of the nonlinearity to the mixedness, since it vanishes (together with the whole expression) for vanishing nonlinear coupling.

6. Applications to optomechanics

Now that we have obtained our general techniques and results, we can apply them to simple scenarios. In particular, we focus on Hamiltonians that resemble more closely that of a standard closed optomechanical system without external drive (i.e., we still consider the evolution through the nonlinear unitary operator \( \hat{U}_{NL}(t) \)).

We will assume that \( \lambda_p^{(\pm)} = \gamma_n^{(\pm)} = 0 \). This implies also that \( \tilde{E}^{(p)}_{c,n} = \tilde{E}_n^{(\pm)} = 0 \).

6.1. Applications to optomechanics: single mode & multi-resonator cavity

We specialise to standard optomechanical scenarios with one cavity mode \( \hat{a}_k \) and an arbitrary number of mechanical resonators \( \hat{b}_p \). This means that the set \( I = \{ \hat{k} \} \), which has one element.
The first quantities we can compute are the mode operators. We find
\[ \hat{a}_k(t) = e^{-i\omega_k t - \sum_p F_k^{(p)} + \sum_n (F_{k+n}^{(p)} + 2F_k^{(p+)} F_{k-n}^{(p-)}))} e^{-i \sum_p F_k^{(p+)} \hat{B}_k^{(p)-} \hat{a}_n^{+}} \]
\[ \hat{b}(t) = e^{-i F_0} \left[ \hat{b} - \sum_n e^{g_n^{(+)}} \hat{a}_n^{+} \hat{a}_n \right] . \] (31)

We can also compute the first-order bipartite quantum coherence \( g_{1}^{(1)} \) for the field mode and an oscillator which reads
\[ g_{1}^{(1)}(t) = \frac{|F_k^{(p)}| \sinh^2 r_p + e^{-2i\phi_k} |\mu_k|^2|}{\sqrt{\sinh^2 r_p + |F_k^{(p)}|^2 |\mu_k|^2 (1 + |\mu_k|^2)}} e^{-2 |\mu_k|^2 \sin^2 \phi_k} e^{-\frac{i}{2} \sum_p \cosh(2 r_p) |F_k^{(p)}|^2} \] (32)

and for two oscillators, which gives us
\[ g_{1}^{(1)}(t) = \frac{|F_k^{(p)}| |F_k^{(p)}| |\mu_k|^2 (1 + |\mu_k|^2)}{\sqrt{\sinh^2 r_p + |F_k^{(p)}|^2 |\mu_k|^2 (1 + |\mu_k|^2)}} e^{-2 |\mu_k|^2 \sin^2 \phi_k} e^{-\frac{i}{2} \sum_p |F_k^{(p)}|^2} \] (33)

Above, we have defined the angle \( \phi_k := \frac{1}{2} \sum_p (F_{k+n}^{(p)} + 2F_k^{(p+)} F_{k-n}^{(p-)}). Note that, for zero temperature we have \( r_p = 0 \) for all \( p \), and the expressions (32) and (33) simplify and reduce to
\[ g_{1}^{(1)}(t) = \frac{|\mu_k|}{1 + |\mu_k|^2} e^{-2 |\mu_k|^2 \sin^2 \phi_k} e^{-\frac{i}{2} \sum_p |F_k^{(p)}|^2} \] (34)

and \( g_{1}^{(1)}(t) = 1 \). This means that, while the mode of light and any resonator mode are coherent with a strength that depends on the parameters of the problem, any pair of resonators is perfectly coherent at zero temperature. We also note that, in this regime, the light and resonator coherence decreases exponentially with \( |\mu_k| \), unless \( \phi_k = 0 \). In the limit \( |\mu_k| \to \infty \) we have that \( g_{1}^{(1)}(t) = \exp[-\frac{1}{2} \sum_p |F_k^{(p)}|^2] \neq 0 \) only at the times \( t_n \) such that \( \phi_k(t_n) = 0 \) and vanishes for all other times. This implies that, to be able to verify the coherence between light and a single resonator it is necessary to reduce the number of photons in the coherent state as much as possible.

For finite temperature the issue becomes more delicate. It is clear from (32) and (33) that an increase in temperature (i.e., an increase in \( r_p \) for all \( p \)) implies that it is more difficult to establish the desired coherence. Therefore, reducing the temperature to levels where the initial phononic population \( \sinh^2 r_p \) for each resonator becomes small is paramount.

We can also compute the mixedness of the reduced state of the oscillators, which employs algebraic manipulations that can be found in Appendix D.1. Finally, we show that it reads
\[ S_N = S_{N}^{(0)} + 2 e^{-2 |\mu_k|^2} \frac{\sum_{m=1}^{+\infty} I_m(2 |\mu_k|^2) \left( 1 - e^{-\sum_p \cosh(2 r_p) |F_k^{(p)}|^2} \right)}{\prod_p \cosh(2 r_p)} \] (35)

In this formula we have introduced the modified Bessel functions \( I_n(z) \). Notice that, when \( F_k^{(p)} = 0 \) we recover immediately \( S_N = S_{N}^{(0)} \) as expected.
7. Conclusion

In this work we solved the time evolution of an arbitrary number of coupled bosonic modes interacting through a time-dependent optomechanical-like Hamiltonian. Despite of the system having potentially an arbitrarily large amount of constituents, and despite of the nonlinearity present in the interaction driven by a time-dependent coupling, we were able to analytically decouple the time-evolution operator using tools developed for this purpose [8,9]. This result does not rely on any approximation and is therefore completely general: the only assumption made is that the Hamiltonian has the form (1). This occurs, in many systems, such as optomechanical cavities [11].

We were able to compute the time evolution of meaningful operators, such as the average photonic and phononic excitation, the first-order quantum bipartite coherence and, more importantly, the mixedness of the reduced state of the mechanical oscillators. Our results allows us to study the coherence induced between the subsystems due to the nonlinear interaction of light and matter. Furthermore, they allow us to clearly quantify, using the linear entropy, the increase of mixedness of the subsystem of the resonators when the nonlinearity is switched on, as a function of time, the initial photonic population, and temperature inside the cavity. Given the lack of a systematic understanding of the nonlinear nature of the interaction, and the nonlinearity induced by the coupling, this analytical insight can help shedding light onto intrinsic nonlinear aspects of quantum (opto) mechanical systems. In the end, these insights can also help in the quest of demonstrating in the laboratory the quantum nature of “macroscopic objects”, such as the mechanical resonators.

Finally, the decoupling obtained here can be applied to many situations of theoretical and practical interest. We leave it to future work to pursue such new directions.

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Appendix A. Decoupling of the Hamiltonian

Here we show how to decouple the time-evolution operator $\hat{U}$ induced by the Hamiltonian \([9]\), which we reprint here

\[
\hat{H} = \sum_n \hbar \omega_{c,n} \hat{a}_n^\dagger \hat{a}_n + \hbar \omega_m \hat{b}_m^\dagger \hat{b}_m + \hbar \lambda^{(\uparrow)} \hat{B}^{(\uparrow)} + \hbar \lambda^{(\downarrow)} \hat{B}^{(\downarrow)} + \sum_n \hbar g^{(\uparrow)}_n \hat{a}_n^\dagger \hat{a}_n \hat{B}^{(\uparrow)} + \sum_n \hbar g^{(\downarrow)}_n \hat{a}_n^\dagger \hat{a}_n \hat{B}^{(\downarrow)}. \tag{A.1}
\]

The procedure to be followed has been developed in the literature \([3]\). All details can be found there. The time-evolution operator is defined by $\hat{U} := T \exp[-i \int_0^t dt' \hat{H}(t')]$. As discussed in the main text, we make the decoupling ansatz

\[
\hat{U}(t) = e^{-i \sum_n F_n \hat{N}_n - i \sum_n \frac{\hbar}{2} \sum_{nm} F_{nm} \hat{N}_n \hat{N}_m} e^{-i \sum_n F_n^{(\uparrow)} \hat{N}_n \hat{B}^{(\uparrow)} + i \sum_n F_n^{(\downarrow)} \hat{N}_n \hat{B}^{(\downarrow)}} \times e^{-i F_- \hat{B}^{(\downarrow)} - i \sum_n F_n^{(\downarrow)} \hat{N}_n \hat{B}^{(\downarrow)}}. \tag{A.2}
\]

We now take the time derivative on both sides of the expression \([A.2]\) and then multiply on the right by $\hat{U}^\dagger(t)$ to obtain

\[
\frac{1}{\hbar} \dot{\hat{H}} = \sum_n \dot{F}_n \hat{N}_n + \dot{F}_b e^{-i \sum_n F_n \hat{N}_n} e^{i \sum_n F_n \hat{N}_n}
+ \sum_{nm} \dot{F}_{nm} e^{-i \sum_n F_n \hat{N}_n} e^{i \sum_n F_n \hat{N}_n} + \ldots \tag{A.3}
\]

We then use similarity relations of the form

\[
e^{i x (\hat{b}_m^\dagger + \hat{b}_m)} \hat{b}_m^\dagger \hat{b}_m e^{-ix (\hat{b}_m^\dagger + \hat{b}_m)} = \hat{b}_m^\dagger \hat{b}_m - i (\hat{b}_m^\dagger - \hat{b}_m) x + x^2 \mathbb{1}; \quad e^{x (\hat{b}_m^\dagger - \hat{b}_m)} \hat{b}_m^\dagger \hat{b}_m e^{-x (\hat{b}_m^\dagger - \hat{b}_m)} = \hat{b}_m^\dagger \hat{b}_m - (\hat{b}_m^\dagger + \hat{b}_m) x + x^2 \mathbb{1};
\]

\[
e^{i x (\hat{b}_m^\dagger - \hat{b}_m)} i (\hat{b}_m^\dagger - \hat{b}_m) e^{-ix (\hat{b}_m^\dagger + \hat{b}_m)} = i (\hat{b}_m^\dagger - \hat{b}_m) - 2 x \mathbb{1}; \quad e^{x (\hat{b}_m^\dagger - \hat{b}_m)} (\hat{b}_m^\dagger + \hat{b}_m) e^{-x (\hat{b}_m^\dagger - \hat{b}_m)} = (\hat{b}_m^\dagger + \hat{b}_m) - 2 x \mathbb{1}. \tag{A.4}
\]

We obtain an explicit expression for \([A.3]\) and, equating coefficients on both sides we obtain the differential equations for the time dependent $F$-functions in terms of the coefficients of the Hamiltonian, which read

\[
\omega_m = \dot{F}_b
\]

\[
\omega_{c,n} = \dot{F}_n - 2 \dot{F}_- F_n^{(\uparrow)} - 2 \dot{F}_+ F_n^{(\downarrow)}
\]

\[
0 = \frac{1}{2} \dot{F}_{nm} - 2 F_n^{(\uparrow)} \dot{F}_m^{(\downarrow)}
\]

\[
\lambda^{(\uparrow)} = \dot{F}_+ \cos(\omega_b t) - \dot{F}_- \sin(\omega_b t)
\]

\[
\lambda^{(\downarrow)} = -\dot{F}_+ \sin(\omega_b t) - \dot{F}_- \cos(\omega_b t)
\]

\[
g_n^{(\uparrow)} = \dot{F}_n^{(\uparrow)} \cos(\omega_b t) - \dot{F}_n^{(\downarrow)} \sin(\omega_b t)
\]

\[
g_n^{(\downarrow)} = -\dot{F}_n^{(\uparrow)} \sin(\omega_b t) - \dot{F}_n^{(\downarrow)} \cos(\omega_b t). \tag{A.5}
\]

Note that the similarity relations \([A.4]\) are not exhaustive nor complete. They are presented to give an intuitive understanding of the type of relations needed to simplify the expression \([A.3]\). A complete list can be found in \([9]\).
Appendix B. Some useful expressions

In this section we provide some useful expression used in this work.

Let us start by the first, defined as $I_{\alpha} := \langle \mu | \exp[-i \alpha \hat{a}^\dagger \hat{a}] | \mu \rangle$, where $\hat{a}|\mu \rangle = \alpha |\mu \rangle$. We have

$$I_{\alpha} = \langle \mu | \exp[-i \alpha \hat{a}^\dagger \hat{a}] | \mu \rangle = \sum_{p,q=0}^{+\infty} e^{-|\mu|^2} \frac{(\mu^*)^p \mu^q}{\sqrt{p!} \sqrt{q!}} \langle p| e^{-i \alpha \hat{a}^\dagger \hat{a}} | q \rangle = \sum_{p=0}^{+\infty} e^{-|\mu|^2} \frac{(|\mu|^2)^p}{p!} e^{-i \alpha p}$$

$$I_{\alpha} = e^{-|\mu|^2} (1-e^{-i \alpha}) \tag{B.1}$$

Notice that the expression $I_{\alpha}^{(n)} := \langle \mu | (\hat{a}^\dagger \hat{a})^n \exp[-i \alpha \hat{a}^\dagger \hat{a}] | \mu \rangle$ can be computed as $I_{\alpha}^{(n)} = i^n \frac{d^n}{d\alpha^n} I_{\alpha}$. Therefore

$$I_{\alpha}^{(1)} = |\mu|^2 e^{-i \alpha} e^{-|\mu|^2} (1-e^{-i \alpha}) \tag{B.2}$$

We continue by computing the expression $J_{\alpha} := \sum_{n=0}^{+\infty} T^{2n} / C^2 \langle n | \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}] | n \rangle$, where $T := \tanh r$ and $C := \cosh r$. We have

$$J_{\alpha} = \sum_{n=0}^{+\infty} \frac{T^{2n}}{C^2} \langle n | e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} | n \rangle$$

$$= e^{\frac{1}{2} |\alpha|^2} \sum_{n=0}^{+\infty} \frac{T^{2n}}{C^2} \langle n | e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} | n \rangle$$

$$= e^{\frac{1}{2} |\alpha|^2} \sum_{n,p,q=0}^{+\infty} \frac{T^{2n} (-\alpha^*)^p \alpha^q}{p! q!} \langle n | \hat{a}^p \hat{a}^\dagger \hat{a}^q | n \rangle$$

$$= e^{\frac{1}{2} |\alpha|^2} \sum_{n,p,q=0}^{+\infty} \frac{T^{2n} (-\alpha^*)^p \alpha^q}{p! q!} \sqrt{\frac{(p+n)!}{n!}} \sqrt{\frac{(q+n)!}{n!}} \langle n + p | n + q \rangle$$

$$= e^{\frac{1}{2} |\alpha|^2} \sum_{n,p=0}^{+\infty} \frac{T^{2n} (-|\alpha|^2)^p (p+n)!}{p! n!}$$

$$= e^{\frac{1}{2} |\alpha|^2} \sum_{n=0}^{+\infty} \frac{T^{2n} (-|\alpha|^2)^p}{p!} F_1(n+1, 1, -|\alpha|^2) = e^{\frac{1}{2} |\alpha|^2} e^{-|\alpha|^2} \sum_{n=0}^{+\infty} \frac{T^{2n}}{C^2} F_1(-n, 1, |\alpha|^2)$$

$$= e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{+\infty} \frac{T^{2n}}{C^2} L_n(|\alpha|^2)$$

$$= e^{\frac{1}{2} |\alpha|^2} \frac{1}{C^2 (1-T^2)} e^{-\frac{2|\alpha|^2}{1-T^2}} \tag{B.3}$$

which gives us the final, simple and compact result

$$J_{\alpha} = e^{-\frac{1}{2} \cosh(2r)} |\alpha|^2 \tag{B.4}$$
In the above computations we have introduced the Confluent hypergeometric function \( {}_1F_1(a; b; z) \) and the Laguerre polynomial \( L_n(z) \). We have used the fundamental property \( {}_1F_1(a; b; z) = \exp[z] \sum_{i=0}^\infty (i!)^b \frac{(a)_i}{(b)_i} \frac{z^i}{i!} \) and the relation \( L_n(z) = {}_1F_1(-n, 1; z) \).

We have used the generating function for the Laguerre polynomial to be able to go to the last line of \([B.3]\) from the second-to-last one.

We are also interested in \( \tilde{J}_\alpha := \sum_{n=0}^{\infty} \frac{T^{2n}}{C^2} \langle n | \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}] | n \rangle \). We can proceed as above and find

\[
\tilde{J}_\alpha = \sum_{n=0}^{\infty} \frac{T^{2n}}{C^2} \langle n | \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}] | n \rangle
= e^{\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \sqrt{n} \frac{T^{2n}}{C^2} \langle n | e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} | n-1 \rangle
= e^{\frac{1}{2} |\alpha|^2} \sum_{p=0}^{\infty} \sqrt{n} \frac{T^{2n}}{C^2} \frac{(-\alpha)^p \alpha^q}{p! q!} \langle n | \hat{a}^p \hat{a}^q | n-1 \rangle
= e^{\frac{1}{2} |\alpha|^2} \sum_{p=0}^{\infty} \sqrt{n} \frac{T^{2n}}{C^2} \frac{(-\alpha)^p \alpha^q}{p! q!} \frac{(p+n)!}{n!} \frac{(q+n-1)!}{(n-1)!} \langle n+p | n+q-1 \rangle
= \alpha e^{\frac{1}{2} |\alpha|^2} \sum_{p=0}^{\infty} \sqrt{n} \frac{T^{2n}}{C^2} \frac{(-|\alpha|^2)^p}{p! q!} \frac{(p+n)!}{n! (n-1)!}
= \alpha e^{\frac{1}{2} |\alpha|^2} \sum_{p=0}^{\infty} \frac{n T^{2n}}{C^2} {}_1F_1(n+1, 2, -|\alpha|^2)
= \alpha e^{\frac{1}{2} |\alpha|^2} \sum_{m=0}^{\infty} \frac{T^{2m}}{C^2} {}_1F_1(-m, 2, |\alpha|^2)
= \alpha e^{\frac{1}{2} |\alpha|^2} \sum_{m=0}^{\infty} \frac{T^{2m}}{C^2} L_m^{(1)}(|\alpha|^2)
= \alpha e^{\frac{1}{2} |\alpha|^2} \sum_{m=0}^{\infty} \frac{T^{2m}}{C^2} \frac{1}{(1 - T^2)^2} e^{-\frac{x^2}{1 - T^2}} |\alpha|^2
\]

which gives us the final, simple and compact result

\[
\tilde{J}_\alpha = \alpha \sinh^2 r e^{-\frac{1}{2} \cosh(2 r) |\alpha|^2}.
\]

Note that here we have introduced the generalised Laguerre polynomials \( L_q^{(a)}(z) \). Here as well, we have used the generating function for the generalised Laguerre polynomial to go to the last line of \([B.5]\) from the second-to-last one.

Finally, we want to compute the following \( \tilde{L}_\alpha := \sum_{l, l' = 0}^{+\infty} \frac{T^{2(l+l')}}{C^4} \langle l | \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}] | l' \rangle \).
We once more proceed using similar techniques as above, skip some passages and find

\[
\hat{L}_\alpha = \sum_{l,l'=0}^{+\infty} \frac{T^{2(l+l')}}{C^4} |\langle l | \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}] |l' \rangle|^2
\]

\[
= 2 e^{\alpha^2} \sum_{l=0}^{+\infty} \frac{T^{4q+2} (l+q)!}{C^4} |\alpha|^{2q} \left| 1_{F_1} \left( l + q + 1, q + 1; -|\alpha|^2 \right) \right|^2
\]

\[
+ e^{-\alpha^2} \sum_{l=0}^{+\infty} \frac{T^{4l}}{C^4} \left| 1_{F_1} \left( l + 1, 1; -|\alpha|^2 \right) \right|^2
\]

\[
= 2 e^{\alpha^2} \sum_{l=0}^{+\infty} \frac{T^{4l+2q} (l+q)!}{C^4} |\alpha|^{2q} \left| 1_{F_1} \left( l + q + 1, q + 1; -|\alpha|^2 \right) \right|^2
\]

\[
+ e^{-\alpha^2} \sum_{l=0}^{+\infty} \frac{T^{4l}}{C^4} \left| 1_{F_1} \left( l + 1, 1; -|\alpha|^2 \right) \right|^2
\]

\[
= 2 e^{\alpha^2} \sum_{l=0}^{+\infty} \frac{T^{4l+2q} l!}{C^4} (l+q)! |\alpha|^{2q} \left| L_l^{(q)} \left( |\alpha|^2 \right) \right|^2 + e^{-\alpha^2} \sum_{l=0}^{+\infty} \frac{T^{4l}}{C^4} \left| L_l \left( |\alpha|^2 \right) \right|^2 ,
\]

(B.7)

where \(1_{F_1}(a, b; x)\) is the Confluent hypergeometric function that satisfies \(1_{F_1}(a, b; x) = \exp[x] 1_{F_1}(b - a, b; -x)\), and the functions \(L_l(x)\) and \(L_l^{(q)}(x)\) are the Laguerre and generalised Laguerre polynomials respectively.

We now use the Hardy-Hille formula for Laguerre polynomials and obtain

\[
\hat{L}_\alpha = \exp \left[ -\frac{T^2}{1 - T^4} |\alpha|^2 \right] \left[ 2 \sum_{q=1}^{+\infty} I_q \left( 2 \frac{T^2}{1 - T^4} |\alpha|^2 \right) + I_0 \left( 2 \frac{T^2}{1 - T^4} |\alpha|^2 \right) \right].
\]

(B.8)

Finally, we use the fundamental Jacobi-Anger expansion, which can be recast as the identity \(\exp[z \cos \theta] = I_0(z) + 2 \sum_{n=1}^{+\infty} I_n(z) \cos(n \theta)\) for modified Bessel functions, and note that in our case we have \(z = 2 \frac{T^2}{1 - T^4} |\alpha|^2\) and \(\theta = 0\). Therefore

\[
\hat{L}_\alpha = \frac{\exp \left[ -\frac{T^2}{\cosh(2 r)} |\alpha|^2 \right]}{\cosh(2 r)}.
\]

(B.9)

Appendix C. Reduced state of the resonators

We want to compute the final reduced state \(\hat{\rho}_m\) of the mechanical resonators. The reduced state is defined by \(\hat{\rho}_m := \text{Tr}_{\text{phot}}(\hat{\rho}_{NL}(t))\). We assume that the initial state is
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We have

\[ \dot{\rho}_m(t) = \sum_{n_1, \ldots, n_N} \langle n_1, \ldots, n_N | \hat{U}_{NL} \hat{\rho}_0 \hat{U}_{NL}^\dagger | n_1, \ldots, n_N \rangle \]

\[ = \sum_{n_1, \ldots, n_N} \langle n_1, \ldots, n_N | \hat{\rho}_c(0) | n_1, \ldots, n_N \rangle \hat{D}_{\{n_k\}} \hat{\rho}_m(0) \hat{D}_{\{n_k\}}^\dagger \]

\[ = \sum_{\{n_k\} \in I} p_{\{n_k\}} \hat{D}_{\{n_k\}} \hat{\rho}_m(0) \hat{D}_{\{n_k\}}^\dagger, \]  

(C.1)

where we have introduced \( \sum_{\{n_k\} \in I} := \sum_{n_1, n_2, \ldots, n_N} \) for \( N \) modes that belong to the set of all possible combinations of excitations \( I \), while \( \sum_{k \in I} P_k := P_1 + P_2 + \ldots + P_N \) for any \( k \)-dependent quantities \( P_k \). We have also introduced

\[ |n_1, \ldots, n_N\rangle := |n_1 \rangle \otimes \ldots \otimes |n_N\rangle \]

\[ p_{\{n_k\}} := \langle n_1, \ldots, n_N | \hat{\rho}_c(0) | n_1, \ldots, n_N \rangle \]

\[ \hat{D}_{\{n_k\}} := \prod_p \exp \left[ -i \sum_p F_m^{(p)} \hat{b}_p^\dagger \hat{b}_p \right] \exp \left[ -i \sum_p \left( F_+^{(p)} + \sum_{k \in I} n_k F_k^{(p,+)} \right) \hat{b}_p^\dagger \right] \]

\[ \times \exp \left[ -i \sum_p \left( F_-^{(p)} + \sum_{k \in I} n_k F_k^{(p,-)} \right) \hat{b}_p \right], \]

\[ = \prod_p e^{i\theta_p} \exp \left[ -i \sum_p F_m^{(p)} \hat{b}_p^\dagger \hat{b}_p \right] \exp \left[ \left( F_+^{(p)} + \sum_{k \in I} n_k F_k^{(p,+)} \right) \hat{b}_p^\dagger - \text{h.c.} \right]. \]  

(C.2)

Finally, we have also introduced \( F^{(p)} := F_+^{(p)} + i F_-^{(p)} \) and \( F_k^{(p)} := F_k^{(p,+)} + i F_k^{(p,-)} \). The phases \( e^{i\theta_p} \) are irrelevant since they cancel out in the expression (17). Note that we have \( \text{Tr}(\hat{\rho}_m(t)) = \sum_{\{n_k\} \in I} p_{\{n_k\}} = 1 \) as expected.

Appendix D. Computing mixedness for initial coherent/thermal states of field modes/resonators

We want to compute an expression for \( \text{Tr}_{\{m_k\},\{n_k\}} \) which is defined as

\[ \text{Tr}_{\{m_k\},\{n_k\}} := \text{Tr} \left( \hat{D}_{\{m_k\}}^\dagger \hat{D}_{\{n_k\}} \hat{\rho}_m(0) \hat{D}_{\{n_k\}}^\dagger \hat{D}_{\{m_k\}} \hat{\rho}_m(0) \right), \]  

(D.1)

where

\[ \hat{D}_{\{m_k\}}^\dagger \hat{D}_{\{n_k\}} = \prod_p e^{i\theta_p} \exp \left[ -i \sum_p F_m^{(p)} \hat{b}_p^\dagger \hat{b}_p \right] \exp \left[ \Delta_{\{k \in I\}}^{(p)} \hat{b}_p^\dagger - \text{h.c.} \right] \]  

(D.2)

and \( \Delta_{\{k \in I\}}^{(p)} := \sum_{k \in I} (n_k - m_k) F_k^{(p)} \) for simplicity of presentation. Again, the phases \( e^{i\theta_p} \) are irrelevant.

Our goal can be easily reached using the useful result (B.19). We first note that we have to consider many \( \hat{L}_{\Delta_{\{k \in I\}}^{(p)}} \) and \( r_p \). We then note the important expression

\[ \text{Tr}_{\{m_k\},\{n_k\}} := \prod_p \hat{L}_{\Delta_{\{k \in I\}}^{(p)}}, \]  

(D.3)
This easily gives us
\[
\mathbf{Tr}\{m_k,\{n_k\}\} = \prod_p \exp\left(\frac{1}{\cosh(2r_p)}|\Delta^{(p)}_{\{k\in I\}}|^2\right). \tag{D.4}
\]

**Appendix D.1. Computing mixedness for initial coherent/thermal states of field modes/resonators**

Here we need to compute the mixedness \(\mathbf{\Lambda}_{\alpha}\). This requires us to compute the simpler contribution
\[
\Lambda_{\alpha} = e^{-2|\mu|^2} + \infty \sum_{n,m} \frac{|\mu|^{2(n+m)}}{n! m!} e^{-\alpha(n-m)^2}. \tag{D.5}
\]

The calculations follow here
\[
\Lambda_{\alpha} = e^{-2|\mu|^2} + \infty \sum_{n,m} \frac{|\mu|^{2(n+m)}}{n! m!} e^{-\alpha(n-m)^2}
\]
\[
= e^{-2|\mu|^2} \left[ 2 \sum_{n=m}^{+\infty} \frac{|\mu|^{2(n+m)}}{n! m!} e^{-\alpha(n-m)^2} + \sum_{n=m}^{+\infty} \frac{|\mu|^{4n}}{n! n!} \right]
\]
\[
= e^{-2|\mu|^2} \left[ 2 \sum_{n=0}^{+\infty} \frac{|\mu|^{4n+2d}}{n! (n+d)!} e^{-\alpha d^2} + \sum_{n=0}^{+\infty} \frac{|\mu|^{4n}}{n! n!} \right]
\]
\[
= e^{-2|\mu|^2} \left[ 2 \sum_{d=1}^{+\infty} I_d(2|\mu|^2) e^{-\alpha d^2} + I_0(2|\mu|^2) \right]. \tag{D.6}
\]

Using the Jacobi-Anger expansion for the modified Bessel functions we obtain
\[
\Lambda_{\alpha} = 1 - 2 e^{-2|\mu|^2} \sum_{d=1}^{+\infty} I_d(2|\mu|^2) \left( 1 - e^{-\alpha d^2} \right). \tag{D.7}
\]