Harnack inequalities for Hunt processes with Green function
(Preliminary version)

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Abstract

Let \((X, \mathcal{W})\) be a balayage space, \(1 \in \mathcal{W}\), or – equivalently – let \(\mathcal{W}\) be the set of excessive functions of a Hunt process on a locally compact space \(X\) with countable base such that \(\mathcal{W}\) separates points, every function in \(\mathcal{W}\) is the supremum of its continuous minorants and there exist strictly positive continuous \(u, v \in \mathcal{W}\) such that \(u/v \to 0\) at infinity. We suppose that there is a Green function \(G > 0\) for \(X\), a metric \(\rho\) on \(X\) and a decreasing function \(g: [0, \infty) \to (0, \infty]\) having the doubling property and a mild upper decay such that \(G \approx g \circ \rho\) (which is equivalent to a 3G-inequality).

Then the corresponding capacity for balls of radius \(r\) is bounded by a constant multiple of \(1/g(r)\). Assuming that reverse inequalities hold as well and that jumps of the process, when starting at neighboring points, are related in a suitable way, it is proven that positive harmonic functions satisfy scaling invariant Harnack inequalities. Provided that the Ikeda-Watanabe formula holds, sufficient conditions for this relation are given. This shows that rather general Lévy processes are covered by this approach.

Keywords: Harnack inequality; Hunt process; balayage space; Lévy process; Green function; 3G-property; equilibrium potential; capacity.

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1 Setting and main result

Our basic setting will be almost as in [7], but assuming that points are polar:

Let \(X\) be a locally compact space with countable base. Let \(\mathcal{C}(X)\) denote the set of all continuous real functions on \(X\) and let \(\mathcal{B}(X)\) be the set of all Borel measurable numerical functions on \(X\). The set of all (positive) Radon measures on \(X\) will be denoted by \(\mathcal{M}(X)\).

Moreover, let \(\mathcal{W}\) be a convex cone of positive lower semicontinuous numerical functions on \(X\) such that \(1 \in \mathcal{W}\) and \((X, \mathcal{W})\) is a balayage space (see [2], [5] or [9, Appendix]). In particular, the following holds:

\((C)\) \(\mathcal{W}\) separates the points of \(X\), for every \(w \in \mathcal{W}\),

\[ w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) : v \leq w\}, \]
and there are strictly positive \( u, v \in \mathcal{W} \cap \mathcal{C}(X) \) such that \( u/v \to 0 \) at infinity.

Then there exists a Hunt process \( \mathfrak{X} \) on \( X \) such that \( \mathcal{W} \) is the set \( E_{\mathcal{P}} \) of excessive functions for the transition semigroup \( \mathcal{P} = (P_t)_{t \geq 0} \) of \( \mathfrak{X} \) (see \[2, IV.7.6\]), that is,

\[
\mathcal{W} = \{ v \in \mathcal{B}^+(X) : \sup_{t > 0} P_t v = v \}.
\]

We note that, conversely, given any sub-Markov right-continuous semigroup \( \mathcal{P} = (P_t)_{t \geq 0} \) on \( X \) such that \( (C) \) is satisfied by its convex cone \( E_{\mathcal{P}} \) of excessive functions, \((X, E_{\mathcal{P}})\) is a balayage space, and \( \mathcal{P} \) is the transition semigroup of a Hunt process (see \[5, Corollary 2.3.8\] or \[9, Corollary A.5\]).

For every subset \( A \) of \( X \), we have reduced functions \( R_u^A \), \( u \in \mathcal{W} \), and reduced measures \( \varepsilon_x^A \), \( x \in X \), defined by

\[
R_u^A := \inf \{ v \in \mathcal{W} : v \geq u \text{ on } A \} \quad \text{and} \quad \int u \, d\varepsilon_x^A = R_u^A(x).
\]

Of course, \( R_u^A \leq u \) on \( X \) and \( R_u^A = u \) on \( A \). The greatest lower semicontinuous minorant \( \hat{R}^A_1 \) of \( R_u^A \) (which is also the greatest finely lower semicontinuous minorant of \( R_u^A \)) is contained in \( \mathcal{W} \), and \( \hat{R}^A_1 = R_u^A \) on \( A^c \) (see \[2, VI.2.3\]). If \( A \) is not thin at any of its points (see \[2, VI.4\]) for the definition), in particular, if \( A \) is open, then \( R_u^A \in \mathcal{W} \). If \( A \) is Borel measurable, then

\[
R_u^A(x) = P^x[T_A < \infty], \quad x \in X,
\]

where \( T_A(\omega) := \inf \{ t \geq 0 : X_t(\omega) \in A \} \) (see \[2, VI.3.14\]) and, for every Borel measurable set \( B \) in \( X \),

\[
\varepsilon_x^A(B) = P^x[X_{T_A} \in B; T_A < \infty].
\]

For every open set \( U \) in \( X \), let \( \mathcal{H}^+(U) \) denote the set of all functions in \( \mathcal{B}^+(X) \) which are \textit{harmonic on} \( U \) (in the sense of \[2\]), that is, such that \( h|_U \in \mathcal{C}(U) \) and

\[
(1.2) \quad H_U h(x) := \varepsilon_x^{V^c}(h) := \int h \, d\varepsilon_x^{V^c} = h(x) \quad \text{if } V \text{ is open and } x \in V \subset U.
\]

If, for example, \( A \subset X \) and \( u \in \mathcal{W} \), then, by \[2, VI.2.6\],

\[
(1.3) \quad R_u^A \in \mathcal{H}^+(X \setminus \overline{A}) \quad \text{provided } u \leq w \text{ for some } w \in \mathcal{W} \cap \mathcal{C}(X).
\]

We note that \( U \mapsto \mathcal{H}^+(U) \) has the following sheaf property: If \( U_i, \ i \in I \), are open sets in \( X \), then

\[
\bigcap_{i \in I} \mathcal{H}^+(U_i) = \mathcal{H}^+(\bigcup_{i \in I} U_i).
\]

In fact, given an open set \( U \) in \( X \), a function \( h \in \mathcal{B}^+(X) \) which is continuous on \( U \) is already contained in \( \mathcal{H}^+(U) \), if, for every \( x \in U \), there exists a fundamental system of relatively compact open neighborhoods \( V \) of \( x \) in \( U \) such that \( \varepsilon_x^{V^c}(h) = h(x) \) (see \[2, III.4.4 \text{ and III.4.5}\] or \[3, Corollary 5.2.8 \text{ and Corollary 5.2.9}\]).

Moreover, let \( \mathcal{H}_b^+(U) \) be the set of all \( h \in \mathcal{B}^+(X) \) satisfying \( (1.2) \). Then

\[
(1.4) \quad \mathcal{H}_b^+(U) = \mathcal{H}_b^+(U).
\]
Indeed, let $V$ be an open set such that $V \subset U$. By\textsuperscript{[1.3]}, $H_V 1 = R^1$ is harmonic on $V$. Moreover, for every $f \in \mathcal{B}_0^+(X)$ with compact support, the function $H_V f$ is continuous on $V$ (see\textsuperscript{[2, III.2.8]}). So, for every $f \in \mathcal{B}_0^+(X)$, both $H_V f$ and $H_V (\|f\| - f)$ are lower semicontinuous on $V$, and hence (due to the continuity of the sum) both are continuous on $V$.

Assuming that we have a metric $\rho$ for $X$ and a Green function $G$ on $X$ such that

(i) $G \approx g \circ \rho$, where $g$ is decreasing with doubling property and weak upper decay,

(ii) for balls $B(x, r) := \{y \in X : \rho(y, x) < r\}$, the corresponding capacity satisfies

$\text{cap } B(x, r) \approx g(r)^{-1},$

(iii) there are constants $c, M \in (1, \infty)$ with

$\varepsilon_x^{B(x, r)c} \leq c \varepsilon_y^{B(x, Mr)c} \quad \text{on } B(x, Mr)^c, \quad y \in B(x, r)$

(see the Assumptions\textsuperscript{[2.2, 4.2] and 5.1}), our main result is the following.

**THEOREM 1.1.** (1) For every open set $U$ in $X$, $\tilde{H}^+(U) = H^+(U)$.

(2) Scaling invariant Harnack inequalities: There exist constants $K, M \in (1, \infty)$ such that the following holds: For all $x_0 \in X$, $R > 0$ such that $\overline{B}(x_0, Mr)$ is a compact proper subset of $X$, and all $h \in H^+(B(x_0, Mr))$,

$\sup h(B(x_0, R)) \leq K \inf h(B(x_0, R)).$

2 Green function and capacity

Before writing down assumptions on a Green function let us note the following (for the short proof and additional remarks see\textsuperscript{[7]}).

**PROPOSITION 2.1.** Let $G : X \times X \to (0, \infty]$ such that $G = \infty$ on the diagonal and $G < \infty$ outside the diagonal. Then the following properties are equivalent:

(G1) $G$ has the triangle property

$G(x, z) \wedge G(y, z) \leq CG(x, y), \quad x, y, z \in X.$

(G2) There exist a metric $\rho$ for $X$, $\gamma > 0$, and $c > 0$ such that

$c^{-1} \rho^{-\gamma} \leq G \leq c \rho^{-\gamma}.$

(G3) There exist a metric $\rho$ for $X$, a continuous decreasing numerical function $g > 0$ on $[0, \infty)$, and $c \geq 1$ such that

$c^{-1} g \circ \rho \leq G \leq c g \circ \rho,$

the function $g$ has the doubling property, that is, there exists $c_D > 1$ such that

$g(r) \leq c_D g(2r), \quad r > 0,$

and there exist $M_0 > 1$ and $\delta_0 \in (0, 1)$ such that

$g(M_0 r) \leq \delta_0 g(r), \quad r > 0.$
By definition, a potential on $X$ is a function $p \in \mathcal{W}$ such that, for every relatively compact open set $U$ in $X$, the function $x \mapsto \varepsilon_x^U (p)$ is continuous and real on $U$ (and hence harmonic on $U$) and
\[
\inf \{ R_p^X \setminus U : \text{$U$ relatively compact open in $X$} \} = 0.
\]
By [5, Proposition 4.2.10], a function $p \in \mathcal{W} \cap C(X)$ is a potential if and only if there exists a strictly positive $q \in \mathcal{W} \cap C(X)$ such that $p/q$ vanishes at infinity. Let $P(X)$ denote the set of all continuous real potentials on $X$.

From now on we assume the following:

**ASSUMPTION 2.2.** There exists a Borel measurable function $G : X \times X \to (0, \infty]$ such that $G = \infty$ on the diagonal, $G < \infty$ off the diagonal, and the following holds:

(i) For every $y \in X$, $G(\cdot, y)$ is a potential which is harmonic on $X \setminus \{y\}$.

(ii) For every potential $p$ on $X$, there exists a measure $\mu$ on $X$ such that
\[
(2.5) \quad p = G\mu := \int G(\cdot, y) \, d\mu(y).
\]

(iii) $(G3)$ holds.

**REMARKS 2.3.**

1. Having (i), each of the following properties implies (ii).

   - $G$ is lower semicontinuous on $X \times X$, continuous outside the diagonal, the potential kernel $V_0 := \int_0^\infty P_t \, dt$ of $X$ is proper, and there is a measure $\mu$ on $X$ such that $V_0 f := \int G(\cdot, y) \, d\mu(y)$ (see [12] and [2, III.6.6]).

   - $G$ is locally bounded off the diagonal, each function $G(x, \cdot)$ is lower semicontinuous on $X$ and continuous on $X \setminus \{x\}$, and there exists a measure $\nu$ on $X$ such that $G\nu \in \mathcal{C}(X)$ and $\nu(U) > 0$, for every finely open $U \neq \emptyset$ (the latter holds, for example, if $V_0(x, \cdot) \ll \nu, x \in X$). See [8, Theorem 4.1].

2. The measure in $(2.5)$ is uniquely determined and, given any measure $\mu$ on $X$ such that $p := G\mu$ is a potential, the complement of the support of $\mu$ is the largest open set, where $p$ is harmonic (see, for example, [8, Proposition 5.2 and Lemma 2.1]).

3. For the special case $X = \mathbb{R}^d$ with $\rho(x, y) = |x - y|$ and isotropic unimodular Green function, covering rather general Lévy processes, see [9, Section 6] and [6].

Suppose that $A$ is a subset of $X$ such that $\hat{R}_1^A$ is a potential. Then there is a unique measure $\mu_A$ on $X$, the equilibrium measure for $A$, such that
\[
\hat{R}_1^A = G\mu_A.
\]
If $A$ is open, then $\hat{R}_1^A = R_1^A \in \mathcal{H}(X \setminus \overline{A})$ and $\mu_A$ is supported by $\overline{A}$. For a general balayage space this may already fail if $A$ is compact (see [2, V.9.1]).

We define inner capacities for open sets $U$ in $X$ by
\[
cap^* U := \sup \{ \|\mu\| : \mu \in \mathcal{M}(X), \mu(X \setminus U) = 0, G\mu \leq 1 \}
\]
and outer capacities for arbitrary sets $A$ in $X$ by

$$\text{cap}^* A := \inf \{\text{cap}_* U : U \text{ open neighborhood of } A\}.$$ 

(2.7)

Obviously, $\text{cap}^* A = \text{cap}_* A$, if $A$ is open. If $\text{cap}_* A = \text{cap}^* A$, we might simply write $\text{cap} A$ and speak of the capacity of $A$. It is easily seen that $U \mapsto \text{cap} U$ is subadditive and $\text{cap} U_n \uparrow \text{cap} U$, for any sequence $(U_n)$ of open sets in $X$ with $U_n \uparrow U$.

The capacity of open sets $U$ is essentially determined by the total mass of equilibrium measures for relatively compact open sets in $U$ (see [7, Lemma 1.6]):

**Lemma 2.4.** For every open set $U$ in $X$,

$$\text{cap} U \geq \sup \{\|\mu_V\| : V \text{ open and } \overline{V} \text{ compact in } U\} \geq c^{-2} \text{cap} U.$$ 

**3 Hitting of sets before leaving large balls**

Let us first recall the following simple fact (see [6, 7]), where, as usual, $\tau_U := T_{U^c}$.

**Lemma 3.1.** Let $A$ be a Borel measurable set in an open set $U \subset X$ and $\gamma > 0$. If $R_1^A \leq \gamma$ on $U^c$, then

$$P^x[T_A < \tau_U] \geq R_1^A(x) - \gamma, \quad \text{for every } x \in U.$$ 

Using Lemma 2.4, this leads to a lower estimate for the probability of hitting a subset of a ball before leaving a much larger ball (see [7, Proposition 4.1]).

For later applications, let us observe that, by (2.4), for every $\delta > 0$, there exists $M > 0$ such that

$$(3.1) \quad g(Mr) \leq \delta g(r), \quad r > 0.$$ 

Indeed, it suffices to choose $k \in \mathbb{N}$ with $\delta_0^k < \delta$ and to take $M := M_0^k$.

**Proposition 3.2.** Let $\eta := (2c^3 \hat{c}_0^2)^{-1}$, $r > 0$, and $M > 3$ with $g((M - 2)r) \leq c \eta g(r)$. Then, for all $x_0 \in X$, $x \in B := B(x_0, 2r)$, and Borel measurable sets $A$ in $B(x_0, 2r)$,

$$(3.2) \quad P^x[T_A < \tau_{B(x_0, Mr)}] \geq \eta g(r) \text{cap}^*(A).$$ 

**Proof.** To prove (3.2) we may assume without loss of generality that $A$ is open (see [2, VI.3.14]). Let $V$ be an open set such that $\overline{V}$ is compact in $A$. Since $\rho(x, \cdot) \leq 4r$ on $\overline{V}$, we have

$$R_1^V(x) = \int G(x, z) d\mu_V(z) \geq c^{-1} g(4r)\|\mu_V\| \geq 2c^2 \eta g(r)\|\mu_V\|.$$ 

If $y \in X \setminus B(x_0, Mr)$, then $\rho(y, \cdot) \geq (M - 2)r$ on $\overline{V}$, and therefore

$$R_1^V(y) = \int G(y, z) d\mu_V(z) \leq c g((M - 2)r)\|\mu_V\| \leq c^2 \eta g(r)\|\mu_V\|.$$ 

So, using Lemma 3.1,

$$P^x[T_A < \tau_{B(x_0, Mr)}] \geq P^x[T_V < \tau_{B(x_0, Mr)}] \geq c^2 \eta g(r)\|\mu_V\|.$$ 

An application of Lemma 2.4 completes the proof. \qed
REMARK 3.3. Let us note that our probabilistic statements and proofs can be replaced by analytic ones using that, for all Borel measurable sets \( A, B \) in an open set \( U \),

\[
P^x[X_{T_A} \in B; T_A < \tau_U] = \varepsilon_x^{A \cup U^c}(B)
\]

(see [2, VI.2.9]) and, for all Borel measurable sets \( B \) in \( X \) and \( B \subset A \subset X \),

\[
\varepsilon_x^B = \varepsilon_x^A|_B + (\varepsilon_x^A|_B)^B.
\]

(If \( x \in B \), then (3.3) holds trivially. If \( x \notin B \) and \( p \in \mathcal{P}(X) \), then, by [2, VI.9.1],

\[
\hat{R}_p^B(x) = R_p^B(x) = \int R_p^B d\varepsilon_x^A = \int_B p \, d\varepsilon_x^A + \int_{B^c} \hat{R}_p^B d\varepsilon_x^A.
\]

4 Equilibrium potential and capacity of balls

Let us recall the following statements (see [7, Proposition 1.10 and Proposition 5.1]).

PROPOSITION 4.1. Let \( x \in X, r > 0, B := B(x, r) \) and \( C \geq 1 \).

(a) The reduced function \( R_1^B \) is a potential (in fact, bounded by a potential \( p \in \mathcal{P}(X) \)),

\[
R_1^B \leq c \frac{G(\cdot, x)}{g(r)} \leq c^2 \frac{g(\rho(\cdot, x))}{g(r)}, \quad \text{cap } B \leq c \frac{1}{g(r)},
\]

\[
R_1^B \geq c^{-1} \text{cap } B \cdot g(\rho(\cdot, x) + r).
\]

(b) If \( \text{cap } B \geq (Cg(r))^{-1} \), then

\[
R_1^B \geq (c^2 c_D C)^{-1} \frac{G(\cdot, x)}{g(r)} \quad \text{on } X \setminus B.
\]

(c) If \( R_1^B \geq (Cg(r))^{-1} G(\cdot, x) \) on \( X \setminus B \) and \( B(x, 4r) \setminus \overline{B(x, 2r)} \neq \emptyset \), then

\[
\text{cap } B \geq (Cc^2 c_D^2)^{-1} \frac{1}{g(r)}.
\]

We assume from now on the following (in addition to the Assumptions 2.2).

ASSUMPTION 4.2. There exists \( c_0 \geq 1 \) such that, for all \( x \in X \) and \( r > 0 \),

\[
\text{cap } B(x, r) \geq c_0^{-1} g(r)^{-1}.
\]

Then, by Proposition 4.1 for all \( x \in X \) and \( r > 0 \),

\[
R_1^{B(x,r)} \geq (cc_0)^{-1} \frac{g(\rho(\cdot, x) + r)}{g(r)}.
\]
EXAMPLES 4.3. 1. Assume for the moment that \((X, \mathcal{W})\) is a harmonic space, that is, \(\mathcal{X}\) is a diffusion. Moreover, suppose that \(X\) is non-compact, but balls are relatively compact. Then Assumption 4.2 is satisfied.

Indeed, let \(x \in X\), \(r > 0\), and \(B := B(x, r)\). Then \(p := G(\cdot, x) \wedge (cg(r)) \in \mathcal{P}(X)\), \(p = G(\cdot, x)\) on \(X \setminus B\), and hence \(p\) is harmonic on \(X \setminus B\). By the minimum principle (see [2, III.6.6]), \(R_1 \supseteq (cg(r))^{-1/p}\). Finally, let \(2r < s < 4r\) and \(y \in X \setminus B(x, s)\). Since we have the minimum principle for \(B(x, s)\) and \(G(\cdot, y) \in \mathcal{H}^+(X \setminus \{y\})\) is strictly positive, we see that \(\emptyset \neq \partial B(x, s) \subset B(x, 4r) \setminus B(x, 2r)\). So the claim follows by Proposition 4.1(c).

2. If \(X = \mathbb{R}^d\) and \(\rho(x, y) = |x - y|\), then Assumption 4.2 is satisfied provided there exists \(C_G \geq 1\) such that \(d \int_0^r s^{d-1} g(s) ds \leq C_G r^d g(r)\) for all \(r > 0\), since then the normalized Lebesgue measure \(\lambda_{B(x, r)}\) on \(B(x, r)\) satisfies \(G \lambda_{B(x, r)} \leq G \lambda_{B(x, r)}(x) \leq c C_G g(r)\) (see [6]). So Assumption 4.2 is satisfied for rather general isotropic unimodular Lévy processes.

5 Two crucial lemmas

Finally, we assume the following on the jumps.

ASSUMPTION 5.1. There exist \(c_j > 0\), \(M > 3\) such that, for all \(x \in X\), \(r > 0\) and \(y \in B(x, r)\),

\[
\varepsilon_x^{B(x, r)c} \leq c_j \varepsilon_y^{B(y, M r)c} \quad \text{on } B(y, M r)c.
\]

REMARKS 5.2. 1. If \(X\) is a diffusion or – equivalently – if \((X, \mathcal{W})\) is a harmonic space, then Assumption 5.1 holds trivially, since the measures in (5.1) do not charge the complement of \(B(y, M r)\).

2. If \(M' \geq M\), then \(B(x, M' r)c \subset B(x, M r)c\), and hence, by (3.3),

\[
\varepsilon_y^{B(y, M r)c} |_{B(x, M' r)c} \leq \varepsilon_y^{B(y, M' r)c}.
\]

Therefore we may replace \(M\) in (5.1) by any larger \(M'\).

3. Similarly, (3.3) implies that, for every \(y \in B(x, r)\), \(\varepsilon_y^{B(y, M r)c} |_{B(x, (M+1) r)c} \leq \varepsilon_y^{B(y, (M+1) r)c}\) and \(\varepsilon_y^{B(y, M r)c} |_{B(x, (M+1) r)c} \leq \varepsilon_y^{B(y, (M+1) r)c}\). Therefore Assumption 5.1 is equivalent to the existence of \(c_j > 0\) and \(M > 3\) such that, for all \(x \in X\), \(r > 0\) and \(y \in B(x, r)\),

\[
\varepsilon_x^{B(x, r)c} \leq c_j \varepsilon_y^{B(x, M r)c} \quad \text{on } B(x, M r)c.
\]

For a proof of Theorem 1.1 we employ essential ideas from [1]. However, not assuming the existence of a volume measure and not having any information on the expectation of hitting times, we shall rely entirely on capacities of sets.

A very similar approach has been used in [14], where the Lévy process on \(\mathbb{R}^d\), \(d \geq 3\), with characteristic exponent \(\phi(\xi) = |\xi|^2 \ln^{-1}(1 + |\xi|^2) - 1\) is considered, and \(g(r) \approx r^{d - 2} \ln(1/r)\) as \(r \to 0\).

As in Proposition 3.2 let \(\eta := (2c_j^3 r^2)^{-1}\). We fix \(M > 3\) such that (5.3) holds and \(g((M - 2) r) \leq c \eta g(r)\) for all \(r > 0\) (see (3.1) and (5.2)) . Moreover, let

\[
\beta := \frac{\eta}{6c_0}, \quad \gamma := \frac{1}{6} \wedge \frac{\beta}{c_j}, \quad \kappa := 3 \beta \gamma = \frac{\eta \gamma}{2c_0}.
\]
We choose \( m_0, m_1 \in \mathbb{N} \) such that
\[
2^{m_0} > 4c_D\beta^{-1}, \quad 2^{m_1} > M^2,
\]
and define
\[
(5.4) \quad K := \kappa^{-1}c_D^{m_0+m_1}.
\]

Now we fix \( x_0 \in X \) and \( R > 0 \) such that \( B(x_0, (M^2 + 2)R) \) is relatively compact. Since \( g \) is continuous, \( \lim_{r \to 0} g(r) = \infty \), and \( \lim_{r \to \infty} g(r) = 0 \), we may choose \( r_j > 0 \), such that
\[
(5.5) \quad \frac{g(r_j)}{g(R/M^2)} = c_D^{m_0}(1 + \beta)^{-1}, \quad j \in \mathbb{N}.
\]
Since \( g \) is decreasing, the sequence \((r_j)\) is strictly decreasing.

The following two lemmas are crucial for the proof of Theorem 1.1.

**Lemma 5.3.** The sum of all \( r_j, j \in \mathbb{N} \), is less than \( R/M^2 \).

**Proof.** Defining \( j_0 := \lfloor c_D/\beta \rfloor \), we have \( (1 + \beta)^{j_0} \geq 1 + \beta > c_D \). Hence, for \( j \in \mathbb{N} \),
\[
g(r_{j+j_0}) > c_D g(r_j) \geq g(r_j/2)
\]
showing that \( r_{j+j_0} < r_j/2 \). Therefore
\[
S := \sum_{j \in \mathbb{N}} r_j \leq j_0 \sum_{k=0}^{\infty} 2^{-k} r_1 \leq 2c_D \beta^{-1} r_1 < 2^{m_0-1} r_1.
\]
So \( g(S) \geq g(2^{m_0-1} r_1) \geq c_D^{1-m_0} g(r_1) = c_D g(R/M^2) > g(R/M^2) \) and \( S < R/M^2 \).

**Lemma 5.4.** Let \( x_0 \in X \) and \( h \in \mathcal{H}_{s+1}(B(x_0, (M^2 + 2)R)) \) such that \( h(y_0) = 1 \) for some \( y_0 \in B(x_0, R) \). If \( j \in \mathbb{N} \) and \( x \in B(x_0, 2R - r_j) \) such that
\[
h(x) > (1 + \beta)^{-1}K,
\]
then there exists \( x' \in B(x, M^2 r_j) \) such that
\[
h(x') > (1 + \beta)^j K.
\]

**Proof.** Let \( j \in \mathbb{N} \), \( r := r_j \), and \( x \in B(x_0, 2R - r_j) \) with \( h(x) > (1 + \beta)^{-1}K \). Let
\[
U_1 := B(x, r) \cap \{ h > \gamma h(x) \} \quad \text{and} \quad U_2 := B(x, r) \cap \{ h < 2\gamma h(x) \}.
\]
Then \( U_1 \) and \( U_2 \) are open sets and \( U_1 \cup U_2 = B(x, r) \subset B(x_0, 2R) \). In particular,
\[
(5.6) \quad \text{cap } B(x, r) \leq \text{cap } U_1 + \text{cap } U_2.
\]
If \( V \) is an open set with \( V \subset U_1 \), then, by Proposition 3.2,
\[
1 = h(y_0) = \varepsilon_{y_0}^{\mathcal{V} \cup B(x_0, MR)^c} (h) \geq \gamma h(x) \varepsilon_{y_0}^{\mathcal{V} \cup B(x_0, MR)^c} (\mathcal{V})
\]
\[
\geq \gamma h(x) P^{y_0}[\mathcal{V} < \tau_{B(x_0, MR)}] \geq \eta \gamma h(x) g(R) \text{cap } V.
\]
So \( \text{cap } U_1 \leq (\eta \gamma h(x) g(R))^{-1} \). By Assumption 4.2, \( \text{cap } B(x, r) \geq (c_0 g(r))^{-1} \). Since \( g(R/M^2) \leq g(2^{-m_1} R) \leq c_D^{m_1} g(R) \), we conclude, by (5.5) and (5.4), that
\[
\frac{\text{cap } U_1}{\text{cap } B(x, r)} \leq \frac{c_D^{m_0}(1 + \beta)^{-1}g(R/M^2)}{2\kappa h(x)g(R)} \leq \frac{(1 + \beta)^{-1}K}{2h(x)} < \frac{1}{2}.
\]
By \([5.6]\), we obtain that
\[
\text{cap } U_2 > (1/2) \text{ cap } B(x, r) \geq (2c_0g(r))^{-1}.
\]
We choose an open set \(W\) such that \(\overline{W} \subset U_2\), \(\text{cap } W > (2c_0g(r))^{-1}\), and define
\[
L := \overline{W}, \quad \nu := \varepsilon_x^{L \cap B(x, Mr)^c}.
\]
Then, by Proposition \(3.2\)
\[
\nu(L) = P^x[T_L < \tau_{B(x, Mr)}] \geq P^x[T_W < \tau_{B(x, Mr)}] \geq \eta g(r) \text{ cap } W > \frac{\eta}{2c_0} = 3\beta.
\]
We next claim that \(H := 1_{B(x,M^2r)^-}h\) satisfies
\[
(5.8) \quad \varepsilon_x^{B(x,M^2r)^c}(H) \leq \beta h(x).
\]
Indeed, if not, then \([5.3]\) implies that, for every \(y \in B(x, r)\),
\[
h_y(h) = \varepsilon_y^{B(x,M^2r)^c}(h) \geq c_1 \varepsilon_x^{B(x,M^2r)^c}(H) > c_1 \beta h(x) \geq \gamma h(x),
\]
contradicting the fact that \(U_1\) is a proper compact of \(B(x, r)\).

Finally, let \(a := \sup h(B(x, M^2r))\). Then
\[
h(x) = \nu(h) \leq 2\gamma h(x)\nu(L) + \int_{X \setminus B(x, Mr)} h \, d\nu,
\]
where
\[
\int_{B(x, M^2r) \setminus B(x, Mr)} h \, d\nu \leq \alpha v(B(x, M^2r) \setminus B(x, Mr)) \leq a(1 - \nu(L))
\]
and, by \((3.3)\),
\[
\int_{B(x, M^2r)^c} h \, d\nu = \nu(H) \leq \varepsilon_x^{B(x,M^2r)^c}(H) \leq \beta h(x).
\]
Therefore
\[
h(x) \leq 2\gamma h(x)\nu(L) + a(1 - \nu(L)) + \beta h(x),
\]
and
\[
(5.9) \quad a \geq \frac{1 - \beta - 2\gamma \nu(L)}{1 - \nu(L)} h(x) > (1 + \beta)h(x) > (1 + \beta)^j K
\]
completing the proof (since \(1 - 2\gamma \geq 2/3\) and \(\nu(L) > 3\beta\), we have \((1 - 2\gamma)\nu(L) > 2\beta\), hence \(1 - \beta - 2\gamma \nu(L) > 1 + \beta - \nu(L) > (1 + \beta)(1 - \nu(L))\).

Finally, we shall use the following little observation.

**Lemma 5.5.** Let \(U := B(x, R)\), \(x \in X\), \(R > 0\), such that \(\overline{U}\) is a proper compact subset of \(X\), and let \(L\) be a compact in \(U\). Then there exists a function \(h \in H^+_b(U)\) such that \(h > 0\) on \(L\).

**Proof.** Let \(y \in X \setminus \overline{U}\), \(0 < r < R\) with \(L \subset V := B(x, r)\). Then, for every \(n \in \mathbb{N}\),
\[
h_n := H_V(G(\cdot, y) \land n) \in H^+_b(U) \quad \text{and} \quad h_n \uparrow H_VG(\cdot, y) = G(\cdot, y),
\]
as \(n \to \infty\). Since \(G(\cdot, y) > 0\), there exists \(n \in \mathbb{N}\) such that \(h_n > 0\) on \(L\). \(\square\)
6 Proof of Theorem 1.1

Let us first give a complete statement of Theorem 1.1. Let $K, M$ be as in Section 3 and let $M := M^2 + 3$.

**THEOREM 6.1.** Let $(X, \mathcal{W})$ be a balayage space, $1 \in \mathcal{W}$, and suppose that the Assumptions 2.2, 4.2, 5.1 are satisfied. Then the following hold.

1. For every open set $U$ in $X$, $\mathcal{H}^+(U) = \mathcal{H}^+(U)$.
2. Scaling invariant Harnack inequalities: Let $x_0 \in X$ and $R > 0$ such that $\overline{B}(x_0, MR)$ is a proper compact subset of $X$. Then, for all $h \in \mathcal{H}^+(B(x_0, MR))$,

\[
\sup h(B(x_0, R)) \leq K \inf h(B(x_0, R)).
\]

Proof. (a) To prove (2), let $B := B(x_0, R)$, $B' := B(x_0, (M^2 + 2)R)$, and let us first consider $h \in \mathcal{H}^+(B')$ with $h(y_0) = 1$ for some point $y_0 \in B$. Then

\[
h \leq K \quad \text{on } B.
\]

Indeed, suppose that $h(x_1) > K$ for some $x_1 \in B$. Then, by Lemmas 5.3 and 5.4, there exist points $x_2, x_3, \ldots$ in $B(x_0, 2R)$ such that $h(x_j) > (1 + \beta)^{j-1}K$, $j \in \mathbb{N}$. This contradicts the boundedness of $h$.

(b) Next let $h$ be an arbitrary function in $\mathcal{H}^+(B')$. By Lemma 5.5, there exists $h_0 \in \mathcal{H}^+(B')$ such that $h_0 > 0$ on $B$. Let $y \in B$ and $\varepsilon > 0$. Applying (6.2) to $(h + \varepsilon h_0)/(h + \varepsilon h_0)(y)$ we get that $h + \varepsilon h_0 \leq K(h + \varepsilon h_0)(y)$ on $B$. Thus (6.1) holds.

(c) Finally, we consider an arbitrary $h \in \mathcal{H}^+(B(x_0, MR))$. For $n \in \mathbb{N}$,

\[
h_n := H_{B'}(h \land n) \in \mathcal{H}^+(B'),
\]

by (1.4) (the relation $H_V H_{B'} = H_{B'}$ for open $V \subset B'$ is a special case of (3.3)). By (b), $\sup h_n(B) \leq K \inf h_n(B)$. Clearly, $h_n \uparrow h$ as $n \to \infty$. Thus $h$ satisfies (6.1).

(d) To prove (1), let $U$ be an arbitrary open set in $X$, $h \in \mathcal{H}^+(U)$, and $x_0 \in U$. We choose $R > 0$ such that the closure of $W := B(x_0, MR)$ is a proper compact subset of $U$. Again, let $B := B(x_0, R)$. We define

\[
h_n := H_W(h \land n), \quad n \in \mathbb{N}.
\]

The functions $h - h_n$ are contained in $\mathcal{H}^+(W)$, and hence, by (c),

\[
h - h_n \leq K(h - h_n)(x_0) \quad \text{on } B
\]

for every $n \in \mathbb{N}$. Since, of course, $h_n \uparrow h$ as $n \to \infty$, we see that the functions $h_n|_W \in \mathcal{C}(W)$ converge to $h$ uniformly on $B$. So $h|_B \in \mathcal{C}(B)$, and we conclude that $h|_U \in \mathcal{C}(U)$ completing the proof. \qed
7 Sufficient conditions for Assumption 5.1

For relatively compact open sets $V$ in $X$, let $G_V$ denote the associated Green function on $V$, that is,

$$G_V(\cdot, y) := G(\cdot, y) - R_{G(\cdot, y)}^{V^c}.$$

(we have $G_V(z, y) = 0$, unless both $y$ and $z$ are points in $V$). If $W$ is open and $W \subset V$, then $R_{G(\cdot, y)}^{V^c} \leq R_{G(\cdot, y)}^{W^c}$, and hence $G_W \leq G_V$.

We shall need the following simple statement.

**LEMMA 7.1.** There exists $M' \geq 3$ such that, for all $y \in X$ and $r > 0$,

$$G_{B(y, M'r)}(\cdot, y) \geq \frac{1}{2} G(\cdot, y) \quad \text{on } B(y, 2r).$$

**Proof.** Let $M' \geq 3$ such that $g(M'r) \leq (2c'g(-1)g(r)$ for every $r > 0$ (see (3.1)). Let $y \in X$, $r > 0$ and $V := B(y, M'r)$. Since $G(\cdot, y) \leq cg(M'r)$ on $V^c$, we obtain that $R_{G(\cdot, y)}^{V^c} \leq cg(M'r) \leq (2ccD)^{-1}g(r)$, whereas $G(\cdot, y) \geq c^{-1}g(2r) \geq (ccD)^{-1}g(r)$ on $B(y, 2r)$. So (7.1) holds. \qed

In this section, let us assume the following estimate of Ikeda-Watanabe type, which by [11] Example 1 and Theorem 1 holds, with $C_N = 1$ and on $X \setminus \overline{B(x, r)}$, for all (temporally homogeneous) Lévy processes.

**ASSUMPTION 7.2.** There exist a measure $\lambda$ on $X$, a kernel $N$ on $X$, $M_N \geq 3$, and $C_N \geq 1$ such that, for all $x \in X$ and $r > 0$,

$$C_N^{-1} \varepsilon_{x}^{B(x, r)^c} \leq \int G_{B(x, r)}(x, z)N(z, \cdot) d\lambda(z) \leq C_N \varepsilon_{x}^{B(x, r)^c} \quad \text{on } B(x, M_N r)^c.$$

**PROPOSITION 7.3.** Suppose that there exist $C \geq 1$ and $a \geq 3$ such that, for all $x \in X$, $r > 0$ and $y \in B(x, r)$,

$$N(x, \cdot) \leq C N(y, \cdot) \quad \text{on } B(y, ar)^c,$$

and

$$\int_{B(x, r)} g(\rho(x, z)) d\lambda(z) \leq C \int_{B(y, 2r)} g(\rho(y, z)) d\lambda(z).$$

Then Assumption 7.1 is satisfied.

**Proof.** Let $M := M' \vee M_N \vee (a + 2)$, $x \in X$, $r > 0$, $y \in B(x, r)$, and let $E$ be a Borel measurable set in $B(y, Mr)^c$. If $z \in B(y, 2r)$, then $E \subset B(z, ar)^c$ and, by (7.2),

$$C^{-1}N(z, E) \leq N(y, E) \leq CN(z, E).$$

Since $B(x, r) \subset B(y, 2r)$, we obtain that

$$\varepsilon_{x}^{B(x, r)^c}(E) \leq C_N \int G_{B(x, r)}(x, z)N(z, E) d\lambda(z) \leq cCC_N N(y, E) \int_{B(x, r)} g(\rho(x, z)) d\lambda(z) \leq cC^2C_N N(y, E) \int_{B(y, 2r)} g(\rho(y, z)) d\lambda(z) \leq 2c^2C^3 C_N \int G_{B(y, Mr)}(y, z)N(z, E) d\lambda(z) \leq 2c^2C^3 C^2 N \varepsilon_{y}^{B(y, Mr)^c}(E).$$
Moreover, let us now assume that \( X = \mathbb{R}^d \), \( \rho(x, y) = |x - y| \), the measure \( \lambda \) in Assumption 7.2 is Lebesgue measure (a case, where clearly (7.1) holds) and that there exists a constant \( C_G \geq 1 \) such that, for all \( x \in X \) and \( r > 0 \),

\[
(7.5) \quad G \lambda_{B(x, r)} \leq C_G g(r).
\]

**PROPOSITION 7.4.** Suppose that there exist a measure \( \tilde{\lambda} \) on \( \mathbb{R}^d \), a function \( n: \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \) and \( C \geq 1 \), \( a \geq 2 \) such that \( N(y, \cdot) = n(y, \cdot) \tilde{\lambda} \) for every \( y \in X \) and, for all \( x \in X \), \( r > 0 \), \( y \in B(x, r) \) and \( \tilde{z} \in B(x, ar)^c \),

\[
(7.6) \quad n(x, \tilde{z}) \leq C n(y, \tilde{z}) \quad \text{provided} \quad |x - \tilde{z}| \geq |y - \tilde{z}|.
\]

Then Assumption 7.2 is satisfied.

**Proof (cf. the proof of [3, Proposition 6]).** Again, let \( M := M' \vee M_N \vee (a + 2) \), \( x \in X \), \( r > 0 \), \( y \in B(x, r) \), and let \( E \) be a Borel measurable set in \( B(y, Mr)^c \). By (7.2),

\[
\varepsilon^B(x, r)^c (E) \leq c C_N \int_{B(x, r)} \int_E g(|x - z|) n(z, \tilde{z}) d\tilde{\lambda}(\tilde{z}) d\lambda(z).
\]

Similarly, since \( B(x, r) \subset B(y, 2r) \) and \( |y - z| \leq 2r \) for every \( z \in B(x, r) \),

\[
\varepsilon^B(x, r)^c (E) \geq C^{-1}_N \int G_B(y, Mr) (y, z) N(z, E) d\lambda(z) \geq (2C_N)^{-1} \int_{B(x, r)} G(y, z) N(z, E) d\lambda(z) \geq (2cc_D C_N)^{-1} g(r) \int_{B(x, r)} \int_E n(z, \tilde{z}) d\tilde{\lambda}(\tilde{z}) d\lambda(z).
\]

Hence it will be sufficient to show that, for every \( \tilde{z} \in B(y, Mr)^c \),

\[
(7.7) \quad \int_{B(x, r)} g(|x - z|) n(z, \tilde{z}) d\lambda(z) \leq C' g(r) \int_{B(x, r)} n(z, \tilde{z}) d\lambda(z)
\]

(with some constant \( C' > 0 \)). So let \( \tilde{z} \in B(y, Mr)^c \).

Let \( B := B(x, r/2) \). Since \( g(|x - z|) \leq g(r/2) \leq c_D g(r) \) for every \( z \in B^c \),

\[
\int_{B(x, r) \setminus B} g(|x - z|) n(z, \tilde{z}) d\lambda(z) \leq c_D g(r) \int_{B(x, r)} n(z, \tilde{z}) d\lambda(z).
\]

Moreover, let

\[
x' := x + \frac{3}{4} \frac{\tilde{z} - x}{|\tilde{z} - x|} r \quad \text{and} \quad B' := B(x', r/4),
\]

so that \( B' \subset B(x, r) \setminus B \). If \( z \in B \) and \( z' \in B' \), then \( |z - \tilde{z}| \geq |z' - \tilde{z}| \), and therefore, by (7.6),

\[
n(z, \tilde{z}) \leq \frac{C}{\lambda(B')} \int_{B'} n(z', \tilde{z}) d\lambda(z') = \frac{2dC}{\lambda(B)} \int_{B'} n(z', \tilde{z}) d\lambda(z').
\]
Hence
\[ \int_B g(|x - z|)n(z, \tilde{z}) \, d\lambda(z) \leq 2^d C \left( \int_{B'} n(z', \tilde{z}) \, d\lambda(z') \right) \cdot \left( \frac{1}{\lambda(B)} \int_B g(|x - z|) \, d\lambda(z) \right), \]
where
\[ \frac{1}{\lambda(B)} \int_B g(|x - z|) \, d\lambda(z) \leq c G \lambda_B(x) \leq c C_G g(r/2) \leq c C_G g(r). \]
Thus (7.7) holds with \( C' := c D (1 + 2^d c C G) \).

If \( y \in B(x, r) \) and \( \tilde{z} \in B(x, 2r)^c \), then \(|x - \tilde{z}| \leq 2|x - y| + 2|y - \tilde{z}| - |x - \tilde{z}| < 2|y - \tilde{z}|\). Hence we have the following result.

**COROLLARY 7.5.** Suppose that there exists a measure \( \tilde{\lambda} \) on \( \mathbb{R}^d \) such that \( N(y, \cdot) = n(y, \cdot) \tilde{\lambda}, y \in X \), where \( n(x, y) \approx n_0(|x - y|) \), and that there exists \( C_0 \geq 1 \) such that
\[ n_0(s) \leq C_0 n_0(r), \quad \text{whenever } 0 < r < s < 2r. \]
Then Assumption 5.1 holds.

Thus rather general Lévy processes may serve as examples for our approach (see, for example, [3, 13, 14, 15, 16, 17]).

**REMARK 7.6.** Localizing our assumptions, in particular, considering only small radii (where the inequality (2.4) in (G3) would be assumed only for small \( M_0 r \)) our approach still yields locally scaling invariant Harnack inequalities and the identity \( \mathcal{H}^+(U) = \mathcal{H}^+(U) \) for every open set \( U \) in \( X \).

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