ON THE *H*-FUNCTION

Anatoly A. Kilbas
Department of Mathematics and Mechanics, Belarusian State University
Minsk 220050, Belarus

Megumi Saigo
Department of Applied Mathematics, Fukuoka University
Fukuoka 814-0180, Japan

Abstract

The paper is devoted to study the *H*-function defined by the Mellin-Barnes integral

\[ H_{m,n}^{p,q}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{m,n}^{p,q}(s) z^{-s} ds, \]

where the function \( \mathcal{H}_{m,n}^{p,q}(s) \) is a certain ratio of products of Gamma functions with the argument \( s \) and the contour \( \mathcal{L} \) is specially chosen. The conditions for the existence of \( H_{m,n}^{p,q}(z) \) are discussed and explicit power and power-logarithmic series expansions of \( H_{m,n}^{p,q}(z) \) near zero and infinity are given. The obtained results define more precisely the known results.

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1. Introduction

This paper deals with the *H*-function \( H_{m,n}^{p,q}(z) \) introduced by Pincherle in 1888 (see [3, Section 1.19]). Interest in this function appeared in 1961, when Fox [4] investigated such a function as symmetrical Fourier kernel. Therefore, the *H*-function is often called Fox’s *H*-function. For integers \( m, n, p, q \) such that

\[ 0 \leq m \leq q, 0 \leq n \leq p \text{ and } \alpha_i, \beta_j \in \mathbb{C} \text{ and } \alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty) \text{ (} 1 \leq i \leq p, 1 \leq j \leq q \text{)} \]

the function is defined by the Mellin-Barnes integral

\[ H_{m,n}^{p,q}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{m,n}^{p,q}(s) z^{-s} ds, \]
where

\[(1.2) \quad \mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}, \]

the contour \( \mathcal{L} \) is specially chosen and an empty product, if it occurs, is taken to be one. The theory of this function may be found in [10, 2], [1, 9, Chapter 2], [12, Chapter 1] and [11, Section 8.3]. We only indicate that most of the elementary and special functions are particular cases of the \( H \)-function \( H_{p,q}^{m,n}(z) \). In particular, if \( \alpha \)'s and \( \beta \)'s are equal to 1, the \( H \)-function (1.1) reduces to Meijer’s \( G \)-function \( G_{p,q}^{m,n}(z) \).

The conditions of the existence of the \( H \)-function can be made by inspecting the convergence of the integral (1.1), which depend on the selection of the contour \( \mathcal{L} \) and the relations between parameters \( a_i, \alpha_i \) \( (i = 1, \cdots, p) \) and \( b_j, \beta_j \) \( (j = 1, \cdots, q) \). Especially, the relations may depend on the numbers \( \Delta, \delta \) and \( \mu \) defined by

\[(1.3) \quad \Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i, \]

\[(1.4) \quad \delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^\beta_j, \]

\[(1.5) \quad \mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p - q}{2}. \]

Such a selection of \( \mathcal{L} \) and the relations on parameters are indicated in the handbook [11, Section 8.3.1], but some of the results there needs correction. In this paper we would like to give such a correction in the following cases:

(a) \( \Delta \geq 0 \) and the contour \( \mathcal{L} = \mathcal{L}_-\infty \) in (1.1) runs from \( -\infty + i\varphi_1 \) to \( -\infty + i\varphi_2 \), \( \varphi_1 < \varphi_2 \), such that the poles of \( \Gamma(b_j + \beta_j s) \) \( (j = 1, \cdots, m) \) lie on the left of \( \mathcal{L}_-\infty \) and those of \( \Gamma(1 - a_i - \alpha_i s) \) \( (i = 1, \cdots, n) \) on the right of \( \mathcal{L}_-\infty \).

(b) \( \Delta \leq 0 \) and the contour \( \mathcal{L} = \mathcal{L}_+\infty \) in (1.1) runs from \( +\infty + i\varphi_1 \) to \( +\infty + i\varphi_2 \), \( \varphi_1 < \varphi_2 \), such that the poles of \( \Gamma(b_j + \beta_j s) \) \( (j = 1, \cdots, m) \) lie on the left of \( \mathcal{L}_+\infty \) and those of \( \Gamma(1 - a_i - \alpha_i s) \) \( (i = 1, \cdots, n) \) on the right of \( \mathcal{L}_+\infty \).

Our results are based on the asymptotic behavior of the function \( \mathcal{H}_{p,q}^{m,n}(s) \) given in (1.2) at infinity. Using the behavior and following [1], we give the series representation of \( H_{p,q}^{m,n}(z) \) via residues of the integrand \( \mathcal{H}_{p,q}^{m,n}(s)z^{-s} \). In this way we simplify the proof of Theorem 1 in [1] by applying the former results to find the explicit series expansions of \( H_{p,q}^{m,n}(z) \). Such power expansions, as corollaries of the results from [1], were indicated in [12, Chapter 2.2] (see also [11, Section 8.3.1]), provided that the poles of Gamma-functions \( \Gamma(b_j + \beta_j s) \) \( (j = 1, \cdots, m) \) and \( \Gamma(1 - a_i - \alpha_i s) \) \( (i = 1, \cdots, n) \) do not coincide

\[(1.6) \quad \beta_j(a_i - 1 - k) \neq \alpha_i(b_j + l) \quad (i = 1, \cdots, n; \quad j = 1, \cdots, m; \quad k, l \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}) \]

in the cases:
(c) \( \Delta > 0 \) with \( z \neq 0 \) or \( \Delta = 0 \) with \( 0 < |z| < \delta \), and the poles of Gamma-functions \( \Gamma(b_j + \beta_j s) \) \((j = 1, \cdots, m)\) are simple:

\[
1.7 \quad \beta_j (b_i + k) \neq \beta_i (b_j + l), \quad (i \neq j, \; i, j = 1, \cdots, m; \; k, l \in \mathbb{N}_0);
\]

(d) \( \Delta < 0 \) with \( z \neq 0 \) or \( \Delta = 0 \) with \( |z| > \delta \), and the poles of Gamma-functions \( \Gamma(1 - a_i - \alpha_i s) \) \((i = 1, \cdots, n)\) are simple:

\[
1.8 \quad \alpha_j (1 - a_i + k) \neq \alpha_i (1 - a_j + l), \quad (i \neq j, \; i, j = 1, \cdots, n; \; k, l \in \mathbb{N}_0).
\]

When the poles of Gamma-functions in (c) and (d) coincide, explicit series expansions of \( H_{m,n}^{p,q}(z) \) should be more complicated power-logarithmic expansions. Such expansions in particular cases of the Meijer’s \( G \)-functions \( G_{0,p}^{p,0} \) and \( H_{p,p}^{p,0} \) and of the \( H \)-functions \( H_{p,0}^{p,0} \) and \( H_{p,p}^{p,0} \) were given in [7] and [8], respectively.

We obtain the explicit expansions of the \( H \)-function of general form \( H_{p,q}^{m,n}(z) \) under the conditions in (1.6). We show that, if the poles of the Gamma-functions \( \Gamma(b_j + \beta_j s) \) \((j = 1, \cdots, m)\) and \( \Gamma(1 - a_i - \alpha_i s) \) \((i = 1, \cdots, n)\) coincide in the cases (c) and (d), respectively, then the \( H \)-function (1.1) has power-logarithmic series expansions. In particular, we give the asymptotic expansions of \( H_{p,q}^{m,n}(z) \) near zero. We note that the obtained results will be different in the cases when either \( \Delta \geq 0 \) or \( \Delta \leq 0 \).

The paper is organized as follows. Section 2 is devoted to the conditions of the existence of the \( H \)-function (1.1) which are based on the asymptotic behavior of \( H_{p,q}^{m,n}(s) \) at infinity. Here we also give the representations of (1.1) via the residues of the integrand. The latter result is applied in Sections 3 and 4 to obtain the explicit power and power-logarithmic series expansions of \( H_{p,q}^{m,n}(z) \) and, in particular, its asymptotic estimates near zero.

2. Existence and Representations of \( H_{p,q}^{m,n}(z) \)

First we give the asymptotic estimate of Gamma function \( \Gamma(z) \), \( z = x + iy \), [3, Chapter 1] at infinity on lines parallel to the coordinate axes.

**Lemma 1.** Let \( z = x + iy \) with \( x, y \in \mathbb{R} = (-\infty, \infty) \). Then the following asymptotic estimates at infinity are valid:

\[
2.1 \quad |\Gamma(x + iy)| \sim \sqrt{2\pi} |x|^{x-1/2} e^{-x-\pi(1-\text{sign } x)y/2} \quad (|x| \to \infty; \; y \neq 0 \text{ if } x < 0)
\]

and

\[
2.2 \quad |\Gamma(x + iy)| \sim \sqrt{2\pi} |y|^{y-1/2} e^{x-\pi y/2} \quad (|y| \to \infty).
\]

**Proof.** Applying the Stirling formula \([3, 1.18(2)]\)

\[
2.3 \quad \Gamma(z) \sim \sqrt{2\pi} e^{(z-1/2) \log z} e^{-z} \quad (|z| \to \infty; \; |\arg(z)| < \pi),
\]
we have

\[\begin{align*}
(2.4) \quad |\Gamma(x + iy)| & \sim \sqrt{2\pi} |e^{(x+iy-1/2)\log|x+iy|+i\arg(x+iy)}| e^{-(x+iy)} \\
& \sim \sqrt{2\pi} |x + iy|^{x-1/2} e^{-x-y\arg(x+iy)} \quad (|x + iy| \to \infty; \; y \neq 0 \text{ if } x \leq 0).
\end{align*}\]

Let \( y \in \mathbb{R} \) be fixed and \(|x| \to \infty\). Then \(|x + iy| \sim |x|\), and \(\arg(x + iy) \to 0\) as \(x \to +\infty\) and \(\arg(x + iy) \to \pi\) as \(x \to -\infty\). Therefore, (2.4) implies

\[\begin{align*}
(2.5) \quad |\Gamma(x + iy)| & \sim \sqrt{2\pi} |x|^{x-1/2} e^{-x} \quad (x \to +\infty)
\end{align*}\]

and

\[\begin{align*}
(2.6) \quad |\Gamma(x + iy)| & \sim \sqrt{2\pi} |x|^{x-1/2} e^{-x-\pi y} \quad (x \to -\infty; \; y \neq 0),
\end{align*}\]

which yield (2.1).

Turning to the case \( x \in \mathbb{R} \) being fixed and \(|y| \to \infty\), we find \(|x + iy| \sim |y|\) and \(\arg(x + iy) \to \pi/2\) as \(y \to \infty\) and \(\arg(x + iy) \to -\pi/2\) as \(y \to -\infty\). Thus (2.4) implies (2.2).

**Remark 1.** The relation \([3, (1.18.6)]\) needs correction with addition of the multiplier \(e^x\) in the left hand side and it must be replaced by

\[\begin{align*}
(2.7) \quad \lim_{|y| \to \infty} |\Gamma(x + iy)| e^{x+\pi|y|/2} |y|^{1/2-x} = \sqrt{2\pi}.
\end{align*}\]

Next assertion gives the asymptotic behavior of \(\mathcal{H}_{p,q}^{m,n}(s)\) defined in (1.2) at infinity on lines parallel to the real axis.

**Lemma 2.** Let \(\Delta, \delta\) and \(\mu\) be given by (1.3) to (1.5) and let \(t, \sigma \in \mathbb{R}\). Then there hold the estimates

\[\begin{align*}
(2.8) \quad |\mathcal{H}_{p,q}^{m,n}(t + i\sigma)| & \sim A \left(\frac{e}{t}\right)^{-\Delta t} \delta^t t^{\Re(\mu)} \quad (t \to +\infty)
\end{align*}\]

with

\[\begin{align*}
A = (2\pi)^{m+n-(p+q)/2} e^{q-m-n} \frac{\prod_{j=1}^{q} [(\beta_j)^{\Re(b_j)} - 1/2 e^{-\Re(b_j)}]}{\prod_{i=1}^{p} [(\alpha_i)^{\Re(a_i)} - 1/2 e^{-\Re(a_i)}]} \frac{\prod_{i=1}^{n} e^{\pi |\sigma a_i + \Im(a_i)|}}{\prod_{j=m+1}^{q} e^{\pi |\sigma \beta_j + \Im(b_j)|}},
\end{align*}\]

and

\[\begin{align*}
(2.10) \quad |\mathcal{H}_{p,q}^{m,n}(t + i\sigma)| & \sim B \left(\frac{e}{|t|}\right)^{\Delta |t|} \delta^{-|t|} |t|^{|\Re(\mu)|} \quad (t \to -\infty)
\end{align*}\]

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with

\[ (2.11) \quad B = (2\pi)^{m+n-(p+q)/2} e^{q-m-n} \frac{\prod_{j=1}^{g} \left( (\beta_j)^{\text{Re}(b_j)} - 1/2 e^{-\text{Re}(b_j)} \right) \prod_{i=1}^{p} e^{\pi [\sigma a_i + \text{Im}(a_i)]}}{\prod_{i=n+1}^{p} \left( (\alpha_i)^{\text{Re}(a_i)} - 1/2 e^{-\text{Re}(a_i)} \right) \prod_{j=1}^{m} e^{\pi [\sigma \beta_j + \text{Im}(b_j)]}}. \]

**Proof.** By virtue of (2.1), we have, for a variable \( s = t + i\sigma \) and a complex constant \( k = c + id \),

\[ (2.12) \quad |\Gamma(s + k)| \sim \sqrt{2\pi} t^{c-1/2} e^{-\pi \left( t^2 + \frac{c^2}{4} \right)} \quad (t \to +\infty) \]

and

\[ (2.13) \quad |\Gamma(s + k)| \sim \sqrt{2\pi} |t|^{c-1/2} e^{-\pi \left( t^2 + \frac{c^2}{4} \right)} e^{-\pi \sigma d} \quad (t \to -\infty). \]

Substituting these estimates into (1.2) and using (1.3) to (1.5), we obtain (2.8) and (2.10).

**Remark 2.** The asymptotic estimate of the function \( \mathcal{H}_{p,q}^{m,n}(s) \) at infinity on lines parallel to the imaginary axis \( \mathcal{H}_{p,q}^{m,n}(\sigma + it) \) as \( |t| \to \infty \) was given in our paper with Shlapakov [5].

By appealing to Lemma 2, we give conditions of the existence of the \( H \)-function (1.1) with the contour \( \Sigma \) being chosen as indicated in (a) and (b) in Section 1.

**Theorem 1.** Let \( \Delta, \delta \) and \( \mu \) be given by (1.3) to (1.5). Then the function \( H_{p,q}^{m,n}(z) \) defined by (1.1) and (1.2) exists in the following cases:

\[ (2.14) \quad \mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta > 0, \quad z \neq 0; \]
\[ (2.15) \quad \mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad 0 < |z| < \delta; \text{ label } 1.2.15 \]
\[ (2.16) \quad \mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta = 0, \quad |z| = \delta, \quad \text{Re}(\mu) < -1; \]
\[ (2.17) \quad \mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta < 0, \quad z \neq 0; \]
\[ (2.18) \quad \mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta = 0, \quad |z| > \delta; \]
\[ (2.19) \quad \mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta = 0, \quad |z| = \delta, \quad \text{Re}(\mu) < -1. \]

**Proof.** Let us first consider the case (a) for which \( \Delta \geq 0 \) and \( \mathcal{L} = \mathcal{L}_{-\infty} \). We have to investigate the convergence of the integral (1.1) on the lines

\[ (2.20) \quad l_1 = \{ t \in \mathbb{R} : t + i\varphi_1 \} \quad \text{and} \quad l_2 = \{ t \in \mathbb{R} : t + i\varphi_2 \} \quad \text{for} \quad \varphi_1 < \varphi_2 \]

as \( t \to -\infty \). According to (2.10), we have the following asymptotic estimate for the integrand of (1.1):

\[ (2.21) \quad |\mathcal{H}_{p,q}^{m,n}(s) z^{-s}| \sim B_1 e^{\varphi_2 \arg z} \left( \frac{e}{|\varphi|} \right)^{|\varphi|} \left( \frac{|z|}{\delta} \right)^{|\varphi|} |t|^{|\varphi|} (t \to -\infty; t \in l_i (i = 1, 2)), \]
where $B_1$ and $B_2$ are given by (2.11) with $\sigma$ being replaced by $\varphi_1$ and $\varphi_2$, respectively. It follows from (2.21) that the integral (1.1) is convergent if and only if one of the conditions in (2.14) to (2.16) is satisfied.

In the case (b), $\Delta \leq 0$ and the contour $\mathcal{L}$ is taken to be $\mathcal{L}_{-\infty}$. Then we have to investigate the convergence of the integral (1.1) on the lines $l_1$ and $l_2$ in (2.20), as $t \to +\infty$. By virtue of (2.8) and (2.9), we have the asymptotic estimate:

$$|\mathcal{H}_{m,n}^{p,q}(s)z^{-s}| \sim A_i e^{\varphi_i \arg z} \left(\frac{e}{t}\right)^{\Delta t} \left(\frac{\delta}{|z|}\right)^t t^\text{Re}(\mu) \quad (t \to +\infty; \ t \in l_i \ (i = 1, 2)), \tag{2.22}$$

where $A_1$ and $A_2$ are given by (2.9) with $\sigma$ being replaced by $\varphi_1$ and $\varphi_2$, respectively. Thus (2.22) implies that the integral (1.1) converges if and only if one of the conditions in (2.17) to (2.19) holds.

**Corollary 1.** The estimate (2.21) holds for $t \to -\infty$ uniformly on sets which have a positive distance to the points

$$b_{jl} = -\frac{b_j + l}{\beta_j} \quad (j = 1, \cdots, m; \ l \in \mathbb{N}_0). \tag{2.23}$$

and do not contain points to the right of $\mathcal{L}_{-\infty}$.

The estimate (2.22) holds for $t \to +\infty$ uniformly on sets which have a positive distance to the points

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i} \quad (i = 1, \cdots, n; \ k \in \mathbb{N}_0). \tag{2.24}$$

and do not contain points to the left of $\mathcal{L}_{+\infty}$.

**Remark 3.** The conditions for the existence of the $H$-function (1.1)

$$\sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j \geq 0, \quad \text{Re}(\mu) < 0 \tag{2.25}$$

given in [11, Section 8.3.1] in the cases when $\mathcal{L} = \mathcal{L}_{-\infty}, \Delta = 0, |z| = \delta \ ([8, Section 8.3.1.3])$ and $\mathcal{L} = \mathcal{L}_{+\infty}, \Delta = 0, |z| = \delta \ ([8, Section 8.3.1.4])$ can be replaced by the condition

$$\text{Re}(\mu) < -1. \tag{2.26}$$

The following statement follows from Theorem 1, Corollary 1 and the theory of residues.

**Theorem 2. (A)** If the conditions in (1.6), and (2.14) or (2.15) are satisfied, then the $H$-function (1.1) is an analytic function of $z$ in the corresponding domain indicated in (2.14) or (2.15), and

$$H_{p,q}^{m,n}(z) = \sum_{j=1}^m \sum_{l=0}^\infty \text{Res}[\mathcal{H}_{p,q}^{m,n}(s)z^{-s}], \tag{2.27}$$
where \( b_{jl} \) are given in (2.23).

**B.** If the conditions in (1.6), and (2.17) or (2.18) are satisfied, then the \( H \)-function (1.1) is an analytic function of \( z \) in the corresponding domain indicated in (2.17) or (2.18), and

\[
H_{p,q}^{m,n}(z) = - \sum_{i=1}^{n} \sum_{k=0}^{\infty} \text{Res}_{s=a_{ik}} [H_{p,q}^{m,n}(s) z^{-s}],
\]

where \( a_{ik} \) are given in (2.24).

**Remark 4.** The first assertion of Theorem 2 was proved in [1, p.278, Theorem 1] for the \( H \)-function represented by the integral obtained from (1.1) and (1.2) after replacing \( s \) by \(-s\). The proof of Theorem 1 in [1] is complicated and based on Lemma 2 there in which the asymptotic estimate at infinity of the functions \( h_0(s) \) defined by

\[
h_0(s) = \frac{\prod_{i=1}^{p} \Gamma(1 - a_i + \alpha_i s)}{\prod_{j=1}^{q} \Gamma(1 - b_j + \beta_j s)}
\]

is given. But our proof of Theorem 2 along the ideas of [1] is more simple and is based on the asymptotic estimate of \( H_{p,q}^{m,n}(s) \) at infinity given in Lemma 2.

### 3. Explicit Power Series Expansions

In this section we apply Theorem 2 to obtain explicit power series expansions of the \( H \)-function (1.1) under the condition (1.6) in the case of (1.7) or (1.8).

First we consider the former case. By Theorem 2(A), we have to evaluate the residues of \( H(s) z^{-s} \) at the points \( s = b_{jl} \) given in (2.23), where and in what follows we simplify \( H_{p,q}^{m,n}(s) \) by \( H(s) \). To evaluate these residues we use the property of the Gamma-function [6, (3.30)], that is, in a neighbourhood of the poles \( z = -k \) (\( k \in \mathbb{N}_0 \)) the Gamma-function \( \Gamma(z) \) can be expanded in powers of \( z + k = \epsilon \)

\[
\Gamma(z) = \frac{(-1)^k}{k! \epsilon} [1 + \psi(1 + k) + O(\epsilon^2)], \quad \text{where} \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.
\]

Since the poles \( b_{jl} \) are simple, i.e., the conditions in (1.7) hold,

\[
\text{Res}_{s=b_{jl}} [H(s) z^{-s}] = h_{jl}^* z^{-b_{jl}} \quad (j = 1, \ldots, m; \ l \in \mathbb{N}_0),
\]

where

\[
h_{jl}^* = \lim_{s \to b_{jl}} ((s - b_{jl}) H(s))
\]

\[
= \frac{(-1)^l}{l! \beta_j} \prod_{i=1, i \neq j}^{m} \Gamma \left( b_i - [b_j + l] \frac{\beta_i}{\beta_j} \right) \prod_{i=1}^{n} \Gamma \left( 1 - a_i + [b_j + l] \frac{\alpha_i}{\beta_j} \right) \prod_{i=n+1}^{p} \Gamma \left( a_i - [b_j + l] \frac{\alpha_i}{\beta_j} \right) \prod_{i=m+1}^{q} \Gamma \left( 1 - b_i + [b_j + l] \frac{\beta_i}{\beta_j} \right).
\]
Thus we obtain

**Theorem 3.** Let the conditions in (1.6) and (1.7) be satisfied and let either \( \Delta > 0, z \neq 0 \) or \( \Delta = 0, 0 < |z| < \delta \). Then the \( H \)-function (1.1) has the power series expansion

\[
H_{p,q}^{m,n}(z) = \sum_{j=1}^{m} \sum_{l=0}^{\infty} h_{jl}^* z^{(b_j+1)/\beta_j},
\]

where the constants \( h_{jl}^* \) are given by (3.3).

**Corollary 2.** If the conditions in (1.6) and (1.7) are satisfied and \( \Delta \geq 0 \), then (3.4) gives the asymptotic expansion of \( H_{p,q}^{m,n}(z) \) near zero and the main terms of this asymptotic formula have the form:

\[
H_{p,q}^{m,n}(z) = \sum_{j=1}^{m} \left[ h_{j0}^* z^{b_j/\beta_j} + O \left( z^{(b_j+1)/\beta_j} \right) \right] \quad (z \to 0),
\]

where

\[
h_{j0}^* \equiv h_{j0}^* = \frac{1}{\beta_j} \prod_{i=1, i \neq j}^{m} \Gamma \left( b_i - \frac{b_j \beta_i}{\beta_j} \right) \prod_{i=1}^{n} \Gamma \left( 1 - a_i + \frac{b_j \alpha_i}{\beta_j} \right) \prod_{i=m+1}^{p} \Gamma \left( a_i - \frac{b_j \alpha_i}{\beta_j} \right) \prod_{i=m+1}^{q} \Gamma \left( 1 - b_j + \frac{b_j \beta_i}{\beta_j} \right).
\]

**Corollary 3.** Let the conditions in (1.6) and (1.7) be satisfied, and let \( \Delta \geq 0 \) and \( j_0 (1 \leq j_0 \leq m) \) be an integer such that

\[
\frac{\text{Re}(b_{j_0})}{\beta_{j_0}} = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right].
\]

Then there holds the asymptotic estimate:

\[
H_{p,q}^{m,n}(z) = h_{j_0}^* z^{b_{j_0}/\beta_{j_0}} + o \left( z^{b_{j_0}/\beta_{j_0}} \right) \quad (z \to 0),
\]

where \( h_{j_0}^* \) is given by (3.6) with \( j = j_0 \). In particular,

\[
H_{p,q}^{m,n}(z) = O(z^\rho) \quad (z \to 0) \quad \text{with} \quad \rho = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right].
\]

Now we consider the case (1.8) when the poles of the Gamma-functions \( \Gamma(1-a_i-\alpha_is) \) \((i = 1, \cdots, n)\) are simple. By (3.1), evaluating the residues of \( \mathcal{H}(s)z^{-s} \) at the points \( a_{ik} \) given in (2.24) we have similarly to the previous argument that

\[
\text{Res}_{s=a_{ik}} \left[ \mathcal{H}(s)z^{-s} \right] = -h_{ik} z^{-a_{ik}} \quad (i = 1, \cdots, n; \ k \in \mathbb{N}_0),
\]

8
where $a_{ik}$ are given by (2.24) and

$$h_{ik} = \lim_{s \to a_{ik}} \left[-(s - a_{ik})\mathcal{H}(s)\right]$$

(3.11)

$$= (-1)^k \frac{k!\alpha_i}{\prod_{j=n+1}^{p} \Gamma \left( a_j + [1 - a_i + k] \frac{\beta_j}{\alpha_i} \right) \prod_{j=m+1}^{q} \Gamma \left( 1 - b_j + [1 - a_i + k] \frac{\beta_j}{\alpha_i} \right) \prod_{j=1, j \neq i}^{m} \Gamma \left( b_j + [1 - a_i + k] \frac{\beta_j}{\alpha_i} \right)}$$

Thus from Theorem 2(B) we have

**Theorem 4.** Let the conditions in (1.6) and (1.8) be satisfied and let either $\Delta < 0$, $z \neq 0$ or $\Delta = 0$, $|z| > \delta$. Then the $H$-function (1.1) has the power series expansion

$$H_{p,q}^{m,n}(z) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} h_{ik} z^{(\alpha_i - k - 1)/\alpha_i},$$

(3.12)

where the constants $h_{ik}$ are given by (3.11).

**Corollary 4.** If the conditions in (1.6) and (1.8) are satisfied and $\Delta < 0$, then (3.12) gives the asymptotic expansion of $H_{p,q}^{m,n}(z)$ near infinity and the main terms of this asymptotic formula have the form:

$$H_{p,q}^{m,n}(z) = \sum_{i=1}^{n} h_i z^{(\alpha_i - 1)/\alpha_i} + O\left(z^{(\alpha_i - 2)/\alpha_i}\right) \quad (|z| \to \infty),$$

(3.13)

where

$$h_i \equiv h_{i0} = \frac{1}{\alpha_i} \frac{\prod_{j=1}^{m} \Gamma \left( b_j - [a_i - 1] \frac{\beta_j}{\alpha_i} \right) \prod_{j=1, j \neq i}^{m} \Gamma \left( 1 - a_j + [a_i - 1] \frac{\alpha_j}{\alpha_i} \right)}{\prod_{j=n+1}^{p} \Gamma \left( a_j - [a_i - 1] \frac{\alpha_j}{\alpha_i} \right) \prod_{j=m+1}^{q} \Gamma \left( 1 - b_j + [a_i - 1] \frac{\beta_j}{\alpha_i} \right)}.$$  

(3.14)

**Corollary 5.** Let the conditions in (1.6) and (1.8) be satisfied, and let $\Delta < 0$ and $i_0 \ (1 \leq i_0 \leq n)$ be an integer such that

$$\frac{\text{Re}(\alpha_{i_0}) - 1}{\alpha_{i_0}} = \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(\alpha_i) - 1}{\alpha_i} \right].$$

(3.15)

Then there holds the asymptotic estimate:

$$H_{p,q}^{m,n}(z) = h_{i_0} z^{(\alpha_{i_0} - 1)/\alpha_{i_0}} + o\left(z^{(\alpha_{i_0} - 1)/\alpha_{i_0}}\right) \quad (|z| \to \infty),$$

(3.16)

where $h_{i_0}$ is given by (3.14) with $i = i_0$. In particular,

$$H_{p,q}^{m,n}(z) = O(z^g) \quad (|z| \to \infty) \quad \text{with} \quad g = \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(\alpha_i) - 1}{\alpha_i} \right].$$  

(3.17)
Remark 5. The relations (3.4) and (3.12) are given in [12, (2.2.4) and (2.2.7)] and [11, 8.3.2.3 and 8.3.2.4].

4. Explicit Power-Logarithmic Series Expansions

Now let us discuss the case when the condition (1.6) holds, but (1.7) or (1.8) is violated:

(e) \( \mathcal{L} = \mathcal{L}_{-\infty}, \Delta \geq 0 \) and some poles of the Gamma-functions \( \Gamma(b_j + \beta_j s) \) \( j = 1, \cdots, m \) coincide.

(f) \( \mathcal{L} = \mathcal{L}_{+\infty}, \Delta \leq 0 \) and some poles of the Gamma-functions \( \Gamma(1-a_i-\alpha_i s) \) \( i = 1, \cdots, n \) coincide.

First we consider the case (e). Let \( b \equiv b_{jl} \) be one of points (2.23) for which some poles of the Gamma-functions \( \Gamma(b_j + \beta_j s) \) \( j = 1, \cdots, m \) coincide and \( n^* \equiv N^* \) be order of this pole. It means that there exist \( j_1, \cdots, j_{N^*} \in \{1, \cdots, m\} \) and \( l_{j_1}, \cdots, l_{j_{N^*}} \in \mathbb{N}_0 \) such that

\[
(4.1) \quad b = b_{jl} = -\frac{b_{j_1} + l_{j_1}}{\beta_{j_1}} = \cdots = -\frac{b_{j_{N^*}} + l_{j_{N^*}}}{\beta_{j_{N^*}}}.
\]

Then \( \mathcal{H}(s)z^{-s} \) has the pole of order \( N^* \) at \( b \) and hence

\[
(4.2) \quad \text{Res}_{s=b}[\mathcal{H}(s)z^{-s}] = \frac{1}{(N^* - 1)!} \lim_{s \to b}(s - b)^{N^*}\mathcal{H}(s)z^{-s} = (N^* - 1)! \lim_{s \to b}(s - b)^{N^*}\mathcal{H}(s)z^{-s} = (N^* - 1)! \lim_{s \to b}(s - b)^{N^*}\mathcal{H}(s)z^{-s}.
\]

We denote

\[
(4.3) \quad \mathcal{H}_1^*(s) = (s - b)^{N^*} \prod_{j=j_1}^{N^*} \Gamma(b_j + \beta_j s), \quad \mathcal{H}_2^*(s) = \frac{\mathcal{H}(s)}{\prod_{j=j_1}^{N^*} \Gamma(b_j + \beta_j s)}.
\]

Using the Leibniz rule, we have

\[
[(s - b)^{N^*}\mathcal{H}(s)z^{-s}]^{(N^*-1)} = \sum_{n=0}^{N^*-1} N^* - 1 \text{ chosen}[\mathcal{H}_1^*(s)]^{(N^*-1-n)}[\mathcal{H}_2^*(s)z^{-s}]^{(n)}
\]

\[
= \sum_{n=0}^{N^*-1} \left( \begin{array}{c} N^* - 1 \\ n \end{array} \right) [\mathcal{H}_1^*(s)]^{(N^*-1-n)} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right)(-1)^i[\mathcal{H}_2^*(s)]^{(n-i)}z^{-s}[\log z]^i
\]

\[
= z^{-s} \sum_{i=0}^{N^*-1} \left\{ \sum_{n=i}^{N^*-1} (-1)^i \left( \begin{array}{c} N^* - 1 \\ n \end{array} \right) \left( \begin{array}{c} n \\ i \end{array} \right) [\mathcal{H}_1^*(s)]^{(N^*-1-n)}[\mathcal{H}_2^*(s)]^{(n-i)} \right\}[\log z]^i.
\]

Substituting this into (4.2), we obtain

\[
(4.4) \quad \text{Res}_{s=b_i}[\mathcal{H}(s)z^{-s}] = z^{(b_{j_i}+l_i)/\beta_j} \sum_{i=0}^{N^*_j-1} H_{j_i}[\log z]^i,
\]

10
where

\[(4.5) \quad H_{jli}^* \equiv H_{jli}^*(N_{jli}^*; b_{jli}) = \frac{1}{(N_{jli}^* - 1)!} \sum_{n=0}^{N_{jli}^* - 1} (-1)^n \binom{N_{jli}^* - 1}{n} \left[ \mathcal{H}_{1}^*(b_{jli}) \right]^{(N_{jli}^* - 1-n)} \left[ \mathcal{H}_{2}^*(b_{jli}) \right]^{(n-i)} \]

In particular, if \( l = 0 \) and \( i = N_{jli}^* - 1 \), then by setting \( N_{jli}^* = N_{jli}^* \) and from (3.1), (4.1) and (4.3), we have

\[(4.6) \quad H_{j}^* \equiv H_{j,0,N_{j}^*}^*(N_{j}^*; b_{j0}) = \frac{(-1)^{N_{j}^* - 1}}{(N_{j}^* - 1)!} \mathcal{H}_{1}^*(b_{j0}) \mathcal{H}_{2}^*(b_{j0})
\]

\[
= \frac{(-1)^{N_{j}^* - 1}}{(N_{j}^* - 1)!} \left\{ \prod_{k=1}^{N_{j}^*} \left( -1 \right)^{j_{k}} \right\} \prod_{i=m+1}^{p} \Gamma \left( a_i - \frac{b_{j} \alpha_i}{\beta_j} \right) \prod_{i=m+1}^{q} \Gamma \left( 1 - b_i + \frac{b_{j} \beta_i}{\beta_j} \right) .
\]

Thus, in view of Theorem 2(A), we have

**Theorem 5.** Let the conditions in (1.6) be satisfied and let either \( \Delta > 0, z \neq 0 \) or \( \Delta = 0, 0 < |z| < \delta \). Then the \( H \)-function (1.1) has the power-logarithmic series expansion

\[(4.7) \quad H_{p,q}^{m,n}(z) = \sum_{j,l} \sum' h_{j,l}^* z^{(b_{j} + l)/\beta_j} + \sum_{j,l} \sum'' H_{j,l}^* z^{(b_{j} + l)/\beta_j} [\log z]^i .
\]

Here \( \sum' \) and \( \sum'' \) are summations taken over \( j, l \) \((j = 1, \ldots, m; l = 0, 1, \ldots)\) such that the \( H \)-functions \( \Gamma(b_{j} + \beta_{j}s) \) have simple poles and poles of order \( N_{j}^* \) at the points \( b_{j} \), respectively, and the constants \( h_{j,l}^* \) are given by (3.3) while the constants \( H_{j,l}^* \) are given by (4.5).

**Corollary 6.** If the conditions in (1.6) are satisfied and \( \Delta \geq 0 \), then (4.7) gives the asymptotic expansion of \( H_{p,q}^{m,n}(z) \) near zero and the main terms of this asymptotic formula have the form:

\[(4.8) \quad H_{p,q}^{m,n}(z) = \sum_{j=1}^{m} \left[ h_{j}^* z^{b_{j}/\beta_j} + O(z^{(b_{j}+1)/\beta_j}) \right] + \sum_{j=1}^{m} \left( H_{j}^* z^{b_{j}/\beta_j} [\log z]^{N_{j}^* - 1} + O(z^{(b_{j} + 1)/\beta_j} [\log z]^{N_{j}^* - 1}) \right) \ (z \rightarrow 0).
\]

Here \( \sum' \) and \( \sum'' \) are summations taken over \( j \) \((j = 1, \ldots, m)\) such that the \( H \)-functions \( \Gamma(b_{j} + \beta_{j}s) \) have simple poles and poles of order \( N_{j}^* \equiv N_{j0}^* \) at the points \( b_{j0} \), respectively, and \( h_{j}^* \) are given by (3.6) while \( H_{j}^* \) are given by (4.6).
Corollary 7. Let the conditions in (1.6) be satisfied, and let \( \Delta \geq 0 \) and \( b_{jl} \) be poles of the Gamma-functions \( \Gamma(b_j + \beta_j s) \) \((j = 1, \ldots, m)\). Let \( j_{01} \) and \( j_{02} \) \((1 \leq j_{01}, j_{02} \leq m)\) be integers such that

\[
\frac{\text{Re}(b_{j_{01}})}{\beta_{j_{01}}} = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right],
\]

when the poles \( b_{jl} \) \((j = 1, \ldots, m; l \in \mathbb{N}_0)\) are simple, and

\[
\frac{\text{Re}(b_{j_{02}})}{\beta_{j_{02}}} = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right],
\]

when the poles \( b_{jl} \) \((j = 1, \ldots, m; l \in \mathbb{N}_0)\) coincide.

a) If \( j_{01} < j_{02} \), then the asymptotic expansion of the \( H \)-function has the form:

\[
H_{p,q}^{m,n}(z) = h^*_{j_{01}} z^{b_{01}/\beta_{j_{01}}} + o \left( z^{b_{01}/\beta_{j_{01}}} \right) \quad (z \to 0),
\]

where \( h^*_{j_{01}} \) is given by (3.6) with \( j = j_{01} \). In particular, the relation (3.9) holds.

b) If \( j_{01} \geq j_{02} \) and \( b_{j_{02},0} \) has the pole of order \( N_{j_{02}}^* \), then the first term in asymptotic expansion of the \( H \)-function has the form:

\[
H_{p,q}^{m,n}(z) = H^*_{j_{02}} z^{b_{02}/\beta_{j_{02}}} [\log(z)]^{N_{j_{02}}^* - 1} + o \left( z^{b_{02}/\beta_{j_{02}}}[\log(z)]^{N_{j_{02}}^* - 1} \right) \quad (z \to 0),
\]

where \( H^*_{j_{02}} \) is given by (4.6) with \( j = j_{02} \). In particular, if \( N^* \) is the largest order of general poles of Gamma-functions \( \Gamma(b_j + \beta_j s) \) \((j = 1, \ldots, m)\), then

\[
H_{p,q}^{m,n}(z) = O \left( z^\rho[\log(z)]^{N^*} \right) \quad (z \to 0) \quad \text{with} \quad \rho = \min_{1 \leq j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right].
\]

Now we consider the case (f). Let \( a = a_{ik} \) be one of points (2.24) for which some poles of \( \Gamma(1 - a_i - \alpha_i s) \) coincide and \( N = N_{ik} \) be order of this pole. It means that there exist \( i_1, \ldots, i_N \in \{1, \ldots, n\} \) and \( k_1, \ldots, k_i \in \mathbb{N}_0 \) such that

\[
a = a_{ik} = \frac{1 - a_{i_1} + k_{i_1}}{\alpha_{i_1}} = \cdots = \frac{1 - a_{i_N} + k_{i_N}}{\alpha_{i_N}}.
\]

Then the integrand \( \mathcal{H}(s)z^{-s} \) of the integral (1.1) has the pole of order \( N \) at \( a \). Similarly to (4.3), we denote

\[
\mathcal{H}_1(s) = (s - a)^N \prod_{i=1}^{i_N} \Gamma(1 - a_i - \alpha_i s), \quad \mathcal{H}_2(s) = \frac{\mathcal{H}(s)}{\prod_{i=1}^{i_N} \Gamma(1 - a_i - \alpha_i s)}.
\]
and, then, find similarly to (4.4) and (4.5) that

\begin{equation}
\text{Res}_{s=a_{ik}} [\mathcal{H}(s) z^{-s}] = z^{(a_{i} - 1 - k)/\alpha_{i}} \sum_{j=0}^{N_{ik}-1} H_{ikj}[\log(z)]^{j},
\end{equation}

where

\begin{equation}
H_{ikj} \equiv H_{ikj}(N_{ik}; a_{ik}) = \frac{1}{(N_{ik} - 1)!} \left\{ \sum_{n=j}^{N_{ik}-1} (-1)^{j} \binom{N_{ik} - 1}{n} \binom{n}{j} [\mathcal{H}_{1}(a_{ik})]^{(N_{ik} - 1 - n)}[\mathcal{H}_{2}(a_{ik})]^{(n-j)} \right\}.
\end{equation}

In particular, if we set \( k = 0, j = N_{i0} - 1 \) and \( N_{i0} \equiv N_{i} \), then, using (4.15) and (3.1), we have

\begin{equation}
H_{i} \equiv H_{i,0,N_{i}-1}(N_{i}; a_{i0}) = \frac{(-1)^{N_{i}-1}}{(N_{i} - 1)!} \mathcal{H}_{1}(a_{i0}) \mathcal{H}_{2}(a_{i0}) = \frac{(-1)^{N_{i}-1}}{(N_{i} - 1)!} \left( \prod_{j=1}^{m} \Gamma \left( b_{j} + \left[ 1 - a_{i} \right] / \alpha_{i} \right) \prod_{j=1, j \neq i_{1}, i_{2}, \ldots, i_{N_{i}}}^{N_{i}} \frac{\Gamma \left( 1 - a_{j} - \left[ 1 - a_{i} \right] / \alpha_{i} \right)}{\Gamma \left( 1 - b_{j} - \left[ 1 - a_{i} \right] / \alpha_{i} \right)} \right).
\end{equation}

Therefore, Theorem 2(B) implies the similar result to Theorem 5:

**Theorem 6.** Let the conditions in (1.6) be satisfied and let either \( \Delta < 0, z \neq 0 \) or \( \Delta = 0, |z| > \delta \). Then the \( H \)-function (1.1) has the power-logarithmic series expansion

\begin{equation}
H_{p,q}^{m,n}(z) = \sum_{i,k}^{'} h_{ik} z^{(a_{i} - 1 - k)/\alpha_{i}} + \sum_{i,k}^{''} \sum_{j=0}^{N_{ik}-1} H_{ikj} z^{(a_{i} - 1 - k)/\alpha_{i}}[\log(z)]^{j}.
\end{equation}

Here \( \sum^{'} \) and \( \sum^{''} \) are summations taken over \( i, k \) (\( i = 1, \ldots, n; k = 0, 1, \ldots \)) such that Gamma-functions \( \Gamma(1 - a_{i} - \alpha_{i}s) \) have simple poles and poles of order \( N_{ik} \) at the points \( a_{ik} \), respectively, and the constants \( h_{ik} \) are given by (3.11) while the constants \( H_{ikj} \) are given by (4.17).

**Corollary 8.** If the conditions in (1.6) are satisfied and \( \Delta \leq 0 \), then (4.19) gives the asymptotic expansion of \( H_{p,q}^{m,n}(z) \) near infinity and the main terms of this asymptotic formula have the form

\begin{equation}
H_{p,q}^{m,n}(z) = \sum_{i=1}^{n}^{'} \left[ h_{i} z^{(a_{i} - 1)/\alpha_{i}} + O(z^{(a_{i} - 2)/\alpha_{i}}) \right] + \sum_{i=1}^{n}^{''} \left( H_{i} z^{(a_{i} - 1)/\alpha_{i}}[\log(z)]^{N_{i}-1} + O(z^{(a_{i} - 2)/\alpha_{i}}[\log z]^{N_{i}-1}) \right) \quad (|z| \to \infty).
\end{equation}
Here $\sum'$ and $\sum''$ are summations taken over $i$ ($i = 1, \ldots, n$) such that the Gamma-functions $\Gamma(1 - a_i - \alpha_i s)$ have simple poles and poles of order $N_i \equiv N_{i0}$ in the points $a_{i0}$ in (2.24), respectively, and $h_i$ are given by (3.14) while $H_i$ are given by (4.18).

**Corollary 9.** Let the conditions in (1.6) be satisfied, and let $\Delta \leq 0$ and $\alpha_{ik}$ be poles of the Gamma-function $\Gamma(1 - a_i - \alpha_i s)$ ($i = 1, \ldots, n$). Let $i_{01}$ and $i_{02}$ ($1 \leq i_{01}, i_{02} \leq n$) be integers such that

\[
\frac{\text{Re}(a_{i01}) - 1}{\alpha_{i01}} = \frac{\text{max}}{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right],
\]

when the poles $a_{ik}$ ($i = 1, \ldots, n; k = 0, 1, \ldots$) are simple, and

\[
\frac{\text{Re}(a_{i02}) - 1}{\alpha_{i02}} = \frac{\text{max}}{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right],
\]

when the poles $a_{ik}$ ($i = 1, \ldots, n; k \in \mathbb{N}_0$) coincide.

(a) If $i_{01} < i_{02}$, then the asymptotic expansion of the $H$-function has the form:

\[
H_{p,q}^{m,n}(z) = h_{i_{01}} z^{(a_{i01} - 1)/\alpha_{i01}} + o \left( z^{(a_{i01} - 1)/\alpha_{i01}} \right) \quad (|z| \to \infty),
\]

where $h_{i_{01}}$ is given by (3.14) with $i = i_{01}$. In particular, the relation (3.17) holds.

(b) If $i_{01} \geq i_{02}$ and $a_{i02,0}$ has the pole of order $N_{i02}$, then the asymptotic expansion of the $H$-function has the form:

\[
H_{p,q}^{m,n}(z) = H_{i02} z^{(a_{i02} - 1)/\alpha_{i02}} \log z)^{N_{i02} - 1} + o \left( z^{(a_{i02} - 1)/\alpha_{i02}} \log(z)^{N_{i02} - 1} \right) \quad (|z| \to \infty),
\]

where $H_{i02}$ is given by (4.18) with $i = i_{02}$. In particular, if $N$ is the smallest order of general poles of Gamma-functions $\Gamma(1 - a_i - \alpha_i s)$ ($i = 1, \ldots, n$), then

\[
H_{p,q}^{m,n}(z) = O \left( z^{\varrho} \log(z)^N \right) \quad (|z| \to \infty) \quad \text{with} \quad \varrho = \min_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right].
\]

In conclusion, we give the following consequence of Corollaries 3, 5, 7 and 9, which unifies the power and power-logarithmic asymptotic behavior of $H_{p,q}^{m,n}(z)$ near zero and infinity.

**Theorem 7.** Let the conditions in (1.6) be satisfied.

(a) If $\Delta \geq 0$ and the poles of the Gamma-function $\Gamma(b_j + \beta_j s)$ ($j = 1, \ldots, m$) are simple, then the $H$-function (1.3) has the asymptotic estimate (3.9) at zero. If some of poles of $\Gamma(b_j + \beta_j s)$ ($j = 1, \ldots, m$) coincide, then $H_{p,q}^{m,n}(z)$ has the asymptotic estimate either (3.9) or (4.13) at zero.

(b) If $\Delta \leq 0$ and the poles of the Gamma-function $\Gamma(1 - a_i - \alpha_i s)$ ($i = 1, \ldots, n$) are simple, then the $H$-function (1.3) has the asymptotic estimate (3.17) at infinity. If some of poles of $\Gamma(1 - a_i - \alpha_i s)$ ($i = 1, \ldots, n$) coincide, then $H_{p,q}^{m,n}(z)$ has the asymptotic estimate
(3.17) or (4.25) at infinity.

**Remark 6.** The power-logarithmic expansions and more complicated results than in (4.7) were indicated in [9, Section 3.7] (see also [8, Section 5.8]) and the particular cases $H_{0,0}^{p,0}(z)$ and $H_{p,0}^{p,0}(z)$ in [7].

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**References**

[1] Braaksma, B.L.G., Asymptotic expansions and analytic continuation for a class of Barnes-integrals, *Compositio Math.* 15(1964), 239-341.

[2] Dixon, A.L. and Ferrar, W.L., A class of discontinuous integrals, *Quart. J. Math., Oxford Ser.* 7(1936), 81-96.

[3] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., *Higher Transcendental Functions, Vol. I*, McGraw-Hill, New York, Toronto and London, 1953.

[4] Fox, C., The G and H functions as symmetrical Fourier kernels., *Trans. Amer. Math. Soc.* 98(1961), 395-429.

[5] Kilbas, A.A., Saigo, M. and Shlapakov, S.A., Integral transforms with Fox’s H-function in spaces of summable functions, *Integral Transform. Spec. Funct.* 1(1993), 87-103.

[6] Marichev, O.I., *Handbook of Integral Transforms of Higher Transcendental Functions. Theory and Algorithmic Tables*, Wiley (Ellis Horwood), New York, Brisbane, Chichester and Toronto, 1982.

[7] Mathai, A.M., An expansion of Meijer’s G-function in the logarithmic case with applications, *Math. Nachr.* 48(1971), 129-139.

[8] Mathai, A.M., A few results on the exact distributions of certain multivariete statistics. II, *Multivariate Statistical Inference* (Proc. Res. Sem. Dalhouse Univ., Halifax, N.S., 1972), 169-181, North-Holland, Amsterdam and New York, 1973.

[9] Mathai, A.M. and Saxena, R.K., *The H-Function with Applications in Statistics and other Disciplines*, Wiley (Halsted P.), New York, London, Sydney and Toronto, 1978.

[10] Mellin, Hj., Abriß einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen, *Math. Ann.* 68(1910), 305-337.

[11] Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I., *Integrals and Series, Vol.3, More Special Functions*, Gordon and Breach, New York, Philadelphia, London, Paris, Montreux, Tokyo and Melbourne, 1990.

[12] Srivastava, H.M., Gupta, K.C. and Goyal, S.P., *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.