The single-indexed exceptional Krawtchouk polynomials

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ABSTRACT
The Darboux transformations of Krawtchouk polynomials are investigated and all possible exceptional Krawtchouk polynomials obtainable from a single-step Darboux transformation are considered. The properties of these exceptional Krawtchouk polynomials including the Diophantine ones and the recurrence relations are obtained.

1. Introduction

Classical orthogonal polynomials (COPs), defined as polynomial eigenfunctions of a second-order differential/\((q-)\)-difference operator, appear and play important roles in many problems [1]. Generalizations of the COPs have been proposed from many viewpoints and an important one is that of the exceptional orthogonal polynomials (XOPs). The definition of the XOPs is almost the same as that of the COPs. The XOPs can be obtained from the COPs by applying iterated Darboux transformations [10,20,21]. The difference lies in the absence of several degrees in XOP families although these ensembles nevertheless provide complete bases for the corresponding Hilbert spaces [8,9]. Due to this correspondence, the Askey-scheme of the COPs has been extended to XOPs and the exceptional Askey-Wilson polynomials have been extensively studied [6,7,19].

We shall focus on the Krawtchouk polynomials which are COPs of a discrete variable and shall consider their exceptional counterparts. The ordinary Krawtchouk polynomials have applications in many areas including signal processing, coding theory and so on [12,22]. The authors recently found that the exceptional Krawtchouk (X-Krawtchouk) polynomials lead to interesting continuous-time classical and quantum walks [14]. With further applications in mind, this study motivates the examination of the properties of these X-Krawtchouk polynomials. It was already pointed out that the X-Krawtchouk polynomials can formally be obtained from the exceptional Meixner (X-Meixner) polynomials by
choosing a different parametrization [5]. In that sense, the properties of the \(X\)-Krawtchouk polynomials are expected to be obtainable from those of the \(X\)-Meixner polynomials. However, for finite orthogonal polynomials, one needs to be careful about the situation where their degree is close to that of the dimension of their finite orthogonality. In particular some type of polynomial eigenfunctions has not been discussed before (see §4, type 2 \(X\)-Krawtchouk polynomials). In our framework, all possible type of Darboux transformations for the exceptional orthogonal polynomials are naturally introduced.

This paper aims to characterize explicitly the \(X\)-Krawtchouk polynomials by applying a Darboux transformation directly to the ordinary Krawtchouk polynomials. It should be noted that multi-step Darboux transformations are usually considered in the construction of XOPs. However, we shall only examine here the single-step transformation so as to work out all details. We shall thus proceed to determine the properties of these \(X\)-Krawtchouk polynomials including the Diophantine ones, the recurrence formulas and so on.

This paper is organized as follows. In Section 2, the basics of the Krawtchouk polynomials are introduced as polynomial eigenfunctions of the Krawtchouk operator and their properties are reviewed from that angle. All eigenfunctions of the Krawtchouk operator will then be considered. In Section 3, we introduce the Darboux transformation and the \(X\)-Krawtchouk operator from the eigenfunctions of the Krawtchouk operator. We then construct the \(X\)-Krawtchouk polynomials by this Darboux transformation in Section 4. The properties of the \(X\)-Krawtchouk polynomials including their orthogonality, recurrence relations, etc. are spelled out in Section 5. We conclude with a summary.

### 2. Krawtchouk polynomials and their difference operator

We first review the definition and basic properties of the Krawtchouk polynomials, which are necessary for the discussion that follows.

Let \(N\) be a positive integer and \(p \in (0, 1)\). For \(n = 0, 1, 2, \ldots\), the Krawtchouk polynomials of \(n\)th degree in monic form are given by

\[
K_n(x) = K_n(x; p, N) = (-N)_n p^n \binom{-x, -n}{-N; 1/p}
= \sum_{j=0}^{n} \frac{(-n)_j(-N+j)n-j}{j!} p^{n-j} (-x)_j
\]

with \((a)_n\) the standard Pochhammer symbol defined by

\[
(a)_n = \begin{cases} 
1 & (n = 0) \\
 a(a+1) \cdots (a+n-1) & (n = 1, 2, \ldots) 
\end{cases}
\]

(2)

It should be stressed that the Krawtchouk polynomials are usually defined for \(n = 0, 1, \ldots, N\). The monic polynomial sequence \(\{K_n(x)\}_{n=0}^{N}\) satisfies the three-term recurrence relation:

\[
x K_n(x) = K_{n+1}(x) + \left( p(N - n) + n(1 - p) \right) K_n(x)
+ (N + 1 - n)p(1 - p)K_{n-1}(x)
\]

(3)
with the initial values \( K_0(x) = 1 \) and \( K_1(x) = x - Np \). It should be remarked that the recurrence relation (3) is valid also for \( n = N + 1, N + 2, \ldots \). In the following, we will omit the dependence of the function on \( p \) and \( N \) and write \( F(x) = F(x; p, N) \) so long as there is no confusion.

The set consisting of \( K_0(x), K_1(x), \ldots, K_N(x) \) is known to verify the discrete orthogonality relation associated to a binomial distribution: for \( n, m \in \{0, 1, 2, \ldots, N\} \),

\[
\sum_{x=0}^{N} w(x) K_n(x) K_m(x) = h_n \delta_{n,m}, \tag{4}
\]

where

\[
w(x) = \frac{\Gamma(N + 1)}{\Gamma(x + 1) \Gamma(N - x + 1)} p^x (1 - p)^{N-x},
\]

\[
h_n = (-1)^n (-N)_n n! p^n (1 - p)^n. \tag{5}
\]

Note that the Krawtchouk polynomials possess several formulas some of which are expressed in the relations recorded below for later usage:

\[
K_n(x; 1 - p, a) = (-1)^n K_n(a - x; p, a) \quad (a \in \mathbb{R}),
\]

\[
(-N)_x p^x K_n(x; p, N) = (-N)_n p^n K_x(n; p, N) \quad (x \in \{0, 1, 2, \ldots, N\}),
\]

\[
\frac{K_{N-n}(x; p, N)}{K_n(N - x; p, N)} = (-1)^n \frac{N(N - n)!}{n!} \frac{(p - 1)^{x-n}}{p^{x+n-N}} \quad (x \in \{0, 1, 2, \ldots, N\}),
\]

\[
K_n(x + 1; p, N) = K_n(x; p, N) + nK_{n-1}(x; p, N - 1),
\]

\[
K_n(x; p, N + 1) = K_n(x; p, N) - n p K_{n-1}(x; p, N),
\]

\[
(x - N)K_n(x + 1; p, N) = K_{n+1}(x; p, N) + (2n - N)(1 - p)K_n(x; p, N)
\]

\[
\quad + n(n - N - 1)(1 - p)^2 K_{n-1}(x; p, N). \tag{6}
\]

There exists a factorization property of the Krawtchouk polynomials when \( n \geq N + 1 \) as

\[
K_n(x) = K_{N+1}(x) Q_{n-N-1}(x), \tag{7}
\]

where

\[
K_{N+1}(x) = K_{N+1}(x; p, N) = x(x - 1)(x - 2) \cdots (x - N),
\]

\[
Q_m(x) = K_m(x - N - 1; p, -N - 2). \tag{8}
\]

It should be mentioned that the relation (7) is usually called ‘Diophantine property’ \([3,4]\) of the Krawtchouk polynomials.
Let $\mathcal{L}$ be the second-order difference operator defined by

$$\mathcal{L} = p(N - x)(T - I) + x(1 - p)(T^{-1} - I)$$

where $T$ is the shift operator $T[f](x) = f(x + 1)$ and $I$ is the identity operator. Then the action of the operator $\mathcal{L}$ on a polynomial $f(x)$ is explicitly given by

$$\mathcal{L}f(x) = p(N - x)(f(x + 1) - f(x)) + x(1 - p)(f(x - 1) - f(x)).$$

The Krawtchouk polynomials are known to be polynomial eigenfunctions of $\mathcal{L}$:

$$\mathcal{L}K_n(x; p, N) = \lambda_n K_n(x; p, N)$$

with $\lambda_n = -n$. We shall henceforth call $\mathcal{L}$ the Krawtchouk operator.

For later use, we shall introduce all the possible eigenfunctions of the Krawtchouk operator $\mathcal{L}$ of the quasi-polynomial form:

$$\mathcal{L}\phi(x) = \lambda\phi(x),$$

$$\phi(x) = \xi(x)P(x) \quad (P(x) \in \mathbb{R}[x]).$$

It is readily seen that the $P(x)$ satisfy

$$(\xi(x))^{-1}\mathcal{L}\xi(x)P(x) = \lambda P(x),$$

which implies that $P(x)$ is a polynomial eigenfunction of the transformed operator $\tilde{\mathcal{L}} = (\xi(x))^{-1}\mathcal{L}\xi(x) + \text{constant}$. Here we choose $\xi(x)$ so that the transformed operator $\tilde{\mathcal{L}}$ is again a Krawtchouk operator with different parameters:

$$\tilde{\mathcal{L}} = \mathcal{L} \big|_{x = \tilde{x}, p = \tilde{p}, N = \tilde{N}},$$

which is related with shape invariance in quantum mechanics [17]. From the ensuing necessary conditions on $\xi(x)$, each of $p(N - x)\xi(x + 1)/\xi(x)$ and $x(1 - p)\xi(x - 1)/\xi(x)$ needs to be a linear polynomial in $x$, which implies that $\xi(x + 1)/\xi(x)$ must be a constant or a $[1/1]$-type rational function, i.e.

$$\frac{\xi(x + 1)}{\xi(x)} = \frac{a_0(x - a_1)}{x - a_2},$$

where $a_0, a_1, a_2$ are some constants. Furthermore, since the constant is an eigenfunction of $\tilde{\mathcal{L}}$, one finds that

$$p(N - x)\left(\frac{\xi(x + 1)}{\xi(x)} - 1\right) + x(1 - p)\left(\frac{\xi(x - 1)}{\xi(x)} - 1\right) - \mu = 0$$

holds for some constant $\mu$. Substituting (15) into (16), we obtain the condition for $a_0, a_1, a_2$ to find the following six candidates of $\xi(x + 1)/\xi(x)$:

$$\frac{p - 1}{p}, \frac{x + 1}{x - N}, \frac{p - 1(x + 1)}{p(x - N)}, \frac{x + 1 - Np}{x - Np}, \frac{(p - 1)(x + 1 + N(p - 1))}{p(x + N(p - 1))}.$$
is shown to be satisfied only by the following four kinds of eigen-pairs \((\lambda, \phi(x))\):

\[
(\lambda, \phi(x)) \in \{(\lambda_n^{(j)}, \phi_n^{(j)}(x)) \mid j \in \{1, 2, 3, 4\}, n \in \mathbb{Z}_{\geq 0}\},
\]

where \(\lambda_n^{(j)}\) and \(\phi_n^{(j)} = \xi^{(j)}(x)P_n^{(j)}(x)\) are given as follows.

\[
(\lambda_n^{(j)}, \xi^{(j)}(x), P_n^{(j)}(x))
= \begin{cases} 
(-n, 1, K_n(x; p, N)) & (j = 1) \\
(-N - n - 1, (x - N)_{N+1}, K_n(x - N - 1; p, -N - 2)) & (j = 2) \\
(-N + n, (p - 1)^{x^2}K_n(x; 1 - p, N)) & (j = 3) \\
(n + 1, (p - 1)^{x^2}K_n(x - N + 1, K_n(x - N - 1; 1 - p, -N - 2)) & (j = 4)
\end{cases} 
\]

(19)

Here, the possible range of \(\lambda\) is that of the integers and for each integer \(\lambda\), depending on the subset to which \(n\) belongs, two cases are related:

\[
\lambda = -n = \begin{cases} 
\lambda_n^{(1)} = \lambda_n^{(2)} = \lambda_{n-N-1} & (N < n) \\
\lambda_n^{(3)} = \lambda_{n-n} & (0 \leq n \leq N) \\
\lambda_{N-n}^{(3)} = \lambda_{-n-1} & (n < 0)
\end{cases} .
\]

(20)

It is immediate to see that

\[
\lambda_n^{(1)} + \lambda_n^{(4)} = -1, \quad \lambda_n^{(2)} + \lambda_n^{(3)} = -2N - 1,
\]

(21)

which implies that the eigenvalues \(\lambda_n^{(1)}\) and \(\lambda_n^{(2)}\) are mirror symmetric to \(\lambda_n^{(4)}\) and \(\lambda_n^{(3)}\) with respect to 1/2 and \(-N - 1/2\), respectively. In the case of Meixner polynomials, this doubling does not occur in general because the constant corresponding to \(N\) is real. It is because of the symmetries of the Krawtchouk polynomials that these situations are possible. Using the Diophantine property (7), one finds the simple expressions for the eigenfunctions:

\[
\phi_n^{(j)}(x) = \begin{cases} 
K_n(x; p, N) & (j = 1) \\
(x - N)_{N+1}K_n(x - N - 1; p, -N - 2) = K_{n+1}(x; p, N) & (j = 2) \\
p^{x}(p - 1)^{x^2}K_n(x; 1 - p, N) & (j = 3) \\
(x - N)_{N+1}(1 - p^{-1})^{x^2}K_n(x; 1 - p, N)
\end{cases} .
\]

(22)

From the above expressions, one finds that

\[
\phi_n^{(2)}(x) = \phi_{n+N+1}^{(1)}(x),
\]

(23)

and also from (6) that

\[
\phi_n^{(3)}(x) = p^{-x}(p - 1)^{x^2}(-1)^n\phi_n^{(1)}(N-x),
\phi_n^{(4)}(x) = \phi_{n+N+1}^{(3)}(x) = p^{-x}(p - 1)^{x^2}(-1)^{n+N+1}\phi_{n+N+1}^{(1)}(N-x).
\]

(24)
3. Exceptional Krawtchouk operator

In this section, we introduce the $X$-Krawtchouk operator which is derived as follows through a Darboux transformation. Let $(\mu, \chi(x))$ be an eigen-pair of the Krawtchouk operator $\mathcal{L}$ such that $\mathcal{L}\chi(x) = \mu \chi(x)$. Then, the Krawtchouk operator $\mathcal{L}$ can be factored as

$$\mathcal{L} = \mathcal{B} \circ \mathcal{F} + \mu, \quad (25)$$

where

$$\mathcal{F} = (\eta(x))^{-1}(\chi(x)T - \chi(x+1)I),$$
$$\mathcal{B} = (\chi(x))^{-1}(p(N-x)I - x(1-p)T^{-1})\eta(x), \quad (26)$$

with an arbitrary decoupling factor $\eta(x)$. We define the new operator $\mathcal{L}$ by

$$\mathcal{\hat{L}} = \mathcal{F} \circ \mathcal{B}. \quad (27)$$

This $\mathcal{\hat{L}}$ is a second-order difference operator which is different from the Krawtchouk operator $\mathcal{L}$. We will call $\mathcal{\hat{L}}$ the exceptional Krawtchouk ($X$-Krawtchouk) operator. It should be remarked here that the eigen-pairs of the $X$-Krawtchouk operator $\mathcal{\hat{L}}$ are given in terms of the eigen-pairs of the Krawtchouk operator such that $\mathcal{L}\hat{\phi} = \hat{\lambda}\hat{\phi}$:

$$\mathcal{\hat{L}}\hat{\phi} = \hat{\lambda}\hat{\phi} \quad (28)$$

with

$$\hat{\phi} = \mathcal{F}\phi, \quad \hat{\lambda} = \lambda - \mu. \quad (29)$$

The eigenfunction $\chi(x)$ plays the role of the seed function of the Darboux transformation. Here and hereafter we shall choose $\chi(x) = \phi_d^{(j)}(x) = \xi^{(j)}(x) p_d^{(j)}(x)$ (and $\mu = \lambda_d^{(j)}$, $j = 0, 1, 2, \ldots$) with $\phi_d^{(j)}(x)$ and $\xi^{(j)}(x)$ as in the previous section. The corresponding operators (26) are explicitly given by

$$\mathcal{F}^{(j,d)} = (\eta(x))^{-1}(\phi_d^{(j)}(x)T - \phi_d^{(j)}(x+1)I),$$
$$\mathcal{B}^{(j,d)} = (\phi_d^{(j)}(x))^{-1}(p(N-x)I - x(1-p)T^{-1})\eta(x). \quad (30)$$

It should be noted that the decoupling factor $\eta(x)$ plays an important role in the construction of the monic $X$-Krawtchouk polynomials, which will be illustrated in the next section.

We write the exceptional Krawtchouk operator by

$$\mathcal{L}^{(j,d)} = \mathcal{F}^{(j,d)} \circ \mathcal{B}^{(j,d)} + \lambda_d^{(j)} I \quad (31)$$

for $j \in \{1, 2, 3, 4\}$ and $n, d \in \mathbb{Z}_{\geq 0}$. 
From the discussion above, for \( \ell, j \in \{1, 2, 3, 4\} \) and \( n, d \in \mathbb{Z}_{\geq 0} \), we find that the solutions to the eigenvalue problem

\[
\mathcal{L}^{(j,d)} \left[ \psi_{\ell,n}^{(j,d)}(x) \right] = \lambda_n^{(\ell)} \psi_{\ell,n}^{(j,d)}(x).
\]

are formally given by

\[
\psi_{\ell,n}^{(j,d)}(x) = \mathcal{F}^{(j,d)}[\phi_n^{(\ell)}(x)],
\]

if \( \mathcal{F}^{(j,d)}[\phi_n^{(\ell)}(x)] \neq 0 \) and otherwise (if \( \mathcal{F}^{(j,d)}[\phi_n^{(\ell)}(x)] = 0 \)), \( \psi_{\ell,n}^{(j,d)}(x) \) is taken from \( \text{Ker}(\mathcal{F}^{(j,d)} \circ \mathcal{B}^{(j,d)}) = \{ f(x) \mid \mathcal{F}^{(j,d)} \circ \mathcal{B}^{(j,d)}[f(x)] = 0 \} \).

4. Exceptional Krawtchouk polynomials

In the previous section, we discussed the four types of seed functions for the Darboux transformation. In order to construct the exceptional Krawtchouk polynomials \( \hat{K}_n^{(j,d)}(x) \), we have to consider the Darboux transformation of the Krawtchouk polynomials, that is, \( \mathcal{F}^{(j,d)}[\phi_n^{(1)}(x)] = \mathcal{F}^{(j,d)}[K_n(x)] \). According to the Casoratian determinantal expression of the exceptional Askey-Wilson polynomials [18,19], if \( \mathcal{F}^{(j,d)}[K_n(x)] \neq 0 \), it is given as follows in terms of the second-order Casoratian determinant:

\[
\hat{K}_n^{(j,d)}(x) = \hat{K}_n^{(j,d)}(x; p, N) = \frac{\mathcal{F}^{(j,d)}[K_n(x)]}{\nu_n} \left| \begin{array}{cc} \phi_n^{(j)}(x) & K_n(x) \\ \phi_n^{(j)}(x + 1) & K_n(x + 1) \end{array} \right|, 
\]

where \( \nu_n \) and \( \eta(x) \) are chosen so that the eigenfunctions are monic polynomials by removing the common factors for any \( n \in \mathbb{Z}_{\geq 0} \). It is straightforward to see from (22) that

\[
\hat{K}_n^{(1,d+N+1)}(x) = \hat{K}_n^{(2,d)}(x), \\
\hat{K}_n^{(3,d+N+1)}(x) = \hat{K}_n^{(4,d)}(x).
\]

Thus, in the rest of this paper, we shall restrict \( d \) to be in the range \( 0 \leq d \leq N \) when \( j = 1 \) or 3. For each of the cases \( j = 1 \) to 4, one can verify that

\[
\begin{align*}
| K_d(x) & \quad K_n(x) \\
K_d(x + 1) & \quad K_n(x + 1) |
= (n - d)x^{n+d-1} + \cdots,
\end{align*}
\]

\[
\begin{align*}
| \phi_d^{(2)}(x) & \quad K_n(x) \\
\phi_d^{(2)}(x + 1) & \quad K_n(x + 1) |
= (n - d - N - 1)(x - N + 1)N(x^{n+d} + \cdots),
\end{align*}
\]

\[
\begin{align*}
| \phi_d^{(3)}(x) & \quad K_n(x) \\
\phi_d^{(3)}(x + 1) & \quad K_n(x + 1) |
= \frac{(p - 1)^x}{p^{x+1}}(x^{n+d} + \cdots),
\end{align*}
\]

\[
\begin{align*}
| \phi_d^{(4)}(x) & \quad K_n(x) \\
\phi_d^{(4)}(x + 1) & \quad K_n(x + 1) |
= \frac{(p - 1)^x}{p^{x+1}}(x - N + 1)N(x^{n+d+1} + \cdots). 
\end{align*}
\]
In order that \( \hat{K}_n^{(j,d)}(x) \) are monic polynomials, we chose \( v_n \) and \( \eta(x) \) as follows:

\[
\begin{align*}
\nu_n = \nu_n^{(j,d)} &= \begin{cases} 
  d - n & (j = 1) \\
  d - n + N + 1 & (j = 2) \\
  1 & (j = 3, 4)
\end{cases}, \\
\eta(x) = \eta^{(j)}(x) &= \begin{cases} 
  -1 = -\xi^{(1)}(x) & (j = 1) \\
  -(x - N + 1)N = (N - x)^{-1}\xi^{(2)}(x) & (j = 2) \\
  p^{-x-1}(p - 1)^x = p^{-1}\xi^{(3)}(x) & (j = 3) \\
  p^{-x-1}(p - 1)^x(x - N + 1)N = p^{-1}(x - N)^{-1}\xi^{(4)}(x) & (j = 4)
\end{cases}.
\end{align*}
\] (37)

In these cases, the degree of \( \hat{K}_n^{(j,d)}(x) \) is given by

\[
\deg[\hat{K}_n^{(j,d)}(x)] = \begin{cases} 
  n + d - 1 & (j = 1) \\
  n + d & (j = 2 \text{ or } 3) \\
  n + d + 1 & (j = 4)
\end{cases}.
\] (38)

Note that the cases where \( \nu_n^{(1,d)} = 0 \) and \( \nu_n^{(2,d)} = 0 \) need to be discussed separately.

For cases 3 and 4, observe that the relations below are satisfied on the grid points \( x \in \{0, 1, 2, \ldots, N - 1\} \), where the orthogonality holds:

\[
\begin{align*}
\hat{K}_n^{(3,d)}(x) &= \gamma_{n,d,N}\hat{K}_n^{(1,N-d)}(x), \quad (39a) \\
\hat{K}_n^{(4,d)}(x) &= \gamma_{n,d,N}^*p^{-x}(p - 1)^x\hat{K}_{N-n}^{(2,d)}(N - x - 1), \quad (39b)
\end{align*}
\]

with \( \gamma_{n,d,N} = \frac{p^{d-N}(p-1)^d(-N)_d}{(-1)^d(-N)_{N-d}}(n + d - N) \) and \( \gamma_{n,d,N}^* = \frac{p^n(p-1)^{n+1}(-N)_n}{(-1)^{d+1}(-N)_{N-n}}(n + d + 1) \).

Therefore, cases 3 and 4 can be obtained from \( \hat{K}_n^{(1,d)}(x) \) and \( \hat{K}_n^{(2,d)}(x) \).

From these results, we have obtained \( \hat{K}_n^{(j,d)}(x) \) as the X-Krawtchouk polynomials which are polynomial eigenfunctions of the X-Krawtchouk operator defined from (31). Since the \( \hat{K}_n^{(j,d)}(x) \) are defined by considering \( F^{(j,d)}[\phi_n^{(1)}(x)] \), the other X-Krawtchouk polynomials are expected to be obtained from \( F^{(j,d)}[\phi_n^{(1)}(x)] \) \( (\ell = 2, 3, 4) \). However, if \( F^{(j,d)}[\phi_n^{(1)}(x)] \) \( \neq 0 \), these functions take zero values at \( x = 1, 2, \ldots, N \) when \( \ell = 2 \) and do not give polynomial eigenfunctions in \( x \) when \( \ell = 3, 4 \). Therefore, in the case of \( F^{(j,d)}[\phi_n^{(1)}(x)] \) \( \neq 0 \), \( \hat{K}_n^{(j,d)}(x) \) are the only X-Krawtchouk polynomials. It is straightforward to see that the X-Krawtchouk polynomials \( \hat{K}_n^{(j,d)}(x) \) are polynomial eigenfunctions of the X-Krawtchouk operator:

\[
F^{(j,d)} \circ B^{(j,d)} \hat{K}_n^{(j,d)}(x) = (-n - \lambda_d^{(j)}(x))\hat{K}_n^{(j,d)}(x).
\] (40)

One can also find other polynomial eigenfunctions to this problem when \( F^{(j,d)}[\phi_n^{(1)}(x)] = 0 \). We have then the following cases:

\[
\{(j, d, \ell, n) \mid F^{(j,d)}[\phi_n^{(1)}(x)] = 0\} = \{(j, d, j, d) \mid 1 \leq j \leq 4, d \geq 0\} \\
\cup \{(2, d, 1, d + N + 1) \mid d \geq 0\} \\
\cup \{(4, d, 3, d + N + 1) \mid d \geq 0\}.
\] (41)
When \((\ell, n) = (j, d)\), we only have to consider \(\ker (B^{(j,d)})\), i.e. \(B^{(j,d)}\psi(x) = 0\), from where we obtain

\[
\psi^{(j,d)}_{j,d}(x) = \begin{cases} 
(1)^{x+N}(1 - p)x^{j}p^{-x}(-x)_{N} & (j = 1) \\
(1)^{x}(1 - p)x^{j}p^{-x} & (j = 2) \\
(1)^{N}(-x)_{N} & (j = 3) \\
1 & (j = 4)
\end{cases}
\]  

(42)

We can determine that \(\psi^{(4,d)}_{4,d}(x) = 1\) belongs to the class of X-Krawtchouk polynomials \({\hat{K}}^{(4,d)}_{n}(x)\) and we then introduce

\[
\hat{K}^{(4,d)}_{-d-1}(x) = \psi^{(4,d)}_{4,d} = 1,
\]  

(43)

where the subscript of \(\hat{K}^{(4,d)}_{-d-1}(x)\) is chosen to keep \(\deg \hat{K}^{(4,d)}_{n}(x) = n + d + 1\). In the case of \(\psi^{(3,d)}_{3,d}(x)\), it should be remarked that \(\psi^{(3,d)}_{3,d}(x) = \hat{K}^{(j,d)}_{N-d}(x) \propto \psi^{(3,d)}_{1,N-d}(x)\). In the other cases, we should directly solve \(\mathcal{F}^{(j,d)} \circ B^{(j,d)} \psi = 0\) to find

\[
\hat{K}^{(2,d)}_{N+d+1}(x) = \psi^{(2,d)}_{1,N+d+1} = \hat{K}^{(2,d)}_{N+d+1}(x; p, N)
\]

\[
= \sum_{0 \leq k, j \leq d} \sum_{\ell = 0}^{N+k+j+1} (-1)^{k+j+\ell} (-d)_{j}(-d)_{k}(1 - p)^{d-k}(-p)^{N+d+1+k-\ell} \frac{1}{j! k! \ell!} \times (N + j + k + 1)! (N - d - 1)_{d-j}(N - d - 1)_{d-k}(-x - k - 1)_{\ell}
\]

\[
= \sum_{0 \leq k, j \leq d} (-d)_{j}(-d)_{k}(1 - p)^{d-k}p^{d-j} \frac{1}{j! k!} \times (N - d - 1)_{d-j}(N - d - 1)_{d-k}(-x - k - 1)_{\ell} \times K^{N+k+j+1}(x + k + 1; p, N + k + j + 1),
\]  

(44)

so that \(B^{(2,d)}[\hat{K}^{(2,d)}_{N+d+1}(x)] = K^{(2,d)}_{N+d+1}(x) \in \ker (\mathcal{F}^{(2,d)})\). In a similar way, the functions \(\psi^{(4,d)}_{3,N+d+1}\) can also be calculated although they are not polynomials in \(x\). It should be remarked here that \(\hat{K}^{(2,d)}_{N+d+1}(x)\), that is a newly found state-adding, can formally be obtained via

\[
\hat{K}^{(2,d)}_{N+d+1}(x) = \lim_{\varepsilon \to 0} \hat{K}^{(2,d)}_{N+d+1}(x; p, N + \varepsilon),
\]  

(45)

although it is difficult to find its explicit expression as in (44). In addition, the Krawtchouk polynomials can be recovered by acting with \(B^{(j,d)}\) on the X-Krawtchouk polynomials \(\hat{K}^{(j,d)}_{n}(x)\) so as to find

\[
B^{(j,d)}[\hat{K}^{(j,d)}_{n}(x)] = \hat{v}^{(j,d)}_{n} K^{(j,d)}_{n}(x),
\]  

(46)

with \(\hat{v}^{(j,d)}_{n} = \delta_{1,j} + \delta_{2,j} + (\delta_{3,j} + \delta_{4,j})(\lambda_{n} - \lambda^{(j)}_{d})\) and \(\delta_{ij}\) be a Kronecker delta, except in the cases where \(\hat{v}^{(j,d)}_{n} = 0\), i.e.

\[
B^{(3,d)}[\hat{K}^{(3,d)}_{N-d}(x)] = B^{(4,d)}[\hat{K}^{(4,d)}_{-d-1}(x)] = 0.
\]  

(47)
Here we give a list of examples of $\hat{K}_n^{(2,2)}$ for $p = 1/2$ and $N = 2$.

\begin{align*}
\hat{K}_0^{(2,2)}(x) &= x^2 - x + 1, \\
\hat{K}_1^{(2,2)}(x) &= (x^2 - x + 1/2)(x - 1/2), \\
\hat{K}_2^{(2,2)}(x) &= (x^2 - x + 1/2)(x^2 - x - 1), \\
\hat{K}_3^{(2,2)}(x) &= (x^4 - 2x^3 - x^2 + 2x + 5/2)(x - 1/2)(x^2 - x - 3/2),
\end{align*}

and all other $\hat{K}_n^{(2,2)}(x)$ for $n \in \mathbb{Z}_{\geq 0}\setminus\{0, 1, 2, 5\}$ has a common factor $(x + 1)x(x - 1)(x - 2)$. Let us denote the set of zeros of $\hat{K}_n^{(2,2)}(x; p, N)$ in the interval $[-1, N]$ by $\zeta_n^{(2,2)}$. Then we have

\begin{align*}
\zeta_{0; 1/2, 2}^{(2,2)} &= \emptyset, \\
\zeta_{1; 1/2, 2}^{(2,2)} &= \left\{ \frac{1}{2} \right\}, \\
\zeta_{2; 1/2, 2}^{(2,2)} &= \left\{ -\frac{\sqrt{5}}{2} + \frac{1}{2}, \frac{\sqrt{5}}{2} + \frac{1}{2} \right\}, \\
\zeta_{5; 1/2, 2}^{(2,2)} &= \left\{ \frac{1}{2}, \frac{\sqrt{7}}{2}, \frac{1}{2} - \frac{\sqrt{7}}{2} \right\}.
\end{align*}

5. Properties of the exceptional Krawtchouk polynomials

We have introduced all classes of $X$-Krawtchouk polynomials $\hat{K}_n^{(j,d)}$ by means of a Darboux transformation. These $\hat{K}_n^{(j,d)}$ are sometimes called the $X$-Krawtchouk polynomials of type $j$. In this section, we shall examine their properties.

5.1. Diophantine property/factorization

Analogues of the Diophantine property of the ordinary Krawtchouk polynomials (7) can be found for the $X$-Krawtchouk polynomials and will be called the Diophantine properties of the $X$-Krawtchouk polynomials. They are listed below. When $n > N$, we have

\begin{align*}
\hat{K}_n^{(1,d)}(x) &= (x - N + 1)_N \hat{K}_n^{(2,d)}(x - N - 1; p, -N - 2) \quad (n \neq d), \\
\hat{K}_n^{(2,d)}(x) &= (x - N)_{N+2} \hat{K}_n^{(1,d)}(x - N - 1; p, -N - 2) \quad (n \neq N + d + 1), \\
\hat{K}_n^{(3,d)}(x) &= (x - N + 1)_N \hat{K}_n^{(4,d)}(x - N - 1; p, -N - 2), \\
\hat{K}_n^{(4,d)}(x) &= (x - N)_{N+2} \hat{K}_n^{(3,d)}(x - N - 1; p, -N - 2),
\end{align*}

and

$$
\hat{K}_{N-d}^{(3,d)}(x) = (x - N + 1)_N.
$$
When $d > N$, we have

$$\hat{K}_n^{(1,d)}(x) = (x - N + 1)_N \hat{K}_n^{(2,d-N-1)}(x),$$

$$\hat{K}_n^{(3,d)}(x) = (x - N + 1)_N \hat{K}_n^{(4,d-N-1)}(x).$$

When $n, d > N$, we have

$$\hat{K}_n^{(1,d)}(x) = (-x)_N (-1 - x)_{N+2} \hat{K}_{n-N-1}^{(1,d-N-1)}(x - N - 1; p, -N - 2) \quad (n \neq d),$$

$$\hat{K}_n^{(3,d)}(x) = (-x)_N (-1 - x)_{N+2} \hat{K}_{n-N-1}^{(3,d-N-1)}(x - N - 1; p, -N - 2),$$

where we have used that for $m \geq 0$ and $d > N$,

$$\hat{K}_m^{(2,d)}(x; p, -N - 2) = (x + 1)_{N+2} \hat{K}_m^{(1,d-N-1)}(x; p, -N - 2),$$

$$\hat{K}_m^{(4,d)}(x; p, -N - 2) = (x + 1)_{N+2} \hat{K}_m^{(3,d-N-1)}(x; p, -N - 2).$$

### 5.2. Orthogonality

The orthogonality relations of the X-Krawtchouk polynomials can be obtained from the orthogonality relations (4) of the (original) Krawtchouk polynomials as follows:

$$(\lambda_n - \lambda_d^{(j)}) h_n \delta_{n,m} = (\lambda_n - \lambda_d^{(j)}) \sum_{x=0}^{N} w(x) K_m(x) K_n(x)$$

$$= \sum_{x=0}^{N} w(x) K_m(x) (B^{(j,d)} \circ F^{(j,d)}) [K_n(x)]$$

$$= \sum_{x=0}^{N} w(x) K_m(x) \frac{v_n^{(j,d)}}{\phi_d^{(j)}(x)} \left( p(N-x) I - x(1-p) T^{-1} \right) \eta^{(j)}(x) \hat{\xi}^{(j,d)}(x)$$

$$= \sum_{x=0}^{N} K_m(x) \frac{v_n^{(j,d)}}{p_d^{(j)}(x)} \left( G_0^{(j)}(x) \hat{\xi}^{(j,d)}(x) - G_1^{(j)}(x) \hat{\xi}^{(j,d)}(x-1) \right),$$

where $\delta_{ij}$ is a Kronecker delta and

$$G_0^{(j)}(x) = p(N-x) w(x) \eta^{(j)}(x)/\xi^{(j)}(x),$$

$$G_1^{(j)}(x) = x(1-p) w(x) \eta^{(j)}(x-1)/\xi^{(j)}(x).$$

For $j = 1, 2, 3, 4$, by using $G_0^{(j)}(x) \xi^{(j)}(x) = G_1^{(j)}(x + 1) \xi^{(j)}(x + 1)$ and

$$G_0^{(1)}(N) = G_1^{(1)}(0) = 0,$$

$$G_0^{(2)}(N+1) = G_1^{(2)}(-1) = 0,$$

$$G_0^{(3)}(N) = G_1^{(3)}(0) = 0,$$

$$G_0^{(4)}(N+1) = G_1^{(4)}(-1) = 0,$$
we can rewrite (55) as

$$\frac{\lambda_n - \lambda_d^{(j)}}{v_n^{(j,d)}} h_n \delta_{n,m}$$

\[= \sum_{x=-e(j)}^{N-1+e(j)} \frac{G_0^{(j)}(x)}{\nu_m^{(j,d)}} \left( \frac{K_m(x)}{P_d^{(j)}} - \frac{\xi^{(j)}(x)}{\xi^{(j)}(x+1)P_d^{(j)}} \right) \hat{K}_n^{(j,d)}(x) \]

\[= \sum_{x=-e(j)}^{N-1+e(j)} w^{(j,d)}(x) \hat{K}_n^{(j,d)}(x) \hat{K}_m^{(j,d)}(x), \quad (58)\]

where \(e(j) = \delta_{2,j} + \delta_{4,j}\) and

$$w^{(j,d)}(x) = \frac{G_0^{(j)}(x) \eta^{(j)}(x)}{\xi^{(j)}(x+1)P_d^{(j)}}. \quad (59)$$

In the subsequent subsections, we will discuss the explicit form of the orthogonality relations for each \(j = 1, 2, 3, 4\).

### 5.2.1. Type 1 and type 3

From the Diophantine property (50a), one finds that for \(x \in \{0, 1, 2, \ldots, N-1\}\), \(\hat{K}_n^{(1,d)}(x)\) can take a non-zero value only when \(n, d \in \{0, 1, 2, \ldots, N\}\). Suppose that \(d \in \{0, 1, 2, \ldots, N\}\) and let \(X_{(1,d)} = \{0, 1, \ldots, d-1, d+1, \ldots, N\}\). In this case, the so-called state-deletion occurs. The set \(\{\hat{K}_n^{(1,d)}(x) \mid n \in X_{(1,d)}\}\) has the discrete orthogonality:

$$\sum_{x=0}^{N-1} \hat{w}^{(1,d)}(x) \hat{K}_n^{(1,d)}(x) \hat{K}_m^{(1,d)}(x) = \hat{h}_n^{(1,d)} \delta_{n,m} \quad (60)$$

for \(n, m \in X_{(1,d)}\), where

$$\hat{w}^{(1,d)}(x) = \frac{w^{(1,d)}(x)}{Np(p-1)} = \frac{N-x}{N(1-p)} \frac{w(x; p, N)}{P_d^{(1)}(x)P_d^{(1)}(x+1)} = \frac{w(x; p, N-1)}{P_d^{(1)}(x)P_d^{(1)}(x+1)},$$

$$\hat{h}_n^{(1,d)} = \frac{(\lambda_d^{(1)} - \lambda_n)h_n}{(v_n^{(1,d)})^2 Np(1-p)} = \frac{h_n}{(n-d)Np(1-p)}. \quad (61)$$

Only when \(d = 0, N\), does the weight function \(\hat{w}^{(1,d)}(x; p, N)\) not change sign on \(x \in \{0, 1, \ldots, N-1\}\). This implies that the exceptional Krawtchouk polynomials of type 1 \(\{\hat{K}_n^{(1,d)}\}_{n \in X_{(1,d)}}\) are positive definite when \(d = 0, N\).

In the same fashion, we observe from (50c) that the X-Krawtchouk polynomials of type 3 \(\hat{K}_n^{(3,d)}\) on \(x \in \{0, 1, \ldots, N-1\}\) take a non-zero value when \(n \in X_{(3,d)} = \{0, 1, \ldots, N-d-1, N-d+1, \ldots, N\}\). Furthermore, from (39a) we see that the orthogonality for the set \(\{X_n^{(3,d)}\}_{n \in X_{(3,d)}}\) is essentially the same as that for the X-Krawtchouk polynomials of type 1.
5.2.2. Type 2 and type 4
In this case, a state-addition occurs. Let \( X_{(2,d)} = \{0, 1, \ldots, N, N + d + 1\} \). The set \( \{\hat{K}^{(2,d)}_n(x) \mid n \in X_{(2,d)}\} \) has the discrete orthogonality:

\[
\sum_{x=-1}^{N} \hat{w}^{(2,d)}(x) \hat{K}^{(2,d)}_n(x) \hat{K}^{(2,d)}_m(x) = \hat{h}^{(2,d)}_n \delta_{n,m}
\] (62)

for \( n, m \in X_{(2,d)} \), where

\[
\hat{w}^{(2,d)}(x) = -(N + 1)w^{(2,d)}(x) = \frac{p(N + 1)}{(x + 1)} \frac{w(x; p, N)}{P^{(2)}_d(x)P^{(2)}_d(x + 1)}
\]

\[
\hat{h}^{(2,d)}_n = \frac{(\lambda_n - \lambda^{(2)}_d)(N + 1)h_n}{(v^{(2,d)}_n)^2} = \frac{(N + 1)h_n}{N + d + 1 - n}
\]

\[
= \binom{N + d + 1}{n} \frac{(N - n + 1)d(n!)^2p^n(1 - p)^n}{(N + 2)_d}
\] (63)

When \( d \) is an even integer, one finds that the weight function \( \hat{w}^{(2,d)}(x) \) is positive on \( x \in X_{(2,d)} \). As in the cases of type 1 and of type 3 X-Krawtchouk polynomials, we find from (50d) that the X-Krawtchouk polynomials of type \( 4 \) \( \hat{K}^{(4,d)}_n(x) \) on \( x \in \{-1, 0, 1, \ldots, N\} \) take a non-zero value when \( n \in X_{(4,d)} = \{-d - 1, 0, 1, \ldots, N\} \). Furthermore, from (39b) we see that the orthogonality for \( \{X_{(4,d)}^n\}_{n \in X_{(4,d)}} \) is essentially the same as that of the X-Krawtchouk polynomials of type 2.

5.3. Recurrence relations
The ordinary orthogonal polynomials are known to satisfy the three term recurrence relations. However XOPs satisfy the recurrence relations with more terms [13]. In order to derive the recurrence formula of the X-Krawtchouk polynomials, we first give the resultant of the Krawtchouk polynomials [11]. For the polynomials

\[
f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0,\]

\[
g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0,\] (64)
we denote the resultant of \( f(x), g(x) \) by \( \text{Res}(f(x), g(x)) \), which is defined as follows:

\[
\text{Res}(f(x), g(x)) = \begin{vmatrix}
 a_m & a_{m-1} & \cdots & \cdots & a_0 \\
 \vdots & \ddots & \ddots & \ddots & \vdots \\
 a_m & a_{m-1} & \cdots & \cdots & a_0 \\
 \vdots & \ddots & \ddots & \ddots & \vdots \\
 b_n & b_{n-1} & \cdots & \cdots & b_0 \\
 \end{vmatrix}.
\]  \ (65)

**Lemma 5.1:** For \( n = 1, 2, \ldots \) and \( a \in \mathbb{R} \), the following relation holds:

\[
\text{Res} \left( K_n(x; p, a), K_n(x + 1; p, a) \right) \\
= (-n)^n \text{Res} \left( K_{n-1}(x; p, a-1), K_n(x + 1; p, a) \right),
\]

\[
\text{Res} \left( K_n(x; p, a-1), K_{n+1}(x + 1; p, a) \right) \\
= ((n-a)np(1-p))^n \text{Res} \left( K_{n-1}(x; p, a-1), K_n(x + 1; p, a) \right) \quad \ \ \ (66)
\]

and

\[
\text{Res} \left( K_n(x; p, a), K_n(x + 1; p, a) \right) \\
= n^n \prod_{k=1}^{n-1} k^k (k - a)^k p^{n(n-1)/2} (1 - p)^{n(n-1)/2}. \quad \ \ \ (67)
\]

**Proof:** Recall that \( K_n(x; p, a) = (-a)_n(p) \binom{2F1}{n, \frac{-x}{a}; \frac{1}{p}} \). By using the contiguous relations of the hypergeometric functions [2]:

\[
2F1 \left[ \begin{array}{c}
-n, 1-x \\
-a
\end{array}; \frac{1}{p} \right] - 2F1 \left[ \begin{array}{c}
-n, -x \\
-a
\end{array}; \frac{1}{p} \right] = \frac{n}{ap} \cdot 2F1 \left[ \begin{array}{c}
1-n, 1-x \\
1-a
\end{array}; \frac{1}{p} \right],
\]

\[
ap \cdot 2F1 \left[ \begin{array}{c}
-n, 1-x \\
-a
\end{array}; \frac{1}{p} \right] + x \cdot 2F1 \left[ \begin{array}{c}
-n, 1-x \\
1-a
\end{array}; \frac{1}{p} \right] = ap \cdot 2F1 \left[ \begin{array}{c}
-n, -x \\
-a
\end{array}; \frac{1}{p} \right],
\]

\[
a \cdot 2F1 \left[ \begin{array}{c}
-n, -x \\
-a
\end{array}; \frac{1}{p} \right] + (n-a) \cdot 2F1 \left[ \begin{array}{c}
-n, 1-x \\
1-a
\end{array}; \frac{1}{p} \right] \\
= n \left( 1 - \frac{1}{p} \right) \cdot 2F1 \left[ \begin{array}{c}
1-n, 1-x \\
1-a
\end{array}; \frac{1}{p} \right], \quad \ \ \ (68)
\]

and \( \text{Res}(K_1(x; p, a), 1) = 1 \), simple calculations show that (66) holds and (67) can be obtained by induction. \( \Box \)

From Lemma 5.1, we can obtain the following corollary.

**Corollary 5.2:** \( K_n(x; p, a) \) and \( K_n(x + 1; p, a) \) have no common zeros if and only if \( p \notin \{0, 1\} \) and \( a \notin \{1, 2, \ldots, n-1\} \).
Proposition 5.3: If \( q(x) \in \mathbb{R}[x] \), then \( B^{(j,d)}[q(x)] \in \mathbb{R}[x] \) if and only if
\[
q(x) \in \text{span}\{\hat{K}_{n}^{(j,d)}(x)\}_{n \in \mathbb{Z}_{\geq 0}^{(j,d)}},
\]
where
\[
\mathbb{Z}_{\geq 0}^{(j,d)} = \begin{cases} 
\mathbb{Z}_{\geq 0} \setminus \{d\} & (j = 1) \\
\mathbb{Z}_{\geq 0} & (j = 2, 3) \\
\mathbb{Z}_{\geq 0} \cup \{-d - 1\} & (j = 4)
\end{cases}
\]

Proof: Assume that \( B^{(j,d)}[q(x)] \in \mathbb{R}[x] \) and write \( r(x) = B^{(j,d)}[q(x)] \). We have
\[
\begin{cases} 
p(x - N)q(x) + (1 - p)xq(x - 1) = P_{d}^{(1)}(x)r(x) & (i = 1) \\
pq(x) + (1 - p)xq(x - 1) = P_{d}^{(2)}(x)r(x) & (i = 2) \\
(N - x)q(x) + xq(x - 1) = P_{d}^{(3)}(x)r(x) & (i = 3) \\
-q(x) + q(x - 1) = P_{d}^{(4)}(x)r(x) & (i = 4)
\end{cases}
\]

If \( r(x) \) can be expanded in terms of the Krawtchouk polynomials \( \{K_{n}(x)\}_{n \in \mathbb{Z}_{\geq 0}} \) as follows
\[
r(x) = \sum_{n=0}^{m} a_{n}^{(j,d)} c_{n}^{(j)} \hat{K}_{n}(x)
\]
with \( m = \deg r(x) \), we can find a particular solution to (71) by using (46) and (47):
\[
q(x) = q_{r}^{(j)}(x) = \sum_{n=0}^{m} c_{n}^{(j)} \hat{K}_{n}^{(j,d)}(x) \in \text{span}\{\hat{K}_{n}^{(j,d)}(x)\}.
\]

Note here that if \( \{c_{n}^{(j)}\} \) are arbitrary constants, then \( r(x) \) is an arbitrary polynomial of degree \( m \) when \( j = 2 \) and \( j = 4 \). But if \( q(x) \) is expanded in terms of the Krawtchouk polynomial and contains \( K_{d}(x) \) when \( j = 1 \), or \( K_{N-d}(x) \) when \( j = 3 \), then the corresponding polynomial \( q(x) \) does not exist and the assumption cannot be satisfied. Hence we can see that by excluding \( K_{d}(x) \) when \( j = 1 \) and \( K_{N-d}(x) \) when \( j = 3 \) from the linear combination of sequence of the Krawtchouk polynomials, \( r(x) \) of the form (72) is a general polynomial of degree \( m \) that can satisfy the assumption (71) for a given \( (j, d) \).

Then, by setting \( q(x) = q_{0}^{(j)}(x) + q_{r}^{(j)}(x) \), the equation of \( q_{0}^{(j)}(x) \) becomes a homogeneous first-order difference equation with \( r(x) = 0 \) in (71), and we see that \( q_{0}^{(j)}(x) \) can be obtained from \( \text{Ker}(B^{(j,d)}) \cap \mathbb{R}[x] \). From (42), we obtain
\[
q_{0}^{(1)}(x) = q_{0}^{(2)}(x) = 0, \quad q_{0}^{(3)}(x) = \alpha_{3} \hat{K}_{N-d}^{(3,d)}(x), \quad q_{0}^{(4)}(x) = \alpha_{4} \hat{K}_{d-1}^{(4,d)}(x),
\]
where \( \alpha_{3} \) and \( \alpha_{4} \) are arbitrary constants, and the general solution of (71) can be written in the following form:
\[
q(x) = q_{0}^{(j)}(x) + q_{r}^{(j)}(x).
\]

Hence we conclude that \( q(x) \in \text{span}\{\hat{K}_{n}^{(j,d)}(x)\}_{n \in \mathbb{Z}_{\geq 0}^{(j,d)}} \). Sufficiency is obvious from (46) and (47).
Proposition 5.4: For \( \pi(x) \in \mathbb{R}[x] \), \( L^{(j,d)}[\pi(x)] \in \mathbb{R}[x] \) if and only if \( B^{(j,d)}[\pi(x)] \in \mathbb{R}[x] \).

Proof: Suppose that \( B^{(j,d)}[\pi(x)] \) is not a polynomial. Then it is given by a rational function of the following form:

\[
B^{(j,d)}[\pi(x)] = \pi_0(x) + \sum_{k=1}^{m} \frac{\pi_k(x)}{(x-a_k)^{\mu_k}}, \quad (75)
\]

where \( \mu_1, \ldots, \mu_m \) are positive integers and \( \pi_0(x), \ldots, \pi_m(x) \) are some polynomials with the conditions \( \pi_k(a_k) \neq 0 \) for \( k = 1, 2, \ldots, m \). We here note that \( a_k \) are the zeros of \( P_d^{(j)}(x) \), i.e. \( P_d^{(j)}(a_k) = 0 \). Then, by applying \( F^{(j,d)} \) to (75), we obtain

\[
L^{(j,d)}[\pi(x)] = \frac{\tilde{\pi}(x)}{\prod_{k=1}^{m}(x-a_k)^{\mu_k}(x+1-a_k)^{\mu_k}}, \quad (76)
\]

where

\[
\tilde{\pi}(x) = (x-a_k)^{\mu_k}(x+1-a_k)^{\mu_k}\pi^*(x) + \rho_j(x)(x-a_k)^{\mu_k}P_d^{(j)}(x)\pi_k(x+1) + \rho_j^*(x)(x+1-a_k)^{\mu_k}P_d^{(j)}(x+1)\pi_k(x) \quad (77)
\]

with \( \pi^*(x) \) a polynomial and

\[
\rho_1(x) = -1, \quad \rho_1^*(x) = 1,
\]

\[
\rho_2(x) = N - x, \quad \rho_2^*(x) = 1 + x,
\]

\[
\rho_3(x) = p, \quad \rho_3^*(x) = 1 - p,
\]

\[
\rho_4(x) = p(x-N), \quad \rho_4^*(x) = (1-p)(1+x).
\]

Using Corollary 5.2, it is shown that if \( P_d^{(j)}(a_k) = 0 \), then \( P_d^{(j)}(a_k + 1) \neq 0 \) and \( P_d^{(j)}(a_k - 1) \neq 0 \), and further that \( \tilde{\pi}(a_k) = 0 \) and \( \tilde{\pi}(a_k - 1) = 0 \) cannot be true at the same time, since

\[
P_d^{(j)}(a_k + 1)\pi_k(a_k) \neq 0, \quad P_d^{(j)}(a_k - 1)\pi_k(a_k) \neq 0 \quad (j \in \{1, 3\}),
\]

\[
(a_k + 1)P_d^{(j)}(a_k + 1)\pi_k(a_k) \neq 0, \quad (a_k - N - 1)P_d^{(j)}(a_k - 1)\pi_k(a_k) \neq 0 \quad (j \in \{2, 4\}).
\]

Thus \( L^{(j,d)}[\pi(x)] \) can never be polynomial if \( B^{(j,d)}[\pi(x)] \) is not polynomial. Hence, if \( L^{(j,d)}[\pi(x)] \) is a polynomial, then \( B^{(j,d)}[\pi(x)] \) is a polynomial.

It is obvious that if \( B^{(j,d)}[p(x)] \) is a polynomial, then \( L^{(j,d)}[p(x)] \) is a polynomial. \( \blacksquare \)

Theorem 5.5: Let \( q_\pi(x) \) be a non-constant polynomial such that

\[
q_\pi(x) \in \begin{cases} 
\text{span}\{\hat{K}_{n}^{(4,d)}(x-N-1,1-p,-N-2)\}_{n \in \mathbb{Z}_{0}^{(4,d)}} (j = 1) \\
\text{span}\{\hat{K}_{n}^{(4,d)}(x,1-p,N)\}_{n \in \mathbb{Z}_{0}^{(4,d)}} (j = 2) \\
\text{span}\{\hat{K}_{n}^{(4,d)}(x-N-1,p,-N-2)\}_{n \in \mathbb{Z}_{0}^{(4,d)}} (j = 3) \\
\text{span}\{\hat{K}_{n}^{(4,d)}(x,p,N)\}_{n \in \mathbb{Z}_{0}^{(4,d)}} (j = 4) 
\end{cases} \quad (78)
\]
Then there exists a sequence \( \{c_{n,\ell}^{(j,d)}\}_{n,\ell} \) satisfying

\[
q_\pi(x) \hat{K}_n^{(j,d)}(x) = \sum_{\ell=n-m}^{n+m} c_{n,\ell}^{(j,d)} \hat{K}_\ell^{(j,d)}(x), \tag{79}
\]

where \( m = \deg q_\pi(x) \geq d + 1 \). In particular, for the lowest degree, \( m = d + 1 \), \( q_\pi(x) \) is given by

\[
q_\pi(x) = \begin{cases} 
K_{d+1}(x + 1; p, N + 1) & (j = 1) \\
K_{d+1}(x - N; p, -N - 1) & (j = 2) \\
K_{d+1}(x + 1; 1 - p, N + 1) & (j = 3) \\
K_{d+1}(x - N; 1 - p, -N - 1) & (j = 4)
\end{cases}. \tag{80}
\]

**Proof:** Let us define

\[
\pi(x) = p(N - x) \frac{\eta^{(j)}(x)}{\xi^{(j)}(x)p_d^{(j)}(x)} (q_\pi(x) - q_\pi(x - 1)). \tag{81}
\]

From (78), one finds that \( \pi(x) \in \mathbb{R}[x] \) and

\[
\deg \pi(x) = m - d - \delta_{2,j} - \delta_{4,j}, \tag{82}
\]

where \( \delta_{i,j} \) is a Kronecker delta. Note that for \( j = 1, 3 \), \( p_d^{(j)}(x) \) does not have a factor of \( x - N \) and then \( \pi(x) \) must have a factor of \( x - N \) in order to satisfy the assumption of this proposition. Thus it follows that \( m \geq d + 1 \), since \( \deg \pi(x) \geq \delta_{1,j} + \delta_{3,j} \). By applying \( B^{(j,d)} \) to \( q_\pi(x) \hat{K}_n^{(j,d)}(x) \), we obtain

\[
B^{(j,d)}[q_\pi(x) \hat{K}_n^{(j,d)}(x)] \\
= \frac{p(N - x)\eta^{(j)}(x)}{\xi^{(j)}(x)p_d^{(j)}(x)} (I - T^{-1})[q_\pi(x)] \hat{K}_n^{(j,d)}(x) + q_\pi(x)B^{(j,d)}[\hat{K}_n^{(j,d)}(x)] \\
= \pi(x) \hat{K}_n^{(j,d)}(x) + q_\pi(x - 1)\nu^{(j,d)}_n K_n(x) \\
= \sum_{\ell=n-m}^{n+m} c_{n,\ell}^{(j,d)} K_\ell(x) \in \mathbb{R}[x], \tag{83}
\]

where we have used (3) and (6). With the help of Propositions 5.3 and 5.4, \( q_\pi(x) \hat{K}_n^{(j,d)}(x) \) can be presented as a linear combination of the exceptional Krawtchouk polynomial \( \hat{K}_n^{(j,d)}(x) \):

\[
q_\pi(x) \hat{K}_n^{(j,d)}(x) = \sum_{\ell} c_{n,\ell}^{(j,d)} \hat{K}_\ell^{(j,d)}(x). \tag{84}
\]

By applying \( B^{(j,d)} \) to both sides of (84), we obtain

\[
B^{(j,d)}[q_\pi(x) \hat{K}_n^{(j,d)}(x)] = \sum_{\ell} c_{n,\ell}^{(j,d)} \nu^{(j,d)}_\ell K_\ell(x), \tag{85}
\]
which can be considered together with (83) to define the range of the sum via \( c_{n,\ell}^{(j,d)} = 0 \) for \(|\ell - n| > d + m\) if \( \tilde{v}_n^{(j,d)} \neq 0 \). Notice that \( \tilde{v}_n^{(4,d)} = \tilde{v}_{n-d}^{(3,d)} = 0 \) and thus we can not find \( c_{n,-d-1}^{(4,d)} \) and \( c_{n,N-d}^{(3,d)} \) from the above method. We can determine \( c_{n,-d-1}^{(4,d)} \) as follows. From the discussion in Section 5.2.2, we recall the orthogonality relation of the type-4 X-Krawtchouk polynomials given by

\[
\left( \hat{K}_n^{(4,d)}(x), \tilde{K}_n^{(4,d)}(x) \right)_4 = \sum_{x=-1}^N \hat{w}_n^{(4,d)}(x) \tilde{K}_n^{(4,d)}(x), \hat{K}_n^{(4,d)}(x) = \hat{h}_n^{(4,d)} \delta_{mn},
\]

with \( \hat{h}_n^{(4,d)} \neq 0 \) for \( m, n \in X_{(4,d)} \). When \( m-d-1 < n \leq N \), one sees from the definition that \( q_\pi(x) \in \text{span}\{ \hat{K}_{n-d-1}, \hat{K}_0, \ldots, \hat{K}_{m-d-1} \} \) and finds that

\[
\left( q_\pi(x), \tilde{K}_n^{(4,d)}(x) \right)_4 = \left( 1, q_\pi(x) \tilde{K}_n^{(4,d)}(x) \right)_4 = \left( \tilde{K}_{n-d-1}(x), q_\pi(x) \tilde{K}_n^{(4,d)}(x) \right)_4 = 0.
\]

Therefore, from (84), one concludes that \( c_{n,-d-1}^{(4,d)} = 0 \ (m-d-1 < n \leq N) \). For \( n > N \), from the Diophantine property (50d) and (84), one sees that the following relation holds:

\[
0 = q_\pi(k) \hat{K}_n^{(4,d)}(k) = \sum_{\ell \in X_{(4,d)}} c_{n,\ell}^{(4,d)} \tilde{K}_\ell^{(4,d)}(k) \ (k = -1, 0, \ldots, N).
\]

From the linear independence of the X-Krawtchouk polynomials \( \{ \tilde{K}_n^{(4,d)}(x) \}_{n \in X_{(4,d)}} \) on \( x = -1, 0, \ldots, N \), one finds that \( c_{n,-d-1}^{(4,d)} = 0 \ (n > N) \).

With respect to \( c_{n,N-d}^{(3,d)} \) from the Diophantine property (50c) and the recurrence relation of the X-Krawtchouk polynomials of type 4, we can immediately see that \( c_{n,N-d}^{(3,d)} = 0 \ (n > m + N - d) \). The relation \( c_{n,N-d}^{(3,d)} = 0 \ (n < -m + N - d) \) is also verified by comparing both sides of (84). Finally, it is shown that (79) holds including for the cases where \( \tilde{v}_n^{(j,d)} = 0 \). Equation (80) is almost obvious from (6) and (34).

If \( F^{(j,d)}[K_n(x)] \neq 0 \), one observes from (83) that

\[
\pi(x) \hat{K}_n^{(j,d)}(x) + q_\pi(x) \tilde{v}_n^{(j,d)} K_n(x)
\]

\[
= \tilde{q}^{(j)}_{m-1}(x)(N-x)K_n(x+1) + \tilde{q}^{(j)}_m(x)K_n(x) + q_\pi(x) \tilde{v}_n^{(j,d)} K_n(x),
\]

where

\[
\tilde{q}^{(j)}_{m-1}(x) = -(\tilde{v}_n^{(j,d)})^{-1} (-p)^{\delta_{xj}+\delta_{kj}} (x-N)^{-\delta_{xj}-\delta_{kj}} \pi(x) P_d^{(j)}(x),
\]

\[
\tilde{q}^{(j)}_m(x) = (\tilde{v}_n^{(j,d)})^{-1} (1-p)^{\delta_{xj}+\delta_{kj}} (x+1)^{\delta_{xj}+\delta_{kj}} \pi(x) P_d^{(j)}(x+1),
\]

with \( \pi(x) \) given by (81). Therefore, using (3) and (6), one can exactly calculate the coefficients of the recurrence relation (79).
Corollary 5.6: Let \( q_\pi(x) \) and \( \pi(x) \) be polynomials satisfying (78) and (81), respectively, and \( X \) be an linear operator which acts on a basis \( \{e_n\}_{n=0}^\infty \) as follows:

\[
X[e_n] = e_{n+1} + b_n e_n + u_n e_{n-1}, \quad n = 0, 1, \ldots
\]

with \( b_n = p(N - n) + n(1 - p), u_n = (N + 1 - n)p(1 - p) \). We introduce the constants \( \{\hat{c}^{(j,d)}_{n,\ell}\} \) defined by the following relation:

\[
\hat{q}^{(j)}_{m-1}(X)\{e_{n+1} + (2n - N)(1 - p)e_n + n(n - N - 1)(1 - p^2)e_{n-1}\}
+ \left(\hat{q}^{(j)}_m(X) + \hat{v}^{(j,d)}_n q_\pi(X)\right)\{e_n\}
= \sum_{\ell = n - m}^{n + m} \hat{c}^{(j,d)}_{n,\ell} \hat{v}^{(j,d)}_\ell e_\ell
\]

(92)

For \((j, n) \neq (2, N + d - 1), (4, -d - 1)\), the following relation holds between \( \{\hat{c}^{(j,d)}_{n,\ell}\} \) and \( \{c^{(j,d)}_{n,\ell}\} \) in (79):

\[
\hat{c}^{(j,d)}_{n,\ell} = c^{(j,d)}_{n,\ell}, \quad (j, \ell) \neq (3, N - d), (4, -d - 1).
\]

(93)

It should be remarked here that \( c^{(3,d)}_{n,N-d} \) and \( c^{(4,d)}_{n,-d-1} \) can be automatically identified from the recurrence relation (84). Furthermore, \( \{c^{(2,d)}_{N+d-1,\ell}\} \) can be calculated in a similar manner by using (44) and \( \{c^{(4,d)}_{-d-1,\ell}\} \) are given from (78) since \( q_\pi(x) \hat{K}^{(4,d)}_{-d-1}(x) = q_\pi(x) \).

5.4. Concrete example: \( \hat{K}^{(2,2)}_n(x) \)

Here we give the explicit form of the properties of the \( X \)-Krawtchouk polynomials \( \{\hat{K}^{(j,d)}_n(x)\} \) with \( j = 2 \) and \( d = 2 \), that were used in [14]. For \( x = -1, 0, \ldots, N \) and \( n \in X_{2,2} = \{0, 1, \ldots, N, N + 3\} \), the type 2 exceptional Krawtchouk polynomials \( \{\hat{K}^{(2,2)}_n(x)\}_{n \in X_{2,2}} \) are defined by

\[
\hat{K}^{(2,2)}_n(x) = \frac{1}{N + 3 - n} \begin{vmatrix} (N - x)K_2(x - N + 1; p, -N - 2) & K_n(x) \\ -(1 + x)K_2(x - N + 2; p, -N - 2) & K_n(x + 1) \end{vmatrix}
\]

(94)

for \( n = 0, 1, \ldots, N \) and

\[
\hat{K}^{(2,2)}_{N+k+1}(x) = \sum_{0 \leq k, j \leq 2} \frac{(-2)^j(-2)_k(p - 1)^{2-k}p^{2-j}}{j!k!(N - 3)_2 - j(N - 3)_2 - k} \times K_{N+k+j+1}(x + k + 1; p, N + k + j + 1).
\]

(95)

The orthogonality relation is given by

\[
\sum_{x=-1}^{N} \hat{w}^{(2,2)}(x) \hat{K}^{(2,2)}_n(x) \hat{K}^{(2,2)}_m(x) = \delta_{n,m},
\]

(96)
where
\[
\hat{w}^{(2,2)}(x) = \frac{w(x + 1; p, N + 1)}{K_2(x - N - 1; p, -N - 2)K_2(x - N; p, -N - 2)},
\]
\[
\hat{h}^{(2,2)}_n = 2! (N + 1)! (N + 3)! (1 - p)^{N+3} p^{N+3}.
\]

It can be easily shown that $K_2(x; p, -N - 2) > 0$ for $x \in (-N - 2, 0)$ and $p \in (0, 1)$, which guarantees the positivity of the weight function $\hat{w}^{(2,2)}(x)$ on $x \in \{-1, 0, \ldots, N\}$. \{\hat{K}^{(2,2)}_n(x)\} satisfies the following 7-term recurrence relation:
\[
q_3(x)\hat{K}^{(2,2)}_n(x) = c_{n,3}\hat{K}^{(2,2)}_{n+3}(x) + c_{n,2}\hat{K}^{(2,2)}_{n+2}(x) + c_{n,1}\hat{K}^{(2,2)}_{n+1}(x)
\]
\[
\quad + c_{n,0}\hat{K}^{(2,2)}_n(x)
\]
\[
\quad + c_{n,-1}\hat{K}^{(2,2)}_{n-1}(x) + c_{n,-2}\hat{K}^{(2,2)}_{n-2}(x) + c_{n,-3}\hat{K}^{(2,2)}_{n-3}(x)
\]
with
\[
q_3(x) = K_3(x - N; p - N - 1) - K_3(-1 - N; p, -N - 1),
\]
\[
c_{n,3} = 1,
\]
\[
c_{n,2} = 3(N - n + 1)(2p - 1),
\]
\[
c_{n,1} = 3(N - n + 2) \left\{ N - n + 1 - (4N - 5n + 2)p(1 - p) \right\},
\]
\[
c_{n,0} = -\sum_{-3 \leq \ell \leq 3, \ell \neq 0} \frac{(3 - N)\ell(-N)\ell}{(1 - N)\ell} p^\ell c_{n,\ell},
\]
\[
c_{n,-1} = 3(N - n + 1)(-n)(p - 1)p(N - n + 4) \left\{ N - n + 2 - (4N - 5n + 7)p(1 - p) \right\},
\]
\[
c_{n,-2} = 3(N - n + 1)(-n)^2(p - 1)^2 p^2(N - n + 5)(2p - 1),
\]
\[
c_{n,-3} = (N - n + 1)(-n)^3(p - 1)^3 p^3(N - n + 6).
\]

Note that $\hat{K}^{(2,2)}_{N+1}(x)$ and $\hat{K}^{(2,2)}_{N+1}(x)$, which are zero-valued at $x = -1, 0, \ldots, N$ on the grid, also appear in (98). Furthermore, when $\frac{1}{2} < p < 1$, the coefficients $\{c_{n,k}\}_{k \neq 0}$ are all positive and when $p = \frac{1}{2}$, $c_{n,2} = c_{n,-2} = 0$ even though the weight function $\hat{w}^{(2,2)}(x)$ still takes positive value at $x = -1, 0, \ldots, N$.

6. Concluding remarks

The exceptional Krawtchouk polynomials derived by a single-step Darboux transformation fall into four classes. Their construction confirmed that, like the other exceptional orthogonal polynomials, the resulting sequence of polynomials has a gap in degree. The weight function which determines their orthogonality is obtained by multiplying the weight function of the Krawtchouk polynomial by an appropriate rational function. Because of the symmetries of the Krawtchouk polynomials, there are various relations among the four classes of exceptional Krawtchouk polynomials. Their factorization or Diophantine properties were also found and revealed the duality between case 1 and case 2 of the exceptional Krawtchouk polynomials, as well as between cases 3 and 4. It was furthermore shown that the space spanned by the exceptional Krawtchouk polynomials can be
characterized as a subspace of polynomials obtained from the action (on polynomials) of the exceptional Krawtchouk operator, thereby showing without using the limiting procedure from the exceptional Meixner polynomials, that there exist $2d+3$-term recurrence relations for the exceptional Krawtchouk polynomials.

In view of the results obtained here for the single-indexed Krawtchouk polynomials, it is natural to expect that interesting properties will occur in the case of the multi-indexed Krawtchouk polynomials resulting from multi-step Darboux transformations. As a follow-up to this paper, further research including the multi-indexed case has been discussed [15,16] and is being actively developed.

Like the Krawtchouk polynomials that have many applications in probability theory, stochastic processes, coding theory, quantum mechanics, etc., the exceptional Krawtchouk polynomials presented in this paper are poised to be similarly useful. This entails fascinating questions, some of which we plan to examine.

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