Solutions to a measured-valued mass evolution problem with flux boundary conditions inspired by crowd dynamics

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Abstract

Motivated by our research on pedestrian flows, we study a non-conservative measure-valued evolution problem posed in a finite interval and explore the possibility of imposing a flux boundary condition. The main steps of our work include the analysis of a suitably scaled regularized problem possessing a boundary layer that accumulates mass and detailed investigations of the boundary layer by means of semigroup techniques in spaces of measures. We consider passage to the singular limit where thickness of the layer vanishes (resembling the fast reaction asymptotics typical for systems with slow transport and rapid reactions). We obtain not only suitable solutions to the measured-value evolution problem, but also derive a convergence rate for the approximation procedure as well as the structure of (flux) boundary conditions for the limit problem.

1 Introduction

This paper discusses the resolution of various analytical issues that arise when one wants to introduce boundary conditions in a measure-valued formulation of an interacting particle system or population dynamical model, in particular flux boundary conditions. The non-smoothness of the setting makes it unclear how to formulate such conditions, since the concept of normal derive cannot be used readily. Even more important from application point of view – which we shall discuss below – is to understand how the boundary conditions, once obtained in any measure-valued formulation, derive from particular types of interaction of particles or individuals at microscopic scale with the boundary in a thin boundary layer. This derivation will involve a limit in which the thickness of this boundary layer shrinks to zero. A natural problem to consider then is to determine in what sense and how well the limit system approximates the microscopic model with boundary interaction layer.

Searching for the correct flux boundary conditions is a topic often addressed in the literature, but this has not yet been treated in the measure-theoretical framework. We refer here for instance to [3, 19, 27, 28] (in the context of reaction and diffusion scenarios) and Gurtin [21] (the shrinking pillbox principle in continuum mechanics). Our motivation stems from insisting on capturing both the direct effect and feedback of granularity (i.e.
presence of a finite set of interacting individuals in a background medium) or of some
local fluctuations (local perturbations by random forces) on the meso- and macro-scale
dynamics of pedestrian flows. As practical goal, we search for the mechanisms responsible
for producing streaming flow patterns able to fluidize locally congested pedestrian flows.
Transporting measures is a tool that perfectly matches this goal especially if one has in
mind infinite domains [13]. What happens at and close to active boundaries (e.g. where
Cauchy fluxes need to be defined) is of primary interest in building design and evacuation
control, but mathematically, coping with boundary (flux) measures in a system of balance
laws is very much open.

Measure-valued formulations gained interest substantially lately, both from fundamen-
tal mathematical perspective (like gradient flows in metric spaces [4]) as in applications
(like evolution of structured cell densities [1 2 11 10 22], crowd dynamics [29 6]). Peo-
desrian crowds and their dynamics in high-density regimes is a modern topic of intense study
not only in security, logistics, and civil engineering (crowd-structure interactions) but also
in non-equilibrium statistical mechanics of social systems; see e.g. [24] (an example of a
very influential paper in the field), and [31] for a detailed overview of the current status
or research, and applied mathematics cf. e.g. [29 6 8] and references cited therein.

In order to focus on the essential mathematical issues, we consider in this paper the
simplest system that we can envision: a system of particles moving in a one dimensional
confined space, say with position \( x \) in the interval \([0,1]\), forced by an externally determined
velocity field \( v(x) \). There is no interaction (yet) among individuals and the boundaries are
‘sticking’ and partially absorbing: once a particle arrives at the boundary its stays there
and can be removed from the system (being ‘absorbed’ or ‘gated’) randomly at a time
after arrival that is exponentially distributed with constant absorption rate \( a \). Formally,
one would expect an equation of the form

\[
\frac{∂}{∂t} \mu_t + \frac{∂}{∂x} (v \mu_t) = -a \mu_t(\{1\}) \delta_1.
\] (1.1)

That is, in the measure-valued formulation for the associated particle system with ‘stick-
ing’ boundary conditions [33] these conditions should be incorporated in the measure-
valued equation (1.1) as a density dependent (point located) sink.

There is an issue with the precise interpretation of (1.1) however. Evaluation of the
measure at the set \( \{1\} \) seems to inhibit a proper interpretation in distributional sense
on \( \mathbb{R}^+ × [0,1] \), which is the point of view taken in e.g. [9 22 10]. The main source of
problems in the analysis of (1.1) is the discontinuity of the map \( \mu \mapsto \mu(\{1\}) \) in the natural
topology on measures for this problem.

Our approach differs from those cited above. It relies on our related preliminary studies
[18 11] (crowd dynamics, balanced mass measures) and [26] (measures in the semigroups
context). Concerning the interpretation of (1.1), we view it as formal expression that the
solution is obtained as perturbation of the semigroup \((P_t)_{t≥0}\) of mass transport operators
on finite Borel measures on \([0,1]\) along characteristics defined by the velocity field \( v \) by
means of a perturbation \( \mu \mapsto -a \mu(\{1\}) \delta_1 \). The solution should satisfy an integral equation
that may be considered as a variation of constants formula for (1.1) when viewed as an
evolutionary equation:

\[
\mu_t = P_t \mu_0 - a \int_0^t \mu_s(\{1\}) \, ds \cdot \delta_1.
\] (1.2)

(see Section 2 for further discussion).

Our approach connects to the approach to structured population models as described
for instance by Diekmann & Getto [14], Gwiazda et al. [22 10] or Ackleh et al. [1 2].
See also Canizo et al. [9] that simplifies part of the approach in [23 10]. Their approach
consists of first studying the situation of independently moving individuals described
by a (semi-)linear system like (1.1). Then use the obtained results to treat the case of forced velocity fields changing in time and finally close the system for a density dependent velocity field, which models interacting individuals. This paper addresses the first step in this ‘program’ for a system with boundary conditions. Work on the remaining steps is in progress.

At points in this paper the reader may find some proofs overly complicated for the particular one-dimensional example that is discussed. However, where possible we have chosen on purpose to give arguments that employ the general structure of the problem rather then the particular. These proofs may then be used as ‘template’ for dealing with the more delicate higher dimensional cases (2D for crowd dynamics, ‘any’ for structured population models) or abstract systems on general Polish spaces with non-empty boundary.

Various technical complications will arise in the higher dimensional setting. The boundary would need to be sufficiently smooth so that a boundary layer can be introduced as a suitable Lipschitz deformation of the boundary. The transport part is expected to become anisotropic, in the sense that some directions will matter more than others. Another generalisation of the currently discussed problem would be a system like

\[ \partial_t \mu_t + \text{div}(v \mu_t) = -aR(\mu_t(\{1\}))Q(\nu_t(\{1\}))\delta_1 \]

\[ \partial_t \nu_t = +aR(\mu_t(\{1\}))Q(\nu_t(\{1\}))\delta_1 \]

(1.3)

and corresponding regularizations, where the partial reaction rates \(Q(\cdot)\) and \(R(\cdot)\) have suitable algebraic structures. (1.3) brings us very close to a more fundamental chemical reaction theme – handling nonlinear chemical kinetics based on the mass-action law. The type of nonlinear structure in (1.3) is the main point of the setting addressed by Fibich et al. in [19].

1.1 Organization of the paper

For convenience of the reader Section 1.2 collects a series of properties and results on finite Borel measures, topologies on spaces of such measures and associated metrising norms and (Bochner) integrals involving measure-valued kernels that will be used throughout the paper. Section 2 sets-up the detailed mathematical model that will be considered. Section 2.1 derives fundamental estimates for the movement of single individuals in the domain, i.e. the individualistic flow, where there is no absorption yet, but sticking boundaries only. Section 2.2 provides a probabilistic underpinning of the fundamental integral equation (1.2). In Section 2.3 we discuss an alternative approach that consists of extension of the domain to an unbounded setting that may seem a way to avoid some technicalities. We show that it suffers from the same type of technical difficulties. Section 3 discusses systems in which there is interaction in a (thin) layer near the boundary. There, well-posedness and positivity for measure-valued solutions is proven for these systems. Finally, Section 4 deals with the integral equation (1.2). Well-posedness is shown in Section 4.1 exploiting the convenient structure of the system. The most important parts are Section 4.2 and 4.3. In the first the unique solution to (1.2) is shown to arise as limit of the solutions of a sequence of thin boundary layer solutions, in the limit of vanishing thickness of the layer. Finally, Section 4.3 computes the rate of convergence of these boundary layer solutions to the solutions of the limit equation.

1.2 Preliminaries on measures

If \(S\) is a topological space, we denote by \(\mathcal{M}(S)\) the space of finite Borel measures on \(S\) and \(\mathcal{M}^+(S)\) the convex cone of positive measures included in it. This cone defines a partial ordering on \(\mathcal{M}(S)\): \(\mu \leq \nu\) iff \(\nu - \mu \in \mathcal{M}^+(S)\). Clearly, \(\mu \leq \nu\) if and only if \(\mu(E) \leq \nu(E)\) for all Borel measurable \(E \subset S\).
For $x \in S$, $\delta_x$ denotes the Dirac measure at $x$. Put
\[
\langle \mu, \phi \rangle := \int_S \phi \, d\mu,
\] (1.4)
the natural pairing between measures $\mu \in M(S)$ and bounded measurable functions $\phi$. If $\Phi : S \to S$ is Borel measurable, the push forward or image measure of $\mu$ under $\Phi$ is the measure
\[
\Phi \# \mu := \mu \circ \Phi^{-1}.
\]

It is easily verified that $\langle \Phi \# \mu, \phi \rangle = \langle \mu, \phi \rangle$. We denote by $C_b(S)$ the Banach space of bounded continuous functions on $S$ equipped with the supremum norm $\| \cdot \|_\infty$. The total variation norm $\| \cdot \|_{TV}$ on $M(S)$ is given by
\[
\| \mu \|_{TV} := \sup \left\{ \langle \nu, \phi \rangle \mid \phi \in C_b(S), \| \phi \|_\infty \leq 1 \right\}.
\] (1.5)

It follows immediately that for $\Phi : S \to S$ continuous, $\| \Phi \# \mu \|_{TV} \leq \| \mu \|_{TV}$.

The total variation norm is too strong for our application, since $\| \delta_x - \delta_y \|_{TV} = 2$ if $x \neq y$. The natural topology to consider is the weak topology induced by $C_b(S)$ through the pairing (1.4). In this topology $x \mapsto \delta_x : S \to M(S)$ is continuous.

In our setting, $S = [0,1]$ is a Polish space. It is well-established (cf. [16, 17]) that in this case the weak topology on the positive cone $M^+(S)$ is metrisable by a metric derived from a norm, e.g. the Fortet-Mourier norm or the Dudley norm, which is also called the dual bounded Lipschitz norm, that we shall introduce now. To that end, let $d$ be a metric on $S$ that metrises the topology, such that $(S,d)$ is separable and complete. Let $BL(S,d) = BL(S)$ be the vector space of real-valued bounded Lipschitz functions on $(S,d)$. For $\phi \in BL(S)$, let
\[
|\phi|_L := \sup \left\{ \frac{|\phi(x) - \phi(y)|}{d(x,y)} \mid x, y \in S, x \neq y \right\}
\] (1.6)
be its Lipschitz constant. Then
\[
\| \phi \|_{BL} := \| \phi \|_\infty + |\phi|_L
\] (1.7)
defines a norm on $BL(S)$ for which it is a Banach space. In fact it makes $BL(S)$ a Banach algebra for pointwise product of functions:
\[
\| \phi \cdot \psi \|_{BL} \leq \| \phi \|_{BL} \cdot \| \psi \|_{BL}.
\] (1.8)

Let $\| \cdot \|_{BL}^*$ be the dual norm on the dual space $BL(S)^*$, i.e. for any $x^* \in BL(S)^*$:
\[
\|x^*\|_{BL}^* := \sup \{|\langle x^*, \phi \rangle| \mid \phi \in BL(S), \|\phi\|_{BL} \leq 1\}.
\] (1.9)

The map $\mu \mapsto I_{\mu}$ with $I_{\mu}(\phi) := \langle \mu, \phi \rangle$ defines a linear embedding of $M(S)$ into $BL(S)^*$ ([16], Lemma 6). Thus $\| \cdot \|_{BL}^*$ induces a norm on $M(S)$, which is denoted by the same symbol. Generally, $\| \mu \|_{BL} \leq \| \mu \|_{TV}$. For positive measures the norms coincide:
\[
\| \mu \|_{BL} = \| \mu \|_{TV} = \| \mu \|_{TV}
\] for all $\mu \in M^+(S)$.

The $\| \cdot \|_{BL}$-norm topology on $M^+(S)$ coincides with the restriction of the weak topology ([16, Theorem 12]). $M(S)$ is not complete for $\| \cdot \|_{BL}$ generally. We denote by $M(S)^{\text{BL}}$ its completion, viewed as closure of $M(S)$ within $BL(S)^*$. $M^+(S)$ is complete for $\| \cdot \|_{BL}^*$, hence closed in $M(S)$ and $M(S)^{\text{BL}}$. Note that $BL(S,d)$ will vary with $d$, hence $\| \cdot \|_{BL}$ on $M(S)$ will depend on $d$ too. The subscript ‘BL’ in the notation for the completion $M(S)^{\text{BL}}$ is intended for reminding to this dependence.
The $\| \cdot \|^{*}_{\text{BL}}$-norm is convenient also for integration. If $(X, \Sigma)$ is a measurable space, then a map $x \mapsto f(x) : X \to \mathcal{M}^{+}(S)_{\text{BL}}$ is Bochner measurable (as a function mapping into the Banach space $\mathcal{M}(S)_{\text{BL}}$) iff $x \mapsto f(x)(E)$ is measurable for each Borel set $E \subset S$ (cf. [23], Proposition 2.5). If $\nu$ is a finite positive measure on $(X, \Sigma)$, $x \mapsto f(x)$ is Bochner measurable and $x \mapsto \|f(x)\|^{*}_{\text{BL}}$ is integrable with respect to $\nu$, then $f$ is Bochner integrable and
\[
\| \int_{X} f(x) \, d\nu(x) \|^{*}_{\text{BL}} \leq \int_{X} \|f(x)\|^{*}_{\text{BL}} \, d\nu(x) \tag{1.11}
\]
(see e.g. [13]). Moreover, for any continuous map $P : \mathcal{M}^{+}(S)_{\text{BL}} \to \mathcal{M}^{+}(S)_{\text{BL}}$ that is additive and positively homogeneous, i.e. $P(a\mu) = aP(\mu)$ for $a \geq 0$, one has
\[
P\left( \int_{S} f(x) \, d\nu(x) \right) = \int_{S} P[f(x)] \, d\nu(x), \tag{1.12}
\]
\[
\left( \int_{S} f(x) \, d\nu(x) \right)(E) = \int_{S} f(x)(E) \, d\nu(x) \tag{1.13}
\]
(cf. [23], Proposition 2.6 for (1.13)). Note that the continuity of the map $x \mapsto \delta_{x} : S \to \mathcal{M}^{+}(S)_{\text{BL}}$ together with [13] yields the useful identity
\[
\mu = \int_{S} \delta_{x} \, d\mu(x). \tag{1.14}
\]

2 Model formulation

An ‘educated guess’ may lead directly to equation (1.1). In cases where the dynamics is more complicated – in higher dimensions, in the interior of the domain, at the boundary – it may not be that evident. In this section we explain how (1.1) can be derived by considering a thin boundary layer in which well-described interactions with the boundary occur, and letting its thickness vanish in a limit. Such an approach would allow to determine the manner in which boundary interactions should be incorporated in the measure-valued formulation. In [19] (in a non-measure setting of concentration functions and surface densities) such an approach was used to establish the right boundary conditions for chemical reactions occurring on a surface in a three dimensional domain.

First, the operator $\mu \mapsto -\frac{d}{ds}(\nu(\mu))$ may be considered as the generator of a strongly continuous semigroup in $\mathcal{M}([0, 1])_{\text{BL}}$: the semigroup $(P_{t})_{t \geq 0}$ of mass transport along characteristics associated to the velocity field $v$, which we shall define rigorously in Section 2.1. Then (1.1), when considered as an evolutionary equation, can be viewed as perturbation of $(P_{t})_{t \geq 0}$ by means of $F : \mu \mapsto -aP(\{1\}) \delta_{1}$. The resulting perturbed solution should satisfy a variation of constants formula of the form
\[
\mu_{t} = P_{t}\mu_{0} + \int_{0}^{t} P_{t-s}F(\mu_{s}) \, ds. \tag{2.1}
\]

Since $\delta_{1}$ is invariant under $(P_{t})_{t \geq 0}$, i.e. $P_{t}\delta_{1} = \delta_{1}$ for all $t$, equation (2.1) reduces to (1.2).

Thus, as (mild) solution to (1.1) we consider any continuous map $\mu : \mathbb{R}^{+} \to \mathcal{M}([0, 1])_{\text{BL}}$ that satisfies (1.2). Note that the integral in (1.2) or (2.1) is well-defined, since $t \mapsto \mu_{t}(\{1\})$ is measurable if $t \mapsto \mu_{t}$ is continuous for the $\| \cdot \|_{\text{BL}}$-topology (see Section 1.2). These integral equations will be the main object of study in the remaining part of the paper.

The main issue with the perturbation formulation (2.1) is, that the map $\mu \mapsto \mu(\{1\})$ is not Lipschitz continuous, not even continuous on $\mathcal{M}([0, 1])_{\text{BL}}$. The standard arguments for
solving such equations use a technique similar to Picard interations, but require Lipschitz continuity of the perturbation term to invoke Banach Fixed Point Theorem.

In Section 4.2 we consider \( F \) as a limit of maps \( F^{(n)} \) that are Lipschitz continuous, using interactions within a boundary layer instead of solely at the boundary. The solution \( \mu^{(n)}_t \) to the regularized integral equations corresponding to \( F^{(n)} \) converges to the solution of (1.2) as \( n \to \infty \) (see Section 1.2). Integral equation (1.2) is closely related to a probabilistic description of the system, see Section 2.2. In Section 2.3 we show that an alternative approach by means of extending the state space \([0,1]\) to \( \mathbb{R}^+ \) meets similar issues as (1.2).

### 2.1 Mass transport along characteristics

We assume that a single particle (‘individual’) is moving in the domain \([0,1]\) deterministically, described by the differential equation for its position \( x(t) \) at time \( t \):

\[
\begin{cases}
  \dot{x}(t) = v(x(t)), \\
  x(0) = x_0,
\end{cases}
\]

where \( v : [0,1] \to \mathbb{R} \) is a Lipschitz function. Thus, a solution to (2.2) is unique, it exists for time up to reaching the boundary 0 or 1 and depends continuously on initial conditions.

Let \( x(\cdot;x_0) \) be this solution and \( I_{x_0} \) be its maximal interval of existence. Put

\[
\tau_0(x_0) := \sup I_{x_0} \in [0,\infty],
\]

i.e. the time at which the solution reaches the boundary (if it happens).

In order to focus on the essence of mathematical issues that arise when having a sticky, partially absorbing boundary, we focus on a situation where there is only one such boundary point can be reached. That is, we assume \( v(0) > 0 \) and \( v(1) > 0 \), such that the boundary point 1 is the only one of interest.

The individualistic flow on \([0,1]\) is the family of maps \( \Phi_t : [0,1] \to [0,1], t \geq 0 \), defined by

\[
\Phi_t(x_0) := \begin{cases}
  x(t;x_0), & \text{if } t \in I_{x_0}, \\
  1, & \text{otherwise.}
\end{cases}
\]

**Lemma 2.1.** \((\Phi_t)_{t \geq 0}\) is a semigroup of Lipschitz transformations of \([0,1]\). Moreover,

(i) \( |\Phi_t|_{L} \leq e^{|v|_{L,t}} \) for \( t \geq 0 \).

(ii) For any \( t, s \in \mathbb{R}^+ \),

\[
\sup_{x \in [0,1]} \left| \Phi_t(x) - \Phi_s(x) \right| \leq ||v||_{L} |t-s|.
\]

**Proof.** (i): The solution \( x(t) = x(t;x_0) \) is given (implicitly) as solution to the integral equation

\[
x(t) = x_0 + \int_0^t v(x(s)) \, ds, \quad \text{for all } t \in I_{x_0}.
\]

Application of Gronwall’s Lemma thus yields

\[
|\Phi_t(x_0) - \Phi_t(x'_0)| \leq |x_0 - x'_0| e^{|v|_{L}t} \quad \text{for all } t \in I_{x_0} \cap I_{x'_0}.
\]

A complication in proving (i) arises from the validity of (2.3) for \( t \) in a set that depends on the initial conditions. Since we assume that \( v(1) > 0 \), there exists \( \varepsilon > 0 \) such that \( t \mapsto \Phi_t(x_0) \) is strictly increasing for \( 1-\varepsilon < x_0 < 1 \). If \( x_0 \in (1-\varepsilon,1) \) and \( x'_0 = 1 \), then by monotonicity,

\[
|\Phi_t(x_0) - 1| \leq |x_0 - 1|
\]
for all \( t \geq 0 \). Note that there exists \( \tau > 0 \) such that for any \( x_0, x'_0 \in [0, 1-e], [0, \tau] \subset I_{x_0} \cap I_{x'_0} \). Thus (2.6) holds for all \( 0 \leq t \leq \tau \) and \( x_0, x'_0 \in [0, 1-e] \). The case \( x_0, x'_0 \in (1-e, 1) \) remains. We may assume \( x_0 < x'_0 \), hence \( I_{x_0} \subset I_{x'_0} \). Put \( t_0 := \sup x_0 \) and \( \tau'_0 := \sup I_{x'_0} \). Then \( \tau'_0 < t_0 \). For \( 0 \leq t \leq t_0 \), estimate (2.6) holds. For \( 0 < t < t_0 \), one has according to (2.7)

\[
|\Phi_t(x_0) - \Phi_t(x'_0)| = |\Phi_{t-t_0}(\Phi_{\tau'_0}(x_0)) - 1| \leq |\Phi_{t-t_0}(x_0) - \Phi_{\tau'_0}(x_0)|
\]

\[
|\Phi_{t-t_0}(x_0) - \Phi_{\tau'_0}(x_0)| \leq |x_0 - x'_0| e^{|v| L t} \leq |x_0 - x'_0| e^{|v| L \tau'}
\]

For \( t \geq t_0 \), \( \Phi_t(x_0) = \Phi_t(x'_0) \). Thus we conclude that (2.6) holds for all \( t \in [0, \tau) \) and \( x_0, x'_0 \in [0, 1] \). By the semigroup property \( \Phi_{t+s} = \Phi_t \circ \Phi_s \), each \( \Phi_t \) is Lipschitz and for \( n \in \mathbb{N} \) large such that \( t/n \leq \tau \),

\[
|\Phi_{t/n} - \Phi_{t/n}| \leq e^{|v| L t}.
\]

(ii): Let \( t, s \in I_{x_0} \). Without loss of generality, assume that \( t > s \).

\[
|x(t) - x(s)| = \left| \int_0^t v(x(\sigma)) d\sigma - \int_0^s v(x(\sigma)) d\sigma \right| \leq \int_s^t |v(x(\sigma))| d\sigma \leq \|v\|_\infty (t-s).
\]

(2.8)

If both \( t, s \in \mathbb{R}^+ \) are not in \( I_{x_0} \), then inequality (2.8) is trivially satisfied. Suppose now that \( s \in I_{x_0} \), while \( t \) is not. Let \( t_0 := \sup I_{x_0} \). Then

\[
|x(t) - x(s)| = |1 - x(s)| = |x(t_0) - x(s)| \leq \|v\|_\infty (t_0 - s),
\]

according to (2.8). Clearly \( t_0 - s \leq t - s \). The estimates are independent of \( x_0 \in [0,1] \).

Thus we obtain (2.4).

Define \( P_t \) to be the lift of \( \Phi_t \) to the space of finite Borel measures \( \mathcal{M}([0,1]) \) by means of push forward under \( \Phi_t \). That is, for all \( \mu \in \mathcal{M}([0,1]) \),

\[
P_t \mu := \Phi_t \# \mu = \mu \circ \Phi_t^{-1}.
\]

(2.9)

Clearly, \( P_t \) maps positive measures to positive measures and \( P_t \) is mass preserving on positive measures. Since the maps \( \Phi_t, t \geq 0 \), form a semigroup, so do the maps \( P_t \) in the space \( \mathcal{M}([0,1]) \). That is, \( (P_t)_{t \geq 0} \) is a Markov semigroup on \([0,1]\). One has \( \|P_t \mu\|_{TV} \leq \|\mu\|_{TV} \) for general \( \mu \in \mathcal{M}([0,1]) \).

**Lemma 2.2.** Let \( \mu \in \mathcal{M}([0,1]) \) and \( t, s \in \mathbb{R}^+ \). Then

(i) \( \|P_t \mu - P_s \mu\|_{BL} \leq \|v\|_\infty \|\mu\|_{TV} |t-s| \).

(ii) \( \|P_t \mu\|_{BL} \leq \max(1, \|\Phi_t\|_{L}) \|\mu\|_{BL} \leq e^{|v| L t} \|\mu\|_{BL} \).

**Proof.** For all \( \phi \in \mathcal{BL}([0,1]) \),

\[
|\langle P_t \mu - P_s \mu, \phi \rangle| = |\langle \mu, \phi \circ \Phi_t - \phi \circ \Phi_s \rangle| \leq \|\mu\|_{TV} \|\phi \circ \Phi_t - \phi \circ \Phi_s\|_{\infty}
\]

\[
\leq \|\mu\|_{TV} \|\phi\|_{L} \sup_{x \in [0,1]} |\Phi_t(x) - \Phi_s(x)| \leq \|\mu\|_{TV} \|\phi\|_{L} \|v\|_{\infty} |t-s|,
\]

where we used Lemma 2.1 (ii) in the last inequality. For the operator norm of \( P_t \) on \( \mathcal{M}([0,1])_{BL} \), use that for any \( \phi \in \mathcal{BL}([0,1]) \),

\[
|\langle P_t \mu, \phi \rangle| = |\langle \mu, \phi \circ \Phi_t \rangle| \leq \|\mu\|_{BL} \|\phi \circ \Phi_t\|_{BL} \leq \|\mu\|_{BL} (\|\phi\|_{\infty} + \|\phi\|_{L} |\Phi_t|_{L}).
\]

The statement in the lemma follows.
We shall need the following result on the time of arrival at the boundary, \( \tau_0(x) \). Put \( S_\partial := \{ x \in [0, 1] \mid \tau_0(x) < \infty \} \), the set of all points that will reach the boundary in finite time.

**Lemma 2.3.** If \( v \) is Lipschitz and \( v(0) > 0 \) and \( v(1) > 0 \), then \( S_\partial \) is non-empty and open in \([0, 1]\) and for each \( x \in S_\partial \) there exists and open neighbourhood \( U \) of \( x \) such that \( \tau_0|_U : U \to \mathbb{R} \) is Lipschitz.

**Proof.** \( S_\partial \) is the connected component of the set \( \{ x \in [0, 1] \ ; v(x) \neq 0 \} \) that contains 1. So it is non-empty and open. Let \( \bar{v} \) be a Lipschitz extension of \( v \) to \( \mathbb{R}^+ \) such that \( |\bar{v}|_L = |v|_L \) and \( ||\bar{v}||_\infty = ||v||_\infty \). Denote the globally existing solutions to \( \bar{x}'(t) = \bar{v}(\bar{x}(t)) \) with initial value \( x \) by \( \bar{x}(t; x) \). The flow map \( (t, x) \mapsto \bar{x}(t; x) \) is locally Lipschitz, because \( v \) is Lipschitz. \( \tau_0(x) \) is implicitly defined by \( \bar{x}(\tau_0(x); x) = 1 \) for \( x \in S_\partial \). Since \( v(1) \neq 0 \), an implicit function theorem for Lipschitzian maps (e.g. [12], Section 7.1 or [30]) yields the type of local Lipschitz continuity of \( \tau_0 \) on \( S_\partial \) as stated. \( \square \)

We will be interested in the part of the time interval \([0, t]\) that an individual spends in the interior of the \([0, 1]\) before possibly sticking to the boundary at 1. This time is given by

\[
\tau_0(x) \wedge t := \min(\tau_0(x), t),
\]

when the individual started at \( x \) at \( t = 0 \).

**Corollary 2.4.** Assume the conditions of Lemma 2.3. Then \( x \mapsto \tau_0(x) \wedge t \) is Lipschitz. Moreover, \( t \mapsto |\tau_0(\cdot) \wedge t|_L \) is non-decreasing. \( t \mapsto \tau_0(x) \wedge t \) is Lipschitz for fixed \( x \in S_\partial \).

**Proof.** The set \( K_t := \{ x \in [0, 1] \mid \tau_0(x) \leq t \} \) is compact, hence it can be covered by finitely many open sets \( U \) as in Lemma 2.3. So \( \tau_0 \) is Lipschitz on this set. The function \( \tau_0(x) \wedge t \) is a Lipschitz extension of this restriction to the full set \([0, 1]\). The sets \( K_t \) are increasing with increasing \( t \), therefore the Lipschitz constants are non-decreasing. For fixed \( x, t \mapsto \tau_0(x) \wedge t \) is the infimum of two Lipschitz functions \( \tau_0(x) \mathbb{1} \) and \( t \) (as a function of \( t \)), hence is Lipschitz. \( \square \)

### 2.2 A probabilistic interpretation yielding the integral equation

Let us consider in this section \( N \) individuals in a confined space, with position \( X_i^t \in [0, 1] \) at time \( t \) say \( (i = 1, \ldots, N) \). We assume there is a sticky boundary at 1 that is partially absorbing. By this we mean that at the absorbing boundary we have a ‘gate’ that absorbs an individual present there a time \( T \) after arrival, which is an exponentially distributed random variable with (constant) rate \( a \). We assume that the individuals are indistinguishable and the absorption of individuals (gating) occurs independently. We denote by \( \pi_i(t) \) the law of \( X_i^t \) when \( X_i^0 \) is distributed according to the probability measure \( \pi_0 \).

Since individuals behave independently from each other, the expected number of individuals in a Borel set \( E \subset [0, 1] \) at time \( t \) say is given by the measure \( \mu_t(E) \), where \( \mu_t \) satisfies

\[
\mu_t(E) = \mathbb{E} \left[ \sum_{i=1}^{N} \mathbb{1}_{X_i^t}(E) \right] = \sum_{i=1}^{N} \pi_i(t)(E). \tag{2.10}
\]

The measures \( \pi_i(t) \) should satisfy

\[
\pi_t(E) = P_t \pi_0(E) - \delta_1(E) \int_0^t a \pi_s(\{1\}) \, ds \cdot \delta_1, \tag{2.11}
\]
such that (2.10) together with (2.11) yields (1.2). To see (2.11), let us introduce the conditional probability

\[ p(t, \Delta t) := \text{Prob} (\text{Individual is gated in } [t, t + \Delta t] \mid X_t = 1) \]

\[ = 1 - e^{-a \Delta t}. \]  

(2.12)

Discretize the time interval [0, t] into \( \nu \) steps of length \( \Delta s_i \) and let \( s_i \) be the left point of the \( i \)-th subinterval. Then the probability that the individual has been gated in [0, t] is approximately

\[ \sum_{i=1}^{\nu} p(s_i, \Delta s_i) \pi(s_i) \{1\} = \sum_{i=1}^{\nu} \frac{p(s_i, \Delta s_i)}{\Delta s_i} \pi(s_i) \{1\} \Delta s_i \]

\[ \rightarrow \int_0^t a \pi_s(\{1\}) ds \text{ as } \nu \to \infty. \]  

(2.13)

Formulating now (2.11) in terms of measures living in \( M^+([0,1])_{BL} \) gives

\[ \pi_t = P_t \pi_0 - \int_0^t a \pi_s(\{1\}) ds. \]  

(2.14)

2.3 An alternative approach by extension

It may seem feasible at first sight to use the extension procedure as used in the proof of Lemma 2.3. That is, extend the velocity field \( v \) on [0, 1] to a field \( \bar{v} \) on \( \mathbb{R}^+ \) such that \( ||\bar{v}||_\infty = ||v||_\infty \) and \( |\bar{v}|_L = |v|_L \). The solutions \( \bar{x}(t) = \bar{x}(t; x_0) \) to

\[ \bar{x}'(t) = \bar{v}(\bar{x}(t)), \quad \bar{x}(0) = x_0 \in \mathbb{R}^+ \]

exist for all time and define a dynamical system \( \Phi_t \) consisting of the solution maps \( \Phi_t(x_0) := \bar{x}(t; x_0) \). Lift \( \Phi_t \) to measures by means of push forward: \( P_t \mu := \Phi_t \# \mu \). We may then project all excess mass in \([1, \infty)\) onto 1 by means of push forward under the map \( \Psi : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto \max(x, 1) \):

\[ P_t := \Psi \# \bar{P}_t \]

\( P_t \) does not depend on the choice of extension \( \bar{v} \), because \( v(1) \geq 0 \). It coincides with the mass transport operators along characteristics of Section 2.1.

One could think, that the extension procedure would also work for describing the partial absorption at the boundary. That is, suppose that at each point \( x \geq 1 \) there is absorption of mass at exponential rate \( a \). A solution to the extended system, \( \bar{\mu}_t \), then should satisfy the variation of constants formula

\[ \bar{\mu}_t = \bar{P}_t \mu_0 - \int_0^t \bar{P}_{t-s}[a \mathbb{1}_{[1, \infty)} \cdot \bar{\mu}_s] ds, \]

(2.15)

where the integral should be interpreted as Bochner integral in \( M(S)_{BL} \), taking values in \( M^+(S) \). Since \( \Psi \) is Lipschitz, the map \( \mu \mapsto \Psi \# \mu \) is continuous on \( M^+(S)_{BL} \). Thus, by applying the latter map to both sides in (2.15) and using \( \Psi \# \bar{P}_t \bar{\mu} = P_t(\Psi \# \bar{\mu}) \), one obtains

\[ \Psi \# \bar{\mu}_t = P_t \mu_0 - \int_0^t P_{t-s}[a(\Psi \# \bar{\mu}_s)(\{1\}) \cdot \delta_1] ds. \]
That is, \( \mu_t := \Psi \# \bar{\mu}_t \) satisfies (2.1).

Apparently, one may consider (2.15) instead of (2.14) with the mentioned technical complications. Integral equation (2.15) suffers from similar problems however. The perturbation \( F : \mu \mapsto -a \mathbb{1}_{[1,\infty)} \cdot \mu \) is continuous as map \( \mathcal{M}(S)_{TV} \to \mathcal{M}(S)_{TV} \), but it is not as map \( \mathcal{M}(S)_{BL} \to \mathcal{M}(S)_{BL} \). This is easily seen by considering a sequence \( x_n \uparrow 1 \): \( F \delta_{x_n} = 0 \) for all \( n \), while \( F \delta_1 = -a \delta_1 \). A serious issue is that the maps \( t \mapsto \bar{P}_t \mu : \mathbb{R}^+ \to \mathcal{M}(S)_{TV} \) are not Bochner measurable for all \( \mu \in \mathcal{M}(S) \). If they were, then these maps would be continuous on \( (0,\infty) \) (cf. [24], Theorem 10.2.3, p.305). This cannot hold for Dirac measures. So (2.15) cannot be interpreted as an integral equation in \( \mathcal{M}(S)_{TV} \). It can as an equation in \( \mathcal{M}(S)_{BL} \), but then failure of (Lipschitz) continuity of the perturbation leads to similar issues as for (2.14) in establishing solutions using ‘standard’ techniques for solving such integral equations.

We prefer to consider solutions to (1.2) instead of (2.15). In the latter situation, one needs to consider a somewhat artificial extension to a larger state space. In fact, in case of a dynamical system \( (\Phi_t)_{t \geq 0} \) is a general Polish space \( S \), there is no canonical candidate for the extension space in which \( S \) embeds. It seems more natural then to use arguments ‘within’ \( S \).

3 Regularized systems

The main problem with integral equation (1.2) (or 2.15) is caused by failure of continuity of the map restricting measures to a measurable subset \( E : \mu \mapsto 1_E \cdot \mu \). Multiplication by a bounded Lipschitz function instead of the discontinuous indicator function improves its properties for the \( \| \cdot \|_{BL} \)-norm topology:

**Lemma 3.1.** Let \( f : [0,1] \to \mathbb{R} \) be measurable and define \( M_f(\mu) := f d\mu \) for \( \mu \in \mathcal{M}([0,1]) \).

(i) If \( f \) is bounded, then \( \| M_f(\mu) \|_{TV} \leq \| f \|_{\infty} \| \mu \|_{TV} \).

(ii) If \( f \in BL([0,1]) \), then \( \| M_f(\mu) \|_{BL} \leq \| f \|_{BL} \| \mu \|_{BL} \).

**Proof.** Part (i) is easily verified. For part (ii), let \( \phi \in BL([0,1]) \). Then

\[
\| \langle M_f(\mu), \phi \rangle \| = \| \langle \mu, f \cdot \phi \rangle \| \leq \| \mu \|_{BL} \| f \cdot \phi \|_{BL} \leq \| \mu \|_{BL} \| f \|_{BL} \| \phi \|_{BL},
\]

because \( BL([0,1]) \) is a Banach algebra.

The integral equation (2.1) in which the perturbation \( F : \mu \mapsto a \mu(\{1\}) \delta_1 \) is replaced by

\[
F_f : \mu \mapsto a f(\cdot) d\mu
\]

with \( f \in BL([0,1]) \) such that \( 0 \leq f \leq 1 \) and \( f(1) = 1 \) will be called a *regularization* of (1.2). In a regularized system particles can be absorbed also in the interior of the interval where \( f > 0 \). The function \( f \) is called the regularizer.

3.1 Well-posedness and positivity of solutions

We consider integral equation (2.1) with \( F = M_f \) for \( f \in BL([0,1]) \).

**Theorem 3.2.** Let \( \mu_0 \in \mathcal{M}([0,1]) \) and \( f \in BL([0,1]) \). Then there exists \( T \in (0,\infty] \) and a unique solution \( \mu_t^f \) to integral equation (2.1) with \( F = M_f \) in \( C([0,T], \mathcal{M}([0,1]_{BL})) \). \( T \) is finite if and only if \( \| \mu_t^f \|_{TV} \to \infty \) as \( t \to \infty \). The solution depends continuously on initial data.
Proof. In [9], Section 2.2, the case of well-posedness for a non-linear and time-dependent perturbation of a transport equation on \( \mathbb{R}^d \) in the space of measures is considered. The line of proof in our bounded setting is completely similar, hence is omitted. Note that our perturbation \( F^{(b)} \) is linear and bounded for \( \| \cdot \|_{TV} \), such that it satisfies the crucial condition (H5) in [9] that ensures solutions starting in \( \mu_0 \in \mathcal{M}([0,1]) \) will remain measures up to time \( T \) where blow-up occurs in \( \| \cdot \|_{TV} \) ([9], Theorem 2.4).

Theorem 3.3. Assume that \( \mu_0 \in \mathcal{M}^+([0,1]) \) and let \( f \in \mathcal{B}L([0,1]) \). The solution \( \mu_t^f \) to (2.1) with \( F = M_f \) exists for all time and \( \mu_t^f \in \mathcal{M}^+([0,1]) \) for all \( t \).

Proof. We will show below that the solution remains a positive measure on its domain of existence \([0, T]\). Then we can apply Gronwall’s Inequality, since for all \( t \in [0, T) \),

\[
\| \mu_t^f \|_{TV} = \| \mu_t^f \|_{BL} \leq \| P_t \mu_0 \|_{BL} + \int_0^t \| P_{t-s} M_f(\mu_s^f) \|_{BL} ds
\]

\[
\leq e^{v|L|t} \| \mu_0 \|_{TV} + \int_0^t e^{v|L|(t-s)} \| f \|_{BL} \| \mu_s^f \|_{TV} ds.
\]

Therefore

\[
\| \mu_t^f \|_{TV} \leq \| \mu_0 \|_{TV} e^{v|L| + \| f \|_{BL} t}
\]

and the total variation norm cannot blow-up in finite time. So \( T = \infty \), according to Theorem 3.2. (Positivity). The arguments used in [9], Lemma 3.2, to prove positivity cannot be transferred simply to our setting, since they use reduction to \( L^1 \)-solutions and approximate measure-valued solutions by the former. However, in the bounded domain there are no \( L^1 \)-solutions (for all time). The idea of proof is essentially the same however. Let \( f^\pm \) be the positive and negative part of \( f \), such that \( f = f^+ - f^- \). Put \( \bar{b} := \| f^- \|_{BL} \). The perturbation \( M_f + \bar{b} I \) is then a positive perturbation, mapping \( \mathcal{M}^+([0,1]) \) into itself. Let \( A \) be the generator of \( P_t \) on \( \mathcal{M}([0,1])_{BL} \). Then \( A - \bar{b} I \) is the generator of a \( C_0 \)-semigroup too: that of the positive semigroup \( \hat{P}_t := e^{-\bar{b}t} P_t \). One can show (e.g. [34], in quite a general context) that a solution to the integral equation

\[
\mu_t = \hat{P}_t \mu_0 + \int_0^t \hat{P}_{t-s}(M_f + \bar{b} I)(\mu_s) ds
\]

is a solution to (2.1) with \( F = M_f \) (and \textit{vice versa}). The perturbation in (3.1) is positive and Lipschitz for \( \| \cdot \|_{BL} \). Hence (3.1) can be solved (on a suitably small time interval \([0, \tau_1]\)) by means of Picard interations:

\[
u_0(t) := \hat{P}_t \mu_0, \quad u_{k+1}(t) := \hat{P}_t \mu_0 + \int_0^t \hat{P}_{t-s}(M_f + \bar{b} I)(u_k(s)) ds.
\]

Since \( u_0(t) \) is a positive measure for each \( t \), each \( u_k(t) \) is, and positivity follows on \([0, \tau_1]\). This solution can then be extended to \([\tau_1, \tau_2]\) as positive solution. Thus continuing one obtains a positive solution on \([0, T)\).

According to 3.3 for \( f \in \mathcal{B}L([0,1]) \) we can define a semigroup of solution operators \( Q_t^f \mu_0 := \mu_t \in \mathcal{M}^+([0,1]) \).
Lemma 3.4. The solution semigroup consist of Lipschitz maps on $\mathcal{M}^+([0,1])_BL$:
\[
\|Q^t_t \mu_0 - Q^t_0 \mu'_0\|_{BL}^* \leq e^{t|\nu|L + \|f\|_{ML}} t \|\mu_0 - \mu'_0\|_{BL}^*.
\]

Proof. The proof consists essentially of repeating the type of estimates and arguments employed in the proof of the first part of Theorem 3.3, so these will be omitted. \[
\square
\]

3.2 Boundary layer approximation by regularized systems

We shall now consider a countable family of regularized systems defined by a decreasing sequence $(f_n) \subset BL([0,1])$ of regularizers. Define $f_n(x) := [n(x - (1 - \frac{1}{n}))]^+$, where $[\cdot]^+$ denotes the positive part of the argument. That is,
\[
f_n(x) := \begin{cases} 
0, & \text{for } x \in [0, 1 - \frac{1}{n}); \\
n(x - (1 - \frac{1}{n})), & \text{for } x \in [1 - \frac{1}{n}, 1].
\end{cases}
\]  

(3.2)

It is easy to see that
\[
\|f_n\|_{BL} = \|f_n\|_{\infty} + |f_n|_L = 1 + n.
\]  

(3.3)

Moreover,

Lemma 3.5. For any $m, n \in \mathbb{N}^+$, satisfying $m \geq n$
\[
\|f_n - f_m\|_{\infty} = 1 - \frac{n}{m}.
\]  

(3.4)

Define $F(n) : \mathcal{M}([0,1]) \rightarrow \mathcal{M}([0,1])$ by means of
\[
F(n)\mu = -a f_n \cdot d\mu.
\]  

(3.5)

Lemma 3.1 and 3.3 yield that $F(n)$ is a continuous linear operator on $\mathcal{M}([0,1])_{BL}$ with operator norm $\|F(n)\| \leq |a|(n+1)$. The norms of the $F(n)$ are not bounded in $n$ unfortunately. This complicates the transition to the limit equation as $n \rightarrow \infty$ and necessitates the more delicate approach presented in Section 4 below. However, if measures are properly ordered, then one has

Lemma 3.6. For any $n \in \mathbb{N}^+$ and $\mu, \nu \in \mathcal{M}^+([0,1])$ satisfying $\mu \geq \nu$, the following inequality holds:
\[
\|F(n)\mu - F(n)\nu\|_{BL}^* \leq a \|\mu - \nu\|_{BL}^*.
\]  

(3.6)

Proof. By assumption, $\mu - \nu \in \mathcal{M}^+([0,1])$, so $-F^{(n)}(\mu - \nu) \in \mathcal{M}^+([0,1])$. Therefore $\|\mu - \nu\|_{BL} = \|\mu - \nu\|_{TV}$ and Lemma 3.1 (i) yields
\[
\|F^{(n)}(\mu - \nu)\|_{BL} = \|F^{(n)}(\mu - \nu)\|_{TV} \leq a \|\mu - \nu\|_{TV} = a \|\mu - \nu\|_{BL}.
\]

\[
\square
\]

We shall denote by $\mu^{(n)}_t$ the mild solution to the regularized system defined by $f_n$ with initial condition $\mu_0 \in \mathcal{M}^+([0,1])$. That is, $t \mapsto \mu^{(n)}_t$ is the unique continuous map $\mathbb{R}^+ \rightarrow \mathcal{M}^+([0,1])_{BL}$ that satisfies
\[
\mu^{(n)}_t = P_t \mu_0 + \int_0^t P_{t-s} F^{(n)} \mu^{(n)}_s \, ds, \quad \text{for all } t \geq 0.
\]  

(3.7)

As a consequence of Theorem 3.3 and Lemma 3.4 we thus have:

Corollary 3.7. For each $n \in \mathbb{N}^+$ and $\mu_0 \in \mathcal{M}^+([0,1])$ there exists a unique mild solution $\mu^{(n)}_t \in C([\mathbb{R}^+, \mathcal{M}^+([0,1])_{BL}])$ to equation (3.7). The solution operators $Q^{(n)}_t$ defined by $Q^{(n)}_t \mu_0 := \mu^{(n)}_t$ are Lipschitz continuous on $\mathcal{M}^+([0,1])$ for $\|\cdot\|_{BL}$. 

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3.3 Ordering of regularized solutions

Lemma 3.6 indicates that ordering of the regularized solutions may be exploited in estimates. One may anticipate such an ordering, since with increasing $n$, the functions $f_n$ decrease. Thus one would expect more mass in the mild solution for larger $n$ at the same time, because less absorption has taken place. We shall make this precise below.

**Lemma 3.8** ('Diracs stay Diracs'). If $\mu_0 = \delta_x$, then $\mu_t^{(n)}$ is concentrated at $\Phi_t(x)$.

**Proof.** On a small interval $[0, T]$ the solution to $\mu_t = P_t \mu_0 + \int_0^t P_{t-s} F(n) \mu_s ds$ is given by the limit of a sequence $u_k(t) \in \mu_t \in C([0, T], M^+([0, 1])_{BL})$ defined by

$$u_0(t) := P_t \mu_0, \quad u_{k+1}(t) := P_t \mu_0 + \int_0^t P_{t-s} F(n) u_k(s) ds. \quad (3.8)$$

If $\mu_0 = \delta_x$, then $P_t \mu_0 = \delta_{\Phi_t(x)}$ is a Dirac distribution located at $\Phi_t(x)$. Suppose now that $u_k(t) = \delta_k(t) \Phi_t(x)$ for some $\delta_k(t) > 0$. Then

$$F(n) u_k(t) = -a f_n \cdot [\delta_k(t) \Phi_t(x)] = -a f_n(\Phi_t(x)) \delta_k(t) \Phi_t(x)$$

is a Dirac located at $\Phi_t(x)$. Consequently,

$$u_{k+1}(t) = \delta_{\Phi_t(x)} + \int_0^t \delta_{\Phi_t(x)} ds$$

$$= 1 - a \int_0^t f_n(\Phi_t(x)) \delta_k(s) ds \cdot \Phi_t(x). \quad (3.9)$$

Thus $\mu_t := \lim_{k \to \infty} u_k(t)$ is a multiple of a Dirac measure positioned at $\Phi_t(x)$.}

Substituting $\mu_t = \alpha_t^{(n)} \cdot \delta_{\Phi_t(x)}$ into the variation of constants formula leads to

$$\alpha_t^{(n)} \delta_{\Phi_t(x)} = [1 - a \int_0^t f_n(\Phi_t(x)) \alpha_s^{(n)}(s) ds] \cdot \delta_{\Phi_t(x)}.$$ 

Evaluating the measures on $\{\Phi_t(x)\}$ (or on $[0, 1]$) yields

$$\alpha_t^{(n)} = 1 - a \int_0^t f_n(\Phi_t(x)) \alpha_s^{(n)}(s) ds. \quad (3.10)$$

Since $t \to \alpha_t^{(n)}$ is continuous, it follows from (3.10) that this is even differentiable, and correspondingly,

$$\frac{d}{dt} \alpha_t^{(n)} = -a f_n(\Phi_t(x)) \alpha_t^{(n)}. \quad (3.11)$$

Therefore

$$\alpha_t^{(n)} = \alpha_0^{(n)} \exp(-a \int_0^t f_n(\Phi_s(x)) ds). \quad (3.12)$$

and we obtain:
Lemma 3.9. Assume that \( \mu_0 = \alpha_0 \delta_{x_0} \) with \( \alpha_0 \in \mathbb{R}^+ \) and \( x_0 \in [0,1] \). Then for any \( n \in \mathbb{N}^+ \), \( \mu^{(n)}_t = \alpha^{(n)}_t \delta_{x^{(n)}} \), where

\[
\alpha^{(n)}_t = \alpha_0 \exp \left( -a \int_0^t f_n(\Phi_s(x_0)) \, ds \right). \tag{3.13}
\]

In particular, if \( m,n \in \mathbb{N}^+ \) and \( n \leq m \), then \( \alpha^{(n)}_t \leq \alpha^{(m)}_t \).

This leads to

Proposition 3.10 (Ordering of regularized solutions). For all \( m,n \in \mathbb{N}^+ \), such that \( n \leq m \), the corresponding regularized solutions satisfy \( \mu^{(n)}_t \leq \mu^{(m)}_t \) for all \( t \geq 0 \).

Proof. Let \( Q^{(n)}_t \) be the solution operator for the regularized system with regularizer \( f_n \).

This operator is Lipschitz continuous for \( \| \cdot \|_{BL} \) (Corollary 3.7). Identity (1.14) and continuity is used to obtain

\[
\mu^{(n)}_t = Q^{(n)}_t(\mu_0) = \int_{[0,1]} Q^{(n)}_t(\delta_x) \, d\mu_0(x) \tag{3.14}
\]

\[
\leq \int_{[0,1]} Q^{(m)}_t(\delta_x) \, d\mu_0(x) = Q^{(m)}_t(\mu_0) = \mu^{(m)}_t, \tag{3.15}
\]

using Lemma 3.9.

Another type of ordering of regularized solutions is provided by the observation that measures can only lose mass over time. There is no source in the system. This yields

Lemma 3.11. If \( \mu_0 \in \mathcal{M}^+([0,1]) \), then for all \( t \geq 0 \), \( n \in \mathbb{N}^+ \), \( \mu^{(n)}_t \leq P_t \mu_0 \). In particular, \( \| \mu^{(n)}_t \|_{TV} \leq \| \mu_0 \|_{TV} \).

Proof. If \( \mu_0 \) is positive, then so is \( \mu^{(n)}_t \) (Corollary 3.7). Using the variation of constants formula (3.7) again, one has

\[
\mu^{(n)}_t = P_t \mu_0 - a \int_0^t P_{t-s}[f_n \mu^{(n)}_s] \, ds \tag{3.16}
\]

The integral term in (3.16) is a positive measure, so \( P_t \mu_0 - \mu^{(n)}_t \geq 0 \). The norm estimate follows directly from the positivity of \( \mu^{(n)}_t \) and this ordering:

\[
\| \mu^{(n)}_t \|_{TV} = \mu^{(n)}_t([0,1]) \leq P_t \mu_0([0,1]) = \| P_t \mu \|_{TV} = \| \mu \|_{TV}.
\]

4 The limit equation

In this section we consider the problem of existence, uniqueness and continuous dependence on initial conditions for the solutions of the integral equation (1.2) for the limit system with interaction at the boundary only. In Section 3.2 we prove this using the particular structure of the system that allows such an approach. In Section 4.2 consider the limit system truly as a limit of regularized systems with interactions in a boundary layer. The latter employs estimates on the integral equations for the regularized systems that may be used for other type of systems in a similar manner.
4.1 Well-posedness

As can be expected, the results follow from careful analysis what happens to mass on the boundary, which is possible in our case, because of the relative simplicity of the one-dimensional situation. Note that the proof of continuous dependence on initial conditions in Proposition 4.3 requires a delicate argument, in particular it employs Corollary 2.4.

**Proposition 4.1** (Uniqueness). A solution to (1.2) in $C(\mathbb{R}^+, \mathcal{M}([0,1]))_{BL}$ is unique, if it exists.

**Proof.** In this case, a modified argument of Gronwall-type shows the uniqueness of solutions. In fact, if (1.2) had two solutions $\mu_t$ and $\hat{\mu}_t$ on $[0, T]$, having the same initial data $\mu_0$, then for all $t \geq 0$,

$$\mu_t - \hat{\mu}_t = -a \int_0^t [\mu_s(\{1\}) - \hat{\mu}_s(\{1\})] ds \cdot \delta_1. \quad (4.1)$$

That is, two solutions can differ by mass concentrated at 1 only. Note that the integrand in (4.1) is a bounded measurable function. Evaluating the latter equation at $\{1\}$ yields:

$$\mu_t(\{1\}) - \hat{\mu}_t(\{1\}) = -a \int_0^t [\mu_s(\{1\}) - \hat{\mu}_s(\{1\})] ds. \quad (4.2)$$

and consequently

$$|\mu_t(\{1\}) - \hat{\mu}_t(\{1\})| \leq a \int_0^t |\mu_s(\{1\}) - \hat{\mu}_s(\{1\})| ds \quad (4.3)$$

A version of Gronwall’s Lemma yields that $|\mu_t(\{1\}) - \hat{\mu}_t(\{1\})| = 0$ for all $t \geq 0$. \qed

Since there is no ‘smoothing’ effect in the dynamics in the interior of the interval $[0,1]$, we expect Dirac masses to stay Dirac masses. These move according to $P_t$. The latter acts simply on Dirac masses: $P_t \delta_x = \delta_{\Phi_t(x)}$. Therefore one may try as particular solution to (1.2) with $\mu_0 = \delta_x$:

$$\mu_t = \alpha_x(t) \delta_{\Phi_t(x)}. \quad (4.4)$$

Substituting (4.4) into (1.2) yields, after evaluation of the measures on the full space $[0,1]$:

$$\alpha_x(t) = \alpha_x(0) - a \int_{\tau_0(x) \wedge t}^t \alpha_x(s) ds,$$

where $\tau_0(x) \wedge t$ is the time in the interval $[0, t]$ that the individual is not at the boundary (see Section 2.1). Thus we obtain that

$$\alpha_x(t) = \alpha_x(0) e^{-a(t - \tau_0(x) \wedge t)}. \quad (4.5)$$

Any initial measure is a superposition of Dirac masses, according to (1.14). Therefore we obtain the following existence result and integral representation for the solution to (1.2):

**Proposition 4.2** (Existence). For each $\mu_0 \in \mathcal{M}([0,1])$ there exists a continuous solution $\mu : \mathbb{R}^+ \to \mathcal{M}([0,1])_{BL}$ to (1.2) defined by

$$\mu_t := \int_{[0,1]} e^{-a[t - \tau_0(x) \wedge t]} \delta_{\Phi_t(x)} d\mu_0(x). \quad (4.6)$$

as Bochner integral in $\mathcal{M}([0,1])_{BL}$. 

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Continuity of the maps $t \mapsto M$ (Corollary 2.4). Thus for Proposition 4.3 (Continuous dependence on initial conditions) Instead we shall use (4.6).

The integrand in (4.6) is a bounded continuous function from $[0, 1]$ into $M^+([0, 1])_{BL}$ (cf. Corollary 2.4) and application of Lebesgue’s Dominated Convergence Theorem yield continuity of $t \mapsto \mu_t$. It is easily verified that $\mu_t$ satisfies (1.2).

A ‘standard’ argument with Gronwall’s Inequality to obtain continuous dependence on initial conditions fails in this setting, because the perturbation is not Lipschitz continuous. Instead we shall use (4.6).

**Proposition 4.3** (Continuous dependence on initial conditions). For each $T > 0$, there exists $C_T > 0$ such that for all initial values $\mu_0, \mu'_0 \in M^+([0, 1])$ the corresponding solutions $\mu$ and $\mu'$ to (1.2) satisfy

$$\|\mu_t - \mu'_t\|_{BL} \leq C_T \|\mu_0 - \mu'_0\|_{BL}$$

(4.7) for all $t \in [0, T]$.

**Proof.** In view of Lemma 2.2 (ii) we need to control the integral term in (1.2). Intuitively, this term is the total amount of mass that disappeared from the system in the time interval $[0, t]$. To be precise, according to (1.2) and (4.6):

$$a \int_0^t \mu_s(\{1\}) \, ds = P_t \mu_0(S) - \mu_1(S) = \|\mu_0\|_{TV} - e^{-at} \int_0^t e^{a[\tau_0(x) \wedge t]} \|\mu_0\|_{BL}$$

$$= \|\mu_0\|_{BL} - e^{-at} \langle \mu_0, e^{a[\tau_0(x) \wedge t]} \rangle.$$ (4.8)

Note that the map $x \mapsto e^{a[\tau_0(x) \wedge t]}$ is bounded Lipschitz (Corollary 2.4):

$$\|e^{a[\tau_0(x) \wedge t]}\|_{L^1} \leq e^{at} \|\tau_0(x) \wedge t\|_{L^1}$$

and

$$\|\mu_0\|_{BL}$$

because $\tau_0(x) \wedge t \leq t$ for all $x$. Therefore, using Lemma 2.2 and (4.8),

$$\|\mu_t - \mu'_t\|_{BL} \leq \|P_t (\mu_0 - \mu'_0)\|_{BL} + a \int_0^t \mu_s(\{1\}) \, ds$$

$$\leq e^{at} \|\mu_0 - \mu'_0\|_{BL} + a \|\mu_0\|_{BL} - \|\mu'_0\|_{BL} + e^{-at} \langle \mu_0 - \mu'_0, e^{a[\tau_0(x) \wedge t]} \rangle$$

$$\leq \|\mu_0 - \mu'_0\|_{BL} (e^{at} + 1 + |\tau_0(x) \wedge t|) \|\mu_0\|_{BL}$$

$$\leq (a + e^{at} + 1 + |\tau_0(x) \wedge t|) \|\mu_0 - \mu'_0\|_{BL}.$$ 

The factor in front of $|\mu_0 - \mu'_0|_{BL}$ is non-decreasing in $t$, according to Corollary 2.4, so it is dominated by a constant $C_T$ for $t$ in an interval $[0, T]$. □
4.2 Approximation by regularized systems

Our main point is, that solutions to \((1.2)\) can be viewed as a limit of solutions to regularized systems, with interaction with the boundary in a thin boundary layer. Well-posedness of the latter systems follows essentially the standard proof for abstract semi-linear equations in Banach spaces, see Section 3.1. Such results are expected to persist in more complex situations (see e.g. [9]). In this section we show how estimation of the variation of constants formula for the regularized systems \((3.7)\) leads to equicontinuity of the family of solutions \(\{\mu^{(n)} | n \in \mathbb{N}\}\) in \(C([0,T],\mathcal{M}([0,1]_{BL}))\), for any \(T \geq 0\). An Arzela-Ascoli type of theorem \([3]\) then yields precompactness, hence the existence of subsequences that converge. The limit of each convergent subsequence yields a solution to \((1.2)\). The latter is unique, so we obtain convergence of the full sequence.

We now arrive at the main result.

**Lemma 4.4.** For all \(s, t \in [0,T]\), satisfying \(s \leq t\), the following estimates hold:

\[
\begin{align*}
(i) \quad \| \int_0^s P_{t-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) - P_{s-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) \, d\sigma \|_{BL}^* & \leq a \|v\|_{TV} \|\mu_0\|_{TV} |t-s|; \\
(ii) \quad \| \int_s^t P_{t-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) \, d\sigma \|_{BL}^* & \leq a \|\mu_0\|_{TV} |t-s|.
\end{align*}
\]

**Proof.** (i) According to Lemmas 2.1, 2.2 and 3.11 and properties of the Bochner integral, we have

\[
\begin{align*}
\| \int_0^s P_{t-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) - P_{s-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) \, d\sigma \|_{BL}^* & \leq \int_0^s \| P_{t-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) - P_{s-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) \|_{BL} \, d\sigma \\
& \leq \|v\|_{TV} |t-s| \int_0^s \| F^{(n)}(\mu^{(n)}_{\sigma}) \|_{TV} \, d\sigma \\
& \leq a \|v\|_{TV} |t-s| \int_0^s \| \mu^{(n)}_{\sigma} \|_{TV} \, d\sigma \\
& \leq a \|v\|_{TV} |t-s| \|\mu_0\|_{TV} s.
\end{align*}
\]

(ii): The second estimate is obtained as follows:

\[
\begin{align*}
\| \int_s^t P_{t-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) \, d\sigma \|_{BL}^* & \leq \int_s^t \| P_{t-\sigma} F^{(n)}(\mu^{(n)}_{\sigma}) \|_{BL} \, d\sigma = a \int_s^t \| P_{t-\sigma} [f_n \mu^{(n)}_{\sigma}] \|_{TV} \, d\sigma \\
& = a \int_s^t \| f_n \mu^{(n)}_{\sigma} \|_{TV} \, d\sigma \leq a \int_s^t \| \mu^{(n)}_{\sigma} \|_{TV} \, d\sigma \\
& \leq a \|\mu_0\|_{TV} |t-s|.
\end{align*}
\]

Here we used that \(f_n \mu^{(n)}_{\sigma}\), hence \(P_{t-\sigma} [f_n \mu^{(n)}_{\sigma}]\), is a positive measure for all \(\sigma\), so \(\|\cdot\|_{BL}^*\) and \(\|\cdot\|_{TV}\) coincide. \(\square\)

We now arrive at the main result.

**Theorem 4.5.** Let \(\mu_0 \in \mathcal{M}^+(\mathbb{R})\) and \(T \geq 0\). The solutions \(\mu^{(n)}_{\cdot}\) to the regularized system \((3.7)\) converge in \(C([0,T],\mathcal{M}^+([0,1]_{BL}))\) to the unique solution \(\mu_{\cdot}\) of \((1.2)\).
Proof. Let $T \geq 0$ and $\mu_0 \in \mathcal{M}^+([0,1])$. First we prove equicontinuity of the family of regularized solutions $\{\mu^{(n)}\} | n \in \mathbb{N}^+ \}$ in $C([0,T],\mathcal{M}([0,1])_{BL})$. To that end, let $n \in \mathbb{N}$ and $t, s \in [0,T]$ such that $t > s$. Then

$$
\|\mu_t^{(n)} - \mu_s^{(n)}\|_{BL} \leq \|P_t\mu_0 - P_s\mu_0\|_{BL} + \| \int_0^t P_{t-s} F(\mu^{(n)})(\mu_s^{(n)}) \, d\sigma - \int_0^s P_{s-s} F(\mu^{(n)})(\mu_s^{(n)}) \, d\sigma \|_{BL}^*
$$

$$
\leq \|\mu_0\|_{TV} \|v\|_\infty |t-s| + \| \int_0^t P_{t-s} F(\mu^{(n)})(\mu_s^{(n)}) - P_{s-s} F(\mu^{(n)})(\mu_s^{(n)}) \, d\sigma \|_{BL}^*
$$

$$
+ \| \int_s^t P_{t-s} F(\mu^{(n)})(\mu_s^{(n)}) \, d\sigma \|_{BL}^*
$$

$$
\leq \|\mu_0\|_{TV} \|v\|_\infty |t-s| + \|v\|_\infty a \|\mu_0\|_{TV} |s| |t-s| + a \|\mu_0\|_{TV} |t-s|.
$$

Here we used Lemma 2.2 in the first step and Lemma 4.4 in the last. Thus there exists $C = C_T > 0$ such that for all $n \in \mathbb{N}$ and $t, s \in [0,T]$,

$$
\|\mu_t^{(n)} - \mu_s^{(n)}\|_{BL} \leq C\|\mu_0\|_{TV} |t-s|.
$$

In particular, $\{\mu^{(n)} | n \in \mathbb{N}\}$ is uniformly equicontinuous in $C([0,T],\mathcal{M}([0,1])_{BL})$.

For each $t \in [0,T]$ and $n \in \mathbb{N}^+$, $\|\mu_t^{(n)}\|_{TV} \leq \|\mu_0\|_{TV}$ (Lemma 3.11). Since $[0,1]$ is compact, the set $\{\mu_t^{(n)} | n \in \mathbb{N}^+\}$, consisting of positive measures, is uniformly tight for each $t$. According to a version of Prohorov’s Theorem (cf. [7], Theorem 8.6.2), it is precompact in $\mathcal{M}^+([0,1])_{BL}$. Ambrosetti’s generalisation of the Arzela-Ascoli Theorem 31 yields that $\{\mu^{(n)} | n \in \mathbb{N}^+\}$ is precompact in $C([0,T],\mathcal{M}([0,1])_{BL})$.

Thus there exists a convergent subsequence $\mu^{(n_k)}$ in $C([0,T],\mathcal{M}([0,1])_{BL})$ with limit $\mu^*$. Since $\mathcal{M}^+([0,1])$ is closed in $\mathcal{M}([0,1])_{BL}$, $\mu^* \in C([0,T],\mathcal{M}^+(\mathcal{S})_{BL})$. Application of Lebesgue’s Dominated Convergence Theorem (for Bochner integration, see [15]) to (3.7) shows that $\mu^*$ satisfies (2.14). Any convergent subsequence of $\{\mu^{(n)}\}$ must have the same limit, by uniqueness of solutions to (2.14) (Lemma 4.1). We conclude that the sequence $\{\mu^{(n)}\}$ itself is convergent in $C([0,T],\mathcal{M}^+([0,1])_{BL})$ with limit $\mu^*$. 

\[\square\]

4.3 Rate of convergence of approximation

We obtain a rate of convergence of regularized solutions to the solution of the limit system for a subset of initial conditions by means of a Cauchy-argument. We show that for initial conditions in this subset (which is sufficiently rich from the point of view of applications), the sequence of regularized solutions is actually a Cauchy sequence.

Reconsider equation (3.7) that is satisfied by the regularized solution $\mu_t^{(n)}$. Let $n > m$. Then, writing $\|\cdot\|$ instead of $\|\cdot\|_{BL^*}$,

$$
\|\mu_t^{(n)} - \mu_t^{(m)}\| = \| \int_0^t P_{t-s} [F(\mu_s^{(n)})(\mu_s^{(n)}) - F(\mu_s^{(m)})(\mu_s^{(m)})] \, ds \|
$$

$$
\leq \int_0^t \|P_{t-s} [F(\mu_s^{(n)})(\mu_s^{(n)}) - F(\mu_s^{(m)})(\mu_s^{(m)})] \| \, ds + \int_0^t \|P_{t-s} [F(\mu_s^{(m)})(\mu_s^{(m)}) - F(\mu_s^{(m)})(\mu_s^{(m)})] \| \, ds
$$

(4.9)

Let us denote the left and right integral terms in (4.9) by $I_1$ and $I_2$ respectively and
start by considering the last term:

\[ I_2 \leq e^{\|v\|L_t} \int_0^t \| F^{(n)}(t) \mu_s^{(m)} - F^{(m)}(t) \mu_s^{(m)} \| ds, \]

according to Lemma 2.2. Put \( E_m := [1 - \frac{1}{m}, 1) \).

**Lemma 4.6.** Let \( m, n \in \mathbb{N}^+ \) such that \( n > m \). Then for all \( \mu \in \mathcal{M}([0, 1]) \),

\[ \| F^{(n)}(t) \mu - F^{(m)}(t) \mu \|_{TV} \leq a \frac{1}{m} \cdot m |\mu|(E_m). \]

**Proof.** Let \( \varphi \in \mathcal{C}_b([0, 1]) \). Then according to Lemma 3.5,

\[ |\langle F^{(n)}(t) \mu, \varphi \rangle| \leq a \int_{[\frac{1}{m}, 1]} |f_n - f_m| \|\varphi\| d\mu \]

\[ \leq a(1 - \frac{m}{n}) \cdot \|\varphi\|_{TV} \cdot |\mu|(E_m). \]  \hspace{1cm} (4.10)

Note that one can remove \( \{1\} \) from integration in \( \text{(4.10)} \), since the integrand is zero at 1. The result follows immediately from \( \text{(4.10)} \). \( \square \)

Since we take positive initial conditions, \( \mu_s^{(m)} \geq 0 \) for all \( s \). Hence

\[ I_2 \leq \alpha e^{\|v\|L_t} \left\{ \frac{1}{m} \right\} \cdot m \int_0^t \mu_s^{(m)}(E_m) ds. \]

Now, \( \mu_s^{(m)} \leq P_s \mu_0 \) (Lemma 3.11), so we obtain an estimate for \( I_2 \) that depends on the initial condition only:

\[ I_2 \leq \alpha e^{\|v\|L_t} \left\{ \frac{1}{m} \right\} \cdot m \left\{ \int_0^t P_s \mu_0 ds \right\} (E_m). \]  \hspace{1cm} (4.11)

We shall now provide conditions on \( \mu_0 \) such that the integral in \( \text{(4.11)} \), evaluated on \( E_m \), is \( O(1/m) \). Let \( \lambda \) denote Lebesgue measure on \([0, 1]\)

**Proposition 4.7.** Let \( \mu_0 \in \mathcal{M}^+(\{0, 1\}) \) be such that \( \mu_0 = \mu_{0,s} + g_0 d\lambda \) is the Lebesgue decomposition of \( \mu_0 \) with \( \mu_{0,s} \), the part of \( \mu_0 \) that is singular with respect to \( \lambda \). Suppose \( g_0 \in L^\infty([0, 1]) \) and \( \mu_{0,s} \) is such that for \( \lambda \)-a.e. \( s \in [0, 1] \),

\[ \frac{P_s \mu_{0,s}(E_m)}{\lambda(E_m)} \rightarrow 0 \text{ as } m \rightarrow \infty. \]  \hspace{1cm} (4.12)

Then there exists \( c > 0 \), independent of \( t \), such that

\[ \limsup_{m \rightarrow \infty} m \int_0^t P_s \mu_0(E_m) ds \leq c \|g_0\|_{\infty}^{1-\|v\|L} (e^{\|v\|L_t} - 1). \]  \hspace{1cm} (4.13)

**Proof.** For each \( s \in [0, t] \) fixed, \( P_s \mu_{0,s} \) is singular with respect to \( \lambda \) (cf. Lemma 3.8). Moreover, \( \Phi_s \) is Lipschitz, hence differentiable almost everywhere by Rademacher’s Theorem. Put \( J_s := D\Phi_s \in L^\infty([0, 1]) \). Then \( \|J_s(x)\|_{TV} \leq c |\Phi_s|_{L} \) for some \( c > 0 \). Furthermore, for each \( s \),

\[ P_s(g_0 d\lambda) = c_0(s) \delta_1 + g_0 d\lambda. \]  \hspace{1cm} (4.14)
where \( g_s \in L^1([0,1]) \cap L^\infty([0,1]) \). Indeed, for any \( E \subset [0,1] \) Borel, we have

\[
P_s(g_0 d\lambda)(E) = \Phi_s \#(g_0 d\lambda)(E) = \int g_0 \, dx + \int_{\Phi^{-1}_s(1) \setminus E} g_0 \, dx = c_0(s) \delta_1(E) + \int_{E \setminus \{1\}} g_0(\Phi_s(x)) J_s(x) \, dx,
\]

where \( c_0 \) is some constant possibly depending on \( s \). Taking \( g_s := (g_0 \circ \Phi_s) \cdot J_s \) gives (4.14).

Based on [20], we note that for \( E_m := [1 - \frac{1}{m}, 1] \), we obtain

\[
\lim_{m \to \infty} \frac{P_s \mu_0(E_m)}{\lambda(E_m)} = \lim_{m \to \infty} \frac{(P_s \mu_{0,s} + c_0(s) \delta_1(E_m) + (g_s d\lambda)(E_m))}{\lambda(E_m)} = \lim_{m \to \infty} \frac{P_s \mu_{0,s}(E_m)}{\lambda(E_m)} + \frac{(g_s d\lambda)(E_m)}{\lambda(E_m)},
\]

(4.15)

The first term in the limit shown in (4.15) is dealt with by (4.12). Note that \( P_s \mu_{0,s}(E_m) \leq P_s \mu_{0,s}([0,1]) \leq P_s \mu_0([0,1]) \). So (4.12) and Lebesgue’s Dominated Convergence Theorem give

\[
\frac{1}{\lambda(E_m)} \int_0^t P_s \mu_{0,s}(E_m) \, ds \to 0 \text{ as } m \to \infty.
\]

(4.16)

Let us now turn the attention of the second term in the last limit of (4.15). For all \( s \in [0, t] \):

\[
\left| \frac{(g_s d\lambda)(E_m)}{\lambda(E_m)} \right| \leq \frac{1}{\lambda(E_m)} \int_{E_m} g_0(\Phi_s(x)) |J_s(x)| \, dx \leq \|g_0 \circ \Phi_s\|_{\infty, E_m} \cdot c|\Phi_s|_L \leq c\|g_0\|_\infty e^{c|\cdot|_L s}.
\]

So, we can conclude that

\[
\frac{1}{\lambda(E_m)} \int_0^t (g_s d\lambda)(E_m) \, ds \leq c\|g_0\|_\infty \cdot \frac{1}{|\cdot|_L} (e^{c|\cdot|_L t} - 1)
\]

(4.17)

Combining (4.16) and (4.17) together yields the desired result. \( \square \)

The first integral in (4.9) may be estimated as follows:

\[
I_1 \leq e^{c|\cdot|_L t} \int_0^t \|F^{(n)} \mu^{(m)}_s - F^{(n)} \mu^{(m)}_s\| \, ds.
\]

Observing that \( \mu^{(n)}_s \geq \mu^{(m)}_s \) for all \( s \in [0, t] \) (Theorem 3.10), Lemma 3.6 yields

\[
I_1 \leq ae^{c|\cdot|_L t} \int_0^t \|\mu^{(n)}_s - \mu^{(m)}_s\| \, ds.
\]

Thus we can apply a version of Gronwall’s Lemma (e.g. [32], Lemma D.2) and obtain

\[
\|\mu^{(n)}_t - \mu^{(m)}_t\| \leq aC_t e^{c|\cdot|_L t} \left| \frac{1}{n} - \frac{1}{m} \right| \cdot \left[ 1 + ats|\cdot|_L \exp \left( ate|\cdot|_L t \right) \right].
\]

(4.18)

Here

\[
C_t = c\|g_0\|_\infty |\cdot|_L^{-1} (e^{c|\cdot|_L t} - 1).
\]

Thus one gets
Theorem 4.8. Let the initial datum $\mu_0$ satisfy the conditions of Proposition \ref{prop:4.7}. Then the sequence of solutions $(\mu^{(n)}_t)$ to the regularized systems defined by $(f_n)$ is a Cauchy sequence in $C([0,T], M^+([0,1]))$ for each $T > 0$. Moreover,

$$\|\mu^{(n)}_t - \mu_t\|_{\text{BL}} = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty,$$

(4.19)

uniformly on compact time intervals.

Note that the rate of convergence is ‘as good as’ that of the sequence $(f_n)$ in view of Lemma \ref{lem:3.5}. A similar type of result could be obtained with regularizations $\hat{f}_n$ different from $f_n$ but still bounded Lipschitz, supported on $E_n$ and such that $\hat{f}_n$ converges monotone (decreasing) pointwise to $\mathbb{I}_{\{1\}}$. The crucial condition then is, the equivalent of Lemma \ref{lem:3.5} namely that for $n > m$,

$$\|\hat{f}_n - \hat{f}_m\|_{\infty} \leq \omega(n,m) \cdot m,$$

where $\omega \geq 0$ such that for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\omega(n,m) < \varepsilon$ for all $n, m \geq N$. The rate of convergence of the approximations towards the limit solution is then controled by $\bar{\omega}(n) := \limsup_{m \to \infty} \omega(n,m)$.

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