Galaxy Clusters in Cosmology: Cluster Abundance as a Probe of Structure Formation

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Abstract. In gaussian theories of structure formation, the galaxy cluster abundance is an extremely sensitive probe of the density fluctuation power spectrum and of the density parameter, Ω. We develop this theme by deriving and studying in detail the mass function of collapsed objects and its relation to these quantities. Application to current data yields constraints which are degenerate between the amplitude of the perturbations and the density parameter; we nevertheless obtain an important limit on the present day mass perturbation amplitude as a function of Ω and can rule out the ‘standard’ cold dark matter (CDM) model. Future observations of the evolution of the cluster abundance will break the degeneracy and provide important constraints on both the power spectrum and the density parameter, individually. We focus primarily on X-ray clusters in the discussion, and finish with a presentation of the promising new field of Sunyaev–Zel’dovich observations of cluster evolution.

1. Introduction

Galaxy clusters are useful to cosmology because they may be studied as individual objects, within the sea of the more general galaxy distribution, with clearly definable dynamics. Their characteristics reveal much about the nature of the mechanism responsible for the formation of large-scale structure in the Universe, the central problem in modern cosmology. In this chapter, we shall consider in detail one cluster property – their abundance and its evolution – and what it can tell us about our theoretical models. I refer the reader to Sadat (these proceedings) for a discussion of some of the implications of other cluster attributes.

Why should the cluster abundance be such a useful tool for studying structure formation? According to the favored scenario, formation by gravitational instability, galaxies and galaxy clusters form where the density contrast, δ, (see Coles, these proceedings) is large enough that the surrounding matter may separate from the general expansion and collapse. It should be no surprise, then, that the abundance of collapsed objects depends on the amplitude of the density perturbations. These latter are modeled statistically and, on a given scale, they follow a probability distribution, \( \text{Prob}(\delta) \) (e.g., a gaussian: \( \text{Prob}(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\delta^2/2\sigma^2} \)). The amplitude of the perturbations on a scale, \( R \), is defined as the variance, \( \sigma(R) \), which is related to the power spectrum, \( P(k) \), a quantity of fundamental importance (see below, and Coles, these proceedings).
Now, the amplitude, $\sigma(R)$, appears to decrease with increasing scale, which implies that the density contrast required to form a large object, such as a galaxy cluster, becomes an increasingly rare event on the tail of the statistical distribution. The present abundance of clusters is therefore expected to be extremely sensitive to small changes in $P(k)$. In addition, the rate of cluster evolution is essentially controlled by the density parameter, $\Omega$ (recall the linear-theory solutions for density perturbations). Thus, by studying the cluster abundance and its evolution, we can constrain both $P(k)$ and $\Omega$.

We begin with an introduction to the mass function, which provides a relation between the power spectrum and the abundance of collapsed objects of a given mass. After a brief look at the general problem of constructing the mass function, the Press-Schechter formula (Press & Schechter 1974) is presented as a sufficiently accurate approximation to the mass function observed in N-body simulations. We then develop in detail its dependence on $P(k)$ and $\Omega$. It is here that we derive the foundations of our approach. In fact, application to real data is little more than finding trustworthy relations between the mass of a cluster and some more readily observable parameter, such as temperature.

In our application to data, we avoid optical cluster samples. This omission is due, paradoxically, to the difficulties of constructing well-defined cluster catalogs as overdensities of galaxies; the projection of galaxies along the line-of-sight can result in cluster misidentifications and erroneous determination of cluster properties, e.g., the velocity dispersion. We instead focus on X-ray observations. The empirical fact that the X-ray luminosity of a cluster scales as a large power of richness (see Sadat, these proceedings, for a definition of richness) eliminates projection effects as a problem. We give our attention primarily to the cluster X-ray temperature distribution function. We shall see why the X-ray temperature should be closely related to the mass, more so than the X-ray luminosity, and how numerical simulations indeed show a tight relation. We are then able to draw some important conclusions from existing data concerning the power spectrum.

We conclude by studying the Sunyaev-Zel’dovich (SZ) effect. In principle, the cluster population may be studied by this method in a manner completely analogous to X-ray observations. At the present time, however, this approach is much less advanced observationally, due to the rather demanding combination of radio telescope sensitivity, resolution and field-of-view requirements; it is, nevertheless, coming into its own right as an exciting new field. We will examine the potential of this approach and also the advantages it offers, even in comparison to X-ray observations.

2. The Mass Function

Our goal is to relate the number density of collapsed, virialized objects to the density perturbation power spectrum. These perturbations are characterized by their density contrast

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - < \rho >}{< \rho >},$$

where the mean cosmic density is denoted by $< \rho >$ and may be written as $< \rho > = \Omega \rho_c = \Omega 3H_0^2/8\pi G$, introducing the density parameter, $\Omega$ (we will also
henceforth refer to the present value of the Hubble constant as \( h \equiv H_0/100 \) km/s/Mpc). We usually work with a smoothed version of the density field:

\[
\bar{\delta}_R(\vec{x}) = \int d^3x' \delta(\vec{x}') W_R(\vec{x} - \vec{x}').
\] (2)

We will only consider a top–hat window function, defined by

\[
W_R(\vec{x}) = W_R(x) = \begin{cases} 
3/4\pi R^3 & x < R \\
0 & \text{otherwise} 
\end{cases}
\] (3)

The smoothing scale, \( R \), may be written in terms of the mass enclosed by the window: \( M = (4\pi/3)R^3 < \rho > \). Notice that the scale \( R \) is written as a comoving coordinate, and so the enclosed mass is related to the mean cosmic density, not the mean density of a collapsed object. We will more often speak of scales in terms of mass, \( M \), than in terms of linear scale, \( R \).

For completeness, I rewrite the expression for the power spectrum, \( P(k) \), in terms of the variance of \( \bar{\delta}_R \):

\[
\sigma^2(M) \equiv \langle \bar{\delta}_R^2(\vec{x}) \rangle = \frac{1}{2\pi^2} \int dk \, k^2 P(k) |W_R(k)|^2.
\] (4)

Strictly speaking, the average is over an ensemble of different realizations (different universes!) of the density field at the point \( \vec{x} \); practically speaking, the average is taken over space (i.e., positions \( \vec{x} \)). The Fourier transform of the top–hat window function used in Eq. (4) is

\[
W_R(k) = \int d^3x \, e^{i\vec{k} \cdot \vec{x}} W_R(\vec{x}) = \frac{3}{(kR)^2} \left( \frac{\sin(kR)}{kR} - \cos(kR) \right).
\] (5)

### 2.1. Derivation of the Press–Schechter Formula

In this section, we follow closely the notation and presentation of Blanchard et al. (1992). Let \( \mathcal{F}(> M) \) be the fraction of material in collapsed objects of mass \( > M \). The mass function, \( n(M) \), giving the comoving density of objects of mass \( M \) per unit of mass, may then be expressed as

\[
n(M)dM \equiv \frac{\langle \rho \rangle}{M} \left| \frac{d}{dM} \mathcal{F}(> M) \right| dM.
\] (6)

In hierarchical models, in which the perturbations strictly decrease with scale, and which include most of the currently favored models (e.g., cold dark matter (CDM)–like models), we have the condition

\[
\int_0^\infty dMMn(M) = \langle \rho \rangle.
\] (7)

This follows from the fact that each particle in the universe must find itself contained in a sphere whose probability for collapse approaches unity as the radius decreases, because the variance becomes larger and larger.
In many models, the density perturbations are gaussian (important exceptions are defect models); this lead Press and Schechter (1974) to propose the following ansatz:

\[ F(> M) = \frac{1}{\sqrt{2\pi}} \int_{\nu_c}^{\infty} d\nu e^{-\nu^2/2}, \]

(8)

where \( \nu \equiv \bar{\delta}/\sigma(M) \) and \( \nu_c \), or \( \delta_c \), is some critical, linear density contrast required for an object to collapse. There are several important remarks concerning this proposition:

1. It is assumed that the linear density contrast, smoothed on a scale of mass \( M \), is the only criterion governing the formation of objects of this same mass, \( M \);
2. only points at the center of top–hat spheres are counted in the fraction \( F \);
3. it is assumed that all objects with mass > \( M \) are counted on the scale \( M \).

The first point is important because it means that we relate the abundance of non–linear regions to the linear theory density field, whose evolution is simple to calculate. Assumptions 2 and 3 lead to the violation of condition 7: \[ \int n(M) dM = 0.5 < \rho > \] – the ansatz misses some mass. Press and Schechter originally corrected this by simply multiplying Eq. (8) by 2. More recently, Bond et al. (1991) showed clearly that the missed mass is associated with objects of mass > \( M \) which are not counted among the regions of mass \( M \) capable of collapsing (remark 3). This mass should also be closely related to particles within collapsed spheres on the scale \( M \) (remark 2). Notice also that it is clear that the ansatz will miss some mass, because as \( M \to 0 \), \( \nu_c \to 0 \) and not \(-\infty\) to provide the necessary normalization; in other words, the mass in underdense regions, \( \nu_c < 0 \), is poorly modeled at the outset.

In any case, once corrected for the factor of 2, the Press–Schechter (PS) mass function becomes

\[ n(M, z) dM = \sqrt{\frac{2}{\pi}} \frac{< \rho > M}{\nu_c(M, z)} \frac{d\Delta(M, z)}{d\ln M} \frac{e^{-\nu_c^2/2} dM}{M}, \]

(9)

where \( \nu_c(M, z) \) is as before, but now explicitly showing its dependence on both mass and redshift (Problem 1). For example, in a critical universe (\( \Omega = 1 \)), linear perturbations grow as \( 1/(1 + z) \), and \( \nu_c = \delta_c(z)/\sigma(M, z) = \delta_c(z)(1 + z)/\sigma_o(M) \), where \( \sigma_o(M) \) is the present day power spectrum and \( \delta_c(z) = 1.68 \), a constant.

The value of this constant (1.68) can be found by considering the collapse of a spherical overdensity (Peebles 1980). More generally, we expect the critical density contrast to depend on the redshift and the cosmological parameters, and we write \( \delta_c(z; \Omega, \lambda) \), where \( \lambda \) is the cosmological constant. Even more importantly, the linear growth factor, \( D_g(z) \), also depends on the cosmology: \( D_g(z; \Omega, \lambda) \). As already stated, for \( \Omega = 1 \), \( D_g = 1/(1 + z) \). With this notation we have

\[ \nu_c(M, z) = \frac{\delta_c(z; \Omega, \lambda)D_g(0; \Omega, \lambda)}{\sigma_o(M)D_g(z; \Omega, \lambda)}. \]

(10)

We now see explicitly the dependence of the mass function on the power spectrum and the cosmological parameters.
2.2. Properties of the Mass Function

Let’s explore in more detail the mass function by considering a simple power–law model for the power spectrum: \( P(k) \propto k^n \). Via Eq. (4), this may also be written as

\[
\sigma_o(M) = \sigma_8 \left( \frac{M}{M_8} \right)^{-\alpha},
\]

\[
= \frac{1}{b} \left( \frac{M}{M_8} \right)^{-\alpha},
\]

in which I introduce the mass contained in spheres of radius \( 8h^{-1} \text{ Mpc} \) as \( M_8 \). The exponent \( \alpha = (n + 3)/6 \). The scale of \( 8h^{-1} \text{ Mpc} \) is often used as a reference for the normalization of the power spectrum, because the variance of galaxy counts in such spheres is about unity (Davis & Peebles 1983; Loveday et al. 1996). This leads to the notion of bias, and it is in the second line, Eq. (12), that I introduce the bias parameter as \( b = 1/\sigma_8 \); this quantifies any possible difference between the mass variance and the galaxy count variance on this scale.

With this power–law model, we can now write the PS mass function in a more compact and illustrative form:

\[
n(M, z) d\ln M = \sqrt{\frac{2}{\pi}} \left\langle \rho \right\rangle M^* \left( \frac{M}{M_8} \right)^{\alpha-1} e^{-0.5(M/M_8)^2} d\ln M. \tag{13}
\]
Figure 2. The influences of the power spectrum and density parameter on the PS mass function. The power spectrum is modeled by a power law – \( P(k) \propto k^n \) – and normalized by the bias parameter \( b = 1/\sigma_8 \). The solid curves show the results for a critical universe, with the labeled parameters. The degeneracy over cluster scales between \( P(k) \) and \( \Omega \) is demonstrated by the dashed curve. In all cases \( h = 1/2 \).

I have simplified things by defining a characteristic mass, \( M_\ast \), such that \( \nu_c(M, z) \equiv \left( \frac{M}{M_\ast(z)} \right)^\alpha \). The characteristic mass depends on redshift; it increases with time, or decreases with \( z \). Eq. (13) is plotted at two different redshifts in Fig. 1. The evolutionary trend is such that small objects merge to form large ones, which progressively moves the “knee” towards large scales and the low–mass end down. This simple power–law model is in fact quite useful and a not–to–bad approximation over limited scales of more realistic power spectra. As we shall soon see, CDM–like models typically have values of \( \sigma_8 \sim [0.5, 1] \) and \( n \sim [-2, -1] \) over cluster scales (\( \gtrsim 10^{14} \) solar masses).

It is remarkable that this simple and appealing PS formula provides an accurate description of the mass function seen in numerical simulations, at least over the interval of mass accessible to the simulations and centered on the “knee” of the distribution. At a given mass, the predicted PS number of objects deviates at worst from the numerical results by only \( \sim 25\% \). This conclusion has been reached by by Efstathiou et al. (1988) for power-law spectra in a critical universe, by Lacey and Cole (1994) using a larger simulation of the same types of models, and more recently by Eke et al (1996), who considered CDM–like power spectra in critical, flat and open geometries. It applies to the range of masses of interest to us – clusters with present–day virial temperatures \( \sim 1 – 10^9 \) keV. Thus, it appears that, perhaps as an happy accident, the PS formula provides a simple and accurate (to \( 25\% \) in number) representation of the mass function over cluster
scales for a variety of power spectra and cosmological models, within the context of gaussian theories.

Before we move on to apply what we have learned about the mass function to real data, let me emphasize the essential elements of our aim. We wish to deduce the mass function from cluster data and thereby constrain $\Omega, n$ and $b$. In Fig. 3 I plot the mass function at redshift zero for several different power-law spectra in a critical and an open universe. The set of solid lines shows the effects of changing the power spectrum index $(n)$ and normalization $(b)$. For the same normalization, “flat” – or “blue” – spectra (more negative values of $n$, represented by $n = -1.85$) push mass up into larger objects than “steep” – or “red” – spectra (represented by $n = -1$). We also see quite clearly the sensitivity of the mass function to the normalization. The dashed line represents an open model ($\Omega = 0.3$) with different values of $n$ and $b$ and which essentially reproduces one of the critical models – there is a degeneracy among $n, b$ and the underlying cosmological model.

Over cluster scales, the $n = -1$ power spectra mimic well the standard CDM spectrum ($\Omega = 1, h = 1/2$). As we shall soon see, this model does not correctly reproduce the observed number of X-ray clusters, regardless of the normalization. The bluer spectrum ($n = -1.85$) fairs better, and is the reason for which I chose this value for the figure (Oukbir et al. 1996). The open model parameters were chosen for the same reason (Oukbir & Blanchard 1996).

The degeneracy between the power spectrum and the density parameter prevents us from drawing any definitive conclusions from the mass function at a single redshift. Fortunately, the evolution of the mass function with redshift breaks this degeneracy. This is demonstrated by Fig. (1), where we see that degenerate models evolve differently towards higher $z$. The key point is that in an open universe, one expects more clusters at large $z$ than in an critical model. The reason is that the linear growth factor, $D_g$, “freezes-out” in an open model when the curvature begins to drive the expansion; this is to say that there is less growth at low redshift compared to a critical model. The consequence is that, normalized to the present day, the cluster abundance evolves less rapidly towards higher redshift in an open model. I emphasize the importance of this result, for it means that the very existence of high $z$ clusters carries important information about the density parameter, $\Omega$.

3. X–ray Clusters

In addition to galaxies, clusters consist of dark matter (perhaps baryonic, perhaps not) and a diffuse, hot gas known as the intra–cluster medium (ICM). The trick is to get at the mass of a cluster by observing these various components. Traditionally, one measures the velocity distribution of a cluster’s galaxies and applies the virial theorem (Sadat, these proceedings). As emphasized by several authors (e.g., Lucey 1983; Frenk et al. 1990), the identification of cluster members is confused by the projection along the line–of–sight of foreground and background galaxies, leading to possible contamination of optical catalogs by false clusters; it seems clear, for example, that the Abell catalog is so contaminated (Dekel et al. 1989; Efstathiou et al. 1992). In addition, the misidentification of cluster members inflates estimates of cluster velocity dispersions, and hence
virial mass estimates. The solution to these problems would seem to be greater care in cluster selection, by using objective computer algorithms (rather than human inspection, as in the case of the Abell catalog), and the acquisition of a large number of redshifts on each cluster to better model its dynamics. Such an approach becomes more observationally accessible thanks to the ever increasing size of multi-object spectrographs. However, because of these intricacies, we shall not consider optical observations in the following.

The ICM presents an alternative. This gas, more or less in equilibrium in the cluster potential, shines in bremsstrahlung emission at the virial temperature. For a typical cluster, this temperature falls around a few keV - clusters emit X-rays. Detailed study of cluster X-ray spectra demonstrates the thermal origin of the emission: the continuum follows a thermal bremsstrahlung spectrum and the existence of the Iron emission lines around 7 keV clenches the conclusion.

Two important quantities characterize the X-ray emission: the total (bolometric) luminosity and the temperature. The luminosity is an integral over the cluster volume:

$$L_X \propto \int dV n_g^2 T^{1/2} \propto f_g M < n_g T^{1/2} >_p . \tag{14}$$

The second proportionality introduces the cluster gas fraction by mass - $f_g$ - and the particle averaged quantity $< n_g T^{1/2} >_p \equiv (1/M) \int dV n_g(n_g T^{1/2})$. The above equation gives us a relation between the observable X-ray luminosity and the cluster virial mass, $M$. However, the application of this relation requires correct modeling of $< n_g T^{1/2} >_p$, and this depends most particularly on the density of the gas. This is to say that the cluster luminosity is sensitive to the ICM’s spatial distribution, a direct consequence of the “n-square” dependence of the bremsstrahlung process. In terms of the King model (Sadat, these proceedings), the X-ray luminosity depends on the core radius, $r_c$, of the gas distribution. The problem is that we really have no understanding of the physics determining $r_c$, and therefore modeling its evolution is difficult. One may, nevertheless, approach the mass function by this direction (Colafrancesco & Vittorio 1994), but I prefer instead to move on to the temperature, which is much easier to deal with from a modeling perspective.

If we assume that the gas simply shock heats on collapse to the virial temperature of the gravitational potential, we would have

$$T \propto \frac{M}{R} \propto M^{2/3}(1 + z), \tag{15}$$

where $R$ is the virial radius of the cluster mass distribution, i.e., the radius beyond which matter is infalling and has not yet been dynamically incorporated into the cluster system. The second proportionality follows from the expectation that $R \sim (M/\rho)^{1/3} \sim M^{1/3}(1 + z)^{-1}$. In problem 2, I ask you to consider in detail the physics of collapse and to show that the form of this relation is general, but that the exact value of the coefficient depends on the mass profile of the collapsing cluster and on the cosmological model. Thus, in general $T = \text{coeff}(\text{cosmology, profile}) M^{2/3}(1 + z)$. As spelled out in the problem, the relation follows from energy conservation only for the particle-weighted temperature, $< T >_p \equiv (1/M) \int dV n_g T$. Unfortunately, the measured X-ray temperature of
a cluster is not this quantity - it is instead an emission-weighted version (this is not true for the Sunyaev–Zel’dovich effect; more on this important point later!). We may also note that what is truly observed is the emission-weighted mean electron energy, and it has been pointed out that this may not be the same as the proton temperature, at least in the outer parts of clusters (Teyssier 1996).

Given these complications, one may nevertheless consider the observed X-ray temperature as a “good” indicator of the mass. This is most thoroughly demonstrated by the simulations presented by Evrard et al. (1996). In this work, the authors demonstrate the existence at $z = 0$ of a tight relation between the observable X-ray temperature and $M^{2/3}$. The simulations also supply the numerical value of the coefficient:

$$T = (6.8h^{2/3} \text{ keV}) M^{2/3}. \quad (16)$$

Simulated clusters in a variety of cosmological models follow this same relation with only a 20% scatter in mass at a given temperature. The numerical value of the coefficient turns out to be very close to the value one finds for a simple spherical collapse with a flat profile (the simulation value is slightly lower, probably due to incomplete thermalization of the shocked gas). The relation is so good, in fact, that Evrard (1997) uses it as a primary mass indicator for observed clusters when considering the baryon fraction over an ensemble of clusters (see also Sadat, these proceedings).

This means that we can now calculate with some confidence the expected cluster X-ray temperature distribution function from the mass function – the T–M relation was exactly what we needed. Comparison with observations then permits us to use our understanding of the mass function to constrain the power spectrum and the density parameter. Unfortunately, the temperature function has only been measured at $z = 0$, and we are currently unable to directly observe its evolution with redshift. It is true, of course, that clusters are detected in X-ray surveys out to redshifts now approaching unity, but the full spectrum needed to find $T$ requires many more photons than a mere detection – hence, the present limitation to $z = 0$. This will change with the next generation X-ray satellites (in particular, ASCA, AXAF and XMM). As per our previous discussion, we expect to be limited by the degeneracy among the cosmological parameters and the power spectrum when fitting our predicted temperature distribution function to the $z = 0$ observations.

In Figures 3 and 4, I compare the observed local temperature function with some theoretical predictions. The data come from Henry & Arnaud (1991) and Edge et al. (1990); it should be borne in mind that the two samples are not independent as they share some of the same clusters. The highest temperature datum is an estimate by Oukbir & Blanchard (1996) based on the existence of A2163. The theoretical calculations have all been made by transforming the PS mass function, with the given parameters, into a temperature function by using the above T–M relation (for $h = 1/2$). Consider the power–law models for $\Omega = 1$ and $\Omega < 1$. We see that in a critical universe clusters demand $\sigma_8 \sim 0.5 < 1$, i.e., their abundance provides evidence for non-zero bias. However, an open model with $\Omega < 1$ and $\sigma_8 = 1$ is also consistent with the data, manifesting the degeneracy we spoke about. Thus, the cluster abundance can be explained by either a critical, biased or open, unbiased scenario, and this degeneracy can
only be broken by observing cluster evolution (more on this in a minute). The values of the best fit parameters for the critical and open models were taken from Oukbir et al. (1996) and Oukbir & Blanchard (1996). Similar results for the critical model, also based on power–law spectra, were obtained by Henry & Arnaud (1991).

How about more realistic power–spectra? Several authors have worked with CDM–like power spectra and used the local cluster abundance to find the normalization, $\sigma_8$ (Bond & Meyers 1991; Lilje 1992; Bahcall & Cen 1993; Bartlett & Silk 1993; White et al. 1993; Colafrancesco & Vittorio 1994; Viana & Liddle 1996; Eke et al. 1996). For a critical universe, the normalization is insensitive to the exact shape of the power spectrum because the scale of $8h^{-1}$ Mpc corresponds to the mass of a rich cluster of galaxies: $M_8 = (4\pi/3)\rho_c(8h^{-1} \text{Mpc})^3 = 5.9 \times 10^{14}\Omega/h$ solar masses. We expect, and it is the case, that the results should therefore be the same as found when using power–law spectra. Notice, though, that for smaller values of $\Omega$ the normalization scale moves out through the low end of the cluster mass range, introducing a correlation between the deduced value of $\sigma_8$ and the shape of the power spectrum; this is clearly demonstrated, for example, by the non–circular contours in the $bias – n$ plane in the work of Oukbir & Blanchard (1996). Thus, in principal, the determination of $\sigma_8$ by the cluster abundance is model dependent in a low density universe – it depends on the power–spectrum adopted.

As a summary of published results, which show a fair degree of agreement, I quote Viana & Liddle (1996):

$$\sigma_8 \sim 0.60\Omega^{-0.59+0.16\Omega-0.06\Omega^2} \quad \text{flat} \quad (17)$$

$$\sigma_8 \sim 0.60\Omega^{-0.36-0.31\Omega+0.28\Omega^2} \quad \text{open} \quad (18)$$

In their work, Viana & Liddle fixed the shape of the power spectrum while varying the density parameter. However, the CDM–like power spectrum depends on $\Omega$ and $h$, and we have just seen that the actual shape of the spectrum does influence the deduced value of $\sigma_8$. For comparison, I quote a result obtained by taking this into consideration (for $h = 1/2)$:

$$\sigma_8 = 1/(0.77 + \Omega - 0.04\Omega^{-0.35}) \quad (19)$$

(Blanchard & Barbosa, private communication); the data used where those of Henry & Arnaud.

The uncertainty in the final result is difficult to gage. The two data sets lead to normalizations differing by $\sim 10–20\%$. Viana & Liddle provide an error analysis based on their approach (which does not take into account the detailed shape of the temperature function). However, a thorough study of the various approaches with the goal of digging–out a more complete understanding of the errors has not yet been performed. As a final remark, I note that Eke et al. (1996) claim that the high and low ends of the observed temperature function are anticorrelated.

The important conclusion is that the current data (limited to $z = 0$) on the cluster abundance permit us to constrain the amplitude of the mass fluctuations at the present epoch. It is clear, for example, that a critical model must be biased. A second key result is that ‘standard’ CDM does not work: the shape of
Figure 3. The present–epoch cluster X–ray temperature function normalized to the data from Henry & Arnaud (1991). The solid line shows a critical model for a power–law power spectrum with $n = -1.85$, $b = 1.65$; the dashed line is for an unbiased open model with $\Omega = 0.3$, $n = -1.4$ (Oukbir et al. 1996; Oukbir & Blanchard 1996). A CDM spectrum normalized to $b = 1.65$ produces the dot-dashed curve. The diamond at 14 keV is an estimate by Oukbir & Blanchard based on A2163 (Arnaud et al. 1992).
Figure 4. The present–epoch cluster X–ray temperature function normalized to the data from Edge et al. (1990). The solid line shows a critical model for a power–law power spectrum with $n = -2.02$, $b = 1.84$; the dashed line is for an unbiased open model with $\Omega = 0.2$, $n = -1.56$ (Oukbir et al. 1996; Oukbir & Blanchard 1996). A CDM spectrum normalized to $b = 1.84$ produces the dot-dashed curve. The diamond at 14 keV is an estimate by Oukbir & Blanchard based on A2163 (Arnaud et al. 1992).
its power spectrum is too steep (too few large, hot clusters relative to the small ones) and, in any case, a COBE normalization implies $\sigma_8 = 1$, leading to far too many clusters at any temperature. There are, however, other models which do work. Low-density models, either flat or open, with $\Omega \sim 0.2-0.7$ and $h \sim 0.5-1$ satisfy both the COBE data and the cluster temperature distribution function; they also have power spectra similar to that seen in the galaxy distribution on large scales (Liddle et al. 1996a, 1996b). A critical model may be made to work by changing its power spectrum. Tilting the primordial spectrum (Cen et al. 1992), adding some hot dark matter (Davis et al. 1992; Schaefer & Shafi 1992) or lowering the Hubble constant (Bartlett et al. 1995) are all ways of achieving the desired result.

Can we say anything yet about evolution? The deepest X–ray sample at the time of writing is the EMSS (Einstein Medium Sensitivity Survey; Gioia et al. 1990), which contains $\sim 100$ clusters serendipitously detected in Einstein pointings. The clusters extend out to redshifts approaching unity. This catalog will soon be supplanted by ROSAT samples based on the same principle - serendipitous discovery in deep pointed observations. Oukbir & Blanchard (1996) have modeled the redshift distribution of the EMSS clusters by employing the locally observed luminosity–temperature relation. This must be done because no direct measurements of the temperature are yet available for these clusters. It appears that if there is no evolution in the L–T relation with redshift, then a high–density model is favored; but with appropriate evolution, any model can be made to fit the data. For example, both the critical, biased model and the unbiased, open model with $\Omega = 0.3$ shown in Figure 1 can be made to work. We must await future observations with ASCA, AXAF and XMM, which can directly measure the temperature of high redshift clusters, before any definite conclusion can be drawn; but the principle is clear and the means will soon be available!

4. Sunyaev-Zel’dovich Effect

The hot intrachuster gas leads to another observational consequence. Via inverse Compton scattering, the electrons in the medium transfer energy to CMB photons passing through the cluster and distort the CMB spectrum, an effect known as the Sunyaev–Zel’dovich effect (SZ) (Sunyaev & Zel’dovich 1972). The effect is quantified by the induced change in sky brightness, $\delta i_\nu$, towards the cluster as compared to the mean CMB intensity:

$$\delta i_\nu = y j_\nu(x), \quad (20)$$

where the Compton $y$–parameter specifies the amplitude in terms of an integral along the line–of–sight –

$$y = \int dl \frac{kT_o}{m_e c^2} n_e \sigma_T \quad (21)$$

– and the spectral dependence is given by

$$j_\nu(x) = 2 \frac{(kT_o)^3}{(h_\nu c)^2} \frac{x^4 e^x}{(e^x - 1)^2} \left[ \frac{x}{\tanh(x/2)} - 4 \right]. \quad (22)$$
In these expressions, $T_e$, $n_e$ and $m_e$ refer to the electron temperature, density and mass, respectively; $\sigma_T = 6.65 \times 10^{-25}$ cm$^2$ is the Thompson cross-section and $x \equiv h\nu/kT$ is the dimensionless frequency of observation in terms of the unperturbed CMB temperature $T = 2.728$ K (Fixen et al. 1996). Note that $y$ is dimensionless and $j_\nu$ carries units of sky brightness - ergs/s/cm$^2$/steradian/Hz. A convenient unit is the Jansky (Jy), defined as $10^{-23}$ ergs/s/cm$^2$/Hz, so that $j_\nu$ may be written in units of Jy/steradian.

I plot the spectral function ($j_\nu$) in Figure 5; the shape is unique, unlike any other astrophysical source. This is the result of the fact that, while increasing the mean photon energy, inverse Compton scattering conserves the number of photons. In essence, the photons just diffuse upwards in energy, overpopulating the Wein region at the expense of the Rayleigh–Jeans. Thus, a cluster appears as a decrement in the sky brightness (a ‘hole’) at wavelengths longer than $\lambda = 1.38$ mm and as a source of excess brightness at shorter wavelengths. I ask you in problem 3 to explore the physics of inverse Compton scattering and to derive Equation (22).

To give you a feeling for the order of magnitude in rich clusters, the effect, which may be described in the Rayleigh–Jeans by a temperature perturbation towards the cluster, is typically $\delta T/T \sim 10^{-5} - 10^{-4}$. Incidentally, its detection proves the cosmological origin of the necessary normalization; in other words, the mass in underdense regions, $\nu_c < 0$, is poorly modeled at the outset. CMB; the fact that this is little doubted in the standard Big Bang model does not diminish the importance of this result - such proofs are always welcome.
Consider now the integrated effect of a cluster, or its flux density, \( S_\nu \):

\[
S_\nu(x, M, z) = j_\nu(x)D_a^{-2}(z) \int dV \frac{kT_e(M, z)}{m_e c^2} n_e(M, z) \sigma_T. \tag{23}
\]

The integral extends over the cluster volume and the angular distance \( D_a(z) = 2cH_0^{-1}[\Omega z + (\Omega - 2)(\sqrt{1 + \Omega z} - 1)]/\Omega^2(1 + z)^2 \) (for \( \lambda = 0 \)). The extremely important point here is that the integrated effect of a cluster, its flux density, depends only on the quantity of gas at temperature \( T_e \), and not on the spatial distribution of the gas. This is in contrast to the X-ray flux. In fact, we may write \( S_\nu \propto f_g M < T_e >_p \), where the temperature is, again in contrast to X-ray observations, the mean, particle-weighted electron energy. Recall that this is the quantity most clearly and directly related to the mass by energy conservation during collapse. We may hope, therefore, that the temperature-mass relation used here is not too grossly affected by any non-isothermality of the ICM; in any case, it should be much less so than the X-ray temperature.

Since the flux density is such a simple function of the total mass and redshift of a cluster, we may, as before, transform the mass function into an SZ distribution function at any redshift to calculate, for example, the mean distortion produced by the entire cluster population and the cluster number counts and redshift distribution (Korolyov et al. 1986; Markevitch et al. 1994; Bartlett & Silk 1994; Barbosa et al. 1996). Figure 8 reproduces a plot from Barbosa et al. (1996). Actually testing these predictions is difficult, for it demands a radio telescope of high resolution, about an arcmin, capable of surveying large quantities of sky; one of the important goals for ESA’s Planck Surveyor mission will be to measure the SZ cluster counts. The relatively simple modeling required to describe the effect and the ability to observe clusters out to arbitrarily large redshifts means that this method should prove very powerful in constraining \( \Omega \) and the history of the ICM.

I refer the reader to Barbosa et al. for a discussion of the implications for the mean \( y \) distortion (see also Cavaliere et al. 1991; Markevitch et al. 1991); it would appear that for low-density cosmologies, the predicted distortion approaches the current limit imposed by the FIRAS spectrum. Finally, discussion of the fluctuations induced in the CMB by unresolved clusters may be found in Shaeffer & Silk 1988, Cole & Kaiser 1989, Bond & Meyers 1991, Markevitch et al. 1992, Bartlett & Silk 1994, Ceballos & Barcons 1994, Colafrancesco et al. 1994.

5. Conclusions

As we have seen, galaxy clusters do indeed provide one of the most important probes of the density perturbation power spectrum and of \( \Omega \). The reason for this is clear: Clusters find themselves on the tail of the rapidly falling mass function expected in gaussian theories of structure formation. One may apply the same approach to non-gaussian models, but only after recalculating the mass function. The texture scenario is an example of a physically motivated non-gaussian model, and an analysis of galaxy clusters in this context may be found in Bartlett et al. (1993).
Figure 6. SZ redshift distribution and source counts. On the left, the thin lines trace the redshift distribution for a critical model, at a given flux density, while the thick lines represent the same for a degenerate open model. The integrated counts for the two models are shown on the right. In all cases, $h = 1/2$. See Barbosa et al. (1996).
Many authors have used current observations of X–ray clusters to constrain various popular theories of structure formation. The results appear to all be in agreement: a critical universe requires a non–zero bias to avoid over–producing the number of X–ray clusters. In particular, ‘standard’ CDM does not work, its most the most severe shortcoming being the high normalization required by the COBE data; one could imagine that gravity waves produce some of the COBE signal, thereby lowering the implied CDM amplitude, but even in this case it appears that the model produces a temperature distribution function which is steeper than the data. Quantitative results for the amplitude, $\sigma_8$, are given in Eqs. (17), (18) and (19). The important point of this lecture should be that this method provides access to the mass fluctuation amplitude, the fundamental theoretical quantity which is observationally rather elusive.

The present data are limited to the local Universe, $z = 0$, which prevents us from separating the influences of the power spectrum from the density parameter. With the ability to measure cluster X–ray temperatures at large redshift, future space missions, such as AXAF and XMM, will provide the means necessary to individually constrain the power spectrum and $\Omega$. I would like to emphasize the fact that clusters offer us a very useful probe of $\Omega$; this should be the second fundamental point of this lecture.

We have not discussed models with non–zero $\lambda$, but the various given literature references cover this topic. In general, a flat, low–density model evolves with a rate somewhere between a critical model and the corresponding open model with the same density. Such a scenario does not present any particular problems from the point of view of cluster modeling.

Statistical cluster studies via the SZ effect are just beginning to become an observational reality. This line of research should be quite analogous to X–ray clusters studies; we thus have an independent way to study cluster evolution. In fact, the SZ effect will permit us to see clusters to larger redshifts than X–ray observations - provided they are out there, a question whose answer has fundamental consequences. In addition, the SZ effect is easier to model than the X–ray emission of a cluster, an important advantage. By providing the first all–sky survey of clusters detected by their SZ effect, ESA’s Planck Surveyor (ex. COBRAS/SAMBA) seems to guarantee an exciting future for this new field of cluster observations.

6. Suggested Problems

6.1. The Mass Function

The mass function of collapsed objects is one of the fundamental quantities of any theory of structure formation. Following the notation of Blanchard et al. (1992), we write this function formally as

$$n(M)dM = -\frac{<\rho>}{M}dF(>M)dM,$$

where the function $F(>M)$ is the fraction of material in objects of mass greater than $M$, and $<\rho>$ is the cosmic mean density.
The Press-Schechter Ansatz  For gaussian theories, Press & Schechter (1974) proposed the following:

\[ F(> M) = \frac{1}{\sqrt{2\pi}} \int_{\nu_c}^{\infty} d\nu e^{-\nu^2/2}, \]

where \( \nu_c \equiv \delta/\sigma(M) \) is some critical, linear threshold for collapse. Using this ansatz, derive the Press-Schechter mass function (to within a factor of 2):

\[ n(M) = \frac{1}{\sqrt{2\pi}} \frac{< \rho > M}{\nu_c} \left( -\frac{d\ln \sigma}{dM} \right) e^{-\nu_c^2/2}. \]

Details  The above result undercounts the amount of mass in collapsed objects. To see this, show that

\[ \int dMMn(M) \neq < \rho > . \]

We must arbitrarily multiply by 2 to correct this expression to find the Press–Schechter mass function. This problem may be understood in two ways: As pointed out by Blanchard et al. (1992), this prescription only counts mass points at the center of top–hat spheres satisfying the threshold; it does not include particles within such spheres which are not at the center. Secondly, the prescription assumes that all collapsed objects with a mass \( > M \) satisfy the threshold criteria at the smoothing scale corresponding to \( M \). Bond et al. (1991) show that this is in fact not the case (at least for a sharp k–space filter); their formalism allowed them to derive the factor of 2.

6.2. The Temperature–Mass Relation

Here we will consider the origin of the temperature–mass relation for virialized objects. We start from the conservation of energy and the virial theorem as applied to a thin shell of matter as it collapses onto the cluster:

\[ K + U = \text{const} = U_{ta} \]  \hspace{1cm} (24)

\[ K + \frac{1}{2}U = 0, \]  \hspace{1cm} (25)

where \( K \) and \( U \) represent, respectively, the kinetic energy and gravitational potential energy of the shell, and \( U_{ta} \) is its potential (and hence total) energy at turn–around. Each collapsing shell then contributes a kinetic energy of \( K = -U_{ta} \) to the cluster, and the total kinetic energy may be written as a sum over shells. To proceed, assume that the collapse is spherical; there may be some reason to hope that this is not too far from reality for clusters, which often represent (in biased theories!) perturbations of several sigma (Bernardeau 1994).

Set-up  Show that an integral for \( K_{tot} \) is

\[ K_{tot} = (4\pi < \rho >)^2 \frac{G}{3} \int_0^{x_0} dx \frac{x^5}{a_{ta} x_{ta}}, \]

where \( < \rho > \) is the mean comoving matter density, \( x \) is a comoving lagrangian radius defined by \( M(< x) = (4\pi/3)x^3 < \rho >, \) while \( x_{ta} \) is the comoving radius of the shell at turn–around, when the expansion factor is \( a_{ta} \).
Specific Case  Consider the collapse of spherical density perturbations with a power–law profile, $\delta(x) \propto x^{-\gamma}$, in a critical universe ($\Omega_0 = 1$). In this case,

$$\delta_{ta} = 1.06 = \delta_o(x)_{ta}$$

relates the radius of a shell to its turn–around epoch, $a_{ta}$. Show that the mean, particle–weighted temperature – defined by $< T >_p \equiv (2/3)k_{tot}/(Nk)$, where $N$ is the total number of particles in the cluster and $k$ is Boltzmann’s constant – of the cluster is

$$< T >_p = (4\pi < \rho >)^{2/3} \frac{2G}{3} \frac{1}{k} \frac{\mu m_p}{\eta} \frac{(1+z)}{5-\alpha} M^{2/3}.$$ 

The mean molecular weight of the particles is $\mu m_p$, and $\eta \equiv x_{ta}/x$. Notice that the temperature has the form $M^{2/3}(1+z)$, and also that the constant of normalization in front depends on the density profile in the linear regime.

Generalization  Prove, regardless of the density profile or the cosmology, that $< T >_p$ always has the dependence $M^{2/3}F(z)$, where $F(z)$ is an unknown function of the redshift only. Once again, apply the spherical collapse model.

Comments  The important point here is that, within the spherical collapse model, $< T >_p$ is always a function of $M^{2/3}F(z)$, but that the coefficient in the relation depends on the cosmology and the density profile. This means that it may depend on the spectrum of density perturbations, something which we ignored during the lecture. The other point is that $< T >_p$ is not what is measured by the X-ray observations – this is rather the emission weighted temperature. If the cluster is not isothermal, they need not be the same. On the other hand, the SZ effect sees $< T >_p$. This is one of the advantageous aspects of the SZ modeling.

6.3. The Sunyaev-Zel’dovich Effect

Inverse Compton scattering of CMB photons by a thermal gas at temperature $T_e$ is described by the Kompaneets equation (Kompaneets 1957) for the photon phase–space distribution function, $n_\gamma$:

$$\frac{\partial n_\gamma}{\partial y} = \frac{1}{x_e^2} \frac{\partial}{\partial x_e} \left( x_e^4 \left( \frac{\partial n_\gamma}{\partial x_e} + n_\gamma + n_\gamma^2 \right) \right)$$

In this equation, $x_e \equiv h\nu/kT_e$, where $T_e$ is the temperature of the electron gas, and the Compton–y parameter is given by $dy \equiv (kT_e/m_e c^2)n_e \sigma_T c dt$. Note that we distinguish $x_e$ from $x \equiv h\nu/kT_o$, where $T_o = 2.728$ is the temperature today of the CMB, as used in the text.

General Solution  Show that the photon number density – $N_\gamma = 4\pi \int d\nu \nu^2 n_\gamma \propto \int dx_e x_e^2 n_\gamma$ – is conserved by this equation; what is the physical reason for this? Given this information, find the most general equilibrium solution to the Kompaneets equation (hint: it is an extension of a blackbody spectrum).
First Order Solution  

Galaxy clusters produce only a small perturbation to the CMB spectrum, of order $y \sim 10^{-5}$. Derive the first order perturbative solution ($y << 1$) around a pure blackbody spectrum:

$$j_{\nu}(x) = 2\frac{(kT_o)^3}{(hpc)^2} \left( \frac{x^4 e^x}{(e^x - 1)^2} \left( \frac{x}{\tanh(x/2)} - 4 \right) \right).$$

Notice that it is $x$ and not $x_e$ which is used in this expression.

Acknowledgments.  I would like to thank my collaborators on this work, from whom I have learned much and with whom I have greatly enjoyed working over the years: D. Barbosa, A. Blanchard, J. Oukbir, and J. Silk.

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