Non-standard Dirac equations for non-standard spinors

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Group-theoretical grounds of ELKO theory are reexamined. A compact four-component equation for ELKO is presented. It is demonstrated that in spite of their non-covariant nature ELKO can serve as a carrier space of representation of Poincaré group. However, the corresponding boost generators are not manifestly covariant and generate non-local momentum dependent transformations.

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I. INTRODUCTION

Dark matter is seemed to be the biggest challenge for human intellect. Being quantitative dominant substance of the universe, it is still waiting for a consistent theoretical framework for its description.

Till now we can indicate only preliminary attempts to create hypothetical elements of future dark matter theory. To create such theory, in essence new classes of fields are requested. One of candidates to form the dark matter is the axion field, see, e.g. [1] and references cited therein. This circumstance was an inspiration for us to analyze group-theoretical grounds of axion electrodynamics and construct exact solutions for the related field equations [2], [3].

Few years ago a new class of spinor fields was introduced [4], [5]. They are the dual-helicity eigen-spinors of the charge conjugation operator (in German: Eigenspinoren des Ladungs- konjugationsoperators, ELKO). The concept of ELKO open new interesting possibilities in constructing of relativistic models, including the models of dark matter (see, e.g., [4]-[7]), and of other cosmological phenomena. In particular, this concept was used to provide a new explanation of the accelerated expansion of the universe [8]-[10]. Higher dimension aspects of ELKO theory were considered in [11] and [12]. We will not discuss the validity and perspectives of all these models, but restrict ourselves to their kinematical backgrounds connected with using ELKO.

Mathematically, ELKO belong to one of non-equivalent classes of bispinor fields classified by Louesto [13]. Namely, they are the so-called flagpole spinor fields [14]. There exist a clear representation of these spinors proposed in papers [4] and [5]. However, the generally accepted description of kinematics of these fields is rather cumbersome. Indeed, starting with papers [4] and [5], everybody believes that such (free) fields should be described by an sixteen component Dirac equation supplemented by two additional

conditions. But it is a too complicated way to describe a four-component wave function.

By construction, ELKO are eigenvectors of the dual helicity operator. Such property is not evidently compatible with Lorentz invariance, since this operator is not a relativistic scalar for massive fields. Moreover, in fact ELKO contain a hidden preferred direction that breaks Lorentz symmetry [15].

A natural question arises whether it is possible to formulate this theory in a more compact and relativistic invariant manner. In paper [16] a manifestly covariant generalization of ELKO concept is proposed. The related ”dark matter spinors” solve a second order field equations supplemented by nonlocal constraints.

In the present paper a simple way to describe ELKO by a four-component generalized Dirac equation is presented. More exactly, a modernized Dirac equation will be used, the mass term of which is not proportional to the unit matrix. In addition, a direct and straightforward connection between ELKO and Dirac spinors will be demonstrated.

We also find the explicit form of generators of Poincaré group which can be realized on ELKO. It appears that these generators do not have a manifestly covariant form and generate momentum dependent and so non-local transformations of vectors from their carrier space. And that is a message that ELKO theory can be treated as Poincaré invariant in spite of that it is not manifestly covariant.

Finally, we present a toy model which, being transparently relativistic invariant, is characterized by the same kinematical equations as ELKO.

II. MULTI-COMPONENT DIRAC EQUATION FOR ELKO

Let us start with the main definitions of ELKO theory. To save a room we will use compact notations presented in what follows.

Like the Dirac spinors, the ELKO \( \lambda^S_{\{\pm,\mp\}} = \psi^\pm, \lambda^A_{\{\pm,\mp\}} = \psi^\mp \) and \( \lambda^A_{\{\mp,\pm\}} = \psi^- \) satisfy the Klein-Gordon equation. However, they do not satisfy Dirac equation, which is changed to the
following system in the momentum representation [4, 3]:
\[\begin{align*}
\gamma^\mu p_\mu \psi^+ + im\psi^+ &= 0, \\
\gamma^\mu p_\mu \psi^+ - im\psi^+ &= 0, \\
\gamma^\mu p_\mu \psi^- + im\psi^- &= 0, \\
\gamma^\mu p_\mu \psi^- - im\psi^- &= 0.
\end{align*}\] (1)

Following [4, 3] we use the Weyl representation of Dirac matrices with diagonal and hermitian matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Then ELKO can be specified in the following way:
\[\psi^\mu(p) = \frac{\sqrt{E + m}}{m} \left(1 - \mu \frac{p}{E + m}\right) \lambda^\mu \] (2)

where
\[\lambda^\mu = \begin{pmatrix} \varepsilon\sigma_2\phi_\mu(0)^* \\ \phi_\mu(0) \end{pmatrix},\]

$\varepsilon, \mu = \pm$, $E = \sqrt{p^2 + m^2}$, $p^2 = p_0^2 + p_1^2 + p_2^2 + p_3^2$, $\sigma_2$ is the Pauli matrix, and $\varepsilon$ is an involution which commutes with generators of Poincaré group. However, this representation is not covariant.

By construction, the four-component bispinors $\psi^\mu$ satisfy the following conditions:
\[C\psi^\mu = \varepsilon\psi^\mu \] (3)

and
\[\sigma_p\psi^\lambda = \lambda\psi^\lambda \] (4)

where $C = \gamma_2\kappa$ is the charge conjugation operator, $\kappa\psi = \psi^*$, and $\sigma_p = \frac{1}{p} \gamma_0\gamma_\alpha p_\alpha \equiv \gamma_5 \frac{\sigma_p}{p}$ is a product of the helicity operator with matrix $\gamma_5$. In other words, $\psi^\epsilon$ are eigenvectors of commuting operators $C$ and $\sigma_p$, and just equations (3) and (4) can be used as a formal definition of ELKO. Notice that such definition is universal and does not depend on concrete realizations of $\gamma$-matrices and spinor components.

Relations (1) specify a sixteen component Dirac equation in momentum representation for $\Psi$ = column($\psi^+_1$, $\psi^+_2$, $\psi^+_+$, $\psi^-_-$), and can be rewritten as:
\[ (\Gamma^\mu p_\mu - m)\Psi = 0 \] (5)

where
\[\Gamma^\mu = \begin{pmatrix} 0 & -i\gamma^\mu & 0 & 0 \\ i\gamma^\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & i\gamma^\mu \\ 0 & 0 & -i\gamma^\mu & 0 \end{pmatrix}\]

are the $16 \times 16$ Dirac matrices. Moreover, this equation should be considered together with the additional constraints (3) and (4) which reduce the number of independent components of $\Psi$ to 4.

Equations (3) have a symmetric form which is transparently relativistic invariant. The same is true for equation (5). However, it is not the case for the constraint (3) which is not relativistic invariant in the generally accepted meaning. Namely, covariant equations (1) and (3) connect modes $\psi^+_+$ and $\psi^-_-$ that are not defined in a covariant manner.

A natural question appears whether ELKO can be treated as a relativistic substance et all. We will see that it is the case since these non-standard spinors form a carrier space for a representation of Poincaré group. However, this representation is not covariant.

One more inspiration to reexamine the grounds of ELKO notion is the general feeling that there are too many equations for a spinor with four independent components. It is naturally to look for a more compact kinematical equations. Just such equation, which also opens a way to give a possible interpretation of Poincaré invariance of ELKO theory, is presented in the following section.

### III. FOUR-COMPONENT EQUATION FOR ELKO

Let us consider the following generalized Dirac equation
\[(\gamma^\mu p_\mu + Im)\psi = 0 \] (6)

where $I$ is an involution which commutes with $\gamma^\mu p_\mu$, or pseudo involution anticommuting with $\gamma^\mu p_\mu$. Solutions of any equation of type (6) satisfy the condition
\[(p_0^2 - p^2 - m^2)\psi = 0 \] (7)

which generate the relativistic dispersion relation. If, in addition, $I$ commutes with generators of Poincaré group, equation (6) is transparently relativistic invariant.

For $I$ being the unity operator equation (6) is reduced to the standard Dirac equation. Choosing $I = \gamma_5$ we obtain a good relativistic equation which is equivalent to the Dirac one.

But there are more involutions $I$ which satisfy the enumerated criteria. In particular, they can be constructed using matrix $\gamma_5$, space inversion $P$, time reflection $T$, charge conjugation $C$ and their products.

Let us consider a more sophisticated example of equation (6) with $I = iC\sigma_p$:
\[(\gamma^\mu p_\mu + iC\sigma_p m)\psi = 0. \] (8)

Just this equation can be used to describe the kinematics of ELKO. Indeed, this compact expression is completely equivalent to the cumbersome system (1), (3).
To prove this statement it is sufficient to act on \( \mathbf{8} \) from the left by projectors \( P^\nu_p = \frac{1}{4} (1 + \varepsilon C)(1 + \nu \sigma_p) \) and use the following identities:

\[
P^\nu_p \gamma^\mu p_\mu = \gamma^\mu p_\mu P^{-\nu}_p, \quad P^\nu_p iC \sigma_p = i \varepsilon \lambda p^{-\nu}_p,
\]

and notations

\[
P^\nu_p \psi = \psi^\nu_0.
\]

In this way we immediately come to equations \( \mathbf{11} \), \( \mathbf{3} \) and \( \mathbf{4} \). On the other hand, summing up all equations included into system \( \mathbf{1} \) and using definitions \( \mathbf{9} \) we come to equation \( \mathbf{8} \). Thus the system \( \mathbf{1} \), \( \mathbf{3} \), \( \mathbf{1} \) admits an equivalent and rather compact formulation \( \mathbf{8} \).

Let us show that equation \( \mathbf{8} \) is mathematically equivalent to the Dirac equation. Indeed, multiplying \( \mathbf{8} \) by \( \gamma_0 \) we transform it to the Schrödinger form:

\[
p_0 \psi = H \psi, \quad H = \gamma_0 \gamma_a p_a + i \gamma_0 \sigma_p m.
\]

Then, making the transformation

\[
\psi \to \psi_D = U \psi, \quad H \to H_D = U H U^{-1}
\]

with

\[
U = \frac{1}{2} (1 - \gamma_5 \sigma_p)(1 - i \gamma_5), \quad U^{-1} = \frac{1}{2} (1 + \gamma_5 \sigma_p)(1 + i \gamma_5)
\]

we reduce \( \mathbf{10} \) to the standard Dirac equation:

\[
p_0 \psi_D = H_D \psi_D, \quad H_D = \gamma_5 \gamma_a p_a + \gamma_0 m.
\]

Operator \( \mathbf{12} \) satisfies the condition \( U U^\dagger = 1 \) and includes the complex conjugation operation. Following Wigner \( \mathbf{17} \) we classify \( \mathbf{11} \) as unitary-antiunitary transformation.

**IV. RELATIVISTIC INVARIANCE**

Equation \( \mathbf{8} \) is completely equivalent to system \( \mathbf{1} \), \( \mathbf{3} \), \( \mathbf{4} \) and generates a relativistic dispersion relation \( \mathbf{7} \). However, the multiplier for the mass term in \( \mathbf{3} \) is not a relativistic scalar, and this fact can be treated as a direct proof that ELKO are not well defined covariant spinors. On the other hand, equation \( \mathbf{8} \) is equivalent to the relativistic Dirac equation, and so it has to inherit its symmetries at least in some more generalized meaning.

In the wide sense, relativistic invariance of a differential equation means that its solutions form a carries space of a representation of Poincaré group. Let us show that equation \( \mathbf{8} \) satisfies this week invariance condition.

It is a common knowledge that the Dirac equation is relativistic invariant. Moreover, the corresponding generators of Poincaré group can be represented in the following form:

\[
P_0 = iH_D, \quad P_a = ip_a, \quad J_{ab} = p_0 \frac{\partial}{\partial p_a} - p_a \frac{\partial}{\partial p_b} + S_{ab},
\]

where \( p_0 = \pm \sqrt{p^2 + m^2} \) and \( S_{\mu \nu} = \frac{1}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \).

On the set of solutions of equation \( \mathbf{13} \) the boost generators \( J_{0a} \) can be rewritten in the following form:

\[
J_{0a} = -\frac{1}{2} (H_D \dot{x}_a + \dot{x}_a H_D)
\]

where \( \dot{x}_a = \frac{\partial}{\partial p_a} - \frac{p_a}{p^2} \).

To find a realization of these generators on the set of solutions of equation \( \mathbf{8} \) it is sufficient to make the transformation \( P_\mu \to P'_\mu = U^{-1} P_\mu U, \quad J_{\mu \nu} \to J'_{\mu \nu} = U^{-1} J_{\mu \nu} U \) where \( P_\mu \) and \( J_{\mu \nu} \) with \( \mu, \nu = 0, 1, 2, 3 \) are generators \( \mathbf{14} \), and \( U \) is operator \( \mathbf{12} \). As a result we obtain:

\[
P_0 = iH, \quad P_a = i p_a, \quad J'_{ab} = J_{ab},
\]

\[
J'_{0a} = -\frac{1}{2} (\dot{x}_a H + H \dot{x}_a) + \frac{m}{p^2} \varepsilon_{abc} \gamma_5 \gamma_0 p_c \gamma_0 m
\]

where \( H \) is hamiltonian fixed in \( \mathbf{10} \) and \( \dot{x}_a \) is the transformed imaginary unit, i.e., \( i = -U^{-1} i U = i \frac{\hat{p}_a}{p^2} C \).

Alternatively, starting with realization \( \mathbf{14} \) for \( J_{0a} \) and using the identities

\[
U^{-1} \frac{\partial}{\partial p_a} U = \frac{\partial}{\partial p_a} + \frac{i}{p} S_{ab} p_b (\gamma_5 \sigma_p C - 1),
\]

\[
U^{-1} S_{0a} U = S_{0a} - \frac{i}{p} S_{ab} p_b (\sigma_p - \gamma_5 C)
\]

we obtain:

\[
J'_{0a} = -p_0 \frac{\partial}{\partial p_a} + \Sigma_{0a}
\]

where

\[
\Sigma_{0a} = S_{0a} + \frac{i}{p^2} S_{ab} p_b (p_0 - \gamma_0 \gamma_a p_a) (1 - \gamma_5 \sigma_p C).
\]

When acting on ELKO \( \psi^\pm_+, \psi^-_+ \) and \( \psi^+_+, \psi^-_+ \) matrix \( \mathbf{19} \) is reduced to the form

\[
\Sigma_{0a} = \begin{pmatrix} \frac{1}{2} \sigma_a & 0 \\
\frac{m}{p^2} \varepsilon_{abc} \sigma_b p_c & -\frac{1}{2} \sigma_a \end{pmatrix}
\]

and

\[
\Sigma_{0a} = \begin{pmatrix} \frac{1}{2} \sigma_a & \frac{m}{p^2} \varepsilon_{abc} \sigma_b p_c \\
0 & -\frac{1}{2} \sigma_a \end{pmatrix}
\]
correspondingly.

Operators (15), (17) and (16), (18) satisfy the following commutation relations

\[ [P'_\mu, P'_\nu] = 0, \quad [P'_\mu, J'_{\lambda\sigma}] = g_{\mu\lambda} P'_\sigma - g_{\mu\sigma} P'_\lambda, \]
\[ [J'_{\mu\nu}, J'_{\lambda\sigma}] = g_{\mu\lambda} J'_{\nu\sigma} + g_{\nu\lambda} J'_{\mu\sigma} - g_{\nu\sigma} J'_{\mu\lambda}, \]

which specify the Lie algebra of Poincaré group.

Thus we find the explicit form of generators of Poincaré group which can be defined on the set of solutions of equation (10) for ELKO. As expected, the angular momentum for ELKO and Dirac fields have the same form.

The boost generators \( J_{0a} \) and \( J'_{0a} \) are different. Moreover, they are qualitatively different. Indeed, using equation (17), generators \( J_{0\lambda} \) can be rewritten in covariant form (12), whereas generators \( J'_{0\lambda} \) do not keep this property.

An important quality of realization (14) is that the matrix term \( S_{0a} \) which generates transformations for the wave function commutes with the term \( p_0 p_a \) responsible for transformations of independent variables. As a result Lorentz transformations for Dirac spinors have the following generic form:

\[ \psi(\hat{p}) \rightarrow D(\Lambda^{-1}) \psi(\Lambda \hat{p}) \]  

(20)

where \( \hat{p} = (p_0, p_1, p_2, p_3) \). \( \Lambda \) is the Lorentz transformation matrix and \( D(\Lambda^{-1}) \) is a numeric matrix dependent on transformation parameters. In particular, for Lorentz boost we have

\[ D(\Lambda^{-1}) = \exp(S_{0a} \theta_a) \]
\[ = \cos \left( \frac{\theta}{2} \right) + 2 S_{0a} \theta_a \theta \sinh \left( \frac{\theta}{2} \right) \]  

(21)

where \( \theta_a \) with \( a = 1, 2, 3 \) are transformation parameters and \( \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2} \).

Notice that transformations for the wave function are the same for all values of independent variables.

The boost generator (18) does not have a covariant form.

Indeed, the matrix term \( \Sigma_{0a} \) is much more complicated than term \( S_{0a} \) present in (14). It depends on \( p_0 \) does not commute with \( p_0 \) and generates dependent on \( p \) transformations for \( \psi \). Nevertheless, integrating the Lie equations generated by operators (17), in principle it is possible to find transformations for ELKO which correspond to Lorentz transformations of independent variables.

Let us note that in spite of its non-covariant form, boost generator (17) gives rise to covariant transformations in the form (20), (21) provided the new inertial reference frame moves parallel to momentum \( p \). Indeed, in this case the transformation parameter vector \( \theta = (\theta_1, \theta_2, \theta_3) \) can be represented as \( \theta = \alpha n \) where \( n = \frac{p}{p_0} \), and so \( S_{0a} \theta_a = S_{0a} \theta \alpha \), exactly as in the case of Dirac equation. Thus it is possible to make standard transformations to the rest frame, study VSR aspects of ELKO (19), etc, etc.

V. ONE MORE NON-STANDARD DIRAC EQUATION

Let us return to equations (9). We will consider their solutions as functions of four independent variables \( p_0, p_1, p_2, p_3 \) which are equal in rights. Then we postulate invariance of (9) with respect to the following discrete transformations

\[ \Psi(p_0, p) \rightarrow \Psi(\gamma)(p_0, p) = \gamma_0 \Psi(p_0, -p), \]
\[ \Psi(p_0, p) \rightarrow \tau \Psi(p_0, p) = \gamma_1 \gamma_3 \Psi^T(-p_0, p), \]
\[ \Psi(p_0, p) \rightarrow \epsilon \Psi(p_0, p) = \gamma_2 \Psi^*(p_0, p). \]

By definition, \( P \) commutes with \( \gamma^\mu p_\mu \) while \( C \) and \( T \) anticommute with this term. Thus, in order to equation (6) be invariant with respect to these transformations, it is necessary to ask for the following conditions for \( I \):

\[ PI = IP, \quad CI = -IC, \quad TI = -IT. \]

In addition, to guarantee correct dispersion relations (7), (pseudo)involution \( I \) should satisfy one of the following relation:

\[ \gamma^\mu p_\mu I = I \gamma^\mu p_\mu, \quad I^2 = 1 \]  

(22)

or, alternatively,

\[ \gamma^\mu p_\mu I = -I \gamma^\mu p_\mu, \quad I^2 = -1. \]  

(23)

These conditions together with the requirement of Lorentz invariance leave the only possibility for \( I \), i.e., \( I = i\gamma_5 PT \). In this case equation (6) takes the following form:

\[ (\gamma^\mu p_\mu + im\gamma_5 PT) \Psi = 0 \]  

(24)

were we change \( \psi \rightarrow \Psi \) to discriminate solutions of (24) from wave functions discussed in the previous sections.

Operator \( i\gamma_5 PT \) commutes with \( \gamma^\mu p_\mu \) and is an involution, i.e., \( (i\gamma_5 PT)^2 = 1 \). Thus acting on equation (24) from the left by \( (\gamma^\mu p_\mu - im\gamma_5 PT) \) we immediately find that equation (24) generates condition (7). Moreover, in contrast with (10), equation (24) is transparently relativistic invariant.

Let show that there exist some intriguing similarities between solutions of equation (24) and ELKO. Indeed, acting to this equation from the left by the projector

\[ \hat{P}_\lambda = \frac{1}{4}(1 + \epsilon C)(1 + \lambda \gamma_5 PCT) \]

and using the identities

\[ \hat{P}_\lambda \gamma^\mu p_\mu = \gamma^\mu p_\mu \hat{P}_\lambda^{-\epsilon}, \quad \hat{P}_\lambda i\gamma_5 PT = i\epsilon \lambda \hat{P}_\lambda^{-\epsilon}, \]
one can make sure that the linearly independent functions
\[ \Psi_+ = \hat{P}_+ \Psi, \quad \Psi_- = \hat{P}_- \Psi, \quad \Psi_{\pm} = \hat{P}_\pm \Psi \]
(25) satisfy the fundamental equations of ELKO theory, given by formulae (11).

In accordance with (25), spinors \( \Psi_{\lambda} \) satisfy the following conditions:
\[ C\Psi_{\varepsilon} = \varepsilon\Psi_{\lambda}, \quad \gamma_5 PCT \Psi_{\varepsilon} = \lambda\Psi_{\lambda}. \]
(26) Since operator \( \gamma_5 PCT \) is nothing but a total reflection of all independent variables, the latter equation can be rewritten in the following form:
\[ \Psi_{\varepsilon}(p_0, -p) = \lambda\Psi_{\varepsilon}(p_0, p). \]
(27) Thus, like ELKO, functions (25) are eigenvectors of the charge conjugation operator, satisfying equations (11) and (3). However, in contrast with (26), they are not eigenstates of the chirality operator, but are eigenstates of \( \gamma_5 PCT \) instead.

The fundamental distinction of the introduced spinors \( \Psi \) from ELKO is that, in contrast with (11), they are transparent relativistic invariant.

Finally, let us represent a non-standard Dirac equation in configuration space:
\[ (i\gamma^\mu \partial_\mu - imR)\Psi(x) = 0 \]
(28) where \( x = (x_0, x_1, x_2, x_3) \) and \( R \) is the total reflection operator whose action on \( \Psi(x) \) is defined as \( R\Psi(x) = \Psi(-x) \).

Acting on (28) from the left by projectors \( P_+ = \frac{1}{2}(1 + R) \) and \( P_- = \frac{1}{2}(1 - R) \), we obtain the following system
\[ i\gamma^\mu \partial_\mu \Psi_+(x) = im\Psi_-(x), \]
\[ i\gamma^\mu \partial_\mu \Psi_-(x) = -im\Psi_+(x) \]
(29) where \( \Psi_\pm = P_\pm \Psi \) are eigenvectors of the total reflection operator.

Up to the meaning of vectors \( \Psi_\pm \) the system (29) coincides with equations for ELKO in configuration space, presented, e.g., in [5].

VI. DISCUSSION

In this paper a new look on grounds of the ELKO theory is presented. First, we give a compact four component formulation (8) of kinematic equation of this theory which is equivalent to the cumbersome system (11), (3) and (3). Then, a simple and straightforward connection between Dirac spinors and ELKO is presented. Finally, the transformation properties of ELKO w.r.t. Lorentz boost are discussed, and Poincaré invariance of ELKO approach is established, but in some restricted sense.

The transformations connecting the Dirac and ELKO were studied in [18]. However, these transformations where made under the supposition that the left handed components of the Dirac and ELKO coincide. In order to this supposition be correct, the Dirac spinors should satisfy one of the additional constraints discussed in [18]. The transformation generated by operator (12) is valid without additional constraints and is seemed to be more convenient.

We show that in spite of that the eigenvectors of dual helicity operator are not covariant subjects, the ELKO form a carrier space of the representation of Poincaré group, whose generators are given by equations (16). The corresponding boost generators do not have a covariant form. Nevertheless, they generate covariant transformations for the case when the new frame of reference moves parallel to particle momentum. These facts can be treated as the last resort of ELKO approach which is appears to be non-covariant in the standard meaning [5].

Equations, presented in Section 5 are just toy models which are seemed to be rather peculiar. In particular, the equality in rights of all variables in equation (21) is a natural but non-standard proposition. Usually \( p_0 \) is considered as a distinguished variable which is not affected by the time reflection.

The formal analogy of these equations with kinematical equations for ELKO is rather curious. And this analogy generates a challenge to search for possible applications of the corresponding fields in non-standard physical theories.

A specific feature of equations (8), (23) and (21) is that they include involutions \( C\sigma_\mu, \gamma_5 PCT \) or \( R \) as essential constructive elements. Such (and other) involutions present additional tools for creating alternatives to Dirac’s factorization of the Klein-Gordon equation. Apparently the first example of such non-standard factorization was proposed long time ago in paper [20] where a two component version of first order equations for a massive spinor field was discussed.

It is interesting to note that equation (10) for ELKO can be decoupled to two subsystems each of which is two-component like equation proposed in [20]. Indeed, hamiltonian \( H \) commutes with diagonal matrix \( \gamma_5 \), and so
\[ H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \]
where
\[ H_\pm = \sigma_a p_a \left( \pm 1 - \frac{im}{p} \sigma_2 \kappa \right) \]
and \( \sigma_a \) are Pauli matrices. Let us note that boost transformations mix eigenvalues of \( H_+ \) and \( H_- \).

Involutive discrete symmetries also have useful applications in construction of exact Foldy-Wouthuysen...
transformations \cite{21} and generating of non-standard realizations of symmetry algebras and superalgebras \cite{22,23,24,25,26}. We see that such involutions can also be effectively used to formulate a compact equation for ELKO.

[1] Leanne D. Duffy, Karl van Bibber, Axions as Dark Matter Particles. New J.Phys. 11 (2009) 105008; arXiv:0904.3346.
[2] A. G. Nikitin and Oksana Kuriksha, Symmetries of field equations of axion electrodynamics, Phys. Rev. D 86 (2012) 025010; arXiv:1201.4935.
[3] A. G. Nikitin and O. Kuriksha, Invariant solutions for equations of axion electrodynamics. Commun. Nonlinear Sci. Numer. Simulat. 17 (2012), 4585.
[4] D. V. Ahluwalia and D. Grumiller, Dark matter: A spin one half fermion field with mass dimension one? Phys.Rev. D 72 (2005) 067701; arXiv:0410192.
[5] D. V. Ahluwalia and D. Grumiller, Spin-half fermions with mass dimension one: theory, phenomenology, and dark matter, JCAP 0507 (2005) 012; arXiv:0412.080.
[6] C.G. Böhmer, The EinsteinCartanELKO system, Annalen der Physik, 16 (2007) 38; arXiv:0607.088.
[7] Christian G. Böhmer, David F. Mota, CMB anisotropies and inflation from non-standard spinors, Phys. Lett. B 663 (2008) 168; arXiv:0710.2003.
[8] A. Basak, J. R. Bhatt, S. Shankaranarayanan, K.V. P. Varma, Attractor behaviour in ELKO cosmology, JCAP 1304 (2013) 025; arXiv:1212.3445.
[9] C. G. Böehmer, The Einstein-ELKO system: Can dark matter drive inflation? Annalen Phys. 16 (2007) 325; ArXiv 0701.087.
[10] L. Fabbri, The most general cosmological dynamics for ELKO matter fields. Phys. Lett. B 704 (2011) 255; arXiv:1011.1637.
[11] Y. X. Liu, X. N. Zhou, K. Yang and F. W. Chen, Localization of 5D Elko spinors on Minkowski branes. Phys. Rev. D 86 (2012): 064012; arXiv:1107.2596.
[12] J. A. Nieto, Higher Dimensional Elko Theory, arXiv:1307.1429.
[13] P. Lounesto, Clifford Algebras and Spinors, 2nd ed., Chapters 11 and 12, p. 152-173, Cambridge Univ. Press, Cambridge 2002.
[14] R da Rocha, WA Rodrigues Jr. Where are ELKO spinor fields in Lounesto spinor field classification? Mod. Phys. Lett. A 21 (2006) 65; arXiv:0506075.
[15] D. V. Ahluwalia, C.-Y. Lee, and D. Schritt, Self-interacting Elko dark matter with an axis of locality, Phys. Rev. D 83 (2011) 065017; arXiv:0911.2947.
[16] C. G. Boehmer, J. Burnett, D. F. Mota and D. J. Shaw, Dark spinor models in gravitation and cosmology, JHEP 1007 (2010) 053; arXiv:1003.3858.
[17] E. P. Wigner, Unitary representations of the inhomogeneous Lorentz group including reflections. Contribution in: Gursey, F. (editor): Group theoretical concepts and methods in elementary particle physics. New York: Gordon and Breach 1964.
[18] R. da Rocha and J. M. Hoff da Silva, From Dirac spinor fields to eigenspinoren des ladungskonjugationoperators, J. Math. Phys. 48 (2007) 123517; arXiv:0711.1103.
[19] D. V. Ahluwalia, S. P. Horvath, Very special relativity as relativity of dark matter: the ELKO connection, JHEP 11 (2010) 078; arXiv:1008.0436.
[20] L. C. Biedenharn, M. Y. Han and H. van Dam, Two-Component Alternative to Dirac’s Equation, Phys. Rev. D 6 (1972) 500.
[21] A. G. Nikitin, On exact Foldy-Wouthuysen transformation, J. Phys. A 31 (1998) 13297.
[22] J. Niederle and A. G. Nikitin, Involutive symmetries, supersymmetries and reductions of the Dirac equation, J. Phys. A 30 (1997) 999.
[23] J. Beckers, N. Debergh, and A. G. Nikitin, Reducibility of supersymmetric quantum mechanics, Int. J. Theor. Phys. 36 (1997) 1991.
[24] A. G. Nikitin, Algebras of discrete symmetries and supersymmetries for the Schrödinger-Pauli equation, Int. J. Mod. Phys. A 14 (1999) 885.
[25] J. Niederle and A. G. Nikitin, Extended supersymmetries for the Schrödinger-Pauli equation, J. Math. Phys. 40 (1999) 1280.
[26] V. X. Genest, J.-M. Lemay, L. Vinet, and A. Zhedanov, The Hahn superalgebra and supersymmetric Dunkl oscillator models, J. Phys. A 46 (2013) 505204; arXiv:1309.1701v1 2013.