The Horizon Energy of a Black Hole

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Abstract
We investigate the energy distribution of a black hole in various space-times as reckoned by a distant observer using the quasi-local energy approach. In each case the horizon mass of a black hole: neutral, charged or rotating, is found to be twice the irreducible mass observed at infinity. This is known as the Horizon Mass Theorem. As a consequence, the electrostatic energy and the rotational energy of a general black hole are all external quantities. Matter carrying charges and spins could only lie outside the horizon. This result could resolve several long-standing paradoxes related to known black hole properties; such as why entropy is proportional to area and not to volume, the information loss problem, the firewall problem, the internal structure and the thin shell model of a black hole.

Keywords: Quasi-local Energy; Horizon Mass; Horizon Mass Theorem.
1. Quasi-local Energy

A black hole has the strongest gravitational field of all gravitational systems and the greatest gravitational potential energy. It is well known that gravitational energy density cannot be defined consistently in general relativity since gravitational field can be transformed away in a local inertial frame. Nevertheless, it is possible to consider the total energy contained in a surface enclosing a black hole at a given coordinate distance. This is based on the quasi-local energy approach [1] obtained from a Hamiltonian-Jacobi analysis of the Hilbert action in general relativity.

The quasi-local energy expression is the most important development in general relativity in recent years to understand the dynamics of the gravitational field, such as energy, momentum and angular momentum [2]. For asymptotically flat spacetime, the quasi-local energy agrees with the Arnowitt-Deser-Misner energy [3] at spatial infinity and for spherically symmetric spacetime, it has the correct Newtonian limit, including negative contribution to gravitational binding. It also agrees with the Komar energy [4] and Bondi energy [5] at null infinity. The quasi-local energy approach is therefore naturally suited for investigating the energy distribution of a black hole.

The expression for the quasi-local energy is given in terms of the total mean curvature of a surface bounding a volume for a gravitational system in four-dimensional spacetime. The Brown and York expression is given in the form of an integral [1]

\[ E = \frac{c^4}{8\pi G} \int_{\partial B} d^2x \sqrt{\sigma} (k - k^0), \]

where \( \sigma \) is the determinant of the metric defined on the two-dimensional surface \( \partial B \); \( k \) is the trace of extrinsic curvature of the surface and \( k^0 \), the trace of curvature of a reference space. For asymptotically flat reference spacetime, \( k^0 \) is zero.
2. Horizon Mass Theorem

The Horizon Mass Theorem is the final outcome of the quasi-local energy expression applied to the black hole. The mass of a black hole depends on where the observer is. The closer one gets to a black hole the less gravitational energy one expects to see. As a result, the mass of a black hole increases as one gets near the horizon. The Horizon Mass Theorem can be stated in the following [6]:

**Theorem.** For all black holes: neutral, charge or rotating, the horizon mass is always twice the irreducible mass observed at infinity.

It is useful to introduce the following definitions of mass in order to understand the energy of a black hole:

1. The *asymptotic mass* is the mass of a neutral, charged or rotating black hole including electrostatic and rotational energy. It is the mass observed at infinity.

2. The *horizon mass* is the mass which cannot escape from the horizon of a neutral, charged or rotating black hole. It is the mass observed at the horizon.

3. The *irreducible mass* is the final mass of a charged or rotating black hole when its charge or angular momentum is removed by adding external particles to the black hole. It is the mass observed at infinity.

3. Schwarzschild Black Hole

The total energy contained in a sphere enclosing the black hole at a coordinate distance \( r \) is given by the expression [1,7,8]

\[
E(r) = \frac{rc^4}{G} \left[ 1 - \sqrt{1 - \frac{2GM}{rc^2}} \right],
\]  

(2)
where \( M \) is the mass of the black hole observed at infinity, \( c \) is the speed of light and \( G \) is the gravitational constant. At the horizon, the Schwarzschild radius is \( r = 2GM/c^2 \). Evaluating the expression in Eq.(2), we find the metric coefficient \( g_{00} = (1 - 2GM/rc^2)^{1/2} \) vanishes identically and the energy at the horizon is therefore

\[
E(r) = \left(\frac{2GM}{c^2}\right)^{\frac{4}{G}} = 2Mc^2. \tag{3}
\]

The horizon mass of the Schwarzschild black hole is simply twice the asymptotic mass \( M \) observed at infinity. The negative gravitational energy outside the black hole is as great as the asymptotic mass.

4. Charged Black Hole

The total energy of a charged black hole contained within a radius at coordinate \( r \) is now given by [7]

\[
E(r) = \frac{rc^4}{G} \left[ 1 - \sqrt{1 - \frac{2GM}{rc^2} + \frac{GQ^2}{r^2c^4}} \right], \tag{4}
\]

where \( M \) is the mass of the black hole including electrostatic energy observed at infinity and \( Q \) is the electric charge. At the horizon radius

\[
r_+ = \frac{GM}{c^2} + \frac{GM}{c^2} \sqrt{1 - \frac{Q^2}{GM^2}}, \tag{5}
\]

the square root in Eq.(4) again vanishes and the horizon energy becomes

\[
E(r_+) = \frac{r_+c^4}{G} = Mc^2 + Mc^2 \sqrt{1 - \frac{Q^2}{GM^2}}. \tag{6}
\]

When this is expressed in terms of the irreducible mass of the charged black hole

\[
M_{irr} = \frac{M}{2} + \frac{M}{2} \sqrt{1 - \frac{Q^2}{GM^2}}, \tag{7}
\]

the horizon energy becomes exactly twice the irreducible energy

\[
E(r_+) = 2M_{irr}c^2. \tag{8}
\]
The horizon mass therefore depends only on the energy of the black hole when it is neutralized by adding oppositely charged particles. There is no electrostatic energy inside the charged black hole.

5. Slowly Rotating Black Hole

The total energy of a slowly rotating black hole with angular momentum \( J \) and angular momentum parameter \( a = J/Mc \) using the quasi-local energy approach is given by the approximate expression [9], \( 0 < a \ll 1 \),

\[
E(r) = \frac{rc^4}{G} \left[ 1 - \sqrt{1 - \frac{2GM}{rc^2} + \frac{a^2}{r^2}} \right] \\
+ \frac{a^2c^4}{6rG} \left[ 2 + \frac{2GM}{rc^2} + \left( 1 + \frac{2GM}{rc^2} \right) \sqrt{1 - \frac{2GM}{rc^2} + \frac{a^2}{r^2}} \right] + \cdots \quad (9)
\]

Again, using the horizon radius in this case

\[
r_h = \frac{GM}{c^2} + \sqrt{\frac{G^2M^2}{c^4} - \frac{J^2}{M^2c^2}} \quad (10)
\]

and the irreducible mass

\[
M^2_{irr} = \frac{M^2}{2} + \frac{M^2}{2} \sqrt{1 - \frac{J^2c^2}{G^2M^4}} \quad (11)
\]

we arrive at a very good approximate relation for the horizon energy

\[
M_h \simeq 2M_{irr} + O(a^2). \quad (12)
\]

6. Black Hole at Any Rotation

For general and fast rotations, the quasi-local energy approach has limitation but the total energy can be obtained very accurately by numerical evaluation in the teleparallel formulation of general relativity [10]. The teleparallel gravity is an equivalent geometric formulation of general relativity in which
the action is constructed purely with torsion without curvature. It has a
gauge field approach. There is a perfectly well-defined gravitational energy
density and the result agrees very well with Eq.(12) at any rotation. The
small discrepancy is due to the axial symmetry of a rotating black hole com-
pared with the exact spherical symmetry of a Schwarzschild black hole. The
result shows that the rotational energy appears to reside almost completely
outside the black hole.

For an exact relationship, however, we have to employ a formula known
for the area of a rotating black hole valid for all rotations in the Kerr metric
[11],

$$A = 4\pi (r_h^2 + a^2) = \frac{16\pi G^2 M_{\text{irr}}^2}{c^4}.$$  \hspace{1cm} (13)

This area is exactly the same as that of a Schwarzschild black hole with
asymptotic mass $M_{\text{irr}}$. Now a local observer who is comoving with the ro-
tating black hole at the event horizon will see only this Schwarzschild black
hole. Since the horizon mass of the Schwarzschild black hole is $2M_{\text{irr}}$, there-
fore the horizon mass of the rotating black hole is exactly $M_h = 2M_{\text{irr}}$. We
have shown in each case, the horizon mass of a black hole is always twice the
irreducible mass observed at infinity.

The Horizon Mass Theorem shows that the electrostatic energy and the
rotational energy of a general black hole are all external quantities. They
are absent inside the black hole. A charged black hole does not have elec-
tric charges inside. A rotating black hole does not rotate; only the exter-
nal space is rotating. The conclusion is surprising. It could resolve several
long-standing paradoxes of black holes such as the entropy problem, the in-
formation problem, the firewall problem and the gravastar thin shell model
since matter carrying charges and spins could only stay outside the horizon.
The quasi-local energy of black holes is one of the profound and fascinating
results known recently in general relativity.
References

[1] J.D. Brown and J.W. York, Jr. *Phys. Rev. D* **47**, 1407 (1993).

[2] M.T. Wang and S.T. Yau, *Phys. Rev. Lett.* **102**, 021101 (2009).

[3] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* **117**, 1595 (1960).

[4] A. Komar, *Phys. Rev.* **113**, 934 (1959).

[5] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Proc. R. Soc. London Ser. A* **269**, 21 (1962).

[6] Y.K. Ha, *Int. J. Mod. Phys. D* **14**, 2219 (2005).

[7] J.W. Maluf, *J. Math. Phys.* **36**, 4242 (1995).

[8] Y.K. Ha, *Gen. Rel. Gra.* **35**, 2045 (2003).

[9] E.A. Martinez, *Phys. Rev. D* **50**, 4920 (1994).

[10] J.W. Maluf, E.F. Martins and A. Kneip, *J. Math. Phys.* **37**, 6302 (1996).

[11] D. Christodoulou and R. Ruffini, *Phys. Rev. D* **4**, 3552 (1971).