BENJAMINI–SCHRAMM CONVERGENCE OF RANDOM PLANAR MAPS

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Abstract. We prove that the uniform random planar map with \( n \) edges converges in the Benjamini–Schramm sense towards an infinite random map as \( n \) tends to infinity.

1. Introduction

The uniform infinite planar map (UIPM) was constructed in [10] as the local weak limit of a planar map \( m_n \) with \( n \) edges, chosen uniformly at random. The limit map \( m_\infty \) describes the asymptotic behaviour near the root-edge. In the present note we build upon this result to construct a Benjamini–Schramm limit \( m_\infty^* \) of this model. Hence the limit describes the asymptotic behaviour of the vicinity of a uniformly selected vertex.

Theorem 1.1. The random map \( m_n \) converges in the Benjamini–Schramm sense towards an infinite random planar map \( m_\infty^* \). The convergence preserves the embedding in the plane.

This result is related to analytic studies of the degree distribution in random planar maps [6, 3, 4, 9]. Specifically, the proof of Theorem 1.1 shows how the limiting constants given in [6] for the number of vertices of a given degree in \( m_n \) are encoded into the UIPM \( m_\infty \). Here we make use of the fact that \( m_n \) is stochastically invariant under re-rooting a uniformly selected corner and that the number of vertices in \( m_n \) concentrates around a constant multiple of \( n \). As a small application we also show that the expected number of occurrences of a given pattern in \( m_n \) is asymptotically proportional to \( n \) (Theorem 4.1). The random map \( m_n \) has also received attention in recent literature [1] concerning the asymptotic global geometric structure.

By similar arguments it is possible to construct Benjamini–Schramm limits for other models of random planar maps where a local weak limit describing the vicinity of the root-corner is known. The main properties required are the re-rooting invariance and a concentration result for the number of vertices. For example, this applies to uniform quadrangulations with \( n \) faces.

Notation and Terminology. Rooting a planar map at a corner is equivalent to specifying and orienting an edge. We say the origin of this edge is the root vertex of the map. We use the convention that the face to the "right" of the oriented

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root edge is the outer face and is drawn as the unique unbounded face in plane representations. The radius of the map is the maximal distance of a vertex from the root vertex. Let $m$ denote a corner-rooted planar map. We let $v(m)$ and $e(m)$ denote its number of vertices and edges. For any integer $r \geq 1$ we let $U^c_r(m)$ denote the corner-rooted submap induced by all vertices with distance at most $r$ from the vertex incident to the root-corner. We let $m^v$ denote the vertex-rooted version of $m$, where we forget about the root-corner and only mark the root-vertex. In particular, $(U^c_r(m))^v := (U^c_r(m))^v$ is the vertex-rooted version of the neighbourhood $U^c_r(m)$. Maps may be re-rooted. Given a vertex $v$ and a corner $c$ of $m$, we let $(m, v)$ and $(m, c)$ represent the result of re-rooting $m$ at this vertex or corner and forgetting about the original root.

2. Preliminaries

We are going to make use of the following re-rooting invariance. Details on the enumerative background of planar maps may be found in [8].

**Lemma 2.1.** The map $m_n$ is stochastically invariant under re-rooting at a uniformly selected corner.

**Proof.** Let $m$ be an arbitrary unrooted map with $2n$ corners. We have to show that any corner-rooted version of $m$ corresponds to the same number of choices among the $n$ corners of $m$. If this holds, then re-rooting $m_n$ at a uniformly selected corner is identically distributed to $m_n$.

To this end, let us label the corners of $m$ with numbers from $1$ to $2n$ to form a corner-labelled unrooted map $\hat{m}$. There are many ways to do this, and we pick an arbitrary one. A permutation $\sigma$ of $\{1, \ldots, 2n\}$ is termed an automorphism of $\hat{m}$ if the result of relabelling $\hat{m}$ according to $\sigma$ is identical to $\hat{m}$. The collection of automorphisms of $\hat{m}$ is its automorphism group.

If rooting the map $\hat{m}$ at $1 \leq i \leq 2n$ or $1 \leq j \leq 2n$ and forgetting about the labels yields two identical unlabelled corner-rooted maps, then there must be an automorphism $\sigma$ of $\hat{m}$ such that $\sigma(i) = \sigma(j)$. Conversely, if there exists an automorphism $\sigma$ with $\sigma(i) = j$, then clearly rooting $\hat{m}$ at $i$ or $j$ yields identical unlabelled corner-rooted maps. Moreover, if $\sigma$ and $\nu$ are automorphisms of $\hat{m}$ that both satisfy $\sigma(i) = j$ and $\nu(i) = j$, then $\sigma \nu^{-1}$ is an automorphism that fixes the label $i$. Hence $\sigma \nu^{-1}$ is an automorphism of a corner-rooted labelled planar map. Corner-rooted maps are asymmetric, so $\sigma \nu^{-1}$ must be the identity permutation, that is, $\sigma = \nu$. Thus, the cardinality of the automorphism group of $\hat{m}$ is equal to the number of corners (among the $n$ choices) such that rooting $\hat{m}$ at this corner yields the same unlabelled corner-rooted map as rooting $\hat{m}$ at the corner $i$. This number does not depend on $i$, so to any corner-rooted version of $m$ corresponds to the same number of choices for root-corners.

The following result seems to be classical in the theory of random planar maps, however, the only explicit reference we found is a lecture by Marc Noy at the ALEA-meeting 2010 in Luminy.

**Lemma 2.2.** The number of vertices $v(m_n)$ satisfies a central limit of the form

$$\frac{v(m_n) - n/2}{\sqrt{25n/32}} \xrightarrow{d} \mathcal{N}(0, 1)$$

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with \( \mathbb{E} [v(\mathbf{m}_n)] = \frac{3}{2} + 1 \) and \( \text{Var}[\mathbf{m}_n] = 25n/32 + O(1) \).

*Proof.* Let \( M(z, x, u) \) denote the generating function of rooted planar maps, where the variable \( z \) corresponds to the number of edges, \( x \) to the number of vertices and \( u \) to the root face valency. Then by the usual combinatorial decomposition of maps we have

\[
M(z, x, u) = x + zu^2M(z, x, u)^2 + zu \frac{M(z, x, 1) - uM(z, x, u)}{1 - u}
\]

and by the quadratic method we can express \( M(z, x, 1) \) as a rational function in

\[
u = u(z, x) \text{ that is given by the solution of the algebraic equation}
\]

\[
4u^4xz + u^4z^2 - 2u^4z - 8u^3x + 4u^3z + 4u^2xz + 2u^3z - 2u^2z - 7u^2 + 8u - 3 = 0,
\]

from which we obtain a singular expansion of the form

\[
u(z, x) = u_0(x) + u_1(x) \sqrt{1 - \frac{z}{\rho(x)}} + u_2(x) \left(1 - \frac{z}{\rho(x)}\right)^{3/2} + \cdots,
\]

where the functions \( u_j(x) \) are analytic at \( x = 1 \), satisfy \( u_0(1) = \frac{6}{25} \), \( u_1(1) = -\frac{6}{25} \), and the function \( \rho(x) \) satisfies the equation

\[
3072x^3z^4 - 4608x^2z^3 - 1536x^2z^3 + 4608xz^4 + 1536x^3z^3 - 1536x^4 + 192xz^2 + 768z^3 - 96z^2 = 0
\]

with \( \rho(1) = \frac{1}{25} \). From this it follows that

\[
M(z, x, 1) = \frac{1 - (4xz - z^2)u^4 - (-8x + 2)zu^3 + (-1 + (4x - 2)z)u^2 - 2u}{4(1 - u)u^3z^2}
\]

has a local representation of the form

\[
M(z, x, 1) = b_0(x) + b_2(x) \left(1 - \frac{z}{\rho(x)}\right)^{3/2} + \cdots,
\]

where the functions \( b_j(x) \) are analytic at \( x = 1 \) and satisfy \( b_0(1) = \frac{3}{4} \), \( b_2(1) = -\frac{1}{4} \), \( b_3(1) = \frac{3}{4} \).

At this stage we can apply standard tools (see [5, Chapter 2]) to obtain a central limit theorem for \( v(\mathbf{m}_n) \) of the form \( (v(\mathbf{m}_n) - \mu n)/\sqrt{\sigma^2 n} \xrightarrow{d} \mathcal{N}(0, 1) \), where

\[
\mu = -\frac{\rho'(1)}{\rho(1)}, \quad \sigma^2 = \mu^2 - \frac{\rho''(1)}{\rho(1)}.
\]

Since \( \rho'(1) = -\frac{1}{25} \) and \( \rho''(1) = -\frac{1}{375} \) we immediately obtain \( \mu = \frac{1}{2} \) and \( \sigma^2 = \frac{25}{12} \). We also have \( \mathbb{E} [v(\mathbf{m}_n)] = \mu n + O(1) \) and \( \text{Var}[\mathbf{m}_n] = \sigma^2 n + O(1) \). In this special case Euler’s relation and duality can be used to obtain (the even more precise representation) \( \mathbb{E} [v(\mathbf{m}_n)] = n/2 + 1 \).

## 3. Proof of Theorem 1.1

Let \( \mathbf{m}_\infty \) denote the UIPM, and let \( \mathbf{m}_n \) be the planar map sampled uniformly at random among all corner-rooted maps with \( n \) edges. Hence for any integer \( r \geq 1 \) and any corner-rooted map \( \mathbf{m} \) it holds that

\[
\lim_{n \to \infty} \mathbb{P}(U_r^c(\mathbf{m}_n) = \mathbf{m}) = \mathbb{P}(U_r^c(\mathbf{m}_\infty) = \mathbf{m}).
\]
Let \( v_n \) denote a uniformly at random selected vertex of \( m_n \). Let \( \kappa(m) \) denote the number of corners \( c \) incident to the root-vertex such that \( m \) is invariant under re-rooting at \( c \). Since \( m_n \) is stochastically invariant under re-rooting at a uniform corner, it holds that

\[
\mathbb{P}(U_r^v(m_n, v_n) = m^v) = \mathbb{E}[X_n/v(m_n)]
\]

with \( X_n \) denoting the number of vertices \( v \) in \( m_n \) such that \( U_r^v(m_n, v) = m^v \). To each such vertex correspond precisely \( \kappa(m) \) corners \( c \) with \( U_r^c(m_n, c) = m \). Thus the total number \( Y_n \) of corners whose corner-rooted \( r \)-neighbourhood equals \( m \) satisfies

\[
Y_n = \kappa(m)X_n.
\]

Hence

\[
(1) \quad \mathbb{P}(U_r^v(m_n, v_n) = m^v) = \mathbb{E}\left[\frac{Y_n}{v(m_n)\kappa(m)}\right].
\]

The map \( m_n \) is stochastically invariant under re-rooting at a uniformly selected corner \( c_n \). Thus

\[
\mathbb{E}[Y_n/(2n)] = \mathbb{P}(U_r^v(m_n, c_n) = m)
= \mathbb{P}(U_r^v(m_n) = m)
\rightarrow \mathbb{P}(U_r^v(m_\infty) = m).
\]

Using Lemma 2.2 and \( Y_n/(v(m_n)\kappa(m)) \leq 1 \) it follows that for large enough \( n \)
\[
\mathbb{E}\left[\frac{Y_n}{v(m_n)\kappa(m)}\right] = o(1) + \mathbb{E}\left[\frac{Y_n}{v(m_n)\kappa(m)}, |v(m_n)/n - \mu_v| \leq n^{-1/3}\right]
\leq o(1) + \mathbb{E}\left[\frac{Y_n}{n(\mu_v - n^{-1/3})\kappa(m)}\right]
= o(1) + \frac{2}{\kappa(m)\mu_v}\mathbb{P}(U_r^v(m_\infty) = m).
\]

Similarly, we obtain a lower bound, as \( Y_n \leq 2n \) implies that
\[
\mathbb{E}\left[\frac{Y_n}{v(m_n)\kappa(m)}\right] \geq o(1) + \mathbb{E}\left[\frac{Y_n}{n(\mu_v + n^{-1/3})\kappa(m)}, |v(m_n)/n - \mu_v| \leq n^{-1/3}\right]
= o(1) + \mathbb{E}\left[\frac{Y_n}{n(\mu_v + n^{-1/3})\kappa(m)}\right].
\]

By Equation (1) this implies
\[
\mathbb{P}(U_r^v(m_n, v_n) = m^v) \rightarrow \frac{2}{\kappa(m)\mu_v}\mathbb{P}(U_r^v(m_\infty) = m)
= \frac{2}{\alpha(m)\kappa(m)\mu_v}\mathbb{P}(U_r^v(m_\infty) = m^v)
\]

with \( \alpha(m) \) denoting the number of different corner-rooted maps that may be obtained by re-rooting \( m \) at a corner incident to the root-vertex.

For any vertex rooted planar map \( m \) let us set
\[
p_{m,n} := \mathbb{P}(U_r^v(m_n, v_n) = m)
\]
and
\[
p_m := \frac{2}{\kappa(m)\alpha(m)}\mathbb{P}(U_r^v(m_\infty) = m).
\]
In order to deduce weak convergence of $U_r^v(m_n, v_n)$ it remains to verify
\begin{equation}
\sum_m p_m = 1
\end{equation}
with the sum index ranging over all vertex-rooted planar maps $\hat{m}$. To this end, let $X_n(\hat{m})$ denote the number of vertices $v$ in $m_n$ with $U_r^v(m_n, v) = m$. Let $Y_n(\hat{m})$ denote the number of corners in $m_n$ whose corner-rooted $r$-neighbourhood equals $\hat{m}$. For any fixed $K \geq 1$ it follows from Lemma 2.2 that
\begin{align*}
\sum_{k \geq K} \sum_{\hat{m}, e(\hat{m}) = k} p_{\hat{m}, n} & = E \left[ \sum_{k \geq K} \sum_{\hat{m}, e(\hat{m}) = k} \frac{X_n(\hat{m})}{v(m_n)} \right] \\
& \leq o(1) + CE \left[ \sum_{k \geq K} \sum_{m, e(m) = k} \frac{Y_n(\hat{m})}{2n} \right] \\
& = o(1) + CP(e(U_r(m_n)) \geq K)
\end{align*}
for some bound $C > 0$ that does not depend on $n$ (or $k$ or $m$) and an $o(1)$ term that converges to zero uniformly in $k$ and $\hat{m}$ as $n$ becomes large. Since $U_r(m_n) \converges_d U_r(m_\infty)$ it follows that for any $\epsilon > 0$ we may select $K \geq 1$ large enough such that
\begin{equation}
\sum_{k \geq K} \sum_{\hat{m}, e(\hat{m}) = k} p_{\hat{m}, n} < \epsilon
\end{equation}
for large enough $n$. This entails
\begin{equation}
\sum_{k < K} \sum_{\hat{m}, e(\hat{m}) = k} p_{\hat{m}} \geq 1 - \epsilon.
\end{equation}
We have thus proved Equation (3). Hence there is a random vertex-rooted planar map $V_r$ with distribution $P(V_r = \hat{m}) = p_{\hat{m}}$ such that
\begin{equation}
U_r^v(m_n, v_n) \converges_d V_r
\end{equation}
as $n$ becomes large. The family $(V_r)_{r \geq 1}$ forms a projective system with respect to the projections $U_r^v(\cdot)$, since for any $1 \leq s \leq r \leq t$ it holds that
\begin{equation}
U_s^v(U_r^v(V_i)) \converges_d U_s^v(V_i).
\end{equation}
It follows by a general result [2, Ch. 9, §4, No. 3, Theorem 2] that there is a random infinite planar map $m_\infty^*$ such that
\begin{equation}
U_r^v(m_\infty^*) \converges_d V_r
\end{equation}
for all $r \geq 1$. By (4) it follows that $m_\infty^*$ is the Benjamini–Schramm limit of the random planar map $m_n$ and the convergence preserves the embedding in the plane.

4. Patterns in random planar maps

Let $\hat{m}$ denote a plane representation of some planar map. We say that $\hat{m}$ occurs in $m_n$ as a pattern, if $\hat{m}$ may be embedded in a face-preserving way into the plane representation of $m_n$. For example, if $\hat{m}$ is a cycle of length $d$ then these embeddings correspond to the inner faces of $m_n$ that are cycles and have degree $d$. 
Theorem 4.1. Let $\hat{m}$ denote a plane representation of some planar map. Then the number $s(\hat{m}, m_n)$ of occurrences of $\hat{m}$ as a pattern in $m_n$ satisfies
\[ E[s(\hat{m}, m_n)] \sim \gamma(\hat{m}) n \]
for some constant $\gamma(\hat{m}) > 0$.

Proof of Theorem 4.1. Let $m$ be fixed corner-rooted version of the plane map $\hat{m}$ such that the plane representation of $m$ that has the unbounded face to the right of the root-edge coincides with $\hat{m}$. We say $m$ occurs as a pattern at a corner $c$ of $m_n$ if $m$ may be embedded into $m_n$ in a face-preserving way such that the root-corner of $m$ gets mapped to the corner $c$.

If we count the number $Z_n$ of corners of $m_n$ where $m$ appears as a pattern, then we over-count the occurrences of the unrooted plane map $\hat{m}$. If $\beta(\hat{m})$ denotes the number of ways that $\hat{m}$ may be rooted at an half-edge of its boundary, then
\[ s(\hat{m}, m_n) = Z_n / \beta(\hat{m}). \]

Lemma 2.1 together with the convergence of $m_n$ towards the UIPM $m_\infty$ implies that $E[Z_n/2n]$ converges to the probability $q(\hat{m})$ that $\hat{m}$ occurs as a pattern at the root of $m_\infty$. Hence
\[ E \left[ \frac{s(\hat{m}, m_n)}{n} \right] \to \frac{2q(\hat{m})}{\beta(\hat{m})}. \]

5. Open Problems

In Theorem 1.1 we have shown that there is a Benjamini-Schramm limit $m_\infty^*$ of random planar maps. However, this limit graph has no explicit description. In particular it is not clear how the probability distribution of some (simple) parameters of $m_\infty^*$ can be computed. For example, it would be nice to have a proper representation of the constants $\gamma(\hat{m})$ in Theorem 4.1.

Problem 5.1. Describe the Benjamini-Schramm limit $m_\infty^*$ of random planar maps in a proper (explicit) way.

Another open question is to make Theorem 4.1 more precise. Actually a central limit theorem is expected (as given, for example, in [7] for random quadrangulations and 2-connected triangulations or in [6] for vertices of degree $k$ in random maps or 2-connected maps). In particular this would imply a law of large numbers.

Problem 5.2. Does the number of occurrences $s(\hat{m}, m_n)$ of a pattern $\hat{m}$ in a random planar map $m_n$ satisfy a central limit theorem (similarly to Lemma 2.2)?

Finally, we remark that it would be highly interesting to establish a Benjamini–Schramm limit for uniform random planar graphs as their number of vertices tends to infinity. Some progress has been made in this direction [11, Theorem 6.39], but the question remains open.

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