On invariant linearization of Lie groupoids

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Abstract

The linearization theorem for proper Lie groupoids organizes and generalizes several results for classic geometries. Despite the various approaches and recent works on the subject, the problem of understanding invariant linearization remains somehow open. We address it here, by first giving a counter-example to a previous conjecture, and then proving a sufficient criterion that uses compatible complete metrics and covers the case of proper group actions. We also show a partial converse that fixes and extends previous results in the literature.

Keywords  Lie groupoids · Linearization · Invariant metrics · Differentiable stacks

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1 Introduction

The linearization theorem for proper Lie groupoids is a cornerstone of the theory. It generalizes classic results such as Ehresmann’s theorem for submersions, Reeb stability for foliations, and the tube theorem of compact group actions. It also serves as a key tool in establishing local models for Poisson geometry. The original source [14] proves the regular case and made important contributions, such as a reduction to the fixed-point case. A first complete proof was given in [15], with some confusion in the statements and the extra assumption of source-locally trivial. The hypothesis and variants were later clarified in [1].

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Given $G \rightrightarrows M$ a Lie groupoid, the linear model around an orbit $O \subset M$, which we review in Proposition 2.1, is the groupoid-theoretic normal bundle $\nu(G, G|_O) \supset \nu(M, O)$. The linearization theorem [1, Thm.1-Cor.2] claims that the groupoid is locally isomorphic to its linear model if it is (s-)proper at a point. This can be replaced by global (s-)proper, as explained in [2, Rmk.5.1.4], by restricting the attention to an invariant neighborhood, namely one that contains every orbit that it meets. We can then restate the theorem as follows:

**Theorem 1.1** [1] If $G \rightrightarrows M$ is a proper Lie groupoid and $O \subset M$ is an orbit, then there are open neighborhoods $O \subset U \subset \nu(M, O)$ and $O \subset V \subset M$ and a linearization isomorphism:

$$\phi : (\nu(G, G|_O)|_U \supset U) \cong (G|_V \supset V).$$

If $G \rightrightarrows M$ is source-proper, then the linearization is invariant, $U, V$ can be taken to be invariant.

In [1] the authors gave a simple proof of the fixed-point case, clarified several other aspects, and proposed as an open problem the characterization of invariant linearization, related to the stability of the corresponding dynamics. As they posed it, while the above theorem implies a large number of related classic results, it is intriguing that it fails to cover proper actions of non-compact groups [8, Thm.2.4.1]. They conjectured as a possible solution the notion of source inv-trivial groupoid, which we can replace by source trivial by restricting to a neighborhood.

**Question 1.2** (cf. [1, Probl.0.1]) Does a proper Lie groupoid $G \rightrightarrows M$ whose source map is trivial admit an invariant linearization around its orbits?

Our first contribution here is a negative answer to this question in Example 2.6. It turns out that [14, Ex.10.1], which combines exotic structures in $\mathbb{R}^4$ and the results on smooth fibrations from [10], is already a counterexample. We made a slight simplification, and use the main result in [3] to insure that a locally trivial submersion over a contractible manifold is trivial.

A new approach to the linearization of groupoids was developed in [5]. Given $G \rightrightarrows M$ a Lie groupoid, denote by $G_2 = G \times_M G$ the manifold of pairs of composable arrows, which identify with commutative triangles. A 2-metric is a metric $\eta_2$ on $G_2$ that is fibered for the multiplication $m : G_2 \to G$ and invariant under the action $S_3 \curvearrowright G_2$ permuting the vertices of a triangle. Such an $\eta_2$ induces a 1-metric $\eta_1$ on $G$, and a 0-metric $\eta_0$ on $M$, which is invariant under the normal representation [13]. The main results in [5] show a recipe to cook up 2-metrics on proper groupoids, called the gauge trick, and show that 2-metrics give linearizations via the exponential maps around full invariant subgroupoids, in particular around orbits.

**Theorem 1.3** [5] If $G \rightrightarrows M$ is proper and $\eta_2$ is a 2-metric, then there are open neighborhoods $O \subset U \subset \nu(M, O)$ and $O \subset V \subset M$ such that the following is an isomorphism:

$$\exp = (\exp^{\eta_1}, \exp^{\eta_0}) : (\nu(G, G|_O)|_U \supset U) \cong (G|_V \supset V).$$
If $G \rightrightarrows M$ is source-proper, then we can take $U$ and $V$ to be invariant.

In this paper, we build over the Riemannian theory of groupoids and stacks [4–6] to characterize proper groupoids that are invariantly linearizable. In Theorem 3.3, we give a sufficient condition for invariant linearization in terms of completeness of groupoid metrics. Then we show in Corollary 3.5 how our criterion easily implies the tube theorem for proper actions of non-compact groups. And in Theorem 4.4, we cook up complete 0-metrics on proper invariantly linearizable groupoids. This can be seen as (i) a partial converse for Theorem 3.3, (ii) a fixed version of [13, Prop.3.14] which is one of the main results there, and (iii) a stacky version of [3, Thm.5], for we build a complete metric on $M \rightarrow [M/G]$. Our proof in fact adapts the ideas presented in [3]. We can summarize our main contributions as follows:

**Theorem (3.3, 4.4)** Let $G \rightrightarrows M$ be a proper groupoid. Then:

(i) If it admits a 2-metric $\eta_2$ such that $\eta_0$ is complete, then $G \rightrightarrows M$ is invariantly linearizable.

(ii) If $G \rightrightarrows M$ is invariantly linearizable, then it admits a complete 0-metric $\eta_0$.

Note that (ii) is not the exact converse of (i), for we do not know if there is a 0-metric $\eta_0$ which actually extends to a 2-metric. The extension problem for metrics may not have a positive answer in general, see [5], and keeping track of completeness along the gauge trick is delicate. Anyway, we conjecture that the converse of (i) holds, and we prove it for regular groupoids in Corollary 4.6, where this extension problem has always a solution.

### 2 Linearization and source-triviality

We review various constructions for the linear model of a Lie groupoid around an orbit, provide examples and basic facts about invariantly linearizable groupoids, and we present the Example 2.6 of a source-trivial groupoid that is not invariantly linearizable, hence giving a partial answer to the open question proposed by [1, Probl.0.1].

Given $G \rightrightarrows M$ a Lie groupoid and given $O \subset M$ an orbit, we denote by $G_O \subset G$ the submanifolds of arrows within objects of $O$, so $G_O = s^{-1}(O) = t^{-1}(O)$. The restriction $G_O \rightrightarrows O$ becomes a Lie subgroupoid, and the linear model of $G \rightrightarrows M$ around $O$ can be defined in any of the following equivalent ways (see [1,2,14]):

**Proposition 2.1** The following groupoids are canonically isomorphic:

(A) The Lie groupoid-theoretic normal bundle $v(G, G_O) \rightrightarrows v(M, O)$, whose objects and arrows are the normal bundles $v(M, O) = TM|_O/T_O$ and $v(G, G_O) = TG|_{G_O}/TG_O$, and whose structure maps are induced by those of the tangent groupoid $TG \rightrightarrows TM$;

(B) The action groupoid $G_O \times O v(M, O) \rightrightarrows v(M, O)$ of the normal representation, which is the linear action of the restriction $G_O \rightrightarrows O$ over the normal bundle $v(M, O) \rightrightarrows O$ given by $g \cdot [v] = [\partial_\epsilon y_\epsilon|_{\epsilon=0}]$, where $y_\epsilon \xleftarrow{g_\epsilon} x_\epsilon$ is any curve satisfying $g_0 = g$ and $[\partial_\epsilon x_\epsilon|_{\epsilon=0}] = [v]$.

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The quotient \[\left(\left(P_x \times P_x \Rightarrow P_x\right) \times (N_x \Rightarrow N_x)\right)/G_x\] of the pair groupoid of a source-fiber \(P_x \Rightarrow G\left(-, x\right) = s^{-1}(x)\) times the unit groupoid of the normal vector space \(N_x = T_x M/T_x O\), by the group action \(G_x \curvearrowright P_x \times P_x \times N_x\), \(\lambda \cdot (h, h', v) = (h\lambda^{-1}, h'\lambda^{-1}, \lambda v)\).

**Proof** Construction (A) is better understood by thinking on the category of VB-groupoids over the restriction \(GO \Rightarrow O\) [2, 3.4]. The linear model is just the cokernel of the VB-groupoid inclusion \((TG_O \Rightarrow TO) \rightarrow (TG|_{GO} \Rightarrow TM|_O)\). Since both normal bundles have the same rank, the core of this VB-groupoid is trivial, and therefore, it is the action groupoid of a representation, namely the normal representation of (B). Finally, since \(GO \Rightarrow O\) is a transitive groupoid, it is Morita equivalent to the isotropy group \(G_x \Rightarrow x\), for \(x \in O\), and then there is a 1–1 correspondence between their representations. The bibundle realizing this Morita equivalence is \(P_x\), and the correspondence between representations follow a universal formula specialized in (C).

The restriction \(G_O \Rightarrow O\) embeds into the linear model as the 0-section. The Lie groupoid \(G \Rightarrow M\) is **linearizable** around \(O\) if there are opens neighborhoods \(O \subset U \subset v(M, O)\) and \(O \subset V \subset M\) and a linearization isomorphism

\[\phi : (v(G, G_O)|_U \Rightarrow U) \cong (G|_V \Rightarrow V)\]

The linearization is **invariant** if \(U\) and \(V\) are so, namely if they contain every orbit they meet.

**Example 2.2** 1. Source-proper groupoids, namely those whose source map \(s : G \to M\) is a proper map, are invariantly linearizable, this is part of Theorem 1.1.
2. Submersion groupoids arising from fiber bundles (locally trivial submersions) are invariantly linearizable, and they are source-proper if and only if the fiber is compact.
3. Action groupoids from proper actions of Lie groups are also invariantly linearizable [8, Thm. 2.4.1], and they are source-proper if and only if the group is compact.

The source-local triviality, which holds in the above examples, is in fact a necessary condition.

**Lemma 2.3** If \(G \Rightarrow M\) is invariantly linearizable, then it is source-locally trivial.

**Proof** Working locally, it is enough to show that the linear model is source-locally trivial. And this follows easily from the description C) in Proposition 2.1.

The main goal of the present paper is to understand which proper groupoids are invariantly linearizable besides the source-proper ones. A first remark in this direction is the following:

**Lemma 2.4** Let \(G \Rightarrow M\) be proper with connected orbit space \(M/G\). If \(G\) is invariantly linearizable, then either it is source-proper, or none of its orbits are compact.
**Proof** For each \( x \in M \) the isotropy bundle \( G(\cdot , x) \to O_x \) is a principal bundle with structure group \( G_x \), which is compact for \( G \) is proper. Then the source-fiber \( G(\cdot , x) \) is compact if and only if the corresponding orbit \( O_x \) is so. Since \( G \) is invariantly linearizable the source map \( s : G \to M \) is locally trivial and a source-fiber is compact if and only if the nearby ones are, and this is the case if and only if the groupoid is source-proper. This shows that both the compact and the non-compact orbits define opens in the orbit space \( M/G \), hence the result. \( \square \)

Source-local triviality is a necessary condition for invariant linearization, and as expressed by Weinstein in [14], it is tempting to think that it is also sufficient, but [14, Ex.10.1] already showed a counter-example. Looking for a sufficient condition to ensure invariant linearization, [1, Def.4.12] proposes the notion of **source inv-trivial**; see also Q. 1.2. We show here that this is also not enough, by relating local and global triviality with the following result.

**Lemma 2.5** A smooth fiber bundle \( p : E \to B \) over a contractible base \( B \) must be trivial.

**Proof** Let \( \{0, 1\} \subset I \subset \mathbb{R} \) be an open interval and let \( h : B \times I \to B \) be a smooth homotopy between \( h_0(x) = x \) and \( h_1(x) = b \) constant. The pullback bundle \( h^*E = (B \times I) \times_B E \to B \times I \) admits a complete Ehresmann connection \( H \) by [3, Thm.3]. The horizontal lift of \( (0, \partial/\partial t) \) on \( B \times I \) with respect to \( H \) is a vector field \( X \) whose flow \( \varphi_1 \) gives an isomorphism between \( h^*E|_{B \times 0} \cong h_0^*E = E \) and \( h^*E|_{B \times 1} \cong h_1^*E = E_b \times B \), proving that \( E \) is indeed trivial. \( \square \)

As a counter-example to Question 1.2, we propose the submersion groupoid associated to a map in [10, Ex.40], which is similar to the one originally given by Weinstein [14, Ex.10.1].

**Example 2.6** Let \( V \subset \mathbb{R}^4 \) be an open subset homeomorphic but not diffeomorphic to \( \mathbb{R}^4 \), and let \( E = \{(v, t) \in \mathbb{R}^4 \times \mathbb{R} : v \in V \text{ or } t \neq 0\} \). Then \( E \) is a five-dimensional Euclidean open, contractible, and simply connected at infinity, and therefore diffeomorphic to \( \mathbb{R}^5 \) [10, Ex.37]. The projection \( \pi : E \to \mathbb{R}, \pi(v, t) = t \), is not locally trivial, for the fiber \( \pi^{-1}(0) = E_0 \cong V \) is not diffeomorphic to the others. Consider the product between the submersion groupoid of \( \pi \) and the group \( SU(2) \).

\[
(G \rightrightarrows M) = (E \times_{\mathbb{R}} E \rightrightarrows E) \times (SU(2) \rightrightarrows *)
\]

It is shown in [10, Ex.40] that the source map \( s : G \to M \) is locally trivial, and it follows from our Lemma 2.5 that is globally trivial too, in particular source inv-trivial. But there is not an invariant linearization of \( G \rightrightarrows M \), for its orbits identify with the fibers \( E_t \) of \( \pi \), and the orbits of the linear model \( \nu(G, G_{E_0}) \) are all diffeomorphic to \( E_0 = V \).
3 Completeness as a sufficient condition

We recall Riemannian submersions and Riemannian groupoids and discuss some preliminaries. Then we present a sufficient condition for invariant linearization, and derive the tube theorem for proper actions of non-compact groups as a corollary.

Given \( \pi : E \to B \) a submersion, a Riemannian metric \( \eta^E \) on \( E \) is fibered if for all \( e, e' \) in the same fiber \( E_b = \pi^{-1}(b) \) the composition \( T_e E_b \to T_b B \leftarrow T'_e E_b \) is an isometry. Equivalently, \( \eta^E \) is fibered if it induces a metric \( \eta^B \) on \( B \) so that \( \pi \) becomes a Riemannian submersion, namely \( d\pi : \nu(e, E_b) \cong T_e E_b \to T_b B \) is an isometry for every \( e \in E \).

In such a Riemannian submersion, if a geodesic \( \gamma \) on \( E \) is orthogonal to a fiber, then it is orthogonal to every fiber, and its projection is a geodesic [12, Cor.2]. In this case, we say that \( \gamma \) is a horizontal geodesics. If \( S \subset B \) is embedded and \( S' = \pi^{-1}(S) \), the exponential maps yield a commutative square,

\[
\begin{array}{cccc}
v(E, S') & \supset U' & \xrightarrow{\exp^E} & E \\
\downarrow d\pi & & \downarrow \pi & \\
v(B, S) & \supset U & \xrightarrow{\exp^B} & B \\
\end{array}
\]

where \( v(E, S') \cong T S'^\perp \subset T E, d\pi = d\pi|_{v(E,S')}, \) and \( U' \) and \( U \) are opens around the 0-sections satisfying \( d\pi(U') \subset U \). The following is a sharper version of [5, Prop.5.9]:

**Lemma 3.1**

(i) If \( \exp^B|_U \) is étale, then so does \( \exp^E|_{U'} \), and the converse holds if \( d\pi(U') = U \).

(ii) If \( \exp^B|_U \) is injective, then so does \( \exp^E|_{U'} \), and the converse holds if \( \eta^E \) is complete and \( U' = d\pi^{-1}(U) \).

**Proof**

Given \((e, w) \in U'\), writing \( d\pi(e, w) = (b, v) \), \( \exp^E(e, w) = \tilde{e} \) and \( \exp^B(b, v) = \tilde{b} \), we have the following map of short exact sequences:

\[
\begin{array}{cccc}
0 & \xrightarrow{\ker(e, w) d\pi} & T(e, w) U' & \xrightarrow{d\pi} & T_b U & \xrightarrow{\exp^B} & 0 \\
\downarrow & & \downarrow d\exp^E & & \downarrow d\exp^B & & \\
0 & \xrightarrow{\ker \tilde{e} d\pi} & T\tilde{e} E & \xrightarrow{d\pi} & T\tilde{b} B & & 0 \\
\end{array}
\]

The first vertical arrow is an isomorphism, it identifies with the differential of the parallel transport \( E_b \cong d\pi^{-1}(b, v) \xrightarrow{\exp^E} E_{\tilde{b}} \) over the geodesic \( \exp^B(b, \epsilon v) \). It follows that the second vertical arrow is an isomorphism if and only if the third one is, hence (i).

Suppose that \( \exp^E(e, w) = \exp^E(e', w') \) with \((e, w), (e', w') \in U'\). Then \( \exp^B(\pi(e), d\pi(w)) = \exp^B(\pi(e'), d\pi(w')) \) and, if \( \exp^B|_U \) is injective, \((\pi(e), d\pi(w)) = (\pi(e'), d\pi(w')) \). The geodesics \( \exp^E(e, \epsilon w) \) and \( \exp^E(e', \epsilon w') \) have then the
same projection and the same value at 1, so they are equal and \((e, w) = (e', w')\), proving that the exponential map \(\exp^E |_{U'}\) is injective. Next we prove the converse.

Consider \((b, v), (b', v') \in U\) with \(\exp^B(b, v) = \exp^B(b', v')\). Picking \((e, w) \in U'\) such that \(d\pi(e, w) = (b, v)\), the geodesic \(\exp^E(e, ew)\) is a horizontal lift of \(\exp^B(b, e\epsilon)\). If \(\eta^E\) is complete we can lift \(\exp^B(b', e\epsilon v')\) to a horizontal geodesic \(\gamma\) such that \(\gamma(1) = \exp^E(e, w)\). Then \((\gamma(0), \dot{\gamma}(0)) \in d\pi^{-1}(U) = U'\) and satisfy \(\exp^E(\gamma(0), \dot{\gamma}(0)) = \gamma(1) = \exp^E(e, w)\). Since \(\exp^E |_{U'}\) is injective we conclude that \((\gamma(0), \dot{\gamma}(0)) = (e, w)\) and that \((b', v') = d\pi(\gamma(0), \dot{\gamma}(0)) = d\pi(e, w) = (b, v)\). □

Given a Lie groupoid \(G \rightrightarrows M\), write \(G_2 = G \times_M G\) for the manifold whose points are pairs of composable arrows, or equivalently commutative triangles. There is a canonical action \(S_3 \curvearrowright G_2\) which permutes the vertices of a triangle. A 2-metric is a metric \(\eta_2\) on \(G_2\) that is fibered for the multiplication \(m : G_2 \to G\) and invariant under the \(S_3\)-action. A 2-metric \(\eta_2\) induces metrics \(\eta_1\) on \(G\) and \(\eta_0\) on \(M\) such that the following hold [5]:

- \(m, \pi_1, \pi_2 : G_2 \to G\) and \(s, t : G \to M\) are Riemannian submersions;
- \(u(M) \subset G\) is totally geodesic;
- \(\eta_0\) is a 0-metric, namely it is invariant under the normal representation, in the sense that for every \(y \overset{\rho_g}{\rightarrow} x\) the linear map \(\rho_g : \nu(M, O)_x \to \nu(M, O)_y\) is an isometry; and
- \(\eta^M\) makes the foliation by orbits a singular Riemannian foliation.

The main theorems on groupoid metrics in [5] are: (i) every proper groupoid admits a 2-metric, and (ii) the exponential maps of \(\eta_1, \eta_0\) yield groupoid linearizations around orbits and, more generally, invariant submanifolds. The normal vectors in \(\nu(G, G_O)\) and \(\nu(M, O)\) give rise to normal geodesics, namely geodesics on \(G\) that are both horizontal for the source and target, and geodesics on \(M\) that are orthogonal to the orbits. We will show here that if a groupoid metric is complete, then the resulting linearization is invariant.

Given \((M, \eta)\) a Riemannian manifold, \(S \subset M\) a submanifold and \(r > 0\), write \(B'(S, r) \subset \nu(M, S)\) for the infinitesimal tube around \(S\) of radius \(r\), namely the normal vectors with norm smaller than \(r\), and write \(B(S, r) \subset M\) for the tube around \(S\) of radius \(r\), namely the points whose distance at \(S\) is smaller than \(r\). If \(\eta\) is complete and \(S\) is closed, then the distance from a point \(s\) to \(S\) can be realized by a geodesic orthogonal to \(S\), and therefore \(\exp(B'(S, r)) = B(S, r)\).

**Proposition 3.2** Given \(G \rightrightarrows M\) a Lie groupoid and \(O \subset M\) an orbit, then:

(i) If \(\eta_0\) is a 0-metric, then the infinitesimal tubes \(B'(O, r)\) are invariant in the linear model:

(ii) If \(\eta_2\) is a 2-metric, then \(B'(G_O, r) = d\pi^{-1}(B'(O, r)) = d\pi^{-1}(B'(O, r)) = \nu(G, G_O)|_{B'(O, r)}\);

(iii) If \(\eta_2\) is a 2-metric, \(G \rightrightarrows M\) is proper, and the induced 0-metric \(\eta_0\) is complete, then the tubes \(\exp^{0b}(B'(O, r)) = B(O, r)\) are invariant in \(G \rightrightarrows M\).

**Proof** (i) is immediate, for the normal representation preserves the norm of the normal vectors. (ii) is also easy, for in this case the source and target map are Riemannian.
submersions, and therefore the norm of normal vectors are preserved. Regarding (iii), let \( r > 0 \) and let \( O' \) be another orbit of \( \tilde{G} \) that intersects \( B(O, r) \) in some \( x \in M \).

Let us show that for any \( y \in B(O, r) \), the point \( y \) is also in the tube \( B(O, r) \). Since \( G \) is proper the orbit \( O' \) is closed, and since \( \eta_0 \) is complete there is a normal geodesic \( a \) realizing the distance between \( O \) and \( x \), so \( a(0) \in O, a(1) = x \) and \( \| \tilde{a}(0) \| < r \). By [4, Prop.10] the normal geodesic \( a \) admits a global lift through the source \( \tilde{a} \) starting at \( g \). Then the projection via the target \( t \circ \tilde{a} \) is a normal geodesic of length smaller than \( r \) and connecting \( O \) and \( y \), hence proving that \( y \in B(S, r) \). □

We are now ready to prove our first main result, giving a sufficient condition for invariant linearization, in terms of completeness of compatible metrics.

**Theorem 3.3** Let \( G \Rightarrow M \) be a proper groupoid and \( \eta_2 \) a 2-metric such that the induced 0-metric \( \eta_0 \) is complete. Then \( G \Rightarrow M \) is invariantly linearizable around its orbits.

**Proof** Let \( O \subset M \) be an orbit. Fix \( x_0 \in O \), take \( x_0 \in W \subset O \) a relatively compact neighborhood, and pick \( 0 < r < \frac{1}{4}d(x_0, O \setminus W) \) such that \( \exp^{\eta_0} \big|_{B'(W, r)} \) is an embedding. We will show now that \( \exp^{\eta_0} \) is then an embedding over the whole infinitesimal tube \( B'(O, r) \).

\[
\exp^{\eta_0} \mid_{B'(O, r)} \text{ is étale:} \quad \text{Given } (x, v) \in B'(O, r), \text{ take } (x_0, v_0) \xrightarrow{(g, w)} (x, v) \text{ in } v(G, G)_{|B'(O, r)} = B'(G, O, r), \text{ see Proposition 3.2.} \text{ The normal geodesics are defined for all time by } [4, \text{ Prop.10}], \text{ and } \exp^{\eta_0} : B'(W, r) \rightarrow M \text{ is an open embedding.} \text{ It follows from Lemma 3.1 applied to the target map that } \exp^{\eta_1} : dt^{-1}(B'(W, r)) \rightarrow G \text{ is also an open embedding. The same Lemma, now applied to the source map, says that } \exp^{\eta_0} : ds(dt^{-1}(B'(W, r))) \rightarrow M \text{ is at least étale, and therefore } d \exp^{\eta_1} : T_{(g, w)}v(G, G)_{|B'(O, r)} \rightarrow T_{\exp^{\eta_1}(g, w)}G \text{ is a linear isomorphism.} \text{ \exp^{\eta_0} \mid_{B'(O, r)} \text{ is injective:} Let } (x, v), (x', v') \in B'(O, r) \text{ be such that } \exp^{\eta_0}(x, v) = \exp^{\eta_0}(x', v'). \text{ As before, take } (x_0, v_0) \xrightarrow{(g, w)} (x, v) \text{ in } B'(G, O, r). \text{ Since } s(\exp^{\eta_1}(g, w)) = \exp^{\eta_0}(x', v'), \text{ we can lift the geodesic } \exp^{\eta_0}(x', v) \text{ through the source to a normal geodesic } \gamma \text{ such that } \gamma(1) = \exp^{\eta_1}(g, w). \text{ We claim that the projection } t \circ \gamma \text{ starts in } W. \text{ This is because } \exp^{\eta_0}(x_0, v_0) = t(\exp^{\eta_1}(g, w)) = t(\gamma(1)) = \exp^{\eta_0}(t \circ \gamma)(0), (t \circ \gamma)(0)), \text{ and therefore } \]

\[
d(x_0, (t \circ \gamma)(0)) \leq d(x_0, \exp^{\eta_0}(x_0, v_0)) + d((t \circ \gamma)(1), (t \circ \gamma)(0)) \leq \| v_0 \| + \| (t \circ \gamma)(0) \| < 2r.
\]

By construction \( \exp^{\eta_0} \) is injective over \( B'(W, r) \), from where \( (x_0, v_0) = ((t \circ \gamma)(0), (t \circ \gamma)(0)) \) and \( \gamma(\epsilon) = \exp^{\eta_1}(g, \epsilon w). \text{ Finally, projecting via the source, we conclude that } \exp^{\eta_0}(x', \epsilon v') = s(\gamma(\epsilon)) = s(\exp^{\eta_1}(g, \epsilon w)) = \exp^{\eta_0}(x, \epsilon v) \text{ and that } (x', v') = (x, v). \]

We have that \( (M, \eta_0) \) is complete, that the geodesics normal to the orbits in \( G \) are defined for all time, see Prop. [4, Prop.10], and we have just seen that \( \exp^{\eta_0} \big|_{B'(O, r)} \) is an open embedding. It follows from the proof in [5, Thm.5.11] that we can actually

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take \( U = B'(O, r) \) and get a linearization isomorphism

\[
\exp : (\nu(G, G|O)|_{B'(O, r)} \cong B'(O, r)) \to (G|_{B(O, r)} \cong B(O, r))
\]

By Proposition 3.2 \( B'(G_O, r) \) and \( \exp^{0\nu}(B'(O, r)) = B(O, r) \) are both invariant, hence the result. \( \square \)

The criterion presented in the previous theorem allows us to derive the linearization of proper actions of non-compact groups as a corollary of the linearization of groupoids. We need the following simple remark on completeness of pushed forward metrics (cf. [9, Thm.1]).

**Lemma 3.4** If \( \pi : (E, \eta^E) \to (B, \eta^B) \) is a Riemannian submersion and \( \eta^E \) is complete, then so does \( \eta^B \). In particular, if \( K \) is a Lie group, \( (M, \eta^M) \) is complete and \( K \bowtie M \) is a free and proper isometric action then the quotient \( M/K \) inherits a complete metric.

**Proof** Given a geodesic \( a \) in \( B \), if \( \tilde{a} \) is some local horizontal lift, then it can be extended to every time because \( E \) is complete, and the projection gives an extension of \( a \). \( \square \)

We will now cook up a complete 2-metric on an action groupoid coming from a proper action, using the gauge-trick from [5,6], which combined with Theorem 3.3, frames the Tube theorem into the theory.

**Corollary 3.5** If \( K \bowtie M \) is a proper Lie group action, then the action groupoid \( K \times M \Rightarrow M \) is invariantly linearizable around its orbits.

**Proof** Let \( \eta^K \) be a right invariant metric on \( K \), which is of course complete, as the right translations of geodesics are again geodesics, and regard \( \eta^K \times \eta^K \times \eta^K \) as a 2-metric on the pair groupoid \( K \times K \Rightarrow K \). Let \( \eta^M \) be a complete \( K \)-invariant metric on \( M \), see e.g. [7, Lemma.4.3.6], and regard it as a 2-metric on the unit groupoid \( M \Rightarrow M \). Then \( (\eta^K \times \eta^K \times \eta^K, \eta^M) \) is a complete 2-metric on the product and it is fibered for the canonical groupoid fibration

\[
(K \times K \Rightarrow K) \times (M \Rightarrow M) \to (K \times M \Rightarrow M)
\]

given on objects, arrows and pairs of composable arrows by the following formulas:

\[
(k, x) \mapsto kx \quad (k_2, k_1, x) \mapsto (k_2k_1^{-1}, k_1x) \quad (k_3, k_2, k_1, x) \mapsto (k_3k_2^{-1}, k_2k_1^{-1}, k_1x)
\]

The pushforward 2-metric is complete by Lemma 3.4 and the result follows from Theorem 3.3. \( \square \)

Our Theorem 3.3 is a groupoid version of the classic result [9, Thm.1], asserting that a complete Riemannian submersion is locally trivial, whose converse was later shown in [3, Thm.5]. Our result should also be compared with [11, Thm.1], where a complete singular Riemannian foliation \((M, F, \eta)\) is shown to be isomorphic to a linear model over a tube around a leaf—the complete hypothesis is missing in their statement but used along the proof. When the foliation is induced by a complete Riemannian groupoid, then the invariant linearization gives a similar result. The problem of comparing both models is left to be explored elsewhere.
4 Cooking up complete invariant metrics

We review here the Morita invariance of 2-metrics and the result metrics on stacks [6]. Then we prove our second main result, which shows the existence of complete 0-metrics on proper invariantly linearizable groupoids. We finally show that in the regular case this 0-metric can be extended to a 2-metric, and pose the question for the general case.

Given a Lie groupoid $G \rightrightarrows M$, we denote by $[M/G]$ it Morita equivalence, or equivalently its orbit differentiable stacks (see e.g. [2,6]). Two 2-metrics $\eta_2, \eta'_2$ on $G \rightrightarrows M$ are equivalent if for every $x \in M$ they induce the same inner product on $\nu(M, O)_x$. The class $[\eta_2]$ is a Morita invariant by [6, Thm.6.3.3], hence it defines a stacky metric. Stacky geodesics were then introduced and studied in [4]. Next we provide a quick review of the concepts that we need, and refer there for further details and examples.

A stacky curve $\alpha : I \to [M/G]$ is described by a sequence of curves of objects $a_i : I_i \to M$, where $I_i \subset I \subset \mathbb{R}$ are connected opens, together with curves of arrows $s_{i+1} : I_{i+1} \cap I_i \to G$ linking them by $ta_{i+1} = a_{i+1}$ and $sa_{i+1} = a_i$. Two collections $(a_i, s_{i+1})$ define the same stacky curve if they induce isomorphic maps $(\coprod I_i : I_{i+1} \cap I_i \to (G \rightrightarrows M)$ over a common refinement. If $\eta$ is a 2-metric on $G \rightrightarrows M$, a stacky geodesic $\alpha$ is a stacky curve which can be represented by geodesics $(a_i, s_{i+1})$ normal to the orbits. If every stacky geodesic can be extended to every time, we say that $[\eta_2]$ is a complete stacky metric on $[M/G]$.

**Proposition 4.1** Let $G \rightrightarrows M$ be a proper groupoid.

(a) If a 2-metric $\eta_2$ induces a complete 0-metric $\eta_0$, then $[\eta_2]$ is complete;
(b) There may not exists a complete 0-metric $\eta_0$;
(c) There always exists a 2-metric $\eta_2$ such that $[\eta_2]$ is complete.

**Proof** A stacky geodesic $\alpha : I \to [M/G]$ is locally represented by a geodesic $\alpha : I_i \to M$ normal to the orbits, this extends to the whole line $\bar{\alpha} : \mathbb{R} \to M$, giving a stacky global extension $[\bar{\alpha}]$ of $\alpha$. This proves (a).

Regarding (b), let $G \rightrightarrows M$ be the submersion groupoid arising from the first projection $\pi_1 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$. The existence of a complete 0-metric $\eta_0$ would imply that $\eta_0 \times \eta_0 \times \eta_0$ is a 2-metric extending it, that $G \rightrightarrows M$ is invariantly linearizable by Theorem 3.3, and that $\pi$ is locally trivial by Lemma 2.3, which is clearly not the case.

Finally, (c) appears as [4, Cor.21], where a product $f\eta$ of an arbitrary metric $\eta$ and a conformal factor measuring the distance to $\infty$ is considered. \hfill $\square$

Note that (a) is a stacky version of Lemma 3.4 and [9, Thm.1] applied to $M \to [M/G]$. The counter-example in (b) is already presented in [4, Rmk.5] and shows that the completeness in [13, Prop.3.14] is not always possible to obtain. We will now correct this result by adding up the key hypothesis of invariantly linearizable, hence acquiring a partial converse for Theorem 3.3. In light of Lemma 2.4, we can split the problem in the source-proper case and in the case where the orbits are non-compact. The first case easily follows from the following result.

**Proposition 4.2** Let $G \rightrightarrows M$ be a s-proper groupoid. A 2-metric $\eta_2$ is complete if and only if the stacky metric $[\eta_2]$ on $[M/G]$ is complete.
Proof If \( \eta_2 \) is complete then we have just proved that \([\eta_2]\) is also complete in Proposition 4.1. Suppose now that \([\eta]\) is complete. Let \( a : (p, q) \to M \) be a maximal geodesic and suppose that \( q < \infty \). Given \( q_n \not\to q \), by [4, Thm.3] we get a Cauchy sequence \( \{a(q - \frac{1}{n})\} \) in \( M/G \). By the stacky Hopf-Rinow Theorem [4, Thm.19] there is \( x \in M \) such that \([a(q - \frac{1}{n})]\) → \([x]\). It follows that \( a(t) \) sits inside some compact tube \( B(O_x, \varepsilon) \) for \( t \) close to \( q \), hence \( a \) is extendable and we reach a contradiction. The proof \( p = -\infty \) is analogous.

Given \( G \rightrightarrows M \) an invariantly linearizable proper groupoid with non-compact orbits, our strategy to cook up an invariant complete metric on it will be the following: (i) set a complete stacky metric on \([M/G]\), (ii) lift it locally to invariantly linearizable opens via the stacky submersion \( U \to [U/G_U] \), and (iii) patch the local pieces together in way inspired by [3, Thm.5]. Step (i) was recalled in Proposition 4.1. Let us address Lemma 4.3 now the Step (ii).

Lemma 4.3 Let \( \eta_2 \) be a 2-metric on the the infinitesimal tube \( B'(G_0, 2r) \rightrightarrows B'(O, 2r) \), viewed as a subgroupoid of the linear model \( \nu(G, G_O) \rightrightarrows \nu(M, O) \). Then there is a new 2-metric \( \eta'_2 \) on \( B'(G_0, 2r) \rightrightarrows B'(O, 2r) \) such that \( \eta'_0 \) is complete and such that \( \eta_2|B'(G_0, r) \sim \eta'_2|B'(G_0, r) \).

Proof Pick \( x \in O \), write \( K = G_x \) and \( N = B(0, 2r) \subset \nu_x(G, G_O) \). Then the infinitesimal tube is Morita equivalent to the action groupoid \( K \times N \rightrightarrows N \) via the following two maps,

\[
(B'(G_0, 2r) \rightrightarrows B'(O, 2r)) \leftarrow (P \times P \rightrightarrows P) \times (K \times N \rightrightarrows N) \to (K \times N \rightrightarrows N)
\]

where the second is just the projection, and the first is the quotient by the subgroupoid \( K \rightrightarrows \ast \). Let \( \eta' \) be a 2-metric on \( K \times N \rightrightarrows N \) corresponding to \( \eta \) by this Morita equivalence [6, Thm.6.3.3]. Let \( f : N \to \mathbb{R} \) be a positive \( K \)-invariant function such that \( f|B'(0, r) \equiv 1 \) and \( f \eta'_0 \) is complete. The composition \( f_2 = f \pi : K \times K \times N \to \mathbb{R} \) satisfy that \( f_2 \eta'_2 \) is a 2-metric on \( K \times N \rightrightarrows N \) and that \((f \eta')|B'(0, r) = f|B'(0, r) \). Let \( \eta''_0 \) be a complete \( K \)-invariant metric on \( P [7, Lemma.4.3.6] \). Then the product \( \eta''_0 \times \eta''_0 \times f_2 \eta'_2 \) is a complete 2-metric on the middle groupoid that is \( K \)-invariant, and its quotient via the first map is the desired 2-metric.

We are finally in conditions to prove our second main theorem.

Theorem 4.4 Let \( G \rightrightarrows M \) be a proper groupoid that is invariantly linearizable around its orbits. Then it admits a complete 0-metric \( \eta_0 \).

Proof By Proposition 4.1 we can consider a 2-metric \( \eta_2 \) on \( G \rightrightarrows M \) such that \([\eta_2]\) is complete. The problem now consists of showing that \([\eta_2]\) can be lifted to a complete metric on \( M \) along the stacky submersion \( M \to [M/G] \). We can work on each connected component of \( M/G \) independently. It follows from Lemma 2.4 that we can either assume that it is source-proper or that none of its orbits are compact. In the first case, it follows from Proposition 4.2 that \( \eta_2 \) is already complete and we are done. In the second case, we will show how to replace \( \eta_2 \) by an equivalent metric \( \eta'_2 \) such that \( \eta'_0 \) is complete.
For each \( x \) in \( M \), since \( G \Rightarrow M \) is invariantly linearizable around \( O_x \), there exist \( r > 0 \) and \( O_x \subset V_x \subset M \) such that \( \exp : (B(G(O_x), 2r_x) \Rightarrow B(O_x, 2r_x)) \cong (G|_V_x \Rightarrow V_x) \) is an isomorphism. Write \( W_x = \exp(B(O_x, r_x)) \). Using Lemma 4.3 we can build a new metric \( \eta_x^1 \) on \( G|_V_x \Rightarrow V_x \) such that \( \eta_x^1|_{W_x} \sim \eta_2|_{W_x} \) and that \( \eta_0^1 \) is complete. The next step will be to merge the several \( \eta_x^2 \) using a smart partition of 1 emulating what is done in [3, Thm.5].

Extract a countable covering \( \{W_{x_i}\}_{i \in \mathbb{N}} \) from \( \{W_x\}_{x \in M} \), and fix \( f : M \to [0, +\infty) \) a smooth proper function. Note that \( \overline{W}_{x_i} \cap \{f > n\} \supset O_{x_i} \cap \{f > n\} \neq \emptyset \) for all \( n \) because we are assuming that the orbits are non-compact. For each pair \( i, n \) such that \( \overline{W}_{x_i} \cap \{f \leq n\} \neq \emptyset \) the set \( B(\overline{W}_{x_i} \cap \{f \leq n\}, 1) \) is compact and therefore we can pick \( l(i, n) > 0 \) satisfying \( d(\overline{W}_{x_i} \cap \{f \leq n\}, \overline{W}_{x_i} \cap \{f > n + l(i, n)\}) > 1 \). We define an \( f \)-tube within \( V_{x_i} \) with inner radius \( n \) to be a set of the form \( T_i(n) = \overline{W}_{x_i} \cap \{n \leq f \leq n + l(i, n)\} \). Using these compact \( f \)-tubes we will merge the 0-metrics \( \eta_{x_i}^0 \) into a complete 0-metric \( \eta_0^0 \) equivalent to \( \eta_0 \).

We first construct a sequence of \( f \)-tubes \( \{T_i(n(i, j))\}_{i \leq j} \) with inner radius defined inductively as follows. We start by setting \( n(1, 1) \) so that \( T_1(n(1, 1)) \neq \emptyset \). After choosing \( n(i, j) \), if \( i < j \), we set \( n(i + 1, j) \) so that the new tube \( T_{i+1}(n(i + 1, j)) \) is not empty and does not meet any of the previous tubes, which is possible because \( f \) has a maximum over the union of them. If \( i = j \) we proceed similarly, setting \( n(1, j + 1) \) so that \( T_1(n(1, j + 1)) \) is not empty and does not meet the previous tubes. This way we end up with a sequence \( \{T_i(n(i, j))\}_{i \leq j} \) such that (i) it contains infinitely many \( f \)-tubes within \( V_{x_i} \), and (ii) the terms of the sequence are pairwise disjoint.

Let \( T_i = \bigcup_j T_i(n(i, j)) \) be the union of the \( f \)-tubes in of the sequence within \( V_{x_i} \). Let \( \{\phi_i\}_{i \in \mathbb{N}} \) be a partition of unity subordinated to \( V_{x_i} \setminus \bigcup_{i \neq k} T_k \). Finally, take the cotangent average of the metrics \( \eta_{x_i}^0 \)

\[
\eta_0' = \left( \sum_i \phi_i (\eta_{0_i}^0)^* \right)^*.
\]

Since each \( \eta_{x_i}^0 \) induce the same inner product on the normal vector spaces \( \nu(M, O)_x \) as \( \eta_0 \), the same holds for \( \eta_0' \), and therefore \( \eta_0' \) is a 0-metric. It only remains to show that \( \eta_0' \) is complete.

Let \( a : (p, q) \to M \) be a maximal unit-speed geodesic, suppose \( q \leq \infty \) and take \( q_n \not\to q \). By the relation between the stacky metric and the distance on \( M/G \) established in [4, Thm.3] we have \( d([a(q_n)], [a(q_m)]) \leq |q_n - q_m| \), so \( ([a(q_n)]) \) is a Cauchy sequence. By the stacky Hopf-Rinow [4, Thm.19] the space \( (M/G, d) \) is complete and we get \( x \in M \) such that \([a(q_n)] \to [x] \). Let \( i \) be such that \( x \in W_{x_i} \), and hence \( a(q - \delta, q) \subset W_{x_i} \) for small \( \delta \). Since \( a(q - \delta, q) \) cannot be extended, it cannot be contained in any compact, and therefore it must go through infinitely many tubes \( T_i(n(i, j)) \). But over each of these tubes the metric \( \eta_0' \) agrees with \( \eta_{0_i}^0 \), and therefore \( a \) needs at least time 1 to go through each of them. This leads to a contradiction proving that \( q = \infty \). The proof of \( p = -\infty \) is analogous. \( \square \)

It should be noted that Theorem 4.4 is not the precise converse of Theorem 3.3, for a priori the constructed 0-metric is not induced by a 2-metric. The problem of extending
a 0-metric to a 2-metric is subtle and we refer to [5] for several examples. We believe that a proper invariantly linearizable groupoid may indeed admit a 2-metric $\eta_2$ with $\eta_0$ complete, but we have not found yet a proof. When working with (Hausdorff) regular groupoids things get simpler:

**Lemma 4.5** If $G \rightrightarrows M$ is proper and regular then every 0-metric $\eta_0$ extends to a 2-metric.

**Proof** Let $F \subset TM$ be the foliation by orbits and let $\eta'$ be an auxiliary metric on $G$. Writing $I = \ker ds \cap \ker dt$, we get the following vector bundle orthogonal decomposition:

$$TG = I \oplus (\ker ds \cap I^\perp) \oplus (\ker dt \cap I^\perp) \oplus (\ker ds + \ker dt)^\perp$$

The maps $ds, dt : (\ker ds + \ker dt)^\perp \to F^\perp$ are fiberwise isomorphism, and the pullbacks of $\eta|_{F^\perp}$ along these two maps agree, for $\eta$ is invariant. We endow $(\ker ds + \ker dt)^\perp$ with this metric. We also endow $\ker ds \cap I^\perp$ and $\ker dt \cap I^\perp$ with the pullback metrics of $\eta_0|_F$ along $ds$ and $dt$, respectively, equipped $I$ with an arbitrary metric, and declare the four terms to be orthogonal. The resulting metric $\eta'$ on $G$ is fibered with respect to the source and the target, and therefore the cotangent average $\frac{1}{2}(\eta'^* + i^*\eta'^*)$ is a 1-metric $\eta'_1$ [5, Prop.2.2] extending $\eta_0$. Since $G \rightrightarrows M$ is proper, we can apply the gauge trick to $\eta'_1$ to get a 2-metric extending $\eta_0$, see [6, Lemma.3.1.5].

**Corollary 4.6** A regular proper groupoid $G \rightrightarrows M$ is invariantly linearizable if and only if it admits a 2-metric $\eta_2$ with $\eta_0$ complete.

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**References**

1. Crainic, M., Struchiner, I.: On the Linearization Theorem for proper Lie groupoids. Ann. Sci. Éc. Norm. Supér. **46**, 723–746 (2013)
2. del Hoyo, M.: Lie Groupoids and their orbispaces. Port. Math. **70**, 161–210 (2013)
3. del Hoyo, M.: Complete connections on fiber bundles. Indag. Math. **27**, 985–990 (2016)
4. del Hoyo, M., de Melo, M.: Geodesics on Riemannian stacks. Transform. Groups (2020). https://doi.org/10.1007/s00031-020-09596-y
5. del Hoyo, M., Fernandes, R.L.: Riemannian Metrics on Lie Groupoids. J. Reine Angew. Math. **735**, 143–173 (2018)
6. del Hoyo, M., Fernandes, R.L.: Riemannian metrics on differentiable stacks. Math. Z. **292**, 103–132 (2019)
7. de Melo, M.: Topics in Riemannian groupoids. Ph.D. thesis, IMPA (2019)
8. Duistermaat, J.J., Kolk, J.A.C.: Lie Groups. Springer, New York (2000)
9. Hermann, R.: A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. Proc. AMS **11**, 236–242 (1960)
10. Meigniez, G.: Submersions, fibrations and bundles. Trans. Am. Math. Soc. **354**, 3771–3787 (2002)
11. Mendes, R., Radeschi, M.: A slice theorem for singular Riemannian foliations, with applications. Trans. Am. Math. Soc. **371**, 4931–4949 (2019)
12. O’Neill, B.: Submersions and geodesics. Duke Math. J. 34, 363–373 (1967)
13. Pflaum, M., Posthuma, H., Tang, X.: Geometry of orbit spaces of proper Lie groupoids. J. Reine Angew. Math. 694, 49–84 (2014)
14. Weinstein, A.: Linearization of regular proper groupoids. J. Inst. Math. Jussieu 1(03), 493–511 (2002)
15. Zung, N.: Proper groupoids and momentum maps: linearization, affinity, and convexity. Ann. Sci. Éc. Norm. Supér. 39, 841–869 (2006)

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