Nature of an intermediate non-Fermi liquid state in Ge-substituted YbRh$_2$Si$_2$: Fermionized skyrmions, Lifshitz transition, Skyrmion liquid, and Gruneisen ratio

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We propose a skyrmion liquid state for the non-Fermi liquid (NFL) phase in Ge-substituted YbRh$_2$Si$_2$, where skyrmions form their Fermi surface, argued to result from the strongly coupled nature between skyrmions and itinerant electrons. The fermionized skyrmion theory identifies the antiferromagnetic (AF) transition with the Lifshitz transition, where the quantum critical point (QCP) is characterized by the dynamical critical exponent $z = 2$. Nonlocal interactions between skyrmions allow a critical line above the AF QCP, which originates from the Kondo-coupling effect with itinerant electrons. This critical line is described by the skyrmion liquid state, which results in Landau damping for spin fluctuations, thus characterized by $z = 3$. As a result, the Gruneisen ratio is predicted to change from $\sim T^{-1}$ at the AF QCP to $\sim T^{-2/3}$ in the NFL phase.

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I. INTRODUCTION

Recent experiments on Yb-based heavy fermion systems of YbRh$_2$(Si$_{0.95}$Ge$_{0.05}$)$_2$ [1], Yb(Rh$_{0.94}$Ir$_{0.06}$)$_2$Si$_2$ [2], YbAgGe [3], and β-YbAlB$_4$ [4] have uncovered the appearance of non-Fermi liquid physics over a finite zero-$T$ region of the magnetic field- or pressure-tuned phase diagram, rather than at a single quantum critical point (QCP). The implication of these experiments casts a doubt on our fundamental understanding for correlated electrons in metals because the present theoretical framework cannot access such a non-Fermi liquid phase away from quantum criticality above one dimension in a rigorous sense [9]. Actually, the analogy between our skyrmion-dynamics problem and the vortex phase diagram serves an essential physical picture in this study. A novel vortex phase was proposed in the region above the vortex lattice state, identified with a vortex liquid phase, where the strongly coupled dynamics between vortices and itinerant electrons gives rise to an exotic quantum liquid state [9].

We propose a skyrmion liquid state for the nature of the non-Fermi liquid phase in Ge-substituted YbRh$_2$Si$_2$, the characteristic feature of which is the power-law behavior in the skyrmion density-density correlation function. A key ingredient in this theoretical proposal is to fermionize skyrmion excitations at the antiferromagnetic QCP. Although this fermionization procedure cannot be justified above one dimension in a rigorous sense a priori, essentially the same idea has been applied to describe non-Fermi liquid transport phenomena in the vortex liquid phase [9], where the Chern-Simons term, which appears from the payment for statistical transmutation, becomes irrelevant at criticality [10]. This fermionic skyrmion conjecture identifies the antiferromagnetic quantum phase transition with the Lifshitz transition for skyrmion excitations, where the skyrmion chemical potential touches the skyrmion band, and thus, the dynamical critical exponent is given by $z = 2$. If
we translate tuning of the magnetic field with control-
ing of the skyrmion chemical potential, we are driven
to conclude that the Fermi surface of skyrmions emerges
and the Landau damping from their particle-hole excitations
gives rise to the dynamical critical exponent \( z = 3 \)
for spin fluctuations. As a result, we predict that the
Gruneisen ratio, given by the ratio between the ther-
mal expansion and specific heat coefficients, changes from
\( \Gamma(T) \propto T^{-1} \) at the antiferromagnetic (Lifshitz) QCP to
\( \Gamma(T) \propto T^{-2/3} \) in the non-Fermi liquid (skyrmion liquid)
phase.

II. SKYRMION LIQUID

A. O(3) nonlinear \( \sigma \) model

We start from an effective Kondo-Heisenberg lattice
Hamiltonian,

\[
H = \sum_{k} (\epsilon_{k} - \mu) c_{k\sigma}^{\dagger} c_{k\sigma} + J_{K} \sum_{i} s_{i} \cdot S_{i} + J \sum_{ij} S_{i} \cdot S_{j},
\]

(1)

\( S_{\text{eff}} = S_{NLsM} + S_{\text{Kondo}}, \quad S_{NLsM} = \frac{c_{r}}{2g_{r}} \int_{0}^{\beta} d\tau \int d^{d}r \left\{ \frac{1}{c_{r}^{2}} [\partial_{\tau} n(r, \tau)]^{2} + |\nabla n(r, \tau)|^{2} \right\} + S_{B}, \]

(2)

\( S_{\text{Kondo}} = -N_{F} \lambda^{2} \sum_{i} \int d^{d}q \left( n(r, \tau) \times [\partial_{\tau} n(r, \tau)] \right) \frac{[\Omega]}{q_{\pi} q_{F}^{'}} \left( n(r', \tau') \times [\partial_{\tau} n(r', \tau')] \right) \) \( q_{\pi} \rightarrow i\Omega. \)

\( S_{NLsM} \) is the typical nonlinear \( \sigma \) model for the
Heisenberg-type model \( [13] \), where the coupling constant
\( g_{r} \) between antiferromagnetic spin waves and the spin-
wave velocity \( c_{r} \) are renormalized by the Kondo-coupling
term. \( S_{B} \) denotes a single-spin Berry phase term, which
originates from the path-integral quantization of a spin
field in the spin-coherent state representation. In the
present study we will not take into account the role of this
Berry phase term. \( S_{\text{Kondo}} \) describes an effective interaction
between ferromagnetic fluctuations with \( \lambda = J_{K} / J \),
where the polarization kernel \( \Pi(q, i\Omega) = \frac{[\Omega]}{q_{\pi} q_{F}^{'}} \) shows the
Landau damping dynamics.

An important remark in this setup is that the Landau
damping form seems to be robust beyond the present one-
loop approximation \( [14][17] \), where the existence itself of
the Fermi surface seems to protect the Landau damping
form. However, we would like to point out that the ex-
act expression for the polarization kernel still remains as
an open question. Within this uncertainty, our problem
is clarified as follows. What is the role of the nonlocal
effective interaction for ferromagnetic spin fluctuations
(\( S_{\text{Kondo}} \)) in the antiferromagnetic quantum phase transition
described by the nonlinear \( \sigma \) model (\( S_{NLsM} \))?  

B. Duality transformation

Performing the duality transformation for this effec-
tive nonlinear \( \sigma \) model, we derive an effective field theory
for skyrmion excitations \( [17] \). First, we rewrite Eq.
(2) in terms of bosonic spinons \( z_{\sigma} \), given by
\( n(r, \tau) = \frac{1}{2} z_{\sigma}^{\dagger}(r, \tau) \sigma_{\sigma\tau} z_{\sigma}(r, \tau) \). Second, we consider an easy-
plane approximation, given by \( z_{\sigma} = \frac{1}{\sqrt{2}} e^{i\phi_{\sigma}} \), where the
O(3) symmetry is reduced to O(2). Resorting to this CP\(^{1}\)
representation with the easy plane anisotropy, we obtain
\[ S_{\text{eff}} = \int_0^{\beta_r} d\tau \int d^2 r \left\{ \frac{1}{2g_2} (\partial_\mu \phi_\sigma - a_\mu)^2 + \frac{1}{2c_2^2} (\partial \times a)^2 \right\} \]

\[ - \int_0^{\beta_r} d\tau \int d^2 r \int_0^{\beta_r} d\tau' \int d^2 r' c_r N_F \lambda^2 [\partial_\tau \phi_\uparrow (r, \tau) - \partial_\tau \phi_\downarrow (r, \tau)] \Pi (r - r', \tau - \tau') [\partial_\tau \phi_\uparrow (r', \tau') - \partial_\tau \phi_\downarrow (r', \tau')] \] (3)

where \( e_a \) is an internal electric charge for bosonic spinons, which results from the U(1) gauge redundancy in the CP\(^1\) representation, and \( \tau \) is scaled into \( c_r \tau \) with \( \beta_r = c_r \beta \).

It is straightforward to perform the duality transformation for Eq. (3) although the Kondo-fluctuation induced term gives rise to complications. The resulting skyrmion field theory is given by

\[ S_{sk} = \int_0^{\beta_r} d\tau \int d^2 r \left\{ \mu_{sk} \phi_s (\partial_\tau - ic_r) \phi_s + \left| (\partial_\mu - ic_\mu) \phi_s \right|^2 + m_s^2 \phi_s^2 \frac{1}{2} N_F |\phi_s|^4 + \frac{1}{2c_2^2} (\partial \times c)^2 \right\} \]

\[ + \int_0^{\beta_r} d\tau \int d^2 r \int_0^{\beta_r} d\tau' \int d^2 r' \frac{1}{2p_F} [\partial \times c (r, \tau)] \Pi_{rr', \tau \tau'} [\partial \times c (r', \tau')] \] (4)

where \( \phi_s (r, \tau) \) represents a skyrmion field and \( \mu_{sk} (r, \tau) \) expresses a spin-wave excitation. \( \frac{q}{2} \rightarrow \frac{1}{2}\eta_c \) and \( \frac{2c_r N_F g^2 \lambda^2}{v_F} \rightarrow \frac{1}{v_F} \) have been done to translate two coupling constants, \( J \) and \( J_K \) of the original model into two kinds of internal charges of skyrmions, \( q_r \) and \( p_r \). \( \Pi_{rr', \tau \tau'} \) represents the effect from Fermi surface fluctuations, given by \( \Pi (q, i\Omega) = |\Omega|/|q| \) in the momentum-frequency space. We would like to emphasize that our formulation can be constructed in any general expression for the polarization kernel. In this respect we can give an answer beyond some limited approximations for the polarization kernel. See appendix A for the derivation from Eq. (3) to Eq. (4).

Several remarks should be given for our effective field theory [Eq. (4)] of skyrmions. Since two species of bosonic spinons exist in the CP\(^1\) representation, there must be two kinds of vortices, \( \Phi_\uparrow \) and \( \Phi_\downarrow \), identified with meron excitations \[\text{[17]}\]. However, we take the limit of \( \epsilon_a \rightarrow \infty \), resulting in the fact that all gauge non-singlet excitations are confined to disappear from the effective field theory \[\text{[18]} \text{[19]}\]. As a result, \( \Phi_s \sim \Phi_\uparrow \Phi_\downarrow \) arises naturally from the confinement ansatz of \( \epsilon_a \rightarrow \infty \), identified with the skyrmion field instead of meron excitations. The effective field theory, Eq. (4) consists of “normal” skyrmions and spin wave excitations in the easy plane approximation, where the U(1) gauge symmetry is associated with the conservation of the \( z \)-component of localized spins. Another important aspect is the presence of the particle-hole symmetry-breaking term, given by the linear time-derivative with \( \mu_{sk} \) in the skyrmion dynamics. As discussed in the introduction, our physical picture is based on the analogy with the vortex phase diagram of type II superconductors \[\text{[8]}\]. We speculate that there exists a skyrmion “lattice”—like region at least at finite temperatures before the antiferromagnetic order disappears, which breaks the particle-hole symmetry for skyrmion dynamics as the vortex lattice phase. Appearance of the particle-hole symmetry breaking is attributed to the low Neel temperature and the Kondo-coupling effect with itinerant electrons \[\text{[20]}\]. Our physical picture is that skyrmions serve effective magnetic fields to itinerant electrons, which can play a role in reducing the kinetic energy of itinerant electrons. As a result, the total energy can be more lowered, where skyrmion fluctuations become more softened by the interplay between itinerant electrons and skyrmions. Indeed, a similar phenomenon has been observed theoretically in the Kondo system on geometrically frustrated lattices, where coplanar ordering of localized spins in the absence of itinerant electrons becomes unstable to turn into spin chiral ordering structures at special fillings of itinerant electrons, which generate effective magnetic fields to itinerant electrons and quench the kinetic energy to lower the total ground-state energy \[\text{[21]}\]. The antiferromagnetic QCP is achieved by proliferation of skyrmion excitations, but we claim that there can exist an intermediate skyrmion liquid state before the skyrmion condensed phase.

An interesting feature of Eq. (4) is the nonlocal term with the polarization kernel in the gauge or spin-wave propagator. This gives rise to the fact that the “charge” \( p_r \) does not renormalize at the QCP when the linear time-derivative term is neglected. It is straightforward to obtain renormalization group equations for two coupling
constants,
\[ \frac{dq_r^2}{d \ln \mu} = q_r^2(\epsilon - \eta_c), \quad \frac{dp_r^2}{d \ln \mu} = p_r^2(\epsilon - \eta_c). \]  
(5)

\( \eta_c = \frac{d \ln Z_c}{d \ln \mu} = C_c q_r^2 \) is the anomalous dimension of the
gauge field, where \( Z_c \) is the wave-function renormalization
constant for the gauge field and \( C_c \) is a positive numerical constant. \( \epsilon \equiv 3 - d \) is with dimension \( d \). A key
point in these renormalization group equations is that the
anomalous dimension \( \eta_c \) appears in the same way, which
originates from the gauge invariance. As a result, we ob-
tain \( \eta_c = \epsilon \) at the charged fixed point, which is exact
for all orders [22]. This defines the critical coupling con-
tant \( g_c \). However, there is no equation to fix \( p_r \), which
remains unrenormalized. This originates from the nonlo-
local interaction generated from the Kondo-coupling effect.
The non-renormalization for \( p_r \) motivates us to search
an appropriate description for the existence of a critical
line instead of a critical point. One way is to fermion-
ize skyrmion excitations. The fermionization procedure
gives rise to non-relativistic dynamics for such fermion-
ized skyrmion excitations. The fermionization procedure
provides rise to non-relativistic dynamics for such fermion-
ized skyrmion excitations, characterized by \( z = 2 \). The
\( z = 2 \) dynamics seems to be inconsistent with the bosonic
description with \( z = 1 \). Our requirement of mathemat-
ical consistency between fermionic and bosonic descrip-
tions for skyrmions forces us to introduce the linear time-
derivative term in Eq. (4). The physical picture for this
particle-hole symmetry breaking term was discussed in the
previous paragraph.

C. Fermionized skyrmions and Lifshitz transition

Possibility of statistical transmutation for vortices has
been discussed in the vortex liquid phase [9] and at quan-
tum criticality in geometrically frustrated spin systems
[10]. Fermionization of skyrmion excitations can be per-
formed in the same way as that of vortices. The key
point is that the Chern-Simons term becomes irrelevant
at quantum criticality [9, 10], implying that the quantum
statistics for skyrmions or vortices may not be well de-
\n
\[
S_{\text{eff}} = \int_0^{\beta_r} d\tau \int d^2r \left\{ \psi_s^\dagger (\partial_\tau - \mu_\psi - ic_\tau) \psi_s + \frac{1}{2m_{sk}} \left[ (\nabla - i\mathbf{c}) \psi_s \right]^2 + \frac{1}{2g_r^2} (\partial \times \mathbf{c})^2 \right\} \\
+ \int_0^{\beta_r} d\tau \int d^2r \int_0^{\beta_r} d\tau' \int d^2r' \frac{1}{2p_r^2} \partial \times \mathbf{c}(r, \tau) \left. \Pi_{\tau\tau',\tau'} [\partial \times \mathbf{c}(r', \tau')] \right|_\tau,
\]
\( \text{[Eq. 6]} \)

where \( \psi_s \) represents the fermionic skyrmion field with
the skyrmion chemical potential \( \mu_\psi \) and the band mass
\( m_{sk} \propto \mu_{sk} \). The antiferromagnetic transition is described
by tuning the chemical potential from \( \mu_\psi < 0 \) to \( \mu_\psi > 0 \).
In this respect the antiferromagnetic QCP is realized at
\( \mu_\psi = 0 \), which means that the chemical potential touches
the lowest position of the band, thus identified with the
Lifshitz transition. The Lifshitz transition is character-
ized by the dynamical critical exponent \( z = 2 \), where
the renormalization effect from gauge fluctuations is not
relevant. As a result, both \( z = 2 \) critical skyrmion fluc-
tuations and \( z = 1 \) gapless spin fluctuations coexist to
contribute to thermodynamics. Increasing the skyrmion
chemical potential from zero to positive, the skyrmion
Fermi surface will arise. Then, Landau damping for
gauge fluctuations emerges from particle-hole excitations
of fermionic skyrmions, resulting in the \( z = 3 \) dynamics
\[ \text{[Eq. 23]} \]

The renormalization group analysis has been per-
formed \[ \text{[Eq. 24, 25]} \], where the renormalization group equa-
tion for \( q_r \) remains essentially the same as that of Eq.
(5). An effective field theory has been constructed right
at the interacting charged fixed point in the absence of \( p_r \).

D. Experimental implication: Gruneisen ratio

Although the skyrmion liquid state can be character-
ized by the power-law correlation function of skyrmion
densities, there are more directly measurable thermody-
namic quantities. Both specific heat and thermal expan-
sion coefficients can be deduced from two critical ex-
ponents, the dynamical critical exponent \( \nu \) and the corre-
alation length exponent \( \nu \). Unfortunately, these two thermo-
dynamic functions depend on dimensionality. Although
our skyrmion description was constructed in two dimen-
sions, we cannot exclude the possible existence of the
skyrmion liquid state in three dimensions. In this respect
it is more reliable to consider a dimension-independent quantity, which is the Gruneisen ratio $\Gamma(T)$, given by the ratio between the thermal expansion and specific heat coefficients. It is straightforward to see $\Gamma(T) \propto T^{-1/\nu z}$ from the scaling argument [26].

|           | $z$ | $\nu$ | $\Gamma(T) \propto T^{-1/\nu z}$ |
|-----------|-----|-------|----------------------------------|
| AF-QCP    | 2   | 1/2   | $T^{-1}$                   |
| NFL       | 3   | 1/2   | $T^{-2/3}$                  |

TABLE I: Gruneisen ratio at the antiferromagnetic quantum critical point (AF-QCP) and in the non-Fermi liquid phase (NFL)

Table I shows how the Gruneisen ratio changes from the antiferromagnetic QCP to the skyrmion liquid phase. As the dynamical critical exponent changes from $z = 2$ at the antiferromagnetic QCP (Lifshitz transition) to $z = 3$ in the non-Fermi liquid phase (skyrmion liquid with its Fermi surface), the Gruneisen ratio will follow from $\sim T^{-1}$ to $\sim T^{-2/3}$, respectively. We recall that the $T^{-1}$ behavior was seen from CeNi$_2$Ge$_2$, regarded as a conventional antiferromagnet, while the $T^{-2/3}$ behavior was observed from YbRh$_2$(Si$_{0.95}$Ge$_{0.05}$)$_2$, expected to belong to a different class of heavy fermion systems [27]. In particular, the $T^{-2/3}$ behavior has been attributed to the Kondo breakdown mechanism, which differs from the present scenario [28, 29].

III. DISCUSSION

A. Role of Berry phase

The role of Berry phase has been discussed intensively in insulating antiferromagnets [17]. Although it does not play any important role in an antiferromagnetic phase, it assigns a nontrivial quantum number to skyrmion excitations in a paramagnetic phase, the condensation of which results in translational symmetry breaking (valence bond solids). Furthermore, interplay between Berry phase and interaction has been proposed to allow exotic excitations at a quantum critical point between an antiferromagnetic phase and a translational symmetry broken paramagnetic state. One dimensional spin-liquid features have been claimed to appear in two dimensions, where spin-1 excitations in both antiferro- and para- magnetic phases become fractionalized into spin-1/2 spinon excitations. However, the role of Berry phase has not been clarified in metallic antiferromagnets. We believe that this problem deserves to investigate intensively near future.

B. On the easy-plane approximation

To take the easy-plane limit reduces the O(3) spin symmetry to O(2) $\times$ Z$_2$. Although it is certainly meaningful to study this type of antiferromagnets coupled to itinerant electrons, an actual question is how much different physics will arise.

First of all, we would like to point out that qualitatively distinguished physics between O(3) and O(2) symmetries are not kept in the present scheme of approximation, where our calculational tool is basically based on the scheme of the 1/N expansion. Since critical exponents depend on $N$, one may claim that the quantum critical point of $N = 2$ differs from that of $N = 3$. However, this difference is just quantitative in the 1/N approximation. Both cases of $N = 2$ and $N = 3$ show continuous phase transitions. On the other hand, if one claims the emergence of fractionalized excitations in the O(N) vector model, possible effective field theories will be given in terms of fractionalized excitations (spinons) and gauge fluctuations [17]. In this case the symmetry reduction can give rise to unexpected errors sometimes, for example, resulting in weak first order transitions instead of expected second order ones [30]. However, such a reduced symmetric model can be manipulated to allow qualitatively similar features, compared with the original symmetric model. In particular, deconfined quantum criticality has been predicted within the symmetry-reduced model, which also appears in the full symmetric model [17]. We would like to emphasize that these exotic excitations are not taken into account in the present study. Instead, we consider "conventional" spin 1 excitations only. In this respect we expect no qualitative different physics between fully symmetric and symmetry reduced spin models.

C. More on particle-hole symmetry breaking in skyrmion dynamics

It is our essential proposal to replace the second-time derivative term with the linear-time derivative. Unfortunately, we cannot prove the emergence of the Galileian invariance (the linear-time derivative) from the relativistic invariance (the second-time derivative) at present. In this respect one can call the emergence of the Galileian invariance in the presence of itinerant electrons as our speculation.

There must be an underlying physical mechanism for this linear-time derivative term, which breaks the particle-hole symmetry for "bosonic" skyrmion excitations. It has been demonstrated that the density of skyrmion excitations is finite at an antiferromagnetic quantum critical point without itinerant electrons, i.e. in an insulating antiferromagnet [31]. Of course, only the second-time derivative term is allowed in this case, i.e., particle-hole symmetric, which means that an equal number of skyrmion and anti-skyrmion excitations exists at this quantum critical point. Our problem is what happens on the particle-hole symmetry in skyrmion excitations at the quantum critical point where itinerant charge carriers are introduced. This is a long-standing problem,
where non-perturbative effects from interactions between itinerant electrons and "many" topologically nontrivial excitations should be taken into account on equal footing. Frankly speaking, we do not have any reliable mathematical tools for the description of such interactions.

Our speculation is that the presence of itinerant electrons will induce the particle-hole symmetry breaking in the skyrmion sector because itinerant electrons favor skyrmion excitations (or anti-skyrmions, i.e., one of the two). The physical mechanism is as follows. When itinerant electrons move in the background of skyrmions, they feel an effective magnetic flux, which quenches the kinetic energy of electrons. Our expectation is that the gain in the electron kinetic energy contribution is larger than the cost in the particle-hole symmetry breaking of the skyrmion sector. As a result, the particle-hole symmetry breaking is favorable in the respect of the total energy. Actually, this mechanism has been realized in the system of frustrated magnets, where the presence of itinerant electrons leads the co-planar ordering in the triangular antiferromagnet to be ordered into an out-of-plane way, which corresponds to a spin chiral order [21]. In the present study we do not consider this time reversal symmetry breaking because our skyrmion excitations are strongly fluctuating at our quantum critical point, which will not form such an exotic static order.

### D. Summary of approximations

It is necessary to summarize our approximations for the procedure from Eq. (1) to Eq. (6). It is basically an exact procedure to derive an effective O(3) nonlinear σ model [Eq. (2)] from the Kondo-Heisenberg lattice Hamiltonian [Eq. (1)] after integrating over itinerant-electron degrees of freedom if we replace the Landau damping form for the polarization kernel with its general expression. Although we keep the Berry phase term in Eq. (2), we do not take into account its role in spin dynamics, particularly, skyrmion dynamics. The easy-plane approximation has been resorted for the derivation from Eq. (2) to Eq. (3). The effective dual Lagrangian [Eq. (4)] in terms of bosonic skyrmions has been found from the duality transformation, as shown in appendix A. An essential aspect is to replace the second-time derivative term with the linear-time derivative. This is our key proposal. The resulting field theory [Eq. (6)] in terms of fermionic skyrmion excitations come from the bosonic skyrmion theory [Eq. (4)], performing the fermionization procedure. In appendix B, we prove irrelevance of the Chern-Simons term at quantum criticality, which supports the emergence of fermionic skyrmions.

### E. On electron dynamics

It is not straightforward to understand the feedback effect from skyrmion excitations to itinerant electrons because we need to calculate spin-spin correlation functions of ferromagnetic fluctuations and this is not easy to perform, in particular, for transverse spin fluctuations. The spin-correlation function for the \( z \)-component is basically given by the flux correlator in terms of \( (\nabla \times \varepsilon) \). However, the spin-correlation function for transverse components is related with "magnetic monopole" correlations which changes \( 2\pi \)-flux in \( (\nabla \times \varepsilon) \) because transverse components flip the \( z \)-component spin and the \( z \)-component spin is identified with the gauge flux in the dual representation.

### F. Comparison with a non-Fermi liquid metallic phase in the slave-fermion theory

We would like to point out that a possible non-Fermi liquid metallic state has been proposed in the context of the slave-fermion theory [23, 24]. However, the non-Fermi liquid state from the slave-fermion theory is characterized by the existence of spin gap while the skyrmion liquid state does not exhibit such a spin gap, where spin fluctuations remain gapless. The skyrmion liquid phase should be distinguished from the spin-gapped non-Fermi liquid state.

### IV. CONCLUSION

In conclusion, we proposed a quantum critical metallic state for a non-Fermi liquid phase of Ge-substituted YbRh\(_2\)Si\(_2\), identified with a skyrmion liquid state with a skyrmion Fermi surface. Nonlocal interactions, originated from Kondo fluctuations, and fermionized skyrmions, argued to be allowed from both nonlocal interactions and coexistence with itinerant electrons, give rise to a stable interacting fixed line. The antiferromagnetic transition has been interpreted by the Lifshitz transition of fermionized skyrmions, characterized by the dynamical exponent \( z = 2 \). The skyrmion liquid state is described by its Fermi surface, resulting in \( z = 3 \) for collective spin fluctuations. We suggested the fingerprint of this scenario from the change of the Gruneisen ratio, where it behaves from \( \sim T^{-1} \) at the antiferromagnetic quantum critical point \( (z = 2) \) to \( \sim T^{-2/3} \) in the non-Fermi liquid phase \( (z = 3) \).

An important aspect of the present study is on the introduction of Fermi-surface fluctuations. Generally speaking, it is not justified to integrate over Fermi-surface fluctuations. An interesting point is that although we cannot estimate the precise form of the polarization kernel, which results from Fermi-surface fluctuations to give nonlocal interactions between collective spin fluctuations, such nonlocal interactions can be irrelevant if we assume that skyrmion excitations become fermions instead of bosons with particle-hole symmetry breaking.

The next question will be, "Can we derive this physics based on a purely diagrammatic way?" The present re-
sult can be translated into the fact that nonlocal interactions from Fermi-surface fluctuations may cause interesting novel physics to dynamics of spin fluctuations. Although this issue has been addressed in the duality picture, it is an important direction to prove the emergence of \( z = 3 \) antiferromagnetic quantum criticality based on the original description for spins.

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Appendix A: Duality transformation

Starting from Eq. (3) and performing the Hubbard-Stratonovich transformation, we obtain

\[
Z = \int D\phi_r(r, \tau) Da_\mu(r, \tau) DJ^\sigma_\mu(r, \tau) Dj_\tau(r, \tau) \exp \left[ - \int_0^{\beta_r} d\tau \int d^2r \left( \frac{g_r}{2} J_\mu^2 - iJ^\sigma_\mu (\partial_\mu \phi_\sigma - a_\mu) \right) \right.
\]

\[
+ \frac{1}{2c^2_a} (\partial \times a)^2 \left. + \int_0^{\beta_r} d\tau \int d^2r \int_0^{\beta_\tau} d\tau' \int d^2r' \frac{1}{4c_\gamma N_F \lambda^2} j_\tau(r, \tau) [\Pi(r - r', \tau - \tau')]^{-1} j_\tau(r', \tau') \right],
\]

(A1)

where \( J^\sigma_\mu \) and \( j_\tau \) are auxiliary fields which correspond to currents.

Separating the phase field into spin-wave (smooth) and vortex (singular) parts and integrating over spin-wave fluctuations, we obtain equations of constraints,

\[
\partial_\mu J^\sigma_\mu + \sigma \partial_\mu j_\tau = 0,
\]

(A2)

where \( \sigma = \pm \) represent spin \( \uparrow \) and \( \downarrow \), respectively. These equations are solved to give

\[
J^\sigma_\mu + \sigma j_\tau = (\partial \times c^\sigma)_\mu,
\]

(A3)

where \( c^\sigma_\mu \) is U(1) gauge field to incorporate spin-wave excitations.

Integrating over \( J^\sigma_\mu \) with Eq. (A3), we obtain

\[
Z = \int D\phi^\sigma_r(r, \tau) Da_\mu(r, \tau) DC^\sigma_\mu(r, \tau) \exp \left[ - \int_0^{\beta_r} d\tau \int d^2r \left( \frac{g_r}{2} [(\partial \times c^\sigma)_\mu - \sigma j_\tau \delta_{\mu \tau}]^2 - i(\partial \times c^\sigma)_\mu (\partial_\mu \phi^\sigma_\sigma - a_\mu) \right) \right.
\]

\[
+ \frac{1}{2c^2_a} (\partial \times a)^2 \left. \right. \left. + \int_0^{\beta_r} d\tau \int d^2r \int_0^{\beta_\tau} d\tau' \int d^2r' \frac{g_r}{4c_\gamma N_F \lambda^2} j_\tau(r, \tau) [\Pi(r - r', \tau - \tau')]^{-1} j_\tau(r', \tau') \right],
\]

(A4)

It is straightforward to perform the integration for \( j_\tau \). Then, we obtain a dual action in the first-quantization expression

\[
Z = \int D\phi^\sigma_r(r, \tau) Da_\mu(r, \tau) DC^\sigma_\mu(r, \tau) \exp \left[ - \int_0^{\beta_r} d\tau \int d^2r \left( \frac{g_r}{2} [(\partial \times c^\sigma)_\mu - \sigma j_\tau \delta_{\mu \tau}]^2 - i(\partial \times c^\sigma)_\mu (\partial_\mu \phi^\sigma_\sigma - a_\mu) \right) \right.
\]

\[
- \int_0^{\beta_\tau} d\tau \int d^2r \int_0^{\beta_\tau} d\tau' \int d^2r' \frac{g_r}{2} [(\partial \times c^\sigma)_\mu - (\partial \times c^\sigma)_\tau] (r, \tau)
\]

\[
\left\{ g_r [\Pi(r - r', \tau - \tau')]^{-1} [(\partial \times c^\sigma)_\tau - (\partial \times c^\sigma)_\tau] (r', \tau') \right\}.
\]

(A5)

We note that dynamics of spin-wave excitations is strongly modified by nonlocal interactions from Fermi-surface fluctuations of itinerant electrons through the Kondo-coupling effect.

Considering the minimal coupling term of \( c^\sigma_\mu \) in Eq. (A5), where \( [J^\sigma]_\mu = (\partial \times \phi^\sigma_\mu)_\mu \) is identified with a vortex current, one can construct the second-quantization form of Eq. (A5), introducing vortex field variables. As a result, we obtain an effective dual field theory

\[
Z = \int D\Phi_\sigma(r, \tau) Da_\mu(r, \tau) DC^\sigma_\mu(r, \tau) \exp \left[ - \int_0^{\beta_r} d\tau \int d^2r \left( |(\partial_\mu - ic^\sigma_\mu) \Phi_\sigma|^2 + m^2 \Phi_\sigma^2 + \frac{\mu_v}{2} |\Phi_\sigma|^4 + \frac{g_r}{2} (\partial \times c^\sigma)^2 \right) \right.
\]

\[
+ i(\partial \times c^\sigma)_\mu a_\mu + \frac{1}{2c^2_a} (\partial \times a)^2 \left. - \int_0^{\beta_\tau} d\tau \int d^2r \int_0^{\beta_\tau} d\tau' \int d^2r' c_\gamma N_F g^2 \lambda^2 [(\partial \times c^\sigma)_\tau - (\partial \times c^\sigma)_\tau] (r, \tau)
\]

\[
\Pi(r - r', \tau - \tau')[(\partial \times c^\sigma)_\tau - (\partial \times c^\sigma)_\tau] (r', \tau') \right].
\]

(A6)
\( \Phi_\sigma \) represents a meron field, which acts as a source of \( c^\sigma_\mu \) (spin-wave field), where the meron current is given by 
\[ [J^\sigma]_\mu = -i[\Phi_\sigma(\partial_\sigma \Phi_\sigma) - (\partial_\sigma \Phi_\sigma)^2][\Phi_\sigma]. \]
\( m_\sigma^2 \) is a mass of a meron, which identifies its quantum critical point with \( m_\sigma^2 = 0 \).
\( u_\sigma \) is a coupling constant for its local self-interaction, determined from non-universal short-distance physics.

As discussed in section II-B, we will not allow meron excitations. Emergence of meron excitations has been discussed intensively in Ref. 17, where the interplay between Berry phase and local self-interactions of merons would make "magnetic" monopole excitations of compact U(1) gauge fields \( a_\mu \) become irrelevant in the renormalization group sense, giving rise to merons as elementary excitations. Emergence of meron excitations implies that of spinons, which carry fractional spin quantum number \( 1 \). On the other hand, we take the limit of \( e_a \rightarrow \infty \), where such fractionalized excitations cannot exist. Merons should be combined with anti-merons, forming skyrmions, which means that spinons should be confined with anti-spinons, resulting in spin 1 excitations, identified with conventional spin fluctuations. It is still extremely difficult to clarify the precise condition for the emergence of deconfinement although its existence seems to be accepted.

In order to realize the confinement ansatz, i.e., \( e_a \rightarrow \infty \), we take the amplitude-frozen limit of \( \Phi_\sigma \sim e^{i\theta_\sigma} \) and obtain
\[ (\partial_\mu \theta_\sigma - c^\sigma_\mu)^2 = \frac{1}{2} \left( |\partial_\mu \theta_\sigma + \partial_\mu \theta_4| - |c^\dot{\mu}_\sigma + c^{\dot{\mu}}_\sigma| \right)^2 + \frac{1}{2} \left( |\partial_\mu \theta_\sigma - \partial_\mu \theta_4| - |c^\dot{\mu}_\sigma - c^{\dot{\mu}}_\sigma| \right)^2. \]
(A7)

At the same time, we rewrite the kinetic energy of the vortex gauge field (spin-wave) as follows,
\[ \frac{g_r}{2}(\partial \times c^\sigma)^2 = \frac{g_r}{4} \left( (\partial \times c^\sigma) - (\partial \times c^\dot{\sigma}) \right)^2. \]
(A8)

Then, one can rewrite Eq. (A6) in terms of newly defined vortices and gauge fields,
\[ \mathcal{L}_v = |(\partial_\mu - ic^\mu_\mu)|\Phi_+|^2 + m^2_+|\Phi_+|^2 + \frac{u_+}{2}|\Phi_+|^4 + ia_\mu \epsilon_{\mu\nu\lambda} \partial_\nu c^\nu_\lambda + \frac{g_r}{4}(\partial \times c^+) + \frac{1}{2c^2_\sigma}(\partial \times a)^2 \]
\[ + |(\partial_\mu - ic^\mu_\mu)|\Phi_-|^2 + m^2_-|\Phi_-|^2 + \frac{u_-}{2}|\Phi_-|^4 + \frac{g_r}{4}(\partial \times c^-)^2 \]
\[ + c_\nu N_F g^2_\lambda \int_{0}^{\beta_\nu} d\tau' \int d^2 r' |\partial \times c^\nu(\mathbf{r}, \tau)| \Pi(\mathbf{r} - \mathbf{r}', \tau - \tau') |\partial \times c^\nu(\mathbf{r}', \tau')|, \]
(A9)

where \( \theta_\pm = \frac{\theta_1 + \theta_2}{\sqrt{2}} \rightarrow \Phi_\pm \) is a "new" vortex field and \( c^\pm_\mu = \frac{c^\mu_\pm + c^{\dot{\mu}}_\pm}{\sqrt{2}} \) is also a "new" vortex-gauge field. An essential aspect of Eq. (A9) is that only the \( \Phi_+ \) sector contains interaction with compact U(1) gauge fluctuations \( a_\mu \) while the dynamics of \( \Phi_- \) decouples from the gauge dynamics completely. If we take \( e_a \rightarrow \infty \), \( \Phi_+ \) cannot appear as elementary excitations. This is the confinement ansatz, which do not allow such fields as carry the U(1) internal gauge charge of \( a_\mu \). This dual language can be interpreted as the fact that either spinons or anti-spinons cannot appear as the physical spectrum.

Integrating over \( a_\mu \) in the limit of \( e_a \rightarrow \infty \), we see that \( \Phi_+ \) and \( c^\mu_+ \) disappear in the physical spectrum. As a result, we reach an effective field theory for skyrmion dynamics in the presence of itinerant electrons
\[ \mathcal{L}_{sk} = |(\partial_\mu - ic^\mu_\mu)|\Phi_-|^2 + m^2_-|\Phi_-|^2 + \frac{u_-}{2}|\Phi_-|^4 + \frac{g_r}{4}(\partial \times c^-)^2 \]
\[ + c_\nu N_F g^2_\lambda \int_{0}^{\beta_\nu} d\tau' \int d^2 r' |\partial \times c^-| \Pi(\mathbf{r} - \mathbf{r}', \tau - \tau') |\partial \times c^-|, \]
(A10)

Appendix B: Irrelevance of the Chern-Simons term at quantum criticality

Starting from Eq. (4) and replacing \( \Phi_\sigma \) with \( \sqrt{\mu_{sk}} \Phi_\sigma \), we obtain
\[ Z_{sk} = \int D\Phi_\sigma DC_\sigma \exp \left[ -\int_{0}^{\beta_\nu} d\tau \int d^2 r \left\{ \hat{\Phi}_\sigma(\partial_\tau - ic_\tau)(\partial_\tau - ic_\tau) \Phi_\sigma + \frac{1}{\mu_{sk}}|\partial_\mu \Phi_\sigma|^2 + \frac{m^2_{sk}}{\mu_{sk}}|\Phi_\sigma|^2 + \frac{u_{sk}}{2\mu_{sk}^2}|\Phi_\sigma|^4 + \frac{1}{2\mu_{sk}^4}(\partial \times c)^2 \right\} \right] \]
\[ - \int_{0}^{\beta_\nu} d\tau \int d^2 r \int_{0}^{\beta_\nu} d\tau' \int d^2 r' \frac{1}{2p_r^2} |\partial \times c(\mathbf{r}, \tau)| \Pi_{\mathbf{r}, \mathbf{r}', \tau, \tau'} |\partial \times c(\mathbf{r}', \tau')|. \]
(B1)
In order to introduce the statistical transmutation for skyrmion excitations, we perform the Chern-Simons gauge transformation, which attaches a flux to a skyrmion. Then, we obtain

\[
Z_{sk} = \int D\Psi_s Dc_{\mu} D\alpha_{\mu} \exp \left[ -\int_{0}^{\beta_{r}} d\tau \int d^{2}r \left\{ \Psi_{s} (\partial_{\tau} - ic_{\tau} - i\alpha_{\tau}) \Psi_{s} + \frac{1}{\mu_{sk}} |(\partial_{\mu} - ic_{\mu} - i\alpha_{\mu})\Psi_{s}|^{2} + \frac{m^{2}}{\mu_{sk}} |\Psi_{s}|^{2} + \frac{u_{s}}{2\mu_{sk}^{2}} |\Psi_{s}|^{4} \right\} + \frac{1}{2q_{r}^{2}} (\partial \times c - \partial \times \alpha)^{2} - \frac{i}{2\pi} \alpha_{\mu} \epsilon_{\mu\nu\lambda} \partial_{\nu} \alpha_{\lambda} \right] - \int_{0}^{\beta_{r}} d\tau \int d^{2}r \int d^{2}r' \left[ \frac{1}{2q_{r}^{2}} [\partial \times c (r, \tau) - \partial \times \alpha (r, \tau), \Pi_{\tau\tau'}] [\partial \times c (r', \tau') - \partial \times \alpha (r', \tau')] \right].
\]

(B2)

where \(\Psi_{s}\) is a fermionized skyrmion field and \(\alpha_{\mu}\) is the Chern-Simons gauge field to guarantee the statistical transmutation. Shifting the skyrmion gauge field (spin wave) as \(c_{\mu} \rightarrow c_{\mu} - \alpha_{\mu}\), we obtain

\[
Z_{sk} = \int D\Psi_s Dc_{\mu} D\alpha_{\mu} \exp \left[ -\int_{0}^{\beta_{r}} d\tau \int d^{2}r \left\{ \Psi_{s} (\partial_{\tau} - ic_{\tau}) \Psi_{s} + \frac{1}{\mu_{sk}} |(\partial_{\mu} - ic_{\mu})\Psi_{s}|^{2} + \frac{m^{2}}{\mu_{sk}} |\Psi_{s}|^{2} + \frac{u_{s}}{2\mu_{sk}^{2}} |\Psi_{s}|^{4} \right\} + \frac{1}{2q_{r}^{2}} (\partial \times c - \partial \times \alpha)^{2} - \frac{i}{2\pi} \alpha_{\mu} \epsilon_{\mu\nu\lambda} \partial_{\nu} \alpha_{\lambda} \right] - \int_{0}^{\beta_{r}} d\tau \int d^{2}r \int d^{2}r' \left[ \frac{1}{2q_{r}^{2}} [\partial \times c (r, \tau) - \partial \times \alpha (r, \tau), \Pi_{\tau\tau'}] [\partial \times c (r', \tau') - \partial \times \alpha (r', \tau')] \right].
\]

(B3)

Integrating over the Chern-Simons gauge field, we reach the following expression

\[
Z_{sk} = \int D\Psi_s Dc_{\mu} \exp \left[ -\int_{0}^{\beta_{r}} d\tau \int d^{2}r \left\{ \Psi_{s} (\partial_{\tau} - ic_{\tau}) \Psi_{s} + \frac{1}{\mu_{sk}} |(\partial_{\mu} - ic_{\mu})\Psi_{s}|^{2} + \frac{m^{2}}{\mu_{sk}} |\Psi_{s}|^{2} + \frac{u_{s}}{2\mu_{sk}^{2}} |\Psi_{s}|^{4} + \frac{1}{2q_{r}^{2}} (\partial \times c)^{2} \right\} 
- \int_{0}^{\beta_{r}} d\tau \int d^{2}r \int d^{2}r' \left[ \frac{1}{2q_{r}^{2}} [\partial \times c (r, \tau), \Pi_{\tau\tau'}] [\partial \times c (r', \tau')] \right] - S_{irr.}[c_{\mu}(r, \tau)] \right].
\]

(B4)

where \(S_{irr.}[c_{\mu}(r, \tau)]\) describes the dynamics of \(c_{\mu}\), which results from the integration of the Chern-Simons gauge field. It turns out to be

\[
S_{irr.}[c_{\mu}(r, \tau)] \propto c_{\mu}(q, \Omega) \Pi_{\mu\nu}(q, \Omega) c_{\nu}(-q, -i\Omega) \propto |q|^{3} c_{\mu}(q, \Omega) P_{\mu\nu}^{T}(q, \Omega) c_{\nu}(-q, -i\Omega),
\]

(B5)

where \(P_{\mu\nu}^{T}(q, \Omega)\) is the projection operator to the transverse direction. As shown in this expression, the proportionality to \(|q|^{3}\) implies the irrelevance of the Chern-Simons interaction at quantum criticality.
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