Quantum gambling using two nonorthogonal states

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We give a (remote) quantum gambling scheme that makes use of the fact that quantum nonorthogonal states cannot be distinguished with certainty. In the proposed scheme, two participants Alice and Bob can be regarded as playing a game of making guesses on identities of quantum states that are in one of two given nonorthogonal states: if Bob makes a correct (an incorrect) guess on the identity of a quantum state that Alice has sent, he wins (loses). It is shown that the proposed scheme is secure against the nonentanglement attack. It can also be shown heuristically that the scheme is secure in the case of the entanglement attack.

A fundamental property of quantum bits (qubits) that differs from those of classical bits is that unknown qubits cannot be copied with unit efficiency (the no-cloning theorem). Another related property of qubits is that nonorthogonal qubits cannot be distinguished with certainty. The no-cloning theorem is the basis for the success of the Bennett-Brassard 1984 quantum-key-distribution scheme. Therefore, it is interesting to search for quantum protocols utilizing the property that nonorthogonal qubits cannot be distinguished with certainty. Bennett’s other quantum-key-distribution scheme indeed utilizes this property. On the other hand, a (remote) quantum gambling scheme has been found by Goldenberg et al. recently.

In this paper we propose another (remote) quantum gambling scheme that makes use of the fact that two nonorthogonal qubits cannot be distinguished with certainty. In the proposed scheme, two participants Alice and Bob can be regarded as playing a game of making guesses on identities of quantum states that are in one of two given nonorthogonal states: Alice randomly sends one of two nonorthogonal qubits, say, $|0\rangle$ and $|0\rangle$ (or $|0\rangle$ and $|1\rangle$). In this paper, $|0\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ and $|1\rangle = (1/\sqrt{2})(|0\rangle - |1\rangle)$. If Bob makes a correct guess, he wins. If not, he loses. Due to the fact that two nonorthogonal qubits cannot be distinguished with certainty, it is easy to see that there is no way for Bob to cheat. Alice might try to increase her gain by sending some qubits other than $|0\rangle$ and $|0\rangle$. There are two kinds of attacks. In nonentanglement attacks, qubits sent to Bob are not entangled with Alice's. In entanglement attacks (or EPR attack), qubits sent to Bob are highly entangled with hers. We show that the scheme is secure in the case of nonentanglement attacks. In the case of entanglement attacks, however, we heuristically show the security of the scheme. It is true that a quantum cryptographic scheme is of little use without security proof against all attacks including entanglement attacks. And what makes it complicated to prove security of a quantum cryptographic scheme is the entanglement attack.

The difference between our scheme and the original one [1] is that the former relies on the fact that two nonorthogonal states cannot be distinguished with certainty while the latter relies on gerneral quantum mechanical laws. Another difference is that no quantum system needs to be additionally sent in checking steps in our scheme while it needs to be in the original scheme. Now, let us describe the scheme more precisely.

1. Alice randomly chooses one between two nonorthogonal qubits $|0\rangle$ and $|0\rangle$, and sends it to Bob.
2. On the qubit he receives, Bob performs a measurement by which he can obtain maximal probability $p$ of correctly guessing the identity of the qubit.
3. On basis of the measurement’s results, he makes a guess on which one the qubit is and announces it to Alice.
4. If he made a correct (an incorrect) guess, Alice announces she has won (lost).
5. When Bob has won, Alice gives him one coin. When he has lost, Bob gives her $p/(1 - p)$ coins.

However, after the first step, Bob follows the following ones instead of steps (2) – (5), at randomly chosen instances with a rate $r$ ($0 < r < 1$).

These checking steps are similar to those of the original work of Goldenberg et al.

2’ Bob performs no measurement on the qubit and stores it.
3’ Bob performs no measurement on the qubit and stores it.
4’ Do the same thing as step (4).
5’ In the previous step, Alice has actually revealed which one she chose to tell him the qubit is (regardless of her honesty). When it is $|0\rangle$, Bob performs $\hat{S}_z$ ($\hat{S}_z$ an orthogonal measurement that composes of two projection operators $|0\rangle\langle 0|)$ and $|1\rangle\langle 1|$ or $\{0\rangle\langle 0|$, $|1\rangle\langle 1|\}$.) If the outcome is $|1\rangle$, Bob announces that he performed $\hat{S}_z$ and got $|1\rangle$ as an outcome. Then Alice must give him $R (R > 1)$ coins. If the outcome is $|0\rangle$, Bob says nothing about which measurement he performed and follows step...
(5). In the case of $|\bar{0}\rangle$, similar things are done with $\hat{S}_x = \{\hat{0}, \hat{0}, \hat{1}, \hat{1}\}$.

In step (2), it is important for Bob to perform the optimal measurement that assures maximal probability $p$ of correctly guessing the identity of the qubit in order to assure his maximal gain. Although it is known that average information gain is constrained by the Levitin-Holevo bound $\tilde{R}$, to find the optimal one is not an easy task. Fortunately, however, in the case of two nonorthogonal qubits, the measurement giving maximal information gain is well known $\tilde{R}$: a measurement $\{\hat{0}\langle \hat{0}|, \hat{1}\langle \hat{1}|\}$, where $|\hat{0}\rangle$ and $|\hat{1}\rangle$ are qubits corresponding to a vector $(1/\sqrt{2})|\hat{z} - \hat{x}\rangle$ and $(1/\sqrt{2})|\hat{z} + \hat{x}\rangle$, respectively, in the Bloch sphere representation, where a single qubit density operator $\rho_B = (1/2)(1+i\hat{\sigma})$. Here $1$ is the identity operator, $\hat{\sigma}$ is a Bloch vector, $\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli operators. (See Fig. 1 of Ref. [3] or Fig. 2 of Ref. [3]) Since information gain is maximal if and only if $p$ is maximal, a measurement with maximal information gain is what maximizes $p$. Thus the measurement $\{\hat{0}\langle \hat{0}|, \hat{1}\langle \hat{1}|\}$ is the optimal one. For maximal $p$, Bob does as the following. When the outcome is $|\hat{0}\rangle$ (|1\rangle), he makes a guess that the qubit is $|\hat{0}\rangle$ (|1\rangle). Then the probability $p$ of correctly guessing the qubit is given by $p = |\langle \hat{0}|\hat{0}\rangle|^2 = |\langle \hat{1}|1\rangle|^2 = \cos^2(\pi/8).

Now let us show how each player’s average gain is assured. First it is clear by definition that Bob can do nothing better than performing the optimal measurement, as long as Alice prepares the specified qubits. In the scheme, the numbers of coins that Alice and Bob pay are adjusted so that no one gains when Bob’s win probability is $p$. Thus Bob’s gain $G_B$ cannot be greater than zero, that is, $G_B \leq 0$. Next let us consider Alice’s strategy. As noted above, we first show the security against nonentanglement attacks. In the most general nonentanglement attacks, Alice randomly generates each qubit in state $|i\rangle$ with a probability $p_i$. Here $|i\rangle$’s are arbitrarily specified states of qubits, $i = 1, 2, ..., N$ and $\sum_i^N p_i = 1$. However, since Bob has no knowledge about which $|i\rangle$ Alice selected at each instance, his treatments on qubits become equal for all qubits. Thus it is sufficient to show the security for a qubit in an arbitrary state $|j\rangle = a|0\rangle + b|1\rangle$. ($a$ and $b$ are some complex numbers with a constraint $|a|^2 + |b|^2 = 1$). First we do it for states within the $z$-x plane, $|j\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$. Later we will generalize the argument to the former case. Let us consider the following. In steps (2′) – (5′), Bob checks at randomly chosen instances whether Alice has really sent $|0\rangle$ or $|\bar{0}\rangle$ by performing measurements $\hat{S}_z$ or $\hat{S}_x$, respectively. If the measurement’s outcomes are $|0\rangle$ or $|\bar{0}\rangle$ (|1\rangle or $|\bar{1}\rangle$), Alice has passed (not passed) the test. When not passed, Alice must give him $R > 1$ coins $\tilde{R}$.

Roughly speaking, Alice can do nothing but preparing either $|0\rangle$ or $|\bar{0}\rangle$ and honestly tell the identity of the state to him later. Otherwise she sometimes must pay $R$ coins to him, decreasing her gain. Let us consider this point more precisely. First we estimate the upper bound of Alice’s gain $G_A$. It is clear that

$$G_A \leq \max\{G_A^0, G_A^0\},$$

(1)

where $\max\{x, y\}$ denotes the maximal one between $x$ and $y$, and $G_A^0, G_A^0$ is Alice’s gain when she insists that $|j\rangle$ is $|0\rangle$ ($|\bar{0}\rangle$). When he has performed the measurements already, he has no way of detecting Alice’s cheating. So, Alice’s maximal gain in this case is $p/(1-p)$. However, it is clear that $G_A$ is bounded by $p/(1-p)$ in any case. When he has preserved the qubits following the checking steps, Alice’s cheating can be statistically detected. This case contributes to Alice’s gain by a largely negative term whose modulus is proportional to the product of the rate $r$ of checking steps, the probability that $|1\rangle$ or $|\bar{1}\rangle$ is detected, and the number of coins $R$ she must pay when it is detected, namely $-r|\langle 1|j\rangle|^2R$ or $-r|\langle \bar{1}|j\rangle|^2R$. However, here we should take into account the fact that Alice obtains partial information about whether Bob has performed the measurement: let $f_u$ be Alice’s estimation of the probability that Bob did not perform the measurement. With no information, $f_u$ is $r$. However, Bob’s announced guess gives her partial information on his measurement’s result if he did. This information can be used to make a better estimate of $f_u$. For example, in the case where Alice sends $|j\rangle$ and Bob performs the optimal measurement $\{\hat{0}\langle \hat{0}|, \hat{1}\langle \hat{1}|\}$, we obtain using the Bayes’s rule that $f_u = (r/2)/(r/2 + (1-r)|\langle 0|j\rangle|^2)$ when his guess is $|0\rangle$. However, it is clear that $f_u \geq r/2$: when Bob did not perform the measurement, he simply guesses it with equal probabilities regardless of what he received. Thus by the Bayes’s rule, Alice can see that there remains a probability greater than $r/2$ that Bob did not perform the measurement. The relation $f_u \geq r/2$ also holds for the entanglement attacks, since it is satisfied for any $|j\rangle$ as shown above (refer to the related discussion on entanglement-attack later). Combining above facts, we obtain

$$G_A^0 \leq \frac{p}{1-p} - \frac{r}{2}|\langle 1|j\rangle|^2R$$

(2)

and

$$G_A^0 \leq \frac{p}{1-p} - \frac{r}{2}|\langle \bar{1}|j\rangle|^2R.$$  

(3)

From Eqs. (1)–(3), we can see, in order that $G_A$ be non-negative the following two conditions must be satisfied. (1) Either $|\langle 1|j\rangle|^2 \sim (1/rR) \ll 1$ (that is, $|j\rangle \sim |0\rangle$) or $|\langle \bar{1}|j\rangle|^2 \sim (1/rR) \ll 1$ (that is, $|j\rangle \sim |\bar{0}\rangle$); (2) Between $|0\rangle$ and $|\bar{0}\rangle$, Alice chooses what is nearer to $|j\rangle$. Then she pretends in the step (4)’s that it is the qubit sent to Bob. Otherwise, $G_A$ will be dominated by the negative second term in the right-hand sides of Eqs. (2) and (3).

Alice might increase her gain by sending a qubit that slightly differs from either $|0\rangle$ or $|\bar{0}\rangle$. However, the gain
can be made negligible by making $R$ large, as we show in the following. Let us consider the case where Alice prepares $|j\rangle$ (\sim |0\rangle) and later tells him that it is $|0\rangle$, for example. In this case, $G_A$ for a given $r$ and $R$ is given by

$$G_A = (1-r)|\langle 0|j\rangle|^2(-1) + |\langle 1|j\rangle|^2 \left\{ \frac{p}{1-p} \right\}$$

$$-r|\langle 0|j\rangle|^2R + r|\langle 0|j\rangle|^2 \left\{ \frac{1}{2}(-1) + \frac{p}{1-p} \right\}$$

$$< (1-r)\left\{ \cos^2 \left( \frac{\pi}{8} + \frac{\theta}{2} \right) \right\}(-1) + \sin^2 \left( \frac{\pi}{8} + \frac{\theta}{2} \right) \left\{ \frac{p}{1-p} \right\}$$

$$-r \sin \frac{\theta}{2} R + 3r.$$  \hspace{1cm} (4)

Here the first (second and third) term in the right-hand sides is due to normal steps (1)–(5) [checking steps (2′)–(5′)]. We can check Eq. (4) by verifying that $G_A < 3r \sim 0$ when $|j\rangle$ equals $|0\rangle$. By the two conditions, we might only consider the case where $\theta \sim 0$, and thus we can neglect higher-order terms in Eq. (4):

$$G_A < \frac{\alpha}{1-p} \theta - \frac{rR}{4} \theta^2 + 3r,$$  \hspace{1cm} (5)

where $\alpha = \cos(\pi/8) \sin(\pi/8)$ and a small term of order $\theta^2$ is also neglected. Alice would maximize her gain for given $r$ and $R$. Maximal value of $G_A$ is obtained when $\theta = (2\alpha/(1-p))(1/rR)$:

$$G_A^{\text{max}} = \frac{\alpha^2}{(1-p)^2} \frac{1}{rR} + 3r.$$  \hspace{1cm} (6)

Bob would minimize $G_A^{\text{max}}$ for a given $R$. $G_A^{\text{max}}$ has its minimal value $2\sqrt{3\alpha/(1-p)} \sqrt{R}$ when $r = \alpha/(1-p) \sqrt{3R}$. Therefore, if Bob chooses $r = \alpha/(1-p) \sqrt{3R}$, then $G_A$ is bounded by the positive term $2\sqrt{3\alpha/(1-p)} \sqrt{\sqrt{R}} \propto 1/\sqrt{R}$ that approaches to zero as $R$ become large, similarly to the case of the scheme of Goldenberg et al. \[3]\.

Now we argue that using a qubit outside the $z$-$x$ plane does not increase Alice’s gain: we can see in Eq. (4) that $G_A$ can only be increased by making the ratio $|\langle 1|j\rangle|^2/|\langle 0|j\rangle|^2$ large while keeping $|\langle 1|j\rangle|^2$ a very small constant. Let us consider some set of $|j\rangle$’s (not confined in the $z$-$x$ plane) that give the same value of $|\langle 1|j\rangle|^2$. Bloch vectors of this set make a circle around that of $|1\rangle$. We can see by inspection that what gives the maximal value of the ratio $|\langle 1|j\rangle|^2/|\langle 0|j\rangle|^2$ lies within the $z$-$x$ plane.

Now, let us heuristically argue that the entanglement attacks \[7,8\] do not work in the proposed scheme. Let us consider the case where Alice prepares pairs of qubits in an entangled state,

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |0\rangle_B),$$  \hspace{1cm} (7)

where $A$ and $B$ denote Alice and Bob, respectively. Alice sends qubits with label $B$ to Bob while storing those with label $A$. If she performs $S_z$, Bob is given a mixture of $|0\rangle$ and $|0\rangle$ with equal frequency. Thus if Alice always performs $S_z$, the attack reduces to a nonentanglement attack where she randomly sends either $|0\rangle$ or $|0\rangle$. Let us consider an example illustrating how performing measurements much different from $S_z$ is not of benefit for Alice; we can rewrite Eq. (1) as

$$|\psi\rangle = \sqrt{\frac{2+\sqrt{2}}{2}} |0\rangle_A |\alpha\rangle_B + \sqrt{\frac{2-\sqrt{2}}{2}} |1\rangle_A |\beta\rangle_B,$$  \hspace{1cm} (8)

where $|\alpha\rangle$ and $|\beta\rangle$ are normalized ones of $(|0\rangle + |\bar{0}\rangle)$ and $(|0\rangle - |\bar{0}\rangle)$, respectively. Thus if Alice performs $S_x$, either $|\alpha\rangle$ or $|\beta\rangle$ is generated at Bob’s site with probabilities given by Eq. (8). However, since all of $|\langle 1|\alpha\rangle|^2$, $|\langle 1|\beta\rangle|^2$, and $|\langle 1|\beta\rangle|^2$ are of order of one, $G_A$ becomes much negative in any case. So Alice would not perform $S_x$. In fact, if Alice is able to change the qubits between $|0\rangle$ and $|0\rangle$ as she likes, her cheating will always be successful. However, she is not allowed to do so, since $|0\rangle |\bar{0}\rangle \neq |0\rangle |\bar{0}\rangle$ and Bob’s reduced density operator $\rho_B = \text{Tr}_A(\rho_{AB})$ cannot be changed even with entanglement attacks.

By appropriately choosing her measurement, Alice can generate at Bob’s site any $\rho_B$ satisfying $\sum_i p_i \langle i| \langle i | = \rho_B$, where $\{ p_i, \langle i| \langle i | \}$ denotes a mixture of pure states $\langle i| \langle i |$ with relative frequency $p_i$ (the theorem of Houghston, Jozsa, and Wootters) \[21\]. Let $\rho_B = (1/2)(1 + \hat{r} \cdot \vec{\sigma})$. Since $\rho_B = \sum_i p_i \langle i| \langle i | = (1/2)(1 + \hat{r} \cdot \vec{\sigma})$ where $\hat{r}_i$ is the corresponding Bloch vector, we have $(1/2)(1 + \hat{r} \cdot \vec{\sigma}) = (1/2)(1 + \sum_i p_i \hat{r}_i \cdot \vec{\sigma})$ and thus

$$\hat{r} = \sum_i p_i \hat{r}_i.$$  \hspace{1cm} (9)

Therefore, for a given $\rho_B$ whose Bloch vector is $\hat{r}$, Alice can prepare at Bob’s site any mixture $\{ p_i, \langle i| \langle i | \}$ as long as its Bloch vectors $\hat{r}_i$ satisfy the Eq. (4). However, if Alice always performs a given measurement, the entanglement attacks reduce to the nonentanglement attacks: outcomes of measurements on entangled pairs do not depend on temporal order of the two participants’ measurements. So we can confine ourselves to the case where Alice measures first. Then the attack reduces to a nonentanglement attack where Alice generates $\langle i |$ with probability $p_i$. Alice can only utilize the entanglement by choosing her measurements according to Bob’s announced guesses. However, the checking steps also prevent Alice from increasing her gain: she must choose the measurement that gives some mixture $\{ p_i, \langle i| \langle i | \}$ at Bob’s site where each $\hat{r}_i$ is nearly the same as either $z$ or $x$. Otherwise $G_A$ becomes dominated by a much negative term involving $rR$. Therefore, Alice’s freedom in the choice of measurements is negligible and thus she can increase her
gain by negligible amounts even with the entanglement attacks.

Although the proposed scheme can be implemented with currently available technologies, it is very sensitive to errors. So before methods for reducing decoherence, e.g., quantum error correcting codes [22] or decoherence-free subspaces [23] are realized with high performance, the proposed scheme seems to be impractical. And even if such methods are available, errors will remain to be generated with a small rate. Alice might insist that all errors are the residual ones and would not give him the R coins. Bob’s practical solution to this problem is that he aborts the whole protocol if the error rate is greater than the expected residual error rate, as suggested in the original work [1]. Despite these difficulties, however, it is worthwhile to have another application of the fundamental property that nonorthogonal qubits cannot be distinguished with certainty [3–8].

In conclusion, we have given another (remote) quantum gambling scheme that makes use of the fact that nonorthogonal states cannot be distinguished with certainty. In the proposed scheme, two participants Alice and Bob can be regarded as playing a game of making guesses on identities of quantum states that are in one of two given nonorthogonal states: if Bob makes a correct (an incorrect) guess on the identity of a quantum state that Alice has sent, he wins (loses). It was shown that the proposed scheme is secure against the nonentanglement attack. It could also be shown heuristically that the scheme is secure in the case of the entanglement attack.

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