Geometric picture for SLOCC classification of pure permutation symmetric three-qubit states

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Abstract

The quantum steering ellipsoid inscribed inside the Bloch sphere offers an elegant geometric visualization of two-qubit states shared between Alice and Bob. The set of Bloch vectors of Bob’s qubit, steered by Alice via all possible local measurements on her qubit, constitutes the steering ellipsoid. The steering ellipsoids are shown to be effective in capturing quantum correlation properties, such as monogamy, exhibited by entangled multiqubit systems. We focus here on the canonical ellipsoids of two-qubit states realized by incorporating optimal local filtering operations by Alice and Bob on their respective qubits. Based on these canonical forms, we show that the reduced two-qubit states drawn from pure entangled three-qubit permutation symmetric states, which are inequivalent under stochastic local operations and classical communication (SLOCC), carry distinct geometric signatures. We provide detailed analysis of the SLOCC canonical forms and the associated steering ellipsoids of the reduced two-qubit states extracted from entangled three-qubit pure symmetric states: We arrive at (i) a prolate spheroid centered at the origin of the Bloch sphere—with longest semiaxis along the z-direction (symmetry axis of the spheroid) equal to 1—in the case of pure symmetric three-qubit states constructed by permutation of 3 distinct spinors and (ii) an oblate spheroid centered at (0, 0, 1/2) inside the Bloch sphere, with fixed semiaxes lengths (1/√2, 1/√2, 1/2), when the three-qubit pure state is con-
structured via symmetrization of 2 distinct spinors. We also explore volume monogamy relations formulated in terms of the volumes of the steering ellipsoids of the SLOCC inequivalent pure entangled three-qubit symmetric states.

**Keywords** Permutation symmetric states · SLOCC canonical form · Majorana representation · Steering ellipsoid · Volume monogamy

### 1 Introduction

The Bloch sphere representation of a single qubit state provides a valuable geometric intuition in basic quantum information processing protocols. A natural generalization of the Bloch sphere picture to visualize two-qubit states, shared between Alice and Bob say, is given by the quantum steering ellipsoid [1–3]. In particular, the set of all points on the surface of the ellipsoid correspond to Bloch vectors to which Bob’s qubit can be steered to via all possible local measurements carried out on Alice’s qubit. Suppose that Alice and Bob carry out local filtering operations on their respective qubits so as to reduce the two-qubit state into its SLOCC canonical forms: One of the SLOCC canonical forms happens to be the Bell diagonal form of two-qubit state and the other a nondiagonal canonical form [4, 5]. The steering ellipsoids associated with the SLOCC canonical forms provide a much simpler geometric picture representing the set of all SLOCC equivalent two-qubit states. In this paper, we explore SLOCC canonical forms of the two-qubit reduced state, extracted from pure entangled three-qubit permutation symmetric state. We show that inequivalent SLOCC families of entangled three-qubit pure permutation symmetric states exhibit distinct canonical steering ellipsoids for the constituent two-qubit subsystem. In other words, SLOCC classes of pure symmetric three-qubit entangled states can be classified based entirely on their canonical steering ellipsoids. We also show that the SLOCC canonical forms of the steering ellipsoids capture monogamy properties of the pure three-qubit entangled state effectively and they mirror insightful information about two-qubit entanglement.

Contents of this paper are organized as follows: In Sec. 2, we give an outline of the SLOCC canonical structure [5] of an arbitrary two-qubit density matrix and the associated canonical steering ellipsoid inscribed inside the Bloch sphere. Section 3 is devoted to a brief review on SLOCC classification of pure permutation symmetric \(n\)-qubit states based on Majorana representation [6–8]. Explicit parametrization of two SLOCC inequivalent families of three-qubit entangled pure symmetric states [9] is given in Sec. 4. We obtain the SLOCC canonical forms and the associated steering ellipsoid of the reduced two-qubit states drawn from entangled three-qubit pure symmetric states in Sec. 5. Discussions on volume monogamy relation obeyed by the three-qubit pure symmetric states belonging to SLOCC inequivalent families and bounds imposed on the concurrence in terms of the obesity of the steering ellipsoid is elucidated in Sec. 6. Summary of our results is presented in Sec. 7.
2 The quantum steering ellipsoid of the two-qubit SLOCC canonical form

We outline the procedure developed in Ref. [5] to obtain SLOCC canonical form and the associated geometric visualization of an arbitrary two-qubit system.

Consider a two-qubit density matrix $\rho_{AB}$, expanded in the Hilbert–Schmidt basis $\{\sigma_\mu \otimes \sigma_\nu, \mu, \nu = 0, 1, 2, 3\}$:

$$\rho_{AB} = \frac{1}{4} \sum_{\mu, \nu=0}^{3} \Lambda_{\mu \nu} (\sigma_\mu \otimes \sigma_\nu),$$

(2.1)

$$\Lambda_{\mu \nu} = \text{Tr} \left[ \rho_{AB} (\sigma_\mu \otimes \sigma_\nu) \right] = \Lambda^*_{\mu \nu}$$

(2.2)

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2.3)

The expansion coefficients $\Lambda_{\mu \nu}$ in (2.1) may be arranged as a $4 \times 4$ real matrix

$$\Lambda = \begin{pmatrix} 1 & r_1 & r_2 & r_3 \\ s_1 & t_{11} & t_{12} & t_{13} \\ s_2 & t_{21} & t_{22} & t_{23} \\ s_3 & t_{31} & t_{32} & t_{33} \end{pmatrix},$$

(2.4)

Here $r = (r_1, r_2, r_3)^T$, $s = (s_1, s_2, s_3)^T$ are respective Bloch vectors of Alice, Bob’s qubits and $T = (t_{ij})$ is the correlation matrix, i.e.,

$$r_i = \Lambda_{i0} = \text{Tr} \left[ \rho_{AB} (\sigma_i \otimes \sigma_0) \right]$$

(2.5)

$$s_j = \Lambda_{0j} = \text{Tr} \left[ \rho_{AB} (\sigma_0 \otimes \sigma_j) \right]$$

(2.6)

$$t_{ij} = \Lambda_{ij} = \text{Tr} \left[ \rho_{AB} (\sigma_i \otimes \sigma_j) \right], \quad i, j = 1, 2, 3.$$  

(2.7)

- Under SLOCC transformation, the two-qubit density matrix

$$\rho_{AB} \longrightarrow \tilde{\rho}_{AB} = \frac{(A \otimes B) \rho_{AB} (A^\dagger \otimes B^\dagger)}{\text{Tr} \left[ \rho_{AB} (A^\dagger A \otimes B^\dagger B) \right]}$$

(2.8)

where $A, B \in \text{SL}(2, \mathbb{C})$ denote $2 \times 2$ complex matrices with unit determinant.

- Corresponding to the SLOCC transformation $\rho_{AB} \longrightarrow \tilde{\rho}_{AB}$ of the two-qubit state, the $4 \times 4$ real matrix $\Lambda$ transforms as

$$\Lambda \longrightarrow \tilde{\Lambda} = \frac{L_A \Lambda L_B^T}{(L_A \Lambda L_B^T)_{00}}$$

(2.9)
where \( L_A, L_B \in SO(3, 1) \) are \( 4 \times 4 \) proper orthochronous Lorentz transformation matrices \([10]\) corresponding to \( A, B \in SL(2, C) \), respectively, and the superscript \('T'\) denotes transpose operation.

- The \( 4 \times 4 \) real symmetric matrix \( \Omega = \Lambda G \Lambda^T \), where \( G = \text{diag} (1, -1, -1, -1) \) denotes the Lorentz metric, undergoes a \textit{Lorentz congruent transformation} under SLOCC (up to an overall factor) \([5]\):

\[
\Omega \rightarrow \tilde{\Omega}_A = \tilde{\Lambda} G \tilde{\Lambda}^T = L_A \Lambda L_B^T G L_B \Lambda^T L_A^T = L_A \Omega L_A^T \quad (2.10)
\]

where the defining property \([10]\) \( L^T G L = G \) of Lorentz transformation is used. The real symmetric matrix \( \Omega \) plays a significant role in obtaining SLOCC canonical form and the corresponding geometrical structure of the two-qubit density matrix \( \rho_{AB} \).

- It is seen that the matrix \( G \Omega \), constructed using the real matrix parametrization \( \Lambda \) of a two-qubit density matrix \( \rho_{AB} \) (see (2.1), (2.2)), exhibits the following properties \([5]\):

(i) The \( 4 \times 4 \) matrix \( G \Omega \) possesses \textit{non-negative} eigenvalues \( \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \).

(ii) Eigenvector \( X \) associated with the highest eigenvalue \( \lambda_0 \) of the matrix \( G \Omega \) obeys one of the following Lorentz invariant conditions:

\[
X^T G X > 0 \quad (2.11)
\]

or

\[
X^T G X = 0. \quad (2.12)
\]

Furthermore, the condition (2.12) is accompanied by the observation that the matrix \( G \Omega \) has only two eigenvalues \( \lambda_0, \lambda_1 \) obeying \( \lambda_0 \geq \lambda_1 \), both of which are doubly degenerate.

(iii) Suppose the eigenvector \( X \) satisfies the Lorentz invariant condition (2.11). Then there exists suitable SLOCC transformations \( A_1, B_1 \in SL(2, C) \) (with corresponding Lorentz transformations \( L_{A_1}, L_{B_1} \in SO(3, 1) \), respectively) such that the matrix \( \Omega \) assumes a diagonal form:

\[
\tilde{\Omega}_{I_c} = L_{A_{I_c}} \Omega L_{A_{I_c}}^T = \text{diag} (\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3). \quad (2.13)
\]

Correspondingly, one obtains the Lorentz canonical form for the real matrix \( \Lambda \):

\[
\Lambda \rightarrow \tilde{\Lambda}_{I_c} = \frac{L_{A_1} \Lambda L_{B_1}^T}{(L_{A_1} \Lambda L_{B_1}^T)_{00}} = \text{diag} \left( 1, \sqrt{\frac{\lambda_1}{\lambda_0}}, \sqrt{\frac{\lambda_2}{\lambda_0}}, \pm \sqrt{\frac{\lambda_3}{\lambda_0}} \right) \quad (2.14)
\]
where the sign $\pm$ is chosen depending on sign $(\det(\Lambda)) = \pm$ and

$$
\left( L_{A_1} \wedge L_{B_1}^T \right)_{00} = \sqrt{\lambda_0}.
$$

Consequently, the two-qubit density matrix

$$
\rho_{AB} \rightarrow \tilde{\rho}_{AB} = \frac{(A_{Ic} \otimes B_{Ic}) \rho_{AB} (A_{Ic}^\dagger \otimes B_{Ic}^\dagger)}{\text{Tr} \left[ \rho_{AB} (A_{Ic}^\dagger A_{Ic} \otimes B_{Ic}^\dagger B_{Ic}^\dagger) \right]}
$$

reduces to the Bell diagonal form

$$
\tilde{\rho}_{AB} = \frac{1}{4} \left( \sigma_0 \otimes \sigma_0 + \sum_{i=1,2} \sqrt{\frac{\lambda_i}{\lambda_0}} \sigma_i \otimes \sigma_i \pm \sqrt{\frac{\lambda_3}{\lambda_0}} \sigma_3 \otimes \sigma_3 \right)
$$

under SLOCC transformations.

(iv) Whenever the eigenvector $X$ obeys the condition (2.12) suitable SLOCC transformations $A_{IIc}, B_{IIc} \in SL(2, C)$ (associated Lorentz transformations denoted, respectively, by $L_{A_{IIc}}, L_{B_{IIc}} \in SO(3, 1)$) exist such that

$$
\tilde{\Omega}_{IIc} = L_{A_{IIc}} \Omega L_{A_{IIc}}^T = \begin{pmatrix}
\phi_0 & 0 & 0 & \phi_0 - \lambda_0 \\
0 & -\lambda_1 & 0 & 0 \\
0 & 0 & -\lambda_1 & 0 \\
\phi_0 - \lambda_0 & 0 & 0 & \phi_0 - 2 \lambda_0,
\end{pmatrix},

(2.17)
$$

where

$$
\phi_0 = \left( L_{A_{IIc}} \Omega L_{A_{IIc}}^T \right)_{00}.
$$

As a consequence, the real matrix $\Lambda$ and the two-qubit density matrix $\rho_{AB}$ assume the following canonical forms:

$$
\Lambda \rightarrow \tilde{\Lambda}_{IIc} = \frac{L_{A_{IIc}} \wedge L_{B_{IIc}}^T}{\left( L_{A_{IIc}} \wedge L_{B_{IIc}}^T \right)_{00}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a_1 & 0 & 0 \\
0 & 0 & -a_1 & 0 \\
1 - a_0 & 0 & 0 & a_0
\end{pmatrix}
$$

(2.19)

where

$$
\left( L_{A_{IIc}} \wedge L_{B_{IIc}}^T \right)_{00} = \sqrt{\phi_0}.
$$

$$
\rho_{AB} \rightarrow \tilde{\rho}_{AB} = \frac{(A_{IIc} \otimes B_{IIc}) \rho_{AB} (A_{IIc}^\dagger \otimes B_{IIc}^\dagger)}{\text{Tr} \left[ \rho_{AB} (A_{IIc}^\dagger A_{IIc} \otimes B_{IIc}^\dagger B_{IIc}^\dagger) \right]}
$$

(2.20)
\[
\begin{align*}
&= \frac{1}{4} \left[ \sigma_0 \otimes \sigma_0 + (1 - a_0) \sigma_3 \otimes \sigma_0 \\
&\quad + a_1 (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + a_0 \sigma_3 \otimes \sigma_3) \right] \\
&\quad + a_1 (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + a_0 \sigma_3 \otimes \sigma_3) \\
&\quad + a_1 (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + a_0 \sigma_3 \otimes \sigma_3) \\
&\quad + a_1 (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + a_0 \sigma_3 \otimes \sigma_3)
\end{align*}
\]

(2.21)

where

\[
\begin{align*}
a_0 &= \frac{\lambda_0}{\phi_0}, \\
a_1 &= \sqrt{\frac{\lambda_1}{\phi_0}}, \\
0 &\leq a_1^2 \leq a_0 \leq 1.
\end{align*}
\]

(2.22)

(v) Denote the set of all projective valued measurements (PVM) on Bob’s qubit by

\[
\left\{ E = \sum_{\mu=0}^{3} e_{\mu} \sigma_{\mu}, \; E \succ 0, \; e_0 = 1, \; e = (e_1, e_2, e_3), \; e \cdot e = e_1^2 + e_2^2 + e_3^2 = 1 \right\}.
\]

Local PVM on the Bob’s qubit leads to collapsed state of Alice’s qubit

\[
\rho_p = \frac{1}{p_E} \text{Tr}_B \left[ \rho_{AB} (\sigma_0 \otimes E) \right] = \frac{1}{2} \sum_{\mu} p_{\mu} \sigma_{\mu}, \; p_{\mu} = \frac{1}{p_E} \sum_{v} \Lambda_{\mu v} e_v,
\]

with probability

\[
p_E = \text{Tr} \left[ \rho_{AB} (\sigma_0 \otimes E) \right] = \sum_{\mu=0}^{3} \Lambda_{0 \mu} e_{\mu} = 1 + r \cdot e.
\]

Steered Bloch vectors of Alice’s qubit lie on and within the Bloch sphere, and they constitute an ellipsoidal surface defined by \( \mathcal{E}_{A|B} = \left\{ p = \frac{s + T^T e}{1 + r \cdot e} \right\} \).

Points inside the steering ellipsoid are accessible when Bob employs convex combinations of PVMs i.e., positive operator values measures (POVMs) to steer Alice’s qubit.

(vi) For the Lorentz canonical form \( \tilde{\mathcal{A}}_{L} \) (see (2.14)) of the two-qubit state \( \tilde{\rho}_{AB} \), one obtains

\[
p = \left( p_1 = \sqrt{\frac{\lambda_1}{\lambda_0}} e_1, \; p_2 = \sqrt{\frac{\lambda_2}{\lambda_0}} e_2, \; p_3 = \sqrt{\frac{\lambda_3}{\lambda_0}} e_3 \right)
\]

(2.23)

as steered Bloch points of Alice’s qubit, which obey the equation

\[
\frac{\lambda_0 p_1^2}{\lambda_1} + \frac{\lambda_0 p_2^2}{\lambda_2} + \frac{\lambda_0 p_3^2}{\lambda_3} = 1
\]

of an ellipsoid with semi-axes \( (\sqrt{\lambda_1/\lambda_0}, \sqrt{\lambda_2/\lambda_0}, \sqrt{\lambda_3/\lambda_0}) \) and center \( (0, 0, 0) \) inside the Bloch sphere. We refer to this as the canonical steering.
ellipsoid of the set of all two-qubit density matrices which are on the SLOCC orbit of the Bell diagonal state $\tilde{\rho}_{AB}^{Ic}$ (see (2.16)).

(vii) Corresponding to the second Lorentz canonical form $\tilde{\Lambda}_{1c}$ (see (2.19)) one obtains canonical steering spheroid inside the Bloch sphere with its semi-axes lengths $(a_1, a_1, a_0)$ and center $(0, 0, 1 - a_0)$. In other words a shifted spheroid, inscribed within the Bloch sphere, represents two-qubit states, which are SLOCC equivalent to $\tilde{\rho}_{AB}^{IIc}$ (see (2.21)). One obtains a prolate spheroid, centered at the origin of the Bloch sphere, with semiaxes length equal to 1 along z-axis when $a_0 = 1$ as the canonical steering ellipsoid offering geometric visualization of the set of all two-qubit states on the SLOCC orbit of $\tilde{\rho}_{AB}^{IIc}$ given by (2.21)).

3 SLOCC classification of pure permutation symmetric states

In 1932, Majorana [6] expressed an arbitrary pure symmetric state $|\Psi_{sym}\rangle$ of spin $j = \frac{N}{2}$ as a symmetrized combination of $N$ constituent spinors (qubits) as follows:

$$|\Psi_{sym}\rangle = \mathcal{N} \sum_{P} \hat{P} \{ |\alpha_1, \beta_1\rangle \otimes |\alpha_2, \beta_2\rangle \otimes \ldots \otimes |\alpha_N, \beta_N\rangle \}$$

where $|\alpha_k, \beta_k\rangle = \cos \frac{\beta_k}{2} |0\rangle + e^{i\alpha_k} \sin \frac{\beta_k}{2} |1\rangle$, $0 \leq \beta_k \leq \pi$, $0 \leq \alpha_k \leq 2\pi$ denote the constituent spinor states (states of the qubits). Here $\hat{P}$ denotes the set of all $N!$ permutations and $\mathcal{N}$ corresponds to an overall normalization factor. The parameters $(\alpha_k, \beta_k)$ characterizing the states $|\alpha_k, \beta_k\rangle$, $k = 1, 2, \ldots, N$ of the constituent qubits offer geometric representation of $N$-qubit pure symmetric state $|\Psi_{sym}\rangle$ in terms of $N$ points on the Bloch sphere. This is referred to as the Majorana representation of an arbitrary $N$-qubit symmetric state $|\Psi_{sym}\rangle$ in terms of the constituent qubit states $|\alpha_k, \beta_k\rangle$, $k = 1, 2, \ldots, N$.

Recall that two $N$-qubit states $|\Phi\rangle$, $|\chi\rangle$ can be obtained from one another by means of stochastic local operations and classical communications (SLOCC) if and only if there exists a local operation $A_1 \otimes A_2 \otimes \ldots \otimes A_N$, where $A_k$, $k = 1, 2, \ldots, N$ denote $2 \times 2$ complex invertible matrices, such that $|\Phi\rangle = A_1 \otimes A_2 \otimes \ldots \otimes A_N |\chi\rangle$. In the special case of permutation symmetric $N$-qubits, it is sufficient to search for identical local operations of the form $A^{\otimes N} = A \otimes A \otimes \ldots \otimes A$ to verify the SLOCC equivalence [7, 8]. The Majorana representation discussed above leads to a natural identification of different SLOCC inequivalent families, depending on the number and arrangement of the distinct qubits constituting the pure symmetric state [7, 8]. From early studies, it was identified that the three-qubit GHZ and W states are inequivalent under SLOCC [11]. Based on Majorana geometric representation, it is realized that GHZ class of three-qubit pure symmetric states are constituted by 3 distinct qubits, whereas W class of states consist of 2 distinct qubits. In Ref. [9], Meill and Meyer proposed a convenient parametrization, based on the Majorana geometric representation [7, 8], and evaluated the algebraically independent local unitary invariant quantities of these SLOCC inequivalent classes of pure entangled three-qubit symmetric states. More
recently, some of us [12] carried out a detailed analysis on the non-locality features of entangled pure symmetric three-qubit states, belonging to SLOCC inequivalent classes using the Meill–Meyer parametrization [9]. In the following sections, we study the explicit structure of steering ellipsoids of SLOCC canonical forms [4, 5] of the reduced two-qubit density matrices, extracted from entangled pure symmetric three-qubit states. We show that the two SLOCC inequivalent families of pure entangled three-qubit symmetric states exhibit distinct steering ellipsoids for their reduced two-qubit density matrices.

4 SLOCC classification of pure symmetric three-qubit states

A pure symmetric three-qubit state can be expressed in the Majorana geometric representation (see (3.1)) as

$$|\Psi_{\text{sym}}^{ABC}\rangle = N \sum_{P} \hat{P}(|\alpha_1, \beta_1\rangle \otimes |\alpha_2, \beta_2\rangle \otimes |\alpha_3, \beta_3\rangle)$$

(4.1)

where $|\alpha_k, \beta_k\rangle \equiv \cos \frac{\beta_k}{2} |0\rangle + e^{i\alpha_k} \sin \frac{\beta_k}{2} |1\rangle$, $k = 1, 2, 3$ are the states of the constituent qubits.

Depending on the number of distinct spinor states $|\alpha_k, \beta_k\rangle$, $k = 1, 2, 3$, there arise two different classes of entangled pure symmetric states of three qubits [7, 8, 12]:

1. Two distinct spinor class: $|\alpha_1, \beta_1\rangle = |\alpha_2, \beta_2\rangle \neq |\alpha_3, \beta_3\rangle$
2. Three distinct spinor class: $|\alpha_1, \beta_1\rangle \neq |\alpha_2, \beta_2\rangle \neq |\alpha_3, \beta_3\rangle$.

Note that if there is only one distinct spinor characterizing the state, i.e., when $|\alpha_1, \beta_1\rangle = |\alpha_2, \beta_2\rangle = |\alpha_3, \beta_3\rangle = |\alpha, \beta\rangle$, the pure state $|\Psi_{\text{sym}}^{ABC}\rangle \equiv |\Psi_{3,1}^{ABC}\rangle$ of (4.1) reduces to

$$|\Psi_{3,1}^{ABC}\rangle = |\alpha, \beta\rangle \otimes |\alpha, \beta\rangle \otimes |\alpha, \beta\rangle = |\alpha, \beta\rangle \otimes^3$$

(4.2)

and hence it is separable.

We denote the two distinct spinor class of pure three-qubit symmetric states by $\mathcal{D}_{3,2}$, the three distinct spinor class by $\mathcal{D}_{3,3}$ and the one distinct spinor class (unentangled) by $\mathcal{D}_{3,1}$. These three classes are evidently inequivalent under SLOCC [7, 8]. Three-qubit W state given by

$$|W\rangle = N \sum_{P} \hat{P}(|0\rangle \otimes |0\rangle \otimes |1\rangle)$$

$$= \frac{1}{\sqrt{3}} (|0_A 0_B 1_C\rangle + |0_A 1_B 0_C\rangle + |1_A 0_B 0_C\rangle)$$

(4.3)

and its obverse state

$$|\overline{W}\rangle = N \sum_{P} \hat{P}(|1\rangle \otimes |1\rangle \otimes |0\rangle)$$
belong to the class $D_{3,2}$. Three-qubit GHZ state is a well-known example of permutation symmetric state belonging to the SLOCC class $D_{3,3}$:

$$|\text{GHZ}\rangle = \mathcal{N} \sum_{P} \hat{P}(|\phi\rangle_1 \otimes |\phi\rangle_2 \otimes |\phi\rangle_3)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle_0 0\rangle_0 C\rangle + |1\rangle_0 1\rangle_0 C\rangle)$$ (4.5)

where $|\phi\rangle_P = \frac{1}{\sqrt{2}}(|0\rangle + \omega^p |1\rangle), \ p = 1, 2, 3$ are the constituent qubit states; $\omega$ denotes the cube root of unity.

An arbitrary three-qubit pure state $|\Psi_{ABC}\rangle$ belonging to the two distinct spinor class $D_{3,2}$ takes the following simple form under local unitary operations [8, 9]:

$$|\Psi_{ABC}\rangle = \mathcal{N}_{3,2} \sum_{P} \hat{P}(|0\rangle \otimes |0\rangle \otimes |\beta\rangle\rangle, \ |\beta\rangle = \cos \frac{\beta}{2} |0\rangle + \sin \frac{\beta}{2} |1\rangle, \ 0 < \beta \leq \pi$$

$$= \frac{1}{\sqrt{2 + \cos \beta}} \left( \sqrt{3} \cos \frac{\beta}{2} |0\rangle_0 0\rangle_0 C\rangle + \sin \frac{\beta}{2} |W\rangle \right).$$ (4.6)

In other words, the states belonging to the two distinct spinor class $D_{3,2}$ are characterized by one real parameter $\beta$.

The reduced two-qubit density matrix extracted from the pure state $|\Psi_{ABC}\rangle$ is expressed (in the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$) by

$$\rho^{(2\text{-qubit})}_{3,2} = \text{Tr}_{A} \langle \Psi_{ABC}\rangle_{3,2} |\Psi_{ABC}\rangle_{3,2} = \text{Tr}_{B} \langle \Psi_{ABC}\rangle_{3,2} |\Psi_{ABC}\rangle_{3,2} = \text{Tr}_{C} \langle \Psi_{ABC}\rangle_{3,2} |\Psi_{ABC}\rangle_{3,2}$$

$$= \begin{pmatrix}
1 - 2A_{3,2} & B_{3,2} & B_{3,2} & 0 \\
B_{3,2} & A_{3,2} & A_{3,2} & 0 \\
B_{3,2} & A_{3,2} & A_{3,2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$ (4.7)

where

$$A_{3,2} = \frac{1 - \cos \beta}{6 (2 + \cos \beta)}, \ B_{3,2} = \frac{\sin \beta}{2 (2 + \cos \beta)}. \quad (4.8)$$

Similarly, any arbitrary three-qubit pure symmetric state $|\Psi_{ABC}\rangle$ belonging to Majorana three distinct spinor class $D_{3,3}$ can be reduced—under local unitary operations—to the simple form given by [9, 12]

$$|\Psi_{ABC}\rangle = \mathcal{N}_{3,3} \langle 0 \rangle^{\otimes 3} + ye^{i\alpha} |\beta\rangle^{\otimes 3}, \ |\beta\rangle = \cos \frac{\beta}{2} |0\rangle + \sin \frac{\beta}{2} |1\rangle,$$

$$0 < y \leq 1, \ 0 \leq \alpha \leq 2\pi, \ 0 < \beta \leq \pi,$$
\[ \mathcal{N}_{3, 3} \left[ \left( 1 + y e^{i\alpha} \cos \frac{\beta}{2} \right) |0_A 0_B 0_C\rangle + y e^{i\alpha} \sin \frac{\beta}{2} |1_A 1_B 1_C\rangle + \frac{\sqrt{3}}{2} y \sin \beta \left( \cos \frac{\beta}{2} |W\rangle + \sin \frac{\beta}{2} |\bar{W}\rangle \right) \right] \]

(4.9)

and is characterized by three real parameters \( y, \alpha \) and \( \beta \).

The two-qubit density matrix drawn from the pure state \( |\Psi_{3, 3}^{ABC}\rangle \) (see (4.9)) is given explicitly by (in the computational basis \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \})

\[ \rho_{3, 3}^{(2-\text{qubit})} = \text{Tr}_A |\Psi_{3, 3}^{ABC}\rangle \langle \Psi_{3, 3}^{ABC}| = \text{Tr}_B |\Psi_{3, 3}^{ABC}\rangle \langle \Psi_{3, 3}^{ABC}| = \text{Tr}_C |\Psi_{3, 3}^{ABC}\rangle \langle \Psi_{3, 3}^{ABC}| \]

\[ = \begin{pmatrix} A_{3, 3} & B_{3, 3} & B_{3, 3} & C_{3, 3} \\ B_{3, 3}^* & D_{3, 3} & D_{3, 3} & E_{3, 3} \\ B_{3, 3}^* & D_{3, 3} & D_{3, 3} & E_{3, 3} \\ C_{3, 3}^* & E_{3, 3}^* & E_{3, 3}^* & F_{3, 3} \end{pmatrix} \]

(4.10)

where

\[ A_{3, 3} = \frac{1 + y^2 \cos^2 \frac{\beta}{2} + 2y \cos \alpha \cos^3 \frac{\beta}{2}}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}}, \]

\[ B_{3, 3} = \frac{y \cos^2 \frac{\beta}{2} \sin \frac{\beta}{2} \left( e^{-i\alpha} + y \cos \frac{\beta}{2} \right)}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}}, \]

\[ C_{3, 3} = \frac{y \sin^2 \frac{\beta}{2} \cos \frac{\beta}{2} \left( e^{-i\alpha} + y \cos \frac{\beta}{2} \right)}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}}, \]

\[ D_{3, 3} = \frac{y^2 \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2}}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}}, \]

\[ E_{3, 3} = \frac{y^2 \sin^2 \frac{\beta}{2} \cos \frac{\beta}{2}}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}}, \]

\[ F_{3, 3} = 1 - A_{3, 3} - 2D_{3, 3}. \]

(4.11)

We make use of the explicit form of the two-qubit density matrices \( \rho_{3, 2}^{(2-\text{qubit})} \) (see (4.7), (4.8)) and \( \rho_{3, 3}^{(2-\text{qubit})} \) (see (4.10), (4.11)) to compute the SLOCC canonical forms and the associated geometrical representations following the detailed analysis given in Ref. [5].

5 Canonical steering ellipsoids of \( \rho_{3, 2}^{(2-\text{qubit})} \) and \( \rho_{3, 3}^{(2-\text{qubit})} \)

In this section, we focus our attention on determining the SLOCC canonical forms and the associated quantum steering ellipsoids of the two-qubit density matrices \( \rho_{3, 2}^{(2-\text{qubit})} \) (see (4.7), (4.8)) and \( \rho_{3, 3}^{(2-\text{qubit})} \) (see (4.10), (4.11)) which are obtained by tracing out one of the qubits from the respective three-qubit pure states \( |\Psi_{3, 2}^{ABC}\rangle, |\Psi_{3, 3}^{ABC}\rangle \) given by (4.6), (4.9).
5.1 SLOCC Canonical form of $\rho_{3, 2}^{(2-qubit)}$

Expressing the density matrix $\rho_{3, 2}^{(2-qubit)}$ given by (4.7), (4.8) in the Hilbert–Schmidt basis $\{\sigma_\mu \otimes \sigma_\nu, \mu, \nu = 0, 1, 2, 3\}$, we obtain

$$\rho_{3, 2}^{(2-qubit)} = \frac{1}{4} \sum_{\mu, \nu=0}^{3} \Lambda_{\mu \nu}^{(3, 2)} (\sigma_\mu \otimes \sigma_\nu),$$

$$\Lambda_{\mu \nu}^{(3, 2)} = \text{Tr} \left[ \rho_{3, 2}^{(2-qubit)} (\sigma_\mu \otimes \sigma_\nu) \right],$$

where the $4 \times 4$ real matrix $\Lambda^{(3, 2)}$ has the explicit structure

$$\Lambda^{(3, 2)} = \begin{pmatrix}
1 & \sin \beta \frac{2}{2+\cos \beta} & 0 & 5+4 \cos \beta \\
\sin \beta \frac{2}{2+\cos \beta} & \frac{3(2+\cos \beta)}{3(2+\cos \beta)} & 0 & \frac{1-\cos \beta}{4+5 \cos \beta} \\
0 & 0 & 1-\cos \beta & \frac{5+4 \cos \beta}{2+\cos \beta} \\
\frac{5+4 \cos \beta}{3(2+\cos \beta)} & \sin \beta \frac{2}{2+\cos \beta} & \frac{1-\cos \beta}{4+5 \cos \beta} & \frac{3(2+\cos \beta)}{3(2+\cos \beta)}
\end{pmatrix} = \left( \Lambda^{(3, 2)} \right)^T. \quad (5.1)$$

Following the procedure outlined in Sec. 2, we evaluate the $4 \times 4$ symmetric matrix $\Omega^{(3, 2)}$ from $\Lambda^{(3, 2)}$:

$$\Omega^{(3, 2)} = \Lambda^{(3, 2)} G \left( \Lambda^{(3, 2)} \right)^T = \Lambda^{(3, 2)} G \Lambda^{(3, 2)} = \begin{pmatrix}
2 u(\beta) & 0 & 0 & u(\beta) \\
0 & -u(\beta) & 0 & 0 \\
u(\beta) & 0 & -u(\beta) & 0 \\
u(\beta) & 0 & 0 & 0
\end{pmatrix}, \quad u(\beta) = \left[ \frac{1-\cos \beta}{3(2+\cos \beta)} \right]^2. \quad (5.2)$$

The matrix $\Omega^{(3, 2)}$ is already in the canonical form given by (2.17) with

$$\lambda_0 = u(\beta) = \lambda_1, \quad \text{and} \quad \phi_0 = 2 u(\beta). \quad (5.3)$$

Indeed, the eigenvalues of the matrix $G \Omega^{(3, 2)}$, $G = \text{diag} (1, -1, -1, -1)$ are four-fold degenerate:

$$\lambda_0 = u(\beta) = \left[ \frac{1-\cos \beta}{3(2+\cos \beta)} \right]^2 = \lambda_1 = \lambda_2 = \lambda_3 \quad (5.4)$$

and one of the eigenvectors $X^{(3, 2)} = (1, 0, 0, -1)^T$ obeys the Lorentz invariant condition (2.12) confirming our observation. Substituting (5.3) in (2.19), (2.22), we find that the Lorentz canonical form $\tilde{\Lambda}^{(3, 2)}$ of the real matrix $\Lambda^{(3, 2)}$ is independent of the state parameter $\beta$.
Fig. 1 (Color online) Steering spheroid inscribed within the Bloch sphere representing the Lorentz canonical form $\tilde{\Lambda}^{(3, 2)}$ (see (5.5)) geometrically. Lengths of the semi-axes of the shifted spheroid, which is centered at $(0, 0, 1/2)$, are given by $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2})$.

\[
\tilde{\Lambda}^{(3, 2)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix}.
\] (5.5)

Thus, we conclude that the reduced two-qubit density matrices extracted from the pure symmetric three-qubit state $|\Psi^{ABC}_{3, 2}\rangle$ given by (4.6) can be reduced to the following SLOCC canonical form (see (2.21))

\[
\tilde{\rho}^{(2\text{-qubit})}_{3, 2} = \frac{1}{4} \left[ \sigma_0 \otimes \sigma_0 + \frac{1}{2} \sigma_3 \otimes \sigma_0 + \frac{1}{\sqrt{2}} (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + \frac{1}{2} \sigma_3 \otimes \sigma_3 \right].
\] (5.6)

The quantum steering ellipsoid associated with the two-qubit state (5.6) is a spheroid centered at $(0, 0, 1/2)$ inside the Bloch sphere, with fixed semiaxes lengths $(1/\sqrt{2}, 1/\sqrt{2}, 1/2)$ (see Fig. 1). It is interesting to note that the SLOCC canonical form is independent of the state parameter $\beta$. 
5.2 SLOCC Canonical form of $\rho_{3, 3}^{(2-qubit)}$

From the explicit form of the density matrix $\rho_{3, 3}^{(2-qubit)}$ given by (4.10), (4.11), we evaluate the elements $\Lambda_{\mu \nu}^{(3, 3)} = \text{Tr} \left[ \rho_{3, 3}^{(2-qubit)} (\sigma_\mu \otimes \sigma_\nu) \right]$, $\mu, \nu = 0, 1, 2, 3$ of the $4 \times 4$ real matrix $\Lambda^{(3, 3)}$:

$$\Lambda_{00}^{(3, 3)} = 1, \quad \Lambda_{01}^{(3, 3)} = A(y, \alpha, \beta) y \sin \beta \left( y + \cos \alpha \cos \frac{\beta}{2} \right) = \Lambda_{10}^{(3, 3)},$$
$$\Lambda_{02}^{(3, 3)} = A(y, \alpha, \beta) y \sin \alpha \sin \beta \cos \frac{\beta}{2} = \Lambda_{20}^{(3, 3)},$$
$$\Lambda_{03}^{(3, 3)} = A(y, \alpha, \beta) \left( 1 + 2 y \cos \alpha \cos^3 \frac{\beta}{2} + y^2 \cos \beta \right) = \Lambda_{30}^{(3, 3)},$$
$$\Lambda_{11}^{(3, 3)} = A(y, \alpha, \beta) y \sin \beta \frac{\beta}{2} \left( \cos \alpha + 2 y \cos \frac{\beta}{2} \right),$$
$$\Lambda_{12}^{(3, 3)} = A(y, \alpha, \beta) y \sin \alpha \sin \beta \frac{\beta}{2} = \Lambda_{21}^{(3, 3)},$$
$$\Lambda_{13}^{(3, 3)} = A(y, \alpha, \beta) y \sin \beta \left( \cos \alpha \cos \frac{\beta}{2} + y \cos \beta \right) = \Lambda_{31}^{(3, 3)},$$
$$\Lambda_{22}^{(3, 3)} = -A(y, \alpha, \beta) y \cos \alpha \sin \beta \frac{\beta}{2},$$
$$\Lambda_{23}^{(3, 3)} = A(y, \alpha, \beta) \sin \alpha \sin \beta \cos \frac{\beta}{2} = \Lambda_{32}^{(3, 3)},$$
$$\Lambda_{33}^{(3, 3)} = \frac{A(y, \alpha, \beta)}{2} \left[ 2 + y^2 + 4 y \cos \alpha \cos^3 \frac{\beta}{2} + y^2 \cos 2 \beta \right],$$

where we have denoted

$$A(y, \alpha, \beta) = \frac{1}{1 + y^2 + 2 y \cos \alpha \cos^3 \frac{\beta}{2}}. \quad (5.7)$$

The real symmetric $4 \times 4$ matrix $\Omega^{(3, 3)}$ constructed from $\Lambda^{(3, 3)}$ is given by

$$\Omega^{(3, 3)} = \Lambda^{(3, 3)} G \left( \Lambda^{(3, 3)} \right)^T$$
$$= B(y, \alpha, \beta) \begin{pmatrix}
3 + \cos \beta & \sin \beta & 0 & 1 + \cos \beta \\
\sin \beta & -(1 + \cos \beta) & 0 & \sin \beta \\
0 & 0 & -(1 + \cos \beta) & 0 \\
1 + \cos \beta & \sin \beta & 0 & -(1 - \cos \beta)
\end{pmatrix} \quad (5.8)$$
where

\[ B(y, \alpha, \beta) = \frac{y^2(1 - \cos \beta)^2}{2 \left(1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}\right)^2}. \]  

(5.9)

Eigenvalues of the matrix \( G \Omega^{(3, 3)} \) are 2-fold degenerate and are given by

\[ \lambda_0 = 2B(y, \alpha, \beta) = \frac{y^2(1 - \cos \beta)^2}{\left(1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}\right)^2}, \]  

(5.10)

\[ \lambda_1 = B(y, \alpha, \beta) (1 + \cos \beta) = \frac{y^2 \sin^2 \beta (1 - \cos \beta)^2}{2 \left(1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}\right)^2}. \]  

(5.11)

Except in the case of \( \beta = \pi \), the eigenvector \( X^{(3, 3)} \) of the matrix \( G \Omega^{(3, 3)} \) associated with the highest eigenvalue \( \lambda_0 \) satisfies the condition \( (X^{(3, 3)})^T G X^{(3, 3)} = 0 \), confirming that the matrix \( \Omega^{(3, 3)} \) can be reduced to the form (2.17) under SLOCC transformation. The Lorentz transformation matrix \( L^{(3, 3)}_{A_2} \), which takes \( \Omega^{(3, 3)} \) to its canonical form \( \tilde{\Omega}^{(3, 3)} \), is then explicitly constructed following the detailed approach of Ref. [5]:

\[
L^{(3, 3)}_{A_2} = \begin{pmatrix}
\frac{3+\cos \beta}{2 \sin \beta} & \frac{1+\cos \beta}{\sin \beta} & 0 & \frac{1+\cos \beta}{2 \sin \beta} \\
-1 & -1 & 0 & -1 \\
0 & 0 & -1 & 0 \\
-1 & -1 & 0 & \frac{1-\cos \beta}{\sin \beta} \\
\end{pmatrix}, \quad \beta \neq \pi.
\]  

(5.12)

We have

\[
\tilde{\Omega}^{(3, 3)} = L^{(3, 3)}_{A_2} \Omega^{(3, 3)} \left(L^{(3, 3)}_{A_2}\right)^T
= 2B(y, \alpha, \beta)
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\cos^2 \frac{\beta}{2} & 0 & 0 \\
0 & 0 & -\cos^2 \frac{\beta}{2} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \beta \neq \pi.
\]  

(5.13)

Note that

\[
\phi_0 = \left(L^{(3, 3)}_{A_2} \Omega^{(3, 3)} \left(L^{(3, 3)}_{A_2}\right)^T\right)_{00} = 2B(y, \alpha, \beta) = \lambda_0,
\]  

(5.14)
and hence, the canonical form $\tilde{\Omega}^{(3,3)}$ is diagonal (see (2.17)). Consequently, the Lorentz canonical form $\tilde{\Lambda}^{(3,3)}$ (see (2.19)) too is diagonal

$$\tilde{\Lambda}^{(3,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \frac{\beta}{2} & 0 & 0 \\ 0 & 0 & 0 & -\cos \frac{\beta}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta \neq \pi$$

(5.15)

and SLOCC canonical form $\tilde{\rho}^{(2\text{-qubit})}_{3,3}$ of the reduced two-qubit density matrix obtained by tracing one of the qubits of the pure symmetric three-qubit state $|\Psi^{ABC}_{3,3} \rangle$ (see (4.9)) reduces to the following simple form:

$$\tilde{\rho}^{(2\text{-qubit})}_{3,3} = \frac{1}{4} \left[ \sigma_0 \otimes \sigma_0 + \cos \frac{\beta}{2} (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + \sigma_3 \otimes \sigma_3 \right].$$

(5.16)

The Lorentz canonical form (5.15) is represented by a prolate spheroid centered at the origin $(0, 0, 0)$ of the Bloch sphere, with longest semiaxis $a_0$ along the $z$-direction (symmetry axis of the spheroid) equal to 1 and radius of the circular cross section, perpendicular to the symmetry axis, given by $a_1 = \cos \frac{\beta}{2}$. The steering spheroid associated with the set of all two-qubit states, which lie on the SLOCC orbit of $\tilde{\rho}^{(2\text{-qubit})}_{3,3}$ (see (5.16)), is depicted in Fig. 2 (for the value $\beta = \pi/2$ of the state parameter).

When the parameters $\beta = \pi$, $\alpha = 0$, $y = 1$, the state $|\Psi^{ABC}_{3,3} \rangle$ (see (4.9)) corresponds to the three-qubit GHZ state (see (4.5)). It can be readily seen from (6.27) that the matrix $\Omega^{(3,3)}$ is diagonal:

$$\Omega^{(3,3)} = \text{diag} (1, 0, 0, 1).$$

From (2.14), it is clear that the Lorentz canonical form of the matrix $\Lambda^{(3,3)}$ is given by

$$\tilde{\Lambda}^{(3,3)} = \text{diag} (1, 0, 0, 1)$$

This corresponds geometrically to a line joining the north and the south poles of the Bloch sphere.

### 6 Volume monogamy relations and entanglement

Monogamy relations restrict shareability of quantum correlations in a multipartite state. They find potential applications in ensuring security in quantum key distribution [13, 14]. Milne et. al. [2] introduced a geometrically intuitive monogamy relation for the volumes of the steering ellipsoids representing the two-qubit subsystems of a pure three-qubit state, which is stronger than the well-known Coffman–Kundu–Wootters monogamy relation [15]. In this section, we explore how volume monogamy
Fig. 2 (Color online) Prolate spheroid centered at the origin of the Bloch sphere representing the SLOCC canonical form of the two-qubit density matrix $\tilde{\rho}_{3,2}^{(2\text{-qubit})}$ given in (5.16). Semiaxes lengths of the spheroid are given by \((\cos \frac{\beta}{2}, \cos \frac{\beta}{2}, 1)\). Here we have chosen $\beta = \pi/2$.

relation [2] imposes limits on the volumes of the quantum steering ellipsoids representing the two-qubit subsystems $\rho_{3,2}^{(2\text{-qubit})} = \text{Tr}_A |\Psi_{3,2}^{ABC}\rangle\langle \Psi_{3,2}^{ABC}|$, $\rho_{3,3}^{(2\text{-qubit})} = \text{Tr}_A |\Psi_{3,3}^{ABC}\rangle\langle \Psi_{3,3}^{ABC}|$ (see (4.7), (4.10)) of the SLOCC inequivalent families $\mathcal{D}_{3,2}, \mathcal{D}_{3,3}$ of pure symmetric three-qubit states. We also study the upper bounds imposed on the concurrence of reduced two-qubit states of pure three-qubit symmetric states and their SLOCC canonical forms discussed in Sec. 5.

The volume of the quantum steering ellipsoid $\mathcal{E}_{A|B} = \{ p = \frac{s + r^T e}{1 + r \cdot e} \}$ of a two-qubit state $\rho_{AB}$ (see (2.1)) is given by [1]

$$V_{B|A} = \left( \frac{4\pi}{3} \right) \frac{|\det \Lambda|}{(1 - r^2)^2}, \quad (6.1)$$

where $r^2 = r \cdot r = r_1^2 + r_2^2 + r_3^2$ (see 2.5). As the steering ellipsoid is constrained to lie within the Bloch sphere, $V_{B|A} \leq V_{\text{unit}} = (4\pi/3)$.

Volume of a steering ellipsoid captures several interesting quantum correlation features [1–3], which we list below:

(a) Volumes $V_{B|A}, V_{A|B}$ of the quantum steering ellipsoids $\mathcal{E}_{B|A}$ and $\mathcal{E}_{A|B}$ are related by [1]

$$\frac{V_{A|B}}{(1 - r^2)^2} = \frac{V_{B|A}}{(1 - s^2)^2}. \quad (6.2)$$
(b) An upper bound for the concurrence $C(\rho_{AB})$ is placed in terms of the obesity $O(\rho_{AB}) = |\det \Lambda|^{1/4}$ of the quantum steering ellipsoid [2]:

$$C(\rho_{AB}) \leq O(\rho_{AB}) = |\det \Lambda|^{1/4}. \tag{6.3}$$

Furthermore, if $\rho_{AB} \rightarrow \tilde{\rho}_{AB} = (A \otimes B)\rho_{AB} (A^\dagger \otimes B^\dagger)/(\text{Tr}(A^\dagger A \otimes B^\dagger B)\rho_{AB})$, $A, B \in SL(2, C)$, it follows that [2]

$$\frac{O(\rho_{AB})}{C(\rho_{AB})} = \frac{O(\tilde{\rho}_{AB})}{C(\tilde{\rho}_{AB})} \tag{6.4}$$

(c) While steering ellipsoids of a two-qubit state lie inside the Bloch sphere, not all ellipsoids contained within the Bloch sphere correspond to bonafide states. Physical states impose limits on the volume of steering ellipsoid for a given center. The extremal steering ellipsoids are those with largest volume for a fixed center $r_0$ that can fit inside the Bloch sphere. Oblate spheroid of volume $V = (4\pi/3)(1-r_0)^2$, with major semi-axes $a = \sqrt{1-r_0}$ and radially oriented minor semi-axis $c = (1-r_0), r_0 \neq 1$ with center $r_0 = (0, 0, r_0)$, corresponds to extremal ellipsoid of entangled two-qubit state [2].

(d) The two-qubit Werner state on the separable entangled boundary has volume of the steering ellipsoid $V_{\text{Werner}} = V = (4\pi/81) = V_\star$, which happens to be the volume of largest possible tetrahedron that can be inscribed inside Bloch sphere [1]. Separable two-qubit systems do not exhibit volume of their steering ellipsoids larger than $V_\star = (4\pi/81)$. Thus, $V > V_\star$ provides an intuitive geometric condition for entanglement in two-qubit states [1].

- Let Alice, Bob, and Charlie share a pure three qubit state. Suppose that Bob performs all possible local measurements on his qubit so as to steer the states possessed by Alice and Charlie. The volumes $V_{A|B}$, $V_{C|B}$ of the resulting steering ellipsoids obey monogamy relation given by [1]

$$\sqrt{V_{A|B}} + \sqrt{V_{C|B}} \leq \sqrt{\frac{4\pi}{3}}. \tag{6.5}$$

### 6.1 Volume monogamy relations governing the states $|\Psi^{ABC}_{3, 2}\rangle$ of the SLOCC class $\mathcal{D}_{3, 2}$

Let Alice, Bob, and Charlie share a pure entangled three-qubit symmetric state $|\Psi^{ABC}_{3, 2}\rangle$ (see (4.6)) of the SLOCC class $\mathcal{D}_{3, 2}$. Bob’s local measurements generate identical steering ellipsoids $E_{A|B} = E_{C|B} = E_{3, 2}$ of Alice and Charlie because the reduced two-qubit density operators drawn from the pure three qubit state $|\Psi^{ABC}_{3, 2}\rangle$ are all same i.e., $\rho_{AB} = \rho_{BC} = \rho_{AC} = \rho_{3, 2}^{(2-\text{qubit})}$ (see (4.7)). Evidently the volumes $V_{A|B}$, $V_{C|B}$ of the steering ellipsoids are equal. Denoting $V_{3, 2} = V_{A|B} = V_{C|B}$, the monogamy
relation (6.5) takes the form,

$$\sqrt{\frac{3 \ V_{3,2}}{\pi}} \leq 1$$  \hspace{1cm} (6.6)$$

for pure symmetric three-qubit states $|\Psi_{3,2}^{ABC}\rangle$ (see (4.6)) of the SLOCC class $D_{3,2}$.

From (6.1), we have

$$V_{3,2} = \left(\frac{4\pi}{3}\right) \frac{|\det \Lambda^{(3,2)}|}{(1 - r^2)^2},$$  \hspace{1cm} (6.7)$$

where $\Lambda^{(3,2)}$ is given in (5.2) and

$$r_1 = \frac{\sin \beta}{2 + \cos \beta}, \quad r_2 = 0, \quad r_3 = \frac{5 + 4 \cos \beta}{3(2 + \cos \beta)}$$  \hspace{1cm} (6.8)$$

Under suitable SLOCC the state $\tilde{\rho}_{3,2}^{(2\text{-qubit})}$ and the associated real matrix $\tilde{\Lambda}^{(3,2)}$ get transformed to their respective canonical forms (see (5.5), (5.6))

$$\tilde{\rho}_{3,2}^{(2\text{-qubit})} = \frac{(A \otimes B) \rho_{3,2}^{(2\text{-qubit})} (A^{\dagger} \otimes B^{\dagger})}{\text{Tr}[\rho_{3,2}^{(2\text{-qubit})} (A^{\dagger} A \otimes B^{\dagger} B)]} = \frac{1}{4} \left[ \sigma_0 \otimes \sigma_0 + \frac{1}{2} \sigma_3 \otimes \sigma_0 + \frac{1}{\sqrt{2}} (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + \frac{1}{2} \sigma_3 \otimes \sigma_3 \right]$$

and

$$\Lambda^{(3,2)} \rightarrow \tilde{\Lambda}^{(3,2)} = \frac{L_A \Lambda L_B^T}{(L_A \Lambda L_B^T)_{00}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}. $$

where (see (2.20), (5.3), (5.4))

$$\left(L_A \Lambda L_B^T\right)_{00} = \sqrt{\phi_0} = \sqrt{2} \left[ \frac{1 - \cos \beta}{3(2 + \cos \beta)} \right].$$  \hspace{1cm} (6.9)$$

Using the property $\det L_A = \det L_B = 1$ of orthochronous proper Lorentz transformations [10] and substituting $|\det \tilde{\Lambda}_{3,2}| = 1/4$, we obtain

$$|\det \tilde{\Lambda}_{3,2}| = \frac{1}{4} = |\det L_A| |\det L_B| \left| \det \left( \frac{\Lambda^{(3,2)}}{\sqrt{\phi_0}} \right) \right| = \frac{|\det \Lambda^{(3,2)}|}{\phi_0^2},$$  \hspace{1cm} (6.10)$$
Substituting (6.8), (6.9) and (6.10) in (6.7), we obtain a simple form for the volume of the corresponding steering ellipsoid associated with the two-qubit state $\rho_{3,2}^{(2-\text{qubit})}$:

$$V_{3,2} = \left(\frac{4\pi}{3}\right) \frac{\phi_0^2}{4(1-r^2)^2} = \frac{\pi}{3}. \quad (6.11)$$

Evidently the volume monogamy relation (6.6) is saturated by the family of three-qubit pure entangled symmetric states $|\Psi_{3,2}^{ABC}\rangle$ belonging to the SLOCC class $D_{3,2}$. Milne et al. have shown that the volume monogamy relation (6.5) is saturated if and only if the steering ellipsoid has maximum volume for a given center. Thus, the reduced two-qubit states of the $D_{3,2}$ class of three-qubit pure states have maximal volume steering ellipsoids. Coincidentally, this feature is also reflected in the canonical steering ellipsoid of $\tilde{\rho}_{3,2}^{(2-\text{qubit})}$ (see (5.6)), which has its center at $r_0 = (0, 0, 1/2)$, semi-major axes $a = b = \frac{1}{\sqrt{2}}$, and semi-minor axis $c = 1/2$.

### 6.2 Volume monogamy relations governing the states $|\Psi_{3,3}^{ABC}\rangle$ of the SLOCC class $D_{3,3}$

Suppose that Alice, Bob, and Charlie possess a qubit each of the permutation symmetric three-qubit pure entangled state $|\Psi_{3,3}^{ABC}\rangle$ (see (4.9)). As the reduced two-qubit states are all identical i.e., $\rho_{AB} = \rho_{BC} = \rho_{AC} = \rho_{3,3}^{(2-\text{qubit})}$ (see (4.10)), the steering ellipsoids of Alice and Charlie generated by Bob’s local measurements are also identical, i.e., $E_{B|A} = E_{C|A} = E_{3,3}$. We thus have $V_{A|B} = V_{C|B} = V_{3,3}$ and the volume monogamy relation (6.5) for pure symmetric three-qubit states $|\Psi_{3,3}^{ABC}\rangle$ (see (4.9)) of the SLOCC class $D_{3,3}$ can be expressed as

$$\sqrt{\frac{3V_{3,3}}{\pi}} \leq 1. \quad (6.12)$$

We evaluate the volume of the steering ellipsoid corresponding to the state $\rho_{3,3}^{(2-\text{qubit})}$ (see (4.10))

$$V_{3,3} = \left(\frac{4\pi}{3}\right) \frac{|\det \Lambda_{3,3}^{(3,3)}|}{(1-r^2)^2} \quad (6.13)$$

where (see (5.7))

$$r_1 = \frac{y \sin \beta \left(y + \cos \alpha \cos \frac{\beta}{2}\right)}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}}, \quad r_2 = \frac{y \sin \alpha \sin \beta \cos \frac{\beta}{2}}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}},$$

$$r_3 = \frac{1 + 2y \cos \alpha \cos^3 \frac{\beta}{2} + y^2 \cos \beta}{1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2}} \quad (6.14)$$
Fig. 3 (Color online) Left-hand side \( \sqrt{\frac{3V_{3,3}}{\pi}} = \frac{2 \cos \frac{\beta}{2}}{(1+\cos^2 \frac{\beta}{2})} \) (see (6.19)) of the volume monogamy relation (6.12) governing the set of all three-qubit pure symmetric states \(|\Psi_{ABC}^{3,3}\rangle\) belonging to the SLOCC family \(D_{3,3}\) by expressing it in terms of the volume \(V_{3,3} = (4\pi/3)\cos^2(\beta/2)\) of the canonical steering ellipsoid specified by its radially aligned longest semi-major axis \(c = 1\) and semi-minor axes \(a = b = \cos(\beta/2)\) (see (5.15)).

Substituting \(|\det \Lambda_{3,3}| = \cos^2(\beta/2)\) for the determinant of the Lorentz canonical form \(\tilde{\Lambda}_{3,3}\) given by (5.15) and simplifying, we obtain

\[
\cos^2 \frac{\beta}{2} = |\det \tilde{\Lambda}_{3,3}| = \left| \det \left( \frac{L_A \Lambda^{(3,3)} L_B^T}{(L_A \Lambda^{(3,3)} L_B^T)_{00}} \right) \right| = \phi_0^{-2} \left| \det \Lambda^{(3,3)} \right|, (6.15)
\]

where (see (2.20), (5.10), (5.14)).

\[
\phi_0 = \frac{y^2(1 - \cos \beta)^2}{(1 + y^2 + 2y \cos \alpha \cos^3 \frac{\beta}{2})^2} \quad (6.16)
\]

Thus,

\[
\left| \det \Lambda^{(3,3)} \right| = \phi_0^2 \cos^2 \frac{\beta}{2}. \quad (6.17)
\]

After simplification (substituting (6.14), (6.17) in (6.13) and using (6.16)), we obtain

\[
\frac{3V_{3,3}}{\pi} = \frac{4 \phi_0^2 \cos^2 \frac{\beta}{2}}{(1 + \cos^2 \frac{\beta}{2})^2} = \frac{4 \cos^2 \frac{\beta}{2}}{(1 + \cos^2 \frac{\beta}{2})^2}. \quad (6.18)
\]

Thus, the volume monogamy relation (6.12) gets reduced to the simple form

\[
\frac{2 \cos \frac{\beta}{2}}{(1 + \cos^2 \frac{\beta}{2})} \leq 1 \quad (6.19)
\]

which is evidently obeyed by the set of all three-qubit pure symmetric states \(|\Psi_{ABC}^{3,3}\rangle\) (see Fig. 3).
6.3 Connection between obesity of the steering ellipsoid and concurrence

The concurrence of a two qubit state $\rho_{AB}$ is given by [16]

$$C(\rho_{AB}) = \max(0, \mu_1 - \mu_2 - \mu_3 - \mu_4), \quad (6.20)$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are the square roots of the eigenvalues, in decreasing order, of the matrix $\rho_{AB} (\sigma_2 \otimes \sigma_2) \rho_{AB}^* (\sigma_2 \otimes \sigma_2)$.

We evaluate the concurrence for the SLOCC canonical state $\tilde{\rho}_{3,2}^{(2-\text{qubit})}$ given by (5.6) to obtain

$$C(\tilde{\rho}_{3,2}^{(2-\text{qubit})}) = \frac{1}{\sqrt{2}}. \quad (6.21)$$

The obesity of the canonical steering ellipsoid of $\tilde{\rho}_{3,2}^{(2-\text{qubit})}$ is given by (see (5.5))

$$O(\tilde{\rho}_{3,2}^{(2-\text{qubit})}) = |\text{det} \tilde{\Lambda}^{(3,2)}|^{1/4} = \frac{1}{\sqrt{2}}. \quad (6.22)$$

We also obtain the obesity $O(\rho_{3,2}^{(2-\text{qubit})})$ using (6.9), (6.10):

$$O(\rho_{3,2}^{(2-\text{qubit})}) = |\text{det} \Lambda^{(3,2)}|^{1/4} = \left[ \frac{1 - \cos \beta}{3(2 + \cos \beta)} \right]. \quad (6.23)$$

Based on the relation (6.4), we compute the concurrence for the two-qubit state $\rho_{3,2}^{(2-\text{qubit})}$ as

$$C(\rho_{3,2}^{(2-\text{qubit})}) = \frac{1 - \cos \beta}{3(2 + \cos \beta)} \quad (6.24)$$

which matches perfectly with the expression derived previously in [9].

It is straightforward to check (see 6.23), (6.24)) that the inequality (6.3) gets saturated in $\rho_{3,2}^{(2-\text{qubit})}$, in accordance with the result that the maximum volume steering ellipsoids maximize concurrence [2]. For the Bell diagonal two-qubit state $\tilde{\rho}_{3,3}^{(2-\text{qubit})}$ (see (5.16), the concurrence is readily evaluated to be

$$C(\tilde{\rho}_{3,3}^{(2-\text{qubit})}) = \frac{\beta}{2} \quad (6.25)$$

and the obesity of the canonical steering ellipsoid of $\tilde{\rho}_{3,3}^{(2-\text{qubit})}$ (see (5.15)) is given by

$$O(\tilde{\rho}_{3,3}^{(2-\text{qubit})}) = |\text{det} \tilde{\Lambda}_{3,3}|^{1/4} = \sqrt{\frac{\cos \beta}{2}} \quad (6.26)$$

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Using (6.15), (6.16), we compute the obesity

\[ \mathcal{O}(\rho_{3,3}^{(2\text{-qubit})}) = \left| \det \Lambda^{(3,3)} \right|^{1/4} = \frac{y (1 - \cos \beta) \sqrt{\cos \beta^2}}{(1 + y^2 + 2y \cos \alpha \cos^3(\beta/2))} \]  

(6.27)

and obtain the concurrence for \( \rho_{3,3}^{(2\text{-qubit})} \) based on (6.4)

\[ C(\rho_{3,3}^{(2\text{-qubit})}) = \frac{y \sin \beta \sin \beta^2}{(1 + y^2 + 2y \cos \alpha \cos^3(\beta/2))}, \]  

(6.28)

which is clearly in agreement with the expression for concurrence given by Meill and Meyer [9]. It is easy to see that the limiting condition (6.3)) imposed on the concurrence (6.28) by the obesity (6.27) of the steering ellipsoid reduces to a simple trigonometric relation \( \sqrt{\cos \beta^2} \leq \cos \beta^2 \).

7 Summary

We have shown that SLOCC inequivalent classes of pure entangled three-qubit symmetric states come with distinct geometric visualization in terms of canonical steering spheroid inscribed within the Bloch sphere. We have evaluated explicit analytical structure for the SLOCC canonical forms associated with the reduced two-qubit states—obtained by tracing out one of the qubits of entangled three-qubit pure symmetric states. Our analysis yields a clear geometric insight: Reduced two-qubit states of a pure permutation symmetric three-qubit state \(|\Psi_{3,2}^{ABC}\rangle\), which is constructed by symmetrizing 2 distinct qubits—is represented by an extremal steering spheroid centered at (0, 0, 1/2) inside the Bloch sphere, with fixed semiaxes lengths (1/√2, 1/√2, 1/2). On the other hand, the two-qubit density matrices drawn from the three-qubit state \(|\Psi_{3,3}^{ABC}\rangle\) constituted by permutation of 3 distinct qubits is geometrically viewed as a prolate spheroid centered at the origin of the Bloch sphere—with length of the semi-major axis along the symmetry direction of the spheroid equal to 1. The canonical steering ellipsoids are shown to offer insightful information on different aspects of quantum correlations. We have illustrated volume monogamy relations obeyed by the three-qubit pure symmetric states belonging to the SLOCC inequivalent classes \(\mathcal{D}_{3,2}, \mathcal{D}_{3,3}\). Furthermore, connection between the obesity of the steering ellipsoid and concurrence of the two-qubit subsystems of three-qubit pure symmetric states is elucidated.

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Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.
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