On the dispersionless Davey-Stewartson system: Hamiltonian vector fields Lax pair and relevant nonlinear Riemann-Hilbert problem for dDS-II system

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Abstract

In this paper, the semiclassical limit of Davey-Stewartson systems are studied. It shows that these dispersionless limited integrable systems of hydrodynamic type, which are defined as dDS (dispersionless Davey-Stewartson) systems, are arisen from the commutation condition of Lax pairs of one-parameter vector fields. The relevant nonlinear Riemann-Hilbert problems with some symmetries for the dDS-II system are also constructed. This kind of Riemann-Hilbert problems are meaningful for applying the formal inverse scattering transform method, recently developed by Manakov and Santini, to study the dDS-II system.

1 Introduction

In 1974 Davey and Stewartson [1] used a multi-scale analysis to derive a coupled system of nonlinear partial differential equations describing the evolution of a three dimensional wave packet in water of a finite depth. The general DS(Davey-Stewartson) systems (parametrized by $\varepsilon > 0$) for a complex (wave-amplitude) field $q(x,y,t)$ and a real (mean-flow) field $\phi(x,y,t)$ are given by

\begin{align}
\varepsilon q_t + \frac{\varepsilon^2}{2}(q_{xx} + \sigma^2 q_{yy}) + \delta q \phi &= 0, \\
\sigma^2 \phi_{yy} - \phi_{xx} + (|q|^2)_{xx} + \sigma^2 (|q|^2)_{yy} &= 0,
\end{align}

(1)
in which the variables $t, x, y \in \mathbb{R}$. We shall refer to (1) with $\sigma = 1$ the DS-I (Davey-Stewartson-I) system and with $\sigma = i$ the DS-II (Davey-Stewartson-II) system. We also refer to (1) with $\delta = 1$ the focusing case and with $\delta = -1$ the defocusing case respectively. The Davey-Stewartson systems (1), as prototype example of classical integrable systems, have been extensively studied and many important results are obtained [2]: $N$-line soliton solutions [3–8]; localized exponentially decaying solitons [9]; an infinite dimensional symmetry group, in fact this involves an infinite dimensional Lie algebra with a Kac-Moody-Virasoro loop structure [10, 11]; similarity reductions to the second and fourth Painlevé equations [12]; a Bäcklund transformation and Painlevé property [13]; an infinite number of commuting symmetries and conservation quantities, a recursion operator and bi-Hamiltonian structure [14, 15].

As we know, the IST (inverse scattering transform) method is a powerful method to identify and solve classes of integrable nonlinear PDEs and integrable dynamical systems. In 1967, Gardner, Greene, Kruskal and Miura pioneered this new method of mathematical physics. They solved the Cauchy problem of the celebrated KdV (Korteweg-de Vries) equation $u_t + uu_x + u_{xxx} = 0$, a model equation for the description of weakly nonlinear, weakly dispersive (1 + 1)-dimensional waves, arising in many physical contexts, by making use of the ideas of direct and inverse scattering for the stationary Schrödinger operator $\hat{L} = -\partial_x^2 + u(x, t)$ [16]. In 1968, Lax generalized these ideas, showing in particular that integrable nonlinear PDEs arise as commutation condition of pairs of linear partial differential operators (now called ”Lax pairs”) [17]. In 1972, Zakharov and Shabat showed that the IST method is not a particular method just for the KdV equation, but it is also applicable to the NLS (nonlinear Schrödinger) equation $iq_t + q_{xx} + \delta|q|^2u = 0$ [18], another important model equation in the description of the amplitude modulation of weakly nonlinear and strongly dispersive waves in nature. Next, in 1973, Ablowitz, Kaup, Newell and Segur developed a method to find a rather wide class of nonlinear evolution equations solvable by these techniques [19, 20], including the sine-Gordon equation $u_{xt} = \sin u$. A localized disturbance evolving according to an integrable nonlinear PDEs associated with the classical IST method evolves into a number of solitons (elastically interacting solitary waves arising from the balance of nonlinearity and dispersion) plus dispersive wave trains, and soliton behavior has been observed in several physical contexts. For these reasons these classical integrable systems are often called “soliton PDEs”. Physically relevant integrable generaliza-
tions of the KdV and NLS equations to (2 + 1) dimensions, respectively the KP (Kadomtsev-Petviashvili) and DS (Davey-Stewartson) equations have also been constructed and solved [21]. To the best of my knowledge, a further generalization to higher dimensions leads to non-integrable systems; indeed, a part from few exceptional cases, it is not possible to construct nonlinear PDEs integrable by the classical IST method in more than 2 + 1 dimensions.

Besides the soliton PDEs, there is another important class of integrable PDEs: the "integrable PDEs of hydrodynamic type", often called "dispersionless PDEs". In some cases, these PDEs are the dispersionless (or semi-classical) limits of integrable soliton equations, and they often arise in various problems of Mathematical Physics and are intensively studied in the recent literature.

Integrable nonlinear PDEs of hydrodynamic type are equivalent to the commutation condition of vector fields Lax pairs and, for this reason, they can be in an arbitrary number of dimensions (as observed long ago by Zakharov and Shabat [22]). In addition, since they do not contain dispersive or dissipative terms, an initial localized disturbance can evolve towards a gradient catastrophe. Therefore such integrable systems can give a good chance to obtain analytical results on the study of wave breaking phenomena in multidimensions.

Manakov and Santini have introduced, at a formal level, a novel IST method for solving integrable PDEs of hydrodynamic type, based on the construction of a direct and inverse spectral problem for one-parameter families of vector fields [23, 24]. The particular nature of vector fields introduces important novelties with respect to the classical IST theory:

1) Since the space of the (zero energy) eigenfunctions of the vector fields is a ring (not only the sum, but also the product of eigenfunctions is an eigenfunction), the direct and inverse spectral theory in essentially nonlinear, as opposed to those associated with the classical IST, in which the space of eigenfunctions is linear. In particular, the inverse problem can be formulated as a nonlinear Riemann-Hilbert problem on a suitable contour of the complex plane of the spectral parameter [23, 24].

2) If the vector field Lax pair consists of Hamiltonian vector fields, the space of the eigenfunctions is not only a ring, but also a Lie algebra, whose commutator is given by the Poisson bracket.

This novel IST method has been applied to solve the Cauchy problem for
The dispersionless Kadomtsev-Petviashvili (dKP) equation
\[ u_{xt} + u_{yy} + (uu_x)_x = 0, \quad u = u(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \quad (2) \]
which is the dispersionless limit of the KP equation, and describes the evolution in nature of small amplitude, nearly one-dimensional waves with negligible dispersion and dissipation. It appeared first in the description of unsteady motion in transonic flow \[25] and in the nonlinear acoustics of confined beams \[26].

The dKP equation arises as the commutation condition \[^{\hat{L}_1, \hat{L}_2} = 0\] of the pair of Hamiltonian vector fields:
\[ \hat{L}_1 = \partial_y + \lambda \partial_x - u_x \partial_\lambda = \partial_y + \{ H_2, \cdot \}_x, \]
\[ \hat{L}_2 = \partial_t + (\lambda^2 + u) \partial_x - \lambda u_x \partial_\lambda = \partial_t + \{ H_3, \cdot \}_x, \quad (3) \]
where \( H_2, H_3 \) are the Hamiltonians
\[ H_2 = \frac{\lambda^2}{2} + u, \]
\[ H_3 = \frac{\lambda^3}{3} + \lambda u - \partial_x^{-1}(u_y), \quad (4) \]
The dKP equation is the first nontrivial member (corresponding to \( m = 2, \ n = 3, \ t_2 = y, \ t_3 = t \)) of the following hierarchy of integrable PDEs:
\[ H_{n-t} - H_{m-t} + \{ H_m, H_n \}_x = 0, \quad H_n \equiv \frac{1}{n} (f^n)_{\geq 0}, \quad (5) \]
where \( f \) is the formal zero energy eigenfunction of the operator \( \hat{L}_1 \) (the solution of equation \( \hat{L}_1 f = 0 \)) with a polar singularity in a neighborhood of \( \lambda = \infty \), with the expansion
\[ f = \lambda + \frac{u}{\lambda} - \frac{\partial_x^{-1}(u_y)}{\lambda^2} + \sum_{j \geq 3, j \in \mathbb{Z}} \frac{q_j}{\lambda^j}, \quad |\lambda| \gg 1, \quad (6) \]
and \((h)_{\geq 0}\) is the non-negative part of the Laurent expansion of function \( h \) \[27\].

In \[28\], by using the IST method developed in \[24\], Manakov and Santini showed, in particular, that a localized initial disturbance evolving according to dKP breaks at finite time and, if small, in the longtime regime, when the solution is described by the formula:
\[ u = \frac{1}{\sqrt{t}} G(x + \frac{y^2}{4t} - 2ut, \frac{y}{2t}) + o(\frac{1}{\sqrt{t}}), \quad t \gg 1, \quad (7) \]
where the spectral function $G$ is connected to the initial datum $u(x, y, 0)$ via the direct spectral transform developed in [24]. According to formula (7), a small and localised initial datum evolves into a parabolic wave front described by equation $x + \frac{x^2}{4t} = \tilde{x}$, in the space-time region:

$$x = \tilde{x} + v_1 t, \quad y = v_2 t, \quad \tilde{x} - 2ut, v_1, v_2 = O(1), \quad v_2 \neq 0, \quad t \gg 1, \quad (8)$$

and breaks in a point of the parabola. Indeed, since the argument of $G$ in (7) depends on $u$ itself through the combination $x + \frac{x^2}{4t} - 2ut$, in full analogy with the case of the general solution $v = v_0(x - vt)$ of the one-dimensional analogue of dKP, the celebrated Hopf equation $v_t + vv_x = 0$, the breaking mechanism for dKP is very similar to that described by the Hopf equation, and the details of such a $(2 + 1)$ dimensional wave breaking have been investigated analytically in a quite explicit manner. Therefore the dKP equation (2) can be viewed as a prototype physical model equation in the description of wave breaking phenomena in $(2 + 1)$ dimensions, exactly as the Hopf equation is the prototype model equation in the description of wave breaking phenomena in $(1 + 1)$ dimensions.

The novel IST method has been used to solve also the Cauchy problem for the second heavenly equation of Plébanski [23,29]:

$$\theta_{xt} - \theta_{yz} + \theta_{xx}\theta_{yy} - \theta_{xy}^2 = 0, \quad (9)$$

an exact 4-dimensional reduction of the Einstein equations of General Relativity [30], and the d2DT(dispersionless two-dimensional Toda) equation [31]:

$$\phi_{\zeta_1\zeta_2} = (e^{\phi_1})_t, \quad \phi = \phi(\zeta_1, \zeta_2, t), \quad (10)$$

whose elliptic and hyperbolic versions are relevant, describing, for instance, integrable H-spaces (heavens) [32,33] and integrable Einstein-Weyl geometries [34,36]. String equations solutions of it are relevant in the ideal Hele-Shaw problem [37,39].

This novel IST method has recently been applied to one distinguished class of equations, the so-called Dunajski hierarchy [40]. The Dunajski hierarchy is a basic example of hierarchy of dispersionless integrable PDEs, including the heavenly and the Manakov-Santini hierarchies as particular cases, the first flow of this hierarchy with the divergence free constraint is the well known Dunajski equation characterizing a general anti-self-dual conformal structure in neutral signature [41]...

\[ \theta_{xt} - \theta_{yz} + \theta_{xx}\theta_{yy} - \theta_{xy}^2 = f, \quad (11a) \]
\[ f_{xt} - f_{yz} + \theta_{yy}f_{xx} + \theta_{xx}f_{yy} - 2\theta_{xy}f_{xy} = 0. \]  

(11b)

As the dKP equation is the integrable physically relevant generalization of the Hopf equation \( u_t + uu_x = 0 \) in (2+1) dimensions, the dDS (dispersionless Davey-Stewartson) system is the integrable physically relevant generalization of the dNLS (dispersionless nonlinear Schrödinger) system of equations

\[
\begin{align*}
  u_t + (uv)_x &= 0, & \quad (12a) \\
  v_t + vv_x - \delta u_x &= 0, & \quad \delta = \pm 1, & \quad (12b)
\end{align*}
\]

the dispersionless (semi-classical) limit \( \varepsilon \to 0 \) of the \( \varepsilon \)-dependent NLS equation

\[
\begin{align*}
  i\varepsilon q_t + \frac{\varepsilon^2}{2} q_{xx} + \delta|q|^2q &= 0, & \quad \delta = \pm 1, & \quad (13)
\end{align*}
\]

where \( u \) is the square modulus and \( v \) is the wave number of \( q \):

\[
q = \sqrt{u} \exp(i\frac{\partial^{-1}(v)}{\varepsilon}), \quad u, v \in \mathbb{R}, \quad u > 0. \quad (14)
\]

In the defocusing \( \delta = -1 \) case, the system is hyperbolic and describes an isentropic gas evolving towards a gradient catastrophe at finite time \( t \) of the type described by the Hopf equation \( u_t + uu_x = 0 \), where the singularity of hodograph solution takes place on a curve; in the focusing \( \delta = 1 \) case, the system is elliptic and evolves towards an elliptic umbilical singularity \[42\]. Therefore the dNLS system is clearly richer than the Hopf equation \( u_t + uu_x = 0 \). It is reasonable to believe that the picture could be even richer in the (2 + 1) dimensional case of the dDS system \[22\]. For this reason, we hope to apply the novel IST method to study the dDS system in a analytical way, to study the wave breaking mechanism for dDS system and to identify the type of singularities.

In fact, The defocusing DS-II system has been shown in numerical experiments to exhibit behavior in the semiclassical limit that qualitatively resembles that of its (1+1) dimensional reduction, the defocusing NLS equation \[43\]. In 2017, Assainova, Klein, Mclaughlin and Miller consider the direct spectral transform for the defocusing DS-II system for smooth initial data in the semiclassical limit. In this paper they show that the direct spectral
transform involves a singularly-perturbed elliptic Dirac system in two dimensions. They introduce a WKB-type method for this problem, prove that it makes sense formally for sufficiently large values of the spectral parameter by controlling the solution of an associated nonlinear eikonal problem, and they give numerical evidence that the method is accurate for such parameter in the semiclassical limit [44].

This paper is organized as follows: first, we construct, following [45], the dDS system and its Hamilton-Jacobi Lax pair by taking the dispersionless (semiclassical) limit of the DS system (1) and of its Lax pair formulation; second, we derive its vector field Lax pair formulation, the basic mathematical tool of the Manakov-Santini theory; third, we construct a nonlinear Riemann-Hilbert problem for dDS-II system with some symmetries.

We plan to use the IST method for vector fields to study the Cauchy problem and study how the dynamics give rise to a wave breaking and then study analytically the nature of such wave breaking with the identification of the type of singularities. The results in this paper are the necessary background for these studies.

2 Semiclassical limit of Davey-Stewartson system and Hamiltonian vector fields Lax pair

2.1 Dispersionless Davey-Stewartson systems and relevant Hamilton-Jacobi type equations

In this section, we consider the following \( \varepsilon \)-parametrized DS(Davey-Stewartson) system

\[
\begin{align*}
    i\varepsilon q_t + \varepsilon^2 (q_{zz} + q_{\hat{z}\hat{z}}) + \delta q \phi &= 0, \quad (15a) \\
    \phi_{zz} - \frac{1}{2} \left[ (|q|^2)_{zz} + (|q|^2)_{\hat{z}\hat{z}} \right] &= 0, \quad (15b)
\end{align*}
\]

which is equivalent to the system (1) by introducing the independent variables:

\[
z = x + \sigma y, \quad \hat{z} = x - \sigma y.
\]
This nonlinear DS system (15) is equivalent to the compatibility condition for the following linear system

\[
\begin{align*}
\varepsilon \begin{pmatrix} \psi_z \\ \varphi_z \end{pmatrix} &= M \begin{pmatrix} \psi \\ \varphi \end{pmatrix}, \quad (17a) \\
\varepsilon \begin{pmatrix} \psi_t \\ \varphi_t \end{pmatrix} &= T \begin{pmatrix} \psi \\ \varphi \end{pmatrix}, \quad (17b)
\end{align*}
\]

where the two matrices \(M\) and \(T\) read as follows

\[
M = \begin{pmatrix} 0 & -\frac{1}{2\delta \bar{q}} \\ \frac{\varepsilon}{2\delta \bar{q}} & 0 \end{pmatrix}, \quad T = i \begin{pmatrix} \partial_{\bar{z}}^2 + \frac{1}{2}\delta W & \frac{1}{2\sigma}(q\partial_z - q_z) \\ \frac{\sigma}{2}(q\partial_{\bar{z}} - \bar{q}_z) & -\partial_{\bar{z}}^2 - \frac{1}{2}\delta V \end{pmatrix}, \quad (18)
\]

in which \(\bar{q}\) is the complex conjugate of \(q\) and

\[
W = (|q|^2)_z, \quad V = (|q|^2)_{\bar{z}}, \quad \phi = \frac{1}{2}(W + V). \quad (19)
\]

The well-known interpretation of the dispersionless (semiclassical) limit is that afforded by the quantum hydrodynamic system that one can derive from (15) by following the ideas of Madelung [46]. Let us assume only that \(|q| > 0\) and represent \(q\) in the "oscillatory wavepacket" or standard WKB form

\[
q = \sqrt{u} \exp\left(i \frac{S}{\varepsilon}\right), \quad u > 0, \quad S \in \mathbb{R}, \quad (20)
\]

where \(u(z, \bar{z}, t) > 0\) is a real amplitude and \(S(z, \bar{z}, t)\) is a real phase. Inserting this form into (15), dividing out the common phase factor and separating it into real and imaginary parts gives the following system governing the three real-valued fields \(u, S\) and \(\phi\)

\[
\begin{align*}
&u_t + 2(uS_z)_z + 2(uS_{\bar{z}})_{\bar{z}} = 0, \quad (21a) \\
&S_t + S_z^2 + S_{\bar{z}}^2 - \delta \phi = \varepsilon^2 \frac{(\sqrt{u})_{zz} + (\sqrt{u})_{\bar{z}\bar{z}}}{\sqrt{u}}, \quad (21b) \\
&\phi_{z\bar{z}} - \frac{1}{2}(u_{zz} + u_{\bar{z}\bar{z}}) = 0. \quad (21c)
\end{align*}
\]
In the dispersionless (semiclassical) limit $\varepsilon \to 0$, the $\varepsilon$-dependent DS system \[21\] reduces to the following dDS (dispersionless Davey-Stewartson) system

\begin{align*}
  u_t + 2(uS_z)_z + 2(uS_{\hat{z}})_{\hat{z}} &= 0, \\
  S_t + S_z^2 + S_{\hat{z}}^2 - \delta \phi &= 0, \\
  \phi_{z\hat{z}} - \frac{1}{2}(u_{zz} + u_{\hat{z}\hat{z}}) &= 0.
\end{align*}

(22a), (22b), (22c)

**Remark 1.** The uniform expression \[22\] include the dDS-I and dDS-II systems. In fact, by considering the equations \[22\] with the origin real-valued independent variables $(x, y, t)$, i.e., $u(z, \hat{z}, t) = u(x + \sigma y, x - \sigma y, t) = S(z, \hat{z}, t) = S(x + \sigma y, x - \sigma y, t)$ (in this paper, we also use $u(x, y, t), S(x, y, t)$ to express the above two functions), we obtain the following two different equations

\begin{align*}
  u_t + (uS_x)_x + (uS_y)_y &= 0, \\
  S_t + \frac{1}{2}(S_x^2 + \sigma^2 S_{\sigma x}^2) - \delta \phi &= 0, \\
  \sigma^2 \phi_{yy} - \phi_{xx} + u_{xx} + \sigma^2 u_{yy} &= 0,
\end{align*}

(23a), (23b), (23c)

in which $\sigma = 1$ corresponds the dDS-I system and $\sigma = i$ corresponds the dDS-II system.

To construct vector fields Lax pair for the dDS system \[22\], by considering the requirement of phases equivalence in the system \[17\], we follow [45] and write the eigenfunctions $\psi$ and $\varphi$ of \[17\] in the form

\begin{align*}
  \psi = i\psi_0 \exp(\frac{i}{\epsilon}f), \quad \varphi = \varphi_0 \exp(\frac{i}{\epsilon}g), \quad S = f - g.
\end{align*}

(24)

Substituting these expressions into the linear system \[17\], taking account of the terms of $O(\varepsilon^0)$, Eliminating $\psi_0$ and $\varphi_0$ from this system, one obtains the following system of three nonlinear equations of Hamilton-Jacobi type

\begin{align*}
  4f_{z\hat{z}} - \delta u &= 0,
\end{align*}

(25a)
\[ f_t + \left( f_z^2 + f_{\bar{z}}^2 - 2f_zg_z - \frac{\delta}{2} W \right) = 0, \quad (25b) \]

\[ g_t - \left( g_z^2 + g_{\bar{z}}^2 - 2f_{\bar{z}}g_{\bar{z}} - \frac{\delta}{2} V \right) = 0, \quad (25c) \]

where \( S = f - g, \ W_z = u_{\bar{z}}, \ V_{\bar{z}} = u_z. \) Since the dDS system \((22)\) arises as the compatibility condition of equations \((25)\), the system \((25)\) should be interpreted as the nonlinear Lax formulation of Hamilton-Jacobi type of dDS system.

### 2.2 Hamiltonian vector fields Lax pair for dispersionless Davey-Stewartson systems

In order to apply the Manakov-Santini method, it is important to construct an alternative vector field formulation of such a Lax pair. In this subsection, we will construct this kind of Hamiltonian vector fields Lax pair for dDS (dispersionless Davey-Stewartson) system.

As we know, there are some basic facts in the classical Hamiltonian mechanics in the real framework. Suppose

\[ \tilde{L} = \partial_t - \{ \tilde{H}, \cdot \}_{(p,x)}, \quad t, x, p \in \mathbb{R}, \quad (26) \]

is a Hamiltonian vector field, with the Hamiltonian

\[ \tilde{H} = \tilde{H}(t, x, p). \quad (27) \]

Any eigenfunction \( \tilde{\Psi}(t, x, p) \) of \( \tilde{L} \), i.e.,

\[ \tilde{L}\tilde{\Psi} = 0, \quad t, x, p \in \mathbb{R}, \quad \tilde{\Psi} \in \mathbb{R}, \quad (28) \]

is exactly a conservation law for the associated dynamical system

\[ \frac{dx}{dt} = \tilde{H}_p(t, x, p), \quad \frac{dp}{dt} = -\tilde{H}_x(t, x, p). \quad (29) \]

Hamiltonian systems can also be studied using the Hamilton-Jacobi equation

\[ \frac{\partial \tilde{K}}{\partial t} = \tilde{H}(t, x, \frac{\partial \tilde{K}}{\partial x}). \quad (30) \]
In classical Hamiltonian mechanics, above expressions give three equivalent formulations of the problem. These facts, especially the connection between vector fields and Hamiltonian-Jacobi equations, give us a way to construct vector fields Lax pair based on some Hamiltonian-Jacobi equations, even if the variables are complex.

In the following, the Poisson bracket defines as \( \{ A, B \}_{(\lambda,z)} = A_\lambda B_z - A_z B_\lambda \). Let’s consider the Hamiltonian vector field

\[
L = \partial_s - \{ H, \cdot \}_{(\lambda,z)}, \quad s, z, \lambda \in \mathbb{C},
\]

with the Hamiltonian

\[
H = H(s, z, \lambda).
\]

For any eigenfunction \( \Psi(s, z, \lambda) \) of \( L \), i.e., \( L\Psi = 0 \), by considering the level sets

\[
\Psi(s, z, \lambda) = k
\]

and solving this relation at the regular points, i.e., \( \Psi_\lambda(s, z, k) \neq 0 \)

we can define a complex value function \( \Lambda(s, z, k) \). The following lemma shows the connection between vector field (complex variables) and Hamiltonian-Jacobi equation is similar as Hamilton mechanics.

**Lemma 1.** The complex-valued function \( \Lambda(s, z, k) \), connected with the Hamiltonian vector field \( L \) by the above formula, satisfies the equation

\[
\Lambda_s = \frac{\partial}{\partial z} [H(s, z, \Lambda(s, z, k))].
\]

If we define \( \Lambda(s, z, k) \equiv K_z(s, z) \), then \( K(s, z) \) satisfies the Hamilton-Jacobi equation

\[
K_s = H(s, z, K_z).
\]

**Proof.** The condition \( \Psi(s, z, \lambda) \) is an eigenfunction of vector field \( L \) reads as

\[
L\Psi = \Psi_s - H_\lambda \Psi_z + H_z \Psi_\lambda = 0.
\]
And the relation \((33)-(34)\) read as

\[ \Psi(s, z, \Lambda(s, z, k)) = k, \]  

\((38)\)

from this one obtains

\[ 0 = \frac{\partial k}{\partial s} = \Psi_s + \Psi_\lambda \Lambda_s, \]

\[ 0 = \frac{\partial k}{\partial z} = \Psi_z + \Psi_\lambda \Lambda_z. \]  

\((39)\)

Then one obtains the following relation by considering both \((37)-(39)\)

\[ 0 = \Psi_s + \Psi_\lambda \Lambda_s - H_\lambda (\Psi_z + \Psi_\lambda \Lambda_z) \]

\[ = (\Lambda_s - H_\lambda \Lambda_z - H_z)\Psi_\lambda \]  

\((40)\)

Taking into account that \(\Psi_\lambda \neq 0\), one obtains

\[ \Lambda_s = H_\lambda \Lambda_z + H_z \]

\[ = \frac{\partial}{\partial z} [H(s, z, \Lambda(s, z, k))]. \]  

\((41)\)

By taking advantage of Lemma 1, we have the following main result in this section.

**Proposition 1.** The dDS system \((22)\) arises as the commutation condition

\[ [L_1, L_2] = 0 \]  

\((42)\)

of the following one-parameter Hamiltonian vector fields Lax pair

\[ L_1 = \partial_{\bar{z}} - \{H_1, \cdot\}_{(\lambda, z)}, \]  

\((43a)\)

\[ L_2 = \partial_t - \{H_2, \cdot\}_{(\lambda, z)}, \]  

\((43b)\)

with the Hamiltonian functions

\[ H_1(\lambda) = S_{\bar{z}} + \frac{\delta u}{4\lambda}, \]  

\((44a)\)
\[ H_2(\lambda) = \lambda^2 - 2S_\lambda + \left( \frac{\delta}{2} W - S_\lambda^2 \right) - \frac{\delta u S_\lambda}{2 \lambda} - \frac{1}{16} \frac{u^2}{\lambda^2}, \]  

(44b)

where the parameter \( \lambda \in \mathbb{C}/\{0\} \).

**Proof.** As we know, the commutation condition \([L_1, L_2] = 0\) is equivalent to the condition that the two vector fields \(L_1\) and \(L_2\) share the same eigenfunctions \(\Psi(t, \hat{z}, z, \lambda)\), i.e., \(L_1 \Psi = 0 \Leftrightarrow L_2 \Psi = 0\). By using the conclusion of Lemma 1, we have the following relations

\[ \Lambda_\hat{z} = \frac{\partial}{\partial \hat{z}}[H_1(\Lambda)] = \frac{\partial}{\partial \hat{z}}[S_\hat{z} + \frac{\delta u}{4 \Lambda}] \]  

(45a)

\[ \Lambda_t = \frac{\partial}{\partial \hat{z}}[H_2(\Lambda)] = \frac{\partial}{\partial \hat{z}} \left[ \Lambda^2 - 2S_\lambda + \left( \frac{\delta}{2} W - S_\lambda^2 \right) - \frac{\delta u S_\lambda}{2 \lambda} - \frac{1}{16} \frac{u^2}{\Lambda} \right], \]  

(45b)

in which \(\Psi(t, \hat{z}, z, \Lambda(t, \hat{z}, z, k)) = k\) as described in Lemma 1.

By noting \(f_{\hat{z}}(t, \hat{z}, z) = \Lambda(t, \hat{z}, z, k) = \lambda\), we have the following equations

\[ f_{\hat{z}} = H_1(f_{\hat{z}}) = S_{\hat{z}} + \frac{\delta u}{4 f_{\hat{z}}}, \]  

(46a)

\[ f_t = H_2(f_{\hat{z}}) = f_{\hat{z}}^2 - 2S_{\lambda} f_{\hat{z}} + \left( \frac{\delta}{2} W - S_{\lambda}^2 \right) - \frac{\delta u S_{\hat{z}}}{2 f_{\hat{z}}} - \frac{1}{16} \frac{u^2}{f_{\hat{z}}}, \]  

(46b)

which are equivalent to the Hamilton-Jacobi equations (25) and lead to the dDS systems (22). □

**Remark 2.** Since \(L_1\) and \(L_2\) are Hamiltonian vector fields, the commutation condition (42) is equivalent to

\[ H_{1,t} - H_{2,\hat{z}} + \{H_1, H_2\}(\lambda, z) = 0. \]  

(47)

So the dDS systems (22) are equivalent to the zero-curvature equation (47) about \(H_1\) and \(H_2\). □

**Remark 3.** An integrable PDE system is usually associated with a hierarchy of PDEs defining infinitely many symmetries. A new hierarchy related to dDS system, which is called the dDS hierarchy, can also be defined. We will show the details in the next separated paper.
Proposition 2. Considering the following vector nonlinear Riemann-Hilbert problem on an arbitrary closed contour $\Gamma$ of the complex $\lambda$-plane

$$\vec{\pi}(\lambda) = \vec{R}(\vec{\pi}^{-}(\lambda)), \quad \lambda \in \Gamma,$$

for the two-dimensional vector functions $\vec{\pi}^{\pm}(\lambda) = (\pi_{1}^{\pm}(\lambda), \pi_{2}^{\pm}(\lambda)) \in \mathbb{C}^{2}$ which are analytical respectively inside and outside the contour $\Gamma$, normalized in the following way ($|\lambda| >> 1$)

$$\pi_{1}^{-}(\lambda) = \lambda - S_{z} - \frac{\delta V}{4\lambda} + O(\lambda^{-2})$$

$$\pi_{2}^{-}(\lambda) = \lambda t + \left(\frac{z}{2} - ts_{z}\right) - \frac{\delta t V}{4\lambda} + O(\lambda^{-2}),$$

where $\vec{R}(s) = (R_{1}(s_{1}, s_{2}), R_{2}(s_{1}, s_{2}))$, $s \in \mathbb{C}^{2}$ are given differentiable spectral data and its Jacobian matrix $J$ is defined as $J_{ij} = \frac{\partial R_{i}}{\partial s_{j}}, i, j = 1, 2$.

Assuming that the above nonlinear Riemann-Hilbert problem and its linearized form $\vec{\sigma}^{\pm}(\lambda) = (\pi_{1}^{\pm}(\lambda), \pi_{2}^{\pm}(\lambda)) \in \mathbb{C}^{2}$ are common eigenfunctions of the vector fields (43): $L_{j}\vec{\pi}^{\pm} = \vec{0}$, $j = 1, 2$. Then the potentials $u, S$ reconstructed from the above solve the dDS systems (22).

Proof. By applying the operators $L_{j}(j = 1, 2)$ defined in (43) to the nonlinear Riemann-Hilbert problem (48), one obtains the linearized Riemann-Hilbert problem

$$L_{j}\vec{\pi}^{\pm}(\lambda) = JL_{j}\vec{\pi}^{-}(\lambda), \quad \lambda \in \Gamma.$$

Since the normalization (49), we have $L_{j}\vec{\pi}^{-} \to \vec{0}$ when $\lambda \to \infty$. It follows that by uniquely solvable assumption, $\vec{\pi}^{\pm}$ are common eigenfunctions of the vector fields, i.e., $L_{j}\vec{\pi}^{\pm} = \vec{0}, j = 1, 2$. Consequently, the vector fields $L_{1}$ and $L_{2}$ share all the eigenfunctions on the whole complex $\lambda$-plane, which is equivalent to the commutation condition $[L_{1}, L_{2}] = 0$. So the potentials $u, S$ reconstructed from the above solve the dDS systems (22). 

Remark 4. The above proposition shows us a general way to construct
the complex-valued solutions solving the dDS systems (22) through a non-
linear Riemann-Hilbert problem. But in fact the real-valued restriction
is required by the original physics meaning (20), so we need to give the
above nonlinear Riemann-Hilbert problem some constraints to guarantee
this. We will show one result for dDS-II system in the following section.

3 The relevant nonlinear Riemann-Hilbert
problem for dDS-II system

In order to apply the Manokov-Santini novel IST method for studying the
dDS system, similar as the previous hydrodynamical systems mentioned,
such as the dKP equation, the Pavlov equation, the second heavenly equation
of Plébanski and d2DT (dispersionless 2D Toda) equation, it is important
to construct relevant nonlinear Riemann-Hilbert problems in the complex λ
plane with some constraints. Particularly, the reality constraint is important,
it gives us the real-valued solutions with clear physics significance.

As in the d2DT case, also in this dDS framework, 0 and ∞ are the singular
points in the complex λ plane. In the d2DT case, the generators of the two
Hamiltonians are two independent eigenfunctions respectively, one with polar
singularity around 0 and the other around ∞. But now in this dDS case,
both eigenfunctions appear in the construction of the Hamiltonian. Therefore
the nonlinear Riemann-Hilbert problem that will be hopefully associated
with the dDS system in the continuation of this research is expected to be
more complicated than that associated with the dKP and d2DT systems.

When σ = i, independent variables z, ẑ read as conjugate complex variables,
this symmetry helps us to construct a symmetric nonlinear Riemann-Hilbert
problem in the complex plane. For this reason, we will study dDS-II systems
(with σ = i) in this section, and note ẑ ≡ ẑ means the conjugate of z. The
construction of the relevant nonlinear Riemann-Hilbert problem for dDS-II
system with reality constraint will be demonstrated in this section.

Proposition 1 shows that the dDS system (22) arises as the commutation
condition [L1, L2] = 0. As we already know, this commutation condition
is equivalent to the condition that the two vector fields L1 and L2 share
the same eigenfunctions Ψ(t, ẑ, z, λ), i.e., L1Ψ = 0 ⇔ L2Ψ = 0. Now we
introduce a new parameter p (here and hereafter √u means the principal
value)

\[ p = \frac{2}{\sqrt{u}} \lambda, \quad (51) \]

and let

\[ \Psi(t, \bar{z}, z, \lambda) = \Psi(t, \bar{z}, z, \frac{\sqrt{u}}{2} p) = \Phi(t, \bar{z}, z, p), \quad (52) \]

Then the description of common eigenfunctions for \( L_1, L_2 \), i.e., \( L_1 \Psi(\lambda) = 0 \) and \( L_2 \Psi(\lambda) = 0 \), read as follows

\[ \frac{2}{\sqrt{u}} L_1[p] \Phi(p) = 0, \quad (53a) \]

\[ \frac{2}{\sqrt{u}} L_2[p] \Phi(p) = 0, \quad (53b) \]

in which

\[ L_1 = \left\{ \frac{\sqrt{u}}{2} p, \cdot \right\}_{(p, \bar{z})} - \left\{ H_1\left( \frac{\sqrt{u}}{2} p \right), \cdot \right\}_{(p, \bar{z})}, \quad (54a) \]

\[ L_2 = \left\{ \frac{\sqrt{u}}{2} p, \cdot \right\}_{(p, t)} - \left\{ H_2\left( \frac{\sqrt{u}}{2} p \right), \cdot \right\}_{(p, z)}, \quad (54b) \]

where the Hamiltonian functions \( H_1(\lambda) \) and \( H_2(\lambda) \) are defined in (44).

**Remark 5.** The commutation condition \([L_1, L_2] = 0\), which arises the dDS-II system, now reads as \( \left[ \frac{2}{\sqrt{u}} L_1[p], \frac{2}{\sqrt{u}} L_2[p] \right] = 0 \) (instead of \([L_1[p], L_2[p]] = 0\)). And this commutation condition is also equivalent to the condition that the two vector fields \( L_1[p] \) and \( L_2[p] \) share the same eigenfunctions \( \Phi(p) \), i.e., \( L_1[p] \Phi(p) = 0 \Leftrightarrow L_2[p] \Phi(p) = 0 \).

By applying the new parameter \( p \), one obtains the following Proposition 3 for constructing the real solutions \( u, S \) for dDS-II system.

**Proposition 3.** Considering the following vector nonlinear Riemann-Hilbert problem on the unit circle \( \gamma \) of the complex \( p \)-plane

\[ \tilde{\psi}^+(p) = R\left( \tilde{\psi}^-(p) \right), \quad p \in \gamma, \quad (55) \]
for the two-dimensional vector functions $\vec{\psi}^{\pm}(p) = (\psi_1^{\pm}(p), \psi_2^{\pm}(p)) \in \mathbb{C}^2$, which are analytical respectively inside and outside the unit circle $\gamma$, normalized in the following way ($|p| >> 1$)

$$\psi_1^{-}(p) = \frac{\sqrt{u}}{2} p - S_z - \frac{\delta}{2} \frac{V}{\sqrt{u}} \frac{1}{p} + O(p^{-2})$$  \hspace{1cm} (56a)$$

$$\psi_2^{-}(p) = \frac{t \sqrt{u}}{2} p + (\frac{z}{2} - t S_z) - \frac{\delta}{2} \frac{t V}{\sqrt{u}} \frac{1}{p} + O(p^{-2})$$  \hspace{1cm} (56b)$$

where $\vec{R}(\vec{s}) = (R_1(s_1, s_2), R_2(s_1, s_2)), \vec{s} \in \mathbb{C}^2$ are given differentiable spectral data and its Jacobian matrix $J$ is defined as $J_{ij} = \partial R_i / \partial s_j, i, j = 1, 2$.

Assuming that:

(1) the above nonlinear Riemann-Hilbert problem (55) and its linearized form $\vec{\sigma}^{+} = J \vec{\sigma}^{-}$ are uniquely solvable;

(2) the spectral data $\vec{R}(\vec{s})$ satisfy the reality constraint

$$\vec{R}(\vec{R}(\vec{s})) = \vec{s}, \ \forall \vec{s} \in \mathbb{C}^2.$$  \hspace{1cm} (57)$$

Then the solutions $\vec{\psi}^{\pm}(p)$ for the nonlinear Riemann-Hilbert problem (55) satisfy

$$\vec{\psi}^{+}(p) = \overline{\vec{\psi}^{-}(-\delta/\bar{p})},$$  \hspace{1cm} (58)$$

and the potentials $u, S$ reconstructed from the above solutions $\vec{\psi}^{\pm}(p) = (\psi_1^{\pm}(p), \psi_2^{\pm}(p))$ are real functions for $(x, y, t)$ and solve the dDS-II system (22) or (23) (with $\sigma = 1$).

**Proof.** By applying the operators $\mathcal{L}_j (j = 1, 2)$ defined in (54) to the nonlinear Riemann-Hilbert problem (55), one obtains the linearized Riemann-Hilbert problem

$$\mathcal{L}_j \vec{\psi}^{+}(p) = J \mathcal{L}_j \vec{\psi}^{-}(p), \quad p \in \gamma.$$  \hspace{1cm} (59)$$

Since the normalization (56), we have $\mathcal{L}_j \vec{\psi}^{-}(p) \to \vec{0}$ when $p \to \infty$. It follows that by uniquely solvable assumption, $\vec{\psi}^{\pm}$ are common eigenfunctions of the vector fields, i.e., $\mathcal{L}_j \vec{\psi}^{\pm} = \vec{0}, \ j = 1, 2$. Consequently, the vector fields $\mathcal{L}_1$ and $\mathcal{L}_2$ share all the eigenfunctions on the whole complex $p$-plane, so the
potentials $u, S$ reconstructed from the solutions of above nonlinear Riemann-Hilbert problem (55) solve the dDS-II system (22) or (23) (with $\sigma = \mathbf{i}$).

By considering the case $\vec{s} = \vec{\psi} - (\alpha/\bar{p})$ (with $|\alpha| = 1$), one obtains the following relation directly from the reality constraint (57) and the uniquely solvable assumption

$$\vec{\psi}^+(p) = \vec{\psi}^-(\alpha/\bar{p}), \quad |p| < 1. \quad (60)$$

From the normalization condition (56), we have

$$\psi_1^+(\alpha/\bar{p}) = \frac{\sqrt{u}}{2} \frac{\delta}{\bar{p}} - S\frac{\alpha}{2\sqrt{u\alpha}} + O(p^2) \quad (61a)$$

$$\psi_2^+(\alpha/\bar{p}) = \frac{t\sqrt{u}}{2} \frac{\delta}{\bar{p}} + \left(\frac{\bar{z}}{2} - tS\right) - \frac{\delta}{2} t\frac{V}{\sqrt{u}} \frac{\bar{p}}{\alpha} + O(p^2) \quad (61b)$$

By considering the condition $L_j \vec{\psi}^+(p) \rightarrow \vec{0}$ when $p \rightarrow 0$, one obtains the following formal series

$$\psi_1^+(p) = -\frac{\sqrt{u}}{2} \frac{\delta}{p} - S\frac{1}{2\sqrt{u}} p + O(p^2) \quad (62a)$$

$$\psi_2^+(p) = -\frac{t\sqrt{u}}{2} \frac{\delta}{p} + \left(\frac{\bar{z}}{2} - tS\right) - \frac{1}{2} \frac{W}{\sqrt{u}} p + O(p^2). \quad (62b)$$

From (60) (61) and (62), by considering the uniquely solvable assumption, one obtains the main results of this proposition: $\alpha = -\delta$ and the solutions $u, S$ are real functions of $(x, y, t)$.

Remark 6. For any real function $F(x, y)$ of real variables $(x, y)$, by noting $F(x, y) = F(\frac{x+i}{2}, \frac{y-i}{2})$, we have the following relation

$$F_x = \frac{1}{2} (F_x - iF_y) \quad (63a)$$

$$F_z = \frac{1}{2} (F_x + iF_y) = \overline{F_z}, \quad (63b)$$

this is how we understand the reality of the functions in this section.
4 Outlook

These results are the necessary background for all the future studies we are planning to make, and that consist of the following steps.

1) The use the Manakov-Santini method for vector fields to study the Cauchy problem for the dDS (dispersionless Davey-Stewartson) system (22), based on the above results, through the following: the identification of the appropriate formal zero energy eigenfunctions of the vector fields Lax pair, analytic in the complex parameter respectively in a neighborhood of $\infty$ and $0$; the identification of the vector nonlinear Riemann-Hilbert inverse problem on a closed curve in the complex plane.

2) The use of the nonlinear Riemann-Hilbert inverse problem to construct the longtime behavior of the solutions of the Cauchy problem.

3) The use of the Riemann-Hilbert inverse problem to construct classes of exact implicit solutions of the dDS system (22).

4) The use of the Riemann-Hilbert inverse problem to study how the above dynamics give rise to a wave breaking, and to study analytically the nature of such wave breaking, with the identification of the type of singularities, in the focusing as well as in the defocusing cases.

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