Two $\theta_{\mu\nu}$-deformed Poincare-Hopf Algebroids and Quantum Poincare Covariance of Noncommutative Space-Time Algebra

Jerzy Lukierski, Mariusz Woronowicz

Institute for Theoretical Physics, University of Wroclaw,
pl. Maxa Borna 9, 50-205 Wroclaw, Poland

Abstract
Starting from well-known $\theta_{\mu\nu}$-deformed $D=4$ Poincare-Hopf algebra $H$ we introduce two Hopf algebroids: the first describing $\theta_{\mu\nu}$-deformed relativistic phase space with canonical NC space-time (constant $\theta_{\mu\nu}$ parameters) and the second incorporating dual to $H$ quantum $\theta_{\mu\nu}$-deformed Poincare-Hopf group algebra $G$, with noncommutativity of space-time translations given by $\Lambda$-dependent $\Theta_{\mu\nu}$ parameters ($\Lambda \equiv\Lambda_{\mu\nu}$ parametrize classical Lorentz group). We show that the canonical $\theta_{\mu\nu}$-deformed space-time algebra is covariant under the quantum Poincare transformations described by $G$.

1 Introduction

In this paper we study new algebraic aspects of the well-known $\theta_{\mu\nu}$-deformed quantum space-times and associated quantum phase spaces. These quantum-deformed algebras are generated by the following Abelian twist [1],[2]

$$\mathcal{F} \equiv \mathcal{F}(1) \otimes \mathcal{F}(2) \exp\left(\frac{i}{2\kappa^2} \theta^{\mu\nu} p_\mu \otimes p_\nu\right),$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ and $\kappa$ is the deformation parameter with dimension of mass (in this paper we shall put $\kappa = 1$).

The twist can be employed in two ways:

1) It defines $\theta_{\mu\nu}$-deformed quantum Poincare-Hopf algebra $H$, with nondeformed Poincare algebra sector however with modified coproducts and antipodes [3]

$$\Delta_{\mathcal{F}}(h) = \mathcal{F} \circ \Delta_0(h) \circ \mathcal{F}^{-1}, \quad h = \{p_\mu, M_{\mu\nu}\},$$

$$S_{\mathcal{F}}(h) = U S_0(h) U^{-1}, \quad U = \mathcal{F}(1) S(\mathcal{F}(2)).$$
By Hopf-algebraic duality one can define the \( \theta_{\mu\nu} \)-deformed quantum Poincare group \( G \), with generators \( \hat{g} = \{ \hat{\xi}_{\mu}, \hat{\Lambda}_{\mu
u} \} \), describing generalized NC coordinates on algebraic \( \theta_{\mu\nu} \)-deformed Poincare group manifold \([1], [4]\). Using the Heisenberg double construction \([5],[6]\) given by smash product \( G \rtimes H \), one obtains \((10+10)\)-dimensional generalized \( \theta_{\mu\nu} \)-deformed quantum phase space \( \mathcal{H}^{(10+10)} = (\hat{\xi}_{\mu}, \hat{\Lambda}_{\mu\nu}, p_{\mu}, M_{\mu\nu}) \). Such phase space can be used for the description of dynamics on \( \theta_{\mu\nu} \)-deformed quantum Poincare group \([1]\), with quantum Lorentz group parameters \( \hat{\Lambda}_{\mu\nu} \) and the NC quantum Poincare group translations satisfying the following algebra \([4]\)

\[
[\hat{\xi}_{\mu}, \hat{\xi}_{\nu}] = i\theta^{\rho\sigma}(\eta_{\mu\rho}\eta_{\nu\sigma} - \hat{\Lambda}_{\mu\rho}\hat{\Lambda}_{\nu\sigma}) := i\Theta_{\mu\nu}(\hat{\Lambda}). \tag{4}
\]

ii) One can introduce the classical Minkowski space-time coordinates \( x_{\mu} \in \mathbb{X} \) as vectorial representation of relativistic symmetries described by classical Poincare algebra \( P \), with the algebraic structure of \( P \otimes \mathbb{X} \) given by the semidirect product \( P \rtimes \mathbb{X} \). In quantum-deformed theory one introduces the NC space-times \( \hat{x}_{\mu} \in \hat{\mathbb{X}} \) as the module algebra (NC algebraic representation) of quantum Poincare-Hopf algebra \( \mathbb{H} \). In the case of \( \theta_{\mu\nu} \)-deformation the quantum space-time coordinates \( \hat{x}_{\mu} \in \hat{\mathbb{X}} \) are obtained from Drinfeld twisting procedure \([3]\) by star product technique \([7],[8]\). Let us consider the algebra \( \hat{A} \) of functions on \( \hat{\mathbb{X}} \) \( (f(\hat{x}) \in \hat{A}) \). In the scheme of twist quantization one can represent the algebra \( (\hat{A}, \cdot) \) by the algebra \( (A, \star_F) \) of classical functions, with their multiplication defined by the nonlocal star product \( \star_F \)

\[
f(\hat{x}) \cdot g(\hat{x}) \simeq f(x) \star_F g(x) = m[F^{-1} \circ (f \otimes g)] = (F^{-1}_1 \triangleright f)(F^{-1}_2 \triangleright g). \tag{5}
\]

Putting in \([5]\) \( f(\hat{x}) = \hat{x}_{\mu}, g(\hat{x}) \equiv 1 \) one gets

\[
\hat{x}_{\mu} = m[F^{-1} \circ (\triangleright \otimes 1)](x_{\mu} \otimes 1)] = (F^{-1}_1 \triangleright x_{\mu})F^{-1}_2. \tag{6}
\]

Formula \([6]\) provides the deformed NC space-time coordinates as expressed by a nonlocal map in undeformed relativistic phase space \( (x_{\mu}, p_{\mu} = \frac{1}{i}\partial_{\mu}) \). If we put \( f(x) = x_{\mu}, g(x) \equiv x_{\nu} \), one can calculate from \([6]\) the commutator

\[
[x_{\mu}, x_{\nu}]_{\ast_F} \equiv x_{\mu} \ast_F x_{\nu} = x_{\nu} \ast_F x_{\mu} - x_{\nu} \ast_F x_{\mu} = i\theta_{\mu\nu}. \tag{7}
\]

It appears that both \( \theta_{\mu\nu} \)-deformed structures presented above are necessary in order to describe in complete way the NC space-time \([1]\) as describing quantum Poincare-covariant \( \mathbb{H} \)—module with generators \( \hat{x}_{\mu} \), transforming under quantum Poincare group \( G \) in the following standard way

\[
\hat{x}_{\mu}' = \hat{\Lambda}_{\mu\nu} \hat{x}_{\nu} + \hat{\xi}_{\mu}. \tag{8}
\]

\(^1\)Because the twist \( F \) is the function of classical Poincare algebra generators \( \hat{g} = (p_{\mu}, M_{\mu\nu}) \), the action \( \hat{g} \cdot f(x) \) in formula \([5]\) is described by the differential realization of classical Poincare algebra on functions of standard Minkowski coordinates \( x_{\mu} \).
If we wish to describe the quantum Poincare transformations \( \mathcal{P}(8) \) using star-product formula one should extend the star multiplication \( \hat{\star} \) to the functions of NC variables \( \hat{x}_\mu \) (NC Minkowski space) and \( \hat{\xi}_\mu \) (NC Poincare translations)

\[
F(\hat{x}_\mu, \hat{\xi}_\mu) \cdot G(\hat{x}_\nu, \hat{\xi}_\nu) \simeq F(x_\mu, \xi_\mu) \hat{\star} \hat{x} G(x_\nu, \xi_\nu)
\]

\[
= m \circ \exp \left[ \frac{i}{2} (\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} + \Theta^{\mu\nu}(\hat{\Lambda}) \frac{\partial}{\partial \xi_\mu} \otimes \frac{\partial}{\partial \xi_\nu}) \right] (F(x_\mu, \xi_\mu) \otimes G(x_\nu, \xi_\nu)).
\]

In (9) both relations (4), (7) have been taken into account, with the undeformed numerical Lorentz group variables \( \hat{\Lambda}^{\mu\nu} \equiv \Lambda^{\mu\nu} \) playing the role of classical parameters. It appears that the quantum Poincare covariance is expressed by the following equality of star products (see also \[10\])

\[
F(\Lambda^{\mu\alpha} x^\alpha + \xi_\mu) \hat{\star} \hat{x} G(\Lambda^{\nu\beta} x^\beta + \xi_\nu)
\]

\[
= F(x_\mu', 0) \hat{\star} \hat{x} G(x_\nu', 0) \bigg|_{x_\mu \rightarrow \Lambda^{\mu\alpha} x^\alpha + \xi_\mu, x_\nu \rightarrow \Lambda^{\nu\beta} x^\beta + \xi_\nu}
\]

\[
= m \circ \exp \left[ \frac{i}{2} (\theta^{\mu\nu} \frac{\partial}{\partial x'^\mu} \otimes \frac{\partial}{\partial x'^\nu}) \right] (F(x'_\mu) \otimes G(x'_\nu)) \equiv F(x'_\mu) \hat{\star} \hat{x} G(x'_\nu).
\]

The plan of our paper is following. In Sect. 2 we recall the \( \theta^{\mu\nu} \)-deformed Poincare-Hopf algebra \( \mathbb{H} \) extended by NC space-time coordinates \( \hat{x}_\mu \in \hat{\mathbb{X}} \) which describe the Lorentz group extension of the relativistic phase space \( (\hat{x}_\mu, p_\mu) \in \hat{\mathbb{X}} \otimes T^4 \)

\[
\mathbb{H} \otimes \hat{\mathbb{X}} = \mathcal{O}(3,1) \times (T^4 \rtimes \mathbb{X}).
\]

In Sect. 3 we describe the \( \theta^{\mu\nu} \)-deformed Poincare quantum group \( \mathbb{G} \) and calculate the \( \theta^{\mu\nu} \)-deformed Heisenberg double \( \mathcal{H}(10+10) = \mathbb{H} \rtimes \mathbb{G} \) with generalized NC Poincare coordinates \( \{ \hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu} \} \in \mathbb{G} \) and generalized momenta \( \{ p_\mu, M_{\mu\nu} \} \in \mathbb{H} \).

In Sect. 4 we specify the data which define two Hopf algebroids, first providing the quantum Poincare-covariant \( \theta^{\mu\nu} \)-deformed phase space with positions (coordinates) described by algebra \( \hat{\mathbb{X}} (\hat{x}_\mu \in \hat{\mathbb{X}}) \) supplemented with Lorentz parameters \( \hat{\Lambda}_{\mu\nu} \) and the second Hopf algebroid describing quantum \( \theta^{\mu\nu} \)-deformed Poincare symmetry transformations (see \[8\]), with coordinate sector described by quantum \( \theta^{\mu\nu} \)-deformed Poincare group \( \mathbb{G} \).

## 2 \( \theta^{\mu\nu} \)-deformed quantum Poincare algebra \( \mathbb{H} \) and dual group \( \mathbb{G} \)

### 2.1 Twist deformed quantum Poincare algebra \( \mathbb{H} \)

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2See \[9\] for the use of star product to represent the quantum group transformations.
The classical $D = 4$ Poincare-Hopf algebra looks as follows

\[
[p_\mu, p_\nu] = 0 \\
[M_{\mu\nu}, p_\rho] = i(\eta_{\rho\nu} p_\mu - \eta_{\rho\mu} p_\nu) \\
[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\rho\nu} M_{\mu\sigma} - \eta_{\rho\mu} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\nu\rho} M_{\mu\sigma})
\]

where $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$ and is supplemented by primitive costructure maps

\[
\Delta_0(h) = h \otimes 1 + 1 \otimes h, \quad S_0(h) = -h, \quad \epsilon_0(h) = 0.
\]

The twist $F$ in an element of $\mathbb{H} \otimes \mathbb{H}$ ($\mathbb{H} = U(P)$) which has an inverse, satisfies the cocycle condition

\[
F_{12}(\Delta_0 \otimes 1) F = F_{23} (1 \otimes \Delta_0) F, \quad F_{12} (\Delta_0 \otimes 1) F = F_{23} (1 \otimes \Delta_0) F
\]

and the normalization condition ($\epsilon \otimes 1) F = (1 \otimes \epsilon) F = 1$ where $F_{12} = F \otimes 1$ and $F_{23} = 1 \otimes F$.

The twist $F$ does not modify the algebraic part and the counit, but changes the coproducts $\Delta : H \rightarrow H \otimes H$ and the antipodes $S : H \rightarrow H$ according to formulae (2)-(3). The quantum $\theta$-deformation is generated by the twist (1).

From formula (2) one gets the coproducts

\[
\Delta_F(p_\mu) = p_\mu \otimes 1 + 1 \otimes p_\mu \\
\Delta_F(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}
\]

\[-\frac{1}{2} \theta^{\alpha\beta} \left( (\eta_{\alpha\mu} p_\nu - \eta_{\alpha\nu} p_\mu) \otimes p_\beta + p_\alpha \otimes (\eta_{\beta\mu} p_\nu - \eta_{\beta\nu} p_\mu) \right).
\]

Because from (3) and (1) follows that $U = 1$ in considered case of $\theta_{\mu\nu}$-deformations the antipodes remain unchanged, i.e. $S_F(h) = S_0(h) = -h$.

### 2.2 Algebra of generalized coordinates $\hat{X}$

For twisted deformed case we can introduce the coordinates algebra $\hat{X} \ni \hat{X}_A = \{ \hat{x}_\mu, \hat{\Lambda}_{\mu\nu} \}$ with the multiplication given by the star product formula

\[
\hat{X}_A \cdot \hat{X}_B \simeq X_A \star_F X_B = m[F^{-1} \circ (X_A \otimes X_B)] = (F^{-1}_{(1)} \triangleright X_A)(F^{-1}_{(2)} \triangleright X_B)
\]

where

\[
h \triangleright X_A = [h, X_A], \quad h = \{ p_\mu, M_{\mu\nu} \}, \quad X_A = \{ x_\mu, \Lambda_{\mu\nu} \}
\]

and in undeformed case we obtain

\[
[p_\mu, x_\nu] = i \eta_{\mu\nu}, \quad [M_{\mu\nu}, x_\rho] = i(\eta_{\rho\nu} x_\mu - \delta_{\rho\mu} x_\nu) \\
[M_{\mu\nu}, \Lambda_{\rho\sigma}] = \eta_{\rho\nu} \Lambda_{\mu\sigma} - \eta_{\rho\mu} \Lambda_{\nu\sigma}.
\]

The formula (16) can be also written as follows

\[
f(X) \star_F k(X') = f(\hat{X}) \triangleright k(X')
\]
where $\hat{f}(X)$ denotes the noncommutative star representation of $f(\hat{X})$ defined by the formula (see also (3))

$$f(\hat{X}) \simeq \hat{f}(X) = m[F^{-1}(\langle \rangle \otimes 1)(f(X) \otimes 1)] \quad (21)$$

For the twist (1) we get from (21) the following explicit formulas describing generalized coordinates $\hat{X}_A = \{\hat{x}_\mu, \hat{\Lambda}_{\mu\nu}\}$ in terms of undeformed relativistic quantum phase space variables $(x_\mu, p_\mu)$

$$\hat{x}_\mu = x_\mu + \frac{1}{2} \theta_\mu^\alpha p_\alpha, \quad \hat{\Lambda}_{\rho\sigma} = \Lambda_{\rho\sigma} \quad (22)$$

Due to (18) one gets the expected algebraic relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \quad (23)$$
$$[\hat{x}_\mu, \hat{\Lambda}_{\rho\sigma}] = [\hat{\Lambda}_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = 0 \quad (24)$$

Due to the relation

$$h \triangleright (\hat{X}_A \hat{X}_B) = (h_{(1)} \triangleright \hat{X}_A)(h_{(2)} \triangleright \hat{X}_B) \quad (25)$$

one can check easily that the commutators (23)-(24) are covariant under the action of Poincare-Hopf algebra.

### 2.3 $\theta_{\mu\nu}$-deformed quantum phase space $H^{(10+10)} = (\hat{x}_\mu, \hat{\Lambda}_{\mu\nu}, p_\mu, M_{\mu\nu})$

Using (22) one can check the following set of cross commutators

$$[p_\mu, \hat{x}_\nu] = i\eta_{\mu\nu} \quad (26)$$
$$[p_\mu, \hat{\Lambda}_{\rho\sigma}] = 0 \quad (27)$$
$$[M_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = -i(\delta_{\rho\mu}\hat{\Lambda}_{\nu\sigma} - \delta_{\rho\nu}\hat{\Lambda}_{\mu\sigma}) \quad (28)$$
$$[M_{\mu\nu}, \hat{x}_\rho] = i\delta_{\rho\nu}(\hat{x}_\mu - \frac{1}{2}\theta_\mu^\alpha p_\alpha) - i\delta_{\rho\mu}(\hat{x}_\nu - \frac{1}{2}\theta_\nu^\alpha p_\alpha) \quad (29)$$
$$-i\frac{1}{2}(\theta_{\mu\rho}p_\nu - \theta_{\rho\nu}p_\mu).$$

Obviously, together with commutators (23)-(24) the set of relations (26)-(29) satisfies the Jacobi identities.

### 3 $\theta_{\mu\nu}$-deformed quantum Poincare matrix group $\hat{G}$ and quantum Poincare covariance

#### 3.1 RTT quantization method and $\theta_{\mu\nu}$-deformed quantum group algebra $\hat{G}$

The universal $\mathcal{R}$-matrix ($(a \wedge b = a \otimes b - b \otimes a)$)

$$\mathcal{R} = \mathcal{F}^T \mathcal{F}^{-1} = \exp[-i\theta_{\mu\nu}p_\mu \otimes p_\nu] \quad (a \otimes b)^T = b \otimes a \quad (30)$$
can be used for the description of 10-generator deformed $D = 4$ Poincaré group. Using the $5 \times 5$ - matrix realization of the Poincaré generators

\[
(M_{\mu\nu})^A_B = \delta^A_{\mu} \eta_{\nu B} - \delta^A_{\nu} \eta_{\mu B} \quad (p_\mu)^A_B = \delta^A_{\mu} \delta^4_B
\]  

we can show that in (30) only the linear term is non-vanishing $R = 1 \otimes 1 - i\theta^{\mu\nu}P_{\mu} \otimes P_{\nu}$.

To find the matrix quantum group which is dual to our Hopf algebra $H$ in matrix realization (31) we introduce the following $5 \times 5$ - matrices

\[
\hat{T}_{AB} = \begin{pmatrix} 0 & \hat{\xi}_\mu \\ \hat{\Lambda}_{\mu\nu} & 1 \end{pmatrix}
\]

where $\hat{\Lambda}_{\mu\nu}$ parametrizes the quantum Lorentz rotation and $\hat{\xi}_\mu$ denotes quantum translations. In the framework of the $\text{FRT}$ procedure, the algebraic relations defining such a quantum group $\hat{G}$ are described by the following relation

\[
R\hat{T}_1 \hat{T}_2 = \hat{T}_2 \hat{T}_1 R
\]

while the composition law for the coproduct remains classical $\Delta(\hat{T}_{AB}) = \hat{T}_{AC} \otimes \hat{T}_{CB}$ with $\hat{T}_1 = \hat{T} \otimes 1$, $\hat{T}_2 = 1 \otimes \hat{T}$ and quantum $R$-matrix given in the representation (31).

In terms of the basis $(\hat{\Lambda}_{\mu\nu}, \hat{\xi}_\mu)$ of $\hat{G}$ the algebraic relations (33), describing the quantum group algebra, can be written as follows

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta^{\rho\sigma} (\eta_{\mu\rho} \eta_{\nu\sigma} - \hat{\Lambda}_{\mu\rho} \hat{\Lambda}_{\nu\sigma}) := i\Theta_{\mu\nu} (\hat{\Lambda}),
\]

\[
[\hat{x}_\mu, \hat{\Lambda}_{\rho\sigma}] = 0, \quad [\hat{\Lambda}_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = 0
\]

while the coproduct takes the well known classical form

\[
\Delta(\hat{\Lambda}_{\mu\nu}) = \hat{\Lambda}_{\mu\rho} \otimes \hat{\Lambda}_{\rho\nu} \quad \Delta(\hat{\xi}_\mu) = \hat{\Lambda}_{\mu\nu} \otimes \hat{\xi}_\nu + \hat{\xi}_\mu \otimes 1.
\]

One can check that coproducts (36) are homomorphic to the algebra (31)-(35) defining $\theta_{\mu\nu}$-deformed quantum Poincare group.

### 3.2 The covariance of $\hat{X}$ under quantum Poincare group $\hat{G}$

We recall that $\hat{X}$ is the algebra of generalized coordinates $\hat{X}_A = \{\hat{\Lambda}_{\mu\nu}, \hat{x}_\mu\}$ and $\hat{G}$ is the algebra of Poincare symmetry parameters $\hat{g} = \{\hat{\Lambda}_{\mu\nu}, \hat{\xi}_\mu\}$. One can perform the quantum Poincare transformations of $\hat{X}_A$ in the following way

\[
\hat{x}_\mu \rightarrow \hat{x}_\mu' = \hat{\Lambda}_{\mu\nu} \hat{x}_\nu + \hat{\xi}_\mu
\]

\[
\hat{\Lambda}_{\mu\nu} \rightarrow \hat{\Lambda}_{\mu'\nu'} = \hat{\Lambda}_{\mu\rho} \hat{\Lambda}_{\rho\nu} = \hat{\Lambda}_{\mu\nu}.
\]

The commutators of algebra $\hat{X}$ (see (23)-(24)) are invariant under a such transformation if the quantum Poincare symmetry parameters $\hat{G} \ni \hat{g} = \{\hat{\Lambda}_{\mu\nu}, \hat{\xi}_\mu\}$
satisfy the relations defining the algebra \( \hat{G} \) (see (34)-(35)) and \([\hat{X}_A, \hat{g}] = 0\). In particular we have
\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \hat{x}_\mu \rightarrow \hat{\Lambda}_{\mu\nu} \hat{x}_\nu + \hat{\xi}_\mu \rightarrow \hat{\Lambda}_{\mu\nu} \hat{x}_\nu + \hat{\xi}_\mu = i\theta_{\mu\nu}. \tag{39}
\]
We see that the generalized coordinates algebra \( \hat{X} \) in different \( \hat{G} \)-frames specified by eq. (37)-(38) transform covariantly under the transformations of quantum Poincare group \( \hat{G} \).

4 Duality between quantum Hopf algebras \( \mathbb{H} \) and \( \hat{G} \) and Heisenberg double \( \mathcal{H} = \mathbb{H} \rtimes \hat{G} \)

Two Hopf algebras \( \mathbb{H}, \hat{G} \) are said to be dual if there exists a nondegenerate bilinear form \( \langle , \rangle : \mathbb{H} \times \hat{G} \rightarrow \mathbb{C}, (h, \hat{g}) \rightarrow \langle h, \hat{g} \rangle \) such that the duality relations
\[
\langle hh', \hat{g} \rangle = \langle h, \Delta(h') \hat{g} \rangle \tag{40}
\]
\[
\langle h, \hat{g} \hat{g}' \rangle = \langle h, \hat{g} \rangle \langle h, \hat{g}' \rangle \tag{41}
\]
are satisfied. In our considerations the following pairing relations
\[
< p_\mu, \hat{\xi}_\nu > = i\eta_{\mu\nu} < M_{\mu\nu}, \hat{\Lambda}_{\alpha\beta} > = -i(\eta_{\mu\alpha} \eta_{\nu\beta} - \eta_{\nu\alpha} \eta_{\mu\beta}) < 1, \hat{\Lambda}_{\mu\nu} > = \eta_{\mu\nu} \tag{42}
\]
are appropriate. The basic action of \( \mathbb{H} \) on \( \hat{G} \) promoting \( \hat{G} \) to the \( \mathbb{H} \)-module is given by the following relation
\[
h \triangleright \hat{g} = \hat{g}_{(1)} \langle h, \hat{g}_{(2)} \rangle. \tag{43}
\]
After using (40) one gets the relation
\[
h \triangleright (\hat{g} \hat{g}') = \hat{g}_{(1)} \langle \Delta h, \hat{g}_{(2)} \hat{g}'_{(2)} \rangle = \hat{g}_{(1)} \langle h, \hat{g}_{(2)} \rangle \langle h_{(1)}, \hat{g}'_{(2)} \rangle \tag{44}
\]
what establishes that algebra \( \hat{G} \) is the \( \mathbb{H} \)-module.

In Heisenberg double framework we can obtain cross commutators between the algebra \( \mathbb{H} \) and group \( \hat{G} \) by the following relation
\[
[h, \hat{g}] = \hat{g}_{(2)} \langle h_{(1)}, \hat{g}_{(1)} \rangle h_{(2)} - \hat{g} h = \{ M_{\mu\nu}, p_\sigma \}; \hat{g} = \{ \hat{\Lambda}_{\mu\nu}, \hat{\xi}_\rho \}. \tag{45}
\]
Using pairing (42) and coproducts (14), (15) and (36) we get
\[
[p_\mu, \hat{\xi}_\nu] = i\eta_{\mu\nu} \tag{46}
\]
\[
[p_\mu, \hat{\Lambda}_{\rho\sigma}] = 0 \tag{47}
\]
\[
[M_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = -i(\delta_{\mu\rho} \hat{\Lambda}_{\nu\sigma} - \delta_{\nu\rho} \hat{\Lambda}_{\mu\sigma}) \tag{48}
\]
\[
[M_{\mu\nu}, \hat{\xi}_\rho] = i\delta_{\rho\mu} (\hat{\xi}_\nu - \frac{1}{2} \theta_\mu^\alpha p_\alpha) - i\delta_{\rho\nu} (\hat{\xi}_\mu - \frac{1}{2} \theta_\nu^\alpha p_\alpha) \tag{49}
\]
\[
-\frac{i}{2}(\theta_{\rho\mu} p_\nu - \theta_{\rho\nu} p_\mu)
\]
The quantum phase space described by Heisenberg double algebra $\mathcal{H}$ describes the dynamical system with the coordinates specified by the NC quantum Poincare group manifold $\mathbb{G}$.

5 Two Hopf algebroids

5.1 Briefly on Hopf algebroids

The Hopf algebroids, introduced in [11], are described by bialgebroids with supplemented antipodes. It has been argued (see e.g. [12]–[17]) that the Hopf algebroids are well adjusted to the description of physically important quantum (canonical and noncanonical) phase space.

The bialgebroid $\mathcal{B}$ is specified by the set of the data $(H, A; s, t; m, \tilde{\Delta}, \epsilon)$ where $H$ is the total algebra with product $m$ and its subalgebra $A \subset H$ is called the base algebra. The source map $s(a) : A \to H$ is a homomorphism and the target map $t(a) : A \to H$ an antihomomorphism, with their images commuting

$$[s(a), t(b)] = 0 \quad a, b \in A \quad s(a), t(b) \in H.$$  

The canonical choice of the source map is $s(a) = a$. One can introduce natural $(A, A)$–bimodule structure on $H$ using source and target maps in the basic $(A, A)$-bimodule formula, namely $a b t(a) s(b)$. The coproducts $\tilde{\Delta}$ are described by the maps $H \to H \otimes_A H$ from $H$ into $(A, A)$ bimodules $H \otimes_A H$, satisfying the co-associativity condition

$$(\tilde{\Delta} \otimes_A id_H)\tilde{\Delta} = (id_H \otimes_A \tilde{\Delta})\tilde{\Delta}.$$  

Because $H \otimes_A H$ as the codomain of coproducts $\tilde{\Delta}$ does not inherit the algebra structure from $H \otimes H$, in order to have well defined multiplication one introduces the submodule $H \times_A H \subset H \otimes H$ defined by the Takeuchi coproduct [18].

The algebra $H$ with the product $m$ and the coalgebra with Takeuchi coproduct $\tilde{\Delta}$ are compatible, i.e

$$\tilde{\Delta}(hh') = \tilde{\Delta}(h)\tilde{\Delta}(h').$$  

The coproduct $\tilde{\Delta}$ in $H \times_A H$ can be expressed in terms of standard tensor product $H \otimes H$ by the equivalence classes satisfying the condition

$$\tilde{\Delta}(h)I_L(a) = 0, \quad I_L(a) = t(a) \otimes 1 - 1 \otimes s(a),$$  

where $I_L$ define the left ideal in $H \otimes H$. The equivalence classes defined by are parametrized by so-called coproduct gauge (see e.g. [13]).

The counit map $\epsilon : H \to A$ is satisfying $\epsilon(1_H) = 1_A$, and

$$(\epsilon \otimes_A id_H)\tilde{\Delta} = (id_H \otimes_A \epsilon)\tilde{\Delta} = id_H.$$  

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We get Hopf algebroids if we are able to introduce an antipode (bijective map) $S : H \rightarrow H$ which is an algebra antihomomorphism and satisfies the following properties

\begin{align*}
S(t) &= s \quad (55) \\
m[(1 \otimes S) \circ \gamma \hat{\Delta}] &= s \epsilon = \epsilon \quad (56) \\
m[(S \otimes 1) \circ \hat{\Delta}] &= t \epsilon S, \quad (57)
\end{align*}

where in general case we need additional linear map $\gamma : H \otimes_A H \rightarrow H \otimes H$, so-called anchor map.

5.2 First choice: the coordinates $\hat{X}$ as the base algebra (following [19])

Let us choose the bialgebroid group coproducts for base algebra generators $X_A = \{ \hat{x}_\mu, \hat{\Lambda}_{\mu
u} \}$

$$\hat{\Delta}(X_A) = X_A \otimes 1 \quad (58)$$

satisfies the group and generalized quantum phase space algebra (26)-(29).

The source $s(X_A)$ and the target $t(X_A)$ maps are the following

\begin{align*}
s(\hat{X}_A) &= m[F^{-1}(\triangleright \otimes 1)(s_0(X_A) \otimes 1)] = \hat{X}_A \quad (59) \\
t(\hat{x}_\mu) &= m[(F^{-1})^\dagger(\triangleright \otimes 1)(t_0(x_\mu) \otimes 1)] \\
&= x_\mu - \frac{1}{2} \theta^a_\mu p_\alpha = \hat{x}_\mu - \theta^a_\mu p_\alpha \\
t(\hat{\Lambda}_{\mu\nu}) &= m[(F^{-1})^\dagger(\triangleright \otimes 1)(t_0(\Lambda_{\mu\nu}) \otimes 1)] = \Lambda_{\mu\nu} = \hat{\Lambda}_{\mu\nu} \quad (60)
\end{align*}

where $s_0(X_A) = X_A$, $t_0(X_A) = X_A$. The maps (59)-(61) satisfy the following relations

\begin{align*}
[s(\hat{X}_A), t(\hat{X}_B)] &= 0 \quad (62) \\
[s(\hat{x}_\mu), s(\hat{x}_\nu)] &= i \theta_{\mu\nu}, \quad [t(\hat{x}_\mu), t(\hat{x}_\nu)] = -i \theta_{\mu\nu} \quad (63) \\
[s(\cdot), s(\cdot)] &= [t(\cdot), t(\cdot)] = 0 \quad \text{(for the other choices of } \hat{X}_A). \quad (64)
\end{align*}

The counit has canonical form

$$\epsilon(\hat{X}_A) = m[F^{-1}(\triangleright \otimes 1)(\epsilon_0(X_A) \otimes 1)] = \hat{X}_A. \quad (65)$$

Using (55) one gets explicit formulae for the antipodes

$$S(\hat{X}_A) = t(\hat{X}_A). \quad (66)$$

In our case we have $S^2 = 1$. One can check that for ideal $I = t \otimes 1 - 1 \otimes s$ it is true that

$$m[(1 \otimes S)]I = m[(S \otimes 1)]I = 0 \quad (67)$$

and it follows that we do not need the anchor map (see (56)).
In order to determine the ideal \( \mathcal{I}_0 \) let us start from the nondeformed ideal (for \( \theta_{\mu\nu} = 0 \))

\[
\mathcal{I}_0(X_A) = X_A \otimes 1 - 1 \otimes X_A.
\]

We can obtain \( \hat{X}_A \) for \( X_A \) by using twisting formula (21). One gets (see (59)-(61))

\[
\mathcal{I}_\mathcal{L}(\hat{X}_A) = \mathcal{F}\mathcal{I}_0(X_A)\mathcal{F}^{-1} = t(\hat{X}_A) \otimes 1 - 1 \otimes s(\hat{X}_A).
\]

In particular one can check the following relations

\[
[\Delta(\hat{X}_A), \mathcal{I}_\mathcal{L}(\hat{X}_A)] = [\mathcal{I}_\mathcal{L}(\hat{X}_A), \mathcal{I}_\mathcal{L}(\hat{X}_B)] = 0.
\]

If we change the bialgebroid group coproduct by introducing the following coproduct gauge transformation

\[
\tilde{\Delta}(\hat{X}_A) \rightarrow \tilde{\Delta}_\lambda(\hat{X}_A) = \tilde{\Delta}(\hat{X}_A) + \lambda \mathcal{I}_\mathcal{L}(\hat{X}_A)
\]

it follows from (70) that new bialgebroid coproduct describe the homomorphism of all the group commutation relations (23)-(24) extended to the phase space commutators (26)-(29).

5.3 Second choice: quantum group \( \hat{G} \) as the base algebra (following [11])

The half-primitive bialgebroid coproducts for \( \hat{g} = \{ \hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu} \} \)

\[
\hat{\Delta}(\hat{g}) = \hat{g} \otimes 1
\]

together with the coproducts (14)-(15) satisfy the Heisenberg double commutators (see (34)-(35) and (46)-(49)).

The source \( s(\hat{g}) \) and the target \( t(\hat{g}) \) maps should be consistent with base algebra in the following sense

\[
[s(\hat{g}), t(\hat{g}')] = 0, \quad [s(\hat{\xi}_\mu), s(\hat{\xi}_\nu)] = i\Theta_{\mu\nu}(s(\hat{\Lambda})), \quad [t(\hat{\xi}_\mu), t(\hat{\xi}_\nu)] = -i\Theta_{\mu\nu}(t(\hat{\Lambda})), \quad [s(\cdot), t(\cdot)] = [t(\cdot), s(\cdot)] = 0 \quad \text{(for the other choices of } \hat{g} \text{)}.
\]

where due to (34) the relations (74) describe quadratic algebras. Subsequently, it can be easily shown that analogously to (59)-(61) one gets

\[
s(\hat{g}) = \hat{g}, \quad t(\hat{\Lambda}_{\mu\nu}) = \hat{\Lambda}_{\mu\nu}, \quad t(\hat{\xi}_\mu) = \hat{\xi}_\mu - \Theta^\alpha_{\mu}(\hat{\Lambda})p_\alpha.
\]

The counit is

\[
\epsilon(\hat{g}) = \hat{g}.
\]
and the antipodes which are given by

\[ S(\hat{g}) = t(\hat{g}) \]  

(79)
satisfy the required relations (55)-(57). Similarly as in Sect 5.2 \( S^2 = 1 \) and we do not need the anchor map.

If we consider the ideal (see also (76)-(77))

\[ \mathcal{I}_L(\hat{g}) = t(\hat{g}) \otimes 1 - 1 \otimes s(\hat{g}) \]  

(80)
one can introduce as well the counterpart of the coproduct gauge transformation (71) however only if \( \lambda = 1 \) for \( \hat{\xi}_\mu \) and \( \lambda = -1 \) for \( \hat{\Lambda}_{\mu\nu} \).

6 Final remarks

In this paper we considered the most popular quantum deformation of space-time coordinates and the corresponding quantum phase-space, described by \( \theta_{\mu\nu} \)-deformation of relativistic Heisenberg algebra. Our aim was to describe the \( \theta_{\mu\nu} \)-deformed phase spaces in the language of Hopf algebroid, with the extension of translational sectors \((\hat{x}_\mu, p_\mu)\) or \((\hat{\xi}_\mu, p_\mu)\) by the rotational Lorentz phase space coordinates \((\hat{\Lambda}_{\mu\nu}, M_{\mu\nu})\).

There were introduced relativistic quantum phase spaces with Hopf-algebroid structure in two ways:

- by twisting of classical canonical bialgebroid, describing undeformed relativistic quantum phase space with space-time coordinates (see e.g. [12], [15]),

- by considering dual pairs of quantum-deformed Poincare-Hopf algebras which define the Poincare-Heisenberg double by introducing the semidirect (smash) product of quantum Poincare group describing generalized coordinate sector and the quantum Poincare algebra which provides the generalized momenta sector.

Both methods can be applied in order to obtain the \( \theta_{\mu\nu} \)-deformed quantum phase spaces. If we consider the pair of 10 + 10-dimensional generalized phase space, with generalized coordinates described respectively by \((\hat{x}_\mu, \hat{\Lambda}_{\mu\nu})\) and \((\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu})\), we get different noncommutativity for \( \hat{x}_\mu \) and \( \hat{\xi}_\mu \): first one is characterized by constant \( \theta_{\mu\nu} \), and second one which leads to \( \Lambda \)-dependent \( \Theta_{\mu\nu}(\hat{\Lambda}) \) (see Sect. 5.2 and 5.3), with commutation relations quadratic in \( \hat{\Lambda} \) (see [34]).

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