On the cosmological effects of the Weyssenhoff spinning fluid in the Einstein-Cartan framework

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Abstract
The effects of non-Riemannian structures in Cosmology have been studied long ago and are still a relevant subject of investigation. In the seventies, it was discovered that singularity avoidance and early accelerated expansion can be induced by torsion in the Einstein-Cartan theory. In this framework, torsion is not dynamical and is completely expressed by means of the spin sources. Thus, in order to study the effects of torsion in the Einstein-Cartan theory, one has to introduce matter with spin. In principle, this can be done in several ways. In this work we consider the cosmological evolution of the universe in the presence of a constant isotropic and homogeneous axial current and the Weyssenhoff spinning fluid. We analyse possible solutions of this model, with and without the spinning fluid.

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1 Introduction
The non-Riemannian generalizations of General Relativity have been played an important role in gravity and particularly in Cosmology. The most simple generalization of General Relativity (preserving metricity) is achieved by the introduction of an asymmetric connection, with torsion as its antisymmetric part. It is possible to consider cosmological models in the framework of more general non-Riemannian structures (see for example the review by Puetzfeld[1]), but these cases will not be treated here. For the introduction of the foundations of the theory, see Ref. 2, and for a more recent review, including the quantum aspects of torsion, see Ref. 3.
According to Kopczynski [4], Trautman [5] and Hehl, von der Heyde, Kerplick [6], the singularity avoidance and accelerated expansion can be induced by torsion in the Einstein-Cartan theory. In this framework, torsion is completely expressed in terms of the spin sources [2], such that one has to introduce matter with spin in the gravitational action. Spin sources can be introduced, for example, by means of the Dirac action, in the light of the variational principle (See, e.g., Ref. [7], where torsion does not prevent the initial singularity, but rather enhances it).

Alternatively, one can consider a fluid with intrinsic spin, not admitting a priori a Lagrangian full description. One has to postulate a spin correction to the usual energy-momentum tensor. For example, Szydlowski and Krawiec [8] have studied the cosmological effects of an exotic perfect fluid known as the Weyssenhoff fluid [9], as well as the constraints from supernovae Ia type observations, concluding that the dust Weyssenhoff fluid provides accelerated expansion but it can not serve as an alternative to Dark Energy. Gasperi [10] considered the Weyssenhoff fluid with its energy momentum tensor (derivable from a Lagrangian) improved by Ray and Smalley [11], with spin as a thermodynamical variable. Obukhov and Korotky [12] formulated a more general variational theory describing the Weyssenhoff fluid and also applied to cosmological models with rotation, shear and expansion.

In Ref. [10], torsion provides singularity avoidance and inflation, but the expansion factor of the cosmological scale, \(a(t)\), is too small, unless the state equation parameter \(w (p = w \rho)\) of the spin fluid is fine tuned in a very special way. In this work we shall consider the Weyssenhoff fluid with the energy-momentum tensor improved by Ray and Smalley [11], in the relativistic regime, and also a constant axial current coupled with torsion, \(J^\mu S_\mu\) (\(S_\mu = \text{totaly antisymmetric torsion}\)).

## 2 Variational Principle

The action in the Einstein-Cartan framework is given by

\[
S = \int \sqrt{-g} d^4x \left\{ -\frac{1}{\kappa^2} \hat{R} + \mathcal{L}_M \right\},
\]

(1)

where metric has signature \((+ - - -)\), \(\kappa^2 = 16\pi G\) (we use units such that \(h = \epsilon = 1\)) and \(\hat{R}\) is the Ricci scalar constructed with the asymmetric connection\(^1\) \(\hat{\Gamma}^{\mu}_{\alpha \beta}\), which, by using the metricity condition \((\nabla_{\alpha} g_{\mu \nu} = 0)\) and the following definition of torsion

\[
T^{\mu}_{\alpha \beta} := \hat{\Gamma}^{\mu}_{\alpha \beta} - \hat{\Gamma}^{\mu}_{\beta \alpha},
\]

can be expressed as

\[
\hat{\Gamma}^{\mu}_{\alpha \beta} = \Gamma^{\mu}_{\alpha \beta} + K^{\mu}_{\alpha \beta},
\]

(2)

\(^1\)All quantities with an upper tilde are constructed with the asymmetric connection, and the corresponding quantities without tilde are constructed with the Riemannian (symmetric) connection.
where $\Gamma^{\mu}_{\alpha\beta}$ is the Riemannian connection (Levi-Civita connection) and the quantity $K^{\mu}_{\alpha\beta}$ is the contortion tensor, given by

$$K^{\mu}_{\alpha\beta} = \frac{1}{2}(T^{\mu}_{\alpha\beta} - T^{\alpha}_{\mu\beta} - T^{\beta}_{\mu\alpha}) .$$

The term $\mathcal{L}_M$ is the Lagrangian describing matter distribution. We consider here the following matter Lagrangian:

$$\mathcal{L}_M = \mathcal{L}_{AC} + \mathcal{L}_{SF} ,$$

where $\mathcal{L}_{SF}$ is the Lagrangian of spin fluid [11] and $\mathcal{L}_{AC}$ is the external source, present in the minimally coupling Dirac sector (see, e.g., Ref. [3]):

$$\mathcal{L}_{AC} = J^{\mu}S_{\mu} ,$$

where $J^{\mu}$ is a constant background axial current[4] and $S_{\mu}$ is the axial part of torsion, defined by $S_{\mu} = \varepsilon_{\lambda\rho\sigma\mu}T^{\lambda\rho\sigma}$ ($\varepsilon_{\lambda\rho\sigma\mu}$ is the Levi-Civita tensor, with $\sqrt{-g}\varepsilon_{0123} = 1$).

In order to vary the action and get the dynamical equations, let us choose $g^{\mu\nu}$ and $T^{\mu}_{\alpha\beta}$ as independent dynamical variables (as was done in Refs. [2, 10]), and $J^{\mu}$ as an external quantity.

The algebraic equation for torsion, coming from variation with respect to $T^{\mu}_{\alpha\beta}$, reads

$$T^{\mu\alpha\beta} = -\kappa^2 \left\{ 2J^{\sigma}\varepsilon^{\mu\sigma\alpha\beta} + \frac{1}{2}S^{\alpha\beta}u^{\mu} \right\} ,$$

where $S^{\alpha\beta}$ is the spin tensor, we have used[10, 11]

$$\tau^{\mu\nu\alpha}_{SF} = \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{SF})}{\delta K^{\mu\nu\alpha}} = \frac{1}{2}S^{\mu\nu}u^{\alpha} ,$$

and $u^{\alpha}$ is the fluid four-velocity. Substitution of Eq. (5) into the dynamical equations coming from variation with respect to $g^{\mu\nu}$ yields, after taking the average,

$$G^{\mu\nu} = \kappa^4 \left\{ -3g_{\mu\nu}J^2 - 6J_{\mu}J_{\nu} + \frac{1}{16}g_{\mu\nu}\sigma^2 - \frac{1}{8}u_{\mu}u_{\nu}\sigma^2 \right\} + g_{\mu\nu}\Lambda$$

$$+ \frac{\kappa^2}{2} \left\{ (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu} \right\} ,$$

where $2\sigma^2 = <S_{\mu\nu}S^{\mu\nu}>$, $J^{\mu} = J_{\mu}J^{\mu}$ and we have used the same energy-momentum tensor for the spinning fluid of Gasperini [10] and included the cosmological constant, $\Lambda$.

\footnote{Here we do not consider $J^{\mu} = \bar{\psi}\gamma^{\mu}\gamma^5\psi$ a priori.}
3 Dynamical Equations and Solutions

The spacetime metric is the spatially flat homogeneous and isotropic metric, \( ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2) \). For the radiation fluid \((\rho = \rho/3)\), the relevant components of Eqs. (6) can be used to get

\[
\frac{\dot{a}}{a} = \frac{\kappa^4 \sigma^2}{24} - \frac{\kappa^2}{6} \rho + \Lambda/3. \tag{7}
\]

It is remarkable that the axial current has no effect in the above equation. Further manipulations give the energy conservation, \( \dot{\rho}a + 4 \dot{a} \rho = 0 \), with solution coinciding with the standard radiation dominated solution, \( \rho = \rho_0 \frac{a^4}{t^4} \) (subscript 0 means present time). In order to find solution for \( a(t) \), one substitutes this result and the assumption\[ \sigma^2 = \gamma \rho^{3/2} \] into the temporal component of Eqs. (6),

\[
\frac{3 \dot{a}^2}{a^2} = \kappa^4 \left( -9J^2 - \sigma^2/16 \right) + \kappa^2 \rho/2 + \Lambda. \tag{8}
\]

Let us choose \( \rho_0 = 10^{-54} \text{ GeV}^4 \) and keep in mind \( \Lambda = 5 \times 10^{-84} \text{ GeV}^2 \) and \( \kappa^2 = 3.38 \times 10^{-37} \text{ GeV}^{-2} \). In the absence of the spin fluid, positivity of \( \dot{a}^2 \) establishes the upper bound \( J_0^2 \simeq 4.863 \times 10^{-12} \text{ GeV}^6 \). For \( J^2 < J_0^2 \) and \( \gamma = 0 \), there is no singularity avoidance and late accelerated expansion begins at \( a = 0.0136 \) (independently on the \( J^2 \)). Similar upper bound occurs if we include the spin fluid. For \( \gamma = 10^{-12} \), the minimum allowed \( a \) is \( 6.50 \times 10^{-39} \), the (very brief) early accelerated expansion ends at \( a = 9.19 \times 10^{-39} \) and late accelerated expansion starts at \( a = 0.0136 \). Actually, these results corresponds to the case \( J^2 = 0 \) (\( J^2 < J_0^2 \) is still too small to give substantial effect). The two different accelerated expansion epochs is guaranteed as far as \( 0 < \gamma < 8 \times 10^{59} \).

It is worth mentioning that Kostelecký, Russell and Tasson [14] have applied the recent experimental searches to find constraints for the constant torsion field. For example, if torsion is minimally coupled with fermions, it is showed (in our notations) that \( |S_{\mu} \xi^\mu| < 8.4 \times 10^{-54} \text{ GeV}^2 \). Of course it does not apply to the present consideration, because torsion depends also on the spin density and so it is not constant. But in the absence of the spin fluid, torsion is entirely written in terms of the axial current, such that it is constant. In this case, the above upper bound can be used to fix \( J^2 \lesssim 10^{18} \text{ GeV}^6 \), which is remarkably much larger than the upper bound dictated by positivity of \( \dot{a}^2 \).

The investigation for time-dependent \( J^2 \) is postponed for a forthcoming work.

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