Hyers-Ulam stability of first order linear differential equation using Fourier transform method

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Abstract. This research paper aims to identify and also tries to prove differential equation of first order that has Hyers-Ulam and Generalized Hyers-Ulam stability by inducing Fourier Transform method. The analysis of the Hyers-Ulam stability had resulted in providing a constant value for homogeneous and non-homogeneous differential equation with implementation of Fourier transform. In order to prove these facts, few application of non-homogeneous linear differential equation in connect with the main results are provided.

Key Words: Fourier Transform, Hyers-Ulam stability, Generalized Hyers-Ulam stability, Differential Equation.

1. Introduction
In recent passed, differential equation have performed an unavoidable role in the field of mathematics. The concept of stability of differential equation is nothing but a function satisfying the differential equation approximately is around the corner of an exact solution of differential equation. The history of Hyers-Ulam stability starts from middle of the 19th century. The class of stability we used in this paper was first formulated by S.M.Ulam [1] for functional equation which was solved by Hyers [2] for additive function defined on Banach space. After this result, the stability concept was investigated and generalized by M.Rassias [3] is called Hyers-Ulam-Rassias or generalized Hyers-Ulam stability.

Further, C.Alsina and R.Ger [4] were established the Hyers-Ulam stability for differential equations by replacing functional equation. Rezaei and S.M.Jung and Rassias investigate the Hyers-Ulam stability of linear differential equation by applying Laplace Transform Method. In 2014, Q.H.Algifiary and S.M.Jung [8] also derived Hyers-Ulam stability of n-th order linear differential equation with the help of Laplace Transform method. The many researcher investigate the stability problems of functional, differential and fractional differential equation in various directions-see, for example[5], [6], [7], [9].

Fourier Transform is one of the oldest and well known method to find solution of differential equation. The Fourier transform named after Joseph Fourier in 1807. It is a mathematical transform which convert a function in time domain into a function in frequency domain. In the literature, there are so many transform to solve differential equations such as Fourier, Laplace, Mellin and Hankel etc. One of the best integral transform for differential equation is Fourier transform. It converts differential equation into simple algebraic equation. After solving the algebraic equation, we can find the solution of the original equation by inverse Fourier technique.
However, the Hyers-Ulam stability of first order differential equation has not been fully reported with the help of Fourier transform method.

Motivated by above results, the aim of this paper is to prove some results related to stability based on Fourier transform approach.

2. Preliminaries

Here, we give some basic definition, properties and theorems to prove our main results.

**Definition 2.1** The linear differential equation

\[ I'(u) - aI(u) = s(u) \]  \hspace{1cm} (2.1)

has stability in sense of ”Hyers-Ulam stability” on \( \mathbb{R} \) if a differentiable mapping \( I: (-\infty, \infty) \rightarrow \mathbb{R} \) satisfies \( |I'(u) - aI(u) - s(u)| \leq \epsilon, \forall u \in \mathbb{R} \), then there exist a solution \( J: (-\infty, \infty) \rightarrow \mathbb{R} \) of Eq.(2.1) such that \( |I(u) - J(u)| \leq \kappa \epsilon, \forall u \in \mathbb{R} \). Where \( \epsilon > 0 \) is an arbitrary constant and \( \kappa > 0 \) is a Hyers-Ulam stability constant (HUS constant) for Eq.(2.1) on \( \mathbb{R} \).

**Definition 2.2** Let \( \phi: (-\infty, \infty) \rightarrow \mathbb{R}_+ \) be a mapping on \( \mathbb{R} \). We say that Eq.(2.1) has Hyers-Ulam Rassias stability if there exist a mapping \( \chi: (-\infty, \infty) \rightarrow \mathbb{R} \), depending on \( \phi \) and Eq.(2.1), such that for every function \( I \in C^1(\mathbb{R}) \) satisfying the condition

\[ |I'(u) - aI(u) - s(u)| \leq \phi(u) \forall u \in \mathbb{R} \]

there exist a solution \( J \in C^1(\mathbb{R}) \) of Eq.(1.1) such that

\[ |I'(u) - aI(u)| \leq \chi(u) \forall u \in \mathbb{R} \]

Suppose that the function \( \phi \) and \( \chi \) values are constant, then Eq.(2.1) is stable in Hyers-Ulam sense. Now, we recall basic definition of Fourier transform

**Definition 2.3** The Fourier transform of a function \( I: \mathbb{R} \rightarrow \mathbb{F} \) is defined by the integral

\[ F(I(u)) = \hat{I}(\omega) = \int_{-\infty}^{\infty} I(u)e^{i\omega u}du \]  \hspace{1cm} (2.2)

The inverse Fourier Transform is given by

\[ F^{-1}(\hat{I}(\omega)) = I(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{I}(\omega)e^{-i\omega u}d\omega \]  \hspace{1cm} (2.3)

The properties of Fourier transform is given by

(i) (Linear) The Fourier is linear transform

\[ F(aI + bJ) = aF(I) + bF(J) \]

(ii) (one-to-one) Let \( f, g: \mathbb{R} \rightarrow \mathbb{F} \) be piecewise continuous and differential functions. If \( F(I(u)) = F(J(u)) \) then \( I(u) = J(u), \forall u \in \mathbb{R} \).

(iii) (Differentiation) Let \( f \) and \( f' \) are continuous and absolutely integrable. Then, \( F(I'(u)) = -i\omega F(I(u)) \)

Remark: We can notice that differential operator is turned into ”Multiplication” operator. Similar behavior is applied to higher number of dimensions. In general,

\[ F(I^n(u)) = (-i\omega)^n F(I(u)) = (-i\omega)^n \hat{I}(\omega) \]
Definition 2.4 (Convolution) Given two functions $f$ and $g$ are both Lebesgue integrable on $\mathbb{R}$. The convolution of $f$ and $g$ is defined as

$$(I * J)(\omega) = \int_{-\infty}^{\infty} I(\omega)J(u - \omega)d\omega$$

Theorem 2.5 The Fourier transform of the convolution $I(u)$ and $J(u)$ is the product of the Fourier transform of functions $I(u)$ and $J(u)$. That is

$$F(I(u) * J(u)) = F(I(u))F(J(u))$$

The convolution is a mathematical operator which computes the amount of overlap between two functions. A small table of Fourier transform are given below and all of these results from using elementary Fourier transform definitions Eq.(2.2) and (2.3)

| Function  | Fourier Transform | Description                  |
|----------|-------------------|------------------------------|
| $\delta(u)$ | 1                 | Delta function in $u$        |
| $e^{-a|u|}$ | $\frac{2a}{a^2 + \omega^2}$ | Exponential in $u$, $a > 0$  |
| $\frac{2a}{a^2 + \omega^2}$ | $2\pi e^{-a|\omega|}$ | Exponential in $\omega$     |
| $I'(u)$  | $-i\omega \hat{I}(\omega)$ | Derivative in $u$           |
| $uI(u)$  | $i\hat{I}(\omega)$ | Derivative in $\omega$      |
| $I(u - a)$ | $e^{-ia\omega} \hat{I}(\omega)$ | Translation in $\omega$   |
| $e^{iau}I(u)$ | $\hat{I}(\omega - a)$ | Translation in $\omega$    |
| $I(au)$  | $\frac{1}{a} \hat{I}\left(\frac{\omega}{a}\right)$ | Dilation in $\omega$        |
| $I(u) * J(u)$ | $\hat{I}(\omega)\hat{J}(\omega)$ | convolution                  |
| $H(u)$   | $\frac{1}{\omega + \pi \delta(u)}$ | Unit step function          |
| $H(u)e^{-au}$ | $\frac{1}{a + i\omega}$ | $a$ is a constant           |
| $H(u)ue^{-au}$ | $\frac{1}{(a+i\omega)^2}$ | $a$ is a constant           |

3. Main result

In this section, we are going to derive our results related to Hyers-Ulam stability of the homogeneous first order differential equation

$$I'(u) - aI(u) = 0 \quad (3.1)$$

Where, $I$ is a continuously differentiable function and $a$ is a constant.

Theorem 3.1 Let $a$ is a scalar in $\mathbb{R}$ and for every $\epsilon > 0$, there exist a constant $K > 0$. If a mapping $I:(-\infty, \infty) \to \mathbb{R}$ satisfies the condition

$$|I'(u) - aI(u)| \leq \epsilon, \quad \forall u > 0 \quad (3.2)$$

Then there exist a solution $Z:(-\infty, \infty) \to \mathbb{R}$ of Eq.(3.1) such that

$$|I(u) - Z(u)| \leq K(\epsilon), \quad \forall t \in \mathbb{R}$$

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Proof. Consider a function \( q: (-\infty, \infty) \rightarrow \mathbb{R} \) such that
\[
q(u) = I'(u) - aI(u) \quad (3.3)
\]
for each \( u \in \mathbb{R} \). Suppose that \( I(u) \) be a continuously differentiable function satisfies the inequality (3.2). We have, \( |q(u)| \leq \epsilon \). The Fourier transform of Eq.(3.3) is
\[
F(q(u)) = F(I'(u)) - aF(I(u))
\]
\[
\hat{Q}(\omega) = (-i\omega)F(I(u)) - aF(I(u))
\]
\[
\hat{T}(\omega) = \frac{\hat{Q}(\omega)}{-i\omega - a} \quad (3.4)
\]
Where, \( \hat{T}(\omega) \) is Fourier Transform of \( I(u) \). We now set \( \hat{R}(\omega) = F(r(u)) = \frac{1}{-i\omega - a} \). By inverse Fourier transform
\[
r(u) = F^{-1}\left(\frac{1}{-i\omega - a}\right) = -e^{au}H(u) \quad (3.5)
\]

Where
\[
H(u) = \begin{cases} 
0 & \text{if } u < 0 \\
1 & \text{if } u \geq 0 
\end{cases}
\]
is unit step function. Let us define a function \( Z: (-\infty, \infty) \rightarrow \mathbb{R} \) such that \( Z(u) = c_1e^{-r_1u} \), where \( c_1 \) is a constant. The Fourier transform of \( Z(u) \) is
\[
F(Z(u)) = \hat{Z}(\omega) = \int_{-\infty}^{\infty} c_1e^{-r_1ue^{i\omega}du} = 0 \quad (3.6)
\]
By differentiation property of Fourier transform and In view of Eq.(3.6)
\[
F(I'(u) - aZ(u)) = (-i\omega - a) \hat{Z}(\omega) = 0
\]
We get, \( I'(u) - aZ(u) = 0 \) since Fourier transform is one to one operator.

Hence \( Z(u) \) satisfies differential Eq.(3.1). Further, from Eq.(3.4) and (3.6) and convolution property of Fourier transform
\[
F(I'(u) - Z(u)) = \hat{T}(\omega) - \hat{Z}(\omega)
\]
\[
= \frac{\hat{Q}(\omega)}{-i\omega - a}
\]
\[
= \hat{Q}(\omega) \hat{R}(\omega)
\]
\[
= F(q(u))F(r(u))
\]
\[
= F(q(u) * r(u))
\]
Thus, \( I'(u) - Z(u) = q(u) * r(u) \) by one to one property of Fourier transform. We have
\[
|I(u) - Z(u)| = |q(u) * r(u)| = \left| \int_{-\infty}^{\infty} q(\omega)r(u - \omega)d\omega \right| \leq \int_{-\infty}^{\infty} |q(\omega)r(u - \omega)| d\omega
\]
By using Eq.(3.2) and Eq.(3.5). We obtain,
\[
|I(u) - Z(u)| \leq \epsilon \int_{-\infty}^{\infty} e^{a(u - \omega)}H(u - \omega)d\omega \leq Ke
\]
where \( K = \int_{-\infty}^{\infty} e^{a(u - \omega)}H(u - \omega)d\omega \) for any value of \( u \). This completes the proof.
The result of Hyers-Ulam stability for first order non-homogeneous differential equation

\[ I'(u) - aI(u) = s(u) \]  \hspace{1cm} (3.7)

is contained in the next theorem.

**Theorem 3.2** Let \( a \in \mathbb{R}, \epsilon > 0 \) and continuous differentiable mapping \( y:(-\infty, \infty) \to \mathbb{R} \) satisfies the condition

\[ |I'(u) - aI(u) - s(u)| \leq \epsilon \quad \forall u > 0 \]  \hspace{1cm} (3.8)

for some \( K > 0 \), there exist a solution \( J: (-\infty, \infty) \to \mathbb{R} \) of Eq. (3.7) such that

\[ |I(u) - J(u)| \leq K \epsilon, \quad \forall u \in \mathbb{R} \]

**proof.** Define

\[ q(u) = I'(u) - aI(u) - s(u) \]  \hspace{1cm} (3.9)

where, \( u \in \mathbb{R} \). Assume that, the function \( I(u) \) satisfies the inequality (3.8). We have, \( |q(u)| \leq \epsilon \). By Fourier transform of Eq. (3.9) is obtained as

\[
F(q(u)) = F(I'(u)) - aF(I(u)) - \hat{S}(\omega) \\
\hat{Q}(\omega) = (-i\omega)F(I(u)) - aF(I(u)) - \hat{S}(\omega) \\
\hat{I}(\omega) = \frac{\hat{Q}(\omega) + \hat{S}(\omega)}{-i\omega - a}
\]  \hspace{1cm} (3.10)

Where \( \hat{I}(\omega) \) is Fourier Transform of \( I(u) \). We now set \( \hat{R}(\omega) = F(r(u)) = \frac{1}{-i\omega - a} \). By inverse Fourier transform

\[ r(u) = F^{-1}\left(\frac{1}{-i\omega - a}\right) = -e^{-au}H(u) \]  \hspace{1cm} (3.11)

Where \( H(u) \) is unit step function. Let \( J \) be given by \( J(u) = c_1e^{-r_1u} + s(u) * r(u) \), where \( c_1 \) is a constant. The Fourier transform to \( J(u) \), we obtain

\[ F(J(u)) = \hat{J}(\omega) = \int_{-\infty}^{\infty} c_1e^{r_1u}e^{i\omega u}dt + \hat{S}(\omega)\hat{R}(\omega) = \hat{S}(\omega)\hat{R}(\omega) \]  \hspace{1cm} (3.12)

By differentiation property of Fourier transform and In view of Eq. (3.12)

\[ F(J'(u) - aJ(u)) = (-i\omega - a)\hat{J}(\omega) = \hat{S}(\omega) \]

By Taking account of property of Fourier transform, we get

\[ J'(u) - aJ(u) = s(u) \]

Hence \( J(u) \) satisfies differential Eq. (3.7). By Eq. (3.10) and (3.12) and convolution property of Fourier transform

\[
F(I(u) - J(u)) = \hat{I}(\omega) - \hat{J}(\omega) \\
= \frac{\hat{Q}(\omega) + \hat{S}(\omega)}{-i\omega - a} - \hat{S}(\omega)\hat{R}(\omega) \\
= \hat{Q}(\omega)\hat{R}(\omega) \\
= F(q(u)) F(r(u)) \\
= F(q(u) * r(u))
\]
We obtain, \( \mathcal{I}(u) - \mathcal{J}(u) = q(u) * r(u) \).

\[
|\mathcal{I}(u) - \mathcal{J}(u)| = |q(u) * r(u)| = \left| \int_{-\infty}^{\infty} q(\omega)r(u - \omega) d\omega \right| \leq \int_{-\infty}^{\infty} |q(\omega)r(u - \omega)| d\omega
\]

In view of Eq.(3.8) and Eq.(3.11). We obtain,

\[
|\mathcal{I}(u) - \mathcal{J}(u)| \leq \epsilon \int_{-\infty}^{\infty} e^{a(u - \omega)} H(u - \omega) d\omega \leq K \epsilon
\]

where, \( K = \int_{-\infty}^{\infty} e^{a(u - \omega)} H(u - \omega) d\omega \) is HUS constant for any \( u \in \mathbb{R} \). The theorem is proved.

Now, the main result of Hyers-Ulam Rassias stability of Eq.(3.1) and Eq.(3.7) is given in the following theorems. Define \( \chi(u) : (-\infty, \infty) \to \mathbb{R} \) by

\[
\chi(u) = e^{-au} \int_{-\infty}^{\infty} e^{au} \phi(u) H(u - \omega) d\omega
\]

**Theorem 3.3** Let \( a \) is a scalar in \( \mathbb{F} \) and for every \( \epsilon > 0 \), there exist a positive constant \( K \) and continuous function \( \phi : (-\infty, \infty) \to \mathbb{R}^+ \) such that \( I : (-\infty, \infty) \to \mathbb{R} \) continuously differentiable function satisfies the property

\[
|I'(u) - aI(u)| \leq \phi(u)
\]

for all \( u > 0 \). Then there exist a solution \( \mathcal{K} : (-\infty, \infty) \to \mathbb{F} \) of differential Eq.(3.1) such that

\[
|\mathcal{I}(u) - \mathcal{K}(u)| \leq \chi(u), \quad \forall u \in \mathbb{R}
\]

**Proof.** Suppose that a function \( I \) satisfies the inequality (3.14). Define \( q : (-\infty, \infty) \to \mathbb{F} \) such that

\[
q(u) = I'(u) - aI(u) \quad u > 0
\]

We have, \( |q(u)| \leq \phi(u) \). Taking Fourier transform of Eq.(3.15),

\[
F(q(u)) = F(I'(u)) - aF(I(u))
\]

\[
\hat{Q}(\omega) = (-i\omega) F(I(u)) - aF(I(u))
\]

\[
= ((-i\omega) - a) \hat{I}(\omega)
\]

\[
\hat{I}(\omega) = \frac{\hat{Q}(\omega)}{-i\omega - a}
\]

Where \( \hat{I}(\omega) \) is consisting frequency component of a function \( I(u) \). We now set \( \hat{R}(\omega) = \frac{1}{-i\omega - a} \). By inverse Fourier transform

\[
r(u) = F^{-1}\left( \frac{1}{-i\omega - a} \right) = -e^{-au} H(u)
\]

Where, \( H(u) \) is unit step function. Let us define \( \mathcal{K} : (-\infty, \infty) \to \mathbb{F} \) as \( K(u) = c_1 e^{-r_1 u} \), where \( c_1 \) is a constant. Taking Fourier transform to \( K(u) \), we obtain

\[
F(K(u)) = \hat{K}(u) = \int_{-\infty}^{\infty} c_1 e^{-r_1 u} e^{iu} du = 0
\]

By differentiation property of Fourier transform and In view of Eq.(3.18)

\[
F(K'(u) - aK(u)) = (-i\omega - a) \hat{K}(\omega) = 0
\]
We get, $\mathcal{K}'(u) - a\mathcal{K}(u) = 0$

Hence $\mathcal{K}(u)$ satisfies differential Eq.\((\ref{eq:3.1})\). Further, it follows from Eq.\((\ref{eq:3.16})\) and \((\ref{eq:3.18})\) and convolution property of Fourier transform

\[
F(\mathcal{I}(u) - \mathcal{K}(u)) = \hat{\mathcal{I}}(\omega) - \hat{\mathcal{K}}(\omega) \\
= \frac{\hat{Q}(\omega)}{-i\omega - a} \\
= \hat{Q}(\omega)\hat{R}(\omega) \\
= F(q(u))F(r(u)) \\
= F(q(u) * r(u))
\]

We obtain, $\mathcal{I}(u) - \mathcal{K}(u) = q(u) * r(u)$.

\[
|\mathcal{I}(u) - \mathcal{K}(u)| = |q(u) * r(u)| = \left| \int_{-\infty}^{\infty} q(\omega)r(u - \omega)d\omega \right| \leq \int_{-\infty}^{\infty} |q(\omega)r(u - \omega)|d\omega
\]

In view of Eq.\((\ref{eq:3.14})\) and Eq.\((\ref{eq:3.17})\). We obtain,

\[
|\mathcal{I}(u) - \mathcal{K}(u)| \leq e\int_{-\infty}^{\infty} e^{a(u - \omega)}\phi(u)H(u - \omega)d\omega \leq K\phi(u) = \chi(u)
\]

Hence the existence of stability for Eq.\((\ref{eq:3.1})\) is proved.

**Theorem 3.4** For every continuously differentiable function $y:\(-\infty, \infty\) \rightarrow \mathbb{R}$ satisfies the condition

\[
|\mathcal{I}'(u) - a\mathcal{I}(u) - \mathcal{I}(u)| \leq \phi(u), \ \forall u \in \mathbb{R} \tag{\ref{eq:3.19}}
\]

there exist a solution $\mathcal{M}: (-\infty, \infty) \rightarrow \mathbb{R}$ of Eq.\((\ref{eq:3.7})\) such that

\[
|\mathcal{I}(u) - \mathcal{M}(u)| \leq \chi(u), \ \forall u \in \mathbb{R}
\]

**proof:** Let $\mathcal{I} \in C^1(\mathbb{R})$ satisfies \((\ref{eq:3.19})\) and define

\[
q(u) = \mathcal{I}'(u) - a\mathcal{I}(u) - s(u) \tag{\ref{eq:3.20}}
\]

for each $u > 0$. We have,

\[
|q(u)| \leq \phi(u)
\]

By the formula of Fourier transform and from Eq.\((\ref{eq:3.20})\)

\[
F(q(u)) = F(\mathcal{I}'(u) - aF(\mathcal{I}(u)) - \mathcal{S}(\omega) \\
\hat{Q}(\omega) = (-i\omega)F(\mathcal{I}(u)) - aF(\mathcal{I}(u)) - \mathcal{S}(\omega) \\
\hat{\mathcal{I}}(\omega) = \frac{\hat{Q}(\omega) + \mathcal{S}(\omega)}{-i\omega - a} \tag{\ref{eq:3.21}}
\]

Where, $\hat{\mathcal{I}}(\omega)$ represents frequency value of $\mathcal{I}(u)$. We now set $R(\omega) = F(r(u)) = \frac{1}{-i\omega - a}$. By inverse Fourier transform

\[
r(u) = F^{-1}\left(\frac{1}{-i\omega - a}\right) = -e^{-au}H(u) \tag{\ref{eq:3.22}}
\]
If we set $\mathcal{M}: (-\infty, \infty) \to \mathbb{R}$ such that $\mathcal{M}(u) = c_1 e^{-r_1 u} + s(u) * r(u)$, then by virtue of Fourier transform to $\mathcal{M}$ and Eq. (3.23), it holds

$$F(\mathcal{M}(u)) = \widehat{\mathcal{M}}(\omega) = \int_{-\infty}^{\infty} c_1 e^{r_1 u} e^{i\omega u} d\omega + \widehat{S}(\omega)\widehat{R}(\omega) = \widehat{S}(\omega)\widehat{R}(\omega)$$

(3.23)

$$F(\mathcal{M}'(u) - a\mathcal{M}(u)) = (-i\omega - a)\widehat{\mathcal{M}}(\omega) = \hat{S}(\omega) = F(s(u))$$

We get, $\mathcal{M}'(u) - a\mathcal{M}(u) = s(u)$ by taken account of one-to-one property.

Thus $\mathcal{M}(u)$ is solution of Eq. (3.7). Further, for every $u \in \mathbb{R}$ and it follows from definition of convolution, Eq. (3.21) and (3.23) that

$$F(\mathcal{I}(u) - \mathcal{M}(u)) = \hat{\mathcal{I}}(\omega) - \hat{\mathcal{M}}(\omega)$$

$$= \hat{\mathcal{Q}}(\omega) + \hat{\mathcal{S}}(\omega) - \hat{\mathcal{S}}(\omega)\hat{R}(\omega)$$

$$= \hat{\mathcal{Q}}(\omega)\hat{R}(\omega)$$

$$= F(q(u)) F(r(u))$$

$$= F(q(u) * r(u))$$

Since Fourier transform satisfies one-to-one property. It holds that, $\mathcal{I}(u) - \mathcal{M}(u) = q(u) * r(u)$. Then,

$$|\mathcal{I}(u) - \mathcal{M}(u)| = |q(u) * r(u)| = \left| \int_{-\infty}^{\infty} q(\omega)r(u - \omega) d\omega \right| \leq \int_{-\infty}^{\infty} |q(\omega)r(u - \omega)| d\omega$$

Moreover, by Eq. (3.19) and Eq. (3.22). We obtain,

$$|\mathcal{I}(u) - \mathcal{M}(u)| \leq \int_{-\infty}^{\infty} e^{\sigma(u - \omega)} \phi(u) H(u - \omega) d\omega$$

$$= \chi(u)$$

(3.24)

Which completes the proof.

4. Application to non-homogeneous linear differential equation

In this section, we provide example to illustrate our main results given above

**Example 4.1** Let $u \in (-\infty, \infty)$ and consider

$$\mathcal{I}'(u) - \frac{1}{3}\mathcal{I}(u) = \frac{2}{3} e^u + \frac{1}{3} e^{-2u} - \frac{2}{3}$$

(4.1)

Set $a = \frac{1}{3}$ and $s(u) = \frac{2}{3} e^u + \frac{1}{3} e^{-2u} - \frac{2}{3}$. Define $q : (-\infty, \infty) \to \mathbb{R}$ by

$$q(u) = \mathcal{I}'(u) - \frac{1}{3}\mathcal{I}(u) - \frac{2}{3} e^u - \frac{1}{3} e^{-2u} + \frac{2}{3}$$

Let $\mathcal{I}(u) = e^u$ and choose $\epsilon = \frac{1}{3}$. Notice that $\mathcal{I}(u) = e^u$ satisfies

$$\left| \mathcal{I}'(u) - \frac{1}{3}\mathcal{I}(u) - \frac{2}{3} e^u - \frac{1}{3} e^{-2u} + \frac{2}{3} \right| = \left| e^u - \frac{1}{3} e^u - \frac{2}{3} e^u - \frac{1}{3} e^{-2u} + \frac{2}{3} \right| = \left| \frac{2}{3} - \frac{1}{3} e^{-2u} \right| \leq \frac{2}{3}$$
That is, $|q(u)| \leq \epsilon$, for any $u \in \mathbb{R}$. The characteristic equation of Eq. (4.1) is $(-i\omega - \frac{1}{3})I(u) = 0$. Now we set $\hat{R}(\omega) = \frac{1}{(-i\omega - \frac{1}{3})}$.

From Eq. (3.11) and by inverse Fourier Transform:

$$r(u) = F^{-1}\hat{R}(\omega) = F^{-1}\left(\frac{1}{(-i\omega - \frac{1}{3})}\right) = -e^{-\frac{1}{3}u}H(u)$$

(4.2)

Where, $H(u)$ is unit step function. The general of solution of Eq. (4.1) is

$$J(u) = c_1 e^{\frac{u}{3}} + e^{u} - \frac{1}{7} e^{-2u} + 2$$

By theorem 3.2, we have

$$|I(u) - J(u)| = |q(u) * r(u)| = \left|\int_{-\infty}^{\infty} q(\omega)r(u - \omega)d\omega\right|$$

$$|I(u) - J(u)| \leq \epsilon \int_{0}^{\infty} e^{(-\frac{1}{3}(u-\omega))}d\omega = \epsilon e^{-\frac{1}{3}u}$$

(4.3)

Hence,

$$|I(u) - J(u)| \leq K\epsilon$$

Where, $K = e^{-\frac{1}{3}u}$ is "HUS Constant" for any $u \in \mathbb{R}$. Hence there exist a unique solution of $J(u)$ of Eqn. (4.1) such that inequality (4.3) is always true for any $u$. Clearly, this implies that the differential equation is Hyers-Ulam stable.

5. Conclusions

The analysis of this paper had resulted in obtaining a stability of the linear differential equation of first order with constant co-efficient in the sense of "Hyers-Ulam and Generalized Hyers-Ulam stability" with the help of Fourier transform method. That is, we proved the approximate solution is nearer to exact solution of differential equation. Hence, the Fourier transform is performed an immodest role to prove stability of homogeneous and non-homogeneous linear differential equation of order one.

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