Correlations among the Riemann zeros: Invariance, resurgence, prophecy and self-duality

W. T. Lu† and S. Sridhar*

Department of Physics and Electronic Materials Research Institute, Northeastern University, Boston, Massachusetts 02115

(Dated: March 30, 2022)

We present a conjecture describing new long range correlations among the Riemann zeros leading to 3 principal features: (i) The spectral auto-correlation is \textit{invariant} w.r.t. the averaging window. (ii) \textit{Resurgence} occurs wherein the lowest zeros appear in all auto-correlations. (iii) Suitably defined correlations lead to \textit{predictions} (\textit{prophecy}) of new zeros. This conjecture is supported by analytical arguments and confirmed by numerical calculations using $10^{22}$ zeros computed by Odlyzko. The results lead to a self-duality of the Riemann spectrum similar to the quantum-classical duality observed in billiards.

PACS numbers: 02.10.De, 03.65.Sq, 05.45.Mt, 05.45.Ac

I. INTRODUCTION

The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1}$$

with product over all prime numbers $p$. The Riemann zeta function has a simple pole at $s = 1$. Besides the trivial zeros at $s = -2, -4, -6, \cdots$, the Riemann hypothesis (RH) states that all the nontrivial zeros are distributed on line as $\frac{1}{2} \pm it_n$ with $t_n > 0$. The Riemann zeta function and its zeros have been studied for over a century \cite{2-13}. The RH still remains unproven and presents a major mathematical challenge \cite{18}.

Motivated by the Hilbert-Polya conjecture of the existence of a Hamiltonian whose quantum spectrum is comprised of the set \{t$_n$\} as the spectra, there has been a continuing quest to find such a Hamiltonian since it would prove the RH. Attempts have been made to construct certain Hamiltonian such as that with a fractal potential \cite{21} for the Riemann zeros (RZ). For a brief review about Hamiltonians and prime numbers, see \cite{20}.

The RZ \{\frac{1}{2} \pm it_n\} can also be considered as a model spectrum for arithmetic or quantum chaos \cite{21}. It is widely believed that the ordinates \{t$_n$\} of the RZ can be interpreted as the quantum spectrum of a chaotic system which does not have time reversal symmetry \cite{4, 22} though other interpretations are also possible. The most notable result in support of this notion is that of Montgomery \cite{23} who showed that the spectral correlation function $R_2(s)$ of the RZ is identical with that of a Gaussian Unitary Ensemble (GUE) which describes a chaotic system with no time-reversal symmetry in random matrix theory. This result has led to a line of inquiry which studies the statistics and correlations among the RZ. The merit of this approach is that the properties of the RZ will lead to insights into other systems where the classical dynamics is well established. These are motivated by the analog of the Riemann zeta function with Gutzwiller’s trace formula and the dynamic zeta function of chaotic systems.

In this paper, we present a conjecture which reveals new long range correlations among the RZ, obtained from a study of the auto-correlation of a smoothed Riemann spectrum. Several new properties emerge: (i) The spectral auto-correlation is \textit{invariant} w.r.t. the averaging window. We demonstrate numerically that this remarkable invariance applies over any part of the spectrum of
10^{22} zeros, and clearly suggests that the same structure is encoded in all parts of the spectrum. (ii) Resurgence occurs wherein the lowest zeros appear in all auto-correlations. This property of resurgence was first pointed out by Berry and Keating [9]. (iii) Suitably defined correlations lead to predictions (prophecy) of new zeros.

The property of resurgence and prophecy suggests a self-duality for the RZ which we represent as $C\{\frac{1}{2} + it_{n}\} \Rightarrow \{\frac{1}{2} + it_{n}\}$. This is very similar to a duality $C\{k_{n} + ik'_{n}\} \Rightarrow \{\gamma'_{i} + i\gamma''_{i}\}$ which is observed from the auto-correlation of the quantum spectrum of open chaotic billiards, and which relates the quantum resonance spectrum $\{k_{n} + ik'_{n}\}$ and classical resonance spectrum $\{\gamma'_{i} + i\gamma''_{i}\}$ of the hyperbolic n-disk repeller [24].

The paper is organized in the following. In Sec. II, we introduce a smoothed spectral function and the spectral correlations and describe a conjecture based upon the spectral correlations. A “semiclassical” derivation of the conjecture is presented in Sec. III. Numerical calculations supporting the properties of invariance, resurgence, prophecy and self-duality embodied in the conjecture are presented in Sec. IV. In Sec. V, we discuss the self-duality of the RZ with the quantum-classical duality observed in several billiard systems, and discuss the implications for a dynamic interpretation of the RZ.

II. CORRELATIONS AND A CONJECTURE

The spectral density is the sum of $\delta$-function $\rho(k) = \sum_{n} \delta(k \pm t_{n})$. In order to study the correlation among the RZ, we use a Lorentzian-smoothed spectral density

$$\rho_{\epsilon}(k) = \frac{1}{\pi} \sum_{n} \frac{\epsilon}{(k \pm t_{n})^{2} + \epsilon^{2}}.$$  

Here $t_{n}$ are the ordinates of the RZ and $\epsilon > 0$ is a small width. In the limit $\epsilon \to 0$, one gets the stick spectrum. It can be written as a continuous part plus fluctuations $\rho_{\epsilon}(k) = \langle \rho_{\epsilon}(k) \rangle + \delta \rho_{\epsilon}(k)$. The continuous part of the spectral density for the nontrivial RZ [4] is $\langle \rho(k) \rangle = (1/2\pi) \ln(k/2\pi)$ in the limit $\epsilon \to 0$. We define the following fluctuation part of the spectral density $\delta \rho_{\epsilon}(k)$ defined above is an even function. Note that the above quantity $\delta \rho_{\epsilon}(k)$ is related to the fluctuation part of the quantum time delay $\tau_{Q}(w)$ considered in [25]; $\delta \rho_{1/2}(k) = \pi \tau_{Q}(k/2)/4$. In practice, $\delta \rho_{\epsilon}(k)$ is a finite sum. For the spectral function $\delta \rho_{\epsilon}(k)$ in a window $k \in [K_{0} - \Delta, K_{0} + \Delta]$ with $K_{0} \gg \Delta \gg 1$, one notices that the contribution of RZ far outside $[K_{0} - \Delta, K_{0} + \Delta]$ is negligible for small $\epsilon$. Thus one may use a finite sum

$$\delta \rho_{\epsilon}(k) \simeq -\frac{1}{2\pi} \ln \frac{k}{2\pi} + \frac{1}{\pi} \sum_{n=N_{0}}^{N} \frac{\epsilon}{(k - t_{n})^{2} + \epsilon^{2}}.$$  

The RZ used in the above expression should be slightly outside the window such as $t_{N_{0}} \lesssim K_{0} - \Delta$ and $t_{N} \gtrsim K_{0} + \Delta$.

Define the following cross correlation of $\delta \rho_{\epsilon}(k)$ among two windows $[K_{1} - \Delta, K_{1} + \Delta]$ and $[K_{2} - \Delta, K_{2} + \Delta]$ with $K_{1}$, $K_{2}$ any real numbers and $K_{1} \leq K_{2}$ as

$$C_{\rho_{\epsilon}}^{c}(s; K_{1}, K_{2}, \Delta) \equiv \{(\delta \rho_{\epsilon}(k + k)\delta \rho_{\epsilon}(K_{2} + k + s))_{k} \mid |s| \leq \Delta, \quad k \in [-\Delta - \min(0, s), \Delta - \max(0, s)]\}.$$


The superscript “c” denotes cross. Here the average over $k$ is defined as

$$\langle f_1(k) f_2(k) \rangle = \frac{1}{b-a} \int_a^b f_1(k) f_2(k) dk.$$ 

There are two special cases

$$C_{ρc}^r(s; K_0, \Delta) \equiv C_{ρc}^r(s; K_0, K_0, \Delta) \equiv (\delta ρc(K_0 + k) \delta ρc(K_0 + k + s))_k, \quad (4)$$

$$C_{ρc}^p(s; K_0, \Delta) \equiv C_{ρc}^p(s; K_0, -K_0, \Delta) \equiv (\delta ρc(K_0 - k) \delta ρc(K_0 + k + s))_k. \quad (5)$$

In Eq. (5), we used $\delta ρc(K_0 - k) = \delta ρc(-K_0 + k)$. The superscript “r” and “p” denotes resurgence and prophecy, respectively. The meaning will be apparent later.

We now summarize the principal results of this paper. We have the following CONJECTURE:

1. **INVARIANCE** the spectral correlation is invariant and independent of averaging window. That is

$$C_{ρc}^c(s; K_1, K_2, \Delta) = C_{ρc}^c(K_2 - K_1 + s), \quad (6)$$

$$C_{ρc}^p(s; K_0, \Delta) = C_{ρc}^p(s), \quad (7)$$

$$C_{ρc}^p(s; K_0, \Delta) = C_{ρc}^p(2K_0 + s) \quad (8)$$

Note that one has $-\Delta \leq s \leq \Delta$ with $\Delta \gg \langle δ = t_n - t_{n-1} \rangle$, the mean spacing between the RZ in a window.

2. The spectral correlation is a sum of functions of the pole and zeros of the Riemann zeta function

$$C^c(s) = \Re \left[ g(1 + 2\epsilon + is, 1) - \sum_{n=1}^{\infty} g(1 + 2\epsilon + is, -2n) - \sum_{n=1}^{\infty} g(1 + 2\epsilon + is, \frac{1}{2} + it_n) \right]. \quad (9)$$

Here $g(z, \alpha)$ is a certain two-variable function. The second (third) term is the summation over trivial (nontrivial) RZ.

3. The function $g(z, \alpha)$ can be broken into diagonal and off-diagonal terms

$$g(z, \alpha) = g_{\text{diag}}(z, \alpha) + g_{\text{off}}(z, \alpha). \quad (10)$$

Here $g_{\text{off}} \ll g_{\text{diag}}$. Then we show that

$$g_{\text{diag}}(z, \alpha) = \frac{1}{2\pi^2} \frac{1}{(z - \alpha)^2} = \frac{1}{2\pi^2} \sum_{n=2}^{\infty} \frac{c_n}{(z - \alpha/n)^2} \quad (11)$$

with $c_n$’s real coefficients, $|c_n| \leq 1/n$, and $c_n \to 0$ for $n \to \infty$. For the correlation function of interest $C_{ρc}(s)$ with $s \in [a, b]$, it can be replaced with a finite sum over the pole and zeros of the Riemann zeta function

$$C^c(s) \simeq \frac{1}{2\pi^2} \Re \left[ \frac{1}{(is + 2\epsilon)^2} - \sum_{n=1}^{\infty} \frac{1}{(is + 2\epsilon + 2n)^2} - \sum_{n=N_0}^{N} \frac{1}{(is - it_n + \frac{1}{2} + 2\epsilon)^2} \right], \quad (12)$$

$$s \in [a, b], \quad t_{N_0} \lesssim a, \quad t_N \gtrsim b.$$ 

The contribution of RZ far outside $[a, b]$ is small.
4. RESURGENCE

\[ C^r_{\rho_e}(s; K_0, \Delta) = C^r(s) \approx -\frac{1}{2\pi^2} \Re \sum_{n=1}^{N} \frac{1}{(is \pm it_n + \frac{1}{2} + 2\epsilon)^2}, \quad |s| < \Delta, \quad t_N \gtrsim \Delta. \]  

5. PROPHECY

\[ C^p_{\rho_e}(s; K_0, \Delta) = C^p(2K_0 + s) \approx -\frac{1}{2\pi^2} \Re \sum_{n=N_0}^{N} \frac{1}{(is + 2iK_0 + it_n + \frac{1}{2} + 2\epsilon)^2}, \quad |s| < \Delta, \quad t_{N_0} \lesssim 2K_0 - \Delta, \quad t_N \gtrsim 2K_0 + \Delta. \]  

A graphic representation of the conjecture is shown in Fig. 1.

A few remarks are in order.

INVARIANCE The invariance of the correlations means that the same structure is encoded in the Riemann spectrum independent of the center and width of the averaging window. The cross correlation \( C^c_{\rho_e}(s; K_1, K_2, \Delta) \) between the spectral function \( \delta_{\rho_e}(k) \) in any two windows \([K_1 - \Delta, K_1 + \Delta]\) and \([K_2 - \Delta, K_2 + \Delta]\) is the same if the center distance \( K_2 - K_1 \) is the same.

RESURGENCE We find that the peaks of \( -C^r_{\rho_e}(s) \) are located at the positions \( s = t_n \). The correlation between RZ in \([K_0 - \Delta, K_0 + \Delta]\) leads to RZ in \([-\Delta, \Delta]\). The auto-correlation \( C^r_{\rho_e}(s) \) in Eq. (11) is essentially the two-level correlation \( R_2(s) \) [6]. Resurgence was first noted by Berry and Keating [7, 9, 10] and has been revisited recently by Leboeuf and collaborators [12, 13, 14]. While the correlations studied in the present paper are related to the above work, the difference
is that here, there is no unfolding and no rescaling by the mean level spacing, and reveals some unexpected new results among the RZ summarized in the equations above.

PROPHECY can lead to the prediction of new zeros in the interval $[2K_0 - \Delta, 2K_0 + \Delta]$ from interval $[K_0 - \Delta, K_0 + \Delta]$.

Resurgence and prophecy can also be observed in the cross correlation $C_{\rho_\epsilon}^c(s; K_1, K_2, \Delta)$.

### III. TRACE FORMULA DERIVATION

The above properties are “derived” using a semiclassical method [28]. The fluctuation part of the spectral density of the RZ is [4, 25]

$$\delta \rho_\epsilon(k) = -\frac{1}{\pi} \sum_p L_p \sum_{r=1}^{\infty} \Lambda_r^{-r(\frac{1}{2} + \epsilon)} \cos(rkL_p)$$

(15)

Note that this is exactly of the same form as the Gutzwiller trace formula for chaotic billiards in wave number $k$ [27], with length of “periodic orbit” equal to the logarithm of prime numbers $L_p = \ln p$ and $\Lambda_p = e^{L_p} = p$. Like that of the chaotic systems such as the chaotic $n$-disk systems [24], the number of “periodic orbits” for the RZ grows asymptotically as $N(L_p < L) \sim e^{L}/L$.

Although the above trace formula converges only for $\epsilon > \frac{1}{2}$, we may still use it formally for $\epsilon \to 0$. The correlation can be written as the sum of diagonal and off-diagonal terms

$$C_{\rho_\epsilon}^c(s) = \langle \delta \rho_\epsilon(k) \delta \rho_\epsilon(k + s) \rangle = C_{\text{diag}}^c(s) + C_{\text{off}}^c(s).$$

The diagonal part is

$$C_{\text{diag}}^c(s) = \frac{1}{2\pi^2} \sum_p L_p^2 \sum_{r=1}^{\infty} \Lambda_r^{-r(1+2\epsilon)} \cos(rsL_p) = -\frac{1}{2\pi^2} \Re \frac{\partial^2}{\partial s^2} \sum_p \sum_{r=1}^{\infty} \frac{1}{r^2} t_p^r$$

(16)

with $t_p = \Lambda_p^{-(1+2\epsilon)} e^{isL_p} = p^{1-2\epsilon+is}$.

$C_{\text{diag}}^c(s)$ can also be broken into two parts [12, 17]

$$C_{\text{diag}}^c(s) = C_\zeta^c(s) - C_{\mathcal{F}}^c(s)$$

(17)

with

$$C_\zeta^c(s) = -\frac{1}{2\pi^2} \Re \frac{\partial^2}{\partial s^2} \ln(1 + 2\epsilon + is)$$

and

$$C_{\mathcal{F}}^c(s) = -\frac{1}{2\pi^2} \Re \frac{\partial^2}{\partial s^2} \ln \mathcal{F}_\epsilon(s).$$

(18)

Here

$$\ln \mathcal{F}_\epsilon(s) = \sum_p \sum_{r=1}^{\infty} \frac{r}{(r+1)^2} t_p^r.$$  

(19)

Alternatively, one has the following expressions by direct summation over $r$

$$C_{\text{diag}}^c(s) = \frac{1}{2\pi^2} \Re \sum_p L_p^2 \frac{t_p}{1 - t_p},$$

$$C_\zeta^c(s) = \frac{1}{2\pi^2} \Re \sum_p L_p^2 \frac{t_p}{(1 - t_p)^2},$$

$$C_{\mathcal{F}}^c(s) = \frac{1}{2\pi^2} \Re \sum_p L_p^2 \frac{t_p^2}{(1 - t_p)^2}.$$  

(20)
These expressions have absolute convergence. Obviously, one has $C_\zeta^\epsilon(s) < \frac{1}{2} C_\zeta(s)$.

In order to obtain a compact form for $C_\zeta^\epsilon(s)$, we notice the following expression for the Riemann zeta function $\zeta(s)$
\[
\zeta(s) = \frac{e^{(\ln 2\pi - \frac{1}{4})s}}{2(s - 1)\Gamma(1 + \frac{s}{2})} \prod_n \left(1 - \frac{s}{\gamma_n}\right) e^{s/\gamma_n}
\]
with $\gamma_n = \frac{1}{2} \pm it_n$. One has
\[
\frac{\partial^2}{\partial s^2} \ln \zeta(1 + is) = \frac{1}{s^2} + \frac{1}{4} \psi_1(\frac{3}{2} + i\frac{1}{2}s) + \sum_n \frac{1}{(1 + is - \gamma_n)^2}.
\]
Here $\psi_n(z) \equiv d^{n+1}\Gamma(z)/dz^{n+1}$ is the polygamma function with
\[
\psi_n(z) = (-1)^{n+1} n! \sum_{j=0}^{\infty} \frac{1}{(j + z)^{n+1}}.
\]
Define the following function
\[
Z(z) \equiv \frac{1}{(z - 1)^2} = \sum_{j=1}^{\infty} \frac{1}{(z + 2j)^2} - \sum_{j=1}^{\infty} \frac{1}{(z - \frac{1}{2} \pm it_j)^2},
\]
one thus has
\[
C_\zeta^\epsilon(s) = \frac{1}{2\pi^2} \Re Z(1 + 2\epsilon + is).
\]
In order to obtain a compact form for the term $C_\zeta^\epsilon(s)$, we use the following expansion for small $t$
\[
\sum_{q=2}^{\infty} \frac{q - 1}{q^2} t^q = \sum_{n=2}^{\infty} \sum_{r=1}^{\infty} c_n \frac{t^{nr}}{r}.
\]
The coefficients $c_n$ can be obtained by equalizing the coefficients of $t^n$ on both sides of the equation. For prime numbers and the power of 2, one has
\[
c_p = (p - 1)/p^2,
\]
\[
c_{2n} = 2^{-2n}, \quad n \geq 1.
\]
The rest of the $c_n$'s are given by the following recursive equation
\[
\sum_{n \geq 2, r \geq 1, nr = q} \frac{1}{r} c_n = \frac{q - 1}{q^2}.
\]
For example, one has $c_6 = -1/18, c_{10} = -1/25, c_{12} = -1/72, c_{14} = -3/98, c_{15} = -8/15^2$.
Using the expansion (24) and function $Z(z)$, one has
\[
C_\zeta^\epsilon(s) = \frac{1}{2\pi^2} \Re \frac{\partial^2}{\partial s^2} \sum_{n=2}^{\infty} c_n \ln [n(1 + 2\epsilon + is)]
\]
\[
= \frac{1}{2\pi^2} \Re \sum_{n=2}^{\infty} n^2 c_n Z[n(1 + 2\epsilon + is)].
\]
One thus gets

\[ C_{\text{diag}}^e(s) = -\frac{1}{2\pi^2} \Re \sum_{n=1}^{\infty} n^2 c_n Z[n(1 + 2\epsilon + is)] \]

\[ = -\sum_{n=1}^{\infty} n^2 c_n C^{141/2 + n\epsilon}(ns) \]  

(28)

with \( c_1 = -1 \). If one ignores the off-diagonal term \( C_{\text{off}}^e(s) \), one obtains \( g(z, \alpha) \) given by Eq. (11).

The value of the correlation at the origin can also be calculated in the following

\[ C_{\text{diag}}^e(0) = \frac{1}{2\pi^2} \sum_p L_p^2 \sum_{r=1}^{\infty} \Lambda_r^{-r(1+2\epsilon)} = \frac{1}{2\pi^2} \sum_p \frac{L_p^2}{e^{(1+2\epsilon)L_p} - 1}. \]

The average density of periodic orbits for large \( L \) is \( \rho(L) = d(e^L/L)dL = (L-1)e^L/L^2 \). Since one has

\[ \int_L^{\infty} \frac{L^2 \rho(L)dL}{e^{(1+2\epsilon)L} - 1} \approx \int_L^{\infty} e^{-2\epsilon L} LdL = \frac{1}{4\epsilon^2} e^{-2\epsilon L}. \]

Simply set \( L = \ln 2 \), one has \( C_{\text{diag}}^e(0) \approx (1 + 2\epsilon \ln 2)/8\pi^2 \epsilon^2 4^\epsilon \). For example for \( \epsilon = 0.5 \), one has \( C_{\text{diag}}^e(0) \approx 0.0429 \). In the limit \( \epsilon \to 0 \), one has \( C_{\text{diag}}^e(0) \approx 1/8\pi^2 \epsilon^2 \).

### IV. NUMERICAL RESULTS

In order to check the validity of the conjecture, we construct \( \delta \rho_\epsilon(k) \) according to Eq. (2) from a finite sequence of \( t_n \) such that \( t_{N_0} \lesssim K_0 - \Delta \) and \( t_N \gtrsim K_0 + \Delta \). Define \( \delta_N = 2\pi/\ln(t_N/2\pi) \) which is the local level spacing in the neighbourhood of \( t_N \). We use \( \epsilon \sim \delta_N \). One has \( \delta_N \approx 0.669, 0.257, 0.141, 0.134 \) for \( N = 10^3, 10^{12}, 10^{21}, 10^{22} \), respectively. The RZ we studied are obtained from Odlyzko [30]. We find the following properties of spectral correlations of the RZ.

#### A. Correlation invariance

For the first \( 10^5 \) Riemann zeros, the fluctuation part of the spectral density \( \delta \rho_\epsilon(k) \) is constructed. We equally divide them into 5 segments : \([K_n - \Delta, K_n + \Delta]\) with \( K_n = (2n-1)\Delta + 10, \Delta = 7490, \) and \( n = 1, 2, \cdots 5 \). Note that \( t_10^5 = 74920.8 \). We then evaluate the auto-correlation \( C_{\rho_\epsilon}^e(s) \) for each segment. We find that without any normalization, all 5 of them collapse to the same curve. As a comparison, we also calculated the auto-correlation for every \( 2 \times 10^4 \) of the first \( 10^5 \) RZ. The correlations are also the same. The relative difference between them is less than 0.1%.

In the same way, we consider four groups of RZ: \( \{1, 10^5\}, \{10^{12} + 1, 10^{12} + 10^4\}, \{10^{21} + 1, 10^{21} + 10^4\}, \) and \( \{10^{22} + 1, 10^{22} + 10^4\} \). For the spectral correlations of the above four segments of RZ with the same width \( \epsilon \), they also all collapse to the same curve without any normalization as shown in Fig.2.

In general, one would expect that the spectral correlation will depend on the choice of \( K_0 \) and \( \Delta \) in Eq.(1). Even though the spectrum is more congested around \( t_n \sim K_0 \) for larger \( n \), they all have almost the same spectral auto-correlation. We find that the main feature of the auto-correlation is only determined by the choice of \( \epsilon \) and independent of \( K_0 \) and \( \Delta \). So no matter how high the RZ located on the critical line, they are all correlated in the same way as the low lying RZ. The
dependence on \( K_0 \) and \( \Delta \) will show up for very small width \( \epsilon \) such that \( \epsilon < \delta \) and very narrow \( \Delta \). This is shown in Fig. 3 where we used \( \epsilon = 0.1 < \delta_{10^{12}} \simeq 0.134 \).

We point out that the diagonal approximation is very accurate for the Riemann zeros. Numerically calculated \( C^r_\rho(\epsilon)(s) \) are almost indistinguishable from the direct evaluation of \( C_{\text{diag}}(s) \) given in Eq. (20) using the first 350,000 primes for \( s < 500 \). For example for \( \epsilon = 0.25, 0.5 \), one has the standard deviation \( \sigma = \langle (C^{2}_\rho - C^{2}_\xi)/\langle C^2_\rho \rangle \rangle^{1/2} \sim 0.022, 0.031 \), respectively for the spectral correlation of the RZ \( \{t_n\} \) with \( n \sim 10^{12} \).

\[ \Delta C_2(s) = C^r_\rho(s; K_0, \Delta) - C^r_\xi(s) + C^r_{\rho^2+2\epsilon}(2s; K_0, \Delta), \]
respectively. To improve the fitting, one can use \( C_s \in K \). The center distance between these two windows is \( \Delta = 620 \sim 10^{22} \) windows \([K, \sigma] \). Further define \( \rho \), one has the standard deviation reduced to \( \sigma = 0.041, 0.036 \) for \( \epsilon = 0.25, 0.5 \), respectively. This is comparable with the off-diagonal term. The behaviors of \( C_s^r - C_s^c, \Delta C_2, \) and \( \Delta C_3 \) for \( \epsilon = 0.25 \) are plotted in Fig.5.

Resurgence can also be observed in the cross correlation \( C_s^c(s; K_1, K_2, \Delta) \) in Eq. 8 for two windows \([K_1 - \Delta, K_1 + \Delta]\) and \([K_2 - \Delta, K_2 + \Delta]\) with \( K_1 \sim 3.0 \). Suppose we consider the set \( \{ t_n \} \) with \( 10^{22} < n \leq 10^{22} + 10^4 \). We use \( K_1 = 0, K_2 = (t_{10^{22}+1} + t_{10^{22}+10^4})/2 \), and choose the window size \( \Delta = 620 \sim (t_{10^{22}+10^4} - t_{10^{22}+1})/2 \) in (8) so that \( K_2 + \Delta \sim t_{10^{22}+10^4} \). The spectral function \( \rho_c(k) \) is constructed according to Eq. (11) in windows \([-\Delta, \Delta]\) and \([K_2 - \Delta, K_2 + \Delta]\). The center distance between these two windows is \( K_2 \). The cross correlation \( C_s^c(s; 0, K_2, \Delta) \) with \( s \in [-\Delta/2, \Delta/2] \) and \( \epsilon = 0.25 \) is shown in Fig.4. One can see that \( C_s^c(s; 0, K_2, \Delta) \) can be well approximated by \( C_s^c(K_2 + s) \) with \( \sigma = \langle (C_s^c - C_s^c)^2 \rangle^{1/2}/(C_s^c)^{1/2} \sim 0.199, 0.171 \) for \( \epsilon = 0.25, 0.5 \), respectively. To improve the fitting, one can use

\[
C_s^c(s; 0, K_2, \Delta) \approx C_s^c(K_2 + s) - C_s^c_{p, \Delta}^{\sigma} (2s; K_2, \Delta).
\]

Here \( C_s^c(s; K_2, \Delta) \) with \(-\Delta \leq s \leq \Delta\) is the cross correlation \( 5 \) between \([-K_2 - \Delta, -K_2 + \Delta]\) and \([K_2 - \Delta, K_2 + \Delta]\). This time, the standard deviation is reduced to \( \sigma = 0.118, 0.079 \) for \( \epsilon = 0.25, 0.5 \), respectively. Similar results are also obtained for \( K_2 \sim t_{10^{12}} \) and \( K_2 \sim t_{10^{21}} \).

We point out that the small \( s \) behavior of \( C_s^r(s) \) is not a Lorentzian, contrary to the claim in 26. According to (12), one has

\[
C_s^r(s) \approx (4\epsilon^2 - s^2)/2\pi^2(s^2 + 4\epsilon^2)^2
\]

for \( s \leq 5 \). So that \( C_s^r(0) \approx 1/(8\pi^2\epsilon^2) \). This value is very close to that at the origin for different auto-correlations. The behavior of \( C_s^r(s) \) for small \( s \) is shown in Fig.7 for different segments of RZ.
FIG. 4: Resurgence of RZ in $C_r^\rho(s; K_0, \Delta)$ for $\epsilon = 0.25$. Here $C_r^\rho(s; K_0; \Delta)$ is the numerically calculated auto correlation in $[K_0 - \Delta, K_0 + \Delta]$ with $K_0 = (t_{10^{12} + 1} + t_{10^{12} + 10^4})/2$ and $\Delta \approx (t_{10^{12} + 10^4} - t_{10^{12} + 1})/2$. The dashed curve is $C_\eta(s)$. Vertical lines are located at $t_n$ and coincide with the valleys.

FIG. 5: Resurgence of RZ in $C_r^\rho(s; K_0, \Delta)$ for $\epsilon = 0.25$. Here $C_r^\rho(s; K_0; \Delta)$ is the numerically calculated auto correlation in $[K_0 - \Delta, K_0 + \Delta]$ with $K_0 - \Delta = t_{10^{12} + 1}$ and $K_0 + \Delta \approx t_{10^{12} + 10^4}$. (a) $C_r^\rho(s; K_0, \Delta) - C_\eta(s)$, (b) $\Delta C_2(s)$, (c) $\Delta C_3(s)$. 

can be better fitted by Eq. (32) with fitting parameter $\beta$ of the time delay $\beta$ with fitting parameter $\delta_{p_{c}}$ of RZ with small $\delta_{p}$ and choose the window size $\Delta = (K + \Delta) / 2$ to prophecy of new RZ from $C_{\rho c}(s; 0, K_2, \Delta)$ discussed in [26] with $\delta_{p} = 0.25$ as shown in Fig.9. As we mentioned before, $\rho_{c}(k)$ is related to the auto-correlation of the time delay $\tau_{n}(w)$, so $C_{\rho c}(s)$ is related to the auto-correlation of $c(\varepsilon; w_{c}, \Delta)$ discussed in [26] with $\varepsilon = 0.5$ and $s = 2\varepsilon$. Thus for small $\varepsilon$, $c(\varepsilon; w_{c}, \Delta)$ can be better fitted by Eq. (32) with $\beta = 0.46$ as shown in Fig.8b.

C. Prophecy

By prophecy of RZ we mean the appearance of RZ $\{1/2 \pm it_{n}\}$ with large $n$ from the correlation of RZ with small $n$. We consider the correlation $C_{\rho c}(s; \Delta, \Delta)$ in Eq. (34) for RZ $\{1, 10^{4}\}$. The prophecy of new RZ from $C_{\rho c}(s; K_2, \Delta)$ with $\varepsilon = 0.25$ is shown in Fig.9. The agreement with Eq. (34) is very good.

Prophecy can also be observed in the cross correlation $C_{\rho c}(s; K_1, K_2, \Delta)$ in Eq. (34) for two windows $[K_1 - \Delta, K_1 + \Delta]$ and $[K_2 - \Delta, K_2 + \Delta]$ with $K_1 + \Delta = 0$. As in the above subsection, we consider the set $\{t_{n}\}$ with $10^{22} < n \leq 10^{22} + 10^{4}$. We use $K_1 = -\Delta$, $K_2 = (t_{10^{22}+1} + t_{10^{22}+10^{4}}) / 2$, and choose the window size $\Delta = (t_{10^{22}+1} + t_{10^{22}+10^{4}}) / 2 - 50 = 620.8$. $\rho_{c}(k)$ is calculated in windows $[-2\Delta, 0]$ and $[K_2 - \Delta, K_2 + \Delta]$. The center distance between these two windows is $K_2 + \Delta \approx t_{10^{22}+10^{4}}$. The cross correlation $C_{\rho c}(s; K_1, K_2, \Delta)$ is shown in Fig.10. If $s \leq 50$, one has resurgence of RZ since $s + \Delta + K_2 \leq t_{10^{22}+10^{4}}$. If $s > 50$, one has prophecy of new RZ. Similar results are also obtained for $K_2 \sim t_{10^{22}}$ and $K_2 \sim t_{10^{21}}$. The above long range correlation can be used to calculate RZ from known ones. When applied to prophecy correlation $C_{\rho c}(s)$, new RZ can be obtained. The Fourier transformation of $C_{\rho c}(s)$ is

$$\tilde{C}(x) = \int C_{\rho c}(s)e^{ixs}ds \simeq -\frac{1}{2\pi}x \sum_{n} e^{it_{n}x - \gamma x} \tag{34}$$
FIG. 7: Small $s$ behavior of $C_r(s)$ with (a) $\epsilon = 0.25$ and (b) $\epsilon = 0.5$ for RZ: \{1, 10^5\} (dashed), \{10^{12} + 1, 10^{12} + 10^4\} (dash-dotted), \{10^{21} + 1, 10^{21} + 10^4\} (dotted), and \{10^{22} + 1, 10^{22} + 10^4\} (solid). The bold solid line is $C_\zeta(s)$ (Eq. (12)).

FIG. 8: Small $s$ behavior of $C_r(s)/C_r(0)$ with (a) $\epsilon = 0.25$, $\beta = 0.24$ and (b) $\epsilon = 0.5$, $\beta = 0.46$ for RZ: \{1, 10^5\} (dashed), \{10^{12}+1, 10^{12}+10^4\} (dash-dotted), \{10^{21}+1, 10^{21}+10^4\} (dotted), and \{10^{22}+1, 10^{22}+10^4\} (solid). Here $\beta$ is the fitting parameter in Eq. (33).
FIG. 9: Prophecy of the RZ in $C_p^\rho(s;K_0,\Delta)$ for RZ $\{1,10^4\}$ with $\epsilon = 0.25$, $K_0 = 4941.4$, and $\Delta = 4920$. The dashed curve is $C_\xi^\rho(2K_0 + s)$. The vertical lines are at $s = t_n - 2K_0$.

FIG. 10: Prophecy of the RZ in the cross correlation $C_p^c(s;\Delta,K_2,\Delta)$ between $[-2\Delta,0]$ and $[K_2-\Delta,K_2+\Delta]$ for $\epsilon = 0.25$. (a) $C_p^c(s;\Delta,K_2,\Delta)$, (b) $C_p^c(s;\Delta,K_2,\Delta) - C_\xi^\rho(s+\Delta+K_2)$. Here $C_\xi^\rho(s+\Delta+K_2)$ is calculated from Eq. (12) with $t_n$ only up to $N = 10^{22} + 10^4$.

with $\gamma = 1 + 2\epsilon$. $\tilde{C}(x)$ will have peaks at $x = rL_p$. Since the above expression is a sum over different exponentially decaying modes, one may use the Prony algorithm to get $t_n$. The above method provides a way to calculate or at least to estimate new RZ.
V. DISCUSSION: SELF-DUALITY OF THE RIEMANN SPECTRUM

The motivation for this work has come from experimental observations in the microwave transmission of open $n$-disk billiards which led to the observation of classical Ruelle-Pollicott (RP) resonances in the auto-correlation of quantum spectra of hyperbolic $n$-disk open billiards [24]. The result established a new approach to quantum-classical correspondence by demonstrating a correspondence between the quantum and classical resonance spectra of an open chaotic system. Applying the same procedures developed there to the Riemann spectrum, we have arrived at the results described in this paper. There are other parallels which are summarized below.

1. In $n$-disk open billiards mentioned above [24], the experimental trace was fitted well by the sum of Lorentzians $T(k) = \sum_n b_n/[(k - k_n)^2 + k_0^2]$. Here $\{k_n + ik_n'\}$ is the spectrum of eigen-wave vector resonances, which are eigenvalues of the Helmholtz (or Laplace-Beltrami) operator, with Dirichlet boundary conditions $(\nabla^2 + k^2)\psi = 0, \psi = 0$ on $\partial D$. We showed that the auto-correlation of the transmission $C(\kappa) = \langle T(k)T(k + \kappa) \rangle$ are well-described as a sum of Lorentzians $C(\kappa) = \sum_i c_i/[(\kappa - \gamma_i)^2 + \gamma_i'^2]$. Here $\gamma_i + i\gamma_i''$ are the classical RP resonances. The coefficients $c_i$ are related to the eigen-functions of the Perron-Frobenius operator. The $c_i$ are all positive, whereas the coefficients are negative for the RZ. For hyperbolic systems, Ruelle has related these eigenvalues to the time-evolution of classical correlations [31]. The above result can be viewed in the context of dynamic Ruelle zeta functions which are relevant to the quantum-classical correspondence by demonstrating a duality between the quantum and classical resonances.

2. For the two-disk billiard (i.e. $n = 2$), the semiclassical resonances in wave vector space are $k_n = [n\pi + i(1/2)\ln \Lambda]/(R - 2a)$ with $\Lambda = \sigma - 1 + \sqrt{\sigma(\sigma - 2)}$ the eigenvalue of the instability matrix in the fundamental domain and $\sigma = R/a$. Here $n$ is odd for $A_1$ representation, $n$ even and $n \neq 0$ for $A_2$ representation [34]. The classical RP resonances of the two-disk system are $\gamma_n = [\ln \Lambda + in\pi]/(R - 2a)$. In this case there is a direct 1-to-1 correspondence between quantum and classical resonances.

3. In the cases of the rectangle billiard and the equal-lateral triangle billiard, the auto-correlation $C^r_{\rho c}(s)$ are shown [35] to correspond to the Fourier transform of the trace of the classical evolution operator, i.e.

$$C_{\rho c}(s) \simeq F \left[ \text{tr} C^t = \sum_p \sum_{r=1}^{\infty} \frac{a_p}{r L_p} \delta(l - r L_p) \right] = \sum_p \sum_{r=1}^{\infty} \frac{a_p}{r L_p} \cos(rs L_p).$$

There again the property of correlation invariance and resurgence are clearly demonstrated.

4. For a particle on a surface of constant negative curvature, a self-duality exists for the quantum momentum spectrum, $\{p_n\}$ and the classical spectrum $\{\gamma_n = \frac{1}{2} \pm ip_n\}$ [30]. The quantum eigen-energy are given by $E_n = \frac{1}{2} + p_n^2$. 

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Motivated by these observations, we examine a dynamic interpretation of the Riemann zeta function. One has the following expression due to Riemann

\[ \sum_p \sum_r \frac{1}{r} \delta(x - p^r) = \frac{1}{\ln x} - \frac{1}{x(x^2 - 1) \ln x} - 2 \sum_n \frac{\cos \left( t_n \ln x \right)}{x^2 \ln x} \]  

(35)

where \( x > 1 \). Define \( l = \ln x \), multiple \( \ln x \) to both sides, one has

\[ \sum_p \sum_r \frac{\ln p}{p^r} \delta(l - r \ln p) = 1 - \frac{e^{-2l}}{2 \sinh l} - \sum_n e^{-\left( \frac{1}{2} \pm i t_n \right) l} \]  

(36)

which is valid for \( l > 0 \). Note the analog with the classical trace formula

\[ \text{tr} \mathcal{L}^t = \sum_p \sum_r \frac{L_p}{\Lambda_p} \delta(l - r \Lambda_p) \]

with \( l = t \sqrt{E} \). Thus one has formally

\[ \text{tr} \mathcal{L}^t = 1 + \sum_n g_n e^{-\gamma_n l} = 1 - \sum_{n=1}^{\infty} e^{-(2n+1)l} - \sum_{n=1}^{\infty} e^{-\left( \frac{1}{2} \pm i t_n \right) l}. \]

(37)

This is very similar to the case for the motion on the surface of constant negative curvature, where the quantum eigen-momentum spectrum is \( \{ p_n \} \). There \( \text{tr} \mathcal{L}^t = 1 + \sum_{n=1}^{\infty} e^{-\left( \frac{1}{2} \pm i t_n \right) l} \). Furthermore Biswas and Sinha have shown that the classical resonances are given by \( \gamma_n = \frac{1}{2} \pm it_n \) by demonstrating that the peaks of the power spectrum of classical correlations indeed coincide with \( \{ p_n \} \).

For the RZ, the result of performing the auto-correlation of the smoothed density of states then leads to a self-dual “classical” spectrum \( \{ \frac{1}{2} \pm it_n \} \). So one might say that the classical resonances are \( \gamma_n = \frac{1}{2} \pm it_n \). This interpretation of RZ as RP resonances was suggested recently \[12, 14\]. However here \( g_n = -1 \) for the RZ while \( g_n = 1 \) for the Riemann zeta function pole and RP resonances. This is related to the well-known sign problem for which a possible explanation has been offered by Connes \[39\].

We have clearly established a self-duality which then strongly suggests the interpretation of the spectrum \( \{ t_n \} \) as momentum or wave vector eigenvalues of a Helmholtz operator on a suitable domain. A similar description has been made in which the Riemann zeta function appears in the scattering matrix of the quantum scattering of a particle on a 2D surface of constant negative curvature \[26, 29\], so that \( \{ \frac{1}{2} t_n \} \) are the real parts of the poles of the scattering matrix.

VI. CONCLUSION

In this paper, we report the observation of new correlations among the RZ that presented as a conjecture supported by analytical arguments and numerical simulations. The spectral correlations possess three important properties: invariance, resurgence and prophecy. The prophecy can be used to progressively obtain new RZ from known ones. The last observation suggests an interesting new strategy for proving the Riemann hypothesis via a bootstrap approach starting with known zeros.

We thank F.Y. Wu and J.V. José for discussions. This work is supported partially by NSF-PHY-0098801.
†w.lu@neu.edu *s.sridhar@neu.edu

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