A Polynomial Time Algorithm for the Hamilton Circuit Problem

Hanlin Liu
Computer science and technology
Jinan University
abc517256409@qq.com

ABSTRACT
In this paper, we introduce two new algorithms to find a Hamilton Circuit in a graph $G = (V, E)$. One algorithm is use a multistage graph as a special NFAs to find all Hamilton Circuit in exponential time; while another is use $O(|V|)$ variant multistage graph as a special state set NFAs to find a fuzzy data of all Hamilton Circuit in polynomial time. The fuzzy data of the data contain those data, and the fuzzy data is not empty, if and only if the data is also not empty. And, the data is also not empty if and only if there are Hamilton Circuit in the graph. And we can find a Hamilton Circuit from the fuzzy data. Our result implies NP=P.

Categories and Subject Descriptors
F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems – Complexity of proof procedures; F.1.3 [Computation by Abstract Devices]: Complexity Measures and Classes – Relations among complexity classes; G.2.2 [Discrete Mathematics]: Graph Theory – Graph algorithms

General Terms
Algorithms, Theory

Keywords
Algorithm, Hamilton Circuit problem, NP complete problem, NP/P, NFAs,

1. INTRODUCTION
The Hamilton Circuit problem is a well-known NP-complete problem [1]. This famous problem can be described as follows:

A Hamiltonian path is a path in an undirected or directed graph that visits each vertex exactly once. A Hamiltonian circuit is a Hamiltonian path that is a cycle.

We can see a path as sequence of vertex, and sequence of vertex as location of vertex, since we know every location of vertex in a path, if and only if we know the path. We just find all possible vertex of every location in a Hamilton Circuit, not all possible Hamilton Circuit in this algorithm. It is common solution of this problem. And the differences between our solution with other is that we use a special NFAs and a special state set NFAs, two new data structure, to store data. A special NFAs can store all possible Hamilton Circuit and all possible vertex of every location in a Hamilton Circuit, while a special state set NFAs just store all possible vertex of every location in a Hamilton Circuit, not store all possible Hamilton Circuit.

The Hamilton circuit of a graph $G = (V, E)$ can be seen as a path whose length is $|V|$, beginning vertex and ending vertex are same vertex and all vertex is in this path.

First step, we suppose the vertex of the first location is vertex $a, a \in V$; and second step we can find all possible vertex of the second location, then all possible vertex of the third location and so on in a path whose vertex is not repeat.

The two different algorithms have same algorithms ideas, but different data structure.

In this paper, we introduce two data structure, one is the special NFAs which is implemented by a multistage graph, while the special state set NFAs which is variance version of the special NFAs are implemented by $O(|V|)$ variant multistage graph.

Two kinds of multistage graph have similar definition: vertex set are all possible vertex of every location, while edge set are all relation between two possible vertexes in neighboring location. But, the difference of two kinds of multistage graph is that the multistage graph just compress all vertex which have same edges from it to the ending vertex or from beginning vertex to it, and variant multistage graph compress all vertex which have same edge from the vertex of prior stage to it.

2. DEFINITION
In this section, we introduce the definition of a special NFAs and a special state set NFAs with some lemmas about those NFAs.

2.1 Definition of Special NFAs
Definition 1: a special NFAs is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. $Q$ is a finite set called the states and $Q$ is union set of $Q_0, Q_1, Q_2, Q_3, \ldots, Q_n$ with the intersection every two subset is empty set.
2. $\Sigma$ is a finite set called the alphabet.
3. $\delta: Q_l \times \Sigma \rightarrow Q(l + 1)$ or null is a function called the transition function with $0 \leq l < (n - 1)$ and every state $q(l + 1), q(l + 1) \in Q(l + 1)$ have same $a, a \in \Sigma$ and different $q_l, q_l \in Q_l$ in $\delta: q_l \times a \rightarrow q(l + 1)$.
4. $q_0$ is an element of $Q_0$ called the start state and $|Q_l| = 1$.
5. $F$ is a subset of $Q_n$ called the accept states.

The string of this special NFAs have limitation of length, and this NFAs can store data as the pattern of data structure. In this algorithms, we find all possible vertex location by location. So, the data in this special NFAs of this algorithms have same length in every step.
**Lemma 1:** these special NFAs can use a multistage graph to store data.

**Proof:** every data can be seen as a set of strings, and every string can be viewed as a sequence of $\delta: q_i \times a \rightarrow q(i+1)$, $q_i \in Q_i$, $q(i+1) \in Q(i+1)$ and $a \in \Sigma$. So, these special NFAs is store data.

But, when we use a multistage graph to implement this special NFAs, there are a problem that there are some vertex while have some meaning. We set a vertex as two things, one is meaning, while another is tag.

In this paper, we use multistage directed graph to implement this special NFAs. We can use this special NFAs to design an algorithm for the Hamilton Circuit Problem with exponential time and exponential space.

### 2.2 Definition of Special State Set NFAs

**Definition 2:** if we have a special NFAs $(Q, \Sigma, \delta, q_0, F)$, then we can get a special state set NFAs which corresponding to this special NFAs $(Q, \Sigma, \delta, q_0, F)$ is a 7-tuple $(Q', \Sigma', \delta', q_0', F', \text{Rel}(q'))$, where

1. $Q'$ is a finite set called the states and $Q'$ is union set of $Q_0', Q_1', Q_2', Q_3', \ldots, Q_n'$ with the intersection every two subset is empty. Combine those state $q(i+1)$, the same input $q_i, q_i \in Q_i$ in $\delta: q_i \times a \rightarrow q(i+1)$ with the same input $a, a \in \Sigma$, different state $q(i+1)=q(i+1)'$ and $q(i+1)'$ is an element of $Q(i+1)'$, $Q_0'=Q_0$.
2. $\Sigma' = \Sigma$.
3. $\delta': Q' \times \Sigma' \rightarrow Q(i+1)'$ or null is a function called the transition function with $0 \leq i < (n-1)$ and $\delta': q_i' \times a \rightarrow q(i+1)'$ if and only if $\delta: q_i \times a \rightarrow q(i+1)$ where $q(i+1)=q(i+1)'$ and $q(i+1)'$ is an element of $Q(i+1)'$, $Q_0'=Q_0$.
4. $q_0'$ is an element of $Q_0'$ called the start state and $|Q_0'| = 1$.
5. $F'$ is a subset of $Q_0'$ called the accept states.
6. $\text{Rel}(q')$ is set of all state set whose element state is in a string from start state to any element state of $q'$ in this special NFAs, $q_i' \in Q_i'$.
7. $\text{RelRel}(p')$ is all $\delta': q_i' \times a \rightarrow q(i+1)'$ which is in a path that go thought $p'$ and from start state to $q'$, $p' \in \text{Rel}(q')$. $q_i' \in Q_i'$, $q(i+1)' \in Q(i+1)'$ and $a \in \Sigma'$.

The special state set NFAs is fuzzy data of this corresponding special NFAs. In other word, the data of a special NFAs is subset of this corresponding special state set NFAs.

**Lemma 2:** the length of the longest string of a special NFAs is equal to the length of the longest string of this corresponding special state set NFAs.

**Proof:** because we combine those state $q(i+1)$ with the same input $a, a \in \Sigma$ in $\delta: q_i \times a \rightarrow q(i+1)$, $q_i \in Q_i$ and $q(i+1) \in Q(i+1)$ with state set $q(i+1)'$, $q(i+1)' \in Q(i+1)'$ and $q(i+1)'$ is an element of $Q(i+1)'$.

Then we know there are maximum number $n, Qn\epsilon Q$ in this special NFAs, if and only if there are maximum number $n, Qn\epsilon Q$ in this corresponding special state set NFAs.

So, the length of the longest string of a special NFAs is equal to the length of the longest string of this corresponding special state set NFAs.

We use $O(|V|)$ variance multistage directed graph to implement the ability of this special state set NFAs, since we just know the ending vertex of that path whose vertex is not repeat. Every ending vertex know which vertex can reach it and if we delete a state set, which function $\delta': q_i' \times a \rightarrow q(i+1)'$ and state $q_i'$ will be deleted.

### 3. NOTION

In this section, we introduce some notions in this paper.

1. graph $G = (V, E)$ is the graph that should be found the Hamilton Circuit by this algorithms.
2. $n=|V|$, where $V$ is the element of graph $G = (V, E)$.
3. graph $G_0 = (V_0, E_0)$ is a multistage directed graph of graph $G = (V, E)$. $G_0$ is equal to all path from a fix vertex $a, a \in V$ to the ending vertex, and those vertex of every path is not repeat. $\forall o$ have two things, one is the meaning of vertex, the corresponding vertex in graph $G_0 = (V_0, E_0)$, while another is tag which differ with other vertex, $\forall o \in V_0$. $E_0$ is from $i$th stage to $(i+1)$ stage, is direction between two vertexes.
4. graph $G_1 = (V_1, E_1)$ is a variant multistage directed graph, and there are $O(|V|)$ variance multistage directed graph. Every variance multistage directed graph correspond to one meaning of ending vertex of the multistage directed graph. $V_1$ is all different vertex which have same meaning and some stage in $V_0$.

### 4. ALGORITHM

In this section, we will introduce those algorithms and some lemmas in detail.

**Lemma 3:** there are a Hamilton Circuit in a graph $G = (V, E)$ if and only if there are some path whose length is from 0 to $|V|$ beginning with a fixed vertex (for example is vertex $a, a \in V$) and vertexes without repetition except that the beginning vertex and the ending vertex.

**Proof:** because Hamilton Circuit can be seen as a path whose length is $|V|$, beginning and ending with a fixed vertex (for example is vertex $a, a \in V$) and vertexes without repetition except that the beginning vertex and the ending vertex. And we can find any path whose length is from 0 to $|V|$ begin with a fixed vertex (for example is vertex $a, a \in V$) and vertexes without repetition except that the beginning vertex and the ending vertex is same in this Hamilton Circuit.

So, there are a Hamilton Circuit in a graph $G = (V, E)$ if and only if there are some path whose length is from 0 to $|V|$ beginning with a fixed vertex (for example is vertex $a, a \in V$) and vertexes without repetition except that the beginning vertex and the ending vertex.

We design algorithm 1 because of Lemma 3

#### 4.1 Algorithm 1

In this section, we use multistage directed graph to implement this special NFAs. We can use this special NFAs to design an algorithm for the Hamilton Circuit Problem in exponential time and exponential space.

Those strings of this data can be seen as a tree. And a tree also can been seen as a multistage directed graph.
This multistage directed graph is structured from 0th level to |V|th level. Every level is mean to every location of path whose vertex is not repeat.

**Lemma 4:** If the vertex b, beV is in the possible vertex in (i + 1)th location of a Hamilton Circuit, then vertex b is necessary in a Hamilton Circuit from 0th location to ith location.

Proof: if the vertex b, beV is in the possible vertex in (i + 1)th location of a Hamilton Circuit. So, there exist a Hamilton Circuit whose (i + 1)th location is vertex b. And all vertex of Hamilton Circuit is not repeat.

So, vertex b is not necessary in a Hamilton Circuit from 0th location to ith location.

When we get i level of multistage directed graph, then we have O(|V|) ending vertex of those path, and they are all in ith level of multistage directed graph. If we should find the possible vertex of (i + 1)th location, we should they whether necessary vertex in path from 0th location to ith location or not. In algorithm 1.1, we will describe a solution in polynomial time.

Supposed we should find a Hamilton Circuit in graph $G = (V, E)$.

**Algorithms 1.1**

**Input:** a multistage directed graph: $G0$ with i level

**Output:** a multistage directed graph: $G0'$ with (i + 1) level

1. loop for every vertex $u$ in i level
2. pick all sub graph $G'$ whose path are from 0th to vertex $u$
3. loop for every vertex $v$ in $V$ of $G$
4. delete all vertex who is to represent vertex $v$ in graph $G'$
5. delete all vertex who are not in path from 0th level to i level
6. end loop
7. if $G'$ is empty
8. then vertex $v$ is necessary vertex from 0th to vertex $u$
9. else
10. $G'$ is sub graph of $G0'$
11. vertex $v'$ is to represent vertex $v$ who is in (i + 1)th level
12. add $(u, v')$ in $G0'$
13. end if
14. end loop

Then, we can use algorithm 1.1 to judge whether a vertex is necessary vertex in a Hamilton Circuit from 0th location to ith location. Because all path without repetition from 0th level to vertex $u$ ith level can be seen as a tree, a multistage graph. Then if we delete all vertex $v'$ which is to represent vertex $v$, $v \epsilon V$ of graph $G = (V, E)$ in sub graph which are all string from 0th to vertex u without repetition, then all path which go thought this vertex can delete easily. The modify graph is all string from 0th to vertex u without repetition and vertex v.

Then we can search a multistage graph with (i + 1) level depend on a multistage graph with i level.

Using those solution, we can get all path whose vertex is not repeat from 0th level to |V|th level in a multistage graph. Then, we just find those |V|th stage ending vertex of those path are whether link with the beginning vertex.

Then, we use a multistage graph as a special NFAs to solve Hamilton Circuit problem in exponential time and exponential space.

### 4.2 Algorithm 2

In this section, we should introduce a polynomial time algorithm for the Hamilton Circuit problem. The algorithm 2 is a variant algorithm of the algorithm 1.

In the algorithm 2, we use $O(|V|)$ variance multistage directed graph as a special state set NFAs to store the data. In the algorithm 1, we should know vertex $u$ is whether necessary vertex in a Hamilton Circuit from 0th location to ith location.

We find a new algorithms to solve this problem in polynomial time and polynomial space.

The algorithm 1 can be seen three main step: first step is find all sub graph which are all string from 0th to vertex $u$ and vertex without repetition; second step is delete all vertex which is to represent vertex $v$, $v \epsilon V$ of graph $G = (V, E)$ in this sub graph which are all string from 0th to vertex $u$ without repetition and third step is delete all vertex who are not in path from 0th level to i level.

Then, we can design the algorithm 2 and also 3 main step.

First step is also find all sub graph which are all string from 0th to vertex $u$ and vertex without repetition. We use $O(|V|)$ variant multistage graph to store this data of $O(|V|)$ ending vertex.

Second step is delete all vertex which is to represent vertex $v$, $v \epsilon V$ of graph $G = (V, E)$ in this sub graph which are all string from 0th to vertex $u$ without repetition. Deleting the state set which is to represent vertex $v$ is equal to deleting the state which is to represent vertex $v$. So, the second step is same as the second step of the algorithm 1.1.

Third step, we should delete all vertex who are not in path from 0th level to ith level. But, in special state set NFAs, We cannot delete a state set since we should delete an element of this state set. So, we just delete edge or $\delta: q_1 \times \Sigma \rightarrow q(i + 1)$ which are not gone through without thought this state set. And, we can delete all relative complement of the union set of other $RelRel(p')$. They all should be deleted when we delete a state set.

**Lemma 5:** the solution of algorithm 2 is equal to the solution of algorithm 1.

Proof: the step 1 and the step 2 of algorithm 1 is same as the step 1 and the step 2 of algorithm 2. So, if the step 3 is same in two algorithms, then the solution of two algorithm is same.

In step 3 of algorithm 1 can be seen as that delete all vertex or state which are not gone through without thought this state set. The step which delete all vertex or state which are not gone through without thought this state set is also equal to the step which delete
all edge or $\delta: q_i \times \Sigma \rightarrow q(i + 1)$ which are not gone through without thought this state set.

Since lemma 2, we can know the power of algorithm 1 is equal to the power of algorithm 2.

5. CONCLUSIONS
In this paper, we find a new method to solve the Hamilton Circuit problem in polynomial time. It is analyze all possible vertexes of every location in Hamilton Circuit, not all possible Hamilton Circuit.

The key solution is the problem whether a vertex is necessary vertex from $0^{th}$ stage to $i^{th}$ stage. When it is necessary vertex, then it cannot appear in the location from $(i + 1)$th stage to $|V|$th stage in a variant multistage graph. So, if we have no choice in one location, then there are not Hamilton Circuit in this graph $G = (V, E)$.

We have two method to solve the key problem. One is store all possible path in detail, while another is store the fuzzy data of those data. Because of the Lemma 2, we can know it is necessary and sufficient condition of Hamilton Circuit that the path from $0^{th}$ stage to $|V|$th stage and the ending vertex adjacent to the beginning vertex in this variant multistage graph.

6. REFERENCES
[1] Michael R. Garey and David S. Johnson 1997. Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman, ISBN 0-7167-1045-5A1.3: GT37 – 39, pp. 199 – 200.