ADHM DATA FOR THE HILBERT SCHEME OF POINTS
OF THE TOTAL SPACE OF $\mathcal{O}_{\mathbb{P}^1}(-n)$

Claudio Bartocci,§ Ugo Bruzzo,¶‡
Valeriano Lanza§ and Claudio L. S. Rava¶

¶Dipartimento di Matematica, Università di Genova, via Dodecaneso 35, 16146 Genova (Italy),
§Scuola Internazionale Superiore di Studi Avanzati, via Bonomea 265, 34136 Trieste (Italy) and
‡Istituto Nazionale di Fisica Nucleare, Sezione di Trieste.

Correspondence to be sent to: bartocci@dima.unige.it

Abstract. Relying on a monadic description of the moduli space of framed sheaves on Hirzebruch surfaces, we construct ADHM data for the Hilbert scheme of points of the total space of the line bundle $\mathcal{O}(-n)$ on $\mathbb{P}^1$.

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§ Email addresses: bartocci@dima.unige.it (C. Bartocci); bruzzo@sissa.it (U. Bruzzo); lanza@dima.unige.it (V. Lanza); clsrava@gmail.com (C. L. S. Rava).
1. Introduction

Let $X$ be a smooth quasi-projective irreducible surface over $\mathbb{C}$. The Hilbert scheme of points $\text{Hilb}^c(X)$, which parameterizes 0-dimensional subschemes of $X$ of length $c$, is well known to be quasi-projective $[6]$ and smooth of dimension $2c$ $[5]$; indeed, the so-called Hilbert-Chow morphism $\text{Hilb}^c(X) \rightarrow S^c X$ onto the $c$-th symmetric product of $X$ is a resolution of singularities. Hilbert schemes of points on surfaces were extensively studied from many perspectives over the past two decades (see e.g. $[10, 8, 9]$), however there are relatively few cases in which they are susceptible of an explicit description. Arguably, the most significant examples are the spaces $\text{Hilb}^c(\mathbb{C}^2)$, which can be described by means of linear data, the so-called ADHM (Atiyah-Drinfel’d-Hitchin-Manin) data $[10]$. Also the Hilbert schemes of points of multi-blowups of $\mathbb{C}^2$ admit an ADHM description, as provided by the work of A.A. Henri $[7]$ specialized to the rank one case.

The goal of this paper is to provide an ADHM-type construction for the Hilbert schemes of points over the total space $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n))$ of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$ on $\mathbb{P}^1$. These spaces are the rank 1 case of the moduli spaces of framed sheaves of the Hirzebruch surface $\Sigma_n$ (by framing to the trivial bundle on a divisor linearly equivalent to the section of $\Sigma_n \rightarrow \mathbb{P}^1$ of positive self-intersection) which were studied in $[3, 2]$. These modules spaces were considered in physics in connection with the so-called D4-D2-D0 brane system in topological string theory (cf. $[11, 1]$ and $[3]$ for a concise discussion).

To construct the ADHM data for the Hilbert scheme of points of $\text{Hilb}^c(\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))$ we identify it with the moduli space $\mathcal{M}^n(1, 0, c)$ of framed sheaves on the Hirzebruch surface $\Sigma_n$ that have rank 1, vanishing first Chern class, and second Chern class $c_2 = c$, and exploit the description of $\mathcal{M}^n(1, 0, c)$ in terms of monads given in $[2]$. Theorem $2.1$ states that the moduli space $\mathcal{M}^n(1, 0, c)$ is isomorphic to the quotient $P^n(c)/\text{GL}(c, \mathbb{C})^\times 2$, where $P^n(c)$ is a quasi-affine variety contained in the linear space $\text{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \text{Hom}(\mathbb{C}^c, \mathbb{C})$. This result relies on the fact that the partial quotient $P^n(c)/\text{GL}(c, \mathbb{C})$ can be assembled glueing $c + 1$ open sets, each one isomorphic to the space of ADHM data for $\text{Hilb}^c(\mathbb{C}^2)$ (Theorem $3.1$). Since the proof of Theorem $3.1$ is based on the description of the moduli spaces of framed sheaves on $\Sigma_n$ worked out in $[2]$, for the reader’s convenience we briefly recall here the fundamental ingredients of that construction.

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Background material. Let $\Sigma_n$ be the $n$-th Hirzebruch surface, i.e., the projective closure of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$; we restrict ourselves to the case $n > 0$. We denote by $F$ the class in $\text{Pic}(\Sigma_n)$ of the fibre of the natural ruling $\Sigma_n \to \mathbb{P}^1$, by $H$ the class of the section of the ruling squaring to $n$, and by $E$ the class of the section squaring to $-n$. We fix a curve $\ell_\infty \simeq \mathbb{P}^1$ in $\Sigma_n$ linearly equivalent to $H$ and think of it as the “line at infinity”.

A framed sheaf on $\Sigma_n$ is a pair $(E, \theta)$, where $E$ is a torsion-free sheaf that is trivial along $\ell_\infty$, and $\theta: E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ is a fixed isomorphism, $r$ being the rank of $E$. A morphism between the framed sheaves $(E, \theta)$, $(E', \theta')$ is by definition a morphism $\Lambda: E \to E'$ such that $\theta'|_{\ell_\infty} \circ \Lambda|_{\ell_\infty} = \theta$. The moduli space parameterizing isomorphism classes of framed sheaves $(E, \theta)$ on $\Sigma_n$ with $\text{ch}(E) = (r, aE, -c - \frac{1}{2}na^2)$ will be denoted by $M_n(r, a, c)$. We assume that the framed sheaves are normalized in such a way that $0 \leq a \leq r - 1$.

A description of the moduli space $M_n(r, a, c)$ in terms of monads was provided in [2], generalizing work by Buchdahl [4]. If $[(E, \theta)]$ lies in $M_n(r, a, c)$, the sheaf $E$ is isomorphic to the cohomology of a monad

\begin{equation}
M(\alpha, \beta): \quad 0 \to \mathcal{U}_{\vec{k}} \xrightarrow{\alpha} \mathcal{V}_{\vec{k}} \xrightarrow{\beta} \mathcal{W}_{\vec{k}} \to 0,
\end{equation}

where $\vec{k} = (n, r, a, c)$; in others words, the terms of (1.1) depend only on the Chern character of $E$. More precisely, if we put

\begin{align}
\begin{cases}
k_1 &= c + \frac{1}{2}na(a - 1) \\
k_2 &= k_1 + na \\
k_3 &= k_1 + (n - 1)a \\
k_4 &= k_1 + r - a,
\end{cases}
\end{align}

we have

\begin{align*}
\mathcal{U}_{\vec{k}} := & \mathcal{O}_{\Sigma_n}(0, -1)^{\oplus k_1} \\
\mathcal{V}_{\vec{k}} := & \mathcal{O}_{\Sigma_n}(1, -1)^{\oplus k_2} \oplus \mathcal{O}_{\Sigma_n}^{\oplus k_4} \\
\mathcal{W}_{\vec{k}} := & \mathcal{O}_{\Sigma_n}(1, 0)^{\oplus k_3}.
\end{align*}

This procedure yields a map

\begin{equation}
(E, \theta) \mapsto \text{Hom}(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}) \oplus \text{Hom}(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}}),
\end{equation}

whose image $L_{\vec{k}}$ is a smooth variety, which can be completely characterized by imposing suitable conditions on the pairs $(\alpha, \beta) \in \text{Hom}(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}) \oplus \text{Hom}(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}})$ [2 §2].
construct a principal $\text{GL}(r, \mathbb{C})$-bundle $P_k$ over $L_k$ whose fibre over a point $(\alpha, \beta)$ is naturally identified with the space of framings for the cohomology of the complex (1.1). Hence, the map (1.3) can be lifted to a map

$$(\mathcal{E}, \theta) \mapsto \theta \in P_k.$$ 

The algebraic group $G_k = \text{Aut}(U_k) \times \text{Aut}(V_k) \times \text{Aut}(W_k)$ of isomorphisms of monads of the form (1.1) acts freely on $P_k$, and the moduli space $\mathcal{M}^n(r, a, c)$ can be described as the quotient $P_k/G_k$ [2, Theorem 3.4]. This space is nonempty if and only if $c + \frac{1}{2} na(a - 1) \geq 0$, and, in this case, is a smooth algebraic variety of dimension $rc + (r - 1)na^2$.

If the sheaf $\mathcal{E}$ has rank $r = 1$, by normalizing we can assume $a = 0$. Hence, the double dual $\mathcal{E}^{**}$ of $\mathcal{E}$, being locally free with $c_1(\mathcal{E}^{**}) = c_1(\mathcal{E}) = 0$, is isomorphic to the structure sheaf $\mathcal{O}_{\Sigma^n}$. As a consequence, since $\mathcal{E}$ is trivial on $\ell_\infty$, the correspondence

$$\mathcal{E} \mapsto \text{schematic support of } \mathcal{E}^{**}/\mathcal{E}$$

yields an isomorphism

$$\mathcal{M}^n(1, 0, c) \cong \text{Hilb}^c(\Sigma_n \setminus \ell_\infty) = \text{Hilb}^c(\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n))).$$

In the following, we shall denote the moduli space $\mathcal{M}^n(1, 0, c)$ simply by $\mathcal{M}^n(c)$.

2. Statement of the Main Theorem

We call $P^n(c)$ the subset of the vector space $\text{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \text{Hom}(\mathbb{C}^c, \mathbb{C})$ whose points $(A_1, A_2; C_1, \ldots, C_n; e)$ satisfy the following conditions:

(P1) \[ \begin{array}{l}
A_1C_1A_2 = A_2C_1A_1 \\
A_1C_q = A_2C_{q+1} \\
C_qA_1 = C_{q+1}A_2
\end{array} \] 

for \( q = 1, \ldots, n - 1 \) when \( n > 1 \); \( \text{when } n = 1 \)

(P2) there exists $[\nu_1, \nu_2] \in \mathbb{P}^1$ such that $\det(\nu_1A_1 + \nu_2A_2) \neq 0$;

(P3) for all values of the parameters $([\lambda_1, \lambda_2], (\mu_1, \mu_2)) \in \mathbb{P}^1 \times \mathbb{C}^2$ such that

$$\lambda_1^n \mu_1 + \lambda_2^n \mu_2 = 0$$
there is no nonzero vector $v \in \mathbb{C}^c$ such that

\[
\begin{align*}
(\lambda_2 A_1 + \lambda_1 A_2) v &= 0 \\
(C_1 A_2 + \mu_1 1_c) v &= 0 \\
(C_n A_1 + (-1)^{n-1} \mu_2 1_c) v &= 0 \\
e v &= 0.
\end{align*}
\]

We define an action of $GL(c, \mathbb{C}) \times \mathbb{C}^2$ on $P^n(c)$ by the equations

\[
\begin{align*}
C_j &\rightarrow \phi_1 C_j \phi_2^{-1} & j &= 1, \ldots, n \\
A_i &\rightarrow \phi_2 A_i \phi_1^{-1} & i &= 1, 2 \\
e &\rightarrow e \phi_1^{-1}
\end{align*}
\]

Theorem 2.1. There is an isomorphism of complex varieties

\[P^n(c)/GL(c, \mathbb{C})^\times \simeq M^n(c) = \text{Hilb}^c(\text{Tot}(O_{\mathbb{P}^1}(-n)))\]

and $P^n(c)$ is a locally trivial principal $GL(c, \mathbb{C})^\times$-bundle over $M^n(c)$.

2.1. A consistency check. Before proving Theorem 2.1 we check its consistency in the simplest case $c = 1$, by verifying that the quotient $P^n(1)/(\mathbb{C}^\times)^2$ is isomorphic to the total space of $O_{\mathbb{P}^1}(-n)$. Indeed, one has $\text{Tot}(O_{\mathbb{P}^1}(-n)) \simeq \tilde{T}_n/\mathbb{C}^\times$, where

\[\tilde{T}_n = \{ ((y_1, y_2), (u_1, u_2)) \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^2 | u_1 y_1^n = u_2 y_2^n \}\]

and the $\mathbb{C}^\times$-action is

\[
\begin{align*}
(y_1, y_2) &\rightarrow \lambda (y_1, y_2) & \lambda &\in \mathbb{C}^\times \\
(u_1, u_2) &\rightarrow (u_1, u_2)
\end{align*}
\]

(cf. eq. (3.1)).

Proposition 2.2. $P^n(1)/(\mathbb{C}^\times)^2 \simeq \text{Tot}(O_{\mathbb{P}^1}(-n))$.

Proof. When $c = 1$, the matrices $A_1, A_2, C_1, \ldots, C_n, e$ are complex numbers, and the condition (P2) is equivalent to requiring that $(A_1, A_2) \neq (0, 0)$. When $n = 1$ the condition (P1) is identically satisfied, while when $n > 1$ it is equivalent to

\[
\begin{align*}
C_q &= \left( \frac{A_2}{A_1} \right)^{n-q} C_n & \text{for } q &= 1, \ldots, n-1 & \text{if } A_1 \neq 0 \\
C_q &= \left( \frac{A_1}{A_2} \right)^q C_1 & \text{for } q &= 2, \ldots, n & \text{if } A_2 \neq 0.
\end{align*}
\]

Using these equations it is possible to show that the condition (P3) reduces to $e \neq 0$. By acting with $(\mathbb{C}^\times)^2$ we can fix $e = 1$, and the maximal subgroup preserving this condition
is clearly \(\{1\} \times \mathbb{C}^*\). We introduce the variety

\[ \tilde{Y}_n = \left\{ \left( (y_1, y_2), (x_1, x_2) \right) \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^2 \mid x_1y_1^{n-1} = x_2y_2^{n-1} \right\}, \]

with \(n \geq 1\), and we let \(\mathbb{C}^*\) act on \(\tilde{Y}_n\) as follows:

\[
\begin{align*}
(y_1, y_2) &\mapsto \lambda(y_1, y_2) \\
(x_1, x_2) &\mapsto \lambda^{-1}(x_1, x_2)
\end{align*}
\]

\(\lambda \in \mathbb{C}^*\).

We cover \(\tilde{Y}_n\) with the two \(\mathbb{C}^*\)-invariant subsets \(\tilde{Y}_{n, i} = \{ y_i \neq 0 \}\), and analogously we cover \(P^n(1)\) with the \((\mathbb{C}^*)^2\)-invariant subsets \(P^n(1)_i = \{ A_i \neq 0 \}, i = 1, 2\). Next, we define the morphisms

\[
\tilde{Y}_{n, i} \longrightarrow P^n(1)_i
\]

\[
\left( (y_1, y_2), (x_1, x_2) \right) \mapsto \begin{cases} 
(y_1, y_2, \left( \frac{y_2}{y_1} \right)^{n-1} x_2, \left( \frac{y_2}{y_1} \right)^{n-2} x_2, \ldots, x_2, 1) & i = 1 \\
(y_1, y_2, x_1, \left( \frac{y_1}{y_2} \right) x_1, \ldots, \left( \frac{y_1}{y_2} \right)^{n-1} x_1, 1) & i = 2
\end{cases}
\]

These glue together providing a \(\mathbb{C}^*\)-equivariant closed immersion \(\tilde{Y}_n \hookrightarrow P^n(1)\), which induces an isomorphism

\[ P^n(1)/((\mathbb{C}^*)^2) \simeq \tilde{Y}_n/\mathbb{C}^*. \]

Finally, the \(\mathbb{C}^*\)-equivariant morphism

\[
\tilde{Y}_n \longrightarrow (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^2
\]

\[
\left( (y_1, y_2), (x_1, x_2) \right) \mapsto \left( (y_1, y_2), (u_1, u_2) \right) = \left( (y_1, y_2), (x_1y_2, x_2y_1) \right)
\]

establishes the required isomorphism. \(\square\)

### 3. Glueing ADHM Data

In this section we provide an ADHM description for each open set of a suitable open cover of \(\mathcal{M}^n(c)\). If we fix \(c+1\) distinct fibres \(f_0, \ldots, f_c \in |\mathcal{F}|\), for any \([E, \theta] \in \mathcal{M}^n(c)\) there exists at least one \(m \in \{0, \ldots, c\}\) such that \(E|_{f_m} \simeq \mathcal{O}_{f_m}\). With this in mind, we choose the fibres \(f_m\) cut in

\[
\Sigma_n = \left\{ \left[ (y_1, y_2), [x_1, x_2, x_3] \right) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid x_1y_1^n = x_2y_2^n \right\}
\]

by the equations

\[
f_m = \{ [y_1, y_2] = [c_m, s_m] \} \quad m = 0, \ldots, c
\]
where

\[(3.2) \quad c_m = \cos \left( \frac{\pi m}{c + 1} \right), \quad s_m = \sin \left( \frac{\pi m}{c + 1} \right).\]

Then we get an open cover \(\{M^n(c)_m\}_{m=0}^c\) for \(M^n(c)\) by letting

\[M^n(c)_m := \left\{ [(E, \theta)] \in M^n(c) \mid \text{the restricted sheaf } E|_{f_m} \text{ is isomorphic to } O_{f_m} \right\}. \]

Each of these spaces is isomorphic to the Hilbert scheme of points of \(\mathbb{C}^2\), so that it admits the ADHM description [10], which we briefly recall. The variety \(T(c)\) of ADHM data is defined as the space of triples \((b_1, b_2, e) \in \text{End}(\mathbb{C}^c)^{\oplus 2} \oplus \text{Hom}(\mathbb{C}^c, \mathbb{C})\) such that

\[(T1) \quad [b_1, b_2] = 0; \]

\[(T2) \quad \text{for all } (z, w) \in \mathbb{C}^2 \text{ there is no nonzero vector } v \in \mathbb{C}^c \text{ such that } \]

\[
\begin{align*}
(b_1 + z 1_c) v &= 0 \\
(b_2 + w 1_c) v &= 0 \\
ev &= 0.
\end{align*}
\]

A \(\text{GL}(c, \mathbb{C})\)-action on \(T(c)\) is naturally defined as follows:

\[(3.3) \quad \begin{cases} 
  b_i &\mapsto \phi b_i \phi^{-1} & i = 1, 2 \\
e &\mapsto e \phi^{-1} & \phi \in \text{GL}(c, \mathbb{C}).
\end{cases}\]

The ADHM data for the open set \(M^n(c)_m\) will be denoted by \((b_{1m}, b_{2m}, e_m)\); the transition functions on the intersections \(M^n(c)_{ml} = M^n(c)_m \cap M^n(c)_l\) are explicitly described in the next Theorem.

**Theorem 3.1.** The intersection \(M^n(c)_{ml} = M^n(c)_m \cap M^n(c)_l\) is characterized by the condition

\[\det (c_m - 1_c + s_m - 1b_{1l}) \neq 0 \quad \text{(or, equivalently, } \det (c_l - 1_c + s_l - 1b_{1m}) \neq 0),\]

where \(c_m\) and \(s_m\) are the numbers defined in eq. \((3.2)\). On any of these intersections, the ADHM data are related by the transition functions

\[\varphi_{lm}: M^n(c)_{ml} \longrightarrow M^n(c)_{ml} \]

\[[(b_{1m}, b_{2m}, e_m)] \mapsto [(b_{1l}, b_{2l}, e_l)],\]
ADHM data for the Hilbert scheme of the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$

\[
\begin{aligned}
\begin{cases}
&b_{1l} = (c_{m-l}1_c - s_{m-l}b_{1m})^{-1}(s_{m-l}1_c + c_{m-l}b_{1m}) \\
&b_{2l} = (c_{m-l}1_c - s_{m-l}b_{1m})^{n}b_{2m} \\
&e_{l} = e_{m}
\end{cases}
\end{aligned}
\]

where

\[
\begin{align*}
\begin{cases}
b_{1l} &= (c_{m-l}1_c - s_{m-l}b_{1m})^{-1}(s_{m-l}1_c + c_{m-l}b_{1m}) \\
b_{2l} &= (c_{m-l}1_c - s_{m-l}b_{1m})^{n}b_{2m} \\
e_{l} &= e_{m}
\end{cases}
\end{align*}
\]

To prove Theorem 3.1 we observe that $\text{GL}(c, \mathbb{C})$ can be embedded as a closed subgroup of $G_{\vec{k}}$ by means of the homomorphism

\[
\iota: \quad \text{GL}(c, \mathbb{C}) \rightarrow G_{\vec{k}}
\]

(3.4)

\[
\phi \quad \mapsto \quad \begin{pmatrix} t_\phi^{-1} & 0 & 0 \\ 0 & t_\phi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Let $\pi: P_{\vec{k}} \rightarrow \mathcal{M}^n(c)$ be the canonical projection. The open subsets

\[
P_{k,m} = \pi^{-1}(\mathcal{M}^n(c)_m), \quad m = 0, \ldots, c,
\]

provide a $G_{\vec{k}}$-invariant open cover of $P_{\vec{k}}$. $\text{GL}(c, \mathbb{C})$ acts on each $P_{k,m}$ via the immersion (3.4).

**Proposition 3.2.** There are $\text{GL}(c, \mathbb{C})$-equivariant closed immersions $j_{m}: T(c) \hookrightarrow P_{k,m}$ for $m = 0, \ldots, c$.

These induce isomorphisms

\[
\eta_{m}: \quad T(c)/\text{GL}(c, \mathbb{C}) \rightarrow P_{k,m}/G_{\vec{k}} \simeq \mathcal{M}^n(c)_m \quad \text{for } m = 0, \ldots, c.
\]

(3.5)

**Proof.** See Section A.2

We introduce the open subsets

\[
T(c)_{m,l} = j_{m}^{-1}\left(\text{Im } j_{m} \cap P_{k,l}\right) \quad \text{for } m, l = 0, \ldots, c.
\]

**Lemma 3.3.** $T(c)_{m,l} = \{(b_{1}, b_{2}, e) \in T(c) | \det(c_{m-l}1_c - s_{m-l}b_{1}) \neq 0\}$.

**Proof.** The intersection $\text{Im } j_{m} \cap P_{k,l}$ is the set of points $(\alpha, \beta, \xi) \in \text{Im } j_{m}$ such that $\det(\beta_{1}|_{f_{l}}) \neq 0$, where $\beta_{1}$ is the first component of $\beta$. From the fact that $(\alpha, \beta, \xi) \in \text{Im } j_{m}$ it follows that

\[
\beta_{1} = 1_{c}y_{1m} + t_{b_{1}}y_{2m} = \begin{pmatrix} 1_{c} & t_{b_{1}} \end{pmatrix} \begin{pmatrix} y_{1m} \\ y_{2m} \end{pmatrix} = \begin{pmatrix} 1_{c} & t_{b_{1}} \end{pmatrix} \begin{pmatrix} c_{m} & s_{m} \\ -s_{m} & c_{m} \end{pmatrix} \begin{pmatrix} y_{1} \\ y_{2} \end{pmatrix}.
\]
ADHM data for the Hilbert scheme of the total space of $O_{\mathbb{P}^1}(-n)$

Since $[y_1, y_2] = [c_l, s_l]$ along $f_l$, the thesis follows.

Note that $\mathcal{T}(c)_{m,l} \neq \mathcal{T}(c)_{l,m}$, but $\mathcal{T}(c)_{m,l}/\text{GL}(c, \mathbb{C}) \simeq \mathcal{T}(c)_{l,m}/\text{GL}(c, \mathbb{C}) \simeq \mathcal{M}^n(c)_{ml}$.

**Proposition 3.4.** The map

$$
\tilde{\varphi}_{lm}: \quad \mathcal{T}(c)_{m,l} \longrightarrow \mathcal{T}(c)_{l,m}
$$

$$
\begin{pmatrix}
    b_1 \\
    b_2 \\
    c
\end{pmatrix}
\longmapsto
\begin{pmatrix}
    (c_{m-l}1_c - s_{m-l}1_c + c_{m-l}b_1)^{-1} (s_{m-l}1_c + c_{m-l}b_1) \\
    (c_{m-l}1_c - s_{m-l}b_1)^n b_2 \\
    c
\end{pmatrix}
$$

is a $\text{GL}(c, \mathbb{C})$-equivariant isomorphism. It induces an isomorphism

$$
\varphi_{lm}: \quad \mathcal{T}(c)_{m,l}/\text{GL}(c, \mathbb{C}) \longrightarrow \mathcal{T}(c)_{l,m}/\text{GL}(c, \mathbb{C})
$$

such that the triangle

$$
\xymatrix{
\mathcal{T}(c)_{m,l}/\text{GL}(c, \mathbb{C}) \ar[r]^-{\varphi_{lm}} \ar[d]_{\eta_{m,l}} & \mathcal{T}(c)_{l,m}/\text{GL}(c, \mathbb{C}) \ar[d]^{\eta_{l,m}} \\
\mathcal{M}^n(c)_{ml} &}
$$

is commutative, where $\eta_{m,l}$ is the restriction of $\eta_m$ to $\mathcal{T}(c)_{m,l}/\text{GL}(c, \mathbb{C})$ (see eq. (3.5)).

**Proof.** See Section A.3

Theorem 3.1 is now a direct consequence of Proposition 3.4.

4. PROOF OF THE MAIN THEOREM

We introduce the matrices

$$
A_{1m} = c_m A_1 - s_m A_2,
$$

$$
A_{2m} = s_m A_1 + c_m A_2,
$$

$$
E_m = \left[ \sum_{q=1}^n \binom{n-1}{q-1} c_m^{n-q} s_m^{q-1} C_q \right] A_{2m},
$$

for $m = 0, \ldots, c$. Since the polynomial $\det(A_1 \nu_1 + A_2 \nu_2)$ has at most $c$ distinct roots in $\mathbb{P}^1$, the $\text{GL}(c, \mathbb{C}) \times \mathbb{C}^2$-invariant open subsets

$$
P^n(c)_m = \{(A_1, A_2; C_1, \ldots, C_n; e) \in P^n(c) | \det A_{2m} \neq 0\}, \quad m = 0, \ldots, c,$$
ADHM data for the Hilbert scheme of the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$

cover $P^n(c)$. On $P^n(c)_m$ one can also define the matrix

\[(4.3) \quad B_m = A_{2m}^{-1} A_{1m} . \]

The matrices $(B_m, E_m, A_{2m}, e)$ provide local affine coordinates for $P^n(c)$.

**Proposition 4.1.** The morphism

\[ \zeta_m: \quad P^n(c)_m \to [\text{End}(\mathbb{C}^c)^{\oplus 2} \oplus \text{Hom}(\mathbb{C}^c, \mathbb{C})] \times \text{GL}(c, \mathbb{C}) \]

\[ (A_1, A_2; C_1, \ldots, C_n; e) \to (B_m, E_m, e; A_{2m}) \]

is an isomorphism onto $\mathcal{T}(c) \times \text{GL}(c, \mathbb{C})$. The induced $\text{GL}(c, \mathbb{C})^{\times 2}$-action is given by

\[(4.4) \quad \begin{cases} 
B_m \mapsto \phi_1 B_m \phi_1^{-1} \\
E_m \mapsto \phi_1 E_m \phi_1^{-1} \\
A_{2m} \mapsto \phi_2 A_{2m} \phi_2^{-1} \\
e \mapsto e \phi_1^{-1} .
\end{cases} \]

We divide the proof of this Proposition into several steps. First we define the matrices $\sigma^h_m = (\sigma^h_{m, pq})_{0 \leq p, q \leq h}$ for all $h \geq 0$ and $m \in \mathbb{Z}$ by means of the equations

\[(4.5) \quad (s_m \mu_1 + c_m \mu_2)^p (c_m \mu_1 - s_m \mu_2)^{h-p} = \sum_{q=0}^{h} \sigma^h_{m, pq} \mu_2^q \mu_1^{h-q} \]

for any $(\mu_1, \mu_2) \in \mathbb{C}^2$ and $p = 0, \ldots, h$. Notice that $\sigma^h_{m, 0} = \sigma^h_{m, h} = \sigma^h_{m, h+1}$ and $\sigma^h_{0, h+1} = 1_{h+1}$. In particular, $\sigma^h_m$ is invertible for all $h \geq 0$ and $m \in \mathbb{Z}$.

To prove the injectivity of $\zeta_m$ — which is trivial only when $n = 1$ — we need the following Lemma.

**Lemma 4.2.** Assume $n > 1$. If the matrices $A_1, A_2 \in \text{End}(\mathbb{C}^c)$ satisfy the condition (P2), the system

\[
\begin{pmatrix}
A_1 & -A_2 \\
& \ddots & \ddots \\
& & A_1 & -A_2
\end{pmatrix}
\begin{pmatrix}
C_1 \\
\vdots \\
\vdots \\
C_n
\end{pmatrix} = 0 ,
\]

is satisfied.
ADHM data for the Hilbert scheme of the total space of $O_{\mathbb{P}^1}(-n)$, has maximal rank, namely, $(n-1)c^2$. In particular, if $\det A_2m \neq 0$, the general solution is

$$
\begin{pmatrix}
C_1 \\
\vdots \\
\vdots \\
C_n
\end{pmatrix} = (\sigma_m^{n-1} \otimes 1_c) \begin{pmatrix}
1_c \\
B_m \\
\vdots \\
B_m^{n-1}
\end{pmatrix} D_m,
$$

where we have chosen as free parameter the matrix

$$
D_m = \sum_{q=1}^n \left( \frac{n-1}{q-1} \right) c_m^{n-q} s_m^{q-1} C_q.
$$

Proof. First we show by induction that the $(n-1)c \times nc$ matrices

$$
\mathcal{A}_n = \begin{pmatrix}
A_1 & -A_2 \\
\vdots & \ddots & \ddots \\
A_1 & -A_2
\end{pmatrix} \quad \mathcal{A}_n' = \begin{pmatrix}
-tA_2 & tA_1 \\
\vdots & \ddots & \ddots \\
-tA_2 & tA_1
\end{pmatrix}
$$

have maximal rank for all $n > 1$. For $n = 2$ the condition (P2) ensures the existence of a point $[\nu_1, \nu_2] \in \mathbb{P}^1$ such that $\det (A_1\nu_1 + A_2\nu_2) \neq 0$; it follows that the columns of $A_1$ and $A_2$ span a vector space of dimension $c$, so that $\text{rk} \mathcal{A}_2 = c$. The case of $\mathcal{A}_2'$ is analogous.

Assume that the claim holds true for some $k > 1$, and observe that

$$
\mathcal{A}_{k+1} = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & t \mathcal{A}_k' & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -A_2
\end{pmatrix}.
$$

Let $v \in \mathbb{C}^{(k+1)c}$, and decompose it as

$$
v = \begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix} \uparrow c \quad (k-1)c.
$$

If $\mathcal{A}_{k+1}v = 0$, we get

$$
\begin{cases}
A_1v_1 + t \mathcal{A}_k'v_2 = 0 \\
t \mathcal{A}_k'v_2 = 0 \\
t \mathcal{A}_k'v_2 - A_2v_3 = 0.
\end{cases}
$$
Since \( \ker A_k' = 0 \) by inductive hypothesis, it follows that \( A_{k+1} \) has maximal rank. The case of \( A_k' \) is analogous. Eq. (4.6) is checked by direct computation and eq. (4.7) is obtained by using the invertibility of \( \sigma_m^{n-1} \).

Since \( E_m = D_m A_{2m} \), the morphism \( \zeta_m \) is injective.

Next we prove that \( \im \zeta_m \subseteq T(c) \times \GL(c, \mathbb{C}) \) via the following two Lemmas.

**Lemma 4.3.** For all \((B_m, E_m, e; A_{2m}) \in \im \zeta_m\) one has

\[
[B_m, E_m] = 0.
\]

**Proof.** For all \( n \geq 1 \) the condition \((P1)\) implies that

\[
A_1 C_q A_2 - A_2 C_q A_1 = 0 \quad \text{for} \quad q = 1, \ldots, n.
\]

By recalling eqs. (4.1) and (4.3), the thesis follows from the identity

\[
A_1 C A_2 - A_2 C A_1 = A_{1m} C A_{2m} - A_{2m} C A_{1m},
\]

which holds true for all \( C \in \End(\mathbb{C}^c) \) and for \( m = 0, \ldots, c \). \qed

**Lemma 4.4.** Let \((A_1, A_2; C_1, \ldots, C_n; e) \in \End(\mathbb{C}^c)^\oplus(n+2) \oplus \Hom(\mathbb{C}^c, \mathbb{C})\) be an \((n+3)\)-tuple which satisfies the condition \((P1)\) and \( \det A_{2m} \neq 0 \). Then

- if \([\lambda_1, \lambda_2] = [c_m, s_m]\), the condition \((P3)\) is trivially satisfied;
- if \([\lambda_1, \lambda_2] \neq [c_m, s_m]\), the condition \((P3)\) holds if and only if the condition \((T2)\) holds for the triple \((B_m, E_m, e)\).

**Proof.** One has

\[
\lambda_2 A_1 + \lambda_1 A_2 = \begin{cases} 
\lambda A_{2m} & \text{if} \quad [\lambda_1, \lambda_2] = [c_m, s_m] \\
\lambda A_{2m}(B_m + z 1_c) & \text{if} \quad [\lambda_1, \lambda_2] \neq [c_m, s_m]
\end{cases}
\]

for some \( \lambda \neq 0 \), where

\[
z = \frac{c_m \lambda_1 + s_m \lambda_2}{-s_m \lambda_1 + c_m \lambda_2}.
\]

This proves the first statement. As for the second statement, eq. (4.6) yields

\[
C_1 = (c_m 1_c - s_m B_m)^{n-1} E_m A_{2m}^{-1}
\]

\[
C_n = (s_m 1_c + c_m B_m)^{n-1} E_m A_{2m}^{-1}.
\]
Moreover, whenever \([\lambda_1, \lambda_2] \neq [c_m, s_m]\), the condition \(\lambda_1 \mu_1 + \lambda_2 \mu_2 = 0\) is satisfied if and only if
\[
\begin{align*}
\mu_1 &= (s_m z + c_m)^n w \\
\mu_2 &= -(c_m z + s_m)^n w
\end{align*}
\]
for some \(w \in \mathbb{C}\). Eqs. (4.8) and (4.9) show the equivalence of the following systems:
\[
\begin{aligned}
&\begin{cases}
(\lambda_2 A_1 + \lambda_1 A_2) v = 0 \\
(C_1 A_2 + \mu_1 1_c) v = 0 \\
(C_n A_1 + (-1)^{n-1} \mu_2 1_c) v = 0
\end{cases} \iff \begin{cases}
(B_m + z 1_c) v = 0 \\
(s_m z + c_m)^n (E_m + w 1_c) v = 0 \\
(-c_m z + s_m)^n (E_m + w 1_c) v = 0
\end{cases}
\end{aligned}
\]
Since the polynomials \(s_m z + c_m\) and \(-c_m z + s_m\) are coprime in \(\mathbb{C}[z]\), the right-hand system is equivalent to
\[
\begin{align*}
(B_m + z 1_c) v &= 0 \\
(E_m + w 1_c) v &= 0.
\end{align*}
\]

Finally we prove that \(T(e) \times \text{GL}(e, \mathbb{C}) \subseteq \text{Im} \zeta_m\). Let \((b_1, b_2, e; A) \in T(e) \times \text{GL}(e, \mathbb{C})\); if we set
\[
\begin{align*}
A_1 &= A(c_m b_1 + s_m 1_c), \\
A_2 &= A(-s_m b_1 + c_m 1_c),
\end{align*}
\]
then \((A_1, A_2; C_1, \ldots, C_n; e) \in P^n(e)_m\) and \(\zeta_m(A_1, A_2; C_1, \ldots, C_n; e) = (b_1, b_2, e; A)\). It is an easy matter to verify by substitution that the condition (P1) holds. Notice now that by substituting (4.10) into eq. (4.11) one gets
\[
A_{1m} = A b_1, \quad A_{2m} = A, \quad E_m = b_2.
\]
This shows that \(A_{2m}\) is invertible, and in particular the condition (P2) holds true. By eq. (13) one has that \(B_m = b_1\), so that the validity of the condition (P3) follows from the condition (T2) by Lemma 4.3. This concludes the proof of Proposition 4.1.
We now compute the transition functions on the intersections $P^n(c)_{ml} = P^n(c)_m \cap P^n(c)_l$, for $m, l = 0, \ldots, c$. First observe that

$$
\zeta_m(P^n(c)_{ml}) = \mathcal{T}(c)_{m,l} \times \text{GL}(c, \mathbb{C}) .
$$

This fact is a consequence of the identity

$$
A_{2l} = \left( s_l 1_c, c_l 1_c \right) \begin{pmatrix} c_m 1_c & s_m 1_c \\ -s_m 1_c & c_m 1_c \end{pmatrix} \begin{pmatrix} A_{1m} \\ A_{2m} \end{pmatrix} = A_{2m}(c_{m-l} 1_c - s_{m-l} B_m) .
$$

(4.11)

**Proposition 4.5.** One has the commutative triangle

$$
\begin{array}{c}
\zeta_{m,l} \\
\zeta_{l,m} \\
\omega_{lm} \\
\mathcal{T}(c)_{m,l} \times \text{GL}(c, \mathbb{C})
\end{array}
\begin{array}{c}
P^n(c)_{ml}
\\
T(c)_{m,l} \times \text{GL}(c, \mathbb{C})
\\
T(c)_{l,m} \times \text{GL}(c, \mathbb{C})
\end{array}
$$

where $\zeta_{m,l}$ and $\zeta_{l,m}$ are the restrictions of $\zeta_m$ and $\zeta_l$, respectively, and

$$
\omega_{lm}(B_m, E_m, e; A_{2m}) = (\tilde{\varphi}_{lm}(B_m, E_m, e), A_{2m}(c_{m-l} 1_c - s_{m-l} B_m)) ,
$$

the functions $\tilde{\varphi}_{lm}$ being defined as in Proposition 3.4. The transition functions $\omega_{lm}$ are $\text{GL}(c, \mathbb{C})^{\times 2}$-equivariant.

**Proof.** We want to express $(B_l, E_l, e; A_{2l})$ in terms of $(B_m, E_m, e; A_{2m})$. We already have eq. (4.11); analogously, one can prove that $A_{1l} = A_{2m}(s_{m-l} 1_c + c_{m-l} B_m)$. So, it follows that $B_l = (c_{m-l} 1_c - s_{m-l} B_m)^{-1}(s_{m-l} 1_c + c_{m-l} B_m)$.

As for $E_l$, one has

$$
E_l = \sum_{p=1}^{n} \sigma_{-l,0,p-1}^{n-1} C_p A_{2l} = \sum_{p=0}^{n-1} \sigma_{m-l,0,p}^{n-1} B_p^m E_m A_{2m} A_{2l} = (c_{l-m} 1_c - s_{l-m} B_m)^n E_m ,
$$

where we have used eq. (4.6), the relation $\sigma_{m-l}^{n-1} = \sigma_{-l}^{n-1} \sigma_{m}^{n-1}$ and Lemma 4.3.

The equivariance of $\omega_{lm}$ is straightforward, and this completes the proof. \qed
By Proposition 4.1 and Lemma A.2 the immersion $T(c) \hookrightarrow T(c) \times \{1\}$ induces an isomorphism

$$P^n(c)/\text{GL}(c, \mathbb{C})^{\times 2} \simeq T(c)/\Delta,$$

where $\Delta \subset \text{GL}(c, \mathbb{C})^{\times 2}$ is the diagonal subgroup. By comparing eqs. (3.3) and (4.4), it turns out that $T(c)/\Delta = T(c)/\text{GL}(c, \mathbb{C})$. It follows that

$$P^n(c)/\text{GL}(c, \mathbb{C})^{\times 2} \simeq \mathcal{M}^n(c).$$

Recall that $T(c)$ is a principal $\text{GL}(c, \mathbb{C})$-bundle over $T(c)/\text{GL}(c, \mathbb{C}) [10]$. Now, by Proposition 4.1 there is an isomorphism $P^n(c) \simeq T(c) \times \text{GL}(c, \mathbb{C})$ which is well-behaved with respect to the group actions; as a consequence, $P^n(c)/\Delta$ is a locally trivial principal $\text{GL}(c, \mathbb{C})^{\times 2}$-bundle over $\mathcal{M}^n(c)$. Finally, Propositions 3.4 and 4.4 ensure that this property globalizes, in the sense that $P^n(c)$ is a locally trivial principal $\text{GL}(c, \mathbb{C})^{\times 2}$-bundle and this completes the proof of Theorem 2.1.

5. SOME GEOMETRICAL CONSTRUCTIONS

The projection $g_n: \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)) \rightarrow \mathbb{P}^1$ induces a morphism

$$p_{n,c}: \text{Hilb}^c(\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n))) \rightarrow \mathbb{P}^c$$

defined as the composition

$$\text{Hilb}^c(\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n))) \xrightarrow{\pi_{n,c}} S^c \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)) \xrightarrow{g_n} S^c \mathbb{P}^1 = \mathbb{P}^c,$$

where $\pi_{n,c}$ is the Hilbert-Chow morphism. This morphism can be described in terms of ADHM data, as the following result essentially shows. Let $N(c)$ be the space of pairs $(A_1, A_2)$ of $c \times c$ complex matrices satisfying the property (P2), see the beginning of Section 2. The group $\text{GL}(c, \mathbb{C})^{\times 2}$ acts on $N(c)$ as in equation (2.1).

**Proposition 5.1.** There is a commutative diagram of scheme morphisms

(5.1) $\begin{array}{ccc} P^n(c) & \xrightarrow{h_{n,c}} & \text{Hilb}^c(\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n))) \\ p_{n,c} \downarrow & & \downarrow p_{n,c} \\ N(c) & \xrightarrow{g_c} & \mathbb{P}^c, \end{array}$

where $g_c$ is the categorical quotient (in the sense of geometric invariant theory), and $h_{n,c}$, with reference to the notation in the beginning of Section 2, is the morphism

$$h_{n,c}(A_1, A_2; C_1, \ldots, C_n; e) = (A_1, A_2).$$
ADHM data for the Hilbert scheme of the total space of $\mathcal{O}_{\mathbb{P}^3}(-n)$

**Proof.** We introduce the open affine cover $\{U_m\}_{m=0,..,c}$ of $\mathbb{P}^c$

$$U_m = \{[x_0, \ldots, x_c] \in \mathbb{P}^c | \sum_{p=0}^c \sigma^c_{m;p} x_p \neq 0\} \simeq \mathbb{C}^c,$$

where the matrices $\sigma^c_m$ are defined in (4.5). The inverse images $N_m = g^{-1}_c(U_m)$ yield an affine open cover of $N(c)$. The open subsets $h^{-1}_{n,c}(N_m) \subset P^n(c)$ are exactly the sets $P^n(c)_m$ defined in equation (4.2). The composition $g_c \circ h_{n,c}$ on $P^n(c)_m$ can be identified with the map that to the quadruple $(B_m, E_m, e; A_{2m})$ (cf. Proposition 4.1) associates the evaluation of the symmetric elementary functions on the eigenvalues of $B_m$. Since checking the commutativity of the diagram (5.1) is a local matter, and locally our ADHM data coincide with those for the Hilbert scheme of $\mathbb{C}^2$, we can proceed as in [10, p. 10]. □

**Appendix A. Proofs of Propositions 3.2 and 3.4**

**A.1. Preliminaries.** As we recalled in the Introduction, for any isomorphism class $[(\mathcal{E}, \theta)]$ in the moduli space $\mathcal{M}^n(r, a, c)$ of framed sheaves on $\Sigma_n$, the underlying sheaf $\mathcal{E}$ is isomorphic to the cohomology of a monad

(A.1) \[ M(\alpha, \beta) : \quad 0 \longrightarrow U_\bar{k} \overset{\alpha}{\longrightarrow} V_\bar{k} \overset{\beta}{\longrightarrow} W_\bar{k} \longrightarrow 0, \]

where $\bar{k} = (n, r, a, c)$. To express the pair of morphisms $(\alpha, \beta)$ as a pair of matrices, we select suitable bases for the space

\[
\text{Hom}(U_{\bar{k}}, V_{\bar{k}}) \oplus \text{Hom}(V_{\bar{k}}, W_{\bar{k}}) = \\
\text{Hom}(C^{k_1}, C^{k_2}) \otimes H^0(\mathcal{O}_{\Sigma_n}(1, 0)) \oplus \text{Hom}(C^{k_3}, C^{k_4}) \otimes H^0(\mathcal{O}_{\Sigma_n}(0, 1)) \oplus \\
\text{Hom}(C^{k_2}, C^{k_3}) \otimes H^0(\mathcal{O}_{\Sigma_n}(0, 1)) \oplus \text{Hom}(C^{k_4}, C^{k_1}) \otimes H^0(\mathcal{O}_{\Sigma_n}(1, 0)),
\]

where the integers $k_1, k_2, k_3, k_4$ are specified in eq. (1.2). To this aim, after fixing homogeneous coordinates $[y_1, y_2]$ for $\mathbb{P}^1$, we introduce additional $c$ pairs of coordinates

\[
[y_1m, y_2m] = [c_my_1 + s_my_2, -s_my_1 + c_my_2], \quad m = 0, \ldots, c,
\]

where $c_m$ and $s_m$ are the real numbers defined in eq. (3.2). The set $\{y_{2m}^{q}y_{1m}^{h-q}\}_{q=0}^{h}$ is a basis for $H^0(\mathcal{O}_{\Sigma_n}(0, h)) = H^0(\pi^*\mathcal{O}_{\mathbb{P}^3}(h))$ for all $h \geq 1$, where $\pi : \Sigma_n \longrightarrow \mathbb{P}^1$ is the canonical projection. Furthermore if we call $s_E$ the (unique up to homotheties) global section of $\mathcal{O}_{\Sigma_n}(E)$, it induces an injection $\mathcal{O}_{\Sigma_n}(0, n) \hookrightarrow \mathcal{O}_{\Sigma_n}(1, 0)$, so that the set $\{((y_{2m}^{q}y_{1m}^{n-q})s_E)_{q=0}^{n}\}$
\( \{ s_\infty \} \) is a basis for \( H^0 (O_{\Sigma_n}(1, 0)) \), where \( s_\infty \) is the section whose zero locus is \( \ell_\infty \). We get
\[
\alpha = \left( \sum_{q=0}^{n} \alpha_{1q}^{(m)} \left( y_{2m} y_{1m}^{-q} s_E \right) + \alpha_{1n+1} s_\infty \right) \\
\beta = \left( \sum_{q=0}^{n} \beta_{2m}^{(m)} \left( y_{2m} y_{1m}^{-q} s_E \right) + \beta_{2n+1} s_\infty \right).
\]
By restricting the display of the monad \( M(\alpha, \beta) \) to \( \ell_\infty \), twisting by \( O_{\ell_\infty}(-1) \) and taking cohomology, one finds the diagram
\[
(A.2) \quad 0 \rightarrow H^0(\mathcal{V}_{k, \infty}(-1)) \rightarrow H^0(\mathcal{A}_{\infty}(-1)) \rightarrow H^1(\mathcal{U}_{k, \infty}(-1)) \rightarrow 0,
\]
where \( \mathcal{A}_{\infty} = (\text{coker} \alpha)_{\ell_\infty} \). One of the conditions that characterize \( L_k \) is the invertibility of \( \Phi \) (see [2, §2, condition (c4)]). By suitably splitting the short exact sequence which appears in (A.2), the morphism \( \Phi \) becomes
\[
\Phi = \begin{cases} \\
\begin{pmatrix}
\beta_{11}^{(m)} \alpha_{10}^{(m)} + \beta_{21}^{(m)} \alpha_{20}^{(m)}
\beta_{10}^{(m)} \\
\beta_{11}^{(m)} \\
\vdots \\
0 \\
\beta_{11}^{(m)} \\
\end{pmatrix} & \text{for } n = 1; \\
\begin{pmatrix}
\beta_{11}^{(m)} \alpha_{10}^{(m)} + \beta_{21}^{(m)} \alpha_{20}^{(m)}
\beta_{22}^{(m)} \alpha_{20}^{(m)} \\
\vdots \\
\beta_{2n-1}^{(m)} \alpha_{20}^{(m)} \\
\beta_{2n}^{(m)} \alpha_{20}^{(m)}
\end{pmatrix} & \text{for } n > 1.
\end{cases}
\]

Let us now consider the principal \( \text{GL}(r, \mathbb{C}) \)-bundle \( \tau: P_{k}^r \rightarrow L_{k} \), whose fibre over a point \((\alpha, \beta)\) is naturally identified with the space of framings for the cohomology of the monad (A.1). By inspecting the display of \( M(\alpha, \beta) \), one sees that fixing a framing in the fibre \( \tau^{-1}(\alpha, \beta) \) is equivalent to choosing a basis for \( H^0 (\ker \beta|_{\ell_\infty}) = \ker H^0 (\beta|_{\ell_\infty}) \). So, \( P_{k}^r \) can be described as the quasi-affine variety of the triples \((\alpha, \beta, \xi)\), where \((\alpha, \beta)\) is a point of \( L_k \) and \( \xi: \mathbb{C}^r \rightarrow V_k := H^0(\mathcal{V}_{k, \infty}) \) is an injective vector space morphism such that \( H^0 (\beta|_{\ell_\infty}) \circ \xi = 0 \).

A.2. Proof of Proposition [3.2]. We now are in the case where \( r = 1 \) (hence, \( a = 0 \)). We begin by constructing the immersion \( j_m \) for any fixed \( m \in \{ 0, \ldots, c \} \). We define the
morphism

\[ \tilde{\mathcal{I}}_m : \text{End}(\mathbb{C}^c) \oplus \text{Hom}(\mathbb{C}^c, \mathbb{C}) \longrightarrow \text{Hom}(\mathcal{U}_E, \mathcal{V}_E) \oplus \text{Hom}(\mathcal{V}_E, \mathcal{W}_E) \oplus \text{Hom}(\mathbb{C}^c, \mathcal{V}_E) \]

\[(b_1, b_2, e) \mapsto (\alpha, \beta, \xi) \]

where

\[ \alpha = \begin{pmatrix}
1_c(y^{2m}_E s_E) + t b_2 s_\infty \\
1_c y_{1m} + t b_1 y_{2m} \\
0
\end{pmatrix} \]

\[ \beta = \begin{pmatrix}
1_c y_{1m} + t b_1 y_{2m} \\
0 \\
0 \\
1
\end{pmatrix} \]

\[ \xi = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix} \]

**Lemma A.1.** The restriction of \( \tilde{\mathcal{I}}_m \) to \( \mathcal{T}(c) \) is a \( \text{GL}(c, \mathbb{C}) \)-equivariant closed immersion into \( P_{k,m} \).

**Proof.** Let \( \mathcal{I}_m \) be the restriction of \( \tilde{\mathcal{I}}_m \) to \( \mathcal{T}(c) \). Since it is clear that \( \mathcal{I}_m \) is a closed immersion, it is enough to prove that

\[ \text{Im} \mathcal{I}_m \cap P_{k,m} = \text{Im} \mathcal{I}_m. \]

Let \( (\alpha, \beta, \xi) = \mathcal{I}_m(b_1, b_2, e) \) be any point in the intersection \( \text{Im} \mathcal{I}_m \cap P_{k,m} \); the equation \( \beta \circ \alpha = 0 \) implies that the triple \( (b_1, b_2, e) \) satisfies the condition (T1), while the fact that \( \beta \otimes k(x) \) has maximal rank for all \( x \in \Sigma_n \) implies the condition (T2). It follows that

\[ \text{Im} \mathcal{I}_m \cap P_{k,m} \subseteq \text{Im} \mathcal{I}_m. \]

To get the opposite inclusion, note that for all \( (\alpha, \beta, \xi) \in \text{Im} \mathcal{I}_m \) the following conditions are satisfied:

1. The morphism \( \alpha \otimes k(x) \) fails to have maximal rank at most at a finite number of points \( x \in \Sigma_n \); hence, \( \alpha \) is injective;
2. The morphisms \( \alpha \otimes k(x) \) and \( \beta \otimes k(x) \) have maximal rank for all points \( x \in \ell_\infty \cup f_m \);
3. The morphism \( \Phi \) is invertible;
4. One has \( \beta_1|_{f_m} = 1_c \);
5. The morphism \( \xi \) has maximal rank.

If \( (\alpha, \beta, \xi) \in \text{Im} \mathcal{I}_m \), the condition (T2) implies that \( \beta \otimes k(x) \) has maximal rank for all \( x \in \Sigma_n \setminus (\ell_\infty \cup f_m) \); by the condition (ii) this is sufficient to ensure that \( \beta \) is surjective. The condition (T1) implies that \( \beta \circ \alpha = 0 \), so that we can define the quotient sheaf
$E = \ker \beta / \text{Im} \alpha$. By the condition (i) $E$ is torsion free, by the conditions (ii) and (iii) it is trivial at infinity, and by the condition (iv) $E|_{f_m}$ is trivial as well. The $\text{GL}(c, \mathbb{C})$-equivariance of $j_m$ is readily checked. □

To prove that $j_m$ induces an isomorphism between the quotients, we need to use a well-known result which we recall for the reader’s convenience. Let $Y \xrightarrow{j} X$ be a closed immersion of complex algebraic varieties, and let $H \xhookrightarrow{i} G$ be an injective homomorphism of complex algebraic groups. Consider a $G$-action on $X$ and a $H$-action on $Y$ such that $j$ is $H$-equivariant.

**Lemma A.2.** If for all $G$-orbits $O_G$ in $X$ the intersection $O_G \cap \text{Im} j$ is nonempty and its stabilizer in $G$ is $\text{Im} \iota$, then $j$ induces an isomorphism of ringed spaces between $Y/H$ and $X/G$. □

We have to show that for any $G_{\vec{k}}$-orbit $O_{G_{\vec{k}}}$ in $P_{\vec{k},m}$ the intersection $O_{G_{\vec{k}}} \cap \text{Im} j_m$ is not empty, and that the stabilizer of this intersection in $G_{\vec{k}}$ is $\text{Im} \iota$. We build up a strictly descending chain of closed subvarieties

$$P_{\vec{k},m} =: P^0 \supsetneq P^1 \supsetneq \cdots \supsetneq P^h = \text{Im} j_m,$$

for a certain $h > 0$, such that there exists a strictly descending chain of subgroups

$$G_{\vec{k}} =: G^0 \supsetneq G^1 \supsetneq \cdots \supsetneq G^h = \text{Im} \iota$$

with the property that $G^i$ is the stabilizer inside $G_{\vec{k}}$ of the intersection $O_{G_{\vec{k}}} \cap P^i$ for all $G_{\vec{k}}$-orbits in $P_{\vec{k},m}$.

Note that for each point $(\alpha, \beta, \xi) \in P_{\vec{k}}$ one has an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\Sigma_n} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where $\mathcal{E} = \mathcal{E}_{\alpha, \beta}$ and $Z$ is the singular locus of $\mathcal{E}$. If we restrict this sequence to $f_m$, twist it by $\mathcal{O}_{f_m}(-1)$ and take cohomology, we find out that $Z \cap f_m = \emptyset$ if and only if $H^i(\mathcal{E}|_{f_m}(-1)) = 0$ for $i = 0, 1$. By using the display of the monad $M(\alpha, \beta)$ one sees that this condition is equivalent to the condition $\det(\beta_{10}^{(m)}) \neq 0$ (the coefficient $\beta_{10}^{(m)}$ is defined in eq. (A.1)).

By acting with $G_{\vec{k}}$ on $(\alpha, \beta, \xi)$ we can assume that

$$\begin{cases} 
\beta_{10}^{(m)} = 1_c \\
\beta_{2q}^{(m)} = 0 & q = 0, \ldots, n - 1.
\end{cases}$$
These equations define the subvariety $P^1$, whose stabilizer $G^1$ is the subgroup of $G_\mathbf{k}$ determined by

$$
\begin{align*}
\psi_{11} &= \chi \\
\psi_{12} &= 0.
\end{align*}
$$

Let $\ell b := \beta_{11}^{(m)}$.

The equation $\beta \circ \alpha = 0$ implies that

$$
\begin{align*}
\alpha_{1q}^{(m)} &= 0 \quad q = 0, \ldots, n - 1 \\
\alpha_{1n}^{(m)} &= -\beta_{2n}^{(m)} \alpha_{20}^{(m)}.
\end{align*}
$$

(A.3)

The invertibility of $\Phi$ is equivalent to the condition $\det \alpha_{1n}^{(m)} \neq 0$, and by acting with $G^1$ we can assume that $\alpha_{1n}^{(m)} = 1_c$. This equation cuts the subvariety $P^2$ inside $P^1$, and the stabilizer $G^2$ is the subgroup of $G^1$ where $\chi = \phi$.

From eq. (A.3) we deduce that

$$
1_c = -\beta_{2n}^{(m)} \alpha_{20}^{(m)}, \quad \text{so that} \quad \text{rk } \beta_{2n}^{(m)} = \text{rk } \alpha_{20}^{(m)} = c.
$$

(A.4)

Therefore, by acting with $G^2$ we can assume that

$$
\alpha_{20}^{(m)} = \begin{pmatrix} 1_c \\ 0 \end{pmatrix}.
$$

This equation cuts the subvariety $P^3$ inside $P^2$, and the stabilizer $G^3$ is the subgroup of $G^2$, where

$$
\psi_{22} = \begin{pmatrix} \phi & g_{12} \\ 0 & g_{22} \end{pmatrix}
$$

for some $g_{12} \in \text{Hom}(\mathbb{C}, \mathbb{C}^c)$ and $g_{22} \in \mathbb{C}^*$.  

Eq. (A.4) implies that $\beta_{2n}^{(m)}$ is of the form $\begin{pmatrix} -1_c & * \end{pmatrix}$, but by acting with $G^3$ we can assume that $\beta_{2n}^{(m)} = \begin{pmatrix} -1_c & 0 \end{pmatrix}$. This equation characterizes $P^4$ inside $P^3$, and the stabilizer $G^4$ is the subgroup of $G^3$ where $g_{12} = 0$. The equation $H^0 (\beta|_{\ell \infty}) \circ \xi = 0$ implies that

$$
\xi^{(m)} = \begin{pmatrix} 0 \\ g^{-1} \end{pmatrix}.
$$

By acting with $G^4$ we can assume that $\theta = 1$: this cuts $P^5$ inside the variety $P^4$, and the stabilizer $G^5$ is the subgroup of $G^4$ where $g_{22} = 1$. It is not difficult to show that $G^5$ coincides with $\text{Im } \iota$. To prove that $P^5 = \text{Im } j_m$ we use once more the constraint $\beta \circ \alpha = 0$.  

and get the system
\[
\begin{align*}
\begin{cases}
\left[ t^i b_1 + (-1_c \quad 0) \alpha_{21}^{(m)} \right] = 0 \\
\alpha_{1,n+1} + \beta_{2,n+1} \left( 1_c \quad 0 \right) = 0 \\
\beta_{11}^{(m)} \alpha_{1,n+1} + \beta_{2,n+1} \alpha_{21}^{(m)} = 0.
\end{cases}
\end{align*}
\]
From the first two equations we deduce that
\[
\alpha_{21}^{(m)} = \begin{pmatrix} t^i b_1 \\ t^e e_2 \end{pmatrix} \quad \text{and} \quad \beta_{2,n+1} = \begin{pmatrix} -\alpha_{1,n+1} & t^e \end{pmatrix}
\]
for some \( e \in \text{Hom}(\mathbb{C}^e, \mathbb{C}) \) and \( e_2 \in \text{Hom}(\mathbb{C}, \mathbb{C}^e) \). Only the last equation is not identically satisfied, and is equivalent to
\[
t^i b_1 t^i b_2 - t^i b_2 b_1 + t^e e_2 = 0,
\]
where we have put \( t^i b_2 = \alpha_{1,n+1} \). Since the morphism \( \beta \otimes k(x) \) has maximal rank for all \( x \in \Sigma_n \), the quadruple \( (t^i b_1, t^i b_2, t^e, t^e e_2) \) satisfies the hypotheses of \([10, \text{Proposition 2.8}]\), which implies \( e_2 = 0 \). It follows that \( P^5 = \text{Im} j_m \).

A.3. Proof of Proposition 3.4.

Lemma A.3. For any \( l, m = 0, \ldots, c \) and for any point \( \vec{b}_m = (b_1, b_2, e_m) \in T(c)_m \), there exists a unique element \( \psi_l(\vec{b}_m) = (\phi, \psi, \chi) \in G_k \) such that
\[
\begin{itemize}
\item \( \chi = 1_c; \)
\item the point \((\alpha', \beta', \xi') = \psi_l(\vec{b}_m) \cdot j_m(\vec{b}_m) \) lies in the image of \( j_l \).
\end{itemize}
\]
If we set \( (b_{1l}, b_{2l}, e_l) = j_l^{-1}(\alpha', \beta', \xi') \), we have
\[
\begin{align*}
\begin{cases}
b_{1l} = (c_m-1_c - s_m - b_{1m})^{-1} (s_m-1_c + c_m - b_{1m}) \\
b_{2l} = (c_m-1_c - s_m - b_{1m})^n b_{2m} \\
e_l = e_m.
\end{cases}
\end{align*}
\]
(A.5)

Proof. If we set \( (\alpha, \beta, \xi) = j_m(\vec{b}_m) \), by expressing \( [y_{1n}, y_{2m}] \) as function of \( [y_{1l}, y_{2l}] \) we get
\[
\alpha = \begin{pmatrix}
\sum_{q=0}^n (\sigma_q 1_c) (y_{2l} y_{1l}^{n-q} s_E) + t^i b_{2m} s_\infty \\
d_1 y_{1m} + d_2 y_{2m} \\
0
\end{pmatrix},
\]
\[
\beta = \begin{pmatrix}
d_1 y_{1m} + d_2 y_{2m} - \sum_{q=0}^n (\sigma_q 1_c) (y_{2l} y_{1l}^{n-q} s_E) - t^i b_{2m} s_\infty & t^e s_\infty
\end{pmatrix};
\]

ADHM data for the Hilbert scheme of the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$

where

\[ d_{1m} = c_{m-1} c - s_{m-l} b_{1m} \quad \text{and} \quad d_{2m} = s_{m-l} c + c_{m-l} b_{1m} \]

and we have put $\sigma_q = \sigma_{l-n; m}^n$ for $q = 0, \ldots, n$ (see eq. (4.5)). The explicit form of $\psi_l(\vec{b}_m)$ is obtained by imposing the equality

\[(A.6) \quad (\phi, \psi, 1_c) \cdot (\alpha, \beta, \xi) = j_l(b_{1l}, b_{2l}, e_l)\]

for some $(b_{1l}, b_{2l}, e_l) \in T(c)_l$. One gets

\[
\phi = d_{1m}^{(n-1)} \\
\psi = \begin{pmatrix}
d_{1m} & \psi_{12,1} & 0 \\
0 & d_{1m}^{n} & 0 \\
0 & 0 & 1_r
\end{pmatrix},
\]

where

\[
\psi_{12,1} = -\sum_{q=0}^{n-1} \sum_{p=0}^{q} \sigma_{q-p} \left(-d_{2m} d_{1m}^{n-1}\right)^p y_{1l} y_{2l}^{n-1-q}.
\]

Eq. (A.5) follows from eq. (A.6).

Since $j_m$ and $j_l$ are injective, the map $\tilde{b}_m \mapsto \psi_l(\tilde{b}_m) \cdot j_m(\tilde{b}_m)$ induces the morphism $\tilde{\varphi}_{lm}$ in eq. (3.6). This completes the proof of Proposition 3.4.

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