ON THE NONLINEAR STATISTICS OF OPTICAL FLOW

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Abstract. In A naturalistic open source movie for optical flow evaluation, Butler et al. create a database of ground-truth optical flow from the computer-generated video Sintel. We study the high-contrast 3 × 3 patches from this video, and provide evidence that this dataset is well-modeled by a torus (a nonlinear 2-dimensional manifold). Our main tools are persistent homology and zigzag persistence, which are popular techniques from the field of computational topology. We show that the optical flow torus model is naturally equipped with the structure of a fiber bundle, which is furthermore related to the statistics of range images.

1. Introduction

A video records a moving three-dimensional world as a sequence of two-dimensional images. The apparent motion of the two-dimensional images, due to changing brightness, is called optical flow. The optical flow at a frame is a vector field, where the vector at each pixel points to where that pixel appears to move for the subsequent frame [10].

A fundamental problem is to estimate optical flow from a video sequence [7, 23]. It is impossible to recover the optical flow field exactly using only a video sequence; for example, if one is given a video of a spinning barber’s pole, one does not know (without prior knowledge) whether the pole is moving up or instead spinning horizontally. Given these difficulties, algorithms estimating optical flow must exploit or make assumptions about the statistics of optical flow, and hence there is interest in understanding these statistics.

As no instrument measures ground-truth optical flow, databases must be generated. One example database is from the 3-D, animated, open source, short film Sintel, which has several desirable features. The scenes are long, and the movements and textures are more complex than in some of the other datasets. Since the film is open source, the optical flow data is available for analysis (see Figure 1). The Sintel optical flow dataset is described in detail in [14].

In this paper we use the topological machinery of [2, 17] to study the nonlinear statistics of optical flow from the Sintel dataset. First, we build a space of high-contrast 3 × 3 optical flow patches. Using Vietoris–Rips complexes and persistent homology, we identify the topologies of dense subsets of this space. The densest patches lie near a circle, the horizontal flow circle [1]. Then, in a more refined analysis, we select out the optical flow patches whose predominant direction of flow is a small bin of angle values. We show that the patches in each such bin are well-modeled by a circle; each such circle is explained by the nonlinear statistics of range image patches. We show that these circles at each angle piece together, via the structure of a fiber bundle, into a torus model for optical flow.
Figure 1. Two sample optical flows extracted from the Sintel database. Horizontal components in (a) and (c); vertical components in (b) and (d). White corresponds to flow in the positive direction (right or up) and black corresponds to the negative direction.

The torus model for the nonlinear statistics of optical flow could also be used for optical flow compression. Indeed, one can express a $3 \times 3$ optical flow patch as an average flow vector, plus a patch on a 2-dimensional torus, plus a $3 \times 3$ error vector whose entries will tend to be small in magnitude.

We survey related work in Section 2, and in Section 3 we introduce our topological methods. We describe the spaces of high-contrast optical flow patches in Section 4, and present our main results in Section 5. Our code is available at [bitbucket.org/Cross_Product/optical_flow/](https://bitbucket.org/Cross_Product/optical_flow/).

2. Related Work

2.1. Optical Flow Datasets. There are a variety of databases that reconstruct ground truth optical flow samples. The Middlebury dataset in [5] ranges from real stereo imagery of rigid scenes to realistic synthetic imagery; the database contains public ground truth optical flow training data along with sequestered ground truth data for the purpose of testing algorithms. The data from [30] consists of twenty different synthetic scenes with the camera and movement information provided. The KITTI Benchmark Suite [24] uses a car mounted with two cameras to film short clips of pedestrians and cars; attached scanning equipment allows one to reconstruct the underlying truth optical flow for data testing and error evaluation.

Another example is the database created by Roth and Black [33] to study the statistics of optical flow. Unlike databases used to test optical flow estimation, the Roth and Black database does not include accompanying video sequences. Freed from this constraint, Roth and Black generate optical flow for a wide variety of natural scenes by pairing range images[1] with camera motions. The resulting optical flow can be calculated from the geometry of the static scene and of the camera motion. The database includes only optical flow from static scenes seen by a moving camera: no objects in the field of view move independently. By contrast, the ground-truth Sintel optical flow database [14], which we study in this paper, is computed directly from the film’s motion vectors; there is no need to reconstruct the flow via computations from laser scans.

2.2. Optical Flow Applications. Optical flow estimation is most commonly used in computer vision tasks. Computer vision is a process where a computer takes in

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1 A range image contains a distance at each pixel.
visual data, analyzes the data via various statistics (for example after estimating optical flow), and then outputs information or a decision based on the data. Computer vision algorithms utilizing the estimation of optical flow can be found in facial recognition software [6], in driving autonomous cars [24], and in robotic tracking [25].

2.3. Optical and Range Image Work from the Topological Perspective. We briefly describe a subset of papers that have analyzed the statistics of optical images, range images, and optical flow from the perspective of computational topology. Foundational papers in this area include [28], which proposes a circular model for $3 \times 3$ optical image patches, and [17], which uses persistent homology to extend this circular model to both a three-circle model and a Klein bottle model for different dense core subsets.

The nonlinear statistics of range image patches (which contain a distance at each pixel) will play an important role in our work. In [28], the authors observe that high-contrast $3 \times 3$ range patches from [27] cluster near binary patches. The paper [2] uses persistent homology to find that the densest range clusters are arranged in the shape of a circle. After enlarging to $5 \times 5$ or $7 \times 7$ patches, the entire primary circle in Figure 2(a) is dense. The patches forming the range primary circle are binary approximations to linear step edges; see Figure 2(b).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{(Left) Range patch primary circle [2]. White regions are far; black regions are near. (Right) The top row contains linear step edges; the bottom row contains their range image binary approximations.}
\end{figure}

The paper [1] uses the nudged elastic band method to propose the horizontal flow circle model for (horizontal) optical flow patches.

3. Topological Machinery

In this section we describe how to use only a finite sampling from some unknown underlying space to estimate the underlying space’s topology. The first step is to build a nested family of simplicial complexes, and the second is to apply persistent homology. This is the same topological approach used to analyze optical and range image patches in [2, 17]. We refer the interested reader to [1, 25] for more information on homology, to [15, 21, 22, 57] for introductions to persistent homology, and to [1, 8, 11, 12, 19, 20, 34, 35, 36] for example applications of persistent homology to sensor networks, machine learning, biology, medical imaging, etc.

3.1. Vietoris–Rips Complexes. Our nested complexes will be Vietoris–Rips simplicial complexes. The main idea is to define all data points to be vertices of the complex, and to define a simplex $\sigma$ on each finite set of vertices within a given diameter. Indeed, let $(X, d)$ denote a metric space, and fix a scale parameter $r \geq 0$. The
Vietoris–Rips simplicial complex with vertex set $X$ and scale parameter $r$, denoted $\text{VR}(X; r)$, is defined as follows. A finite subset $\sigma = \{x_1, \ldots, x_n\} \subseteq X$ is a face of $\text{VR}(X; r)$ whenever $\text{diam}(\sigma) \leq r$ (i.e., whenever $\sup_{1 \leq i \leq j \leq n} \{d(x_i, x_j)\} \leq r$). By definition, $\text{VR}(X; r) \subseteq \text{VR}(X; r')$ whenever $r \leq r'$, so this family is indeed nested.

Let us consider an example. Let $X$ be 21 points which (unknown to us) are sampled with noise from a circle. Figure 3 contains four nested Vietoris–Rips complexes built from $X$, with $r$ increasing from left to right. The black dots denote $X$. In (a), $r$ is small enough that a loop has not yet formed. In (b), $r$ is such that we recover instead a figure-eight. In (c), $\text{VR}(X; r)$ recovers a circle. In (d), $r$ is large enough that the loop has filled to a disk.

**Figure 3.** Four nested Vietoris–Rips complexes, with $\beta_0$ equal to 1 in all four complexes, and with $\beta_1$ equal to 0, 2, 1, and 0.

![Vietoris–Rips complexes](image)

**Figure 4.** (Top) The 0-dimensional persistence barcode associated to the dataset in Figure 3. (Bottom) The 1-dimensional persistence barcode associated to the same dataset.

3.2. **Persistent Homology.** Betti numbers are one way of distinguishing between different topological spaces: a necessary condition for two spaces to be homotopy equivalent is for all of their Betti numbers to be equal. The $k$-th Betti number of a topological space, denoted $\beta_k$, is the rank of the $k$-th homology group. Roughly speaking, $\beta_k$ is the number of “$k$-dimensional holes” in a space, where the number of 0-dimensional holes is the number of connected components. For an $n$-dimensional sphere with $n \geq 1$, we have $\beta_0 = 1$ and $\beta_n = 1$.

If we want to estimate the topology of the underlying space by the topology of $\text{VR}(X; r)$, the choice of $r$ is important. However, without knowing the underlying space, we do not know how to make this choice. Hence, we use persistent homology [22, 37], which allows us to compute the Betti numbers over a range of $r$-values and to display the result as a persistent homology barcode. See Figure 4.
Persistent homology depends on the fact that the map from a topological space $Y$ to its $k$-th homology group $H_k(Y)$ is a functor. This means that for $r \leq r'$, the inclusion $\text{VR}(X;r) \hookrightarrow \text{VR}(X;r')$ of topological spaces induces a map $H_k(\text{VR}(X;r)) \rightarrow H_k(\text{VR}(X;r'))$ between homology groups $[21]$.

The horizontal axis in Figure 3 contains the varying $r$-values. At a given scale $r$, the Betti number $\beta_k$ is the number of intervals in the dimension $k$ plot that intersect the vertical line through scale $r$. In the dimension 0 plot, we see the 21 disjoint spaces joining into one connected component as $r$ increases. The two intervals in the dimension 1 plot correspond to the two loops that appear: each interval begins when a loop forms and ends when that loop fills to a disk.

The topological profile of this example, $\beta_0 = 1$ and $\beta_1 = 1$, is obtained for a long range of $r$-values in Figure 3. The idea of persistent homology is that long intervals in the persistence barcodes correspond to real topological features of the underlying space. We often disregard short intervals as noise. Hence, this barcode reflects the fact that our points $X$ were noisily sampled from a circle.

3.3. Zigzag Persistent Homology. Zigzag persistence $[16, 18]$ provides a generalization of the theory of persistent homology. In zigzag persistence, the direction of maps along a sequence of topological spaces is arbitrary, as opposed to the unidirectional sequence of maps in persistent homology. Given a large dataset $Y$, one may attempt to estimate the topology of $Y$ by instead estimating the topology of a number of smaller subsets $Y_i \subseteq Y$. Toward that end, consider the following diagram of inclusion maps between subsets of the data.

\begin{equation}
Y_1 \leftrightarrow Y_1 \cup Y_2 \leftrightarrow Y_2 \leftrightarrow Y_2 \cup Y_3 \leftrightarrow Y_3 \leftrightarrow \cdots \leftrightarrow Y_n.
\end{equation}

Applying the Vietoris–Rips construction at scale parameter $r$ and $k$-dimensional homology, we obtain an induced sequence of linear maps

\[ H_k(\text{VR}(Y_1;r)) \rightarrow H_k(\text{VR}(Y_1 \cup Y_2;r)) \leftrightarrow H_k(\text{VR}(Y_2;r)) \rightarrow \cdots \leftrightarrow H_k(\text{VR}(Y_n;r)) \]

which is an example of a zigzag diagram. Crucially, such a sequence of linear maps provides the ability to track features contributing to homology among the samples $Y_i$. In other words, generators for homology of two spaces $\text{VR}(Y_i;r)$ and $\text{VR}(Y_{i+1};r)$ which map to the same generator of $H_k(\text{VR}(Y_i \cup Y_{i+1};r))$ indicate a feature common to both $Y_i$ and $Y_{i+1}$. Hence, by tracking features common to all samples $Y_i$, one may estimate the topology of $Y$ without explicitly computing the persistent homology of the entire dataset.

3.4. Fiber Bundles. Our identification of a torus model for the MPI-Sintel optical flow dataset is guided by the notion of a fiber bundle. Precisely, a fiber bundle is a tuple $(E, B, f, F)$, where $E$, $B$, and $F$ are topological spaces and $f: E \rightarrow B$ is a continuous map satisfying a so-called local triviality condition, described below. We call $B$ the base space, $E$ the the total space, and $F$ the fiber. The local triviality condition on $f$ is as follows: given $b \in B$, there exists an open set $U \subseteq B$ containing $b$ and a homeomorphism $\varphi: f^{-1}(U) \rightarrow U \times F$ such that $\text{proj}_U \circ \varphi = f|_{f^{-1}(U)}$, where $\text{proj}_U$ denotes the projection onto the $U$–component. In other words, we require $f^{-1}(U)$ to be homeomorphic to $U \times F$ in a particular way. Therefore, for any $p \in B$, we have $f^{-1}((p)) \cong F$. Locally, the total space $E$ looks like $B \times F$, while globally, a fiber bundle contains information about how these copies of the fiber $F$ may be “twisted” in a particular way.
As an example, both the cylinder and the Möbius band may be realized as fiber bundles with base space the circle $S^1$, and with fibers the unit interval $[0, 1]$. In the case of the Möbius band, the global structure of the fiber bundle gives a “half twist” as one loops around the circle, whereas the global structure of the cylinder does not contain a twist. Locally, however, both spaces look the same, as each have the same fiber above each point of $S^1$.

Analogously, both the torus and the Klein bottle may be realized as fiber bundles over $S^1$, with fibers homeomorphic to $S^1$. In this case, the fibers of the Klein bottle “twist” in a particular way, whereas the fibers of the torus do not.

In Section 5, we use persistent homology to provide evidence that the MPI-Sintel dataset is naturally equipped with the structure of a fiber bundle over a circle, with each fiber being a circle. It is not a priori clear whether this fiber bundle model should be the orientable torus or the nonorientable Klein bottle; indeed, the space of optical image patches as studied in [17] is well-modeled by a Klein bottle. However, in Section 5.2, we provide evidence that this optical flow fiber bundle is a torus.

4. Spaces of Flow Patches

The MPI-Sintel optical flow dataset [14] contains 1041 optical flow fields, each $1024 \times 436$ pixels. The dataset originates from the open-source animated film Sintel [32], which contains a variety of realism-enhancing effects, including widely varied motion, illumination, and blur. This data is extracted from 23 scenes of indoor and outdoor environments, with up to 49 frames per scene.

We create spaces of high-contrast optical flow patches, $X(k, p)$ or $X_\theta(k, p)$. The version $X_\theta(k, p)$ includes only those optical flow patches whose predominant angle is near $\theta \in [0, \pi)$. Our preprocessing is similar to that done in [2, 17, 28].

Step 1: We randomly choose $4 \cdot 10^5$ size $3 \times 3$ optical flow patches from the MPI-Sintel database. Each patch is a matrix of ordered pairs, where $u_i$ and $v_i$ are the horizontal and vertical components of the flow vector at pixel $i$.

\[
\begin{bmatrix}
(u_1, v_1) & (u_4, v_4) & (u_7, v_7) \\
(u_2, v_2) & (u_5, v_5) & (u_8, v_8) \\
(u_3, v_3) & (u_6, v_6) & (u_9, v_9)
\end{bmatrix}
\]

For convenience, we rearrange each patch $x$ to be a length-18 vector, $x = (u_1, u_2, \ldots, u_9, v_1, \ldots, v_9) \in \mathbb{R}^{18}$. We define $u$ and $v$ to be the vectors of horizontal and vertical flow: $u = (u_1, u_2, \ldots, u_9)^T$ and $v = (v_1, v_2, \ldots, v_9)^T$.

Step 2: Let $i \sim j$ denote that pixels $i$ and $j$ are adjacent in the $3 \times 3$ patch. For each patch $x$, we compute the contrast norm $\|x\|_D$ by summing the squared differences between all adjacent pixels and then taking the square root, namely:

$$\|x\|_D^2 = \sum_{i \sim j} \|(u_i, v_i) - (u_j, v_j)\|^2 = \sum_{i \sim j} (u_i - u_j)^2 + (v_i - v_j)^2 = u^T D u + v^T D v.$$

Matrix $D$, which stores the adjacency information of the pixels in a $3 \times 3$ patch, is a symmetric positive definite $9 \times 9$ matrix given in [28].

Step 3: We select out the patches that have a contrast norm in the top 20% of the entire sample; hence we are only studying high-contrast flow patches, which we expect to follow a different distribution than low-contrast patches. After doing so, we replace each patch $x$ with its contrast-normalized patch $x/\|x\|_D$: this places each patch on the surface of an ellipsoid. We need not worry about dividing by contrast norm zero, as such patches are not high-contrast.
Step 4: We further normalize the patches to have zero average flow. For a patch $x$, let $ar{u} = \frac{1}{9} \sum_{i=1}^{9} u_i$ be the average horizontal flow, and let $\bar{v} = \frac{1}{9} \sum_{i=1}^{9} v_i$ be the average vertical flow. We replace each contrast-normalized vector $x$ with $(u_1 - \bar{u}, \ldots, u_9 - \bar{u}, v_1 - \bar{v}, \ldots, v_9 - \bar{v})^T$. The purpose of studying mean-centered optical flow patches is that one can represent any optical flow patch as its mean vector plus a mean-centered patch.

Step 5: If we are computing $X_\theta(k,p)$ (as opposed to $X(k,p)$), then we compute the predominant direction of each mean-centered flow patch, as follows. For each $3 \times 3$ patch, construct a $9 \times 2$ matrix $X$ whose $i$-th row is $(u_i, v_i) \in \mathbb{R}^2$. We apply principal component analysis (PCA) to $X$ in order to retrieve the principal component with the greatest component variance (i.e., the direction that best approximates the deviation from the mean). The angle of this line (in $[0, \pi]$ or $\mathbb{RP}^1$) is defined to be the predominant direction of the patch. We select out only those patches whose predominant direction is in the range of angles on $\mathbb{RP}^1 = [0, \pi)$ from $\theta - \frac{\pi}{12}$ to $\theta + \frac{\pi}{12}$.

Step 6: If we have more than 50,000 patches, then we randomly subsample down to 50,000 random patches for the sake of computational feasibility.

Step 7: We now have at most 50,000 high-contrast normalized optical flow patches. Instead of trying to approximate the topology of such a diverse space, we restrict to dense core subsets. We use the density estimator $\rho_k$, where $\rho_k(x)$ is the distance from $x$ to its $k$-th nearest neighbor. Note that $\rho_k$ is inversely proportional to density. Decreasing (or increasing) the choice of $k$ produces a more local (or global) estimate of density. We select out the top $p\%$ densest points, based on the density estimator $\rho_k$. We denote this set of patches by $X(k,p)$ (or by $X_\theta(k,p)$, in the case where Step 5 is performed).

5. Results

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{(Left) Projection of $X(300,30)$ onto $e_1^u$ and $e_2^u$. (Right) The horizontal flow circle. The patch at angle $\alpha$ is $\cos(\alpha)e_1^u + \sin(\alpha)e_2^u$.}
\end{figure}

5.1. The Horizontal Flow Circle. The dataset $X(300,30)$ is well-modeled by the horizontal flow circle, as identified in [1] via the nudged elastic band method. We locate this circle by projecting onto suitable basis vectors. Let $e_1, e_2, \ldots, e_8$ be the discrete cosine transform (DCT) basis for $3 \times 3$ scalar patches, normalized to have mean zero and contrast norm one. This basis is given in [28]. For convenience, we rearrange each $e_i$ to be a length-9 vector. Now, let $z$ be the length-9 zero vector.
For each $i = 1, 2, \ldots, 8$, we define optical flow vectors $e_i^u = \begin{pmatrix} e_i^u \cr z \end{pmatrix}$ and $e_i^v = \begin{pmatrix} z \cr e_i^v \end{pmatrix}$.

Note that $e_i^u, e_i^v \in \mathbb{R}^{18}$ correspond respectively to optical flow in the horizontal and vertical directions. We change coordinates from the canonical basis for $\mathbb{R}^{18}$ to the 16 basis vectors $e_1^u, \ldots, e_8^u, e_1^v, \ldots, e_8^v$. Some of these basis vectors are in Figure 6 (left). The projection of $X(300, 30)$ onto basis vectors $e_1^u$ and $e_2^u$, shown in Figure 5 (left), reveals the circular topology.

![Figure 6](image)

**Figure 6.** (Left) In the $e_1$ and $e_2$ DCT patches, white pixels are positive and black negative. The arrows in the flow patches $e_1^u$, $e_2^u$, $e_1^v$, and $e_2^v$ show the optical flow vector field patch. (Right) Camera axes.

Let $S^1$ denote the interval $[0, 2\pi]$ with endpoints identified. The patches in $X(300, 50)$ lie near \( \{ \cos(\alpha)e_1^u + \sin(\alpha)e_2^u \mid \alpha \in S^1 \} \), which we call the **horizontal flow circle**. We will use the statistics of both the camera motion database and the range image database to explain why the horizontal circle is high-density.

Camera motion can be decomposed into six sub-motions. The first three are translation in the $x$, $y$, or $z$ direction, commonly referred to as right-left, up-down, or inward-outward translation. The remaining three are rotation about the $x$, $y$, or $z$ axis, commonly referred to as pitch, yaw, or roll. See Figure 6 (right). We will refer to $\theta \in S^1$ camera translation, by which we mean translation of $\cos(\alpha)$ units to the right, $\sin(\alpha)$ units up, and no units inwards or outwards. Some of the most common camera translations are when $\theta = 0$ or $\pi$, i.e. when the camera is translated to the left or right, for example if the camera is mounted on a horizontally moving car or held by a horizontally walking human.

In [2] the authors find that high-contrast range patches are dense near the range patch primary circle, defined as \( \{ \cos(\alpha)e_1 + \sin(\alpha)e_2 \mid \alpha \in S^1 \} \) and depicted in Figure 2(a). Negative and positive pixel coordinates correspond, respectively, to near and far.

Let us consider pairing the common $\theta = 0$ or $\pi$ camera translations with primary circle range patches. Under camera translation in the $xy$ plane, the flow vector at a foreground pixel has the same direction but greater magnitude than at a background pixel. So after the mean-centering normalization in Step 4 of Section 2 a $\theta = 0$ camera translation over the range patch $\cos(\alpha)e_1 + \sin(\alpha)e_2$ produces the optical flow patch $\cos(\alpha)e_1^u + \sin(\alpha)e_2^u$. Similarly, $\theta = \pi$ translation produces flow patch $-\cos(\alpha)e_1^u - \sin(\alpha)e_2^u$. Hence $\theta = 0$ or $\pi$ translation applied to all primary circle range patches produces the horizontal flow circle \( \{ \cos(\alpha)e_1^u + \sin(\alpha)e_2^u \mid \alpha \in S^1 \} \) (Figure 5 (right)).

### 5.2. A Torus Model for Optical Flow

Define the map $f : S^1 \times S^1 \rightarrow \mathbb{R}^{18}$ as follows. Given $(\alpha, \theta) \in S^1 \times S^1$, let $f(\alpha, \theta)$ be the optical flow patch produced from $\theta$ camera translation over the primary circle range patch $\cos(\alpha)e_1 + \sin(\alpha)e_2$. More
optical flow patches. We obtain information that is consistent with this model by considering directional components of optical flow.

Through we have only presented evidence for the horizontal flow, though we have only presented evidence for the horizontal flow, in other words, antipodal points in Figure 12 produce the same flow patch. For instance, the horizontal flow, the right edge of Figure 13 is identified by shifting one upwards by half its length. So

\[
\beta = \angle \text{of the line separating these regions. The black arrow (} > \text{, } \vee, \text{, } < \text{, or } \wedge \text{) in the white rectangle is the direction } \theta \text{ of camera translation. Together, the black and white arrows show the induced optical flow } f(\alpha, \theta) \text{. In Figure 7(left), parameter } \alpha \text{ varies in the horizontal direction, and parameter } \theta \text{ varies in the vertical direction.}
\]

Note that for two points \((\alpha, \theta)\) and \((\alpha', \theta')\) on the torus \(S^1 \times S^1\), we have

\[
f(\alpha, \theta) = f(\alpha', \theta') \iff (\alpha, \theta) = (\alpha', \theta') \text{ or } (\alpha, \theta) = (-\alpha', -\theta').
\]

explicitly,

\[
f(\alpha, \theta) = \cos(\theta) \left( \cos(\alpha) e_1^u + \sin(\alpha) e_2^u \right) + \sin(\theta) \left( \cos(\alpha) e_1^v + \sin(\alpha) e_2^v \right).
\]

The horizontal flow circle is obtained by restricting to common camera motions \(\theta \in \{0, \pi\}\) and allowing \(\alpha\) to vary. We hypothesize that when neither parameter is restricted, a good model for flow patches is obtained. Hence we ask, what is the image space \(\text{im}(f)^2\)? Consider Figure 7(left), which shows the domain of \(f\), namely \(\{ (\alpha, \theta) \in S^1 \times S^1 \}\}. This space is a torus, obtained by identifying the outside edges of the figure as indicated by the arrows. A sample patch on this torus is shown in the insert to the right. The black and white rectangles are the foreground and background regions, respectively, of the underlying range patch. Parameter \(\alpha\) is the angle of the line separating these regions. The black arrow (\(>\), \(\vee\), \(<\), or \(\wedge\)) in the white rectangle is the direction \(\theta\) of camera translation. Figure 7(left), parameter \(\alpha\) varies in the horizontal direction, and parameter \(\theta\) varies in the vertical direction.

Consider Figure 7(left), which shows the domain of \(f\), namely \(\{ (\alpha, \theta) \in S^1 \times S^1 \}\}. This space is a torus, obtained by identifying the outside edges of the figure as indicated by the arrows. A sample patch on this torus is shown in the insert to the right. The black and white rectangles are the foreground and background regions, respectively, of the underlying range patch. Parameter \(\alpha\) is the angle of the line separating these regions. The black arrow (\(>\), \(\vee\), \(<\), or \(\wedge\)) in the white rectangle is the direction \(\theta\) of camera translation. Together, the black and white arrows show the induced optical flow \(f(\alpha, \theta)\). In Figure 7(left), parameter \(\alpha\) varies in the horizontal direction, and parameter \(\theta\) varies in the vertical direction.
Figure 9. The 1-dimensional persistent homology of Vietoris–Rips complexes of $X_\theta(300, 30)$, computed in Ripser [9], confirms that these data sets are well-modeled by a circle (one significant 1-dimensional feature in the top left of each plot). These diagrams contain the same content as persistence intervals, just in a different format: each point is a topological feature with birth scale and death scale given by its $x$ and $y$ coordinates. Above we plot only two sample angles: $\theta = \frac{3\pi}{12}$ (left) and $\theta = \frac{7\pi}{12}$ (right).

In other words, antipodal points in Figure 7 (left) produce the same flow patch under the map $f$. For instance, the horizontal flow circle in red appears twice (note the top and bottom edges are identified).

The image space $\text{im}(f)$ is homeomorphic to the quotient space $\{(\alpha, \theta) \in S^1 \times S^1\}/\sim$, where $\sim$ denotes the identification $(\alpha, \theta) \sim (-\alpha, -\theta)$. A torus with antipodal points identified remains a torus, and we refer to $\text{im}(f)$ as the flow torus. See Figure 7. The right and left edges of the middle image are identified by shifting one upwards by half its length (not by twisting) before gluing. This suggests a change of coordinates: in Figure 7 (right) we plot the same flow torus, except we replace the vertical parameter with $\theta - \alpha$. The horizontal flow circle in red now wraps once around one circular direction and twice around the other.

We hypothesize that $\text{im}(f)$, the flow torus, is a good model for high-contrast optical flow. This is confirmed in part by Figures 8 and 9, which show that for any angle $\theta$, the patches $X_\theta(300, 30)$ (with predominant flow in direction $\theta$) form a circle. Together these circles group together to form a torus, equipped with the structure of a fiber bundle. Indeed, the map from the torus to the predominant angle $\theta$ of each patch is a fiber bundle with total space a torus, with base space the circle of all possible predominant angles $\theta$, and with each fiber a circle (arising from the primary circle of range images in Figure 2).

We do a zigzag persistence computation in order to confirm that the circular fibers glue together to form a torus. Consider the following zigzag diagram.

$$X_0(300, 50) \hookrightarrow X_0(300, 50) \cup X_{\frac{\pi}{12}}(300, 50) \hookrightarrow X_{\frac{2\pi}{12}}(300, 50) \hookrightarrow \ldots \hookrightarrow X_{\frac{11\pi}{12}}(300, 50).$$

The one-dimensional zigzag persistence computation of Vietoris–Rips complexes built on top of these datasets (see Figure 10) shows that the circles piece together compatibly into a fiber bundle structure.

In more detail, we construct the dense core subsets $X_\theta(300, 50)$ in twelve different angle bins $\theta \in \left\{0, \frac{\pi}{12}, \ldots, \frac{11\pi}{12}\right\}$. For computational feasibility, we then apply sequential maxmin downsampling [20] to reduce each set $X_\theta(300, 50)$ to a subset
of 50 data points. Based on Ripser computations we observe that the persistent homology is robust with regard to this downsampling procedure. We then build a zigzag filtration as described above, and use Dionysus [31] to compute the zigzag homology barcodes in Figure 10. The long interval confirms that the circles indeed piece together compatibly. Furthermore, by checking that the orientation on a generator for the 1-dimensional homology of $X_0(300,50)$ is preserved after looping once around the circle, we confirm that this fiber bundle structure is that of a torus (instead of a Klein bottle). We remark that another way to verify that this fiber bundle is a torus instead of a Klein bottle would be to use persistence for circle-valued maps [13] (on the map from the total space to the circle that encodes the predominant angle $\theta$ of each flow patch).

![Figure 10. A 1-dimensional zigzag persistence computation, showing that the circles in Figure 8 glue together in the structure of a fiber bundle.](image)

We would like to emphasize that the 2-dimensional flow torus model does not model all common optical flow patches, such as zooming in, zooming out, or roll (rotation around the $z$-axis in Figure 6 (right)).

6. Conclusions

Using topological machinery, including persistent homology and zigzag persistence, we explore the nonlinear statistics of high-contrast $3 \times 3$ optical flow patches from the computer-generated video short Sintel. We find that with a global estimate of density, the densest patches lie near a circle. Furthermore, after selecting the optical flow patches whose predominant direction of flow lies in a small bin of angle values, we find that the patches in each such bin are well-modeled by a circle. Combining these bins together provides a torus model for optical flow, which furthermore is naturally equipped with the structure of a fiber bundle over a circular base space of range image patches. As no instrument can measure ground-truth optical flow, an understanding of the nonlinear statistics of flow is needed in order to serve as a prior for optical flow estimation algorithms.

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