Convergence of Non-symmetric Diffusion Processes on RCD Spaces

Kohei Suzuki *

Abstract
We construct non-symmetric diffusion processes associated with Dirichlet forms consisting of uniformly elliptic forms and derivation operators with killing terms on RCD spaces by aid of non-smooth differential structures introduced by Gigli [17]. After constructing diffusions, we investigate conservativeness and the weak convergence of the laws of diffusions in terms of a geometric convergence of the underlying spaces and convergences of the corresponding coefficients.

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*Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, D-53115 Bonn. E-mail: suzuki@iam.uni-bonn.de
1 Introduction

1.1 Motivation and Overview

The aim of this paper is to investigate non-symmetric diffusion processes and their convergence on varying metric measure spaces under Riemannian Curvature-Dimension (RCD) conditions. We first construct non-symmetric diffusion processes on metric measure spaces under RCD conditions, which are constructed by certain Dirichlet forms consisting of uniformly elliptic operators and derivation operators with killing terms. Then we investigate conservativeness and the weak convergence of these diffusions in terms of a geometric convergence of the underlying spaces and convergences of the corresponding coefficients.

The notions of CD/RCD conditions on metric measure spaces are generalizations of the notion of lower Ricci curvature bounds in the framework of metric measure spaces, which are stable under geometric convergences such as the measured Gromov–Hausdorff (GH) convergence. They therefore contain various (finite- and infinite-dimensional) singular spaces such as Ricci limit spaces (Sturm [33, 34], Lott–Villani [26]), Alexandrov spaces (Petrunin [30], Zhang–Zhu [43]), warped products and cones (Ketterer [23, 24]), quotient spaces (Galaz-García–Kell–Mondino–Sosa [16]) and infinite-dimensional spaces such as Hilbert spaces with log-concave measures (Ambrosio–Savare–Zambotti [7]) (related to various stochastic partial differential equations). The main point is that the notion of lower Ricci curvature bounds can be completely characterized by convexity of entropy functionals on Wasserstein spaces, for which only metric measure structures are essential (Sturm [33, 34], Lott–Villani [26], Ambrosio-Gigli-Savare [4], Ambrosio–Gigli–Mondino–Rajala [1], Ambrosio–Mondino-Savare [6] and Erbar–Kuwada–Sturm [11]).

A natural issue in probability theory is whether one can construct diffusion processes on these non-smooth spaces, and if one can construct them, what properties these diffusion processes have. By recent developments of analysis on metric measure spaces, we can construct Brownian motions on RCD spaces by using a certain quadratic form, what is called Cheeger energy. This is a generalization of Dirichlet energy on smooth manifolds and induces a quasi-regular strongly local conservative symmetric Dirichlet form (Ambrosio-Gigli-Savare [3, 4] and Ambrosio–Gigli–Mondino–Rajala [1]). Since the Cheeger energy is determined only by the underlying metric measure structure, theoretically speaking, every property of Brownian motions should be derived from the geometric properties of the underlying spaces. With this motivation, in [35], the author focused on the relation between geometric and stochastic convergences: the former is the pointed measured Gromov (pmG) convergence of the underlying spaces, and the latter is the weak convergence of Brownian motions. The main results in [35] state that the pmG convergence of the underlying spaces implies the weak convergence of Brownian motions on RCD($K,\infty$) spaces (and under more strict conditions, these two convergences are equivalent).

In this paper, as a next step of [35], we construct non-symmetric diffusion processes and investigate their convergences on varying RCD spaces. To construct non-symmetric diffusions, we utilize linear transformations between $L^2$-vector fields (called tangent mod-
ule in Gigli [17]) as second-order perturbations, and derivation operators (Weaver [42]) as first-order perturbations corresponding to vector fields on metric measure spaces. We take advantage of a Dirichlet form approach to construct diffusion processes on these non-smooth spaces (see Remark 3.3 for different approaches). Next we investigate conservativeness and the weak convergence of these diffusion processes. For the weak convergence, we utilize the notion of convergence of non-symmetric forms according to Hino [21] with a slight modification for varying metric measure spaces. We show the convergence of non-symmetric forms under convergences of uniformly elliptic operators and derivations whereby the convergence of derivations was introduced by Ambrosio–Stra–Trevisan [8]. Consequently, we obtain the convergence of non-symmetric forms in the case of non-compact spaces. In the case of compact spaces, we use heat kernel estimates.

We remark that every result in this paper can be applied also to the case of time-dependent coefficients $A_t$ (diffusion coefficients), $b_t$ (drift coefficients) and $c_t$ (killing coefficients) with slight modifications, but we only deal with the time-independent case in this paper.

### 1.2 Main Results

In this section, we briefly present our main results, referring to Section 2 for more details on notation. We consider a pointed metric measure (p.m.m.) space $\mathcal{X} = (X, d, m, \overline{x})$, whereby we always assume that $(X, d)$ is a complete separable geodesic metric space with non-negative and non-zero Borel measure $m$ which is finite on all bounded sets with $\text{supp}[m] = X$, and $\overline{x}$ is a fixed point in $X$.

\begin{equation}
\text{(1.1)}
\end{equation}

Here $\text{supp}[m]$ denotes the support of the measure $m$. We also assume the following volume growth condition: there exist constants $c_1, c_2 > 0$ depending only on $K$ satisfying

\begin{equation}
m(B_r(\overline{x})) \leq c_1 e^{c_2 r^2}, \quad \forall r > 0.
\end{equation}

\begin{equation}
\text{(1.2)}
\end{equation}

We always assume that $(X, d, m, \overline{x})$ satisfies the RCD$(K, \infty)$ condition, which means that the Ricci curvature is bounded from below by $K$ and the space admits a linear gradient structure in this generality (see Section 2.7).

We first construct non-symmetric diffusion processes associated with the following bilinear form $\mathcal{E} : \text{Lip}_{bs}(X) \times \text{Lip}_{bs}(X) \rightarrow \mathbb{R}$:

\begin{equation}
\mathcal{E}(f, g) := \frac{1}{2} \int_X \langle A \nabla f, \nabla g \rangle dm + \int_X b_1(f)g dm + \int_X f b_2(g) dm + \int_X fgcdm.
\end{equation}

\begin{equation}
\text{(1.3)}
\end{equation}

In this generality, defining the above formula (1.3) is a non-trivial issue, but it is possible according to non-smooth differentiable structures developed by Gigli [17]: $\text{Lip}_{bs}(X)$ denotes the set of bounded Lipschitz functions with bounded support on $X$, and $|\nabla f|$ denotes the minimal weak upper gradient of $f$. Let $A : L^2(TX) \rightarrow L^2(TX)$ denote a (not necessarily symmetric) module morphism on $L^2$-vector fields (tangent module) so that there exists $H \in L^1_{loc}(X, m)$ satisfying $|A \nabla f| \leq H|\nabla f|$ for any $f \in \text{Lip}_{bs}(X)$. We write $|A|$ for the minimal element among such $H$. Here $|Y| \in L^2(X, m)$ denotes the point-wise norm for $Y \in L^2(TX)$. 

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We denote by \( \langle \cdot, \cdot \rangle \) the point-wise scalar product on \( L^2(TX) \). The notation \( b_i \) \((i = 1, 2)\) means a derivation operator (see Weaver [42], Fitzsimmons [14], Gigli [17]), which is a linear map \( b_i : \text{Lip}_0(X) \to L^1_\text{loc}(X, m) \) so that there exists \( h \in L^1_\text{loc}(X, m) \) satisfying \( b_i(f) \leq h |\nabla f| \) for any \( f \in \text{Lip}_0(X) \). We write \( |b_i| \) for the minimal element among such \( h \). Every notion in this paragraph is explained in Section 2 in more detail.

Under suitable assumptions (Assumption 3.1), we show that \( \mathcal{E} \) is closable and the smallest closed extension \( (\mathcal{E}, \mathcal{F}) \) is a (quasi-)regular local Dirichlet form (Proposition 3.2). Therefore, there exist diffusion processes corresponding to the Dirichlet form \( (\mathcal{E}, \mathcal{F}) \).

We now focus on the weak convergence of the laws of the corresponding diffusion processes. Let \( S'' = (\mathbb{P}'', S) \) (resp. \( \hat{S}' = (\hat{\mathbb{P}}'', \hat{S}) \)) denote the diffusion process (resp. its dual process) with the initial distribution \( \nu \) associated with the Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) (resp. \( (\hat{\mathcal{E}}, \hat{\mathcal{F}}) \)). Let \( \zeta_{S} \) and \( \zeta_{\hat{S}} \) be lifetimes for \( S \) and \( \hat{S} \) respectively. Let \( \zeta := \min\{\zeta_{S}, \zeta_{\hat{S}}\} \), and \( S''_T \) (resp. \( \hat{S}'_T \)) denotes the diffusion process \( S'' \) (resp. \( \hat{S}' \)) restricted on \( \{\zeta > T\} \) for \( T > 0 \). We assume the following:

**Assumption 1.1** Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of p.m.m. spaces satisfying RCD\((K, \infty)\) condition with \( m_n(X_n) = 1 \), or \( \text{RCD}^*(K, N) \). Let us suppose the following conditions:

(i) \( X_n \to X_\infty \) in the pmG sense;

(ii) \( \sup_{n \in \mathbb{N}} (\|A_n\|_\infty + \|b_1^n\|_\infty + \|b_2^n\|_\infty + \|\text{div}b_1^n\|_\infty + \|\text{div}b_2^n\|_\infty + \|c_n\|_\infty) < \infty \), and \( A_n \) is symmetric and there exists \( \lambda > 0 \) so that

\[
\langle A_n \nabla f, \nabla f \rangle \geq \lambda (\nabla f, \nabla f), \quad m_n \text{-a.e.,} \quad \forall f \in \text{Lip}_0(X_n), \forall n \in \mathbb{N};
\]

(iii) for any non-negative \( f \in \text{Lip}_0(X_n), i = 1, 2, \forall n \in \mathbb{N},

\[
\int_{X_n} (b_i^n(f) + c_n f) dm_n \geq 0;
\]

(iv) \( A_n \to A_\infty \), and \( b_1^n \to b_1^\infty, \quad \text{div} b_1^n \to \text{div} b_1^\infty \) \((i = 1, 2)\) and \( c_n \to c_\infty \) strongly in \( L^2 \), respectively;

(v) The initial distribution \( \nu_n \in \mathcal{P}(X_n) \) satisfies \( \nu_n(dx) = \phi_n m_n(dx) \) for \( n \in \mathbb{N} \) with \( \sup_{n \in \mathbb{N}} \|\phi_n\|_{B_r(\pi_n), \infty} < \infty \) and \( \phi_n \to \phi_\infty \) weakly in \( L^2 \).

The notion of the pmG convergence was introduced by Gigli–Mondino–Savaré [19] and is recalled in Section 2. The notion of a convergence of \( A_n \) on varying metric measure spaces is introduced in Definition 4.7. The \( L^2 \)-strong convergence of derivation operators \( b_i^n \) was introduced by Ambrosio–Strat-Trevisan [8] and the precise definition is recalled in Section 4. The divergence of a derivation \( b \) is denoted by \( \text{div} b \), which is recalled in Section 4 (when we write \( \text{div} b \), we assume implicitly the existence of \( \text{div} b \)). We mean \( \|\phi_n\|_{B_r(\pi_n), \infty} := \text{ess-sup}_{x \in B_r(\pi_n)} |\phi_n(x)| \), whereby \( B_r(\pi_n) \) means the open ball centered at \( \pi_n \) with radius \( r \).

The notion of \( L^p \)-convergence of functions on varying metric measure spaces is according to Gigli–Mondino–Savaré [19] and stated in Section 4. The space \( \mathcal{P}(X_n) \) denotes the set of Borel probability measures on \( X_n \). Note that, since the RCD\((K, \infty)\) condition is stable under the pmG convergence (see [19, Theorem 7.2]), the limit space \( X_\infty \) also satisfies the RCD\((K, \infty)\) condition. Therefore, the diffusion process associated with \( (\mathcal{E}_\infty, \mathcal{F}_\infty) \) and the
initial distribution \( \nu_{\infty} \) corresponding to (1.3) can be defined also on the limit space \( X_{\infty} \) and the corresponding diffusion restricted on \( \{ \zeta_{\infty} > T \} \) is denoted by \( S^{\nu_{\infty}}_{\infty, T} \) (resp. \( \hat{S}^{\nu_{\infty}}_{\infty, T} \)).

Under the pmG convergence, we can embed each space \( X_n \) to a common ambient space \( X \) isometrically and thus, we may consider each \( X_n \) to be a subset of \( X \). Let \( C([0,T];X) \) denote the space of continuous paths from \([0,T] \) to \( X \) with uniform topology on compact sets. Now we state the following two main theorems.

**Theorem 1.2** Under Assumption 1.1, the laws of \( S^{\nu_n}_{n,T} \) and \( \hat{S}^{\nu_n}_{n,T} \) converge weakly to \( S^{\nu_{\infty}}_{\infty, T} \) and \( \hat{S}^{\nu_{\infty}}_{\infty, T} \), respectively in the space \( \mathcal{P}_{\leq 1}(C([0,T];X)) \).

Here \( \mathcal{P}_{\leq 1}(C([0,T];X)) \) denotes the set of all Borel sub-probability measures (i.e., measures whose total mass is less than or equal to 1) on \( C([0,\infty);X) \). The next theorem requires stronger conditions than Theorem 1.2, but the initial distribution can be improved to dirac measures \( \delta_{x_n} \).

**Theorem 1.3** Suppose Assumption 1.1 and \( \text{RCD}^*(K,N) \) with \( \sup_{n \in \mathbb{N}} \text{diam}(X_n) < \infty \). If \( \text{div} b^n_1 = c_n \) (resp. \( \text{div} b^n_2 = c_n \)), then the law of \( \hat{S}^{\nu_n}_{\infty,n} \) (resp. \( S^{\nu_n}_{\infty,n} \)) converges weakly to \( \hat{S}^{\nu_{\infty}}_{\infty, \infty} \) (resp. \( S^{\nu_{\infty}}_{\infty, \infty} \)) in \( \mathcal{P}(C([0,\infty);X)) \).

**Remark 1.4** We give two remarks about the main results.

(i) The elliptic constant \( \lambda > 0 \) in (ii) of Assumption 1.1 needs to be uniform in \( n \in \mathbb{N} \), which is used to prove the convergence of Dirichlet forms and appears in (4.1) in Section 4.

(ii) Under the assumption of Theorem 1.3, the underlying space \( X_n \) is compact for every \( n \in \mathbb{N} \).

Finally we give a criterion for conservativeness of forms associated with (1.3) (Proposition 7.1).

1.3 Organization of the Paper

The paper is structured as follows. First, the notation is fixed and preliminary facts are recalled in Section 2 (no new results are included): basic notations and definitions from metric geometry (Subsection 2.1); pmG convergence (Subsection 2.2); \( L^2 \)-normed modules (Subsection 2.3); Tangent module (Subsection 2.4); Dirichlet forms (Subsection 2.5); \( \text{RCD}(K,\infty) \) and \( \text{RCD}^*(K,N) \) spaces (Subsection 2.7). In Section 3, we prove Proposition 3.2 to construct a Dirichlet form corresponding to (1.3). In Section 4, we show convergence of non-symmetric forms. We first recall \( L^p \)-convergence of functions on varying metric measure spaces. Secondly, we introduce convergence of \( A_n \) and recall a notion of convergence of derivations according to Ambrosio–Stra–Trevisan [8]. Finally, we show convergence of non-symmetric forms with a modification for varying spaces. In Section 5, we prove the weak convergence of finite-dimensional distributions of diffusions under Assumption 1.1. In Section 6, we give proofs for the tightness of diffusions under Assumption 1.1 and complete the proofs of Theorem 1.2 and 1.3. In Section 7, we show Proposition 7.1, which is a criterion for conservativeness. Finally in Section 8, we give examples for which Assumption 1.1 is satisfied.
2 Notation & Preliminary Results

2.1 Preliminary from Metric Measure Geometry

Let \( \mathbb{N} = \{0, 1, 2, \ldots \} \) and \( \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\} \) be the set of natural numbers and the set of extended natural numbers, respectively. Let \((X, d)\) be a complete separable metric space. We write \( B_r(x) = \{y \in X : d(x, y) < r\} \) for an open ball centered at \( x \in X \) with radius \( r > 0 \). By using \( \mathcal{B}(X) \), we denote the family of all Borel sets in \((X, d)\); and by \( \mathcal{B}_b(X) \), the set of real-valued bounded Borel-measurable functions on \( X \). Let \( C(X) \) denote the set of real-valued continuous functions on \( X \), while \( C_b(X), C_0(X) \) and \( C_{bs}(X) \) denote the subsets of \( C(X) \) consisting of bounded functions, functions with compact support, and bounded functions with bounded support, respectively. Let \( \text{Lip}(X) \) denote the set of real-valued Lipshitz continuous functions on \( X \). Let \( \text{Lip}_b(X) \) and \( \text{Lip}_{bs}(X) \) denote the subsets of \( \text{Lip}(X) \) consisting of bounded functions, and bounded functions with bounded supports, respectively. For \( f \in \text{Lip}_{bs}(X) \), the global Lipschitz constant \( \text{Lip}(f) \) is defined as the infimum of \( L > 0 \) satisfying \( |f(x) - f(y)| \leq Ld(x, y) \) for any \( x, y \in X \). The set \( \mathcal{P}(X) \) denotes all Borel probability measures on \( X \). The set of continuous functions on \([0, \infty)\) valued in \( X \) is denoted by \( C([0, \infty); X) \).

Let \( \text{supp}[m] = \{x \in X : m(B_r(x)) > 0, \forall r > 0\} \) denote the support of \( m \). Let \((Y, d_Y)\) be a complete separable metric space. For a Borel measurable map \( f : X \rightarrow Y \), let \( f \# m \) denote the push-forward measure on \( Y \):

\[
f \# m(B) = m(f^{-1}(B)) \quad \text{for any Borel set } \quad B \in \mathcal{B}(Y).
\]

For a measure space \((X, m)\) with a Borel measure \( m \), we denote by \( L^p(X, m) \) (\( L^p(m) \) for brevity if no confusion occurs) \( (1 \leq p \leq \infty) \) the space of \( m \)-equivalence classes of Borel measurable functions \( f : X \rightarrow \mathbb{R} \cup \{\infty\} \) so that \( \|f\|_{L^p(X, m)}^p := \int_X |f|^p dm < \infty \) if \( 1 \leq p < \infty \), and \( \|f\|_{L^\infty(X, m)} = \text{ess-sup}_{x \in X} |f(x)| < \infty \) in the case of \( p = \infty \). We sometimes write \( \|\cdot\|_p \) for brevity. Let \( L^0(X, m) \) denote the set of equivalent classes of Borel measurable functions \( f : X \rightarrow \mathbb{R} \). For \( f, g \in L^2(X, m) \), let \( (f, g)_{L^2(X, m)} \) (simply \( (f, g) \)) denote the inner product \( \int_X f g dm \). For a measurable set \( A \subset X \), let us denote the indicator function by \( 1_A \), which is equal to 1 for \( x \in A \) and 0 otherwise. For any two functions \( f, g : X \rightarrow \mathbb{R} \), we write \( f \lor g = \max\{f, g\} \) and \( f \land g = \min\{f, g\} \).

A curve \( \gamma : [0, 1] \rightarrow X \) is absolutely continuous if there exists a function \( f \in L^1(0, 1) \) so that

\[
d(\gamma_t, \gamma_s) \leq \int_t^s f(r)dr, \quad \forall t, s \in [0, 1], \quad t < s. \quad (2.1)
\]

The metric speed \( t \mapsto |\dot{\gamma}|_t \in L^1(0, 1) \) is defined as the essential infimum among all the functions \( f \) satisfying (2.1).

A Borel probability measure \( \pi \) on \([0, 1]; X\) is a test plan if there exists a constant \( C(\pi) \) so that

\[
(e_t)_{\# \pi} \leq C(\pi)m, \quad \forall t \in [0, 1], \quad \text{with} \quad \int_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) < \infty.
\]

Here \( e_t(\gamma) := \gamma(t) \in X \) is the evaluation map.

The set of Sobolev functions \( S^2(X, d, m) \) (or, simply \( S^2(X) \)) is defined to be the space of all functions in \( L^0(X, m) \) so that there exists a non-negative \( G \in L^2(m) \) for which it holds

\[
\int |f(\gamma_t) - f(\gamma_0)| d\pi(\gamma) \leq \int \int_0^1 G(\gamma_t)|\dot{\gamma}_t| dtd\pi(\gamma), \quad \forall \text{test plan } \pi.
\]
It turns out (see [4]), that for \( f \in \mathcal{S}^2(X) \) there exists a minimal \( G \) in the \( m \)-a.e. sense for which the above inequality holds. We denote by \(|\nabla f|\) such \( G \) and call it minimal weak upper gradient. Let us define \( W^{1,2}(X, d, m) := \mathcal{S}^2(X, d, m) \cap L^2(X, m) \) (or, simply \( W^{1,2}(X) \)). A functional Cheeger energy \( \text{Ch} : W^{1,2}(X, d, m) \to \mathbb{R} \) is defined as follows

\[
\text{Ch}(f) = \frac{1}{2} \int_X |\nabla f|^2 dm, \quad f \in W^{1,2}(X, d, m).
\]

**Remark 2.1** Note that \( \text{Ch} : L^2(X, m) \to [0, +\infty] \) is a lower semi-continuous and convex functional, but not necessarily a quadratic form. This means that \( (W^{1,2}(X, d, m), \sqrt{2\text{Ch}(\cdot) + \|\cdot\|_2^2}) \) is a Banach space, but not necessarily a Hilbert space.

We say that \( (X, d, m) \) satisfies the infinitesimal Hilbertian (IH) condition if \( \text{Ch} \) is a quadratic form, i.e.,

\[
2\text{Ch}(u) + 2\text{Ch}(v) = \text{Ch}(u + v) + \text{Ch}(u - v),
\]

for any \( u, v \in W^{1,2}(X, d, m) \). Let us define the point-wise scalar product as follows

\[
\langle \nabla f, \nabla g \rangle := \lim_{\varepsilon \to 0} \frac{|\nabla (f + \varepsilon g)|^2 - |\nabla f|^2}{\varepsilon}, \quad f, g \in W^{1,2}(X, d, m),
\]

whereby the limit is with respect to \( L^1(m) \). If the Cheeger energy \( \text{Ch} \) is quadratic, the point-wise inner product becomes a \( L^1(m) \)-valued bilinear form (see [4, Definition 4.12], and [8, Theorem 2.7]). We set \( \text{Ch}(f, g) := (1/2) \int_X \langle \nabla f, \nabla g \rangle dm \).

### 2.2 Pointed Measured Gromov Convergence

We recall the definition of pmG convergence introduced in Gigli-Mondino-Savaré [19].

**Definition 2.2 ([19]) (pmG Convergence)** A sequence of p.m.m. spaces \( X_n = (X_n, d_n, m_n, \pi_n) \) satisfying (1.1) is said to be convergent to \( X_\infty = (X_\infty, d_\infty, m_\infty, \pi_\infty) \) in the pointed measured Gromov (pmG) sense if there exist a complete separable metric space \( (X, d) \) and isometric embeddings \( \iota_n : X_n \to X \) (\( n \in \mathbb{N} \)) satisfying

\[
\iota_n(\pi_n) \to \iota_\infty(\pi_\infty) \in X_\infty, \quad \text{and} \quad \int_X f \, d(\iota_n \# m_n) \to \int_X f \, d(\iota_\infty \# m_\infty),
\]

for any bounded continuous function \( f : X \to \mathbb{R} \) with bounded support.

**Remark 2.3** We give two remarks for Definition 2.2.

(i) The pmG convergence is weaker than the pointed measured Gromov-Hausdorff (pmGH) convergence ([19, Theorem 3.30, Example 3.31]). If \( \{X_n\}_{n \in \mathbb{N}} \) satisfies a uniform doubling condition, then pmG and pmGH coincide [19, Theorem 3.33].

(ii) The pmG convergence is metrizable by a distance \( pG_W \) on the collection \( X \) of all isomorphism classes of p.m.m. spaces ([19, Definition 3.13]). The space \( (X, pG_W) \) is a complete and separable metric space ([19, Theorem 3.17]).
2.3 $L^p(m)$-normed Module

In this subsection, we recall the notion of $L^p$-normed module by following [17, §1.2].

**Definition 2.4 ([17]) ($L^\infty(m)$-premodule)** An $L^\infty(m)$-premodule is a Banach space $(M, \| \cdot \|_M)$ equipped with a bilinear map $L^\infty(m) \times M \ni (f, v) \mapsto f \cdot v \in M$ satisfying

$$(fg) \cdot v = f \cdot (g \cdot v), \quad 1 \cdot v = v, \quad \|f \cdot v\|_M \leq \|f\|_{L^\infty(m)} \|v\|_M,$$

for any $v \in M$ and $f, g \in L^\infty(m)$, whereby $1 := 1_X \in L^\infty(m)$.

**Definition 2.5 ([17]) ($L^\infty(m)$-module/Hilbert Module)** An $L^\infty(m)$-module is an $L^\infty(m)$-premodule $M$ satisfying the following two conditions:

(i) (Locality) For any $v \in M$, $A_n \in \mathcal{B}(X)$ and $n \in \mathbb{N}$,

$$1_{A_n} \cdot v = 0, \quad \forall n \in \mathbb{N} \quad \text{implies} \quad 1_{\bigcup_{n \in \mathbb{N}} A_n} \cdot v = 0.$$

(ii) (Gluing) For any sequence $\{v_n\}_{n \in \mathbb{N}} \subset M$ and $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{B}(X)$ so that

$$1_{A_i \cap A_j} \cdot v_i = 1_{A_i \cap A_j} \cdot v_j, \quad \forall i, j \in \mathbb{N},$$

and

$$\limsup_{n \to \infty} \left\| \sum_{n=1}^\infty 1_{A_n} \cdot v_i \right\|_M < \infty,$$

there exists $v \in M$ so that

$$1_{A_i} \cdot v = 1_{A_i} \cdot v_i, \quad \forall i \in \mathbb{N}, \quad \text{and} \quad \|v\|_M \leq \liminf_{n \to \infty} \left\| \sum_{i=1}^n 1_{A_i} \cdot v_i \right\|_M.$$

If furthermore $(M, \| \cdot \|_M)$ is a Hilbert space, then $M$ is called Hilbert module.

**Example 2.6** One of the typical examples for $L^\infty(m)$-modules is $M = L^p(X, m)$ with the norm $\| \cdot \|_M := \| \cdot \|_{L^p(m)}$ for $p \in [1, \infty]$. If the underlying space $(X, d, m)$ is a Riemannian manifold, then an $L^p(X, m)$-vector field for $p \in [1, \infty]$ is also an $L^\infty(m)$-module.

For two given $L^\infty(m)$-modules, $M_1, M_2$, a map $T : M_1 \to M_2$ is called a module morphism provided that it is a bounded linear map from $M_1$ to $M_2$ as a map between Banach spaces and satisfies

$$T(f \cdot v) = f \cdot T(v), \quad \forall v \in M_1, f \in L^\infty(m). \quad (2.5)$$

The set of all module morphisms is denoted by $\text{Hom}(M_1, M_2)$. It is known that $\text{Hom}(M_1, M_2)$ has a canonical $L^\infty(m)$-module structure.

**Definition 2.7 ([17]) (Dual Module)** For an $L^\infty(m)$-module $M$, the dual module $M^*$ is defined as $\text{Hom}(M, L^1(m))$.

**Definition 2.8 ([17]) ($L^p(m)$-normed module)** Let $p \in [1, \infty]$. An $L^p(m)$-normed module is an $L^\infty(m)$-module $M$ endowed with a map $| \cdot | : M \to L^p(m)$ with non-negative values so that

$$\| |v||_{L^p(m)} = \|v\|_M, \quad |f \cdot v| = |f||v|, \quad m\text{-a.e},$$

for every $v \in M$ and $f \in L^\infty(m)$. The map $| \cdot |$ is called point-wise norm.
2.4 Tangent Module

In this subsection, following [17, §2], we recall the tangent module $L^2(TX)$ on $(X, d, m)$, which is an $L^2(m)$-normed module in the sense of Definition 2.8 and a generalized notion of the space of $L^2$-sections of the tangent bundle on smooth manifolds.

We recall the set \textit{Pre-cotangent module} $Pcm$, which is defined as follows:

\textbf{Definition 2.9 ([17]) (Pre-cotangent module)} The set $Pcm$ defined as follows is called \textit{Pre-cotangent module}:

\[ Pcm := \left\{ \{(f_i, A_i)\}_{i \in \mathbb{N}} : \{A_i\}_{i \in \mathbb{N}} \subset \mathcal{B}(X) \text{ is a partition of } X \right\}, \]

where $f_i \in S^2(X), \forall i \in \mathbb{N}$, and $\sum_{i \in \mathbb{N}} \int_{A_i} |\nabla f_i|^2 dm < \infty$.

An equivalence relation between two elements in $Pcm$ $\{(f_i, A_i)\}_{i \in \mathbb{N}} \sim \{(g_j, B_j)\}_{j \in \mathbb{N}}$ is defined as follows:

\[ |\nabla (f_i - g_j)| = 0, \text{ m-a.e. on } A_i \cap B_j, \forall i, j \in \mathbb{N}. \]

A vector space structure can be endowed with the quotient space $Pcm/\sim$ by defining the sum and the scalar multiplication as follows: for any $\lambda \in \mathbb{R}$,

\[ [(f_i, A_i)_i] + [(g_j, B_j)_j] := [(f_i + g_j, A_i \cap B_j)_{i,j}], \quad \lambda [(f_i, A_i)_i] := [\lambda f_i, A_i]_i. \]

The product operation $(\cdot : Sf(m) \times Pcm/\sim \rightarrow Pcm/\sim$ can be defined by the following manner. Let $Sf(m) \subset L^\infty(m)$ denote the set of all simple functions $f$, which means that $f$ attains only a finite set of values. Given $[(f_i, A_i)_i] \in Pmc/\sim$ and $h = \sum_j a_j 1_{B_j} \in Sf(m)$ with $\{B_j\}_{j \in \mathbb{N}}$ a partition of $X$, the product $h \cdot [(f_i, A_i)_i]$ is defined as follows:

\[ h \cdot [(f_i, A_i)_i] := [(a_j f_i, A_i \cap B_j)_{i,j}]. \]

We now recall the point-wise norm $| \cdot |_*$ (we use the notation $| \cdot |_*$ as a point-wise norm for the sake of consistency with the definition of tangent modules given later): Define $| \cdot |_*$ on $Pcm/\sim \rightarrow L^2(X, m)$ by

\[ |[(f_i, A_i)_i]|_* := |\nabla f_i|, \text{ m-a.e. on } A_i \text{ for all } i \in \mathbb{N}. \]

Then the map $\| \cdot \|_{L^2(T^*X)} : Pcm/\sim \rightarrow [0, \infty)$ is defined as follows:

\[ \|[(f_i, A_i)_i]\|_{L^2(T^*X)}^2 := \int \|[(f_i, A_i)_i]\|_*^2 dm = \sum_{i \in \mathbb{N}} \int_{A_i} |\nabla f_i|^2 dm. \]

Then $\| \cdot \|_{L^2(T^*X)}$ is a norm on $Pcm/\sim$.

\textbf{Definition 2.10 ([17]) (Cotangent Module)} The cotangent module $(L^2(T^*X), \| \cdot \|_{L^2(T^*X)})$ is defined as the completion of $(Pcm/\sim, \| \cdot \|_{L^2(T^*X)})$.

It can be checked that the cotangent module $(L^2(T^*X), \| \cdot \|_{L^2(T^*X)})$ is an $L^2$-normed module with the product $(\cdot : L^\infty(m) \times L^2(T^*X) \rightarrow L^2(T^*X))$, and the point-wise norm $| \cdot |_*$ (see [17, §2.2] for more details).
Definition 2.11 (Tangent Module) The tangent module \((L^2(TX), \| \cdot \|_{L^2(TX)})\) is defined as the dual module of \((L^2(T^*X), \| \cdot \|_{L^2(T^*X)})\). The point-wise norm associated with the dual of \(\cdot \) is written as \(\cdot \).

Under the condition (IH), the tangent module \((L^2(TX), \| \cdot \|_{L^2(TX)})\) is a Hilbert module and the point-wise norm \(\cdot \) satisfies the parallelogram identity. Therefore, we can define the point-wise inner product \(\langle \cdot , \cdot \rangle\).

Now we recall the notions of differential and gradient for a function in Sobolev class.

Definition 2.12 (Differential) Let \(f \in S^2(X)\). The differential \(df \in L^2(T^*X)\) is defined as

\[
df := [(f, X)] \in \text{Pcm}/\sim \subset L^2(T^*X).
\]

Here \([(f, X)]\) means \([(f_i, A_i)_{i \in \mathbb{N}}]\) for \(f_0 = f, A_0 = X\) and \(A_i = \emptyset\) for \(i \geq 1\).

By definition, we have \(|df|_\ast = |\nabla f|\). The notion of gradient of a Sobolev function is defined through duality with the notion of the differential.

Definition 2.13 (Gradient) Let \(f \in S^2(X)\). We say that \(X \in L^2(TX)\) is a gradient of \(f\) if

\[
df(X) = |X|^2 = |df|_\ast^2.
\]

The set of all gradients of \(f\) is denoted by \(\text{Grad}(f)\).

Under condition (IH), the set \(\text{Grad}(f)\) has a unique element, which is denoted by \(\nabla f\). In this case, the gradient \(\nabla f\) satisfies the following linearity ([17, Proposition 2.3.17]):

\[
\nabla (f + g) = \nabla f + \nabla g, \quad m\text{-a.e.}, \quad f, g \in S^2(X).
\]

Let \((X, d, m)\) satisfy (IH) and \(\langle \cdot , \cdot \rangle : L^2(TX) \to L^1(X, m)\) be the point-wise inner product, which is induced by the structure of \(L^2\)-normed module with the point-wise norm \(\cdot \) in \(L^2(TX)\). Under condition (IH), \(\langle \nabla f, \nabla g \rangle\) can be identified in the \(m\text{-a.e.}\) sense with the same expression defined in (2.3) in Subsection 2.1.

2.5 Derivation

In this subsection, we briefly explain derivations on metric measure spaces by following [8, §3].

Definition 2.14 (Derivation) A linear functional \(b : \text{Lip}_{bs}(X) \to L^0(m)\) is said to be a derivation if there exists \(h \in L^0(m)\) so that

\[
|b(f)| \leq h|\nabla f|, \quad m\text{-a.e. in } X, \text{ for all } f \in \text{Lip}_{bs}(X).
\]

The \(m\text{-a.e.}\) smallest function \(h\) satisfying the above inequality is denoted by \(|b|\). The space of all derivations is denoted by \(\text{Der}(X, d, m)\). We denote by \(\text{Der}^p(X, d, m)\) (resp. \(\text{Der}_{loc}^p(X, d, m)\)) the space of derivations \(b\) so that \(|b| \in L^p(m)\) (resp. \(L^p_{loc}(m)\)).
Derivation operators satisfy the local property: for any \( f, g \in \text{Lip}_{bs}(X) \),

\[
|b(f - g)| \leq h|\nabla (f - g)| = 0, \quad m\text{-a.e. on } \{f = g\}.
\]

By the local property, the chain rule holds

\[
b(\phi(f)) = (\phi' \circ f)b(f), \quad \phi \in \text{Lip}(\mathbb{R}), \quad m\text{-a.e.},
\]

and the Leibniz rule also holds:

\[
b(fg) = b(f)g + fb(g), \quad m\text{-a.e.}
\]

See [17, §2.2, 2.3] and [8, §3] for more details.

**Remark 2.15 (Der^2(X, d, m) and L^2(TX))** The space Der^2(X, d, m) is an L^2(m)-normed module with the map \( b \to |b| \). If W^{1,2}(X, d, m) is reflexive as a Banach space, then Der^2(X, d, m) can be canonically and isometrically identified with the tangent module L^2(TX) recalled in Definition 2.11. See [8, Remark 3.5] for more details.

Now we recall the notion of divergence of derivations.

**Definition 2.16 ([8]) (Divergence)** A derivation \( b \in \text{Der}^1_{loc}(X, d, m) \) has divergence in \( L^1_{loc}(X, m) \) if there exists \( g \in L^1_{loc}(X, m) \) so that

\[
-\int_X b(f)dm = \int_X fgdm, \quad \forall f \in \text{Lip}_{bs}(X).
\]

Such a \( g \) is uniquely determined if it exists, and we denote it by \( \text{div} b \). The existence of such \( g \) is not necessarily true for general \( b \) but when we write \( \text{div} b \), we implicitly assume the existence of such \( g \). Let \( \text{Div}^P_{loc}(X, d, m) := \{b \in \text{Der}^1_{loc}(X, d, m) : \text{div} b \in L^1_{loc}(X, d, m)\} \) and \( \text{Div}^P(X, d, m) := \{b \in \text{Der}^1_{loc}(X, d, m) : \text{div} b \in L^p(X, d, m)\} \).

By using the Leibniz rule, we have

\[
\int_X b(f)\phi dm = -\int_X b(\phi)f dm - \int_X f\phi \text{div} b dm, \quad \forall f, \phi \in \text{Lip}_{bs}(X).
\]

**2.6 Dirichlet Forms**

In this subsection, following [27], we recall basic notions concerning Dirichlet forms.

Let \( \mathcal{F} \subset L^2(X, m) \) be a dense linear subspace and \( \mathcal{E} \) be a bilinear form on \( \mathcal{F} \). We write \( \mathcal{E}_\alpha(f, g) := \mathcal{E}(f, g) + \alpha(f, g)L^2(X, m) \) and \( \mathcal{E}_\alpha(f) := \mathcal{E}_\alpha(f, f) = \mathcal{E}(f, f) + \alpha\|f\|^2_2 \) for \( \alpha \in [0, \infty) \) in short. The symmetric part of \( \mathcal{E} \) is defined by \( \tilde{\mathcal{E}}(f, g) = (1/2)(\mathcal{E}(f, g) + \mathcal{E}(g, f)) \) and the anti-symmetric part of \( \mathcal{E} \) by \( \hat{\mathcal{E}}(f, g) = (1/2)(\mathcal{E}(f, g) - \mathcal{E}(g, f)) \). The bilinear form \( (\mathcal{E}, \mathcal{F}) \) is a coercive closed form if the following three conditions hold:

\[(\mathcal{E}.1) \text{ } \mathcal{E} \text{ is non-negatively definite: } \mathcal{E}(f) \geq 0 \text{ for all } f \in \mathcal{F}.
\]

\[(\mathcal{E}.2) \text{ } \mathcal{E} \text{ satisfies the weak sector condition: there exists a constant } C \geq 1 \text{ so that}
\]

\[
|\mathcal{E}_1(f, g)| \leq C \mathcal{E}_1(f)\mathcal{E}_1(g), \quad \forall f, g \in \mathcal{F}.
\]
(ℰ.3) ℱ is a Hilbert space with respect to the symmetric part \( \tilde{E}_1^{1/2} \).

Let \( D \subset L^2(X, m) \) be a dense linear subspace. A bilinear form \((ℰ, D)\) satisfying (ℰ.1) and (ℰ.2) is \textit{closable} if, for any \( ℰ\)-Cauchy sequence \( f_n \in D \) with \( \lim_{n \to \infty} \| f_n \|_{L^2(m)} = 0 \), it holds that \( \lim_{n \to \infty} ℰ( f_n ) = 0 \). We say that \((ℰ, ℱ)\) is \textit{symmetric} if \( ℰ(f, g) = ℰ(g, f) \) for all \( f, g \in ℱ \).

The \textit{dual form} \( ℰ \) is defined to be \( \hat{ℰ}(f, g) = ℰ(g, f) \) for \( f, g \in ℱ \).

If \((ℰ, ℱ)\) is a coercive closed form, then there exist the corresponding semigroups \( \{ T_t \}_{t \geq 0} \) and \( \{ \hat{T}_t \}_{t \geq 0} \) on \( L^2(X, m) \) so that \( (T_t f, g) = (g, \hat{T}_t f) \) for any \( t \geq 0 \) and \( f, g \in L^2(X, m) \), and the corresponding resolvents \( G_α \) and \( \hat{G}_α \), which are defined as \( G_α f = \int_0^∞ e^{-αt} T_t f dt \) and \( \hat{G}_α f = \int_0^∞ e^{-αt} \hat{T}_t f dt \), satisfy

\[
ℰ_α(G_α f, g) = (f, g) = ℰ_α(g, \hat{G}_α f), \quad \forall f \in L^2(X, m), \quad g \in ℱ, \quad α > 0.
\]

Concerning the Markovian property, the following statements are known to be equivalent (e.g., [29, Theorem 1.1.5]):

(ℰ.4) For all \( f \in ℱ \), it holds that

\[
f^+ \land 1 \in ℱ, \quad ℰ(u + u^+ \land 1, u - u^+ \land 1) \geq 0, \quad ℰ(u - u^+ \land 1, u + u^+ \land 1) \geq 0. \tag{2.7}
\]

Here \( u^+ := u \lor 0 \).

(M) \( \{ T_t \}_{t \geq 0} \) and \( \{ \hat{T}_t \}_{t \geq 0} \) are Markovian: If \( f \in L^2(X, m) \) satisfies \( 0 \leq f \leq 1 \) \text{-a.e.}, then \( 0 \leq T_t f \leq 1 \) and \( 0 \leq \hat{T}_t f \leq 1 \) \text{-a.e.}.

A bilinear form \((ℰ, ℱ)\) is called \textit{Dirichlet form} if (ℰ.1)–(ℰ.4) hold.

Now we recall the property of regularity/quasi-regularity for Dirichlet forms, which is a sufficient condition for the existence of Hunt processes/m-tight special standard processes and their dual processes (see [27, Theorem 3.5 Chapter IV]) corresponding to Dirichlet forms.

An increasing sequence \( \{ E_n \}_{n \in \mathbb{N}} \) of closed subsets of \( X \) is called \textit{ℰ-nest} if

\[
\bigcup_{n \in \mathbb{N}} ℱ|_{E_n} \text{ is dense in } ℱ \text{ with respect to } \tilde{E}_1^{1/2}.
\]

Here we mean that \( ℱ|_A := \{ u \in ℱ : u = 0 \text{ m-a.e. on } A^c \} \). A subset \( N \subset X \) is called \textit{ℰ-exceptional} if

\[
N \subset \bigcap_{n \in \mathbb{N}} E_n^c \text{ for some } ℰ\text{-nest } \{ E_n \}_{n \in \mathbb{N}}.
\]

We say that a property of points in \( X \) holds \( ℰ\)-quasi-everywhere (ℰ-q.e.) if the property holds outside some \( ℰ\)-exceptional set. A function \( f \) ℰ-q.e. defined on \( X \) is called \textit{ℰ-quasi-continuous} if there exists an ℰ-nest \( \{ E_n \}_{n \in \mathbb{N}} \) so that \( f \in C(\{ E_n \}) \) whereby

\[
C(\{ E_n \}) := \{ f : A \to \mathbb{R} : \bigcup_{n \in \mathbb{N}} E_n \subset A \subset X, \, f|_{E_n} \text{ is continuous } \forall n \in \mathbb{N} \}.
\]

A Dirichlet form \((ℰ, ℱ)\) on \( L^2(X, m) \) is called \textit{quasi-regular} if the following three conditions hold:

(i) There exists an ℰ-nest \( \{ E_n \}_{n \in \mathbb{N}} \) consisting of compact sets.
(ii) There exists an $\mathcal{E}^{1/2}$-dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi-continuous $m$-versions.

(iii) There exist $u_n \in \mathcal{F}$ for $n \in \mathbb{N}$ having $\mathcal{E}$-quasi-continuous $m$-versions $\tilde{u}_n$ and an $\mathcal{E}$-exceptional set $N \subset X$ so that $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ separates points of $X \setminus N$.

Let $(X, d)$ be a locally compact separable metric space with a Radon measure $m$. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ is called regular with a core $C_1$ if $C_1 \subset C_0(X) \cap \mathcal{F}$ is dense both in $C_0(X)$ with the uniform norm $\| \cdot \|_{\infty}$ and in $\mathcal{F}$ with $\mathcal{E}^{1/2}$, respectively. We note that $(\mathcal{E}, \mathcal{F})$ is quasi-regular if it is regular ([27, Chapter IV Section 4 a])

Let $\{T_t\}_{t \geq 0}$ be the semigroup corresponding to $(\mathcal{E}, \mathcal{F})$. An important fact ([27, Theorem 3.5 Chapter IV]) is that if a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is quasi-regular, then there exists an $m$-tight special standard process $(\Omega, \mathcal{M}, \{\mathcal{M}_t\}_{t \geq 0}, \{S_t\}_{t \geq 0}, \{\mathbb{P}^x\}_{x \in X})$ ([27, Definition 1.13 in Chapter IV]) so that, for all $t \geq 0$ and $f \in B_b(X) \cap L^2(X, m)$, 

$$T_t f(x) = E^x(f(S_t)), \quad \mathcal{E}\text{-q.e. } x.$$ 

Here $E^x(f(S_t)) := \int_0^t f(S_t(\omega))d\mathbb{P}^x(d\omega)$.

We adjoin an extra point $\partial$ (the cemetery point) to $X$ as an isolated point to obtain a Hausdorff topological space $X_\partial$ with Borel $\sigma$-algebra $\mathcal{B}(X_\partial) = \mathcal{B}(X) \cup \{B \cup \partial : B \in \mathcal{B}(X)\}$. Any function $f : X \to \mathbb{R}$ can be considered as a function from $X_\partial$ by defining $f(\partial) = 0$. If $X$ is locally compact, we consider the one-point compactification (Alexandroff compactification) for $X_\partial$. We say that a stochastic process $(\Omega, \mathcal{M}, \{\mathcal{M}_t\}_{t \geq 0}, \{S_t\}_{t \geq 0}, \{\mathbb{P}^x\}_{x \in X})$ has a lifetime $\zeta$ if

(i) $S_t : \Omega \to X_\partial$ is $\mathcal{M}/\mathcal{B}(X_\partial)$-measurable;

(ii) $\zeta : \Omega \to [0, \infty]$ is $\mathcal{M}$-measurable;

(iii) for any $\omega \in \Omega$, $S_t(\omega) \in X$ whenever $t < \zeta(\omega)$ and $S_t = \partial$ for all $t \geq \zeta(\omega)$.

We say that $(\mathcal{E}, \mathcal{F})$ is local if $\mathcal{E}(f, g) = 0$ whenever $f, g \in \mathcal{F}$ with $\text{supp}[f] \cap \text{supp}[g] = \emptyset$. We say that $(\mathcal{E}, \mathcal{F})$ is strongly local if for any $f, g \in \mathcal{F}$, the following holds: if $g$ is constant on a neighbourhood of $\text{supp}[f]$, then $\mathcal{E}(f, g) = 0$. If a quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is local, then the corresponding processes have continuous paths on $[0, \zeta]$ $\mathbb{P}^x$-almost surely for $\mathcal{E}$-q.e. $x \in X$ (see [27, Theorem 1.11]).

### 2.7 RCD Spaces

In this subsection, we recall $\text{RCD}(K, \infty)/\text{RCD}^*(K, N)$ spaces. Recall that $\text{Ch}$ denotes the Cheeger energy and the property of infinitesimal Hilbertianity (IH) was defined in (2.2). Under (IH), $\text{Ch}$ becomes a strongly local symmetric Dirichlet form ([3, 4]). By the third paragraph in Subsection 2.6, there exists the corresponding semigroup $\{H_t\}_{t \geq 0}$ (called heat semigroup) and the infinitesimal generator $\Delta$. Let us consider the following condition:

Every function $f \in W^{1,2}(X, d, m)$ with $|\nabla f| \leq 1$ m-a.e., admits a continuous 1-Lipschitz representative $\tilde{f}$. (2.8)
We recall gradient estimates of the heat semigroups: for every \( f \in W^{1,2}(X, d, m) \) with \( |\nabla f| \leq 1 \) \( m \)-a.e., and every \( t > 0 \), we have

\[
H_tf \in \text{Lip}_b(X), \quad |\nabla H_tf|^2 \leq e^{-2Kt}H_t(|\nabla f|^2), \quad m\text{-a.e. in } X. \tag{2.9}
\]

The gradient estimate with dimensional upper bounds is as follows:

\[
H_tf \in \text{Lip}_b(X), \quad |\nabla H_tf|^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)}|\Delta H_tf|^2 \leq e^{-2Kt}H_t(|\nabla f|^2), \quad m\text{-a.e. in } X. \tag{2.10}
\]

According to a sequence of results \([2, 3, 11]\), \( \text{RCD}(K, \infty)/\text{RCD}^*(K, N) \) conditions can be identified with \((2.9)/(2.10)\) if we assume \((1.1), (1.2), (\text{IH})\) and \((2.8)\).

A metric measure space \((X, d, m)\) is called an \( \text{RCD}(K, \infty) \) (resp. \( \text{RCD}^*(K, N) \)) space if \((2.9)\) (resp. \((2.10)\)) holds under the assumptions \((1.1), (1.2), (\text{IH})\) and \((2.8)\).

**Remark 2.17** Note that \( \text{RCD}(K, \infty)/\text{RCD}^*(K, N) \) conditions have been originally defined in terms of \( K \)-convexity of the relative entropy in the \( L^2 \)-Wasserstein space with condition \((\text{IH})\) \([1, 3, 11]\)).

The class of \( \text{RCD} \) spaces contains various (finite- and infinite-dimensional) singular spaces such as Ricci limit spaces \([33, 34]\), Lott–Villani \([26]\), Alexandrov spaces \([1, 3, 11]\), warped products and cones \([23, 24]\), quotient spaces \([16]\) and infinite-dimensional spaces such as Hilbert spaces with log-concave measures \([7]\) (related to various stochastic partial differential equations). See these references for concrete examples.

An important property of \( \text{RCD}(K, \infty)/\text{RCD}^*(K, N) \) spaces is their stability under the \( \text{pmG} \) convergence.

**Theorem 2.18** \([1, 4, 11, 19, 33, 34]\) **(Stability of the \( \text{RCD}(K, \infty)/\text{RCD}^*(K, N) \))**

Let \( \mathcal{X}_n \) be an \( \text{RCD}(K, \infty)/\text{RCD}^*(K, N) \) space for \( n \in \mathbb{N} \). If \( \mathcal{X}_n \) converges to \( \mathcal{X}_\infty \) in the \( \text{pmG} \) sense, then the limit space \( \mathcal{X}_\infty \) also satisfies the \( \text{RCD}(K, \infty)/\text{RCD}^*(K, N) \) condition.

### 3 Construction of Non-Symmetric Dirichlet Forms

In this section, we construct a non-symmetric Dirichlet form consisting of a uniformly elliptic operator, derivations and a killing term. Let us consider the following bilinear form \( \mathcal{E} : \text{Lip}_{ba}(X) \times \text{Lip}_{ba}(X) \to \mathbb{R} \):

\[
\mathcal{E}(f, g) := \frac{1}{2} \int_X \langle A \nabla f, \nabla g \rangle dm + \int_X b_1(f)g dm + \int_X f b_2(g) dm + \int_X fg dm.
\]

We write \( \int_X \langle A \nabla f, \nabla g \rangle dm = 2\text{Ch}_A(f, g) \) in short. Recall that \( L^2(TX) \) denotes the tangent module as in Definition 2.11. Let \( A : L^2(TX) \to L^2(TX) \) denote a (not necessarily \( L^2(TM) \))-symmetric) module morphism satisfying that there exists \( H \in L^1_{\text{loc}}(X, m) \) so that \( |AY| \leq H|Y| \) for any \( Y \in L^2(TX) \). We write \( |A| \) for the minimal element among such \( H \). Let \( \hat{A} := 1/2(A + A^*) \) and \( \hat{A} := 1/2(A - A^*) \) where \( A^* \) is the \( L^2(TX) \)-adjoint operator of \( A \). Suppose the following conditions:
Assumption 3.1 Let \((X, d, m)\) be a metric measure space with (1.1), (1.2) and (IH). Suppose that

(i) \(|\hat{A}| \in L^\infty(X, m)\) and there exists a constant \(\lambda > 0\) so that for any \(f \in \text{Lip}_{bs}(X)\),
\[
\langle A\nabla f, \nabla f \rangle \geq \lambda \langle \nabla f, \nabla f \rangle, \ m\text{-a.e.}; \tag{3.1}
\]

(ii) \(|b_1 + b_2|, c \in L^\infty_{loc}(X, m), |b_1 - b_2| \in L^\infty(X, m)\);

(iii) for any non-negative \(f \in \text{Lip}_{bs}(X), i = 1, 2,\)
\[
\int_X (b_i(f) + cf)dm \geq 0. \tag{3.2}
\]

Then we can show the following proposition.

Proposition 3.2 Suppose Assumption 3.1.

(a) If \((X, d, m)\) is an \(RCD^*(K, N)\) space, then the form (1.3) is closable and the closed form \((\mathcal{E}, \mathcal{F})\) is a regular local Dirichlet form.

(b) Let \((X, d, m)\) be an \(RCD(K, \infty)\) space with \(m(X) < \infty\). If \(|A|, |b_1|, |b_2|, c \in L^\infty(X, m)\),
then the form (1.3) is closable and the closed form \((\mathcal{E}, \mathcal{F})\) is a quasi-regular local Dirichlet form.

Proof of Proposition 3.2. Non-negativity (\(\mathcal{E}.1\)): By \(|A| \in L^1_{loc}(X, m)\), the integrand \(\langle A\nabla f, \nabla g \rangle\) for \(f, g \in \text{Lip}_{bs}(X)\) is \(m\)-integrable. For \(f \in \text{Lip}_{bs}(X)\), by Leibniz formula of \(b_i\), (3.1) and (3.2), we have
\[
\mathcal{E}(f) = \text{Ch}_A(f) + \int_X \left( b_1(f^2) + b_2(f^2) + 2f^2c \right)dm \geq \lambda \text{Ch}(f) \geq 0. \tag{3.3}
\]

Closability: Let \(\{f_n\}_{n \in \mathbb{N}} \subset \text{Lip}_{bs}(X)\) be an \(\mathcal{E}\)-Cauchy sequence so that \(\|f_n\|_2 \to 0\). Since \(\text{Ch}\) is closable by [3, 4], we have \(\text{Ch}(f_n) \to 0\), which implies that there exists a subsequence \(f_{n'}\) so that \(|\nabla f_{n'}|\) converges to zero \(m\)-a.e.. By \(|A\nabla f_n| \leq |A||\nabla f_n|\), we have that \(|A\nabla f_{n'}| \to 0\) \(m\)-a.e. Noting that \(\text{Ch}_A(f) \leq 2\mathcal{E}(f)\) by (3.3), and by using Fatou’s lemma, we have
\[
2\text{Ch}_A(f_n) = \lim_{n' \to \infty} \int_X \liminf_{n' \to \infty} \left( A\nabla(f_n - f_{n'}) - \nabla f_n - \nabla f_{n'} \right)dm \leq \liminf_{n' \to \infty} \int_X \left( A\nabla(f_n - f_{n'}) - \nabla(f_n - f_{n'}) \right)dm \leq \liminf_{n' \to \infty} \mathcal{E}(f_n - f_{n'}). \]

Since \(\{f_n\}_{n \in \mathbb{N}}\) is an \(\mathcal{E}\)-Cauchy sequence, the R.H.S. above can be arbitrarily close to zero as \(n\) is sufficiently large. Thus, \(\text{Ch}_A(f_n) \to 0\) as \(n \to \infty\). We next show the closability of the remaining part of \(\mathcal{E}\). By replacing \(f_n\) to \(f_nh\) for any \(h \in \text{Lip}_{bs}(X)\) with \(h \geq 0\), we may assume that \(\text{supp}(f_n) \subset K \subset X\) for some bounded open set \(K\). Noting that \(b_i(f_n) \leq |b_i||\nabla f_n|\) for \(i = 1, 2\) and \(\text{Ch}(f_n) \to 0\) by the closability of \(\text{Ch}\), we have
\[
\left| \int_X b_1(f_n)dm + \int_X b_2(f_n)dm + \int_X f_n^2dm \right| \leq \left( |b_1 + b_2|1_K \right)^{1/2} \text{Ch}(f_n)^{1/2} + |c1_K|f_n^2 \leq \left( |b_1 + b_2|1_K \right)^{1/2} \|f_n\|^2/2 + |c1_K|\|f_n\| \to 0 \ (n \to \infty).
\]
Thus, we have $E(f_n) \to 0$ and we have proved the closability of $E$.

Weak sector condition ($E$.2): It suffices to show that there exists a constant $C > 0$ so that $E(f, g) \leq CE_1(f)E_1(g)$, whereby $E(f, g) = 1/2(E(f, g) - E(g, f))$ (see [27, Chapter I, §2]). Then

$$2E(f, g) = \int_X \left( \frac{1}{2} \langle \Delta f, \nabla g \rangle + b_1(f)g - b_2(f)g - f b_1(g) + f b_2(g) \right) dm$$

$$\leq \frac{1}{2} \| \nabla f \|_2 \| \nabla g \|_2 + \| b_1(f) - b_2(f) \|_2 \| g \|_2 + \| b_2(g) - b_1(g) \|_2 \| f \|_2$$

$$\leq \frac{1}{2} \| \nabla f \|_2 \| \nabla g \|_2 + \| b_1 - b_2 \| \| f \|_2 + \| b_1 - b_2 \| \| g \|_2. \quad (3.4)$$

Since $|b_1 - b_2| \in L^\infty(X; m)$ and $2\text{Ch}(f) = \| \nabla f \|_2^2 \leq (2/\lambda)E(f)$ by (3.3), we have

$$\frac{1}{2} \| A \|_\infty \| \nabla f \|_2 \| \nabla g \|_2 + \| b_1 - b_2 \| \| f \|_2 + \| b_1 - b_2 \| \| g \|_2$$

$$\leq \frac{1}{\lambda} \| A \|_\infty \| \nabla f \|_2 \| \nabla g \|_2 + \sqrt\frac{2}{\lambda} \| (b_1 - b_2) \| \| f \|_2 \| g \|_2$$

$$+ \sqrt\frac{2}{\lambda} \| (b_1 - b_2) \| \| f \|_2 \| g \|_2$$

$$\leq C\| \nabla f \|_2 \| \nabla g \|_2.$$

We finished to prove that $E$ satisfies the weak sector condition.

Markov Property ($E$.4): Let $\phi_\varepsilon : \mathbb{R} \to [-\varepsilon, 1 + \varepsilon]$ be an infinitely differentiable function so that $0 \leq \phi_\varepsilon(t) - \phi_\varepsilon(s) \leq t - s$ for all $t, s \in \mathbb{R}$ with $t \geq s$ and

$$\phi_\varepsilon(t) = \begin{cases} t & \text{for } t \in [0, 1], \\ 1 + \varepsilon & \text{for } t \in [1 + 2\varepsilon, \infty), \\ -\varepsilon & \text{for } t \in (-\infty, -2\varepsilon]. \end{cases}$$

By [27, Proposition 4.7, 4.10 in Chapter I], it suffices to show that for all $f \in \text{Lip}_{ba}(X)$ and $\varepsilon > 0$, it holds that $\phi_\varepsilon(f) \in F$ and

$$\liminf_{\varepsilon \to 0} E(\phi_\varepsilon(f), f - \phi_\varepsilon(f)) \geq 0, \quad \liminf_{\varepsilon \to 0} E(f - \phi_\varepsilon(f), \phi_\varepsilon(f)) \geq 0. \quad (3.5)$$

It is clear that $\phi_\varepsilon(f) \in \text{Lip}_{ba}(X) \subset F$. We only show the L.H.S. side of (3.5) (the proof of the R.H.S. is similar). The diffusion part $\text{Ch}_A$ clearly satisfies (3.5) by the chain rule, $0 \leq \phi_\varepsilon' \leq 1$, (2.5) and (3.3). In fact,

$$\int_X \langle A\nabla \phi_\varepsilon(f), \nabla (f - \phi_\varepsilon(f)) \rangle dm = \int \phi_\varepsilon'(f)(1 - \phi_\varepsilon'(f)) \langle A\nabla f, \nabla f \rangle dm \geq 0.$$

For the remaining part, we have

$$\int_X b_1(f - \phi_\varepsilon(f))(\phi_\varepsilon(f)) dm + \int_X (f - \phi_\varepsilon(f))b_2(\phi_\varepsilon(f)) dm + \int_X (f - \phi_\varepsilon(f))(\phi_\varepsilon(f)) c dm$$

$$= \int_X (b_1(\phi_\varepsilon(f)(f - \phi_\varepsilon(f)) - \phi_\varepsilon(f)(f - \phi_\varepsilon(f)) c) dm + \int \phi_\varepsilon'(f - \phi_\varepsilon(f))(b_1 - b_2)(f) dm.$$
The first term in the second line above is non-negative since (3.2) and \((f - \phi_\varepsilon(f))\phi_\varepsilon(f) \geq 0\).

The second term converges to zero since

\[
\phi'_\varepsilon(f)(f - \phi_\varepsilon(f)) \leq \left(1_{[-2\varepsilon,1+2\varepsilon]}(f)(f - \phi_\varepsilon(f))\right) \rightarrow 0 \quad \varepsilon \downarrow 0.
\]

Thus, we finished to prove the Markovian property (E.4).

In the case of \(\text{RCD}^*(K, N)\) spaces, the underlying space \(X\) is locally compact by the local volume doubling property according to the Bishop–Gromov inequality [11, Proposition 3.6] (see also [34, Corollary 2.4]). Therefore, \((\mathcal{E}, \mathcal{F})\) is regular since \(\text{Lip}_b(X)\) is dense in both in \(C_0(X)\) with \(\|\cdot\|_\infty\) and \(\mathcal{F}\) with \(\tilde{\mathcal{E}}^{1/2}\). In the case of \(\text{RCD}(K, \infty)\) spaces, \(X\) is generally not locally compact and therefore we need to show that there exists an \(\mathcal{E}\)-nest of compact sets \(\{E_k\}_{k\in\mathbb{N}}\), which is called tightness of capacity.

Tightness of Capacity: We show that there exists an \(\mathcal{E}\)-nest of compact sets \(\{E_k\}_{k\in\mathbb{N}}\), i.e., an increasing sequence of compact sets \(E_k \subset E_{k+1}\) so that \(\text{Cap}_\mathcal{E}(X \setminus E_k) \rightarrow 0\). Let \(\{x_k\}_{k\in\mathbb{N}} \subset X\) be a countable dense subset. Define

\[
w_n(x) := \min\{1, \min_{1 \leq k \leq n} d(x_k, x)\}.
\]

We see that \(0 \leq w_n \leq 1\) and \(w_n \downarrow 0\) as \(n \rightarrow \infty\). Thus, \(w_n \rightarrow 0\) in \(L^2(m)\). By the definition of \(\text{Ch}\), it is easy to see that \(2\text{Ch}(w_n) \leq \text{Lip}(w_n) \leq 1\). Noting \(|\nabla w_n| \leq 1\), we have that

\[
\mathcal{E}(w_n) = \text{Ch}_A(w_n) + \int_X (b_1(w_n) w_n) dm + \int_X (w_n b_2(w_n)) dm + \int_X w_n^2 cd m \\
\leq \left\| \frac{|A|}{2}\right\| + \left\| b_1 + b_2 \right\|_\infty \| \nabla w_n \|_2 \| w_n \|_2 + \| c \|_\infty \| w_n \|_2^2 \\
\leq \left\| \frac{|A|}{2}\right\| + \left\| b_1 + b_2 \right\|_\infty + \| c \|_\infty.
\]

(3.6)

Therefore, \(w_n\) is a uniformly bounded sequence in \(\mathcal{F}\) with respect to \(\tilde{\mathcal{E}}_1\) where \(\tilde{\mathcal{E}}_1(f, g) := 1/2(\mathcal{E}_1(f, g) + \mathcal{E}_1(f, g))\). Thus, \(w_n \rightarrow 0\) weakly in \(\mathcal{F}\) with respect to \(\tilde{\mathcal{E}}_1\) by [27, Lemma 2.12 in Chapter I]. By the Banach-Saks theorem, there exists a subsequence \(\{n(i)\}_i\) so that the Cesàro means

\[
v_l = \frac{1}{l} \sum_{i=1}^{l} w_{n(i)}
\]

converges in \(\mathcal{F}\), thus in \((L^2(X, m), \|\cdot\|_2)\). By [27, Proposition III. 3.5], we have that there exists a subsequence \(\{n(i)\}_j\) so that \(v_{n(j)} \rightarrow 0\) quasi-uniformly, i.e., for any \(k\) there exists a closed set \(G_k \subset X\) so that \(\text{Cap}_\mathcal{E}(X \setminus G_k) \leq 1/k\) and \(v_{n(i)} \rightarrow 0\) uniformly on \(G_k\). Since \(w_{n(i)}) \leq v_{n(i)}\), setting \(F_k = \bigcap_{i \leq k} G_i\), we have that \(w_{n(i)} \rightarrow 0\) uniformly on \(F_k\) for all \(k\) and \(\text{Cap}_\mathcal{E}(X \setminus F_k) \leq 1/k\). Set \(\varepsilon > 0\) and \(n\) be an integer so that \(w_n < \varepsilon\), and the definition of \(w_n\) implies

\[
F_k \subset \bigcup_{k=1}^{n} B(x_k, \varepsilon).
\]

Since \(\varepsilon\) is arbitrarily small, we have that \(F_k\) is totally bounded and thus compact. The other conditions for quasi-regularity in [27, (ii) and (iii) in Definition 3.1 in Chapter IV] are easy to check since \(\text{Lip}_b(X)\) separates points and is dense in \(\mathcal{F}\) with respect to \(\tilde{\mathcal{E}}^{1/2}\).
The local property is obvious according to the locality of $Ch$ and derivations $b_i$ for $i = 1, 2$. Thus, we have finished the proof of Proposition 3.2.

**Remark 3.3** Construction of non-symmetric diffusion on metric measure spaces (including RCD spaces) has been already considered in Fitzsimmons [14] and Trevisan [40] with different approaches and different scopes.

(i) In the former paper, the Girsanov transform was used to produce drift perturbations from symmetric diffusions.

(ii) In the latter paper, the martingale problem was developed to construct diffusion processes in this generality.

An advantage of the Dirichlet form approach adopted in this paper is to make the issue of convergence simpler. This is mainly because the domains of Dirichlet forms corresponding to (1.3) has the common core $\text{Lip}_b(X)$, which is useful especially to show tightness of diffusion processes in Section 5.

### 4 Convergence of Non-symmetric Forms

In this section, we show the convergence of non-symmetric forms. We first modify the definition in [21] for varying metric measure spaces and prove this modified convergence under Assumption 1.1. We recall the definition of the $L^p$-convergence on varying metric measure spaces in the sense of pmG following [19].

**Definition 4.1** (See [19, Definition 6.1]) Let $(X_n, d_n, m_n, \mathfrak{m}_n)$ be a sequence of p.m.m for $n \in \mathbb{N}$, spaces. Assume that $(X_n, d_n, m_n, \mathfrak{m}_n)$ converges to $(X_\infty, d_\infty, m_\infty, \mathfrak{m}_\infty)$ in the pmG sense. Let $(X, d)$ be a complete separable metric space and $\iota_n : \text{supp}[m_n] \to X$ be isometries as in Definition 2.2. We identify $(X_n, d_n, m_n)$ with $(\iota_n(X_n), d, \iota_n#m_n)$ and omit $\iota_n$.

(i) We say that $f_n \in L^2(X, m_n)$ converges weakly to $f_\infty \in L^2(X, m_\infty)$ if the following hold:

$$\sup_{n \in \mathbb{N}} \int |f_n|^2 \, dm_n < \infty \quad \text{and} \quad \int \phi f_n \, dm_n \to \int \phi f_\infty \, dm_\infty \quad \forall \phi \in C_0(X).$$

(ii) We say that $f_n \in L^2(X, m_n)$ converges strongly to $f_\infty \in L^2(X, m_\infty)$ if $f_n$ converges weakly to $f_\infty$ and the following holds:

$$\limsup_{n \to \infty} \int |f_n|^2 \, dm_n \leq \int |f_\infty|^2 \, dm_\infty.$$
Definition 4.2 (See also [39, Definition 7.11]) We say that \((E_n, F_n)\) converges to \((E_\infty, F_\infty)\) if the following two conditions hold:

(F1) If a sequence \(f_n \in L^2(X, m_n)\) converges weakly to \(f_\infty \in L^2(X, m_\infty)\) with \(\liminf_{n \to \infty} \Phi_n(f_n) < \infty\), then it holds that

\[ f_\infty \in F_\infty. \]

(F2) For any sequence \(f_n \in F_n\) converging weakly in \(L^2\) to \(f_\infty \in F_\infty\), and any \(w_\infty \in F_\infty\), there exists a sequence \(w_n \in F_n\) converging strongly in \(L^2\) to \(w_\infty \in F_\infty\) so that

\[ \lim_{n \to \infty} E_n(f_n, w_n) = E_\infty(f_\infty, w_\infty). \]

Remark 4.3 In Tölle [38, 39], he introduced a notion of a convergence of non-symmetric forms whose basic Banach spaces vary. The difference of his approach and this paper is the notion of the \(L^2\)-convergence on varying metric measure spaces whereby in this paper we follow [19]. If the Hilbert spaces \(\{L^2(X; m_n)\}_{n \in \mathbb{N}}\) have an asymptotic relation in [39], these two notions of the \(L^2\)-convergence are equivalent, so the following Theorem 4.4 corresponds to [39, Theorem 7.15].

Verifying (F2) is not always easy and we introduce another condition:

(F2') For any sequence \(\{n_k\} \uparrow \infty\) and any sequence \(f_k \in L^2(X; m_{n_k})\) weakly convergent in \(L^2\) to \(f_\infty \in F_\infty\) with \(\sup_{k \in \mathbb{N}} \Phi_{n_k}(f_k) < \infty\), there exists a dense subset \(C \subset F_\infty\) for the topology with respect to \(\tilde{E}_1^\infty\) so that every \(w \in C\) has a sequence \(\{w_k\}\) with \(w_k \in F_{n_k}\) converging to \(w\) strongly in \(L^2\) with

\[ \liminf_{k \to \infty} E_{n_k}(w_k, f_k) \leq E_\infty(w, f_\infty). \]

We also define (F1*) by replacing \(\Phi_n(f_n)\) with \(\tilde{E}_1^\alpha(f_n)^{1/2}\) in (F1), and (F2'*') by replacing \(\Phi_{n_k}(f_k)\) with \(\tilde{E}_1^{\alpha_k}(f_k)^{1/2}\) in (F2').

We now study the relation between the convergence of forms and \(L^2\)-convergences of the corresponding semigroups and resolvents. Let \(\{T^n_t\}_{t \geq 0}\) and \(\{G_\alpha^n\}_{\alpha \geq 0}\) be the \(L^2\)-contraction semigroup and resolvent associated with \(E_n\).

(R) For any sequence \(f_n\) converging to \(f_\infty\) strongly in \(L^2\), the resolvent \(G_{\alpha}^n f_n\) converges to \(G_{\alpha}^\infty f_\infty\) strongly in \(L^2\) for any \(\alpha > 0\).

(S) For any sequence \(f_n\) converging to \(f_\infty\) strongly in \(L^2\), \(T^n_t f_n\) converges to \(T^\infty_t f_\infty\) strongly in \(L^2\). The convergence is uniform on any compact time interval \([0, T]\).

Theorem 4.4 The following statements hold:

(i) \((F1)(F2) \iff (F1)(F2') \iff (R) \iff (S)\);

(ii) \((F1_*)(F2_*) \implies (R)\).
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Proof. The proof is just a modification of [21, Theorem 3.2, Corollary 3.3] for varying metric measure spaces in the sense of the pmG so that the notion of $L^2$-convergence is replaced by Definition 4.1. So we omit the proof.

Hereafter in this section, we focus on the convergence of Dirichlet forms corresponding to (1.3). To characterize the convergence of these forms in terms of convergences coefficients, we introduce a convergence of $A_n$ and recall a convergence of derivation operators $b^i_n$ ([8]). Let $\mathcal{A} \subset \text{Lip}_0(X_\infty)$ denote the smallest algebra containing the following functions:

$$\min\{d(\cdot, x), k\}, \quad k \in \mathbb{Q}, \; x \in D \subset X_\infty,$$

dense subset.

The algebra $\mathcal{A}$ becomes a vector space over $\mathbb{Q}$. Let $\mathcal{A}_{bs}$ be a subalgebra consisting of bounded support functions. Let $\{H_t^\infty\}_{t \geq 0}$ be the heat semigroup associated with Cheeger energy $\text{Ch}_\infty$ (note that $\{H_t^\text{sym}\}_{t \geq 0}$ is not the semigroup associated with the non-symmetric form $\mathcal{E}$). Let

$$H^\infty_{Q_+} \mathcal{A}_{bs} := \{H^\infty f : f \in \mathcal{A}_{bs}, s \in \mathbb{Q}_+\} \subset \text{Lip}_0(X),$$

whereby $\text{Lip}_0(X)$ denotes the set of bounded Lipschitz functions on $X$. Recall that $\text{Der}_{loc}^p(X, d, m)$ and $\text{Der}^p(X, d, m)$ be the set of derivation operators $b$ in $(X, d, m)$ with $|b| \in L^p_{loc}(X, d, m)$ and $|b| \in L^p(X, d, m)$, respectively.

Definition 4.5 ([8, Definition 4.3, 5.3]) Let $(X_n, d_n, m_n, \tau_n)$ converge to $(X_\infty, d_\infty, m_\infty, \tau_\infty)$ in the pmG sense. Let $(X, d)$ be a complete separable metric space and $\iota_n : \text{supp}[m_n] \rightarrow X$ be isometries as in Definition 2.2.

1. (Weak Convergence) We say that $b_n \in \text{Der}_{loc}^1(X, d, m_n)$ converges weakly to $b_\infty \in \text{Der}_{loc}^1(X, d, m_\infty)$ in duality with $H^\infty_{Q_+} \mathcal{A}_{bs}$ if, for all $f \in H^\infty_{Q_+} \mathcal{A}_{bs}$,

$$\int_X b_n(f) hdm_n \rightarrow \int_X b_\infty(f) hdm_\infty \quad \forall h \in C_{bs}(X).$$

2. (Strong Convergence) We say that $b_n \in \text{Der}_{loc}^1(X, d, m_n)$ converges strongly to $b_\infty \in \text{Der}_{loc}^1(X, d, m_\infty)$ if, for all $f \in H^\infty_{Q_+} \mathcal{A}_{bs}$, the function $b_n(f)$ converges in measure to $b_\infty(f)$, i.e.,

$$\int_X \Phi(b_n(f)) hdm_n \rightarrow \int_X \Phi(b_\infty(f)) hdm_\infty \quad \forall h \in C_{bs}(X), \quad \forall \Phi \in C_b(\mathbb{R}).$$

3. ($L^p$-strong Convergence) Let $p \in [1, \infty)$. We say that $b_n \in \text{Der}_{loc}^p(X, d, m_n)$ converges $L^p_{loc}$-strongly to $b_\infty \in \text{Der}_{loc}^p(X, d, m_\infty)$ if $b_n$ converges strongly to $b_\infty$ and for all $f \in H^\infty_{Q_+} \mathcal{A}_{bs}$ and for $R > 0$,

$$\limsup_{n \rightarrow \infty} \int_{B_R(\tau_n)} |b_n(f)|^p hdm_n \leq \int_{B_R(\tau_\infty)} |b_\infty(f)|^p hdm_\infty.$$

Analogously we say that $b_n \in \text{Der}_{loc}^p(X, d, m_n)$ converges $L^p$-strongly to $b_\infty \in \text{Der}_{loc}^p(X, d, m_\infty)$ if, $b_n$ converges strongly to $b_\infty$ and, for all $f \in H^\infty_{Q_+} \mathcal{A}_{bs}$,

$$\limsup_{n \rightarrow \infty} \int_X |b_n(f)|^p hdm_n \leq \int_X |b_\infty(f)|^p hdm_\infty.$$
Now we recall the $W^{1,2}$-convergence of functions on varying metric measure spaces in the sense of pmG.

**Definition 4.6** ([5, Definition 5.2]) Let $(X_n, d_n, m_n, \overline{\nu}_n)$ be a sequence of p.m.m. spaces. Assume that $(X_n, d_n, m_n, \overline{\nu}_n)$ converges to $(X_\infty, d_\infty, m_\infty, \overline{\nu}_\infty)$ in the pmG sense. Let $(X, d)$ be a complete separable metric space and $\iota_n : \text{supp}[m_n] \to X$ be isometries as in Definition 2.2. We identify $(X_n, d_n, m_n)$ with $(\iota_n(X_n), d, \iota_n m_n)$ and omit $\iota_n$.

(i) We say that $f_n \in W^{1,2}(X, m_n)$ converges weakly to $f_\infty \in W^{1,2}(X, m_\infty)$ in $W^{1,2}$ if $f_n \to f_\infty$ weakly in $L^2$ in the sense of Definition 4.1 and $\sup_{n \in \mathbb{N}} \text{Ch}_n(f_n) < \infty$;

(ii) We say that $f_n \in W^{1,2}(X, m_n)$ converges strongly to $f_\infty \in W^{1,2}(X, m_\infty)$ in $W^{1,2}$ if $f_n$ converges strongly to $f_\infty$ in $L^2$ in the sense of Definition 4.1 and $\lim_{n \to \infty} \text{Ch}_n(f_n) = \text{Ch}_\infty(f_\infty)$.

Now we introduce a convergence of $A_n$.

**Definition 4.7** We say that $A_n$ converges to $A_\infty$ if for any $u_n \to u_\infty$ weakly in $W^{1,2}$ and $v_n \to v_\infty$ strongly in $W^{1,2}$,

\[
\int_X (\nabla u_n, A_n \nabla v_n) d\mu_n \to \int_X (\nabla u_\infty, A_\infty \nabla v_\infty) d\mu_\infty,
\]

\[
\int_X (A_n \nabla u_n, \nabla v_n) d\mu_n \to \int_X (A_\infty \nabla u_\infty, \nabla v_\infty) d\mu_\infty.
\]

Now we show the main theorem in this section.

**Theorem 4.8** Under Assumption 1.1, $(\mathcal{E}_n, \mathcal{F}_n)$ (resp. $(\hat{\mathcal{E}}_n, \mathcal{F}_n)$) converges to $(\mathcal{E}_\infty, \mathcal{F}_\infty)$ (resp. $(\hat{\mathcal{E}}_\infty, \mathcal{F}_\infty)$).

**Proof.** By Theorem 4.4, it suffices to show $(F1_\lambda)$ and $(F2'_\lambda)$.

$(F1_\lambda)$: Let $u_n \to u_\infty$ weakly in $L^2$ and we may assume $\lim \inf_{n \to \infty} \mathcal{E}_n(u_n) < \infty$. Since $(\text{Ch}_n, \mathcal{F}_n)$ converges to $(\text{Ch}_\infty, \mathcal{F}_\infty)$ in the Mosco sense [19, Theorem 6.8], we have, by (3.3),

\[
\text{Ch}_\infty(u_\infty) \leq \lim \inf_{n \to \infty} \text{Ch}_n(u_n) \leq \frac{1}{\lambda} \lim \inf_{n \to \infty} \mathcal{E}^n(u_n) < \infty. \tag{4.1}
\]

This implies $u_\infty \in W^{1,2}(m_\infty)$ by the definition of $W^{1,2}(m_\infty)$.

$(F2'_\lambda)$: Let $n_k \uparrow \infty$ and $u_k \to u_\infty$ weakly in $L^2$ with $\sup_{k \in \mathbb{N}} \mathcal{E}_n(u_k) < \infty$ and $u_\infty \in W^{1,2}(m_\infty)$. Then we have that $u_k \to u_\infty$ weakly in $W^{1,2}$ by definition. Let us take $\mathcal{C} = \ldots$
We first show (I) the dual forms can be shown in the same manner. Next we show (II) $k$ we identify $\iota$ In this section, we show the weak convergence of finite-dimensional distributions. Recall that

\[ H_{Q, \mathscr{A}_b}. \text{ Take } w \in \mathcal{C}. \] By (2.6), we have

\[
|\mathcal{E}_{n_k}(u_k, w) - \mathcal{E}_\infty(u, w)| = \left| \text{Ch}_A^{n_k}(u_k, w) - \text{Ch}_A^{\infty}(u, w) \right| + \left| \int_{X_{n_k}} b_{1}^{n_k}(u_k) w \, dm_{n_k} - \int_{X_{\infty}} b_{1}^{\infty}(u) w \, dm_{\infty} \right| \\
+ \left| \int_{X_{n_k}} u_k b_{2}^{n_k}(w) \, dm_{n_k} - \int_{X_{\infty}} u_{\infty} b_{2}^{\infty}(w) \, dm_{\infty} \right| + \left| \int_{X_{n_k}} u_k w c_{n_k} \, dm_{n_k} - \int_{X_{\infty}} u_{\infty} w c_{\infty} \, dm_{\infty} \right|
\]

\[
= \left| \text{Ch}_A^{n_k}(u_k, w) - \text{Ch}_A^{\infty}(u, w) \right| + \left| \int_{X_{n_k}} u_k b_{1}^{n_k}(w) \, dm_{n_k} - \int_{X_{\infty}} u_{\infty} b_{1}^{\infty}(w) \, dm_{\infty} \right| \\
+ \left| \int_{X_{n_k}} u_k w \text{div} b_{2}^{n_k} \, dm_{n_k} - \int_{X_{\infty}} u_{\infty} w \text{div} b_{2}^{\infty} \, dm_{\infty} \right| + \left| \int_{X_{n_k}} u_k b_{2}^{n_k}(w) \, dm_{n_k} - \int_{X_{\infty}} u_{\infty} b_{2}^{\infty}(w) \, dm_{\infty} \right| \\
+ \left| \int_{X_{n_k}} u_k w c_{n_k} \, dm_{n_k} - \int_{X_{\infty}} u_{\infty} w c_{\infty} \, dm_{\infty} \right|
\]

\[ := (I)_k + (II)_k + (III)_k + (IV)_k + (V)_k. \]

We first show $(I)_k \to 0$ as $k \to \infty$. By the convergence of $A_n$ to $A_\infty$, we have

\[
|\text{Ch}_A^{n_k}(u_k, w) - \text{Ch}_A^{\infty}(u, w)| = \frac{1}{2} \left| \int_X \langle A_{n_k} \nabla u_k, \nabla w \rangle \, dm_{n_k} - \int_X \langle A_\infty \nabla u_{\infty}, \nabla w \rangle \, dm_{\infty} \right| \xrightarrow{k \to \infty} 0.
\]

Next we show $(II)_k \to 0$ as $k \to \infty$. Combining $\sup_{n \in \mathbb{N}} |b^n| < \infty$ with $L^2$-convergence of $b^n_k$ to $b^\infty_k$, we have $b_{2}^{n_k} \to b_{2}^{\infty}$ strongly in $L^2$, especially we have $b_{2}^{n_k}(w) \to b_{2}^{\infty}(w)$ strongly in $L^2$. Since $u_k \to u_{\infty}$ weakly in $L^2$, we have that $(II)_k \to 0$. The quantity $(IV)_k \to 0$ in the same proof. Since $|\text{div} b^n_k|$ is uniformly bounded in $n$ and $\text{div} b_{2}^{n_k} \to \text{div} b_{2}^{\infty}$ in $L^2$, the quantity $(III)_k \to 0$ also goes to zero. It is easy to check that $(V)_k \to 0$. The convergence of the dual forms can be shown in the same manner.

\section{Convergence of Finite-dimensional Distributions}

In this section, we show the weak convergence of finite-dimensional distributions. Recall that we identify $\iota_n(X_n)$ with $X_n$ and we omit $n$ for simplifying the notation. We first show the weak convergence of finite-dimensional distributions under the assumptions in Theorem 1.3 in the case that the initial distribution is the Dirac measure $\delta_{x_n}$.

\textbf{Lemma 5.1 (Convergence of Finite-dimensional Distributions)} Suppose the conditions assumed in Theorem 1.3. Then, for any $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \infty$ and $f_1, f_2, \ldots, f_k \in C_{b}(X)$, the following holds:

\[
\mathbb{E}_{s^n} \left[ f_1(S^n_{t_1}) \cdots f_k(S^n_{t_k}) \right] \xrightarrow{n \to \infty} \mathbb{E}_{s} \left[ f_1(S_{t_1}) \cdots f_k(S_{t_k}) \right].
\]

For the dual process $\tilde{s}^{n}$, the same statement holds.

\textbf{Proof.} We omit the proof for the dual process which is the same as that of $s^{n}$. Let $\{T_t\}_{t \geq 0}$ be the semigroup associated with the Dirichlet form $(\mathbf{E}, \mathcal{F})$ corresponding to (1.3). According to the Gaussian heat kernel estimate [25, Theorem 5.4] (see also (6.19)), under the assumptions

\[ \text{we have } \mathbb{E}_{s^n} \left[ f_1(S^n_{t_1}) \cdots f_k(S^n_{t_k}) \right] \xrightarrow{n \to \infty} \mathbb{E}_{s} \left[ f_1(S_{t_1}) \cdots f_k(S_{t_k}) \right]. \]
in Theorem 1.3, we can easily show that \( \{T^n_t\}_{t \geq 0} \) is a Feller semigroup, which implies the uniqueness of the corresponding diffusions for every starting point. Therefore, we have the following equality: for every \( f \in C_b(X) \cap L^2(X; m_\infty) \),

\[
\mathbb{E}^n_t(f(S^n_t)) = T^n_t f(x), \tag{5.2}
\]

for every \( x \in X_n \). By using the Markov property, for all \( n \in \mathbb{N} \), we have

\[
\begin{align*}
\mathbb{E}^n_t[f_1(S^n_{t_1}) \cdots f_k(S^n_{t_k})] &= T^n_{t_1-t_0} \left( f_1 T^n_{t_2-t_1} \left( f_2 \cdots T^n_{t_k-t_{k-1}} f_k \right) \right)(x) \\
&=: P^n_t(x).
\end{align*}
\]

By [25, Corollary 4.18], the action of the semigroup \( T^n_t f \) for \( f \in L^\infty(m_n) \) is a Hölder continuous function whose Hölder constant and exponent are independent of \( n \) (depending only on \( N, K, D, \sup_{n \in \mathbb{N}} \|A_n\|_\infty, \sup_{n \in \mathbb{N}} \|b_n\|_\infty, \sup_{n \in \mathbb{N}} \|c_n\|_\infty \)).

For later arguments, we extend \( P^n_k \) to the whole space \( X \) by the McShane extension ([28, Corollary 1.2]) (note that \( P^n_k \) is defined only on each \( X_n \)). The key point is to extend \( P^n_k \) to the whole space \( X \) preserving its Hölder regularity and bounds. Let \( \hat{P}^n_k \) be the following function on the whole space \( X \):

\[
\hat{P}^n_k(x) := \left( \sup_{a \in X_n} \{P^n_k(a) - Hd(a, x)^\beta\} \wedge \sup_{a \in X_n} P^n_k(a) \right) \vee \inf_{a \in X_n} P^n_k(a), \quad x \in X, \tag{5.3}
\]

whereby \( H \) and \( \beta \) are the Hölder constant and exponent of the original function \( P^n_k \). Then we have that \( \hat{P}^n_k \) is a bounded Hölder continuous function on the whole space \( X \) with the same Hölder constant \( H \), exponent \( \beta \), and the same bound \( \|\hat{P}^n_k\|_\infty \), and satisfies \( \hat{P}^n_k = P^n_k \) on \( X_n \).

Coming back to the proof of Lemma 5.1, we have that

\[
\begin{align*}
&\left| \mathbb{E}^n_t[f_1(S^n_{t_1}) \cdots f_k(S^n_{t_k})] - \mathbb{E}^\infty_t[f_1(S^\infty_{t_1}) \cdots f_k(S^\infty_{t_k})] \right| \\
&= |P^n_k(x) - P^\infty_k(x)| \\
&\leq |P^n_k(x) - \hat{P}^n_k(x)| + |\hat{P}^n_k(x) - P^\infty_k(x)| \\
&=: (I)_n + (II)_n. \tag{5.4}
\end{align*}
\]

Thus, it suffices to show that \( (I)_n \) and \( (II)_n \) converge to zero as \( n \) goes to infinity. We first show that \( (I)_n \) converges to zero as \( n \) goes to infinity. By the McShane extension, we have

\[
(I)_n = |P^n_k(x) - \hat{P}^n_k(x)| = |\hat{P}^n_k(x) - P^\infty_k(x)| \leq Hd(x, x)^\beta \to 0 \quad (n \to \infty).
\]

Now we show that \( (II)_n \) goes to zero as \( n \) tends to infinity. Since

\[
\|T^n_t f\|_\infty = \|f\|_\infty \int_{X_n} p_n(t, x, y) m_n(dy) \leq \|f\|_\infty,
\]

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for any \( f \in C_b(X_n) \cap L^2(X;m_\infty) \), we have
\[
\sup_{n \in \mathbb{N}} \|P^n_k\|_\infty \leq k \prod_{i=1}^k \|f_i\|_\infty < \infty.
\]
(5.5)
Therefore, by the property of the McShane extension, we also have that
\[
\sup_{n \in \mathbb{N}} \|\tilde{P}^{n'}_k\|_\infty < \infty.
\]
(5.6)
By the uniform boundedness (5.6) and the equi-continuity of \( \{\tilde{P}^{n'}_k\}_{n \in \mathbb{N}} \), we can apply the Ascoli–Arzelà theorem to \( \{\tilde{P}^{n'}_k\}_{n \in \mathbb{N}} \) so that \( \{\tilde{P}^{n'}_k\}_{n \in \mathbb{N}} \) is relatively compact with respect to the uniform convergence so that for any subsequence \( \{\tilde{P}^{n'}_k\}_{n' \subset \{n\}} \), there exists a further subsequence \( \{\tilde{P}^{n''}_k\}_{n'' \subset \{n'\}} \) satisfying
\[
\tilde{P}^{n''}_k \rightarrow F'' \text{ uniformly in } X.
\]
(5.7)
On the other hand, we have that \( P^n_k \) converges to \( P^\infty_k \) strongly in the sense of Definition 4.1. We give a proof below.

**Lemma 5.2** \( P^n_k \) converges to \( P^\infty_k \) in the \( L^2 \)-strong sense in Definition 4.1.

**Proof.** By Theorem 4.4, 4.8, the statement is true for \( k = 1 \). Assume that the statement is true when \( k = l \). By noting \( P^n_{l+1} = T^{n}_{t_{l+1} - t_l}(f^{(l)}_{l+1} P^n_l) \), by Theorem 4.4, 4.8 it suffices to show \( f^{l+1} P^n_l \rightarrow f^{l+1} P^\infty_l \) strongly in \( L^2 \). This is easy to show because \( P^n_l \rightarrow P^\infty_l \) strongly (the assumption of induction), \( f^{l+1} \in C_b(X) \) and \( P^n_l \) is bounded uniformly in \( n \) because of (5.6). Thus, the statement is true for any \( k \in \mathbb{N} \).

**Proof of Lemma 5.1 (Conclusion).** By using Lemma 5.2 and the uniform convergence (5.7), it is easy to see that
\[
F''|_{X_\infty} = P^\infty_k,
\]
whereby \( F''|_{X_\infty} \) means the restriction of \( F'' \) into \( X_\infty \). The R.H.S. \( P^\infty_k \) of the above equality is clearly independent of choices of subsequences and thus, the limit \( F''|_{X_\infty} \) is independent of choice of subsequences. Therefore, we conclude that
\[
\tilde{P}^{n}_k \rightarrow P^\infty_k \text{ uniformly in } X_\infty.
\]
(5.8)
Going back to showing that \( (\Pi)_n \) goes to zero, we have that
\[
(\Pi)_n = |\tilde{P}^{n}_k(\pi_\infty) - P^\infty_k(\pi_\infty)| \leq \|\tilde{P}^{n}_k - P^\infty_k\|_{\infty,X_\infty} \rightarrow 0 \quad (n \rightarrow \infty).
\]
Here \( \|\cdot\|_{\infty,X_\infty} \) means the uniform norm on \( X_\infty \). Thus, we finish the proof of Lemma 5.1.

We now show the weak convergence of finite-dimensional distributions of \( S_n \) under Assumption 1.1 with initial distributions \( \nu_n \). Let us recall that \( \zeta_{S_n} \) and \( \zeta_{\hat{S}_n} \) denote lifetimes for \( S^n \) and \( \hat{S}^n \) respectively. Let \( \xi^n := \min\{\zeta_{S^n}, \zeta_{\hat{S}^n}\} \).
Lemma 5.3 Under Assumption 1.1, for any \( k \in \mathbb{N} \), \( 0 = t_0 < t_1 < t_2 < \cdots < t_k < \zeta_\infty < \infty \) and \( f_1, f_2, \ldots, f_k \in C_b(X) \), the following holds:

\[
E^{\nu_n}[f_1(S_{t_1}^n) \cdots f_k(S_{t_k}^n)] \xrightarrow{n \to \infty} E^{\nu_\infty}[f_1(S_{t_1}^\infty) \cdots f_k(S_{t_k}^\infty)],
\]

(5.9)

For the dual process \( \hat{S} \), the same statement holds.

**Proof.** We omit the proof for \( \hat{S} \) since the proof is the same as the case of \( S \). Since the limit diffusion \( S^\infty \) is conservative, it suffices to show the statement only for \( f_1, f_2, \ldots, f_k \in C_b(X) \cap L^2(X; m_\infty) \). In fact, for any \( \varepsilon > 0 \) and \( \zeta_\infty > T > 0 \), there exists \( R = R(\varepsilon, T) \) so that the open ball \( B_R(\overline{x}) \) satisfies

\[
P^{\nu_\infty}(S_{t_1}^\infty \in B_R(\overline{x})) < \varepsilon \quad \forall t \in [0, T],
\]

whereby \( A^c := X_\infty \setminus A \). By the strong \( L^2 \)-convergence of the semigroup \( \{P_t^n\}_{t \geq 0} \) following from Theorem 4.8, we have that

\[
\lim_{n \to \infty} P^{\nu_n}(S_{t_1}^n \in B_R(\overline{x})) < \varepsilon \quad \forall t \in [0, T].
\]

Therefore, for any \( f_1, \ldots, f_k \in C_b(X) \), for arbitrarily small \( \delta > 0 \), we can take \( R > 0 \) so that

\[
\lim_{n \to \infty} E^{\nu_n}(f_1(S_{t_1}^n) \cdots f_k(S_{t_k}^n)) = \lim_{n \to \infty} E^{\nu_n}\left(f_1(S_{t_1}^n) \cdots f_k(S_{t_k}^n) : \bigcap_{j=1}^k \{S_{t_j}^n \in B_R(\overline{x})\}\right)
+ \lim_{n \to \infty} E^{\nu_n}\left(f_1(S_{t_1}^n) \cdots f_k(S_{t_k}^n) : \bigcap_{j=1}^k \{S_{t_j}^n \in B_R(\overline{x})\}\right)^c
= \lim_{n \to \infty} E^{\nu_n}\left(f_11_{B_R(S_{t_1}^n)} \cdots f_k1_{B_R(S_{t_k}^n)}\right) + \delta.
\]

Here we mean that, for an event \( A \subset \Omega \), \( E^\varepsilon(f(S_t) : A) := \int_{\Omega \cap A} f(S_t(\omega))d\mathbb{P}^\varepsilon(\omega) \). Thus, we may show the proof only for \( f_1, f_2, \ldots, f_k \in C_b(X) \cap L^2(X; m_\infty) \). Since \( \nu_n \) converges weakly to \( \nu_\infty \) in \( \mathcal{P}(X) \), for any \( \varepsilon > 0 \), there exists a compact set \( K \subset X \) so that

\[
\sup_{n \in \mathbb{N}} \nu_n(K^c) < \varepsilon.
\]

Thus, by (5.5), for any \( \delta > 0 \), there exists a compact set \( K \subset X \) so that

\[
\sup_{n \in \mathbb{N}} \left| \int_{X_n} \mathcal{P}_k^n \, d\nu_n - \int_K \mathcal{P}_k^n \, d\nu_n \right| = \sup_{n \in \mathbb{N}} \left| \int_{X_n} \mathcal{P}_k^n(1_{X_n} - 1_{K \cap X_n}) \, d\nu_n \right|
\leq \sup_{n \in \mathbb{N}} \|\mathcal{P}_k^n\|_{2,n}^{1/2} \nu_n(K^c)
\leq \left( \prod_{i=1}^k \|f_i\|_{2,n} \right) \sup_{n \in \mathbb{N}} \nu_n(K^c) < \delta.
\]

(5.10)
We note that $\sup_{n \in \mathbb{N}} \prod_{i=1}^{k} \| f_{i} \|_{2,n} < \infty$ because $f_{1}, f_{2}, ..., f_{k} \in C_{b}(X) \cap L^{2}(X; m_{\infty})$ and $X_{n}$ converges to $X_{\infty}$ in the pmG sense. Take $r > 0$ so that $K \subset B_{r}(\xi_{n}) := \{ x \in X : d(\xi_{n}, x) < r \}$. Let $\tilde{1}_{r}^{T}$ denote the following function: ($r < R$)

$$\tilde{1}_{r}^{T}(x) = \begin{cases} 1 & x \in B_{r}(\xi_{n}), \\ 1 - \frac{d(x, B_{r}(\xi_{n}))}{R - r} & x \in B_{R}(\xi_{n}) \setminus B_{r}(\xi_{n}), \\ 0 & \text{o.w.} \end{cases}$$

Then $\tilde{1}_{r}^{T} \in C_{b}(X)$. Thus, by Theorem 4.4, 4.8 and (5.10), for any $\delta > 0$, there exists $r > 0$ so that

$$\left| \mathbb{E}^{\nu_{0}}[f_{1}(S_{t_{1}}^{n}) \cdots f_{k}(S_{t_{k}}^{n})] - \mathbb{E}^{\nu_{\infty}}[f_{1}(S_{t_{1}}^{\infty}) \cdots f_{k}(S_{t_{k}}^{\infty})] \right|$$

$$= \left| \int_{X_{n}} P_{t_{k}}^{n} d\nu_{n} - \int_{X_{\infty}} P_{t_{k}}^{\infty} d\nu_{\infty} \right|$$

$$= \left| \int_{X_{n}} P_{t_{k}}^{n} d\nu_{n} - \int_{X_{n}} \tilde{1}_{r}^{T} P_{t_{k}}^{n} d\nu_{n} + \int_{X_{n}} \tilde{1}_{r}^{T} P_{t_{k}}^{n} \phi_{n} d\nu_{n} - \int_{X_{n}} \tilde{1}_{r}^{T} P_{t_{k}}^{\infty} \phi_{n} d\nu_{\infty} \right|$$

$$+ \int_{X_{n}} \tilde{1}_{r}^{T} P_{t_{k}}^{\infty} d\nu_{\infty} - \int_{X_{\infty}} P_{t_{k}}^{\infty} d\nu_{\infty} \right|$$

$$\leq \delta + \left| \int_{X} \tilde{1}_{r}^{T} P_{t_{k}}^{n} \phi_{n} d\nu_{n} - \int_{X} \tilde{1}_{r}^{T} P_{t_{k}}^{\infty} \phi_{\infty} d\nu_{\infty} \right| + \delta$$

Here, in the fifth line above, in the first $\delta$, we used (5.10) and in the second $\delta$, we used the tightness of the single measure $m_{\infty}$. The middle term in the fifth line converges to zero because of the $L^{2}$-strong convergence of the semigroup $T_{t}^{n}$ by Theorem 4.4 and Theorem 4.8, and $L^{2}$-weak convergence of $\phi_{n}$ to $\phi_{\infty}$. Thus, we have completed the proof.

6 Tightness

In this section, we investigate the tightness of the diffusion processes associated with $(\mathcal{E}_{n}, \mathcal{F}_{n})$. According to [36] and [41], we have a decomposition of additive functionals for non-symmetric forms, which is called Lyons-Zheng decomposition. Suppose Assumption 3.1. Let $\mathcal{S} = (\Omega, \{ \mathcal{M}_{t} \}_{t \geq 0}, \{ \mathcal{S}_{t} \}_{t \geq 0}, \{ \mathbb{P}^{x} \}_{x \in X})$ be a diffusion process on $\Omega = D([0, \infty); X_{0})$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ corresponding to (1.3). We take $S_{t}(\omega) = \omega(t)$ as a coordinate process for $\omega \in \Omega$. Let $\hat{\mathcal{S}} = (\Omega, \{ \mathcal{M}_{t} \}_{t \geq 0}, \{ \mathcal{S}_{t} \}_{t \geq 0}, \{ \hat{\mathbb{P}}^{x} \}_{x \in X})$ be a dual process associated with the dual form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$. Let $r_{t}$ be a time reversal operator defined as follows:

$$r_{t}(\omega)(t) = \begin{cases} \omega((s - t) -) & 0 \leq t \leq s < \infty, \\ \omega(0) & s < t. \end{cases}$$

Here $\omega(t-) := \lim_{s \uparrow t} \omega(s)$. Since $(\mathcal{E}, \mathcal{F})$ is local, the corresponding processes are diffusive and jump only to the cemetery point $\hat{\delta}$. Thus, we may omit to write $\omega(t-)$ before lifetime and simply write $\omega(t)$. By the Fukushima decomposition, we have that for $f \in \mathcal{F}$

$$f(S_{t}) - f(S_{0}) = M_{t}^{[f]} - N_{t}^{[f]}, \quad \mathbb{P}^{x} \text{-a.e., q.e. } x,$$

(6.1)
whereby $M^{[f]}$ is a martingale and $N^{[f]}$ is a zero-energy process. Here we mean by zero-energy $e(N^{[f]}) = 0$, in which the energy of $N^{[f]}$ (generally, the energy of additive functionals) is defined as follows:

$$e(N^{[f]}) := \lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t}(N^t) f dt \right].$$

Similarly, we have that for $f \in \mathcal{F}$,

$$f(S_t) - f(S_0) = \hat{M}^{[f]}_t - \hat{N}^{[f]}_t, \quad \mathbb{P}_x\text{-a.e., q.e. } x. \quad (6.2)$$

Let $\zeta_\mathcal{S}$ and $\zeta_\hat{\mathcal{S}}$ be lifetimes for $\mathcal{S}$ and $\hat{\mathcal{S}}$ respectively. Let $\zeta := \min\{\zeta_\mathcal{S}, \zeta_\hat{\mathcal{S}}\}$. We note that, on $\{\zeta > T\}$, we have that, for an $\mathcal{M}_T$-measurable function $\mathcal{F}$,

$$\hat{\mathbb{E}}^m(F(r_T \omega)) = \mathbb{E}^m(F(\omega)).$$

By [41], for $f \in \mathcal{F}$, we have that on $\{\zeta > T\}$,

$$\tilde{f}(S_t) - \tilde{f}(S_0) = \frac{1}{2} M^{[f]}_t - \frac{1}{2} (\hat{M}^{[f]}_t - r_T) + \frac{1}{2} (\hat{N}^{[f]}_t - \hat{N}^{[f]}_0), \quad (6.3)$$

for $0 \leq t \leq T$ $\mathbb{P}^m$-a.e. Here $\tilde{f}$ means a quasi-continuous modification of $f$.

Now we estimate $\frac{1}{2}(N^{[u]}_t - \hat{N}^{[u]}_t)$.

**Lemma 6.1** Suppose Assumption 3.1, $|A|, |b_1|, |b_2|, c \in L^\infty(X, m)$, $b_i \in \text{Div}^2(X, d, m)$ ($i = 1, 2$) and symmetry of $A$. For $f \in \mathcal{F}$ and $t \geq 0$, it holds that on $\{\zeta > T\}$,

$$\hat{N}^{[f]}_t - N^{[f]}_t = \int_0^t \left(2b_1(f) - 2b_2(f) - f \text{div} b_1 + f \text{div} b_2 \right)(S_s)ds,$$

for $0 \leq t \leq T$ $\mathbb{P}^m$-a.e.

**Proof.** First we prove the statement for $f \in \mathcal{D}(\hat{L})$, whereby $\hat{L}$ denotes the generator associated with $(\hat{\mathcal{E}}, \mathcal{F})$ and $\mathcal{D}(\hat{L})$ denotes the domain of $\hat{L}$. In this case, we have $\hat{N}^{[f]}_t = \int_0^t \hat{L} f(X_s)ds$ and thus, we see $\hat{N}^{[f]}_t(r_i) = \hat{N}^{[f]}_t$. Then for $f \in \mathcal{F}$, we have on $\{\zeta > T\}$,

$$\mathbb{E}^m[\hat{N}^{[f]}_t] = \hat{\mathbb{E}}^m[\hat{N}^{[f]}_t(r_i) \hat{g}(S_t)]$$

$$= \hat{\mathbb{E}}^m[\hat{N}^{[f]}_t \hat{g}(S_t)]$$

$$= \hat{\mathbb{E}}^m[\hat{N}^{[f]}_t \hat{g}(S_t)] + \hat{\mathbb{E}}^m[\hat{N}^{[f]}_t (\hat{g}(S_t) - \hat{g}(S_0))], \quad \forall g \in \mathcal{F},$$

whereby $\hat{g}$ denotes a quasi-continuous modification of $g$. We have

$$\alpha^2 \left| \hat{\mathbb{E}}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}^{[f]}_t \hat{g}(S_t) - \hat{g}(S_0) \right] \right|$$

$$\leq \left( \alpha^2 \hat{\mathbb{E}}^m \left[ \int_0^\infty e^{-\alpha t} (\hat{N}^{[f]}_t)^2 dt \right] \right)^{1/2} \left( \alpha^2 \hat{\mathbb{E}}^m \left[ \int_0^\infty e^{-\alpha t} (\hat{g}(X_t) - \hat{g}(X_0))^2 dt \right] \right)^{1/2} \alpha \to \infty \hat{e}(\hat{N}^{[f]}_t)^{1/2} \hat{e}(\hat{g}(X_t) - \hat{g}(X_0))^{1/2} = 0.$$
Therefore, by [29, Theorem 5.3.1], we have
\[
\lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}_t^{|f|} dt \right]
= \lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}_t^{|f|} \tilde{g}(S_t) dt \right]
= \lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}_t^{|f|} \tilde{g}(S_t) dt \right] + \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}_t^{|f|} (\tilde{g}(S_t) - \tilde{g}(S_0)) \right]
= \lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}_t^{|f|} dt \right]
= -\hat{E}(f, g).
\]

Since it holds that (recall symmetry of \(A\))
\[
\hat{E}(f, g) + \int_X (2b_1(f) - 2b_2(f) - f \text{div} b_1 + f \text{div} b_2) g dm = \mathcal{E}(f, g),
\]
by [29, Theorem 5.3.1], we obtain
\[
\lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}_t^{|f|} dt \right] - \int_0^t \left( 2b_1(f) - 2b_2(f) - f \text{div} b_1 + f \text{div} b_2 \right) (S_s) ds dt
= \lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} \hat{N}_t^{|f|} dt \right].
\]
(6.4)

On the other hand, we can calculate the energy of \(\hat{N}_t^{|f|}\) as follows:
\[
e(\hat{N}_t^{|f|}) = \lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} (\hat{N}_t^{|f|})^2 dt \right]
= \lim_{\alpha \to \infty} \alpha^2 \mathbb{E}^m \left[ \int_0^\infty e^{-\alpha t} (\hat{N}_t^{|f|})^2 dt \right]
= \hat{e}(\hat{N}_t^{|f|})
= 0.
\]
(6.5)

Thus, \(\hat{N}_t^{|f|} - \int_0^t (2b_1(f) - 2b_2(f) - f \text{div} b_1 + f \text{div} b_2) (S_s) ds\) is an additive functional of \(S\) with zero energy. By (6.4), (6.5), and [29, Theorem 5.3.1], we have the desired result for \(f \in \mathcal{D}(\hat{A})\).

For general \(f \in \mathcal{F}\), we can take a sequence \(f_n \in \mathcal{D}(\hat{L})\) so that \(f_n\) converges to \(f\) with respect to the norm of the symmetric part \(\hat{E}_1\) and for q.e. \(x\) ([29, Theorem 5.1.3]),
\[
\hat{\pi}^x[\hat{\Gamma}_T] = 1,
\]
where
\[
\hat{\Gamma}_T = \{ \omega \in \Omega : \hat{N}_{t_0}^{|f_n|}(\omega) \text{ converges to } \hat{N}_t^{|f|}(\omega) \text{ uniformly in } t \text{ on } [0, T] \}.
\]
Since we have that on \(\{ \zeta > T \}\),
\[
\hat{N}_t^{|f_n|}(r_T \omega) = \int_{T-t}^T \hat{L} f_n(S_s(\omega)) ds = \hat{N}_T^{|f_n|}(\omega) - \hat{N}_{T-1}^{|f_n|}(\omega),
\]
(6.6)
the set $\hat{\Gamma}_T$ is $r_T$-invariant, i.e., \( \{r_T\omega \in \hat{\Gamma}_T\} = \hat{\Gamma}_T \). Therefore, the complement $\hat{\Gamma}_T^c$ of $\hat{\Gamma}_T$ is also $r_T$-invariant. Thus, we obtain
\[
P^m(\hat{\Gamma}_T^c) = \hat{P}^m(r_T\omega \in \hat{\Gamma}_T^c) = \hat{P}^m(\hat{\Gamma}_T^c) = 0.
\]
Therefore, we can conclude the desired result for general $f \in \mathcal{F}$.

By the previous lemma, by easy calculation, we have that, for $f \in \mathcal{F}$, on $\{\zeta > T\}$,
\[
\bar{f}(S_n) - \bar{f}(S_0) = \frac{1}{2}M_t^{[f]} - \frac{1}{2}(\dot{M}_T^{[f]} - \dot{M}_{T-t}(r_T)) - \int_0^t \left( b_1(f) - b_2(f) - \frac{1}{2}f\text{div}b_1 + \frac{1}{2}f\text{div}b_2 \right)(S_n)ds,
\]
for $0 \leq t \leq T$ $\mathbb{P}^m$-a.e.

**Lemma 6.2** Under Assumption 1.1, $\{S^n\} n \in \mathbb{N}$ and $\{\hat{S}^\nu, 1_{\{\zeta > T\}} \} n \in \mathbb{N}$ are tight in $P_{\leq 1}(C([0,T];X))$ for any $T > 0$.

**Proof.** We only show the tightness of $\{S^n\} n \in \mathbb{N}$ since the proof for the dual processes is the same. Let us denote the law of $h(B^n)$ for $h \in \text{Lip}_{bs}(X)$ as follows:
\[
S^{\nu, h} = (h(S^n), \mathbb{P}^\nu).
\]
Here we set $h(\partial) = 0$. It is easy to show that $\text{Lip}_{bs}(X)$ strongly separates points in $C_b(X)$, that is, for every $x$ and $\epsilon > 0$, there exists a finite set $\{h_i\}_{i=1}^k \subset \text{Lip}_{bs}(X)$ so that
\[
\inf_{y : d(y,x) \geq \epsilon} \max_{1 \leq i \leq k} |h_i(x) - h_i(y)| > 0.
\]
By [12, Theorem 3.9.1, Corollary 3.9.2] (we can apply these statements also to the space $P_{\leq 1}(C([0,T];X))$ of sub-probability measures) and Lemma 5.3, the following (i) follows from (ii): For any $T > 0$,
\[
\begin{align*}
(i) \quad & \{S^n\} n \in \mathbb{N} \text{ is tight in } P_{\leq 1}(C([0,T];X)); \\
(ii) \quad & \{S^{\nu, h}, 1_{\{\zeta > T\}} \} n \in \mathbb{N} \text{ is tight in } P_{\leq 1}(C([0,T];\mathbb{R})) \text{ for } \forall h \in \text{Lip}_{bs}(X).
\end{align*}
\]
In fact, we can show the compact containment condition [12, (9.1) in Theorem 3.9.1] according to (ii) and Lemma 5.3 by a proof similar to [12, Corollary 3.9.2]. We note that, although [12, Theorem 3.9.1, Corollary 3.9.2] gives sufficient conditions for tightness only in the càdlàg space $D([0,T];X)$, since the laws of each diffusions $S^n_\circ$ and $S^{\nu, \circ}$ have their support on the space of continuous paths $C([0,T];X)$ before lifetime because of the locality of $(\mathcal{E}_n, \mathcal{F}_n)$, the tightness in $D([0,T];X)$ implies the tightness in $C([0,T];X)$. See, e.g., [13, Lemma 5 in Appendix] for this point. Thus, we will show that (ii) holds, i.e., for any $T$ and any $h \in \text{Lip}_{bs}(X)$,
\[
\{S^{\nu, h}, 1_{\{\zeta > T\}} \} n \in \mathbb{N} \quad \text{is tight in } P_{\leq 1}(C([0,T];X)).
\]
By the equality (6.7), we have that, on \( T >\) for any \( \varepsilon > 0 \), there exists \( R > 0 \) so that

\[
\int_{X_n} L_{n,T}^{h,n} \nu_n(dx) = \| \phi_n 1_{B_R(x_n)} \|_{\infty} \int_{X_n} L_{n,T}^{h,n} 1_{B_R(x_n)} dm_n + \nu_n(B_R^n(x_n)) < \| \phi_n 1_{B_R(x_n)} \|_{\infty} \int_{X_n} L_{n,T}^{h,n} 1_{B_R(x_n)} dm_n + \varepsilon.
\]

Let \( m_n,R := 1_{B_R(x_n)} m_n \). We have

\[
\int_{X_n} L_{n,T}^{h,n} dm_{n,R} = \mathbb{P}_n^{m,R}(\sup_{0 \leq s,t \leq T, |t-s| \leq \eta} |h(S^n_t) - h(S^n_s)| > \delta : \{ \zeta^n > T \})
\]

(6.10)

\[
:= (1)_{n,\eta}.
\]

It suffices to show that, for any \( T, R > 0 \),

\[
\sup_{n \in \mathbb{N}} (1)_{n,\eta} \to 0, \quad \eta \to 0.
\]

By the equality (6.7), we have that, on \( \{ \zeta^n > T \} \),

\[
\begin{align*}
\frac{1}{2} M_{t}^{[h],n} - \frac{1}{2} (\hat{M}_{t}^{[h],n} - \hat{M}_{t-s}^{[h],n} (r_T)) &
\quad & - \int_{0}^{t} \left( b_1^n(h) - b_2^n(h) - \frac{1}{2} \text{div} b_1^n + \frac{1}{2} \text{div} b_2^n \right) (S^n_s) ds,
\end{align*}
\]

(6.11)

for \( 0 \leq t \leq T \) \( \mathbb{P}^m \)-a.e. Thus, we have

\[
(1)_{n,\eta} = \mathbb{P}^{m,n,R}(\sup_{0 \leq s,t \leq T, |t-s| \leq \eta} |h(S^n_t) - h(S^n_s)| > \delta : \{ \zeta^n > T \})
\]

(6.12)

\[
\leq \mathbb{P}^{m,n,R}(\sup_{0 \leq s,t \leq T, |t-s| \leq \eta} |M_{t}^{[h],n} - M_{t-s}^{[h],n}| > \delta : \{ \zeta^n > T \})
\]

\[
+ \mathbb{P}^{m,n,R}(\sup_{0 \leq s,t \leq T, |t-s| \leq \eta} \hat{M}_{t-s}^{[h],n} (r_T) - \hat{M}_{t-s}^{[h],n} (r_T) > \delta : \{ \zeta^n > T \})
\]

\[
+ \mathbb{P}^{m,n,R}(\sup_{0 \leq s,t \leq T, |t-s| \leq \eta} |\int_{s}^{t} \left( b_1^n(h) - b_2^n(h) - \frac{1}{2} \text{div} b_1^n + \frac{1}{2} \text{div} b_2^n \right) (S^n_l) dl | > \delta : \{ \zeta^n > T \}.
\]

(6.12)
Noting that \( \hat{E}^m(F(r_T \omega)) = E^m(F(\omega)) \) for an \( \mathcal{M}_T \)-measurable function \( F \) on \( \{\zeta^n > T\} \), we have

\[
\text{(R.H.S. of (6.12))} = \mathbb{P}^{m,R}( \sup_{0 \leq s,t \leq T} |M_t^{[h],n} - M_s^{[h],n}| > \delta : \{\zeta^n > T\})
\]

\[
+ \hat{\mathbb{P}}^{m,R}( \sup_{0 \leq s,t \leq T} |\hat{M}_t^{[h],n} - \hat{M}_s^{[h],n}| > \delta : \{\zeta^n > T\})
\]

\[
+ \mathbb{P}^{m,R}( \sup_{0 \leq s,t \leq T} \left| \int_s^t (b_2^n(h) - b_1^n(h) - \frac{1}{2} b_{1}^n + \frac{1}{2} b_{1}^n) (S_t dl) \right| > \delta : \{\zeta^n > T\})
\]

\[
\leq \mathbb{P}^{m,R}( \sup_{0 \leq s,t \leq T} |M_t^{[h],n} - M_s^{[h],n}| > \delta : \{\zeta^n > T\})
\]

\[
+ \hat{\mathbb{P}}^{m,R}( \sup_{0 \leq s,t \leq T} |\hat{M}_t^{[h],n} - \hat{M}_s^{[h],n}| > \delta : \{\zeta^n > T\})
\]

\[
+ \mathbb{P}^{m,R}\left( (\|b_1^n\|_{\infty} + \|b_2^n\|_{\infty} + \|\text{div} b_1^n\|_{\infty} + \|\text{div} b_2^n\|_{\infty}) \eta > \delta : \{\zeta^n > T\}\right). \tag{6.13}
\]

We first estimate the martingale part. Since \( M_t^{[h],n} \) is a continuous martingale, by the martingale representation theorem, there exists a one-dimensional Brownian motion \( B^n(t) \) on an extended probability space \((\tilde{\Omega}, \tilde{\mathcal{M}}, \tilde{\mathbb{P}}^x)\), whereby \( M_t^{[h],n} \) is represented as a time-changed Brownian motion with respect to the quadratic variation \( \tilde{\mathbb{P}}^x_t \)-a.s. q.e. \( x \in X_n \) (see, e.g., Ikeda–Watanabe [22, Chapter II Theorem 7.3]). That is, for q.e. \( x \in X_n \),

\[
M_t^{[h],n} = B^n((M_t^{[h],n})_t) = B^n\left( \int_0^t \frac{d\mu^n(h)}{dm^n} (S^n_u) du \right) = B^n\left( \int_0^t (A_n \nabla h, \nabla h)(S^n_u) du \right) \quad \tilde{\mathbb{P}}^x_t\text{-a.s.}
\tag{6.14}
\]

Here \( \mu^n_t \) means the energy measure associated with \( Ch_n \): \( Ch_n(f) = \int_{X_n} \mu^n_t(dx) \) for \( f \in F^n \).

Since \( |\nabla h| \leq \text{Lip}(h) \), we have

\[
\{ \omega \in \tilde{\Omega} : \sup_{0 \leq s,t \leq T} |M_t^{[h],n} - M_s^{[h],n}| > \delta \}
\]

\[
\subset \{ \omega \in \tilde{\Omega} : \sup_{0 \leq s,t \leq T} |B^n\left( \int_0^t \|A_n\|_{\infty} |\nabla h|^2(S^n_u) du \right) - B^n\left( \int_0^s \|A_n\|_{\infty} |\nabla h|^2(S^n_u) du \right) | > \delta \}
\]

\[
\subset \{ \omega \in \tilde{\Omega} : \sup_{0 \leq s,t \leq \|A_n\|_{\infty} \text{Lip}(h)^2 T} \|B^n(t) - B^n(s)\| > \delta \}.
\]

Let \( \mathbb{W} \) be the standard Wiener measure on \( C([0, \infty); \mathbb{R}) \). Let

\[
\theta(\eta, h) := \mathbb{W}_n\left( \sup_{0 \leq s,t \leq \|A_n\|_{\infty} \text{Lip}(h)^2 T} |\omega(t) - \omega(s)| > \delta \right).
\]
By (6.13) and noting $\sup_{n \in \mathbb{N}} m_n(B_R(\overline{x}_n)) < \infty$ because of the weak convergence of $m_n$, we have, for any $T > 0$,

$$
\sup_{n \in \mathbb{N}} \mathbb{P}^{m_{n,R}} \left( \sup_{0 \leq s,t \leq T} \left| M_t^{[h],n} - M_s^{[h],n} \right| > \delta : \{ \zeta^n > T \} \right) < \infty.
$$

Provided that

$$
\eta \sup_{n \in \mathbb{N}} (\| b^n_1 \|_{\infty} + \| b^n_2 \|_{\infty} + \| \text{div} b^n_1 \|_{\infty} + \| \text{div} b^n_2 \|_{\infty}) \rightarrow 0. \quad (6.15)
$$

The dual martingale part can be estimated in a similar way, so, we omit the proof. For the remaining part, we can see that the following uniform estimate in $n$:

$$
\sup_{n \in \mathbb{N}} \mathbb{P}^{m_{n,R}} \left( \| b^n_1 \|_{\infty} + \| b^n_2 \|_{\infty} + \| \text{div} b^n_1 \|_{\infty} + \| \text{div} b^n_2 \|_{\infty} \right) \rightarrow 0,
$$

provided that

$$
\eta \sup_{n \in \mathbb{N}} (\| b^n_1 \|_{\infty} + \| b^n_2 \|_{\infty} + \| \text{div} b^n_1 \|_{\infty} + \| \text{div} b^n_2 \|_{\infty}) < \delta.
$$

Thus, we obtain $\sup_{n \in \mathbb{N}} (I)_{n,\eta} \rightarrow 0$ as $\eta \rightarrow 0$. We have finished the proof. \hfill \Box

**Lemma 6.3** Suppose the conditions assumed in Theorem 1.3. Then $\{S_T^n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([0, \infty), X))$.

**Proof.** Since $\overline{x}_n$ converges to $\overline{x}_{\infty}$ in $(X,d)$, the laws of the initial distributions $\{S^n_0\}_{n \in \mathbb{N}} = \{\delta_{\overline{x}_n}\}_{n \in \mathbb{N}}$ are clearly tight in $\mathcal{P}(X)$. Thus, it suffices to show the following (see [9, Theorem 12.3]): for each $T > 0$, there exist $\beta > 0$, $C > 0$ and $\theta > 1$ such that, for all $n \in \mathbb{N}$,

$$
\mathbb{E}^n[\tilde{d}^\beta(S^n_t, S^n_{t+h})] \leq C h^\theta, \quad \text{for every } 0 \leq t \leq T \text{ and } 0 \leq h \leq 1, \quad (6.16)
$$

whereby $\tilde{d}(x,y) := d(x,y) \wedge 1$. By the Markov property, we have

L.H.S. of (6.16)

$$
= \int_{X_n \times X_n} \int_{X_n} p_n(t,x_n,y)p_n(h,y,z) \tilde{d}^\beta(\iota_n(y),\iota_n(z)) m_n(dy) m_n(dz).
$$

By the generalized Bishop–Gromov inequality [11, Proposition 3.9], we have the following volume growth estimate: there exist positive constants $\nu = \nu(N,K,D) > 0$ and $c = c(N,K,D) > 0$ such that, for all $n \in \mathbb{N}$

$$
m_n(B_r(x)) \geq cr^{2\nu} \quad (0 \leq r \leq 1 \wedge D). \quad (6.18)
$$

On the other hand, the volume doubling property ([33]) and the Poincaré inequality ([20, 31, 32]) hold under $\text{RCD}^*(K,N)$ condition. According to [25, Theorem 5.4] and (6.18), we
have that there exist positive constants \( C_1 \) and \( C_2 \) depending only on \( N, K, D, \sup_{n \in \mathbb{N}} \{ ||A_n||_\infty + \|b^1_n\|_\infty + \|b^2_n\|_\infty + \|c_n\|_\infty \} \) and \( T \) so that
\[
p(t, x, y) \leq \frac{C_1}{t^\nu} \exp\left\{ -C_2 \frac{d(x, y)^2}{t} \right\},
\] (6.19)
for all \( x, y \in X \) and \( 0 < t \leq D^2 \). Thus, we have
\[
\int_{X_n} p_n(s, y, z) d^3(\tau_n(y), \tau_n(z)) m_n(dz)
\]
\[
\leq \frac{C_1}{c^\nu} \int_{X_n} \exp\left( -C_2 \frac{d_n(y, z)^2}{s} \right) d^3(\tau_n(y), \tau_n(z)) m_n(dz)
\]
\[
\leq \frac{C_1}{c^\nu} \int_{X_n} \exp\left( -C_2 \frac{d_n(y, z)^2}{s} \right) d^3_n(y, z) m_n(dz)
\]
\[
\leq C_1 c^{-1} C_2^{2/\beta} C_3 s^{\beta/2-\nu} m_n(X_n) \sup_{y, z \in X_n} \left\{ \left( C_2 \frac{d_n(y, z)^2}{s} \right)^{\beta/2} \exp\left( -C_2 \frac{d_n(y, z)^2}{s} \right) \right\}
\]
\[
\leq C_1 c^{-1} C_2^{2/\beta} M_\beta s^{\beta/2-\nu}
\]
\[
= C_4 s^{\beta/2-\nu}.
\] (6.20)

By (6.20), we have
\[
\text{R.H.S. of (6.17)} \leq C_4 h^{\beta/2-\nu} \int_{X_n} p_n(t, x_n, y) m_n(dy)
\]
\[
\leq C_4 h^{\beta/2-\nu}.
\] (6.21)

Thus, we finish the proof by taking \( \beta > 0 \) such that \( \beta/2 - \nu > 1 \), and set \( \theta = \beta/2 - \nu \). \( \square \)

By Lemma 5.3, 6.2, we can finish the proof of Theorem 1.2. By Lemma 5.1, 6.3, we finish the proof of Theorem 1.3.

## 7 Conservativeness

In this section, under Assumption 3.1 with \( \alpha_0 = 0 \), we give a criterion for the conservativeness of \((E, F)\) and \((\bar{E}, \bar{F})\). In the case of finite mass \( m(X) < \infty \), if
\[
\text{div} b_i = c, \quad i = 1, 2,
\]
then it is easy to check the conservativeness since \( 1 \in F \) and \( E(1, g) = 0 \) for any \( g \in F \) (see e.g., [29, Theorem 5.6.1]). We focus only on the case of RCD\(^*(K, N)\) with infinite mass \( m(X) = \infty \).

Let \((X, d, m, F)\) be an RCD\(^*(K, N)\) space. Note that \((X, d, m)\) becomes locally compact because of the RCD\(^*(K, N)\) condition. Recall that \( \partial \) denotes a cemetery point jointed to \( X \) as one-point compactification. Let \( A := \{ \rho \in F_{\text{loc}} \cap C(X) : \lim_{x \to \partial} \rho(x) = \infty, \{ x \in X : \rho(x) \leq r \} \text{ is compact for any } r > 0 \} \). Let \( B^\rho_\theta := \{ x \in X : \rho(x) \leq r \} \) and \( M^\rho(r) := \text{ess-sup}_{x \in B^\rho_\theta} \langle A \nabla \rho, \nabla \rho \rangle(x) \).
Proposition 7.1 Let $(X, d, m, \mathcal{F})$ be an RCD$^*$ $(K, N)$ space. Suppose Assumption 3.1 and
\[
\text{div} b_i = c, \quad \text{for } i = 1, 2. \tag{7.1}
\]
Assume that there exists $\rho \in A$ so that, for any $R > 0$,
\[
\lim_{r \to \infty} \frac{m(B^\rho_{R+r}) \text{Erfc} \left( \frac{r}{\sqrt{M^\rho(R+r)}T} \right)}{\rho} = 0, \tag{7.2}
\]
whereby \( \text{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy \) and there exists a constant $c > 0$ so that with the above $\rho$, it holds that
\[
|b_1 - b_2| \sqrt{\rho} |1_{B^\rho_r}| \leq c(1 + r) \quad \text{m.-a.e., for any } r > 0. \tag{7.3}
\]
Then the form $(\mathcal{E}, \mathcal{F})$ and the dual form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ are conservative.

Proof of Proposition 7.1. The idea of the proof is similar to the case of Euclidean diffusions discussed in [37, Section 4]. We only prove the statement for $(\mathcal{E}, \mathcal{F})$ since the dual case can be shown in the same proof. Let us write $m_R = m1_{B^\rho_R}$. Let $\{\{S_t\}_{t \geq 0}, \mathbb{P}^x_t\}$ denote a part process on $B_t(\mathcal{F})$ of $\{\{S_t\}_{t \geq 0}, \mathbb{P}^x\}$ with $x \in B^\rho_0$, which is a stopped process when it hits the boundary $\partial B_t(\mathcal{F})$. Let $T > 0$. If $S_0 \in B^\rho_R$, then, by the locality of $(\mathcal{E}, \mathcal{F})$,
\[
E_r = \{ \sup_{t \in [0, T]} (\rho(S_t) - \rho(S_0)) \geq r \} = \{ \sup_{t \in [0, T], t < \tau_{R+r}} (\rho(S_t) - \rho(S_0)) \geq r \} \quad \text{under } S_0 \in B^\rho_R.
\]
Here $\tau_{R+r} = \inf\{t \geq 0 : S_t \in \partial B^\rho_{R+r}\}$. Thus, we have $\mathbb{P}_m(E_r) = \mathbb{P}_{m_R}(E_r)$. By (7.1), the form $\mathcal{E}$ has no killing term, which implies that the corresponding process has no inside killing. Therefore,
\[
\mathbb{P}_m \left( \sup_{t \in [0, T]} (\rho(S_t) - \rho(S_0)) = \infty \right) = \lim_{r \to \infty} \mathbb{P}_m(E_r) = \lim_{r \to \infty} \mathbb{P}_{m_R}(E_r) = \lim_{r \to \infty} \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} (\rho(S_t) - \rho(S_0)) \geq r).
\]
The goal for the proof is to show
\[
\lim_{r \to \infty} \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} (\rho(S_t) - \rho(S_0)) \geq r) = 0.
\]
It is easy to check that the function $\rho_{R+r} := ((\rho - (R + r)) \land 0) + R + r$ belongs to $\mathcal{F}_{B^\rho_R}$, and $\rho_{R+r} = \rho$ on $B^\rho_{R+r}$. Thus, by (6.3), we have
\[
\mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} (\rho(S_t) - \rho(S_0)) \geq r)
\]
\[
\leq \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} \frac{1}{2} M_t^\rho \geq \frac{r}{4}) + \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} -\frac{1}{2} \tilde{M}_t^\rho (r_T) \geq \frac{r}{4}) + \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} -\frac{1}{2} (N_t^\rho - \tilde{N}_t^\rho) \geq \frac{r}{4}).
\]
\[
\leq \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} \frac{1}{2} M_t^\rho \geq \frac{r}{4}) + \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} \frac{1}{2} \tilde{M}_t^\rho (r_T) \geq \frac{r}{4}) + \mathbb{P}_{m_R}(\sup_{t \in [0, T], t < \tau_{R+r}} \frac{1}{2} (N_t^\rho - \tilde{N}_t^\rho) \geq \frac{r}{4}).
\]
By using (7.2), the martingale parts go to zero as $r \to \infty$ in a similar way to [15, §5.7], so we omit the proof. We just need to estimate the zero-energy parts. By the equality (6.7) and (7.1), we have
\[
\frac{1}{2}(N_t^\rho - \tilde{N}_t^\rho) = \int_0^t (b_1 - b_2)(\rho)(S_s)\,ds.
\]
Therefore, by (7.3), we have
\[
P_{mR}^{R+r} \left( \sup_{t \in [0,T], t < \tau_{R+r}} \frac{1}{2}(N_t^\rho - \tilde{N}_t^\rho) \geq \frac{r}{4} \right)
\]
\[
= P_{R+r}^{R+r} \left( \sup_{t \in [0,T], t < \tau_{R+r}} \int_0^t (b_1 - b_2)(\rho)(S_s)\,ds \geq \frac{r}{4} \right)
\]
\[
= P_{R+r}^{R+r} \left( |b_1 - b_2||\nabla \rho|T \geq \frac{r}{4} \right)
\]
\[
\leq P_{R+r}^{R+r} \left( c(1 + r + R)T \geq \frac{r}{4} \right)
\]
\[
\to 0 \text{ as } r \to \infty \text{ if } T < \frac{1}{4c}.
\]
Thus, the desired result is true for $T < \frac{1}{4c}$. By using the Markov property, we can extend the result for any $T \geq 0$. Thus, we have finished the proof.

As a corollary, we obtain that diffusion processes constructed in Proposition 3.2 are conservative if $\text{div} \, b_i = c$ for $i = 1, 2$.

**Corollary 7.2** Let $X$ be an RCD$^+(K, N)$ space and $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated with (1.3). Under Assumption 3.1 and $\text{div} \, b_i = c$ for $i = 1, 2$, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative.

**Proof.** By (1.2), we have the following volume growth estimate: there exist positive constants $c_1, c_2$ depending only on $K$ so that $m(B_r(\bar{y})) \leq c_1 e^{c_2 r^2}$. Take $\rho = d(\cdot, \bar{y})$. Then we obtain
\[
m(B_{R+r}^\rho) \text{ Erfc}(\frac{r}{\sqrt{M^\rho(R+r)T}}) \leq c_3 \exp\{c_2(R + r)^2\} \sqrt{M^\rho(R+r)T} \exp\{-\frac{r^2}{2M^\rho(R+r)T}\}.
\]
Here $c_3$ is a positive constant depending only on $K$. By $|\tilde{A}| \in L^\infty(X;m)$, it holds that $M^\rho(\cdot) \in L^\infty(X;m)$. Therefore, R.H.S. of (7.4) goes to zero as $r \to \infty$ and we obtain (7.2). The inequality (7.3) holds immediately because $|b_1 - b_2| \in L^\infty(X;m)$.

8 Examples

In this section, several specific examples satisfying Assumption 1.1 are given. There are various concrete examples of non-smooth metric measure spaces satisfying RCD conditions. See Ricci limit spaces (Sturm [33, 34], Lott–Villani [26], Cheeger–Colding [10, Example 8]), Alexandrov spaces (Petrunin, Zhang–Zhu [30, 43]), warped products and cones (Ketterer [23, 24]), quotient spaces (Galaz-García–Kell–Mondino–Sosa [16]) and infinite-dimensional spaces.
such as Hilbert spaces with log-concave measures (Ambrosio–Savaré–Zambotti [7]). Also, in [35, Section 4], the author explained various examples relating to the weak convergence of Brownian motions such as weighted Riemannian manifolds whose weighted Ricci curvature is bounded below, its pmG limit spaces, Alexandrov spaces, and Hilbert spaces with log-concave probability measures. Those examples are also available for this paper, and we omit the descriptions of those examples and we refer the reader to those references. What we discuss in this section is how to construct concrete examples of coefficients $A_n, b^n, c_n$ satisfying Assumption 1.1.

For any $f \in W^{1,2}(X, d, m)$, recall that we set a gradient derivation operator $b_f$ so that

$$b_f(g) := \langle \nabla f, \nabla g \rangle, \quad g \in W^{1,2}(X, d, m).$$

Then we can check that $b_f$ is an $L^2$-derivation and $|b_f| = |\nabla f|$ (see e.g., Gigli [17] for detail).

Recall a sufficient condition for the $L^2$-strong convergence for gradient derivations according to [8, Theorem 6.4].

**Theorem 8.1** ([8, Theorem 6.4]) Let $(X_n, d_n, m_n, \bar{\nu}_n)$ be a sequence of p.m.m. spaces with RCD$(K, \infty)$ condition. Assume that $(X_n, d_n, m_n, \bar{\nu}_n)$ converges to $(X_{\infty}, d_{\infty}, m_{\infty}, \bar{\nu}_{\infty})$ in the pmG sense. If $f_n \in W^{1,2}(m_n)$ converges strongly in $W^{1,2}$ to $f_{\infty} \in W^{1,2}(m_{\infty})$, then $b_{f_n}$ converges strongly in $L^2$ to $b_{f_{\infty}}$.

Using Theorem 8.1, we give an example of gradient derivations which is given by the resolvent of the Cheeger energy and satisfies Assumption 1.1, according to [8, Example 6.6].

**Example 8.2** (Derivation associated with resolvents) Let $(X_n, d_n, m_n, \bar{\nu}_n)$ be a sequence of p.m.m. spaces with RCD$(K, \infty)$ condition with $m_n(X_n) = 1$ or RCD$(K, N)$. Assume that $(X_n, d_n, m_n, \bar{\nu}_n)$ converges to $(X_{\infty}, d_{\infty}, m_{\infty}, \bar{\nu}_{\infty})$ in the pmG sense. Let $\{G_{\lambda}\}_{\lambda \geq 0}$ and $\{H^i\}_{i \geq 0}$ be the resolvent and the semigroup associated with Cheeger energy $\text{Ch}_n$. Let $h \in H^1(Q, \omega_{\alpha}(X_{\infty})$ with $h \geq 0$. Let $g^n_i \in L^{\infty}(m_n) \cap L^2(m_n)$ ($i = 1, 2$) satisfying $\sup_{n \in \mathbb{N}} \|g^n_i\|_{L^\infty} < \infty$ and $g^n_i$ converges to $g^n_{\infty} \in L^{\infty}(m_{\infty}) \cap L^2(m_{\infty})$ strongly in $L^2$ for $i = 1, 2$. Set $A_n := h|_{X_n} + a$ with a constant $a > 0$, thus

$$\langle A_n \nabla f, \nabla f \rangle = (h|_{X_n} + a)|\nabla f|^2,$$

whereby $\sim$ means the McShane extension of a function on $X_{\infty}$ to the whole space $X$ and $|X_n|$ denotes the restriction of functions to $X_n$. Let $f^n_i := G_{\lambda} g^n_i, b^n_i := b_{f^n_i}$ for $i = 1, 2$ and take $c_n \in L^{\infty}(m_n) \cap L^2(m_n)$ ($n \in \mathbb{N}$) converging in $L^2$ strongly to $c_{\infty}$ and $c_n \geq \max\{\text{div} b_{f^n_1}, \text{div} b_{f^n_2}\}$ for all $n \in \mathbb{N}$. Then Assumption 1.1 is satisfied.

**Proof of Example 8.2.** By [5, Theorem 5.7], it is easy to see that $A_n$ converges to $A_{\infty}$ in the sense of Definition 4.7. We only discuss $g^n_1$ and $f^n_1$, and write $g^n_i = g^n_{\lambda}$ and $f^n_i = f_{\lambda}$ for simplicity of notation in this paragraph. Since $g^n_{\lambda} \in L^{\infty}(m_n) \cap L^2(m_n)$ and $f_{\lambda} := G_{\lambda} g^n_{\lambda}$, we have that $H^n_{\lambda} g^n_{\lambda} \in \mathcal{D}(\Delta_n) \subset W^{1,2}(X, d, m) \cap L^{\infty}(m_n)$ where $\mathcal{D}(\Delta_n)$ denotes the domain of the infinitesimal generator associated with $\text{Ch}_n$. Moreover $G_{\lambda} g^n_{\lambda} \in \text{Lip}_b(X)$. In fact, since we know that, by [4, Theorem 6.5],

$$H^n_{\lambda} g^n_{\lambda} \in L^{\infty}(X, m_n), \quad \text{Lip}(H^n_{\lambda} g^n_{\lambda}) \leq \frac{\|g^n\|_{L^\infty}}{\sqrt{2t_2K(t)}}.$$
where \( I_K(t) := \int_0^t e^{Ks}ds \). Noting \( G^u_t g_n = \int_0^\infty e^{-\lambda t}H^n_t g_n \), we have that \( \|G^u_t g_n\|_\infty < \|g_n\|_\infty/\lambda \).

and

\[
|G^u_{\lambda} g_n(x) - G^u_{\lambda} g_n(y)| \leq \int_0^\infty e^{-\lambda t}|H^n_{\lambda} g_n(x) - H^n_{\lambda} g_n(y)|dt \\
\leq \int_0^\infty e^{-\lambda t} \frac{||g_n||_\infty}{\sqrt{2tK(t)}}d(x,y)dt \\
\leq \int_0^\infty e^{-\lambda t} \frac{||g_n||_\infty}{\sqrt{\int_0^t e^{2Ks}ds}}d(x,y)dt \\
\leq ||g_n||_\infty d(x,y) \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{\frac{2K(t)-1}{2}}}dt.
\]

Therefore, \( G^u_{\lambda} g_n \in \text{Lip}_p(X_n) \) and its Lipschitz constant is uniformly bounded in \( n \). Thus, \( \sup_{n \in \mathbb{N}} ||\mathbf{b}_{f_n}||_\infty = \sup_{n \in \mathbb{N}} ||\nabla f_n||_\infty = \sup_{n \in \mathbb{N}} ||\nabla G_{\lambda} g_n||_\infty < \infty \). Since \( \sup_{n \in \mathbb{N}} ||g_n||_\infty < \infty \) and \( g_n \to g_\infty \) in \( L^2 \) strongly, then \( \sup_{n \in \mathbb{N}} ||f_n||_\infty < \infty \), and by the result of the Mosco convergence of \( \text{Ch}_n \) to \( \text{Ch}_\infty \) [19, Theorem 6.8], we have that \( f_n \to f_\infty \) strongly in \( W^{1,2} \) (see also, [19, Corollary 6.10]). Therefore, by Theorem 8.1, the gradient derivation \( \mathbf{b}_{f_n} \) converges to \( \mathbf{b}_{f_\infty} \) strongly in \( L^2 \), and \( \sup_{n \in \mathbb{N}} ||\mathbf{b}_{f_n}||_\infty < \infty \). Since \( \text{div} \mathbf{b}_{f_n} = \Delta_n f_n = \lambda f_n - g_n \), we have \( \sup_{n \in \mathbb{N}} ||\text{div} \mathbf{b}_{f_n}||_\infty < \infty \) and \( \text{div} \mathbf{b}_{f_n} \to \text{div} \mathbf{b}_{f_\infty} \) strongly in \( L^2 \).

We give another example, which is given in terms of eigenfunctions of Laplacian according to [8, Example 6.7].

**Example 8.3 (Derivation associated with eigenfunctions of Laplacian)**

Let \( X_n = (X_n,d_n,m_n,\overline{\tau}_n) \) be an RCD\((K,\infty)\) space for all \( n \in \mathbb{N} \) converging to \( X_\infty = (X_\infty,d_\infty,m_\infty,\overline{\tau}_\infty) \) in the pmG sense. Let \( u_n \) be a normalized eigenfunction \( \int_{X_n} u_n^2dm_n = 1 \) of the generator \(-\Delta_n\) associated with \( \text{Ch}_n \) with \(-\Delta_nu_n = \lambda u_n \) for some \( \lambda \in \mathbb{R}_{\geq0} \). Assuming \( K > 0 \), or \( m_n(X_n) = 1 \), by [19, Proposition 6.7], we have that \(-\Delta_n\) has discrete spectra \( \{\lambda^k\}_{k=1}^\infty \) (non-decreasing order) with the eigenfunctions \( \{u^k_n\}_{k=1}^\infty \). By [19, Theorem 7.8], we know that \( \lambda^k_n \) converges to \( \lambda^k_\infty \), and \( u^k_n \) converges to \( u^k_\infty \) strongly in \( L^2 \) if the limit eigenvalue is simple (if not simple, we can extract a convergence subsequence). By \( \text{Ch}_n(u^k_n) = \lambda^k_n \to \lambda^k_\infty = \text{Ch}_\infty(u^k_\infty) \), it holds that \( u^k_n \) converges to \( u^k_\infty \) strongly in \( W^{1,2} \), which implies \( \mathbf{b}_{u^k_n} \) converges to \( \mathbf{b}_{u^k_\infty} \) strongly in \( L^2 \) by Theorem 8.1. On the other hand, the action of the heat semigroup \( H^n_t u^k_n \) is also the \( k\)-th eigenfunction since \( \Delta_n H^n_t u^k_n = H^n_t \Delta_n u^k_n = \lambda^k_n H^n_t u^k_n \). Since \( u^k_n \in W^{1,2}(X_n,d_n,m_n) \), by Lipschitz regularization of \( H^n_t \) for \( W^{1,2} \) elements (see e.g., [18, Theorem 4.3]), the proof is available also for the case without the condition of Alexandrov spaces), if \( \{H_t\}_{t \geq 0} \) is ultra-contractive and \( \|H^n_t\|_{1 \to \infty} \) is uniformly bounded from above in \( n \), we have that \( u^k_n \) has a Lipschitz representation \( \hat{u}^k_n \) and therefore we may assume that \( u^k_n \) is Lipschitz continuous and its Lipschitz constant is dominated by

\[
e^{-Kt} \sqrt{\|H^n_t\|_{1 \to \infty} \text{Ch}_n(u^k_n)} = e^{-Kt} \sqrt{\|H^n_t\|_{1 \to \infty} \lambda^k_n}.
\]

For instance, if \( X_n \) is RCD\((K,N)\) with \( \sup_{n \in \mathbb{N}} \text{diam}(X_n) < \infty \), or with \( \inf_{x \in X_n} m_n(B_r(x)) > 0 \) for any fixed \( r > 0 \), \( \{H_t\}_{t \geq 0} \) is ultra-contractive and \( \|H^n_t\|_{1 \to \infty} \) is uniformly bounded from above in \( n \). This is because the volume doubling property and the Poincaré inequality imply
the Gaussian heat kernel estimate whose constants only depend on the constants appearing in
the doubling and Poincaré inequalities, which yields the desired uniform ultra-contractivity.
Thus, we obtain
\[
\sup_{n \in \mathbb{N}} \|b_{u_n}^k\|_\infty = \sup_{n \in \mathbb{N}} \|\nabla u_n^k\|_\infty < e^{-Kt} \sup_{n \in \mathbb{N}} \sqrt{\|H^n_m\|_{1 \to \infty} \lambda_n^k} < \infty.
\]
Also we have
\[
\sup_{n \in \mathbb{N}} \|\text{div} b_{u_n}^k\|_\infty = \sup_{n \in \mathbb{N}} \|\Delta_n u_n^k\|_\infty < \sup_{n \in \mathbb{N}} \lambda_n^k < \infty.
\]
Take \(b_1^\infty = b_{u_n}^k\), \(b_2^\infty = b_{u_n'}^k\) and \(c_n \in L^\infty(m_n) \cap L^2(m_n) (n \in \mathbb{N})\) converging in \(L^2\) strongly
to \(c_\infty\) and satisfying \(c_n \geq \max\{\text{div} b_{u_n}^k, \text{div} b_{u_n'}^k\}\) for all \(n \in \mathbb{N}\). Thus, by taking \(A_n\) as in
Example 8.2, we have given an example satisfying Assumption 1.1.

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