On the basis of surface curvature-dependent terms in 6D

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Abstract. Total derivative terms play an important role in the integration of conformal anomaly. In four dimensional space $4D$ there is only one such term, namely $\Box R$. In the case of six dimensions $6D$ the structure of surface terms is more complicated, and it is useful to construct a basis of linear independent total derivative terms. We briefly review the general scheme of integrating anomaly and present the reduction of the minimal set of the surface terms in $6D$ from eight to seven. Furthermore, we discuss the comparison with the previously known equivalent reduction based on the general covariance and obtain it also from the conformal symmetry. Our results confirms that the anomaly induced effective action in $6D$ really has a qualitatively new (compared to previously elaborated $2D$ and $4D$ cases) ambiguity, which is parameterized by the two parameters $\xi_1$ and $\xi_2$.

Keywords: Total derivatives terms, Higher dimensions, Conformal anomaly

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1 Introduction

The integration of trace anomaly is the simplest way to derive the effective action (EA) of vacuum. The anomaly-induced action proved being a powerful tool due to the compact and useful form of the result, which finds many applications (see, e.g. [1] for the review). The integration of anomaly was originally done in the two dimensional space $2D$ in the
important work of Polyakov [2]. The generalization for 4D was done by Riegert [3] and Fradkin and Tseytlin [4]. There are interesting general features of anomaly, which can not be seen in 2D and can be merely noticed in 4D. The reason is that in 4D there is only one possible surface term $\Box R$ in the anomaly, while in 2D there are no such terms at all.

Things change dramatically in 6D, where we meet a bunch of the surface terms, which make integration of anomaly quite a challenging task. As we have discussed in the previous papers [3, 6], the number of possible covariant surface terms with the proper dimension coming from the derivatives of the metric may be larger [7], but it can be reduced to eight [6]. In the present contribution we show how this number can be reduced further to seven terms and discuss the relation of the corresponding identity to the diffeomorphism invariance from one side and with the conformal property of the Gauss-Bonnet term in 6D from another side. Indeed, the identity itself has been known previously [8], but in what follows we present its direct derivation and also show the relation to conformal transformation of the metric.

The paper is organized as follows. In Sec. 2 we briefly review the general scheme of integrating anomaly in even dimension and present the result of anomaly integration in 6D. In Sec. 3 we describe the reduction of the basis of surface terms. Sec. 4 describes how the main reduction formula is related to the general covariance and conformal invariance of the term which is topological invariant in 6D, and why this identity is valid beyond this particular dimension. Finally, in Sec. 5 we draw our conclusions and discuss the implications of this work for the integration of anomaly.

## 2 Anomaly induced effective action

The general structure of conformal anomaly in an arbitrary even dimension $D = 2n$ includes the following three types of terms:

i) Conformal invariant structures, such as $C_{\mu\nu\alpha\beta}^2 = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ in 4D. In the simplest case of 2D there are no conformal term, while in higher dimensions there may be much more such terms $\sum c_r W^D_r$, with the sum over $r$. For instance, there are three of them in 6D [9].

ii) The topological invariant

$$E_{(2n)} = \frac{1}{2^n} \varepsilon^{\alpha_1\beta_1 \ldots \alpha_n\beta_n} \varepsilon^{\gamma_1\delta_1 \ldots \gamma_n\delta_n} R_{\alpha_1\beta_1\gamma_1\delta_1} \ldots R_{\alpha_n\beta_n\gamma_n\delta_n}. $$

(1)

iii) The set of surface terms $\Xi_D = \sum \gamma_k \chi_k$. In 4D there is only one surface term $\Box R$, and in higher dimensions there are always much more such terms. The main subject of
the present communication is the reduction of the minimal set of surface terms in $6D$.

The reason for the described classification of constituents of the trace anomaly (see, e.g., [10, 11, 12]) is that the anomaly reflects the form of the one-loop divergences in the vacuum sector, and the last satisfy conformal Noether identity in case when quantized matter fields are conformal. Thus the anomaly can be presented in the universal form

$$T = \langle T_{\mu}^{\mu} \rangle = c_r W^r_D + a E_D + \Xi_D. \quad (2)$$

For the integration of anomaly it proves useful to start from the conformal transformation of the metric tensor $g_{\mu \nu}$,

$$g_{\mu \nu} = e^{2\sigma(x)} \bar{g}_{\mu \nu}. \quad (3)$$

The main ingredient of the scheme described in [3, 4] is the transformation rule for the corrected topological invariant,

$$\sqrt{-\bar{g}} \tilde{E}_D = \sqrt{-\bar{g}} (\bar{E}_D + \kappa \bar{\Delta}_D \sigma), \quad (4)$$

where $D = 2, 4, 6, \cdots$ and

$$\tilde{E}_D = E_D + \sum_i \alpha_i \Xi_i \quad (5)$$

is the modified Euler density which is a sum of the original topological term $E_D$ and a special linear combination of the total derivatives of the curvature-dependent terms $\Xi_i$ with the coefficients $\alpha_i$. Finally, $\Delta_D$ is the conformal operator, which is a $D$-dimensional generalization of the Paneitz operator in $4D$, [13, 14]. The $6D$ solution for $\alpha_i$ is [6]

$$\begin{align*}
\alpha_1 &= \frac{3}{5}, \\
\alpha_2 &= -\frac{9}{10} - \frac{5}{4} \xi_1 + \frac{3}{8} \xi_2, \\
\alpha_3 &= \xi_1, \\
\alpha_4 &= 0, \\
\alpha_5 &= \frac{84}{5} + 3 \xi_1 + \frac{11}{2} \xi_2, \\
\alpha_6 &= -\frac{36}{5} - 2 \xi_1 - 5 \xi_2, \\
\alpha_7 &= -\frac{18}{5} - \xi_1 - \frac{7}{2} \xi_2, \\
\alpha_8 &= \xi_2. \quad (6)
\end{align*}$$

Here $\xi_1$ and $\xi_2$ are two arbitrary parameters which can be fixed only if we find more than one identically vanishing linear combination of the surface terms. Indeed, the two-parameter ambiguity in the conformal operator $\bar{\Delta}_D$ has been found in the paper [15].

As far as the coefficients $\alpha_i$ in Eq. (5) are established, the integration of conformal anomaly becomes a relatively simple exercise, and the general answer can be written in the form [6]

$$\Gamma_{ind} = S_c + \iint_{xy} \left\{ \frac{1}{4} c_r W^r_D + \frac{a}{8} \tilde{E}_D(x) \right\} G(x, y) \tilde{E}_D(y)$$

$$+ \sum_k (\gamma_k - \alpha_k) \sum_i c_{ik} \int_x L_i. \quad (7)$$
Here $S_c = S_c[g_{\mu\nu}]$ is an arbitrary conformal invariant functional, $\int_x \equiv \int d^Dx \sqrt{-g}$, $G(x, y)$ is the Green function of the conformal operator $\Delta_D$ and, finally, $\mathcal{L}_i$ are local Lagrangians which generate the surface terms in the anomaly through the relations,

$$-\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \sum_i c_{ik} \int_x \mathcal{L}_i = \chi_k.$$  \hspace{1cm} (8)

One can see that surface terms $\chi_k$ play a decisive role in the integration of anomaly. Therefore it is very desirable to establish a minimal set of linear independent surface terms. According to the previous publications, e.g. [7] or [6] there are eight such terms. In the next section we show how this number can be reduced to seven.

3 Reduction of six derivative surface terms

The set of six derivative surface terms which was used in [6] looks as follows:

$$\Xi_1 = \Box^2 R, \quad \Xi_2 = \Box R^2_{\mu\nu\alpha\beta}, \quad \Xi_3 = \Box R^2_{\mu\nu}, \quad \Xi_4 = \Box R^2, \quad \Xi_5 = \nabla_\mu \nabla_\nu (R^\mu_{\lambda\alpha\beta} R^\nu_{\lambda\alpha\beta}),$$

$$\Xi_6 = \nabla_\mu \nabla_\nu (R_{\alpha\beta} R^{\mu\nu})_{\alpha\beta}, \quad \Xi_7 = \nabla_\mu \nabla_\nu (R^\mu_{\alpha\beta} R^{\nu\alpha}), \quad \Xi_8 = \nabla_\mu \nabla_\nu (R R^{\mu\nu}).$$  \hspace{1cm} (9)

Let us start with the following statement which can be obtained by direct calculation. Performing the conformal transformations of the structures $\Xi_k$ one can prove, with the help of the software Mathematica [16], that the following linear combination of surface terms is conformal invariant:

$$\sqrt{-g}(\Xi_2 - 4\Xi_3 + \Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8) = \sqrt{-\bar{g}}(\bar{\Xi}_2 - 4\bar{\Xi}_3 + \bar{\Xi}_4 - 4\bar{\Xi}_5 + 8\bar{\Xi}_6 + 8\bar{\Xi}_7 - 4\bar{\Xi}_8).$$  \hspace{1cm} (10)

Could the above combination be identically vanishing, indicating linear dependence of the set [9]? The answer to this question is positive, and we demonstrate this in what follows. For the sake of this proof, we introduce the following notations:

$$\Sigma_1 = \Box^2 R \quad \Sigma_2 = (\nabla_\lambda R^\lambda_{\mu\nu})_{\mu\nu}^2 \quad \Sigma_3 = R_{\mu\nu\beta} \nabla_\mu \nabla_\nu R^{\alpha\beta}$$

$$\Sigma_4 = R_{\mu\nu} R^{\mu\alpha\beta} R^\nu_{\lambda\alpha\beta} \quad \Sigma_5 = R^{\mu\nu}_{\alpha\beta} R^{\nu\lambda\tau}_{\alpha\beta} R^\lambda_{\mu\nu} \quad \Sigma_6 = R^{\mu\nu}_{\alpha\beta} \nabla_\mu \nabla_\nu R^{\alpha\beta}$$

$$\Sigma_7 = (\nabla_\lambda R^\lambda_{\mu\nu})^2 \quad \Sigma_8 = R_{\mu\nu} \Box R^{\mu\nu} \quad \Sigma_9 = (\nabla_\mu R)^2$$

$$\Sigma_{10} = R \Box R \quad \Sigma_{11} = (\nabla_\alpha R_{\mu\nu}) \nabla_\mu R^{\nu\alpha} \quad \Sigma_{12} = R^{\mu\nu} \nabla_\mu \nabla_\nu R$$

$$\Sigma_{13} = R_{\mu\nu} R^{\mu\nu}_{\alpha\beta} R^{\mu\alpha\beta} \quad \Sigma_{14} = R_{\mu\nu} R^{\mu\alpha} R^\alpha_{\nu}.$$  \hspace{1cm} (11)
Let us elaborate each of the terms \( \Xi_i \) using notations (11). We start from the trivial simplest case and then go the more complicated part.

\[
\Box^2 R = \Sigma_1. \tag{12}
\]

\[
\Box R_{\mu\nu\alpha\beta} = 2(\nabla_\lambda R_{\mu\nu\alpha\beta})^2 + 2R_{\mu\nu\alpha\beta}\Box R_{\mu\nu\alpha\beta}. \tag{13}
\]

Using the properties of the Riemann tensor and Bianci identities, the second term in the last relation can be rewritten as \([17]\)

\[
R_{\mu\nu\alpha\beta}\Box R_{\mu\nu\alpha\beta} = 4R_{\mu\lambda\nu\alpha}R_{\lambda \alpha\nu\beta} + 2R_{\mu\nu\mu\lambda\nu\alpha\beta}R_{\lambda \alpha\nu\beta} - R_{\mu\nu\alpha\beta}R_{\lambda \alpha\nu\beta}R_{\lambda \nu \beta}. \tag{14}
\]

Thus we arrive at the first relations

\[
\Box R_{\mu\nu\alpha\beta} = 2\Sigma_2 + 8\Sigma_3 + 4\Sigma_4 - 2\Sigma_5 - 8\Sigma_6, \tag{15}
\]

\[
\Box R_{\mu\nu} = 2(\nabla_\lambda R_{\mu\nu})^2 + 2R_{\mu\nu}\Box R_{\mu\nu} = 2\Sigma_7 + 2\Sigma_8, \tag{16}
\]

\[
\Box R^2 = 2(\nabla_\lambda R)^2 + 2R\Box R = 2\Sigma_9 + 2\Sigma_{10}. \tag{17}
\]

Furthermore,

\[
\nabla_\mu \nabla_\nu R_{\lambda\alpha\beta} R^\nu R^\lambda = \nabla_\mu \left[ (\nabla_\nu R_{\lambda\alpha\beta}) R^\nu R^\lambda + R_{\lambda\alpha\beta} \nabla_\mu R^\nu R^\lambda \right] 
\]

\[
= \nabla_\mu \left[ (\nabla_\nu R_{\lambda\alpha\beta}) R^\nu R^\lambda + R_{\lambda\alpha\beta} \nabla_\mu R^\nu R^\lambda \right] 
\]

\[
= (\nabla_\mu \nabla_\nu R_{\lambda\alpha\beta}) R^\nu R^\lambda + (\nabla_\nu R_{\lambda\alpha\beta}) \nabla_\mu R^\nu R^\lambda + (\nabla_\mu R_{\lambda\alpha\beta}) \nabla_\nu R^\lambda + R_{\mu\lambda\alpha\beta} \nabla_\mu R^\nu R^\lambda 
\]

\[
= (\nabla_\mu \nabla_\nu R_{\lambda\alpha\beta}) R^\nu R^\lambda + (\nabla_\nu R_{\lambda\alpha\beta}) \nabla_\mu R^\nu R^\lambda + 2(\nabla_\mu R_{\lambda\alpha\beta}) \nabla_\alpha R^\lambda + 2\Sigma_3. \tag{18}
\]

The first term in the expression (18) can be transformed as follows:

\[
(\nabla_\mu \nabla_\nu R_{\lambda\alpha\beta}) R^\nu R^\lambda = (\nabla_\nu \nabla_\mu R_{\lambda\alpha\beta}) R^\nu R^\lambda + R_{\mu\lambda\alpha\beta} \nabla_\mu R^\nu R^\lambda 
\]

\[
= (\nabla_\nu \nabla_\mu R_{\lambda\alpha\beta}) R^\nu R^\lambda + (\nabla_\nu \nabla_\mu R_{\lambda\alpha\beta}) R^\nu R^\lambda + R_{\lambda\alpha\beta} \nabla_\mu R^\nu R^\lambda 
\]

\[
= R_{\lambda\mu\nu} R_{\kappa\alpha\beta} R^\nu R^\lambda - R_{\kappa\mu\nu} R_{\lambda\alpha\beta} R^\nu R^\lambda - R_{\lambda\mu\nu} R_{\kappa\alpha\beta} R^\nu R^\lambda 
\]

\[
= 2\Sigma_3 + \Sigma_4 - R_{\lambda\mu\nu} R_{\kappa\alpha\beta} R^\nu R^\lambda - 2R_{\kappa\mu\nu} R_{\lambda\alpha\beta} R^\nu R^\lambda. \tag{19}
\]

At this moment we remember that

\[
R_{\lambda\mu\nu} R_{\kappa\alpha\beta} R^\nu R^\lambda = R_{\kappa\lambda\mu\nu} R_{\alpha\beta} R^\gamma R^\lambda - R_{\kappa\lambda\mu\nu} R_{\alpha\beta} R^\gamma R^\lambda. \tag{20}
\]
By making change of indices $\nu \leftrightarrow \lambda$ in the first of these expressions, we arrive at

$$R^\kappa_{\lambda \mu \nu} R^\mu_{\kappa \nu \beta} R^\nu_{\lambda \kappa \beta} = -R^\kappa_{\kappa \lambda \mu \nu} R^\mu_{\kappa \alpha \beta} R^\alpha_{\beta \lambda \nu} + \Sigma_5 = \frac{1}{2} \Sigma_5. \quad (21)$$

Next, the last term of (19) can be transformed as

$$R^\kappa_{\alpha \mu \nu} R^\mu_{\lambda \kappa \beta} R^\nu_{\lambda \alpha \beta} = R^\kappa_{\alpha \mu} R^\lambda_{\kappa \beta} R^\alpha_{\lambda \beta} = R^\mu_{\nu \alpha} R^\nu_{\alpha \beta} R^\lambda_{\mu \beta} = \Sigma_6, \quad (22)$$

thus we get

$$(\nabla_\mu \nabla_\nu R^\mu_{\lambda \alpha \beta}) R^\nu_{\lambda \alpha \beta} = 2\Sigma_3 + \Sigma_4 - \frac{1}{2} \Sigma_5 - 2 \Sigma_6. \quad (23)$$

The second term of the second expression of (18) can be developed as

$$(\nabla_\nu R^\mu_{\mu \lambda \alpha \beta}) \nabla_\mu R^\nu_{\lambda \alpha \beta} = -(\nabla_\lambda R^\nu_{\nu \mu \alpha \beta}) \nabla_\mu R^\mu_{\nu \lambda \alpha \beta} - (\nabla_\mu R^\nu_{\nu \lambda \alpha \beta}) \nabla_\mu R^\nu_{\nu \mu \alpha \beta} = \Sigma_7 - \Sigma_{11}. \quad (24)$$

By using the index exchange $\nu \leftrightarrow \lambda$ in the last formula we get

$$(\nabla_\nu R^\mu_{\mu \lambda \alpha \beta}) \nabla_\mu R^\nu_{\lambda \alpha \beta} = \frac{1}{2} \Sigma_2. \quad (25)$$

Using the first reduced Bianchi identity, we develop the third term of (18) such that

$$(\nabla_\nu R^\mu_{\lambda \alpha \beta}) \nabla^\alpha R^\lambda_{\beta \mu} = (\nabla_\alpha R^\lambda_{\beta \mu}) - (\nabla_\beta R^\lambda_{\alpha \mu}) = \Sigma_7 - \Sigma_{11}. \quad (26)$$

Replacing (23), (26) e (27) into (18) we obtain

$$\nabla_\mu \nabla_\nu (R^\mu_{\alpha \beta} R^\nu_{\lambda \alpha \beta}) = \frac{1}{2} \Sigma_2 + 4 \Sigma_3 + \Sigma_4 - \frac{1}{2} \Sigma_5 - 2 \Sigma_6 + 2 \Sigma_7 - 2 \Sigma_{11}. \quad (28)$$

The next step is to consider

$$\nabla_\mu \nabla_\nu (R^\mu_{\lambda \alpha \beta} R^\nu_{\lambda \alpha \beta}) = \nabla_\mu \left[ (\nabla_\nu R^\mu_{\lambda \alpha \beta}) R^\nu_{\nu \mu \alpha \beta} + R^\kappa_{\alpha \beta} \nabla_\nu R^\mu_{\nu \mu \beta \alpha} \right]$$

$$= \nabla_\mu \left[ (\nabla_\nu R^\mu_{\lambda \alpha \beta}) R^\mu_{\nu \mu \beta \alpha} + R^\kappa_{\alpha \beta} \nabla_\nu R^\mu_{\nu \mu \beta \alpha} - R^\kappa_{\alpha \beta} \nabla_\alpha R^\mu_{\nu \nu \beta \mu} \right]$$

$$= (\nabla_\mu \nabla_\nu R^\mu_{\lambda \alpha \beta}) R^\mu_{\nu \mu \beta \alpha} + (\nabla_\nu R^\mu_{\lambda \alpha \beta}) \nabla_\mu R^\mu_{\nu \mu \beta \alpha} + (\nabla_\mu R^\mu_{\lambda \alpha \beta})^2$$

$$+ R^\kappa_{\alpha \beta} \nabla_\alpha R^\mu_{\nu \nu \beta \mu} - (\nabla_\mu R^\mu_{\lambda \alpha \beta}) \nabla_\beta R^\beta_{\alpha \mu} - R^\kappa_{\alpha \beta} \nabla_\mu \nabla_\alpha R^\beta_{\nu \nu \mu}$$

$$= \Sigma_3 + (\nabla_\beta R^\beta_{\alpha \mu})^2 - (\nabla_\nu R^\mu_{\lambda \alpha \beta}) \nabla_\beta R^\mu_{\nu \nu \beta \alpha} + \Sigma_7 + \Sigma_8 - \Sigma_{11}$$

$$- \frac{1}{2} R^\alpha_{\beta \alpha} \nabla_\alpha \nabla_\beta R - R^\alpha_{\beta \alpha} [\nabla_\mu, \nabla_\alpha] R^\beta_{\nu \nu \mu}$$

$$= \Sigma_3 + 2 \Sigma_7 + 2 \Sigma_8 - 2 \Sigma_{11} - \frac{1}{2} \Sigma_{12} - R^\kappa_{\nu \nu \beta \mu} R^\mu_{\lambda \kappa \alpha} R^\alpha_{\nu \nu \beta \mu} - R^\kappa_{\beta \beta \alpha} R^\alpha_{\beta \kappa \alpha} R^\beta_{\mu \mu \kappa}, \quad (29)$$
that means

$$\nabla_\mu \nabla_\nu (R_{\alpha\beta} R^{\mu\nu\alpha\beta}) = \Sigma_3 + 2\Sigma_7 + \Sigma_8 - 2\Sigma_{11} - \frac{1}{2}\Sigma_{12} + \Sigma_{13} - \Sigma_{14}. \quad (30)$$

Next,

$$\nabla_\mu \nabla_\nu (R^\mu_\alpha R^{\nu\alpha}) = \nabla_\mu \left[ (\nabla_\nu R^\mu_\alpha) R^{\nu\alpha} + \frac{1}{2} R^\mu_\alpha \nabla^{\alpha} R \right]$$

$$= \left( \nabla_\mu \nabla_\nu R^\mu_\alpha \right) R^{\nu\alpha} + (\nabla_\nu R^\mu_\alpha) \nabla_\mu R^{\nu\alpha} + \frac{1}{4} (\nabla_\alpha R)^2 + \frac{1}{2} R^\mu_\alpha \nabla_\mu \nabla^{\alpha} R$$

$$= \frac{1}{2} (\nabla_\nu \nabla_\alpha R) R^{\nu\alpha} + \left( [\nabla_\mu, \nabla_\nu] R^\mu_\alpha \right) R^{\nu\alpha} + \frac{1}{4} \Sigma_9 + \Sigma_{11} + \frac{1}{2} \Sigma_{12}$$

$$= \frac{1}{4} \Sigma_9 + \Sigma_{11} + \Sigma_{12} - \Sigma_{13} + \Sigma_{14}. \quad (31)$$

Finally, simpler operations provide the last ingredients,

$$\nabla_\mu \nabla_\nu (RR^{\mu\nu}) = \frac{1}{2} R^2 + (\nabla_\mu R)^2 + R^{\mu\nu} \nabla_\mu \nabla_\nu R, \quad (32)$$

$$\nabla_\mu \nabla_\nu (RR^{\mu\nu}) = \Sigma_9 + \frac{1}{2} \Sigma_{10} + \Sigma_{12}. \quad (33)$$

Now we possess all what is needed to solve the equation of our interest,

$$a \Xi_1 + b \Xi_2 + c \Xi_3 + d \Xi_4 + e \Xi_5 + f \Xi_6 + g \Xi_7 + h \Xi_8 \equiv 0. \quad (34)$$

The solution for the coefficients of this equation is as follows:

$$a = 0 \quad b = \beta \quad c = -4\beta \quad d = \beta$$

$$e = -4\beta \quad f = 8\beta \quad g = 8\beta \quad h = -4\beta, \quad (35)$$

where $\beta$ is an arbitrary number which can be equal to one. Therefore, we have proved the identity

$$\Xi_2 - 4\Xi_3 + \Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8 = 0. \quad (36)$$

Eq. (36) resolves the main problem which we posed at the beginning of this contribution. Namely, it reduce the number of linearly independent six-derivative surface terms from eight to seven. Still this is not a complete solution of all relevant issues which one meets in the part of surface terms, and one can find the description of remaining problems in the next section.
4 Two sides of the identity (36)

After sending the first version of this manuscript to arXiv we learned about the well known paper [8], where the identity (36) has been used for deriving other relations between the equations of motion of the six-derivative actions in 6D. The way this identity has been obtained in the mentioned work came from the similar consideration in [18] for the Gauss-Bonnet invariant in 4D. The relation can be obtained as a Noether identity for the diffeomorphism invariance of the corresponding topological action (in some formulas we avoid using condensed notation for a D dimensional integral, just to stress its dimension),

\[ S_{GB}^{(D)} = \int d^D x \sqrt{-g} E. \] (37)

The Gauss-Bonnet term (1) in 6D can be cast into the form

\[ E_6 = \frac{1}{8} \epsilon^{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3} \epsilon^{\gamma_1 \delta_1 \cdots \gamma_3 \delta_3} R_{\alpha_1 \beta_1 \gamma_1 \delta_1} \cdots R_{\alpha_3 \beta_3 \gamma_3 \delta_3} \]

\[ = -8 \mathcal{L}_1 + 4 \mathcal{L}_2 - 24 \mathcal{L}_3 + 24 \mathcal{L}_4 + 16 \mathcal{L}_5 + 3 \mathcal{L}_6 - 12 \mathcal{L}_7 + \mathcal{L}_8. \] (38)

It is interesting that the first of these presentations does not admit simple generalization to an arbitrary dimension D, while for the second one it is not an obstacle. In what follows we will assume that E_6 means the expression in the r.h.s. when it is considered in D \neq 6.

The Noether identity for the general covariance of the action has the form

\[ \nabla_\mu \left[ \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_GB^{(D)}}{\delta g_{\mu \nu}} \right] = 0. \] (39)

The last identity reflects only the covariance of the action (37) and does not not use the topological nature of this expression. Therefore this identity is going to hold even for the dimension D where this action is not topological. At the same time, since in the “proper” dimension the equation of motion for the topological action is supposed to vanish (see [19] and the book [20] for detailed discussion), its trace is also vanishing [5]. One can anticipate that in the case of the action (37) this can produce another identity, which can be related to (39) due to the topological nature of the action in in the “proper” dimension. Let us check the situation in the case of 6D.

In what follows we will need the list of the six-derivative actions which are not full derivatives. One can define these actions in the form \[ I_n = \int_x \mathcal{L}_n, \] where

\[ \mathcal{L}_1 = R^\alpha_\lambda R^\lambda_\rho R^\rho_\sigma R^\sigma _\alpha, \quad \mathcal{L}_2 = R^\alpha_\lambda R^\lambda_\tau R^\tau_\rho R^\rho_\sigma, \quad \mathcal{L}_3 = R^\alpha_\lambda R^\lambda_\tau R^\tau_\gamma R^\gamma_\lambda, \]
\[ \mathcal{L}_4 = R^\alpha_\lambda R^\lambda_\tau R^\alpha_\tau, \quad \mathcal{L}_5 = R^\alpha_\lambda R^\lambda_\beta, \quad \mathcal{L}_6 = R R^2, \quad \mathcal{L}_7 = R R^2, \quad \mathcal{L}_8 = R^3, \]
\[ \mathcal{L}_9 = R^\alpha_\beta \square R^\alpha_\beta, \quad \mathcal{L}_{10} = R \square R. \] (40)
Furthermore, let us give the list of the corresponding equations of motion [17] (see also [21, 8])

\[ \Phi_n^{\mu \nu} = \frac{1}{\sqrt{-g}} \frac{\delta I_n}{\delta g_{\mu \nu}} \]
and their traces \( \Phi_n = g_{\mu \nu} \Phi_n^{\mu \nu} \),

which have the following form:

\[ \Phi_1^{\mu \nu} = \frac{1}{2} g^{\mu \nu} R^{\alpha \beta \gamma \lambda} \nabla_\alpha R^{\beta \gamma \lambda \nu} - R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda \nu} - 2R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda \nu} - 2D - 6 R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda \nu} - \frac{3}{2} \Xi_5 - 3 \Xi_6, \]

\[ \Phi_2 = \frac{D - 6}{2} R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda \nu} - 6 \Xi_5, \]

\[ \Phi_3 = \frac{D - 6}{2} R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda \nu} - \frac{1}{2} \Xi_2 - \frac{D - 2}{2} \Xi_5 - 2 \Xi_6 - 2 \Xi_7, \]

\[ \Phi_4 = \frac{D - 6}{2} R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda \nu} - \Xi_3 - (D - 2) \Xi_6 + \Xi_7 - \Xi_8, \]

\[ \Phi_5^{\mu \nu} = \frac{1}{2} g^{\mu \nu} R^{\alpha \beta \gamma \lambda} R^{\alpha \beta \gamma \lambda} - 3R^{\alpha \beta \gamma \lambda} R^{\alpha \beta \gamma \lambda} - 3 \nabla^\alpha \nabla_\alpha (R^{\mu \nu \lambda} R^{\nu \lambda}) - 3 \nabla^\alpha \nabla_\alpha (R^{\mu \nu \lambda} R^{\nu \lambda}) \]

\[ - \frac{3}{2} g^{\mu \nu} \nabla_\alpha \nabla_\beta (R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda}) - \frac{3}{2} \square (R^{\mu \nu \lambda} R^{\nu \lambda}), \]

\[ \Phi_5 = \frac{D - 6}{2} R^{\alpha \beta \gamma \lambda} R^{\beta \gamma \lambda \nu} - \frac{3}{2} \Xi_3 - 3 \frac{D - 2}{2} \Xi_7, \]

\[ \Phi_6^{\mu \nu} = \frac{1}{2} g^{\mu \nu} R^{\alpha \beta \gamma \lambda} - R^{\mu \nu \lambda} R^{\alpha \beta \gamma \lambda} - 2R R^{\alpha \beta \gamma \lambda} + \nabla^\nu \nabla_\nu R^{\alpha \beta \gamma \lambda} - 8 \nabla^\lambda (R R^{\alpha \beta \gamma \lambda}), \]

\[ \Phi_6 = \frac{D - 6}{2} R^{\alpha \beta \gamma \lambda} - (D - 1) \Xi_2 - 4 \Xi_8, \]
\[
\Phi_{\mu\nu} = \frac{1}{2} g^{\mu\nu} R^2_{\alpha\beta} - R^{\mu\nu} R_{\alpha\beta} \nonumber - 2 RR_{\lambda}^{\mu} R_{\alpha\beta} + \nabla^{(\mu} \nabla^{(\nu} (R^2_{\alpha\beta}) \nonumber - g^{\mu\nu} \Box R^2_{\alpha\beta} \nonumber + 2 \nabla^{(\mu} \nabla^{(\nu} R_{\alpha\beta} \nonumber - g^{\mu\nu} \nabla^{(\gamma} (RR^2_{\alpha\beta}) - \Box RR_{\alpha\beta}, \quad (48)
\]

\[
\Phi_7 = \frac{D - 6}{2} RR^2_{\alpha\beta} - (D - 1) \Xi_3 - \Xi_4 - (D - 2) \Xi_8, \quad (48)
\]

\[
\Phi^{\mu\nu} = \frac{1}{2} g^{\mu\nu} R^3 - 3 R^{\mu\nu} R^2 + 3 \nabla^{\mu} \nabla^{\nu} R^2 - 3 g^{\mu\nu} \Box R^2, \quad (49)
\]

\[
\Phi_8 = \frac{\delta}{2} R^3 - 3 (D - 1) \Xi_4, \quad (49)
\]

\[
\Phi^{\mu\nu} = \frac{1}{2} g^{\mu\nu} R^3_{\alpha\beta} - R^{\alpha\beta} \nabla^{(\mu} R_{\alpha\beta} - 2 R^{(\mu} R^{\nu)\alpha} + 2 \nabla^{\mu} \nabla^{(\nu} R_{\alpha\beta} \nonumber - g^{\mu\nu} \nabla^{\alpha} (R_{\alpha\beta} \nabla^{(\mu} R^{\nu)\beta} - 2 \nabla^{\alpha} (R^{\beta} R_{\alpha\beta}) \nonumber + \nabla^{(\mu} R_{\alpha\beta} \nabla^{\nu) R_{\alpha\beta}} - \frac{1}{2} g^{\mu\nu} \nabla^{\lambda} (R_{\alpha\beta} \nabla^{\lambda} R_{\alpha\beta}), \quad (50)
\]

\[
\Phi_9 = \frac{\delta}{2} R^3_{\alpha\beta} R_{\alpha\beta} - \frac{\delta}{2} \Xi_1 - \frac{\delta}{2} \Xi_3 + \frac{\delta}{2} \Xi_4 + 2 (D - 2) \Xi_6 - 2 \Xi_7 - (D - 4) \Xi_8, \quad (50)
\]

\[
\Phi^{\mu\nu} = - \frac{1}{2} g^{\mu\nu} (\nabla R)^2 + (\nabla^{\mu} R) (\nabla^{\nu} R) + 2 \nabla^{\mu} \nabla^{\nu} \Box R - 2 g^{\mu\nu} \Box R - 2 RR_{\alpha\beta} \nonumber \Box R, \quad (51)
\]

\[
\Phi_{10} = - \frac{\delta}{2} R \Box R - 2 (D - 1) \Xi_4 - \frac{\delta}{2} \Xi_4, \quad (51)
\]

In the derivation of traces we used expressions given in Appendix B.

Taking the last observation and new notation into account, by combining equations (42) - (49) we arrive at the following relation:

\[
\frac{1}{\sqrt{-g}} g_{\mu\nu} \delta \frac{\delta}{\delta g_{\mu\nu}} \int E_6 = \frac{D - 6}{2} E_6 \nonumber - 3 (D - 5) (\Xi_2 - 4 \Xi_3 + \Xi_4 - 4 \Xi_5 + 8 \Xi_6 + 8 \Xi_7 - 4 \Xi_8). \quad (52)
\]

The first term in the r.h.s. of the Eq. (52) obviously vanish in $D = 6$. At the same time, the l.h.s. also vanish in $D = 6$, because in this specific dimension it is the trace of the variational derivative of the topological term\footnote{One can prove this even without taking trace (see, e.g., [20]), but such a proof requires choosing a special coordinate system. In general coordinates this equation does not look trivial, as it was discussed in [19] [22].}. In this sense the relation (52) proves that the remaining term in the r.h.s. also vanish in $D = 6$. However, this term is exactly
an identity \((36)\) which we proved directly in the previous section. It is worth noticing
that the proof which we presented there does not depend on the dimension.

Taking the identity \((36)\) into account, we arrive at the simple rule of conformal shift
of the term under consideration, namely

\[
g_{\mu\nu} \frac{\delta}{\sqrt{-g}} g_{\mu\nu} \int E_6 = \frac{D-6}{2} E_6,
\]

which is perfectly consistent to the main relation of integrating anomaly \((5)\).

To close the story, let us mention that there is yet another equivalent form of our
identity \((36)\)

\[
\Xi_2 - 4\Xi_3 + 4\Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8 = \frac{1}{4!} \delta_{\mu\nu\xi\eta\kappa\chi} \nabla_\mu \nabla_\nu \left( R^\xi_{\alpha\beta} R^{\kappa\chi}_{\lambda\tau} \right),
\]

where (in Euclidean signature)

\[
\delta_{\mu\nu\xi\eta\kappa\chi} = \epsilon_{\rho\mu\alpha\beta\lambda\tau} \epsilon_{\rho\nu\xi\eta\kappa\chi} = 5! \delta_{\mu\nu} \delta_{\alpha\beta} \delta_{\lambda\tau}.
\]

The proof of the relation

\[
\delta_{\mu\nu\xi\eta\kappa\chi} \nabla_\mu \nabla_\nu \left( R^\xi_{\alpha\beta} R^{\kappa\chi}_{\lambda\tau} \right) \equiv 0
\]

can be found in Appendix \[7\].

5 Conclusions and Discussions

As we have just mentioned above, Eq. \((36)\) reduce the number of surface terms which is
needed to construct the full basis of such terms in \(6D\). Let us discuss the importance of
this formula in the general context.

The complete and consistent integration of trace anomaly in \(6D\) requires several math-
ematical results which are all not very easy to accomplish, mainly because the practical
calculations in \(6D\) are essentially more involved than the ones in \(4D\). At the first place
one needs the main formula \((5)\) which immediately produce the non-local part of the
anomaly-induced action \([10, 6]\). The general formal expression for this action \((7)\) for an
arbitrary even dimension has been constructed in Ref. \([6]\), where we also reported on the
explicit realization of the key formula \((5)\) in the case of \(6D\). Then, looking at the general
expression \((7)\) we can see that the remaining part of the effective action is related to the
integration of total derivative terms.
Usually the importance of total derivative terms in the anomaly is underestimated, since it is supposed that they can be modified or eliminated by adding finite local counterterms. Such an addition is a mathematically legal procedure, because the gravitational vacuum action is not quantized in the framework of semiclassical theory. Therefore, even if the local nonconformal terms are not needed for renormalization, one can add them without changing the general structure of quantum theory in curved space.

In some cases such an addition can be pretty well justified. The main example of this sort is the Starobinsky inflation \cite{23,24}, where the $R^2$ term with a very large coefficient is required to provide the control over perturbations and, in general, correspondence with the existing observational data. The attempts to explain the magnitude of this coefficient from quantum field theory arguments are currently at the rudimentary level (see, e.g., \cite{25}) and hence the introduction of the large coefficient of $R^2$ is a phenomenological operation. In general, and especially in $6D$, there are no observational evidence which can be used to fix the coefficients of the total derivative terms. Therefore for us the importance of these terms is certain and without doubts\footnote{Further arguments concerning the ambiguities related to local terms in the induced action can be found in Ref. \cite{25}, where one can see also the relation to the non-local structures in the case of almost vanishing masses of quantum fields.}

In this situation the formula defining the part of effective action which comes from the total derivative terms in the anomaly is \cite{8}. Then the reduction of the number of the total derivatives $\chi_k$ in the r.h.s. of this equation from eight to seven increases our chances to find the solution. And, from the general perspective, it would be interesting to have an independent, new and nontrivial confirmation of the possibility to integrate total derivatives with the local gravitational terms, according to Eq. \cite{5}.

Two concluding observations are in order. First of all, since in $6D$ the structure \cite{3,7} is topological, its variational derivative is zero. At the same time, in $6D$ even the identity for the trace is not easy to prove explicitly, as the reader could ensure from Sec. \cite{8}. The second aspect is that the topological term \cite{11} is unique and, therefore, the vanishing linear combination \cite{36} is also unique\footnote{We are grateful to Dr. Sourya Ray for stressing this point to us.} The important consequence of this uniqueness is that further reduction of the solution \cite{3} is impossible, because \cite{11} was already taken into account in \cite{3}. Thus the main result of the present work is that now we can affirm that the fundamental difference between the $2D$ and $4D$ formulas \cite{4} from one side and similar formula in $6D$ from another side is that in the last case this important formula has two-parameter ambiguity. The changes of $\xi_1$ or $\xi_2$ do not produce a change of conformal functional $S_C$, which is the unique ambiguous part of the effective action in $2D$ and $4D$. 


cases. Therefore, now we can claim that in 6D we meet a qualitatively new kind of ambiguity, that is something which does not take place in 2D and 4D cases.

6 Appendix A. Useful relations for total derivatives.

Let us give useful list of relations for total derivatives,

\[ \nabla_\mu \nabla_\nu (R^\mu_{\alpha\beta\lambda} R^{\nu\lambda\beta\alpha}) = \frac{1}{2} \Xi_5, \quad (57) \]
\[ \nabla_\mu (R^{\mu\nu} \nabla_\nu R) = -\frac{1}{4} \Xi_4 + \Xi_8, \quad (58) \]
\[ \nabla_\alpha (R_{\mu\nu} \nabla^\alpha R^{\mu\nu}) = \frac{1}{2} \Xi_3, \quad (59) \]
\[ \nabla_\mu (R^\mu_{\chi\lambda} \nabla_\nu R^{\mu\lambda}) = \frac{1}{8} \Xi_4 + \Xi_7 - \frac{1}{2} \Xi_8, \quad (60) \]
\[ \nabla_\mu (R^{\epsilon\alpha\beta\gamma} \nabla_\nu R_{\alpha\beta}) = -\frac{1}{2} \Xi_3 + \frac{1}{8} \Xi_4 + \Xi_6 + \Xi_7 - \frac{1}{2} \Xi_8, \quad (61) \]
\[ \nabla_\mu \nabla_\nu \Box R^{\mu\nu} = \frac{1}{2} \Xi_1 + \frac{1}{2} \Xi_3 - \frac{1}{4} \Xi_4 - 2 \Xi_6 + \Xi_8. \quad (62) \]

7 Appendix B. Proof of the relation (56)

Let us denote the object of our interest \( \Omega \) and take one of the derivatives,

\[ \Omega = \delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\chi} \nabla_\mu \nabla_\nu (R^\xi_{\alpha\beta} R^{\eta\chi \lambda\tau}) \]
\[ = \delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\chi} \nabla_\mu \left[ R^{\epsilon\chi \lambda\tau} \nabla_\nu R^\eta_{\alpha\beta} + R^\xi_{\alpha\beta} \nabla_\nu R^{\epsilon\chi \lambda\tau} \right]. \quad (63) \]

Using antisymmetry of the object (55) and the Bianchi identity, the last expression transforms into

\[ \Omega = 2 \nabla_\mu \left( \delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\chi} R^{\epsilon\chi \lambda\tau} \nabla_\nu R^\eta_{\alpha\beta} \right) \]
\[ = -2 \nabla_\mu \left[ \delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\chi} \left( R^{\epsilon\chi \lambda\tau} \nabla_\nu R^\eta_{\alpha\beta} + R^\epsilon_{\alpha\beta} \nabla_\nu R^{\epsilon\chi \lambda\tau} \right) \right]. \quad (64) \]

Once again using antisymmetry of (55) we arrive at

\[ \Omega = -4 \nabla_\mu \left( \delta^{\mu\alpha\beta\lambda\tau}_{\nu\xi\eta\chi} R^{\epsilon\chi \lambda\tau} \nabla_\nu R^\eta_{\alpha\beta} \right). \quad (65) \]

Comparing (64) and (65) one can check that

\[ \Omega = -2 \Omega, \quad (66) \]

which is equivalent to Eq. (56).
Let us stress that the analog of this result can be found in [18] for 4D and can be also found in [8] for 6D. The derivation of this identity in both cases was based on the relation (39) which reflects diffeomorphism invariance of the action (37) with \( D = 6 \) and \( E_6 \), defined as in the r.h.s. of Eq. (38). For this reason the identity is valid in any dimension \( D \). At the same time the same identity can be also obtained in exactly \( D = 6 \) as a Noether identity of the conformal symmetry (53).

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