THE SHARP LOG-SOBOLEV INEQUALITY ON A COMPACT INTERVAL

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ABSTRACT. We provide a proof of the sharp log-Sobolev inequality on a compact interval.

1. INTRODUCTION

The Gaussian log-Sobolev inequality, due to A. J. Stam [S, 1959, Eqn. 2.3] or Paul Federbush [F, 1969, Eqn. (14)], although often attributed to L. Gross [G, 1975, Cor. 4.2], played a crucial role in Perelman’s [P, 2002] proof of the Poincaré Conjecture. We consider log-Sobolev inequalities for finite Lebesgue measure. F. Maggi [M, 2009] observed that the sharp log-Sobolev inequality on the interval follows from an isoperimetric conjecture of Díaz et al. [DHHT, 2010], which remains open, but provided no proof. We found it in Wang [Wa, 1999], who cited Deuschel and Stroock [DS, 1990], who gave a proof of the sharp log-Sobolev inequality on the circle. We then traced this result back to Emery and Yukich [EY, 1987, p. 1], Rothaus [R1, 1980, Thm. 4.3], and Weissler [We, 1980, Thm. 1]. Our Theorem 2.2 shows that the interval case follows quickly from the circle case.

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2. LOG-SOBOLEV INEQUALITY ON A COMPACT INTERVAL

In considering the isoperimetric problem in sectors of the plane with density \( r^p \), Díaz et al. [DHHT, Cor. 4.24, Conj. 4.18] conjectured the following inequality:

\[
\left[ \int_0^1 r^p \, d\alpha \right]^{1/q} \leq \int_0^1 \sqrt{r^2 + (q-1) \frac{r^2}{\pi^2}} \, d\alpha,
\]

where \( 1 < q \leq 2 \). F. Maggi [M] observed that (1) implies the log-Sobolev inequality of Theorem 2.2 below. Here we observe that Theorem 2.2 follows from the following proposition of Weissler.

Proposition 2.1. [We, Thm. 1] Let \( f \) be a non-negative \( C^1 \) function on the circle \( S^1 \) of length 1. Suppose \( \int_{S^1} f^2 = 1 \). Then we have the following sharp inequality:

\[
4\pi^2 \int_{S^1} f^2 \log f \leq \int_{S^1} f^2.
\]

Various proofs are discussed in Section 3.

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Theorem 2.2. Let $f$ be a non-negative $C^1$ function on the interval $[0,1]$. Suppose $\int_0^1 f^2 = 1$. Then we have the following inequality:

\begin{equation}
\pi^2 \int_0^1 f^2 \log f \leq \int_0^1 f^2.
\end{equation}

Proof. Let $f$ be any non-negative $C^1$ function on $[0,1]$ such that $\int_0^1 f^2 = 1$. Define a non-negative piecewise $C^1$ function $g$ on $S^1$ such that

\begin{align*}
g(x) &= \begin{cases} 
    f(2x), & \text{if } 0 \leq x \leq \frac{1}{2} \\
    f(2 - 2x), & \text{if } \frac{1}{2} < x \leq 1
\end{cases}.
\end{align*}

Then $\int_{S^1} g^2 = 1$. By smoothing, Proposition 2.1 applies to $g$. By simple computation, we have that

\begin{align*}
\int_{S^1} g^2 \log g &= \int_0^1 f^2 \log f \\
\int_{S^1} g^2 &= 4 \int_0^1 f^2.
\end{align*}

The conclusion follows. \(\square\)

Remark 2.3. Feng-Yu Wang [Wa, Ex. 1.2] suggested an alternative proof of 2.2(1), but we don’t understand his proof. He considered densities $C_0 \exp(\epsilon \cos \pi x)$ and functions $f_\epsilon = \exp(-\epsilon \cos \pi x)$, with $C_0$ chosen to make the integral of $f_\epsilon^2$ equal to 1. Then $f_\epsilon$ satisfies the differential equation

\begin{equation}
f_\epsilon'' - \pi \epsilon \sin \pi x f_\epsilon' = -\pi^2 f_\epsilon \log f_\epsilon.
\end{equation}

He said that it follows that 2.2(1) holds for those functions and densities with sharp constant $\pi^2$. This might follow if it were known that functions realizing equality exist, but Wang himself [Wa, p. 655] admits that "the author is not sure yet whether there always exists [such a function]." Indeed, in the case of the circle with unit density, there apparently is no such function. Of course, the sharp inequality for density 1 would follow as $\epsilon$ approaches 0.

A similar result holds on any interval for a function with root mean square $m$:

Corollary 2.4. Let $f$ be a non-negative $C^1$ function on the interval $[a,b]$. Suppose

\begin{equation}
\frac{1}{b-a} \int_a^b f^2 = m^2
\end{equation}

$(m > 0)$. Then we have the following inequality:

\begin{equation}
\frac{\pi^2}{(b-a)^2} \left( \int_a^b f^2 \log f - (b-a)m^2 \log m \right) \leq \int_a^b f^2.
\end{equation}

Proof. Let $f$ be a non-negative $C^1$ function on the interval $[a,b]$ such that

\begin{equation}
\frac{1}{b-a} \int_a^b f^2 = m^2 > 0
\end{equation}

$(m > 0)$. Define a function $g$ on the interval $[0,1]$ as

\begin{equation}
g(x) = \frac{1}{m} f((b-a)x + a).
\end{equation}
Then $g$ is non-negative and $C^1$. Moreover, we have
\[
\int_0^1 g(x)^2 \, dx = \int_0^1 \frac{1}{m^2} f((b-a)x+a)^2 \, dx = \frac{1}{(b-a)m^2} \int_0^a f(y)^2 \, dy = 1.
\]
Therefore, we can apply Theorem 2.2 to the function $g$. We have
\[
(2) \quad \frac{\pi^2}{b-a} \int_0^1 g^2 \log g \leq (b-a) \int_0^1 g'^2.
\]
Note that
\[
g'(x) = \frac{b-a}{m} f((b-a)x+a).
\]
By direct calculation, we have
\[
\int_0^1 g'(x)^2 \, dx = \frac{(b-a)^2}{m^2} \int_0^1 f'(((b-a)x+a)^2 \, dx = \frac{(b-a)^2}{m^2} \int_a^b f'(x)^2 \, dx.
\]
We also have that
\[
\int_0^1 g(x)^2 \log g(x) \, dx = \frac{1}{m^2} \int_0^1 f((b-a)x+a)^2 \log \frac{f((b-a)x+a)}{m} \, dx
\]
\[
= \frac{1}{(b-a)m^2} \int_a^b f(x)^2 \log \frac{f(x)}{m} \, dx
\]
\[
= \frac{1}{(b-a)m^2} \left( \int_a^b f^2 \log f - (b-a)m^2 \log m \right).
\]
Therefore, by plugging these identities into (2), we have
\[
\frac{\pi^2}{(b-a)m^2} \left( \int_a^b f^2 \log f - (b-a)m^2 \log m \right) \leq b-a \int_a^b f'^2.
\]
This is equivalent to the desired inequality (1).

Corollary 2.4 can be written in the following form:

**Corollary 2.5.** Let $f$ be a non-negative $C^1$ function on the interval $[a,b]$. Suppose
\[
\frac{1}{b-a} \int_a^b f = m > 0.
\]
Then we have the following inequality:
\[
\frac{2\pi^2}{(b-a)^2} \left( \int_a^b f \log f - m \log m \right) \leq \int_a^b f'^2.
\]

**Proof.** Define a non-negative piecewise $C^1$ function $g$ on the interval $[a,b]$ as $g = \sqrt{f}$. Plugging $g$ into Corollary 2.4 yields the desired result.

**Proposition 2.6.** In Theorem 2.2 $\pi^2$ is the best possible constant.
Proof. For any $0 < \varepsilon < 1$, define
\[ f_\varepsilon(x) = \sqrt{1 - \varepsilon^2} + \sqrt{2\varepsilon} \cos \pi x. \]
Then by direct computation, we have
\[ \lim_{\varepsilon \to 0^+} \int_0^1 f_\varepsilon^2(x) \log f_\varepsilon(x) = \pi^2. \]
Therefore, the constant $\pi^2$ cannot be replaced by a larger constant.

Remark 2.7. The function $\cos \pi x$ comes from the equality case of a Wirtinger inequality which follows from the log-Sobolev inequality \[M].

3. PROOFS OF THE SHARP LOG-SOBOLEV INEQUALITY ON THE CIRCLE

We summarize three proofs of Proposition 2.1 given by Rothaus \[RI\] Thm. 4.3, 1980, Weissler \[We\] Thm. 1, 1980, Emery and Yukich \[EY\] p. 1, 1987, and Deuschel and Stroock \[DS\] Rmk. 1.14(i)].

3.1. Weissler's Proof. Weissler proved a stronger result than Proposition 2.1 by using Fourier expansion of functions of period $2\pi$.

Proposition 3.1. \[We, 1980, Theorem 1\] Let $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be in $L^2$ and suppose $f(\theta) \geq 0$ almost everywhere. Then
\[ \int f^2 \log f \leq \sum_{n=-\infty}^{\infty} |n|^2 |a_n|^2 + \|f\|_2^2 \log \|f\|_2 \]
in the sense that if the right hand side is finite, then so is the left hand side and the inequality holds.
(0$^2 \log 0$ is taken to be 0.)

Obviously the above inequality is stronger than the following inequality:
\[ \int f^2 \log f \leq \sum_{n=-\infty}^{\infty} |n|^2 |a_n|^2 + \|f\|_2^2 \log \|f\|_2 \]
which is equivalent to Proposition 2.1 by change of variables as in Corollary 2.4.

Weissler \[We\] cited Rothaus’s previous 1978 paper \[R2\] but did not have Rothaus’s 1980 paper \[RI\] where Rothaus gave his proof of Proposition 2.1.

3.2. Rothaus’s Proof. Rothaus proved Proposition 2.1 by a variational method. He considered an equivalent problem with a positive parameter $\rho$ \[RI\] Section 4. If a related constant $b_\rho$ is zero then the log-Sobolev inequality on the circle with the constant $2/\rho$ holds. Therefore, Proposition 2.1 is equivalent to showing that $b_{1/2\pi^2}$ is zero.

Rothaus \[RI\] proved that a minimizing function exists, is positive and satisfies a related differential equation \[RI\] Thm. 4.2]. Moreover, for $\rho > 1/2\pi^2$, the only positive solution to the differential equation is the constant function 1 \[RI\] Thm. 4.3] and hence $b_\rho$ is zero. Therefore in the limit $b_{1/2\pi^2}$ is zero, and Proposition 2.1 follows.

Rothaus \[RI\] cited Weissler’s paper \[We\] and said that "A result related to Theorem 6.3 appears in \[We]."

3.3. Emery and Yukich’s Proof. Emery and Yukich \[EY\] 1987, p. 1] proved Proposition 2.1 by using estimates deploying the Brownian motion semi-group.

Emery and Yukich \[EY\] cited both Weissler \[We\] and Rothaus \[RI\].
3.4. **Deuschel and Stroock's Proof.** Deuschel and Stroock considered the log-Sobolev inequality in general spaces with densities. As a special case they proved [DS, Rmk. 1.14(i)] that the log-Sobolev constant for the circle of length 1 with Lebesgue measure is the first eigenvalue of the Laplacian, namely $4\pi^2$ (corresponding to the first eigenfunction $\sin 2\pi x$).

Deuschel and Stroock [DS] cited Emery and Yukich [EY].

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