Application of the Dual Space of Gelfand-Shilov Spaces of Beurling Type

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ABSTRACT: Using a previously obtained structure theorem of Gelfand-Shilov spaces \( \Sigma^\beta_\alpha \) of Beurling type of ultradistributions, we prove that these ultradistributions can be represented as an initial values of solutions of the heat equation by describing the action of the Gauss-Weierstrass semigroup on the dual space \( (\Sigma^\beta_\alpha)' \).

Key Words: Short-time Fourier transform, Tempered Ultradistributions, Structure Theorem, Gelfand-Shilov spaces.

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1. Introduction

The theory of generalized functions devised by L. Schwartz was to provide a satisfactory framework for the Fourier transform (see [11]). They are objects which generalize functions, and they extend the concept of derivative to all integrable functions and beyond, and used to formulate generalized solutions of partial differential equations (see [5]).

Gelfand and Shilov have introduced other types of distributions called ultradistributions in the study of the uniqueness of the Cauchy problems of partial differential equations (see [3]). These spaces are invariant under Fourier transform, closed under differentiation and multiplication by polynomials, moreover, it contains Schwartz space of tempered distributions as a subspace. This makes the Gelfand Shilov spaces appropriate domains for quantum field theory. S. Pilipovic obtained structural theorems and defined the convolution for Gelfand-Shilov spaces of Roumieu and Beurling type (see [9], [10], [4]).

In this paper, we use the characterization of Gelfand-Shilov spaces of Beurling type of test functions of tempered ultradistribution in terms of their Fourier transform obtained in [2] and the structure theorem for functionals in dual space \( (\Sigma^\beta_\alpha)' \) equipped with the weak topology, to study the action of Gauss-Weierstrass semigroup on the dual space \( (\Sigma^\beta_\alpha)' \). Consequently, we prove that these ultradistributions can be represented as an initial values of solutions of the heat equation \( u_t - Au = 0 \).

Throughout the paper the symbols \( C^\infty, C^\infty_0, L^p, \) etc., denote the usual spaces of functions defined on \( \mathbb{R}^n \), with complex values. We denote \( |\cdot| \) the Euclidean norm on \( \mathbb{R}^n \), while \( \|\cdot\|_p \) indicates the \( p \)-norm in the space \( L^p \), where \( 1 \leq p \leq \infty \). In general, we work on the Euclidean space \( \mathbb{R}^n \) unless we indicate otherwise as appropriate. The Fourier transform of a function \( f \) will be denoted by \( \mathcal{F}(f) \) or \( \hat{f} \) and it will be defined as \( \int_{\mathbb{R}^n} e^{-2\pi ix\xi} f(x) dx \). A Fréchet space are a locally convex topological vector spaces that are completely metrizable.

2. Preliminary definitions and results

In this section, we introduce basic notations and recalling some facts concerning Gelfand-Shilov spaces.

Remark 2.1. For \( \alpha > 1 \), the function \( |\cdot|^{1/\alpha} : [0, \infty) \rightarrow [0, \infty) \) has the following properties:

1. \( |\cdot|^{1/\alpha} \) is increasing, continuous and concave,

2. \( |x|^{1/\alpha} \geq a + b \ln (1 + x) \) for some \( a \in \mathbb{R} \) and some \( b > 0 \).

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Remark 2.2. Property 2 of Remark 2.1 implies that the function $e^{-N|x|^{1/\alpha}}$ is integrable for some positive constant $N$. In fact, if $N > \frac{b}{\alpha}$ is an integer, then
\[ C_N = \int_{\mathbb{R}^n} e^{-N|x|^{1/\alpha}} \, dx < \infty, \text{ for all } \alpha > 1, \]
where $b$ is the constant in Property 2 of Remark 2.1. Moreover, Property 1 in Remark 2.1 implies that $|\bullet|^{1/\alpha}$ is subadditive.

In the following theorem we state a symmetric characterization of the Gelfand-Shilov spaces $\Sigma^b_\alpha$ in terms of the Fourier transformations.

Theorem 2.3. The space $\Sigma^b_\alpha$ can be described as a set as well as topologically by
\[ \Sigma^b_\alpha = \left\{ \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all } k = 0, 1, 2, ..., p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty \right\}, \]
where $p_{k,0}(\varphi) = \left\| e^{k|x|^{1/\alpha}} \varphi \right\|_\infty$, $\pi_{k,0}(\varphi) = \left\| e^{k|x|^{1/\alpha}} \hat{\varphi} \right\|_\infty$.

The space $\Sigma^b_\alpha$, equipped with the family of semi-norms
\[ N = \{ p_{k,0}, \pi_{k,0} : k \in \mathbb{N}_0 \}, \]
is a Fréchet space.

The proof of Theorem 2.3 mimics the proof of Theorem 3.1 in [7] so we omit it. In the other hand, we can employ the above theorem to prove the following structure theorem for functionals $T \in (\Sigma^b_\alpha)'$.

Theorem 2.4. If $T \in (\Sigma^b_\alpha)'$, then there exist two regular complex Borel measures $\mu_1$ and $\mu_2$ of finite total variation and $k \in \mathbb{N}_0$ such that
\[ T = e^{k|\bullet|^{1/\alpha}} \mu_1 + \mathcal{F}(e^{k|\bullet|^{1/\beta}} \mu_2) \] (2.1)
in the sense of $(\Sigma^b_\alpha)'$.

The following Lemma will be useful in the proofs later.

Lemma 2.5. ([7]) Let $\varphi \in \Sigma^b_\alpha$. Then $\varphi(x + y) \in \Sigma^b_\alpha$ for each $y \in \mathbb{R}^n$.

Proof: Fix $y \in \mathbb{R}^n$ and let $\varphi \in \Sigma^b_\alpha$. First, let us prove that
\[ \left\| e^{k|x|^{1/\alpha}} \varphi(x + y) \right\|_\infty < \infty. \]
To do so, we use concavity property of $|\bullet|^{1/\alpha}$ as follows:
\[ e^{k|x|^{1/\alpha}} |\varphi(x + y)| \leq e^{k|x|^{1/\alpha}} e^{-2k|x+y|^{1/\alpha}} e^{2k|x+y|^{1/\alpha}} |\varphi(x + y)| \]
\[ \leq Ce^{2k\left(\frac{x|^{1/\alpha}}{|x|} - 2|x+y|^{1/\alpha}\right)} \leq Ce^{2k\left(\frac{|x|^{1/\alpha}}{|x|} - 1\right)} \]
\[ \leq Ce^{-k|y|^{1/\alpha}} < \infty. \]
This proves that $\left\| e^{k|x|^{1/\alpha}} \varphi(x + y) \right\|_\infty < \infty$. Similarly, $\left\| e^{k|x|^{1/\alpha}} \hat{\varphi}(x + y) \right\|_\infty < \infty$. This completes the proof of Lemma 2.5. \qed

Given two functionals $T$ and $S$ that are integrable functions, the classical definition of convolution of $T$ and $S$ is given by
\[ \langle T \ast S, \phi \rangle = \langle T_x, S_y, \phi(y + x) \rangle. \]

Using this definition, Definition 1.6.11, and results from Section 1.7 of [1], it is easy to show that if $T \in (\Sigma^b_\alpha)'$ and $\varphi \in \Sigma^b_\alpha$, then the functional $T \ast \varphi \in (\Sigma^b_\alpha)'$. 
**Theorem 2.6.** If $T \in (\Sigma_\alpha^\beta)'$ and $\varphi \in \Sigma_\alpha^\beta$, then the functional $T \ast \varphi \in (\Sigma_\alpha^\beta)'$ and given by $\langle T, \varphi(x - \cdot) \rangle$.

We end this section with the definition of operator semigroup on a Banach space that we will use in application in the next section.

**Definition 2.1.** [8] Let $\mathfrak{B}$ be a Banach space. An operator semigroup on $\mathfrak{B}$ is a family $(T_t : t \in \mathbb{R}^+)$ of bounded linear operators on $\mathfrak{B}$ such that

i) $T_0 = I$,

ii) $T_s T_t = T_{s+t}$ for all $t, s \in \mathbb{R}^+$.

### 3. Applications

In this section, we study some applications on the structure theorem of $\Sigma_\alpha^\beta$ tempered ultradistributions stated in Theorem 2.4 by proving some results on a semi-group acting on the Fréchet space $\Sigma_\alpha^\beta$ and extend it to its dual $(\Sigma_\alpha^\beta)'$. We start this section by recalling a previously proved result which says that the convolution in Theorem 2.6 coincides with classical definition of convolution of two integrable functionals.

**Theorem 3.1.** If $T \in (\Sigma_\alpha^\beta)'$ and $\varphi \in \Sigma_\alpha^\beta$, then the functional $T \ast \varphi$ defined by

$$
\langle T \ast \varphi, \phi \rangle = \langle T_y, (\varphi_z, \phi(x+y)) \rangle
$$

coincides with the functional given by integration against the function $\psi(x) = \langle T_y, \varphi(x-y) \rangle$.

**Proof:** Using (2.1) in Theorem 2.4, we can write for each $x$

$$
\psi(x) = \langle T_y, \varphi(x-y) \rangle = \int_{\mathbb{R}^n} e^{ik|y|^{1/\alpha}} \varphi(x+y) d\mu_1(y) + \int_{\mathbb{R}^n} e^{|\xi|^{1/\beta}} e^{-2\pi i y \cdot \xi} \mathcal{F}^{-1}(\varphi)(\xi) d\mu_2(\xi).
$$

So,

$$
\langle T \ast \varphi, \phi \rangle = \langle T_y, \varphi_z, \phi(x+y) \rangle
$$

$$
= \int_{\mathbb{R}^n} e^{ik|y|^{1/\alpha}} \int_{\mathbb{R}^n} \varphi(x-y) \phi(y) d\mu_1(y) + \int_{\mathbb{R}^n} e^{ik|\xi|^{1/\beta}} \mathcal{F}^{-1}(\varphi)(\xi) \phi(\xi) d\mu_2(\xi)
$$

$$
= \int_{\mathbb{R}^n} e^{ik|y|^{1/\alpha}} \int_{\mathbb{R}^n} \varphi(x-y) \phi(y) d\mu_1(y) + \int_{\mathbb{R}^n} e^{ik|\xi|^{1/\beta}} \mathcal{F}(\varphi \ast \phi)(\xi) d\mu_2(\xi)
$$

$$
= \langle e^{ik|\cdot|^{1/\alpha}} \mu_1(y), \varphi(x-y) \phi(x) \rangle + \langle \mathcal{F}(e^{ik|\cdot|^{1/\beta}} \mu_2)(y), \varphi(x-y), \phi(x) \rangle
$$

$$
= \langle e^{ik|\cdot|^{1/\alpha}} \mu_1(y), \mathcal{F}(e^{ik|\cdot|^{1/\beta}} \mu_2)(y), \varphi(x-y), \phi(x) \rangle
$$

$$
= \langle T_y, \varphi(x-y), \phi(x) \rangle
$$

for all $\phi \in \Sigma_\alpha^\beta$. This completes the proof of Theorem 3.1.

Now we employ the above theorem to describe the action of the semi-group defined by the convolution kernel $t^{-n}T(\frac{x-y}{t})$, where $t > 0$ on $(\Sigma_\alpha^\beta)'$.

**Theorem 3.2.** Let $T \in \Sigma_\alpha^\beta$ and $\{P_t\}_{t \geq 0}$ be a semi-group defined by the convolution kernel $t^{-n}T(\frac{x-y}{t})$, where $t > 0$. Then, the action of $P_t$ on $(\Sigma_\alpha^\beta)'$ is given by the integration against the function

$$
\rho(x) = \langle S_y, t^{-n}T(\frac{x-y}{t}) \rangle,
$$

where $S_y \in (\Sigma_\alpha^\beta)'$ and $y$ indicates on which variable the functional $S$ acts.
We begin estimating \( I \) in Theorem 3.2.

**Proof:** Using Lemma 2.5 and Theorem 3.1, it is enough to show that \( T(\frac{x}{t}) \in \Sigma^\beta_\alpha \) for each \( t > 0 \). Note that

\[
\left| e^{k|x|^{\frac{1}{\alpha}}} T(\frac{x}{t}) \right| \leq \left| e^{k|\xi|^{\frac{1}{\alpha}}} T(\frac{x}{t}) \right|
\]

\[
\leq \left| e^{((|k|+1)|\xi|^{\frac{1}{\alpha}}) T(\frac{x}{t})} \right|
\]

\[
= \left| e^{m|\xi|^{\frac{1}{\alpha}}} T(\frac{x}{t}) \right|
\]

\[
\leq \left| e^{m|\xi|^{\frac{1}{\alpha}}} T \right|_\infty
\]

and

\[
\left| e^{k|\xi|^{\frac{1}{\beta}}} \hat{T}(\frac{x}{t})(\xi) \right| = \left| e^{k|\xi|^{\frac{1}{\beta}}} t\hat{T}(\xi) \right| = C_t \left| e^{k|\xi|^{\frac{1}{\beta}}} \hat{T}(\xi) \right|.
\]

Now if \( t \geq 1 \), then \( |\xi|^{\frac{1}{\beta}} \leq |t\xi|^{\frac{1}{\beta}} \) and therefore

\[
\left| e^{k|\xi|^{\frac{1}{\beta}}} \hat{T}(\xi) \right| \leq \left| e^{k|\xi|^{\frac{1}{\beta}}} \hat{T}(\xi) \right|
\]

\[
\leq \left| e^{k|\xi|^{\frac{1}{\beta}}} T \right|_\infty.
\]

For \( 0 < t < 1 \), we have

\[
\left| e^{k|\xi|^{\frac{1}{\beta}}} \hat{T}(\xi) \right| \leq \left| e^{kN|\xi|^{\frac{1}{\beta}}} \hat{T}(\xi) \right|
\]

\[
\leq \left| e^{kN|\xi|^{\frac{1}{\beta}}} T \right|_\infty,
\]

where \( N \) is an integer such that \( N \geq \frac{1}{\alpha} \). This completes the proof of Theorem 3.2. \( \square \)

**Theorem 3.3.** Let \( B \) be a bounded subset of \( \Sigma^\beta_\alpha \). Then

\[
\varphi_t(x) = (t^{-n}T(\frac{x-y}{t}), \varphi(x)) = \int_{\mathbb{R}^n} t^{-n}T(\frac{x-y}{t}) \varphi(y) dy \to \varphi
\]

in \( \Sigma^\beta_\alpha \) as \( t \to 0^+ \) uniformly on \( B \).

**Proof:** We note that \( \varphi_t \in \Sigma^\beta_\alpha \subset (\Sigma^\beta_\alpha)' \) for each \( t > 0 \). If \( 0 < t < 1 \) and \( z = \frac{x-y}{t} \), then for any \( \delta > 0 \), we can write

\[
e^{k|y|^{\frac{1}{\alpha}}} |\varphi_t(x) - \varphi(y)| = \int_{\mathbb{R}^n} e^{k|y|^{\frac{1}{\alpha}}} T(z) |\varphi(y + tz) - \varphi(y)| dz \leq I_1 + I_2 + I_3,
\]

where

\[
I_1 = \int_{|y| \leq \delta} e^{k|y|^{\frac{1}{\alpha}}} T(z) |\varphi(y + tz) - \varphi(y)| dz,
\]

\[
I_2 = \int_{|y| \geq \delta} e^{k|y|^{\frac{1}{\alpha}}} T(z) |\varphi(y + tz)| dz,
\]

\[
I_3 = \int_{|y| \geq \delta} e^{k|y|^{\frac{1}{\alpha}}} T(z) |\varphi(y)| dz.
\]

We begin estimating \( I_1 \). For each \( 0 < t < 1 \) and \( z \in \mathbb{R}^n \), there exists \( C > 0 \) such that

\[
e^{k|y|^{\frac{1}{\alpha}}} |\varphi(y + tz) - \varphi(y)| \leq Ct |z|.
\]
We note that property 2 in Remark 2.1 implies that there exist \( N \in \mathbb{N} \) and \( C > 0 \) such that \( |z| \leq C e^{N|z|^{1/\alpha}} \). Substituting this into \( I_1 \), we obtain the estimate

\[
I_1 \leq \int_{|y| \leq \delta} C t |z| T(z) dz 
\leq C \int_{|y| \leq \delta} t e^{N|z|^{1/\alpha}} T(z) dz 
\leq C \delta t \left\| e^{N \cdot} T \right\|_{\infty}.
\]

Next, we estimate \( I_2 \). Using the subadditivity of \( \cdot^{1/\alpha} \) and \( 0 < t < 1 \), we obtain

\[
I_2 \leq \int_{|y| \geq \delta} e^{k|y|^{1/\alpha}} T(z) |\varphi(y + tz)| dz 
= \int_{|y| \geq \delta} e^{k|y + tz - tz|^{1/\alpha}} T(z) |\varphi(y + tz)| dz 
\leq \int_{|y| \geq \delta} e^{k|tz|^{1/\alpha}} T(z) \left| e^{k|y + tz|^{1/\alpha}} \varphi(y + tz) \right| dz 
\leq \left\| e^{N \cdot} T \right\|_{\infty} \int_{|z| \geq \delta} e^{k|z|^{1/\alpha}} T(z) dz 
\leq C \int_{|z| \geq \delta} e^{k|z|^{1/\alpha}} T(z) dz.
\]

Finally, let us estimate \( I_3 \). We have

\[
I_3 = \int_{|z| \geq \delta} e^{k|z|^{1/\alpha}} T(z) |\varphi(y)| dz 
\leq \left\| e^{k \cdot} \varphi \right\|_{\infty} \int_{|z| \geq \delta} e^{k|z|^{1/\alpha}} T(z) dz.
\]

Therefore, if we choose \( \delta \) to be sufficiently large and \( t \) sufficiently small then the estimates in (3.2), (3.3) and (3.4) imply that \( \left\| e^{k|y|^{1/\alpha}} (\varphi_t(x) - \varphi(y)) \right\|_{\infty} \) converges to 0 as \( t \to 0^+ \).

Now to prove that \( \left\| e^{k|\xi|^{1/\beta}} \mathcal{F}(\varphi_t(x) - \varphi(y)) (\xi) \right\|_{\infty} \) converges to 0 as \( t \to 0^+ \), we consider

\[
ee^{k|\xi|^{1/\beta}} \left| \mathcal{F}(\varphi_t(x) - \varphi(y)) (\xi) \right| = e^{k|\xi|^{1/\beta}} \left| \mathcal{F} \left( \int_{\mathbb{R}^n} t^{-n} T(\frac{x - y}{t}) \varphi(y) dy \right)(\xi) - \mathcal{F}(\varphi(y))(\xi) \right| 
= e^{k|\xi|^{1/\beta}} \left| \mathcal{F} \left( \int_{\mathbb{R}^n} t^{-n} T(\frac{x - y}{t}) \varphi(y) dy \right)(\xi) - \mathcal{F}(\varphi(y))(\xi) \right| 
= e^{k|\xi|^{1/\beta}} \hat{\varphi}(\xi) \left| \mathcal{F}(T)(\xi) - 1 \right| 
\leq \left\| e^{k \cdot} \hat{\varphi} \right\|_{\infty} \left| \mathcal{F}(T)(\xi) - 1 \right| \leq C \left| \mathcal{F}(T)(\xi) - 1 \right|.
\]

Now by uniform continuity of \( \mathcal{F}(T)(\xi) \), we observe that \( \mathcal{F}(T)(\xi) \to \mathcal{F}(T)(0) = \left\| t^{-n} T(\xi) \right\|_1 = 1 \), which implies that \( C \left| \mathcal{F}(T)(\xi) - 1 \right| \to 0 \) as \( t \to 0^+ \) uniformly on compact subsets of \( \mathbb{R}^n \). Thus

\[
\left\| e^{k|\xi|^{1/\beta}} \mathcal{F}(\varphi_t(x) - \varphi(y)) \right\|_{\infty}
\]

converges to 0 uniformly on \( B \). This completes the proof of Theorem 3.3. \( \square \)
Example 3.4. Consider the heat kernel

\[ E(x, t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}, & \text{for } t > 0, \\ 0, & \text{for } t < 0. \end{cases} \]

It is known that \( \|E(\cdot, t)\|_1 = 1 \) for \( t > 0 \) (see [6]). Also, consider the Gauss-Weierstrass semigroups \( \{T_t\}_{t \geq 0} \) defined by the integration with respect to the heat kernel

\[ T_\sqrt{t}(\varphi)(x) = \langle E(x - y, t), \varphi(y) \rangle = \langle t^{-n/2}T\left(\frac{x - y}{\sqrt{t}}\right), \varphi(y) \rangle. \]

This semigroup generated by the Laplacian on \( \mathbb{R}^n \) and the function \( u(x, t) = T_\sqrt{t}(\varphi)(x) \) is a solution of the equation \( u_t - \Delta u = 0 \) with \( u(x, 0) = \varphi(x) \) for an appropriate \( \varphi \). That is, the convolution

\[ u(x, t) = E * \varphi \]

is the solution to the heat equation and

\[ u(x, 0) = \varphi(x) = \lim_{t \to 0^+} T_\sqrt{t}(\varphi)(x) \]

and the convergence is uniform on bounded subsets of \( \mathbb{R}^n \). Now it is clear that \( E(x, t) \in \Sigma^\beta_\alpha \) for all \( |\cdot|^{1/\alpha}, |\cdot|^{1/\beta} \) satisfying the properties in Remark 2.1 since \( E(x, t) \) is exponentially decreasing and using Theorem 3.2. Moreover, Theorem 3.2 implies that the action of \( T_\sqrt{t} \) on \( L \in (\Sigma^\beta_\alpha)' \) for all such \( |\cdot|^{1/\alpha}, |\cdot|^{1/\beta} \) can be defined by the integral against the function \( \rho(x) \) given in (3.1) and by using Theorem 3.1, we conclude that this is equivalent to

\[ T_\sqrt{t}(T) = \langle L_y, \langle t^{-n/2}T\left(\frac{x - y}{\sqrt{t}}\right), \phi(x) \rangle \rangle \]

which implies that \( \lim_{t \to 0^+} T_\sqrt{t}(T) = T \) in the sense of \( (\Sigma^\beta_\alpha)' \) and this is equivalent to

\[ \langle t^{-n/2}T\left(\frac{x - y}{\sqrt{t}}\right), \phi(x) \rangle \rangle \rightarrow \varphi \text{ in } (\Sigma^\beta_\alpha)' \text{ as } t \to 0^+. \]

As a result, the \( (\Sigma^\beta_\alpha)' \) tempered ultradistributions can be realized as boundary values of the equation \( u_t - Au = 0 \).

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