Abstract

This article, in a first step, considers two Bayes estimators for the relativity premium of a given Bonus–Malus system. It then develops a linear relativity premium that closes, in the sense of weighted mean square error loss, to such Bayes estimators. In a second step, it supposes that the claim size distribution for a given Bonus–Malus system can be formulated as a finite mixture distribution. It then evaluates the base premium under a Bayesian framework for such a finite mixture distribution. The Loimaranta efficiency of such a linear relativity premium, for several Bonus–Malus systems, has been compared with two Bayes and ordinary linear relativity premiums.

Keywords: Bonus–Malus system; Relativity premium; Bayes estimator; Weighted mean square error; Loimaranta efficiency.

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1. Introduction

A Bonus–Malus system is a popular actuarial tool that is based on the true risk of policyholders and categorizes them into a finite number of levels, numbered from 1 to $s$. The main purpose of a Bonus–Malus system is to determine the next year’s premium of classified policyholders based on their number of claims as well as their current level. A Bonus–Malus system is a commercial tool.
and practical version of a more general actuarial system, well known as the rate-making system. The rate-making system determines the next year’s premium of each policyholder based on its average number of claims in the last $t$ years. An “optimal” Bonus–Malus system can be designed through (1) pricing—i.e., evaluating both base and relativity premiums—of a given Bonus–Malus system; (2) determining an “appropriate” statistical model for random variables involved in a given Bonus–Malus system; (3) determining an “optimal” transition rule for a new Bonus–Malus system; and (4) a combination of the above tasks. There is considerable attention from researchers on the design of an optimal Bonus–Malus system (or rate-making system). For instance, Lange (1969) provided some useful actuarial and mathematical tools to determine the premium of a given rate-making system. Dionne & Vanasse (1992) employed Poisson and negative binomial regression models to consider available asymmetrical information whenever an actuary wants to estimate the accident distribution in an insurance rate-making system. Lemaire & Zi (1994) compared 30 Bonus–Malus systems with respect to 4 different criteria. Namely, they considered the stationary average premium level, the coefficient of variation of premiums, the efficiency of Bonus–Malus systems, and average retention under the Bonus–Malus system as 4 appropriated measures to study the optimality of a Bonus–Malus system. Based on these comparison studies, they provided several practical suggestions to design an appropriate Bonus–Malus system. Lemaire (1995) modelled the claim frequency of a given Bonus–Malus system by a negative binomial distribution and derived an estimate for premiums under a quadratic loss function. De nuit (1997) employed a Poisson–Goncharov distribution (introduced by Lefevre & Picard, 1996) to model the annual number of reported claims under a given Bonus–Malus system. Pinquet (1997) suggested the use of claim severity in designing a Bonus–Malus system. Moreover, the consideration of different types of claims has been suggested by Pinquet (1998). Walhin & Paris (1999) considered a finite mixture Poisson distribution for random claim frequency of a given Bonus–Malus system and derived a Bayesian premium. Denuit & Dhaene (2001) used an exponential loss function to calculate the relativity premiums of a given Bonus–Malus system. Frangos & Vrontos (2001) considered both
claim frequency and severity in designing an optimal Bonus–Malus system. Morillo & Bermúdez (2003) considered a Poisson–inverse Gaussian model to provide a Bayesian relativity premium under an exponential loss function. Boucher & Denuit (2006) developed a Bayesian relativity premium under a zero-inflated count model for panel data. In 2008, Boucher & Denuit extended Boucher & Denuit’s (2006) findings under quadratic and exponential loss functions. Bermúdez & Morata (2009) considered a Bonus–Malus system with two different types of claims. They employed a bivariate Poisson regression model to price such a Bonus–Malus system. In 2011, Bermúdez & Karlis, based on work of Bermúdez & Morata (2009), developed a situation in which the Bonus–Malus system has more than one type of claim and there exists a non-ignorable correlation between such types of claims. They used a Bayesian multivariate Poisson model to price Bonus–Malus systems. Chen & Li (2014) derived an optimal linear relativity premium from the surplus of insurers’ viewpoints. Their linear relativity premium, somehow, can be restated as a smoothing version of a Bayesian relativity premium under a quadratic loss function. Payandeh Najafabadi et al. (2015) employed the Payandeh Najafabadi (2010) method to derive a credibility formula for the relativity premium of a given rate-making system whenever count data have been sampled from a zero-inflated Poisson gamma distribution. Teimourian et al. (2015) employed the maximum entropy approach to determine a linear relativity premium for a given Bonus–Malus system under both long-run and short-run situations. In a Bonus–Malus system, each level’s premium is determined by multiplication of the base premium and the corresponding relativity premium of that level. Therefore, to determine the value of the premium in a Bonus–Malus system, one must determine both base and relativity premiums. The base premium has been evaluated using the size of the claim regardless of the given Bonus–Malus system. To evaluate the relativity premium, one must involve true but unobserved risk characteristics (risk parameters) of the levels of the given Bonus–Malus system. From a decision theory point of view, the Bayes estimator offers an intellectual and acceptable estimation for the relativity premium. Unfortunately, the Bayes estimator suffers from the following disadvantages: (1) In most cases, it cannot be restated as a convex combination of prior
and current observation means (see Payandeh Najafabadi 2010 for more details). Therefore, for such cases, the Bayesian relativity premium, from a computational viewpoint, is very time-consuming. (2) There is no guarantee that the Bayesian relativity premium—say, $r_{j}^{Bays}$ for $j = 1, \cdots, s$—satisfies the logical condition $a \leq r_{1}^{Bays} \leq r_{2}^{Bays} \leq \cdots \leq r_{s}^{Bays} \leq b$, where $a$ and $b$ are two given positive values. To eliminate the above disadvantages, Gilde & Sundt (1989), among others, suggested the following linear class of estimators for the relativity premium:

$$\mathcal{C} := \{r_{i}^{Lin} := \alpha + \beta l, \text{ such that } \alpha \& \beta \geq 0 \text{ and } l = 1, \cdots, s\}. \quad (1)$$

The two coefficients, $\alpha$ and $\beta$, have been determined by an optimal criterion, such as minimizing the average square distance between the relativity premium and the risk parameter. The goals of this article are twofold. First, it is supposed that for random claims, the given risk parameter $\eta$ under a Bonus–Malus system can be reformulated as a finite mixture model. It then provides an approximation for the Bayes estimator for the risk parameter $\eta$. This approximated Bayes estimator is very easy to compute. Moreover, it does not suffer from the “label-switching problem”. Second, it considers two Bayes estimators for the relativity premium, which are developed from the distribution of the number of reported claims and the steady-state distribution of the Bonus–Malus system. Within the class of linear relativity premiums (1), it then develops a linear relativity premium that simultaneously minimizes the square distances between the linear estimator and these Bayes estimators. A practical application of our finding, along with a comparison study, has been given for some Bonus–Malus systems. The rest of this article organized as follows. Section 2 collects some preliminary result that play a vital role in this article. The main results are represented in Section 3. Section 4 compares Loimaranta’s efficiency for the optimal linear relativity premium with the ordinary linear relativity premium and Bayes relativity premiums for some given Bonus–Malus systems. Concluding remarks along with some suggestions for future research are given in Section 5.
2. Preliminaries

This section collects some primary results that will be used in the future. For a complex-valued and integrable function $f$, the Fourier transform, say $\mathcal{F}(f)$, and the inverse Fourier transform, say $\mathcal{F}^{-1}(f)$, are defined by

$$\mathcal{F}(f; x; \omega) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x)e^{-ix\omega}dx$$

$$\mathcal{F}^{-1}(f; x; \omega) = \int_{\mathbb{R}} f(x)e^{ix\omega}dx,$$

where $\omega \in \mathbb{R}$. It is worth mentioning that the well-known characteristic function for a random variable may be viewed as the Fourier transform of the density/probability function of this random variable. The Hausdorff–Young theorem states that an $L^q(\mathbb{R})$ function $s$ and its corresponding Fourier transform $\mathcal{F}(s) \in L^{q^*}(\mathbb{R})$ satisfy $||\mathcal{F}(s)||_{q^*} \leq (2\pi s)^{-1/q}||s||_q$, where $\pi_s$ stands for the Pi number and $1 < q \leq 2$ and $1/q + 1/q^* = 1$; see Pandey (1996) for more details.

Based upon standard distributions, the mixture models provide statistical models that illustrate most aspects of complex systems; see Tallis (1969) and McLachla & Peel (2004), among others, for more details on mixture models. Unfortunately, most mixture models are not identifiable because they are invariant under permutations of the indices of their components. This identifiability problem is well known as the “label-switching problem”. The posterior distribution may also inherit the “label-switching problem” from a prior distribution that is also invariant under permutations (Rufo et al., 2007). Under the “label-switching problem”, there is a positive probability that one of the components in the mixture model does not contribute to any of the observations. Therefore, the sample $x_1, \cdots, x_n$ has no information about this component. Consequently, unknown parameter(s) of such a component cannot be estimated under either classical or Bayesian frameworks. A naïve solution to the “label-switching problem” is to impose some constraint on the parameter space for the classical approach (Maroufy & Marriott, 2015), and for the Bayesian approach, some constraints have been added to the prior distribution that lead to a posterior distribution that does not suffer from the “label-switching problem” (Marin et al., 2005). Unfortunately, insufficient care
in the choice of suitable identifiability constraints can lead to other problems (Rufo et al., 2006). A random variable $X$, given parameter $\eta$, has a finite mixture distribution with $k$ components if its corresponding density function can be reformulated as

$$f_X(x|\eta) = \sum_{i=1}^{k} v_i g_i(x|\eta),$$

(2)

where $g_i(x|\eta)$-s are some given density functions, $v_i \in [0, 1]$, for $i = 1, \cdots, k, \sum_{i=1}^{k} v_i = 1$. Many authors have employed the mixture distribution in an actuarial setting. For instance, Feldmann & Whitt (1998) showed that a large class of distributions, including several heavy tail distributions, can be approximated by a finite mixture of exponential distributions. Zhang & Kwok (2010) approximated a given mixture model with a simpler mixture model. They showed that this approach increases computational time and, in several cases, improves the results compared with other approximation methods. Bouguila (2011) employed a finite mixture approximation method to model count data. Payandeh Najafabadi (2015) approximated claim size distributions by a finite mixture exponential distribution. He then provided an accurate approximation for finite- and infinite-time ruin probabilities for compound Poisson processes. Suppose that there is a continuous random variable $X$, where the given risk parameter $\eta$ stands for the random claim size of a policyholder under an insurance contract. Moreover, suppose that (1) the policyholder under this insurance contract can be categorized into $s$ different risk levels and (2) prior information about risk parameter $\eta$ can be restated in terms of the following prior distribution function: $\pi(\eta) = \sum_{i=1}^{s} \omega_i \pi_i(\eta)$. Therefore, the posterior distribution for $\eta$ given $X = x$ can be restated as the following mixture posterior distribution.

$$\pi(\eta|X = x) = \frac{f_{X|\eta}(x)\pi(\eta)}{\int_{0}^{\infty} f_{X|\eta}(x)\pi(\eta)d\mu(\eta)} = \frac{\int_{0}^{\infty} f_{X|\eta}(x)\pi(\eta)d\mu(\eta)}{\int_{0}^{\infty} f_{X|\eta}(x)\sum_{i=1}^{s} \omega_i \pi_i(\eta) d\mu(\eta)}$$

$$= \sum_{i=1}^{s} \frac{\omega_i m_i(x)}{\sum_{i=1}^{s} \omega_i m_i(x)} \pi_i(\eta|x) = \sum_{i=1}^{s} \rho_i \pi_i(\eta|x),$$
where \( m_i(x) = \int_0^\infty \pi_i(\eta)f_X(x|\eta)d\mu(\eta) \) and \( \rho_i = \omega_i m_i(x)/(\sum_{i=1}^s \omega_i m_i(x)) \). Consequently, the Bayes estimator for risk parameter \( \eta \) under the squared error loss function is

\[
\delta_{\text{Bayes}}(x) = \mathbb{E}(\eta|x) = \int_0^\infty \eta \sum_{i=1}^s \rho_i \pi_i(\eta|x)d\mu(\eta) = \sum_{i=1}^s \rho_i \delta_{\text{Bayes}}^\pi_i(x).
\]

From the above result, it is concluded that the Bayes estimator for risk parameter \( \eta \) can be represented as a weighted combination of the Bayes estimator for each category. The above result cannot be generalized to a random sample size \( X_1, \ldots, X_n, n(>1) \). In this situation, one must employ an MCMC, a missing method, or a nonparametric Bayesian approach to estimate the parameters of a mixture model under a Bayesian framework. Unfortunately, all 3 of these approaches suffer from the “label-switching problem” and are computationally very time-consuming; see Marin et al. (2005) and Lin et al. (2014, §25), among others, for more details. Finding a closed form for the likelihood function based on random sample \( X_1, \ldots, X_n \) is the main problem. The following provides a likelihood function (joint distribution) for random sample \( X_1 \cdots, X_n \), under a finite mixture model.

**Theorem 1.** Suppose random sample \( X_1 \cdots, X_n \), given risk parameter \( \eta \), is sampled from the finite mixture density function \( f(t|\eta) = \sum_{i=1}^k \nu_i g_i(t|\eta) \), where \( 0 \leq \nu_i \leq 1 \) and \( \sum_{i=1}^k \nu_i = 1 \). The joint distribution function of random sample \( X_1 \cdots, X_n \), given risk parameter \( \eta \), can then be restated as

\[
\prod_{j=1}^n f(x_j|\eta) = \sum_{j_1=0}^n \cdots \sum_{j_k=0}^n \nu_{j_1}^{j_1} g_1(I_{j_1}|\eta) \cdots \nu_{j_k}^{j_k} g_k(I_{j_k}|\eta),
\]

where \( I_{j_1}, \ldots, I_{j_k} \) (for \( j_1, \cdots, j_k = 0, \cdots, n \), where \( j_1 + \cdots + j_k = n \)) are distinct partitions of random sample \( X_1 \cdots, X_n \), with \( j_1, \cdots, j_k \) elements, respectively.

**Proof.** The desired result is obtained by partitioning random sample \( X_1 \cdots, X_n \) into distinct partitions \( I_{j_1}, \ldots, I_{j_k} \). \( \square \)
Theorem (1) provided an exact joint density function of random sample $X_1 \cdots , X_n$, which are sampled from a finite mixture distribution. Certainly, the above finding cannot be employed in practical situations.

The following theorem studies a situation in which this joint density function was approximated by a finite mixture distribution. Hereafter, without loss of generality, we assume that the weights of our finite mixture density functions are equal. In a situation where some density functions have more weight, such density functions can be repeated to achieve this assumption.

**Theorem 2.** Suppose that random sample $X_1 \cdots , X_n$, given risk parameter $\eta$, is sampled from the finite mixture density function $f(t|\eta) = \sum_{i=1}^{k} g_i(t|\eta)/k$. Moreover, suppose that the joint distribution function of random sample $X_1 \cdots , X_n$, given risk parameter $\eta$, can be approximated by

$$n \prod_{j=1}^{n} f(x_j|\eta) \approx \frac{1}{k} \sum_{i=1}^{k} g_i(x_1, \cdots , x_n|\eta).$$

The error bound for the above approximation then satisfies

$$\left| n \prod_{j=1}^{n} f(x_j|\eta) - \frac{1}{k} \sum_{i=1}^{k} g_i(x_1, \cdots , x_n|\eta) \right| \leq \frac{M^k}{k^n} P(n),$$

where $M = \max\{f_x, g_1, \cdots , g_k\}$ and $P(\cdot)$ stands for the partition function.

**Proof.** Using the result of Theorem (1), observe that

$$\left| n \prod_{j=1}^{n} f(x_j|\eta) - \frac{1}{k} \sum_{i=1}^{k} g_i(x_1, \cdots , x_n|\eta) \right|$$

$$= \left| \sum_{j_1=0}^{n-1} \cdots \sum_{j_k=0}^{n-1} \frac{1}{k^n} \sum_{I_{j_1}, \cdots , I_{j_k}} g_1(I_{j_1}|\eta) \cdots g_k(I_{j_k}|\eta) - \frac{1}{k} \sum_{i=1}^{k} g_i(x_1, \cdots , x_n|\eta) \right|$$

$$\leq \frac{M^k}{k^n} \sum_{j_1=0}^{n-1} \cdots \sum_{j_k=0}^{n-1} \sum_{I_{j_1}, \cdots , I_{j_k}} \left| g_1(I_{j_1}|\eta) \cdots g_k(I_{j_k}|\eta) \right| = \frac{M^k}{k^n} P(n). \square$$
The following theorem studies a situation in which the density function of a continuous random variable has been approximated by a finite mixture density function.

**Theorem 3.** Suppose that random sample $X_1 \cdots, X_n$, given risk parameter $\eta$, is sampled from the density function $f(\cdot | \eta)$, and the density function $f(\cdot | \eta)$ is approximated by the finite mixture density function $\frac{1}{k} \sum_{i=1}^{k} g_i(\cdot | \eta)$. Moreover, suppose that the joint distribution function of random sample $X_1 \cdots, X_n$, given risk parameter $\eta$, is approximated by

$$\prod_{j=1}^{n} f(x_j | \eta) \approx \frac{1}{k} \sum_{i=1}^{k} g_i(x_1, \cdots, x_n | \eta).$$

An $L^p(\mathbb{R})$-norm of the error bound for the above approximation then satisfies

$$\left\| \prod_{j=1}^{n} f(x_j | \eta) - \prod_{j=1}^{n} \left( \frac{1}{k} \sum_{i=1}^{k} g_i(x_j | \eta) \right) \right\|_p \leq \frac{nM^{n-1}}{k} \sum_{i=1}^{k} ||\psi_X(\cdot | \eta) - \psi_i(\cdot | \eta)||_q,$$

where $1 < p \leq 2$, $1/q + 1/q = 1$, $M = \max\{f_x, g_1, \cdots, g_k\}$, and $\psi_X(\cdot | \eta), \psi_1(\cdot | \eta) \cdots \psi_k(\cdot | \eta)$ are the characteristic functions corresponding to density functions $f_X(\cdot | \eta), g_1(\cdot | \eta), \cdots, g_k(\cdot | \eta)$, respectively.

**Proof.** For briefness, set $f^*(\cdot | \eta) := \frac{1}{k} \sum_{i=1}^{k} g_i(\cdot | \eta)$. To obtain the desired result, employ the inequality $|\prod_{j=1}^{n} f(x_j | \eta) - \prod_{j=1}^{n} f^*(x_j | \eta)| \leq M^{n-1} \sum_{j=1}^{n} |f(x_j | \eta) - f^*(x_j | \eta)|$ (see Durrett, 2010, Lemma 3.4.3.) along with the triangle inequality, and observe that

$$\left\| \prod_{j=1}^{n} f(x_j | \eta) - \prod_{j=1}^{n} f^*(x_j | \eta) \right\|_p \leq M^{n-1} \sum_{j=1}^{n} ||f(x_j | \eta) - f^*(x_j | \eta)||_p.$$

An application of the Hausdorff–Young theorem completes the desired proof. \(\square\)

It is worth mentioning that an appropriate and practical approximation for the density function $f(\cdot | \eta)$ by the finite mixture density function $\frac{1}{k} \sum_{i=1}^{k} g_i(\cdot | \eta)$ arrives at a situation in which each $g_i(\cdot | \eta)$, for $i = 1, \cdots, k$, has a dimensional minimal sufficient statistic, say $T_i(n)$, for risk parameter $\eta$ based on random sample $X_1 \cdots, X_n$. Suppose that BMS stands for an ordinary Bonus–Malus system in which a given policyholder moves between its $s$ levels, numbered from 1 to $s$, according to the number of last year’s reported claims and the transition probability matrix $A$. Moreover,
suppose that $N_t$, given risk parameter $\theta$, stands for a counting process that represents the number of reported claims by a policyholder at year $t$. Assuming $N_t$, the given risk parameter $\theta$ is independent of the level of a policyholder. Denuit et al. (2007, §4) showed that the transition probability matrix $A$ can be reformulated as

$$A(\theta) = \sum_{n=0}^{\infty} T(n) P(N_t = n|\theta),$$

where $T(n) = [t_{ij}(n)]$, for $i, j = 1, \cdots, s$, stands for a matrix that describes the transition rules of the Bonus–Malus system as follows: $t_{ij}(k) = 1$ if by $k$ claims in a year, a policyholder goes from level $i$ to $j$, and $t_{ij}(k) = 0$ otherwise; see Denuit et al. (2007, §4) for more details. In the situation where $N_t$, given risk parameter $\theta$, is dependent on the level of a policyholder, the above result may be extended as follows.

**Corollary 1.** Suppose that the number of reported claims at year $t$ for a policyholder whose true level in a Bonus–Malus system $\text{BMS}$ is $i$ is distributed according to the counting process $N_t$ with the parameter $\theta_i$. Then, $i \times j$ element of the transition rules of the Bonus–Malus system $A$, say $a_{ij}(\theta_i)$, is $\sum_{n=0}^{\infty} t_{ij}(n) P(N_t = n|\theta_i)$, for $i, j = 1, \cdots, s$.

Assume that $L_t$ represents the level of a policyholder in year $t$. Because the Markovian condition is met by stochastic process $L_t$, one may consider $L_t$ as a Markov chain with transition probability matrix $A(\theta)$. Several authors discussed the appropriateness of the ordinary Markov chain to model a given Bonus–Malus system. For instance, Korolkiewicz & Elliot (2008) and Payandeh Najafabadi & Kanani Dizaji (2011) employed a hidden Markov model to study the behaviour of a given Bonus–Malus system. Hereafter, we consider $L_t$ to be a Markov chain with the transition probability matrix $A(\theta)$. The steady-state distribution for the Bonus–Malus system $\text{BMS}$ is presented as the long-run probabilistic behaviour of $\text{BMS}$. The steady-state distribution is a left-hand eigenvector of probability matrix $A(\theta)$ with eigenvalue 1, $\pi^{ss}(\theta) = (\pi_1^{ss}(\theta), \cdots, \pi_s^{ss}(\theta))'$. Suppose that $L$ stands for the level occupied by a randomly selected policyholder whenever the steady-state distribution is met by the Bonus–Malus system $\text{BMS}$. Norberg (1976) showed that the probability mass function
for random variable $L$ can be restated as

$$P(L = l) = \int_0^\infty \pi_i^{ss}(\theta) dF_\Theta(\theta),$$

(5)

where $\pi_i^{ss}(\cdot)$ and $F_\Theta(\cdot)$ stand for the steady-state distribution of level $l$ and the prior distribution (structural function) for risk parameter $\theta$, respectively.

The relativity for a policyholder who occupied level $l$, denoted by $r_l$, represents the amount of the base premium to be paid by this policyholder. Certainly, the relativity premium for low-risk policyholders is less than 1 (they received a bonus from the insurance company), and it is greater than 1 for high-risk policyholders who received a malus from the company. However, the relativity premium $r_l$ must be satisfied: $a \leq r_1 \leq r_2 \leq \cdots \leq r_s \leq b$, where $a$ and $b$ are two given constants determined by the insurance company to control the lowest and highest premiums under the Bonus–Malus system. The use of linear estimators and the use of Bayesian estimators are two well-known approaches to estimating the relativity premium. The linear estimator is the estimator within the class of $\Pi$ whose coefficients have been estimated under optimal criteria. Under the mean squared error optimality criteria, Gilde & Sundt (1999) showed that these coefficients are

$$\hat{\alpha}^{Ord.\,Lin} = E(\Theta) - Cov(\Theta, L)E(L)/Var(L)$$  

and

$$\hat{\beta}^{Ord.\,Lin} = Cov(\Theta, L)/Var(L).$$

The Bayes estimator, under the squared error loss function for the relativity premium, is obtained by minimizing the expectation of the squared distance between the true relativity premium $\Theta$ and its estimator $r_l$. Such minimization can be achieved by conditioning on either a random level of $L$ or a random number of reported claims $N$. The following provides such Bayes estimators.

**Lemma 1.** Suppose that BMS stands for an $s$-level Bonus–Malus system with the transition probability matrix $A(\theta)$. Moreover, suppose the following:

1) information on the number of reported claims $N_l$, given $\theta_l$, is available, and the true relativity premium for a policyholder at level $l$ is $\Theta_l$. The Bayes estimator with respect to the prior distribution $F_\Theta_l$ and under the squared error loss function is then

$$r_l^{(1)} := E(\Theta_l|N_l = n) = \frac{\int_0^\infty \theta P(N_l = n|\Theta_l = \theta \lambda) dF_\Theta_l(\theta)}{\int_0^\infty P(N_l = n|\Theta_l = \theta \lambda) dF_\Theta_l(\theta)}.$$  

(6)
2) $\Theta_l$ stands for the true relativity premium for a randomly selected policyholder in level $l = 1, \ldots, s$.

The Bayes estimator with respect to the prior distribution $F_{\Theta_l}$ and under the squared error loss function is then

$$r_l^{(2)} := E(\Theta_l | L = l) = \frac{\int_0^\infty \theta \pi_{lss}(\theta \lambda) dF_{\Theta_l}(\theta)}{\int_0^\infty \pi_{lss}(\theta \lambda) dF_{\Theta_l}(\theta)}$$

whenever information on random level $L$ is considered, and $\lambda$ stands for the a priori expected claim frequency.

Proof. Part (1) The desired results are obtained by conditioning $E((\Theta_L - r_L)^2)$ on the random variable $N_l$. For part (2), one must find the Bayes estimator by minimizing $E((\Theta_L - r_L)^2)$. This estimator is obtained by conditioning on the random variable $L$. □

In the situation where (1) $N_l$, given risk parameter $\theta_l$, is distributed according to either a Poisson distribution or a zero-inflated Poisson distribution and (2) information about risk parameter $\theta_l$ can be reformulated as $\text{Gamma}(a_l, b_l)$, the above Bayes estimator $r_l^{(1)}$ can be simplified as $r_l^{(1)} = (n + a_l)/(\lambda + b_l)$ for a Poisson distribution and $r_l^{(1)} = [pb_l^{-a_l-1} + (1-p)(\lambda + b_l)^{-a_l-1}]/[pb_l^{-a_l} + (1-p)(\lambda + b_l)^{-a_l}] 1_{\{0\}}(n) + [n + a_l]/[\lambda + b_l] 1_{\{1,2,\ldots\}}(n)$ for a zero-inflated Poisson distribution. Moreover, under these assumptions, the Bayes estimator $r_l^{(2)}$ can be simplified as

$$r_l^{(2)} = \frac{\int \theta e^{-\lambda \theta} (\lambda \theta)^n e^{-b_l \theta} \theta^{a_l-1} \pi_{lss}(\theta) d\theta}{\int e^{-\lambda \theta} (\lambda \theta)^n e^{-b_l \theta} \theta^{a_l-1} \pi_{lss}(\theta) d\theta}$$

The Loimaranta efficiency is a statistical tool that measures the change of an expected premium paid by a policyholder subject to a Bonus–Malus system as a function of its annual expected claim frequency. The Loimaranta efficiency of an optimal Bonus–Malus system increases with increasing annual expected claim frequency. Greater response to increases in the annual expected claim frequency represents greater appropriateness of the Bonus–Malus system. The Loimaranta efficiency $Eff_{\text{Loi}}$ for the annual expected claim frequency $\vartheta$ is given by

$$Eff_{\text{Loi}}(\vartheta) = \frac{d \ln R(\vartheta)}{d \ln (\vartheta)},$$

(8)
where $\bar{R}(\vartheta) = \sum_{i=1}^{s} r_i \pi_i^{s}(\vartheta)$; see Loimaranta (1972) for more details. The Loimaranta efficiency measures how the average relativity premium that must be paid by a policyholder who stays in a Bonus–Malus system for a long time responds to the change of annual expected claim frequency. An ideal efficiency should be close to 1 for the most common values of annual expected claim frequency $\vartheta$. It is necessary to say that the Loimaranta efficiency can be greater than 1; see De Pril (1978) for more details.

3. Main Results

This section develops the base and relativity premiums for the given Bonus–Malus system. Namely, the base premium has been evaluated from a Bayesian framework, whereas the relativity premium is determined through a linear approach. To develop a Bayes estimator for the base premium, we suppose that the claim size random variable $X$, given risk parameter $\eta$, can be restated (approximately or exactly) as a finite mixture distribution. Moreover, we suppose that the prior information on risk parameter $\eta$ can be reformulated as $s$ different prior distributions for $s$ classes of the Bonus–Malus system. More precisely, the prior information on the risk parameter $\eta$ can be restated as a mixture distribution function with $s$ components.

3.1. Bayesian approach to the base premium

The Bonus–Malus system, based on the risk of policyholders, categorized them into $s$ different risk classes. As mentioned above, the premium of each class is determined by multiplying the estimate of the risk parameter for the claim size by the estimate of the risk parameter for the number of reported claims. In the ordinary approach to evaluating the risk parameter for the claim size, say $\eta$, the level of the Bonus–Malus system is not considered, so we suppose that the random claim size $X$, given risk parameter $\eta$, is distributed according to a single (even unimodal) density function.
Moreover, we suppose that the prior information on risk parameter \( \eta \) can be reformulated as a single (even unimodal) prior distribution. Certainly, policyholders’ risk levels impact their claim size and risk parameters. Therefore, these two assumptions will, almost certainly, be violated in practice. To eliminate these two barriers, this section supposes that both the claim size distribution and the prior information of the risk parameter are two finite mixture distributions. It then develops the Bayes estimator for risk parameter \( \eta \). Unfortunately, for the joint distribution function of random sample \( X_1, \ldots, X_n \), the given risk parameter \( \eta \) cannot be restated in closed form whenever the common density function is a finite mixture distribution. Theorem (3) provides an approximation for this joint distribution function.

The Bayes estimator for the finite mixture model cannot be found in a closed form, and one must employ an MCMC method, such a Gibbs sampler (McLachlan & Peel, 2004, §4); a missing method; or a nonparametric Bayesian approach to evaluate it numerically (Marin et al., 2005 and Lin et al., 2014, §25). All three of these approaches suffer from the “label-switching problem” and are computationally very time-consuming; see Marin et al. (2005) for more details.

The following provides an approximation for the Bayes estimator under a finite mixture model. This approximated Bayes estimator is very easy to compute and does not suffer from the “label-switching problem”.

**Theorem 4.** Suppose that nonnegative random sample \( X_1, \ldots, X_n \), given risk parameter \( \eta \), is sampled from the density function \( f(\cdot|\eta) \). Moreover, suppose that the joint density function \( f_{X_1, \ldots, X_n}(\cdot \cdot \cdot |\eta) \) is approximated by the finite mixture density function \( f_{X_1, \ldots, X_n}(\cdot \cdot \cdot |\eta) = \frac{1}{k} \sum_{i=1}^{k} g_i(x_1, \ldots, x_n|\eta) \), where \( T_i(n) \), for \( i = 1, \ldots, k \), is a one-dimensional minimal sufficient statistic for risk parameter \( \eta \) based on random sample \( X_1, \ldots, X_n \) with respect to the density function \( g_i(\cdot|\eta) \). Under the mixture prior distribution \( \pi(\eta) = \sum_{l=1}^{s} \omega_l \pi_l(\eta) \) and the squared-error loss function, we have the following:
(1) The Bayes estimator for $\eta$ can be approximated by

$$\delta_{\pi,f}^{Bayes}(x_1 \cdots , x_n) \approx \delta_{\pi,f^*}^{Bayes}(x_1 \cdots , x_n)$$

$$= \sum_{i=1}^{k} \sum_{l=1}^{s} \rho_{i,l}(x_1, \cdots , x_n) \delta_{\pi_1;\xi_l}(T_i(n)),$$

where $\rho_{i,l}(x_1, \cdots , x_n) = \omega_i m_{i,l}(x_1, \cdots , x_n)/ \left(\sum_{i=1}^{k} \sum_{l=1}^{s} \omega_i m_{i,l}(x_1, \cdots , x_n) \right)$ and $m_{i,l}(x_1, \cdots , x_n) = \int_0^\infty g_i(x_1, \cdots , x_n) \eta \pi_l(\eta) d\eta$.

(2) The $L_p(\mathbb{R})$-norm for the error bound of this approximation satisfies

$$||\delta_{\pi,f}^{Bayes} - \delta_{\pi,f^*}^{Bayes}||_p \leq \frac{nM^2}{km^2_p / \sqrt{2\pi_s}} \sum_{i=1}^{k} \sum_{l=1}^{s} \omega_l \int_0^\infty \eta \pi_l(\eta) ||\psi(\cdot) - \psi_l(\cdot)||_q d\eta$$

$$+ \frac{nM^2}{km^2_p / \sqrt{2\pi_s}} \sum_{i=1}^{k} \sum_{l=1}^{s} \omega_l \int_0^\infty \eta \pi_l(\eta) ||\psi(\cdot) - \psi_l(\cdot)||_q d\eta,$$

where $m_p = \min\{||\int_0^\infty f(\cdot) \pi(\eta) d\eta||_p, ||\int_0^\infty g_1(\cdot) \pi(\eta) d\eta||_p, \cdots, ||\int_0^\infty g_k(\cdot) \pi(\eta) d\eta||_p\}$, $a = \int_0^\infty \eta \pi(\eta) d\eta$, $M = \max\{f, g_1, \cdots, g_k\}$, $\pi_* = 3.141592654 \cdots$, $1/p + 1/q = 1$, and $1 \leq p \leq 2$.

**Proof.** An application of Theorem (3) completes the proof of Part (i). Using Jensen’s inequality (with an absolute-valued function) along with the integral version of Minkowski’s inequality (Beckenbach & Bellman, 2012, Page 22), one may conclude that

$$||\delta_{\pi,f}^{Bayes} - \delta_{\pi,f^*}^{Bayes}||_p \leq \left|\left| \int_0^\infty \eta \pi(\eta) \left| \frac{f_{X_1,\cdots,X_n}(\cdots |\eta)}{\pi(\eta)f_{X_1,\cdots,X_n}(\cdots |\eta)} - \frac{f^*_{X_1,\cdots,X_n}(\cdots |\eta)}{\pi(\eta)f^*_{X_1,\cdots,X_n}(\cdots |\eta)} \right| d\eta \right|_p \right.$$}

$$\leq \int_0^\infty \eta \pi(\eta) \left|\left| \frac{f_{X_1,\cdots,X_n}(\cdots |\eta)}{\pi(\eta)f_{X_1,\cdots,X_n}(\cdots |\eta)} - \frac{f^*_{X_1,\cdots,X_n}(\cdots |\eta)}{\pi(\eta)f^*_{X_1,\cdots,X_n}(\cdots |\eta)} \right| d\eta \right|_p.$$

Using the extended Jensen’s inequality for $L_p$-norm (with $\phi(t) = 1/t$ for $t>0$) as well as the triangle inequality, the above inequality can be simplified as

$$||\delta_{\pi,f}^{Bayes} - \delta_{\pi,f^*}^{Bayes}||_p \leq \frac{M}{m^2_p} \int_0^\infty \eta \pi(\eta) \left|\left| f_{X_1,\cdots,X_n}(\cdots |\eta) - f^*_{X_1,\cdots,X_n}(\cdots |\eta) \right| d\eta \right|_p$$

$$+ \frac{Ma}{m^2_p} \int_0^\infty \eta \pi(\eta) \left|\left| f_{X_1,\cdots,X_n}(\cdots |\eta) - f^*_{X_1,\cdots,X_n}(\cdots |\eta) \right| d\eta \right|_p.$$
The desired results will now be obtained by an application of Theorem (3). □

To show the practical application of Theorem (4), two examples are now provided.

**Example 1.** Suppose that the random sample claim size $X_1, \ldots, X_n$, given risk parameter $\eta$, is distributed according to the following finite mixture distribution.

$$f_X(x) = \frac{1}{3}\text{LogNormal}(\eta, 1) + \frac{1}{3}\text{LogNormal}(\eta, 1) + \frac{1}{3}\text{Normal}(\eta, 1).$$

Moreover, suppose that the prior information about risk parameter $\eta$ can be reformulated as $\pi(\eta)$, with support $[0, \infty)$.

Using Theorem (4), one may show that the Bayes estimator for the risk parameter $\eta$ (and consequently the base premium) is

$$\delta_{\pi,f}^\text{Bayes}(x_1, \ldots, x_n) = \frac{2 \int_0^\infty \eta^2 \pi(\eta) \exp\{-\frac{1}{2}(T_1 - 2\eta T_2 + 2T_2 + n\eta^2)\}d\eta + \int_0^\infty \eta \pi(\eta) \exp\{-\frac{1}{2}(T_3 - 2\eta T_4 + n\eta^2)\}d\eta}{2 \int_0^\infty \pi(\eta) \exp\{-\frac{1}{2}(T_1 - 2\eta T_2 + 2T_2 + n\eta^2)\}d\eta + \int_0^\infty \pi(\eta) \exp\{-\frac{1}{2}(T_3 - 2\eta T_4 + n\eta^2)\}d\eta},$$

where $T_1 = \sum_{j=1}^n \ln^2(x_j)$, $T_2 = \sum_{j=1}^n \ln(x_j)$, $T_3 = \sum_{j=1}^n x_j^2$, and $T_4 = \sum_{j=1}^n x_j$.

**Example 2.** Suppose that the random sample claim size $X_1, \ldots, X_n$, given risk parameter $\eta$, is distributed according to the following finite mixture distribution.

$$f_X(x) = \frac{1}{2}\text{Gamma}(2, \eta) + \frac{1}{2}\text{ParetoTypeI}(0.3, \eta).$$

Moreover, suppose that the prior information about risk parameter $\eta$ can be reformulated as $\pi(\eta)$, with support $[0, \infty)$.

Using Theorem (4), one may show that the Bayes estimator for the risk parameter $\eta$ (and consequently the base premium) is

$$\delta_{\pi,f}^\text{Bayes}(x_1, \ldots, x_n) = \frac{\int_0^\infty \eta^2 \pi(\eta) \exp\{\frac{1}{2}(T_2 - \eta T_4)\}d\eta + \int_0^\infty \eta \pi(\eta) \exp\{(-\eta + 1)T_2\} I_{[0,3,\infty)}(x(1))d\eta}{\int_0^\infty \eta^2 \pi(\eta) \exp\{\frac{1}{2}(T_2 - \eta T_4)\}d\eta + \int_0^\infty \eta \pi(\eta) \exp\{(-\eta + 1)T_2\} I_{[0,3,\infty)}(x(1))d\eta},$$

where $T_2 = \sum_{j=1}^n \ln(x_j)$, $T_4 = \sum_{j=1}^n x_j$, $x(1) = \min\{x_1, \ldots, x_n\}$, and $I_A(x)$ stands for the indicator function.
3.2. An optimal linear relativity premium

From a decision theory point of view, the Bayes estimator offers an intellectual and acceptable estimation for the relativity premium. Unfortunately, two Bayes estimators for the relativity premium, given by Lemma (1), are computationally very time-consuming, and there is no guarantee that such estimators satisfy logical condition $a \leq r_1 \leq r_2 \leq \cdots \leq r_s \leq b$; see Denuit et al. (2007) for more details. To eliminate the above restrictions, Gilde & Sundt (1999) suggested the linear estimator $r^\text{Lin}_l$, within class $C$ given by (1), for the relativity premium, which is the minimized mean square error $E(\Theta - r^\text{Lin}_L)^2$. The following theorem employs the weighted mean square error method to provide the linear estimator for the relativity premium, which is simultaneously close to both Bayes estimators given by Lemma (1).

**Theorem 5.** Suppose that BMS stands for an $s$-level Bonus–Malus system with transition probability matrix $A(\theta)$. Moreover suppose (1) that the number of reported claims by a policyholder in level $l$, say $N_l$, with given risk parameter $\theta_l$, is distributed according to the given probability mass function $P(N_l = n|\theta_l)$ and (2) prior information about the risk parameter $\theta_l$ is reformulated by the cumulative distribution $F_{\Theta_l}$. Within the class of linear estimator $C$, the linear relativity premium

$$r^\text{opt}_l = \alpha^\text{opt} + \beta^\text{opt}l$$

minimized the weighted mean square distance between Bayesian relativity estimators $r^\text{opt}_l$ and $r^\text{Lin}_l$, where

$$\alpha^\text{opt} = \xi E(r^\text{opt}_L) + (1 - \xi) E(r^{\text{Lin}}_L) - \frac{E(L)}{\text{Var}(L)} \left[ \xi \text{Cov}(L, r^\text{opt}_L) + (1 - \xi) \text{Cov}(L, r^{\text{Lin}}_L) \right]$$

$$\beta^\text{opt} = \frac{1}{\text{Var}(L)} \left[ \xi \text{Cov}(L, r^\text{opt}_L) + (1 - \xi) \text{Cov}(L, r^{\text{Lin}}_L) \right]$$

$E(r^\text{opt}_1) = \sum_{l=1}^s r^\text{opt}_l P(L = l)$, $E(r^{\text{Lin}}_1) = \sum_{l=1}^s r^{\text{Lin}}_l P(L = l)$, $\text{Cov}(L, r^\text{opt}_L) = \sum_{l=1}^s l \int_0^\infty r^\text{opt}_l \pi^\text{ss}_l(\theta) dF_{\Theta_l}(\theta) - E(r^\text{opt}_1) E(L)$, $\text{Cov}(L, r^{\text{Lin}}_L) = \sum_{l=1}^s l \int_0^\infty r^{\text{Lin}}_l \pi^\text{ss}_l(\theta) dF_{\Theta_l}(\theta) - E(r^{\text{Lin}}_1) E(L)$, $\xi$ is a given number in $[0,1]$, and $r^\text{opt}_1$ and $r^{\text{Lin}}_1$ are given by Lemma (1).
Proof. The weighted mean square distance between Bayesian relativity estimators $r_l^{(1)}$, $r_l^{(2)}$ and linear relativity premium $r_L^{Lin}$ within class $C$, given by (1), can be restated as

$$WMSE(\alpha, \beta) = \xi E(r_L^{(1)} - r_L^{Lin})^2 + (1 - \xi)E(r_L^{(2)} - r_L^{Lin})^2,$$

where $\xi$ is a given number in $[0, 1]$. Setting the derivative of $WMSE(\alpha, \beta)$ with respect to $\alpha$ (and with respect to $\beta$) equal to 0 yields $\alpha^{opt}$ and $\beta^{opt}$. To show that these $\alpha^{opt}$ and $\beta^{opt}$ minimize $WMSE(\alpha, \beta)$, one must show that its corresponding Hessian matrix is positive semi-definite. This can be achieved by showing that the trace and determinate of the Hessian matrix are nonnegative. Because $\partial^2 WMSE(\alpha, \beta)/\partial \alpha^2 = 1$, $\partial^2 WMSE(\alpha, \beta)/\partial \alpha \partial \beta = \partial^2 WMSE(\alpha, \beta)/\partial \beta \partial \alpha = E(L)$, $\partial^2 WMSE(\alpha, \beta)/\partial \beta^2 = E(L^2)$, one may show that the trace and determinate of the Hessian matrix are $1 + E(L^2) \geq 0$ and $E(L^2) - E^2(L) = Var(L) \geq 0$, respectively. This observation completes the desired results. □

4. Practical applications

This section considers the Bonus–Malus system of Ireland (Hong Kong), Kenya, and Brazil to show the application of our findings. Table 1 shows such Bonus–Malus systems.

| Country          | Number of classes | Starting level | Scale   |
|------------------|-------------------|----------------|---------|
| Ireland (Hong Kong) | 6                 | 6              | -1/ +3  |
| Kenya            | 7                 | 7              | -1 / Top|
| Brazil           | 7                 | 7              | -1/+1   |

Two base and relativity premiums for the Bonus–Malus systems given in Table 1 have been evaluated using the methods developed above.

3Because these three Bonus–Malus systems have been studied by Lemaire & Zi (1994), we reconsider them for our study. It is worth mentioning that our results can be employed for any Bonus–Malus system in which policyholders move between its levels according to their number of reported claims.
Relativity premium

To evaluate the relativity premium for these Bonus–Malus systems, we suppose that the number of reported claims for a policyholder at level $l$, say $N_l$, given risk parameter $\theta_l$, has been distributed according to either a Poisson distribution or a zero-inflated Poisson distribution. We then evaluate the relativity premium using the ordinary linear approach (given by Gilde & Sundt, 1999), both Bayes estimators (given by Lemma 1) and the optimal linear relativity premium (given by Theorem 5) whenever $\hat{\lambda} = 0.1474$ (Denuit et al., 2007, Page 91). These four estimators for different values of $\lambda$ have been compared using the Loimaranta efficiency, given by Equation (8). Tables 2 to 4 show four such estimators.
Table 2: Relativity premium under Kenya’s Bonus–Malus system.

| $l$ | $\omega_l$ | $\pi_l$ | $P(L = l)$ | $N_l|\theta_l \sim \text{Poisson}(\theta_l)$ | $N_l|\theta_l \sim \text{ZIPoisson}(\theta_l)$ |
|-----|-------------|---------|-------------|---------------------------------|---------------------------------|
|     |             |         |             | $r^{(1)}_l$ | $r^{(2)}_l$ | $r^{(opt)}_l$ | $P(L = l)$ | $r^{(1)}_l$ | $r^{(2)}_l$ | $r^{(opt)}_l$ |
| 1   | $\Gamma(1,7)$ | 0.486   | 0.143       | 0.796       | 0.139       | 0.635       | 0.143       | 0.133       | 0.901       | 0.142       |
| 2   | $\Gamma(3,7)$ | 0.051   | 0.429       | 0.513       | 0.894       | 0.436       | 0.045       | 0.429       | 0.534       | 0.972       | 0.440       |
| 3   | $\Gamma(5,7)$ | 0.061   | 0.714       | 0.783       | 0.992       | 0.732       | 0.050       | 0.714       | 0.810       | 1.043       | 0.737       |
| 4   | $\Gamma(7,7)$ | 0.073   | 1.000       | 1.065       | 1.90        | 1.029       | 0.056       | 1.000       | 1.093       | 1.113       | 1.035       |
| 5   | $\Gamma(9,7)$ | 0.088   | 1.286       | 1.358       | 1.188       | 1.326       | 0.063       | 1.286       | 1.383       | 1.184       | 1.333       |
| 6   | $\Gamma(11,7)$ | 0.108  | 1.571       | 1.663       | 1.287       | 1.622       | 0.071       | 1.571       | 1.679       | 1.254       | 1.631       |
| 7   | $\Gamma(13,7)$ | 0.133  | 1.857       | 1.980       | 1.385       | 1.919       | 0.080       | 1.857       | 1.980       | 1.325       | 1.928       |

Table 3: Relativity premium under Hong Kong’s Bonus–Malus system.

| $l$ | $\omega_l$ | $\pi_l$ | $P(L = l)$ | $N_l|\theta_l \sim \text{Poisson}(\theta_l)$ | $N_l|\theta_l \sim \text{ZIPoisson}(\theta_l)$ |
|-----|-------------|---------|-------------|---------------------------------|---------------------------------|
|     |             |         |             | $r^{(1)}_l$ | $r^{(2)}_l$ | $r^{(opt)}_l$ | $P(L = l)$ | $r^{(1)}_l$ | $r^{(2)}_l$ | $r^{(opt)}_l$ |
| 1   | $\Gamma(1,6)$ | 0.699   | 0.167       | 0.158       | 0.165       | 0.917       | 0.870       | 0.167       | 0.163       | 0.166       | 0.985       |
| 2   | $\Gamma(3,6)$ | 0.090   | 0.500       | 0.630       | 0.535       | 1.036       | 0.045       | 0.500       | 0.645       | 0.50        | 1.041       |
| 3   | $\Gamma(5,6)$ | 0.109   | 0.833       | 0.961       | 0.904       | 1.154       | 0.054       | 0.833       | 0.989       | 0.914       | 1.096       |
| 4   | $\Gamma(7,6)$ | 0.052   | 1.167       | 1.425       | 1.273       | 1.273       | 0.016       | 1.167       | 1.460       | 1.288       | 1.152       |
| 5   | $\Gamma(9,6)$ | 0.017   | 1.500       | 1.701       | 1.643       | 1.392       | 0.005       | 1.500       | 1.768       | 1.662       | 1.208       |
| 6   | $\Gamma(11,6)$ | 0.033  | 1.833       | 2.168       | 2.012       | 1.510       | 0.009       | 1.833       | 2.156       | 2.036       | 1.263       |

Table 4: Relativity premium under Brazil’s Bonus–Malus system.

| $l$ | $\omega_l$ | $\pi_l$ | $P(L = l)$ | $N_l|\theta_l \sim \text{Poisson}(\theta_l)$ | $N_l|\theta_l \sim \text{ZIPoisson}(\theta_l)$ |
|-----|-------------|---------|-------------|---------------------------------|---------------------------------|
|     |             |         |             | $r^{(1)}_l$ | $r^{(2)}_l$ | $r^{(opt)}_l$ | $P(L = l)$ | $r^{(1)}_l$ | $r^{(2)}_l$ | $r^{(opt)}_l$ |
| 1   | $\Gamma(1,7)$ | 0.819   | 0.143       | 0.140       | 0.972       | 0.141       | 0.860       | 0.143       | 0.140       | 0.983       | 0.142       |
| 2   | $\Gamma(3,7)$ | 0.117   | 0.429       | 0.561       | 1.068       | 0.499       | 0.099       | 0.429       | 0.563       | 1.068       | 0.495       |
| 3   | $\Gamma(5,7)$ | 0.038   | 0.714       | 0.981       | 1.163       | 0.846       | 0.027       | 0.714       | 0.983       | 1.154       | 0.848       |
| 4   | $\Gamma(7,7)$ | 0.015   | 0.999       | 1.399       | 1.259       | 1.199       | 0.009       | 1.000       | 1.402       | 1.240       | 1.201       |
| 5   | $\Gamma(9,7)$ | 0.006   | 1.286       | 1.812       | 1.354       | 1.551       | 0.003       | 1.286       | 1.817       | 1.325       | 1.555       |
| 6   | $\Gamma(11,7)$ | 0.003  | 1.571       | 2.218       | 1.449       | 1.904       | 0.001       | 1.571       | 2.228       | 1.411       | 1.908       |
| 7   | $\Gamma(13,7)$ | 0.002  | 1.857       | 2.362       | 1.545       | 2.257       | 0.001       | 1.857       | 2.368       | 1.496       | 2.261       |
Figure 1 illustrates the behaviour of the Loimaranta efficiency for four relativity premiums against the a priori expected claim frequency $\lambda$.

Form Figure 1, one may observe that the Loimaranta efficiency of the linear relativity premium is improved by using the optimal linear relativity premium for all $\lambda \in [0,1]$. Moreover, for some $\lambda$, the Loimaranta efficiency of the optimal linear relativity is relativity close to both Bayes relativity premiums.

**Base premium**

To derive the base premium, we suppose that random sample claim size $X_1, \ldots, X_n$, given risk parameter $\eta$, has been distributed according to one of the following four models. Moreover, we suppose that risk parameter $\eta$ has prior distribution $\pi_1(\cdot)$ or $\pi_2(\cdot)$ for the Bonus–Malus system that has 7 or 6 levels, respectively.

**Model 1:** Consider the mixture density function given by Example (1) with $n = 20, T_1 = 188.7745, T_2 = 56.95046, T_3 = 86422.7, \text{ and } T_4 = 691.2832$. Moreover, suppose that the number of reported claims for a policyholder at level $l$, say $N_l$, given risk parameter $\theta_l$, is distributed according to a Poisson distribution.

**Model 2:** Consider the mixture density function given by Example (2) with $n = 200, T_2 = 201.1964, T_4 = 676.6038, \text{ and } x_{(1)} = 0.3159083$. Moreover, suppose that the number of reported claims for a policyholder at level $l$, say $N_l$, given risk parameter $\theta_l$, is distributed according to a Poisson distribution.

**Model 3:** Consider the mixture density function given by Example (1) with $n = 20, T_1 = 188.7745, T_2 = 56.95046, T_3 = 86422.7, \text{ and } T_4 = 691.2832$. Moreover, suppose that the number of
reported claims for a policyholder at level \( l \), say \( N_l \), given risk parameter \( \theta_l \), is distributed according to a zero-inflated Poisson distribution.

**Model 4:** Consider the mixture density function given by Example (2) with \( n = 200 \), \( T_2 = 201.1964 \), \( T_4 = 676.6038 \), and \( x_{(1)} = 0.3159083 \). Moreover, suppose that the number of reported claims for a policyholder at level \( l \), say \( N_l \), given risk parameter \( \theta_l \), is distributed according to a zero-inflated Poisson distribution.

\[
\begin{align*}
\pi_1(\eta) &= \frac{1}{7} \text{Gamma}(1, 7) + \frac{1}{7} \text{Gamma}(3, 7) + \frac{1}{7} \text{Gamma}(5, 7) + \frac{1}{7} \text{Gamma}(7, 7) + \frac{1}{7} \text{Gamma}(9, 7) \\
&\quad + \frac{1}{7} \text{Gamma}(11, 7) + \frac{1}{7} \text{Gamma}(13, 7), \\
\pi_2(\eta) &= \frac{1}{6} \text{Gamma}(1, 6) + \frac{1}{6} \text{Gamma}(3, 6) + \frac{1}{6} \text{Gamma}(5, 6) \\
&\quad + \frac{1}{6} \text{Gamma}(7, 6) + \frac{1}{6} \text{Gamma}(9, 6) + \frac{1}{6} \text{Gamma}(11, 6).
\end{align*}
\]

Under the above conditions, Tables 5 to 7 report the base premium along with the optimal relativity and level premiums.
Figure 1: The Loimaranta efficiency for the four relativity premiums under: Kenya’s Bonus–Malus system, whenever $N_l|\theta_l \sim Poisson(\theta_l)$ (Part a); Kenya’s Bonus–Malus system, whenever $N_l|\theta_l \sim ZIPoisson(\theta_l)$ (Part b) Hong Kong’s Bonus–Malus system, whenever $N_l|\theta_l \sim Poisson(\theta_l)$ (Part c); Hong Kong’s Bonus–Malus system, whenever $N_l|\theta_l \sim ZIPoisson(\theta_l)$ (Part d); and Brazil’s Bonus–Malus system, whenever $N_l|\theta_l \sim Poisson(\theta_l)$ (Part e); Brazil’s Bonus–Malus system, whenever $N_l|\theta_l \sim ZIPoisson(\theta_l)$ (Part f)
Table 5: Relativity, Base and level premiums under Kenya’s Bonus–Malus system, for such four models.

|   | Model 1 |          | Model 2 |          | Model 3 |          | Model 4 |          |
|---|---------|----------|---------|----------|---------|----------|---------|----------|
|   | $r_1$   | Base     | $r_1$   | Base     | $r_1$   | Base     | $r_1$   | Base     |
| 1 | 0.139   | 2.705    | 0.376   | 0.139    | 0.984   | 0.137    | 0.142   | 2.705    |
| 2 | 0.436   | 2.705    | 1.179   | 0.436    | 0.984   | 0.429    | 0.440   | 2.705    |
| 3 | 0.732   | 2.705    | 1.983   | 0.732    | 0.984   | 0.721    | 0.737   | 2.705    |
| 4 | 1.029   | 2.705    | 1.786   | 1.029    | 0.984   | 1.014    | 1.035   | 2.705    |
| 5 | 1.326   | 2.705    | 3.590   | 1.326    | 0.984   | 1.306    | 1.333   | 2.705    |
| 6 | 1.622   | 2.705    | 4.393   | 1.622    | 0.984   | 1.598    | 1.631   | 2.705    |
| 7 | 1.919   | 2.705    | 5.196   | 1.919    | 0.984   | 1.890    | 1.928   | 2.705    |

Table 6: Relativity, Base and level premiums under Hong Kong’s Bonus–Malus system, for such four models.

|   | Model 1 |          | Model 2 |          | Model 3 |          | Model 4 |          |
|---|---------|----------|---------|----------|---------|----------|---------|----------|
|   | $r_1$   | Base     | $r_1$   | Base     | $r_1$   | Base     | $r_1$   | Base     |
| 1 | 0.917   | 2.719    | 2.480   | 0.917    | 0.983   | 0.902    | 0.985   | 2.719    |
| 2 | 1.036   | 2.719    | 2.802   | 1.036    | 0.983   | 1.019    | 1.041   | 2.719    |
| 3 | 1.154   | 2.719    | 3.124   | 1.154    | 0.983   | 1.137    | 1.096   | 2.719    |
| 4 | 1.273   | 2.719    | 3.446   | 1.273    | 0.983   | 1.254    | 1.152   | 2.719    |
| 5 | 1.392   | 2.719    | 3.768   | 1.392    | 0.983   | 1.371    | 1.208   | 2.719    |
| 6 | 1.510   | 2.719    | 4.090   | 1.510    | 0.983   | 1.488    | 1.263   | 2.719    |

Table 7: Relativity, Base and level premiums under Brazil’s Bonus–Malus system, for such four models.

|   | Model 1 |          | Model 2 |          | Model 3 |          | Model 4 |          |
|---|---------|----------|---------|----------|---------|----------|---------|----------|
|   | $r_1$   | Base     | $r_1$   | Base     | $r_1$   | Base     | $r_1$   | Base     |
| 1 | 0.141   | 2.705    | 0.381   | 0.141    | 0.984   | 0.139    | 0.142   | 2.705    |
| 2 | 0.499   | 2.705    | 1.350   | 0.499    | 0.984   | 0.491    | 0.495   | 2.705    |
| 3 | 0.846   | 2.705    | 2.318   | 0.846    | 0.984   | 0.843    | 0.848   | 2.705    |
| 4 | 1.199   | 2.705    | 3.287   | 1.199    | 0.984   | 1.196    | 1.201   | 2.705    |
| 5 | 1.551   | 2.705    | 4.255   | 1.551    | 0.984   | 1.548    | 1.555   | 2.705    |
| 6 | 1.904   | 2.705    | 5.223   | 1.904    | 0.984   | 1.900    | 1.908   | 2.705    |
| 7 | 2.257   | 2.705    | 6.192   | 2.257    | 0.984   | 2.252    | 2.261   | 2.705    |
5. Conclusion and suggestions

This article designs an optimal Bonus–Malus System by evaluating relativity and base premiums. To estimate the relativity premium, this article considers a class of linear relativity premiums and determines an optimal premium within this class such that the estimator is simultaneously close to both possible Bayes relativity premiums. The base premium is evaluated under a Bayesian framework and two finite mixture models for both random claim size and risk parameter $\eta$. The Loimaranta efficiency shows that the efficiency of the new linear relativity premium is drastically improved compared with the ordinary relativity premium.

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