RIEMANNIAN STOCHASTIC APPROXIMATION ALGORITHMS

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Abstract. We examine a wide class of stochastic approximation algorithms for solving (stochastic) nonlinear problems on Riemannian manifolds. Such algorithms arise naturally in the study of Riemannian optimization, game theory and optimal transport, but their behavior is much less understood compared to the Euclidean case because of the lack of a global linear structure on the manifold. We overcome this difficulty by introducing a suitable Fermi coordinate frame which allows us to map the asymptotic behavior of the Riemannian Robbins–Monro (RRM) algorithms under study to that of an associated deterministic dynamical system. In so doing, we provide a general template of almost sure convergence results that mirrors and extends the existing theory for Euclidean Robbins–Monro schemes, despite the significant complications that arise due to the curvature and topology of the underlying manifold. We showcase the flexibility of the proposed framework by applying it to a range of retraction-based variants of the popular optimistic / extra-gradient methods for solving minimization problems and games, and we provide a unified treatment for their convergence.

1. Introduction

Consider a nonlinear system of equations of the general form

Find $x^* \in \mathcal{M}$ such that $v(x^*) = 0$ (Root)

where $\mathcal{M}$ is a smooth $d$-dimensional manifold and $v$ is a vector field on $\mathcal{M}$. Root-finding problems of this type play a crucial role in many areas of mathematical programming and data science, from Riemannian optimization and game theory to reinforcement learning, signal processing, and information theory. In particular, in addition to standard minimization problems – that is, when $v = -\nabla f$ for some smooth function $f$ on $\mathcal{M}$ – the general form of (Root) includes bilevel and saddle-point problems, dynamic programming, games in normal form, and many other equilibrium problems that arise in practice. For a range of applications and a comprehensive introduction to the topic, see Absil et al. [1], Boumal [10], and references therein.

The vast majority of methods for solving (Root) are iterative in nature, and they rely on building a successively finer “model” function which is applied to the last computed approximation of a root in order to get a new approximation. Usually, this model function is
based on the value of $v$ at a candidate solution; in many cases however, even this first-order model is too costly or even impossible to compute — e.g., if $v(x) = \mathbb{E}[V(x; \omega)]$ for some random variable $\omega$ with unknown distribution. In this case, a popular alternative is to rely on stochastic approximation algorithms that only require oracle access to a random — and possibly incomplete — approximation of $v$ at the queried point.

In this general context, when $\mathcal{M} = \mathbb{R}^d$, the method of choice for solving (Root) is the Robbins–Monro (RM) algorithm

$$X_{n+1} = X_n + \gamma_n V_n$$

(RM)

where $\gamma_n > 0$ is a variable step-size sequence and $V_n$ is a random estimate of $v(X_n)$. This method was introduced in the seminal papers of Robbins and Monro [55] and Kiefer and Wolfowitz [33], and the first general convergence results were obtained by Ljung [43, 44] for gradient problems. This has subsequently led to substantial activity on the topic, with major contributions by Benaïm and Hirsch [6], Benveniste et al. [8], Kushner and Clark [37], Kushner and Yin [39], and many others. However, the linear structure of $\mathbb{R}^d$ is deeply ingrained in all these works — and the method itself — preventing its use for solving (Root) in a manifold setting — e.g., the $d$-dimensional torus for a robotic arm with $d$ joints, Grassman or Stiefel manifolds for robust principal component analysis, hyperbolic space for text and graph embeddings, etc. Because of this, the applicability of RM methods to general root-finding problems is significantly more narrow than one might expect, even when $\mathcal{M}$ has a relatively simple structure (like a matrix group or a Grassmannian).

**Our contributions in the context of related work.** In view of the above, our main objective is to bridge the gap between Euclidean and Riemannian stochastic approximation schemes by replacing the “+” operation in (RM) with the Riemannian exponential map $\exp_{X_n}(\cdot)$ on $\mathcal{M}$ — or, more generally (and often more tractably), a retraction on $\mathcal{M}$ based at $X_n$. In Riemannian optimization, this approach was pioneered by Bonnabel [9], who examined the case where $v$ is the Riemannian gradient of some objective function $f$. Subsequent works [11, 15, 41, 61, 63, 65] expanded on the results of Bonnabel [9] for Riemannian stochastic gradient descent, while similar results were obtained in [7, 22, 28, 42] for Riemannian proximal point methods.

All these works focus exclusively on the case where $v$ is a gradient field, so they do not apply to general, non-gradient instances of (Root). A partial extension to the non-gradient case was provided by a concurrent line of works which examined the use of Riemannian extragradient methods under the assumption of (geodesic) monotonicity [14, 21, 23, 32, 50, 60]. This is a strong, convexity-type assumption which posits that $v$ globally points towards its (necessarily connected) root system in a suitable, geodesic sense; convergence is then obtained following a similar line of reasoning as in the case of monotone operator theory in Hilbert spaces [3].

Our paper does not make any such assumptions and directly examines the dynamics of Riemannian Robbins–Monro methods for general vector fields $v$. In this regard, our main contributions can be summarized as follows:

1. We introduce a **generalized Riemannian Robbins–Monro** template which includes as special cases all methods mentioned above (Riemannian stochastic gradient descent, extragradient methods, etc.).

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1 More sophisticated methods — like Newton’s algorithm — rely on a first-order Taylor approximation of $v$ around an iterate; Halley’s method employs a second-order model, and the hierarchy continues with the general class of Householder methods that use $k$-th order derivatives to build more precise polynomial models of $v$ around an iterate. However, because these tensor methods involve the computation of higher-order derivatives of $v$, they may become highly impractical for moderate-to-high values of $d$. 


extra-gradient, proximal point methods, etc.), as well as a number of new stochastic approximation (SA) schemes for (Root).

(2) Under some mild technical conditions on \( \mathcal{M} \), we show that the sequence of generated points forms an “approximate solution” – an asymptotic pseudotrajectory (APT) to be exact – of an associated deterministic dynamical system (Theorem 2).

This stochastic approximation result extends the seminal theory of Benaïm and Hirsch [6] for Euclidean Robbins–Monro schemes to a Riemannian setting, and allows us to infer almost sure convergence of Riemannian Robbins–Monro (RRM) schemes to the internally chain-transitive (ICT) sets of the underlying Riemannian dynamics (cf. Corollaries 1 and 2). In gradient and strictly monotone problems, these ICT sets are precisely the roots of \( v \), so we readily recover many of the asymptotic convergence results mentioned above (often under much weaker assumptions). In addition, as we show in Section 4, our framework applies to several settings beyond gradient or monotone systems – such as ordinal potential games, supermodular games, and cooperative dynamics – and covers a significantly wider class of SA schemes.

**Tools, techniques, and related approaches.** In the absence of a linear structure on \( \mathcal{M} \), the major challenge we have to overcome is the lack of a suitable coordinate frame within which to analyze the trajectories of Riemannian SA algorithms. This reflects the dichotomy that, unlike in the case of \( \mathbb{R}^d \), points and vectors on manifolds obey fundamentally different rules and have to be compared using different moving frames. To circumvent this obstacle, we introduce an *extended Fermi coordinate* frame inspired by Manasse and Misner [45], and we use it to prove that Riemannian SA schemes enjoy similar error bounds as in Euclidean spaces (up to some high-order terms that vanish in the long run). The aggregation and propagation of these errors can then be controlled using arguments from martingale limit theory which ultimately yield the convergence properties mentioned above.

A concurrent approach to establish the APT property in Riemannian SA schemes is due to Shah [56], who assumes the existence of a local diffeomorphism mapping geodesic interpolations to linear interpolations in a Euclidean space. However, the existence of such a diffeomorphism on every point of \( \mathcal{M} \) implies that the manifold is globally flat, i.e., essentially Euclidean [29]; this assumption is too restrictive for bona fide Riemannian applications, so the analysis of [56] is not relevant for our purposes. An additional issue is that the error bounds employed by Shah [56, p. 1131] rule out vector fields with a rotational component – such as \( v(x, y) = (-y, x) \) on \( \mathbb{R}^2 \) – further limiting the applicability of their techniques to the setting under consideration.

Finally, Durmus et al. [18, 19] also recently consider a generic version of Robbins–Monro schemes, with both vanishing and constant step-sizes. The analysis of the latter type of schemes cannot lead to convergence with probability 1, so the results of [19] are necessarily ergodic in nature and hence beyond our paper’s scope. The setting of [18] is closer in spirit to our own, and it also accounts for the effects of bias in the queries to \( v \); however, the results obtained therein concern dynamics that admit a *Lyapunov function* – the so-called “gradient-like” case [4] – so there is no overlap with our analysis.

**Basic notions and notation.** Throughout our paper, \( \mathcal{M} \) will denote a \( d \)-dimensional, geodesically complete, Riemannian manifold. We will write \( \langle \cdot, \cdot \rangle_x \), \( x \in \mathcal{M} \), for its underlying metric, \( \| \cdot \|_x \) for the induced norm, and we will assume that the sectional curvatures of \( \mathcal{M} \) are bounded from above and below by \( K_{\text{up}} \) and \( K_{\text{low}} \) respectively.

For any curve \( \gamma \) on \( \mathcal{M} \), the notation \( \dot{\gamma}(t) \) will always denote its velocity at time \( t \). Given any pair of points \( x, x' \in \mathcal{M} \) and a vector \( v \) in the tangent space at \( x \), \( T_x \mathcal{M} \), we denote
by \( \Gamma_{x \to x'}(v) \in \mathcal{T}_x \mathcal{M} \) the vector obtained by parallel transporting \( v \) along the minimizing geodesic connecting \( x \) and \( x' \); if minimizing geodesics are not unique, \( \Gamma_{x \to x'}(v) \) will be understood as parallel transport along any of them. We also say that a vector field \( v \) on \( \mathcal{M} \) is (geodesically) \( L \)-Lipschitz if, for all \( x, x' \in \mathcal{M} \), we have

\[
\| \Gamma_{x \to x'}(v(x)) - v(x') \|_{x'} = \| v(x) - \Gamma_{x' \to x}(v(x')) \|_x \leq L \text{dist}(x, x'),
\]

where \( \text{dist}(\cdot, \cdot) \) denotes the distance function on \( \mathcal{M} \) induced by \( \langle \cdot, \cdot \rangle \). All vector fields in this paper are assumed to be \( L \)-Lipschitz, complete and bounded, i.e., \( G := \sup_{x \in \mathcal{M}} \| v(x) \|_x < \infty \).

Finally, the Riemannian gradient of a smooth function \( f : \mathcal{M} \to \mathbb{R} \) will be denoted by \( \nabla f \).

For a detailed account, we refer the reader to the masterful treatment of Lee [40].

2. Riemannian Robbins–Monro algorithms: Definitions and assumptions

2.1. The Riemannian Robbins–Monro template. We begin by discussing the basic template of Riemannian Robbins–Monro (RRM) algorithms. As we stated before, the main difference with their Euclidean counterpart is that addition along “straight lines” is replaced with the Riemannian exponential mapping. This leads to the abstract update rule

\[
X_{n+1} = \exp_{X_n}(\gamma_n(v(X_n) + W_n))
\]

where

1. \( X_n \in \mathcal{M} \) denotes the state of the algorithm at each iteration counter \( n = 1, 2, \ldots \)
2. \( W_n \in \mathcal{T}_{X_n} \mathcal{M} \) is an abstract error term (described in detail below).
3. \( \gamma_n > 0 \) is the method’s step-size (also discussed below).

In the above, we tacitly assume that the error term \( W_n \) is generated after \( X_n \), so it is not adapted to the history \( \mathcal{F}_n := \sigma(X_1, \ldots, X_n) \) of \( X_n \) – that is, \( W_n \) is random relative to \( \mathcal{F}_n \). In addition, to differentiate between “random” (zero-mean) and “systematic” (non-zero-mean) errors, it will be convenient to further decompose \( W_n \) as

\[
W_n = U_n + b_n
\]

where \( U_n = W_n - \mathbb{E}[W_n | \mathcal{F}_n] \) captures the random, zero-mean part of \( W_n \), while \( b_n = \mathbb{E}[W_n | \mathcal{F}_n] \) represents the systematic component thereof.

To quantify all this, we will assume in the sequel that \( U_n \) and \( b_n \) are bounded as

\[
\mathbb{E}[\| U_n \|_{X_n}^2 | \mathcal{F}_n] \leq \sigma_n^2 \quad \text{and} \quad \mathbb{E}[\| b_n \|_{X_n} | \mathcal{F}_n] \leq B_n \quad \text{a.s.}
\]

where \( \sigma_n \) and \( B_n \) are to be construed as upper bounds on the noise and bias of the error terms entering (RRM). Finally, for concision, we will also write

\[
V_n = v(X_n) + W_n
\]

so \( V_n \) can be seen as a noisy – and potentially biased – estimator of \( v(X_n) \) in (RRM). Obviously, \( V_n = v(X_n) \) whenever \( B_n = \sigma_n = 0 \); we will refer to this case as “deterministic”.

2.2. Stochastic approximation. Our main goal in the sequel will be to connect the asymptotic behavior of the trajectories of (RRM) to a continuous-time dynamical system on \( \mathcal{M} \). To do so, in analogy with the ODE method for standard, Euclidean Robbins–Monro schemes [4, 39], we will view (RRM) as an inexact (forward) Euler discretization of the Riemannian mean dynamics

\[
\dot{\theta}(t) = v(\theta(t))
\]

and we will try to establish a measure of “closeness” between the discrete-time sequence \( X_n \) generated by (RRM) and the continuous-time orbits \( \theta(t) \) of (RMD).
In the Euclidean case, this is provided by taking an affine interpolation \( X(t) \) of \( X_n \) that agrees with \( X_n \) at all instances of the “effective time” variable \( \tau_n = \sum_{k=1}^{n-1} \gamma_k \), viz.

\[
X(t) = X_n + \frac{t - \tau_n}{\tau_{n+1} - \tau_n} (X_{n+1} - X_n) \quad \text{for all } t \in [\tau_n, \tau_{n+1}], \quad n = 1, 2, \ldots
\]  

(5)

Of course, this definition is not meaningful in a Riemannian setting because of the lack of an affine structure on \( \mathcal{M} \). Instead, noting that the increments in (5) can be rewritten as \( (X_{n+1} - X_n)/(\tau_{n+1} - \tau_n) = V_n \), a natural Riemannian analogue would be to follow the geodesic emanating from \( X_n \) along \( V_n \) until reaching \( X_{n+1} \). With this in mind, we define the \textit{geodesic interpolation} \( X(t) \) of \( X_n \) as

\[
X(t) = \exp_{X_n}((t - \tau_n) V_n) \quad \text{for all } t \in [\tau_n, \tau_{n+1}], \quad n \geq 1,
\]

(GI)

so, by construction, (a) \( X(\tau_n) = X_n \) for all \( n \); and (b) each segment of \( X(t) \) is a geodesic.

Now, to compare \( X(t) \) to the solution orbits of \( \text{(RMD)} \), let \( \Theta: \mathbb{R}_+ \times \mathcal{M} \to \mathcal{M} \) denote the \textit{flow} of \( \text{(RMD)} \), i.e., \( \Theta_h(x) \) is simply the position at time \( h \geq 0 \) of the solution orbit of \( \text{(RMD)} \) that starts at \( x \in \mathcal{M} \) (recall here that \( v \) is complete). We then have the following notion of “asymptotic closeness”:

\textbf{Definition 1} (Benaïm and Hirsch, 1996). We say that \( X(t) \) is an \textit{asymptotic pseudotrajectory} \( \text{(APT)} \) of the mean dynamics \( \text{(RMD)} \) if, for all \( T > 0 \), we have

\[
\lim_{t \to \infty} \sup_{0 \leq h \leq T} \text{dist}(X(t+h), \Theta_h(X(t))) = 0.
\]

(6)

Intuitively, Definition 1 states that one cannot distinguish between the geodesically interpolated process \( X(t+h) \) and the orbit of \( \text{(RMD)} \) that starts at \( X(t) \) as \( t \to \infty \). This is a highly non-trivial requirement, so much of the analysis to follow hinges on establishing exactly this property; we carry all this out in Section 3, where we also describe the precise connection between the asymptotic behavior of \( \text{(RRM)} \) and that of \( \text{(RMD)} \).

\textbf{2.3. Technical assumptions.} We conclude this section with the basic assumptions that underlie the rest of our paper. These are as follows:

\textbf{Assumption 1} (Robbins–Monro step-sizes). The step-size sequence \( \gamma_n, \quad n = 1, 2, \ldots, \) of \( \text{(RRM)} \) satisfies the Robbins–Monro summability conditions \( \sum_n \gamma_n = \infty \) and \( \sum_n \gamma_n^2 < \infty \).

\textbf{Assumption 2} (Error bounds). The bounds on the noise and bias in (3) satisfy

\[
\sum_{n=1}^{\infty} \gamma_n^2 \mathbb{E}[\sigma_n^2] < \infty, \quad \sum_{n=1}^{\infty} \gamma_n \mathbb{E}[B_n] < \infty, \quad \text{and} \quad B_n \to 0 \quad \text{(a.s.).}
\]

(7)

Variants of the above assumptions are standard in the context of (Euclidean) Robbins–Monro methods, cf. Benaïm [4], Kushner and Clark [37], and references therein. Nonetheless, it is worth noting that Assumption 1 lies under the explicit control of the algorithm designer, while Assumption 2 is \textit{implicit} and depends on the specific problem at hand – the mechanism providing access to \( v \), the specific form of \( \text{(RRM)} \), etc. In this regard, Assumption 2 is more delicate than Assumption 1; we discuss this issue in detail in Section 4, where we show that Assumption 2 is indeed satisfied for a wide range of practical algorithms that adhere to the general template \( \text{(RRM)} \).

Moving forward, to exclude cases where \( \text{(RRM)} \) becomes unstable over time, a standard practice in the literature is to assume that the sequence \( X_n \) is contained in a compact subset of \( \mathcal{M} \), a property known as \textit{precompactness}. Formally, we have:
Assumption 3 (Precompactness). The set of iterates \{X_n : n = 1, 2, \ldots\} has compact closure in \(\mathcal{M}\).

Albeit standard in the literature [4, 8, 37], Assumption 3 may be difficult to verify if \(\mathcal{M}\) is not itself compact (e.g., if \(\mathcal{M} = \mathbb{R}^d\)). To account for this, we introduce in Section 3 a set of structural hypotheses on \(\mathcal{M}\) and \(v\) which guarantee that Assumption 3 holds with probability 1.

With all this in hand, our last blanket assumption is a technical requirement that interfaces between the geometry of \(\mathcal{M}\) and the dynamics of (RMD). To state it, recall first that \(x \in \mathcal{M}\) is said to be conjugate to \(x' = \exp_x(v)\) along the geodesic \(\exp_x(tv)\) if the exponential map \(\exp_x(\cdot)\) is not a diffeomorphism in a neighborhood of \(v\) [40]. In addition, define the Picard flow \(\lambda : \mathbb{R}_+ \to \mathcal{M}\) associated with \(X(t)\) to be the dynamical system
\[
\lambda(h) = \Gamma_{X(t+h) \to X(0)}(v(X(t+h)))
\]
with initial condition \(\lambda(0) = X(t)'\).

Finally, given some \(T, t > 0\), consider the sets
\[
\mathcal{C}_\lambda(t, T) := \{h \in [0, T] : X(t + h) \text{ is conjugate to } \lambda(h)\}, \quad (8a)
\]
\[
\mathcal{C}_\theta(t, T) := \{h \in [0, T] : X(t + h) \text{ is conjugate to } \Theta_h(X(t))\}, \quad (8b)
\]
and let \(\mathcal{C}(t, T) = \mathcal{C}_\lambda(t, T) \cup \mathcal{C}_\theta(t, T)\). We then make the following assumption:

Assumption 4 (Nowhere dense conjugates). The set \(\mathcal{C}(t, T)\) is nowhere dense in \([0, T]\) for sufficiently large \(t > 0\) and for all \(T > 0\).

At first sight, Assumption 4 may appear quite opaque but it is otherwise quite mild. Indeed, since the set of points conjugate to \(X(t + h)\) is at most one-dimensional [64], the only way that Assumption 4 can fail is if the curves \(X(t + h)\) and \(\Theta(h)\) or \(\lambda(h)\) simultaneously traverse the same one-dimensional submanifold of \(\mathcal{M}\) – a fact which occurs with probability 0 if the distribution of \(W_n\) is non-singular. It is also worth noting that Assumption 4 is automatically satisfied on negatively curved spaces by the Cartan–Hadamard theorem (cf. Proposition 1 in the next section) and, finally, it is straightforward to verify Assumption 4 manually on many of the manifolds that arise in practical applications – such as spheres, Grassmannians, Stiefel manifolds and fixed-rank spectrahedra, cf. [1, 40, 46] and references therein. We discuss all this in more detail in the next section.

3. Analysis and Main Results

3.1. Statement and discussion of main results. The connecting tissue between asymptotic pseudotrajectories and the mean dynamics (RMD) is provided by the internally chain-transitive (ICT) sets of (RMD). These are defined as follows:

Definition 2 (Benaïm, 1999). Let \(\mathcal{S}\) be a nonempty compact subset of \(\mathcal{M}\). Then:

1. \(\mathcal{S}\) is invariant under (RMD) if \(\Theta_t(S) = S\) for all \(t \in \mathbb{R}\).
2. \(\mathcal{S}\) is an attractor of (RMD) if it is invariant under (RMD) and there exists a compact neighborhood \(\mathcal{K}\) of \(\mathcal{S}\) such that \(\lim_{t \to -\infty} \text{dist}(\Theta_t(x), S) = 0\) uniformly in \(x \in \mathcal{K}\). 
3. \(\mathcal{S}\) is internally chain-transitive (ICT) if it is invariant and \(\Theta|_{\mathcal{S}}\) admits no attractors other than \(\mathcal{S}\).

The term “Picard flow” stems from the fact that the integral \(\int_0^T v(X(t + s)) \, ds\) is the basic iteration in Picard’s method of successive approximations for solving an ODE in Euclidean spaces. In the case of (PFlow), the parallel transport is the extra ingredient required to express the idea of “integrating \(v\) along \(X(t)\)”, so (PFlow) can be seen as a bona fide generalization of the Picard iteration to Riemannian manifolds.
In words, ICT sets are the “terminal objects” of the dynamics (RMD): the orbits that converge to an ICT set cannot be contained in a smaller subset thereof, so ICT sets may be viewed as minimal connected periodic orbits up to arbitrarily small numerical errors [4, Prop. 5.3]. The importance of this property for the theory of stochastic approximation is owed to the following theorem:

**Theorem 1** (Benaïm and Hirsch, 1996). Let \( X(t), t \geq 0, \) be a precompact asymptotic pseudotrajectory of (RMD). Then the limit set \( L(X) \coloneqq \bigcap_{t \geq 0} \text{cl}\{X(s) : s \geq t\} \) of \( X \) is an ICT set of (RMD).

Remark. Theorem 1 above was originally stated in the context of abstract metric spaces; we have adapted it here to the Riemannian setting of (RMD) for concision and concreteness.

The limit set theorem above provides a fundamental link between APTs and the long-run behavior of (RMD) as captured by its ICT sets. That being said, the APT property itself may be difficult to verify from first principles, so the application of Theorem 1 to Riemannian Robbins–Monro algorithms can be just as difficult. In the Euclidean case (\( M = \mathbb{R}^d \)), Benaïm and Hirsch [6] address this issue via a series of criteria under which standard (Euclidean) Robbins–Monro methods give rise to an APT of the associated mean dynamics [4, Props. 4.1 and 4.2]. Unfortunately however, in a bona fide Riemannian setting, these criteria cannot be used because they are inextricably tied to the affine structure of \( \mathbb{R}^d \); as a result, it is not clear how to leverage Theorem 1 to obtain a theory of stochastic approximation for Robbins–Monro methods on Riemannian manifolds. We tackle this question below:

**Theorem 2.** Suppose that Assumptions 1–4 hold. Then, with probability 1, the geodesic interpolation \( X(t) \) of the sequence of iterates \( X_n \) of (RRM) is an APT of (RMD).

Theorem 2 plays a pivotal role in our paper and is the cornerstone for the analysis to follow. In particular, by invoking Theorems 1 and 2 in tandem, we readily obtain the following characterization of the limiting behavior of (RRM):

**Corollary 1.** Suppose that Assumptions 1–4 hold. Then \( X_n \) converges to an ICT set of (RMD) w.p.1.

An important consequence of Corollary 1 is that the analysis of the stochastic, discrete-time system (RRM) boils down to that of the deterministic, continuous-time system (RMD). In this way, Theorem 2 allows us to employ the same high-level strategies as the classic literature on stochastic approximation: For example, if \( v \) admits a potential or is geodesically (strictly) monotone, it is easy to verify that the only ICT sets of (RMD) are the roots of \( v \) [57], so we readily recover the series of asymptotic convergence results mentioned in the introduction. In Section 4, we present a wide range of problems whose ICT sets coincide with the solutions of (Root), and we further illustrate how Theorem 2 captures a series of Riemannian stochastic approximation algorithms – old and new – in a unified fashion.

The proof of Theorem 2 is fairly arduous, so we defer it to the end of this section; instead, we proceed to discuss here in more detail the theorem’s precompactness and conjugacy requirements (Assumptions 3 and 4 respectively). First, as mentioned above, Assumption 3 is common in the stochastic approximation literature but, in general, it is impractical to verify directly from the primitives of the problem at hand – that is, the ambient manifold \( M \) and the defining vector field \( v \). Likewise, if the cut loci of different points on \( M \) happen to have a complicated topological structure, verifying Assumption 4 could also be impractical. To account for all this, we provide below a set of structural conditions on \( M \) and \( v \) which guarantee that Assumptions 3 and 4 both hold with probability 1:
Proposition 1. Suppose that the following hypotheses hold:

(H1) \( \mathcal{M} \) is a Hadamard manifold with the Heine-Borel property. \(^3\)

(H2) \( v \) satisfies the weak coercivity condition

\[
\langle v(x), \nabla \text{dist}^2(p,x) \rangle \leq 0 \tag{WC}
\]

for some base point \( p \in \mathcal{M} \) and for all \( x \in \mathcal{M} \setminus B_R(p) \) outside a closed geodesic ball \( B_R(p) := \{ x \in \mathcal{M} : \text{dist}(p,x) \leq R \} \) of radius \( R > 0 \) and centered at \( p \).

Then, with probability 1, Assumptions 3 and 4 hold as stated – with Assumption 4 only requiring (H1).

Corollary 2. Suppose that Assumptions 1 and 2 and Hypotheses (H1) and (H2) hold. Then \( X_n \) converges to an ICT set of \( \mathcal{RMD} \) w.p.1.

To streamline our discussion, we postpone the proof of Proposition 1 to Appendix A. For now, we simply note that the Hadamard requirement (H1) is fairly common in applications to Riemannian optimization and includes Stiefel and Grassmann manifolds (as homogeneous spaces), standard matrix manifolds, hyperbolic spaces, etc. \([13, 26]\).

In this regard, the most delicate requirement in Proposition 1 is Hypothesis (H2). This hypothesis may be viewed as a Riemannian relaxation of the coercivity condition

\[
\lim_{x \to \infty} \langle v(x), x \rangle / \|x\|_2 = -\infty,
\]

which posits that the “inward-pointing” component of \( v \) grows unbounded at infinity, a property which is frequently used to ensure the stability of Euclidean iterative algorithms \([20, 52]\). In our Riemannian setting, the role of the radial field is played by the gradient of the squared distance function \( \nabla \text{dist}^2(p,x) \), which is itself equal to \( -2 \log_x(p) \) if \( p \) lies in the injectivity radius of \( x \) (and hence for all \( x \in \mathcal{M} \) if \( \mathcal{M} \) is Hadamard). In addition, it is important to bear in mind that (WC) does not impose any growth requirements on the radial component of \( v \); it only requires that \( v \) does not have a consistent outward-pointing component that could lead the process to diverge, so it is significantly lighter in that respect (hence the adjective “weak”).

3.2. Proof of Theorem 2. Because the proof of Theorem 2 and the geometric scaffolding required are quite delicate, we begin with a high-level outline outlining the main difficulties and technical challenges involved. In brief, the basic obstacle that we have to overcome is as follows:

- On the one hand, we need a coordinate system to compare and compute distances between different points on \( \mathcal{M} \). This can be done efficiently in normal coordinates \([40]\).
- On the other hand, we also need to compare vectors living on different tangent spaces.

In general, this comparison is very difficult to carry out in a normal coordinate frame, but it is much easier in the parallel frame system (PFS) that we describe in detail in Appendix B.1.

Intuitively, the normal coordinate system is where distances behave as if the ambient space were Euclidean, and the parallel frame system is where vectors behave as in the Euclidean setting. Unfortunately however, the only regime where these two systems can coexist is when \( \mathcal{M} \) is flat, i.e., the problem is “essentially Euclidean” to begin with. To circumvent this obstacle, we take the following technical approach:

\(^3\)A Hadamard manifold is a simply connected Riemannian manifold with non-positive sectional curvatures, and the Heine-Borel property simply posits that every closed bounded subset thereof is compact.
(1) Based on the notion of Fermi coordinates [45] – which can be intuitively understood as “normal coordinates along a curve”, cf. Fig. 1 – we begin by constructing an extended Fermi coordinate frame that allows us to focus on a neighborhood of $X(t)$ containing all the information we need. [This is a challenging, but otherwise mostly technical, step that does not affect the big picture.]

(2) Using the extended Fermi coordinates constructed above, we can reduce the task of comparing the distance between two Riemannian curves to comparing several Euclidean, albeit individually intractable, vector fields. This step incurs an error term that is not present in the analysis of Euclidean stochastic approximation schemes, and which is difficult to control in regions of high sectional curvature, cf. Eqs. (25) and (27).

(3) To obtain expressions of vector fields that are more amenable to computation, we will switch from the extended Fermi coordinates to the parallel frame system and bound the difference between the two. This step invariably introduces an additional error relative to the Euclidean analysis, cf. Eqs. (37) and (38).

(4) Serendipitously, these additional error terms can be managed without any further assumptions, and a series of arguments in the spirit of Benaïm and Hirsch [6] concludes our proof.

To formalize all this, we will need the following notions and results from Riemannian geometry:

- The concept of a parallel frame system.
- A technical lemma by Fujita and Kotani [24] and Takahashi and Watanabe [59] which bounds the distortion of velocities measured by a moving observer on a manifold relative to flat spaces.
- A comparison lemma to estimate the difference between parallel transport and the pushforward of the Riemannian exponential map.

These elements are quite technical and involved, so we proceed directly to the proof of Theorem 2 and we relegate all precise statements regarding the above to Appendix B.
Step 1: Discrete-to-continuous time comparisons. Following Benaïm [4], consider the “continuous-to-discrete” counter

\[ m(t) = \sup\{n \geq 1 : t \geq \tau_n\} \tag{9} \]

which measures the number of iterations required for the effective time \( \tau_n = \sum_{k=1}^{n-1} \gamma_k \) to reach a given timestamp \( t \geq 0 \). Moreover, given an arbitrary sequence \( A_n \), we will denote its piecewise-constant interpolation as

\[ \tilde{A}(t) = A_n \quad \text{for all} \quad t \in [\tau_n, \tau_{n+1}), \quad n \geq 1. \tag{10} \]

Using this notation, we may express the geodesic interpolation \( X(t) \) of \( X_n \) in differential form as

\[ \dot{X}(t + h) = \Gamma_{X(t+h) \to X(t+h)} \left( v \left( \dot{X}(t + h) \right) + \bar{W}(t + h) \right) \tag{11} \]

and we further let

\[ \tilde{\gamma}^*(t) := \sup_{t \leq h \leq t + T} \tilde{\gamma}(h) \quad \text{and} \quad \tilde{B}^*(t) := \sup_{t \leq h \leq t + T} \tilde{B}(h) \quad \text{for all} \quad t, h \geq 0. \tag{12} \]

By the Stolz-Cesàro theorem, Assumptions 1 and 2 readily imply that \( \lim_{n \to \infty} B_n = 0 \), so we also have \( \lim_{n \to \infty} \tilde{B}^*(t) = 0 \); we will use this fact freely in the sequel.

As a last comparison step, we will also need a noise stability criterion. To that end, let \( \{e_k(n)\}_{k=1}^{d} \) be an arbitrary sequence of orthonormal bases for \( T_{X_n}M \), and let \( U^\pi_n \) be the (Euclidean) noise vector composed of components of the noise \( U_n \) in the basis \( \{e_k(n)\}_{k=1}^{d} \), viz.

\[ U^\pi_{k,n} := \langle U_n, e_k(n) \rangle_{X_n}. \tag{13} \]

It is then easy to see that \( \mathbb{E}[U^\pi_n | \mathcal{F}_n] = 0 \), and, moreover

\[ \mathbb{E} \left[ \|U^\pi_n\|^2_2 \mid \mathcal{F}_n \right] = \mathbb{E} \left[ \|U_n\|^2_{X_n} \mid \mathcal{F}_n \right] \leq \sigma_n^2 \tag{14} \]

by Assumption 2. Then, letting

\[ \Delta(t; T) := \sup_{0 \leq h \leq T} \left\| \frac{1}{t} \int_{t}^{t+h} \tilde{U}^\pi(s) \, ds \right\|_2, \tag{15} \]

a classical argument by Benaïm [4, cf. Eq. (13) and onwards] readily gives

\[ \lim_{t \to \infty} \Delta(t; T) = 0 \quad \text{for all} \quad T > 0 \quad (\text{a.s.}). \tag{16} \]

Step 2: Preliminary error bounds. We first note that, by Assumption 3, \( \sup_h r(X(t)) < \infty \) where \( r(\cdot) := \text{dist}(\cdot, p) \) is the radial function defined in (WC). We claim that Assumption 3 also implies the boundedness of the Picard flow. To see this, recall that the parallel transport is an isometry, so

\[ \left\| \dot{\lambda}(h) \right\|_{\lambda(h)} = \left\| \Gamma_{X(t+h) \to X(\lambda(h))} (v(X(t + h))) \right\|_{\lambda(h)} = \left\| v(X(t + h)) \right\|_{X(t+h)}, \tag{17} \]

and hence

\[ \sup_{0 \leq h \leq T} \text{dist}(\lambda(0), \lambda(h)) \leq T \cdot \sup_{0 \leq h \leq T} \left\| \dot{\lambda}(h) \right\|_{\lambda(h)} < \infty, \tag{18} \]
which implies $\sup_{0 \leq h \leq T} r(\lambda(h)) < \infty$. On the other hand, the boundedness for the flow follows readily from the fact that $v$ and dist are both $L$-Lipschitz, so

\[
\frac{d}{dh} r(\theta(h)) \leq \left\| \dot{\theta}(h) \right\|_{\theta(h)} = \| v(\theta(h)) \|_{\theta(h)} \\
\leq \left\| v(\theta(h)) - \Gamma_{p-\theta(h)} (v(p)) \right\|_{\theta(h)} + \| \Gamma_{p-\theta(h)} (v(p)) \|_{\theta(h)} \\
\leq L r(\theta(h)) + \| v(p) \|_p.
\]

An application of Grönwall’s inequality then gives $\sup_{0 \leq h \leq T} r(\theta(h)) < \infty$. Therefore, all computations in the sequel can be restricted to a compact set, and we may assume without loss of generality that $\tilde{\gamma}^*(t) < 1$ for all $t \geq 0$ and, in addition, there exists some $R \equiv R(T, L, G)$ such that

\[
\max \{ \text{dist}(X(t+h), \theta(h)), \text{dist}(X(t+h), \lambda(h)) \} \leq R \quad \text{for all } h \in [0, T].
\]  

**Step 3: Constructing the extended Fermi coordinates.** Let $\tilde{U}$ be the neighborhood defined in Appendix B.2 restricted to $[t, t + T]$, i.e., $\tilde{U} = \bigcup_{h=0}^{t+T} U_h$ with $U_h$ given by (B.5). Clearly, $\tilde{U}$ contains $\{X(t+h) : h \in [0, T]\}$; moreover, by construction, $\tilde{\Phi}$ carries a system of orthonormal frames $\{e_k(h)\}_{k=1}^{d}$, one for each $X(t+h)$. In what follows, all quantities will be expressed in these frames unless explicitly mentioned otherwise.

Now, fix some $h_0 \in [0, T]$ and let $\gamma_\theta$ and $\gamma_\lambda$ be two minimizing geodesics such that $\gamma_\theta(0) = \gamma_\lambda(0) = X(t+h_0)$, $\gamma_\theta(1) = \theta(h_0)$ and $\gamma_\lambda(1) = \lambda(h_0)$. Our first goal will be to extend $\tilde{U}$ to an open set of $\mathcal{M}$ that contains the geodesics $\gamma_\theta$ and $\gamma_\lambda$, while retaining the exponential mapping as a local diffeomorphism. This will serve a dual purpose:

1. It enables us, for a fixed $h_0$, to consider the parallel frame systems at $X(t+h_0)$ and $\theta(h_0)$ -- or $X(t+h_0)$ and $\lambda(h_0)$, depending on the context -- so that we can easily compare the vector fields at these points; see Appendix B.1.

2. We want to apply Lemma B.1 to the curves $\theta(\cdot)$ and $\lambda$; however, for $\tilde{\theta}(h_0)$ and $\tilde{\lambda}(h_0)$ to make sense, $\tilde{U}$ must contain both curves for at least an open time interval that includes $h_0$.

This is where Assumption 4 comes into play: Since the conjugate points arise by definition when the exponential map ceases to be local diffeomorphisms [40], it is reasonable to expect that, away from the time points where $X(t+h)$ is conjugate to $\theta(h)$ or $\lambda(h)$, it is always possible to extend $\tilde{U}$ to include the geodesics connecting $X(t+h)$ to $\theta(h)$ and $\lambda(h)$. In this regard, Assumption 4 simply posits that there cannot be “too many” such conjugate points.

To formalize this, fix some $h_0 \notin C(t, T)$ where $C(t, T)$ is defined as in (4), and assume also that $t + h_0 \neq \tau_n$ for all $n$ -- i.e., $t + h_0$ is not a “kink point” of (GI). Since the exponential mapping is a local diffeomorphism away from conjugate points [40], it follows that $\exp_{X(t+h_0)} : T_{X(t+h_0)} \mathcal{M} \to \mathcal{M}$ is a local diffeomorphism at $\lambda(h_0)$ and $\tilde{\theta}(h_0)$, where $\tilde{\lambda}(h_0)$ and $\tilde{\theta}(h_0)$ are the normal coordinates of $\lambda(h_0)$ and $\theta(h_0)$ with center $X(t+h_0)$ respectively. By the continuity of the Picard flow and the frame system $\{e_k(\cdot)\}_{k=1}^{d}$, there exists an open interval $(h_{\text{init}}, h_{\text{fin}})$ containing $h_0$ such that, for all $h \in (h_{\text{init}}, h_{\text{fin}})$, $\exp_{X(t+h)} : T_{X(t+h)} \mathcal{M} \to \mathcal{M}$ is a local diffeomorphism at $\lambda(h)$ and $\tilde{\theta}(h)$, where $\tilde{\lambda}(h)$ and $\tilde{\theta}(h)$ are the normal coordinates of $\lambda(h)$ and $\theta(h)$ with center $X(t+h)$ respectively.

---

[4] Due to the smoothness of the flow, the function $r(\theta(h))$ is always differentiable in $h$ in the metric space sense [2], even though $\theta(h)$ might reach the cut locus of $p$. 

On that account, let \( \gamma^h_0 \) be a family of minimizing geodesics such that \( \gamma^h_0(0) = X(t + h) \) and \( \gamma^h_0(1) = \theta(h) \), and define \( \gamma^h_\lambda \) similarly. Since both \( \gamma^h_0 \) and \( \gamma^h_\lambda \) are minimizing geodesics and \( \theta(h) \) and \( \lambda(h) \) are not conjugate to \( X(t + h) \), [40, Theorem 10.15] ensures that no point on \( \gamma^h_0 \) or \( \gamma^h_\lambda \) is conjugate to \( X(t + h) \). Summing up, we have shown that the exponential mapping is a local diffeomorphism at any point of the set

\[
\{ \gamma^h_0 \}_{h \in (h_{\text{init}}, h_{\text{fin}})} \cup \{ \gamma^h_\lambda \}_{h \in (h_{\text{init}}, h_{\text{fin}})}.
\]

The final step in our construction is to consider the union of all such \( (h_{\text{init}}, h_{\text{fin}}) \) for all \( h_0 \not\in C(t, T) \) and \( t + h_0 \neq \tau_n \); we denote the set obtained in this way by \( \mathcal{H} \). More precisely, we claim that a) \( \mathcal{H} \) is a dense open subset of \([0, T]\); and b) \( \mathcal{H} \) can be written as a countable union of disjoint open intervals, i.e., \( \mathcal{H} = \bigcup_k (h_k, h_{k+1}) \). Indeed, the first claim follows readily from Assumption 4 and the fact that the set \( \{ h_0 : t + h_0 = \tau_n \text{ for some } n \} \) is countable. The second claim is due to the compactness of \([0, T]\) and the fact that every nonempty open interval in \( \mathbb{R} \) contains a rational number.

In view of the above, it is possible to extend \( \tilde{\mathcal{U}} \) and \( \tilde{\Phi} \) to an open set containing \( \{ \gamma^h_0 \}_{h \in (h_k, h_{k+1})} \cup \{ \gamma^h_\lambda \}_{h \in (h_k, h_{k+1})} \), which, in turn, obviously contains \( \bigcup_{h \in (h_k, h_{k+1})} \{ \theta(h), \lambda(h) \} \).

We call such a pair \( (\tilde{\Phi}, \tilde{\mathcal{U}}) \) the extended Fermi coordinate frame because it not only contains the central curve \( h \to X(t + h) \) as in the classical case, but also \( \theta(h) \) and \( \lambda(h) \) for almost every \( h \in [0, T] \).

**Step 4: Controlling the distance by decomposition.** From this point forward, we will assume that all computations take place in the extended Fermi coordinate system \( (\tilde{\Phi}, \tilde{\mathcal{U}}) \). By the definition of \( \theta(t) \), and given that \( \tilde{\theta}(h) \) is the normal coordinate of \( \theta(h) \) with center \( X(t + h) \), we have, for all \( h \in \mathcal{H} \),

\[
\text{dist}(X(t + h), \Theta_h(X(t))) = \text{dist}(X(t + h), \theta(h)) = ||\tilde{\theta}(h)||_2 \leq ||\tilde{\theta}(h) - \tilde{\lambda}(h)||_2 + ||\tilde{\lambda}(h)||_2.
\]

Since \( \mathcal{H} \) is a dense open subset of \([0, T]\) and since it is a countable union of open intervals, it follows that \( \tilde{\theta}(h) \) and \( \tilde{\lambda}(h) \) are differentiable except on a set of measure zero. We may thus write

\[
||\tilde{\theta}(h) - \tilde{\lambda}(h)||_2 = \left\| \int_0^h [\dot{\tilde{\theta}}(u) - \dot{\tilde{\lambda}}(u)] \, du \right\|_2.
\]

Moreover, by Lemma B.1 and the definition of (RMD) and (PFlow), we have

\[
\begin{align*}
\dot{\theta}^k(u) &= \dot{\theta}_k(u) - \dot{X}_k(t + u) + \mathcal{O}(||\dot{\theta}(u)||_2^2) = \tilde{v}_k(u, \dot{\theta}(u)) - \dot{X}_k(t + u) + \mathcal{O}(||\ddot{\theta}(u)||_2^2), \\
\dot{\lambda}^k(u) &= \dot{\lambda}_k(u) - \dot{X}_k(t + u) + \mathcal{O}(||\dot{\lambda}(u)||_2^2) = \tilde{\Lambda}_k \left( u, \dot{\lambda}(u) \right) - \dot{X}_k(t + u) + \mathcal{O}(||\ddot{\lambda}(u)||_2^2),
\end{align*}
\]

where \( \dot{X}_k \) is defined in Lemma B.1 and \( \tilde{v}_k(u, \dot{\theta}(u)) \) and \( \tilde{\Lambda}_k \left( u, \dot{\lambda}(u) \right) \) are, respectively, the \( k \)-th components of the vectors \( v(\theta(u)) \) and \( \Gamma_{X(t+u)\rightarrow\lambda(u)}(v(X(t+u))) \) in the frame induced by the normal coordinate with center \( X(t + u) \) and frame \( \{e_k(u)\}_{k=1}^d \). Now, denoting by \( \tilde{v}(u, \dot{\theta}(u)) \) the (Euclidean) vector with components \( \tilde{v}_k(u, \dot{\theta}(u)) \) and, by \( \tilde{\Lambda}(u, \dot{\lambda}(u)) \) the vector with components \( \tilde{\Lambda}_k \left( u, \dot{\lambda}(u) \right) \), we may write

\[
||\dot{\theta}(h) - \dot{\lambda}(h)||_2 \leq \left\| \int_0^h \left( \tilde{v}(u, \dot{\theta}(u)) - \tilde{\Lambda}(u, \dot{\lambda}(u)) \right) \, du \right\|_2 + \int_0^h R_\lambda(u) \, du
\]
where the remainder term \( R_1(u) \) is of order \( O \left( \| \hat{\theta}(u) \|_2^2 + \| \hat{\lambda}(u) \|_2^2 \right) \). Noting that, by (20),

\[
\| \hat{\theta}(u) \|_2 = \text{dist} (X(t + u), \Theta_u(X(t))) \leq R,
\]

\[
\| \hat{\lambda}(u) \|_2 = \text{dist} (X(t + u), \lambda(t + u)) \leq R,
\]

we have \( \| \hat{\theta}(u) \|_2^2 + \| \hat{\lambda}(u) \|_2^2 \leq R \left( \| \hat{\theta}(u) \|_2 + \| \hat{\lambda}(u) \|_2 \right) \), we get \( R_1(u) = O_R \left( \| \hat{\theta}(u) \|_2 + \| \hat{\lambda}(u) \|_2 \right) \) where \( O_R(\cdot) \) includes any constants depending on \( R \).

In the same vein, denoting by \( \hat{X}(t + u) \) the Euclidean vector whose \( k \)-th component is \( \hat{X}_k(t + u) \), we have

\[
\| \hat{\lambda}(h) \|_2 \leq \left\| \int_0^h \left( \hat{\lambda} \left( u, \hat{\lambda}(u) \right) - \hat{X}(t + u) \right) du \right\|_2 + \int_0^h R_2(u) \, du
\]

with \( R_2(u) = O \left( \| \hat{\lambda}(u) \|_2^2 \right) = O_R \left( \| \hat{\lambda}(u) \|_2 \right) \).

**Step 5: From Fermi to parallel coordinates.** So far, we have reduced the proof to comparing the vectors in Eqs. (25) and (27). However, these vectors are not amenable to further computation as they are expressed in the frames induced by the normal coordinates, and these frames may not even be orthonormal.

On the other hand, when expressed in the parallel frame system (see Appendix B.1) with a common base point \( X(t + u) \), the vectors \( v(\theta(u)) \) and \( \Gamma_{X(t+u) \rightarrow \lambda(u)}(v(X(t + u))) \) possess some favorable properties. To see this, parallel transport the frame \( \{e_k(u)^d \}_{k=1} \) along the geodesic from \( X(t + u) \) to form an orthonormal frame \( \{e_k(u)^d \}_{k=1} \) of \( T_{\lambda(u)}M \), and consider the Euclidean vector \( \hat{\lambda}(u) \) whose components are defined as

\[
\hat{\lambda}_k(u, \hat{\lambda}(u)) := \langle \Gamma_{X(t+u) \rightarrow \lambda(u)}(v(X(t + u))), e_k(u) \rangle_{X(t+u)} - \langle v(\theta(u)), e_k(u) \rangle_{X(t+u)}.
\]

Similarly, parallel transport the frame \( \{e_k(u)^d \}_{k=1} \) along the geodesic from \( X(t + u) \) to form an orthonormal frame \( \{e_k''(u)^d \}_{k=1} \) of \( T_{\theta(u)}M \), and let \( \hat{v}_k(u, \hat{\theta}(u)) := \langle v(\theta(u)), e_k''(u) \rangle_{\theta(u)} \).

Since the parallel transport is an isometry, we get

\[
\hat{v}_k(u, \hat{\theta}(u)) - \hat{\lambda}_k(u, \hat{\lambda}(u)) = \langle v(\theta(u)), e_k''(u) \rangle_{\theta(u)} - \langle \Gamma_{\theta(u) \rightarrow X(t+u)}(v(\theta(u))), e_k(u) \rangle_{X(t+u)} - \langle v(X(t + u)), e_k(u) \rangle_{X(t+u)}
\]

\[
= \langle \Gamma_{\theta(u) \rightarrow X(t+u)}(v(\theta(u))), e_k(u) \rangle_{X(t+u)} - \langle v(X(t + u)), e_k(u) \rangle_{X(t+u)}
\]

and hence

\[
\| \hat{v}^0(u, \hat{\theta}(u)) - \hat{\lambda}(u) \|_2^2 = \| \Gamma_{\theta(u) \rightarrow X(t+u)}(v(\theta(u))) - v(X(t + u)) \|_{X(t+u)}
\]

\[
\leq L \text{dist}(\theta(u), X(t + u)) = L \| \hat{\theta}(u) \|_2
\]

where we have used the fact that \( v \) is \( L \)-Lipschitz and that \( \hat{\theta}(u) \) are normal coordinates with center \( X(t + u) \).

We would therefore like to replace \( \hat{v}(u, \hat{\theta}(u)) - \hat{\lambda}(u) \) with \( \hat{v}^0(u, \hat{\theta}(u)) - \hat{\lambda}^0(u, \hat{\lambda}(u)) \) in (25). To this end, consider the difference in the \( k \)-th component of \( \hat{v}(u, \hat{\theta}(u)) \) and
where \( \frac{\partial}{\partial x_i} \big|_{\theta(u)} \) is the \( k \)-th basis in the frame induced by the normal coordinate with center \( X(t+u) \) and frame \( \{e_k(u)\}_{k=1}^d \). More specifically, denote by \( v \) the vector \( \sum_{k=1}^d \tilde{v}_k(u)e_k(u) \in T_{X(t+u)}M \) and consider the family of geodesics

\[
\gamma(s, s') := \exp_{X(t+u)}(s(v + s'e_k(u)))
\]

so

\[
\frac{\partial}{\partial x_i} \big|_{\theta(u)} = \frac{\partial}{\partial s} \gamma(1, 0) = d\exp_{X(t+u)}(v)(e_k(u)).
\]

Invoking Cauchy-Schwartz, Lemma B.2, and the fact that \( \{e_k(u)\}_{k=1}^d \) is orthonormal, we get

\[
\left| \tilde{v}_k(u, \tilde{\theta}(u)) - \tilde{v}_k(u, \tilde{\theta}(u)) \right| = \left| \left< v(\theta(u)), e_k'(u) - \frac{\partial}{\partial x_k} \big|_{\theta(u)} \right> \right|
\leq \|v(\theta(u))\|_{\theta(u)} \cdot \|e_k'(u) - \frac{\partial}{\partial x_k} \big|_{\theta(u)}\|_{\theta(u)}
\leq G\cdot K_{max} \cdot f_{-K_{max}}(v) \cdot \|e_k(u)\|_{\theta(u)}
\leq G\cdot K_{max} \cdot f_{-K_{max}}(v) = G\cdot K_{max} \cdot f_{-K_{max}}(\|\tilde{\theta}(u)\|_2)
\]

where \( K_{max} = \max\{|K_{up}|, |K_{low}|\} \). Since \( \sinh x/x - 1 \leq \cosh x - e^x \) for all \( x \geq 0 \), (B.9) and (B.10) yield

\[
f_{-K_{max}}(\|\tilde{\theta}(u)\|_2) \leq \frac{1}{K_{max}} \cdot \exp\left( \sqrt{K_{max}}\|\tilde{\theta}(u)\|_2 \right) \leq \frac{e^{-R\sqrt{K_{max}}}}{R \cdot K_{max}} \cdot \|\tilde{\theta}(u)\|_2
\]

where the last inequality follows from \( \|\tilde{\theta}(u)\|_2 = \text{dist}(X(t+u), \Theta_u(X(t))) \leq R \). Combining (34) and (35), we thus get

\[
\left| \tilde{v}_k(u, \tilde{\theta}(u)) - \tilde{v}_k(u, \tilde{\theta}(u)) \right| \leq \frac{Ge^{-R\sqrt{K_{max}}}}{R} \cdot \|\tilde{\theta}(u)\|_2.
\]

In short, we have shown that

\[
\tilde{v} \left( u, \tilde{\theta}(u) \right) = \tilde{v}^{u} \left( u, \tilde{\theta}(u) \right) + R_3(u)
\]

where \( R_3(u) = O_{G, K_{max}, R} \left( \|\tilde{\theta}(u)\|_2 \right) \) collects constants that depend on \( G, K_{max} \) and \( R \). Exactly the same computation shows that, for some \( R_4(u) = O_{G, K_{max}, R} \left( \|\tilde{\lambda}(u)\|_2 \right) \),

\[
\tilde{\lambda} \left( u, \tilde{\lambda}(u) \right) = \tilde{\lambda}^{u} \left( u, \tilde{\lambda}(u) \right) + R_4(u).
\]
Step 6: Putting everything together. With all these preliminaries in hand, we are finally in a position to complete our proof.

Proof of Theorem 2. We will proceed by bounding (25) and (27). Using (37), (38), and (30) in (25), we obtain:

\[ \| \hat{\theta}(h) - \tilde{\lambda}(h) \|_2 \leq \left\| \int_0^h \left( \hat{v}(u, \hat{\theta}(u)) - \hat{\Lambda}(u, \hat{\lambda}(u)) \right) du \right\|_2 + \int_0^h R_1(u) du \]

\[ \leq \int_0^h \left\| \hat{v}(u, \hat{\theta}(u)) - \hat{\Lambda}(u, \hat{\lambda}(u)) \right\|_2 du + \int_0^h R_1(u) du \]

\[ \leq \int_0^h \left\| \hat{v}(u, \hat{\theta}(u)) - \hat{\Lambda}(u, \hat{\lambda}(u)) \right\|_2 du + \int_0^h (R_1 + R_3 + R_4)(u) du \]

\[ \leq L \int_0^h \left\| \hat{\theta}(u) \right\|_2 du + \int_0^h (R_1 + R_3 + R_4)(u) du. \quad (39) \]

We next turn to (27). Our first task is to obtain an expression for \( \hat{X}(t+u) \), i.e., the Euclidean vector whose \( k \)-th component is \( X_k(t+u) \). To this end, fix an iteration count \( n \), and consider all \( u \) such that \( t + u \in [\tau_n, \tau_{n+1}) \) (that is, consider only the interpolation between \( X_n \) and \( X_{n+1} \)). We claim that \( \hat{X}_k(t+u) \) is constant throughout all such \( u \), and, in particular, \( \hat{X}_k(t+u) = X_k(\tau_n) \). This readily follows by noticing that

1. The curve \( X(t) \) is a geodesic segment when restricted to \( [\tau_n, \tau_{n+1}) \); see (GI).
2. The Fermi coordinates along \( X(\cdot) \), when restricted to \( \{X(s) : s \in [\tau_n, \tau_{n+1})\} \), is simply a parallel frame system; this follows from the fact that the frame \( \{e_k(u)\}_{k=1}^d \) is obtained from parallel transporting \( \{e_k(\tau_n)\}_{k=1}^d \) along \( X(\cdot) \) for all \( u \) such that \( t + u \in [\tau_n, \tau_{n+1}) \).

In this way, a simple calculation akin to (B.2) yields, for all such \( u \),

\[ \hat{X}_k(t+u) = \langle \hat{X}(t+u), e_i(u) \rangle_{X(t+u)} \]

\[ = \langle \Gamma_{X(t+u) \to X(\tau_n)} \left( \hat{X}(t+u) \right), e_i(\tau_n) \rangle_{X(\tau_n)} \]

\[ = \langle v(X(\tau_n)) + W_n, e_i(\tau_n) \rangle_{X(\tau_n)} = \langle v(X(\tau_n)) + U_n + b_n, e_i(\tau_n) \rangle_{X(\tau_n)} \quad (40) \]

where we have used (11) and the definition of \( W_n \) in the last equality.

Armed with the above, we can obtain a succinct expression for \( \hat{X}(t+u) \) as follows. First, let \( \hat{x}(u) \) be the normal coordinate of \( X(t+u) \) with center \( X(t+u) \) (i.e., \( \langle u, \hat{x}(u) \rangle = \Phi(u, X(t+u)) \)), and define an Euclidean vector \( \tilde{v}^i(u, \hat{x}(u)) \) by setting its \( k \)-th component to

\[ \tilde{v}^i_k(u, \hat{x}(u)) := \langle v(\hat{X}(t+u)), e_i(m(t+u)) \rangle_{X(t+u)} \quad (41) \]

where the mapping \( m(\cdot) \) is defined in (9). Define also the Euclidean noise and bias vectors \( U_n^i \) and \( b_n^i \) by setting their components to

\[ U_{k,n}^i := \langle U_n, e_i(m(t+u)) \rangle_{X(t+u)} \quad (42a) \]

\[ b_{k,n}^i := \langle b_n, e_i(m(t+u)) \rangle_{X(t+u)} \quad (42b) \]

Then (40) states precisely that

\[ \hat{X}(t+u) = \tilde{v}^i(u, \hat{x}(u)) + \bar{U}^i(t+u) + \tilde{b}^i(t+u). \quad (43) \]
Substituting (43) into (27) and invoking (38), (12), and (15), we obtain

\[
\|\tilde{\lambda}(h)\|_2 \leq \left\| \int_0^h \left( \tilde{\Lambda} \left( u, \tilde{\lambda}(u) \right) - \tilde{v}(u, \tilde{x}(u)) - \tilde{U}(t + u) - \tilde{b}'(t + u) \right) du \right\| \leq \int_0^h R_2(u) \, du \\
\leq \int_0^h \left\| \tilde{\Lambda} \left( u, \tilde{\lambda}(u) \right) - \tilde{v}(u, \tilde{x}(u)) \right\|_2 \, du \\
+ \left\| \int_0^h \tilde{U}(t + u) \, du \right\| \leq \left\| \int_0^h \tilde{b}'(t + u) \, du \right\| + \int_0^h [R_2(u) + R_4(u)] \, du \\
\leq \int_0^h \left\| \tilde{\Lambda} \left( u, \tilde{\lambda}(u) \right) - \tilde{v}(u, \tilde{x}(u)) \right\|_2 \, du \\
+ \Delta(t, T) + \tilde{B}^*(t) \cdot h + \int_0^h [R_2(u) + R_4(u)] \, du. \tag{44}
\]

To bound the first term in (44), recall (28) and (41). An identical argument leading to (B.4) shows that

\[
\left\| \tilde{\Lambda} \left( u, \tilde{\lambda}(u) \right) - \tilde{v}(u, \tilde{x}(u)) \right\|_2 = \left\| v(X(t + u)) - \Gamma_{\tilde{X}(t+u) \rightarrow X(t+u)} (v(\tilde{X}(t + u))) \right\|_{X(t+u)} \\
\leq L \cdot \text{dist} (\tilde{X}(t + u), X(t + u)). \tag{45}
\]

Since \(X(\cdot)\) is a (not necessarily minimizing) geodesic on \([m(t + u), t + u]\), Eqs. (12) and (43) yield

\[
\text{dist}(\tilde{X}(t + u), X(t + u)) \leq \left\| \int_{m(t+u)}^{t+u} \dot{\tilde{X}}(s) \, ds \right\|_2 \\
= \left\| \int_{m(t+u)}^{t+u} \tilde{v}(s, \tilde{x}(s)) + \tilde{U}(s) + \tilde{b}'(s) \, ds \right\|_2 \\
\leq \left\| \int_{m(t+u)}^{t+u} \tilde{v}(s, \tilde{x}(s)) \, ds \right\|_2 \\
+ \left\| \int_{m(t+u)}^{t+u} \tilde{U}(s) \, ds \right\|_2 + \left\| \int_{m(t+u)}^{t+u} \tilde{b}'(s) \, ds \right\|_2 \\
\leq (G + \tilde{B}^*(t)) \cdot \left( \int_{m(t+u)}^{t+u} ds \right) + \left\| \int_{m(t+u)}^{t+u} \tilde{U}(s) \, ds \right\|_2 \\
\leq (G + \tilde{B}^*(t)) \cdot \hat{\gamma}^*(t) + \left\| \int_{m(t+u)}^{t+u} \tilde{U}(s) \, ds \right\|_2. \tag{46}
\]

For \(t\) large enough, we have \(\hat{\gamma}^*(t) < 1\), and hence

\[
\left\| \int_{m(t+u)}^{t+u} \tilde{U}(s) \, ds \right\|_2 \leq \left\| \int_{m(t+u)}^{m(t+u)} \tilde{U}(s) \, ds \right\|_2 + \left\| \int_{m(t+u)}^{t+u} \tilde{U}(s) \, ds \right\|_2 \leq 2\Delta(t - 1, T + 1). \tag{47}
\]

Combining Eqs. (44)–(47) then gives

\[
\|\tilde{\lambda}(h)\|_2 \leq L\hat{h}[G + \tilde{B}^*(t)]\hat{\gamma}^*(t)
\]
Recall that the injectivity radius of \( p \) is the radius of the largest geodesic ball on which \( \exp_p \) is a diffeomorphism, and the injectivity radius of \( \mathcal{M} \) is the infimum over all such radii \([40]\). In this regard, Assumption 5 serves to ensure that the exponential map is invertible at consecutive iterates of \( \text{RRM} \) so no local topological complications can arise;
we will in fact prove in Proposition 2 that \( \log_{x_n} := \exp_{x_n}^{-1} \) is well-defined for all sufficiently large \( n \) in the algorithms to follow.

For concreteness, we will also assume that the algorithms considered in this section have black-box access to \( v \) via a \textit{stochastic first-order oracle} (SFO) \cite{49}. Specifically, when called at \( x \in \mathcal{M} \) with random seed \( \omega \in \Omega \), an SFO returns a random vector \( V(x; \omega) \in T_x \mathcal{M} \) of the form

\[
V(x; \omega) = v(x) + \text{Err}(x; \omega)
\]  

(SFO)

where the error term \( \text{Err}(x; \omega) \in T_x \mathcal{M} \) is zero-mean and bounded in \( L^2 \), i.e.,

\[
\mathbb{E}[\text{Err}(x; \omega)] = 0 \quad \text{and} \quad \mathbb{E}[\|\text{Err}(x; \omega)\|^2] \leq \sigma^2 \quad \text{for all} \quad x \in \mathcal{M}.
\]  

(54)

Finally, we will also assume that all algorithms under study are run with a step-size policy \( \gamma_n \) such that

\[
\frac{A}{n} \leq \gamma_n \leq \frac{B}{n^{1/2}(\log n)^{1/2+\varepsilon}} \quad \text{for some} \quad A, B, \varepsilon > 0 \quad \text{and all} \quad n = 1, 2, \ldots
\]  

(55)

Clearly, Assumption 1 is satisfied automatically under (55).

Remark. To facilitate comparisons with the Riemannian optimization literature, we will sometimes refer to queries of the SFO \( V(x; \omega) \) as (stochastic) “gradients”; we stress however that, in general, \( v \) is not assumed to admit a potential.

4.1. Basic examples. We now proceed to outline below a – highly incomplete – series of algorithms that can be seen as special cases of the general template (RRM).

Algorithm 1 (Riemannian stochastic gradient methods). The simplest \textit{Riemannian stochastic gradient method} (RSGM) \cite{9} queries (SFO) and proceeds as

\[
X_{n+1} = \exp_{X_n} (\gamma_n \, V(X_n; \omega_n)),
\]  

(RSGM)

As such, (RSGM) is an RRM scheme with \( U_n = \text{Err}(X_n; \omega_n) \) and \( b_n = 0 \).

Algorithm 2 (Riemannian proximal point methods). The (deterministic) \textit{Riemannian proximal point method} (RPPM) \cite{22} is an implicit update rule of the form

\[
\exp_{X_n}^{-1} (X_n) = -\gamma_n \, v(X_{n+1}).
\]  

(RPPM)

The RRM representation of (RPPM) is then obtained by taking \( b_n = \Gamma_{X_{n+1}} \rightarrow X_n (v(X_{n+1})) - v(X_n) \) and \( U_n = 0 \) in the decomposition (2) of the error term \( W_n \) of (RRM).

Algorithm 3 (Riemannian stochastic extra-gradient). Inspired by the original work of Korpelevich \cite{35}, the \textit{Riemannian stochastic extra-gradient} (RSEG) method \cite{50, 60} proceeds as

\[
X_{n+1/2} = \exp_{X_n} (\gamma_n \, V(X_n; \omega_n)),
\]

\[
X_{n+1} = \exp_{X_n} (\Gamma_{X_{n+1/2}} \rightarrow X_n (\gamma_n \, V(X_{n+1/2}; \omega_{n+1/2})))
\]  

(RSEG)

where \( \omega_n \) and \( \omega_{n+1/2} \) are independent seeds for (SFO). Thus, to recast (RSEG) in the RRM framework, simply take \( U_n = \Gamma_{X_{n+1/2}} \rightarrow X_n (\text{Err}(X_{n+1/2}; \omega_{n+1/2})) \) and \( b_n = \Gamma_{X_{n+1/2}} \rightarrow X_n (v(X_{n+1/2})) - v(X_n) \).
Algorithm 4 (Riemannian optimistic gradient). Compared to (RSGM), the scheme (RSEG) involves two oracle queries per iteration. Building on an original idea by Popov [53], the last oracle query can be “recycled”, leading to the Riemannian optimistic gradient method

\[
X_{n+1/2} = \exp_{X_n}(\gamma_n V(X_{n-1/2}; \omega_{n-1})), \\
X_{n+1} = \exp_{X_n}(\Gamma_{X_{n+1/2} \rightarrow X_n}(\gamma_n V(X_{n+1/2}; \omega_n))). \tag{ROG}
\]

With this in mind, (ROG) may be seen as a special case of (RRM) by taking \(U_n = \Gamma_{X_{n+1/2} \rightarrow X_n}(\text{Err}(X_{n+1/2}; \omega_n))\) and \(b_n = \Gamma_{X_{n+1/2} \rightarrow X_n}(v(X_{n+1/2})) - v(X_n)\).

In view of Theorems 1 and 2, the convergence analysis of Algorithms 1–4 essentially boils down to verifying Assumptions 1–4. Proposition 2 below does much of the heavy lifting for this:

**Proposition 2.** Suppose that Assumption 5 holds. If Algorithms 1–4 are run with oracle and step-size parameters satisfying (54) and (55), then:

1. Assumption 2 holds as stated.
2. With probability 1, \(X_{n+1}\) lies in the injectivity radius of \(X_n\) for all sufficiently large \(n\), so Assumption 4 also holds as stated.

**Corollary 3.** Suppose that Hypotheses (H1) and (H2) hold. If Algorithms 1–4 are run with oracle and step-size parameters satisfying (54) and (55), then, with probability 1, the generated sequence \(X_n, n = 1, 2, \ldots,\) converges to an ICT set of (RMD).

To streamline our discussion, we postpone the proof of Proposition 2 to Appendix C.

4.2. Algorithmic variants and modifications. To increase the computational efficiency of Riemannian iterative schemes, several important operations are routinely tacked on to the base algorithms described in the previous section. In view of this, we proceed below to illustrate a range of algorithmic variants and modifications that can be readily incorporated in the general framework of (RRM).

**Retraction-based methods.** When the exponential map is expensive to compute, a popular alternative is to employ a so-called retraction map [1, 10], defined here as a smooth mapping \(R : T_M \rightarrow M\) that agrees with the exponential map up to first order, i.e.,

\[
R_x(0) = x \quad \text{and} \quad \left. \frac{d}{dt} \right|_{t=0} R_x(tv) = v \quad \text{for all } (x, v) \in TM. \tag{Rtr}
\]

As it turns out, to replace the exponential map in Algorithms 1–4 with a retraction, we only need to slightly strengthen our assumptions on the noise in (SFO):

**Proposition 3.** Suppose that the error term \(\text{Err}(x; \omega)\) of (SFO) is bounded in \(L^4\), i.e., \(E[\|\text{Err}(x; \omega)\|^4] \leq \kappa^2\) for some \(\kappa > 0\) and all \(x \in M\). Then Proposition 2 holds as stated if the exponential map in Algorithms 1–4 is replaced by a retraction.

**Corollary 4.** Suppose that Hypotheses (H1) and (H2) hold. If a retraction-based variant of Algorithms 1–4 is run with a step-size schedule satisfying (55) and an SFO with bounded fourth moments, then, with probability 1, the generated sequence \(X_n, n = 1, 2, \ldots,\) converges to an ICT set of (RMD).

**Remark 1.** Albeit relatively light, the condition \(\sup_x E[\|\text{Err}(x; \omega)\|^4] < \infty\) cannot be relaxed. This is true even in the Euclidean case with \(R_x(v) = x + (e^v - 1)\) and \(\text{Err}(x; \omega) \overset{\text{law}}{=} \sqrt{X - E[\sqrt{X}]}\), where \(P(X > x) = x^{-3/2}\) if \(x \geq 1\), and \(P(X > x) = 1\) otherwise.
Remark 2. In many practical settings, the map $R_x(v)$ satisfies the stronger requirement of being a “second-order retraction”, i.e., it agrees with $\exp$ up to second order \cite{1, 10}. In this case, the proof technique in Appendix C.2 can be used to show that it suffices to have $\sup_{x \in M} E[|\text{Err}(x; \omega)|^3_x] < \infty$ in Proposition 3.

Alternating variants. The next set of variants concerns the case where $v$ is generated from a 2-player, min-max game (the $N$-player general-sum case is similar, but the notation is more involved so we omit it).

To state it, consider a min-max game of the form

$$\min_{y \in Y} \max_{z \in Z} \ell(y, z)$$

played over the smooth manifolds $Y$ and $Z$ (for the min and max player respectively). Then, instead of performing simultaneous updates for each player in Algorithms 1–4, a common variant is to alternate variables according to the basic recursion

$$
Y_{n+1} = \exp_{Y_n}(\gamma_n [v_y(Y_n, Z_n) + W_{y,n}]) \\
Z_{n+1} = \exp_{Z_n}(\gamma_n [v_z(Y_{n+1}, Z_n) + W_{z,n}])
$$

(RRM-alt)

where $(v_y, v_z) := (-\nabla_y \ell, \nabla_z \ell)$, and $W_{y,n}$ and $W_{z,n}$ respectively denote the error terms entering (RRM) for the min and max player respectively. Our next result shows that the alternating variants of Algorithms 1–4 retain the convergence properties of their simultaneous counterparts:

**Proposition 4.** Suppose that Algorithms 1–4 are run with alternating updates as per (RRM-alt) and oracle and step-size parameters satisfying (54) and (55). Then Proposition 2 holds as stated.

**Corollary 5.** Suppose that Hypotheses (H1) and (H2) hold. If an alternating variant of Algorithms 1–4 is run with oracle and step-size parameters satisfying (54) and (55), then, with probability 1, the generated sequence $X_n$, $n = 1, 2, \ldots$, converges to an ICT set of (RMD).

Remark. A simple but useful observation is that compositions of RRM schemes do not change its asymptotic behavior: specifically, for any two RRM schemes $\mathcal{RM}_1$ and $\mathcal{RM}_2$, the update $X_{n+1} = \mathcal{RM}_2 \circ \mathcal{RM}_1(X_n)$ is equivalent to a new RRM scheme $\tilde{X}_n$ where $\tilde{X}_{2n} = X_n$ and $\tilde{X}_{2n+1} = \mathcal{RM}_1(\tilde{X}_{2n})$. This allows us to “mix-and-match” Propositions 2–4 to prove, for instance, the convergence of alternating (RSEG) minimizer vs. retraction-based (RPPM) maximizer in Riemannian two-player games.

4.3. Implications for optimization and learning. We conclude this section with some concrete implications of our general theory when $v$ is specialized to specific instances that arise in optimization and learning theory. This allows us to extend and unify several existing results, but also to obtain completely new ones altogether.

Optimization on manifolds. Perhaps the most common task for learning on manifolds is the basic minimization problem

$$\min_{x \in M} f(x) \quad \text{(Opt)}$$

where $f : M \to \mathbb{R}$ is a $C^d$-smooth function (not necessarily geodesically convex). In this case, applying our general theory to $v = -\nabla f$ yields:

**Proposition 5.** Suppose that Algorithms 1–4 are run against (Opt) with oracle and step-size parameters satisfying (54) and (55). Assume further that the problem satisfies (a) Assumptions 3 and 5; or, alternatively, (b) Hypotheses (H1) and (H2). Then, with probability 1, the
induced sequence of iterates \( X_n, n = 1, 2, \ldots \), converges to a component of critical points of \( f \) on which \( f \) is constant.

If, in addition, \( \sup_x \mathbb{E}[\|\text{Err}(x; \omega)\|_F^2] < \infty \), the above conclusions apply to all retraction-based variants of Algorithms 1–4.

Proof. By Corollary 1, Proposition 2 and Corollary 3 (or Proposition 3 and Corollary 4 for the retraction-based case), it follows that \( X_n \) converges to an ICT set of (RMD) with probability 1. Moreover, by Sard’s theorem [57], the set of critical values of \( f \) has empty interior, so Proposition 6.4 of Bonnabel [9] implies that every ICT of (RMD) is contained in a set of critical points of \( f \) on which \( f \) is constant. Our assertion then follows by combining the above.

Proposition 5 contains as a special case the analysis of Bonnabel [9] who established the almost sure convergence of Algorithm 1 (and its retraction-based variants) under the assumptions that (a) \( X_n \) is precompact (Assumption 3); (b) the injectivity radius of \( M \) is uniformly bounded from below (Assumption 5); and (c) the oracle \( V(x; \omega) \) is uniformly bounded in both \( x \in M \) and \( \omega \in \Omega \), i.e., \( \text{ess sup}_{x, \omega} \|V(x; \omega)\|_x < \infty \). In this regard, Proposition 5 not only relaxes the bounded oracle requirement of Bonnabel [9], but it provides a straightforward way to establish the convergence of a wide array of Riemannian algorithms which cannot otherwise be covered by the tailor-made analysis of Bonnabel [9]. We are not aware of a comparable convergence result for Algorithms 2–4 (or their retraction-based variants).

In addition, we note that Proposition 5 includes the celebrated natural policy gradient (NPG) method of Kakade [31] for reinforcement learning – which, as noted by Bonnabel [9], is a particular case of retraction-based Algorithm 1. We stress that our result applies not only to “vanilla” NPG methods, but also to its optimistic/extra-gradient variants. To our knowledge, there are no comparable results in the NPG literature.

Games on manifolds. We next move on to the game setting, i.e., when \( M \) decomposes as a product of the form \( M = M_1 \times \cdots \times M_N \) for some \( N \in \mathbb{N} \) and, likewise, the \( i \)-th component of \( v = (v_1, \ldots, v_N) \) is of the form \( v_i = \nabla x_i, u_i(x_1, \ldots, x_N) \), where \( u_i : M \to \mathbb{R} \) is the payoff function of the \( i \)-th player – for applications and a detailed discussion, cf. Ratliff et al. [54] and references therein.

There are many solution concepts of games on manifolds that can be seen as a natural generalization of Nash equilibria. In this work, we will consider the so-called Nash–Stampacchia equilibrium (NSE) [36, 47]:

Definition 3. We say that \( x^* = (x_1^*, \ldots, x_N^*) \in M \) is a Nash–Stampacchia equilibrium of \( v \) if

\[
\langle v(x^*), \exp_{x_i^*}^{-1}(x) \rangle_x := \sum_{i=1}^N \langle v_i(x^*), \exp_{x_i^*}^{-1}(x_i) \rangle_{x_i^*} \geq 0, \quad \text{for all } x \in M.
\]

Our first result below concerns the convergence of Algorithms 2–4 in a general class of (Riemannian) monotone games known as \( \alpha \)-accretive games [62]:

Proposition 6 (Riemannian \( \alpha \)-accretive games). Let \( v = [-\nabla x, \ell_i] \) be an \( \alpha \)-accretive game field, i.e., all \( r \geq 0 \), we have

\[
(1 + \alpha r) \text{dist}(x, x') \leq \text{dist}(\exp_x(rv(x)), \exp_x(rv(x'))), \quad \alpha > 0.
\]

Then Algorithms 1–4, as well as their alternating/retraction-based versions, converge to the game’s set of NSE.

Proof. This immediately follows by combining Propositions 2–4 with the main result of [36].
To the best of our knowledge, most of the algorithms we consider are new in the setting of Riemannian monotone games except for the deterministic gradient and extra-gradient methods [14, 21, 32, 50, 60].

While being quite general, accretivity is a strong, convexity-like assumption about the games. In our next result, we prove general convergence for a class of non-convex games.

**Proposition 7** (Riemannian potential games). Let \( v = [-\nabla x, \ell_i] \) be a game field associated with a Riemannian potential game [34, 48]. Then Algorithms 1–4, as well as their alternating/retraction-based versions, converge to the critical points of the game potential.

**Proof.** Simply combine Propositions 2–4. ■

For Riemannian potential games, the convergence of the continuous-time dynamics (RMD) is well known, but we are not otherwise aware of a similar result for stochastic, discrete-time RRM methods. Our theory bridges this gap by showing that the same guarantees are in fact achieved by a wide array of RRM schemes – not only by their continuous-time ancestor.

**Limit cycles in Riemannian manifolds.** We conclude this section by showing that, in complement to the pointwise convergence results above, our theory can also be used to derive convergence to limit cycles that arise in more general Riemannian settings.

**Example 4.1** (Cycling of RRM schemes). The following example is taken from [17]: Consider the vector field on \( \mathbb{S}^2 := \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \} \), defined by

\[
v(x) = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} + \left( x_3^2 - \frac{1}{4} \right) \begin{bmatrix} x_1x_3 \\ x_2x_3 \\ -x_1^2 - x_2^2 \end{bmatrix}. \tag{56}
\]

Then it is known that the associated (RMD) cycles in the sense that the ICT sets of \( v \) in (56) contain attracting periodic orbits; see Fig. 2. Our Propositions 2–4 then imply that any RRM scheme driven by \( v \) also cycles. To the best of our knowledge, this is the first rigorous example of a cycling problem for Riemannian stochastic approximation in the literature.

5. **Concluding remarks**

Our theory provides a unified analysis for the convergence of Riemannian Robbins–Monro schemes that might seem vastly different from each other at first sight; instead, by verifying certain simple criteria on the error terms \( W_n \) in Assumption 2, our analysis shows that we can study the deterministic dynamics (RMD) to directly infer the algorithm’s long-run behavior. At the same time, our results offer but a glimpse of the flexibility of (RRM), and several important research directions remain open:
In many applications (especially to game theory and sequential online learning), oracle access to \( v \) may be out of reach, and one would need to employ zeroth-order – or bandit – optimization methods. In this case, a key question that arises is whether a Riemannian Kiefer–Wolfowitz algorithm can be analyzed as a special case of (RRM) – and, if so, whether there are any fundamental differences relative to the Euclidean setting.

The diminishing step-size assumption is indispensable for our analysis, which covers many practically relevant settings. However, another common strategy involves constant step-sizes, and this is not covered by our theory. It would thus be interesting to see if the techniques for analyzing constant step-size SA schemes [38, 39] can be generalized to a manifold setting.

Finally, several Riemannian algorithms are known to escape undesirable solutions [15, 58]. We conjecture that the general avoidance theory of [5, 12, 27, 51] can be likewise extended to Riemannian manifolds; if true, this would imply that many iterative Riemannian methods (including retraction-based ones) converge with probability 1 only to local minimizers.

We defer the study of the above questions to future work.

Acknowledgments

This research was supported by the SNSF grant 407540_167212 through the NRP 75 Big Data program, and by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme grant agreement No 815943. YPH acknowledges funding through an ETH Foundations of Data Science (ETH-FDS) postdoctoral fellowship. Part of this work was done while PM was visiting the Simons Institute for the Theory of Computing. PM is also grateful for financial support by the French National Research Agency (ANR) in the framework of the “Investissements d’avenir” program (ANR-15-IDEX-02), the LabEx PERSYVAL (ANR-11-LABX-0025-01), MIAI®Grenoble Alpes (ANR-19-P3IA-0003), and the grant ALIAS (ANR-19-CE48-0018-01).

Appendix A. Stability analysis

The purpose of this appendix is to prove Proposition 1. To that end, let \( p \in \mathcal{M} \) be as in (WC), define the radial distance function \( r(x) := \text{dist}(x, p) \), and let \( k(x) := r^2(x)/2 \). The following theorem makes it clear under which assumptions \( k \) is smooth and provides a control on its Hessian.

**Theorem 3** (Jost [30, Theorem 6.6.1]). With notation as above, suppose that the exponential map \( \exp_p \) is a diffeomorphism on \( \{ v \in T_p \mathcal{M} : \|v\|_p \leq \rho \} \). Moreover, suppose that the sectional curvature of \( \mathcal{M} \) is nonpositive and bounded below by \(-\kappa^2\) on a geodesic ball of radius \( \rho > 0 \) centered at \( p \). Then \( k \) is smooth on the punctured ball \( \mathbb{B}_p^\ast(p) := \{ x \neq p : \text{dist}(x, p) \leq \rho \} \) and, in particular,

\[
\nabla k(x) = -\log_p(x)
\]  

(A.1)

In addition, \( \|\nabla k(x)\| = r(x) \), and

\[
\text{Hess} \ k(x)[v, v] \leq \kappa r(x) \text{ctgh}(\kappa r(x))\|v\|_x^2, \quad \text{for all } x \in \mathbb{B}_p^\ast(p) \text{ and } v \in T_x \mathcal{M}.
\]

(A.2)

**Remark.** Since \( \mathcal{M} \) is simply connected and complete – by Hypothesis (H1) – we may take \( \rho = \infty \) in the theorem above [30, Corollary 6.9.1].
Now, to proceed, recall that \( v \) satisfies the weak coercivity condition (WC) if, for some \( R > 0 \), we have
\[
\langle v(x), \nabla k(x) \rangle_x \leq 0
\]
whenever \( x \in \mathcal{M} \) lies outside a geodesic \( \mathcal{B}_R(p) \) contained in the interior of \( \mathcal{M} \). Our proof relies on constructing a suitable “energy function” that serves as an easy-to-control proxy for the distance of the iterates of (RRM) from the origin. This function will be of the form
\[
\Phi(x) = f(r(x))
\]
where \( f \) is a \( C^\infty \) non-negative function with \( f(x) = 0 \) for all \( x \leq R \) and satisfies
\[
0 \leq f'(x) \leq C_1, \quad f''(x) \leq C_2
\]
for all \( x \geq R \). Moreover, we require \( f(x) = \Omega(x) \) as \( x \to \infty \) so that controlling \( f \) implies control of \( x \) (for a concrete example of such a function, cf. Lemma A.2). Our first result is that \( \Phi = f \circ r \) has a bounded Hessian and is smooth.

**Lemma A.1.** Let \( \Phi \) be defined as above. Then \( \Phi \) is negatively correlated with \( v \) in the sense that
\[
\langle \nabla \Phi(x), v(x) \rangle_x \leq 0 \quad \text{for all } x \in \mathcal{M}.
\]
Moreover, there exists a constant \( C > 0 \) such that \( \text{Hess} \Phi(x)[v, v] \leq C\|v\|_x^2 \), and hence
\[
\Phi(x') \leq \Phi(x) + \langle \nabla \Phi(x), \log_x(x') \rangle + \frac{C}{2} \text{dist}^2(x, x') \quad \text{for all } x, x' \in \mathcal{M}.
\]

**Proof.** We begin by noting that the gradient of \( \Phi \) is given by
\[
\nabla \Phi(x) = \begin{cases} 0 & \text{if } r(x) \leq R, \\ \frac{f'(r(x))}{r(x)} \nabla k(x) & \text{if } r(x) > R. \end{cases}
\]
By assumption, \( f'(r(x))/r(x) \geq 0 \) so \( \langle \nabla \Phi(x), v(x) \rangle_x \) has the same sign as \( \langle \nabla k(x), v \rangle_x \) if \( r(x) \geq R \) and is otherwise zero if \( r(x) \leq R \). We thus conclude that \( \Phi \) and \( v \) are negatively correlated, as claimed.

Now, to compute the Hessian of \( \Phi \), notice that \( \text{Hess} \Phi(x)[v, v] = \langle \nabla_v \nabla \Phi(x), v \rangle_x \). Hence,
\[
\text{Hess} \Phi(x)[v, v] = \nabla_v \left( \frac{f'(r(x))}{r(x)} \cdot \langle \nabla k(x), v \rangle_x + \frac{f''(r(x))}{r(x)} \langle \nabla_v \nabla k(x), v \rangle_x \right)
\]
\[
= \left( \frac{f'(r(x))}{r(x)} \right) \langle \nabla \left( \nabla k(x), v \right)_x \rangle_I + \frac{f''(r(x))}{r(x)} \text{Hess} k(x)[v, v].
\]

Here we use the same notation for directional derivative of a scalar function and the covariant derivative. With this in mind, the first step in computing \( I \) is the observation that
\[
\nabla \frac{f'(r(x))}{r(x)} = \left( f''(r(x)) - \frac{f'(r(x))}{r(x)} \right) \frac{1}{r^2(x)} \nabla k(x),
\]
and hence
\[
I = \left( f''(r(x)) - \frac{f'(r(x))}{r(x)} \right) \frac{1}{r^2(x)} \langle \nabla k(x), v \rangle_x^2 \leq \frac{C_2}{r^2(x)} \| \nabla k(x) \|_x^2 \| v \|_x^2 = C_2 \| v \|_x^2.
\]
For \( \Pi \), as \( \text{ctgh} x \leq 1 + x \) for \( x \geq 0 \), we obtain
\[
\Pi \leq \frac{f'(r(x))}{r(x)} (1 + \kappa r(x)) \| v \|_x^2 \leq C_1 (1/R + \kappa) \| v \|_x^2.
\]
Summing up everything, we obtain
\[ \text{Hess} \Phi(x)[v, v] \leq (C_2 + C_1/R + C_1\kappa)\|v\|^2 =: C\|v\|^2, \quad (A.13) \]
that is, \( \Phi \) has bounded Hessian. Moreover, \( \Phi \) is smooth as a composition of smooth functions. Let \( x, x' \in \mathcal{M} \) be arbitrary, and let \( \gamma : [0, 1] \to \mathcal{M} \) be a geodesic connecting the two. By Taylor’s remainder theorem, there exists some \( t \in (0, 1) \) such that
\[ \Phi(x') = \Phi(x) + \langle \nabla \Phi(x), \gamma'(0) \rangle_x + \frac{1}{2} \text{Hess} \Phi(t)[\gamma', \gamma'], \quad (A.14) \]
Thus, invoking (A.13) and noting that \( x \) and \( \gamma'(0) \) are bounded, we deduce
\[ \Phi(x') \leq \Phi(x) + \|\nabla \Phi(x)\|_{\mathcal{M}} \|x\|_{\mathcal{M}} + \frac{C}{2} \|x\|^2_{\mathcal{M}}, \]
and hence, by Assumption 2 and the dominated convergence theorem, we infer that
\[ \Phi(x_n) \to \Phi(x) \text{ (a.s.)}, \]
and the proof is complete.

We now proceed to the main argument, where we show how to use \( \Phi \) to control the iterates \( X_n \). Letting \( \Phi_n = \Phi(X_n) \) and using Lemma A.1 we get
\[
\Phi_{n+1} = \Phi(\exp_{X_n}(\gamma_n V_n))
\leq \Phi_n + \gamma_n \langle \nabla \Phi(X_n), V_n \rangle_{X_n} + \frac{C_2 \gamma_n^2}{2} \|V_n\|^2_{X_n}
\leq \Phi_n + \gamma_n \langle \nabla \Phi(X_n), U_n + b_n \rangle_{X_n} + \frac{3C_2 \gamma_n^2}{2} \left[ \|v(X_n)\|^2_{X_n} + \|U_n\|^2_{X_n} + \|b_n\|^2_{X_n} \right]
\quad (A.15)
\]
where the second line follows from the negative correlation of \( \Phi \) and \( V_n \), and the Cauchy-Schwarz inequality. Conditioning on \( \mathcal{F}_n \) and taking expectations, and invoking Cauchy-Schwarz and the fact that \( \|\nabla \Phi(X_n)\| \leq \frac{C}{2} \|\nabla k(X_n)\| = C_1 \), we obtain:
\[ E[\Phi_{n+1} | \mathcal{F}_n] \leq \Phi_n + \gamma_n C_1 B_n + \frac{3C_2}{2} C_1 \gamma_n^2 \left[ G^2 + B_n^2 + \sigma_n^2 \right], \quad (A.16) \]
where we have bounded the second moments by their respective upper bounds.

To proceed, let \( \varepsilon_n = \gamma_n C_1 B_n + (3/2)C_2 \gamma_n^2 \left[ G^2 + B_n^2 + \sigma_n^2 \right] \) denote the “residual” term in (A.16). Notice that
\[ \sum_{n=1}^{\infty} \varepsilon_n \leq C_1 \sum_{n=1}^{\infty} \gamma_n B_n + \frac{3C_1}{2} \sum_{n=1}^{\infty} \gamma_n^2 \left( G^2 + B_n^2 + \sigma_n^2 \right), \quad (A.17) \]
and hence, by Assumption 2 and the dominated convergence theorem, we infer that \( E[\sum_{n=1}^{\infty} \varepsilon_n] < \infty \).

Next, consider the auxiliary process \( \hat{\Phi}_n = \Phi_n + E[\sum_{k=n}^{\infty} \varepsilon_k | \mathcal{F}_n] \), adapted to the same filtration. By (A.16), we have \( E[\hat{\Phi}_{n+1} | \mathcal{F}_n] \leq \Phi_n + E[\sum_{k=n}^{\infty} \varepsilon_k | \mathcal{F}_n] = \hat{\Phi}_{n-1} \), i.e., \( \hat{\Phi}_n \) is a supermartingale with respect to \( \mathcal{F}_n \). This shows that \( E[\hat{\Phi}_n] \leq E[\Phi_1] < \infty \), i.e., \( \hat{\Phi}_n \) is uniformly bounded in \( L^1 \). Hence, by Doob’s supermartingale convergence theorem [25, Theorem 2.5], it follows that \( \hat{\Phi}_n \) converges with probability 1 to some finite random limit \( \hat{\Phi}_\infty \). In turn, since \( \sum_{n=1}^{\infty} \varepsilon_n < \infty \) (a.s.), this implies that \( \Phi_n = \hat{\Phi}_n - E[\sum_{k=n}^{\infty} \varepsilon_k | \mathcal{F}_n] \) also converges to some (random) finite limit (a.s.). From this and the fact that \( \Phi_n = \Omega(r(X_n)) \), we deduce \( \limsup_n r(X_n) < \infty \) as claimed.

**Lemma A.2.** Let \( h : \mathbb{R} \to \mathbb{R} \) be the function
\[
h(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
\frac{e^{-x}}{1 - e^{-1/x}} & \text{if } x \in (0, 1) \\
1 & \text{otherwise}
\end{cases}
\quad (A.18)
\]
and, for \( R > 0 \), let \( f(x) = \int_0^x h(s-R) \, ds \). Then \( f \) is \( C^\infty \) and it satisfies the conditions (A.5) with \( C_1 = 1 \) and \( C_2 = 2 \). In addition, one has \( f(x) \geq x - (R + 1) \), and hence \( f(x) = \Omega(x) \).
Proof. As $h(x) \in [0, 1]$, we obtain that $f'(x) \in [0, 1]$. By a straightforward computation, one observes that the first derivative of $h$ is bounded as $0 \leq h'(x) \leq 2$, so $f''(x) = h'(x - R) \leq 2$. Then, to complete our proof, simply notice that, for $x \geq R + 1$, we have $f(x) = \int_0^x h(s - R) \, ds \geq \int_{R+1}^x 1 \, ds = x - (R + 1)$, as claimed.

Appendix B. Geometric preliminaries for the proof of Theorem 2

In this appendix, we collect the necessary technical prerequisites for the proof of Theorem 2.

B.1. The parallel frame system. We begin with the definition of the frame system that we use to compare vectors on different tangent spaces. To that end, fix any two points $x, x' \in \mathcal{M}$, and consider two arbitrary vectors $v \in T_x \mathcal{M}$ and $w \in T_{x'} \mathcal{M}$. There is a convenient frame system (i.e., a set of bases for $T_x \mathcal{M}$ and $T_{x'} \mathcal{M}$) for comparing $v$ and $w$, defined as follows: Pick an arbitrary orthonormal frame $\{e_k\}_{k=1}^d$ for $T_x \mathcal{M}$. Since the parallel transport map is an isometry, the vectors $\{e'_k\}_{k=1}^d := \{\Gamma_{x' \to x}(e_k)\}_{k=1}^d$ form an orthonormal basis for $T_{x'} \mathcal{M}$. Now consider the components of $v, w$ in these two frames, namely

$$v'_k := \langle v, e'_k \rangle_x \text{ and } w'_k := \langle w, e'_k \rangle_{x'} \text{ for all } k = 1, \ldots, d. \quad (B.1)$$

We shall call (B.1) the parallel frame system for vectors $v$ and $w$ (the dependence on the initial frame $\{e_k\}_{k=1}^d$ is suppressed).

By virtue of parallel transport, in the parallel frame system we have

$$\Gamma_{x' \to x}(v)_x = (\Gamma_{x' \to x'}(v), e'_k)_{x'} = \langle v, \Gamma_{x' \to x}(e_k) \rangle_x = v'_k. \quad (B.2)$$

In the same vein, we have $\Gamma_{x' \to x}(w)_x = w'_k$. Thus, since $\{e_k\}_{k=1}^d$ is orthonormal, we may write:

$$\|w - \Gamma_{x' \to x}(v)\|_x = \|v - \Gamma_{x' \to x}(w)\|_{x'} = \|v' - w'\|_2. \quad (B.3)$$

In other words, in the parallel frame system, the comparison of vectors living on different tangent spaces is reduced to simply comparing the Euclidean norms of their components. For instance, the $L$-Lipschitzness of $v$ can be rephrased as

$$\|v'(x') - v'(x)\|_2 \leq L \text{dist}(x, x') \quad \text{for all } x, x' \in \mathcal{M}. \quad (B.4)$$

B.2. The Fermi coordinate system. For any $h$, let $U_h \subset T_{X(t+h)} \mathcal{M} \simeq \mathbb{R}^d$ be a neighborhood of 0 on which the mapping

$$\exp_{X(t+h)}: U_h \to \mathcal{M} \quad (B.5)$$

is a diffeomorphism between $U_h$ and $\exp_{X(t+h)}(U_h)$. It is well-known that such a neighborhood exists, and that the exponential map $\exp_{X(t+h)}$ along with an arbitrary orthonormal frame at $T_{X(t+h)} \mathcal{M}$ induces a local coordinate system on $\exp_{X(t+h)}(U_h)$, called the normal coordinate frame with center $X(t+h)$ [40]. Normal coordinates are best-suited for comparing distances of points on manifolds: for instance, if $\tilde{x}'$ is the normal coordinate of $x'$ with center $x$, then $\text{dist}(x, x') = \|\tilde{x}'\|_2$.

The Fermi coordinate system (FCS) [45], roughly speaking, is a system of normal coordinates “along a curve”. To define it, fix an arbitrary orthonormal frame $\{e_k(0)\}_{k=1}^d$ for $T_{X(t)} \mathcal{M}$. We can obtain a system of orthonormal frames $\{e_k(h)\}_{k=1}^d$ by parallel transporting $\{e_k(0)\}_{k=1}^d$ from $T_{X(t)} \mathcal{M}$ to $T_{X(t+h)} \mathcal{M}$ along the curve $h \mapsto X(t+h)$. Let $U_h \subset T_{X(t+h)} \mathcal{M}$ be a neighborhood of 0 defined as in (B.5), and set $\tilde{U} := \bigcup_h \{\exp_{X(t+h)}(U_h)\} \subset \mathcal{M}$. Finally, consider the mapping

$$\tilde{\Phi}: \mathbb{R}_+ \times \tilde{U} \to \mathbb{R}_+ \times \mathbb{R}^d \quad (B.6)$$
that sends a point \((h, x) \in \mathbb{R}^+ \times \mathcal{U}\) to \((h, \tilde{x}) \in \mathbb{R}^+ \times \mathbb{R}^d\), where \(\tilde{x}\) is the normal coordinate of \(x\) with center \(X(t + h)\) and frame \(\{e_k(h)\}_{k=1}^d\). By virtue of the normal coordinates, we know that \(\Phi\) is a diffeomorphism between \(\mathbb{R}^+ \times \mathcal{U}\) and a neighborhood of \(\mathbb{R}^+ \times \{0\}\). The mapping \(\Phi\) and its inverse is called the Fermi coordinate system along the curve \(h \mapsto X(t + h)\). In the sequel, we will abuse the notation and simply call it the Fermi coordinates along \(X\).

The following property of the Fermi coordinate system plays a key role in our analysis and we will use it freely:

**Lemma B.1** (Takahashi and Watanabe, 1981; Fujita and Kotani, 1982). Let \(\gamma\) be a differentiable curve on \(\mathcal{M}\) such that \(\gamma(h) \in \mathcal{U}_h\) for all \(h \in \mathbb{R}^+\), and let \(\tilde{\gamma}\) be the curve of \(\gamma\) in the FCS along \(X\) (i.e., \((h, \tilde{\gamma}(h)) = \Phi(h, \gamma(h))\)). Then

\[
\dot{\tilde{\gamma}}_k(h) = \dot{\gamma}_k(h) - \dot{X}_k(t + h) + O(\|\dot{\gamma}(h)\|_2^2)
\]

(B.7)

where \(\dot{X}_k(t + h) := (\dot{X}(t + h), e_k(h))_{X(t+h)}\) is the \(k\)-th component of \(\dot{X}(t + h)\) in the frame \(\{e_k(h)\}_{k=1}^d\), and \(\dot{\gamma}_k(h)\) is the \(k\)-th component of \(\dot{\gamma}(h)\) in the (possibly non-orthonormal) frame induced by the normal coordinate system with center \(X(t + h)\) and frame \(\{e_k(h)\}_{k=1}^d\).

**B.3. Comparing the differential of \(\exp\) and parallel transport.** As will become clear in the proof, the PFS is convenient for comparing vectors at different points, whereas the FCS is best suitable for comparing vectors at nearby points. To this end, we will need the following technical lemma, whose proof can be found in [41, Theorem 3.12] or [16, Proposition A.1]:

**Lemma B.2** (Comparing \(d \exp\) and parallel transport). Let \(\mathcal{M}\) be a Riemannian manifold whose sectional curvatures are in the interval \([K_{\text{low}}, K_{\text{up}}]\), and let \(K = \max(\|K_{\text{low}}\|, |K_{\text{up}}|)\). For \(v \in T_x\mathcal{M}\), consider the geodesic \(\gamma(t) = \exp_x(tv)\). If \(\gamma\) is defined and has no interior conjugate point on the interval \([0, 1]\), then

\[
\forall w \in T_x\mathcal{M}, \quad \|T_v(w) - \Gamma_v(w)\|_{\gamma(1)} \leq K \cdot f_{K_{\text{low}}}(|v|_x) \cdot \|w_x\|_x
\]

(B.8)

where \(w_x := w - \langle v, w \rangle_x v\) is the component of \(w\) orthogonal to \(v\), \(T_v = d\exp_x(v)\) is the differential of the exponential map, and \(\Gamma_v\) denotes the parallel transport along \(\gamma\) from \(\gamma(0)\) to \(\gamma(1)\). The function \(f_{K_{\text{low}}}\) in (B.8) is defined as

\[
f_{K_{\text{low}}}(a) = \begin{cases} 
\frac{r^2 \sinh(a/r)}{a/r} - 1 & \text{if } K_{\text{low}} = 0, \\
\frac{r^2}{a/r} \left(1 - \frac{\sinh(a/r)}{a/r}\right) & \text{if } K_{\text{low}} = \frac{1}{r^2} > 0, \\
\frac{r^2}{a/r} \left(\frac{\sinh(a/r)}{a/r} - 1\right) & \text{if } K_{\text{low}} = -\frac{1}{r^2} < 0.
\end{cases}
\]

(B.9)

Moreover, the function \(f_{K_{\text{low}}}\) is dominated by the case \(K_{\text{low}} < 0\): For all \(K \geq |K_{\text{low}}|\) and \(a \in \mathbb{R}^+\),

\[
f_{K_{\text{low}}}(a) \leq f_{-K}(a).
\]

(B.10)
Appendix C. Proofs for Section 4

C.1. Proof of Proposition 2. We prove Proposition 2 in this appendix. To this end, we first provide a convenient lemma which shows that, almost surely, the effect of the noise is asymptotically annihilated by the step-size:

**Lemma C.1.** Under the assumptions in Proposition 2, we have, with probability 1,
\[
\lim_{n \to \infty} \| \gamma_n V(X_n; \omega_n) \|_{X_n} = 0. \tag{C.1}
\]

**Proof.** By definition,
\[
\| \gamma_n V(X_n; \omega_n) \|_{X_n} = \| \gamma_n v(X_n) + \text{Err}(X_n; \omega_n) \|_{X_n} \leq \gamma_n G + \gamma_n \| \text{Err}(X_n; \omega_n) \|_{X_n}. \tag{C.2}
\]
The first term goes to 0 by choice of step-sizes. To control the second term, note that by Chebyshev’s inequality and (54), we have
\[
P \left( \| \text{Err}(X_n; \omega_n) \|_{X_n} \geq \sqrt{n \log^{1+\frac{1}{2}} n} \right) \leq \frac{\sigma^2}{n \log^{1+\frac{1}{2}} n} \tag{C.3}
\]
where \( \varepsilon \) is the same as in our choice of step-size in Proposition 2. In turn, this implies that
\[
\sum_{n=2}^{\infty} P \left( \| \text{Err}(X_n; \omega_n) \|_{X_n} \geq \sqrt{n \log^{1+\frac{1}{2}} n} \right) < \infty
\]
so, by the Borel-Cantelli lemma, we have \( \| \text{Err}(X_n; \omega_n) \|_{X_n} = O \left( \sqrt{n \log^{1+\frac{1}{2}} n} \right) \) with probability 1. Hence, by our assumptions for the method’s step-size, we get
\[
\gamma_n \| \text{Err}(X_n; \omega_n) \|_{X_n} = O \left( \frac{\sqrt{n \log^{1+\frac{1}{2}} n}}{\sqrt{n \log^{1+\frac{1}{2}} n}} \right) = O \left( \frac{1}{\log^{\frac{1}{2}} n} \right)
\]
which, combined with (C.2), implies (C.1). \( \blacksquare \)

We are now ready for the full proof.

The second claim of Proposition 2 is a direct consequence of Lemma C.1 and Assumption 5. As for the first, note that \( \sum_n \gamma_n^2 < \infty \) by our choice of step-sizes so Assumption 1 holds by construction and we are left to establish the summability conditions (7) of Assumption 2. To this end, by Assumption 1, (54), and Lemma C.1, it suffices to show that \( \| b_n \|_{X_n} = O(\gamma_n \cdot \text{noise}) \) where “noise” is a query from (SFO) at the appropriate state. We proceed method-by-method:

**Algorithm 1: Riemannian stochastic gradient method.** For (RSGM), we have \( W_n = U_n = \text{Err}(X_n; \omega_n) \) and \( b_n = 0 \), so (7) follows from the stated assumptions for (SFO).

**Algorithm 2: Riemannian proximal point method.** For (RPPM), we have \( U_n = 0 \) and
\[
\| b_n \|_{X_n} = \| \Gamma_{X_{n+1} \rightarrow X_n} (v(X_{n+1})) - v(X_n) \|_{X_n} \leq L \text{dist}(X_n, X_{n+1}) = \gamma_n L \| v(X_n) \|_{X_n} \leq \gamma_n LM = O(\gamma_n)
\]
where we have used the \( L \)-Lipschitzness and \( G \)-boundedness of \( v \), and the distance-minimizing property of \( \exp \) within the injectivity radius.
Algorithm 3: Riemannian stochastic extra-gradient. For the RSEG algorithm, the recurrence (RSEG) gives $U_n = \Gamma_{X_{n+1/2}} X_{n} \left( \text{Err}(X_{n+1/2}; \omega_{n+1/2}) \right)$ so

\[ \mathbb{E}\left[ \|U_n\|_{X_n}^2 \right] = \mathbb{E}\left[ \|\text{Err}(X_{n+1/2}; \omega_{n+1/2})\|_{X_{n+1/2}}^2 \right] \leq \sigma^2 \]  

(C.5)

by (54) and the fact that the parallel transport map is a linear isometry. Finally, for the second part of (7), the definition of (RSEG) yields

\[ \|b_n\| = \|\Gamma_{X_{n+1/2}} X_{n} \left( v(X_{n+1/2}) \right) - v(X_n) \|_{X_n} \leq L \text{dist}(X_{n+1/2}, X_n) = \gamma_n L \|V(X_n; \omega_n)\|_{X_n} \]  

(C.6)

so our claim follows from the assumptions on $V$.

Algorithm 4: Riemannian optimistic gradient. For the ROG algorithm, the recurrence (ROG) gives $U_n = \Gamma_{X_{n+1/2}} X_{n} \left( \text{Err}(X_n; \omega_{n+1/2}) \right)$ and $b_n = \Gamma_{X_{n+1/2}} X_{n} \left( v(X_{n+1/2}) \right) - v(X_n)$, so Assumption 2 can be checked exactly as in the case of Algorithm 3 above.

C.2. Proof of Proposition 3. By definition, $R_x(v)$ is a smooth map and hence satisfies $\lim_{n \to 0} R_x(v) = x$. Then Lemma C.1 readily implies that $X_{n+1}$ lies in the injectivity radius of $X_n$ with probability 1 for $n$ large enough.

We first consider the retraction-based Algorithm 1:

\[ X_{n+1} = R_{X_n}(\gamma_n V(X_n; \omega_n)). \]  

(C.7)

Let $\tilde{v}_n \in \mathcal{T}_{X_n} \mathcal{M}$ be the vector such that $\exp_{X_n}(\gamma_n \tilde{v}_n) = X_{n+1}$, i.e.,

\[ \gamma_n \tilde{v}_n = \exp_{X_n}^{-1}(R_{X_n}(\gamma_n V(X_n; \omega_n))). \]  

(C.8)

Then (C.7) is an RRM scheme with $W_n = \tilde{v}_n - v(X_n)$ where $\tilde{v}_n$ is defined in (C.8). We will show that, under the assumption $\mathbb{E}[ \|\text{Err}(x; \omega)\|_2^4 ] < \infty$, the following holds with probability 1:

\[ b_n = \mathbb{E}[W_n | \mathcal{F}_n] \to 0, \quad \sup_n \mathbb{E}[ \|W_n\|_{X_n}^2 ] < \infty \]  

(C.9)

which obviously implies (7).

Consider the curve $c(t) := R_{X_n}(tV(X_n; \omega_n))$. By Lemma C.1, for $n$ large enough, $c(t)$ lies in the injectivity radius of $X_n$ almost surely for all $t \in [0, \gamma_n]$. Let $\hat{c}(t)$ be the smooth curve of $c(t)$ in the normal coordinate with base $X_n$ and an arbitrary orthonormal frame, and let $\hat{X}_{n+1}$ be the normal coordinate of $X_{n+1}$. Also, let $\hat{v}_n^N$ be the (Euclidean) vector of $\tilde{v}_n$ expanded in the chosen orthonormal basis, and define $V^N(X_n; \omega_n)$ and $\text{Err}^N(X_n; \omega_n)$ similarly. By definition, $X_{n+1}$ is nothing but $\gamma_n \hat{v}_n^N$.

Since $X_n = c(0)$ and $X_{n+1} = c(\gamma_n)$, by properties of a retraction map we must have

\[ \gamma_n \hat{v}_n^N = \hat{c}(\gamma_n) = \hat{c}(0) + \gamma_n \hat{c}(0) + \mathcal{O}\left( \gamma_n^2 \|\hat{c}(0)\|_2^2 \right) = \gamma_n V^N(X_n; \omega_n) + \mathcal{O}\left( \gamma_n \|V(X_n; \omega_n)\|_{X_n}^2 \right) =: \gamma_n V^N(X_n; \omega_n) + \gamma_n b_n \]  

(C.10)

where $\hat{b}_n = \mathcal{O}(\gamma_n \|V(X_n; \omega_n)\|_{X_n}^2)$. Therefore, since $\mathbb{E}[ \|\text{Err}(x; \omega)\|_2^4 ] < \infty$ for all $x \in \mathcal{M}$, we have

\[ \mathbb{E}\left[ \|W_n\|_{X_n}^2 \right] = \mathbb{E}\left[ \|\text{Err}^N(X_n; \omega_n) + \hat{b}_n\|_2^2 \right] < \infty. \]  

(C.11)
On the other hand,
\[ \|b_n\|_{X_n} = \|E[W_n | F_n]\|_{X_n} = \|E[\tilde{b}_n | F_n]\|_2 = O(\gamma_n \|V(X_n; \omega_n)\|_{X_n}). \] (C.12)

By Chebyshev’s inequality and the fact that \( E[\|\text{Err}(x; \omega)\|_4^4] \leq \kappa^2 < \infty \), we have
\[ P(\|\text{Err}(X_n; \omega_n)\|_{2X_n} \geq \sqrt{n \log \frac{1}{\varepsilon}}) \leq \kappa^2 \frac{1}{n \log \frac{1}{\varepsilon}} \]
where \( \varepsilon \) is the same as in our choice of step-size in Proposition 2. Using an calculation identical to Lemma C.1, we conclude that
\[ \gamma_n \|\text{Err}(X_n; \omega_n)\|_{2X_n} = O\left( \sqrt{\frac{n \log \frac{1}{\varepsilon}}{n \log \frac{1}{\varepsilon}}} \right) = O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right) \]
which concludes the proof of (C.9).

For the retraction-based variants of Algorithms 2–4, by the above analysis, we may replace \( R_x(\gamma_n V(\cdot; \omega_n)) \) with \( \exp_x(\gamma_n (V(\cdot; \omega_n) + \tilde{b}_n)) \) where \( \tilde{b}_n \to 0 \) almost surely. The rest is the same as in the proof of Proposition 2. \( \blacksquare \)

C.3. Proof of Proposition 4. Similar to Proposition 2, Lemma C.1 guarantees that all geodesics are minimizing and invertible. Hence, by the \( L \)-Lipschitzness of \( v \) and (RRM-alt), we have, with probability 1,
\[ \|v_z(Y_{n+1}, Z_n) - v_z(Y_n, Z_n)\|_{z_n} \leq L \text{dist}\left( \left[ \begin{array}{c} Y_{n+1} \\ Z_n \end{array} \right], \left[ \begin{array}{c} Y_n \\ Z_n \end{array} \right] \right) \]
\[ = \gamma_n \|v_y(Y_n, Z_n) + U_y,n + b_y,n\|_{Y_n} \]
\[ \to 0 \] (C.13)
by Lemma C.1 and Proposition 2. Therefore, we may rewrite (RRM-alt) as
\[ X_{n+1} = \exp_{X_n} \left( \gamma_n \left[ v_y(Y_n, Z_n) + U_y,n + b_y,n \right] \right) \] (C.14)
where \( b'_{z,n} = b_{z,n} + v_z(Y_{n+1}, Z_n) - v_z(Y_n, Z_n) \). By (C.13) and Proposition 2, \( b'_{z,n} \) satisfies (7), so our proof is complete. \( \blacksquare \)

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