Swimmers in thin films: from swarming to hydrodynamic instabilities

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We investigate theoretically the collective dynamics of a suspension of low Reynolds number swimmers that are confined to two dimensions by a thin fluid film. Our model swimmer is characterized by internal degrees of freedom which locally exert active stresses (force dipoles or quadrupoles) on the fluid. We find that hydrodynamic interactions mediated by the film can give rise to spontaneous continuous symmetry breaking (swarming), to states with either polar or nematic homogeneous order. For dipolar swimmers, the stroke averaged dynamics are enough to determine the leading contributions to the collective behaviour. In contrast, for quadrupolar swimmers, our analysis shows that detailed features of the internal dynamics play an important role in determining the bulk behaviour. In the broken symmetry phases, we investigate fluctuations of hydrodynamic variables of the system and find that these destabilize order. Interestingly, this instability is not generic and depends on length-scale.

In nature, baths of micron-scale swimmers are found to show remarkable out of equilibrium phenomena ranging from anomalous diffusion and viscosity enhancement, to turbulent and swirl-like motion or self organization into complex dissipative structures\cite{1–5}. They act as the inspiration for man-made devices able to control and mix fluids on micron scales. The search for the design principles of such devices\cite{6,7} remains in its infancy. However, simplified models of low Reynolds number swimmers should prove to be a useful starting point for the theoretical understanding of this class of collective phenomena.

A very simple picture\cite{8} then, of a swimmer is an internally driven vector oriented towards its direction of motion. Theoretically the dynamics of a collection of such objects can be described, on long length- and time-scales, by vector and tensor equations which are natural generalizations of equilibrium liquid crystalline hydrodynamics\cite{10}. These active fluids involve the study of conserved and broken symmetry variables that are non-equilibrium analogues\cite{11} of Goldstone modes. Previous studies on active suspensions\cite{12,13,14} have shown that fluctuations in these modes, destabilize ordered states. In 3d this instability is termed generic, as it is independent of length-scale. However, these descriptions are in fundamental sense, phenomenological since another physical mechanism\cite{14,15} must be invoked to generate the ordered states which have subsequently been shown to be unstable\cite{12}.

In this letter, we introduce and study a self-contained and microscopically defined physical system in which it is possible to both generate homogeneously ordered states and to examine their stability. This is provided by a ‘suspension’ of swimmers confined to two dimensions by a viscous thin film\cite{5}. In addition the thin film geometry is particularly accessible to experiments both from the point of view of ease of observation and external activation. We study analytically a model that directly links the collective behaviour to the microscopic dynamics. It combines fluctuations, both active and passive, with the deterministic motion due to activity and hydrodynamic interactions.

We find that the purely physical coupling mediated by the thin film can give rise to local spontaneous breaking of symmetry and we identify the possible ordered states. The system can have both nematic order; characterized by a macroscopic axis of mean orientation $\mathbf{n}$ and symmetry $\mathbf{n} \rightarrow -\mathbf{n}$, or polar order; with mean orientation axis $\mathbf{p}$ for which $\mathbf{p} \neq -\mathbf{p}$. We then examine the stability of each of the homogeneous phases to hydrodynamic fluctuations. We find that the isotropic phase is stable. On the other hand, as for the bulk 3d system, the homogeneous broken symmetry states are destabilized by the hydrodynamic modes of the system. Here, however the thin film weakens this effect and the instabilities observed are length scale dependent (i.e. not generic).

We restrict ourselves to a dilute solution, and we make use of a mean field approximation and study the one-particle distribution function: $c(\mathbf{R}, \bar{u}, t) = \sum_i \delta(\mathbf{R}_i - \mathbf{R}) \delta(\bar{u}_i - \bar{u})$, the probability of finding a swimmer with average orientation $\bar{u}$ at position $\mathbf{R}$. This satisfies a dynamic equation

$$\partial_t c = -\nabla \cdot \mathbf{J}_T - \mathbf{R} \cdot \mathbf{J}_R \quad ;$$

$$\mathbf{J}_T = -D_T \cdot \nabla c + (\bar{v} \bar{u} + \mathbf{V})c, \quad \mathbf{J}_R = -D_R \mathbf{R} c + \mathbf{\Omega} c,$$

in terms of translational $\mathbf{J}_T(\mathbf{R}, \bar{u}, t)$ and rotational $\mathbf{J}_R(\mathbf{R}, \bar{u}, t)$ currents, with $\mathbf{\mathcal{R}} := \bar{u} \wedge \Omega \mathbf{u}$\cite{10}. $D_T$ and $D_R$ are respectively the translational (rotational) diffusion tensor (constant) of the swimmer and represent both passive and active fluctuations. The deterministic quantities are active, describing self-propulsion: $\bar{v} \bar{u}$, and the translational $\mathbf{V}$ and rotational $\mathbf{\Omega}$ velocities induced on a swimmer due to the activity of others by hydrodynamic interactions mediated by the film.

Thin film hydrodynamics: The film is described as an infinite incompressible two dimensional layer of fluid with $(2d)$ viscosity $\eta$ filling the plane $z = 0$ and coupled hydrodynamically to another incompressible bulk fluid of $(3d)$
viscosity \( \eta_c \) which fills the region \( z \neq 0 \). To distinguish between them we indicate the three-dimensional quantities with a prime. We consider the fluid dynamics in the vanishing Reynolds number (Stokes) limit where inertia can be neglected [17]. For the in-plane quantities, given an in-plane force density \( \mathbf{F}(x, y) \) the velocity \( \mathbf{v}(x, y) \) and pressure \( p(x, y) \) satisfy [18]

\[
\eta \nabla_x^2 \mathbf{v} + \nabla_x p + \sigma^+ - \sigma^- = - \mathbf{F}; \quad \nabla_x \cdot \mathbf{v} = 0
\] (2)

where \( \nabla_x = (\partial_x, \partial_y) \) is the 2d gradient operator. \( \sigma^\pm := \eta \partial_x v_i |_{y^\pm} \) is the shear stress of the bulk fluid at the top/bottom of the thin film; in the external region, \( z \neq 0 \), the velocity \( \mathbf{v}'(x, y, z) \) and pressure \( p'(x, y, z) \) satisfy the Stokes equation: 

\[
\eta \nabla_x^2 \mathbf{v}' + \nabla_x p' = 0; \quad \nabla_x \cdot \mathbf{v}' = 0,
\]

where \( \nabla = (\nabla_x, \partial_z) \) is a 3d gradient operator. The ratio of the two and three dimensional viscosities introduces a length-scale \( s := \eta/(2\eta_c) \) that governs two asymptotic regimes. For \( r \ll s \) dissipation occurs almost entirely in plane and hydrodynamic flow fields are quasi-two dimensional while at lengths \( r \gg s \) dissipation is mostly due to flow out of plane, and the hydrodynamics is similar (but not identical) to that in three dimensions.

The viscous drag coefficient on a flat disk of radius \( a \) embedded in the film is \( \gamma = 4\pi \eta b / a \), with \( b \) a function of \( s/a \) [18]. On length scales large compared to \( a \), the interaction between several disks lying in the film can be approximated using point-like forces at their centers and the Green's function \( \mathbf{H} \) of eq. (2), corresponding to the flow, \( \mathbf{v}(r) = \mathbf{H}(r-r_0) \cdot \mathbf{f}_0 \) generated by a point-like force \( \mathbf{f}_0 \) at \( r_0 \). The tensor \( \mathbf{H}(r) = \frac{\eta}{2\pi} \int \frac{d^2k}{(2\pi)^2} e^{i \mathbf{k} \cdot \mathbf{r}} \frac{\tau_{ij} \hat{e}^{ij\mathbf{k}}} {s^2 + \mathbf{k}^2} \) is the thin film equivalent of the Oseen tensor [16]. In the following, we work in the limit \( r \gg s \) [10, 20].

We consider swimmers with an average speed \( \overline{v} \) moving in a direction \( \hat{u} \) which can be represented at large length-scales as a time-dependent force dipole that generates an associated velocity field. We also consider swimmers for which the force dipole is zero and the leading behaviour is determined by a force quadrupole. Hence the average force density of a swimmer with mean position, \( r_0 \), is of the form \( \mathbf{f}_0(r) = -\mathbf{F}_j \nabla_j \hat{u} \delta(r-r_0) + \frac{1}{2} \mathbf{J}_j \nabla_j \hat{u} \nabla_i \delta(r-r_0) + \cdots \), representing the dipole and quadrupole contributions respectively where \( \mathbf{J} \) is the typical dimension of the swimmer.

For a concrete calculation, an explicit microscopic model of a swimmer is required and we have used a three-disk model of swimmer [21]. This is characterized by a minimum number (2) of degrees of freedom, with typical length \( l \) that move in a non-reciprocal fashion in time with frequency \( \omega = \frac{2\pi}{T} \) to achieve locomotion in the Stokes limit. In the following we indicate the time-average over a swimmer cycle period \( T \) with an over-bar \( \overline{\mathbf{F}} = \frac{1}{T} \int_0^T \mathbf{F}(t) dt \). In the limit of small sinusoidal oscillations of amplitude \( d \) around \( l \) is possible to obtain [21] the self-propulsion velocity as \( \overline{\mathbf{V}} = \frac{a_0 \omega d^2}{6 \eta} [1 + \frac{1}{\kappa} - \frac{1}{(1+\kappa)^2}] \) where \( a_0 := \frac{4a}{3\eta} \)

and our convention is that the swimmer move in the same direction of \( \hat{u} \). For the average force we find \( \overline{\mathbf{F}} = l \) and \( \overline{\mathbf{J}}_d = \frac{d}{12} \left[ \frac{1}{\eta} - 2 \left( \frac{1}{\eta} - \kappa \right) \right] \) \( \overline{\mathbf{J}}_q = \frac{d}{6\eta} [-1 + \frac{2}{\eta} + 2 \kappa + \kappa \eta + 2 \kappa^2 + \frac{1}{(1+\kappa)^2}] \) Here \( f := \gamma \omega d \) is the force scale on each degree of freedom of the swimmer. The parameter \( \kappa \), which is the average ratio of the internal lengths, controls the nature of the swimmer: the dipole is positive (pusher) for \( \kappa > 1 \), negative (puller) for \( \kappa < 1 \) and zero (quadrupole) for \( \kappa = 1 \).

The interaction between two such swimmers, which is the origin of \( \mathbf{V}, \mathbf{\Omega} \) in eq. (1), is complex as each one is characterized by periodic internal dynamics whose cycles may also have different phases [22]. Here we will restrict ourselves to swimmers that all have the same phase [22], and the limit where the typical separation \( r \) between the swimmers centres \( \mathbf{R}^A \) and \( \mathbf{R}^B \) is much larger than the typical dimension \( l \) of the object. Dynamical quantities depending on hydrodynamic interactions, such as translational \( \mathbf{v}_{2B} := \mathbf{R}^A - \mathbf{R}^B \) and angular \( \omega_{2B} := \mathbf{u}_A \times \mathbf{u}_B \) velocities felt by each swimmer, for \( \alpha = A, B \), are described to good approximation by the first few terms of the expansion in spherical harmonics [13, 22] of the tensor \( \mathbf{H} \). To leading order, for swimmer \( A \) we find \( \mathbf{v}_{2B} = \frac{\omega_{2B}(\lambda)}{2\pi \eta} \theta \mathbf{F}(r) \mathbf{B} + \mathbf{O}(1/r^3) \) and \( \omega_{2B} = \frac{\omega_{2B}(\lambda)}{2\pi \eta} \theta (3(B_2 - 1) \mathbf{F} - B \mathbf{F}^B) + \mathbf{O}(1/r^3) \) where \( \lambda := (\mathbf{r} \cdot \mathbf{u}_A + \mathbf{r} \cdot \mathbf{u}_B) / \mathbf{r} \cdot \mathbf{F}_d \) and \( \theta := \hat{R} \cdot \mathbf{F} / \mathbf{r} \cdot \mathbf{F}_d \) are averaged quantities.

The collective dynamics is obtained by a coarse graining procedure, first in time and then in space. The velocities \( \mathbf{V}, \mathbf{\Omega} \) in eqs. (1) are obtained from the two body velocities for swimmers with positions \( \mathbf{R} \) and \( \mathbf{R}' \) and orientations \( \hat{u} \) and \( \hat{u}' \) as

\[
\begin{align*}
\mathbf{V} & = \int \hat{u}' \cdot \mathbf{R}' \left[ \mathbf{v}_{2B}(\mathbf{R}, \mathbf{R}', \hat{u}, \hat{u}') \right] c(\mathbf{R}', \hat{u}') / c(\mathbf{R}, \hat{u})
\end{align*}
\] (3)

where, as before, the over-bar denotes time average over the swimmer cycle. Order parameters such as local density \( \rho \), polarization \( \mathbf{P} \) and nematic orientation tensor \( \mathcal{S} \) are defined as moments of \( c \) : 

\[
\begin{align*}
\rho & = \int \mathbf{R} \left[ \frac{1}{c} \hat{u} \right] c(\mathbf{R}, \hat{u}, t) / c(\mathbf{R}, \hat{u}),
\mathcal{P} & = \int \mathbf{R} \left( \frac{1}{c} \hat{u} \otimes \hat{u} - 1/2 \right) c(\mathbf{R}, \hat{u}, t).
\end{align*}
\] (4)

Homogeneous ordered states These are states in which \( c \) and its moments defined above do not vary with position (denoted by \( c^0, \rho^0, \mathcal{P}^0, \mathcal{S}^0 \) respectively). Under these conditions, the mean field velocities are \( \mathbf{V}^0 = \beta_0 \hat{u} \wedge \mathcal{P}^0 + \beta_1 \hat{u} \wedge \mathcal{S}^0 \cdot \hat{u} \) and \( \mathbf{V}' = \beta_T \mathcal{P}^0 \) where the coefficients \( \beta_0, \beta_1 \) and \( \beta_T \) are averaged quantities.
that depend on the microscopic details of each swimmer [23]. To leading order \( \beta_0 \approx f \left( \frac{1}{2} \right) \lambda \frac{1}{\beta_0} \) and 
\[ \beta_1 \approx f \left( \frac{1}{2} \right) \lambda \frac{1}{\beta_0} \left[ \frac{1}{2} \left( \kappa - 1 \right) + \frac{1}{2} \left( \kappa^2 - \frac{1}{4} \right) \right] + \frac{3}{2} \left( \kappa^3 - \frac{1}{12} \right). \]
They are both positive for \( \kappa > 1 \) (pushers) and negative for \( \kappa < 1 \) (pullers). Inserting the expressions of \( \Omega^0 \) and \( V^0 \) in eq (1), taking the time derivative of eq (4), we obtain dynamic equations for the moments of \( \delta \). Density is conserved, hence \( \partial_t \rho^0 = 0 \). The others are,
\[ \partial_t \Pi^0 = -D_R \Pi^0 + \frac{\beta_0}{2} \rho^0 \Pi^0 + \left( \frac{\beta_1}{2} - \beta_0 \right) \delta S^0 \cdot \Pi^0; \]
\[ \partial_t \delta S^0_{ab} = -4D_R \delta S^0_{ab} + \beta_0 \left( \Pi^0 \right)^{ab} - \frac{1}{2} \delta_{ab} \left( \Pi^0 \right)^2 + \frac{\beta_1}{2} \rho^0 \delta S^0_{ab}. \]
Analysis of eqs (5) shows that, when \( \beta_0 \) or \( \beta_1 \) are positive, the system can undergo a bifurcation, that signals the appearance of order. Hence pushers can give rise to order whereas pullers cannot. The conditions
\[ \begin{align*}
- D_R + \rho^0 \beta_0 / 2 = 0 \\
- 4D_R + \rho^0 \beta_1 / 2 = 0
\end{align*} \]
deﬁne two critical lines in the space of parameters \( \rho^0 \) and \( f \), above which the instability occurs [24]. The ratio \( \beta_1/(4\beta_0) \).

As in many active systems, we ﬁnd that their effect is to destabilize order [12]. In the following we consider the deviations of the ﬁelds from their homogeneous values given by \( \delta \rho = \rho - \rho^0 \), similarly for \( \Pi \) and \( S \).

We introduce Fourier transforms in the usual way as
\[ f(r) = \int_{k} e^{-ik \cdot r} f(k), \]
\[ \Pi(k) = \int_{k} e^{ik \cdot r} \Pi(r). \]

**Isotropic state** In the homogeneous isotropic state \( \rho = \rho^0, \Pi = \Pi^0 = 0 \) and \( \rho \) is the only hydrodynamic variable. Variables \( \delta \Pi_{ab} = 0 \), splay \( \delta S_{ab} \) and bend (\( \delta S_{ab} \)) show diffusive behaviour. From the resulting set of equations density ﬂuctuations are linearly stable [12].

**Polar state** In the homogeneous polar state \( \rho = \rho^0, \Pi = \Pi^0 \) and the hydrodynamic variables are \( \rho \) and the director \( \Pi \). The orientation tensor is slaved to \( \Pi \), and given by \( \Pi = \Pi^0 \Pi^2 (\Pi(\Pi - 1/2) \) where \( \Pi \) is determined by eqn. (6). The magnitude \( \Pi \) is not a hydrodynamic variable and relaxes to a constant value on microscopic time-scales. We set \( \Pi = 1 \) in the following and study linear perturbations around this state. Decomposing \( \kappa = \cos \phi \Pi^0 + \sin \phi \Pi^0, \) with \( \kappa = \Pi(\Pi - 1/2) \) leading order in \( k \) we ﬁnd that ﬂuctuations in density and director are coupled via splay \( (\kappa = \partial \delta \Pi, \) giving a growing mode \( \Gamma \) with real part \( \Re(\Gamma) \sim k^2 S \delta \phi^{(2)} \cos 2\phi (2 + \cos 2\phi) \) and imaginary part, which determines the propagation speed, \( \Im(\Gamma) \sim \pm k \gamma \sin \phi \), as shown in Figure (2a). When \( \Re(\Gamma) \) is positive fluctuations grow exponentially, signalling an instability of the ordered state due to hydrodynamic interactions, analogous to those found in 3d [12]. However unlike those, the growth rate here scales as \( k \) in the \( k \to 0 \) limit.

**Nematic state** A similar analysis can be performed around the homogeneous nematic state (finite \( \rho^0, \Pi^0 \)). In this case hydrodynamic variables are density, \( \rho \) and nematic orientation tensor, \( \Pi = \Pi^0 \Pi^2 (\Pi \Pi - 1/2) \) and \( \kappa = 1 \). The magnitude \( \Pi \) remains constant fast and in the following will be set to \( \Pi = 1 \). To leading order in \( k \) ﬂuctuations of density \( \delta \rho \) and nematic director \( \delta \Pi \) are decoupled and splay ﬂuctuations have a real mode \( \Gamma \sim k^2 S \delta \phi^{(2)} \cos 2\phi (2 + \rho^0 \cos 2\phi) \). For angles above \( \pi/4 \) destabilize the order, as shown in Figure (2b). For quadrupolar swimmers the relevant terms describing hydrodynamic interactions scale as \( k^2 \) instead of \( k \). Their analysis is not reported here [21].

It is instructive to compare our results with previous studies of swimmers in a 3d fluid [12,14,15]. Our analysis starts from a microscopic model, and ‘integrates out the fluid degrees of freedom to see the effect on the other hydrodynamic modes. Alternatively one may perform a phenomenological analysis of an ordered state as in [12] replacing the Stokes equation by eq (2), this yields qualitatively the same hydrodynamic instabilities that we have presented above. In essence, the thin ﬁlm model changes the Fourier spectrum of the hydrodynamic kernel from \( k^2 \) to \( k + sk^2 \) which in the limit \( sk \ll 1 \) reduces to \( k \). This is the origin of the different scaling of the in-

![Fig. 1: Phase diagram showing homogeneous states and their hydrodynamic stability. Isotropic-Nematic and Nematic-Polar transitions for dipolar swimmers.](image)
are treated on the same level, have to be considered. In which hydrodynamic and non-hydrodynamic variables rather than stationary, or the development of a formalism successful in the study of phase transitions in equilib-

Landau-Ginzburg framework [12, 15], which has been so non-equilibrium active systems illustrate the limits of the instabilities [11, 24, 25]. In contrast, here we ‘derive’ such conditions.

rules from a particular microscopic model using a coarse-

between swimmers are used to study aspects of collective behaviour [11, 24, 25]. In contrast, here we ‘derive’ such rules from a particular microscopic model using a coarse-graining procedure which is valid under a precise set of conditions.

The question of how to characterise the system beyond the instability [14, 25] remains open. This highlights a fundamental theoretical issue. The instabilities of homogeneous ordered states seen here and in other non-equilibrium active systems illustrate the limits of the Landau-Ginzburg framework [12, 15], which has been so successful in the study of phase transitions in equilib-

uum systems. It might be that alternative approaches, such as describing the homogeneous state as dynamical rather than stationary, or the development of a formalism in which hydrodynamic and non-hydrodynamic variables are treated on the same level, have to be considered.

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FIG. 2: Angular dependence of the growing mode \( \Gamma \) in the splay fluctuations around homogeneous order for pushers. A positive sign indicates instability. (a) polar state. \( \Re(\Gamma) \) is measured in units of \( k (s S^2(\Theta^{(2)}))/(8 \eta) \) and \( \Im(\Gamma) \) in units of \( k \tilde{v}/2 \); (b) nematic state. \( \Re(\Gamma) \) is measured in units of \( k (s(\Theta^{(2)}))/(4 \eta) \) and we have used \( \rho^2 = 0.3 \).