Chaotic dynamics of the contact interaction of two beams described of Euler-Bernoulli and Pelekh-Sheremetyev-Reddy hypotheses

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Abstract. Beam structures are widely used as the main elements of structures in aircraft and rocket engineering, mechanical engineering and instrument making. During operation, beams can experience external influences of various types, which leads to contact of the beams. In connection with this, the construction of a mathematical model of the contact interaction of beams is an actual problem. The aim of the work are to construct mathematical models based on the kinematic hypotheses of the first (Euler-Bernoulli) and the third (Pelekh-Sheremetyev-Reddy) approximations, the creation of methods for calculating the highly nonlinear (geometric and constructive nonlinearities) mechanical structures under the action of transversal harmonic loads. The resulting system of nonlinear partial differential equations in a dimensionless form by the method of finite differences of the second order of accuracy reduces to the Cauchy problem, which is solved by the Runge-Kutta method of the fourth order. The convergence of the obtained solutions is investigated depending on the intervals of the partitioning over the spatial and temporal coordinates. A rigid limitation was imposed on the coincidence of the basic functions in chaotic vibrations for n and 2n partitions of the interval of integration over the spatial coordinate. As an example, we consider the nonlinear dynamics of two beams, the gap between which is equal to unity. It is shown that the transition of the beam structure vibrations from harmonic to chaotic occurs through a subharmonic cascade of bifurcations. Chaotic phase synchronization based on the Morlet wavelet is investigated. The values of the highest Lyapunov exponent were calculated by the methods of Wolff, Kantz, and Rosenstein.

Keywords - Euler-Bernoulli beam, Peleha-Sheremetyev-Reddy beam, chaotic dynamics, contact interaction, Lyapunov exponents, chaotic phase synchronization of oscillations, finite difference method.

1. Introduction
At the moment, the mechanics of contact interaction is one of the most rapidly developing topics of the mechanics of a deformable solid and is widely used in various fields of science. Many of the nodes and structures used in engineering, construction, medicine and other fields operate under conditions of contact interaction. In this connection, mathematical modeling of the contact interaction of beams, plates and shells is a topical task of modern science. The objects of contact interaction can be described by different kinematic hypotheses. Among the works on the study of the dynamics of contact interaction we mention the works [1], [2]. Work [3] is devoted to the problems of contact interaction of elements, taking into account the kinematic hypothesis of the first approximation. The chaotic dynamics of two Timoshenko beams are explore in [4]. In [5] was investigated chaotic dynamics of functionally graded size dependent beams with transversal shifts. The kinematic hypothesis of the second approximation is taken into account in [6], [7]. Work [8] is devoted to the chaotic dynamics of two beams described by the kinematic hypothesis of the third approximation. In [9] the size-dependent non-linear micro-beam was studied. However, a review of the scientific literature on this subject allows us to conclude that there is no work to take into account the contact interaction of the beams described by different kinematic hypotheses. This issue is devoted to this work.
2. Formulation of the problem
In this paper we first study the nonlinear dynamics of the contact interaction of two beams, described by the kinematic hypotheses of the first (Euler -Bernoulli) and the third (Pelekh-Sheremetyev-Reddy) [10] approximation. An external alternating load acts on the upper beam. The lower beam (beam 2) is understood as an elastic base for the beam 1 and is set in motion by contact with the beam 1.

We are focused on the investigation of two beams structure being as 2D object in space $R^2$ with rectangular co-ordinates introduced in the following way. In the body of beam 1 a certain arbitrary reference curve $z = 0$ is fixed; the OX goes along main curvature of the reference curve, whereas the axis OZ is directed to the reference curvature center. In the given coordinates the beam structure as a 2D object $\Omega$ is defined as $\Omega = \{x \in [0,a], -h \leq z \leq h_k + 3h, \ 0 \leq t \leq \infty\}$, where $[0,a]$ defines a straight beam line.

![Figure 1. The settlement scheme](image)

The equations of beams motion, as well as the boundary and initial conditions, are obtained from the Hamilton-Ostrogradski principle. The geometric nonlinearity of the beams is assumed by the T. von Karman model, and the contact interaction is described by the B. Ya. Cantor model. [11]

3. Theory
The equations of motion of the beams, taking into account the contact interaction according to the model of B. Cantor, will take the form:

\[
\frac{1}{\lambda^2} \left[ \frac{4}{5} \frac{\partial^3 \gamma_{x1}}{\partial x^3} - \frac{1}{4} \frac{\partial^4 w_1}{\partial x^4} \right] + k^2 \frac{G_{13}}{E_1} \left[ \frac{\partial^2 \gamma_{x1}}{\partial x^2} + \frac{\partial^2 w_1}{\partial x^2} \right] + \\
+ \frac{1}{\lambda^2} \left[ L_1(w_1,u_1) + L_1(w_1,u_1) + \frac{3}{2} L_2(w_1,u_1) \right] + ( -1)^{1} K(w_1 - w_2 - h_k) \Psi - \\
+ q(x,t) - \frac{\partial^2 w_1}{\partial t^2} - \varepsilon \frac{\partial w_1}{\partial t} = 0; \\
\frac{\partial^2 u_1}{\partial x^2} + L_4(w_1,u_1) - \frac{\partial^2 u_1}{\partial t^2} = 0; \\
\]  

(1)
We must add boundary and initial conditions to the system of differential equations (1). We begin with detection of beams in the equations, where \( d \) describes the contact pressure between beams. The function \( \Psi = \frac{1}{2} \left[ 1 + \text{sign}(w_i - w_j - h_k) \right] \) describes the contact pressure between beams. We begin with detection of contact between beams, i.e., \( \Psi = 1 \) if \( w_i > w_j + h_k \) and there is a contact between beams; otherwise \( \Psi = 0 \). We must add boundary and initial conditions to the system of differential equations (1). Beams are rigidly fixed from both ends.

Boundary conditions for Pelek-Shemeretev-Riddy model:

\[
\begin{align*}
  w_i(0,t) &= w_i(1,t) = 0; \quad u_i(0,t) = u_i(1,t) = 0; \quad \gamma_{ai}(0,t) = \gamma_{ai}(1,t) = 0; \\
  \frac{\partial w_i(0,t)}{\partial x} &= \frac{\partial w_i(1,t)}{\partial x} = 0; \quad \frac{\partial u_i(0,t)}{\partial x} = \frac{\partial u_i(1,t)}{\partial x} = 0; \\
  \frac{16}{5} \frac{\partial^2 \gamma_{ai}(0,t)}{\partial x^2} - \frac{16}{5} \frac{\partial^3 w_i(1,t)}{\partial x^3} = 0; \quad \frac{136}{315} \frac{\partial \gamma_{ai}(0,t)}{\partial x} - 0.038 \frac{\partial^2 w_i(1,t)}{\partial x^2} = 0.
\end{align*}
\]

(2)

Initial conditions for Pelek-Shemeretev-Riddy model:

\[
\begin{align*}
  w_i(x,t) \big|_{t=0} &= u_i(x,t) \big|_{t=0} = \gamma_{ai}(x,t) \big|_{t=0} = 0, \\
  \frac{\partial w_i(x,t)}{\partial t} \big|_{t=0} &= \frac{\partial u_i(x,t)}{\partial t} \big|_{t=0} = \frac{\partial \gamma_{ai}(x,t)}{\partial t} \big|_{t=0} = 0.
\end{align*}
\]

(3)

Boundary conditions for Euler-Bernoulli model:

\[
\begin{align*}
  w_2(0,t) &= w_2(1,t) = u_2(0,t) = u_2(1,t) = \frac{\partial w_2(0,t)}{\partial x} = \frac{\partial w_2(1,t)}{\partial x} = 0
\end{align*}
\]

(4)

Initial conditions for Euler-Bernoulli model:

\[
\begin{align*}
  w_2(x) \big|_{t=0} &= 0, \quad u_2(x) \big|_{t=0} = 0, \quad \frac{\partial w_2(x)}{\partial t} \big|_{t=0} = 0, \quad \frac{\partial u_2(x)}{\partial t} \big|_{t=0} = 0
\end{align*}
\]

(5)

The system of governing equations (1), together with the boundary and initial conditions are reduced to the non-dimensional form using the following relations:
Here $E$ - is the Young's modulus, $g$ - is the acceleration due to gravity, $\gamma$ - the specific gravity of the material, $\rho$ - the density, $2h$ - the height, $2h_0$ - the thickness of the beam in the center, $a$ - the length of the beam. In equations (1) – (5) bars are omitted.

The following harmonic load is applied on the beam 1:

$$ q = q_0 \sin(\omega_p t) $$

(7)

Where $q_0$ is the amplitude, $\omega_p$ is the excitation frequency.

The resulting system of nonlinear partial differential equations (1), together with the boundary and initial conditions, reduces to a system of ordinary differential equations by the finite differences method with approximation $O(c^2)$. Where $c$ is a step along the spatial coordinate. In each grid node we obtain the following system of ordinary differential equations. The Cauchy problem is solved by Runge-Kutta type methods. A software package has been created for solve the task posed depending on the control parameters $\{q_0, \omega_p\}$. Much attention was paid to the problem of non-penetration of structural elements into each other.

To analyze the contact interaction of a beam structure, methods of the qualitative theory of differential equations and methods for investigating nonlinear dynamics are applied. Signals, Fourier power spectra, Morlet wavelet spectra, phase portraits, Poincaré pseudo-mappings are constructed, the values of the highest Lyapunov exponent are calculated using three different algorithms – Wolff, Kantz, and Rosenstein.

4. Experimental results

To carry out a numerical experiment, we put: $\omega_p = 5.1$, $h_k = 1$, $\lambda = a / 2h = 40$, $\varepsilon = 1$. The $\omega_p$ is a natural beam frequency. As the control parameter, the amplitude of the load – $q_0$.

For $q_0 = 9000$ before the contact interaction, the beam 1 vibrations are chaotic, at the frequency of the driving vibrations $\omega_p = 5.1$ and the frequencies, linearly dependent on it $3\omega_p / 16$, $2\omega_p / 5$, $11\omega_p / 14$ (Table 1).

**Table 1.** Dynamic characteristics of beam 1 for $q_0 = 9000$.
In Tables 2 and 3, we give power spectra, Poincaré pseudo-mappings, Morlet wavelet spectra, phase portraits of beams at $q_0 = 12750$ and $q_0 = 27750$, respectively. Table 4 shows the Morlet wavelet spectra of chaotic phase synchronization of beams.

**Table 2.** Dynamic characteristics of beams for $q_0 = 12750$

| Beam 1 | Beam 1 |
|--------|--------|
| FFT    | Poincaré pseudo-mappings |
|        | FFT    | Poincaré pseudo-mappings |
| Wavelet spectrum | 2D phase portrait $w(w')$ | Wavelet spectrum | 2D phase portrait $w(w')$ |

| LLe   | Wolf | Rosenstein | Kantz |
|-------|------|------------|-------|
|       | 0.00081 | 0.07033 | 0.01921 |

**Table 3.** Dynamic characteristics of beams for $q_0 = 27750$

| Beam 1 | Beam 1 |
|--------|--------|
| FFT    | Poincaré pseudo-mappings |
|        | FFT    | Poincaré pseudo-mappings |
| Wavelet spectrum | 2D phase portrait $w(w')$ | Wavelet spectrum | 2D phase portrait $w(w')$ |

| LLe   | Wolf | Rosenstein | Kantz |
|-------|------|------------|-------|
|       | 0.00154 | 0.07178 | 0.01872 |
5. Discussion of the results
On the FFT for $q_0 = 12750$ (Table 2), seven frequencies are identified, which are a linear combination of frequency $\omega_p = 5.1$. In this case, there is a subharmonic cascade of bifurcations of stable cycles of period seven. When $q_0 = 27750$ (Table 3), on the FFT for both beams, three frequencies are singled out, which are a linear combination of frequency $\omega_p$ and we can speak of a subharmonic cascade of bifurcations of stable limit cycles of period three. Thus, with an increase in the amplitude of the load, the oscillatory process stabilizes. All the Lyapunov exponents are positive, which indicates a chaotic nature of structural oscillations. Wavelet spectra, by means of which it is possible to investigate the change in frequency characteristics in time, show the appearance of a frequency $\omega_p / 7$, $t > 400$ for loads $q_0 = 12750$, which is characteristic for both beams. The Poincaré pseudo-mappings, like the phase portraits, represent a breaking torus for beam 1, for beam 2 a strange attractor (Table 2). In three-frequency vibrations (Table 3), the pseudo Poincaré map and the phase portrait represent three stable limit cycles, wavelet spectra also reflect three frequencies over the entire time interval.

The wavelet spectra of chaotic phase synchronization are significantly different (Table 4). If $q_0 = 12750$, the synchronization is time-reversal, for $t > 400$ the vibrations are synchronized at frequencies $\omega_p / 7$ and $5\omega_p / 7$. When $t < 200$ vibrations are synchronized to the frequencies of the driving vibrations. An increase in the load results in synchronization of the vibrations at three fundamental frequencies.

6. Conclusions
The transition of the vibrations of the beam structure (beam 1-the Pelekh-Sheremetyev-Reddy model, beam 2-the Euler-Bernoulli model) from harmonic vibrations to chaotic ones is studied. The scenario is characterized as a transition to chaos through a subharmonic cascade of bifurcations. Chaotic phase synchronization of structure oscillations is investigated. The phenomenon of tuning of the vibration process in time is revealed.

The phenomenon of stabilization of the vibration is shown with increasing amplitude of the load.

### Table 4. Wavelet spectrum of chaotic phase synchronization

|        | Wolf | Rosenstein | Kantz | Wolf | Rosenstein | Kantz |
|--------|------|------------|-------|------|------------|-------|
| $0.00164$ | $0.03041$ | $0.02005$ | $0.00175$ | $0.04217$ | $0.02134$ |
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