Diplomarbeit

Topological Entropy of Formal Languages

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Preface

In this thesis we will introduce topological automata and the topological entropy of a topological automaton, which is the topological entropy of the dynamical system contained in the automaton. We will use these notions to define a measure of complexity for formal languages. We assign to every language the topological entropy of the unique minimal topological automaton accepting it.

We contribute several new results. We use a preexisting characterization of the topological entropy of a formal language in terms of Myhill-Nerode congruence classes to compute the topological entropy of several new example languages. We determine the entropy of the Dyck languages and the deterministic palindrome language. Also we will further develop an idea from Schneider and Borchmann [5] to solve the previously open question of whether the entropy function is surjective. Furthermore, we show that all languages accepted by deterministic real-time multi-counter automata have zero entropy and all languages accepted by deterministic real-time multi-push-down automata have finite entropy, bounded in terms of the sizes of the stack alphabets of the automaton. In particular this proves that all deterministic real-time context-free languages have finite entropy. We also give an example of a deterministic context-free language with infinite entropy, proving that not all context-free languages have finite entropy. We show that there are encodings of SAT, 3COLORING, and CLIQUE such that these languages have infinite entropy.
Danksagung

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1 Introduction

One goal of complexity theory is to be able to assign a degree of complexity to a given formal language. A classical and simple way to do this is to use the Chomsky hierarchy, which distinguishes five classes: regular, context-free, context-sensitive, decidable and undecidable languages. But this classification is very coarse as for example 2SAT and QBF are both in the same class. Furthermore, this separation depends on different models of computation, namely finite automata, push-down automata, linearly bounded automata, and Turing machines, and there is no particular reason to prefer these models over other computation models. There are also some counterintuitive classifications: for example the language $a^n b^n c^n$ is context-sensitive yet it seems quite simple.

Another way to measure the complexity of a language is by using the best possible time- or space-complexity of any Turing machine recognizing the language. This way we can connect problems that seemingly have no connection like the knapsack problem and the traveling salesperson problem. Also we get a more detailed classification of formal languages than with the Chomsky hierarchy. But in practice it is often nearly impossible to either show that a given machine is optimal or to find the optimal machine. For example nobody was so far able to show that either the guess and check algorithm for SAT is optimal ($P \neq NP$) nor to find a fundamentally better one ($P = NP$).

The goal of this thesis is to present a different way to measure the complexity of a language and investigate some properties of this new definition. We do not say that this approach is superior, but propose it as a different way to look at the complexity of a language. We want to define the complexity of a language in terms of the complexity of a minimal automaton accepting it. To do this we need a computation model with three important properties:

- universality: for every language there is an automaton recognizing it,
- minimalizability: there is a unique minimal automaton for each language,
- measurability: we can assign a degree of complexity to each automaton.

Furthermore, we want measureability to be monotone, i.e., the minimal automaton for a language has the minimal degree of complexity among all automaton accepting the language.

Turing machines are not suitable for this task because they are not universal: there are no Turing machines for undecidable problems like the halting problem. Deterministic finite automata are not universal either, but they have the other
two properties: there are minimal automata and an obvious degree of complexity would be the number of states \[7\]. When we allow the number of states to be infinite, then this model becomes universal. Then, however, all non-regular languages require automata with infinitely many states, and thus we need a new way to measure the complexity of an automaton. One approach to do this is to add some more structure to the states by equipping them with a topology. This yields the computation model of topological automata. As seen in \[6\], these automata still allow the notion of a minimal automaton accepting a language. Furthermore, we can also define a sensible measure of complexity for these kinds of automata. But before we can define this measure we need to introduce some notions from the realm of dynamical systems.

A typical example of a dynamical system is the flow of water particles in a container. An example is indicated in the picture.

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[-----]
[-----]
[-----]
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It is described by the possible positions and a function that determines how each position changes over time. But what does this have to do with formal languages? Well, in general a dynamical system is a continuous semigroup action on a compact Hausdorff space. In the water example time acts on the set of positions. The set of all words \(\Sigma^*\) is a monoid and in a finite automaton this monoid acts on the set of states. Equipped with the discrete topology this action is also continuous. Obviously, the discrete topology is Hausdorff and since there are only finitely many states it is also compact.

For topological automata with infinitely many states we need to be more careful with the choice of the topology, because the discrete topology on an infinite set is not compact. On the other hand, we will, for the most part, not need to worry about the concrete topology used and once these issues are fixed, we can see topological automata as dynamical systems and are able to apply the mathematics of dynamical systems to them. Note that there are also definitions of dynamical systems which demand more structure on the underlying space, e.g., a metric, but since there is no natural structure on the states of an automaton we want a definition that demands as little structure as possible.

A well established notion to measure the complexity for dynamical systems is topological entropy. It measures how chaotic or random a dynamical system is. Using this we can define the complexity of a language to be the topological entropy of the minimal topological automaton accepting it.

Steinberg introduced the notion of a topological automaton in 2013 \[6\]. Then in 2016, Schneider and Borchmann used this notion to define the topological entropy of a formal language. They gave a characterization of the topological entropy of a formal language in terms of Myhill-Nerode congruence classes and determined the entropy of some example languages \[5\].
The purpose of this thesis is to summarize these findings, to solve previously open problems in this field, and to expand the variety of example languages. We construct a language for every possible entropy, in other words we show that the entropy function is surjective. We show that every language accepted by a deterministic real-time counter automaton with an arbitrary amount of counters has zero entropy, which generalizes the fact that all regular languages have zero entropy. Furthermore, we give a finite upper bound for the entropy of languages accepted by deterministic real-time push-down automata with an arbitrary amount of stacks, which shows that every deterministic \(\varepsilon\)-free context-free language has finite entropy. We determine the entropy of the Dyck languages, the deterministic palindrome language, and of some other new example languages. Among them is also a deterministic context-free language with infinite entropy.

This thesis is structured as follows. First we give a short introduction to dynamical systems and topological entropy. Then we introduce the notion of a topological automaton and its topological entropy. We proceed by summarizing the preexisting results about topological entropy of formal languages. Most noticeably, this includes a characterization of topological entropy in terms of Myhill-Nerode congruence classes. In Chapter 3 we will compute the entropy of some well known example languages. Once we have established a better understanding of the concepts we will venture on to more general results. Firstly, we will have a brief discussion on the effect different encodings have on the entropy of a language in Section 4.1. This will motivate Section 4.2, where we will show that for every possible entropy there is also a language with that entropy. In Chapter 5 we will bound the entropy of languages accepted by certain kinds of counter automata and push-down automata. There we will also explore the connection between topological entropy and the Chomsky hierarchy. Finally, in Chapter 6 we will take a glimpse on the connection between topological entropy and complexity theory by calculating the entropy of some decision problems. In the end we will give an outline for future work.
2 Preliminaries

In this chapter we will give a brief introduction to the Myhill-Nerode congruence relation, topology, and dynamical systems. Then we will introduce the notion of a topological automaton and its topological entropy. We will summarize the relevant results for this thesis from [5, 6]. These results show that topological automata are universal and minimalizable. Furthermore, they prove that topological entropy is a monotone measure of the complexity of a topological automaton and give us a characterization of topological entropy in terms of Myhill-Nerode congruence classes.

2.1 Myhill-Nerode congruence relation

The Myhill-Nerode congruence relation is a very basic concept of formal languages, but as it is essential to this thesis we will give a short recapitulation. Let $L$ be a formal language over some alphabet $\Sigma$. The Myhill-Nerode right-congruence relation of $L$, denoted by $\Theta(L)$, is

$$\Theta(L) = \{(u, v) \mid \forall w \in \Sigma^* \cdot uw \in L \iff vw \in L\}.$$ 

Firstly, note that $\Theta(L)$ is indeed a congruence relation of the algebra $(\Sigma^*, \cdot)$, since it is an equivalence relation and $(a, b), (c, d) \in \Theta(L)$ implies $(ac, bd) \in \Theta(L)$. For two words $u$ and $v$ we say that $w$ witnesses $(u, v) \notin \Theta(L)$ if $uw \in L \iff vw \in L$. For a word $u$, we call all words $w$ with $uw \in L$ positive witnesses of $u$ and all words $w$ with $uw \notin L$ negative witnesses of $u$. Note that the words in the congruence class $[w]$ are exactly the words with the same positive and negative witnesses as $w$.

If a language $L$ has only finitely many congruence classes, then the minimal finite automaton accepting it can be constructed using $\Theta(L)$: take the set $\Sigma^*/\Theta(L)$ as states, where the final states are the congruence classes contained in $L$, $[e]$ as initial state, and $([w], a) \mapsto [wa]$ as transition function.

2.2 Topology

Here we will introduce the topological definitions needed to understand this chapter. For a more comprehensive introduction see [4], but be assured that a deeper understanding of topology is not necessary to read this thesis. A topology $T$ on a set $X$ is a subset of $\mathcal{P}(X)$ that has the following three properties:

- $\emptyset, X \in T$,
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- if $A, B \in T$, then $A \cap B \in T$, and
- if $(A_i)_{i \in I} \subseteq T$, then $\bigcup_{i \in I} A_i \in T$.

The sets in $T$ are called open and every complement $A^c$ of a set $A$ in $T$ is called closed. For a subset $A$ of $X$ we denote the closure of $A$, i.e., the smallest closed set containing $A$, as $\overline{A}$. On any set $X$ there is always the trivial topology, which contains only $X$ and the empty set and the discrete topology, which contains every subset of $X$. A typical example for a topology is the set of all open subsets of $\mathbb{R}$.

The tuple $X = (X, T)$ is a topological space if $T$ is a topology on the set $X$. Let $X$ be a topological space. An open cover of $A \subseteq X$ is a family $(U_i)_{i \in I}$ of open subsets of $X$ such that $\bigcup_{i \in I} U_i \supseteq A$. A set $A \subseteq X$ is called compact if for every open cover of $A$ there exists a finite subcover. The topological space $X$ is called compact if the set $X$ is compact. We say that $X$ is Hausdorff if for every two distinct elements $u$ and $v$ from $X$ there are open subsets $U$ and $V$ such that $U$ and $V$ are disjoint and $u \in U$ and $v \in V$.

The product of a family $(X_i)_{i \in I}$ of topological spaces is the topological space consisting of the set $\prod_{i \in I} X_i$ equipped with the product topology, which consists of all unions of sets of the form $\prod_{i \in I} U_i$ where $U_i$ is an open subset of $X_i$ and there are only finitely many $U_i \neq X_i$. It is the smallest topology on $\prod_{i \in I} X_i$ such that all projections $\pi_i: X \to X_i$ are continuous. It is well known that the product of arbitrarily many Hausdorff spaces is again a Hausdorff space. As stated in Tychonoff’s theorem, the product of compact spaces is also a compact space.

In the following we will write $X$ for the set $X$ as well as for the topological space $X$.

### 2.3 Topological automata

In this section we will first introduce dynamical systems. Then we will define topological automata and introduce the notion of a minimal topological automaton accepting a language.

Let $S = (S, \cdot)$ be a semigroup and $X$ a topological space. A semigroup action of $S$ on $X$ is mapping $\delta: X \times S \to X$ such that $\delta(x, a \cdot b) = \delta(\delta(x, a), b)$ for all $a, b \in S$ and all $x \in X$.

A dynamical system is a triple $(X, S, \delta)$, where $\delta$ is a continuous semigroup action of $S$ on the compact Hausdorff space $X$. We can interpret the discrete movement of water particles in a compact container as a dynamical system, where $S = \mathbb{N}$ and $X$ is the set of possible positions. In this case the time acts on the position of the particle. Every deterministic finite automaton $A$ contains a dynamical system, where the monoid $\Sigma^*$ acts on the state of $A$. In the following definition we generalize this idea to automata with infinitely many states.

**Definition 2.1.** A topological automaton is a 5 tuple $A = (X, \Sigma, \delta, x_0, F)$ where

- $X$ is a compact Hausdorff space (the states),
2.3 Topological automata

- \( \Sigma \) is an alphabet (the input alphabet),
- \( \delta : X \times \Sigma^* \to X \) is a continuous action of \( \Sigma^* \) on \( X \) (the transition function),
- \( x_0 \) is an element from \( X \) (the initial state), and
- \( F \) is a clopen, i.e., closed and open, subset of \( X \) (the final states).

The language accepted by \( A \) is

\[
L(A) = \{ w \in \Sigma^* \mid \delta(x_0, w) \in F \}.
\]

Note that, because \( F \) is clopen, the set \( \{ F, F^c \} \) is an open cover of \( X \). This will be important later in Section 2.5. We say that \( A \) is trim if \( \delta(x_0, \Sigma^*) = X \). If \( \delta(x_0, \Sigma^*) = \delta(x_0, \Sigma^c) \), then this just means that every state in \( X \) is reachable from the initial state. Clearly, every automaton can be transformed into a trim automaton accepting the same language by replacing \( X \) with \( \delta(x_0, \Sigma^*) \) and \( F \) with \( F \cap \delta(x_0, \Sigma^*) \). The new set of final states is clopen because the intersection of two clopen sets is again clopen.

Note that for a topological automaton \( A = (X, \Sigma, \delta, x_0, F) \), the triple \( (X, \Sigma^*, \delta) \) is a dynamical system. We will denote this system again by \( A \).

For example, every deterministic finite automaton \( (Q, \Sigma, \delta, q_0, F) \) can be interpreted as the topological automaton \( (Q, \Sigma, \delta^*, q_0, F) \), where \( Q \) is equipped with the discrete topology and \( \delta^* \) is the usual extension of \( \delta \) to \( \Sigma^* \) defined by

\[
\delta^*(q, \varepsilon) = q \quad \text{and} \quad \delta^*(q, wa) = \delta(\delta^*(q, w), a).
\]

However, as the number of states in a topological automaton does not need to be finite, they can recognize much more languages than finite automata. In fact, for any language \( L \) over some alphabet \( \Sigma \) we can construct a topological automaton \( A_L \) with \( L(A_L) = L \). For this, let \( \chi_L : \Sigma^* \to \{0, 1\} \) be the characteristic function of \( L \) and equip \( X = \{0, 1\}^{\Sigma^*} \) with the product topology, where the topology on \( \{0, 1\} \) is the discrete topology \( \mathcal{P}(\{0, 1\}) \), which clearly is compact and Hausdorff. Then \( X \) is a compact Hausdorff space, since the product of compact Hausdorff spaces is again a compact Hausdorff space. Define the topological automaton \( (X, \Sigma, \delta, \chi_L, F) \) by

\[
\delta(f, u) = (w \mapsto f(uw)) \quad \text{and} \quad F = \{ f \in X \mid f(\varepsilon) = 1 \}.
\]

This automaton already accepts \( L \). But we define \( A_L \) to be the trim version of this automaton, i.e., \( A_L = (\delta(\chi_L, \Sigma^*), \Sigma, \chi_L, F \cap \delta(\chi_L, \Sigma^*)) \) and \( L(A_L) = L \). Hence topological automata have the first of the three desired properties, namely universality.
Example 2.2. Consider as example the language $L = \{a^{3n} \mid n \in \mathbb{N}\}$. Then the automaton $A_L$ is the tuple $(\{f_0, f_1, f_2\}, \{a\}, \delta, f_0, \{f_0\})$, where

$$f_i(u) = \begin{cases} 1 & \text{if } |u| + i \mod 3 = 0 \\ 0 & \text{otherwise} \end{cases} \quad \delta(f_i, w) = f_{i+|w| \mod 3}.$$

The language $L$ has the following three congruence classes

$$\{w \mid |w| \mod 3 = 0\} \quad \{w \mid |w| \mod 3 = 1\} \quad \{w \mid |w| \mod 3 = 2\}.$$

For any word $w$ we can define a function

$$f_w(u) = \begin{cases} 1 & \text{if } wu \in L \\ 0 & \text{otherwise} \end{cases}$$

and these functions characterize the congruence relation of $L$, i.e., $f_u = f_v$ if and only if $|u| = |v|$. Note that for any $w$ with $w \mod 3 = i$ and any $u \in \{a\}^*$ we have that

$$f_w(u) = 1 \iff wu \in L \iff |u| + i \mod 3 = 0 \iff f_i(u) = 1.$$

Consequently, the states of $A_L$ coincide with the congruence classes of $\Theta(L)$, which in turn coincide with the states of the minimal deterministic finite automaton accepting $L$. Hence $A_L$ seen as a deterministic finite automaton is minimal.

Note that this is the case for any language $L$. Every state reachable from the initial state by some word $w$ is the function $f_w$ and coincides with the congruence class $[w]$. Hence for regular languages $L$ the topological automaton $A_L$ is the minimal deterministic finite automaton accepting $L$. More generally, also for non-regular languages the automaton $A_L$ is minimal in the following sense.

Lemma 2.3 (Theorem 2.2 from [6]). Let $A = (Y, \Sigma, \alpha, y_0, G)$ be a trim topological automaton accepting the language $L$ and $A_L = (X, \Sigma, \delta, x_0, F)$. Then there is a unique surjective continuous function $\varphi: Y \to X$ satisfying $\varphi(\alpha(y, w)) = \delta(\varphi(y), w)$ and $\varphi(y_0) = x_0$. Moreover, in this case $\varphi(G) = F$.

This lemma shows that topological automata have the second of our three desired properties, for every language there is a unique minimal automaton accepting it. Our next goal is to define a measure of complexity of a topological automaton, its topological entropy.

2.4 Topological entropy

In this section we will first introduce some notions for open covers. Then we will use these notions to define the topological entropy of a dynamical system and of a formal language.
Recall that an open cover of a topological space $X$ is a family $(U_i)_{i \in I}$ of open subsets of $X$ such that $\bigcup_{i \in I} U_i = X$. We denote the set of all finite open covers of $X$ by $\mathcal{C}(X)$. Note that if $U \in \mathcal{C}(X)$ and $f : X \to X$ is continuous, then $f^{-1}(U)$ is also a finite open cover of $X$. For $U \in \mathcal{C}(X)$ define

$$N(U) = \inf \{ |V| \mid V \in \mathcal{C}(X), V \subseteq U \}.$$ 

If $U$ contains $X$, then $N(U) = 1$, since $\{X\}$ is a finite open cover of $X$. If $U$ is a partition of $X$, then $N(U) = |U|$. For a family $(U_i)_{i \in I}$ from $\mathcal{C}(X)$ define the refinement of $(U_i)_{i \in I}$ as

$$\bigvee_{i \in I} U_i = \{ \bigcap_{i \in I} U_i \mid (U_i)_{i \in I} \in \prod_{i \in I} U_i \}.$$ 

Note that if $I$ is finite, then the resulting set is also in $\mathcal{C}(X)$. To understand this formula we consider an example.

**Example 2.4.** For the sake of readability we will in all the following examples only consider partitions from $\mathcal{C}(X)$. Consider the two finite open covers $U_1$ and $U_2$ indicated in the picture, then

$$U_1 \vee U_2 = U_1 \vee U_2$$

and $N(U_1) + N(U_2) = 3 + 4 \geq 6 = N(U_1 \vee U_2)$.

For two covers $U, V \in \mathcal{C}(X)$ we say that $U$ is refined by $V$, denoted $U \leq V$, if

$$\forall V \in \mathcal{C}(X) \exists U \in U, V \subseteq U.$$ 

Some basic observations are: $\{X\} \leq V$ for all $V \in \mathcal{C}(X), U \leq U \vee V$, i.e., $U$ and $V$ are refined by the refinement of $U$ and $V$, and $U \leq V$ implies $N(U) \leq N(V)$.

Now we are prepared to define the topological entropy of a dynamical system $\mathcal{D} = (X, S, \delta)$. For every finite generating subset $F$ of $S$ and $U \in \mathcal{C}(X)$ define the topological entropy of the dynamical system $\mathcal{D}$ with respect to $F$ and $U$ as

$$\eta(\mathcal{D}, F, U) = \limsup_{n \to \infty} \frac{\log_2 \left( N(\bigvee_{w \in F^n} w^{-1}(U)) \right)}{n},$$

where $w^{-1}(U) = \{ x \mid \delta(x, w) \in U \}$. Intuitively, this formula measures how fast $N(\bigvee_{w \in F^n} w^{-1}(U))$ grows. We will give an example in a moment. The topological entropy of the dynamical system $\mathcal{D}$ with respect to $F$, denoted by $\eta(\mathcal{D}, F)$, is the supremum over all $U$ in $\mathcal{C}(X)$ of $\eta(\mathcal{D}, F, U)$. The entropy still depends on the particular choice of $F$, but as we can see in the following lemma from [5], this choice is not essential.
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Lemma 2.5 (Proposition 2.6 from [5]). Suppose $F, E \subseteq S$ are finite generating subsets of $S$. Then

$$\frac{1}{m} \cdot \eta(D, F) \leq \eta(D, E) \leq n \cdot \eta(D, F),$$

where $m = \inf\{k \in \mathbb{N} \mid F \subseteq E^k\}$ and $n = \inf\{k \in \mathbb{N} \mid E \subseteq F^k\}$.

Example 2.6. To better understand this definition, we again look at the water particles from the example in the introduction. Recall the dynamical system $D = (X, \mathbb{N}, \delta)$, where $\delta$ describes the discrete movement of water particles in $X$. Choose $F = \{0, 1\}$, then $F^n = \{0, \ldots, n\}$. This is clearly a finite generating subset of $\mathbb{N}$. For the finite open cover pick $U = \{U_l, U_r\}$ as indicated in the picture.

\[
\begin{array}{c|c}
U_l & U_r \\
\hline
\end{array}
\]

Note that every set $U$ in $\bigvee_{t \in F^n} t^{-1}(U)$ is of the form

$$\bigcap_{t=0}^{n} t^{-1}(U_t) \text{ where } U_t \in \{U_l, U_r\}.$$ 

If $U$ is nonempty, then there is a position $x$ in $U$ and we know that it is possible for a particle to take a path such that it is at time point $t$ in $U_t$. If the particle is at time point 0 in $U_l$, and at time point 1 in $U_r$, then we denote its path by $lr$. In our example the paths $lllr, lrrr$, and $rrrr$ are possible, but since all particles move from left to right there is no particle with the path $rlrr$. Hence $N(\bigvee_{t \in F^n} t^{-1}(U))$ counts how many different paths are possible in the first $n$ steps. For $n = 2$ there are the 4 possible paths $lll, llr, lrr, rrr$. In general $N(\bigvee_{t \in F^n} t^{-1}(U)) = n + 2$. Because of this

$$\eta(D, F, U) = \limsup_{n \to \infty} \frac{\log_2(n + 2)}{n} = 0.$$ 

If we take a more fine grained partition $U' = \{U_1, U_2, U_3, U_4\}$, given by

\[
\begin{array}{c|c|c|c}
\hline
& & & \\
\hline
\end{array}
\]

then every possible path is a monotone increasing sequence over $\{1, \ldots, 4\}$. By Lemma 8.1 there are at most $\sum_{i=1}^{4} \binom{4}{i} \cdot \binom{n-i}{1-1}$ paths of length $n$. Hence

$$N(\bigvee_{t \in F^n} t^{-1}(U')) \leq 4 + 6 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + \binom{n}{3} \leq 4 \cdot 6 \cdot n^3.$$
for \( n \geq 1 \). For the entropy we get
\[
\eta(D, F, U') \leq \limsup_{n \to \infty} \frac{\log_2(24 \cdot n^3)}{n} = \limsup_{n \to \infty} \frac{\log_2(24) + 3 \cdot \log_2 n}{n} = 0.
\]
The same argument can be applied for all \( U' \), and thus \( \eta(D, F) = 0 \).

Another way to look at \( N(\bigvee_{t \in F} t^{-1}(U)) \) is the following: given the information about the positions of a particle in the first \( n - 1 \) steps, how accurately can we predict where the particle will be in the next step? The fewer uncertainty in the prediction the lower the entropy of the system.

Our intention was to measure the complexity of a topological automaton. Since topological automata are dynamical systems with some additional structure, we can define the entropy of an automaton \( A \) to be the topological entropy of the dynamical system \( A \) with respect to some finite generating set \( F \). We choose \( \Sigma^{(1)} \), the smallest possible generating set, for \( F \). The entropy of a language \( L \) is defined as the entropy of the minimal automaton \( A_L \). Define
\[
\eta(A, U) = \eta(A, \Sigma^{(1)}, U) \quad \eta(A) = \eta(A, \Sigma^{(1)}) \quad \eta(L) = \eta(A_L).
\]

Now topological automata fulfill all three properties, on the other hand it is very hard to actually compute the entropy of a given language. This issue is addressed in the following section.

### 2.5 Characterization

There also is another problem with the definition. Besides the fact that it is very hard to compute the entropy of a concrete language using this definition, we do not yet know whether \( A_L \) is the automaton with the least entropy accepting \( L \).

Both problems are solved by means of a beautiful characterization of topological entropy with Myhill-Nerode congruence classes from [5]. Here we will just repeat the important statements without giving any proofs.

For a topological automaton \( A = (X, \Sigma, \delta, x_0, F) \), \( L = L(A) \), and \( E \subseteq \Sigma^* \) we define the following two equivalence relations
\[
\Theta_E(L) = \{ (u, v) \mid \forall w \in E. uw \in L \iff vw \in L \} \text{ and } \\
\Lambda_E(A) = \{ (x, y) \mid \forall w \in E. \delta(x, w) \in F \iff \delta(y, w) \in F \}.
\]
The relation \( \Theta_E(L) \) is an approximation of the Myhill Nerode congruence relation of \( L \) in the sense that it allows only words from \( E \) as witnesses. Hence if we choose \( E = \Sigma^* \), then \( \Theta_{\Sigma^*}(L) \) is the Myhill Nerode congruence relation of \( L \). The second relation \( \Lambda_E(A) \) is the counterpart of \( \Theta_E(L) \) for the states of \( A \).

For \( n \in \mathbb{N} \cup \{\ast\} \) we denote
\[
\Theta_{\Sigma^n}(L) \text{ by } \Theta_n(L) \text{ and } \Lambda_{\Sigma^n}(A) \text{ by } \Lambda_n(A).
\]
Recall that the usual way to minimize a deterministic finite automaton is to first remove all unreachable states and then incrementally compute an equivalence relation on the states such that \( p \) and \( q \) are in the relation if for all words \( w \) either \( p \) and \( q \) reach a final state after reading \( w \) or neither of the two states reach a final state. We then factor the states by this relation to obtain the minimal automaton. The states of this minimal automaton correspond to the congruence classes of the language. Note that \( \Lambda_n(\mathcal{A}) \) is exactly the relation from this construction and \( \Lambda_*(\mathcal{A}) \) is the resulting relation. Consequently, if \( \mathcal{A} \) is trim and finite, then \( \Sigma^*/\Theta_n(L) \) corresponds to \( X/\Lambda_n(\mathcal{A}) \). We can lift this observation to topological automata with infinitely many states and connect these relations with \( \bigvee_{w \in E} w^{-1}(\{F, F^c\}) \). Here we need that \( F \) is clopen because otherwise \( \{F, F^c\} \) would not be an open cover of \( X \).

**Lemma 2.7** (Lemma 3.6 from [5]). Let \( \Phi : \Sigma^* \to X \) with \( \Phi(w) = \delta(x_0, w) \). Then the following statements hold:

1. \( X/\Lambda_E(\mathcal{A}) = \left( \bigvee_{w \in E} w^{-1}(\{F, F^c\}) \right) \setminus \{\emptyset\} \).
2. \( \Theta_E(L) = (\Phi \times \Phi)^{-1}(\Lambda_E(\mathcal{A})) \).
3. If \( \mathcal{A} \) is trim, then \( \Phi^{-1}(V) \neq \emptyset \) for all \( V \in X/\Lambda_E(\mathcal{A}) \).

The intuition behind the first statement is that two states are in the same class of the partition on the right side if for both the words \( w \) in \( E \) for which the state is in \( w^{-1}(F) \) are the same. But that is exactly what the partition on the left side does, too.

As a consequence of this lemma we obtain that if \( \mathcal{A} \) is trim, then

\[
\Sigma^*/\Theta_E(L) = \Phi^{-1}\left( \left( \bigvee_{w \in E} w^{-1}(\{F, F^c\}) \right) \setminus \{\emptyset\} \right).
\]

The first two statements already imply that the nonempty sets on both sides are the same and the third statement implies that every set in the right hand side is nonempty. Because of this, \( \Sigma^*/\Theta_E(L) \) has at least as many classes as \( \left( \bigvee_{w \in E} w^{-1}(\{F, F^c\}) \right) \setminus \{\emptyset\} \) and if \( \mathcal{A} \) is trim, then they have the same number of classes.

The following theorem is an easy consequence from this observation.

**Theorem 2.8** (Theorem 3.5 from [5]). Let \( \mathcal{A} \) be a topological automaton. Then

\[
\limsup_{n \to \infty} \frac{\log_2(\text{ind} \Theta_n(L))}{n} \leq \eta(\mathcal{A}, \{F, F^c\}) \leq \eta(\mathcal{A}),
\]

where the index of an equivalence relation, denoted by \( \text{ind} \), is the number of its classes. If \( \mathcal{A} \) is trim, then the first inequality becomes an equality.
2.5 Characterization

The second inequality can in general be strict, but if $\mathcal{A}$ is minimal, then it is also an equality. For this we need to show that for minimal automata $\{F, F^c\}$ gives the largest entropy among all finite open covers.

**Lemma 2.9** (Lemma 3.9 from [5]). If $\mathcal{A}$ is minimal, then for any finite open cover $\mathcal{U}$ of $X$ there exists some $n \in \mathbb{N}$ such that $\mathcal{U} \preceq \bigvee_{w \in \Sigma^n} w^{-1}(\{F, F^c\})$.

This lemma says that for any two states in a minimal automaton there is some word after reading of which only one of these states becomes a final state. As a consequence, any finite open cover is eventually refined by $\bigvee_{w \in \Sigma^n} w^{-1}(\{F, F^c\})$ for large enough $n$. If $\mathcal{A}$ is trim but not minimal, then there are at least two states that can be merged, and if a finite open cover $\mathcal{U}$ separates these two states, then it can never be refined by $\bigvee_{w \in \Sigma^n} w^{-1}(\{F, F^c\})$ for any $n$.

Using Theorem 2.8, Lemma 2.9 and some additional computations the main theorem follows.

**Theorem 2.10** (Theorem 3.10 from [5]). If $\mathcal{A}$ is minimal, then $\eta(\mathcal{A}) = \eta(\mathcal{A}, \{F, F^c\})$ and for all languages $L$ we have that

$$\eta(L) = \limsup_{n \to \infty} \frac{\log_2(\text{ind}_n(L))}{n}.$$  

Note that this together with Theorem 2.8 implies that for any topological automaton $\mathcal{A}$ accepting $L$ we have

$$\eta(L) = \eta(\mathcal{A}_L) \leq \eta(\mathcal{A}).$$

Thus $\mathcal{A}_L$ is also minimal in the sense that it has the minimal possible entropy among all automata accepting $L$. To get a better understanding of this characterization we will look at some example languages next.
3 Entropy of Example Languages

The purpose of this chapter is to give entropies of selected example languages. Some of these examples were already discussed in [5], the results from Examples 3.3, 3.5, and 3.10 are new and have to our knowledge not been discussed before.

**Example 3.1.** Let $L$ be a regular language. Then $\Theta^*(L)$ is finite. Hence

$$\eta(L) = \limsup_{n \to \infty} \frac{\log_2 \text{ind}_{\Theta_n}(L)}{n} \leq \limsup_{n \to \infty} \frac{\log_2 \text{ind}_{\Theta_n}(L)}{n} = 0.$$

All regular languages have zero entropy, which goes with the intuition that regular languages are simple.

Conversely, in [5] we have already seen that the standard example of a context-sensitive language $\{a^n b^n c^n \mid n \in \mathbb{N}\}$ has zero entropy. Hence not every language with zero entropy is also regular.

**Example 3.2.** We will now calculate the entropy of the typical context-free language $L = \{a^n b^n \mid n \in \mathbb{N}\}$, which is unsurprisingly also zero. To calculate the entropy we determine an upper bound for number of classes of $\Theta_n(L)$. There are three types of classes, namely $[a^k], [a^k b^l]$ and $[b]$ for $k \leq n$ and $1 \leq l \leq k$. Hence $\text{ind}_{\Theta_n}(L) \leq (n + 1) + (n + 1)^2 + 1 \leq 2 \cdot n^2$ for all $n \geq 4$ and

$$\eta(L) \leq \limsup_{n \to \infty} \frac{\log_2 (2 \cdot n^2)}{n} = 0.$$

Before we come to the next example, note that for any language $L$, $n \in \mathbb{N}$, and any word $w$ the equivalence class of $\Theta_n(L)$ generated by $w$ is characterized by the set $U_w = \{v \in \Sigma^{(n)} \mid wv \in L\}$ consisting of all positive witnesses of length at most $n$. Since for all $w, w' \in \Sigma^*$

$$[w] = [w'] \iff (wv \in L \iff w'v \in L) \text{ for all } v \in \Sigma^{(n)}$$

$$\iff (v \in U_w \iff v \in U_{w'}) \text{ for all } v \in \Sigma^{(n)}$$

$$\iff U_w = U_{w'}.$$

**Example 3.3.** Consider the unary language of all words with quadratic length $L = \{a^{n^2} \mid n \in \mathbb{N}\}$. From the observation above it follows that $\text{ind}_{\Theta_n}(L)$ is the same as the size of the set $\{U_w \subseteq \Sigma^{(n)} \mid w \in \Sigma^*\}$. This set can be split into the three distinct sets

1. $\{U_w \mid w \in \Sigma^*, |U_w| = 0\}$,
2. \( \{ U_w \mid w \in \Sigma^*, |U_w| = 1 \} \), and 3. \( \{ U_w \mid w \in \Sigma^*, |U_w| = 2 \} \).

The first of these sets has one element, the empty set, and second one has \( n + 1 \) elements, all singletons. Thus let \( U_w \) be from the third set and let \( a^{l_1}, a^{l_2} \) be the two shortest elements from \( U_w \). Our goal is now to use \( l_1 \) and \( l_2 \) to determine the rest of the elements of \( U_w \). Since \( wa^{l_1} \) and \( wa^{l_2} \) are in \( L \) and \( wa^{l} \not\in L \) for all \( l_1 < l < l_2 \) we know that \( |w| + l_1 \) and \( |w| + l_2 \) are two consecutive squares. Therefore
\[
(|w| + l_2) - (|w| + l_1) = l_2 - l_1 = (k + 1)^2 - k^2 = 2 \cdot k + 1
\]
for some \( k \in \mathbb{N} \). The \( k \) is uniquely determined by \( k = \frac{l_2 - l_1 - 1}{2} \). Hence we know that \( |w| = k^2 - l_1 \). As the alphabet is unary there is just one word of length \( k^2 - l_1 \) and \( |w| = \{ w \} \). With this information we can also determine \( U_w \) to be \( \{ a^{m^2 - k^2 + l_1} \mid m \geq k, m^2 - k^2 + l_1 \leq n \} \). There are at most \( \binom{n + 1}{2} \) many possibilities for \( l_1 \) and \( l_2 \), and thus
\[
\text{ind} \, \Theta_n(L) \leq \binom{n + 1}{0} + \binom{n + 1}{1} + \binom{n + 1}{2} + (n + 1) \cdot n \leq 2 \cdot n^2 + 2
\]
and
\[
\eta(L) = \limsup_{n \to \infty} \frac{\log_2 (\text{ind} \, \Theta_n(L))}{n} \leq \limsup_{n \to \infty} \frac{\log_2 (2 \cdot n^2 + 2)}{n} \leq \limsup_{n \to \infty} \frac{\log_2 3 + 2 \cdot \log_2 n}{n} = 0.
\]

In conclusion, \( \eta(L) = 0 \).

We start to wonder whether there even are languages with nonzero entropy and the answer is: Yes there are. For an alphabet \( \Sigma \) define the palindrom language over \( \Sigma \) to be
\[
\text{Pali}_\Sigma = \{ w w^R \mid w \in \Sigma^* \}.
\]

Schneider and Borchmann show in [5] that \( \log_2 |\Sigma| \leq \eta(\text{Pali}_\Sigma) \leq \log_2 |\Sigma| + 1 \).

Example 3.4. We will consider here the deterministic palindrom language. Assume that \( \# \) is not in \( \Sigma \) and define
\[
\text{DPali}_\Sigma = \{ w \# w^R \mid w \in \Sigma^* \}.
\]
We will show that as in the nondeterministic case, \( \log_2 |\Sigma| \) is a lower bound for the entropy of \( \text{DPali}_\Sigma \). Consider the set \( \Sigma^n \) of words of length \( n \). For two words \( u, v \in \Sigma^n \) we have that \( u \#^R v \in \text{DPali}_\Sigma \) iff \( u = v \). Since \( v \#^R v \in \text{DPali}_\Sigma \), every word in \( \Sigma^n \) is in a different class of \( \Theta_{n+1}(\text{DPali}_\Sigma) \). Hence \( |\Sigma|^n \) is a lower bound for \( \text{ind} \Theta_{n+1}(\text{DPali}_\Sigma) \) and

\[
\eta(\text{DPali}_\Sigma) \geq \limsup_{n \to \infty} \frac{\log_2 |\Sigma|^n}{n + 1} = \log_2 |\Sigma|.
\]

Later in Corollary 5.16 we will show that \( \log_2 |\Sigma| \) is also an upper bound.

**Example 3.5.** Another example considered in [5] is the Dyck language with \( k \) sorts of parenthesis, which consists of all balanced strings over \( \{(1,)_1, \ldots, (k,)_k\} \). More generally, let \( \Gamma \) be an alphabet and \( \Gamma = \{a \mid a \in \Gamma\} \). Then \( \Gamma \to \Gamma \) is a bijection. Now the Dyck language over \( \Gamma \), denoted by \( \text{Dyck}_\Gamma \), is the set of all words \( w \) such that successively replacing \( \overline{a} \) in \( w \) by \( \varepsilon \) results in \( \varepsilon \).

In [5] it was already shown that \( \log_2 |\Gamma| \leq \eta(\text{Dyck}_\Gamma) \leq \log_2 (2|\Gamma| - 1) \). In the following we will improve upon this result using term rewriting systems [1]. We will use some notions from term rewriting systems without further introduction as most of them are self explanatory. First we will define a unique normal form for each word and then we will use this normal form to determine representatives for all classes of \( \Theta_n(\text{Dyck}_\Gamma) \).

Let \( (\Sigma^*, \to) \) be an abstract reduction system, where the reduction relation \( \to \) is \( \{(ua\overline{a}v, uv) \mid u, v \in \Sigma^*, a \in \Gamma\} \). Clearly, the reduction \( \to \) is terminating as every rule application strictly reduces the length of the word. Hence \( \to \) is also normalizing, i.e., every word has at least one normal form. To see that it is also confluent (even has the diamond property) consider the following diagram:

\[
\begin{array}{c}
ua\overline{a}v\overline{a}w \\
\downarrow \\
vwbw
\end{array}
\quad
\begin{array}{c}
uvbw \\
\downarrow \\
uawv \end{array}
\quad
\begin{array}{c}
uawv \\
\downarrow \\
uvw \end{array}
\]

The reduction is normalizing and confluent, hence we can define a function \( \text{red}: \Sigma^* \to \Sigma^* \), which maps every word \( u \) in \( \Sigma^* \) to its unique normal form \( \text{red}(u) \). Note that \( \text{Dyck}_\Gamma = \text{red}^{-1}(\varepsilon) \).

**Lemma 3.6.** The function \( \text{red}: \Sigma^* \to \Sigma^* \) has the following properties:

(P1) \( \text{red}(uv) = \text{red}(\text{red}(u)v) = \text{red}(u \text{red}(v)) \) for all \( u, v \in \Sigma^* \).

(P2) If \( \text{red}(u) \) is of the form \( u_1\overline{a}u_2 \), then \( \text{red}(uv) = u_1\overline{a} \text{red}(u_2v) \) for all \( v \in \Sigma^* \).

(P3) For all words \( u \) and \( v \): \( \text{red}(uv) = \varepsilon \) iff \( \text{red}(u) = \overline{\text{red}(v)} \in \Gamma^* \), where \( \overline{u_1 \ldots u_n} = \overline{u_n} \ldots \overline{u_1} \) and \( \overline{a} = a \).
3 Entropy of Example Languages

Proof. The first property follows from $uv \red \Rightarrow \red(u)v$ and $uv \red \Rightarrow u\red(v)$ together with the fact that $\red$ is confluent.

For the second property note that $\red(u) = u_1\overline{a}u_2$ implies that there is no rule which can be applied to $u_1\overline{a}$. Let $v$ be a word in normal form. Then $u_1\overline{a}v$ is also in normal form since no rule can be applied to either $u_1\overline{a}$ nor $v$ and a rule can only remove $\overline{a}a$ but not $\overline{a}a$. Because of this, the right hand side of $uv \red \Rightarrow u_1\overline{a}\red(u_2)v$ is in normal form. Hence the claim follows from the uniqueness of the normal form.

To prove the third property we show the implications in both directions. First consider the $\Leftarrow$ direction. We proceed by induction on the length of $\red(u)$. If $|\red(u)| = 0$, then $\red(u) = \red(v) = \epsilon$ and the claim trivially holds. In the case that $|\red(u)| > 0$ let $\red(u) = u_1 \ldots u_n$. Then

$$u\overline{a}v \red \Rightarrow \red(u) \red(v) = u_1 \ldots u_{n-1}u_n\overline{a}_{n-1}\ldots\overline{a}_1 \red \Rightarrow u_1 \ldots u_{n-1}w_{n-1}\ldots w_1 \red \Rightarrow \epsilon$$

by induction hypothesis. For the $\Rightarrow$ direction we again use induction, this time on the number $n$ of rule applications necessary to reduce $uv$ to $\epsilon$. The base case is trivial so let $n = n' + 1$. There are $w_1, w_2 \in \Sigma^*$ such that

$$uv = w_1\overline{a}w_2 \red \Rightarrow w_1w_2 \red \Rightarrow \overline{a} \epsilon.$$

If $u$ is a prefix of $w_1$ or $v$ is a suffix of $w_2$ then the claim follows by induction hypothesis. In the remaining case $u = w_1a$, $v = \overline{a}w_2$, and by induction hypothesis $\red(w_1) = \red(w_2)$. Hence $\red(u) = \red(w_1)a$ and $\red(v) = \overline{a}\red(w_2)$ and therefore $\red(u) = \red(v) \in \Gamma^*$.

Intuitively, the function $\red$ just successively eliminates matching pairs of letters from a word until there are none left.

Lemma 3.7. Two words $u$ and $v$ are in the same class of $\Theta_n(\text{Dyck}_T)$ if and only if $\red(u)$ and $\red(v)$ are equal and in $\Gamma^{(n)}$, or if both $\red(u)$ and $\red(v)$ are not in $\Gamma^{(n)}$.

Proof. First we show the implication from right to left. If $\red(u) = \red(v)$, then by $[P1]$

$$\red(uw) = \red(u)\red(w) = \red(v)\red(w) = \red(vw)$$

for all $w \in \Sigma^*$. Therefore $(u, v) \in \Theta_n(\text{Dyck}_T)$.

If $\red(u)$ is not in $\Gamma^{(n)}$ then either $|\red(u)| > n$ and $\red(u) \notin \Gamma^*$, or $\red(u) \notin \Gamma^*$. In the first case $\red(u)w \neq \epsilon$ for all $w \in \Sigma^{(n)}$ (e.g. “$u = (w^{n+1})$”). In the latter case there is an $\overline{a} \in \overline{T}$ and $u_1, u_2 \in \Sigma^*$ such that $\red(u) = u_1\overline{a}u_2$. Hence $[P2]$ implies $\red(uw) = u_1\overline{a}\red(u_2w) \neq \epsilon$ for all $w \in \Sigma^*$ (e.g. “$u = (w^{n+1})$”). Consequently, $\red(u), \red(v) \notin \Gamma^{(n)}$ implies $uw, vw \notin \text{Dyck}_T$ for all $w \in \Sigma^{(n)}$. It follows that $(u, v) \in \Theta_n(\text{Dyck}_T)$.

For the $\Rightarrow$ direction let $(u, v) \in \Theta_n(\text{Dyck}_T)$. Then for all $w \in \Sigma^{(n)}$ we have that $uw \in \text{Dyck}_T \iff vw \in \text{Dyck}_T$. There are two cases to consider:
If $uw, vw \notin \text{Dyck}_\Gamma$ for all $w \in \Sigma^{(n)}$, then $\text{red}(u) \notin \Gamma^*$ or $|\text{red}(u)| > n$. Similarly, $\text{red}(v) \notin \Gamma^*$ or $|\text{red}(v)| > n$. Hence $\text{red}(u)$ and $\text{red}(v)$ are not in $\Gamma^{(n)}$.

Otherwise $uw, vw \in \text{Dyck}_\Gamma$ for some $w \in \Sigma^{(n)}$. Then $\text{red}(uw) = \text{red}(vw) = \varepsilon$ and $(\text{P3})$ implies $\text{red}(v) = \text{red}(w) = \text{red}(u) \in \Gamma^*$. Since $w$ has length at most $n$ we know that the length of $\text{red}(w)$ is also at most $n$. Therefore $\text{red}(v) = \text{red}(u) \in \Gamma^{(n)}$.

Using this lemma we can easily determine the entropy of $\text{Dyck}_\Gamma$.

**Lemma 3.8.** For all alphabets $\Gamma$ we have $\eta(\text{Dyck}_\Gamma) = \log_2 |\Gamma|$.

**Proof.** Note that $\text{red}$ is the identity on $\Gamma^*$. Together with the previous lemma we conclude that the set $\Gamma^{(n)} \cup \{\varepsilon\}$ contains exactly one representative of each class of $\Theta_n(\text{Dyck}_\Gamma)$. Therefore $\text{ind} \Theta_n(\text{Dyck}_\Gamma) = |\Gamma^{(n)}| + 1$. We can compute

$$\eta(\text{Dyck}_\Gamma) = \limsup_{n \to \infty} \frac{\log_2 (|\Gamma^{(n)}| + 1)}{n}$$

$$= \limsup_{n \to \infty} \frac{\log_2 (\sum_{i=0}^{n} |\Gamma|^i + 1)}{n}$$

For $|\Gamma| = 1$, the sum $\sum_{i=0}^{n} |\Gamma|^i$ equals $n + 1$, and we have

$$\eta(\text{Dyck}_\Gamma) = \limsup_{n \to \infty} \frac{\log_2 (n + 1 + 1)}{n} = 0 = \log_2 |\Gamma|.$$ 

For $|\Gamma| > 1$, the sum $\sum_{i=0}^{n} |\Gamma|^i$ is a geometric sequence, which by Lemma 8.3 equals $\frac{|\Gamma|^{n+1} - 1}{|\Gamma| - 1}$. Thus

$$\eta(\text{Dyck}_\Gamma) = \limsup_{n \to \infty} \frac{\log_2 \left( \frac{|\Gamma|^{n+1} - 1}{|\Gamma| - 1} + 1 \right)}{n}$$

$$= \limsup_{n \to \infty} \frac{n \cdot \log_2 |\Gamma|}{n} = \log_2 |\Gamma|.$$ 

This finishes the proof. 

Now we have populated the number line with languages of arbitrarily large entropy. What is a bit upsetting is that all these languages are just Dyck languages, which means they are relatively simple. So what information could we possibly get from knowing that the entropy of some language lies between $\text{Dyck}_\Gamma$ and $\text{Dyck}_{\Gamma \cup \{a\}}$? We will revisit this issue in Section 4.1.

We have shown that there are languages of arbitrarily large entropy, so the next question to ask is: are there languages with infinite entropy?
Example 3.9. Let $\Sigma$ be an alphabet such that $|\Sigma| \geq 2$ and for all $n \in \mathbb{N}$ choose mappings $\varphi_n: \Sigma^{2^n} \to \mathcal{P}(\Sigma^n)$ such that $|\text{im} \varphi_n| \geq 2^{2^n}$. Then define the following language:

$$\text{Inf}_{\Sigma, \varphi_n} = \{uv \mid \exists n \in \mathbb{N}, v \in \varphi_n(u)\}.$$ 

The rigorous proof that $\eta(\text{Inf}_{\Sigma, \varphi_n}) = \infty$ can be found in [5]. We will not repeat the proof. Instead let us consider an example to better understand the idea. Take the alphabet $\Sigma = \{0, 1\}$. Now each subset of $\Sigma^n$ can be interpreted as a binary string over 0 and 1 of length $2^n$, where each 1 marks an element in the subset. To assign all $2^n$ words in $\Sigma^n$ a position we order them lexicographically, and give the first position to the first word, the second position to the second word and so on. Then for example the set $\{000, 001, 111\}$ corresponds to the string $11000001$. We define $\varphi_n$ to map each word $u$ to its corresponding subset:

$$\varphi_n(a_0 \ldots a_{2^n-1}) = \{w \in \Sigma^n \mid a_{\text{fromBin}(w)} = 1\},$$

where $\text{fromBin}(w)$ is the unique natural number $n$ whose binary representation is $w$, where we add leading zeros if necessary. We will denote the resulting language $\text{Inf}_{\Sigma, \varphi_n}$ by $\text{Inf}$. Now consider a word $u$ of length $2^n$. The witnesses of length $n$ are sufficient to uniquely determine $u$. Hence a lower bound for $\text{ind} \Theta_n(\text{Inf})$ is the number of subsets of $\Sigma^n$, which is $2^{2^n}$. Because of this $\eta(\text{Inf}) = \infty$.

Note that $\text{Inf}$ is context-sensitive. Naturally the question arises whether there is also a context-free language with infinite entropy or whether they all have finite entropy. We will answer this question in Section 5.2.

Finally, we will discuss the mathematically very interesting language of all prime numbers.

Example 3.10. Let $\text{Prime} = \{a^p \mid p \text{ is prime}\}$ be the unary encoding of all prime numbers. To determine an upper bound for $\eta(\text{Prime})$ we use the same trick as for the squares and find an upper bound for the number of possible $U_w$. Since the alphabet is unary we only care about the lengths of the words in $U_w$. Hence the classes of $\Theta_n(\text{Prime})$ are also characterized by

$$U_m = \{k \mid a^k \in U_m\} = \{k \mid a^{m+k} \in \text{Prime}\}.$$ 

For $m > 2$ there cannot be two consecutive numbers in $U_m$ since the only $k$ for which $k$ and $k+1$ is prime is 2. Because of this, $U_m$ contains only even numbers or only odd numbers if $m$ is odd or even, respectively. As a consequence, $\text{ind} \Theta_n(\text{Prime}) \leq 2 \cdot 2^{\lceil n/2 \rceil} + 2$ and $\eta(\text{Prime}) \leq \frac{1}{2}$. We can make a similar argument for the prime 3. For $m > 3$ there cannot be a $k$ such that $k, k+2, k+4 \in U_m$ because one of these numbers is divisible by 3. Since

$$\{k, k+2, k+4\} \mod 3 = \{k, k+2, k+1\} \mod 3 = \{0, 1, 2\}.$$ 

Thus $\text{ind} \Theta_n(\text{Prime}) \leq 2 \cdot 3 \cdot 2^{\lceil \frac{n}{3} \rceil} + 2$ and $\eta(\text{Prime}) \leq \frac{1}{3}$. These observations lead to the following definition:
**Definition 3.11.** A set $A \subseteq \{0, \ldots, n-1\}$ represents a plausible sequence of primes of length $n$ if for all primes $p \leq n$

$$A \mod p \neq \{0, \ldots, p-1\}.$$ 
The set $A$ represents an occurring sequence of primes of length $n$ if there is a $k \in \mathbb{N}$ such that for all $l \in \{0, \ldots, n-1\}$

$$k + l \text{ is prime } \iff l \in A.$$ 

We denote the number of plausible sequences of length $n$ by $s_n$.

Note that for example the set $\{0, 1\}$ occurs for $k = 2$, but is not plausible since $\{0,1\} \mod 2 = \{0,1\}$. This can happen because the idea behind the plausible sequences is that every number in the sequence divisible by $p$ is not a prime, which holds for all numbers except for $p$ itself. But if we say that the starting point $k$ of the sequence is greater than the length $n$ of the sequence, than $p$ cannot occur in $k + \{0, \ldots, n-1\}$. Hence if $A$ represents an occurring sequence of length $n$, which occurs for some $k > n$, then $A$ also represents a plausible sequence.

With this observation we can bound $\text{Ind} \Theta_n(\text{Prime})$ by

$$|\{|a^k| \mid k \leq n + 1\}| + |\{|a^k| \mid k > n + 1\}| \leq n + 2 + s_{n+1}.$$ 

The obvious question is whether $s_{n+1}$ is a lower bound, i.e., whether every plausible sequence also occurs.

**Conjecture 3.12.** Every plausible sequence of primes occurs at least once.

Whether this conjecture holds is unknown to the author. If it holds, then

$$s_{n+1} \leq \text{Ind} \Theta_n(\text{Prime}) \leq s_{n+1} + n + 2.$$ 

This would determine the entropy of $\text{Prime}$, since the lower and the upper bound have the same limit. Unfortunately, it is not only very hard to prove this conjecture, but we were also not able to determine a formula for $s_n$. We have algorithmically calculated $s_n$ up to $n = 96$ and verified the conjecture for sequences up to length 29, where the last found sequence of length 29 is represented by $\{0,2,12,14,18,20,24\}$ and occurs first for $k = 3153569$. Some of the values for $s_n$ we calculated are

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 24 | 25 | 26 | ... | 96 |
|-----|---|---|---|---|---|---|----|----|----|-----|----|
| $s_n$ | 1 | 2 | 3 | 5 | 7 | 10 | 1275 | 1658 | 2041 | ... | 719623329 |

Even though it is not relevant for the entropy of $\text{Prime}$, it is simply too beautiful not to mention the following result.

**Lemma 3.13.** The following statements are equivalent:

1. Every plausible sequence of primes occurs at least once.
3 Entropy of Example Languages

2. Every plausible sequence of primes occurs infinitely often.

Proof. Assume the sequence represented by $A \subseteq \{0, \ldots, n\}$ occurs only $k$ times. Then let $N \in \mathbb{N}$ such that there are more than $k$ plausible sequences represented by $A_1, \ldots, A_l$ of length $N$ with the initial sequence as prefix, i.e., $A_i \cap \{0, \ldots, n\} = A$. Hence at least one of these $A_i$ does not represent an occurring sequence, a contradiction. \qed

Note that if we can show that the sequence $\{0, 2\}$ of length 3 occurs infinitely often, then we have proven the twin prime conjecture. As a consequence, Conjecture 3.12 is a generalization of the twin prime conjecture and the Green-Tao Theorem.
4 Encodings and Surjectivity

In this chapter we will first discuss how encodings effect the entropy of a language. In Section 4.2 we will solve the previously open problem of whether the entropy functions is surjective by constructing a language for every possible entropy. Finally, we will discuss the entropy of languages over unary alphabets.

4.1 Encoding

Let's resume our discussion of the Dyck languages from Chapter 3. Remember that \( \eta(Dyck) = \log_2 |\Gamma| \). We took issue with the fact that Dyck languages can have arbitrarily large entropy. But we also noted that the higher entropy comes at the price of larger alphabets. So what happens if we take a fixed alphabet? We can encode all the Dyck languages over say, a two element alphabet. But will this encoding preserve the entropy? In this section we will give upper and lower bounds for the entropy of an encoded language. Then we will apply these results to the Dyck languages, to show that their entropy is bounded if we encode them over a two letter alphabet.

**Definition 4.1.** An encoding of symbols in \( \Sigma \) over \( \Gamma \) is a mapping \( \text{enc}: \Sigma \to \Gamma^+ \) with the prefix property, i.e., there is no word in the image of \( \text{enc} \) which is a prefix of another word in the image.

The prefix property is necessary because otherwise the encoding might not be invertible. For now we will restrict ourselves to encodings where all encoded words have the same length and we will generalize the result afterwards.

**Lemma 4.2.** Let \( L \) be a language over \( \Sigma \), \( k \geq 1 \), and \( \text{enc}: \Sigma \to \Gamma^k \) an encoding of symbols in \( \Sigma \) over \( \Gamma \). Then \( k \cdot \eta(\text{enc}(L)) = \eta(L) \).

**Proof.** First we show the following two inequalities:

\[
\begin{align*}
\text{ind } \Theta_{n,k}(\text{enc}(L)) & \geq \text{ind } \Theta_n(L) \quad \text{(*)} \\
\text{ind } \Theta_{n,k}(\text{enc}(L)) & \leq |\Gamma^{(k-1)}| \cdot \text{ind } \Theta_n(L) + 1. \quad \text{(**) }
\end{align*}
\]

To prove (\( * \)) we show that \( [w] \mapsto [\text{enc}(w)] \) is an injective well defined map. Obviously, \( (w,w') \in \Theta_n(L) \) implies \( (\text{enc}(w),\text{enc}(w')) \in \Theta_{n,k}(\text{enc}(L)) \), because of this the map is well defined. Furthermore, if \( w \) witnesses \( (u,v) \notin \Theta_n(L) \), then \( |\text{enc}(w)| = k \cdot |w| \leq k \cdot n \). Consequently, \( \text{enc}(w) \) is a witness for the fact that \( (\text{enc}(u),\text{enc}(v)) \notin \Theta_{n,k}(\text{enc}(L)) \). Therefore our map is also injective and (\( * \)) holds.
4 Encodings and Surjectivity

For (**) we consider the map

$$\Sigma^* / \Theta_n(L) \times \Gamma^{(k-1)} \rightarrow \Gamma^* / \Theta_{n,k}(\text{enc}(L))$$

$$([w], v) \mapsto [\text{enc}(w) \cdot v].$$

Firstly, we want to show that it is well defined. Consider $([w], v)$ and $([w'], v)$ with $[w] = [w']$. Then for all $u \in \Gamma^{(n \cdot k)}$ we have

$$\text{enc}(w)vu \in \text{enc}(L) \iff \exists x. \text{enc}(x) = vu \text{ and } wx \in L$$

$$\iff \exists x. \text{enc}(x) = vu \text{ and } w'x \in L$$

$$\iff \text{enc}(w')vu \in \text{enc}(L).$$

Therefore $[\text{enc}(w) \cdot v] = [\text{enc}(w') \cdot v]$, and thus our map is well defined.

Next we show that it is almost surjective since there is at most one class which is not in the image. For example if $L = \{a, b, c\}^*$ and $\text{enc}(a) = 00$, $\text{enc}(b) = 01$, $\text{enc}(c) = 10$. Then the class which consists of all faulty encodings (i.e., words containing 11 where the first 1 is at an even position) is not in the image. For this example

$$\text{ind} \Theta_{n,k}(\text{enc}(L)) = 3 = 2 \cdot 1 + 1 = |\Gamma^{k-1}| \cdot \text{ind} \Theta_n(L) + 1.$$ 

We deduce that the class $\{ w \in \Gamma^* \mid wu \notin \text{enc}(L) \text{ for all } u \in \Gamma^{(k \cdot n)} \}$ might not have a preimage. But for every other class $[w'v]$ with $w' \in (\Gamma^*)^k$ and $v \in \Gamma^{(k-1)}$ where there exists a $u$ such that $w'vu \in \text{enc}(L)$, we have that $w'vu$ is an encoding of some word over $\Sigma$. Hence there is some $w \in \Sigma^*$ such that $\text{enc}(w) = w'$ and $([w], v)$ lies in the preimage of $[w'v]$. Therefore (**) holds.

Using these inequalities and Lemma 8.2 we can now infer

$$k \cdot \eta(\text{enc}(L)) = k \cdot \limsup_{n \to \infty} \frac{\log_2 \text{ind} \Theta_n(\text{enc}(L))}{n}$$

$$= k \cdot \limsup_{n \to \infty} \frac{\log_2 \text{ind} \Theta_{n,k}(\text{enc}(L))}{n \cdot k} \quad \text{(Lemma 8.2)}$$

$$\geq \limsup_{n \to \infty} \frac{\log_2 (\text{ind} \Theta_n(L))}{n}$$

$$= \eta(L).$$
The other direction follows similarly

\[
\begin{align*}
    k \cdot \eta(\text{enc}(L)) &= k \cdot \limsup_{n \to \infty} \log_2 \Theta_n(\text{enc}(L)) \\
    &= k \cdot \limsup_{n \to \infty} \log_2 \Theta_{n \cdot k}(\text{enc}(L)) \\
    &\leq \limsup_{n \to \infty} \frac{\log_2(2^{(k-1)} \cdot \ind \Theta_n(L) + 1)}{n} \quad \text{(Lemma 8.2)} \\
    &= \limsup_{n \to \infty} \frac{\log_2 2^{(k-1)} + \log_2 \Theta_n(L)}{n} \\
    &= \eta(L).
\end{align*}
\]

This concludes the proof. \(\square\)

For general encodings we can only bound the entropy of the encoded language. The upper bound and lower bound depend on the length of the shortest and the longest string used to encode a single symbol, respectively.

**Lemma 4.3.** Let \(L\) be a language over \(\Sigma\) and \(\text{enc}: \Sigma \to \Gamma^+\) an encoding of symbols in \(\Sigma\) over \(\Gamma\). Then \(\frac{n(L)}{k_1} \leq \eta(\text{enc}(L)) \leq \frac{2(L)}{k_2}\), where \(k_1 = \max\{|w| \mid w \in \text{im enc}\}\) and \(k_2 = \min\{|w| \mid w \in \text{im enc}\}\).

**Proof.** Note that \(\epsilon\) is not in the image of \(\text{enc}\), and as a consequence \(k_2\) is at least 1. We have the slightly adapted inequalities from the previous proof.

\[
\begin{align*}
    \ind \Theta_{n \cdot k_1}(\text{enc}(L)) &\geq \ind \Theta_n(L) \quad \text{(**)} \\
    \ind \Theta_{n \cdot k_2}(\text{enc}(L)) &\leq |A| \cdot \ind \Theta_n(L) + 1 \quad \text{(**) where \(A\) contains all real prefixes of words in the image of \(\text{enc}\).}
\end{align*}
\]

Showing these inequalities works similar as in the previous proof, but we need to make some adaptations. In particular the map \([w] \mapsto [\text{enc}(w)]\), we used to show (**), might not be well defined. It is possible that there are two words \(w_1, w_2\) in \([w]\) and some word \(u\) that witnesses \((w_1, w_2) \notin \Theta_n(L)\). Then \(|u| > n\), but it could be that \(|\text{enc}(u)| \leq n \cdot k_1\), which would imply \(|\text{enc}(w_1)| \neq |\text{enc}(w_2)|\).

To remedy this we consider the map

\[
[w] \mapsto \{[\text{enc}(w')] \mid w' \in [w]\},
\]

which is clearly well defined. For this map injectivity does not suffice to show (**). We need to show that the images of two different classes \([w_1]\) and \([w_2]\) are disjoint. Let \([\text{enc}(w_1')]\) and \([\text{enc}(w_2')]\) be elements from sets corresponding to \([w_1]\) and \([w_2]\), respectively. Because \(w_1' \in [w_1]\) and \(w_2' \in [w_2]\) there is a \(u\) that witnesses \((w_1', w_2') \notin \Theta_n(L)\) and \(|\text{enc}(u)| \leq k_1 \cdot |u| \leq k_1 \cdot n\). Hence \(\text{enc}(u)\) is a witness for \((\text{enc}(w_1'), \text{enc}(w_2')) \notin \Theta_{n \cdot k_1}(\text{enc}(L))\).
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For (**) we consider the same map as before, namely

\[ \Sigma^*/\Theta_n(L) \times A \rightarrow \Gamma^*/\Theta_{n,k_2}(\text{enc}(L)) \]
\[ ([w], v) \mapsto [\text{enc}(w) \cdot v]. \]

Note that every word in \( \Gamma^* \) has at most one decomposition into a \( w' \in \text{enc}(\Sigma^*) \) and a \( v \in A \). Thus let \( [w'v] \) be a class of \( \Theta_{n,k_2}(\text{enc}(L)) \) where there exists a \( u \) such that \( w'vu \in \text{enc}(L) \). Then \( w'vu \) is the encoding of some word in \( \Sigma^* \). Therefore there is a \( w \in \Sigma^* \) such that \( \text{enc}(w) = w' \). As a consequence, \( ([w], v) \) lies in the preimage of \([w'v] \) from this (**) follows.

Finally, the computation of the upper and lower bound of the entropy of \( L \) is analogous to the previous proof. \( \square \)

Note that if we choose an encoding with \( k_1 = k_2 \), then we obtain Lemma 4.2.

We see that encodings cannot increase the entropy of a language. This seems reasonable because in the encoded language longer witnesses are required to uncover new congruence classes. For example, we can use this idea to make the entropy of every language \( L \) over \( \Sigma \) with finite entropy arbitrarily small. For this we encode \( \Sigma \) over itself with \( \text{enc}(a) = a^k \) for all \( a \in \Sigma \). Then \( \eta(\text{enc}(L)) = \frac{\eta(L)}{k} \).

Note that if a language has infinite entropy, then every encoding of this language also has infinite entropy. On the other hand if \( \eta(L) \) is zero, then the entropy of \( \text{enc}(L) \) will also be zero.

We can encode every language over \( \Sigma \) over a two element alphabet \( \Gamma \) with an encoding \( \text{enc}: \Sigma \rightarrow \Gamma^k \) for some \( k \in \mathbb{N} \). Note that the minimal possible value of \( k \) for which we can define such an encoding is \( \lceil \log_2 |\Sigma| \rceil \). Hence we call the encoding \( \text{enc} \) efficient if \( k = \lceil \log_2 |\Sigma| \rceil \).

**Corollary 4.4.** If \( L \) is a language over \( \Sigma \) and \( \text{enc}: \Sigma \rightarrow \Gamma^k \) efficiently encodes \( \Sigma \) over a two letter alphabet \( \Gamma \), then \( \eta(\text{enc}(L)) = \frac{\eta(L)}{|\log_2 |\Sigma||}. \)

Now we can answer our initial question: What happens if we encode the Dyck languages over a two element alphabet?

**Corollary 4.5.** Let \( \text{enc}: \Sigma \rightarrow \Delta^k \) be an efficient encoding of \( \Sigma = \Gamma \cup \Gamma \) over a two letter alphabet \( \Delta \). Then

\[ \eta(\text{enc}(\text{Dyck}_1)) = \frac{\log_2 |\Gamma|}{\log_2 |\Gamma| + 1}. \]

For example, if we encode \( \text{Dyck}_{\{(1,2)\}} \) over \{0, 1\} with

\begin{align*}
(1 & \mapsto 00) \\
(2 & \mapsto 01)
\end{align*}

then the encoded language has entropy \( \frac{1}{2} \).

Interestingly, this corollary implies that any encoded Dyck language has entropy less than one. As this argument also affects the palindrome languages, we
4.2 Every entropy has its language

In this section we will show that the entropy function is surjective over any alphabet with at least two letters. We have already seen that there are languages with zero entropy and we have also encountered a language with infinite entropy. Recall the definition of $\text{Inf}$ from Example 3.9:

$$\text{Inf} = \{uv \mid \exists n \in \mathbb{N}, v \in \varphi_n(u)\}.$$ 

The idea was that the witnesses of length $n$ determine the set $\varphi_n(u)$ and therefore $\text{ind} \Theta_n(\text{Inf}) \geq |\text{im} \varphi_n| = 2^{2^n}$. We can generalize this idea and construct a language $L$ with $\text{ind} \Theta_n(L) \approx 2^{kn}$ for a suitable sequence $(k_n)_{n \in \mathbb{N}}$. By choosing $k_n = \lceil x \cdot n \rceil$ we obtain a language with entropy $x$ for any $x \in (0, \infty)$.

For the remainder of this section we will fix the alphabet $\Sigma = \{0, 1\}$. We construct $L$ in the following way:

$$L = \{uv \mid \exists n \in \mathbb{N}, f_n(v) \in \varphi_n(u)\}.$$ 

This generalizes the definition of $\text{Inf}$. To see why, we can define the functions $f_n$ and $\varphi_n$ as follows:

$$\varphi_n : \Sigma^{2^n} \rightarrow \mathcal{P}(\{1, \ldots, 2^n\}) \quad a_1 \ldots a_{2^n} \mapsto \{i \in \{1, \ldots, 2^n\} \mid a_i = 1\},$$

$$f_n : \Sigma^n \rightarrow \{1, \ldots, 2^n\} \quad a_1 \ldots a_n \mapsto 1 + \sum_{i=1}^{n} a_i \cdot 2^{n-i}.$$ 

The word $a_1 \ldots a_{2^n}$ is a representation of the characteristic function of a subset $A$ of $\{1, \ldots, 2^n\}$ and $\varphi_n$ maps this word to $A$. The function $f_n$ maps a word $v$ to $m + 1$, where $v$ is a binary representation of $m$.

Let us call a sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers suitable if it is monotone increasing, $k_0 = 1$, and $k_n \leq 2 \cdot k_{n-1}$. Note that in this case $k_n \leq 2^n$. We define for all $n \in \mathbb{N}$

$$\varphi_n : \Sigma^{2^n} \rightarrow \mathcal{P}(\{1, \ldots, k_n\}) \quad a_1 \ldots a_{2^n} \mapsto \{i \in \{1, \ldots, k_n\} \mid a_i = 1\}.$$ 

For $n \in \mathbb{N}$ we define the function $f_n : \Sigma^n \rightarrow \{1, \ldots, k_n\}$ recursively. Let us fix $f_0(\varepsilon) = 1$. For $n \in \mathbb{N}$ let $m = k_{n+1} - k_n$ and define

$$f_{n+1}(0w) = f_n(w) \quad f_{n+1}(1w) = \begin{cases} f_n(w) & \text{if } f_n(w) > m \\ f_n(w) + k_n & \text{if } f_n(w) \leq m. \end{cases}$$
4 Encodings and Surjectivity

Before we continue, let us make some observations. Firstly, we always have $|\text{im } \varphi_n| = 2^k$. Secondly, if we choose $k_n = 2^n$, then $\varphi_n$ and $f_n$ are exactly the functions we used for $\text{Inf}$. For $\varphi_n$ this is clear, but for $f_n$ a small inductive proof is necessary:

\[
\begin{align*}
    f_{n+1}(0w) &= f_{n+1}(0a_2 \ldots a_{n+1}) = f_n(a_2 \ldots a_{n+1}) \\
    f_{n+1}(1w) &= f_{n+1}(1a_2 \ldots a_{n+1}) = f_n(a_2 \ldots a_{n+1}) + 2^n \\
    &\quad \overset{\text{IH}}{=} 1 + \sum_{i=2}^{n+1} a_i \cdot 2^{n+1-i} \\
    a_1 = 0 &\quad \overset{\text{IH}}{=} 1 + \sum_{i=1}^{n+1} a_i \cdot 2^{n+1-i} \\
\end{align*}
\]

The idea of $f_n$ is that if $k_n + 1 < 2 \cdot k_n$, then we do not need all possible new words to distinguish all elements in $\text{im } \varphi_{n+1}$. This is because the number of words doubles since every word $w$ is split into $0w$ and $1w$. As a consequence, depending on the way we look at it, $f_n$ fuses some of these words together, or only splits as much words as needed.

**Example 4.6.** For example, if we choose $k_n = n + 1$, then $k_n + 1 - k_n = 1$ and

\[
\begin{array}{c|cccccccc}
\text{f}_n(w) & 00 & 01 & 10 & 11 & 000 & 010 & 100 & 110 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

The pattern is that a word is mapped to 1 if it contains only 0’s and is mapped to the number of 0’s behind the rightmost 1 plus 2 otherwise. This is because $f_{n+1}$ only splits one word. This word is 0...0 and it is split into 00...0 and 10...0, which are mapped to 0 and $n + 1$, respectively.

Now it is time to take a closer look at the properties of $\varphi_n$ and $f_n$. Clearly, $\varphi_n$ and $f_n$ are surjective. More interesting are the following two properties:

(P1) the following are equivalent for all $n \in \mathbb{N}$ and all $u, v \in \Sigma^n$

a) $f_n(u) = f_n(v)$

b) $f_{n+1}(au) = f_{n+1}(av)$ for all $a \in \Sigma$

c) $f_{n+1}(au) = f_{n+1}(av)$ for some $a \in \Sigma$

(P2) the following are equivalent for all $n, k \in \mathbb{N}$, $w_1, w_2 \in \Sigma^k$ with $k \leq 2^n$:

a) $\varphi_n(w_1v) = \varphi_n(w_2v)$ for some $v \in \Sigma^{2^n-k}$

b) $\varphi_n(w_1v) = \varphi_n(w_2v)$ for all $v \in \Sigma^{2^n-k}$.
Let us check that these properties hold. For [P1] consider first (a) ⇒ (b). We either have
\[
\begin{align*}
    f_{n+1}(au) &= f_n(u) = f_n(v) = f_{n+1}(av), \\
    f_{n+1}(au) &= f_n(u) + k_n = f_n(v) + k_n = f_{n+1}(av)
\end{align*}
\]
for all \( a \in \Sigma \). The implication from (b) to (c) is trivial. For the last implication consider the case \( a = 0 \), then \( f_{n+1}(0u) = f_n(u) = f_n(v) \) by definition. If \( f_{n+1}(1u) = f_{n+1}(1v) \), then either \( f_{n+1}(1u) > k_{n+1} - k_n \) and \( f_{n+1}(1u) = f_n(u) = f_n(v) = f_{n+1}(1v) \) or \( f_{n+1}(1u) \leq k_{n+1} - k_n \), which again implies \( f_n(u) = f_n(v) \).

The property [P2] is clear from the definition of \( \varphi_n \). If we assume an additional property on \( k_n \), then we are able to bound \( \text{ind} \Theta_n(L) \) for large \( n \).

**Lemma 4.7.** If \( k_n \) grows sufficiently fast, i.e., there is an \( N \in \mathbb{N} \) such that \( n \cdot k_n \leq k_{2^n} \) for all \( n \geq N \), then
\[
2^{k_n} \leq \text{ind} \Theta_n(L) \leq 3 \cdot (n + 1)^2 \cdot 2^{k_n} \text{ for all } n \geq 2^N.
\]

**Proof.** Let \( n \in \mathbb{N} \) with \( n \geq 2^N \). First we will determine the number of classes of \( \Theta_n(L) \) generated by words of length \( 2^n \). For \( u, v \in \Sigma^{2^n} \), if \( \varphi_n(u) \neq \varphi_n(v) \), then fix some element \( k \in \varphi_n(u) \triangle \varphi_n(v) \). Since \( f_n \) is surjective there is a \( w \in \Sigma^n \) such that \( f_n(w) = k \), and the word \( w \) witnesses the fact that \( u \) and \( v \) are not in the same class. Vice versa, if \( \varphi_n(u) = \varphi_n(v) \), then \( (u, v) \in \Theta_n(L) \). Now the lower bound immediately follows, i.e., \( \text{ind} \Theta_n(L) \geq \{|w| \mid w \in \Sigma^{2^n} \} = |\text{im} \varphi_n| = 2^{k_n} \).

For the upper bound there is much more work to do. First we look at the different types of words there are and then we will bound the number of equivalence classes generated by words of each type separately.

![Figure 4.1: The four different types of words](image)

- **Words of type I** are of the form \( uv \) where \( u \in \Sigma^{2^k} \) and \( v \in \Sigma^{(k)} \) for some \( k \geq \log_2 n \). All words \( v' \in \Sigma^{(n)} \) for which \( uvv' \in L \) have to be of length \( k - |v| \). Recall \( uvv' \in L \) iff \( f_k(vv') \in \varphi_k(u) \). Therefore the witnesses can only be used to determine \( \varphi_k(u) \). Note that the decomposition into \( u \) and \( v \) is unique.

- **Every word** \( u \) of type II is in \( \Sigma^{2^l-k} \) for some \( l \) and \( k \) with \( \log_2 n \leq l \leq n \) and \( k + l \leq n \). All words \( w \in \Sigma^{(n)} \) with \( uw \in L \) are of the form \( u'v \) with \( u' \in \Sigma^k \) and \( v \in \Sigma^l \). The difference to words of type I is that now \( \varphi_k(uu') \) depends on the choice of the witness. Note that this could potentially lead to a lot of new equivalence classes.
The words of type III are all words with length at most \( n - 1 \). Some of these words \( u \) are short enough such that there can be positive witnesses of \( u \) in \( \Sigma^{(n)} \) with different lengths.

Finally, for all words \( w \) of type IV we have \( uv \notin L \) for all \( v \in \Sigma^{(n)} \). Hence these words are all in the same equivalence class.

We denote number of equivalence classes the words of type I generate, i.e., \(|\{[w] \mid w \text{ is of type I}\}|\), by \( B_I \). Analogues we define \( B_{II} \), \( B_{III} \), and \( B_{IV} \). Since any word in \( \Sigma^* \) is of one of the four types we have that

\[
\text{ind } \Theta_n(L) \leq B_I + B_{II} + B_{III} + B_{IV}.
\]

We have already seen that \( B_{IV} = 1 \). Before we determine an upper bound for \( B_I \), \( B_{II} \), and \( B_{III} \) we will look at sets of the form \( \Sigma^{2k} \cdot \{0\}^k \) and determine an upper bound for the size of \( \{[w] \mid w \in \Sigma^{2k} \cdot \{0\}^k\} \) for \( k \in \{0, \ldots, n\} \).

Let \( w_1, w_2 \in \Sigma^{2k} \cdot \{0\}^k \). Then \( w_1 = u_1v, w_2 = u_2v \) for some \( u_1, u_2 \in \Sigma^{2n} \) and \( v = 0^k \). If \( (w_1, w_2) \notin \Theta_n(L) \), then there is some \( w' \in \Sigma^{n-k} \) which bares witness to this fact. Now \( vw' \) witnesses \( (u_1, u_2) \notin \Theta_n(L) \). Hence the words in \( \Sigma^{2n} \cdot 0^k \) give rise to at most as many classes as the words in \( \Sigma^{2k} \), which, as we have already shown, decompose into \( 2^{k_n} \) classes. Therefore we conclude

\[
|\{[w] \mid w \in \Sigma^{2k} \cdot \{0\}^k\}| \leq 2^{k_n} \text{ for all } n \in \mathbb{N}, k \in \{0, \ldots, n\}.
\] (\

Recall the definition of \( U_w \). For \( w \in \Sigma^* \) we denote \( \{v \in \Sigma^{(n)} \mid vw \in L\} \), the set of positive witnesses of \( w \) of length at most \( n \), by \( U_w \). Note that \( U_w \) is just a different representation of \([w]\), i.e., \([w_1] = [w_2]\) iff \( U_{w_1} = U_{w_2} \).

To find an upper bound for \( B_I \) we will show that every word of type I is already in the same class as some word in \( \Sigma^{2k} \cdot \{0\}^{(n)} \). Let \( uv \) be a word of type I with \( u \in \Sigma^{2k} \) and \( v \in \Sigma^{(k)} \). We have already noticed that then \( U_{uv} \subseteq \Sigma^{k+|v|} \). Define \( v' = 0^{n-k+|v|} \). Since \( n - k + |v| + k - |v| = n \) and \( \phi_n \) is surjective there is an \( u' \) such that \( \phi_n(u') = f_n(v'U_{uv}) \). Note that \( u'v' \in \Sigma^{2k} \cdot \{0\}^{(n)} \). Our next goal is to show that \( uv \) and \( u'v' \) are in the same equivalence class, i.e.,

\[
w \in U_{uv} \iff f_n(v'w) \in \phi_n(u') \implies w \in U_{u'v'}.
\]

The \( \Rightarrow \) direction is straightforward. If \( w \in U_{uv} \), then \( v'w \in v'U_{uv} \) and therefore \( f_n(v'w) \in f_n(v'U_{uv}) = \phi_n(u') \). For the \( \Leftarrow \) direction take some word \( w \) with \( f_n(v'w) \in \phi_n(u') \). By choice of \( u' \) there is a \( w' \in U_{uv} \) such that \( f_n(v'w) = f_n(v'w') \). Then

\[
f_n(v'w) = f_n(v'w') \iff f_{k-|v|}(w) = f_{k-|v|}(w')
\]

\[
\iff f_k(vw) = f_k(vw'). \quad \text{(P1)}
\]

Because \( w' \in U_{uv} \) implies \( f_k(vw) = f_k(vw') \in \phi_n(u) \) we conclude \( w \in U_{uv} \). Thus \( uv \) is in the same class as \( u'v' \).
Consequently, we can finally give an upper bound for $B_I$:

\[
B_I \leq |\{ [w] \mid w \in \Sigma^{2^n} \cdot \{0\}^{(n)} \}|
\leq \sum_{k=0}^{n} |\{ [w] \mid w \in \Sigma^{2^n} \cdot 0^k \}|
\leq (n + 1) \cdot 2^{k_u}.
\]

For $B_{II}$ we will consider all words of type II with the same length $2^l - k$ for some $l, k \in \mathbb{N}$ with $l + k \leq n$. Note that since $l + k \leq n$ there are at most $(n + 1)^2$ possible lengths. For fixed $l$ and $k$ we want to bound the size of $\{ U_u \mid u \in \Sigma^{2^l - k} \}$ by $2^{k_l}$. To prove this let $u' \in \Sigma^k$. We show that

\[
\{ U_u \mid u \in \Sigma^{2^l - k} \} \rightarrow \text{im} \; \varphi_l
\]

\[
U_u \mapsto \varphi_l(uu')
\]

is a well defined injective map. Firstly, we tackle the problem of well definedness. Let $u_1, u_2 \in \Sigma^{2^l - k}$ such that $U_{u_1} = U_{u_2}$. Then

\[
f_l(v) \in \varphi_l(u_1u') \iff u'v \in U_{u_1}
\]

\[
\iff u'v \in U_{u_2}
\]

\[
\iff f_l(v) \in \varphi_l(u_2u')
\]

and $\varphi_l(u_1u') = \varphi_l(u_2u')$. For injectivity take two words $u_1, u_2 \in \Sigma^{2^l - k}$ with $\varphi_l(u_1u') = \varphi_l(u_2u')$. Let $w \in U_{u_1}$. Then decompose $w$ into $uu''v'$ with $u'' \in \Sigma^k$ and $v' \in \Sigma^l$. Now, by definition, $v' \in \varphi_l(u_1u')$. Furthermore, from (P2) we can deduce $\varphi_l(u_1u'') = \varphi_l(u_2u'')$, and thus $v' \in \varphi_l(u_2u'')$. Because of this $w \in U_{u_2}$ and $U_{u_1} \subseteq U_{u_2}$. By exchanging the roles of $u_1$ and $u_2$ equality of $U_{u_1}$ and $U_{u_2}$ follows. Therefore

\[
|\{ [u] \mid u \in \Sigma^{2^l - k} \}| = |\{ U_u \mid u \in \Sigma^{2^l - k} \}| \leq | \text{im} \; \varphi_l | = 2^{k_l} \leq 2^{k_u}.
\]

Recall that there are at most $n + 1$ many possible $l$ and $k$. Consequently,

\[
B_{II} \leq (n + 1)^2 \cdot 2^{k_u}.
\]

To bound $B_{III}$ we use the same method we used for $B_{II}$. Fix a $k < n$ and consider all classes generated by words in $\Sigma^k$. Firstly, note that for an $u \in \Sigma^k$ there can be words of different lengths in $U_u$, but we still know that $U_u \subseteq \bigcup_{l \leq n} \Sigma^{2^l + l - k}$ where $m = \max \{ l \in \mathbb{N} \mid 2^l + l - k \leq n \}$ and $\Sigma^{-i} = \emptyset$. Furthermore, $m \leq \lceil \log_2 n \rceil$, because

\[
2^{\lceil \log_2 n \rceil} + \lceil \log_2 n \rceil + 1 - k \geq 2 \cdot n + 1 - (n - 1) > n.
\]
4 Encodings and Surjectivity

Denote $U_u \cap \Sigma^{|l-i-k|}$ by $U_{u,l}$. To these sets we can then apply the same argument as before and obtain $|\{U_{u,l} \mid u \in \Sigma^k\}| \leq |\text{im } \varphi| \leq 2^k$. Now we can compute

$$\left| \{U_u \mid u \in \Sigma^k\} \right| \leq \prod_{l=1}^{m} 2^{k_l} \leq 2^m \cdot 2^{m-k_m}.$$  

(k_0 = 1)

We know that $m \leq \log_2 n$ and $n \geq 2^N$ implies $\log_2 n \geq N$. Furthermore, if we assume for the sake of readability that $\log_2 n$ is a natural number, then

$$m \cdot k_m \leq \log_2 n \cdot k_{\log_2 n} \leq k_n.$$ 

As there are $n$ many possible values for $k$ we obtain $B_{\text{III}} \leq 2 \cdot n \cdot 2^{k_n}$.

Now we can finally give an upper bound for $\text{ind } \Theta_n(L)$:

$$\text{ind } \Theta_n(L) \leq B_1 + B_{\text{II}} + B_{\text{III}} + B_{\text{IV}} \leq (n + 1) \cdot 2^{k_n} + (n + 1)^2 \cdot 2^{k_n} + 2 \cdot n \cdot 2^{k_n} + 1 \leq 3 \cdot (n + 1)^2 \cdot 2^{k_n}.$$ 

This finishes the proof.

Note that this lemma can be applied to any surjective functions $\varphi_n$ and $f_n$ fulfilling the properties [P1] and [P2], not just the $\varphi_n$ and $f_n$ we defined. Now we can, using this lemma, show the main result from this section.

**Theorem 4.8.** For any alphabet $\Sigma$ with at least two letters the entropy function

$$\eta : \mathcal{P}(\Sigma^*) \rightarrow [0, \infty]$$

$$L \mapsto \eta(L)$$

is surjective.

**Proof.** For 0 and $\infty$ we have already seen that there are languages with that entropy. Thus let $x$ be a positive real number. We would like to use the sequence $k_n = \lceil n \cdot x \rceil$, but it is not suitable, because $k_0 = 0$. Furthermore, $k_1$ could be larger than 2. Because of this we define $k_n = \max\{\min\{\lceil n \cdot x \rceil, 2^n\}, 1\}$. Now the sequence $(k_n)_{n \in \mathbb{N}}$ is suitable. Since $\Sigma$ has at least two letters we can define $f_n$, $\varphi_n$, and $L$ as above. Clearly, there is an $N_1 \in \mathbb{N}$ such that $k_n = \lceil n \cdot x \rceil$ for all $n \geq N_1$. 

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Furthermore, there is an $N_2 \in \mathbb{N}$ such that
\[
\begin{align*}
  n \cdot k_n &= n \cdot \lceil n \cdot x \rceil \\
  &\leq n \cdot (n \cdot x + 1) \\
  &= n \cdot (n + \frac{1}{x}) \cdot x \\
  &\leq 2^n \cdot x \quad \text{(for all } n \geq N_2) \\
  &\leq k_{2^n}.
\end{align*}
\]
Hence we apply Lemma 4.2 with $N = \max\{N_1, N_2\}$ to obtain
\[
2^{k_n} \leq \ind \Theta_n(L) \leq 3 \cdot (n + 1)^2 \cdot 2^{k_n}
\]
for all $n \geq 2^N$. Now we easily compute
\[
\begin{align*}
  \eta(L) &\geq \limsup_{n \to \infty} \frac{\log_2(2^{k_n})}{n} \\
  &= \limsup_{n \to \infty} \frac{k_n}{n} \\
  &= \limsup_{n \to \infty} \frac{\lceil n \cdot x \rceil}{n} \\
  &= x
\end{align*}
\]
and
\[
\begin{align*}
  \eta(L) &\leq \limsup_{n \to \infty} \frac{\log_2(3 \cdot (n + 1)^2 \cdot 2^{k_n})}{n} \\
  &= \limsup_{n \to \infty} \frac{\log_2 3 + 2 \cdot \log_2(n + 1) + k_n}{n} \\
  &= \limsup_{n \to \infty} \frac{\log_2 3 + 2 \cdot \log_2(n + 1) + \lceil n \cdot x \rceil}{n} \\
  &= 0 + 0 + x.
\end{align*}
\]
Therefore, $\eta(L) = x$, and we conclude that $\eta$ is surjective.

Naturally the question arises whether same holds for unary alphabets. In the next section we will address this question.

### 4.3 Unary Languages

For the remainder of this section let $\Sigma = \{a\}$ be a unary alphabet. We shall write $n$ instead of $a^n$. Then we can view $\Sigma^*$ as $\mathbb{N}$ and a unary language $L$ is just a subset of $\mathbb{N}$. We will show that the entropy of a language over a unary alphabet can be
4 Encodings and Surjectivity

bounded by one and we will show that this upper bound is tight. Before we continue let us look at some examples of unary languages. We have already seen some examples in Chapter 3, namely the set of all square numbers \( \{n^2 \mid n \in \mathbb{N}\} \) and the set of all prime numbers \( \{p \in \mathbb{N} \mid p \text{ is prime}\} \). Both of these examples are decidable and not very complicated. But if we take an enumeration \( M_1, M_2, \ldots \) of all Turing machines and define \( L = \{i \in \mathbb{N} \mid M_i \text{ halts on input } \varepsilon\} \), then we have encoded the halting problem as a unary language. Consequently, there are hard unary languages, which makes the following result even more interesting.

**Theorem 4.9.** Let \( L \) be a unary language. Then \( \eta(L) \leq 1 \).

**Proof.** Let \( L \subseteq \mathbb{N} \) be a unary language. We already know that \( [w] \) corresponds to \( U_w \) and therefore \( \text{ind } \Theta_n(L) = |\{U_w \mid w \in \Sigma^*\}|. \) For all \( w \) we have that \( U_w \subseteq \Sigma^{(n)} \). Hence \( \text{ind } \Theta_n(L) \leq 2^{2^n} \). Since \( \Sigma \) is unary, we have

\[
\eta(L) = \limsup_{n \to \infty} \frac{\log_2(\text{ind } \Theta_n(L))}{n} \\
\leq \limsup_{n \to \infty} \frac{\log_2(2^{n+1})}{n} \\
= \limsup_{n \to \infty} \frac{(n+1) \cdot \log_2 2}{n} \\
= 1.
\]

This finishes the proof. \( \square \)

As in the case of \( \text{Inf} \) we are interested whether this upper bound is tight. It turns out that with an idea similar to the construction of \( \text{Inf} \), again using power sets, we can show that it is.

**Example 4.10.** For any \( n \in \mathbb{N} \) let \( \varphi_n: \{0, \ldots, 2^{n+1} - 1\} \to \mathcal{P}(\{0, \ldots, n\}) \) be a bijection. Note that if we know that a number is of the form \( 2^n + k \) and \( k < 2^n \), then we can uniquely determine \( k \) and \( n \) from that number. With this in mind we define:

\[
\text{UInf} = \{2^{n+2^m} + k \mid n \in \mathbb{N}, m \leq 2^n + 1, k \in \varphi_n(m)\}.
\]

To decompose any given natural number \( n_1 \) uniquely into \( 2^{n_1+2^m} + k \), find \( n_2 \), the largest power of 2 less or equal than \( n_1 \). Then \( n_1 = 2^{n_2} + k \) for some \( k < 2^{n_2} \). Use the same method to decompose \( n_2 = 2^n + n \) with \( n < 2^n \) and set \( m = n_3 - n \). Note that \( m \) can be negative and \( k \) can be greater than \( n \). For example, the unique decomposition of 1025 is \( 2^{10} + 1 = 2^{2+2^{10}} + 1 \), so \( n = 2, m = 1, \) and \( k = 1 \) and \( 1025 \in \text{UInf} \) if \( 1 \in \varphi_2(1) \). The reason we need \( n + m \) in the exponent instead of just \( m \) is that the value \( 2^m \) is for small \( m \) not necessarily larger than \( n \), which would make the decomposition impossible. Note that \( \lceil \log_2(n+1) \rceil + m \) would have been sufficient but for readability we take \( n + m \) instead.
4.3 Unary Languages

To determine the entropy of $\text{Unif}$ let $n \in \mathbb{N}$. Firstly, we denote $2^{n+2^n} + m$ by $w_{n,m}$. Now let us consider the set $\{w_{n,m} \mid m \leq 2^{n+1} - 1\}$. If $w_{n,m} \neq w_{n,m'}$, then $m \neq m'$ and since $\varphi_n$ is injective we have that there is a $k \in \varphi_n(m) \triangle \varphi_n(m')$. This $k$ witnesses that $(w_{n,m}, w_{n,m'}) \notin \Theta_n(\text{Unif})$. Hence

$$\text{ind} \Theta_n(\text{Unif}) \geq \left|\{w_{n,m} \mid m \leq 2^{n+1} - 1\}\right| = 2^{n+1}.$$ 

Therefore

$$\eta(\text{Unif}) \geq \limsup_{n \to \infty} \frac{\log_2 2^{n+1}}{n} = \limsup_{n \to \infty} \frac{(n+1) \cdot \log_2 2}{n} = 1.$$ 

Together with Theorem 4.9 we can conclude that $\eta(\text{Unif}) = 1$.

We have seen in the previous section that the entropy function is surjective over every alphabet with at least two letters. A natural next question to ask is whether there is a similar statement for unary alphabets. We conjecture that the following holds.

**Conjecture 4.11.** For any unary alphabet $\Sigma$ we have that the entropy function

$$\eta : \mathcal{P}(\Sigma^*) \to [0,1]$$

$$L \mapsto \eta(L)$$

is surjective.

Unfortunately, we were not able to prove this conjecture. We tried to adapt the construction from the previous section and define a language $L$ with

$$L = \{2^{n+2^n} + k \mid n \in \mathbb{N}, k \in \{0, \ldots, n\}, m \leq 2^{n+1} - 1, f_n(k) \in \varphi_n(m)\}$$

for suitable functions $f_n$ and $\varphi_n$. This is a generalization of $\text{Unif}$. To see why choose $\varphi_n$ as before and define $f_n(k) = k$.

We say that a sequence $(k_n)_{n \in \mathbb{N}}$ is unary suitable if it is monotone increasing, $k_0 = 1$, and $k_{n+1} \leq k_n + 1$. Note that in this case $k_n \leq n + 1$. We define $\varphi_0(0) = \emptyset$, $\varphi_0(1) = \{0\}$ and for all $n \in \mathbb{N}^+$

$$\varphi_n : \{0, \ldots, 2^n - 1\} \to \mathcal{P}(\{1, \ldots, k_n\})$$

$$m \mapsto \begin{cases} \varphi_{n-1}(m) & \text{if } m < 2^n \\ \varphi_{n-1}(m - 2^n) \cup \{k_n\} & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$ define

$$f_n : \{0, \ldots, n\} \to \{1, \ldots, k_n\}$$

$$m \mapsto k_m.$$
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Note that both mappings are surjective and \( f_{n+1}|_{\{0,\ldots,n\}} = f_n \).

Now we would like to prove a unary version of Lemma 4.7 but the proof seems to require the additional condition on the sequence \((k_n)_{n \in \mathbb{N}}\) that there is some sort of regularity in the sequence. This only seems to hold for a sequence \((\lceil n \cdot x \rceil)_{n \in \mathbb{N}}\) if \(x\) is a rational number, but it seems very unlikely that all values of the entropy function for languages over a unary alphabet are rational. Hence we leave this as an open problem for future work.
4.3 Unary Languages
5 Topological Entropy and Automata

In this chapter we will introduce the notion of $k$-counter automata. We will show that languages accepted by deterministic $\varepsilon$-free $k$-counter automata have zero entropy. In Section 5.2 we will introduce $k$-stack push-down automata and generalize our findings about counter automata to obtain an upper bound for the entropy of a language recognized by a deterministic $\varepsilon$-free $k$-stack push-down automata. Finally, we will show that the restriction to deterministic $\varepsilon$-free automata is in most cases necessary.

5.1 Counter Automata

We have already seen that all languages accepted by finite automata have zero entropy, but that there are also non-regular languages like $\{a^n b^n \mid n \in \mathbb{N}\}$ and $\{a^n b^n c^n \mid n \in \mathbb{N}\}$, which also have zero entropy. Interestingly, these languages can be recognized by a finite automaton with one, respectively two, counters. Hence Schneider and Borchmann suspected in [5] that all languages that can be recognized by a one-way finite automaton equipped with a fixed number of counters and an acceptance condition that does only require to check local conditions have zero entropy. In this section we will show that this conjecture holds if we require the automaton to also be $\varepsilon$-free and deterministic.

In Chapter 2 we have already observed that deterministic finite automata can be seen as a special kind of topological automata with the same set of states. Note that this observation could also be used to show that regular languages have zero entropy. For any topological automaton $A = (X, \Sigma, \delta, x_0, F)$ we have that $\bigvee_{t \in F^n} t^{-1}(\{F, F^c\})$ is refined by $\{\{x\} \mid x \in X\}$. If $A$ represents a finite automaton, then $N(\{\{x\} \mid x \in X\}) = |X|$ is finite, and thus

$$\eta(A) = \limsup_{n \to \infty} \frac{\log_2(N(\bigvee_{w \in \Sigma^n} w^{-1}(\{F, F^c\})))}{n} \leq \limsup_{n \to \infty} \frac{\log_2 |X|}{n} = 0$$

and $\eta(L(A)) = 0$ as well.

We will use a similar approach for languages accepted by deterministic $\varepsilon$-free finite counter automata. First we will interpret them as a special kind of topological automata and then refine $\bigvee_{t \in F^n} t^{-1}(\{F, F^c\})$ to bound $N(\bigvee_{t \in F^n} t^{-1}(\{F, F^c\}))$ and to show that the entropy of the automaton is zero.

We give a formal definition of counter automata equivalent to the one Hopcroft and Ullman used [3]. To ease both reading and writing we shall adopt the following useful convention: we will abbreviate the tuple $(n_1, \ldots, n_k)$ by $n$, and apply functions component wise to $n$. For example $n - m = (n_1 - m_1, \ldots, n_k - m_k)$. 

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5 Topological Entropy and Automata

Definition 5.1. A \( k \)-counter automaton is a 5-tuple \( A = (Q, \Sigma, \delta, q_0, F) \) where

- \( Q \) is a finite set (of states),
- \( \Sigma \) is an alphabet (the input alphabet),
- \( \delta \subseteq \{0, 1\}^k \times Q \times (\Sigma \cup \{\varepsilon\}) \times Q \times \{-1, 0, 1\}^k \) (the transition relation),
- \( q_0 \) is from \( Q \) (the initial state), and
- \( F \) is a subset of \( Q \) (the set of final states).

A configuration of \( A \) is a tuple \((q, n, w)\) consisting of the current state \( q \in Q \), the values of the counters \( n \in \mathbb{N}^k \), and the rest of the input word \( w \in \Sigma^* \). Let \((q, n, aw), (p, m, w)\) be two configurations of \( A \), where \( a \in \Sigma \cup \{\varepsilon\} \). We define \((q, n, aw) \vdash (p, m, w)\) if \((\text{sgn}(n), q, a, p, m - n) \in \delta\). As usual, we will denote the reflexive transitive closure of \( \vdash \) by \( \vdash^* \).

Now we can define the language accepted by \( A \), written \( L(A) \), as

\[
L(A) = \{w \in \Sigma^* \mid \exists p \in F, n \in \mathbb{N}^k. (q_0, 0, w) \vdash^* (p, n, \varepsilon)\}.
\]

The automaton \( A \) is deterministic if for all \((q, n, w)\) there is at most one \((p, m, w')\) with \((q, n, w) \vdash (p, m, w')\).

Furthermore, we call \( A \) \( \varepsilon \)-free (a real-time counter automaton) if there is no tuple of the form \((n, q, \varepsilon, p, m)\) in \( \delta \).

Note that a tuple of the form \((0, q, a, p, -1)\) is not forbidden to be contained in \( \delta \), but can never be used for a transition, because the counter values cannot drop below zero. It is clear that a 0-counter automaton is just an ordinary finite automaton, and while counter automata do not seem to be that powerful, we have the following surprising result.

Theorem 5.2 (Theorem 7.9 from [3]). A 2-counter automaton can simulate an arbitrary Turing machine.

The main idea of the proof is to use one counter to cleverly store the tape and the other one to do secondary calculations. But our goal was to use these automata to show that the entropy of a large family of languages is zero and Turing machines can recognize languages with infinite entropy. Hence we have to use a more restrictive version. It turns out that \( \varepsilon \)-transitions are essential to the computational power of counter automata. As topological automata are inherently deterministic and \( \varepsilon \)-free, we will in the following consider mainly deterministic \( \varepsilon \)-free counter automata.

Before we come to counter automata as topological automata we shall discuss some example languages and some augmentations for our automata to get a better understanding of their capabilities.
Example 5.3. Let us construct a 1-counter automaton for the context-free language \( \{ a^n b^n \mid n \in \mathbb{N} \} \). One would expect that two states should suffice, one that reads \( a \)'s and increases the counter and one that reads \( b \)'s and decreases the counter:

Here an arrow from \( q \) to \( p \) with label \( n, a, m \) means that \((n, q, a, p, m) \in \delta \) and the \* in \( *, a, 1 \) means that this transition can be used regardless of whether the counter is zero or not. One problem with this automaton remains, namely that we want the states \( q \) and \( p \) to accept only if the counter is zero, but the final states of our automaton can not check whether the counter is zero at the end of the word. We would have to decide whether \( p \) or \( q \) are final regardless of the value in the counter. This can not lead to the desired language.

A solution to this problem would be to store in the state whether the counter is zero or not. We can imagine this as taking the lowest cell from the counter and storing it in the state.

Then a configuration \((q, n, w)\) in the original automaton would correspond to \(((q, 1), n - 1, w)\) in the modified one if \( n \geq 1 \) and to \(((q, 0), 0, w)\) if \( n = 0 \). The following automaton implements this idea.

We can generalize this idea to counter automata with more than one counter. More formally, a \( k \)-counter automaton where final states can check the counters is again a 5-tuple \( A = (Q, \Sigma, \delta, q_0, F) \) with the only difference that \( F \subseteq Q \times \{0, 1\}^k \) and the language recognized by \( A \) is now

\[
L(A) = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}^k, p \in Q, (p, \text{sgn}(n)) \in F, (q_0, 0, w) \vdash^* (p, n, \epsilon) \}. 
\]
5 Topological Entropy and Automata

Lemma 5.4. For every $k$-counter automaton where final states can check the counters there exists a $k$-counter automaton accepting the same language.

Proof. We will first consider the case where $k = 1$. Thus let $A = (Q, \Sigma, \delta, q_0, F)$ be a 1-counter automaton where final states can check the counters, and define $B = (Q \times \{0, 1\}, \Sigma, \delta', (q_0, 0), F)$, where $\delta'$ contains

$\delta'$ contains

$$(0, (q, 0), a, (p, 1), 0) \quad \text{for } (0, q, a, p, 1) \in \delta,$$

$$(0, (q, 0), a, (p, 0), 0) \quad \text{for } (0, q, a, p, 0) \in \delta,$$

$$(\ast, (q, 1), a, (p, 1), 1) \quad \text{with } \ast \in \{0, 1\} \quad \text{for } (1, q, a, p, 1) \in \delta,$$

$$(\ast, (q, 1), a, (p, 1), 0) \quad \text{with } \ast \in \{0, 1\} \quad \text{for } (1, q, a, p, 0) \in \delta,$$

$$(0, (q, 1), a, (p, 0), 0) \quad \text{and } (1, (q, 1), a, (p, 1), -1) \quad \text{for } (1, q, a, p, -1) \in \delta.$$

With the idea of this construction in mind, configurations of $B$ of the form $((q, 0), n, w)$ with $n \geq 1$ do not make sense. But note that every configuration of $B$ reachable from a configuration of the form $((q, 0), 0, w)$ or $((q, 1), n, w)$ is again of the form $((q, 0), 0, w)$ or $((q, 1), n, w)$. Now consider a run $\kappa'_1 \vdash \cdots \vdash \kappa'_m$ of $B$ where $\kappa_1$ is not of the form $((q, 0), n, w')$ with $n \geq 1$. It corresponds to the run $\kappa_1 \vdash \cdots \vdash \kappa_m$ of $A$ where $\kappa_i = (q, n + l, w)$ if $\kappa'_i = ((q, 1), n, w)$. Vice versa, a run $\kappa_1 \vdash \cdots \vdash \kappa_m$ of $A$ corresponds to the run $\kappa'_1 \vdash \cdots \vdash \kappa'_m$ of $B$, where

$$\kappa'_i = \begin{cases} ((q, 1), n - 1, w) & \text{if } \kappa_i = (q, n, w) \text{ with } n \geq 1 \\ ((q, 0), 0, w) & \text{if } \kappa_i = (q, 0, w) \end{cases}$$

for all $i \in \{1, \ldots, m\}$. Every initial configuration of $B$ is of the form $((q_0, 0), 0, w)$. Hence $((q_0, 0), 0, w) \vdash^* ((p, l), n, \varepsilon)$ with $(p, l) \in F$ iff $(q_0, 0, w) \vdash^* (p, n + l, \varepsilon)$ with $(p, \text{sgn}(n + l)) = (p, l) \in F$. Because of this $L(A) = L(B)$.

For $k$-counter automata we just apply the construction from above $k$ times. \hfill \Box

Note that repeated application of the construction of the proof would allow all the states of our counter automaton to check the counters for arbitrarily large values, not just zero.

The language $\{a^n b^n \mid n \in \mathbb{N}\}$ is context-free, but counter automata can also recognize some context-sensitive languages.

Example 5.5. For the context-sensitive language $\{a^n b^n c^n \mid n \in \mathbb{N}^+\}$ we can easily give a 2-counter automaton

![2-counter automaton diagram]

where $q_c$ accepts if both counters are zero.
Note that there is no 1-counter automaton accepting this language because every 1-counter automaton is apparently equivalent to a push-down automaton (PDA). Hence if there was a 1-counter automaton accepting \( \{a^n b^n c^n \mid n \in \mathbb{N}\} \), then this language would be context-free. Also, 1-counter automata are strictly weaker than PDA’s, as they can for example not recognize Dyck\(_{(1,1)}\), among others. Interestingly, there is also no deterministic \( \varepsilon \)-free 2-counter automaton for this language, which we shall show later.

Next we look at a language for which it is not immediately clear that there is a counter automaton recognizing it.

**Example 5.6.** Consider the language from Example 3.3 which contains all words of quadratic length \( L = \{ n^2 \mid n \in \mathbb{N}\} \). We use the fact that \( (n+1)^2 = n^2 + 2n + 1 \) to construct the following automaton:

![Automaton Diagram](image)

Note that we sometimes used words in the transitions for readability.

The automaton works as follows. One counter is used to store the largest \( n \) such that the automaton has already read \( n^2 \) symbols from the input and the other is used to check the length of the word. Each time it passes a square the roles of the counters swap. As an example, for \( a^9 \) we have the following run:

\[
(q, 0, 0, a^9) \vdash (q', 1, 0, a^8) \vdash (p', 0, 1, a^6) \vdash (q, 0, 2, a^5) \vdash (p, 1, 1, a^3) \vdash \\
(p, 2, 0, a^1) \vdash (q', 3, 0, \varepsilon).
\]

In general for odd \( n \) we have

\[
(q, 0, 0, a^{n^2}) \vdash^* (q', 1, 0, a^{n^2-1^2}) \vdash^* (q, 0, 2, a^{n^2-2^2}) \vdash^* (q', 3, 0, a^{n^2-3^2}) \vdash^* \\
(q, 0, 4, a^{n^2-4^2}) \vdash^* (q', 5, 0, a^{n^2-5^2}) \vdash^* \ldots \vdash^* (q', n, 0, a^{n^2-n^2}).
\]

We can generalize this idea. There are also counter automata for the languages \( \{a^n \mid n \in k\mathbb{N}\} \), \( \{a^n \mid n \in k^{2\mathbb{N}}\} \), and even \( \{a^n \mid n \in k^{2\mathbb{N}}\} \) for any \( k \in \mathbb{N} \).

**Example 5.7.** We will first give a counter automaton for \( \{a^n \mid n \in 2\mathbb{N}\} \).
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Where \((p,0,1)\) and \((q,1,0)\) are the final states. We use that \(2^{n+1} = 2^n + 2^n\) and swap the roles of the counters whenever we pass a \(2^n\) for some \(n \in \mathbb{N}\). For example the run for \(a^8\) would be

\[
(s,0,0,a^8) \vdash (p,0,1,a^7) \vdash (q,2,0,a^6) \vdash (p,1,2,a^5) \vdash (p,0,4,a^4) \vdash (q,2,3,a^3) \vdash (q,4,2,a^2) \vdash (q,6,1,a^1) \vdash (q,8,0,\varepsilon).
\]

And in general for even \(n\) we have

\[
(s,0,0,a^{2^n}) \vdash^* (p,0,1,a^{2^n-2^n}) \vdash^* (q,2,0,a^{2^n-2^{n-1}}) \vdash^* (p,0,4,a^{2^n-2^{n-2}}) \vdash^* (q,8,0,a^{2^n-2^{n-3}}) \vdash^* (p,0,16,a^{2^n-2^{n-4}}) \vdots \vdash^* (p,0,2^n,a^{2^n-2^n}).
\]

For larger \(k\) we have \(k^{n+1} = k^n + (k-1) \cdot k^n\). We can generalize the idea from \(k = 2\) and give the following automaton for \(\{a^n \mid n \in k\mathbb{N}\}\):

\[
\begin{align*}
0,0,0,1 & \xrightarrow{0,1,a^k-1,k} p \\
1,0,1,0 & \xrightarrow{1,0,a^k-1,k} q
\end{align*}
\]

where again \((p,0,1)\) and \((q,1,0)\) are the final states.

Automata for the other two languages can be constructed in a similar fashion with the use of more than two counters.

In the previous examples we used transitions where counters were increased by more than 1 in one step. Usually we would achieve this by adding \(\varepsilon\)-transition, but we do not want to use those. Hence we need a slightly more complicated workaround.

**Lemma 5.8.** For every \(k\)-counter automaton with arbitrary counter increase there exists a \(k\)-counter automaton accepting the same language.

**Proof.** We will first consider the case \(k = 1\). Let \(A = (Q, \Sigma, \delta, q_0, F)\) be such an automaton and \(n\) the maximal value by which a counter is increased. The idea is now to store \((k \mod n)\) in the counter instead of \(k\) and store the remainder of \(k/n\) in the state. Define \(B = (Q \times \{0, \ldots, n-1\}, \Sigma, \delta', q_0, F \times \{0, \ldots, n-1\}\) where \(\delta'\) contains

\[
\begin{align*}
(0, (q,0), a, (p,l), 0) & \quad \text{for } (0, q, a, p, l) \in \delta, l < n, \\
(0, (q,0), a, (p,0), 1) & \quad \text{for } (0, q, a, p, n) \in \delta, \\
(*, (q,k), a, (p,k + l \mod n), m) & \quad \text{with } * \in \{0,1\} \text{ for } (1, q, a, p, l) \in \delta, k < n,
\end{align*}
\]

where \(m = \begin{cases} 1 & \text{if } k + l \geq n \\ -1 & \text{if } k + l < 0 \\ 0 & \text{otherwise} \end{cases}\)
A configuration \((q, l, k, w)\) in \(B\) corresponds to the configuration \((q, k \cdot n + l, w)\) in \(A\). It is easy to verify that \(A\) and \(B\) recognize the same language.

In conclusion we can equip counter automata with the abilities to increase counters by arbitrary values and final states with the ability to check the counters for zeros without increasing their computational power. We have also seen that counter automaton can recognize much more than just the regular languages.

After this excursion we can finally come back to our main goal: showing that all languages recognized by deterministic \(\varepsilon\)-free counter automata are simple in the sense that they have zero entropy.

**Theorem 5.9.** Let \(k \in \mathbb{N}\) and \(A\) be a deterministic \(\varepsilon\)-free \(k\)-counter automaton. Then \(\eta(L(A)) = 0\).

Proving this theorem requires some preparations. The first step is to interpret counter automata as a special kind of topological automata, a deterministic \(\varepsilon\)-free \(k\)-counter automaton \(A = (Q, \Sigma, \delta, q_0, F)\) can be seen as the topological automaton \(B = (Q \times \mathbb{N}^k, \Sigma, a, (q_0, 0), F \times \mathbb{N}^k)\) where \(a((q, n), w) = (p, m)\) if \((q, n, w) \vdash^* (p, m, \varepsilon)\). Clearly, both automata recognize the same language.

We will now discuss the topology on the states of \(B\). Recall that the product of compact Hausdorff spaces is again a compact Hausdorff space. Because of this it is sufficient to consider \(Q\) and \(\mathbb{N}\) separately. Since \(Q\) is finite we can take the discrete topology to obtain a compact Hausdorff space. For \(\mathbb{N}\) we consider the Alexandroff compactification \(N_\infty\) of \(\mathbb{N}\) equipped with the discrete topology \([2]\), i.e., the set \(N_\infty = \mathbb{N} \cup \{\infty\}\) equipped with the topology

\[
\{ A \subseteq N_\infty \mid \infty \in A \implies N \setminus A \text{ is finite} \}.
\]

Therefore, the correct topological automaton \(B\) is \((Q \times N_{\infty}^k, \Sigma, a, (q_0, 0), F \times N_{\infty}^k)\), where \(a\) is extended to infinite counter values in the obvious way. As neither the topology nor the additional states of \(B\) are essential to the following proof, we will ignore them and pretend that the states of \(B\) are just \(Q \times \mathbb{N}^k\).

Now we can find an upper bound for the entropy of the language \(L\) of \(A\) using its corresponding topological automaton \(B\). Define \(\mathcal{U} = \{F \times \mathbb{N}^k, (Q \setminus F) \times \mathbb{N}^k\}\) and denote \(\bigvee_{w \in \Sigma^*} w^{-1}(\mathcal{U})\) by \(\mathcal{U}_n\). Furthermore, recall from Chapter \([2]\) that

\[
\eta(L) \leq \eta(B) = \limsup_{n \to \infty} \frac{\log_2(N(\mathcal{U}_n))}{n}.
\]

We find an upper bound for \(\eta(B)\) by giving a refinement of \(\mathcal{U}_n\). Recall that \(\mathcal{V}\) refines \(\mathcal{U}\), denoted by \(\mathcal{U} \preceq \mathcal{V}\), if for all \(V \in \mathcal{V}\) there is a \(U \in \mathcal{U}\) such that \(V \subseteq U\). For \(n \in \mathbb{N}\) define

\[
\mathcal{H}_n = \{ \{(q, m)\} \mid m \in \{0, \ldots, n - 1, N_{\geq n}\}^k \}
\]

where for example \(\{(q, 4, \ldots, N_{\geq n}, \ldots, N_{\geq n}, \ldots, 2)\}\) is a shorthand for

\[
\{(q, 4, \ldots, m_1, \ldots, m_2, \ldots, 2) \mid m_1, m_2 \in N_{\geq n}\}.
\]
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Note that \( \{(q, m)\} \) is only a singleton if \( m \in \{0, \ldots, n - 1\}^k \). The intuition is that \( H_n \) can distinguish \((q, m)\) and \((q', m')\) if \( q \neq q' \) or if \( m \) and \( m' \) differ in at least one counter, where at least one of these two different values is less than \( n \).

With all the necessary definitions in place we can start with the proof of Theorem 5.9.

Proof of Theorem 5.9. Let \( A = (Q, \Sigma, \delta, q_0, F) \) be a deterministic \( \varepsilon \)-free \( k \)-counter automaton and \( B = (Q \times \mathbb{N}^k, \Sigma, \alpha, (q_0, 0), F \times \mathbb{N}^k) \) its corresponding topological automaton. We begin by proving that \( H_n \) works as intended, using induction on \( n \) we show

\[
U_n \subseteq H_n \text{ for all } n \in \mathbb{N}.
\]

For the sake of readability we will for this claim only consider the case where \( k = 1 \). The construction, however, can easily be adapted for larger values of \( k \).

If \( n = 0 \), then by definition

\[
U_0 = U = \{ F \times \mathbb{N}, (Q \setminus F) \times \mathbb{N} \} \quad \text{and} \quad H_0 = \{ \{ q \} \mid q \in Q \} \times \mathbb{N}.
\]

Any set \( \{ q \} \times \mathbb{N} \) from \( H_0 \) is either contained in \( F \times \mathbb{N} \) if \( q \in F \), or in \( (Q \setminus F) \times \mathbb{N} \) if \( q \notin F \), respectively. Hence, \( U_0 \) is refined by \( H_0 \).

If \( n = n' + 1 \), then the claim follows from

\[
U_n = \bigvee_{w \in \Sigma^n} w^{-1}(U) \quad (\ast)
\]

\[
= \bigvee_{a \in \Sigma^n} \bigvee_{w' \in \Sigma^{n-1}} (aw')^{-1}(U)
\]

\[
= \bigvee_{a \in \Sigma^n} \left( \bigvee_{w' \in \Sigma^{n-1}} w'^{-1}(U) \right) \quad \text{(Lemma 8.4)}
\]

\[
= \bigvee_{a \in \Sigma^n} a^{-1}(U_{n-1}) \quad (\text{Induction Hypothesis})
\]

\[
\subseteq \bigvee_{a \in \Sigma^n} a^{-1}(H_{n-1}) \quad (**)
\]

Let us take a closer look at the steps. The \((\ast)\) follows from

\[
\Sigma^n = \Sigma^n \cup \Sigma^{n-1}
\]

\[
= \Sigma \cdot \Sigma^n \cup \{ \varepsilon \} \cdot \Sigma^{n-1}
\]

\[
= \Sigma^1 \cdot \Sigma^{n-1}
\]

and the fact that \( \vee \) is idempotent.

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To prove that (**) is correct we have to show that for every \( V \in \mathcal{L}_n \) there is a \( U \in \bigvee_{a \in \Sigma \cup \{\varepsilon\}} a \cdot a^{-1}(\mathcal{H}_{n-1}) \) such that \( V \subseteq U \). Now we look at the forms \( V \) and \( U \) can have. By definition every \( U \) is of the form \( \bigcap_{a \in \Sigma \cup \{\varepsilon\}} a \cdot a^{-1}(\mathcal{H}_{n-1}) \), hence it suffices to find for every \( a \in \Sigma \cup \{\varepsilon\} \) an \( U_a \) such that \( V \subseteq a^{-1}(U_a) \), i.e., \( a(V) \subseteq U_a \). The set \( V \) is either of the form \( \{q\} \times \mathbb{N}_{\geq n} \) or \( \{(q,l)\} \) for some \( l < n \). In the latter case the statement is trivial. Since \( a\{(q,l)\} = \{a((q,l),a)\} \) is again a singleton set, that is surely contained in some \( U_a \in \mathcal{H}_{n-1} \) because \( \mathcal{H}_{n-1} \) is a cover of \( Q \times \mathbb{N} \).

In the other case \( V = \{q\} \times \mathbb{N}_{\geq n} \) for some \( q \in Q \). If \( a = \varepsilon \), then we can choose \( U_\varepsilon = \{q\} \times \mathbb{N}_{\geq n} \) and obtain
\[
\varepsilon(V) = V = \{q\} \times \mathbb{N}_{\geq n} \subseteq \{q\} \times \mathbb{N}_{\geq n} = U_\varepsilon.
\]
Otherwise \( a \) is in \( \Sigma \). Let \((p,k) = a((q,1),a)\) for some \( p \in Q \) and some \( k \in \mathbb{N} \). Then by definition \((1,q,a,p,k-1) \in \delta_\varepsilon \). Now we can define \( U_a = \{p\} \times \mathbb{N}_{\geq n-1} \).

And for any \((q,l) \in V \) and \( a((q,l),a) = (p',l') \) we have the following:

\[
a((q,l),a) = (p',l') \iff (q,l,a) \vdash^* (p',l',\varepsilon) \\
(\iff (q,l,a) \vdash (p',l',\varepsilon) \\
(\iff (\text{sgn}(l),q,a,p',l'-l) \in \delta \\
(\iff (1,q,a,p',\nu(l)-l) \in \delta \\
(\iff p = p' \text{ and } l' = l + k - 1 \\
(\iff p = p' \text{ and } l' = l + k - 1 \geq 1
\]

We conclude \( p' = p \) and \( l' \geq l - 1 \geq n - 1 \). Hence, \((p',l') \in U_a \) and \( a(V) \subseteq U_a \) as desired.

Now that we have shown \( U_n \subseteq \mathcal{H}_n \), we can bound \( N(U_n) \) by
\[
N(U_n) \leq N(\mathcal{H}_n) = |\mathcal{H}_n| = |Q| \cdot (n + 1)^k.
\]

We can now easily compute the entropy of \( L(A) \).

\[
\eta(L(A)) \leq \limsup_{n \to \infty} \frac{\log_2(|Q| \cdot (n + 1)^k)}{n}
\]
\[
= \limsup_{n \to \infty} \frac{\log_2 |Q| + k \cdot \log_2 (n + 1)}{n}
\]
\[
= 0
\]

This shows the theorem. \( \square \)

We have seen that all languages accepted by counter automata have zero entropy. Since \( \eta(\text{Dyck}_{\{(1,2)\}}) = 1 \), there is no deterministic \( \varepsilon \)-free counter automaton recognizing \( \text{Dyck}_{\{(1,2)\}} \). However, there is a counter automaton for \( \text{Dyck}_{\{(1)\}} \).

Before we will generalize these results in the next section we shall discuss the \( \varepsilon \)-freeness condition. As we know from Theorem 5.2 there is no finite upper
bound for the entropy of $k$-counter automata with $\varepsilon$-transitions for $k \geq 2$. But what if $k = 1$? If there are two counters, then the automaton can use $\varepsilon$-transitions to make computations. But with just one counter all a deterministic automaton can do with $\varepsilon$-transitions is

- calculating the counter value modulo some $n$ and setting the value to zero in the process,
- increasing the counter by some fixed value (instead of just one),
- setting the counter to zero.

As we have seen in the proof of Lemma 5.8, increasing the counter value and computing it modulo some $n$ do not give any new computational power. Hence only setting the counter to zero is interesting and we can transform every 1-counter automaton with $\varepsilon$-transitions into a 1-counter automaton where there is a new type of transition, namely $(1, q, a, p, \perp)$, which sets the counter to zero. For these types of automata the entropy is still zero. To proof this we just have to add the case in the proof of Theorem 5.9 that $\alpha((q, 1, a)) = (p, 0)$. In this case we choose $U_a$ to be $\{(p, 0)\}$. This is sufficient since $\alpha((q, n, a)) = (p, 0)$ for all $n \geq 1$. We sum this up in the following corollary.

**Corollary 5.10.** All languages recognized by a deterministic 1-counter automaton with $\varepsilon$-transitions have zero entropy.

To see why 1-counter automaton with $\varepsilon$-transitions are interesting consider the following example.

**Example 5.11.** Define

$$L = \{a^{n_1}i_1a^{n_2}\#\ldots\#a^{n_k}i_k a^{m_k} | k \geq 1, n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}, i_1, \ldots, i_k \in \{0, 1\}, i_l = 1 \text{ implies } n_l = m_l \text{ for all } l \leq k\}.$$ 

We can easily find a 1-counter automaton with $\varepsilon$-transitions for $L$.

The states $p$ and $q$ accept if the counter is zero.

On the other hand we suspect that there is no deterministic $\varepsilon$-free $k$-counter automaton for $L$ no matter how many counters we can use, because whenever there is a block of the form $a^n0a^m$ with $m < n$ the automaton does not have enough time to clear its counter. However, we were not able to prove this statement, hence it remains an open problem.
5.2 Push-down Automata

A $k$-stack push-down automaton ($k$-stack PDA) is a finite automaton equipped with $k$ stacks. The counter automata from last section are $k$-stack PDA’s where every stack has just one type of stack symbol. Then the height of the stack represents the value of the counter.

Before we start with the formal definition of $k$-stack PDA’s, we introduce some functions and conventions. We will write $v$ for the tuple $(v_1, \ldots, v_k)$ from the set $\Gamma_1^* \times \cdots \times \Gamma_k^*$ and $\epsilon$ for the tuple containing only $\epsilon$ in each entry. For a word $w = a_1 \ldots a_n \in \Sigma^n$ and $I \subseteq \mathbb{N}$ let $\pi_I(w)$ be the projection of $w$ onto $I$:

$$\pi_I(w) = a_{i_1} \ldots a_{i_k} \text{ where } i_1 < \cdots < i_k \text{ and } \{i_1, \ldots, i_k\} = I \cap \{1, \ldots, n\}.$$  

Let $\pi_u = \pi_{\{1, \ldots, n\}}$, head = $\pi_{\{1\}}$, and tail = $\pi_{\{2, 3, \ldots\}}$. Dually we define the functions bottom$(w)$ = $\pi_{\{|w|\}}(w)$ and front$(w)$ = $\pi_{|w|-1}(w)$. For two words $w, v \in \Sigma^*$ where $v$ is a postfix of $w$ define $w - v$ as $w_1$, where $w = w_1v$. For tuples $v$ the functions head, tail, and $-$ are applied componentwise. Note that if $w$ is a word over a unary alphabet, then head$(w)$ corresponds to $\text{sgn}(|w|)$.

Definition 5.12. A $k$-stack PDA is a tuple $A = (Q, \Sigma, \Gamma_1, \ldots, \Gamma_k, \delta, q_0, F)$ where

- $Q$ is a finite set (of states),
- $\Sigma$ is an alphabet (the input alphabet),
- $\Gamma_1, \ldots, \Gamma_k$ are alphabets (the stack alphabets),
- $\delta \subseteq \Gamma_1^{(1)} \times \cdots \times \Gamma_k^{(1)} \times Q \times (\Sigma \cup \{\epsilon\}) \times Q \times \Gamma_1^* \times \cdots \times \Gamma_k^*$ (the transition relation),
- $q_0$ is from $Q$ (the initial state), and
- $F$ is a subset of $Q$ (the set of final states).

A configuration of $A$ is a tuple $(q, v, w)$ with the current state $q$, the values stored in the stacks $v$, and the remaining input $w$. For $a \in \Sigma^{(1)}$ we can make the transition $(q, v, aw) \vdash (p, u, w)$ if $(\text{head}(v), q, a, p, u - \text{tail}(v)) \in \delta$. Beware that a transition of the form $(\epsilon, p, a, q, u)$ can only be used if the stack is empty. Also the symbol on top of a stack represented by $\hat{v}$ is the leftmost symbol of $v$. Hence the stacks grow to the left. The language accepted by $A$ is defined as

$$L(A) = \{w \in \Sigma^* \mid \exists p \in F. (q_0, \epsilon, w) \vdash^* (p, v, \epsilon)\}.$$  

The definitions of $\epsilon$-freeness and determinicity are the same as for counter automata.
Note that if all of our stack alphabets are unary, then we get a counter automaton with arbitrary counter increase. If we try to construct a PDA for Dyck$_k$, it becomes clear that it would be useful if the final states of our PDA’s could check whether stacks are empty. As we shall see we will even give them the power to see the bottom of the stacks, similar to the case of counter automata.

We define a $k$-stack PDA where final states can check the stacks to be a tuple $A = (Q, \Sigma, \Gamma_1, \ldots, \Gamma_k, \delta, q_0, F)$ where all elements are the same as they would be for a normal $k$-stack PDA, except $F$ is now a subset of $Q \times \Gamma_1^{(1)} \times \cdots \times \Gamma_k^{(1)}$. The language recognized by $A$ is

$$L(A) = \{ w \in \Sigma^* \mid \exists p, \nu. (p, \text{head}(\nu)) \in F, (q_0, \epsilon, w) \vdash^* (p, \nu, \epsilon) \}.$$ 

**Lemma 5.13.** For every $k$-stack PDA where final states can check the stacks, there exists a $k$-stack PDA recognizing the same language.

**Proof.** The idea for the construction is the same as for counter automata: we store the stack symbols at the bottom of each stack in the state. This image illustrates the idea for a 2-stack PDA.

First we consider the case where $k = 1$. Let $A = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a 1-stack PDA where final states can check the stacks. We define the PDA $B$ as the tuple $(Q \times \Gamma^{(1)}, \Sigma, \Gamma, \delta', (q_0, \epsilon), F)$, where $\delta'$ contains

- $(\epsilon, (q, \epsilon), a, (p, \text{bottom}(u)), \text{front}(u))$ for $(\epsilon, q, a, p, u) \in \delta$,
- $(\epsilon, (q, b), a, (p, \text{bottom}(u)), \text{front}(u))$ for $(b, q, a, p, u) \in \delta$,
- $(b, (q, c), a, (p, c), u)$ for $(b, q, a, p, u) \in \delta, c \in \Gamma$.

A configuration $((q, a), u, w)$ of $B$ corresponds to the configuration $(q, ua, w)$ in $A$. With this in mind it is easy to verify that $L(A) = L(B)$. For PDA’s with more than one stack we can apply this construction for each stack.

Now we can give PDA’s for some languages from Chapter 3

**Example 5.14.** A 1-stack PDA where final states can check the stacks for Dyck$_\Gamma$ would be $\langle \{q\}, \Gamma \cup \overline{\Gamma}, \Gamma, \delta, q, \{q, \epsilon\} \rangle$ with

$$\delta = \{(X, q, a, q, aX) \mid a \in \Gamma, X \in \Gamma \cup \{\epsilon\}\} \cup \{(a, q, \overline{\epsilon}, q, \epsilon) \mid a \in \Gamma\}.$$ 

Also for the palindrome languages we can give a PDA. The 1-stack PDA

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where \( q \) and \( p \) accept if the stack is empty recognizes \( \text{Pali}_\varepsilon \). With the label \( X, a, aX \) we mean that we read an \( a \in \Sigma \) and put an \( a \) on top of the stack while leaving the rest of the stack untouched. Note that this automaton is nondeterministic and has \( \varepsilon \)-transitions.

The automaton for \( \text{DPali}_\varepsilon \) is almost the same, we just replace \( X, \varepsilon, X \) by \( X, \# X \). The resulting automaton is deterministic and \( \varepsilon \)-free.

Now we come back to our main goal in this section, finding an upper bound for the entropy of the language recognized by some \( k \)-stack PDA \( A \). The proof has the same structure as for counter automata. First we will interpret PDA’s as topological automata. Then we will use a refinement of \( \text{Pali}^k \) for the entropy of the language recognized by some \( \varepsilon \)-transitions.

A deterministic \( \varepsilon \)-free \( k \)-stack PDA \( (Q, \Sigma, \Gamma_1, \ldots, \Gamma_k, \delta, q_0, F) \) corresponds to the topological automaton \( (Q \times \Gamma^*_1 \times \cdots \times \Gamma^*_k, \Sigma, (q_0, \varepsilon), F \times \Gamma^*_1 \times \cdots \times \Gamma^*_k) \) where \( \delta((q, v), w) = (p, u) \) if \( (q, v, w) \vdash^* (p, u, \varepsilon) \).

We will now discuss the topology on the states of \( B \). For readability assume that \( \Gamma_1 = \cdots = \Gamma_k = \Gamma \). It is sufficient to consider the topologies on \( Q \) and \( \Gamma^* \) separately. Since \( Q \) is finite we can take the discrete topology to obtain a compact Hausdorff space. For \( \Gamma^* \) we want, as for counter automata, to find a compactification of \( \Gamma^* \) equipped with the discrete topology. Therefore, we define the set \( \Gamma^\infty = \Gamma^* \cup \Gamma^N \), where \( \Gamma^N \) denotes the set of all (right) infinite words over \( \Gamma \). We equip \( \Gamma^\infty \) with the topology defined by the following basis of open sets

\[
\{\{w\} \mid w \in \Gamma^*\} \cup \{w\Gamma^\infty \mid w \in \Gamma^*\},
\]

i.e., the open sets are exactly the sets that can be expressed as unions of sets from the basis. As a consequence, the correct topological automaton \( B \) would be \( (Q \times \Gamma^\infty \times \cdots \times \Gamma^\infty, \Sigma, (q_0, \varepsilon), F \times \Gamma^\infty \times \cdots \times \Gamma^\infty) \), where \( \delta \) is extended to infinite words in the stacks in the obvious way. As neither the topology nor the additional states of \( B \) are essential to the following proof, we will ignore them and pretend that the states of \( B \) are just \( Q \times \Gamma^*_1 \times \cdots \times \Gamma^*_k \).

Let \( A = (Q, \Sigma, \Gamma_1, \ldots, \Gamma_k, \delta, q_0, F) \) be a deterministic \( \varepsilon \)-free \( k \)-stack PDA and \( B \) its corresponding topological automaton. Define

\[
\mathcal{U} = \{F \times \Gamma^*_1 \times \cdots \times \Gamma^*_k, F^c \times \Gamma^*_1 \times \cdots \times \Gamma^*_k\}
\]

and as before \( \mathcal{U}_n = \bigvee_{w \in \Sigma^n} w^{-1}(\mathcal{U}) \). This time the candidate for a refinement of \( \mathcal{U}_n \) is

\[
\mathcal{H}_n = \{\{(q, u)\} \mid u \in \bigcap_{i=1}^k (F_i^{(n-1)} \cup \Gamma^*_i \cdot \{\Gamma_i^*\})\}
\]
where $\Gamma(-1) = \emptyset$ and for example $\{(q, ab, \ldots, aaa_G^*, \ldots, bb)\}$ is a shorthand for $\{(q, ab, \ldots, aau, \ldots, bb) \mid u \in \Gamma_1^*\}$.

Note that $|\mathcal{H}_n| = |Q| \cdot |\Gamma_1^{(n)}| \cdot \ldots \cdot |\Gamma_k^{(n)}|$. This leads to the following theorem.

**Theorem 5.15.** Let $\mathcal{A} = (Q, \Sigma, \Gamma_1, \ldots, \Gamma_k, \delta, q_0, F)$ be a deterministic $e$-free $k$-stack PDA. Then

$$\eta(L(\mathcal{A})) \leq \log_2 |\Gamma_1| + \cdots + \log_2 |\Gamma_k|.$$  

**Proof.** As the proof is very similar to the one of Theorem 5.9, we shall be a bit less detailed than before. Let $\mathcal{A} = (Q, \Sigma, \Gamma_1, \ldots, \Gamma_k, \delta, q_0, F)$ be a deterministic $e$-free $k$-stack PDA, $\mathcal{B}$ its corresponding topological automaton, and $\mathcal{U}_n$, $\mathcal{H}_n$ as defined above. First we show

$$\mathcal{U}_n \preceq \mathcal{H}_n \text{ for all } n \in \mathbb{N}.$$  

For the sake of readability we will only consider the case $k = 1$. The construction, however, can easily be adapted for larger values of $k$.

If $n = 0$, then $\mathcal{U}_0 = \mathcal{U} = \{F \times \Gamma^*, F^c \times \Gamma^*\}$ and $\mathcal{H}_0 = \{(q, \Gamma^*) \mid q \in Q\}$. Clearly, $\mathcal{U}_0$ is refined by $\mathcal{H}_0$.

If $n = n' + 1$, then

$$\mathcal{U}_n = \bigvee_{a \in \Sigma \cup \{\varepsilon\}} a^{-1}(\mathcal{U}_{n-1})$$

$$\preceq \bigvee_{a \in \Sigma \cup \{\varepsilon\}} a^{-1}(\mathcal{H}_{n-1}) \quad \text{(Induction Hypothesis)}$$

$$\preceq \mathcal{H}_n \quad \text{(**)}$$

For (**) it suffices to show that for every $V \in \mathcal{H}_n$ and every $a \in \Sigma \cup \{\varepsilon\}$ there is an $U_a \in \mathcal{H}_{n-1}$ such that $a(V) \subseteq U_a$. For $a = \varepsilon$ this statement is trivial because $\mathcal{H}_{n-1} \preceq \mathcal{H}_n$. If $V$ is of the form $\{(q, v)\}$ for some $v \in \Gamma^{(n-1)}$, then $a(V)$ is again a singleton set. Hence it is contained in some $U_a$.

Thus consider the remaining case where $a \in \Sigma$ and $V = \{(q, v')\}$ for some $v' \in \Gamma^n$ and $q \in Q$. There are $p \in Q$ and $u \in \Gamma^*$ such that $a((q, v), a) = (p, u)$ and by definition $(\text{head}(v), q, a, p, w) \in \delta$, where $w \text{tail}(v) = u$. Define

$$U_a = \{(p, \pi_{n-1}(u) \cdot \Gamma^*)\}.$$  

Since $|u| \geq |\text{tail}(v)| = n - 1$, we get that $U_a$ is indeed an element of $\mathcal{H}_{n-1}$. For any $(q, vv') \in V$ with $a((q, vv'), a) = (p', u')$ we have

$$a((q, vv'), a) = (p', u') \iff (\text{head}(vv'), q, a, p', u' - \text{tail}(vv')) \in \delta$$

$$\iff (\text{head}(v), q, a, p', u' - \text{tail}(v)v') \in \delta \quad (|v| = n \geq 1)$$

$$\iff p = p' \text{ and } u' - \text{tail}(v)v' = w$$

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Hence $p = p'$ and $u' = w \text{tail}(v)v' = uv'$. Remember $|u| \geq n - 1$, as a consequence, $\pi_{n-1}(u') = \pi_{n-1}(u)$ and thus $(p', u') \in \{(p, \pi_{n-1}(u) \cdot \Gamma')\} = \mathcal{U}_a$. Therefore $a(V) \subseteq \mathcal{U}_a$. This concludes the proof of $\mathcal{U}_n \preceq \mathcal{H}_n$ for all $n \in \mathbb{N}$.

Now we can bound the entropy of $L(A)$.

$$\eta(L(A)) \leq \limsup_{n \to \infty} \log_2(N(\mathcal{U}_n))$$

$$\leq \limsup_{n \to \infty} \log_2 |\mathcal{H}_n|$$

$$= \limsup_{n \to \infty} \log_2(|Q| \cdot |\Gamma_1^{(n)}| \cdot \ldots \cdot |\Gamma_k^{(n)}|)$$

$$= \limsup_{n \to \infty} \log_2 |Q| + \log_2 \left( \sum_{i=0}^{n} |\Gamma_1|^i + \ldots + \sum_{i=0}^{n} |\Gamma_k|^i \right)$$

$$= \limsup_{n \to \infty} \log_2 |Q| + \log_2 \left( \sum_{i=0}^{n} |\Gamma_1|^i + \ldots + \sum_{i=0}^{n} |\Gamma_k|^i \right)$$

$$= \limsup_{n \to \infty} \log_2 |Q| + \log_2 \left( \sum_{i=0}^{n} |\Gamma_1|^i + \ldots + \sum_{i=0}^{n} |\Gamma_k|^i \right)$$

$$= \limsup_{n \to \infty} \log_2 |Q| + \log_2 \left( \sum_{i=0}^{n} |\Gamma_1|^i + \ldots + \sum_{i=0}^{n} |\Gamma_k|^i \right)$$

$$\leq \limsup_{n \to \infty} \left( n + 1 \right) \cdot \left( \log_2 |\Gamma_1| + \ldots + \log_2 |\Gamma_k| \right) + 0$$

$$= \log_2 |\Gamma_1| + \ldots + \log_2 |\Gamma_k|$$

This finishes the proof. \qed

Note that for the special case that all stack alphabets are unary we have

$$\eta(L(A)) = 0.$$ 

So this theorem is a generalization of Theorem 5.9. As a first result we can finally determine the entropy of the deterministic palindrome language from Example 3.4.

Corollary 5.16. The entropy of the deterministic palindrome language $\text{DPali}_\Sigma$ is $\log_2 |\Sigma|$ and every 1-stack PDA accepting it needs at least $|\Sigma|$ many stack symbols.

Proof. We have already seen in Example 3.4 that $\log_2 |\Sigma|$ is a lower bound for $\eta(\text{DPali}_\Sigma)$. In Example 5.14 we have seen a deterministic $\epsilon$-free 1-stack for $\text{DPali}_\Sigma$ with $\Sigma$ as stack alphabet. Hence, Theorem 5.15 implies that $\log_2 |\Sigma|$ is also an upper bound. If there were a deterministic $\epsilon$-free 1-stack PDA with stack alphabet $\Gamma$ and $|\Gamma| < |\Sigma|$, then $\log_2 |\Gamma|$ would be an upper bound for $\eta(L(A))$ that would be smaller than the lower bound $\log_2 |\Sigma|$, a contradiction. \qed
Whether the same holds for the nondeterministic palindrome language is unknown to the author.

We can generalize the example of deterministic palindrome languages to show that our upper bound for the entropy is tight, i.e., for every $\Gamma_1, \ldots, \Gamma_k$ there is a deterministic $\varepsilon$-free $k$-stack PDA with stack alphabets $\Gamma_1, \ldots, \Gamma_k$ whose language has entropy $\log_2 |\Gamma_1| + \cdots + \log_2 |\Gamma_k|.$

**Example 5.17.** We construct a product palindrome language. Let $\Gamma_1, \ldots, \Gamma_k$ be alphabets and consider the language $\text{DPal}_1 \times \cdots \times \Gamma_k.$ As we have just seen the entropy of this language is

$$\eta(\text{DPal}_1 \times \cdots \times \Gamma_k) = \log_2 |\Gamma_1 \times \cdots \times \Gamma_k|$$

$$= \log_2 (|\Gamma_1| \cdot \ldots \cdot |\Gamma_k|)$$

$$= \log_2 |\Gamma_1| + \cdots + \log_2 |\Gamma_k|.$$  

Obviously, there is a deterministic $\varepsilon$-free $k$-stack PDA with stack alphabets $\Gamma_1, \ldots, \Gamma_k$ accepting the language from Example 5.17. Therefore the upper bound given in Theorem 5.15 is tight.

Note that there also is a deterministic $\varepsilon$-free 1-stack PDA accepting the language. Since 2-stack PDA’s with $\varepsilon$-transitions are already Turing complete, using more than two stacks does not increase their computational power. But this does not seem to be the case for deterministic $\varepsilon$-free PDA’s. Therefore we want to give examples for languages $T_k$ for which there exists a $\varepsilon$-free $k$-stack PDA, but no deterministic $\varepsilon$-free $l$-stack PDA for $l < k.$

**Example 5.18.** Define for $k \in \mathbb{N}^+$

$$T_k = \{a^{n_1} \# a^{n_2} \# \ldots \# a^{n_k} i_1 a^{m_1} \ldots i_2 a^{m_2} i_1 a^{m_1} | n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N},$$

$$i_1, \ldots, i_k \in \{0, 1\},$$

$$i_l = 1 \text{ implies } n_l = m_l \text{ for all } l \leq k\}.$$  

The idea is that if $i_l = 1$, then the automaton has to check whether $n_l = m_l$, but if $i_l = 0$, then it can just delete $n_l$. The following automaton accepts if it stops in $c$ or in $d$ with an empty stack. It is a deterministic 1-stack PDA with $\varepsilon$-transitions recognizing $T_k.$
Can we also give an automaton without $\varepsilon$-transitions? One idea to remove the $\varepsilon$-transitions would be to delete the unnecessary stack symbols while just continuing to read the input. But this runs into problems when $n_i > m_i$, because then the automaton is unable to delete $n_i$ before starting to check $n_{i+1}$. There is no obvious way to fix this without increasing the number of stacks, because if we can use $k$ stacks, then the automaton can store each $n_i$ in a different stack. So, intuitively it seems clear that there cannot be a deterministic $\varepsilon$-free $l$-stack PDA recognizing $T_k$ for $l < k$. However, we were not able to prove this statement, hence it remains an open problem to show that there is no deterministic $\varepsilon$-free $l$-stack PDA recognizing $T_k$ for $l < k$.

Following this logic there should not be a $k$ such that there is a deterministic $\varepsilon$-free $k$-stack PDA recognizing the language $T_\infty = \bigcup_{k \in \mathbb{N}^+} T_k$.

In contrast to the claim for $T_k$, we can actually prove this statement by computing the entropy of $T_\infty$, which is infinite. This result is quite surprising because there is a $k$-counter automaton recognizing $T_k$, and thus $\eta(T_k) = 0$ for every $k \in \mathbb{N}^+$.

Note that with witnesses of the form $0^k1a^l0^m$ we can determine the size of a lot of blocks in a word. More precisely, let $n \in \mathbb{N}$ and consider the set

$$\{a^{k_1} a^{k_2} \ldots a^{k_n} \mid k_1, \ldots, k_n \in \{0, \ldots, n-1\}\}.$$ 

If we take $w_1 \neq w_2$ from this set, then there is some $i \in \{1, \ldots, n\}$ such that the $i^{th}$ blocks have different sizes $k$ and $k'$. Then $0^{n-i}1a^k0^{i-1}$ witnesses that $w_1$ and $w_2$ are not in the same congruence class. Note that

$$|0^{n-i}1a^k0^{i-1}| = n - i + 1 + k + i - 1 = n + k \leq 2n.$$

Hence, all these words are in different classes of $\Theta_{2n}(T_\infty)$ and

$$\text{ind} \Theta_{2n}(T_\infty) \geq |\{a^{k_1} # a^{k_2} \ldots # a^{k_n} \mid k_1, \ldots, k_n \in \{0, \ldots, n-1\}\}| = n^n.$$

Thus we can compute the entropy of $T_\infty$

$$\eta(T_\infty) = \limsup_{n \to \infty} \frac{\log_2(\text{ind} \Theta_n(T_\infty))}{n}$$

$$\geq \limsup_{n \to \infty} \frac{\log_2(\text{ind} \Theta_{2n}(T_\infty))}{2n}$$

$$\geq \limsup_{n \to \infty} \frac{\log_2 n^n}{2n}$$

$$= \limsup_{n \to \infty} n \cdot \frac{\log_2 n}{2n}$$

$$= \limsup_{n \to \infty} \frac{\log_2 n}{2}$$

$$= \infty.$$
Therefore it follows from Theorem 5.15 that there is no deterministic $\varepsilon$-free $k$-stack PDA recognizing $T_\infty$.

On the other hand the language from Example 5.18 can be recognized by a nondeterministic 1-stack PDA (by guessing which blocksizes need to be stores), or a 1-stack PDA with $\varepsilon$-transitions (by deleting unnecessary blocksizes). The following corollary connects what we have learned so far about the entropy of PDA’s with the Chomsky hierarchy.

**Corollary 5.19.** All deterministic $\varepsilon$-free context-free languages have finite entropy. But deterministic context-free languages can have infinity entropy.

Note that $\varepsilon$-free nondeterministic context-free languages are the same as nondeterministic context-free languages. However, as we have just proven, $\varepsilon$-free deterministic context-free languages are not the same as deterministic context-free languages. Because of this, for 1-stack PDA’s nonterde rminicity is strictly stronger than $\varepsilon$-freeness. Interestingly, for $k$-stack PDA’s with $k \geq 2$ it is the other way around, as deterministic 2-stack PDA’s are already as powerful as Turing machines.

Recall from the end of the previous section that the entropy of languages accepted by deterministic 1-counter automata with $\varepsilon$-transitions is zero. We summarize all the upper bounds for the entropy we have found so far in Table 5.1. The automata models in the underlined region are Turing complete.

|                         | $\varepsilon$-transitions | $\varepsilon$-free |
|-------------------------|---------------------------|-------------------|
|                         | nondet.       | deterministic | nondet.        | deterministic |
| 1-counter automaton     | ?             | 0             | ?              | 0             |
| 1-stack PDA             | $\infty (T_\infty)$ | $\infty (T_\infty)$ | $\infty (T_\infty)$ | $\log_2 |\Gamma|$ |
| $k$-counter automaton   | $\infty$      | $\infty$     | ?              | 0             |
| $k$-stack PDA           | $\infty$      | $\infty$     | $\infty (T_\infty)$ | $\sum_{i=1}^{k} \log_2 |\Gamma_i|$ |

Table 5.1: Upper bounds for the entropy with example languages.

Our goal in the rest of this section is to determine the unknown values from this table. Using modified versions of the language $T_k$ we shall show that all “?” in the table can be replaced by $\infty$.

**Example 5.20.** Clearly, $T_\infty$ cannot be recognized by a $k$-counter automaton, even if it is nondeterministic, because we cannot store the values for multiple blocks in the same counter as we did with the PDA’s. But note that we only needed witnesses with one 1 to show that $\eta(T_\infty)$ is infinite. To decrease the number of necessary counters we also no longer demand that both sides of the word have
5.2 Push-down Automata

the same number of blocks. Combining these two ideas we define
\[ T'_{k,l} = \{ \#a^{n_1}\#a^{n_2}\# \ldots \#a^{n_k}i_1 a^{m_1}i_2 a^{m_2}i_3 a^{m_3} \mid n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}, \]
\[ i_1, \ldots, i_k \in \{0,1\}, i_1 + \cdots + i_k = 1, \]
\[ i_{r+1} + \cdots + i_k = 0 \]
\[ i_j = 1 \text{ implies } n_j = m_j \text{ for all } j \leq r \}

where \( r = \min\{k,l\} \) and
\[ T'_{\infty} = \bigcup_{k,l \in \mathbb{N}^+} T'_{k,l} \]

We still have \( \eta(T'_{\infty}) = \infty \), but now there is also a nondeterministic 2-counter automaton recognizing \( T'_{\infty} \), see Figure 5.1.

![Figure 5.1: Nondeterministic 2-counter automaton for \( T'_{\infty} \)](image)

In the automaton in Figure 5.1, the states \( c \) and \( p \) accept if both counters are zero. The automaton guesses the block that should be saved and stores the size of the block in the first counter. It uses the second counter to store which block was saved. Therefore the smallest upper bound for the entropy of nondeterministic \( \varepsilon \)-free \( k \)-counter automaton is \( \infty \) for \( k \geq 2 \).

For nondeterministic 1-counter automaton we need to modify this example even further because they cannot store the size of a block and its position at the same time. The new idea is to store only the size and omit the position.

Example 5.21. First we look at the language
\[ B_a = \{ \#a^{n_1}\#a^{n_2}\# \ldots \#a^{n_k} \# \mid k \geq 1, n_1, \ldots, n_k \in \mathbb{N}, m = n_i \text{ for some } i \} \]

A simpler version of the counter automaton from the previous example recognizes \( B_a \).
The state $c$ accepts if the counter is zero.

Next we will compute the entropy of the language $B_a$. Let us consider the set 
\[ \{ w_A \mid A \subseteq \{0, \ldots, k - 1\} \}, \]
where $w_A = \#a_{n_1}^1#a_{n_2}^1\ldots#a_{n_k}^1$ with \( n_1, \ldots, n_k = A \) and \( n_1 < n_2 < \cdots < n_i = \cdots = n_k \). If $A, B \subseteq \{0, \ldots, k - 1\}$ distinct, then there is some \( m \in A \Delta B \) and $a^m$ witnesses that $w_A$ and $w_B$ are not in the same class. Hence \( (w_A, w_B) \notin \Theta_k(B_a) \) and $\ind_k(B_a) \geq |\mathcal{P}(\{0, \ldots, k - 1\})| = 2^k$. Using this we can find a lower bound for the entropy of $B_a$:

\[ \eta(B_a) \geq \limsup_{k \to \infty} \frac{\log_2 2^k}{k} = \log_2 2 = 1. \]

We want to show that there is no finite upper bound for the entropy, thus $B_a$ is not enough. But we can generalize this power-set idea. For larger alphabets $\Sigma = \{a_1, \ldots, a_l\}$ define

\[ B_\Sigma = \{ #a_{n_1}^1#a_{n_2}^1\ldots#a_{n_k}^1 #a_{n_1}^2#a_{n_2}^2\ldots#a_{n_k}^2 \cdots #a_{n_1}^j#a_{n_2}^j\ldots#a_{n_k}^j \mid i \}
\]

\[ k_1, \ldots, k_l, n_1^1, \ldots, n_l^j \in \mathbb{N}, j \in \{1, \ldots, l\}, \]

\[ m = n_i^j \text{ for some } i \in \{1, \ldots, k\} \} \]

There is still a nondeterministic $\varepsilon$-free 1-counter automaton for $B_\Sigma$. It stores $j$ in the state and works otherwise similar to the one for $B_a$. To find a lower bound for the entropy of $B_\Sigma$ we consider the set 
\[ \{ w_{A_1, \ldots, A_l} \mid A_1, \ldots, A_l \subseteq \{0, \ldots, k - 1\} \}, \]
where

\[ w_{A_1, \ldots, A_l} = \#a_{n_1^1}^1#a_{n_2^1}^1\ldots#a_{n_k^1}^1#a_{n_1^2}^2\ldots#a_{n_k^2}^2 \cdots #a_{n_1^j}^j\ldots#a_{n_k^j}^j \]

with \( n_1^j, \ldots, n_k^j = A_j \) and \( n_1^j < n_2^j < \cdots < n_i^j = \cdots = n_k^j \) for all $j \in \{1, \ldots, l\}$. 
If \( (A_1, \ldots, A_l) \neq (B_1, \ldots, B_l) \), then there is some $j \in \{1, \ldots, l\}$ and $m \in A_j \Delta B_j$. As a consequence, $a^m$ witnesses \( (w_{A_1, \ldots, A_l}, w_{B_1, \ldots, B_l}) \notin \Theta_k(B_\Sigma) \). Because of this

\[ \ind_k(B_\Sigma) \geq 2^k \ldots 2^k = 2^{k-l} \]

and

\[ \eta(B_\Sigma) \geq \limsup_{k \to \infty} \frac{\log_2 2^{k/l}}{k} = l \cdot \log_2 2 = l = |\Sigma|. \]

Now we got languages of arbitrarily high entropy, but the entropy depends on the size of the input alphabet. To make it independent on the size of the
input alphabet, we use encodings from Section 4.1. Let \( \text{enc}: \Sigma \cup \{\#, \epsilon\} \rightarrow \Gamma^k \) be an encoding over some binary alphabet \( \Gamma \) with \( k = \lceil \log_2(|\Sigma| + 2) \rceil \). Then by Corollary 4.4 we have

\[
\eta(\text{enc}(B_{\Sigma})) \geq \frac{|\Sigma|}{\log_2(|\Sigma| + 2)}.
\]

Finally, note that \( \text{enc}(B_{\Sigma}) \) can still be recognized by a nondeterministic \( \epsilon \)-free 1-counter automaton. Hence, there are languages that can be recognized by a nondeterministic \( \epsilon \)-free 1-counter automaton with arbitrarily large entropy, even over a two element alphabet.

This shows that \( \infty \) is the smallest upper bound for the entropy of languages recognized by nondeterministic \( \epsilon \)-free 1-counter automata. But it is still an open question whether all these languages have finite entropy or whether there is a language with infinite entropy.

In Table 5.2 we see the updated table with all upper bounds.

|                  | \( \epsilon \)-transitions | \( \epsilon \)-free |
|------------------|-----------------------------|---------------------|
| \( 1 \)-counter automaton | \( \infty \) (\( B_{\Sigma} \)) | \( \infty \) (\( B_{\Sigma} \)) | 0 |
| \( 1 \)-stack PDA    | \( \infty \) (\( T_\infty \)) | \( \infty \) (\( T_\infty \)) | \( \log_2 |\Gamma| \) |
| \( k \)-counter automaton | \( \infty \) | \( \infty \) | \( \infty \) (\( T_\infty \)) | 0 |
| \( k \)-stack PDA    | \( \infty \) | \( \infty \) | \( \infty \) (\( T_\infty \)) | \( \sum_{i=1}^k \log_2 |\Gamma_i| \) |

Table 5.2: Upper bounds for the entropy with example languages.

For all these upper bounds, except for the ones for nondeterministic 1-counter automata, we have seen languages that reach these bounds.
6 Entropy of Decision Problems

We have just seen a connection between topological entropy and the Chomsky hierarchy. In this chapter we will connect topological entropy with decision problems. First we will compute the entropy of some \textbf{NP}-complete problems. Then we will use padding to show that the entropy of any language can be reduced to zero. In particular this shows that there are undecidable languages with zero entropy.

Let us take a look at the problems \textbf{SAT}, \textbf{3COLORING}, and \textbf{CLIQUE}. To be able to compute the entropy of these languages, we need to define a suitable encoding. We encode \textbf{SAT} over the alphabet \{0,1,\land,\lor,\neg\}. To encode a formula \(\phi\) we replace every variable \(x_i\) by \(\text{bin}(i)\), the binary representation of \(i\). We denote the encoded formula by \(\langle \phi \rangle\). For example \(\langle x_1 \land x_2 \rangle = 1 \lor 10\).

Let \(G = (V,E)\) be an undirected graph with vertices \(V = \{v_1,\ldots,v_n\}\) and edges \(E = \{\{u_1,u'_1\},\ldots,\{u_k,u'_k\}\}\) and \(f: V \rightarrow \{1,\ldots,n\}\) a bijection. Then \(G\) is encoded as

\[
\text{bin}(f(v_1)),\ldots,\text{bin}(f(v_n)),(\text{bin}(f(u_1)),\text{bin}(f(u'_1))),\ldots,(\text{bin}(f(u_k)),\text{bin}(f(u'_k)))
\]

and

\[
\text{3COLORING} = \{\langle G \rangle \mid G \text{ is 3 colorable}\}.
\]

Similarly we define \textbf{CLIQUE} as

\[
\text{CLIQUE} = \{\langle G \rangle \mid G \text{ contains a clique of size } k\}.
\]

\textbf{Lemma 6.1.} The languages \textbf{SAT}, \textbf{3COLORING}, and \textbf{CLIQUE} have infinite entropy.

\textbf{Proof.} For \textbf{SAT} consider the set

\[
\{\langle L_1 \land \cdots \land L_{2^n} \rangle \mid L_1 \in \{x_1,\neg x_1\},\ldots,L_{2^n} \in \{x_{2^n},\neg x_{2^n}\}\}.
\]

Take two words \(w_1 = \langle \varphi_1 \rangle\) and \(w_2 = \langle \varphi_2 \rangle\) from this set. If \(w_1 \neq w_2\), then there is some \(k \in \{1,\ldots,2^n\}\) such that the \(k\)-th literal of \(\varphi_1\) and \(\varphi_2\) differ. Without loss of generality assume that the \(k\)-th literal of \(\varphi_1\) is \(x_k\) and the \(k\)-th literal of \(\varphi_2\) is \(\neg x_k\). Then \(\varphi_1 \land x_k\) is satisfiable and \(\varphi_2 \land x_k\) is not. Note that \(|\land \text{bin}(k)| \leq 1 + \log_2 2^n = 1 + n\).
Therefore $\wedge \text{bin}(k)$ witnesses $(w_1, w_2) \notin \Theta_{n+1}(\text{SAT})$. Since the set contains $2^2^n$ words we can now show that infinity is a lower bound for the entropy of $\text{SAT}$

$$\eta(\text{SAT}) \geq \limsup_{n\to\infty} \frac{\log_2(2^{2^n})}{n+1}$$

$$= \limsup_{n\to\infty} \frac{2^n}{n+1}$$

$$= \infty.$$ 

Next we will show that $\text{2COLORING}$ has infinite entropy. We can give an analogous proof for $\text{3COLORING}$. We define the set

$$\{ \langle G^k_n \rangle \mid k \in \{1, \ldots, 2^{2^n} \} \},$$

where

$$G^k_n = (\{0, \ldots, 2^n + 1\}, \{\{i, i+1\} \mid i \in \{1, \ldots, 2^n\}, \text{the } i\text{th symbol of bin}(k) \text{ is } 1\}).$$

As example $G^3_{185}$ is

```
  0 1 2 3 4 5 6 7 8 9
```

the edges coincide with the binary representation of 185. Note that all these graphs are 2 colorable. Take two graphs $G^k_n$ and $G^l_n$ with $l \neq k$ Then without loss of generality there is some $i$ such that $\{i, i+1\}$ is an edge in $G^k_n$ but not in $G^l_n$. If we add the edges $\{0, i\}$ and $\{0, i+1\}$ to $G^k_n$, then the resulting graph is no longer 2 colorable. But if we do the same with $G^l_n$ we end up with a 2 colorable graph. As a consequence, the word $(0, \text{bin}(i)), (0, \text{bin}(i+1))$ witnesses $(\langle G^k_n \rangle, \langle G^l_n \rangle) \notin \Theta_{2\cdot n+10}(\text{2COLORING})$. We conclude

$$\eta(\text{2COLORING}) \geq \limsup_{n\to\infty} \frac{\log_2(2^{2^n})}{2 \cdot n + 10}$$

$$= \limsup_{n\to\infty} \frac{2^n}{2 \cdot n + 10}$$

$$= \infty.$$ 

For $\text{3COLORING}$ we can use the same idea to determine $\eta(\text{3COLORING}) = \infty$. To show that the entropy of $\text{CLIQUE}$ is infinite consider the set

$$\{ \text{bin}(3), \langle G^k_n \rangle \mid k \in \{1, \ldots, 2^{2^n} \} \}.$$ 

Using the same witnesses as before we can deduce that all words are in different classes of $\Theta_{2\cdot n+10}(\text{CLIQUE})$. As a consequence, $\eta(\text{CLIQUE}) = \infty$. 

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6 Entropy of Decision Problems
Note that 2COLORING is in \( P \) and the same construction we used for SAT could also be used for 2SAT. Hence there does not seem to be an obvious distinction between \( \text{NP} \)-complete problems and problems in \( P \) from the point of view of topological entropy.

Next we will discuss the effect padding has on the complexity of a language. In complexity theory padding can be used to decrease the complexity of a language. What happens for topological entropy? For a language \( L \) over \( \Sigma \) define

\[
\text{PAD}(L) = \{uv \mid u \in L, v \in \Sigma^{2|w|}\}.
\]

Note that if \( L \) is in \( \text{EXPTIME} \), then \( \text{PAD}(L) \) is in \( P \). So \( \text{PAD}(L) \) is much easier than \( L \), and this decrease in complexity is also reflected in the topological entropy of \( \text{PAD}(L) \).

**Lemma 6.2.** Let \( L \) be a language over \( \Sigma \) with \( |\Sigma| \geq 2 \). Then

\[
\eta(\text{PAD}(L)) = 0.
\]

**Proof.** Note that every word in \( \text{PAD}(L) \) has a length of the form \( 2^{2k} + k \). Consider the set \( \{U_w \mid |w| \geq \log_2 n\} \). We will show that every \( U_w \) is either the empty set or \( \Sigma^k \) for some \( k \in \{0, 1, \ldots, n\} \). Let \( U_w \) be from this set. For every \( k \geq \log_2 n \) we have:

\[
(2^{k+1} + k + 1) - (2^k + k) = 2 \cdot 2^k - 2^k + 1 = 2^k + 1 \geq n + 1.
\]

Hence the lengths of words in \( \text{PAD}(L) \) are so far apart that \( U_w \subseteq \Sigma^k \) for some \( k \in \{0, 1, \ldots, n\} \). If \( U_w \) is not empty, then there is some \( v \in U_w \) and \( vw \in \text{PAD}(L) \).

We know that \( vw \) is of the form \( u'v' \) for some \( u' \in L \) and \( v' \in \Sigma^* \) with \( |v'| = 2|u'| \).

Next we show that \( |u'| \leq |w| \). Assuming that it is not, we have

\[
|u'| > |w| \geq \log_2 n
\]

and

\[
k = |v| > |v'| = 2|u'| > 2^{\log_2 n} = n,
\]

a contradiction, since \( k \leq n \). Hence, \( |u'| \leq |w| \) and \( v \) is a postfix of \( v' \).

By definition \( u'v'' \in \text{PAD}(L) \) for all \( v'' \) with \( |v''| = 2|u'| \). As a consequence, \( wv'' \in \text{PAD}(L) \) for all \( v'' \in \Sigma^k \) and therefore \( U_w = \Sigma^k \). Hence we can bound the number of classes in \( \Theta_n(\text{PAD}(L)) \) by

\[
\text{ind } \Theta_n(\text{PAD}(L)) \leq |\{U_w \mid |w| < \log_2 n\}| + |\{U_w \mid |w| \geq \log_2 n\}|
\]

\[
\leq |\Sigma|^{\log_2 n} + n + 2
\]

\[
\leq |\Sigma|^{|\log_2 n| + 1} + n + 2. \quad (\text{Lemma 8.3})
\]
Now we can determine the entropy

\[ \eta(PAD(L)) \leq \limsup_{n \to \infty} \frac{\log_2 (|\Sigma|^{\log_2 n + 1} + n + 2)}{n} \]

\[ \leq \limsup_{n \to \infty} \frac{(\log_2 n + 1) \cdot \log_2 |\Sigma| + \log_2 n + \log_2 2}{n} \]

\[ = 0. \]

This finishes the proof.

Note that this works for any language \( L \), even if it is undecidable, and since \( L \) can be reconstructed from \( PAD(L) \) we know that \( PAD(L) \) is also undecidable.

**Corollary 6.3.** There are undecidable languages with zero entropy.

This is a rather strange result, because the barrier of undecidability cannot be breached in classical complexity theory. Undecidable languages are always complicated.
7 Conclusion

In this thesis we introduced the notion of a topological automaton from Steinberg [6]. We defined topological entropy of topological automata and used this to define the topological entropy of a formal language as the entropy of the minimal topological automaton accepting it. We summarized a result from Schneider and Borchmann, who showed in [5] that the topological entropy of a language can be computed using an approximation of the Myhill-Nerode congruence relation.

We further investigated the notion of topological entropy of formal languages and its suitability as a measure of the complexity of formal languages. Previously, there were only three examples of languages with nonzero entropy known, the Dyck languages, the palindrome language, and $\text{Inf}_{\varphi_n}$. Schneider and Borchmann already gave bounds for the entropy of the Dyck languages and the palindrome language [5]. We were able to improve upon this result and calculate the entropy of the Dyck languages. The exact entropy of the palindrome language remains an open problem, but we determined the entropy of the deterministic version of the palindrome language and provided many other new examples.

We modified an example from [5] to show that the entropy function

$$\eta: \mathcal{P}(\Sigma^*) \to [0, \infty]$$

is surjective for every $\Sigma$ with $|\Sigma| \geq 2$. We showed that 1 is an upper bound for the entropy of unary languages. Naturally, the question arises whether there the entropy of a unary language can have every value in $[0, 1]$. We believe this to be the case but were not able to prove it.

Our second main result concerns a conjecture from Schneider and Borchmann [5]. They suspected that all languages accepted by a one-way finite automaton equipped with a fixed number of counters and an acceptance condition that does only require to check local conditions have zero entropy. We showed that this conjecture holds if we assume the automaton to be deterministic and $\epsilon$-free and we were even able to generalize this result to deterministic $\epsilon$-free push-down automata. We showed that the entropy of a language accepted by such an automaton is bounded in terms of the sizes of the stack alphabets of the automaton. This result proved that all deterministic $\epsilon$-free context-free languages have finite entropy.

On the other hand, we also saw that the definition of entropy is not very robust, since we can use padding to decrease the entropy of any language to zero.
7 Conclusion

Consequently, there are also undecidable languages with zero entropy. It is also counterintuitive that the entropy of a language is not the same as the entropy of the reversed language. Hence we suggest to define something like the entropy of the core of a language with the following properties:

- the entropy of the core of a language is at least as large as the entropy of the language,
- padding a language does not influence the entropy of the core of this language,
- reversing a language does not influence the entropy of the core of the language,
- the entropy of the core of the languages we used to show surjectivity should be infinite, and
- encoding the language should not change the entropy of the core of the language.

We propose to define the entropy of the core of a language $L$ in the following way:

$$\eta_{\text{core}}(L) = \sup \{ \eta(L') \mid L' \in \text{core}(L) \},$$

where $\text{core}(L)$ should contain at least $L$, $L^R = \{ w^R \mid w \in L \}$, and every $L'$ such that there is an encoding $\text{enc}$ with $\text{enc}(L') = L$. But a suitable definition of $\text{core}(L)$ remains to be found.
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8 Appendix

Here we will give proofs for a few simple statements used in this thesis.

**Lemma 8.1.** The number of monotone increasing sequences over \( \{1, \ldots, m\} \) with length \( n \) is
\[
\sum_{i=1}^{m} \binom{m}{i} \cdot \binom{n-1}{i-1}.
\]

**Proof.** First we determine how many monotone sequences of length \( n \) that use exactly the numbers \( \{1, \ldots, m\} \) there are. Note that if \( m = 2 \), then there are \( n - 1 \) sequences because there are \( n - 1 \) positions at which the sequence can change from 1 to 2. For \( m = 3 \) there are \( \binom{n-1}{2} \) sequences. Therefore, in general there are \( \binom{n-1}{m-1} \) sequences of length \( n \) that use exactly the numbers \( \{1, \ldots, m\} \). The lemma now follows from the fact that in an increasing sequence over \( \{1, \ldots, m\} \) with length \( n \) there occurs at least one value from \( \{1, \ldots, m\} \) and at most \( m \).

**Lemma 8.2.** Let \( (a_n)_{n \in \mathbb{N}} \) be a monotone increasing series. Then
\[
\limsup_{n \to \infty} \frac{a_n}{n} = \limsup_{n \to \infty} \frac{a_{kn}}{k \cdot n}
\]
for all \( k \in \mathbb{N}^+ \).

**Proof.** Clearly, \( \{k \cdot i \mid i \geq n\} \subseteq \{i \mid i \geq n\} \) for all \( n \in \mathbb{N} \). Therefore
\[
\limsup_{n \to \infty} \frac{a_n}{n} \leq \limsup_{n \to \infty} \frac{a_{kn}}{k \cdot n}.
\]
The other direction is more cumbersome. Firstly, there is a subsequence such that
\[
\lim_{m \to \infty} \frac{a_{n_m}}{n_m} = \limsup_{n \to \infty} \frac{a_n}{n}.
\]
Decompose \( n_m \) into \( k \cdot n_m' + l_m \) such that \( l_m < k \).
\[
\lim_{m \to \infty} \frac{a_{n_m}}{n_m} = \lim_{m \to \infty} \frac{a_{kn_m' + l_m}}{k \cdot n_m' + l_m} \cdot \frac{k \cdot n_m' + k}{k \cdot n_m' + l_m}
\]
\[
= \lim_{m \to \infty} \frac{a_{kn_m' + l_m}}{k \cdot n_m' + l_m} \cdot \lim_{m \to \infty} \frac{k \cdot n_m' + k}{k \cdot n_m' + l_m}
\]
\[
= \lim_{m \to \infty} \frac{a_{kn_m' + l_m}}{k \cdot n_m' + k} \cdot 1 \quad (l_m \leq k)
\]
\[
\leq \lim_{m \to \infty} \frac{a_{k \cdot n_m' + k}}{k \cdot n_m' + k} \quad (a_n \text{ monotone increasing})
\]
Hence
\[ \limsup_{n \to \infty} \frac{a_n}{n} \leq \limsup_{n \to \infty} \frac{a_{k \cdot n}}{k \cdot n}. \]
This finishes the proof. \( \square \)

**Lemma 8.3.** For all \( n \in \mathbb{N} \) and \( v \in \mathbb{R} \) with \( v \neq 1 \):

\[ \sum_{i=0}^{n} v^i = \frac{v^{n+1} - 1}{v - 1}. \]

**Proof.** Using telescope sums we obtain
\[
\sum_{i=0}^{n} v^i = \frac{1}{v - 1} \sum_{i=0}^{n} (v - 1) \cdot v^i \\
= \frac{1}{v - 1} \sum_{i=0}^{n} (v^{i+1} - v^i) \\
= \frac{v^{n+1} - 1}{v - 1}.
\]
This finishes the proof. \( \square \)

**Lemma 8.4.** Let \( X \) be a topological space and \((U_i)_{i \in I}\) be a family from \( \mathcal{C}(X) \). Then for every continuous \( f : X \to X \):

\[ f^{-1}(\bigvee_{i \in I} U_i) = \bigvee_{i \in I} f^{-1}(U_i). \]

**Proof.** The proof is just a straightforward application of the definitions:
\[
f^{-1}(\bigvee_{i \in I} U_i) = f^{-1}(\{ \bigcap_{i \in I} U_i \mid (U_i)_{i \in I} \in \prod_{i \in I} U_i \}) \\
= \{ f^{-1}(\bigcap_{i \in I} U_i) \mid (U_i)_{i \in I} \in \prod_{i \in I} U_i \} \\
= \{ \bigcap_{i \in I} f^{-1}(U_i) \mid (U_i)_{i \in I} \in \prod_{i \in I} U_i \} \\
= \{ \bigcap_{i \in I} U_i \mid (U_i)_{i \in I} \in \prod_{i \in I} f^{-1}(U_i) \} \\
= \bigvee_{i \in I} f^{-1}(U_i).
\]
This finishes the proof. \( \square \)
ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema “Topological Entropy of Formal Languages” selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Datum  Unterschrift