On the equivalence of the Impulse Approximation and the Gersch-Rodriguez-Smith series for structure functions

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Abstract

We derive for a non-relativistic system an approximation for Final State Interactions in a form, resembling a DWIA which corrects the structure function computed in the PWIA. We then compare the Gersch-Rodriguez-Smith and the IA series for structure functions and prove that to order $O(1/q^2)$ the above DWIA representation is contained in the GRS series to the same order. There is an additional term in the GRS series that is missing in the DWIA due to the eikonal approximations in the latter. This strongly suggests that the two approaches, when treated exactly, produce identical structure function to arbitrary order in $1/q$.

I. INTRODUCTION

Virtually all computations of structure functions of nuclei, as measured by inclusive scattering of high-energy electrons, use relativistic generalizations of either the non-relativistic (NR), perturbative impulse approximation (IA) series\textsuperscript{1,2}, or of a non-perturbative theory, formulated by Gersch, Rodriguez and Smith (GRS)\textsuperscript{4}. 
We start with the latter. It produces an expansion of the response in inverse powers of the momentum transfer $q$ with coefficient functions, depending on many-body density-matrices which are diagonal in all, but one coordinate. Terms in that series are the asymptotic limit for $q \to \infty$ and a series of Final State Interaction (FSI) terms for decreasing, finite $q$.

The Impulse Approximation (IA) series is one in the interaction $\bar{V}(r_1) = \sum_{k \geq 2} V(r_1 - r_k)$ between the struck nucleon and the core. To lowest order, i.e. in the Plane Wave Impulse Approximation (PWIA), one neglects $\bar{V}(r_1)$. FSI interactions for the IA series are thus perturbatively calculated contributions of increasing order in the initially neglected interaction.

The formal GRS and the IA series appear very dissimilar, yet those provide two representations of the same structure function. Consequently an exact treatment of each ought to produce identical results. A frequently raised question is then, which approach is better when treated approximately. To our knowledge not even a criterion, to be followed in principle, has been formulated in the past. The main purpose of the present note is just such a formulation, followed by a proof of equivalence.

The above quest is encumbered by the fact that we do not know of an exact, manageable evaluation of FSI in the IA series, as exists for the GRS theory. A pre-requisite for a comparison is therefore a realistic model for FSI, replacing the IA series.

As to the nature of such a model one is guided by the fact that the relative weight of FSI in the response diminishes with increasing $q$. It is therefore natural to consider kinematic conditions, generally reached for scattering with high beam energies. Those are by necessity accompanied by effects due to relativity, particle production and the like, whose treatment can never be exact. As a starting point we suggest a well-defined non-relativistic (NR) model, based on a hamiltonian for point-particles, i.e. for particles which cannot be excited. The model can be treated exactly and provides insight which later can be incorporated in realistic situations.

We start in Section II with the GRS theory, recapitulate some formally exact expressions for the lowest order terms of the GRS series and cite results for partial summations of
selected higher order terms. In Section III we consider the response of a semi-inclusive (SI) \( A(e,e'p)X_{A-1} \) reaction in the PWIA, which features the one-hole spectral function. We then suggest a realistic form for FSI which in nuclear parlance is called, the Distorted Wave Impulse Approximation (DWIA). Integrating the SI response over the momenta of the outgoing nucleon, produces for that model the totally inclusive (TI) cross section. We then demonstrate in Section IV that the GRS to \( \mathcal{O}(1/q^2) \) contains the DWIA terms to the same order and attribute the absence of an extra term to the approximate nature of the chosen DWIA. We conclude that Section by comparing ours with work of similar scope. In Section V we briefly discuss the embedding of the above in a relativistic theory.

II. THE GRS SERIES AND SOME RESUMMATIONS.

We consider the structure function for, or the response \( S(q,\omega) \) of, a NR many-body system to a scalar perturbation, defined as the ratio of the inclusive scattering by a projectile and the elementary projectile-constituent cross section. The kinematic variables \((q,\omega)\) are the momentum and energy, transferred by the projectile to the target. A form for the response per particle reads

\[
S(q,\omega) = (2\pi A)^{-1}\langle \Phi^0_A | \rho_q \delta(\omega - H_A) \rho_q | \Phi^0_A \rangle, \tag{1}
\]

where \( H_A, \Phi^0_A, E^0_A \) are the exact \( A \)-body hamiltonian, its ground state wave function and the corresponding energy.

For large \( q \) it is useful to introduce the reduced response \( \phi(q,y) = (q/M)S(q,\omega) \), with \( M \) the mass of a particle and \( y \) a kinematic variable, replacing the energy loss \( \omega \).

\[
y = \frac{M}{q} \left( \omega - \frac{q^2}{2M} \right), \tag{2}
\]

Substitution of \( \rho_q = \sum_j e^{iq \cdot r_j} \) into (1) produces incoherent and coherent components. When considering high-\( q \) responses, it suffices to consider the dominant incoherent part, where a single particle is tracked in its propagation through the medium. We cite Ref. for a derivation of the GRS series.
\[
\phi(q, y) = \sum_{n \geq 0} (1/v_q)^n F_n(y),
\]

where \(v_q = q/M\) is the recoil velocity, corresponding to a momentum transfer \(q\). The coefficient functions \(F_n(y)\) are functionals of the inter-particle interaction \(V\) and density matrices \(\rho_n(1'; j, 1)\), \(j \geq 2\). Those are diagonal in all coordinates \(j = r_j\), except that of the struck nucleon, which is chosen to be '1'. All derive from \(\rho_A(1', k; 1, k)\), \(A \geq k \geq 2\) and satisfy in our convention the relations

\[
\rho_n(1', 2...n; 1, 2...n) = \frac{1}{(A - n)!} \left( \prod_{j=n+1}^{A} \int d\eta \right) \rho_A(1'j; 1j) \\
\rho_A(1'k; 1k) = A! \Phi_A^*(1', k) \Phi_A(1, k)
\]

The appearance of exact many-body densities shows that from the onset the theory accounts for correlations of the target nucleons.

For our purposes it suffices to mention \(\rho_n\) for \(n = 1, 2, 3\) which enter expressions for the asymptotic limit \(F_0\) and the two dominant FSI corrections \(F_1, F_2\)

\[
F_0(y) = \frac{1}{A!} \int \frac{ds}{2\pi} e^{isy} \int d1 \left( \Pi_{k \geq 2} A \int dk \right) \rho_A(1 - s, k; 1, k) \\
= \frac{1}{A} \int \frac{ds}{2\pi} e^{isy} \int d1 \rho_1(1 - s; 1) = \frac{1}{4\pi^2} \int_{|y|}^{\infty} dp n(p)
\]

\[
\frac{1}{v_q} F_1(y) = \frac{i}{A!} \int \frac{ds}{2\pi} e^{isy} \int d1 \left[ \Pi_{k \geq 2} A \int dk \right] \rho_A(1 - s, k; 1k) \sum_{k \geq 2} \tilde{\chi}_q(1 - k, s) \\
= \frac{i}{A} \int \frac{ds}{2\pi} e^{isy} \int d1 d2 \rho_2(1 - s, 2; 12) \tilde{\chi}_q(1 - 2, s)
\]

\[
\frac{1}{v_q^2} F_2(y) = -\frac{1}{2A!} \int \frac{ds}{2\pi} e^{isy} \int d1 \left[ \Pi_{k \geq 2} A \int dk \right] \rho_A(1 - s, k; 1k) \left[ \int_0^s d\sigma \sum_{k \geq 2} \tilde{\chi}_q(1 - k, s) \right]^2 + \frac{1}{v_q^2} F_2^{(r)}(y)
\]

\[
\frac{1}{v_q^2} F_2^{(r)}(y) = -\frac{1}{A!} \int \frac{ds}{2\pi} e^{isy} \int d1 \left[ \Pi_{k \geq 2} A \int dk \right] \rho_A(1 - s, k; 1k) \left[ \frac{1}{2} \frac{\partial^2}{\partial s^2} \left( \sum_{k \geq 2} \int_0^s d\sigma \tilde{\chi}_q(1 - k, \sigma) \right)^2 \\
- \left( \sum_{k \geq 2} \tilde{\chi}_q(1 - k, s) \right)^2 \right]
\]

with \(n(p)\) the single-particle momentum distribution. The expression for \(F_2^{(r)}\) is easily derived from Eq. (14) in Ref. [4]. Above we also introduced

\[
\tilde{\chi}_q(1, s) = \tilde{\chi}_q^{(1)}(1, s) + \tilde{\chi}_q^{(2)}(1, s) \\
= -(1/v_q) \int_0^s d\sigma [V(1 - \sigma) - V(1)],
\]
where we write symbolically

\[ V = V^{(1)} + V^{(2)} \]

\[ V^{(1)} \rightarrow V^{(1)}_\sigma = V(1 - \sigma) \]

\[ V^{(2)} \rightarrow -V(1) \]  \hfill (7)

Eq. (\ref{eq:coordinate_representation}) defines the coordinate representation of the off-shell eikonal phase \( \tilde{\chi}(1, s) \) corresponding to the total \( V \) and its components \( \tilde{\chi}^{(a)}(1, s) ; a = 1, 2 \) which are characteristic of the GRS theory or of path integral methods.\cite{footnote}

It is frequently useful to make resummations within the GRS series. We consider first a ladder sum of repeated interactions \( V \) which results in the replacement \( V \rightarrow t = V_{\text{eff}} \). This replacement is mandatory if the bare interaction \( V \) is singular. The corresponding change in the phase \( \tilde{\chi} \) is

\[ i\tilde{\chi} \rightarrow \tilde{\Gamma} = e^{i\tilde{\chi}} - 1, \]  \hfill (8)

with \( \tilde{\Gamma} \), the total off-shell profile function.

Next we consider a cumulant resummation which to lowest order reads

\[ \phi(q, y) = \frac{1}{A} \int \frac{ds}{2\pi} e^{isy} \int d\rho_1(1 - s; 1) \exp \left[ \frac{\int d^2\rho_2(1 - s, 1, 2)\tilde{\Gamma}_q(1 - 2, s)}{\rho_1(1 - s; 1)} + \ldots \right] \]  \hfill (9)

When expanded, it reproduces the lowest order terms in the GRS series, as well as selected higher order contributions.\cite{footnote}

III. FSI CORRECTIONS TO THE PWIA RESPONSE.

By way of introduction we consider first SI scattering. The corresponding response per nucleon is

\[ S^{SI}(q, \omega; p) = \frac{1}{A} \sum_m \left| \langle \Phi^0_A | \rho^\dagger_q | \Psi^{(-)}_{(A-1)m} ; p+q \rangle \right|^2 \delta \left( \omega - \Delta_m - \frac{(p + q)^2}{2M} \right), \]  \hfill (10)

where \( p \) is the momentum of the struck, and \( p + q \) that of the detected outgoing nucleon after absorbing the momentum transfer \( q \). \( \Psi^{(-)}_{(A-1)m} ; p+q \) is the state of that nucleon, scattered
from a nucleus of $A - 1$ particles in state $m$. $\Delta_m$ is the separation energy of a nucleon in the ground state $A$-body system, with the daughter nucleus in the state $\Phi_{A - 1}^m$. We write the total Hamiltonian as

$$H_A(1; k) = H_{A - 1}(k) + T(1) + \bar{V}(1)$$

(11)

with $\bar{V}(1) = \sum_{k \geq 2} V(1 - k)$, the interaction of particle 1 with the core. Neglect of the latter defines the PWIA

$$\left[\Psi_{(A - 1)_m; p + q}^-(r_1; r_k)\right]^{PWIA} \rightarrow \Phi_{A - 1}^m(r_k) e^{-i(p + q).r_1}$$

(12)

When substituted into Eq. (10), it produces the standard PWIA approximation for the SI response

$$S_{SI; PWIA}^{(q, \omega; p)} = \int dE P(p, E) \delta\left(E - \omega - \frac{(p + q)^2}{2M}\right)$$

(13a)

$$\phi_{SI; PWIA}^{(q, y_0; p)} \approx \delta(y_0 - p_z)n(p)$$

(13b)

$$n(p) = \int dE P(p, E)$$

(13c)

Here $P(p, E)$ is the single-hole spectral function, dependent on the separation-energies of each of the daughter states $m$. Eq. (13b) results from the assumption that those separation energies may be replaced by an average $\Delta_m \rightarrow \langle \Delta \rangle$. One may then replace the energy loss $\omega$ by the IA scaling variable, also in terms of $\langle \Delta \rangle$

$$y_0 = -q + \sqrt{2M(\omega - \langle \Delta \rangle)}$$

(14)

FSI corrections to the PWBA result (13b) are by definition, contributions due to the residual interaction $V$, treated perturbatively. With no practical way to do so systematically, we proceed in an approximative manner and assume that the outgoing nucleon scatters from an initially frozen configuration of $k$ nucleons

$$\Psi_{(A - 1)_m; p + q}^-(1; k) \approx \Phi_{A - 1}^m(k) \psi^-(1; \langle k \rangle),$$

(15)

Such an approximation is justified if, with respect to the Fermi momentum $p_F$
\[ p \approx p_F \]
\[ |p + q| \approx q \gg p_F, \]

One notes that, contrary to the perturbative nature of the actual IA series, the approximation (13) for it is non-perturbative.

Substituting (15) into (10) and replacing again state-dependent separation energies by an average, one performs closure over states of the daughter nucleus and obtains

\[
\phi_{SI}^0(q, y_0; p) \approx \delta(y_0 - p_z) \left\langle \Phi_A^0(1'; k) | e^{-i q \cdot r_1} | \psi_{p+q}(1'; k) \right\rangle \left\langle \psi_{p+q}(-) (1'; k) | e^{i q \cdot r_1} | \Phi_A^0(1; k) \right\rangle^* \]

\[ \approx \delta(y_0 - p_z) \left\langle \Phi_A^0(1'; k) | e^{-i p \cdot r_1} | \xi_{p+q}(1'; k) \right\rangle \left\langle \xi_{p+q}(-) (1; k) | e^{i p \cdot r_1} | \Phi_A^0(1; k) \right\rangle^* \quad (16) \]

In Eq. (16) we used the standard eikonal expression for the state describing scattering from a static field with fixed coordinates \( \langle k \rangle \)

\[
\psi_{\pm}^\pm(1; \langle k \rangle) = e^{i \kappa \zeta} \xi_{\pm}^\pm(1; \langle k \rangle) \]

(17)
due to the static non-central field \( \sum_{k \geq 2} V(1 - \langle k \rangle) \). The distortion function \( \xi \) in the approximation \( |p + q| \approx q \) reads\( \xi \)

\[
\xi_{q}^(-)(1; \langle k \rangle) = \exp \left[ - \frac{i}{v_q} \sum_k \int_{z_1}^{\infty} d\zeta V(1 - \langle k \rangle - \zeta) \right] \]

(18)

After substitution into (10) one restores the core coordinates to their dynamical status and obtains for real \( V \) the following expression for the SI response in the DWIA

\[
\phi_{SI,DWIA}^0(q, y_0; p) = \frac{1}{A!} \delta(y_0 - p_z) \int ds e^{i p \cdot s} \int d1 \Pi_{k \geq 2} dk \rho_A(1 - s, k; 1, k) \exp \left[ - \frac{i}{v_q} \sum_k \int_{z_1}^{\infty} d\zeta V(1 - k - \zeta) \right] \]

(19)

At this stage one exploits the absence of degrees of freedom, others than point-particles. The TI response is then obtained by integrating the SI response over missing momenta \( p \).

As a result \( s = r_1 - r_1' = s \hat{q} \) lies in the direction of \( q \) and one finds

\[
\phi_{TI,DWIA}^0(q, y_0) = \frac{1}{A!} \int \frac{ds}{2\pi} e^{i y_0 s} \int d1 \Pi_{k \geq 2} dk \rho_A(1 - s, k; 1, k) \exp[i \sum_{k \geq 2} \chi_q^0(1 - k, s)] \]

(20)
The above result has still the full complexity of a many-body problem, present in the $A$-body density matrix. That complexity is considerably reduced in an independent-pair (Kirkwood) approximation

$$
\rho_A(1k; 1'k) \approx \frac{(A - 1)!}{(A - 1)^{A-1}} \Pi_{k \geq 2} \rho_2(1k; 1'k),
$$

which respects the sum rules \( \mathbb{3} \). Substitution in \( \mathbb{20} \) produces for the reduced TI response per nucleon in the DWIA

$$
\phi_{TI,DWIA}(q, y_0) \approx \frac{1}{A(A - 1)^{A-1}} \int \frac{ds}{2\pi} e^{iy_0s} \int \frac{d1}{[\rho_1(1-s; 1)]^{A-2}} \left[ \Pi_{k \geq 2} \int dk \rho_A(1-s,k; 1,k) \tilde{\Gamma}^{(1)}(1-k,s) \right]
$$

$$
\approx \frac{1}{A} \int \int \frac{ds}{2\pi} e^{iy_0s} d1 \rho_1(1-s; 1) \exp \left[ \int \frac{d2 \rho_2(1-s, 2; 1, 2) \tilde{\Gamma}^{(1)}(1-2,s)}{\rho_1(1-s; 1)} \right].
$$

(22a)

$$
\approx \frac{1}{A} \int dy_0 F_0(y_0 - y'_0) \mathcal{R}_q(y_0)
$$

(22b)

$$
\mathcal{R}_q(y_0) \approx \int \frac{ds}{2\pi} e^{iy_0s} \int d1 \exp \left[ \int \frac{d2 \rho_2(1-s, 2; 1, 2) \tilde{\Gamma}^{(1)}(1-2,s)}{\rho_1(1-s; 1)} \right].
$$

(22c)

For later use we expressed the response \( \mathbb{22a} \) as a convolution of the asymptotic limit and a generalized FSI factor (cf. Eq. \( \mathbb{5a} \), of the last Ref. \( \mathbb{3} \)).

There clearly is a formal similarity in the expressions \( \mathbb{9} \) and \( \mathbb{22a} \) for the TI response in, respectively, the first cumulant expression in the GRS theory, and the approximate IA series. The apparent differences amount to i) the appearance of $y_0$ instead of $y = y_w$ and ii) the presence in the DWIA of a profile function $\tilde{\Gamma}^{(1)}$, related to the first potential in \( \mathbb{6} \) and not to both, as is the case in the GRS theory. In the following Section we shall investigate whether, and to what extent these apparently similar expressions coincide.

**IV. MEASURE OF EQUIVALENCE OF GRS AND APPROXIMATE IA SERIES.**

There are two, in principle equivalent ways to compare the exact IA and GRS series for the response, namely by isolating and counting powers in either the residual interaction $\bar{V}(1)$ or in $1/q$. However, in view of the fact that the IA series is treated approximately, the
exact GRS series becomes the natural standard. Both approaches shall be traced to terms up, and including $\mathcal{O}(1/q^2)$.

We start with the GRS series (3)

$$\phi(q, y; \mathcal{V}) = \sum_{n \geq 0} (1/v_q)^n F_n(y; \mathcal{V})$$

(23)

Using (7) we make explicit the two components of the ‘total’ interaction action in (5)

$$F_0(y) = \frac{1}{4\pi^2} \int_{|y|}^{\infty} dp \rho_0(p)$$

(24a)

$$F_1(y) = -\frac{i}{A!} \int \frac{ds}{2\pi} e^{iys} \int d1 \left[ \prod_{k \geq 2} \int dk \right] \rho_A(1-s, k; 1k)$$

$$\sum_{k \geq 2} \int_0^s d\sigma [V(1-2-\sigma) - V(1-k)]$$

$$= -\frac{1}{A} \int \frac{ds}{2\pi} e^{iys} \int \int d1 d2 \rho_2(1-s, 2; 12) \int_0^s d\sigma [V(1-2-\sigma) - V(1-2)]$$

(24b)

$$F_2(y) = \frac{i^2}{A!} \int \frac{ds}{2\pi} e^{isy} \int d1 \left[ \prod_{k \geq 2} \int dk \right] \rho_A(1-s, k; 1k)$$

$$\frac{1}{2} \left[ \int_0^s d\sigma \sum_{k \geq 2} \int_0^s d\sigma [V(1-k-\sigma) - V(1-k)] \right]^2 + F_2^{(r)}(y)$$

(24c)

Next we get to the approximate DWIA expression (22a), which is of non-perturbative nature but, which using (3) may be formally expanded

$$\phi^{TI,DWIA}(q, y_0; \mathcal{V}^{(1)}) = \frac{1}{A} \int \frac{ds}{2\pi} e^{isy_0} \int d1 \left[ \rho_1(1-s; 1) -$$

$$\frac{i}{v_q} \int d2 \rho_2(1-s, 2; 1, 2) \int_0^s d\sigma V(1-2-\sigma) - \frac{1}{2v_q^2} \int d2 \left( \int_0^s d\sigma \mathcal{V}^{(1)}(1-2-\sigma) \right)^2 -$$

$$\frac{1}{2v_q^2} \int d2 \int d3 \rho_3(1-s, 2, 3; 1, 2, 3) \int_0^s d\sigma V(1-2-\sigma) \int_0^s d\sigma' V(1-3-\sigma') \right] + \mathcal{O}(1/v_q^3)$$

(25)

It is then our quest to investigate whether, and to what extent, the terms (24a) - (24c) of the GRS series contain the DWBI counterparts (25). We do so by the following technique:

i) Separate in Eqs. (24) terms which depend exclusively on $\mathcal{V}^{(1)}$. The former we expect to meet in (25).

ii) Track in the remainder of (24) parts where $\mathcal{V}^{(2)}$ acts on the groundstate in $\rho_A$. Using (4) one has
\[
\left[ \sum_{l \geq 2} V(1 - k) \right] \Phi(1, k) = [H_A - H_{A-1} - T(1)] \Phi(1, k)
\]
\[
\approx - \int \frac{dp}{2\pi^3} e^{ip \cdot r} \left( \langle \Delta \rangle + \frac{p^2}{2M} \right) \Phi(p, k),
\]

where, in line with assumption made above, separation energies are again replaced by an average.

iii) Collect terms, which enable the replacement of the GRS-West scaling variable by the IA one, making use of

\[
y(= y_w) = y_0 + \frac{1}{v_q} \left( \frac{y_0^2}{2M} + \langle \Delta \rangle \right),
\]

We start with

\[
F_1(\mathcal{V}) = F_1^{(1)}(\mathcal{V}^{(1)}) + F_1^{(2)}(\mathcal{V}^{(2)}),
\]

where the superscripts indicate dependence on \(\mathcal{V}^{(1)}, \mathcal{V}^{(2)}\) and following i) we consider the part

\[
\frac{1}{v_q} F_1^{(2)}(y) = -i \frac{\partial}{A! \partial y} \int \frac{ds}{2\pi} e^{isy} \int d[l] \rho_A(1 - s, k; 1, k) V(l)
\]

\[
= \frac{\partial}{\partial y} \int \frac{dp}{2\pi^3} \delta(p - y) \frac{1}{v_q} \left( \langle \Delta \rangle + \frac{p^2}{2M} \right) n(p) = (y - y_0) \frac{dF_0(y)}{dy}
\]

Continuing with \(F_2\) we write (cf. Eqs. (25c), (24d))

\[
F_2(\mathcal{V}) = F_2^{(1)} + F_2^{(2)} + F_2^{(1,2)} + F_2^{(r)},
\]

with \(F_2^{(1,2)}\) containing mixed \(\mathcal{V}^{(1)}, \mathcal{V}^{(2)}\) terms. The reasoning which leads to (29) produces

\[
\frac{1}{v_q^2} F_2^{(2)}(y) = \frac{1}{2A! \partial y^2} \int \frac{ds}{2\pi} e^{isy} \int d[l] \rho_A(1 - s, k; 1, k) V(1 - k) \frac{1}{v_q} \sum_{l \geq 2} V(1 - l)
\]

\[
= \frac{1}{2} \left( \frac{1}{v_q} \left( \langle \Delta \rangle + \frac{y^2}{2M} \right) \right)^2 \frac{d^2 F_0(y)}{dy^2} = \frac{1}{2} (y - y_0)^2 \frac{d^2 F_0(y)}{dy^2}
\]

Finally for the mixed term

\[
\frac{1}{v_q} F_2^{(1,2)}(y) = -\frac{1}{A! \partial y} \int \frac{ds}{2\pi} e^{isy} \int d[l] \rho_A(1 - s, k; 1, k) V(1 - k)
\]

\[
= \frac{1}{v_q} \sum_{l \geq 2} \int_0^s d\sigma V(1 - l - \sigma)]
\]

\[
= \frac{1}{v_q} \left( \frac{y^2}{2M} + \langle \Delta \rangle \right) \frac{dF_1^{(1)}(y)}{dy} = (y - y_0) \frac{dF_1^{(1)}(y)}{dy}
\]
Assembling the last three results and using (27) one finds

\[ \phi(q, y; \mathcal{V}) = F_0(y) + \frac{1}{v_q} F_1(y; \mathcal{V}) + \frac{1}{v_q^2} F_2(y; \mathcal{V}) + \mathcal{O}(1/v_q^3) \]  

(33a)

\[ = \left[ F_0(y) + (y - y_0) \frac{dF_0(y)}{dy} + \frac{1}{2} (y - y_0)^2 \frac{d^2F_0(y)}{dy^2} \right] \]

\[ + \frac{1}{v_q} \left[ F_1^{(1)}(y) + (y - y_0) \frac{dF_1^{(1)}(y)}{dy} \right] + \frac{1}{v_q^2} \left[ F_2^{(1)}(y) + \frac{1}{v_q} F_2^{(r)}(y) \right] + \mathcal{O}(1/v_q^3) \]  

(33b)

\[ = \left[ F_0(y_0) + \frac{1}{v_q} F_1^{(1)}(y_0) + \frac{1}{v_q^2} F_2^{(1)}(y_0) \right] + \frac{1}{v_q^2} F_2^{(r)}(y) + \mathcal{O}(1/v_q^3) \]  

(33c)

\[ = \phi(q, y; \mathcal{V}^{(1)}) + \frac{1}{v_q^2} F_2^{(r)}(y; \mathcal{V}) + \mathcal{O}(1/v_q^3) \]  

(33d)

In Eqs. (33a), (33d) we reinstated for greater clarity the dependence on \( \mathcal{V} \) and its component \( \mathcal{V}^{(1)} \). The above demonstrates that all terms of \( \mathcal{O}(1/v_q^2) \) of the IA series are contained in the GRS series of the same order, which however, has one additional term, not reproduced in the DWIA. This can be traced to the approximation (15). Higher order eikonal terms to the distortion function \( \xi \) in (18) are at least of order \( 1/v_q^2 \) and are expected to account for the above difference to that order.

Eqs. (29), (31) and (32) are truly remarkable in that the \( \mathcal{V}^{(2)} \) dependence of coefficient functions of given order can be expressed in derivatives of functions \( F \) of lower order \( F \), and which are free of that interaction component. Grouped terms ultimately produce the replacement \( y \rightarrow y_0 \) and bring about significant cancellations in the GRS series.

The above completes the equivalence proof of the two expressions of the structure function for any NR many-body system. Were it not for the use of average separation energies, Eq. (33d) would be exact. In a way this approximation is unavoidable, because the appearance of an essentially kinematic IA scaling variable \( y_0 \) as in Eq. (18) requires an average separation energy. An earlier attempt to keep actual separation energies invites other approximations (cf. Eqs. (2.23) e.v. in Ref. 14), but we shall not pursue that extension here.

We conclude this section, by mentioning previous incorporations of FSI interactions for the IA series. In particular Benhar and coworkers advocated a convolution of the PWIA
spectral function and some FSI, specifically in the energy loss variable \( \omega \). Their FSI features the component \( V^{(1)} \) as in (22a), based on our DWIA, and not the full \( V \).

We emphasize that a convolution of the lowest order asymptotic limit and a FSI factor has been proved for the GRS theory and its series, and the appropriate variable is the GRS-West scaling variable \( y \) and not \( \omega \). A presumed generalization, valid for the IA series certainly requires a proof, which to our knowledge has not been provided. Such a proof would select the convolution variable.

Let us put aside the ad hoc convolution and attempt to replace \( \omega \) by a scaling variable. That is possible for any candidate, built from purely kinematic variables as is \( y_w \), Eq. (2). The result is clearly neither (4) nor (22a): The latter manifestly requires the IA variable (27), but should be ruled out, because it assumes the existence of an average separation energy, which counters the emphasis on the exact spectral function with its state-dependent \( \Delta_m \). The latter is just demonstrated by (22a), which may be written as a convolution (22b) in the 'natural' IA variable \( y_0 \).

Finally we mention a, as yet unpublished conference report which also uses the above folding procedure. A discussion should await its publication.

V. SUMMARY AND CONCLUSION.

Our major goal above was a comparison of the structure function of a NR system of point-particles, when computed by means of the GRS and IA series. Whereas for the former there exists a formally exact expression, we do not know of a similar, manageable one for the IA series beyond the PWIA. Any comparison therefore requires first an approximation for FSI corrections to the PWIA. Having described a representative DWIA for the IA response, we could perform the above comparison.

Our demonstration starts with of the GRS series up to and including \( \mathcal{O}(1/v_q^2) \), with coefficients, functions of the GRS-West scaling variable \( y \). We then proved striking cancelations, producing the same lowest order terms from the DWIA expressed in the parallel IA scal-
ing variable $y_0$. One unretrieved term in the GRS series is undoubtedly due to the chosen DWIA, approximating the FSI in the actual IA series. Similar cancelations are expected to occur to any order.

The above success naturally elicits the question of a relativistic extension. There clearly is no hope to derive results with comparable rigor. It is nevertheless of interest to recall here some models where nuclear and nucleon structure functions are related by a generalized convolution

$$F^A = f^{PN} * F_N,$$

with $f^{PN} \propto \phi$. Here $\phi$ is in principle the structure function of a nucleus, composed of point-particles, where inter-nucleon potentials as in (8) are replaced by scattering amplitudes which have also meaning in a relativistic theory. We refer to Ref. 3 where a generalization of the effectively 2-component interaction in the above spirit is discussed.

It is our hope that this paper will lay to rest a long-lingering, and occasionally controversial, issue in the study of responses.

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