Fostering mathematical thinking in the learning of multivariable calculus through computer-based tools

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Abstract

Calculus as a prerequisite course to other advanced mathematics courses is one of the important and difficult courses for undergraduate students in many fields of study. Mathematical thinking is an important method to support students in the learning of calculus and specifically multivariable calculus. Researchers endeavour to support students’ mathematical thinking in calculus with or without computer-based tools. The main goal of this paper is to illustrate the importance of using computer-based tools for fostering students’ mathematical thinking to overcome their obstacles in multivariable calculus.

Keywords: Computer-based Learning, Mathematical Thinking, Multivariable Calculus, Students’ Obstacles

1. Introduction

Calculus, particularly multivariable calculus, is one of the most important parts of mathematics syllabus for undergraduate students. It is offered as a prerequisite course to other advanced mathematics courses and even other courses. However, calculus is one of the most difficult courses for most undergraduate students to study in their field (Tall, 1993; Artigue and Ervynck, 1993; Yudariah and Roselainy, 2001; Willcox and Bounova, 2004; Kashefi, Zaleha, and Yudariah, 2010, 2011a). Various problematic areas have been identified in basic calculus and multivariable calculus. Some of these were, the difficulty of learning some specific mathematical topics, the difficulty in coordinating procedures and manipulating concepts, the particular events that the students experienced in the past, poor problem solving skills, the inability to select and use appropriate mathematical representations, the translation of real-world problems into calculus formulations, absorbing complex new ideas in a limited time, the students’ beliefs and their learning styles (Kashefi, Zaleha, and Yudariah, 2011b).

There have been several attempts to improve students’ learning and the teaching of mathematics in basic calculus and multivariable calculus through moving away from remedial classes towards teaching to increase understanding. Improving students’ learning through the enhancement of their problem solving and mathematical thinking skills as well as through using technological tools to support conceptual understanding and problem solving methods are now...
thought to be more appropriate. It enables them to cope with the mathematical needed for solving problems in their fields of study.

Researchers that promote mathematical thinking with computer or without it try to support students for the understanding of mathematical concepts and solving real problems in face-to-face classroom. Some researchers like Schoenfeld (1992), Yudariah (1995), Watson and Mason (1998), and Roselainy (2009) tried to promote mathematical thinking in face-to-face classroom without using computer, while other researchers like Dubinsky and his colleagues (Dubinsky, 1991, 1994, 1995; Dubinsky and Yiparaki, 1996; Asiala et al., 1996) and Tall in many researches (1986, 1989, 1990, 1993, 2000, 2003) tried to promote mathematical thinking with computers.

In a study of multivariable calculus, Roselainy and her colleagues (Roselainy, 2009; Roselainy, Yudariah, and Mason, 2007; Roselainy, Yudariah, and Sabariah, 2007) presented a model of active learning in face-to-face multivariable calculus classroom. The model was based on invoking students’ mathematical thinking powers, supporting mathematical knowledge construction, and promoting generic skills such as communication, teamwork, and self-directed learning, but without using computer. In this study, the ways of supporting students’ mathematical thinking through Roselainy et al.’s method are put forward. Subsequently, we will show how by using computer-based tools can foster students’ mathematical thinking in Roselainy et al.’s method.

2. Supporting Mathematical Thinking in Multivariable Calculus

In the study of multivariable calculus, Roselainy and her colleagues (Roselainy, 2009; Yudariah and Roselainy, 2004; Roselainy, Yudariah, and Mason, 2005, 2007; Roselainy, Yudariah, and Sabariah, 2007) adopted the theoretical foundation of Tall (1995) and Gray et al. (1999) and used frameworks from Mason, Burton, and Stacey (1982) and Watson and Mason (1998) to develop the mathematical pedagogy for classroom practice. They highlighted some strategies in order to support students to empower themselves with their own mathematical thinking powers and help them in constructing new mathematical knowledge and generic skills, particularly, communication, teamwork, and self-directed learning (Yudariah and Roselainy, 2004).

Roselainy (2009) used the ideas of mathematical thinking as proposed by Mason, Burton, and Stacey (1982). In presenting those ideas, Burton (1984) described mathematical thinking as a way to improve understanding and extending control over the study of mathematics. In particular, he described mathematical thinking from three aspects, the operations, processes and dynamics of mathematical thinking. Certain operations were identified as mathematical such as enumeration, iteration, ordering, making correspondence, forming equivalence classes, combining or substituting one from another to transform into a new state. These operations were independent of content area but very necessary for understanding and using mathematical ideas. Four processes were identified as central to mathematical thinking, specializing, conjecturing, generalizing, and convincing. Specializing is the exploration of meaning by looking at particular cases to make clear some common properties. Conjecturing should naturally follow as a student search for relationships that connects the examples and tries to express and substantiate any underlying patterns. Generalization was the ability to recognize those patterns or regularity and making an attempt in expressing it mathematically. Convincing oneself and then another about the conjecture of the generalization that has been made encourages students to examine their ideas and explicitly communicate it first to themselves and then to others.

In proposing strategies to provoke learners to become aware of mathematical thinking processes, Watson and Mason (1998) described a framework to generate and organized generic questions which can be asked about mathematical topics in various contexts. These questions reflected the internal structures of mathematics and mathematical thinking and thus served the objectives of increasing learners’ awareness of their own powers of thinking. Their framework for generating questions is the most important guide in developing teaching strategies, in turning ideas into classroom tasks and activities. For designing sufficient prompts and questions, first they grouped the various kinds of mental activity such as specializing and generalizing, imagining and expressing, conjecturing and convincing, organizing and characterizing that represent mathematical thinking. Then, they developed several general questions and prompts that can be used to encourage the development of a “sense of Mathematics” among the students.
On the other hand, Roselainy and her colleagues also grouped various forms of mathematical structures which could be made for any mathematics topic under eight collective heading as: (i) Definitions, (ii) Facts, Theorems and Properties, (iii) Examples and Counter-examples, (iv) Techniques and Instructions, (v) Conjectures and Problems, (vi) Representation and Notation, (vii) Explanations, Justifications, Proofs and Reasoning, (viii) Links, Relationships and Connections. They had tried to connect explicitly the processes of mathematical thinking with these different types of mathematical structures (Roselainy, Yudariah, and Sabariah, 2007).

Roselainy and her colleagues (Yudariah and Roselainy, 2004; Roselainy, Yudariah, and Mason, 2005) used mathematical themes through specially designed prompts and questions to provide linkages between mathematical ideas, to expose the structures of the mathematics, and to support students’ generic skills. Some of the themes that they used were, invariance amidst change, which forms the basis for many mathematical theorems and technique, and doing and undoing, which can help students to identify features or structures that should be the focus of attention. They endeavored to design “prompts and questions” based on Watson and Mason (1998) in order to draw students’ attention to the mathematical processes and structures involved in facilitating their understanding of the concepts learnt. Some examples of prompts that they commonly used were: Give me one or more examples, Find a counter-example, and Compare examples (Roselainy, 2009). The questions such as What is the same?, What is different?, What can change and what stays the same?, What connects the different examples? and What happens in general?. In this way, the students’ attention was focused and directed to the prompts and questions in the beginning until students were aware of the questions asked in the class and became increasingly directed over time as they gradually use the prompts and questions themselves (Sabariah, Yudariah, and Roselainy, 2008).

For instance, in teaching the definitions of two-variable functions and the domain and range, two examples from the textbook is chosen to demonstrate the focus of attention that students should attend to. Table 1 below showed an extract of one of the structured example.

| Example 1:                                                                 | Questions and Prompts:                                                                                                                                                                                                 |
|---------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| **Given** \( z = 1 - x^2 - y^2 \)                                          | • Which pairs of variables are the input variables?                                                                                                                                                                      |
| i. Evaluate \( f(2,1), f(-4, 3), f(0,-5) \) and \( f(u,v) \).             | • Which variable is the output variable?                                                                                                                                                                                  |
| ii. Find the domain and range.                                            | • Is there any restriction on the input variables for which the function is defined?                                                                                                                                     |
| iii. Sketch the domain of \( f \)                                         | • How do you represent the set of all inputs graphically?                                                                                                                                                               |

For this problem, the following themes and powers (see Table 2) were identified for the students to focus on.

| Table 2. Themes, powers and mathematical activities of Example 1 (from Roselainy (2009)) |
|------------------------------------------------------------------------------------------|
| Theme: Invariance amidst Change                                                          |
| Sub-theme: Range of Change                                                                 |
| Activities: Specialising and Generalising, Characterising, Expressing                    |

| **Problem:** Finding the domain of a function | **Focus of Attention:** property of function, values of domain and range; graph of function |
### Example 1:

Given \( z = 4 - x^2 - y^2 \)

i. Evaluate \( f(2,1), f(-4,3), f(0,-5) \) and \( f(u,v) \).

ii. Find the domain and range.

iii. Sketch the domain of \( f \)

The Questions and Prompts were to direct students’ attention to the roles of the independent and dependent variables as well as to the property of the function, \( z \).

The following example (Table 3) is provided to help students in moving from a few instances to making conjecture about a wide class of cases (Mason, Burton, and Stacey, 1982). In fact, by using some specific examples and also the students’ own examples, will help students to see the “general in the particular” and also to see the “particular in the general” (Roselainy, 2009).

### Table 3. Specialising and generalising (from Roselainy (2009))

| Sub-theme: Range of Change | Activities: Specialising and Generalising, Characterising, Expressing |
|---------------------------|---------------------------------------------------------------------|
| Example 2:                | Questions and Prompts:                                              |

Let \( f(x,y) = \sqrt{4-x^2-y^2} \)

i. Find the domain and range of \( f \).

ii. Sketch the graph of the domain

(iii) Could you give one example that is like Examples 1 or 2?

(iv) Please give another example?

(v) Can you give a general example?

### 3. Mathematical Thinking and Computer-Based Tools

An earlier study using Roselainy et al.’s method, found that students still faced difficulties when they encountered with non-routine problems in multivariable calculus (Kashefi, Zaleha, and Yudariah, 2010, 2011a). For most students, imaging and sketching in 3-dimensions were the greatest difficulties that they encountered when doing non-routine problems in multivariable calculus (Kashefi, Zaleha, and Yudariah, 2010b). These findings indicate that Roselainy et al.’s method will help in making the mathematical thinking processes an explicit learning. They also highlight the students’ struggle as they encounter new mathematical ideas and concepts.

As mentioned before, in the Roselainy et al.’s method, the theoretical foundation on the development of the strategies for mathematical knowledge construction and the enhancement of students’ mathematical thinking were based on Tall (1995) and Grey et al. (1999). On the other hand, Tall’s theory underlies the creation of computer software which Tall called generic organizer and it was used in his researches (Tall, 1986, 1989, 1993, 2000, 2003)
to support students’ mathematical construction and build embodied approach to mathematical concepts. In fact, the computer brings a new dimension into the “didactic triangle” model including pupil, teacher, and mathematics in the face-to-face learning environment. However, Roselainy and her colleagues did not use any computer-based tools in their method.

On the other hand, using prompts and questions as an important strategy in Roselainy et al.’s method, that was based on Mason, Burton, and Stacey (1982) and Watson and Mason (1998), is not easy in solving some non-routine problems. For instance, to solve, *If g is a function of one variable, how is the graph of \( f(x, y) = g(\sqrt{x^2 + y^2}) \) ?*, the prompts and questions that were prepared for this problem were as follows.

- Take specific function such as \( g_1, g_2, \) and \( g_3 \).
- What is the same between them?
- What is the different between them?
- Sketch the graphs of the \( g_1(\sqrt{x^2 + y^2}), g_2(\sqrt{x^2 + y^2}), \) and \( g_3(\sqrt{x^2 + y^2}) \).

Considering different \( g_1, g_2, \) and \( g_3 \) sketching \( g_1(\sqrt{x^2 + y^2}), g_2(\sqrt{x^2 + y^2}), \) and \( g_3(\sqrt{x^2 + y^2}) \) are not an easy task for the students. As a result, they could not find the similarity and the difference between the graphs of these two-variable functions. To have a better understanding of this difficulty, consider the vertical or horizontal shifts of two-variable functions as in the following problem. *Explore the results graphically of the transformation \( g_1(x, y) = f(x, y) + c, g_2(x, y) = f(x + c, y), \) and \( g_3(x, y) = f(x, y + c). \) Describe what changes occur when the constant is added.* The prompts and questions for this problem were as follows.

- Take a specific function and sketch the graphs of the transformed functions \( g_1, g_2, \) and \( g_3 \) for different \( c \).
- What remains the same?
- What has changed?

In this problem, sketching \( g_1, g_2, \) and \( g_3 \) are also not an easy task for the students. However, by using computer web-based tools can support students in sketching \( g_1, g_2, \) and \( g_3 \). For example, by considering \( f(x, y) = x^2 + y^2 \) as a specific two-variable function and by using computer tools, it can help students to sketch it correctly. See Figure 1.

![Figure 1. Sketching \( f(x, y) = x^2 + y^2 \) by a web-based tool (Knisley, 2001)](image)

In addition, by using computer tools, students can also sketch \( g_1(x, y) = x^2 + y^2 + c \) for different \( c \). By comparing these graphs, student can find the responses for the prompts and questions such as: “What remains the same?” and “What has changed?”. Thus, they can find the answer of the following prompts and questions in order to solve this problem. Figure 2 represent the graph of \( g_4(x, y) = x^2 + y^2 - 2 \) as a typical example.

- Try negative constant.
- What is the same between them?
- What is the difference?
Test that for other examples and compare the results.
- What remains the same?
- What has changed?
- What patterns do you see in your answers?
- Test your hypothesis for other examples.
- Can the above rule be used to find the graphs of the transformed functions?

![Figure 2](image2.png)

Figure 2. Sketching $f(x, y) = x^2 + y^2 - 2$ by a web-based tool (Knisley, 2001)

4. Conclusion

This study investigated the importance of computer-based tools on students’ learning in multivariable calculus through a mathematical thinking approach. Previous studies revealed that although mathematical thinking played an important role in supporting students’ mathematical thinking in the learning of multivariable, students still had difficulties when they encountered new ideas or non-routine problems. The findings of these studies and the theoretical foundation of mathematical thinking, specifically from Tall approach, had identified the need to use computer-based tools for fostering students’ mathematical thinking powers. Using sufficient computer-based tools during the process of solving problems can help students in getting the ideas of prompts and questions as an important strategy the Roselainy et al.’s method. The results obtained from this study are expected to be useful in designing activities and tools to teach multivariable calculus based on mathematical thinking approach. Thus, it will support students to overcome their obstacles in this course.

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