An \( L_1 \) Representer Theorem for Multiple-Kernel Regression

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Abstract

The theory of RKHS provides an elegant framework for supervised learning. It is the foundation of all kernel methods in machine learning. Implicit in its formulation is the use of a quadratic regularizer associated with the underlying inner product which imposes smoothness constraints. In this paper, we consider instead the generalized total-variation (gTV) norm as the sparsity-promoting regularizer. This leads us to propose a new Banach-space framework that justifies the use of generalized LASSO, albeit in a slightly modified version. We prove a representer theorem for multiple-kernel regression (MKR) with gTV regularization. The theorem states that the solutions of MKR have kernel expansions with adaptive positions, while the gTV norm enforces an \( \ell_1 \) penalty on the coefficients. We discuss the sparsity-promoting effect of the gTV norm which prevents redundancy in the multiple-kernel scenario.

Keywords: Multiple-kernel regression, representer theorem, vector-valued learning, multiple-kernel learning, generalized LASSO.

1. Introduction

The determination of an unknown function from a series of samples is a classical problem in machine learning and falls under the category of “supervised learning”, in which there exists a rich literature. For a review, we recommend Wahba (1990); Bishop (2006); Hastie et al. (2009).

Mathematically speaking, the goal of supervised learning is to recover a target function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) from its \( M \) noisy samples \( y_m = f(x_m) + \epsilon_m, \ m = 1, 2, \ldots, M \), where \( \epsilon_m \) is a disturbance term. A general way to formulate supervised learning is through the minimization problem

\[
\min_{f} \sum_{m=1}^{M} E(f(x_m), y_m) + \lambda R(f), \tag{1}
\]

where the cost function is made of two terms. The first one (data fidelity) measures how well \( f \) fits the given training data set while the second one (regularization) tries to impose the prior knowledge about the function model. The parameter \( \lambda \in \mathbb{R}^+ \) balances the terms.

1.1 RKHS in Machine Learning

The simplest form of (1) is the least-squares problem with Tikhonov regularization (Tikhonov, 1963)

\[
\min_{f \in \mathcal{H}_L} \sum_{m=1}^{M} |f(x_m) - y_m|^2 + \lambda \|L\{f\}\|_{L_2}^2, \tag{2}
\]

where \( L \) is the regularization operator and where \( \mathcal{H}_L \), known as the native space of \( L \), is the space of functions \( f \) such that \( L\{f\} \in L_2(\mathbb{R}^d) \). It is a classical quadratic minimization problem that
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has a closed-form solution (Tikhonov, 1963). One important assumption in this formulation is the continuity of the sampling functionals $f \mapsto f(x_m)$ for $m = 1, 2, \ldots, M$. This is equivalent to $H_L$ being a reproducing-kernel Hilbert space (RKHS) (Aronszajn, 1950; De Boor and Lynch, 1966; Wahba, 1990). The theory of RKHS have been used in supervised learning, function approximation, and statistical learning in the last decades (Schölkopf and Smola, 2001; Steinke and Schölkopf, 2008; Berlinet and Thomas-Agnan, 2011; Fasshauer and Ye, 2011).

Assume that $H(R^d)$ is a Hilbert space consisting of functions from $R^d$ to $R$ with the inner product $\langle \cdot, \cdot \rangle_H$. The space $H(R^d)$ is an RKHS if there exists a bivariate function $k : R^d \times R^d \to R$ such that

$$k(x, \cdot) \in H(R^d) \quad \text{for all } x \in R^d$$

and

$$f(x) = a^T G a, \quad a \in R^M$$

which can be computed exactly using standard numerically techniques.

1.2 Toward Sparse Kernel Expansions

In the solution form (4), the kernels are located on the data samples and hence fixed. This is elegant but it can become cumbersome when the number of samples $M$ becomes too large. Hence, several schemes have been developed to reduce the number of active kernels; these include the use of a sparsity-enforcing loss such as the $\epsilon$-insensitive norm of support-vector machine (SVM) regression (Vapnik, 1998; Evgeniou et al., 2000), or the substitution of the quadratic regularization $a^T G a$ in (5) by a sparsity-promoting penalty such as $\|a\|_{\ell_1}$. The latter transforms (5) into

$$\min_{a \in R^M} \sum_{m=1}^M E(Ga_m, y_m) + \lambda \|a\|_{\ell_1}, \quad (6)$$

which is called the generalized LASSO (Roth, 2004).
In this paper, we propose an alternative Banach-space framework for supervised learning based on the generalized total-variation (gTV) regularization which promotes sparsity in the continuous domain. Our formulation is the continuous counterpart of the generalized LASSO, since it results in the same type of solution with the fundamental difference that the kernel centers $x_m$ in (3) now become adaptive. This result is deducible from the existing theory in Unser et al. (2017). We shall discuss it in more details in Section 4.

Moreover, we study a multicomponent model for the target function $f$ where it is assumed that $f$ lies in the sum of $N$ prescribed native spaces. In other words, we consider the decomposition $f = \sum_{n=1}^{N} f_n$ for the target function, where each $f_n$ belongs to a different predetermined space; the goal is to fit the training data to this hybrid search space. Note that this formulation is more general than the existing framework of multiple-kernel learning. Our main result shows that the solution of this problem has a kernel expansion with several remarkable properties:

- The number of active kernels is upperbounded by the number of samples $M$. This justifies the use of multiple kernels since the flexibility of the model will be increased while the problem remains well-posed.

- The gTV norm translates into an $\ell_1$ penalty on the kernel coefficients, which justifies the use of $\ell_1$ techniques (like the generalized LASSO) in this framework.

- The kernel positions are adaptive and will be chosen such that the solution becomes sparse.

Since a generalization of the existing theory of (Unser et al., 2017) to vector-valued settings is needed, our second objective is to derive the vector-valued counterpart of supervised learning with gTV regularization. It can also be thought of as a non-reflexive Banach-space extension of the RKHS theory of Micchelli and Pontil (2006a). Again, we show that this will yield some adaptive and sparse variant of the traditional RKHS solution.

To summarize, our contributions in this paper will be as follows.

- Providing a Banach-space framework for multiple-kernel regression with gTV regularization and proving a representer theorem for this problem.

- Proving a representer theorem for vector-valued learning with gTV regularization.

- Identifying useful families of multidimensional single and multi-kernels that are compatible with our framework.

The paper is organized as follows: We briefly mention the mathematical preliminaries required throughout the paper in Section 2. In Section 3, we define and study suitable linear operators that are acting on vector-valued functions. We state our vector-valued representer theorem in Section 4. In Section 5, we apply our vector-valued framework to the multiple-kernel regression scheme. Finally, we provide examples of useful kernels in Section 6.

2. Preliminaries and Notations

In this section, we specify the notations and recall relevant mathematical concepts. A vector-valued function from $\mathbb{R}^d$ to $\mathbb{R}^d$ is denoted by a bold small letter, for example $f : \mathbb{R}^d \to \mathbb{R}^d$. An operator $L : f \mapsto L\{f\}$ that acts on vector-valued functions is denoted by a capital bold letter. Finally, we use $e_n \in \mathbb{R}^d$ to denote the $n$th element of the canonical basis of $\mathbb{R}^d$ with $[e_n]_m = \delta[n-m]$, where $\delta[\cdot]$ is the Kronecker delta. A smooth and slowly growing function $f : \mathbb{R}^d \to \mathbb{R}$ is a function in $C_\infty(\mathbb{R}^d)$ with all of its derivatives growing slower than a polynomial at infinity. Similarly, all the derivatives of a rapidly decaying function decay faster than the inverse of any polynomial at infinity.

Schwartz space of smooth and rapidly decaying functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its topological dual is $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions (Gelfand and Shilov, 1969).
The space of continuous functions over \( \mathbb{R}^d \) that vanish at infinity is \( C_0(\mathbb{R}^d) \). It is a Banach space, equipped with the supremum norm \( \| \cdot \|_\infty \). Since \( S(\mathbb{R}^d) \) is a dense subspace of \( C_0(\mathbb{R}^d) \), one can define the topological dual of \( C_0(\mathbb{R}^d) \) as

\[
\mathcal{M}(\mathbb{R}^d) = \{ w \in S'(\mathbb{R}^d) : \| w \|_\mathcal{M} \triangleq \sup_{\phi \in S(\mathbb{R}^d)} |\langle w, \phi \rangle| < +\infty \}.
\]

Here, \( \mathcal{M}(\mathbb{R}^d) \) is the space of Radon measures over \( \mathbb{R}^d \). It includes the shifted Dirac impulses \( \delta(\cdot - x_0) \), with \( \| \delta(\cdot - x_0) \|_\mathcal{M} = 1 \). It can be seen that \( L_1(\mathbb{R}^d) \subseteq \mathcal{M}(\mathbb{R}^d) \) from the definition of Radon measures with \( \| f \|_{L_1} = \| f \|_{\mathcal{M}} \). Therefore, one can interpret \( \| \cdot \|_{\mathcal{M}} \) as a generalization of the \( L_1 \) norm.

Since we shall be dealing with vector-valued functions, we generalize the spaces by defining \( S(\mathbb{R}^d; \mathbb{R}^d) \) as the space of vector-valued functions such that each entry belongs to \( S(\mathbb{R}^d) \), so that \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in S(\mathbb{R}^d; \mathbb{R}^d) \) if and only if \( \varphi_1, \varphi_2, \ldots, \varphi_d \in S(\mathbb{R}^d) \). By definition, \( S(\mathbb{R}^d; \mathbb{R}^d) = S(\mathbb{R}^d) \oplus S(\mathbb{R}^d) \), where \( \oplus \) stands for the direct sum of topological vector spaces. More generally,

\[
S(\mathbb{R}^d; \mathbb{R}^{d'}) = S(\mathbb{R}^d) \oplus S(\mathbb{R}^d) \oplus \cdots \oplus S(\mathbb{R}^d) \quad (d' \text{ times}).
\]

The dual of \( S(\mathbb{R}^d; \mathbb{R}^d) \) is \( S'(\mathbb{R}^d; \mathbb{R}^d) \). It is the space of vector distributions \( w = (w_1, w_2, \ldots, w_d) \) such that \( w_1, w_2, \ldots, w_d \in S'(\mathbb{R}^d) \). Indeed, the dual of \( X \oplus Y = \{(x, y) : x \in X, y \in Y\} \) is \( X' \oplus Y' \) for any pair of topological spaces \( X \) and \( Y \).

We define \( C_0(\mathbb{R}^d; \mathbb{R}^d) \) as the space of continuous functions \( f : \mathbb{R}^d \to \mathbb{R}^d \) that vanish at infinity, which means that, for every sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d \) that satisfies \( \lim_{n \to \infty} \|x_n\| = +\infty \), one has that \( \lim_{n \to \infty} \|f(x_n)\| = 0 \). This space is equipped with a norm equal to the sum of the supremum norms of its components, with \( \|f\|_\infty = \sum_{i=1}^{d'} \|f_i\|_\infty \) for \( f = (f_1, f_2, \ldots, f_{d'}) \in C_0(\mathbb{R}^d; \mathbb{R}^d) \).

Likewise, the dual of \( C_0(\mathbb{R}^d; \mathbb{R}^d) \) is defined as

\[
\mathcal{M}(\mathbb{R}^d; \mathbb{R}^d) = \{ w = (w_1, w_2, \ldots, w_{d'}) \in S'(\mathbb{R}^d; \mathbb{R}^d) : \| w \|_{\mathcal{M}} < +\infty \},
\]

where

\[
\| w \|_{\mathcal{M}} = \sum_{i=1}^{d'} \|w_i\|_{\mathcal{M}}.
\]

### 3. Vector-Valued Regularization Operators

In this section, we introduce the regularization operator \( \mathcal{L} \) which needs to fulfill some fundamental stability and invertibility requirements.

We assume that \( \mathcal{L} : S'(\mathbb{R}^d; \mathbb{R}^d) \to S'(\mathbb{R}^d; \mathbb{R}^d) \) is a linear shift-invariant (LSI) operator, with \( \mathcal{L}\{f(\cdot - x_0)\} = \mathcal{L}\{f(\cdot - x_0)\} \) for any \( f \in S'(\mathbb{R}^d; \mathbb{R}^d) \) and any shift \( x_0 \in \mathbb{R}^d \). We also assume that \( \mathcal{L} \) is an isomorphism over \( S'(\mathbb{R}^d; \mathbb{R}^d) \), meaning that it is continuous and invertible with its inverse being the continuous operator \( \mathcal{L}^{-1} : S'(\mathbb{R}^d; \mathbb{R}^d) \to S'(\mathbb{R}^d; \mathbb{R}^d) \).

Define the set of operators \( \mathcal{L}_{i,j} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) as \( \mathcal{L}_{i,j}\{\cdot\} = \{\mathcal{L}\{e_j\}\}_i \) for \( (i, j) \in \{1, 2, \ldots, d'\} \times \{1, 2, \ldots, d'\} \). By the linearity of \( \mathcal{L} \), we have that

\[
\forall w \in S'(\mathbb{R}^d; \mathbb{R}^d) : [\mathcal{L}\{w_1, w_2, \ldots, w_{d'}\}]_i = \sum_{j=1}^{d'} \mathcal{L}_{i,j}\{w_j\}.
\]

In this way, any linear operator \( \mathcal{L} : S'(\mathbb{R}^d; \mathbb{R}^d) \to S'(\mathbb{R}^d; \mathbb{R}^d) \) can be expressed by a matrix of operators \( \mathcal{L}_{i,j} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d) \).

Since \( \mathcal{L}_{i,j} \) are continuous LSI operators over \( S'(\mathbb{R}^d) \), we can consider their effect in the Fourier domain. The Fourier transform is a well-defined and continuous operator over \( \mathcal{F} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d) \).
One defines the frequency response of the LSI operator $L : S'(\mathbb{R}^d) \to S'(\mathbb{R}^{d'})$ as the Fourier transform of its impulse response

$$\hat{L}(\omega) \triangleq \mathcal{F}\{L(\delta)\}(\omega).$$

(7)

Likewise, for the LSI operator $L : S'(\mathbb{R}^d; \mathbb{R}^d) \to S'(\mathbb{R}^d; \mathbb{R}^{d'})$ that acts on vector-valued functions, the frequency response can be specified component-wise as

$$\hat{L}(\omega) = \begin{pmatrix}
\hat{L}_{1,1}(\omega) & \cdots & \hat{L}_{1,d'}(\omega) \\
\vdots & \ddots & \vdots \\
\hat{L}_{d',1}(\omega) & \cdots & \hat{L}_{d',d'}(\omega)
\end{pmatrix},$$

(8)

where $L_{i,j}$ is the $(i,j)$th element of the matrix $L$.

Finally, the determinant of the matrix $L : S'(\mathbb{R}^d; \mathbb{R}^d) \to S'(\mathbb{R}^d)$ is the LSI operator $\det(L) : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ whose Fourier transform is $\hat{\det(L)} : \mathbb{R}^d \to \mathbb{R}$.

In Proposition 1 we provide a necessary and sufficient condition for $L$ to be an isomorphism over $S'(\mathbb{R}^d; \mathbb{R}^{d'})$. The proof is given in Appendix A.

**Proposition 1** The operator $L$ is an isomorphism over $S'(\mathbb{R}^d; \mathbb{R}^{d'})$ if and only if

- The $\hat{L}_{i,j}(\omega)$ are smooth and slowly growing functions for all $(i,j) \in \{1, 2, \ldots, d\} \times \{1, 2, \ldots, d'\}$.
- The frequency response of the determinant never vanishes, with $\hat{\det(L)}(\omega) \neq 0$ for all $\omega \in \mathbb{R}^d$.
- Its reciprocal $\frac{1}{\hat{\det(L)}(\omega)}$ is a slowly growing function.

The native space of $L$ is the set of all functions $f$ with finite gTV norm. Formally,

$$\mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'}) = \{ f \in S'(\mathbb{R}^d; \mathbb{R}^{d'}) : \|L\{f\}\|_{\mathcal{M}} < +\infty \}.$$ 

It is a Banach space equipped with the gTV norm

$$\text{gTV}(f) \triangleq \|f\|_{\mathcal{M}_L} = \|L\{f\}\|_{\mathcal{M}} < +\infty.$$ 

Note that the restriction of $L$ to its native space results in the isomorphism $L : \mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'}) \to \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d'})$. Therefore, the adjoint operator $L^*$ is well-defined over $C_0(\mathbb{R}^d; \mathbb{R}^{d'})$ and its image is the Banach space $\mathcal{C}_L(\mathbb{R}^d; \mathbb{R}^{d'})$ with the norm $\|f\|_{\mathcal{C}_L} \triangleq \|L^{-1}\{f\}\|_{\mathcal{M}}$. It is the predual of $\mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'})$, meaning that $\mathcal{C}_L(\mathbb{R}^d; \mathbb{R}^{d'})' = \mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'})$.

Similarly to the theory of RKHS, we are interested in the case where the sampling operator is weak*-continuous in $\mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'})$. This means that the shifted Dirac impulses at each component should be included in the predual of the native space, so that $\forall x \in \mathbb{R}^d, e_i\delta(\cdot - x) \in C_L(\mathbb{R}^d; \mathbb{R}^{d'})$ for $i = 1, 2, \ldots, d$. This is equivalent to $L^{-1}\{e_i\delta\} \in C_0(\mathbb{R}^d; \mathbb{R}^{d'})$ for $i = 1, 2, \ldots, d$.

As an example, we take $d = 1$ and $d' = 2$ and consider the matrix operator $L$

$$L = \begin{pmatrix}
L_{1,1} & L_{1,2} \\
L_{2,1} & L_{2,2}
\end{pmatrix} = \begin{pmatrix}
D^2 - I & 0 \\
0 & \frac{1}{2}D^2 - 1
\end{pmatrix},$$

where $D : S'(\mathbb{R}) \to S'(\mathbb{R})$ is the generalized derivative operator and $I$ is the identity. The inverse of this operator is

$$L^{-1} = \begin{pmatrix}
(D^2 - I)^{-1} & 0 \\
0 & 4(D^2 - 4I)^{-1}
\end{pmatrix}.$$
One can readily check that both $L$ and $L^{-1}$ are continuous maps in $S'(\mathbb{R}; \mathbb{R}^2)$ (see Proposition 1) and that the operators are all self-adjoint. Moreover, we have that

$$L^{-1}\{e_1\delta\} = L^{-1}\{e_1\delta\} = \left( \begin{array}{c} -\exp(-|\cdot|) \\ 0 \end{array} \right) \in C_0(\mathbb{R}; \mathbb{R}^2)$$

$$L^{-1}\{e_2\delta\} = L^{-1}\{e_2\delta\} = \left( \begin{array}{c} 0 \\ -\exp(-2|\cdot|) \end{array} \right) \in C_0(\mathbb{R}; \mathbb{R}^2),$$

which implies the weak*-continuity of the sampling operator in $\mathcal{M}_L(\mathbb{R}; \mathbb{R}^2)$. Therefore, $L$ could be one of the possible candidates for the regularization operator. A detailed discussion with additional examples will be provided in Section 5.

4. Vector-Valued Learning with gTV Regularization

The problem of learning a vector-valued function from a finite number of samples has been addressed in the machine learning community under the title of multi-task learning (Caruana, 1997). Empirical results show that learning multiple tasks jointly is beneficial, as compared to learning them individually (Baxter, 1997; Thrun and O’Sullivan, 1998; Bakker and Heskes, 2003).

The vector-valued learning in an RKHS has been studied thoroughly in the literature (Micchelli and Pontil, 2005a,b,c). It is based on vector-valued RKHS, which has been originally studied by Burbea and Masani (1984). Consider the vector-valued RKHS $H(\mathbb{R}^d; \mathbb{R}^{d'})$ and the minimization problem

$$\min_{f \in H(\mathbb{R}^d; \mathbb{R}^{d'})} \sum_{m=1}^M E(f(x_m), y_m) + \lambda \|f\|_{H}^2. \quad (9)$$

The solution of this problem is given by the vector-valued extension of the representer theorem for RKHS.

**Theorem 2 (Micchelli and Pontil (2005a))** If Problem (9) has a solution, then it admits the form

$$f(\cdot) = \sum_{m=1}^M \mathbf{K}(\cdot, x_m) a_m, \quad (10)$$

where $a_m \in \mathbb{R}^{d'}$ and $\mathbf{K}: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d' \times d'}$ is the unique reproducing kernel of $H(\mathbb{R}^d; \mathbb{R}^{d'})$.

In this section, we propose a representer theorem that specifies the form of the solution for the vector-valued learning problem with gTV regularization. First, let us recall the representer theorem of Unser et al. (2017) under the assumption that the regularization operator $L: S'(\mathbb{R}^d) \rightarrow S'(\mathbb{R}^d)$ is invertible. Consider the minimization problem

$$\beta = \min_{f \in \mathcal{M}_L(\mathbb{R}^d)} \|L(f)\|_{\mathcal{M}} \quad \text{s.t.} \quad \nu_m(f) = y_m, \quad m = 1, 2, \ldots, M, \quad (11)$$

where $\nu_m : \mathcal{M}_L(\mathbb{R}^d) \rightarrow \mathbb{R}$ are weak*-continuous linear measurement functionals for $m = 1, 2, \ldots, M$.

In the case of the regression problem, $\nu_m$ would typically be Dirac impulses at the sampled locations.

**Theorem 3 (Unser et al. (2017))** The solution set of (11) is nonempty, convex, and weak*-compact with the extreme points in the form

$$f(\cdot) = \sum_{n=1}^N a_n k(\cdot - z_n),$$

where $k = L^{-1}\{\delta\} : \mathbb{R}^d \rightarrow \mathbb{R}$ is the kernel function, $a_n \in \mathbb{R}$ are kernel weights, and $z_n \in \mathbb{R}^d$ are adaptive kernel positions. In addition, we have that $N \leq M$ and $\beta = \sum_{n=1}^N |a_n|$. 
Theorem 3 ensures that Problem (11) admits a solution that has a “sparse” kernel expansion with adaptive centers \( z_n \) and \( \text{gTV}(f) = \|a\|_{\ell_1} \). It is worth mentioning that the kernel function in this Banach-space theory is stationary — that is a map \((x, y) \mapsto k(x - y)\) — but not necessarily positive-definite (as opposed to the RKHS theory).

In the case of sampling, \( \nu_m \) are Dirac impulses at the sampled locations. Then, Theorem 3 translates the original problem (11) into the discrete minimization

\[
\min_{a \in \mathbb{R}^M, Z \in \mathbb{R}^{d \times N}} \|a\|_{\ell_1} \quad \text{s.t.} \quad G_Z a = (y_1, y_2, \ldots, y_m),
\]

where \( Z = (z_1, z_2, \ldots, z_N) \) is the kernel-position matrix and \( G_Z \in \mathbb{R}^{M \times N} \) is a matrix with \([G_Z]_{m,n} = k(x_m, z_n)\). The unconstrained form of (12) is

\[
\min_{a \in \mathbb{R}^M, Z \in \mathbb{R}^{d \times N}} \sum_{m=1}^{M} E([G_Z a]_m, y_m) + \lambda \|a\|_{\ell_1}.
\]

This is the same formulation as the generalized LASSO (6), with the difference that the minimization is through the positions as well. Hence, an important aspect of Theorem 3 is a justification for the use of \( \ell_1 \) techniques in kernel methods.

We are interested in characterizing the solution set of a supervised vector-valued learning problem with \( \text{gTV} \) regularization. In our formulation, we consider the minimization problem

\[
\min_{f \in \mathcal{M}_L(\mathbb{R}^{d};\mathbb{R}^{d'})} \sum_{m=1}^{M} E(\nu_m(f), y_m) + \lambda \|L\{f\}\|_{\mathcal{M}},
\]

where \( \nu_m : \mathcal{M}_L(\mathbb{R}^{d};\mathbb{R}^{d'}) \to \mathbb{R} \) are weak*-continuous linear measurements. Note that, in regression, each sample of \( f \) can be expressed as \( d' \) linear measurements.

A key element behind Theorem 3 is the generalized version of the Fisher-Jerome theorem (Fisher and Jerome 1975) that was proven in (Unser et al., 2017). Here, we shall first generalize the Fisher-Jerome theorem for vector-valued functions. Using its generalized version, we then prove our representor theorem for the constrained case. Finally, by showing an equivalence between the constrained and unconstrained problems, we obtain a representor theorem for the unconstrained problem (14) as well.

4.1 The Vector-Valued Fisher-Jerome Theorem

Let \( B = \mathcal{M}(\mathbb{R}^{d};\mathbb{R}^{d'}) \bigoplus \mathcal{N} \), where \( \mathcal{N} \) is an \( N_0 \)-dimensional normed space (The space \( \mathcal{N} \) is not needed here; we have included it in order to maintain the compatibility with the scalar framework).

**Theorem 4 (Vector-valued Fisher-Jerome) Let us assume that \( F : B \to \mathbb{R}^M \) is a linear and weak*-continuous functional \((M \geq N_0)\) such that**

\[
\exists B > 0: \quad \forall p \in \mathcal{N} \setminus \{0\}, \quad B \leq \frac{\|F(0, p)\|_2}{\|p\|_{\mathcal{N}}}
\]

**and that the minimization problem**

\[
\mathcal{V} = \arg \min_{f = (w, p) \in B} \|w\|_\mathcal{M} \quad \text{s.t.} \quad F(f) \in \mathcal{C}
\]

**is feasible for a convex and compact set \( \mathcal{C} \subseteq \mathbb{R}^M \). Then, \( \mathcal{V} \) is a nonempty, convex, weak*-compact subset of \( B \) while the components of its extreme points \((w_1, w_2, \ldots, w_{d'}, p)\) are all of the form**

\[
w_i = \sum_{j=1}^{M_i} a_{i,j} \delta(\cdot - z_{i,j}), \quad i = 1, 2, \ldots, d',
\]

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where \( a_{i,j} \in \mathbb{R} \) and \( z_{i,j} \in \mathbb{R}^d \). Moreover, \( \sum_{i=1}^{d'} M_i \leq M \) and the minimum \( \mathcal{M} \)-norm obtained for the problem is equal to \( \sum_{i=1}^{d'} \sum_{j=1}^{M_i} |a_{i,j}| \).

The proof is given in Appendix B.

Remark 5 The result is called the Fisher-Jerome theorem in recognition of the pioneering work of the authors Fisher and Jerome (1975). They proved a classical scalar theorem for a compact domain \( \Omega \subseteq \mathbb{R}^d \) (instead of \( \mathbb{R}^d \)) and a box feasibility set \( \mathcal{C} = \prod_{m=1}^M [a_m, b_m] \subseteq \mathbb{R}^M \). Unser et al. (2017) extended the scalar theorem which we can recover in the case \( d' = 1 \) of our theorem.

4.2 Representer Theorems for Constrained and Unconstrained Forms

In this section, we state and prove our representer theorem for the problem of learning a vector-valued function using gTV regularization. We start with the constrained problem.

Theorem 6 (Minimum gTV-norm fitting) Assume that there exists a function \( f \in \mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'}) \) such that \( \nu_m(f) = y_m \) for \( m = 1, 2, \ldots, M \), where the \( \nu_m \) are weak*-continuous measurement functionals. Then, the solution set of the minimum-norm interpolation problem

\[
\beta = \min_{f \in \mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'})} \| L\{f\} \|_\mathcal{M} \quad \text{s.t.} \quad \nu_m(f) = y_m, \quad m = 1, 2, \ldots, M, \tag{16}
\]

is nonempty, convex, and weak*-compact with extremal points of the form

\[
f(\cdot) = \sum_{i=1}^{d'} \sum_{j=1}^{M_i} a_{i,j} k_i(\cdot - z_{i,j}), \tag{17}
\]

where

\[
k_i : \mathbb{R}^d \to \mathbb{R}^{d'}, \quad k_i = \mathbf{L}^{-1}\{e_i\delta\}, \quad i = 1, 2, \ldots, d' \tag{18}
\]

are vector-valued kernels, \( a_{i,n} \in \mathbb{R} \) are kernel weights, and \( z_{i,n} \in \mathbb{R}^d \) are adaptive kernel positions. Moreover, \( \sum_{i=1}^{d'} M_i \leq M \) and \( \beta = \sum_{i=1}^{d'} \sum_{j=1}^{M_i} |a_{i,j}|. \)

Proof By defining \( w = L\{f\} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d'}) \) and by invoking the invertibility of \( \mathbf{L} \), we rewrite the minimization problem (16) as

\[
\beta = \min_{w \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d'})} \| w \|_\mathcal{M} \quad \text{s.t.} \quad \nu_m(\mathbf{L}^{-1}\{w\}) = y_m, \quad m = 1, 2, \ldots, M. \tag{19}
\]

We then apply Theorem 4, which shows that the solution set of (19) is a nonempty, convex, and weak*-compact subset of \( \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d'}) \). The components of the extremal point \( w = (w_1, w_2, \ldots, w_{d'}) \) take the form

\[
w_i = \sum_{j=1}^{M_i} a_{i,j} \delta(\cdot - z_{i,j}), \quad i = 1, 2, \ldots, d'. \tag{20}
\]

By applying \( \mathbf{L}^{-1} \) on both sides of (20), we deduce the form (17) for the extreme points of (16). The last part is then obtained by explicitly calculating

\[
\beta = \| L\{f\} \|_\mathcal{M} = \| w \|_\mathcal{M} = \sum_{i=1}^{d'} \sum_{j=1}^{M_i} |a_{i,j}|.
\]

As the final part of this section, we state our representer theorem for the unconstrained problem (LASSO regression).
Theorem 7 (Vector-valued regression with gTV) Consider the vector-valued learning problem

\[
\min_{f \in \mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'})} \sum_{m=1}^{M} E(\nu_m(f), y_m) + \lambda \|L(f)\|_{\mathcal{M}_L},
\]

where \(E(x, y)\) is a nonnegative function that measures the dissimilarity between \(x\) and \(y\) such that \(E(\cdot, y)\) is proper, coercive, lower-semicontinuous and strictly convex for all \(y \in \mathbb{R}\). Then, the solution set for (21) is a nonempty, convex, weak*-compact set and its extreme points admit the same form as (17).

Proof We write the cost functional as \(H(f) + \|f\|_{\mathcal{M}_L}\), where \(H(f) = \sum_{m=1}^{M} E(\nu_m(f), y_m)\). To show that the solution set is nonempty, we apply a standard technique in convex analysis which is to show that the cost functional is coercive and weakly lower-semicontinuous (Kurdila and Zabarankin, 2006). This also works when the latter property is replaced by weak* lower-semicontinuity (see Gupta et al., 2018, Proposition 8).

The cost functional is the sum of a nonzero functional \(H(\cdot)\) and a coercive functional \(\|\cdot\|_{\mathcal{M}_L}\). This ensures its overall coercivity.

The sampling operator is weak*-continuous by assumption. Its composition with a lower-semicontinuous functional \(E(\cdot, y)\) and summation over \(m\) yields a cost functional \(H(f)\) that is weak* lower-semicontinuous as well.

The scalar \(\mathcal{M}\) norm is weak*-continuous on \(\mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d'})\). Therefore, \(\|\cdot\|_{\mathcal{M}}\) is also weak*-continuous on \(\mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d'})\). Consequently, since \(L\) is an isomorphism, the gTV norm is a weak*-continuous functional on \(\mathcal{M}_L\). Therefore, the cost functional \(H(f) + \|f\|_{\mathcal{M}_L}\) is weak* lower-semicontinuous. Together with the coercivity of the cost functional, this proves the existence of the solution.

Now, consider two solutions \(f_1\) and \(f_2\) of the problem and denote the minimum value as \(\beta^*\). Assume that \(\nu_m(f_1) \neq \nu_m(f_2)\) for some \(m\). Since \(E(\cdot, y)\) is a strictly convex function for any \(y \in \mathbb{R}\), then we have that

\[
H\left(\frac{f_1 + f_2}{2}\right) > \frac{H(f_1) + H(f_2)}{2}
\]

and, therefore

\[
H\left(\frac{f_1 + f_2}{2}\right) + \lambda \|L\left(\frac{f_1 + f_2}{2}\right)\|_{\mathcal{M}} < \frac{H(f_1) + H(f_2) + \lambda \|L(f_1)\|_{\mathcal{M}} + \|L(f_2)\|_{\mathcal{M}}}{2} = \beta^*,
\]

which contradicts the assumption that \(f_1\) and \(f_2\) are solutions of (7). Consequently, \(\nu_m(f_1) = \nu_m(f_2) = z_m\) for \(m = 1, 2, \ldots, M\) and one can rewrite the problem as

\[
\min_{f \in \mathcal{M}_L(\mathbb{R}^d; \mathbb{R}^{d'})} \|L(f)\|_{\mathcal{M}} \quad \text{s.t.} \quad \nu_m(f) = z_m, \quad m = 1, 2, \ldots, M.
\]

We then get the desired property for the solution set of (21) by applying Theorem 6. \(\blacksquare\)

It is instructive to compare the solution considered in Theorem 7 with the outcome of the RKHS theory of vector-valued learning. To that end, we rewrite the solution form of RKHS (10) by denoting \(a_m = (a_{1,m}, a_{2,m}, \ldots, a_{d',m})\) and \(K = [k_1 \ k_2 \ \cdots \ k_{d'}]\) as

\[
f(\cdot) = \sum_{i=1}^{d'} \sum_{m=1}^{M} a_{i,m} k_i(\cdot, x_m),
\]

which is similar to (17). However, the fundamental difference is that the kernel locations are prescribed in the RKHS solution form (they are located on the data points) and adaptive in the gTV solution form. The adaptability of the kernel locations helps us to find the “sparsest” representation of the target function for which \(\|a\|_{\ell_1}\) is minimized.
5. Multiple-Kernel Regression

In this section, we prove our main result: the representer theorem of multiple-kernel regression with gTV regularization. In effect, the gTV norm will force the learned function to use the fewest active kernels.

**Theorem 8 (Multiple kernel regression with gTV)** Assume that our training data set consists of $M$ distinct pairs $(x_m, y_m)$ for $m = 1, 2, \ldots, M$. Consider the minimization problem

$$
\min_{f_n \in \mathcal{M}_N(\mathbb{R}^d)} \sum_{m=1}^{M} E(f(x_m), y_m) + \lambda \sum_{n=1}^{N} \|L_n \{f_n\}\|_\mathcal{M} \quad \text{s.t.} \quad f = \sum_{n=1}^{N} f_n, \tag{23}
$$

where $E(\cdot, y)$ is a proper, coercive, lower-semicontinuous and strictly convex functional. Then, the solution set of this problem is nonempty, convex, and weak*-compact. Any of its extreme points $(f_1, f_2, \ldots, f_N)$ results in a function $f = \sum_{n=1}^{N} f_n$ that takes the form

$$
f(\cdot) = \sum_{n=1}^{N} \sum_{j=1}^{M_n} a_{n,j} k_n(\cdot - z_{n,j}) \tag{24}
$$

for $a_{n,j} \in \mathbb{R}$, $z_{n,j} \in \mathbb{R}^d$, and $k_n = L_n^{-1}\{\delta\}$. Moreover, $\sum_{n=1}^{N} M_n \leq M$ and $\sum_{n=1}^{N} \|L_n \{f_n\}\|_\mathcal{M} = \sum_{n=1}^{N} \sum_{j=1}^{M_n} |a_{n,j}|$.

**Proof** Define the operator $L = \text{diag}(L_1, L_2, \ldots, L_N)$ and the measurement functionals $\nu_m$ as

$$
\nu_m(f) = \sum_{n=1}^{N} f_n(x_m).
$$

Then, it can be seen that (23) is equivalent to (21). The result then follows from Theorem 7.

Theorem 8 suggests a kernel expansion for the target function with adaptive positions and sparse coefficients. Again, the kernel functions are stationary, but not necessarily positive-definite. Moreover, the number of nonzero coefficients in this expansion is upper-bounded by the number of data points $M$ and, hence, does not depend on the number of kernels $N$. This suggests that increasing $N$ which improves the flexibility of the model does not necessarily add redundancy.

We now like to make a link with the multiple kernel learning framework mentioned in the introduction, which is a refinement of the RKHS technique. The underlying principle there is to learn a positive-definite kernel of the form $k = \sum_{n=1}^{N} \mu_n k_n$ (Bach et al., 2004). By replacing the learned kernel into the standard RKHS formula (41), one gets

$$
f(\cdot) = \sum_{m=1}^{M} a_m \left( \sum_{n=1}^{N} \mu_n k_n(\cdot, x_m) \right) = \sum_{n=1}^{N} \sum_{m=1}^{M} \mu_n a_m k_n(\cdot, x_m) \tag{25}
$$

which is similar to (24). The fundamental differences, however are as follows:

- The kernel positions in (25) are fixed on the data points by contrast with (24) where the centers are adaptive.
- The kernel coefficients in (25) are separable with respect to the variables $n$ and $m$.
- The kernel functions in (25) are constrained to be positive-definite and not necessarily stationary.
- In (25), the $a_m$’s are constrained with a quadratic regularization, while in (24), the $a_{n,j}$’s are regularized with an $\ell_1$ penalty, which typically results in a sparse expansion with less than $M$ active terms.
Last but not least, it is instructive to mention that one can be tempted to use the idea of generalized LASSO in multiple-kernel models by trying to remove the unnecessary kernels in (24). Again, the main difference between the generalized LASSO and our formulation is having adaptive positions. Once the positions are fixed, one can use the classical $\ell_1$ minimization algorithms in order to find a sparse kernel coefficients.

6. Examples of Admissible Kernels

In this section, we reverse the perspective by identifying classes of adaptive-kernel expansions (with $\ell_1$ regularization) that admit a min $g$TV variational interpretation. The correspondence between the kernel functions and the regularization operators allows us to classify the admissible kernels in our theory by specifying their corresponding operators. Since the operators are LSI and defined over $S'(\mathbb{R}^d; \mathbb{R}^d)$, they can be entirely specified in terms of their frequency response.

First, we provide a sufficient condition for a kernel function to ensure its admissibility.

**Proposition 9** Assume that the function $k : \mathbb{R}^d \rightarrow \mathbb{R}$ has the following properties:

- Its frequency response, $\hat{k}(\omega)$ is a nonvanishing, smooth and slowly growing function of $\omega \in \mathbb{R}^d$.
- It has a heavy-tailed frequency response; that is, there exist $\alpha, C > 0$ such that $\hat{k}(\omega) \geq C(\|\omega\| + 1)^{-\alpha}$.

Then, the operator $L$ with the frequency response $\hat{L}(\omega) = \frac{1}{\hat{k}(\omega)}$ is an isomorphism over $S'(\mathbb{R}^d)$.

**Proof** Due to Proposition 1, it is sufficient to show that both $\hat{k}(\omega)$ and $\frac{1}{\hat{k}(\omega)}$ are non-zero, smooth and slowly growing functions of $\omega \in \mathbb{R}^d$. The only nontrivial part is to show that $\frac{1}{\hat{k}(\omega)}$ is a slowly growing function.

First, note that if $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ are smooth and slowly growing functions and moreover, $g$ is non-zero and heavy-tailed, then

$$\frac{\partial}{\partial x_i} \left( \frac{f}{g} \right) = \frac{\frac{\partial f}{\partial x_i} g - \frac{\partial g}{\partial x_i} f}{g^2}$$  \hspace{1cm} (26)

is a quotient whose nominator is a smooth and slowly growing function and whose denominator $g^2$ is a non-zero, heavy-tailed, smooth and slowly growing function. Hence, the quotient itself is a smooth function whose growth is bounded by a polynomial.

Now, one can deduce from induction that all the arbitrary order derivatives of $\frac{1}{\hat{k}(\omega)}$ can be expressed by a quotient with a slowly growing nominator and a heavy-tailed denominator. This shows that $\frac{1}{\hat{k}(\omega)}$ is a smooth and slowly growing function.  \hspace{1cm} \blacksquare

Proposition 9 ensures that any strictly positive-definite kernel with a heavy-tail, smooth and slowly growing Fourier transform is included in our theory. We now use this condition to specify several classes of kernel functions that can be used in practice.

The case of having one space (scalar learning) is interesting on its own right. We can easily construct separable kernels from a scalar kernel function $k : \mathbb{R} \rightarrow \mathbb{R}$ by considering

$$k(x, y) = \prod_{i=1}^{d} k(x_i - y_i).$$  \hspace{1cm} (27)

An important class of this family are the sub-Gaussian kernels which are specified as

$$k_\alpha(x, y) = \exp(-\|x - y\|_\alpha^\alpha), \hspace{1cm} \alpha \in (0, 2).$$  \hspace{1cm} (28)
These sub-Gaussian kernels are known to be positive-definite (Unser and Tafti, 2014, Appendix B) and their inverse Fourier transforms (the so-called $\alpha$-stable distributions) are heavy-tailed and infinitely smooth with algebraically decaying derivatives of any order (Sato, 1999, Chapter 5). Hence, they satisfy the required conditions of Proposition 9. Note that the classical Gaussian kernels are excluded because their frequency response are not heavy-tailed. However, one can get arbitrarily close by letting $\alpha$ tend to its critical value 2.

To construct a non-separable kernel, one can consider an invertible non-diagonal mixture matrix $A$ along with the kernel $k(Ax, Ay)$. The corresponding regularization operator has the frequency response $\frac{|\text{det}(A)|}{k(A^{-1}\omega)}$.

Another important class of non-separable kernels are the Bessel potentials that have been used in kernel estimation (Aronszajn and Smith, 1961). For a positive real number $s > 0$, we consider the operator $(I - \Delta)^{-\frac{s}{2}} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$, where $\Delta$ is the Laplacian operator. The Bessel potentials $G_s$ are the kernels of these class of operators. They are defined as

$$G_s(x) = F^{-1}\left\{\frac{1}{(1 + \|\omega\|^2)^\frac{s}{2}}\right\}(x).$$

We can use the mentioned kernels in our theory to better fit the target function. Due to Theorem 8, the total number of active kernels is upper-bounded by the number of samples. Moreover, the regularization term for the solution is equal to $\sum_{i=1}^{N} \sum_{j=1}^{M_i} |a_{i,j}|$. This imposes sparsity on the vector of coefficients and enables us to use a plethora of kernels.

One way of considering multiple kernels is to select a parametric kernel family (like the Bessel potentials) and choose $N$ different parameter values within that family (different values of $s$). This allows us to can control the smoothness and also the decay rate of the kernel functions. In Figure 1 and 2, we have plotted Bessel potentials and sub-Gaussian kernels for different parameter settings.

7. Conclusion

In this paper, we have provided a theoretical foundations on multiple-kernel regression with gTV regularization. We have derived a representer theorem that shows that the learned function can be written as a linear combination of kernels with adaptive centers. Our representer theorem provides an upperbound to the number of active elements, which allows us to use as many kernels as desired. We have also derived a representer theorem for the vector-valued learning with gTV regularization. Our solution forms motivate us for further studies on developing algorithms to find the kernel coefficients and positions.

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Appendix A. Proof of Proposition 1

Proof Before going into the details of the proof, let us mention a theorem that concerns the continuous operators defined over $S'(\mathbb{R}^d)$ (Schwartz, 1957).

**Theorem 10 (Theorem IX, pp. 244 in Schwartz (1957))** If $L : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ is a linear, shift-invariant, and continuous operator over $S'(\mathbb{R}^d)$, then $\hat{L}(\omega)$ is a smooth and slowly growing function.

Additionally, any smooth and slowly growing function $\hat{L}$ defines an LSI and continuous operator $L : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ via

$$L\{f\} = F^{-1}\{\hat{L}\hat{f}\}.$$  

(30)
Theory of Multiple-Kernel Regression

Figure 1: Bessel-potential kernels for different values of $s$.

Now, assume that $L$ satisfies the conditions of Theorem 10. Then, we deduce the continuity of $L_{i,j}$ for all $(i,j) \in \{1,2,\ldots,d\} \times \{1,2,\ldots,d'\}$ and, hence, the continuity of $L$.

Let us denote $\text{adj}(L)$ as the adjugate matrix of $L$. If $\hat{\det}(L)(\omega) \neq 0$, then we can define the inverse operator $L^{-1}$ in the Fourier domain as

$$
\hat{L}^{-1} = \frac{1}{\hat{\det}(L)(\omega)} \text{adj}(\hat{L}).
$$

On one hand, the smoothness of $\hat{L}_{i,j}$ for all $(i,j) \in \{1,2,\ldots,d\} \times \{1,2,\ldots,d'\}$ implies the smoothness of each element of $\hat{L}^{-1}$. On the other hand, since $\frac{1}{\hat{\det}(L)(\omega)}$ has a slow growth, we can easily deduce that each element of $\hat{L}^{-1}$ has also a slow growth. That yields the continuity of the inverse operator $L^{-1} : S'(\mathbb{R}^d;\mathbb{R}^{d'}) \rightarrow S'(\mathbb{R}^{d};\mathbb{R}^{d'})$.

Assume now that $L$ is an isomorphism over $S'(\mathbb{R}^{d};\mathbb{R}^{d'})$. Therefore, $[L(e_j)]_i$ is a continuous operator over $S'(\mathbb{R}^{d})$, which implies that $\hat{L}_{i,j}(\omega)$ are smooth functions with slow growth for all $(i,j) \in \{1,2,\ldots,d'\} \times \{1,2,\ldots,d'\}$. Therefore, $\hat{\det}(L)(\omega)$ is also a smooth function.

Since $L^{-1}$ is well-defined and continuous, each entry of its Fourier matrix has to be a smooth function with slow growth. Therefore, the determinant of its Fourier matrix $\frac{1}{\hat{\det}(L)(\omega)}$ is also a smooth function dominated by a polynomial. That completes the proof. $\blacksquare$
Appendix B. Proof of Theorem 4

Proof Let us first remark that any closed cube in $\mathcal{B}$ is weak*-compact. This also means that any bounded sequence in $\mathcal{B}$ has a subsequence that converges in $\mathcal{B}$ in the weak* sense. These are consequences of the Banach-Alaoglu theorem (Rudin, 1991, Theorem 3.15).

The proof is in two parts. First, we show that the solution set is nonempty, weak*-compact, and convex. Then, we explore the form of its extreme points to complete the theorem. The first part is directly deducible from the arguments that were given in Unser et al. (2017) to prove their scalar version of the Fisher-Jerome theorem. However, the second part requires technical details and it is not a direct generalization of the scalar theorem.

Existence of a Solution: Define $\mathcal{U} = F^{-1}(\mathcal{C})$. It is a weak*-closed (due to weak*-continuity of $F$ and closedness of $\mathcal{C}$) and convex (due to linearity of $F$ and convexity of $\mathcal{C}$) subset of $\mathcal{B}$. We can reformulate the problem as

$$ \mathcal{V} = \arg \min_{(w, p) \in \mathcal{U}} \|w\|_M. $$

Define $\beta = \inf_{(w, p) \in \mathcal{U}} \|w\|_M$. Consider a sequence $\{f_n\}_{n \in \mathbb{N}} \in \mathcal{U}$ with $f_n = (w_n, p_n)$ such that $\|w_n\|_M$ decreases monotonically to $\beta$. Since $\mathcal{C}$ is compact, $A = \max_{x \in \mathcal{C}} \|x\|_2$ will be a finite
constant. Therefore, on one hand we have that
\[ \|p_n\|_F \leq \frac{1}{B} \|F(0, p_n)\|_2 = \frac{1}{B} \|F(w_n, p_n) - F(w_n, 0)\|_2 \]
\[ \leq \frac{1}{B}(\|F(w_n, p_n)\|_2 + \|F(w_n, 0)\|_2) \leq \frac{1}{B}(A + \|w_n\|_M). \]

On the other hand,
\[ \|f_n\|_B = \|w_n\|_M + \|p_n\|_F \leq \|w_n\|_M(1 + \frac{1}{B}) + \frac{A}{B} \]
\[ \leq \|w_0\|_M(1 + \frac{1}{B}) + \frac{A}{B} < +\infty, \]

which implies that \(\{f_n\}\) is a bounded sequence in \(U\) and therefore has a convergent subsequence \(\{f_{n_k}\}_{k \in \mathbb{N}}\) that converges to \(f^* \in B\) in the weak* topology. Therefore, \(\|F\|_B = \beta\) due to the weak*-continuity of the norm. By the weak*-closedness of \(U\), we also deduce that \(f^* \in U\), which implies that \(f^* \in \mathcal{V}\). Therefore, \(\mathcal{V}\) is not empty.

We can rewrite \(\mathcal{V}\) as \(\mathcal{V} = U \cap \{f = (w, p) \in B : \|w\|_M \leq \beta\}\). Therefore, one can interpret \(\mathcal{V}\) as the intersection of two weak*-closed and convex sets. This results in \(\mathcal{V}\) being weak*-closed and convex as well.

We also know that, for any \(f = (w, p) \in \mathcal{V}\), \(\|p\|_F \leq \frac{A + A\beta}{B} = \gamma\). Therefore,
\[ \mathcal{V} \subseteq \{f = (w, p) \in B : \|w_i\|_M \leq \beta \quad i = 1, 2, \ldots, d' \quad \|p\|_F \leq \gamma\}. \]

Finally, \(\mathcal{V}\) is a weak*-closed subset of a bounded cube in \(B\), which we know to be weak*-compact. This implies that \(\mathcal{V}\) is also a weak*-compact subset of \(B\). Using the Krein-Milman theorem (Rudin (1991), Theorem 3.23), we deduce that \(\mathcal{V}\) is the convex hull of its extreme points.

**Form of Extreme Points:** Consider an arbitrary extreme point of \(\mathcal{V}\) such as \((w, p)\), where \(w = (w_1, w_2, \ldots, w_{d'})\).

First, we show that it is not possible to have disjoint Borelian sets \(E_{i,j} \subseteq \mathbb{R}^d\) such that \(\langle w_i, 1_{E_{i,j}} \rangle \neq 0\), where \(i = 1, 2, \ldots, d'\) and \(j = 1, 2, \ldots, M_i\) with \(\sum_{i=1}^{d'} M_i \geq M + 1\). We prove the result by contradiction.

Assume such disjoint sets exist. Define \(v_{i,j} = w_i 1_{E_{i,j}}, v_{i,j} = e; v_{i,j}, E_{i,c} = (\bigcup_{j=1}^{M_i} E_{i,j})^c, v_{i,c} = w_i 1_{E_{i,c}}, \) and let \(w_c = (v_{1,c}, v_{2,c}, \ldots, v_{d',c})\). It can be seen that \(w = w_c + \sum_{i=1}^{d'} \sum_{j=1}^{M_i} v_{i,j}\). Define \(y_{i,j} = F(v_{i,j}, p)\). Since the \(y_{i,j}\) are at least \(M + 1\) vectors in \(\mathbb{R}^M\), they are linearly dependent. Consequently, there exist constants \(\alpha_{i,j} \in \mathbb{R}\) such that
\[ \sum_{i=1}^{d'} \sum_{j=1}^{M_i} \alpha_{i,j} y_{i,j} = 0. \]

For \(i = 1, 2, \ldots, d'\), define \(\mu_i = \sum_{j=1}^{M_i} \alpha_{i,j} v_{i,j}\) and \(\mu = (\mu_1, \mu_2, \ldots, \mu_{d'})\). Also, denote \(\epsilon_{\max} = \frac{1}{\max_{i,j} |\alpha_{i,j}|} > 0\). For any \(\epsilon \in (-\epsilon_{\max}, \epsilon_{\max})\), we have that \(1 + \epsilon \alpha_{i,j} > 0\) for all \(i = 1, 2, \ldots, d'\) and \(j = 1, 2, \ldots, M_i\). We can also see that
\[ F(\mu, p) = \sum_{i=1}^{d'} \sum_{j=1}^{M_i} \alpha_{i,j} y_{i,j} = 0. \]

Now, for any \(\epsilon \in (-\epsilon_{\max}, \epsilon_{\max})\), we have that
\[ F(w + \epsilon \mu, p) = F(w, p) \in C \]
and, therefore, \((w + \epsilon \mu, p) \in \mathcal{U}\). Moreover,

\[
    w + \epsilon \mu = w_c + \sum_{i=1}^{d'} \sum_{j=1}^{M_i} (1 + \epsilon \alpha_{i,j}) v_{i,j}.
\]

Note that the \(i\)th element of \(w_c\) has support \(E_{i,c}\). Moreover, the \(i\)th element of \(v_{i',j}\) has support \(E_{i',j}\) for \(i' = i\) and has empty support otherwise. Therefore, the \(i\)th entries have disjoint supports, which allow us to write that

\[
    \|w + \epsilon \mu\|_\mathcal{M} = \sum_{i=1}^{d'} \|v_{i,c} + \sum_{j=1}^{M_i} (1 + \epsilon \alpha_{i,j}) v_{i,j}\|_\mathcal{M}
\]

\[
    = \sum_{i=1}^{d'} \|v_{i,c}\|_\mathcal{M} + \sum_{i=1}^{d'} \sum_{j=1}^{M_i} (1 + \epsilon \alpha_{i,j}) \|v_{i,j}\|_\mathcal{M}
\]

\[
    = \beta + \epsilon \sum_{i=1}^{d'} \sum_{j=1}^{M_i} \alpha_{i,j} \|v_{i,j}\|_\mathcal{M}.
\]

For sufficiently small values of \(\epsilon\), this gives either \(\|w + \epsilon \mu\|_\mathcal{M} < \beta\) or \(\|w - \epsilon \mu\|_\mathcal{M} < \beta\). Therefore, \(\sum_{i=1}^{d'} \sum_{j=1}^{M_i} \alpha_{i,j} \|v_{i,j}\|_\mathcal{M} = 0\), which yields that \(\|w + \epsilon \mu\|_\mathcal{M} = \|w - \epsilon \mu\|_\mathcal{M} = \beta\). This shows that \((w + \epsilon \mu, p), (w - \epsilon \mu, p) \in \mathcal{V}\), which contradicts that \((w, p)\) is an extreme point. Therefore \(w\), is nonzero at most in \(M\) points, which yields the form of (15). Computing the norm of such an extreme point results in

\[
    \|w\|_\mathcal{M} = \sum_{i=1}^{d'} \sum_{j=1}^{M_i} |a_{i,j}| \|\delta(\cdot - x_{i,j})\|_\mathcal{M} = \sum_{i=1}^{d'} \sum_{j=1}^{M_i} |a_{i,j}|,
\]

which completes the proof.

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