Massive scalar field on (A)dS space from a massless conformal field in $\mathbb{R}^6$

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We show how the equations for the scalar field (including the massive, massless, minimally and conformally coupled cases) on de Sitter and Anti-de Sitter spaces can be obtained from both the SO(2,4)-invariant equation $\Box \phi = 0$ in $\mathbb{R}^6$ and two geometrical constraints defining the (A)dS space. Apart from the equation in $\mathbb{R}^6$, the results only follow from the geometry. We also show how an interaction term in (A)dS space can be taken into account from $\mathbb{R}^6$.

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I. INTRODUCTION

The relation between SO(2,4)-invariant free fields in $\mathbb{R}^6$ and fields in Minkowski space has long been known [1]. Nowadays, the space $\mathbb{R}^6$ together with the conformal symmetry, SO(2,4)-invariance and/or Weyl rescaling invariance, provides a tool in working properties of (conformal) fields in four dimensional spaces [2, 3]. Besides its usefulness, the existence of conformal symmetry as a (spontaneously broken) symmetry of Nature remains an open question (see for instance [4] and references herein). This point provides a strong motivation for investigations on conformal symmetry.

In this paper, we are interested in the relation between the free (and interacting) scalar fields in $\mathbb{R}^6$ and in four dimensional de Sitter (dS) or Anti-de Sitter (AdS) space. Indeed, in previous works [2, 6] we already considered the case of the conformal scalar. Other scalar fields, in particular the familiar massive scalar on de Sitter space, were not considered. Here, a new result is obtained: we show that the equations for the free scalar field in dS or AdS space, including the massless conformally coupled and the massive scalars as special cases, follow from the equation $\Box \phi = 0$ in $\mathbb{R}^6$ supplemented by two constraints. Precisely, we realize the (A)dS space as the intersection between the null cone of $\mathbb{R}^6$ and a hyperplane. These geometrical constraints naturally define operators which can be diagonalized simultaneously with the Laplace-Beltrami operator leading to the restriction of $\Box \phi = 0$ to the (A)dS space, that is the scalar equation on that space. The solutions are shown to be homogeneous functions in $\mathbb{R}^6$, whose degree of homogeneity can be freely set to obtain each scalar representation of the (A)dS group. A remarkable point is that the results come purely from the geometry: once the constraints defining the space (dS or AdS) as a sub-manifold in $\mathbb{R}^6$ are given, the whole properties of the field, including its homogeneity, follow.

The paper is organized as follows. The geometrical framework is summarized in Sec. II. We remind in particular how the (A)dS space can be obtained from two geometrical constraints in $\mathbb{R}^6$. We then briefly examine in Sec. III how these two constraints affect the free massless particle in $\mathbb{R}^6$. This allows us to make apparent in Sec. IV that the restriction of the Laplace-Beltrami operator of $\mathbb{R}^6$ to that of the (A)dS space comes, in our formalism, from purely geometrical constraints. We then obtain from a massless scalar field in $\mathbb{R}^6$ the free Klein-Gordon scalar field on de Sitter space, and we found that this field belongs to well known representations of SO$_0(1,4)$, including in particular the familiar massive one. We then examine the AdS case, and the representations of SO$_0(2,3)$, along the same lines. We exhibit a set of modes satisfying all these constraints. We then consider the action of the conformal group SO$_0(2,4)$. Finally, we show how an interaction term for the (A)dS field can be obtained from $\mathbb{R}^6$. We then conclude. A reminder of the scalar representations of both de Sitter and Anti-de Sitter groups are given for reference in appendix A.

II. GEOMETRY

A. Conventions

Our conventions for indexes are: $\alpha, \beta, \gamma, \delta, \ldots = 0, \ldots, 5$, and $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$, and concerning differential geometry we follow those of Fecko [7]. In addition, except otherwise stated, a superscript (or subscript) $f$ indicates a quantity on (A)dS space.

B. The (A)dS space in $\mathbb{R}^6$

The space $\mathbb{R}^6$ is provided with the natural orthonormal coordinates $\{y^a\}$ and the metric
η = diag(+, −, −, −, −, +) which is left invariant under the group O(2, 4) and its connected subgroup SOo(2, 4).

A four dimensional manifold \( X_f \) is obtained as the intersection of the five dimensional null cone of \( \mathbb{R}^6 \), \( C := \{ y \in \mathbb{R}^6 : C(y) = 0 \} \), where \( C(y) := y^α y_α \), and a surface \( P_f := \{ y \in \mathbb{R}^6 : f(y) = 1 \} \), in which \( f \) is homogeneous of degree one. Conformally flat spaces, including Robertson-Walker spaces, can be obtained by suitable choices of the homogeneous function \( f \). Details about this construction may be found in [3]. Here we focus on dS and AdS spaces. The de Sitter space is defined using the replacement \( \delta p_α \mapsto \delta α \). This justifies and replaces the usually postulated rule \( p_α \mapsto \partial_α \). We denote these vectors with the same symbols as before for simplicity: \( D = y^α \partial_α \) and \( F = (\partial^α f) \partial_α \).

The two additional constraints \( F \) and \( D \) now are realized as operators. Consequently, the restriction of \( \Box_6 \) to \( X_f \) appears as a particular case of the problem of finding the set of functions on which the operators \( \Box_6, D, \) and \( F \) are constants, that is to diagonalize them simultaneously. Then, the commutators

\[
\left[ \Box_6, D \right] = 2 \Box_6, \quad \left[ \Box_6, F \right] = 0, \quad \left[ F, D \right] = F,
\]

show us that the only possibility is \( \Box_6 \phi = F \phi = 0 \), \( D \phi \) being unspecified, or more precisely that the function \( \phi \) is homogeneous of some unspecified weight \( r \), namely \( D \phi = r \phi \). Note that the homogeneity of the field is not assumed, it is a consequence of the geometry. To summarize, we will verify that to restrict \( \Box_6 \phi = 0 \) to the space \( X_f \), corresponds to satisfy in \( \mathbb{R}^6 \) the system

\[
\begin{aligned}
\Box_6 \phi &= 0 \\
F \phi &= 0 \\
D \phi &= r \phi.
\end{aligned}
\]

Now, the explicit expression of the restriction of \( \Box_6 \) to the de Sitter manifold is obtained by a straightforward calculation as follows. We express \( \Box_6 \), \( D \) and \( F \) in the coordinate system (see for instance [2])

\[
x^α = \frac{y^α y_α}{(y^+ y^-)^2}, \quad x^μ = \frac{2y^μ}{y^+ y^-}, \quad x^+ = f(y).
\]

In it the two constraints \( C \) and \( f \) defining the de Sitter space \( X_f \) are respectively \( x^c = 0 \) and \( x^+ = 1 \), the dilation operator \( D \) reads \( x^+ \partial_+ \) and can be used to make

IV. THE CLASSICAL FREE SCALAR FIELD

We now consider the free Klein-Gordon equation in \( \mathbb{R}^6 \), namely \( \Box_6 \phi = 0 \), in which \( \Box_6 \) is the Laplace-Beltrami operator. As in the previous section we look at the restriction of this equation to the space \( X_f \). We will see that the whole scalars in the unitary irreducible representations (UIR) of the dS and AdS groups, including both massless and massive case, can be obtained from the geometrical constraints defined above.

A. Equations and constraints

As for the free particle we need some constraints in addition to \( f \) and \( C \) to perform the restriction of \( \Box_6 \phi = 0 \) to \( X_f \). One can expect that due to their geometrical nature the two constraints \( D \) and \( F \) will play a part in the restriction process. Indeed, the maps \( p \mapsto D \) and \( p \mapsto F \) define linear forms on the phase space \( T^* \mathbb{R}^6 \), they belong to \( T^* \mathbb{R}^6 \), and thus they naturally identify with vectors of \( T \mathbb{R}^6 \) (namely \( \frac{1}{2} \delta dC \) and \( \delta df \) respectively). This justifies and replaces the usually postulated rule \( p_α \mapsto \partial_α \). We denote these vectors with the same symbols as before for simplicity: \( D = y^α \partial_α \) and \( F = (\partial^α f) \partial_α \).

The two additional constraints \( F \) and \( D \) now are realized as operators. Consequently, the restriction of \( \Box_6 \) to \( X_f \) appears as a particular case of the problem of finding the set of functions on which the operators \( \Box_6, D, \) and \( F \) are constants, that is to diagonalize them simultaneously. Then, the commutators

\[
\left[ \Box_6, D \right] = 2 \Box_6, \quad \left[ \Box_6, F \right] = 0, \quad \left[ F, D \right] = F,
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show us that the only possibility is \( \Box_6 \phi = F \phi = 0 \), \( D \phi \) being unspecified, or more precisely that the function \( \phi \) is homogeneous of some unspecified weight \( r \), namely \( D \phi = r \phi \). Note that the homogeneity of the field is not assumed, it is a consequence of the geometry. To summarize, we will verify that to restrict \( \Box_6 \phi = 0 \) to the space \( X_f \), corresponds to satisfy in \( \mathbb{R}^6 \) the system

\[
\begin{aligned}
\Box_6 \phi &= 0 \\
F \phi &= 0 \\
D \phi &= r \phi.
\end{aligned}
\]
apparent the degree of homogeneity $r$ of the scalar field $\phi$. After some algebra, the expression of $\Box_0$ becomes

$$\Box_0 \phi^f = \Box_f \phi^f + 2(r+1)(F\phi)^f - r(r+3)H^2 \phi^f,$$

where $\Box_f$ is the Laplace-Beltrami operator of de Sitter space and $(A)^f$ is the restriction to $X_f$ of the scalar function $A$. We first note, that this expression does not depend on coordinates, as it is written in terms of geometrical objects, it is therefore a consequence of the geometry. Following the discussion of the previous paragraph $F\phi$ is now set to zero while the homogeneity $r$ is left unspecified. Finally, we obtain the equation on the de Sitter space $X_f$

$$\Box_f \phi^f - r(r+3)H^2 \phi^f = 0.$$  

(3)

It is now obvious that this equation yields the free Klein-Gordon equation with a suitable choice of $r$. Besides that, one realizes that as far as the Minkowski spacetime, for which $H^2 = 0$, is considered only the massless conformally invariant field fulfilling $\Box_f \phi^f = 0$ can be recovered. On (A)dS spaces, however, thanks to a nonvanishing curvature a wider class of fields can be reached, encompassing massive non-conformally invariant fields. Hereafter, such cases are inspected on (A)dS space and are linked to the relevant representations of group of isometries.

B. Scalar representations of the (Anti-)de Sitter group

The freedom left in the degree of homogeneity $r$ can be used to obtain various couplings. In fact, since on de Sitter space, the Laplace-Beltrami operator $\Box_f$, and the first order Casimir operator $Q_1$, are related through $\Box_f = -H^2 Q_1$, all scalar UIRs can be retrieved. Since we are in particular interested in the massive scalar let us set the parametrization

$$-r(r+3)H^2 \phi^f = (m^2 + 12\xi H^2)\phi^f,$$

$m$ and $\xi$ being parameters which in the massive representations of $SO_0(1,4)$, respectively identify with the mass of the field and the coupling constant to the scalar curvature $R = -12H^2$. The above equation can be solved for $r$ and the symmetry $r \rightarrow -(r+3)$ allows us to choose the positive root without loss of generality

$$r = \frac{3}{2} + \sqrt{\frac{9}{4} - (12\xi - \frac{m^2}{H^2}).}$$

This degree of homogeneity $r$, or more exactly the value of $12\xi + (m/H)^2$, can be used in order to classify the scalar UIRs of the de Sitter group, they are reminded in appendix A.

The case of the AdS space is obtained through the replacement $H \rightarrow iH$ in the de Sitter formulas. The equation (3) becomes

$$\Box_f \phi^f + r(r+3)H^2 \phi^f = 0,$$

and the corresponding reparametrization in terms of the “mass” $m$ and the coupling constant $\xi$ now reads:

$$r(r+3)H^2 \phi^f = (m^2 - 12\xi H^2)\phi^f,$$

a solution of which is

$$r = \frac{3}{2} + \sqrt{\frac{9}{4} - (12\xi - \frac{m^2}{H^2}),}$$

where we kept the positive root only, since it gives the right homogeneity degree $r = -1$ for the massless conformally coupled scalar field.

The value of $12\xi - (m/H)^2$ relates the field to scalar UIRs of $SO_0(2,3)$, the procedure is reminded in appendix A.

C. Modes solutions

We know already that the set of functions satisfying the system (1) is non-empty: it contains at least the conformal scalar for which $r = -1$ (2). In this section we note that modes solutions for other values of $r$ can also be found: the set of functions $\phi_{k,r}(y) := (k \cdot y)^r$, where $k$ is a constant vector of $\mathbb{R}^3$, satisfy

$$\Box_0 \phi_{k,r} = (r+1)k^2 \phi_{k,r-2},$$

$$F\phi_{k,r} = rf(k)\phi_{k,r-1},$$

$$D\phi_{k,r} = r\phi_{k,r},$$

from which it is apparent that $\{\phi_{k,r}(y)\}$ with $k^2 = 0$ and $f(k) = [1 + H^2]k^3 + (1 - H^2)k^4]/2 = 0$, is a set of modes for Eq. (3) on de Sitter space, that is for $y \in X_f$.

The study of these modes is out of the scope of the present paper. We just remark that when $(k \cdot y)$ vanishes these modes are ill-defined, but that this problem can be circumvented following a method analogous to that used in [13] where the modes are seen as boundary values of their complexified versions in a suitable domain of analyticity.

D. Group action

Since the constraint $F\phi = 0$ refers explicitly to the space $X_f$ we expect the invariance of the set of solutions of system (1) to be restricted to the (A)dS group. This is indeed the case, with the known exception of the conformal scalar field.

Let us recall the algebra of $so(2,4)$

$$[X_{\alpha\beta}, X_{\gamma\delta}] = \eta_{\beta\gamma}X_{\alpha\delta} + \eta_{\alpha\delta}X_{\beta\gamma} - \eta_{\alpha\gamma}X_{\beta\delta} - \eta_{\beta\delta}X_{\alpha\gamma},$$

which realizes on functions with $X_{\alpha\beta} = y_\alpha \partial_\beta - y_\beta \partial_\alpha$. One can check that these generators commute with both $\Box_0$ and $D$. By contrast, they do not commute with the operator $F$ except for $X_{\mu\nu}$ and the combination

$$Y_\mu := \frac{1}{2}(1 - H^2)X_\delta\mu - \frac{1}{2}(1 + H^2)X_\delta\mu,$$
which together form a (A)dS sub-algebra. We thus conclude that the set of solutions of the system \([1]\) is at least invariant under the (A)dS group.

For the special case of the conformal scalar, whose degree of homogeneity is \(r = -1\), the equation \([2]\) shows that the effect of the constraint \(F\) disappears. Consequently, the field remains invariant under the whole conformal group \(SO_0(2, 4)\). The explicit group action in that case includes a multiplicative conformal weight \(\omega_f\) which accounts for the group elements which are not in the (A)dS group. This point as already been described in our geometrical framework in \([3, 6]\).

### Appendix A: Scalar representations of (A)dS group

For the de Sitter \(SO_0(1, 4)\) the scalar irreducible representations are labeled as (see for instance \([\S]\)):

- \(12\xi + \frac{m^2}{H^2} \geq \frac{9}{4}\): Principal series of representations.
- \(0 < 12\xi + \frac{m^2}{H^2} < \frac{9}{4}\): Complementary series of representations with the special case of the massless conformally coupled field obtained for \(\xi = 1/6, \ m = 0\) and \(r = -1\).
- \(12\xi + \frac{m^2}{H^2} = 0\): The massless minimally coupled field, that is the first term of the discrete series of representations \([9]\).

The case of the scalar irreducible representations of the AdS \(SO_0(2, 3)\) group is performed as follows.

On the one hand, the UIRs of \(SO_0(2, 3)\) are labelled as \(\mathcal{D}(E_0, s)\), see \([8, 11]\). In the scalar case, here considered, one has \(\mathcal{D}(E_0 \geq 1/2, 0)\) for which the spectrum of the first order Casimir operator \(Q_1\) reduces to \(Q_1 = E_0 - 3\). The spectrum of \(Q_1\) is consequently bounded below with \(Q_1 \geq -9/4\), since \(E_0 \geq 1/2\).

On the other hand, on AdS in the scalar representation one can relate \(Q_1\) to the wave equation through \(\Box f = H^2 Q_1\). Then, the above bound translates into the condition

\[
-\frac{9}{4} \leq \left(12\xi - \frac{m^2}{H^2}\right).
\]

This condition is in direct relation to the Breitenlohner-Freedman bound \([11]\), for which a conserved convergent scalar product might be defined upon the space of solutions to the wave equation.

Thus, provided the above condition is fulfilled the relevant UIRs can be retrieved:

- \(-\frac{9}{4} < 12\xi - \frac{m^2}{H^2} \leq -\frac{5}{4}\). Two UIRs might be involved, namely \(\mathcal{D}\left(3/2 \pm \sqrt{9/4 + 12\xi - (m/H)^2}, 0\right)\). Those UIRs can be distinguished by their lowest eigenvalue of the “conformal energy” \(E_0\). In this specific range lies the massless conformally coupled field, which belongs to the reducible representation \(\mathcal{D}(1, 0) \oplus \mathcal{D}(2, 0)\), obtained for \(\xi = 1/6, \ m = 0\) and \(r = -1\). At the edge of this interval, for \(12\xi - (m/H)^2 = -5/4\), lies the, so-called, \(\mathcal{D}(1/2, 0) = \text{Rac representation } [12]\). In this interval both the “regular” and “irregular” modes are allowed, see \([11]\).

- \(-\frac{5}{4} < 12\xi - \frac{m^2}{H^2}\). Only one UIR might be involved, namely: \(\mathcal{D}\left(3/2 + \sqrt{9/4 + 12\xi - (m/H)^2}, 0\right)\).

There only the “regular” modes are allowed.
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