Peek Arc Consistency

Manuel Bodirsky     Hubie Chen

Abstract
This paper studies peek arc consistency, a reasoning technique that extends the well-known arc consistency technique for constraint satisfaction. In contrast to other more costly extensions of arc consistency that have been studied in the literature, peek arc consistency requires only linear space and quadratic time and can be parallelized in a straightforward way such that it runs in linear time with a linear number of processors. We demonstrate that for various constraint languages, peek arc consistency gives a polynomial-time decision procedure for the constraint satisfaction problem. We also present an algebraic characterization of those constraint languages that can be solved by peek arc consistency, and study the robustness of the algorithm.

1 Introduction

Background. A basic knowledge reasoning task that has been studied in many incarnations is to decide the satisfiability of given relationships on variables, where, for instance, variables may represent objects such as temporal events or spatial regions, and relationships may express precedence, containment, overlap, disjointness, and so forth. Instances of this reasoning task can typically be modeled using the constraint satisfaction problem (CSP), a computational problem in which the input consists of a set of constraints on variables, and the question is whether or not there is an assignment to the variables satisfying all of the constraints. While the CSP is in general NP-hard, researchers have, in numerous settings, aimed to identify restricted sets of relationships under which the CSP is polynomial-time decidable; we refer to sets of relationships as constraint languages.

Arc consistency is an algorithmic technique for constraint satisfaction that has been heavily studied and for which highly efficient implementations that are linear in both time and space are known. Arc consistency provides a one-sided satisfiability check. It may detect an inconsistency, which always implies that the input instance is unsatisfiable. While the converse does not hold in general, it has been shown to hold for some particular constraint languages, that is, arc consistency provides a decision procedure for satisfiability for these languages. Examples include the language of boolean Horn clauses; various graph homomorphism problems, for example, homomorphisms to orientations of finite paths [1]; and all constraint languages where satisfiability is first-order definable [1].

Curiously, arc consistency typically cannot be used as a decision procedure for infinite-domain constraint languages, by which we mean constraint languages under which variables can take on infinitely many values. In many cases, a reason for this is that arc consistency performs inference by considering unary (arity 1) projections of relations, and all such projections are already equal to the full domain of the language. As an example, consider the binary relations ≤ and ≠ interpreted over the domain of rational numbers Q. For each of these relations, both of the two possible unary projections are equal to Q, and arc consistency in fact will not perform any inference.

Path consistency is a more powerful algorithmic technique that provides a polynomial-time decision procedure for further finite and infinite domain constraint languages. However, the greater power comes at the price of worse time and space complexity: the best known implementations require cubic space and quadratic time. Unfortunately, this makes the path consistency procedure prohibitive for many applications where one has to deal with large instances of the constraint satisfaction problem.

Peek arc consistency. In this paper, we study a general algorithmic technique for constraint satisfaction that we call peek arc consistency. Here, we describe the idea of the algorithm for finite-domain constraint
satisfaction, although, as we show in the paper, this algorithm can be effectively applied to infinite-domain constraint satisfaction as well. The algorithm performs the following. For each variable-value pair \((x, a)\), the variable \(x\) is set to the value \(a\), and then the arc consistency procedure is run on the resulting instance of the CSP. If there is a variable \(x\) such that for all values \(a\) the arc consistency procedure detects an inconsistency on \((x, a)\), then the algorithm reports an inconsistency. As with arc consistency, this algorithm provides a one-sided satisfiability check. One might conceive of this algorithm as being a step more sophisticated than arc consistency; it invokes arc consistency as it takes a “peek” at each variable.

Peek arc consistency has many practical and theoretical selling points. Arc consistency can be implemented in linear space, for any fixed finite-domain constraint language and many infinite-domain constraint languages; the same holds for peek arc consistency. The time complexity of peek arc consistency is quadratic in the input size, which is still much better than the path-consistency algorithm, where the best known implementations have a running time that is cubic in the input size. Moreover, peek arc consistency can be parallelized in a straightforward way: for each variable-value pair, the arc consistency procedure can be performed on a different processor. Hence, with a linear number of processors, we achieve a linear running time, for a fixed constraint language. We would also like to remark that implementing peek arc consistency is straightforward if one has access to an implementation of arc consistency as a subroutine.

We demonstrate that the class of constraint languages solvable by peek arc consistency is a considerable extension of that which can be solved by arc consistency, and in particular contains many infinite-domain constraint languages. Examples are the constraint satisfaction problem for the point algebra in temporal reasoning \([15]\), and tractable set constraints \([8]\). But also, several finite-domain constraint languages where previously the “best” known algorithm was the path-consistency procedure can be solved by our peek arc consistency procedure. For example, this is the case for homomorphism problems to unbalanced orientations of cycles \([9]\). Other examples that can be solved by peek arc consistency but not by arc consistency are 2-SAT, and many other CSPs where the relations are closed under a dual discriminator or a median operation.

Our study of peek arc consistency employs universal algebraic techniques which have recently come into focus in the complexity of constraint satisfaction. In addition to obtaining results showing that languages are tractable by this algorithm, we develop an algebraic characterization of the constraint languages solvable by the algorithm. The characterization is exact–necessary and sufficient–for all finite and infinite domain constraint languages. We also exhibit closure properties on the class of constraint languages tractable by the algorithm.

A notable feature of this work is the end to which universal algebraic techniques are applied. Thus far, in constraint satisfaction, such techniques have primarily been used to demonstrate complexity class inclusion results, such as polynomial-time decidability results, and completeness results, such as NP-completeness results. Here, we utilize such techniques to investigate the power of a particular efficient and practical algorithm. That is, we differentiate among constraint languages depending on whether or not they are solvable via a specific algorithmic method, as opposed to whether or not they are contained in a complexity class. To our knowledge, this attitude has only been adopted in a limited number of previous papers that studied arc consistency and extensions thereof \([7, 5]\).

2 Preliminaries

Our definitions and notation are fairly standard.

**Structures.** A tuple over a set \(B\) is an element of \(B^k\) for a value \(k \geq 1\) called the arity of the tuple; when \(T\) is a tuple, we use the notation \(T = (t_1, \ldots, t_k)\) to denote its entries. A relation over a set \(B\) is a subset of \(B^k\) for a value \(k \geq 1\) called the arity of the relation. A signature \(\sigma\) is a finite set of symbols, each of which has an associated arity. We use \(\pi_i\) to denote the operator that projects onto the \(i\)th coordinate: \(\pi_i(T)\) denotes the \(i\)th entry \(t_i\) of a tuple \(T = (t_1, \ldots, t_k)\), and for a relation \(R\) we define \(\pi_i(R) = \{\pi_i(T) \mid T \in R\}\).

A structure \(B\) over signature \(\sigma\) consists of a universe \(B\), which is a set, and a relation \(R^B \subseteq B^k\) for each symbol \(R\) of arity \(k\). (Note that in this paper, we are concerned only with relational structures, which we refer to simply as structures.) Throughout, we will use the bold capital letters \(A, B, \ldots\) to denote
structures, and the corresponding non-bold capital letters A, B, . . . to denote their universes. We say that a structure B is finite if its universe B has finite size.

For two structures A and B over the same signature σ, the product structure A × B is defined to be the structure with universe A × B and such that R A×B = \{ (a 1 , b 1 ), . . . , (a k , b k ) \} | π ∈ RA , \bar{b} ∈ RB \} for all R ∈ σ. We use A n to denote the n-fold product A × · · · × A.

We say that a structure B over signature σ′ is an expansion of another structure A over signature σ if (1) σ′ ⊇ σ, (2) the universe of B is equal to the universe of A, and (3) for every symbol R ∈ σ, it holds that R B = R A. We will use the following non-standard notation. For any structure A (over signature σ) and any subset S ⊆ A, we define [A, S] to be the structure with the signature σ ∪ \{U\} where U is a new symbol of arity 1, defined by U[A, S] = S and R[A, S] = R A for all R ∈ σ.

For two structures A and B over the same signature σ, we say that A is an induced substructure of B if A ⊆ B and for every R ∈ σ of arity k, it holds that R A = A k ∩ R B. Observe that for a structure B and a subset B′ ⊆ B, there is exactly one induced substructure of B with universe B′.

**Homomorphisms and the constraint satisfaction problem.** For structures A and B over the same signature σ, a homomorphism from A to B is a mapping h : A → B such that for every symbol R of σ and every tuple (a 1 , . . . , a k ) ∈ RA, it holds that (h(a 1 ), . . . , h(a k )) ∈ R B. We use A → B to indicate that there is a homomorphism from A to B; when this holds, we also say that A is homomorphic to B.

The homomorphism relation → is transitive, that is, if A → B and B → C, then A → C.

For any structure B (over σ), the constraint satisfaction problem for B, denoted by CSP(B), is the problem of deciding, given as input a finite structure A over σ, whether or not there exists a homomorphism from A to B. In discussing a problem of the form CSP(B), we will refer to B as the constraint language.

There are several equivalent definitions of the constraint satisfaction problem for a constraint language, most notably the definition used in artificial intelligence. In logic, the constraint satisfaction problem can be formulated as the satisfiability problem for primitive positive formulas in a fixed structure B. Homomorphism problems as defined above have been studied independently from artificial intelligence in graph theory, and the connection to constraint satisfaction problems has been observed in [10].

**pp-definability.** Let σ be a signature; a primitive positive formula over σ is a formula built from atomic formulas \( R(w_1, \ldots, w_n) \) with R ∈ σ, conjunction, and existential quantification. A relation R ⊆ B k is primitive positive definable (pp-definable) in a structure B (over σ) if there exists a primitive positive formula \( \phi(v_1, \ldots, v_k) \) with free variables v 1 , . . . , v k such that

\[ (b_1, \ldots, b_k) \in R \iff B, b_1, \ldots, b_k \models \phi. \]

**Automorphisms.** An isomorphism between two relational structures A and B over the same signature σ is a bijective mapping from A to B such that \( \mathcal{T} \in R^A \) if and only if \( f(\mathcal{T}) \in R^B \) for all relation symbols R in σ. An automorphism of A is an isomorphism between A and A. An orbit of A is an equivalence class of the equivalence relation ≡ that is defined on A by x ≡ y iff \( \alpha(x) = y \) for some automorphism \( \alpha \) of A.

**Polymorphisms.** When \( f : B^n → B \) is an operation on B and \( \mathcal{T}_1 = (t_{11}, \ldots, t_{1k}), \ldots, \mathcal{T}_n = (t_{n1}, \ldots, t_{nk}) \in B^k \) are tuples of the same arity k over B, we use \( f(\mathcal{T}_1, \ldots, \mathcal{T}_n) \) to denote the arity k tuple obtained by applying \( f \) coordinatewise, that is, \( f(\mathcal{T}_1, \ldots, \mathcal{T}_n) = (f(t_{11}, \ldots, t_{n1}), \ldots, f(t_{1k}, \ldots, t_{nk})) \). An operation \( f : B^n → B \) is a polymorphism of a structure B over σ if for every symbol R ∈ σ and any tuples \( \mathcal{T}_1, \ldots, \mathcal{T}_n \in R^B \), it holds that \( f(\mathcal{T}_1, \ldots, \mathcal{T}_n) \in R^B \). That is, each relation R B is closed under the action of \( f \). Equivalently, an operation \( f : B^n → B \) is a polymorphism of B if it is a homomorphism from B n to B. Note that every automorphism is a unary polymorphism.

**Categoricity.** Several of our examples for constraint languages over infinite domains will have the following property that is of central importance in model theory. A countable structure is \( \omega \)-categorical if all countable models of its first-order theory\(^\text{1}\) are isomorphic. By the Theorem of Ryll-Nardzewski (see

\(^\text{1}\)The first-order theory of a structure is the set of first-order sentences that is true in the structure.
e.g. [12]) this is equivalent to the property that for each $n$ there is a finite number of inequivalent first-order formulas over $\Gamma$ with $n$ free variables. A well-known example of an $\omega$-categorical structure is $(\mathbb{Q}, <)$; for many more examples of $\omega$-categorical structures and their application to formulate well-known constraint satisfaction problems, see [2].

3 Arc Consistency

In this section, we introduce the notion of arc consistency that we will use, and review some related notions and results. The definitions we give apply to structures with relations of any arity, and not just binary relations. The notion of arc consistency studied here is sometimes called hyperarc consistency. Our discussion is based on the paper [7].

For a set $B$, let $\varphi(B)$ denote the power set of $B$. For a structure $B$ (over $\sigma$), we define $\varphi(B)$ to be the structure with universe $\varphi(B) \setminus \{\emptyset\}$ and where, for every symbol $R \in \sigma$ of arity $k$, $R^{\varphi(B)} = \{ (\pi_1 S, \ldots, \pi_k S) \mid S \subseteq R^B, S \neq \emptyset \}$.  

**Definition 1** An instance $A$ of CSP($B$) has the arc consistency condition (ACC) if there exists a homomorphism from $A$ to $\varphi(B)$.

**Definition 2** We say that arc consistency (AC) decides CSP($B$) if for all finite structures $A$, the following holds: $(A, B)$ has the ACC implies that $A \rightarrow B$.

Note that the converse of the condition given in this definition always holds. By a singleton, we mean a set containing exactly one element.

**Proposition 3** For any structures $A$ and $B$, if $h$ is a homomorphism from $A$ to $B$, then the mapping that takes $a$ to the singleton $\{h(a)\}$ is a homomorphism from $A$ to $\varphi(B)$.

Hence, when AC decides CSP($B$), an instance $A$ of CSP($B$) is a “yes” instance if and only if $A$ has the ACC with respect to $B$. That is, deciding whether an instance $A$ is a “yes” instance can be done just by checking the ACC. It was observed in [10] that, for any finite structure $B$, there is an algebraic characterization of AC: AC decides CSP($B$) if and only if there is a homomorphism from $\varphi(B)$ to $B$.

It is well-known that for a finite structure $B$, whether or not instances $A$ of CSP($B$) have the ACC can be checked in polynomial-time. The algorithm for this is called the arc consistency procedure, and it can be implemented in linear time and linear space in the size of $A$; note that we consider $B$ to be fixed. The same holds for many infinite-domain constraint languages, for example for all $\omega$-categorical constraint languages. Since this is less well-known, and requires a slightly less standard formulation of the arc consistency procedure, we present a formal description of the algorithm that we use; this algorithm can be applied for any (finite- and infinite-domain) constraint language that has finitely many pp-definable unary relations in $B$.

We assume that $B$ contains a relation for each unary primitive positive definable relation in $B$. This is not a strong assumption, since we might always study the expansion $B'$ of $B$ by all such unary relations. Then, if we are given an instance $A$ of CSP($B$), we might run the algorithm for CSP($B'$) on the expansion $A'$ of $A$ that has the same signature as $B'$ and where the new unary relations are interpreted by empty relations. It is clear that a mapping from $A$ to $B$ is a homomorphism from $A$ to $B$ if and only if it is a homomorphism from $A'$ to $B'$.

To conveniently formulate the algorithm, we write $R_{\phi(x)}$ for the relation symbol of the relation that is defined by a pp-formula $\phi(x)$ in $B$. We write $Q_A(\{a_1, \ldots, a_l\})$ for the conjunction over all formulas of the form $S(a_i)$ where $S$ is a unary relation symbol such that $a_i \in S^A$. The pseudo-code of the arc consistency procedure can be found in Figure[1]

The space requirements of the given arc consistency procedure are clearly linear. It is also well-known and easy to see that the procedure can be implemented such that its running time is linear in the size of the input.

**Proposition 4** Let $B$ be a structure with finitely many primitive-positive definable unary relations. Then a given instance $A$ of CSP($B$) has the ACC if and only if the arc consistency procedure presented in Figure[1] does not reject.
Arc Consistency\(_B(A)\)
Input: a finite relational structure \(A\).

Do
For every relation symbol \(R\), every tuple \((a_1, \ldots, a_l) \in R^A\), and every \(i \in \{1, \ldots, l\}\)
Let \(\phi\) be the formula
\[
\exists a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_l. R(a_1, \ldots, a_l) \land Q_A(\{a_1, \ldots, a_l\})
\]
If \(\phi\) defines the empty unary relation over \(B\) then reject
Else, add \(a_i\) to \(R^A\)
Loop until no relation in \(A\) is changed

Figure 1: The arc consistency procedure for CSP\(_B\), where \(B\) contains all primitive positive definable unary relations in \(B\).

In particular, we can apply the algorithm shown in Figure 1 to all constraint satisfaction problems with an \(\omega\)-categorical constraint language. However, it was shown that in this case the algorithm cannot be used as a decision procedure for CSP\(_B\) (i.e., that rejects an instance \(A\) if and only if it does not homomorphically map to \(B\)), unless \(B\) is homomorphically equivalent to a finite structure [4].

4 Peek Arc Consistency

We present basic definitions and results concerning peek arc consistency. The following two definitions are analogous to Definitions [1] and [2] of the previous section.

**Definition 5** An instance \((A, B)\) of the CSP has the peek arc consistency condition (PACC) if for every element \(a \in A\), there exists a homomorphism \(h\) from \(A\) to \(\varphi(B)\) such that \(h(a)\) is a singleton.

**Definition 6** We say that peek arc consistency (PAC) decides CSP\(_B\) if for all finite structures \(A\), the following holds: \((A, B)\) has the PACC implies that \(A \to B\).

The converse of the condition given in this definition always holds. Suppose that \(A \to B\); then, the mapping taking each \(a \in A\) to the singleton \(\{h(a)\}\) is a homomorphism from \(A\) to \(\varphi(B)\) (Proposition [3]), and hence \((A, B)\) has the PACC.

We now present an algorithm that decides for a given instance \(A\) of CSP\(_B\), whether \((A, B)\) has the PACC. We assume that \(B\) has a finite number of orbits and pp-definable binary relations. This holds in particular for all \(\omega\)-categorical structures. The following lemma then allows us to use the arc consistency procedure presented in Figure 1 for every expansion of \(B\) by singletons.

**Lemma 7** Let \(B\) be a structure with finitely many pp-definable binary relations. Then every expansion of \(B\) by a constant has finitely many pp-definable unary relations.

**Proof.** Suppose for contradiction that for a constant \(b\), there are infinitely many pairwise distinct unary relations with a pp-definition in the expansion of \(B\) with \(\{b\}\). For each such definition, if we replace the occurrences of the relation symbol for the singleton \(\{b\}\) by a new variable, we obtain formulas that are pp-definitions in \(B\) of pairwise distinct binary relations. □

**Proposition 8** Let \(B\) be a structure with finitely many orbits and finitely many pp-definable binary relations. Then a given instance \(A\) of CSP\(_B\) has the PACC if and only if the algorithm presented in Figure 2 does not reject \(A\).

**Proof.** Suppose that \(A\) is an instance of CSP\(_B\) that has the PACC. We have to show that for any element \(a\) from \(A\) there exists an orbit \(O\) of \(B\) such that for any choice of \(b \in O\) the arc consistency procedure that is called in the inner loop of the algorithm in Figure 2 does not reject the instance \([A, \{a\}]\) of CSP\((B, \{b\})\).
Because \( \mathbf{A} \) has the PACC, there exists a homomorphism from \( \mathbf{A} \) to \( \wp(\mathbf{B}) \) such that \( h(a) \) is a singleton \( \{c\} \). Let \( O \) be the orbit of \( c \), and let \( b \) be the element from \( O \) that is used by the algorithm. We know that there exists an automorphism \( \alpha \) that maps \( c \) to \( b \). Clearly, the mapping \( h' \) defined by \( x \mapsto \alpha(h(x)) \) is a homomorphism from \( \mathbf{A} \) to \( \wp(\mathbf{B}) \) such that \( h'(a) = \{b\} \) is a singleton. By Lemma 7 the structure \( \wp(\mathbf{B}, \{b\}) \) has finitely many pp-definable unary relations. Proposition 4 then shows that the arc consistency procedure does not reject the instance \( \mathbf{A}, \{a\} \) of \( \text{CSP}(\wp(\mathbf{B}, \{b\})) \). All the implications in this argument can be reversed, which shows the statement of the proposition. \( \square \)

**Theorem 9** Let \( \mathbf{B} \) be a structure with finitely many orbits and finitely many pp-definable binary relations, and suppose that PAC solves \( \text{CSP}(\mathbf{B}) \). Then there exists a quadratic-time and linear-space algorithm that decides \( \text{CSP}(\mathbf{B}) \). Moreover, \( \text{CSP}(\mathbf{B}) \) can be decided in linear time with a linear number of processors.

**Proof.** Let \( \mathbf{A} \) be an instance of \( \text{CSP}(\mathbf{B}) \). If the algorithm in Figure 2 rejects \( \mathbf{A} \), then it does not have the PACC and hence \( \mathbf{A} \) does not homomorphically map to \( \mathbf{B} \). If the algorithm in Figure 2 does not reject \( \mathbf{A} \), then \( \mathbf{A} \) has the PACC. By assumption, PAC solves \( \text{CSP}(\mathbf{B}) \), and therefore there exists a homomorphism from \( \mathbf{A} \) to \( \mathbf{B} \).

Because the arc consistency procedure uses linear space, the algorithm in Figure 2 can be implemented in linear space as well. The arc consistency procedure is called a linear number of times (recall that it is finite, and not part of the input). Because the arc consistency procedure can be implemented such that it uses linear time, the overall running time on a sequential machine is quadratic in the worst case. However, note that each application of the arc consistency procedure can be performed on a different processor. \( \square \)

### 5 Algebraic Characterization

In this section we present a general algebraic characterization of those constraint languages where PAC decides \( \text{CSP}(\mathbf{B}) \) for an arbitrary finite or infinite structure \( \mathbf{B} \).

We use the notation \( \text{Ind}(\wp(\mathbf{B})^n) \) to denote the induced substructure of \( \wp(\mathbf{B})^n \) whose universe contains an \( n \)-tuple of \( \wp(\mathbf{B})^n \) if and only if at least one coordinate of the tuple is a singleton.

**Theorem 10** Let \( \mathbf{B} \) be a structure. PAC decides \( \text{CSP}(\mathbf{B}) \) if and only if for all \( n \) there is a homomorphism from all finite substructures of \( \text{Ind}(\wp(\mathbf{B})^n) \) to \( \mathbf{B} \).

**Proof.** (\( \Leftarrow \)): Suppose that \( (\mathbf{A}, \mathbf{B}) \) has the PACC. Then, by definition of the PACC, for all \( a \in A \), there is a homomorphism \( h_a \) from \( [\mathbf{A}, a] \) to \( \wp(\mathbf{B}), \{\{b\} \mid b \in B\} \). Let \( n = |A| \). Now consider the homomorphism \( h \) from \( \mathbf{A} \) to \( \wp(\mathbf{B})^n \) defined by \( h(x) = \Pi_{a \in A} h_a(x) \). Notice that for every \( a \in A \), the element \( h_a(a) \) of the tuple \( h(a) \) is a singleton, and hence \( h \) is in fact a homomorphism from \( \mathbf{A} \) to \( \text{Ind}(\wp(\mathbf{B})^n) \). Let \( \mathbf{C} \) be the structure that is induced by the image of \( h \) in \( \text{Ind}(\wp(\mathbf{B})^n) \). Since \( \mathbf{C} \) is finite, it is by assumption homomorphic to \( \mathbf{B} \), and by composing homomorphisms we obtain that there is a homomorphism from \( \mathbf{A} \) to \( \mathbf{B} \).
(\Rightarrow): Let \( n \geq 1 \), and let \( C \) be a finite substructure of \( \text{Ind}(\varphi(B)^n) \). We have to show that \( C \), viewed as an instance of CSP(B), has the PACC, which suffices by assumption. Let \( a \) be any element of the universe of \( C \). By definition of \( C \), we have that \( a \) is an \( n \)-tuple such that some coordinate, say the \( i \)th coordinate, is a singleton. The projection function \( \pi_i \) is a homomorphism from \( [C, \{a\}] \) to \( [\varphi(B), \{\{b\} \mid b \in B\}] \). \( \square \)

6 Robustness

In this section, we demonstrate that the class of structures \( B \) such that PAC decides CSP(B) is robust in that it satisfies certain closure properties.

We first investigate expansion by pp-definable relations. Say that a structure \( B' \) (over \( \sigma' \)) is a pp-expansion of \( B \) (over \( \sigma \)) if \( B' \) is an expansion of \( B \) and for every symbol \( R \in \sigma' \setminus \sigma \), it holds that \( R^{B'} \) is pp-definable over \( B \).

**Theorem 11** Suppose that PAC decides CSP(B). Then for any pp-expansion \( B' \) of \( B \), it holds that PAC decides CSP(B').

**Proof.** It suffices to show that the theorem holds for an expansion of \( B \) by \( (1) \) an intersection of existing relations, \( (2) \) a projection of an existing relation, \( (3) \) a product of an existing relation with \( B \), or \( (4) \) the equality relation. In each of these cases, we will consider an expansion \( B' \) of \( B \) where the signature of \( B' \) has an additional symbol \( T \). We will use \( \sigma \) to denote the signature of \( B \), and so the signature of \( B' \) will be \( \sigma \cup \{T\} \).

By Theorem10 it suffices to show that for every \( n \geq 1 \) and for all finite substructures \( C' \) of \( \text{Ind}(\varphi(B')^n) \) there exists a homomorphism \( h \) from \( C' \) to \( B' \). Let \( C \) be a finite subset of the universe of \( \text{Ind}(\varphi(B)^n) \), let \( C \) be the induced substructure of \( \text{Ind}(\varphi(B)^n) \) with universe \( C \), and let \( C' \) be the induced substructure of \( \text{Ind}(\varphi(B')^n) \) with universe \( C \). By Theorem10 there is a homomorphism \( h \) from \( C' \) to \( B' \). Since \( B' \) is an expansion of \( B \) with just one additional symbol \( T \), it suffices to show that \( h(T^{C'}) \subseteq T^{B'} \).

(1): Suppose that \( T^{B'} = R^{B} \cap S^{B} \) for \( R, S \in \sigma \). It follows that \( T^{\varphi(B')} \subseteq R^{\varphi(B)} \cap S^{\varphi(B)} \), from which we obtain \( T^{C'} \subseteq R^{C} \cap S^{C} \). For any tuple \( \overline{t} \in T^{C'} \), we thus have \( h(\overline{t}) \in R^{B} \cap S^{B} \), and hence \( h(\overline{t}) \in T^{B'} \).

(2): Suppose that \( T^{B'} = \{ (t, t) \mid t \in R^{B} \} \). Then for any tuple \( \overline{t} \in T^{C'} \), we have that \( h(\overline{t}) \) is the projection of a tuple in \( R^{B} \), and hence in \( T^{B'} \).

(3): Suppose that \( T^{B'} = R^{B} \times B \) for \( R \in \sigma \). Let \( \overline{t} = (t_1, \ldots, t_{k+1}) \) be any tuple in \( T^{C'} \). We have that \( (t_1, \ldots, t_k) \in R^{C} \), and hence \( h(t_1, \ldots, t_k) \in R^{B} \). Since we have \( h(t_{k+1}) = b \), it follows that \( h(t_1, \ldots, t_{k+1}) \in T^{B'} \).

(4): Suppose that \( T^{B'} = \{ (b, b) \mid b \in B \} \). For any tuple \( (t_1, t_2) \in T^{\varphi(B')} \), we have \( t_1 = t_2 \). For any tuple \( (t_1, t_2) \in T^{C'} \) we thus also have \( t_1 = t_2 \), and we have \( h(t_1, t_2) \in T^{B'} \). \( \square \)

We now consider homomorphic equivalence.

**Theorem 12** Let \( B \) be a structure. Suppose that PAC decides CSP(B) and that \( B' \) is a structure that is homomorphically equivalent to \( B \), that is, \( B \rightarrow B' \) and \( B' \rightarrow B \). Then PAC decides CSP(B').

We first establish the following lemma.

**Lemma 13** Let \( f \) be a homomorphism from \( B' \) to \( B \). The map \( f' \) defined on \( \varphi(B') \setminus \{\emptyset\} \) by \( f'(U) = \{ f(u) \mid u \in U \} \) is a homomorphism from \( \varphi(B') \) to \( \varphi(B) \).

**Proof.** Let \( R \) be a symbol, and let \( \overline{t} \) be a tuple in \( R^{\varphi(B')} \). We have \( \overline{t} = (\pi_1 S, \ldots, \pi_k S) \) where \( S \subseteq R^B \) and \( S \neq \emptyset \). Define \( S' = \{ f(\overline{s}) \mid \overline{s} \in S \} \). We have \( S' \subseteq R^{B'} \). As \( f'(\overline{t}) = (\pi_1 S', \ldots, \pi_k S') \), the conclusion follows. \( \square \)
Example 14 Let $B$ be a set, and let $d : B^3 \rightarrow B$ be the operation such that $d(x, y, z)$ is equal to $x$ if $x = y$, and $z$ otherwise. This operation is known as the dual discriminator on $B$, and is an example of a slice-semilattice operation. For examples of constraint languages that have a dual discriminator polymorphism, see e.g. [14].

Example 15 Let $B$ be a subset of the rational numbers, and let $\text{median} : B^3 \rightarrow B$ be the ternary operation on $B$ that returns the median of its arguments. (Precisely, given three arguments $x_1$, $x_2$, and $x_3$ in ascending order so that $x_1 \leq x_2 \leq x_3$, the median operation returns $x_2$.) This operation is an example of a slice-semilattice operation.

Theorem 16 Let $B$ be a finite structure that has a slice-semilattice polymorphism. Then, the problem $\text{CSP}(B)$ is tractable by PAC.

Proof. Let $f$ denote the slice-semilattice polymorphism. By Theorem [10] it suffices to show that for every finite substructure $C$ of $\text{Ind}(\varphi(B)^n)$ there is a homomorphism $h$ from $C$ to $B$.

For each element $(S_1, \ldots, S_n)$ in $C$ we define $h(S_1, \ldots, S_n)$ as follows. Let $g$ be the maximum index such that $S_g$ is a singleton; we are guaranteed the existence of such an index by the definition of $\text{Ind}(\varphi(B)^n)$. We define a sequence of values $b_g, \ldots, b_n \in B$ inductively. Set $b_g$ to be the value such that $\{b_g\} = S_g$. For $i$ with $g < i \leq n$, define $b_i = \bigoplus_{b_j \leq i} S_j$. We define $h(S_1, \ldots, S_n) = b_n$.

We now show that $h$ is in fact a homomorphism from $C$ to $B$. Let $R$ be any symbol of arity $k$. Suppose that $((S_1^1, \ldots, S_1^n), \ldots, (S_k^l, \ldots, S_k^n)) \in R^{\text{Ind}(\varphi(B)^n)}$. We define a sequence of tuples $\overline{t_1}, \ldots, \overline{t_k} \in R^B$ in the following way. Let $\overline{t_i}$ be any tuple such that $\overline{t_i} \in (S_1^1 \times \cdots \times S_k^l) \cap R^B$. For $i$ with $1 < i \leq n$, we define $\overline{t_i} = \bigoplus_{t_{i-1} \leq i} S_i^1, \ldots, \bigoplus_{t_{i-1} \leq i} S_i^n$. Given that $\overline{t_{i-1}}$ is in $R^B$, we prove that $\overline{t_i}$ is in $R^B$. Let $C_i \subseteq R^B$ be a set of tuples such that $\varphi(C_1) = (S_1^1, \ldots, S_k^n)$. Let $\overline{c_1}, \ldots, \overline{c_m}$ with $m \geq 2$ be a sequence of tuples such that $\{\overline{c_1}, \ldots, \overline{c_m}\} = C_i$. We have $\overline{t_i} = f(\overline{c_1}, \ldots, \overline{c_m}, f(\overline{c_1}, \overline{c_2}, \overline{t_{i-1}}), \overline{t_{i-1}})$. Since $f$ is a polymorphism of $R^B$, we obtain $\overline{t_i} \in R^B$.

Observe now that for each tuple $(S_1^1, \ldots, S_n^n)$, the values $b_g, \ldots, b_n$ that were computed to determine $h(S_1^1, \ldots, S_n^n) = b_n$ have the property that for each $i$ with $g \leq i \leq n$, $b_i = t_{ij}$. It follows that $h$ is the desired homomorphism. □

It is well-known that the problem 2-SAT can be identified with the problem CSP($B$) for the structure $B$ with universe $B = \{0, 1\}$ and relations

$$
\begin{align*}
R^B_{(0,0)} &= \{(0, 1)^2 \setminus \{(0, 0)\} \\
R^B_{(0,1)} &= \{0, 1\}^2 \setminus \{(0, 1)\} \\
R^B_{(1,1)} &= \{0, 1\}^2 \setminus \{(1, 1)\}
\end{align*}
$$
It is known, and straightforward to verify, that the dual discriminator operation on \( \{0, 1\} \) is a polymorphism of this structure \( B \). We therefore obtain the following.

**Theorem 17** The problem 2-SAT is tractable by PAC.

Let \( \sigma \) be the signature \( \{E\} \) where \( E \) is a symbol having arity 2. We call a structure \( G \) over \( \sigma \) an undirected bipartite graph if \( E^G \) is a symmetric relation, the universe \( G \) of \( G \) is finite, and \( G \) can be viewed as the disjoint union of two sets \( V_0 \) and \( V_1 \) such that \( E^G \subseteq (V_0 \times V_1) \cup (V_1 \times V_0) \).

**Theorem 18** Let \( G \) be an undirected bipartite graph. The problem \( \text{CSP}(G) \) is tractable by PAC.

**Proof.** Let \( G' \) be the bipartite graph with universe \( \{0, 1\} \) and where \( E^{G'} = \{(0, 1), (1, 0)\} \). As \( E^{G'} \) is pp-definable over the structure \( B \) corresponding to 2-SAT above, by \( \phi(v_1, v_2) \equiv R_{(0,1)}(v_1, v_2) \wedge R_{(0,1)}(v_2, v_1) \), we have that PAC decides \( \text{CSP}(G') \) by Theorem 11.

If \( E^G \) is empty, the claim is trivial, so assume that \( (s, s') \in E^G \). We claim that \( G \) and \( G' \) are homomorphically equivalent, which suffices by Theorem 11. The map taking \( 0 \mapsto s \) and \( 1 \mapsto s' \) is a homomorphism from \( G' \) to \( G \). The map taking all elements in \( V_0 \) to 0 and all elements in \( V_1 \) to 1 is a homomorphism from \( G \) to \( G' \). \( \square \)

We call a finite structure \( D \) over signature \( \{A\} \) where \( A \) is a binary relation symbol an orientation of a cycle if \( D \) can be enumerated as \( d_1, \ldots, d_n \) such that \( D^D \) contains either \( (d_i, d_{i+1}) \) or \( (d_{i+1}, d_i) \) for all \( 1 \leq i < n \), contains either \( (d_n, d_1) \) or \( (d_1, d_n) \), and contains no other pairs. The orientation of a cycle is called unbalanced if the number of elements \( D^D \) of the form \( (d_i, d_{i+1}) \) or \( (d_n, d_1) \) is distinct from \( n/2 \). It has been shown in [9] that for every unbalanced orientation of a cycle \( D \) there is a linear order on \( D \) such that \( D \) is preserved by the median operation with respect to this linear order.

We therefore have the following result.

**Theorem 19** Let \( D \) be an unbalanced orientation of a cycle. Then \( \text{CSP}(D) \) is tractable via PAC.

**The Point Algebra in Temporal Reasoning.** The structure \( (\mathbb{Q}; \leq, \neq) \) is known as the point algebra in temporal reasoning. The problem \( \text{CSP}(\mathbb{Q}; \leq, \neq) \) can be solved by the path-consistency procedure [16].

**Theorem 20** \( \text{CSP}(\mathbb{Q}; \leq, \neq) \) is tractable via PAC.

**Proof.** Clearly, the structure \( (\mathbb{Q}; \leq, \neq) \) has only one orbit. It is well-known that it is also \( \omega \)-categorical [12], and therefore has in particular a finite number of pp-definable binary relations. To apply Theorem 9 we only have to verify that PAC decides \( \text{CSP}(\mathbb{Q}; \leq, \neq) \).

Let \( A \) be an instance of \( \text{CSP}(\mathbb{Q}; \leq, \neq) \). We claim that if there is a sequence \( a_1, \ldots, a_k \in A \) such that \( (a_i, a_{i+1}) \in \leq A \) for all \( 1 \leq i < k \), \( (a_k, a_1) \in \leq A \), and \( (a_p, a_q) \in \neq A \) for some \( p, q \in \{1, \ldots, k\} \), then there is no homomorphism from \( A \) to \( \varphi(B) \) such that \( h(a_1) \) is a singleton \( \{b_1\} \). Suppose otherwise that there is such a homomorphism \( h \). By the definition of \( \varphi(B) \) there must be a sequence \( b_1, \ldots, b_k \) such that \( b_i = h(a_i) \) for all \( 1 \leq i \leq k \) and \( (b_i, b_{i+1}) \in \leq B \) for all \( 1 \leq i < n \). Moreover, \( (b_k, b_1) \in \leq B \), and hence \( b_1 = \cdots = b_k \). But then we have \( (h(a_p), h(a_q)) = (b_1, b_1) \in \neq B \), a contradiction. Hence, the structure \( A \) does not have the PACC if \( A \) is such a sequence \( a_1, \ldots, a_k \). It is known [16] that if \( A \) does not contain such a sequence, then \( A \to (\mathbb{Q}; \leq, \neq) \). This shows that PAC decides \( \text{CSP}(\mathbb{Q}; \leq, \neq) \). \( \square \)

**Set Constraints.** Reasoning about sets is one of the most fundamental reasoning tasks. A tractable set constraint language has been introduced in [8]. The constraint relations in this language are containment \( X \subseteq Y \) ("every element of \( X \) is contained in \( Y \)"), disjointness \( X \upharpoonright Y \) (\( X \) and \( Y \) do not have common elements’), and disequality \( X \neq Y \) (\( X \) and \( Y \) are distinct’). In the CSP for this constraint language we are given a set of constraints and a set of containment, disjointness, and disequality constraints between variables, and we want to know whether it is possible to assign sets (we can without loss of generality assume that we are looking for subsets of the natural numbers; note that we allow the empty set) to these variables such that all the given constraints are satisfied. It was shown in [3] that this problem can be modeled as \( \text{CSP}(\downarrow D; \subseteq, \upharpoonright, \neq) \), where \( D \subseteq 2^\mathbb{N} \) is a countably infinite set of subsets of \( \mathbb{N} \), and such that \( (D; \subseteq, \upharpoonright, \neq) \) is \( \omega \)-categorical and has just two orbits (the orbit for \( \emptyset \), and the orbit for all other points).
Theorem 21. CSP\((\mathcal{D}; \subseteq, ||, \neq)\) is tractable via PAC.

**Proof.** Because \((\mathcal{D}; \subseteq, ||, \neq)\) is \(\omega\)-categorical, it suffices as in the proof of Theorem 20 to verify that PAC decides CSP\((\mathcal{D}; \subseteq, ||, \neq)\) in order to apply Theorem 9. Let \(\mathcal{A}\) be an instance of CSP\((\mathcal{D}; \subseteq, ||, \neq)\). We claim that if there are four sequences \((a_1^i, \ldots, a_k^i)\), \(\ldots, (a_1^j, \ldots, a_k^j)\) of elements from \(\mathcal{A}\) such that

- \(a_1^i = a_2^i = a_3^i = a_4^i\),
- \((a_1^i, a_1^{i+1}) \subseteq A\) for all \(1 \leq j \leq 4, 1 \leq i < k_j\),
- \((a_1^i, a_2^i) \notin A\), and
- \((a_3^j, a_4^j) \notin || A\).

then there is no homomorphism \(h\) from \(\mathcal{A}\) to \(\varphi(\mathcal{B})\) such that \(h(a_1^i)\) is a singleton \(\{b_1^i\}\). Suppose otherwise that there is such a homomorphism \(h\). By the definition of \(\varphi(\mathcal{B})\) there must be sequences of elements

- \((b_1^i, \ldots, b_k^i)\), \(\ldots, (b_1^j, \ldots, b_k^j)\)

such that

- \(b_1^i \in h(a_1^i)\) for all \(1 \leq j \leq 4, 1 \leq i \leq k_j\),
- \(b_1^{k_i} = b_2^{k_i} = b_3^{k_i} = b_4^{k_i}\),
- \((b_1^i, b_1^{i+1}) \subseteq B\) for all \(1 \leq j \leq 4, 1 \leq i < k_j\),
- \((b_1^i, b_2^i) \notin B\), and
- \((b_3^j, b_4^j) \notin || B\).

The third item and the fourth item together imply that \(b_1^{k_i} = b_2^{k_i}, \neq 0\) (any set that contains two distinct sets cannot be empty). The third item and the fifth item together imply that \(b_1^{k_i} = b_2^{k_i}, \neq 0\) (any set that is contained in two disjoint subsets must be the empty set), a contradiction.

It follows from Lemma 3.7. in [8] that if \(\mathcal{A}\) does not contain such sequences, then \(\mathcal{A} \rightarrow (\mathcal{D}; \subseteq, ||, \neq)\). This shows that PAC decides CSP\((\mathcal{D}; \subseteq, ||, \neq)\). \(\square\)

PAC tractability results can also be shown for the basic binary relations in the spatial reasoning formalism of RCC-5 [13], which is closely related to set constraints, but also for other known tractable spatial constraint satisfaction problems in qualitative spatial reasoning, e.g., in [6].

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Peek Arc Consistency

Manuel Bodirsky       Hubie Chen

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Abstract

This paper studies peek arc consistency, a reasoning technique that extends the well-known arc consistency technique for constraint satisfaction. In contrast to other more costly extensions of arc consistency that have been studied in the literature, peek arc consistency requires only linear space and quadratic time and can be parallelized in a straightforward way such that it runs in linear time with a linear number of processors. We demonstrate that for various constraint languages, peek arc consistency gives a polynomial-time decision procedure for the constraint satisfaction problem. We also present an algebraic characterization of those constraint languages that can be solved by peek arc consistency, and study the robustness of the algorithm.

1 Introduction

Background. A basic knowledge reasoning task that has been studied in many incarnations is to decide the satisfiability of given relationships on variables, where, for instance, variables may represent objects such as temporal events or spatial regions, and relationships may express precedence, containment, overlap, disjointness, and so forth. Instances of this reasoning task can typically be modeled using the constraint satisfaction problem (CSP), a computational problem in which the input consists of a set of constraints on variables, and the question is whether or not there is an assignment to the variables satisfying all of the constraints. While the CSP is in general NP-hard, researchers have, in numerous settings, aimed to identify restricted sets of relationships under which the CSP is polynomial-time decidable; we refer to sets of relationships as constraint languages.

Arc consistency is an algorithmic technique for constraint satisfaction that has been heavily studied and for which highly efficient implementations that are linear in both time and space are known. Arc consistency provides a one-sided satisfiability check. It may detect an inconsistency, which always implies that the input instance is unsatisfiable. While the converse does not hold in general, it has been shown to hold for some particular constraint languages, that is, arc consistency provides a decision procedure for satisfiability for these languages. Examples include the language of boolean Horn clauses; various graph homomorphism problems, for example, homomorphisms to orientations of finite paths [11]; and all constraint languages where satisfiability is first-order definable [1].

Curiously, arc consistency typically cannot be used as a decision procedure for infinite-domain constraint languages, by which we mean constraint languages under which variables can take on infinitely many values. In many cases, a reason for this is that arc consistency performs inference by considering unary (arity 1) projections of relations, and all such projections are already equal to the full domain of the language. As an example, consider the binary relations ≤ and ≠ interpreted over the domain of rational numbers ℚ. For each of these relations, both of the two possible unary projections are equal to ℚ, and arc consistency in fact will not perform any inference.

Path consistency is a more powerful algorithmic technique that provides a polynomial-time decision procedure for further finite and infinite domain constraint languages. However, the greater power comes at the price of worse time and space complexity: the best known implementations require cubic space and quadratic time. Unfortunately, this makes the path consistency procedure prohibitive for many applications where one has to deal with large instances of the constraint satisfaction problem.
**Peek arc consistency.** In this paper, we study a general algorithmic technique for constraint satisfaction that we call *peek arc consistency*. Here, we describe the idea of the algorithm for finite-domain constraint satisfaction, although, as we show in the paper, this algorithm can be effectively applied to infinite-domain constraint satisfaction as well. The algorithm performs the following. For each variable-value pair \((x, a)\), the variable \(x\) is set to the value \(a\), and then the arc consistency procedure is run on the resulting instance of the CSP. If there is a variable \(x\) such that for all values \(a\) the arc consistency procedure detects an inconsistency on \((x, a)\), then the algorithm reports an inconsistency. As with arc consistency, this algorithm provides a one-sided satisfiability check. One might conceive of this algorithm as being a step more sophisticated than arc consistency; it invokes arc consistency as it takes a “peek” at each variable.

Peek arc consistency has many practical and theoretical selling points. Arc consistency can be implemented in linear space, for any fixed finite-domain constraint language and many infinite-domain constraint languages; the same holds for peek arc consistency. The time complexity of peek arc consistency is quadratic in the input size, which is still much better than the path-consistency algorithm, where the best known implementations have a running time that is cubic in the input size. Moreover, peek arc consistency can be parallelized in a straightforward way: for each variable-value pair, the arc consistency procedure can be performed on a different processor. Hence, with a linear number of processors, we achieve a linear running time, for a fixed constraint language. We would also like to remark that implementing peek arc consistency is straightforward if one has access to an implementation of arc consistency as a subroutine.

We demonstrate that the class of constraint languages solvable by peek arc consistency is a considerable extension of that which can be solved by arc consistency, and in particular contains many infinite-domain constraint languages. Examples are the constraint satisfaction problem for the point algebra in temporal reasoning [15], and tractable set constraints [8]. But also, several finite-domain constraint languages where previously the “best” known algorithm was the path-consistency procedure can be solved by our peek arc consistency procedure. For example, this is the case for homomorphism problems to unbalanced orientations of cycles [9]. Other examples that can be solved by peek arc consistency but not by arc consistency are 2-SAT, and many other CSPs where the relations are closed under a dual discriminator or a median operation.

Our study of peek arc consistency employs universal algebraic techniques which have recently come into focus in the complexity of constraint satisfaction. In addition to obtaining results showing that languages are tractable by this algorithm, we develop an algebraic characterization of the constraint languages solvable by the algorithm. The characterization is exact–necessary and sufficient–for all finite and infinite domain constraint languages. We also exhibit closure properties on the class of constraint languages tractable by the algorithm.

A notable feature of this work is the end to which universal algebraic techniques are applied. Thus far, in constraint satisfaction, such techniques have primarily been used to demonstrate complexity class inclusion results, such as polynomial-time decidability results, and completeness results, such as NP-completeness results. Here, we utilize such techniques to investigate the power of a particular efficient and practical algorithm. That is, we differentiate among constraint languages depending on whether or not they are solvable via a specific algorithmic method, as opposed to whether or not they are contained in a complexity class. To our knowledge, this attitude has only been adopted in a limited number of previous papers that studied arc consistency and extensions thereof [7, 5].

## 2 Preliminaries

Our definitions and notation are fairly standard.

**Structures.** A tuple over a set \(B\) is an element of \(B^k\) for a value \(k \geq 1\) called the *arity* of the tuple; when \(T\) is a tuple, we use the notation \(T = (t_1, \ldots, t_k)\) to denote its entries. A relation over a set \(B\) is a subset of \(B^k\) for a value \(k \geq 1\) called the *arity* of the relation. A signature \(\sigma\) is a finite set of symbols, each of which has an associated arity. We use \(\pi_i\) to denote the operator that projects onto the \(i\)th coordinate: \(\pi_i(T)\) denotes the \(i\)th entry \(t_i\) of a tuple \(T = (t_1, \ldots, t_k)\), and for a relation \(R\) we define \(\pi_i(R) = \{\pi_i(T) \mid T \in R\}\).

A *structure* \(B\) over signature \(\sigma\) consists of a universe \(B\), which is a set, and a relation \(R^B \subseteq B^k\) for each symbol \(R\) of arity \(k\). (Note that in this paper, we are concerned only with relational structures,
which we refer to simply as structures.) Throughout, we will use the bold capital letters $A$, $B$, . . . to denote structures, and the corresponding non-bold capital letters $a$, $b$, . . . to denote their universes. We say that a structure $B$ is finite if its universe $B$ has finite size.

For two structures $A$ and $B$ over the same signature $\sigma$, the product structure $A \times B$ is defined to be the structure with universe $A \times B$ such that $R^{A \times B} = \{(a_1, b_1), \ldots, (a_n, b_k) \mid \pi \in R^A, \exists \in R^B\}$ for all $R \in \sigma$. We use $A^n$ to denote the $n$-fold product $A \times \cdots \times A$.

We say that a structure $B$ over signature $\sigma'$ is an expansion of another structure $A$ over signature $\sigma$ if (1) $\sigma' \supseteq \sigma$, (2) the universe of $B$ is equal to the universe of $A$, and (3) for every symbol $R \in \sigma$, it holds that $R^B = R^A$. We will use the following non-standard notation. For any structure $A$ (over signature $\sigma$) and any subset $S \subseteq A$, we define $[A, S]$ to be the structure with the signature $\sigma \cup \{U\}$ where $U$ is a new symbol of arity 1, defined by $U^{[A, S]} = S$ and $R^{[A, S]} = R^A$ for all $R \in \sigma$.

For two structures $A$ and $B$ over the same signature $\sigma$, we say that $A$ is an induced substructure of $B$ if $A \subseteq B$ and for every $R \in \sigma$ of arity $k$, it holds that $R^A = A^k \cap R^B$. Observe that for a structure $B$ and a subset $B' \subseteq B$, there is exactly one induced substructure of $B$ with universe $B'$.

**Homomorphisms and the constraint satisfaction problem.** For structures $A$ and $B$ over the same signature $\sigma$, a homomorphism from $A$ to $B$ is a mapping $h : A \to B$ such that for every symbol $R \in \sigma$ and every tuple $(a_1, \ldots, a_k) \in R^A$, it holds that $(h(a_1), \ldots, h(a_k)) \in R^B$. We use $A \to B$ to indicate that there is a homomorphism from $A$ to $B$; when this holds, we also say that $A$ is homomorphic to $B$. The homomorphism relation $\to$ is transitive, that is, if $A \to B$ and $B \to C$, then $A \to C$.

For any structure $B$ (over $\sigma$), the constraint satisfaction problem for $B$, denoted by $\text{CSP}(B)$, is the problem of deciding, given as input a finite structure $A$ over $\sigma$, whether or not there exists a homomorphism from $A$ to $B$. In discussing a problem of the form $\text{CSP}(B)$, we will refer to $B$ as the constraint language.

There are several equivalent definitions of the constraint satisfaction problem for a constraint language, most notably the definition used in artificial intelligence. In logic, the constraint satisfaction problem can be formulated as the satisfiability problem for primitive positive formulas in a fixed structure $B$. Homomorphism problems as defined above have been studied independently from artificial intelligence in graph theory, and the connection to constraint satisfaction problems has been observed in [10].

**pp-definability.** Let $\sigma$ be a signature; a primitive positive formula over $\sigma$ is a formula built from atomic formulas $R(w_1, \ldots, w_n)$ with $R \in \sigma$, conjunction, and existential quantification. A relation $R \subseteq B^k$ is primitive positive definable (pp-definable) in a structure $B$ (over $\sigma$) if there exists a primitive positive formula $\phi(v_1, \ldots, v_k)$ with free variables $v_1, \ldots, v_k$ such that

$$(b_1, \ldots, b_k) \in R \iff B, b_1, \ldots, b_k \models \phi.$$  

**Automorphisms.** An isomorphism between two relational structures $A$ and $B$ over the same signature $\sigma$ is a bijective mapping from $A$ to $B$ such that $f \in R^A$ if and only if $f(\exists) \in R^B$ for all relation symbols $R$ in $\sigma$. An automorphism of $A$ is an isomorphism between $A$ and $A$. An orbit of $A$ is an equivalence class of the equivalence relation $\equiv$ that is defined on $A$ by $x \equiv y$ iff $\alpha(x) = y$ for some automorphism $\alpha$ of $A$.

**Polymorphisms.** When $f : B^n \to B$ is an operation on $B$ and $\overline{t_1} = (t_{11}, \ldots, t_{1k}), \ldots, \overline{t_n} = (t_{n1}, \ldots, t_{nk}) \in B^k$ are tuples of the same arity $k$ over $B$, we use $f(\overline{t_1}, \ldots, \overline{t_n})$ to denote the arity $k$ tuple obtained by applying $f$ coordinatewise, that is, $f(\overline{t_1}, \ldots, \overline{t_n}) = (f(t_{11}, \ldots, t_{n1}), \ldots, f(t_{1k}, \ldots, t_{nk}))$. An operation $f : B^n \to B$ is a polymorphism of a structure $B$ over $\sigma$ if for every symbol $R \in \sigma$ and any tuples $\overline{t_1}, \ldots, \overline{t_n} \in R^B$, it holds that $f(\overline{t_1}, \ldots, \overline{t_n}) \in R^B$. That is, each relation $R^B$ is closed under the action of $f$. Equivalently, an operation $f : B^n \to B$ is a polymorphism of $B$ if it is a homomorphism from $B^n$ to $B$. Note that every automorphism is a unary polymorphism.

**Categoricity.** Several of our examples for constraint languages over infinite domains will have the following property that is of central importance in model theory. A countable structure is $\omega$-categorical if all countable models of its first-order theory\(^1\) are isomorphic. By the Theorem of Ryll-Nardzewski (see

\(^1\)The first-order theory of a structure is the set of first-order sentences that is true in the structure.
e.g. [12]) this is equivalent to the property that for each \( n \) there is a finite number of inequivalent first-order formulas over \( \Gamma \) with \( n \) free variables. A well-known example of an \( \omega \)-categorical structure is \((\mathbb{Q}, <)\); for many more examples of \( \omega \)-categorical structures and their application to formulate well-known constraint satisfaction problems, see [2].

3 Arc Consistency

In this section, we introduce the notion of arc consistency that we will use, and review some related notions and results. The definitions we give apply to structures with relations of any arity, and not just binary relations. The notion of arc consistency studied here is sometimes called hyperarc consistency. Our discussion is based on the paper [7].

For a set \( B \), let \( \wp(B) \) denote the power set of \( B \). For a structure \( B \) (over \( \sigma \)), we define \( \wp(B) \) to be the structure with universe \( \wp(B) \setminus \{\emptyset\} \) and where, for every symbol \( R \in \sigma \) of arity \( k \), \( R^{\wp(B)} = \{(\pi_1 S, \ldots, \pi_k S) \mid S \subseteq R^B, S \neq \emptyset\} \).

**Definition 1** An instance \( A \) of CSP\( \left( B \right) \) has the arc consistency condition (ACC) if there exists a homomorphism from \( A \) to \( \wp(B) \).

**Definition 2** We say that arc consistency (AC) decides CSP\( \left( B \right) \) if for all finite structures \( A \), the following holds: \( \langle A, B \rangle \) has the ACC implies that \( A \rightarrow B \).

Note that the converse of the condition given in this definition always holds. By a singleton, we mean a set containing exactly one element.

**Proposition 3** For any structures \( A \) and \( B \), if \( h \) is a homomorphism from \( A \) to \( B \), then the mapping that takes \( a \) to the singleton \( \{h(a)\} \) is a homomorphism from \( A \) to \( \wp(B) \).

Hence, when AC decides CSP\( \left( B \right) \), an instance \( A \) of CSP\( \left( B \right) \) is a “yes” instance if and only if \( A \) has the ACC with respect to \( B \). That is, deciding whether an instance \( A \) is a “yes” instance can be done just by checking the ACC. It was observed in [10] that, for any finite structure \( B \), there is an algebraic characterization of AC: AC decides CSP\( \left( B \right) \) if and only if there is a homomorphism from \( \wp(B) \) to \( B \).

It is well-known that for a finite structure \( B \), whether or not instances \( A \) of CSP\( \left( B \right) \) have the ACC can be checked in polynomial-time. The algorithm for this is called the arc consistency procedure, and it can be implemented in linear time and linear space in the size of \( A \); note that we consider \( B \) to be fixed. The same holds for many infinite-domain constraint languages, for example for all \( \omega \)-categorical constraint languages. Since this is less well-known, and requires a slightly less standard formulation of the arc consistency procedure, we present a formal description of the algorithm that we use; this algorithm can be applied for any (finite- and infinite-domain) constraint language that has finitely many pp-definable unary relations in \( B \).

We assume that \( B \) contains a relation for each unary primitive positive definable relation in \( B \). This is not a strong assumption, since we might always study the expansion \( B' \) of \( B \) by all such unary relations. Then, if we are given an instance \( A \) of CSP\( \left( B \right) \), we might run the algorithm for CSP\( \left( B' \right) \) on the expansion \( A' \) of \( A \) that has the same signature as \( B' \) and where the new unary relations are interpreted by empty relations. It is clear that a mapping from \( A \) to \( B \) is a homomorphism from \( A \) to \( B \) if and only if it is a homomorphism from \( A' \) to \( B' \).

To conveniently formulate the algorithm, we write \( R_{\phi(x)} \) for the relation symbol of the relation that is defined by a pp-formula \( \phi(x) \) in \( B \). We write \( Q_A(\{a_1, \ldots, a_l\}) \) for the conjunction over all formulas of the form \( S(a_i) \) where \( S \) is a unary relation symbol such that \( a_i \in S^A \). The pseudo-code of the arc consistency procedure can be found in Figure 1.

The space requirements of the given arc consistency procedure are clearly linear. It is also well-known and easy to see that the procedure can be implemented such that its running time is linear in the size of the input.

**Proposition 4** Let \( B \) be a structure with finitely many primitive-positive definable unary relations. Then a given instance \( A \) of CSP\( \left( B \right) \) has the ACC if and only if the arc consistency procedure presented in Figure 1 does not reject.
Arc Consistency_B(A)
Input: a finite relational structure A.

Do
For every relation symbol R, every tuple \((a_1, \ldots, a_l) \in R^A\), and every \(i \in \{1, \ldots, l\}\)
Let \(\phi\) be the formula \(\exists a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_l. R(a_1, \ldots, a_l) \land Q_A(\{a_1, \ldots, a_l\})\)
If \(\phi\) defines the empty unary relation over B then reject
Else, add \(a_i\) to \(R^A_{\phi}\)
Loop until no relation in A is changed

Figure 1: The arc consistency procedure for CSP(B), where B contains all primitive positive definable unary relations in B.

In particular, we can apply the algorithm shown in Figure 1 to all constraint satisfaction problems with an \(\omega\)-categorical constraint language. However, it was shown that in this case the algorithm cannot be used as a decision procedure for CSP(B) (i.e., that rejects an instance A if and only if it does not homomorphically map to B), unless B is homomorphically equivalent to a finite structure [4].

4 Peek Arc Consistency

We present basic definitions and results concerning peek arc consistency. The following two definitions are analogous to Definitions 1 and 2 of the previous section.

Definition 5 An instance \((A, B)\) of the CSP has the peek arc consistency condition (PACC) if for every element \(a \in A\), there exists a homomorphism \(h\) from \(A\) to \(\wp(B)\) such that \(h(a)\) is a singleton.

Definition 6 We say that peek arc consistency (PAC) decides CSP(B) if for all finite structures A, the following holds: \((A, B)\) has the PACC implies that \(A \rightarrow B\).

The converse of the condition given in this definition always holds. Suppose that \(A \rightarrow B\); then, the mapping taking each \(a \in A\) to the singleton \(\{h(a)\}\) is a homomorphism from \(A\) to \(\wp(B)\) (Proposition 3), and hence \((A, B)\) has the PACC.

We now present an algorithm that decides for a given instance A of CSP(B), whether \((A, B)\) has the PACC. We assume that B has a finite number of orbits and pp-definable binary relations. This holds in particular for all \(\omega\)-categorical structures. The following lemma then allows us to use the arc consistency procedure presented in Figure 1 for every expansion of B by singletons.

Lemma 7 Let B be a structure with finitely many pp-definable binary relations. Then every expansion of B by a constant has finitely many pp-definable unary relations.

Proof. Suppose for contradiction that for a constant \(b\), there are infinitely many pairwise distinct unary relations with a pp-definition in the expansion of B with \(\{b\}\). For each such definition, if we replace the occurrences of the relation symbol for the singleton \(\{b\}\) by a new variable, we obtain formulas that are pp-deﬁnitions in B of pairwise distinct binary relations. □

Proposition 8 Let B be a structure with ﬁnitely many orbits and ﬁnitely many pp-deﬁnable binary relations. Then a given instance A of CSP(B) has the PACC if and only if the algorithm presented in Figure 2 does not reject A.

Proof. Suppose that A is an instance of CSP(B) that has the PACC. We have to show that for any element \(a\) from A there exists an orbit \(O\) of B such that for any choice of \(b \in O\) the arc consistency procedure that is called in the inner loop of the algorithm in Figure 2 does not reject the instance \([A, \{a\}]\) of CSP([B, \{b\}]).
Because \( A \) has the PACC, there exists a homomorphism from \( A \) to \( \wp(B) \) such that \( h(a) \) is a singleton \( \{c\} \). Let \( O \) be the orbit of \( c \), and let \( b \) be the element from \( O \) that is used by the algorithm. We know that there exists an automorphism \( \alpha \) that maps \( c \) to \( b \). Clearly, the mapping \( h' \) defined by \( x \mapsto \alpha(h(x)) \) is a homomorphism from \( A \) to \( \wp(B) \) such that \( h'(a) = \{b\} \) is a singleton. By Lemma 7 the structure \([B, \{b\}]\) has finitely many pp-definable unary relations. Proposition 4 then shows that the arc consistency procedure does not reject the instance \([A, \{a\}]\) of \( \text{CSP}([B, \{b\}]) \). All the implications in this argument can be reversed, which shows the statement of the proposition. \( \Box \)

**Theorem 9** Let \( B \) be a structure with finitely many orbits and finitely many pp-definable binary relations, and suppose that PAC solves \( \text{CSP}(B) \). Then there exists a quadratic-time and linear-space algorithm that decides \( \text{CSP}(B) \). Moreover, \( \text{CSP}(B) \) can be decided in linear time with a linear number of processors.

**Proof.** Let \( A \) be an instance of \( \text{CSP}(B) \). If the algorithm in Figure 2 rejects \( A \), then it does not have the PACC and hence \( A \) does not homomorphically map to \( B \). If the algorithm in Figure 2 does not reject \( A \), then \( A \) has the PACC. By assumption, PAC solves \( \text{CSP}(B) \), and therefore there exists a homomorphism from \( A \) to \( B \).

Because the arc consistency procedure uses linear space, the algorithm in Figure 2 can be implemented in linear space as well. The arc consistency procedure is called a linear number of times (recall that \( B \) is fixed and not part of the input). Because the arc consistency procedure can be implemented such that it uses linear time, the overall running time on a sequential machine is quadratic in the worst case. However, note that each application of the arc consistency procedure can be performed on a different processor. \( \Box \)

## 5 Algebraic Characterization

In this section we present a general algebraic characterization of those constraint languages where PAC decides \( \text{CSP}(B) \) for an arbitrary finite or infinite structure \( B \).

We use the notation \( \text{Ind}(\wp(B)^n) \) to denote the induced substructure of \( \wp(B)^n \) whose universe contains an \( n \)-tuple of \( \wp(B)^n \) if and only if at least one coordinate of the tuple is a singleton.

**Theorem 10** Let \( B \) be a structure. PAC decides \( \text{CSP}(B) \) if and only if for all \( n \) there is a homomorphism from all finite substructures of \( \text{Ind}(\wp(B)^n) \) to \( B \).

**Proof.** \((\Rightarrow)\): Suppose that \( (A, B) \) has the PACC. Then, by definition of the PACC, for all \( a \in A \), there is a homomorphism \( h_a \) from \([A, a] \) to \([\wp(B), \{b\} \mid b \in B] \). Let \( n = |A| \). Now consider the homomorphism \( h \) from \( A \) to \( \wp(B)^n \) defined by \( h(x) = \Pi_{a \in A} h_a(x) \). Notice that for every \( a \in A \), the element \( h_a(a) \) of the tuple \( h(a) \) is a singleton, and hence \( h \) is in fact a homomorphism from \( A \) to \( \text{Ind}(\wp(B)^n) \). Let \( C \) be the structure that is induced by the image of \( h \) in \( \text{Ind}(\wp(B)^n) \). Since \( C \) is finite, it is by assumption homomorphic to \( B \), and by composing homomorphisms we obtain that there is a homomorphism from \( A \) to \( B \).
(⇒): Let $n ≥ 1$, and let $C$ be a finite substructure of $\text{Ind}(\varphi(B)^n)$. We have to show that $C$, viewed as an instance of CSP($B$), has the PACC, which suffices by assumption. Let $a$ be any element of the universe of $C$. By definition of $C$, we have that $a$ is an $n$-tuple such that some coordinate, say the $i$th coordinate, is a singleton. The projection function $\pi_i$ is a homomorphism from $[C, \{a\}]$ to $[\varphi(B), \{\{b\} | b ∈ B\}]$. □

6 Robustness

In this section, we demonstrate that the class of structures $B$ such that PAC decides CSP($B$) is robust in that it satisfies certain closure properties.

We first investigate expansion by pp-definable relations. Say that a structure $B'$ (over $\sigma'$) is a pp-expansion of $B$ (over $\sigma$) if $B'$ is an expansion of $B$ and for every symbol $R ∈ \sigma' \setminus \sigma$, it holds that $R^{B'}$ is pp-definable over $B$.

**Theorem 11** Suppose that PAC decides CSP($B$). Then for any pp-expansion $B'$ of $B$, it holds that PAC decides CSP($B'$).

**Proof.** It suffices to show that the theorem holds for an expansion of $B$ by (1) an intersection of existing relations, (2) a projection of an existing relation, (3) a product of an existing relation with $B$, or (4) the equality relation. In each of these cases, we will consider an expansion $B'$ of $B$ where the signature of $B'$ has an additional symbol $T$. We will use $\sigma$ to denote the signature of $B$, and so the signature of $B'$ will be $\sigma \cup \{T\}$.

By Theorem 10, it suffices to show that for every $n ≥ 1$ and for all finite substructures $C'$ of $\text{Ind}(\varphi(B')^n)$ there exists a homomorphism $h$ from $C'$ to $B$. Let $C$ be a finite subset of the universe of $\text{Ind}(\varphi(B)^n)$, let $C$ be the induced substructure of $\text{Ind}(\varphi(B)^n)$ with universe $C$, and let $C'$ be the induced substructure of $\text{Ind}(\varphi(B')^n)$ with universe $C$. By Theorem 10, there is a homomorphism $h$ from $C$ to $B$. Since $B'$ is an expansion of $B$ with just one additional symbol $T$, it suffices to show that $h(T^C) ⊆ T^{B'}$.

1: Suppose that $T^{B'} = R^B \cap S^B$ for $R, S ∈ \sigma$. It follows that $T^{\varphi(B')} ⊆ R^{\varphi(B')} \cap S^{\varphi(B)}$, from which we obtain $T^{C'} ⊆ R^C \cap S^C$. For any tuple $t ∈ T^{C'}$, we thus have $h(t) ∈ R^B \cap S^B$, and hence $h(t) ∈ T^{B'}$.

2: Suppose that $T^{B'}$ is the relation

$$\{(t_1, \ldots, t_k) | \exists t_{k+1}, \ldots, t_{k+l} \text{ so that } (t_1, \ldots, t_{k+l}) ∈ R^B\}$$

that is, $T^{B'}$ is the projection of $R^B$ onto the first $k$ coordinates; we denote the arity of $R ∈ \sigma$ by $k + l$. (We assume that the projection is onto an initial segment $\{1, \ldots, k\}$ of coordinates for the sake of notation; a similar argument holds for an arbitrary set of coordinates.) We have that $T^{\varphi(B')}$ is the projection of $R^{\varphi(B)}$ onto the first $k$ coordinates. Thus, for any tuple $t ∈ T^{C'}$, we have that $h(t)$ is the projection of a tuple in $R^B$, and hence in $T^{B'}$.

3: Suppose that $T^{B'} = R^B \times B$ for $R ∈ \sigma$. Let $\bar{t} = (t_1, \ldots, t_{k+1})$ be any tuple in $T^{C'}$. We have that $(t_1, \ldots, t_k) ∈ R^C$, and hence $h(t_1, \ldots, t_k) ∈ R^B$. Since we have $h(t_{k+1}) ∈ B$, it follows that $h(t_1, \ldots, t_{k+1}) ∈ T^{B'}$.

4: Suppose that $T^{B'} = \{\{b, b\} | b ∈ B\}$. For any tuple $(t_1, t_2) ∈ T^{\varphi(B')}$, we have $t_1 = t_2$. For any tuple $(t_1, t_2) ∈ T^{C'}$ we thus also have $t_1 = t_2$, and we have $h(t_1, t_2) ∈ T^{B'}$. □

We now consider homomorphic equivalence.

**Theorem 12** Let $B$ be a structure. Suppose that PAC decides CSP($B$) and that $B'$ is a structure that is homomorphically equivalent to $B$, that is, $B → B'$ and $B' → B$. Then PAC decides CSP($B'$).

We first establish the following lemma.

**Lemma 13** Let $f$ be a homomorphism from $B'$ to $B$. The map $f'$ defined on $\varphi(B') \setminus \{∅\}$ by $f'(U) = \{f(u) | u ∈ U\}$ is a homomorphism from $\varphi(B')$ to $\varphi(B)$.

**Proof.** Let $R$ be a symbol, and let $\bar{t}$ be a tuple in $R^{\varphi(B')}$. We have $\bar{t} = (π_1 S, \ldots, π_k S)$ where $S ⊆ R^B$ and $S ≠ ∅$. Define $S' = \{f(π) | π ∈ S\}$. We have $S' ⊆ R^B$. As $f'(\bar{t}) = (π_1 S', \ldots, π_k S')$, the conclusion follows. □
Proof. (Theorem 12) Suppose that \((A, B')\) has the PACC. We want to show that \(A \rightarrow B'\).

We first show that \((A, B)\) has the PACC. Let \(a\) be an element of \(A\). There exists a homomorphism \(h\) from \(A\) to \(\wp(B')\) such that \(h(a)\) is a singleton. The mapping \(f'\) given by Lemma 13 is a homomorphism from \(\wp(B')\) to \(\wp(B)\) that maps singletons to singletons. Hence, the map \(a \rightarrow f'(h(a))\) is a homomorphism from \(A\) to \(\wp(B)\) mapping \(a\) to a singleton. We thus have that \((A, B)\) has the PACC.

Since PAC decides CSP\((B)\), there is a homomorphism from \(A\) to \(B\). By hypothesis, there is a homomorphism from \(B\) to \(B'\), and so we obtain that \(A\) is homomorphic to \(B\). □

7 Tractability by PAC

Slice-semilattice operations. We first study a class of ternary operations. Recall that a semilattice operation is a binary operation that is associative, commutative, and idempotent, and that a semilattice operation \(\oplus\) is well-defined on finite sets, that is, for a finite set \(S = \{s_1, \ldots, s_n\}\) we may define \(\oplus(S) = \oplus(\ldots \oplus(\oplus(s_1, s_2), s_3), \ldots, s_n)\). We say that a ternary operation \(t : B^3 \rightarrow B\) is a slice-semilattice operation if for every element \(b \in B\), the binary operation \(\oplus_b\) defined by \(\oplus_b(x, y) = t(x, y, b)\) is a semilattice operation. These ternary operations have been studied in [5]; there, the following examples were presented.

Example 14 Let \(B\) be a set, and let \(d : B^3 \rightarrow B\) be the operation such that \(d(x, y, z)\) is equal to \(x\) if \(x = y\), and \(z\) otherwise. This operation is known as the dual discriminator on \(B\), and is an example of a slice-semilattice operation. For examples of constraint languages that have a dual discriminator polymorphism, see e.g. [14].

Example 15 Let \(B\) be a subset of the rational numbers, and let \(\median : B^3 \rightarrow B\) be the ternary operation on \(B\) that returns the median of its arguments. (Precisely, given three arguments \(x_1, x_2, x_3\) in ascending order so that \(x_1 \leq x_2 \leq x_3\), the median operation returns \(x_2\).) This operation is an example of a slice-semilattice operation.

Theorem 16 Let \(B\) be a finite structure that has a slice-semilattice polymorphism. Then, the problem CSP\((B)\) is tractable by PAC.

Proof. Let \(f\) denote the slice-semilattice polymorphism. By Theorem 10, it suffices to show that for every finite substructure \(C\) of \(\Ind(\wp(B)^n)\) there is a homomorphism \(h\) from \(C\) to \(B\).

For each element \((S_1, \ldots, S_n)\) in \(C\) we define \(h(S_1, \ldots, S_n)\) as follows. Let \(g\) be the maximum index such that \(S_g\) is a singleton; we are guaranteed the existence of such an index by the definition of \(\Ind(\wp(B)^n)\). We define a sequence of values \(b_1, \ldots, b_n \in B\) inductively. Set \(b_g\) to be the value such that \(\{b_g\} = S_g\). For \(i\) with \(g < i \leq n\), define \(b_i = \oplus_{h_{i-1}} S_i\). We define \(h(S_1, \ldots, S_n) = b_n\).

We now show that \(h\) is in fact a homomorphism from \(C\) to \(B\). Let \(R\) be any symbol of arity \(k\). Suppose that \((S_1\), \ldots, \(S_n\))\) \((S_1, \ldots, S_n)\) \(\in R^{\Ind(\wp(B)^n)}\). We define a sequence of tuples \(t_1, \ldots, t_n \in R^B\) in the following way. Let \(t_i\) be any tuple such that \(t_i \in (S_1 \times \cdots \times S_k) \cap R^B\). For \(i\) with \(1 < i \leq n\), we define \(t_i = (\oplus_{t_{i-1}}, S_1, \ldots, \oplus_{t_{i-1}} S_k)\). Given that \(t_{i-1}\) is in \(R^B\), we prove that \(t_i\) is in \(R^B\). Let \(C_1 \subseteq R^B\) be a set of tuples such that \(\pi_1(C_1), \ldots, \pi_k(C_1) = (S_1, \ldots, S_k)\). Let \(c_1, \ldots, c_m\) with \(m \geq 2\) be a sequence of tuples such that \(\{c_1, \ldots, c_m\} = C_1\). We have \(t_i = f(c_1, \ldots, t_{i-1}, c_{i-1}) \cdots \cdots t_1\). Since \(f\) is a polymorphism of \(R^B\), we obtain \(t_i \in R^B\).

Observe now that for each tuple \((S'_1, \ldots, S'_n)\), the values \(b_g, \ldots, b_n\) that were computed to determine \(h(S'_1, \ldots, S'_n) = b_n\) have the property that for each \(i\) with \(g \leq i \leq n\), \(b_i = t_{ij}\). It follows that \(h\) is the desired homomorphism. □

It is well-known that the problem 2-SAT can be identified with the problem CSP\((B)\) for the structure \(B\) with universe \(B = \{0, 1\}\) and relations

\[
\begin{align*}
R_{(0,0)}^B &= \{0, 1\}^2 \setminus \{(0, 0)\} \\
R_{(0,1)}^B &= \{0, 1\}^2 \setminus \{(0, 1)\} \\
R_{(1,1)}^B &= \{0, 1\}^2 \setminus \{(1, 1)\}
\end{align*}
\]
It is known, and straightforward to verify, that the dual discriminator operation on \( \{0, 1\} \) is a polymorphism of this structure \( B \). We therefore obtain the following.

**Theorem 17** The problem 2-SAT is tractable by PAC.

Let \( \sigma \) be the signature \( \{E\} \) where \( E \) is a symbol having arity 2. We call a structure \( G \) over \( \sigma \) an undirected bipartite graph if \( E^G \) is a symmetric relation, the universe \( G \) of \( G \) is finite, and \( G \) can be viewed as the disjoint union of two sets \( V_0 \) and \( V_1 \) such that \( E^G \subseteq (V_0 \times V_1) \cup (V_1 \times V_0) \).

**Theorem 18** Let \( G \) be an undirected bipartite graph. The problem CSP(\( G \)) is tractable by PAC.

**Proof.** Let \( G' \) be the bipartite graph with universe \( \{0, 1\} \) and where \( E^{G'} = \{(0, 1), (1, 0)\} \). As \( E^{G'} \) is pp-definable over the structure \( B \) corresponding to 2-SAT above, by \( \phi(v_1, v_2) \equiv R_{0,1}(v_1, v_2) \wedge R_{0,1}(v_2, v_1) \), we have that PAC decides CSP(\( G' \)) by Theorem 11.

If \( E^G \) is empty, the claim is trivial, so assume that \( (s, s') \in E^G \). We claim that \( G \) and \( G' \) are homomorphically equivalent, which suffices by Theorem 12. The map taking \( 0 \to s \) and \( 1 \to s' \) is a homomorphism from \( G \) to \( G' \). The map taking all elements in \( V_0 \) to 0 and all elements in \( V_1 \) to 1 is a homomorphism from \( G \) to \( G' \).

We therefore have the following result.

**Theorem 19** Let \( D \) be an unbalanced orientation of a cycle. Then CSP(\( D \)) is tractable via PAC.

**The Point Algebra in Temporal Reasoning.** The structure \( (\mathbb{Q}, \leq, \neq) \) is known as the point algebra in temporal reasoning. The problem CSP(\( \mathbb{Q}, \leq, \neq \)) can be solved by the path-consistency procedure [16].

**Theorem 20** CSP(\( \mathbb{Q}, \leq, \neq \)) is tractable via PAC.

**Proof.** Clearly, the structure \( (\mathbb{Q}, \leq, \neq) \) has only one orbit. It is well-known that it is also \( \omega \)-categorical [12], and therefore has in particular a finite number of pp-definable binary relations. To apply Theorem 9, we only have to verify that PAC decides CSP(\( \mathbb{Q}, \leq, \neq \)).

Let \( A \) be an instance of CSP(\( \mathbb{Q}, \leq, \neq \)). We claim that if there is a sequence \( a_1, \ldots, a_k \in A \) such that \((a_i, a_{i+1}) \in \leq^A \) for all \( 1 \leq i < k \), \((a_k, a_1) \in \leq^A \), and \((a_p, a_q) \in \neq^A \) for some \( p, q \in \{1, \ldots, k\} \), then there is no homomorphism from \( A \) to \( \phi(B) \) such that \( h(a_1) \) is a singleton \( \{b_1\} \). Suppose otherwise that there is such a homomorphism \( h \). By the definition of \( \phi(B) \) there must be a sequence \( b_1, \ldots, b_k \) such that \( b_i \in h(a_i) \) for all \( 1 \leq i \leq k \) and \( (b_i, b_{i+1}) \in \neq^B \) for all \( 1 \leq i < n \). Moreover, \((b_k, b_1) \in \leq^B \) and hence \( b_1 = \cdots = b_k \). But then we have \( (h(a_p), h(a_q)) = (b_1, b_1) \in \neq^B \), a contradiction. Hence, the structure \( A \) does not have the PACC if \( A \) has such a sequence \( a_1, \ldots, a_k \). It is known [16] that if \( A \) does not contain such a sequence, then \( A \rightarrow (\mathbb{Q}, \leq, \neq) \). This shows that PAC decides CSP(\( \mathbb{Q}, \leq, \neq \)). □

**Set Constraints.** Reasoning about sets is one of the most fundamental reasoning tasks. A tractable set constraint language has been introduced in [8]. The constraint relations in this language are containment \( X \subseteq Y \) ("every element of \( X \) is contained in \( Y \"), disjointness \( X\setminus Y \) ("\( X \) and \( Y \) do not have common elements"), and disequality \( X \not= Y \) ("\( X \) and \( Y \) are distinct"). In the CSP for this constraint language we are given a set of constraints and a set of containment, disjointness, and disequality constraints between variables, and we want to know whether it is possible to assign sets (we can without loss of generality assume that we are looking for subsets of the natural numbers; note that we allow the empty set) to these variables such that all the given constraints are satisfied. It was shown in [3] that this problem can be modeled as CSP(\( (D; \subseteq, ||, \neq) \)), where \( D \supseteq 2^N \) is a countably infinite set of subsets of \( N \), and such that \((D; \subseteq, ||, \neq) \) is \( \omega \)-categorical and has just two orbits (the orbit for the \( \emptyset \), and the orbit for all other points).
Theorem 21 \( \text{CSP}((D; \subseteq, ||, \neq)) \) is tractable via PAC.

**Proof.** Because \((D; \subseteq, ||, \neq)\) is \(\omega\)-categorical, it suffices as in the proof of Theorem 20 to verify that PAC decides \(\text{CSP}((D; \subseteq, ||, \neq))\) in order to apply Theorem 9.

Let \( A \) be an instance of \(\text{CSP}((D; \subseteq, ||, \neq))\). We claim that if there are four sequences \((a^1_{k_1}, \ldots, a^1_{k_1})\), \((a^2_{k_2}, \ldots, a^2_{k_2})\), \((a^3_{k_3}, \ldots, a^3_{k_3})\), \((a^4_{k_4}, \ldots, a^4_{k_4})\) of elements from \( A \) such that

- \( a^1_{k_1} = a^2_{k_2} = a^3_{k_3} = a^4_{k_4} = a^1 \)
- \( (a^j_i, a^{j+1}_i) \subseteq A \) for all \( 1 \leq j \leq 4, 1 \leq i < k_j \)
- \( (a^1_1, a^2_1) \notin A \), and
- \( (a^3_{k_3}, a^4_{k_4}) \in || A \)

then there is no homomorphism \( h \) from \( A \) to \( \varphi(B) \) such that \( h(a^1_{k_1}) \) is a singleton \( \{b^1_{k_1}\} \). Suppose otherwise that there is such a homomorphism \( h \). By the definition of \( \varphi(B) \) there must be sequences of elements \((b^1_{k_1}, \ldots, b^1_{k_1})\), \((b^2_{k_2}, \ldots, b^2_{k_2})\), \((b^3_{k_3}, \ldots, b^3_{k_3})\), \((b^4_{k_4}, \ldots, b^4_{k_4})\) such that

- \( b^j_i \in h(a^j_i) \) for all \( 1 \leq j \leq 4, 1 \leq i \leq k_j \)
- \( b^1_{k_1} = b^2_{k_2} = b^3_{k_3} = b^4_{k_4} = b^1 \)
- \( (b^j_i, b^{j+1}_i) \subseteq B \) for all \( 1 \leq j \leq 4, 1 \leq i < k_j \)
- \( (b^1_1, b^2_1) \notin B \), and
- \( (b^3_{k_3}, b^4_{k_4}) \in || B \)

The third item and the fourth item together imply that \( b^1_{k_1} = b^2_{k_2} = b^3_{k_3} = b^4_{k_4} \neq \emptyset \) (any set that contains two distinct sets cannot be empty). The third item and the fifth item together imply that \( b^1_{k_1} = b^2_{k_2} = \emptyset \) (any set that is contained in two disjoint subsets must be the empty set), a contradiction.

It follows from Lemma 3.7. in [8] that if \( A \) does not contain such sequences, then \( A \to (D; \subseteq, ||, \neq) \). This shows that PAC decides \( \text{CSP}(D; \subseteq, ||, \neq) \). \( \square \)

PAC tractability results can also be shown for the basic binary relations in the spatial reasoning formalism of RCC-5 [13], which is closely related to set constraints, but also for other known tractable spatial constraint satisfaction problems in qualitative spatial reasoning, e.g., in [6].

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