Exact BPS domain walls at finite gauge coupling

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Bogomol’nyi–Prasad–Sommerfield solitons in models with spontaneously broken gauge symmetry have been intensively studied at the infinite gauge coupling limit, where the governing equation—the so-called master equation—is exactly solvable. Except for a handful of special solutions, the standing impression is that analytic results at finite coupling are generally unavailable. The aim of this paper is to demonstrate, using domain walls in Abelian–Higgs models as the simplest example, that exact solitons at finite gauge coupling can be readily obtained if the number of Higgs fields \(N_F\) is large enough. In particular, we present a family of exact solutions, describing \(N\) domain walls at arbitrary positions in models with at least \(N_F \geq 2N + 1\). We have also found that adding together any pair of solutions can produce a new exact solution if the combined tension is below a certain limit.

1. Introduction

Domain walls are planar-like objects, that separate two distinct phases/states of matter. Perhaps the most iconic examples are walls in ferromagnets, where “domains” of uniform orientation of magnetic moments are punctuated by thin layers, within which moments change rapidly to accommodate the relative difference. Numerous other examples can be found in solid-state physics. The concept, however, found its place in various other fields. In theoretical physics, domain walls are often used as low-energy toy models of both D-branes \([8,22]\) in string theory and “branes” in the brane world scenario \([18]\). The lowest common denominator in all their incarnations, however, is the existence of distinct phases/states of matter, which makes domain walls prototypes of topological solitons \([14]\).

To model a domain wall in field theory, in principle all that is required is discrete and degenerate vacua that the domain wall separates. The simplest example is a scalar field theory with the double-well potential, the well-known \(Z_2\) kink \([23]\). However, in generic scalar theories, a configuration of more than one domain wall cannot be static, due to attractive scalar forces. Dynamical multi-soliton configurations are therefore difficult to study in such theories, other than via numerical or perturbative methods. Only in very special cases, we are able to write down exact dynamical multi-soliton solutions, such as in the famous sine-Gordon model.

Hence, to understand multi-domain walls (and many other topological solitons as well), without the need to deal with dynamics, it is advantageous to use supersymmetric (SUSY) gauge theories. In these models, attractive scalar forces between solitons can be exactly compensated by repulsive massive vector interactions (originating from spontaneously broken gauge symmetry). Thus, arbitrary configurations of solitons can be static. Solitons with such a property are called Bogomol’nyi–Prasad–Sommerfield (BPS) solitons. They can be shown to possess the least amount...
of energy in their topological sector (so-called Bogomol’nyi bound [6]) and they also partially preserve SUSY.

BPS solitons have a further advantage. Instead of dealing with complicated second-order equations of motion, BPS solitons can be shown to satisfy a much simpler set of first-order equations, correspondingly called the BPS equations. These equations have many interesting properties and attract scientific interest on their own.

If the gauge symmetry is only partially broken by scalar fields—the so-called Coulomb phase—prototypal solitons are magnetic monopoles [17,20] and to them related instantons [5]. Their BPS equations can be solved exactly via the famous Nahm equations [15] and Atiyah-Drinfeld-Hitchin-Manin construction [3], respectively, providing an analytic form for static multi-monopole and multi-instanton configurations. This reflects the underlying integrability of BPS equations in the Coulomb phase.

If the gauge symmetry is fully broken by scalar fields—the so-called Higgs phase—the prototype solitons are vortices [1,4,9,16] and domain walls [12,21]. In contrast with the Coulomb phase, similar analytic techniques for solving BPS equations in the Higgs phase are not available, as they are not integrable [10]. Although isolated exact solutions were found for domain walls [11,13], they only seem to prove the general rule.

This paper is dedicated to extending the knowledge about exact solutions for solitons in the Higgs phase. In particular, we concentrate on domain walls, but our results have obvious implications to other BPS solitons in the Higgs phase.

We consider a part of the bosonic sector of Abelian \( N = 2 \) SUSY gauge theory with \( N_F \) Higgs fields. Such a model has \( N_F \) discrete vacua and consequently supports configurations of up to \( N_F - 1 \) elementary domain walls. We present the model in detail in Sect. 2, where we also discuss properties of walls and of multi-wall configurations at some length.

To construct BPS domain walls we employ the so-called moduli matrix approach (for a review see Ref. [7]). This technique involves a change of variables that solves all BPS equations identically, except one. The remaining equation called the \textit{master equation}, has the following general form:

\[
\frac{1}{g^2} \partial_x^2 u = 1 - \Omega_0(x) e^{-u}, \tag{1}
\]

where \( g \) is proportional to the gauge coupling constant and \( u \) is the moduli field related to Higgs fields and gauge fields via the moduli matrix approach. We call \( \Omega_0(x) \) the “source term” since it specifies number, mass, and positions of walls. In fact, together with the boundary condition \( u \to \log \Omega_0(x) \) at the spatial infinity, \( \Omega_0 \) fully determines the solution.

Not surprisingly, the nonlinear second-order differential equation (1) is very hard to solve exactly. Indeed, up to now, there are only a handful of exact solutions for special choices of \( \Omega_0 \). Chronologically, the first exact solution was a junction of domain walls reported in Ref. [13]. From this solution, one can extract a single wall solution, which was added to two other exact single wall solutions and one double wall solution in Ref. [11]. Outside these important findings, the master equation of type (1) is believed to be impregnable by analytic means.

In this paper, we will present many new exact solutions for multi-domain walls. In particular, we will show that some exact solutions of Eq. (1), each with different \( \Omega_0 \), can be combined to gain novel solutions, with new \( \Omega_0 \)’s. These solutions, which we call \textit{chains}, can under certain conditions generate an unlimited wealth of exact solutions, which we specify in Sect. 3. Furthermore, we will show that beyond chains there is a bewildering maze of exact solutions based upon a family of ansatzes, which...
we present and study in Sect. 4. Both of these findings reveal unexpected analytic structure within the master equation for domain walls and strongly implies the same for other topological solitons. We will discuss these implications in Sect. 5.

2. Master equation for Abelian domain walls

2.1. Abelian-Higgs theory

Let us consider a $U(1)$ gauge theory in (3+1)-dimensions with $N_F$ complex scalar fields assembled into the row vector $H \equiv (H_1, H_2, \ldots, H_{N_F})$ and a real scalar field $\sigma$. The Lagrangian is given as

$$L = -\frac{1}{4g^2}(F_{\mu\nu})^2 + \frac{1}{2g^2}(\partial_\mu \sigma)^2 + |D_\mu H|^2 - V, \tag{2}$$

$$V = g^2 \left( v^2 - HH^\dagger \right)^2 + |\sigma H - HM|^2, \tag{3}$$

where $g$ is the gauge coupling constant and the parameter $v$ is the vacuum expectation value of the Higgs field. In the context of supersymmetry, $v^2$ can be identified with the so-called Fayet–Iliopoulous parameter. The mass matrix is given as $M = \text{diag}(m_1, \ldots, m_{N_F})$. Notice that the trace part of $M$ can be changed arbitrarily by shifting $\sigma \rightarrow \sigma + k$, $\text{Tr}[M] \rightarrow \text{Tr}[M] - k$, which leaves the Lagrangian unaffected. We will use this freedom to make all entries in $M$ negative (or zero) by setting $k = N_F \max\{m_i\}$. The benefit of this choice will be apparent when we discuss positions of domain walls. Also, we can always demand that the masses are ordered $m_1 \geq \cdots \geq m_{N_F}$, without loss of generality. This makes $m_1 = 0$.

We adopt following conventions:

$$\eta_{\mu\nu} = \text{diag}(+, -, -, -), \tag{4}$$

$$F_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu, \tag{5}$$

$$D_\mu H = \partial_\mu H + iw_\mu H. \tag{6}$$

Nongeneric values of coupling constants in front of $(\partial \sigma)^2$ and in the potential $V$ allow us to embed this model into a supersymmetric model with eight supercharges by adding appropriate bosonic and fermionic fields. This “effective” supersymmetry enables us to reduce the full equations of motion by one order. The reduced equations are known as 1/2 BPS equations.

Vacua of Eq. (2) are constant field configurations with vanishing potential $V = 0$. There are precisely $N_F$ such configurations, which we label as $\langle 1 \rangle, \ldots, \langle N_F \rangle$, with the representative values

$$H^{(k)} = (0, \ldots, v_{k}, \ldots, 0), \quad \sigma^{(k)} = m_k, \quad (k = 1, \ldots, N_F). \tag{7}$$

The existence of discrete vacua is what makes domain walls possible. To be specific, since there are $N_F$ vacua we expect there to be a domain wall for every transition between a pair of vacua $\langle i \rangle \rightarrow \langle j \rangle$, where $i < j$ (if $i > j$ such transitions are called anti-walls). Thus, there are $\binom{N_F - 1}{2}$ different domain walls. Let us denote each transition as $\langle ij \rangle$ for brevity. Out of these, only transitions between successive vacua $\langle i \rangle \rightarrow \langle i + 1 \rangle$ are considered to be “elementary”, while other transitions can be shown to be composite configurations of elementary walls, as we will discuss in Sect. 2.3.

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1 The number of spatial dimensions plays no role in our discussions. We keep it at 3 for simplicity, but all results presented in subsequent sections are valid in any number of spatial dimensions.
2.2. 1/2 BPS equations

Let us construct 1/2 BPS domain walls along, say, the first direction $x^1$. To that goal we will assume that all fields depend only on the $x^1$ coordinate and, as a gauge choice, we take all gauge fields to be zero, $w_{\mu} = 0$. The BPS equations are found using the well-known technique of bounding the energy density from below due to Bogomol’nyi [6]:

$$
\mathcal{E} = \frac{1}{2g^2} (\partial_1 \sigma)^2 + \frac{g^2}{2}(v^2 - |H|^2)^2 + |\partial_1 H|^2 + |\sigma H - HM|^2
= \frac{1}{2g^2} \left( \partial_1 \sigma - \eta g^2 (v^2 - |H|^2) \right)^2 + |\partial_1 H + \eta (\sigma H - HM)|^2 + \eta \nu^2 \partial_1 \sigma
- \eta \partial_1 (|H|^2 \sigma - H M H) \geq \eta T - \eta \partial_1 J,
$$

(8)

where $T = v^2 \partial_1 \sigma$ is the energy density of the domain wall and $J = |H|^2 \sigma - H M H$ is a boundary term. Thus, the energy per unit area (tension) of the domain wall is never less than $\eta T$, where

$$
T = v^2 \int_{-\infty}^{\infty} dx^1 \partial_1 \sigma = v^2 \Delta m
$$

(9)

and $\Delta m$ is a difference of masses of respective vacua.\(^2\) The minimum $E = \eta T$ is achieved if the following set of first-order equations is obeyed:

$$
\partial_1 H + \eta (\sigma H - HM) = 0,
\partial_1 \sigma = \eta g^2 (v^2 - |H|^2).
$$

(10) (11)

Parameter $\eta^2 = 1$ labels whether the solution is either a wall ($\eta = 1$) or an anti-wall ($\eta = -1$).

BPS equations (10)–(11) can be reduced to a single second-order partial differential equation by using the moduli matrix approach [7]. First, let us solve Eq. (10) identically using the ansatz

$$
H = ve^{-u/2}H_0e^{\eta M x^1}, \quad \eta \sigma = \frac{1}{2} \partial_1 u,
$$

(12)

where $u$ is a new field variable and $H_0$ is a row vector of $N_F$ constants, generically called the moduli matrix. Notice that the pair $\{u, H_0\}$ does not determine $\{H, \sigma\}$ uniquely, since the so-called $V$-transformation $\{u, H_0\} \to \{u + 2c, e^c H_0\}$, where $c \in \mathbb{C}$, leaves assignment (12) unaltered. The second BPS equation (11) turns into the master equation

$$
\frac{1}{2g^2 v^2} \partial_1^2 u = 1 - \Omega_0 e^{-u}, \quad \Omega_0 = H_0 \exp (2\eta M x^1) H_0^*.
$$

(13)

2.3. The master equation

As an explicit example, let us consider the case $N_F = 2$, $M = \text{diag}(0, -m)$. There is only one domain wall, $\langle 1, 2 \rangle$, that interpolates between the second vacuum $\sigma \to -m$ at $x^1 \to -\infty$ and the first vacuum $\sigma \to 0$ at $x^1 \to \infty$. Its tension is $T_{\langle 1, 2 \rangle} = v^2 m$. The generic moduli matrix can be written as $H_0 = (1, e^{m R})$ by fixing the first entry via the $V$-transformation. Parameter $R$ represents

\(^2\) Notice that due to the ordering of masses this quantity is always positive for walls $\eta = 1$ and negative for anti-walls $\eta = -1$. In this way the combination $\eta \Delta m = |\Delta m|$ is always positive.
the position of the wall and we can set $R = 0$ by translation symmetry. The master equation (13) in such a case reduces to ($x^1 \equiv x$, $\eta = 1$)

$$
\frac{1}{2g^2v^2} \partial_x^2 u = 1 - \left(1 + e^{-2mx}\right) e^{-u}.
$$

The general solution of this equation is unknown, although three exact solutions for specific values of the ratio $m/(gv)$ have been found [11]. We will discuss these solutions in the next section.

Apart from these solutions (or solving the master equation numerically), one can develop a qualitative feeling about the shape of the solution via the so-called infinite gauge coupling limit $g \to \infty$. Notice that in this limit the left-hand side of Eq. (14) vanishes and right-hand side can be solved algebraically as

$$
uu = \log\left(1 + e^{-2mx}\right).
$$

The difference between an exact solution and $\nuu$ is generally confined to a finite region around the domain wall as illustrated in Fig. 1. Thus, the general shape of the solution is faithfully represented by $\nuu$. This is also the reason why the infinite gauge coupling limit is used almost exclusively in studies about BPS domain walls, or other solitons in the Higgs phase.

Let us now consider the case $N_F = 3$ with mass matrix $M = \text{diag}(0, -m, -2m)$. It is easy to see that there are two elementary walls representing the transitions $\langle 1 \ 2 \rangle$ and $\langle 2 \ 3 \rangle$ given by moduli matrices $H_0 = (1, 1, 0)$ and $H_0 = (0, 1, 1)$, respectively. Indeed, these choices would produce essentially identical master equations as in Eq. (14). On the other hand, if we choose $H_0 = (1, e^{R/2}, 1)$ we obtain a new master equation

$$
\frac{1}{2g^2v^2} \partial_x^2 u = 1 - \left(1 + e^{R} e^{-2mx} + e^{-4mx}\right) e^{-u},
$$

which is the first example of the so-called composite wall. This configuration describes the transition $\langle 1 \ 3 \rangle$. Depending on the value of the parameter $R$, the configuration on its way from $\langle 1 \rangle$ vacuum at $x \to \infty$ to $\langle 3 \rangle$ vacuum at $x \to -\infty$ can approach the $\langle 2 \rangle$ vacuum arbitrarily closely. For positive and large $R$ we can interpret the configuration as a pair of elementary walls separated by the distance...
\( \sim R/m \). Indeed, in the vicinity of the point \( x = R/(2m) + \tilde{x} \), where \(|\tilde{x}| \ll R/(2m)\) we have

\[
\frac{1}{2g^2v^2} \partial_x^2 u \approx 1 - \left( 1 + e^{-2m\tilde{x}} + e^{-2R} \right) e^{-u},
\]

which is nearly the transition (1.2) up to the negligible term \( e^{-2R} \ll 1 \), while near the point \( x = -R/(2m) + \tilde{x} \) we get

\[
\frac{1}{2g^2v^2} \partial_x^2 u \approx 1 - \left( 1 + e^{2R(e^{-2m\tilde{x}} + 1)} \right) e^{-u},
\]

We can shift the field \( u \rightarrow u + 2R \) to make the first term in the parenthesis negligible, thus approximately obtaining an equation for the elementary wall (2.3).

In other words, for sufficiently big \( R \) the solution describes a pair of elementary walls located at

\[
x_1 = \frac{R}{m}, \quad x_2 = -\frac{R}{m}.
\]

In the region between the walls, the fields \( H \) and \( \sigma \) are nearly at the second vacuum and in the limit \( R \to \infty \), the whole configuration becomes (2). On the other hand, when \( R \) is close to zero or negative the two walls merge together: they form a so-called compressed wall. In the extreme limit \( R \to -\infty \), the master equation (20) reduces to Eq. (14) and we have pure transition (1.3) with a single domain wall. An exact solution of Eq. (16) was reported in Ref. [11]. Again, we will discuss it in the next section.

Let us now make a few comments about the general case. For ease of reference let us repeat here the master equation (13) in a fully unpacked way (\( x^1 \equiv x, \eta = 1 \)):

\[
\frac{1}{2g^2v^2} \partial_x^2 u = 1 - e^{-u} \left( \sum_{i=1}^{N_F} h_i^2 e^{-2m_i x} \right),
\]

where \( h_i \) are absolute values of moduli matrix \( H_0 \) and \( m_i \) are diagonal elements of the mass matrix \( M \). Notice that not all moduli parameters and mass parameters are independent. We can always make the transformation

\[
u(x) \rightarrow u(x + \alpha) + \beta + \gamma x,
\]

\[
m_i \rightarrow m_i - \gamma/2,
\]

\[
h_i \rightarrow \exp(\gamma \alpha/2 - m_i \alpha + \beta),
\]

without changing the equation. This redundancy is a manifestation of \( V \)-transformation, translation invariance, and reparametrization freedom of the mass matrix.

Generalizing what we learned in particular cases, there are \( N_F - 1 \) elementary walls (\( i + 1 \)), which are obtained by setting \( h_i = 1, h_{i+1} = 1 \) for \( i = (1, \ldots, N_F - 1) \) and \( h_j = 0 \) otherwise. On the other hand, if there are three or more nonzero moduli parameters, we have a configuration of two or more elementary walls. In the maximal case where all moduli are nonzero, we have everything between a system of well-separated \( N_F - 1 \) elementary walls and a single domain wall (1 \( N_F \)).

Can we tell which elementary walls are compressed, which are isolated, and at what positions walls (compressed or elementary) roughly are just from moduli parameters? Yes. Let us concentrate on the \( i \)th elementary domain wall. We can estimate its position by comparing the factors \( h_i^2 e^{2m_i x} \) and \( h_{i+1}^2 e^{2m_{i+1} x} \) on the right-hand side of Eq. (20). If neither of these numbers (at given \( x \)) is significantly

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This is so because ordering of masses close to the wall. Obviously, both factors must be equally dominant over remaining terms, in order to be close to the corresponding vacuum, but again not close to the vacuum and hence we are also far away from the \(i\)th wall. The point where both factors are equal is

\[
x_i = \log\left(\frac{h_{i+1}/h_i}{m_i - m_{i+1}}\right).
\]

But \(x_i\) will faithfully represent the position of the wall only if the numbers \(h_i^2 e^{2m_i x_i}\) and \(h_{i+1}^2 e^{2m_{i+1} x_i}\) really are much bigger than all remaining factors. This happens when the \(i\)th wall is isolated from its neighbors, i.e., if \(x_{i-1} \ll x_i \ll x_{i+1}\). If, e.g., \(x_{i-1}\) is close to or bigger than \(x_i\) then both elementary walls are compressed with each other and located at the center of mass of the two-wall system:

\[
\tilde{x}_i = \frac{(m_i - m_{i+1})x_i + (m_{i-1} - m_i)x_{i-1}}{m_{i-1} - m_i} = \frac{1}{m_{i-1} - m_i} \log\left(\frac{h_{i+1}/h_{i-1}}{m_i - m_{i+1}}\right).
\]

This is so because ordering of masses \(m_i \geq m_{i+1}\) implies that vacua, which are visited by the configuration along the \(x\)-axis, must be ordered as well. Therefore, the elementary domain walls cannot have arbitrary positions and in fact the \(i\)th wall must be placed on the right side of the \((i+1)\)th wall.

In general, if we work out the values of all \(x_i, i = 1, \ldots, N_F - 1\) given in Eq. (24), the ordering of these points tells us which walls are isolated and which are compressed. If they are ordered as \(x_1 \gg \cdots \gg x_{N_F - 1}\), there are \(N_F - 1\) isolated walls located approximately at those points. At the other extreme, if the center of mass of the first \(N_F - 2\) elementary walls \(\tilde{x}\) is smaller than \(x_{N_F - 1}\), then there is only one compressed domain wall located at their mutual center of mass.

From Eq. (24) we see that not all \(h\)'s are necessary to describe positions of walls. We can use the symmetry transformation (21)–(23) to reduce their number by two. Let us fix this freedom and rewrite the moduli \(h_i\) in the standard form, where their meaning becomes more transparent:

\[
h_i = \prod_{j=1}^{i-1} e^{R_j}, \quad i = (1, \ldots, N_F - 1).
\]

By virtue of this assignment, \(h_1 = 1\). New parameters \(R_i \equiv x_i(m_i - m_{i+1})\) are weighted positions of the walls. Since the position of the center-of-mass of the entire system is not very interesting moduli, let us fix it at \(x = 0\). In other words we demand \(R_1 + \cdots + R_{N_F - 1} = 0\). This makes \(h_{N_F} = 1\). Furthermore, we will fix the remaining symmetry by demanding that \(m_1 = 0\). We will call a configuration of domain walls with center of mass fixed at the origin and \(m_1 = 0\) a balanced configuration. Any initially unbalanced configuration of domain walls, i.e., \(h_1 \neq 1\), \(h_{N_F} \neq 1\), and \(m_1 \neq 0\), can be turned into a balanced one via the transformation of the solution \(u(x) \rightarrow u(x + \alpha) + \beta + 2m_1 x\), where

\[
\alpha = \frac{1}{m_1 - m_{N_F}} \log\left(h_{N_F}/h_1\right),
\]

\[
\beta = \frac{2m_1}{m_1 - m_{N_F}} \log\left(h_{N_F}\right) - \frac{2m_{N_F}}{m_1 - m_{N_F}} \log(h_1).
\]
2.4. Crude model of the domain wall

The domain wall is known to possess a three-layered structure [7,19]. The outer two layers of the domain wall are customarily called the “skin” and here we call the inner layer the “core”. The skin is characterized by the rapid decay of the corresponding Higgs field, from the asymptotic value $H = v$ in the vacuum outside the domain wall to (almost) zero. The core is a region where all Higgs fields remain very close to zero. This means that in the core the gauge symmetry is almost restored, and we can understand the skin as a transitional region from the spontaneously broken phase outside the domain wall to the unbroken phase in the core.

The core, however, is not present for all domain walls, but it develops only when the total tension is sufficiently big. Let us again consider the $N_F = 2$ example with the mass matrix $M = \text{diag}(0, -m)$ and $H_0 = (1, 1)$. In Fig. 2 we show energy density profiles of three domain walls with increasing values of total tension. The three-layered structure is most visible for the heaviest wall, where the plateau in the energy density is clearly present. The lightest wall is visibly core-less and has only skin, while the middle wall is just at the point where core starts to develop.

The existence of the core can be partially understood from the BPS equations (10)–(11). In particular, the second equation (11) can be seen as a condition that the value of the tension density $\nu^2 \partial_x \sigma$ cannot be at any point higher than $g^2 v^4$ (in Fig. 2 this value is exactly 0.125). Notice that this upper bound can be reached only when Higgs fields are zero, which is the unbroken phase. Hence, if we assume that the (maximal) skin energy is independent of the mass (as we will show is the case) and that the total tension $\nu^2 \Delta m$ is significantly higher than $g v^3$, the core of the domain wall must develop, in order to store the excess energy.

This observation implies that the effective radius of the core is linearly proportional to the tension, $R_c \sim \Delta m$. In fact, ignoring skin altogether we easily estimate $R_c$ by comparing energy inside the core $g^2 v^4 R_c$ to the total energy $\nu^2 \Delta m$, thus obtaining $R_c \sim \Delta m / g^2 v^2$.

We can gain much more insight into the structure of the domain wall by imagining a very crude model as depicted in Fig. 3. There, we take the Higgs fields as piecewise linear functions. Outside the domain wall they are exactly fixed at given vacua ($(1)$ on the right side and $(2)$ on the left side).
Fig. 3. The crude model of the domain wall, where Higgs fields are piecewise linear functions.

Inside the skin they fall linearly to zero across the interval of size $R_s$ and remain vanishing in the core of width $R_c$.

The $\sigma$ field is determined by demanding that it is a continuous function and inside every region a solution of the second BPS equation (11). Carrying out the integration of Eq. (11) in each region we find

\[ \sigma_{\text{core}} = g^2 v^2 x - \frac{m}{2}, \]  \hspace{1cm} (29)

\[ \sigma_{\text{right skin}} = g^2 v^2 \left( x + \frac{(R_c - 2x)^3}{24 R_s^2} \right) - \frac{m}{2}, \]  \hspace{1cm} (30)

\[ \sigma_{\text{left skin}} = g^2 v^2 \left( x - \frac{(R_c + 2x)^3}{24 R_s^2} \right) - \frac{m}{2}, \]  \hspace{1cm} (31)

\[ \sigma(1) = 0, \]  \hspace{1cm} (32)

\[ \sigma(2) = -m. \]  \hspace{1cm} (33)

The integration constants are fixed in the following way. In the core we demand $\sigma_{\text{core}}(0) = -m/2$ by reflection symmetry. In both skins we determine the integration constants by sewing the $\sigma_{\text{right skin}}$ and $\sigma_{\text{left skin}}$ with $\sigma_{\text{core}}$ at the points $x = \pm R_c/2$. Interestingly, the remaining continuity conditions $\sigma_{\text{right skin}} = \sigma(1)$ and $\sigma_{\text{left skin}} = \sigma(2)$ at the points $x = \pm R_c/2 \pm R_s$ give a unique relation between $R_s$ and $R_c$ in the form

\[ R_c = \frac{m}{g^2 v^2} - \frac{4}{3} R_s. \]  \hspace{1cm} (34)

Using this relation we can calculate the tension of the right skin to be

\[ T_{\text{right skin}} = v^2 \int_{R_c/2}^{R_c/2 + R_s} \partial_x \sigma_{\text{right skin}} \, dx = v^2 \sigma_{\text{right skin}} \bigg|_{R_c/2}^{R_c/2 + R_s} \]  \hspace{1cm}

\[ = v^2 \left( 0 - g^2 v^2 R_c/2 + m/2 \right) = \frac{v^2 m}{2} - \frac{g^2 v^4 R_c}{2}. \]  \hspace{1cm} (35)
By symmetry we have $T_{\text{skin}} = 2T_{\text{right skin}} = v^2 m - g^2 v^2 R_c$. Combining this with the tension of the core $T_{\text{core}} = g^2 v^2 R_c$ we see that in total we have $T = T_{\text{skin}} + T_{\text{core}} = v^2 m$, as it should be.

However, not all of the energy has been accounted for. Notice that at the core and in the vacua our fields are exact BPS solutions, while in the skin only the second BPS equation (11) is satisfied. This means that besides the tension $T$ there is an additional energy contribution $E_{\text{extra}}$ coming from the violation of the first BPS equation:

$$E_{\text{extra}} = \int_{\text{skin}} |\partial_x H + \sigma H - HM|^2 = \frac{2v^2}{R_s} - \frac{4g^2 v^4}{15} R_s + \frac{34g^4 v^6}{2835} R_s^3,$$  \hspace{1cm} (36)

where we have used the relation (34). We want this term to be as small as possible. Remarkably, $E_{\text{extra}}$ is strictly positive with a unique minimum at

$$R_s = \frac{c_s}{g v}, \quad c_s = \left( \frac{63 + 3\sqrt{2226}}{17} \right)^{1/2}.$$  \hspace{1cm} (37)

Thus, using our crude model we managed to find formulas relating the width of the core $R_c$ and of the skin $R_s$ to the parameters of the model as

$$R_c = \frac{m}{g^2 v^2} - \frac{4c_s}{3g v}, \quad R_s = \frac{c_s}{g v}, \quad c_s \approx 3.47,$$  \hspace{1cm} (38)

which are in full accordance with the previous estimates found in Refs. [7,19]. As per our intuitive understanding, the core develops only when the tension of the domain wall is sufficiently big. More precisely, the crude model gives us an estimate $T_{\text{min}} \approx 4.63 g v^3$, which corresponds to a value
$R_c = 0.3$ The numerical analysis confirms that the formulas in Eq. (38) are working. For illustration, we show a comparison of the crude model with a particular numerical solution in Fig. 4.

3. Chains of walls

Any domain wall solution of the master equation (13) is fully specified by the source term $\Omega_0$, which in turn depends on the mass matrix $M$ and on (the absolute value of) the moduli matrix $H_0$. However, given $\Omega_0$ it is not possible to reconstruct $M$ and $H_0$ unambiguously, as we can always include any number of zero elements inside $H_0$ or degenerate entries in $M$. This effectively lifts the solution to a model with an arbitrarily large number of Higgs fields $N_F$. Thus, in the rest of the paper, when we specify a mass matrix $M$ or a moduli matrix $H_0$, we always refer to the minimal model, which can be ascertained unambiguously.

In this section, we present an infinite amount of exact solutions for the master equation (13). We do this by first presenting two new single wall solutions—or 1-chains—and then we show that simple addition of these yields new multi-wall solutions—or multi-chains—provided that certain restriction on parameters hold.

The existence of such solutions might be surprising, given the nonlinearity of the master equation. However, it is important to keep in mind that for each solution the source term $\Omega_0$ is different. To be precise, if we link a single wall solution to an existing chain, two things happen. First, the (minimal) model, where the solution lives, is enlarged. If we denote the mass matrix of the $n$-chain as $M^{(n)}$ and the mass matrix of the link as $M^{(1)}$, the new chain’s mass matrix is given as $M^{(n+1)} = M^{(n)} \oplus M^{(1)}$, where by $\oplus$ we mean all possible pairings of elements from both matrices. The moduli matrices of the chain and newly added link combine in a much more complicated fashion.

Second, the constraint on the parameters, which is present to ensure the validity of the solution, becomes more strict. Roughly speaking, the maximum tension per individual wall is decreased. As we will argue, this tightening occurs primarily to prevent a violation of the no-go policy:

No cores for exact walls.

This is the key observation of this paper. As far as we know there is no exact solution of the domain wall with a core. If such solutions exist at all, they are certainly out of the reach of methods used in this paper. The search for an exact domain wall solution with a core remains an interesting open problem.

Another peculiarity of exact solutions presented in this paper is that there are always nonelementary walls. Except for a few cases, which we show in the next subsection and which have already been reported elsewhere [11], we have found no exact solution with isolated elementary walls. In fact, the number of separable walls through the moduli parameters in the solutions themselves is always

---

3 One can imagine modified versions of the crude model, where the Higgs fields in the skin can deviate from a straight line. For example, if we assume a sine-like profile and follow the same procedure as in the linear case, we obtain the formulas

$$R_c = \frac{m}{g^2 v^2}, \quad R_s = \frac{c_s}{g v}, \quad c_s \approx \pi \left( \frac{2}{75\pi^2 - 705 (13 + (1200\pi^2 - 11111)^{1/2})} \right)^{1/2},$$

which give a slightly larger estimate $T_{\min} \approx 5.9gv^3$. Intuitively, closer to the actual solution the profile in the skin is, the more energy is shifted from the core to the skin, giving us higher values for $T_{\min}$. In that sense, the value of the crude model $T_{\min} \approx 4.63gv^3$ can be seen as a lower bound.

4 That is, e.g., $\text{diag}(m_1, m_2) \oplus \text{diag}(n_1, n_2) = \text{diag}(m_1 + n_1, m_1 + n_2, m_2 + n_1, m_2 + n_2)$.
less than the number of elementary domain walls of the model. In short, our solutions cover only subspaces of the full moduli space.

In the following two subsections, we present all 1-chains, 2-chains, and 3-chains. We will shortly discuss their properties, but we mainly focus on three characteristics: (1) the minimal number of flavors, (2) the condition that ensures their validity, and (3) a check that they have no cores. Based on these particular examples, in the third subsection we provide general formulas for these characteristic for arbitrary $N$-chains.

### 3.1. Exact 1-chains

In Ref. [11] three exact single wall solutions were presented. In our notation, they read

\[
  u^{2F}_1(x) = 2 \log \left(1 + e^{-mx}\right), \quad T = \sqrt{2}gv, \tag{39}
\]

\[
  u^{2F}_{11}(x) = 2 \log \left(1 + \sqrt{2}e^{-mx} + e^{-2mx}\right), \quad T = \sqrt{8}gv, \tag{40}
\]

\[
  u^{2F}_{111}(x) = 3 \log \left(1 + e^{-mx}\right), \quad T = \frac{3}{2} \sqrt{2}gv. \tag{41}
\]

Here we have indexed each solution by the number of flavors in the superscript and by a roman numeral in the subscript for ease of reference. We have also kept the parameter $m$, although $m = \sqrt{2}gv$ holds for all three solutions, to keep the formulas simple.

Solutions (39)–(41) are remarkable for many reasons, two of which are particularly relevant here. First, they are the only instances of exact elementary domain walls. As we stated, all other exact solutions presented in this paper are nonelementary walls. Second, all walls are core-less, since $T < T_{\text{min}} \approx 4.63gv$. In other words, these walls are not heavy enough to have cores, as we discussed in Sect. 2.4.

Can we make chains out of $u^{2F}_1$, $u^{2F}_{11}$, or $u^{2F}_{111}$? In general, no. The only exception is a remarkable double wall solution also reported in Ref. [11], which in our notation reads

\[
  u^{3F}_1(x) = 2 \log(1 + (6 + e^R)^{1/2} e^{-mx} + e^{-2mx}), \tag{42}
\]

\[
  \Omega_0 = 1 + e^R e^{-2mx} + e^{-4mx}, \quad m = \sqrt{2}gv. \tag{43}
\]

The parameter $R/m$ can be interpreted as a separation of walls if $R \gg m$, while for $R \sim m$ or less, the walls are compressed. As already noted in Ref. [11], $u^{3F}_1$ can be rewritten as

\[
  u^{3F}_1(x) = u^{2F}_1(x + S) + u^{2F}_1(x - S), \quad S = \frac{1}{2m} \arccosh(2 + e^R/2). \tag{44}
\]

This is a first example of a 2-chain. Indeed, $u^{3F}_1$ is a solution of a 3F model, written as the sum of two 2F domain walls and the mass matrix equal to the direct sum $M = \text{diag}(0, m, 2m) = \text{diag}(0, m) \oplus \text{diag}(0, m)$. A peculiar feature of $u^{3F}_1$ is that it is undefined for $|S| < |S_{\text{min}}| \equiv \frac{1}{2m} \arccosh(2)$, which would cause $R$ to be complex. In fact, at $|S| = |S_{\text{min}}|$ the solution $u^{3F}_1$ becomes $u^{2F}_{11}$, which is an elementary wall. This peculiarity, however, seems to be confined to this 2-chain.

With the exception of Eq. (44), no other chains can be constructed out of Eqs. (39)–(41). Let us illustrate this on $u^{2F}_{111}$. Quick calculation reveals that the function $u^{2F}_{111}(x - S) + u^{2F}_{111}(x + S)$ is a solution to the master equation for the source term

\[
  \Omega_0 = 1 - 9e^{-2mx} + e^R e^{-3mx} - 9e^{-4mx} + e^{-6mx}, \tag{45}
\]
Although very similar to \( u^{2F}_{II} \), the mass parameter in \( u^{3F}_{II} \) can be chosen arbitrarily within the range \( 0 < m^2 \leq 2g^2v^2 \). This solution lives in a 3F model with the mass matrix \( M^{(1)} = -\text{diag}(0, m, 2m)/2 \). From Eq. (24) we can calculate (naive) positions of elementary walls to be

\[
x_1 = \frac{1}{m} \log \left( 2 - \frac{m^2}{g^2v^2} \right), \quad x_2 = -\frac{1}{m} \log \left( 2 - \frac{m^2}{g^2v^2} \right).
\]  

(48)

But since \( x_1 < x_2 \), both walls are compressed and there is, in fact, only a single wall with the tension \( T^{3F}_{II} \equiv mv^2 \leq \sqrt{2gv}^3 \) located at \( mx_1 + mx_2 = 0 \). In the limit \( m^2 \rightarrow 2g^2v^2 \) the compression becomes infinite \((x_1 \rightarrow -\infty, x_2 \rightarrow \infty)\) and we return to the original solution \( u^{2F}_{II} \).

Unlike \( u^{3F}_{II} \), our new solution does not have a modulus that controls separation of elementary walls. In other words, this solution covers only a subspace of the full moduli space of the (minimal) model, where it lives. Also, notice that the amount of compression for both elementary walls is a very specific \( m\)-dependent number. If the numbers \( x_1 \) and \( x_2 \) in Eq. (48) are only slightly different, no corresponding exact solution is known to the author. The reason—if there is one—why only such a special arrangement of elementary walls allows an exact solution, while other arrangements do not, remains elusive.

The second exact solution is constructed similarly by relaxing mass constraint in \( u^{2F}_{III} \):

\[
u^{4F}_{II}(x) = 3 \log(1 + e^{-mx}), \quad m \leq \sqrt{2gv},\]

\[
\Omega_0 = 1 + 3e^{-mx}(1 - \frac{m^2}{2g^2v^2}) + 3(1 - \frac{m^2}{2g^2v^2})e^{-2mx} + e^{-3mx}.
\]  

(49) (50)

This solution lives in a 4F model with mass matrix \( M^{(2)} = -\text{diag}(0, m, 2m, 3m)/2 \). The ordering of elementary walls

\[
x_1 = \frac{1}{m} \log \left( 3 - \frac{3m^2}{2g^2v^2} \right) < 0, \quad x_2 = 0, \quad x_3 = -\frac{1}{m} \log \left( 3 - \frac{3m^2}{2g^2v^2} \right) > 0,
\]  

(51)

again implies that there is only one compressed wall at the origin. The original solution is restored in the limit \( m^2 \rightarrow 2g^2v^2 \), as before. The tension is never greater than \( T^{4F}_{II} \equiv 3mv^2/2 \leq 3\sqrt{2gv} \). Both our new solutions \( u^{3F}_{II} \) and \( u^{4F}_{II} \) are obviously core-less, since they are no heavier than \( u^{2F}_{II} \) and \( u^{2F}_{III} \), respectively. But before we discuss 2-chains made of \( u^{3F}_{II} \) and \( u^{4F}_{II} \), let us introduce a simple check that a given exact solution has no core. The trick is to find the maximum value of the tension density

\[
T_{\text{max}} = \frac{1}{2} \left. \nu''(0) \right|_{\text{coincident walls}}
\]  

(52)

with all walls fixed at, say, the origin \( x = 0 \). Since inside the core (as discussed in Sect. 2.4) the tension density is practically equal to \( g^2v^4 \), it is sufficient to show that \( T_{\text{max}} \) is lower than that value.

\[13/38\]
Indeed, a short calculation reveals that all exact single wall solutions given in this subsection have their maximal values,

\[ T^{2\text{F}}_I(0) = 0.5g^2v^4, \quad T^{2\text{F}}_\Pi(0) \approx 0.89g^2v^4, \quad T^{2\text{F}}_{\text{III}}(0) = 0.75g^2v^4, \quad (53) \]

\[ T^{3\text{F}}_\Pi(0) = \frac{m^2v^2}{4} \leq 0.5g^2v^4, \quad T^{3\text{F}}_I(0) = \frac{3m^2v^2}{8} \leq 0.75g^2v^4, \quad (54) \]

markedly lower than \( g^2v^4 \).

### 3.2. Exact 2-chains and 3-chains

Let us first investigate a 2-chain made of equal tension \( u^{3\text{F}}_\Pi \) walls. That is,

\[ u^{3\text{F}}_I(x) = u^{3\text{F}}_\Pi(x + S; m) + u^{3\text{F}}_\Pi(x - S; m), \quad (55) \]

\[ \Omega_0 = 1 + 4e^{-mx} \cosh(mS)(1 - \frac{m^2}{2g^2v^2}) + 2e^{-2mx}(2 + \cosh(2mS) - \frac{2m^2}{g^2v^2}) \]

\[ + 4e^{-3mx} \cosh(mS)(1 - \frac{m^2}{2g^2v^2}) + e^{-4mx}. \quad (56) \]

This 2-chain lives in a 5F model with mass matrix

\[ M = M^{(1)}(m) \oplus M^{(1)}(m) = -\text{diag}(0, m, 2m, 3m, 4m)/2. \]

Notice that the (minimal) number of flavors of \( u^{3\text{F}}_I \) is not always 5, but can be lower for special values of \( m \) or \( S \) (or both). For example, taking the limit \( m^2 \to 2g^2v^2 \) eliminates two factors from its \( \Omega_0 \), which means that the resulting solution has only 3 flavors. Incidentally, this solution is \( u^{3\text{F}}_I \). Also, by choosing \( S = \frac{1}{2m} \arccosh(2 - \frac{2m^2}{g^2v^2}) \), we can nullify the third term in Eq. (55), which gives us a new 4F solution

\[ u^{4\text{F}}_\Pi(x) = 2 \log \left( 1 + 2e^{-mx} \left( \frac{m^2}{2g^2v^2} - \frac{1}{2} \right)^{1/2} + e^{-2mx} \right), \quad \frac{1}{4} \leq \frac{m^2}{2g^2v^2} \leq 1, \quad (57) \]

\[ \Omega_0 = 1 + 4e^{-mx} \left( 1 + e^{-2mx} \right) \left( 1 - \frac{m^2}{2g^2v^2} \right) \left( \frac{m^2}{2g^2v^2} - \frac{1}{2} \right)^{1/2} + e^{-4mx}, \quad (58) \]

and again taking the limit \( m^2 \to 2g^2v^2 \) in this solution gives us \( u^{2\text{F}}_\Pi \).

Generally, all chains can be reduced in a similar manner to a sequence of new exact solutions, which, however, possess fewer parameters than their parent solutions. We will investigate these “reductions” in detail in Sect. 4, since their exact structure becomes quickly very complicated for a higher number of flavors. In the remainder of this section, we will skip these considerations entirely, for brevity.

Before going further, let us discuss a methodology, how we determine under which condition any chain is a valid solution of the master equation (13). For simple solutions such as \( u^{3\text{F}}_I \) and \( u^{4\text{F}}_\Pi \) this can be done by inspection, but for more complicated ones, we need to establish a clear criterion.

As we saw in the example of a false 2-chain made out of \( u^{2\text{F}}_\Pi \) in Eq. (45), the issue is that we have to guarantee that the moduli matrix \( H_0 \) is well defined or, more simply, that all coefficients multiplying exponential factors in \( \Omega_0 \) are positive numbers. These coefficients, however, are generally complicated functions of distances between walls, which makes the analysis very cumbersome.

Fortunately, in order to obtain a general condition for any arrangement of walls, we need only to look at the coincident case, where all walls sit on the same point on the \( x \)-axis, say the origin. At this limit, all coefficients in \( \Omega_0 \) reach their lowest values.
Why it is so? Let us illustrate this on an example of a generic 2-chain made of single-wall solutions $u_1$ and $u_2$, where

$$
\frac{1}{2g^2v^2} \partial_x^2 u_1 = 1 - \Omega_0^{(1)} e^{-u_1}, \quad \frac{1}{2g^2v^2} \partial_x^2 u_2 = 1 - \Omega_0^{(2)} e^{-u_2}.
$$

(59)

The master equation for $u_1 + u_2$ can be rewritten as

$$
\frac{1}{2g^2v^2} \partial_x^2 (u_1 + u_2) = 2 - \Omega_0^{(1)} e^{-u_1} - \Omega_0^{(2)} e^{-u_2}
= 1 - \Omega_0^{(1)} \Omega_0^{(2)} e^{-u_1-u_2} + (1 - \Omega_0^{(1)} e^{-u_1})(1 - \Omega_0^{(2)} e^{-u_2})
= 1 - \Omega_0^{(1)} \Omega_0^{(2)} e^{-u_1-u_2} + \frac{1}{4g^4v^4} \partial^2_x u_1 \partial^2_x u_2
= 1 - \left( \Omega_0^{(1)} \Omega_0^{(2)} - \frac{1}{4g^4v^4} \partial^2_x u_1 \partial^2_x u_2 e^{u_1+u_2} \right) e^{-u_1-u_2}.
$$

(60)

Thus, the 2-chain $u_1 + u_2$ solves the master equation with the source term

$$
\Omega_0^{(1+2)} = \Omega_0^{(1)} \Omega_0^{(2)} - \frac{1}{4g^4v^4} \partial^2_x u_1 \partial^2_x u_2 e^{u_1+u_2}.
$$

(61)

Note that $\Omega_0^{(1+2)}$ has two parts. The first part is manifestly positive since $\Omega_0^{(1)} \Omega_0^{(2)}$ has only positive coefficients. The second part is proportional to tension densities of both solutions and hence (given the minus sign) is manifestly negative. Thus, the coefficients in $\Omega^{(1+2)}$ are lowest if the second term is (in an absolute sense) largest. But this happens precisely when domain walls from the first and second solutions are coincident because the product of their tension densities, $\sim \partial^2_x u_1 \partial^2_x u_2$, is largest there. Hence, our claim holds. Notice that this argument can be trivially extended to any number of walls. Therefore, the criterion that ensures well-defined $H_0$ for any exact solution, is established by looking at the lowest coefficient in $\Omega_0$ at the coincident point.

In the case of $u_1^{SF}$ the coincident limit reads

$$
\Omega_0 \xrightarrow{S \to 0} 1 + 4 \left( 1 - \frac{m^2}{2g^2v^2} \right) e^{-mx} + 2 \left( 3 - \frac{2m^2}{g^2v^2} \right) e^{-2mx} + 4 \left( 1 - \frac{m^2}{2g^2v^2} \right) e^{-3mx} + e^{-4mx},
$$

(62)

from which we see

$$
m^2 \leq 3g^2v^2/2.
$$

(63)

Notice that this is more stringent than for individual 1-chains, where the condition was $m^2 \leq 2g^2v^2$. This is intuitively understandable from the point of view of the no-cores-for-exact-solutions philosophy. Since the tensions of individual links in the chain add up linearly, the restriction on them must be larger, compared to restrictions on the links themselves. If that were not the case, a sufficiently long chain would eventually break the threshold for developing a core. And indeed, with the more stringent restriction, the maximum tension density for our 2-chain is well below the threshold:

$$
T_1^{SF}(0) = \frac{v^2 m^2}{2} \leq \frac{3g^2 v^4}{4}.
$$

(64)

---

5 But see the discussion below Eq. (145).
Thus, the solution

\[ \text{2-chain is a special case of a more general solution designated as} \]

\[ \text{taking limit} \]

\[ N \]

\[ \text{collect all the relevant information about them into Table 1. There we show the number of flavors} \]

\[ \text{constituents. At the coincident point} \]

\[ \text{is again given by considering all possible pairings between matrix elements of its single wall} \]

\[ \text{dealt with both cases separately. In fact, many characteristics of the chain change discontinuously,} \]

\[ \text{which is again more severe than the equivalent for 1-chains. The limit on the tension density is} \]

\[ \text{limit on the tension density is} \]

\[ \text{Notice that taking limit} \]

\[ \text{reproduces neither condition (63) nor (64). This is why we have} \]

\[ \text{for different levels of mass-degeneracies. For example, the number of elementary walls is now eight,} \]

\[ \text{when we look at each} \]

\[ \text{when} \]

\[ \text{the act of combining both solutions changes their nature. Nothing of this sort happens in equal tension case.} \]

\[ \text{Instead of continuing the discussion for 2-chains involving combinations of} \]

\[ \text{we rather collect all the relevant information about them into Table 1. There we show the number of flavors} \]

\[ \text{condition on parameter(s), and maximum value of the tension density} \]

\[ \text{Table 1. All 1-chains and 2-chains.} \]

| chain | \( N_F \) | condition | \( T_{\text{max}}/g^2 v^4 \) |
|-------|---------|-----------|-----------------|
| 1(m)  | 3F      | \( m^2 \leq 2g^2 v^2 \) | 1/2             |
| 2(m)  | 4F      | \( m^2 \leq 2g^2 v^2 \) | 3/4             |
| 1(m) \( \oplus \) 1(m) | 5F | \( m^2 \leq 3g^2 v^2 /2 \) | 3/4             |
| 1(m) \( \oplus \) 2(m) | 6F | \( m^2 \leq 4g^2 v^2 /3 \) | 5/6             |
| 2(m) \( \oplus \) 2(m) | 7F | \( m^2 \leq 10g^2 v^2 /9 \) | 5/6             |
| 1(m_1) \( \oplus \) 1(m_2) | 9F | \( m_1^2 + m_2^2 \leq 2g^2 v^2 \) | 1/2             |
| 1(m_1) \( \oplus \) 2(m_2) | 12F | \( m_1^2 + m_2^2 \leq 2g^2 v^2 \) | 3/4             |
| 2(m_1) \( \oplus \) 2(m_2) | 16F | \( m_1^2 + m_2^2 \leq 2g^2 v^2 \) | 3/4             |

\[ \text{Let us now consider a 2-chain made of two} \]

\[ u_{\text{II}}^{3F} \]

\[ \text{walls with unequal tensions:} \]

\[ u_{\text{IV}}^{3F}(x) = u_{\text{II}}^{3F}(x - S_1; m_1) + u_{\text{II}}^{3F}(x - S_2; m_2). \] (65)

\[ \text{This 2-chain is a special case of a more general solution designated as} u_{\text{IV}}^{3F}(x), \text{which is studied in} \]

\[ \text{detail in Sect. 4. We count 9 flavors. The mass matrix} \]

\[ M = M^{(1)}(m_1) \oplus M^{(1)}(m_2) \]

\[ = - \text{diag}(0, m_1, m_2, m_1 + m_2, 2m_1, 2m_2, 2m_1 + m_2, m_1 + 2m_2, 2m_1 + 2m_2)/2 \]

is again given by considering all possible pairings between matrix elements of its single wall constituents. At the coincident point \( S_1 = S_2 = 0 \) we have

\[ \Omega_0 = 1 + 2 \left( 1 - \frac{m_1^2}{2g^2 v^2} \right) \left( 1 + e^{-2m_1 x} \right) e^{-m_1 x} + 2 \left( 1 - \frac{m_2^2}{2g^2 v^2} \right) \left( 1 + e^{-2m_2 x} \right) e^{-m_2 x} \]

\[ + e^{-2m_1 x} + e^{-2m_2 x} + 4 \left( 1 - \frac{m_1^2 + m_2^2}{2g^2 v^2} \right) e^{-(m_1 + m_2)x} + e^{-2(m_1 + m_2)x}. \] (66)

Thus, the solution \( u_{\text{IV}}^{3F} \) is valid if

\[ m_1^2 + m_2^2 \leq 2g^2 v^2, \] (67)

which is again more severe than the equivalent for 1-chains. The limit on the tension density is

\[ T_{\text{IV}}^{3F}(0) = v^2 \frac{m_1^2 + m_2^2}{4} \leq 0.5g^2 v^4. \] (68)
Table 2. All 3-chains.

| Chain | N_F | Condition | \( T_{\text{max}} / g^2 v^4 \) |
|-------|-----|-----------|--------------------------------|
| 1(m) ⊕ 1(m) ⊕ 1(m) | 7F | \( m^2 \leq 10 g^2 v^2 / 9 \) | 5/6 |
| 1(m) ⊕ 1(m) ⊕ 2(m) | 8F | \( m^2 \leq g^2 v^2 \) | 7/8 |
| 1(m) ⊕ 2(m) ⊕ 2(m) | 9F | \( m^2 \leq 7 g^2 v^2 / 8 \) | 7/8 |
| 2(m) ⊕ 2(m) ⊕ 2(m) | 10F | \( m^2 \leq 4 g^2 v^2 / 5 \) | 9/10 |
| 1(m_1) ⊕ 1(m_3) ⊕ 1(m_2) | 15F | \( 4 m_1^2 + 3 m_2^2 \leq 6 g^2 v^2 \) | 3/4 |
| 1(m_1) ⊕ 1(m_3) ⊕ 2(m_1) | 18F | \( 3 m_1^2 + 2 m_2^2 \leq 4 g^2 v^2 \) | 5/6 |
| 1(m_1) ⊕ 1(m_3) ⊕ 2(m_2) | 20F | \( 4 m_1^2 + 3 m_2^2 \leq 6 g^2 v^2 \) | 3/4 |
| 1(m_1) ⊕ 2(m_2) ⊕ 2(m_2) | 21F | \( 5 m_1^2 + 9 m_2^2 \leq 10 g^2 v^2 \) | 5/6 |
| 1(m_1) ⊕ 2(m_3) ⊕ 2(m_2) | 24F | \( 3 m_1^2 + 2 m_2^2 \leq 4 g^2 v^2 \) | 5/6 |
| 1(m_1) ⊕ 1(m_3) ⊕ 1(m_3) | 27F | \( m_1^2 + m_2^2 + m_3^2 \leq 2 g^2 v^2 \) | 1/2 |
| 2(m_1) ⊕ 2(m_1) ⊕ 2(m_2) | 28F | \( 9 m_1^2 + 5 m_2^2 \leq 10 g^2 v^2 \) | 5/6 |
| 1(m_1) ⊕ 2(m_2) ⊕ 2(m_3) | 36F | \( m_1^2 + m_2^2 + m_3^2 \leq 2 g^2 v^2 \) | 3/4 |
| 1(m_1) ⊕ 2(m_3) ⊕ 2(m_3) | 48F | \( m_1^2 + m_2^2 + m_3^2 \leq 2 g^2 v^2 \) | 3/4 |
| 2(m_1) ⊕ 2(m_2) ⊕ 2(m_3) | 64F | \( m_1^2 + m_2^2 + m_3^2 \leq 2 g^2 v^2 \) | 3/4 |

In Table 1 we introduce the compact notation \( 1(m) = u_{\text{II}}^3F(x; m) \) and \( 2(m) = u_{\text{I}}^4F(x; m) \) for the single wall solutions. To indicate a 2-chain we use the \( \oplus \) symbol, which represents a plain sum for solutions, but a direct sum for mass matrices \( M^{(1)}(m) \) and \( M^{(2)}(m) \). We do not show \( \Omega_0 \), since the formulas are too long. However, one can easily reconstruct both the solution itself and the corresponding source term from the prescription given in the first column in the table and from the identity

\[
\Omega_0 = e^{\mu(x)} \left( 1 - \frac{1}{2 g^2 v^2} \partial_x^2 u(x) \right),
\]

which holds for any solution \( u(x) \) of the master equation.

We list the same results for the 3-chains in Table 2.

### 3.3. Exact N-chain

Let us define an arbitrary \( N \)-chain as a sum of \( N \) single wall solutions formally denoted by

\[
u[X] = \bigoplus_{i=1}^{N} X_i(m_i),
\]

where \( X \in \{ 1 = u_{\text{II}}^3F, 2 = u_{\text{I}}^4F \} \). The parameters \( m_i \) can either be the same or different for each link \( X_i(m_i) \). But, as was the case for 2-chains and 3-chains, all characteristics of interest, such as a number of flavors \( N_F \), the condition ensuring the validity of a solution, and \( T_{\text{max}} \), change discontinuously across degeneracies. This means that taking limits of parameters \( m_i \) in formulas found for the nondegenerate case does not give correct formulas for partially degenerate cases. Therefore, the goal of this subsection is to develop general formulas for all three observables, that apply for all possible degeneracies of parameters \( m_i \) from the beginning.

Let us start with the number of flavors \( N_F \). The rule of thumb turns out to be that for the links with the same \( m_i \), the number of flavors \textit{minus one} add up, while for links with different \( m_i \), flavors multiply. Let us illustrate this on two extreme examples. First, consider a totally degenerate chain
made out of $N - l$ 1-links and $l$ 2-links:

$$\left( \bigoplus_{i=1}^{N-l} \mathbf{1}(m_i) \right) \oplus \left( \bigoplus_{j=1}^{l} \mathbf{2}(m_j) \right) \equiv \mathbf{1}^{N-l}(m) \oplus \mathbf{2}^{l}(m).$$

Here we introduced a shorthand notation $\mathbf{X}^n(m) \equiv \bigoplus_{i=1}^{n} \mathbf{X}(m)$. The solution at the coincident point is given as

$$u = \log (1 + e^{-mx})^{2N+l}.$$ 

The number of flavors $N_F$ is simply the number of distinct exponentials in $\Omega_0$, or equivalently in $e^u$. This gives us the result

$$N_F = 2N + l + 1,$$

which is simply the number of different powers of $e^{-mx}$ in $(1 + e^{-mx})^{2N+l}$, including the zero power. Notice that the number of elementary domain walls can be rewritten as $N_F - 1 = 2(N - l) + 3l$. At the same time, a 1-link is composed of two, while a 2-link is composed of three elementary domain walls. Therefore, in totally degenerate case the elementary walls simply add up.

For a second example let us consider a totally nondegenerate chain:

$$\left( \bigoplus_{i=1}^{N-l} \mathbf{1}(m_i) \right) \oplus \left( \bigoplus_{j=1}^{l} \mathbf{2}(m_j) \right) \Rightarrow u = \log \left( \prod_{i=1}^{N-l} \left( 1 + e^{-m_i x} \right)^2 \prod_{j=1}^{l} \left( 1 + e^{-m_j x} \right)^3 \right).$$

Counting the number of distinct exponentials in $e^u$ we obtain

$$N_F = 3^{N-l}4^l.$$ 

Here we see the multiplication rule: each 1-link contributes 3 flavors and each 2-link gives 4 flavors.

A generic case can be characterized by partial degeneracy of the parameters $m_i$ into $k \leq N$ groups of size $d_i$, where $N = \sum_{i=1}^{k} d_i$. Within each group, $d_i - l_i$ are 1-links and $l_i$ are 2-links. The solution at the coincident point of such a chain reads

$$\bigoplus_{i=1}^{k} \left( \mathbf{1}^{d_i-l_i}(m_i) \oplus \mathbf{2}^{l_i}(m_i) \right) \Rightarrow u = \log \left( \prod_{i=1}^{k} \left( 1 + e^{-m_i x} \right)^{2d_i+l_i} \right).$$

Within a degenerate group, the number of flavors follows the additive rule of Eq. (73), i.e., we have $2d_i + l_i + 1$ distinct exponentials for each group. These numbers must then be multiplied as per the rule in Eq. (75). Thus, the most general formula for an arbitrary $N$-chain reads

$$N_F = \prod_{i=1}^{k} (2d_i + l_i + 1).$$

It is easy to check that the results in the second columns in Tables 1–2 are consistent with the above formula.

Let us now establish the condition under which an arbitrary $N$-chain is a valid solution of the master equation (13). Following the discussion in the previous subsection, we must investigate the
lowest coefficient in $\Omega_0$ at the coincident point. For the most general $N$ chain given in Eq. (76) we have

$$\Omega_0 = \prod_{j=1}^{k} \left(1 + e^{-m_0} \right)^{2d_j + l_j} \left(1 - \sum_{i=1}^{k} \frac{(2d_i + l_i)m_i^2}{2g^2v^2} \frac{e^{-m_0}}{1 + e^{-m_0}} \right)^2.$$  (78)

Now we extract the lowest coefficient. Let us first write down the largest factor coming from the negative half (the second term in the large parenthesis) of $\Omega_0$:

$$- \sum_{i=1}^{k} \left( \frac{2d_i + l_i - 2}{d_i + \left\lfloor \frac{l_i}{2} \right\rfloor - 1} \prod_{j \neq i} \left( \frac{2d_j + l_j}{d_j + \left\lfloor \frac{l_j}{2} \right\rfloor} \right) \frac{(2d_i + l_i)m_i^2}{2g^2v^2} \exp \left( - \sum_{n=1}^{k} (d_n + \left\lfloor \frac{l_n}{2} \right\rfloor)m_nx \right) \right),$$  (79)

where $\left\lfloor x \right\rfloor$ is the floor function. Simply put, we have expanded each bracket in the product and picked out the largest factor, which is in the middle of the binomial series. The factor with the same power in the exponential in the positive half reads

$$\prod_{i}^{k} \left( \frac{2d_i + l_i}{d_i + \left\lfloor \frac{l_i}{2} \right\rfloor} \right) \exp \left( - \sum_{n=1}^{k} (d_n + \left\lfloor \frac{l_n}{2} \right\rfloor)m_nx \right).$$  (80)

Combining both factors together and demanding that the overall coefficient is nonnegative we find the condition for arbitrary $N$-chain to be

$$\sum_{i=1}^{k} \left( d_i + \left\lceil \frac{l_i + 1}{2} \right\rceil \right) \left( d_i + \left\lceil \frac{l_i}{2} \right\rceil \right) m_i^2 \leq 2g^2v^2,$$  (81)

Let us consider a few examples. The totally degenerate case is given by setting $k = 1, m_1 \equiv m$, and $d_1 = N$. Taking $l_1 \equiv l$ out of $N$ walls to be 2-links we have the conditions

(l is even) \quad m^2 \leq \frac{4g^2v^2}{N + \frac{l}{2}} \left( 1 - \frac{1}{2N + l} \right), \quad (l is odd) \quad m^2 \leq \frac{4g^2v^2}{N + \frac{l+1}{2}}. \quad (82)

On the other hand in the fully nondegenerate case $k = N, d_i = 1, l_i \in \{0, 1\}$ the general condition (81) reduces to

$$\sum_{i=1}^{N} m_i^2 \leq 2g^2v^2,$$  (83)

which is curiously independent of all $l_i$’s. Again, the validity of condition (81) for 2-chains and 3-chains can be checked by looking at the result in Tables 1–2.

The remaining task is to find a general formula for $T_{\text{max}}$, defined as

$$T_{\text{max}} \equiv \frac{v^2}{2} u''(0) \bigg|_{\text{coincident walls}}.$$  (84)

For the generic $N$-chain of Eq. (76) we find

$$T_{\text{max}} = \frac{v^2}{8} \sum_{i=1}^{k} (2d_i + l_i)m_i^2.$$  (85)
We need to find the maximum of this. From formula (81) we see that the upper bound for an \(i\)th factor in the above sum is
\[
(2d_i + l_i)m_i^2 \leq 2g^2v^2 \frac{(2d_i + l_i - 1)(2d_i + l_i)}{(d_i + \lfloor \frac{l_i + 1}{2} \rfloor)(d_i + \lfloor \frac{l_i}{2} \rfloor)}.
\] (86)
This can be simplified to
\[
(2d_i + l_i)m_i^2 \leq 8g^2v^2 \left(1 - \frac{1}{2(d_i + \lfloor \frac{l_i + 1}{2} \rfloor)}\right).
\] (87)
Obviously, the maximum of the sum of these numbers amounts to picking \(i \equiv n\) for which \(2d_n + l_n \geq 2d_i + l_i\) for \(i = 1, \ldots, k\). Then, the sum in Eq. (85) is maximized by taking \(m_i = 0\) for \(i \neq n\). In summary, the formula for maximum of the tension density for a generic \(N\)-chain is
\[
T_{\text{max}} = g^2v^4 \left(1 - \frac{1}{2(d_n + \lfloor \frac{l_n + 1}{2} \rfloor)}\right).
\] (88)

Its validity for 2-chains and 3-chains can be checked by comparing with the findings in the third columns of Tables 1–2. We see that \(T_{\text{max}}\) is always less than \(g^2v^4\) as expected. The closest approach is achieved by fully degenerate 2 chains, i.e., \(d_n = N\) and \(l = N\):
\[
T_{\text{max}} = g^2v^4 \left(1 - \frac{1}{3N}\right) \quad \text{(N is even),}
\]
\[
T_{\text{max}} = g^2v^4 \left(1 - \frac{1}{3N+1}\right) \quad \text{(N is odd).}
\]
(89) (90)

In other words, \(\left|T_{\text{max}} - g^2v^4\right| \sim 1/N\) for \(N \gg 1\). It might seem that for sufficiently big chains, the core should appear as the gap between \(T_{\text{max}}\) and \(g^2v^4\) closes. However, this is misleading, because we are looking only at the height of the peak. But its full shape for a fully degenerate 2-chain is
\[
T = \frac{v^2}{2} \partial_x^2 u = \frac{m^2v^2}{8} \frac{3N}{\cosh^2(mx/2)} \leq g^2v^4 \frac{1}{\cosh^2 \left(\frac{2}{3N}\right)^{1/2}x}, \quad N \gg 1.
\] (91)
In other words, an increase in \(N\) causes only widening of the peak and not development of the plateau as seen in Fig. 2. This clearly confirms the no-core rule for the exact solution, as far as chains are concerned.

Let us close this section by commenting on the issue of infinite chains. Periodic domain wall solutions, or other exact infinite wall configurations, are potentially very interesting objects, worthy of study on their own. However, the chains presented here cannot be extended to the infinite case. The obstacle can be clearly seen from the condition (81), which implies that as \(N \to \infty\), the parameters vanish \(m_i \to 0\). More simply, if we look again at a fully degenerate 2 chain as an illustrative example, we see that the amount of tension per wall decreases:
\[
T_{\text{max}}/N \sim \left(\frac{6}{N}\right)^{1/2} gv^3, \quad N \gg 1.
\] (92)
In other words, taking \(N\) to infinity, the chain disappears.
4. Hierarchy of exact solutions

Rather than focusing on the particular type of exact solutions—like the chains of the previous section—in this part we want to *exhaust* all exact solutions, which can be written as the logarithm of a sum of exponentials. To that goal, in the first subsection, we present a convenient ansatz, or rather a class of ansätze. We devote later subsections to investigating the first few smallest domain wall configurations as examples, highlighting the main point of this section. That is, that all exact solutions form a hierarchy, where any particular solution might be seen as a special limit in parameter space of another solution, living in the larger model. Unlike for chains, these relations turn out to be increasingly complicated as the minimal number of flavors of the underlying model increases. In particular, in this section we explore these relations in detail up to $N_F = 10$ flavors.

For simplicity, throughout this section we fix the effective gauge coupling parameter $\tilde{g} \equiv \sqrt{2} \tilde{g} v$ appearing in the master equation (20) to $\tilde{g} = 1$ unless otherwise stated. Any solution $u(x; m_i)$ at unit effective gauge coupling can be rescaled to the arbitrary case as $u(\tilde{g} x; m_i / \tilde{g})$.

4.1. Ansatz

Let us consider the function

$$u_1^{(N)}(x) \equiv 2 \log F_N(x) \equiv 2 \log \left( 1 + \sum_{i=1}^{N} \prod_{j=1}^{i} e^{R_i - m_j x} \right).$$

That is, in particular,

$$u_1^{(1)}(x) = 2 \log \left( 1 + e^{R_1 - m_1 x} \right),$$

$$u_1^{(2)}(x) = 2 \log \left( 1 + e^{R_1 - m_1 x} + e^{R_1 + R_2 - (m_1 + m_2) x} \right),$$

$$u_1^{(3)}(x) = 2 \log \left( 1 + e^{R_1 - m_1 x} + e^{R_1 + R_2 - (m_1 + m_2) x} + e^{R_1 + R_2 + R_3 - (m_1 + m_2 + m_3) x} \right).$$

The reason why we choose the ansatz (93) in this way is that the vanishing of any of the $m_i$'s reduces the $N$th solution to the $N - 1$ case. Therefore, it is sufficient to study just strictly positive values of the $m_i$'s. Also, all powers of $e^{-x}$ are automatically ordered since $0 < m_1 < m_1 + m_2 < \cdots$. This allows us to easily estimate the positions of all (separable) walls, using similar arguments to those in Sect. 2: we look at points where a pair of neighboring exponentials equals

$$\tilde{x}_1 = \frac{R_1}{m_1}, \quad \tilde{x}_2 = \frac{R_2}{m_2}, \quad \ldots \quad \tilde{x}_N = \frac{R_N}{m_N}.$$  

Of course, these numbers represents actual locations of domain walls if they are ordered $\tilde{x}_1 \gg \tilde{x}_2 \gg \cdots \gg \tilde{x}_N$, in the same way as we discussed for positions of elementary walls in Eq. (24). Also notice that parameters $m_i$ entering Eq. (93) are not equal to the masses of the model, but they are related to them (see Eq. (108)).

The crucial detail in our ansatz is the factor 2 in front of the logarithm since it makes $\Omega_0$ automatically a finite sum of exponentials. Indeed, one can easily see that the general form is given as

$$\Omega_0^{(N)} = F_N^2 + 2(F_N')^2 - 2F_N'^2 F_N.$$  

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Let us expand the first few lowest cases

\[ \Omega_0^{(1)} = 1 + 2e^{R_1}e^{-m_1 x}(1 - m_1^2) + e^{2R_1}e^{-2m_1 x}, \]
\[ \Omega_0^{(2)} = 1 + 2e^{R_1}e^{-m_1 x}(1 - m_1^2) + e^{2R_1}e^{-2m_1 x} + 2e^{R_1+R_2}e^{-(m_1+m_2)x}(1 - (m_1 + m_2)^2) \]
\[ + 2e^{2R_1+R_2}e^{-(2m_1+m_2)}(1 - m_2^2) + e^{2R_1+2R_2}e^{-(2m_1+2m_2)x}, \]
\[ \Omega_0^{(3)} = 1 + 2e^{R_1}e^{-m_1 x}(1 - m_1^2) + e^{2R_1}e^{-2m_1 x} + 2e^{R_1+R_2}e^{-(m_1+m_2)x}(1 - (m_1 + m_2)^2) \]
\[ + 2e^{2R_1+R_2}e^{-(2m_1+m_2)}(1 - m_2^2) + 2e^{R_1+R_2+R_3}e^{-(m_1+m_2+m_3)x}(1 - (m_2 + m_3)^2) \]
\[ + 2e^{2R_1+2R_2+R_3}e^{-(2m_1+2m_2+m_3)x}(1 - m_3^2) + e^{2R_1+2R_2+2R_3}e^{-(m_1+m_2+m_3)x}. \]

We count 3, 6, and 10 independent flavors (terms) respectively, assuming generic \( m_i \)'s. The (unordered) mass matrix and the real part of the moduli matrix are given in each case as

\[ M^{(1)} = -\text{diag}(0, m_1, 2m_1)/2, \]
\[ M^{(2)} = -\text{diag}(0, m_1, 2m_1, m_1 + m_2, 2m_1 + m_2, 2m_1 + 2m_2)/2, \]
\[ M^{(3)} = -\text{diag}(0, m_1, 2m_1, m_1 + m_2, 2m_1 + m_2, m_1 + m_2 + m_3, \]
\[ 2m_1 + m_2 + m_3, 2m_1 + 2m_2 + m_3, 2m_1 + 2m_2 + 2m_3)/2, \]

and

\[ H_0^{(1)} = \left( 1, e^{R_1/2} \left( 2 - 2m_1^2 \right)^{1/2}, e^{R_1} \right), \]
\[ H_0^{(2)} = \left( 1, e^{R_1/2} \left( 2 - 2m_1^2 \right)^{1/2}, e^{R_1}, e^{(R_1+R_2)/2} \left( 2 - 2(m_1 + m_2)^2 \right)^{1/2}, e^{R_1+R_2/2} \left( 2 - 2m_2^2 \right)^{1/2}, \right. \]
\[ \left. e^{R_1+R_2} \right), \]
\[ H_0^{(3)} = \left( 1, e^{R_1/2} \left( 2 - 2m_1^2 \right)^{1/2}, e^{R_1}, e^{(R_1+R_2)/2} \left( 2 - 2(m_1 + m_2)^2 \right)^{1/2}, \right. \]
\[ e^{R_1+R_2/2} \left( 2 - 2m_2^2 \right)^{1/2}, e^{R_1+R_2}, e^{(R_1+R_2+R_3)/2} \left( 2 - 2(m_1 + m_2 + m_3)^2 \right)^{1/2}, \]
\[ e^{R_1+R_2+R_3/2} \left( 2 - 2(m_2 + m_3)^2 \right)^{1/2}, e^{R_1+R_2+R_3} \left( 2 - 2m_3^2 \right)^{1/2}, e^{R_1+R_2+R_3} \right). \]

Inspection of the above moduli matrices reveals that in order to have a well-defined solution, the parameters \( m_1, m_2, \) and \( m_3 \) must be constrained. In the \( N = 1 \) case, only \( m_1 \leq 1 \) is allowed due to the square root in the second entry. In the \( N = 2 \) case the same condition reads \( m_1 + m_2 \leq 1 \), while in the \( N = 3 \) case it is \( m_1 + m_2 + m_3 \leq 1 \). In other words, within the parameter \( m \)-space, the regions corresponding to sensible solutions are \( N \)-simplexes.
For the general case, the number of flavors is $N_F = \binom{N+2}{2}$. The mass matrix and the real part of the moduli matrix can be schematically written as (restoring generic $\tilde{g} \neq 1$)

$$M = - \text{diag}\left\{ \sum_{j=0}^{k_1} m_j + \sum_{j=0}^{k_2} m_j \right\} / 2,$$

(108)

$$H_0 = \left\{ \exp\left( \sum_{j=0}^{k_1} R_j + \sum_{j=k_1+1}^{k_2} R_j / 2 \right) \left( 2 - \left( \sum_{j=k_1+1}^{k_2} \frac{m_j}{g v^2} \right)^2 \right)^{1/2} \right\},$$

(109)

where $k_1, k_2$ are whole numbers in the range $k_1 \leq k_2 \in [0, N]$. The condition on the $m_i$’s that ensures validity of the solution is

$$\sum_{i=1}^{N} m_i \leq \sqrt{2} g v.$$

(110)

Notice that this condition restricts the total tension $T = v^2 \sum_i m_i$ to be at most $T \leq \sqrt{2} g v^3$. This confirms, at least for generic parameters, that no domain wall in Eq. (93) has a core (see the discussion in Sect. 2.4).

In summary, the solution (93) describes a configuration of $N$ domain walls with tensions $T_i = v^2 m_i$ at (naive) positions $x_i$ given in Eq. (97), provided that Eq. (110) is satisfied. The minimal model, where the solution lives, has $N_F = \binom{N+2}{2}$ flavors, and mass matrix $M$ and moduli matrix $H_0$ given in Eqs. (108)–(109), respectively.

Interestingly, we could view any $N > 1$ solution in Eq. (93) as a combination of $N - 1$ building blocks $u_1^{(1)}(x) \equiv u_{11}^{3F}(x)$. Indeed, we can rewrite Eqs. (95)–(96) as

$$u_1^{(2)}(x) = 2 \log \left( \exp(u_{11}^{3F}(x; m_1)/2) + \exp(u_{11}^{3F}(x; m_1 + m_2)/2) - 1 \right),$$

(111)

$$u_1^{(3)}(x) = 2 \log \left( \exp(u_{11}^{3F}(x; m_1)/2) + \exp(u_{11}^{3F}(x; m_1 + m_2 + m_3)/2) - 1 \right)$$

$$= 2 \log \left( \exp(u_{11}^{3F}(x; m_1)/2) + \exp(u_{11}^{3F}(x; m_1 + m_2)/2)$$

$$+ \exp(u_{11}^{3F}(x; m_1 + m_2 + m_3)/2) - 2 \right).$$

(112)

And similar relations apply for cases of higher $N$. We can understand this as a kind of “nonlinear” chain. However, if we rephrase everything in terms of $v(x) \equiv e^{u(x)/2} - 1$ everything becomes linear too. In particular, the master equation turns into

$$\frac{1}{g^2 v^2} \left( \partial_x^2 v + v \partial_x^2 v - (\partial_x v)^2 \right) = (v + 1)^2 - \Omega_0(x),$$

(113)

and defining $v_1^{(1)}(x) \equiv v_1(x) \equiv \exp(u_{11}^{3F}(x)/2) - 1$ we can see that solutions corresponding to the $N = 2$ and $N = 3$ cases are simply given as

$$v_1^{(2)}(x) = v_1(x; m_1) + v_1(x; m_1 + m_2),$$

(114)

$$v_1^{(3)}(x) = v_1(x; m_1) + v_1(x; m_1 + m_2) + v_1(x; m_1 + m_2 + m_3),$$

(115)

and analogously for higher $N$. 

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The ansatz (93) is not the only one we can consider. We may also take
\[ u_2^{(N)}(x) = 3 \log F_N(x), \tag{116} \]
where \( F_N \) is the same function as in Eq. (93). The corresponding source term is given as
\[ \Omega_0 = F_N^3 + 3(F_N')^2 F_N - 3F_N'' F_N. \tag{117} \]
The minimal model has \( N_F = \binom{N+3}{3} \) flavors. This solution also describes a configuration of \( N \) domain walls at (naive) positions \( x_i \) of Eq. (97). However, the tension of each wall is now \( T_i = 3v^2 m_i / 2 \) and the condition of validity is again
\[ \sum_{i=1}^{N} m_i \leq \sqrt{2} g v, \tag{118} \]
which places an upper bound on the total tension \( T \leq 3g v^3 / \sqrt{2} \). The solutions \( v_2^{(N)}(x) = \exp(u_2^{(N)}(x)) - 1 \) can also be written as chains of single-wall solutions \( v_2^{(1)}(x) = \exp(u_2^{(1)}(x)) - 1 \) in complete analogy to the previous results:
\[ v_2^{(2)}(x) = v_2(x; m_1) + v_2(x; m_1 + m_2), \tag{119} \]
\[ v_2^{(3)}(x) = v_2(x; m_1) + v_2(x; m_1 + m_2) + v_2(x; m_1 + m_2 + m_3). \tag{120} \]

The most generic configuration, however, can be achieved by a combination of both:
\[ u_2^{(N, \tilde{N})}(x) = u_1^{(N)}(x) + u_2^{(\tilde{N})}(x) = 2 \log F_N(x) + 3 \log \tilde{F}_\tilde{N}(x). \tag{121} \]
This solution lives in a model with \( N_F = \binom{N+2}{2} \binom{\tilde{N}+3}{3} \) flavors. It depicts a configuration of \( N \) domain walls with tensions \( T_i = v^2 m_i \) and \( \tilde{N} \) domain walls with tensions \( T_i = 3v^2 \tilde{m}_i / 2 \). The condition now reads
\[ \left( \sum_{i=1}^{N} m_i \right)^2 + \left( \sum_{j=1}^{\tilde{N}} \tilde{m}_j \right)^2 \leq 2 g^2 v^2. \tag{122} \]
Interestingly, the solutions \( v_2^{(N, \tilde{N})} = \exp(u_2^{(N, \tilde{N})}(x)) - 1 \) cannot be written as linear chains.

We believe that the class of functions (121) exhaust all exact solutions given as a logarithm of a sum of exponentials. This is the main result of this paper.

4.2. Exact multi-wall solutions up to \( N_F = 10 \) flavors

In this subsection we will investigate solutions \( u_1^{(N)} \) for \( N = 1, N = 2, \) and \( N = 3 \) in detail. We are not going to study any solutions in the families \( u_2^{(N)} \) or \( u_3^{(N)} \) for the sake of simplicity.

The main focus of our analysis is to find special relations between parameters \( m_i \) and \( R_i \) that reduce the given solution to another solution in the model with a lesser number of flavors. These reductions can be achieved in two complementary ways. First, we can make some entries of the moduli matrix (109) vanish. In \( m \)-space, such instances correspond to the (part of) the boundary of the \( N \)-simplex within which the solutions are valid. The second way is to make some entries of the mass matrix (108) equal. As we will see, these relations represent special “cuts” of the \( N \)-simplex.
Fig. 5. Allowed region of mass parameter \( m \) for the \( N = 1 \) case. The corresponding solution for each region is indicated.

If two such cuts intersect, further reduction occurs along this intersection, etc. The biggest reduction within this approach occurs when all \( m_i \)'s are equal. In such a case, the ansatz (93) is reduced to a degenerate 1 \( N \)-chain, described in the previous section.

The combination of both ways leads to an intricate web of exact solutions that we call the hierarchy. We can view each solution in the family (93) as a special parent solution, which gives many child solutions by applying these reductions. Furthermore, each child can have their own children. Each reduction effectively represents a loss of one freedom, either the freedom to control tensions of individual walls (given by parameters \( m_i \)) or their mutual separations (parameters \( R_i \)). Solutions that do not have any free parameters are irreducible and they are terminating points of the hierarchy.

Let us illustrate these general remarks on the concrete examples. Throughout this subsection, we will revive the practice, which we introduced below Eq. (41), that we index each new solution by the number of flavors in the superscript and by a roman numeral in the subscript (distinguishing solutions of the same flavor number) for ease of reference.

4.2.1. \( N = 1 \)

Let us repeat the solution and \( \Omega_0 \) for reference:

\[
\begin{align*}
\bar{u}^{1F}_i(x) & = 2 \log(1 + e^{-mx}), \quad \Omega_0 = 1 + 2(1 - m^2)e^{-mx} + e^{-2mx}.
\end{align*}
\]

(123)

For simplicity, we define \( m_1 \equiv m \) and set the moduli \( R_1 = 0 \). The mass matrix and (real part of) the moduli matrix reads

\[
\begin{align*}
M & = - \text{diag}(0, m, 2m)/2, \\
H_0 & = (1, (2 - 2m^2)^{1/2}, 1).
\end{align*}
\]

(124)

(125)

The allowed range of the parameter \( m \) is the 1-simplex, the interval \([0, 1]\). The (naive) positions of elementary walls are

\[
\begin{align*}
x_1 & = \frac{1}{m} \log(2 - 2m^2), \\
x_2 & = -\frac{1}{m} \log(2 - 2m^2).
\end{align*}
\]

(126)

(127)

Within the 1-simplex \( x_1 \leq x_2 \), so elementary walls are compressed into a single wall located at the origin (their center of mass). The only reduction of the number of flavors can be achieved by setting \( m = 1 \), which makes the second entry in the moduli matrix zero. From the above formulas we can see that in this limit the compression becomes infinite (\( x_1 \to -\infty, x_2 \to \infty \)) and the solution \( \bar{u}^{1F}_i(m = 1) = \bar{u}^{2F}_i \) can be identified with the previously known 2F solution. We visualize these findings diagrammatically in Fig. 5.
4.2.2. $N = 2$

The solution and $\Omega_0$ are written

$$u_1^{(2)} = u_1^{FE}(x) = 2 \log(1 + e^R e^{-m_1 x} + e^{-(m_1 + m_2) x}),$$

$$\Omega_0 = 1 + 2 e^R (1 - m_1^2) e^{-m_1 x} + e^{2R} e^{-(m_1 + m_2) x} (1 - m_2^2) + 2 e^{-(m_1 + m_2) x} (1 - (m_1 + m_2)^2) + e^{-2(m_1 + m_2) x}.$$

We have set $R_2 = -R_1$ to put the center of mass at the origin and relabel $R_1 = R$ for simplicity.

First, let us verify that the solution $u_1^{FE}(x)$ describes a configuration of two nonelementary walls located at

$$\tilde{x}_1 = \frac{R}{m_1}, \quad \tilde{x}_2 = -\frac{R}{m_2}.$$  \hfill (130)

Contrary to the previous case, the mass matrix cannot be ordered in the same way for all values of the parameters $m_1$ and $m_2$. In fact, there are two possible orderings:

$$M = - \text{diag}(0, m_1, 2m_1, m_1 + m_2, 2m_1 + m_2, 2m_1 + 2m_2)/2, \quad m_2 > m_1,$$

$$M = - \text{diag}(0, m_1, m_1 + m_2, 2m_1, 2m_1 + m_2, 2m_1 + 2m_2)/2, \quad m_2 \leq m_1.$$  \hfill (131, 132)

The positions of elementary walls in the first case read

$$x_1 = \frac{1}{m_1} \left[R + \log(2 - 2m_1^2)/2\right],$$

$$x_2 = \frac{1}{m_1} \left[R - \log(2 - 2m_1^2)/2\right],$$

$$x_3 = \frac{1}{m_2 - m_1} \left[-2R + \log(2 - 2(m_1 + m_2)^2)/2\right],$$

$$x_4 = \frac{1}{m_1} \left[R + \log\left(\frac{1 - m_2^2}{1 - (m_1 + m_2)^2}\right)/2\right],$$

$$x_5 = \frac{1}{m_2} \left[-R - \log(2 - 2m_2^2)/2\right].$$  \hfill (133, 134, 135, 136, 137)

The allowed range of the parameters $m_1$ and $m_2$ is a 2-simplex (a rectangular triangle) defined by the relation $m_1 + m_2 \leq 1$. This means that $m_1 \leq 1$ and hence $x_1 \leq x_2$. Thus, the first two elementary walls form a compressed wall located at their center of mass $\tilde{x}_1 = R/m_1$ as we claimed. Assuming that $R > 0$ we see that $x_4 > 0$ and $x_3 < 0$, which means that the third and fourth walls are compressed with each other. Their center of mass is located at $-R/m_2 + \log(2 - 2m_2^2)/2$, which is always smaller than $x_5$, indicating that the fifth wall is compressed as well. Their total center of mass lies at $\tilde{x}_2 = -R/m_2$, in confirmation of Eq. (130).
Fig. 6. Allowed region of mass parameters $m_1$ and $m_2$ for the $N = 2$ case. On the left, the edges and vertices are specified, while on the right we show which solution each segment represents. The points are formatted as $[m_1, m_2]$.

On the other hand, in the second case $m_2 \leq m_1$, the positions of elementary walls are

\[
x_1 = \frac{1}{m_1} \left[ R + \log \left( 2 - 2m_1^2 \right)/2 \right],
\]

\[
x_2 = \frac{1}{m_2} \left[ -R + \log \left( \frac{1 - (m_1 + m_2)^2}{1 - m_1^2} \right)/2 \right],
\]

\[
x_3 = \frac{1}{m_1 - m_2} \left[ 2R - \log \left( 2 - 2(m_1 + m_2)^2 \right)/2 \right],
\]

\[
x_4 = \frac{1}{m_2} \left[ -R + \log \left( 2 - 2m_2^2 \right)/2 \right],
\]

\[
x_5 = \frac{1}{m_2} \left[ - R - \log \left( 2 - 2m_2^2 \right)/2 \right].
\]

The analysis is completely analogous but now the roles are reversed. The last two walls are compressed with each other (since $x_4 < x_5$) and the first three as well (since for $R > 0$ we have $x_3 > 0$, $x_2 < 0$, and $(m_1 + m_2)x_3 + m_2x_2 > m_1x_1$). The center of mass of the triplet is located at $\tilde{x}_1 = R/m_1$, while that of the doublet is at $\tilde{x}_2 = -R/m_2$, again in full accordance with Eq. (130).

It is clear that for higher $N$ the positions of the elementary wall will be increasingly laborious to ascertain due to the ever higher number of possible orderings of the elements in the mass matrix $M$. The previous analysis, however, confirmed that we can be confident of locations of compressed walls given in Eq. (97), which we extracted directly from the solution (93).

In Fig. 6 we show two versions of the 2-simplex. On the left, we show relations among the $m_i$'s that define its boundary and on the right we put a label on each face, edge, and vertex specifying which solution a given region represents. This figure summarizes where reductions of the number of flavors occur. As discussed in the previous subsection, sources of reduction are twofold. The first is where some entries of the moduli matrix $H_0$ vanish. There are only three possibilities, which defines the boundary of the triangle. The edges $m_1 = 0$ and $m_2 = 0$ return us to the $N = 1$ case, therefore we already know the answer there. The last edge of the triangle $m_2 = 1 - m_1$ represents
The solution and escape the 2-simplex and still give a well-defined solution for arbitrary separation of walls. But in other words, we set

\[ N = 3 \]

where we have relabeled the parameters \( m_1 = m \) and \( m_2 = 1 - m \) for simplicity. This solution can be reduced further by setting \( m = 1 \) or \( m = 0 \), which leads to the \( u_1^{2F}(x) \) solution.

The second way to reduce the number of flavors is to make some elements of the mass matrix \( M \) identical. In the present case, there is only one interesting possibility: \( m_1 = m_2 \equiv m \). This is the 2-chain that we encountered in Eq. (55):

\[
\begin{align*}
  u_1^{5F}(x) &= 2 \log (1 + e^R e^{-mx} + e^{-x}), \\
  \Omega_0 &= 1 + 2e^R e^{-mx}(1 - m^2) + e^{2R} e^{-2mx} + 2e^R e^{-(1 + m)x} m(2 - m) + e^{-2x},
\end{align*}
\]

where \( S = \text{arccosh}(R/2)/m \). The curious property of this solution is that the parameter \( m \) can escape the 2-simplex and still give a well-defined solution for arbitrary separation of walls. But when \( m \geq 1/2 \) it is preferable to redefine parameter \( R \) as

\[
e^R \to (e^{2R} + 8m^2 - 2)^{1/2},
\]

so that every coefficient in \( \Omega_0 \) is manifestly positive. We can reduce the number of flavors further by setting \( m = 1 \), which produces a known 3F solution \( u_1^{3F}(m = 1) = u_1^{3F} \).

There is, however, still one option left. We can also reduce the number of flavors of \( u_1^{4F}(x) \) by fixing the positions of the wall to specific values, which will make the third factor in Eq. (146) disappear. In other words, we set \( R = \log(8m^2 - 2)/2 \). This can be done only for \( m \in [1/2, 1] \) and the resulting solution is

\[
\begin{align*}
  u_1^{4F}(x) &= 2 \log (1 + (8m^2 - 2)^{1/2} e^{-mx} + e^{-2mx}), \\
  \Omega_0 &= 1 + 2e^{-mx}(8m^2 - 2)^{1/2} (1 - m^2) + 2e^{-3mx}(8m^2 - 2)^{1/2} (1 - m^2) + e^{-4mx}.
\end{align*}
\]

In Fig. 6 we have indicated this special solution by a dashed line. Lastly, by setting \( m = 1 \) in \( u_1^{4F}(x) \) we obtain \( u_1^{4F}(m = 1) = u_1^{2F} \). With this observation, we have exhausted all possible reductions in the \( N = 2 \) case. We summarize the structure of reductions in Fig. 7.

### 4.2.3. \( N = 3 \)

The solution and \( \Omega_0 \) read

\[
\begin{align*}
  u_1^{(3)} &= u_1^{10F}(x) = 2 \log (1 + e^{R_1} e^{-m_1 x} + e^{R_1 + R_2} e^{-(m_1 + m_2)x} + e^{-(m_1 + m_2 + m_3)x}), \\
  \Omega_0 &= 1 + 2e^{R_1} e^{-m_1 x}(1 - m_1^2) + 2e^{R_1 + R_2} e^{-(m_1 + m_2)x}(1 - (m_1 + m_2)^2)
  + 2e^{-(m_1 + m_2 + m_3)x}(1 - (m_1 + m_2 + m_3)^2) + 2e^{2R_1 + R_2} e^{-(2m_1 + m_2)x}(1 - m_2^2)
  + 2e^{R_1} e^{-(2m_1 + m_2 + m_3)x}(1 - (m_2 + m_3)^2) + 2e^{R_1 + R_2} e^{-(2m_1 + 2m_2 + m_3)x}(1 - m_3^2)
  + e^{2R_1 + 2R_2} e^{-(2m_1 + 2m_2 + m_3)x} + e^{-(m_1 + m_2 + m_3)x} + e^{2R_1} e^{-2m_1 x}.
\end{align*}
\]

We have set \( R_3 = -R_1 - R_2 \) to put the center of mass at the origin. The solution \( u_1^{10F}(x) \) describes three domain walls located at

\[
\tilde{x}_1 = \frac{R_1}{m_1}, \quad \tilde{x}_2 = \frac{R_2}{m_2}, \quad \tilde{x}_3 = -\frac{R_1 + R_2}{m_3}.
\]

\[ 28/38 \]
Fig. 7. A schematic representation of reductions of the $u_1^6F(x)$ solution. The labels follow the structure of Fig. 6. The notation $R^*$ indicates that a special choice of positions of domain walls is taken.

Fig. 8. The allowed parameter region for the $N = 3$ case is a rectangular tetrahedron. In the left panel, we see the generic solutions attached to each face. Faces where one of the parameters vanishes correspond to the $N = 2$ case, while the top face is a new 9F solution. In the right panel, we “remove” the top face to reveal the inner structure of the tetrahedron, which is cut by five planes. These cuts represent mass matrix degeneracies of the general solution. The points shown on the pictures are formatted as $[m_1,m_2,m_3]$.

The allowed region in $m$-space is a rectangular tetrahedron $m_1 + m_2 + m_3 \leq 1$ (see Fig. 8). The boundary of this 3-simplex is where some entries of the moduli matrix vanish. The faces that are attached to the origin—rectangular triangles $m_1 = 0$, $m_2 = 0$, and $m_3 = 0$—are where we return to the $N = 2$ case.
We can further lower the number of flavors of this solution by investigating linear relations among parameters $m_1$ and $m_2$. The full structure of reductions is displayed in Fig. 9.

The relations

$$m_1 = m_2, \quad m_1 = 1 - 2m_2,$$  

give us new 8F solutions:

$$u_1^{8F}(x) = 2 \log(1 + e^{R_1} e^{-mx} + e^{R_1+R_2} e^{-(m_1+m_2)x} + e^{-x}),$$  

$$\Omega_0 = 1 + 2e^{R_1} e^{-m_1x} (1 - m_1^2) + 2e^{R_1+R_2} e^{-(m_1+m_2)x} (1 - m_1 + m_2^2) + 2e^{R_1+R_2} e^{-(m_1+m_2)x} (1 - m_2^2) + 2e^{R_1} e^{-(1+m_1)x} m_1 (2 - m_1) + e^{-2x} + 2e^{R_1+R_2} e^{-(m_1+m_2)x} (1 - m_1 + m_2^2) + e^{2R_1} e^{-2m_1x} + e^{-2x},$$  

We can further lower the number of flavors of this solution by investigating linear relations among parameters $m_1$ and $m_2$. The full structure of reductions is displayed in Fig. 9.

The relations

$$m_1 = m_2, \quad m_1 = 1 - 2m_2,$$  

give us new 8F solutions:

$$u_1^{8F}(x) = 2 \log(1 + e^{R_1} e^{-mx} + e^{R_1+R_2} e^{-(m_1+m_2)x} + e^{-x}),$$  

$$\Omega_0 = 1 + 2e^{-mx} e^{R_1} (1 - m_1^2) + e^{-2mx} e^{R_1} \left(e^{R_1} + 2e^{R_2} (1 - 4m^2)\right) + 2e^{-3mx} e^{2R_1+R_2} (1 - m^2) + 2e^{-(1+m)x} e^{R_1} m (2 - m) + e^{-4mx} e^{2R_1+2R_2} + 8e^{-(1+2m)x} e^{R_1+R_2} m (1 - m) + e^{-2x},$$  

$$u_\Omega^{8F}(x) = 2 \log(1 + e^{R_1} e^{-(1-2m)x} + e^{R_1+R_2} e^{-(1-m)x} + e^{-x}),$$  

$$\Omega_0 = 1 + 8e^{-(1-2m)x} e^{R_1} m (1 - m) + 2e^{-(1-m)x} e^{R_1+R_2} m (2 - m) + 2e^{-(2-3m)x} e^{R_1+R_2} (1 - m^2) + 2e^{-(2-m)x} e^{R_1+R_2} (1 - m^2) + e^{-2(1-m)x} e^{R_1+R_2} + e^{-2x},$$
The above relations define planes that cut the rectangular tetrahedron. Along these cuts, the solution given in Eq. (186). However, solutions \( u_{III}^{SF}(x) \) and \( u_{IV}^{SF}(x) \) are obviously not a part of the top face, since for both \( m_3 \neq 1 - m_1 - m_2 \). In fact, they are situated on planes \( m_1 = m_2 + m_3 \) and \( m_3 = m_1 + m_2 \) respectively, which are inside the tetrahedron and correspond to reductions of mass matrix elements. We will discuss such planes shortly. Both solutions \( u_{III}^{SF}(x) \) and \( u_{IV}^{SF}(x) \) can be reduced by one flavor by taking \( R_1 = \log(8m^2 - 2)/2 \) and \( R_2 = -R_1 + (8m^2 - 2)/2 \), respectively. These new 7F solutions,

\[
\begin{align*}
    u_{I}^{7F}(x) &= 2 \log\left(1 + (8m^2 - 2)^{1/2} e^{-mx} + R_2 (8m^2 - 2)^{1/2} e^{-x} + e^{-2mx}\right), \\
    u_{II}^{7F}(x) &= 2 \log\left(1 + e^{R_1} e^{-(2m-1)x} + (8m^2 - 2)^{1/2} e^{-mx} + e^{-2mx}\right),
\end{align*}
\]

cannot be reduced further.

At the point on the top face \( m_1 = m_2 = 1/3 \), where the two 8F solutions \( u_{III}^{SF} \) and \( u_{IV}^{SF} \) meet, the number of flavors is reduced by one. This is not, however, a new solution; rather it is a special point of the \( u_{I}^{7F}(x) \) solution given in Eq. (186).

With this observation, we have finished the analysis of the upper face of the tetrahedron and we can now move to explore relations between mass matrix elements. These are

\[
\begin{align*}
    m_1 &= m_2, & m_1 &= m_2 + m_3, & m_1 &= m_3, & m_3 &= m_1 + m_2, & m_2 &= m_3.
\end{align*}
\]

The above relations define planes that cut the rectangular tetrahedron. Along these cuts, the solution is reduced by one flavor. We display these cuts in Fig. 8. The corresponding solutions are (in order)

\[
\begin{align*}
    u_{II}^{9F}(x; m_1, m_3) &= 2 \log\left(1 + e^{R_1} e^{-m_1 x} + e^{R_1 + R_2 e^{-m_1 x}} + e^{-(2m_1 + m_3) x}\right), \\
    u_{III}^{9F}(x; m_2, m_3) &= 2 \log\left(1 + e^{R_1} e^{-(m_2 + m_3) x} + e^{R_1 + R_2 e^{-(m_2 + m_3) x}} + e^{-(2m_2 + m_3) x}\right), \\
    u_{IV}^{9F}(x; m_1, m_2) &= 2 \log\left(1 + e^{R_1} e^{-m_1 x} + e^{R_1 + R_2 e^{-(m_1 + m_2) x}} + e^{-(2m_1 + m_2) x}\right), \\
    u_{V}^{9F}(x; m_1, m_2) &= 2 \log\left(1 + e^{R_1} e^{-m_1 x} + e^{R_1 + R_2 e^{-(m_1 + m_2) x}} + e^{-(m_1 + m_2) x}\right), \\
    u_{V_1}^{9F}(x; m_1, m_2) &= 2 \log\left(1 + e^{R_1} e^{-m_1 x} + e^{R_1 + R_2 e^{-(m_1 + m_2) x}} + e^{-(m_1 + 2m_2) x}\right).
\end{align*}
\]
Not all of these solutions are independent. Solutions $u^{\text{9F}}_{\text{II}}(x)$, $u^{\text{9F}}_{\text{III}}(x)$, and $u^{\text{9F}}_{\text{V}}(x)$ are related by redefinition of parameters

$$u^{\text{9F}}_{\text{III}}(x; m_1 + m_3, -m_3) \sim u^{\text{9F}}_{\text{V}}(x; 2m_1 + m_3, -m_1 - m_3) \sim u^{\text{9F}}_{\text{II}}(x; m_1, m_3),$$

(173)

where $\sim$ means up to inconsequential rebalancing involving shifts and redefinitions of parameters $R_1$ and $R_2$. Furthermore, solution $u^{\text{9F}}_{\text{VI}}(x)$ is related to $u^{\text{9F}}_{\text{II}}(x)$ by another reflection identity,

$$u^{\text{9F}}_{\text{VI}}(-x; m_3, m_1) - 2(m_3 + 2m_1)x \sim u^{\text{9F}}_{\text{II}}(x; m_1, m_2).$$

(174)

Notice that both $u^{\text{9F}}_{\text{I}}(x)$ and $u^{\text{9F}}_{\text{IV}}(x)$ are self-dual under reflection:

$$u^{\text{9F}}_{\text{I}}(-x; 1 - m_1, -m_2) - 2x = u^{\text{9F}}_{\text{I}}(x; m_1, m_2),$$

(175)

$$u^{\text{9F}}_{\text{IV}}(-x; m_1, m_2) - 2(2m_1 + m_2)x \sim u^{\text{9F}}_{\text{IV}}(x; m_1, m_2).$$

(176)

Given these identities we need to closely examine only, say, $u^{\text{9F}}_{\text{II}}(x)$, $u^{\text{9F}}_{\text{VI}}(x)$, and $u^{\text{9F}}_{\text{IV}}(x)$.

In Fig. 10 we display the structure of solutions $u^{\text{9F}}_{\text{II}}(x)$ and $u^{\text{9F}}_{\text{VI}}(x)$. There are two new 8F solutions:

$$u^{\text{8F}}_{\text{V}}(x) = 2 \log \left( 1 + e^{R_1} e^{-mx} + e^{R_1+R_2} e^{-2mx} + e^{-4mx} \right),$$

(177)

$$u^{\text{8F}}_{\text{VI}}(x) = 2 \log \left( 1 + e^{R_1} e^{-2mx} + e^{R_1+R_2} e^{-3mx} + e^{-4mx} \right),$$

(178)

where in the first one $m_1 = m_2 = m_3/2 \equiv m$ and in the second one $m_1/2 = m_2 = m_3 \equiv m$. They are related to each other via the reflection identity

$$u^{\text{8F}}_{\text{V}}(-x) - 8mx \sim u^{\text{8F}}_{\text{VI}}(x),$$

(179)

which is inherited from the more general identity in Eq. (174). Both of these solutions can be reduced by one flavor, either by taking $m = 1/3$ or assuming special positions of domain walls, namely $R_1 = -R_2 + \log(32m^2 - 2)/2$ for $u^{\text{8F}}_{\text{V}}(x)$ and $R_1 = \log(32m^2 - 2)/2$ in the case of $u^{\text{8F}}_{\text{V}}(x)$. 

![Fig. 10. Cuts of the tetrahedron defined by the relations $m_1 = m_2$ and $m_2 = m_3$.](https://academic.oup.com/ptep/article-abstract/2017/1/013B01/2932490)
Both approaches gives us new 7F solutions:

\[
\begin{align*}
    u_{1V}^{7F}(x) &= 2 \log(1 + e^{-R_2(32m^2 - 2)^{1/2} e^{-mx} + (32m^2 - 2)^{1/2} e^{-2mx} + e^{-4mx}}), \\
    u_{1Y}^{7F}(x) &= 2 \log(1 + e^{R_1 - x/3} + e^{R_1 + R_2 e^{-2x/3} + e^{-4x/3}}), \\
    u_{1II}^{7F}(x) &= 2 \log(1 + e^{R_1(32m^2 - 2)^{1/2} e^{-2mx} + e^{-R_1(32m^2 - 2)^{1/2} e^{-3mx} + e^{-4mx}}}, \\
    u_{1III}^{7F}(x) &= 2 \log(1 + e^{R_1 e^{-2x/3} + e^{R_1 + R_2 e^{-x} + e^{-4x/3}}}).
\end{align*}
\]

Furthermore, taking both \( m = 1/3 \) and special positions of domain walls yields new 6F solutions:

\[
\begin{align*}
    u_{1I}^{6F}(x) &= 2 \log(1 + e^{-R_2 \sqrt{14/9} e^{-x/3} + \sqrt{14/9} e^{-2x/3} + e^{-4x/3}}), \\
    u_{1II}^{6F}(x) &= 2 \log(1 + e^{R_1 \sqrt{14/9} e^{-2x/3} + e^{-R_1 \sqrt{14/9} e^{-x} + e^{-x/3}}},
\end{align*}
\]

which cannot be reduced further.

As we can see in Fig. 8, the planes \( m_1 = m_2 \) and \( m_2 = m_3 \) intersect at a line \( m_1 = m_2 = m_3 \), which gives a 7F solution:

\[
\begin{align*}
    u_{1V}^{7F}(x) &= 2 \log(1 + e^{R_1 e^{-mx} + e^{R_1 + R_2 e^{-2mx} + e^{-3mx}}}, \\
    \Omega_0 &= 1 + 2e^{R_1 e^{-mx}}(1 - m^2) + e^{-2mx}(e^{2R_1} + 2e^{R_1 + R_2}(1 - 4m^2)) \\
    &\quad + 2e^{-3mx}(e^{2R_1 + R_2} - 1 - 9m^2) + e^{-4mx}(e^{2R_1 + 2R_2} + 2e^{R_1}(1 - 4m^2)) \\
    &\quad + 2e^{R_1 + R_2} e^{-5mx}(1 - m^2) + e^{-6mx}.
\end{align*}
\]

This solution is a degenerate 3-chain

\[
\begin{align*}
    u_{1V}^{7F}(x) &= u_{1II}^{3F}(x - S_1) + u_{1II}^{3F}(x - S_2) + u_{1II}^{3F}(x + S_1 + S_2),
\end{align*}
\]

where

\[
e^{R_1} = e^{mS_1} + e^{mS_2} + e^{-m(S_1 + S_2)}, \quad e^{R_1 + R_2} = e^{-mS_1} + e^{-mS_2} + e^{m(S_1 + S_2)}.
\]

We cannot reduce this solution further unless we fix the position parameters \( R_1, R_2 \). This can be done in several ways. If \( m > 1/3 \) then we can fix

\[
R_2 = -2R_1 + \log \frac{9m^2 - 1}{1 - m^2},
\]

to get a 6F solution

\[
\begin{align*}
    u_{1V}^{6F}(x) &= 2 \log(1 + e^{R_1 e^{-mx} + \frac{9m^2 - 1}{1 - m^2} e^{-R_1 e^{-2mx} + e^{-3mx}}}).
\end{align*}
\]

Furthermore, if \( m > 1/2 \) we can fix \( R_1 \) in two ways:

\[
\begin{align*}
    R_1 &= R_2 + \log(8m^2 - 2), \\
    R_1 &= -2R_2 + \log(8m^2 - 2).
\end{align*}
\]

Each choice reduces the solution by one flavor:

\[
\begin{align*}
    u_{1V}^{6F}(x) &= 2 \log(1 + e^{R_2 e^{-mx}(8m^2 - 2) + e^{2R_2 e^{-2mx}(8m^2 - 2) + e^{-3mx})}, \\
    u_{1V1}^{6F}(x) &= 2 \log(1 + e^{-2R_2 e^{-mx}(8m^2 - 2) + e^{-R_2 e^{-2mx}(8m^2 - 2) + e^{-3mx})}. \\
\end{align*}
\]
We can combine the above relations and fix both $R_1$ and $R_2$ in three independent ways:

\[
R_1 = \frac{1}{3} \log \frac{(9m^2 - 1)(8m^2 - 2)}{1 - m^2}, \quad R_2 = \frac{1}{3} \log \frac{9m^2 - 1}{(1 - m^2)(8m^2 - 2)^2}, \tag{196}
\]

\[
R_1 = \frac{1}{3} \log \frac{(9m^2 - 1)^2}{(1 - m^2)^2(8m^2 - 2)^2}, \quad R_2 = \frac{1}{3} \log \frac{(1 - m^2)(8m^2 - 2)^2}{9m^2 - 1}. \tag{197}
\]

\[
R_1 = \log(8m^2 - 2), \quad R_2 = 0. \tag{198}
\]

with corresponding 5F solutions

\[
u_{iii}^{5F}(x) = 2 \log \left( 1 + \left( \frac{(9m^2 - 1)(8m^2 - 2)}{1 - m^2} \right)^{1/3} e^{-mx} + \left( \frac{(9m^2 - 1)2}{(1 - m^2)(8m^2 - 2)} \right)^{1/3} e^{-2mx} + e^{-3mx} \right), \tag{199}
\]

\[
u_{iv}^{5F}(x) = 2 \log \left( 1 + \left( \frac{(9m^2 - 1)^2}{(1 - m^2)^2(8m^2 - 2)^2} \right)^{1/3} e^{-mx} + \left( \frac{(9m^2 - 1)(8m^2 - 2)}{1 - m^2} \right)^{1/3} e^{-2mx} + e^{-3mx} \right), \tag{200}
\]

\[
u_{i}^{5F}(x) = 2 \log \left( 1 + (8m^2 - 2)e^{-mx} + (8m^2 - 2)e^{-2mx} + e^{-3mx} \right). \tag{201}
\]

And finally, if we choose special values for $m$ for which all three relations are satisfied, we obtain the most flavor-reduced solutions. These special values are the roots of the equation $(8m^2 - 1)^2(1 - m^2) = 9m^2 - 1$. It turns out that there are two for which $m \in [1/2, 1]$. Let us denote these roots by $\tilde{m}$ and $\bar{m}$. Their approximate values are $\tilde{m} \approx 0.883$ and $\bar{m} \approx 0.597$. If we choose the first root we obtain $R_1 = \log(8\tilde{m}^2 - 2) \approx 1.444$, $R_2 = 0$ and the corresponding solution is given as

\[
u_{iii}^{5F}(x) = 2 \log \left( 1 + (8\tilde{m}^2 - 2)e^{-\tilde{m}x} + (8\tilde{m}^2 - 2)e^{-2\tilde{m}x} + e^{-3\tilde{m}x} \right), \tag{202}
\]

\[
\Omega_0 = 1 + 2(8\tilde{m}^2 - 2)(1 - \tilde{m}^2)e^{-\tilde{m}x}(1 + e^{-4\tilde{m}x}) + e^{-6\tilde{m}x}. \tag{203}
\]

For the other root we have $R_1 = \log(8\bar{m}^2 - 2) \approx 0.616$ and $R_2 = 0$. The corresponding solution

$\nu_{iv}^{5F}(x)$ is functionally the same as $\nu_{iv}^{5F}(x)$ with $\tilde{m}$ replaced by $\bar{m}$.

Lastly, let us comment on the reduction structure of the solution $\nu_{iv}^{5F}(x)$, which is shown in Fig. 11. The solution and corresponding $\Omega_0$ read

\[
u_{iv}^{5F}(x) = 2 \log \left( 1 + e^{R_1e^{-m_1x}} + e^{R_1+R_2e^{-(m_1+m_2)x}} + e^{-(2m_1+m_2)x} \right), \tag{204}
\]

\[
\Omega_0 = 1 + 2e^{R_1e^{-(m_1+m_2)x}}(1 - m_1^2) + 2e^{R_1+R_2e^{-(m_1+m_2)x}}(1 - (m_1 + m_2)^2)
\]

\[
+ 2e^{-(2m_1+m_2)x}\left[ e^{2R_1+R_2(1 - m_2^2)} + 1 - (2m_1 + m_2)^2 \right] + e^{2R_1+2R_2e^{-2(m_1+m_2)x}}
\]

\[
+ 2e^{R_1e^{-(3m_1+m_2)x}}(1 - (m_1 + m_2)^2) + 2e^{R_1+R_2e^{-(3m_1+2m_2)x}}(1 - m_1^2) + e^{-2(2m_1+m_2)x}. \tag{205}
\]

The values of parameters that produce meaningful solutions are constrained as $m_1 + m_2 \leq 1$, but for $m_2 \geq 1 - 2m_1$ we see that $R_2 \geq R^* = -2R_1 + \log((2m_1 + m_2)^2 - 1)/(1 - m_2^2)$ must hold so that the fourth term in $\Omega_0$ is nonnegative. We indicate that by a dashed line in Fig. 11. If $R_2 = R^*$ we
We can reduce this solution further by setting \( R_2 = -2R_1 + \log(2 + m)/(2 - m) \) to obtain a new 6F solution

\[
u_{\text{VII}}^6(x) = 2 \log \left(1 + e^{R_1} e^{-m_1 x} + e^{-R_1} e^{-(m_1 + m_2)x} \frac{(2m_1 + m_2)^2 - 1}{1 - m_2^2} + e^{-(2m_1 + m_2)x}\right),
\]

\[
\Omega_0 = 1 + 2e^{R_1} e^{-m_1 x}(1 - m_1^2) + 2e^{-R_1} e^{-(m_1 + m_2)x}(1 + (m_1 + m_2)^2) \frac{(2m_1 + m_2)^2 - 1}{1 - m_2^2}
+ e^{-2R_1} e^{-(2m_1 + m_2)x} \left(\frac{(2m_1 + m_2)^2 - 1}{1 - m_2^2}\right) + 2e^{R_1} e^{-(3m_1 + m_2)x}(1 + (m_1 + m_2)^2)
+ 2e^{-R_1} e^{-(3m_1 + m_2)x}(1 - m_1^2) \frac{(2m_1 + m_2)^2 - 1}{1 - m_2^2} + e^{-2(2m_1 + m_2)x} + e^{2R_1} e^{-2m_1 x}.
\]

The upper edge of the triangle in Fig. 11 defined by relations \( m_2 = 1 - m, m_1 = m \) gives a new 7F solution

\[
u_{\text{VIII}}^7(x) = 2 \log \left(1 + e^{R_1} e^{-m_1 x} + e^{R_1 + R_2} e^{-x} + e^{-(1+m)x}\right),
\]

\[
\Omega_0 = 1 + 2e^{R_1} e^{-m_1 x}(1 - m_1^2) + 2me^{-(1+m)x}(e^{2R_1 + R_2} (2 - m) - 2 - m)
+ e^{2R_1} e^{-2mx} + e^{2R_1 + 2R_2} e^{-2x} + 2e^{R_1 + R_2} e^{-(2+m)x}(1 - m^2) + e^{-2(1+m)x}.
\]

We can reduce this solution further by setting \( R_2 = -2R_1 + \log(2 + m)/(2 - m) \) to obtain a new 6F solution

\[
u_{\text{VII}}^6(x) = 2 \log \left(1 + e^{R_1} e^{-mx} + e^{-R_1} e^{-x} \frac{2+m}{2-m} + e^{-(1+m)x}\right),
\]

\[
\Omega_0 = 1 + 2e^{R_1} e^{-mx}(1 - m^2)
+ e^{2R_1} e^{-2mx} + e^{-2R_1} e^{-2x} \frac{(2+m)^2}{(2-m)^2} + 2e^{-R_1} e^{-(2+m)x}(1 + m^2) \frac{2+m}{2-m} + e^{-2(1+m)x},
\]

which cannot be reduced further. With this observation, we have exhausted all possible reductions in the \( N = 3 \) case. We summarize the structure of reductions in Fig. 12.
exhausts all exact solutions of the master equation for domain walls (13) that can be written as a

\[ \text{mass parameters described in the text.} \]

\( \star \) indicates a special choice of positions of domain walls is taken and \( m^\ast \) indicates a special choice of

5. Conclusions

In this paper, we have shown many new exact solutions of the master equation (13). Our main claim

\[ 2 \log \left( 1 + \sum_{i=1}^{N} \prod_{j=1}^{i} e^{R_{i} - m_{i} x} \right) + 3 \log \left( 1 + \sum_{i=1}^{N} \prod_{j=1}^{i} e^{R_{i} - \tilde{m}_{i} x} \right) \]  (212)

exhausts all exact solutions of the master equation for domain walls (13) that can be written as a logarithm of the sum of exponentials. Of course, since we cannot prove this statement, we regard it as a conjecture.

Fig. 12. A schematic representation of reductions of the \( \alpha_{10}^0(x) \) solution. Only the most important reductions are shown and the \( N = 2 \) solutions are not included. The labels follow the structure of Fig. 10. The notation \( R^* \) indicates that a special choice of positions of domain walls is taken and \( m^* \) indicates a special choice of mass parameters described in the text.
In this paper we have studied two subgroups of the class (212) in detail. In Sect. 3 we developed the concept of chains, originally spotted in Ref. [11]. These are the solutions, which can be written as a sum of single-wall solutions, or links, which we denoted by $u^{3F}_{II}(x)$ and $u^{4F}_{I}(x)$, and their definitions can be found in Eqs. (46) and (49). We discussed general properties of arbitrarily long chains and derived general formulas for both the number of flavors of the minimal model in Eq. (77) and the restriction on parameters in Eq. (81), which must be satisfied in order to have a well-defined moduli matrix.

In Sect. 4 we gave a general discussion of solutions, which corresponds to taking the $\tilde{N} = 0$ limit in Eq. (212). In particular, we showed that, in terms of new variables $v(x) = e^{u(x)} - 1$, these solutions can be understood as chains as well. More importantly, we pointed out that solutions are related to each other via limits in their parameter space and thus form hierarchies. We showcased this hierarchy in detail for solutions up to $N_F = 10$ flavors.

Throughout this paper, we have argued that our exact solutions have two common features. First, they are all core-less and second, they are always nonelementary.

Let us briefly discuss the first property. The lack of cores is intimately linked to the restriction on parameters in Eq. (122). Indeed, this condition is in place to ensure that the lowest coefficient in $\Omega_0$ is never negative. At most, it can be zero, in which case the remaining coefficients are positive. On the other hand, the core can develop only when all coefficients in $\Omega_0$ are very close to zero, since if $\Omega_0 = 0$ the solution $u = g^2 v^2 x^2$ is the unbroken phase inside the core. Given this observation, it is no surprise that our solutions cannot develop cores. Another, more heuristic reason why our solutions are always core-less domain walls is that the functions we are working with, i.e., combinations of $\log(x)$ and $\exp(x)$, seem not to be sufficient to produce the shape of core-full domain walls, visualized in Fig. 2.

The fact that our solutions are nonelementary walls is harder to understand. More precisely, we claim that none of the multi-wall solutions of this paper have moduli, which could isolate elementary walls. For the chains of Sect. 3 this is easy to see because the single-wall links of which chains are made of $u^{3F}_{II}$ and $u^{4F}_{I}$ are nonelementary walls from the beginning. To confirm the same for the broadest class of solutions in Eq. (212) is much harder to do in general. Intuitively, however, given that the constraint (122) makes any generic multi-wall configuration lighter than the lightest exact elementary wall $u^{2F}_{I}$, it is natural to conclude that our solutions are all nonelementary. Despite this observation, the very fact that exact description of elementary walls seems to be a much more difficult task than description of compressed ones remains very puzzling.

In light of these findings, we are left with two challenges: the first is to find solutions for walls with well-developed cores and the second is to understand the disproportion between the number of exact elementary domain walls—which is three: $u^{2F}_{I}$, $u^{3F}_{I}$, and $u^{3F}_{III}$—and the number of exact compressed domain walls—which is potentially unlimited if the number of Higgs fields $N_F$ is arbitrarily large.

In closing, let us briefly comment on the impact of our findings on other topological solitons. The most direct implications can be made for composite solitons containing domain walls, such as wall–wall junctions and wall–vortex junctions. The reason is that their master equations include Eq. (13) as a subset, which hints that a similar richness of exact solutions could be found there as found for the domain walls. Our preliminary results strongly favor this possibility and we plan to explore this fully in a future study.

Another straightforward implication is that multi-flavor non-Abelian domain walls (studied in Ref. [12]) should possess a large number of exact solutions too. This can be seen easily by acknowledging the fact that solutions in the Abelian model can be embedded into a non-Abelian case.
precisely, for a class of so-called $U(1)$-factorizable solutions, the non-Abelian analog of the master equation can be shown to decompose into a direct sum of independent Abelian master equations. Therefore any solution presented here can be used to construct $U(1)$-factorizable solutions in a non-Abelian model. The question of whether our approach can help to find exact solutions that are not $U(1)$-factorizable provides another interesting direction for future work.

Lastly, an indirect implication of our results is that it raises expectations about the number and abundance of exact solutions in any master equation. In particular, in Ref. [2], F.B. and others found several new exact solutions of semilocal vortices, hinting at a very rich structure of exact solutions there. We plan to elaborate on these findings in the future.

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