THE MANY FACES OF THE
QUANTUM LIOUVILLE EXPONENTIALS

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Abstract

First, it is proven that the three main operator-approaches to the quantum Liouville exponentials — that is the one of Gervais-Neveu (more recently developed further by Gervais), Braaten-Curtright-Ghandour-Thorn, and Otto-Weigt — are equivalent since they are related by simple basis transformations in the Fock space of the free field depending upon the zero-mode only. Second, the GN-G expressions for quantum Liouville exponentials, where the $U_q(sl(2))$ quantum group structure is manifest, are shown to be given by $q$-binomial sums over powers of the chiral fields in the $J = 1/2$ representation. Third, the Liouville exponentials are expressed as operator tau functions whose chiral expansion exhibits a $q$ Gauss decomposition, which is the direct quantum group analogue of the classical solution of Leznov and Saveliev. It involves $q$ exponentials of quantum group generators with group “parameters” equal to chiral components of the quantum metric. Fourth, we point out that the OPE of the $J = 1/2$ Liouville exponential provides the quantum version of the Hirota bilinear equation.
1 INTRODUCTION

The problem of understanding the structure of 2d gravity from the continuum point of view can look back upon a history of more than ten years. During this time, various frameworks have been put forward to construct the quantum Liouville field, either in a canonical setup, or more recently, by path integral methods. For the classical theory itself, our understanding was much deepened by studying its generalization to the infinite (Toda) hierarchy of integrable systems where the powerful machinery of the Kyoto school is at work.

The main quantum canonical schemes\(^{1,2,3}\) that have been proposed look rather different at first glance. In particular two of them\(^{2,3}\) aimed at establishing quantum Bäcklund-type transformations into a particular free field, while the third\(^{1}\) worked more symmetrically, and developed an operatorial scheme that is close to the spirit of the BPZ method. It was more recently realized\(^{3,4}\) that the basic principle behind this third scheme (ref.\(^{1}\)) is the \(U_q(sl(2))\) quantum-group structure of the theory, which determines the form of the relevant chiral braiding and fusing matrices in terms of quantum group symbols. This quantum group aspect seems rather remote at first sight from the free-field calculations of the other two (refs.\(^{2,3}\)) and the question arises, whether, and in which sense the three schemes just mentioned are equivalent. This problem is also of relevance for the relation between the continuum and matrix model approaches, in view of the recent result\(^{4}\) that the approach of refs.\(^{1,7}\) is able to reproduce the matrix-model three-point functions. In the first part of this article, we shall show that the powers of the two-dimensional metric do indeed agree in all three approaches up to a simple equivalence transformation. In the second part we push further the quantum group aspect of the approach of refs.\(^{1,7}\). Indeed, simple manipulations of the expression of the Liouville exponential derived in ref.\(^{7}\) in terms of quantum group covariant chiral vertex operators will allow us to rewrite it as a q binomial expansion, or as a kind of operator tau-function that obeys an operator Hirota equation, in close parallel to the classical Leznov-Saveliev\(^{10}\) solution, and Kyoto school approach\(^{12,13}\).

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\(^{1}\) We consider only the case of finite volume — which is relevant e.g. for the string application — and thus will not discuss the approaches of refs.\(^{4,5}\).

\(^{2}\) The recent article of Kazama and Nicolai\(^{8}\) follows the scheme of ref.\(^{3}\), so that we do not treat it separately.
The paper is organized as follows. Sections 2 and 3 display a detailed comparison of the three schemes of refs. [1, 7, 2, 3]. Since the main ingredients in all constructions are locality and conformal invariance, the analysis of their equivalence also provides evidence for the problem whether these two requirements suffice to determine the structure of the quantum theory uniquely. It may seem surprising that no such investigation has been carried out to date. In the present analysis we shall demonstrate therefore in a very explicit manner the equivalence of the approaches put forward by Gervais and Neveu [1] and more recently [7] by Gervais [7], Braaten, Curtright, Ghandour and Thorn [3] (BCGT) and Otto and Weigt [3] (OW). The relevant conformal objects being exponentials of the Liouville field, we shall show that there exist what one may regard as basis-transformation operators $S_1$, $S_2$ depending only on the free-field zero mode $\varpi$ such that (up to trivial field redefinitions, see below)

$$e^{-\varphi} = S_1(\varpi)e^{-J\varphi}S_1^{-1}(\varpi) = S_2(\varpi)e^{-J\varphi}S_2^{-1}(\varpi) \quad (1.1)$$

A correspondence of this type could be expected since all three frameworks are based on a transformation of the Liouville field $\varphi$ onto a set of free fields, such that the energy-momentum tensor takes the same form in all approaches, and impose the same locality condition. However, the unexpected point is that the operators $S_1$ only involve the zero-mode. Thus the correspondence is basically rather simple, and as a consequence, the exponentials can all be written in the form (see below for details)

$$e^{-\varphi} = \sum_{J+m=0,1,2,\ldots} a_m^{(J)}(\varpi)V_m^{(J)}\bar{V}_m^{(J)} \quad (1.2)$$

where $a_m^{(J)}(\varpi)$ are suitable coefficients, and $V_m^{(J)}$ resp. $\bar{V}_m^{(J)}$ are the same left- resp. right-moving primaries in all approaches, given in terms of free fields, with a conformal dimensions characterized by $J$ (equal, of course, to the dimension of the exponential itself) and a definite shift of the zero mode given by $m$:

$$\varpi V_m^{(J)} = V_m^{(J)}(\varpi - 2m) \quad (1.3)$$

and likewise for $\bar{V}_m^{(J)}$. The equivalence Eq. (1.1) is easily seen by use of Eq. (1.3) to imply (for integer $2J$) the following relation between the coefficients $a_m^{(J)}$.

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5GN only considered open strings in detail. Therefore we compare with the more recent expression of ref. [3].
in the 3 frameworks:

\[ a^{(J)}_m(\varpi)\big|_{GN-G} = C_1^{2J} \prod_{r=0}^{2m-1} f_1(\varpi + r) a^{(J)}_m(\varpi)\big|_{BCGT} \]

\[ = C_2^{2J} \prod_{r=0}^{2m-1} f_2(\varpi + r) a^{(J)}_m(\varpi)\big|_{OW} \]  

(1.4)

with suitable functions \( f_{1,2}(\varpi) \). In Eq. (1.4), we have accounted also for the possibility of trivial redefinitions \( \varphi \to \varphi + \ln C^2 \); this is the origin of the constants \( C_1, C_2 \). The products are defined in the standard fashion when \( 2m \) is a negative integer (see e.g. ref. [7]). Using the recently obtained result [14] that the GN-G construction can be extended to noninteger \( 2J \), we shall show that the OW and GN-G operators are actually equivalent for arbitrary \( J \) (in the BCGT framework, only the exponential with \( J = 1/2 \) was constructed explicitly). The correspondences BCGT vs. GN-G, and OW vs. GN-G are the subject of sections 2, and 3, respectively. To facilitate comparison with the literature, the notations of BCGT resp. OW will be used in the corresponding sections, with comprehensive translation tables presented in an appendix. In the rest of the paper we use the notation of GN-G.

In section 4, we next briefly recall the group theoretical approach to the classical Liouville exponential, based on the solution of ref. [10]. The exponential \( e^{-\varphi/2} \) describing the inverse square root of the metric plays a special role for the integrability structure of the theory. In the standard Lax pair approach, it appears as the monodromy invariant solution of the auxiliary linear system [6, 15]. We use the group-theoretic viewpoint to derive a bilinear equation of the Hirota type for this exponential — of course equivalent to the Liouville equation — which clearly shows that it is a tau function. In section 5, we start from an expression of the Liouville exponential in terms of a different set of fields \( \xi^{(J)}_M \) that are quantum group covariant. First we show that the expansion of the quantum Liouville exponential in terms of the \( \xi \) fields is a field-theory analogue of the expansion of a q binomial. Then a group-theoretic method is developed, based on the q deformation of the classical \( sl(2) \) group of section 3, which is shown to be directly connected with the \( U_q(sl(2)) \) quantum group structure of the GN-G approach. The quantum Liouville exponentials are written using a generalized Gauss decomposition of the quantum group where the group “parameters” are the covariant \( \xi \) fields just mentioned. This shows how the quantum Liouville exponential may be
regarded as a q tau function — which is an operator. The quantum Hirota equation is established by observing that the quantum equivalent of the classical bilinear equations is the fact that a particular term in the short-distance operator-product expansion of the quantum inverse square root of the metric is a constant, since it is given by its zeroth power. From this viewpoint the derivation of the quantum Hirota equation becomes a straightforward consequence of the relation between Clebsch-Gordan coefficients and fusing matrix of the $\xi$ fields.

2 BCGT vs. GN-G

In the BCGT formalism, the Liouville field is expressed in terms of free chiral fields $\psi_L, \psi_R$ by means of a Bäcklund transformation. On the quantum level, the authors were able to explicitly construct the inverse square root of the metric, but could only obtain approximate results for other Liouville exponentials. Following ref.\[2\], this particular power is represented as

$$e^{-g\Phi(\tau, \sigma)} = \zeta me^{-g\tilde{\psi}^-(\tau, \sigma)}Z(\tau, \sigma)e^{-g\tilde{\psi}^+(\tau, \sigma)}$$

Here, the fields $\tilde{\psi}^\pm$ are just the annihilation resp. creation parts of

$$\tilde{\psi}(\tau, \sigma) = \psi_L(\tau + \sigma) - \psi_R(\sigma - \tau)$$

with

$$\psi_L(\tau + \sigma) = \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} A_n \frac{e^{-in(\tau+\sigma)}}{n}, \quad [A_n, A_m] = n\delta_{n,-m}$$

$$\psi_R(\tau - \sigma) = \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} B_n \frac{e^{-in(\tau-\sigma)}}{n}, \quad [B_n, B_m] = n\delta_{n,-m}$$

The definition of the constants $\zeta, m$ and $g$ is given in an appendix, and $\tau, \sigma$ are world sheet coordinates on the cylinder as usual. The nontrivial part of the structure of $e^{-g\Phi}$ resides in the nonlocal periodic operator $Z(\tau, \sigma)$. It can be written as an integral over free field exponentials of dimension 1:

$$Z(\tau, \sigma) = \int_0^{2\pi} d\sigma' f(\sigma - \sigma') :F(P)e^{gPd(\sigma'-\sigma)}\cosh(g\psi(\tau, \sigma'))e^{g\tilde{\psi}(\tau, \sigma')} :$$
where
\[
\psi(\tau, \sigma) = Q + P \frac{\tau}{2\pi} + \psi_L(\tau + \sigma) + \psi_R(\tau - \sigma), \quad [Q, P] = i
\]
\[
F(P) = (\sinh^2(gP/2) + \sin^2(g^2/4))^{-1/2}
\]
\[
f(\sigma - \sigma') = \left(4 \sin^2 \frac{\sigma - \sigma'}{2}\right)^{g^2/4\pi}
\]
and \(d(\sigma - \sigma')\) is a periodic antisymmetric function given by
\[
d(\sigma' - \sigma) = \frac{\sigma' - \sigma}{2\pi} - \frac{1}{2} \epsilon(\sigma' - \sigma),
\]
with \(\epsilon(\sigma' - \sigma)\) the stair-step function, equal to the sign of \(\sigma' - \sigma\) for \(\sigma' - \sigma \in [-2\pi, 2\pi]\). The normal ordering prescription is just the standard one for the harmonic oscillator modes, whereas for the zero modes one has
\[
: e^{\alpha Q} g(P) : \equiv e^{\alpha Q/2} g(P) e^{\alpha Q/2}
\]
(hermitian normal ordering). Eq.(2.1) is fully normal-ordered. We shall rewrite it as a product of normal-ordered operators which are primary fields and periodic up to a multiplicative constant. Since everything is given in terms of chiral free fields, we can w.l.o.g. put \(\tau = 0\) in the following. Decomposing the cosh and reorganizing the zero modes, we have
\[
e^{-g \Phi(\tau, \sigma)} = \zeta m \int_0^{2\pi} d\sigma' \left(\frac{1}{2} Y_L(\sigma, \sigma') + \frac{1}{2} Y_R(\sigma, \sigma')\right),
\]
\[
Y_L(\sigma, \sigma') = f(\sigma - \sigma')F(\sigma) e^{i(gP+ig/2)d(\sigma' - \sigma)} e^{+gQ} : e^{g\psi_R(\sigma)} e^{-g\psi_L(\sigma)} e^{+2g\psi_L(\sigma')} :
\]
\[
Y_R(\sigma, \sigma') = f(\sigma - \sigma')F(\sigma') e^{i(gP-ig/2)d(\sigma' - \sigma)} e^{-gQ} : e^{g\psi_R(\sigma)} e^{-g\psi_L(\sigma)} e^{-2g\psi_R(\sigma')} :.
\]
Using the relations Eqs.(A.4,A.6) between the free fields of BCGT and GN-G, we see immediately that
\[
: e^{g\psi_R(\sigma)} e^{-g\psi_L(\sigma)} e^{+2g\psi_L(\sigma')} \equiv \tilde{V}_{1/2}^{(1/2)}(\sigma) \hat{V}_{-1/2}^{(1/2)}(\sigma) \tilde{V}_1^{(-1)}(\sigma') :
\]
\[
: e^{g\psi_R(\sigma)} e^{-g\psi_L(\sigma)} e^{-2g\psi_R(\sigma')} \equiv \tilde{V}_{1/2}^{(1/2)}(\sigma) \hat{V}_{-1/2}^{(1/2)}(\sigma) \tilde{V}_1^{(-1)}(\sigma') :.
\]
where the hats denote the oscillator parts of the corresponding chiral primaries. In the following, we discuss only $Y_L(\sigma, \sigma')$ explicitly; the treatment of $Y_R$ is totally analogous. The RHS of Eq. (2.7) can be written as

\[ : \hat{V}(1/2)(\sigma) \hat{V}_{-1/2}(\sigma) \hat{V}_{1}^{-1}(\sigma') := (1 - z'/z)^{-g^2/2\pi} \hat{V}(1/2)(\sigma) \hat{V}_{-1/2}(\sigma) \hat{V}_{1}^{-1}(\sigma') \]

with

\[ z := e^{i\sigma} \quad z' := e^{i\sigma'} \]

and we have

\[ (1 - z'/z)^{-g^2/2\pi} = \left( 4 \sin^2 \frac{\sigma - \sigma'}{2} \right)^{-g^2/4\pi} e^{-\frac{1}{2} \theta^2 d(\sigma' - \sigma)} \]

(2.8)

(2.9)

The function $f(\sigma - \sigma')$ is thus absorbed by the removal of the external normal ordering. It remains to reinstate the zero mode dependence needed to complete $\hat{V}(1/2)(\sigma), \hat{V}_{-1/2}(\sigma), \hat{V}_{1}^{-1}(\sigma')$ to the full primary fields. Here we have to take into account that in the GN-G formalism, one works in a CFT-adapted notation where two formally independent sets of zero modes $\bar{q}_0, p_0$ and $\bar{q}_0, \bar{p}_0$ are used for left and right movers, allowing the chiral factorization of the basic conformal operators. It is well known that one obtains in this way the same amplitudes as from the a priori representation with only one set of zero modes, provided of course one restricts to operators with equal left and right zero mode shifts and puts $p_0 = \bar{p}_0$ in all matrix elements. In the Liouville context, the distinction between $p_0$ and $\bar{p}_0$ is in fact slightly more than formal due to the possibility of a winding number in the elliptic sector. However, in the hyperbolic sector considered by BCGT and OW -corresponding to regular solutions of the Liouville equation- one really has $\bar{p}_0 = p_0$, with real $p_0$. Passing from the BCGT to the GN-G notation, we thus have to replace in $Y_L$:

\[ e^{gQ} \rightarrow e^{gQ} e^{\bar{g}Q} \quad, \quad e^{gPd(\sigma' - \sigma)} \rightarrow e^{-gP\sigma/4\pi} e^{gP/4\pi} e^{-gP/4\pi} \]

(2.10)

In view of the relations (cf. Eqs. [A.4], [A.6])

\[ V_{-1/2}(\sigma) = e^{-gQ - gP\sigma/4\pi} \hat{V}_{-1/2}(\sigma) \]

\[ \hat{V}_{1/2}(\sigma) = e^{gQ - gP\sigma/4\pi} \hat{V}_{1/2}(\sigma) \]

\[ \hat{V}_{1}^{-1}(\sigma) = e^{gQ - gP\sigma/4\pi} \hat{V}_{1}^{-1}(\sigma) \]
we can then rewrite $Y_L(\sigma, \sigma')$ in the form

$$Y_L(\sigma, \sigma') = F(P + ig/2)e^{-\frac{1}{2}gP\varepsilon(\sigma' - \sigma)}V_{1/2}^{(1/2)}(\sigma)V_{1/2}^{(1/2)}(\sigma)V_1^{(-1)}(\sigma')$$

and analogously

$$Y_R(\sigma, \sigma') = F(\bar{P} - ig/2)e^{-\frac{1}{2}g\bar{P}\varepsilon(\sigma' - \sigma)}\bar{V}_{1/2}^{(1/2)}(\sigma)\bar{V}_{1/2}^{(1/2)}(\sigma)\bar{V}_1^{(-1)}(\sigma')$$

Thus we obtain altogether

$$e^{-g\Phi(\sigma)} = \frac{\zeta m}{2} \int_0^{2\pi} d\sigma' \left\{ F(P + ig/2)e^{-\frac{1}{2}gP\varepsilon(\sigma' - \sigma)}V_{1/2}^{(1/2)}(\sigma)V_{1/2}^{(1/2)}(\sigma)V_1^{(-1)}(\sigma') + F(\bar{P} - ig/2)e^{-\frac{1}{2}g\bar{P}\varepsilon(\sigma' - \sigma)}\bar{V}_{1/2}^{(1/2)}(\sigma)\bar{V}_{1/2}^{(1/2)}(\sigma)\bar{V}_1^{(-1)}(\sigma') \right\}$$

In view of the periodicity of $e^{-g\Phi}$, we can take $\sigma \in [0, 2\pi]$. The integrals are then recognized to coincide up to a prefactor with the screening charges of refs. [13, 14]:

$$\int_0^{2\pi} d\sigma' e^{-\frac{1}{2}g(P + ig)e(\sigma' - \sigma)}V_1^{(-1)}(\sigma') = e^{-\frac{1}{2}g(P + ig)}S(\sigma),$$

where

$$S(\sigma) = e^{2ih(\sigma + 1)} \int_0^\sigma d\sigma' V_1^{(-1)}(\sigma') + \int_\sigma^{2\pi} d\sigma' V_1^{(-1)}(\sigma')$$

and similarly for the right-moving part. Notice, however, that here everything is expressed in terms of the free field $\phi_1$. Therefore, $\bar{S}(\sigma)$ differs from the corresponding expression in ref. [14] by the replacement $\bar{\phi}_2 \rightarrow \bar{\phi}_1$. The operators $V_{1/2}^{(1/2)}$ and $\bar{V}_{1/2}^{(1/2)}$ formed from the fields $\phi_2, \bar{\phi}_2$ are related to $V_{1/2}^{(1/2)}, \bar{V}_{1/2}^{(1/2)}$ by the application of $S$ resp. $\bar{S}$:

$$V_{1/2}^{(1/2)}(\sigma) = N(\bar{\sigma})V_{-1/2}^{(1/2)}(\sigma)S(\sigma)$$

$$\bar{V}_{1/2}^{(1/2)}(\sigma) = \bar{N}(\bar{\sigma})\bar{V}_{1/2}^{(1/2)}(\sigma)\bar{S}(\sigma)$$

$$\bar{V}_{1/2}^{(1/2)}(\sigma) = \bar{N}(\bar{\sigma})\bar{V}_{1/2}^{(1/2)}(\sigma)\bar{S}(\sigma)$$

$$\bar{V}_{-1/2}^{(1/2)}(\sigma) = \bar{N}(\bar{\sigma})\bar{V}_{-1/2}^{(1/2)}(\sigma)\bar{S}(\sigma)$$

$$\bar{V}_{-1/2}^{(1/2)}(\sigma) = \bar{N}(\bar{\sigma})\bar{V}_{-1/2}^{(1/2)}(\sigma)\bar{S}(\sigma)$$
with (cf. Eq. A.4)

\[
N(\omega) = \frac{1}{2} \frac{e^{-ih\omega}}{\sin(h\omega)} \frac{\Gamma[1 + (1 + \omega)h/\pi]}{\Gamma(1 + h/\pi)\Gamma(h\omega/\pi)}
\]

\[
\tilde{N}(\omega) = -\frac{1}{2} \frac{e^{-ih\omega}}{\sin(h\omega)} \frac{\Gamma[1 + (1 - \omega)h/\pi]}{\Gamma(1 + h/\pi)\Gamma(-h\omega/\pi)}
\]  

(2.17)

Hence, setting \( \bar{\omega} = \omega \) again eventually, the final form of \( e^{-g\Phi} \) becomes

\[
e^{-g\Phi(\tau, \sigma)} = \sum_{m=\pm 1/2} a_m^{(1/2)}(\omega)|_{BCGT} V_m^{(1/2)}(\tau + \sigma) \bar{V}_m^{(1/2)}(\tau - \sigma)
\]  

(2.18)

with

\[
a_{\pm 1/2}^{(1/2)}(\omega)|_{BCGT} = \frac{\zeta_m}{2} F(P + ig/2)e^{-gP/2} \frac{1}{N(\omega)}
\]

\[
a_{-1/2}^{(1/2)}(\omega)|_{BCGT} = \frac{\zeta_m}{2} F(P - ig/2)e^{-gP/2} \frac{1}{N(\omega)}
\]  

(2.19)

We note that \( V_{\pm 1/2}^{(1/2)} \) resp. \( \bar{V}_{\pm 1/2}^{(1/2)} \) are nothing but the (normalized) quantum versions of the fields \( AA^{1/2} \) and \( A \) resp. \( B^{1/2} \) and \( BB^{1/2} \), with \( A(\tau + \sigma) \) and \( B(\tau - \sigma) \) the arbitrary functions parametrizing the general classical solution

\[
2g\Phi = \ln \left[ \frac{8A'B'}{\mu^2(A-B)^2} \right]
\]  

(2.20)

Let us now compare this with the corresponding expressions in the GN-G construction. There we have for general \( J [4], (\mu^2 = 1) \)

\[
a_m^{(J)}(\omega)|_{GN-G} = \frac{(-1)^{J-m}}{2J} \prod_{r=0}^{2J} [\omega + m - J + r],
\]

\[
\lambda_m^{(J)}(\omega)N_m^{(J)}(\omega)\bar{N}_m^{(J)}(\omega),
\]

where

\[
\lambda_m^{(J)}(\omega) = \binom{2J}{J+m} \prod_{r=0}^{2J} [\omega + m - J + r],
\]

\[
\binom{2J}{J+m} := \frac{\prod_{r=1}^{J+m} [J - m + r]}{\prod_{r=1}^{J+m} [r]},
\]

\[
N_m^{(J)}(\omega) = \bar{N}_m^{(J)}(\omega) = \frac{\prod_{r=1}^{J-m} [1 + rh/\pi]}{\prod_{r=1}^{J+m} [1 + rh/\pi]} 
\]
\[ J^{-m} \prod_{r=1}^{J-m} \Gamma[(\omega - r)h/\pi] J^{+m} \prod_{r=1}^{J+m} \Gamma[(-\omega - r)h/\pi] \frac{\sqrt{\Gamma[(\omega + r - 1)h/\pi]}}{\sqrt{\Gamma[(-\omega - r)h/\pi]}} \] (2.21)

For \( J = 1/2 \), this reduces to
\[ a^{(1/2)}_{-1/2}(\omega)|_{GN-G} = -\sqrt{|\omega|} |\omega - 1| \Gamma(-\omega h/\pi) \Gamma[(\omega - 1)h/\pi] \]
\[ a^{(1/2)}_{+1/2}(\omega)|_{GN-G} = +\sqrt{|\omega|} |\omega + 1| \Gamma(-\omega h/\pi) \Gamma[(\omega + 1)h/\pi] \] (2.22)

The equivalence conditions Eq.1.4 read in this case
\[ a^{(1/2)}_{-1/2}(\omega)|_{GN-G} = \frac{C_1}{f_1(\omega - 1)} a^{(1/2)}_{-1/2}(\omega)|_{BCGT} \]
\[ a^{(1/2)}_{+1/2}(\omega)|_{GN-G} = C_1 f_1(\omega) a^{(1/2)}_{+1/2}(\omega)|_{BCGT} \] (2.23)

Eqs.2.23 are solved by
\[ C_1 = \pm \frac{\sqrt{8\Gamma(-h/\pi)} \zeta}{\xi}, \quad f_1(\omega) = \pm i \] (2.24)

The appearance of the imaginary unit is due to the fact that the hermiticity properties of \( e^{-\alpha - \Phi_{GN}} \) and \( e^{-g\Phi_{BCGT}} \) are adjusted to the elliptic and hyperbolic sector, respectively. This completes the proof of equivalence of the BCGT and GN-G constructions of the inverse square root of the metric. We mention that the same kind of equivalence can be established for an earlier construction of \( e^{-\alpha - \Phi/2} \) in the GN-G formalism [18].

### 3 OW vs. GN-G

Following ref.[3], we start from the expression for the Liouville exponentials given by Otto and Weigt[6], written in a factorized form[7]

\[ e^{\lambda \varphi(\tau, \sigma)} = \sum_{n=0}^{\infty} \frac{(-\mu^2)^n}{[n]!} \frac{2\lambda}{n} Z^{(\lambda,n)}(\tau, \sigma) \]

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6 In fact we consider here the expression given in the later papers of Weigt, which differs from the original formula by a factor \((\sin(h)/h)^n\) in the n-th term; the two versions are obviously equivalent in the sense of Eq.1.4.

7 In the first two papers of ref.[3], the authors consider also slightly different normal ordering prescriptions which do not allow a factorization into powers of screening charges. We will not discuss these possibilities here.
with

\[ [2\lambda]_n \equiv \prod_{j=0}^{n-1} [2\lambda + j] \quad , \quad [n]! \equiv \prod_{j=1}^{n} [j] \]

\[ Z_{\eta}^{(\lambda,n)}(\tau,\sigma) = F^{(\lambda,n)}(\hat{P} + ih(n + \lambda)) \hat{Z}_{\eta}^{(\lambda,n)}(\tau,\sigma), \]

\[ F^{(\lambda,n)}(\hat{P} + ih(n + \lambda)) = \prod_{j=0}^{n-1} \frac{1}{\sinh(\hat{P} - ih(j - n)) \sinh(\hat{P} + ih(j + n + 2\lambda))} \]

\[ \hat{Z}_{\eta}^{(\lambda,n)}(\tau,\sigma) =: e^{\lambda \eta \psi(\tau,\sigma)} \cdot \prod_{j=1}^{n} \int^2_{-\pi} d\sigma_j d\bar{\sigma}_j e^{ih(\bar{\omega} + 1)(\epsilon(\sigma_j - \bar{\sigma}_j) - \epsilon(\sigma - \bar{\sigma}))/\pi} \times \]

\[ V_1^{(-1)}(\tau + \sigma_j) \bar{V}_1^{(-1)}(\tau - \bar{\sigma}_j) \]

(3.1)

The normal ordering is the same as in the BCGT construction, and \( h \) is the same as in the GN-G notation. Table A.6 gives for the oscillator part of \( e^{\eta \psi^+ + \eta \psi^-} \):

\[ : e^{\eta \psi^+ + \eta \psi^- + (\tau + \sigma_j)} : \text{osc} = \hat{V}_1^{(-1)}(\tau + \sigma_j) \bar{V}_1^{(-1)}(\tau - \bar{\sigma}_j) \]

(3.2)

Notice the appearance of the operator \( \hat{V}_1^{(-1)} \) in place of \( \hat{V}_1^{(-1)} \) in the BCGT scheme, as a consequence of the use of the free field \( \psi_{OW} = \ln A' + \ln \frac{B'}{\beta^2} \) instead of \( 2g\psi_{BCGT} = \ln A' - \ln B' \), with \( A, B \) as in Eq.2.20. The zero mode dependence of the \( j \)th factor can be written as

\[ : e^{\hat{P}(\epsilon(\sigma - \bar{\sigma}_j) - \epsilon(\sigma - \bar{\sigma}))} e^{\eta \psi^+(\tau + \sigma_j) + \eta \psi^-(\tau - \bar{\sigma}_j)} : | \text{zero mode} = e^{(\hat{P} + ih)(\epsilon(\sigma - \bar{\sigma}_j) - \epsilon(\sigma - \bar{\sigma}))} e^{\hat{Q}(\tau + \sigma_j)/\pi + \hat{P}(\tau - \bar{\sigma}_j)/\pi} \]

(3.3)

This is just the required dependence to complete \( \hat{V}_1^{(-1)} \) and \( \bar{V}_1^{(-1)} \) to \( V_1^{(-1)} \) and \( \bar{V}_1^{(-1)} \), if we again take into account the zero mode doubling as explained above. Thus we have

\[ \hat{Z}_{\eta}^{(\lambda,n)}(\tau,\sigma) =: e^{\lambda \eta \psi(\tau,\sigma)} \cdot \prod_{j=1}^{n} \int^2_{-\pi} d\sigma_j d\bar{\sigma}_j e^{ih(\omega + 1)(\epsilon(\sigma_j - \bar{\sigma}_j) - \epsilon(\sigma - \bar{\sigma}))/\pi} \times \]

\[ V_1^{(-1)}(\tau + \sigma_j) \bar{V}_1^{(-1)}(\tau - \bar{\sigma}_j) \]

(3.4)
Introducing screening charges $S$, $\tilde{S}$ with $S$ as in Eq.2.13 and

$$\tilde{S} = e^{-2i\hbar(\varphi+1)} \int_0^\sigma d\sigma' \tilde{V}_1^{(-1)}(\sigma') + \int_\sigma^{2\pi} d\sigma' \tilde{V}_1^{(-1)}(\sigma'),$$

we get

$$\hat{Z}^{(\lambda,\eta)}(\tau,\sigma) =: e^{\lambda \eta \psi(\tau,\sigma)} (\tilde{S} S)^n$$

(3.5)

It is equally easy to see that

$$: e^{\lambda \eta \psi(\tau,\sigma)} : = V^{(-\lambda)}_\lambda(\tau + \sigma) \tilde{V}^{(-\lambda)}_\lambda(\tau - \sigma)$$

(3.6)

Thus, with the identification $\lambda = -J$, the form of $e^{\lambda \varphi}$ agrees with the GN-G expression as written in the form of ref.[14]. As in ref.[14], we pass from $V^{(\lambda)}_\lambda S^n, \tilde{V}^{(\lambda)}_\lambda \tilde{S}^n$ to the canonically normalized operators $V^{(J)}_m, \tilde{V}^{(J)}_m$ with

$$\langle \varpi | V^{(J)}_m | \varpi + 2m \rangle = 1 = \langle \tilde{\varpi} | \tilde{V}^{(J)}_m | \tilde{\varpi} + 2m \rangle$$

(3.8)

by means of the normalization factors $I^{(J)}_m, \tilde{I}^{(J)}_m$ given in appendix, so that

$$e^{\lambda \varphi} = \sum_{J+m=0,1,2,...} a^{(J)}_m(\varpi) |_{OW} V^{(J)}_m \tilde{V}^{(J)}_m$$

with

$$a^{(J)}_m(\varpi) |_{OW} = \frac{(-\mu^2)^n \lambda^n \Gamma(\lambda,\varpi + 1)(\lambda + n) J^{(J)}_m(\varpi) I^{(J)}_m(\varpi)}{[n]!}$$

(3.9)

Again, we have to find $C$ and $f(\varpi)$ relating the coefficients $a^{(J)}_m$ of GN-G resp. OW. The calculation is straightforward and we shall suppress the details. One finds

$$C_2 = -\frac{\Gamma(-h/\pi)}{2} f_2(\varpi) = \frac{1}{2} \sqrt{\frac{[\varpi + 1] \Gamma[1 + (\varpi + 1)h/\pi]}{\Gamma(\varpi h/\pi)}}$$

(3.10)

The present discussion applies directly to the case where $J$ is half-integer positive. However, it is in fact possible to generalize to arbitrary real $J$. Indeed, it was shown in ref.[14] that the GN-G construction can be extended straightforwardly to continuous $J$, with essentially the same expression to be used for the Liouville exponentials as for the half-integer case (details will
be given in a forthcoming publication). Correspondingly, Eq. 3.10 possesses a natural continuation to arbitrary $J$:

$$
\prod_{r=0}^{2m-1} f_2(\omega + r) = 2^{-2m} \prod_{r=0}^{2m-1} \sqrt{\frac{[\omega + r + 1]}{[\omega + r]} \Gamma[1 + (\omega + r + 1)h/\pi]} \frac{\Gamma[1 + (\omega + r)h/\pi]}{\Gamma((\omega + r)h/\pi)} \prod_{r=1}^{2m} (\omega + r) = (h/2\pi)^{2m} \sqrt{\frac{[\omega + 2m]}{[\omega]} \Gamma(\omega + 2m)h/\pi}] \Gamma((\omega + 2m)h/\pi)} \Gamma(\omega + 2m + 1) \Gamma(\omega + 1)} \right) \tag{3.11}
$$

The last expression makes sense for arbitrary $2m$, and we read off

$$
S_2(\omega) = \left(\frac{2\pi}{h} \right)^{\omega} \frac{1}{\sqrt{\frac{[\omega]}{\Gamma(\omega h/\pi)} \Gamma(\omega + 1)}} \tag{3.12}
$$

In ref. [3], the locality of the Liouville exponentials was verified only up to the third power of the cosmological constant; the above analysis thus provides at the same time a proof to all orders that the result of Otto and Weigt is correct.

A general word of caution should be added, however, in the case of non-integer $2J$. The expansion Eq. 1.2 then becomes an infinite sum which certainly does not converge in any naive sense. Consequently, the equivalence of Eqs. 1.1 and 1.4 may be formal. Nevertheless, it can be shown that whenever $a_{jn}$ solves the locality conditions [14], so will any other member of the equivalence class defined by transformations of type Eq. 1.4. Thus, the GN-G resp. OW Liouville exponentials are local as formal power series in the screening charges, as a consequence of the underlying chiral algebra [1, 7, 14].

4 Group-theoretic approach to the classical Liouville exponentials

As a preparation of the coming section, it is pedagogical to temporarily return to the classical case. Then we have\textsuperscript{8} the general solution of ref. [10]:

$$
e^{-j\Phi(z, \bar{z})} = \frac{\langle j, j | M^{-1}(\bar{z})M(z) | j, j \rangle}{(s(z)s(\bar{z}))^j} \tag{4.1}
$$

\\textsuperscript{8}In this section, we redefine the Liouville field, for simplicity, so that the coupling constant need not be written any more.
\[ \frac{dM}{dz} = s(z)Mj_-, \quad \frac{d\bar{M}}{d\bar{z}} = \bar{s}(\bar{z})\bar{M}j_+ , \]  

(4.1)

where \( s \) and \( \bar{s} \) are arbitrary functions of a single variable. The symbols \( j_\pm \) represent \( sl(2) \) generators satisfying \( [j_+, j_-] = 2j_3 \), and \( |j, j> \) are highest-weight states \( j_+ |j, j> = 0 \). In Eq.4.1 and below, we use Euclidean coordinates on the sphere, \( z = e^{\tau+i\sigma} \), \( \bar{z} = e^{\tau-i\sigma} \). To establish the connection with the approaches of the last section, note first that \( s, \bar{s} \) are related to the arbitrary functions \( A, B \) appearing in the general classical solution Eq.2.20 by

\[ s(z) = A'(z), \quad \bar{s}(\bar{z}) = -B'/(B)^2 \]  

(4.2)

and thus \( s \) and \( \bar{s} \) can be identified with the classical equivalents of the screening charge densities \( V^{(-1)}_1, \bar{V}^{(-1)}_1 \) (up to normalization). It is then immediate to verify that Eq.4.1 reduces to (the classical limit of) Eq.1.2 after evaluation of the matrix element. Actually, the \( SL(2, C) \) symmetry of the theory,

\[ A \to \frac{aA + b}{cA + d}, \quad B \to \frac{aB + b}{cB + d}, \quad ad - bc = 1 \]  

(4.3)

allows us just as well to identify \( s^{-1/2}, \bar{s}^{-1/2} \) with any linear combination of the quasiperiodic (Bloch wave) fields \( V^{(1/2)}_{-1/2} \) and \( V^{(1/2)}_{+1/2} \) resp. \( \bar{V}^{(1/2)}_{-1/2} \) and \( \bar{V}^{(1/2)}_{+1/2} \). For the classical considerations below, the assignment chosen is irrelevant; however, on the quantum level the situation will be quite different. Looking at Eq.4.1, one may wonder why the Liouville exponential should be given by highest-weight matrix elements. The basic reason is that Eq.4.1 must be such that

\[ e^{-j_1\Phi(z, \bar{z})}e^{-j_2\Phi(z, \bar{z})} = e^{-(j_1 + j_2)\Phi(z, \bar{z})} \]  

(4.4)

by the very definition of the classical exponential function. In order to verify this, we introduce the rescaled exponentials\(^9\)

\[ E^{(j)}(s, j-) = s^{-j}(z)M(z) = s^{-j}(z)e^{\int_{\bar{z}}^{z}s(z')dz'} j_- \]

resp.

\[ \bar{E}^{(j)}(\bar{s}, j_+) = \bar{s}^{-j}(\bar{z})\bar{M}^{-1}(\bar{z}) = \bar{s}^{-j}(\bar{z})e^{-\int_{\bar{z}}^{z}\bar{s}(z')dz'} j_+ \]  

(4.5)

\(^9\)For the conceptual considerations here, the precise normalization of the \( E^{(j)} \) is not important. Therefore we don’t specify the lower integration limits in Eq4.5.
Thus the left-hand side of Eq.4.4 can be written as

\[ e^{-j_1\Phi(z, \bar{z})} e^{-j_2\Phi(z, \bar{z})} = (\langle j_1, j_1 | j_2, j_2 \rangle \bar{E}^{(j_1)}(s, j_+) \otimes \bar{E}^{(j_2)}(\bar{s}, j_+) \times E^{(j_1)}(s, j_-) \otimes E^{(j_2)}(s, j_-)(|j_1, j_1 > |j_2, j_2 >) \\
= (\langle j_1, j_1 | j_2, j_2 \rangle \bar{E}^{(j_1+j_2)}(\bar{s}, j_+ \otimes 1 + 1 \otimes j_+ \times E^{(j_1+j_2)}(s, j_- \otimes 1 + 1 \otimes j_-)(|j_1, j_1 > |j_2, j_2 >) \right) \tag{4.6}
\]

The highest weight-states considered are the only ones such that the tensor product gives a single irreducible representation. Its spin is \(j_1 + j_2\), and \(|j_1, j_1 > |j_2, j_2 >\) is the highest-weight vector. Since the matrix element of \(\bar{M}^{-1}M\) is determined solely by the group structure, it only depends upon the spin of the representation, and not upon the way it is realized; hence Eq.4.4 follows. In particular, we have

\[ e^{-j\Phi(z, \bar{z})} = \left( e^{-(1/2)\Phi(z, \bar{z})} \right)^{2j} \tag{4.7} \]

and thus Eq.4.1 may be re-written using binomial coefficients (more on this below).

It is well-known (see e.g. ref. [11]) that the Liouville solution and its Toda generalization are actually tau-functions in the sense of the Kyoto group[12]. A characteristic feature of tau-functions is to involve highest-weight states. We shall not dwell into the precise connection, since it is not directly evident from Eq.4.4. We shall rather recall the existence of bilinear equations of the Hirota type, which was the original motivation to introduce tau functions. The method of derivation we will use is not the same as the standard ones of ref. [12] or ref. [13]. Its interest is that it will carry over to the quantum case.

One may obtain a closed equation for the \(j = 1/2\) Liouville exponential as follows. Making use of Eq.4.4, let us compute the antisymmetric bilinear expression

\[ e^{-\Phi/2} \partial_x \partial_x e^{-\Phi/2} - \partial_x e^{-\Phi/2} \partial_x e^{-\Phi/2} = \\
- \langle \frac{1}{2}, \frac{1}{2} | \bar{M}^{-1}M | \frac{1}{2}, \frac{1}{2} > - \langle \frac{1}{2}, \frac{1}{2} | \bar{M}^{-1}M | \frac{1}{2}, -\frac{1}{2} > \\
+ \langle \frac{1}{2}, -\frac{1}{2} | \bar{M}^{-1}M | \frac{1}{2}, \frac{1}{2} > - \langle \frac{1}{2}, -\frac{1}{2} | \bar{M}^{-1}M | \frac{1}{2}, -\frac{1}{2} > \right) \tag{4.8} \]
The point of this particular combination of derivatives is that the functions \( s \) and \( \bar{s} \) disappear, and the result is given by the matrix element of \( \mathcal{M}^{-1} \mathcal{M} \) in the \( j = 0 \) representation, where \( \mathcal{M} = \mathcal{M} = 1 \). Thus we get

\[
e^{-\Phi/2} \partial_x e^{-\Phi/2} - \partial_x e^{-\Phi/2} \partial_x e^{-\Phi/2} = -1.
\] (4.9)

Of course it is trivial to rederive this equation directly from the Liouville equation; however this form – the simplest example of Hirota bilinear equations – which only makes use of the Liouville exponentials and not of the field \( \Phi \) itself, will be much easier to generalise to the quantum case. For later use we note that this Hirota equation is equivalent to the following relation in the Taylor expansion for \( z' \to z, \bar{z}' \to \bar{z} \).

\[
e^{-\Phi/2(z',\bar{z}')} e^{-\Phi/2(z,\bar{z})} - e^{-\Phi/2(z',\bar{z})} e^{-\Phi/2(z,\bar{z})'} \sim -(z' - z)(\bar{z}' - \bar{z})
\] (4.10)

Clearly we may obtain other bilinear equations for \( \exp(-j\Phi) \) with \( j \neq 1/2 \) by again projecting out the \( j = 0 \) component of the product.

5 Liouville exponentials and q-deformations

5.1 The quantum group structure

The deep connection of the quantum Liouville theory with \( U_q(sl(2)) \) was there from the beginning\[19\], but in disguise. It was elucidated more recently in refs.\[6, 7, 20, 21, 22\]. The quantum group-parameter \( \hbar \) is precisely the parameter\[10\] \( 2\pi\eta^2 \) of GN-G. Up to coupling constants, the fusing and braiding matrices of the \( V_m^{(J)} \) fields are given by \( q \)-6j symbols. However, the \( V_m^{(J)} \) fields are not quantum group covariants; there exists another basis of chiral vertex operators \( \xi_M^{(J)} \), related to the \( V_m^{(J)} \) by a linear transformation, which transform as spin \( J \) representations of \( U_q(sl(2)) \). Their fusing and braiding matrices are given by \( q \) Clebsch-Gordan and universal R-matrix elements respectively. In particular, the operator-product algebra of the \( \xi_M^{(J)} \) corresponds to making \( q \) tensor products of representations. We anticipate therefore that it is this basis which should be used when trying to extend

\[10\] There is actually a sort of doubling, and the complete structure is of the type \( U_q(sl(2)) \odot U_q^{-1}(sl(2)) \), corresponding to the existence of more general operators \( V_m^{J_J \bar{J}} \). We do not consider this possibility in this article.
the considerations of the previous section to the quantum level, so that the arbitrariness mentioned below Eq. 4.3 in the precise assignment of $s(z), \bar{s}(\bar{z})$ is lifted. It was shown in ref.[9] that the Liouville exponentials take the form

$$e^{-J\alpha_\omega \Phi(z, \bar{z})} = \sum_{M=-J}^{J} (-1)^{J+M} e^{i\hbar(J+M)} \xi^{(J)}(z) \bar{\xi}^{(J)}(\bar{z})$$  \hspace{1cm} (5.1)$$

Since they play no role we do not indicate the factors $S(\omega), S^{-1}(\omega)$ any more. Of course they should be put back in order to establish the precise connection between this last formula and the expressions of BCGT and OW.

The point of this section is to show that with this last expression for the Liouville exponential, its quantum properties are directly connected with their classical analogues by standard q deformations.

5.2 Connection with q-binomials

We shall use the notations of refs.[7],[20]–[22]. One introduces

$$[x] = \frac{\sin(hx)}{\sin(h)}.$$  \hspace{1cm} (5.2)$$

The q-deformed binomial coefficients noted $\binom{P}{Q}$ are defined by

$$\binom{P}{Q} := \frac{[P]!}{[Q]! [P-Q]!}, \quad [n]! := \prod_{r=1}^{n} [r],$$  \hspace{1cm} (5.3)$$

They are binomial coefficients for the expansion of $(x+y)^{2J}$, with $x$ and $y$ non-commuting variables such that

$$xy = yxe^{-2ih}.$$  \hspace{1cm} (5.4)$$

Indeed, it is easy to verify that they satisfy

$$\binom{m+1}{n} = e^{i\hbar n} \binom{m}{n} + e^{-i\hbar (m-n+1)} \binom{m}{n-1}.$$  \hspace{1cm} (5.5)$$

As a result one sees that if one lets

$$(x+y)^N = \sum_{r=0}^{N} \binom{N}{r} e^{i\hbar r(N-r)} x^r y^{N-r}$$  \hspace{1cm} (5.6)$$
one has, as required,

\[(x + y)^{N+1} = (x + y)(x + y)^N. \tag{5.7}\]

Recall the leading-order fusion of the \(\xi\) fields\textsuperscript{23}: to leading order in the short distance singularity at \(z' \rightarrow z\), the product of \(\xi\) fields behaves as

\[\xi_M^{(J)}(z) \xi_{M'}^{(J')}(z') \sim (z' - z)^{-2J' J'h/\pi} \lambda(J, M; J', M') \xi_{M+M'}^{(J+J')}(\sigma), \tag{5.8}\]

\[\lambda(J, M; J', M') = \sqrt{\frac{2J}{J+M} \frac{2J'}{J'+M'} e^{ih(J'-M'J)}}. \tag{5.9}\]

Thus, if we redefine

\[\eta_M^{(J)} \equiv \xi_M^{(J)}/\sqrt{\left(\frac{2J}{J+M}\right)} \tag{5.10}\]

we have

\[\eta_M^{(J)} \eta_{M'}^{(J')} \sim \eta_{M+M'}^{(J+J')} e^{ih(M'J-J'M')}\]

and thus

\[\eta_M^{(J)} \sim (\eta_{1/2}^{(1/2)})^{J+M} (\eta_{-1/2}^{(1/2)})^{J-M} e^{ih(J^2-M^2)} \tag{5.11}\]

In the above formulae and hereafter the symbol \(\sim\) means leading term of the short-distance expansion, divided by the singular short distance factor appearing in Eq.5.8. In terms of the \(\eta\) fields, the Liouville exponential takes the form

\[e^{-J\alpha_\Phi(z, \bar{z})} = \sum_{J+M=0}^{2J} \binom{2J}{J+M} (-1)^{J+M} e^{ih(J+M)} \eta_M^{(J)}(z) \bar{\eta}_{-M}^{(J)}(\bar{z}) \tag{5.12}\]

or, using Eq.5.11

\[e^{-J\alpha_\Phi(z, \bar{z})} \sim \sum_{J+M=0}^{2J} \binom{2J}{J+M} e^{ih(J+M)(J-M)} \times\]

\[-e^{ih} \eta_{1/2}^{(1/2)} \bar{\eta}_{-1/2}^{(1/2)} \binom{J+M}{(1/2)} \binom{J-M}{(1/2)} \tag{5.13}\]

\textsuperscript{11}Recall we are working here on the sphere, rather than on the cylinder.
which is completely analogous to the q-binomial expansion Eq.\ref{5.6}, if we identify 
\( x = -e^{i\hbar \eta_{1/2}}(z) \eta_{-1/2}(\bar{z}) \), \( y = \eta_{1/2}(z) \eta_{1/2}(\bar{z}) \). To avoid any possible confusion, we remark that Eq.\ref{5.4} does not imply that the fields \( \eta_{1/2}(z) \eta_{1/2}(\bar{z}) \) and \( \eta_{1/2}(\bar{z}) \eta_{1/2}(z) \) commute up to a factor; this is true only in the limit \( z' \to z, \bar{z}' \to \bar{z} \). We remark that Eq.\ref{5.12} can even be interpreted for arbitrary real \( J \), in view of the result of \cite{14} that the fields \( V_m^{(J)} \) – and hence \( \eta_m^{(J)} \) – can be defined for any \( J \). In this case, of course, the sum in Eqs.\ref{5.12}, \ref{5.13} runs from zero to infinity.

5.3 Liouville exponentials as quantum tau-functions

As is well known, the binomial coefficients are closely related to representations of \( U_q(sl(2)) \). Consider group-theoretic states \( |J, M \rangle \), \( -J \leq M \leq J \); together with operators \( J_\pm, J_3 \) such that:

\[
J_\pm |J, M \rangle = \sqrt{[J \mp M][J \pm M + 1]} |J, M \pm 1 \rangle, \quad J_3 |J, M \rangle = M |J, M \rangle. \tag{5.14}
\]

These operators satisfy the \( U_q(sl(2)) \) commutation relations

\[
[J_+, J_-] = [2J_3], \quad [J_3, J_\pm] = \pm J_\pm. \tag{5.15}
\]

It is elementary to derive the formulae

\[
<J, N|(J_+)^P|J, M \rangle = \frac{[J + N][J - M]}{[J - N][J + M]} \delta_{N,M+P} \tag{5.16}
\]

\[
<J, N|(J_-)^P|J, M \rangle = \frac{[J - N][J + M]}{[J + N][J - M]} \delta_{N,M-P}
\]

Recall further that the co-products of representations are defined by

\[
\Lambda(J)_\pm = J_\pm \otimes e^{i\hbar J_3} + e^{-i\hbar J_3} \otimes J_\pm, \quad \Lambda(J)_3 = J_3 \otimes 1 + 1 \otimes J_3 \tag{5.17}
\]

We will show now that the group-theoretical classical formulae of section 4 possess direct quantum equivalents, obtained by replacing \( sl(2) \) by \( U_q(sl(2)) \). We start from the general classical solution and consider Eq.\ref{4.1} as the classical tau function. The quantum tau function should then be given by a

\footnote{We assume for simplicity that \( \hbar/\pi \) is not rational.}
representation of type Eq. 4.1 of the operator \( e^{-J\alpha-\Phi} \); that is, the q tau function should actually be an operator instead of a function. Let us introduce the (rescaled) q-exponentials

\[
E_q^{(J)}(\eta(z), J_-) = \sum_{J+M=0}^{\infty} \eta_M^{(J)} (J_-)^{J+M} [J + M]!
\]

\[
\bar{E}_q^{(J)}(\bar{\eta}(\bar{z}), J_+) = \sum_{J+M=0}^{\infty} e^{ih(J+M)} (-1)^{J+M} \bar{\eta}_M^{(J)} (J_+)^{J+M} [J + M]!
\]

which we take to be the quantum equivalents of the classical (rescaled) exponentials \( E^{(j)}(s, j_-), \bar{E}^{(j)}(\bar{s}, j_+) \). Indeed, in the limit \( h \to 0 \), we see that \( E_q^{(J)}(\eta, J_-), \bar{E}_q^{(J)}(\bar{\eta}, J_+) \) reduce to \( E^{(j)}(s, j_-), \bar{E}^{(j)}(\bar{s}, j_+) \) if we identify classically

\[
\eta_M^{(J)} = s^{-J}(z)(\int z dz')^{J+M} = \eta^{(J)}_1(\eta^{(0)}_1)^{J+M} = \eta^{(J)}_0 \eta^{(0)}_1
\]

\[
\bar{\eta}_M^{(J)} = \bar{s}^{-J}(\bar{z})(\int \bar{z} d\bar{z}')^{J+M} = \bar{\eta}_M^{(J)}(\bar{\eta}^{(0)}_M)^{J+M} = \bar{\eta}_1^{(J)} \eta^{(0)}_M
\]

which corresponds to the assignment

\[
s(z) = \eta^{(-1)}(z) \quad \bar{s}(\bar{z}) = \bar{\eta}^{(-1)}(\bar{z})
\]

Furthermore we will show now, following closely the calculation used to derive Eq. 4.4 group-theoretically, that indeed the \( \bar{E}_q^{(J)} \) obey a composition law appropriate for q-deformed exponentials. The argument was inspired by ref. [25] (with an important difference- see below). In ref. [22], the fusion algebra of the \( \xi \) fields was determined using the general scheme of Moore and Seiberg. One has

\[
\xi^{(J_1)}_{M_1}(z_1) \xi^{(J_2)}_{M_2}(z_2) = \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} g_{J_{12}}^{J_1 J_2} (J_1, M_1; J_2, M_2 | J_{12}) \times
\]

\[
\sum_{\{\nu\}} \xi^{(J_{12}, \{\nu\})} (z_2) \ll \omega_{J_{12}}, \{\nu\} | V^{(J_1)}_{J_{12}}(z_1 - z_2) \omega_{J_2} \gg,
\]

where \( \{\nu\} \) is a multi-index that labels the descendants, and \( \omega_J, \{\nu\} \gg \) denotes the corresponding state in the Virasoro Verma-module. Similar equations hold for the \( \bar{\xi} \) fields. The explicit expression of the coupling constant \( g \)
is not needed in the present argument. The symbol \((J_1, M_1; J_2, M_2|J_{12})\) denotes the q-Clebsch-Gordan coefficients. It follows from their very definition (see, e.g. ref. [26]) that
\[
\sum_{M_1+M_2=M_{12}} (J_1, M_1; J_2, M_2|J_{12})|J_1, M_1 > \otimes |J_2, M_2 > = \frac{(\Lambda(J)_-)^{J_{12}-M_{12}}}{|J_{12} - M_{12}|! \sqrt{\binom{2J_{12}}{J_{12} - M_{12}}}}|J_{12}, J_{12} >.
\]
(5.22)
and one finds
\[
E^{(J_1)}_q(\eta(z_1), J_- \otimes 1) E^{(J_2)}_q(\eta(z_2), 1 \otimes J_-) (|J_1, J_1 > |J_2, J_2 >) = \sum_{J_{12}=|J_1 - J_2|} \sum_{\nu} g_{J_1 J_2}^{J_{12}} \times
\sum_{\nu} E^{(J_{12}, \nu)}_q(\eta(z_2), \Lambda(J)_-) |J_{12}, J_{12} > <\nu J_{12}, \nu V^{(J_{12})}_{J_2-J_{12}} (z_1 - z_2) |\nu J_2 >.
\]
(5.23)
In particular, to leading order one has
\[
E^{(J_1)}_q(\eta(z_1), J_- \otimes 1) E^{(J_2)}_q(\eta(z_2), 1 \otimes J_-) (|J_1, J_1 > |J_2, J_2 >) \sim \frac{\Lambda(J)_-}{J_{12} + J_1 + J_2 >}.
\]
(5.24)
which is the natural multiplication law for q-exponentials involving quantum group generators. The coproduct is non-symmetric between the two representations. On the left hand side this comes from the non-commutativity of the \(\eta\) fields as quantum field operators. Thus the present definition of q exponentials is conceptually different from the usual one where the group “parameters” are \(c\) numbers. Let us recall the latter for completeness. If one defines
\[
e_q(X) \equiv \sum_{r=0}^{\infty} \frac{X^r}{[r]!} e^{-ihr(r+1)/2},
\]
(5.25)
one has
\[
e_q(x, J_\pm \otimes e^{ihJ_3}) e_q(x e^{-ihJ_3} \otimes J_\pm) = e_q(x \Lambda(J)_\pm).
\]
(5.26)
Since they transform the $q$ sum of infinitesimal generators into products, the $q$ exponentials are the natural way to exponentiate $q$ Lie algebras. The last equation should be compared with Eq. 5.24. Now the non-symmetry of the co-product is taken care of by exponentiating non-commuting group elements $(J_+ \otimes e^{ihJ_3}, e^{-ihJ_3} \otimes J_+)$ on the left-hand side, with $x$ a number. This is in contrast with Eq. 5.24, where $J_+ \otimes 1$, and $1 \otimes J_+$ are used instead.

Finally, we can rewrite Eq. 5.12 under the form

$$e^{-J(\alpha - \Phi(z, \bar{z}))} = \langle J, J \mid e^{q(\bar{\eta}(\bar{z}), J_+)} e^{q(\eta(z), J_-)} \mid J, J > = \tau_q(\eta, \bar{\eta}).$$

(5.27)

Indeed we will see that it possesses the obvious $q$-analogues of the properties Eq. 4.6 and Eq. 4.7. Hence, Eq. 5.27 should be viewed as the quantum version of the Leznov-Saveliev formula Eq. 4.1. Since $\eta$ and $\bar{\eta}$ commute, we deduce that

$$e^{-J_1(\alpha - \Phi(z_1, \bar{z}_1))} e^{-J_2(\alpha - \Phi(z_2, \bar{z}_2))} =$$

$$\langle J_1, J_1 | < J_2, J_2 \rangle E_q^{(J_1)}(\bar{\eta}(\bar{z}_1), J_+ \otimes 1) E_q^{(J_2)}(\eta(z_2), 1 \otimes J_-) (\mid J_1, J_1 > | J_2, J_2 >).$$

(5.28)

Making use of the fusion algebra Eq. 5.21 together with its counterpart for the $\bar{\eta}$ fields gives back the fusion algebra of the Liouville exponentials derived in ref.[9]. To leading order one has

$$e^{-J_1(\alpha - \Phi(z_1, \bar{z}_1))} e^{-J_2(\alpha - \Phi(z_2, \bar{z}_2))} \sim e^{-(J_1 + J_2)\alpha - \Phi}.$$

(5.29)

Clearly these properties are natural generalizations of the classical features recalled in section 4.

5.4 The quantum Hirota equation

We retain the basic idea of the calculation which led to Eq. 4.9 or 4.10. Clearly, Eq. 4.10 should be replaced by a relation for the operator-product expansion. The OPE of the $J = 1/2$ Liouville exponential has the form

$$e^{-\frac{1}{2}\alpha - \Phi(z_1, \bar{z}_1)} e^{-\frac{1}{2}\alpha - \Phi(z_2, \bar{z}_2)} \sim [(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)]^{-h/2\pi} e^{-\alpha - \Phi(z_2, \bar{z}_2)}$$

$$+ [(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)]^{1+3h/2\pi} c_0 + \text{descendants},$$

(5.30)

\[^{13}\text{Related formulae are already given in ref. [24], p. 27.}\]
where \( c_0 \) is a constant that can be changed by a global shift of the Liouville field. The second Liouville exponential is equal to a constant, since its spin \( J \) is equal to zero. This property is the quantum equivalent of Eq.4.10. Thus, the quantum Hirota equation simply results from the fact that for \( J = 0 \), \( \exp(-J\alpha_\Phi) = \text{cst} \). One sees that choosing a particular combination of derivatives in a bilinear classical expression of \( J = 1/2 \) Liouville exponentials is replaced by picking up the spin zero term in the operator-product expansion of \( \exp(-(1/2)\alpha_\Phi) \) with itself. Note that due to the quantum effects, the difference of the powers of \((z_1 - z_2)(\bar{z}_1 - \bar{z}_2)\) between the first and second term is not equal to one — it is equal to \( 1 + 2\hbar \) — so that a simple antisymmetrization is not enough, as in the classical case, to remove the first term. Clearly, Eq.5.30 gives non-trivial equations relating the matrix elements of the quantum Liouville exponentials.

6 Conclusions

Starting from three manifestly different frameworks for the quantum Liouville theory, we have shown that, apart from mere notational differences, the Liouville exponentials obtained from them coincide up to a simple equivalence transformation. This implies in particular that in all cases the quantum structure is determined by the underlying \( U_q(sl(2)) \) symmetry, though this is manifest only in the GN-G approach. Note that in the work of GN-G, the \( U_q(sl(2)) \) symmetry was not used as an input to define the quantum construction, but arose as a consequence of the fundamental requirements of locality and conformal invariance. Together with the results obtained here this provides a very strong indication that the Liouville quantum theory is unique.

The frameworks of BCGT and OW were formulated only for the hyperbolic sector corresponding to regular solutions of the Liouville equation and real zero modes. However, their equivalence with the GN-G theory in this sector shows how they should be extended to the elliptic sector with imaginary zero modes, since the GN-G operators by construction apply simultaneously to both cases. As regards correlation functions, an equivalence of type Eq.1.1 implies, in principle, that the corresponding n-point functions should agree, possibly up to an overall n-independent factor related to the treatment of the end points[9]. At present, however, this is completely clear.
only in the case $J$ half-integer positive where the Liouville exponentials can be represented as finite sums over chiral primaries. In the general case of arbitrary $J$, the meaning of the infinite sum has yet to be clarified and the equivalence of Eqs.1.1 and 1.4 is nontrivial; this is the subject of ongoing investigations.

If the quantum Liouville theory is indeed unique, and governed by the $U_q(sl(2))$ quantum group symmetry, then it is natural to expect that it should be given by a q-deformation of the group-theoretic approach of Leznov and Saveliev to the classical theory. We have shown that in fact the quantum version of the Leznov-Saveliev formula can be obtained by a combination of standard q-deformation techniques and results of the quantum group analysis of [7]. It turns out that the fusion properties of the Liouville exponentials and their chiral constituents can be neatly described in terms of the multiplication law for field theoretic q-exponentials of a new type. We also presented an interpretation of the quantum Leznov-Saveliev representation in terms of q tau functions which satisfy a bilinear Hirota-type equation, for the simplest case $J = 1/2$. Defining the quantum equations of motion in this way, rather than by a direct quantization of the classical Liouville equation as in [2, 3] has the obvious advantage that only conformal objects are involved. On the other hand, one expects in general (see ref. [25], for instance) that the q-Hirota equations should take the form of difference equations, which, however, is not immediately evident from Eq.5.30. It would be desirable to have a better understanding of this point, and also to generalize the above picture to the case of arbitrary Toda theories.

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A appendix

Notations

| GN-G | BCGT | OW |
|------|------|----|

23
\[ c = 1 + \frac{6\pi}{h}(1 + \frac{h}{\pi})^2 \]
\[ c = 1 + \frac{2\pi}{g^2}(1 + \frac{g^2}{2\pi})^2 \]
\[ c = 1 + \frac{48\pi}{\gamma^2} \]
\[ h \equiv 2\pi\hbar \eta^2, \quad \hbar = \frac{3}{c-1} \]
\[ \eta = \eta_\pm = \frac{1}{4h}(1 - \sqrt{1 - 8h}) \]
\[ \zeta = \sqrt{\frac{2\sin g^2/2}{g}}, \quad 8m^2 = \mu^2 \]
\[ (\mu^2 = \text{cosm. constant}) \]
\( (A.1) \)

Liouville exponentials

\[ \begin{align*}
\text{GN-G} & : & \text{BCGT} & : & \text{OW} \\
\exp(-J\alpha_+ \Phi) & : & \exp(\beta \Phi) & : & \exp(\lambda \varphi) \\
(\alpha_+ = \sqrt{\frac{2\hbar}{\pi}}) & : & \Delta = \beta \left(1 + \frac{q^2}{2\pi}\right) - \frac{\beta^2}{8\pi} & : & \Delta = -\frac{1}{2}(\lambda \eta + \hbar \lambda^2 \eta^2) \\
\Delta = -J - \frac{h}{\pi} J(J+1) & : & \Phi_{BCGT} = \frac{\alpha}{2g} \Phi_G \\
\sqrt{g} = e^{\alpha_- \Phi} & : & \beta = -2gJ & : & \varphi = \alpha_- \Phi_G \\
\sqrt{g} = e^{2g \Phi} & : & \lambda = -J & : & \sqrt{\gamma} = e^\varphi \\
\end{align*} \]
\( (A.2) \)

Free fields

\[ \begin{align*}
\phi_j(u) &= q_0^{(j)} + p_0^{(j)} u + i \sum_{n \neq 0} e^{-inu} \frac{p_n^{(j)}}{n} \\
\tilde{\phi}_j(v) &= \tilde{q}_0^{(j)} + \tilde{p}_0^{(j)} v + i \sum_{n \neq 0} e^{-inu} \tilde{p}_n^{(j)} \frac{p_n^{(j)}}{n} \\
\end{align*} \]

\footnote{We use here the notation of the more recent papers of Gervais. The OW notation in fact agrees directly with the older GN notation.}
with
\[ u = \tau + \sigma, \quad \tau = \tau - \sigma, \quad j = 1, 2, \]
\[ [q^{(j)}, p^{(j)}] = i = [q^{(j)}, p^{(j)}], \quad [p^{(j)}_n, p^{(j)}_m] = n\delta_{n,-m} = [p^{(j)}_n, p^{(j)}_m] \]

It is often convenient to use the rescaled zero modes
\[ \varpi := \frac{i}{\sqrt{\hbar}} p^{(1)}_0 \quad \bar{\varpi} := \frac{i}{\sqrt{\hbar}} \bar{p}^{(1)}_0 \]

The above fields are related to the functions \( A(u), B(v) \) describing the general solution of the Liouville equation by
\[ \phi_1 = \frac{1}{\sqrt{\hbar}} \ln A^\prime - 1/2 \quad \phi_2 = \frac{1}{\sqrt{\hbar}} \ln AA^\prime - 1/2 \]
\[ \bar{\phi}_1 = -\frac{1}{\sqrt{\hbar}} \ln B^\prime - 1/2 \quad \bar{\phi}_2 = -\frac{1}{\sqrt{\hbar}} \ln BB^\prime - 1/2 \]

(A.3)

up to additive constants depending only on the zero modes \( p_0, \bar{p}_0 \).

BCGT :
\[ \psi(\tau, \sigma) = Q + P \frac{\tau}{2\pi} + \psi_L(u) + \psi_R(v), \quad \bar{\psi}(\tau, \sigma) = \psi_L(u) - \psi_R(v) \]

with
\[ \psi_L(u) = \frac{i}{\sqrt{4\pi}} \sum_{n\neq 0} \frac{A_n}{n} e^{-inu} \quad \psi_R(v) = -\frac{i}{\sqrt{4\pi}} \sum_{n\neq 0} \frac{B_n}{n} e^{-inv} \]
\[ [Q, P] = i \quad [A_n, A_m] = n\delta_{n,-m} = [B_n, B_m] \]

relation to GN-G :
\[ \psi_L(u) = -\frac{1}{\sqrt{4\pi}} \phi^{(osc)}_1(u) \quad \psi_R(v) = -\frac{1}{\sqrt{4\pi}} \bar{\phi}^{(osc)}_1(v) \]

\[ \text{We remind the reader of the remarks made below Eq.2.9 about the doubling of the zero modes in the GN-G notation} \]
\[ P = -\sqrt{4\pi} p_0^{(1)} \quad Q = -\frac{1}{\sqrt{4\pi}} q_0^{(1)} \quad A_n = -p_n^{(1)} \quad B_n = -\bar{p}_n^{(1)} \quad (n \neq 0) \]

\[ (\phi_1^{(osc)} \equiv \text{oscillator part of } \phi_1) \]
\[ \psi(\tau, \sigma) \text{ is related to the functions } A(u) \text{ and } B(v) \text{ by} \]
\[ 2g\psi(\tau, \sigma) = \ln A'(u) - \ln B'(v) \]

(A.4)

**OW:**

\[ \psi(\tau, \sigma) = \psi^+ (u) + \psi^- (v) \]

with

\[ \psi^+ (u) = \gamma \left( \frac{1}{2} Q + \frac{1}{4\pi} P u + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{a_n^+}{n} e^{-inu} \right) \]

\[ \psi^- (v) = \gamma \left( \frac{1}{2} Q + \frac{1}{4\pi} P v + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{a_n^-}{n} e^{-inv} \right) \]

\[ [Q, P] = i \quad [a_n^+, a_m^+] = n\delta_{n,-m} = [a_n^-, a_m^-] \]

relation to GN-G :

\[ \psi^{+(osc)} (u) = -2\sqrt{\hbar} \phi_1^{(osc)} (u) \quad \psi^{-(osc)} (u) = +2\sqrt{\hbar} \phi_2^{(osc)} (v) \]

\[ P = -\sqrt{4\pi} p_0^{(1)} \quad Q = -\frac{1}{\sqrt{4\pi}} q_0^{(1)} \quad a_n^+ = -p_n^{(1)} \quad a_n^- = +\bar{p}_n^{(2)} \quad (n \neq 0) \]

It is convenient to use the abbreviations \[\hat{P}, \hat{Q}\]

\[ \hat{P} \equiv \sqrt{\pi} \hbar \eta P = -2\pi \sqrt{\hbar} \eta p_0^{(1)} , \quad \hat{Q} \equiv 4\sqrt{\pi} \hbar \eta Q = -2\sqrt{\hbar} q_0^{(1)} , \]

\[ [\hat{Q}, \hat{P}] = 2i\hbar \]

\[ \psi(\tau, \sigma) \text{ is related to the functions } A(u) \text{ and } B(v) \text{ by} \]

\[ \psi(\tau, \sigma) = \ln A'(u) + \ln \frac{B'(v)}{B^2} \]
Free field exponentials

\begin{align*}
V_{-1/2}^{(1/2)}(u) &= e^{+\sqrt{\hbar/2}\phi_1(u)} & & e^{-g\psi_L} \quad e^{\eta\psi^+/2} \\
V_{+1/2}^{(1/2)}(u) &= e^{-\sqrt{\hbar/2}\phi_2(u)} & & / \\
\bar{V}_{-1/2}^{(1/2)}(v) &= e^{-\sqrt{\hbar/2}\phi_1(\bar{v})} & & / \quad e^{-\eta\psi^-/2} \\
\bar{V}_{+1/2}^{(1/2)}(v) &= e^{\sqrt{\hbar/2}\phi_1(v)} & & e^g\psi_R \\
V_{-1}^{(-1)}(u) &= e^{-2\sqrt{\hbar/2}\phi_1(u)} & & e^{2g\psi_L} \quad e^{\eta\psi^+} \\
V_{+1}^{(-1)}(u) &= e^{-2\sqrt{\hbar/2}\phi_2(u)} & & / \\
\bar{V}_{+1}^{(-1)}(v) &= e^{+2\sqrt{\hbar/2}\phi_2(v)} & & / \quad e^{\eta\psi^-} \\
\bar{V}_{-1}^{(-1)}(v) &= e^{+2\sqrt{\hbar/2}\phi_1(v)} & & e^{-2g\psi_R} \\
\end{align*}

The above correspondences are identities for the oscillator parts of the operators; the zero mode dependences are discussed in the text. Furthermore, we remark that we can replace everywhere \( \phi_1 \) by \( \phi_2 \) since this is nothing but a particular \( \text{SL}(2) \) transformation (corresponding to the exchange \( A \to -1/A, \ B \to -1/B \) in the classical solution). The particular assignment we have chosen above in comparing the free fields resp. free field exponentials is thus to this extent a matter of convention.

Finally, we note that the relation between the fields \( V_\lambda^{(-\lambda)}S^n \) and the normalized operators \( V^{(J)}_m, \ J = -\lambda, \ m = n - J \) is given by

\[ V_\lambda^{(-\lambda)}S^n = I_m^{(J)}(\omega)V_m^{(J)} \]

with

\[ I_m^{(J)}(\omega) = i^n \prod_{l=1}^n \left\{ e^{i\pi\beta(l-1)}(1 - e^{2\pi i(\gamma + \beta(l-1))}) \right\} \times \prod_{l=1}^n \left\{ \frac{\Gamma(1-\beta)\Gamma(1+\gamma+(l-1)\beta)\Gamma(1+\alpha+(l-1)\beta)}{\Gamma(1-l\beta)\Gamma(2+\gamma+\alpha+(n-2+l)\beta)} \right\} \]
\[ \alpha = 2 J h, \quad \beta = -\frac{h}{\pi}, \quad \gamma = \frac{h}{\pi} (\varpi + 2m - 1) - 1. \]  

(A.7)

A completely analogous relation connects \( \tilde{V}^{(-\lambda)} \tilde{S}^{n} \) with \( \tilde{V}^{(J)} \tilde{S}^{n} \), the only change being that \( \varpi \) is to be replaced by \( \varpi \) and the imaginary unit \( i \) by \( -i \).

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