A GENERALIZATION OF LIFTING NON-PROPER TROPICAL INTERSECTIONS

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ABSTRACT. Let $X$ and $X'$ be closed subschemes of an algebraic torus $T$ over a non-archimedean field. We prove the rational equivalence as tropical cycles in the sense of [Mey11] between the tropicalization of the intersection product $X \cdot X'$ and the stable intersection $\text{trop}(X) \cdot \text{trop}(X')$, when restricted to (the inverse image under the tropicalization map of) a connected component $C$ of $\text{trop}(X) \cap \text{trop}(X')$. This requires possibly passing to a (partial) compactification of $T$ with respect to a suitable fan. We define the compactified stable intersection in a toric tropical variety, and check that this definition is compatible with the intersection product in loc.cit.. As a result we get a numerical equivalence between $\tilde{X} \cdot \tilde{X}'$ and $\text{trop}(X) \cdot \text{trop}(X')$ via the compactified stable intersection, where the closures are taken inside the compactifications of $T$ and $\mathbb{R}^n$. In particular, when $X$ and $X'$ have complementary codimensions, this equivalence generalizes [OR11, Theorem 6.4], in the sense that $X \cap X'$ is allowed to be of positive dimension. Moreover, if $X \cap X'$ has finitely many points which tropicalize to $\tilde{C}$, we prove a similar equation as in [OR11, Theorem 6.4] when the ambient space is a reduced subscheme of $T$ (instead of $T$ itself).

Keywords. Stable intersections, compactification, refined Gysin homomorphism.

1. INTRODUCTION

Let $K$ be an algebraically closed field with a valuation $\text{val}: K \rightarrow \mathbb{R}$. Let $T$ be an algebraic torus of dimension $n$ over $K$. There is a tropicalization map $\text{trop}: T \rightarrow \mathbb{R}^n$ defined by taking the valuation of every coordinate. Under this map, the image of a pure dimensional subscheme $X$ of $T$ is a balanced polyhedral complex of the same dimension, which is denoted by $\text{trop}(X)$. It is natural to consider under what conditions does the intersection commute with tropicalization, namely, given subschemes $X, X' \subseteq T$ when do we have $\text{trop}(X \cap X') = \text{trop}(X) \cap \text{trop}(X')$. This reduces to a lifting problem since we always have $\text{trop}(X \cap X') \subseteq \text{trop}(X) \cap \text{trop}(X')$. Results in this direction have been applied in [CJ15], studying the connection between the theta characteristics of a $K_4$-curve and the theta characteristics of its minimal skeleton; and in [CP12], showing that the tropicalization of an irreducible subvariety of an algebraic torus is connected through codimension one; and in [CJP15], discussing the lifting of divisors on a chain of loops as the skeleton of a smooth projective curve, etc.

When $\text{trop}(X)$ intersects $\text{trop}(X')$ properly this problem is studied thoroughly by Osserman and Payne in [OP13]. They proved that $\text{trop}(X \cap X') = \text{trop}(X) \cap \text{trop}(X')$, which generalizes a well-known result [BJS+07, Lemma 3.2] concerning the lifting when $\text{trop}(X)$ and $\text{trop}(X')$ intersect transversely. Moreover, they gave a lifting formula for the intersection multiplicity of $\text{trop}(X) \cdot \text{trop}(X')$ along a maximal face of $\text{trop}(X) \cap \text{trop}(X')$, where the ambient space is a closed subscheme of $T$ (instead of $T$ itself). See [OP13, §5] for details.

The commutativity does not hold when $\text{trop}(X) \cap \text{trop}(X')$ is nonproper. For example one can take hyperplanes $X = \{x = 1\}$ and $X' = \{x = 1 + a\}$ where $\text{val}(a) > 0$, then $X$ and $X'$ have empty intersection and same tropicalizations. However, one can still ask about the connections between the intersection cycles $X \cdot X'$ and $\text{trop}(X) \cdot \text{trop}(X')$. As an example, assume
$K$ is nonarchimedean, Morrison [Mor15] proved that when $X$ and $X'$ are plane curves that intersect properly, the tropicalization of the intersection cycle $\text{trop}(X \cdot X')$ is rationally equivalent to $\text{trop}(X) \cdot \text{trop}(X')$ as divisors on the (possibly degenerated) tropical curve $\text{trop}(X) \cap \text{trop}(X')$. In the higher dimensional case, Osserman and Rabinoff proved in [OR11] Theorem 6.4] that when $X$ and $X'$ are of complementary codimension, the number of points of $X \cap X'$, after a suitable compactification of the torus, that tropicalize to (the closure in the corresponding compactification of $\mathbb{R}^n$ of) a connected component $C$ of $\text{trop}(X) \cap \text{trop}(X')$ is the same as the number of points in $\text{trop}(X) \cdot \text{trop}(X')$ supported on $C$, where both numbers are assumed to be finite and are counted with multiplicities.

In this paper we generalize the result of [OR11] in several directions. First assume $X$ and $X'$ do not necessarily intersect properly, hence the intersection multiplicity is not well defined. Instead of counting points of their intersection, we look at the refined intersection product $\overline{X} \cdot \overline{X}'$ on $\overline{X} \cap \overline{X}'$, which as a cycle class is represented by a formal sum of points supported on $\overline{X} \cap \overline{X}'$ (see Example 4.3), where closures are taken inside the toric variety $X(\Delta)$ associated to a certain unimodular fan $\Delta$ (hence $X(\Delta)$ is smooth). Restricting to the closure of a component of $\text{trop}(X) \cap \text{trop}(X')$ in the corresponding compactification, denoted by $N_{\mathbb{R}}(\Delta)$, of $\mathbb{R}^n$ we have:

**Theorem 1.1.** Let $X$ and $X'$ be closed subschemes of $T$ of complementary codimensions, $C$ a connected component of $\text{trop}(X) \cap \text{trop}(X')$. Then there exists a fan $\Delta$ such that the degree of the subset of $\overline{X} \cdot \overline{X}'$ that tropicalizes to $\overline{C}$ is the same as that of $\text{trop}(X) \cdot \text{trop}(X')$ supported on $C$.

In particular we are allowed to consider self-intersections of subschemes of $T$ (see Example 4.8), which is not mentioned in [OR11]. The idea of proof is similar to [OR11] Theorem 6.4], namely, we show that the intersection cycle $\overline{X} \cdot \overline{X}'$ can be approached by the intersection of $\overline{X}$ and a perturbation of $\overline{X}'$, when restricted to a neighborhood of $\overline{C}$. Here by perturbation we mean $t \overline{X}'$ for some $t \in T$ with $\text{val}(x_i(t))$ small enough. The argument requires passing to nonarchimedean analytic spaces. This case will be discussed in Section 4, see Theorem 4.3. Moreover, a sufficient condition for the fan $\Delta$ will be given.

Theorem 1.1 is easily generalized to multiple intersections (see Theorem 4.8) which plays an important role in our next approach of generalization, namely testing higher dimensional intersections. Let $\iota_c: \overline{X} \cap \overline{X}' \to \overline{X} \cap \overline{X}'$ be the inclusion of the union of irreducible components of $\overline{X} \cap \overline{X}'$ that tropicalize to $\overline{C}$; note that this is an open and closed subset inclusion. Assuming $\dim(X) + \dim(X')$ is greater than or equal to $n$, we prove that, after restricting to $\overline{C}$, we have $\text{trop}(\overline{X} \cdot \overline{X}')$ rationally equivalent to the closure of $\text{trop}(X) \cdot \text{trop}(X')$ as tropical cycles in $N_{\mathbb{R}}(\Delta)$. Specifically, we have:

**Theorem 1.2.** Let $X$ and $X'$ be closed subschemes of $T$ of pure dimensions $k$ and $l$, and $C$ a connected component of $\text{trop}(X) \cap \text{trop}(X')$. Then there is a family of fans $\Delta$ such that

$$[\text{trop}(\iota_c(\overline{X} \cdot \overline{X}'))] = [\text{trop}(X) \cdot \text{trop}(X')|_{\overline{C}}] \in A_{k+l-n}(N_{\mathbb{R}}(\Delta)).$$

See Section 2.3 or [Mey11] §2 for the definitions of tropical cycles and rational equivalence on $N_{\mathbb{R}}(\Delta)$. In particular, rational equivalence preserves the degrees of zero cycles. In Section 3 we develop a compactified stable intersection “$\cdot_c$” on $N_{\mathbb{R}}(\Delta)$ of two tropical cycles of certain type which extends the stable intersection on $N_{\mathbb{R}}$. We check that this compactified stable intersection is compatible with the intersection product defined in loc.cit., which we denote by “$\cdot$”, and as a result we have:

**Theorem 1.3.** Let $X$ and $X'$ be as in Theorem 1.2. For a certain family of tropical cycles $F$ in $\mathbb{R}^n$ we have

$$\deg(\text{trop}(\iota_c(\overline{X} \cdot \overline{X}')) \cdot_c F) = \deg(\text{trop}(X) \cdot \text{trop}(X')|_{\overline{C} \cdot_c F}).$$
Theorem 1.2 will be restated and proved as Theorem 5.7 and Theorem 1.3 as Corollary 5.9 where the notations are explained. The main idea of the proof of Theorem 1.2 is, knowing that there is a natural isomorphism (induced by the tropicalization map) between the Chow rings of $X(\Delta)$ and $\mathbb{N}(\Delta)$ through Minkowski weights on $\Delta$ (see Lemma 5.6), one actually only need to check the equality (between degrees) above. This can be accomplished by using the “multiple intersection” version of Theorem 1.1 since we can restrict to cycles which is a sum of products of the tropicalizations of hypersurfaces in $X(\Delta)$.

Note that Theorem 1.2 generalizes Theorem 1.1 except that the fan condition is more restrictive, and it is different from just saying that $\text{trop}(X) \cap \text{trop}(X')$ only has one component, see Remark 4.5 and Example 4.6.

In a complementary direction, motivated by the work of Osserman and Payne, in Section 6 we generalize [OR11] Theorem 6.4 in the case of an ambient space which is a reduced closed subscheme of $T$:

**Theorem 1.4.** Let $Y$ be a reduced closed subscheme of $T$ and $X$ and $X'$ be subschemes of $Y$ of complementary codimension, let $C$ be a connected component of $\text{trop}(X) \cap \text{trop}(X')$ which is contained in the relative interior of a maximal face $\iota$ of $\text{trop}(Y)$ of multiplicity one. For a certain family of fans $\Delta$, if there are only finitely many points of $X \cap X'$ that tropicalize to $C$ then we have:

$$\sum_{x \in Z} i(x, X \cap X'; Y) = \sum_{u \in C} i(u, \text{trop}(X) \cap \text{trop}(X'); \text{trop}(Y)).$$

See below for explanations of the notations. In the proof of Theorem 1.4 we consider the analyticification of $Y$. Assuming $Y$ is of dimension $d$, the key point is that locally at a point that tropicalizes to a simple point in $\iota$ we have that $Y_{\text{an}}$ is isomorphic to the analytic torus of dimension $d$. Hence one can essentially replace $Y$ by an algebraic torus, and all the informations that “lie in” $\iota$ will be preserved. Therefore the theorem becomes a corollary of [OR11] Theorem 6.4. See Theorem 6.3 for details.

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**Notations.** In the sequel we fix an algebraically closed non-archimedean field $K$ with nontrivial valuation group $G = \text{val}(K)$. Let $N$ be a lattice of dimension $n$. Let $T_N$ be the algebraic torus of dimension $n$ whose lattice of characters, usually denoted by $\mathcal{M}$, is dual to $N$. Denote $N_\mathbb{R} = N \otimes \mathbb{R}$. We always identify $N_\mathbb{R}$ with $\mathbb{R}^n$ when $N$ is specified. For a polyhedron (resp. polyhedral complex) $P$ we denote the recession cone (resp. recession fan) of $P$ by $\rho(P)$, and the relative interior by $\text{relint}(P)$. Denote $P(m)$ the set of faces of $P$ of dimension $m$.

Let $\Sigma$ be a fan in $N_\mathbb{R}$ and $\tau \in \Sigma$. We denote $\text{Star}(\tau) = \{\sigma \subset \mathbb{R}^n/\mathbb{R}\tau \mid \sigma \prec \tau \in \Sigma\}$ as a fan in $\mathbb{R}^n/\mathbb{R}\tau$. We also set $\text{Star}(\tau, \Sigma)$ a fan in $\mathbb{R}^n$ whose cones are of the form $\hat{\sigma} = \{\lambda(x - y) \mid \lambda \geq 0, x \in \sigma, y \in \tau\}$ for all $\sigma \in \Sigma$ which contains $\tau$ as a face.

Let $X$ and $X'$ be closed subschemes of $Y$ such that $\dim(X) + \dim(X') = \dim(Y)$. We denote $i(x, X \cap X'; Y)$ the intersection multiplicity of $X$ and $X'$ at an isolated point $x$ of $X \cap X'$ at which $Y$ is smooth. In addition $Y$ is a closed subscheme of an algebraic torus, we denote $i(u, \text{trop}(X) \cap \text{trop}(X'); \text{trop}(Y))$ the multiplicity of $\text{trop}(X)_u \cdot \text{trop}(X')_u$ at the origin in $\text{trop}(Y)_u$ for $u \in \text{trop}(X) \cap \text{trop}(X')$, where $\text{trop}(X)_u$ is the star of $\text{trop}(X)$ at $u$, constructed
by translating \( \text{trop}(X) \) so that \( u \) is at the origin and taking the cones spanned by faces of \( \text{trop}(X) \) that contain \( u \). We usually omit \( \text{trop}(Y) \) in \( i(u, \text{trop}(X) \cdot \text{trop}(X'); \text{trop}(Y)) \) when \( Y = T_N \) is a given torus.

Let \( X \) be a scheme of finite type over \( K \), and \( Z \) is a union of connected components of \( X \) such that \( Z \) is proper, and \( \alpha \) a cycle class on \( X \) of dimension zero. We denote \( A_k(X) \) the \( k \)-th Chow group of \( X \), denote \( \int_Z \alpha \) the degree of the restriction of \( \alpha \) on \( Z \).

2. Preliminaries

In this section we recall some facts which are useful for later arguments. Note that most facts works for more general fields \( K \).

2.1. Refined intersection product. Let \( i_X : X \to Y \) be a regular imbedding of codimension \( d \) of schemes of finite type over \( K \), and \( f : Y' \to Y \) be a morphism. For any fiber square:

\[
\begin{array}{ccc}
X' & \xrightarrow{i_{X'}} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{i_X} & Y,
\end{array}
\]

according to [Ful98] there is a well-defined pull back map \( i_X^! : A_k(Y') \to A_{k-d}(X') \), called the refined Gysin homomorphism. Note that if \( i_X' \) is also a regular embedding of codimension \( d \) and we have fiber squares:

\[
\begin{array}{ccc}
X'' & \xrightarrow{i_{X''}} & Y'' \\
\downarrow & & \downarrow \\
X' & \xrightarrow{i_X'} & Y',
\end{array}
\quad
\begin{array}{ccc}
X'' & \xrightarrow{i_{X''}} & Y'' \\
\downarrow & & \downarrow \\
X & \xrightarrow{i_X} & Y,
\end{array}
\]

then the excess intersection formula implies that \( i_X^! = i_X'^! : A_k(Y') \to A_{k-d}(X') \).

Let \( Y \) be a smooth variety. Then the diagonal embedding \( \delta : Y \to Y \times Y \) is a regular embedding of codimension \( \dim(Y) \). Let \( X \) and \( X' \) be subvarieties of \( Y \). The refined intersection product \( X \cdot X' \) is then defined as \( \delta^!([X \times X']) \in A_*(X \cap X') \) with respect to the following square:

\[
\begin{array}{ccc}
X' \cap X & \xrightarrow{} & X \times X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\delta} & Y \times Y.
\end{array}
\]

If \( i_X : X \to Y \) is a regular embedding, then \( X \cdot X' = i_X^![X'] \).

Now consider families of cycle classes. Let \( Z \) be an irreducible variety of dimension \( m \), let \( t \in Z \) be a regular closed point, which implies that the inclusion \( i_t : t \hookrightarrow Z \) is a regular embedding of codimension \( m \). Given a morphism \( p : Y \to Z \) and a \((k+m)\)-cycle \( \alpha \) on \( Y \), we get a family of \( k \)-cycle classes \( i_t^!([\alpha]) \in A_k(Y_t) \) for all \( t \in Z \), where \( i_t^! : A_{k+m}(Y) \to A_k(Y_t) \) is the refined Gysin homomorphism defined by the following fibre square.

\[
\begin{array}{ccc}
Y_t & \xrightarrow{p_t} & Y \\
\downarrow & & \downarrow \\
t & \xrightarrow{i_t} & Z.
\end{array}
\]
Note that by the construction of refined Gysin homomorphism, we actually get a cycle class in $A_k(\{\alpha\})$ where $\{\alpha\}$ is the support of $\alpha$.

**Lemma 2.1.** Let $Z$ be a non-singular variety of dimension $m$, assume $t \in Z$ is rational over the ground field, and $Y$ is smooth over $Z$ of relative dimension $n$. If $\alpha \in A_{k+m}(Y)$ and $\beta \in A_{l+m}(Y)$ then

$$i^*_t(\alpha) \cdot i^*_t(\beta) = i^*_t(\alpha \cdot \beta)$$

in $A_{k+l-n}(\{\alpha\} \cap \{\beta\})$.

**Proof.** A similar result is proved in [Ful98, Corollary 10.1] where both sides of the equation are in $A_{k+l-n}(Y_t)$, the same argument works in $A_{k+l-n}(\{\alpha\} \cap \{\beta\})$. □

2.2. Divisors on toric varieties. Let $X(\Sigma)$ be the toric variety associated to a fan $\Sigma$ in $N_\mathbb{R}$. The set of $T_N$-invariant divisors on $X(\Sigma)$ has an explicit description by the combinatorial informations of $\Sigma$. In the following we list some useful conclusions for later arguments, for more details see [CLST11] §4.8.6. Denote the group of $T_N$-invariant Weil divisors on $X(\Sigma)$ by $\text{Div}_{T_N}(X(\Sigma))$.

Any ray $\rho \in \Sigma(1)$ gives a codimension one orbit $O(\rho)$ whose closure is a $T_N$-invariant prime divisor on $X(\Sigma)$. We denote this divisor by $D_\rho$. Let $u_\rho$ be the minimal lattice generator of $\rho$. Recall that $M$ is the lattice of characters of $T_N$. We then have:

**Proposition 2.2.** (1) Divisors of the form $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ are precisely the divisors that is invariant under the torus action on $X(\Sigma)$. In other words we have:

$$\text{Div}_{T_N}(X(\Sigma)) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \subseteq \text{Div}(X(\Sigma)).$$

(2) For $m \in M$, the corresponding character $\chi^m$ is a rational function on $X(\Sigma)$ and the associated divisor is given by:

$$\text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

(3) We have an exact sequence:

$$0 \to M \to \text{Div}_{T_N}(X(\Sigma)) \to \text{Cl}(X(\Sigma)) \to 0$$

where the first map is $m \to \text{div}(\chi^m)$. In particular every Weil divisor is rationally equivalent to a $T_N$-invariant divisor.

(4) The space of global sections of $\mathcal{O}(D)$ is given by

$$\Gamma(X(\Sigma), \mathcal{O}(D)) = \bigoplus_{m \in P_D \cap M} K \cdot \chi^m$$

where $P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}$.

(5) The divisor $D$ is Cartier if and only if for each $\sigma \in \Sigma$ there is a $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$.

(6) Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a cartier divisor on $X(\Sigma)$, let $m_\sigma$ be as in (5). Then $D$ is ample if and only if $\langle m_\sigma, u_\rho \rangle > -a_\rho$ for all $\sigma \in \Sigma(n)$ and $\rho \in \Sigma(1) \setminus \sigma(1)$.

On the other hand, for a full dimensional lattice polytope $P$, we denote by $X_P$ the toric variety of the normal fan of $P$. Let $P_1$ be the set of full dimensional lattice polytopes and $P_2$ be the set of pairs $(X(\Sigma), D)$ such that $\Sigma$ is a complete fan and $D$ is a torus-invariant ample divisor on $X(\Sigma)$. We then have the following relation:
Theorem 2.3. There is an one-to-one correspondence between $P_1$ and $P_2$ given by maps $P \mapsto (X_P, D_P)$ and $(X(\Sigma), D) \mapsto P_D$ that are inverses to each other. In particular, if $D$ is an ample divisor on $X(\Sigma)$ then $\Sigma$ is the normal fan of $P_D$.

Here $P_D$ is as in Proposition 2.2.4. We won’t use the definition of $D_P$ in the sequel, for more details see [CLS11] §6.1.

2.3. Tropical varieties and the extended tropicalization map. Let $\Delta$ be a unimodular fan in $N_\mathbb{R}$. Associated to $\Delta$ there is a tropical variety $N_\mathbb{R}(\Delta) = \bigcup_{\tau \in \Delta} N_\mathbb{R}(\tau)$ with topology induced from natural gluing, where $N_\mathbb{R}(\tau) = \text{Hom}_{\mathbb{R}_{\geq 0}}(\tau^\vee, \mathbb{R} \cup \{+\infty\})$. The space $N_\mathbb{R}(\Delta)$ satisfies the following conditions:

1. $N_\mathbb{R}(\Delta)$ contains $N_\mathbb{R} = N_\mathbb{R}({\{0\}})$ as an open dense subset and the addition on $N_\mathbb{R}$ extends to an action of $N_\mathbb{R}$ on $N_\mathbb{R}(\Delta)$, of which the orbits are of the form $\text{Hom}_{\mathbb{R}_{\geq 0}}(\tau^\perp, \mathbb{R}) = N_\mathbb{R}/\mathbb{R}\tau$. This gives $N_\mathbb{R}(\Delta)$ a stratification by affine linear spaces: $N_\mathbb{R}(\Delta) = \bigsqcup_{\tau \in \Delta} N_\mathbb{R}/\mathbb{R}\tau$. We denote the orbits by $O(\tau)$ for convenience.

2. Let $x \in N_\mathbb{R}$ be a finite point, and $v \in N_\mathbb{R}$ be a direction vector. Then $x + \lambda v$ converges for $\lambda \rightarrow +\infty$ to a point $x_\tau \in N_\mathbb{R}/\mathbb{R}\tau$ precisely when $v \in |\Delta|$, and $\tau$ is the unique face which contains $v$ as an (relative) interior point, in which case $x_\tau$ is the image of $x$ under the projection $\pi_\tau: \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{R}\tau$.

3. Let $X(\Delta)$ be the toric variety associated to $\Delta$. The tropicalization map on $T_N$ extends naturally to trop: $X(\tau) \rightarrow N_\mathbb{R}(\tau)$ for $\tau \in \Delta$ by the formula $\langle u, \text{trop}(\xi) \rangle = \text{val}(x^u(\xi))$ (note that when $\tau = \{0\}$ we get the original tropicalization map trop: $T_N \rightarrow N_\mathbb{R}$ which is just taking the valuation of each coordinate). This is compatible with the orbits stratifications of $X(\Delta)$ and $N_\mathbb{R}(\Delta)$, namely we have $O(\tau) = O(\sigma)$ where $O(\tau)$ denotes the torus orbit of $X(\Delta)$ that corresponds to $\tau$. In particular we have the extension trop: $X(\Delta) \rightarrow N_\mathbb{R}(\Delta)$. Note that for a subscheme $X$ of $T_N$ we have trop$(X) = \text{trop}(X)$ where the closures are taken in $N_\mathbb{R}(\Delta)$ and $X(\Delta)$ respectively.

Example 2.4. Given $e_1, \ldots, e_n$ a basis of $N$, and $e'_1, \ldots, e'_n \in M$ the dual basis. Let $\tau = \mathbb{R}_{\geq 0} e_1 + \cdots + \mathbb{R}_{\geq 0} e_n$. Then $\tau^\vee = \mathbb{R}_{\geq 0} e'_1 + \cdots + \mathbb{R}_{\geq 0} e'_n$, hence $X(\tau) = \mathbb{A}^n$ and $N_\mathbb{R}(\tau) = (\mathbb{R} \cup \{+\infty\})^n$, as in the following diagram when $n = 2$.

\[
\begin{array}{ccc}
(0, +\infty) & \rightarrow & (+\infty, +\infty) \\
\downarrow & & \downarrow \\
O & \rightarrow & (+\infty, 0)
\end{array}
\]

For $(x_1, \ldots, x_n) \in \mathbb{A}^n$ we have trop$(x_1, \ldots, x_n) = (\text{val}(x_1), \ldots, \text{val}(x_n))$, where we set \text{val}(0) = +\infty.

For future reference we also denote $V(\tau)$ and $V(\tau) = \text{trop}(V(\tau))$ the closed orbits correspond to $\tau$ in $X(\Delta)$ and $N_\mathbb{R}(\Delta)$ respectively.

2.4. Miscellaneous. We next list some basic definitions of [ORTT] which will be frequently used later.
Definition 2.5. Let $P$ be a finite collection of polyhedra in $N_{R}$ and let $\Delta$ be a pointed fan:

1. The fan $\Delta$ is said to be compatible with $P$ provided that, for all $P \in P$ and all cones $\sigma \in \Delta$, either $\sigma \subseteq \rho(P)$ or $\text{relint}(\sigma) \cap \rho(P) = \emptyset$.

2. The fan $\Delta$ is said to be a compactifying fan for $P$ provided that, for all $P \in P$ the recession cone $\rho(P)$ is a union of cones in $\Delta$.

Definition 2.6. Let $P = \bigcap_{i=1}^{r} \{ v \in N_{R} \mid (v, u_{i}) \leq a_{i} \}$, where $u_{i} \in M$ and $a_{i} \in G$, be an integral $G$-affine polyhedron in $N_{R}$. A thickening of $P$ is a polyhedron of the form $P' = \bigcap_{i=1}^{r} \{ v \in N_{R} \mid (v, u_{i}) \leq a_{i} + \epsilon \}$ for some $\epsilon > 0$ in $G$. If $P$ is a finite collection of integral $G$-affine polyhedra, a thickening of $P$ is a collection of (integral $G$-affine) polyhedra of the form $P' = \{ P' \mid P \in P \}$ where $P'$ denotes a thickening of $P$.

Note that if $P'$ is a thickening of $P$ then $\rho(P) = \rho(P')$.

Definition 2.7. Let $\Delta$ be an integral fan and let $P$ be a finite union of integral $G$-affine polyhedra. A refinement of $P$ is a finite collection of integral $G$-affine polyhedra $P'$ such that every polyhedron of $P'$ is contained in some polyhedron of $P$, and every polyhedron of $P$ is a union of polyhedra in $P'$. A $\Delta$-decomposition of $P$ is a refinement $P'$ of $P$ such that $\rho(P) \in \Delta$ for all $P \in P'$. A $\Delta$-thickening of $P$ is a thickening of a $\Delta$-decomposition of $P$.

Note that if $P'$ is a $\Delta$-thickening of $P$ then $|P| \subseteq |P'|$, where the closures are taken inside $N_{R}(\Delta)$.

Definition 2.8. Let $X$ and $X'$ be closed subschemes of $T_{N}$, let $C$ be a connected component of trop$(X') \cap \text{trop}(X')$.

1. A compactifying datum for $X, X'$ and $C$ consists of a pair $(\Delta, P)$, where $P$ is a finite collection of integral $G$-affine polyhedra in $N_{R}$ such that trop$(X') \cap \text{trop}(X') \cap |P| = |C|$ and $\Delta$ is an integral compactifying fan for $P$ which is compatible with trop$(X)' \cap P$.

2. ([OR11] Lemma 4.7) If in addition we have codim$(X) + \text{codim}(X') = n$ then there exists a $\Delta$-thickening $P'$ of $P$, a number $\epsilon > 0$ and a cocharacter $\upsilon \in N$ such that: (a) $(\Delta, P')$ is a compactifying datum for $X, X'$ and $C$. (b) For all $r \in [-\epsilon, 0) \cup (0, \epsilon]$ the set $(\text{trop}(X) + r \cdot \upsilon) \cap \text{trop}(X') \cap |P'|$ is finite and contained in $|P'|$, and each point lies in the interior of facets of trop$(X) + r \cdot \upsilon$ and trop$(X')$. We call the tuple $(P', \epsilon, \upsilon)$ a tropical moving data.

3. TROPICAL INTERSECTION THEORY

Let $\Delta$ be a complete unimodular fan. In this section we define the stable intersection of two tropical cycles in the tropical variety $\mathcal{X} = N_{R}(\Delta)$, where one of them is compatible with $\Delta$ (or equivalently $\Delta$ is a compactifying fan of one of them). We then prove that our definition of stable intersection is compatible with the intersection product defined in [Mey11] (up to rational equivalence).

3.1. Tropical Intersection Product. We first recall some concepts from [Mey11], §2] about intersection theory on $\mathcal{X}$. For a treatment of the intersection theory on $N_{R}$ (i.e. $\Delta = \{0\}$) where the intersection product agrees with the stable intersection see [AR10] and [AHR14].

\footnote{Note that the compatibility is not symmetric. Fix a fan $\Delta$, we usually say “$P$ is compatible with $\Delta$” but mean that “$\Delta$ is compatible with $P$”. This should be clear when $P$ is not necessarily a fan.}
Definition 3.1. Let $\mathcal{X} = N_\mathbb{R} (\Delta)$ be a tropical variety. A $k$-cycle on $\mathcal{X}$ is a collection $\overline{\alpha} = (\overline{\alpha}_\tau)_{\tau \in \Delta}$ where each $\overline{\alpha}_\tau$ is the closure of a (formal sum of) balanced weighted polyhedral complex $\alpha_\tau \subset O(\tau)$ of dimension $k$.

Definition 3.2. A tropical rational function on $N_\mathbb{R}$ is a continuous piecewise linear function $r : N_\mathbb{R} \to \mathbb{R}$ such that there is a finite cover $N_\mathbb{R} = \bigcup P_i$ with polyhedra with rational slopes such that $r$ is integral affine on each $P_i$. A tropical rational function on a tropical variety $\mathcal{X}$ is a tropical rational function on its main torus. We denote $\text{Rat}(\mathcal{X})$ the set of tropical rational functions on $\mathcal{X}$.

Definition 3.3. Let $r$ be a tropical rational function on $\mathcal{X}$ and $\tau \in \Delta$. We say $r$ restricts to $O(\tau)$ if the assignment $z \to \lim_{x \to z} r(x)$ defines a tropical rational function on $O(\tau)$, which we denote by $r^\tau (z)$.

Definition 3.4. Let $\overline{\alpha}$ be a $k$-cycle on $\mathcal{X}$. A Cartier divisor on $\overline{\alpha}$ is a finite family $\varphi = (U_i, r_i)$ of pairs of open subsets $U_i$ of $\overline{\alpha}$ and tropical rational functions $r_i$ on $\mathcal{X}$ satisfying:

(1) The union of all $U_i$ covers $\overline{\alpha}$.

(2) For every component $\overline{\alpha}_\tau$ of $\overline{\alpha}$ with $\alpha_\tau \subset O(\tau)$ and every chart $U_i$ such that $U_i \cap |\overline{\alpha}_\tau| \neq \emptyset$ the function $r_i$ must restrict to $O(\tau)$.

(3) For every component $\overline{\alpha}_\tau$ of $\overline{\alpha}$ with $\alpha_\tau \subset O(\tau)$ and all charts $U_i$ and $U_j$ such that $U_i \cap U_j \cap |\overline{\alpha}_\tau| \neq \emptyset$ there is an affine linear tropical rational function $d$ on $O(\tau)$ such that $r_i^\tau - r_j^\tau = d$ on $U_i \cap U_j \cap |\alpha_\tau|$ and $d$ extends to a continuous function on $U_i \cap U_j$.

Note that a tropical rational function is a Cartier divisor on $\mathcal{X}$.

Lemma 3.5. (Mey11, Lemma 2.46) Let $P$ be a rational polyhedron and $\tau \in \Delta$ such that $P' = \overline{P} \cap O(\tau)$ is non-empty and $\dim P = \dim P' + 1$. Then there exists a unique primitive lattice vector $v_{P/P'}$ such that $-v_{P/P'} \in N \cap \rho(P) \cap \tau$.

We call $P'$ an infinite cell of $P$ in the case of Lemma 3.5.

Definition 3.6. Let $\varphi$ be a Cartier divisor on a $(k + 1)$-cycle $\overline{\alpha}$ with $\alpha_\tau \subset O(\tau)$ as defined in Definition 3.4. We construct the intersection product $\varphi \cdot \overline{\alpha}$ as follows: Choose a polyhedral structure on $\alpha_\tau$ such that $\varphi$ is linear on every cell of $\alpha_\tau$. For each cell $P$ choose rational functions $r_P$ in open charts $U_P$ containing $P$. Let $r'_P$ denote the linear part of the restriction of $r_P$ to $P$. We first get a component $\overline{E}_\tau$ with $\overline{E}_\tau \subset O(\tau)$ whose cells are the codimension one cells $Q$ of $\alpha_\tau$ with weight

$$w(Q) = \sum_{Q \subseteq P \in \alpha_\tau} w(P) r'_P (v_{P/Q}) - r'_Q (\sum_{Q \subseteq P \in \alpha_\tau} w(P) v_{P/Q})$$

where $v_{P/Q}$ is a primitive lattice generator of $\text{Star}_P (Q)$ and $w(P)$ is the weight of $P$ in $\alpha_\tau$. For each $\sigma \in \Delta$ with $\tau \subseteq \sigma$ we get a component $\overline{E}_\sigma$ where $E_\sigma \subset O(\sigma)$ consists of infinite cells $P' = \overline{P} \cap O(\sigma)$ of $P$ for all $P \subset \alpha_\tau$ such that $\dim (\overline{P} \cap O(\sigma)) = \dim (\alpha_\tau) - 1$ with weight:

$$w(P') = w(P) [N(\sigma)_{P'} : N_P(\sigma)] r'_P (v_{P/P'})$$

Here $N(\sigma)_{P'} = \pi_\sigma (N) \cap \text{span}(P')$ and $N_P(\sigma) = \pi_\sigma (\pi_\tau (N) \cap \text{span}(P))$. Then $\varphi \cdot \overline{\alpha}$ is defined as $\varphi \cdot \overline{\alpha} = \sum_{\tau \subseteq \sigma} \overline{E}_\sigma$. This is well defined according to [Mey11, Theorem 2.48]. We extend this definition by linearity to the set of all $(k + 1)$-cycles $\overline{\alpha}$ on $\mathcal{X}$.

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1In [Mey11] a tropical cycle on $\mathcal{X}$ is defined as a collection of cycles $\alpha_\tau$ on each $O(\tau)$ without taking closures. Here we use the closures $|\overline{\alpha}_\tau|$ just to be consistent with our later argument. This will not change the intersection theory on $\mathcal{X}$.

2In [Mey11, Lemma 2.46] the vector $v_{P/P'}$ is defined as the unique lattice vector such that $v_{P/P'} \in N \cap \rho(P) \cap \tau$. However, according to [Mey11, Definition 2.47] and [Mey11, Example 2.49] it is more natural to require $-v_{P/P'} \in N \cap \rho(P) \cap \tau$. 
Example 3.7. Let $\Delta$ be the complete fan in $N_\mathbb{R} = \mathbb{R}^2$ whose facet $\sigma_i$ is the $i$-th quadrant for $i = 1, 2, 3, 4$. Let $\rho_x, \rho'_x, \rho_y, \rho'_y$ be the rays of $\Delta$ as below (on the right). We have $\mathcal{X} = (\mathbb{R} \cup \{-\infty, +\infty\})^2$.

Let $\varphi$ be the tropical rational function on $\mathcal{X}$ such that $\varphi(x, y) = 0$ if $x + y \leq 0$ and $\varphi(x, y) = x + y$ if $x + y \geq 0$. We calculate $\varphi \cdot \mathcal{X} = \varphi \cdot \alpha_\tau$ where $\alpha_\tau = N_\mathbb{R}$ using symbols in Definition 3.6. Take the polyhedral structure on $\alpha_\tau$ which contains two facets $P_+ = \{(x, y) | x + y \leq 0\}$ and $P_- = \{(x, y) | x + y \geq 0\}$ and a codimension one cell $Q = \{(x, y) | x + y = 0\}$. We have $U_{P_+} = U_{P_-} = N_\mathbb{R}(\Delta)$ and $r_{P_-} = r_{P_+} = \varphi$, and $r'_{P_+}(x, y) = x + y$ and $r'_{P_-}(x, y) = 0$. Restricting to $Q$ we have $r'_{P_+} = r'_{P_-} = r'_{Q} = 0$.

We have $E_x = Q$. To calculate the weight of $Q$ note that $w(P_+) = w(P_-) = 1$, and we can pick $v_{P_+}/Q = (0, 1)$ and $v_{P_-}/Q = (0, -1)$. It follows that $w(Q) = r'_{P_+}(v_{P_+}/Q) + r'_{P_-}(v_{P_-}/Q) = 1$.

There are four infinite cells to consider: $P_x = O(\rho_x)$ and $P_y = O(\rho_y)$ are contained in the closure of $P_+$ while $P'_x = O(\rho'_x)$ and $P'_y = O(\rho'_y)$ are contained in the closure of $P_-$. In order to calculate $w(P_x)$ and $w(P_y)$, we first take $v_{P_+}/P_+ = (-1, 0)$ and $v_{P_-}/P_- = (0, -1)$. One also checks that $N(\rho_x)_{P_x} = \pi_{\rho_x}(N)$ and $N(\rho_y)_{P_y} = \pi_{\rho_y}(\text{span}(P_+)) = \pi_{\rho_x}(N \cap \text{span}(P_+)) = N(\rho_x)_{P_x}$, and similarly $N(\rho_y)_{P_y} = N(\rho_y)_{P_y}$. Thus we have $w(P_x) = r'_{P_+}((-1, 0)) = -1$ and $w(P_y) = r'_{P_-}((0, -1)) = -1$. Similarly, taking $v_{P_+}/P'_+ = (1, 0)$ and $v_{P_-}/P'_y = (0, 1)$, we have $w(P'_x) = w(P'_y) = 0$.

It follows that $E_{\rho_x} = -O(\rho_x)$ and $E_{\rho_y} = -O(\rho_y)$ and $E_{\rho'_x} = 0$ and $E_{\rho'_y} = 0$. Hence $\varphi \cdot \mathcal{X} = \overline{Q} - V(\rho_x) - V(\rho_y)$.

Remark 3.8. In [AR10] a similar notion of the intersection product between a tropical rational function on $O(\tau)$ (up to translation by a linear function) and a cycle in $O(\tau)$ is defined, which we denote by “$\cdot \tau$”. In Definition 3.6 if $\varphi$ is a Cartier divisor on $V(\tau)$, then it induces a tropical rational function $\varphi_{\tau}$ on $O(\tau)$ (again up to translation by a linear function). We then have $E_\tau = \varphi_{\tau} \cdot \tau \alpha_\tau$, which is also equal to the stable intersection of $\varphi_{\tau} \cdot \tau O(\tau)$ and $\alpha_\tau$ inside $O(\tau)$.  

Definition 3.9. Let $\mathcal{X} = N_\mathbb{R}(\Delta)$ be a tropical variety. Let $Z_k(\mathcal{X})$ be the group of $k$-cycles on $\mathcal{X}$. We define subgroups:

$$R_k(\mathcal{X}) = \text{span}_\mathbb{Z}\{r \cdot C | r \in \text{Rat}(\mathcal{X}), C \in Z_{k+1}(\mathcal{X})\}$$

$$R'_k(\mathcal{X}) = \text{span}_\mathbb{Z}\{f_*(C) | f : \mathcal{Y} \to \mathcal{X} \text{ toric morphism}, C \in R_k(\mathcal{Y})\}.$$

Then the $k$-th Chow group of $\mathcal{X}$ is $A_k(\mathcal{X}) = Z_k(\mathcal{X})/R'_k(\mathcal{X})$. A tropical $k$-cycle $\overline{\sigma}$ is rationally equivalent to $\overline{\tau}$, denoted by $[\overline{\sigma}] = [\overline{\tau}]$, if there exists $\overline{\sigma} - \overline{\tau} \in R'_k(\mathcal{X})$.

Here $f_*$ is the pushforward of tropical cycles, see [Mey11, Definition 2.51] for its definition. According to [Mey11, Remark 2.68] every cycle on $\mathcal{X}$ is rationally equivalent to a sum of products of Cartier divisors (with $\mathcal{X}$). This defines an intersection product:

$$\ast : A_{n-k}(\mathcal{X}) \times A_{n-l}(\mathcal{X}) \to A_{n-k-l}(\mathcal{X})$$

which makes $A_\ast(\mathcal{X})$ a graded ring. More precisely, if $[\overline{\sigma}] = [\varphi_1 \cdots \varphi_k \cdot \mathcal{X}] \in A_{n-k}(\mathcal{X})$ and $[\overline{\tau}] \in A_{n-l}(\mathcal{X})$, then $[\overline{\sigma}] \ast [\overline{\tau}] = [\varphi_1 \cdots \varphi_k \cdot \mathcal{X}] \ast [\overline{\tau}] \in A_{n-k-l}(\mathcal{X})$.

Note that this intersection product of two cycles $\overline{\sigma}$ and $\overline{\tau}$ with $\alpha, \beta \subset \mathbb{R}$ does not necessarily coincide with the closure of the stable intersection $\alpha \cdot \beta$, except that $\Delta$ is compatible with one of $\alpha$ and $\beta$ (Lemma 3.18). The main reason is that, unlike stable intersection, passing to the product with a cartier divisor may create cycles supported on the boundary. See the example below.

Example 3.10. Consider the same $\Delta$ and same figure as in Example 3.7. Let $\alpha = Q \subset \mathbb{R}$ be the line defined by $x+y = 0$ with multiplicity one. We have $[\overline{\sigma}] = [V(\rho_x)] + [V(\rho_y)]$ by loc.cit.. Obviously the closure of the stable intersection of $\alpha$ and itself is empty. We now calculate $[\overline{\sigma}] \ast [\overline{\tau}]$.

Consider the Cartier divisor $\varphi = (U_i, r_i)_{1 \leq i \leq 4}$ where $U_i = \cup_{r \subset \sigma_i} O(\tau) = N_\mathbb{R}(\sigma_i)$. Let $r_1(x, y) = -x-y$ and $r_2(x, y) = -y$ and $r_3(x, y) = 0$ and $r_4(x, y) = -x$. Straightforward calculation shows that $\varphi \cdot \mathcal{X} = V(\rho_x) + V(\rho_y)$, hence $[\overline{\sigma}] \ast [\overline{\tau}] = [\varphi \cdot \mathcal{X}] = \{(-\infty, +\infty) + (+\infty, -\infty)\}$.

In addition, every cycle on $\mathcal{X}$ is rationally equivalent to a formal sum of closed $\mathbb{R}$-orbits ([Mey11, Theorem 2.59]) as well as a formal sum of closures of subfans of $\Delta$ ([Mey11, Lemma 2.66]). As a result we have:

Theorem 3.11. ([Mey11, Corollary 2.67]) We have group isomorphisms:

$$A_\ast(X(\Delta)) \cong MW_\ast(\Delta) \to A_\ast(N_\mathbb{R}(\Delta)).$$

Here $MW_\ast(\Delta)$ is the ring of Minkowski weights on $\Delta$, see [FS94] for the construction of $\varphi$. See also [OP13, §2] for an illustration of $\phi$. Note that $\phi$ is actually a ring isomorphism (Lemma 3.18) and that the composition $\phi \circ \varphi$ is indeed (at least when $X(\Delta)$ is projective) induced by the tropicalization map, see Lemma 5.6.

3.2. Compactified stable intersection with certain tropical cycles. In general it is not obvious to define the stable intersection for two arbitrary cycles in $\mathcal{X}$. A basic issue is that taking limits of perturbed intersections does not work in the compactified case. For example any translation of a line in $\mathbb{TP}^3$, the tropical projective space of dimension 3, which passes through $(+\infty, +\infty, +\infty)$ still passes through the same point, but we expect them to have empty intersection.

However, we are able to define the compactified stable intersection for a restricted class of cycles, namely cycles that are compatible with $\Delta$, hence have the same codimension in every affine strata $O(\tau) \subset \mathcal{X}$. Before we state the definition some balancing and compatibility conditions need to be checked:

\footnote{If $\Delta$ is not complete we may restrict to cycles of which $\Delta$ is a compactifying fan.}
Lemma 3.12. Let $\Sigma$ be a polyhedral complex in $\mathbb{R}^n$ which is compatible with $\Delta$ and let $\tau \in \Delta$. Then one can choose polyhedral structures on $\Sigma$ and $\Sigma \cap O(\tau)$ such that there is a one-to-one correspondence induced by $\pi_\tau$ between faces of $\Sigma$ whose recession cones contain $\tau$ and faces of $\Sigma \cap O(\tau)$.

Proof. First we take a refinement of $\Sigma$ such that for any $\sigma \in \Sigma$ we have $\rho(\sigma) \in \Delta$. This is possible since $\Delta$ is compatible with $\Sigma$, and we can refine the structure on $\Sigma$ such that $\sigma \cap \delta$ is a face of $\Sigma$ for all $\sigma \in \Sigma$ and $\delta \in \Delta$. It follows from Lemma 3.9 of [OR1]1 that

$$\Sigma \cap O(\tau) = \bigcup_{\sigma \in \Sigma, \tau \subset \rho(\sigma)} \pi_\tau(\sigma).$$

Take $\sigma \in \Sigma$ a face such that $\tau \subset \rho(\sigma)$, then we have $\sigma = \bigcap_{u_i \in I} \{x \in \mathbb{R}^n | \langle x, u_i \rangle \geq b_i\}$ where $I \subset \tau^\perp$ is a finite set. It follows that $\sigma + \mathbb{R}\tau = \bigcap_{u_i \in I \cap \tau^\perp} \{x \in \mathbb{R}^n | \langle x, u_i \rangle \geq b_i\}$ and

$$\pi_\tau(\sigma) = \pi_\tau(\sigma + \mathbb{R}\tau) = \bigcap_{u_i \in I \cap \tau^\perp} \{y \in \mathbb{R}^n / \mathbb{R}\tau | \langle y, u_i \rangle \geq b_i\}$$

This is a polyhedron in $\mathbb{R}^n / \mathbb{R}\tau$ and will be considered as a face of $\pi_\tau(\Sigma)$. We need to show that $\pi_\tau(\sigma)$ are compatible for all $\sigma \in \Sigma$ such that $\tau \subset \rho(\sigma)$.

Note that faces of $\pi_\tau(\sigma)$ are of the form

$$F_J = \bigcap_{u_i \in (I \cap \tau^\perp) \setminus J} \{y \in \mathbb{R}^n / \mathbb{R}\tau | \langle y, u_i \rangle \geq b_i\} \bigcap_{u_i \in J} \{y \in \mathbb{R}^n / \mathbb{R}\tau | \langle y, u_i \rangle = b_i\}$$

where $J$ is a subset of $I \cap \tau^\perp$. The preimage of $F_J$ in $\sigma$ is

$$E_J = \bigcap_{u_i \in I \setminus J} \{x \in \mathbb{R}^n | \langle x, u_i \rangle \geq b_i\} \bigcap_{u_i \in J} \{x \in \mathbb{R}^n | \langle x, u_i \rangle = b_i\}.$$ 

This is an one-to-one correspondence from faces of $\pi_\tau(\sigma)$ to faces of $\sigma$ whose recession cone contains $\tau$.

To complete the proof notice that if $\sigma_1, \sigma_2 \in \Sigma$ are two faces then $\pi_\tau(\sigma_1) \cap \pi_\tau(\sigma_2) = \pi_\tau(\sigma_1 \cap \sigma_2)$ since their recession cones all contain $\tau$. It follows that $\tau \subset \rho(\sigma_1) \cap \rho(\sigma_2) = \rho(\sigma_1 \cap \sigma_2)$ and hence $\pi_\tau(\sigma_1 \cap \sigma_2)$ is a face of both $\pi_\tau(\sigma_1)$ and $\pi_\tau(\sigma_2)$ due to the argument above. \[\square\]

Note that the correspondence in Lemma 3.12 preserves codimension of every face, in particular we get an induced weight for every maximal face of $\Sigma \cap O(\tau)$. The following proposition follows directly:

**Proposition 3.13.** Let $\Sigma \subset \mathbb{R}^n$ be a tropical cycle which is compatible with $\Delta$, let $\tau \in \Delta$ be a face. Then $\Sigma \cap O(\tau)$ is a tropical cycle in $O(\tau)$ with the induced structure from $\Sigma$.

We are now able to define the compactified stable intersections in this special situation:

**Definition 3.14.** Let $\Sigma$ be a tropical cycle in $\mathbb{R}^n$ such that $\Delta$ is compatible with $\Sigma$. Let $\gamma \subset O(\tau)$ for some $\tau \in \Delta$. We define the **compactified stable intersection** of $\Sigma$ and $\gamma$ and denote by $\Sigma \cdot_{c} \gamma$ to be the closure of the stable intersection of $\Sigma \cap O(\tau)$ and $\gamma$ as tropical cycles on $O(\tau)$.

Intuitively the stable intersection is a right action of the multiplicative group of tropical cycles compatible with $\Delta$ on the group of usual tropical cycles. Next we check the associativity of this action.
Proposition 3.15. Let $\Sigma_1, \Sigma_2$ be tropical cycles in $\mathbb{R}^n$ compatible with $\Delta$ and of codimensions $l$ and $m$ respectively. Let $\tau \in \Delta$ be a face of dimension $d$. Then we have
\[ \overline{\bigcup_{\sigma_i \in \Sigma_i} \sigma_i \cap O(\tau)} = (\overline{\bigcup_{\sigma_i \in \Sigma_i} \sigma_i \cap O(\tau)}) \cap (\overline{\bigcup_{\sigma_i \in \Sigma_i} \sigma_i \cap O(\tau)}) \]
as tropical cycles in $O(\tau)$. In particular for any tropical cycle $\gamma \subset X$ we have the associativity:
\[ (\gamma \circ \Sigma_1) \circ \Sigma_2 = \gamma \circ (\Sigma_1 \circ \Sigma_2). \]

Proof. Take refinement of $\Sigma_i$ such that for $\sigma_i \in \Sigma_i$ we have $\rho(\sigma_i) \in \Delta$ and $\sigma_1 \cap \sigma_2$ is a face of both $\Sigma_1$ and $\Sigma_2$ (first refine $\Sigma_1$ as in Lemma 3.12 and then take intersections of faces of $\Sigma_1$ and $\Sigma_2$). The induced structure on $\Sigma_i \cap O(\tau)$ also satisfies that $\delta_1 \cap \delta_2$ is a face of $\Sigma_i \cap O(\tau)$ for all $\delta_i \in \Sigma_i \cap O(\tau)$ and $i = 1, 2$.

It is easy to check that, with the induced structure, the correspondence in Lemma 3.12 between $\Sigma_1 \cap \Sigma_2$ and $(\Sigma_1 \cap O(\tau)) \cap (\Sigma_2 \cap O(\tau))$ is still valid. We then have (set-theoretically):
\[ \bigcup_{\sigma_i \in \Sigma_i \text{ and } \dim(\sigma_1 + \sigma_2) = n} \sigma_i \cap O(\tau) = \bigcup_{\sigma_i \in \Sigma_i \text{ and } \tau \subset \rho(\sigma_i) \text{ and } \dim(\sigma_1 + \sigma_2) = n} \pi_{\tau}(\sigma_1 \cap \sigma_2) \]
\[ \bigcup_{\sigma_i \in \Sigma_i \text{ and } \tau \subset \rho(\sigma_i) \text{ and } \dim(\sigma_1 + \sigma_2) = n} \pi_{\tau}(\sigma_1 \cap \sigma_2) = \bigcup_{\delta_i \in \Sigma_i \cap O(\tau) \text{ and } \dim(\delta_1 + \delta_2) = n - d} \delta_1 \cap \delta_2 \]
\[ = (\overline{\bigcup_{\sigma_i \in \Sigma_i \cap O(\tau)} \sigma_i \cap O(\tau)}) \cap (\overline{\bigcup_{\sigma_i \in \Sigma_i \cap O(\tau)} \sigma_i \cap O(\tau)}). \]

On the other hand, for any $\delta \in (\Sigma_1 \cap O(\tau)) \cap (\Sigma_2 \cap O(\tau))$ of codimension $l + m$ in $O(\tau)$ and $\sigma \in \Sigma_1 \cap \Sigma_2$ with $\pi_{\tau}(\sigma) = \delta$, we have $\text{Star}(\sigma, \Sigma_i) = \text{Star}(\delta, \Sigma_i \cap O(\tau)) \times \mathbb{R} \tau$ and hence
\[ i(\sigma, \Sigma_1 \cap \Sigma_2) = i(\delta, \Sigma_i \cap O(\tau)) \cap (\Sigma_2 \cap O(\tau)). \]

Given a tropical cycle $\Sigma \subset N_\mathbb{R}$ such that $\sigma \in \Sigma$ implies $\rho(\sigma) \in \Delta$, according to [AHR14, Definition 5.1], there is a natural balanced structure on $\rho(\Sigma)$ where the weight of $\tau \in \rho(\Sigma)$ is defined by:
\[ m(\tau) = \sum_{\sigma \in \Sigma, \rho(\sigma) = \tau} m(\sigma). \]

Lemma 3.16. If $\Delta$ is compatible with $\Sigma$, then $\rho(\Sigma \cap O(\tau)) = \overline{\rho(\Sigma) \cap O(\tau)}$ as tropical cycles on $O(\tau)$.

Proof. Let $\sigma \in \Sigma$, note that if $\tau \subset \rho(\sigma)$ then $\pi_{\tau}(\rho(\sigma)) = \rho(\pi_{\tau}(\sigma))$. The lemma follows directly from Lemma 3.12 and the definition of structures on the recession fans. \qed

It turns out that, in the compactified sense, intersecting with $\Sigma$ is numerically the same as intersecting with $\rho(\Sigma)$ when the two tropical cycles have complementary codimension:

Proposition 3.17. Let $\Sigma$ and $\gamma$ be as in Definition 3.14. Assume $\dim(\Sigma) + \dim(\gamma) = n$. We have $\deg(\gamma \circ \Sigma) = \deg(\gamma \circ \rho(\Sigma))$. 

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Proof. Follows from Lemma 3.16 and [AHR14, Theorem 5.3].

We are now able to check the compatibility of the compactified stable intersection “∗” and the intersection product “∗c”:

Lemma 3.18. If β ⊂ NR is a tropical cycle which is compatible with Δ and α ⊂ O(τ), then [α] * [β] = [α ·c β].

Proof. First assume |β| = |Δ(n − 1)|. Then we can find a cartier divisor φ such that φ · X = β. Indeed, assume [β] = ∑ρ∈Δ(1) aρ[V(ρ)]. Let f be the support function on Δ such that f(αρ) = aρ where uρ as in §2.2. Then f · X = F − ∑ aρV(ρ) where F is a codimension one subfan of Δ. Hence [F] = ∑ aρ[V(ρ)], and Theorem 3.11 implies that |β| = F as tropical cycles. For all σ ∈ Δ(n) take a (unique) mσ ∈ M such that f|σ = 〈·, mσ〉. Now let φ = (Uσ, rσ)σ∈Δ(n) where Uσ = Uρ<σO(τ) and rσ = f − 〈·, mσ〉. Note that rσ is zero on σ. Straightforward calculation shows that |β| = φ · X (all infinite cells have weight zero) and, using symbols in Remark 3.8, we have |β| ∩ O(τ) = φ · τ O(τ) as tropical cycles on O(τ). Now by loc.cit. we have

|β| · [α] = [φ · τ] = [∑ τ⊂σ Eσ] = [Eτ] = [φ · τ] = [β ·c α].

The associativity of “∗” and “∗c” guarantees that the conclusion is true for β any cycle of subfans of Δ. In particular the morphism φ in Theorem 3.11 is a ring isomorphism.

For the general case note that Theorem 3.11 and Proposition 3.17 shows that |β| = |ρ(β)|. Hence we have:

|α| · |β| = |α| · |ρ(β)| = |α ·c ρ(β)| = |α ·c β|.

One can check the last equality by counting the degree of the intersection products of both side with all cycles γ where γ is a subfan of Δ of suitable codimension, which, as we already showed, is the same as the degree of α ·c ρ(β) ·c γ and α ·c β ·c γ respectively.

4. The Complementary Codimension Case

In this section we assume that we are given two closed subschemes X and X′ of T X which are of complementary codimension but possibly have positive dimensional intersection. In this case X · X′ is a zero dimensional cycle class whose degree is not well-defined in general. We show that, after a suitable compactification and restriction to a component C of trop(X) ∩ trop(X′), the degree above is invariant under rational equivalence, and is equal to the corresponding intersection number on the tropical side.

Let (Δ, P) be a compactifying datum for X, X′ and C. Let (P′, ε, v) be a tropical moving data for (Δ, P). We then have ([ORT11, Corollary 4.8]):

Lemma 4.1. In the situation above we have

trop(X) ∩ trop(X′) ∩ |P′| = C ⊂ |P| ⊂ |P′|°

and for all r ∈ [−ε, 0) ∪ (0, ε] we have

trop(X) + r · v ∩ trop(X′) ∩ |P′| = (trop(X) + r · v) ∩ trop(X′) ∩ |P′| ⊂ |P′|° ⊂ |P′|°

where all closures are taken in NR(Δ).

Let ZC be the union of irreducible components of X ∩ X′ whose tropicalization intersects C, where X and X′ are closures of X and X′ in X(Δ) respectively. We claim that ZC is a union of connected components of X ∩ X′, hence has a natural scheme structure which is in particular
flat over $\overline{X} \cap \overline{X'}$. Indeed, according to Lemma 4.1 we know $C$ is a connected component of 
$trop(X) \cap trop(X')$, and the claim follows from the fact that the tropicalization of every irreducible component of $\overline{X} \cap \overline{X'}$ is a connected set. In particular $trop(Z) \subseteq C$.

Moreover, [OR11, Corollary 4.17] shows that $Z$ is proper over the ground field, hence there is a well-defined degree of every cycle class of dimension zero on $Z$. In the following we take $\mathcal{P} = C$ with the induced polyhedral complex structure. It follows that $(\Delta, \mathcal{P})$ is a compactifying datum if $\Delta$ is a compactifying fan for $C$:

**Situation 4.2.** Let $X$ and $X'$ be closed subschemes of $T_N$ of pure dimension $k$ and $l$ respectively, where $k + l = n$. Let $C$ be a connected component of $trop(X) \cap trop(X')$. Let $\Delta$ be a unimodular compactifying fan for $C$, let $i_Z: Z \to \overline{X} \cap \overline{X'}$ be the inclusion.

**Theorem 4.3.** In Situation 4.2 we have
\[
\int_{Z} i_\Delta^*$
\[
\int_{Z} i_\Delta^*(X \cdot \overline{X'}) = \sum_{u \in C} i(u, trop(X) \cdot trop(X'))
\]
where $X \cdot \overline{X'}$ is the refined intersection product.

**Proof of Theorem 4.3.** As in [OR11] the idea is to approach $\overline{X} \cdot \overline{X'}$ (resp. $trop(X) \cdot trop(X')$) by $\overline{X} \cdot z\overline{X'}$ (resp. $trop(X) \cdot trop(zX')$) where $z \in T_N$ such that $trop(zX') = \text{val}(z) + trop(X')$ is a small perturbation of $trop(X')$.

Consider the action $\mu: G_m \times X(\Delta) \to X(\Delta)$ of $G_m$ on $X(\Delta)$ given by $\mu(t, x) = v(t) \cdot x$. Let $p_1: G_m \times X(\Delta) \to G_m$ and $p_2: G_m \times X(\Delta) \to X(\Delta)$ be the two projections. Denote $X = (p_1, \mu)(G_m \times \overline{X})$ and $X' = G_m \times \overline{X'}$.

Note $G_m$ is a nonsingular variety of dimension 1, and $Z = G_m \times X(\Delta)$ is smooth over $G_m$ of relative dimension $n$, and $[X] \in A_{k+1}(Z)$ and $[X'] \in A_{l+1}(Z)$. Now by Lemma 2.1 we have for $t \in G_m$ the equality $i_t^*[X] \cdot i_t^*[X'] = i_t^*([X] \cdot [X']) \in A_0(X_t \cap X'_t)$ where $i_t: t \to G_m$ is the inclusion.

Since $X \to G_m$ and $X' \to G_m$ are flat morphisms, we know by excess intersection formula that $i_t^*[X] = [X_t]$ and $i_t^*[X'] = [X'_t]$. On the other hand, assume $[X] \cdot [X'] = \sum \alpha_i \cdot [\alpha_i]$ where $\alpha_i \in A_1(X \cap X')$ are subvarieties. If $\alpha_i \subset Z_{t_0} = X(\Delta)$ for some $t_0 \in G_m$, then $(\alpha_i)_t = \alpha_i$ or $\emptyset$, in both cases we have $i_t^*\alpha_i = 0$ by definition. Moreover since $Z$ is flat over $G_m$, we have that $i_{Z_t}: Z_t \to Z$ is a regular embedding of codimension one and $i_t^*(\gamma) = i_{Z_t}(\gamma) = Z_t \cdot \gamma$ in $A_*(\gamma_t)$ for $\gamma$ any subscheme of $Z$. Put these all together we get:
\[
[X_t] \cdot [X'_t] = \sum_{i=1}^m \alpha_i \cdot [Z_i] \text{ in } A_0(X_t \cap X'_t)
\] (1)

where $\alpha_i$ are subvarieties of $X \cap X'$ of dimension 1 which intersect properly with $Z_t$ for all $t \in G_m$.

We next show that $\int_{(X_t \cap X'_t)\cap \mathcal{P}'} X_t \cdot X'_t$ is constant for $t$ with valuation in $[-\epsilon, \epsilon]$, where $(X_t \cap X'_t)\cap \mathcal{P}'$ is the part of $X_t \cap X'_t$ that tropicalize to $[\mathcal{P}']$. Again this number is well-defined according to the discussion below Lemma 4.1. More generally we claim that under the same assumption of $t$, the number $\int_{(X_t \cap X'_t)\cap \mathcal{P}'} \alpha \cdot [Z_t]$ is constant for subvarieties $\alpha$ of $X \cap X'$ of dimension one which intersect properly with $Z_t$ for all $t \in G_m$.

Let $\mathcal{S}_e = \text{val}^{-1}[-\epsilon, \epsilon] \subset G_m$ and $U' = trop^{-1}([\mathcal{P}']) \subset X(\Delta)^n$. Let $\mathcal{Y}_\alpha = \alpha^n \cap (\mathcal{S}_e \times U')$. We have:
\[
\mathcal{Y}_\alpha = \alpha^n \cap (\mathcal{S}_e \times X(\Delta)^n) \cap (\mathcal{S}_e \times U') = \alpha^n \cap (\mathcal{S}_e \times X(\Delta)^n) \cap (trop \circ p_2)^{-1}([\mathcal{P}'])
\]
since $\alpha$ is supported in $\mathcal{X} \cap \mathcal{X}'$. Thus $\mathcal{Y}_\alpha$ is a union of connected components of $\alpha^{an} \cap (S_\epsilon \times X(\Delta)^{an})$, and is Zariski closed in $S_\epsilon \times X(\Delta)^{an}$ and hence proper over $S_\epsilon$ by [OR11 Proposition 4.16]. Denote $S_\epsilon \times U^{P'}$ by $Z$, it follows from [OR11 Proposition 5.7] that:

$$\int_{(X \cap X')^{\frac{1}{P,\{P\}}} \cap \mathcal{X}_t} \alpha \cdot Z_t = \sum_{x \in \mathcal{Y}_\alpha \cap Z_t} i(x, \mathcal{Y}_\alpha \cdot Z_t; Z) .$$

We will use [OR11 Proposition 5.8]. Look at the projection map $f : S_\epsilon \times Z \to S_\epsilon$ and the Zariski closed subspaces $\mathcal{X} = S_\epsilon \times \mathcal{Y}_\alpha$ and $\mathcal{X}' = \Delta(S_\epsilon) \times U^{P'}$ where $\Delta(S_\epsilon)$ is the diagonal of $S_\epsilon \times S_\epsilon$. It is easy to see that both $\mathcal{X}$ and $\mathcal{X}'$ are flat over $S_\epsilon$, and $Z$ is an analytic domain in $S_\epsilon \times X(\Delta)^{an}$, hence quasi-smooth. It follows that $f$ is quasi-smooth and so is $S_\epsilon \times Z$ since $X_\epsilon$ is quasi-smooth.

To show $\mathcal{X} \cap \mathcal{X}'$ is finite over $S_\epsilon$, according to the following fibered diagram we have $X \cap X' \simeq \mathcal{Y}_\alpha$.

$$\begin{array}{ccc}
\mathcal{X} \cap \mathcal{X}' & \longrightarrow & S_\epsilon \times \mathcal{Y}_\alpha \\
\downarrow & & \downarrow \\
\Delta(S_\epsilon) \times U^{P'} & \longrightarrow & S_\epsilon \times S_\epsilon \times U^{P'} \\
\downarrow & & \downarrow \\
S_\epsilon \times S_\epsilon \times U^{P'} & \longrightarrow & S_\epsilon \times U^{P'} .
\end{array}$$

Hence $\mathcal{X} \cap \mathcal{X}'$ is proper over $S_\epsilon$. Moreover $(\mathcal{X} \cap \mathcal{X}')^{\frac{1}{P,\{P\}}} \simeq \mathcal{Y}_\alpha \cap Z_t$ is finite, since $\alpha \cap Z_t$ is finite. It follows that $\mathcal{X} \cap \mathcal{X}'$ is finite over $S_\epsilon$. Now by [OR11 Proposition 5.8] we know that $\int_{(X \cap X')^{\frac{1}{P,\{P\}}} \cap \mathcal{X}_t} \alpha \cdot Z_t$ is constant for $t \in \text{val}^{-1}([-\epsilon, \epsilon])$. The theorem follows from [OR11 Theorem 6.3] since $\text{trop}(X_t)$ intersects $\text{trop}(X'_t)$ properly for $t \in \text{val}^{-1}((-\epsilon, 0) \cup (0, \epsilon])$. \hfill \Box

Example 4.4. Let $X = X'$ be the plane curves in $T = \text{Spec}K[x^\pm, y^\pm]$ defined by the equation $f(x, y) = ax^n + by + 1 = 0$ with $\text{val}(a) = \text{val}(b) = 0$. The tropicalization $\text{trop}(X)$ equals the union of rays $R_1 = \mathbb{R}_{\geq 0} \cdot e_1$ and $R_2 = \mathbb{R}_{\geq 0} \cdot e_2$ and $R_3 = \mathbb{R}_{\geq 0} \cdot (-e_1 - ne_2)$. These rays have multiplicities $1, n$ and $1$ respectively as showed in the diagram below.

Let $\Delta$ be the complete fan generated by the rays $\rho_1 = (1, 0), \rho_2 = (0, 1), \rho_3 = (-1, -n), \rho_4 = (0, -1)$ and let $D_i, i = 1, 2, 3, 4$ be the corresponding torus-invariant divisor. Then $\Delta$ is a smooth compactifying fan for $\text{trop}(X)$ and $X(\Delta)$ is a Hirzebruch Surface. For generic choices of $a$ and $b$ we have $\text{div}(f) = \overline{X} - nD_3 - D_4$ and hence $[\overline{X}] = nD_3 + D_4$ as cycle classes. According to Theorem 4.3 we have

$$i_K([X : X(\Delta)]) = i(0, \text{trop}(X) \cdot \text{trop}(X)) = n$$

which is the same as the degree of $(nD_3 + D_4) \cdot (nD_3 + D_4)$.

Remark 4.5. Note that if $\text{trop}(X) \cap \text{trop}(X')$ contains only one component $C$, and $\Delta$ is also compatible with either $\text{trop}(X)$ or $\text{trop}(X')$ and $X(\Delta)$ is a smooth projective variety, as in Example
4.4 Then Theorem 4.3 is the same as saying that deg([X] · [X]) = deg([trop(X)] * [trop(X')]) (Lemma 5.6). For two reasons: (1) all points of X ∩ X' tropicalize to C (cf. [OR11, Proposition 3.12], take P = trop(X) or trop(X')), and (2) the degree of [trop(X)] * [trop(X')] is the same as the degree of trop(X) · trop(X') (cf. Lemma 3.18). This is not true in general, see the example below.

Example 4.6. Let X, X' be curves in T^2 defined by \( f(x, y) = ax^2 + xy + ay^2 + x + y + a \) and \( g(x, y) = x + y + a \) respectively, where \( a \in K \) such that \( \text{val}(a) = 1 \). Then trop(X) and trop(X') are as in the following graph, where the red part is the unique component C of trop(X) ∩ trop(X').

Let Δ be the complete fan whose facets are all quadrants of the plane, which is not compatible with trop(X) or trop(X'). Then X(Δ) = \( \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \overline{X} \) and \( \overline{X} ' \) are curves of type (2, 2) and (1, 1) respectively, and \( N(R) = (R \cup \{-\infty, +\infty\})^2 \). Straightforward calculation shows that \( \text{deg}([\text{trop}(X)] * [\text{trop}(X')]) = \text{deg}([X] \cdot [X']) = 4 \) but \( \int_{Z(\Sigma)} i(\overline{X} \cdot \overline{X'}) = \text{deg}(\text{trop}(X) \cdot \text{trop}(X')) = 2 \). One checks easily that X ∩ X' contains two points, counted with multiplicity, that tropicalize to (1, 1), while \( (X \cap X') \setminus T^2 \) contains two points that tropicalize to \(( -\infty, -\infty) \notin C \).

Using the same definition and argument as in [OR11, Theorem 6.10] we generalize Theorem 4.3 as follows:

Situation 4.7. Let \( X_1, \ldots, X_m \) be pure-dimensional closed subschemes of T with \( \sum \text{codim}(X_i) = n \) and \( m \geq 2 \). Let C be a connected component of trop(X_1) ∩ ... ∩ trop(X_m) and Δ a unimodular compactifying fan for C.

Theorem 4.8. In Situation 4.7 let Z(\Sigma) be the union of irreducible components of \( \bigcap_i \overline{X}_i \) which tropicalize to C, and \( i_C : Z(\Sigma) \hookrightarrow \bigcap_i \overline{X}_i \) be the inclusion. We have:

\[
\int_{Z(\Sigma)} i_C^*(\overline{X}_1 \cdots \overline{X}_m) = \sum_{u \in C} i(u, \text{trop}(X_1) \cdots \text{trop}(X_m))
\]

where \( \overline{X}_1 \cdots \overline{X}_m \) is the refined intersection product.

Remark 4.9. Note that Theorem 4.3 (resp. Theorem 4.8) can be easily generalized to the case where C is a collection of connected components of trop(X) ∩ trop(X') (resp. \( \bigcap_i \text{trop}(X_i) \)).
5. The Higher Dimensional Case

In this section we restate and prove Theorem 1.2, where \( X \cdot X' \) is a cycle class of possibly positive dimension. For technical reasons we require the ambient space \( X(∆) \) to be projective, in which case the \((n-1)\)-skeleton of \( ∆ \) is a tropical hypersurface. This is possible up to a refinement of \( ∆ \) due to the toric Chow lemma and the projective resolution of singularities (Theorem 11.1.9)). In this case as a first approach we can consider the degree of \( X \cdot X' \) (after restricting to a component of \( \text{trop}(X) \cap \text{trop}(X') \)) as a cycle class in a projective space, this is indeed equal to the degree of \( \text{trop}(X) \cdot \text{trop}(X') \) (after restricting to the same component of \( \text{trop}(X) \cap \text{trop}(X') \)) intersecting with certain tropical hypersurfaces in \( N_\mathbb{R} \). Note that in this case \( ∆ \) is complete, hence being compatible with a polyhedron is equivalent to being a compactifying fan of that polyhedron.

**Situation 5.1.** Let \( X \) and \( X' \) be closed subschemes of \( T_N \) of pure dimensions \( k \) and \( l \) respectively. Let \( C \) be a connected component of \( \text{trop}(X) \cap \text{trop}(X') \). Let \( ∆ \) be a compactifying fan for \( C \) such that \( X(∆) \) is smooth and projective. Let \( i_{C\Gamma^-} : Z_{CΓ^-} \to X \cap X' \) be the inclusion of the union of irreducible components of \( X \cap X' \) whose tropicalization intersects \( C \) (as in Situation 4.2).

Given \( ∆ \) a compactifying fan for \( P = C \), we can still find a \( ∆ \)-thickening \( P' \) of \( P \) such that \( (∆, P') \) is a compactifying datum for \( X, X' \) and \( C \). It follows that we still have

\[
\text{trop}(X) \cap \text{trop}(X') \cap [P'] = C \subset [P] \subset [P']^\circ.
\]

Hence as in Situation 4.2 \( Z_{CΓ^-} \) is an open and closed subscheme of \( X \cap X' \) and the restriction \( i_{C\Gamma^-} : X \cap X' \to A_{k+l-n}(Z_{CΓ^-}) \) is well-defined.

Take a fan \( ∆ \) with \( X(∆) \) smooth and projective. Let \( D \) be a torus-invariant ample divisor\(^5\) on \( X(∆) \). We can write \( D = \sum_{ρ ∈ Σ(1)} a_ρ D_ρ \). For \( σ ∈ ∆(n) \) we take \( m_σ ∈ M \) which satisfies the condition of Proposition 2.2(5), hence \( m_σ \) is uniquely determined by \( D \) and \( D \) is also determined by \( \{m_σ \} \) \( \sigma ∈ ∆(n) \). It follows from Proposition 2.2(6) that \( \{m_σ \} \) \( \rho ∈ ∆(n) \) are all vertices of \( P_D \).

Now the set of sections \( \{x^m \mid m ∈ P_D \cap M \} \) gives a closed immersion of \( X(∆) \) into \( \mathbb{P}^{s-1} \) where \( s \) is the order of this set. In order to calculate the degree of \( i_{C\Gamma^-} : X \cap X' \), note that this is essentially the same as intersecting \( k+l-n \) times with a hyperplane in \( \mathbb{P}^{s-1} \), hence with a hyperplane section \( L \subset X(∆) \) that intersects \( T_N \). We can use Theorem 4.3 to get a combinatorial result, as long as that \( ∆ \) is a compactifying fan of (components of) \( C \cap \text{trop}(L \cap T_N) \).

**Lemma 5.2.** Given \( ∆ \) and \( D \) as above, there exists a hypersurface \( H = V(f) ⊆ T_N \) such that \( \overline{H} \) is a hyperplane section of \( X(∆) \) with respect to \( D \) and \( \text{trop}(H) = |∆(n-1)| \).

**Proof.** Consider the regular functions

\[
f = \sum_{σ ∈ ∆(n)} a_σ x_σ^{m_σ} ∈ K[M] \cap \Gamma(X(Σ), O(D)) \text{ where } \text{val}(a_σ) = 0 \text{ for all } σ ∈ ∆(n).
\]

Let \( H = V(f) ⊆ T \). According to Proposition 2.2(6), for generic choice of \( \{a_σ \} \) we have

\[
0 = \text{div}(f) = \overline{H} - \sum_{ρ ∈ ∆(1)} a_ρ D_ρ = \overline{H} - D ∈ \text{Cl}(X(∆)).
\]

Hence \( \overline{H} = D \) as a divisor class.

\(^5\)Note that on a smooth complete toric variety ample is equivalent to very ample.
On the other hand, since $\text{val}(a_\sigma) = 0$, the corresponding regular subdivision of Newton polygon $P_D$ of $f$ is trivial. Thus $\text{trop}(H)$ is dual to $P_D$ by [MST5 Lemma 3.4.6], in other words $\text{trop}(H) = |\Delta(n-1)|$.

**Remark 5.3.** Note that according to loc.cit. for $\sigma \in \Delta(n-1)$ the multiplicity of $\sigma$ in $\text{trop}(H)$ is the lattice length of the edge with end points $m_{\sigma_1}$ and $m_{\sigma_2}$, where $\sigma \subseteq \sigma_1, \sigma_2 \in \Delta(n)$.

**Lemma 5.4.** In Situation 5.1, for any $T$-invariant ample divisors $D_1, \ldots, D_{k+l-n}$ on $X(\Delta)$ and the corresponding hyperplane sections $\overline{\Pi}_1, \ldots, \overline{\Pi}_{k+l-n}$ constructed as above, we have

$$
\int_{X(\Delta)} i^*_C(\mathbf{x} \cdot \mathbf{x'}) \cdot \overline{\Pi}_1 \cdots \overline{\Pi}_{k+l-n} = \sum_{u \in C} i(u, \text{trop}(X) \cdot \text{trop}(X') \cdot \text{trop}(H_1) \cdots \text{trop}(H_{k+l-n}))
$$

where the product on the left is taken inside $X(\Delta)$, and $i^*_C(\mathbf{x} \cdot \mathbf{x'})$ is considered as a cycle class on $X(\Delta)$ by pushing forward from $Z_C$.

**Proof.** We first claim that

$$
\int_{X(\Delta)} i^*_C(\mathbf{x} \cdot \mathbf{x'}) \cdot \overline{\Pi}_1 \cdots \overline{\Pi}_{k+l-n} = \int_{X(\Delta)} i^*_C(\mathbf{x} \cdot \mathbf{x'} \cdot \overline{\Pi}_1 \cdots \overline{\Pi}_{k+l-n}).
$$

Indeed, consider the following diagram:

$$
\begin{array}{ccc}
Z_C \cap \overline{\Pi}_1 & \longrightarrow & Z_C \\
\downarrow \pi_1 & & \downarrow \pi \\
\mathbf{x} \cap \mathbf{x'} \cap \overline{\Pi}_1 & \longrightarrow & \mathbf{x} \cap \mathbf{x'} \\
\downarrow & & \downarrow \\
\overline{\Pi}_1 & \longrightarrow & X(\Delta)
\end{array}
$$

and note that $i_C$ is an open immersion, hence flat, we have

$$
i^*_C(\mathbf{x} \cdot \mathbf{x'}) \cdot \overline{\Pi}_1 = i^*_1(i^*_C(\mathbf{x} \cdot \mathbf{x'})) = i^*_1(i^*_1(\mathbf{x} \cdot \mathbf{x'})) = i^*_1(\mathbf{x} \cdot \mathbf{x'} \cdot \overline{\Pi}_1).
$$

Since $i_\pi$ is also an open immersion and flat, we get the desired equation by induction.

Now note that $Z_C \cap (\cap_{i=1}^{k+l-n} \overline{\Pi}_i) = \{ x \in \mathbf{x} \cap \mathbf{x'} \cap (\cap_{i=1}^{k+l-n} \text{trop}(H_i)) \}$ by [ORT11 Lemma 3.10]. This is the closure of a union of connected components of $\text{trop}(X) \cap \text{trop}(X') \cap (\cap_{i=1}^{k+l-n} \text{trop}(H_i))$. It is easy to check that $\Delta$ is a compactifying fan for $C \cap [\Delta(n-1)]$. Now the conclusion follows from Theorem 4.3.

$$
\int_{X(\Delta)} i^*_C(\mathbf{x} \cdot \mathbf{x'} \cdot \overline{\Pi}_1 \cdots \overline{\Pi}_{k+l-n}) = \sum_{u \in C \cap [\Delta(n-1)]} i(u, \text{trop}(X) \cdot \text{trop}(X') \cdot \text{trop}(H_1) \cdots \text{trop}(H_{k+l-n}))
$$

$$
= \sum_{u \in C} i(u, \text{trop}(X) \cdot \text{trop}(X') \cdot \text{trop}(H_1) \cdots \text{trop}(H_{k+l-n})).
$$

□

Setting $H_1 = H_2 = \cdots = H_{k+l-n}$ we have the following corollary, which is an analogue of Theorem 4.3.
Corollary 5.5. In Situation \([\mathcal{S}]\) for any \(T\)-invariant ample divisor \(D\) on \(X(\Delta)\), under the embedding \(X(\Delta) \to \mathbb{P}^{s-1}\) induced by \(D\) we have
\[
\deg(i^*_{\mathcal{I}}(X \cdot X')) = \sum_{u \in C} i(u, \trop(X) \cdot \trop(X') \cdot (\Delta(n - 1))^{k + l - n})
\]
where the multiplicities of facets of \(\Delta(n - 1)\) is given as in remark \([\mathcal{S}]3\).

Proof. Let \(H\) be the corresponding hypersurface in \(T_X\) as before. Let \(\tilde{H}\) be a hyperplane in \(\mathbb{P}^{s-1}\) such that \(\tilde{H} \cap X(\Delta) = \tilde{H}\) (the closure of \(H\) in \(X(\Delta)\)). Then intersecting with \(\tilde{H}\) in \(\mathbb{P}^{s-1}\) is essentially the same as intersecting with \(\overline{\mathcal{I}}\) in \(X(\Delta)\): consider the fiber product:
\[
\begin{array}{ccc}
\mathcal{I} & \xrightarrow{\pi} & X(\Delta) \\
\downarrow & & \downarrow \\
\tilde{H} & \xrightarrow{i_{\tilde{H}}} & \mathbb{P}^{s-1},
\end{array}
\]
for a closed subvariety \(\alpha\) of \(X(\Delta)\) we have \(\alpha \cdot \tilde{H} = \frac{i_{\tilde{H}}^*(\alpha)}{\mathcal{I}} = \frac{i_{\mathcal{I}}^*(\alpha)}{\mathcal{I}} = \alpha \cdot \overline{\mathcal{I}}\) where the first intersection product is in \(\mathbb{P}^{s-1}\) while the last one is in \(X(\Delta)\). Hence the corollary follows from Lemma \([\mathcal{S}]4\).

The following lemma is a complement of Theorem \([3.1]\).

Lemma 5.6. If \(X(\Delta)\) is smooth and projective then the isomorphism
\[
\phi \circ \varphi : \mathcal{A}_s(X(\Delta)) \to \mathcal{A}_s(N_\mathbb{R}(\Delta))
\]
in Theorem \([3.1]\) is induced by the tropicalization map\([\mathcal{S}]3\).

For \([\alpha]\) \(\in\mathcal{A}_s(X(\Delta))\) by \(\trop([\alpha])\) we mean the tropicalization of any representative of \(\alpha\) as a sum of irreducible closed subschemes of \(X(\Delta)\). More specifically, let \(\beta\) be an irreducible closed subscheme of \(X(\Delta)\) and \(\mathcal{V}(\tau) \cong X(\text{Star}_{\Delta}(\tau))\) be the maximal closed torus orbit that contains \(\beta\) as a closed subset. Write \([\beta] = a[\beta']\) where \(\beta'\) is reduced irreducible, hence a closed subscheme of \(\mathcal{V}(\tau)\). We then define \(\trop([\beta]) = \trop(\beta')\) with multiplicities equal to \(a\) times the multiplicities of \(\trop(\beta' \cap \mathcal{O}(\tau))\). Note that although our notation is non-standard, the following theorem/corollary does not depend on the choice of representative.

Proof of Lemma \([5.6]\) Let \(\alpha\) be an reduced irreducible closed subscheme of \(X(\Delta)\), we need to show that \(\varphi([\alpha]) = \phi^{-1}([\trop(\alpha)])\). First assume \(\alpha = D_{\rho_\alpha}\) is a torus-invariant divisor, hence \(\trop(\alpha) = V(\rho_\alpha)\). We have \(\varphi([\alpha]) = m_\alpha\) where \(m_\alpha(\tau) = \deg([\alpha \cdot |\mathcal{V}(\tau)|])\) for all \(\tau \in \Delta(n - 1)\). On the other hand let \(f\) be the support function on \(\Delta\) defined by \(f|_{\sigma} = \langle \cdot, m_\sigma\rangle\) for \(\sigma \in \Delta(n)\) and \(m_\sigma \in M\) such that \(f(u_\rho) = -1\) if \(\rho = \rho_\alpha\) and \(f(u_\rho) = 0\) otherwise, where \(u_\rho\) is the lattice generator of \(\rho \in \Delta(1)\). Then \(f \cdot N_\mathbb{R}(\Delta) = V(\rho_\alpha) - F\) where \(F\) is a subfan of \(\Delta\) of codimension one with weights
\[
m'_\alpha(\tau) = \begin{cases} 
0 & \text{if } \rho_\alpha + \tau \not\in \Delta(n), \\
1 & \text{if } \rho_\alpha + \tau \in \Delta(n), \\
\langle u_{\rho_1}, m_{\sigma_2}\rangle & \text{if } \rho_\alpha \in \tau.
\end{cases}
\]

\[\text{Note that a more general case, where } \Delta \text{ is only assumed to be complete unimodular, is considered in } \text{[Mey11, Theorem 3.21]. However, it appears that the argument in its proof is using (without proof) that } \deg(\trop(X) \cdot \trop(X')) = \deg(\mathcal{I}_1 \cdot \mathcal{I}_2) \text{ when } \trop(X) \text{ and } \trop(X') \text{ are of complementary codimension and intersect transversally and the intersection } \trop(X) \cap \trop(X') \text{ is supported on } N_\mathbb{R}. \text{ This is not obvious to the author, hence we sketch the proof in our special situation for completeness.}\]
where $\sigma_1$ and $\sigma_2$ are the two maximal faces of $\Delta$ that contain $\tau$ and $\rho_i \in \sigma_i(1) \setminus \tau(1)$. It then follows that $\phi^{-1}(\{\trop(\alpha)\}) = m'_\alpha$ and one checks directly that $m_\alpha = m'_\alpha$.

Next let $\alpha = \overline{H}$ be as in Lemma 5.2. We have $[\alpha] = \sum a_\rho D_\rho$ and $[\trop(\alpha)] = \sum a_\rho [V(\rho)]$ (by Remark 5.3 and looking at the support function defined by $\{m_\alpha\}$ corresponding to $D$ in Lemma 5.2). Thus $\varphi([\alpha]) = \phi^{-1}(\{\trop(\alpha)\})$. The same is true for $\alpha = g\overline{H}$ for any $g \in T_N$.

Now let $\alpha$ be an arbitrary irreducible closed subscheme of $X(\Delta)$ of dimension $k$ that intersects $T_N$. Given ample divisors $\overline{H}_1, \ldots, \overline{H}_{n-k}$ as in Lemma 5.2. Take $g_1, \ldots, g_{n-k} \in T_N$ such that $\trop(g_1H_1), \ldots, \trop(g_{n-k}H_{n-k})$ intersect properly, and $\trop(\alpha) \cap \trop(g_1\overline{H}_1) \cap \cdots \cap \trop(g_{n-k}\overline{H}_{n-k}) = \trop(\alpha) \cap \trop(g_1H_1) \cap \cdots \cap \trop(g_{n-k}H_{n-k})$ (OR11 Lemma 3.10). It follows from [OP13 Theorem 5.2.3] that

$$\deg([\alpha] \cdot [g_1\overline{H}_1] \cdots [g_{n-k}\overline{H}_{n-k}]) = \deg(\trop(\alpha \cap T_N) \cdot \trop(g_1H_1) \cdots \trop(g_{n-k}H_{n-k})).$$

By Lemma 3.18 and the fact that $\trop(g_iH_i)$ is rationally equivalent to $\trop(H_i)$, the equation above is equivalent to:

$$\deg([\alpha] \cdot [\overline{H}_1] \cdots [\overline{H}_{n-k}]) = \deg(\{\trop(\alpha)\} \ast [\trop(\overline{H}_1)] \ast \cdots \ast [\trop(\overline{H}_{n-k})]).$$

Since $A_*(X(\Delta))$ is generated by divisors of form $\overline{H}$ as in Lemma 5.2 and $\phi \circ \varphi$ in Theorem 3.11 is a ring isomorphism, this implies $\varphi([\alpha]) = \phi^{-1}(\{\trop(\alpha)\})$.

For arbitrary $\alpha$ let $V(\tau)$ be the maximal closed orbit that contains $\alpha$. We can write $[\alpha] = \sum_{\tau < \sigma \in \Delta(n-k)} a_{\sigma}[V(\sigma)]$. The discussion above shows that $[\trop(\alpha)] = \sum a_{\sigma}[V(\sigma)]$ as tropical cycle classes on $V(\tau)$, hence they are in the same class of cycles on $N_{\mathbb{R}}(\Delta)$. It follows that $\varphi([\alpha]) = \phi^{-1}(\{\trop(\alpha)\})$. \qed

We now state the main theorem of this section, which comes as a corollary of Lemma 5.4 when we let $H_i$ vary. Let $\trop(X) \cdot \trop(X')|_C$ be the part of $\trop(X) \cdot \trop(X')$ that is supported on $C$, which is a tropical cycle in $N_{\mathbb{R}}$.

**Theorem 5.7.** With the same notation as in Lemma 5.4 we have that $[\trop(\overline{i}^*_C(X \cdot X'))] = [\trop(X) \cdot \trop(X')|_C]$ as cycle classes in $A_{k+1-n}(N_{\mathbb{R}}(\Delta))$.

**Proof.** According to Lemma 5.6 and Lemma 5.4 and Lemma 3.18 we have:

$$\deg([\trop(\overline{i}^*_C(X \cdot X'))] \ast [\trop(\overline{H}_1)] \ast \cdots \ast [\trop(\overline{H}_{k+1-n})])$$

$$= \deg([\trop(X) \cdot \trop(X')|_C] \ast [\trop(\overline{H}_1)] \ast \cdots \ast [\trop(\overline{H}_{k+1-n})]).$$

Since the cycles of the form $\trop(\overline{H})$ generate $A_*(N_{\mathbb{R}}(\Delta))$ where $H$ is defined as in Lemma 5.2 we got the desired conclusion. \qed

**Remark 5.8.** When $\trop(X)$ and $\trop(X')$ intersect properly taking $\Delta = \{0\}$ is enough. It follows from [OP13 Corollary 5.1.2] that we actually have $\trop(\overline{i}^*_C(X \cdot X')) = \trop(X) \cdot \trop(X')|_C$ as tropical cycles in $N_{\mathbb{R}}$.

**Corollary 5.9.** With the same notation as in Lemma 5.4 we have:

$$\deg(\trop(\overline{i}^*_C(X \cdot X')) \cdot F) = \deg(\trop(X) \cdot \trop(X')|_C \cdot F),$$

or equivalently

$$\deg(\trop(\overline{i}^*_C(X \cdot X')) \cdot F) = \deg(\trop(X) \cdot \trop(X')|_C \cdot F)$$

for all representatives of $\overline{i}^*_C(X \cdot X')$ and $F \subset N_{\mathbb{R}}$ such that $F$ is compatible with $\Delta$. 

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Note that the corollary is not necessarily true if we ignore the compatibility condition for \( F \) and use the stable intersection on \( N_\mathbb{R} \) for the left hand side of the equation.

**Example 5.10.** Let \( X = X' \) be planes in \( T^3 \) defined by the equation \( x + y + z + 1 = 0 \). Then \( \text{trop}(X) = \text{trop}(X') \) is the standard tropical plane \( P \) in \( \mathbb{R}^3 = \mathbb{R}e_x + \mathbb{R}e_y + \mathbb{R}e_z \) where \( e_x, e_y, e_z \in N \) are dual to \( x, y, z \in M \). The intersection of the tropicalizations has only one component \( P \) and the stable intersection \( \text{trop}(X) \cdot \text{trop}(X') \) is the standard tropical line in \( \mathbb{R}^3 \). Let \( \Delta \) be the complete fan whose 2-skeleton agrees with \( P \), then \( X(\Delta) = \mathbb{P}^3 \) and \( \mathbb{X} \cdot \mathbb{X}' \) is a projective line in \( \mathbb{X} \). First let \( F \) be the plane \( \mathbb{R}e_y + \mathbb{R}e_z \). Take a representative of \( \alpha = \iota_{\mathbb{X}}(\mathbb{X} \cdot \mathbb{X}') \) by the line \( x = a, y + z + 1 - a = 0 \) with \( \text{val}(a) = \text{val}(1 - a) = 0 \). Then \( \text{trop}(\alpha \cap T^3) \) is the standard tropical line in \( F \). We have

\[
\deg(\text{trop}(\alpha \cap T^3 \cdot F)) = 0 \neq 1 = \deg(\text{trop}(X) \cdot \text{trop}(X')|_P \cdot F).
\]

On the other hand if we take \( F = P \) then one checks easily that both sides of the equation in Corollary 5.9 are equal to 1.

### 6. Lifting within Nontoric Ambient Spaces

In this section we assume \( X \) and \( X' \) are of complementary codimension in a subscheme \( Y \) of an algebraic torus, which is not necessarily smooth. Even if \( Y \) is smooth, there is no obvious condition for \( \Delta \) such that the closure \( \overline{Y} \) in \( X(\Delta) \) is still smooth. In this case the intersection cycle \( \mathbb{X} \cdot \mathbb{X}' \) in \( \mathbb{Y} \) is not well-defined, however the intersection multiplicity is still valid at isolated points of \( \mathbb{X} \cap \mathbb{X}' \) at which \( \mathbb{Y} \) is regular, and indeed it is possible to choose \( \Delta \) such that \( \mathbb{Y} \) is at least smooth at points of \( \mathbb{X} \cap \mathbb{X}' \) that we are interested in.

**Situation 6.1.** Assume \( Y \) is a reduced closed subscheme of \( T_N \) of pure dimension \( d \). Let \( X \) and \( X' \) be closed subschemes of \( Y \) of pure dimensions \( k \) and \( l \) such that \( k + l = d \). Let \( C \) be a connected component of \( \text{trop}(X) \cap \text{trop}(X') \) that is contained in the relative interior of a maximal face \( \iota \) of \( \text{trop}(Y) \) of multiplicity one. Let \( W_\mathbb{R} \) be the affine subspace in \( N_\mathbb{R} \) of dimension \( d \) parallel to \( \iota \) and \( W = W_\mathbb{R} \cap N \) the corresponding sublattice. Let \( \Delta \) be a unimodular fan contained in \( W_\mathbb{R} \) which is a compactifying fan for \( \text{trop}(Z) \cap \text{trop}(Z') \cap C \), where \( Z \) and \( Z' \) run over all irreducible components of \( X \) and \( X' \) respectively. Assume there are finitely many points of \( \mathbb{X} \cap \mathbb{X}' \) which tropicalize to \( \overline{C} \), in other words \( Z_\overline{C} \) is a finite set.

We first check that in the above situation \( \mathbb{Y} \) is smooth at points which tropicalize to \( \overline{C} \). Let \( X(\Delta) \) be as before and \( Y(\Delta) \) the toric variety associated to \( \Delta \) where \( \Delta \) is considered as a fan in \( W_\mathbb{R} \). Note that \( X(\Delta) = Y(\Delta) \times T^{n-d} \) and \( N_\mathbb{R}(\Delta) = W_\mathbb{R}(\Delta) \times \mathbb{R}^{n-d} \). We denote by \( \pi: X(\Delta) \to Y(\Delta) \) and \( \overline{\pi}: N_\mathbb{R}(\Delta) \to W_\mathbb{R}(\Delta) \) the projections, which are induced by a splitting of \( N \to N/W \), and \( \text{trop}_Y: Y(\Delta) \to W_\mathbb{R}(\Delta) \) the tropicalization map on \( Y(\Delta) \). Take \( P \subset \text{relint}(i) \) an integral \( G \)-affine polyhedron such that \( \rho(P) \in \Delta \) is a face. Let \( P' \) be the image of \( P \) in \( W_\mathbb{R} \) under the projection \( \overline{\pi} \).

As in [BPR12] §4.29 we let \( U_P \) be the preimage of \( \overline{Y} \) under the map \( \text{trop}: X(\Delta)^{an} \to N_\mathbb{R}(\Delta) \). This is the same as the preimage under \( \text{trop} : X(\rho(P))^{an} \to N_\mathbb{R}(\rho(P)) \), hence according to [Rab12] Proposition 6.9) \( U_P \) is an affinoid domain in \( X(\Delta)^{an} \) whose ring of global sections is integral. Similarly we define \( U_{P'} \) to be the preimage of \( \overline{Y} \) under \( \text{trop}_{P'} : Y(\Delta)^{an} \to W_\mathbb{R}(\Delta) \), which is an affinoid domain in \( Y(\Delta)^{an} \). Let \( \mathcal{Y} = \mathbb{Y}^{an} \cap U_P \) and \( \pi_P: \mathcal{Y} \to U_P \) the map induced by the projection.

**Lemma 6.2.** \( \pi_P \) is an isomorphism.

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Proof. Assume \( \mathcal{Y}^P = \mathcal{M}(\mathcal{B}) \) and \( \mathcal{U}^{P'} = \mathcal{M}(\mathcal{A}) \) where \( \mathcal{A} \) is integral. According to Theorem 4.30 and Corollary 4.32 of [BPR12] \( \pi_P \) is a finite morphism of pure degree one in the sense of [BPR12, §3.15]. Look at the induced map \( \phi : \text{Spec}(\mathcal{B}) \to \text{Spec}(\mathcal{A}) \), which is also finite of pure degree one, so \( \text{Spec}(\mathcal{B}) \) is irreducible. We first claim that \( \text{Spec}(\mathcal{B}) \) is integral. Indeed, since \( \overline{Y}^\text{an} \) is reduced, we know \( \overline{Y}^\text{an} \) is reduced, hence so is \( \mathcal{Y}^P \) as an affinoid domain in \( \overline{Y}^\text{an} \), therefore \( \text{Spec}(\mathcal{B}) \) is reduced.

On the other hand, since \( Y(\Delta)^\text{an} \) is quasi-smooth, so is \( \mathcal{U}^{P'} \), in particular \( \text{Spec}(\mathcal{A}) \) is smooth, and hence \( \mathcal{A} \) is an integral normal ring. Now we have a finite morphism of degree one between integral domains whose source is normal, it must be an isomorphism. Therefore, \( \pi_P \) is an isomorphism. □

Note that Lemma 6.2 generalizes easily to the case where \( P = |\mathcal{P}| \) for a finite collection \( \mathcal{P} \) of integral \( G \)-affine polyhedra in \( \text{relint}(\iota) \) of which \( \Delta \) is a compactifying fan. In this case \( \mathcal{Y}^P \) as an analytic domain in \( \overline{Y}^\text{an} \) is quasismooth, hence \( \overline{Y}^\text{an} \) is regular at points tropicalize to \( \overline{P} \), therefore \( \overline{Y} \) is regular at points tropicalize to \( \overline{P} \). We now have a well-defined intersection multiplicity \( i(x, \overline{X} \cdot \overline{X}' \cdot \overline{Y}) \) for \( x \) such that \( \text{trop}(x) \in \overline{C} \), and are able to state the theorem for non-toric ambient space:

**Theorem 6.3.** In situation 6.1 we have

\[
\sum_{x \in Z_{\mathcal{Y}}} i(x, X \cdot X' \cdot Y) = \sum_{u \in C} i(u, \text{trop}(X) \cdot \text{trop}(X') \cdot \text{trop}(Y)).
\]

**Proof.** We show by passing to analytic spaces that in the above equality, we can replace \( Y \) with \( T_{\mathcal{W}} \) and replace \( X \) and \( X' \) with subschemes of \( T_{\mathcal{W}} \). Since intersection multiplicities and tropicalizations are additive with respect to cycles, we may assume both \( X \) and \( X' \) are reduced. First assume \( \dim(C) \leq \min\{k - 1, l - 1\} \). Choose a splitting of \( N \to N/W \) such that the corresponding projection yields

\[
\overline{\pi}^{-1}(\overline{\pi}(C)) \cap \text{trop}(X) = \overline{\pi}^{-1}(\overline{\pi}(C)) \cap \text{trop}(X') = C.
\]

Take also a \( \Delta \)-thickening \( \mathcal{P} \) of \( C \) such that \( P = |\mathcal{P}| \cap \text{trop}(Y) \) is contained in \( \text{relint}(\iota) \) and that

\[
\overline{\pi}^{-1}(\overline{\pi}(P)) \cap \text{trop}(X) = \overline{\pi}^{-1}(\overline{\pi}(P)) \cap \text{trop}(X) \cap \iota
\]

\[
\overline{\pi}^{-1}(\overline{\pi}(P)) \cap \text{trop}(X') = \overline{\pi}^{-1}(\overline{\pi}(P)) \cap \text{trop}(X') \cap \iota.
\]

Let \( \tilde{X} \) and \( \tilde{X}' \) be the scheme theoretic image of \( X \) and \( X' \) under \( \pi \) in \( Y(\Delta) \), which can be identified as the closure of \( \pi(X) \) and \( \pi(X') \) in \( Y(\Delta) \) where \( \pi(X)(\text{resp. } \pi(X')) \) is the scheme theoretic image of \( X \) (resp. \( X' \)) in \( T_{\mathcal{W}} \). Let \( \mathcal{X}' = \tilde{X}' \cap U^P \) and \( \mathcal{X}' = \tilde{X}' \cap U^P \), let \( \tilde{X}^P = \tilde{X}^\text{an} \cap U^P \) and \( \tilde{X}'^P = \tilde{X}'^\text{an} \cap U^P \). We then have \( \varphi : \mathcal{X}' \to \tilde{X}^P \) induced by the projection. To show that \( \varphi \) is an isomorphism.

We can assume \( P \) is a polyhedron and \( \rho(P) \in \Delta \). First check that \( \varphi \) is surjective. Since \( X \to \tilde{X} \) is dominant, the induced \( \overline{X}^\text{an} \to \overline{X}^\text{an} \) is also dominant. There exists a \( \Delta \)-thickening \( Q' \subset \text{relint}(\iota') \) of \( P' \), such that \( \overline{\pi}^{-1}([Q']) \cap \text{trop}(X) = [Q] \cap \text{trop}(X) \), where \( Q \) is the preimage under \( \overline{\pi} \) of \( Q' \) in \( \iota \). It follows that the map

\[
\overline{X}^\text{an} \cap \text{trop}^{-1}(\overline{\pi}^{-1}([Q']) \cap [Q]) = \overline{X}^\text{an} \cap \text{trop}^{-1}(\overline{\pi}^{-1}([Q']) \cap [Q]) = \overline{X}^\text{an} \cap \text{trop}^{-1}(\overline{\pi}^{-1}([Q']) \cap [Q])
\]

is dominant. As \( \pi_Q \) is isomorphism, the image of the map above is closed, hence the map is surjective, it follows that \( \varphi \) is surjective. On the other hand since \( \pi_P \) is isomorphism, \( \varphi \) is a closed immersion, also the same argument as in Lemma 6.2 shows that both \( \mathcal{X}' \) and \( \tilde{X}'^P \) are reduced, so \( \varphi \) is an isomorphism.

\[\text{[In [BPR12] Theorem 4.30] the conclusion is proved for } \Delta = \{0\} \text{ and } P \text{ a polytope which has trivial recession cone, their argument still works in our case where } \Delta = \rho(P).\]
Note that the same argument shows that $\mathcal{X}' \cdot P$ is isomorphic to $\mathcal{X}' \cdot P'$ under the projection. Now for every point $x \in X_\pi$ we have

$$i(x, \mathcal{X} \cdot \mathcal{X}; Y) = i(x, \mathcal{X}^{\an} \cdot \mathcal{X}^{\an}; Y^{\an}) = i(x, \mathcal{X}' \cdot \mathcal{X}' \cdot P \cdot Y)$$

$$= i(\pi^{an}(x), \mathcal{X}' \cdot \mathcal{X}' \cdot P' \cdot Y') = i(\pi(x), \mathcal{X} \cdot \mathcal{X}; Y(D))$$

by [ORT1] Proposition 5.7. We next check that $\text{trop}(X)$ and $\text{trop}_Y(\pi(X))$ have the same multiplicities at faces contained in $P$ and $P'$ respectively.

Let $\sigma \subset P$ be a bounded face of $\text{trop}(X)$ such that $\tilde{\pi}(\sigma)$ is also a face of $\text{trop}_Y(\pi(X))$. Let $L$ be the sublattice of $W$ of rank $k$ such that $L_\mathbb{R}$ is parallel to $\sigma$. Let $\pi_W: T_W \to T_L$ be the projection corresponds to $\tilde{\pi}_W: W_\mathbb{R} \to L_\mathbb{R}$. Let $\pi_N = \pi_W \circ \pi: T_N \to T_L$ and $\tilde{\pi}_N = \pi_W \circ \tilde{\pi}: N_\mathbb{R} \to L_\mathbb{R}$. For any $w \in \text{relint}(\sigma)$ we have:

$$X^{an} \cap U^w \xrightarrow{\phi} \pi(X)^{an} \cap U^{\tilde{w}(w)} \xrightarrow{\pi^{\tilde{w}}_N} U^{\tilde{\pi}_N(w)}.$$  

Since $\phi$ is an isomorphism, we have $\deg(\pi^{an}_N) = \deg(\pi^{\tilde{w}}_N)$ (when restricted to the above diagram). By [BPRT2] Corollary 4.32 we have

$$m_{\text{trop}(X)}(\sigma) = m_{\text{trop}_Y(\pi(X))}(\tilde{\pi}(\sigma)).$$

Now we can replace $Y$ with $T_W$ and replace $X$ and $X'$ with $\pi(X)$ and $\pi(X')$, then the theorem follows from [ORT1] Theorem 6.4.

For the general case we take $N_+ = N \oplus \mathbb{Z} \oplus \mathbb{Z}$, hence $T_{N_+} = T_N \times G^2_m$. Take also $Y_+ = Y \times G^2_m$ and $X_+ = X \times G_m \times \{1\}$ and $X'_+ = X' \times \{1\} \times G_m$. Then $\text{trop}(Y_+) = \text{trop}(Y) \times \mathbb{R}^2$ and $\text{trop}(X_+) = \text{trop}(X) \times \mathbb{R} \times \{0\}$ and $\text{trop}(X'_+) = \text{trop}(X') \times \{0\} \times \mathbb{R}$ as tropical cycles, and $\text{trop}(X_+) \cap \text{trop}(X'_+) = \text{trop}(X) \cap \text{trop}(X')$. Let $X(\Delta)_+$ be the toric variety associated to $\Delta$ as a fan in $(N_+)_R$. We also have $\mathcal{Y}_+ = \mathcal{Y} \times G^2_m$ and $\mathcal{X}_+ = \mathcal{X} \times G_m \times \{1\}$ and $\mathcal{X}'_+ = \mathcal{X}' \times \{1\} \times G_m$ and $\mathcal{X}_+ \cap \mathcal{X}'_+ = \mathcal{X} \cap \mathcal{X}'$. It follows from the projection formula that for all isolated points $x \in X_+ \cap X'_+$ we have:

$$i(x, \mathcal{X}_+ \cdot \mathcal{X}_+; Y_+) = i(x, \mathcal{X}_+ \cdot (\mathcal{X}' \times \{1\}); Y \times G_m) = i(x, \mathcal{X} \cdot \mathcal{X}; Y).$$

Also for $u \in C$ we have

$$i(u, \text{trop}(X_+) \cdot \text{trop}(X'_+); \text{trop}(Y_+)) = i(u, \text{trop}(X_+) \cdot (\text{trop}(X') \times \{0\}); \text{trop}(Y) \times \mathbb{R})$$

$$= i(u, \text{trop}(X) \cdot \text{trop}(X'); \text{trop}(Y')).$$

Note that $\dim C \leq \dim(X) = \dim(X_+) - 1$ and $\dim C \leq \dim(X') = \dim(X'_+) - 1$, hence we reduced to the case above and the theorem is proved.

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