THE ALGEBRA OF CONJUGACY CLASSES IN
SYMMETRIC GROUPS, AND PARTIAL PERMUTATIONS

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1. Introduction. This article was originally published in Russian in "Representation Theory, Dynamical Systems, Combinatorial and Algorithmical Methods III" (A. M. Vershik, ed.), Zapiski Nauchnyh Seminarov POMI 256 (1999), 95–120 (this text in Russian is available via http://www.pdmi.ras.ru/znsl/1999/v256.html).

The main purpose of this note is to prove a convolution formula for conjugacy classes in symmetric groups suggested in [7] (formula (2.2), see also [8]).

Given a partition $\rho \vdash r$ of a positive integer $r$, where $r \leq n$, denote by $\tilde{\rho} = \rho \cup (1^{n-r})$ the partition of $n$ obtained from $\rho$ by adding an appropriate number ofunities. Let $C_{\rho; n}$ be the sum of permutations $w \in \mathfrak{S}_n$ of cycle type $\tilde{\rho}$. We also define “normalized” classes

$$A_{\rho; n} = \left( \binom{n - r + m_1(\rho)}{m_1(\rho)} \right) C_{\rho; n},$$ (1.1)

where $m_1(\rho)$ is the number of unities in the partition $\rho$. We will show that there exist integer constants $g_{\sigma, \tau}^\rho$ which define the convolution of normalized conjugacy classes in the symmetric group $\mathfrak{S}_n$:

$$A_{\sigma; n} * A_{\tau; n} = \sum_\rho g_{\sigma, \tau}^\rho A_{\rho; n}. \quad (1.2)$$

Formula (1.2) is valid for $n$ great enough. More exactly, $n$ must be not less than $|\sigma| + |\tau|$, where $|\rho|$ denotes the number such that $\rho$ is a partition of $|\rho|$; otherwise the summands in the right-hand side which are not realized by permutations from $\mathfrak{S}_n$ should be omitted. Note that the elements $C_{\rho; n}$ do not form a basis in the algebra of central functions on $\mathfrak{S}_n$: they may coincide for different partitions $\rho$.

Formula (1.2) immediately implies an old result [5]:

$$C_{\sigma; n} * C_{\tau; n} = \sum_\rho q_{\sigma, \tau}^\rho(n) C_{\rho; n}, \quad (1.3)$$

where $\sigma, \tau, \rho$ run only over partitions without unity summands, and the coefficients $q_{\sigma, \tau}^\rho(n)$ depend on $n$ in a polynomial way.

To prove (1.2), we introduce semigroups $\mathcal{P}_n$ of partial permutations of the set $\{1, 2, \ldots, n\}$. The semigroup algebras $\mathbb{C}[\mathcal{P}_n]$ are semi-simple and form a projective...
family with limit $B_\infty = \lim_{B} C[P_n]$. The group of finite permutations $S_\infty$ acts in $B_\infty$ by conjugations. Its orbits $A_\rho$ are indexed by all partitions of positive integers, and form a basis in the algebra of invariants $A_\infty = B_\infty^{S_\infty}$. The numbers $g_{\rho,\tau}^\rho$ arise as the multiplication structure constants of the algebra $A_\infty$ in this basis.

We show that the algebra $A_\infty$ is naturally isomorphic to the algebra of shifted symmetric functions $\Lambda^*$ introduced in [3]. This isomorphism plays the same role for convolution of central elements in the algebras $C[S_n]$ as the characteristic mapping $ch$ plays for the multiplication — inducing of characters of symmetric groups, see [9, I.7]. We also give examples of filtrations on the algebra $A_\infty$.

2. The semigroup of partial permutations. Denote by $P_n = \{1, \ldots, n\}$ a segment of positive integers, and by $S_n$ — the group of all permutations of $P_n$. A partial permutation of the set $P_n$ is a pair $\alpha = (d, w)$ consisting of an arbitrary subset $d \subset P_n$ and an arbitrary bijection $w: d \rightarrow d$ of this subset. The set $d$ will be referred to as the support of $\alpha$. Denote by $P_n$ the set of all partial permutations of the set $P_n$.

Obviously, the number of elements in $P_n$ equals

$$s_n = \sum_{k=0}^{n} \binom{n}{k} k! = \sum_{k=0}^{n} (n \upharpoonright k), \tag{2.1}$$

where $(n \upharpoonright k) = n(n-1) \ldots (n-k+1)$ is the falling factorial power. These numbers satisfy a recurrence relation $s_n = ns_{n-1} + 1$. Several first values are $s_0 = 1, s_1 = 2, s_2 = 5, s_3 = 16, s_4 = 65$.

Given a partial permutation $(d, w) \in P_n$, denote by $\tilde{w}$ the permutation of the whole set $P_n$ coinciding with $w$ on $d$ and identical outside $d$. The permutation $\tilde{w}$ is well defined on all subsets containing its support. This allows us to introduce a natural multiplication on the set of partial permutations.

Given two partial permutations $(d_1, w_1), (d_2, w_2)$, we define their product as the pair $(d_1 \cup d_2, w_1w_2)$. With this multiplication, $P_n$ becomes a semigroup. The partial permutation $(\emptyset, e_0)$, where $e_0$ is the trivial permutation of the empty set $\emptyset$, is the unity in $P_n$, and it is a unique invertible element in the semigroup $P_n$.

3. The semigroup algebra. Denote by $B_n = C[P_n]$ the complex semigroup algebra of the semigroup $P_n$. Let us check that this algebra is semi-simple and find its irreducible representations.

Fix a subset $x \subset P_n$ of size $|x| = k$ and denote by $S_x$ the group of permutations of the subset $x$. It is easy to see that the formula

$$\varphi_x(d, w) = \begin{cases} \tilde{w}, & \text{if } d \subset x; \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

defines a homomorphism of algebras $B_n \rightarrow C[S_x]$. This homomorphism is obviously surjective.

Lemma 3.1. Let

$$b = \sum_{k=0}^{n} \sum_{|d|=k} \sum_{w \in S_d} b_{d, w} (d, w)$$

be an element of the algebra $B_n$. Then the following conditions are equivalent:

1. $\varphi_y(b) = 0$ for all $y \subset x$;
2. $b_{d, w} = 0$ for all $d \subset x$. 


Proof. Obviously, (2) implies (1). To prove that (1) implies (2), we use induction on the size \( k = |x| \) of the set \( x \).

If \( k = 0 \), we have \( \varphi_\emptyset(b) = b_{\emptyset, e_0} e_0 \), and the statement is obvious.

Let \( x \subset \mathbb{P}_n \) and let \( v \) be a permutation of the set \( x \). Denote by \( \overline{d} \) the set of non-fixed points of the permutation \( v \), and by \( \overline{v} \) — the restriction of \( v \) on \( \overline{d} \). Partial permutations \((d, w)\) with \( \varphi_x(d, w) = v \) are characterized by the following conditions:

1. \( \overline{d} \subset d \subset x \);
2. the restriction of \( w \) on \( \overline{d} \) coincides with \( \overline{v} \);
3. all points of \( d \setminus \overline{d} \) are fixed for \( w \).

Thus the coefficient of \( v \) in the decomposition of \( \varphi_x(b) \) equals

\[
\sum_{d \subset \overline{d} \subset x} \sum_{\overline{w} = \overline{v}} b_{d, w},
\]

where \( \overline{w} \) denotes the restriction of a permutation \( w \) on the set of its non-fixed points. The unique summand in this sum with \( d = x \) equals \( b_{x,v} \). But by the induction hypothesis all summands with \( d \nsubsetneq x \) are zero, hence this coefficient equals \( b_{x,v} \).

The Lemma follows. \( \square \)

**Corollary 3.2.** The algebra \( \mathcal{B}_n \) is semi-simple, and it is isomorphic to the direct sum of the group algebras of symmetric groups,

\[
\bigoplus_{x \subset \mathbb{P}_n} \varphi_x : \mathcal{B}_n \cong \bigoplus_{x \subset \mathbb{P}_n} \mathbb{C}[\mathfrak{S}_x].
\] (3.2)

**Proof.** By Lemma 3.1, the homomorphism \( \varphi = \bigoplus \varphi_x \) is injective, and the dimension of the right-hand side equals

\[
\sum_{x \subset \mathbb{P}_n} |x|! = \sum_{k=0}^n \binom{n}{k} k! = |\mathbb{P}_n|
\]

and coincides with the dimension of \( \mathcal{B}_n \). Thus \( \varphi \) is an isomorphism. \( \square \)

Let

\[
\epsilon_d = \sum_{y : d \subset y \subset \mathbb{P}_n} (-1)^{|y| - |d|} (y, e),
\] (3.3)

where \( e \) is the identity permutation. It is easy to see that \( \varphi_d(\epsilon_d) = e \in \mathfrak{S}_d \) and \( \varphi_x(\epsilon_d) = 0 \) for \( x \neq d \). Thus the element \( \epsilon_d \) of the algebra \( \mathcal{B}_n \) is a central projection. Minimal central projections are of the form \( \epsilon_d \delta \lambda \), where \( d \subset \mathbb{P}_n \) and \( \delta \lambda \) runs over minimal central projections in the algebra \( \mathbb{C}[\mathfrak{S}_d] \). The centre of the algebra \( \mathcal{B}_n \) is of the form

\[
Z(\mathcal{B}_n) \cong \bigoplus_{d \subset \mathbb{P}_n} Z(\mathbb{C}[\mathfrak{S}_d]),
\]

where \( Z(\mathbb{C}[\mathfrak{S}_d]) \) is the centre of the group algebra \( \mathbb{C}[\mathfrak{S}_d] \). The dimension of the centre equals

\[
\dim Z(\mathcal{B}_n) = \sum_{k=0}^n \binom{n}{k} p(k),
\] (3.4)

where \( p(k) \) is the number of partitions of \( k \).
4. Conjugacy classes in $\mathcal{P}_n$. The symmetric group $\mathfrak{S}_n$ acts on the semigroup $\mathcal{P}_n$ by automorphisms $(d, w) \mapsto (vd, vwv^{-1})$. The orbits of this action will be referred to as conjugacy classes in $\mathcal{P}_n$. It is obvious that two partial permutations are conjugate if and only if the sizes of their supports coincide as well as their cycle types. Thus the conjugacy classes $A_{\rho;n} \subset \mathcal{P}_n$ are indexed by partial partitions of $n$, i.e. by partitions $\rho \vdash r$ of any integers $0 \leq r \leq n$. In particular, $A_{\emptyset;0} = \{(\emptyset, e)\}$.

Given a partial partition $\rho \vdash r \leq n$, denote by $\tilde{\rho} = \rho \cup \{1^{n-r}\}$ the partition of $n$ obtained by adding an appropriate number of unities. Let $C_{\rho;n}$ be the conjugacy class in the group $\mathfrak{S}_n$ consisting of permutations of cycle type $\tilde{\rho}$. As usual, denote by $m_k = m_k(\rho)$ the number of rows of length $k$ in the partition $\rho$. The complement $\mathbb{P}_n \setminus d$ of the support of the partial permutation $(d, w)$ contains $n - r$ points, thus the total number of fixed points of $\tilde{w}$ equals $n - r + m_1(\rho)$.

Denote by $\psi : (d, w) \mapsto \tilde{w}$ the homomorphism of forgetting the support of a partial permutation.

Let a permutation $v \in \mathfrak{S}_n$ have cycle type $\tilde{\rho}$. The set $\psi^{-1}(v) \cap A_{\rho;n}$ consists exactly of partial permutations $(d, w)$ such that the support $d$ contains all non-fixed points of the permutation $v$. In the set $d$, one may arbitrarily choose $m_1(\rho)$ fixed points from the total number of fixed points of the permutation $v$ which is equal to $n - r + m_1(\rho)$. Hence the numbers of elements in the class $A_{\rho;n}$ and in the conjugacy class $C_{\rho;n}$ of the group $\mathfrak{S}_n$ are related by the formula

$$|A_{\rho;n}| = \binom{n - r + m_1(\rho)}{m_1(\rho)} |C_{\rho;n}|.$$  \hfill (4.1)

The action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{P}_n$ can be continued by linearity to an action of $\mathfrak{S}_n$ on the algebra $\mathcal{B}_n$. Denote by $A_n = B_n^{\mathfrak{S}_n}$ the subalgebra of invariant elements for this action.

The homomorphism $\psi : \mathcal{P}_n \to \mathfrak{S}_n$ can also be continued to a surjective homomorphism of algebras $\psi : \mathcal{B}_n \to \mathbb{C}[\mathfrak{S}_n]$. It commutes with the action of the group $\mathfrak{S}_n$ by conjugations on the algebras $\mathcal{B}_n$ and $\mathbb{C}[\mathfrak{S}_n]$. Thus $\psi(A_n) = Z(\mathbb{C}[\mathfrak{S}_n])$, where $Z(\mathbb{C}[\mathfrak{S}_n])$ is the centre of the group algebra $\mathbb{C}[\mathfrak{S}_n]$.

Let us identify the conjugacy class $A_{\rho;n}$ with the element

$$A_{\rho;n} = \sum_{(d, w) \in A_{\rho;n}} (d, w)$$  \hfill (4.2)

of the algebra $\mathcal{B}_n$. In particular, if $|\rho| > n$, then $A_{\rho;n} = 0$. It follows from our definitions that the elements $A_{\rho;n}$, where $|\rho| \leq n$, form a linear basis of the algebra $\mathcal{A}_n$. It is clear that

$$\psi(A_{\rho;n}) = \binom{n - r + m_1(\rho)}{m_1(\rho)} C_{\rho;n}.$$  \hfill (4.3)

In Sect. 12 we construct all irreducible representations $\pi_{x,\lambda}$ of the algebra $\mathcal{B}_n$. Note that in any irreducible representation $\pi_{x,\lambda}$ of the algebra $\mathcal{B}_n$ the element $A_{\rho;n}$ acts as a scalar operator. Thus $A_{\rho;n} \in Z(\mathcal{B}_n)$, and the algebra $\mathcal{A}_n$ lies in the centre $Z(\mathcal{B}_n)$. This inclusion is strict for $n \geq 2$.

5. Algebras $\mathcal{B}_\infty$ and $\mathcal{A}_\infty$. Let $m \leq n$. We introduce a mapping $\theta_n : \mathcal{B}_n \to \mathcal{B}_m$ by the formula

$$\theta_n(d, w) = \begin{cases} (d, w), & \text{if } d \subset \mathbb{P}_m, \\ 0 & \text{otherwise.} \end{cases}$$  \hfill (5.1)
The mapping $\theta_m$ is a homomorphism of algebras and it commutes with the action of the group $S_m$ on $B_n$ and on $B_m$. Hence $\theta_m(A_n) = A_m$.

Define the degree of a partial permutation $(d, w) \in P_n$ as $\deg(d, w) = |d|$. Given $b = \sum_{\alpha \in P_n} b_{\alpha} \alpha \in B_n$, let $\deg(b) = \max \deg \alpha$, where the maximum is over all $\alpha$ with $b_{\alpha} \neq 0$. The function $\deg$ defines a filtration on the algebra $B_n$. Note that $\deg(\theta_m(b)) \leq \deg b$ for all $b \in B_n$.

Denote by $B_\infty$ the projective limit of the algebras $B_n$ with respect to the morphisms $\theta_n$, and by $A_\infty$ — the projective limit of the algebras $A_n$. Both limits are taken in the category of filtered algebras.

Let $S_\infty$ be the infinite symmetric group, i.e. the group of finite permutations of positive integers. The group $S_\infty$ acts naturally on $B_\infty$, and $A_\infty = B_\infty^{S_\infty}$ is the subalgebra of invariants for this action.

6. Structure constants of the algebra $A_\infty$. Denote by $\theta_n$ the natural homomorphism $\theta_n : B_\infty \rightarrow B_n$ as well as its restriction on $A_\infty$. The natural inclusion of algebras $i_n : B_n \rightarrow B_\infty$ accords with the projection $\theta_n$: $\theta_n \circ i_n = id_{B_n}$. It is convenient to write elements of $B_\infty$ as formal infinite sums,

$$b = \sum_{n=0}^{\infty} \sum_{|d|=n} \sum_{w \in S_d} b_{d,w} (d, w).$$  \hspace{1cm} (6.1)

Given a partition $\rho \vdash r$, let $A_\rho = \sum (d, w)$; the sum extends to partial permutations $(d, w) \in P_\infty$ such that $|d| = r$ and $w$ has cycle type $\rho$. The elements $A_\rho$, where $\rho$ runs over all partitions, form a linear basis in $A_\infty$.

Denote by $g_{\sigma, \tau}^\rho$ the structure constants of the algebra $A_\infty$ in the basis $\{A_\rho\}$,

$$A_\sigma A_\tau = \sum_{\rho} g_{\sigma, \tau}^\rho A_\rho.$$  \hspace{1cm} (6.2)

Note that $\theta_n(A_\rho) = A_{\rho;n}$, where $A_{\rho;n}$ is the element of the algebra $A_n$ introduced in Sect. 4. Since $\theta_n : A_\infty \rightarrow A_n$ is a homomorphism, we obtain the following statement.

**Proposition 6.1.**

$$A_{\sigma;n} A_{\tau;n} = \sum_{\rho} g_{\sigma, \tau}^\rho A_{\rho;n}.$$  \hspace{1cm} (6.3)

For $|\rho| \leq n$, by definition $A_{\rho;n} = 0$. Let us illustrate Proposition 6.1 by an example. The simplest non-trivial multiplication formula in the algebra $A_\infty$ is

$$A_{(2)} A_{(2)} = A_{(1^2)} + 3A_{(3)} + 2A_{(2^2)}.$$  

In the algebras $A_2, A_3, A_4$ we have

$$A_{(2);2} A_{(2);2} = A_{(1^2);2},$$

$$A_{(2);3} A_{(2);3} = A_{(1^2);3} + 3A_{(3);3},$$

$$A_{(2);4} A_{(2);4} = A_{(1^2);4} + 3A_{(3);4} + 2A_{(2^2);4}.$$  \hspace{1cm} (6.4)

Let us give a useful combinatorial interpretation of the structure constants $g_{\sigma, \tau}^\rho$. 


Proposition 6.2. Given a partition $\rho \vdash r$, let $d_{\rho} = \mathbb{P}_r$ and let
\[
w_{\rho} = (1, \ldots, \rho_1)(\rho_1 + 1, \ldots, \rho_1 + \rho_2) \ldots (|\rho| - \rho_{\ell(\rho)} + 1, \ldots, |\rho|)
\]
be a fixed permutation of the set $d_{\rho}$ of cycle type $\rho$. Consider the set $G_{\sigma,\tau}^{\rho}(n)$ of pairs $((d_1, w_1), (d_2, w_2)) \in \mathcal{P}_n \times \mathcal{P}_n$ such that
1. $(d_1, w_1) \in A_{\sigma;n}$, $(d_2, w_2) \in A_{\tau;n};$
2. $d_1 \cup d_2 = d_{\rho}$, $w_1 w_2 = w_{\rho}$.

Then for $n \geq r$ the number of elements $|G_{\sigma,\tau}^{\rho}(n)|$ equals $g_{\sigma,\tau}^{\rho}$. □

Proof. The partial permutation $(d_{\rho}, w_{\rho})$ belongs to the class $A_{\rho}$. By definition, the number $|G_{\sigma,\tau}^{\rho}(n)|$ is the coefficient of the element $(d_{\rho}, w_{\rho})$ in the product $A_{\sigma;n} A_{\tau;n}$. If $n \geq |\rho|$, then $A_{\rho;n} \neq 0$ and using Proposition 6.1 we obtain $|G_{\sigma,\tau}^{\rho}(n)| = g_{\sigma,\tau}^{\rho}$. □

Proposition 6.3. If $g_{\sigma,\tau}^{\rho} \neq 0$, then $|\rho| \leq |\sigma| + |\tau|$.

Proof. If $g_{\sigma,\tau}^{\rho} \neq 0$, then it follows from Proposition 6.2 that $G_{\sigma,\tau}^{\rho}(|\rho|) \neq \emptyset$. Hence there exist sets $d_1, d_2$ with $|d_1| = |\sigma|$, $|d_2| = |\tau|$, $|d_1 \cup d_2| = |\rho|$. Thus $|\rho| \leq |\sigma| + |\tau|$. □

Remark 6.4. Denote by $\sigma \cup \tau$ the partition whose parts are obtained by uniting all parts of partitions $\sigma$ and $\tau$. It follows from Proposition 6.2 that $\rho = \sigma \cup \tau$ is the unique partition with $|\rho| = |\sigma| + |\tau|$ and $g_{\sigma,\tau}^{\rho} \neq 0$, the coefficient being equal to
\[
g_{\sigma \cup \tau}^{\rho} = \prod_{k \geq 1} \left( \frac{m_k(\sigma) + m_k(\tau)}{m_k(\sigma)} \right).
\]

7. Convolutions of conjugacy classes in symmetric groups. Given an arbitrary partition $\rho \vdash r$, denote by $\overline{\rho}$ the partition obtained from $\rho$ by removing all its unity parts (if they existed). Thus $m_1(\overline{\rho}) = 0$ and $\rho = \overline{\rho} \cup 1^{r-|\overline{\rho}|}$.

Recall that we have selected in the center of the group algebra of the symmetric group $Z(\mathbb{C}[\mathfrak{S}_n])$ the elements $C_{\rho;n}$ indexed by partitions $\rho \vdash \{\rho\}$ of any numbers $0 \leq r \leq n$,
\[
C_{\rho;n} = \sum_{w \text{ is of type } \overline{\rho}} w.
\] (7.1)

In particular, $C_{\rho;n} = 0$, if $|\rho| > n$. Assume now that $|\sigma|, |\tau| \leq n$; then $C_{\sigma;n} = C_{\tau;n}$ if and only if $\overline{\sigma} = \overline{\tau}$. Let us say that a partition $\rho \vdash r$ is proper, if $m_1(\rho) = 0$. Denote the set of all proper partitions by $\overline{\mathfrak{Y}}$. The elements $\{C_{\rho;n} : \rho \in \overline{\mathfrak{Y}}, |\rho| \leq n\}$ form a linear basis in $Z(\mathbb{C}[\mathfrak{S}_n])$.

Theorem 7.1. Given a partition $\rho \vdash r \leq n$, consider the images $\psi(A_{\rho;n})$
\[
\psi(A_{\rho;n}) = \left( \frac{n - r + m_1(\rho)}{m_1(\rho)} \right) C_{\rho;n}
\] (7.2)
of the elements $A_{\rho;n} \in \mathfrak{A}_n$ in the centre of the group algebra of the symmetric group $\mathfrak{S}_n$ under the “forgetting support” mapping from Sect. 4. Then

a) for every $n$, the following equality holds,
\[
\psi(A_{\sigma;n}) \psi(A_{\tau;n}) = \sum_{\rho} g_{\sigma,\tau}^{\rho} \psi(A_{\rho;n}).
\] (7.3)
where $g^\rho_{\sigma,\tau}$ are the structure constants of the algebra $A_\infty$ which do not depend on $n$.

b) for $n \geq |\sigma| + |\tau|$, the sum in the right-hand side of the equality (7.3) is stable, i.e. the non-zero summands are indexed by the same partitions $\rho$.

**Proof.** a) Equality (7.3) follows from the fact that $\psi$ is a homomorphism and from Proposition 6.1.

b) The collection of non-zero summands in the right-hand side of equality (7.3) is indexed by partitions $\rho$ such that $|\rho| \leq n$ and $g^\rho_{\sigma,\tau} \neq 0$. By Proposition 6.3, this collection is fixed for $n \geq |\sigma| + |\tau|$. $\square$

**Proposition 7.2.** Let numbers $h^\rho_{\sigma,\tau}$ satisfy

$$\psi(A_\sigma;n) \psi(A_\tau;n) = \sum_\rho h^\rho_{\sigma,\tau} \psi(A_\rho;n)$$

for every $n$. Then $h^\rho_{\sigma,\tau} = g^\rho_{\sigma,\tau}$ for all $\rho, \sigma, \tau$.

**Proof.** Suppose the contrary. Choose arbitrary partitions $\sigma$ and $\tau$. Let $\rho$ be a partition such that $h^\rho_{\sigma,\tau} \neq g^\rho_{\sigma,\tau}$ and $h^\nu_{\sigma,\tau} = g^\nu_{\sigma,\tau}$ for any partitions $\nu$ with $|\nu| < |\rho|$. Let $|\rho| = n$; then

$$0 = \sum_\nu (g^\nu_{\sigma,\tau} - h^\nu_{\sigma,\tau}) \psi(A_\nu;n) = \sum_{|\nu| = n} (g^\nu_{\sigma,\tau} - h^\nu_{\sigma,\tau}) \psi(A_\nu;n).$$

On the other hand, the set $\{\psi(A_\nu;n) : |\nu| = n\}$ is linearly independent. The obtained contradiction proves the Proposition. $\square$

Thus $\{\psi(A_\rho;n)\}$ is a family of elements proportional to conjugacy classes (with proportionality coefficients depending on $\rho$ and on $n$), and the multiplication structure constants do not depend on $n$. A similar family was introduced earlier in [7, 8]. The notion of such elements is also close to [1].

Let us come back to multiplication of conjugacy classes $C_\rho;n$.

**Proposition 7.3.** Let $\sigma, \tau, \rho \in \mathbb{Y}$ be proper partitions (i.e. without unity parts). We define polynomials $q^\rho_{\sigma,\tau}(n)$ as

$$q^\rho_{\sigma,\tau}(n) = \sum_{k \geq 0} g^{\rho \cup (1^k)}_{\sigma,\tau} \binom{n - |\rho|}{k}. \quad (7.4)$$

Then

$$C_\sigma;n \ast C_\tau;n = \sum_{|\rho| \leq n, m_1(\rho) = 0} q^\rho_{\sigma,\tau}(n) C_\rho;n \quad (7.5)$$

for all $\sigma, \tau \in \mathbb{Y}$.

The fact that the coefficients $q^\rho_{\sigma,\tau}(n)$ in (7.5) are polynomials on $n$ assuming only integer values in integer points was first obtained in [5, Theorem 2.2].

**Proof.** For a proper partition $\sigma$, formula (4.3) becomes simpler and reduces to $\psi(A_\sigma;n) = C_\sigma;n$. Hence

$$C_\sigma;n C_\tau;n = \psi(A_\sigma;n) \psi(A_\tau;n) = \sum_{|\rho| \leq n} g^\rho_{\sigma,\tau} \psi(A_\rho;n) =$$

$$\sum_{|\rho| \leq n} g^\rho_{\sigma,\tau} \binom{n - |\rho| + m_1(\rho)}{m_1(\rho)} C_\rho;n.$$
Collecting similar summands of the form $C_{\rho;n} = C_{\rho,(1^k)n}$ for $k \geq 0$, we obtain formula (7.4) for coefficients in (7.5). □

It follows from Remark 6.4 that $\deg q_{\sigma,\tau}^{(\cup)} = 0$.

Remark 7.4. The sum participating in formula (7.4) may contain more than one non-zero summand. Consider, for example, $\sigma = \tau = (3)$. Formulae from Sect. 11 show that both $g_{\sigma,\tau}^{(3)} \neq 0$ and $g_{\sigma,\tau}^{(3,1)} \neq 0$.

Remark 7.5. Let $\Phi$ be the ring of polynomials in one variable assuming only integer values in integer points, and let $\Phi_C$ be the ring of polynomials of the form $\tilde{P}(t) = cP(t)$, where $P \in \Phi$ and $c \in \mathbb{C}$. In [5] the authors deal with a $\Phi$-algebra $K$ freely generated over $\Phi$ by a basis $C_\rho$, where $\rho \in \mathcal{Y}$ runs over proper partitions (without unity parts). A formal relation of our algebra $A_\infty$ and the algebra $K$ is given by the fact that the mapping

$$A_\rho \mapsto \left(n - |\rho| + m_1(\rho)\right)C_{\rho}$$

defines an epimorphism of $\Phi_C$-algebras $A_\infty \otimes_\Phi \Phi_C$ and $K \otimes_\Phi \Phi_C$.

8. The semigroup of fillings. Given a partition $\rho \vdash r$, denote by

$$z_\rho = \prod_{i \geq 1} i^{m_i(\rho)} m_i(\rho)!$$

the size of the centralizer of a permutation of cycle type $\rho$. Note that

$$\frac{z_\rho}{m_1(\rho)!} = z_\sigma = \frac{z_{\tilde{\rho}}}{(n - r + m_1(\rho))!}.$$

Along with the introduced above elements $A_\rho$, we consider a related basis in the algebra $A_\infty$ consisting of the elements $a_\rho = z_\rho A_\rho$. Let $f_{\sigma,\tau}^\rho$ be the multiplication structure constants in this new basis,

$$a_{\sigma;n} a_{\tau;n} = \sum_\rho f_{\sigma,\tau}^\rho a_{\rho;n}.$$

They are related to the constants $g_{\sigma,\tau}^\rho$ by an obvious formula

$$f_{\sigma,\tau}^\rho = \frac{z_\sigma z_\tau}{z_\rho} g_{\sigma,\tau}^\rho.$$

We will show that all numbers $f_{\sigma,\tau}^\rho$ are non-negative integers and give their combinatorial interpretation similar to Proposition 6.2.

Definition. Let $\lambda$ be a Young diagram with $k$ boxes, and $d$ be the set of $k$ distinct positive integers. Any bijection $R : \lambda \rightarrow d$ will be referred to as a filling of shape $\lambda$ and weight $d$, and the set $d$ will be called the support of the filling $R$. Define a permutation $w_R \in \mathfrak{S}_d$ by declaring the rows of the filling $R$ to be cycles, so that the cycle type of $w_R$ is $\lambda$. Obviously, the number of fillings of shape $\lambda$ and a fixed weight equals $n! = z_\lambda |C_\lambda|$.
For example, given a filling

\[
R = \begin{pmatrix}
4 & 3 & 1 \\
9 & 2 & 7 \\
6 & 5
\end{pmatrix}
\]

of the diagram \( \lambda = (3, 3, 2) \) with support \( d = \{1, 2, 3, 4, 5, 6, 7, 9\} \), we have \( w_R = (1, 4, 3) \ (2, 7, 9) \ (5, 6) \).

We define the convolution \( R = S * T \) of fillings \( S, T \) by the following rules.

Let \( d_R = d_S \cup d_T \) be the union of the supports of fillings \( S, T \). Order the set \( d_R \) by reading first the elements of \( S \) and then the elements of \( T \) from left to right along each row and from top to bottom. Repeating elements of the filling \( T \) are to be ignored.

The first element of \( d_R \) is the first element \( s \) of the first row of the filling \( S \). Form a row of the filling \( R \), which we want to define, as the cycle of the product of permutations \( \tilde{w}_S \) and \( \tilde{w}_T \) containing \( s \) and beginning with \( s \). The tilde means that the permutations \( w_S, w_T \) are trivially continued to be defined on the set \( d_R \): all points outside the former domain are assumed fixed.

Passing to constructing other rows of the filling \( R \), assume that a part of rows of \( R \) is already constructed. If the elements of the set \( d_R \) are not all used, denote by \( s \) the first of the remained elements (in the above-mentioned order). The next row of \( R \) is the cycle of the product of permutations \( \tilde{w}_S \) and \( \tilde{w}_T \) containing \( s \). The first element of the row is chosen to be \( s \).

The lengths of rows formed according to these rules do not necessarily decrease. Let us reorder the rows by decreasing of lengths without changing the respective order of rows of equal length. The obtained filling \( R \) is the convolution of fillings \( S \) and \( T \).

We illustrate the definition of the convolution by an example. Let

\[
S = \begin{pmatrix}
3 & 4 & 5 & 6 & 9 \\
2 & 1 & 7
\end{pmatrix}; \quad T = \begin{pmatrix}
4 & 3 & 2 \\
1 & 9 & 6 \\
8
\end{pmatrix}.
\]

Then \( d_R = \mathbb{P}_9 \), and the product of the permutations \( \tilde{w}_S \) and \( \tilde{w}_T \) is equal to \( (1, 3)(2, 5, 6, 7)(4)(8)(9) \). Hence

\[
S * T = \begin{pmatrix}
3 & 1 & 4 & 5 & 6 & 7 & 2 \\
5 & 6 & 7 & 2 \\
9 & 8
\end{pmatrix} = \begin{pmatrix}
5 & 6 & 7 & 2 \\
3 & 1 & 4 \\
9 & 8
\end{pmatrix}.
\]

**Proposition 8.1.** Fix Young diagrams \( \sigma, \tau \). Given a partition \( \rho \vdash r \), put \( d_\rho = \mathbb{P}_r \) and let \( R_\rho \) be the canonical filling of the Young diagram \( \rho \) in which the boxes are indexed with the numbers \( 1, 2, \ldots, r \) successively from left to right and from top to bottom. Consider the set \( F_{\sigma, \tau}^\rho \) of pairs of fillings \((S, T)\) such that

1. \( S \) is of shape \( \sigma \), and \( T \) is of shape \( \tau \);
2. \( S * T = R_\rho \).
Then the number of elements \(|F_{\sigma, \tau}^\rho|\) equals \(f_{\sigma, \tau}^\rho\). In particular, the structure constants in (8.2) are non-negative integers.

**Proof.** The set \(F_r\) of fillings \(R\) with support \(dR \subset \mathbb{P}_r\) forms a semigroup with respect to the introduced above convolution operation. Associating a partial permutation \((d_r, w_R)\) with a filling \(R\), we obtain an epimorphism of semigroups \(u : F_r \to \mathcal{P}_r\).

The group \(\mathcal{G}_r\) acts in an obvious way by automorphisms of the semigroup \(F_r\), and the homomorphism \(u\) is equivariant under this action. Each partial permutation \((d, w) \in \mathcal{P}_r\) of cycle type \(\rho\) has exactly \(z_\rho\) inverse images in \(F_r\). Thus the multiplication structure constants \(|F_{\sigma, \tau}^\rho|\) and \(g_{\sigma, \tau}^\rho\) are related by a formula of type (8.3), and the Proposition follows. \(\square\)

One can easily deduce from Proposition 8.1 formulae for coefficients \(f_{\sigma, \tau}^\rho\) in the simplest cases.

**Corollary 8.2.** If \(\rho = \sigma \cup \tau\), then \(f_{\sigma, \tau}^\rho = 1\).

**Corollary 8.3.** Let \(\rho = \sigma \cup \tau\) be obtained from Young diagrams \(\sigma, \tau\) by (1) replacing a row of \(\sigma\) of length \(i\) and a row of \(\tau\) of length \(j\) by a row of length \(i + j - 1\), and (2) uniting the remained rows (with subsequent ordering by decreasing).

Let \(m_i(\sigma)\) be the multiplicity of rows of length \(i\) in \(\sigma\), and \(m_j(\tau)\) be the multiplicity of rows of length \(j\) in \(\tau\). Then \(f_{\sigma, \tau}^\rho = \text{im}_i(\sigma) \text{jm}_j(\tau)\).

9. Isomorphism of the algebra \(A_\infty\) and the algebra of shifted symmetric functions. The algebra \(\Lambda^*\) of shifted symmetric functions was introduced and studied in [10, 8, 3]. In this section we establish an isomorphism of this algebra with the algebra \(A_\infty\) and indicate the elements of \(A_\infty\) corresponding to the shifted Schur functions \(s_\lambda^*\) and shifted analogues of the Newton power sums \(p_\mu^s\) introduced in [3, (1.6) and (14.9)].

The algebra \(\Lambda^*\) is defined as follows. Denote by \(\Lambda^*(n)\) the algebra of polynomials with complex coefficients in \(x_1, \ldots, x_n\) that become symmetric in new variables \(x'_i = x_i - i\). The algebra \(\Lambda^*(n)\) is filtered by the degree of polynomials. The specification \(x_{n+1} = 0\) defines a homomorphism of filtered algebras \(\Lambda^*(n + 1) \to \Lambda^*(n)\). Denote by \(\Lambda^*\) the projective limit of the algebras \(\Lambda^*(n)\) with respect to these homomorphisms (in the category of filtered algebras). The algebra \(\Lambda^*\) is called the algebra of shifted symmetric functions. The ring \(\Lambda^*\) can also be defined over \(\mathbb{Z}\), but for our purposes it is more convenient to assume that \(\Lambda^*\) is an algebra over the field \(\mathbb{C}\).

Given an element \(f \in \Lambda^*\) and a partition \(\lambda\), we denote by \(f(\lambda)\) the value \(f(\lambda_1, \ldots, \lambda_{\ell(\lambda)})\). Elements of the algebra \(\Lambda^*\) are uniquely defined by their values on partitions.

The key point of the paper [3] is the basis of the shifted Schur functions \(\{s_\mu^*\}\) of the space \(\Lambda^*\) indexed by partitions \(\mu \in \mathcal{Y}\). The paper [3] contains explicit formulae and many other remarkable facts for these functions, but we use below only the following two properties of the functions \(s_\mu^*\). If \(\rho \vdash r \leq n\), then

\[
\sum_{\mu \vdash r} s_\mu^*(\lambda) \chi_\mu^\rho = \frac{1}{\dim \lambda} \sum_{\mu \vdash r} \dim(\lambda/\mu) \chi_\mu^\rho = \frac{1}{\dim \lambda} \chi_\rho^\lambda
\]

(9.1)

according to [3, Theorem 7.1]. And if \(\mu \vdash r > |\lambda|\), then

\[
s_\mu^*(\lambda) = 0
\]

(9.2)
by [3, Theorem 3.1]. Given a partition $\lambda \vdash n$, we denote by $\chi^\lambda$ the irreducible character of the symmetric group $S_n$. If $\rho$ is a partition of a number $r \leq n$, then $\chi^\lambda_{\tilde{\rho}}$ is the value of the character $\chi^\lambda$ on an element of cycle type $\tilde{\rho}$.

Following [3, §14.2] (see also [8]), we introduce another basis $\{p^\#_\rho\}$ in $\Lambda^*$ by the formula

$$p^\#_\rho = \sum_{\mu \vdash r} \chi^\mu_{\tilde{\rho}} s^*_\mu.$$  

Note that $\deg p^\#_\rho = |\rho|$. It follows from (9.1) and (9.2) that

$$\frac{1}{z_\rho} p^\#_\rho (\lambda) = \begin{cases} \frac{1}{\dim \lambda} \chi^\lambda_{\tilde{\rho}}, & \text{if } n \geq r, \\ 0, & \text{otherwise}. \end{cases} \quad (9.3)$$

**Theorem 9.1.** The linear mapping $F: A_\infty \to \Lambda^*$ defined on the basis elements of $A_\rho$ by the formula

$$F(A_\rho) = \frac{p^\#_\rho}{z_\rho}$$  

is an isomorphism of algebras $A_\infty$ and $\Lambda^*$.

**Proof.** Let $\rho$ be a partition of a number $r$ and $m_1 = m_1(\rho)$. According to (4.3), the image of the element $A_\rho \in A_\infty$ in the centre of the group algebra $Z(\mathbb{C}[S_n])$ equals

$$(\psi \circ \theta_n)(A_\rho) = \left(\frac{n-r+m_1}{m_1}\right) C_{\rho;n}.$$ 

If $r \leq n$, then

$$\chi^\lambda (\psi \circ \theta_n (A_\rho)) = \left(\frac{n-r+m_1}{m_1}\right) n! \chi^\lambda_{\tilde{\rho}} = \frac{(n \upharpoonright r)}{z_\rho} \chi^\lambda_{\tilde{\rho}},$$

thus formula (9.3) implies

$$\frac{1}{z_\rho} p^\#_\rho (\lambda) = \frac{(n \upharpoonright r)}{z_\rho} \frac{1}{\dim \lambda} \chi^\lambda_{\tilde{\rho}} = \frac{1}{\dim \lambda} \chi^\lambda (\psi \circ \theta_n (a_\rho)).$$

If $r > n$, the last formula is valid too, since both sides are zero in view of (9.2). Thus the mapping $F: A_\infty \to \Lambda^*$ defined in (9.4) may be defined by an equivalent formula

$$F(a)(\lambda) = \frac{1}{\dim \lambda} \chi^\lambda (\psi \circ \theta |\lambda| (a)); \quad a \in A_\infty. \quad (9.5)$$

The mapping $\psi$ is a homomorphism of the algebra $A_n$ onto $Z(\mathbb{C}[S_n])$, and irreducible normalized characters $\chi^\lambda/\dim \lambda$ of the group $S_n$ define homomorphisms $Z(\mathbb{C}[S_n]) \to \mathbb{C}$. Hence,

$$F(ab)(\lambda) = (F(a) F(b))(\lambda)$$  

for all $a, b \in A_\infty$, and formulae (9.4), (9.5) define an isomorphism of the algebra $A_\infty$ onto the algebra $\Lambda^*$. The Theorem follows.  \[\square\]
Proposition 9.2. Associate with a partition \( \mu \vdash m \) an element

\[
x_\mu = \sum_{|d|=m} \sum_{w \in \mathcal{S}_d} \chi^\mu(w)(d,w)
\]  

(9.7)

of the algebra \( A_\infty \). Then \( F(x_\mu) = s^*_\mu \).

Proof. If \(|\mu| > |\lambda|\), then \( F(x_\mu)(\lambda) = 0 = s^*_\mu(\lambda) \). If \(|\mu| \leq |\lambda|\), then

\[
\psi \circ \theta_n(x_\mu) = \sum_{|d|=m} \sum_{d \subseteq \mathbb{P}_n} \psi(d,w) = \sum_{|d|=m} \sum_{d \subseteq \mathbb{P}_n} \widetilde{w}.
\]

From (9.1) and the orthogonality relations for irreducible characters of the symmetric group \( \mathcal{S}_m \), we obtain a chain of equalities

\[
F(x_\mu)(\lambda) = \sum_{|d|=m} \sum_{w \in \mathcal{S}_d} \chi^\mu(w) \frac{\chi^\lambda(\widetilde{w})}{\dim \lambda} =
\]

\[
= \binom{n}{m} \sum_{w \in \mathcal{S}_m} \chi^\mu(w) \sum_{\nu \vdash m} \chi^\nu(w) \frac{s^*_\nu(\lambda)}{(n \downarrow m)} =
\]

\[
= \binom{n}{m} \sum_{\nu \vdash m} s^*_\nu(\lambda) \frac{m!}{(n \downarrow m)} \sum_{w \in \mathcal{S}_m} \chi^\mu(w) \chi^\nu(w) =
\]

\[
= \binom{n}{m} m! s^*_\nu(\lambda) \frac{m!}{(n \downarrow m)} = s^*_\nu(\lambda).
\]

The Proposition follows. \( \square \)

10. Filtrations of the algebra \( A_\infty \). There is an obvious filtration on the algebra \( A_\infty \),

\[
\text{deg}_1(A_\rho) = |\rho|.
\]

(10.1)

In the decomposition of the convolution \( A_\sigma A_\tau \), there is the unique summand \( A_{\sigma \cup \tau} \) of the highest degree \( \text{deg}_1(A_{\sigma \cup \tau}) = |\sigma| + |\tau| \). Thus the generators \( A_{(1)}, A_{(2)}, \ldots \) are algebraically independent, and the adjoined graded algebra is naturally isomorphic to the algebra of polynomials in \( A_{(1)}, A_{(2)}, \ldots \) (over the field \( \mathbb{C} \)).

Another filtration on \( A_\infty \) was introduced in [7].

Proposition 10.1. The function

\[
\text{deg}_2(A_\rho) = |\rho| + m_1(\rho)
\]

(10.2)

defines a filtration on the algebra \( A_\infty \).

Proof. Consider partial permutations \((d_1, w_1), (d_2, w_2)\) and break the union of their supports \( d_1 \cup d_2 \) into disjoint parts as follows:

\[
d_1 \setminus d_2 = d^1_{mf} \cup d^1_{ff}
\]

\[
d_2 \setminus d_1 = d^2_{mf} \cup d^2_{ff}
\]

\[
d_1 \cap d_2 = d_{mm} \cup d_{fm} \cup d_{mf} \cup d_{ff}.
\]
The first index equals $f$, if the points of the corresponding domain are fixed for the permutation $w_1$, and equals $m$, if they are non-fixed. The second index has a similar sense with respect to the permutation $w_2$. By definition,

$$\deg_2(w_1) = |d_{m,f}^{12}| + |d_{m,f}| + |d_{f,f}| + 2 |d_{f,f}^{12}| + 2 |d_{f,f}| + 2 |d_{f,m}|$$

$$\deg_2(w_2) = |d_{f,m}^{21}| + |d_{m,f}| + |d_{f,m}| + 2 |d_{f,m}^{21}| + 2 |d_{f,f}| + 2 |d_{f,m}|.$$

The permutation $w = w_1w_2$ of the set $d = d_1 \cup d_2$ has no fixed points in the domains $d_{m,f}^{12}$, $d_{m,f}$, $d_{f,m}$, $d_{f,m}^{21}$. On the contrary, all points of the domains $d_{f,f}^{12}$, $d_{f,f}$, $d_{f,f}^{21}$ are fixed for $w$. The domain $d_{m,m}$ may contain both non-fixed and fixed points of $w$. Hence

$$\deg_2(w_1w_2) \leq |d_{m,f}^{12}| + |d_{m,f}| + |d_{f,m}| +$$

$$+ |d_{f,m}^{21}| + 2 |d_{f,f}^{12}| + 2 |d_{f,f}| + 2 |d_{f,m}^{21}| + 2 |d_{f,m}|$$

$$\leq |d_{m,f}^{12}| + 3 |d_{m,f}| + 3 |d_{f,m}| +$$

$$+ |d_{f,m}^{21}| + 2 |d_{f,f}^{12}| + 4 |d_{f,f}| + 2 |d_{f,m}^{21}| + 2 |d_{f,m}|$$

$$= \deg_2(w_1) + \deg_2(w_2),$$

and the Proposition follows. \(\square\)

Let $T_2$ be the set of transpositions, i.e. of all permutations from $S_\infty$ with a unique non-trivial cycle of length 2. The length $\deg(w)$ of a permutation $w$ with respect to a family of generators $T_2$ is called the Cayley metric. It is clear that the function $\deg_3(d, w) = \deg(w)$ defines a filtration on the algebra $A_\infty$. The Cayley filtration was studied in [5], [9, Chap. I, §7, examples 24, 25], [6]. In particular, it is known that

$$\deg_3(A_\rho) = |\rho| - \ell(\rho). \quad (10.3)$$

Note that according to [5, Lemma 3.9], $\deg_3(A_\sigma) + \deg_3(A_\tau) = \deg_3(A_\rho)$ if and only if the polynomial $g^\rho_{\sigma, \tau}(n)$ introduced in Proposition 7.3 is a constant not depending on $n$.

One may set a problem of general description of filtrations on the algebra $A_\infty$. Not having a general answer, we make here only several observations. First of all, let us give some definitions.

Denote by $Y$ the set of all partitions. A function $\theta : Y \to \mathbb{Z}_+$ is called a filtration of the algebra $A_\infty$, if each triple of partitions $\sigma, \tau, \rho$ with $g^\rho_{\sigma, \tau} > 0$ satisfies the inequality $\theta(\rho) \leq \theta(\sigma) + \theta(\tau)$. We say that a filtration $\theta$ is additive, if

$$\theta(\sigma \cup \tau) = \theta(\sigma) + \theta(\tau) \quad (10.4)$$

for all $\sigma, \tau \in Y$. Condition (10.4) means that

$$\theta(\rho) = \sum_{k \geq 1} \gamma_k m_k(\rho), \quad (10.5)$$

where $\gamma_k = \theta((k))$ are the degrees of one-cycle permutations.
Example 10.1. All above-mentioned filtrations $\deg_1$, $\deg_2$, $\deg_3$ are additive. The constants $\gamma_k$ are of the form

\[
\begin{align*}
\gamma_1 &= 1, \quad \gamma_2 = 2, \quad \gamma_3 = 3, \quad \gamma_4 = 4, \ldots \quad \text{for} \quad \deg_1 \\
\gamma_1 &= 2, \quad \gamma_2 = 2, \quad \gamma_3 = 3, \quad \gamma_4 = 4, \ldots \quad \text{for} \quad \deg_2 \\
\gamma_1 &= 0, \quad \gamma_2 = 1, \quad \gamma_3 = 2, \quad \gamma_4 = 3, \ldots \quad \text{for} \quad \deg_3.
\end{align*}
\]

Let us mention some common properties of the constants $\gamma_k$.

**Proposition 10.2.** The following properties of the numbers $\gamma = \{\gamma_k\}_{k=1}^{\infty}$ are common for all additive filtrations:

\[
\begin{align*}
0 &\leq \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4 \leq \ldots \quad (10.6) \\
\gamma_{i+j+1} &\leq \gamma_i + \gamma_{j+1}. \quad (10.7) \\
k \gamma_1 &\leq 2 \gamma_k \quad (10.8) \\
\gamma_{k+1} &\leq k \gamma_2 \quad (10.9) \\
\gamma_{2k+1} &\leq 2 \gamma_{k+1}. \quad (10.10)
\end{align*}
\]

The limit

\[
\lim_{k \to \infty} \frac{\gamma_{k+1}}{k} = \inf_k \frac{\gamma_{k+1}}{k} =: L(\gamma), \quad (10.11)
\]

always exists, and $\gamma_1 \leq 2L(\gamma) \leq 2 \gamma_2$.

**Proof.** Since $A_{(1)} \ast A_{(1)} = 2A_{(12)} + A_{(1)}$, we have $\gamma_1 \leq 2\gamma_1$, and hence $\gamma_1 \geq 0$.

Multiplying cycles that intersect by a common pair of neighbour elements, we have

\[
(b_1, b_2, \ldots, b_i, a_1, a_2) \quad (a_1, a_2, c_1, c_2, \ldots, c_j) = (b_1, b_2, \ldots, b_i, a, c_1, c_2, \ldots, c_j), \quad (10.12)
\]

which implies that $g_{(i+2),(j+2)}^{(i+1),(j+1)} > 0$ and $\gamma_{i+1} + \gamma_{j+1} \leq \gamma_{i+2} + \gamma_{j+2}$. In particular, for $i = j$ we obtain $\gamma_j \leq \gamma_{j+1}$ which proves (10.6).

To prove (10.7), note that

\[
(b_1, b_2, \ldots, b_i, a) \quad (a, c_1, c_2, \ldots, c_j) = (b_1, b_2, \ldots, b_i, a, c_1, c_2, \ldots, c_j),
\]

thus $g_{(i+1),(j+1)}^{(i+j+1)} > 0$. Statement (10.11) is a standard corollary of inequalities (10.7) (see, for example, [4, problem 98]).

Formula (10.8) follows from

\[
(b_1, b_2, \ldots, b_k) \quad (b_k, b_{k-1}, \ldots, b_1) = (b_1) \quad (b_2) \quad \ldots \quad (b_k),
\]

and (10.10) is a particular case of (10.7) for $i = j = k$.

Since

\[
\begin{align*}
(a_1, a_2) \quad (a_3, a_4) \ldots \quad (a_{2k-1}, a_{2k}) &\quad (a_2, a_3) \quad (a_4, a_5) \ldots \quad (a_{2k}, a_{2k+1}) = \\
&\quad (a_2, a_4, \ldots, a_{2k}, a_{2k+1}, a_{2k-1}, \ldots, a_3, a_1); \\
(a_1, a_2) \quad (a_3, a_4) \ldots \quad (a_{2k+1}, a_{2k+2}) &\quad (a_2, a_3) \quad (a_4, a_5) \ldots \quad (a_{2k}, a_{2k+1}) = \\
&\quad (a_2, a_4, \ldots, a_{2k}, a_{2k+2}, a_{2k+1}, a_{2k-1}, \ldots, a_3, a_1),
\end{align*}
\]
we obtain inequalities (10.9) for even and odd \( k \) respectively. The last statement of the Proposition follows immediately from the definition of the limit \( L(\gamma) \) and formulae (10.8), (10.9). \( \square \)

**Remark.** Conditions (10.6) — (10.11) do not guarantee that the function defined by the numbers \( \gamma \) via formula (10.5) is a filtration. For example, the number of non-trivial cycles \( \theta(\rho) = \ell(\rho) - m_1(\rho) \) corresponds to the constants \( \gamma_1 = 0, \gamma_k = 1 \) for \( k \geq 2 \). Inequalities (10.6) — (10.11) are satisfied, but the function \( \theta \) is not a filtration:

\[
(1, 2, 3, 4)(1, 5, 4, 6, 3) = (1, 5)(2, 3)(4, 6).
\]

**Remark.** Formula (10.12) may be generalized as follows:

\[
(b_1, b_2, \ldots, b_i, a_1, a_2, \ldots, a_{2k}) \ (a_1, a_2, \ldots, a_{2k}, c_1, c_2, \ldots, c_j) = \\
= (b_1, b_2, \ldots, b_i, a_1, a_3, \ldots, a_{2k-1})(a_2, a_4, \ldots, a_{2k}, c_1, c_2, \ldots, c_j)
\]

for an even number of common elements of multiplied cycles. For an odd number of common elements we have

\[
(b_1, b_2, \ldots, b_i, a_1, a_2, \ldots, a_{2k}, a_{2k+1})(a_1, a_2, \ldots, a_{2k}, a_{2k+1}, c_1, c_2, \ldots, c_j) = \\
= (b_1, b_2, \ldots, b_i, a_1, a_3, \ldots, a_{2k-1}, a_{2k+1}, c_1, c_2, \ldots, c_j, a_2, a_4, \ldots, a_{2k})
\]

thus \( \gamma_{i+j+2k+1} \leq \gamma_{i+2k+1} + \gamma_{j+2k+1} \). Note also that

\[
(b_1, b_2, \ldots, b_i, a_0, a_1, \ldots, a_{k-1}) \ (a_{k-1}, \ldots, a_1, a_0, c_1, c_2, \ldots, c_j) = \\
= (b_1, b_2, \ldots, b_i, a_0, c_1, c_2, \ldots, c_j)(a_1) (a_2) \ldots (a_{k-1}),
\]

thus \( \gamma_{i+j+1} + (k-1)\gamma_1 \leq \gamma_{i+k} + \gamma_{j+k} \) for \( k \geq 1 \).

**Remark.** The equality

\[
(a_1, \ldots, a_i, w, c_1, \ldots, c_k, v, b_1, \ldots, b_j, u) \\
(\alpha_1, \ldots, \alpha_I, u, \beta_1, \ldots, \beta_J, v, \gamma_1, \ldots, \gamma_K, w) = \\
= (a_1, \ldots, a_i, w, \alpha_1, \ldots, \alpha_I)(b_1, \ldots, b_j, u, \beta_1, \ldots, \beta_J)\times \\
x(c_1, \ldots, c_k, v, \gamma_1, \ldots, \gamma_K)
\]

implies the inequality \( \gamma_{I+i+1} + \gamma_{J+j+1} + \gamma_{K+k+1} \leq \gamma_{I+J+K+3} + \gamma_{i+j+k+3} \) and, in particular, \( 3\gamma_{2n-1} \leq 2\gamma_{3n} \).

**Proposition 10.3.** Let \( J \subset \mathbb{N} \). Put

\[
\theta_J(\rho) = |\rho| + \sum_{k \in J} m_k(\rho).
\]

Then the function \( \theta_J \) is an additive filtration of the algebra \( A_\infty \).

**Proof.** Consider partitions \( \sigma, \tau, \rho \) such that \( g_{\rho}^{\sigma, \tau} > 0 \). Then there exist elements \( (d_1, \omega_\sigma) \in A_\sigma, (d_2, \omega_\tau) \in A_\tau \) such that \( (d_1 \cup d_2, \omega_\sigma \omega_\tau) \in A_\rho \).

Consider the decompositions of the permutations \( \omega_\sigma, \omega_\tau, \omega_\sigma \omega_\tau \) into products of disjoint cycles. If \( 1 \notin J \), then denote by \( M \) the set of cycles of the permutation \( \omega_\sigma \omega_\tau \) that are contained neither in the decomposition of \( \omega_\sigma \), nor in the decomposition
of \( \omega \), and the length of each such cycle belongs to \( J \). If \( 1 \in J \), then we also include in the set \( M \) fixed points (informally speaking, “cycles of length one”) of the permutation \( \omega \tau \) in the set \( d_1 \cup d_2 \) that are fixed neither with respect to \( \omega \), nor with respect to \( \omega \). Then

\[
|M| \geq \sum_{k \in J} (m_k(\rho) - m_k(\sigma) - m_k(\tau)).
\]

Let \( \alpha \) be a cycle from \( M \). Then there exists at least one element in this cycle that belongs to \( d_1 \cap d_2 \). In a similar way, if \( x \) is a fixed point belonging to \( M \), then \( x \in d_1 \cap d_2 \). Hence

\[
|M| \leq |d_1 \cap d_2| = |d_1| + |d_2| - |d_1 \cup d_2| = |\sigma| + |\tau| - |\rho|.
\]

It follows from (10.13) and (10.14) that

\[
|\sigma| + |\tau| + \sum_{k \in J} (m_k(\sigma) + m_k(\tau)) \geq |\rho| + \sum_{k \in J} m_k(\rho).
\]

\( \square \)

11. **Examples of convolutions of classes in the algebra \( A_\infty \).** We present below the simplest formulae for multiplication of basis elements \( a_\rho \) of the algebra \( A_\infty \). In view of Theorem 9.1, one may regard the same formulae as examples of multiplication of the functions \( p_\rho^\# \) in the algebra \( \Lambda^* \).

One may also use similar formulae for calculating the convolution of conjugacy classes in symmetric groups. For example, substituting \( a_\rho = z_\rho A_\rho \) we obtain from the corresponding row the formula

\[
A_{(3)} * A_{(3)} = 2 A_{(3^2)} + 5 A_{(5)} + 8 A_{(2^3)} + 3 A_{(31)} + A_{(3)} + 2 A_{(1^3)}.
\]

Passing to the homomorphic images \( \psi(A_\rho) \) and substituting

\[
\psi(A_\rho) = \left( t - r + m_1(\rho) \right) A_\rho,
\]

we obtain an example of the formula from [5],

\[
C_{(3)} * C_{(3)} = 2 C_{(3^2)} + 5 C_{(5)} + 8 C_{(2^3)} + (3t - 8) C_{(3)} + \frac{t(t - 1)(t - 2)}{3} C_{\emptyset}.
\]

Convolutions of conjugacy classes are obtained from this formula by substituting different positive integer values of the variable \( t \). For example,

\[
C_{(3);3} * C_{(3);3} = 2 C_{\emptyset;3} + C_{(3);3}
\]
\[
C_{(3);4} * C_{(3);4} = 8 C_{\emptyset;4} + 4 C_{(3);4} + 8 C_{(2^3);4}
\]
\[
C_{(3);5} * C_{(3);5} = 20 C_{\emptyset;5} + 7 C_{(3);5} + 8 C_{(2^3);5} + 5 C_{(5);5}
\]
\[
C_{(3);6} * C_{(3);6} = 40 C_{\emptyset;6} + 10 C_{(3);6} + 8 C_{(2^3);6} + 5 C_{(5);6} + 2 C_{(3^2);6}.
\]

So, examples of multiplication formulae:

\[
a_{(2)} a_{(2)} = a_{(2^2)} + 4 a_{(3)} + 2 a_{(1^2)}
\]
12. Irreducible representations of the semigroup $\mathcal{P}_n$ and characters of the algebra $B_n$. Fix a subset $x \subset \mathbb{P}_n$ of size $|x| = k$ and let $\lambda$ be a Young diagram with $k$ boxes. Then the formula

$$
\pi_{x,\lambda}(d, w) = \begin{cases} 
\pi_\lambda(\varphi_x(d, w)), & \text{if } d \subset x; \\
0, & \text{otherwise}
\end{cases} \tag{12.1}
$$

defines an irreducible representation of the semigroup $\mathcal{P}_n$. Here $0$ is the zero matrix of order $\dim \pi_\lambda$.

**Proposition 12.1.** The representations $\pi_{x,\lambda}$ (where $x \subset \mathbb{P}_n$, $|x| = k$ and $\lambda \in \mathbb{Y}_k$ are a subset and a Young diagram of common size $0 \leq k \leq n$) are irreducible, pairwise non-equivalent and form a complete list of irreducible representations of the semigroup algebra $B_n$.

**Proof.** The matrices $\pi_{x,\lambda}(d, w)$ of the representation $\pi_{x,\lambda}$ are non-zero exactly for elements $(d, w)$ such that $d \subset x$. Thus the representations indexed by different sets $x$ are non-equivalent. It is obvious that the representations $\pi_{x,\lambda}$, $\pi_{x,\mu}$ for $\lambda \neq \mu$ are non-equivalent too. Completeness of the list follows from Corollary 3.2. $\square$

Let us describe the branching rule for irreducible representations of the semigroup $\mathcal{P}_n$ when restricting on $\mathcal{P}_{n-1} \subset \mathcal{P}_n$.

We write $\mu \not\succ \lambda$, if a diagram $\lambda$ is obtained from a diagram $\mu$ by adding one box. Let $\Gamma_n$ be the set of pairs $(x, \lambda)$, where $x \subset \mathbb{P}_n$ is a subset, $\lambda$ is a Young diagram, and $x$ and $\lambda$ are assumed to have the same size, $|x| = |\lambda|$. The set $\Gamma_n$ indexes irreducible representations $\pi_{x,\lambda}$ of the semigroup $\mathcal{P}_n$. 

\[a_{(3)} a_{(2)} = a_{(32)} + 6 a_{(4)} + 6 a_{(21)}
\]
\[a_{(4)} a_{(2)} = a_{(42)} + 8 a_{(5)} + 4 a_{(22)} + 8 a_{(31)}
\]
\[a_{(2^2)} a_{(2)} = a_{(2^2)} + 8 a_{(32)} + 8 a_{(4)} + 4 a_{(21^2)}
\]
\[a_{(3)} a_{(3)} = a_{(32)} + 9 a_{(5)} + 9 a_{(22)} + 9 a_{(31)} + 3 a_{(3)} + 3 a_{(1^3)}
\]
\[a_{(5)} a_{(2)} = a_{(52)} + 10 a_{(6)} + 10 a_{(32)} + 10 a_{(41)}
\]
\[a_{(32)} a_{(2)} = a_{(32^2)} + 6 a_{(42)} + 4 a_{(32^2)} + 12 a_{(5)} + 6 a_{(22^1)} + 2 a_{(31^2)}
\]
\[a_{(4)} a_{(3)} = a_{(43)} + 12 a_{(6)} + 24 a_{(32)} + 12 a_{(44)} + 12 a_{(4)} + 12 a_{(21^2)}
\]
\[a_{(2^2)} a_{(3)} = a_{(32^2)} + 12 a_{(42)} + 24 a_{(5)} + 12 a_{(22^1)} + 24 a_{(31)}
\]
\[a_{(6)} a_{(2)} = a_{(62)} + 12 a_{(7)} + 12 a_{(42)} + 6 a_{(3^2)} + 12 a_{(51)}
\]
\[a_{(42)} a_{(2)} = a_{(4^2)} + 8 a_{(52)} + 4 a_{(43)} + 16 a_{(6)} + 4 a_{(23^2)} + 8 a_{(32)} + 2 a_{(1^4)}
\]
\[a_{(3^2)} a_{(2)} = a_{(3^2)} + 12 a_{(43)} + 18 a_{(6)} + 12 a_{(32^1)}
\]
\[a_{(2^3)} a_{(2)} = a_{(2^4)} + 12 a_{(32^2)} + 24 a_{(42)} + 6 a_{(2^21^2)}
\]
\[a_{(5)} a_{(3)} = a_{(53)} + 15 a_{(7)} + 30 a_{(42)} + 15 a_{(3^2)} + 15 a_{(51)} + 30 a_{(5)} + 15 a_{(2^21)} + 15 a_{(31^2)}
\]
\[a_{(32)} a_{(3)} = a_{(32^2)} + 9 a_{(52)} + 6 a_{(43)} + 36 a_{(6)} + 9 a_{(23)} + 15 a_{(321)} + 21 a_{(32)} + 36 a_{(41)} + 3 a_{(21^3)}
\]
\[a_{(4)} a_{(4)} = a_{(4^2)} + 16 a_{(7)} + 32 a_{(42)} + 24 a_{(3^2)} + 16 a_{(51)} + 48 a_{(5)} + 32 a_{(22^1)} + 4 a_{(2^2)} + 16 a_{(32^2)} + 16 a_{(31)} + 4 a_{(1^4)}
\]
\[a_{(2^2)} a_{(2^2)} = a_{(2^4)} + 16 a_{(32^2)} + 32 a_{(42)} + 32 a_{(3^2)} + 64 a_{(5)} + 8 a_{(2^21^2)} + 32 a_{(31^2)} + 16 a_{(22)} + 8 a_{(1^4)}
\]
Proposition 12.2. Assume that a set $x \subset \mathbb{P}_n$ does not contain $n$. Then the restriction of the representation $\pi_{x,\lambda}$ on the subsemigroup $\mathcal{P}_{n-1}$ remains irreducible (and is indexed by the same pair $(x, \lambda)$ regarded as an element of $\Gamma_{n-1}$). And if $n \in x$, then the restriction of the irreducible representation $\pi_{x,\lambda}$ on $\mathcal{P}_{n-1}$ is of the form

$$\text{Res} \pi_{x,\lambda} = \bigoplus_{\mu: \mu \not\succ \lambda} \pi_{y,\mu},$$

where $y = x \setminus \{n\}$.

Proof. Follows immediately from the construction of representations and a well known branching rule for irreducible representations of symmetric groups. □

Thus the set of vertices of the branching graph of irreducible representations of the semigroups $\mathcal{P}_n$ is $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$. Let $(y, \mu), (x, \lambda)$ be vertices of neighbour levels $\Gamma_{n-1}$ and $\Gamma_n$. They are joined by an edge (of multiplicity 1) if and only if $\mu = \lambda$ or $\mu \not\succ \lambda$ (i.e. if the diagram $\lambda$ coincides with $\mu$ or is obtained from $\mu$ by adding one box). In the first case $x = y$, in the second case $x = y \cup \{n\}$.

Applying the ergodic method (see [2]) to the branching graph $\Gamma$, one can easily obtain a description of non-negative indecomposable harmonic functions on $\Gamma$ or, equivalently, of characters of the algebra $B_\infty$. Not considering this question in details, we point out only the parameterization of characters.

Let $X$ be an arbitrary subset in $\mathbb{P}_\infty = \{1, 2, \ldots\}$. Denote by $\Gamma_X$ the set of vertices $(x, \lambda) \in \Gamma$ with $x \subset X$. By Proposition 12.2, $\Gamma_X$ is a coideal in the branching graph $\Gamma$, i.e.

$$(y, \mu) \not\succ (x, \lambda) \in \Gamma_X \implies (y, \mu) \in \Gamma_X.$$

Each harmonic function on a coideal can be canonically continued (by zero) to a harmonic function on the whole branching graph. Note that if the set $X$ is infinite, then the graph $\Gamma_X$ differs from the Young graph $\mathcal{Y}$ only by trivial doubling of some levels. As for the Young graph, its indecomposable harmonic functions $\varphi^{(\alpha;\beta;X)}$ are indexed by the points of the Thoma simplex $\Delta$. By definition, $\Delta$ consists of pairs of non-increasing non-negative sequences $\alpha = (\alpha_1, \alpha_2, \ldots), \beta = (\beta_1, \beta_2, \ldots)$ with

$$\sum_{n=1}^{\infty} \alpha_n + \sum_{n=1}^{\infty} \beta_n \leq 1.$$ 

The functions $\varphi^{(\alpha;\beta;X)}$ are of the form

$$\varphi^{(\alpha;\beta;X)}(y, \mu) = \begin{cases} s_\mu(\alpha; \beta), & \text{if } y \subset X \\ 0, & \text{otherwise,} \end{cases}$$

where $s_\mu(\alpha; \beta)$ are the extended Schur functions, see [1].

If $X$ is a finite set, $|X| = k$, then indecomposable harmonic functions $\varphi^{(\lambda;X)}$ on the coideal $\Gamma_X$ are indexed by Young diagrams $\lambda \in \mathcal{Y}_k$ with $k$ boxes and are of the form

$$\varphi^{(\lambda;X)}(y, \mu) = \begin{cases} \dim(\mu, \lambda) / \dim \lambda, & \text{if } y \subset X \text{ and } \mu \subset \lambda \\ 0, & \text{otherwise,} \end{cases}$$

where $\dim(\mu, \lambda)$ is the number of standard skew Young tableaux of shape $\lambda/\mu$ and $\dim \lambda$ is the number of all standard tableaux of shape $\lambda$.

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