The Existence of Graph whose Vertex Set Can be Partitioned into a Fixed Number of Domination Strong Critical Vertex-sets*

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Abstract

Let $\gamma(G)$ denote the domination number of a graph $G$. A vertex $v \in V(G)$ is called a critical vertex of $G$ if $\gamma(G - v) = \gamma(G) - 1$. A graph is called vertex-critical if every vertex of it is critical. In this paper, we correspondingly introduce two such definitions: (i) a set $S \subseteq V(G)$ is called a strong critical vertex-set of $G$ if $\gamma(G - S) = \gamma(G) - |S|$; (ii) a graph $G$ is called strong $l$-vertex-sets-critical if $V(G)$ can be partitioned into $l$ strong critical vertex-sets of $G$. Whereafter, we give some properties of strong $l$-vertex-sets-critical graphs by extending the previous results of vertex-critical graphs. As the core work, we study on the existence of this class of graphs and obtain that there exists a strong $l$-vertex-sets-critical connected graph if and only if $l \notin \{2, 3, 5\}$.

Keywords: Domination; Critical vertex; Strong critical vertex-set; Vertex-critical graph

AMS 2010 Mathematics Subject Classification: 05C69

1 Introduction

The graphs considered in this paper are finite, undirected and simple. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $X \subseteq V(G)$, denote by $G[X]$ the subgraph of $G$ induced by $X$. For any $v \in V(G)$, let $d_G(v)$, $N_G(v)$ and $N_G[v]$ denote the degree, open and closed neighborhood of vertex $v$ in $G$, respectively. Furthermore, for any $U \subseteq V(G)$, the open and closed neighbourhood of $U$ are defined as $N_G(U) = \bigcup_{v \in U} N_G(v)$ and $N_G[U] = N_G(U) \cup U$, respectively. Two graphs are disjoint if they have no common vertices. The union of graphs $G$ and $H$, denoted by $G \cup H$, is a graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

*This work is supported by NSFC (12061047); Undergraduate Innovation Training Project of Hubei Province (2021Byb004); Foundation of Cultivation of Scientific Institutions of Jianghan University (06210033).

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A set $M \subseteq V(G)$ is called a 2-packing of graph $G$ if $d_G(x, y) > 2$ for every pair of distinct vertices $x, y \in M$. A set $D \subseteq V(G)$ is called a dominating set of $G$ if every vertex of $G$ is either in $D$ or adjacent to a vertex of $D$. The domination number $\gamma(G)$ is the cardinality of a minimum dominating set of $G$. We denote by $\text{MDS}(G)$ the set of all the minimum dominating sets. That is, $\text{MDS}(G) = \{D \mid D$ is a minimum dominating set of $G\}$.

1.1 Domination vertex-critical

**Definition 1.1.** A vertex $v \in V(G)$ is called a critical vertex of $G$ if $\gamma(G - v) = \gamma(G) - 1$.

**Observation 1.2.** For any $v \in V(G)$,
\[\gamma(G - v) = \gamma(G) - 1 \iff \gamma(G - v) \leq \gamma(G) - 1.\] (1.1)

**Definition 1.3.** A graph $G$ is called vertex-critical if every vertex of $G$ is critical.

The research on vertex-critical graph was early in [4, 5]. Afterwards, authors studied on its diameter [8], connectivity [1], existence of perfect matching [1, 2, 15] and factor critical property [2, 20, 21] in succession. Moreover, based on the right and the left of Formula (1.1), Brigham et al. [6] and Phillips et al. [19] extended the notion of vertex-critical graphs by introducing $(\gamma, k)$-critical graphs and $(\gamma, t)$-critical graphs, respectively.

**Definition 1.3◦ [6] A graph $G$ is called $(\gamma, k)$-critical if $\gamma(G - S) < \gamma(G)$ for every $S \subseteq V(G)$ with $|S| = k$.**

**Definition 1.3⋄ [19] A graph $G$ is called $(\gamma, t)$-critical if $\gamma(G - S) = \gamma(G) - t$ for every 2-packing $S$ of $G$ with $|S| = t$.**

In Definition 1.3◦, if $k = 2$, then $G$ is called to be domination bicritical. For more information of $(\gamma, k)$-critical or domination bicritical graphs, readers are suggested to refer to [7, 9, 10, 16–18].

Now, again based on the left of Formula (1.1), we introduce the definition of strong critical vertex-set to extend the notion of critical vertex in the following Definition 1.1’. (It is easy to get that a strong critical vertex-set of $G$ is also a 2-packing of $G$.) To compare Definition 1.3 we give Definition 1.3.

**Definition 1.1’ [23] A set $S \subseteq V(G)$ is called a strong critical vertex-set (or just st-critical vertex-set) of $G$ if $\gamma(G - S) = \gamma(G) - |S|$.**

**Definition 1.3.** A graph $G$ is called strong $l$-vertex-sets-critical if $V(G)$ can be partitioned into $l$ (non-empty) strong critical vertex-sets of $G$. 

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1.2 On strong critical vertex-set and two-colored $\gamma$-set

When we talk about st-critical vertex-set, we would like to mention another related notion —— two-colored $\gamma$-set. The present authors think that both of them are important on the problem of building family of graphs that make the equality hold in Vizing’s Conjecture [12, 23].

**Definition 1.4.** [12] Let $D \in \text{MDS}(G)$. $D$ is called a two-colored $\gamma$-set of $G$ if $D$ partitions as $D = D_1 \cup D_2$ such that $V(G) - N_G[D_1] = D_2$ and $V(G) - N_G[D_2] = D_1$.

In Definition 1.4, since $V(G) - N_G[D_1] = D_2$, we can deduce that $D_1 \in \text{MDS}(G - D_2)$. So $\gamma(G - D_2) = |D_1| = |D| - |D_2| = \gamma(G) - |D_2|$, which implies that $D_2$ is an st-critical vertex-set of $G$, and so is $D_1$ symmetrically. Because two-colored $\gamma$-set is not the motif of this paper, we just introduce a proposition and a conjecture about it below, where in Proposition 1.5, “□” represents the cartesian product and a nontrivial connected graph $G$ is called a generalized comb if each vertex of degree greater than one is adjacent to exactly one 1-degree-vertex of $G$.

**Proposition 1.5.** [12] If $G$ is a generalized comb and $H$ has a two-colored $\gamma$-set, then $\gamma(G \square H) = \gamma(G) \gamma(H)$.

**Conjecture 1.6.** [13] If $G$ is a connected bipartite graph such that $V(G)$ can be partitioned into two-colored $\gamma$-sets, then $G$ is the 4-cycle or $G$ can be obtained from $K_{2t, 2t}$ by removing the edges of $t$ vertex-disjoint 4-cycles.

At last, we account for the coming two sections. We will compare the properties of vertex-critical graphs and strong $l$-vertex-sets-critical graphs in Section 2, for the reason that st-critical vertex-set is a generalization of critical vertex as well as strong $l$-vertex-sets-critical graphs is a special kind of vertex-critical graphs. Let $\mathcal{C}_l = \{G \mid G$ is a strong $l$-vertex-sets-critical connected graph$\}$. We will obtain that $\mathcal{C}_l \neq \emptyset$ if and only if $l \notin \{2, 3, 5\}$ in Section 3.

2 To compare vertex-critical and strong $l$-vertex-sets-critical

Brigham et al. [5] studied on the vertex-critical graphs, and listed the following Theorems 2.2 and 2.3 without proofs because they thought the proofs are cumbersome but straightforward. In order to state these two theorems, we have to introduce the notion of vertex coalescence first.

**Definition 2.1.** [5, 14] Let $G$ and $H$ be two disjoint graphs with $g \in V(G)$ and $h \in V(H)$. The vertex coalescence $G \cdot gh H$ (or just $G \cdot H$ if $g$ and $h$ are arbitrary) of $G$ and $H$ via $g$ and $h$, is the graph obtained from the union of $G$ and $H$ by identifying the vertices $g$ and $h$ as one vertex $g$.

(Note: 1. An example of vertex coalescence is shown in Figure 1; 2. By the way, for the edge coalescence, readers can refer to [11].)
Figure 1: The vertex coalescence of graphs $H_{4,8}$ and $C_4$.

**Theorem 2.2.** Let $G$ and $H$ be two disjoint graphs. Form any vertex coalescence $G \cdot H$. Then 
\[ \gamma(G) + \gamma(H) - 1 \leq \gamma(G \cdot H) \leq \gamma(G) + \gamma(H). \]
Furthermore, if both $G$ and $H$ are vertex-critical or $G \cdot H$ is vertex-critical, then \[ \gamma(G \cdot H) = \gamma(G) + \gamma(H) - 1. \]

**Theorem 2.3.** The graph $G \cdot H$ is vertex-critical if and only if both $G$ and $H$ are vertex-critical.

To compare Brigham’s results, we give the corresponding results on strong $l$-vertex-sets-critical graphs one to one (see Definition 2.1′, Theorems 2.2′ and 2.3′). For the mathematical rigor, we are going to prove them without the supporting of Theorems 2.2 and 2.3, where in fact, our proofs include the derivation of Brigham’s results. Before this, we need to display four observations, two definitions and one lemma.

**Observation 2.4.** Let $G$ be a graph. If $G_1$ and $G_2$ are vertex-induced subgraphs of $G$ such that $V(G) = V(G_1) \cup V(G_2)$. Then $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$ with the equality holds if $G_1$ and $G_2$ are two components of $G$.

**Observation 2.5.**
(a) For any $S \subseteq V(G)$, $\gamma(G - S) \leq \gamma(G) - |S| \iff \gamma(G - S) = \gamma(G) - |S|$.
(b) For any $S \subseteq V(G)$, $\gamma(G - S) \geq \gamma(G) - |S|$.

**Observation 2.6.** A subset of an st-critical vertex-set of $G$ is still an st-critical vertex-set of $G$.

**Observation 2.7.** Let $S$ be an st-critical vertex-set of $G$, and $S_1, S_2 \subseteq S$ with $S_1 \cap S_2 = \emptyset$. Then $S_1$ is an st-critical vertex-set of $G - S_2$.

**Definition 2.8.** Let $S_1, S_2, \ldots, S_l$ be non-empty strong critical vertex-sets of $G$. If $\{S_1, S_2, \ldots, S_l\}$ is a partition of $V(G)$, then we call $\{S_1, S_2, \ldots, S_l\}$ or $S_1 \cup S_2 \cup \cdots \cup S_l$ as a strong critical vertex-sets partition of $G$.

**Definition 2.9.** Let $J$ be a graph with $x, y \in V(J)$. $x, y$ are called mutually compatible in $J$ if there exists $D_0 \in MDS(J)$ such that $\{x, y\} \subseteq D_0$, and mutually incompatible in $J$ if $|\{x, y\} \cap D| < 2$ for any $D \in MDS(J)$. 

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Lemma 2.10. Let $J$ be a graph with $x, y \in V(J)$, and $J'$ be the graph obtained from $J$ by identifying the two vertices $x$ and $y$ as one vertex $x$. Then $\gamma(J) - 1 \leq \gamma(J') \leq \gamma(J)$ with the second equality holds if and only if $x, y$ are mutually incompatible and neither $x$ nor $y$ is critical in $J$.

Proof. Let $D' \in MDS(J')$. Then $D' \cup \{y\}$ is a dominating set of $J$, and so $\gamma(J) \leq |D' \cup \{y\}| \leq \gamma(J') + 1$. Let $D \in MDS(J)$ and

$$D'_0 = \begin{cases} D, & \text{if } y \notin D; \\ (D - \{y\}) \cup \{x\}, & \text{if } y \in D. \end{cases}$$

Then $D'_0$ is a dominating set of $J'$, and so $\gamma(J') \leq |D'_0| \leq |D| = \gamma(J)$.

Furthermore, suppose that $x, y$ are mutually incompatible and neither of $x$ nor $y$ is critical in $J$. We need to prove that $\gamma(J') = \gamma(J)$. Let $D' \in MDS(J')$. There are two cases. If $x \in D'$, then $D' \cup \{y\}$ is a dominating set of $J$. Since $x, y$ are mutually incompatible in $J$, we have $\gamma(J') + 1 = |D' \cup \{y\}| \geq \gamma(J) + 1$, which implies that $\gamma(J') = \gamma(J)$. If $x \notin D'$, then $N_J(x) \cap D' \neq \emptyset$. So one of $x$ and $y$, say $x$, can be dominated by $D'$ in $J$. Thus $D'$ is a dominating set of $J - y$. Since $y$ is not critical in $J$, we have $\gamma(J) \leq \gamma(J - y) \leq |D'| = \gamma(J')$, which also implies that $\gamma(J') = \gamma(J)$.

Conversely, if $\gamma(J') = \gamma(J)$, we prove firstly that $x, y$ are mutually incompatible in $J$. Otherwise, let $D_{xy} \in MDS(J)$ with $\{x, y\} \subseteq D_{xy}$. Then $D_{xy} - \{y\}$ is a dominating set of $J'$ with cardinality $\gamma(J) - 1 = \gamma(J') - 1 < \gamma(J')$, a contradiction. We prove secondly that neither of $x$ nor $y$ is critical in $J$. Otherwise, we have that one of $x$ and $y$, say $x$, is critical in $J$. Let $D^- \in MDS(J - x)$ and $D_x = D^- \cup \{x\}$. Then $D_x \in MDS(J)$. Since $x, y$ are mutually incompatible in $J$, we have $y \notin D_x$. So $N_J(y) \cap D_x \neq \emptyset$, which implies that $N_J(x) \cap D_x \neq \emptyset$. Thus $D_x - \{x\}$ is a dominating set of $J'$. Hence $\gamma(J') \leq |D_x - \{x\}| = \gamma(J) - 1$, also a contradiction. \hfill \Box

Definition 2.1. Let $G$ and $H$ be two disjoint graphs with $\emptyset \neq X \subseteq V(G)$, $\emptyset \neq Y \subseteq V(H)$ and $|X| = |Y|$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. The vertex-set coalescence $G \cdot_{XY} H$ of $G$ and $H$ via $X$ and $Y$, is the graph obtained from the union of $G$ and $H$ by identifying the vertices $x_i$ and $y_i$ as one vertex $x_i$ for every $1 \leq i \leq m$. (Refer to Figure 2)

![Figure 2: Illustration for the vertex-set coalescence $G \cdot_{XY} H$.](image)

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Theorem 2.2. Let \( G \) and \( H \) be two disjoint graphs with \( \emptyset \neq X \subseteq V(G) \), \( \emptyset \neq Y \subseteq V(H) \) and \( |X| = |Y| \). Then

(a) \( \gamma(G) + \gamma(H) - |X| \leq \gamma(G : XY H) \leq \gamma(G) + \gamma(H) \);

(b) \( X \) and \( Y \) are st-crITICAL vertex-sets of \( G \) and \( H \) respectively if and only if \( X \) is an st-crITICAL vertex-set of \( G : XY H \);

(c) if \( X \) is an st-crITICAL vertex-set of \( G : XY H \), then \( \gamma(G : XY H) = \gamma(G) + \gamma(H) - |X| \).

Proof. (a) Firstly, let \( D' \in \mathbf{MDS}(G : XY H) \). Then \( D' \cup Y \) is a dominating set of \( G \cup H \), and so \( \gamma(G) + \gamma(H) = \gamma(G \cup H) \leq |D' \cup Y| = \gamma(G : XY H) + |Y| \), which implies that \( \gamma(G) + \gamma(H) - |X| \leq \gamma(G : XY H) \). Secondly, let \( H' \) be the subgraph of \( G : XY H \) induced by \( V(H - Y) \cup X \). Then \( H' \cong H \). Let \( D_G \in \mathbf{MDS}(G) \) and \( D_{H'} \in \mathbf{MDS}(H') \). Then \( D_G \cup D_{H'} \) is a dominating set of \( G : XY H \), and so \( \gamma(G : XY H) \leq |D_G \cup D_{H'}| \leq \gamma(G) + \gamma(H) \).

(b) \( (\Rightarrow) \) Let \( D_G' \in \mathbf{MDS}(G - X) \) and \( D_{H'} \in \mathbf{MDS}(G - Y) \). Then \( D_G' \cup D_{H'} \) is a dominating set of \( G : XY H \). So \( \gamma(G : XY H - X) \leq |D_G' \cup D_{H'}| = \gamma(G - X) + |H - Y| = \gamma(G) + \gamma(H) - 2|X| \).

By Item (a), we have \( \gamma(G) + \gamma(H) - 2|X| \leq \gamma(G : XY H) - |X| \). Thus \( X \) is an st-crITICAL vertex-set of \( G : XY H \).

(\( \Leftarrow \)) We are going to prove the sufficiency by induction on \( |X| \). When \( |X| = 1 \), we let \( X = \{x\} \), \( Y = \{y\} \), and \( J = G \cup H \). If \( \gamma(G : xy H) = \gamma(G) + \gamma(H) \), then by Lemma 2.10, neither \( x \) nor \( y \) is critical in \( J \). So \( \gamma(G) + \gamma(H) - 1 = \gamma(G : xy H) - 1 = \gamma(G : xy H - x) = \gamma(G - x) + \gamma(H - y) \geq \gamma(G) + \gamma(H) \), a contradiction. Thus we have \( \gamma(G : xy H) = \gamma(G) + \gamma(H) - 1 \) by Item (a). So \( \gamma(G) - 1 + \gamma(H) - 1 = \gamma(G : xy H) - 1 = \gamma(G : xy H - x) = \gamma(G - x) + \gamma(H - y) \geq \gamma(G) - 1 + \gamma(H) - 1 \), from which we get \( \gamma(G - x) = \gamma(G) - 1 \) and \( \gamma(H - y) = \gamma(H) - 1 \), and so the sufficiency holds.

Suppose that the sufficiency holds when \( |X| = n + 1 \) below. We consider the case when \( |X| = n + 1 \) below. Let \( x \in X \), \( y \in Y \), \( X_0 = X - \{x\} \), \( Y_0 = Y - \{y\} \), \( J = G : X_0 Y_0 H \) and \( J' = G : XY H \). Let \( D_1 \in \mathbf{MDS}(G - X) \), \( D_2 \in \mathbf{MDS}(H - Y) \) and \( D = D_1 \cup X \cup D_2 \). Since \( X \) is an st-crITICAL vertex-set of \( J' \), it follows that \( D \in \mathbf{MDS}(J') \). Also, \( D \) is a dominating set of \( J - y \). So \( \gamma(J - y) \leq |D| = \gamma(J') \). By Definition 1.1 and Lemma 2.10, we have

\[
\gamma(J - y) = \begin{cases} 
\gamma(J) - 1 = \gamma(J'), & \text{if } y \text{ is a critical vertex of } J; \\
\geq \gamma(J) \geq \gamma(J'), & \text{if } y \text{ is not a critical vertex of } J,
\end{cases}
\]

from which we know \( \gamma(J - y) \geq \gamma(J') \). Thus \( \gamma(J - y) = \gamma(J') \). Therefore \( \gamma((J - y) - X) = \gamma(G - X) + \gamma(H - Y) = \gamma(J' - X) = \gamma(J') - |X| = \gamma(J - y) - |X| \), which implies that \( X \) is, and so \( X_0 \) is, an st-crITICAL vertex-set of \( J - y \). By Observation 2.7, we see that \( x \) is a critical vertex of \( J' - X_0 \). Note that \( J - y = G : X_0 Y_0 (H - y) \) and \( J' - X_0 = (G - X_0) : xy (H - Y_0) \). By the inductive hypothesis, we have \( X_0 \) is an st-crITICAL vertex-set of \( G \) as well as \( x \) is critical in \( G - X_0 \). Hence \( \gamma(G - X) = \gamma((G - X_0) - x) = \gamma(G - X_0) - 1 = \gamma(G) - |X_0| - 1 = \gamma(G) - |X| \).

That is to say, \( X \) is an st-crITICAL vertex-set of \( G \). Symmetrically, one can prove that \( Y \) is an st-crITICAL vertex-set of \( H \). Thus the result is true when \( |X| = n + 1 \). Item (b) follows.
(c) Let $D^o \in \text{MDS}(G \cdot X_Y H - X)$. Then by Item (b), we have $\gamma(G \cdot X_Y H) = |D^o| + |X| = \gamma(G - X) + \gamma(H - Y) + |X| = \gamma(G) - |X| + \gamma(H) - |Y| + |X| = \gamma(G) + \gamma(H) - |X|$.

**Theorem 2.3.** Let $G$ and $H$ be two disjoint graphs. Let $\emptyset \neq X_i \subseteq V(G)$ for $i = 1, 2, \ldots, k$ and $\emptyset \neq Y_j \subseteq V(H)$ for $j = 1, 2, \ldots, l$ with $|X_i| = |Y_j|$. Then $\{X_1, X_2, \ldots, X_k\}$ and $\{Y_1, Y_2, \ldots, Y_l\}$ are partitions of st-critical vertex-sets of $G$ and $H$ respectively if and only if $\{X_1, X_2, \ldots, X_k, Y_2, Y_3, \ldots, Y_l\}$ is a partition of st-critical vertex-sets of $G, X_1 Y_1 H$.

**Proof.** Let $X = \{X_1, X_2, \ldots, X_k\}$, $Y = \{Y_1, Y_2, \ldots, Y_l\}$ and $X, Y = \{X_1, X_2, \ldots, X_k, Y_2, X_3, \ldots, Y_l\}$.

$(\Rightarrow)$ Clearly, $X, Y$ is a partition of $V(G \cdot X_1 Y_1 H)$. For any $S \in X, Y$, we have $S \in X$ or $S \in Y$. If $S \in X$, then by Theorem 2.2 (c), we have $\gamma(G \cdot X_1 Y_1 H - S) \leq \gamma(G - S) + \gamma(H - Y_1) = \gamma(G) - |S| + \gamma(H) - |X_1| = \gamma(G \cdot X_1 Y_1 H) - |S|$. Similarly, we can also prove that $\gamma(G \cdot X_1 Y_1 H - S) \leq \gamma(G \cdot X_1 Y_1 H) - |S|$. If $S \in Y$. The necessity follows.

$(\Leftarrow)$ Clearly, $X$ and $Y$ are partitions of $V(G)$ and $V(H)$, respectively. Firstly, by Theorem 2.2 (b), $X_1$ and $Y_1$ are st-critical vertex-sets of $G$ and $H$, respectively. Secondly, for any $S \in X - \{X_1\}$, we let $D^- \in \text{MDS}(G \cdot X_1 Y_1 H - S)$, $L = X_1 - (X_1 \cap D^-)$ and $L_G$ be the subset of $L$ that can be dominated by $D^- \cap V(G)$. Let $H'$ be the subgraph of $G \cdot X_1 Y_1 H$ induced by $V(H - Y_1) \cup X_1$. Then $D^- \cap V(G)$ and $D^- \cap V(H')$ are dominating sets of $(G - S) - (L - L_G)$ and $H' - L_G$, respectively. So

$$|D^-| = |D^- \cap V(G)| + |D^- \cap V(H')| - |D^- \cap X_1|$$

$$\geq \gamma((G - S) - (L - L_G)) + \gamma(H' - L_G) - |X_1 \cap D^-|$$

$$\geq \gamma(G - S) - |L - L_G| + \gamma(H') - |L_G| - |X_1 \cap D^-| \quad \text{(By Observation 2.2 (b))}$$

$$\geq \gamma(G) - |S| + \gamma(H) - |X_1|$$

$$= \gamma(G \cdot X_1 Y_1 H) - |S| \quad \text{(By Theorem 2.2 (c))}$$

$$= |D^-|.$$

By the forth equality, we have $\gamma(G - S) = \gamma(G) - |S|$. Thirdly, for any $S \in Y - \{Y_1\}$, we can similarly prove that $\gamma(H - S) = \gamma(H) - |S|$. From these three observations, the sufficiency follows, too.

$$\square$$

3 The existence

In this section, we write $d_G(\cdot) = d(\cdot)$, $N_G(\cdot) = N(\cdot)$ and $N_G[\cdot] = N[\cdot]$, as well as $C_4 \cdot C_4 = (C_4)^2$, $C_4 \cdot 4 \cdot C_4 = (C_4)^3$ and so on for belief.

**Lemma 3.1.** An st-critical vertex-set of a graph $G$ is a 2-packing of $G$.

**Lemma 3.2.** If $d(u) = 1$ and $v \in N(u)$, then $v$ is not a critical vertex of $G$. (This implies that a vertex-critical graph has no vertices of degree one.)
Lemma 3.3. \[\text{Let } S \text{ be an st-critical vertex-set of } G. \text{ If } D_G \in MDS(G - S), \text{ then } |D_G^-| = \gamma(G) - |S| \text{ and } D_G^- \cap N(S) = \emptyset.\]

Proof. Firstly, from the definition of st-critical vertex-set, we have $|D_G^-| = \gamma(G) - |S|$. Secondly, if $D_G^- \cap N(S) \neq \emptyset$, let $L = N(D_G^-) \cap S$. Then $D_G^- \cap N(S)$ is a dominating set of $G - (S - L)$, and so $\gamma(G - (S - L)) \leq |D_G^-| = \gamma(G) - |S| < \gamma(G) - |S - L|$. However, we have $S - L$ is an st-critical vertex-set of $G$ by Observation 2.6, which implies that $\gamma(G - (S - L)) = \gamma(G) - |S - L|$, a contradiction.

Lemma 3.4. Let $S$ be an st-critical vertex-set of $G$ and $w \in V(G - S)$.

(a) If $z \in N(w) \cap S$, then there exists $v_0 \in N(w) - \{z\}$ such that $N(v_0) \cap S = \emptyset$.
(b) Let $uvwz$ be a path or a cycle in $G$ (i.e. $u = z$ is possible). If $u, z \in S$, then $d(w) > 2$.
(c) Let $X = N(w)$. If $2 \leq |X| \leq 3$ and $N(x) \cap S \neq \emptyset$ for every $x \in X$, then $|N(X) \cap S| = 1$.
(d) Let $uvwz$ be a trail in $G$. If $u, z \in S$ and $d(w) = 2$, then $u = z$.

Proof. (a) Suppose to the contrary that $N(v) \cap S \neq \emptyset$ for every $v \in N(w) - \{z\}$. Then $N[w] - \{z\} \subseteq N(S)$. By Lemma 3.3 there exists $D_G^- \in MDS(G - S)$ such that $D_G^- \cap (N[w] - \{z\}) = \emptyset$. But now, we see that $D_G^-$ can not dominate $w$ in $G - S$, a contradiction.

(b) It is an immediate result of Item (a).

(c) Suppose to the contrary that $|N(X) \cap S| \neq 1$. By Lemma 3.1 we have $|N(X) \cap S| \leq |X|$. This implies $|N(X) \cap S| = 2$ or $3$. Let $\{r, s\} \subseteq N(X) \cap S$. Then $N(r) \cap N(s) = \emptyset$. So we must have that at least one of $r$ and $s$, say $r$, is adjacent to only one element of $X$. Thus we may suppose that $\{r\} = N(x') \cap S$, where $x' \in X$. Note that $N(x) \cap S \neq \emptyset$ for every $x \in X$ implies $X \subseteq N(S)$.

By Lemma 3.3 there exists $D_G^- \in MDS(G - S)$ such that $D_G^- \cap X = \emptyset$ and $|D_G^-| + |S| = \gamma(G)$. In order to dominate $w$ in $G - S$, we have $w \in D_G^-$. But then $(D_G^- - \{w\}) \cup (S - \{r\}) \cup \{x'\}$ is a dominating set of $G$ with cardinality $\gamma(G) - 1$, a contradiction.

(d) It is an immediate result of Item (c).

Theorem 3.5. There exists a connected graph $G$ such that $V(G)$ can be partitioned into $l$ strong critical vertex-sets if and only if $l \notin \{2, 3, 5\}$.

Proof. ($\Rightarrow$) Let $k \in Z^+$ and $H_{4,8}$ be the (Harary) graph as shown in Figure 1. Based on the fact that $Z^+ - \{2, 3, 5\} = \{1\} \cup \{3k \mid k \geq 2\} \cup \{3k + 1 \mid k \geq 1\} \cup \{3k + 2 \mid k \geq 2\}$, we let

\[G = \begin{cases} K_1, & \text{if } l = 1; \\ (C_4)^k, & \text{if } l \in \{3k \mid k \geq 2\} \cup \{3k + 1 \mid k \geq 1\}, \\ H_{4,8} \cdot (C_4)^{k-2}, & \text{if } l \in \{3k + 2 \mid k \geq 2\}. \end{cases}\]

Noting that $V(C_4)$ and $V(H_{4,8})$ can be partitioned into 4 and 8 st-critical vertex-sets respectively, we can recursively deduce that $V((C_4)^k)$ and $V(H_{4,8} \cdot (C_4)^{k-2})$ can be partitioned into $3k + 1$ and $3k + 2$ ($k \geq 2$) st-critical vertex-sets respectively by Theorem 2.6. Also, note that $V((C_4)^2)$
can be partitioned into 6 st-critical vertex-sets. So \( V((C_4)^k) \) can be partitioned into 3k \((k \geq 2)\) st-critical vertex-sets. The sufficiency follows.

\( (\Rightarrow) \) Suppose to the contrary that \( l \in \{2, 3, 5\} \). If \( l = 2 \), then by Lemma 3.4, we get that \( d(h) = 1 \) for every \( h \in V(G) \), which implies that \( G \cong K_2 \), contradicting the fact that \( K_2 \) is not a vertex-critical graph. If \( l = 3 \), then by Lemmas 3.2 and 3.1, we deduce that \( d(h) = 2 \) for every \( h \in V(G) \), which implies that \( G \) is a cycle. However, one can check that this is impossible. (According to the two well-known facts that \( \gamma(C_n) = \lceil \frac{n}{2} \rceil \) and \( \gamma(P_n) = \lceil \frac{n}{2} \rceil \), we deduce that a cycle of order at least 4 can not own an st-critical vertex-set of cardinality 2.)

If \( l = 5 \), then we let \( V(G) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \) be a partition of st-critical vertex-sets of \( G \). By Lemmas 3.2 and 3.1, we get that \( 2 \leq d(g) \leq 4 \) for every \( g \in V(G) \).

**Claim 1.** Let \( \{j, k, l, m, n\} = \{1, 2, 3, 4, 5\} \). If \( N(s_n) = \{s_j, s_k, s_l\} \), where \( s_i \in S_i \) for every \( i \in \{j, k, l, m, n\} \), then \( |N(\{s_j, s_k, s_l\}) \cap S_m| = 1 \).

For convenience, suppose without loss of generality that \( \{j, k, l, m, n\} = \{1, 2, 3, 4, 5\} \). We use reduction to absurdity. Assume that \( |N(\{s_1, s_2, s_3\}) \cap S_4| \neq 1 \). Then since

\[
N(s_5) = \{s_1, s_2, s_3\},
\]

by the contrapositive of Lemma 3.4(c), at least one of \( s_1, s_2 \) and \( s_3 \), say \( s_1 \), satisfies \( N(s_1) \cap S_4 = \emptyset \). Thus \( N(s_1) \setminus \{s_5\} \subseteq S_2 \cup S_3 \). To combine this with Lemma 3.4(b), we must have \( d(s_1) \neq 2 \), which implies that \( d(s_1) = 3 \). So \( N(s_1) \cap S_2 \neq \emptyset \) and \( N(s_1) \cap S_3 \neq \emptyset \).

Since \( s_1 \in N(s_5) \setminus S_1 \), by Lemma 3.4(a), one of \( N(s_2) \cap S_1 \) and \( N(s_3) \cap S_1 \), say \( N(s_2) \cap S_1 \), is empty set. By Lemma 3.1, we have \( (N(s_2) \setminus \{s_5\}) \cap (S_2 \cup S_5) = \emptyset \). Since \( s_3 \in N(s_3) \) and \( N(s_1) \cap S_3 \neq \emptyset \), by Lemma 3.4(a), we obtain that \( N(s_2) \cap S_3 = \emptyset \). So by Lemma 3.2 we get \( N(s_2) \cap S_4 \neq \emptyset \). Let \( N(s_2) \cap S_4 = \{s_4\} \). Then

\[
N(s_2) = \{s_4, s_5\}.
\]

So we have \( N(s_4) \cap S_5 = \emptyset \) by the contrapositive of Lemma 3.4(b). Thus \( N(s_4) \setminus \{s_2\} \subseteq S_1 \cup S_3 \).

Since \( d(s_2) = 2 \), we have \( s_4 s_3 \in E(G) \) or \( s_4 s_1 \in E(G) \) by Lemma 3.4(d). But we have supposed that \( N(s_1) \cap S_4 = \emptyset \) in the third sentence of the first paragraph. Thus, only \( s_4 s_3 \in E(G) \) holds, which implies that

\[
N(s_4) = \{s_2, s_3\}.
\]

So by Lemma 3.4(b), we get

\[
N(s_3) \cap S_2 = \emptyset.
\]

Now, if \( N(s_3) \cap S_1 = \emptyset \) (refer to Figure 3 (i-a)) or \( N(s_3) \cap S_1 = \{s_1\} \) (refer to Figure 3 (i-b)), then \( s_1 \) is a cut-vertex of \( G \). Thus by Theorem 2.3 \( G[\{s_1, s_2, s_3, s_4, s_5\}] \) is vertex-critical. But one can check that it is not true. So we can let \( N(s_3) \cap S_1 = \{r_1\} \) with \( r_1 \neq s_1 \). However, by Lemma 3.3 there exists \( D_G^+ \in MD\{(G \setminus \{r_1, s_1\})\} \) such that \( D_G^+ \cap \{s_3, s_5\} = \emptyset \) and \( |D_G^+| + 2 = \gamma(G) \). In order to dominate \( \{s_2, s_4\} \) in \( G \setminus \{r_1, s_1\} \), we have \( \{|s_2, s_4| \cap D_G^+| = 1 \). But then \( (D_G^+ \setminus \{s_2, s_4\}) \cup \{s_3, s_5\} \) is a dominating set of \( G \) with cardinality \( \gamma(G) - 1 \), a contradiction.
that either

This implies that

by Lemma 3.4 (a). There are two subcases.

Then by Lemma 3.4 (b), we have $N(s_2) \cap S_1 = \emptyset$ and $N(s_3) \cap S_1 = \emptyset$, as well as $N(s_2) \cap S_3 = \emptyset$ and $N(s_3) \cap S_2 = \emptyset$. So we must have $N(s_2) \cap S_4 \neq \emptyset$ and $N(s_3) \cap S_4 \neq \emptyset$ by Lemma 3.2 By (3.1), we get $N(s_2) \cap S_4 = \{s_4\} = N(s_3) \cap S_4$. Again by Lemma 3.4 (b), we have $N(s_4) \cap S_5 = \emptyset$.

If $N(s_4) \cap S_1 \neq \emptyset$, then by Lemma 3.4 (d), we have $N(s_4) \cap S_1 = \{s_1\}$. From this, we see that either $G = G[\{s_1, s_2, s_3, s_4, s_5\}]$, or $s_1$ is a cut-vertex of $G$ (no matter $N(s_4) \cap S_1 = \emptyset$ or not). Altogether, we have $G[\{s_1, s_2, s_3, s_4, s_5\}]$ is vertex-critical by Theorem 2.3. But clearly this is not true. (Refer to Figure 3 (ii-A).)

**Case A.** At least two of $s_1$, $s_2$ and $s_3$, say $s_2$ and $s_3$, have degree 2 in $G$.

Then by Lemma 3.4 (a), at least one of $N(s_2) \cap S_1 = \emptyset$ and $N(s_3) \cap S_1 = \emptyset$, say $N(s_2) \cap S_1 = \emptyset$, holds. So $N(s_2) \subseteq S_3 \cup S_4 \cup S_5$, and thus $d(s_2) = 3$. This implies that $N(s_2) \cap S_4 \neq \emptyset$ and $N(s_2) \cap S_3 \neq \emptyset$. From the former, we get $N(s_2) \cap S_4 = \{s_4\}$ while by the latter we can let $N(s_2) \cap S_3 = \{r_3\}$. ($r_3 = s_3$ is possible.) Since $s_3 \in N(s_5)$, we get

$$N(s_1) \cap S_3 = \emptyset$$

(3.2) by Lemma 3.4 (a). There are two subcases.

When $N(s_1) \cap S_2 = \emptyset$, we have $N(s_1) \cap S_4 \neq \emptyset$ since $d(s_1) \geq 2$. By (3.1), we have $N(s_1) \cap S_4 = \{s_4\}$. So $N(s_1) = \{s_4, s_5\}$. Thus by Lemma 3.4 (b), we have $N(s_4) \cap S_5 = \emptyset$. Since $r_3 \in N(s_2)$, we get $N(s_4) \cap S_3 = \emptyset$ by Lemma 3.4 (a), and so $N(s_4) = \{s_1, s_2\}$. If $r_3 = s_3$, then $s_3$ is a cut-vertex of $G$, and so $G[\{s_1, s_2, s_3, s_4, s_5\}]$ is vertex-critical, which is not true. If $r_3 \neq s_3$, then $\{r_3, s_3\}$ is a vertex-cut of $G$. (Refer to Figure 4 (ii-B1).) By Observation 2.6 and Theorem 2.3, $G[\{s_1, s_2, s_3, s_4, s_5, r_3\}]$ is vertex-critical, which is also not true.

When $N(s_1) \cap S_2 \neq \emptyset$, by (3.2) and Lemma 3.4 (b), we have $N(s_1) \cap S_4 \neq \emptyset$, which implies that $N(s_1) \cap S_4 = \{s_4\}$. Since $s_2 \in N(s_5)$, we have $N(s_3) \cap S_2 = \emptyset$ by Lemma 3.4 (a). So $d(s_3) = 3$, and thus $N(s_3) \cap S_1 \neq \emptyset$ and $N(s_3) \cap S_4 \neq \emptyset$. By (3.1), we have $N(s_3) \cap S_4 = \{s_4\}$. (Refer to
Now, we have \( r_3 \in N(s_2), N(s_4) \cap S_3 = \{s_3\} \) and \( N(s_5) \cap S_3 = \{s_3\} \). However, according to Lemma 3.4 (a), this is impossible.

**Claim 3.** \( d(g) \neq 4 \) for every \( g \in V(G) \).

Without loss of generality, suppose to the contrary that there exists some \( s_5 \in S_5 \) such that \( N(s_5) = \{s_1, s_2, s_3, s_4\} \), where \( s_i \in S_i \), \( i = 1, 2, 3, 4 \). For every \( 1 \leq i \leq 4 \), by Lemma 3.4 (b) and Claim 2, we have \( d(s_i) \neq 2 \) and \( d(s_i) \neq 3 \), which implies that \( d(s_i) = 4 \). (Refer to Figure 4 (iii).) However, by Lemma 3.4 (a), this is impossible.

By Claims 2 and 3, we get that \( d_H(g) = 2 \) for every \( g \in V(G) \), which implies that \( G \) is a cycle, a contradiction. The necessity follows, too.

**4 Conclusion**

In [23], the authors got the following Proposition 4.1 which tells us that \( \mathcal{C}_4 = \{C_4\} \), where \( \mathcal{C}_4 \) was defined in the last paragraph of Section 1. It is easy to see that the circulant graph \( C_{12}(1, 5) \), the vertex coalescence \( C_4 \cdot C_4 \) and the Harary graph \( H_{4,6} \) (see Figure 5) belong to \( \mathcal{C}_6 \). To compare Proposition 4.1, we want to know whether \( \mathcal{C}_6 \) is a finite set. So we present Problem 4.2.

![Figure 4: Illustration for the proofs of Claim 2-B and Claim 3.](image)

**Proposition 4.1.** [23] Let \( H \) be a connected graph. Then \( V(H) \) can be partitioned into 4 strong critical vertex-sets if and only if \( H \cong C_4 \).

**Problem 4.2.** Give a constructive characterization of the connected graphs \( G \) such that \( V(G) \) can be partitioned into 6 strong critical vertex-sets of \( G \).
It is known that each graph has a degree sequence, but a given sequence may be not a degree sequence of any simple graph. For instance, the sequence \( (7,6,5,4,3,3,2) \) can not become a degree sequence of a simple graph (see [3], Ex. 1.5.6). If \( V(G) = S_1 \cup S_2 \cup \cdots \cup S_l \) is a strong critical vertex-sets partition of a graph \( G \), then we call the sequence \( (|S_1|, |S_2|, \ldots, |S_l|) \) as a strong critical vertex-sets sequence of \( G \). It is noteworthy that even a connected graph may own different strong critical vertex-sets sequences. For example, both \( (3,2,2,1,1,1,1,1,1) \) and \( (2,2,2,2,1,1,1,1,1) \) are strong critical vertex-sets sequences of the graph depicted in Figure 6. Also, for connected graphs, it follows from Theorem 5.3 that the strong critical vertex-sets sequence \( (1,1,1,1) \) exists but \( (1,1,1,1,2) \) does not exist.

![Figure 6: A graph with more than one strong critical vertex-sets sequences.](image)

**Problem 4.3.** What kinds of strong critical vertex-sets sequences do exist? Or to be concrete about it, if \( (|S_1|, |S_2|, \ldots, |S_l|) \) is a strong critical vertex-sets sequence of a connected graph \( G \), then what are the relations of \( |S_1|, |S_2|, \ldots, |S_l|, l \) and \( \gamma(G) \)?

5 **Acknowledgement**

Thank Professor D. F. Rall for his suggestion on the definition of strong critical vertex-set, and e-mailing the article (ref. [19]) to us!

**References**

[1] N. Ananchuen and M. D. Plummer, Matchings in 3-vertex-critical graphs: The even case, Networks, 45(4) (2005) 210-213.

[2] N. Ananchuen and M. D. Plummer, Matchings in 3-vertex-critical graphs: The odd case, Discrete Math. 307(13) (2007) (1651-1658).

[3] J. A. Bondy and U. S. R. Murty, Graph theory with applications, Macmillan Press, London, 1976.

[4] R. C. Brigham, P. Z. Chinn, and R. D. Dutton, A Study of vertex domination critical graphs. Department of Mathematics Technical Report M-2, University of Central Florida, 1984.

[5] R. C. Brigham, P. Z. Chinn, and R. D. Dutton, Vertex domination-critical graphs, Networks, 18(3) (1988) 173-179.
[6] R. C. Brigham, T. W. Haynes, M.A. Henning, D. F. Rall, Bicritical domination, Discrete Math. 305 (2005) 18-32.

[7] X. G. Chena, S. Fujita, M. Furuya and M. Y. Sohn, Constructing connected bicritical graphs with edge-connectivity 2, Discrete Appl. Math. 160 (2012) 488-493.

[8] J. Fulman, D. Hanson and G. Macgillivray, Vertex domination-critical graphs, Networks, 25(2) (1995) 41-43.

[9] M. Furuya, Construction of $(\gamma, k)$-critical graphs, Australas. J. Combin. 53 (2012) 53-65.

[10] M. Furuya, On the diameter of domination bicritical graphs, Australas. J. Combin. 62 (2015) 184-196.

[11] P. J. P. Grobler and A. Roux, Coalescence and criticality of graphs, Discrete Math. 313(10) (2013) 1087-1097.

[12] B. L. Hartnell and D. F. Rall, On Vizing’s conjecture, Congr. Numer. 82 (1991) 87-96.

[13] B. L. Hartnell and D. F. Rall, On dominating the Cartesian product of a graph and $K_2$, Discuss. Math. Graph Theory 24 (2004) 389-402.

[14] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, eds., Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.

[15] A. P. Kazemi, Every $K_{1,7}$ and $K_{1,3}$-free, 3-vertex-critical graph of even order has a perfect matching, J. Discrete Math. Sci. Cryptography, 13(6) (2010) 583-591.

[16] M. Krzywkowski and D. A. Mojdeh, Bicritical domination and double coalescence of graphs, Georgian Math. J. 23(3) (2016) 399-404.

[17] D. A. Mojdeh and P. Firoozi, Characteristics of $(\gamma, 3)$-critical graphs, Appl. Anal. Discrete Math. 4 (2010) 197-206.

[18] D. A. Mojdeh, P. Firoozi and R. Hasni, On connected $(\gamma, k)$-critical graphs, Australas. J. Combin. 46 (2010) 25-35.

[19] J. B. Phillips, T. W. Haynes, P. J. Slater, A generalization of domination critical graphs, Utilitas Math. 58 (2000) 129-144.

[20] T. Wang and Q. L. Yu, Factor-critical property in 3-dominating-critical graphs, Discrete Math. 309(5) (2009) 1079-1083.

[21] T. Wang and Q. L. Yu, A conjecture on $k$-factor-critical and 3-$\gamma$-critical graphs, Sci. China Math. 53(5) (2010) 1385-1391.

[22] Y. Wang, F. Wang and W. S. Zhao, Construction for trees without domination critical vertices, AIMS Math. 6(10) (2021) 10696-10706.

[23] W. S. Zhao, R. Z. Lin and J. Q. Cai, On Construction for Trees Making the Equality Hold in Vizing’s Conjecture, J. Graph Theory, (2022) (DOI: 10.1002/jgt.22833).