1 Preliminary Results and Notations

A Matrix-Sequence $\{A_n\}_n$ is an ordered collection of complex matrices such that $A_n \in \mathbb{C}^{n \times n}$. We will denote by $E$ the space of all matrix-sequences, 

$$E := \{\{A_n\}_n : A_n \in \mathbb{C}^{n \times n}\}.$$ 

It is often observed in practice that matrix-sequences $\{A_n\}_n$ arising from the numerical discretization of linear differential equations possess a Spectral Symbol (see [5] and references therein), that is, a measurable function describing the asymptotic distribution of the eigenvalues of $A_n$ in the Weyl sense [3, 5, 9]. We recall that a spectral symbol associated with a sequence $\{A_n\}_n$ is a measurable function $f : D \subseteq \mathbb{R}^q \rightarrow \mathbb{C}$, $q \geq 1$, satisfying 

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{l(D)} \int_D F(x)dx$$

for every continuous function $F : \mathbb{C} \rightarrow \mathbb{C}$ with compact support, where $D$ is a measurable set with finite Lebesgue measure $l(D) > 0$ and $\lambda_i(A_n)$ are the eigenvalues of $A_n$. In this case we write 

$$\{A_n\}_n \sim \lambda f.$$ 

Another important and linked concept is the notion of Spectral Measure. Given a matrix sequence $\{A_n\}_n$ and a measure $\mu$ on $\mathbb{C}$, we say that $\mu$ is the spectral measure associated to $\{A_n\}_n$ if 

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \int_{\mathbb{C}} F(x) d\mu \quad \forall F \in C_c(\mathbb{C})$$

and in this case, we can write $\{A_n\}_n \sim \lambda \mu$. In [1] we showed that the space of sequences that admit a spectral symbol coincide with the space of sequences admitting a spectral measure that is also a probability measure. In particular, since $(\mathbb{C}, \mu)$ is a Standard Probability Space, the following lemma can be proved.

Lemma 1. Given any probability measure $\mu$ on $\mathbb{C}$, there exists a measurable function $k : [0, 1] \rightarrow \mathbb{C}$ such that 

$$\int_0^1 F(k(x))dx = \int_{\mathbb{C}} F(x) d\mu$$

for every $F \in C_c(\mathbb{C})$. In particular, the essential range of $k$ is contained in the support of the measure $\mu$, and if $\mu$ is supported on $\mathbb{R}$ then $k(x)$ can be chosen monotone increasing.

We want to give some result of existence for the spectral symbols, given particular hypothesis on the eigenvalues of $A_n$. Here we report two theorems that give necessary and sufficient conditions on the existence of a measure given its moments (also known as moments problem).

Theorem 1 (Hausdorff). [3] Given a sequence of real numbers $\{m_n\}_n$, let 

$$(\Delta m)_n := m_{n+1} - m_n.$$ 

We say that the sequence $\{m_n\}_n$ is a Completely Monotonic Sequence if 

$$(-1)^k(\Delta^k m)_n \geq 0 \quad \forall n, k \in \mathbb{N}.$$ 

$\{m_n\}_n$ is a completely monotonic sequence if and only if there exists a measure $\mu$ over $[0, 1]$ satisfying 

$$m_n = \int_0^1 x^n d\mu \quad \forall n \in \mathbb{N}.$$ 

In this case the determined measure is unique.
In order to state the second theorem, we have to recall that a Laurent polynomial is an object in \( \mathbb{C}[z, z^{-1}] \), that is also denoted as \( \mathbb{C}[\mathbb{Z}] \). This space is a unit \( * \)-algebra with

\[
q(z) = \sum_{i=-M}^{M} a_i z^i \in \mathbb{C}[[z]] \implies q^*(z) = \sum_{i=-M}^{M} \overline{a_i} z^{-i} \in \mathbb{C}[\mathbb{Z}].
\]

Given a sequence of complex numbers \( s = (s_i)_{i \in \mathbb{N}} \) and its prolongation to \( \mathbb{Z} \) by \( s_{-i} := \overline{s_i} \), we can define the linear operator

\[
q(z) = \sum_{i=-M}^{M} a_i z^i \implies L_s(q) = \sum_{i=-M}^{M} a_i s_i.
\]

**Theorem 2** (Carathéodory–Toeplitz). \([7]\) Given a complex sequence \( (s_i)_{i \in \mathbb{N}} \), the following are equivalent:

1. There exists a Radon measure on \( \mathbb{T} \) such that \( s_k = \int_{\mathbb{T}} z^k d\mu(z) \quad \forall k \in \mathbb{N} \).

2. \( L_s(q^* q) \geq 0 \) for all \( q \in \mathbb{C}[\mathbb{Z}] \).

3. The infinite Toeplitz matrix \( H(s) = (s_{j-i})_{i,j=0}^{\infty} \) is positive semidefinite.

Eventually, we have to recall a pair of classical approximation results that let us approximate continuous functions over compact sets with polynomials or trigonometric polynomials.

**Theorem 3** (Stone-Weierstrass).

1. Suppose \( f \) is a continuous real-valued function defined on the real interval \( [a, b] \). For every \( \varepsilon > 0 \), there exists a polynomial \( p \) such that \( \| f - p \|_{\infty} \leq \varepsilon \).

2. Suppose \( f \) is a continuous complex-valued function defined on the unitary circle \( \mathbb{T} \). For every \( \varepsilon > 0 \), there exists a Laurent polynomial \( q \) such that \( \| f - q \|_{\infty} \leq \varepsilon \).

**Theorem 4.** If \( f \) is an absolutely continuous real-valued function and \( 2\pi \)-periodic, then the Fourier series of \( f \) converges uniformly.

### 2 Polynomial Limits

From now on, we will work with double indexed sequences \( \{x_{i,n}\}_{i,n} \) of real or complex values, where \( i, n \in \mathbb{N}, \quad 0 < i < n \). Usually, we will deal with convergent sequences

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F_k(x_{i,n}) \in \mathbb{C}, \quad \forall k \in \mathbb{N}
\]

for specific class of functions \( F_k \) and we want to know if this is enough to conclude that the limit is described asymptotically by a measure, that is

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(x_{i,n}) = \int_{\mathbb{C}} F(x) d\mu
\]

for every \( F \in C_b(\mathbb{C}) \). In particular, we want to know if it holds when \( F(x) \) is locally a polynomial, so that we can use both Weierstrass and Hausdorff theorems.

**Lemma 2.** Let \( \{x_{i,n}\}_{i,n} \) be a sequence of real numbers with \( 1 \leq i \leq n \) and

- \( |x_{i,n}| \leq M \) for every \( i, n \),
- \( h_k = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_{i,n}^k \in \mathbb{R} \quad \forall k \in \mathbb{N} \).

Then there exists a function \( k : [0, 1] \to [-M, M] \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(x_{i,n}) = \int_{0}^{1} F(k(x)) dx
\]

for every \( F \in C[-M, M] \).
Given a sequence $\{A_n\}_n$ of Hermitian matrices with uniformly bounded by $M$, the sequence $\{A_n\}_n$ admits a spectral symbol if and only if

$$\lim_{n \to \infty} \frac{1}{n} Tr(A_n^k) \in \mathbb{R} \quad \forall k \in \mathbb{N}.$$
for every \( F \in C_c(\mathbb{C}) \), that coincides with the definition of spectral symbol for \( \{A_n\}_n \).

Vice versa, if we suppose that \( \{A_n\}_n \) admits a spectral symbol \( k \), then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_{i,n} = \frac{1}{|D|} \int_D k(x) dx \in \mathbb{R}, \quad \forall h \in \mathbb{N}.
\]

\[\square\]

Let us now consider \( \lambda_{i,n} \) eigenvalues of \( A_n \) on a complex circle, meaning that they have all the same norm. There’s a simple way to prove that if the averages of \( \lambda_{i,n}^k \) converge, then \( \{A_n\}_n \) admit a spectral measure that is also absolutely continuous with respect to the Lebesgue measure, but only if we assume a decay assumption.

**Lemma 3.** For all \( n \in \mathbb{N} \) consider a set of complex numbers \( \{\lambda_{i,n}\}_{i=1:n} \) with polar representations \( \lambda_{i,n} = ce^{ikx_{i,n}} \) such that \( x_{i,n} \in [-\pi, \pi] \) and \( c \in \mathbb{R}^+ \) does not depend on \( i, n \). Suppose moreover that

\[
s_k := \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \lambda_{i,n}^k, \quad \forall k \in \mathbb{Z}
\]

and define the function

\[
M(x) := \sum_{k=-\infty}^{\infty} \frac{s_k}{c^k} e^{-ikx}.
\]

If \( \sum_{k=-\infty}^{\infty} \frac{|s_k|}{c^k} < \infty \), then for every function \( F \in C[-\pi, \pi] \) the following ergodic relation holds:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(x_{i,n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)M(x)dx.
\]

**Proof.** First, we notice that \( M(x) \) is well defined and continuous, since the series

\[
\sum_{k=-\infty}^{\infty} \frac{s_k}{c^k} e^{ikx}
\]

converges uniformly. The trigonometric polynomials are dense in the space of periodic continuous functions on \([-\pi, \pi]\), so we first test the ergodic relation on functions of the type \( e^{ikx} \).

\[
\frac{1}{n} \sum_{i=1}^{n} \exp(ikx_{i,n}) = \frac{1}{c^n} \sum_{i=1}^{n} \lambda_{i,n}^k \xrightarrow{n \to \infty} \frac{s_k}{c^k}
\]

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(ikx)M(x)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{s_k}{c^k} dx = \frac{s_k}{c^k}.
\]

The ergodic relation thus holds for every \( F \in C_{per}[-\pi, \pi] \). Let now \( G \) be any continuous function on \([\pi, \pi]\) and \( F \) a continuous periodic function on the same domain that coincides with \( G \) on \([-\pi + \varepsilon, \pi - \varepsilon]\) and such that \( \|F\|_{\infty} \leq \|G\|_{\infty} \).

\[
\left| \frac{1}{n} \sum_{i=1}^{n} G(x_{i,n}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x)M(x)dx \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} G(x_{i,n}) - \frac{1}{n} \sum_{i=1}^{n} F(x_{i,n}) \right| + \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)M(x)dx \right| + \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (F(x) - G(x))M(x)dx \right|
\]

\[
\leq 2\|G\|_{\infty} \frac{\# \left\{ i \mid \|x_{i,n} - \pi\| \leq \varepsilon \right\}}{n} + o_n(1) + 2\|G\|_{\infty} \|M\|_{\infty} \frac{\varepsilon}{\pi}.
\]

All that is left to show is that the first term is proportional to \( \varepsilon \). Let \( H \) be a periodic continuous nonnegative function on \([-\pi, \pi]\) such that \( H = 1 \) on \([-\pi - \varepsilon, -\pi + \varepsilon] \cup [\pi - \varepsilon, \pi] \), \( H = 0 \) on \([-\pi + 2\varepsilon, \pi - 2\varepsilon] \), and \( \|H\|_{\infty} = 1 \). We know that

\[
\limsup_{n \to \infty} \frac{\# \left\{ i \mid \|x_{i,n} - \pi\| \leq \varepsilon \right\}}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} H(x_{i,n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(x)M(x)dx \leq \|M\|_{\infty} \frac{2\varepsilon}{\pi}.
\]
The last relation shows that definitively on $n$,

$$
\left| \frac{1}{n} \sum_{i=1}^{n} G(x_i,n) - \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x)M(x)dx \right| \leq 2\|G\|_\infty \left( \frac{2\varepsilon}{\pi} + o(n(1) + 2\|G\|_\infty \frac{\varepsilon}{\pi} \right)
$$

holds for every $\varepsilon$, leading to the wanted conclusion. \qed

In the case when $\sum_{k=-\infty}^{\infty} \frac{|x_k|}{c} = \infty$, there may not be an absolutely continuous spectral measure, but $\{A_n\}_n$ possesses a probability spectral measure anyway, and thus a spectral symbol.

**Lemma 4.** Suppose $\{A_n\}_n$ is a sequence of matrices $A_n \in \mathbb{C}^{n \times n}$ with eigenvalues $\{\lambda_{i,n}\}_{i=1:n}$. Suppose that

- $|\lambda_{i,n}| = c$ is independent from $i, n$,
- $s_k = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \lambda_{i,n}^k \in \mathbb{C}$, $\forall k \in \mathbb{Z}$.

In this case, $\{A_n\}_n$ admits a spectral symbol.

**Proof.** Let $\omega_{i,n} := \lambda_{i,n}/c$ be complex numbers of unitary norm. If $d_k := s_k/c^k$, then

$$
d_k = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \omega_{i,n}^k \in \mathbb{C}, \quad \forall k \in \mathbb{Z}
$$

and moreover $d_{-k} = \overline{d_k}$. Let $\mu_n$ be the atomic probability measure on $T$ induced by $\omega_{i,n}$ and call $r_{k,n}$ its moments

$$
\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\omega_{i,n}}, \quad r_{k,n} = \int x^k d\mu_n = \frac{1}{n} \sum_{i=1}^{n} \omega_{i,n}^k.
$$

The sequence $r_n := (r_{k,n})_k$ represents the moments of a measure on the unit circle, so we can apply theorem\(^2\) and obtain that for every $q \in \mathbb{C}[\mathbb{Z}]$, the quantity $L_{r_n}(q^*q)$ is nonnegative. We know that $r_{k,n}$ converges to $d_k$ if $n$ goes to infinity, so we can prove that also $L_{r_n}$ converges punctually to $L_d$:

$$
q^*q = \sum_{i=-\infty}^{\infty} a_i z^i \quad \Rightarrow \quad L_{r_n}(q^*q) = \sum_{i=-\infty}^{\infty} a_i r_{i,n} \xrightarrow{n \to \infty} \sum_{i=-\infty}^{\infty} a_i d_i = L_d(q^*q).
$$

Since $L_{r_n}(q^*q)$ is nonnegative, also $L_d(q^*q)$ will not be negative, and applying theorem again, we obtain that $(d_k)_k$ represents the moments of a measure $\mu$ on $T$, that is also a probability measure since $d_0 = 1$.

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \omega_{i,n}^k = d_k = \int_T z^k d\mu(z), \quad \forall k \in \mathbb{Z}
$$

Stone-Weierstrass Theorem\(^3\) let us approximate any continuous function on $T$ with a Laurent polynomial, so

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} G(\omega_{i,n}) = \int_T G(z)d\mu(z)
$$

holds for every $G \in C_c(\mathbb{C})$. Going back to the eigenvalues $\lambda_{i,n}$, suppose that $F$ is a continuous function defined on $cT$, and $G(x) := F(cx)$.

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_{i,n}) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} G(\omega_{i,n}) = \int_T G(z)d\mu(z) = \int_{cT} F(z)d\mu \left( \frac{z}{c} \right)
$$

The last relation let us conclude that $\{A_n\}_n$ admits a spectral measure, and consequently admits also a spectral symbol. \qed
3 Curves and Weil Systems

Given a curve $X$ of genus $g$ on a finite field $\mathbb{F}_q$, we can define its Zeta function as

$$Z(t) = \frac{P(t)}{(1-t)(1-qt)}$$

where $P(t)$ is a polynomial with the following properties (Weil Conjecture):

- $P(t) \in \mathbb{Z}[t]$,
- $P(t) = \prod_{i=1}^{2g}(1 - \lambda_it)$ where $|\lambda_i| = \sqrt{q}$ and if $\lambda \in \mathbb{R}$ it has even multiplicity,
- There exists natural numbers $B_m$ such that
  $$P(t) = (1-t)(1-qt) \prod_{m=1}^{\infty} (1 - t^m)^{-B_m},$$
- There exist natural numbers $N_m$ such that
  $$P(t) = (1-t)(1-qt) \exp \left( \sum_{m=1}^{n} N_m \frac{t^n}{m} \right),$$
- $P(t)$ is the characteristic polynomial of the Frobenius endomorphism induced to the Tate module (or the first Étale cohomology group) of the curve.

Given this properties one can prove relations between $\lambda_i$, $N_m$ and $B_m$. Namely

$$N_m = q^m + 1 - \sum_{i=1}^{2g} \lambda_i^m = \sum_{d|m} dB_d, \quad mB_m = \sum_{d|m} \mu \left( \frac{m}{d} \right) \left( q^d + 1 - \sum_{i=1}^{2g} \lambda_i^d \right) = \sum_{d|m} \mu \left( \frac{m}{d} \right) N_d.$$  

An *asymptotically exact family of Weil systems* is a sequence of curves $\{X_n\}$, where $X_n$ has genus $n$, and such that

$$\beta_m = \lim_{n \to \infty} \frac{B_m(X_n)}{n} \in \mathbb{R}$$

exists for every $m \in \mathbb{N}$. Thanks to the relations above, we can see that this condition is equivalent to require that

$$\nu_m = \lim_{n \to \infty} \frac{\sum_{i=1}^{2g} \lambda_i^m}{n} \in \mathbb{R}$$

for every $m \in \mathbb{N}$, where $\lambda_{i,n}$ are the roots of the polynomials $P_n(t)$ associated to $X_n$.

Here we have all the hypothesis to apply Lemma $\text{[4]}$, with the exception for $A_n \in \mathbb{C}^{n \times n}$. In fact, $\lambda_{i,n}$ are eigenvalues of the Frobenius endomorphisms, that has elements in the $p$-adic field $\mathbb{Q}_p$. We can notice, though, that the result does not depend on the matrices $A_n$ themselves, but only on their eigenvalues, and we can conclude nonetheless that there exists a limit measure

$$\mu = \lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_{i,n}}.$$  

In the case of asymptotically exact families, it is possible to give an estimate of $\nu_k$, that follows from the proved bound on $\beta_m$

$$\sum_{m=1}^{\infty} \frac{m\beta_m}{q^{m/2} + 1} \leq 1.$$  

In fact, using

$$-\nu_m = -\lim_{n \to \infty} \frac{\sum_{i=1}^{2g} \lambda_i^m}{n} = \lim_{n \to \infty} \frac{N_m(X_n)}{n} = \lim_{n \to \infty} \frac{\sum_{d|m} dB_d(X_n)}{n} = \sum_{d|m} dB_d \geq 0$$

it is easy to prove that

$$0 \leq -\sum_{m=1}^{\infty} \nu_m q^{-m/2} = \sum_{m=1}^{\infty} \frac{m\beta_m}{q^{m/2} + 1} \leq 1.$$  

Owing to Lemma $\text{[3]}$ we can conclude that the spectral measure $\mu$ has also a density $M(x)$ with respect to the Lebesgue measure. These results were already proven in $\text{[11]}$ and $\text{[12]}$. 
3.1 Food for Thought

One can notice that we used only particular cases of the results we proved in Section 2, so one can ask if a generalization of ideas could lead to something more.

For example, consider \( \lambda_{i,n} \) complex numbers of the same norm \( c \), with \( 1 \leq i \leq 2n \). It’s fairly easy to see that, in analogy with the curves theory, the following assertions hold.

- We can define \( N_m, B_m \) with
  \[
  N_m := a^m + 1 - \sum_{i=1}^{2n} \lambda_{i,n}^m, \quad mB_m = \sum_{d|m} \mu \left( \frac{m}{d} \right) N_d
  \]
  where \( a \in \mathbb{N} \).

- If \( P_n(x) = \prod_{i=1}^{2n} (1 - \lambda_{i,n}x) \), then
  \[
  P_n(x) = (1 - ax)(1 - x) \exp \left( \sum_{k=1}^{\infty} N_k \frac{x^k}{k} \right) = (1 - ax)(1 - x) \prod_{k=1}^{\infty} (1 - x^m)^{-B_m}
  \]

If we require also that the averages of the powers of \( \lambda_{i,n} \) converge, then we can apply Lemma 4 and conclude that there exists an asymptotic measure for \( \lambda_{i,n} \).

In general \( N_m, B_m \) will be complex numbers, so it does not seem an interesting generalization. We thus consider an ulterior hypothesis that brings the setting closer to Weil systems. If we suppose that \( P_n(x) \in \mathbb{Z}[x] \), then we can prove that \( N_m \) are integer numbers using properties of symmetric functions, and therefore \( B_m \) are rational numbers. The set of asymptotic measures that describe the polynomials \( P_n(x) \) strictly contains the measures describing asymptotically exact family of Weil systems, and in particular, it includes some atomic measures.

It may thus be interesting to study the exact Weil system through their asymptotic measures by looking at it from a more general context, or considering the spectral symbols associated to the measures.

Eventually, we report an example of \( \lambda_{i,n} \) for which \( N_m, B_m \) are integers, but the asymptotic measure is yet atomic.

**Example 1.** Let \( \lambda_{i,n} = \sqrt{k}(-1)^i \), where \( k \) is a natural number different from 0, and let \( a \) be an integer number. In this case, it is easy to see that the asymptotic measure \( \mu \) on the circle with ray \( \sqrt{k} \) is atomic \( \mu = (\delta_\sqrt{k} + \delta_{-\sqrt{k}})/2 \).

The polynomial \( P_n(x) \) has integer coefficients, since
\[
P_n(x) = \prod_{i=1}^{n} (1 - \sqrt{k}x)(1 + \sqrt{k}x) = (1 - kx^2)^n
\]
and thus \( N_m \) are integers
\[
N_m = \begin{cases} 
  a^m + 1 & m \ odd, \\
  a^m + 1 - 2nk^m/2 & m \ even.
\end{cases}
\]

Exploiting the properties of Moebius function, we can also prove that \( B_m \) is integer, in fact
\[
\sum_{d|m} \mu \left( \frac{m}{d} \right) N_d \equiv 0 \pmod{m}, \quad \forall m \in \mathbb{N}.
\]

As one can see from the relations above, \( N_m \) is negative when \( m \) is even and \( n \) is large enough. In fact, \( \nu_n \) are all non-negative integers and
\[
-\sum_{m=1}^{\infty} \nu_m e^{-m} = \sum_{m=1}^{\infty} \frac{m\beta_m}{e^m - 1} = -\infty.
\]

References

[1] Barbarino G. *Spectral Measures*. Proceedings of Cortona Meeting, Springer INdAM Series (to appear 2018).

[2] Barbarino G., Sierra-Capizzano S. *Non-Hermitian perturbations of Hermitian matrix-sequences and applications to the spectral analysis of approximated PDEs*. (2018).

[3] Böttcher A., Silbermann B. *Introduction to Large Truncated Toeplitz Matrices*. Springer, New York (1999).
[4] Donatelli M., Neytcheva M., Serra-Capizzano S. Canonical eigenvalue distribution of multilevel block Toeplitz sequences with non-Hermitian symbols. Spectral theory, mathematical system theory, evolution equations, differential and difference equations, Oper. Theory Adv. Appl. 221, Birkhäuser/Springer Basel AG, Basel, (2012) 269–291.

[5] Garoni C., Serra-Capizzano S. Generalized Locally Toeplitz Sequences: Theory and Applications (Volume I). Springer, Cham (2017).

[6] Golinskii L., Serra-Capizzano S. The asymptotic properties of the spectrum of nonsymmetrically perturbed Jacobi matrix sequences. J. Approx. Theory 144 (2007), no. 1, 84–102.

[7] Schmüdgen K. The Moment Problem. Graduate Texts in Mathematics, 277, Springer International Publishing (2017).

[8] Shohat, J.A., Tamarkin, J.D. The problem of moments. American Mathematical Society, Mathematical surveys (1943).

[9] Tlli P. Locally Toeplitz sequences: spectral properties and applications. Linear Algebra Appl. 278 (1998) 91–120.

[10] Tlli P. Some results on complex Toeplitz eigenvalues. J. Math. Anal. Appl. 239 (1999), no. 2, 390–401.

[11] Tsfasman M.A. Some Remarks on the Asymptotic Number of Points. Lect. Notes Math. 1518 (1970) 178–192.

[12] Tsfasman M.A., Vlăduţ S.G. Asymptotic Properties of Zeta-Functions. J. Math. Sci. 84 (1997), No. 5, 1445–1467.