List-decodable Codes and Covering Codes

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Abstract

The list-decodable code has been an active topic in theoretical computer science since the seminal papers of M. Sudan and V. Guruswami in 1997-1998. There are general results about the Johnson radius and the list-decoding capacity theorem. However, few results about general constraints on rates, list-decodable radius and list sizes for list-decodable codes have been obtained. List-decodable codes are also considered in rank-metric, subspace metric, cover-metric, pair metric and insdel metric settings. In this paper we show that rates, list-decodable radius and list sizes are closely related to the classical topic of covering codes. We prove new general simple but strong upper bounds for list-decodable codes in general finite metric spaces based on various covering codes. The general covering code upper bounds can apply to the case when the volumes of the balls depend on the centers, not only on the radius case. Then any good upper bound on the covering radius or the size of covering code imply a good upper bound on the sizes of list-decodable codes. Hence the list-decodability of codes is a strong constraint from the view of covering codes on general finite metric spaces. Our results give exponential improvements on the recent generalized Singleton upper bound of Shangguan and Tamo in STOC 2020 for Hamming metric list-decodable codes, when the code lengths are very large. The asymptotic forms of covering code bounds can partially recover the Blinovsky bound and the combinatorial bound of Guruswami-Håstad-Sudan-Zuckerman in Hamming metric setting. We also suggest to study the combinatorial covering list-decodable codes as a natural generalization of combinatorial list-decodable codes. We apply our general covering code upper bounds for list-decodable rank-metric codes, list-decodable subspace codes, list-decodable insertion

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codes and list-decodable deletion codes. Some new better results about non-list-decodability of rank-metric codes and subspace codes are obtained.

1 Introduction

For a vector \( \mathbf{a} \in \mathbb{F}_q^n \), the Hamming weight \( \text{wt}(\mathbf{a}) \) of \( \mathbf{a} \) is the number of non-zero coordinate positions. The Hamming distance \( d_H(\mathbf{a}, \mathbf{b}) \) between two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is defined to be the Hamming weight of \( \mathbf{a} - \mathbf{b} \). For a (linear) code \( C \subset \mathbb{F}_q^n \) of dimension \( k \), its Hamming distance (or weight) \( d_H \) is the minimum of Hamming distances \( d_H(\mathbf{a}, \mathbf{b}) \) between any two different codewords \( \mathbf{a} \) and \( \mathbf{b} \) in \( C \). It is well-known that the Hamming distance (or weight) of a linear code \( C \) is the minimum Hamming weight of its non-zero codewords. The famous Singleton bound \( d_H \leq n - k + 1 \), see [80], is the basic upper bound for linear error-correcting codes. A linear code attaining this bound is called a MDS (maximal distance separable) code.

For a code \( C \subset \mathbb{F}_q^n \), we define its covering radius by

\[
R_{\text{covering}}(C) = \max_{\mathbf{x} \in \mathbb{F}_q^n} \min_{\mathbf{c} \in C} \{\text{wt}(\mathbf{x} - \mathbf{c})\}
\]

Hence the Hamming balls \( B(x, R_{\text{covering}}(C)) \) centered at all codewords \( x \in C \) with the radius \( R_{\text{covering}}(C) \) cover the whole space \( \mathbb{F}_q^n \). Actually the covering radius \( R_{\text{covering}}(C) \) of a linear \([n, k]_q \) code \( C \subset \mathbb{F}_q^n \) can be determined as follows. If \( H \) is any \((n - k) \times n \) parity check matrix of \( C \), \( R_{\text{covering}}(C) \) is the least integer such that every vector in \( \mathbb{F}_q^{n-k} \) can be represented as \( R_{\text{covering}}(C) \) or fewer columns of \( H \), see [53, 14]. We refer to the excellent book [11] on this classical topic of coding theory. Let \( n \) be a fixed positive integer and \( q \) be a fixed prime power, for a given positive integer \( R < n \), we denote \( K_q(n, R) \) the minimal size of a code \( C \subset \mathbb{F}_q^n \) with the covering radius smaller than or equal to \( R \). Set \( \frac{\log_q K_q(n, R)}{n} = k_n(q, \rho) \).

The following asymptotic bound is well-known,

\[
1 - H_q(\rho) \leq k_n(q, \rho) \leq 1 - H_q(\rho) + O\left(\frac{\log n}{n}\right),
\]

We refer to Chapter 12 of [11] and [12]. In particular when each vector in \( \mathbb{F}_q^n \) is in exactly one Hamming ball centered in codewords in \( C \) with the radius of \( R \), we call this code a perfect codes. The existence and determination of perfect codes is is a fascinating topic in coding theory related to many other
topics of mathematics. Hamming code and Golay code are basic examples of perfect codes, see Chapter 11, [11].

Let $F_q$ be an arbitrary finite field, $P_1, \ldots, P_n$ be $n \leq q$ elements in $F_q$. The Reed-Solomon codes $RS(n, k)$ is defined by

$$RS(n, k) = \{ (f(P_1), \ldots, f(P_n)) : f \in F_q[x], \deg(f) \leq k - 1 \}.$$  

This is a $[n, k, n - k + 1]_q$ linear MDS codes from the fact that a degree $\deg(f) \leq k - 1$ polynomial has at most $k - 1$ roots.

A length $n$ code $C \subset F_q^n$ is called (combinatorial) $(d_{\text{list}}, L)$ list-decodable if the Hamming ball of the radius $d_{\text{list}}$ centered at any $x \in F_q^n$ contains at most $L$ codewords of $C$. Since the classical papers of P. Elias and J. M. Wozencraft [19, 20, 93], the list-decoding has been an interesting topic in coding theory. The Johnson bound claims that any length $n$ code over $F_q$ with the minimum Hamming distance $\delta n$ is $(\sqrt[n]{1 - \delta} n, q^{\frac{n^2}{2}})_{\text{list-decodable}}$, see [54, 42]. The 1997 paper of M. Sudan [83] gives a beautiful list-decoding algorithm for the Reed-Solomon codes with the rate less or equal to $\frac{1}{2}$ beyond half the minimum distances (attaining $1 - \sqrt{2R}$). Then an improved list-decoding algorithm matching the Johnson radius $1 - \sqrt{R}$ were given for Reed-Solomon codes and algebraic-geometric codes in [30, 31]. The list-decodability of Reed-Solomon codes beyond the Johnson radius efficiently or combinatorially has been a central question in the theory. For the recent progress we refer to [8, 75, 78].

For random codes, the list-decoding capacity theorem asserts that for any positive real number $\epsilon > 0$, there exists codes $C \subset F_q^n$ with the rate $R(C) \geq 1 - H_q(r) - \epsilon$ such that $C$ is $(rn, \frac{1}{\epsilon})_{\text{list-decodable}}$, where $H_q(r) = r \log_q(q - 1) - r \log_q r - (1 - r) \log_q(1 - r)$ is the $q$-ary entropy function. If the rate is bigger than $1 - H_q(r) + \epsilon$, the code is list-decodable with the exponential (in $n$) list size, we refer to [20, 95], Theorem 2.1 and Theorem 2.2 in [74]. For code families achieving the list-decoding capacity we refer to [35, 67]. The existence of Reed-Solomon codes of the rate $\Omega(\epsilon)$, which are $(1 - \epsilon, O(\frac{1}{\epsilon})$ list-decodable, was proved in [24], we refer to [44] for the other method. For other results about the list-decodability of Reed-Solomon codes beyond the Johnson radius, we also refer to [33, 43, 39, 37, 38, 79, 92].

It is obvious that list-decodable codes can be considered in other finite metric spaces. For some works on list-decoding and list-decodable
codes in rank-metric spaces, subspace metric spaces, cover metric spaces, pair metric spaces, and list-decodable insertion-deletion codes, we refer to \[38, 40, 87, 73, 17, 6, 72, 76, 69, 89, 88, 63, 64, 70, 47, 48, 50, 43, 65\].

2 Related works and our contribution

2.1 Related works

Though there have been active research on list-decodable codes since the 1997, it is strange that very few general bounds on list-decodable codes have been obtained. We refer to the two early papers \[55, 32\] of Justesen-Hoeholdt and Guruswami-Håstad-Sudan-Zuckerman for the asymptotical combinatorial bounds of list decodable codes and the recent paper \[78\] for the generalized Singleton upper bound. The classical Singleton bound

\[|C| \leq q^{n-2d_{\text{list}}}\]

for the \((d_{\text{list}}, 1)\) list-decodable codes in \[80\] was generalized to

\[|C| \leq Lq^{n-\left\lfloor \frac{(L+1)d_{\text{list}}}{L} \right\rfloor}\]

for \((d_{\text{list}}, L)\) list-decodable codes in the recent paper \[78\]. Several existence results about families of optimal list-decodable Reed-Solomon codes beyond the Johnson radius with the list sizes \(L = 2\) or \(L = 3\) were proved and in the case \(L = 2\), some explicit Reed-Solomon codes list-decodable beyond the Johnson radius were given. It was conjectured that their Singleton type bound is tight for the Reed-Solomon codes over large enough fields in \[78\]. Hence there are indeed optimal Reed-Solomon codes list-decodable beyond the Johnson radius with the constant list size 2 and 3, attaining the generalized Singleton bound in \[78\].

On the other hand combinatorial bounds for list-decoding have been studied in \[55, 32\]. Let \(Z^+\) be the set of positive integers and \(l : Z^+ \rightarrow Z^+\) be a function of the list size. As in \[32\],

\[
\text{radius}(C, l) = \max \{ e | \forall x \in F_q^n, |B(x, e) \cap C| \leq l \},
\]

\[
\text{Rad}(C_i, l) = \inf \left\{ \frac{\text{radius}(C_i, l(n_i))}{n_i} \right\}.
\]
For a given rate $R$,

$$U_l(R) = \sup_{C, R(C) \geq R} \text{Rad}(C, l),$$

$U_c^{\text{const}}$ and $U_c^{\text{poly}}$ means the maximal of $U_l(R)$ for constant list size $c$ and the polynomial list size $c_1 n^c$, $U^{\text{const}}(R) = \limsup_{c \to \infty} U_c^{\text{const}}(R)$, $U^{\text{poly}}(R) = \limsup_{c \to \infty} U_c^{\text{poly}}(R)$. In [32], only the One main result in [32] is

$$H^{-1}(1 - \frac{1}{c} - R) \leq U_c^{\text{const}}(R) \leq H^{-1}(1 - R).$$

The equality

$$U^{\text{const}}(R) = U^{\text{poly}}(R) = H^{-1}(1 - R),$$

was proved in [95], where $H(x)$ is 2-ary entropy function. The quantity $U_q^{\text{const}}(R)$ and $U_q^{\text{poly}}(R)$ can be defined similarly for any fixed prime power $q$.

Similarly

$$L_l(\delta) = \inf_{C, \text{relative-distance}(C) \geq \delta} \text{Rad}(C, l)$$

is defined in [32]. $L_c^{\text{const}}(\delta)$ is the $L_l(\delta)$ for the constant list size $l(n) = c$, $L_c^{\text{poly}}(\delta) = \sup_{c_1} L_l(\delta)$ for $l(n) \leq c_1 n^c$, $L^{\text{const}}(\delta) = \limsup_{c \to \infty} L_c^{\text{const}}(\delta)$, $L^{\text{poly}}(\delta) = \limsup_{c \to \infty} L_c^{\text{poly}}(\delta)$. Another main result in [32] asserts

$$L_c^{\text{poly}}(\delta) < \delta,$$

and

$$L^{\text{poly}}(\delta) \leq \frac{1}{2}(1 - (1 - 2\delta)^{\frac{1}{2} + \epsilon}),$$

for $\delta = \frac{1}{2}(1 - \Theta((\text{long}n)^{\epsilon - 1})$. It was conjectured in [32] for $0 < \delta < \frac{1}{2}$,

$$L^{\text{const}}(\delta) = L^{\text{poly}}(\delta) = \frac{1}{2}(1 - \sqrt{1 - 2\delta}).$$

The quantity $L_q^{\text{const}}(\delta)$ and $L_q^{\text{poly}}(\delta)$ can be defined similarly for any fixed prime power $q$.

In [71] the problem of packing Hamming balls of the same radius in $\mathbb{F}_2^n$ with the constraint these balls covering each point of $\mathbb{F}_2^n$ with the multiplicities at most $L$ is considered. Then asymptotic upper bounds on the list-decodable radius of binary codes improving Blinovskiy bound was given.
For list-decodable binary codes, we also refer to [2, 34].

On the other hand there have been active research on list-decodability and list-decoding of rank-metric codes, subspace codes, cover-metric codes, pair metric spaces, and insertion-deletion codes, we refer to [38, 40, 87, 73, 89, 88, 0, 72, 17, 63, 64, 69, 70, 76, 47, 50, 48, 43, 65]. For example it was shown that Gabidulin codes can not be list-decodable beyond the Johnson radius, see [87, 17, 72]. The constraint on list-decodable insertion-deletion code codes over general alphabets was given in [48]. Efficient binary codes list-decodable to $1 - \epsilon$ insertion and deletion errors was constructed in [47, 43].

2.2 Our contribution

The main contribution of this paper is as follows.

1) We introduce the concept of $(R_{list}, L_1, L_2)$ (combinatorial) covering list-decodable codes in general finite metric spaces. The (combinatorial) $(d_{list}, L)$ list-decodable codes are just $(d_{list}, L_1 = 0, L_2)$ covering list-decodable codes. We suggest to determine the $(R_{list}, L_1 = L_2 \geq 1)$ covering codes. These codes are the generalization of perfect codes.

2) We look at list-decodable codes from the view of covering codes and then give covering code upper bound for the size of $(d, L)$ list-decodable codes in general finite metric spaces. This upper bound can apply to the case that the volumes of balls in this metric space $X$ depend on centers. Then any good upper bound on the covering radius and any good upper bound on the size of covering codes with the radius $d$ would lead to a good upper bound on the size of arbitrary $(d, L)$ list-decodable codes.

3) As far as our knowledge there have been very few bounds on list-decodable codes when the length is not going to the infinity, for example, the length is polynomial size of the field size. Our covering upper bounds give a lot of constraints in this range of lengths if the covering code upper bounds are known in this range of length. For Hamming error-correcting code case, from a lot of fascinating classic results on covering codes in Hamming metric setting, we can give many strong constraints on list-decodable codes in Hamming metric spaces. For example, for moderate length list-decodable codes over small fields, we have good upper bounds for the sizes
of list-decodable and good lower bounds for the list sizes from the previous results about $K_q(n, R)$, for example the Litsyn table [62] of covering codes.

4) For Hamming error-correcting codes, even for $(\lfloor \frac{d-1}{2} \rfloor, 1)$ list-decodable codes, our covering code upper bounds give highly nontrivial upper bounds on the sizes of codes with the given minimum Hamming distances.

5) For Hamming error-correcting codes, when code lengths are very large, an exponential improvement on the generalized Singleton upper bound in [78], when the code lengths are large. Actually in the case $L \geq d_{list}$ the generalized Singleton bound is just an easy corollary of our covering code upper bound plus the almost trivial redundancy upper bound $R_{covering} \leq n - k$ for linear $[n, k]_q$ codes. We show that even in some cases that the code length is small the generalized Singleton bound is not tight. This is different when comparing with the classical Singleton bound. Asymptotic forms of our covering code upper bounds partially recover some previous asymptotic bounds due to Blinovsky and Guruswami-Håstad-Sudan-Zuckerman.

6) We suggest the method to lower bound the list size for $(d_{list}, L)$ list-decodable codes by find small covering codes with the fixed covering radius $R_{covering} = d_{list}$. Hence now the main problem for (combinatorial) list-decodable codes is transformed to the determination of the smallest size $K_q(n, R)$ of covering codes.

7) We apply our general covering upper bounds for list-decodable rank-metric codes, subspace codes and insertion-deletion codes. Then some new results about non-list-decodability of rank-metric codes, subspace codes and insertion-deletion codes are obtained. Some previous results in [87, 89, 50, 114, 72, 65] can be partially recovered. In some cases our results about non-list-decodability are better.

7a) We give a sufficient condition for the non-list-decodability of rank-metric codes and constant subspace codes, from which the non-list-decodability rooted in the sizes of the codes, not depending on other properties. Hence Gabidulin codes and lifted Gabidulin codes which can not be list-decodable to any positive radius were given.

7b) An asymptotic rate sufficient condition for the non-list-decodability for subspace code in $Grass(F_q^n)$ of subspaces of all dimensions, with the subspace metric or the injection metric, are given. As far as our knowledge,
there is no results about combinatorial list-decodability about these sub-space code. For efficient list-decoding of these codes, we refer to the paper [70] of Mahdavifar and Vardy.

7c) We give some upper bounds on list-decodable insertion or deletion codes from previous insertion-covering codes or deletion-covering codes.

3  Covering code upper bounds

Combinatorial list-decodable codes can also be defined for general finite metric space. Let \((X, d)\) is a finite metric space, we assume that \(d\) takes values in the set of non-negative integers. We say that a code \(C \subset X\) has the covering radius \(R\), covering \((C)\), if the balls centered at all codewords with the radius \(R\) covering \((C)\) cover the whole space \(X\), and this \(R\) covering \((C)\) is the smallest radius with this property. A code \(C \subset X\) is called (combinatorial) \((d, L)\) list-decodable if each ball centered at any element in \(X\) of the radius \(d\) contains at most \(L\) codewords of \(C\). Let \(K_X(r)\) be the minimum size of covering code on \(X\) with the radius \(r\). For a sequence of finite metric spaces \(\{(X_n, d_n)\}\), and a fixed covering ratio \(r\), we define \(k(r) = \lim_{n \to \infty} \frac{\log_q(K_{X_n}(r_n))}{\log_q(|X_n|)}\). Here \(q\) is a prime power depending on the metric setting, that is, the code rate is defined as \(\text{rate}(C) = \frac{\log_q(|C|)}{\log_q(|X|)}\). This parameter is important to derive the asymptotic bound of non-list-decodability.

Notice that in insertion-deletion error-correcting setting studied in [89, 50, 47, 48, 43, 65], shorter or longer received words with different lengths are allowed in their definition of list-decodable codes. Hence we need to use the finite metric spaces of strings with different lengths.

**Definition 3.1** Let \((X, d)\) be a general finite metric space. A code \(C \subset X\) is called \((R_{\text{list}}, L_1, L_2)\) covering list-decodable if every ball in \(X\) of the radius \(R_{\text{list}}\) contains at least \(L_1\) codewords and at most \(L_2\) codewords. An \((d_{\text{list}}, L)\) locally decodable code is an \((d_{\text{list}}, 0, L)\) covering list-decodable code. A \((R_{\text{list}}, L_1 \geq 1, L_2)\) covering list-decodable code \(C\) has its covering radius \(R_{\text{covering}}(C) \leq R_{\text{list}}\). A perfect code of minimum distance \(d\) is a \((d-1, 1, 1)\) covering list-decodable code. Hence it would be interesting to study \((R_{\text{list}}, L_1 = L_2)\) covering list-decodable codes. For such codes, each ball of radius \(R_{\text{list}}\) contains exactly \(L_1 = L_2\) codewords. These codes can be
thought as generalized perfect codes. For diameter perfect constant weight codes we refer to [23], we refer to [11] for previous generalizations of perfect codes.

For the view of efficient decoding, if we consider any covering code $C$ as an $(R_{\text{covering}}(C), 1, L)$ covering list-decodable code, it seems not trivial to ask an efficient decoding for some well-structured covering list-decodable codes. For example for a perfect code $C$ in Hamming metric space as an $(\frac{d_H(C) - 1}{2}, L_1 = L_2 = 1)$ covering list-decodable code, the decoding to the covering radius is just the unique decoding up to $\frac{d_H(C) - 1}{2}$.

The main result of this paper is the following covering code upper bound for the list-decodable codes in general finite metric spaces.

**Theorem 3.1 (General covering code upper bounds).** 1) Let $(X, d)$ be general finite metric space. Let $C \subset X$ be an $(d, L)$ list-decodable code. Suppose that $C' \subset X$ is a code with the covering radius $R_{\text{covering}} \leq d$, then we have

$$|C| \leq L|C'|.$$ 

Moreover if $C_1 \subset X$ is an $(R, L_1 \geq 1, L_2)$ covering list-decodable code. Let $C'' \subset X$ be a code of the minimum Hamming distance $d(C'') \geq 2R + 1$. Then

$$L_1|C''| \leq |C_1| \leq L_2|C'|.$$ 

2) Asymptotically for any fixed small positive $\epsilon$, if a code family $C_i, i = 1, 2, \ldots, k(r) \geq \epsilon$ is $(rn, L_n)$ list-decodable, the list size $L_n$ has to be exponential in $n$.

**Proof.** This covering code upper bound is almost obvious. The balls of the radius $d \geq R_{\text{covering}}(C')$ centered at the codewords of $C'$ cover the whole space $X$. Then in each such ball there are at most $L$ codewords of $C$. We have $|C| \leq L \cdot |C'|$. The other conclusions follow directly.

This covering code upper bound is strong since we can take any code $C'$ with the covering radius $R_{\text{covering}}(C') \leq d$. Actually any good upper bound for the covering radius implies a good upper bound on the size of $(d, L)$ list-decodable codes from our covering code upper bound. For a given code $C$, we also can use our covering code bounds to lower bound the list sizes if
4 Hamming metric

4.1 Covering code upper bounds for Hamming error-correcting codes

From Theorem 3.1 we have the following result for \((d_{\text{list}}, L)\) list-decodable codes in Hamming metric setting.

**Theorem 4.1 (Covering code upper bounds for Hamming error-correcting codes).** Let \(C \subset \mathbb{F}_q^n\) be an \((d_{\text{list}}, L)\) list-decodable code. Suppose that \(C' \subset \mathbb{F}_q^n\) is a code with the covering radius \(R_{\text{covering}} \leq d_{\text{list}}\), then we have

\[
|C| \leq L|C'|.
\]

Hence we have

\[
|C| \leq LK_q(n, d_{\text{list}}).
\]

Moreover if \(C_1 \subset \mathbb{F}_q^n\) is an \((R_{\text{list}}, L_1 \geq 1, L_2)\) covering list-decodable code. Then

\[
K_q(n, R_{\text{list}}) \leq |C_1| \leq L_2|C'|.
\]

Let \(C'' \subset \mathbb{F}_q^n\) be a code of the minimum Hamming distance \(d_H(C'') \geq 2R_{\text{list}} + 1\). Then

\[
L_1|C''| \leq |C_1| \leq L_2|C'|.
\]

The covering upper bounds for a \((d_{\text{list}}, L)\) list-decodable code \(C \subset \mathbb{F}_q^n\) are weaker than the sphere-packing upper bound

\[
|C| \leq L \cdot \frac{q^n}{\sum_{j=0}^{d_{\text{list}}} \binom{n}{j} (q - 1)^j}.
\]

This sphere-packing upper bound was formulated in \[66\] for binary codes. However when \(n\) is large, it is computational infeasible to get clear and explicit upper bounds from this expression. Hence by using various results in covering codes, we can give explicit upper bounds for list-decodable codes in Hamming metric setting.
There are a lot of classical results about the binary covering codes, we refer to [11, 62, 18, 5]. From the table [62] of S. Litsyn, we have the following upper bound on the length 16 binary $(3, L)$ list decodable code $C \subset \mathbb{F}_2^{16}$,

$$|C| \leq 192L,$$

since $K_2(16, 3) \leq 192$. Thus if the linear $[16, 9, 4]_2$ code is $(3, L)$ list-decodable, the list size has to be at least 3. The following upper bound on the length 32 binary $(6, L)$ list decodable code $C \subset \mathbb{F}_2^{32}$,

$$|C| \leq 24576L,$$

since $K_2(32, 6) \leq 24576$. If the linear $[32, 16, 8]_2$ code is $(6, L)$ list-decodable then the list size is at least 3. Similarly we have $|C| \leq 384L$ for an $(10, L)$ list-decodable binary codes with the length 32. It is clear we have many such kind of upper bounds for list-decodable codes, and this kind of upper bounds from our covering code upper bounds are much stronger than the Singleton type bound in [78] for any list size $L$.

The almost trivial redundancy upper bound $R_{covering} \leq n - k$ for the covering radius of the linear $[n, k]_q$ code, [11] page 217, and our covering code upper bounds imply an upper bound which is close to the Singleton type upper bound in [78].

**Corollary 4.1.** Let $C'$ be a linear $[n, k]_q$ code in $\mathbb{F}_q^n$, then $R_{covering}(C') \leq n - k$. Hence an $(d_{list}, L)$ list-decodable code $C \subset \mathbb{F}_q^n$ have to satisfy

$$|C| \leq Lq^{n - d_{list}}.$$

**Proof.** The first conclusion follows that for any $k$ information set coordinate positions $1 \leq i_1 < i_2 \cdots < i_k \leq n$, the coordinates at these positions of codewords in $C'$ can be arbitrary vectors in $\mathbb{F}_q^k$. For the second conclusion, we take a linear $[n, n - d_{list}]_q$ code $C'$, then $R_{covering}(C') \leq d_{list}$. The conclusion follows immediately.

This is weaker than the Singleton upper bound

$$|C| \leq Lq^{n - \left[\frac{(L+1)d_{list}}{L}\right]}.$$ 

in [78]. However when the list size $L \geq d_{list}$, this almost trivial upper bound from our covering code upper bounds plus the redundancy upper bound for
the covering radius is equivalent to the Singleton type bound in [78]. We should mention that the covering radius of a \([n,k]_q\) Reed-Solomon code with the length \(n \leq q\) is \(n-k\), we refer to page 281, [11]. However in many cases of \(q\) and \(n\), there is a linear \([n,k]_q\) code with the covering radius \(n-k-1\), see [4]. We have the following result.

**Corollary 4.2.** Let \(q = p^h\) be a prime power with \(p \geq 3\), \(m\) be a prime factor of \(q-1\) satisfying

\[
\max\{7, \frac{(n-k-2)^2}{2}, \frac{3(n-k-2)^2-n+k-12}{2}\} \leq m \leq \frac{1}{8}q^2
\]

let \(n\) and \(k\) be two positive integers satisfying

\[
\left\lfloor \frac{n}{2} \right\rfloor \leq (\left\lfloor \frac{n-k}{2} \right\rfloor -1)(\left\lceil \frac{q-2\sqrt{q}+1}{m} \right\rceil +30) + \frac{2(m+1)}{n-k-2} + n-k-\frac{7}{2}
\]

Then an \((n-k-1,L)\) list decodable code \(C \subset F^n_q\) satisfies

\[
|C| \leq Lq^k.
\]

Since \(k = n - d_{\text{list}} - 1\) in the above upper bound, it is stronger than the Singleton type bound in [78] when \(L \geq d_{\text{list}} = n - k - 1\).

From the known linear perfect codes such as linear \([q^{m-1}/q-1, q^{m-1}/q-1-m, 3]_q\) Hamming perfect code, binary Golay \([23,12,7]_2\) perfect code and ternary Golay \([11,6,5]_3\) perfect code, see, Chapter 11, [11], we have the following corollary.

**Corollary 4.3.** 1) Let \(n = \frac{a^{m-1}}{q-1}\), where \(q\) is a fixed prime power and \(m = 2,3,4,\ldots\). Let \(k > n-m\) be a positive integer. Then if a linear \([n,k]_q\) code is \((1,L)\) list-decodable, then the list size has to satisfies \(L \geq q^{k-n+m}\).
2) If a binary linear \([23,k]_2\) code is \((3,L)\) list-decodable, then the list size has to satisfies \(L \geq 2^{k-12}\).
3) If a ternary linear \([11,k]_3\) code is \((2,L)\) list-decodable, then the list size have so satisfies \(L \geq 3^{k-6}\).

We show that our covering code upper bound is much stronger than the Singleton type bound due to Shangguan and Tamo in [78] when the lengths are large. It is well-known that the covering radius of the first order \([2^m,m + 1,2^{m-1}]_2\) Reed-Muller code is \(2^{m-1} - 2^{m-2}\) when \(m\) is even, see page 243, [11]. Then we have the following upper bound for binary \((rn,L)\)
list-decodable codes, where $\frac{1}{2} - \frac{1}{2^{m+2}} \leq r < \frac{1}{2}$.

**Corollary 4.4.** Let $n = 2^m$ and $m = 2, 4, 6, \ldots$. Let $r$ be a positive real number satisfying $\frac{1}{2} - \frac{1}{2^{m+2}} \leq r < \frac{1}{2}$. If a binary length $n = 2^m$ code $C$ is $(rn, L)$ list-decodable, then

$$|C| \leq L2^{m+1}.$$  

Hence a length $2^m$ binary linear code with the minimum Hamming weight $2^m - 2^m + 1$ has its dimension at most $m + 1$.

For the positive list size $L \geq 2$ we have $\frac{L}{4} \leq \frac{2}{2}$, hence the generalized Singleton upper bound in [78] for this case is at least

$$L2^{2m-3-2m-2} = L2^{2m-2}.$$  

Our upper bound is an exponential $2^{\frac{3}{2} \log n + 1}$ improvement of the bound in [78] for binary codes.

For a code family of length $n = 2^m$, $m = 2, 4, 6, \ldots$ with the rate $R > 0$, then if it is $(\left( \frac{1}{2} - \frac{1}{2^{m+2}} \right)n, L)$ list-decodable, then the list size $L$ has to be exponential. If we use the list-decoding capacity theorem, $R$ has to bigger than $1 - H(\frac{1}{2} - \frac{1}{2^{m+2}})$. Hence Corollary 3.4 is a little stronger than the second half of the list-decoding capacity theorem in the binary case. If we look at the first half of the list-decoding capacity theorem, $1 - H(\frac{1}{2} - \frac{1}{2^{m+2}})$ is going to zero.

Over an arbitrary finite field $\mathbb{F}_q$ by using the first order $q$-ary Reed-Muller code we can also give an exponential improvement on the generalized Singleton bound in [78] when the list-decodable codes are very long. From the main result of [56] the covering radius $R_{covering}(1, m, q)$ of the first order $q$-ary Reed-Muller code is upper bounded by

$$R_{covering}(1, m, q) \leq (q - 1)q^{m-1} - q^{\frac{m}{2}-1}.$$  

Hence we have the following result.

**Corollary 4.5.** Let $q$ be an arbitrary prime power. Let $n = q^m$ and $m = 1, 2, 3, \ldots$. Let $r$ be a positive real number satisfying $\frac{a-1}{q} - \frac{1}{q^{\frac{m}{2}}} \leq r < \frac{a-1}{q}$. If a $q$-ary code $C \subset \mathbb{F}_q^n$ is $(rn, L)$ list-decodable, then

$$|C| \leq Lq^{m+1}.$$
When the list size is at least $L \geq q\sqrt{n}$, the generalized Singleton upper bound in [78] in this case is at least $q^{n - \frac{n}{q}}$. In this case our bound is an exponential improvement of their bound. This proves that any $((\frac{q-1}{q} - \frac{1}{\sqrt{n}})n, L)$ list-decodable code for $L \leq \text{poly}(n)$, when $n = q^m$, has its rate at most $\frac{\log q n + 1}{n}$.

We can use the previous known result about the covering radius of the $\frac{m}{2}$-th length $2^m$ Reed-Muller code, when $m$ is even, see Chapter 9 of [11] to get the further upper bound as follows. From the result in page 257 of [11], the covering radius satisfies $R_{\text{covering}}(\frac{m}{2}, m) \leq 2^{m-2} - 2\frac{m}{2} + 2$. Then we have the following result from our covering code upper bounds.

**Corollary 4.6.** Let $n = 2^m$ and $m = 2, 4, 6, \ldots$. Let $r$ be a positive real number satisfying $\frac{1}{4} - \frac{1}{2^m} \leq r < \frac{1}{4}$. If a binary length $n = 2^m$ code $C$ is $(rn, L)$ list-decodable, then

$$|C| \leq L2^{\sum_{i=0}^{m} \binom{m}{i}}.$$

Actually this upper bound is $L2^n - \binom{m}{1}$. For give two positive integers $k$ and $d$, let $n_q(k, d)$ be the minimum length $n$ such that there is a linear $[n, k, d]_q$ code. Then the covering radius $R$ of this linear $[n_q(k, d), k, d]_q$ code satisfies

$$R_{\text{covering}}(n_q(k, d)) \leq d - \lceil \frac{d}{q^k} \rceil,$$

see Corollary 8.1 in [53]. Then the following result follows immediately from our covering code upper bounds.

**Corollary 4.7.** Let $k$ and $d$ be given two positive integers and $n_q(k, d)$ defined as above. Then for an $(d - \lceil \frac{d}{q^k} \rceil, L)$ list decodable code $C \subset \mathbb{F}_q^{n_q(k, d)}$, we have

$$|C| \leq Lq^k.$$
For example for \( k = 4 \) and \( d = 12 \), \( n_2(4, 12) = 23 \), see [29]. Then an \((12, L)\) list-decodable code \( C \subset \mathbb{F}_2^{23} \) has its size \(|C| \leq 16L\). This is better than the upper bound from [78] for list size \( L \geq 2 \) again.

Let \( r \) and \( R \) be two fixed positive integers satisfying \( r \geq R \). Let \( l_q(r, R) \) be the smallest length of a \( q \)-ary code with the covering radius \( R \) and the redundancy \( r = n - k \). Then it is clear that there are \([n, n - r]_q\) code with the covering radius \( R \) for each integer \( n \geq l_q(r, R) \). Actually this can be proved directly from the characterization of the covering radius of linear codes from its parity check matrix. There are a lot of results about this topic, we refer to [13, 14, 5] and references therein. From our covering code upper bounds the size of an \((R, L)\) list-decodable code \( C \subset \mathbb{F}_n^q \) has to satisfy

\[ |C| \leq Lq^{n-r}. \]

When the list size is not one, and \( r \geq \frac{3}{2}R \), the upper bound is better than the Singleton bound in [78]. For example for \( r = 5 > \frac{3}{2} \cdot 3 \), the size of a length \( n \geq l_q(5, 3) \) \((3, L)\) list-decodable code satisfies \(|C| \leq Lq^{n-5}\). In this case \( l_q(5, 3) < 2.884q^\frac{5}{3}(lnq)^{\frac{1}{3}} \) was proved, see [5]. More generally the following result follows from the bound in [14].

**Corollary 4.8.** Let \( q \geq 8 \) be an even prime power. Let \( R \geq 4 \) be a positive integer. Set \( m = \lceil \log_q(R + 1) \rceil + 1 \). For any given positive integer \( t \geq 3m + 2 \), and the length \( n \geq Rq^{(t-1)R} + 2q^{t-2} + \sum_{j=3}^{m+2} q^{t-j} \), the size of an \((R, L)\) list-decodable code \( C \subset \mathbb{F}_n^q \) has to satisfy

\[ |C| \leq Lq^{n-tR}. \]

**Proof.** The conclusion follows from Theorem 8 in [14].

Comparing with the generalized Singleton bound in [78], the upper bound is \(|C| \leq Lq^{n-\lceil \frac{t-2}{t-R} \rceil} \) is much weaker. Hence when the code length is large the bound in [78] is far away from the tight. If we consider the case of \( L = 1 \), Corollary 3.8 implies that when \( n \geq Rq^{(t-1)R} + 2q^{t-2} + \sum_{j=3}^{m+2} q^{t-j} \), the size of a length \( n \) code with the Hamming distance \( 2R + 1 \) has at most \( q^{n-tR} \) codewords. Thus a length \( n \) linear code with the minimum Hamming
weight \(2R + 1\) has its dimension \(k \leq n - tR\). This is nontrivial upper bound on the size of codes with the given minimum Hamming weight.

The following result gives a lower bound for \(L_2 - L_1\) for an arbitrary code as an \((R_{\text{list}}, L_1, L_2)\) covering list-decodable code.

**Corollary 4.9.** Let \(C \subset \mathbb{F}_q^n\) be an \((R_{\text{list}}, L_1, L_2)\) covering list-decodable code. Suppose that \(C' \subset \mathbb{F}_q^n\) is a covering code with the covering radius \(R_{\text{list}}\) and \(C'' \subset \mathbb{F}_q^n\) is a code with the minimum Hamming distance \(2R_{\text{list}} + 1\). Then

\[
L_2 - L_1 \geq |C| \left( \frac{1}{|C'|} - \frac{1}{|C''|} \right).
\]

Let \(A_q(n, 2R_{\text{list}} + 1)\) be the size of the maximal code with the Hamming distance \(2R_{\text{list}} + 1\). Then we have

\[
L_1 A_q(n, 2R_{\text{list}} + 1) \leq |C| \leq L_2 K_q(n, R_{\text{List}}).
\]

Hence for a given length \(n\) code as a \((R_{\text{list}}, L_1, L_2)\) covering list-decodable code, then

\[
L_2 - L_1 \geq |C| \left( \frac{1}{K_q(n, R_{\text{list}})} - \frac{1}{A_q(n, 2R_{\text{list}} + 1)} \right).
\]

This section can be continued without stop by collecting previous upper bound results on the covering radius of codes, we now turn to the covering radius upper bounds from dual codes.

### 4.2 Covering code dual upper bounds for list-decodable codes

One existing point of the classical theory of covering codes is the upper bounds on the covering radius of a code from its dual codes. This was originated from the work of [15, 85, 86]. We refer to [15, 85, 86, 3, 7, 81] for some upper bounds on the covering radius from dual code. The Delsarte upper bound on the covering radius of a linear code \(C\) asserts that

\[
R_{\text{covering}}(C) \leq s,
\]

where \(s\) is the number of nonzero weights of its dual code. Hence we have the following result.
Corollary 4.10 (Delsarte type bound). Let $C \subset \mathbb{F}_q^n$ be an $(d_{\text{list}}, L)$ list-decodable code. Suppose that $C_1 \subset \mathbb{F}_q^n$ is a linear code such that the total number of nonzero weights of $C_1^\perp$ is smaller than or equal to $d_{\text{list}}$, then we have

$$|C| \leq L|C_1|.$$  

We refer the Delsarte upper bound on the covering radius of linear codes to [15, 86, 52]. Actually there are many results about linear codes with few weights, for example we refer to [16]. From the Delsarte upper bound on the covering radius, we have a lot of covering code upper bounds for $(d_{\text{list}}, L)$ list-decodable codes for small $d_{\text{list}}$. Let $q = p^m$ be an odd prime power with the even exponent $m$, then a linear $[\frac{q-1}{2}, m, (\frac{p-1)(q-\sqrt{q})}{2p}]_p$ code with two weights is explicitly given in [16] Corollary 4. Hence for any $(2, L)$ list-decodable code $C \subset \mathbb{F}_p^{\frac{q-1}{2}}$, we have

$$|C| \leq Lp^{\frac{q-1}{2} - m},$$  

from the above Delsarte upper bound. This is again much stronger than the generalized Singleton upper bound in [78].

Set $\binom{x}{j} = \frac{(x(x-1)\cdots(x-j+1)}{j!}$, the Krawchouk polynomial

$$K_k(x, q, n) = \sum_{j=0}^{k} \binom{x}{j} \binom{n-x}{k-j} (q-1)^{k-j}.$$  

For example $K_2(x, q, n) = \frac{1}{2}(q^2x^2 - q(2qn - q - 2n + 2)x + (q-1)^2n(n-1))$. Let $x(k, q, n)$ be the smallest zero of $K_k(x, q, n)$. Then for a length $n$ linear code with the dual distance $d^\perp$ the upper bound for the covering radius is as follows, $R_{\text{covering}}(C) \leq x(u, q, n - 1)$ if $d^\perp = 2u - 1$, and $R_{\text{covering}}(C) \leq x(u, q, n)$ if $d^\perp = 2u$, see [86]. For the roots of Krawchouk we refer to [60, 61]. For example when $q = 2$ and $2 \leq u \leq \frac{n}{2}$,

$$x(u, 2, n) \leq \frac{n}{2} - \sqrt{(n-u+2)(u-2)}.$$  

Corollary 4.11 (Tietävän type bound). Suppose that $C_1$ is a linear $[n, k]_q$ code with the dual distance $d^\perp = 2u$. Let $d_{\text{list}} \leq x(u, q, n)$ be a positive integer. Then the size of an $(d_{\text{list}}, L)$ list-decodable code $C \subset \mathbb{F}_q^n$ has to satisfy $|C| \leq Lq^k$. In particular a length $n$ code with the Hamming
distance \(2x(u, q, n) + 1\) has its size at most \(q^k\).

Even in the case of \((\lceil d - 1/2 \rceil, 1)\) list-decodable case, our covering code bound gives highly nontrivial upper bound on the sizes of codes of the fixed minimum distance as follows.

**Corollary 4.12.** For a given minimum distance \(d\) let \(u\) be the smallest positive integer such that \(x(u, q, n) \leq \frac{d - 1}{2}\). Then any linear \([n, k, d]_q\) code has its dimension

\[
k \leq n - \lfloor \log_q \left( \frac{q^n}{\sum_{j=0}^{2u-1} \binom{n}{j} (q - 1)^j} \right) \rfloor.
\]

**Proof.** This is direct from the Tietäväinen type bound and the Gilbert-Varshamov bound for linear codes.

**Corollary 4.13.** For two given positive integers \(d < n\), Let \(u\) be the smallest positive integer in the range \(2 \leq u \leq \frac{n}{2}\) satisfying \(\frac{n-d+1}{2} \leq \sqrt{n-u+2}(u-2)\), then the size of a length \(n\) binary code \(C \subset \mathbb{F}_2^n\) with the given minimum Hamming distance \(d\) satisfies

\[
|C| \leq 2(\sum_{j=0}^{2u-1} \binom{n}{j}).
\]

This result can be compared with Corollary 2.7 in [61].

### 4.3 Asymptotic bounds

From the asymptotic bound \(k_n(q, \rho) \leq 1 - H_q(\rho) + O\left(\frac{\log m}{n}\right)\), see [12], and the above covering code upper bound, it follows immediately that if the code family with the rate \(R \geq 1 - H_q(\rho) + \epsilon\), is \((\rho n, L)\) list-decodable then the list size \(L\) has to be the exponential. This recovers the classical result due to Elias and Zyablov-Pinsker, see Theorem 2.3, page 18 of [74]. This is the second half of the list-decoding capacity theorem.
The definition of $U^\text{const}_q(R)$ and $U^\text{poly}_q(R)$ are similarly as the definitions in [32] for binary codes. We have

\[ R \leq 1 - H_q(U^\text{const}_q(R)), \]

and

\[ R \leq 1 - H_q(U^\text{poly}_q(R)), \]

from the asymptotic form of our covering code bound Theorem 3.1. Hence if $U^\text{const}_q(R)$ and $U^\text{poly}_q(R)$ are in the range $(0, \frac{q-1}{q})$, we have

\[ U^\text{const}_q(R) \leq H_q^{-1}(1 - R), \]

and

\[ U^\text{poly}_q(R) \leq H_q^{-1}(1 - R). \]

This partially recovers the bound in [31], see Theorem 6 in [32].

On the other hand for given relative Hamming distance $\delta$ from Gilbert-Varshamov bound there exists a code family with the relative Hamming distance $\delta$ and rate $R \geq 1 - H_q(\delta)$. From the covering code upper bound we have

\[ 1 - H_q(\delta) \leq 1 - H_q(L^\text{const}_q(\delta)), \]

and

\[ 1 - H_q(\delta) \leq 1 - H_q(L^\text{poly}_q(\delta)). \]

**Corollary 4.14 (asymptotic bound from covering codes).** If $U^\text{const}_q(R)$ and $U^\text{poly}_q(R)$ are in the range $(0, \frac{q-1}{q})$, we have $U^\text{const}_q(R) \leq H_q^{-1}(1 - R)$, and $U^\text{poly}_q(R) \leq H_q^{-1}(1 - R)$. If $\delta$ and $L^\text{const}_q(\delta)$ (resp. $L^\text{poly}_q(\delta)$) are in the range $(0, \frac{q-1}{q})$, then $L^\text{const}_q(\delta) \leq \delta$ and $L^\text{poly}_q(\delta) \leq \delta$.

### 4.4 List sizes

We can use the cover code upper bound to lower bound the list size by the size of an $(R, L)$ list decodable code in $C \subset \mathbb{F}^n_q$. Let $C_1 \subset \mathbb{F}^n_q$ be a covering code with the radius $R_{\text{covering}}(C_1) \leq R$ with the smallest possible size, then the list size is at list

\[ \frac{|C|}{|C_1|} \leq L. \]
The asymptotic case of this argument implies the second half of the list-decoding theorem.

We consider the beyond Johnson radius case $R > n - \sqrt{n(n-d)}$. In this case $|C| \leq q^{n-d+1}$ from the Singleton bound in [80]. Since $n-2d > k-1$ we have $\sqrt{n(n-d)} \geq k-1$. Let $t < k$ be a positive integer. Then $(n-k+t, L)$ list-decodability is beyond the Johnson radius. The following result is direct from our covering code upper bound.

**Corollary 4.15.** If a linear $[n, k]_q$ code over $\mathbb{F}_q$ is $(n - k + t, L)$ list-decodable, the list size $L \geq q^t$.

**Proof.** We take any linear $[n, k-t]_q$ code as $C'$ in Theorem 3.1 then the covering radius of this code is not bigger than $n - k + t$ from the redundancy bound. Hence the conclusion follows directly.

From the covering code upper bounds we have the following result directly.

**Corollary 4.16.** Let $C$ be a length $n$ code over $\mathbb{F}_q$ of the covering radius $R_{\text{covering}}$. Then it is an $(R_{\text{covering}}, L_1 \geq 1, L_2)$ covering list-decodable code satisfying $L_2 \geq K_q(n, R_{\text{covering}})$. 

If there is a good upper bound for $K_q(n, d_{\text{list}})$ then we can lower bound the list sizes of $(d_{\text{list}}, L)$ list-decodable codes.

## 5 Non-list-decodability of large rank-metric codes

List-decoding and list-decodability of rank-metric codes have been studied since 2012, we refer to [38, 57, 40, 17, 72]. The rank-metric on the space $\mathbb{M}_{m \times n}(\mathbb{F}_q)$ of size $m \times n$ matrices over $\mathbb{F}_q$ is defined by the rank of matrices, i.e., $d_r(A, B) = \text{rank}(A - B)$. The minimum rank-distance of a code $C \subset \mathbb{M}_{m \times n}(\mathbb{F}_q)$ is defined as

$$d_r(C) = \min_{A \neq B} \{d_r(A, B) : A \in C, B \in C\}$$

The rate of this code $C$ the rate is $\text{rate}(C) = \log_q |C|$. For a code $C$ in $\mathbb{M}_{m \times n}(\mathbb{F}_q)$ with the minimum rank distance $d_r(C) \geq d$, it is well-known that
the number of codewords in $C$ is upper bounded by $q^{\max\{m,n\}(\min\{m,n\}−d+1)}$, see [25]. A code attaining this bound is called a maximum rank-distance (MRD) code.

The Gabidulin code in $M_n(F_q)$ consisting of $F_q^n$ linear mappings on $F_q^n \cong F_q^n$ defined by $q$-polynomials $a_0x + a_1x^q + \cdots + a_i x^{q^i} + \cdots + a_t x^{q^t}$, where $a_t, \ldots, a_0 \in F_q^n$ are arbitrary elements in $F_q^n$, is an MRD code, see [25].

The rank-distance is $n−t$ since there are at most $q^t$ roots in $F_q^n$ for each such $q$-polynomial. There are $q^{n(t+1)}$ such $q$-polynomials in this Gabidulin code. For the case $m \neq n$, let $h$ be a non-negative integer and $\phi : F_q^n \rightarrow F_{q^{n+h}}$ be a $q$-linear embedding. Then

$$a_t \phi(x^{q^t}) + a_{t-1} \phi(x^{q^{t-1}}) + \cdots + a_1 \phi(x^q) + a_0 \phi(x)$$

is a $q$-linear mapping from $F_q^n$ to $F_{q^{n+h}}$, where $a_i \in F_{q^{n+h}}$ for $i = 0, 1, \ldots, t$. It is clear that the dimension of the kernel of any such mapping is at most $t$. Then we have Gabidulin MRD code for the case $m = n + h$ with rank distance $d_R = n − t$ and $q^{(n+h)(n−d_R+1)}$ elements.

It was shown in [87, 72] that Gabidulin codes cannot be list-decodable beyond the Johnson radius. The similar result was also obtained for rank-metric codes containing Gabidulin codes. In this section we prove that from our covering code upper bounds, the non-list-decodability of rank-metric codes is rooted in the size of these codes. The following result is for general rank-metric codes, not only Gabidulin codes. Without loss of generality we only restrict to the case $m = n$. Our results essentially rely on the results of Gadouleau-Yan [28] on covering rank-metric codes.

**Theorem 5.1.** Let $n$ and $\rho$ be two positive even numbers satisfying $\rho < n$. Suppose that $C \subset M_n(F_q)$ is an $(\rho, L)$-list decodable rank-metric code. Then

$$|C| \leq L \cdot q^{\frac{n(n−\rho)}{2}}.$$  

Then the size is at least

$$L \geq \frac{|C|}{q^{\frac{n(n−\rho)}{2}}}.$$  

**Proof.** As in [28] we set $K_R(q^n, n, \rho)$ the minimal size of covering rank-metric codes in $M_{m \times n}(F_q)$ with the radius $\rho$. From Proposition 11 we have

$$K_R(q^n, n, \rho) \leq q^{\frac{n(n−\rho)}{2}}.$$  

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The conclusion follows from Theorem 3.1 immediately.

For example $K_R(2^6, 4, 2) \leq 256$ was proved in the Table I of [28], then any $(2, L)$ list-decodable code $C \subset M_{6 \times 4}(F_2)$ has to satisfy $|C| \leq 256L$. We can get many such upper bounds on the code sizes and the lower bounds on the list sizes from Table 1 in [28].

From Theorem 5.1 we have the following result.

**Corollary 5.1.** If a rank-metric code $C \subset M_n(F_q)$ of the cardinality at least $q^{nk}$, where $k$ is positive integer satisfying that $k - \frac{n - \rho}{2} \geq c$ for a fixed positive real number $c$, is $(\rho, L)$ list-decodable for even $\rho$ and $n$, then the list size is at least $q^n$, exponential in $n$. For Gabidulin codes in $M_n(F_q)$ with the cardinality $q^{nk}$ cannot be list-decodable to the radius $\rho$, if $\rho$ is a positive integer satisfying $\rho \geq n - 2k + \epsilon$, where $\epsilon > 0$ is fixed.

Gabidulin codes with the rate $\frac{1}{4} < rate \leq \frac{1}{2}$ cannot be list-decodable to $1 - 2rate$. It is clear $1 - 2rate \leq 1 - \sqrt{rate}$, this result improves the previous results in [87, 72]. If we consider the case $2k = n$, that is the rate $\frac{1}{2}$, when $n$ goes to the infinity, rate $\frac{1}{2}$ Gabidulin codes cannot be list-decodable to the radius $\epsilon n$ where $\epsilon$ is any positive fixed real number. Some such kind of Gabidulin codes have been presented in [72].

We consider the asymptotic case. Suppose that $\lim \frac{n}{m} = b$ where $b \in (0, 1)$ is a fixed real number. It was shown in [28]

$$k(r) = \lim_{n \to \infty} \inf \log_K R(q^m, n, r) = (1 - r)(1 - br).$$

Hence any rank-metric $(rn, L)$ list-decodable code family in $M_{m \times n}(F_q)$ has its rate at most $(1 - r)(1 - br)$ from Theorem 3.1 2). This recovers the first half of the main result in [17].

## 6 Non-list-decodability of subspace codes

List-decoding for subspace codes have been studied in [73, 6, 17, 72, 69, 67]. The research on subspace codes including constant dimension codes and mixed dimension codes was originated from the paper [68] of R. Kötter and F. R. Kschischang. It was proposed to correct errors and erasures in network transmissions of information. A set $C$ of $M$ subspaces of the
dimension $k \in T$ in $\mathbb{F}_q^n$, where $T$ is a subset of $\{1, 2, \ldots, n-1\}$, is called an $(n, M, d, T)_q$ subspace code if $d_S(U, V) = \dim U + \dim V - 2 \dim(U \cap V) \geq d$ is satisfied for any two different subspaces $U, V$ in $\mathbb{C}$. The main problem of the subspace coding is to determine the maximal possible size $A_q(n, d, T)$ of such a code for given parameters $n, d, T, q$. When $T$ is the whole set $\{1, 2, \ldots, n\}$, we write $A_q(n, d)$ for the maximal possible size of the set of subspaces in $\mathbb{F}_q^n$ such that the subspace distances between any different subspaces in this set are at least $d$. Let $(n\choose k)_q = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}$ be the $q$-ary Gauss coefficient, which is the number of $k$-dimensional subspaces in $\mathbb{F}_q^n$. It is clear

$$A_q(n, d, T) \leq \sum_{k \in T} {n\choose k}_q$$

and

$$A_q(n, d) \leq \sum_{k=1}^{n-1} {n\choose k}_q.$$ 

When $T = \{k\}$ contains only one dimension this is a constant dimension subspace code, otherwise it is called a mixed dimension subspace code. There have been some upper and lower bounds for $A_q(n, d, k)$. We refer to papers [21, 22, 94].

We present some asymptotic results about non-list-decodability of codes in the space $\text{Grass}(\mathbb{F}_q^n)$, that is the space of subspaces of $\mathbb{F}_q^n$ with the dimension $k \in \{1, 2, \ldots, n\}$, endowed with the subspace distance $d_S$ as above. In this setting the list-decoding of Kötter-Kschischang codes was given in [99]. The rate of a code $C \subset \text{Grass}(\mathbb{F}_q^n)$ is defined as $\frac{\log_2|C|}{\log_2(\text{Grass}(\mathbb{F}_q^n))}$, see page 2100 in [26]. Notice that the volumes of balls in this finite metric space of the same radius are different when the center subspaces have different dimensions. Set $K_S(q, n, \rho)$ the minimal size of covering codes in this space with the radius $\rho$, and $k_S(r) = \liminf_{n \to \infty} \frac{\log_2 K_S(q, n, \rho n)}{\log_2(\text{Grass}(\mathbb{F}_q^n))}$. It was shown in Proposition 10 of [28],

$$k_S(r) = 1 - 2r,$$

for $r \in [0, \frac{1}{2}]$.

The injection metric on $\text{Grass}(\mathbb{F}_q^n)$ is defined as $d_I(U, V) = \frac{1}{2}(d_S(U, V) + |\dim U - \dim V|)$. In the constant dimension case $d_I = d_S$. In general $\frac{1}{2}d_S \leq d_I \leq d_S$. Set $K_I(q, n, \rho)$ the minimal size of covering codes in
this space endowed with the injection metric, of the radius \( \rho \), and 
\[
k_I(r) = \lim_{n \to \infty} \frac{\log K_I(q,n,rn)}{\log |\text{Grass}(F_q^n)|}.
\]
It was shown in Proposition 10 of \[25\],
\[
k_I(r) = (1 - 2r)^2,
\]
for \( r \in [0, \frac{1}{2}] \).

We have the following asymptotic results from the above asymptotic results about covering subspace codes under the subspace metric or injection metric.

**Theorem 6.1.** 1) If \( C \subset (\text{Grass}(F_q^n), d_S) \) is \((\rho, L)\) list-decodable then its size has to satisfy \( |C| \leq L K_S(q,n,k,\rho) \). If \( C \subset (\text{Grass}(k,F_q^n), d_I) \) is \((\rho, L)\) list-decodable then its size has to satisfy \( |C| \leq L K_I(q,n,k,\rho) \).

2) Asymptotically any \((rn, L)\) list-decodable code family in \((\text{Grass}(F_q^n), d_S)\) has its rate at most \(1 - 2r\). Asymptotically any \((rn, L)\) list-decodable code family in \((\text{Grass}(F_q^n), d_I)\) has its rate at most \((1 - 2r)^2\).

We can also consider the constant dimension subspace code case. The list-decoding in this setting was studied in \[?, 72\], in particular the list-decoding of lifted Gabidulin codes. Let \( \text{Grass}(k,F_q^n) \) be the set of all dimension \( k \) subspace of \( F_q^n \). Then \( |\text{Grass}(k,F_q^n)| = \binom{n}{k} \). Let \( K_C(q,n,k,\rho) \) be the minimal size of covering codes in \((\text{Grass}(k,F_q^n), d_S)\). Several upper bounds on \( K_C(q,n,k,\rho) \) were proved in \[27\]. We have the following result.

**Proposition 6.1.** If \( C \subset \text{Grass}(k,F_q^n) \) is \((\rho, L)\) list-decodable then its size has to satisfy \( |C| \leq L K_C(q,n,k,\rho) \). Hence \( L \geq \left( \frac{|C|}{\binom{n}{k}} \right) \cdot K_R(q^{n-k}, k, \rho) \).

**Proof.** It follows from Proposition 4 of \[27\] immediately.

We consider the case \( \text{Grass}(n,F_{2n}^n) \), from Proposition 6.1 we have \( |C| \leq L q^{n(n-\rho)} \). Then the following result follows.

**Theorem 6.2.** Let \( k, \rho n \) be three even positive integer satisfying \( \rho < n \) and \( k < n \). We assume that \( k - \frac{\rho n}{2} \geq c \) for a fixed positive integer \( c \). If a constant dimension subspace code of the size at least \( q^{nk} \) is \((\rho, L)\) list-decodable, then the list size is at least \( q^{cn} \), exponential in \( n \).
It follows that lifted Gabidulin codes in $\text{Grass}(n, \mathbb{F}_q^{2n})$ can not be list-decodable to any positive radius, when $n$ goes to the infinity. This improves the previous results in [73, 72].

7 Upper bounds on list-decodable insertion codes, deletion codes, cover metric codes and symbol-pair codes

The covering code upper bound in Theorem 3.1 is general in the following sense that any upper bound on covering codes or covering radius with respect to any metric can be translated to upper bounds on the sizes of list-decodable codes or the lower bounds on the list sizes. However not like the Hamming metric case, there are few results about covering codes in insdel metric, cover metric and pair metric. Hence we can only get weak results in these three cases.

7.1 Generalized Singleton bounds for list-decodable cover metric codes and list-decodable symbol pair codes

The list-decodability and list-decoding of cover metric codes have been studied in [88, 63]. The cover metric $d_{\text{cover}}$ on the space $\mathbf{M}_{m \times n}(\mathbb{F}_q)$ of $m \times n$ matrices over $\mathbb{F}_q$, $m \geq n$, is defined as follows. Let $A \in \mathbf{M}_{m \times n}(\mathbb{F}_q)$, a subset pair $(I, J)$, $I \subset \{1, 2, \ldots, m\}$, $J \in \{1, 2, \ldots, n\}$, is called a cover of $A$ if for any entry of $A$ at the $(i, j)$ position $a_{ij} \neq 0$, then $i \in I$ or $j \in J$. The cover weight $wt_C(A)$ of $A$ is defined as the minimum $|I| + |J|$ for all such covers. The distance of $A, B \in \mathbf{M}(\mathbb{F}_q)$ is defined by

$$d_{\text{cover}}(A, B) = wt_C(A - b).$$

This is a metric on $\mathbf{M}(\mathbb{F}_q)$, see [88, 63].

It is obvious there is a covering code in $(\mathbf{M}_{m \times n}(\mathbb{F}_q)$ with the cardinality $q^{mnk}$ and the covering radius $n - k$. Hence we have the following generalized Singleton bound on the $(d, L)$ list-decodable codes with the cover metric from Theorem 3.1.
Proposition 7.1. If $C \subset (M_{m \times n}(F_q), d_{cover})$ be an $(d, L)$ list-decodable code, then we have
\[
|C| \leq L|q^{n-d}|
\]
This is essentially same as Theorem 1 in [63].

The pair metric on $F_q^n$ is defined as follows. For $x = (x_1, \ldots, x_n) \in F_q^n$, we define $\text{pair}(x) = ((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0))$. The pair distance $d_{pair}$ is defined by
\[
d_{pair}(x, y) = d_H(\text{pair}(x), \text{pair}(y)),
\]
see [?, 10, 64]. The list-decodability and list-decoding of symbol-pair codes have been studied in [64]. The following generalized Singleton bound for $(d, L)$ list-decodable codes with the pair metric follows from our general covering code upper bound Theorem 3.1.

Proposition 7.2. If $C \subset (F_q^n, d_{pair})$ be an $(d, L)$ list-decodable code, then we have
\[
|C| \leq L|q^{n-d+2}|
\]
Proof. We construct a linear $[n, k]_q$ covering code with the first $k$ positions as information set positions. Then at the first $k$ positions, for any given vector in $F_q^k$, there is codeword whose the first $k$ coordinate vector equal to this vector. Then the covering radius of this code with the pair metric is at most $n - k + 2$. The conclusion follows immediately from Theorem 3.1.

This recovers the result in [10], see [10, 64].

7.2 List decodable deletion codes and insertion codes

It has been a notorious difficult problem to construct efficient codes to deal with insertion or deletion errors, see [49] for a nice survey. Haeupler and Shahrasbi introduced the concept to synchronization strings and obtained near-Singleton rate-distance tradeoff by using indexing based on their synchronization strings in [45, 46]. The codes constructed in [45, 46] have efficient encoding and efficient unique decoding within the half insdel distances.
Let $A$ be an alphabet with $v$ elements. The insdel distance $d_{\text{insdel}}(a, b)$ between two strings $a \in A^m$ and $b \in A^n$ is the number of insertions and deletions which are needed to transform $a$ into $b$. Actually $d_{\text{insdel}}(a, b) = m + n - 2l$ where $l$ is the length of the longest common substring of $a$ and $b$. This insdel distance $d_{\text{insdel}}$ is indeed a metric. For $a$ and $b$ in $A^n$, it is clear 

$$d_{\text{insdel}}(a, b) \leq 2d_H(a, b)$$

since $l \geq n - d_H(a, b)$ is valid for arbitrary two different vectors $a$ and $b$ in $F_q^n$. The insdel distance of a code $C \subset F_q^n$ is the minimum of the insdel distances of two different codewords in this code. Hence the Singleton upper bound 

$$|C| \leq q^{n - \frac{d_{\text{insdel}}}{2} + 1}$$

follows from the Singleton bound for codes in the Hamming metric directly, see [45, 65]. The relative insdel distance is defined as $\delta = \frac{d_{\text{insdel}}}{2n}$ since $d_{\text{insdel}}$ takes non-negative integers up to $2n$. From the Singleton bound $|C| \leq q^{n - \frac{d_{\text{insdel}}}{2} + 1}$. For insertion-deletion codes the ordering of coordinate positions strongly affects the insdel distances of codes.

Efficient binary efficient list-decodable codes have been studied in [47, 43]. For general bounds on list-decodable insertion-deletion codes, we refer to [89, 50, 48, 65]. For efficient list-decodable code construction over general fields, see [65]. We refer to [59, 77, 90, 11, 58] for the covering codes with insertions or deletions.

**Definition 7.1.** A code $C \subset A^n$ is called $(d, L)$ list-decodable deletion code if for each string $x$ in $A^{n-d}$, $x$ is substrings of at most $L$ strings in $C$. A code $C \subset A^n$ is called $(d, L)$ list-decodable insertion code if for each string $x$ in $A^{n+d}$, $x$ contains at most $L$ substrings in $C$. A code $C \subset A^n$ is called $(d_1, d_2, L)$ list-decodable insertion-deletion code, if for each string $x$ in $A^{n-u+v}$, where $u \leq d_2$ and $v \leq d_1$, from arbitrary $u$ insertions and arbitrary $v$ deletions on this string $x$, at most $L$ strings in $C$ can be gotten. This condition is hold for any given positive integers $u$ and $v$ satisfying $u \leq d_2$ and $v \leq d_1$.

Notice that from the above definition of $(d, L)$ list-decodable deletion code, each string in $A^{n-d'}$, where $d' \leq d$, is substrings of at most $L$ strings in $C$. For $(d, L)$ list-decodable insertion code, each string in $A^{n+d'}$, where $d' \leq d$, contains at most $L$ substrings in $C$. 

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Definition 7.2. A code $C \subset A^n$ is called a deletion-covering code of the radius $d$ if for each string $x$ in $A^{n-d}$, $x$ is substring of some strings in $C$. A code $C \subset A^n$ insertion-covering code of the radius $d$, if for each string $x$ in $A^{n+d}$, $x$ contains some substrings in $C$. A sequence of codes $C_{u,v} \subset A^{n-u+d}$, where $u$ and $v$ are any given two positive integers satisfying $u \leq d_2$ and $v \leq d_1$, is called a $(d_1, d_2, L)$ insertion-deletion covering code sequence of the radius $(d_1, d_2)$, if each string in $A^n$ can be gotten from $u$ insertions and $v$ deletions on some string in $C_{u,v}$. This condition is hold for any given positive integers $u$ and $v$ satisfying $u \leq d_2$ and $v \leq d_1$. We define the minimum size of this covering code sequence as

$$\min_{u \leq d_2, v \leq d_1} \{|C_{u,v}|\}.$$

Example. In the binary case $A = F_2$, let $C$ be the code consisting of the all-zero string and the all-one string. This is a insertion-covering code in $C \subset F_2^n$ of the radius $n$, since each string in $F_2^{2n}$ have a substring of length $n$ with all one or all zero coordinates. Similarly let $C \subset F_2^n$ be the code consisting of one string (0101...01). This is a deletion-covering code of the radius $n$. Then it is direct that we have a cardinality $2^{n-t+1}$ insertion-covering code in $F_2^n$ of the radius $t$ and a cardinality $2^{n-2t}$ in $F_2^n$ deletion-covering code of the radius $t$.

The following result follows from our general covering code upper bounds Theorem 3.1.

Theorem 7.1. 1) Let $C \subset A^n$ be an $(d,L)$ list-decodable deletion code. Suppose that $C_1 \subset A^{n-d}$ is a insertion-covering code of the radius $d$. Then

$$|C| \leq L|C_1|.$$

2) Let $C \subset A^n$ be an $(d,L)$ list-decodable insertion code. Suppose that $C_2 \subset A^{n+d}$ is a deletion-covering code of the radius $d$. Then

$$|C| \leq L|C_2|.$$

3) Let $C \subset A^n$ be an $(d_1, d_2, L)$ list-decodable insertion-deletion code. Suppose that $C_{u,v} \subset A^{n-d_1}$, where $u \leq d_2$ and $v \leq d_1$, is a insertion-deletion
covering code sequence of the radius \((d_1, d_2)\). Then

\[
|C| \leq L \min_{u \leq d_2, v \leq d_1} \{|C_{u,v}|\}.
\]

We have the following result from the previous result in [1]. Notice that the deletion-covering code in [1] is the insertion-covering codes in definitions in [59, 77, 90, 58].

Corollary 7.1 follows from Theorem 6.14 of [1].

**Corollary 7.1.** Let \(C \subset A^n\) be an \((1, L)\) list-decodable deletion code. Suppose \(\frac{n}{\log n} \geq 48v\), then we have

\[
|C| \leq LW \left(\frac{v^n \log n}{n}\right),
\]

where \(W\) is a fixed positive constant.

The following result can be proved from the main results in [90].

**Corollary 7.2.** Let \(C \subset A^5\) be an \((2, L)\) list-decodable insertion code. Suppose that \(C_1 \subset A^7\) is the deletion-covering code of the radius 2 determined in [90]. Then we have

\[
|C| \leq L|C_1|.
\]

The following result can be proved from the above example.

**Corollary 7.3** If \(C \subset A^n\) is an \((d_1, d_2, L)\) list-decodable deletion code, then we have \(|C| \leq L \min\{v^{n-2d_2}, v^{n-d_1}\} \}

**8 Conclusion**

In this paper we propose to look at \((d_{\text{list}}, L)\) list-decodable codes in a general finite metric space \((X, d)\) from various covering codes \(C'\) of the same length with the covering radius \(R_{\text{covering}}(C') \leq d_{\text{list}}\). Then any such small
\textbf{C'} gives a good upper bound on the sizes of \((d, L)\) list-decodable codes in \((X, d)\). The corresponding asymptotic covering code upper bounds are also given. This kind of simple upper bounds is a strong constraint on the list-decodability of codes in \((X, d)\). For example in Hamming-metric setting we prove that that any \(((\frac{q-1}{q} - \frac{1}{\sqrt{n}})n, L)\) list-decodable code, when \(n = q^m\), has its rate at most \(\frac{\log q + 1}{n}\). Various upper bounds on the sizes of list-decodable codes or the lower bound on the list sizes from various known results about covering codes in Hamming metric are given, to illustrate that if we could find suitable small covering codes with \(R_{covering} \leq d_{list}\) the upper bound on the size and the lower bound on the list size can be given.

Our general covering code upper bound can apply to the finite metric space in which the volumes of balls depend not only on radius, but also center. Actually from our this covering-code point of view about \((d_{list}, L)\) list-decodability, the key ingredient is the size of covering code with the radius \(d_{list}\), not the volumes of balls of the radius \(d_{list}\). Our general covering code upper bounds are also applied to list-decodable rank-metric codes, list-decodable subspace codes, list-decodable insertion codes and list-decodable deletion codes. The covering code with the Hamming metric has been a classical topic in coding theory, see [11] and references therein. However there are few results on covering codes with other metrics. The basic point of this paper is that upper bounds for covering code sizes and covering radius with various metrics are constraints on list-decodable codes with these metrics.

We introduce combinatorial \((R_{list}, L_1, L_2)\) covering list-decodable codes in a general finite metric space and suggest to study \((R_{list}, L_1 = L_2)\) covering list-decodable codes as generalized perfect codes. In particular we ask the following question. Can we find \(n < q\) special evaluation points in \(\mathbb{F}_q\) such that the corresponding Reed-Solomon \([n, k]_q\) code is a nice \((R_{list}, L_1 \geq 1, L_2)\) covering list-decodable code with the smallest possible \(L_2 - L_1\)? Notice that in this case \(R_{list} \geq n - k \geq n - \sqrt{n(k - 1)}\) is larger than the Johnson radius.

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