Quasi-spin wave quantum memories with dynamic symmetry

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For the two-mode exciton system formed by the quasi-spin wave collective excitation of many Λ atoms fixed at the lattice sites of a crystal, we discover a dynamic symmetry depicted by the semi-direct product algebra $SU(2)\otimes h_2$ in the large $N$ limit with low excitations. With the help of the spectral generating algebra method, we obtain a larger class of exact zero-eigenvalue states adiabatically interpolating between the initial state of photon-type and the final state of quasi-spin wave exciton-type. The conditions for the adiabatic passage of dark states are shown to be valid, even with the presence of the level degeneracy. These theoretical results can lead to propose new protocol of implementing quantum memory robust against quantum decoherence.

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Recent progresses in quantum information have stimulated the development of new concept technologies, such as quantum computation, quantum cryptography and quantum teleportation [1]. The practical implementation of these quantum protocols relies on the construction of both quantum memories (QMEs) and quantum carriers (QCAs) free of quantum decoherence. While photons can be generally taken as quantum carriers, quantum memories should correspond to localized systems capable of storing and releasing quantum states reversibly. Moreover, to control the coherent transfer of information, there should be a time dependent mechanism for turning on and turning off the interaction between QME and QCA at appropriate instants of time.

A single atom in a cavity QED system [2] seems to satisfy the above mentioned requirements for QME using the Raman adiabatic passage mechanism [3]. To achieve the strong coupling required for a practical QME a very elegant method has been proposed recently [4–6], where ensembles of Λ-type atoms are used to store and transfer the quantum information of photons by the collective atomic excitations through electromagnetically induced transparency (EIT) [7]. Some experiments [8,9] have already demonstrated the central principle of this technique - the group velocity reduction. The recent success in experiment also shows the power of such an atomic ensemble QME [11] and motivates additional theoretical works [10]. But there still exists the decoherence problem. An ensemble consists of many moving atoms and atoms in different spatial positions may experience different couplings to the controlling external fields. This results in decoherence in quantum information processing [12]. To avoid the spatial-motion induced decoherence, one naturally considers the case that each Λ atom is fixed at a lattice site of a crystal. The most recent experiment of the ultraslow group velocity of light in a crystal of $Y_2SIO_5$ [14] proposes the possibility of implementing robust quantum memories by utilizing the solid state exciton system.

In this letter, we study a system consisting of the quasi-spin wave collective excitations of many Λ-type atoms. In this system, most spatially-fixed atoms stay in the ground state, the two quasi-spin collective excitations to two excited states behave as two types of bosons and thus a two mode exciton system forms. We prove that in the large $N$ limit with low excitations, this excitonic system possess a hidden dynamic symmetry described by the semi-direct product $SU(2)\otimes h_2$ of quasi-spin $SU(2)$ and the exciton Heisenberg-Weyl algebra $h_2$. With the help of the spectrum generating algebra theory [16] based on $SU(2)\otimes h_2$, we can construct the eigen-states of the two mode exciton-photon system including the collective dark states as a special class. Since the external classical field is controllable, the quantum information can be coherently transferred from the cavity photon to the exciton system and vice versa. Therefore, the two mode quasi-spin wave exciton system can serve as a robust quantum memory.

The model system we consider consists of a crystal with $N$ lattice sites as shown in Fig. 1. There are $N$ 3-level atoms of Λ-type with the excited state $|a\rangle$, the relative ground state $|b\rangle$ and the meta-stable lower state $|c\rangle$. They interact with two single-mode optical fields. The transition from $|a\rangle$ to $|b\rangle$ of each atom is approximately resonantly coupled to a quantized radiation mode (with coupling constant $g$ and annihilation operator $a$), and the transition from $|a\rangle$ to $|c\rangle$ is driven by an exactly resonant classical field of Rabi-frequency $\Omega$. In recent years, for the similar exciton system in a crystal slab with spatially fixed two level atoms, both quantum decoherence and fluorescence process have been extensively studied [13].

For convenience we introduce the notation $\mathbf{j} = (a_x, a_y, a_z)$ to denote the position of the $j$–th site where $a_u$ is the length of the crystal cell along $u$–direction and $j_u = 0, 1, 2, \cdots, N_u$ for $u = x, y, z$. Then the quantum dynamics of the total system is described by the following Hamiltonian in the interaction picture:

$$H = ga \sum_{j=1}^{N} \exp(i\mathbf{K}_{ab} \cdot \mathbf{j}) a_j^a b_j^b.$$
\[ + \Omega \sum_{j=1}^{N} \exp(iK_{ca} \cdot j) \sigma_{ac}^j + h.c., \]  
(1)

where \( N = N_x N_y N_z \), and \( K_{ba} \) and \( K_{ca} \) are respectively the wave vectors of the quantum and classical light fields. The quasi-spin operators \( \sigma_{\alpha\beta}^j = |\alpha\rangle \langle \beta| \) \( (\alpha, \beta = a, b, c) \) for \( \alpha \neq \beta \) describe the transition between the levels of \( |a\rangle, |b\rangle \) and \(|c\rangle\).

The quasi-spin operators \( \sigma_{\alpha\beta}^j \) that in the large \( N \) limit with the low excitation condition that there are only a few atoms occupying \( |a\rangle \) or \(|c\rangle\) [15], the two mode quasi-spin wave excitations behave as two bosons since in this case they satisfy the bosonic commutation relations \( [A, A^\dagger] = 1, [C, C^\dagger] = 1 \). In the same limit, it is worth to point out that \( [A, C] = 0 \) and \( [A, C^\dagger] = -T_- / N \rightarrow 0 \), thus these quasi-spin wave low excitations are independent of each other.

In terms of these two mode exciton operators, the interaction Hamiltonian reads

\[ H = g \sqrt{N} a A^\dagger + \Omega T_+ + h.c., \]  
(5)

where the collective operators

\[ T_- = \sum_{j=1}^{N} e^{-iK_{ca} \cdot j} \sigma_{ca}^j, \quad T_+ = (T_-)^\dagger, \]  
(6)

generate the \( SU(2) \) algebra together with the third generator \( T_3 = \sum_{j=1}^{N} (\sigma_{aa}^j - \sigma_{cc}^j)/2 \). It is very interesting to observe that the exciton operators and the \( SU(2) \) generators span a larger Lie algebra. By a straightforward calculation we obtain

\[ [T_-, C] = -A, [T_+, A] = -C. \]  
(7)

Denote by \( h_2 \) the Lie algebra generated by \( A, A^\dagger, C, \) and \( C^\dagger \). It then follows that \( [SU(2), h_2] \subset h_2 \). This means that in the large \( N \) limit with the low excitation condition the operators \( A, A^\dagger, C, C^\dagger, T_3, T_\pm \) and the identity \( 1 \) span a semidirect product Lie algebra \( SU(2) \oplus h_2 \). In the following discussion we will focus on this case except otherwise explicitly specified.

Since the Hamiltonian \( H \) can be expressed as a function of the generators of \( SU(2) \oplus h_2 \), one says that the two-mode exciton system possesses a dynamic symmetry governed by the dynamic “group” (or dynamic algebra) \( SU(2) \oplus h_2 \). The discovery of this dynamic symmetry leads us, by the spectrum generating

![FIG. 1. Configuration of the proposed quantum memory with A-type atoms. (a) located at lattice sites of crystal. (b) resonantly coupled to a control classical field and a quantized probe field.](image)

![FIG. 2. Illustration of the second order process \(|b\rangle \rightarrow |a\rangle \rightarrow |c\rangle\) induced by the classical and quantized lights.](image)

\[ [T_-, C] = -A, [T_+, A] = -C. \]  
(7)
finite and each energy level possesses a very large degeneracy. The dressed two-mode exciton system resembles that of a two eigen-states. Physically the spectral structure of the vacuum of the electromagnetic field, we find that for each given pair of indices \((m, k)\), \(|e(m, k; n)|\)\(n = 0, 1, 2, \cdots\) defines a degenerate set of eigen-states. Physically the spectral structure of the dressed two-mode exciton system resembles that of a two mode harmonic oscillator, but its energy level number is finite and each energy level possesses a very large degeneracy.

To that cause, we define

\[
D = a \cos \theta - C \sin \theta
\]  

with \(\theta(t)\) satisfying \(\tan \theta(t) = \frac{g\sqrt{N}}{\Omega(t)}\). It mixes the electromagnetic field and collective atomic excitations of quasi spin wave. Evidently, \(\{D, D\dagger\} = 1\) and \(\{D, H\} = 0\). Thus the Heisenberg-Weyl group \(h\) generated by \(D\) and \(D\dagger\) is a symmetry group of the two-mode exciton-photon system. We introduce the state \(|0\rangle = |v\rangle \otimes |0_i\rangle\) where \(|0_i\rangle\) is the vacuum of the electromagnetic field, we find \(D|0\rangle = 0\) and it is an eigen-state of \(H\) with zero eigen-value. Consequently, a degenerate class of eigen-states of \(H\) with zero eigen-value can be constructed naturally as follows:

\[
|d_m\rangle = |n!\rangle^{-1/2} D^n|0\rangle.
\]  

Physically, the above dressed state is cancelled by the interaction Hamiltonian and thus is called a dark state or a dark-state polariton (DSP). A DSP traps the electromagnetic radiation from the excited state due to quantum interference cancelling. For the case with an ensemble of free moving atoms, the similar DSP was obtained in refs. [4–6] to clarify the physics of the state-preserving slow light propagation in EIT associated with the existence of quasi-particles.

Now starting from these dark states \(|d_m\rangle\), we can use the spectrum generating algebra method to build other eigenstates for the total system. To this end we introduce the bright-state polariton operator

\[
B = a \sin \theta + C \cos \theta.
\]  

It is easy to check that \([B, B\dagger] = 1\), and \([D, B\dagger] = [D, B] = 0\). Evidently \([A, B] = [A, B\dagger] = 0\), this amounts to the fact that \(A\) commutes with \(C\) and \(C\dagger\) in the large \(N\) limit with low excitations. What is crucial for our purpose is the commutation relations \([H, Q_{\pm}k] = \pm \epsilon Q_{\pm}k\) for \(Q_{\pm} = \frac{1}{\sqrt{2}}(A \pm B)\) where \(\epsilon = \sqrt{g^2 N + \Omega^2}\). Thanks to these commutation relations we can construct the eigenstates

\[
|e(m, k; n)\rangle = |m!k!\rangle^{-1/2} Q_{+}^{m} Q_{-}^{k}|d_m\rangle,
\]  

as the dressed states of the two-mode exciton system. The corresponding eigen-values are

\[
E(m, k) = (m - k)\epsilon, m, k = 0, 1, 2, \cdots
\]  

We notice that for each given pair of indices \((m, k)\), \(|e(m, k; n)|\)\(n = 0, 1, 2, \cdots\) defines a degenerate set of eigen-states. Physically the spectral structure of the dressed two-mode exciton system resembles that of a two mode harmonic oscillator, but its energy level number is finite and each energy level possesses a very large degeneracy.

The above equations show that there exists a larger class \(S\) : \(|e(m, m; n)| = 0, n = 0, 1, \cdots\) of states of zero-eigen-value \(E(m, m) = 0\). They are constructed by acting \(Q_{+}^{m} Q_{-}^{m}\) \(m\) times on \(|d_m\rangle\):

\[
|d(m, n)\rangle = \sum_{k=0}^{m} A^{2(m-k)} B^{2k} 2^{m(m-k)!}|d_m\rangle.
\]  

This larger degeneracy is physically rooted in the larger symmetry group \(h_2\) generated by \(Q_{+}^{m} Q_{-}^{m}\) and \(Q_{+}Q_{-}\) together with \(D\) and \(D\dagger\). The original quantum memory defined by \{|d_m\rangle = |d(0, n)\rangle\} in ref. [4–6] actually is a special subset of the larger class.

Now we consider whether these states of zero-eigenvalue can work well as a quantum memory by the adiabatic manipulation [4–6]. The quantum adiabatic theorem [17,18] for degenerate cases tells us that, under the adiabatic condition

\[
|\langle e(m, k; n)|D_{t}|d(m, n)\rangle\rangle| \sim \frac{g\sqrt{N} |\Omega(t)|}{\epsilon^3} \ll 1,
\]  

the adiabatic evolution of any degenerate system will keep itself within the block \(S\) of dark states with the same instantaneous eigen-value 0. However, it does not forbid transitions within states in this block \(S\), such as those between \{|d_m\rangle = |d(0, n)\rangle\} and \{|d(m, n)\rangle(m \neq 0)\}. So it is important to consider whether there exists any dynamic mechanism to forbid such transitions. Actually this issue has been uniformly ignored in all previous studies even for the degenerate set \{|d_m\rangle\}.

We can generally consider this problem by defining the zero-eigenvalue subspace \(S^{[m]} : \{|d(m, n)\rangle\} n = 0, 1, 2, \cdots\}, S^{[0]} = S\). The complementary part of the direct sum \(DS = S^{[0]} \oplus S^{[1]} \oplus \cdots\) of all dark state subspaces is \(CS = \{|e(m, k; n)\rangle | k \neq m, n = 0, 1, 2, \cdots\}\) in which each \(|e(m, k; n)\rangle\) has non-zero eigenvalue. Any state \(|\phi^{[m]}(t)\rangle = \sum_{n} c_{n}^{[m]}(t)|d(m, n)\rangle\) in \(S^{[m]}\) evolves according to

\[
i \frac{d}{dt} c_{n}^{[m]}(t) = \sum_{m',n'} D_{mn'}^{nm} c_{n'}^{[m']}(t) + F^{[CS]}_{m},
\]  

where \(F^{[CS]}\), which can be ignored under the adiabatic conditions [17,18], represents a certain functional of the complementary states and \(D_{mn'}^{nm} = -i \langle d(m', n')|\partial_{d}|d(m, n)\rangle\). Considering \(\partial_{d} B = D\) and \(\partial_{d} D = -B\), we have \(D_{mn'}^{nm} = -i \partial_{d}\langle d(m', n')|\partial_{d}|d(m, n)\rangle\). The equation about \(\partial_{d}|d(m, n)\rangle\) contains 4 terms: \(|e(m, m \pm 1; n \pm 1)\rangle\) and \(|e(m \mp 1, m; n \pm 1)\rangle\). This implies the exact result \(\langle d(m', n')|\partial_{d}|d(m, n)\rangle = 0\), showing there is indeed no mixing among the dark states during the adiabatic evolution. Viewed from physical aspect, this can also be understood as the adiabatic change of external parameters do not lead the system to enter into the complementary space \(CS\). Notice that only for the non-adiabatic evolution, will the non-zero matrix elements
\[ \langle e(m', k, n') | \partial \eta | d(m, n) \rangle \] will be a cause for state mixing.

The same physics has been considered in the context of the Abelianization of the non-Abelian gauge structure induced by an adiabatic process [17,18]. This argument gives a necessary theoretical support for the practical realization of the original scheme of quantum memory by Fleischhauer, Lukin and their collaborators [4–6].

Based on the above consideration, we thus claim that, for each fixed \( m \neq 0 \), each subspace \( S^{|m|} \) can work formally as a quantum memory different from that in ref. [4–6]. We introduce the notation

\[
|A, P, m\rangle = \frac{1}{2^{|m|!}} (A^{2} - P^{2/m}|0\rangle
\]

for \( P = a, c \). Both the initial state \( |d(m, n)\rangle_{\theta=0} = |A, C, m\rangle \otimes |n\rangle_{L} \) and the final state \( |d(m, n)\rangle_{\theta=\pi/2} = |n\rangle_{C} \otimes (-1)^{m} |A, a, m\rangle \) have factorization structure. Thus we can use the general initial state \( |s(0)\rangle = \sum_{n} c_{n} |n\rangle_{L} \) of single-mode light to record quantum information and prepare the exciton in a paired state \( |A, C, m\rangle \).

When one rotates the mixing angle \( \theta \) from 0 to \( \pi/2 \) by changing the coupling strength \( \Omega(t) \) adiabatically, the total system will reach the final state \( |S(t)\rangle = (\sum_{n} c_{n} |n\rangle_{C}) \otimes |A, a, m\rangle \) with the \( \epsilon \)-mode quasi-spin wave decoupling with the other parts. From the viewpoint of quantum measurement the decoding process is then to average over the states of photon and \( A \)-exciton and to obtain the pure state density matrix \( \rho_{C} = \sum_{n, m} c_{n} c_{m}^{*} |n\rangle_{C} \langle m| \), which is the same as that for the initial photons. Therefore, the above discussion suggests a new protocol of storing quantum information when the decay of excited state is enough small during adiabatic manipulation.

Before concluding, we would like to address that, the individual atoms in the generalized states \( |d(m, n)\rangle \) have excited state components and therefore \( |d(m, n)\rangle \) is not totally dark in practice. If the excited state decays faster, the generalized states \( |d(m, n)\rangle \) would also decay during slow adiabatic manipulation. This metastable nature would lead to an undesirable effect for memory application. We also point out that the present treatment is only valid for the low density excitation regime where the bosonic modes of the quasi-spin wave excitations can be used effectively. Therefore the above down Fock state formally written as \( A^{m}C^{n}|0\rangle \) doesn’t make sense when \( m \) or \( n \) is large. By the mathematical duality, the situation with extremely-high excitation can be dealt with in a similar manner. In fact, the serious difficulty only lies in the region where the excitation is neither very low nor very high. In that case, it turns out that the boson commutation relation of the excitator operators must be modified, for example, to the \( q \)-deformed one \( (q = 1 - O(\hat{\omega})) \) [15]. Physically this modification will cause quantum decoherence of the collective degrees of freedom in the exciton system. Finally we emphasize that, though in our model system assumed to be located at regular lattice sites as in a crystal, our results (at least in mathematical formulation ) remain valid for an ensemble of atoms with random spatial positions, provided that we can ignore the kinetic energy terms (of the center of mass motion) of the atom. It seems that the ensemble of free atoms can function as quantum memory of same kind. However the strict treatment of the atomic ensemble based quantum memory should include the kinetic energy terms of the atom center of mass. The momentum transfer of atomic center of mass can induce additional quantum decoherence [12].

In our present protocol, this decoherence effect is partly overcome by fixing atoms at lattice sites and thus neglecting the kinetic energy terms.

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