ON STABILITY OF THE FIBRES OF HOPF SURFACES AS HARMONIC MAPS AND MINIMAL SURFACES

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ABSTRACT. We construct a family of Hermitian metrics on the Hopf surface $S^3 \times S^1$, whose fundamental classes represent distinct cohomology classes in the Aeppli cohomology group. These metrics are locally conformally Kähler. Among the toric fibres of $\pi: S^3 \times S^1 \to \mathbb{C}P^1$ two of them are stable minimal surfaces and each of the two has a neighborhood so that fibres therein are given by stable harmonic maps from 2-torus and outside, far away from the two tori, there are unstable harmonic ones that are also unstable minimal surfaces.

1. Introduction

It is interesting to know when a holomorphic map from a compact Riemann surface to a non-Kählerian Hermitian manifold is area minimizing. Besides mappings between Kähler manifolds, little seems to be known about the second variation of holomorphic maps between Hermitian manifolds, even in the case that they are also harmonic maps, for the energy functional. A strong reason for this phenomenon is that the Riemannian properties of a Hermitian metric are considerably more complicated than those of a Kähler metric. It demands specific information on the Riemannian curvature of the Hermitian metric to determine stability or instability from the second variation formulas.

In this paper, we construct explicitly a continuous family of Hermitian metrics on $S^3 \times S^1$; with these metrics, we study stability of the fibres of $\pi: S^3 \times S^1 \to \mathbb{C}P^1$ as harmonic maps for the energy functional and as minimal surfaces for the area functional in $S^3 \times S^1$.

A classical theorem of Kodaira [16] states that a compact complex surface homeomorphic to $S^3 \times S^1$ is complex analytically diffeomorphic to a Hopf surface. Such surface cannot carry any Kähler metric as its second Betti number is 0. Hermitian metrics on the Hopf surfaces have been constructed explicitly; for example, Gauduchon and Ornea produced locally conformally Kähler metrics [13]. We demonstrate, in Section 2, a construction of Hermitian metrics $g_\epsilon$ via a sequence of Hermitian metrics $\tilde{g}_\epsilon$ on the Calabi-Eckmann 3-fold $S^3 \times S^1$ induced from $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$. Each $\tilde{g}_\epsilon$ induces a Hermitian metric $g_\epsilon$ on $S^3 \times S^1$ for $\epsilon > 0$. The collapsing sequence $(S^3 \times S^1, \tilde{g}_\epsilon)$ converges to $(S^3 \times S^1, g_0)$ in the Gromov-Hausdorff distance as $\epsilon \to 0$, and $g_0$ is a smooth Hermitian metric on $S^3 \times S^1$. The fundamental classes associated with these non-Kähler metrics belong to distinct cohomology classes in the Aeppli cohomology $H_{A}^{1,1}$, which is isomorphic to the Bott-Chern cohomology $H_{BC}^{1,1}$. The metrics $g_\epsilon$ are different from those in [13], see Remark 2.7.
Throughout the paper, we use the complex structure defined by Calabi and Eckmann [7] on $S^3 \times S^1$ and denote the complex coordinates by $w_0 = x_0 + i x_2, w_1 = x_1 + i x_3$, where $w_0$ is on the fibre and $w_1$ on the base $\mathbb{C}P^1$, see Section 2.1.

Holomorphic maps may not be harmonic in a non-Kähler setting, as one can multiply the target Hermitian metric by positive functions to obtain new Hermitian metrics resulting in new harmonic map equations. Lichnerowicz gave a sufficient condition in [20]; however, it is not satisfied by our $g_\epsilon$. Harmonicity of the holomorphic inclusions in Theorem 1.2 can be verified from the harmonic map equation as written by Sampson [27] or from the fact that $T^2_p$ is totally geodesic in $(S^3 \times S^1, g_\epsilon)$, both rely on explicit computation of geometric quantities of $g_\epsilon$, see Section 3. For mappings between Kähler manifolds, it was also observed in [20] that the energy-minimizing maps are precisely the $\overline{\partial}$-energy minimizing maps, because the energy and the $\overline{\partial}$-energy differ by a homotopy invariant, namely the Kähler class of the target Kähler manifold evaluated at the homology class of the image of the mapping (cf. [80, p.192]). This is not necessarily true for mappings into Hermitian manifolds.

Our analysis of stability relies strongly on the vanishing of particular curvature components of the Riemannian curvature (cf. Lemma 3.1), together with delicate control on the positive terms (unfavorable for stability) of curvature in the second variation formula by certain part of the covariant derivatives of vector fields. This allows us to verify nonnegativity for all variation fields or exhibit negativity along certain variation fields of the Morse index form, according to whether the harmonic maps are close to or far from the two special tori (see below).

Another interesting fact of the Riemannian curvature of $g_\epsilon$ is $R(X, Y, X, Y) \geq 0$ for all $X, Y$ in $T^{1,0}(\mathbb{S}^3 \times \mathbb{S}^1)$ and it vanishes precisely on two distinguished tori

$$T^2_{p_0} = \{0\} \times \mathbb{S}^1 \times \mathbb{S}^1, \quad T^2_{p_1} = \mathbb{S}^1 \times \{0\} \times \mathbb{S}^1$$

in $\mathbb{C}^2 \times \mathbb{C}$, corresponding to the fibres over $p_0, p_1 \in \mathbb{C}P^1$, respectively. Vectors in the holomorphic tangent bundle $T^{1,0}(\mathbb{S}^3 \times \mathbb{S}^1)$ are isotropic vectors but not vice versa. We do not know whether the curvature is nonnegative on all isotropic 2-planes, see [24] (Compact Hermitian surfaces with nonnegative isotropic curvatures were discussed in [11]), so we shall use the real form of the second variation formula for energy. The curvature term in the second variation for area vanishes at the two minimal surfaces $T^2_{p_0}, T^2_{p_1}$. Although it is difficult to identifying zeros of curvature in general, a link between stability and zero locus of the ambient curvature was already revealed four decades ago by Bourguignon and Yau in [6] stating that the Riemannian curvature operator of a $K3$ surface vanishes along any nonconstant closed stable geodesic.

The main results of the paper are summarized as follows. For existence of geometrically different Hermitian metrics:

**Theorem 1.1.** There is a family of Hermitian metrics $g_\epsilon$ on the Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$ for constant $\epsilon \in [0, 1]$ with the following properties:

1. Each $g_\epsilon$ is locally conformally Kähler and is a Gauduchon metric with non-parallel Lee form.
(2) The fundamental classes $\omega_\epsilon$ of $g_\epsilon$ represent different elements in $H^{1,1}_A(S^3 \times S^1)$ for different $\epsilon$.

(3) For any linearly independent $X, Y \in T^{1,0}_x(S^3 \times S^1)$, $R(X, Y, \overline{X}, \overline{Y}) \geq 0$ where $R$ is the Riemannian curvature operator and $x \in S^3 \times S^1$, it equals to zero at $x \in S^3 \times S^1$ if and only if $x \in T^2_{p_0} \cup T^2_{p_1}$.

For stability of the fibres as harmonic maps and as minimal embeddings:

**Theorem 1.2.** Let $\pi : S^3 \times S^1 \to \mathbb{CP}^1$ be the Hopf fibration and $T^2_p = \pi^{-1}(p)$ with $p \in \mathbb{CP}^1$ and $p_0, p_1$ correspond to the two special tori. Then there exist neighbourhoods $U_0, U_1$ of $p_0, p_1$ in $\mathbb{CP}^1$, respectively, such that the holomorphic inclusion $f_\pi : T^2_p \to (S^3 \times S^1, g_\epsilon)$ is a stable harmonic map when $p \in U_0 \cup U_1$, and $f_{p_0}, f_{p_1}$ are stable minimal embeddings. Furthermore, $f_p$ is an unstable harmonic map and an unstable minimal embedding, when either $x_1 = 0, |x_3| > \sqrt{2}$ or $x_3 = 0, |x_1| > \sqrt{2}$, where $w_1 = x_1 + i x_3$ is the holomorphic coordinate on $\mathbb{CP}^1$.

Let $E^\mathbb{C}_p = f_\pi^* T(S^3 \times S^1) \otimes \mathbb{C}$ be the complexified pullback bundle over $T^2_p$. There is a unique holomorphic structure $\mathcal{E}$ on $E^\mathbb{C}_p$ (for more detail, see p.21). Let $H^{\mathcal{E}}_\partial(T^2_p, E^\mathbb{C}_p)$ be the linear space of holomorphic sections of $E^\mathbb{C}_p$. A useful curvature property is that $R(W, \partial_{x_0}, W, \partial_{x_1}) \geq 0$ for any smooth section $W$ of $E^\mathbb{C}_p$, which arises in the complexified second variation formula. We have

**Theorem 1.3.** Let $U_0, U_1$ be the two neighbourhoods in Theorem 1.2 and $p \in U_0 \cup U_1$. If $p \neq p_0, p_1$ then $H^{\mathcal{E}}_\partial(T^2_p, E^\mathbb{C}_p) = \text{Span}_\mathbb{C} \{ \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1} \}$. If $p = p_0, p_1$ then $H^{\mathcal{E}}_\partial(T^2_p, E^\mathbb{C}_p) = \text{Span}_\mathbb{C} \{ \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \}$.

Theorem 1.3 is concluded from knowing the harmonic mappings $f_p$ are stable already. The usual practice goes in the opposite direction: estimating the Morse index from existing holomorphic (or almost holomorphic) sections, see [24], [10], [11]. On the other hand, stability together with positivity (nonnegativity) of curvature of the ambient manifold force rigidity results for harmonic maps in many situations, as demonstrated, for example, in [4], [5], [8], [10], [11], [19], [18], [22], [24], [25], [26], [29], [30], [31], et al. We also include in Appendix a known result for $S^3 \times S^1$ equipped with the product of standard metrics.

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### 2. Construction of a Family of Hermitian Metrics on $S^3 \times S^1$

In this section, we construct a family of Hermitian metrics on the Hopf surface $S^3 \times S^1$ from Hermitian metrics on the Calabi-Eckmann complex 3-manifold $S^3 \times S^3$ via an adiabatic limit type procedure, i.e., scaling down a factor of a product.
We then explore properties of these metrics and explain why they are not those constructed in [13].

2.1. Complex structures on $S^3 \times S^3$ and $S^3 \times S^1$. We recall Calabi-Eckmann’s construction of an integrable complex structure on a product of two odd dimensional unit spheres and follow the notations in [7]. We only consider $S^3 \times S^3$ and $S^3 \times S^1$. Let $E$ and $E'$ be the complex Euclidean space $C^2$ and $C^p$ respectively, where $p = 1, 2$, with complex affine coordinates $z_0, z_1$ on $E$ and $z'_0, z'_1$ on $E'$ for $p = 2$ and $z'_0$ for $p = 1$. Let $V_{a\beta}$ be the open subset of $S^3 \times S^{2p-1}$ defined by

$$V_{a\beta} = \{(z_0, z_1, z'_0, z'_{p-1}) : (z_0, z_1) \in S^3 \subset C^2, (z'_0, z'_{p-1}) \in S^{2p-1} \subset C^p, z_{a\alpha} z'_{\beta} \neq 0\}$$

where $\alpha = 0, 1$ and $\beta = 0, p - 1$, with $(z'_0, z'_{p-1}) := z'_0$ for $p = 1$. The family $\{V_{a\beta}\}$ is an open cover of $S^3 \times S^{2p-1}$. On $V_{a\beta}$, with $\alpha, j = 0, 1$ and $\beta, k = 0, p - 1$ define

$$w_{a\alpha} = \frac{z_j}{z_0}, \quad j \neq \alpha$$

(2.2)$$w'_{\beta k} = \frac{z'_j}{z'_0}, \quad k \neq \beta$$

(2.3)$$t_{a\beta} = \frac{1}{2\pi i} \left( \log z_0 + i \log z'_0 \right) \mod (1, i).$$

Let $T^2$ be the square torus $C/[1, i]$ with a coordinate chart defined by (2.3). When $p = 2$, the pair of complex numbers $(w_{a\alpha}, w'_{\beta k})$ is a local inhomogeneous coordinates of $C P^1 \times C P^1$ and $(w_{a\alpha}, w'_{\beta k}, t_{a\beta})$ is a differentiable map from $V_{a\beta}$ into $C^2 \times T^2$; when $p = 1$, (2.2) is vacuous and $C P^1 \times C P^0 := C P^1$ so $w_{a\alpha}$ is a coordinate of $C P^1$ and $(w_{a\alpha}, t_{a\beta})$ maps $V_{a\beta}$ differentially into $C^1 \times T^2$. Furthermore, as shown in [7, p.496], given any $w_{a\alpha}, w'_{\beta k} \in C$ and a point $[t_{a\beta}] \in T^2$ which is represented by a complex number $t_{a\beta}$ in the congruence class of the lattice mod $(1, i)$, the quadruple $z_0, z_1, z'_0, z'_1$ are uniquely solvable in $V_{a\beta}$. In fact

Proposition 2.1. (Calabi-Eckmann) Each $V_{a\beta}$ is homeomorphic to $C^p \times T^2$. On

$$U_{a\beta} = \{(w_{a\alpha}, w'_{\beta k}, t_{a\beta}) \in C^{p+1} : 0 < \Re t_{a\beta}, \Im t_{a\beta} < 1\} \subset V_{a\beta}, (w_{a\alpha}, w'_{\beta k}, t_{a\beta})$$

is a complex coordinate system of $V_{a\beta} \subset S^3 \times S^{2p-1}$. For this complex structure, the fibre bundle $S^3 \times S^{2p-1} \rightarrow C P^1 \times C P^{p-1}$ is complex analytic and each fibre is a holomorphic nonsingular torus.

2.2. Hermitian Metrics on $S^3 \times S^3$. Let $U_{a\beta}$ be the coordinate neighbourhood in Proposition 2.1 which forms an open coordinate cover of $S^3 \times S^3$. Without loss of generality, we consider $U_{00}$ on which we set

$$(w_0, w_1, w_2) = (t_{00}, w_{01}, w'_{01})$$

for simplicity. The inclusion map

$$\iota : S^3 \times S^3 \rightarrow C^2 \times C^2$$

...
can be written locally in these coordinates as
\[
\begin{align*}
\begin{cases}
\zeta_0 &= A^{-\frac{i}{2}} e^{i \left( \pi w_0 + i \pi w_1 \right) + \frac{i}{2} \log B}, \\
\zeta_0' &= B^{-\frac{i}{2}} e^{i \left( \pi w_0 - i \pi w_0 \right) - \frac{i}{2} \log A}, \\
\zeta_1 &= z_0 w_1 = w_1 A^{-\frac{i}{2}} e^{i \left( \pi w_0 + i \pi w_0 \right) + \frac{i}{2} \log B}, \\
\zeta_1' &= z_0' w_2 = w_2 B^{-\frac{i}{2}} e^{i \left( \pi w_0 - i \pi w_0 \right) - \frac{i}{2} \log A}
\end{cases}
\end{align*}
\]
(2.4)
where
\[
A = 1 + |w_1|^2, \quad B = 1 + |w_2|^2.
\]

To explicitly write down Hermitian metrics on \(\mathbb{S}^3 \times \mathbb{S}^3\), we compute \(\partial t\):
\[
\begin{align*}
\frac{\partial \zeta_0}{\partial w_0} &= i \pi z_0, & \frac{\partial \zeta_0'}{\partial w_0} &= \pi z_0', & \frac{\partial \zeta_1}{\partial w_0} &= i \pi z_1, & \frac{\partial \zeta_1'}{\partial w_0} &= \pi z_1', \\
\frac{\partial \zeta_0}{\partial w_1} &= \frac{1}{2} \overline{w}_1 z_0, & \frac{\partial \zeta_0'}{\partial w_1} &= -\frac{i}{2} \overline{w}_1 z_0', & \frac{\partial \zeta_1}{\partial w_1} &= z_0 - \frac{1}{2} \overline{w}_1 z_1, & \frac{\partial \zeta_1'}{\partial w_1} &= -\frac{i}{2} \overline{w}_1 z_1', \\
\frac{\partial \zeta_0}{\partial w_2} &= \frac{i}{2} \overline{w}_2 z_0, & \frac{\partial \zeta_0'}{\partial w_2} &= -\frac{i}{2} \overline{w}_2 z_0', & \frac{\partial \zeta_1}{\partial w_2} &= \frac{i}{2} \overline{w}_2 z_1, & \frac{\partial \zeta_1'}{\partial w_2} &= \frac{i}{2} \overline{w}_2 z_1'.
\end{align*}
\]

Now, we take Hermitian metrics on \(\mathbb{C}^2 \times \mathbb{C}^2\) defined by
\[
h_{\epsilon} = \sum_{i=0,1} dz_i \otimes d\bar{z}_i + \epsilon^2 \sum_{i=0,1} dz_i' \otimes d\bar{z}_i', \quad \epsilon \searrow 0.
\]

For any \(X \in T^{1,0}(\mathbb{S}^3 \times \mathbb{S}^3)\), let \(\iota_*(X)^{1,0} \in T^{1,0}(\mathbb{C}^2 \times \mathbb{C}^2)\) be the \((1, 0)\) part of the push forward \(\iota_* X\). Then
\[
\tilde{g}_{\epsilon}(X, Y) = h_{\epsilon} \left( \iota_*(X)^{1,0}, \iota_*(Y)^{1,0} \right), \quad X, Y \in T^{1,0}(\mathbb{S}^3 \times \mathbb{S}^3)
\]
is a Hermitian metric on the complex manifold \(\mathbb{S}^3 \times \mathbb{S}^3\). Its components
\[
\tilde{g}_{\epsilon,ij} = h_{\epsilon} \left( \iota_* \left( \frac{\partial}{\partial \bar{w}_j} \right)^{1,0}, \iota_* \left( \frac{\partial}{\partial \bar{w}_j} \right)^{1,0} \right)
\]
are given by the Hermitian matrix
\[
(\tilde{g}_{\epsilon,ij}) = \begin{pmatrix}
(1 + \epsilon^2)^2 \pi^2 & (1 + \epsilon^2) i \pi \overline{w}_1 A & (1 + \epsilon^2) \pi w_2 B \\
(1 + \epsilon^2) i \pi \overline{w}_1 B & \frac{1 + \epsilon^2}{4} + \frac{3 |w_1|^2}{A^2} & -\frac{1 + \epsilon^2}{4} \frac{w_1 w_2}{A B} \\
(1 + \epsilon^2) i \pi w_1 A & -\frac{1 + \epsilon^2}{4} \frac{w_1 w_2}{A B} & \frac{1 + \epsilon^2}{4} + \frac{3 |w_2|^2}{B^2}
\end{pmatrix}.
\]
(2.6)

For example,
\[
\iota_* \left( \frac{\partial}{\partial \bar{w}_0} \right)^{1,0} = \sum_{i=0,1} \frac{\partial \zeta_i}{\partial w_0} \frac{\partial}{\partial \zeta_i} + \sum_{i=0,1} \frac{\partial \zeta_i'}{\partial w_0} \frac{\partial}{\partial \zeta_i'}
\]
and
\[
\bar{g}_\epsilon \left( \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_0} \right) = h_\epsilon \left( \iota_* \left( \frac{\partial}{\partial w_0} \right)^{1,0}, \iota_* \left( \frac{\partial}{\partial w_0} \right)^{1,0} \right)
= \sum_{i=0,1} \left| \frac{\partial z_i}{\partial w_0} \right|^2 + \epsilon^2 \sum_{i=0,1} \left| \frac{\partial \bar{z}_i}{\partial w_0} \right|^2
= \left( 1 + \epsilon^2 \right) \pi^2.
\]

For \( \epsilon > 0 \), \( \bar{g}_\epsilon \) induces a Hermitian metric \( g_\epsilon \) on \( S^3 \times S^1 \) given in the coordinates \((w_0, w_1)\) by
\[
(g_{\epsilon, ij}) = \begin{pmatrix}
(1 + \epsilon^2) \pi^2 & \frac{1 + \epsilon^2}{2} \pi \frac{w_1}{A} \\
- \frac{1 + \epsilon^2}{2} \pi \frac{w_1}{A} & \frac{1}{A} + \frac{3}{4A^2} + \frac{\epsilon^2 w_1^2}{4A^2}
\end{pmatrix}.
\]

When \( \epsilon = 0 \) the symmetric 2-tensor \( \bar{g}_0 \) is nonnegative but not positive definite; however, we will show that it still yields a Hermitian metric \( g_0 \) on \( S^3 \times S^1 \).

**Proposition 2.2.** With respect to the Calabi-Eckmann complex structure, the complex surface \( S^3 \times S^1 \) is a complex submanifold of \( S^3 \times S^3 \). The 2-tensor \( g_0 \), defined by \( g_0(X, Y) = h_0(\iota_* (X)^{1,0}, \iota_* (Y)^{1,0}) \) for \( X, Y \in T^{1,0}(S^3 \times S^3) \), is a Hermitian metric on \( S^3 \times S^1 \).

**Proof.** Consider the map \( \tau : S^3 \times S^1 \to S^3 \times S^3 \) given by \( \tau(z_0, z_1, z'_0) = (z_0, z_1, z'_0, 0) \in \mathbb{C}^2 \times \mathbb{C}^2 \) for any \((z_0, z_1, z'_0) \in \mathbb{C}^2 \times \mathbb{C} \) with \( |z_0|^2 + |z_1|^2 = 1 \) and \( |z'_0| = 1 \). It is obvious that \( \tau(S^3 \times S^1) = (S^3 \times S^3) \cap \{ z'_1 = 0 \} \) and \( \tau : S^3 \times S^1 \to \tau(S^3 \times S^1) \) is diffeomorphic. Moreover, \( \tau \) is a holomorphic map: we may assume \( z_0 \neq 0 \) without loss of generality. Then, \( \tau(w_0, w_1) = (w_0, w_1, 0) \) is holomorphic.

To see \( g_0 \) is a Hermitian metric on \( S^3 \times S^1 \), without loss of generality, we consider it in the coordinate chart \( U_{00} \) of \( S^3 \times S^1 \). For \( \epsilon = 0 \)
\[
(g_{0, ij}) = \begin{pmatrix}
\pi^2 & \frac{\pi}{2} \frac{w_1}{A} \\
- \frac{\pi}{2} \frac{w_1}{A} & \frac{1}{4A} + \frac{3}{4A^2}
\end{pmatrix},
\]
therefore
\[
g_0 = \sum_{i,j=0,1} g_{0, ij} dw_i \otimes d\bar{w}_j,
\]
where
\[
(g_{0, ij}) = \begin{pmatrix}
\pi^2 & \frac{\pi}{2} \frac{w_1}{A} \\
- \frac{\pi}{2} \frac{w_1}{A} & \frac{1}{4A} + \frac{3}{4A^2}
\end{pmatrix}
\]
is a positive definite Hermitian matrix. \( \square \)
2.3. Gromov-Hausdorff convergence.

**Theorem 2.3.** The Hermitian manifolds \((S^3 \times S^3, g_e)\) converge to the Hermitian manifold \((S^3 \times S^1, g_0)\) in the Gromov-Hausdorff distance as \(e \searrow 0\).

**Proof.** As previously done, we identify \(S^3 \times S^1\) with \((S^3 \times S^3) \cap \{z_2' = 0\}\). Let \(g_e\) be the pullback of \(g_r\) by \(S^3 \times S^1 \hookrightarrow S^3 \times S^3\). Direct computation shows that \(\det g_0 = 0\), and in \(U_{00}\) the degenerating direction is

\[
\alpha = \frac{w_2}{2\pi} \frac{\partial}{\partial w_0} + B \frac{\partial}{\partial w_2},
\]

namely, \(g_0(\alpha, X) = 0\) for all \(X \in T^{1,0}(S^3 \times S^1)\). Write \(\alpha = \xi_1 + i \xi_2\), where

\[
\xi_1 = B \frac{\partial}{\partial x_2} - \frac{x_2}{2\pi} \frac{\partial}{\partial x_0} + \frac{x_5}{2\pi} \frac{\partial}{\partial x_3}, \quad \xi_2 = B \frac{\partial}{\partial x_5} + \frac{x_5}{2\pi} \frac{\partial}{\partial x_0} + \frac{x_2}{2\pi} \frac{\partial}{\partial x_3}
\]

where \(w_0 = x_0 + i x_1, w_1 = x_1 + i x_4, w_2 = x_2 + i x_5\). Since \(g_r\) is Hermitian,

\[
(2.9) \quad \tilde{g}_e(\xi_1, \xi_1) = \tilde{g}_e(\xi_2, \xi_2) = \frac{1}{2} \tilde{g}_e(\alpha, \alpha) = \frac{1}{2} \epsilon^2 \quad \text{and} \quad \tilde{g}_e(\xi_1, \xi_2) = 0.
\]

Now, we will find the integral curve \(\gamma(t)\) of \(a\xi_1 + b\xi_2\) in \(S^3 \times S^3\), where \(a, b\) are real constants satisfying \(a^2 + b^2 = 1\) to be determined later. In the local chart \(U_{00}\), assume \(\gamma(t) = (x_0(t), \cdots, x_5(t))\). From \(d\gamma/dt = a\xi_1 + b\xi_2\), we have

\[
(2.10) \quad \begin{cases}
\frac{dx_3}{dt} = AB, \\
\frac{dx_5}{dt} = bB, \\
\frac{dx_0}{dt} = \frac{b}{2\pi} x_5 - \frac{a}{2\pi} x_2, \\
\frac{dx_1}{dt} = -\frac{a}{2\pi} x_5 + \frac{b}{2\pi} x_2, \\
\frac{dx_4}{dt} = 0.
\end{cases}
\]

Let \(x_2 = r \cos \theta, x_5 = r \sin \theta\). Then

\[
\begin{cases}
\frac{dx_2}{dt} = a(1 + r^2), \\
\frac{dx_5}{dt} = b(1 + r^2).
\end{cases}
\]

Define \(v_{ab}\) such that \(\cos v_{ab} = b, \sin v_{ab} = a\). It follows

\[
\begin{cases}
\frac{dr}{dt} = \sin(\theta + v_{ab})(1 + r^2), \\
\frac{d\theta}{dt} = \cos(\theta + v_{ab})(1 + r^2).
\end{cases}
\]

When \(\cos(\theta(0) + v_{ab}) = 0\), it is easy to check

\[
\begin{cases}
r = \sin(\theta + v_{ab}) \tan(t + \sin(\theta + v_{ab}) a_1), \\
\theta = \theta(0)
\end{cases}
\]
where \( a_1 = \arctan r(0) \) is a solution. As long as \( x_2, x_5 \) are found, the functions \( x_0, x_1, x_3, x_4 \) can be solved from (2.10) uniquely for the initial data.

For any given \((w_0(0), w_1(0), w_2(0))\), we may choose \( a, b \) so that \( \cos(v_{ab} + \theta(0)) = 0, \sin(v_{ab} + \theta(0)) = -1 \). Let \( \gamma(t) \) be the integral curve solving (2.10) with \( \gamma(0) = (w_0(0), w_1(0), w_2(0)) \). From the discussion above, \( r = -\tan(t - a_1) \) and it follows that \( r(a_1) = 0 \), in turn, \( \gamma(a_1) \in \mathbb{S}^3 \times \mathbb{S}^1 \). By (2.9)

\[
\left| \frac{dy}{dt} \right|_{g_\epsilon} \leq C \epsilon.
\]

Hence the length of \( \gamma(t) \) for \( 0 \leq t \leq a_1 \) is less than \( C \epsilon \). Therefore, the Gromov-Hausdorff distance between \((\mathbb{S}^3 \times \mathbb{S}^3, \tilde{g}_\epsilon)\) and \((\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon)\) is less than \( C \epsilon \).

Next, we show that \( g_\epsilon \to g_0 \) as \( \epsilon \to 0 \) in the \( C^0 \) norm in the metric \( g_1 \) on \( \mathbb{S}^3 \times \mathbb{S}^1 \). In fact, by (2.7)

\[
g_\epsilon - g_0 = \epsilon^2 \left( -\frac{\pi^2}{2} + \frac{i \pi}{A} - \frac{i \pi w_1}{2 A} \frac{i \pi w_1}{2 A} \right).
\]

Direct computation exhibits

\[
\begin{align*}
(g_{\epsilon,0j} - g_{0,0j}) g_0^{kj} &= \epsilon^2 \delta_{k0}, \\
(g_{\epsilon,1j} - g_{0,1j}) g_0^{ij} &= -\epsilon^2 \frac{i}{2} \frac{w_1}{A}, \\
(g_{\epsilon,1j} - g_{0,1j}) g_0^{ij} &= 0,
\end{align*}
\]

where

\[
g_0^{-1} = \frac{1}{\pi^2} \begin{pmatrix}
\frac{\delta}{A} + \frac{3}{4} - \frac{i \epsilon A w_1}{2 A^2} \\
\frac{i \epsilon A w_1}{2 A^2} & \pi^2 A^2
\end{pmatrix}.
\]

Hence

\[
|g_\epsilon - g_0|^2_{g_0} = g_0^{ij} g_0^{kl} (g_{\epsilon,ij} - g_{0,ij}) (g_{\epsilon,kj} - g_{0,kj}) = g_0^{ij} g_0^{kl} (g_{\epsilon,ij} - g_{0,ij}) (g_{\epsilon,0j} - g_{0,0j}) = \epsilon^4 \to 0, \text{ as } \epsilon \to 0.
\]

So \((\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon)\) converge to \((\mathbb{S}^3 \times \mathbb{S}^1, g_0)\) in the Gromov-Hausdorff distance.

Finally, by the triangle inequality,

\[
d_{GH}(\mathbb{S}^3 \times \mathbb{S}^3, \tilde{g}_\epsilon, (\mathbb{S}^3 \times \mathbb{S}^3, g_\epsilon)) \\
\leq d_{GH}(\mathbb{S}^3 \times \mathbb{S}^3, \tilde{g}_\epsilon, (\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon)) + d_{GH}(\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon, (\mathbb{S}^3 \times \mathbb{S}^1, g_0)) \\
\leq C \epsilon
\]

where \( d_{GH} \) denotes the Gromov-Hausdorff distance. We conclude that \((\mathbb{S}^3 \times \mathbb{S}^3, g_\epsilon)\) converge to \((\mathbb{S}^3 \times \mathbb{S}^1, g_0)\) in the Gromov-Hausdorff topology as \( \epsilon \to 0^+ \). \qed
2.4. Cohomology classes of the fundamental forms $\omega_e$. The Aeppli cohomology on a compact complex manifold of complex dimension $n$ is defined by

$$H^{p,q}_A = \frac{\langle \text{Ker} \ i \partial \bar{\partial} \rangle \cap \Omega^{p,q}}{(\text{Im} \ \partial + \text{Im} \ \bar{\partial}) \cap \Omega^{p,q}}$$

where $\Omega^{p,q}$ is the space of $(p, q)$ forms. The Hodge $*$ operator associated to a Hermitian metric is isomorphic from the Bott-Chern cohomology $H^{p,n-p}_{BC}$ to $H^{p,q}_A$ (cf. [2], Theorem 2.5). On a complex manifold with a Hermitian metric $g$, the Chern Ricci curvature (of first type) is

$$\text{Ric}(g) := -i \partial \bar{\partial} \log \det g$$

and the Chern scalar curvature is

$$R(g) := g^{ij} \text{Ric}_{ij}.$$ 

Both $g$ and the complex structure $J$ on $\mathbb{S}^3 \times \mathbb{S}^1$ are parallel with respect to the Chern connection. More information can be found, for example, in [21].

A Hermitian metric on a complex $n$-dimensional manifold is called a Gauduchon metric if its fundamental class $\omega$ satisfies $i \partial \bar{\partial} (\omega^{n-1}) = 0$. We have

**Proposition 2.4.** Let $\omega_e = i g_{e,ij} dz_i \wedge d\bar{z}_j$ be the Kähler form associated to $g_e$. Then

1. $\omega_e$ is a Gauduchon metric, i.e., $i \partial \bar{\partial} \omega_e = 0$, and represents an element in $H^{1,1}_A$.
2. $\text{Ric}(g_e) = 2i A^{-2} d\omega_1 \wedge d\bar{\omega}_1$ and $R(g_e) = 2$.
3. $[\omega_e] \neq 0$ in $H^{1,1}_A$ and $[\omega_1] = \left[ \frac{2}{1+\epsilon} \omega_e \right]$. In particular, $[\omega_{e_1}] \neq [\omega_{e_2}]$ if $e_1 \neq e_2$.

**Proof.** (1) From (2.6), when $(i_0, j_0) = (0, 0)$ or $(1, 1)$, it is obvious that

$$i \partial \bar{\partial} (g_{e,ij_0} dw_{i_0} \wedge d\bar{w}_{j_0}) = 0,$$

when $(i_0, j_0) = (0, 1)$, we have

$$i \partial \bar{\partial} (g_{e,01} dw_0 \wedge d\bar{w}_1) = \partial_1 \bar{\partial} \left( \frac{(1+\epsilon^2) i \pi w_1}{2 A} dw_0 \wedge d\bar{w}_1 \right) = 0$$

as $g_{e,01}$ only depends on $w_1, \bar{w}_1$; similarly,

$$i \partial \bar{\partial} (g_{e,10} dw_1 \wedge d\bar{w}_0) = 0.$$

(2) From (2.7) direct computation leads to

$$\text{det}(g_e) = (1 + \epsilon^2) \pi^2 A^{-2},$$

$$\text{Ric}(g_e) = -i \partial \bar{\partial} \log \text{det}(g_e) = 2 i A^{-2} d\omega_1 \wedge d\bar{\omega}_1.$$

By (2.7), the inverse matrix is

$$g^{11}_e = \frac{A^2}{(1+\epsilon^2) \pi^2} \left( \frac{1}{A^2} + \frac{1}{4A^2} + \frac{1}{4A^2} \left( 1 + \frac{1+\epsilon^2}{\pi w_1} \right) \frac{1}{A} \right) = \frac{g_{e,11}}{2}.$$

Hence, the Chern scalar curvature is constant for the family of metrics $g_e$:

$$R(g_e) = g^{ij}_e \text{Ric}(g_{e,ij}) = 2.$$
(3) The global (1,1) form
\[ \omega_1 - \frac{2}{1 + e^2} \omega_e = \frac{e^2 - 1}{1 + e^2} i A^{-2} dw_1 \wedge \overline{w}_1 = \frac{e^2 - 1}{2(1 + e^2)} \text{Ric}(\omega_1) \]
is $d$-closed; hence, it is $d$-exact since $d = \partial + \overline{\partial}$ and the cohomology group $H^2(\mathbb{S}^3 \times \mathbb{S}^1) = 0$. So $[\omega_1] = \frac{2}{1 + e^2}[\omega_e]$ in $H^1_A$.

By [13, Proposition 37], $[\omega_1] \neq 0$ in $H^1_A$. In fact, if the real (1, 1) form $\omega_e$ were in $Im \partial + Im \overline{\partial}$ then it could be expressed as $\partial \alpha + \overline{\partial} \beta$ for some $\alpha \in \Omega^{0,1}, \beta \in \Omega^{1,0}$. Then taking $S = \alpha + \beta$ and applying Stokes’ theorem
\[
0 = \int_{\mathbb{S}^3 \times \mathbb{S}^1} d(S \wedge \overline{dS}) = \int_{\mathbb{S}^3 \times \mathbb{S}^1} dS \wedge \overline{dS} = \int_{\mathbb{S}^3 \times \mathbb{S}^1} \partial \omega_e \wedge \overline{\omega}_e + \int_{\mathbb{S}^3 \times \mathbb{S}^1} dS^{0,2} \wedge \overline{dS}^{0,2} + \int_{\mathbb{S}^3 \times \mathbb{S}^1} dS^{2,0} \wedge \overline{dS}^{2,0}.
\]
Since all three integrals are non-negative, and $\omega_e \wedge \overline{\omega}_e \geq 0$, we must have $\omega_e \wedge \overline{\omega}_e \equiv 0$, but this contradicts that $\omega_e$ is the fundamental class. It follows that $[\omega_e] - [\omega_{e_1}] = [\frac{e^2 - e^2}{2} \omega_1] \neq 0$ in $H^1_A$, whenever $e_2 \neq e_1$.

We say a Hermitian metric $g$ is a locally conformally Kähler metric on a complex manifold $M$ if for each point $x \in M$ there exists an open neighbourhood $U$ of $x$ and a function $f$ on $U$ so that $e^{-f} g$ is Kähler. It is equivalent to that the Lee form, which is a real 1-form defined by
\[
(2.12) \quad d\omega = -2\theta \wedge \omega,
\]
is closed [13]. We have

**Theorem 2.5.** Each metric $g_\epsilon$ is locally conformal Kähler.

**Proof.** By [13], it suffices to prove that the Lee form $\theta$ is closed. Let $\omega_e$ be the Kähler form corresponding to $g_\epsilon$ and $\theta_e$ be its Lee form. Then
\[
\omega_e = (1 + e^2) \pi^2 i dw_0 \wedge \overline{dw}_0 + \frac{(1 + e^2) i \pi w_1}{2A} i dw_0 \wedge \overline{dw}_1 - \frac{(1 + e^2) i \pi w_1}{2A} i dw_1 \wedge \overline{dw}_0 + \left( \frac{1}{4A} + \frac{3}{4A^2} + \frac{e^2 |w_1|^2}{4A^2} \right) i dw_1 \wedge \overline{dw}_1.
\]
Hence, we have
\[
(2.13) \quad d\omega_e = i \frac{(1 + e^2) \pi}{2} \left( \frac{1}{A} - \frac{|w_1|^2}{A^2} \right) dw_1 \wedge dw_0 \wedge \overline{dw}_1 - i \frac{(1 + e^2) \pi}{2} \left( \frac{1}{A} - \frac{|w_1|^2}{A^2} \right) \overline{dw}_1 \wedge dw_1 \wedge \overline{dw}_0
\]
\[
= i \frac{(1 + e^2)}{2} \pi \left( \frac{1}{A} - \frac{|w_1|^2}{A^2} \right) dw_0 \wedge dw_1 \wedge \overline{dw}_1 + i \frac{(1 + e^2)}{2} \pi \frac{1}{A^2} \overline{dw}_0 \wedge dw_1 \wedge \overline{dw}_1.
\]
Assume
\[
\theta_e = a dw_0 + \overline{a} \overline{dw}_0 + b dw_1 + \overline{b} \overline{dw}_1.
\]
We have
\[
\theta_e \wedge \omega_e = -a \left( \frac{(1 + e^2) i \pi \overline{w}_1}{2} i dw_0 \wedge dw_1 \wedge d\overline{w}_0 + a \left( \frac{A + 3 + e^2 |w_1|^2}{4A^2} \right) i dw_0 \wedge dw_1 \wedge d\overline{w}_1 \right.
\]
\[
+ \overline{a} \left( \frac{(1 + e^2) i \pi w_1}{2} i d\overline{w}_0 \wedge dw_0 \wedge d\overline{w}_1 + \overline{a} \left( \frac{A + 3 + e^2 |w_1|^2}{4A^2} \right) i d\overline{w}_0 \wedge dw_1 \wedge d\overline{w}_1 \right)
\]
\[
+ b (1 + e^2) \pi^2 i dw_1 \wedge dw_0 \wedge d\overline{w}_0 + b \left( \frac{(1 + e^2) i \pi w_1}{2} \right) i dw_1 \wedge dw_0 \wedge d\overline{w}_1
\]
\[
+ \overline{b} (1 + e^2) \pi^2 i d\overline{w}_1 \wedge dw_0 \wedge d\overline{w}_0 - \overline{b} \left( \frac{(1 + e^2) i \pi \overline{w}_1}{2} \right) i d\overline{w}_1 \wedge dw_1 \wedge d\overline{w}_0.
\]
It follows
\[
\theta_e \wedge \omega_e = \left( a \left( \frac{(1 + e^2) i \pi \overline{w}_1}{2} \right) i dw_0 \wedge d\overline{w}_0 \wedge dw_1 \right.
\]
\[
+ \left( -\overline{a} \left( \frac{(1 + e^2) i \pi w_1}{2} \right) i d\overline{w}_0 \wedge dw_0 \wedge d\overline{w}_1 \right)
\]
\[
+ \left( \frac{A + 3 + e^2 |w_1|^2}{4A^2} \right) i dw_0 \wedge dw_1 \wedge d\overline{w}_1
\]
\[
+ \left( \frac{A + 3 + e^2 |w_1|^2}{4A^2} \right) i d\overline{w}_0 \wedge dw_1 \wedge d\overline{w}_1.
\]
By (2.12) and (2.13), we have
\[
a \left( \frac{(1 + e^2) i \pi \overline{w}_1}{2} \right) + b (1 + e^2) \pi^2 = 0,
\]
\[
a \left( \frac{A + 3 + e^2 |w_1|^2}{4A^2} \right) - b \left( \frac{(1 + e^2) i \pi w_1}{2} \right) = \frac{(1 + e^2) i \pi}{4} \frac{1}{A^2}.
\]
Then
\[
a = (1 + e^2) \frac{i \pi}{4}, \quad b = (1 + e^2) \frac{\overline{w}_1}{8A}.
\]
We obtain
\[
(2.14) \quad \theta_e = (1 + e^2) \left( \frac{i \pi}{4} dw_0 - \frac{i \pi}{4} d\overline{w}_0 + \frac{w_1}{8A} dw_1 + \frac{\overline{w}_1}{8A} d\overline{w}_1 \right).
\]
Therefore
\[
d\theta_e = \frac{1}{8} (1 + e^2) \left( \left( \frac{1}{A} - \frac{|w_1|^2}{A^2} \right) d\overline{w}_1 \wedge dw_1 + \left( \frac{1}{A} - \frac{|w_1|^2}{A^2} \right) dw_1 \wedge d\overline{w}_1 \right) = 0.
\]
\[
\square
\]
Now, we demonstrate that the metrics \(g_x\) are different from those in [13].

First we recall the construction of the Hermitian metrics in [13] and adopt the notations therein. The Hopf surface \(M_{a_1, a_2}\) is \((\mathbb{C}^2 \setminus \{(0, 0)\})/\sim\), where
\[
(u, v) \sim (a_1 u + A \nu^m, a_2 v), \quad (u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}
\]
and $\alpha_1, \alpha_2, \lambda$ are complex numbers and $m$ is a nonnegative integer such that
\[|\alpha_1| \geq |\alpha_2| > 1\]
and
\[(\alpha_1 - \alpha_2^m)\lambda = 0.\]

Let $\Phi_{\alpha_1, \alpha_2}$ be a function that satisfies
\[|u|^{-2} \Phi_{\alpha_1, \alpha_2}^{-2k_1} + |v|^{-2} \Phi_{\alpha_1, \alpha_2}^{-2k_2} = 1,\]
where $k_i = \ln |\alpha_i|, i = 1, 2$. It is shown that $\frac{1}{\Phi_{\alpha_1, \alpha_2}} \partial \bar{\partial} \Phi_{\alpha_1, \alpha_2}$ are well defined and locally conformally Kähler metrics on $M_{\alpha_1, \alpha_2}$ with parallel Lee form.

However, for our $g_{\epsilon}$, it holds

**Proposition 2.6.** The Lee form $\theta_{\epsilon}$ is not parallel.

**Proof.** Assume $w_0 = x_0 + i x_2, w_1 = x_1 + i x_3$. By (2.14), we have
\[\theta_{\epsilon} = (1 + \epsilon^2) \left( -\frac{\pi}{2} dx_2 + \frac{1}{4A} (x_1 dx_1 + x_3 dx_3) \right).\]

By (3.8), $g_{ij}$ are independent of $x_0, x_2$ and $g_{ij}, i, j = 0, 2$ are constant, we have
\begin{align*}
\nabla \frac{\theta_{\epsilon}}{w_0} &= -(1 + \epsilon^2) \sum_{p=0}^{3} \left( \frac{x_1}{4A \rho_0} + \frac{x_3}{4A \rho_0} \right) dx_p \\
&= -(1 + \epsilon^2) \frac{1}{8A} \sum_{p,q=0}^{3} \left( x_1 g_{1q} (g_{0q,p} - g_{0p,q}) + x_3 g_{3q} (g_{0q,p} - g_{0p,q}) \right) dx_p \\
&= -(1 + \epsilon^2) \frac{1}{8A} \sum_{p,q=0}^{3} \left( x_1 g_{1q} + x_3 g_{3q} \right) (g_{0q,p} - g_{0p,q}) dx_p \\
&= -(1 + \epsilon^2) \frac{1}{8A} \sum_{q=1}^{3} \left( x_1 g_{1q} + x_3 g_{3q} \right) (g_{0q,1} - g_{01,q}) dx_1 \\
&= -(1 + \epsilon^2) \frac{1}{8A} \sum_{q=1}^{3} \left( x_1 g_{1q} + x_3 g_{3q} \right) (g_{0q,3} - g_{03,q}) dx_3.
\end{align*}

By (3.8) and $g_{13} = 0$ given by (3.4), we have
\[\sum_{q=1}^{3} \left( x_1 g_{1q} + x_3 g_{3q} \right) (g_{0q,1} - g_{01,q}) = (x_1 g_{13} + x_3 g_{33}) (g_{01,1} - g_{03,1}) = x_3 g_{33} \left( 1 + \frac{\epsilon^2}{4} \frac{2\pi}{A^2} (x_3^2 - x_1^2) \right).
\]

Then $\nabla \frac{\theta_{\epsilon}}{w_0} \neq 0$, as required. \hfill \Box

**Remark 2.7.** First, by [13] Theorem 1, the Lee form of $\omega_{\alpha_1, \alpha_2} = \frac{1}{\Phi_{\alpha_1, \alpha_2}} \partial \bar{\partial} \Phi_{\alpha_1, \alpha_2}$ is parallel. Therefore, there is no diffeomorphism pulling back the metric $\omega_{\alpha_1, \alpha_2}$ to $\omega_{\epsilon}$. Second, if $\omega_{\alpha_1, \alpha_2}$ is a Gauduchon metric, then its conformal class must be
different from that of our metric \( \omega_\epsilon \). In fact, by Proposition 2.4 \( \omega_\epsilon \) is Gauduchon. However, the Gauduchon metric is unique up to a constant scaling in a conformal class (by [12] or [23] Theorem 1.2.4).

3. Stability of the toric fibres as harmonic map and minimal surface

Continue to let \( \pi : \mathbb{S}^3 \times \mathbb{S}^1 \to \mathbb{C}P^1 \) be the Hopf fibration. For any point \( p \) in the base \( \mathbb{C}P^1 \), denote \( \mathbb{T}_p^2 = \pi^{-1}(p) \) and \( p_0, p_1 \) for the two special tori defined in the introduction. As the fibre \( \mathbb{T}_p^2 \) is a complex submanifold, let \( c_p \) be the conformal structure on it such that the inclusion mapping \( f_p : \mathbb{T}_p^2 \to \mathbb{S}^3 \times \mathbb{S}^1 \) is holomorphic with respect to \( c_p \). It is well known that harmonicity of a mapping from a Riemann surface \( \Sigma \) only depends on the conformal class \([c]\) of \( \Sigma \), not on specific metrics within \([c]\). This leads to further simplification of (3.10), when the domain is a Riemann surface, since we can use the isothermal coordinates associated with \([c]\).

3.1. Curvature of \((\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon)\). We have constructed a family of Hermitian metrics \( g_\epsilon \) for \( \epsilon \geq 0 \) on \( \mathbb{S}^3 \times \mathbb{S}^1 \). We will be interested in the isotropic curvatures of the isotropic 2-planes spanned by \( T_1^1,0 \)-vectors. The observation is that each of these curvatures is nonnegative and vanish exactly on the two special tori \( \mathbb{T}_p^2, \mathbb{T}_p^2 \). The properties of the Riemannian curvature tensor contained in Lemma 3.1 are crucial for our stability analysis in Section 4.

Let us first consider a local picture. In \( V_{10} \), the inclusion map \( \mathbb{S}^3 \times \mathbb{S}^1 \hookrightarrow \mathbb{C} \times \mathbb{C} \) can be written as

\[
\begin{aligned}
&z_0 = A^{-\frac{1}{2}} e^{i \pi (w_0 + \overline{w}_0)}, \\
&z_1 = z_0 w_1 = w_1 A^{-\frac{1}{2}} e^{i \pi (w_0 + \overline{w}_0)}, \\
&z_1' = e^{i \pi (w_0 - \overline{w}_0)^0} \log A
\end{aligned}
\]

with \( A = 1 + |w_1|^2 \). When \( w_1 = 0 \), we get \((z_0, z_1, z_1') = (e^{i \pi (w_0 + \overline{w}_0)}, 0, e^{i \pi (w_0 - \overline{w}_0)})\) tracing a square torus \( \mathbb{S}^1 \times \{0\} \times \mathbb{S}^1 \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C} \). Similarly, in \( V_{10} \) there is another square torus \( \{0\} \times \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C} \). In fact, these are the only tori in the holomorphic toric fibration \( \pi \) as a direct product of a great circle in \( \mathbb{S}^3 \) with \( \mathbb{S}^1 \). It turns out that the isotropic curvature vanishes on and only on these two square tori in \( \mathbb{S}^3 \times \mathbb{S}^1 \subset \mathbb{C}^2 \times \mathbb{C} \).

Assume \( w_0 = x_0 + i x_2, w_1 = x_1 + i x_3 \). Set \( R_{ijkl} = R \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right) \), where

\[
R_{ijkl} = -g \left( \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_l} - \nabla_{\frac{\partial}{\partial x_l}} \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_l}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} \right)
= -\frac{1}{2} \left( g_{ik, jl} + g_{jl, ik} - g_{jk, il} - g_{il, jk} \right) - \Gamma^p_{ik} \Gamma^q_{jl} g_{pq} + \Gamma^p_{ik} \Gamma^q_{jl} g_{pq}.
\]

First we prove
Lemma 3.1.

\[ R_{0101} = \Gamma^p_{10} \Gamma^q_{10} g_{pq} = \frac{1}{4} g^{33} (g_{031} - g_{013})^2, \]
\[ R_{0303} = \Gamma^p_{30} \Gamma^q_{30} g_{pq} = \frac{1}{4} g^{11} (g_{301} - g_{103})^2, \]
\[ R_{2121} = \Gamma^p_{12} \Gamma^q_{12} g_{pq} = \frac{1}{4} g^{33} (g_{213} - g_{321})^2, \]
\[ R_{2323} = \Gamma^p_{32} \Gamma^q_{32} g_{pq} = \frac{1}{4} g^{11} (g_{213} - g_{321})^2, \]
\[ R_{0103} = \Gamma^p_{01} \Gamma^q_{03} g_{pq} = 0, \]
\[ R_{2123} = \Gamma^p_{21} \Gamma^q_{23} g_{pq} = 0 \]

and

\[ R_{0213} = 0, R_{0321} = 0, R_{0123} = 0. \]

**Proof:** The underlying Riemannian metric of the Hermitian metric \( g_x \) is given locally in \( U_0 \) by the \( 4 \times 4 \) real matrix (now \( 0 \leq i, j \leq 3 \))

\[
(g_{ij}) = \frac{1}{2} \begin{pmatrix}
(1 + \epsilon^2)\pi^2 & -\frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & 0 & -\frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} \\
-\frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} & \frac{1}{4A} + \frac{3}{4A^2} + \frac{\epsilon^2 |w_j|^2}{4A^2} & \frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & 0 \\
0 & \frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & (1 + \epsilon^2)\pi^2 & -\frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} \\
-\frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & 0 & -\frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} & \frac{1}{4A} + \frac{3}{4A^2} + \frac{\epsilon^2 |w_j|^2}{4A^2}
\end{pmatrix}.
\]

We also have

\[
(g_{ij})^{-1} = \frac{2A^2}{(1 + \epsilon^2)\pi^2} \begin{pmatrix}
\frac{1}{4A} + \frac{3}{4A^2} + \frac{\epsilon^2 |w_j|^2}{4A^2} & \frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} & 0 & \frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} \\
\frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & \frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} & -\frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & 0 \\
0 & -\frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & \frac{1}{4A} + \frac{3}{4A^2} + \frac{\epsilon^2 |w_j|^2}{4A^2} & \frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} \\
\frac{1+\epsilon^2}{2} \pi \frac{x_i}{A} & 0 & \frac{1+\epsilon^2}{2} \pi \frac{x_j}{A} & (1 + \epsilon^2)\pi^2
\end{pmatrix}.
\]

Now, we compute the five curvature terms in (3.7). First, we calculate \( R_{0101} \). Evidently the entries \( g_{ij} \) are independent of \( x_0 \) and \( x_2 \). Therefore

\[
\Gamma^i_{00} = \frac{1}{2} g^{il} (g_{0l,0} + g_{0l,0} - g_{00,l}) = \frac{1}{2} g^{il} g_{00,l} = 0, \\
\Gamma^i_{02} = \frac{1}{2} g^{il} (g_{0l,2} + g_{0l,0} - g_{02,l}) = \frac{1}{2} g^{il} g_{02,l} = 0, \\
\Gamma^i_{22} = \frac{1}{2} g^{il} (2g_{2l,2} - g_{22,l}) = -\frac{1}{2} g^{il} g_{22,l} = 0.
\]
Then, as $g_{pq}$ are constant for $p, q = 0, 2$ and $g_{ij,0}, g_{ij;2} = 0$, we have

$$R_{0101} = -\frac{1}{2} (g_{00,11} + g_{11,00} - g_{10,10} - g_{10,01}) - \Gamma_{00}^p \Gamma_{11}^q g_{pq} + \Gamma_{01}^p \Gamma_{10}^q g_{pq}$$

$$= \Gamma_{10}^p \Gamma_{10}^q g_{pq}$$

$$= \frac{1}{4} \sum_{p,q=0,1,2,3} g_{pq} (g_{1p,0} + g_{0p,1} - g_{01,p})(g_{1q,0} + g_{0q,1} - g_{01,q})$$

$$= \frac{1}{4} \sum_{p,q=1,3} g_{pq} (g_{0p,1} - g_{01,p})(g_{0q,1} - g_{01,q})$$

$$= \frac{1}{4} g^{33} (g_{03,1} - g_{01,3})^2.$$

By the symmetry of the indices, we have

$$R_{0303} = \Gamma_{30}^p \Gamma_{30}^q g_{pq} = \frac{1}{4} g^{11} (g_{30,1} - g_{10,3})^2,$$

$$R_{2121} = \Gamma_{12}^p \Gamma_{12}^q g_{pq} = \frac{1}{4} g^{33} (g_{21,3} - g_{32,1})^2,$$

$$R_{2323} = \Gamma_{32}^p \Gamma_{32}^q g_{pq} = \frac{1}{4} g^{11} (g_{23,1} - g_{32,1})^2.$$

In addition, by $g^{13} = 0$, we get

$$R_{0213} = -\Gamma_{01}^p \Gamma_{23}^q g_{pq} + \Gamma_{12}^p \Gamma_{03}^q g_{pq}$$

$$= \frac{1}{4} g^{ps} (g_{0s,1} + g_{1s,0} - g_{01,s})g_{pq}$$

$$= \frac{1}{4} g^{ps} (g_{1s,2} + g_{2s,1} - g_{12,s})g_{pq}$$

$$= \frac{1}{4} g^{ps} (g_{0s,1} - g_{01,s})(g_{2p,3} - g_{23,p}) + \frac{1}{4} g^{ps} (g_{2s,1} - g_{12,s})(g_{0p,3} - g_{03,p})$$

$$= \frac{1}{4} g^{13} (g_{03,1} - g_{01,3})(g_{21,3} - g_{23,1}) + \frac{1}{4} \sum_{s=1,3} g^{1s} (g_{2s,1} - g_{12,s})(g_{01,3} - g_{03,1})$$

$$= \frac{1}{4} g^{11} (g_{21,1} - g_{12,1})(g_{01,3} - g_{03,1})$$

$$= 0.$$

Similarly, we obtain $R_{0103} = 0, R_{2123} = 0, R_{0321} = 0, R_{0123} = 0$. □

We also have:

**Proposition 3.2.** For any linearly independent $X, Y \in T_p^{1,0}(S^3 \times S^1)$, $R(X, Y, \bar{X}, \bar{Y}) \geq 0$ where $R$ is the Riemannian curvature operator and $p \in S^3 \times S^1$, it equals to zero at $p \in S^3 \times S^1$ if and only if $p \in \left(S^3 \cap \{z_0 = 0\} \cup \{z_1 = 0\}\right) \times S^1$.

**Proof.** Here and below we omit $\epsilon$ for simplicity of expressions. We first confine to the complex coordinate chart $(U_{00}, (w_0, w_1))$. 
Assume \( X = a \frac{\partial}{\partial w_0} + b \frac{\partial}{\partial w_1} \) and \( Y = c \frac{\partial}{\partial w_0} + d \frac{\partial}{\partial w_1} \), for some complex numbers \( a, b, c, d \) with \( ab - cd \neq 0 \). Then
\[
R \left( X, Y, \bar{X}, \bar{Y} \right) = |ad - bc|^2 R \left( \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1} \right).
\]

It suffices to prove that \( R \left( \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1} \right) \geq 0 \).

Note that
\[
(3.7)
\]
\[
R \left( \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1} \right) = R \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1} \right) + R \left( \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) - 2R \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3} \right)
\]
where \( w_0 = x_0 + i x_2, w_1 = x_1 + i x_3 \).

To locate the zero locus of the curvature, we observe
\[
(3.8)
\]
\[
g_{30,1} = \frac{1 + \epsilon^2}{4} \left( \frac{\pi}{A} + \frac{2\pi}{A^2} x_1^2 \right),
\]
\[
g_{10,3} = \frac{1 + \epsilon^2}{4} \left( \frac{\pi}{A} + \frac{2\pi}{A^2} x_3^2 \right),
\]
\[
g_{21,3} = -g_{32,1} = -\frac{1 + \epsilon^2}{2} \frac{\pi}{A^2} x_1 x_3.
\]

Hence, by Lemma 3.3,
\[
(3.9)
\]
\[
R \left( \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1} \right) = \frac{1}{2} g_{11} \left( \left( g_{30,1} - g_{01,3} \right)^2 + \left( g_{21,3} - g_{32,1} \right)^2 \right) \geq 0.
\]

It follows \( R \left( \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_0}, \frac{\partial}{\partial w_1} \right) = 0 \) if and only if
\[
x_1^2 + x_3^2 = 0
\]
that is \( w_1 = 0 \) hence \( z_1 = 0 \). Similarly, we have \( z_0 = 0 \) on \( V_{10} \).

Therefore, the curvature vanishes on and only on \( S^3 \cap \{ z_1 = 0 \} \times S^1 \) and \( S^3 \cap \{ z_0 = 0 \} \times S^1 \). Both are embedded complex tori in \( S^3 \times S^1 \). \( \square \)

Theorem 1.1 follows from results established in this section.

3.2. Stability of harmonic maps from \( \mathbb{T}^2 \) to the Hermitian Hopf surfaces. Let \( f : M \to N \) be a smooth mapping from a Kähler manifold \( M \) with metric tensor \( h_{\alpha\beta} dz^\alpha d\bar{z}^\beta \) to a Riemannian manifold \( N \) with metric tensor \( g_{ij} dy^i dy^j \). Using the complex coordinates on \( M \), Sampson wrote the harmonic map equation \([27, p.129]\) as
\[
(3.10)
\]
\[
h_{\alpha\beta} \left( \frac{\partial^2 f^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma^i_{jk} \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial \bar{z}^\beta} \right) = 0
\]
where $\Gamma^i_{jk}$ are the Christoffel symbols of the Levi-Civita connection of $(N, g)$. This is convenient for verification of harmonicity of the holomorphic maps from tori in Theorem 1.2.

**Theorem 3.3.** Each $f_p$ is a harmonic map from $(\mathbb{P}^2, c_p)$ to $(\mathbb{S}^3 \times \mathbb{S}^1, g_e)$. There exist neighbourhoods of $p_0, p_1$ respectively such that $f_p$ is a stable harmonic map when $p$ belongs to the neighbourhoods.

**Proof:** First, we show that $f_p$ is harmonic for any fixed $p$. It suffices to verify the harmonic map equation locally. In the complex coordinate chart $(U_{00}; w_0, w_1)$, the inclusion map may be written as $f_p(w_0, w_1)$. Now $w_0$ is a complex coordinate on $\mathbb{T}^2_p$ for the conformal structure $c_p$, the harmonic map equation (3.10) is

$$\frac{\partial^2 f_p}{\partial w_0 \partial \bar{w}_0} + \Gamma^i_{jk} \frac{\partial f_p}{\partial w_0} \frac{\partial f_p}{\partial \bar{w}_0} = 0$$

where the indices $0 \leq i, j, k \leq 3$ arise from $w_0 = x_0 + i x_2, w_1 = x_1 + i x_3$. Since $f_p$ is holomorphic, the functions $f^i_j$ are harmonic for any metric in the conformal class of $c_p$, hence the first term above vanishes. To see the second term also vanishes, since $f^i_0, f^i_3$ are constant it suffices to show $\Gamma^i_{00}, \Gamma^i_{02}, \Gamma^i_{22} = 0$. Therefore, by (3.5), $f_p$ is harmonic.

Next, we examine the stability of these harmonic tori. Let $E_p = f_p^* T(\mathbb{S}^3 \times \mathbb{S}^1)$ be the pullback bundle over $\mathbb{T}^2_p$. Let $\nabla$ denote the pullback Riemannian connection of $g_e$. For convenience, we denote $g_e$ by $g$. The second derivative of energy at a critical point $f_p$ along a variation field $V$ is given (cf. [24], (2.1), [28]) by the index form

$$I(V, V) = \int_{\mathbb{T}^2_p} \left\{ \left| \nabla \frac{\partial f}{\partial x_0} V \right|^2 + \left| \nabla \frac{\partial f}{\partial x_2} V \right|^2 - R \left( \frac{\partial f}{\partial x_0}, V, \frac{\partial f}{\partial x_0}, V \right) - R \left( \frac{\partial f}{\partial x_2}, V, \frac{\partial f}{\partial x_2}, V \right) \right\} d x_0 d x_2$$

where the Riemannian metric on $\mathbb{T}^2_p$ is compatible with the conformal structure $c_p$. Here we regard $\frac{\partial f}{\partial x_0}$ as a section of $T(\mathbb{S}^3 \times \mathbb{S}^1)$ defined by $\frac{\partial f}{\partial x_0}(p) = f_*(\frac{\partial}{\partial x_0}|_p)$. For convenience, we use $\mathbb{T}^2$ to denote $\mathbb{T}^2_p$.

Since the tangent bundle of $\mathbb{T}^2$ is trivial, $\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_2}$ are global vector fields on $\mathbb{T}^2$. In addition, when $p$ is in a neighbourhood of $p_0$, $f_p(\mathbb{T}^2) \subset V_{00}$. Note that the tangent bundle of $V_{00}$ is trivial, we have $E_p$ is a trivial vector bundle on $\mathbb{T}^2$ and $\frac{\partial}{\partial x_i}, i = 0, 1, 2, 3$ are global sections of $E_p$.

For any smooth section $V$ of $E_p$, we may write $V = \sum_{i=0}^3 a_i \frac{\partial}{\partial x_i}$ for some functions $a_i \in C^\infty(\mathbb{T}^2)$. Associated to $V$, let

$$V_1 = \sum_i a_i \nabla \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_i}, \quad V_2 = \sum_i \frac{\partial a_i}{\partial x_0} \frac{\partial}{\partial x_i},$$

$$V_3 = \sum_i a_i \nabla \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_i}, \quad V_4 = \sum_i \frac{\partial a_i}{\partial x_2} \frac{\partial}{\partial x_i}.$$
Then we have
\begin{equation}
\nabla \frac{\partial}{\partial x_0} V = V_1 + V_2, \quad \nabla \frac{\partial}{\partial x_2} V = V_3 + V_4.
\end{equation}

As \( \frac{\partial f_p}{\partial x_0} = \frac{\partial}{\partial x_0} \) for the inclusion \( f_p \), by expansion,
\begin{equation}
R \left( \frac{\partial f_p}{\partial x_0}, V, \frac{\partial f_p}{\partial x_0}, V \right) = \sum_{i,j=0}^{3} a_i a_j R_{0i0j}.
\end{equation}

We will show that for \( i, j = 0, 2 \),
\begin{equation}
R_{000j} = 0, \quad R_{002j} = 0, \quad R_{222j} = 0.
\end{equation}

Reasoning similarly as in the proof of Lemma 3.1,
\begin{align*}
R_{000j} &= -\frac{1}{2} \left( g_{00,ij} + g_{ij,00} - g_{00,j0} - g_{00,0i} \right) - \Gamma_{00}^{p} \Gamma_{ij}^{q} g_{pq} + \Gamma_{00}^{p} \Gamma_{j0}^{q} g_{pq} \\
&= \frac{1}{4} \epsilon^{ij} \left( g_{0i,j} - g_{0j,i} \right) \left( g_{00,j} - g_{00,i} \right) = 0
\end{align*}

and
\begin{align*}
R_{002j} &= \frac{1}{2} \left( g_{02,ij} + g_{ij,02} - g_{02,j0} - g_{02,0i} \right) - \Gamma_{02}^{p} \Gamma_{ij}^{q} g_{pq} + \Gamma_{02}^{p} \Gamma_{j0}^{q} g_{pq} \\
&= \frac{1}{4} \epsilon^{ij} \left( g_{0i,j} - g_{0j,i} \right) \left( g_{00,j} - g_{00,i} \right) = 0.
\end{align*}

Similarly, \( R_{222j} = 0 \). It then follows from (3.2) and (3.12) that
\begin{equation}
R \left( \frac{\partial f}{\partial x_0}, V, \frac{\partial f}{\partial x_0}, V \right) = a_1^2 R_{0101} + a_3^2 R_{0303} + 2a_1 a_3 R_{0103} = a_1^2 \Gamma_{01}^{p} \Gamma_{01}^{q} g_{pq} + a_3^2 \Gamma_{03}^{p} \Gamma_{03}^{q} g_{pq} + 2a_1 a_3 \Gamma_{01}^{p} \Gamma_{03}^{q} g_{pq}.
\end{equation}

On the other hand, by (3.5), we have
\begin{equation}
V_1 = a_i \nabla \frac{\partial}{\partial x_0} = \left( a_1 \Gamma_{01}^{p} + a_3 \Gamma_{03}^{p} \right) \frac{\partial}{\partial x_p}.
\end{equation}

By (3.14), we obtain
\begin{equation}
\left| V_1 \right|_g^2 = a_1^2 \Gamma_{01}^{p} \Gamma_{01}^{q} g_{pq} + a_3^2 \Gamma_{03}^{p} \Gamma_{03}^{q} g_{pq} + 2a_1 a_3 \Gamma_{01}^{p} \Gamma_{03}^{q} g_{pq} = R \left( \frac{\partial f}{\partial x_0}, V, \frac{\partial f}{\partial x_0}, V \right).
\end{equation}

By (3.11), we have
\begin{equation}
\left| \nabla \frac{\partial}{\partial x_0} V \right|_g^2 = \left| V_1 \right|_g^2 + \left| V_2 \right|_g^2 + 2g(V_1, V_2).
\end{equation}

We handle the third term above as
\begin{align*}
g(V_1, V_2) &= \sum_{i=0}^{3} a_i \nabla \frac{\partial}{\partial x_0} a_i \nabla \frac{\partial}{\partial x_1} + a_3 \nabla \frac{\partial}{\partial x_3} \\
&= \sum_{i=0}^{3} a_i \nabla \frac{\partial}{\partial x_0} g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_1} \right) + \sum_{i,j=1,3} a_i a_j \nabla \frac{\partial}{\partial x_0} g \left( \frac{\partial}{\partial x_i}, \nabla \frac{\partial}{\partial x_j} \right).
\end{align*}
By $\frac{\partial}{\partial x_0} g_{ij} = 0$ and (3.15),
\[
\sum_{i=0,2; j=1,3} a_j \frac{\partial a_i}{\partial x_0} g \left( \frac{\partial}{\partial x_i}, \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_j} \right) = \sum_{i=0,2; j=1,3} a_j \frac{\partial a_i}{\partial x_0} \frac{\partial}{\partial x_i} g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)
\]
\[- \sum_{i=0,2; j=1,3} a_j \frac{\partial a_i}{\partial x_0} g \left( \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = 0
\]
and
\[
g \left( \frac{\partial}{\partial x_i}, \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_j} \right) + g \left( \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{1}{2} \frac{\partial}{\partial x_0} g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = 0, \quad i = 1, 3.
\]
Therefore,
\[
g(V_1, V_2) = a_3 \frac{\partial a_1}{\partial x_0} g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_3} \right) + a_1 \frac{\partial a_3}{\partial x_0} g \left( \frac{\partial}{\partial x_3}, \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1} \right)
\]
\[
= \left( a_3 \frac{\partial a_1}{\partial x_0} - a_1 \frac{\partial a_3}{\partial x_0} \right) g \left( \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \right). \tag{3.17}
\]
In conclusion, by (3.15), (3.16) and (3.17), we have
\[
|\nabla \frac{\partial}{\partial x_0} V^2 |^2 = R \left( \frac{\partial f}{\partial x_0}, V, \frac{\partial f}{\partial x_0}, V \right) = |V_2|^2 + 2 \left( a_3 \frac{\partial a_1}{\partial x_0} - a_1 \frac{\partial a_3}{\partial x_0} \right) g \left( \nabla \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \right). \tag{3.18}
\]
By (3.19), we get
\[
g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} \right) = \Gamma_{03}^{01} = \frac{1}{2} (g_{01,3} - g_{03,1}) = (1 + \varepsilon^2) \frac{\pi}{4A^2} \left( x_3^2 - x_1^2 \right). \tag{3.19}
\]
Therefore, we have
\[
\int_{T^2} \left| \nabla \frac{\partial}{\partial x_0} V^2 \right|^2 = \frac{1}{C} \int_{T^2} \left( \frac{\partial a_1}{\partial x_0} \right)^2 + \left( \frac{\partial a_3}{\partial x_0} \right)^2 + \left( \frac{\partial a_1}{\partial x_1} \right)^2 + \left( \frac{\partial a_3}{\partial x_1} \right)^2 \right) \cdot dx_0dx_2 \tag{3.20}
\]
where we have used (3.19) and integration by parts.

There is a uniform positive constant $C$ arising from the largest eigenvalue of the symmetric matrix $(g_{ij})$ for $|x_1|, |x_3| \leq 1$ such that
\[
\int_{T^2} |V_2|^2 g dx_0dx_2 + \int_{T^2} \left( \sum_{i=0,2} g_{ij} \frac{\partial^2}{\partial x_i} \right) dx_0dx_2 \geq \int_{T^2} \left( \frac{\partial a_1}{\partial x_0} \right)^2 + \left( \frac{\partial a_3}{\partial x_0} \right)^2 \right) \cdot dx_0dx_2.
Note that
\[ \int_{\mathbb{T}^2} \frac{\partial a_1}{\partial x_0} dx_0 dx_2 = 0. \]

By Poincaré's inequality and (3.19), if $|x_1|, |x_3|$ small enough, we have
\[ (3.21) \]
\[
4 \left| \int_{\mathbb{T}^2} a_3 \frac{\partial a_1}{\partial x_0} g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_3} \right) dx_0 dx_2 \right|
= 4 \left| g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_3} \right) \int_{\mathbb{T}^2} \frac{\partial a_1}{\partial x_0} \left( a_3 - \frac{1}{\text{Vol}(\mathbb{T}^2)} \int_{\mathbb{T}^2} a_3 \right) dx_0 dx_2 \right|
\leq 2 \left| g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_3} \right) \left( \int_{\mathbb{T}^2} \left| \frac{\partial a_1}{\partial x_0} \right|^2 dx_0 dx_2 + \int_{\mathbb{T}^2} \left| a_3 - \frac{1}{\text{Vol}(\mathbb{T}^2)} \int_{\mathbb{T}^2} a_3 \right|^2 dx_0 dx_2 \right) \right|
\leq C \left( \frac{\pi}{4A^2} (x_3^2 - x_1^2) \right) \left( \int_{\mathbb{T}^2} \left| \frac{\partial a_1}{\partial x_0} \right|^2 dx_0 dx_2 + \int_{\mathbb{T}^2} \left| a_3 \right|^2 dx_0 dx_2 \right)
\leq \frac{1}{2} \left( \int_{\mathbb{T}^2} |V_2|_g^2 dx_0 dx_2 + \int_{\mathbb{T}^2} |V_4|_g^2 dx_0 dx_2 \right).
\]

Similarly with (3.20), we have
\[ (3.22) \]
\[
\int_{\mathbb{T}^2} \left\{ \left| \nabla \frac{\partial}{\partial x_2} \right|_g^2 - R \left( \frac{\partial f}{\partial x_2}, V, \frac{\partial f}{\partial x_2}, V \right) \right\} dx_0 dx_2
= \int_{\mathbb{T}^2} \left\{ |V_4|_g^2 + 4a_3 \frac{\partial a_1}{\partial x_2} g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_3} \right) \right\} dx_0 dx_2.
\]

By (3.8), we see that
\[ g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_3} \right) = g_{1p} \Gamma_{23}^p = \frac{1}{2} (g_{21,3} - g_{32,1}) = - \left( \frac{1}{2} + \epsilon^2 \right) \frac{\pi}{2A^2} x_1 x_3.
\]

When $|x_1|, |x_3|$ small enough, as argued above, we have
\[
4 \left| \int_{\mathbb{T}^2} a_3 \frac{\partial a_1}{\partial x_2} g \left( \frac{\partial}{\partial x_1}, \nabla \frac{\partial}{\partial x_3} \right) dx_0 dx_2 \right|
\leq \frac{1}{2} \left( \int_{\mathbb{T}^2} |V_2|_g^2 dx_0 dx_2 + \int_{\mathbb{T}^2} |V_4|_g^2 dx_0 dx_2 \right).
\]

Combining with (3.20), (3.21) and (3.22), we conclude $I(V, V) \geq 0$. Hence $f_p$ is a stable harmonic map when $p$ is in some neighbourhood of $p_0$ or $p_1$.

\[ \square \]

**Theorem 3.4.** The harmonic map $f_p$ is unstable, when $x_1 = 0$ and $|x_3| > \sqrt{2}$. 
Proof. Choose $V = a_1 \frac{\partial}{\partial x_1} + a_3 \frac{\partial}{\partial x_3}$ with $a_1 = \frac{1}{2\pi} \cos(2\pi x_0)$, $a_3 = \frac{1}{2\pi} \sin(2\pi x_0)$. Then, by (3.18), (3.8) and (3.3), when $x_1 = 0$

$$|\nabla_{\frac{\partial}{\partial x_0}} V|^2_g - R \left( \frac{\partial f}{\partial x_0}, V, \frac{\partial f}{\partial x_0}, V \right) = |V|^2_g - 2 g^p_{1p} T^p_{03} = \sin^2(2\pi x_0) g_{11} + \cos^2(2\pi x_0) g_{33} - (g_{01,3} - g_{03,1})$$

$$= \frac{1}{2} \left( 1 + \frac{3}{4} \right) \frac{\epsilon^2 |w|^2}{4A^2} - 2 \left( 1 + \epsilon^2 \right) \frac{\pi}{4A^2} x_3^2$$

$$= \frac{1}{2} \left( 1 + \frac{3}{4} \right) \frac{\epsilon^2 |w|^2}{4A^2} - 2 \left( 1 + \epsilon^2 \right) \frac{\pi}{4A^2} + \frac{1}{A^2}$$

$$< - \frac{1}{A} + \frac{3}{A^2} < 0$$

when $|x_3| > \sqrt{2}$.

Note $\frac{\partial f_{10}}{\partial x_3} = \frac{\partial f_{30}}{\partial x_3} = 0$. Hence $V_4 = 0$. By (3.22),

$$\int_{T^2} \left( |\nabla_{\frac{\partial}{\partial x_0}} V|^2_g - R \left( \frac{\partial f}{\partial x_0}, V, \frac{\partial f}{\partial x_0}, V \right) \right) d x_0 d x_2 = 0.$$  

It follows $I(W, W) < 0$. \hfill \Box

Let $E^C_p = f^* p T(S^3 \times S^1) \otimes \mathbb{C}$ be the complexified pullback bundle over $T^2_p$. The pullback metric $f^* p g_\epsilon$ makes $E^C_p$ a Hermitian bundle. Let $\nabla$ denote the pullback Riemannian connection of $g_\epsilon$ extended to a complex connection on $E^C_p$, which decomposes into $\nabla = \nabla' + \nabla''$ where

$$\nabla' : \mathcal{A}^{0,0}(E^C_p) \rightarrow \mathcal{A}^{1,0}(E^C_p), \quad \nabla'' : \mathcal{A}^{0,0}(E^C_p) \rightarrow \mathcal{A}^{0,1}(E^C_p)$$

and $\mathcal{A}^{r,s}(E^C_p)$ is the space of $E^C_p$-valued $(r, s)$-forms on $T^2_p$. The curvature 2-form is of type $(1, 1)$ as the base $T^2_p$ is of complex dimension 1. It is well known from [17], [8, Theorem 5.1] and [15, Proposition 1.3.7] that there is a unique holomorphic structure $\hat{\partial}$ on $E^C_p$ so that $\nabla'' = \hat{\partial}$ and with respect to which a section $W$ of $E^C_p$ is holomorphic if and only if

$$\nabla_{\hat{\partial}} W = 0.$$

The discussion above is contained in [24].

Let $H^0_\partial(T^2_p, E^C_p)$ be the linear space of holomorphic sections of $E^C_p$.

**Theorem 3.5.** When $p \neq p_0, p_1$ is in $U_0 \cup U_1$ where the neighbourhoods $U_0, U_1$ of $p_0, p_1$ are defined as in Theorem 3.2, $H^0_\partial(T^2_p, E^C_p) = \text{Span}_\mathbb{C} \left\{ \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_3} \right\}$. At $p_0, p_1$, we have $H^0_\partial(T^2_p, E^C_p) = \text{Span}_\mathbb{C} \left\{ \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3} \right\}$.

Proof: Assume that $p$ is in the neighbourhood $U_0$ of $p_0$. So $f_p(T^2_p) \subset V_{00}$. As explained in the proof of Theorem 3.2, $\frac{\partial}{\partial x_0}, \cdots, \frac{\partial}{\partial x_3}$ are global sections of $E_p$ hence...
they are also global sections of $E_p$. First, we show $R(\frac{\partial f_\rho}{\partial w_0}, W, \overline{\frac{\partial f_\rho}{\partial w_0}}, \overline{W}) \geq 0$ for any smooth section $W \in \Gamma(E_p)$. Assume $W = \sum_{i=0}^{3} a_i \frac{\partial}{\partial x_i}$ where $a_i \in C^\infty(T^2, \mathbb{C})$. By Lemma 3.1, 3.13 and 3.8, (3.23)

$$R \left( \frac{\partial f_\rho}{\partial w_0}, W, \frac{\partial f_\rho}{\partial w_0}, \overline{W} \right) = |a_1|^2 R_{0101} + |a_3|^2 R_{0303} + i \left( |a_1|^2 R_{0121} + |a_3|^2 R_{0323} \right)$$

$$- i \left( |a_1|^2 R_{2101} + |a_3|^2 R_{2303} \right) + |a_1|^2 R_{2121} + |a_3|^2 R_{2323}$$

$$= \frac{1}{4} (|a_1|^2 + |a_3|^2) \left( g^{33}(g_{03,1} - g_{01,3})^2 + g^{11}(g_{21,3} - g_{32,1})^2 \right)$$

$$= \frac{1}{4} (\pi^2 + 2)(|a_1|^2 + |a_3|^2) g^{33} \left( \left( \frac{\pi}{2A^2}(x_1^2 - x_3^2) \right)^2 \right) \geq 0.$$

The second derivative of energy at a critical point $f_\rho$ along $W$ is given in [24, 2.3] by the index form

$$I(W, W) = 2 \int_{T^2} \left[ \left\| \nabla_\rho W \right\|^2 - R \left( W, \frac{\partial f_\rho}{\partial w_0}, \overline{W}, \overline{\frac{\partial f_\rho}{\partial w_0}} \right) \right] dw_0 \wedge d\overline{w}_0.$$ 

Assume $W$ is holomorphic. Since $f_\rho$ is stable by Theorem 3.3

$$I(W, W) = 2 \int_{T^2} \left[ -R \left( W, \frac{\partial f_\rho}{\partial w_0}, \overline{W}, \overline{\frac{\partial f_\rho}{\partial w_0}} \right) \right] dw_0 \wedge d\overline{w}_0 \geq 0.$$ 

By (3.23), we have

$$R \left( W, \frac{\partial f_\rho}{\partial w_0}, \overline{W}, \overline{\frac{\partial f_\rho}{\partial w_0}} \right) \equiv 0.$$ 

When $p \neq p_0, p_1$, we have $a_1 = a_3 = 0$. Then

$$0 = \nabla_\rho \rho W = \frac{\partial a_0}{\partial w_0} \frac{\partial}{\partial x_0} + \frac{\partial a_2}{\partial w_0} \frac{\partial}{\partial x_2},$$

as $\Gamma^{k}_{\alpha \beta}, \Gamma^{k}_{\alpha \gamma}, \Gamma^{k}_{\beta \gamma}$ are all zero by (3.5), this implies $a_0, a_2$ are constant.

At $p_0$, we have $\Gamma^{k}_{ij} = 0, i = 0, 2$ and all $j$. In fact, since $g_{ij}$ is independent of $x_0, x_2$ and $g_{is}$ with $r, s = 0, 2$ are constant

$$\Gamma^{k}_{01}(p_0) = \frac{1}{2} g^{kj} (g_{0,1} - g_{1,0})(p_0) = \frac{1}{2} g^{kj} (g_{30,1} - g_{31,0})(p_0) = 0$$

by (3.8); the Christoffel symbols vanish for other $i, j$ similarly. So at $p_0$

$$\nabla_\rho \rho \frac{\partial}{\partial x_j} = \sum_{k=0}^{3} \Gamma^{k}_{ij} \frac{\partial}{\partial x_k} = 0$$

for $i = 0, 2, j = 0, 1, 2, 3$. It follows

(3.24) \[ \nabla_\rho \rho \frac{\partial}{\partial x_j} = 0, \]
i.e., $\frac{\partial}{\partial x_j}$ are holomorphic sections for $j = 0, 1, 2, 3$. Thus $\text{Span}_\mathbb{C}\{\frac{\partial}{\partial x_i} : 0 \leq i \leq 3\} \subseteq H^0(\mathbb{T}^2_p, \mathbb{E}_p^\mathbb{C})$. On the other hand, for any $W \in H^0(\mathbb{T}^2_p, \mathbb{E}_p^\mathbb{C})$

$$0 = \nabla_{\frac{\partial}{\partial x_0}} W = \sum_{i=0}^3 \frac{\partial a_i}{\partial w_0} \frac{\partial}{\partial x_i}$$

by (3.24). So $a_i$ are holomorphic functions on $\mathbb{T}^2_p$ and they must be constant. In turn, $W \in \text{Span}_\mathbb{C}\{\frac{\partial}{\partial x_i} : 0 \leq i \leq 3\}$.

**Theorem 3.6.** Each fibre $\mathbb{T}^2_p$ of the Hopf fibration is a flat and totally geodesic minimal surface in $(\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon)$. It is a stable minimal surface if $p = p_0, p_1$ and it is unstable if $p$ is as in Theorem 3.4.

**Proof.** Reading from (3.3), $g_{\epsilon,00} = g_{\epsilon,22} = (1 + \epsilon^2)\pi$ and $g_{\epsilon,02} = g_{\epsilon,20} = 0$. It follows that $\mathbb{T}^2_p$ is flat for the induced metric from $g_\epsilon$. The Gauss formula together with (3.3) asserts that $\mathbb{T}^2_p$ is totally geodesic in $(\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon)$.

According to a theorem of Ejiri and Micallef in [9], the Morse index $i_A$ of the area functional for the minimal surface $\mathbb{T}^2_p$ is no smaller than the Morse index $i_E$ of the harmonic map $f_p$. So $\mathbb{T}^2_p$ is unstable as a minimal surface if $f_p$ is unstable as a harmonic map.

We now consider $p = p_0$. Let $X$ be a normal vector field of $\mathbb{T}^2_{p_0}$ and $\nabla^\perp$ be the normal connection of $\mathbb{T}^2_{p_0}$. The second variation formula of area for the minimal surface $\mathbb{T}^2_{p_0}$ is

$$\delta^2|S^2|_{\mathbb{T}^2_{p_0}}(X, X) = \int_{\mathbb{T}^2_{p_0}} |\nabla^\perp X|^2 - \text{tr}_{g_\epsilon} R_{g_\epsilon}(\cdot, X, \cdot, X) - g_\epsilon(A, X)^2$$

$$= \int_{\mathbb{T}^2_{p_0}} |\nabla^\perp X|^2 - \text{tr}_{g_\epsilon} R_{g_\epsilon}(\cdot, X, \cdot, X)$$

where $A \equiv 0$ is the second fundamental form of the totally geodesic $\mathbb{T}^2_{p_0}$ in $(\mathbb{S}^3 \times \mathbb{S}^1, g_\epsilon)$.

We claim that at $w_1 = 0$, i.e., at $p_0$,

$$R_{00j} = 0, \quad R_{02j} = 0, \quad R_{2i2j} = 0, \quad 0 \leq i, j \leq 3.$$  

Reasoning similar as in the proof of Lemma 3.1

$$R_{00j} = -\frac{1}{2} \left(g_{00j} + g_{ij,00} - g_{j0,0} - g_{j0,0i}\right) - \Gamma_{00}^p \Gamma_{ij}^q g_{pq} + \Gamma_{00}^q \Gamma_{j0}^p g_{pq}$$

$$= \frac{1}{4} \delta^{ij} \left(g_{0i,j} - g_{0i,j}\right) \left(g_{0j,i} - g_{0j,i}\right)$$

and

$$R_{02j} = -\frac{1}{2} \left(g_{02j} + g_{j2,02} - g_{j0,2i} - g_{j0,2i}\right) - \Gamma_{02}^p \Gamma_{ij}^q g_{pq} + \Gamma_{02}^q \Gamma_{j0}^p g_{pq}$$

$$= \frac{1}{4} \delta^{ij} \left(g_{2i,j} - g_{2i,j}\right) \left(g_{0j,i} - g_{0j,i}\right).$$
By (3.3) we see $g_{0,1} - g_{1,0} = 0$ at $w_1 = 0$; further, (3.3) is independent of $x_0$ and $x_2$ and the constancy of the related entries, we have $g_{0,s} - g_{0,s} = 0$ for all $j, s$. Therefore, $R_{00j} = 0$ and $R_{02j} = 0$. Similarly, $R_{22j} = 0$.

Therefore, by (3.26), $\text{tr}_{g_i} R_{g_i} (\cdot, X, \cdot, X) = 0$. Hence $\mathbb{T}^2_{p_0}$ is a stable minimal surface. Similarly, so is $\mathbb{T}^2_{p_1}$. \hfill $\square$

4. Appendix

The following result mentioned Introduction should be well known. A proof may not be explicitly documented in the literature, we include one below.

**Proposition 4.1.** There is no stable branched minimal immersion of a compact Riemann surface $\Sigma$ in $\mathbb{S}^3 \times \mathbb{S}^1$ equipped with the product metric $g = g_1 \oplus g_2$ where $g_1$ is the constant curvature 1 metric on $\mathbb{S}^3$ and $g_2$ is any Riemannian metric on $\mathbb{S}^1$.

**Proof:** Suppose $f : \Sigma \to \mathbb{S}^3 \times \mathbb{S}^1$ is a branched conformal map that is harmonic as well. We write $f = (f_1, f_2)$ where $f_1 : \Sigma \to \mathbb{S}^3, f_2 : \Sigma \to \mathbb{S}^1$. Due to the product structure of the metric, $f_1$ is a harmonic map into the round 3-sphere $\mathbb{S}^3$ (similarly for $f_2$). By a result of Leung [19] the harmonic map $f_1$ must be unstable or constant. The latter is impossible since a nonconstant conformal map cannot have image in $(q) \times \mathbb{S}^1$ for some $q \in \mathbb{S}^3$.

There is a variation $V_1$ of the unstable harmonic map $f_1$ making second variation of the energy negative. In other words, there is a family of $f_1^t \in C^2(\Sigma, \mathbb{S}^3)$ parametrized by $t$ with $f_1^0 = f_1$, so that $V_1 = \frac{\partial f_1^t}{\partial t}{|_{t=0}}$, and

$$\int_{\Sigma} \left[ |\nabla_{g_1} V_1|^2_{g_1} - \text{tr}_{g_1} R_{g_1} \left( V_1, \frac{\partial f_1}{\partial w_0}, V_1, \frac{\partial f_1}{\partial w_0} \right) \right] \, d\mu_{g_1} < 0$$

where $\nabla_{g_1}$ is the pullback connection on the bundle $f_1^\ast T \mathbb{S}^3$ over $\Sigma$ by $f_1$.

Set $V = (V_1, 0)$. For $f^t = (f_1^t, f_2)$, it is clear that $f^0 = f$ and $V = \frac{\partial f^t}{\partial t}{|_{t=0}}$. As $g = g_1 \oplus g_2$, the pullback connection $\nabla^g$ on the pullback bundle $f^\ast T (\mathbb{S}^3 \times \mathbb{S}^1)$ over $T^2$ by $f$ splits into $\nabla^{g_1} + \nabla^{g_2}$, and the Riemannian curvature simplifies according to

$$R_g (X, Y, X, Y) = R_{g_1} (X_1, Y_1, X_1, Y_1) + R_{g_2} (X_2, Y_2, X_2, Y_2) = R_{g_1} (X_1, Y_1, X_1, Y_1)$$

where $X = X_1 + X_2, Y = Y_1 + Y_2$ and $X_1, Y_1 \in T \mathbb{S}^3, X_2, Y_2 \in T \mathbb{S}^1$. Then

$$\int_{\Sigma} \left[ |\nabla_{g_1^t} V|^2_{g_1^t} - \text{tr}_{g_1^t} R_{g_1^t} \left( V, \frac{\partial f}{\partial w_0}, V, \frac{\partial f}{\partial w_0} \right) \right] \, d\mu_{g_1^t}$$

$$= \int_{\Sigma} \left[ |\nabla_{g_1^t} V|^2_{g_1^t} - \text{tr}_{g_1^t} R_{g_1^t} \left( V_1, \frac{\partial f_1}{\partial w_0}, V_1, \frac{\partial f_1}{\partial w_0} \right) \right] \, d\mu_{\Sigma} < 0.$$

Therefore $f$ is unstable as a harmonic map. In fact, the argument up to this point holds for harmonic maps from any compact manifold to $\mathbb{S}^3 \times \mathbb{S}^1$ with the product metric $g$; so they are all unstable.

Now, since $i_E > 0$, the inequality $i_E \leq i_A$ in [9] implies that $f$ is unstable for area as a minimal surface. \hfill $\square$
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