Conformal fields and the quantum state of the universe

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Abstract. The creation of a quantum Universe is described by a density matrix which yields an ensemble of universes with the cosmological constant limited to a bounded range \( \Lambda_{\text{min}} \leq \Lambda \leq \Lambda_{\text{max}} \). The domain \( \Lambda < \Lambda_{\text{min}} \) is ruled out by a cosmological bootstrap requirement (the self-consistent back reaction of hot matter). The upper cutoff results from the quantum effects of vacuum energy and the conformal anomaly mediated by a special ghost-avoidance renormalization. The cutoff \( \Lambda_{\text{max}} \) establishes a new quantum scale – the accumulation point of an infinite sequence of garland-type instantons. The cosmological evolution starting with these initial conditions also have some new features: the stage of cosmic acceleration can be followed by a big boost singularity – a rapid growth up to infinity of the scale factor acceleration parameter. A correspondence between the 4-dimensional modified quantum Friedmann equations and the Friedmann equations arising in the context of 5-dimensional classical cosmological models was established.

1. Introduction
The idea of the quantum state of the universe, prescribing initial conditions for its classical evolution was realized in the quantum cosmology, the branch of quantum theory treating the Universe as a unique quantum object. The basic equation of quantum cosmology - the Wheeler-DeWitt equation was formulated in the sixties [1]. However, certain prescriptions for the wave function of the universe, satisfying this equation were suggested only in the early eighties in papers [2, 3, 4, 5, 6]. In papers mentioned above the two related approaches were used: the analogy with the tunneling processes in quantum mechanics [3] and the apparatus of the Euclidean field theory [2]. In both cases the phenomenon of the so called “quantum birth of the universe from nothing” was employed. Both approaches used the instanton solutions of the Euclidean Einstein equations, however their physical predictions were different because the Euclidean action entered with different signs the exponential of the wave function of the universe, calculated in the semiclassical approximation. Namely, the Hartle-Hawking or “no-boundary” wave function of the universe [2] which behaves in the lowest order of the WKB approximation as \( \psi_{\text{NB}} \sim \exp(-\Gamma) \), where \( \Gamma \) is the Euclidean action on the underlying instanton, predicts the quantum birth of a universe with a very large (infinite) initial radius, which looks quite counter-intuitive. The tunneling or Vilenkin wave function of the universe [3] behaves as \( \psi_{\text{T}} \sim \exp(+\Gamma) \) and predicts the birth of a universe with an infinitely small radius. Besides, both of these
functions are non-normalizable and it is hardly possible to prescribe to them the traditional quantum-mechanical probabilistic interpretation.

Considering solutions of the Wheeler-DeWitt equation in the one-loop approximation, one can achieve (imposing some constraints on the particle content of the theory) the normalizability of the wave function of the universe in both the tunneling and no-boundary prescriptions [7]. Moreover, for the tunneling wave function of the universe one can predict a peak of the probability of the quantum birth of the universe with reasonable initial parameters [8].

However, the traditional approach to quantum cosmology limited to the consideration of only pure quantum states and associated with them instantons looks too restrictive. It appears that relaxing the requirement of the “purity” of possible quantum states of the universe and taking into account the possibility of existence of the gravitational instantons with more complicated geometries than those considered in the above works on quantum cosmology, one can obtain some, at first glance, unexpected results. In our papers [9, 10] we have generalized the traditional scheme of quantum cosmology. The main goals of our approach were the following:

(i) Description of the birth of the universe from nothing in a mixed state and the use of the density matrix instead of the wave function of the universe.
(ii) Prediction of initial conditions for the cosmological evolution, which we call “cosmological landscape” in analogy with a very popular string landscape [11].
(iii) Elimination of “infrared catastrophe” (an infinitely large probability of the birth of the universe of an infinitely large size) in the Hartle-Hawking prescription.
(iv) Establishing connections with string theory.

The tools which we have used were

(i) Quantum theory of tunneling: Euclidean quantum gravity.
(ii) Quantum field theory: renormalization in curved spacetime; effective action formalism.
(iii) The account of non-local effects due to back reaction of the conformal anomaly of quantum fields and their radiation.

Our main results can be formulated as follows:

(i) The closed system of equations describing the quantum birth of the universe is derived: the generalized Friedmann equation with the quantum radiation source and the “bootstrap” equation for the latter.
(ii) The solution of these equations gives the families of acceptable parameters, characterizing initial conditions for cosmological evolution - “cosmological landscape”.
(iii) The problem of “infrared catastrophe” in the Hartle-Hawking prescription is resolved.
(iv) The scenarios of the future evolution of the Universe are studied.

2. Density matrix, radiation and instantons

The idea to consider the density matrix of the universe instead of the wave function of the universe was put forward in [12] where it was also noticed that such a density matrix is based on an instanton with two disjoint boundaries (see Fig.1). The density matrix describes a mixed state which might account for the presence of radiation [13]. For the pure quantum state [2] the instanton bridge between Σ and Σ' breaks down (see Fig.2). However, the radiation stress tensor prevents these half instantons from closure. Indeed, the Euclidean Friedmann equation for a closed universe with the metric

\[ ds^2 = N^2(\tau) d\tau^2 + a^2(\tau) d^2 \Omega^{(3)} \]  

1
Figure 1. Picture of instanton representing the density matrix. Dashed lines depict the Lorentzian Universe nucleating from the instanton at the minimal surfaces Σ and Σ′.

Figure 2. Density matrix of the pure Hartle-Hawking state represented by the union of two vacuum instantons.

Figure 3. Calculation of the partition function represented by compactification of the instanton to a torus with periodically identified Euclidean time.

in the presence of a cosmological constant Λ = 3H^2 and radiation characterized by a constant C

\[ \frac{\dot{a}^2}{a^2} = \frac{1}{a^2} - H^2 - \frac{C}{a^4} \]  \hspace{1cm} (2)

has the solution

\[ a = \frac{1}{\sqrt{2H}} \sqrt{1 - (1 - 4CH^2)^{1/2} \cos 2H\tau} \]

with two turning points, neither of them vanishing, \( a_{\pm} = \frac{1}{\sqrt{2H}} \sqrt{1 \pm (1 - 4CH^2)^{1/2}}, \quad 4H^2C \leq 1 \).

The relevant density matrix is the path integral

\[ \rho[\varphi, \varphi'] = e^{\Gamma} \int_{\varphi, \phi | \Sigma, \Sigma'} D[g, \phi] \exp\left(-S_E[g, \phi]\right). \]  \hspace{1cm} (3)

with the partition function \( e^{-\Gamma} \) which follows from integrating out the field \( \varphi \) in the coincidence \( \varphi' = \varphi \) corresponding to the identification of \( \Sigma' \) and \( \Sigma \), the underlying instanton acquiring the toroidal topology (see Fig.3).

3. Conformal anomaly and ghosts

The metric of the instanton introduced above

\[ ds^2 = a^2(\eta)(d\eta^2 + d^2\Omega^{(3)}), \]  \hspace{1cm} (4)

is conformally equivalent to the metric of the Einstein static universe:

\[ d\tilde{s}^2 = d\eta^2 + d^2\Omega^{(3)}, \]  \hspace{1cm} (5)
where $\eta$ is the conformal time parameter. We shall consider conformally invariant fields. As is well known, the quantum effective action for such fields has a conformal anomaly first studied in cosmology in [14, 15]. It has the form

$$g_{\mu\nu} \delta \Gamma_{1-\text{loop}}[\bar{g}] = \frac{1}{4(4\pi)^2} g^{1/2} \left( \alpha \Delta R + \beta E + \gamma C^{2}_{\mu\nu\alpha\beta} \right),$$

(6)

where $E = R^{2}_{\mu
u\rho\gamma} - 4R^{2}_{\mu\nu} + R^2$ and $\Delta$ is the four-dimensional Laplacian. This anomaly, when integrated functionally along the orbit of the conformal group, gives the relation between the actions on conformally related backgrounds [16].

$$\Gamma_{1-\text{loop}}[g] = \Gamma_{1-\text{loop}}[\bar{g}] + \delta \Gamma[g, \bar{g}],$$

(7)

$$g_{\mu\nu}(x) = e^{\sigma(x)} \bar{g}_{\mu\nu}(x),$$

(8)

where

$$\delta \Gamma[g, \bar{g}] = \frac{1}{2(4\pi)^2} \int d^4x \bar{g}^{1/2} \left\{ \frac{1}{2} \left[ \frac{\gamma}{2} \bar{C}^{2}_{\mu\nu\alpha\beta} \right. \right.$$

$$+ \beta \left( E - \frac{2}{3} \Delta R \right) \sigma$$

$$+ \frac{\beta}{2} \left[ (\Delta \sigma)^2 + \frac{2}{3} \bar{R} (\nabla_{\mu} \sigma)^2 \right] \left\} \right.$$

$$- \frac{1}{2(4\pi)^2} \left( \frac{\alpha}{12} + \frac{\beta}{18} \right)$$

$$\times \int d^4x \left( g^{1/2} R^2(g) - \bar{g}^{1/2} R^2(\bar{g}) \right).$$

(9)

One can show that the higher-derivative in $\sigma$ terms are all proportional to the coefficient $\alpha$. The $\alpha$-term can be arbitrarily changed by adding a local counterterm $\sim g^{1/2} R^2$. We fix this local renormalization ambiguity by an additional criterion of the absence of ghosts. The conformal contribution to the renormalized action on the minisuperspace background equals

$$\delta \Gamma[g, \bar{g}] = \Gamma_{R}[g] - \Gamma_{R}[\bar{g}]$$

$$= m_{p}^{2} B \int d\tau \left( \frac{\dot{a}^2}{a} - \frac{1}{6} \frac{\dot{a}^4}{a} \right),$$

(10)

$$m_{p}^{2} B = \frac{3}{4} \beta,$$

(11)

with the constant $m_{p}^{2} B$ which for scalars, two-component spinors and vectors equals respectively $1/240$, $11/480$ and $31/120$.

4. Effective action on a static Einstein instanton

For a conformal scalar field

$$S[\bar{g}, \phi] = \frac{1}{2} \sum_{\omega} \int_{0}^{\eta} d\eta' \left( \left( \frac{d\phi_{\omega}}{d\eta'} \right)^2 + \omega^2 \phi_{\omega}^2 \right),$$

(12)

where $\omega = n, n = 0, 1, 2, ...$, labels a set of eigenmodes and eigenvalues of the Laplacian on a unit 3-sphere. Thus

$$e^{-\Gamma_{1-\text{loop}}[\bar{g}]}$$

$$= \prod_{\omega} d\phi_{\omega} \int D[\phi] \exp \left( - S[\bar{g}, \phi] \right)$$

$$= \text{const} \prod_{\omega} \left( \sinh \frac{\omega \eta}{2} \right)^{-1},$$

(13)
and the effective action equals the sum of contributions of the vacuum energy \( E_0 \) and free energy \( F(\eta) \) with the inverse temperature played by \( \eta \) — the circumference of the toroidal instanton in units of a conformal time,

\[
\Gamma_{\text{1-loop}}[\bar{g}] = \sum_\omega \left[ \frac{\eta \omega}{2} + \ln\left(1 - e^{-\omega \eta}\right) \right]
\]

\[
= m_P^2 E_0 \eta + F(\eta),
\]

\[
m_P^2 E_0 = \sum_\omega \frac{\omega^2}{2} = \sum_{n=1}^{\infty} \frac{n^3}{2},
\]

\[
F(\eta) = \sum_\omega \ln\left(1 - e^{-\omega \eta}\right)
\]

\[
= \sum_{n=1}^{\infty} n^2 \ln\left(1 - e^{-n \eta}\right).
\]

Similar expressions hold for other conformally invariant fields of higher spins. In particular, the vacuum energy (an analog of the Casimir energy) on Einstein static spacetime is

\[
m_P^2 E_0 = \frac{1}{960} \times \begin{cases} 4 \\ 17 \\ 88 \end{cases}
\]

respectively for scalar, spinor and vector fields.

We should take into account the effect of the finite ghost-avoidance renormalization denoted below by a subscript \( R \), which results in the replacement of \( E_0 \) above by a new parameter \( C_0 \):

\[
\Gamma_R[\bar{g}] = m_P^2 C_0 \eta_0 + F(\eta),
\]

\[
m_P^2 C_0 = m_P^2 E_0 + \frac{3}{16} \alpha.
\]

A direct observation indicates the following universality relation for all conformal fields of low spins

\[
m_P^2 C_0 = \frac{1}{2} m_P^2 B.
\]

**5. Effective Friedmann and bootstrap equations**

Now we can write down the effective Friedmann equation governing the Euclidean evolution of the universe. First of all, the full conformal time on the instanton is

\[
\eta = 2 \int_{\tau_-}^{\tau_+} \frac{d\tau}{a(\tau)},
\]

where \( \tau_{\pm} \) label the turning points for \( a(\tau) \) — its minimal and maximal values.

The effective action is \( (m_P^2 \equiv 3/4\pi G) \)

\[
\Gamma[a(\tau), N(\tau)] = 2m_P^2 \int_{\tau_-}^{\tau_+} d\tau \left( -\frac{\dot{a}^2}{N} - Na + NH^2 a^3 \right)
\]

\[
+ 2Bm_P^2 \int_{\tau_-}^{\tau_+} d\tau \left( \frac{\dot{a}^2}{Na} - \frac{1}{6} \frac{\dot{a}^4}{N^3 a} \right)
\]

\[
+ F \left( 2 \int_{\tau_-}^{\tau_+} \frac{d\tau}{a} \right) + Bm_P^2 \int_{\tau_-}^{\tau_+} \frac{d\tau}{a},
\]

\[\text{(23)}\]
and the effective Friedmann equation reads

\[
\frac{\delta \Gamma}{\delta \mathcal{N}} = 2m_P^2 \left( \frac{a\dot{a}^2}{N^2} - a + H^2a^3 \right) + 2Bm_P^2 \left( -\frac{\dot{a}^2}{N^2a} + \frac{1}{2} \frac{\dot{a}^4}{N^4a} \right) + \frac{2}{a} \left( \frac{dF(\eta)}{d\eta} + \frac{B}{2} m_P^2 \right) = 0.
\]

(24)

In the gauge \( N = 1 \) this equation takes form

\[
\frac{\dot{a}^2}{a^2} + B \left( \frac{1}{2} \frac{\dot{a}^4}{a^4} - \frac{\dot{a}^2}{a^2} \right) = \frac{1}{a^2} - H^2 - \frac{C}{a^4},
\]

(25)

where the amount of radiation constant \( C \) is given by the bootstrap equation

\[
m_P^2 C = m_P^2 \frac{B}{2} + \frac{dF(\eta)}{d\eta} = \frac{B}{2} m_P^2 + \sum \omega \frac{\omega}{e^{\omega \eta} - 1}.
\]

(26)

The Friedmann equation can be rewritten as

\[
\dot{a}^2 = \sqrt{\frac{(a^2 - B)^2}{B^2} + \frac{2H^2}{B} (a_+^2 - a^2)(a^2 - a_-^2) - \frac{(a^2 - B)}{B}}
\]

(27)

and has the same two turning points \( a_\pm \) as in the classical case provided

\[ a^2 \geq B. \]

(28)

This requirement is equivalent to

\[ C \geq B - B^2H^2, \quad BH^2 \leq \frac{1}{2}. \]

(29)

Together with

\[ CH^2 \leq \frac{1}{4}, \]

the admissible domain for instantons reduces to the curvilinear wedge below the hyperbola and above the straight line to the left of the critical point (see Figure 4)

\[ C = \frac{B}{2}, \quad H^2 = \frac{1}{2B}. \]

For a scalar field the numerical analysis of the Friedmann and bootstrap equations shows that the one-parameter family of instantons interpolates between the point on the lower line boundary with the parameters

\[ H^2 \approx 2.997 m_P^2, \quad C \approx 0.004 m_P^{-2}, \quad \Gamma_0 \approx -0.1559, \]

(30)

and the point on the upper hyperbolic boundary

\[ H^2 \approx 12.968 m_P^2, \quad C \approx 0.0193 m_P^{-2}, \quad \Gamma_0 \approx -0.0883. \]

(31)
Figure 4. The instanton domain in the \((H^2, C)\)-plane is located between bold segments of the upper hyperbolic boundary and lower straight line boundary. The first one-parameter family of instantons is labeled by \(k = 1\). Families of garlands are qualitatively shown for \(k = 2, 3, 4\). \((1/2B, B/2)\) is the critical point of accumulation of the infinite sequence of garland families.

The last instanton describes the creation of a static Einstein Universe of the constant size

\[
a = a_+ = a_- = 1/(\sqrt{2H})
\]

with the hot gas of a conformally-invariant scalar field particles in the equilibrium state with the temperature

\[
T = \frac{1}{a\eta} = \frac{H}{\pi \sqrt{1 - 2BH^2}}.
\]

(32)

6. Infrared catastrophe is eliminated.

The suggested approach allows to resolve the problem of the so-called infrared catastrophe for the no-boundary state of the Universe based on the Hartle-Hawking instanton. This problem is related to the fact that the Euclidean action on this instanton is negative and inverse proportional to the value of the effective cosmological constant. This means that the probability of the universe creation with an infinitely big size is infinitely high. We shall show now that the conformal anomaly effect allows one to avoid this counter-intuitive conclusion.

Indeed, outside of the admissible domain for the instantons with two turning points, obtained above, one can also construct instantons with one turning point which smoothly close at \(a_- = 0\) with \(\dot{a}(\tau_-) = 1\). Such instantons correspond to the Hartle-Hawking pure quantum state. However, in this case the on-shell effective action, which reads for the set of solutions obtained above as

\[
\Gamma_0 = F(\eta) - \eta \frac{dF(\eta)}{d\eta}
\]
\[ +4m_P^2 \int_{a_-}^{a_+} \frac{da}{a} \left( B - a^2 - \frac{B \dot{a}^2}{3} \right), \quad (33) \]
diverges to plus infinity. Indeed, for \( a_- = 0 \) and \( \dot{a}_- = 1 \)
\[ \eta = \int_0^{a_+} \frac{da}{\dot{a} a} = \infty, \quad F(\infty) = F'(\infty) = 0, \quad (34) \]
and hence the effective Euclidean action diverges at the lower limit to \( +\infty \). Thus,
\[ \Gamma_0 = +\infty, \quad \exp(-\Gamma_0) = 0, \]
and this fact completely rules out all pure-state instantons, and only mixed quantum states of
the universe, described by the cosmological density matrix appear to be admissible.

7. Instanton garlands

One should consider also the multiple instanton configurations, which could be called “Instanton
garlands” (see Figure 5). The total conformal time for such an instanton garland is
\[ \eta_0^{(k)} = 2k \int_{\tau_-}^{\tau_+} d\tau = 2k \int_{a_-}^{a_+} \frac{da}{\dot{a} a}, \quad (35) \]
where \( k \) is the number of simple instanton folds in a garland.

Numerical analysis for \( k = 2 \) shows the existence of the one-parameter family of instantons
similar to the case of \( k = 1 \). It interpolates between the point on the lower boundary of \((C, H^2)\)-
plane
\[ H_2^{(2)} \approx 45.89 m_P^2, \quad C_2^{(2)} \approx 0.0034 m_P^{-2}, \quad \Gamma_0^{(2)} \approx -0.0113, \quad (36) \]
and the point on the upper (hyperbolic) boundary
\[ H_2^{(2)} \approx 61.12 m_P^2, \quad C_2^{(2)} \approx 0.0041 m_P^{-2}, \quad \Gamma_0^{(2)} \approx -0.0145. \quad (37) \]
Such families exist for all \( k, 1 \leq k \leq \infty \), and their infinite sequence is saturated at the critical
point,
\[ \eta_0^{(k)} \approx \ln k^2, \quad (38) \]
\[ H_2^{(k)} \approx \frac{1}{2B} \left( 1 - \ln^2 k^2 \frac{2}{2k^2 \pi^2} \right), \quad (39) \]
\[ C_2^{(k)} \approx \frac{B}{2} \left( 1 + \ln^2 k^2 \frac{2}{2k^2 \pi^2} \right), \quad (40) \]
\[ \Gamma_0^{(k)} \approx -m_P^2 B \ln^3 k^2 \frac{4k^2 \pi^2}{4k^2 \pi^2}. \quad (41) \]
The length of instanton families decreases as \( 1/k^4 \). Infinite garlands \( k \to \infty \) do not dominate
the instanton distribution because their action grows with \( k \) rather than decreases to \(-\infty\).

A growing spin of a conformal particle decreases the instanton size and makes its probability
weight higher. For \( N \) fields
\[ C \to NC, \quad (42) \]
\[ B \to NB, \quad (43) \]
\[ \eta_0 \to \eta_0, \quad (44) \]
\[ F(\eta_0) \to NF(\eta_0), \quad (45) \]
\[ H^2 \to \frac{H^2}{N}. \quad (46) \]
The initial size of the universe grows with the growing spin and number of fields.
8. Where Euclidean Quantum Gravity and Cosmology comes from?

In the preceding sections we have described a new approach to the problem of initial conditions in cosmology based on the use of the combination of two ideas: the density matrix formalism and Euclidean quantum gravity. A natural question arises: where Euclidean quantum gravity comes from? The answer can be formulated briefly as follows: from the Lorentzian quantum gravity (LQG) [17]. Namely, the density matrix of the Universe for the microcanonical ensemble in Lorentzian quantum cosmology of spatially closed universes describes an equipartition in the physical phase space of the theory, but in terms of the observable spacetime geometry this ensemble is peaked about a set of cosmological instantons (solutions of the Euclidean quantum cosmology) limited to a bounded range of the cosmological constant. These instantons obtained above as fundamental in Euclidean quantum gravity framework, in fact, turn out to be the saddle points of the LQG path integral, belonging to the imaginary axis in the complex plane of the Lorentzian signature lapse function [17].

9. Cosmological evolution and Big Boost singularity

Now let us consider the cosmological evolution of the universe starting from the initial conditions described above. Making the transition from the Euclidean time to the Lorentzian one, $\tau = it$, we can write the modified Lorentzian Friedmann equation as [18]

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{1}{B} \left( 1 - \sqrt{1 - \frac{16\pi G}{3} B \varepsilon} \right),$$

(47)

$$\varepsilon = \frac{3}{8\pi G} \left( H^2 + \frac{C}{a^4} \right),$$

(48)

$$C \equiv C - \frac{B}{2},$$

(49)

where $\varepsilon$ is a total gravitating matter density in the model (including at later stages also the contribution of particles created during inflationary expansion and thermalized at the inflation exit). A remarkable feature of this equation is that the Casimir energy is totally screened here and only the thermal radiation characterized by $C$ weighs.

If one wants to compare the evolution described by Eq. (49) with the real evolution of the universe, first of all it is necessary to have a realistic value for an effective cosmological constant $\Lambda = 3H^2$. The only way to achieve this goal is to increase the number of conformal fields and the corresponding parameter $B$, (11), of the conformal anomaly (6). The mechanisms for growing number of the conformal fields exist in some string inspired cosmological models with extra dimensions [17]. If some of these mechanisms work we can encounter an interesting phenomenon: if the $B$ grows with $a$ faster than the rate of decrease of the energy density $\varepsilon$ one encounters a new type of the cosmological singularity - Big Boost. This singularity is characterized by finite values of the cosmological radius $a_{BB}$ and of its time derivative $\dot{a}_{BB}$, while the second time variable $\ddot{a}$ has an infinite positive value. The universe reaches this singularity at some finite moment of cosmic time $t_{BB}$:

$$a(t_{BB}) = a_{BB} < \infty,$$

(50)
\[ \dot{a}(t) = a_B \leq \infty, \quad \lim_{t \to t_B} \dot{a}(t) = \infty. \]  

10. The dualities between 4-dimensional quantum theories and 5-dimensional classical theories

It was found that there exist some correspondences between quantum 4-dimensional equations of motion and some classical 5-dimensional equations of motion [18, 19]. We have considered two five-dimensional models: the Randall-Sundrum model [20] and the generalized Dvali-Gabadadze-Porrati (DGP) model [21].

The Randall-Sundrum braneworld model is a 4-dimensional spacetime braneworld embedded into the 5-dimensional anti-de Sitter bulk with the radius \( L \). In the limit of small energy densities the modified quantum Friedmann equations coincide with the modified 4-dimensional Friedmann equations of the Randall-Sundrum model provided

\[ \beta G = \frac{\pi L^2}{2}. \]  

The 5-dimensional action of the generalized DGP model includes the 5-dimensional curvature term, the 5-dimensional cosmological constant and the 4-dimensional curvature term on the brane. If we require the spherical symmetry, when we have the Schwarzschild-de Sitter solution, which depends also on the Schwarzschild radius \( R_S \). The effective 4-dimensional Friedmann equations on the 4-brane coincide with the modified Friedmann equations in quantum model, provided the quantity of the radiation is expressed through the Schwarzschild radius as

\[ C = R_S^2 \]  

If we add the condition of the regularity of the Schwarzschild-de Sitter instanton, (i.e. the condition of the absence of conical singularities), we obtain an additional relation for the parameters of the quantum cosmological model and the set of admissible values for the effective cosmological constant becomes discrete.

The Schwarzschild - de Sitter 5-dimensional instanton is presented on Fig. 6.

\[ \text{Figure 6. A part, } a_- < R < a_+, \text{ of the 5D Schwarzschild-de Sitter Euclidean bulk, scanned by 4D brane with the induced geometry of the } k = 1 \text{ garland instanton. Every point of this cigar instanton represents a 3-sphere of radius } R. \]

11. Conclusion

In the presented series of works it was shown that

(i) The density matrix is a fundamental object.
(ii) Appearance of the density matrix is not a result of our ignorance.
(iii) The universe in the framework of quantum cosmology is born in a mixed and not in a pure quantum state.
There exists some correspondence between the 4-dimensional modified quantum Freidmann equations and the Friedmann equations arising in the context of 5-dimensional classical cosmological models.

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